On Grothendieck’s Riemann-Roch Theorem

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Abstract

We prove that, for smooth quasi-projective varieties over a field, the $K$-theory $K(X)$ of vector bundles is the universal cohomology theory where $c_1(L \otimes \bar{L}) = c_1(L) + c_1(\bar{L}) - c_1(L)c_1(\bar{L})$. Then, we show that Grothendieck’s Riemann-Roch theorem is a direct consequence of this universal property, as well as the universal property of the graded $K$-theory $GK^\bullet(X) \otimes \mathbb{Q}$.

INTRODUCTION: In 1957 Grothendieck introduced the $K$-group $K(X)$ of vector bundles on an algebraic variety $X$ and the $K$-group $G(X)$ of coherent sheaves. He showed that the $K$-theory $K(X)$ has a cohomological behavior whereas $G(X)$ has a homological behavior. For example, $K(X) = G(X)$ when $X$ is smooth, so that we have an unexpected direct image $f_!: K(Y) \to K(X)$ for any projective map $f: Y \to X$ between smooth varieties. Then Grothendieck gave his astonishing formulation of the Riemann-Roch theorem as a determination of the relationship between this direct image $f_!$ on $K$-theory and the usual direct image $f_*$ at the level of the Chow ring $CH^\bullet(X)$ (or the singular cohomology, etc.). The relationship involved two series: the Chern character $ch$ and the Todd class $Td$.

Grothendieck’s Riemann-Roch Theorem: Let $f: Y \to X$ be a projective morphism between smooth quasi-projective algebraic varieties and denote $T_X$ and $T_Y$ their tangent bundles. Then we have a commutative square

$$
\begin{array}{c}
K(Y) \xrightarrow{f_!} K(X) \\
\downarrow \text{Td}(T_Y) \cdot ch \\
CH^\bullet(Y) \otimes \mathbb{Q} \xrightarrow{f_*} CH^\bullet(X) \otimes \mathbb{Q}
\end{array}
$$

In this article we prove that Grothendieck’s $K$-theory $K(X)$ is the universal cohomology theory where Chern classes of line bundles fulfil that

$$c_1(L \otimes \bar{L}) = c_1(L) + c_1(\bar{L}) - c_1(L)c_1(\bar{L}).$$

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Even if this universal property is clearly implicit in the work of Levine, Morel, Panin, Pimenov and Röndigs on the algebraic cobordism ([8],[13]), this result seems to be new.

After that, we show that Grothendieck’s Riemann-Roch theorem is a direct consequence of this universal property of the \( K \)-theory. We also obtain the universal property of the graded \( K \)-theory \( GK^\bullet(X) \otimes \mathbb{Q} \), when the ring \( K(X) \) is filtered by the subgroups \( F_d(X) \) generated by the coherent sheaves with support of codimension \( \geq d \).

Let us comment the plan of this article. In section 1 we review the theory of Chern classes in a cohomology theory \( A(X) \). In section 2 we prove the universal property of \( K \)-theory: for any cohomology theory \( A(X) \) following the multiplicative law \( c_1(L \otimes \bar{L}) = c_1(L) + c_1(\bar{L}) - c_1(L)c_1(\bar{L}) \) there exists a unique functorial ring morphism \( \varphi: K(X) \to A(X) \) preserving direct images. The key point, due to Panin in [11], is to prove that any functorial ring morphism preserving Chern classes of the tautological line bundles also preserves direct images, \( \varphi(f_!(y)) = f_!(\varphi(y)) \). For the sake of completeness, we include Panin’s proof, greatly simplified in the case of the projection \( p: \mathbb{P}^n \to \text{pt} \) using the approach of Déglise ([3]).

In section 3 we consider cohomology theories \( A(X) \) following the additive law \( c_1(L \otimes \bar{L}) = c_1(L) + c_1(\bar{L}) \). In any such a theory we may modify the direct image with a formal series with rational coefficients (essentially an exponential) so that it follows the multiplicative law of the \( K \)-theory and hence, due to the universal property of \( K \)-theory, we have a functorial ring morphism \( \text{ch}: K(X) \to A(X) \otimes \mathbb{Q} \) compatible with the new direct image. This compatibility is just Grothendieck’s Riemann-Roch theorem. We also show that the graded \( K \)-theory \( GK^\bullet(X) \otimes \mathbb{Q} \) is the universal graded cohomology with rational coefficients (with values in the category of graded \( \mathbb{Q} \)-algebras). Finally, in section 4 we present some direct consequences of these theorems.

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1 Cohomology theories

Throughout this paper, let us fix a base field $k$.

**Definition:** With Panin ([11]), we define a **cohomology theory** to be a contravariant functor $A$ from the category of smooth quasi-projective varieties over $k$ into the category of commutative rings, endowed with a functorial morphism of $A(X)$-modules $f_* : A(Y) \to A(X)$, called direct image, for any projective morphism $f : Y \to X$ (that is to say, $\text{Id}_* = \text{Id}$, $(fg)_* = f_*g_*$ and the projection formula $f_*(f^*(x)y) = xf_*(y)$ holds). Hence we have a **fundamental class** $[Y]_A := i_*(1) \in A(X)$ for any smooth closed subvariety $i : Y \to X$, and a **Chern class** $c_1^A(L) := s^*(s_0^*(1)) \in A(X)$ for any line bundle $L \to X$ (where $s_0 : X \to L$ is the zero section).

These data are assumed to satisfy the following conditions:

1. The ring morphism $i^1_* + i^2_* : A(X_1 \amalg X_2) \to A(X_1) \oplus A(X_2)$ is an isomorphism, where $i_j : X_j \to X_1 \amalg X_2$ is the natural immersion.

   Hence $A(\emptyset) = 0$.

2. The ring morphism $\pi_* : A(X) \to A(P)$ is an isomorphism for any affine bundle $\pi : P \to X$ (i.e., a torsor over a vector bundle $E \to X$).

   Therefore, if $\pi : L \to X$ is a line bundle, then $c^A_1(L) = s^*(s_0^*(1))$ for any section $s$, because $s^* = s_0^*$ both being the inverse of $\pi^* : A(X) \to A(L)$.

3. For any smooth closed subvariety $i : Y \to X$ we have an exact sequence $A(Y) \xrightarrow{i_*} A(X) \xrightarrow{j^*} A(X - Y)$.

