Linearly Constrained Kalman Filter For Linear Discrete State-Space Models

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Abstract

For linear discrete state-space (LDSS) models, under certain conditions, the linear least mean squares filter estimate has a convenient recursive predictor/corrector format, akin to the Kalman filter (KF). The aim of the paper is to introduce the general form of the linearly constrained KF (LCKF) for LDSS models, which encompasses the linearly constrained minimum variance estimator (LCMVE). Thus the LCKF opens access to the abundant literature on LCMVE in the deterministic framework which can be transposed to the stochastic framework. Therefore, among other things, the LCKF may provide alternative solutions to $\hat{x}_k$ filter and unbiased finite impulse response filter to robustify the KF, which performance are sensible to misspecified noise or uncertainties in the system matrices.

Key words: State estimation; unbiased filter; linear constraints; filtering; minimum mean-squared error upper bound.

1 Introduction

We consider the general class of linear discrete state-space (LDSS) models represented with the state and measurement equations, respectively,

$$
x_k = F_k x_{k-1} + w_k
$$

(1a)

$$
y_k = H_k x_k + v_k
$$

(1b)

where the time index $k \geq 1$, $x_k$ is the $P_k$-dimensional state vector, $y_k$ is the $N_k$-dimensional measurement vector. The state and measurement noise sequences $\{w_k\}$ and $\{v_k\}$, as well as the initial state $x_0$ are random vectors with arbitrary distributions. The noise sequences $\{w_k\}$ and $\{v_k\}$ have zero-mean values and the initial state $x_0$ has a finite known mean value. The system matrices $\{F_k, H_k\}$ and the covariance and cross-covariance matrices of $\{w_k, v_k, x_0\}$ contain elements with finite modulus and are either known or specified according to known parametric models. The objective is to estimate $x_k$ based on the measurements and our knowledge of the model dynamics. If the estimate of $x_k$ is based on measurements up to and including time $l$, we denote the estimator as $\hat{x}_{k|l} \triangleq \hat{x}_{k|l}(y_1, \ldots, y_l)$ and we use the term estimator to refer to the class of algorithms that includes filtering, prediction, and smoothing. A filter estimates $x_k$ based on measurements up to and including time $k$. A predictor estimates $x_k$ based on measurements prior to time $k$. A smoother estimates $x_k$ based on measurements prior to time $k$, at time $k$, and later than time $k$.

Since the seminal paper of Kalman [1], it is known that, provided that the system matrices $\{F_k, H_k\}$ and the covariance and cross-covariance matrices of $\{w_k, v_k, x_0\}$ are known, if $\{w_k, v_k, x_0\}$ verify certain uncorrelation conditions [2, (18)] and are Gaussian, the minimum variance or minimum mean squared error (MSE) filter estimate for LDSS models has a convenient recursive predictor/corrector form\(^2\), $\forall k \geq 2$:

$$
\hat{x}_{k|k} = F_k \hat{x}_{k-1|k-1} + H_k (y_k - H_k \hat{x}_{k-1|k-1})
$$

(2)

so-called the Kalman filter (KF) [1]. Even if the noise is non-Gaussian, the KF is the linear least mean squares (LLMS) filter (LLMSF) estimate [3]. As the computation of the KF depends on prior information on the first

\(^1\) This assumption is equivalent to the assumption of nonzero but known noises mean values [4, §3.2.4].

\(^2\) The superscript $^b$ is used to remind the reader that the value under consideration is the "best" one according to a criterion previously defined.

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and second order statistics of the initial state \( x_0 \) \([4][5][6]\), the KF can be looked upon as an "initial state first and second order statistics" matched filter [2]. However in numerous applications first and second order statistics of \( x_0 \) may be unknown. A commonly used solution to circumvent this lack of prior information on \( x_0 \) is the Fisher initialization \([7][8][9]\). The Fisher initialization consists in initializing the KF recursion at time \( k = 1 \) with the best linear unbiased estimator (BLUE) of \( x_1 \) associated to the measurement model (1b), where \( x_1 \) is regarded as a deterministic unknown parameter vector. In the deterministic framework, the BLUE of \( x_1 \) is also known as the linear minimum variance distortionless response (LMVDR) estimator of \( x_1 \) \([9][10][11]\), and coincides with the weighted least squares estimator (WLSE) of \( x_1 \). If \( H_1 \) is full rank and the covariance matrix of \( v_1 \) (\( C_{v_1} \)) is invertible, the Fisher initialization yields:

\[
\begin{align*}
\hat{x}_{1|1} &= P_{1|1} H_1^H C_{v_1}^{-1} y_1, \\
P_{1|1} &= (H_1^H C_{v_1}^{-1} H_1)^{-1}.
\end{align*}
\]

Actually, under mild regularity conditions on the noises covariance matrices, the Fisher initialization (3) yields the stochastic LMVDR filter (LMVDRF), which shares the same recursion as the KF, except at time \( k = 1 \) \([2][12]\). Although the LMVDRF is sub-optimal in MSE sense and is an upper bound on the performance of the KF, it is an infinite impulse response distortionless filter which performance is robust to an unknown initial state. However since the LMVDRF shares the same recursion as the KF, it also shares the same sensitivity to misspecified covariance matrices \([5][6][7][10][11]\) or uncertainties in the system matrices \([16][17][18][19]\). This sensibility of the performance achievable by the LMVDR estimator to misspecifications or uncertainties is also well documented in deterministic parameters estimation \([9][10][11]\). For instance, in array processing, the performance of MVDR beamformers are not particularly robust in the presence of various types of differences between the model and the actual environment (array perturbation, direction of arrival mismatch, inaccurate estimation of \( C_{v_1} \) \([9][6][11]\). Thus linearly constrained minimum variance (LCMV) beamformers have been developed in which additional linear constraints are imposed to make the MVDR beamformer more robust \([9][6][11]\).

The aim of the paper is to introduce the general form of the linearly constrained KF (LCKF) for LDSS models. Among other things, the LCKF can be used to robustify the KF, where robustness is understood as an ability to achieve high performance in the situations with imperfect, incomplete, or erroneous knowledge about the system under consideration and its environment. So far, in many applications where the statistical properties of state and measurement noises are not accurately known, it has been common practice to use a \( H_\infty \) filter \([5][10][20][21][22]\), also called the minimax filter, since it does not make any assumptions about the noise, and it minimizes the worst-case estimation error. Lately another possible way to robustify the KF to the presence of noises mismodeling via unbiased finite impulse response (UFIR) \([23]\), p-shift FIR \([24][25][11]\) or minimum variance UFIR \([26]\) filters, has been introduced. These algorithms have the same predictor/corrector format as the KF, often ignore initial estimations errors and the statistics of the noise, and become virtually optimal as the length of the FIR window increases. Therefore, since the LCMV estimator (LCMVE) is a special case of the LCKF, the use of LCKF opens access to the abundant literature on LCMVE in the deterministic framework \([11]\) which can be transposed to the stochastic framework in order to provide alternative solutions to \( H_\infty \) filter and UFIR filter to robustify the KF. As an example, we show how linear constraints can be used to robustify the KF in the presence of parametric modelling errors in the system matrices \( \{ F_k, H_k \} \). However, the disadvantage of using multiple linear constraints is that additional degrees of freedom are used by the LCKF in order to satisfy these constraints which increases the minimum MSE achieved. Last, it is noteworthy that linear constraints can be taken into account in any existing generalizations of the KF \([5][7]\), whether to deal with correlated state and measurement noise, colored state noise, colored measurement noise, for filtering with fading memory, to incorporate state constraints, for prediction, for smoothing, ....

