COMMUTATOR ESTIMATES FROM A VIEWPOINT OF REGULARITY STRUCTURES

MASATO HOSHINO

Abstract. First we introduce the Bailleul-Hoshino’s result [4], which links the theory of regularity structures and the paracontrolled calculus. As an application of their result, we give another algebraic proof of the multicomponent commutator estimate [3], which is a generalized version of the Gubinelli-Imkeller-Perkowski’s commutator estimate [11, Lemma 2.4].

1. Introduction

In this paper, we introduce the recent research by Bailleul and Hoshino [4] and show its application to the commutator estimate, one of the important tools in the analysis of singular SPDEs.

Singular SPDEs often involve ill-defined products of distributions. The theory of regularity structures by Hairer [12] and the paracontrolled calculus by Gubinelli, Imkeller and Perkowski [11] provide general approaches to give a meaning to such SPDEs. Both of them are extensions of the rough path theory, which was originally introduced by Lyons [14] and reformulated by Gubinelli [9]. The latter version consists of a Lie group of enhanced noises (called rough paths) and a fiber bundle of enhanced solution spaces (called controlled paths). The Itô-Lyons map is the continuous mapping from a given rough path to the solution of SDE in the class of controlled paths. This part is purely deterministic. The only probabilistic part is how to lift the stochastic noise (typically, a Brownian motion) to the rough path. Both of the regularity structures and the paracontrolled calculus share the same spirit. The only difference between them is the definition of enhanced noises and enhanced solutions. The rough meaning of this difference is whether the function/distribution is described locally or globally. For example, consider the typical SDE

\[ dX_t = f(X_t)dB_t. \]

In the theory of regularity structures, its solution \( X \) is assumed to have the local structure

\[ X_t - X_s = X'_s(B_t - B_s) + O(|t - s|^{1-}). \]

(This is nothing but the definition of controlled path.) Here the global description of the solution is not given, so we need a reconstruction operator explained later. In the paracontrolled calculus, the solution \( X \) is assumed to have the global structure

\[ X = X' \prec B + (C^1), \]

where \( \prec : S' \times S' \to S' \) is a bilinear continuous operator defined below, called paraproduct. These two kinds of definitions have their own merits. In the regularity structures, the way to give a meaning to singular SPDEs is automated in [7] [8] [6] by using the language of Hopf algebra. In the paracontrolled calculus, since the solution is given globally, we can study more detailed properties of specific singular SPDEs [16] [1] [10] [13] by using well-known techniques in the real analysis. Hence we can use either of them according to the situation.

The equivalence between the two theories is not well studied. One of the studies is in the Gubinelli-Imkeller-Perkowski’s original paper [11]. In its last section, the authors showed that the reconstruction operator is represented by an integral operator like a paraproduct. By using such operator, Martin and Perkowski [15] introduced an intermediate notion between the local and global descriptions as above and show the equivalence between the local and intermediate forms. However, these studies still depend on the local form of the solution. Bailleul and Hoshino [4] introduced the “paracontrolled remainders” and showed the paracontrolled form of the reconstruction operator. They also showed a partly equivalent relation between the local and global forms.

In this paper, first we briefly introduce their result and next apply it to the proof of the commutator estimate [11, Lemma 2.4]. Their commutator is defined by

\[ C(f,g,h) = (f \prec g) \circ h - f (g \circ h) \]
for smooth inputs \((f, g, h)\). The commutator estimate implies that \(C\) is uniquely extended to the continuous trilinear operator from \(C^\alpha \times C^\beta \times C^\gamma\) to \(C^{\alpha+\beta+\gamma}\), with \(\alpha \in (0, 1), \beta + \gamma < 0\), and \(\alpha + \beta + \gamma > 0\). In the above SDE, such estimate shows that the product of \(f(X)\) and \(\hat{B} = \frac{dB}{\sqrt{t}}\) can be decomposed as follows.

\[
\hat{f}(X) \hat{B} = f'(X)X'(B \circ \hat{B}) + \text{(continuous function of } (X, X', B, \hat{B})\).
\]

Hence we have only to define the explicit distribution \(B \circ \hat{B}\) to define the product \(\hat{f}(X) \hat{B}\), even though there is an unknown function \(X\). This is the key point in the Fourier approach to the rough path theory \(\textbf{[11]}\). In the analysis of SPDEs, we often need more iterated versions

\[
C(f_1, f_2, f_3, h) = C(f_1 \times f_2, f_3, h) - f_1C(f_2, f_3, h), \\
C(f_1, f_2, f_3, f_4, h) = C(f_1 \times f_2, f_3, f_4, h) - f_1C(f_2, f_3, f_4, h),
\]

Bailleul and Bernicot \(\textbf{[3]}\) studied many kinds of such operators. As in \(\textbf{[11, 3]}\), the commutator estimate is usually proved by the Fourier analysis technique, but in this paper we show a different type of proof as an application of the Bailleul-Hoshino’s result. Our approach is more algebraic and automatic than the direct computation. Moreover, it would make it clear the role of the “paracontrolled remainder” in \(\textbf{[4]}\). I conjecture that other kinds of operators in \(\textbf{[3]}\) can also be reformulated from an algebraic viewpoint.

This paper is organized as follows. In Section 2, we recall some notions and facts from the regularity structures and the paracontrolled calculus and introduce the Bailleul-Hoshino’s result. In Section 3, as a preparation for Section 4, we construct a canonical Hopf algebra associated with iterated paraproducts. In Section 4, we show the algebraic proof of the multicomponent commutator estimate.

2. Preliminaries and Bailleul-Hoshino’s result

From now on, we consider the functions and distributions on \(\mathbb{R}^d\). Denote by \(S'(\mathbb{R}^d)\) the space of tempered distributions.

2.1. Besov space and paraproduct. We recall the Littlewood-Paley theory. Fix a smooth radial functions \(\chi\) and \(\rho\) on \(\mathbb{R}^d\) such that,

- \(\text{supp}(\chi) \subset \{x; |x| < \frac{4}{3}\}\) and \(\text{supp}(\rho) \subset \{x; \frac{4}{3} < |x| < \frac{5}{3}\}\),
- \(\chi(x) + \sum_{j=0} L^{\sim j} x = 1\) for any \(x \in \mathbb{R}^d\).

