Hidden functional relation in Large-\(N\) Quark-Monopole system at finite temperature

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Abstract

The quark-monopole potential is computed at finite temperature in the context of AdS/CFT correspondence. It is found that the potential is invariant under \(g \rightarrow 1/g\) and \(U_T \rightarrow U_T/g\). As in the quark-quark case there exists a maximum separation between quark and monopole, and \(L\)-dependence of the potential exhibits a bifurcation behavior. We find a functional relation

\[
dE_{QM}^{\text{Reg}}/dL = \left[ \left( 1/E_{(1,0)}^{\text{Reg}}(U_0) \right)^2 + \left( 1/E_{(0,1)}^{\text{Reg}}(U_0) \right)^2 \right]^{-1/2}
\]

which is responsible for the bifurcation. The remarkable property of this relation is that it makes a relation between physical quantities defined at the AdS boundary through a quantity defined at the bulk. The physical implication of this relation for the existence of the extra dimension is speculated.

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One of the most important effects of AdS/CFT correspondence \cite{1,2} is that it makes it possible to extract the highly non-trivial quantum effect in the large N super Yang-Mills theories from the classical configuration of string in the 10-dimensional $AdS_5 \times S^5$ background. In fact, the expectation value of the rectangular Wilson loop calculated in the context of AdS/CFT correspondence \cite{3} yields an analytical expression for the interquark potential $E_{QQ}$ which falls off as Coulomb potential, indicating that the theory at the AdS boundary is conformally invariant.

Subsequently, Witten \cite{4} suggested the generalization of AdS/CFT correspondence at finite temperature where temperature is defined as an inverse of the compactified Euclidean time. In this context the Wilson loop is examined at finite temperature \cite{5,6}. The main difference of the finite temperature case from the zero temperature one is the appearance of a cusp (or bifurcation point) in the plot of interquark potential-vs-interquark distance.

Same kind of bifurcation point is also realized in the Euclidean point particle theory when the finite temperature generalization is considered. The appearance of the bifurcation point at the point particle theories is discussed at quantum mechanical \cite{7} and field theoretical \cite{8} levels. In this case the simple functional relation $dS_E/dP = \mathcal{E}$, where $S_E$, $P$, and $\mathcal{E}$ are Euclidean action, period, and energy of the classical particle, is responsible for the appearance and disappearance of the bifurcation point. In fact, the appearance of the bifurcation point indicates the instability of the upper branch in the action-temperature diagram. This means there are multiple zero modes at the bifurcation point regardless of the symmetry of the underlying theory. This fact is shown explicitly by computing the spectrum of the fluctuation operator numerically \cite{9}. It is also possible to prove the multiple zero modes at the bifurcation point analytically \cite{10}.

Hence, it is natural to ask what kind of functional relation governs the bifurcation point realized at the Wilson loop calculation in the context of AdS/CFT correspondence. The answer to this question for the rectangular Wilson loop is given at Ref. \cite{11}, where the relation
\[
\frac{dE_{QQ}}{dL} = \frac{\sqrt{U_0^4 - U_T^4}}{2\pi R^2}
\]  

is explicitly derived. Of course, \(E_{QQ}\) and \(L\) are interquark potential and interquark distance, respectively. In Ref. [11] the righthand side of Eq. (1) is interpreted as a regularized energy of the trivial string configuration \(U = U_0\), i.e. \(E_{\text{Reg}}(U_0)\).

It is worthwhile noting that \(U_0\) and \(U_T\) are locations of minimum point of string configuration and the horizon respectively. Hence, they are not defined at the AdS boundary, i.e. \(U = \infty\), while \(E_{QQ}\) and \(L\) are physical quantities defined at the boundary. The remarkable feature of Eq. (1) is that it is a relation between the physical quantities defined at the AdS boundary via a quantity defined at the bulk space. This means it is impossible to derive Eq. (1) if we confine ourselves in the 4-dimensional world volume without perception of the fifth dimension. In this sense the relation (1) implies the physical importance of the extra dimension. Same kind of the functional relation is derived at the two circular Wilson loop case [12] and is used for the analysis of the finite temperature Gross-Ooguri phase transition [13,14].

In this paper we will show that there is similar functional relation at quark-monopole system. The existence of the same kind of functional relation makes us to expect that a formula of this type has some universal validity.