The rest of the paper is organized as follows. Notations and signal model (joint proper complex) are introduced in Section II. In Section III, for sake of clarity, we give the main points of background knowledge on linear filters (including LLMSF and LMVDFR) required to discuss the filtering equations in the next Section. In section IV, we derive the general form of the LCKF for LDSS models and provide some analysis on the various forms of the LCKF recursion which depends on linear constraints combination. In section V, we show that the LCMVE in deterministic parameters estimation is a special case of the LCKF, which opens access to the abundant literature on LCMVE. Last, an example of the possible transposition of the LCMVE’s literature to the stochastic framework is given in Section VI.

## 2 Notations and signal model

The notational convention adopted is as follows: we shall use italic, small boldface and capital boldface letters to denote respectively scalars, column vectors and matrices. \( \mathcal{M}_C (N, P) \) denotes the vector space of complex matrices with \( N \) rows and \( P \) columns. The scalar/matrix/vector transpose conjugate is indicated by the superscript \( \dagger \). \( \mathbf{I} \) is the identity matrix. \( [\mathbf{A} \ \mathbf{B}] \) and \( [\mathbf{A} \mathbf{B}]^H \) denote the matrix resulting from the horizontal and the vertical concatenation of \( \mathbf{A} \) and \( \mathbf{B} \), respectively. The matrix resulting from the vertical concatenation \( k \) matrices \( \mathbf{A}_1, ..., \mathbf{A}_k \) of same column number is denoted
\( \overline{A}_k \). \( E[\cdot] \) denotes the expectation operator. If \( \mathbf{x} \) and \( \mathbf{y} \) are two complex random vectors: a) \( \mathbf{C}_x, \mathbf{C}_y \) and \( \mathbf{C}_{x,y} \) are respectively the covariance matrices of \( \mathbf{x} \), of \( \mathbf{y} \) and the cross-covariance matrix of \( \mathbf{x} \) and \( \mathbf{y} \); b) if \( \mathbf{C}_y \) is invertible, then \( \mathbf{C}_{x,y} \triangleq \mathbf{C}_x - \mathbf{C}_{x,y} \mathbf{C}_y^{-1} \mathbf{C}_{x,y}' [3] \).

As in [4, §3] and [10, §5.1], we adopt a joint proper (proper and cross-proper) complex signals assumption for the set of vector \( \{ \mathbf{x}_n, \{ w_k \}, \{ v_k \} \} \) which allows to resort to standard estimation in the MSE sense defined on the Hilbert space of complex random variables with finite second-order moment. A proper complex random variable is uncorrelated with its complex conjugate [10], and a zero mean proper complex random vector is said to be second-order circular [4, §3.2.5]. Moreover, any result derived with joint proper complex random vectors are valid for real random vectors provided that one substitutes the matrix/vector transpose conjugate for the matrix/vector transpose [4, §3.2.5][10, §5.4.1].

2.1 Equivalent linear observation model

Here, \( \mathbf{F}_{k-1} \in \mathcal{M}_C (P_k, P_{k-1}) \) and \( \mathbf{H}_k \in \mathcal{M}_C (N_k, P_k) \). First, as (1a) can be rewritten as, \( \forall k \geq 2 \):

\[
\mathbf{x}_k = \mathbf{B}_{k,1} \mathbf{x}_1 + \sum_{l=1}^{k-1} \mathbf{B}_{k,l+1} \mathbf{w}_l, \mathbf{B}_{k,l} = \begin{bmatrix} \mathbf{F}_{k-1} \mathbf{F}_{k-2} \cdots \mathbf{F}_1, k > l \\ 1, \quad k = l \\ 0, \quad k < l \end{bmatrix}
\]

an equivalent form of (1b) is:

\[
\mathbf{y}_k = \mathbf{A}_k \mathbf{x}_1 + \mathbf{n}_k, \quad \mathbf{A}_k = \mathbf{H}_k \mathbf{B}_{k,1}, \quad \mathbf{n}_k = \sum_{l=1}^{k-1} \mathbf{H}_k \mathbf{B}_{k,l+1} \mathbf{w}_l + \mathbf{v}_k, \quad k \geq 2.
\]

Second, (1b) can be extended on a horizon of \( k \) points from the first observation as:

\[
\mathbf{y}_k = \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_k \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_k \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{n}_k \end{bmatrix} = \overline{\mathbf{A}}_k \mathbf{x}_1 + \overline{\mathbf{p}}_k, \quad (5)
\]

\( \mathbf{y}_k, \overline{\mathbf{p}}_k \in \mathcal{M}_C (N_k, 1), \overline{\mathbf{A}}_k \in \mathcal{M}_C (N_k, P_k), N_k = \sum_{l=1}^{k} N_l. \)

3 Background on linear filters

3.1 Linear least-mean-squares estimator (LLMSE)

Let us consider two joint zero mean proper complex random vectors \( \mathbf{x} \) and \( \mathbf{y} \). The error between the sig-

\( \mathbf{C}_{x,y} \) is the covariance matrix of \( \mathbf{x} \) given \( \mathbf{y} \).

\( \text{If } \mathbf{x} \text{ and } \mathbf{y} \text{ are (proper) normal complex random vectors, then } \mathbf{C}_{x,y} \text{ is the covariance matrix of } \mathbf{x} \text{ given } \mathbf{y}. \)

\[ \mathbf{P}(\mathbf{K}) = E [\mathbf{e e}^H] = E \left[ (\hat{\mathbf{y}}(\mathbf{y}) - \mathbf{x}) (\hat{\mathbf{y}}(\mathbf{y}) - \mathbf{x})^H \right]. \]

(6)

If \( \mathbf{C}_y \) is invertible, then (6) can be rewritten as [3][10, p121]:

\[ \mathbf{P}(\mathbf{K}) = \mathbf{C}_{x,y} + (\mathbf{K} - \mathbf{C}_{x,y} \mathbf{C}_y^{-1}) \mathbf{C}_y (\mathbf{K} - \mathbf{C}_{x,y} \mathbf{C}_y^{-1})^H, \]

(7)

yielding:

\[ \mathbf{K}^b = \arg \min_{\mathbf{K}} \{ \mathbf{P}(\mathbf{K}) \} = \mathbf{C}_{x,y} \mathbf{C}_y^{-1}, \]

\[ \mathbf{P}(\mathbf{K}^b) = \mathbf{C}_{x,y}, \quad (8a) \]

\[ \hat{\mathbf{y}}^b = \mathbf{C}_{x,y} \mathbf{C}_y^{-1} \mathbf{y}. \quad (8b) \]

3.2 Linear least-mean-squares filter (LLMSF)

Therefore, the LLMSF of \( \mathbf{x}_k \) based on measurements up to and including time \( k, k \geq 2 \), is simply [1][3]:

\[ \hat{\mathbf{x}}^b_{k|k} = \left[ \mathbf{G}_{k-1}^b - \mathbf{K}_k^b \right] \mathbf{y}_k + \left[ \mathbf{G}_{k-1}^b - \mathbf{K}_k^b \right] \mathbf{C}_y = \mathbf{C}_{x,y} \mathbf{y}_k. \quad (9a) \]

where \( \mathbf{G}_{k-1}^b \in \mathcal{M}_C (P_k, N_{k-1}) \) and \( \mathbf{K}_k^b \in \mathcal{M}_C (P_k, N_k). \)

A few lines of algebra allows to rewrite (9a) as [12]:

\[ \hat{\mathbf{x}}^b_{k|k} = \hat{\mathbf{x}}^b_{k|k-1} + \mathbf{K}_k^b \left( \mathbf{y}_k - \bar{\mathbf{y}}^b_{k|k-1} \right); \]

(9b)

which is the general form of the so-called predictor/corrector format of LLMSF (\( \hat{\mathbf{x}}^b_{k|k-1} \) is also known as the a priori estimate of \( \mathbf{x}_k \)). Moreover, (9b) can be recasted as [2][12]:

\[ \hat{\mathbf{x}}^b_{k|k} = (\mathbf{I} - \mathbf{K}_k^b \mathbf{H}_k) \mathbf{F}_{k-1} \hat{\mathbf{x}}^b_{k-1|k-1|k-1} + \mathbf{K}_k^b \mathbf{y}_k + (\mathbf{I} - \mathbf{K}_k^b \mathbf{H}_k) \bar{\mathbf{y}}^b_{k-1|k-1|k-1} - \mathbf{K}_k \bar{\mathbf{y}}^b_{k|k-1}, k \geq 2. \]