Set \(\rho_1 := \chi\) and \(\rho_j := \rho(2^{-j} \cdot)\) for \(j \geq 0\). We define the Littlewood-Paley blocks

\[
\Delta_j f := \mathcal{F}^{-1}(\rho_j \mathcal{F} f), \quad f \in S'(\mathbb{R}^d),
\]

where \(\mathcal{F}\) is the Fourier transform on \(\mathbb{R}^d\) and \(\mathcal{F}^{-1}\) is the inverse transform. For \(\alpha \in \mathbb{R}\), we define the (nonhomogeneous) Besov space

\[
C^\alpha := \{f \in S'(\mathbb{R}^d); \|f\|_\alpha := \sup_{j \geq -1} 2^{j\alpha} \|\Delta_j f\|_{L^\infty} < \infty\}.
\]

For any smooth functions \(f, g\) on \(\mathbb{R}^d\), we decompose the product \(fg\) as follows.

\[
fg = \sum_{j, k \geq -1} \Delta_j f \Delta_k g
\]

\[
= \sum_{j < k - 1} \Delta_j f \Delta_k g + \sum_{|j - k| \leq 1} \Delta_j f \Delta_k g + \sum_{j = k - 1} \Delta_j f \Delta_k g + \sum_{j > k} \Delta_j f \Delta_k g
\]

\[
=: f \prec g + f \circ g + f \succ g.
\]

\(f \prec g = g \succ f\) is called a paraproduct, and \(f \circ g\) is called a resonant. The following estimates are basic.

**Proposition 2.1 (\(\textbf{[3]}\)).** Let \(\alpha, \beta \in \mathbb{R}\).

1. If \(\alpha \neq 0\), then the map \(C^\alpha \times C^\beta \ni (f, g) \mapsto f \prec g \in C^{\alpha+0+\beta}\) is well-defined and continuous.
2. If \(\alpha + \beta > 0\), then the map \(C^\alpha \times C^\beta \ni (f, g) \mapsto f \circ g \in C^{\alpha+\beta}\) is well-defined and continuous.
2.2. Regularity structures. We recall some important notions of the theory of regularity structures [12].

**Definition 2.1.** A (concrete) regularity structure \((T^+, T)\) is a pair of graded vector spaces

\[
T^+ = \bigoplus_{\alpha \in A^+} T^+_\alpha, \quad T = \bigoplus_{\beta \in A} T_\beta,
\]

where each \(T^+_\alpha\) and \(T_\beta\) are finite dimensional spaces and such that,

- \(A^+, A \subset \mathbb{R}\) are countable sets bounded from below and without any accumulation point. In particular, \(0 = \min A^+ + A^+ \subset A^+\),
- \(T^+\) is a graded algebra \((T^+_\alpha, T^+_\alpha T^+_\alpha \subset T^+_\alpha + \alpha_0)\) with unit \(1\) and \(T^+_0 = \mathbb{R}1\),
- \(T^+\) is a graded Hopf algebra with coproduct \(\Delta: T^+ \rightarrow T^+ \otimes T^+\) such that, \(\Delta^+1 = 1 \otimes 1\) and

\[
\Delta^+ \tau \in \tau^1 + 1 \otimes \tau + \bigoplus_{0 < \beta < \alpha} T^+_{\beta} \otimes T^+_{\alpha - \beta}
\]

for any \(\tau \in T^+_{\alpha}\) with \(\alpha > 0\),
- \(T\) has a coproduct \(\Delta: T \rightarrow T \otimes T^+\) with the right comodule property \((\Delta \otimes 1)\Delta = \Delta(1 \otimes \Delta^+)\Delta\) and with

\[
\Delta \tau \in \tau^1 + \bigoplus_{\beta < \alpha} T_{\beta} \otimes T^+_{\alpha - \beta}
\]

for \(\tau \in T_{\alpha}\).

By definition, there exists \(\min A \in \mathbb{R}\), which is called a regularity of \(T\). From now on, we set \(\alpha_0 := (\min A) \land 0\). We denote by \(\|\cdot\|\), the equivalent norm on the finite dimensional space \(T_{\alpha}\). For an arbitrary \(\tau \in T\), we write \(\|\tau\|_\alpha\) for the norm of the projection of \(\tau\) into \(T_{\alpha}\). We fix the bases \(B^+\) and \(B\) of \(T^+\) and \(T_\beta\), respectively. We set \(B^+ = \bigcup_{\alpha \in A^+} B^+_\alpha\) and \(B = \bigcup_{\beta \in A} B_\beta\). For any \(\tau \in B^\alpha\) and \(\sigma \in B_\beta\), we write \(|\tau| = \alpha\) and \(|\sigma| = \beta\).

Next we define models on \((T^+, T)\). Note that the space of all nonzero algebra homomorphisms \(f: T^+ \rightarrow \mathbb{R}\) forms a character group \(G = \text{ch}(T^+)\) by the product

\[
f \ast g := (f \otimes g)\Delta^+.
\]

We define the class of test functions

\[
\Phi := \left\{ \varphi: \mathbb{R}^d \rightarrow \mathbb{R} : \begin{array}{l}
\varphi \text{ is a bounded \(C^r\) function with compact support,} \\
|\varphi| \leq 1,
\end{array} \right\}.
\]

Given \(\varphi \in \Phi\), \(x \in \mathbb{R}^d\), and \(0 < \lambda \leq 1\), we set \(\varphi^\lambda_x(\cdot) := \lambda^{-d}\varphi(\lambda^{-1} \cdot - x)\).

**Definition 2.2.** For a function \(g: \mathbb{R}^d \rightarrow G\) and a linear map \(\Pi: T \rightarrow \mathcal{C}^{\alpha_0}\), we set

\[
\|g\| := \sup_{\tau \in B^+, x \in \mathbb{R}^d} |g(x)(\tau)|, \quad \|\Pi\|_\alpha := \sup_{\tau \in \Pi} \|\tau\|_{\alpha_0} + \sup_{\lambda \in \mathbb{R}^d} \sup_{\tau \in B} |\lambda^{-1}\tau| |(\Pi_x^\lambda(\tau), \varphi^\lambda_x)| < \infty.
\]

We set \(\|g\| := \|g\| + \|\Pi\|_\alpha\). Although the set of all models is not linear, we can define the metric

\[
d(Z, Z') := \|Z - Z'\|\text{ on it, by replacing } g, \Pi, \Pi_x^\lambda \text{ by } g - g', \Pi - \Pi', \Pi_x^\lambda - (\Pi_x^\lambda)^\prime \text{ in the above definition}.
\]

We define the class of modelled distributions.

