MULTIVARIATE EXPANSIVITY THEORY AND PIERCE-BIRKHOFF CONJECTURE

T. AGAMA

Abstract. Motivated by the Pierce-Birkhoff conjecture, we launch an extension program for single variable expansivity theory. We study this notion under tuples of polynomials belonging to the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. As an application we develop some class of inequalities to study the Pierce-Birkhoff conjecture.

1. Introduction

The notion of the single variable expansivity theory had been developed quite extensively by the author. This notion turns out to be an important tool in studying Sendov’s conjecture. This theory also has wide range of applications in determining the insolubility of certain systems of differential equations. In the current paper we launch an extension program where the study is carried out for polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$ with real number base field $\mathbb{R}$. It turns out that various basic notion studied under the single variable theory carry over to this setting.

Throughout this paper, we keep the usual standard notion $S$ for all tuples whose entries belong to the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. Occasionally we might choose to index these tuples by $S_j$ over the natural numbers $\mathbb{N}$ if we have two or more and we want to keep them distinct from each other. The tuples $S_0 = (0, 0, \ldots, 0)$ and $S_e = (1, 1, \ldots, 1)$ are still reserved for the null and the unit tuple respectively. Further to the above requirements any tuple of polynomial will be assumed to contain exactly $n$ entries and two tuples under the operation of addition or subtraction will be assumed to contain the same number of entries.

2. Expansion in mixed and specified directions

In this section we introduce the notion of an expansion in a mixed and specific directions. We launch the following extension program.

Definition 2.1. Let $\mathcal{F} := \{S_i\}_{i=1}^\infty$ be a collection of tuples of polynomials $f_k \in \mathbb{R}[x_1, x_2, \ldots, x_n]$. Then by an expansion on $S \in \mathcal{F} := \{S_i\}_{i=1}^\infty$ in the direction $x_i$ for $1 \leq i \leq n$, we mean the composite map

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]} : \mathcal{F} \rightarrow \mathcal{F}$$
Proof. Let $\gamma : \{S_i\}_{i=1}^{\infty} \rightarrow \{S_i\}_{i=1}^{\infty}$ be a fixed direction $\{x_i\}$, $\gamma : \{S_i\}_{i=1}^{\infty} \rightarrow \{S_i\}_{i=1}^{\infty}$ and $\beta \circ \gamma : \{S_i\}_{i=1}^{\infty} \rightarrow \{S_i\}_{i=1}^{\infty}$ be linear, since the map $\gamma : \{S_i\}_{i=1}^{\infty} \rightarrow \{S_i\}_{i=1}^{\infty}$ is bijective. Let $S_a = (f_1, f_2, \ldots, f_n)$, $S_b = (g_1, g_2, \ldots, g_n) \in \mathcal{F} = \{S_i\}_{i=1}^{\infty}$ and let $\lambda, \mu \in \mathbb{R}$, then it follows that

\[
\nabla_{[x_i]}(\lambda S_a + \mu S_b) = \nabla(\lambda(f_1, f_2, \ldots, f_n) + \mu(g_1, g_2, \ldots, g_n))
\]

\[
= \nabla_{[x_i]}((\lambda f_1, \lambda f_2, \ldots, \lambda f_n) + (\mu g_1, \mu g_2, \ldots, \mu g_n))
\]

\[
= \nabla_{[x_i]}((\lambda f_1 + \mu g_1, \lambda f_2 + \mu g_2, \ldots, \lambda f_n + \mu g_n))
\]

\[
= (\frac{\partial (\lambda f_1 + \mu g_1)}{\partial x_1}, \frac{\partial (\lambda f_2 + \mu g_2)}{\partial x_1}, \ldots, \frac{\partial (\lambda f_n + \mu g_n)}{\partial x_1})
\]

\[
= (\lambda \frac{\partial f_1}{\partial x_1} + \mu \frac{\partial g_1}{\partial x_1}, \lambda \frac{\partial f_2}{\partial x_1} + \mu \frac{\partial g_2}{\partial x_1}, \ldots, \lambda \frac{\partial f_n}{\partial x_1} + \mu \frac{\partial g_n}{\partial x_1})
\]

\[
= \lambda \frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_1} \ldots \frac{\partial f_n}{\partial x_1} + \mu \frac{\partial g_1}{\partial x_1} \frac{\partial g_2}{\partial x_1} \ldots \frac{\partial g_n}{\partial x_1}
\]

\[
= \lambda \nabla_{[x_i]}(S_a) + \mu \nabla_{[x_i]}(S_b).
\]

The value of the $l$ th expansion at a given value $a$ of $x_i$ is given by

\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S_a)
\]

where $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S_a)$ is a tuple of polynomials in $\mathbb{R}[x_1, \ldots, x_i-1, x_i+1, \ldots, x_n]$. Similarly by an expansion in the mixed direction $\otimes_{i=1}^{l} x_{\sigma(i)}$ we mean

\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l} x_{\sigma(i)}}(S_a) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=2}^{l} x_{\sigma(i)}}(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l-1} x_{\sigma(i)}}(S_a)
\]

for any permutation $\sigma : \{1, 2, \ldots, l\} \rightarrow \{1, 2, \ldots, l\}$. The value of this expansion on a given value $a_i$ of $x_{\sigma(i)}$ for all $i \in [\sigma(1), \sigma(l)]$ is given by

\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l} x_{\sigma(i)}}(S_a)
\]

where $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l} x_{\sigma(i)}}(S_a)$ is a tuple of real numbers $\mathbb{R}$.

Proposition 2.2. A multivariate expansion is linear.
Similarly,\[\gamma(\lambda S_a + \mu S_b) = \left(\begin{array}{c}
\lambda f_1 + \mu g_1 \\
\lambda f_2 + \mu g_2 \\
\vdots \\
\lambda f_n + \mu g_n
\end{array}\right) = \left(\begin{array}{c}
\lambda f_1 \\
\lambda f_2 \\
\vdots \\
\lambda f_n
\end{array}\right) + \left(\begin{array}{c}
\mu g_1 \\
\mu g_2 \\
\vdots \\
\mu g_n
\end{array}\right) = \lambda \gamma(S_a) + \mu \gamma(S_b)\]

Similarly\[\beta \circ \gamma(\lambda S_a + \mu S_b) = \left(\begin{array}{cccc}
0 & 1 & \cdots & 1 \\
1 & 0 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 0
\end{array}\right) \left(\begin{array}{c}
\lambda f_1 + \mu g_1 \\
\lambda f_2 + \mu g_2 \\
\vdots \\
\lambda f_n + \mu g_n
\end{array}\right) = \lambda \beta \circ \gamma(S_a) + \mu (\beta \circ \gamma)(S_b).

This proves the linearity of expansion. \[\square\]

**Remark 2.3.** Next we prove a fundamental result which shows that an expansion is commutative. This reinforces the very notion that there is no need to give precedence to the direction of an expansion. In essence, it gives some flexibility to the way and manner an expansion could be carried out.

**Proposition 2.4.** An expansion is commutative.

**Proof.** Consider \(\mathcal{F} := \{S_i\}_{i=1}^\infty\) the collection of tuples in the ring \(\mathbb{R}[x_1, x_2, \ldots, x_n]\). It suffices to show that for any \(S \in \mathcal{F}\) then
\[\gamma^{-1} \circ \beta \circ \gamma \circ \nabla|_{[x_i] \otimes [x_j]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_j] \otimes [x_i]}(S).\]

First we can write\[\gamma^{-1} \circ \beta \circ \gamma \circ \nabla|_{[x_i]}(S) = \left(\sum_{t \in [1,n]} \sum_{k=t}^{1} \frac{\partial f_k}{\partial x_1}, \ldots, \sum_{t \in [1,n]} \sum_{k=t}^{1} \frac{\partial f_k}{\partial x_1}\right)\]
and make the assignment

\[ S_{g_k} = (g_{k1}, g_{k2}, \ldots, g_{kn}) = \left( \left( \sum_{t \in [1,n]} \sum_{k \neq t} \frac{\partial f_k}{\partial x_i} \right), \ldots, \left( \sum_{t \in [1,n]} \sum_{k \neq t} \frac{\partial f_k}{\partial x_i} \right) \right) \]

for \( g_{ki} \in \mathbb{R}[x_1, x_2, \ldots, x_n] \). Next we carry out the second expansion on \( S_{g_k} \) and we get

\[ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_{g_k}) = \left( \sum_{s \in [1,n]} \sum_{k = s}^{s \neq 1} \frac{\partial g_{ks}}{\partial x_j}, \ldots, \sum_{s \in [1,n]} \sum_{k = s}^{s \neq n} \frac{\partial g_{ks}}{\partial x_j} \right) \]

so that by combining the two expansions in both directions, we have

\[ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_i] \circ [x_j](S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_{g_k}) = \left( \left( \sum_{s \in [1,n]} \sum_{k = s}^{s \neq 1} \sum_{t \in [1,n]} \sum_{k = t}^{k \neq s} \frac{\partial^2 f_k}{\partial x_j \partial x_i} \right), \ldots, \left( \sum_{s \in [1,n]} \sum_{k = s}^{s \neq n} \sum_{t \in [1,n]} \sum_{k = t}^{k \neq s} \frac{\partial^2 f_k}{\partial x_j \partial x_i} \right) \right) \]

by appealing to the linearity of the operator \( \frac{\partial}{\partial x_i} \). By carrying out the expansion in the opposite direction and appealing to the linearity of the operator

\[ \frac{\partial}{\partial x_i} \]

we have

\[ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j] \circ [x_i](S) = \left( \left( \sum_{s \in [1,n]} \sum_{k = s}^{s \neq 1} \sum_{t \in [1,n]} \sum_{k = t}^{k \neq s} \frac{\partial^2 f_k}{\partial x_j \partial x_i} \right), \ldots, \left( \sum_{s \in [1,n]} \sum_{k = s}^{s \neq n} \sum_{t \in [1,n]} \sum_{k = t}^{k \neq s} \frac{\partial^2 f_k}{\partial x_j \partial x_i} \right) \right) \]

by exploiting the condition

\[ \frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial^2}{\partial x_j \partial x_i} \]

for each polynomial \( g_i, f_i \in \mathbb{R}[x_1, x_2, \ldots, x_n] \). By comparing the result of both expansions in reverse directions, the claim follows immediately. \( \square \)
3. The area of an expansion

In this section we introduce the notion of the area of a multivariate expansion. This is an extension of the area of an expansion under the single variable theory. We exploit some applications in this context.

Definition 3.1. Let $\mathcal{F} := \{S_i\}_{i=1}^{\infty}$ be a collection of tuples of the ring $\mathbb{C}[x_1, x_2, \ldots, x_l]$. Then by the area induced by the expansion

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{i=1}^{l} \{[x_{\sigma(i)}]_i \}^{(S)}.$$

denoted $A_{\vec{a}, \vec{b}}(S)$, from the point $\vec{a}$ to the point $\vec{b}$ we mean the quantity

$$A_{\vec{a}, \vec{b}}(S) := O \Delta_{\vec{a}}^{\vec{b}} \left[ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{i=1}^{l} \{[x_{\sigma(i)}]_i \}^{(S)} \right] \cdot OS_c$$

where

$$\Delta_{\vec{a}}^{\vec{b}} \left[ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{i=1}^{l} \{[x_{\sigma(i)}]_i \}^{(S)} \right] = \left( \prod_{j=1}^{n} \int_{a_{\sigma(j)}}^{b_{\sigma(j)}} \int_{a_{\sigma(j-1)}}^{b_{\sigma(j-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} g_j dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \right)$$

and $S_c = (1, 1, \ldots, 1)$ is the unit tuple and $g_l \in \mathbb{C}[x_1, x_2, \ldots, x_l]$ for $1 \leq i \leq n$.

Next we examine some properties of the area induced by an expansion between any two points in space.

Proposition 3.2. The area of an expansion between points in space is linear.

Proof. Let $\omega, \mu \in \mathbb{R}$ and $S_1, S_2$ be a tuple of polynomials whose entry belongs to the ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$ with $S_1 = (f_1, f_2, \ldots, f_n)$ and $S_2 = (g_1, g_2, \ldots, g_n)$. Let $\vec{a}, \vec{b} \in \mathbb{R}^l$, then we can write

$$A_{\vec{a}, \vec{b}}(\omega S_1 + \mu S_2) = O \Delta_{\vec{a}}^{\vec{b}} \left[ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{i=1}^{l} \{[x_{\sigma(i)}]_i \}^{(S_1)} + \{[x_{\sigma(i)}]_i \}^{(S_2)} \right] \cdot OS_c$$

where

$$\Delta_{\vec{a}}^{\vec{b}} \left[ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{i=1}^{l} \{[x_{\sigma(i)}]_i \}^{(S_1)} \right] = \omega \sum_{j=1}^{n} \int_{a_{\sigma(j)}}^{b_{\sigma(j)}} \int_{a_{\sigma(j-1)}}^{b_{\sigma(j-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} (\omega f_j + \mu g_j) dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)}$$

and

$$\Delta_{\vec{a}}^{\vec{b}} \left[ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{i=1}^{l} \{[x_{\sigma(i)}]_i \}^{(S_2)} \right] = \mu \sum_{j=1}^{n} \int_{a_{\sigma(j)}}^{b_{\sigma(j)}} \int_{a_{\sigma(j-1)}}^{b_{\sigma(j-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} g_j dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)}$$

Next we can write

$$A_{\vec{a}, \vec{b}}(\omega S_1 + \mu S_2) = \omega A_{\vec{a}, \vec{b}}(S_1) + \mu A_{\vec{a}, \vec{b}}(S_2).$$
This proves that the area of an expansion between points in space is a linear map. □

Remark 3.3. Next we examine some immediate applications of the notion of the area of a multivariate expansion of tuples of polynomials with entries belonging to the ring $R[x_1, x_2, \ldots, x_l]$. The offshoot of this notion is an inequality controlling the 2-norm of any $l$-th fold integration by the $l$-th fold integration of the 2-norm of the corresponding integrand.

Theorem 3.4. Let $g_j \in R[x_1, x_2, \ldots, x_l]$ for $1 \leq j \leq n$. Then there exists some constant $C := C(l) > 0$ such that the inequality holds

$$\left| A_{\vec{a}, \vec{b}}(S) \right| \geq C(l) \sum_{j=1}^{n} \left( \int_{a_{\sigma}(i)}^{b_{\sigma}(i)} \int_{a_{\sigma}(i-1)}^{b_{\sigma}(i-1)} \cdots \int_{a_{\sigma}(1)}^{b_{\sigma}(1)} g_j \, dx_{\sigma(1)} \, dx_{\sigma(2)} \cdots dx_{\sigma(l)} \right)^2$$

for $b_i > a_i$ for each $1 \leq i \leq l$.