4. If a morphism $f : \bar{X} \to X$ is transversal to a smooth closed subvariety $i : Y \to X$ of codimension $d$ (that is to say, $f^{-1}(Y) = \emptyset$ or $\bar{Y} = Y \times_X \bar{X}$ is a smooth subvariety of $\bar{X}$ of codimension $d$, so that the natural epimorphism $f^*N^*_Y/X \to N^*_\bar{Y}/\bar{X}$ is an isomorphism, where $N_{Y/X}$ denotes the normal bundle), then we have a commutative square

$$
\begin{array}{ccc}
A(Y) & \xrightarrow{f^*} & A(\bar{Y}) \\
\downarrow i_* & & \downarrow i_* \\
A(X) & \xrightarrow{f^*} & A(\bar{X}).
\end{array}
$$
5. Let \( \pi : \mathbb{P}(E) \to X \) be a projective bundle. For any morphism \( f : Y \to X \) we have a commutative square

\[
\begin{array}{ccc}
A(\mathbb{P}(E)) & \xrightarrow{f^*} & A(\mathbb{P}(f^*E)) \\
\downarrow \pi & & \downarrow \pi \\
A(X) & \xrightarrow{f^*} & A(Y).
\end{array}
\]

6. Let \( \pi : \mathbb{P}(E) \to X \) be the projective bundle associated to a vector bundle \( E \to X \) of rank \( r + 1 \) and let \( \xi_E \to \mathbb{P}(E) \) be the tautological line bundle. Consider the structure of \( A(X) \)-module in \( A(\mathbb{P}(E)) \) defined by the ring morphism \( \pi^* : A(X) \to A(\mathbb{P}(E)) \), and put \( x_E = c_1^A(\xi_E) \), then

\[
\begin{align*}
A(\mathbb{P}(E)) &= A(X) \oplus A(X)x_E \oplus \ldots \oplus A(X)x_E^r,
\end{align*}
\]

Given cohomology theories \( A, \hat{A} \) on the smooth quasi-projective \( k \)-varieties, a morphism of cohomology theories \( \varphi : A \to \hat{A} \) is a natural transformation preserving direct images. That is to say, for any smooth quasi-projective variety \( X \) we have a ring morphism \( \varphi : A(X) \to \hat{A}(X) \) such that \( \varphi(f^*(a)) = f^*(\varphi(a)) \), \( a \in A(Y) \), for any morphism \( f : Y \to X \), and \( \varphi(f_*(b)) = f_*(\varphi(b)) \), \( b \in A(Y) \), for any projective morphism \( f : Y \to X \).

**Remarks:**

1. Both \( i_1 \) and \( i_2 \) are transversal to \( i_1 \) and \( i_2 \); hence \( i_1^*i_1 = 0, i_2^*i_2 = 0, i_1^*i_2 = 0 \). 
   \( i_1^* \) and \( i_2^* \) are transversal to \( i_1 \) and \( i_2 \); hence \( i_1^*i_1 = 0, i_2^*i_2 = 0, i_1^*i_2 = 0 \).
   So that the inverse of the isomorphism \( i_1^* + i_2^* \) of axiom 1 is just \( i_1 + i_2 : A(X_1) \oplus A(X_2) \to A(X_1 \times X_2) \), and for any projective morphism \( f = f_1 \times f_2 : X_1 \times X_2 \to Z \) we have that \( f_* = f_1^* + f_2^* : A(X_1) \oplus A(X_2) \to A(Z) \).

2. Let \( E \to X \) be a vector bundle of rank \( r \). In \( \mathbb{P}(E) \) we have an exact sequence \( 0 \to \xi_E \to \pi^*E \to Q \to 0 \), where \( Q \) is a vector bundle of rank \( r - 1 \), and the ring morphism \( \pi^* : A(X) \to A(\mathbb{P}(E)) \) is injective by axiom 6. Proceeding by induction on the rank we see that the following splitting principle holds: There exists a base change \( \pi : X' \to X \) such that \( \pi^*E \) admits a filtration \( 0 = E_0 \subset E_1 \subset \ldots \subset E_r = \pi^*E \) whose quotients \( E_i/E_{i-1} \), \( i = 1, \ldots, r \), are line bundles, and \( \pi^* : A(X) \to A(X') \) is injective.

3. Given a line bundle \( L \to X \) and a morphism \( f : Y \to X \), the induced morphism \( f \times 1 : f^*L = Y \times_X L \to X \times_X L = L \) is transversal to the null section \( s_0 : X \to L \). Hence

\[
f^*s_0s_0(1) = s_0(f \times 1)^*s_0(1) = s_0s_0(1),
\]

where \( s_0 \) stands for the null section of \( f^*L \), and we see that Chern classes are functorial,

\[
c_1(f^*L) = f^*c_1(L).
\]
4. Let $L_Y \to X$ be the line bundle defined by a smooth closed hypersurface $i: Y \to X$ (dual to the line bundle defined by the ideal of $Y$). It admits a section $s: X \to L_Y$ vanishing just on $Y$ and transversal to the null section $s_0$. Hence

$$c_1(L_Y) = s^*(s_0^*(1)) = i_*(i^*(1)) = i_*(1) = [Y] \in A(X).$$

In particular, if $\xi_d \to \mathbb{P}^d$ is the tautological line bundle of the projective space of dimension $d$, the Chern class of the dual bundle $\xi_d^*$ is just the fundamental class of an hyperplane, $c_1(\xi_d^*) = [\mathbb{P}^{d-1}]$.

In general, $L_x$ will denote a line bundle with Chern class $x \in A(X)$, and we say that a cohomology theory $A$ follows the additive group law $x + y$ when $c_1^A(L_x \otimes L_y) = x + y$, that it follows the multiplicative group law $x + y - xy$ when $c_1^A(L_x \otimes L_y) = x + y - xy$, and so on.

**Examples:**

1. The $K$-theory $K(X)$ is a cohomology theory ([5], [16]). The fundamental class of a smooth closed subvariety $i: Y \to X$ is

$$[Y]^K = i_1(1) = \mathcal{O}_Y \in K(X),$$

where $\mathcal{O}_Y$ is the structural sheaf of $Y$. The Chern class of a line bundle $L \to X$ is

$$c^K_1(L) = s_0^1(s_0^*(1)) = 1 - L^* \in K(X).$$

In general $c^K_1(L^*) \neq -c^K_1(L)$; but we have

$$1 - (L \otimes \bar{L})^* = (1 - L^*) + (1 - \bar{L}^*) - (1 - L^*)(1 - \bar{L}^*),$$

so that the $K$-theory follows the multiplicative group law $x + y - xy$,

$$c^K_1(L \otimes \bar{L}) = c^K_1(L) + c^K_1(\bar{L}) - c^K_1(L)c^K_1(\bar{L}).$$

2. Denote $F^d(X)$ the subgroup of $K(X)$ generated by the coherent sheaves with support of codimension $\geq d$. The graded $K$-theory $G K^*(X) = \oplus_d F^d(X)/F^{d+1}(X)$ is a cohomology theory ([3], [10]). The fundamental class of a smooth closed subvariety $i: Y \to X$ of codimension $d$ is

$$[Y]^{G K} = [\mathcal{O}_Y] \in F^d(X)/F^{d+1}(X),$$

and the Chern class of a line sheaf $L$ is

$$c_1^{G K}(L) = [1 - L^*] = [L - 1] \in F^1(X)/F^2(X).$$

Since $(1 - L^*)(1 - \bar{L}^*) \in F^2(X)$, the graded $K$-theory follows the additive law $c_1^{G K}(L \otimes \bar{L}) = c_1^{G K}(L) + c_1^{G K}(\bar{L})$.

