4d ensembles of percolating center vortices and monopole defects: the emergence of flux tubes with N-ality and gluon confinement

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Ensembles of magnetic defects represent quantum variables that have been detected and extensively explored in lattice SU(N) pure Yang-Mills theory. They successfully explain many properties of confinement and are strongly believed to capture the (infrared) path-integral measure. In this work, we initially motivate the presence of magnetic non-Abelian degrees of freedom in these ensembles. Next, we consider a simple Gaussian model to account for fluctuations. In this case, both center vortices and monopoles become relevant degrees in Wilson loop averages. These physical inputs are then implemented in an ensemble of percolating center vortices in four dimensions by proposing a measure to compute center element averages. Introducing phenomenological information such as monopole tension, stiffness, and fusion, the ensemble integration leads to an effective YMH model with adjoint Higgs fields. If monopoles also condense, then the gauge group undergoes SU(N) \rightarrow Z(N) SSB. This pattern has been proposed as a strong candidate to describe confinement. In the presence of external quarks, these models are known to be dominated by classical solutions, formed by flux tubes with N-ality as well as by confined dual monopoles (gluons).

I. INTRODUCTION

Based on lattice simulations, it is well established that a confining linear potential between a fundamental quark and antiquark is generated in the infrared regime of pure SU(N) Yang-Mills theory [1]. For other quark representations, the asymptotic potential depends solely on how the center of SU(N) is realized [2]. This is one of the properties that favors a quark confinement mechanism based on an ensemble of center vortices [3]-[6]. When the quark Wilson loop is linked by a center vortex, it gains a center element. Thus, in the percolating phase, the obtained area law naturally displays N-ality. This idea has gained momentum over the last many years, settling these degrees as essential infrared quantum variables to capture the path-integral measure [7]-[17]. On the other hand, Monte Carlo simulations also show subleading contributions, which coincide with universal Lüscher corrections due to the transverse (quantum) fluctuations of a string [18]. Moreover, the action and field distributions measured around the confining string are nontrivial, revealing a chromoelectric flux tube [19]-[24]. While center vortices are essential to describe an area law with N-ality, Lüscher terms and flux tubes have not yet been observed in these ensembles.

In contrast, dual superconductivity [25]-[30] is suitable to accommodate stringlike behavior. The idea of Abelian projection [25] and associated ensembles of monopole defects were analyzed in the lattice [31]-[33]. The understanding of confinement in compact QED, and how the proliferation of monopoles induce observable surfaces attached to a quark loop, was obtained in Refs. [34], [35]. In addition, the profile of the confining Yang-Mills flux tube has been fitted using vortex solutions in effective Abelian Higgs models [19]-[24]. However, Abelian scenarios cannot describe N-ality. For example, when applied to double Wilson loops in SU(2) they lead to the sum of areas, instead of the difference-in-areas law observed in the lattice and accommodated by center vortices [36].

Based on the complementary properties of center vortices and monopole defects, it is natural to infer that an appropriate combination of both could capture the whole physical picture. Indeed, in lattice calculations of pure SU(N) Yang-Mills (YM) theory based on center gauges, both center vortices and attached monopoles were detected, forming chains. In fact, they account for 97% of the cases [12]. In the continuum, the description and topological aspects of these arrays, in which
the Lie algebra flux orientation changes at the monopoles, were worked out in Ref. [16]. In Abelian
gauges, the possibility that integrating off-diagonal fluctuations could lead to collimated chains was
suggested in Ref. [37], and references therein.

Another scenario to accommodate $N$-ality has been proposed at the level of possible dual
descriptions. The properties of SU($N$) YM confining strings have been sought in classical topological
solutions by exploring a variety of models. Flux tubes with $N$-ality and confined dual monopoles
are known to exist in SU($N$) Yang-Mills-Higgs (YMH) models when the gauge group is sponta-
neously broken to $Z(N)$ [38]-[45]. Another possibility to accommodate these states is provided by
non-supersymmetric models with $N$ fundamental Higgs fields [46], SU($N$) $\times$ U(1) gauge group, and
a color-flavor locking phase that equips flux tubes and dual monopoles with a non-Abelian moduli
space [47]-[49]. The connection of color-flavor locking with monopole condensates that fit into the
Goddard-Nyuts-Olive classification scheme [50] was extensively analyzed, mainly in a supersym-
metric context. The present status can be found in Ref. [51]. A color-flavor locking phase could
also be present in SU($N$) YMH models with $N^2 - 1$ (real) adjoint Higgs fields [52]-[54]. Confined
dual monopoles were interpreted as gluons in Refs. [46], [55], [56] (see also [52], [53]).

The aim of this work is to combine the different ideas into a possible unified mechanism. Chains
were visualized as magnetic defects of a local color frame in Ref. [57]. Their relation to the
observability of surfaces attached to the quark loop was discussed in Ref. [58]. In Ref. [59],
the derivation of a 3d effective field model for chains allowed to relate the monopole (instanton)
component with the $Z(N)$-symmetric terms in the 't Hooft model [3]. Therefore, when center
vortices condense, monopoles are essential to drive magnetic $Z(N)$ SSB and generate an observable
domain wall with $N$-ality, attached to the quark loop. In 4d, the relation between ensembles
of monopoles that carry adjoint charges and models based on a set of adjoint Higgs fields was
suggested in Ref. [52]. This idea was further elaborated in Ref. [60], where we applied polymer
techniques to an ensemble of worldlines with non-Abelian d.o.f.

In four dimensions, while monopoles are naturally described by effective field models [61]-[64],
the consideration of center vortices would be related to string field theories, which poses important
difficulties. We suggest that when center vortices percolate, the effect of linking numbers could be
captured in an effective field theory. This is motivated by the low-energy effective description of
higher dimensional defect condensates [65]-[68], a recent study about ensembles of center vortices
in 3d [69], and by a simple model based on a smoothed Gaussian Wilson loop.

In sections II and III, we consider a gauge fixing in the continuum [70], motivated by lattice
center gauges [71]-[76], to show the presence of non-Abelian d.o.f. in configurations with thin center
vortices and monopole defects. In section IV, relying on the Petrov-Diakonov representation of
the Wilson loop, we present a simple example, showing that center vortices and monopoles with
non-Abelian d.o.f. may have a combined effect on Wilson loop averages. Then, an ensemble
measure that mixes percolating center vortices and chains is proposed in section V. In section VI,
fusion rules between monopole adjoint lines are associated to effective Feynman diagrams, and the
ensemble partition function is rewritten in terms of a dual SU($N$) YMH model. Finally, in section
VII, we present our conclusions.

Throughout this work, we shall use the internal product between a pair of Lie algebra elements
$X,Y \in \text{su}(N),$

$$ (X,Y) = \text{tr}(\text{Ad}(X)\text{Ad}(Y)), \quad (1) $$

where $\text{Ad}(\cdot)$ refers to the adjoint representation, and shall denote $(X,X) \equiv (X)^2$. The main
properties of this product are the cyclic and group invariances, which are a consequence of the
defining property of a representation,
\( (X, [Y, Z]) = (Z, [X, Y]) \) , \( (UXU^{-1}, UYU^{-1}) = (X, Y) \) \( (2) \)
\( \text{Ad}([X, Y]) = [\text{Ad}(X), \text{Ad}(Y)] \) , \( \text{Ad}(UXU^{-1}) = R(U)\text{Ad}(X)R^{-1}(U) \) \( (3) \)
\( R(U) = \text{Ad}(U) \) is the \( \mathbb{D}_{\text{Ad}} \times \mathbb{D}_{\text{Ad}} \) matrix that represents \( U \) in the adjoint (\( \mathbb{D}_{\text{Ad}} = N^2 - 1 \)). We shall also adopt an orthonormal Lie basis \( T_A, A = 1, \ldots, N^2 - 1, \)
\( (T_A, T_B) = \delta_{AB} \) \( \) , \( [T_A, T_B] = if_{ABC}T_C \) \( (4) \)
\( \text{Ad}(T_A)|_{BC} = -if_{ABC} \) \( ) \( , \) \( f_{ABC}f_{DBC} = \delta_{AD} \) \( (5) \)
Matrices such as \( U \), with no explicit reference to the irrep, are understood to be in the fundamental representation of \( SU(N) \).

