ON THE QUENCHING BEHAVIOR OF THE MEMS WITH FRINGING FIELD

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Abstract. The singular parabolic problem $u_t - \triangle u = \lambda \frac{1+\delta|\nabla u|^2}{(1-u)^2}$ on a bounded domain $\Omega$ of $\mathbb{R}^n$ with Dirichlet boundary condition, models the Microelectromechanical systems (MEMS) device with fringing field. In this paper, we focus on the quenching behavior of the solution to this equation. We first show that there exists a critical value $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$, all solutions exist globally; while for $\lambda > \lambda_0$, all the solution will quench in finite time. The estimate of the quenching time in terms of large voltage $\lambda$ is investigated. Furthermore, the quenching set is a compact subset of $\Omega$, provided $\Omega$ is a convex bounded domain in $\mathbb{R}^n$. In particular, if the domain $\Omega$ is radially symmetric, then the origin is the only quenching point. We not only derive the one-side estimate of the quenching rate, but also further study the refined asymptotic behavior of the finite quenching solution.

1. Introduction. Micro- and nanoelectromechanical systems (MEMS and NEMS) are indubitably the hottest topic in engineering nowadays. These devices have been playing important roles in the development of many commercial systems, such as accelerometers, optical switches, microgrippers, micro force gauges, transducers, micropumps, etc. Yet it remains many researches to be done. A deeper understanding of basic phenomena will advance the design in MEMS and NEMS.

The simplified physical model of MEMS is the idealized electrostatic device. The upper part of this device consists of a thin and deformable elastic membrane that is held fixed along its boundary and which lies above a rigid grounded plate. This elastic membrane is modeled as a dielectric with a small but finite thickness. The upper surface of the membrane is coated with a negligibly thin metallic conducting film. When a voltage $V$ is applied to the conducting film, the thin dielectric membrane deflects towards the bottom plate, and when $V$ is increased beyond a certain critical value $V^*$, which is known as pull-in voltage, the steady-state of the elastic membrane is lost, and proceeds to quench or touch down at finite time.

In designing almost any MEMS or NEMS device based on the interaction of electrostatic forces with elastic structures, the designers will always confront the “pull-in” instability. This instability refers to the phenomena of quenching or touch down as we...
described previously when the applied voltage is beyond certain critical value \( V^* \). It is easy to see that this instability severely restricts the stable range of operation of many devices [22]. Hence many researches have been done in understanding and controlling the instability. Most investigations of MEMS and NEMS have followed Nathanson’s lead [20] and use some sort of small aspect ratio approximation to simplify the mathematical model. An overview of the physical phenomena of the mathematical models associated with the rapidly developing field of MEMS technology is given in [22].

The instability of the simplified mathematical model (cf. [14]) has also been observed and analyzed in [14], [6], [13], etc. This model is described by a partial differential equation:

\[
\begin{cases}
  u_t - \triangle u = \frac{\lambda}{(1-u)^2} & \text{for } (x,t) \in \Omega_T \\
  u(x,t) = 0 & x \in \partial\Omega_T \\
  u(x,0) = 0 & x \in \Omega,
\end{cases}
\]

(1.1)

where \( \sigma_T = \sigma \times [0,T) \), \( T \) is the maximal time of existence of the solution. The study of (1.1) starts from its stationary equation. It is shown in [5] that there exists a pull-in voltage \( \lambda^* := \lambda^*(\Omega) > 0 \) such that

- a. If \( 0 \leq \lambda < \lambda^* \), there exists at least one solution to the stationary equation of (1.1).
- b. If \( \lambda > \lambda^* \), there is no solution to the stationary equation of (1.1).

Concerning the evolutionary equation (1.1), [6] dealt with the issues of global convergence as well as finite and infinite time quenching of (1.1). It asserts that for the same \( \lambda^* \) above, the followings hold:

1. If \( \lambda \leq \lambda^* \), then there exists a unique solution \( u(x,t) \) to (1.1) which globally converges pointwisely as \( t \to +\infty \) to its unique minimal steady-state.
2. If \( \lambda > \lambda^* \), then a unique solution \( u(x,t) \) to (1.1) must quench in finite time.

More refined analysis of the quenching behavior of (1.1) is in [6], [13] and the references therein.

As pointed out in [23], (1.1) is only a leading-order outer approximation of an asymptotic theory based on expansion in the small aspect ratio. The fringing term \( \delta |\nabla u|^2 \) is the first-order correction. The model (1.1) is modified as:

\[
\begin{cases}
  u_t - \triangle u = \frac{\lambda + \delta |\nabla u|^2}{(1-u)^2}, & (x,t) \in \Omega_T \\
  u(x,t) = 0, & (x,t) \in \partial\Omega_T \\
  u(x,0) = 0, & x \in \Omega,
\end{cases}
\]

\((F_{\lambda,\delta})\)

In this paper, we are aim to understand how the fringing term affects the behavior of the solution to \((F_{\lambda,\delta})\), including the pull-in voltage, quenching time, quenching behavior, etc.

The stationary equation of \((F_{\lambda,\delta})\)

\[
\begin{cases}
  -\triangle u = \frac{\lambda + \delta |\nabla u|^2}{(1-u)^2}, & x \in \Omega \subset \mathbb{R}^n \\
  u(x) = 0, & x \in \partial\Omega.
\end{cases}
\]

\((SF_{\lambda,\delta})\)

has been studied in [26]. The authors show that for fixed \( \delta > 0 \), there exists a pull-in voltage \( \lambda_0^\delta > 0 \) such that for \( \lambda > \lambda_0^\delta \) there are no solution to \((SF_{\lambda,\delta})\); for \( 0 < \lambda < \lambda_0^\delta \) there are at least two solutions; and when \( \lambda = \lambda_0^\delta \) there exists a unique solution. Furthermore, for \( \lambda < \lambda_0^\delta \) the equation \((SF_{\lambda,\delta})\) has a minimal solution \( u_\lambda \) and \( \lambda \mapsto u_\lambda \) is increasing for \( \lambda \in (0, \lambda_0^\delta) \).

The instability of \((F_{\lambda,\delta})\) is stated in the following theorem.
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THEOREM 1.1 (Theorem 2.3, [25]). For fixed $\delta > 0$, suppose $\lambda_0^*\delta$ is the pull-in voltage in [26], then the following hold:

1. If $\lambda \leq \lambda_0^*$, then there exists a unique global solution $u(x, t)$ of $F_{\lambda, \delta}$ which converges as $t \to \infty$ monotonically and pointwisely to its unique minimal steady state.

2. If $\lambda > \lambda_0^*$, then the unique solution $u(x, t)$ for $F_{\lambda, \delta}$ must quench in finite time.

In the literature, we say the solution $u$ quenches if it reaches $u = 1$. Although the proof of this theorem has been briefly sketched in [25] with the right-hand side of $F_{\lambda, \delta}$ to be, rather than $\frac{1+\delta|\nabla u|^2}{(1-u)^2}$, even more general nonlinearity $g(u)(1+\delta|\nabla u|^2)$, where $g : [0, 1) \to \mathbb{R}_+$ satisfying $g$ is a $C^2$, positive, nondecreasing and convex function such that $\lim_{u \to 1^-} g(u) = +\infty$, $\int_0^1 g(s)ds = +\infty$.

We believe the argument there is not rigorous, since when passing to the limit, it is not clear why $\lim_{t \to \infty} \nabla k(x, t) = m(x)$ and $\lim_{t \to \infty} \Delta k(x, t) = \Delta m(x)$. Instead, in this paper we adopt the argument in [1] to give a detailed proof.

The pull-in voltage $\lambda_0^*$ has been estimated in [25]:

$$\frac{4}{27} \left| |\xi| + \delta |\Delta \xi| \right|^2 \leq \lambda_0^* \leq \lambda^*.$$  \hspace{1cm} (1.2)

We show in this paper that $\lim_{\delta \to \infty} \lambda_0^* = 0$. This improves the upper bound in (1.2) dramatically for $\delta \gg 1$.

From Theorem 1.1 we know that the solution quenches in finite time when $\lambda \geq \lambda_0^*$, denoted $T = T(\lambda, \delta) < \infty$. The precise definition of quenching time $T$ is

$$T = \sup \{ t > 0 : \| u(\cdot, \tau) \|_\infty < 1, \forall \tau \in [0, t] \}.$$ 

It has been shown in [25] that $T = O \left( (\lambda - \lambda_0^*)^{-\frac{1}{2}} \right)$, provided that $\lambda > \lambda_0^*$ is sufficiently close to $\lambda_0^*$ and $\delta \ll 1$. For $\lambda \gg \lambda_0^*$, we show that the following result:

**THEOREM 1.2.** The quenching time $T = T(\lambda, \delta)$ for the solution $u$ of $F_{\lambda, \delta}$ verifies

$$\lim_{\lambda \to \infty} \sup_{\lambda \to \infty} \lambda T = \frac{1}{3}.$$ 

This result is valid for $F_{\lambda, \delta}$ with or without fringing term. However, it is known that the quenching time for $F_{\lambda, \delta}$ without fringing term satisfies

$$\lim_{\lambda \to \infty} \lambda T = \frac{1}{3}.$$ 

The numerical results in section 6.1 suggest that $\lim_{\lambda \to \infty} \lambda T = 0$. Actually, with the similar argument in [27], we show that

$$T \leq \frac{||\phi||_1}{3\lambda ||\phi||_1 + ||\Delta \phi||_1},$$

where $\phi \geq 0$ is any $C^2$ function in $\Omega$ and $\phi = 0$ on $\partial \Omega$, and $|| \cdot ||_1$ is the $L^1$ norm. This implies that

$$T \lesssim \frac{1}{\lambda},$$

if $\lambda \gg 1$. The notation $a \lesssim b$ means there exists some constant $C > 0$ such that $a \leq Cb$. This is a finer decaying rate than $O(\lambda^{-\frac{1}{2}})$, which obtained in [25]. Besides the quenching time, we are also interested in the quenching set. The mathematical definition of quenching set is

$$\Sigma = \{ x \in \Omega : \exists (x_n, t_n) \in \Omega_T, s.t. x_n \to x, t_n \to T, u(x_n, t_n) \to 1 \}.$$
We assume $\Omega \subset \mathbb{R}^n$ is a convex bounded domain. By the moving plane argument, we assert that the quenching set is a compact subset of $\Omega$. And if $\Omega = B_R$, the ball centered at the origin with the radius $R$, then the quenching solution is radially symmetric (cf. \[8\]) and the only quenching point is the origin.

**Theorem 1.3.** Suppose $\Omega = B_R$. If $\lambda > \lambda^*_\Omega$, then the solution quenches only at $r = 0$. That is, the origin is the unique quenching point.

To understand the quenching behavior of the finite time quenching solution to $(F_{\lambda,\delta})$, we begin with the one-side quenching estimate, which has been derived in [8] for only one dimensional case.

**Lemma 1.4 (One-side quenching estimate).** If $\Omega \subset \mathbb{R}^n$ is a convex bounded domain, and $u(x,t)$ is a quenching solution of $(F_{\lambda,\delta})$ in finite time, then there exists a bounded positive constant $M > 0$ such that

$$M(T - t)^{\frac{1}{2}} \leq 1 - u(x,t),$$

for all $\Omega \times (0,T]$. Moreover, $u_t \to +\infty$ as $u$ touches down.

Actually, we show in this paper that under certain condition (namely (1.6)), the solution quenches in finite time $T$ with the rate

$$1 - u(x,t) \sim (3\lambda(T - t))^{\frac{1}{2}},$$

as $t \to T^-$, provided $\Omega \in \mathbb{R}$ or $\Omega \in \mathbb{R}^n$, $n \geq 2$, is radially symmetric domain.

This result comes from the similarity variables, which is first suggested in [9]–[11]. Let us make the similarity transformation at some point $a \in \Omega_\eta$ as in [9] and [13]:

$$y = \frac{x - a}{\sqrt{T - t}}, \quad s = -\log(T - t), \quad u(x,t) = 1 - (T - t)^{\frac{1}{2}} w_a(y,s), \quad (1.3)$$

where $\Omega_\eta = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \eta\}$, for some $\eta \ll 1$. First, the point $a$ can be identified as a non-quenching point, if $w_a(y,s) \to \infty$, as $s \to +\infty$ uniformly in $|y| \leq C$, for any constant $C > 0$. This is called the nondegeneracy phenomena in [11]. This property is not difficult to derive. It follows immediately from the comparison principle and the nondegeneracy of (1.1) obtained in [13].

The basis of the method, the similarity variables in [13], is the scaling property of (1.1), the fact that if $u$ solves it near $(0,0)$, then so do the rescaled functions

$$1 - u_\gamma(x,t) = \gamma^{-\frac{1}{2}} \left[1 - u(\gamma x, \gamma^2 t)\right], \quad (1.4)$$

for each $\gamma > 0$. If $(0,0)$ is a quenching point, then the asymptotics of the quenching are encoded in the behavior of $u_\gamma$ as $\gamma \to 0$. Unfortunately, compared with (1.1), $(F_{\lambda,\delta})$ doesn’t possess the nice property. That is, it is not rescale-invariant. This is where the difficulty in analysis arises and the condition (1.6) comes from. Essentially, we characterize the asymptotic behavior near a singularity, assuming a certain upper bound on the rate of the gradient’s blow-up. The condition (1.6) in some degree forces the solution of $(F_{\lambda,\delta})$ converges to the self-similar solution of (1.1) as $t \to T^-$. We call $u$ is the self-similar solution to (1.1), if $u$ defines on $\mathbb{R}^n \times (0, +\infty)$ and $u_\gamma = u$ for every $\gamma$ (see (1.4)).

Hence, the study of the asymptotic behavior of $u$ near the singularity is equivalent to understand the behavior of $w_a(y,s)$, as $s \to +\infty$, which satisfies the equation:

$$\frac{\partial w_a}{\partial s} = \triangle w_a - \frac{y}{2} \cdot \nabla w_a + \frac{1}{3} w_a - \frac{\lambda}{w_a^2} - \lambda \delta e^{\frac{\lambda}{w_a^2}}, \quad (1.5)$$
Theorem 1.5. Suppose $w_{a}$ is the solution to \[1.5\] quenching at $x = a$ in finite time $T$. Assume further that
\[
\int_{s_{0}}^{\infty} se^{\frac{1}{2}t} \int_{B_{s}} \rho|\nabla w_{a}|^{2} dy ds < \infty,
\]
for some $s_{0} \gg 1$, where $\rho(y) = e^{-\frac{|y|^{2}}{4}}$, $B_{s}$ is defined in \[5.10\]. Then $w_{a}(y, s) \to w_{a}(y)$, as $s \to \infty$ uniformly on $|y| \leq C$, where $C > 0$ is any bounded constant, and $w_{a}(y)$ is a bounded positive solution of
\[
\Delta w - \frac{1}{2} y \cdot \nabla w + \frac{1}{3} w - \frac{\lambda}{w^{2}} = 0
\]
in $\mathbb{R}^{n}$. Moreover, if $\Omega \in \mathbb{R}$ or $\Omega \in \mathbb{R}^{n}$, $n \geq 2$, is a convex bounded domain, then we have
\[
\lim_{t \to T^{-}} (1 - u(x, t))(T - t)^{-\frac{1}{2}} = (3\lambda)^{\frac{1}{2}}
\]
uniformly on $|x - a| \leq C\sqrt{T - t}$ for any bounded constant $C$.

