A projection based approach for interactive fixed effects panel data models

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Abstract

This paper presents a new approach to estimation and inference in panel data models with interactive fixed effects, where the unobserved factor loadings are allowed to be correlated with the regressors. A distinctive feature of the proposed approach is to assume a nonparametric specification for the factor loadings, that allows us to partial out the interactive effects using sieve basis functions to estimate the slope parameters directly. The new estimator adopts the well-known partial least squares form, and its $\sqrt{NT}$-consistency and asymptotic normality are shown. Later, the common factors are estimated using principal component analysis (PCA), and the corresponding convergence rates are obtained. A Monte Carlo study indicates good performance in terms of mean squared error. We apply our methodology to analyze the determinants of growth rates in OECD countries.

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\textit{Keywords:} cross-sectional dependence, semiparametric factor models, principal components, sieve approximation, large panels.

1 Introduction

The increased availability of panel data sets enriches the complexity of the empirical studies. Especially if one is interested in capturing the unobserved heterogeneity with great statistical precision. Usually, panel data models assume that the
unobserved heterogeneity is accounted by additive effects (i.e., individual and/or time-specific components that can be correlated with the explanatory variables, the so-called “fixed effects”) and observations are independent in the cross-sectional dimension (see Arellano (2003), Baltagi (2015), Hsiao (2014), and the references therein). However, economic agents are typically interdependent because of globalization and the growing importance of individuals’ economic and social interconnections. Ignoring this dependence can seriously damage the empirical results, since standard panel data techniques can lead to inconsistent and inefficient estimators, as it was shown in Phillips and Sul (2003) or Hsiao and Tahmiscioglu (2008) among others.

Recently, panel data models with interactive effects (or factor structure) have become one of the most popular and successful tools to handle the cross-sectional dependence. The multiplicative specification of the unobserved heterogeneity is very useful. Since it contains the conventional additive effects as special cases but, at the same time, is considerably more flexible since it allows time-varying individual effects. In other words, the common time specific effects $f_t$, called common factors, are allowed to have a different impact on the individual-specific effects $\lambda_i$, called factor loadings. Further, this regression model is more realistic from the empirical point of view since the unobserved effects are allowed to exhibit an arbitrary degree of correlation with the explanatory variables. Some of the regressors are decision variables affected by the unobserved heterogeneity. For instance, macroeconomic models use the interactive effects to capture aggregate shocks which might have heterogeneous impacts on the agents (see Bernanke et al. (2005) for example); microeconomic models (e.g. Carneiro et al. (2003)) use the factor structure to study individuals’ education decisions to take into account the time-varying multidimensional individual heterogeneity. In finance, Ross (1976) and Fama and French (1993), among others, use the factor structure to capture the unobserved heterogeneity in a more flexible way for asset pricing, whereas Stock and Watson (2002), Bai and Ng (2006), and Fan et al. (2021) use it for forecasting. While the setting with interactive fixed effects is more realistic and flexible, estimation becomes more complicated due to the non-existence of simple linear transformations to remove the fixed effects. Thus, the pooled least-squares estimator or the within-group estimator for the slope coefficients are inconsistent.

In this paper, we propose a novel semiparametric approach to estimate the regression parameters in a panel model with interactive fixed effects. The basic idea is to extend the projected-PCA approach originally proposed in Fan et al. (2016) for a pure factor model, to a more general case where the regressors play a crucial role. More precisely, we propose a more general specification to control for the dependence between the regressors and factor terms using an unknown smooth function of the time average of covariates, up to an error term independent of the regres-
sors. By incorporating this information into the model of interest, the regression model takes the form of a partially linear model (see, e.g. Härdle et al. (2012) and a new estimation approach can be proposed based on partialling out the interactive effects by using sieve basis functions for the factor loadings. Therefore, this new approach is very appealing since it implies that we do not have to estimate the latent factors in a first step as in Bai (2009). The new estimator takes the very simple and intuitive structure of a partial least squares estimator and is robust to the heteroskedasticity over the cross-section or serial correlation. We show the $\sqrt{NT}$-consistency and asymptotic normality of our estimator. Later, even though the main focus of the paper is the estimation of the regression parameters, consistent estimators for the latent factors and corresponding factor loadings are also provided using the projected-PCA approach. Further, we propose a consistent estimator for the asymptotic covariance matrix of the slope coefficients to conduct valid inference on the estimated parameters.

Several alternative estimation approaches have been developed recently (see Fan and Liao (2021) and the references therein for an intensive review). For panels with a large cross-sectional dimension ($N$) but fixed time dimension ($T$), Holtz-Eakin et al. (1988) propose a quasi-differenced transformation of the original model to remove the factor loadings and treat $f_t$ as a fixed number of parameters to estimate. However, if there are too many parameters to estimate the incidental parameters problem may arise, see Neyman and Scott (1948). To overcome it, Ahn et al. (2001) and Ahn et al. (2013) impose various second moment restrictions on $\lambda_i$ and estimate the resulting moment conditions using a generalized method of moments (GMM) estimator.

More recent literature considers panels with large $N$ and $T$, where the two most common approaches to deal with the interactive effects are the common correlated effects (CCE) approach of Pesaran (2006), and the principal components approach (PCA) of Bai (2009). In general, both procedures are based on estimating first the unknown factors and then running ordinary least squares conditional on the first-step factors estimates, but they exhibit relevant differences. On the one hand, Pesaran (2006) proposes to augment the original model with the cross-sectional averages of the dependent variable, $y_{it}$, and the independent variables, $x_{it}$, that are used as control functions for the interactive fixed effects and provide an estimate for $f_t$. On the other hand, Bai (2009) develops an iterative procedure where a consistent estimator for the slopes is obtained once the factor loadings and common factors are estimated using the PC method to the residuals obtained from an OLS estimator that ignores the interactive effects. Finally, Connor and Linton (2007) and Connor et al. (2012) consider the nonparametric modelling of factor loadings in the pure factor model, and Fan et al. (2016) apply the principal component algorithm to a projected (or smoothed) data matrix. However, they rule out the
potential relationship between the factor loadings and the time-varying regressors and the resulting estimators may be subject to an omitted variable bias. For recent accounts on the estimation of interactive fixed effects models, see Ando and Bai (2017), Bai and Li (2014), Li et al. (2016), Moon and Weidner (2015), Moon and Weidner (2017), Peng et al. (2021), Chen et al. (2021), and Su et al. (2012), among others.

Unlike the aforementioned literature, the estimation procedure proposed in this paper is very appealing for several reasons. Firstly, misspecification problems related to linear structures among factor loadings and regressors are avoided using the more flexible Mundlak-Chamberlain projection (see Mundlak (1978) and Chamberlain (1984)). Secondly, the omitted variable bias is avoided allowing individual and time-varying regressors. Thirdly, the resulting estimator for the slope parameter is unaffected by the number of latent factors and does not require knowledge of them. This is a very interesting feature since the incidental parameter problem that usually characterized this type of models is avoided for \( T \) fixed or large (see Bai (2009) for further details). Therefore, we are able to show the \( \sqrt{NT} \)-consistency and the underlying limiting distribution of the resulting estimator for the slope parameter is centered at zero, so it is proved that the proposed estimation technique enables us to circumvent the issue of having bias terms similar to Bai (2009) in the case of heteroskedastic and correlated error terms. These asymptotic results hold irrespectively of the variance of the idiosyncratic part of the factor loadings being zero, close to zero or much larger than zero. Finally, under the condition that the loadings can be explained completely by the nonparametric functions, it is shown that our estimator reaches the semiparametric efficiency bound.

We validate the theoretical results in a simulation study. In the case, where the time averages of regressors have a non-vanishing explaining power on the factor loadings, our estimator outperforms alternative estimators which do not account for the relation. Later, we apply our method to the identification of the determinants of economic growth. We obtain growth rates and country-specific variables from the Penn World Table and from the World Bank Development Indicators. Lu and Su (2016) argued that the GDP growth rates per capita might not only be determined by observed factors, but might also be influenced by latent factors or shocks. Our projection-based interactive fixed effects estimator is well suited for such a setting. Indeed, our empirical findings suggest an important role of these latent effects. Especially, when only concentrating on the subset of OECD countries, the factor loadings can be well explained by the time averages of regressors.

The paper is organized as follows. In Section 2, we present the model setup and derive our projection-based interactive fixed effects estimator. Also, consistent estimators for the latent factors and corresponding factor loadings are provided. Section 3 states our assumptions and studies the asymptotic properties of the proposed esti-
mators. Based on these results, a consistent estimator of the standard errors of the estimated parameters is also provided. In Section 4, we examine the performance of our estimator in a Monte Carlo study. We apply our method to analyze the determinants of economic growth in Section 5. Section 6 concludes. All proofs are provided in Appendix A.

Notation: Through this paper, for a real matrix $A$, let $\|A\|_F = \{\text{tr}(A^T A)\}^{1/2}$ denote its Frobenius norm and $\|A\|_2 = \{\lambda_{\text{max}}(A^T A)\}^{1/2}$ denote its spectral norm, where $\lambda_{\text{min}}(\cdot)$ and $\lambda_{\text{max}}(\cdot)$ denote the minimum and maximum eigenvalues of a square matrix. For a vector $v$, let $\|v\|$ denote its Euclidean norm. We write $p \to$ and $L \to$ to denote convergence in probability and convergence in distribution, respectively. Let $(a_n)$ and $(b_n)$ be positive-valued sequences. We write $a_n = O(b_n)$ if there exists a positive constant $C$ such that $a_n/b_n \leq C$ for large $n$. For two sequences of random variables, $(X_n)$ and $(Y_n)$, we write $Y_n = O_p(X_n)$ if $Y_n/X_n$ is bounded in probability, and $Y_n = o_p(X_n)$ if $Y_n/X_n \overset{p}{\to} 0$, as $n$ goes to infinity.

2 Model and Estimation Procedure

2.1 Panel Data with Interactive Fixed Effects

Following Bai (2009), we consider the following panel data model with interactive fixed effects,

\[
\begin{align*}
    y_{it} &= x_{it}^\top \beta + \varepsilon_{it}, \\
    \varepsilon_{it} &= \lambda_i^\top f_t + u_{it} \\
\end{align*}
\]

where $y_{it}$ denotes the response variable of the individual $i$ in period $t$, $x_{it}$ is the $Q$-dimensional covariate vector, $\beta$ is the $Q$-dimensional vector of parameters to be estimated, and $\varepsilon_{it}$ has a factor structure. More precisely, the relationship between $x_{it}$ and $y_{it}$ described in (1) contains $K$ unobserved common factors, $f_t = (f_{t1}, \ldots, f_{tK})^\top$, and the corresponding factor loadings for individual $i$, $\lambda_i = (\lambda_{i1}, \ldots, \lambda_{iK})$, so that $\lambda_i^\top f_t = \lambda_{i1} f_{t1} + \ldots + \lambda_{iK} f_{tK}$. Further, these quantities are distorted by the idiosyncratic error term, $u_{it}$, which is assumed to be independent of the regressors and the interactive effect components. Throughout the paper, $\lambda_i$, $f_t$, and $u_{it}$ are all unobserved, and the dimension $K$ of the factor loadings does not depend on the cross-section size $N$ or the time series length $T$. This interactive effects setting generalizes the usual additive individual and time effects. For example, if $\lambda_i \equiv 1$, then $\lambda_i^\top f_t = f_t$, whereas if $f_t \equiv 1$ we get $\lambda_i^\top f_t = \lambda_i$.

