Absence of anomalous interactions in the quantum theory of constrained charged particles in presence of electrical currents

Carmine Ortix\textsuperscript{1,2} and Jeroen van den Brink\textsuperscript{1,2}

\textsuperscript{1}Institute Lorentz for Theoretical Physics, Leiden University, Leiden, The Netherlands \\
\textsuperscript{2}Leibniz-Institute for Solid-State and Materials Research, IFW Dresden, Germany

(Dated: December 23, 2010)

The experimental progress in synthesizing low dimensional nanostructures where carriers are confined to bent surfaces has boosted the interest in the theory of quantum mechanics on curved two-dimensional manifolds. The current theoretical paradigm relies on a thin wall quantization method introduced by Da Costa \textsuperscript{2,4}. In Ref. \textsuperscript{3} Jensen and Dandolof find that when employing the thin wall quantization in presence of externally applied electric and magnetic fields, an unphysical orbital magnetic moment appears. This anomalous result is taken as evidence for the breakdown of the thin wall quantization. We show, however, that in a proper treatment within the Da Costa framework: (i) this anomalous contribution is absent and (ii) there is no coupling between external electromagnetic field and surface curvature \textsuperscript{3}.

PACS numbers:

To derive the thin wall quantization in presence of external electromagnetic field we start with the Schrödinger equation that is minimally coupled with the four component vector potential in a generic curved three-dimensional space. Adopting Einstein summation convention and tensor covariant and contravariant components, we have

\[
\hbar \left[ \partial_t - i Q A_0 \right] \psi = -\frac{\hbar^2}{2m} G^{ij} \left[ D_i - i Q A_i \right] \psi
\]

where \( Q \) is the particle charge, \( G^{ij} \) are the contravariant components of the metric tensor and \( A_i \) are the covariant components of the vector potential \( \mathbf{A} \) with the scalar potential defined by \( V = -A_0 \). The covariant derivative \( D_i \) is as usual defined by

\[
D_i \psi_j = \partial_i \psi_j - \Gamma^k_{ij} \psi_k,
\]

where \( \psi_j \) are the covariant components of a vector field \( \mathbf{v} \) and \( \Gamma^k_{ij} \) are the Christoffel symbols

\[
\Gamma^k_{ij} = \frac{1}{2} G^{kl} \left[ \partial_j G_{li} + \partial_i G_{lj} - \partial_l G_{ij} \right].
\]

From Eq. (1) one obtains

\[
i\hbar D_0 \psi = -\frac{1}{2m} \left[ \hbar^2 G^{ij} \partial_i \partial_j \psi - \hbar^2 G^{ij} \Gamma^k_{ij} \partial_k \psi \right. \\
+ i Q \hbar G^{ij} \partial_i (A_j \psi) + i Q \hbar G^{ij} \Gamma^k_{ij} A_k \psi \left. \\
- i Q \hbar G^{ij} A_i \partial_j \psi - \hbar^2 G^{ij} A_i \partial_j \psi \right],
\]

where for convenience we introduced the time gauge covariant derivative \( D_0 = \partial_t - i Q A_0 / \hbar \). To proceed further, it is useful to define a coordinate system. As in Ref. \textsuperscript{2,4} we consider a surface \( S \) with parametric equations \( \mathbf{r} = \mathbf{r}(q_1, q_2) \). The portion of the 3D space in the immediate neighborhood of \( S \) can be then parametrized as \( \mathbf{R}(q_1, q_2) = \mathbf{r}(q_1, q_2) + q_3 \hat{N}(q_1, q_2) \) with \( \hat{N}(q_1, q_2) \) the unit vector normal to \( S \). We then find, in agreement with previous studies \textsuperscript{2,4}, the relations among \( G_{ij} \) and the covariant components of the 2D surface metric tensor \( g_{ij} \) to be

\[
G_{ij} = g_{ij} + \left[ \alpha g + (\alpha g)^T \right] q_3 + \left[ \alpha g \alpha^T \right] q_3^2 \quad i, j = 1, 2
\]

with \( \alpha \) indicating the Weingarten curvature tensor of the surface \( S \) \textsuperscript{2,4}. We recall that the mean curvature \( M \) and the Gaussian curvature \( K \) of the surface \( S \) are related to the Weingarten curvature tensor by

\[
M = \frac{\text{Tr}(\alpha)}{2} \\
K = \text{Det}(\alpha)
\]