From Theorem 1.5, one hardly tells the effects of the fringing term $\delta|\nabla u|^{2}$ on the asymptotic behavior near the singularity. Therefore, it seems to be necessary to find the refined asymptotic expansion near the singularity. As the first attempt in this direction, we derive a formal expansion as in \[15\] and \[14\]. Let us consider $\Omega \subset \mathbb{R}^{n}$ be a radially symmetrical domain. Then, for $r \ll 1$ and $T - t \ll 1$, we have
\[
u \sim 1 - [3\lambda(T - t)]^{\frac{1}{2}} \left( 1 - \frac{3^{2}n}{8\delta \lambda^{\frac{1}{2}}} (T - t)^{\frac{1}{2}} + \frac{3^{2}}{4\delta \lambda^{\frac{1}{2}}} \frac{r^{2}}{(T - t)^{\frac{1}{2}}} + \cdots \right)^{\frac{1}{2}} .
\]
(1.8)
This expansion is quite different from the one for \[14\]:
\[
u \sim -1 + [3\lambda(T - t)]^{\frac{1}{2}} \left( 1 - \frac{3n}{4|\log (T - t)|} + \frac{3r^{2}}{8|\log (T - t)|} + \cdots \right)^{\frac{1}{2}} .
\]
We believe the difference is due to the fringing term, which can be clearly seen from the method of dominant balance, see detailed analysis in section 5.6.

Finally, as the supplements, we numerically compute the pull-in voltages of \[F_{\lambda, \delta}\] with various $\delta$ and the quenching times of \[F_{\lambda, \delta}\] with various $\delta$ and $\lambda > \lambda_{+}$ using bvp4c in Matlab. Furthermore, we solve \[F_{\lambda, \delta}\] numerically using an appropriate finite difference scheme. The numerical simulations validate the results obtained in the previous sections.

2. Global existence or quenching in finite time. Motivated by \[20\], we make the following transformation
\[
v(x, t) := \zeta_{\lambda, \delta}(u(x, t)) = \int_{0}^{u(x, t)} e^{\frac{-\lambda}{1 - u}} ds ,
\]
then $v(x, t)$ satisfies
\[
\begin{aligned}
v_{t} - \Delta v &= \lambda \rho_{\lambda, \delta}(v), & & (x, t) \in \Omega_{T} \\
v(x, t) &= 0, & & (x, t) \in \partial \Omega_{T} \\
v(x, 0) &= 0, & & x \in \Omega,
\end{aligned}
\]
(\[V_{\lambda, \delta}\])
where $\rho_{\lambda, \delta} = \xi_{\lambda, \delta} \circ \zeta_{\lambda, \delta}^{-1}(v)$, $\xi_{\lambda, \delta}(u) := \frac{x_{+}^{\frac{1}{1 - u}}}{(1 - u)^{\frac{1}{1 - u}}}$. Since $\xi_{\lambda, \delta}$ and $\zeta_{\lambda, \delta}$ are increasing in $[0, 1)$ and $\lim_{u \to 1^{-}} \zeta_{\lambda, \delta}(u) = \infty$, $\rho_{\lambda, \delta}$ is also increasing in $\mathbb{R}_{+}$. It is also not difficult to check that $\rho_{\lambda, \delta}(v)$ satisfies the following properties:
\[
(1) \quad \rho_{\lambda, \delta}(v), \rho_{\lambda, \delta}'(v) \text{ and } \rho_{\lambda, \delta}''(v) > 0, \text{ for } v \in \mathbb{R}_{+}.
\]
In fact, through direct computations we get
\[
\rho_{\lambda, \delta}'(v) = \frac{1}{(1 - u)^{3}} \left( 2 + \frac{\lambda \delta}{1 - u} \right) ; \quad \rho_{\lambda, \delta}''(v) = \frac{2}{(1 - u)^{4}} e^{-\frac{\lambda \delta}{1 - u}} \left( 3 + \frac{2\lambda \delta}{1 - u} \right) .
\]
(2.2)
LEMMA 2.1 (Uniqueness). Suppose \( u_1(x,t) \) and \( u_2(x,t) \) are solutions of \((F_{\lambda,\delta})\) on \( \Omega_T := \Omega \times [0,T] \) such that \( ||u_i||_{L^\infty(\Omega_T)} < 1 \) for \( i = 1, 2 \), then \( u_1 = u_2 \).

Proof. Let us denote \( v_i(x,t) \) the solutions of \((V_{\lambda,\delta})\), i.e. \( v_i = \zeta_{\lambda,\delta}(u_i), i = 1, 2 \). Then \( \hat{v} = v_1 - v_2 \) satisfies

\[
\hat{v}_t - \Delta \hat{v} = \lambda[\rho_{\lambda,\delta}(v_1) - \rho_{\lambda,\delta}(v_2)] = \lambda \frac{\rho_{\lambda,\delta}(v_1) - \rho_{\lambda,\delta}(v_2)}{\hat{v}} \hat{v} = \lambda f \hat{v}.
\]

The condition \( ||u_i||_{L^\infty(\Omega_T)} < 1 \) is equivalent to \( ||v_i||_{L^\infty(\Omega_T)} < 1, i = 1, 2 \). This implies that \( ||\rho_{\lambda,\delta}(v_i)||_{L^\infty(\Omega_T)} < \infty, i = 1, 2 \). Therefore, \( ||f||_{L^\infty(\Omega_T)} < \infty \).

We now fix \( T_1 \in (0,T) \) and consider the solution \( \phi \) of the problem

\[
\begin{align*}
\phi_t + \Delta \phi + \lambda f \phi &= 0, \quad (x,t) \in \Omega_T \\
\phi(x,T_1) &= \theta(x) \in C_0(\Omega), \quad x \in \Omega \\
\phi(x,t) &= 0, \quad (x,t) \in \partial \Omega_{T_1}.
\end{align*}
\]

The standard linear theory gives the unique and bounded solution (cf. Theorem 8.1, [16]).

Multiplying \( \phi \) to \((2.3)\) and integrating in \( \Omega_{T_1} \) on both sides, it yields by integration by parts that

\[
\int_\Omega \hat{v}(x,T_1) \theta(x) dx = 0,
\]

for arbitrary \( T_1 \in (0,T) \) and \( \theta(x) \in C_0(\Omega) \). This implies that \( \hat{v} \equiv 0 \). \( \square \)

2.1. Global existence.

THEOREM 2.2 (Global existence). For every \( \lambda \leq \lambda_*^t \), there exists a unique global solution \( u(x,t) \) of \((F_{\lambda,\delta})\), which monotonically converges as \( t \to \infty \) to the minimal solution \( u_\lambda \) of \((SF_{\lambda,\delta})\).

Proof. This is standard and follows from the maximum principle combined with the existence of the regular minimal steady-state solutions for \( \lambda \in (0,\lambda_*^t) \). Indeed, for any \( 0 < \lambda \leq \lambda_*^t \), from Theorem 1 and Theorem 5, [20], there exists a unique minimal solution \( u_\lambda(x) \) of \((SF_{\lambda,\delta})\). It is clear that \( 0 \) and \( u_\lambda \) are sub- and super-solutions to \((F_{\lambda,\delta})\), respectively. This implies that there exists a unique global solution \( u(x,t) \) of \((F_{\lambda,\delta})\) such that \( 1 > u_\lambda(x) \geq u(x,t) \geq 0 \) in \( \Omega \times (0,\infty) \). Let us denote \( v_\lambda = \zeta_{\lambda,\delta}(u_\lambda) < \infty \). Then, \( 0 \leq v(x,t) \leq v_\lambda < \infty \), where \( v = \zeta_{\lambda,\delta}(v) \).

By differentiating \((V_{\lambda,\delta})\) in time and setting \( w = v_t \), we get for any fixed \( t_0 > 0 \)

\[
\begin{align*}
w_t - \Delta w &= \left[ \frac{\lambda \delta}{(1-u)^3} + \frac{2}{(1-u)^3} \right] w, \quad (x,t) \in \Omega_{t_0} \\
w(x,t) &= 0, \quad (x,t) \in \partial \Omega \times (0,t_0) \\
w(x,0) &\geq 0, \quad x \in \Omega.
\end{align*}
\]

Here \( \left[ \frac{\lambda \delta}{(1-u)^3} + \frac{2}{(1-u)^3} \right] \) is a locally bounded non-negative function, and by the strong maximum principle, we get that \( v_t = w > 0 \) for \( \Omega_{t_0} \) or \( w = 0 \). The second case can’t happen, otherwise \( u(x,t) = u_\lambda(x) \) for any \( t > 0 \). It follows that \( w = v_t > 0 \) for all \( \Omega \times (0,\infty) \). Moreover, since \( v(x,t) \) is bounded, the mononicity in time implies that the unique solution \( v(x,t) \) converges to some steady state, denoted as \( u_{ss}(x) \), as \( t \to \infty \), i.e. \( u(x,t) \to u_{ss}(x) \), as \( t \to \infty \). Hence, \( 1 > u_\lambda(x) \geq u_{ss}(x) > 0 \) in \( \Omega \).
Next, we claim that \( u_{ss}(x) \) is a solution of \((SF_{\lambda,\delta})\). Let us consider \( v_1(x) \) satisfying

\[
\begin{aligned}
-\Delta v_1 &= \lambda \rho_{\lambda,\delta}(u_{ss}), & x \in \Omega \\
v_1(x) &= 0, & x \in \partial \Omega.
\end{aligned}
\]

Let \( \bar{v}(x,t) = v(x,t) - v_1(x) \), then \( \bar{v} \) satisfies \( \bar{v}(x,0) = -v_1(x), \ \bar{v}|_{\partial \Omega \times (0, \infty)} = 0 \) and

\[
\bar{v}_t - \Delta \bar{v} = \lambda \left[ \frac{e^{\frac{\lambda}{1-u^2}}}{(1-u^2)^2} - \frac{e^{\frac{\lambda}{1-u_{ss}^2}}}{(1-u_{ss}^2)^2} \right],
\]

in \( \Omega \). The right-hand side of (2.5) tends to zero in \( L^2(\Omega) \) as \( t \to \infty \), which follows from

\[
\left| \frac{e^{\frac{\lambda}{1-u^2}}}{(1-u^2)^2} - \frac{e^{\frac{\lambda}{1-u_{ss}^2}}}{(1-u_{ss}^2)^2} \right| \leq \frac{\lambda \delta}{(1-u_{ss}^2)(1-u^2)^2} + \frac{2}{(1-u_{ss}^2)^2} \left| u - u_{ss} \right|
\]

and Hölder’s inequality. A standard eigenfunction expansion implies that \( w(x,t) \) converges to zero in \( L^2(\Omega) \) as \( t \to \infty \). That is \( v(x,t) \to v_1(x) \), as \( t \to \infty \). Combined with the fact that \( v(x,t) \to v_{ss}(x) \) pointwisely as \( t \to \infty \). We deduce that \( v_1(x) = v_{ss}(x) \) in \( L^2(\Omega) \), which implies \( v_{ss}(x) \) is also a solution to the stationary equation of \([F_{\lambda,\delta}]\) and the corresponding \( u_{ss}(x) \) is also a solution to \((SF_{\lambda,\delta})\). The minimal property of \( u_\lambda \) yields that \( u_\lambda \equiv u_s \) in \( \Omega \), from which follows that for every \( x \in \Omega \), we have \( u(x,t) \uparrow u_\lambda(x) \), as \( t \to \infty \).

2.2. Finite-time quenching.

THEOREM 2.3 (Finite-time quenching). For every \( \lambda > \lambda^*_\delta \), there exists a finite time \( T = T(\lambda, \delta) \) at which the unique solution \( u(x,t) \) of \([F_{\lambda,\delta}]\) quenches.

Proof. By contradiction, let \( \lambda > \lambda^*_\delta \) and suppose there exists a solution \( u(x,t) \) of \([F_{\lambda,\delta}]\) in \( \Omega \times (0, \infty) \).

Claim: given any \( \varepsilon \in (0, \lambda - \lambda^*_\delta) \), \((F_{\lambda-\varepsilon,\delta})\) has a global solution \( u_\varepsilon \), which is uniformly bounded in \( \Omega \times (0, \infty) \) by some constant \( C_\varepsilon < 1 \).

