Strong fusion control and stable equivalences

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Abstract

This article is dedicated to the proof of the following theorem. Let \( G \) be a finite group, \( p \) be a prime number, and \( e \) be a \( p \)-block of \( G \). Assume that the centraliser \( C_G(P) \) of an \( e \)-subpair \( (P, e_P) \) “strongly” controls the fusion of the block \( e \), and that a defect group of \( e \) is either abelian or (for odd \( p \)) has a non-cyclic center. Then there exists a stable equivalence of Morita type between the block algebras \( \mathcal{O}Ge \) and \( \mathcal{O}C_G(P)e_P \), where \( \mathcal{O} \) is a complete discrete valuation ring of residual characteristic \( p \). This stable equivalence is constructed by gluing together a family of local Morita equivalences, which are induced by bimodules with fusion-stable endo-permutation sources.

Broué had previously obtained a similar result for principal blocks, in relation with the search for a modular proof of the odd \( Z^*_p \)-theorem. Thus our theorem points towards a block-theoretic analogue of the \( Z^*_p \)-theorem, which we state in terms of fusion control and Morita equivalences.

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Introduction

Let $G$ be a finite group, $p$ be a prime number, and $P$ be a $p$-subgroup of $G$. Assume that the centraliser $C_G(P)$ controls the $p$-fusion in $G$, i.e., the subgroup $C_G(P)$ contains a Sylow $p$-subgroup $D$ of $G$ and $N_G(Q) \leq C_G(Q)C_G(P)$ for any subgroup $Q$ of $D$. Then the famous $Z^*_p$-theorem asserts that the group $G$ admits the factorisation $G = O_p(G)C_G(P)$, where $O_p(G)$ is the largest normal subgroup of $G$ with order coprime to $p$. This theorem has been proven originally for $p = 2$ by Glauberman [11], and later deduced, for $p$ odd, from the classification of finite simple groups ([2, Theorem 1], [12, Remark 7.8.3]).

Let $\mathcal{O}$ be a complete discrete valuation ring with algebraically closed residue field $k$ of characteristic $p$. It is well-known, and elementary, that the factorisation $G = O_p(G)C_G(P)$ is satisfied if, and only if, the restriction functor $\text{Res}_{C_G(P)}^G$ induces a Morita equivalence between the principal blocks of the groups $G$ and $C_G(P)$ over the ring $\mathcal{O}$. Therefore, the $Z^*_p$ theorem can be stated in terms of Morita equivalences, and one should expect it to admit a “modular proof”, relying on local representation theory. Such a proof is known for $p = 2$ but, as of today, not for odd $p$.

The investigation of a putative minimal counter-example to the $Z^*_p$-theorem leads to a finite group $G$ and a $p$-subgroup $P$ such that the centraliser $C_G(P)$ “strongly” controls the $p$-fusion in $G$, i.e., $C_G(P)$ contains a Sylow $p$-subgroup $D$ of $G$, and $N_G(Q) \leq O_p(C_G(Q))C_G(P)$ for any non-trivial $p$-subgroup $Q$ of $D$. In this context, Broué has proven that the restriction functor $\text{Res}_{C_G(P)}^G$ induces a stable equivalence between the principal blocks of the groups $G$ and $C_G(P)$. His proof relies on the following statement, which appears in [22, Theorem 5.6]: for any two finite groups $G$ and $H$ with the same local structure, a $p$-permutation bimodule that induces Morita equivalences between the principal block algebras of the “local” subgroups of $G$ and $H$ must induce a stable equivalence between the principal block algebras of $G$ and $H$ themselves.

On the one hand, proving the $Z^*_p$-theorem amounts to proving that Broué’s stable equivalence is actually a Morita equivalence. On the other hand, quoting [14], “it seems to be a general intuition that there should be some block-theoretic analogue of Glauberman’s $Z^*$-theorem”. Such an analogue could be stated as follows.

**$Z^*_e$-conjecture.** Let $G$ be a group, $e$ be a block of the group algebra $\mathcal{O}G$, and $(P,e_P)$ be an $e$-subpair of the group $G$. Assume that the centraliser $C_G(P)$ controls the $e$-fusion in $G$ with respect to a maximal $e$-subpair $(D,e_D)$ that contains $(P,e_P)$. Then there exists a Morita equivalence between the block algebras $\mathcal{O}Ge$ and $\mathcal{O}C_G(P)e_P$.

The main theorem of this article is the following generalisation of Broué’s stable equivalence to the context of the above $Z^*_e$-conjecture, for a non-principal block $e$.

**Theorem 1.** Let $G$ be a group, $e$ be a block of the group algebra $\mathcal{O}G$, and $(P,e_P)$ be an $e$-subpair of the group $G$. Assume that the centraliser $C_G(P)$
"strongly controls the e-fusion in G" with respect to a maximal e-subpair \((D,e_D)\) that contains \((P,e_P)\). Assume moreover that the defect group \(D\) is abelian, or that \(p\) is odd and the poset \(A_{\geq 2}(D)\) of elementary abelian subgroups of \(D\) of rank at least 2 is connected (e.g., the center \(Z(D)\) is non-cyclic). Then there exists a stable equivalence of Morita type between the block algebras \(OGe\) and \(OC_G(P)e_P\).

Let us sketch our proof of this theorem. Although we have two blocks \(e\) and \(e_P\) with the same local structure, and a family of Morita equivalences between the local block algebras attached to these blocks, we cannot use the theorem of Rouquier quoted above. Indeed, the bimodules that define those local Morita equivalences are not \(p\)-permutation bimodules; they admit non-trivial endo-permutation sources. Moreover, we have no given bimodule at the “global” level that would induce those local bimodules. Thus we need to construct the global bimodule by a gluing procedure, which roughly follows the method initiated by Puig in [18].

To complete this task, we use the language of Brauer-friendly modules, as defined in [4]. For the reader’s convenience, we gather in the first two sections the definitions and results that are needed in the present article. In Section 3 we specialise these tools to the situation where the centraliser of a \(p\)-subgroup controls the fusion.

With the assumptions of Theorem 1, we have, for any non-trivial subgroup \(Q\) of the defect group \(D\), the following “local” situation: \(e_Q\) is a block of a group \(G_Q\) that factorises as \(G_Q = O_p'(G_Q)C_{G_Q}(P)\) so, by [13], there is a Morita equivalence \(kG_Q e_Q \sim kC_{G_Q}(P) b_{P}(e_Q)\). In Section 4 we prove an equivariant version of this Morita equivalence and give a new construction of the Brauer-friendly module \(M_Q\) that induces it. Moreover, we identify a vertex subpair of \(M_Q\) and provide an explicit description of its source \(V_Q\).

The heart of our proof is the definition of a “global” source \(V\) from the family of local sources \((V_Q)\) for any non-trivial subgroup \(Q\) of the defect group \(D\), which is achieved in Section 5. In the non-abelian defect case, this is an application of the main theorem of [17]; the obstruction group that appears in this theorem explains the technical condition on the defect group \(D\) that we require in Theorem 1. We hope that this technical condition can be lifted in the future. Finally, in Section 6 we consider the unique indecomposable \((OGe,OC_G(P)e_P)\)-bimodule \(M\) with vertex subpair \((\Delta_D,e_D \otimes e_D')\) and source \(V\) such that the slashed module \(M\langle \Delta_P,e_P \otimes e_P' \rangle\) is isomorphic to the block algebra \(kC_G(P)e_P\), and we use the main result of [16] to prove that \(M\) defines a stable equivalence between the blocks \(e\) and \(e_P\).

1 General definitions and notations

We let \(O\) be a complete discrete valuation ring with maximal ideal \(p\) and algebraically closed residue field \(k\) of characteristic \(p\). This includes the case \(O = k\), so that every result that is proven over the ring \(O\) remains true over the field \(k\).

For any finite group \(G\), we denote by \(\Delta G = \{(g,g); g \in G\}\) the diagonal subgroup of the direct product \(G \times G\). We denote by \(O_p'(G)\) the largest normal
subgroup of $G$ with order coprime to $p$. For an element $g \in G$ and an object $X$, the notation $gX$ stands for the object $gXg^{-1}$ whenever this makes sense. Let $e$ be a block of the group $G$, i.e., a primitive central idempotent of the group algebra $\mathcal{O}G$. We denote by $\bar{e} \in kG$ its reduction modulo $p$, and by $e^0$ its image by the isomorphism $(\mathcal{O}G)^{op} \to \mathcal{O}G, g \mapsto g^{-1}$. For any two groups $G$ and $H$, we may consider an $(\mathcal{O}G, \mathcal{O}H)$-bimodule $M$ as an $\mathcal{O}(G \times H)$-module. If $f$ is a block of the group $H$ such that $eMf = M$, then the $\mathcal{O}(G \times H)$-module $M$ belongs to the block $e \otimes f^0$, where we have implicitly identified the algebras $\mathcal{O}(G \times H)$ and $\mathcal{O}G \otimes_{\mathcal{O}H} \mathcal{O}H$ via the natural isomorphism.

Let $G$ be a finite group and $S$ be a normal subgroup of $G$. An $S$-interior $G$-algebra over the ring $\mathcal{O}$ is a triple $(A, \gamma, \iota)$, where $A$ is an $\mathcal{O}$-algebra and $\gamma : G \to \text{Aut}_{\text{Alg}}(A)$, $\iota : S \to A^\times$ are group morphisms such that, for any $s \in S$, $g \in G$ and $a \in A$,

$$
\gamma(s)(a) = \iota(s)a\iota(s)^{-1} \quad \text{and} \quad \iota(gsg^{-1}) = \gamma(g)(\iota(s)).
$$

With these notations, $A$ has a natural structure of $\mathcal{O}(S \times S)\Delta G$-module. Let $H$ a subgroup of $G$, and $T$ be a normal subgroup of $H$ contained in $S$. Let $B$ be a $T$-interior $H$-algebra, hence an $\mathcal{O}(T \times T)\Delta H$-module. Then the induced module $\text{Ind}_{(T \times T)\Delta H}^{(S \times S)\Delta G} B$ has a natural structure of $S$-interior $G$-algebra (cf. [13] for details about partly interior algebras).

