REPRESENTATIONS OF FIELD AUTOMORPHISM GROUPS

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Abstract. This is a common introduction to math.AG/0011176, math.RT/0101170, math.RT/0306333, math.RT/0506043, math.RT/0601028.

In these papers one studies the automorphism group $G$ of an extension $F/k$ of algebraically closed fields, especially in the case of countable transcendence degree and characteristic zero, its smooth linear and semi-linear representations, and their relations to algebraic geometry (birational geometry, motives, differential forms and sheaves).

Compared to the above references there are some new results including

- a description of a separable closure of an extension of transcendence degree one of an algebraically closed field (Proposition A.1, p. 11);
- a “Künneth formula” for the products with curves;
- the semi-simplicity of the $G$-module $\Omega^n_{F/k, reg}$ of regular differential forms of top degree.

The study of field automorphism groups is an old subject. Without any attempt of describing its complicated history, let me just mention that many topological groups are field automorphism groups. Besides the usual Galois groups we meet here (discrete, $p$-adic for $p < \infty$, or finite adelic) groups of points of algebraic groups.

Let $F/k$ be a field extension of countable (this will be the principal case) or finite transcendence degree $n, 0 \leq n \leq \infty$, and $G = G_{F/k}$ be its automorphism group. Following [Jac, PSS, Sh, I] (and generalizing the case [KI] of algebraic extension), consider $G$ as a topological group with the base of open subgroups given by the stabilizers of finite subsets of $F$. Then $G$ is a totally disconnected Hausdorff group, and for any intermediate subfield $k \subseteq K \subseteq F$ the topology on $G_{F/K}$ coincides with the restriction of the topology on $G$. There are maps: from the set of intermediate subfields in $F/k$ to the set of closed subgroups of $G$, $K \mapsto G_{F/K} := \text{Aut}(F/K)$, and from the set of closed subgroups in $G$ to the set of intermediate subfields in $F/k$, $H \mapsto F^H$. They are mutually inverse to each other in the Galois extension case. If $n < \infty$ then $G$ is locally compact.

Following the very general idea, not only in Mathematics, that a “sufficiently symmetric” system is determined by a representation of its symmetry group, one tries to compare various “geometric categories over $k$” with various categories of representations of $G$.

To ensure that the representation theory of $G$ is rich enough, $F$ should be “big enough”, e.g. algebraically closed. So $F$ is “the function field of the universal tower of $n$-dimensional $k$-varieties”, if $n < \infty$. In that case each perfect subfield $L$ of $F$ containing $k$ is the fixed field of the subgroup $G_{F/L}$ of $G$, and $G$ contains, in particular, the groups $G_{L/k}$ as its sub-quotients.

Usually (unless it is not stated otherwise), $k$ will be algebraically closed (in order to avoid already complicated enough Galois theory) of characteristic zero.

One of the main motivations is the calculation of integrals of meromorphic differential forms $\omega$ on projective complex varieties. To calculate such an integral, one can transfer $\omega$ to other varieties via correspondences. In coordinates this looks as an algebraic change of variables. We may suppose that all function fields are contained in a common field $F$. Then the problem of description of the properties of the (iterated) integrals of $\omega$ (of $\omega_1, \ldots, \omega_N$) becomes related with
determining the structure of the $G$-submodule in the algebra of Kähler differentials $Ω_{F/k}^*$ (resp., in $Ω_{F/k}^* ⊗_k · · · ⊗_k Ω_{F/k}^*$), generated by $ω$ (resp., by $ω_1 ⊗ · · · ⊗ ω_N$).

For example, the irreducible subquotients of the $G$-module $Ω_{F/k,\text{closed}}^1$ of closed Kähler differentials are related to the simple algebraic commutative groups over $k$, cf. Proposition 2.10

In the opposite direction, to clarify relations, so far conjectural, between the motives and the cohomologies, one has to link the most interesting (from the geometric point of view) representations – admissible and “homotopy invariant” – and the Kähler differentials. Conjecturally, the irreducible ones among them are contained in the algebra of differential forms $Ω_{F/k}^*$, if $n = ∞$.

0.1. Some general notations, conventions and goals. Let $F/k$ be an extension of countable or finite transcendence degree $n$, $1 ≤ n ≤ ∞$, of algebraically closed fields of characteristic zero (by default), and $G = G_{F/k}$ be its automorphism group endowed with the above topology.

We study the structure of $G$, its linear and semi-linear representations (with open stabilizers), and their relations to algebraic geometry (birational geometry, motives, differential forms and sheaves) and to automorphic representations. In particular, we look for analogues of known results for $p$-adic (and more generally, locally compact) groups.

1. Structure of $G$

It is well-known (Jac, PSS, Sh, I) that the group $G$ is locally compact if and only if $n < ∞$.

Theorem 1.1 (R1). (1) The subgroup $G^\circ$ of $G$, generated by the compact subgroups, is open and topologically simple, if $n < ∞$. If $n = ∞$ then $G^\circ$ is dense in $G$.

(2) Any closed normal proper subgroup of $G$ is trivial, if $n = ∞$, i.e. $G$ is topologically simple.

Remarks. 1. In fact, Theorem 1.1 holds for any extension $F/k$ of algebraically closed fields of arbitrary characteristic, cf. R1. Moreover, if $n = 1$ and $\text{char}(k) \neq 0$ then the separable closure of $k(x)$ in $F$ is generated by the $G^\circ$-orbit of $x$ for any $x ∈ F\backslash k$, cf. R5.

2. An argument of La shows that $G$ is simple as a discrete group provided that transcendence degree $F$ over $k$ is not countable.

3. If $n < ∞$, the left $G$-action on the one-dimensional oriented $Q$-vector space of right-invariant measures on $G$ gives rise to a surjective homomorphism, the modulus, $χ : G → Q^X_+$, which is trivial on $G^\circ$. However, I do not know even, whether the discrete group $\ker χ/G^\circ$ is trivial. If it is trivial for $n = 1$ then it is trivial in general, cf. R1.

1.1. Closed, open and maximal proper subgroups; Galois theories. The classical morphism $β : \{\text{subfields } F \text{ over } k\} ↦ \{\text{closed subgroups of } G\}$, given by $K ↦ G_{F/K}$, is injective, inverts the inclusions, transforms the compositum of subfields to the intersection of subgroups, and respects the units: $k ↦ G$. The image of $β$ is stable under the passages to sup-/sub- groups with compact quotients; $β$ identifies the subfields over which $F$ is algebraic with the compact subgroups of $G$ (Jac, PSS, Sh, I).

In particular, the proper subgroups in the image of $β$ are the compact subgroups if $n = 1$.

The map $H ↦ F^H$, left inverse of $β$, inverts the order, but does not respect the monoid structure.

In R6, a morphism of partially ordered commutative associative unitary monoids (transforming the intersection of subgroups to the algebraic closure of the compositum of subfields)

$$α : \{\text{open subgroups of } G\} ↦ \{\text{algebraically closed subfields of } F \text{ of finite transcendence degree over } k\}$$

is constructed. It is determined uniquely by the condition $G_{F/α(U)} ⊆ U$ and the transcendence degree of $α(U)$ over $k$ is minimal.
It is shown in [R6] that for any non-trivial algebraically closed extension $L \neq F$ of $k$ of finite transcendence degree in $F$ the normalizer in $G$ of $G_{F/L}$ (which is evidently open) is maximal among the proper subgroups of $G$. In the case $n = \infty$ any maximal open proper subgroup is of this type.

As a consequence, one gets a complete, though not very explicit, Galois theory of algebraically closed extensions of countable transcendence degree (a question of Krull, [K2]), i.e. a construction of all subgroups $H$ of $G$ coincident with the automorphism groups of $F$ over the fixed subfields $F^H$.

Another type of closed non-open maximal proper subgroups is given by the stabilizers of rank one discrete valuations in the case of arbitrary transcendence degree. They are useful in relating representations of $G$ to functors on categories of smooth $k$-varieties, cf. [R1].

1.2. Automorphisms of $G$. The group $G$ is quite rigid in the sense that the group of its continuous automorphisms is “of the same size” as $G$. Namely, it coincides with the group of field automorphisms of $F$ preserving $k$. If $n \geq 2$ this follows from results of F.A.Bogomolov. If $n = 1$ this is shown in [R7].

It would be highly interesting to identify the class of “rational” representations of $G$, i.e. those whose isomorphism class does not change under any continuous automorphism of $G$. In particular, if $L \subset F$ is a field of automorphic functions (of all levels) the functor $H^0(G_{F/L}, -)$ should relate representations of $G$ to automorphic representations.

2. How to translate geometric questions to the language of representation theory?

Depending on type of geometric questions we shall consider one of the following threeer categories of representations of $G$: $SmG \supset I_G \supset Adm$, roughly corresponding to birational geometry over $k$, birational motivic questions (like on the structure of Chow groups of 0-cycles) and “finite-dimensional” birational motivic questions (such as description of “classical” motivic categories).

2.1. $SmG$. Usually an “algebro-geometric datum” $D$ over $F$, a universal domain over $k$ in the sense of Weil, consists of a finite number of polynomial equations involving a finite number of coefficients $a_1, \ldots, a_N \in F$, and the group $G$ acts on the set of “similar” data. Then the stabilizer of $D$ in $G$ is open, since contains $G_{F/k(a_1, \ldots, a_N)}$.

For a $k$-variety $X$, its $F$-subvarieties are examples of such data.

In particular, the $\mathbb{Q}$-vector space $\mathbb{Q}[X(F)]$ of 0-cycles on $X \times_k F$ is a $G$-module, Such representation is huge, but this is just a starting point.

Note that it is smooth, i.e. its stabilizers are open, so all representations we are going to consider will be smooth.

Conversely, as it follows from [R1], Lemma 3.3, any smooth representation of $G$ with cyclic vector is a quotient of the $G$-module $\mathbb{Q}[[k(X)_{/k} F]]$ of “generic” 0-cycles on $X_F$ (equivalently, formal $\mathbb{Q}$-linear combinations of embeddings of the function field $k(X)$ into $F$ over $k$), i.e., 0-cycles outside of the union of the divisors on $X$ defined over $k$, for an appropriate irreducible variety $X$ of dimension $\leq n$ over $k$.

Remarks. 1. One has $\mathbb{Q}[X(F)] = \bigoplus_{x \in X} \mathbb{Q}[[k(x)_{/k} F]]$, so $\mathbb{Q}[X(F)]$ reflects rather the class of $X$ in the Grothendieck group $K_0(Var_k)$ of partitions of varieties over $k$ than $X$ itself.