We follow the similar argument as in [1] or [7]. Let

\[
\begin{aligned}
g(u) &= \frac{1}{(1-u)^2}, & h(u) &= \int_0^u \frac{ds}{g(s)}, & 0 \leq u \leq 1; \\
\tilde{g}(u) &= \frac{\lambda - \varepsilon}{\lambda(1-u)^2}, & \tilde{h}(u) &= \int_0^u \frac{ds}{\tilde{g}(s)}, & 0 \leq u \leq 1;
\end{aligned}
\]

and

\[
\Phi_\varepsilon(u) = \tilde{h}^{-1} \circ h(u).
\]

Direct computations yield that

\[
\Phi_\varepsilon(u) = 1 - \left[ \frac{\varepsilon}{\lambda} + \frac{\lambda - \varepsilon}{\lambda} \right] \frac{1}{(1-u)^3} \leq C_\varepsilon < 1,
\]

for \( 0 \leq u \leq 1 \), where \( C_\varepsilon = 1 - \left( \frac{1}{3} \right)^{\frac{1}{3}} \). Moreover, it is easy to check that \( \Phi_\varepsilon(0) = 0, \ 0 \leq \Phi_\varepsilon(s) < s, \) for \( s \geq 0 \), and \( \Phi_\varepsilon(s) \) is increasing and concave with

\[
\Phi_\varepsilon'(s) = \frac{\tilde{g} \circ \Phi_\varepsilon(s)}{g(s)} > 0.
\]
Setting $w_\epsilon = \Phi_\epsilon(u)$, we have

$$-\Delta w_\epsilon = -\Phi_\epsilon''(u)|\nabla u|^2 - \Phi_\epsilon'(u)\Delta u \geq \Phi_\epsilon'(u) \left[ \frac{\lambda (1 + \delta |\nabla u|^2)}{(1 - u)^2} - u_t \right]$$

$$= \lambda (1 + \delta |\nabla u|^2) \frac{\Phi_\epsilon'(u)}{(1 - u)^2} - (w_\epsilon)_t = \lambda (1 + \delta |\nabla u|^2) \tilde{g}(w_\epsilon) - (w_\epsilon)_t$$

$$= \frac{(\lambda - \epsilon)(1 + \delta |\nabla u|^2)}{(1 - w_\epsilon)^2} - (w_\epsilon)_t.$$

Notice that

$$\Phi_\epsilon'(u) \leq \frac{\lambda - \epsilon (1 - u)^2}{\lambda (1 - u)^2} \leq \frac{\lambda - \epsilon (1 - u)^2}{\lambda} = \left( \frac{\lambda - \epsilon}{\lambda} \right)^2 < 1.$$

Hence,

$$|\nabla w_\epsilon|^2 = (\Phi_\epsilon'(u))^2 |\nabla u|^2 \leq |\nabla u|^2.$$

Furthermore,

$$-\Delta w_\epsilon \geq \frac{(\lambda - \epsilon)(1 + \delta |\nabla w_\epsilon|^2)}{(1 - w_\epsilon)^2} - (w_\epsilon)_t.$$

This means that $w_\epsilon = \Phi_\epsilon(u) \leq C_\epsilon$ is the supersolution to $(F_{\lambda, \epsilon, \delta})$. Since zero is a subsolution of $(F_{\lambda, \epsilon, \delta})$, we deduce that there exists a unique global solution $u_\epsilon$ for $(F_{\lambda, \epsilon, \delta})$ satisfying $0 \leq u_\epsilon \leq w_\epsilon \leq C_\epsilon < 1$ uniformly in $\Omega \times (0, \infty)$.

Let $v_\epsilon = \zeta_{\lambda, \epsilon, \delta}(u_\epsilon)$. It is clear to see that $v_\epsilon$ is a global classical solution to $(V_{\lambda, \epsilon, \delta})$. And it has been checked previously that $\rho_{\lambda, \epsilon, \delta}$ is a nondecreasing, convex function, and there exists some $v_0 > 0$ such that $\rho_{\lambda, \epsilon, \delta}(v_0) > 0$ and

$$\int_{v_0}^\infty \frac{ds}{\rho_{\lambda, \delta}(s)} < \infty.$$

Therefore, from Theorem 1, [11], we obtain a weak solution to the stationary equation of $(V_{\lambda, \epsilon, \delta})$, where $0 < \epsilon < \lambda - \lambda_3^*$. In fact, using Sobolev embedding theorem and a bootstrap argument, any weak solution to the stationary equation of $(V_{\lambda, \epsilon, \delta})$ satisfying $\rho_{\lambda, \epsilon, \delta}(v) \in L^1(\Omega)$ is indeed smooth. This contradicts with the nonexistence result in [26]. □

### 3. Estimates for the pull-in voltage and the finite quenching time.

A lower bound of $\lambda_3^*$ is given in Theorem 2.2, [25], i.e.,

$$\lambda_t := \frac{4}{27} \frac{|\xi||\xi|_\infty}{||\xi||_\infty^2 + \delta ||\Delta \xi||_\infty^2} \leq \lambda_t^*,$$

where $\xi$ is the solution to $-\Delta \xi = 1$, $x \in \Omega$ with the Dirichlet boundary condition. And it is not difficult to see that $\lambda_t^*$, the pull-in voltage for $[1.1]$, is an upper bound for $\lambda_3^*$, due to the comparison principle. From [13], $\lambda_t^* \leq \frac{4}{27} \mu_0$, where $\mu_0 > 0$ is the first eigenvalue of $-\Delta \phi_0 = \mu_0 \phi_0$, $x \in \Omega$ with Dirichlet boundary condition. We shall derive an upper bound for $\lambda_3^*$ to show explicit dependence of $\delta$, if $\delta \ll 1$:

**Proposition 3.1 (Upper bound for $\lambda_3^*$, $\delta \ll 1$).** The pull-in voltage $\lambda_3^* \ll \infty$ of $[F_{\lambda, \delta}]$ has the upper bound

$$\lambda_3^* \leq \lambda_{u,1} := \frac{4}{27} \mu_0 \left( 1 - \frac{1}{27 ||\xi||_\infty} \delta + O(\delta^2) \right),$$

where $\xi$ is the solution to $-\Delta \xi = 1$, $x \in \Omega$ with Dirichlet boundary condition, if $\delta \ll 1$. 

Proof. This argument is used in many estimates of the pull-in voltage (cf. Theorem 3.1, [21] or Theorem 2.1, [14]). Let \( \mu_0 > 0 \) and \( \phi_0 > 0 \) be the first eigen pair \(-\triangle \phi_0 = \mu_0 \phi_0 \) in \( \Omega \) with Dirichlet boundary condition. We multiply the stationary equation of \( V_{\lambda, \delta} \), by \( \phi_0 \), integrate the resulting equation over \( \Omega \), and use Green’s identity to get

\[
\int_\Omega \left[ -\mu_0 u + \lambda \rho_{\lambda, \delta}(u) \right] \phi_0 \, dx = \int_\Omega \left( -\mu_0 \int_0^u e^{\frac{\lambda l}{1-u}} \, ds + \lambda \frac{e^{\frac{\lambda l}{1-u}}}{(1-u)^2} \right) \, dx = 0.
\]

Noting that \( \int_0^u e^{\frac{\lambda l}{1-u}} \, ds \leq e^{\frac{\lambda l}{1-u}} u \), we get that if \( \lambda > \frac{4}{27} \mu_0 \), then

\[
-\mu_0 \int_0^u e^{\frac{\lambda l}{1-u}} \, ds + \lambda \frac{e^{\frac{\lambda l}{1-u}}}{(1-u)^2} \geq -\mu_0 e^{\frac{\lambda l}{1-u}} + \lambda \frac{e^{\frac{\lambda l}{1-u}}}{(1-u)^2} > 0, \tag{3.2}
\]

for any \( u \in [0, 1] \). Therefore, there is no solution to the stationary equation of \( V_{\lambda, \delta} \), so does \( SF_{\lambda, \delta} \), if \( \lambda > \frac{4}{27} \mu_0 \). That is, \( \lambda^*_1 \leq \frac{4}{27} \mu_0 \). This is the upper bound obtained in [14].

In this way, we ignore the effect \( \delta \) completely. Let us go back to (3.2) and we see that if

\[
\lambda \geq \max_{u \in [0,1]} \left[ \mu_0 (1-u)^2 \int_0^u e^{\lambda \phi(u)} \phi(u) \, ds \right],
\]

then (3.2) holds, where \( \lambda_i \) is in (3.1). Let us estimate the maximum in the following:

\[
\int_0^u e^{\lambda \phi(u)} \phi(u) \, ds \leq \frac{1}{2} u \left[ 1 + e^{\lambda \phi(1-u)} \right] ,
\]

due to the convexity of the integrand \( e^{\lambda \phi(u)} \). Therefore, if

\[
\lambda \geq \max_{u \in [0,1]} \frac{1}{2} \mu_0 (1-u)^2 u \left[ 1 + e^{\lambda \phi(1-u)} \right] = \frac{4}{27} \mu_0 - \frac{4 \mu_0}{27 \mu_0} \delta + O(\delta^2),
\]

then (3.2) holds, where \( \xi \) is the solution to \(-\triangle \xi = 1\) for \( x \in \Omega \) with Dirichlet boundary condition, provided \( \delta \ll 1 \).

Next, we show the behavior of \( \lambda^*_1 \) as \( \delta \to \infty \).

Proposition 3.2 (\( \lambda^*_1 \) for \( \delta \gg 1 \)). The pull-in voltage \( \lambda^*_1 \) of \( G_{\lambda, \delta} \) tends to 0, as \( \delta \to \infty \). That is,

\[
\lim_{\delta \to \infty} \lambda^*_1 = 0.
\]

Proof. As shown in Theorem 2.2 and Theorem 2.3, the pull-in voltage \( \lambda^*_1 \) of \( G_{\lambda, \delta} \) is the same one as that of \( SF_{\lambda, \delta} \). Let us multiply \( SF_{\lambda, \delta} \) by \( \phi_0 > 0 \), the first eigenfunction of \(-\triangle \phi_0 = \mu_0 \phi_0 \) in \( \Omega \) with Dirichlet boundary condition, integrate over \( \Omega \), and use Green’s identity to get

\[
0 = \int_\Omega \left( -\mu_0 u + \frac{\lambda}{(1-u)^2} \right) \phi_0 \, dx + \lambda \delta \int_\Omega \frac{\left| \nabla u \right|^2}{(1-u)^2} \phi_0 \, dx. \tag{3.3}
\]

By integration by parts, the third term in the above equation gets

\[
\int_\Omega \frac{\left| \nabla u \right|^2}{(1-u)^2} \phi_0 \, dx = \int_\Omega \nabla u \cdot \frac{\nabla u}{(1-u)^2} \phi_0 \, dx = \int_\Omega \nabla u \cdot \nabla \left( \frac{1}{1-u} \right) \phi_0 \, dx
\]

\[
= -\int_\Omega \triangle u \frac{1}{1-u} \phi_0 \, dx - \int_\Omega \nabla u \cdot \nabla \phi_0 \frac{1}{1-u} \, dx
\]

\[
= \lambda \int_\Omega \frac{1 + \delta |\nabla u|^2}{(1-u)^3} \phi_0 \, dx + \int_\Omega \nabla (\ln (1-u)) \cdot \nabla \phi_0 \, dx
\]

\[
= \lambda \int_\Omega \frac{1 + \delta |\nabla u|^2}{(1-u)^3} \phi_0 \, dx + \mu_0 \int_\Omega \ln (1-u) \phi_0 \, dx. \tag{3.4}
\]
Furthermore, for $p \geq 3$, we have

\[
\int_{\Omega} \left| \nabla u \right|^2 \phi_0 dx = \frac{1}{p-1} \int_{\Omega} \nabla u \nabla \left( \frac{1}{(1-u)^{p-1}} \right) \phi_0 dx
\]

\[
= -\frac{1}{p-1} \int_{\Omega} \nabla u \nabla \left( \frac{1}{(1-u)^{p-1}} \right) \phi_0 dx - \frac{1}{p-1} \int_{\Omega} \nabla u \nabla \left( \frac{1}{(1-u)^{p-1}} \right) \phi_0 dx
\]

\[
= \frac{\lambda}{p-1} \int_{\Omega} \nabla u \nabla \left( \frac{1}{(1-u)^{p-1}} \right) \phi_0 dx
\]

\[
- \frac{1}{(p-1)(p-2)} \int_{\Omega} \nabla \phi_0 \nabla \left( \frac{1}{(1-u)^{p-2}} \right) dx
\]

\[
+ \frac{\lambda}{p-1} \int_{\Omega} \nabla u \nabla \left( \frac{1}{(1-u)^{p-1}} \right) \phi_0 dx
\]

\[
- \frac{\mu_0}{(p-1)(p-2)} \int_{\Omega} \nabla \phi_0 \nabla \left( \frac{1}{(1-u)^{p-2}} \right) dx
\]

\[
- \frac{1}{(p-1)(p-2)} \int_{\partial \Omega} \frac{\partial \phi_0}{\partial \nu} \frac{1}{(1-u)^{p-2}} dS,
\]

(3.5)

where $\nu$ is the outward unit normal vector of $\partial \Omega$. Substitute (3.4) and (3.5) to (3.3), we get

\[
0 = \int_{\Omega} \left\{ -\mu_0 u + \frac{\lambda}{(1-u)^2} + \lambda \delta \left[ \mu_0 \ln (1-u) + \frac{\lambda}{(1-u)^3} \right] \right\} \phi_0 dx
\]

\[
+ \sum_{p=3}^P \frac{\lambda p-1 \delta^{p-1}}{(p-1)!} \left[ \frac{\lambda}{p-1} \int_{\Omega} \frac{1}{(1-u)^{p+1}} \phi_0 dx - \frac{\mu_0}{p-2} \int_{\Omega} \frac{1}{(1-u)^{p-2}} \phi_0 dx
\]

\[
- \frac{1}{p-2} \int_{\partial \Omega} \frac{\partial \phi_0}{\partial \nu} \frac{1}{(1-u)^{p-2}} dS \right]
\]

\[
+ \frac{\lambda p \delta^p}{(P-1)!} \int_{\Omega} \frac{\left| \nabla u \right|^2}{(1-u)^{p+1}} \phi_0 dx,
\]

(3.6)

for arbitrary $P > p$. By the boundary point lemma, we have $\frac{\partial \phi_0}{\partial \nu} < 0$ on $\partial \Omega$. Hence, the term $-\int_{\partial \Omega} \frac{\partial \phi_0}{\partial \nu} \frac{1}{(1-u)^{p-2}} dS$ is positive, so does the term $\int_{\Omega} \frac{\left| \nabla u \right|^2}{(1-u)^{p+1}} dx$. If $\delta > 1$, then $O(\delta^{p-1})$ is the leading order term, except the last term in (3.6). The equality (3.6) can’t hold when

\[
\frac{\lambda}{(1-u)^{p+1}} - \frac{\mu_0}{(P-2)(1-u)^{p-2}} > 0
\]

holds for all $u \in [0,1]$. That is,

\[
\lambda > \frac{\mu_0}{P-2} \max_{u \in [0,1]} (1-u)^3 = \frac{\mu_0}{P-2},
\]

(3.7)

where $P = P(\delta) \to \infty$, if $\delta \to \infty$. Our result follows immediately. \( \square \)

**Proposition 3.3 (Upper bound of $T$).** Let $\phi$ be any nonnegative $C^2$ function such that $\phi \neq 0$ and $\phi = 0$ on $\partial \Omega$. Then for $\lambda$ large enough, the quenching time $T$ for the solution to $(F_{\lambda, \delta})$ satisfies

\[
T \leq \frac{||\phi||_1}{3\lambda||\phi||_1 - ||\Delta \phi||_1},
\]

(3.8)

where $|| \cdot ||_1$ is the $L^1$ norm of $\cdot$ in $\Omega$. 
Proof. Using $\varphi(1-u)^2$ as the test function to \([F_{\lambda,\delta}]\) and integrating over $\Omega$,

\[
\left(\int_\Omega \frac{1}{3} [1-(1-u)^3] \varphi dx\right)_t = \int_\Omega \Delta u \varphi (1-u)^2 dx + \lambda \int_\Omega (1+\delta |\nabla u|^2) \varphi dx \\
\geq -\int_\Omega \nabla u \nabla \varphi (1-u)^2 dx + 2 \int_\Omega |\nabla u|^2 \varphi (1-u) \\
+ \lambda \int_\Omega \varphi dx \\
\geq \frac{1}{3} \int_\Omega [1-(1-u)^3] \Delta \varphi dx + \lambda \int_\Omega \varphi dx - \frac{1}{3} \int_\Omega |\Delta \varphi| dx.
\]

Hence, for any $t < T$, integrating from 0 to $t$, we obtain that

\[
\frac{1}{3} \int_\Omega \varphi dx \geq \frac{1}{3} \int_\Omega [1-(1-u(x,t))^3] \varphi dx \\
\geq \frac{1}{3} \int_0^t \int_\Omega [1-(1-u)^3] \Delta \varphi dx + \lambda t \int_\Omega \varphi dx \geq \lambda t \int_\Omega \varphi dx - \frac{1}{3} t \int_\Omega |\Delta \varphi| dx.
\]

By tending $t$ to $T$, we are done. \(\square\)

We compare the quenching time $T = T(\lambda, \delta)$ with different $\lambda$:

**Proposition 3.4.** Suppose $u_1 = u_1(x,t)$ and $u_2 = u_2(x,t)$ are solutions of \([F_{\lambda,\delta}]\) with $\lambda = \lambda_1 \text{ and } \lambda_2$, respectively. And the corresponding finite quenching times are $T_{\lambda_1}$ and $T_{\lambda_2}$, respectively. If $\lambda_1 > \lambda_2$, then $T_{\lambda_1} < T_{\lambda_2}$.

