On a non-local problem for parabolic-hyperbolic equation with three lines of type changing
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MSC 2000: 35M10
Keywords: parabolic-hyperbolic equation; non-local condition; Volterra integral equation

Abstract. In the present work we investigate a boundary problem with non-local conditions, connecting values of seeking function on various characteristics for parabolic-hyperbolic equation with three lines of type changing. The considered problem is equivalently reduced to the system of Volterra integral equations of the second kind.

Consider an equation

\[
\begin{cases}
  u_{xx} - u_{y}, \quad (x,y) \in \Omega_0, \\
  u_{xx} - u_{yy}, \quad (x,y) \in \Omega_i \quad (i = 1,3)
\end{cases}
\]

in the domain \( \Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup AB \cup AA_0 \cup BB_0 \).

![Figure 1: Domain \( \Omega \)](image)

Problem AS. Find a regular solution of the equation (1) in the domain \( \Omega \), satisfying the following conditions:

\[
a_1(t) u(-t,t) + a_2(t) u(t,-t) = a_3(t), \quad 0 \leq t \leq \frac{1}{2},
\]

\[
b_1(t) u(t, t - 1) + b_2(t) u(2 - t, 1 - t) = b_3(t), \quad \frac{1}{2} \leq t \leq 1,
\]

\[
c_1(t) (u_x + u_y) (t - 1, t) + c_2(t) (u_x - u_y) (2 - t, t) = c_3(t), \quad \frac{1}{2} < t < 1.
\]
Here $a_i(t), b_i(t), c_i(t)$ ($i = 1, 3$) are given functions, such that

$$a_1(0) + a_2(0) \neq 0, \quad b_1(1) + b_2(1) \neq 0, \quad a_1^2(t) + a_2^2(t) > 0, \quad b_1^2(t) + b_2^2(t) > 0,$$

$$c_1^2(t) + c_2^2(t) > 0, \quad a_1^2 + b_1^2 > 0, \quad a_2^2 + b_2^1 > 0.$$

Note, boundary problems for parabolic-hyperbolic equations with two lines of type changing were investigated in the works [1-4], and with three lines of type changing in the papers [5-6]. Distinctive side of the present work is non-local condition, which connect values of seeking function on various characteristics. It makes very difficult the reduction of the considered problem to the system of integral equations and we need special algorithm for solving this problem.

In the domain $\Omega_1$ solution of the Cauchy problem with initial data $u(x, 0) = \tau_1(x)$, $u_y(x, 0) = \nu_1(x)$ can be represented as

$$2u(x, y) = \tau_1(x + y) + \tau_1(x - y) + \int_{x-y}^{x+y} \nu_1(z) dz.$$  \hfill (5)

Assuming in condition (2)

$$u(-t, t) = \varphi_1(t), \quad 0 \leq t \leq \frac{1}{2},$$  \hfill (6)

as given, from (5) we find

$$\tau'_1(t) = \nu_1(t) + \left(\frac{2[a_3(t) - a_1(t) \varphi_1(t)]}{a_2(t)}\right)', \quad 0 < t < 1.$$  \hfill (7)

In condition (3) introduce designation

$$u(2-t, 1-t) = \varphi_2(t), \quad \frac{1}{2} \leq t \leq 1,$$  \hfill (8)

and from (5) we get

$$\tau'_1(t) = -\nu_1(t) + \left(\frac{2[b_3(t) - b_2(t) \varphi_2(t)]}{b_1(t)}\right)', \quad 0 < t < 1.$$  \hfill (9)

From (7) and (9) it follows that

$$\tau'_1(t) = \left(\frac{a_3(t) - a_1(t) \varphi_1(t)}{a_2(t)}\right)' + \left(\frac{b_3(t) - b_2(t) \varphi_2(t)}{b_1(t)}\right)', \quad 0 < t < 1.$$  \hfill (10)