Proof. Using the notion of the area of an expansion we obtain an equivalent expression

$$\left| A_{\vec{a}, \vec{b}}(S) \right| = \left| O\Delta^k_x \left[ \gamma^{-1} \circ \beta \circ \gamma \circ \nabla \right]_{\otimes_{i=1}^{l} [x_{\sigma(i)}]}(S) \right| \cdot \Theta^x$$

$$= \sqrt{n} |\cos(\alpha)| \sum_{j=1}^{n} \left( \int_{a_{\sigma}(i)}^{b_{\sigma}(i)} \int_{a_{\sigma}(i-1)}^{b_{\sigma}(i-1)} \cdots \int_{a_{\sigma}(1)}^{b_{\sigma}(1)} g_j \, dx_{\sigma(1)} \, dx_{\sigma(2)} \cdots dx_{\sigma(l)} \right)^2$$
On the other hand, we observe that by changing the order of summation we can write, using the notion of the area of an expansion, the following

\[
\left| A_{\sigma,\beta}(S) \right| = \left| O\Delta^{[\sigma]}_{\beta} \left[ (\gamma^{-1} \circ \beta \circ \gamma)_{\otimes_{i=1}^{l} [x_{\sigma(i)}]}(S) \right] \cdot \text{O}S_{e} \right|
\]

\[
= \sum_{j=1}^{n} \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} \int_{a_{\sigma(1-1)}}^{b_{\sigma(1-1)}} \cdots \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} g_{j} dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)}
\]

\[
= \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} \int_{a_{\sigma(1-1)}}^{b_{\sigma(1-1)}} \cdots \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \int_{a_{\sigma(l)}}^{b_{\sigma(l)}} g_{j} dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)}
\]

\[
\leq \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} \int_{a_{\sigma(1-1)}}^{b_{\sigma(1-1)}} \cdots \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \left( \sum_{j=1}^{n} g_{j}^{2} \right)^{\frac{1}{2}} dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)}
\]

\[
= \sqrt{n} \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} \int_{a_{\sigma(1-1)}}^{b_{\sigma(1-1)}} \cdots \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \left( \sum_{j=1}^{n} g_{j}^{2} \right)^{\frac{1}{2}} dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)}
\]

and the inequality follows by combining this upper bound with the area of expansion. \(\square\)

**Remark 3.5.** It is important to notice that Theorem 3.4 is a generalization of the usual integral inequality for bounded functions \(h\); in particular, polynomials. That is to say, if we take \(\sum_{j=1}^{n} g_{j}^{2} = h^{2}\) then

\[
\sqrt{\sum_{j=1}^{n} g_{j}^{2}} = h
\]

and the result is the usual integral inequality with \(h\) now on the finite supports \([a_{1}, b_{1}], [a_{2}, b_{2}], \ldots, [a_{l}, b_{l}]\). Next we obtain an inequality relating the minimum gap between the limits of integration to their corresponding \(l\)-th fold integration of the function.

**Corollary 3.6.** Let \(g_{j} \in \mathbb{R}[x_{1}, x_{2}, \ldots, x_{l}]\) for \(1 \leq j \leq n\) such that \(g_{j} \neq 1\) with

\[
\sum_{j=1}^{n} g_{j}^{2} \leq 1
\]
on $\bigcup_{i=1}^{l} [a_i, b_i]$. Then there exist some constant $C := C(l) > 0$ such that the inequality holds

$$\frac{1}{C(l)} \prod_{i=1}^{l} |b_{\sigma(i)} - a_{\sigma(i)}| \geq \sum_{j=1}^{n} \left( \int_{a_{\sigma(j)}(1)}^{b_{\sigma(j)}(1)} \int_{a_{\sigma(j-1)}(1)}^{b_{\sigma(j-1)}(1)} \cdots \int_{a_{\sigma(1)}(1)}^{b_{\sigma(1)}(1)} g_j dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \right)^2.$$ 

There is no obvious reason why an analogous inequality should not hold for functions continuous on the supports $\bigcup_{i=1}^{l} [a_i, b_i]$, by replacing the space of real multivariate polynomials $\mathbb{R}[x_1, x_2, \ldots, x_n]$ with the general space of real multivariate functions $\mathbb{F}[x_1, x_2, \ldots, x_l]$ continuous on the interval $\bigcup_{i=1}^{l} [a_i, b_i]$.

4. The volume of an expansion

In this section we introduce the notion of the **volume** induced by $n$ points in space. We launch the following formal language and exploit some applications.

**Definition 4.1.** Let $\mathcal{F} := \{ S_i \}_{i=1}^{\infty}$ be a collection of tuples of polynomials with entries $f_k \in \mathbb{C}[x_1, x_2, \ldots, x_n]$. Then by the **volume** induced by the expansion

$$\left( \gamma^{-1} \circ \beta \circ \gamma \circ \nabla \right)_{\otimes_{i=1}^{l} [x_{\sigma(i)}]}(S),$$

denoted $\mathcal{V}_{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n}(S)$, at the linearly independent spots $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ such that $\bar{a}_i \neq \hat{O}$ for $1 \leq i \leq n$, we mean the sum

$$\mathcal{V}_{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n}(S) := \sum_{\substack{s, t \in [1, n] \\
\kappa \leq t \\
\kappa \in [1, n]}} \frac{O \Delta_{\bar{a}_s}^{\bar{a}_t}}{k \neq s, t} \left[ \left( \gamma^{-1} \circ \beta \circ \gamma \circ \nabla \right)_{\otimes_{i=1}^{l} [x_{\sigma(i)}]}(S) \right] \cdot \overline{\Delta_{\bar{a}_s}^{\bar{a}_t} \circ \bar{a}_t \circ \cdots \circ \bar{a}_k \circ \bar{a}_n}$$

where $\bar{a}_s \circ \bar{a}_t \circ \cdots \circ \bar{a}_k \circ \bar{a}_n$ is the cross product of any of $n - 1$ spots including the spots $\bar{a}_s, \bar{a}_t$ and $\sigma : [1, \ell] \rightarrow [1, \ell]$ is a permutation.

**Proposition 4.2.** The volume of an expansion between spots is a linear operator.

**Proof.** This is an easy consequence of Proposition 5.2. \qed

**Remark 4.3.** As an immediate application we deduce an average version of the inequality in Theorem 3.4.

**Theorem 4.4.** Let $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ be any linearly independent vectors in the space $\mathbb{R}^l$ and $\sigma : \{1, 2, \ldots, l\} \rightarrow \{1, 2, \ldots, l\}$ be any permutation. If for each $g_k \in \mathbb{R}[x_1, x_2, \ldots, x_l]$ with $1 \leq k \leq n$ and

$$\int_{a_{\sigma(j)}(1)}^{b_{\sigma(j)}(1)} \int_{a_{\sigma(j-1)}(1)}^{b_{\sigma(j-1)}(1)} \cdots \int_{a_{\sigma(1)}(1)}^{b_{\sigma(1)}(1)} g_k dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} > 0$$

then

$$\int_{a_{\sigma(j)}(1)}^{b_{\sigma(j)}(1)} \int_{a_{\sigma(j-1)}(1)}^{b_{\sigma(j-1)}(1)} \cdots \int_{a_{\sigma(1)}(1)}^{b_{\sigma(1)}(1)} \frac{1}{C(l)} \prod_{i=1}^{l} |b_{\sigma(i)} - a_{\sigma(i)}| \geq \sum_{j=1}^{n} \left( \int_{a_{\sigma(j)}(1)}^{b_{\sigma(j)}(1)} \int_{a_{\sigma(j-1)}(1)}^{b_{\sigma(j-1)}(1)} \cdots \int_{a_{\sigma(1)}(1)}^{b_{\sigma(1)}(1)} g_j dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \right)^2.$$
Corollary 4.5. Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ be a linearly independent vectors in the space $\mathbb{R}^l$ and $\sigma : \{1,2,\ldots, l\} \rightarrow \{1,2,\ldots, l\}$ be any permutation. If for each $g_k \in \mathbb{R}$,

then there exist a constant $C > 0$ such that

$$
\sum_{i,j \in [1,n]} \left\| \vec{a}_i \odot \vec{a}_j \odot \cdots \odot \vec{a}_n \right\| \sum_{k=1}^n \int_{a_{j_1}(1)}^{a_{j_2}(1)} \cdots \int_{a_{j_l}(1)}^{a_{j_l}(1)} g_k dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} 
\leq 2C \left( \frac{n}{2} \right) \times \sqrt{n} \times 
\int_{a_{j_1}(1)}^{a_{j_2}(1)} \cdots \int_{a_{j_l}(1)}^{a_{j_l}(1)} \sqrt{\sum_{k=1}^n (\text{max}(g_k))^2} dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)}
$$

Proof. First we let

$$
\int_{a_{j_1}(1)}^{a_{j_2}(1)} \cdots \int_{a_{j_l}(1)}^{a_{j_l}(1)} g_k dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)}
$$

be the $k$th entry of the vector $O\Delta_{\vec{a}_i} \left[ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla) \otimes_{i=1}^l [x_{\sigma(i)}] (S) \right]$ for $1 \leq k \leq n$. Then unpacking the definition of the volume induced by an expansion at $n$ spots yields the left-hand side expression. On the other hand, we observe that each outer sum is determined by their spots and in each of these we maintain two distinct spots twice so that we have $2 \times \binom{n}{2}$ for the number of such possible distinct sums. Under the main assumption the maximum function taken is then absorbed by the $l$-fold integral and the result follows via the interpolation

$$
\frac{\text{max}(g_k)}{O\Delta_{\vec{a}_i} \left[ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla) \otimes_{i=1}^l [x_{\sigma(i)}] (S) \right]} \times \sqrt{\sum_{k=1}^n (\text{max}(g_k))^2} dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)}
$$

We deduce from this result another inequality that controls the average of any $l$-fold integral by the $l$-fold integral of the $2$-norm of the corresponding integrand.

Corollary 4.5. Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ be a linearly independent vectors in the space $\mathbb{R}^l$ and $\sigma : \{1,2,\ldots, l\} \rightarrow \{1,2,\ldots, l\}$ be any permutation. If for each $g_k \in \mathbb{R}$,
\[\mathbb{R}[x_1, x_2, \ldots, x_l] \text{ with } 1 \leq k \leq n \text{ and the norm of the cross products } \|\vec{a}_i \odot \vec{a}_j \odot \cdots \odot \vec{a}_v \| \geq K \text{ for } K \in \mathbb{R} \text{ for all } 1 \leq i, j, \ldots, v \leq n, \text{ then there exist some constant } C > 0 \text{ such that if} \]

\[
\begin{align*}
\int_{a_{\sigma(1)}}^{a_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{a_{\sigma(l-1)}} g_k dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} > 0
\end{align*}
\]

then

\[
\begin{align*}
\sum_{k=1}^{n} \int_{a_{\sigma(1)}}^{a_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{a_{\sigma(l-1)}} g_k dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \leq C \times \frac{1}{K} \sqrt{n} \times \int_{a_{\sigma(1)}}^{a_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{a_{\sigma(l-1)}} \sqrt{\left(\sum_{k=1}^{n} (\max(g_k))^2\right)} dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)}
\end{align*}
\]

5. The totient and residue of an expansion

In this section we introduce the notion of the \textbf{residue} and the \textbf{totient} of an expansion. These two notions are analogous to the notion of the rank and the degree of an expansion under the single variable theory. We launch more formally the following languages.

\textbf{Definition 5.1.} Let \( \mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty} \) be a collection of tuples of polynomials in the ring \( \mathbb{R}[x_1, x_2, \ldots, x_n] \). Then we say the expansion \( (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x|}(\mathcal{S}) \) is \textbf{free} with \textbf{totient} \( k \), denoted \( \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x|}(\mathcal{S})] \), if

\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{|x|}(\mathcal{S}) = \mathcal{S}_0
\]

where \( k > 0 \) is the smallest such number. We call the expansion \( (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k-1}_{|x|}(\mathcal{S}) \) the \textbf{residue} of the expansion, denoted by \( \Theta[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k-1}_{|x|}(\mathcal{S})] \). Similarly by the totient of the mixed expansion \( (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}|x_{\sigma(i)}|}(\mathcal{S}) \), we mean the smallest value of \( k \) such that

\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{\otimes_{i=1}^{l}|x_{\sigma(i)}|}(\mathcal{S}) = \mathcal{S}_0.
\]

We denote the totient of the mixed expansion with

\[
\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}|x_{\sigma(i)}|}(\mathcal{S})].
\]

\textbf{Proposition 5.2.} Let \( \mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty} \) be a collection of tuples of polynomials in the ring \( \mathbb{R}[x_1, x_2, \ldots, x_n] \). If the expansions \( (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x|}(\mathcal{S}_k) \) and \( (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x|}(\mathcal{S}_l) \) are free with totients \( s \) and \( t \), respectively. Then the expansion

\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x|}(\mathcal{S}_k + \mathcal{S}_l)
\]

is also free with totient \( \max\{s, t\} \).
Proof. Suppose the expansions \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}(S_k)\) and \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}(S_l)\) are free with totients \(s\) and \(t\), respectively. Then it follows that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}^s(S_k) = S_0
\]
with \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}^{s-m}(S_k) \neq S_0\) for all \(m \leq s\) and
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}^t(S_l) = S_0
\]
with \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}^{t-m}(S_l) \neq S_0\) for all \(m \leq t\). Now let us apply \(\max\{s, t\}\) copies of the expansion maps to the tuple \(S_k + S_l\) so that we have by appealing to the linearity of an expansion map we have
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}^\max\{s, t\}(S_k + S_l) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}^\max\{s, t\}(S_k) + (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}^\max\{s, t\}(S_l)
\]
\[
= S_0
\]
since \(s, t \leq \max\{s, t\}\). Next we see that for any \(1 \leq r \leq \max\{s, t\}\) then by appealing to the linearity of the expansion map
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}^\max\{s, t\}-r(S_k + S_l) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}^\max\{s, t\}-r(S_k) + (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}^\max\{s, t\}-r(S_l)
\]
\[
\neq S_0
\]
since at least one of the inequality \(\max\{s, t\} - r < s\) or \(\max\{s, t\} - r < t\) must hold. Thus \(\max\{s, t\}\) is the totient of the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}(S_k + S_l)\). This completes the proof of the proposition. \(\square\)

Remark 5.3. Next we expose an important relationship that exists between the totient of the mixed expansion and the underlying expansion in specific directions. One could view this result as a sub-additivity property of the totient of an expansion.