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1Since $(1 - x)(1 - y) = 1 - (x + y - xy)$, if we declare $x$ to be the coordinate of $1 - x$ in the multiplicative group $k - \{0\}$, then the group law is just $x + y - xy$. 

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3. When $k = \mathbb{C}$, the singular cohomology classes of even degree define a cohomology theory $H^{2*}(X, \mathbb{Z}) = \oplus_d H^{2d}(X, \mathbb{Z})$ following the additive law $x + y$.

4. The Chow ring $CH^*(X)$ of rational equivalence classes of cycles is a cohomology theory ([I]) also following the additive law $x + y$. The fundamental class of a subvariety $Z$ is given by the rational equivalence class $[Z]$.

5. Given a field extension $k \rightarrow K$ and a cohomology theory $A$ on the smooth quasi-projective $K$-varieties, then $X \rightarrow A(X \times_k K)$ is a cohomology theory on the smooth quasi-projective $k$-varieties. Hence, if $k$ is a field with a fixed embedding $k \rightarrow \mathbb{C}$, then we have a cohomology theory $H^{2*}(X \times_k \mathbb{C}, \mathbb{Z})$ on the smooth quasi-projective $k$-varieties.

**Definition:** Let $E \rightarrow X$ be a vector bundle of rank $r$. With Grothendieck ([II]) we define the Chern classes $c_i^A(E) \in A(X)$ of $E$ to be the coefficients of the characteristic polynomial $c(E) = x^r - c_1^A(E)x^{r-1} + \ldots + (-1)^r c_r^A(E)$ of the endomorphism of the free $A(X)$-module $A(\mathbb{P}(E))$ defined by the multiplication by $x_E = c_1^A(\xi_E)$,

$$x_E^r - c_1^A(E)x_E^{r-1} + \ldots + (-1)^r c_r^A(E) = 0,$$

and we put $c_n(E)$ when the cohomology theory $A$ is clear.

**Remark:** The signs are introduced so that the Chern class of a line bundle $L$ coincides with the former one. In fact, $\mathbb{P}(L) = X$ and $\xi_L = L$, so that $x_L = c_1(\xi_L) = c_1(L)$.

**Theorem 1.1** For any morphism $f: Y \rightarrow X$, we have $c_n(f^*E) = f^*(c_n(E))$.

**Proof:** Any morphism $f: Y \rightarrow X$ induces a morphism $f: \mathbb{P}(f^*E) \rightarrow \mathbb{P}(E)$ such that $f^*\xi_E = \xi_{f^*E}$. Hence $f^*(x_E) = x_{f^*E}$ by ([II]). Applying $f^*$ to (2), we have

$$x_{f^*E}^r - (f^*c_1(E))x_{f^*E}^{r-1} + \ldots + (-1)^r f^*c_r(E) = 0,$$

and we conclude that $c_n(f^*E) = f^*c_n(E)$.

**Theorem 1.2** Chern classes are additive. That is to say, $c(E) = c(E_1) \cdot c(E_2)$ for any exact sequence $0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$ of vector bundles,

$$c_n(E) = \sum_{i+j=n} c_i(E_1) \cdot c_j(E_2) \quad ; \quad n, i, j \in \mathbb{N}.$$

**Proof:** Assume that $E_1$ is a line bundle. Then $i: X = \mathbb{P}(E_1) \rightarrow \mathbb{P}(E)$ is a section of $\mathbb{P}(E) \rightarrow X$, so that $i_*$ is injective. Moreover $i^*\xi_E = \xi_{E_1}$, $i^*(x_E) = x_{E_1}$.

Denote $j: U \rightarrow \mathbb{P}(E)$ the complement of $\mathbb{P}(E_1)$. The natural projection $p: U \rightarrow \mathbb{P}(E_2)$ is an affine bundle (of associated vector bundle $\text{Hom}(\xi_{E_2}, p^*E_1)$) and $j^*\xi_E = p^*\xi_{E_2}$, so that $j^*(x_E^p) = p^*(x_{E_2}^n)$. 
Hence \( j^* : A(\mathbb{P}(E)) \to A(U) \overset{\varphi}{\longrightarrow} A(\mathbb{P}(E_2)) \) is surjective and, by axiom 3, we have a commutative diagram with exact rows (the first square commutes by the projection formula)

\[
\begin{array}{ccc}
0 & \longrightarrow & A(\mathbb{P}(E_1)) \\ & \downarrow^{x_{E_1}} & \downarrow^{j^*} \\ 0 & \longrightarrow & A(\mathbb{P}(E)) \\ & \downarrow^{x_E} & \downarrow^{j^*} \\ & \longrightarrow & A(\mathbb{P}(E_2)) \\ & \downarrow^{x_{E_2}} & \longrightarrow \\ & 0 & \\
\end{array}
\]

In the general case, by the splitting principle, we may assume that we have a line bundle \( L \subset E_1 \) such that \( E_1 = E_1/L \) and \( E = E/L \) are vector bundles, so that we have an exact sequence \( 0 \to E_1 \to E \to E_2 \to 0 \) and we conclude by induction on the rank of \( E_1 \).

\[
c(E) = c(L)c(\bar{E}) = c(L)c(\bar{E}_1)c(E_2) = c(E_1)c(E_2).
\]

\[\square\]

**Remark:** By the above theorem \( 1 + c_1^A(E)t + c_2^A(E)t^2 + \ldots + c_r^A(E)t^r \) is an additive function on the vector bundles over \( X \), with values in the multiplicative group of invertible formal series with coefficients in \( A(X) \). Hence it extends to the \( K \)-group, and we obtain Chern classes \( c_d^r : K(X) \to A(X) \).

Now, given a vector bundle \( E \to X \) of rank \( r \), by the splitting principle there is a base change \( \pi : X' \to X \) such that \( \pi^* : A(X) \to A(X') \) is injective and \( \pi^*E = L_{\alpha_1} + \ldots + L_{\alpha_r} \) is a sum of line bundles in \( K(X') \). Therefore, \( c_n(E) \) is the \( n \)-th elementary symmetric function of the "roots" \( \alpha_1, \ldots, \alpha_r \),

\[
c_n(E) = \sum_{1 \leq i_1 < \ldots < i_n} \alpha_{i_1} \ldots \alpha_{i_n}. \quad (3)
\]

For example, in the \( K \)-theory the Chern class of a line bundle \( L \) is just \( c^K_1(L) = 1 - L^* \); hence the first Chern class of a vector bundle \( E \) of rank \( r \) is

\[
c^K_r(E) = (1 - L_{\alpha_1}^*) + \ldots + (1 - L_{\alpha_r}^*) = r - E^* \quad (4)
\]

and \( c_i^K(E) = (1 - L_{\alpha_1}^*) \cdot \ldots \cdot (1 - L_{\alpha_i}^*) = \sum_i (-1)^i \alpha_i^*E^*. \)

**Corollary 1.3** The cohomology ring of the projective spaces \( \mathbb{P}^d \) is

\[
A(\mathbb{P}^d) = A(pt)[x]/(x^{d+1}) = A(pt)[y]/(y^{d+1}),
\]

where \( x \) corresponds to \( x_d = c_1(x_d) \) and \( y \) corresponds to \( y_d = c_1(x_d^*) = [\mathbb{P}^{d-1}] \).