For instance, let $T$ be a normal subgroup of a finite group $G$, and $S$ be any normal subgroup of $G$ that contains $T$. Let $b$ be a block of the group $T$, let $H = G_b$ be the stabiliser of $b$ in $G$, and $b' = \text{Tr}_{G_b}(b)$ be the sum of all $G$-conjugates of $b$. Then the block algebra $\mathcal{O}Tb$ is naturally a $T$-interior $G_b$-algebras, via the map $\iota : t \mapsto tb$ and the conjugation action of $G_b$. Moreover the interior structure map $\iota : S \to (\text{Ind}_{(T \times T)\Delta G} \mathcal{O}Tb)^{\times}$ induces an isomorphism of $S$-interior $G$-algebras

$$
\mathcal{O}Sb' \simeq \text{Ind}_{(T \times T)\Delta G}^{(S \times S)\Delta G} \mathcal{O}Tb.
$$

Let $P$ be a $p$-subgroup of a finite group $G$. For any $\mathcal{O}G$-module $M$, we denote by $\text{Br}_P(M)$ the Brauer quotient of $M$, i.e., the $kN_G(P)$-module

$$
\text{Br}_P(M) = M^P / \left( \sum_{Q \not< P} \text{Tr}_Q^P(M^Q) + pM^P \right),
$$

where $M^P$ is the submodule of $P$-fixed points in $M$, $\text{Tr}_Q^P : M^Q \to M^P$ is the relative trace map (as defined, e.g., in [11]). We denote by $\text{br}_P : M^P \to \text{Br}_P(M)$ the projection map. Any morphism of $\mathcal{O}G$-modules $u : L \to M$ induces a morphism of $kN_G(P)$-modules $\text{Br}_P(u) : \text{Br}_P(L) \to \text{Br}_P(M)$. This defines a functor

$$
\text{Br}_P : \mathcal{O}G\text{Mod} \to kN_G(P)\text{Mod}.
$$

Notice that we write the Brauer functor $\text{Br}_P$ with a capital B, and the Brauer map $\text{br}_P$ with a lowercase b. If $A$ is a $G$-interior algebra (e.g., $A = \text{End}_{\mathcal{O}}(M)$ for some $\mathcal{O}G$-module $M$), then the Brauer quotient $\text{Br}_P(A)$ has a natural structure of $C_G(P)$-interior $N_G(P)$-algebra over the field $k$. 

4
2 Brauer-friendly modules and the slash construction

This section gathers definitions and results from [1], [23] and [4]. Notice that the latter reference uses a functorial approach that we do not need here.

Let $G$ be a finite group. The Frobenius category $\text{Fr}(G)$ is defined as follows: an object is a $p$-subgroup; an arrow $\phi : P \to Q$ is a group morphism that is induced by an inner automorphism of the group $G$. Let $P$ be a $p$-subgroup of $G$, and let $V$ be an indecomposable $\mathcal{OP}$-module that is capped, i.e., with vertex $P$. We say that $(P, V)$ is a fusion-stable endo-permutation source pair if, for any $p$-subgroup $Q$ of $G$ and any two arrows $\phi_1, \phi_2 : Q \to P$ in the category $\text{Fr}(G)$, the direct sum $\text{Res}_{\phi_1} V \oplus \text{Res}_{\phi_2} V$ is an endo-permutation $\mathcal{OQ}$-module (i.e., the restrictions $\text{Res}_{\phi_1} V$ and $\text{Res}_{\phi_2} V$ are compatible endo-permutation $\mathcal{OQ}$-modules). Let $M$ be an indecomposable $\mathcal{OG}$-module with vertex $P$ and source $V$. We know from [25, Theorem 1.5] that $M$ is an endo-$p$-permutation $\mathcal{OG}$-module if, and only if, the source pair $(P, V)$ is a fusion-stable endo-permutation source pair.

These ideas admit the following generalisation to blocks. Let $e$ be a block of $G$. A subpair of the group $G$ is a pair $(P, e_P)$, where $P$ is a $p$-subgroup of $G$ and $e_P$ is a block of the group $C_G(P)$. The idempotent $e_P$ is actually a block of the group $H$ whenever $H$ is a subgroup of $G$ such that $C_G(P) \leq H \leq N_G(P, e_P)$. The subpair $(P, e_P)$ is an $e$-subpair if $e_P \text{br}_P(e) \neq 0$, where $\text{br}_P : \mathcal{OG}^P \to kC_G(P)$ denotes the Brauer morphism. Let $(P, e_P)$ and $(Q, e_Q)$ be two $e$-subpairs of $G$. One writes $(P, e_P) \lessdot (Q, e_Q)$ if $(Q, e_Q)$ is an $e_P$-subpair of the group $N_G(P, e_P)$ such that $P \leq Q$. The antisymmetric relation $\lessdot$ generates an order on the set of $e$-subpairs, and the group $G$ acts by conjugation on the resulting poset. The Brauer category $\text{Br}(G, e)$ is defined as follows: an object is an $e$-subpair $(P, e_P)$; an arrow $\phi : (P, e_P) \to (Q, e_Q)$ is a group morphism $\phi : P \to Q$ of the form $x \mapsto \phi x$ for some element $g \in G$ such that $\phi(\phi_P) \leq (Q, e_Q)$. This category is equivalent to a fusion system $\mathcal{F}$ of the block $e$, as defined in [3].

Let $M$ be an indecomposable $\mathcal{OG}e$-module, and $P$ be a vertex of $M$. Let $M'$ be an indecomposable $\mathcal{OG}(P)$-module that is a Green correspondent of $M$, and $f$ be the block of $N_G(P)$ such that $fM \neq 0$. Let $e_P$ be a block of $C_G(P)$ such that $f e_P \neq 0$. The subpair $(P, e_P)$ is called a vertex subpair of the indecomposable module $M$. It follows from Nagao’s theorem that $(P, e_P)$ is an $e$-subpair of the group $G$. Any source $V$ of the indecomposable $\mathcal{OG}(P, e_P)$-module $L = e_P M'$ with respect to the vertex $P$ is called a source of $M$ with respect to the vertex subpair $(P, e_P)$. A source triple $(P, e_P, V)$ of $M$ is well-defined up to conjugation in the group $G$.

Let $(P, e_P)$ be an $e$-subpair of the group $G$, and let $V$ be a capped indecomposable $\mathcal{OP}$-module. We say that $(P, e_P, V)$ is a fusion-stable endo-permutation source triple if, for any $e$-subpair $(Q, e_Q)$ and any two arrows $\phi_1, \phi_2 : (Q, e_Q) \to (P, e_P)$ in the Brauer category $\text{Br}(G, e)$, the direct sum $\text{Res}_{\phi_1} V \oplus \text{Res}_{\phi_2} V$ is an endo-permutation $\mathcal{OQ}$-module. We say that two fusion-stable endo-permutation...
source triples \((P_1, e_1, V_1)\) and \((P_2, e_2, V_2)\) are compatible if, for any \(e\)-subpair \((Q, e_Q)\) and any two arrows \(\phi_1 : (Q, e_Q) \to (P_1, e_1), \phi_2 : (Q, e_Q) \to (P_2, e_2)\) in the Brauer category \(\text{Br}(G, e)\), the direct sum \(\text{Res}_{\phi_1} V_1 \oplus \text{Res}_{\phi_2} V_2\) is an endo-permutation \(OG\)-module.

We say that an \(OG\)-module \(M\) is Brauer-friendly if it is a direct sum of indecomposable \(OG\)-modules with compatible fusion-stable endo-permutation source triples. The following two lemmas are straightforward from \([4, \text{Lemma 9, Theorem 15, and proof of Lemma 18]}\).

**Lemma 2.** Let \(M\) be a Brauer-friendly \(OG\)-module, and \((P, e_P)\) be an \(e\)-subpair of the group \(G\). Any capped indecomposable direct summand of the \(OP\)-module \(e_PM\) is an endo-permutation \(OP\)-module, and there is at most one isomorphism class of such \(OP\)-modules.

**Lemma 3.** Let \(M\) be a Brauer-friendly \(OG\)-module. Let \((P, e_P)\) be an \(e\)-subpair of the group \(G\), and \(H\) be a subgroup of \(G\) such that \(PC_G(P) \trianglelefteq H \trianglelefteq NC_G(P, e_P)\).

(i) There exists a Brauer-friendly \(kH\bar{e}_P\)-module \(M_0\) and an isomorphism of \(C_G(P)\)-interior \(H\)-algebras
\[
\theta_0 : \text{Br}_P(e_P \text{End}_G(M)e_P) \to \text{End}_k(M_0).
\]

(ii) If \((M'_0, \theta'_0)\) is another such pair, then there exists a linear character \(\chi : H/PC_G(P) \to k^\times\) and an isomorphism of \(kH\bar{e}_P\)-modules \(\phi : \chi_* M_0 \to M'_0\) (where \(\chi_* M_0\) means the \(kH\)-module \(M_0\) twisted by \(\chi\)), which induces a commutative diagram

\[
\begin{tikzcd}
\text{Br}_P(e_P \text{End}_G(M)e_P) \ar[r, \theta'_0] \ar[d, \theta_0] & \text{End}_k(M'_0) \\
\text{End}_k(M_0) \ar[r, \phi_* \phi^{-1}] & \text{End}_k(M'_0) \ar[u, \phi \phi^{-1}]
\end{tikzcd}
\]

(iii) If \((Q, e_Q, V)\) is a source triple of an indecomposable direct summand of \(M_0\), then there is a source triple \((Q', e'_Q, V')\) of an indecomposable direct summand of \(M\) such that \((P, e_P) \trianglelefteq (Q, e_Q) \trianglelefteq (Q', e'_Q)\), and that \(V\) is a direct summand of the \(P\)-slashed module \([\text{Res}^Q_{Q'} V'](P)\).

The pair \((M_0, \theta_0)\), or just the \(kH\bar{e}_P\)-module \(M_0\), is called a \((P, e_P)\)-slashed module attached to \(M\) over the group \(H\). We will usually denote it by \(M\langle P, e_P \rangle\). If \(M\) is a \(p\)-permutation module, then there is a canonical choice of \((P, e_P)\)-slashed module attached to \(M\): the Brauer quotient \(\text{Br}_{(P, e_P)}(M) = \text{Br}_P(e_PM)\), together with the natural isomorphism \(\text{Br}_P(e_P \text{End}_G(M)e_P) \simeq \text{End}_k(\text{Br}_P(e_PM))\).

In general, there is no such canonical choice.