The above specification is a useful modelling paradigm since it allows individuals to respond differently to common factors (i.e., technological, institutional, environmental, or health factors, among others) (see Andrews (2005)). Similarly, it allows
time-invariant regressors (such as education, gender or ability) to affect heterogeneously along the time. Therefore, a panel data model with interactive fixed effects is more realistic and can be applied in a great variety of studies in economics and other social sciences. For example, in labor economics this specification allows to take into account unobserved features or unmeasured abilities ($\lambda_i$) when we analyze the effect of observed regressors ($x_{it}$) such as education, experience, gender, or marital status on the wage of individual $i$ at period $t$. In macroeconomic studies, the interactive effects enable to account for the heterogeneous impact of unobservable common shocks (i.e., technological shocks, financial crisis) as it is shown in Gian-none and Lenza (2005), where $y_{it}$ is the output growth rate for country $i$ in year $t$, $x_{it}$ is the vector that contains the production inputs (i.e., labor and capital), and $u_{it}$ is the country-specific error term. In finance, $y_{it}$ can be the return of the stock $i$ in period $t$, $x_{it}$ is the vector of observed regressors such as dividend yields, dividend payout ratio, and consumption gap, $f_t$ represents unobserved common factors such as systematic risk, $\lambda_i$ is the exposure to these risks, and $u_{it}$ is the idiosyncratic part of the return.

2.2 Semiparametric Interactive Fixed Effects Model

Our interest is focused on the estimation of the slope coefficient $\beta$. Nevertheless, in most of the above empirical studies, some of the regressors $x_{it}$ are decision variables that are influenced by the unobserved individual heterogeneities ($\lambda_i$) (see Bai and Li (2014)). Therefore, any attempt to estimate $\beta$ directly through standard panel data estimation techniques will lead to inconsistent estimators since the standard panel data transformations (i.e., within or first differences transformation) are unable to remove the unobserved heterogeneities from the statistical model given the multiplicative form of these terms.

To avoid the more than likely omitted variable bias, one method is to control the dependence between the regressors and factor terms using the Mundlak-Chamberlain projection and incorporate this information into the model of interest. More precisely, as noted in Bai (2009), when $\lambda_i$ is correlated with the regressors, it can be projected onto the regressors such that

$$\lambda_i = \theta \bar{x}_i + \gamma_i,$$

where $\bar{x}_i$ is the time average of $x_{it}$, $\theta$ is a $K \times Q$ matrix of unknown parameters, and $\gamma_i$ is a $K \times 1$ idiosyncratic component of the loading coefficients that cannot be explained by the covariates $\bar{x}_i$ (i.e., the projection residuals) and is assumed to have zero mean and is independent of $\bar{x}_i$ and $u_{it}$.

However, there is a high degree of uncertainty surrounding the way in which the regressors may affect the factor loadings and there can be several situations in which
tight functional specifications such as (2) can lead to an inconsistent estimator and misleading inference. To deal with this potential functional misspecification, Connor et al. (2012) propose to model \( \lambda_i \) through some smooth functions of time-invariant regressors (i.e., \( \lambda_i = g(x_i) \), where \( g(\cdot) \) is an unknown smooth function and \( x_i \) can be individual or firm-specific characteristics). Nevertheless, this modelling can be restrictive from the empirical point of view since the threat of omitted variable bias continues if the loading factors are also explained by the time-varying regressors (\( x_{it} \)).

To overcome the above problems, in this paper we propose an alternative nonparametric specification to control the dependence between \( x_{it} \) and \( \lambda_i \) in a more flexible way. In particular, we assume that this dependence can be modelled as

\[
\lambda_i = g(\bar{x}_i) + \gamma_i,
\]

where \( g(\bar{x}_i) = (g_1(\bar{x}_i), \ldots, g_K(\bar{x}_i))^T \) is a \( K \times 1 \) vector of twice continuously differentiable functions. Plugging (3) into (1) and rearranging terms, we get

\[
y_{it} = x_{it}^\top \beta + g(\bar{x}_i)^\top f_t + \gamma_i^\top f_t + u_{it},
\]

which reduces to model (1) when \( g(\cdot) = 0 \). The above regression model still has a factor error structure and it is possible to define the following composed error term,

\[
v_{it} \defeq \sum_{k=1}^K \gamma_{ik} f_{tk} + u_{it}.
\]

Given that \( v_{it} \) is uncorrelated with \( x_{it} \), consistent estimators for \( \beta \) in (4) can be obtained by using standard estimation techniques for panel data models. Let \( y_t \) and \( v_t \) be \( N \times 1 \) vectors of \( y_{it} \) and \( v_{it} \), respectively, \( X_t \) the \( N \times Q \) matrix of regressors, and \( G(\bar{X}) \) the \( N \times K \) matrix of nonparametric functions, \( g_k(\bar{x}_i) \). Then, the model (4) can be written in matrix form as

\[
y_t = X_t \beta + G(\bar{X}) f_t + v_t, \quad t = 1, \ldots, T.
\]

In this situation, instead of projecting on the space of factors \( f_t \), one can consider a sieve estimation for \( G(\bar{X}) \) to estimate \( \beta \) and project on the space expanded by the sieve basis. This has the advantage that we are able to estimate the slope parameter directly circumventing the necessity to estimate the latent factors in a first step.

Before introducing the estimation procedure that we propose in this paper, we recall as an example the polynomial spline function. Let \( \mathcal{X} \) be an interval with end points \( \zeta_0 < \zeta_{M+1} \). A polynomial spline of degree \( d \geq 0 \) on \( \mathcal{X} \) with knot sequence \( \zeta_0 < \zeta_1 < \ldots < \zeta_{M+1} \) is a function that is a polynomial of degree \( d \) on each of the intervals \( [\zeta_0, \zeta_1), \ldots, [\zeta_{M-1}, \zeta_M), [\zeta_M, \zeta_{M+1}) \), and globally has continuous \( d - 1 \)
derivatives for \( d \geq 1 \). The collection of spline functions of a particular degree and knot sequence form a linear space. Specifically, a piecewise constant function, linear spline, quadratic spline, and cubic spline corresponds to \( d = 0, 1, 2, 3 \), respectively. We refer to de Boor (1978) and Schumaker (1980) as a good overview for spline functions.

In order to address the curse of dimensionality in the nonparametric estimation of \( g_k(\cdot) \) related to the dimension of \( \bar{x}_i \), it will be assumed that for each \( k \), \( g_k(\cdot) \) is an additive function of the form

\[
g_k(\bar{x}_i) = \sum_{q=1}^{Q} g_{kq}(\bar{x}_{i,q}). \tag{6}
\]

Suppose that, for each \( k \) and \( q \), the function \( g_{kq}(\cdot) \) can be approximated by some spline function, that is

\[
g_{kq}(\bar{x}_{i,q}) = \sum_{\ell=1}^{J_g} b_{\ell,kq} \phi_\ell(\bar{x}_{i,q}) + R_{kq}(\bar{x}_{i,q}), \quad k = 1, \ldots, K, \quad q = 1, \ldots, Q, \tag{7}
\]

where \( \phi_\ell(\cdot) \)'s are the sieve basis functions. For \( \ell = 1, \ldots, J_g \), \( b_{\ell,kq} \)'s are the sieve coefficients of the \( q \)th additive component of \( g_k(\bar{x}_i) \) corresponding to the \( k \)th factor loading, and \( R_{kq} \) is a “remainder function” that represents the approximation error. Also, \( J_g \) denotes the number of sieve terms which grows slowly as \( N \to \infty \).

For the sake of simplicity, we take the same basis functions in (7). For each \( k \leq K \), \( q \leq Q \) and \( i \leq N \), let us define

\[
b^T_k = (b_{1,k1}, \ldots, b_{J_g,k1}, \ldots, b_{1,kQ}, \ldots, b_{J_g,kQ}) \in \mathbb{R}^{J_gQ},
\]

\[
\phi(\bar{x}_i)^T = (\phi_1(\bar{x}_{i,1}), \ldots, \phi_{J_g}(\bar{x}_{i,1}), \ldots, \phi_1(\bar{x}_{i,Q}), \ldots, \phi_{J_g}(\bar{x}_{i,Q})) \in \mathbb{R}^{J_gQ}.
\]

Thus, the above equation can be rewritten as

\[
g_k(\bar{x}_i) = \phi(\bar{x}_i)^T b_k + \sum_{q=1}^{Q} R_{kq}(\bar{x}_{i,q}). \tag{8}
\]

By considering (8) in matrix form we obtain

\[
G(\bar{X}) = \Phi(\bar{X})B + R(\bar{X}), \tag{9}
\]

where \( \Phi(\bar{X}) = (\phi(\bar{x}_1), \ldots, \phi(\bar{x}_N))^T \) is a \( N \times J_gQ \) matrix of basis functions, \( B = (b_1, \ldots, b_K) \) is a \( J_gQ \times K \) matrix of sieve coefficients, and \( R(\bar{X}) \) is a \( N \times K \) matrix with the \((i,k)\)th element \( \sum_{q=1}^{Q} R_{kq}(\bar{x}_{i,q}) \).
Then, substituting (9) into (5) leads to

\[ y_t = X_t \beta + \Phi(\bar{X}) F_t + \mathcal{R}(\bar{X}) f_t + v_t, \quad t = 1, \ldots, T. \]  

(10)

We want to point out that the residual term of this regression model consists of two parts, the sieve approximation error \( \mathcal{R}(\bar{X}) f_t \) and the idiosyncratic error \( v_t \) which is of the form \( v_t = \Gamma f_t + u_t \), where \( \Gamma = (\gamma_1, \ldots, \gamma_N) \top \) is a \( N \times K \) matrix of unknown loading coefficients.

With the aim of estimating \( \beta \), and taking as a benchmark the idea in Fan et al. (2016), we define \( P_\Phi \) as the projection matrix onto \( \mathcal{X} \), where \( \mathcal{X} \) is the sieve space spanned by the basis functions of \( \bar{X} \). More precisely, \( P_\Phi \) is the \( N \times N \) projection matrix of the form

\[ P_\Phi(X) = \Phi(\bar{X}) \left( \Phi(\bar{X}) \top \Phi(\bar{X}) \right)^{-1} \Phi(\bar{X}) \top. \]  

(11)

Therefore, one can obtain the estimator of \( \beta \) by partialling out the nonparametric part of the factor loadings, i.e.,

\[ \hat{\beta} = \left[ \sum_{t=1}^{T} X_t \top \left\{ I_N - P_\Phi \bar{X} \right\} X_t \right]^{-1} \sum_{t=1}^{T} X_t \top \left\{ I_N - P_\Phi \bar{X} \right\} y_t, \]  

(12)

where \( X_t \top \left\{ I_N - P_\Phi \bar{X} \right\} X_t \) is assumed to be asymptotically nonsingular. It is worth noting that the resulting estimator of \( \beta \) appears as the solution of a partially linear model (see Härdle et al. (2012) for a comprehensive review of the literature), where the nonparametric part is “partialled out”. Although the asymptotic properties of this estimator have been already studied under many alternative sets of assumptions, it is worthwhile to establish these conditions in our context and obtain its asymptotic distribution.

