SUPERLOGIC MANIFOLDS AND GEOMETRIC APPROACH TO
QUANTUM LOGIC

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ABSTRACT. The main purpose of this paper is to present a new approach to logic
or what we will call superlogic. This approach constitutes a new way of looking
at the connection between quantum mechanics and logic. It is a geometrisation
of the Quantum logic. Note that this superlogic is not distributive reflecting
a good propriety to describe quantum mechanics, non commutative spaces and
contains a nilpotent element.

1. INTRODUCTION

In 1666, G.W. Leibniz envisaged a universal scientific language, the characteristica
universalis, together with a symbolic calculus, the calculus ratiocinator, for formal
logical deduction within this language. Leibniz soon turned his attention to other
matters, including the creation of the calculus of infinitesimals, and only partially
developed his logical calculus. Nearly two centuries later, in Mathematical Analysis
of Logic (1847) and Laws of Thought (1854), G. Boole took the first decisive steps
toward the realization of Leibniz’s projected calculus of scientific reasoning.
The genesis of Quantum logic began with J. von Neumann in 1932 [1]. His main
argument was that certain linear operators, the projections defined on a Hilbert
space, could be regarded as representing experimental propositions affiliated with
the properties of a quantum mechanical system. He wrote,

"...the relation between the properties of a physical system on the
one hand, and the projections on the other, makes possible a sort
of logical calculus with these."

Later on, in 1936, von Neumann published with G. Birkhoff a definitive article on
the logic of quantum mechanics [4, 5]. In this paper, Birkhoff and von Neumann
proposed that the specific Quantum logic of projection operators on a Hilbert

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space should be replaced by a general class of Quantum logics governed by a set of axioms, much in the same way that Boolean algebras had already been characterized axiomatically. They observed that, for propositions $P, Q, R$ pertaining to a classical mechanical system, the distributive law

$$P \land (Q \lor R) = (P \land Q) \lor (P \land R)$$

holds, they gave an example to show that this law can fail for propositions affiliated with a quantum mechanical system, and they concluded that,

"...whereas logicians have usually assumed that properties of negation were the ones least able to withstand a critical analysis, the study of mechanics points to the distributive identities as the weakest link in the algebra of logic."

Birkhoff and von Neumann went on to argue that a Quantum logic ought to satisfy only a weakened version of the distributive law called the modular law; however, they pointed out that projection operators on a Hilbert space can fail to satisfy even this attenuated version of distributivity. Much of von Neumann’s subsequent work on continuous geometries [2] and rings of operators [3] was motivated by his desire to construct logical calculi satisfying the modular law. In 1937, K. Husimi [5] discovered that projection operators on a Hilbert space satisfy a weakened version of the modular law, now called the orthomodular identity.

The other interesting breakthrough was taking in 1957 by G. Mackey, who wrote an expository article on quantum mechanics [7] based on lectures he was giving at Harvard. In 1963, he published an expanded version of these lectures in the form of an influential monograph [8]. Note that Mackey’s questions form an orthomodular lattice. The simplicity and elegance of Mackey’s formulation and the natural and compelling way in which it gave rise to a system of experimental propositions inspired a renewed interest in the study of Quantum logic, now identified with the study of orthomodular lattices.

In 1964, C. Piron introduced an alternative to Mackey’s approach in which questions again band together to form an orthomodular lattice, but this time possessing more of the special features of the lattice of projection operators on a Hilbert space [9].

A list of more or less "natural conditions" on generalized Hilbert spaces was soon proposed in the hopes of singling out the "true" Hilbert spaces. In 1980, H. Keller dashed these hopes by constructing an example of a generalized Hilbert space satisfying all of the proposed natural conditions, but that is not a standard Hilbert space [10].

Even more, in orthodox quantum mechanics, the combined system (when systems are combined or coupled to form composite systems) is represented mathematically by a so-called tensor product of Hilbert spaces. Very early many researchers
realized that the entire Quantum logic program would falter unless a suitable version of tensor product could be found for the more general logical structures then under consideration.