**Definition 2.3.** For a model \(Z = (g, \Pi)\), we define the operator on \(T\) by

\[
\hat{g}_{yx} := (\Pi \otimes g_{yx})\Delta.
\]

Let \(\gamma \in \mathbb{R}\). A function \(f: \mathbb{R}^d \rightarrow T_{< \gamma} := \bigoplus_{\alpha < \gamma} T_\alpha\) is called a \(\gamma\)-class modelled distribution if

\[
\|f\| := \sup_{\alpha < \gamma} \sup_{x \in \mathbb{R}^d} \|f(x)\|_{\alpha} + \sup_{\alpha < \gamma} \sup_{x, y \in \mathbb{R}^d} \frac{\|f(y) - \hat{g}_{yx}f(x)\|_{\alpha}}{|y - x|^{\gamma - \alpha}} < \infty.
\]

Denote by \(\mathcal{D}^\gamma(g)\) the space of all \(\gamma\)-class modelled distributions.
For two elements $f \in D^\gamma(g)$ and $f' \in D^\gamma(g')$ modelled by different models $g$ and $g'$ respectively, we define the quantity $d_\gamma(f, f')$ by replacing $f(x)$ and $f(y) - g_{yx} f(x)$ by $f(x) - f'(x)$ and $(f(y) - g_{yx} f(x)) - (f'(y) - g_{yx} f'(x))$ respectively, in the above definition.

The following theorem is the so-called reconstruction theorem.

**Theorem 2.2 ([12] Theorem 3.10).** Let $\gamma > 0$. For any model $Z$, there exists a continuous operator $R^\gamma : D^\gamma(g) \to C^{\gamma_0}$ uniquely determined by the following property. For any $f \in D^\gamma(g)$ and $\lambda \in (0, 1]$, one has

$$\sup_{x \in \mathbb{R}^d} \sup_{\varphi \in \Phi} |(R^\gamma f - \Pi_\varphi f(x))(\varphi_x^\gamma)| \lesssim \|\varphi\|^\gamma \|f\|_{\gamma_0}.$$

Moreover, the mapping $(Z, f) \mapsto R^\gamma f \in C^{\gamma_0}$ is continuous with respect to the metric $d((Z, f), (Z', f')) = d(Z, Z') + d_\gamma(f, f')$.

2.3. Bailleul-Hoshino’s result. The Bailleul-Hoshino’s result [4] consists of “from local to global” part and “from global to local” part. The first part provides the transformation of models into paracontrolled remainders. For any $\tau, \sigma \in B$ (or $B^+_\lambda$), we define $\tau/\sigma \in T^\lambda$ by the formula

$$\Delta \tau \text{ or } \Delta^+ \tau = \sum_{\sigma \in \Phi \text{ (or } B^+)} \sigma \ominus (\tau/\sigma).$$

We write $\sigma < \tau$ if $\tau/\sigma \neq 0$ and $\sigma \neq \tau$. By definition, $\sigma < \tau$ implies $|\sigma| < |\tau|$.

**Theorem 2.3 ([4] Proposition 11).** For any model $Z = (g, \Pi)$, we define the family of functions (or distributions) $(C_0^\gamma \tau)_{\tau \in B^+_\lambda}$ and $(C^\gamma \sigma)_{\sigma \in B}$ by the formulas

$$g(\tau) = \sum_{\nu \in \mathcal{B}^+_{1 < \nu < \tau}} g(\tau/\nu) \prec [\nu]^\gamma + [\tau]^\gamma, \quad \tau \in B^+, \quad \Pi_\sigma = \sum_{\mu \in \mathcal{B}^+, \mu < \sigma} g(\sigma/\mu) \prec [\mu]^\gamma + [\sigma]^\gamma, \quad \sigma \in B.$$

Then one has

$$[\tau]^\gamma \in C^{|\tau|}, \quad [\sigma]^\gamma \in C^{|\sigma|},$$

for any $\tau \in B^+_\lambda$ and $\sigma \in B$. Moreover, the mappings $g \mapsto [\tau]^\gamma \in C^{|\tau|}$ and $Z \mapsto [\sigma]^\gamma \in C^{|\sigma|}$ are continuous.

They also showed that the reconstruction has the paracontrolled form.

**Theorem 2.4 ([4] Theorem 13).** Let $\gamma > 0$. For any model $Z$, there exists a continuous operator $\| \| : D^\gamma(g) \to C^\gamma$ such that, for any $f = \sum_{|\tau| < \gamma} f_\tau \tau \in D^\gamma(g)$,

$$R^\gamma f = \sum_{|\tau| < \gamma} f_\tau \prec [\tau]^\gamma + [f]^\gamma.$$

Moreover, the mapping $(Z, f) \mapsto [f]^\gamma \in C^\gamma$ is continuous.

The second part of [4] shows how to recover the model from a given family of paracontrolled remainders. Set $B_\lambda := \{ \tau \in B; |\tau| \leq 0 \}$

**Proposition 2.5 ([4] Corollary 14).** Assume that a function $g : \mathbb{R}^d \to G$ with $\|g\| < \infty$ is given. Then for any given family $(\{\tau\}_{\tau \in B_\lambda})$, there exists a unique model $\Pi$ determined by the formula

$$\Pi_\tau = \sum_{\sigma \in \mathcal{B}_\lambda, \sigma < \tau} g(\tau/\sigma) \prec [\sigma] + [\tau], \quad \tau \in B_\lambda.$$

Moreover, the mapping $(g, \{\tau\}_{\tau \in B_\lambda}) \mapsto \Pi$ is continuous.

3. Local behaviors of paraproducts

For a sequence $f_1, f_2, \ldots$ of distributions, we define the *iterated paraproducts*

$$(f_1)^{<} \triangleq f_1, \quad (f_1, \ldots, f_n)^{<} \triangleq (f_1, \ldots, f_{n-1})^{<} \prec f_n.$$

Obviously, $(f, g)^{<} = f \prec g$. The aim of this section is to show the following local behavior of iterated paraproduct, which has an important role in the next section.
In this case, we need additional structures associated with "polynomials" and "derivatives". We do not proof. Only to apply called a model on $W$.