Solution of the Cauchy problem in the domain $\Omega_2$ with given data $u(0, y) = \tau_2(y)$, $u_x(0, y) = \nu_2(y)$ we write as follows

$$2u(x, y) = \tau_2(y + x) + \tau_2(y - x) + \int_{y-x}^{y+x} \nu_2(z) dz.$$  \hfill (11)
Considering (6) from (11) we obtain
\[ \tau'_2(t) = \nu_2(t) + \varphi'_1\left(\frac{t}{2}\right), \quad 0 < t < 1. \] (12)

In condition (4) introduce another designation
\[ (u_x - u_y)(2 - t, t) = \varphi_3(t), \quad \frac{1}{2} < t < 1. \] (13)

Then from (11) we get
\[ \frac{c_3 \left( \frac{t+1}{2} \right) - c_2 \left( \frac{t+1}{2} \right)}{c_1 \left( \frac{t+1}{2} \right)} \varphi_3 \left( \frac{t+1}{2} \right) = \tau'_2(t) + \nu_2(t), \quad 0 < t < 1. \] (14)

From (12) and (14) we deduce
\[ 2\tau'_2(t) = \varphi'_1\left(\frac{t}{2}\right) + \frac{c_3 \left( \frac{t+1}{2} \right) - c_2 \left( \frac{t+1}{2} \right)}{c_1 \left( \frac{t+1}{2} \right)} \varphi_3 \left( \frac{t+1}{2} \right), \quad 0 < t < 1. \] (15)

Solution of the Cauchy problem with data \( u(1,y) = \tau_3(y), \ u_x(1,y) = \nu_3(y) \) in the domain \( \Omega_3 \) has a form
\[ 2u(x, y) = \tau_3(y + x - 1) + 2\tau_2(y - x + 1) + \int_{y-x+1}^{y+x-1} \nu_3(z)\, dz. \] (16)

Using (8) and (13) from (16), after some evaluations one can get
\[ 2\tau'_3(t) = -\varphi'_2\left(\frac{2-t}{2}\right) - \varphi_3\left(\frac{t+1}{2}\right), \quad 0 < t < 1. \] (17)

Further, from the equation (1) we pass to the limit at \( y \to +0 \) and considering (7) we find
\[ \tau''_1(t) - \tau'_1(t) = -\left(\frac{2 \left[ a_3\left(\frac{t}{2}\right) - a_1\left(\frac{t}{2}\right) \varphi_1\left(\frac{t}{2}\right) \right]}{a_2\left(\frac{t}{2}\right)}\right)'. \] (18)

Solution of the equation (18) together with conditions
\[ \tau_1(0) = \frac{a_3(0)}{a_1(0) + a_2(0)}, \quad \tau_1(1) = \frac{b_3(1)}{b_1(1) + b_2(1)}, \] (19)

which reduced from (2) and (3), can be represented as
\[ \tau_1(x) = \frac{a_3(0)}{a_1(0) + a_2(0)} + x \left[ \frac{b_3(1)}{b_1(1) + b_2(1)} - \frac{a_3(0)}{a_1(0) + a_2(0)} \right] + \int_0^1 G(x, t) \left[ \frac{b_3(1)}{b_1(1) + b_2(1)} - \frac{a_3(0)}{a_1(0) + a_2(0)} \right] dt - \int_0^1 G(x, t) \left( \frac{2 \left[ a_3\left(\frac{t}{2}\right) - a_1\left(\frac{t}{2}\right) \varphi_1\left(\frac{t}{2}\right) \right]}{a_2\left(\frac{t}{2}\right)} \right)' dt, \quad 0 \leq x \leq 1, \] (20)
where $G(x,t)$ is Green’s function of the problem (18)-(19).