Composite physical systems were studied from the perspective of Quantum logic in an important and influential sequence of papers by D. Aerts [11]. In parallel with the development of Quantum logic, and starting as early as 1970 [12], Davies, Lewis, Holevo, Ludwig, Prugovecki, Ali, Busch, Lahti, Mittelstaedt, Schroek, 'Bujagski, Beltrametti [13], et al worked out a theory of quantum statistics and quantum measurement based on so-called effect operators on a Hilbert space. Every projection operator is an effect operator, but not conversely, and the effect operators do not even form a lattice, let alone an orthomodular lattice, or even an orthoalgebra.

In fact we can consider that during the years 1938-1994 Quantum logic has been under development for roughly half a century. The history of Quantum logic has been a story of more and more general mathematical structures - Boolean algebras, orthomodular lattices, orthomodular posets, orthoalgebras, and effect algebras - being proposed as basic models for the logics affiliated with physical systems.

In the rest of the paper we will present a new approach to the question of Quantum logic. The main idea is to introduce the correlation or interaction already at the level of the elements of the logic or superlogic. Thus this paper presents a kind of 'geometrization' of Quantum logic [3] that could be extended to all logic [4]. This is fulfilled through the introduction of a new structure $n$ such that $n^2 = 0$ and sheaves of fields on the superlogic manifolds. So it codes the non commutativity and non distributivity of the quantum formalism [5].

Logically, the novelty of this approach consists in a new way of treating propositions. It is inspired by the analogy between propositions and measurements in

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3 Or 2-dimensional logic. Indeed, as in the case of real number where $\mathbb{R}$ is considered as one dimension, $\mathbb{C}$ as 2-dimensions ($\mathbb{R}, \mathbb{R}$) with the structure $i^2 = -1$ and $\mathbb{Q}$ (quaternion) as 4 dimensions with three $i, j, k$ complexes structures. So we can think of the superlogic as a couple of classical logic with a structure $n^2 = 0$, i.e. as 2-dimensional logic. It is even more relevant to think this superlogic as in the case of supersymmetry where we have a commuting and anticommuting (or Grassmann) coordinates.

4 See [23] for introduction to the geometrization of classical logic and the propositional manifolds.

5 Moreover, in some manner our superlogic is linked with a generalized version of Bohr Topos or more precisely a certain presheaf of Bohr toposes. Indeed, as we know, one might think of a Bohr topos as (part of) a formalization of the coordination of the physical theory of quantum mechanics, providing a formalized prescription of how to map the theory to propositions about (experimental) observables of the system, i.e. a “topos-theoretic formulation of physics”. However, Bohr toposes currently formalize but one aspect of quantum mechanics, namely “the quantum mechanical phase space” in the form of the quantum observables and the quantum states. The plain Bohr topos does not encode any dynamics, though in the spirit of AQFT a certain presheaf of Bohr toposes on spacetime does encode dynamics [21, 22].
physics, in particular quantum measurements. Suppose a physicist performs a measurement of a certain physical quantity $A$ pertaining to a certain physical system. Suppose this system is in a certain state $x$. The physicist will then in general formulate the result of his measurement as a proposition of the form $A = \mu$, where $\mu$ is the value of $A$ measured, and he will then claim this proposition to be a true statement about the system under investigation. In this there is no difference between classical and quantum physics. There exists, however, an essential difference between the classical and the quantum case which seems to us of fundamental importance from the logical point of view. Namely, the meanings of the physicist’s assertion that $A = \mu$ is true differ in the two cases, classical and quantum. In classical mechanics the proposition $A = \mu$ is a true statement about the physical system in state $x$. In the quantum case it is a true statement too. The crucial difference, however, is that in the quantum case the proposition $A = \mu$ is in general no longer a true statement about the state $x$ but about a certain state $y$ distinct from $x$, namely about the state of the system ‘after measurement”. The reason for this is that quantum measurements generally involve, in contrast to ‘classical’ measurements, a change of state of the system measured. Logically speaking, the situation we have in classical mechanics is this. Given a state $x$ (state of affairs, state of the world…) and some proposition $\alpha$. Then $\alpha$ has some truth value in state $x$. In bivalent logic these truth values are ‘true’ and ‘false’. In multi-valued logic there are more truth values, possibly even infinitely many. The situation in quantum mechanics is different. Given a state $x$ and a proposition $\alpha$. Then $\alpha$ does not necessarily possess any truth value in $x$. Rather it is only in some other state distinct from $x$, namely in the state ‘after measurement’, that it acquires a truth value.