For a model $(W, \Delta)$, if the pair $(\mathbf{g}, \mathbf{g})$ is a model on $(W^{<1}, W^{<1})$.

Lemma 3.4. For a model $g : \mathbb{R}^d \to G := \text{ch}(W^{<1})$ is called a model on $W^{<1}$, if the pair $(\mathbf{g}, \mathbf{g})$ is a model on $(W^{<1}, W^{<1})$.

Remark 3.3. To consider words with homogeneities $\geq 1$, we need more structures. Indeed, if $\alpha > 0$, then for any $f \in C^\alpha$ we have

$$f(y) = \sum_{|\alpha|<\alpha} \frac{\partial^\alpha f(x)}{\alpha!} (y-x) + O(|y-x|^\alpha).$$

In this case, we need additional structures associated with "polynomials" and "derivatives". We do not consider such structures in this paper.

We consider models on the regularity structure $(W^{<1}, W^{<1})$. A function $g : \mathbb{R}^d \to G := \text{ch}(W^{<1})$ is called a model on $W^{<1}$, if the pair $(\mathbf{g}, \mathbf{g})$ is a model on $(W^{<1}, W^{<1})$.

Lemma 3.4. For a model $g : \mathbb{R}^d \to G$, the operator $g_{yx}$ is given by the formula

$$g_{yx}(i_1 \ldots i_k) = g_y(i_1 \ldots i_k) - g_x(i_1 \ldots i_k) - \sum_{\ell=1}^{k-1} g_x(i_1 \ldots i_\ell)g_{yx}(i_{\ell+1} \ldots i_k).$$

proof. Only to apply $g_y = (g_{yx} \otimes g_x)\Delta$ to the word $(i_1 \ldots i_k)$.

A model $g$ is determined by the family of functions with the following properties.
Definition 3.1. For a family $F = \{ f^{i_1 \cdots i_k} : \mathbb{R}^d \to \mathbb{R} \}_{(i_1, \ldots, i_k) \in \mathcal{V}}$ of functions, define
\[
\omega^{i_1 \cdots i_k}_{yx} (F) := f^{i_1 \cdots i_k}_y - f^{i_1 \cdots i_k}_x - \sum_{t=1}^{k-1} f^{i_1 \cdots i_t}_x \omega^{i_{t+1} \cdots i_k}_{yx} (F).
\]

The family $\{ f^{i_1 \cdots i_k} \}$ is called a seed of model, if
\[
\| F \| := \sup_{(i_1, \ldots, i_k) \in \mathcal{V}} \| f^{i_1 \cdots i_k}_x \| + \sup_{(i_1, \ldots, i_k) \in \mathcal{V}} \sup_{x, y \in \mathbb{R}^d} \frac{|\omega^{i_1 \cdots i_k}_{yx} (F)|}{|y-x|^{|\alpha_{i_1} + \cdots + \alpha_{i_k}|}} < \infty.
\]

Remark 3.5. It is easy to show that $\|(f^{i_1 \cdots i_k})_j\|_{L^\infty} \lesssim 2^{-j \alpha_{i_k}} \|f_{i_1}\|_{\alpha_{i_1}} \cdots \|f_{i_k}\|_{\alpha_{i_k}}$. Hence the above series converges absolutely and defines an element of $L^\infty$. Moreover, we also have $f^{i_1 \cdots i_k} \in C^{\alpha_{i_k}}$ by [2] Lemma 2.84.

Proposition 3.6. $F = \{ f^{i_1 \cdots i_k} \}$ is a seed of model on $\mathcal{V}$. Precisely, if $(i_1 \cdots i_k) \in \mathcal{V} \Leftrightarrow \alpha_{i_1} + \cdots + \alpha_{i_k} < 1$, then one has the bound
\[
|\omega^{i_1 \cdots i_k}_{yx} (F)| \lesssim \|f_{i_1}\|_{\alpha_{i_1}} \cdots \|f_{i_k}\|_{\alpha_{i_k}} |y-x|^{\alpha_{i_1} + \cdots + \alpha_{i_k}}.
\]

Before we turn to the proof, we prove some lemmas.

Lemma 3.7. Let $\{ X^j_{yx} = \sum_{j=1}^{\infty} X^j_{yx} \}_{x, y \in \mathbb{R}^d}$ be a family of absolutely convergent series. Assume that for some $C > 0$ and $\alpha > 0$, the bound
\[
|X^j_{yx}| \lesssim C |y-x|^\theta, \quad x, y \in \mathbb{R}^d
\]
holds for any $\theta$ in a neighborhood of $\alpha$, then one has the bound
\[
|X_{yx}| \lesssim C |y-x|^\alpha, \quad x, y \in \mathbb{R}^d.
\]

Proof. Without loss of generality, we can assume $|y-x| \leq 1$. Fix $\epsilon > 0$ such that the assumption holds for $\theta = \alpha \pm \epsilon$. For any $N \in \mathbb{N}$, we have the bounds
\[
\sum_{j \leq N} |X^j_{yx}| \leq C |y-x|^\alpha + \epsilon \sum_{j \leq N} 2^{j \epsilon} \lesssim C 2^N |y-x|^\alpha + \epsilon,
\]
\[
\sum_{j > N} |X^j_{yx}| \leq C |y-x|^\alpha - \epsilon \sum_{j > N} 2^{-j \epsilon} \lesssim C 2^{-N} |y-x|^\alpha - \epsilon.
\]
Since $|y-x| \leq 1$, we can choose a large $N$ such that $2^N |y-x| \simeq 1$. \hfill \Box

Next we show the useful recursive formula for $\omega^{i_1 \cdots i_k}_{yx} (F)$. We omit the proof because it is an easy induction.
Lemma 3.8. Define
\[
(\omega^{i_1\ldots i_k}_{yx})_j := (f^{i_1\ldots i_k})_j(y) - (f^{i_1\ldots i_k})_j(x) - \sum_{\ell=1}^{k-1} f^{i_1\ldots i_{\ell-1}i_\ell}(x)(\omega^{i_{\ell+1}\ldots i_k}_{yx})_j,
\]
\[
(C^{i_1\ldots i_k}_x)_j := (f^{i_1\ldots i_k})_j(x) - \sum_{\ell=1}^{k-1} (f^{i_1\ldots i_{\ell-1}i_\ell})(x)(C^{i_{\ell+1}\ldots i_k}_x)_j.
\]