**Proof.** Let $\hat{v} = v_1 - v_2$, where $v_i$, $i = 1, 2$, are the corresponding solution of \((V_{\lambda_i,\delta})\), $i = 1, 2$, respectively. Then $\hat{v} |_{\partial \Omega}(x,t) = \hat{v}(x,0) = 0$ and

$\hat{v}_t - \Delta \hat{v} = \lambda_1 \rho_{\lambda_1,\delta}(v_1) - \lambda_2 \rho_{\lambda_2,\delta}(v_2) > \lambda_2 [\rho_{\lambda_2,\delta}(v_1) - \rho_{\lambda_2,\delta}(v_2)] = \lambda_2 \rho'_{\lambda_2,\delta}(\theta) \hat{v}$,

with $\rho'_{\lambda_2,\delta}(\theta) > 0$, for some function $\theta$. Hence, $v_1 > v_2$ in $\Omega \times (0, \min \{T_{\lambda_1}, T_{\lambda_2}\})$. Thus, $T_{\lambda_1} < T_{\lambda_2}$. \(\square\)

**Remark 3.5.** Fix the voltage $\lambda > \max\{\lambda^*_1, \lambda^*_2\}$, if $\delta_1 > \delta_2 > 0$, then $T_{\delta_1} < T_{\delta_2}$, where $T_{\delta}$ are the finite quenching time corresponding to $\delta_i$, $i = 1, 2$. This observation follows immediately from

$\partial_t u_1 - \Delta u_1 = \lambda \frac{1+\delta_1 |\nabla u|^2}{(1-u)^2} > \lambda \frac{1+\delta_2 |\nabla u|^2}{(1-u)^2}$,

which means that $u_1 > u_2$ in $\Omega \times (T_{\delta_1}, T_{\delta_2})$. Hence, $T_{\delta_1} < T_{\delta_2}$.

4. Quenching set. In this section, we assume that $\Omega$ is a bounded convex subset of $\mathbb{R}^n$. It is followed by the moving-plane argument that the quenching set of any finite-time quenching solution to \([F_{\lambda,\delta}]\) is a compact subset of $\Omega$.

**Theorem 4.1 (Compactness of the quenching set).** Suppose $\Omega \subset \mathbb{R}^n$ is convex, and $u(x,t)$ is a solution to \([F_{\lambda,\delta}]\) which quenches in finite time $T$. Then the set of the quenching points is a compact subset of $\Omega$.

**Proof.** (Adaptation of moving-plane argument) It is equivalent to show that the set of the blow-up points of $u$ in \((V_{\lambda,\delta})\) is a compact subset of $\Omega$.

Let us denote $x = (x_1, x') \in \mathbb{R}^n$, where $x' = (x_2, x_3, \cdots, x_n) \in \mathbb{R}^{n-1}$. Take any point $y_0 \in \partial \Omega$, and assume without loss of generality that $y_0 = 0$ and that the half space $\{x_1 > 0\}$ is tangent to $\Omega$ at $y_0$.

Let $\Omega^+ = \Omega \cap \{x_1 > \alpha\}$, $\alpha < 0$, $|\alpha|$ small, and $\Omega^- = \{x = (x_1, x') \in \mathbb{R}^n : (2\alpha - x_1, x') \in \Omega^+\}$, the reflection of $\Omega^+$ with respect to $\{x_1 = \alpha\}$.
First, from the maximum principle, we observe that
\[ v \geq 0, \tag{4.1} \]
for \((x, t) \in \Omega_T\) and \(\frac{\partial v}{\partial n}(t_0) < 0\) on \(\partial \Omega\) for some small \(t_0 \in (0, T)\).

Let us consider
\[ \bar{v}(x, t) = v(x_1, x', t) - v(2\alpha - x_1, x', t), \]
for \(x \in \Omega_\alpha^-,\) then \(\bar{v}\) satisfies
\[ \partial_t \bar{v} - \Delta \bar{v} = \lambda \left[ \rho_{\lambda, \delta}(v(x_1, x', t)) - \rho_{\lambda, \delta}(v(2\alpha - x_1, x', t)) \right] = \lambda c(x, t) \bar{v}, \]
where \(c(x, t)\) is a bounded function. It is clear that \(\bar{v} = 0\) on \(\{x_1 = \alpha\}\) and \(\bar{v} = v(x_1, x', t) \geq 0\) on \(\partial \Omega_\alpha^- \cap \{x_1 < \alpha\} \times (0, T]\). If \(\alpha\) is small enough, then \(\bar{v}(x, t_0) = v(x_1, x', t_0) - v(2\alpha - x_1, x', t_0) \geq 0,\) for \(x \in \Omega_\alpha^-\). Applying maximum principle, we conclude that
\[ \bar{v} > 0 \quad \text{in} \quad \Omega_\alpha^- \times (t_0, T) \quad \text{and} \quad \frac{\partial \bar{v}}{\partial x_1} = -2\frac{\partial v}{\partial x_1} > 0 \quad \text{on} \quad \{x_1 = \alpha\}. \]

Since \(\alpha\) is arbitrary, it follows by varying \(\alpha\) that
\[ \frac{\partial v}{\partial x_1} < 0, \tag{4.2} \]
for \(x \in \Omega_\alpha^{0},\) \(t_0 < t < T\), provided that \(\alpha_0\) is small enough.

Let us consider
\[ J = v_{x_1} + \varepsilon_1(x_1 - \alpha_0) \]
in \(\Omega_\alpha^+ \times (t_0, T),\) where \(\varepsilon_1 = \varepsilon_1(\alpha_0, t_0) > 0\) is a constant to be determined later. Through direct computations, we obtain that
\[ \partial_t J - \Delta J = \Delta(v_{x_1}) + \lambda \varepsilon_1^2 \frac{e^{\frac{\lambda t}{1-u^3}}}{(1-u)^3} u_{x_1} \left[ 2 + \frac{\lambda \delta}{1-u} \right] - \Delta(v_{x_1}) = \frac{\lambda u_{x_1}}{(1-u)^3} \left[ 2 + \frac{\lambda \delta}{1-u} \right] \leq 0, \]
in \(\Omega_\alpha^+ \times (t_0, T),\) where \(u\) is the corresponding solution to \((F_{\lambda, \delta})\). Therefore, \(J\) can’t obtain positive maximum in \(\Omega_\alpha^+ \times (t_0, T).\) Next, \(J < 0\) on \(\{x_1 = \alpha_0\}\) by \((4.2)\). From \((4.1),\)
\[ \frac{\partial v(x, t_0)}{\partial x_1} \leq C < 0. \]

If we can show \(J < 0\) on \(\Gamma \times (t_0, T),\) where \(\Gamma = \partial \Omega_\alpha^+ \cap \partial \Omega,\) then
\[ J < 0, \tag{4.3} \]
in \(\Omega_\alpha^+ \times (t_0, T).\) To show \((4.3),\) we compare \(v\) with the solution \(z\) of the heat equation
\[ \begin{cases} 
    z_t - \Delta z = 0 & \text{in} \quad \Omega \times (t_0, T) \\
    z(x, t) = 0 & \text{on} \quad \partial \Omega \times (t_0, T) \\
    z(x, t_0) = 0 & \text{in} \quad \Omega \times (t_0, T).
\end{cases} \tag{4.4} \]

Since \(\lambda \rho_{\lambda, \delta}(v) \geq 0,\) we have \(v \geq z\). Consequently, \(\frac{\partial v}{\partial \nu} - \frac{\partial z}{\partial \nu} \leq -C_0 < 0\) on \(\partial \Omega \times (t_0, T).\) It follows that, if \(x \in \Gamma,\)
\[ J \leq -C_0 \frac{\partial x_1}{\partial \nu} + \varepsilon_1(x_1 - \alpha_0) < 0, \]
provided \(\varepsilon_1\) small enough. Now, the maximum principle yields that there exists \(\varepsilon_1 = \varepsilon_1(\alpha_0, t_0)\) small enough such that \(J \leq 0\) in \(\Omega_\alpha^+ \times (t_0, T),\) i.e.
\[ -v_{x_1} = |v_{x_1}| \geq \varepsilon_1(x_1 - \alpha_0), \]
if \(x' = 0, \alpha_0 \leq x_1 < 0.\) Integrating with respect to \(x_1,\) we get for any \(\alpha_0 < y_1 < 0,\)
\[ -v(y_1, 0, t) + v(\alpha_0, 0, t) \geq \frac{\varepsilon_1}{2} |y_1 - \alpha_0|^2. \]
It follows that
\[
\liminf_{t \to T} v(0, t) = \liminf_{t \to T} \lim_{y_1 \to 0} v(y_1, 0, t) \\
\leq \liminf_{t \to T} \lim_{y_1 \to 0} \left[ v(\alpha_0, 0, t) - \frac{c_1}{2} |y_1 - \alpha_0|^2 \right] < \infty.
\]
Thus, every point in \( \{ x' = 0, \; \alpha_0 < x_1 < 0 \} \) is not a blow-up point. The above proof shows that \( \alpha_0 \) can be chosen independent of \( y_0 \in \partial \Omega \). Hence, by varying \( y_0 \in \partial \Omega \), we conclude that there is an \( \Omega \)-neighborhood \( \Omega' \) of \( \partial \Omega \) such that each point \( x \in \Omega' \) is not a blow-up point. Since the blow-up points lie in a compact subset of \( \Omega \), it is clearly a closed set.

In addition, if \( \Omega = B_R(0) \) is a ball of radius \( R \) centered at the origin, then according to \( [8] \) we conclude that any solution \( u(x, t) \) is indeed radial symmetric, i.e. \( u(x, t) = u(r, t) \), with \( r = |x| \in [0, R] \). Furthermore, we can show that the only possible quenching point is the origin.

**Theorem 4.2.** Suppose \( \Omega = B_R \). If \( \lambda > \lambda_\delta^1 \), then the solution quenches only at \( r = 0 \). That is, the origin is the unique quenching point.

**Lemma 4.3.** \( v_r < 0 \) in \( \Omega_T \cap \{ r > 0 \} \).

*Proof.* Set \( \bar{v} = r^{n-1}v_r \). Then (4.4) becomes
\[
v_t - \frac{1}{r^n-1} v_r = \lambda \rho_{\lambda, \delta}(v).
\]
Differentiating with respect to \( r \), we get
\[
\bar{v}_t + \frac{n-1}{r} \bar{v}_r - \bar{v}_{rr} - \frac{\lambda}{(1-u)^2} \left( 2 + \frac{\lambda \delta}{1-u} \right) \bar{v} = 0.
\]
Since \( \bar{v} = r^{n-1}v_r < 0 \) on \( \partial \Omega \times (0, T) \) (by maximum principle) and \( \bar{v}(r, 0) = 0 \) by (4.4), we deduce by maximum principle that \( v_r < 0 \) in \( \Omega_T \cap \{ r > 0 \} \).

*Proof of Theorem 4.2.* Let us consider as in Theorem 2.3, [1]
\[
J = \bar{v} + c(r) F(v),
\]
where \( \bar{v} = r^{n-1}v_r \) is defined as in Lemma 4.3 \( F, c \) are positive functions to be determined and \( F' \geq 0, F'' \geq 0 \). We aim to show \( J \leq 0 \) in \( \Omega_T \). Through direct computations, we have
\[
J_t + \frac{n-1}{r} J_r - J_{rr} = \frac{\lambda}{(1-u)^2} \left[ 2 + \frac{\lambda \delta}{1-u} \right] \bar{v}
+ \frac{2(n-1)}{r} F' v_r + \frac{n-1}{r} c F - c F' v_r - 2 c' F v_r - c'' F
\leq \left\{ \frac{\lambda}{(1-u)^2} \left[ 2 + \frac{\lambda \delta}{1-u} \right] + \frac{2(n-1)}{r^n} c F' - \frac{2 c' F}{r^{n-1}} \right\} J
+ \left\{ - \frac{\lambda}{(1-u)^2} \left[ 2 + \frac{\lambda \delta}{1-u} \right] \right\} F + \frac{c F}{(1-u)^2} \frac{\lambda e^{\frac{\lambda \delta}{1-u}}}{(1-u)^2}
- \frac{2(n-1)}{r^n} c^2 F' F + \frac{n-1}{r} c' F + \frac{2}{r^{n-1}} c' F F' - c'' F \right\}
:= AJ + B,
\]
by using \( \bar{v} = J - c F \) and \( \bar{v} = r^{n-1}v_r \). It is easy to see that \( A \) is a bounded function for \( 0 < r < R \). Let us choose
\[
c(r) = \varepsilon r^n \quad \text{and} \quad F(v) = \frac{e^{\frac{\lambda \delta}{1-u}}}{(1-u)^2},
\]
where \( u = \zeta_{\lambda,\delta}^{-1}(v) \), \( \gamma \geq 0 \) is some constant to be determined later. Direct computations yield that
\[
B = c(r)e^{\frac{\lambda}{r}}\left\{ \frac{\lambda(\gamma - 2)}{(1 - u)^{\gamma+3} + 2\varepsilon} + \frac{\lambda\gamma}{(1 - u)^{2\gamma+2} + \frac{\gamma}{(1 - u)^{2\gamma+1}}} \right\} \leq 0,
\]
if \( \gamma < 1 \) and \( \varepsilon \ll 1 \). \( J = 0 \) at \( r = 0 \), due to \( c(0) = 0 \) and it follows that \( J \) can’t obtain positive maximum in \( \Omega_T \) or on \( \{t = T\} \).