II. DETECTING MAGNETIC DEFECTS IN THE CONTINUUM

In the lattice, gauge fixings designed to avoid the Gribov problem and detect center vortices were proposed in refs. \([71]-[76]\) (for a review, see ref. \([77]\)). They are based on the lowest eigenfunctions \( (f_1, f_2, \ldots) \) of the adjoint covariant Laplacian,
\( D_\mu D_\mu (A) f_I = \lambda_I f_I \) \( \) , \( D_\mu (A) = \partial_\mu - i [A_\mu, \, ] \) \( (6) \)
using them to fix a prescribed orientation in color space. For example, in the direct Laplacian center gauge \([75, 76]\), a map \( \text{Ad}(S) \) is constructed in a covariant way, that is, under a chromoelectric gauge transformation \( A_\mu U_e \), the associated map is \( \text{Ad}(U_e S) \). For \( N = 2 \), this is obtained from the polar decomposition of the real \( 3 \times 3 \) matrix formed by the color entries of \( (f_1, f_2, f_3) \). As this procedure is based on the lowest eigenfunctions, it cannot be directly implemented in the continuum. However, in Ref. \([70]\), we introduced a modified version where the assignment,
\( A_\mu \rightarrow f_I \rightarrow \text{Ad}(S) \) \( , \) \( (7) \)
is based on the adjoint fields \( f_I \in \mathfrak{su}(N) \) that solve a set of coupled differential equations,
\( \frac{\delta S_{\text{aux}}}{\delta f_I} = D_\mu D_\mu (A) f_I + \cdots = 0 \) \( (8) \)
In order for \( (f_1, f_2, \ldots) \) to be strongly correlated with \( \text{Ad}(S) \), the auxiliary action \( S_{\text{aux}} \) possesses \( SU(N) \rightarrow Z(N) \) SSB. Considering \( N^2 - 1 \) fields, \( I = 1, \ldots, N^2 - 1, \) the desired map was extracted from a polar decomposition of the tuple \( (f_1, f_2, \ldots) \) in terms of “modulus” \( (q_1, q_2, \ldots) \) and “phase” \( S \)-variables,
\( f_I = Sq_I S^{-1} \) \( \) , \( \sum_I [q_I, T_I] = 0 \) \( (9) \)
The last condition amounts to looking for the rotated \( f_I \)’s that form the tuple which minimizes,
\( \sum_I (q_I - vT_I)^2 \),
where \( (vT_1, vT_2, \ldots) \) is a prescribed point in the vacuum manifold of \( S_{\text{aux}} \). For \( SU(2) \), this makes contact with the polar decomposition of a real \( 3 \times 3 \) matrix. Again, because of covariance, for the gauge transformed field
\( A_\mu^{U_e} = U_e A_\mu U_e^{-1} + i U_e \partial_\mu U_e^{-1} \) \( , \)
the extracted phase is $U_eS,$

$$A^U_e \rightarrow U_e f_I U_e^{-1} \rightarrow \text{Ad}(U_eS) = \text{Ad}(U_e) \text{Ad}(S).$$

(10)

Although $A_\mu$ is a well-defined variable, $S$ could contain defects. Therefore, the equivalence relation given by,

$$S \sim S' \quad \text{if} \quad S' = U_e S,$$

(11)

induces a nontrivial partition of mappings into classes $[S]$, and of configurations into sectors $\mathcal{V}(S)$: two variables $A_\mu, A'_\mu$ are in the same sector if they are mapped to $S, S'$ that are equivalent in the sense given by Eq. (11). This should not be confused with the equivalence relation,

$$A_\mu \sim A'_\mu \quad \text{if} \quad A'_\mu = U_e A_\mu,$$

(12)

In each sector $\mathcal{V}(S)$, there are infinitely many physically inequivalent configurations. For example, there is a perturbative sector formed by those $P_\mu$ mapped to a regular $S$. Other sectors will be related to mappings with different number, types, and locations of defects. Equivalence classes of mappings will be denoted by $[S_0]$, where the label $S_0$ refers to a choice of representative. The gauge fixed variables in $[S_0]$ satisfy,

$$A_\mu \rightarrow \text{Ad}(S_0).$$

In the perturbative sector, $S_0$ can be chosen as the identity map, and the gauge fixed perturbative variables satisfy, $P_\mu \rightarrow I$. The total partition function is a sum over sectors,

$$Z_{YM} = \sum_{S_0} Z^{(S_0)}_{YM},$$

(13)

where $Z^{(S_0)}_{YM}$ are the gauge fixed partial contributions. They are obtained from the path integral over $\mathcal{V}(S)$, $S = U_e S_0$, by using an identity to introduce the equations of motion (8),

$$1 = \int [Df_I] \frac{\delta}{\delta f_I} \left( \frac{\delta S_{\text{aux}}}{\delta f_I} \right) \det \left( \frac{\delta^2 S_{\text{aux}}}{\delta f_I \delta f_J} \right),$$

(14)

then changing to polar variables $q_I$, and finally factorizing the regular part $U_e$ by means of a gauge transformation. On each sector there is a BRST symmetry that transforms $A_\mu, q_I$, auxiliary fields, and ghosts. The latter can be grouped as $b_I, c_I$, needed to exponentiate the constraint and determinant in Eq. (14), and $b, c$, originated from the pure modulus condition in Eq. (9). The BRST symmetry has a sector-independent algebraic structure that cannot be extended globally, due to specific regularity conditions in each sector [70]. This is a welcome property as each BRST can be used to show that partial contributions to observables do not depend on gauge parameters, but not to conclude that the asymptotic space of states is formed by gluons.