(9c)

where \( \bar{\mathbf{y}}^b_{k|k-1} = \mathbf{C}_{v_k} \bar{\mathbf{y}}^b_{k-1|k-1} \mathbf{C}_{v_k}^{-1} \bar{\mathbf{y}}^b_{k-1|k-1} \) and \( \bar{\mathbf{w}}^b_{k-1|k-1} = \mathbf{C}_{w_{k-1}} \mathbf{C}_{w_{k-1}}^{-1} \bar{\mathbf{y}}^b_{k-1|k-1} \). Thus, the general assumptions required to obtain the recursive form (2) of the LLMSF (9c), aka the KF, are:

\[ \bar{\mathbf{w}}^b_{k-1|k-1} = 0, \quad \bar{\mathbf{y}}^b_{k|k-1} = 0, \quad k \geq 2. \quad (10a) \]

that is:

\[ \mathbf{C}_{w_{k-1}} \bar{\mathbf{y}}^b_{k-1} = 0, \quad \mathbf{C}_{v_k} \bar{\mathbf{y}}^b_{k-1} = 0, \quad k \geq 2. \quad (10b) \]

Another noteworthy point is that under the general assumptions (10b), the MSE of any linear filter
\[ \hat{x}_{k|k} = [G_{k-1}K_k] \hat{y}_k, \quad G_{k-1} \in \mathcal{M}_C(P_k,N_{k-1}) \text{ and } K_k \in \mathcal{M}_C(P_k,N_k), \]  
that is (8b):

\[ \mathbf{P}_{k|k} (G_{k-1}, K_k) = E \left[ (\hat{x}_{k|k} - x_k) \left( \hat{x}_{k|k} - x_k \right)^H \right], \quad (11) \]

breaks down into [12]:

\[ \mathbf{P}_{k|k} (G_k, K_k) = Q_{k|k} (G_{k-1}, K_k) \]

\[ + (1 - K_k H_k) \left( C_{w_{k-1}} + F_{k-1} C^H_{w_{k-1}} x_{k-1} \right) \right) (1 - K_k H_k)^H \]

\[ - (1 - K_k H_k) C_{x_k,v_k} K^H_k - K_k C_{x_k,v_k} (1 - K_k H_k)^H \]

\[ + K_k C_{v_k} K^H_k, \quad (12a) \]

where:

\[ Q_{k|k} (G_{k-1}, K_k) = \]

\[ E \left[ \left( G_{k-1} y_{k-1} - (1 - K_k H_k) F_{k-1} x_{k-1} \right) \times \left( G_{k-1} y_{k-1} - (1 - K_k H_k) F_{k-1} x_{k-1} \right)^H \right], \quad (12b) \]

which is a key result in order to derive the general form of the KF and LMVDR filter recursion (without extension of the state and measurement equations). Indeed, from (12a), it is obvious that:

\[ G^b_{k-1} = \arg \min_{G_{k-1}} \{ Q_{k|k} (G_{k-1}, K_k) \}, \quad (13a) \]

that is (8b):

\[ G_{k-1}^{b_{k-1}} = C_{(1 - K_k H_k) F_{k-1} x_{k-1}, y_{k-1}} C_{y_{k-1}, y_{k-1}}^{-1} C_{y_{k-1}, y_{k-1}} \]

\[ = (1 - K_k H_k) F_{k-1} x_{k-1}^{b_{k-1}|k-1}, \quad (13b) \]

leading to the general form of the Joseph stabilized version of the covariance measurement update equation [2][12]:

\[ \mathbf{P}_{k|k} (G_{k-1}, K_k) = (1 - K_k H_k) \]  

\[ D^b_{k-1} (1 - K_k W_k)^H \]

\[ - (1 - K_k H_k) C_{x_k,v_k} K^H_k - K_k C_{x_k,v_k} (1 - K_k H_k)^H \]

\[ + K_k C_{v_k} K^H_k \]

\[ (14) \]

where:

\[ \mathbf{P}^{b}_{k|k-1} = F_{k|k-1} P^{b}_{k-1|k-1} F_{k|k-1}^H + C_{w_{k-1}} \]

\[ + F_{k|k-1} C_{w_{k-1}, x_{k-1}} + C_{w_{k-1}, x_{k-1}} F_{k|k-1}^H \]

\[ = E \left[ \left( \hat{x}_{k|k-1} - x_k \right) \times \left( \hat{x}_{k|k-1} - x_k \right)^H \right], \]

\[ \hat{x}_{k|k-1} = \arg \min_{G_{k-1}} \{ P_{k|k} (G_{k-1}, K_k) \} \quad \text{s.t.} \quad \mathbf{K} \Lambda = \mathbf{T}. \quad (18) \]

The solution of the minimization of (14) is well known, since (14) can be reformulated as [4][5][6]:

\[ \mathbf{P}_{k|k} (G_{k-1}, K_k) = E \left[ \left( K_k \varepsilon_k - (x_k - \hat{x}_{k|k-1}^b) \right) \times \right] \]

\[ \left( K_k \varepsilon_k - (x_k - \hat{x}_{k|k-1}^b) \right)^H, \quad (15a) \]

where:

\[ \varepsilon_k = x_k - \hat{x}_{k|k-1}^b + v_k = y_k - H_k \hat{x}_{k|k-1}^b \]

\[ = y_k - y_{k|k-1}^b, \quad (15b) \]

is the innovations vector. Thus, according to (8a), \( K_k \) is computed according to the following recursion for \( k \geq 2 \):

\[ \mathbf{P}^{b}_{k|k-1} = F_{k|k-1} P^{b}_{k-1|k-1} F_{k|k-1}^H + C_{w_{k-1}} \]

\[ + F_{k|k-1} C_{w_{k-1}, x_{k-1}} + C_{w_{k-1}, x_{k-1}} F_{k|k-1}^H \]

\[ (16a) \]

\[ S_{k|k} = H_k P^{b}_{k|k-1} H_k^H + C_{v_k} + H_k C_{v_k,v_k} H_k^H \]

\[ K^b_k = \left( P^{b}_{k|k-1} H_k^H + C_{v_k,v_k} \right)^{-1} S_{k|k} \]

\[ (16b) \]

\[ \mathbf{P}^{b}_{k|k} = (I - K^b_k H_k) P^{b}_{k|k-1} - K^b_k C_{v_k,v_k} \]

\[ \]

\[ (16c) \]

where \( S_{k|k} = C_{v_k,v_k} \) and:

\[ \mathbf{P}^{b}_{k|k-1} = \min_{(G_{k-2}, K_k-1)} \{ \mathbf{P}_{k-1|k-1} (G_{k-2}, K_{k-1}) \} \]

\[ (16d) \]

The above recursion is also valid for \( k = 1 \) provided that \( \mathbf{P}^{b}_{0|0} = C_{x_0,v_0} \) and \( \hat{x}_{0|0}^b = (1) \) [2]. As already stressed in [2][12], the so-called ”standard LDSS model” mentioned in monographs [4, §9.1][5, §7.1][6, §8.2], which satisfies:

\[ C_{x_0,v_0} = 0, \quad C_{x_0,v_0} = 0, \quad C_{w_i,v_i} = C_{w_i \delta_i}, \]

\[ C_{v_i,v_i} = C_{v_i \delta_i}, \quad C_{w_i,v_i} = C_{w_{i-1} v_i \delta_{i-1} + 1}, \]

which has been regarded so far as leading to the general form of the KF (without extension of the state and measurement equations) including correlated state and measurement noise, is in fact a special case of (10b) yielding simplified expressions of (16a-16c). However, a thorough characterization of the subset of LDSS models compliant with (10b) is out of the scope of the paper and is left for future research.