2. It is not clear, whether the birational type of $X$ is determined by the $G$-module of generic 0-cycles on $X_F$. There exist pairs of non-birational varieties $X$ and $Y$, whose $G$-modules of generic 0-cycles have the same irreducible subquotients, cf. [R1]. Namely, $X = Z \times \mathbb{P}^1$ and $Y = Z' \times \mathbb{P}^1$, where $Z'$ is a twofold cover of $Z$, is such a pair. What is in common between $X$ and $Y$ in this example, is that their primitive motives (see below) coincide (and vanish).
However, one can extract “birational motivic” invariants “modulo isogenies”, such as \( \text{Alb}(X), \ \Gamma(X, \Omega^*_{X/k}) \), out of \( \mathbb{Q}[\{ k(X) \hookrightarrow F \}] \), cf. Theorem 2.5 [5] [5].

Denote by \( S_{mG} \) the category of smooth representations of \( G \) over \( \mathbb{Q} \).

It follows from the topological simplicity of \( G \) (Theorem 1.1) that in the case \( n = \infty \) any finite-dimensional smooth representation of \( G \) is trivial.

2.2. \textit{Adm}. Now consider a more concrete geometric category: the category of motives.

(Effective) pure covariant motives are pairs \((X, \pi)\) consisting of a smooth projective variety \( X \) over \( k \) with irreducible components \( X_j \) and a projector \( \pi = \pi_i^j \in \bigoplus_j B^{\dim X_j}(X_j \times_k X_j) \) in the algebra of correspondences on \( X \) modulo numerical equivalence. The morphisms are defined by \( \text{Hom}((X', \pi'), (X, \pi)) = \bigoplus_{i,j} \pi_j \cdot B^{\dim X_j}(X_j \times_k X'_j) \cdot \pi'_i \). The category of pure covariant motives carries an additive and a tensor structures:

\[
(X', \pi') \boxplus (X, \pi) := (X' \coprod X, \pi' \oplus \pi), \quad (X', \pi') \otimes (X, \pi) := (X' \times_k X, \pi' \times_k \pi).
\]

A primitive \( q \)-motive is a pair \((X, \pi)\) as above with \( \dim X = q \) and \( \text{Hom}(Y \times \mathbb{P}^1, (X, \pi)) := \pi \cdot B^q(Y \times_k Y \times \mathbb{P}^1) = 0 \) for any smooth projective variety \( Y \) over \( k \) with \( \dim Y < q \). For instance, due to the Lefschetz theorem on \((1,1)\)-classes, the category of the pure primitive 1-motives is equivalent to the category of abelian varieties over \( k \) with morphisms tensored with \( \mathbb{Q} \). It is a result of Jannsen [4] that pure motives form a semi-simple abelian category, and it follows from this that any pure motive admits a “primitive” decomposition \( \bigoplus_{i,j} M_{ij} \otimes \mathbb{L}^i \), where \( M_{ij} \) is a primitive \( j \)-motive and \( \mathbb{L} = (\mathbb{P}^1, \mathbb{P}^1 \times \{ 0 \}) \) is the Lefschetz motive.

\textbf{Definition.} A representation \( W \) of a topological group is called \textit{admissible} if it is smooth and the fixed subspaces \( W^U \) are finite-dimensional for all open subgroups \( U \).

Denote by \( \text{Adm} \) the category of admissible representations of \( G \) over \( \mathbb{Q} \).

\textbf{Theorem 2.1 (R1)}. \textit{Adm} is a Serre subcategory in \( S_{mG} \).

In other words, \( \text{Adm} \) is abelian, stable under taking subquotients (this is the point in the case \( n = \infty! \)) in the category of representations of \( G \), and under taking extensions in \( S_{mG} \).

\textbf{Theorem 2.2 (R1)}. There is a fully faithful functor \( \mathbb{B}^* \) when \( n = \infty \):

\[
\{ \text{pure covariant motives over } k \} \rightarrow \mathbb{B}^* \rightarrow \{ \text{graded semi-simple admissible } G\text{-modules of finite length} \}.
\]

The grading corresponds to powers of the motive \( \mathbb{L} \) in the “primitive” decomposition above.

Roughly speaking, the functor \( \mathbb{B}^* \) is defined by spaces of 0-cycles defined over \( F \) modulo “numerical equivalence over \( k' \)”. More precisely, \( \mathbb{B}^* = \bigoplus_j \lim_{\rightarrow} \text{Hom}([L]^{\text{prim}} \otimes \mathbb{L}^j, -) \) is a graded direct sum of pro-representable functors. Here \( L \) runs over all subfield of \( F \) of finite type over \( k \), and \( [L]^{\text{prim}} \) is the quotient of the motive of any smooth projective model of \( L \) over \( k \) by the sum of all submotives of type \( M \otimes \mathbb{L} \) for all effective motives \( M \).

\textbf{Examples}. The motive of the point \( \text{Spec}(k) \) is sent to the trivial representation \( \mathbb{Q} \) in degree 0. The motive of a smooth proper curve \( C \) over \( k \) is sent to \( \mathbb{Q} \oplus J_C(F)/J_C(F) \oplus \mathbb{Q}[1] \), where \( J_C \) is the Jacobian of \( C \) and \( \mathbb{Q}[1] \) denotes the trivial representation in degree 1.

So, this inclusion is already a good reason to study admissible representations.
Moreover, it is expected that

**Conjecture 2.3.** The functor $\mathbb{B}^\bullet$ is an equivalence of categories.

Of course, it would be more interesting to describe in a similar way the abelian category $\mathcal{M}$ of mixed motives over $k$, whose semi-simple objects are pure. This is one more reason to study the category $\mathcal{Adm}$ of admissible representations of $G$.

**Proposition 2.4 (RI).** Assuming $\nu = \infty$, for any $W \in \mathcal{Adm}$, any abelian variety $A$ over $k$ and, conjecturally, for any effective motive $M$ one has

\[
\begin{align*}
\text{Ext}^0_{\mathcal{Adm}}(A(k), W) &= 0 \\
\text{Ext}^1_{\mathcal{Adm}}(A(k), W) &= \text{Hom}_G(A(k), W^G) \\
\text{Ext}^2_{\mathcal{Adm}}(A(k), W) &= 0 \\
\text{Ext}^1_{\mathcal{M}}(H^1(A), M) &= 0 \\
\text{Ext}^2_{\mathcal{M}}(H^1(A), M) &= 0
\end{align*}
\]

As $A(F)/A(k)$ is a canonical direct “$H_1$”-summand of $\mathbb{B}^\bullet(A)$, we see that admissible representations of finite length should be related to effective motives. At least the Ext’s between some irreducible objects are dual.

2.3. $\mathcal{I}_G$. The formal properties of $\mathcal{Adm}$ are not very nice. In particular, to prove Theorem 2.1 and Proposition 2.4 and to give an evidence to Conjecture 2.3 one uses the inclusion of $\mathcal{Adm}$ to a bigger full subcategory in the category of smooth representations of $G$.

**Definition.** An object $W \in \mathcal{Sm}_G$ is called “homotopy invariant” (in birational sense) if $W^{G_F/L} = W^{G_{F/L'}}$ for any purely transcendental subextension $L'/L$ in $F/k$. Denote by $\mathcal{I}_G$ the full subcategory in $\mathcal{Sm}_G$ with “homotopy invariant” objects.

**Remark.** In this definition it suffices to consider only $L$’s of finite type over $k$, cf. [RI], §6.

A typical object of $\mathcal{I}_G$ is the $\mathbb{Q}$-space $CH^q(X_F)_{\mathbb{Q}}$ of cycles of codimension $q$ on $X \times_k F$ modulo rational equivalence, for any smooth variety $X$ over $k$.

**Theorem 2.5 (RI, $n = \infty$).**

1. The category $\mathcal{I}_G$ is a Serre subcategory in $\mathcal{Sm}_G$.
2. $\mathcal{Adm} \subset \mathcal{I}_G$, i.e. any admissible representation of $G$ is “homotopy invariant”.
3. The inclusion $\mathcal{I}_G \hookrightarrow \mathcal{Sm}_G$ admits a left and a right adjoints $\mathcal{I}, -^\vee : \mathcal{Sm}_G \rightarrow \mathcal{I}_G$.
4. The objects $C_{k(X)} := \mathcal{I}[k(X) \hookrightarrow F]$ for all irreducible varieties $X$ over $k$ form a system of projective generators of $\mathcal{I}_G$.
5. For any smooth proper variety $X$ over $k$ there is a canonical filtration $C_{k(X)} \supset \mathcal{F}^1 \supset \mathcal{F}^2 \supset \ldots$, canonical isomorphisms $C_{k(X)}/\mathcal{F}^1 = \mathbb{Q}$ and $\mathcal{F}^1/\mathcal{F}^2 = \text{Alb}(X_F)_{\mathbb{Q}}$, and a non-canonical splitting $C_{k(X)} \cong \mathbb{Q} \oplus \text{Alb}(X_F)_{\mathbb{Q}} \oplus \mathcal{F}^2$. The term $\mathcal{F}^2$ is determined by these conditions together with $\text{Hom}_G(\mathcal{F}^2, \mathbb{Q}) = \text{Hom}_G(\mathcal{F}^2, A(F)/A(k)) = 0$ for any abelian variety $A$ over $k$.
6. For any smooth proper variety $X$ over $k$ there is a canonical surjection $C_{k(X)} \rightarrow CH_0(X_F)_{\mathbb{Q}}$, which is injective if $X$ unirational over a curve.\(^1\)
7. There exist (co-) limits in $\mathcal{I}_G$.

The following two conjectures link $\mathcal{I}_G$ with algebraic geometry and topology.

**Conjecture 2.6 (RI).** If $n = \infty$ then the natural surjection $C_{k(X)} \rightarrow CH_0(X \times_k F)_{\mathbb{Q}}$ is an isomorphism for any smooth proper variety $X$ over $k$.

\(^1\)and in some other cases when $CH_0(X)$ is “known”
Remarks. 1. One deduces from Theorem 2.5 (6) a description of the category of abelian varieties over \( k \) with the groups of morphisms tensored with \( \mathbb{Q} \) as a full subcategory of \( \text{Adm}_G \subset I_G \) in terms of a functorial increasing “level” filtration \( N \) on smooth \( G \)-modules introduced in \( R4 \).