**III. MAGNETIC DEFECTS AND NON-ABELIAN DEGREES OF FREEDOM**

Configurations $A_\mu \in \mathcal{V}(S)$ are created on top of perturbative (topologically) trivial ones, $P_\mu \in \mathcal{V}(I)$, by means of a singular transformation $[3]$

$$\text{Ad}(A_\mu) = R(S) \text{Ad}(P_\mu) R(S)^{-1} + i R(S) \partial_\mu R(S)^{-1} = R(S) \text{Ad}(P_\mu - Z_\mu) R(S)^{-1},$$

$$R(S) = \text{Ad}(S), \quad \text{Ad}(Z_\mu) = i R(S)^{-1} \partial_\mu R(S).$$

(15, 16)
The use of the adjoint representation $\text{Ad}(S)$ eliminates unphysical terms, localized on three-volumes, that would be present when computing
\[ SP_\mu S^{-1} + i S \partial_\mu S^{-1}. \] (17)

An equivalent procedure to get rid of these terms was introduced in Ref. [14]. Besides the usual covariant field strength,
\[ F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \] (18)
it will be useful to define an invariant object,
\[ G_{\mu\nu}(A) = S^{-1}F_{\mu\nu}(A)S, \quad (G_{\mu\nu}(A), T_A) = (F_{\mu\nu}(A), n_A), \quad n_A = ST_A S^{-1}. \] (19)

Magnetic defects are manifested in field strengths through the commutators of ordinary derivatives $[\partial_\mu, \partial_\nu]$, which are nontrivial when applied on singular mappings,
\[ F_{\mu\nu}(A) = S(F_{\mu\nu}(P) - F_{\mu\nu}(Z))S^{-1}, \quad G_{\mu\nu}(A) = F_{\mu\nu}(P) - F_{\mu\nu}(Z), \quad \text{Ad}(F_{\mu\nu}(Z)) = i R(S)^{-1}[\partial_\mu, \partial_\nu] R(S). \] (20) (21)

A. Chains

Let us briefly review some examples. A center vortex worldsheet $\Sigma$ can be created by,
\[ \bar{S} = e^{i\chi \bar{\beta} \cdot \bar{T}}, \quad \bar{\beta} \cdot \bar{T} \equiv \bar{\beta}_q T_q, \quad \bar{\beta} = 2N\bar{w}, \] (22)
where $\chi$, $\partial^2 \chi = 0$, is a multivalued phase that changes by $2\pi$ when going around a path linking $\Sigma$, and $T_q$, $q = 1, \ldots, N - 1$, are the Cartan generators. The magnetic weight $\bar{\beta}$ is $2N$ times a weight $\bar{w}$ of $\text{su}(N)$, see Eq. (A13). The simplest case corresponds to the fundamental representation $\bar{w} = \bar{w}_1$, $i = 1, \ldots, N$ [44], which will be considered from now on. In chains, pairs of center vortex branches are matched by monopoles [16], [57], [52]. In this case, we can write,
\[ \bar{S} = e^{i\chi \bar{\beta} \cdot \bar{T}} W, \] (23)
where the single-valued $W$ creates a closed monopole worldline $C_m$ on $\Sigma$. For example, a pair of semi-infinite center vortices is created by using $\chi = \varphi$, $W = e^{i\varphi \sqrt{NT}_\alpha}$, where $\varphi$, $\theta$ are polar angles centered at the monopole, and $T_\alpha$ $(\bar{\alpha} = \bar{w} - \bar{w}_\sigma)$ is a combination of root vectors (cf. Eq. A15). Since the map $W(\pi)$ is a Weyl transformation,
\[ W(\pi)^{-1} \bar{\beta} \cdot \bar{T} W(\pi) = \bar{\beta} \cdot \bar{T}, \] (24)
it interpolates between two different behaviors, $\bar{S} \sim e^{i\varphi \bar{\beta} \cdot \bar{T}}$ and $\bar{S} \sim e^{i\pi \sqrt{NT}_\alpha} e^{i\varphi \bar{\beta} \cdot \bar{T}}$, around $\theta = 0$ and $\theta = \pi$, respectively. The factor $e^{i\pi \sqrt{NT}_\alpha}$ has no effect on gauge invariant quantities, so that the branches are along $\bar{\beta} \cdot \bar{T}$ and $\bar{\beta} \cdot \bar{T}$, respectively. The contribution to $G_{\mu\nu}$ is [14], [16],
\[ -F_{\mu\nu}(Z) = 2\pi \bar{\beta} \cdot \bar{T} \int d^2\sigma_{\mu\nu} \delta^{(4)}(x - y(\sigma_1, \sigma_2)) + 2\pi \bar{\beta} \cdot \bar{T} \int d^2\sigma_{\mu\nu} \delta^{(4)}(x - y(\sigma_1, \sigma_2)) \]
\[ F_{\mu\nu}(Z) = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}(Z), \quad d^2\sigma_{\mu\nu} = d\sigma_1 d\sigma_2 \left( \frac{\partial y_\mu}{\partial \sigma_1} \frac{\partial y_\nu}{\partial \sigma_2} - \frac{\partial y_\mu}{\partial \sigma_2} \frac{\partial y_\nu}{\partial \sigma_1} \right), \] (25)
where the integrals are done over branches with common border at $C_m$ and whose union is $\Sigma$. 

B. Chains with monopole fusion

In order to discuss possible monopole matchings, let us consider a simple example for \( N \geq 3 \). At a given time \( t \), on a section \( \mathbb{R}^3 \) of the 4d Euclidean spacetime, we can take three points placed on a line at positions \( \mathbf{x}_A, \mathbf{x}_B, \mathbf{x}_C \) (in that order). The map

\[
S = e^{i\varphi_1 \bar{T} W(\gamma, \gamma')} , \quad W(\gamma, \gamma') = W_{12}(\gamma)W_{13}(\gamma') , \quad W_{12}(\theta) = e^{i\theta \sqrt{\mathcal{N}} T_{\alpha\beta}} ,
\]

where \( \gamma \) (resp. \( \gamma' \)) is the angle that \( \mathbf{x}_A, \mathbf{x}_B \) (resp. \( \mathbf{x}_B, \mathbf{x}_C \)) subtend from the observation point \( \mathbf{x} \), describes three monopoles joined by center vortices. In effect, close to the line, to the left of \( \mathbf{x}_A \) and to the right of \( \mathbf{x}_C \), \( \gamma \) and \( \gamma' \) tend to zero, i.e., \( \bar{S} \sim e^{i\varphi_3 \bar{T}} \). The same behavior is verified away from the three points. When the segments between \( \mathbf{x}_A, \mathbf{x}_B (\gamma \to \pi, \gamma' \to 0) \) and between \( \mathbf{x}_B, \mathbf{x}_C (\gamma \to 0, \gamma' \to \pi) \) are approached, it is obtained,

\[
\bar{S} \sim e^{i\varphi_1 \bar{T} W_{12}(\pi)} = W_{12}(\pi) e^{i\varphi_2 \bar{T}} \quad \text{and} \quad \bar{S} \sim e^{i\varphi_1 \bar{T} W_{13}(\pi)} = W_{13}(\pi) e^{i\varphi_3 \bar{T}} .
\]

Hence, \( \bar{S} \) describes center vortex worldsheets meeting at three worldlines \( \mathbf{x}_A(t), \mathbf{x}_B(t), \mathbf{x}_C(t) \) with common endpoints, that carry adjoint weights

\[
\bar{\delta}_1 = \bar{w}_1 - \bar{w}_2 , \quad \bar{\delta}_2 = \bar{w}_2 - \bar{w}_3 , \quad \bar{\delta}_3 = \bar{w}_3 - \bar{w}_1 .
\]

This array describes a creation annihilation process with the fusion rule, \( \bar{\delta}_1 + \bar{\delta}_2 + \bar{\delta}_3 = 0 \). Four monopole worldlines can be fused in a similar way. In the general case, the field tensor is a sum over open surface contributions,

\[
-\mathcal{F}_{\mu\nu}(Z) = 2\pi \sum_j \bar{\beta}_j \cdot \bar{T} \int d^2 \sigma_{\mu\nu} \delta^{(4)}(x - y_j(\sigma_1, \sigma_2)) .
\]