Considering now the estimation of the nonparametric functions, \( g_k(\cdot) \), we have to deal with an identification problem since the elements in \( F_t \) of equation (10) cannot be identified separately. To overcome it, we follow the projected-PCA approach proposed in Fan et al. (2016) to estimate the latent factors and corresponding factor loadings. More precisely, owing to potential correlations between the unobservable effects and the regressors, we treat the matrix of common factors \( F = (f_1, \ldots, f_T) \top \) as fixed-effects parameters to be estimated. Therefore, \( F \) and \( B \) can be estimated jointly through the projected residuals, \( \hat{Y}_t = P_\Phi \tilde{Y}_t \), where \( \tilde{Y}_t \) are the regression residuals defined as

\[ \tilde{Y}_t = Y_t - X_t \hat{\beta}. \]  

(13)

Let \( \hat{Y} = (\hat{Y}_1, \ldots, \hat{Y}_T) \top \), we know that applying PCA to the projected data \( \tilde{Y} = P_\Phi \tilde{Y} \)
is an approximately noiseless problem and its sample covariance has the following approximation:

\[
\frac{1}{T} \hat{Y}^\top \hat{Y} = \frac{1}{T} (Y - X \hat{\beta})^\top P_\Phi (Y - X \hat{\beta}) \approx \frac{1}{T} F G(\bar{X})^\top G(\bar{X}) F^\top.
\]

Therefore, following Fan et al. (2016), we can argue that \( F \) and \( G(\bar{X}) \) can be recovered from the projected data \( \hat{Y} \) under the following identification restrictions: (i) \( T^{-1} F^\top F = I_K \) and (ii) \( G(\bar{X})^\top G(\bar{X}) \) is a diagonal matrix with distinct entries. Then, \( \sqrt{T} \hat{F} \) can be estimated by the eigenvectors associated with the largest \( K \) eigenvalues of the matrix \( T^{-1} \hat{Y}^\top P_\Phi (\bar{X}) \hat{Y} \). In other words, we utilize the projected-PCA approach proposed in Fan et al. (2016) to obtain an estimator for the latent factors in our particular setting with regressors that vary across the individuals and time.

Once we estimated \( \hat{F} \), it is possible to obtain an estimator for the sieve coefficients using a least squares procedure that leads to

\[
\hat{B} = (\hat{b}_1, \ldots, \hat{b}_K) = \frac{1}{T} \left\{ \Phi(\bar{X})^\top \Phi(\bar{X}) \right\}^{-1} \Phi(\bar{X})^\top \hat{Y} \hat{F},
\]

and replacing (14) into (8), we can construct an estimator for the nonparametric functions \( g_k(\cdot) \) of the form

\[
\hat{g}_k(x) = \phi(x)^\top \hat{b}_k, \quad k = 1, \ldots, K.
\]

Let \( \Lambda \) be the \( N \times K \) matrix of factor loadings, then we can estimate \( \hat{\Lambda} = \hat{Y} \hat{F}/T \) (see Fan et al. (2016) for further details). Therefore, the part of the factor loadings in (3) that can be explained by \( \bar{X} \) can be estimated by \( \hat{G}(\bar{X}) = \frac{1}{T} P_\Phi (\bar{X}) \hat{Y} \hat{F} \), whereas the idiosyncratic part can be calculated as \( \hat{\Gamma} = \hat{\Lambda} - \hat{G}(\bar{X}) = \frac{1}{T} (I_N - P_\Phi (\bar{X})) \hat{Y} \hat{F} \).

Finally, note that the number of latent factors, \( K = \text{dim}(f_t) \), is unknown but can be estimated from the data. We follow the approach of Fan et al. (2016) to select \( \hat{K} \) according to the largest ratio of eigenvalues of the matrix \( \hat{Y}^\top P_\Phi (\bar{X}) \hat{Y} \),

\[
\hat{K} = \arg \max_{0 < k < J_q/2} \frac{\lambda_k \left\{ \hat{Y}^\top P_\Phi (\bar{X}) \hat{Y} \right\}}{\lambda_{k+1} \left\{ \hat{Y}^\top P_\Phi (\bar{X}) \hat{Y} \right\}},
\]

where \( \lambda_k \left\{ \hat{Y}^\top P_\Phi (\bar{X}) \hat{Y} \right\} \) denotes the \( k \)-largest eigenvalue of the matrix.

Note that the condition on the true number of factors, i.e., \( 0 < K < J_q/2 \) in (16), is fulfilled naturally since the sieve dimension \( J_q \) grows slowly with the sample size. Interestingly, our estimator for the regression parameters \( \hat{\beta} \) does not require any knowledge of \( K \). However, the number of factors is crucial to the estimation of the factor components as well as to the estimation of standard errors for the regression parameters.
3 Asymptotic Properties

3.1 Assumptions

In this section we analyze the main asymptotic properties of the proposed estimators. With this aim, the following assumptions are considered. We start by introducing some assumptions related to the data generating process in (5). Specifically, about the vector of explanatory variables \( x_{it} \), we assume that, as common in the partially linear regression models, \( x_{it} \) and \( \bar{x}_i \) are related in the following way.

**Assumption 3.1**

\[
x_{itq} = \sum_{q'} h_{qq'}(\bar{x}_{i,q'}) + \pi_{itq}, \quad i = 1, \ldots, N; \quad t = 1, \ldots, T; \quad q = 1, \ldots, Q,
\]

where the \( h_{qq'}(\cdot) \)'s are twice continuously differentiable unknown functions and the \( \pi_{itq} \)'s are random variables with zero mean.

We also need to characterize the asymptotic behavior of \( \bar{x}_i \). We will assume that,

**Assumption 3.2**

\[
\frac{1}{T} \sum_{t} x_{itq} = \mu_{iq} + O_p \left( T^{-1/2} \right), \quad i = 1, \ldots, N; \quad q = 1, \ldots, Q,
\]

as \( T \) tends to infinity. Furthermore, the deterministic sequence of design points, \( \mu_{iq} \), has bounded support \( \mathcal{M} \) and is generated by a design density \( f(\mu) \) that is bounded below and above in \( \mathcal{M} \).

On the smoothness of the nonparametric functions \( h_{qq'}(\cdot) \) and \( g_{kq}(\cdot) \), and on the sieve approximation error, we impose the following assumptions.

**Assumption 3.3**

(i) For all \( k = 1, \ldots, K \) and \( q, q' = 1, \ldots, Q \), the functions \( h_{qq'}(\cdot) \) and \( g_{kq}(\cdot) \) belong to a Hölder class \( \mathcal{G} \) with Hölder coefficient \( 0 < \alpha \leq 1 \),

\[
\mathcal{G} = \left\{ g : |g^{(r)}(s) - g^{(r)}(t)| \leq L|s - t|^{\alpha} \right\}
\]

for some \( L > 0 \).
For the sieve approximation error we assume, for \( \kappa = 2(r + \alpha) \geq 4 \),
\[
\rho_{g,N} = \sup_{\mu \in \mathcal{M}} \left| g_{kq} - \sum_{\ell=1}^{J_g} b_{\ell,kq} \phi_{\ell}(\mu_{i,q}) \right|^2 = \mathcal{O} \left( J_g^{-\kappa} \right),
\]
\[
\rho_{h,N} = \sup_{\mu \in \mathcal{M}} \left| h_{qq'} - \sum_{\ell=1}^{J_h} c_{\ell,qq'} \phi_{\ell}(\mu_{i,q'}) \right|^2 = \mathcal{O} \left( J_h^{-\kappa} \right),
\]
(iii) \( \max_{\ell,k,q} b_{\ell,kq}^2 < \infty \) and \( \max_{\ell,q,q'} c_{\ell,qq'}^2 < \infty \).

As it is remarked in Fan et al. (2016), Assumption 3.3 (ii) is satisfied by the use of common basis functions such as polynomial basis or B-splines. Lorentz (1986) and Chen (2007) show that (i) implies (ii) in this case. We impose the following assumptions on the random matrices \( \pi_t \).

**Assumption 3.4** Let \( \pi_t = (\pi_{1t}, \ldots, \pi_{Nt})^\top \), for \( t = 1, \ldots, T \), be independent random matrices with zero mean and \( \mathbb{E} \| \pi_t \|_F^4 \leq a_0 < \infty \), where \( \pi_{it} = (\pi_{1it}, \ldots, \pi_{qit})^\top \), for \( i = 1, \ldots, N \).

Assumption 3.4 requires serial independence of the \( \pi_t \) and imposes a moment restriction. We impose a strong mixing condition on the factors and on the idiosyncratic error terms. Let \( \mathcal{F}_t^{-\infty} \) and \( \mathcal{F}_t^{\infty} \) denote the \( \sigma \)-algebras generated by \( \{(f_t, u_t) : t \leq 0\} \) and \( \{(f_t, u_t) : t \geq T\} \) respectively. Define the mixing coefficient
\[
\alpha(T) = \sup_{A \in \mathcal{F}_t^{-\infty}, B \in \mathcal{F}_t^{\infty}} |P(A)P(B) - P(AB)|.
\]

**Assumption 3.5**

(i) Let \( \{u_t, f_t\} \) be strictly stationary. In addition, \( \mathbb{E}(u_{it}) = 0 \) and \( \{u_t\} \) is independent of \( \{\pi_t, f_t\} \). Further, \( \{f_t\} \) is independent of \( \{\pi_t\} \).

(ii) There exist constants \( \alpha_1, C_1 > 0 \) such that
\[
\alpha(T) < \exp(-C_1 T^{\alpha_1}).
\]
(iii) There exists a constant $C_2 > 0$, such that
\[
\max_{j \leq N} \sum_{i=1}^{N} |E(u_i u_{ij})| < C_2,
\]
\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} |E(u_i u_{js})| < C_2,
\]
\[
\max_{i \leq N} \frac{1}{NT} \sum_{k=1}^{N} \sum_{m=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} |Cov (u_{i k}, u_{i s})| < C_2.
\]

(iv) There exist constants $\alpha_1, \alpha_2, \alpha_3 > 0$, such that for any $x > 0$, $i \leq N$ and $k \leq K$,
\[
P (|u_{it}| > x) \leq \exp \{- (x/b_1)^{\alpha_2}\}, \]
\[
P (|f_{kt}| > x) \leq \exp \{- (x/b_2)^{\alpha_2}\}.
\]

Assumption 3.5 is standard in factor analysis (Bai, 2003; Stock and Watson, 2002; Fan et al., 2016). Part (ii) is a strong mixing condition for the weak temporal dependence of $\{u_t, f_t\}$, whereas (iii) imposes weak cross-sectional dependence in $\{u_{it}\}_{1 \leq i \leq N, t \leq T}$. Note that the latter condition is usually satisfied when the covariance matrix of the error term $u_{it}$ is sufficiently sparse under the strong mixing condition and is commonly imposed for high-dimensional factor analysis. Finally, (iv) ensures that the tails of $u_{it}$ and $f_{kt}$ are exponential and thus sufficiently light. Now, let $\nu_N$ be
\[
\nu_N = \max_{k \leq K} \frac{1}{N} \sum_{i \leq N} \text{Var} (\gamma_{ik}).
\]

On the random part of the factor loadings, $\gamma_{ik}$, we need to assume the independence from the random part of the covariates. However, we do not need to impose a restrictive i.i.d. assumption. Instead, we only require weak cross-sectional dependence.

**Assumption 3.6**

(i) $E (\gamma_{ik}) = 0$, $\nu_N < \infty$ and
\[
\max_{k \leq K, j \leq N} \sum_{i \leq N} |E \gamma_{ik} \gamma_{jk}| = O (\nu_N).
\]

(ii) $\pi_{it}$ is independent of $\gamma_{ik}$, for $i = 1, \ldots, N$, $t = 1, \ldots, T$ and $k = 1, \ldots, K$.

(iii) $\max_{k \leq K, i \leq N} E (|\gamma_{ik}|)^{2+\delta} < \infty$ for some $\delta > 0$.

For the identification of the factors $F$ and the part of the loadings explained by the covariates, $G (\cdot)$, we need the following assumption.
Assumption 3.7

(i) Almost surely, \( T^{-1}F^\top F = I_K \) and \( G(\mu)^\top G(\mu) \) is a \( K \times K \) diagonal matrix with distinct entries.