The quark-monopole system at zero temperature is considered in Ref. [15] by considering 3-string junction [16–21]. The quark-monopole-dyon system is also discussed at finite temperature [22] with highlighting the issues of screening and clustering. Due to the thermodynamic nature of our hidden functional relation we believe that it plays more important role when many particles are involved in the system. In this context it is interesting to examine the role of this relation in the multi-particle system located in the external temperature background. We hope to visit this issue in the near future.

We start with the classical action of the \((p,q)\) string worldsheet

\[
S_{(p,q)} = \frac{1}{2\pi} \int d\tau d\sigma \sqrt{\left( p^2 + \frac{q^2}{g^2} \right) \det G_{MN} \partial_\alpha X^M \partial_\beta X^N}
\]
in the near extremal Euclidean Schwarzschild-AdS$_5 \times S^5$ background \cite{23}

$$ds_E^2 = \frac{U^2}{R^2} \left[ f(U)dt^2 + dx^i dx^i \right] + \frac{R^2 f(U)^{-1}}{U^2} dU^2 + R^2 d\Omega_5^2 \quad (3)$$

where \( f(U) = 1 - U^4_T/U^4 \) and \( R = (4\pi g_N)^{1/4} \). Here, we have chosen \( \alpha' = 1 \) for simplicity. The horizon parameter \( U_T \) is proportional to the external temperature \( T \) defined by \( T = U_T/(\pi R^2) \) \cite{6}.

After identifying the world sheet variables as \( \tau = t \) and \( \sigma = x \), one can show easily that for the static case the action \( S_{(p,q)} \) becomes

$$S_{(p,q)} = \frac{\bar{\tau}}{2\pi} \int dx \sqrt{\left( p^2 + \frac{q^2}{g^2} \right) \left[ U'^2 + \frac{U^4 - U^4_T}{R^4} \right]} \quad (4)$$

where the prime denotes differentiation with respect to \( x \), and \( \bar{\tau} \) is the entire Euclidean time interval.

The string configuration we consider is as follows: F-string and D-string starting from AdS boundary meet with each other at \( U = U_0 \). For the charge to be conserved \cite{24,25} we need (1,1) string starting at the string junction and ending at the horizon \cite{26}. In summary, the configuration is shown in Fig. 1.

From the world sheet action (4) it is easy to find the string configurations of the F-string and D-string:

$$\frac{U^4 - U^4_T}{\sqrt{U'^2 + \frac{(U^4 - U^4_T)^2}{R^4}}} = R^2 \sqrt{U^4 - U^4_T} \quad i = 1, 2 \quad (5)$$

where \( i = 1(= 2) \) is the result for the F-string(D-string). As noticed in Ref. \cite{13} \( U_i \) is not necessarily equal to \( U_0 \). Using Eq.(3) it is straightforward to derive

$$\Delta L = \frac{\alpha R^2 \sqrt{1 - \alpha^4_T}}{U_0} \int_\alpha^\infty \frac{dy}{\sqrt{(y^4 - \alpha^4_T)(y^4 - 1)}} \quad (6)$$

$$L - \Delta L = \frac{\beta R^2 \sqrt{1 - \beta^4_T}}{U_0} \int_\beta^\infty \frac{dy}{\sqrt{(y^4 - \beta^4_T)(y^4 - 1)}}$$

where
\[ \alpha = \frac{U_0}{U_1} \quad \alpha_T = \frac{U_T}{U_1} \quad (7) \]

\[ \beta = \frac{U_0}{U_2} \quad \beta_T = \frac{U_T}{U_2}. \]

Proceeding as quark-quark case \[3,5,6,11\] one can compute the contribution of F- and D-strings \( E_{(1,0)} \) and \( E_{(0,1)} \) to the quark-monopole potential:

\[ E_{(1,0)} = \frac{U_0}{2\pi\alpha} \int_\alpha^\infty dy \sqrt{\frac{y^4 - \alpha^4}{y^4 - 1}} \]

\[ E_{(0,1)} = \frac{U_0}{2\pi g\beta} \int_\beta^\infty dy \sqrt{\frac{y^4 - \beta^4}{y^4 - 1}}. \]