C. Non-Abelian d.o.f.

Consider a label \( S_0 \) in the gauge fixed partial contribution (13). The left action \( S_0 \to U e^{S_0} \) simply corresponds to a chromoelectric gauge transformation. On the other hand, the right action \( S_0 \to U^{-1} \) generally leads to a new class \( [S_0 U^{-1}] \neq [S_0] \). Of course, starting with perturbative configurations \( (S_0 = \mathcal{I}) \) no new class is generated, \( [U^{-1}] = [\mathcal{I}] \). In the other cases, the transformed labels represent a continuum of different sectors \( \mathcal{Y}(S_0 U^{-1}) \) modulo the equivalence relation in Eq. (11). Now, as \( U \) is regular, it cannot change the number nor the location of magnetic defects. Then, for each possible distribution of defects, there is a continuum of partial contributions. This leads to an important observation: defects possess physical non-Abelian degrees of freedom. Their relevance can be related to the fact that \( F_{\mu\nu}(Z) \), the second term in the chromoelectric gauge invariant tensor \( G_{\mu\nu} \) is generally modified. For the above-mentioned examples, it is verified,

\[
F_{\mu\nu}(Z) = \bar{U} F_{\mu\nu}(Z) U^{-1} , \quad S = S \bar{U} U^{-1} .
\]

Thus, for a chain and the example with fusion, the monopole currents are, respectively,

\[
K_\mu = -D_\nu(\mathbf{L}) F_{\mu\nu}(Z) = 2\pi \sum_j \bar{U} \bar{\delta}_j \cdot \bar{T} \bar{U}^{-1} \int_{\gamma_j} dy_\mu \delta^{(4)}(x - y) ,
\]

\[
K_\mu = -D_\nu(\mathbf{L}) F_{\mu\nu}(Z) = 2\pi \sum_j \bar{U} \bar{\delta}_j \cdot \bar{T} \bar{U}^{-1} \int_{\gamma_j} dy_\mu \delta^{(4)}(x - y) , \quad \sum_j \bar{\delta}_j = 0 ,
\]

which are covariantly conserved. Note also that the second term in the usual field strength continues to be along the Cartan sector, \( SF_{\mu\nu}(Z) S^{-1} = \mathcal{F}_{\mu\nu}(Z) \) (cf. Eqs. (20), (21)).
IV. GAUSSIAN GAUGE INVARIANT SMOOTHING

The Wilson loop for quarks in an irreducible \( \mathcal{D} \)-dimensional representation \( \mathcal{D} \) is

\[
\mathcal{W}_e[A] = \frac{1}{\mathcal{D}} \text{tr} \mathcal{D} \left( P \left\{ e^{i \oint_C A_\mu(x)} \right\} \right) .
\]  

(32)

When thin configurations are considered, i.e., \( P_\mu = 0 \) in Eqs. (15), (17), the result for a chain coincides with that for a center vortex placed at the same location. The non-Abelian d.o.f. do not play a role either. Indeed, the Wilson loop is given by,

\[
\mathfrak{z}(\mathcal{C}_e) = \frac{1}{\mathcal{D}} \text{tr} \mathcal{D}(S_\tau S_i^{-1}) ,
\]  

(33)

\[
S = e^{i \chi \bar{\beta} \cdot \bar{T} W \bar{U}^{-1}} , \quad S_\tau S_i^{-1} = e^{i(\chi_i - \chi_1) \bar{\beta} \cdot \bar{T}} = e^{i2\pi \bar{\beta} \cdot \bar{T} L(\mathcal{C}_e, \Sigma)} .
\]  

(34)

This only depends on the linking number \( L(\mathcal{C}_e, \Sigma) \) between \( \mathcal{C}_e \) and \( \Sigma \). However, the ensemble measure would in principle be generated by path-integrating general field fluctuations \( P_\mu \) around magnetic defects, which might differentiate between center vortices and chains. Answering if this is the case in the YM context is a difficult task. Instead, in the next subsection, we shall discuss a simple example to get some insight about possible effects.

A. Dual representation

For general configurations in \( \mathcal{V}(S) \) (cf. Eqs. (15), (17)) we have,

\[
\mathcal{W}_e[A] = \mathcal{W}_e[P] \mathfrak{z}(\mathcal{C}_e) .
\]  

(35)

The linking number can be equated to the intersection number \( I(S(\mathcal{C}_e), \Sigma) \) between \( S(\mathcal{C}_e) \) (a surface whose border is \( \mathcal{C}_e \) and \( \Sigma \),

\[
I(S(\mathcal{C}_e), \Sigma) = \frac{1}{2} \int d^2 \sigma_{\mu
u} \int d^2 \sigma_{\mu\nu} \delta(4)(w(s, \tau) - y(\sigma_1, \sigma_2)) ,
\]  

(36)

\[
d^2 \sigma_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} d\sigma d\tau \left( \frac{\partial w_\alpha}{\partial \tau} \frac{\partial w_\beta}{\partial s} - \frac{\partial w_\alpha}{\partial s} \frac{\partial w_\beta}{\partial \tau} \right) ,
\]  

(37)

where \( w(s, \tau) \) is a parametrization of \( S(\mathcal{C}_e) \) [14]. Since both vortex orientations will be taken into account, the antifundamental weights can be disregarded. The topological contribution can be written in terms of \( \mathcal{F}_{\mu\nu}(Z) \) as follows: i) consider a general configuration \( \bar{S} \) with defects such that \( \mathcal{F}_{\mu\nu}(Z) \) is along the Cartan sector, ii) note that if a chain links \( \mathcal{C}_e \), then one of the associated vortex branches crosses \( S(\mathcal{C}_e) \), iii) use that for any magnetic weight \( \bar{\beta}_i \), it is verified,

\[
\mathcal{D} \left( e^{-i \frac{\varphi}{2} \bar{\beta}_i \cdot \bar{T}} \right) = \mathcal{D} \left( e^{i \frac{\varphi}{2} \bar{\beta}_i \cdot \bar{w}_e} \right) I_D , \quad \mathfrak{z}(\mathcal{C}_e) = \left( e^{i2\pi \bar{\beta}_i \cdot \bar{w}_e} \right)^{I(S(\mathcal{C}_e), \Sigma)} ,
\]  

(38)

where the tuple \( \bar{w}_e \) is any weight of the quark representation (we can choose the highest) and \( I_D \) is the \( \mathcal{D} \times \mathcal{D} \) identity matrix. Therefore, we can write,

\[
\mathfrak{z}(\mathcal{C}_e) = e^{-i \frac{\varphi}{2} \int d^4 x \left( s_{\mu\nu} \bar{U} \bar{\alpha}_e \cdot \bar{T} \bar{U}^{-1}, \mathcal{F}_{\mu\nu}(Z) \right)} , \quad s_{\mu\nu}(x) = \int_{S(\mathcal{C}_e)} d^2 \bar{\sigma}_{\mu\nu} \delta(4)(x - w(s, \tau)) ,
\]  

(39)

where \( \bar{U}(x) \) is any regular single-valued configuration defined on \( \mathbb{R}^4 \). From Eq. (28), this quantity is \( \bar{U} \)-independent, and using i)-iii) we recover Eq. (38) as long as the monopoles do not touch
where $\Phi$. Now, let us replace the observable in Eq. (41) by the smoothed variable, decomposition,

$$D$$

performed imposing these regularity conditions. Still, it is possible to integrate intrinsic properties. Introducing a Lie algebra-valued tensor field $\Lambda$ that the limit rewrite the effect of fluctuations as an integral over periodic paths,

$$\int [dg] P e^{i \int d^4x \left( g^{-1} F_\mu g + i g^{-1} \partial_\mu g, \bar{\omega}_e \right) T_3} ,$$

(40)