2. The conjecture of Bloch and Beilinson \( [B2] \) and \( [B3] \), Lecture 1) on the “motivic” filtration on the Chow groups together with the semi-simplicity “standard” conjecture of Grothendieck (asserting that numerical and homological equivalence coincide for smooth proper varieties), imply that “numerical” equivalence coincides with rational equivalence on the cycles on \( \text{Spec} \) of the tensor product of two fields over a common subfield \( ([B1] \) \([R4]) \). If combined with Conjecture 2.6, this would give that \( \mathbb{B}^* \) is an equivalence of categories (Conjecture 2.7, cf. also “Corollary” 2.7.1 below.

3. Define a binary non-associative operation \( \otimes_I \) on \( Sm_G \) by \( W_1 \otimes_I W_2 := I(W_1 \otimes W_2) \).

It follows from Conjecture 2.6 that there is a canonical isomorphism, the “Künneth formula”: \( C_k(X \times_k Y) \cong C_k(X) \otimes_I C_k(Y) \) for any pair of irreducible \( k \)-varieties \( X, Y \). An evidence (and an unconditional proof in the case when \( X \) is a curve) for this can be found in \( R5 \).

It would follow from the “Künneth formula” that the restriction of \( \otimes_I \) to \( I_G \) is a commutative associative tensor structure, and that the class of projective objects is stable under \( \otimes_I \), cf. \( R4 \).

It would be interesting to find a “semi-simple graded” version of \( \otimes_I \) to make \( \mathbb{B}^* \) a tensor functor.

**Conjecture 2.7** \( [R2] \). Any irreducible object of \( I_G \) is contained in the algebra \( \Omega^*_{F/k} \) if \( n = \infty \).

“Corollary” 2.7.1 \( [R2] \).

- If numerical equivalence coincides with homological then \( \mathbb{B}^* \) is an equivalence of categories.
- Any irreducible object of \( I_G \) is admissible if \( n = \infty \). So “\( I_G \approx \text{Adm} \)”.

Conjecture 2.7 is one of the main motivations for the study of semi-linear representations of \( G \), cf. \( R3 \). It has also the following geometric corollary, conjectured by Bloch.

“Corollary” 2.7.2 \( [R2] \). If \( \Gamma(X, \Omega^2_{X/k}) = 0 \) for a smooth proper variety \( X \) over \( k \) then the Albanese map induces an isomorphism \( CH_0(X)^0 \cong \text{Alb}(X) \). In that case \( C_k(X) = CH_0(X_F) \).

Remarks. 1. There is a locally compact group \( H \) and a continuous injective homomorphism with dense image \( H \longrightarrow G \) such that \( I_G \) admits an explicit description as a full subcategory of \( Sm_H \) stable under taking subquotients (but not extensions).

The category \( Sm_H \) may be useful in the study of left derivatives of additive functors.

2. \( I_G \) is equivalent to the category of non-degenerate modules over an associative idempotented algebra, \( R3 \).

2.4. Differential forms. In an attempt to compare various cohomology theories \( H^* \), one can associate with them some \( G \)-modules, like \( H^*(F) := \lim \rightarrow \ (H^*(U)) \), where \( U \) runs over spectra of smooth subalgebras in \( F \) of finite type over \( k \), or the image \( H^*_c(F) \) in \( H^*(F) \) of \( \lim \rightarrow \ (H^*(U)) \), where \( X \) runs over smooth proper models of subfields in \( F \) of finite type over \( k \).

Clearly, \( H^*_c(F) \) is an admissible representation of \( G \) over \( H^*(k) \). It would follow from the semi-simplicity standard conjecture that it is semi-simple. For instance, it replaces reference to the semi-simplicity standard conjecture in Remark 2 on p. 5.

In the case \( H^* = H^{dR/k}_* \) of the de Rham cohomology the graded quotients of the (descending) Hodge filtration on \( H^{dR/k,c}_*(F) \) are \( H^{p,q-p}_{F/k} = \lim \rightarrow \ (H^{p-1}(D, \Omega^{q-p-1}_D/k)) \longrightarrow H^p(X, \Omega^{q-p}_X/k) \), where \( (X, D) \) runs over pairs consisting of a smooth proper variety \( X \) with \( k(X) \subset F \) and a normal crossing divisor \( D \) on \( X \) with smooth irreducible components. More particularly, \( H^{q,0}_{F/k} = \Omega^{q}_{F/k, \text{reg}} \subset H^{q}_{dR/k,c}(F) \) is the \( G \)-submodule spanned by the spaces \( \Gamma(X, \Omega^*_X/k) \) of regular differential forms on all smooth projective \( k \)-varieties \( X \) with the function fields embedded into \( F \).
Proposition 2.8 ([R5]). Suppose that the cardinality of \( k \) is at most continuum. Fix an embedding \( \iota : k \hookrightarrow \mathbb{C} \) to the field of complex numbers. Then

- there is a \( \mathbb{C} \)-anti-linear canonical isomorphism (depending on \( \iota \)) \( H_{dR}^{p,q}(F) \otimes_{k,\iota} \mathbb{C} \cong H_{F/k}^{p,q} \otimes_{k,\iota} \mathbb{C} \);
- the representation \( H_{dR/k,c}^{n}(F) \) (and thus, \( \Omega_{F/k,reg}^{n} \)) is semi-simple for any \( 1 \leq n < \infty \).

Recall (Theorem 2.5(3)), that for any \( W \in \text{Sm}_G \) its maximal subobject of \( W \) in \( \mathcal{I}_G \), the “homotopy invariant” part of \( W \), is denoted by \( W^{(0)} \). The following fact gives one more evidence for the cohomological nature of the objects of \( \mathcal{I}_G \), since \( \Omega_{F/k,reg}^{*} \) is the “cohomological part” of \( \bigotimes_{F} \Omega_{F/k}^{*} \).

Proposition 2.9 ([R2]). If \( n = \infty \) then \( (\bigotimes_{F} \Omega_{F/k}^{1})^{(0)} = \Omega_{F/k,reg}^{*} \).

Proposition 2.10 ([R5]). For any \( 1 \leq n \leq \infty \) the representation \( H_{dR/k}^{n}(F) \) modulo the sum of submodules isomorphic to \( F^{\times}/k^{\times} \) is a direct sum of \# \( k \) copies of \( A(F)/A(k) \) for all isogeny classes \( A \) of simple abelian \( k \)-varieties. In particular, \( \Omega_{F/k,reg}^{*} \) is semi-simple.

This suggests that the isomorphism classes of irreducible subquotients of \( H_{dR}^{*}(F) \) can be naturally identified with the irreducible effective primitive motives, and that the isomorphism classes of irreducible subquotients of \( H^{*}(F) \) are related to more general irreducible effective motives, such as the Tate motive \( \mathbb{Q}(-1) \) in the case of \( H_{dR/k}^{1}(F) \).

3. FROM LINEAR TO SEMI-LINEAR REPRESENTATIONS

The representation \( \Omega_{F/k}^{*} \) of \( G \) is also an \( F \)-vector space endowed with a semi-linear \( G \)-action.

Definition. A semi-linear representation of \( G \) over \( F \) is an \( F \)-vector space \( V \) endowed with an additive \( G \)-action \( G \times V \to V \) such that \( g(fv) = gf \cdot gv \) for any \( g \in G \), \( v \in V \) and \( f \in F \).

Denote by \( \mathcal{C} \) the category of smooth semi-linear representations of \( G \) over \( F \).

It is well-known after Hilbert, Tate, Sen, Fontaine... that the semi-linear representations is a powerful tool in the study of Galois representations. We try to use them in non-Galois context.

In some respects \( \mathcal{C} \) is simpler than \( \text{Sm}_G \). In particular, it follows from Hilbert’s Theorem 90 that the category \( \mathcal{C} \) admits a countable system of cyclic generators: \( F[G/G_{F/K_m}] \), where \( K_m \) is a purely transcendental extension of \( k \) in \( F \) of transcendence degree \( m \).

Once again, we are interested in linear representations of \( G \), especially in irreducible ones, and more particularly, in irreducible “homotopy invariant” representations, i.e. objects of \( \mathcal{I}_G \).

The problem of describing (the irreducible objects of) \( \text{Sm}_G \) could be split into describing (the irreducible objects of) \( \mathcal{C} \) and their linear submodules.

For example, all representations \( A(F)/A(k) \) of \( G \) for all abelian \( k \)-varieties \( A \) (i.e. corresponding to all pure 1-motives) are contained in the irreducible object \( \Omega_{F/k}^{1} \) of \( \mathcal{C} \).

Suppose from now on that \( n = \infty \).

One has the faithful forgetful functor \( \mathcal{C} \to \text{Sm}_G(k) \) admitting a left adjoint functor of extending of coefficients to \( F \): \( \text{Sm}_G(k) \to \mathcal{C} \), where \( \text{Sm}_G(k) \) is the category of smooth representations of \( G \) over \( k \), so \( W \to (W \otimes_k F) \). The functor \( F \otimes_k \) is not full and does not respect the irreducibility.

However, if \( W \) is irreducible, there is an irreducible semi-linear quotient \( V \) of \( W \otimes F \) with an inclusion \( W \subset V \), so any irreducible object of \( \text{Sm}_G \) is contained in an irreducible object of \( \mathcal{C} \).

This gives a hint that it might be sufficient for the study of some categories of \( \mathbb{Q} \)-linear representations of \( G \) to know the structure of some “relatively small” full sub-category of \( \mathcal{C} \).
The following claim suggests the category $C$ is "more complicated" than $\mathcal{I}_G(k)$. However, this should be compared with Lemma 3.3.

Lemma 3.1 (R3, Lemma 0.1). The functor $\mathcal{I}_G(k) \xrightarrow{\phi} C$ is fully faithful.

Another, though a weaker, but a little bit more explicit condition on the semi-linear quotients of $W \otimes F$ for $W \in \mathcal{I}_G$ is given in the next section 3.1.

3.1. Valuations and associated functors (R4). In order to associate functors on categories of $k$-varieties to representations of $G$ one can try to "approximate" rings by their subfields. Evidently, this does not work literally, but apparently works in the case of discrete valuation rings of $F$.

Let $v : F^\times/k^\times \longrightarrow \mathbb{Q}$ be a discrete valuation, $\mathcal{O}_v$ be the valuation ring, $\mathfrak{m}_v = \mathcal{O}_v - \mathcal{O}_v^\circ$ be the maximal ideal, and $\kappa(v)$ be the residue field. Denote by $\mathcal{P}_F$ the set of all such valuations.