Now we can apply the thin-layer procedure introduced by Da Costa \textsuperscript{2} and take into account the effect of a confining potential \( V_\lambda(q_3) \), where \( \lambda \) is a squeezing parameter that controls the strength of the confining potential. When \( \lambda \) is large, the total wavefunction will be localized in a narrow range close to \( q_3 = 0 \). This allows one to take the \( q_3 \to 0 \) limit in the Schrödinger equation. For the sake of clarity, it is better to introduce the indices \( a, b \) which can assume the values 1, 2 alone. From the structure of the metric tensor, one finds the following limiting relations:

\[
\lim_{q_3 \to 0} \Gamma^a_{33} = 0 \\
G^{ab} \Gamma^3_{ab} \equiv -2M
\]

With this, the Schrödinger equation Eq. (2) can be then recast as

\[
i\hbar D_0 \psi = \mathcal{H}_S \psi + \mathcal{H}_N \psi
\]
where $\mathcal{H}_S$ is an effective 2D Hamiltonian that depends on the surface $S$ alone and reads

$$\mathcal{H}_S = -\frac{1}{2m} \left[ \hbar^2 g^{ab} \partial_a \partial_b \psi - \hbar^2 g^{ab} \Gamma_{ab}^k \partial_k \psi ight]$$

$$-i Q \hbar g^{ab} \partial_a (A_b \psi) + i Q \hbar g^{ab} \Gamma_{ab}^k A_k \psi$$

$$-i Q \hbar g^{ab} A_a \partial_b \psi - Q^2 g^{ab} A_a A_b \psi \right].$$

The Hamiltonian $\mathcal{H}_N$ that depends on the degrees of freedom in the normal direction to the surface $S$ is

$$\mathcal{H}_N = -\frac{1}{2m} \left[ \hbar^2 g^{ab} \partial_a \partial_b \chi - \hbar^2 g^{ab} \Gamma_{ab}^k \partial_k \chi ight]$$

$$-2i Q \hbar M A_3 \chi - i Q \hbar A_3 \partial_3 \chi - Q^2 A_3^2 \chi \right],$$

where we left out the confining potential term. In the equation above, the linear coupling between the $A_3$ component of the vector potential and the mean curvature of the surface through the term $Q M A_3$ corresponds to the anomalous contribution to the orbital angular momentum of the charged particle found in Ref. [3]. Now we show that this term identically cancels by considering the effective Schrödinger equation for a well-defined surface wavefunction. In agreement with Ref. [4], we subsequently find that there is no coupling between an external magnetic field and the curvature of the surface, independent of the shape of the surface.

In order to obtain a surface wavefunction depending on $(q_1, q_2)$ alone [2, 3], we introduce a new wavefunction $\chi(q_1, q_2, q_3)$ that in the case of separability decomposes as $\chi(q_1, q_2, q_3) = \chi_S(q_1, q_2) \times \chi_N(q_3)$. Conservation of the norm requires

$$\psi(q_1, q_2, q_3) = \left[ 1 + 2M q_3 + K q_3^2 \right]^{-1/2} \chi(q_1, q_2, q_3).$$

In the $q_3 \to 0$ limit, the following relations hold:

$$\lim_{q_3 \to 0} \psi = \chi$$

$$\lim_{q_3 \to 0} \partial_3 \psi = \partial_3 \chi - M \chi$$

$$\lim_{q_3 \to 0} \partial_3^2 \psi = \partial_3^2 \chi - 2M \partial_3 \chi + 3M^2 \chi - K \chi.$$ 

With these limiting relations, $\mathcal{H}_N$ reduces to

$$\mathcal{H}_N = -\frac{\hbar^2}{2m} \left[ \partial_3^2 \chi + (M^2 - K) \chi - 2i Q \hbar A_3 \partial_3 \chi \right.$$

$$\left. -i Q \hbar \chi \partial_3 A_3 - Q^2 A_3^2 \chi \right]$$

The second term in the square brackets corresponds to Da Costa’s curvature induced scalar potential [2].