As expected both \( E_{(1,0)} \) and \( E_{(0,1)} \) are divergent. For the regularization we have to substract the quark mass \( M_q \equiv U_{\text{max}}/2\pi \) and the monopole mass \( M_m \equiv U_{\text{max}}/2\pi g \) from \( E_{(1,0)} \) and \( E_{(0,1)} \) respectively:

\[ E_{\text{Reg}}^{(1,0)} = \frac{U_0}{2\pi\alpha} \left[ \int_\alpha^\infty dy \left( \sqrt{\frac{y^4 - \alpha^4}{y^4 - 1}} - 1 \right) - \alpha \right] \]

\[ E_{\text{Reg}}^{(0,1)} = \frac{U_0}{2\pi g\beta} \left[ \int_\beta^\infty dy \left( \sqrt{\frac{y^4 - \beta^4}{y^4 - 1}} - 1 \right) - \beta \right]. \]

Hence, combining the contribution of (1,1) string the regularized quark-monopole potential \( E_{QM} \) becomes the following form:

\[ E_{QM} = E_{\text{Reg}}^{(1,0)} + E_{\text{Reg}}^{(0,1)} + \frac{U_0 - U_T}{2\pi} \sqrt{\frac{1 + g^2}{g}}. \]

The integration in Eq.(8) and (9) are analytically carried out in Appendix I and II in terms of the various elliptic functions. The final result of \( L \) and \( E_{QM} \) are

\[ L = \frac{R^2}{U_0} \left[ l(\alpha, \alpha_T) + l(\beta, \beta_T) \right] \]

\[ E_{QM} = \frac{U_0}{2\pi} \left[ h(\alpha, \alpha_T) + \frac{1}{g} h(\beta, \beta_T) \right] + \frac{U_0 - U_T}{2\pi} \frac{\sqrt{1 + g^2}}{g} \]

where

\[ l(x, y) = \frac{x}{4} \frac{\sqrt{2(1-y^2)}}{y^2} [F(\Phi(x, y), \kappa(y)) - F(\Phi(x, y), \kappa'(y))] \]

\[ h(x, y) = \frac{1}{4} \frac{\sqrt{2(1-y^2)}}{y^2} [\Phi(x, y) - \kappa'(y)] \]
\[
h(x, y) = -\sqrt{\frac{(x^2 - 1)(x^2 - y^2)}{(x^2 + 1)(x^2 + y^2)}} + \sqrt{\frac{2(1 + y^2)}{4x}} \left[ (1 - y)F(\Phi(x, y), \kappa(y)) + (1 + y)F(\Phi(x, y), \kappa'(y)) \right.
\]
\[\left. - 2E(\Phi(x, y), \kappa(y)) - 2E(\Phi(x, y), \kappa'(y)) \right] \]

\[
\Phi(x, y) = \sin^{-1} \sqrt{\frac{2x^2(1 + y^2)}{(x^2 + 1)(x^2 + y^2)}}
\]

\[
\kappa(y) = \frac{1 + y}{\sqrt{2(1 + y^2)}} \equiv \sqrt{1 - \kappa'^2(y)}
\]

and, \(F(\phi, k)\) and \(E(\phi, k)\) are usual elliptic integral of the first and second kinds. It is easy to show that \(L\) and \(E_{QM}\) in Eq. (11) have the correct zero-temperature limit.

Now, let us determine \(\alpha\) and \(\beta\) from the condition that the net force at the string junction is zero. It was conjectured by Schwarz [16] that such 3-string junctions with a zero net force corresponds to BPS saturated state and subsequently it is verified by world-sheet [18] and space-time [19] approaches.

Using string tensions \(T^{(1,0)} = \sqrt{U_0^4 - U_1^4}/(2\pi RU_0)\) and \(T^{(0,1)} = \sqrt{U_0^4 - U_1^4}/(2\pi gRU_0)\) one can show directly the condition for the zero net force is

\[
T^{(1,0)} \left( - \frac{U_2^4 - U_1^4}{U_0^4 - U_1^4}, \frac{U_0^4 - U_1^4}{U_0^4 - U_1^4} \right) + T^{(0,1)} \left( \frac{U_0^4 - U_2^4}{U_0^4 - U_1^4}, \frac{U_0^4 - U_2^4}{U_0^4 - U_1^4} \right)
\]
\[\sqrt{(1 + g^2)(U_0^4 - U_1^4)} \equiv \sqrt{(0, -1)} = 0. \tag{13}\]

From Eq. (13) one can show \(\tan(\theta - \pi/2) = 1/g\) where \(\theta\) is an angle between D-string and (1,1) string. It is interesting to realize the fact that the temperature does not affect the relative angles between strings.