(ii) There are two constants, \( c_{\text{min}} \) and \( c_{\text{max}} > 0 \), so that, as \( N \) tends to infinity,

\[
c_{\text{min}} < \lambda_{\text{min}} \left\{ \frac{1}{N} G(\mu)^\top G(\mu) \right\} < \lambda_{\text{max}} \left\{ \frac{1}{N} G(\mu)^\top G(\mu) \right\} < c_{\text{max}}.
\]

Part (i) ensures the separate identification of the factors and factor loadings. This condition is commonly used in the estimation of factor models and corresponds to condition PC1 of Bai and Ng (2013) in the case of un-projected data. Note that part (ii) of Assumption 3.7 ensures that the covariates have a non-vanishing explanatory power on the systemic part of the loadings, \( G(\cdot) \). Finally, for the sieve basis functions, we assume the following.

Assumption 3.8

(i) There are \( c'_{\text{min}} \) and \( c'_{\text{max}} > 0 \), as \( N \) tends to infinity,

\[
c'_{\text{min}} < \lambda_{\text{min}} \left\{ \frac{1}{N} \Phi(\mu)^\top \Phi(\mu) \right\} < \lambda_{\text{max}} \left\{ \frac{1}{N} \Phi(\mu)^\top \Phi(\mu) \right\} < c'_{\text{max}}.
\]

(ii) \( \max_{\ell,q} \phi'_{\ell}(\mu_q) < \infty \), where \( \phi'_{\ell}(\cdot) \) is the first-order derivative function of \( \phi_{\ell}(\cdot) \) and \( q \) stands for the covariates index, \( i \) indicates the cross-sectional dimension, \( \ell \) represents the basis function index.

As remarked by Fan et al. (2016), part (i) of Assumption 3.8 can be achieved through proper normalization of commonly used basis functions, such as splines.

3.2 Limiting Theory

In this section, we present the main theoretical results of the paper. Following a similar reasoning as in the proof of Theorem 3.1 in the Appendix, the estimator (12) is shown to have the following representation,

\[
\hat{\beta} - \beta = \left[ \sum_t X_t^\top \{ I_N - P_{\Phi}(\mu) \} X_t \right]^{-1} \sum_t X_t^\top \{ I_N - P_{\Phi}(\mu) \} (\Lambda f_t + u_t) + O_p \left( \frac{1}{\sqrt{NT}} \right),
\]

so there is a direct dependence of \( \hat{\beta} \) on the unobserved factor loadings through \( (NT)^{-1} \sum_t X_t^\top \{ I_N - P_{\Phi}(\mu) \} \Lambda f_t \). Nevertheless, using (3) and given that it can be proved \( (NT)^{-1/2} \sum_t X_t^\top \{ I_n - P_{\Phi}(\mu) \} G(\bar{X}) f_t = O_p(\rho_{g,N}) + O_p(\rho_{h,N} \rho_{g,N}) + O_p(N^{-1}T^{-1/2}) + \)
\( o_p \left( N^{-1}T^{-1/2} \right) \) (see the proof of Lemma A1 in the Appendix), we have that
\[
\sqrt{NT}(\hat{\beta} - \beta) = \left[ (NT)^{-1} \sum_t X_t^\top \{ I_N - P_{\Phi}(\mu) \} X_t \right]^{-1} \\
\times \left( NT \right)^{-1/2} \sum_t X_t^\top \{ I_N - P_{\Phi}(\mu) \} (\Gamma f_t + u_t) + o_p(\rho_g,N) \\
+ o_p(\rho_h,N \rho_g,N) + o_p \left( N^{-1}T^{-1/2} \right) + o_p \left( N^{-1}T^{-1/2} \right) + o_p(1).
\]

In this situation we can conclude that, for \( N \) and \( T \) sufficiently large, the limiting distribution of \( \sqrt{NT}(\hat{\beta} - \beta) \) only depends on idiosyncratic terms (related to both the error term and factor loadings). In order to analyze this result in detail, the following corollary and theorem provide the \( \sqrt{NT} \)-consistency and asymptotic normality, respectively, of the projection-based interactive fixed effects estimator \( \hat{\beta} \) allowing cross-section and serial correlations.

**COROLLARY 3.1 (Convergence rate)** Under assumptions 3.1-3.8 and if \( J_g = o(N^{1/2}) \), \( J_h = o(N^{1/2}) \), \( T/N \rightarrow 0 \). We have that, as both \( N \) and \( T \) tend to infinity,
\[
\sqrt{NT}(\hat{\beta} - \beta) = o_p(\rho_g,N) + o_p(\rho_h,N \rho_g,N) + o_p \left( N^{-1}T^{-1/2} \right) + o_p \left( N^{-1}T^{-1/2} \right) + o_p(1).
\]

**THEOREM 3.1 (Limiting distribution)** Under the assumptions of Corollary 3.1, as both \( N \) and \( T \) tend to infinity,
\[
\sqrt{NT}(\hat{\beta} - \beta) \overset{\mathcal{L}}{\rightarrow} N \left( 0, \hat{V} \right),
\]
where
\[
\hat{V} \overset{\text{def}}{=} \hat{V}^{-1}_\pi \left( \hat{V}_T + \hat{V}_u \right) \hat{V}^{-1}_\pi,
\]
with
\[
\hat{V}_\pi \overset{\text{def}}{=} \lim_{N,T \rightarrow \infty} \frac{1}{NT} \sum_{t=1}^N \sum_{i=1}^T E \left( \pi_{it} \pi_{it}^\top \right), \\
\hat{V}_T \overset{\text{def}}{=} \lim_{NT \rightarrow \infty} \frac{1}{NT} \sum_{t=1}^T E \left( \pi_t^\top \Gamma \pi_t \right), \\
\hat{V}_u \overset{\text{def}}{=} \lim_{N,T \rightarrow \infty} \frac{1}{NT} \sum_{t=1}^T E \left( \pi_t^\top u_t u_t^\top \pi_t \right).
\]

Therefore, in Corollary 3.1 it is shown that, for \( N \) and \( T \) sufficiently large, \( \hat{\beta} \) is a consistent estimator for \( \beta \) even in the presence of heteroskedasticities over the cross section or serial correlation. Furthermore, in Theorem 3.1 it is also shown that
\( \sqrt{NT}(\hat{\beta} - \beta) \) is asymptotically normally distributed centered at zero.

**Remark 3.1** In contrast to the estimator of Bai (2009), our estimator is unbiased even in the more general case of heteroskedasticity and cross-sectionally and serially correlated error terms. As a comparison, the iterative estimator proposed by Bai (2009), \( \hat{\beta}_{\text{Bai}} \), has bias terms of the following form,

\[
\lim_{N,T \to \infty} \sqrt{NT}E(\hat{\beta}_{\text{Bai}} - \beta) = \left( \frac{T}{N} \right)^{1/2} B_0 + \left( \frac{N}{T} \right)^{1/2} C_0,
\]

whereas \( B_0 = 0 \) under cross-sectional homoskedasticity and uncorrelatedness and \( C_0 = 0 \) under the absence of serial correlation and heteroskedasticity. (See Theorem 3 in Bai (2009) for the specific expressions of \( B_0 \) and \( C_0 \) and for further details). The reason for this result is that our estimator does not require estimating the latent factors in a first step.

**Remark 3.2** It is worth noting that the above asymptotic normality result can be extended to hold uniformly over all data generating processes satisfying our conditions. For this purpose, let \( \{P_{N,T}\} \) be a sequence of probability laws, such that \( \{(y_{it}, x_{it}, \gamma_i, f_t)\} \sim P_{N,T} \) for which \( N,T \to \infty \). Suppose conditions 3.1- 3.8 hold for all \( P \in P_{N,T} \). Statements similar to Corollary 2 of Belloni et al. (2015) can be made. In particular, we have

\[
\max_{q=1,\ldots,Q} \sup_{P \in P_{N,T}} \sup_{t \in \mathbb{R}} \left| P_P \left\{ (NT)^{1/2} \tilde{V}_{jj}^{-1/2} (\hat{\beta}_j - \beta_j) \leq t \right\} - \Phi(t) \right| = o(1),
\]

as \( N,T \to \infty \), where \( \tilde{V}_{jj} \) is the \( j \)-th diagonal element of the asymptotic covariance matrix \( \tilde{V} \) and \( \Phi(\cdot) \) is the cumulative distribution function of the standard normal distribution.

The proof of Theorem 3.1 is provided in Appendix A and the proof of Corollary 3.1 can be obtained from it in a straightforward manner. The key component of the proof is following the Frisch-Waugh Theorem to partial out the effect of the latent factors and corresponding loadings. A second key component of the proof is using results from approximation theory for linear sieves to approximate the nonparametric functions.

Looking at (17), the reader can notice that the asymptotic covariance matrix of \( \hat{\beta} \) has the typical sandwich structure. There are two distinct error terms entering the inner term of the covariance matrix, the idiosyncratic part of the factor loadings, \( \gamma_i \), and the idiosyncratic error term, \( u_{it} \). In the case that both of these terms are i.i.d., with \( \text{Var}(\gamma_{ik}) = \sigma_\gamma^2 \) and \( \text{Var}(u_t) = \sigma_u^2 \), the asymptotic covariance matrix of the estimator simplifies to \( \tilde{V} = \tilde{V}_\pi^{-1}(K\sigma_\gamma^2 + \sigma_u^2) \).
All the previously presented asymptotic results hold regardless of the variance of the idiosyncratic factor loading terms being zero, close to zero or far from zero. If we further assume that the latent factor loadings can be completely explained by the non-parametric functions, i.e. $\Gamma = 0$, our estimator reaches the semiparametric efficiency bound (see Newey (1990) for an overview on semiparametric efficiency bounds). Denote $h_q(x_i) = \sum^Q_{q'=1} h_{qq'}(x_{i,q'})$ and let $H(x_i) = (h_1(x_{i,1}), \ldots, h_Q(x_{i,Q}))^\top$. Then, using the results of Chamberlain (1992) and Li (2000), the efficiency bound of the regression model (5) is given by

$$ J_0 = \mathbb{E} \left\{ (x_{it} - H(\bar{x}_i)) \text{Var} (u_{it})^{-1} (x_{it} - H(\bar{x}_i))^\top \right\}. \quad (18) $$

Using Assumption 3.1 and under i.i.d. errors the bound simplifies to

$$ J_0 = \frac{1}{\sigma^2_u} \mathbb{E} \left( \pi_{it} \pi_{it}^\top \right). \quad (19) $$

Since the inverse of our asymptotic covariance matrix attends this bound, our estimator $\hat{\beta}$ is asymptotically semiparametric efficient in the case of homoskedastic errors.

The asymptotic normality of the estimator enables us to conduct inference on the estimated coefficients. For this purpose, we will propose a consistent estimator for the asymptotic covariance matrix in the next subsection. Furthermore, while the primary focus of this paper is the estimation of the regression parameters, we can also consistently estimate the interactive fixed effects components. The consistency result is established in the following theorem. The proof is adapted from the Theorem 4.1 of Fan et al. (2016) to our case of time-varying covariates.