Continuing to assume the function $\varphi_1$ as known, using the formula (10) we represent function $\varphi_2$ via $\varphi_1$. Then using the solution of the first boundary problem for the equation (1) in the domain $\Omega_0$ and functional relations between functions $\tau_j$ and $\nu_j$ ($j = 2, 3$), we get the following:

$$
\tau'_2(y) = \int_0^y \tau'_3(\eta) N(0, y, 1, \eta) \, d\eta - \int_0^y \tau'_2(\eta) N(0, y, 0, \eta) \, d\eta + F_1(y),
$$

$$
\tau'_3(y) = \int_0^y \tau'_3(\eta) N(1, y, 1, \eta) \, d\eta - \int_0^y \tau'_2(\eta) N(1, y, 0, \eta) \, d\eta + F_2(y),
$$

where

$$
F_1(y) = \int_0^1 \tau_1(\xi) \overline{G}_x(o, y, \xi, 0) \, d\xi - \frac{a_3(0)}{a_1(0) + a_2(0)} N(0, y, 0, 0) +
$$

$$
+ \frac{b_3(1)}{b_1(1) + b_2(1)} N(0, y, 1, 0) + \varphi'_1\left(\frac{y}{2}\right),
$$

$$
F_2(y) = \int_0^1 \tau_1(\xi) \overline{G}_x(1, y, \xi, 0) \, d\xi - \frac{a_3(0)}{a_4(0) + a_2(0)} N(1, y, 0, 0) +
$$

$$
+ \frac{b_3(1)}{b_1(1) + b_2(1)} N(1, y, 1, 0) - \varphi_3\left(\frac{y + 1}{2}\right),
$$

$$
\overline{G}(x, y, \xi, \eta) = \frac{1}{2\sqrt{\pi (y-\eta)}} \sum_{n=-\infty}^{\infty} \left[ e^{-\frac{(x-\xi+2n\eta)^2}{4(y-\eta)}} - e^{-\frac{(x+\xi+2n\eta)^2}{4(y-\eta)}} \right]
$$

is Green’s function of the first boundary problem,

$$
N(x, y, \xi, \eta) = \frac{1}{2\sqrt{\pi (y-\eta)}} \sum_{n=-\infty}^{\infty} \left[ e^{-\frac{(x-\xi+2n\eta)^2}{4(y-\eta)}} + e^{-\frac{(x+\xi+2n\eta)^2}{4(y-\eta)}} \right].
$$

From the first equation of (21) we represent function $\varphi_3$ via $\varphi_1$ and further, from the second equation of (21) we find the function $\varphi_1$.

After the finding function $\varphi_1$, using appropriate formulas we find functions $\varphi_2, \varphi_3, \tau_i, \nu_i, (i = 1, 3)$. Solution of the problem AS can be established in the domain $\Omega_0$ as a solution of the first boundary problem, and in the domains $\Omega_i (i = 1, 3)$ as a solution of the Cauchy problem.

**Theorem.** If functions $a_i, b_i, c_i$ are continuously differentiable on the segment, and have continuous second order derivatives on interval, where they given, then the problem AS have the unique regular solution.

**References**

1. Egamberdiev U. Boundary problems for mixed parabolic-hyperbolic equation with two lines of type changing. PhD thesis, Tashkent, 1984.
2. *Abdullaev A.S.* On some boundary problems for mixed parabolic-hyperbolic type equations// Equations of mixed type and problem with free boundary. Tashkent: Fan, 1987, pp. 71-82.

3. *Eleev V.A.*, *Lesev V.N.* On two boundary problems for mixed type equations with perpendicular lines of type changing// Vladikavkaz math.journ. 2001. Vol. 3. Vyp. 4, pp.9-22.

4. *Nakusheva V.A.* First boundary problem for mixed type equation in a characteristic polygon// Dokl.AMAN, 2012. Vol.14, No 1, pp.58-65.

5. *Berdyshev A.S.*, *Rakhmatullaeva N.A.* Nonlocal problems with special gluing for a parabolic-hyperbolic equation. "Further Progress in Analysis”. Proceedings of the 6th ISAAC Congress. Ankara, Turkey, 13-18 August, 2007, pp. 727-734.

6. *Berdyshev A.S.*, *Rakhmatullaeva N.A.* Non-local problems for parabolic-hyperbolic equations with deviation from the characteristics and three type-changing lines //Electronic Journal of Differential Equations. Vol. (2011) 2011, No 7, pp.1-6.