Solving Eq. (13) one can obtain

\[
\alpha = \alpha_0 \equiv \nu \left( \frac{1 + g^2}{\nu^4 + g^2} \right)^{\frac{1}{4}} \tag{14}\]

\[
\beta = \beta_0 \equiv \nu \left( \frac{1 + g^2}{1 + g^2 \nu^4} \right)^{\frac{1}{4}}
\]
where $\nu = U_0/U_T$. It is worthwhile noting that $\alpha_0 \leftrightarrow \beta_0$ under the S-duality transformation $g \leftrightarrow 1/g$ as zero temperature case. At zero temperature this is the origin of S-duality. However, there is a subtle point in S-duality at finite temperature which we will return momentarily.

Using Eq.(14) the final form of the quark-monopole distance $L$ and potential $E_{QM}$ become

$$L = \frac{R^2}{U_0} \left[ l \left( \frac{\alpha_0}{\nu} \right) + l \left( \frac{\beta_0}{\nu} \right) \right]$$

$$E_{QM} = \frac{U_0}{2\pi} \left[ h \left( \frac{\alpha_0}{\nu} \right) + \frac{1}{g} h \left( \frac{\beta_0}{\nu} \right) + \left( 1 - \frac{1}{\nu} \right) \sqrt{1 + g^2} \right].$$

Now, let us discuss S-duality in detail. It is more intuitive to consider the zero temperature case first. Taking a zero-temperature limit ($\nu \to \infty$) in Eq.(15) yields

$$L^{(T=0)} = \frac{R^2}{U_0} \left[ l_0 \left( 1 + g^2 \right) \right]$$

$$E_{QM}^{(T=0)} = \frac{U_0}{2\pi} \left[ h_0 \left( 1 + g^2 \right) \right] + \frac{1}{g} h_0 \left( 1 + g^2 \right) + \sqrt{1 + g^2}.$$
zero temperature \[3\]. Furthermore, the coefficient of the Coulomb potential \(\xi_0\) is invariant under the S-duality transformation \(g \rightarrow 1/g\), so that \(E_{QM}^{(T=0)}\) is also invariant under the same transformation if and only if \(L^{(T=0)}\) is invariant, i.e. \(U_0 \rightarrow U_0/g\) under \(g \rightarrow 1/g\). This statement is also easily verified from Eq.(16) directly.

Now, let us consider the finite temperature case. Unlike the zero temperature case it is impossible to remove \(U_0\) from Eq.(15) directly. But it is easy to show that \(E_{QM}\) and \(L\) are invariant under the transformation \(g \rightarrow 1/g\) and \(U_T \rightarrow U_T/g\). It is interesting that not only the coupling constant but also the temperature parameter are transformed. This transformation might be the generalized S-duality transformation at the finite temperature.

From Eq.(15) one can plot the \(U_0\)-dependence of \(L\) which is shown at Fig. 2. As in the quark-quark case \(U_0\)-dependence of \(L\) exhibits monotonic and non-monotonic behaviors at zero and finite temperature cases respectively. Fig. 3 shows the \(L\)-dependence of \(E_{QM}\) at various temperature. As in the quark-quark case there exists an maximum separation \(L_*\) at finite temperature, which results in the bifurcation. Another interesting feature which Fig. 3 indicates is that there are two branches of \(E_{QM}\) at nonzero temperature, which merge smoothly at \(L = L_*\). If the distance between the quark and the monopole is greater than \(L_*\), the classical string configuration becomes unstable and, as a result, the strings attached to these particles are dropped on horizon separately. The regularized potential energy of this two non-interacting particle system is

\[
E_{iso}^{(Reg)} = -\frac{U_T}{2\pi} \left( 1 + \frac{1}{g} \right).
\]

It is interesting to note that \(E_{iso}^{(Reg)}\) is equal to \(E_{QM}\) in Fig. 3 at \(L = 0\) in the upper branch. It is easily proved by inserting the \(L = 0\) condition \(U_T = U_0\) into Eq.(15). One should note that there exists \(L^{**}\) at each value of nonzero \(U_T\) where \(E_{QM}\) has same value as that of the isolated system in the lower branch, which indicates a transition to free particle system. The interesting points \(L_*\) and \(L^{**}\) are explicitly depicted in Fig. 3 at \(U_T = 1.0\). If the system is given initially at the upper branch of \(E_{QM}\) with \(L < L^{**}\), thermal transition should take place to the lower branch. If, on the other hand, the system is given at either the upper or
the lower branch with \( L_{*} < L < L_{*} \), the thermal transition takes place to the two isolated particle system due to its energetical favor. In fact, this is an exactly same situation with the case of Gross-Ooguri phase transition between the catenoid and the two disconnected circular Wilson loops [12]. Also, same kind of thermal transition is discussed in Ref. [22] in the more complicated three particle system.