The Hamiltonian $\mathcal{H}_N$ governing the dynamics along the direction normal to the surface is thus coupled to the dynamics on the surface $S$ only via the term $A_3 \partial_3 \chi$ [4]. One finds that, contrary to Ref. [3], a coupling between the surface curvature and the external vector potential is lacking. This implies absence of the fundamental problem that was signaled in Ref. [3] in Da Costa’s thin wall quantization method—instead the method is well-founded and can be employed without restriction.

This research is supported by the Dutch Science Foundation FOM.

[1] This Comment was communicated to the authors of Ref. [3], B. Jensen and R. Dandoloff, on December 10th 2009.

[2] R.C.T. da Costa, Phys. Rev. A 23, 1982 (1981).

[3] B. Jensen and R. Dandoloff, Phys. Rev. A 80, 052109 (2009).

[4] G. Ferrari and G. Cuoghi, Phys. Rev. Lett. 100, 230403 (2008).
Absence of anomalous couplings in the quantum theory of constrained electrically charged particles

Carmine Ortix and Jeroen van den Brink

Institute for Theoretical Solid State Physics, IFW Dresden, 01069 Dresden, Germany

(Dated: December 23, 2010)

The experimental progress in synthesizing low-dimensional nanostructures where carriers are confined to bent surfaces has boosted the interest in the theory of quantum mechanics on curved two-dimensional manifolds. It was recently asserted that constrained electrically charged particles couple to a term linear in $A_3 M$, where $A_3$ is the transversal component of the electromagnetic vector potential and $M$ the surface mean curvature, thereby making a dimensional reduction procedure impracticable in the presence of fields. Here we resolve this apparent paradox by providing a consistent general framework of the thin-wall quantization procedure. We also show that the separability of the equation of motions is not endangered by the particular choice of the constraint imposed on the transversal fluctuations of the wavefunction, which renders the thin-wall quantization procedure well-founded. It can be applied without restrictions.

PACS numbers: 03.65.-w, 02.40.-k, 73.21.-b

Introduction – A proper understanding of quantum physics on surfaces in ordinary three-dimensional (3D) space has become immediate due to the present drive in constructing low-dimensional nanostructures such as sheets and tubes that can be bent into curved, deformable objects such as tori and spirals. The current theoretical paradigm relies on a thin wall quantization method of the two-dimensional (2D) manifold introduced by Da Costa. The quantum motion in the 2D surface is treated as a limiting case of a particle in an ordinary 3D space subject to a confining force acting in the normal direction to its 2D manifold. Because of the lateral confinement, quantum excitation energies in the normal direction are raised far beyond those in the tangential direction. Henceforth, the quantum motion in the normal direction can be safely neglected. On the basis of this, one then deduces an effective dimensionally reduced Schrödinger equation. The thin-wall quantization procedure has been widely employed since. From the experimental point of view, the realization of an optical analogue of the curvature-induced geometric potential can be taken as empirical evidence for the validity of Da Costa’s squeezing procedure.

But in spite of its immediate relevance to constrained nanostructures, the thin-wall quantization procedure is still theoretically debated, particularly in the presence of externally applied electric and magnetic fields. Indeed, it has been asserted that a charged particle of charge $Q$ couples to a term linear in $QA_3 M$ with $A_3$ the transverse component of the electromagnetic potential and $M$ the mean curvature of the 2D manifold. Even more, it was argued in Ref. that, independent of the size of the charge $Q$, the essence of the thin-wall quantization procedure, i.e. the decoupling of the transversal quantum fluctuations from the motion along the surface, is necessarily undermined when constraints different from Dirichlet ones are imposed on the normal quantum degrees of freedom. In this Brief Report, we resolve this paradoxical situation and show that: (i) there is no coupling among the mean surface curvature and the external electromagnetic field independently of the gauge choice and (ii) the thin-wall quantization procedure is well founded and can be safely applied even when non-Dirichlet type of constraints are considered for the transversal motion of the quantum particle.