Set $G_v := \{\sigma \in G \mid \sigma(\mathcal{O}_v) = \mathcal{O}_v\}$. This is a closed subgroup in $G$. The $G_v$-action on $\kappa(v)$ induces a homomorphism $G_v \longrightarrow G_{\kappa(v)/k}$.

**Proposition 3.2.** For any discrete valuation $v \in \mathcal{P}_F$ the additive functor $(-)_v : \mathcal{S}m_G \longrightarrow \mathcal{S}m_{G_v}$, $W \mapsto W_v := \bigoplus_{F \subset \mathcal{O}_v} W^{G_v/F} \subseteq W$, is fully faithful and preserves surjections and injections.

Then the additive subfunctor $\Gamma : \mathcal{S}m_G \longrightarrow \mathcal{S}m_G$ of the identity functor, defined by $W \mapsto \Gamma(W) := \bigcap_{v \in \mathcal{P}_F} W_v$, preserves injections.

**Example.** $\Gamma(\Omega^{1}_{F/k}) \cong \bigoplus_{A} (A(F)/A(k)) \otimes_{\text{End}(A)} \Gamma(A, \Omega_{A/k}^{1})$, where $A$ runs over the set of isogeny classes of simple abelian varieties over $k$.

Lemma 3.3. The compositions $\mathcal{I}_G(k) \xrightarrow{\phi} C \xrightarrow{\Gamma \text{for}} \mathcal{S}m_G(k)$ and $\mathcal{I}_G \xrightarrow{\Gamma} \mathcal{S}m_G \xrightarrow{\Gamma} \mathcal{S}m_G$ are identical.

**Remark.** This implies that any semi-linear quotient $V$ of $W \otimes F$ with $W \in \mathcal{I}_G$ (in particular, any irreducible semi-linear representation $V$ containing a "homotopy invariant" representation), is "globally generated", i.e., $\Gamma(V) \otimes F \longrightarrow V$ is surjective.

This is the condition one can impose on the class of "interesting" semi-linear representations. There are some reasons to expect that $(-)_v$ is exact, cf. R6. This would imply some nice properties of the category of "globally generated" semi-linear representations.

3.2. Admissible semi-linear representations. In the study of representations of any group, it is natural to start with the finite-dimensional representations.

**Theorem 3.4 (R1).** Any finite-dimensional smooth semi-linear representation of $G$ over $F$ is trivial, if $n = \infty$.

A natural extension of the notion of finite-dimensional semi-linear representation in the case $n = \infty$ is the notion of admissible semi-linear representation.

**Definition.** A smooth semi-linear representation $V$ of $G$ over $F$ is called admissible if, for any open subgroup $U \subseteq G$, the fixed subspace $V^U$ is finite-dimensional over the fixed subfield $F^U$ (or equivalently, $\dim_K V^{G/F} < \infty$ for any subfield $L \subseteq F$ of finite type over $k$).

**Theorem 3.5 (R2, R3).** The admissible semi-linear representations of $G$ over $F$ form an abelian tensor (but not rigid) category, denoted by $A$.

The functor $H^0(G_{F/L}, -)$ is exact on $A$ for any subfield $L \subseteq F$, so $F$ is a projective object of $A$.

The latter give an example of an admissible semi-linear representation.
Example. Let the ideal $\mathfrak{m} \subset F \otimes_{\mathbb{Q}} F$ be the kernel of the multiplication map $F \otimes_{\mathbb{Q}} F \to F$. Consider the powers of the ideal $\mathfrak{m}$ as objects of $\mathcal{C}$ with the $F$-multiplication via $F \otimes_{\mathbb{Q}} \mathbb{Q}$. Then $\mathfrak{m}^s/\mathfrak{m}^{s+1} = \text{Sym}^s_k \Omega^1_F$, and the objects $\text{Sym}^s_k \Omega^1_F/k$ are admissible for all $s \geq 1$.

In the case $k = \overline{\mathbb{Q}}$, the field of algebraic complex numbers, the category $\mathcal{A}$ is equivalent to the category of “coherent” sheaves in smooth topology $\mathfrak{S}m_k$ on $k$. (The underlying category of $\mathfrak{S}m_k$ is the smooth morphisms of smooth $k$-varieties, and the coverings are coverings by images; “coherent” means: a sheaf of $\mathcal{O}$-modules such that its restriction to the small Zariski site of any smooth $k$-variety is coherent.) Moreover, $\mathcal{A}$ admits the following explicit description, cf. [R3].

- The sum of the images of the $F$-tensor powers $\bigotimes_F^\bullet \mathfrak{m}$ under all morphisms in $\mathcal{C}$ defines a decreasing filtration $W^\bullet$ on the objects of $\mathcal{A}$ such that its graded quotients $gr^q_V$ are finite direct sums of direct summands of $\bigotimes^j_F \Omega^1_F$. This filtration is evidently functorial and multiplicative: $(W^pV_1) \otimes_F (W^qV_2) \subseteq W^{p+q}(V_1 \otimes_F V_2)$ for any $p, q \geq 0$ and any $V_1, V_2 \in \mathcal{A}$.
- $\mathcal{A}$ is equivalent to the direct sum of the category of finite-dimensional $k$-vector spaces and its abelian full subcategory $\mathcal{A}^0$ with objects $V$ such that $V^G = 0$.
- Any object $V$ of $\mathcal{A}^0$ is a quotient of a direct sum of objects (of finite length) of type $\bigotimes^q_F (\mathfrak{m}/\mathfrak{m}^s)$ for some $q, s \geq 1$.
- If $V \in \mathcal{A}$ is of finite type then it is of finite length and $\dim_k \text{Ext}^j_{\mathcal{A}}(V, V') < \infty$ for any $j \geq 0$ and any $V' \in \mathcal{A}$; if $V \in \mathcal{A}$ is irreducible and $\text{Ext}^1_{\mathcal{A}}(\mathfrak{m}/\mathfrak{m}^q, V) \neq 0$ for some $q \geq 2$ then $V \cong \text{Sym}^q_F \Omega^1_F$ and $\text{Ext}^1_{\mathcal{A}}(\mathfrak{m}/\mathfrak{m}^q, V) \cong k$.
- $\mathcal{A}^0$ has no projective objects, but $\bigotimes^q_F \mathfrak{m}$ are its “projective pro-generators”: the functor $\text{Hom}_C(\bigotimes^q_F \mathfrak{m}, -) = \lim \text{Hom}_A(\bigotimes^q_F (\mathfrak{m}/\mathfrak{m}^N), -)$ is exact on $\mathcal{A}$ for any $q$.

Representations of particular interest are admissible ones. Though tensoring with $F$ does not transform them to admissible semi-linear representations, there exists a similar functor in the opposite direction, faithful at least if $k = \overline{\mathbb{Q}}$.

It is explained in [R2], that when $k = \overline{\mathbb{Q}}$, for any object $V$ of $\mathcal{A}$ and any smooth $k$-variety $Y$, embedding of the generic points of $Y$ into $F$ determines a locally free coherent sheaf $\mathcal{V}_Y$ on $Y$ with the generic fibre $V^{G_F/k(Y)}$. Moreover, for any dominant morphism $X \to Y$ of smooth $k$-varieties, the inclusion of the generic fibres $k(X) \otimes_{k(Y)} V^{G_F/k(Y)} \subseteq V^{G_F/k(X)}$ induces an injection of the coherent sheaves $\pi^* \mathcal{V}_Y \hookrightarrow \mathcal{V}_X$ on $X$, which is an isomorphism if $\pi$ is étale.

For any $V \in \mathcal{A}$ the space $\Gamma(Y, \mathcal{V}_Y)$ is a birational invariant of smooth proper $Y$. Then we get a left exact functor $\mathcal{A} \xrightarrow{\Gamma} \mathfrak{S}m_C(k)$ given by $V \mapsto \lim \Gamma(Y, \mathcal{V}_Y)$, where $Y$ runs over the smooth proper models of subfields in $F$ of finite type over $k$. In general, $\Gamma(V)$ is not admissible.

The functor $\Gamma$ coincides with the composition of the forgetful functor to the category of smooth representations of $G$ with the functor $\Gamma$ from [3.1]. The functor $\Gamma$ is faithful, but it is not full, and the objects in its image are highly reducible, cf. Example on page 8.

**Conjecture 3.6.**

1. The functor $\text{Hom}_C(\bigotimes^q_F \mathfrak{m}, -)$ is exact on $\mathcal{A}$ for any $q \geq 0$.
2. Irreducible objects of $\mathcal{A}$ are direct summands of the tensor algebra $\bigotimes_F \Omega^1_F/k$.
3. $\mathcal{A}$ is equivalent to the category of “coherent” sheaves on $\mathfrak{S}m_k$.

As another evidence for Conjecture 3.6, in addition to the case $k = \overline{\mathbb{Q}}$, it is shown in [R2] that for any $L \subset F$ purely transcendental of degree $m$ over $k$ and any $V \in \mathcal{A}$ any irreducible subquotient of the $L$-semi-linear representation $V^{G_F/L}$ of $\text{PGL}_{m+1}$ is a direct summand of $\bigotimes_L \Omega^1_L/k$. 


As there exist smooth non-admissible irreducible semi-linear representations, cf. [R2], §4.2, one cannot replace the category $\mathcal{A}$ in the part (2) of Conjecture 3.6 by the whole category $\mathcal{C}$, and has to put some additional conditions, e.g. the one mentioned in §3.1.

**Remark.** Assuming the part (2) of Conjecture 3.6, one can reformulate Conjecture 2.7 in the following linguistically more convincing form:

Any irreducible object of $\text{Adm}$ (and of $\mathcal{I}_G$) is contained in an irreducible object of $\mathcal{A}$.

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Appendix A. A construction of a separable closure of a transcendence degree one extension of an algebraically closed field of positive characteristic

The following claim is a refinement of Proposition 4.1 of [R1].

Proposition A.1. The $G^o$-modules $F/k$ and $F^\times/k^\times$ are irreducible if either $\text{char}(k) = 0$, or $2 \leq n \leq \infty$. If $n = 1$ and $\text{char}(k) \neq 0$ then the $G^o$-orbit of $x$ generates the separable closure of $k(x)$ in $F$ for any $x \in F$.