For completeness, and to settle notation and conventions, group coherent states and the path-integral representation of holonomies are briefly reviewed in Appendix A. After extending the paths to $\bar{U}(x) | g(s) = \bar{U}(x(s))$, we can apply Stokes’ theorem and join $W_\varepsilon[P]$ with the center element in Eq. (39), thus obtaining,

$$W_\varepsilon[A] \propto \int [\mathcal{D}\bar{U}] e^{\frac{i}{\hbar} \int d^4x \left( Y_{\mu\nu}(P,\bar{U}) - \mathcal{F}_{\mu\nu}(Z) \right) s_{\mu\nu} \bar{U} \bar{\omega}_e \bar{T} \bar{U}^\dagger} ,$$

(41)

$$Y_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} Y_{\rho\sigma}, \quad Y_{\mu\nu}(P,\bar{U}) = D_{\mu\nu}(\bar{L}) (P_{\nu} - \bar{L}_{\nu}) - D_{\nu\nu}(\bar{L}) (P_{\mu} - \bar{L}_{\mu}) ,$$

(42)

where $\bar{L}_\mu$ was defined in Eq. (29) and

$$\partial_\mu P_{\nu}^{\dagger} \bar{U} - \partial_\nu P_{\mu}^{\dagger} = \bar{U}^\dagger Y_{\mu\nu}(P,\bar{U}) \bar{U} .$$

(43)

Now, let us replace the observable in Eq. (41) by the smoothed variable,

$$W[P,\bar{Z}] = \int [\mathcal{D}\bar{U}] e^{-i \int d^4x \left( Y_{\mu\nu}(P,\bar{U}) - \mathcal{F}_{\mu\nu}(Z) \right) s_{\mu\nu} \bar{U} \bar{\omega}_e \bar{T} \bar{U}^\dagger} ,$$

(44)

where we have included a gauge invariant mass term. Unlike $F_{\mu\nu}(P)$, the tensor $Y_{\mu\nu}(P,\bar{U}) = \partial_\mu P_{\nu} - \partial_\nu P_{\mu} + \ldots$ is linear in $P_\mu$. In order to obtain a single valued $A_\mu$ in Eq. (15), the components of $P_\mu$ rotated by $S$ should vanish at the defects. Accordingly, the path-integral over $P_\mu$ has to be performed imposing these regularity conditions. Still, it is possible to integrate $W[P,\bar{Z}]$ in Eq. (44) without them, and relate both results by means of a factor $R[\bar{Z}]$,

$$\int [\mathcal{D}P_\mu]_{\text{r.c.}} W[P,\bar{Z}] = R[\bar{Z}] \int [\mathcal{D}P_\mu] W[P,\bar{Z}] .$$

The ratio $R[\bar{Z}]$ contains information about the distribution of center vortices, providing their intrinsic properties. Introducing a Lie algebra-valued tensor field $\Lambda_{\mu\nu}$, we can rewrite,

$$\int [\mathcal{D}P_\mu] W[P,\bar{Z}] = \int [\mathcal{D}A_{\mu\nu}] W[P,\bar{Z}] e^{-i \frac{1}{\alpha^2} \int d^4x (\Phi_\mu, \Phi_\mu)}$$

$$\times e^{-i \int d^4x \left( \Lambda_{\mu\nu} - 2\pi s_{\mu\nu} \bar{U} \bar{\omega}_e \bar{T} \bar{U}^\dagger \right)^2} e^{-i \frac{4\pi}{\alpha^2} \int d^4x (\Lambda_{\mu\nu}, \mathcal{F}_{\mu\nu}(Z))} , \quad g = 4\pi N/g_0 ,$$

where $\Phi_\mu = \epsilon_{\mu\nu\rho\sigma} D_{\nu}(\bar{L}) \Lambda_{\rho\sigma}, \quad \bar{\omega}_e = 2N \bar{\omega}_e$. Then, using a gauged version of the usual Hodge decomposition,

$$\Lambda_{\mu\nu} = Y_{\mu\nu} + B_{\mu\nu} , \quad Y_{\mu\nu}(\Lambda,\bar{U}) = D_{\mu}(\bar{L}) (\Lambda_{\nu} - \bar{L}_{\nu}) - D_{\nu}(\bar{L}) (\Lambda_{\mu} - \bar{L}_{\mu}) ,$$

(45)

$D_{\nu}(\bar{L}) B_{\mu\nu} = 0$, we see that $B_{\mu\nu}$ couples with the curl of $s_{\mu\nu}$, which is localized on $C_e$. Also note that the limit $\eta \to 0$ enforces the constraint $\Phi_\mu = 0$, whose solution is,

$$\Lambda_{\mu\nu} = Y_{\mu\nu}(\Lambda,\bar{U}) .$$

(46)
In this respect, $\tilde{L}_\mu$ defined in Eq. (29) has the form of a pure gauge, so that\(^1\)

$$\epsilon_{\sigma\rho\mu\nu}D_\rho(\tilde{L})D_\mu(\tilde{L})X_\nu = (1/2)\epsilon_{\sigma\rho\mu\nu}[D_\rho(\tilde{L}), D_\mu(\tilde{L})]X_\nu = 0.$$\(\)

Thus, for small $\eta$, the Gaussian smoothing leads to,

$$\int [\mathcal{D}P]\mathcal{W}[P, Z] \approx R[Z] \int [\mathcal{D}A] \mathcal{V}[A, Z], \quad (47)$$

$$\mathcal{V}[A, Z] = \int [\mathcal{D}\tilde{U}] e^{-\int d^4x \frac{1}{4\tau} \left((\sigma_{\mu\nu}(A, \tilde{U}) - 2\pi\sigma_{\mu\nu}\tilde{\beta}_e\tilde{T})\tilde{U}\right)^2} e^{-\int d^4x \left(A_\mu - \tilde{A}_\mu + D_\nu(\tilde{L})F_{\mu\nu}(Z)\right)}. \quad (48)$$

In particular, for a given distribution of monopole loops, using Eq. (30) and

$$\left(A_\mu - \tilde{A}_\mu, \tilde{U} (\tilde{\alpha} \cdot \tilde{T}) \tilde{U}^{-1}\right) = \left(\tilde{U}^{-1}A_\mu \tilde{U} + i \tilde{U}^{-1}\partial_\mu \tilde{U}, \tilde{\alpha} \cdot \tilde{T}\right), \quad (49)$$

we get,

$$\mathcal{V}[A, Z] = \int [\mathcal{D}\tilde{U}] e^{-\int d^4x \frac{1}{4\tau} \left((\sigma_{\mu\nu}(A, \tilde{U}) - 2\pi\sigma_{\mu\nu}\tilde{\beta}_e\tilde{T})\tilde{U}\right)^2}$$

$$\times e^{i\sum_{k=1}^n \int dx_k \left(\tilde{U}^{-1}A_\mu \tilde{U} + i \tilde{U}^{-1}\partial_\mu \tilde{U}, \tilde{\alpha} \cdot \tilde{T}\right)}. \quad (50)$$

Likewise, monopole fusion (cf. Eq. (31)) implies additional factors involving,

$$e^{i\sum_{j=1}^m \int dx_j \left(\tilde{U}^{-1}A_\mu \tilde{U} + i \tilde{U}^{-1}\partial_\mu \tilde{U}, \tilde{\alpha} \cdot \tilde{T}\right)}, \quad \sum_j \tilde{\alpha}_j = 0. \quad (51)$$

Under non-Abelian magnetic gauge transformations,

$$\Lambda_\mu \rightarrow U_m \Lambda_\mu U_m^{-1} + i U_m \partial_\mu U_m^{-1}, \quad (52)$$

$\mathcal{V}[A, Z]$ is in principle invariant, as the replacement can be absorbed by the change $\tilde{U} \rightarrow U_m \tilde{U}$. In this case, the non-Abelian d.o.f. are coupled to $\Lambda_\mu$ on the whole spacetime, which hinders the integration over the group. Yet we can get some insights from the formal expressions. In particular, were it possible to decouple the group path-integrals, monopole loops and lines would be associated with adjoint Wilson loops and holonomies, respectively. This type of relation will be addressed in the next section in connection with ensembles of percolating center vortices.