Let \(M\) be a Brauer-friendly \(OG\)-module, \((P, e_P) \trianglelefteq (Q, e_Q)\) be two \(e\)-subpairs of \(G\), and \(H, K\) be two subgroups of \(G\) such that \(PC_G(P) \trianglelefteq H \trianglelefteq NC_G(P, e_P)\).
$N_G(P, e_P)$ and $QC_G(Q) \leq K \leq N_H(Q, e_Q)$. Let the pair $(M_0, \theta_0)$ be a $(P, e_P)$-slashed module attached to $M$ over the group $H$, and the pair $(M_1, \theta_1)$ be a $(Q, e_Q)$-slashed module attached to $M_0$ over the group $K$. As appears in the proof of [4, Theorem 19], there is a natural isomorphism $\psi_1 : Br_{PQ}(e_{PQ} End_G(M)e_{PQ}) \to Br_{PQ}(e_{PQ} Br_Q(e_Q M e_Q)e_{PQ})$. Set $\theta'_1 = \theta_1 \circ Br_{PQ}(e_{PQ} \theta_0 e_{PQ}) \circ \psi_1 : Br_{PQ}(e_{PQ} End_G(M)e_{PQ}) \to End_k(M_1)$. The following lemma expresses the transitivity of the slash construction.

**Lemma 4.** With the above notations, the pair $(M_1, \theta'_1)$ is a $(PQ, e_{PQ})$-slashed module attached to $M$ over the group $K$.

The next lemma will allow us to lift certain indecomposable direct summands through the slash construction.

**Lemma 5.** Let $M$ be a Brauer-friendly OG-$e$-module, and $(P, e_P)$ be an $e$-subpair. Let $(M_0, \theta_0)$ be a $(P, e_P)$-slashed module attached to $M$ over the group $N_G(P, e_P)$. If the $kN_G(P, e_P)$-module $M_0$ admits an direct summand with vertex $P$, then the OG-module $M$ admits an indecomposable direct summand with vertex subpair $(P, e_P)$.

**Proof.** Let $X_0$ be an indecomposable direct summand of $M_0$ with vertex $P$. Then there exists a primitive idempotent $i_0$ of the algebra $\text{End}_kN_G(P, e_P)(M_0)$ such that $X_0 = i_0 M_0$, and moreover $i_0$ lies in the ideal $\text{Tr}_{P}^{N_G(P, e_P)}(\text{End}_k(M_0))$.

We consider the projection map $\beta : \text{End}_G(P) \to \text{End}_G(M_0)$, defined by $\beta(u) = \theta_0 \circ \text{br}_P(e_P u e_P)$. For any element $u \in \text{End}_G(M)$, we have

$$\beta \circ \text{Tr}_{P}^{G}(e_P u e_P) = \sum_{g \in N_G(P, e_P) \setminus G/P} \theta_0 \circ \text{br}_P \circ \text{Tr}_{N_S(P, e_P)}^{N_G(P, e_P)}(e_P g u g P e_P)$$

$$= \text{Tr}_{P}^{N_G(P, e_P)} \circ \beta(u)$$

This computation proves that the map $\beta$ sends the ideal $\text{Tr}_{P}^{G}(\text{End}_G(P))$ of the algebra $\text{End}_G(M)$ onto the ideal $\text{Tr}_{P}^{N_G(P, e_P)}(\text{End}_k(M_0))$ of the algebra $\text{End}_kN_G(P, e_P)(M)$. Thus we know from [3] that the primitive idempotent $i_0$ can be lifted through the map $\beta$, i.e., there exists a primitive idempotent $i$ of the algebra $\text{End}_G(M)$ such that $i \in \text{Tr}_{P}^{G}(\text{End}_G(P))$ and $i = \beta(i)$. Then the indecomposable OG-module $X = i M$ is a relatively $P$-projective direct summand of $M$. Moreover, $X$ admits $X_0$ as a $(P, e_P)$-slashed module over the group $N_G(P, e_P)$, so $(P, e_P)$ is a vertex subpair of $X$. \hfill $\square$

### 3 Slashed modules and centrally controlled blocks

Let $e$ be a block of a finite group $G$. We say that a subgroup $H$ of $G$ controls (resp. strongly controls) the $e$-fusion in $G$ with respect to a given maximal subpair $(D, e_D)$ if the defect group $D$ is contained in $H$ and, for any non-trivial $e$-subpair $(Q, e_Q)$ contained in $(D, e_D)$,

$$N_G(Q, e_Q) \leq C_G(Q) H \quad \text{(resp. } N_G(Q, e_Q) \leq O_{p'}(C_G(Q)) H \text{)}.$$
If \( H = C_G(P) \) is the centraliser of an \( e \)-subpair \((P, e_P)\) contained in \((D, e_D)\), then both conditions imply that the Brauer categories \( \text{Br}(G, e) \) and \( \text{Br}(C_G(P), e_P) \) are equivalent and, in particular, that \( N_G(P, e_P) = C_G(P) \).

In the next two lemmas, we assume that \( e \) is a block of a finite group \( G \), and that \((P, e_P)\) be an \( e \)-subpair of \( G \) such that the centraliser \( C_G(P) \) controls the \( e \)-fusion in \( G \) with respect to a maximal subpair \((D, e_D)\) that contains \((P, e_P)\). In this context of centrally controlled blocks, the ambiguity of the definition of slashed modules in the previous section can be lifted for well-behaved Brauer-friendly modules.

**Lemma 6.** Let \((Q, e_Q)\) be a subpair of \((D, e_D)\), and \( H \) be a subgroup of \( G \) such that \( QC_G(Q) \leqslant H \leqslant N_G(Q, e_Q) \). Denote by \( e_{PQ} \) the unique block of the group \( C_G(PQ) \) such that \((PQ, e_{PQ}) \leqslant (D, e_D)\). Let \( M \) be a Brauer-friendly \( \mathcal{O}G_e \)-module. Assume that the slashed module \( M(P, e_P) \) is a \( p \)-permutation \( kC_G(P)e_{PQ} \)-module, and that a slashed module \( M(PQ, e_{PQ}) \) is non-zero. Then there exists a unique isomorphism class of \((Q, e_Q)\)-slashed module \( M(Q, e_Q) \) over the group \( H \) such that

\[
M(Q, e_Q)(PQ, e_{PQ}) \simeq \text{Br}(PQ, e_{PQ})(M(P, e_P)).
\]

**Proof.** Let the pair \((M_0, \theta_0)\) be any \((Q, e_Q)\)-slashed module over the group \( H \) attached to \( M \). Let the pair \((M_1, \theta_1)\) be a \((P, e_{PQ})\)-slashed module attached to \( M_0 \). This slashed module is defined over the centraliser \( C_H(P) = N_H(P, e_{PQ}) \), hence it is uniquely defined up to isomorphism. Let the pair \((M_2, \theta_2)\) be a \((P, e_P)\)-slashed module attached to \( M \). Similarly, this slashed module is uniquely defined over the centraliser \( C_G(P) \). Set \( M_3 = \text{Br}(PQ, e_{PQ})(M_2) \), restricted to a \( kC_H(P) \)-module, and let \( \theta_3 : \text{Br}_P\mathcal{O}(\mathcal{O}PQ)(M_2) \simeq \text{End}_K(M_3) \) be the natural isomorphism.

As in the discussion before Lemma 4, define from \( \theta_0 \) an \( \theta_1 \) a map \( \theta_1' \) such that the pair \((M_1, \theta_1')\) is a \((PQ, e_{PQ})\)-slashed module attached to \( M \) over the group \( C_G(P) \). Define similarly from \( \theta_2 \) and \( \theta_3 \) an isomorphism \( \theta_3' \) such that the the pair \((M_3, \theta_3')\) is a \((PQ, e_{PQ})\)-slashed module attached to \( M \) over the group \( C_G(P) \).

By Lemma 3 (ii), there exists a linear character \( \chi : C_H(P)/C_G(PQ) \to k^\times \) and an isomorphism of slashed modules \( \chi^{}(M_1, \theta_1') \simeq (M_3, \theta_3') \). By control of fusion, the inclusion map \( C_H(P) \to H \) induces an isomorphism \( C_H(P)/C_G(P) \simeq H/C_G(Q) \). Thus twisting the non-zero \( kC_H(P)e_{PQ} \)-module \( M_1 \) by a linear character of the group \( C_H(P)/C_G(P) \) amounts to twisting the \( kH\mathcal{O}Q \)-module \( M_0 \) by a linear character of the group \( H/C_G(Q) \). This proves the existence and uniqueness, up to isomorphism, of a pair \((M_0, \theta_0)\) such that the corresponding \((PQ, e_{PQ})\)-slashed module \((M_1, \theta_1')\) is isomorphic to \((M_3, \theta_3') \). \( \square \)

In the next lemma, the normal subgroup \( H \) could be the centraliser of a normal \( e \)-subpair of \( G \), or just \( G \) itself.

**Lemma 7.** Let \( V \) be a capped indecomposable endo-permutation \( \mathcal{O}D \)-module such that the class \( \text{Defres}^P_{D/P}[V] \) in the Dade group \( D(D/P) \) is trivial. Assume
that the triple \((D, e_D, V)\) is fusion-stable in \(G\), and identify \(V\) to an \(O\Delta D\)-module through the diagonal isomorphism. Let \(H\) be a normal subgroup of \(G\) such that, for any subpair \((Q, e_Q) \leq (D, e_D)\), the idempotent \(e_Q\) lies in the subalgebra \(kC_H(Q)\) of \(kC_G(Q)\). Then, up to isomorphism, there is a unique Brauer-friendly \(O(H \times C_H(P))\Delta C_G(P)\)-module \(M\) with source triple \((\Delta D, e_D \otimes e_P^o, V)\) such that a slashed module \(M(\Delta P, e_P \otimes e_P^o)\) admits a direct summand isomorphic to the \(k(C_H(P) \times C_H(P))\Delta C_G(P)\)-module \(kC_H(P)e_P\).

**Proof.** By Lemma \(3\) (iii), whenever \(M\) is an indecomposable Brauer-friendly \(k(H \times C_H(P))\Delta G\)-module with source triple \((\Delta D, e_D \otimes e_P^o, V)\), the slashed module \(M(\Delta P, e_P \otimes e_P^o)\) is a \(p\)-permutation module. Thus we can set

\[
M'\langle \Delta D, e_D \otimes e_P^o \rangle = Br(\Delta D, e_D \otimes e_P^o)(M'\langle \Delta P \rangle),
\]

in accordance with Lemma \(3\). Once this specific slash construction has been chosen, we know from \([4, \text{Theorem 20}]\) that the mapping \(M \mapsto M(\Delta D, e_D \otimes e_P^o)\) induces a one-to-one correspondence between the isomorphism classes of indecomposable Brauer-friendly \(k(H \times C_H(P))\Delta G\)-modules with source triple \((\Delta D, e_D \otimes e_P^o, V)\) and the isomorphism classes of indecomposable Brauer-friendly \(k(C_H(D) \times C_H(D))\Delta N_G(D, e_D)\)-modules with source triple \((\Delta D, e_D \otimes e_P^o, k)\). The same correspondence, applied to \(k(C_H(P) \times C_H(P))\Delta C_G(P)\)-modules with source triple \((\Delta D, e_D \otimes e_P^o, k)\), implies that the slashed module \(M(\Delta P, e_P \otimes e_P^o)\) admits \(kC_H(P)e_P\) as a direct summand if, and only if, \(M(\Delta D, e_D \otimes e_P^o) \simeq kC_H(D)e_D\). This proves the lemma.