Proof. Let $A$ be the additive subgroup of $F$ generated by the $G^o$-orbit of some $x \in F - k$. For any $y \in A - k$ one has $\frac{y^2 - 1}{y - 1} = \frac{1}{y - 1} - \frac{1}{y + 1}$. As $\frac{1}{y - 1}$ and $\frac{1}{y + 1}$ are in the $G^o$-orbit of $y$, this implies that $y^2 \in A$. As for any $y, z \in A$ one has $yz = \frac{1}{4}((y + z)^2 - (y - z)^2)$, the group $A$ is a subring of $F$, if $\text{char}(k) \neq 2$.

Let $M$ be the multiplicative subgroup of $F^\times$ generated by the $G^o$-orbit of some $x \in F - k$. Then for any $y, z \in M$ one has $y + z = z(y/z + 1)$, so if $y/z \notin k$ then $y + z \in M$, and thus, $M \cup \{0\}$ is a $G^o$-invariant subring of $F$.

Since the $G^o$-orbit of an element $x \in F - k$ contains all elements of $F - \overline{k(x)}$, if $n \geq 2$ then each element of $F$ is the sum of a pair of elements in the orbit. Any $G^o$-invariant subring in $F$, but not in $k$, is a $k$-subalgebra, so if $n = 1$ then $\text{Gal}(F/k^*(G^o x)) \subset G^o$ is a compact subgroup normalized by $G^o$. Then by Theorem 2.9 of [R1] we have $\text{Gal}(F/k(G^o x)) = \{1\}$, i.e., the extension $F/k(G^o x)$ is purely inseparable. As any element of $Q(G^o x)$ is the fraction of a pair of elements in $Z[G^o x]$ and for any $y \in F - k$ the element $1/y$ belongs to the $G^o$-orbit of $y$, the $Z$-subalgebra generated by the $G^o$-orbit of $x$ coincides with $F$, if $\text{char}(k) = 0$, or $2 \leq n \leq \infty$.

Let us show that $k(G^o x)$ is a separable extension of $k(x)$, equivalently, that if $s^N x = x$ for some $N \geq 1$ then $k(x, \sigma x)$ is a separable extension of $k(x)$. Let $P(x, \sigma x)$ be a minimal polynomial. Then $P_1 dx + P_1 d(\sigma x) = 0 \in Q[k(x, \sigma x)/k]$, where either $P_1 \neq 0$, or $P_1 \neq 0$ as otherwise $P = Q^p$ for another polynomial $Q$. If $P_1 \neq 0$ then $k(x, \sigma x)$ is a separable extension of $k(x)$. If $P_1 \neq 0$ then $k(x, \sigma x)$ is a separable extension of $k(\sigma x)$, and thus, $k(x, \sigma^{-1} x)$ is a separable extension of $k(x)$. Then $k(x, \sigma^{-1} x, \ldots, \sigma^{-(N-1)} x = \sigma x)$ is a separable extension of $k(x)$. □

Appendix B. The “Künneth formula” for products with curves

Define a $G$-homomorphism

$$Q[k(X) \otimes k(Y) \overset{\alpha}{\rightarrow} C_{k(X)} \otimes C_{k(Y)}] \xrightarrow{\tau} \tau |_{k(X)} \otimes \tau |_{k(Y)}.$$  

It is shown in [R1] that $\alpha$ is surjective, which gives a surjection $C_{k(X \times_k Y)} \twoheadrightarrow C_{k(X)} \otimes \tau C_{k(Y)}$.

For arbitrary $A \in C_{k(X)}$ and $B \in C_{k(Y)}$ choose some liftings $\tilde{A} \in Q[k(X) \overset{\alpha}{\rightarrow} F]$ and $\tilde{B} \in Q[k(Y) \overset{\alpha}{\rightarrow} F]$ such that all embeddings from $\tilde{A}$ and from $\tilde{B}$ are pairwise general position.2

One has to check that the class of $\tilde{A} \times \tilde{B} \in Q[k(X \times_k Y) \overset{\alpha}{\rightarrow} F]$ in $C_{k(X \times_k Y)}$ is independent of the choice of $\tilde{A}$ and $\tilde{B}$. If some other liftings $\tilde{A} \in Q[k(X) \overset{\alpha}{\rightarrow} F]$ and $\tilde{B} \in Q[k(Y) \overset{\alpha}{\rightarrow} F]$ are defined similarly, choose some lifting $\tilde{B}'' \in Q[k(Y) \overset{\alpha}{\rightarrow} F]$ of $B$ such that all embeddings from $\tilde{A}$ and from $\tilde{B}'$, as well as from $\tilde{A}$ and from $\tilde{B}''$, are in pairwise general position. Then $\tilde{A} \times \tilde{B} - \tilde{A'} \times \tilde{B}' = (\tilde{A} - \tilde{A'}) \times \tilde{B}'' + \tilde{A} \times (\tilde{B} - \tilde{B}'') + \tilde{A'} \times (\tilde{B}'' - \tilde{B}')$.

2First, choose arbitrary $\tilde{A}$ and $\tilde{B}$. For each point $P$ of the support of $\tilde{B}$ choose a generic curve $C$ passing through $P$, on which $P$ is a generic point with respect to a field of definition of $C$. Replace $P$ by a linearly equivalent linear combination of points of $C$ in general position with respect to $A$. Then we get the desired $\tilde{B}$. 

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Thus, one has to check the following condition \( \star_{X,Y} \): if the class of \( \sum_{i=1}^{N} a_i\tau_i \in \mathbb{Q}[[k(X) \rightarrow k F]] \) in \( C_{k(X)} \) is zero and all \( \tau_i \) are in general position with respect to \( \sigma : k(Y) \rightarrow k F \) then the class of \( \sum_{i=1}^{N} a_i(\tau_i, \sigma) \in \mathbb{Q}[[k(X \times_k Y) \rightarrow k F]] \) in \( C_{k(X \times_k Y)} \) is zero. Also, one has to check the condition \( \star_{Y,X} \).

By definition of the functor \( \mathcal{I} \), there exist purely transcendental extensions \( L_j'/L_j \), elements \( \alpha_j \in \mathbb{Q}[[k(X) \rightarrow k F]]_{G_F/L_j} \) and \( \xi_j \in G_{F/L_j} \) such that \( \sum_{i=1}^{N} a_i\tau_i = \sum_j (\xi_j\alpha_j - \alpha_j) \).

If \( \sigma \) is in general position with respect to the compositum \( L \) of all \( \tau_i(k(X)) \) then there exists \( \kappa \in G_{F/L} \) such that \( \kappa\sigma =: \sigma' \) is in general position with respect to the compositum of all \( L_j' \). Then \( \gamma' := \kappa\gamma = \sum_i a_i(\tau_i, \sigma') = \sum_j (\xi_j\alpha_j - \alpha_j) \otimes \sigma' \). Set \( K_j := L_j\sigma'(k(Y)) \) and \( K'_j := L_j'\sigma'(k(Y)) \). Then \( \alpha_j \otimes \sigma' \in \mathbb{Q}[[k(X \times_k Y) \rightarrow k F]]_{G_F/K_j}, \ K'_j \) is a purely transcendental extension of \( K_j \), and there exist \( \xi_j' \in G_{F/\sigma'(k(Y))} \) such that \( \xi_j'|_{L_j} = \xi_j|_{L_j'} \). This implies that \( \gamma' = \sum_j (\xi_j'(\alpha_j \otimes \sigma') - \alpha_j \otimes \sigma') \) belongs, by definition of the functor \( \mathcal{I} \), to the kernel of the projection \( \mathbb{Q}[[k(X \times_k Y) \rightarrow k F]] \to C_{k(X \times_k Y)} \), and therefore, the same is true for \( \gamma \).

Let us check that the conditions \( \star_{X,Y} \) and \( \star_{Y,X} \) are equivalent. Consider a generic curve \( C \) on \( Y \), passing through \( \sigma \), defined over a field containing the compositum of all \( \tau_i(k(X)) \). Then \( \sigma \) is linearly equivalent to a linear combination \( \beta \) of generic points of \( C \) (which are therefore generic points of \( Y \)). Then the image of \( \gamma \) in \( C_{k(X \times_k Y)} \) coincides with the image of \( \sum_i a_i\tau_i \times (\sigma - \beta) \), which shows the implication \( \star_{Y,X} \Rightarrow \star_{X,Y} \).

**Example.** Let us check the condition \( \star_{X,Y} \) in the case, when \( X \) is a smooth proper curve. Let \( K = \sigma(k(Y)) \). Then \( \sum_i a_i\tau_i \) is a generic divisor on the curve \( X_K \) over \( K \), linearly equivalent to zero. According to Lemma 6.18 from [R1], the \( G_{F/K} \)-module of generic divisors on \( X_K \) over \( K \), linearly equivalent to zero, is generated by the elements \( w_M = \sum_{j=1}^{M} (\sigma_j - \sigma'_j) \) for all \( M \gg 0 \), where \( (\sigma_1, \ldots, \sigma_M; \sigma'_1, \ldots, \sigma'_M) \) is a generic \( F \)-point of the fibre over 0 of the morphism \( X_K^M \times_k X_K^M \to \text{Pic}^0 X_K \), sending \((x_1, \ldots, x_M; y_1, \ldots, y_M)\) to the class of \( \sum_{j=1}^{M} (x_j - y_j) \). Clearly, the compositum of all \( \sigma_j(k(X))\sigma'_j(k(X)) \) is in general position with respect to \( K \). The same is true for any other element in the \( G_{F/K} \)-orbit of \( w_M \). Therefore, as we have already seen above, the image of \( \sum_i a_i(\tau_i, \sigma) \) in \( C_{k(X \times_k Y)} \) is zero.

Thus, one has a canonical \( G \)-module surjection \( C_{k(X)} \otimes C_{k(Y)} \twoheadrightarrow C_{k(X \times_k Y)} \), at least if \( X \) is a curve,\(^3\) and the composition \( C_{k(X \times_k Y)} \to C_{k(X)} \otimes \mathcal{I} C_{k(Y)} \to C_{k(X \times_k Y)} \) is identical.