### 3.3 Linearly constrained LLMSE (LCLLMSE)

The linearly constrained LLMSE is the solution of:

\[ K^b = \arg \min_{K} \{ \mathbf{P}(K) \} \quad \text{s.t.} \quad K \Lambda = T. \quad (18) \]

\[ \]
To stress the fact that the LCLLMSE is different from the LLMSE, we adopt the notation used in the
deterministic framework for the MVDR estimator (MVDRE) and its extension, aka the LCMV estimator
(LCMVE) [9, §6][10, §5.6]. Indeed, if \( x \) is a state vector and \( y \) is a
measurement vector, one can define a "state-former" in the same way as a beamformer in array processing or
a frequency-bin former in spectral analysis \([9, \text{s6-7}][10, \text{§5.6}]\), that is \( W \in M_C (\dim(y), \dim(x)) \) yielding the
state vector \( W^H y \). Furthermore, this common notation will help the reader to transpose the abundant literature
on LCMVE in the deterministic framework \([11] \) to the stochastic framework, since the recursive LCMVE is a
special case of the recursive LCLLMSE for LDSS models, as shown in Section 5. All in all it simply amounts
to set \( W = W^H \). Then (18) becomes:

\[
W^b = \arg\min_{W} \{ P(W) \} \quad \text{s.t.} \quad W^H A = T, \\
P(W) = E\left[ (W^H y - x) (W^H y - x)^H \right]. \tag{19}
\]

If \( C_x \) is invertible and \( A \) is a full rank matrix, it can easily be shown that \([27, (2.113)]\):

\[
W^b = C^{-1}_x A (A^H C^{-1}_y A)^{-1} T^H + \left( I - C^{-1}_y A (A^H C^{-1}_y A)^{-1} A^H \right) W, \tag{20a}
\]

and:

\[
P(W^b) = P(W) + (T^H - A^H W)^H (A^H C^{-1}_y A)^{-1} (T^H - A^H W), \tag{20b}
\]

where \( W \) is the best unconstrained state-former:

\[
W = (K^b)^H = C^{-1}_x C_y, X, P(W) = C_{x|y}. \tag{20c}
\]

The LCLLMSE coincides with the LLMSE iff: \( T = W^H A = K^b \).

### 3.4 Linear minimum variance distortionless response filter (LMDVRDF)

Let \( \overline{W}_k = [\overline{x}_{k-1}] \) where \( \overline{D}_{k-1} \in M_C (N_{k-1}, P_k) \) and
\( W_k \in M_C (N_k, P_k) \). Since:

\[
W^H_k y_k = \left( (W^H_k A_k) x_1 + G_k W_{k-1} \right) + W^H_k \bar{m}_k - G_k \bar{w}_{k-1}, \tag{21a}
\]

where \( G_k \bar{w}_{k-1} = \sum_{l=1}^{k-1} B_{k,l+1} W_l, G_k \in M_C (P_k, P_{k-1}), \)
\( P_k = \sum_{l=1}^{k} P_l \), a state-former \( W_k \) is distortionless iff:

\[
W^H_k A_k = B_{k,1} \Leftrightarrow W^H_k y_k = x_k + W^H_k \bar{m}_k - G_k \bar{w}_{k-1}. \tag{21b}
\]

If \( H_k \) is full rank, there exists a best distortionless state-former in the MSE sense, aka the LMVDRF, defined by
\([12]\):

\[
W^H_k = \arg\min_{\overline{W}_k} \{ P_k(k) (W_k) \} \quad \text{s.t.} \quad W^H_k A_k = B_{k,1}, \tag{21c}
\]

where \( P_k(k) (W_k) = E \left[ (W^H_k y_k - x_k) (W^H_k y_k - x_k)^H \right] \).

The MSE breakdown (12a-b) is also valid for any distortionless state-former, therefore:

\[
Q_{k-1} (D_{k-1} - W^H_k H_k) F_{k-1} = B_{k-1,1}, \tag{22a}
\]

\[
Q_{k-1} (D_{k-1} - W^H_k H_k) F_{k-1} = E \left[ (D^H_{k-1} y_{k-1} - (I - W^H_k H_k) F_{k-1} x_{k-1}) \times \right] \times \left[ (D^H_{k-1} y_{k-1} - (I - W^H_k H_k) F_{k-1} x_{k-1}) \times \right]. \tag{22b}
\]

Furthermore, since an equivalent form of the set of linear constraints \( W^H_k A_k = B_{k,1} \) is:

\[
D^H_{k-1} \bar{m}_{k-1} = (I - W^H_k H_k) F_{k-1} B_{k-1,1}, \tag{22c}
\]

one can notice that the solution of (22a) is \([12]\):

\[
Q_{k-1} (D_{k-1} - W^H_k H_k) F_{k-1} = B_{k-1,1}, \tag{22d}
\]

provided that \( C_{\pi_{k-1}} \) is invertible, and yields:

\[
Q_{k-1} (D_{k-1} - W^H_k H_k) F_{k-1} = (I - W^H_k H_k) F_{k-1} \times \tag{22e}
\]

where \( P^b_{k-1,k-1} = P_{k-1,k-1} (W^H_{k-1}) \). Finally, \( \forall k \geq 2 \),
the MSE breakdown (12a-b) \( P_{k|k} (D^b_{k-1}, W_k) \) shares the same general form of the Joseph stabilized version of
the covariance measurement update equation (14), provided that one substitutes \( W^H_k \) for \( K_k \). Therefore, if
\( H_k \) is full rank, the LMDVRDF shares the same recursion as the KF:

\[
\bar{x}^b_{k|k} = F_{k-1} \bar{x}^b_{k-1|k-1} + (W^H_k) \times (y_k - H_k F_{k-1} \bar{x}^b_{k-1|k-1}), \tag{23}
\]
where $W_k$ is updated according to (16a-c) provided that one substitutes $\hat{x}_{1|1}^b$ for $k_1$, except at time $k = 1$ where, if $C_{1|1}$ is full rank:

$$\hat{x}_{1|1}^b = (W_1^b)^H y_1, \quad W_1^b = C_{1|1}^{-1} H_1 (H_1^H C_{1|1}^{-1} H_1)^{-1}, \quad P_{1|1}^b = (H_1^H C_{1|1}^{-1} H_1)^{-1}.$$  

4 Linearly constrained KF for LDSS models

A linearly constrained LLLMSF (LCLLMSF) is the solution of:

$$\overline{W}_k^b = \arg \min_{W_k} \{ P_{k|k} (W_k) \} \quad \text{s.t.} \quad \overline{W}_k^H \Lambda_k = \Gamma_k. \quad (24)$$