Next, we observe that \( J \) can’t obtain positive maximum on \( \{r = R\} \), if \( J_r \leq 0 \) on \( \{r = R\} \). Since
\[
J_r(R) = \bar{v}_r + cF'\bar{v}_r + c'F \leq \bar{v}_r + c'F \overset{4.5}{=} -R^{n-1}\lambda e^{\lambda} + c'(R)F(0) \leq 0,
\]
provided that \( \varepsilon \ll 1 \). Finally, by maximum principle, there exists \( 0 < t_0 < T \) such that \( v_r(r,t_0) < 0 \) for \( 0 < r \leq R \) and \( v_{rr}(0, t_0) < 0 \). Thus, \( J(r, t_0) < 0 \) for \( 0 \leq r < R \), provided \( \varepsilon \ll 1 \).

Therefore, by maximum principle, we conclude that \( J \leq 0 \) in \( B_R \times [t_0, T] \), for any \( 0 < t_0 < T \). That is,
\[
-r^{n-1}e^{\frac{\lambda}{r}}u_r = -r^{n-1}\bar{v}_r \geq c(r)F(v) = \frac{\varepsilon r^n e^{\frac{\lambda}{r}}}{(1 - u)^\gamma},
\]
for \( 0 \leq \gamma < 1 \). It deduces that
\[
\frac{d}{dr}\left[ \frac{1}{\gamma + 1}(1 - u(r,t))^{\gamma + 1} \right] \geq \varepsilon r.
\]
Integrating from 0 to \( r \), we obtain that
\[
\frac{1}{\gamma + 1}(1 - u(r,t))^{\gamma + 1} - \frac{1}{\gamma + 1}(1 - u(0,t))^{\gamma + 1} \geq \frac{1}{2}\varepsilon r^2.
\]
It is known that \( 0 \) is in the set of quenching points. So,
\[
(1 - u(r,t))^{\gamma + 1} \geq \frac{\gamma + 1}{2}\varepsilon r^2. \tag{4.6}
\]
If for any \( 0 < r < R \), \( u(r,t) \to 1 \), as \( t \to T \), then the left-hand side tends to 0. This contradicts with (4.6). Therefore, \( 0 \) is the only quenching point. \( \square \)

5. Quenching behavior.

5.1. Upper bound estimate. We first obtain an one-side quenching estimate. The similar result has been obtained in [18] for only one dimension case, i.e., \( x \in \mathbb{R} \).

**Lemma 5.1** (One-side quenching estimate). If \( \Omega \subset \mathbb{R}^n \) is a bounded convex domain, and \( u(x,t) \) is a quenching solution of \( (F_{\lambda,\delta}) \) in finite time, then there exists a bounded positive constant \( M > 0 \) such that
\[
M(T - t)^{\frac{1}{2}} \leq 1 - u(x,t),
\]
for all \( \Omega_T \). Moreover, \( u_t \to +\infty \) as \( u \) quenches.

**Proof.** Since \( \Omega \) is a convex bounded domain, we show in Theorem 4.1 that the quenching set of \( u \) is a compact subset of \( \Omega \). It is now suffices to discuss the point \( x_0 \) lying in the interior domain \( \Omega_\eta = \{ x \in \Omega : \text{dist}(x, \partial\Omega) > \eta \} \), for some small \( \eta > 0 \), i.e. there is no quenching point in \( \Omega_\eta' := \Omega \setminus \Omega_\eta \).

For any \( t_1 < T \), we recall the maximum principle gives \( u_t > 0 \), for all \( (x, t) \in \Omega \times (0, t_1) \). Furthermore, the boundary point lemma shows that the exterior normal derivative of \( u_t \) on \( \partial\Omega \) is negative for \( t > 0 \). This implies that for any small \( 0 < t_0 < T \), there exists a positive constant \( C = C(t_0, \eta) \) such that \( u_t(x, t_0) \geq C > 0 \), for all \( x \in \Omega_\eta \). For any \( 0 < t_0 < t_1 < T \), we claim that
\[
J_\varepsilon(x, t) = u_t - \varepsilon\rho_{\lambda,\delta}(v) \geq 0,
\]
for all \((x,t) \in \Omega_\eta \times (t_0,t_1)\), where \(v\) is the corresponding solution to \(\{V_{\lambda,\delta}\}\). In fact, it is clear that there exists \(C_\eta = C(t_0,t_1,\eta) > 0\) such that \(v_t = \varepsilon \frac{M - u_t}{1 - u} \geq C_\eta\) on \(\Omega_\eta \times (t_0,t_1)\). And further, we can choose \(\varepsilon = \varepsilon(t_0,t_1,\eta) > 0\) small enough, so that \(J^e \geq 0\) on the parabolic boundary of \(\Omega_\eta \times (t_0,t_1)\), due to the local boundedness of \(\rho_{\lambda,\delta}(v)\) on \(\partial \Omega_\eta \times (t_0,t_1)\). Then the claim is followed by the maximum principle and the direct computations:

\[
J_t^e - \Delta J^e = \lambda \rho_{\lambda,\delta}^e(v)J^e + \varepsilon \rho_{\lambda,\delta}^e(v)|\nabla v|^2 \geq \lambda \rho_{\lambda,\delta}^e(v)J^e,
\]
due to the convexity of \(\rho_{\lambda,\delta}\). This yields that for any \(0 < t_0 < t_1 < T\), there exists \(\varepsilon = \varepsilon(t_0,t_1,\eta) > 0\) such that

\[
u_t \geq \frac{\varepsilon}{(1 - u)^2},
\]

for all \(\Omega_\eta \times (t_0,t_1)\). This inequality implies that \(u_t \to \infty\) as \(u\) touches down, and there exists \(M > 0\) such that

\[
M(T - t)^\frac{1}{2} \leq 1 - u(x,t),
\]

in \(\Omega_\eta \times (0,T)\), due to the arbitrary of \(t_0\) and \(t_1\), where \(M = M(\lambda,\delta,\eta)\). Furthermore, one can obtain \((5.1)\) for \(\Omega \times (0,T]\), due to the boundedness of \(u\) on \(\Omega^e\).

\[ \square \]

5.2. Gradient estimate. We shall study the quenching rate for the higher derivatives of \(u\). The idea of the proof is similar to Proposition 1, [4] and Lemma 2.6, [13].

**Lemma 5.2.** Suppose \(u\) is a quenching solution of \((F_{\lambda,\delta})\) in finite time \(T\). For any point \(x = a \in \Omega_\eta\), for some small \(\eta > 0\). Then there exists a positive constant \(M'\) such that

\[
|\nabla^m u(x,t)|(T-t)^{-\frac{1}{2} + \frac{m}{2}} \leq M',
\]

\(m = 1,2\), holds for \(Q_R = B_R \times (T - R^2, T)\), for any \(R > 0\) such that \(a + R \in \Omega_\eta\).

**Proof.** It suffices to consider the case \(a = 0\) by translation. We may focus on some fixed \(r\), such that \(\frac{1}{2}R^2 < r^2 < R^2\) and denote \(Q_r = B_r \times \left(T \left(1 - \left(\frac{r}{R}\right)^2\right), T\right)\).

Let us first show that \(|\nabla u|\) and \(|\nabla^2 u|\) are uniformly bounded on compact subset of \(Q_R\). Indeed, since \(\rho_{\lambda,\delta}(v)\) is bounded on any compact subset \(D\) of \(Q_R\), standard \(L^p\) estimates for heat equations (see [16]) give

\[
\int_D \left(|\nabla v|^p + |v_t|^p\right) dx dt < C,
\]

for \(1 < p < \infty\) and any cylinder \(D\) with \(D \subset Q_R\). And it also holds for \(u\), i.e.

\[
\int_D \left(|\nabla^2 u|^p + |u_t|^p\right) dx dt < C,
\]

\(1 < p < \infty\), where \(C\) is a generic constant and may vary from line to line. Choosing \(p\) large, by Sobolev embedding theorem, we conclude that \(u\) is Hölder continuous on \(D\), so does \(\rho_{\lambda,\delta}(v)\). Therefore, Schauder’s estimates for heat equation (see [16]) show that \(|\nabla u|\) and \(|\nabla^2 u|\) are bounded on any compact subsets of \(D\), so do \(|\nabla u|\) and \(|\nabla^2 u|\). In particular, there exists \(M_1\) such that

\[
|\nabla u| + |\nabla^2 u| \leq M_1,
\]

for \((x,t) \in B_r \times \left(T \left(1 - \left(\frac{r}{R}\right)^2\right), T \left(1 - \frac{1}{2} \left(1 - \frac{r}{R}\right)^2\right)\right)\), where \(M_1\) depends on \(R, n\) and \(M\) given in \((5.1)\).

We next prove \((5.3)\) for \(B_r \times \left[T \left(1 - \frac{1}{2} \left(1 - \frac{r}{R}\right)^2\right), T\right]\). For fixed point \((x,t) \in B_r \times \left[T \left(1 - \frac{1}{2} \left(1 - \frac{r}{R}\right)^2\right), T\right]\), we consider

\[
\tilde{u}(z,\tau) = 1 - \mu^{-\frac{3}{2}} \left[1 - u \left(x + \mu z, T - \mu^2(T - \tau)\right)\right],
\]

(5.3)
where \( \mu = \left[ 2 \left( 1 - \frac{r}{T} \right) \right]^\frac{1}{2} \), which satisfies

\[
\begin{align*}
\bar{u}_t - \Delta \bar{u} &= \lambda \frac{1 + \delta \mu^{-\frac{3}{4}} |\nabla z \bar{u}|^2}{(1 - \bar{u})^2}, \quad (z, \tau) \in \mathcal{O}_T \\
\bar{u}(z, \tau) &= 1 - \mu^{-\frac{3}{4}} < 0, \quad (z, \tau) \in \partial \mathcal{O}_T \\
\bar{u}(z, 0) &= \bar{u}_0(z), \quad z \in \mathcal{O},
\end{align*}
\]

(5.4)

where \( \bar{u}_0(z) = 1 - \mu^{-\frac{3}{4}} \left[ 1 - u(x + \mu z, T(1 - \mu^2)) \right] \) and \( \Delta z \bar{u}_0 + \lambda \frac{1 + \delta \mu^{-\frac{3}{4}} |\nabla z \bar{u}_0|^2}{(1 - \bar{u}_0)^2} > 0 \) on \( \mathcal{O} \). For the fixed point \((x, t)\), we define \( \mathcal{O} := \{ z : x + \mu z \in \Omega \} \). It is implied by (5.3) that \( T \) is also the finite quenching time of \( \bar{u} \), and the domain of \( \bar{u} \) includes \( Q_{T, 0} \) for some \( r_0 = r_0(R) > 0 \). Since the quenching set of \( u \) is a compact subset of \( \Omega \), due to Theorem 4.1, so does that of \( \bar{u} \). Therefore, the argument of Lemma 5.1 can be applied to (5.4), yielding that there exists a constant \( M_2 > 0 \) such that

\[
1 - \bar{u}(z, \tau) \geq M_2 (T - t)^{\frac{1}{2}},
\]

where \( M_2 \) depends on \( R, \lambda, \delta \) and \( \Omega \). Applying the interior \( L^p \) estimates and Schauder’s estimates to \( \bar{u} \) as before, there exists \( M'_2 = M'_2(R, \lambda, \delta, n, M_2) > 0 \) such that

\[
|\nabla z \bar{u}| + |\nabla^2 \bar{u}| \leq M'_2,
\]

(5.5)

for \((z, \tau) \in B_r \times \left( T \left( 1 - \left( \frac{z}{r_0} \right)^2 \right), T \left( 1 - \frac{1}{2} \left( 1 - \frac{z}{r_0} \right)^2 \right) \right) \), where we assume that \( \frac{1}{2} r_0^2 < r^2 < r_0^2 \). Applying (5.3) and taking \((z, \tau) = (0, \frac{T}{2})\), (5.5) gives

\[
\mu^{-\frac{1}{4} + 1} |\nabla \bar{u}| + \mu^{-\frac{1}{4} + 2} |\nabla^2 \bar{u}| \leq M'_2.
\]

Thus, (5.2) follows immediately from \( \mu = \left[ 2 \left( 1 - \frac{r}{T} \right) \right]^\frac{1}{2} \). \( \square \)

5.3. **Lower bound estimate.** First, we note the following local lower bound estimate.

**Proposition 5.3.** Suppose \( u(x, t) \) is a quenching solution of \( \{P_{\lambda, \delta}\} \) in finite time \( T \). Then, there exists a bounded constant \( C = C(\lambda, \Omega) > 0 \) such that

\[
\max_{x \in \Omega} u(x, t) \geq 1 - C(T - t)^{\frac{1}{2}},
\]

(5.6)

for \( 0 < t < T \).