**Proof:** The projective space \( \mathbb{P}^d \) is just the projective bundle of a trivial vector bundle of rank \( d+1 \) over a point. By additivity, trivial bundles have null Chern classes; hence \( x_{d+1} = 0 \) in \( A(\mathbb{P}^d) = A(pt) \oplus A(pt)x_d \oplus \ldots \oplus A(pt)x_d \).

Now, in \( \mathbb{P}^1 \) we have an exact sequence \( 0 \to \xi_1 \to 1 \oplus 1 \to \xi_1^* \to 0 \), where \( 1 \oplus 1 \) stands for the trivial vector bundle of rank 2. Hence \( y_1 = -x_1 \) in \( A(\mathbb{P}^1) \). Considering a line \( \mathbb{P}^1 \to \mathbb{P}^d \), we see that \( y_d = -x_d + a_2x_d^2 + \ldots + a_dx_d^d \) in \( A(\mathbb{P}^d) \). Hence \( y_{d+1} = 0 \) and we conclude. \[\square\]
Corollary 1.4 Chern classes are always nilpotent.

Proof: Let \( L \to X \) be a line bundle. By Jouanolou’s trick (II) there is an affine bundle \( p: P \to X \) such that \( P \) is an affine variety.

Now \( p^*L \) is generated by global sections; hence \( p^*L = f^*(\xi_d) \) for some morphism \( f: P \to \mathbb{P}^d \). It follows that \( p^*c_1(L) = f^*(y_d) \) is nilpotent, since so is \( y_d \), and we see that \( c_1(L) \) is nilpotent, because \( p^* \) is an isomorphism. We conclude since any Chern class, after a base change injective in cohomology, is a sum of products of Chern classes of line bundles. \( \square \)

2 Universal Property of the \( K \)-theory

Theorem 2.1 If a cohomology theory \( A \) follows the group law \( x + y - xy \) of the \( K \)-theory, there is a unique morphism of cohomology theories \( \varphi: K \to A \).

Proof: Let \( E \to X \) be a vector bundle. Due to (II), we have \( E = \text{rk } E - c_1^K(E^*) \) in \( K(X) \); hence the unique possible morphism \( \varphi: K \to A \) is

\[
\varphi(E) := \text{rk } E - c_1^A(E^*). \tag{5}
\]

Now, since \( \text{rk} \) and \( c_1^A \) are additive, \( \varphi \) is an additive function on the vector bundles over \( X \). Therefore \( \varphi \) defines a group morphism \( \varphi: K(X) \to A(X) \), and \( \varphi \) commutes with inverse images because so do the rank, and \( c_1^A \) by (II).

This group morphism \( \varphi \) preserves products of line bundles because \( A \) follows the law \( x + y - xy \),

\[
\varphi(L_1 \otimes L_2) = 1 - c_1^A(L_1^* \otimes L_2^*) = 1 - c_1^A(L_1^*) - c_1^A(L_2^*) + c_1^A(L_1^*)c_1^A(L_2^*)
\]

\[
= (1 - c_1^A(L_1^*)) \cdot (1 - c_1^A(L_2^*)) = \varphi(L_1) \cdot \varphi(L_2).
\]

Hence \( \varphi \) is a ring morphism, \( \varphi(ab) = \varphi(a)\varphi(b) \), since we may assume (by the splitting principle) that \( a, b \in K(X) \) are sums and differences of line bundles.

It is only left for us to prove that \( \varphi \) preserves direct images. It preserves Chern classes of line bundles,

\[
\varphi(c_1^K(L)) = \varphi(1 - L^*) = 1 - \varphi(L^*) = 1 - (1 - c_1^A(L)) = c_1^A(L).
\]

Therefore \( \varphi \) preserves fundamental classes of hypersurfaces, and the theorem follows from the following result. \( \square \)

Panin’s Lemma: (II) Let \( A \) and \( \tilde{A} \) be two cohomology theories on smooth quasi-projective varieties. If a natural transformation \( \varphi: A \to \tilde{A} \) preserves the first Chern class of the tautological line bundles \( \xi_d \to \mathbb{P}^d \) (i.e. \( \varphi(c_1^A(\xi_d)) = c_1^{\tilde{A}}(\xi_d) \)) then it preserves direct images:

\[
\varphi(f_*(a)) = \tilde{f}_*(\varphi(a)) \tag{6}
\]

for any projective morphism \( f: Y \to X \), and any element \( a \in A(Y) \).
**Proof:** Let $Y$ be a hypersurface of a smooth quasi-projective variety $X$. By the same argument as in Corollary 1.4, we deduce from Jouanolou’s trick that the natural transformation $\varphi : A \to \overline{A}$ preserves fundamental classes of hypersurfaces, $\varphi([Y]^A) = [Y]^A$.

Now let $f : Y \to X$ be a projective morphism. By definition, $f$ is the composition of a closed immersion $i : Y \to \mathbb{P}^n \times X$ with the natural projection $\pi_X : \mathbb{P}^n \times X \to X$. If (6) holds for $i$ and $\pi_X$, then it also holds for the composition $f = \pi_X \circ i$, and it is enough to prove (6) for a closed immersion $i : Y \to X$ and the canonical projection $\pi_X : \mathbb{P}^n \times X \to X$.

1. If equation (6) holds for the zero section $s : Y \to \bar{N} = \mathbb{P}(1 \oplus N_{Y/X})$ of the projective closure of the normal bundle $N_{Y/X}$, then it also holds for the closed immersion $i : Y \to X$.

**Proof:** Let $X'$ be the blow-up of $X \times \mathbb{A}^1$ along $Y \times 0$, so that we have a commutative diagram

$$
\begin{array}{c}
\bar{N} \xrightarrow{i_0} X' \xleftarrow{i_1} X \\
\downarrow s_0 \quad \quad \quad \quad \quad \quad \downarrow i \\
Y \xrightarrow{i_0} Y \times \mathbb{A}^1 \xrightarrow{i_1} Y
\end{array}
$$

Put $U = X' - (Y \times \mathbb{A}^1)$. By axiom 4, we have a commutative diagram

$$
\begin{array}{c}
\tilde{A}(U) \xrightarrow{j^*} \\
\downarrow \\
\tilde{A}(\bar{N}) \xleftarrow{i_0^*} \tilde{A}(X') \\
\downarrow s_0^* \quad \quad \quad \quad \quad \quad \downarrow i_* \\
\tilde{A}(Y) \xleftarrow{i_0^*} \tilde{A}(Y \times \mathbb{A}^1)
\end{array}
$$