\(\Box\)

### 4 Understanding the local situation

In this section, we work directly over the residue field \(k\), i.e., we set \(O = k\). We explore an equivariant version of Morita equivalences, the existence of which is proven in \([15]\). Those are the building blocks that we will glue together to obtain a stable equivalence in the Section \(6\).

Let us fix a few notations that will hold throughout the present section. Let \(P\) be a \(p\)-subgroup of a finite group \(G\), and \(e\) be a block of \(G\) such that \(br_P(e) \neq 0\). We choose, once and for all, a maximal \(e\)-subpair \((D, e_D)\) such that \(P \leq D\). For any subgroup \(Q\) of \(D\), we denote by \(e_Q\) the unique block of \(C_G(Q)\) such that \((Q, e_Q) \leq (D, e_D)\). Let \(H\) be a normal subgroup of \(G\) such that, for any subgroup \(Q\) of \(D\), the block \(e_Q\) of the group algebra \(kC_G(Q)\) lies in the subalgebra \(kC_H(Q)\) (for instance, \(H\) may be the centraliser of a normal \(e\)-subpair of \(G\)). Assume that

\[
G = O_p'(H) C_G(P).
\]

By elementary group theory, this factorisation implies that \(P\) is an abelian \(p\)-group, and that the centraliser \(C_G(P)\) controls the \(p\)-fusion in the group \(G\). In particular, we have \(N_G(D) \leq C_G(P)\). By Brauer’s first main theorem, it follows that the idempotent \(e_P = br_P(e)\) is a block of the group \(C_G(P)\). Thus \((P, e_P) \leq (D, e_D)\) is an \(e\)-subpair of the group \(G\), and the centraliser \(C_G(P)\)
controls the e-fusion with respect to the maximal subpair \((D, e_D)\). The main result of this section is the following.

**Theorem 8.** With the above notations,

(i) Let \(S\) be any \(p'\)-subgroup of \(H\) such that \(S \trianglelefteq G\) and \(G = SC_G(P)\); let \(b\) be any \(D\)-stable block of the group \(S\) such that \(e_D br_D(b) \neq 0\). The \(\Delta D\)-algebra \(kSb \otimes kC_S(P) br_P(b)^e\) defines a class \(v\) in the Dade group \(D(\Delta D)\) that is independent of the choice of \(S\) and \(b\).

(ii) Let \(V\) be a capped indecomposable endo-permutation \(k\Delta D\)-module that belongs to the class \(v\). The source triple \((\Delta D, e_D \otimes e_{\Delta D}^S, V)\) is fusion-stable in the group \((H \times C_H(P))\Delta C_G(P)\).

(iii) There exists a unique indecomposable \(k(\Delta D, e_D \otimes e_{\Delta D}^S, V)\)-module \(M\) with source triple \((\Delta D, e_D \otimes e_{\Delta D}^S, V)\) such that the slashed module \(M(\Delta P)\) is isomorphic to the \(k(C_H(P) \times C_H(P))\Delta C_G(P)\)-module \(kc_H(P)e_P\).

(iv) The module \(M\) induces a \(G/H\)-equivariant Morita equivalence \(kHe \sim kC_H(P) e_P\).

We reach the proof of this theorem through a series of lemmas. We choose, once and for all, a normal \(p'\)-subgroup \(S\) of \(H\) such that \(S \trianglelefteq G\) and \(G = SC_G(P)\). For instance, we could choose \(S = O_{p'}(H)\). Let \(b_D\) be a block of the algebra \(kC_S(D)\) such that \(e_D b_D \neq 0\), i.e., the block \(e_D\) of \(C_S(D)\) covers the block \(b_D\) of the normal subgroup \(C_S(D)\) (as defined in [9]).

We may consider \((D, b_D)\) as a maximal subpair of the nilpotent group \(SD\). Let \(b\) be the block of \(SD\) such that \((D, e_D)\) is a maximal \(b\)-subpair. In other words, \(b\) is a \(D\)-stable block of the \(p'\)-group \(S\), and \(b_D = br_D(b)\). Similarly, for any subgroup \(Q\) of \(D\), the Brauer map \(br_Q\) defines a one-to-one correspondence between the set of \(Q\)-stable blocks of \(S\) and the set of all blocks of \(C_S(Q)\), by Brauer’s first main theorem. In particular, the idempotent \(b_Q = br_Q(b)\) is a block of the group \(C_S(Q)\).

**Lemma 9.** For any subpair \((Q, e_Q) \trianglelefteq (D, e_D)\), the block \(e_Q\) of the group \(C_S(Q)\) covers the block \(b_Q = br_Q(b)\) of the normal subgroup \(C_S(Q)\). In particular, the block \(e\) covers the block \(b\).

**Proof.** By construction, the block \(e_D\) covers the block \(b_D\). We use descending induction to generalise this to any subpair of \((D, e_D)\). Let \((Q, e_Q)\) be a proper subpair of \((D, e_D)\). We assume that, for any subpair \((R, e_R)\) with \((Q, e_Q) < (R, e_R) \trianglelefteq (D, e_D)\), the block \(e_R\) of the group \(C_G(R)\) covers the block \(b_R\) of the normal subgroup \(C_S(R)\). Then we let \((R, e_R)\) be the normaliser subpair of \(Q\) in \((D, e_D)\), which strictly contains \((Q, e_Q)\). Thus we have \(e_R br_R(e_Q) = e_R\) and, by induction, \(e_R b_R \neq 0\). These imply \(br_R(e_Q)b_R \neq 0\), hence \(br_R(e_Q) = br_R(e_Q)b_R \neq 0\). Thus we obtain \(e_Q b_Q \neq 0\) and the block \(e_Q\) covers the block \(b_Q\). This completes the induction step. \(\Box\)
We denote by $G_b$ and $H_b$ the stabilisers of the block $b$ in the groups $G$ and $H$ respectively, and by $b' = \Tr_{T_b}^G(b)$ the sum of all $G$-conjugates of $b$. Then $eb' = e$ and $e \in kHB'$; moreover $e_b = eb$ is a block of the algebra $kG_b$, it lies in the subalgebra $kH_b$, and the $(kHe, kHe_b)$-bimodule $kHe_b$ induces a $(G_b/H_b)$-equivariant) Morita equivalence $kHe \sim kHb$ as is proven for example in [13].

**Lemma 10.** The subgroup $C_{G_b}(P)$ controls the e-fusion in $G$ with respect to the maximal subpair $(D, e_D)$.

**Proof.** Let $(Q, e_Q)$ be a subpair of $(D, e_D)$ and let $g \in G$ be such that $g(Q, e_Q) \leq (D, e_D)$. The centraliser $C_G(P)$ controls the $p$-fusion in $G$, so we may suppose $g \in C_G(P)$. Then we obtain $g(Q, e_Q) \leq (D, e_D)$, so we may suppose $P \leq Q$. The inclusion $g(Q, e_Q) \leq (D, e_D)$ implies $ge_Q = e_Q$. So the block $gQ$ of $C_G(Q)$ covers the block $bQ$ of $C_S(Q)$, and the block $e_Q$ of $C_G(Q)$ covers the block $g^{-1}bQ$ of $C_S(Q)$. As the blocks $bQ$ and $g^{-1}bQ$ of $C_S(Q)$ are covered by the same block $e_Q$ of $C_G(Q)$, they must be conjugate in $C_G(Q)$: there exists $k \in C_G(Q)$ such that $bQ = kg^{-1}bQ$. Then we get

$$br_Q(b) = bQ = kg^{-1}bQ = kg^{-1}br_Q(b) = br_Q(kg^{-1}b).$$

As we have already mentioned, the correspondence $b \leftrightarrow br_Q(b)$ is one-to-one, so we obtain $b = kg^{-1}b$. Hence the element $h = gk^{-1}$ lies in $G_b$. Notice that $g$ and $k$ both centralise the $p$-group $P$ by assumption, so we have $h \in C_{G_b}(P)$, and $g \in C_G(Q)C_{G_b}(P)$. Thus $N_G(Q, e_Q) \leq C_G(Q)C_{G_b}(P)$.

For any element $x \in P$, we denote by $K_x \in kG$ the class sum of $x$, i.e., $K_x = \Tr_{C_G(x)}^G(x)$ (see [14] for a definition of the relative trace map). We have supposed $G = SC_G(P)$ hence $G = SC_G(x)$. So, for any subgroup $T$ of $G$ that contains $S$, the natural map $T/C_T(x) \rightarrow G/C_T(x)$ is a bijection and $K_x = \Tr_{C_T(x)}^G(x)$. We have in particular $K_x = \Tr_{C_S(x)}^G(x)$. Since the subgroup $S$ is normal in $G$, it follows that the class sum $K_x$ lies in the subset $kSx = xkS$ of the algebra $kG$, and that the element $xK_x$ lies in $kS$.

In [20] and [21], Robinson makes great use of the central unit $K_x e \in kGe$ to deal, respectively, with the situation $G = O_p(G)C_G(P)$ and with a minimal counter-example to the odd $Z_p^*$-theorem. The following lemmas highlight once again the importance of the class sum $K_x$. In order to deal efficiently with it, we need more notations.

We will consider the group $G$ as a subgroup of the direct product $G \times P$, via the embedding $g \mapsto (g, 1)$. Since the $p$-group $P$ is abelian, we can consider the $p$-subgroup $P_1 = \{(x, x^{-1}) \colon x \in P\}$ of $G \times P$. For an element $x \in P$, we will usually write $x_1 = (x, x^{-1}) \in P_1$; conversely, for an element $x_1 \in P_1$, we will write $x$ for the unique element of $P$ such that $x_1 = (x, x^{-1})$. We will see the group $G \times P$ as the semi-direct product $GP_1$. Notice that any subgroup of $G$ that is normalised by $P$ is also normalised by $P_1$. Thus we can consider the subgroups $HP_1, SP_1, GbP_1, etc$. The group $P_1$ centralises the defect group $D$ and the blocks $e, e_D, b, etc.$
Lemma 11. (i) The maps \( \iota_b : SP_1 \to (kSb)^x \) and \( \gamma_b : GP_1 \to \text{Aut}_{\text{Alg}}(kSb) \) defined by
\[
\iota_b(sx_1) = sxK_{x^{-1}}b \quad ; \quad \gamma_b(gx_1)(a) = (gx)^{-1}a(gx)
\]
for \( s \in S \), \( g \in G_b \), \( x_1 \in P_1 \), \( a \in kSb \), make \( kSb \) an \( SP_1 \)-interior \( GP_1 \)-algebra.