## V. PERCOLATING CENTER VORTICES AND CHAINS

The above example illustrates a case where the effect of center vortices on Wilson loops (linking numbers) is manifested in a dual gauge field theory. This is encoded in the coupling of the dual gauge field $\Lambda_\mu$ (cf. Eq. (50)) to the source $\tilde{\beta}_e\tilde{s}_{\mu\nu}$ needed to write the initial linking numbers. We begin this section by discussing a possible measure to compute center element averages in 4d ensembles of percolating center vortices. For this aim, let us recall the situation in 3d Euclidean spacetime, where the confining and deconfining phases can be described by an effective complex vortex field $V$ [69]. In this case, the average of the fundamental Wilson loop over an ensemble of

\(^1\) The general theory to deal with this type of non-Abelian Hodge decomposition, with zero curvature connections, was developed in Ref. [79].
center vortices, with small (positive) stiffness $1/\kappa$ and repulsive contact interactions, is represented by,

$$
\frac{Z_v^{(3)}[s_{\mu}]}{Z_v^{(3)}[0]} = \int [DV][D\bar{V}] e^{-\int d^4x \left[ \frac{1}{2} D_\mu V D_\mu V + \frac{1}{2} (\bar{V} V - v^2) \right]},
$$

(53)

$$
D_\mu = \partial_\mu - \frac{2\pi}{N} s_\mu, \quad s_\mu = \int_{S(C_v)} d\bar{\sigma}_\mu \delta^{(3)}(x - w(s, \tau)) .
$$

(54)

In a normal phase ($v^2 < 0$), as the vacuum is at $V = 0$, we have to deal with the complete complex field $V$. In the percolating phase ($v^2 > 0$), the computation of the Wilson loop is a hard problem due to the large quantum fluctuations of the Goldstone modes $\gamma(x)$, $V(x) = \rho(x) e^{i\gamma(x)}$. In order to discuss this case, we kept the soft degrees of freedom $V = v e^{i\gamma}$,

$$
Z_v^{(3)}[s_{\mu}] \approx \int [D\gamma] e^{-\int d^4x \frac{2\pi}{g} D_\mu (e^{-i\gamma}) D_\mu (e^{i\gamma})},
$$

(55)

and switched to the lattice, where the finite spacing takes care of possible phase singularities in $\gamma$. This amounts to considering the frustrated 3d XY and Villain models,

$$
S^{(3)}_{latt} = \beta \sum_{x,\mu} \text{Re} \left[ 1 - e^{i\gamma(x+\hat{\mu})} e^{-i\gamma(x)} e^{-i\alpha_\mu(x)} \right],
$$

(56)

where the frustration $\alpha_\mu(x)$ takes the value $\frac{2\pi}{N}$, if the surface $S(C_v)$ is crossed by the link (in the direction of the link to the normal $S(C_v)$), and is zero otherwise. For a discussion in the context of superfluids, see Ref. [80]. As is well-known, the continuum limit is attained at $\beta_c \approx 0.454$, where a Wilson loop area law with $N$-ality is obtained as an extensive property of the ensemble [69]. Summarizing, while closed worldlines are naturally described by a complex field $V$, in a condensate their description can be approximated by a different object, namely, a compact real field $\gamma$ representing the Goldstone modes.

Now, in 4d Euclidean spacetime, a similar simplification could occur. While the effective description for an ensemble of worldsheets would be a string field theory, maybe that for a condensate it could be approximated by a different (hopefully simpler) object. In this direction, it is known that the Goldstone modes for Abelian condensates of closed surfaces are represented by Abelian gauge fields [65]-[68]. Let us ponder the possibility that the partition function

$$
Z_v[s_{\mu\nu}] \approx \int [D\Lambda_\mu] e^{-\int d^4x \frac{1}{g_\rho} \left( F_{\mu\nu}(\Lambda) - 2\pi s_{\mu\nu} \beta_\rho \bar{T} \right)^2},
$$

(57)

eventually extended with a monopole sector, represent an ensemble of percolating center vortices in four dimensions. $F_{\mu\nu}(\Lambda)$ is the usual field strength (18), computed for a non-Abelian gauge field $\Lambda_\mu$, and $g \sim 1/g_\rho$ is a dual coupling. The external source resembles the one derived in Eq. (50) when dealing with linking numbers in the simple Gaussian model. In fact, the lattice version of Eq. (57) can be directly related to an average of center elements. The Wilson action for the gauge field $\Lambda_\mu$ is,

$$
S_{latt} = \beta \sum_{x,\mu<\nu} \text{Re Tr} \left[ I - V_\mu(x) V_\nu(x + \hat{\mu}) V_\mu^\dagger(x + \hat{\nu}) V_\nu^\dagger(x) e^{-i\alpha_{\mu\nu}(x)} \right], \quad V_{\mu}(x) = e^{i\alpha_{\mu}(x)}
$$

($\beta \sim \frac{1}{g^2}$). The frustration $\alpha_{\mu\nu}$ is only nontrivial on plaquettes $x, \mu, \nu$ that intersect $S(C_v)$, where it satisfies $e^{-i\alpha_{\mu\nu}} = e^{-i2\pi \beta_\rho \bar{T}}$. Now, the usual properties of ordinary integrals over the group imply that, for an arbitrary order in powers of $\beta$, the contribution to $Z_v[s_{\mu\nu}]$ is originated from plaquettes.
that form closed surfaces \cite{81}. Surfaces that link \( C_e \) will intersect \( S(C_e) \) a number of times, gaining a factor \( e^{\pm i 2\pi k \beta \vec{w}_i} I \) for every intersection point. In effect, acting with \( e^{\pm i 2\pi k \beta \vec{w}_i} \) on a basis of the fundamental representation formed by weight vectors \( |\phi_{w_i}\rangle \), \( i = 1, \ldots, N \), we get \( i \)-independent quantities \( e^{-i 2\pi k \beta \vec{w}_i} = e^{-i 2\pi \vec{w}_i \cdot \vec{w}_i} \), that is, the eigenvalues of \( D(e^{i 2\pi k \beta \vec{w}_i}) \) (cf. Eq. (38)).

Then, the fingerprints of center vortices are present in Eq. (57). Its lattice version involves the same center elements \( s(C_e) \) that are generated when a quark Wilson loop in representation \( D \) is linked by a center vortex, averaged over an ensemble of plaquettes distributed on closed surfaces. Note that in 3d, considering the representation in Eq. (56), we can conclude that the nontrivial contributions to \( \prod_x \int_{-\pi}^{\pi} d\gamma(x) \) are originated from links distributed along closed loops accompanied by a center element. In that case, the difference is that there is an effective description \((53)\) that includes the normal phase, which can be derived and used to associate the continuum limit of Eq. (56) with a center vortex condensate. In 4d, for larger \( \beta \) values, larger and multiple closed worldsheets become more important, so that performing the average is a hard problem. As \( \beta \) is associated with the dual coupling \( g \), this regime corresponds to stronger coupling \( g_e \).