**Proof.** Let \( U(t) = \max_{x \in \Omega} u(x, t) \), \( 0 < t < T \), and let \( U(t_i) = u(x_i, t_i), i = 1, 2 \) with \( h = t_2 - t_1 > 0 \). Then,

\[
\begin{align*}
U(t_2) - U(t_1) &\geq u(x_1, t_2) - u(x_1, t_1) = hu_t(x_1, t_1) + o(1); \\
U(t_2) - U(t_1) &\leq u(x_2, t_2) - u(x_2, t_1) = hu_t(x_2, t_2) + o(1).
\end{align*}
\]

It follows that \( U(t) \) is Lipschitz continuous. Hence, for \( t_2 > t_1 \), we have

\[
\frac{U(t_2) - U(t_1)}{t_2 - t_1} \leq u_t(x_2, t_2) + o(1).
\]

On the other hand, since \( \nabla u(x_2, t_2) = 0 \) and \( \Delta u(x_2, t_2) \leq 0 \), we obtain

\[
u_t(x_2, t_2) \leq \frac{\lambda}{(1 - u(x_2, t_2))^2} = \frac{\lambda}{(1 - U(t_2))^2},
\]

for \( 0 < t_2 < T \). Consequently, at any differentiable point of \( U(t) \), it deduces from the above inequalities that

\[
(1 - U)^2 U_t \leq \lambda,
\]

(5.7)

for a.e. \( 0 < t < T \). (5.6) is obtained by integrating (5.7) from \( t \) to \( T \). \( \square \)
5.4. Nondegeneracy of quenching solution. For the quenching solution \( u(x, t) \) of \( (F_{\lambda, \delta}) \) in finite time \( T \), we now introduce the associated similarity variables

\[
y = \frac{x - a}{\sqrt{T - t}}, \quad s = -\log (T - t), \quad u(x, t) = 1 - (T - t)^{\frac{1}{2}} w_a(y, s),
\]

(5.8)
a is any point in \( \Omega_\eta \), for some small \( \eta > 0 \). The form of \( w_a \) defined in (5.8) is motivated by Lemma 5.1 and Proposition 5.3. Then \( w_a(y, s) \) is defined in

\[
W_a := \{(y, s) : a + ye^{-\frac{s}{2}} \in \Omega, \ s > s' = -\log T\},
\]

and it solves

\[
\frac{\partial w_a}{\partial s} = \Delta w_a - \frac{y}{2} \cdot \nabla w_a + \frac{1}{3} w_a - \frac{\lambda}{w_a^2} - \lambda \delta e^s \frac{\|\nabla w_a\|^2}{w_a^2}. \tag{5.9}
\]

Here \( w_a \) is always strictly positive in \( W_a \). The slice of \( W_a \) at a given time \( s = s_0 \) will be denoted as \( \Omega_a(s_0) \):

\[
\Omega_a(s_0) := W_a \cap \{s = s_0\} = e^{-\frac{s_0}{2}} (\Omega - a).
\]

For any \( a \in \Omega_\eta \), there exists \( s_0 = s_0(\eta, a) > 0 \) such that

\[
B_s := \{y : |y| < s\} \subset \Omega_a(s), \tag{5.10}
\]

for \( s \geq s_0 \).

Equation (5.9) could also be written in divergence form:

\[
\rho w_a = \nabla (\rho \cdot \nabla w) + \frac{1}{3} \rho w - \frac{\lambda \rho}{w^2} - \lambda \delta e^s \frac{\|\nabla w\|^2}{w^2}, \tag{5.11}
\]

with \( \rho(y) = e^{-\frac{|y|^2}{2}}. \)

We shall reach the nondegeneracy of the quenching behavior. The conclusion is obtained by the comparison principle [3] and results in [13].

**Theorem 5.4.** Suppose \( u \) is a quenching solution of \( (F_{\lambda, \delta}) \) in finite time \( T \) and \( a \) is any point in \( \Omega_\eta \), for some \( \eta > 0 \). If \( w_a(y, s) \to \infty \) as \( s \to \infty \) uniformly for \( |y| \leq C \), where \( C \) is any positive constant, then \( a \) is not a quenching point of \( u \).

**Proof.** It is easy to see that \( w_a \) in (5.9) is a subsolution of

\[
\frac{\partial}{\partial s} \tilde{w} = \Delta \tilde{w} - \frac{y}{2} \cdot \nabla \tilde{w} + \frac{1}{3} \tilde{w} - \frac{\lambda}{\tilde{w}^2}
\]

in \( B_{s_0} \times (s_0, \infty) \). From the comparison principle (cf. [3]), we get \( w_a \leq \tilde{w} \) in \( B_{s_0} \times (s_0, \infty) \). If \( w_a(y, s) \to \infty \), as \( s \to \infty \) uniformly in \( |y| \leq C \), so does \( \tilde{w}(y, s) \). Our conclusion follows immediately from Theorem 2.12, [13], where \( f \equiv 1 \) and \( \tilde{w} \) is the \( w_a \) in [13]. \( \square \)

Remark 5.5. The proof of Theorem 5.4 also implies that the quenching set of the solution to \( (F_{\lambda, 0}) \) is a subset of that of \( u \), the solution to \( (F_{\lambda, \delta}) \), \( \delta > 0 \).

5.5. Asymptotics of quenching solution. In this subsection, we shall omit all the subscript \( a \) of \( w_a, W_a \) and \( \Omega_a \) if no confusion will arise.

In view of (5.8), one combine Lemma 5.1 and Lemma 5.2 to reach the following estimates on \( w, \nabla w \) and \( \Delta w \):

**Corollary 5.6.** Suppose \( u \) is a quenching solution to \( (F_{\lambda, \delta}) \) in finite time \( T \). Then the rescaled solution \( w \) satisfies

\[
M \leq w \leq e^{\frac{s}{2}}, \quad |\nabla w| + |\Delta w| \leq M', \quad \text{in } W,
\]

where \( M \) and \( M' \) are constants in Lemma 5.1 and Lemma 5.2 respectively. Moreover, it satisfies

\[
M \leq w(y_1, s) \leq w(y_2, s) + M'|y_1 - y_2|, \tag{5.12}
\]

for any \( (y_i, s) \in W, \ i = 1, 2. \)
Lemma 5.7. Let $s_j$ be an increasing sequence such that $s_j \to +\infty$, and $w(y, s + s_j)$ is uniformly convergent to a limit $w_\infty(y, s)$ in compact sets. Then either $w_\infty(y, s) \equiv \infty$ or $w_\infty(y, s) < \infty$ in $\mathbb{R}^n$.

Proof. Inequality (5.12) implies that

$$w_\infty(y_1, s) \leq w_\infty(y_2, s) + M'|y_1 - y_2|$$

and the conclusion follows. \qed

Proposition 5.8. Suppose $w$ is the solution of (5.9) quenching at $x = a$ in finite time $T$. Assume further that

$$\int_{s_0}^{\infty} \int_{B_s} \rho |\nabla w|^2 dy ds < \infty,$$

(5.13)

for some $s_0 \gg 1$, where $\rho(y) = e^{-|y|^2 / 2}$, $B_s$ is defined in (5.10). Then $w(y, s) \to w_\infty(y)$, as $s \to \infty$ uniformly on $|y| \leq C$, where $C > 0$ is any bounded constant, and $w_\infty(y)$ is a bounded positive solution of

$$\Delta w - \frac{1}{2} y \cdot \nabla w + \frac{1}{3} w - \frac{\lambda}{w^2} = 0$$

(5.14)

in $\mathbb{R}^n$.

Proof. Let us adapt the arguments in the proofs of Proposition 6 and 7 [9] or Lemma 3.1 [14]. Let $\{s_j\}$ be an increasing sequence tending to $\infty$ and $s_{j+1} - s_j \to \infty$. Let us denote $w_j(y, s) = w(y, s + s_j)$. Applying Arzela-Ascoli theorem on $z_j(y, s) = \frac{\int w_j(y, s)}{w_j(y, s)}$ with Corollary 5.6 there is a subsequence of $\{z_j\}$, still denoted as $\{z_j\}$, such that

$$z_j(y, s) \to z_\infty(y, s)$$

uniformly on compact sets of $W$ and

$$\nabla z_j(y, m) \to \nabla z_\infty(y, m)$$

for almost all $y$ and for each integer $m$. That is, $w_j(y, s) \to w_\infty(y, s)$ uniformly on the compact sets of $W$ and $\nabla w_j(y, m) \to \nabla w_\infty(y, m)$ for almost all $y$ and for each integer $m$. From Lemma 5.7 we get that either $w_\infty(y, s) \equiv \infty$ or $w_\infty(y, s) < \infty$ in $y \in \mathbb{R}^n$. The case that $w_\infty(y, s) \equiv \infty$ could be excluded by Theorem 5.4 since $a$ is the quenching point.

Let us define the associate energy of $w$ at time $s$:

$$E[w](s) = \frac{1}{2} \int_{B_s} \rho \nabla w \cdot \nabla w_s dy - \frac{1}{6} \int_{B_s} \rho w^2 dy - \lambda \int_{B_s} \frac{\rho}{w} dy.$$

Direct computations yield that

$$\frac{d}{ds} E[w](s) = \int_{B_s} \rho \nabla w \cdot \nabla w_s dy - \frac{1}{3} \int_{B_s} \rho w w_s dy + \lambda \int_{B_s} \frac{\rho}{w^2} w_s dy$$

+ \frac{1}{2} \int_{\partial B_s} \rho |\nabla w|^2 (y \cdot \nu) dS - \frac{1}{6} \int_{\partial B_s} \rho w^2 (y \cdot \nu) dS - \lambda \int_{\partial B_s} \frac{\rho}{w} (y \cdot \nu) dS$$

$$= - \int_{B_s} \nabla (\rho \cdot \nabla w) w_s dy - \frac{1}{3} \int_{B_s} \rho w w_s dy + \lambda \int_{B_s} \frac{\rho}{w^2} w_s dy$$

+ \int_{\partial B_s} \rho (\nabla w \cdot \nu) w_s dS + \frac{1}{2} \int_{\partial B_s} \rho |\nabla w|^2 (y \cdot \nu) dS$$

- \frac{1}{6} \int_{\partial B_s} \rho w^2 (y \cdot \nu) dS - \lambda \int_{\partial B_s} \frac{\rho}{w} (y \cdot \nu) dS$$

$$= - \int_{B_s} \rho |w_s|^2 dy - \lambda \delta e^k \int_{B_s} \rho \frac{|\nabla w|^2}{w^2} w_s dy + G(s),$$

(5.15)
where
\[ G(s) := \int_{\partial B_s} \rho(\nabla w \cdot \nu) w_s dS + \frac{1}{2} \int_{\partial B_s} \rho|\nabla w|^2(y \cdot \nu) dS \]
\[ - \frac{1}{6} \int_{\partial B_s} \rho w^2(y \cdot \nu) dS - \lambda \int_{\partial B_s} \frac{\rho}{w}(y \cdot \nu) dS, \]
\( \nu \) is the exterior unit normal vector to \( \partial \Omega \) and \( dS \) is the surface area element. The first equality in (5.15) is followed by Lemma 2.3 [17]. Let us estimate \( G(s) \) as in Lemma 2.10 [13]:
\[ G(s) \leq \int_{\partial B_s} \rho(\nabla w \cdot \nu) w_s dS + \frac{1}{2} \int_{\partial B_s} \rho|\nabla w|^2(y \cdot \nu) dS \]
\[ \leq C_1 s^n e^{-\frac{s^2}{n}} + C_2 s^{n-1} e^{-\frac{s^2}{n}} \lesssim s^n e^{-\frac{s^2}{2}}, \quad (5.16) \]
since
\[ |w_s| \leq C(1 + |y|) + \frac{w}{3} \leq \tilde{C}(1 + s), \quad (5.17) \]
due to Lemma 5.7 and the fact that \( a \) is the quenching point. Hence, by integrating (5.15) in time from \( a \) to \( b \), we have that
\[ \int_a^b \int_{B_x} \rho|w_s|^2 dy ds \leq E[w](a) - E[w](b) + C \int_a^b \int_{B_x} \rho|\nabla w|^2 dy ds + \tilde{C} \int_a^b G(s) ds \quad (5.18) \]
for any \( a < b \). Now we shall show that \( w_\infty \) is independent of \( s \). Let \( a = m + s_j \), \( b = m + s_j+1 \) and \( w = w_j \) in (5.18):
\[ \int_m^{m+s_j+1} \int_{B_{s+j}} \rho \left| \frac{\partial w_j}{\partial s} \right|^2 dy ds \]
\[ \leq E[w_j](m) - E[w_{j+1}](m) + C \int_{m+s_j}^{m+s_j+1} \rho|\nabla w|^2 dy ds + \tilde{C} \int_{m+s_j}^{m+s_j+1} G(s) ds \quad (5.19) \]
for any integer \( m \). Since \( s_j + m \to \infty \) as \( j \to \infty \), the third and the last term on the right-hand side of (5.19) tend to zero, due to (5.13) and (5.16), respectively. Since \( \nabla w_j(y, m) \) is bounded and independent of \( j \), and \( \nabla w_j(y, m) \to \nabla w_\infty(y, s) \) a.e. as \( j \to \infty \), we have
\[ \lim_{j \to \infty} E[w_j](m) = \lim_{j \to \infty} E[w_{j+1}](m) := E[w_\infty], \quad (5.20) \]
according to the dominated convergence theorem. Thus, the right-hand side of (5.19) tends to zero as \( j \to \infty \). Therefore
\[ \lim_{j \to \infty} \int_m^M \int_{B_{s+j}} \rho \left| \frac{\partial w_j}{\partial s} \right|^2 dy ds = 0 \quad (5.21) \]
for each pair of \( m \) and \( M \). Now, from (5.17) where \( \tilde{C} \) is independent of \( j \), we get \( \frac{\partial w_j}{\partial s} \) converges weakly to \( \frac{\partial w_\infty}{\partial s} \). Since \( \rho \) decreases exponentially as \( |y| \to \infty \) the integral in (5.21) is lower-semicontinuous, and we conclude that
\[ \int_m^M \int_{\mathbb{R}^n} \left| \frac{\partial w_\infty}{\partial s} \right|^2 dy ds = 0. \]
Since \( m \) and \( M \) are arbitrary, we show that \( w_\infty \) is independent of \( s \).
Since \( \frac{\partial w_j}{\partial s} \) and \( \nabla w_j \) are locally bounded in \( \mathbb{R}^n \times (s_0, \infty) \) for some \( s_0 > 1 \), by Corollary 5.6 \( w_\infty \) is locally Lipschitzian. Each \( w_j \) solves (5.9) and condition (5.13) forces \( e^t|\nabla w|^2 \to 0 \), as \( s \to +\infty \), so \( w_\infty \) is a stationary weak solution to (5.14). Schauder’s
estimates (cf. [3]) yields the desired regularity of \( w_\infty \), i.e. \( w_\infty \) is actually a strong solution.

The solution to (5.14) in one dimension has been investigated in [2]. And [12] studied the radially symmetric solution to this equation of dimension \( n \geq 2 \). Combining Proposition 5.8 and their results, we assert that

**Theorem 5.9.** Suppose \( u \) is a solution to \( \left[ F_{\lambda, \delta} \right] \) quenching at \( x = a \) in finite time \( T \). Assume further that condition (5.13) is satisfied. Then we have

\[
\lim_{t \to T^-} (1 - u(x,t))(T - t)^{-\frac{1}{4}} = (3\lambda)^{\frac{3}{4}}
\]

uniformly on \( |x - a| \leq C(T - t) \) for any bounded constant \( C \).

**Proof.** It is shown in Theorem 2.1, [2] and Theorem 1.6, [12] that every non-constant (radially symmetric in dimension \( n \geq 2 \)) solution \( w(y) \) to (5.14) in \( \mathbb{R}^n \) must be strictly increasing for sufficiently large \( |y| \), and \( w(y) \to \infty \), as \( |y| \to \infty \). Therefore, \( w_\infty \) has to be a constant solution, i.e. \( w_\infty \equiv (3\lambda)^{\frac{3}{4}} \).