As emphasized in Ref. [11,12] the non-monotonic behavior of \( L \) in Fig. 2 and the appearance of the cusp in Fig. 3 strongly suggest that there is hidden functional relation in the Y-junction string system. This hidden functional relation is explicitly derived at Appendix III, which is

\[
\frac{dE_{QM}}{dL} = \frac{1}{2\pi R^2} \sqrt{\frac{U_0^4 - U_T^4}{g^2 + 1}}. \tag{20}
\]

It is intuitive to compare Eq.(20) with the functional relation of the quark-quark case (1) explicitly derived at Ref. [11]: The right-hand side of Eq.(1) is interpreted as the regularized energy of the trivial string configuration \( U = U_0 \), i.e. \( E_{Reg}(U_0) \). By the same way the right-hand side of Eq.(20) can be expressed in terms of the trivial F- and D-string configurations:

\[
\left[ \left( \frac{1}{E_{(1,0)}(U_0)} \right)^2 + \left( \frac{1}{E_{(0,1)}(U_0)} \right)^2 \right]^{-\frac{1}{2}} \tag{21}
\]

where

\[
E_{(1,0)}^{Reg}(U_0) = \sqrt{\frac{U_0^4 - U_T^4}{2\pi R^2}} \tag{22}
\]

\[
E_{(0,1)}^{Reg}(U_0) = \sqrt{\frac{U_0^4 - U_T^4}{2\pi gR^2}}.
\]

Eq.(21) might be a kind of sum rule in the point particle analogy of the string.

As in the quark-quark case Eq.(21) represents a functional relation between the physical quantities \( E_{QM} \) and \( L \) defined at the AdS boundary via a quantity which is not defined at the same boundary. This is the reason why it is impossible to derive this kind of relation when we work at four-dimensional world volume. The functional relations derived at Ref. [11,12] and this paper may give some insight into the physical importance of the fifth dimension.
and yield a conjecture that the fundamental phenomena in our world may be intricately related to the existence of the extra-dimension.

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Appendix I

In this appendix we derive $L$ in Eq.(11) by integrating Eq.(6) analytically in terms of elliptic functions. Let us define $I(\alpha, \alpha_T)$ such that

$$I(\alpha, \alpha_T) \equiv \int_{\alpha}^{\infty} \frac{dy}{\sqrt{(y^4 - \alpha_T^4)(y^4 - 1)}}.$$  \hspace{1cm} (A.1)

Then Eq.(6) shows

$$L = \frac{R^2}{U_0} [l(\alpha, \alpha_T) + l(\beta, \beta_T)]$$ \hspace{1cm} (A.2)

where

$$l(x, y) = x \sqrt{1 - y^4} I(x, y).$$ \hspace{1cm} (A.3)

Hence, computation of $I(\alpha, \alpha_T)$ completely determines the quark-monopole distance.

The first step for the computation of $I(\alpha, \alpha_T)$ is to split it into two parts as follows:

$$I(\alpha, \alpha_T) = \frac{1}{\alpha_T} [I_1(\alpha_T) - I_2(\alpha, \alpha_T)]$$ \hspace{1cm} (A.4)

where

$$I_1(\alpha_T) = \int_{\alpha}^{\infty} \frac{dy}{\sqrt{(1 - y^4)(1 - y^4/\alpha_T^4)}}$$ \hspace{1cm} (A.5)

$$I_2(\alpha, \alpha_T) = \int_{\alpha}^{\alpha_T} \frac{dy}{\sqrt{(1 - y^4)(1 - y^4/\alpha_T^4)}}.$$ \hspace{1cm}

Since $I_1(\alpha_T)$ can be read directly from $I_2(\alpha, \alpha_T)$ by taking a $\alpha \to \infty$ limit, it had better calculate $I_2(\alpha, \alpha_T)$ first.