Theorem 5.4. Let \(\mathcal{F} = \{S_i\}_{i=1}^\infty\) be a collection of tuples of polynomials in the ring \(\mathbb{R}[x_1, x_2, \ldots, x_n]\). Then we have the inequality
\[
\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(i)}]}(S)] \leq \frac{1}{\ell} \sum_{i=1}^{\ell} \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(i)}]}(S)] + K
\]
where \(K(\ell) = K > 0\).

It is easily noticeable that the inequality allows us to control the totient of a mixed expansion by the average of the totient of expansions in specific directions involved in the mixed expansion. We relegate the proof of this to latter sections, where we develop the required tools needed. It is fair to say that this inequality is crude; However, we will obtained a much stronger version in the sequel that gives much information.
6. The dropler effect induced by an expansion

In this section we introduce the notion of the dropler effect induced by an expansion. This phenomena is mostly induced by expansion on several other expansions in a specific direction.

**Definition 6.1.** Let $F = \{S_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. Then the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla) \otimes \otimes_{i=1}^{\infty} S_{x_i}(S)$ is said to induce a dropler effect with intensity $k$, denoted $I[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}(S)] = k$, on the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}(S)$ if $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}(S)$ is free with $k < \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}(S)]$ and $k$ is the smallest such number. In other words, we say the expansion admits a dropler effect from the source $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}(S)$ with intensity $k$. The energy saved $E[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}(S)]$ by the expansion under the dropler effect is given by $E[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}(S)] = \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}(S)] - I[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}(S)]$.

We call this equation the energy-dropler effect intensity equation.

**Proposition 6.2.** Let $F = \{S_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. If the expansions $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}(S)$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}(S)$ each admits a dropler effect from the same source with intensities $k_1$ and $k_2$, respectively, then the expansion $[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}] + (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}(S)$ also admits a dropler effect from the same source with intensity $\max\{k_1, k_2\}$.

**Proof.** Suppose the expansions $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}(S)$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}(S)$ each admits a dropler effect from the same source with intensities $k_1$ and $k_2$, respectively. Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes \otimes_{i=1}^{\infty} S_{x_i}(S)}$ be their source, then it follows by virtue of Definition 6.1

\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}^{k_1} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes \otimes_{i=1}^{\infty} S_{x_i}(S)} = S_0 \quad (6.1)\]

and

\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}^{k_2} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes \otimes_{i=1}^{\infty} S_{x_i}(S)} = S_0. \quad (6.2)\]

Let us consider the expansion map $[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}^{k_1} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes \otimes_{i=1}^{\infty} S_{x_i}(S)}]$ and apply $\max\{k_1, k_2\}$ copies to the source $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes \otimes_{i=1}^{\infty} S_{x_i}(S)}$. It follows by the linearity of an expansion and further appealing to (6.1) and (6.2)

\[([(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}^{\max\{k_1, k_2\}} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes \otimes_{i=1}^{\infty} S_{x_i}(S)} = S_0.\]

It is easy to observe that

\[([(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x_i}^{\max\{k_1, k_2\}-r} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes \otimes_{i=1}^{\infty} S_{x_i}(S)} \neq S_0 \quad (6.3)\]
for any $r \geq 1$, by appealing to the linearity of a multivariate expansion and exploiting the fact that at least one of the inequality must hold

$$\max\{k_1, k_2\} - r \leq k_1 \quad \text{or} \quad \max\{k_1, k_2\} - r \leq k_2.$$  

Thus $\max\{k_1, k_2\}$ is the intensity of the dropler effect induced on the concatenations of the expansions under the same source. \hfill \Box

**Remark 6.3.** One could ask whether an analogue of this result exists for expansions with concatenated directions. While a general answer to this question may seem very baffling, we can somehow obtain a variant by imposing some conditions that ensure expansion in one direction does not wear off and interfere with the direction of the other. We make this assertion more precise in the following proposition.

**Proposition 6.4.** Let $\mathcal{F} = \{ S_i \}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. Let the expansions $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S)$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S)$ each admits a dropler effect with intensities $k_1$ and $k_2$, respectively, from the source $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S)$. If the expansions $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S)$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S)$ admits no dropler effect from the sources

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S)$$

and

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S)$$

respectively, then the mixed expansion

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S)$$

also admits a dropler effect from the same source with intensity $\min\{k_1, k_2\}$.

**Proof.** Suppose the expansions $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S)$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S)$ each admits a dropler effect from the same source with intensities $k_1$ and $k_2$, respectively. Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S)$ be their source, then it follows by virtue of Definition \ref{def:dropler}

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S) = S_0 \quad (6.4)$$

and

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S) = S_0. \quad (6.5)$$

Let us apply $\min\{k_1, k_2\}$ copies of the mixed expansion operator $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S)$ to the source $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S)$ then we see that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S) = S_0$$

by appealing to the commutative property of the expansion operator and \ref{6.3} and \ref{6.4}. Again by appealing to the commutative property of an expansion operator the relation holds

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S) \neq S_0$$

for any $\min\{k_1, k_2\} \geq r \geq 1$, since $\min\{k_1, k_2\} - r < k_1$ and $\min\{k_1, k_2\} - r < k_2$ and $k_1, k_2$ are the intensities of the dropler effects and the expansions $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S)$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S)$ admit no dropler effect from the sources

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)|_{[x_i]}(S)$$
and
\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)\]

respectively. This proves that \(\min\{k_1, k_2\} = I[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)]\), the intensity of the droper effect induced on the mixed expansion. \(\square\)

**Proposition 6.5.** Let \(\mathcal{F} = \{S_i\}_{i=1}^\infty\) be a collection of tuples of polynomials in the ring \(\mathbb{R}[x_1, x_2, \ldots, x_n]\). Let \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)\) and \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)\) be expansions with totients \(k_1\) and \(k_2\), respectively. If the expansions \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)\) and \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)\) admits no droper effect from the source
\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^u(S)\]
for \(u < k_2\) and
\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^v(S)\]
for \(v < k_1\), respectively, then
\[\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)] = \min\{k_1, k_2\}\].

**Proof.** Suppose \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)\) and \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)\) are expansions with totients \(k_1\) and \(k_2\), respectively. Then it follows that
\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{k_1}(S) = S_0\]
(6.6)
and
\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{k_2}(S) = S_0\]
(6.7)
where \(k_1, k_2\) are the smallest such number. Appealing to the commutative property of the expansion operator, we can write by virtue of (6.6) and (6.7)
\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{\min\{k_1, k_2\}}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{\min\{k_1, k_2\}}(S) = S_0\]
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^u(S)\)
for \(v < k_2\) and
\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^u(S)\]
for \(u < k_1\), respectively, then it certainly follows that \(\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)]\) and \(\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)]\) are the smallest numbers, respectively, such that
\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^u(S) = S_0\]
and
\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^u(S) = S_0\]
so that for any \(\min\{k_1, k_2\} \geq c \geq 1\) then
\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{\min\{k_1, k_2\} - c}(S) \neq S_0\]
by exploiting the linearity of an expansion operator. This proves that \(\min\{k_1, k_2\} = \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)]\), the totient of the mixed expansion. \(\square\)
7. Destabilization of an expansion

In this section we introduce the notion of destabilization induced by an expansion. This notion will form an essential toolbox in proving some result in this sequel. We launch more formally the following languages.

**Definition 7.1.** Let $\mathcal{F} = \{S_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $R[x_1, x_2, \ldots, x_n]$. We say the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}(S)$ is said to undergo natural destabilization if $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}(0) \neq S_0$. We say it undergoes destabilization at stage $k \geq 1$ if $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}^{k}(S) = S_0$ for all $1 \leq j \leq k - 1$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}^{k+1}(S) \neq S_0$. In other words, we say the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}(S)$ admits a destabilization at stage $k \geq 1$. We say it is strongly destabilized if the vector

$$O(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}^{k}(S)$$

has no zero entry.

**Remark 7.2.** Next we prove a result that tells us that destabilization should by necessity happen in an expansion. The following result confines this stage to a certain range.

**Proposition 7.3.** Let $\mathcal{F} = \{S_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $R[x_1, x_2, \ldots, x_n]$. Then the stage of destabilization $k$ of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}(S)$ satisfies the inequality

$$0 \leq k < \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}(S)].$$

**Proof.** If the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}(S)$ admits a natural destabilization then the stage $k = 0$. Thus we may assume the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}(S)$ do not admit a natural destabilization. Let us suppose to the contrary that the stage of destabilization of some expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}(S_m)$ satisfies $k \geq \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}(S_m)]$ so that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}^{\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}(S_m)]-1}(S_m) = S_0.$$ 

This is a contradiction, since the expansion

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}^{\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}(S_m)]-1}(S_m)$$

is the residue of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_1]}(S_m)$ and thus has no direction of form $[x_i]$. □

**Theorem 7.4.** Let $\mathcal{F} = \{S_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $R[x_1, x_2, \ldots, x_n]$. Then for all directions $[x_j]$ with $1 \leq j \leq n$ every expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S)$ is strongly destabilized at the stage $\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S)]-1$

**Proof.** Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S)$ be any expansion in an arbitrary direction $[x_j]$. Then by virtue of Definition 5.1 the expansion

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S)]-1}(S)$$


is the residue of the expansion \((γ^{-1} o β o γ o ∇)_{[x_j]}(S)\). Let us suppose to the contrary the vector
\[
O(γ^{-1} o β o γ o ∇)_{[x_j]}φ[(γ^{-1} o β o γ o ∇)_{[x_j]}(S)]^{-1}(S)
\]
has at least a zero entry. Then it follows that the expansion \((γ^{-1} o β o γ o ∇)_{[x_j]}(S)\) contains the direction \([x_j]\) and hence
\[
(γ^{-1} o β o γ o ∇)_{[x_j]}φ[(γ^{-1} o β o γ o ∇)_{[x_j]}(S)](S) \neq S_0
\]
which contradicts the fact that \(Φ[(γ^{-1} o β o γ o ∇)_{[x_j]}(S)]\) is the totient of the expansion \((γ^{-1} o β o γ o ∇)_{[x_j]}(S)\). This completes the proof. □

**Remark 7.5.** Next we relate the notion of the dropper effect induced by a mixed expansion on expansions in a specific direction to the notion of destabilization. We show that these two notions are somewhat related.

8. Diagonalization and sub-expansion of an expansion

In this section we introduce the notion of diagonalization of an expansion and sub-expansion of an expansion. This notion is mostly applied to expansions in mixed directions. We launch the following languages to ease our work.

**Definition 8.1.** Let \(F = \{S_i\}_{i=1}^\infty\) be a collection of tuples of polynomials in the ring \(R[x_1, x_2, \ldots, x_n]\). We say the mixed expansion \((γ^{-1} o β o γ o ∇)_{[x_i]}(S)\) is diagonalizable in the direction \([x_j]\) \((1 \leq j \leq n)\) at the spot \(S_r \in F\) with order \(k\) if
\[
(γ^{-1} o β o γ o ∇)_{[x_i]}(S_r) = (γ^{-1} o β o γ o ∇)_{[x_j]}^k(S_r).
\]
We call the expansion \((γ^{-1} o β o γ o ∇)_{[x_j]}(S_r)\) the diagonal of the mixed expansion \((γ^{-1} o β o γ o ∇)_{[x_i]}(S)\) of order \(k \geq 1\). We denote with \(O[(γ^{-1} o β o γ o ∇)_{[x_j]}(S)]\) the order of the diagonal.

**Proposition 8.2.** Let \(F = \{S_i\}_{i=1}^\infty\) be a collection of tuples of polynomials in the ring \(R[x_1, x_2, \ldots, x_n]\). Let the expansions \((γ^{-1} o β o γ o ∇)_{[x_i]}(S)\) and \((γ^{-1} o β o γ o ∇)_{[x_i]}(S)\) both diagonalizable in the fixed direction \([x_i]\) at the spot \(S_u\) with order \(u\) and \(S_k\) with order \(v\), respectively. If \(u > v\) (resp. \(v > u\) then
\[
(γ^{-1} o β o γ o ∇)_{[x_i]}(S_u + S_k)
\]
is also diagonalizable at the spot \((γ^{-1} o β o γ o ∇)_{[x_i]}(S_u) + S_k\) with order \(v\), respectively
\[
(γ^{-1} o β o γ o ∇)_{[x_i]}^u - v(S_k) + S_u
\]
with order \(u\).
Proposition 8.6. We say the sub-expansion is proper if 

\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\gamma(i)}]}(S_i)\]

be both diagonalizable in the fixed direction \([x_i]\) at the spots \(S_a\)
with order \(u\) and \(S_k\) with order \(v\), respectively. Then it follows by virtue of Definition 16.5

\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\gamma(i)}]}(S_a) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S_a)\]

and

\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\gamma(i)}]}(S_k) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S_k).\]

Then by concatenating the two mixed expansion and appealing to the linearity of an expansion with \(u > v\), we have

\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\gamma(i)}]}(S_i + S_r) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\gamma(i)}]}(S_i) + (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\gamma(i)}]}(S_r)\]

\[= (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S_a) + (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S_k).\]

Under the assumption \(u > v\) and appealing to the linearity of an expansion operator, we deduce

\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\gamma(i)}]}(S_i + S_r) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{u-v}(S_a) + S_k.\]

The claim follows by choosing the spot

\[S_f = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{u-v} (S_a) + S_k.\]

\[\Box\]

Remark 8.3. Next we launch the notion of the sub-expansion of an expansion. The same notion under the single variable theory still carries over to this setting.

Definition 8.4. Let \(F = \{S_i\}_{i=1}^{\infty}\) be a collection of tuples of polynomials in the ring \(\mathbb{R}[x_1, x_2, \ldots, x_n]\). We say the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_a)\) is a sub-expansion of the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_i)\), denoted \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_a) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_i)\) if there exist some \(0 \leq m\) such that

\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_a) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k+m}_{[x_i]}(S_i).\]

We say the sub-expansion is proper if \(m + k = l\). We denote this proper sub-expansion by \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_a) < (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_i).\) On the other hand, we say the sub-expansion is ancient if \(m + k > l\).