5.6. Local expansion near the singularity. In this subsection, we shall construct the local expansion of the solution \( u = u(x,t) \) near the quenching point and the quenching time, provided \( \Omega \in \mathbb{R}^n \) is a radially symmetric domain. It has been shown in Theorem 1.2 that the origin is the only quenching point. Let us make the following nonlinear transformation as motivated by [15] and [14]:

\[
\zeta = \frac{1}{3\lambda}(1 - u)^3.
\]

Notice that \( u = 1 \) maps to \( \zeta = 0 \). In terms of \( \zeta \), \( \left[ F_{\lambda, \delta} \right] \) transforms to

\[
\begin{cases}
\zeta_t = \Delta \zeta - \frac{2}{3} |\nabla \zeta|^2 - \frac{\delta \lambda^2}{3^\frac{3}{2}} \frac{|\nabla \zeta|^2}{\zeta^{\frac{3}{2}}} - 1, & (x,t) \in \Omega_T, \\
\zeta(x,t) = \frac{1}{3\lambda}, & (x,t) \in \partial \Omega_T, \\
\zeta(x,0) = \frac{1}{3\lambda}, & x \in \Omega.
\end{cases}
\]

We shall find a formal power series solution to (5.23) near \( \zeta = 0 \). As in [15] and [14] we look for a locally radially symmetric solution to (5.23) in the form

\[
\zeta(r,t) = \zeta_0(t) + \frac{r^2}{2} \zeta_2(t) + \frac{r^4}{4!} \zeta_4(t) + \cdots,
\]

where \( r = |x| \). Substituting (5.24) into (5.23) and collecting the coefficients in \( r \), we obtain the following coupled ODEs for \( \zeta_0 \) and \( \zeta_2 \):

\[
\begin{align*}
\zeta_0' & = -1 + n \zeta_2, \\
\zeta_2' & = \frac{n + 2}{3} \zeta_4 - \frac{4}{3} \zeta_0^2 - \frac{26 \lambda^\frac{3}{4}}{3^\frac{3}{2}} \frac{\zeta_2^2}{\zeta_0^2}.
\end{align*}
\]

We are interested in the solution with \( \zeta_0(T) = 0 \), \( \zeta_0' < 0 \) and \( \zeta_2 < 0 \) for \( T - t > 0 \) and \( T - t \ll 1 \). We shall assume that \( \zeta_4 \ll \frac{\zeta_2^2}{\zeta_0^2} \) near the singularity. And it is clear that \( \frac{\zeta_2^2}{\zeta_0} \ll \frac{\zeta_2^2}{\zeta_0^2} \), since \( \zeta_0 \ll 1 \). Hence, (5.25) reduces to

\[
\begin{align*}
\zeta_0' & = -1 + n \zeta_2, \\
\zeta_2' & = -\frac{26 \lambda^\frac{3}{4}}{3^\frac{3}{2}} \frac{\zeta_2^2}{\zeta_0^2}.
\end{align*}
\]

Now we solve the system (5.26) asymptotically as \( t \to T^- \). We first assume that \( n \zeta_2 \ll 1 \) near \( T \). This leads to \( \zeta_0 \sim T - t \) and the following differential equation for \( \zeta_2 \):

\[
\zeta_2' \sim -\frac{26 \lambda^\frac{3}{4}}{3^\frac{3}{2}} \frac{\zeta_2^2}{(T - t)^\frac{3}{2}}.
\]
By integrating (5.27), we obtain that
\[ \zeta_2 \sim \frac{3^\frac{1}{2}}{2\delta \lambda^\frac{3}{2}} (T-t)^{\frac{1}{2}} + A \frac{(T-t)^{\frac{1}{2}}}{\log (T-t)} + \cdots, \] (5.28)
for some unknown constant A. From (5.28), we observe that the consistency condition that \( n \zeta_2 \ll 1 \) as \( t \to T^- \) is indeed satisfied. Substitute (5.28) into (5.26) for \( \zeta_0 \), we obtain for \( t \to T^- \) that
\[ \zeta'_0 \sim -1 + n \left( \frac{3^\frac{1}{2}}{2\delta \lambda^\frac{3}{2}} (T-t)^{\frac{1}{2}} + A \frac{(T-t)^{\frac{1}{2}}}{\log (T-t)} + \cdots \right). \] (5.29)

Using the method of dominant balance, we look for a solution to (5.29) as \( t \to T^- \) in the form
\[ \zeta_0 \sim (T-t) + (T-t) \left( B_0(T-t)^{\frac{1}{2}} + B_1 \frac{(T-t)^{\frac{1}{2}}}{\log (T-t)} + \cdots \right), \]
for some constants \( B_0 \) and \( B_1 \). A simple calculation yields that
\[ \zeta_0 \sim (T-t) + (T-t) \left[ -\frac{3^\frac{1}{2} n}{8\delta \lambda^\frac{3}{2}} (T-t)^{\frac{1}{2}} - \frac{3}{4} n A \frac{(T-t)^{\frac{1}{2}}}{\log (T-t)} + \cdots \right], \] \( t \to T^- \). (5.30)

The local form for \( \zeta \) near quenching point is \( \zeta \sim \zeta_0 + \frac{r^2}{T} \zeta_2 \). Using the leading term in \( \zeta_2 \) from (5.28) and the first two terms in \( \zeta_0 \) from (5.30), we obtain the local form
\[ \zeta \sim (T-t) \left[ 1 - \frac{3^\frac{1}{2} n}{8\delta \lambda^\frac{3}{2}} (T-t)^{\frac{1}{2}} + \frac{3^\frac{1}{2}}{4\delta \lambda^\frac{3}{2}} \frac{r^2}{(T-t)^{\frac{1}{2}}} + \cdots \right], \] \( r \ll 1 \) and \( T-t \ll 1 \). Finally, using the nonlinear mapping (5.22) relating \( u \) and \( \zeta \), we conclude that
\[ u \sim 1 - [3\lambda(T-t)]^{\frac{1}{2}} \left( 1 - \frac{3^\frac{1}{2} n}{8\delta \lambda^\frac{3}{2}} (T-t)^{\frac{1}{2}} + \frac{3^\frac{1}{2}}{4\delta \lambda^\frac{3}{2}} \frac{r^2}{(T-t)^{\frac{1}{2}}} + \cdots \right)^{\frac{1}{2}}. \] (5.32)

6. Numerical simulations.

6.1. Numerical experiments on pull-in voltage and quenching time. In section 3, we investigate the pull-in voltages \( \lambda^*_\delta \) and the finite quenching time \( T \) of (\( F_{\lambda, \delta} \)). We shall verify our results in section 3 by numerically computing \( \lambda^*_\delta \) and \( T \) for some choice of domain \( \Omega \). Let us consider the following two choices of \( \Omega \):
\[ \Omega : \begin{bmatrix} -\frac{1}{2} & 1 \end{bmatrix} \] (slab),
\[ \Omega : |x| \leq 1, \ x \in \mathbb{R}^2 \] (unit disk).
To obtain \( \lambda_1 \) and \( \lambda_{n,1} \) in (3.1) and Proposition 3.1, we numerically solve \( -\Delta \xi = 1 \) in \( \Omega \) with Dirichlet boundary condition, yielding that \( \|\xi\|_\infty \approx 0.125 \) for the slab and \( \|\xi\|_\infty \approx 0.712 \) for the unit disk in \( \mathbb{R}^2 \). The first eigen pairs \((\mu_0, \phi_0)\) of the operator \( -\Delta \) with Dirichlet boundary condition in \( \Omega \) and with the normalization \( \int_\Omega \phi_0 dx = 1 \) are explicitly given below
\[ \mu_0 = \pi^2, \ \phi_0 = \frac{\pi}{2} \sin \left( \pi \left( x + \frac{1}{2} \right) \right) \] (slab), \] (6.1)
\[ \mu_0 = z_0^2 \approx 5.783, \ \phi_0 = \frac{z_0}{J_1(z_0)} J_0(z_0(|x|)) \] (unit disk), \] (6.2)
where \( J_0 \) and \( J_1 \) are Bessel functions, and \( z_0 \approx 2.4048 \) is the first zero of \( J_0(z) \).

We first compute the pull-in voltage \( \lambda^*_\delta \) for various \( \delta \) in Table 1 for both slab (left column) and unit disk (right column). We use bvp4c in MatLab to determine \( \lambda^*_\delta \) (cf. [24]). It is shown that \( \lambda^*_\delta \) decreases as \( \delta \) increases. And \( \lambda_{n,1} \) for the case \( \delta = 0.7 \) and
Table 1 Pull-in voltages $\lambda_{\delta}^*$ of $(F_{\lambda,\delta})$ with $\delta = 0, 0.1$ and 0.7 for both the slab and the unit disk. The lower bound $\lambda_i$ in (3.1) and the upper bound $\lambda_{u,1}$ in Proposition 3.7 are also shown. Left: slab; Right: unit disk.

| $\delta$ | $\lambda_{\delta}^*$ (slab) | $\lambda_i$ | $\lambda_{u,1}$ | $\delta$ | $\lambda_{\delta}^*$ (slab) | $\lambda_i$ | $\lambda_{u,1}$ |
|----------|----------------|------------|----------------|----------|----------------|------------|----------------|
| 0        | 1.440          | 1.1852     | 1.4622         | 0        | 0.8030         | 0.2080     | 1.4622        |
| 0.1      | 1.391          | 0.9581     | 1.4578         | 0.1      | 0.7890         | 0.2065     | 0.8523        |
| 0.7      | 1.196          | 0.4457     | 1.4314         | 0.7      | 0.712          | 0.1979     | 0.8255        |

1.a: the slab  
1.b: the unit disk

Table 2 The pull-in voltages $\lambda_{\delta}^*$ tend to zero, as $\delta \to \infty$ for both the slab and the unit disk.

| $\delta$ | $\lambda_{\delta}^*$ (slab) | $\lambda_{\delta}^*$ (unit disk) |
|----------|----------------|-------------------------------|
| 0        | 1.440          | 0.8030                        |
| 0.7      | 1.196          | 0.712                         |
| 7        | 0.706          | 0.472                         |
| 70       | 0.301          | 0.218                         |
| 700      | 0.109          | 0.081                         |
| 7000     | 0.036          | 0.028                         |

$\Omega$ is the slab provides a better upper bounds than the natural bound $\lambda_0^*$ (given by the comparison principle, as in [25]).

Next, we verify the result in Proposition 3.2 by numerically computing the pull-in voltage for various $\delta = 0, 0.7, 7, 70, 700$ and 7000. The pull-in voltage $\lambda_{\delta}^*$ is also located by bvp4c in MatLab. It is clearly verified in Table 2 that $\lambda_{\delta}^* \to 0$, as $\delta \to \infty$, for both the slab and the unit disk. We numerically verify in Table 3 that

$$\lim_{\delta \to \infty} \lambda T = \frac{1}{3}$$

for the case without the fringing term. It is also shown numerically that (6.3) no longer holds, for $\delta > 0$. Proposition 3.4 has also been verified by various $\delta$ and domains in Table 3. Moreover, we observe from the results that $\lim_{\lambda \to \infty} \lambda T = 0$ and the rate of convergence is independent of the fringing term $\delta$.

6.2. Numerical solution to $(F_{\lambda,\delta})$. To numerically solve $(F_{\lambda,\delta})$, as suggested in [14], the transformed problem (5.23) is more suitable for implementation. In fact, if we use the local behavior

$$\zeta \sim (T - t) + \frac{3}{4\delta^2 \lambda_T^2} (T - t)^{\frac{3}{2}} r^2,$$

we get that

$$\frac{|\nabla \zeta|^2}{\zeta^4} \sim \frac{3}{4\delta^2 \lambda_T^2} \left( \frac{T - t}{r^2} + \frac{3}{4\delta^2 \lambda_T^2} (T - t)^{\frac{3}{2}} r^2 \right)^{\frac{3}{2}},$$

$$\frac{|\nabla \zeta|^2}{\zeta^4} \sim \frac{3}{4\delta^2 \lambda_T^2} (T - t)^{\frac{3}{2}}.$$

Hence, the two terms $\frac{|\nabla \zeta|^2}{\zeta^4}$ and $\frac{|\nabla \zeta|^2}{\zeta^4}$ in (5.23) is bounded in $r$, for any fixed $t$, even when $t$ is close to $T$. This allows us to use a simple finite-difference scheme to compute the numerical solutions to (5.23).

Experiment 1. Let us first consider the domain slab $[-\frac{1}{2}, \frac{1}{2}]$ in one dimension with $\lambda = 1$, 1.35 or 3 and $\delta = 0$ or 0.7. This interval is discretized into $N + 1$ pieces with $N = 200$, i.e., $h = \frac{1}{N+1} \approx 4.97512 \times 10^{-3}$ is the spatial mesh size. And the time step is labelled as $dt = 6 \times 10^{-6}$. $\zeta_j^n$, for $j = 1, \cdots, N + 2$, is defined to be the discrete
The time-step $dt$ and the experimental stop time is $T$. The quenching times $T_{1}$ and $T_{2}$ are suggested by the numerical simulation that the pull-in voltage of (1.1) should be of an existing solution.

The numerically verifies Remark 3.5. The quenching times $T_{1}$ and $T_{2}$ are chosen to satisfy $T_{1} = 0$ and $T_{2} = 0$ for finite time quenching solution or $\max_{j=1,\ldots,N+2} (\zeta_{j}^{m} - \zeta_{j}^{m}) < 10^{-10}$ for finite time quenching solution or $\max_{j=1,\ldots,N+2} (\zeta_{j}^{m+1} - \zeta_{j}^{m}) < 10^{-10}$ for the globally existing solution.

In Fig. 1, we plot $\zeta$ v.s. $x$ (left) and $u$ v.s. $x$ (right) from the discrete approximation (6.4) at a series of times. The solution to $(F_{\lambda,0})$ with $\delta > 0$ is drawn in blue; while that of $(F_{\lambda,0})$ (cf. (1.1)) is in red. Three different voltages are chosen $\lambda = 1$, 1.35 and 3. It is suggested by the numerical simulation that the pull-in voltage of (1.1) should be $1.35 < \lambda^{*} < 3$; while that of $(F_{\lambda,0,7})$ is between 1 and 1.35. The estimate of $\lambda^{*}$ matches well with the results in Table 1, where $\lambda_{0} = 1.440$ and $\lambda_{0,7} = 1.196$. As to the profiles of the solutions to $(F_{\lambda,\delta})$ with $\delta = 0.7$ and $\delta = 0$, the behavior is similar, if they both globally exist, see Fig. 1a. The quenching profile of $(F_{\lambda,0,7})$ is much flatter than that of (1.1), if they both quench in finite time, see Fig. 1c. The quenching times $T$ for both $\delta = 0$ and $\delta = 0.7$ in Fig. 1c are numerically obtained to be around 0.1515 and 0.134262, respectively. This numerically verifies Remark 3.5.