In order to make use of (20a-20b), we limit ourselves to the case where $\Lambda_k$ is full rank and $C_{1|1}$, $1 \leq l \leq k$, are invertible. As shown in Subsection 3.4, the LMVDRF is an example of LCLLMSF (obtained where $\Lambda_k = \overline{\Lambda}_k$ and $\Gamma_k = B_{k|1}$) with a recursive predictor/corrector format (23). From the derivation of the LMVDRF outlined above, a sensible generalization of the set of constraints (22b) compatible with the predictor/corrector recursion is the set $\Lambda_k$, $k \geq 2$, defined by:

$$C_{1|1}^1: \overline{W}_k^H \left[ \begin{array}{c} \Lambda_{k-1} \\ H_k F_{k-1} \Gamma_{k-1} \end{array} \right] = [F_{k-1} \Gamma_{k-1} T_k],$$

that is:

$$C_{1|1}^1: \overline{D}_{k-1}^H \Lambda_{k-1} = (I - W_k^H H_k) F_{k-1} \Gamma_{k-1}, \quad (25a)$$

$$C_{1|2}^2: \quad \overline{W}_k^H \Delta_k = T_k, \quad (25c)$$

where both $\Lambda_{k-1}$ and $\Delta_k$ are full rank. Indeed, since the MSE breakdown (12a-b) is valid for any state-former, therefore under (25a):

$$\overline{D}_{k-1}^b = \arg \min_{\overline{D}_{k-1}} \{ Q_{k-1} (\overline{D}_{k-1}, W_k) \} \quad \text{s.t.} \quad \overline{D}_{k-1}^H \Lambda_{k-1} = (I - W_k^H H_k) F_{k-1} \Gamma_{k-1},$$

that is according to (20a):

$$\overline{D}_{k-1}^b = W_{k-1}^b (I - W_k^H H_k) F_{k-1} \Gamma_{k-1}^H, \quad (26a)$$

$$W_{k-1}^b = \arg \min_{\overline{W}_{k-1}} \{ P_{k-1|k-1} (W_{k-1}) \} \quad \text{s.t.} \quad \overline{W}_{k-1}^H \Lambda_{k-1} = \Gamma_{k-1}, \quad (26b)$$

and yielding:

$$Q_{k-1} (\overline{D}_{k-1}^b, W_k) = (I - W_k^H H_k) \times \quad \text{F}_{k-1} F_{k-1|k-1} \overline{D}_{k-1}^b (I - W_k^H H_k)^H. \quad (26c)$$

It is then worth noticing that, likewise:

$$\overline{W}_{k-1}^b F_{k-1}^H = \quad \arg \min_{\overline{W}_{k-1}^b} \left\{ E \left[ \left( \overline{W}_{k-1}^H \overline{y}_{k-1} - F_{k-1} x_{k-1} \right) \times \right] \right\} \quad \text{s.t.} \quad \overline{W}_{k-1}^H \Lambda_{k-1} = F_{k-1} \Gamma_{k-1}, \quad (27)$$

since then:

$$E \left[ \left( \overline{W}_{k-1}^H \overline{y}_{k-1} - x_k \right) \times \right] = \quad E \left[ \left( \overline{W}_{k-1}^H \overline{y}_{k-1} - F_{k-1} x_{k-1} \right) \times \right] + \overline{C}_{w_{k-1}} + F_{k-1} \overline{C}_{x_{k-1}, w_{k-1}} + \overline{C}_{w_{k-1}, x_{k-1}} \overline{F}_{k-1}^H. \quad (28a)$$

Thus:

$$\overline{x}_{k|k-1}^b = F_{k-1} \overline{x}_{k|k-1}, \quad (28a)$$

$$P_{k|k-1}^b = F_{k-1} P_{k-1|k-1}^b F_{k-1}^H + \overline{C}_{w_{k-1}} + F_{k-1} \overline{C}_{x_{k-1}, w_{k-1}} + \overline{C}_{w_{k-1}, x_{k-1}} \overline{F}_{k-1}^H, \quad (28b)$$

where $\overline{x}_{k|k-1}^b$ is the solution of (27). Therefore, under (25a), the MSE breakdown (12a-b) $P_{k|k} (\overline{D}_{k-1}^b, W_k)$ yields the general form of the linearly constrained Joseph stabilized version of the covariance measurement update equation (14):

$$W_k^b = \arg \min_{\overline{W}_k} \left\{ E \left[ \left( \overline{W}_k^H \varepsilon_k - \left( x_k - \overline{x}_{k|k-1}^b \right) \right) \times \right] \right\} \quad \text{s.t.} \quad \overline{W}_k^H \Delta_k = T_k, \quad (29)$$
where \( \varepsilon_k \) still stands for the innovations vector (15b) since (from a similar derivation as the one for \( \hat{x}_k^{k|k-1} \)):

\[
\begin{align*}
\mathbf{W}^b_{k-1} & = F^H_{k-1} \mathbf{H}^H_{k-1} = \\
& \arg \min_{\mathbf{W}_{k-1}} \left\{ E \left[ \begin{pmatrix} \mathbf{W}^H_{k-1} \mathbf{v}_{k-1} - \mathbf{y}_{k-1} \\ \mathbf{W}^H_{k-1} \mathbf{y}_{k-1} - \mathbf{y}_{k-1} \end{pmatrix}^H \right] \right\} \\
& \text{s.t. } \mathbf{W}^H_{k-1} \mathbf{A}_{k-1} = \mathbf{H}_k \mathbf{F}_{k-1} \mathbf{A}_{k-1}. \quad (30)
\end{align*}
\]

The solution of (29) is given by (20a-20b), which yields the following linearly constrained KF (LCKF) recursion at time \( k \):

\[
\hat{x}_k^{k|k} = F_{k-1} \hat{x}_k^{k|k-1} + (W_k^b)^H \left( y_k - H_k F_{k-1} \hat{x}_k^{k|k-1} \right),
\]

(31a)

\[
\begin{align*}
\mathbf{P}^b_{k-1} & = F_{k-1} \mathbf{P}^b_{k-1} F_{k-1}^H + C_{w_{k-1}} + \\
& F_{k-1} \mathbf{C}_{x_{k-1},w_{k-1}} + C_{w_{k-1},x_{k-1}} \mathbf{P}^b_{k-1} \\
\mathbf{S}_{k|k} & = H_k \mathbf{P}^b_{k-1} \mathbf{H}_k^H + C_{v_k} + H_k \mathbf{C}_{x_k,v_k} + C_{v_k,x_k} \mathbf{H}_k^H \\
\mathbf{W}_k & = \mathbf{S}^{-1}_{k|k} \left( \mathbf{H}_k \mathbf{P}^b_{k-1} F_{k-1} + C_{v_k,x_k} \right) \\
\mathbf{P}^b_{k|k} & = \left( \mathbf{I} - \mathbf{W}_k \mathbf{H}_k \right) \mathbf{P}^b_{k-1} \left( \mathbf{I} - \mathbf{W}_k \mathbf{H}_k \right) + \\
& \mathbf{C}_{v_k,x_k} - \mathbf{W}_k \mathbf{C}_{v_k,x_k} + \\
& \mathbf{T}_k - \mathbf{W}_k \mathbf{A}_{k-1} \left( \mathbf{D}_{k-1} \right)^H \left( \mathbf{T}_k - \mathbf{W}_k \mathbf{A}_{k-1} \right)^H \\
\text{and: }
\mathbf{P}^b_{k-1|k-1} & = \min_{\mathbf{W}_{k-1}} \left\{ \mathbf{P}_{k-1|k-1} \left( \mathbf{W}_{k-1} \right) \right\} \\
& \text{s.t. } \mathbf{W}_{k-1}^H \mathbf{A}_{k-1} = \mathbf{H}_{k-1}. \quad (31g)
\end{align*}
\]

4.1 The general case

In the general case, we look for the solution of (24) where:

\[
\begin{align*}
\begin{bmatrix} \mathbf{D}_{k-1}^H \\ \mathbf{W}_k \end{bmatrix}^H \begin{bmatrix} \mathbf{F}_{k-1} \\ \mathbf{H}_k \end{bmatrix} = \mathbf{G}_k \iff \mathbf{D}_{k-1}^H \mathbf{F}_{k-1} = \mathbf{G}_k - \mathbf{W}_k^H \mathbf{H}_k. \\
\end{align*}
\]

(32)

Since the MSE breakdown (12a-b) is valid for any state-former, therefore under (32):

\[
\begin{align*}
\mathbf{D}^b_{k-1} &= \arg \min_{\mathbf{D}_{k-1}} \left\{ \mathbf{Q}_{k-1} \left( \mathbf{D}_{k-1}, \mathbf{W}_k \right) \right\} \\
& \text{s.t. } \mathbf{D}_{k-1}^H \mathbf{F}_{k-1} = \mathbf{G}_k - \mathbf{W}_k^H \mathbf{H}_k.
\end{align*}
\]