Using formulas 587.00 and 237.00 of Ref. [27], it is straightforward to calculate $I_2(\alpha, \alpha_T)$:

$$I_2(\alpha, \alpha_T) = \frac{1}{4} \sqrt{\frac{2\alpha_T^2}{1 + \alpha_T^2}} \left[ F \left( \sin^{-1} \left( \sqrt{\frac{(\alpha^2 - 1)(\alpha^2 - \alpha_T^2)}{\alpha^2 - \alpha_T}} \right), \sqrt{\frac{a + 2}{2a}} \right) - \sqrt{\frac{a - 2}{2a}} \right]$$ \hspace{1cm} (A.6)

where $a = (1 + \alpha_T^2)/\alpha_T \geq 2.$
Taking $\alpha \to \infty$ limit in Eq. (A.6) one can obtain $I_1(\alpha_T)$ as follows:

$$I_1(\alpha_T) = \frac{1}{4} \sqrt{\frac{2\alpha_T^2}{1 + \alpha_T^2}} \left[ K\left(\sqrt{\frac{\alpha + 2}{2a}}\right) - K\left(\sqrt{\frac{\alpha - 2}{2a}}\right) \right]$$  \hspace{1cm} (A.7)

where $K(\kappa)$ is complete elliptic integral of first kind.

Inserting Eq. (A.6) and (A.7) into (A.4) one can describe $I(\alpha, \alpha_T)$ as a combination of complete and incomplete elliptic integrals. Using an addition formula

$$F(\theta, \kappa) + F(\phi, \kappa) = K(\kappa)$$  \hspace{1cm} (A.8)

$$\cot \phi = \sqrt{1 - \kappa^2} \tan \theta,$$

it is possible to express $I(\alpha, \alpha_T)$ in terms of only incomplete elliptic integrals as follows:

$$I(\alpha, \alpha_T) = \frac{1}{4} \sqrt{\frac{2}{\alpha_T^2(1 + \alpha_T^2)}} \left[ F(\Phi(\alpha, \alpha_T), \kappa(\alpha_T)) - F(\Phi(\alpha, \alpha_T), \kappa'(\alpha_T)) \right]$$  \hspace{1cm} (A.9)

where $\Phi(\alpha, \alpha_T)$ and $\kappa(\alpha_T)$ are defined at Eq. (12). Combining Eq. (A.2) (A.3) and (A.9) it is easy to derive $L$ in Eq. (13).

Appendix II

In this appendix we derive $E_{QM}$ in Eq. (11) by integrating Eq. (9). Firstly, let us define

$$J(\Lambda, \alpha, \alpha_T) \equiv \int_{\alpha}^{\Lambda} dy \sqrt{\frac{y^4 - \alpha_T^4}{y^4 - 1}}$$  \hspace{1cm} (B.1)

where we introduced a cutoff $\Lambda$ which will be taken to infinity at the final stage of calculation. The first step for the computation of $J(\Lambda, \alpha, \alpha_T)$ is also split it into two parts as follows:

$$J(\Lambda, \alpha, \alpha_T) = \alpha_T^2 \left[ J_1(\Lambda, \alpha_T) - J_2(\alpha, \alpha_T) \right]$$  \hspace{1cm} (B.2)

where

$$J_1(\Lambda, \alpha_T) = \int_{\alpha_T}^{\Lambda} dy \sqrt{\frac{1 - \frac{y^4}{\alpha_T^4}}{1 - y^4}}$$  \hspace{1cm} (B.3)

$$J_2(\alpha, \alpha_T) = \int_{\alpha_T}^{\alpha} dy \sqrt{\frac{1 - \frac{y^4}{\alpha_T^4}}{1 - y^4}}.$$

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As before $J_1(\Lambda, \alpha_T)$ can be read directly from $J_2(\alpha, \alpha_T)$ by taking $\alpha \to \Lambda$ limit. Hence, it had better calculate $J_2(\alpha, \alpha_T)$ first.

Next step is to split $J_2(\alpha, \alpha_T)$ into two parts again:

$$J_2(\alpha, \alpha_T) = -I_2(\alpha, \alpha_T) + \frac{1}{\alpha_T^4} J_3(\alpha, \alpha_T)$$  \hspace{1cm} \text{(B.4)}

where $I_2(\alpha, \alpha_T)$ is introduced in Eq.(A.3) and

$$J_3(\alpha, \alpha_T) = \int_{\alpha_T}^\alpha dy \frac{y^4}{\sqrt{(1-y^4) \left(1 - \frac{y^4}{\alpha_T^4}\right)}}.$$  \hspace{1cm} \text{(B.5)}

Since $I_2(\alpha, \alpha_T)$ is calculated explicitly in Appendix I, let us calculate $J_3(\alpha, \alpha_T)$ here.