Remark 8.5. Next we relate the notion of the sub-expansion of an expansion to the notion of Diagonalization of a mixed expansion. We expose this profound relationship in the following proposition

Proposition 8.6. Let \(F = \{S_i\}_{i=1}^{\infty}\) be a collection of tuples of polynomials in the ring \(\mathbb{R}[x_1, x_2, \ldots, x_n]\). If the mixed expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\gamma(i)}]}(S)\) is diagonalizable in the direction \([x_j]\) (\(1 \leq j \leq n\)) at the spots \(S_a, S_r \in F\) such that \(S_i - S_r\) is not a tuple of \(\mathbb{R}\) with orders \(k_t\) and \(k_r\), respectively and \(k_r > k_t\). Then

\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_t}_{[x_j]}(S_i) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_r}_{[x_j]}(S_r).\]
Proof. Let $F = \{S_i\}_{i=1}^\infty$ be a collection of tuples of the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$ and let
the mixed expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes i=1}^{l}[x_{\sigma(i)}](S)$ be diagonalizable in the direction
$[x_j]$ ($1 \leq j \leq n$) at the spots $S_t, S_r \in F$ such that $S_t - S_r$ is not a tuple of $\mathbb{R}$ with
orders $k_t$ and $k_r$, so that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes i=1}^{l}[x_{\sigma(i)}](S_t) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes i=1}^{l}[x_{\sigma(i)}](S_r) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes i=1}^{l}[x_{\sigma(i)}](S_k)
\] (8.1)
and
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes i=1}^{l}[x_{\sigma(i)}](S_t) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes i=1}^{l}[x_{\sigma(i)}](S_r) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes i=1}^{l}[x_{\sigma(i)}](S_k)
\] (8.2)
It follows by combining (8.1) and (8.2) the relation
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_t}(S_t) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_r}(S_r)
\]
since $S_t - S_r$ is not a tuple of $\mathbb{R}$. Since $k_r > k_t$, it follows that there exist some $m \geq 1$ such that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_t+m}(S_r) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_r}(S_t)
\]
so that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_r}(S_t) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_t}(S_r).
\]

Remark 8.7. The converse of Proposition 8.6 may not necessarily hold because the
sub-expansion may be ancient. But we can be certain the converse will hold if we
allow the sub-expansion to be a proper sub-expansion. This relation is espoused in
the following result as a weaker converse of the above result.

Proposition 8.8. Let $F = \{S_i\}_{i=1}^\infty$ be a collection of tuples of polynomials in the
ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. If the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_t}(S_t)$ is a diagonal with
order $k$ of the mixed expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes i=1}^{l}[x_{\sigma(i)}](S)$ and
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{l}(S_t) < (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_t}(S_t)
\]
then the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{l}(S_t)$ is also a diagonal with order $l$ of the
same mixed expansion.

Proof. Let us suppose the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{l}(S_t)$ is the diagonal with
order $k$ of the mixed expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes i=1}^{l}[x_{\sigma(i)}](S)$. Then it follows by
virtue of Definition 13.3
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes i=1}^{l}[x_{\sigma(i)}](S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{l}(S_t).
\]
Since
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{l}(S_t) < (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_t}(S_t)
\]
it follows by appealing to Definition 13.1
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{l+m}(S_t) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{l+m}(S_t).
\]
for some $0 \leq m$ with $l + m = k$ so that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_t}(S_t) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k_t}(S_t).
\]
The result follows from this relation, since $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{l+m}(S_t)$ is a diagonal with order $k$ of the mixed expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes i=1}^{l}[x_{\sigma(i)}](S)$.
Lemma 8.11. Let \( \Phi \) admit dropler effect from the source \( \Phi[\gamma^{-1} \circ \beta \circ \gamma \circ \nabla]_{\phi_{i=1}^{\infty} [x_{\tau(i)}]}(S) \) that we have. Now let us suppose there exist some \( r \) with \( k < \phi_{i=1}^{\infty} [x_{\tau(i)}] \) and \( k \) is the smallest such number. Then the assumption the expansion \( \gamma^{-1} \circ \beta \circ \gamma \circ \nabla[x_{j}] \) and \( \phi_{i=1}^{\infty} [x_{\tau(i)}] \) is diagonalizable in the direction \( x_{j} \) at the spot \( S_{t} \) with order \( s \), then the expansion
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_{j}](S_{t})
\]
is free with totient
\[
\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\phi_{i=1}^{\infty} [x_{\tau(i)}]}(S)] = k + s.
\]

Proof. First let us suppose the mixed expansion \( \gamma^{-1} \circ \beta \circ \gamma \circ \nabla[x_{j}] \) induces dropler effect with intensity \( k \) on the expansion \( \gamma^{-1} \circ \beta \circ \gamma \circ \nabla[x_{j}](S) \). Then it follows by virtue of Definition 6.1
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}[x_{j}] \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\phi_{i=1}^{\infty} [x_{\tau(i)}]}(S) = S_{0}
\]
with \( k < \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_{j}](S)] \) and \( k \) is the smallest such number. Under the assumption the expansion \( \gamma^{-1} \circ \beta \circ \gamma \circ \nabla[x_{j}](S) \) and is diagonalizable in the direction \( x_{j} \) at the spot \( S_{t} \) with order \( s \), it follows that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\phi_{i=1}^{\infty} [x_{\tau(i)}]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{s}[x_{j}](S_{t})
\]
so that we have
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k+s}[x_{j}](S_{t}) = S_{0}.
\]
Now let us suppose there exist some \( r \leq k + s \) such that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k+s-r}[x_{j}](S_{t}) = S_{0}
\]
then it follows that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k+s-r}[x_{j}](S_{t}) = S_{0}.
\]
This is a contradiction, since \( k = \mathcal{I}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_{j}](S)] \) is the intensity of the dropler effect and is the smallest such number. It follows that
\[
\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_{j}](S_{t})] = \mathcal{I}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_{j}](S)] + \mathcal{O}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_{j}](S_{t})]
\]
\[
= k + s
\]
and the claim follows immediately.

Remark 8.9. The notion of the totient, the dropler effect and the diagonalization of an expansion may seem to be quite separate disparate notion of the theory but the following Proposition indicates a subtle connection among these three.

Proposition 8.10. Let \( \mathcal{F} = \{S_{i}\}_{i=1}^{\infty} \) be a collection of tuples of polynomials in the ring \( \mathbb{R}[x_{1}, x_{2}, \ldots, x_{n}] \). If the mixed expansion \( \langle \gamma^{-1} \circ \beta \circ \gamma \circ \nabla \rangle_{\phi_{i=1}^{\infty} [x_{\tau(i)}]}(S) \) induces a dropler effect with intensity \( k \) on the expansion \( \langle \gamma^{-1} \circ \beta \circ \gamma \circ \nabla \rangle_{\phi_{i=1}^{\infty} [x_{\tau(i)}]}(S) \) and is diagonalizable in the direction \( x_{j} \) at the spot \( S_{t} \) with order \( s \), then the expansion
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_{j}](S_{t})
\]
is free with totient
\[
\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\phi_{i=1}^{\infty} [x_{\tau(i)}]}(S)] = k + s.
\]

Lemma 8.11. Let \( \mathcal{F} = \{S_{i}\}_{i=1}^{\infty} \) be a collection of tuples of polynomials in the ring \( \mathbb{R}[x_{1}, x_{2}, \ldots, x_{n}] \). Then the expansion \( \langle \gamma^{-1} \circ \beta \circ \gamma \circ \nabla \rangle_{\phi_{i=1}^{\infty} [x_{\tau(i)}]}(S) \) for all \( 1 \leq i \leq l \) admits dropler effect from the source
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\phi_{i=1}^{\infty} [x_{\tau(i)}]}(S).
\]
Proof. Let us consider an arbitrary expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(i)}]}(S)\) for all \(1 \leq j \leq l\). Then by appealing to the commutative property of an expansion, we can rewrite

\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(i)}]}(S) - \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{\bigcup_{i=1}^{l} [x_{\sigma(i)}]}(S).
\]

It follows that

\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(i)}]}(S) = S_0.
\]

Next let us consider the residual mixed expansion

\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(j)}]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(j)}]}(S) - \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{\bigcup_{i=1}^{l} [x_{\sigma(i)}]}(S) = S_0.
\]

This proves the claim that each expansion of the form \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(i)}]}(S)\) for all \(1 \leq j \leq l\) admits a dropler effect from the source \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(i)}]}(S)\). \(\square\)

Remark 8.12. Next we show that the notion of diagonalization exist for mixed expansion in each direction involved in the mixed expansion. The proof is quite iterative in nature and will be employed in the sequel.

Proposition 8.13. Let \(F = \{S_i\}_{i=1}^{\infty}\) be a collection of tuples of polynomials belonging to the ring \(\mathbb{R}[x_1, x_2, \ldots, x_n]\). Then the mixed expansion

\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(i)}]}(S)
\]

is diagonalizable in each direction \([x_{\sigma(i)}]\) for \(1 \leq i \leq l\).

Proof. Let us consider the mixed expansion

\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(j)}]}(S)
\]

and let \([x_{\sigma(j)}]\) for \(1 \leq j \leq l\) be our targeted direction, then by appealing to the commutative property of an expansion we have

\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(i)}]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(j)}]}(S) - \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{\bigcup_{i=1}^{l} [x_{\sigma(i)}]}(S).
\]

Next let us consider the residual mixed expansion

\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(i)}]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(i)}]}(S) - \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(i)}]}(S).
\]

If there exist some tuple \(S_a \in F\) such that

\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(i)}]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(j)}]}(S_a)
\]

then we make a substitution and obtain two copies of the expansion operator \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(j)}]}\) by virtue of the commutative property of an expansion. Otherwise we choose

\[
S_b = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)\big|_{[x_{\sigma(i)}]}(S)
\]
and apply the remaining operators on it. By repeating the iteration in this manner, we will obtain
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\sigma^{i} \in [x_{\pi(i)}]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_{\pi(i)}]}(S_{i})
\]
for \( k \geq 1 \) and for some \( S_{i} \in \mathcal{F} \). This completes the proof of the proposition. \( \square \)

Remark 8.14. We are now ready to prove the inequality announced at the outset of the paper. We bring together the tools developed in the foregone section to obtain a stronger version of the inequality.

Theorem 8.15. Let \( \mathcal{F} = \{ S_{i} \}_{i=1}^{\infty} \) be a collection of tuples of polynomials in the ring \( \mathbb{R}[x_{1}, x_{2}, \ldots, x_{n}] \). Then we have the inequality
\[
\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\sigma^{i} \in [x_{\pi(i)}]}(S)] \leq \frac{1}{l} \sum_{i=1}^{l} \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\pi(i)}]}(S)] \\
+ \frac{1}{l} \sum_{1 \leq i \leq l} O[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\pi(i)}]}(S_{i})]
\]
where \( \text{Diag}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\sigma^{i} \in [x_{\pi(i)}]}(S)] \) is the set of all diagonals of the expansion
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\sigma^{i} \in [x_{\pi(i)}]}(S).
\]

Proof. Let us consider the mixed expansion
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\sigma^{i} \in [x_{\pi(i)}]}(S).
\]
Then by appealing to Proposition 16.6 then for each direction \([x_{\pi(i)}] \) for \( 1 \leq i \leq l \) there exist some spot \( S_{i} \) and a number \( k \geq 1 \) such that we can write
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\sigma^{i} \in [x_{\pi(i)}]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_{\pi(i)}]}(S_{i}).
\]
Again appealing to Lemma 8.11 each of the expansions \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_{\pi(i)}]}(S)\) admits a dropler effect from the source
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\sigma^{i} \in [x_{\pi(i)}]}(S).
\]
The upshot is that we can write for each direction \([x_{\pi(i)}] \) the relation
\[
\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\pi(i)}]}(S)] = I[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\pi(i)}]}(S)] + O[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\pi(i)}]}(S_{i})].
\]
By appealing to Definition 6.11 we obtain further the inequality
\[
\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\pi(i)}]}(S)] < \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\pi(i)}]}(S)] + O[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\pi(i)}]}(S_{i})].
\]
Again we see that the inequality is valid
\[
\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\sigma^{i} \in [x_{\pi(i)}]}(S)] = \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_{\pi(i)}]}(S)] \\
\leq \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_{\pi(i)}]}(S_{i})]
\]
so that we have the refined inequality
\[
\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\sigma^{i} \in [x_{\pi(i)}]}(S)] < \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\pi(i)}]}(S)] + O[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\pi(i)}]}(S_{i})].
\]
Since there are $l$ directions under consideration, we add $l$ such chains of the inequality and obtain
\[
\begin{align*}
\Phi[(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_1(1)]}(S) &< \sum_{i=1}^{l} \Phi[(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_1(1)]}(S) \\
&+ \sum_{1 ≤ i < j ≤ l} O[(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_1(1)]}(S_i)
\end{align*}
\]

This completes the proof of the theorem. □

**Corollary 8.16.** Let $F := \{S_i\}_{i=1}^∞$ be a collection of tuples of polynomials in the ring $R[x_1, x_2, \ldots, x_n]$. If $O[(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_1(1)]}(S_i) = 1$ for each
\[
(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_1(1)]}(S_i) ∈ \text{Diag}[(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_1(1)]}(S)
\]
then
\[
\Phi[(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_1(1)]}(S) < \frac{1}{l} \sum_{i=1}^{l} \Phi[(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_1(1)]}(S_i) + 1.
\]

**Proof.** This is a consequence of the inequality in Theorem 8.15 by taking
\[
O[(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_1(1)]}(S_i) = 1
\]
for each
\[
(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_1(1)]}(S_i) ∈ \text{Diag}[(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_1(1)]}(S).
\]

Appealing further to the energy dropler effect-intensity equation in Definition 6.1
\[
E[(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_j]}(S) = \Phi[(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_j]}(S) - I[(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_j]}(S)
\]
and Theorem 8.15 we obtain a refined inequality
\[
\begin{align*}
\Phi[(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_1(1)]}(S) &< \frac{1}{l} \sum_{i=1}^{l} E[(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_1(1)]}(S) \\
&+ \frac{1}{l} \sum_{i=1}^{l} I[(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_1(1)]}(S) \\
&+ \frac{1}{l} \sum_{1 ≤ i < j ≤ l} O[(γ^{-1} ◦ β ◦ γ ◦ ∇)]_{[x_1(1)]}(S_i)
\end{align*}
\]

We call this inequality the totient, energy, dropler effect intensity inequality.
9. Hybrid expansions

In this section we introduce and study the notion of hybrid expansions and explore some connections.