**THEOREM 3.2** Under assumptions 3.1-3.8 and if $J_g = \mathcal{O}(N^{1/2})$, $J_h = \mathcal{O}(N^{1/2})$, $T/N \to 0$. We have that, as both $N$ and $T$ tend to infinity,

$$ \frac{1}{T} \| \hat{F} - F \|_2^2 = \mathcal{O}_p \left( \frac{1}{N} + \frac{1}{J_g^2} \right), $$

$$ \frac{1}{N} \| \hat{G}(\bar{X}) - G(\bar{X}) \|_2^2 = \mathcal{O}_p \left( \frac{J_g}{N^2} + \frac{J_g}{NT} \right), $$

$$ \max_{k=1,...,K} \sup_{x \in \mathcal{K}} | \hat{g}_k(x) - g_k(x) | = \mathcal{O}_p \left( \frac{J_g}{N} + \frac{J_g}{\sqrt{NT}} + \frac{J_g}{J_g^{3/2}} + J_g \left( \frac{\nu}{N} \right) \right). $$

As it can be seen in Theorem 3.2, the convergence rates of the estimators of the interactive fixed effects components are not affected $\hat{\beta}$. That is, they are identical to the pure factor model case of Fan et al. (2016). This follows from the $\sqrt{NT}$ convergence rate of $\hat{\beta}$. Furthermore, the above results have been obtained assuming $T$ is large, but that is not required and similar results can be obtained for small $T$. 17
REMARK 3.3 In our asymptotic setting with both $N$ and $T$ going to infinity, we inherit the result of Fan et al. (2016) of having a more precise estimate of the factor loadings in the case in which $\gamma_{ik}$ vanishes, that is, $\lambda_{ik} = g_k(\tilde{x}_i)$. In this setting, we have

$$\frac{1}{N} \sum_{i=1}^{N} |\hat{\lambda}_{ik} - \lambda_{ik}| = O_p\left(\frac{1}{(NT)^{1-1/\kappa}} + \frac{1}{N^{2-2/\kappa}}\right).$$

By condition 3.3 (ii) on the smoothness of the nonparametric function, we assume that $\kappa \geq 4$. Therefore the convergence rate is faster than the rate of the conventional PCA estimator $\tilde{\lambda}_{ik}$. Stock and Watson (2002) showed the following rate for the conventional estimator,

$$\frac{1}{N} \sum_{i=1}^{N} |\tilde{\lambda}_{ik} - \lambda_{ik}| = O_p\left(\frac{1}{T} + \frac{1}{N}\right).$$

As mentioned before, the number of factors, $K$, is unknown in practice. We now show that the selection procedure based on the maximal ratio of eigenvalues, described in equation (16), consistently estimates the true number of factors. Following Ahn and Horenstein (2013) and Fan et al. (2016), we need to impose an additional assumption on the dependence structure of idiosyncratic error term. Let $U = (u_1, \ldots, u_T)$ be a $N \times T$ matrix and let $\Sigma_u = E(u_t u_\top t)$ denote the covariance matrix of $u_t$ with dimension $N \times N$.

**Assumption 3.9** The matrix $U$ can be decomposed by

$$U = \Sigma_u^{1/2} EM^{1/2},$$

where

(i) the eigenvalues of $\Sigma_u$ are bounded away from zero and infinity,

(ii) $M$ is a $T \times T$ positive semidefinite deterministic matrix, with eigenvalues bounded away from zero and infinity,

(iii) $E = (e_{it})_{N \times T}$ is a stochastic matrix, where $e_{it}$ are independent across $i$ and $t$, and $e_{it} = (e_{i1}, \ldots, e_{NT})^\top$ are i.i.d. isotropic sub-Gaussian vectors, i.e. there exists $C > 0$, for all $s > 0$, such that

$$\sup_{\|v\| = 1} P \left(|v_\top e_t| > s\right) \leq \exp \left(1 - Cs^2\right).$$

Under this assumption, the matrix $U$ is allowed to exhibit dependence both in the cross-sectional as well as in the time dimension. Note that the cross-sectional
dependence is captured by the matrix \( \Sigma \), while the serial dependence is determined by the matrix \( M \). The following theorem shows the consistency of \( \hat{K} \).

**THEOREM 3.3** Under assumptions 3.1 to 3.9 and if \( J_g = o(\min(\sqrt{N}, T)) \) and \( K < J_h Q/2 \), we have as both \( N \) and \( T \) go to infinity,

\[
P(\hat{K} = K) \to 1.
\]

The theorem adapts Theorem 6.1 of Fan et al. (2016) from the pure factor model to the regression case. The result follows from the \( \sqrt{NT} \)-consistency of \( \hat{\beta} \), so the consistency for the selection of the number of factors is not invalidated by the inclusion of regressors in the model. Consistent estimation of \( K \) is crucial for the estimation of standard errors and for the estimation of the interactive fixed effects components. However, it should be reiterated that our projection-based estimator does not require any knowledge about the true number of factors, that is, it is independent of \( K \).

### 3.3 Consistent Estimation of Standard Errors

To conduct valid inference on the estimated parameters, we present a consistent estimator for the asymptotic covariance matrix \( \tilde{V} \), which is defined in (17). We restrict our attention to the case of heteroskedasticity, assuming that the error terms are cross-sectionally and serially independent. An extension to the case of serial dependence could be easily achieved by following the approach of Newey and West (1986).

In order to get a consistent estimator of \( \tilde{V} \), we are required to have consistent estimators for all components of the covariance matrices. To estimate the inner part of the covariance matrix, \( \tilde{V}_\pi \), we define the estimator,

\[
\hat{V}_\pi = \frac{1}{NT} \sum_{t=1}^{T} X_t^\top \{ I_N - P_{\Phi(\bar{X})} \} X_t.
\]

Analogously, we define the estimators for \( \hat{V}_T \) and \( \hat{V}_u \) as,

\[
\hat{V}_T = \frac{1}{NT} \sum_{t=1}^{T} X_t^\top \{ I_N - P_{\Phi(\bar{X})} \} \hat{\Theta} \hat{\Theta}^\top \{ I_N - P_{\Phi(\bar{X})} \} X_t,
\]

\[
\hat{V}_u = \frac{1}{NT} \sum_{t=1}^{T} X_t^\top \{ I_N - P_{\Phi(\bar{X})} \} \text{diag} \{ \hat{u}_{it}^2, \ldots, \hat{u}_{Nt}^2 \} \{ I_N - P_{\Phi(\bar{X})} \} X_t,
\]

where \( \hat{u}_{it} = y_{it} - X_{it}^\top \hat{\beta} - \hat{\lambda}_i \hat{f}_i \) are the fitted residuals of the projected interactive fixed effects estimator. Then, we have the final estimator for the asymptotic covariance
matrix of the form

\[ \hat{V} = \hat{V}_\pi^{-1} \left( \hat{V}_T + \hat{V}_u \right) \hat{V}_\pi^{-1}. \]  
(21)

The following Proposition shows the consistency of \( \hat{V} \).

**Proposition 3.1** Assume that the conditions of Theorem 3.1 hold. Then \( \hat{V}_\pi \overset{p}{\longrightarrow} \tilde{V}_\pi \). In addition, if \( u_{it} \) are serially and cross-sectionally uncorrelated, we have \( \hat{V}_T \overset{p}{\rightarrow} \tilde{V}_T \) and \( \hat{V}_u \overset{p}{\rightarrow} \tilde{V}_u \). As a consequence, \( \hat{V} \overset{p}{\rightarrow} \tilde{V} \).

The proof of Proposition 3.1 is provided in Appendix A.4.

Using the asymptotic normality of \( \hat{\beta} \), the estimator of the asymptotic variance-covariance matrix (21) enables us to construct asymptotically valid confidence intervals. Additionally, we can conduct Wald-type tests on \( d < Q \) linear restrictions on the parameter vector of the form \( H_0 : R\beta = r \), where \( R \) is a \( d \times Q \) matrix and \( r \) is a \( d \times 1 \) vector. By the asymptotic normality of \( \hat{\beta} \) and the consistency result of Proposition 3.1, the Wald test statistic has the usual asymptotic \( \chi^2 \) distribution with \( d \) degrees of freedom.

## 4 Numerical Studies

In this section, we evaluate the finite-sample performance of our estimator in a simulation study. We are interested both in the estimation of the vector of regression parameters, \( \beta \), and the interactive fixed effects parameter matrices, \( F \) and \( G(X) \) (in the following abbreviated with \( G \)). Throughout the study, we set the number of factors, \( K = 3 \), and the dimension of covariates, \( Q = 3 \). The true regression coefficients are set to \( \beta = (2, 1, -1)^\top \). The covariates are generated by setting \( x_{itq} = \bar{x}_{iq} + \pi_{itq} \), where \( \bar{x}_{iq} \sim N(1, 0.5) \) and \( \pi_{itq} \sim N(0, 0.5) \) are both i.i.d. For the factors, we assume independent \( f_{kt} \sim N(0, 1) \).

The factor loadings are set to \( \lambda_{ik} = g_k(\bar{x}_{i*}) + \gamma_{ik} \), where \( g_1(\bar{x}_{i*}) = a_1\bar{x}_{i,1}^2 + b_1\bar{x}_{i,2}, \)
\( g_2(\bar{x}_{i*}) = a_2\bar{x}_{i,2}^2 + b_2\bar{x}_{i,3} \) and \( g_3(\bar{x}_{i*}) = a_3\bar{x}_{i,3}^2 + b_3\bar{x}_{i,1} \) with \( a_k, b_k \sim U[-1, 1] \). In order to satisfy the identification condition on the interactive effects components (Assumption 3.7 (i)), we further transform the factors and loadings. We set \( F_0 \) to \( \sqrt{T} \) times the \( K \) eigenvectors of the matrix \( FG^\top GF^\top \). We proceed with setting \( G_0 = \frac{1}{2}GF^\top F_0 \).

Finally, for the idiosyncratic terms we consider the case of normally distributed errors, i.e., \( u_{it} \sim N(0, 1) \) and \( \gamma_i \sim N(0, 0.05) \). As a robustness check, we consider a second setting in which \( u_{it} \sim t_{10} \) and \( \gamma_i \sim t_{10}/20 \). In this numerical study, we rely on polynomial spline basis functions and \( J_g = 3 \). For each setting, \( S = 1000 \) simulations are conducted.
As for performance measures, we consider the root mean square error (RMSE) and the bias,

\[
RMSE = \sqrt{\frac{1}{SQ} \sum_{s=1}^{S} \sum_{q=1}^{Q} (\hat{\beta}_{q,s} - \beta_q)^2},
\]

\[
Bias = \frac{1}{Q} \sum_{q=1}^{Q} \left| \frac{1}{S} \sum_{s=1}^{S} \hat{\beta}_{q,s} - \beta_q \right|.
\]

We compare the performance of our projection-based interactive fixed effects (P-IFE) estimator for \( \beta \) with the pooled OLS (POLS) estimator and with the principal component-based interactive fixed effects (PC-IFE) estimator of Bai (2009). The simulation results under Gaussian disturbances for different values of \( N \) and \( T \) are reported in Table 1.