(ii) The maps \( \iota'_b : HP_1 \to (kHb')^x \) and \( \gamma'_b : GP_1 \to \text{Aut}_{\text{Alg}}(kHb') \) defined by
\[
\iota'_b(hx_1) = hxK_{x^{-1}}b' \quad ; \quad \gamma'_b(gx_1)(a) = (gx)^{-1}a(gx)
\]
for \( h \in H \), \( g \in G \), \( x_1 \in P_1 \), \( a \in kHb' \), make \( kHb' \) an \( HP_1 \)-interior \( GP_1 \)-algebra.

(iii) The maps \( \iota_e : HP_1 \to (kHe)^x \) and \( \gamma_e : GP_1 \to \text{Aut}_{\text{Alg}}(kHe) \) defined by
\[
\iota_e(hx_1) = hxK_{x^{-1}}e \quad ; \quad \gamma_e(gx_1)(a) = (gx)^{-1}a(gx)
\]
for \( h \in H \), \( g \in G \), \( x_1 \in P_1 \), \( a \in kHe \), make \( kHe \) an \( HP_1 \)-interior \( GP_1 \)-algebra.

Proof. We consider the idempotents \( b \) and \( b_P = br_P(b) \) as respective blocks of the \( p \)-nilpotent groups \( SP \) and \( C_{\mathcal{S}}(P)P \), both with defect group \( P \). It is well known that the block algebras \( kSPb \) and \( kC_{\mathcal{S}}(P)Pb_P \) are both Morita equivalent to \( kP \), so the centers \( Z(kSPb) \) and \( Z(kC_{\mathcal{S}}(P)Pb_P) \) are both isomorphic to \( Z(kP) = kP \); in particular, they have the same dimension. The Brauer map \( br_P \) induces an algebra morphism \( \beta : Z(kSPb) \to Z(kC_{\mathcal{S}}(P)Pb_P) \). We have \( kPb_P \subset Z(kC_{\mathcal{S}}(P)Pb_P) \). Moreover the natural map \( kC_{\mathcal{S}}(P) \otimes kP \to kC_{\mathcal{S}}(P)P \) is an isomorphism, so \( \dim_k(kPb_P) = |P| = \dim_k Z(kC_{\mathcal{S}}(P)Pb_P) \). Hence \( Z(kC_{\mathcal{S}}(P)Pb_P) = kPb_P \).

Let \( x \) in \( P \) be fixed. Since \( C_{\mathcal{G}}(P) \) controls the \( p \)-fusion, no proper conjugate of \( x \) lies in \( C_{\mathcal{G}}(P) \). So \( \beta(K_\chi b) = br_P(K_\chi)br_P(b) = xPb \), which proves that the morphism \( \beta \) is onto. Since its domain and codomain have the same dimension over \( k \), \( \beta \) is an isomorphism. Thus the element \( K_\chi b = \beta^{-1}(xb_P) \) is invertible in \( Z(kSPb) \) and the map \( x \mapsto K_\chi b \) is a group morphism \( P \to Z(kSPb)^x \).

Moreover the group \( P \) is abelian, so the map \( \iota_b : SP_1 \to (kSb)^x \) of (i) is indeed well-defined and a group morphism. The rest of the statement in (i) is straightforward.

Furthermore, the algebra \( kSb' \) is the direct product of \( kSc \) where \( c \) runs over the set of \( G \)-conjugates of \( b \), and \( xK_{x^{-1}}b' = \sum_c xK_{x^{-1}}c \) for any \( x \in P \). So \( xK_{x^{-1}}b' \) is invertible in \( kSb' \) and the map \( x_1 \mapsto xK_{x^{-1}}b' \) is a group morphism \( P_1 \to (kSb')^x \), which extends to the group morphism \( \iota'_b : HP_1 \to (kHb')^x \) of (ii). Notice that cutting off the central idempotent \( c \) cannot harm, so (iii) follows immediately. \( \square \)

Lemma 12. There is a natural isomorphism of \( HP_1 \)-interior \( GP_1 \)-algebras
\[
\phi : kHb' \xrightarrow{\sim} \text{Ind}_{(SP_1 \times SP_1) \Delta G_b} (SP_1 \times SP_1) \Delta G_b kSb.
\]
Proof. Let us write $A = \text{Ind}_{(S \times S)}^{(G \times G)} kSb$. On the one hand, we have from Section 1 an isomorphism of $H$-interior $G$-algebras $kHb' \to \text{Ind}_{(G \times G)}^{(S \times S)} kSb$. On the other hand, the natural map $G/S \to GP_1/S$ is bijective so the natural map $\text{Ind}_{(S \times S)}^{(G \times G)} kSb \to \text{Ind}_{(G \times G)}^{(S \times S)} kSb$ is an isomorphism of $H$-interior $G$-algebras. By composition, we obtain the map $\phi : kHb' \to A$, which appears to be an isomorphism of $H$-interior $G$-algebras. By definition, we have $\iota_{h'}(x_1) = xK_{x'}b'$, where the element $xK_{x'}$ lies in $S$. Since $\phi$ is an isomorphism of left $kS$-modules, we obtain $\phi(\iota_{h'}(x_1)) = xK_{x'} - 1_{A}$. Then it follows from the definition of induced interior algebras that $\iota_{A}(x_1) = xK_{x'} - 1_{A}$. Thus $\phi$ is also an isomorphism of $P_1$-interior algebras, and the lemma is proven. \hfill ∎

We now consider the $HP_1$-interior $GP_1$-algebra $kHe$ as a $k(HP_1 \times HP_1)\Delta G$-module.

Lemma 13. The indecomposable $k(HP_1 \times HP_1)\Delta G$-module $kHe$ is Brauer-friendly with source triple $((P_1 \times P_1)\Delta D, eD \otimes eD, W)$, where the source $W$ is any capped indecomposable direct summand of the restriction $\text{Res}_{P_1 \times P_1}^{G \times G}(kSb)$. Proof. Let us write $K = (SP_1 \times SP_1)\Delta Gb$. The field $k$ is algebraically closed and $S$ is a $p'$-group, so the block algebra $kSb$ is a matrix algebra. It follows that the structure map of the $kSb$-bimodule $kSb$ is an isomorphism of $K$-algebras $kSb \otimes kSb^0 \simeq \text{End}_K(kSb)$. In particular, this proves that $kSb$ is an endo-$p$-permutation $kK$-module.

For any element $(g, h)$ of the group $(H_kP_1 \times H_kP_1)\Delta Gb$, let $R$ be a Sylow $p$-subgroup of the intersection $K \cap (g, h) K$. The $(kS \times S)R$-module $\text{Res}_{(kS \times S)R}^{K} kSb$ is simple and belongs to the block $b \otimes b^0$ of the $p$-nilpotent group $(S \times S)R$. Since the pair $(g, h)$ stabilises the block $b \otimes b^0$, the $(kS \times S)R$-module $\text{Res}_{(S \times S)R}^{K} g(kSb)h^{-1}$ is simple and belongs to the same block $b \otimes b^0$. Since a block of a $p$-nilpotent group contains only one isomorphism class of simple modules, there must be an isomorphism of $(kS \times S)R$-modules

$$\text{Res}_{(S \times S)R}^{K} kSb \simeq \text{Res}_{(S \times S)R}^{(g, h)K} g(kSb)h^{-1}.$$  

It follows that the restrictions $\text{Res}_{(g, h)K}^{K} kSb$ and $\text{Res}_{(g, h)K}^{K} g(kSb)h^{-1}$ are compatible endo-$p$-permutation $k(K \cap (g, h) K)$-modules. By Ulrich's criterion [25, Lemma 1.3] for the induction of endo-$p$-permutation modules, we deduce that the $k(H_kP_1 \times H_kP_1)\Delta Gb$-module

$$kHb \simeq \text{Ind}_{(S \times S)}^{(G \times G)} kSb$$

is an endo-$p$-permutation module. Then its direct summand $kHb e_b$ is also an endo-$p$-permutation $k(H_kP_1 \times H_kP_1)\Delta Gb$-module. We now determine a vertex subpair of this indecomposable module. The commutation of induction and the Brauer functor brings an isomorphism of $(C_{H_k}(P_1) \times C_{H_k}(P_1)\Delta C_{Gb}(P))$-interior algebras

$$\text{Br}_{P_1 \times P_1} (\text{End}_k(kHb)) \to \text{Ind}_{(C_{H_k}(P_1) \times C_{H_k}(P_1)\Delta C_{Gb}(P))}^{(C_{H_k}(P_1) \times C_{H_k}(P_1)\Delta C_{Gb}(P))} \text{Br}_{P_1 \times P_1} (\text{End}_k(kSb)).$$
The natural isomorphism of \((C_S(P))P_1 \times C_S(P)P_1\) \(\Delta C_{G_h}(P)\)-interior algebras
\(\text{Br}_{P_1 \times P_1} (\text{End}_k(kSb)) \simeq \text{End}_k(kC_S(P)b_P)\) and the isomorphism of Lemma 12
then bring an isomorphism of \((C_{H_b}(P)P_1 \times C_{H_b}(P)P_1)\) \(\Delta C_{G_b}(P)\)-interior algebras
\[\text{Br}_{P_1 \times P_1} (\text{End}_k(kH_b b)) \simeq \text{End}_k(kC_{H_b}(P)b_P).\]

It follows that the slashed module \(kH_b e_b(P_1 \times P_1)\) is isomorphic to \(kC_{H_b}(P)b_P e_b(P_1)\).
A vertex of the indecomposable \(k(H_bP_1 \times H_bP_1)\) \(\Delta G_b\)-module \(kH_b e_b\) contains the \(p\)-group \(P_1 \times P_1\). Since \(kC_{H_b}(P)b_P e_b(P_1)\) is a \(p\)-permutation \(kC_{H_b}(P) \times C_{H_b}(P))\) \(\Delta G_b\)(-module, the slash construction may coincide with the Brauer functor from this point on. The images of the block algebra \(kC_{H_b}(P)b_P e_b(P_1)\) by Brauer functors are well-known, so we can use the transitivity of the slash construction for \(p\)-permutation modules, and conclude that a vertex subpair of \(kH_b e_b\) is \(((P_1 \times P_1) 1, \Delta D, e_D 0, e_{D}^0)\).