**Definition 9.1.** Let $\mathcal{F} = \{S_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. We say the expansions $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{[x_i]}(S_a))$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{[x_j]}(S_b))$ with $i \neq j$ are **hybrid** if

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{[x_i]}(S_a)) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{[x_j]}(S_b)).$$

We denote this relationship with

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{[x_i]}(S_a)) \bowtie (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{[x_j]}(S_b)).$$

**Proposition 9.2.** Let $\mathcal{F} = \{S_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. If the mixed expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{\otimes_{i=1}^{t}[x_{\sigma(i)}]}(S))$ is diagonalizable at the spot $S_a$ with order $k$ in the direction $[x_i]$ and

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{[x_i]}(S_a)) \bowtie (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{[x_j]}(S_b))$$

then the mixed expansion is also diagonalizable at the spot $S_b$ with order $t$ in the direction $[x_j]$.

**Proof.** Suppose the mixed expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{\otimes_{i=1}^{t}[x_{\sigma(i)}]}(S))$ is diagonalizable at the spot $S_a$ with order $k$ in the direction $[x_i]$, then by appealing to Definition 9.1 we have

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{\otimes_{i=1}^{t}[x_{\sigma(i)}]}(S)) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{[x_i]}(S_a)).$$

Under the assumption the expansions are hybrid, it follows by appealing to Definition 9.2

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{\otimes_{i=1}^{t}[x_{\sigma(i)}]}(S)) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{[x_j]}(S_b))$$

and the claim follows immediately. \qed

**Proposition 9.3.** Let $\mathcal{F} = \{S_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomial in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{[x_i]}(S_a))$ be a diagonal of the mixed expansion

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{\otimes_{i=1}^{t}[x_{\sigma(i)}]}(S))$$

with order $k \geq 1$. If

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{[x_i]}(S_a)) \bowtie (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{[x_j]}(S_b))$$

then

$$\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{[x_j]}(S_b))] < \max_{S_a \in \text{Diag}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{\otimes_{i=1}^{t}[x_{\sigma(i)}]}(S))]} \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla \mid_{[x_i]}(S_a))].$$
Proof. Let us suppose the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j]\)(\mathcal{S}_a)\) is a diagonal of the mixed expansion
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})
\]
with order \(k \geq 1\). Then it follows that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_j]}(\mathcal{S}_a).
\]
Since
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_a) \otimes (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_b)
\]
it follows that we can write
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_a) \otimes (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_b)
\]
so that by appealing to Theorem [8.15] we obtain the inequality
\[
\Phi((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_b)) = \Phi((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_a))
\]
and the claim follows by further controlling the two sums on the right hand-side of the inequality. \(\square\)

Remark 9.4. Next we express the relationship between hybrid expansion and the notion of diagonalization of a mixed expansion.

Proposition 9.5. Let \(\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^\infty\) be a collection of tuples of polynomials in the ring \(\mathbb{R}[x_1, x_2, \ldots, x_n]\). If
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_j]}(\mathcal{S}_a) \otimes (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^t_{[x_j]}(\mathcal{S}_b)
\]
then the mixed expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j] \otimes (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](\mathcal{S}_a)\) respectively \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j] \otimes (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_j]}(\mathcal{S}_a)\) is diagonalizable at the spots \(\mathcal{S}_a\) with order \(k + 1\) respectively \(\mathcal{S}_b\) with order \(t + 1\).

Proof. Let us suppose
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_j]}(\mathcal{S}_a) \otimes (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^t_{[x_j]}(\mathcal{S}_b)
\]
then it follows that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_j]}(\mathcal{S}_a) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^t_{[x_j]}(\mathcal{S}_b)
\]
so that by applying a copy of the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j]\) respectively \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j]\) on both sides, we have the following relations
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j] \otimes (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k+1}_{[x_j]}(\mathcal{S}_a) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{t+1}_{[x_j]}(\mathcal{S}_b)
\]
and
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j] \otimes (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_j]}(\mathcal{S}_b) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{t+1}_{[x_j]}(\mathcal{S}_b)
\]
and the claim follows immediately from these two relations. \(\square\)
10. Applications of the totient inequality

In this section we explore some applications of the theory. We obtain an inequality that will be useful for the study of the Pierce-Birkhoff conjecture. We first make the following terminologies more precise.

**Definition 10.1.** Let \( f_k \in \mathbb{R}[x_1, x_2, \ldots, x_n] \) be a polynomial. By the index of \( x_i \) for \( 1 \leq i \leq n \) relative to \( f_k \), denoted \( \text{Ind}_{f_k}(x_i) \), we mean the largest power of \( x_i \) in the polynomial \( f_k \).

**Lemma 10.2.** Let \( S = (f_1, f_2, \ldots, f_s) \) be a tuple of polynomials such that \( f_i \in \mathbb{R}[x_1, x_2, \ldots, x_n] \) for \( 1 \leq i \leq s \). Then for any \( 1 \leq j \leq n \)

\[
\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)]_{x_j}(S) = \max\{\text{Ind}_{f_i}(x_j)\}_{i=1}^s + 1.
\]

**Proposition 10.3.** Let \( f_1, f_2, \ldots, f_s \in \mathbb{R}[x_1, x_2, \ldots, x_n] \) be polynomials. Then there exist some \( J := J(l) \geq 0 \) such that

\[
\min\{\max\{\text{Ind}_{f_k}(x_{\sigma(i)})\}_{k=1}^s + 1\}_{i=1}^l < \frac{1}{l} \sum_{i=1}^l \max\{\text{Ind}_{f_k}(x_{\sigma(i)})\}_{k=1}^s + 2 + J.
\]

*Proof.* First let us consider the tuple \( S = (f_1, f_2, \ldots, f_s) \). Next, we break the proof into two special cases: The case were each of the expansions \( (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)]_{x_{\sigma(i)}}(S) \) for \( 1 \leq i \leq l \) does not admit a dropler effect from the source

\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)]_{x_{\sigma(i)}}(S).
\]

In the case each of the expansions admit no dropler effect from the underlying source then by appealing to Proposition 6.4 and Lemma 18.3 we can write

\[
\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)]_{x_{\sigma(i)}}(S) = \min\{\max\{\text{Ind}_{f_k}(x_{\sigma(i)})\}_{k=1}^s + 1\}_{i=1}^l = \min\{\max\{\text{Ind}_{f_k}(x_{\sigma(i)})\}_{k=1}^s + 1\}_{i=1}^l.
\]

Again by appealing to Lemma 18.3 we can write

\[
\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)]_{x_{\sigma(i)}}(S) = \max\{\text{Ind}_{f_k}(x_{\sigma(i)})\}_{k=1}^s + 1
\]

so that by appealing to Corollary 8.16 we can write

\[
\min\{\max\{\text{Ind}_{f_k}(x_{\sigma(i)})\}_{k=1}^s + 1\}_{i=1}^l < \frac{1}{l} \sum_{i=1}^l \max\{\text{Ind}_{f_k}(x_{\sigma(i)})\}_{k=1}^s + 2.
\]

We now turn to the case where at least one of the expansions \( (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)]_{x_{\sigma(i)}}(S) \) for \( 1 \leq i \leq l \) admits a dropler effect. In this case we would have by appealing to Proposition 6.4

\[
\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)]_{x_{\sigma(i)}}(S) = \min\{\max\{\text{Ind}_{f_k}(x_{\sigma(i)})\}_{k=1}^s + 1\}_{i=1}^l - J
\]

for some \( J := J(l) > 0 \). The right hand side expression is not impacted in this case. By combining the inequalities in both cases, the claim inequality follows as a consequence. \( \square \)
11. Exact expansion

In this section we introduce the notion of an exact expansion.

**Definition 11.1.** Let \(\mathcal{F} = \{S_i\}_{i=1}^{\infty}\) be a collection of tuples of polynomials in the ring \(\mathbb{R}[x_1, x_2, \ldots, x_n]\). Then we say the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{F}}(S)\) is exact in the directions \([x_{\sigma(1)}], \ldots, [x_{\sigma(n)}]\) each with multiplicity \(1 \leq l \leq n\) and \(\sigma : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}\) at the spot \(S_i\) if there exists a number \(s \in \mathbb{N}\), called the **degree** of the exactness, such that

\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{F}}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{F}}(S_i).
\]

In general, we say the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{F}}(S)\) is exact in the directions \([x_{\sigma(1)}], \ldots, [x_{\sigma(n)}]\) each with multiplicity \(k_1, \ldots, k_l \in \mathbb{N}\) for \(1 \leq l \leq n\) with degree \(s\) of exactness if

\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{F}}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{F}}(S_i)(k_1 \text{ times}).
\]

The following web shows the commutative diagram of a typical exact expansion

\[
\begin{array}{ccc}
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{F}}(S_1) & \xrightarrow{\phi_2} & (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{F}}(S_1)(k_1) \\
\downarrow{\phi_3} & & \downarrow{\phi_4} \\
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{F}}(S) & \xrightarrow{\eta_1} & (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{F}}(S_1)(k_1)
\end{array}
\]

with degree 3 of exactness, where \(\phi_2 = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{F}}(S_1)\) and \(\phi_4 = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{F}}(S)\) and \(\eta_1 = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{F}}(S_1)(k_1)\) for \(1 \leq l \leq n\). One can also construct more expanded commutative diagrams for exact expansion with arbitrarily large degrees. The notion of an exact expansion provides alternative paths to model an expansion in a specific direction. These type of expansion could conceivably be difficult and often delicate, so that a little distortion in the choice of directions may not guarantee the targeted expansion.

**Proposition 11.2.** The expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{F}}(S)\) is exact in the directions \([x_{\sigma(1)}], \ldots, [x_{\sigma(n)}]\) for \(1 \leq l \leq n\) and \(\sigma : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}\) at the spot \(S_1\) with degree \(s \in \mathbb{N}\) if and only if the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{F}}(S_1)\) is diagonalizable in the direction \([x_{\sigma}]\) at the spot \(S\) with order \(s\).

It is worth noting that Proposition 11.2 expresses the relationship between the notion of an exactness of an expansion and the diagonalization of an expansion. These two notions are quite similar except that the notion of an exactness is applied to expansions in a specific direction where as the notion of diagonalization is appropriate for expansions in a mixed directions. However one perceives these notions as different, they both can be considered as notions orthogonal to each other. Next we show that the notion of exactness in directions can be extended to other directions.

**Proposition 11.3.** If the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{F}}(S)\) is exact in the directions \([x_{\sigma(1)}], \ldots, [x_{\sigma(n)}]\) for \(1 \leq l \leq n\) and \(\sigma : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}\) at the spot \(S_1\) with degree \(s \in \mathbb{N}\), then it also exact in the directions \([x_{\sigma(1)}], \ldots, [x_{\sigma(n)}], [x_k]\) at the spot \(S_1\) with degree \(s + 1\).
Proof. By appealing to definition \[11\] we can write
\[
(γ^{-1} o β o γ o ∇)^{s}_{[x_k]}(S) = (γ^{-1} o β o γ o ∇)_{S_{1}}(S_1).
\]
The claim follows by applying an extra copy of the expansion operator \((γ^{-1} o β o γ o ∇)_{[x_k]}\) on both sides of the equation.

**Remark 11.4.** Next we show that we can extend the notion of an exactness to proper sub-expansions of an expansion.

**Proposition 11.5.** Let \((γ^{-1} o β o γ o ∇)_{[x_k]}(S) < (γ^{-1} o β o γ o ∇)_{[x_k]}(S_a)\) a sub-expansion of the expansion. If \((γ^{-1} o β o γ o ∇)_{[x_k]}(S)\) is exact in the directions \([x_{σ(1)}],...,[x_{σ(l)}]\) for \(1 ≤ l ≤ n\) and \(σ : \{1, 2, ..., n\} → \{1, 2, ..., n\}\) at the spot \(S_1\) with degree \(s \in N\), then there exists some fixed \(m \in N\) such that the expansion \((γ^{-1} o β o γ o ∇)_{[x_k]}(S_a)\) is exact with degree \(s + m - 1\) in the directions \([x_{σ(1)}],...,[x_{σ(l)}]\) for \(1 ≤ l ≤ n\) and \(σ : \{1, 2, ..., n\} → \{1, 2, ..., n\}\) at the spot \(S_1\).

Proof. Under the condition \((γ^{-1} o β o γ o ∇)_{[x_k]}(S) < (γ^{-1} o β o γ o ∇)_{[x_k]}(S_a)\), then there exists some fixed \(m \in N\) such that we can write
\[
(γ^{-1} o β o γ o ∇)_{[x_k]}(S) = (γ^{-1} o β o γ o ∇)^{m}_{[x_k]}(S_a)
\]
so that by applying \((s - 1)\) copies of the expansion operator \((γ^{-1} o β o γ o ∇)_{[x_k]}\) on both sides of the equation, we have
\[
(γ^{-1} o β o γ o ∇)^{s}_{[x_k]}(S) = (γ^{-1} o β o γ o ∇)^{s+m-1}_{[x_k]}(S_a).
\]
The claim follows since the expansion \((γ^{-1} o β o γ o ∇)_{[x_k]}(S)\) is exact with degree \(s\) in the directions \([x_{σ(1)}],...,[x_{σ(l)}]\) for \(1 ≤ l ≤ n\) at the spot \(S_1\). □

Although it is fairly easy to pass the notion of exactness of a sub-expansion to an expansion, the converse is actually difficult. We can only carry out this task under certain underlying condition on an expansion and their sub-expansion. The follow-up result underscores this discussion.

**Proposition 11.6.** Let \((γ^{-1} o β o γ o ∇)_{[x_k]}(S) < (γ^{-1} o β o γ o ∇)_{[x_k]}(S_a)\) a proper sub-expansion of the expansion. If \((γ^{-1} o β o γ o ∇)_{[x_k]}(S_a)\) is exact in the directions \([x_{σ(1)}],...,[x_{σ(l)}]\) for \(1 ≤ l ≤ n\) and \(σ : \{1, 2, ..., n\} → \{1, 2, ..., n\}\) at the spot \(S_1\) with degree \(s \in N\) and \((γ^{-1} o β o γ o ∇)_{[x_k]}(S_a) < (γ^{-1} o β o γ o ∇)_{[x_k]}(S)\), then there exists some \(j \in N\) such that the proper sub-expansion \((γ^{-1} o β o γ o ∇)_{[x_k]}(S)\) is exact with degree \(j < s\) in the directions \([x_{σ(1)}],...,[x_{σ(l)}]\) for \(1 ≤ l ≤ n\) and \(σ : \{1, 2, ..., n\} → \{1, 2, ..., n\}\) at the spot \(S_1\).