Note that $(\text{Ker} \tilde{i}_0^*) \cap (\text{Ker} \tilde{j}^*) = 0$. Indeed the column is exact by axiom 3, and $\tilde{s}_0^*$ is injective because $s_0$ is a section of the natural projection $\bar{N} \to Y$. Now consider the commutative diagram

$$
\begin{array}{c}
\tilde{A}(U) \xrightarrow{j^*} \\
\downarrow \\
\tilde{A}(\bar{N}) \xleftarrow{i_0^*} \tilde{A}(X') \xrightarrow{i_1^*} \tilde{A}(X) \\
\downarrow \psi_1 \quad \quad \quad \quad \quad \quad \downarrow \psi_2 \quad \quad \quad \quad \quad \quad \downarrow \psi_3 \\
\tilde{A}(Y) \xleftarrow{i_0^*} \tilde{A}(Y \times \mathbb{A}^1) \xrightarrow{\sim} \tilde{A}(Y)
\end{array}
$$

where the vertical arrows are the difference of the morphisms that (6) asserts the coincidence ($\psi_1 = \tilde{s}_0^* \varphi - \varphi \tilde{s}_0^*$ and so on). The morphism $\psi_1$
is zero by hypothesis, the morphism $\Psi_2$ is zero because $\bar{i}_0^*\Psi_2 = 0$, $\bar{j}^*\Psi_2 = 0$ and $(\text{Ker } i_0^*) \cap (\text{Ker } j^*) = 0$, and we conclude that $0 = \Psi_3 = \bar{i}_*\varphi - \bar{\varphi}i_*$.  

2. Equation (6) holds for the zero section $s: Y \to \bar{E} = \mathbb{P}(1 \oplus E)$ of the projective closure of any vector bundle $E \to Y$.  

Proof: When $E = L$ is a line bundle, note that $\varphi(s_*(1)) = \bar{s}_*(1)$ since $Y$ is an hypersurface in $\bar{L}$ and that $s^*: A(\bar{L}) \to A(Y)$ is surjective. If $a = s^*b \in A(Y)$, then equation (6) holds:  

\[ \varphi(s_*a) = \varphi(s_*s^*b) = \varphi(bs_*(1)) = \varphi(b)s_*(1) = \bar{s}_*(s^*\varphi(b)) = \bar{s}_*(\varphi(a)). \]

Therefore by the previous point it also holds for the immersion of any closed hypersurface.

Now, if $E$ admits a filtration $\{E_i\}$ such that the quotients $E_i/E_{i-1}$ are line bundles, then (6) holds for the zero section $Y \to E_1$ and for the morphisms $E_1 \to E_2 \to \ldots \to E_r = E$; hence it holds for the composition $s: Y \to E$.

In general, we have a morphism $\pi: Y' \to Y$ such that $\bar{\pi}^*: \bar{A}(Y) \to \bar{A}(Y')$ is injective and $E' = \pi^*E$ admits such filtration. Then (6) holds for the zero section $s': Y' \to E'$, and we conclude applying axiom 4 to the morphisms $\pi: E' \to E$ and $s: Y \to E$,

\[ \bar{\pi}^*\bar{s}'_*\varphi = \bar{s}'_*\bar{\pi}^*\varphi = \bar{s}'_*\varphi\bar{\pi}^* = \varphi\bar{s}'_*\bar{\pi}^* = \varphi\bar{s}'_*s_* = \bar{s}'_*\varphi s_. \]

3. If equation (6) holds for the projection $p: \mathbb{P}^n \to \text{pt}$ onto a point, then it also holds for the canonical projection $\pi_X: \mathbb{P}^n \times X \to X$.  

Proof: It follows from axiom 5.

4. Equation (6) holds for the projection $p: \mathbb{P}^n \to \text{pt}$ onto a point.  

Consider the closed immersion $i: \mathbb{P}^{n-1} \to \mathbb{P}^n$ and set

\[ A = A(\text{pt}), \ x_n = c_1^A(\xi_n^*) = i_*(1) \in A(\mathbb{P}^n), \]

\[ \bar{A} = \bar{A}(\text{pt}), \ \bar{x}_n = c_1^\bar{A}(\bar{\xi}_n^*) = \bar{i}_*(1) \in \bar{A}(\mathbb{P}^n). \]

By hypothesis $\varphi(x_n) = \bar{x}_n$; hence $\varphi(x_n^*) = \bar{x}_n^*$ and by axiom 6 the ring morphism $\varphi: A(\mathbb{P}^n) \to \bar{A}(\mathbb{P}^n)$ induces an isomorphism of $\mathcal{A}$-algebras $A(\mathbb{P}^n) \otimes_A \bar{A} = \bar{A}(\mathbb{P}^n)$.

We have to check that the $\mathcal{A}$-linear map $\bar{p}_*: \bar{A}(\mathbb{P}^n) \to \bar{A}$ is obtained by base change of the $\mathcal{A}$-linear map $p_*: A(\mathbb{P}^n) \to A$.

Let $\Delta_n = \Delta_*(1) \in A(\mathbb{P}^n \times \mathbb{P}^n) = A(\mathbb{P}^n) \otimes_A A(\mathbb{P}^n)$ be the fundamental class of the diagonal immersion $\Delta: \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$. We have

\[ (p_* \otimes 1)(\Delta_n) = (p \times 1)_*\Delta_*(1) = \text{Id}_*(1) = 1 \in A(\mathbb{P}^n), \]

where $p \times 1: \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ is the second projection. That is to say, $p_*$ corresponds to the unity, by means of the polarity $A(\mathbb{P}^n)^* \to A(\mathbb{P}^n)$, $\omega \mapsto (\omega \otimes 1)(\Delta_n)$, defined by the diagonal.
According to the next lemma, the linear map \( p_* : \mathbb{A}(\mathbb{P}^n) \to \mathbb{A} \) is fully determined by this condition. Since the fundamental class of the diagonal is stable by the base change \( A \to \bar{A} \), because (6) holds for closed immersions, we conclude that \( p_* \) also is stable.

**Lemma 2.2** The metric \( \Delta_n \in \mathbb{A}(\mathbb{P}^n) \otimes_{\mathbb{A}} \mathbb{A}(\mathbb{P}^n) \) of the diagonal is non-singular.

**Proof:** By induction on \( n \) we prove that

\[
\Delta_n = \sum_{r,s=0}^{n} a_{rs} x_r^n \otimes x_s^n = \begin{pmatrix}
0 & \ldots & 0 & 1 \\
& & \ddots & \\
& & & \ddots & \bullet \\
& & & & \ddots & \bullet \\
1 & \bullet & \ldots & \bullet
\end{pmatrix}
\]

where \( a_{rs} = 0 \) when \( r + s < n \), and \( a_{rs} = 1 \) when \( r + s = n \). Indeed,

\[
i_*(x_{n-1}^r) = i_* i^*(x_n^r) = x_n^r \cdot i_*(1) = x_n^{r+1},
\]

and by axiom 4 we have that the fundamental class \((1 \otimes i_*)(\Delta_{n-1})\) of the diagonal of \( \mathbb{P}^{n-1} \) in \( \mathbb{P}^{n-1} \times \mathbb{P}^n \) is just \((i^* \otimes 1)(\Delta_n)\). Note that

\[
(i^* \otimes 1)(\Delta_n) = \sum_{r=0}^{n-1} \sum_{s=0}^{n} a_{rs} x_{n-1}^r \otimes x_n^s,
\]

\[
(1 \otimes i_*)(\Delta_{n-1}) = \sum_{r=0}^{n-1} \sum_{s=0}^{n-1} a'_{rs} x_{n-1}^r \otimes x_{n-1}^s,
\]

where \( \Delta_{n-1} = \sum_{r,s} a'_{rs} x_{n-1}^r \otimes x_{n-1}^s \).