Then one has the following formulas.
(1) \( (\omega^{i_1\ldots i_k}_{yx})_j = \Delta_j f_j(y) - \Delta_j f_j(x) \), and for \( k \geq 2 \),
\[
(\omega^{i_1\ldots i_k}_{yx}) = (\omega^{i_1\ldots i_{k-1}}_{yx}) \prec_{j-1} (f^{i_k})_j(y) - (C^{i_1\ldots i_{k-1}}_x) \geq_{j-1} (\omega^{i_k}_{yx})_j.
\]
(2) \( (C^{i_1\ldots i_k}_x) = \Delta_j f_j(x) \), and for \( k \geq 2 \),
\[
(C^{i_1\ldots i_k}_x) = -(C^{i_1\ldots i_{k-1}}_x) \geq_{j-1} (f^{i_k})_j(x).
\]

Proof of Proposition 3.7. Since \( \omega^{i_1\ldots i_k}_{yx}(F) = \sum_j (\omega^{i_1\ldots i_k}_{yx})_j \), we apply Lemma 3.7. First we show the estimate
\[
|C^{i_1\ldots i_k}_x| \lesssim 2^{-j(\alpha_1 + \cdots + \alpha_k)} \|f_1\|_{\alpha_1} \cdots \|f_k\|_{\alpha_k}.
\]
If \( k = 1 \), it holds because \( f_i \in C^{\alpha_i} \). Let \( k \geq 2 \). If \( C^{i_1\ldots i_{k-1}}_x \) satisfies the estimate, then \( C^{i_1\ldots i_{k-1}}_x \geq_{j-1} \) satisfies the same estimate. Hence we have the estimate of \( C^{i_1\ldots i_k}_x \) by the second formula of Lemma 3.8.

Next we show the estimate
\[
|\omega^{i_1\ldots i_k}_{yx}| \lesssim 2^j(\alpha_1 + \cdots + \alpha_k) \|f_1\|_{\alpha_1} \cdots \|f_k\|_{\alpha_k} |y - x| \theta
\]
for \( \theta \in (\alpha_1, \ldots, \alpha_{k-1}, 1] \). For \( k = 1 \), it holds because \( f_1 \in C^{\alpha_1} \). Indeed, since the differentiation \( C^{\alpha_i} \ni f_i \mapsto \nabla f_i \in (C^{\alpha_i-1})^d \) is continuous [2], Proposition 2.78, we have
\[
|\omega^{i_1}_{yx}| \leq 2\|\Delta_j f_j\|_{L^\infty} \lesssim 2^{-j\alpha_1} \|f_1\|_{\alpha_1},
\]
\[
|\omega^{i_1}_{yx}| \leq \|\Delta_j(\nabla f_i)\|_{L^\infty} |y - x| \lesssim 2^{-j(\alpha_1-1)} \|f_1\|_{\alpha_1}|y - x|.
\]
By the interpolation, we have
\[
|\omega^{i_1}_{yx}| \lesssim 2^j(\alpha_1) \|f_1\|_{\alpha_1} |y - x| \theta
\]
for any \( \theta \in [0, 1] \). Let \( k \geq 2 \). If \( \omega^{i_1\ldots i_{k-1}}_{yx} \) satisfies the estimate
\[
|\omega^{i_1\ldots i_{k-1}}_{yx}| \lesssim 2^j(\alpha_1 + \cdots + \alpha_{k-1}) \|f_1\|_{\alpha_1} \cdots \|f_{k-1}\|_{\alpha_{k-1}} |y - x| \theta
\]
for \( \theta \in (\alpha_1, \ldots, \alpha_{k-2}, 1] \), then \( \omega^{i_1\ldots i_{k-1}}_{yx} \prec_{j-1} \) satisfies the same estimate for \( \theta \in (\alpha_1, \ldots, \alpha_{k-1}, 1] \). Hence we have the estimate of \( \omega^{i_1\ldots i_k}_{yx} \) by the first formula of Lemma 3.8.

Next we show that each \( f^{i_1\ldots i_k} \) can be replaced by the iterated paraproduct. The following claim is a reformulation of Theorem 3.1.

Proposition 3.9. The family of iterated paraproducts
\[
F^\prec = \{(f_1, \ldots, f_k)^\prec\}
\]
is a seed of model on \( W^{<1} \). Precisely, if \( \alpha_1 + \cdots + \alpha_k < 1 \), then one has the bound
\[
|\omega^{i_1\ldots i_k}_{yx}(F^\prec)| \lesssim \|f_1\|_{\alpha_1} \cdots \|f_k\|_{\alpha_k} |y - x|^\alpha_1 + \cdots + \alpha_k.
\]

Remark 3.10. We assume \( \|f_i\|_{\alpha_i} \leq 1 \) for \( i = 1, \ldots, n \) and show the uniform bound for such \( f_1, \ldots, f_n \). The general result is obtained by applying the uniform bound to normalized functions \( \frac{f_i}{\|f_i\|_{\alpha_i}} \).

We prove some lemmas. We call \( \Pi = \{\tau_1, \ldots, \tau_m\} \) a partition of the word \( (i_1 \ldots i_k) \) if there are \( 1 = p_1 < p_2 < \cdots < p_m < p_{m+1} = k + 1 \) such that \( \tau_\ell = (i_{p_\ell} \ldots i_{p_{\ell+1}-1}) \) for \( \ell = 1, \ldots, m \).
Lemma 3.11. There are continuous functions
\[(f_1, \ldots, f_k) \mapsto [(i_1 \ldots i_k)] \in C^{\alpha_1 + \cdots + \alpha_k}\]
such that, one has the formula
\[f_1^{i_1} \cdots f_k^{i_k} = \sum_{\ell=1}^{k-1} f_1^{i_1} \cdots f_\ell^{i_\ell} \prec [(i_{\ell+1} \ldots, i_k)] + [(i_1, \ldots, i_k)].\]
and moreover, one has the atomic decomposition
\[f_1^{i_1} \cdots f_k^{i_k} = \sum_{\Pi = \{\tau_1, \ldots, \tau_m\}; \text{a partition of } (i_1 \ldots i_k)} (\tau_1, \ldots, \tau_m) \prec.(\cdot)^\prec\].