Since the indecomposable \(k(H_bP_1 \times H_bP_1)\) \(\Delta G_b\)-module \(kH_b e_b\) is a direct summand of the \(\Delta G\)-module \(\text{Ind}_1^{(H_bP_1 \times H_bP_1)\Delta G_b kSb}\), a source \(W\) of \(kH_b e_b\) with respect to the above vertex subpair is isomorphic to any capped indecomposable direct summand of the restriction \(\text{Res}_1^{(P_1 \times P_1)\Delta D} kSb\). As a consequence of the vertex-preserving Morita equivalence of [13, Theorem 1.6], the induced module
\[kH e \simeq \text{Ind}_1^{(H_bP_1 \times H_bP_1)\Delta G} kH_b e_b\]
is indecomposable and admits the source triple \(((P_1 \times P_1) \Delta D, e_D 0, e_{D}^0, W)\).
Moreover, \(kH_b e_b\) is an endo-\(\Delta G\)-permutation module, so the endo-\(\Delta G\)-permutation source pair \(((P_1 \times P_1) \Delta D, W)\) is fusion-stable in the group \((H_bP_1 \times H_bP_1)\) \(\Delta G_b\).

By Lemma 10, the subgroup \((H_bP_1 \times H_bP_1)\) \(\Delta G_b\) controls the \(e \otimes e^0\)-fusion in the group \((HP_1 \times HP_1)\) \(\Delta G\). Thus the source triple \(((P_1 \times P_1) \Delta D, e_D 0, e_{D}^0, W)\) is fusion-stable in the group \((HP_1 \times HP_1)\) \(\Delta G\), and the indecomposable module \(kH e\) is Brauer-friendly.

Let \(M = kH e(1 \times P_1, e \otimes e^0_P)\) be a slashed module attached to the Brauer-friendly \(k(HP_1 \times HP_1)\) \(\Delta G\)-module \(kH e\). Remember that \(N_G(P, e_P) = C_G(P)\), so the slash construction is unambiguous as long as only the \(p\)-groups \(P\) and \(P_1\) are concerned. Since \(e_P = br_P(e)\), we may also omit the blocks in subpairs concerned only with \(P\) and \(P_1\). For instance, we may write \(M = kH e(1 \times P_1)\). From now on, we will consider \(M\) as a \((H \times C_H(P))\) \(\Delta C_G(P)\)-module, thus forgetting the remaining left action of \(P_1\).

**Lemma 14.** The \((H \times C_H(P))\) \(\Delta C_G(P)\)-module \(M\) induces a \(G/H\)-equivariant Morita equivalence \(kH e \sim kC_H(P)e_P\).

**Proof.** We apply the slash construction to the \((SP_1 \times SP_1)\) \(\Delta G_b\)-module \(kSb\) and the \((HP_1 \times HP_1)\) \(\Delta G\)-module \(kH b\) to define a \((S \times C_S(P))\) \(\Delta G_b\)-module \(L = kSb(1 \times P_1)\) and a \((H \times C_H(P))\) \(\Delta C_G(P)\)-module \(L' = kH b(1 \times P_1)\).

We know from Lemma 12 that there is an isomorphism of \((HP_1 \times HP_1)\) \(\Delta G\)-modules \(kH b' \simeq \text{Ind}_1^{(HP_1 \times HP_1)\Delta G} kSb\). Moreover \(G = SC_G(P)\), so the commutation of induction and the Brauer functor brings an isomorphism of \((HP_1 \times (H \times C_H(P))\) \(\Delta C_G(P)\)-module
Then the non-zero direct summand \( kSb \) interior algebras.

\[
\text{Ind}_{\left( H P_1 \times C_B(P) P_1 \right)} \Delta C_G(P)^2 \otimes \text{Br}_{1 \times P_1} \text{End}_k(kSb)
\]

\[
\rightarrow \text{Br}_{1 \times P_1} \circ \text{Ind}_{\left( H P_1 \times H P_1 \right)} \Delta G_2^2 \text{End}_k(kSb).
\]

Notice that the \( p \)-subgroup \( P_1 \) can be omitted from the induction functors without changing the result. Thus we have an isomorphism of \( k(H \times C_B(P)) \Delta C_G(P) \)-modules

\[
\text{Ind}_{\left( H \times C_B(P) \right)} \Delta C_G(P)^2 \otimes kSb \sim L'.
\]

Then we look closer at the definition of \( L \). Since \( kSb \) is a matrix algebra, the structure map of the \( (kSb, kSb) \)-bimodule \( kSb \) is an isomorphism of \( (S \times S) \)-interior algebras \( kSb \otimes (kSb)^{op} \rightarrow \text{End}_k(kSb) \). Applying the Brauer functor \( \text{Br}_{1 \times P_1} \) turns this into an isomorphism of \( (S \times C_B(P)) \)-interior algebras \( kSb \otimes (kC_S(P)b_P)^{op} \rightarrow \text{End}_k(L) \). So we have an isomorphism of \( k(S \times C_B(P)) \)-modules \( L \simeq X \otimes Y^* \), where \( X \) is a simple module for the matrix algebra \( kSb \) and \( Y^* \) is the \( k \)-dual of a simple module for the matrix algebra \( kC_S(P)b_P \).

We deduce that the \( (S \times C_B(P)) \)-module \( L \) induces a Morita equivalence \( kSb \sim kC_S(P)b_P \). By [17, Theorem 3.4] and [13, Theorem 1.6], it follows that the induced module \( L' \) induces a Morita equivalence \( kHb' \sim kC_H(P)b_P \). Then the non-zero direct summand \( eL'e_P \) induces a Morita equivalence \( kHe \sim kC_H(P)e_P \).

**Lemma 15.** The indecomposable \( k(H \times C_B(P)) \Delta C_G(P) \)-module \( M \) is Brauer-friendly with source triple \( (\Delta D, e_D \otimes e'_D, V) \), where the endo-permutation \( k\Delta D \)-module \( V \) belongs to the class of the Dade group \( D(\Delta D) \) that is defined by the Dade \( \Delta D \)-algebra \( kSb \otimes kC_S(P)b_P^{\delta} \).

**Proof.** The indecomposable \( k(H P_1 \times H P_1) \Delta G \)-module \( kHe \) is Brauer-friendly with source triple \( ((P_1 \times P_1) \Delta D, e_D \otimes e'_D, W) \). We know from Lemma 3 that the slashed module \( M \) is Brauer-friendly. For any subpair \((R,f)\) of the maximal \( e \otimes e'_D \)-subpair \((\Delta D, e_D \otimes e'_D)\) in the group \((H \times C_B(P)) \Delta C_G(P)\), the transitivity of the slash construction shows that an \((R,f)\)-slashed module \( M(R,f) \) attached to \( M \) is also a \((1 \times P_1)R, \text{br}_{1 \times P_1}(f)\)-slashed module attached to \( kHe \). It follows that \( M(R,f) \) is non-zero if, and only if, the subpair \((R,f)\) is contained in \((\Delta D, e_D \otimes e'_D)\) up to conjugation. Thus \( (\Delta D, e_D \otimes e'_D) \) is a vertex subpair of \( M \).

Let \( V \) be the source of \( M \) with respect to the above vertex subpair. By Lemma 3 (iii), the endo-permutation \( k\Delta D \)-module \( V \) is compatible with the slashed module \( \text{Res}_{\Delta D}^{P_1 \times P_1} W \langle 1 \times P_1 \rangle \). Moreover, we know from Lemma 13 that \( W \) is a capped indecomposable direct summand of the \( k\Delta D \)-module \( kSb \). We have \( \text{End}_k(kSb) \simeq kSb \otimes kSb^* \), so \( \text{Br}_{1 \times P_1}(\text{End}_k(kSb)) \simeq kSb \otimes kC_S(P)b_P^{\delta} \).

Thus the \( k\Delta D \)-module \( V \) is isomorphic to a direct summand of a simple module for the matrix algebra \( kSb \otimes kC_S(P)b_P^{\delta} \).

Finally, the uniqueness statement of Theorem 8 (iii) follows from Lemma 7.
5 Gluing sources

In this section, we work over the local ring $\mathcal{O}$. Let $e$ be a block of a finite group $G$, and $(P,e_P)$ be an $e$-subpair of $G$. We choose, once and for all, a maximal $e$-subpair $(D,e_D)$ that contains $(P,e_P)$, and we assume that the centraliser $C_G(P)$ strongly controls the $e$-fusion in $G$ with respect to the maximal subpair $(D,e_D)$. For any subgroup $Q$ of $D$, we denote by $e_Q$ the unique block of the centraliser $C_G(Q)$ such that $(Q,e_Q) \leq (D,e_D)$.

Let $Q \neq 1$ be a non-trivial subgroup of $D$. Then the $e$-subpair $(N_D(Q), e_{N_D(Q)})$ may be seen as an $e_Q$-subpair of the group $N_G(Q)$, although it needs not be maximal. By assumption, we have $N_G(Q,e_Q) \leq O_P(C_G(Q))C_G(P)$. Let $S_Q$ be any normal $p'$-subgroup of $N_G(Q,e_Q)$ such that $S_Q \leq C_G(Q)$ and $N_G(Q,e_Q) \leq S_Q C_G(P)$. Let $b_Q$ be an $N_D(Q)$-stable block of the group $S_Q$ such that the block $e_{N_D(Q)}$ of $C_G(N_D(Q))$ covers the block $br_{N_D(Q)}(b_Q)$ of $C_S(N_D(Q))$. The diagonal conjugation action of the group $N_D(Q)$ on the matrix algebra $kS_Q b_Q \otimes kC_{S_Q}(P) br_{N_D(Q)}(b_Q)^o$ makes it a Dade $N_D(Q)/Q$-algebra, since the normal subgroup $Q$ acts trivially. Let $v_Q$ be the corresponding class in the Dade group $\mathcal{D}(N_D(Q)/Q)$.

**Lemma 16.** With the notations of [7],

(i) The class $v_Q \in \mathcal{D}(N_D(Q))$ is independent of the choice of $S_Q$ and $b_Q$.