Schrödinger equation – To derive the thin wall quantization in the presence of externally applied electromagnetic field we follow the procedure of Refs. and start out with the Schrödinger equation minimally coupled with the four component vector potential in a generic curved three-dimensional space. Adopting Einstein summation convention and tensor covariant and contravariant components, we have

$$i\hbar \left[ \partial_t - \frac{iQA_0}{\hbar} \right] \psi = -\frac{\hbar^2}{2m} G^{ij} \left[ D_i - \frac{iQA_i}{\hbar} \right] \times \left[ D_j - \frac{iQA_j}{\hbar} \right] \psi$$ (1)

where $Q$ is the particle charge, $G^{ij}$ is the inverse of the metric tensor $G_{ij}$ and $A_i$ are the covariant components of the vector potential $A$ with the scalar potential defined by $V = -A_0$. The covariant derivative $D_i$ is as usual defined as $D_i v_j = \partial_i v_j - \Gamma^k_{ij} v_k$, where $v_j$ are the covariant components of a 3D vector field $v$ and $\Gamma^k_{ij}$ is the affine connection related to the 3D metric tensor by

$$\Gamma^k_{ij} = \frac{1}{2} G^{kl} [\partial_j G_{li} + \partial_l G_{ij} - \partial_i G_{lj}] .$$

The gauge invariance of Eq. (1) can be made explicit by considering the gauge transformations $A_j \rightarrow A_j + \partial_j \omega$, $A_0 \rightarrow A_0 + \partial_0 \omega$ and $\psi \rightarrow \psi \exp (iQ \omega/\hbar)$ with $\omega$ a scalar function. To proceed further, it is useful to define a coordinate system. As in Ref. we consider a surface $S$ with parametric equations $r = r(q_1, q_2)$. The portion of the 3D space in the immediate neighborhood of $S$ can be then parametrized as $R(q_1, q_2) = r(q_1, q_2) + q_3 N(q_1, q_2)$ with $N(q_1, q_2)$ the unit vector normal to $S$. 

Absence of anomalous couplings in the quantum theory of constrained electrically charged particles

Carmine Ortix and Jeroen van den Brink

Institute for Theoretical Solid State Physics, IFW Dresden, 01069 Dresden, Germany

(Dated: December 23, 2010)

The experimental progress in synthesizing low-dimensional nanostructures where carriers are confined to bent surfaces has boosted the interest in the theory of quantum mechanics on curved two-dimensional manifolds. It was recently asserted that constrained electrically charged particles couple to a term linear in $A_3 M$, where $A_3$ is the transversal component of the electromagnetic vector potential and $M$ the surface mean curvature, thereby making a dimensional reduction procedure impracticable in the presence of fields. Here we resolve this apparent paradox by providing a consistent general framework of the thin-wall quantization procedure. We also show that the separability of the equation of motions is not endangered by the particular choice of the constraint imposed on the transversal fluctuations of the wavefunction, which renders the thin-wall quantization procedure well-founded. It can be applied without restrictions.

PACS numbers: 03.65.-w,02.40.-k,73.21.-b

Introduction – A proper understanding of quantum physics on surfaces in ordinary three-dimensional (3D) space has become immediate due to the present drive in constructing low-dimensional nanostructures such as sheets and tubes that can be bent into curved, deformable objects such as tori and spirals. The current theoretical paradigm relies on a thin wall quantization method of the two-dimensional (2D) manifold introduced by Da Costa. The quantum motion in the 2D surface is treated as a limiting case of a particle in an ordinary 3D space subject to a confining force acting in the normal direction to its 2D manifold. Because of the lateral confinement, quantum excitation energies in the normal direction are raised far beyond those in the tangential direction. Henceforth, the quantum motion in the normal direction can be safely neglected. On the basis of this, one then deduces an effective dimensionally reduced Schrödinger equation. The thin-wall quantization procedure has been widely employed since. From the experimental point of view, the realization of an optical analogue of the curvature-induced geometric potential can be taken as empirical evidence for the validity of Da Costa’s squeezing procedure.