**proof.** First formula is a consequence of the Bailleul-Hoshino’s result (Theorem 2.3). Second formula is obtained recursively. \(\square\)

Lemma 3.12. We have the formula
\[\omega_{yx}^{i_1 \cdots i_k}(F) = \sum_{\Pi = \{\tau_1, \ldots, \tau_m\}} \omega_{yx}^\prec(\tau_1, \ldots, \tau_m)).\]

**proof.** We prove the formula by an induction on \(k\). For a single word \((i)\), since \(f^i = [(i)]\), we have \(\omega_{yx}^\prec(F) = \omega_{yx}^\prec([(i)]) = f^i(y) - f^i(x)\). Let \(k \geq 2\). By definition,
\[\omega_{yx}^\prec(\tau_1, \ldots, \tau_m) = (\tau_1, \ldots, \tau_m) \prec(x) - (\tau_1, \ldots, \tau_m) \prec(x) - \sum_{\ell=1}^{m-1} (\tau_1, \ldots, \tau_\ell) \prec(x)\omega_{yx}^\prec(\tau_{\ell+1}, \ldots, \tau_m)\].

We obtain the formula by summing them over all partitions \(\Pi\). Indeed, for the paraproducts \((\tau_1, \ldots, \tau_m) \prec\), the sum is equal to \(f_1^{i_1} \cdots f_{\ell+1}^{i_{\ell+1}} \prec\) because of the atomic decomposition (Lemma 3.11). For the difference terms \(\omega_{yx}^\prec(\tau_{\ell+1}, \ldots, \tau_m)\), its sum is equal to \(\omega_{yx}^{i_1 \cdots i_k}(F)\) by the inductive assumption. Hence we have
\[\sum_{\Pi} \omega_{yx}^\prec(\tau_1, \ldots, \tau_m) = f_1^{i_1} \cdots f_k^{i_k}(y) - f_1^{i_1} \cdots f_k^{i_k}(x) = \sum_{\ell=1}^{k-1} f_1^{i_1} \cdots f_{\ell}^{i_\ell}(x) \omega_{yx}^{i_{\ell+1} \cdots i_k}(F) = \omega_{yx}^{i_1 \cdots i_k}(F)\]. \(\square\)

We turn to the proof of the local behavior of paraproduct.

**Proof of Theorem 3.12.** We prove the bound by an induction on the number of components of \(\omega_{yx}^\prec\). By Lemma 3.12
\[\omega_{yx}^\prec(f_1, \ldots, f_k) = \omega_{yx}^{i_1 \cdots i_k}(F) - \sum_{\Pi = \{\tau_1, \ldots, \tau_m\}; m < k} \omega_{yx}^\prec(\tau_1, \ldots, \tau_m)).\]

Since Theorem 3.1 holds for \((k-1)\)-components case, we have
\[|\omega_{yx}^\prec(\tau_1, \ldots, \tau_m)| \lesssim |y - x|^{|\tau_1| + \cdots + |\tau_m|} = |y - x|^{\alpha_1 + \cdots + \alpha_k}.
\]
By Proposition 3.6 we have
\[|\omega_{yx}^\prec(f_1, \ldots, f_k)| \lesssim |y - x|^{\alpha_1 + \cdots + \alpha_k},\]
where the implicit constant is uniform over \(|f_1|_{\alpha_1}, \ldots, |f_k|_{\alpha_k} \leq 1\). \(\square\)
3.3. Conclusions. We define the canonical model on the word Hopf algebra $W^{<1}$.

Theorem 3.13. Let $\alpha_1, \ldots, \alpha_n \in (0,1)$ and $f_i \in C^{\alpha_i}$, $i = 1, \ldots, n$. Let $W^{<1}$ be the concrete regularity structure generated by alphabets $\{1, \ldots, n\}$ with homogeneities $|i| = \alpha_i$. Then the function $g: \mathbb{R}^d \to G = \text{ch}(W^{<1})$ defined by

$$g_x(i_1 \ldots i_k) = (f_{i_1}, \ldots, f_{i_k})^\prec$$

is a model on $W^{<1}$. Moreover, the mapping $(f_1, \ldots, f_n) \mapsto g$ is continuous.

**proof.** Equivalent formulation to Proposition 3.9 \qed

We define a canonical modelled distribution on such a model.

Theorem 3.14. Consider the setting of Theorem 3.13. Let $\beta \in (0,1)$ and $g \in C^\beta$. If $\alpha_1 + \cdots + \alpha_n + \beta < 1$, then the function $g: \mathbb{R}^d \to W^{<1}$ defined by

$$g(x) = \sum_{k=0}^n (g, f_1, \ldots, f_k)^\prec ((k+1) \ldots n)$$

belongs to the space $D^{\alpha_1 + \cdots + \alpha_n + \beta}(g)$. Moreover, the mapping $(f_1, \ldots, f_n, g) \mapsto (g, g)$ is continuous.

**proof.** By definition, $\hat{g}_{yx}(k+1) \ldots n) = \sum_{k=0}^n \hat{g}_{yx}(k+1) \ldots k((k+1) \ldots n)$. Since $\hat{g}_{yx}(k+1) \ldots k = \hat{\omega}_{yx}(f_{k+1}, \ldots, f_k)$, we have

$$g(y) - \hat{g}_{yx}g(x) = \sum_{k=0}^n \left( (g, f_1, \ldots, f_k)^\prec (y) - \sum_{k=0}^n (g, f_1, \ldots, f_k)\hat{\omega}_{yx}(f_{k+1}, \ldots, f_k) \right) ((k+1) \ldots n)$$

$$= \sum_{\ell=0}^n \hat{\omega}_{yx}^\prec ((g, f_1, \ldots, f_\ell)((\ell+1) \ldots n).$$

By Theorem 3.1 we have that $g \in D^{\alpha_1 + \cdots + \alpha_n + \beta}(g)$ \qed

4. Commutator

Finally we show how to apply Bailleul-Hoshino’s result to the commutator estimate.