The RMSE and the bias of the P-IFE estimator can be effectively reduced with increasing \( N \) and \( T \). In comparison to the alternative estimators, our estimator achieves the lowest RMSE in all the settings we consider, except for the setting with \( N = 20 \) and \( T = 50 \). The efficiency gain relative to the PC-IFE estimator is the largest in cases with small time-series dimension \( T \). The advantage seems to vanish in settings with large \( N \) and \( T \). In terms of bias, the P-IFE estimator performs slightly better in situations where \( T \) is small and slightly worse than the PC-IFE estimator in intermediate sample sizes. Again, the performance of the estimators converges if both \( N \) and \( T \) are large. The simulation results for the \( t \)-distributed error terms are reported in Table 2. As in the case of normal errors, the P-IFE estimator performs best in terms of RMSE among all the estimators we consider, except again from the setting with \( N = 20 \) and \( T = 50 \). The same holds for the bias, while as before, the performances of the P-IFE and PC-IFE estimators converge with increasing \( N \) and \( T \).

| \( N \) | \( T \) | RMSE P-IFE | RMSE POLS | RMSE PC-IFE | Bias P-IFE | Bias POLS | Bias PC-IFE |
|---|---|---|---|---|---|---|---|
| 20 | 10 | 0.1743 | 0.2260 | 0.2390 | 0.0030 | 0.0056 | 0.0072 |
| 50 | 10 | 0.0967 | 0.2117 | 0.1315 | 0.0051 | 0.0092 | 0.0072 |
| 100 | 10 | 0.0664 | 0.2071 | 0.0871 | 0.0011 | 0.0020 | 0.0022 |
| 20 | 50 | 0.0805 | 0.1091 | 0.0773 | 0.0030 | 0.0014 | 0.0014 |
| 50 | 50 | 0.0426 | 0.0926 | 0.0444 | 0.0018 | 0.0048 | 0.0018 |
| 100 | 50 | 0.0291 | 0.0947 | 0.0299 | 0.0013 | 0.0063 | 0.0010 |
| 100 | 100 | 0.0207 | 0.0635 | 0.0211 | 0.0004 | 0.0024 | 0.0004 |
| 200 | 100 | 0.0147 | 0.0645 | 0.0150 | 0.0004 | 0.0021 | 0.0004 |
| 500 | 100 | 0.0090 | 0.0666 | 0.0092 | 0.0002 | 0.0030 | 0.0002 |

Table 1. Simulation results for the P-IFE estimator, the POLS estimator and the PC-IFE estimator under Gaussian error terms, averaged over 1000 iterations.
| N  | T  | RMSE | Bias |
|----|----|------|------|
| 20 | 10 | 0.1986 | 0.0016 |
| 50 | 10 | 0.1107 | 0.0039 |
| 100| 10 | 0.0744 | 0.0015 |
| 20 | 50 | 0.0897 | 0.0008 |
| 50 | 50 | 0.0498 | 0.0012 |
| 100| 50 | 0.0334 | 0.0002 |
| 100| 100| 0.0230 | 0.0003 |
| 200| 100| 0.0165 | 0.0001 |
| 500| 100| 0.0102 | 0.0004 |

Table 2. Simulation results for the P-IFE estimator, the POLS estimator and the PC-IFE estimator under $t$-distributed error terms, averaged over 1000 iterations.

We now evaluate the estimation performance for the interactive fixed effects components, $F$ and $G$. Following the simulation design of Fan et al. (2016), we set $\Gamma = 0$. Our estimators are obtained by applying the projected principal component method (PPCA) on the residuals of our P-IFE estimator. We compare the performance to that of the standard principal component (PCA) method without projection. As for performance measures, we choose the max norm and the Frobenius norm. The idiosyncratic error terms are again normally distributed, $u_{it} \sim N(0, 1)$. We consider the case of $T = 10$ or 50 with $N$ varying from 25 to 500. Additionally, we consider the case of $N = 200$ being fixed and $T$ ranging from 25 to 500. The simulation results for the factors, reported in Figure 1, reveal that the PPCA method outperforms standard PCA without projecting the data in all scenarios considered. We also report the results for the factor loadings in Figure 2. As expected, in any of our settings, the PPCA method dominates the standard PCA method. The simulation results also show that the estimation error for the factors and the loadings can be effectively reduced with increasing $T$.

5 Application: Determinants of Economic Growth

5.1 Data and Descriptive Statistics

As an empirical application for our interactive fixed effects estimator, we study the determinants of economic growth. We refer to Durlauf et al. (2005) for a comprehensive review of the growth literature. While many studies focus on a cross-sectional analysis (see for instance Barro (1991)), there are also numerous studies employing a panel data approach with country-specific fixed effects (Acemoglu et al., 2019; Islam, 1995). However, Lu and Su (2016) argue that economic growth rates might
Figure 1. Estimation error of factors $F$, estimated via Projected PCA (solid red line) and PCA (dashed blue line), averaged over 1000 iterations. Upper two panels: $T$ fixed, $N$ grows, bottom panels: $N$ fixed, $T$ grows.
Figure 2. Estimation error of factor loadings $G$, estimated via Projected PCA (solid red line) and PCA (dashed blue line), averaged over 1000 iterations. Upper two panels: $T$ fixed, $N$ grows, bottom panels: $N$ fixed, $T$ grows.
not be solely determined by observable regressors, but could also be influenced by latent factors or shocks. Our projection-based interactive fixed-effect estimator is well suited as it is flexible enough to model such latent factors.

The yearly data of GDP growth rates and the country-specific characteristics are obtained from the Penn World Table (PWT) and from the World Bank World Development Indicators (WDI). Our sample contains 129 countries in a time period from 1991–2019, \( N = 129 \) and \( T = 29 \). Countries with incomplete data availability or which did not exist yet in 1991 are excluded from our analysis. Our dependent variable is the real GDP growth rate per capita. The set of regressors is identical to the regressors in Lu and Su (2016). Summary statistics of all dependent and independent variables can be found in Table 3. Figure 3 shows the time series of the mean growth rates, averaged over all countries in our sample. We also visualize the time series of the cross-sectional 5% and 95%-quantiles of the growth rates in the same figure.

| Variable  | Description                          | Mean   | Median  | Min    | Max    | Data   |
|-----------|--------------------------------------|--------|---------|--------|--------|--------|
| Growth    | Annual GDP growth per capita         | 2.96   | 2.54    | -67.29 | 141.63 | PWT    |
| Young     | Age dependency ratio                 | 54.13  | 49.92   | 14.92  | 107.40 | WDI    |
| Fert      | Fertility rate                       | 3.23   | 2.69    | 1.09   | 7.7    | WDI    |
| Life      | Life expectancy                      | 68.30  | 71.21   | 26.17  | 84.36  | WDI    |
| Pop       | Population growth                    | 1.70   | 1.51    | -6.54  | 19.14  | PWT    |
| Invpri    | Price level of investment            | 0.54   | 0.50    | 0.01   | 7.98   | PWT    |
| Con       | Consumption share                    | 0.64   | 0.65    | 0.09   | 1.56   | PWT    |
| Gov       | Government consumption share         | 0.17   | 0.17    | 0.01   | 0.75   | PWT    |
| Inv       | Investment share                     | 0.22   | 0.22    | 0.00   | 0.92   | PWT    |

Table 3. Summary statistics and data sources of dependent and independent variables.

Figure 3. Time series of average annual real GDP growth rate per capita (solid line) and time series of 5% and 95%-quantiles (dashed lines).
5.2 Estimation Results

We first fit our projection-based interactive fixed effects model using the complete sample of \( N = 129 \) countries and the complete list of regressors. The estimation results can be found in Table 4. Consumption share, government consumption share and fertility rate have a significant negative impact on the growth rates, while the age dependency ratio has a significant positive influence. We also estimate a restricted model which only considers the significant variables from the full model. The parameter estimates and standard errors do not change substantially compared to the full model. Our list of significant variables overlaps with those identified by Lu and Su (2016), however they also include the investment share but do not include the fertility rate.

|       | Estimate | Std. Errors |       | Estimate | Std. Errors |
|-------|----------|-------------|-------|----------|-------------|
| Con   | -0.0570  | 0.0134      |       | -0.0633  | 0.0125      |
| Gov   | -0.0865  | 0.0281      |       | -0.0937  | 0.0265      |
| Inv   | 0.0211   | 0.0220      |       | -        | -           |
| Invpri| -0.0009  | 0.0066      |       | -        | -           |
| Young | 0.0008   | 0.0002      |       | 0.0008   | 0.0002      |
| Fert  | -0.0124  | 0.0038      |       | -0.0113  | 0.0039      |
| Life  | -0.0003  | 0.0004      |       | -        | -           |
| Pop   | 0.1127   | 0.1878      |       | -        | -           |

Table 4. Estimation results for the projected IFE estimator based on the whole sample. *, **, *** indicate the significance at 5%, 1% and 0.1% level.

In contrast to standard panel models such as a country-specific fixed effects model, we are also able to estimate the latent factors and corresponding factor loadings. We select \( K = 4 \) as the number of factors, according to the procedure based on the ratio of eigenvalues described in equation (16). The three estimated latent factors can be found in Figure 4. They can be interpreted as unobserved macro risk factors and the loadings measure the exposure of a given country to these risk factors.

We now restrict our analysis to the subset of countries which are members of the OECD (Organisation for Economic Cooperation and Development). The estimation results can be found in Table 5. The signs of the estimated parameters are consistent with the previous regression based on the full sample. However, the coefficient of the population growth now has a negative sign. The list of significant variables additionally includes the investment share, whereas the government consumption share becomes insignificant. The results of the restricted model, for which we only include the significant variables, again does not deviate strongly from the full model.
The number of factors for the OECD sample is $K = 4$, based on the ratio of eigenvalues. Figure 5 shows the estimated first and second latent factors of the OECD sample. The first factor clearly represents a risk factor for the overall market condition. It increases after the bust of the dot-com bubble in 2000 and it has another sharp peak in the aftermath of the financial crisis in 2009. For all 30 OECD countries in our sample, the sign of the loading parameters associated with the first factor is negative. This implies that a positive-valued shock to the first factor leads to an overall reduction in the GDP growth rates for all OECD countries. The interpretation of the second factor requires a little more attention, as the factor loadings have different signs for different countries. Interestingly, three of the four countries with the largest estimated loading parameters (Ireland, Greece and Spain) belong to the list of PIIGS states, which suffered the most during the Euro crisis. Accordingly, we can observe that the second factor takes a negative value at the beginning of the Euro crisis in 2010.

The fundamental tenet of our model is that the covariates are assumed to have sufficient explanatory power on the latent factor loadings. It is thus a crucial task to check whether this is indeed the case. For this purpose, we take a closer look at the two components of the matrix of estimated loading coefficients. Recall the following decomposition, $\hat{\Lambda} = \hat{G}(\bar{X}) + \hat{\Gamma}$, where $\hat{G}(\bar{X})$ represents the estimated systematic part of the loadings, i.e., the part which is explained by the covariates, and $\hat{\Gamma}$ represents the estimated random part. In Table 6, we calculate the Frobenius norm and the max norm for both matrices as measures of their relative importance. We calculate the norms both for the full sample and the sample of OECD countries. It is evident that the systematic part dominates the random part. Namely, the larger proportion of the factor loadings can be explained by the nonparametric functions. This provides evidence for the validity of our projection-based approach to interactive fixed effects.
Table 6. Estimation results for the two components of the factor loadings, the systemic part \( \hat{G} \) and the random part \( \hat{\Gamma} \).

| All countries | OECD countries |
|---------------|----------------|
| \( \| \cdot \|_F \| \cdot \|_{\max} \) | \( \| \cdot \|_F \| \cdot \|_{\max} \) |
| \( \hat{G} \) | 0.5448 0.2308 |
| \( \hat{\Gamma} \) | 0.3978 0.0942 |

Figure 4. Estimated four factors for the whole sample, \( \hat{F}_1 \), \( \hat{F}_2 \), \( \hat{F}_3 \) and \( \hat{F}_4 \).

Figure 5. Left panel: estimated first factor with shaded blue areas indicating the dot-com bubble and the financial crisis. Right panel: estimated second factor with shaded blue area indicating the Euro crisis.

6 Conclusion

We propose a new estimator for the regression parameters in a panel data model with interactive fixed effects. The key idea of our estimator is to partial out the interactive effects by explicitly modelling the factor loadings as nonparametric functions of the time averages of covariates. Following the Frisch-Waugh theorem, the estimator takes the form of a partial least squares estimator, which partials out the effect of the
latent factors. We show that our estimator is $\sqrt{NT}$-consistent with an asymptotic normal distribution. In the special case of homoskedasticity and if the loadings can be completely explained by the covariates, our estimator reaches the semiparametric efficiency bound. An important feature of our estimator is that it does not require the estimation of the number of factors in advance.

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**A APPENDIX: proofs for Section 3**

In order to show Theorem 3.1, we first present a Lemma.