But in spite of its immediate relevance to constrained nanostructures, the thin-wall quantization procedure is still theoretically debated, particularly in the presence of externally applied electric and magnetic fields. Indeed, it has been asserted that a charged particle of charge $Q$ couples to a term linear in $QA_3 M$ with $A_3$ the transverse component of the electromagnetic potential and $M$ the mean curvature of the 2D manifold. Even more, it was argued in Ref. that, independent of the size of the charge $Q$, the essence of the thin-wall quantization procedure, i.e. the decoupling of the transversal quantum fluctuations from the motion along the surface, is necessarily undermined when constraints different from Dirichlet ones are imposed on the normal quantum degrees of freedom. In this Brief Report, we resolve this paradoxical situation and show that: (i) there is no coupling among the mean surface curvature and the external electromagnetic field independently of the gauge choice and (ii) the thin-wall quantization procedure is well founded and can be safely applied even when non-Dirichlet type of constraints are considered for the transversal motion of the quantum particle.

Schrödinger equation – To derive the thin wall quantization in the presence of externally applied electromagnetic field we follow the procedure of Refs. and start out with the Schrödinger equation minimally coupled with the four component vector potential in a generic curved three-dimensional space. Adopting Einstein summation convention and tensor covariant and contravariant components, we have

$$i\hbar \left[ \partial_t - \frac{iQA_0}{\hbar} \right] \psi = -\frac{\hbar^2}{2m} G^{ij} \left[ D_i - \frac{iQA_i}{\hbar} \right] \times \left[ D_j - \frac{iQA_j}{\hbar} \right] \psi$$ (1)

where $Q$ is the particle charge, $G^{ij}$ is the inverse of the metric tensor $G_{ij}$ and $A_i$ are the covariant components of the vector potential $A$ with the scalar potential defined by $V = -A_0$. The covariant derivative $D_i$ is as usual defined as $D_i v_j = \partial_i v_j - \Gamma^k_{ij} v_k$, where $v_j$ are the covariant components of a 3D vector field $v$ and $\Gamma^k_{ij}$ is the affine connection related to the 3D metric tensor by

$$\Gamma^k_{ij} = \frac{1}{2} G^{kl} [\partial_j G_{li} + \partial_l G_{ij} - \partial_i G_{lj}] .$$

The gauge invariance of Eq. (1) can be made explicit by considering the gauge transformations $A_j \rightarrow A_j + \partial_j \omega$, $A_0 \rightarrow A_0 + \partial_0 \omega$ and $\psi \rightarrow \psi \exp (iQ \omega/\hbar)$ with $\omega$ a scalar function. To proceed further, it is useful to define a coordinate system. As in Ref. we consider a surface $S$ with parametric equations $r = r(q_1, q_2)$. The portion of the 3D space in the immediate neighborhood of $S$ can be then parametrized as $R(q_1, q_2) = r(q_1, q_2) + q_3 N(q_1, q_2)$ with $N(q_1, q_2)$ the unit vector normal to $S$. 

We then find, in agreement with previous studies\cite{6,16,18}, the relations among $G_{ij}$ and the covariant components of the 2D surface metric $g_{ij}$ to be

$$G_{ij} = g_{ij} + \left[ \alpha g + (\alpha g)^T \right]_{ij} q_3 + (\alpha g a^T)_{ij} q_3^2 \quad i, j = 1, 2$$

$$G_{i,3} = G_{3,i} = 0 \quad i = 1, 2; \quad G_{3,3} = 1$$

where $\alpha$ indicates the Weingarten curvature tensor of the surface $S$\cite{6,16,18}. We recall that the mean curvature $M$ and the Gaussian curvature $K$ of the surface $S$ are related to the Weingarten curvature tensor by

$$M = \frac{Tr(\alpha)}{2},$$

$$K = Det(\alpha).$$

Now we can apply the thin-layer procedure introduced by Da Costa\cite{6} and take into account the effect of a confining potential $V_{\lambda}(q_3)$, where $\lambda$ is a squeezing parameter that controls the strength of the confining potential. When $\lambda$ is large, the total wavefunction will be localized in a narrow range close to $q_3 = 0$. This allows one to take the $q_3 \to 0$ limit in the covariant derivative appearing in the Schrödinger equation Eq. (1). From the structure of the metric tensor, it is straightforward to show the following limiting relations for the affine connection to hold