4.1. Commutator estimate. The commutator in [11] is defined by

$$C(f, g, h) = (f \circ g) \circ h - f(g \circ h).$$

However, in this paper, we consider the commutator with respect to the bilinear operator $\geq := \circ + \succ$ instead of the resonant $\circ$. Such operators are sufficient in applications.

**Definition 4.1.** For any smooth functions $g, f_1, f_2, \ldots, f_n, \xi$ on $\mathbb{R}^d$, we define

$$C(g, \xi) := g \geq \xi = g \circ \xi + g \succ \xi,$$

$$C(g, f_1, \xi) := C(g \geq f_1, \xi) - gC(f_1, \xi),$$

$$C(g, f_1, f_2, \ldots, f_n, \xi) := C(g \geq f_1, f_2, \ldots, f_n, \xi) - gC(f_1, f_2, \ldots, f_n, \xi).$$

Our aim is to show the following commutator estimate. Let $C_0^\alpha$ be the closure of the smooth functions in $C^\alpha$.

**Theorem 4.1.** Let $\beta, \alpha_1, \ldots, \alpha_n \in (0,1)$ and $\gamma < 0$ be such that

$$\beta + \alpha_1 + \cdots + \alpha_n < 1,$$

$$\alpha_1 + \cdots + \alpha_n + \gamma < 0 < \beta + \alpha_1 + \cdots + \alpha_n + \gamma.$$

Then there exists a unique multilinear continuous operator

$$\tilde{C}: C_0^\beta \times C_0^{\alpha_1} \times \cdots \times C_0^{\alpha_n} \times C_0^\gamma \to C_0^{\beta+\alpha_1+\cdots+\alpha_n+\gamma}$$

such that,

$$\tilde{C}(g, f_1, \ldots, f_n, \xi) = C(g, f_1, \ldots, f_n, \xi)$$

for any smooth inputs $(g, f_1, \ldots, f_n, \xi)$. 
In this paper, we show only the existence of the continuous map \( \tilde{\mathcal{C}} \). The uniqueness of \( \tilde{\mathcal{C}} \) and its multilinearity follows by the denseness argument.

**Remark 4.2.** Reading the proof carefully, we can see that the operator norm of \( \tilde{\mathcal{C}} \) is locally uniform over regularity parameters. Then by the similar argument to [11 Lemma 2.4], we also obtain the unique multilinear continuous operator

\[
\tilde{\mathcal{C}} : \mathcal{C}^\beta \times \mathcal{C}^{\alpha_1} \times \ldots \times \mathcal{C}^{\alpha_n} \times \mathcal{C}^\gamma \to \mathcal{C}^{\beta + \alpha_1 + \ldots + \alpha_n + \gamma},
\]

such that \( \tilde{\mathcal{C}} = \mathcal{C} \) on any smooth inputs. However, we do not prove it in this paper, because the estimate on the space \( \mathcal{C}^\alpha \) is sufficient in applications.

4.2. **Proof of Theorem 4.1.** First we introduce another concrete regularity structure. Let \( \mathcal{W}^{<1} \) be the graded word Hopf algebra generated by alphabets \( \{1, \ldots, n\} \) with homogeneities \( |i| = \alpha_i \). We introduce another alphabet \( \Xi \) with the homogeneity \( |\Xi| = \gamma \), which represents the distribution \( \xi \in \mathcal{C}^\gamma \).

**Definition 4.2.** Let \( T \) be a linear space spanned by abstract variables

\[
\Xi, \quad (k \ldots n)\Xi \quad (k = 1, \ldots, n),
\]

with homogeneities \( |(k \ldots n)\Xi| = \alpha_k + \cdots + \alpha_n + \gamma \). The coproduct \( \Delta : T \to T \otimes \mathcal{W}^{<1} \) is defined by

\[
\Delta \Xi := \Xi \otimes 1, \quad \Delta(k \ldots n)\Xi := \sum_{\ell=k-1}^n ((\ell + 1) \ldots n)\Xi \otimes (k \ldots \ell).
\]

It is easy to show the following fact.

**Lemma 4.3.** \((T, \Delta)\) is a comodule over \( \mathcal{W}^{<1} \). Thus \((\mathcal{W}^{<1}, T)\) is a concrete regularity structure.

We consider a model \((g, \Pi)\) on \((\mathcal{W}^{<1}, T)\). For given \( f_i \in \mathcal{C}^{\alpha_i} \), \( i = 1, \ldots, n \), let \( g \) be the canonical model on \( \mathcal{W}^{<1} \) defined by

\[
g_x(i_1 \ldots i_k) = (f_{i_1}, \ldots, f_{i_k})^x.
\]

By Theorem 2.13 we have \( \|g\| < \infty \). Then by Proposition 2.23 we can define a linear map \( \Pi : T \to S'(\mathbb{R}^d) \) for any given data \([\tau]\) for the bases \( \tau \) with negative homogeneities, i.e., \( \Xi \) and \((k \ldots n)\Xi\) for \( k = 1, \ldots, n \).

**Lemma 4.4.** Let \( \xi \in \mathcal{C}^\gamma \). We define a model \( Z^0 = (g, \Pi^0) \) on \((\mathcal{W}^{<1}, T)\) by

\[
\|\Xi\|^{Z^0} = \xi, \quad \|(k \ldots n)\Xi\|^{Z^0} = 0.
\]

Then the map

\[
(f_1, \ldots, f_n, \xi) \mapsto Z^0
\]

is continuous.

If all inputs \((f_1, \ldots, f_n, \xi)\) are smooth, we can define another model.

**Definition 4.3.** If all of \( f_1, \ldots, f_n, \xi \) are smooth, then we define a smooth model \( Z^s = (g, \Pi^s) \) on \((\mathcal{W}^{<1}, T)\) by

\[
\Pi^s \Xi = \xi, \quad \Pi^s(k \ldots n)\Xi = (f_k, \ldots, f_n)^x\xi.
\]

We can check the bound \( \|\Pi^s\|^\xi < \infty \) because \( \Pi^s \) maps \( T \) into smooth functions.