Proof. Suppose \((γ^{-1} o β o γ o ∇)_{[x_k]}(S_a)\) is exact in the directions \([x_{σ(1)}],...,[x_{σ(l)}]\) for \(1 ≤ l ≤ n\) and \(σ : \{1, 2, ..., n\} → \{1, 2, ..., n\}\) at the spot \(S_1\) with degree \(s \in N\). Then it follows that
\[
(γ^{-1} o β o γ o ∇)^{s}_{[x_k]}(S_a) = (γ^{-1} o β o γ o ∇)_{S_{1}}(S_1)
\]
so that under the requirement \((γ^{-1} o β o γ o ∇)^{s}_{[x_k]}(S_a) < (γ^{-1} o β o γ o ∇)_{[x_k]}(S)\) there exists some \(j \in N\) such that
\[
(γ^{-1} o β o γ o ∇)^{j}_{[x_k]}(S_a) = (γ^{-1} o β o γ o ∇)^{j}_{[x_k]}(S).
\]
Since \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_k]}(S) < (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_k]}(\mathcal{S}_n)\) a proper sub-expansion of the expansion, there exists some \(m \in \mathbb{N}\) such that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_k]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m_{[x_k]}(\mathcal{S}_n).
\]

By combining both equations, it follows that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n_{[x_k]}(\mathcal{S}_n) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{1+m}_{[x_k]}(\mathcal{S}_n)
\]
so that \(j < s\) and the claim follows from this assertion. \(\square\)

12. Sequences of an exact expansion

In this section we examine the structure and a commutative diagram of an exact expansion.

**Definition 12.1.** Let \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_k]}(S)\) be an exact expansion in the directions \([x_{\sigma(1)}], \ldots, [x_{\sigma(l)}]\) for \(1 \leq l \leq n\) and \(\sigma : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}\) with degree \(s\). Then we call the chain
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(1)}]}(S_1) \rightarrow (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(1)}] \otimes [x_{\sigma(2)}]}(S_1) \rightarrow \ldots
\]

an exact sequence of the exact expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_k]}(S)\) - respectively, \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_k]}(S)\) is an exact expansion of the exact sequence - where \(\phi_l = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(l)}]}\) and \(\phi_l \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_k]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_k] \otimes [x_{\sigma(l)}]}(S)\) for \(1 \leq l \leq n\).

**Definition 12.2.** We say the exact sequence
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\lambda(1)}]}(S_1) \rightarrow (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\lambda(1)}] \otimes [x_{\lambda(2)}]}(S_1) \rightarrow \ldots
\]
is a sub-sequence of the exact sequence
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\lambda(1)}]}(S_2) \rightarrow (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\lambda(1)}] \otimes [x_{\lambda(2)}]}(S_2) \rightarrow \ldots
\]
if the first chain is contained in the second chain.

**Remark 12.3.** Next we use the notion of an exactness of an expansion to study sub-expansions of an expansion. We first extend the notion of a sub-expansion of an expansion in a multivariate sense.
Definition 12.4. We say the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_k|}(S_a)\) is a sub-expansion of the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_k|}(S_b)\) along the directions \([x_\sigma(1)], \ldots, [x_\sigma(t)]\) each with multiplicity \(k_i\) for \(1 \leq i \leq l \leq n\), where \(\sigma : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}\) if and only if
\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_k|}(S_a) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\bigotimes_{i=1}^{t} [x_\sigma(i)]^{k_i}} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_j|}(S_b).

We denote this sub-expansion by
\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_k|}(S_a) \leq_{|x_\sigma(1)|, \ldots, |x_\sigma(t)|} \ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_j|}(S_b).

One could think of this notion as an extended notion of sub-expansions of an expansion. Indeed the underlying intuition remains, that a sub-expansion of an expansion is an outcome of several expansions on the mother expansion. This also provides some flexibility to the manner in which sub-expansions can be obtained from their mother expansion in the setting of an expansion in a mixed directions.

Theorem 12.5. If the exact sequence

\[
\begin{array}{c}
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_\sigma(1)|}(S_1) \\
\rightarrow (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_\sigma(1)|} \otimes [x_\sigma(2)](S_1) \\
\rightarrow \cdots
\end{array}
\]

of the exact expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_j|}(S_a)\) with degree \(u\) of exactness is a sub-sequence of the exact sequence

\[
\begin{array}{c}
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_\sigma(1)|}(S_2) \\
\rightarrow (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_\sigma(1)|} \otimes [x_\sigma(2)](S_2) \\
\rightarrow \cdots
\end{array}
\]

of the exact expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_j|}(S_b)\) with degree \(v\) of exactness then
\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_j|}(S_a) \leq_{|x_\sigma(1)|, |x_\sigma(2)|, \ldots, |x_\sigma(t)|} \ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_j|}(S_b).

for some \(s \in \mathbb{N}\) with \(s \leq n\) and where \(\lambda : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}\).

Proof. Under the main assumption, we can embed the chain

\[
\begin{array}{c}
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_\sigma(1)|}(S_1) \\
\rightarrow (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_\sigma(1)|} \otimes [x_\sigma(2)](S_1) \\
\rightarrow \cdots
\end{array}
\]

into the chain

\[
\begin{array}{c}
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_\sigma(1)|}(S_2) \\
\rightarrow (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_\sigma(1)|} \otimes [x_\sigma(2)](S_2) \\
\rightarrow \cdots
\end{array}
\]

\[
\begin{array}{c}
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_\sigma(1)|}(S_2) \\
\rightarrow (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_\sigma(1)|} \otimes [x_\sigma(2)](S_2) \\
\rightarrow \cdots
\end{array}
\]
so that by the commutative property of an expansion, we can write

\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{r}[x_{\sigma(i)}]}(S_2) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{r}[x_{\lambda(i)}]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{r}[x_{\sigma(i)}]}(S_1) \]

\[= (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{r+1}[x_{\sigma(i)}]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{r+1}[x_{\lambda(i)}]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{r}[x_{\sigma(i)}]}(S_1) \]

\[= (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{r+1}[x_{\sigma(i)}]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{r+1}[x_{\lambda(i)}]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{r}}(S_1) \]

\[= (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{r}}(S_1) \]

since \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{r}[x_{\sigma(i)}]}(S_1) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{r}[x_{\sigma(i)}]}(S_1)\). Under the exactness condition \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{r}[x_{\sigma(i)}]}(S_1) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{r}[x_{\sigma(i)}]}(S_2)\), we obtain

\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_0) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{r}[x_{\sigma(i)}]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{r}[x_{\lambda(i)}]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_0) \]

and it follows that

\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_0) \leq_{[x_{\sigma(1)}],[x_{\sigma(2)}],...,[x_{\sigma(r)}]}(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_0). \]

\[\square \]

### 13. Sub-expansion

**Definition 13.1.** Let \(F = \{S_i\}_{i=1}^{\infty}\) be a collection of tuples of polynomials in the ring \(R[x_1, x_2, ..., x_n]\). We say the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2)\) is a sub-expansion of the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1)\) if it is proper and satisfies the following conditions for all \(k \leq m\) such that

\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{k+m}(S_0). \]

We say the sub-expansion is proper if \(m = k = 1\). We denote this proper sub-expansion by \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2) < (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1)\). On the other hand, we say the sub-expansion is **ancient** if \(m + k > 1\). In general, we say the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_0)\) is a sub-expansion of the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1)\) along the directions \([x_{\sigma(1)}], ..., [x_{\sigma(l)}]\) each with multiplicity \(k_i\) for \(1 \leq i \leq l \leq n\), where \(\sigma : \{1, 2, ..., n\} \rightarrow \{1, 2, ..., n\}\) if and only if

\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_0) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{r}[x_{\sigma(i)}]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_0). \]

We denote this sub-expansion by

\[\text{with } \ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_0) \leq_{[x_{\sigma(1)}], ..., [x_{\sigma(l)}]}(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_0). \]

**Definition 13.2.** Let \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2)\) and \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1)\) be expansions. By the index of the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1)\) in the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1)\), denoted \([(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{r}(S_1)\] , we mean the value of \(r \in \mathbb{N}\) such that

\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{r}(S_1) \]
and we write
\[
\left(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla\right)_{x,j}(S_t) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x,j}(S_z) = r.
\]
We say the index is finite if and only if it exists and we write
\[
\left(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla\right)_{x,j}(S_t) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x,j}(S_z) < \infty.
\]
On the other hand, if no such value exists then we say the index is infinite and we write
\[
\left(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla\right)_{x,j}(S_t) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x,j}(S_z) = \infty.
\]

**Proposition 13.3.** Let \(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla\)_{x,j}(S_z) and \(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla\)_{x,j}(S_t) be expansions. Then
\[
\left(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla\right)_{x,j}(S_t) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x,j}(S_z) < \infty
\]
if and only if \(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla\)_{x,j}(S_z) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x,j}(S_t).

*Proof.* This is a simple consequence of the notion sub-expansions of an expansion and the index of an expansion. \(\square\)

**Proposition 13.4.** Let \(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla\)_{x,j}(S_z), \(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla\)_{x,j}(S_t) and \(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla\)_{x,j}(S_s) be expansions. If
\[
\left(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla\right)_{x,j}(S_t) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x,j}(S_z) < \infty \quad \text{and} \quad \left(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla\right)_{x,j}(S_z) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x,j}(S_z) < \infty,
\]
then
\[
\left(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla\right)_{x,j}(S_t) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x,j}(S_s) < \infty.
\]

*Proof.* Suppose
\[
\left(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla\right)_{x,j}(S_t) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x,j}(S_z) = \infty \quad \text{and} \quad \left(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla\right)_{x,j}(S_z) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{x,j}(S_z) = \infty.
\]
Then there exist some \(r, s \in \mathbb{N}\) such that we can write
\[
\gamma^{-1} \circ \beta \circ \gamma \circ \nabla\)_{x,j}(S_t) \quad \text{and} \quad \gamma^{-1} \circ \beta \circ \gamma \circ \nabla\)_{x,j}(S_s).\]
Remark 13.5. Next we show that the index of a sub-expansion in an expansion decreases with further expansions.

**Proposition 13.6.** Let \((\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_1) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_2)\). If there exists an \(l \in \mathbb{N}\) such that \((\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_1) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_2)\) then
\[
\left[ (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_2) : (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_1) \right] = (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_1).
\]

**Proof.** Suppose \((\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_1) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_2)\) then there exists some \(s \in \mathbb{N}\) such that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_1) = (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_2).
\]
Under the regularity condition \((\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_1) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_2)\) there exists some \(u \in \mathbb{N}\) such that we can write
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_1) = (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_2)
\]
so that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_2) = (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_2)
\]
and \(u < u + l = s\). The claimed inequality follows by making the substitutions
\[
[ (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_2) : (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_1) ] = u \quad \text{and} \quad [ (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_2) : (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_1) ] = s.
\]

**Remark 13.7.** Next we relate the index of the smallest sub-expansion in a collection of chains of sub-expansion in their mother expansion to the index of other sub-expansions in other sub-expansion.

**Theorem 13.8.** Let \((\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_1) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_2) \leq \cdots \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_n)\) - a chain of sub-expansions of the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_n)\). Then
\[
[ (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_n) : (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_1) ] = \sum_{i=1}^{n-1} \left[ (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_i) : (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_{i+1}) \right] - (n - 2).
\]

**Proof.** By appealing to Proposition 13.6 then \( (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_{i+1}) : (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_i) \) \(\infty\) for all \(1 \leq i \leq n - 1\) and there must exist some \(r_1 \in \mathbb{N}\) such that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_{n-1}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_n).
\]
Again there exists some \(r_2 \in \mathbb{N}\) such that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_{n-2}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \partial)[x_j](S_{n-1})
\]
so that
\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_{n-2}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_{n-1}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}^2(S_n).\]

Similarly there exists some \(r_3 \in \mathbb{N}\) such that
\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_{n-3}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}^{r_3}(S_{n-2})\]
so that
\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_{n-3}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}^{r_3}(S_{n-2}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}^{r_1+r_2+r_3-2}(S_n).\]

By repeating this argument and taking cognisance of the fact \([\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_{i+1}) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_i) < \infty\) for all \(1 \leq i \leq n-1\), we obtain
\[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_1) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}^{r_1+r_2+r_3+\cdots+r_{n-1}-(n-2)}(S_n)\]
and it follows that
\[\left[\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_n) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_1)\right] = \sum_{i=1}^{n-1} r_{n-i} - (n-2).

The claim follows since \[\left[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_{i+1}) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_i)\right] = r_{n-i}\]
for \(1 \leq i \leq n-1\).

We now obtain an important inequality as a consequence of Theorem 13.8 relating the index of the smallest sub-expansion in their mother expansion to local indices in each sub-expansion of the sub-expansions in the chain.

**Corollary 13.9** (The index inequality). Let \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_1) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_2) \leq \cdots \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_n) - a chain of sub-expansions of the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_n)\). Then
\[\left[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_n) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_1)\right] \leq \sum_{i=1}^{n-1} \left[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_{i+1}) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_i)\right].\]

**Theorem 13.10.** Let \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_1) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_2) - a sub-expansion of the expansion. If there exists some \(s \in \mathbb{N}\) such that \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_2) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_1)\), then
\[s + 1 = \left[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_2) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_1)\right] + \left[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_1) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x]}(S_2)\right].\]
Remark 14.2. We relate the notion of domination of an expansion to the notion of sub-expansion of an expansion. In fact these two notion are somewhat equivalent.
Proposition 14.3. Let \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)\) and \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)\). Then
\[
\mathbb{D}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1) \mid (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)] < \mathbb{D}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2) \mid (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)]
\]
if and only if \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1)\).

**Proof.** Let us suppose that
\[
\mathbb{D}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1) \mid (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)] < \mathbb{D}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2) \mid (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)]
\]
and let \(\mathbb{D}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1) \mid (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)] = u\) and \(\mathbb{D}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2) \mid (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)] = v\). Then we have \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2)\) and \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1)\). Since \(v\) and \(u\) are the smallest such numbers and \(v > u\), we obtain the chain of sub-expansions
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1)
\]
so that we obtain \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1)\). Conversely, suppose that \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1)\). Then it follows that
\[
\mathbb{D}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1) \mid (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)] < \mathbb{D}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2) \mid (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)]
\]
since \(\mathbb{D}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1) \mid (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)]\) and \(\mathbb{D}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2) \mid (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)]\) are the smallest such numbers. □

The notion of the totient introduced provides a specific time frame for an expansion to run into some sort of extinction. In other words, the totient is the time taken for an expansion to come to a complete halt. The next result controls the total dominating number of any chain of sub-expansion of an expansion by an expression involving the totient.