2. Definition of Superlogic

Superlogic manifolds\(^6\) have a structure analogue to $\mathbb{R}^{m|n}$ which is $\mathbb{Z}/2$-graded vector spaces with $\mathbb{R}^m$ as the even subspace and $\mathbb{R}^n$ as the odd subspace or Grassmann’s numbers. We will use the notation $\mathbb{L}^{m|n}$. This superlogic codes the non commutativity and non distributivity of the Quantum mechanics theories and non commutative spaces.

We start by defining the language of super propositional logic which is built up from the following symbols. As we said our superlogic has the $\mathbb{Z}/2\mathbb{Z}[n]$ structure. The elements of this superlogic are the couples $(P, Q)$ with the structure $n$ where $n^2 = 0$. The elements of $\mathbb{L}$ can be writing as $L = P + nQ$.The property of $n^2 = 0$ means that all we can have are terms of at most degree 1 in $n$.

Some remarks are in order. Here, we should think of the symbol ‘+’ as in the case of complex (and quaternion numbers) where $z \in \mathbb{C}$ can be writing as $z = \ldots$
It means that each point of our supermanifolds is described by a couple of coordinates or said differently composed from two entities, the first one is a commuting coordinate (or propositions) and the second one an anti commuting coordinate. Now, by commuting coordinates we should understand that the order in this case is not an issue, i.e. $P_i P_j = P_j P_i$ as in the classical case. Regarding the anti commuting coordinates (or propositions), here we should be careful about the order, as in the case of the commutators in quantum mechanics and quantum fields theories. More precisely $Q_i Q_j = - Q_j Q_i$. For instance, in quantum mechanics the position and momentum would be described by an anti commuting coordinates in our supermanifolds. For instance, this can help us to describe, from logical point of view, the non abelian gauge theories or/and theories unifying the space-time and non abelian gauge formalism and their quantization.

Let $P$ and $Q$ be an element of propositional logic or predicate logic, this superlogic can generate the others logical systems by differentiation, Note that this time we have four values of truth $0, 1, n, 1 + n$. As an illustration we can think of $P$ as property of an object or probability to have such property and $nQ$ the probability to follow some path. As we will see this property will be useful to describe interference in quantum mechanics.

3. The rules

\[ \neg L = \neg P + n\neg Q \]

Regarding the negation operation it should be understood as map which associate to each proposition $P$ its opposite $\neg P$. Sometimes, as in the case of measurement of the spin for instance, we can instantiate the negation map in terms of symmetry: measuring the Spin $S$ or $-S$.

\[ L \land L' = (P + nQ) \land (P' + nQ') = P \land P' + nP \land Q' + nQ \land P' + nQ \land nQ' = P \land P' + nP \land Q' + nQ \land P' = P \land P' + n(P \land Q' + Q \land P') \]

because $n^2 = 0$.

\[ L \lor L' = (P + nQ) \lor (P' + nQ') = P \lor P' + nP \lor Q' + nQ \lor P' + nQ \lor nQ' = P \lor P' + nP \lor Q' + nQ \lor P' = P \lor P' + n(P \lor Q' + Q \lor P') \]
because \( n^2 = 0 \). We will call the second term : the superposition term, it is an essential difference with the other logic. Indeed the superposition terms it come naturally