$$\lim_{q_3 \to 0} G^{ij} \Gamma_{ij}^{3} = -2M,$$

$$\lim_{q_3 \to 0} G^{ij} \Gamma_{ij}^{k} = g^{ij} \gamma_{ij}^{k}$$

where we introduced $\gamma_{ij}^{k}$, the affine connection related to the 2D surface metric tensor $g_{ij}$. With this, the effective Schrödinger equation in the portion of the 3D space close to the surface $S$ is

$$\psi - \hbar^2 2m \left[ \frac{\partial^2}{\partial^2} - \frac{iQA_3}{\hbar} \right] \psi = -\hbar^2 2m \left[ \frac{\partial^2}{\partial^2} - \frac{iQA_3}{\hbar} \right] \psi - \hbar^2 2m M \partial_3 \psi + \frac{i\hbar}{m} M A_3 \psi, \quad (2)$$

where we left out the confining potential term and defined the 2D covariant derivatives of the surface metric $g_{ij}$ as $\partial_i v_j = \partial_i v_j - \gamma_{ij}^k v_k$ where $v_i$ now indicates the covariant components of a generic 2D vector field. In the equation above, the term $M \partial_3 \psi$ yields a coupling among the transversal fluctuations of the wavefunction and the surface curvature. Similarly, the linear coupling between the $A_3$ component of the vector potential and the mean curvature of the surface through the term $Q MA_3$ yields an anomalous curvature contribution to the orbital magnetic moment of the charged particle\cite{16}. Now we show that both these terms vanish by considering the effective Schrödinger equation for a well-defined surface wavefunction. In agreement with Ref. [18], we subsequently find that in an arbitrary gauge, there is no coupling between an external magnetic field and the curvature of the surface, independent of the shape of the surface.

In order to find a surface wavefunction with a definable surface density probability, we are led to introduce a new wavefunction $\chi(q_1, q_2, q_3)$ for which in the event of separability the surface density probability is $|\chi(q_1, q_2)|^2 \int dq_3 |\chi(q_3)|^2$. Conservation of the norm requires

$$\psi(q_1, q_2, q_3) = \left[ 1 + 2 M q_3 + K q_3^2 \right]^{-1/2} \chi(q_1, q_2, q_3).$$

In the immediate neighborhood of the surface ($q_3 \to 0$) the original wavefunction and its corresponding derivatives in the normal direction are related to the new wavefunction $\chi$ by

$$\lim_{q_3 \to 0} \psi = \chi,$$

$$\lim_{q_3 \to 0} \partial_3 \psi = \partial_3 \chi - M \chi,$$

$$\lim_{q_3 \to 0} \partial_3^2 \psi = \partial_3^2 \chi - 2 M \partial_3 \chi + 3 M^2 \chi - K \chi.$$

With the above relations, the effective Schrödinger equation Eq. (2) takes the following form

$$\frac{i\hbar}{\hbar} \partial_3 \chi = -\hbar^2 2m \left[ \frac{\partial^2}{\partial^2} - \frac{iQA_3}{\hbar} \right] \chi - \hbar^2 2m \left[ \frac{\partial^2}{\partial^2} - \frac{iQA_3}{\hbar} \right] \chi - \hbar^2 2m \left[ \frac{\partial^2}{\partial^2} - \frac{iQA_3}{\hbar} \right] \chi$$

$$= -\hbar^2 2m \left[ \frac{\partial^2}{\partial^2} - \frac{iQA_3}{\hbar} \right] \chi - \hbar^2 2m \left[ \frac{\partial^2}{\partial^2} - \frac{iQA_3}{\hbar} \right] \chi$$

$$= -\hbar^2 2m \left[ \frac{\partial^2}{\partial^2} - \frac{iQA_3}{\hbar} \right] \chi$$

(3)

The purely quantum potential $\propto \hbar^2$ in the equation above corresponds to the curvature induced geometric potential originally found by Da Costa\cite{6}. Apart from that, Eq. (3) represents the gauge invariant Schrödinger equation minimally coupled to the four component vector potential in a curved three-dimensional space with metric tensor\cite{18}

$$\tilde{G} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

(4)