**LEMMA A.1** Let

$$\tilde{\beta} = \left[ \sum_{t=1}^{T} X_t^\top \{I_N - P_\Phi(\mu)\} X_t \right]^{-1} \sum_{t=1}^{T} X_t^\top \{I_N - P_\Phi(\mu)\} y_t,$$

(22)

where

$$P_\Phi(\mu) = \Phi(\mu) \left\{ \Phi(\mu)^\top \Phi(\mu) \right\}^{-1} \Phi(\mu)^\top,$$

$$\Phi(\mu) = (\phi(\mu_1), \ldots, \phi(\mu_N))^\top,$$

$$\phi(\mu_i)^\top = (\phi_1(\mu_{i1}), \ldots, \phi_Jg(\mu_{i1}), \ldots, \phi_1(\mu_{iQ}), \ldots, \phi_Jg(\mu_{iQ})).$$

Then, under assumptions 3.1 to 3.8 and if $J_g = O(N^{1/2})$ and $J_h = O(N^{1/2})$ we have that, as both $N$ and $T$ tend to infinity,

$$\sqrt{NT} \left( \tilde{\beta} - \beta \right) \xrightarrow{d} N \left( 0, \tilde{V} \right),$$

29
where

$$\tilde{V} = \tilde{V}_\pi^{-1} \left( \tilde{V}_T + \tilde{V}_u \right) \tilde{V}_\pi^{-1},$$

(23)

with

$$\tilde{V}_\pi = \lim_{N,T \to \infty} \frac{1}{NT} \sum_{t=1}^T \sum_{i=1}^N E \left( \pi_{it} \pi_{it}^\top \right),$$

$$\tilde{V}_T = \lim_{N,T \to \infty} \frac{1}{NT} \sum_{t=1}^T E \left( \pi_{tT} \Gamma \Gamma^\top \pi_{tT} \right),$$

$$\tilde{V}_u = \lim_{N,T \to \infty} \frac{1}{NT} \sum_{t=1}^T E \left( \pi_{tT} \text{ diag} \left\{ u_{tT} u_{tT}^\top \right\} \pi_{tT} \right).$$

A.1 Proof of Lemma A.1

If we substitute (5) into (22) and rearrange terms, we obtain:

$$\tilde{\beta} - \beta = \left[ \sum_{t=1}^T X_t^\top \{ I_N - P_\Phi(\mu) \} X_t \right]^{-1} \sum_{t=1}^T X_t^\top \{ I_N - P_\Phi(\mu) \} G(\bar{X}) f_t$$

$$+ \left[ \sum_{t=1}^T X_t^\top \{ I_N - P_\Phi(\mu) \} X_t \right]^{-1} \sum_{t=1}^T X_t^\top \{ I_N - P_\Phi(\mu) \} \Gamma f_t$$

$$+ \left[ \sum_{t=1}^T X_t^\top \{ I_N - P_\Phi(\mu) \} X_t \right]^{-1} \sum_{t=1}^T X_t^\top \{ I_N - P_\Phi(\mu) \} u_t.$$

We first show that

$$\frac{1}{NT} \sum_{t=1}^T X_t^\top \{ I_N - P_\Phi(\mu) \} X_t = \frac{1}{NT} \sum_t \pi_{tT} \pi_{tT} + O_p(1).$$

(24)

Note that, by Assumption 3.1,

$$X_t = H(\bar{X}) + \pi_t, \quad t = 1, \ldots, T,$$

(25)

and an element \((i,q)\) of \(H(\bar{X})\) is

$$h_q(\bar{x}_i) = \sum_{q'} h_{qq'}(\bar{x}_{i,q'}).$$

Now, by Assumption 3.3 and a Taylor expansion we have that

$$X_t = H(\mu) + \pi_t + O_p \left( T^{-1/2} \right), \quad t = 1, \ldots, T,$$

(26)
with \((i, q)\)-th element \(h_q(\mu_i) = \sum_{q'} h_{qq'}(\mu_{iq'})\). Hence, substituting (26) into the left hand side of (24) and making \(M_\Phi(\mu) = I_N - P_\Phi(\mu)\), we obtain that

\[
\frac{1}{NT} \sum_{t=1}^T X_t^\top M_\Phi(\mu) X_t = \frac{1}{N} H(\mu)\top M_\Phi(\mu) H(\mu) + \frac{1}{NT} \sum_t \pi_t^\top M_\Phi(\mu) \pi_t
\]

\[+ 2H(\mu)\top M_\Phi(\mu) \frac{1}{NT} \sum_t \pi_t + O_p \left( N^{-1}T^{-1/2} \right) + O_p \left( N^{-1}T^{-1/2} \right). \tag{27} \]

Using Assumption 3.3, \(h_{qq'}(\cdot)\) can be approximated by a linear combination of basis functions, that is,

\[
h_{qq'}(\mu_{iq'}) = \sum_{\ell=1}^{J_h} c_{\ell,qq'} \phi_\ell(\mu_{iq'}) + R_{qq'}(\mu_{iq'}), \quad q, q' = 1, \ldots, Q. \tag{28} \]

where \(\phi_\ell(\cdot)\)'s are the basis functions. The \(c_{\ell,qq'}\)'s are the sieve coefficients of the \(q\)-th additive component of \(h_{qq'}(\cdot)\), and \(R_{qq'}(\cdot)\) is the remainder term that represents the approximation error. Also, \(J_h\) denotes the number of sieve terms which grows slowly as \(N \to \infty\).

For the sake of simplicity, we take the same basis functions in (7). For each \(q, q' \leq Q\) and \(i \leq N\), let us define

\[
c_q^\top = (c_{1,q1}, \ldots, c_{J_h,q1}, \ldots, c_{1,qQ}, \ldots, c_{J_h,qQ}) \in \mathbb{R}^{J_hQ},
\]

\[
\phi(\mu_i)^\top = (\phi_1(\mu_{i1}), \ldots, \phi_{J_h}(\mu_{i1}), \ldots, \phi_1(\mu_{iQ}), \ldots, \phi_{J_h}(\mu_{iQ})) \in \mathbb{R}^{J_hQ}.
\]

Thus, equation (28) can be rewritten as

\[
h_q(\mu_i) = \phi(\mu_i)^\top c_q + \sum_{q'=1}^Q R_{qq'}(\mu_{iq'}). \tag{29} \]

By considering (29) in matrix form we obtain

\[
H(\mu) = \Phi(\mu) C + R(\mu), \tag{30} \]

where \(\Phi(\mu) = (\phi(\mu_1), \ldots, \phi(\mu_N))^\top\) is a \(N \times J_hQ\) matrix of basis functions, \(C = (c_1, \ldots, c_Q)\) is a \(J_hQ \times Q\) matrix of sieve coefficients, and \(R(\mu)\) is a \(N \times Q\) matrix with the \((i, q)\)-th element \(\sum_{q'=1}^Q R_{qq'}(\mu_{iq'})\).

Using (30), the first term of the right hand side of (27) is then

\[
\frac{1}{N} H(\mu)\top M_\Phi(\mu) H(\mu) = \frac{1}{N} \{ H(\mu) - \Phi(\mu) C \}\top M_\Phi(\mu) \{ H(\mu) - \Phi(\mu) C \}.
\]

Let \(H(\mu) = (H_1, \ldots, H_Q)\). It is easy to see that the \(q\)-th element of the right hand
side of the above term can be written as
\[
\frac{1}{N} \left\{ H_q(\mu) - \Phi(\mu)c_q \right\}^\top M\Phi(\mu) \left\{ H_q(\mu) - \Phi(\mu)c_q \right\}
\]
\[
\leq \lambda_{max} \left\{ M\Phi(\mu) \right\} \frac{1}{N} \sum_{i=1}^{N} \left\{ h_q(\mu_i) - \phi(\mu_i)^\top c_q \right\}^2
\]
\[
\leq \frac{1}{N} \sum_{i=1}^{N} \left\{ h_q(\mu_i) - \phi(\mu_i)^\top c_q \right\}^2
\]
\[
\leq \max_i \left\{ h_q(\mu_i) - \phi(\mu_i)^\top c_q \right\}^2 = \mathcal{O}\left( \rho_{h,N}^2 \right),
\]
where \( \rho_{h,N} = \sup_{\mu \in \mathcal{M}} \left| h_q(\mu) - \phi(\mu)^\top c_q \right| \). Note that under assumption 3.3 \( \rho_{h,N} = \mathcal{O}\left( J_h^{-2} \right) \) (see Schumaker (1980), Theorem 6.27) and therefore,
\[
\frac{1}{N} H(\mu)^\top M\Phi(\mu) H(\mu) = \mathcal{O}\left( J_h^{-4} \right).
\]
Furthermore, following the same line as in the proof before and assuming that \( \frac{J_g}{N} \to 0 \), we have that
\[
\frac{1}{NT} \sum_{t} \pi_t^\top M\Phi(\mu)\pi_t = \frac{1}{NT} \sum_{t} \pi_t^\top \pi_t + \mathcal{O}_p(1).
\]
This follows from Assumptions 3.4 and 3.5 and the weak law of large numbers. Finally, noting that for the \( q \)-th element of the right hand side of equation (27)
\[
\text{Var} \left[ \left\{ H_q(\mu) - \Phi(\mu)c_q \right\}^\top M\Phi(\mu) \frac{1}{NT} \sum_{t} \pi_t \right] = \mathcal{O}\left( \frac{1}{NT} \rho_{h,N}^2 \right),
\]
this implies that
\[
H(\mu)^\top M\Phi(\mu) \frac{1}{NT} \sum_{t} \pi_t = \mathcal{O}_p\left( \frac{1}{\sqrt{NT}} \rho_{h,N} \right).
\]
This closes the proof of (24). Following the same line we can show that,
\[
\frac{1}{\sqrt{NT}} \sum_{t=1}^{T} X_t^\top \left\{ I_N - P\Phi(\mu) \right\} G(\bar{X}) f_t = \mathcal{O}_p(1).
\]
Note that, by Assumptions 3.2 and 3.3 and Taylor expansion
\[
\frac{1}{\sqrt{NT}} \sum_{t=1}^{T} X_t^\top M\Phi(\mu) G(\bar{X}) f_t
\]
\[
= \frac{1}{\sqrt{NT}} \sum_{t=1}^{T} X_t^\top M\Phi(\mu) G(\mu) f_t + \mathcal{O}_p\left( N^{-1}T^{-1/2} \right) + \mathcal{O}_p\left( N^{-1}T^{-1/2} \right).
\]
Furthermore, by (26) we have that
\[
\frac{1}{\sqrt{NT}} \sum_{t=1}^{T} X_t^\top M_\Phi(\mu) G(\mu) f_t
\]
\[
= \frac{1}{\sqrt{NT}} \sum_{t=1}^{T} \{H(\mu) - \Phi(\mu) C\}^\top M_\Phi(\mu) \{G(\mu) - \Phi(\mu) B\} f_t
\]
\[
+ \frac{1}{\sqrt{NT}} \sum_{t=1}^{T} \pi_t^\top M_\Phi(\mu) \{G(\mu) - \Phi(\mu) B\} f_t + o_p(1).
\]

It is easy to show that
\[
\frac{1}{\sqrt{NT}} \sum_{t=1}^{T} \{H(\mu) - \Phi(\mu) C\}^\top M_\Phi(\mu) \{G(\mu) - \Phi(\mu) B\} f_t = O(\rho_{h,N} \rho_{g,N}).
\]

where \(\rho_{g,N} = \sup_M \left| g_k(\mu_i) - \phi(\mu_i)^\top b_k \right| \) and \(b_k = (b_{1,k1}, \ldots, b_{I_g,k1}, \ldots, b_{1,kQ}, \ldots, b_{I_g,kQ}) \in \mathbb{R}^{J_g Q} \),

\(\phi(\mu_i)^\top = (\phi_1(\mu_{i1}), \ldots, \phi_{I_g}(\mu_{i1}), \ldots, \phi_1(\mu_{iQ}), \ldots, \phi_{I_g}(\mu_{iQ})) \in \mathbb{R}^{J_g Q} \).