Distributivity I

\[
L \lor (L' \land L'') = (P + nQ) \lor [(P' + nQ') \land (P'' + nQ'')] \\
= (P + nQ) \lor [(P' \land P'') + (P' \land nQ'')] \\
+ (nQ' \land P'') + (nQ' \land nQ'') \tag{4}
\]

in case \( n = 0 \), we found the old equality:

\[
L \lor (L' \land L'') = (P \lor (P' \land P''))
\]

and

\[
(L \lor L') \land L'' = (P \lor P') \land P''
\]

Distributivity II

\[
(L \lor L') \land L'' = (P \lor P') \land P'' + (P \lor P') \land nQ'' \\
+ (P \lor nQ') \land P'' + n(Q \lor P') \land P'' \tag{5}
\]

in case \( n = 0 \), we found the old equality:

\[
L \lor (L' \land L'') = (P \lor (P' \land P''))
\]

4. Propositional (or Predicates) Supermanifolds

Let \( P_i \) and \( Q_j \), \((i, j) \in I \times J\), \( I \) and \( J \) are set of indexes) be two sets of open Boole algebras of propositions (or predicates), a propositional supermanifold is defined as following:

**Definition 1.** \( \forall(i, j) \in (I \times J), I \times J \rightarrow \{0, 1\}^{P_i} \times \{0, 1\}^{P_j}, \)

where \( \{0, 1\}^{P_i} \) (\( \{0, 1\}^{P_j} \)) denote the maps of \( P_i \) (\( P_j \)) into \( \{0, 1\}^{P_i} \) (\( \{0, 1\}^{P_j} \))(see [23] for more details and definition of transitions functions, cocycle conditions and for the equivalence relation).

**Definition 2.** we provide the Boolean supermanifold \( \mathbb{L} \) with the following operations:

- \( \neg : \mathbb{L} \rightarrow \mathbb{L} \)
- \( \lor : \mathbb{L}^2 \rightarrow \mathbb{L} \)
- \( \land : \mathbb{L}^2 \rightarrow \mathbb{L} \)
- \( \Delta : \mathbb{L} \rightarrow \mathbb{L}^2 \)
⊢: \mathbb{L} \to \{0, 1\}

n : \mathbb{L} \to n\mathbb{L}, \text{ with } n^2 = 0

5. Vectors fields

Definition 3. A vector field over a supermanifolds of propositions \(\mathbb{L}\) is a continuous application \(X:\)

\[
X(L) = X(P + nQ) = X(P) + nX(Q),
\]

\[
X(\neg L) = \neg X(L),
\]

\[
X(L \lor K) = [X(L) \lor K] \land [L \lor X(K)],
\]

\[
X(L \land K) = [X(L) \land K] \lor [L \land X(K)]
\]

where \(\lor\) is 'OR' and \(\land\) is "AND".

Such a set of vectors admit the following operations:

\[
[\neg X](L) = \neg [X(L)]
\]

\[
[X \lor Y](L) = X(L) \lor Y(L)
\]

\[
[X \land Y](L) = X(L) \land Y(L).
\]

6. Superfields space

One may define functions from this vector space to itself, which are called superfields. The above algebraic relations imply that, if we expand our superfield as a power series in \(n\) and then we will only find terms at the zeroth and first orders, because \(n^2 = 0\). Therefore superfields may be written as arbitrary functions of \(P\) multiplied by the zeroth and first order terms in Grassmann coordinates

\[
\Phi(L) = \Phi(P) + nQ\Phi(P) + \Phi(nQ)
\]

If \(\mathcal{M}\) is a supermanifold of dimension \((k, l)\), then the underlying space \(M\) inherits the structure of a differentiable manifold whose sheaf of smooth functions is \(O_{\mathcal{M}}/I\), where \(I\) is the ideal generated by all odd functions. Thus \(M\) is called the underlying space, or the body, of \(\mathcal{M}\). The quotient map \(O_{\mathcal{M}} \to O_{\mathcal{M}}/I\) corresponds to an injective map \(M \to \mathcal{M}\); thus \(M\) is a submanifold of \(\mathcal{M}\).
7. Superlogical Cohomology