We therefore obtain a mapping of the original metric tensor $G_{ij}$ into $\tilde{G}_{ij}$ preserving the gauge invariance. If we
now choose to fix the gauge, the new metric tensor Eq. (1) has to be taken into account. For instance by imposing the Lorentz gauge\textsuperscript{16,17}, the condition reads

\[ \nabla \cdot A \equiv \nabla_\parallel \cdot A_\parallel + \partial_3 A_3 \equiv 0. \]

It is then clear that independent of the electromagnetic gauge, a quantum charged particle does not couple to the mean surface curvature and the gauge choice is free of pathologies. This allows us to apply a gauge transformation to Eq. (3) such to cancel \( A_3 \) thereby reaching a separability of the dynamics along to direction normal to the surface \( S \) from the tangential one.

**General action** – Next, we reinforce our conclusions by analyzing in the same spirit of Ref. 16,19,20, a canonical action for the Schrödinger field in the embedding space

\[ S = \int \frac{\hbar^2}{2m} \left[ \left( D^j - iQA^j \right)^\dagger \left( D_j - iQA_j \right) \right] \psi \]

\[ \psi^\dagger \left( \partial_t - iQA_0 \right) \psi \]  

By performing an integration by parts we can separate a volume integral contribution

\[ S_V = \int -\frac{\hbar^2}{2m} \psi^\dagger \left( D^j - iQA^j \right) \left( D_j - iQA_j \right) \psi \]

\[ -i\hbar \psi^\dagger \left( \partial_t - iQA_0 \right) \psi \]  

and a total spatial derivative term

\[ \int \frac{\hbar^2}{2m} D^j \left[ \left( D^j - iQA^j \right) \psi^\dagger \left( D_j - iQA_j \right) \right]. \]  

By varying \( S_V \) alone, we can easily reach the gauge invariant Schrödinger equation Eq. (11). It is also clear that under confinement the volume integral Eq. (6) transforms to

\[ S_V = \int -i\hbar \chi^\dagger \left( \partial_t - iQA_0 \right) \chi - \frac{\hbar^2}{2m} \chi^\dagger \left( D^j - iQA^j \right) \]

\[ \chi \left( D_j - iQA_j \right) \chi \]

where we have introduced the covariant derivative \( \vec{D} \) related to the metric tensor \( \vec{G}_{ij} \) and to conserve the norm we are considering the rescaled scalar field \( \chi \). Variation of the action written above gives precisely the confined Schrödinger equation of Eq. (3). Since the dynamical Schrödinger equation comes entirely from a volume contribution, it directly follows that the total derivative term Eq. (7) acts as a boundary condition which is stipulated by the vanishing of the three-dimensional field

\[ \psi^\dagger \left( D_j - iQA_j \right) \psi \]

along the surface \( \partial \Omega \) of the region of integration. As pointed out in Ref. 16, this boundary condition is fulfilled for construction if the wavefunction \( \psi \) identically vanishes on \( \partial \Omega \) – Dirichlet type of constraints are imposed. They can be achieved by considering a squeezing potential \( V_\lambda(q_3) \) in the form of an infinite potential well. This assumption, however, seems not to be the most physically sound one since under confinement \((q_3 \to 0)\) it would break the natural limits set by the uncertainty principle. That the wavefunction should vanish along the surface of integration is too restrictive a condition. In the confinement procedure, indeed, it is natural to consider region of integration that are symmetrical to the 2D manifold \( S \) in the normal direction. Therefore the boundary condition reads

\[ \lim_{\epsilon \to 0} \left[ (\psi^\dagger \partial_3 \psi)_\epsilon - (\psi^\dagger \partial_3 \psi)_-\epsilon \right] = 0 \]  

where we have considered a region of integration with a \( 2\epsilon \) width in the normal direction and we have set the transversal component of the electromagnetic field \( A_3 \equiv 0 \). By referring to the rescaled scalar field \( \chi \), Eq. (8) becomes

\[ \lim_{\epsilon \to 0} \left[ (\chi^\dagger \partial_3 \chi - M|\chi|^2)_\epsilon - (\chi^\dagger \partial_3 \chi - M|\chi|^2)_-\epsilon \right] = 0. \]  