Using formulas 587.03 and 237.00 of Ref. [27] it is straightforward to calculate $J_3(\alpha, \alpha_T)$ after tedious procedure:

$$J_3(\alpha, \alpha_T) = \frac{\alpha_T^2 \sqrt{\alpha_T}}{4} \left[ \sqrt{\frac{2}{a}(a-1)} F\left(\sin^{-1} \frac{\sqrt{(\alpha^2 - 1)(\alpha^2 - \alpha_T^2)}}{\alpha^2 - \alpha_T}, \sqrt{\frac{\alpha + 2}{2a}}\right) \right.$$

$$+ \sqrt{\frac{2}{a}(a+1)} F\left(\sin^{-1} \frac{\sqrt{(\alpha^2 - 1)(\alpha^2 - \alpha_T^2)}}{\alpha^2 + \alpha_T}, \sqrt{\frac{\alpha - 2}{2a}}\right)$$

$$- 2\sqrt{2}aE \left(\sin^{-1} \frac{\sqrt{(\alpha^2 - 1)(\alpha^2 - \alpha_T^2)}}{\alpha^2 - \alpha_T}, \sqrt{\frac{\alpha + 2}{2a}}\right)$$

$$- 2\sqrt{2}aE \left(\sin^{-1} \frac{\sqrt{(\alpha^2 - 1)(\alpha^2 - \alpha_T^2)}}{\alpha^2 + \alpha_T}, \sqrt{\frac{\alpha - 2}{2a}}\right)$$

$$+ \frac{2}{\sqrt{\alpha_T}} \frac{\sqrt{(\alpha^4 - 1)(\alpha^4 - \alpha_T^4)}}{\alpha(\alpha^2 - \alpha_T)} + \frac{2}{\sqrt{\alpha_T}} \frac{\sqrt{(\alpha^4 - 1)(\alpha^4 - \alpha_T^4)}}{\alpha(\alpha^2 + \alpha_T)} \right].$$  \hspace{1cm} \text{(B.6)}

where $a$ is introduced in Appendix I. Inserting Eq.(A.6) and (B.4) into (B.4) it is possible to compute $J_2(\alpha, \alpha_T)$ which is not described here due to its lengthy expression.

Taking a $\alpha \to \Lambda$ limit, it is also straightforward to compute $J_1(\Lambda, \alpha_T)$ which consists of some contribution of complete elliptic integrals and $\Lambda$-dependent term $\Lambda/\alpha_T^2$. Of course $\Lambda$-dependent term will be removed by regularization. Also using the addition formula (A.8) appropriately, one obtains the following expression finally:

$$J(\Lambda, \alpha, \alpha_T) = \Lambda - \alpha \sqrt{\frac{(\alpha^2 - 1)(\alpha^2 - \alpha_T^2)}{(\alpha^2 + 1)(\alpha^2 + \alpha_T^2)}}$$  \hspace{1cm} \text{(B.7)}
\[
+ \sqrt{2(1 + \frac{\alpha^2}{\nu^2})} \left[ (1 - \alpha_T) F(\Phi(\alpha, \alpha_T), \kappa(\alpha_T)) + (1 + \alpha_T) F(\Phi(\alpha, \alpha_T), \kappa'(\alpha_T)) \right. \\
\left. - 2E(\Phi(\alpha, \alpha_T), \kappa(\alpha_T)) - 2E(\Phi(\alpha, \alpha_T), \kappa'(\alpha_T)) \right]
\]

where \( \Phi(\alpha, \alpha_T) \) and \( \kappa(\alpha_T) \) are defined at Eq. (13). From Eq. (9), (B.6) and (B.7) it is easy to derive \( E_{QM} \) in Eq. (11).