Relying on the Gaussian smoothing of the Wilson loop, we showed that general fluctuations induce a combined effect of center vortices and monopoles with non-Abelian d.o.f. (cf. Eqs. \((47),(50)\)). Here, we shall consider an effect on center element averages that distinguish between percolating center vortices and chains, which will be included as a phenomenological property that YM ensembles might have. For center vortex branches attached in pairs to fixed closed worldlines \( C_k, k = 1, \ldots, n \), the partial contribution is proposed to be,

\[
Z_{mix}[s_{\mu\nu}]_{\text{partial}} \propto \int [D\Lambda]\mu e^{-\int d^4x \frac{1}{4g^2} \left( F_{\mu\nu}(\Lambda) - 2\pi s_{\mu\nu} \vec{b}_e \cdot \vec{T} \right)^2} W_{Ad}^{(1)}[\Lambda] \cdots W_{Ad}^{(n)}[\Lambda] , \tag{58}
\]

\[
W_{Ad}^{(k)}[\Lambda] = \frac{1}{N^2 - 1} \text{tr} \text{Ad} \left( P \left\{ e^{i \int_{0}^{4\pi} dx \cdot \Lambda_{\mu}(x) } \right\} \right) . \tag{59}
\]

Again, the effect of the adjoint Wilson loops can be directly understood in the lattice. Since the representation \( N \otimes \bar{N} \) contains the adjoint, the combination \( \text{Ad}(g)|_{AB} g_{ij} g^{-1} |_{kl} \), \( g \in SU(N) \), contains a singlet, so that the integral over the group is nontrivial in this case. In other words, the contribution to the lattice version of Eq. (58) derives from plaquettes distributed on open surfaces that meet in pairs at the adjoint loops, forming closed two-dimensional arrays with disconnected closed parts. Whenever the surface \( S(C_e) \) is intersected, the configuration will be accompanied by a center element. That is, the closed surfaces and arrays can be identified with center vortices and chains, respectively. For stronger chromoelectric coupling \( g_e \), percolating branches (with fixed boundaries \( C_k \)) are expected. At weak coupling, the leading contribution is given by plaquettes distributed on the faces of elementary cubes with edge at \( C_k \) (see Ref. \cite{81}, p. 62). In section VI, we shall include monopole fusion, for now, the average of center elements over the ensemble mixture is,

\[
\frac{Z_{mix}[s_{\mu\nu}]}{Z_{mix}[0]} , \quad Z_{mix}[s_{\mu\nu}] = \int [D\Lambda]\mu e^{-\int d^4x \frac{1}{4g^2} \left( F_{\mu\nu}(\Lambda) - 2\pi s_{\mu\nu} \vec{b}_e \cdot \vec{T} \right)^2} Z_{\text{loops}}[\Lambda] , \tag{60}
\]

\[
Z_{\text{loops}}[\Lambda] = 1 + Z_1 + Z_2 + \ldots , \quad Z_n \text{ represents a gas of } n \text{ closed worldlines},
\]

\[
Z_n[\Lambda] = \int [Dm]_n \prod_{k=1}^{n} e^{-\int_{0}^{L_k} ds_k \left[ \frac{1}{2} \hat{u}^{(k)}_{\mu} \hat{u}^{(k)}_{\mu} + \mu \right]} W_{Ad}^{(k)}[\Lambda] , \tag{61}
\]

\[
u_{\mu}(s) = \frac{du_{\mu}}{ds} \in S^3 , \quad \dot{\nu}_{\mu}(s) = \frac{d
u_{\mu}}{ds} . \tag{62}
\]

The configurations were weighted by phenomenological inputs such as tension \( \mu \) and stiffness \( 1/\kappa \). In Abelian ensembles, the presence of stiffness is supported by lattice calculations that show a
strong correlation between different link orientations on monopole loops [82, 83]. The measure \([Dm]_0\) implements the integral over paths starting and ending at \(x_k\), with tangent vector \(u_k\). Therefore,

\[
Z_{\text{loops}}[\Lambda] = \sum_n \frac{1}{n!} \prod_{k=1}^n \int_0^\infty \frac{dL_k}{L_k} \int dv_k \int [dx]_{v_k,v_k}^{L_k} e^{-\int_0^{L_k} ds \left[ \frac{1}{2\pi} u \cdot u + e^{i\Lambda} \right]} W_{\text{Ad}} [\Lambda] ,
\]

where \(v\) stands for the pair of variables \(x, u\) and \([dx]_{v, u}^{L}\) path-integrates over a closed worldline \(x(s)\) with fixed length \(L\), starting and ending at \(v\). Writing,

\[
Q(x, u, x_0, u_0, L) = \int [dx(s)]_{v, v_0}^L e^{-\int_0^L ds \left[ \frac{1}{2\pi} u \cdot u + e^{i\Lambda} \right]} \text{Ad} (\Gamma[\Lambda]) ,
\]

\[
\Gamma[\Lambda] = P \left\{ e^{i \int dx \Lambda(x)} \right\} ,
\]

where \(\Gamma[\Lambda]\) is the holonomy for an open path \(x(s)\) with initial and final conditions \(v_0, v\), the partition function adds up to,

\[
Z_{\text{loops}}[\Lambda] = e^{\int_0^\infty \frac{dL}{L} \int dv \tr Q(v, v, L)} .
\]

In order to go further, we can follow Refs. [60], [84]. Let us summarize the main steps adapted to the present scenario. As usual, \(\text{Ad}(\Gamma)\) can be associated with an “evolution” operator,

\[
P \left\{ e^{-\int_0^L ds H(s)} \right\} , \quad H(s) = H(x(s), u(s)) , \quad H(x, u) = -i u \text{Ad} (\Lambda(x)) .
\]

The path-ordering is obtained from the discretized expression

\[
P \left\{ e^{-\int_0^L ds H(s)} \right\} \bigg|_d = e^{-H(x_M, u_M) \Delta \Lambda} \ldots e^{-H(x_2, u_2) \Delta \Lambda} e^{-H(x_1, u_1) \Delta \Lambda} ,
\]

by taking the \(\Delta \Lambda \to 0\), \(M \to \infty\) limit, with \(L = M \Delta \Lambda\). Accordingly, \(Q(v, v_0, L)\) is obtained from,

\[
Q_M(x, u, x_0, u_0) = Q_M(x, x_0, u_0) , \quad x = x_M , u = u_M ,
\]

\[
Q_M(x_M, u_M, x_0, u_0) = \\
\int d^3 x_k dk \prod_{n=1}^M \psi(u_n - u_{n-1}) \delta(x_n - x_{n-1} - u_n \Delta \Lambda) \\
\times e^{-\sum_{n=1}^M (u + e^{i\Lambda}) \Delta \Lambda} e^{-H(x_M, u_M) \Delta \Lambda} \ldots e^{-H(x_2, u_2) \Delta \Lambda} e^{-H(x_1, u_1) \Delta \Lambda} ,
\]

where the differential \(du\) integrates over \(S^3\) and

\[
\psi(u - u') = N e^{-\frac{1}{2\pi} \frac{e^{i\Lambda} - e^{i\Lambda'}}{\Delta \Lambda}} .
\]