This implies that the smoothness of the rescaled wavefunction and of its first derivative as they pass through the curved surface are enough to fulfill the canonical action boundary condition. With this other types of squeezing potentials can be also considered in the thin-wall quantization scheme. As an example we can consider an harmonic trap \( V_\lambda(q_3) = m\lambda^2 q_3^2 / 2 \). Since, after gauge fixing, the variation of the canonical action leads to the separable Schrödinger equation Eq. (3), we may write the rescaled wavefunction as

\[ \chi(q_1,q_2,q_3) = \chi_\parallel(q_1,q_2) \times \left( \frac{m\lambda}{\pi \hbar} \right)^{1/4} e^{-\frac{m\lambda q_3^2}{2\hbar}}, \]  

which readily satisfies Eq. (9). It is worth noticing that the harmonic trap potential corresponds to Neumann type of boundary conditions for which a coupling of the quantum particle to the mean surface curvature was put forward\textsuperscript{16}.

**Conclusions** – Here we have provided a consistent framework of the thin-wall quantization procedure for charged particles in the presence of externally applied electric and magnetic field. Contrary to previous claims\textsuperscript{16} we have shown that the mean surface curvature does not couple to the transversal component of the vector potential neither explicitly in the effective dimensionally reduced Schrödinger equation\textsuperscript{16} nor implicitly in the gauge fixing procedure\textsuperscript{17}. We have also considered a canonical Schrödinger action and shown that the thin-wall quantization procedure is not endangered by the particular constraints imposed on the transverse fluctuations of the wavefunction. Therefore the Da Costa’s method is well-founded and can be applied without restrictions.
1. L. Liu, C. S. Jayanthi, and S. Y. Wu, Phys. Rev. B 64, 033412 (2001).
2. M. Zheng and C. Ke, Small 6, 1647 (2010).
3. O. G. Schmidt and K. Eberl, Nature (London) 410, 168 (2001).
4. C. Deneke, J. Schumann, R. Engelhard, J. Thomas, C. Muller, M. S. Khatri, A. Malachias, M. Weisser, T. H. Metzger, and O. G. Schmidt, Nanotechnology 20, 045703 (2009).
5. C. Ortix and J. van den Brink, Phys. Rev. B 81, 165419 (2010).
6. R. C. T. da Costa, Phys. Rev. A 23, 1982 (1982).
7. G. Cantele, D. Ninno, and G. Iadonisi, Phys. Rev. B 61, 13730 (2000).
8. H. Aoki, M. Koshino, D. Takeda, H. Morise, and K. Kuroki, Phys. Rev. B 65, 035102 (2001).
9. M. Encinosa and L. Mott, Phys. Rev. A 68, 014102 (2003).
10. N. Fujita and O. Terasaki, Phys. Rev. B 72, 085459 (2005).
11. M. Koshino and H. Aoki, Phys. Rev. B 71, 073405 (2005).
12. J. Gravesen and M. Willatzen, Phys. Rev. A 72, 032108 (2005).
13. A. V. Chaplik and R. H. Blick, New J. Phys. 6, 33 (2004).
14. A. Marchi, S. Reggiani, M. Rudan, and A. Bertoni, Phys. Rev. B 72, 035403 (2005).
15. A. Szameit, F. Dreisow, M. Heinrich, R. Keil, S. Nolte, A. Tünnermann, and S. Longhi, Phys. Rev. Lett. 104, 150403 (2010).
16. B. Jensen and R. Dandoloff, Phys. Rev. A 80, 052109 (2009).
17. B. Jensen and R. Dandoloff, Phys. Rev. A 81, 049905 (2010).
18. G. Ferrari and G. Cuoghi, Phys. Rev. Lett. 100, 230403 (2008).
19. S. Matsutani, Phys. Rev. A 47, 686 (1993).
20. M. Burgess and B. Jensen, Phys. Rev. A 48, 1861 (1993).