Provided that \( \mathbf{F}_{k-1} \) is full rank, then according to (20a-20b):

\[
\begin{align*}
\mathbf{D}^b_{k-1} &= \mathbf{C}^{-1}_{y_{k-1}} \mathbf{F}_{k-1} \left( \mathbf{C}^{-1}_{y_{k-1}} \mathbf{F}_{k-1} \right)^{-1} (\mathbf{G}_k - \mathbf{W}_k^H \mathbf{H}_k) \\
& + \left( \mathbf{I} - \mathbf{C}^{-1}_{y_{k-1}} \mathbf{F}_{k-1} \left( \mathbf{C}^{-1}_{y_{k-1}} \mathbf{F}_{k-1} \right)^{-1} \mathbf{F}_{k-1} \right) \mathbf{W}_k \\
& \times \left( (\mathbf{I} - \mathbf{W}_k^H \mathbf{H}_k) \mathbf{F}_{k-1} \right)^H, \quad (33)
\end{align*}
\]

where \( \mathbf{W}_k^b = \mathbf{C}^{-1}_{y_{k-1}} \mathbf{F}_{k-1} \mathbf{F}_{k-1}^H \mathbf{W}_k \). It is noteworthy that (33) can be recasted as:

\[
\begin{align*}
\mathbf{D}^b_{k-1} &= \mathbf{W}_k^b \left( (\mathbf{I} - \mathbf{W}_k^H \mathbf{H}_k) \mathbf{F}_{k-1} \right)^H \\
& + \mathbf{C}^{-1}_{y_{k-1}} \mathbf{F}_{k-1} \left( \mathbf{C}^{-1}_{y_{k-1}} \mathbf{F}_{k-1} \right)^{-1} \times \\
& (\mathbf{G}_k - \mathbf{W}_k^H \mathbf{H}_k) \mathbf{F}_{k-1} \mathbf{F}_{k-1} \mathbf{G}_{k-1},
\end{align*}
\]

where:

\[
\begin{align*}
\mathbf{W}^b_{k-1} &= \arg \min_{\mathbf{W}_{k-1}} \left\{ \mathbf{P}_{k-1|k-1} \left( \mathbf{W}_{k-1} \right) \right\} \\
& \text{s.t. } \mathbf{W}_{k-1}^H \mathbf{A}_{k-1} = \mathbf{H}_{k-1}.
\end{align*}
\]

Therefore, the solution of (24) follows a predictor/corrector recursion (31a) iff:

\[
\mathbf{G}_k - \mathbf{W}_k^H \mathbf{H}_k - (\mathbf{I} - \mathbf{W}_k^H \mathbf{H}_k) \mathbf{F}_{k-1} \mathbf{G}_{k-1} = 0,
\]

that is iff:

\[
\mathbf{D}^H_{k-1} \mathbf{F}_{k-1} = (\mathbf{I} - \mathbf{W}_k^H \mathbf{H}_k) \mathbf{F}_{k-1} \mathbf{G}_{k-1},
\]

which is \( C^{1,1}_k \) (25b). As a consequence, the most general form of (32) leading to a solution following a predictor/corrector recursion is \( C^{1}_k \) (25a).

4.2 Constraints variants

Obviously the set of constraints \( C^2_k \) defined as the restriction of \( C^{1}_k \) to \( C^{1,2}_k \) (25c):

\[
C^2_k : \mathbf{W}^H_k \begin{bmatrix} 0 \\ \mathbf{D}_{k-1}^H \end{bmatrix} = \mathbf{T}_k \iff \mathbf{W}_k^H \mathbf{A}_k = \mathbf{T}_k, \quad (35a)
\]
follows as well the recursion (31a-31f), except that (31g) must be replaced with:

$$P_{k-1|k-1}^b = \min_{\mathbf{W}_{k-1}} \{P_{k-1|k-1} (\mathbf{W}_{k-1})\}, \quad (35b)$$

which means that $\mathbf{W}_{k-1}$ is unconstrained. In the same vein, the set of constraints $C_k^3$ defined as the restriction of $C_k^1$ (25a) to $C_k^1$ (25b):

$$C_k^3: \mathbf{W}_{k-1}^H \left[ \begin{array}{c} \Lambda_{k-1} \\ H_k^T \mathbf{F}_{k-1} \Gamma_{k-1} \end{array} \right] = \mathbf{F}_{k-1} \Gamma_{k-1}$$

$$\Rightarrow \mathbf{D}_{k-1}^H \Lambda_{k-1} = (\mathbf{I} - \mathbf{W}_{k-1}^H \mathbf{H}_k) \mathbf{F}_{k-1} \Gamma_{k-1}, \quad (36a)$$

follows the standard recursion (16a-16c) provided that one substitutes $(\mathbf{W}_{k}^H)^H$ for $\mathbf{K}_{k}^1$ and that $P_{k-1|k-1}^b$ is the solution of (31g):

$$P_{k-1|k-1}^b = \min_{\mathbf{W}_{k-1}} \{P_{k-1|k-1} (\mathbf{W}_{k-1})\}$$

s.t. $\mathbf{W}_{k-1} \Lambda_{k-1} = \Gamma_{k-1}$. \quad (36b)

### 4.3 Constraints combination

Actually, the introduction of the first set of constraints at a given time $k$ is provided by:

- $C_k^3$ if $k = 2$, since then $P_{k-1|k-1}^b$ (36b) results from a LCKF at time $k = 1$.
- $C_k^2$ if $k > 2$, since then $P_{k-1|k-1}^b$ (35b) results from an unconstrained KF at time $k - 1$.

For time $k + 1$, $C_k^2/C_k^3$ propagates via (26a) either in the form of $C_{k+1}^3$ leading to:

$$\begin{bmatrix} \mathbf{D}_k \\ \mathbf{W}_{k+1} \end{bmatrix} = \mathbf{F}_{k} \mathbf{T}_k,$$

or in the form of $C_{k+1}^1$ leading to:

$$\begin{bmatrix} \mathbf{D}_k \\ \mathbf{W}_{k+1} \end{bmatrix} \Lambda_{k+1} = \mathbf{F}_{k} \mathbf{T}_{k+1}.$$

For instance, the set of linear constraints (24) associated to the sequence $\{C_k^3, \ldots, C_k^3\}$ is characterized by:

$$\mathbf{A}_k = \begin{bmatrix} \Delta_1 \\ H_k^T \mathbf{F}_{k-1} \mathbf{T}_1 \\ \vdots \\ H_k^T \mathbf{F}_{k-1} \mathbf{T}_k \end{bmatrix}, \quad \Gamma_k = \mathbf{B}_{k,1} \mathbf{T}_1. \quad (37a)$$

If $\Delta_1 = \mathbf{H}_1$ and $\mathbf{T}_1 = \mathbf{I}$, then $\mathbf{A}_k = \mathbf{\bar{A}}_k$ and $\Gamma_k = \mathbf{B}_{k,1}$, which means that the sequence $\{C_k^3, \ldots, C_k^3\}$ yields the LMVDF. Another example is given by the set of constraints (24) associated to the sequence $\{C_k^1, \ldots, C_k^1\}$ which is characterized by:

$$\mathbf{A}_k = \begin{bmatrix} \Delta_1 \\ H_k^T \mathbf{F}_{k-1} \mathbf{T}_1 \\ \vdots \\ H_k^T \mathbf{F}_{k-1} \mathbf{T}_{k-1} \end{bmatrix} \Delta_{k-1}$$

$$\Gamma_k = \begin{bmatrix} \mathbf{B}_{k,1} \mathbf{T}_1 \\ \mathbf{B}_{k,2} \mathbf{T}_2 \\ \vdots \\ \mathbf{B}_{k,k-1} \mathbf{T}_{k-1} \end{bmatrix} \mathbf{T}_k \quad (37b)$$

Looking at (37a) and (37b), it seems difficult to have a clear understanding of the equivalent system of constraints, i.e. $\mathbf{W}_{k}^H \mathbf{A}_{k} = \mathbf{\Gamma}_{k}$, associated with any combination of $\{C_l^{i_l}, \ldots, C_l^{l'}\}$, $i_l, i_l' \in \{1, 2, 3\}, 2 \leq l \leq l' \leq k$.