\(^3\)and also if \( C_{k(X)} = CH_0(X_F) \otimes \mathbb{Q} \) and the transcendence degree of \( k \) is infinite (or if the same is true for algebraic closures of all extensions of \( k \) of finite type): \( \sum_i a_i\tau_i \in \mathbb{Q}[[k(X) \rightarrow k F]] \) is rationally equivalent to zero; as \( \text{tr.deg}(k) = \infty \), one has \( \mathcal{I}_{/k} \mathbb{Q}[[k(X) \rightarrow k F]] = CH_0(X_F) \otimes \mathbb{Q} \) (identify \( k \) with \( K \) by an automorphism of \( F \) identical on the field of definition of \( X \)), i.e. \( \sum_i a_i(\tau_i, \sigma) \) belongs to the kernel of the composition \( \mathbb{Q}[[k(X) \rightarrow k F]] \overset{\star}{\to} \mathbb{Q}[[k(X \times_k Y) \rightarrow k F]] \to \mathcal{I}_{/k} \mathbb{Q}[[k(X \times_k Y) \rightarrow k F]] \to C_{k(X \times_k Y)} \).
Appendix C. Differential forms

Proposition C.1. Let \( k_0 \subseteq k \) be a subfield. If \( i < n \) then \( \text{Hom}_G(\Omega^i_F, \Omega^j_{F/k_0}) = \Omega^{j-i}_k \oplus \Omega^{j-i-1}_k \cdot d; \)
\( \text{Hom}_G(\Omega^q_F, F) = 0 \) for any \( n \geq 1 \); \( \text{Hom}_G(\Omega^q_{F/k_0, \log}, \Omega^q_{F/k_0}) = \Omega^q_k \); \( \text{End}_G(\Omega^q_{F/k_0, \log}) = \mathbb{Q} \) for any \( 1 \leq q \leq n \).

Proof. For \( i < n \) the \( G \)-module \( \Omega^i_F \) is cyclic and generated by \( \eta := x_0 dx_1 \wedge \cdots \wedge dx_i \) for some algebraically independent elements \( x_0, x_1, \ldots, x_i \in F - \bar{k} \). The space \( \mathbb{Q} \cdot \eta \subseteq \Omega^i_F \) is stable under the subgroup \( H := G_{F, \mathbb{Q}^* x_0, \mathbb{Q}^* x_1, \ldots, \mathbb{Q}^* x_i, \mathbb{Q}^* x_0, \mathbb{Q}^* x_1, \ldots, \mathbb{Q}^* x_i}/k \subset G \) and \( H \) acts via its quotient \( (\mathbb{Q}^*)^{i+1} \).

For any \( \varphi \in \text{Hom}_G(\Omega^q_F, \Omega^q_{F/k_0}) \) the form \( \omega = \varphi(\eta) \) is \( G_{F,x_1,\ldots,x_i,\mathbb{Q}^* x_0, \mathbb{Q}^* x_1, \ldots, \mathbb{Q}^* x_i}/k(x_0) \)-invariant, so \( \omega = \sum_{S=(0\leq s_1<\cdots<s_i)} \xi_S \wedge dx_{s_1} \wedge \cdots \wedge dx_{s_i} \) for some \( \xi_S \in k(x_0) \otimes_k \Omega^{j-t}_{F/k_0} \).

The \((1, \ldots, 1)\) weight subspace in \( \bigoplus_{S=(0\leq s_1<\cdots<s_i)} k(x_0) \otimes_k \Omega^{j-t}_{F/k_0} \wedge dx_{s_1} \wedge \cdots \wedge dx_{s_i} \) with respect to the action of homotheties \( (\mathbb{Q}^*)^{i+1} \) coincides with \( x_0 \Omega^{j-i}_{k/k_0} \wedge dx_1 \wedge \cdots \wedge dx_i \otimes \Omega^{j-i-1}_{k/k_0} \wedge dx_0 \wedge dx_i \), so \( \omega = \xi_1 \wedge \eta + \xi_2 \wedge d\eta \) for some \( \xi_1 \in \Omega^{j-i}_{k/k_0} \) and \( \xi_2 \in \Omega^{j-i-1}_{k/k_0} \), and therefore, \( \varphi = \xi_1 + \xi_2 + d \) (modulo \( \Omega^1_{k/k_0} \wedge \Omega^{j-i-1}_{F/k_0} \)).

It follows from the above that \( \text{Hom}_G(k \otimes \mathbb{Q} \Omega^n_{F,\text{exact}}, \Omega^q_{F/k_0}) = 0 \). Check now that any \( \varphi \in \text{Hom}_G(\Omega^n_F, \Omega^{q<n}_{F,k_0}) \) factors through \( \Omega^n_{F/k} \rightarrow \Omega^q_{F/k_0} \). For any \( a \in k \) and \( x_1, \ldots, x_n \in F \) algebraically independent over \( k \) one has \( \varphi(a \cdot dx_1 \otimes \cdots \otimes dx_n) = \varphi(d(a x_1) \otimes \cdots \otimes dx_n) = 0 \), so \( \varphi(x_1 \cdot dx_1 \otimes \cdots \otimes dx_n) = 0 \). Clearly, \( x_1 \cdot dx_1 \otimes \cdots \otimes dx_n \) for all \( a \in a \) are generators of the \( G \)-submodule \( dx_1 \wedge \cdots \wedge dx_n \) of \( \Omega^n_F \). This implies that \( \text{Hom}_G(\Omega^n_{F, \Omega^{<n}_{F,k_0}}) = \text{Hom}_G(\Omega^1_{DR/k}(F), \Omega^{<n}_{F/k_0}) \).

More particularly, let \( \varphi \in \text{Hom}_G(\Omega^q_{F,k}, F) \). For any \( L \subset F \) over which \( F \) is of transcendence degree one \( \text{Hom}_G(\Omega^q_{F,k}, F) \subseteq \text{Hom}_{F/k}(\Omega^q_{F/k}, F) \subseteq \text{Hom}_{F/k}(\Omega^{q-1}_{L} \otimes L \Omega^1_F) \), so we may assume that \( L = k \) and \( n = 1 \). Let \( \text{Div}^q := \lim_{L \rightarrow [L]_Q} \text{Div}^q([L]_Q) \), \( \text{Pic}^q := \lim_{L \rightarrow [L]_Q} \text{Pic}^q([L]_Q) \) and \( H^1_{DR/k,c} := \lim_{L \rightarrow [L]_Q} H^1_{DR/k}([L]_Q) \), where \([L]_Q\) denotes a smooth proper model of \( L \) over \( k \), cf. [1], p.182. Then \( \text{Hom}_G(\text{Pic}^q_k, F) = \text{Hom}_{k}(H^1_{DR/k,c}(F), F) = 0 \), since if \( \omega \) from \( \text{Pic}^q_k \) or from \( H^1_{DR/k,c} \) is fixed by some \( G_{F/L} \) and sent to \( f \in L \) then \( \text{tr}_{L/L}[\omega] = \text{tr}_{L/L}[f] = [L : L] \cdot f \) for any \( f \in L' \subset L \) purely transcendental over \( k \), over which \( L \) is algebraic, but \( \text{tr}_{L/L}[\omega] = 0 \). So from the short exact sequences \( 0 \rightarrow H^1_{DR/k,c} \rightarrow H^1_{DR/k}(F) \xrightarrow{Res} \text{Div}^q \wedge k ightarrow 0 \) and \( 0 \rightarrow F^x/k \rightarrow \text{Div}^q \rightarrow \text{Pic}^q ightarrow 0 \), we get \( \text{Hom}_G(H^1_{DR/k}(F), F) = 0 \).

The form \( \omega = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \) is a generator of the \( G \)-module \( \Omega^1_{F/k_0, \log} \). A \( G \)-homomorphism to \( \Omega^1_{F/k_0} \) sends it to an element \( \omega' \) of \( \left( \Omega^1_{F/k_0} \right)^{\text{Stab}_\omega} \). As \( \text{Stab}_\omega \supset U_{k(x_1, \ldots, x_q)} \cdot \left( \Omega^1_{F/k_0} \right)^{\text{Stab}_\omega} \subset \Omega^1_{k(x_1, \ldots, x_q)/k} \). As \( \omega' = \sum_I \eta_I \wedge \frac{dx_i}{x_i} \) for some \( \eta_I \in k(x_1, \ldots, x_q) \otimes k \Omega^1_{k/k_0} \) is fixed by \( \mathbb{Q}^q \) one has \( \eta_I \in \Omega^1_{k/k_0} \). As \( \omega' \) is fixed by \( \text{SL}_q(Z) \) one has \( \omega' = \eta_1 \wedge dx_1 \wedge \cdots \wedge \frac{dx_n}{x_n} \), where \( \eta_1, \ldots, q \in \Omega^1_{k/k_0} \).

According to [4], \( H^q_{DR/k}(F) \) := coker[\( \Omega^{q-1}_{F/k} \xrightarrow{d} \Omega^q_{F/k, \text{closed}} \rightarrow \lim_{\rightarrow} H^q_{(X, \Omega^p_{X/k}(\log D))} \)], where \( (X, D) \) runs over pairs consisting of a smooth proper variety \( X \subset k \) and a normal crossing divisor \( D \) on \( X \) with smooth irreducible components. Moreover, the Hodge filtration on \( \Omega^p_{X/k}(\log D) \) induces a descending filtration on \( H^q_{DR/k}(F) \) by \( k \)-linear representations of \( G_{F/k} \) with the graded quotients \( \lim_{\rightarrow} H^p(X, \Omega^p_{X/k}(\log D)) \), where \( (X, D) \) runs over pairs as above.

The weight filtration on \( \Omega^p_{X/k}(\log D) \) induces an increasing filtration \( W^q H^q_{DR/k}(F) \) on \( H^q_{DR/k}(F) \). In particular, \( H^q_{DR/k,c}(F) := W^q H^q_{DR/k}(F) \) is the image in \( H^q_{DR/k}(F) \) of \( H^q_{DR/k}(X) \), where \( X \) runs over smooth proper models of subfields in \( F \) of finite type over \( k \). Clearly, this is an admissible
representation over \(k\). Again, the Hodge filtration on \(\Omega^*_{X/k}\) induces a descending filtration on \(H^{q}_{dR/k,c}(F)\) with the graded quotients \(H^{p,q}_{dR/k}(F) = \lim_{\to} \text{coker}[H^{p-1}(D, \Omega^{q-p-1}_{D/k}) \rightarrow H^p(X, \Omega^{q-p}_{X/k})]\), where \((X, D)\) runs over pairs as above. More particularly, \(H^{q}_{dR/k}(F) = \Omega^{q}_{F/k,\text{reg}} \subset H^{q}_{dR/k,c}(F)\).

**Proposition C.2.** Suppose that the cardinality of \(k\) is at most continuum. Fix an embedding \(i : k \hookrightarrow \mathbb{C}\) to the field of complex numbers. Then

- there is a non-canonical \(\mathbb{Q}\)-linear isomorphism \(H^{p,q}_{dR/k}(\mathcal{M}) \cong H^{p,q}_{dR/k}(\mathcal{N})\), and a \(\mathbb{C}\)-anti-linear canonical isomorphism (depending on \(i\)) \(H^{p,q}_{dR/k}(\mathcal{M}) \cong H^{p,q}_{dR/k}(\mathcal{N})\);
- the representation \(H^{p,q}_{dR/k,c}(F)\) is semi-simple for any \(1 \leq n < \infty\).