Experiment 2. When we consider the unit disk in two dimension, a second-order accurate in space and first-order accurate in time discrete approximation for (5.23), with

| $\delta$ | $\lambda$ | $T_{\text{slab}}$ | $\lambda T_{\text{slab}}$ | $T_{\text{disk}}$ | $\lambda T_{\text{disk}}$ |
|--------|--------|------------------|--------------------------|------------------|--------------------------|
| 0      | 1.5    | 1.073664         | 1.610496                | 0.292764         | 0.439146                 |
| 10     | 0.034122 | 0.34122         | 0.033348                | 0.03348         | 0.33348                  |
| 50     | 0.0066666 | 0.3333          | 0.006666                | 0.333           | 0.333                    |
| 100    | 0.0033333 | 0.3333          | 0.003333                | 0.333           | 0.333                    |
| 0.1    | 2      | 0.30837          | 0.6167                  | 0.19011         | 0.38022                  |
| 20     | 0.016692 | 0.3338          | 0.016668                | 0.033336        |                         |
| 200    | 0.000816 | 0.1632          | 0.000816                | 0.1632          |                         |
| 2000   | 0.000048 | 0.0960          | 0.000048                | 0.0960          |                         |
| 1      | 2      | 0.24009          | 0.4802                  | 0.18327         | 0.36654                  |
| 20     | 0.008658 | 0.1732          | 0.008778                | 0.17556         |                         |
| 200    | 0.00198  | 0.0396          | 0.00198                 | 0.0396          |                         |
| 10     | 2      | 0.098892         | 0.1978                  | 0.101538        | 0.203076                 |
| 20     | 0.001392 | 0.02784         | 0.001398                | 0.02796         |                         |
| 200    | 0.000066 | 0.0132          | 0.000066                | 0.0132          |                         |

Table 3 The quenching time $T_{\text{slab}}$ and $T_{\text{disk}}$ for $\delta = 0, 0.1, 1$ and 10 with various $\lambda$ have been numerically computed, where $T_{\text{slab}}$ and $T_{\text{disk}}$ represent the quenching time for the slab $[-\frac{1}{2}, \frac{1}{2}]$ and the unit disk $|x| \leq 1$ in $\mathbb{R}^{2}$.
1.a: \( \lambda = 1 \). We plot at times \( t = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 2.0 \) and the experimental stop time. Both solutions to \((F_{\lambda, \delta})\) and (1.1) increase towards a steady-state solution as \( t \) increases.

1.b: \( \lambda = 1.35 \). We plot at times \( t = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 1.0, 3.0 \) and the experimental stop time for \( \delta = 0 \); while at times \( t = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.66 \) and the experimental stop time for \( \delta = 0.7 \). The solution to (1.1) still globally exists; while that of \((F_{\lambda, \delta})\) quenches in finite time.

1.c: \( \lambda = 3 \). We plot at times \( t = 0, 0.01, 0.03, 0.05, 0.06, 0.07, 0.09, 0.12 \) and the experimental stop time. Both solutions to (1.1) and \((F_{\lambda, \delta})\) quench in finite time.

FIG. 1 Experiment 1: For the slab domain \([-1/2, 1/2]\) with different \( \lambda \). We plot \( \zeta \) and \( u \) versus \( x \) at a sequential times from the finite difference scheme (6.4) with \( N = 200 \) and \( dt = 0.6 \times 10^{-5} \) and \( \delta = 0 \) or 0.7. Left: \( \zeta \) versus \( x \); Right: \( u \) versus \( x \); Blue: solution of \((F_{\lambda, \delta})\); Red: that of (1.1).
Fig. 2: Experiment 2: For the unit disk domain in two dimension with $\lambda = 1$. We plot $\zeta$ and $u$ versus $x$ at times $t = 0.1, 0.2, 0.3, 0.4, 0.5$ and the experimental stop time from the finite difference scheme (6.4) with $N = 200$ and $\delta = 0$ or 0.7. Left: $\zeta$ versus $|x|$; Right: $u$ versus $|x|$. Blue: solution of $(F_{\lambda,\delta})$; Red: that of $(1.1)$.

spatial mesh size $h$, on $0 \leq r \leq 1$ and $t \geq 0$ is

$$\zeta_j^{m+1} = \zeta_j^m + dt \left( \frac{\zeta_{j+1}^m - 2\zeta_j^m + \zeta_{j-1}^m}{h^2} + \frac{\zeta_{j+1}^m - \zeta_{j-1}^m}{2hr_j} - \frac{(\zeta_j^m - \zeta_{j+1}^m)^2}{6\zeta_j^m h^2} - \frac{\delta \lambda^2}{3} \left( \frac{(\zeta_j^m - \zeta_{j+1}^m)^2}{4} \right) - 1 \right), \quad (6.5)$$

where $r_j = jh$. According to [19], the discrete approximation for $\zeta_1$ at the origin $r = 0$ is

$$\zeta_1^{m+1} = \zeta_1^m + 4dt \frac{\lambda^2}{h^2} (\zeta_2^m - \zeta_1^m).$$

The condition at $r = 1$ is $\zeta_{N+2}^m = \frac{1}{3^8}$, and the initial condition is $\zeta_j^0 = \frac{1}{3^8}$ for $j = 1, \cdots, N + 2$. The experimental stop time is $T_{ex} = m \times dt$, where the $m$ is such that $\min_{j=1, \cdots, N+2}(\zeta_j^m - 0) < 10^{-10}$ for finite time quenching solution or $\max_{j=1, \cdots, N+2}(\zeta_j^{m+1} - \zeta_j^m) < 10^{-10}$ for the globally existing solution.

In Fig. 3 we plot $\zeta$ v.s. $|x|$ (left) and $u$ v.s. $|x|$ (right) from the discrete approximation (6.5) with the voltage chosen to be $\lambda = 1$ at times $t = 0.1, 0.2, 0.3, 0.4, 0.5$ and the experimental stop time $T_{ex}$. The solution to $(F_{\lambda,0})$ with $\delta > 0$ is drawn in blue; while that of $(F_{\lambda,0})$ or $(1.1)$ is in red. It is suggested by the numerical simulations that both the pull-in voltage $\lambda^*$ of $(F_{\lambda,0})$ and that $\lambda^*_{0.7}$ of $(F_{0.7})$ are less than 1. This coincides with $\lambda^* = 0.8030$ and $\lambda^*_{0.7} = 0.712$ in Table 1 or Table 2. And the quenching times $T$ with $\delta = 0$ and 0.7 are numerically obtained to be around 0.7076 and 0.578232, respectively.

Experiment 3. Let us examine the local approximation constructed in (5.32) numerically. From Experiment 1, the numerically obtained the quenching time for $(F_{3,0.7})$ in the slab domain $[-\frac{1}{2}, \frac{1}{2}]$ is 0.134262; and from Experiment 2, the quenching time for $(F_{1,0.7})$ in the unit disk of dimension two is around 0.578232. In Fig. 4 we plot $\zeta$ v.s. $x$ and $|x|$ of the discrete approximation (6.4) with $\lambda = 3$ and (6.5) with $\lambda = 1$ at time $t = 0.134004$ and $t = 0.57822$, respectively, in blue. At the same time, we plot the local approximation obtained in (5.31) in black. From Fig. 4 the local approximation (5.31) matches the numerical solutions well.
7. Conclusion. In this paper, we study the equation \((F_{\lambda,\delta})\) modelling the MEMS device with the fringing term \(\delta > 0\). We first show that the pull-in voltage \(\lambda_\delta^* > 0\) obtained in [26] is the watershed of globally existing solution and the finite time quenching solution of \((F_{\lambda,\delta})\). To be more precisely, if \(\lambda \leq \lambda_\delta^*\), then the unique solution to \((F_{\lambda,\delta})\) exists globally; otherwise, the solution will quench in finite time \(T < \infty\).

According to the comparison principle, a natural upper bound of \(\lambda_\delta^*\) is \(\lambda^*\), the pull-in voltage of \((F_{\lambda,0})\). In this paper, it has been slightly improved in Proposition 3.1 for \(\delta \ll 1\) and numerically verified in Table 1. Moreover, we prove that \(\lim_{\delta \to \infty} \lambda_\delta^* = 0\). This has been validated numerically in Table 2.

About the quenching time \(T\), for \(\lambda > \lambda_\delta^*\), we show that it satisfies \(T \lesssim \frac{1}{\lambda}\), which differs from that corresponding to \((F_{\lambda,0})\) where \(\lim_{\lambda \to \infty} \lambda T = \frac{1}{3}\). We conjecture from Table 3 that \(\lim_{\lambda \to \infty} \lambda T = 0\) and the rate of convergence is independent of \(\delta\).

By adapting the moving-plane argument as in [3], we show that the quenching set of \((F_{\lambda,\delta})\) is a compact set in \(\Omega\), if \(\Omega \subset \mathbb{R}^n\) is a bounded convex set. Furthermore, if \(\Omega = B_R(0)\), the ball centered at the origin with the radius \(R\), then the origin is the only quenching point. This is clearly seen from Fig. 1 and Fig. 2.

Finally, we investigate the quenching behavior of the solution to \((F_{\lambda,\delta})\) with \(\lambda > \lambda_\delta^*\). It is shown in this paper that, under certain condition, if \(u\) is the solution to \((F_{\lambda,\delta})\) quenching at \(x = a\) in finite time \(T\), then it satisfies

\[
\lim_{t \to T^-} (1 - u(x,t))(T - t)^{-\frac{1}{3}} = (3\lambda)^{\frac{1}{3}}.
\]

More refined asymptotic expansion is given in (5.32) and it has been verified numerically in Fig. 3 that this is a good local approximation.

References
[1] Brezis, H., Cazenave, T., Martel, Y., Ramiandrisoa, A. (1996). Blow up for \(u_t - \Delta u = g(u)\) revised. Adv. Differential Equations 1:73-90.
[2] Fila, M., Hušková, J. (1991). A note on the quenching rate. Proc. Amer. Math. Soc. 112(2):473-477.
[3] Friedman, A. (1964). Partial Differential Equations of Parabolic Type. New Jersey: Prentice-Hall.
[4] Friedman, A., McLeod, B. (1985). Blow-up of positive solutions of semilinear heat equations. Indiana Univ. Math. J. 34(2):425-447.
[5] Ghoussoub, N., Guo, Y. (2007). On the partial differential equations of electrostatic MEMS devices: Stationary case. SIAM J. Math. Anal. 38:1423-1449.
[6] Ghoussoub, N., Guo, Y. (2008). On the partial differential equations of electrostatic MEMS devices II: Dynamic case. *NoDEA Nonlinear Differential Equations App.* 15(1-2):115-145.

[7] Ghoussoub, N., Guo, Y. (2008). Estimates for the quenching time of a parabolic equation modeling electrostatic MEMS. *Methods Appl. Anal.* 15(3):361-376.

[8] Gidas, B., Ni, W.-M., Nirenberg, L. (1979). Symmetry and related properties via the maximum principle. *Comm. Math. Phys.* 68(3):209-243.

[9] Giga, Y., Kohn, R. V. (1985). Asymptotically self-similar blow-up of semilinear heat equations. *Comm. Pure Appl. Math.* 38:297-319.

[10] Giga, Y., Kohn, R. V. (1987). Characterizing blow-up using similarity variables. *Indiana Univ. Math. J.* 36:1-40.

[11] Giga, Y., Kohn, R. V. (1989). Nondegeneracy of blow-up for semilinear heat equations. *Comm. Pure Appl. Math.* 42:845-884.

[12] Guo, J. S. (1991). On the semilinear elliptic equation $\Delta w + \frac{1}{2}y \cdot \nabla w - \lambda w - w^{-\beta} = 0$ in $\mathbb{R}^n$. *Chinese J. Math.* 19:355-377.

[13] Guo, Y. (2008). On the partial differential equations of electrostatic MEMS devices III: Refined touchdown behavior. *J. Differential Equations* 244:2277-2309.

[14] Guo, Y., Pan, A., Ward, M. J. (2005). Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties. *SIAM J. Appl. Math.* 66(1):309-338.

[15] Keller, J. B., Lowengrub, J. (1993). Asymptotic and numerical results for blowing-up solutions to semilinear heat equations, in Proceedings of the meeting on Singularities in Fluids, Plasmas, and Optics (Heraklion 1992), NATO Adv. Sci. Instl. Ser. C Math. Phys. Sci. 404, Kluwer Academic Publisher, Dordrecht, The Netherlands, pp. 11-38.

[16] Ladyzenskaja, O. A., Solonnikov, V. A., Uralceva, N. N. (1968). *Linear and quasilinear equations of parabolic type*. Amer. Math. Soc.: Transl. Math. Monographs 23.

[17] Liu, W. (1989). The blow-up rate of solutions of semilinear heat equations. *J. Differential Equations* 77:104-122.

[18] Liu, Z., Wang, X. (2012). On a parabolic equation in MEMS with fringing field. *Arch. Math.* 98:373-381.

[19] Morton, K. W., Mayers, D. F. (1994). *Numerical solution of partial differential equations*, Cambridge, UK: Cambridge University Press.

[20] Nathanson, H. C., Newell, W. E., Wickstrom, R. A. (1967). The resonant gate transistor. *IEEE Trans. on Electron Devices* 14:117-133.

[21] Pelesko, J. A. (2002). Mathematical modeling of electrostatic MEMS with tailored dielectric properties. *SIAM J. Appl. Math.* 62:888-908.

[22] Pelesko, J. A., Bernstein, D. H. (2002) *Modeling MEMS and NEMS*, Chapman Hall and CRC Press.

[23] Pelesko, J. A., Driscoll, T. A. (2005). The effect of the small-aspect-ratio approximation on canonical electrostatic MEMS models. *J. Engrg. Math.* 53:239-252.

[24] Shampine, L., Kierzenka, J., Reichelt, M. Solving boundary value problems for ordinary differential equations in MATLAB with bvp4c, available at [http://www.mathworks.com/bvp_tutorial](http://www.mathworks.com/bvp_tutorial)

[25] Wang, Q. (2013). Estimates for the quenching time of a MEMS equation with fringing field. *J. Math. Anal. Appl.* 405(1):135-147.

[26] Wei, J., Ye, D. (2010). On MEMS equation with fringing field. *Proc. Amer. Math. Soc.* 138(5):1693-1699.

[27] Ye, D., Zhou, F. (2010). On a general family of nonautonomous elliptic and parabolic equations. *Calc. Var. Partial Differential Equations* 37:259-174.