Appendix III

In this appendix we derive the functional relation Eq. (20). As noticed in Ref. [11,12], the functional relation is a thermodynamical analogy. Since the thermodynamical relations are usually realized on the level of first derivative, we compute \( \frac{dL}{d\nu} \) and \( \frac{dE_{QM}^{Reg}}{d\nu} \) whose explicit form can be derived using the various derivative formulas of the elliptic integrals:

\[
\frac{\partial L}{\partial \nu} = \frac{\sqrt{2} R^2}{4 \nu U_T} \Xi_L \\
\frac{\partial E_{QM}^{Reg}}{\partial \nu} = \frac{U_T}{2\pi} \Xi_E
\]

where

\[
\Xi_L = -2 \left[ \frac{2(\nu^2 + \alpha_0^2)}{(\alpha_0^2 + 1)(\nu^2 + 1)(\nu^2 - \alpha_0^2)} \left[ \alpha_0 \alpha_0' \sqrt{\nu^2 - 1} - \nu \sqrt{\frac{\alpha_0^2 - 1}{\nu^2 - 1}} \right] \\
- \frac{\nu \alpha_0' - \alpha_0}{\sqrt{\nu^2 - \alpha_0^2}} \left[ 2E \left( \Phi \left( \alpha_0, \frac{\alpha_0}{\nu} \right), \kappa \left( \frac{\alpha_0}{\nu} \right) \right) + 2E \left( \Phi \left( \alpha_0, \frac{\alpha_0}{\nu} \right), \kappa' \left( \frac{\alpha_0}{\nu} \right) \right) \right. \\
\left. - \left( 1 - \frac{\alpha_0(\nu^2 - \alpha_0^2)}{\nu(\nu^2 + \alpha_0^2)} \right) F \left( \Phi \left( \alpha_0, \frac{\alpha_0}{\nu} \right), \kappa \left( \frac{\alpha_0}{\nu} \right) \right) \right] \left( \alpha_0 \rightarrow \beta_0, \alpha_0' \rightarrow \beta_0' \right)
\]

\[
\Xi_E = \frac{\sqrt{1 + g^2}}{g}
+ \left[ (\nu \alpha_0' - \alpha_0) \left[ \frac{\sqrt{2(\nu^2 + \alpha_0^2)}}{2 \alpha_0^2 \nu} \left[ E \left( \Phi \left( \alpha_0, \frac{\alpha_0}{\nu} \right), \kappa \left( \frac{\alpha_0}{\nu} \right) \right) + E \left( \Phi \left( \alpha_0, \frac{\alpha_0}{\nu} \right), \kappa' \left( \frac{\alpha_0}{\nu} \right) \right) \right] \\
- \frac{\sqrt{2}}{4} \frac{\nu^3 - \alpha_0 \nu^2 + \alpha_0^2 \nu + \alpha_0^3}{\alpha_0^2 \nu^2} F \left( \Phi \left( \alpha_0, \frac{\alpha_0}{\nu} \right), \kappa \left( \frac{\alpha_0}{\nu} \right) \right) \right]
\]

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\[
- \frac{(\nu^2 - 1)(\nu^2 + \alpha_0^2)}{\alpha_0 \nu \sqrt{\alpha_0^4 - 1}} \alpha_0' - \left[ \frac{(\alpha_0^2 - 1)(\nu^2 - 1)}{(\alpha_0^2 + 1)(\nu^2 + 1)} + \frac{1}{g} [\alpha_0 \rightarrow \beta_0, \alpha_0' \rightarrow \beta_0'] \right]
\]

In Eq. (C.2) prime denotes a differentiation with respect to \( \nu \). Although not immediately obvious from the expression, \( \Xi_L \) is amazingly proportional to \( \Xi_E \):

\[
\frac{\Xi_E}{\Xi_L} = \frac{\sqrt{2}}{4\nu} \sqrt{\frac{\nu^4 - 1}{1 + g^2}}.
\]

(C.3)

In fact, Eq. (C.3) can be proved by comparing all coefficients of \( \Xi_L \) and \( \Xi_E \) one by one. From Eq. (C.3) it is easy to verify Eq. (20).
FIGURES

FIG. 1. String configuration we considered in this paper. The (1,1) string is necessary to conserve (p,q) charge. Since it is known that string cannot penetrate into the horizon, it is attached to D3-brane at $U = U_T$.

FIG. 2. $U_0$–dependence of $L$ exhibiting the monotonic and non-monotonic behaviors at zero and finite temperature.

FIG. 3. The $L$–dependence of $E_{QM}^{Reg}$ exhibiting an bifurcated behavior at finite temperature. This behavior strongly suggests that there is hidden relation between physical quantities which is explicitly derived in Eq. (21).
Fig. 1
Fig. 2