**Proof.**

- The complexification of the projection \(F^pH^{p,q}_{dR/k}(X) \rightarrow H^q(X, \Omega^p_{X/k})\) identifies the space \(F^pH^{p,q}_{dR/k}(X) \otimes_{k,\mathcal{O}} \mathbb{C} \cap \overline{F^qH^{p,q}_{dR/k}(X) \otimes_{k,\mathcal{O}} \mathbb{C}}\) with \(H^q(X, \Omega^p_{X/k}) \otimes_{k,\mathcal{O}} \mathbb{C}\), where \(F^p\) is the Hodge filtration. Then the complex conjugation on \(H^{p,q}(X, \mathcal{M}), \mathbb{C}) = H^{p,q}(X, \mathcal{N}), \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}\) identifies \(H^q(X, \Omega^p_{X/k}) \otimes_{k,\mathcal{O}} \mathbb{C}\) with \(H^p(X, \Omega^q_{X/k}) \otimes_{k,\mathcal{O}} \mathbb{C}\).
- The semi-simplicity of \(H^{p,q}_{dR/k,c}(F)\) is equivalent to the semi-simplicity of the representation \(C \otimes_{k,\mathcal{O}} H^{p,q}_{dR/k,c}(F) = \bigoplus_{p+q=n} C \otimes_{k,\mathcal{O}} H^{p,q}_{dR/k}\) of \(G\). For the latter note that there is a positive definite \(G\)-equivariant hermitian form \((C \otimes_{k,\mathcal{O}} H^{p,q}_{dR/k}) \otimes_{\text{id},\mathbb{C},c} (C \otimes_{k,\mathcal{O}} H^{p,q}_{dR/k}) \rightarrow \mathbb{C}(\chi)\), where \(c\) is the complex conjugation and \(\chi\) is the modulus of \(G\), given by \((\omega, \eta) = \int_{X,\mathbb{C}} i^{n/2} \omega \wedge \eta\). Here \(H^{p,q}_{dR/k,c}(F)\) identifies the space orthogonal to the sum of the images of all Gysin maps \(H^{p,q}_{dR/k,c}(F)\) for all desingularizations \(D\) of all divisors on \(X_i,\mathbb{C}\).

Let us check that the sequence \(0 \rightarrow H^1_{dR/k}(X) \oplus k \otimes (k(X)^* / k^*) \rightarrow i \rightarrow H^1_{dR/k}(k(X)) \rightarrow 0\) is exact, where for any divisor \(x \in X^1\) the residue \(\text{res}_x\) is defined, and gives rise to the map \(H^1_{dR/k}(k(X)) \rightarrow k \otimes \text{Pic}^0(X)\).

The map \(i\) sends a pair \((\omega, a \otimes f)\) to \(\omega + a \log f\), and has trivial kernel, since the residue map is trivial on \(H^1_{dR}(k(X)/k)\) and injective on \(k \otimes (k(X)^* / k^*)\). If the residues of \(\omega \in H^1_{dR/k}(k(X))\) are zero then integration along a loop depends only on its homology class. There is an element \(\eta\) of \(H^1_{dR/k}(X)\) with the same periods as \(\omega\), so integration of \(\omega - \eta\) along a path joining a fixed (rational) point with the variable one is independent of a chosen path, and defines a meromorphic (i.e. rational) function. This gives exactness in the middle term: \(\text{Im}(\text{res}_{\omega}) = \ker(\text{res}_{\omega})\).

For any pair \(D_1, D_2\) of algebraically equivalent effective divisors on \(X\) there is a smooth projective curve \(C\), and an effective divisor \(D\) on \(X \times C\), such that \(\text{pr}_X : D \rightarrow X\) is generically finite and for some points \(P, Q \in C\) one has \(D_P = D_Q = D_1 - D_2\).

Since \(\dim_k \Gamma(C, \Omega^1_{\mathcal{O}(P + Q)}) = \dim_k \Gamma(C, \Omega^1_{\mathcal{O}}) + 1\), \(\Gamma(C, \Omega^1_{\mathcal{O}(P)}) = \Gamma(C, \Omega^1_{\mathcal{O}})\), there exists a 1-form \(\omega_{P,\mathcal{O}}\) such that \(\text{Res}(\omega_{P,\mathcal{O}}) = P - Q\).

Set \(\omega = \text{pr}_{X*}(\text{pr}_{X*}^* \omega_{P,\mathcal{O}})|_D\). Then \(\text{Res}(\omega) = D_1 - D_2\). So the image of \(\text{Res}\) contains the algebraically trivial part of the group of divisors with coefficients in \(k\). This also shows that \(\omega \in N^1_{dR/k}(k(X))\). Since \(\text{Res}\) commutes with restriction to a curve, \(\text{Res}(\omega) \cdot C = \text{Res}(\omega|_C) \in CH_0(X)\), \(\deg(\text{Res}(\omega) \cdot C) = 0\) by Cauchy theorem, \(NS(X)_Q \otimes CH_1(X)_\mathbb{Q} / \text{hom} \rightarrow \mathbb{Q}\) is non-degenerate, \(\text{Res}(\omega) = 0 \in H^2(X, \mathbb{Q})\). Thus, the map \(\text{Res}\) is well-defined and surjective.
Proposition C.3. The representation $\Omega^1_{F/k,\text{closed}}$ admits the following description for any $1 \leq n \leq \infty$. Let $H^1_{\text{dR}/k,c}(F) := \ker[H^1_{\text{dR}/k}(F) \xrightarrow{\text{Res}} k \otimes \text{Div}_Q^\circ];$ $\text{Pic}_Q^\circ := \coker[F^\times/k^\times \xrightarrow{\text{div}} \text{Div}_Q^\circ].$ Then $\text{Pic}_Q^\circ = \bigoplus_A A^1(k) \otimes_{\text{End}(A)} (A/F)/A(k))$, where $A$ runs over the isogeny classes of simple abelian varieties over $k$.

- The maximal semi-simple subrepresentation of $G$ in $\Omega^1_{F/k,\text{closed}}$ is canonically isomorphic to
  \[ \bigoplus_A \Gamma(A, \Omega^1_{A/k})^\wedge \otimes_{\text{End}(A)} (A(F)/A(k)) = (F/k) \otimes (F^\times/k^\times) \otimes \Omega^1_{F/k,\text{reg}}, \]
  where $A$ runs over the isogeny classes of simple commutative algebraic $k$-groups.

- The maximal semi-simple subrepresentation of $G$ in $H^1_{\text{dR}/k}(F)$ is canonically isomorphic to
  \[ \bigoplus_A H^1_{\text{dR}/k}(A) \otimes_{\text{End}(A)} (A(F)/A(k)) = k \otimes (F^\times/k^\times) \otimes H^1_{\text{dR}/k,c}(F), \]
  where $A$ runs over the isogeny classes of simple commutative algebraic $k$-groups (with the zero summand corresponding to $\mathbb{G}_a$).

- The representation $H^1_{\text{dR}/k}(F)/(k \otimes (F^\times/k^\times))$ of $G$ is canonically isomorphic to
  \[ \bigoplus_A [H^1_{\text{dR}/k}(k(A))/(k \otimes (k(A)^\times/k^\times))] \otimes_{\text{End}(A)} (A(F)/A(k)), \]
  where $A$ runs over the isogeny classes of simple abelian $k$-varieties.

Proof. Follows from the above and from (evidently modified) Proposition 3.11 of [R1].

Corollary C.4. Let $\mathbb{Q} \subseteq k_0 \subseteq k$. Then $\text{End}_{k_0[G]}(\Omega^1_{F/k,\text{closed}})$ contains

\[ \text{Hom}_{k_0[G]}(\Omega^1_{F/k,\text{closed}}, \Omega^1_{F/k,\text{reg}}) = \prod_A \text{Hom}_{k_0}\left(H^1_{\text{dR}/k}(k(A))/k \otimes (k(A)^\times/k^\times), \Gamma(A, \Omega^1_{A/k})\right) \]

and $\text{End}_{k_0[G]}(H^1_{\text{dR}/k}(F))$ contains (properly, if $k$ is transcendental over $\mathbb{Q}$ and $k_0 \subseteq \overline{\mathbb{Q}}$)

\[ \text{Hom}_{k_0[G]}(H^1_{\text{dR}/k}(F), H^1_{\text{dR}/k,c}(F)) = \prod_A \text{Hom}_{k_0}\left(H^1_{\text{dR}/k}(k(A))/k \otimes (k(A)^\times/k^\times), H^1_{\text{dR}/k,c}(A)\right), \]

where $A$ runs over isogeny classes of simple abelian varieties over $k$.

Proof. As there are no subobjects of $H^1_{\text{dR}/k,c}(F)$ isomorphic to $F^\times/k^\times$,

\[ \text{Hom}_{k_0[G]}(H^1_{\text{dR}/k}(F), H^1_{\text{dR}/k,c}(F)) = \text{Hom}_{k_0[G]}(H^1_{\text{dR}/k}(F)/k \otimes (F^\times/k^\times), H^1_{\text{dR}/k,c}(F)), \]

so applying $\text{Hom}_{k_0[G]}(H^1_{\text{dR}/k}(F), -)$ to $0 \rightarrow H^1_{\text{dR}/k,c}(F) \rightarrow H^1_{\text{dR}/k}(F) \xrightarrow{\text{Res}} k \otimes \text{Div}_Q^\circ \rightarrow 0$ we get the inclusion. On the other hand, the Gauss–Manin connection induces an embedding $\text{Der}(k) \hookrightarrow \text{End}_{k_0[G]}(H^1_{\text{dR}/k}(F))$, which does not factor through $\text{Hom}(H^1_{\text{dR}/k}(F), H^1_{\text{dR}/k,c}(F))$ unless it is zero, i.e. $k \subseteq \overline{\mathbb{Q}}$.

Remark. Clearly, the projection $\Omega^1_{F/k,\text{closed}} \rightarrow \Omega^1_{F/k,\text{closed}}/\Omega^1_{F/k,\text{reg}}$ is split. However, there is no natural splitting.