However two properties of the predictor/corrector recursion (31a) are worth noting in order to grasp the general effect of some constraints. First, it is known that rewriting (31a) as:

$$\mathbf{x}_{k+|k-1} = (\mathbf{I} - (\mathbf{W}_k^b)^H \mathbf{H}_k) \mathbf{F}_{k-1} \left( \mathbf{x}_{k-1|k-1} - \mathbf{x}_k \right)$$

$$- (\mathbf{I} - (\mathbf{W}_k^b)^H \mathbf{H}_k) \mathbf{w}_{k-1} + (\mathbf{W}_k^b)^H \mathbf{v}_k, \quad (38)$$

allows to prove the unbiasedness property propagation:

$$\forall \mathbf{W}_k^b: E \left[ \mathbf{x}_{k-1|k-1} - \mathbf{x}_{k-1} \right] = 0 \Rightarrow E \left[ \mathbf{x}_{k|k} - \mathbf{x}_k \right] = 0. \quad (39)$$

Second, if $\mathbf{x}_{k-1|k-1}$ is a linear distortionless response filter, i.e. (21b):

$$\mathbf{x}_{k-1|k-1} = (\mathbf{W}_k^b)^H \mathbf{v}_k - \mathbf{G}_{k-1} \mathbf{w}_{k-2},$$

$$= \mathbf{x}_k + (\mathbf{W}_k^b)^H \mathbf{v}_k - \mathbf{G}_{k-1} \mathbf{w}_{k-2}.$$
then (38) becomes:

\[
\hat{x}_{k|k}^b - x_k = (W_k^b)^H v_k - (I - (W_k^b)^H H_k) w_{k-1}
+ \left( I - (W_k^b)^H H_k \right) F_{k-1} \left( (\hat{W}_{k-1}^b)^H \hat{n}_{k-1} - G_{k-1} \hat{w}_{k-2} \right)
\]

that is:

\[
\begin{align*}
\hat{x}_{k|k}^b - x_k &= (W_k^b)^H v_k - (I - (W_k^b)^H H_k) G_k \hat{w}_{k-1} \\
&\quad + \left( I - (W_k^b)^H H_k \right) F_{k-1} \left( (\hat{W}_{k-1}^b)^H \hat{n}_{k-1} - G_{k-1} \hat{w}_{k-2} \right)
\end{align*}
\]

since \( G_k \hat{w}_{k-1} = F_{k-1} G_{k-1} \hat{w}_{k-2} + w_{k-1} \). Moreover, whatever the constraint \( C_k^1 \) or \( C_k^2 \) considered, according to (26a):

\[
(\hat{D}_{k-1}^b)^H = \left( I - (W_k^b)^H H_k \right) F_{k-1} \left( (\hat{W}_{k-1}^b)^H \hat{n}_{k-1} \right).
\]

Thus:

\[
\begin{align*}
\hat{x}_{k|k}^b - x_k &= (\hat{D}_{k-1}^b)^H \hat{n}_{k-1} + (W_k^b)^H (H_k G_k \hat{w}_{k-1} + v_k) \\
&\quad - G_k \hat{w}_{k-1},
\end{align*}
\]

that is:

\[
\begin{align*}
\hat{x}_{k|k}^b - x_k &= (\hat{D}_{k-1}^b)^H \hat{n}_{k-1} + (W_k^b)^H \hat{n}_k - G_k \hat{w}_{k-1} \\
&\quad = (W_k^b)^H \hat{n}_k - G_k \hat{w}_{k-1}, \quad (40)
\end{align*}
\]

since \( \hat{n}_k = H_k G_k \hat{w}_{k-1} + v_k \), which proves the distortionless property propagation.

### 4.4 From LCKF to linearly constrained LMVDRE

As mentioned in the introduction, the Fisher initialization (3) yields the stochastic LMVDRE, which shares the same recursion as the KF, except at time \( k = 1 \), as recalled in Subsection 3.4. Although the LMVDRE is sub-optimal in terms of MSE, it has a number of merits [12]: a) it does not depend on the prior knowledge (first and second order statistics) on the initial state, b) it may outperform the KF in case of misspecification of the prior knowledge on \( x_0 \) [2, Section VI]. In other words, the LMVDRE can be pre-computed and its behaviour can be assessed in advance independently of the prior knowledge on \( x_0 \). Interestingly enough, since the predictor/corrector recursion (31a) propagates the distortionless property (40), these results are still valid regarding the LCKF, which can be looked upon as a linearly constrained "initial state first and second order statistics" matched filter. Indeed one can transform a LCKF into a linearly constrained LMVDRE (LCLMVDRE) provided that \( H_1 \) and \( C_{\nu_1} \) are full rank, and \( N_1 \) is large enough to incorporate the distortionless constraints if the LCKF already verifies some linear constraints at time \( k = 1 \):

\[
W_1^H \Delta_1 = T_1, \quad \Delta_1 \text{ full rank} \rightarrow W_1^H \Delta_1 H_1 = [T_1 I], \quad \Delta_1 \text{ full rank}.
\]

These features are quite interesting for filtering performance analysis and design of a LDSS system since they allow to synthesize a wide variety of linearly constrained infinite impulse response (IIR) distortionless filters which performance is robust to an unknown initial state.

### 5 Deterministic parameters estimation

If for LDSS models the focus has always been on the LLMSF, in deterministic parameters estimation, the maximum likelihood estimator (MLE) is the most used because of its nearly optimal properties in the asymptotic regime [28][29]. However the MLE suffers from a large computational cost as it generally requires solving a non-linear multidimensional optimization problem, which has led to the development of various sub-optimal techniques to reduce the computational burden [9]. For instance, in the fields of radar, sonar, and wireless communication, it is common place to design a LMVDRE for the most studied estimation problem: that of separating the components of data formed from a linear superposition of individual signals to noisy data [9][10]. This is the reason why, sometimes, the LMVDRE is also called a deconvolution filter [9, §6][10, §5.6]. In the case of LDSS models (1a-b), the stochastic filtering problem turns into a deterministic estimation problem if \( y_k = x_{k-1} = \ldots = x_1 \), where \( x_1 \) is deterministic and unknown (i.e. \( w_k = 0, k \geq 0 \)). In this instance, the assumptions (10b) reduces to \( C_{\nu_k} \Delta_{\nu_k} = C_{\nu_k} \Delta_{\nu_k} = 0, k \geq 2 \), which means that the measurement noise sequence is temporally uncorrelated: \( C_{\nu_k} \Delta_{\nu_k} = C_{\nu_k} \Delta_{\nu_k} = 0 \). Thus, as already noticed in [2], under temporally uncorrelated measurement noise and provided that \( H_1 \) and \( C_{\nu_1} \) are full rank [12], the LMVDRE:

\[
\hat{x}_{1|k}^b = (W_k^b)^H \hat{y}_k, \quad \hat{x}_{1|k}^b = C_{\nu_k}^{-1} A_k \left( A_k^H C_{\nu_k}^{-1} A_k \right)^{-1}, \quad (41)
\]

is the special case of the LMVDRE for which the predictor/corrector recursion is of the form:

\[
\begin{align*}
\hat{x}_{1|k}^b &= \hat{x}_{1|k-1}^b + (W_k^b)^H (y_k - H_k \hat{x}_{1|k-1}^b) \quad (42a) \\
S_{1|k} &= H_k \mathbf{P}_{1|k-1}^b H_k^H + C_{\nu_k}, \quad (42b) \\
\mathbf{W}_k^b &= S_{1|k}^{-1} H_k^b \mathbf{P}_{1|k-1}^b \quad (42c) \\
\mathbf{P}_{1|k}^b &= \left( I - (W_k^b)^H H_k \right) \mathbf{P}_{1|k-1}^b \quad (42d)
\end{align*}
\]