Furthermore, for the \(q\)-th element of \(\frac{1}{\sqrt{NT}} \sum_{t=1}^{T} \pi_t^\top M_\Phi(\mu) \{G(\mu) - \Phi(\mu) B\} f_t \) we have
\[
\text{Var} \left[ \frac{1}{\sqrt{NT}} \sum_{t=1}^{T} \pi_t^\top M_\Phi(\mu) \{G(\mu) - \Phi(\mu) B\} f_t \right] = O(\rho_{g,N}^2),
\]
and therefore,
\[
\frac{1}{\sqrt{NT}} \sum_{t=1}^{T} \pi_t^\top M_\Phi(\mu) \{G(\mu) - \Phi(\mu) B\} f_t = o_p(\rho_{g,N}).
\]

Finally, using (30) we show that
\[
\frac{1}{\sqrt{NT}} \sum_{t=1}^{T} X_t^\top M_\Phi(\mu) \Gamma f_t
\]
\[
= \frac{1}{\sqrt{NT}} \sum_{t=1}^{T} \{H(\mu) - \Phi(\mu) C\}^\top M_\Phi(\mu) \Gamma f_t
\]
\[
+ \frac{1}{\sqrt{NT}} \sum_{t=1}^{T} \pi_t^\top M_\Phi(\mu) \Gamma f_t,
\]
and
\[
\frac{1}{\sqrt{NT}} \sum_t X_t^\top M_\Phi (\mu) u_t = \frac{1}{\sqrt{NT}} \sum_t \{H (\mu) - \Phi (\mu) C\}^\top M_\Phi (\mu) u_t + \frac{1}{\sqrt{NT}} \sum_t \pi_t^\top M_\Phi (\mu) u_t.
\]

Following the same lines as before, by assumptions 3.5 and 3.6,
\[
\frac{1}{\sqrt{NT}} \sum_t \{H (\mu) - \Phi (\mu) C\}^\top M_\Phi (\mu) \Gamma_f t = O_p (\rho_h, N\nu_N), \tag{34}
\]
and
\[
\frac{1}{\sqrt{NT}} \sum_t \{H (\mu) - \Phi (\mu) C\}^\top M_\Phi (\mu) u_t = O_p (\rho_h, N). \tag{35}
\]

Finally,
\[
\frac{1}{\sqrt{NT}} \sum_t X_t^\top M_\Phi (\mu) \{\Gamma_f t + u_t\} = \frac{1}{\sqrt{NT}} \sum_t \pi_t^\top \{\Gamma_f t + u_t\} + o_p(1). \tag{36}
\]

Finally, by assumptions 3.4, 3.5 (iv) and 3.6 (iii) we have that \(E(|\pi_t q(\gamma^T t f_t + u_t)|^{2+\delta}) < \infty\) for some \(\delta > 0\). This follows from the fact that for constants \(b_1, \alpha_2 > 0\),
\[
E \left( |u_t|^{2+\delta} \right) = \int_0^\infty (2 + \delta)t^{1+\delta} P (|u_t| > x) \, dt \tag{37}
\]
\[
\leq \int_0^\infty (2 + \delta)t^{1+\delta} \exp \left\{ - \left( \frac{t}{b_1} \right)^{\alpha_2} \right\} \, dt < \infty. \tag{38}
\]

In a similar way, it can be shown that \(E(|f_{kt}|^{2+\delta}) < \infty\). By the stationarity and mixing rate assumption in condition 3.5 we can now apply a central limit theorem for stationary sequences of random variables satisfying a strong mixing condition (Bradley, 1985; Ibragimov, 1962). We have that
\[
\frac{1}{\sqrt{NT}} \sum_t \pi_t^\top \{\Gamma_f t + u_t\} \overset{L}{\rightarrow} N \left( 0, \tilde{\Gamma} + \tilde{V}_u \right),
\]
where
\[
\tilde{\Gamma} = \lim_{N \rightarrow \infty} \frac{1}{N} E \left( \pi_t^\top \Gamma \pi_t \right),
\]
\[
\tilde{V}_u = \lim_{N,T \rightarrow \infty} \frac{1}{NT} \sum_{t=1}^T E \left( \pi_t^\top u_t u_t^\top \pi_t \right).
\]
A.2 Proof of Theorem 3.1

According to the definitions of \( \hat{\beta} \) and \( \tilde{\beta} \), rearranging terms it can be written

\[
\hat{\beta} - \beta = \tilde{\beta} - \beta + \left[ \left( \sum_{t=1}^{T} X_t^\top M_\Phi(\bar{X})X_t \right)^{-1} - \left( \sum_{t=1}^{T} X_t^\top M_\Phi(\mu)X_t \right)^{-1} \right] \times \sum_{t=1}^{T} X_t^\top [M_\Phi(\bar{X}) - M_\Phi(\mu)] [G(\bar{X})f_t + \Gamma f_t + u_t]
\]

+ \left( \sum_{t=1}^{T} X_t^\top M_\Phi(\mu)X_t \right)^{-1} \sum_{t=1}^{T} X_t^\top [M_\Phi(\bar{X}) - M_\Phi(\mu)] [G(\bar{X})f_t + \Gamma f_t + u_t].
\]

In this situation, we will have shown the desired result if we prove that

\[
\hat{\beta} - \beta = \tilde{\beta} - \beta + o_p \left( \frac{1}{\sqrt{NT}} \right).
\]

(39)

We already showed in the proof of Lemma A.1,

\[
\frac{1}{NT} \left\{ \sum_{t=1}^{T} X_t^\top M_\Phi(\mu)X_t \right\} = O_p(1),
\]

\[
\frac{1}{NT} \left\{ \sum_{t=1}^{T} X_t^\top M_\Phi(\mu)G(\bar{X})f_t \right\} = o_p \left( \frac{1}{\sqrt{NT}} \right),
\]

\[
\frac{1}{NT} \left[ \sum_{t=1}^{T} X_t^\top M_\Phi(\mu) \{ \Gamma f_t + u_t \} \right] = O_p \left( \frac{1}{\sqrt{NT}} \right).
\]

It remains to show the following results,

\[
\frac{1}{NT} \sum_{t=1}^{T} X_t^\top [M_\Phi(\bar{X}) - M_\Phi(\mu)]X_t = O_p(1),
\]

\[
\frac{1}{NT} \sum_{t=1}^{T} X_t^\top [M_\Phi(\bar{X}) - M_\Phi(\mu)]G(\bar{X})f_t = o_p \left( \frac{1}{\sqrt{NT}} \right),
\]

\[
\frac{1}{NT} \sum_{t=1}^{T} X_t^\top [M_\Phi(\bar{X}) - M_\Phi(\mu)] \{ \Gamma f_t + u_t \} = o_p \left( \frac{1}{\sqrt{NT}} \right).
\]

The first results follows from Assumption 3.2 and the continuous mapping theorem,

\[
\frac{1}{NT} \sum_{t=1}^{T} X_t^\top [M_\Phi(\bar{X}) - M_\Phi(\mu)]X_t = O_p \left( \frac{1}{\sqrt{T}} \right).
\]

35
For the second result it is sufficient to show that
\[
\frac{1}{\sqrt{NT}} \sum_{t=1}^{T} X_t^\top M_\Phi(\bar{X}) G(\bar{X}) f_t = O_p(1).
\]

We can again write,
\[
\frac{1}{\sqrt{NT}} \sum_{t=1}^{T} X_t^\top M_\Phi(\bar{X}) G(\bar{X}) f_t
\]
\[
= \frac{1}{\sqrt{NT}} \sum_{t=1}^{T} \left\{ \left( H(\bar{X}) - \Phi(\bar{X}) C \right)^\top M_\Phi(\bar{X}) \left\{ G(\bar{X}) - \Phi(\bar{X}) B \right\} \right\} f_t
\]
\[
+ \frac{1}{\sqrt{NT}} \sum_{t=1}^{T} \pi_t M_\Phi(\bar{X}) \left\{ G(\bar{X} - \Phi(\bar{X}) B) \right\} f_t.
\]

The first term on the right hand side is of order $O_p(\rho_{g,N} \rho_{h,N})$, the second term is of order $O_p(\rho_{g,N})$. Finally, for the third result, we have to consider the variance of the $q$-th element,
\[
\text{Var} \left[ \frac{1}{\sqrt{NT}} \sum_{t=1}^{T} X_{tq} \left\{ M_\Phi(\bar{X}) - M_\Phi(\mu) \right\} \left( \Gamma f_t + u_t \right) \right] = O_p(1/T).
\]

By Cauchy-Schwarz the term is of order $O_p(1/\sqrt{T}) = o_p(1)$.

\[\square\]

### A.3 Proof of Theorem 3.2

By the $\sqrt{NT}$-consistency result of Theorem 3.1 we have that
\[
\tilde{y}_t = y_t - X_t \hat{\beta}
\]
\[
= \left\{ G(\bar{X}) + \Gamma \right\} f_t + u_t + O_p(1/\sqrt{NT}).
\]

Since the last term is of small order, we have the same situation as in the pure projected factor model. The result follows from Theorem 4.1 of Fan et al. (2016).

### A.4 Proof of Proposition 3.1

By proof the Proof of Lemma A.1, and Assumption 3.4 we have
\[
\hat{V}_\pi = \frac{1}{NT} \sum_{t=1}^{T} \pi_t^\top \pi_t + o_p(1) \xrightarrow{p} \bar{V}_p.
\]
Similarly,
\[
\hat{V}_u = \frac{1}{N} \sum_{t=1}^{T} \pi_t^\top \text{diag}\{\hat{u}_{t1}^2, \ldots, \hat{u}_{Nt}^2\} \pi_t + o_P(1).
\]

Without loss of generality, assume that \(Q = K = 1\). Then, plugging in \(\hat{u}_{it}\),
\[
\hat{V}_u = \frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \pi_{it}^2 \hat{u}_{it}^2 \\
= \frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \pi_{it}^2 u_{it}^2 + \pi_{it}^4 (\beta - \hat{\beta})^2 + 2\pi_{it}^3 (\beta - \hat{\beta}) u_{it} \\
+ \pi_{it}^2 (\lambda_i f_t - \hat{\lambda}_i \hat{f}_t) + 2\pi_{it}^3 (\beta - \hat{\beta})(\lambda_i f_t - \hat{\lambda}_i \hat{f}_t) + 2\pi_{it}^2 (\lambda_i f_t - \hat{\lambda}_i \hat{f}_t) u_{it}.
\]

By Theorem 3.1, \(\hat{\beta} \overset{p}{\to} \beta\), by Theorem 3.2, \(\hat{\lambda}_i \overset{p}{\to} \lambda_i\) and \(\hat{f}_t \overset{p}{\to} f_t\). The desired result follows from Assumption 3.4 and the continuous mapping theorem.
\[
\hat{V}_u = \frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \pi_{it}^2 u_{it}^2 + o_P(1) \overset{p}{\to} \hat{V}_u.
\]

Finally, we have
\[
\hat{V}_\Gamma = \frac{1}{NT} \sum_{t=1}^{T} \pi_t^\top \text{diag}\{\hat{\Gamma}^\top \hat{\Gamma}\} \pi_t + o_P(1).
\]

By Theorem 3.2 we have \(\hat{\Gamma} \overset{p}{\to} \Gamma\), and by Assumption 3.4 we have
\[
\frac{1}{NT} \sum_{t=1}^{T} \pi_t^\top \text{diag}\{\hat{\Gamma}^\top \Gamma - \Gamma \Gamma^\top\} \pi_t \overset{p}{\to} 0.
\]

Therefore \(\hat{V}_\Gamma \overset{p}{\to} \hat{V}_\Gamma\).

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