**Proposition 14.4.** Let \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2) \leq \cdots \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)\) be a chain of sub-expansions of the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)\). Then
\[
\sum_{i=1}^{n} \mathbb{D}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_i) \mid (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)] \leq \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)] - 1 \times (\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)] - 2).
\]

**Proof.** Let us insert the chain \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_1) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_2) \leq \cdots \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)\) of sub-expansions of the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_n)\)
into the full chain of sub-expansion of the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_n)\). Then we note that the sum on the left-hand side is bounded by the sum
\[
\sum_{i=2}^{\Phi[\gamma^{-1} \circ \beta \circ \gamma \circ \nabla][x_j](S_n)} - \frac{1}{i}.
\]
\[\square\]

One could sense that the notion of dominating number is intimately connected to the notion of the index of an expansion. We can in most cases compare these two numbers in the following simple way.

**Proposition 14.5.** \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_1) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_2)\), then
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_2) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_1) \leq \mathbb{D}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_1) | (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_2)].
\]

**Proof.** Let \(v = \left(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_2) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_1) \right)\) and \(u = \mathbb{D}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_1) | (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_2)]\) so that we can write \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_1) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_2)\) and \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_1) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_2)\). It follows that \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_2)\) is a sub-expansion of the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_2)\) so that there exits some \(t \geq 0\) such that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_2) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_2).
\]

It follows that \(\mathbb{D}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_1) | (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_2)] = u + v + t \geq v = \left(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_2) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_1) \right)\). \[\square\]

**Remark 14.6.** Next we generalize the inequality in proposition 14.5 to arbitrary sub-expansions in a chain.

**Theorem 14.7.** Let \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_1) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_2) \leq \cdots \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_n)\) be a chain of sub-expansions of the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_n)\). Then
\[
\sum_{i=k+1}^{n} \left(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_i) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_{i-1}) \right) \leq \mathbb{D}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_k) | (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_n)] + (n - k).
\]

**Proof.** Let \(\mathbb{D}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_k) | (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_n)] = u\) then
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_n) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_k)
\]
so that there exists some \(r_k \in \mathbb{N}\) such that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_n) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_k).
\]

Again by exploiting the chain of sub-expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_k) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_{k+1})\), it follows that there exists some \(r_{k+1} \in \mathbb{N}\) such that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_k) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)[x_j](S_{k+1})
\]

\[\square\]
so that
\[
\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]^n} (S_n) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{r_k}_{|_{[x]}} (S_k)
\]
\[
= (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{r_k+r_{k+1}}_{|_{[x]}} (S_{k+1}).
\]
Again by exploiting further the chain \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]}) (S_{k+1}) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]})(S_{k+2})\), it follows that there exists some \(r_{k+2} \in \mathbb{N}\) such that
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]}) (S_{k+1}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{r_k}_{|_{[x]}} (S_{k+2})
\]
so that we can write
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]^n}) (S_n) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{r_k}_{|_{[x]}} (S_k)
\]
\[
= (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{r_k+r_{k+1}+r_{k+2}}_{|_{[x]}} (S_{k+2})
\]
By continuing this argument, we obtain
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]})(S_n) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{r_k+r_{k+1}+r_{k+2}+\ldots+r_n} (S_n).
\]
It follows that
\[
u = r_k + \sum_{i=k+1}^{n} r_i - (n - k)
\]
with \(r_i = \left[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]}) (S_i) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]})(S_{i-1}) \right]\) and \(\mathbb{D}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]})(S_k) \mid (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]})(S_n)] = u\) and the claimed inequality follows immediately.

**Corollary 14.8.** Let \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]}) (S_1) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]}) (S_2) \leq \ldots \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]})(S_n)\) be a chain of sub-expansions of the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]})(S_n)\). Then
\[
\sum_{i=k+1}^{n} \left[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]}) (S_i) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]})(S_{i-1}) \right] > n - k.
\]

15. **Analytic expansions and application to function theory**

In this section we introduce and study the notion of singularity, the kernel and analytic expansions. We provide an application to the existence of singularities of solutions to certain polynomial equations.

15.1. **Some notions from single variable expansivity theory.** In this section we recall some notions under single variable expansivity theory developed earlier in [I]. These notions will serve as a model to proving the main result in this section.

**Definition 15.1.** Let \(\mathcal{F} = \{S_m\}_{m=1}^{\infty}\) be a family of tuples of polynomials in the ring \(\mathbb{R}[x]\), each having at least two entries with distinct degrees. Then the value of \(n\) such that the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]})(S) \neq S_0\) and \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]})(S) = S_0\) where \(S_0 = (0, 0, \ldots, 0)\) is called the degree of expansion and \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla |_{[x]})(S)\) is the rank of expansion, denoted by \(\mathcal{R}(S)\).
Theorem 15.2. Let $S_1$ and $S_2$ be tuples of polynomials in the ring $\mathbb{R}[x]$, with $\deg(S_1) > \deg(S_2)$, satisfying certain initial conditions at each phase of expansion. If $\mathcal{R}(S_1) = \mathcal{R}(S_2)$, then there exist some $j$ satisfying $1 \leq j < \deg(S_1)$ such that $S_1^j = S_2$.

Proof. Suppose $S_1$ and $S_2$ are tuples of polynomials in the ring $\mathbb{R}[x]$. Let $\deg(S_1) = k_1$ and $\deg(S_2) = k_2$. By definition 15.1, we can write $\mathcal{R}(S_1) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_1}(S_1)$ and $\mathcal{R}(S_2) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_2}(S_2)$. Under the assumption that $\mathcal{R}(S_1) = \mathcal{R}(S_2)$, we must have that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_2}(S_2) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_1}(S_1)$ if and only if $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_1-k_2}(S_1) = S_2$. Since $1 \leq k_1 - k_2 < k_1 = \deg(S_1)$, the result follows immediately.

Definition 15.3. Let $\{S^m\}_{m=1}^{\infty}$ be a family of expanded tuples of $S$, having at least two entries with distinct degrees. Then the limit of expansion of $S$ is the first expanded tuple $S^1 = (g_1, g_2, \ldots, g_n)$ such that $\deg(g_1) = \deg(g_2) = \cdots = \deg(g_n)$ for $n \geq 3$ and $1 \leq j \leq m$. Notation-wise, we denote simply by

$$\lim(S^n) = S^j,$$

the limit of the expansion.

Theorem 15.4. Let $\{S^m\}_{m=1}^{\infty}$ be a family of expansions of the tuple $S$ of polynomials in the ring $\mathbb{R}[x]$, such that at least two entries have distinct degree. Then the limit of expansions $\lim(S^n)$ of $S$ exists.

Proof. Let $\{S^m\}_{m=1}^{\infty}$ be a family of expansions of the tuple $S$ of polynomials in the ring $\mathbb{R}[x]$, having at least two entries with distinct degree. Suppose the limit of expansion does not exist, and let $S^1 = (f_1, f_2, \ldots, f_n)$ be the first phase expansion of $S$, then it follows that $\deg(f_i) \neq \deg(f_j)$ for some $1 \leq i, j \leq n$ with $i \neq j$. It follows in particular that $S^1 \neq \mathcal{R}(S)$ and $S^1 \neq S_0$. Thus the second phase expansion exists and let $S^2 = (g_1, g_2, \ldots, g_n)$ be the second phase expanded tuple. Again, it follows from the hypothesis that $\deg(g_i) \neq \deg(g_j)$ for some $1 \leq i, j \leq n$, with $i \neq j$, and it follows in particular that $S^2 \neq \mathcal{R}(S)$ and $S^2 \neq S_0$. Thus the third phase expansion exists. By induction it follows that the tuple $S$ of $\mathbb{R}[x]$ admits infinite number of expansions, thereby contradicting Proposition 15.1.

Theorem 15.5. Let $\{S^n\}_{n=1}^{\infty}$ be a family of expanded tuples of the tuple $S$ of polynomials in the ring $\mathbb{R}[x]$, such that at least two entries have distinct degrees and satisfying certain initial conditions at each phase of expansion. Then there exist some number $k$ called the dimension of expansion ($\dim(S)$), such that $\lim(S^n) = (\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^k(\mathcal{R}(S))$ for some $k < \deg(S)$.

Proof. Let $S$ be any tuple of polynomials in the ring $\mathbb{R}[x]$ that can be expanded, with at least two entries having distinct degree. Then, the limit exists by Theorem 15.4 and since an expansion can only be applied at a finite number of time and the map $\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma$ is a recovery which exist, it is clear there will exist such number $k$, so that $\lim(S^n) = (\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^k(\mathcal{R}(S))$. We only need to show that $k$ lies in the stated range. In anticipation of a contradiction, let us suppose $\lim(S^n) = (\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^k(\mathcal{R}(S))$ for any $k \geq \deg(S)$. Since the map is a bijection, it follows that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k(\lim(S^n)) = \mathcal{R}(S)$. It is easy to see that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k(\lim(S^n)) = S_0$, in which case we have that $\mathcal{R}(S) = S_0$, and so the rank of an expansion is null, which is a contradiction by definition 15.1. \qed
**Definition 15.6.** Let $\mathcal{S}$ be a tuple of polynomial in the ring $\mathbb{R}[x]$ and $\{S^n\}_{m=1}^\infty$ the family of expanded tuple of $\mathcal{S}$. Then by the local number of expansion, denoted $\mathcal{L}(\mathcal{S})$, we mean the value of $n$ such that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}) = \lim(\mathcal{S}^n)$.

Invoking Theorem 15.5. It follows from the above definition that for any tuple of polynomial in the ring $\mathbb{R}[x]$ satisfying certain initial conditions at each phase of expansion,

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}) = (\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^k(\mathcal{R}(\mathcal{S}))$$

if and only if

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n+k}(\mathcal{S}) = \mathcal{R}(\mathcal{S}).$$

By the definition of the rank of an expansion, it follows that

$$n + k = \deg(\mathcal{S})$$

which we call the principal equation and where $\mathcal{L}(\mathcal{S}) = n$, $\dim(\mathcal{S}) = k$ and $\deg(\mathcal{S})$ are the local number, the dimension and the degree of expansion, respectively, on $\mathcal{S}$. It is interesting to recognize that the value of the local number $\mathcal{L}(\mathcal{S})$ in any case is bounded cannot be more than the dimension of expansion. This assertion is confirmed in the following sequel.

**Lemma 15.7.** Let $\mathcal{S}$ be a tuple of polynomials in the ring $\mathbb{R}[x]$, satisfying certain initial conditions at each phase with $\deg(\mathcal{S}) \geq 4$. If $\dim(\mathcal{S}) > 2$, then the local number $\mathcal{L}(\mathcal{S})$ must satisfy the inequality

$$0 \leq \mathcal{L}(\mathcal{S}) \leq 2.$$ 

**Proof.** Let us suppose on the contrary $\mathcal{L}(\mathcal{S}) > 2$. Then it follows from the principal equation that $\dim(\mathcal{S}) < \deg(\mathcal{S}) - 2$, so that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{\dim(\mathcal{S})+2}(\mathcal{S}) \neq \mathcal{R}(\mathcal{S})$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{\dim(\mathcal{S})+2}(\mathcal{S}) \neq \mathcal{S}_0$. It follows that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{\dim(\mathcal{S})+2}(\mathcal{S}) = \mathcal{S}_1$. Then, using the principal equation, we find that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{\deg(\mathcal{S})}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{\deg(\mathcal{S})}(\mathcal{S}_1),$$

if and only if

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{\deg(\mathcal{S})-\deg(\mathcal{S}_1)}(\mathcal{S}) = \mathcal{S}_1.$$ 

It follows therefore that $\deg(\mathcal{S}) - \deg(\mathcal{S}_1) = \dim(\mathcal{S}) + 2$. Again, using the principal equation, we find that

$$\mathcal{L}(\mathcal{S}) = \deg(\mathcal{S}_1) + 2.$$ 

It follows from the above equation that $\deg(\mathcal{S}_1) + 2 = \mathcal{L}(\mathcal{S}) = \deg(\mathcal{S}) - \dim(\mathcal{S}) < \deg(\mathcal{S}) - 2$, so that $\deg(\mathcal{S}_1) + 4 < \deg(\mathcal{S})$. Since $\deg(\mathcal{S}) \geq 4$, it must be that $\deg(\mathcal{S}_1) + 4 \leq 4$, and we have that $\deg(\mathcal{S}_1) \leq 0$. This leaves us with the only choice that $\deg(\mathcal{S}_1) = 0$, contradicting the fact that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{\dim(\mathcal{S})+2}(\mathcal{S}) \neq \mathcal{R}(\mathcal{S})$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{\dim(\mathcal{S})+2}(\mathcal{S}) \neq \mathcal{S}_0$, and the proof is complete. $\square$
16. The kernel of an expansion

In this section we introduce the notion of the kernel of an expansion. One could draw some parallels with this notion and the notion of the boundary points of an expansion under the single variable theory. This choice of terminology is appropriate for this context, since we are no longer considering points as being solutions to our tuple equation but instead tuples consisting of solutions to certain partial differential equation. We launch formally the following languages.

**Definition 16.1.** Let \( F = \{S_i\}_{i=1}^{\infty} \) be a collection of \( l \)-tuples of polynomials in the ring \( \mathbb{R}[x_1, x_2, \ldots, x_n] \). By the kernel of the expansion \( (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S) \), denoted \( \text{Ker}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S)] \) we mean

\[
\text{Ker}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S)] = \left\{ (f_1, f_2, \ldots, f_l) \mid \right. \\
\left. f_r \in \mathbb{F}_C[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n], \right. \\
\left. (1 \leq r \leq l), \text{Id}_r[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(f_r)](S) = 0 \right\}
\]

where \( \mathbb{F}_C \) is a function field with complex number \( \mathbb{C} \) base space. We call each tuple in the kernel an annihilator of the given expansion.

**Definition 16.2.** Let \( F = \{S_i\}_{i=1}^{\infty} \) be a collection of \( l \)-tuples of polynomials in the ring \( \mathbb{C}[x_1, x_2, \ldots, x_n] \). We denote by

\[
\text{Id}_r[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S)](f_r)_{x_i}
\]

the value of

\[
\text{Id}_r[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S)]
\]

at \( x_i = f_r \).