However, in deterministic parameters estimation, it is well known that the performance achievable by the LMVDRE [9, § 6.7] strongly depends on the accurate knowledge on the parametric models of the measurement equations (1b), that is on \( H_k \) and \( C_{\nu_k} \). For instance, in array processing, the performance of MVDR
beamformers are not particularly robust in the presence of various types of differences between the model and the actual environment (array perturbation, direction of arrival mismatch, inaccurate estimation of $C_{vA}$, ...) [9, § 6.6]. Thus LCMV beamformers have been developed in which additional linear constraints are imposed to make the MVDR beamformer more robust [9, § 6.7][11]. Interestingly enough, the existence of recursive LCKFs, and more specifically of recursive LCMVDRFs, also proves the existence of recursive LCMVEs under temporally uncorrelated measurement noise, which are obtained by adding at each (or at some) recursion a set of linear constraints:

$$\hat{x}_{1|k} = \hat{x}_{1|k-1} + (W_k^b)^H (y_k - A_k \hat{x}_{1|k-1})$$

s.t. $(W_k^b)^H \Delta_k = T_k,$

provided that $\Delta_k$ is full rank and $W_k^b$ is computed as follows (31a-31f):

$$S_{1|k} = H_k P_{1|k-1}^b H_k^H + C_{vA}, \quad W_k = S_{1|k}^{-1} H_k P_{1|k-1}^b,$$

$$W_k^b = W_k + S_{1|k}^{-1} \Delta_k \left(\Delta_k^H S_{1|k}^{-1} \Delta_k\right)^{-1} \left(T_k - W_k^H \Delta_k\right)^H,$$

$$P_{1|k}^b = (I - W_k^H H_k) P_{1|k-1}^b + \left(T_k - W_k^H \Delta_k\right) \left(\Delta_k^H S_{1|k}^{-1} \Delta_k\right)^{-1} \left(T_k - W_k^H \Delta_k\right)^H.$$

However, the disadvantage of using multiple linear constraints is that additional degrees of freedom are used by the LCMVE or the LCKF in order to satisfy these constraints, which increases the minimum MSE achieved.

5.1 The deterministic least-squares problem

For sake of completeness, let us recall that, under temporally uncorrelated measurement noise, the LMVDRE (41) and the WSE:

$$\hat{x}_{1|k} = \arg \min_{x_1} \left\{ \sum_{i=1}^k (y_i - H_i x_1)^H C_{v_i}^{-1} (y_i - H_i x_1) \right\},$$

are identical (duality) [4, §3.4][6, §4]. As a consequence of this, provided that $H_1$ and $C_{vA}$ are full rank, the WSE and the regularized WSE (RWLSE) are primarily special cases of the LMVDRE [2], and their relation to the KF highlighted in [30] is actually purely formal. The extension of this result to the RWLSE [4, §2.4][30]:

$$\hat{x}_{1|k} = \arg \min_{x_1} \left\{ (c - x_1)^H \Sigma^{-1} (c - x_1) + \sum_{i=1}^k (y_i - H_i x_1)^H C_{v_i}^{-1} (y_i - H_i x_1) \right\},$$

where $\Sigma$ is an Hermitian invertible matrix, is simply obtained by adding a fictitious observation at time $k = 0$: $y_0 = H_0 x_1 + v_0, C_{v_0} = \Sigma, y_0 = c$, $H_0 = I$, and by starting the recursion at time $k = 0$: $\hat{x}_{0|0} = P_{0|0}^b H_0^H C_{v_0}^{-1} y_0 = c$, $P_{0|0}^b = (H_0^H C_{v_0}^{-1} H_0)^{-1} = \Sigma [2].$

6 An illustrative example

In the case of LDSS model (1a-b), turning the KF into the LMVDRE thanks to the Fisher initialization (3), can be regarded as a first step towards the robustification of KF, namely to an unknown initial state. To some extent, the LCKF can also robustify the KF in the presence of parametric modelling errors in system matrices: $F_k \triangleq F_k(\omega) = [f_{1k}(\omega), \ldots, f_{nk}(\omega)]$ and $H_k \triangleq H_k(\theta) = [h_{1k}(\theta), \ldots, h_{nk}(\theta)],$ where $\omega$ and $\theta$ are supposed to be deterministic vector values determined via an ad hoc calibration process. In many cases, such calibration process provides estimates $\hat{\omega} = \omega + d\omega$ and $\hat{\theta} = \theta + d\theta$ of the true values $\omega$ and $\theta$, which means that the predictor/corrector recursion (31a) is updated according to $F_{k-1}(\hat{\omega})$ and $H_k(\hat{\theta})$ i.e. $W_k^b \triangleq W_k^b(\hat{\omega}, \hat{\theta}).$ If the calibration process is accurate enough, i.e. $d\omega$ and $d\theta$ are small, then the true state and measurement matrices $F_k(\omega)$ and $H_k(\theta)$ differ from the assumed ones via first order Taylor series. In this circumstance, the following additional linear constraints:

$$W_k^H H_k(\theta) \left[ \frac{\partial h_{1k}^b(\theta)}{\partial \omega} \ldots \frac{\partial h_{nk}^b(\theta)}{\partial \omega} \right] = 0,$$

$$W_k^H H_k(\theta) \left[ \frac{\partial f_{1k-1}^1(\hat{\omega})}{\partial \omega} \ldots \frac{\partial f_{nk-1}^1(\hat{\omega})}{\partial \omega} \right] = 0,$$

yields:

$$W_k^H y_k \simeq W_k^H \left( H_k(\hat{\theta}) (F_{k-1}(\hat{\omega}) x_{k-1} + w_{k-1}) + v_k \right),$$

which means that the predictor/corrector recursion (31a) has become robust to (small) parametric modelling error on the state and measurement matrices. Note that the proposed approach encompasses the LDSS model introduced in [17][18]:

$$x_k = (F_k + \Delta F_k) x_{k-1} + w_{k-1},$$

$$y_k = (H_k + \Delta H_k) x_k + v_k.$$
where \( x_k \in \mathbb{R}^P \), \( y_k \in \mathbb{R}^N \). The matrices \( \Delta F_k \) and \( \Delta H_k \) represent the parameter uncertainties and have the following structure \([17]\): \( \Delta F_k = A_{1,k} B_k C_k \) and \( \Delta H_k = A_{2,k} B_k C_k \), where \( A_{1,k} \), \( A_{2,k} \), and \( C_k \) are known matrices of the appropriate dimensions, and \( B_k \) are unknown matrices satisfying \( B_k B_k^T \leq I \). Then the LCKF may provide an alternative design of a linear filter such that the variance of the filtering error is guaranteed to be within a certain bound for all admissible uncertainties. Indeed, provided that:

\[
W^H_k [A_{2,k} H_k A_{1,k}] = 0,
\]

has a non trivial solution, that is \( N > \text{rank}( [A_{2,k} H_k A_{1,k}] ) \) and \( [A_{2,k} H_k A_{1,k}] \) is full rank, then:

\[
W^H_k y_k = W^H_k (H_k (F_k x_{k-1} + w_{k-1}) + v_k)
\]

and the LCKF does not depends on \( \Delta F_k \) and \( \Delta H_k \) any longer.

7 Conclusion

We introduced the general form of the LCKF for LDSS models. Since the LCME is a special case of the LCKF, the use of LCKF, among other things, opens access to the abundant literature on LCME in the deterministic framework which can be transposed to the stochastic framework in order to provide alternative solutions to the \( H_\infty \) filter and UFIR filter to robustify the KF.

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