By induction on \( n \), we obtain the result for the coefficients \( a_{rs}, r < n \).

By symmetry we also obtain it for \( a_{rs}, s < n \), and we conclude.

With Panin’s lemma we can also compute the possible direct images on a given cohomology theory.

Let us consider the “roots” \( \alpha_1, \ldots, \alpha_r \) of a vector bundle \( E \to X \) of rank \( r \). For any formal power series \( F(t) = \sum a_n t^n \in \mathbb{A}(pt)[[t]] \) we put

\[
F_+(E) = F(\alpha_1) + \ldots + F(\alpha_r) \in \mathbb{A}(X),
\]

where we consider \( F(\alpha_1) + \ldots + F(\alpha_r) \) as a power series in the elementary symmetric functions, which are just the Chern classes of \( E \). It is well defined since Chern classes are nilpotent, and it is an additive function on the vector bundles over \( X \), so defining a functorial group morphism \( F_+ : K(X) \to \mathbb{A}(X) \), named **additive extension** of \( F \).

Analogously, let \( F(t) = a_0 + a_1 t + \ldots \) be an invertible series. We put

\[
F_+(E) = F(\alpha_1) \cdot \ldots \cdot F(\alpha_r) \in \mathbb{A}(X)^*,
\]

and we obtain the **multiplicative extension** \( F_+ : K(X) \to \mathbb{A}(X)^* \) of \( F \).
Theorem 2.3 Let $A$ be a cohomology theory, and $f^\text{new}_*\in A$. For any projective morphism $f: Y \to X$ denote $T_f := T_Y - f^*T_X \in K(Y)$ the virtual relative tangent bundle. Then there exist an invertible series $F(t) \in A[[t]]$ such that

$$f^\text{new}_*(a) = f_*(F_*(T_f)^{-1} \cdot a).$$

Moreover, every invertible series defines new direct images by the preceding formula.

Proof: From Corollary 1.3 we have that $A(\mathbb{P}^d) = A(\mathbb{P}^d) = A(\text{pt})[c_1(\xi_d)]/(c_1(\xi_d)^{d+1})$ for all $d$. Therefore it follows that there exists a series $b_0 + b_1 t + \cdots \in A(\text{pt})[[t]]$ such that

$$c_1^\text{new}(\xi_d) = b_0 + b_1 c_1(\xi_d) + \cdots$$

for all $d$. For $d = 0$ we have $\mathbb{P}^d = \text{pt}$ so that $b_0 = 0$. For $d = 1$ consider the closed immersion $i: \text{pt} \to \mathbb{P}^1$ with open complement $j: \mathbb{A}^1 \to \mathbb{P}^1$. There are exact sequences

$$A(\text{pt}) \xrightarrow{i_*^\text{new}} A(\mathbb{P}^1) \xrightarrow{j^*} A(\mathbb{A}^1)$$

where $\text{Ker } j^* = A(\text{pt}) \cdot c_1(\xi_1) = A(\text{pt}) \cdot c_1^\text{new}(\xi_1)$. Since $c_1^\text{new}(\xi_1) = b_1 \cdot c_1(\xi_1)$ we conclude that $b_1$ is invertible. We set

$$F(t) = b_1 + b_2 t + \cdots .$$

Note that $c_1^\text{new}(L) = c_1(L)F(c_1(L))$.

For any projective morphism $f: Y \to X$ consider the map $f_*(F_*(T_f)^{-1}) : A(Y) \to A(X)$. The pair $(A, f_*(F_*(T_f)^{-1}))$ is also a cohomology theory. Indeed, all the axioms are easy to check, except the last one. Put $x = x_E \in A(\mathbb{P}(E))$ and $y = x^\text{new}_E = xF(x) = x + \ldots \in A(\mathbb{P}(E))$, so that $y^n = x^n + \ldots$, and we have $0 = x^n = y^n$ for some exponent $d$. Since the powers of $x$ generate the $A(X)$-module $A(\mathbb{P}(E))$, so do the powers of $y$. Now, since $A(\mathbb{P}(E))$ is a free $A(X)$-module of rank $r + 1$, we conclude that $1, \ldots, y^r$ define a basis (just consider the characteristic polynomial of the endomorphism defined by $y$).

We can explicitly compute in $(A, f_*(F_*(T_f)^{-1}))$ fundamental classes of a smooth closed hypersurface $i: Y \to X$:

$$i_*\left(F_*(N_{Y/X}) \cdot 1\right) = i_*\left(F_*(i^*L_Y)\right) = F_*(L_Y)i_*(1) = [Y] \cdot F([Y]).$$

Therefore the first Chern class of line sheaf $L$ is $s_0^*s_0_*(F_*(s_0^*L) \cdot 1) = c_1(L)F(c_1(L))$, because $c_1(L) = s_0^*s_0_*(1)$.

Consider the identity $A \to A$. Applying Panin’s lemma to $(A, f^\text{new}_*)$ and $(A, f_*(F_*(T_f)^{-1}))$ we conclude.

\[\square\]
3 Riemann-Roch Theorem

Let $A$ be cohomology theory following the additive law. When we consider formal series with rational coefficients, the additive and multiplicative formal groups are isomorphic, because of the exponential series, and we may modify the direct image of $A \otimes \mathbb{Q}$ with an exponential so that the new theory follows the multiplicative law $x + y - xy$ of the $K$-theory.

Since $e^{at} = 1 - (1 - e^{at})$, we must fix a formal series $F(t)$ such that

$$c_{1}^{\text{new}}(L_x) = 1 - e^{ax} = xF(x). \quad (7)$$

Now, $1 - e^{at} = -at + \ldots$, so that it is convenient to fix $a = -1$.