(ii) If $Q \leq R$ are non-trivial subgroups of $D$, then

$$\text{Def} \text{res}_{N_D(Q)/Q}^{N_D(R)/R} v_Q = \text{Res}_{N_D(Q)/R}^{N_D(R)/R} v_R.$$  

(iii) If $g \in G$ is such that $g(N_D(Q),e_Q) \leq (D,e_D)$, then

$$\text{Res}_{N_D(D^g(Q))}^{N_D(Q)} v_Q = \text{Res}_{N_D(D^{g^{-1}}Q)}^{N_D(Q)} g^{-1} \cdot v_Q.$$  

**Proof.** Let us fix a non-trivial $p'$-subgroup $Q$ of the defect group $D$. We write $G_Q = N_G(Q,e_Q)$ and $H_Q = C_G(Q)$. By assumption, we have the factorisation $G_Q = S_Q C_G(Q)$, so that all the assumptions of Section 4 are satisfied. We denote by $M_Q$ the indecomposable Brauer-friendly $k(H_Q \times C_{H_Q}(P)) \Delta C_{G_Q}(P)$-module of Theorem 8. Let $V_Q$ be a capped indecomposable direct summand of the $k\Delta N_D(Q)$-module $e_{N_D(Q)} M_Q \epsilon_{N_D(Q)}$, which we identify to a $k N_D(Q)$-module through the diagonal isomorphism. We know from Theorem 8 that $V_Q$ exists and is an endo-permutation $kN_D(Q)$-module that belongs to the class $\text{Inf}_{N_D(Q)/Q}^{N_D(Q)} v_Q$, and we know from Lemma 2 that the isomorphism class of $V_Q$ depends only on the Brauer-friendly-module $M_Q$ and the subpair $(Q,e_Q)$. This proves (i).

We now take $1 \neq Q \leq R \leq D$. On the one hand, $S_Q$ is a normal $p'$-subgroup of $G_Q$ such that $S_Q \leq H_Q$ and $G_Q = S_Q H_Q$, and $b_Q$ is an $N_D(Q)$-stable block of $kS_Q$ such that the block $e_{N_D(Q)}$ covers $br_{N_D(Q)}(b_Q)$. We set $G_{Q,R} = N_G(Q,R,e_R)$, $S_{Q,R} = C_{S_Q}(R)$ and $b_{Q,R} = br_{R}(b_Q)$. Then $S_{Q,R}$ is a
normal $p'$-subgroup of $G_{Q,R}$ such that $S_{Q,R} \leq H_R$ and $G_{Q,R} = S_{Q,R}C_{G_{Q,R}}(P)$, and $b_{Q,R}$ is an $N_D(Q,R)$-stable block of $S_{Q,R}$ such that the block $e_{N_D(Q,R)}$ covers $br_{N_D(Q,R)}(b_{Q,R})$. Let $v_{Q,R} \in D(N_D(Q,R)/R)$ be the class defined by the Dade $N_D(Q,R)$-algebra $kS_{Q,R}b_{Q,R} \otimes kC_{S_{Q,R}}(P) br_{b_{Q,R}}$, i.e., $v_{Q,R} = \text{Defres}_{N_D(Q,R)/R}^{N_D(Q)/Q}$. On the other hand, let $S_R$ be a normal $p'$-subgroup of $G_R$ such that $S_R \leq H_R$ and $G_R = S_RC_{G_R}(P)$, and $b_R$ be an $N_D(R)$-stable block of $S_R$ such that the block $e_{N_D(R)}$ covers $br_{N_D(R)}(b_R)$. Then $S_R$ is also a normal $p'$-subgroup of $G_{Q,R}$ such that $S_R \leq C_G(R)$ and $G_{Q,R} = S_RC_{G_{Q,R}}(P)$, and $b_R$ is an $N_D(Q,R)$-stable block of $S_R$ such that the block $e_{N_D(Q,R)}$ covers $br_{N_D(Q,R)}(b_R)$. Since the class $v_{Q,R}$ is independent of the choice of the subgroup $S_{Q,R}$ and of the block $b_{Q,R}$, it follows that $v_{Q,R} = \text{Res}_{N_D(Q,R)/R}^{N_D(R)/R}v_R$, and (ii) is proven. The proof of (iii) is essentially the same. □

The rest of this article depends on the following assumption.

**Assumption 17.** There exists a capped indecomposable endo-permutation $OD$-module $V$ such that the triple $(D,e_D,V)$ is fusion-stable in $G$ and that, for any non-trivial subgroup $Q$ of $D$,

$$\text{Defres}_{N_D(Q)/Q}^{N_D(Q)/Q}[k \otimes_D V] = v_Q.$$

**Lemma 18.** If $e$ is the principal block of the group $G$, or if the defect group $D$ is abelian, or if the prime $p$ is odd and the poset $\mathcal{A}_{32}(D)$ of elementary abelian subgroups of $D$ of rank at least $2$ is connected, then Assumption [17] is satisfied.

**Proof.** Firstly, we suppose that $e$ is the principal block of the group $G$. For any non-trivial subgroup $Q$ of the defect group $D$, the principal block $e_Q$ of the group $C_G(Q)$ covers the principal block $b_Q$ of the $p'$-group $S_Q$, so $v_Q$ is the trivial class in the Dade group $D(N_D(Q)/Q)$. Thus we can choose $V$ to be the trivial $OD$-module.

Secondly, we suppose that the defect group $D$ is abelian. Then we have $N_D(Q) = D$ for any subgroup $Q$ of $D$. Following [18], we consider the function $\mu$ on the set of non-trivial subgroups of $D$ such that $\sum_{R \notin Q} \mu(R) = 1$ for any non-trivial subgroup $Q$ of $D$. We consider the class

$$v = \sum_{1 \neq Q \leq D} \mu(Q) \text{Inf}_{D/Q}^{D} v_Q \in D(D).$$

By Lemma [16] the family $(v_Q)_{1 \neq Q \leq D}$ satisfies the assumptions of [18] Proposition 3.6. Thus the class $v$ is $N_G(D,e_D)$-stable, and $\text{Defres}_{D/Q}^{D} v = v_Q$ for any non-trivial subgroup $Q$ of $D$. By [19] Corollary 8.5 and [24] Lemma 28.1, there exists a unique isomorphism class of capped indecomposable endo-permutation $OD$-module $V$ with determinant 1 (i.e., with a structure map that sends the group $D$ into $SL(V)$) such that $v = [k \otimes_D V]$. Since the defect group $D$ is abelian, the normaliser $N_G(D,e_D)$ controls the $e$-fusion in the group $G$.
respect to the maximal subpair \((D, e_D)\), so the triple \((D, e_D, V)\) is fusion-stable in \(G\).

Thirdly, we suppose that the prime \(p\) is odd and that the poset \(A_{\geq 2}(D)\) is connected. For any subgroup \(Q\) of \(D\), the class \(v_Q \in D(N_D(Q)/Q)\) contains the source of a simple module for the \(p\)-nilpotent group \((S_Q \times C_{S_Q}(P)) \times N_D(Q)/Q\).

Thus we know from [5, Proposition 4.4] that the class \(v_Q\) lies in the torsion part \(D_1(N_D(Q)/Q)\) of the Dade group \(D(N_D(Q)/Q)\). By [2, Theorem 1.1], there is an exact sequence

\[
0 \to D_1(D) \to \lim_{1 \neq Q \subseteq D} D_1(N_D(Q)/Q) \to \hat{H}^0(A_{\geq 2}(D)) \to 0.
\]

By lemma[16] the family \((v_Q)_{1 \neq Q \subseteq D}\) lies in the direct limit \(\lim_{1 \neq Q \subseteq D} D_1(N_D(Q)/Q)\) of the above exact sequence. By assumption, the additive group \(\hat{H}^0(A_{\geq 2}(D), \mathbb{F}_2)\) of locally constant \(\mathbb{F}_2\)-valued functions on \(A_{\geq 2}(D)\), modulo constant functions, is trivial. Thus there exists a unique class \(v\) in the torsion Dade group \(D_1(D)\) such that \(\text{Defres}_{N_D(Q)/Q}^D v = v_Q\) for any non-trivial subgroup \(Q\) of \(D\). As above, there is a unique capped indecomposable endo-permutation \(O\)-module \(V\) with determinant 1 such that the reduction \(k \otimes_{O} V\) belongs to the class \(v\).

Let \((R, e_R)\) be a subpair of \((D, e_D)\) and let \(g \in G\) be such that \(\sigma(R, e_R) \leq (D, e_D)\). Set \(w = \text{Res}_{R}^{D} v\) and \(w' = \text{Res}_{R}^{D} g^{-1} \cdot v\). For any non-trivial subgroup \(Q\) of \(R\), we have

\[
\text{Defres}_{N_R(Q)/Q}^R w = \text{Res}_{N_R(Q)/Q}^{N_D(Q)/Q} v_Q = \text{Res}_{N_R(Q)/Q}^{N_D(Q)/Q} g^{-1} \cdot v_{\sigma Q} = \text{Defres}_{N_R(Q)/Q}^R w'.
\]

Then the injectivity of the deflation-restriction map \(D_1(R) \to \lim_{1 \neq Q \subseteq R} D_1(N_R(Q))\) implies that \(w = w'\). Let \(W\) (resp. \(W'\)) be a capped indecomposable direct summand of the restriction \(\text{Res}_{R}^{D} V\) (resp. \(w' = \text{Res}_{R}^{D} g^{-1} V\)). Since the prime \(p\) is odd, the endo-permutation \(O\)-modules \(W\) and \(W'\) must have determinant 1; moreover, the reductions \(k \otimes_{O} W\) and \(k \otimes_{O} W'\) belong to the same class \(w = w' \in D(R)\). Thus \(W\) and \(W'\) are isomorphic, and the triple \((D, e_D, V)\) is fusion-stable in the group \(G\).

For a general defect group \(D\), the obstruction group \(\hat{H}^0(A_{\geq 2}(D), \mathbb{F}_2)\) needs not be trivial. However, we know from the classification of finite simple groups that the \(Z_p^*\)-theorem is always true. This implies that Assumption[17] is satisfied, at least when the centraliser \(C_G(P)\) controls the \(p\)-fusion in the group \(G\) (and not only the \(e\)-fusion). We do hope that a careful study of the direct image of the family \((v_Q)_{1 \neq Q \subseteq D}\) in the obstruction group \(\hat{H}^0(A_{\geq 2}(D), \mathbb{F}_2)\) will show that this direct image is always trivial. This would allow one to prove Theorem[1] without any restriction on the defect group \(D\).
6 Obtaining a stable equivalence

With all the conventions of the previous section, we now suppose that Assumption \[17\] is satisfied. We identify \( V \) with an \( \mathcal{O}\Delta D \)-module. By Lemma \[7\] there is a unique indecomposable Brauer-friendly \( \mathcal{O}(G \times C_G(P)) \)-module \( M \) with source triple \( (\Delta D, e_D \otimes e_D^0, V) \) such that the slashed module \( M(\Delta P, e_P \otimes e_P^0) \) admits the \( k(C_G(P) \times C_G(P)) \)-module \( kC_G(P) e_P \) as a direct summand.