**Lemma 4.5.** If all of \( f_1, \ldots, f_n, \xi \) are smooth, then we have the equalities

\[
((\Pi^s)^x\Xi)(x) = \xi(x), \quad ((\Pi^s)^x(k \ldots n)\Xi)(x) = 0.
\]

**Proof.** Since \( \Pi^s(\tau\Xi)(x) = g_x(\tau)\xi(x) \) for each word \( \tau \),

\[
(\Pi^s)^x(\tau\Xi)(x) = (\Pi^s \otimes g^{-1}_x)\Delta(\tau\Xi)(x)
\]

\[
= (g_x \otimes g^{-1}_x)\Delta(\tau\xi)(x) = 1^*(\tau)\xi(x).
\]

Since \((g, \Pi^s)\) and \((g, \Pi^0)\) are models, \((g, \Pi) := (g, \Pi^s - \Pi^0)\) is also a model.

**Lemma 4.6.** If all of \( f_1, \ldots, f_n, \xi \) are smooth, then we have the equalities

\[
(\Pi^s_2\Xi)(x) = 0, \quad (\Pi^s_2(k \ldots n)\Xi)(x) = C(f_k, \ldots, f_n, \xi)(x).
\]
**Proof.** Recall that $\Pi_x^\delta = (\Pi \otimes g_x^{-1}) \Delta$. We have the first equality because $\Delta \Xi = \Xi \otimes 1$ and $\Pi \Xi = \Pi^* \Xi - \Pi^0 \Xi = \xi - \xi = 0$. The second equality is proved by an induction on $k$. By the formula $\Pi = (\Pi_x^\delta \otimes g_x) \Delta$, we have

$$\Pi_x^\delta (k \ldots n) \Xi = \Pi (k \ldots n) \Xi = n \sum_{\ell=k}^n g_x (k \ldots \ell) \Pi_x^\delta ((\ell + 1) \ldots n) \Xi.$$ 

We calculate the term $\Pi (k \ldots n) \Xi$. Since the Bailleul-Hoshino’s result (Theorem 2.3) shows

$$\Pi^0 (k \ldots n) \Xi = \sum_{\ell=k}^n g (k \ldots \ell) \preceq [(k \ldots n) \Xi]^{Z^0} + [(k \ldots n) \Xi]^{Z^0}$$

$$= (f_k, \ldots, f_n) \prec \xi,$$

thus we have

$$\Pi (k \ldots n) \Xi = (f_k, \ldots, f_n) \prec \xi - (f_k, \ldots, f_n) \prec \xi = C((f_k, \ldots, f_n) \prec \xi, \xi).$$

If $k = n$, we have $(\Pi_x^\delta (n) \Xi)(x) = C(f_n, \xi)(x)$. If the required formula holds for $m + 1 \leq k \leq n$, then we have

$$(\Pi_x^\delta (m \ldots n) \Xi)(x) = C((f_m, \ldots, f_n) \prec \xi, \xi)(x) - n \sum_{\ell=m}^{n-1} (f_m, \ldots, f_\ell) \prec \xi(x) C(f_{n+1}, \ldots, f_n, \xi)(x)$$

$$= C(f_m, \ldots, f_n, \xi)(x).$$

The last equality follows by the definition of commutator. $\square$

Finally we turn to the proof of commutator estimate.

**Proof of Theorem 4.4.** By Theorem 3.14 a $W^{<1}$-valued function

$$g(x) = \sum_{k=0}^n (g, f_1, \ldots, f_k) \prec (k + 1 \ldots n)$$

belongs to $D^{\beta + \alpha_1 + \ldots + \alpha_n}$ ($W^{<1}; g$). It is easy to show that a $T$-valued function

$$(g \Xi)(x) = \sum_{k=0}^n (g, f_1, \ldots, f_k) \prec (k + 1 \ldots n) \Xi$$

belongs to $D^{\beta + \alpha_1 + \ldots + \alpha_n + \gamma}$ ($T; g$). (Here we specify the range space to distinguish two different classes.) This is a consequence of Theorem 4.7, but it is not difficult to show directly it because

$$\| (g \Xi)(y) - \hat{g}_{x \xi} (g \Xi)(x) \|_{\alpha_{k+1} + \ldots + \alpha_n + \gamma} = \| g(y) - \hat{g}_{x \xi} g(x) \|_{\alpha_{k+1} + \ldots + \alpha_n}$$

$$\leq \| g \|_{\beta + \alpha_1 + \ldots + \alpha_n} \| y - x \|_{\beta + \alpha_1 + \ldots + \alpha_n}.$$

Applying the Bailleul-Hoshino’s reconstruction theorem (Theorem 2.4), we have

$$R^0 (g \Xi) = (g, f_1, \ldots, f_n) \prec \xi + [g \Xi]^{Z^0},$$

and the map

$$(g, f_1, \ldots, f_n, \xi) \mapsto [g \Xi]^{Z^0}$$

is continuous. It turns out that this is the required map $\hat{\mathcal{C}}$. It remains to show that

$$[g \Xi]^{Z^0} = C(g, f_1, \ldots, f_n, \xi)$$

for any smooth inputs $(g, f_1, \ldots, f_n, \xi)$. Let $\Pi^*$ be the smooth model as above. Since $R^\pi = R^z - R^o$, we have

$$R^\pi (g \Xi) = R^z (g \Xi) - R^o (g \Xi).$$

Note that, for any model $Z = (g, \Pi)$ such that $\Pi$ maps $T$ into smooth functions, we have

$$R^z (g \Xi)(x) = (\Pi^\delta (g \Xi)(x))(x)$$
by the uniqueness of the reconstruction operator (see [12, Remark 3.15]). For $Z = Z^*$, by Lemma 4.5 we have

$$R^{Z^*}(g\Xi) = (g, f_1, \ldots, f_n)^\prec_\xi.$$ 

For $Z = Z$, by Lemma 4.6 we have

$$R^{Z}(g\Xi) = \sum_{k=0}^{n-1} (g, f_1, \ldots, f_k)^\prec_\xi C(f_{k+1}, \ldots, f_n, \xi).$$

Hence we have

$$[g\Xi]^{Z_0} = C((g, f_1, \ldots, f_n)^\prec_\xi, \xi) - \sum_{k=0}^{n-1} (g, f_1, \ldots, f_k)^\prec_\xi C(f_{k+1}, \ldots, f_n, \xi) = C(g, f_1, \ldots, f_n, \xi).$$

\hfill\Box

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Faculty of Mathematics, Kyushu University
E-mail address: hoshino@math.kyushu-u.ac.jp