Hence, so as to transform the additive law of $A \otimes \mathbb{Q}$ into a multiplicative law, just modify the direct image with the formal series

$$F(t) = \frac{1 - e^{-t}}{t} = 1 - \frac{t}{2!} + \frac{t^2}{3!} + \ldots \quad (8)$$

so that the new cohomology theory $A_{\mathbb{Q}}^{\text{new}} = (A \otimes \mathbb{Q}, f_{*}^{\text{new}})$ follows the multiplicative group law $x + y - xy$. By the universal property of the $K$-theory, there exists a unique morphism of cohomology theories

$$\text{ch}: K \rightarrow A_{\mathbb{Q}}^{\text{new}}$$

and this is just Grothendieck’s Riemann-Roch theorem. In fact this ring morphism $\text{ch}: K(X) \rightarrow A(X) \otimes \mathbb{Q}$ is the Chern character, the additive extension of the series $e^t$, because by (8) we have

$$\text{ch}(L_x) = 1 - c_{1}^{\text{new}}(L_x) = 1 - c_{1}^{\text{new}}(L_{-x}) = 1 - (1 - e^x) = e^x.$$

If we consider the multiplicative extension Td of the series

$$F(t)^{-1} = \frac{t}{1 - e^{-t}} = 1 + \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{720} + \ldots,$$

usually named Todd class, then we obtain

**Grothendieck’s Riemann-Roch Theorem:** Let $A$ be a cohomology theory on the smooth quasi-projective varieties over a field following the additive law. For any projective morphism $f: Y \rightarrow X$, we have a commutative square

$$\begin{array}{ccc}
K(Y) & \xrightarrow{f_{*}} & K(X) \\
\downarrow \text{Td}(T_Y) \cdot \text{ch} & & \downarrow \text{Td}(T_X) \cdot \text{ch} \\
A(Y) \otimes \mathbb{Q} & \xrightarrow{f_{*}} & A(X) \otimes \mathbb{Q}
\end{array}$$

**Proof:** Since $\text{ch}: K \rightarrow A_{\mathbb{Q}}^{\text{new}}$ preserves direct images,

$$\text{ch}(f_{*}(y)) = f_{*}^{\text{new}}(\text{ch}(y)) = f_{*} [F(f_{*}^{*}T_X - T_Y) \text{ch}(y)]$$

$$= F(T_X) f_{*} [F(T_Y)^{-1} \text{ch}(y)] = \text{Td}(T_X)^{-1} f_{*} [\text{Td}(T_Y) \text{ch}(y)].$$
Definition: A graded cohomology theory is a cohomology theory with values in the category of graded commutative rings (remark that elements of negative degree are assumed to be null)

\[ A^\bullet (X) = \bigoplus_{n \geq 0} A^n (X), \]

such that, for any projective morphism \( f : Y \to X \) between connected smooth quasi-projective varieties, the direct image \( f_* : A^n (Y) \to A^{n+d} (X) \) changes the degree in the codimension \( d = \dim X - \dim Y \).

Morphisms of graded cohomology theories are defined to be homogeneous (degree preserving) morphisms of cohomology theories.

Remark that the fundamental class of a smooth closed subvariety \( Y \to X \) of codimension \( d \) is in \( A^d (X) \), that the Chern class of a line bundle \( L \to X \) is in \( A^1 (X) \) and that, in general, \( c_n (E) \) is in \( A^n (X) \).

Examples: The graded \( K \)-theory, the Chow ring and the singular cohomology ring in the complex case, are graded cohomology theories.

Lemma 3.1 Any graded cohomology theory follows the additive group law

\[ c_1 (L_x \otimes L_y) = x + y. \]

Proof: Let us consider the line bundles \( \pi_1^* \xi_m \) and \( \pi_2^* \xi_n \) on \( \mathbb{P}^m \times \mathbb{P}^n \), and the component \( A^0 \) of degree 0 in the cohomology ring of a point.

According to axiom 6 we have \( A^1 (\mathbb{P}^m \times \mathbb{P}^n) = A^0 (\pi_1^* x_m) \oplus A^0 (\pi_2^* x_n) \), so that \( c_1 (\pi_1^* \xi_m \otimes \pi_2^* \xi_n) = a (\pi_1^* x_m) + b (\pi_2^* x_n) \), and it is easy to show that \( a = b = 1 \).

Now, by Jouanolou’s trick we have \( p^* L_x = f^* (\pi_1^* \xi_m) \) and \( p^* L_y = f^* (\pi_2^* \xi_n) \) for some morphism \( f : P \to \mathbb{P}^m \times \mathbb{P}^n \), so that

\[ p^* c_1 (L_x \otimes L_y) = f^* c_1 (\pi_1^* \xi_m \otimes \pi_2^* \xi_n) = f^* (c_1 (\pi_1^* \xi_m) + c_1 (\pi_2^* \xi_n)) = p^* (x + y). \]

Since \( p^* \) is an isomorphism, we conclude. □

Theorem 3.2 Let \( A^\bullet \) be a graded cohomology theory on smooth quasi-projective varieties over a perfect field. There exists a unique morphism of graded cohomology theories \( G K^\bullet \to A^\bullet \otimes \mathbb{Q} \).

Proof: Let \( X \) be a smooth quasi-projective variety and let \( Y \to X \) be a closed subvariety of codimension \( d \). If \( Y \) is smooth, by the Riemann-Roch theorem for the closed immersion \( Y \to X \), we have

\[ \text{ch}(\mathcal{O}_Y) = \text{ch}(\mathcal{i}_! (1)) = [Y] + \ldots \in \bigoplus_{n \geq d} A^n (X) \otimes \mathbb{Q}. \]

In general, since \( k \) is perfect, any closed subvariety \( Y \) is smooth outside a closed set \( Y_{\text{sing}} \) of bigger codimension.
If we consider the open immersion \( j: U = X - Y_{\text{sing}} \to X \), then by axiom 3 we have injective morphisms (recall that \( A^i(Y_{\text{sing}}) = 0 \) when \( i < 0 \))

\[
j^*: A^n(X) \otimes \mathbb{Q} \to A^n(U) \otimes \mathbb{Q}, \ n \leq d.
\]

Since \( j^*(\text{ch}(\mathcal{O}_Y)) = \text{ch}(j^!\mathcal{O}_Y) = [Y \cap U] + \ldots \in A^*(U) \otimes \mathbb{Q} \), we see that \( \text{ch}: K^0(X) \to A^*(X) \otimes \mathbb{Q} \) preserves filtrations, \( \text{ch}(F^d(X)) \subseteq \oplus_{n \geq d} A^n(U) \otimes \mathbb{Q} \).

Hence it induces a homogeneous ring morphism \( \varphi: GK^*(X) \to A^*(X) \otimes \mathbb{Q} \) preserving inverse images and, by Panin’s lemma, to prove that it preserves direct images, we have to show that it preserves the Chern class of any line sheaf \( L \).

Now, if we put \( \alpha = c_1(L) \in A^1(X) \otimes \mathbb{Q} \), then

\[
\varphi(e_1^{GK}(L)) = \varphi([L - 1]) = [\text{ch}(L - 1)] = [e^\alpha - 1] = [\alpha + \ldots] = \alpha.
\]

This is the unique possible morphism of graded theories \( \varphi: GK^* \to A^* \otimes \mathbb{Q} \), since we have \( \varphi([\mathcal{O}_Y]) = \varphi([Y]^{GK}) = [Y]^A \) for any smooth closed subvariety \( Y \to X \), and the above argument shows that this condition fully determines \( \varphi([\mathcal{O}_Y]) \) when \( Y \) is an arbitrary closed subvariety of \( X \) (recall that these classes \([\mathcal{O}_Y]\) generate \( GK^*(X) \) as a group). □

### 4 Applications

1. Theorem 3.2 shows that any numerical invariant (global intersection numbers, Chern numbers,...) defined in \( A^*(X) \otimes \mathbb{Q} \) coincides with the number obtained in \( GK^*(X) \otimes \mathbb{Q} \). For example, in the case of the singular cohomology of a complex projective smooth variety \( X \), the topological self-intersection number of the diagonal in \( X \times X \) is just the topological Euler-Poincaré characteristic \( \chi_{\text{top}}(X) \), so that it coincides with the algebraic self-intersection number,

\[
\chi_{\text{top}}(X) = \sum_{p,q} (-1)^{p+q} \dim H^p(X, \Omega^q_X),
\]

and, in the case of a smooth projective curve \( C \) we obtain the coincidence of the topological genus, \( \chi_{\text{top}}(C) = 2 - 2g_{\text{top}} \), with the algebraic genus \( g = \dim H^0(C, \Omega^1_C) \).