**Proposition 16.3.** Let \( F = \{S_i\}_{i=1}^{\infty} \) be a collection of \( l \)-tuples of polynomials in the ring \( \mathbb{R}[x_1, x_2, \ldots, x_n] \). If for \( S_1, S_2 \in F \)

\[
\text{Ker}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_1)] = \text{Ker}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_2)]
\]

then \( S_1 = S_2 + S_{C[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]} \) and where \( S_{C[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]} \) is an \( l \)-tuple of polynomials in the ring \( \mathbb{C}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] \).

**Proof.** Let us suppose \( S_1, S_2 \in F \) and

\[
\text{Ker}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_1)] = \text{Ker}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_2)].
\]

For any \( (f_1, f_2, \ldots, f_l) \in \text{Ker}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_1)] \) then \( (f_1, f_2, \ldots, f_l) \in \text{Ker}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_2)] \) so that we can write

\[
\text{Id}_r[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_1)] = \text{Id}_r[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_2)] = 0
\]

for \( 1 \leq r \leq l \). Appealing to Definition [16.2] we can write

\[
\text{Id}_r[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_1)](f_r)_{x_i} = \text{Id}_r[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(f_r)](S_1)
\]

\[
= 0
\]

\[
= \text{Id}_r[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_2)](f_r)_{x_i}
\]

\[
= \text{Id}_r[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_2)](f_r)_{x_i}
\]

\[
= \text{Id}_r[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S_2)](f_r)_{x_i}
\]
for \(1 \leq r \leq l\). It follows that 
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{\delta}_{[x_{\sigma(i)}]}(S_1) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{\delta}_{[x_{\sigma(i)}]}(S_2)
\]
in 
\[
\text{Ker}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{\delta}_{[x_{\sigma(i)}]}(S_1)] = \text{Ker}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{\delta}_{[x_{\sigma(i)}]}(S_2)]
\]
so that by the linearity of an expansion in a specific direction, it follows that 
\[
S_1 = S_2 + S_{\mathbb{R}[x_{1}, \ldots, x_{i+1}, \ldots, x_n]}
\]
since an expansion in a specific direction is uniquely determined by their kernel. \(\square\)

**Remark 16.4.** Next we highlight the possibility two separate disparate expansions in separate directions at spots not quite equivalent can have the same kernel. The proposition below underscores this possibility. Put it differently, any two expansions need not happen in the same direction to have the chance of having the same kernel. That is to say, all hybrids expansions should in principle have the same kernel of their expansions.

**Definition 16.5.** Let \(\mathcal{F} = \{S_i\}_{i=1}^{\infty}\) be a collection of tuples of polynomials in the ring \(\mathbb{R}[x_1, x_2, \ldots, x_n]\). We say the mixed expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{L} = [x_{\sigma(i)}]}(S)\) is diagonalizable in the direction \([x_j]\) (1 \(\leq j \leq n\)) at the spot \(S_r \in \mathcal{F}\) with order \(k\) if 
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{L} = [x_{\sigma(i)}]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{\delta}_{[x_{\sigma(i)}]}(S_r).
\]
We call the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_r)\) the diagonal of the mixed expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{L} = [x_{\sigma(i)}]}(S)\) of order \(k \geq 1\). We denote with \(\mathcal{O}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S_r)]\) the order of the diagonal.

**Lemma 16.6.** Let \(\mathcal{F} = \{S_i\}_{i=1}^{\infty}\) be a collection of tuples of polynomials belonging to the ring \(\mathbb{R}[x_1, x_2, \ldots, x_n]\). Then the mixed expansion 
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{L} = [x_{\sigma(i)}]}(S)
\]
is diagonalizable in each direction \([x_{\sigma(i)}]\) for \(1 \leq i \leq l\).

**Proof.** Let us consider the mixed expansion 
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{L} = [x_{\sigma(i)}]}(S)
\]
and let \([x_{\sigma(j)}]\) for \(1 \leq j \leq l\) be our targeted direction, then by appealing to the commutative property of an expansion we have 
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{L} = [x_{\sigma(i)}]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(i)}]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{L} = [x_{\sigma(i)}]}(S).
\]
Next let us consider the residual mixed expansion 
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{L} = [x_{\sigma(i)}]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\mathcal{L} = [x_{\sigma(i)}]}(S)
\]
\[
\circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(i)}]}(S).
\]
If there exist some tuple \(S_n \in \mathcal{F}\) such that 
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(i)}]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(i)}]}(S_n)
\]
then we make a substitution and obtain two copies of the expansion operator \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(i)}]}\) by virtue of the commutative property of an expansion. Otherwise we choose

\[ S_b = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(i)}]}(S) \]

and apply the remaining operators on it. By repeating the iteration in this manner, we will obtain

\[ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\bigoplus_{i=1}^l[x_{\sigma(i)}]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(l)}]}^k(S) \]

for \(k \geq 1\) and for some \(S_i \in F\). This completes the proof of the proposition. \(\square\)

**Proposition 16.7.** Let \(F = \{S_i\}_{i=1}^\infty\) be a collection of tuples of polynomials in the ring \(\mathbb{R}[x_1, x_2, \ldots, x_n]\). If for \(k \neq j\) with \(1 \leq j, k \leq l\), then there exists some \(S_i, S_r \in F\) with \(S_i - S_r \neq S_R\) and some \(u, v \geq 1\) such that

\[ \text{Ker}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(i)}]}^u(S_i)] = \text{Ker}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(j)}]}^v(S_r)]. \]

**Proof.** Appealing to Lemma 16.4 then under the assumption \(k \neq j\) with \(1 \leq j, k \leq l\) there exists some \(u, v \geq 1\) and \(S_i, S_r \in F\) with \(S_i - S_r \neq S_R\) such that we can write for \(S \in F\)

\[ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\bigoplus_{i=1}^l[x_{\sigma(i)}]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(l)}]}^u(S) \]

and

\[ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\bigoplus_{i=1}^l[x_{\sigma(i)}]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(j)}]}^v(S) \]

so that

\[ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(l)}]}^u(S_i) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(j)}]}^v(S_r) \]

and the claim follows immediately. \(\square\)

17. Singularity and singular points of an expansion

In this section we introduce the notion of **singularity** and associated **singular** points of an expansion in a specific direction. We launch the following terminology.

**Definition 17.1.** Let \(F = \{S_i\}_{i=1}^\infty\) be a collection of \(l\)-tuples of polynomials in the ring \(\mathbb{R}[x_1, x_2, \ldots, x_n]\) and \(\text{Ker}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{ij}]}^k(S)]\) be the **kernel** of the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{ij}]}^k(S)\). Then by a **singular** point of the expansion we mean a tuple \(S = (a_1, a_{i-1}, a_{i+1}, \ldots, a_n)\) with \(a_j \in \mathbb{C}\) such that for some annihilator

\[ (f_1, f_2, \ldots, f_l) \in \text{Ker}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{ij}]}^k(S)] \]

then

\[ f_i[(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)] = \infty \]

for some \(1 \leq i \leq l\). We call the collection of all such points the singularity of the expansion and denote with

\[ \text{Sing}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{ij}]}^k(S)] = \left\{ (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in \mathbb{C}^{n-1} \mid f_i[(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)] = \infty \right\}. \]
18. Analytic Expansions

In this section we introduce and study the notion analytic expansions in specified directions in a particular domain in space.

**Definition 18.1.** Let $\mathcal{F} = \{S_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$ and $D \subset \mathbb{C}^{n-1}$. Then we say the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)$ is analytic in $D$ if

$$D \cap \text{Sing}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)] = \emptyset.$$ 

If the expansion is analytic in the entire $\mathbb{C}^{n-1}$ then we say for simplicity it is analytic.

**Definition 18.2.** Let $f_k \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ be a polynomial. By the index of $x_i$ for $1 \leq i \leq n$ relative to $f_k$, denoted $\text{Ind}_{f_k}(x_i)$, we mean the largest power of $x_i$ in the polynomial $f_k$.

**Lemma 18.3.** Let $S = (f_1, f_2, \ldots, f_s)$ be a tuple of polynomials such that $f_i \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ for $1 \leq i \leq s$. Then for any $1 \leq j \leq n$

$$\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(S)] = \max\{\text{Ind}_{f_k}(x_j)\}_{i=1}^{s} + 1.$$

**Definition 18.4.** Let $\mathcal{F} = \{S_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)$ be an expansion. Then by the **unionization** stage of the expansion, we mean the least of value of $j$ such that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^\ell(S) = S_o.$$

**Definition 18.5.** Let $\mathcal{F} = \{S_i\}_{i=1}^{\infty}$ be a collection of $l$-tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)$ be an expansion. By the **normalization** stage of the expansion, denoted $\overline{\Phi}(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)$, we mean the smallest value of $k$ such that

$$\text{Ind}_{Id_{\ell}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}]}^k(S)(x_i) = \text{Ind}_{Id_{\ell}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}]}^k(S)(x_i)$$

for all $1 \leq r, s \leq l$ with $r \neq s$. We call the corresponding expansion

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_i]}(S)$$

the **fibre** of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)$.

**Remark 18.6.** Throughout this paper we will assume the normalization stage of an expansion satisfies $\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)] > 0$ by working with tuples of multivariate polynomials with at least entries of distinct degree of the underlying direction. We will also assume quite implicitly for any two entry $f_i, f_j \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ of the tuple $S$ then $f_j \neq g \cdot f_i$ for any $g \in \mathbb{R}[x_1, x_2, \ldots, x_n]$.

**Proposition 18.7.** Let $\mathcal{F} = \{S_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)$ be an expansion. Then the **unionization** stage of the expansion satisfies the inequality

$$j \geq \frac{\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)]}{\overline{\Phi}(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)}.$$
Proof. The normalization stage of the expansion \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)\) is given by \(\varrho(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)\) so that the unionization stage is the index of the normalization stage of the expansion obtained as
\[
j \geq \left\lfloor \frac{\Phi(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)}{\varrho(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)} \right\rfloor.
\]

It is important to notice that the notion of the \textbf{normalization} stage is analogous to and has parallels with the notion of the limit of an expansion under the single variable theory. As such the notion of the local number runs exactly parallel to the notion of the \textbf{normalization} stage in multivariate expansivity theory. Next we prove the following Proposition, which will eventually feature in closing our argument.

**Proposition 18.8.** Let \(\{(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S_i)\}_{i=1}^{\infty}\) be a collection of expansions in the direction \(x_i\) of \(l\)-tuples of polynomials in the ring \(\mathbb{R}[x_1, x_2, \ldots, x_n]\) and \(\{(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)(S'_i)\}_{i=1}^{\infty}\) be a collection of expansions of tuples of polynomials in the ring \(\mathbb{R}[x]\). Then the map
\[
\chi(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) : \{\{(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S_i)\}_{i=1}^{\infty}\} \longrightarrow \{\{(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)(S'_i)\}_{i=1}^{\infty}\}
\]
for a fixed \((a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in \mathbb{R}^{n-1}\) such that for any \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S) \in \{(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S'_i)\}_{i=1}^{\infty}\) then
\[
\chi(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)(S)(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in \{(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)(S'_i)\}_{i=1}^{\infty}\)
\]
is an isomorphism. We denote the isomorphism by
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S_i) \simeq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)(S'_i).
\]

**Proposition 18.9.** Let \(\mathcal{F} = \{S_i\}_{i=1}^{\infty}\) be a collection of \(l\)-tuples of polynomials in the ring \(\mathbb{R}[x_1, x_2, \ldots, x_n]\) and \((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)\) be an expansion. Then
\[
\varrho((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)) \leq 2.
\]

**Proof.** Let \(S \in \mathcal{F} = \{S_i\}_{i=1}^{\infty}\) then we fix all other directions \(x_j\) for all \(j \neq i\) so that the expansion
\[
(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S) \simeq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)(S)(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)
\]
where \((S)(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)\) is a tuple of polynomials in the ring \(\mathbb{R}[x_i]\). Appealing to Lemma [5.7] we obtain the inequality
\[
\mathcal{L}\left((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)(S)(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)\right) \leq 2
\]
and by appealing to Proposition [18.8] we recover the following inequality
\[
\varrho((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(S)) \leq 2.
\]

**Remark 18.10.** Next we show that the unionization stage of a typical expansion cannot be too big. In other words it cannot possibly be the case that the totient of an expansion in a specific direction coincides with the unionization stage. We show that it can in fact be a lot smaller than the expected value in the following result.
Theorem 18.11 (Analytic range). Let \( F = \{ S_i \}_{i=1}^\infty \) be a collection of tuples of polynomials in the ring \( \mathbb{R}[x_1, x_2, \ldots, x_n] \) and \( (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_i|}(S) \) be an expansion. Then the expansion is analytic in the range

\[
j \geq \left\lfloor \frac{\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_i|}(S)]}{2} \right\rfloor.
\]

Proof. Every expansion \( (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_i|}(S) \) is always analytic at the unionization stage so that by appealing to Proposition 18.7, we note that

\[
j \geq \left\lfloor \frac{\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_i|}(S)]}{2} \right\rfloor
\]

by appealing to Proposition 18.9 so that the expansion \( (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_i|}(S) \) is analytic in the range

\[
j \geq \left\lfloor \frac{\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_i|}(S)]}{2} \right\rfloor.
\]

\[\square\]

19. An application to function theory

In this section we illustrate how these notion could be used to study the certain statistics about functions. We show that we can use these notions to study the existence of singularities of certain multivariate functions which are solutions to certain polynomial equations.

Corollary 19.1. Let \( F = \{ S_i \}_{i=1}^\infty \) be a collection of tuples of polynomials in the ring \( \mathbb{R}[x_1, x_2, \ldots, x_n] \) and \( (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_i|}(S) \) be an expansion. Then for

\[
j < \left\lfloor \frac{\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_i|}(S)]}{2} \right\rfloor
\]

there exist some \( (f_1, f_2, \ldots, f_l) \in \text{Ker}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{|x_i|}(S)] \) and \( (a_1, a_2, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) \in \mathbb{C}^{n-1} \)

such that

\[
f_i[(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)] = \infty
\]

for some \( 1 \leq i \leq l \).

References

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Department of Mathematics, African Institute for Mathematical sciences, Ghana. Email address: Theophilus@aims.edu.gh/emperordagama@yahoo.com