**Lemma 19.** Let \( Q \) be a non-trivial subgroup of the defect group \( D \). Then the slashed module \( M(\Delta Q, e_Q \otimes e_{PQ}) \) induces a Morita equivalence

\[
k_{G}(Q) e_{Q} \sim k_{C_{G}(PQ)} e_{PQ}
\]

**Proof.** The class \( \operatorname{Defres}_{D/P}^D v \in \mathcal{D}(D/P) \) is trivial, so the slashed module \( M(\Delta P, e_P \otimes e_P^0) \) is a \( p \)-permutation \( k(C_G(P) \times C_G(P)) \)-module. Thus we may use, from now on, the slash construction that we have defined in Lemma \[6\]. For the sake of shortness, whenever \( Q \) is a subgroup of the defect group \( D \), we write

\[
\begin{align*}
C(Q) &= (C_G(Q) \times C_G(PQ)) ; \\
N(Q) &= C(Q) \Delta N_G(Q, e_Q) ; \\
M(Q) &= M(\Delta Q, e_Q \otimes e_{PQ}) ,
\end{align*}
\]

where the latter is a \( kN(Q) \)-module. For any \( Q \leq D \) and any \( g \in G \) such that \( g(Q, e_Q) \leq (D, e_D) \), the uniqueness part of Lemma \[3\] implies that there is an isomorphism of \( kN(Q) \)-modules \( M(Q) \simeq (g^{-1}, g^{-1}) : M(gQ) \). Thus, up to replacing the subgroup \( Q \) by a \( G \)-conjugate, we may suppose that the subpair \( (Q, e_Q) \) is fully normalised in \( (D, e_D) \), i.e., that the normaliser subpair \( (N_D(Q), e_{N_D(Q)}) \) is a maximal \( e_Q \)-subpair of the group \( N_G(Q, e_Q) \). Similarly, for any two subgroups \( Q \leq R \) of \( D \), the \( kN(Q, R) \)-modules \( M(Q) \langle R \rangle \) and \( \operatorname{Res}^{N(R)}_{N(Q, R)} M(R) \) are isomorphic.

By construction of \( M \), we know that the \( kC(P) \)-module \( M(P) \) admits \( kC_G(P) e_P \) as a direct summand. As a consequence, the Brauer quotient \( M(P) \langle Q \rangle \) admits the \( kN(P, Q) \)-module \( kC_G(PQ) e_{PQ} \) as a direct summand. By the above remark on the transitivity of the slash construction, it follows that the slashed module \( M(Q) \langle P \rangle \) also admits \( kC_G(PQ) e_{PQ} \) as a direct summand. Thus there exists an indecomposable direct summand \( M_Q^0 \) of the \( kN(Q) \)-module \( M(Q) \) such that the slashed module \( M_Q^0 \langle P \rangle \) admits the \( kN(P, Q) \)-module \( kC_G(PQ) e_{PQ} \) as a direct summand.

Let \( (R, f) \) be a maximal \( e_Q \)-subpair of the group \( N_G(Q, e_Q) \). Then the Brauer quotient \( \operatorname{Br}_{(\Delta R, f \otimes f^0)}(kC_G(PQ) e_{PQ}) \simeq kC_G(R) f \) is non-zero. By transitivity of the slash construction, it follows that the slashed module \( M_Q^0 \langle \Delta R, f \otimes f^0 \rangle \) is non-zero. Moreover, a vertex subpair of \( M_Q^0 \) must be contained in a conjugate of the vertex subpair \( (\Delta D, e_D \otimes e_D^0) \) of \( M \). Thus \( (\Delta R, f \otimes f^0) \) is a vertex subpair of \( M_Q^0 \). Assuming that the subpair \( (Q, e_Q) \) is fully normalised in \( (D, e_D) \), we deduce from Lemma \[3\] (iii) that \( (\Delta N_D(Q), e_{N_D(Q)} \otimes e_{N_D(Q)}^0, V_Q) \) is a source triple of the indecomposable \( kN(Q) \)-module \( M_Q^0 \). Now it follows from Lemma
and Theorem 8 that the $kN(Q)$-module $M_Q^0$ induces an $N_G(Q, e_Q)/C_G(Q)$-equivariant Morita equivalence

$$kC_G(Q)e_Q \sim kC_G(PQ)e_{PQ}.$$ 

The next step uses descending induction on the order of the group $Q$ to prove that $M(\langle Q \rangle) = M_Q^0$. We know from the proof of Lemma 7 that the slashed module $M(D)$ is isomorphic to the indecomposable $kN(D)$-module $kC_G(D)e_D$, so $M(D) = M^0_D$. Then let $Q$ be a proper subgroup of $D$ and suppose that $M(R) = M^0_R$ for any $p$-group $R$ such that $Q < R \leq D$. We consider a Krull-Schmidt decomposition

$$M(Q) = M_Q^0 \oplus \ldots \oplus M_Q^n,$$

of the $kN(Q)$-module $M(Q)$, and we suppose that $n \geq 1$. Let $(R, f)$ be a vertex subpair of the $kN(Q)$-module $M^1_Q$. Once again, we may assume that the subpair $(Q, e_Q)$ is fully normalised in $(D, e_D)$. We may suppose that $(R, f)$ is contained in the maximal $(e_Q \otimes e_Q^0)$-subpair $((C_D(Q) \times C_D(Q))\Delta N_D(Q), e_{ND}(Q) \otimes e_{ND}(Q))$. By Lemma 3 (iii), the subpair $(R, f)$ must be contained in a $(G \times C_G(P))$-conjugate of the vertex subpair $(\Delta D, e_D \otimes e_{P}^0)$ of $M$. Thus we have $\text{Br}_{(R,f)}(kGe) \neq 0$. Since the subpair $(\Delta Q, e_Q \otimes e_{P}^0)$ is normalised by $(R, f)$ and $\text{Br}_{(\Delta Q,e_Q \otimes e_{P}^0)}(kGe) \simeq kC_G(Q)e_Q$, it follows that $\text{Br}_{(R,f)}(kC_G(Q)e_Q) \neq 0$. So the subpair $(R, f)$ is contained in a $(C_G(Q) \times C_G(Q))\Delta N_D(Q, e_Q)$-conjugate of the vertex subpair $(\Delta N_D(Q), e_{ND}(Q) \otimes e_{ND}(Q))$ of the indecomposable $k(C_D(Q) \times C_D(Q))\Delta N_D(Q, e_Q)$-module $kC_G(Q)e_Q$. Moreover the subgroup $N(P, Q)$ controls the $\langle e_Q \otimes e_{P}^0 \rangle$-fusion in the group $(C_G(Q) \times C_G(Q))\Delta N_D(Q, e_Q)$. So $(R, f)$ is contained in an $N(P, Q)$-conjugate of $(\Delta N_D(Q), e_{ND}(Q) \otimes e_{ND}(Q))$. We may choose $(R, f) = (\Delta R', e_{R'} \otimes e_{P,R'}^0)$ for some subgroup $R'$ of $N_D(Q)$. If $Q < R'$, then we obtain

$$M(R') \simeq M(\langle Q \rangle)\langle R' \rangle = M_Q^0\langle R' \rangle \oplus \ldots \oplus M_Q^n\langle R' \rangle,$$

where at least the direct summands $M_Q^0\langle R' \rangle$ and $M_Q^n\langle R' \rangle$ are non-zero. This contradicts the indecomposability of the $kC(R')$-module $M(R') = M^n_{R'}$. If $Q = R'$, then Lemma 5 implies that the $k(G \times C_G(P))$-module $M$ has an indecomposable direct summand with vertex subpair $(\Delta Q, e_Q \otimes e_{P}^0)$, another contradiction. So the lemma is proven. 

For the reader’s convenience, we quote [16, Theorem 1.1], which is not published yet. We slightly adapt the notations to fit those of the present chapter.

**Theorem** (Linckelmann). Let $A$, $B$ be (almost) source algebras of blocks of finite group algebras over $O$ having a common defect group $D$ and the same fusion system $\mathbf{F}$ on $D$. Let $V$ be an $\mathbf{F}$-stable indecomposable endo-permutation $OD$-module with vertex $D$, viewed as an $O\Delta D$-module through the canonical isomorphism $\Delta D \simeq D$. Let $M$ be an indecomposable direct summand of the $(A, B)$-bimodule

$$A \otimes_{OD} \text{Ind}_{\Delta D} V \otimes_{OD} B$$

20
Suppose that \( M \otimes_B M^* \neq 0 \). Then, for any non-trivial fully \( \mathbf{F} \)-centralised subgroup \( Q \) of \( D \), there is a canonical \( (\text{Br}_Q(A), \text{Br}_Q(B)) \)-bimodule \( M(\Delta Q) \) satisfying \( \text{End}_k(M(\Delta Q)) \cong \text{Br}_{\Delta Q}(\text{End}_Q(M)) \). Moreover, if for all non-trivial fully \( \mathbf{F} \)-centralised subgroups \( Q \) of \( D \) the bimodule \( M(\Delta Q) \) induces a Morita equivalence between \( \text{Br}_{\Delta Q}(A) \) and \( \text{Br}_{\Delta Q}(B) \), then \( M \) and its dual \( M^* \) induce a stable equivalence of Morita type between \( A \) and \( B \).

We now have all the tools that we need to prove our main result.

**Proof of Theorem** Let \( i \in (\text{OGe})^D \) be a source idempotent of the block \( e \) such that \( \bar{e}_D \text{br}_D(i) \neq 0 \), and let \( i_P \in (\text{OCG}(P)e_P)^D \) be a source idempotent of the block \( e_P \) such that \( \bar{e}_D \text{br}_D(i_P) \neq 0 \). Set \( A = i\text{OGi} \) and \( B = i_P\text{OCG}(P)i_P \). Then \( iMi_P \) is an indecomposable direct summand of the \( (A,B) \)-bimodule \( A \otimes_{kD} \text{Ind}_{\Delta D} V \otimes_{kD} B \), where \( V \) is an endo-permutation \( OD \)-module that is fusion-stable for the common fusion system of the source algebras \( A \) and \( B \) on the defect group \( D \). Moreover, by Lemma [19] the slashed module \( iMi_P(\Delta Q) \) induces a Morita equivalence \( \text{Br}_{\Delta Q}(A) \sim \text{Br}_{\Delta Q}(B) \) for any subgroup \( Q \) of the defect group \( D \). Then Linckelmann’s theorem asserts that the \( (A,B) \)-bimodule \( iMi_P \) induces a stable equivalence \( A \sim B \). In terms of block algebras, this means exactly that the \( (\text{OGe},\text{OCG}(P)e_P) \)-bimodule \( M \) induces a stable equivalence

\[
\text{OGe} \sim \text{OCG}(P)e_P.
\]

\( \square \)

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