Moreover, if \( X \) is connected of dimension \( d \), then \( \chi_{\text{top}}(X) \) is the degree of the topological Chern class \( c_d(T_X) \), so that it also coincides with the degree of the algebraic Chern class \( c_d^{GK}(T_X) \), which is just the number of zeroes of any algebraic tangent vector field with isolated singularities.

2. The Chern character and Todd class of a locally free sheaf \( E \) of rank \( r \) are

\[
\text{ch}(E) = \sum_i e^{\alpha_i} = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \ldots
\]

\[
\text{Td}(E) = \prod_i (1 + \frac{c_i}{r} + \frac{c_i^2}{2r^2} + \ldots) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{72}c_1c_2 + \ldots
\]
Let $C$ be a smooth projective irreducible curve over an algebraic closed field and $K := c_1(\Omega_C) = -c_1(T_C) \in A^1(C) \otimes \mathbb{Q}$ be the class of a canonical divisor. Then the Riemann-Roch theorem for the projection $\pi: C \to p$ onto a point gives

$$\chi(C, E) = \pi_!(E) = \pi_* (\text{Td}(T_C) \text{ch}(E)) = \deg c_1(E) - \frac{1}{2} \deg K.$$  

If we apply it to $E = \mathcal{O}_C$ we obtain

$$1 - g = \chi(C, \mathcal{O}_C) = -\frac{1}{2} \deg K.$$  

If we take $E = L_D$ then $c_1(L_D) = [D]$ so that $\deg c_1(L_D) = \deg D$. We obtain

$$\dim H^0(C, L_D) - \dim H^1(C, L_D) = \deg D + 1 - g.$$  

By Serre’s duality $\dim H^1(C, L_D) = \dim H^0(C, K - D)$ and we obtain the classic formula for curves.

If $S$ is a smooth projective surface, $K := c_1(\Omega_S^2) = -c_1(T_S) \in A^1(S) \otimes \mathbb{Q}$ is the class of a canonical divisor, and we put $\chi_{\text{top}} := \deg c_2(T_S)$. The Riemann-Roch theorem for the projection $\pi: C \to p$ onto a point gives

$$\chi(S, E) = \pi_!(E) = \pi_* (\text{Td}(T_S) \text{ch}(E))$$

$$= \frac{1}{12}(K^2 + \chi_{\text{top}}) - \frac{1}{2} K \cdot c_1(E) + \frac{1}{2} c_1(E)^2 - \deg c_2(E).$$

When $E$ is the trivial line bundle, we obtain Noether’s equality

$$\chi(S, \mathcal{O}_S) = \frac{1}{12}(K^2 + \chi_{\text{top}}).$$

3. In the case of a closed smooth hypersurface $i: Y \to X$, the Riemann-Roch theorem gives

$$i_*(\text{Td}(T_Y)) = \text{ch}(\mathcal{O}_Y) \cdot \text{Td}(T_X) = (1 - e^{-Y}) \cdot \text{Td}(T_X)$$

$$Y - \frac{1}{2} i_* (K_Y) + \ldots = (Y - \frac{1}{2} Y^2 + \ldots) (1 - \frac{1}{2} K_X + \ldots)$$

and we obtain the adjunction formula $i_* (K_Y) = Y (K_X + Y)$.

4. In the case of a closed smooth subvariety $i: Y \to X$ of codimension $d$, the Riemann-Roch theorem gives $\text{ch}(\mathcal{O}_Y) = i_* (\text{Td}(N_{Y/X})) = Y + \ldots$. Since

$$a_1^r + \ldots + a_n^r = p_r(s_1, \ldots, s_{r-1}) + (-1)^{r+1} rs_r$$

for some polynomial $p_r$, where $s_i$ is the $i$-th elementary symmetric function of $a_1, \ldots, a_n$ (just consider the roots of the polynomial $x^r - 1$), we see that

$$c_i(\mathcal{O}_Y) = 0 \quad \forall \ i < d, \quad c_d(\mathcal{O}_Y) = (-1)^d (d - 1)! \cdot Y$$

modulo torsion; that is to say, in $A^\bullet (X) \otimes \mathbb{Q}$.  

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5. If $\omega$ is a rational 1-form on a connected smooth projective surface $S$, let us determine $c_2(T_S) \in A^2(S) \otimes \mathbb{Q}$ in terms of the singularities of $\omega$. Let $\Sigma$ be the field of rational functions on $S$. Then

$$(\omega)(U) = \{ f\omega \in \Omega^1_S(U) : f \in \Sigma \}$$

is a line sheaf, and $(\omega) \simeq L_D$, where $D$ is the divisor of zeros and poles of the differential form $\omega$. We have an exact sequence

$$(*) \quad 0 \rightarrow (\omega) \rightarrow \Omega^1_S \xrightarrow{\wedge \omega} \Omega^2_S \otimes L_{-D} \rightarrow \mathcal{E} \rightarrow 0$$

If $x, y$ are parameters at a point $p$, and $\omega = h(dx + gdy)$, where $f, g \in \mathcal{O}_p$ have no common factor, a local equation of $D$ is $h = 0$, and $\mathcal{E}_p = \mathcal{O}_p/(f, g)$. Now, the Riemann-Roch theorem for the immersion $p \hookrightarrow S$ gives that $\text{ch}(\mathcal{O}_S/m_p) = p$, where $m_p$ denotes the maximal ideal of $p$; hence we have

$$\text{ch}(\mathcal{E}) = \sum_p l(\mathcal{E}_p) \cdot p,$$

where $l(\mathcal{E}_p)$ stands for the length of the $\mathcal{O}_{S,p}$-module $\mathcal{E}_p$.

If $K := c_1(\Omega^2_S) = c_1(\Omega^1_S) \in A^1(S) \otimes \mathbb{Q}$ denotes the class of a canonical divisor, the exact sequence $(*)$ gives the **Zeuthen-Segre invariant**:

$$\text{ch}(L_D) + \text{ch}(L_{K-D}) = \text{ch}(\Omega^1_S) + \text{ch}(\mathcal{E}),$$

$$\epsilon^D + \epsilon^{K-D} = 2 + K + \frac{1}{2}K^2 - c_2(T_S) + \text{ch}(\mathcal{E}),$$

$$c_2(T_S) = D(K - D) + \sum_p l(\mathcal{E}_p) \cdot p.$$

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