Definition 4. A character for a set of propositions $\mathbb{L}$ is a function $\chi$ with values in $\{0, 1\}$ such that:

\[
\begin{align*}
\chi(L) &= \chi(P + nQ) = \chi(P) + \chi(nQ) \\
\chi(\neg L) &= \chi^{op}(L), \\
\chi(L \lor K) &= \chi(L) + \chi(K) \\
\chi(L \land K) &= \chi(L)\chi(K) \\
[L \implies K] &= [\chi(L) \leq \chi(K)]
\end{align*}
\]

Let $\mathcal{X}$ be the set of all characters. We have the following diagrams:

\[
\Delta : \mathcal{X} \to \mathcal{X} \otimes \mathcal{X}, \quad x \otimes x
\]

\[
\epsilon : \mathcal{X} \to \mathcal{F}_2 \mapsto 1,
\]

\[
\mu : \mathcal{X} \otimes \mathcal{X} \to \mathcal{F},
\]

$\mathcal{F}$, are the functions over $\mathbb{L}$ with values in $\mathcal{F}_2$, with $\mathcal{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

Definition 5. Let $\mathcal{A}$ denote the algebra generated by the characters of $\mathbb{L}$. The Cohomology of $\mathbb{L}$ is: $H^*(\Omega(\mathcal{A}, \mathcal{F}_2)$, where $\mathcal{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

An essential property is the compatibility with the exact sequences of Boolean manifolds.

8. Applications to Quantum Mechanics

Suppose we prepare a set of electrons to pass through the two-slit experiment. Part of those electrons are prepared to pass through the first hole and the others through the second hole. But quantum mechanics teach us that the probability of the first part to pass through the second hole is not zero, and vice versa for the second part so we have the situation:

($P_1 + nQ_{12}$) is the probability $P_1$ that the part I of the electrons pass through the first hole with the probability $nQ_{12}$ to pass through the second hole.

($P_2 + nQ_{21}$) is the probability $P_2$ that the second part of the electrons pass through the second hole with the probability $nQ_{21}$ to pass through the first hole.

So when we send the set altogether, we’ll have

($P_1 + nQ_{12}$) $\land$ ($P_2 + nQ_{21}$) = $P_1 \land P_2 + P_1 \land nQ_{21} + nQ_{12} \land P_2 + (\text{terms order } n^2 = 0)$

So our Superlogic, describe exactly the two-slit experience: The first term $P_1 \land P_2$ describe the probability that the first part of the electrons pass through the first hole and the second part of the electrons pass through the
second hole, the second term \((P_1 \land nQ_{21} + nQ_{12} \land P_2)\), the interference terms, describe the superposition of the two parts. So here we get the result from the first order without the need to a density of probability. Our Superlogic describe the two-slit experiment naturally. This same approach can be applied to the EPR experience to describe entanglement.

We can consider the second part (or Grassmaniann coordinate) \((nQ)\) as term which describe the correlation between the objects of our experience, more precisely, it is a way to introduce the possibility of interaction already at the probability level.

9. CONCLUSION

Modern logic studies logical systems as formal systems based on a precisely defined formal language. The concept most central to logic is that of logical consequence. Logical consequence is a relation between two statements \(\alpha\) and \(\beta\) or, more generally, a relation between a set of statements \(\Sigma\) and a statement \(\alpha\). One may synonymously say "\(\alpha\) is a logical consequence of \(\Sigma\)" or "\(\alpha\) follows (logically) from \(\Sigma\)" or "\(\alpha\) can be deduced logically (or is deducible) from \(\Sigma\)". Logical deduction is a vital part of our competence as human beings in both everyday and scientific discourse, and it is one of the seminal achievements of modern mathematical logic to have provided the tools for a mathematically rigorous analysis of the intuitive concept of logical.

In this paper we present an axiomatization which give up the distributivity; and allow to combine the individual phenomena and the interaction one. A probability which take infinitesimal values (positive, but smaller than every positive real number); that allow probabilities to be imprecise (interval-valued, or more generally represented with sets of numerical values). It is a extension of the standard logic by introduce a structure coding the non commutativity of certains spaces and theories.

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