\footnote{cf. p.182 before Proposition 3.11 of [R1]}
D.1. “Homotopy invariant” representations as non-degenerate modules. We set $D_E := \lim_{\leftarrow U} E[\Gamma/U]$, where, for a field $E$ of characteristic zero, the inverse system is formed with respect to the projections $E[\Gamma/V] \xrightarrow{r_{UV}} E[\Gamma/U]$ induced by inclusions $V \subseteq U$ of open subgroups in $\Gamma$. For any $\nu \in D_E$, any $\sigma \in \Gamma$ and an open subgroup $U$ we set

$$\nu(\sigma U) := \text{coefficient of } [\sigma U] \text{ of the image of } \nu \text{ in } E[\Gamma/U].$$

The support of $\nu$ is the minimal closed subset $S$ in $\hat{\Gamma}$ such that $\nu(\sigma U) = 0$ if $\sigma U \cap S = \emptyset$. Define a pairing $D_E \times W \rightarrow W$ for each smooth $E$-representation $W$ of $\Gamma$ by $(\nu, w) \mapsto \sum_{\sigma \in \Gamma/V} \nu(\sigma V) \cdot w$, where $V$ is an arbitrary open subgroup in the stabilizer of $w$. When $W = E[\Gamma/U]$ this pairing is compatible with the projections $r_{UV}$, so we get a pairing $D_E \times \lim_{\leftarrow U} E[\Gamma/U] \rightarrow \lim_{\leftarrow U} E[\Gamma/U] = D_E$, and thus an associative multiplication $D_E \times D_E \rightarrow D_E$ extending the convolution of compactly supported measures.

If $n = \infty$ then the action of the associative algebra $D_E$ on any object of $\mathcal{I}_G(E)$ factors through the action of its quotient $C_E := \lim_{\leftarrow L} C_L$, since the morphism $E[\Gamma/G_{F/L}] \otimes_E W^{G_{F/L}} \rightarrow W$ of representations of $G$ factors through $\mathcal{I}(E[\Gamma/G_{F/L}] \otimes_E W^{G_{F/L}}) = C_L \otimes W^{G_{F/L}} \rightarrow W$.

For any compact subgroup $U$ in $\Gamma$ the action of its Hecke algebra $\mathcal{H}_E(U) := h_U \ast D_E \ast h_U$ on $W_U$ factors through the action of its quotient $C_E(U) := h_U \ast C_E \ast h_U$ in $C_E$ for any $W \in \mathcal{I}_G(E)$. E.g., if $F^U$ is purely transcendental over $L$ and $L$ is of finite type over $K$ then $C_E(U) = C_L^U \otimes E$.

Let $\mathcal{H}_U := \lim_{\leftarrow K} \lim_{\leftarrow L} C_L^{G_{F/K}}$ be the associative idempotented algebra without unity. The images $h_K$ of the Haar measures on $G_{F/K}$ for purely transcendental extensions $K$ of subfields of finite type over $K$ in $F$ over which $F$ is algebraic, are projectors in the algebra $\mathcal{H}_U$. Then the category $\mathcal{I}_G$ is equivalent to the category of non-degenerate modules over $\mathcal{H}_U$, i.e. such modules $W$ that $W = \mathcal{H}_U W$. The algebra $\mathcal{H}_U$ is isomorphic to the Hecke algebra (of locally invariant measures with compact support) of neither locally compact group, since any, e.g. finite-dimensional, subspace in $\lim_{\leftarrow L} C_L^U$ is a left ideal in $\mathcal{H}_U$, which never happens in the Hecke algebras.\(^5\)

D.2. A locally compact “dense subgroup” of $\Gamma$ and a description of $\mathcal{I}_G$, etc. In the case $n = \infty$ the category $\mathcal{I}_G$ admits also a description in terms of a locally compact group.

For a descending sequence $L_{\bullet} = (L_1 \supset L_2 \supset L_3 \supset \ldots)$ of subfields in $F$ set $H = H_{L_{\bullet}} = \bigcup_{m \geq 1} G_{F/L_m}$. We take the subgroups $G_{F/L_1(S)} \subseteq H$ for all finite subsets $S$ in $F$ as a base of open subgroups.

- We want $H$ to be a dense “subgroup” of $\Gamma$, so we ask that $\bigcap_{m \geq 1} L_m = k$. This implies that the forgetful functor $\text{Sm}_{\Gamma} \rightarrow H$-mod is fully faithful.\(^6\)
- Further, we want $H$ to be locally compact and therefore we ask $F$ to be of finite transcendence degree over $L_1$. Then $H$ is indeed locally compact, but not unimodular.
- If $L^*_{\bullet}$ is an infinite subset in $L_{\bullet}$ then clearly $H_{L^*_{\bullet}} = H_{L^*_1}$. We want the topologies on $H_{L^*_{\bullet}}$ and on $H_{L_{\bullet}}$ to be the same. For that we ask $L_1$ to be of finite type over $L_m$ for all $m > 1$.

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\(^5\) Let $H$ be a locally compact group. If there is a non-zero finite-dimensional left ideal $a$ in the Hecke algebra of $H$ then the common support of the measures in $a$ is compact and left-invariant, and therefore, $H$ is compact. Then the smooth representations of $H$ are semi-simple.

\(^6\) Proof. Let $W, \alpha \in \text{Sm}_{\Gamma}, \alpha \in \text{Hom}_H(W, W')$, $v \in W$ and $\sigma \in \Gamma$. Let $U$ be the common stabilizer of $v$ and $\alpha(v)$. Choose some $\sigma' \in H \cap \sigma U$. Then $\alpha(\sigma v) = \alpha(\sigma' v) = \sigma' \alpha(v) = \sigma \alpha(v)$. \(\square\)
• The inclusion of \( H \) into \( G \) is a continuous homomorphism, so the forgetful functor \( S m_G \to S m_H \) factors through \( S m_G \to S m_H \). It admits a right adjoint \( W \mapsto \lim \bigcap_{m \geq 1} W^{G_F/L L_m} \), where \( L \) runs over subfields of finite type over \( k \) in \( F \). The \( G \)-action is defined as follows. If \( w \in W^{G_F/L L_m} \) and \( \sigma \in G \) then \( \sigma w := \sigma' w \), where \( \sigma' \in H \) and \( \sigma'|_L = \sigma|_L \). Clearly, this is independent of \( \sigma' \).

• Suppose that \( L_j \) is purely transcendental over \( L_{j+1} \) for any \( j \gg 1 \). As any admissible representation of \( G \) is ‘homotopy invariant’, the forgetful functor induces \( A d m_G \to A d m_H \).

In particular, the effective motives modulo numerical equivalence form a full subcategory in the category of graded semi-simple admissible \( H \)-modules. Note, that as the category of graded semi-simple admissible \( H \)-modules of finite length is self-dual, arbitrary motives (not necessarily effective) modulo numerical equivalence can probably be realized in that category.

Let \( I_H \) be the full subcategory in \( S m_H \) whose objects \( W \) satisfy the “homotopy invariance” condition \( W^{G_F/L L_m} = W^{G_F/L L_m(S)} \) for any \( m \geq 1 \), any extension \( L \) of \( k \) of finite type in \( F \) and any transcendence basis \( S \) of \( F \) over \( L L_m \).

Example. Choose a transcendence basis \( \{x_1, x_2, \ldots \} \) of \( F \) over \( k \) and set \( L_m = k(x_m, x_{m+1}, \ldots) \).

Geometrically, this corresponds to an inverse system of infinite-dimensional irreducible varieties given by finite systems of equations. They are related by dominant morphisms affecting only finitely many coordinates.

It follows from Lemma 2.15 of [R1] that \( H = G_{F/I_H} \cdot H^0 \).

**Proposition D.1.** The forgetful functor \( S m_G \to S m_H \) induces the following equivalences of categories: \( I_G \xrightarrow{\sim} I_H \) and \( A d m_G \xrightarrow{\sim} I_H \cap A d m_H \).

**Proof.** We need to construct an inverse functor \( I_H \to I_G \). In particular, for a given \( W \in I_H \), \( v \in W \) and \( \sigma \in G \), we want to define \( \sigma v \).

There exist a subfield \( L \subset F \) of finite type over \( k \) and an integer \( m \geq 1 \) such that the stabilizer of \( v \) contains \( G_{F/L L_m} \). Let \( L L_m = L'/L_m' \), where \( L' \subset F \) is of finite type over \( k \), and \( L' \) and \( L_m' \) are algebraically independent over \( k \).

Let \( N > m' \) be an integer such that \( L' \sigma(L') \) and \( L_N \) are algebraically independent over \( k \). Take any \( \sigma' \in G_{F/L_N} \) such that \( \sigma'|_{L'} = \sigma|_{L'} \) and set \( \sigma v := \sigma' v \). One has \( v \in W^{G_{F/L' L_m'}} = W^{G_{F/L' L N}} \), so \( \sigma v \) is independent of particular choices of \( N \) and of \( \sigma' \).

Now we check independence of \( L' \). Suppose that \( v \in W^{G_{F/L' L_m'}} \cap W^{G_{F/L'' L_m''}} \). Since \( v \in W^{G_{F/L' L'' L_{m'+m''}}} \), it suffices to treat the case \( L' \subset L'' \). As above, we choose an integer \( N > m'' \) such that \( L'' \sigma(L'') \) and \( L_N \) are algebraically independent over \( k \), and some \( \sigma'' \in G_{F/L_N} \) such that \( \sigma''|_{L''} = \sigma|_{L''} \). Then \( \sigma'' \) can also serve as a \( \sigma' \), i.e., \( \sigma'' v = \sigma' v \).

This gives us a map \( G \times W \to W \). Clearly, this is a linear action, and the stabilizer of \( v \) contains the open subgroup \( G_{F/L'} \), and thus, \( W \) becomes an object of \( I_G \).  

**Remarks.**

1. There are admissible representations of \( H \) outside of \( I_H \), e.g. \( Q(\rho) \) for any non-trivial character \( \rho \) of \( H \).

2. \( I_H \) is closed under subquotients and direct products,\(^7\) but not under extensions in \( S m_H \). As any morphism from \( W \in S m_H \) to an object of \( I_H \) factors through the canonical map to the direct product over all morphisms from \( W \) to representatives of all isomorphism classes in \( I_H \), there is a functor \( I : S m_H \to I_H \) left adjoint to the inclusion \( I_H \hookrightarrow S m_H \).

3. \( A d m_H \) is a Serre subcategory in \( S m_H \).

\(^7\)the direct product of a family of smooth representations is the smooth part of their set-theoretic direct product.
REFERENCES

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