\(Q_M\) can be obtained by iterating a Chapman-Kolmogorov recurrence relation that relates polymers with \(j\) and \(j - 1\) monomers, starting from an initial condition,

\[
Q_0(x, u, x_0, u_0) = \delta(x - x_0) \delta(u - u_0) D_{\text{Ad}} .
\]

As a result, when \(j = M\), it is obtained,

\[
Q_M(x, u, x_0, u_0) = \int du' \psi(u - u') e^{-\mu \Delta \Lambda} e^{-H(x, u) \Delta \Lambda} Q_{M-1}(x - u \Delta \Lambda, x_0, u', u_0) .
\]
Expanding to first order in $\Delta L$ with finite $\kappa$, and taking the continuum limit, we arrive at the Fokker-Plank equation,

$$\partial_L Q = - \left[ \mu - \frac{\kappa}{\pi} \hat{L}^2 + u_\mu \partial_\mu + H(x, u) \right] Q,$$

where $u_\mu \partial_\mu$ gets combined with $H(x, u)$ in Eq. (67) to form the non-Abelian covariant derivative,

$$\left[ \partial_L - \left( \frac{\kappa}{\pi} \right) \hat{L}^2 + u_\mu \left( \partial_\mu - i \text{Ad}(\Lambda_\mu) \right) \right] Q(x, u, x_0, u_0, L) = 0; \quad (73)$$

$$Q(x, u, x_0, u_0, 0) = \delta(x - x_0) \delta(u - u_0) I_{D\text{Ad}}. \quad (74)$$

In the flexible limit (small stiffness), there is almost no correlation between the initial and final tangent vectors. The weak dependence on these directions allows to consistently solve the equations by keeping the smaller angular momenta (see Refs. [60], [85]). In the present case, we get,

$$Q(x, u, x_0, u_0, L) \approx Q_0(x, x_0, L), \quad \partial_L Q_0(x, x_0, L) = -O Q_0(x, x_0, L), \quad (75)$$

where $c = \frac{\pi}{12\kappa}$ and $\Omega_3$ is the solid angle on $S^3$. Using this information in Eq. (66) yields,

$$\int d^4x du Q(x, u, u, u, L) \approx \text{Tr} \left( e^{-LO} \right).$$

This trace is over the adjoint matrix indices and the spacetime coordinate $x$. Therefore, the loop sector is approximated by

$$Z_{\text{loops}}[\Lambda] = e^{\int_0^\infty e^{\frac{dL}{\kappa}} \int d\text{tr} Q(v, v, L)} \approx e^{-\text{Tr} \ln O} = (\text{Det } O)^{-1}, \quad (77)$$

which can be represented by an effective complex field in the adjoint (see section VIC).

### A. Interpretation in terms of non-Abelian d.o.f.

By construction, the factors $W_{\text{Ad}}^{(k)}[\Lambda]$ in Eq. (58) attach pairs of center vortex branches to closed worldlines. Their interpretation as monopoles with non-Abelian d.o.f. relies on the Petrov-Diakonov representation of the adjoint loop (see Appendix A),

$$W_{\text{Ad}}^{(k)}[\Lambda] = \int [dg][P] e^{i \int_{C_k} dx_\mu \left( g^{-1} \Lambda_\mu g + ig^{-1} \partial_\mu g, \bar{\alpha} \cdot \bar{T} \right)}, \quad (78)$$

where $\bar{\alpha}$ is a root. This leads to,

$$Z_n[\Lambda] \approx \int [Dm]_n \prod_{k=1}^n \int [dg^{(k)}][P] e^{- \sum_{k=1}^n S[x^{(k)}]},$$

$$S[x] = \int_0^L ds \left[ \frac{1}{2\kappa} \hat{u}_\mu(s) \hat{u}_\mu(s) + \mu - iu_\mu(s) \left( g^{-1} \Lambda_\mu(x(s)) g + ig^{-1} \partial_\mu g, \bar{\alpha} \cdot \bar{T} \right) \right], \quad (79)$$

which can be thought of as a decoupled version of the group integrals originated from $D_{\nu}(\tilde{L}) F_{\mu\nu}(Z)$ in configurations with chain defects, by identifying $g^{(k)}(s_k) = \tilde{U}(x^{(k)}(s_k))$ (cf. Eqs. (30), (50)).
B. Comparision with linear coherent state variables

The monopole action (79) can be written using the components $z_A$ of group coherent states $|z\rangle = \text{Ad}(\xi)|\varepsilon_\alpha\rangle$ in the adjoint representation,

$$S[x] = \int_0^L ds \left[ \frac{1}{2\kappa} \ddot{u}_\mu(s) \ddot{u}_\mu(s) + \mu + \frac{1}{2} (\dddot{z}_C \dot{z}_C - \dot{z}_C \dddot{z}_C) - i u_\mu(s) \Lambda^A_\mu(x(s)) D(T_A)|_{CD} z_D z_C \right].$$

Here, the path on the group was decomposed in the form $g(s) = \xi(s) h(s)$, where $\text{Ad}(h)$ leaves the root vector $|\varepsilon_\alpha\rangle$ unchanged up to a phase, and $\xi$ belongs to a nonlinear space, namely, the coset of SU($N$) by the invariance subgroup (see Appendix A). A similar coupling was considered in Ref. [60], with the difference that in that case the variables were in a linear space formed by any set of complex numbers $z_A$ [86]. This led to end-to-end probabilities $\langle z| \tilde{Q}(v,v_0,L)|z_0\rangle$,

$$\dot{Q}(v,v_0,L) = \int [dx(s)]_0^L e^{-\int_0^L ds \left[ \frac{1}{\kappa} \ddot{u}_\mu(s) \ddot{u}_\mu(s) + \mu + \phi(x(s)) \right] P \left\{ e^{-\int_0^L ds \tilde{H}(s)} \right\}},$$

$$\tilde{H}(s) = -i u_\mu(s) D(\Lambda_\mu(x))|_{CD} \tilde{a}_C \tilde{a}_D , \quad x = x(s) ,$$

where $\tilde{a}_C^\dagger$ and $\tilde{a}_C$ are the raising and lowering operators for the color mode $C$, acting on an infinite dimensional space of states. In that case, we introduced a projected ensemble by hand, keeping probabilities between color states with occupation number 1, thus obtaining,

$$Z_{\text{proj}}[\Lambda] = e^{\int_0^\infty \frac{dL}{L} \int dv \sum_A \langle A|\tilde{Q}(v,v_0,L)|A\rangle} , \quad |A\rangle = (\tilde{a}_A)\dagger |0\rangle .$$

Using the Fokker-Plank equation for the operator $\dot{Q}$,

$$\partial_L \dot{Q} = -\left[ \mu + \phi(x) - \frac{\kappa}{\pi} L_u^2 + u_\mu \partial_\mu + \tilde{H}(x,u) \right] \dot{Q} ,$$

$$\dot{Q}(v,v_0,0) = \delta(x-x_0)\delta(u-u_0) \dot{I} ,$$

we easily see that the $D_{\text{Ad}} \times D_{\text{Ad}}$ matrix for the projected $\dot{Q}$ satisfies Eq. (73). In other words $\langle B|\tilde{Q}(v,v_0,L)|A\rangle = Q(v,v_0,L)|_{BA}$, so that the projected ensemble is naturally implemented when moving from linear to nonlinear coherent state variables, originated from the magnetic non-Abelian d.o.f. introduced in this work.

VI. EFFECTIVE FEYNMAN DIAGRAMS AND MONOPOLE FUSION

In Ref. [60], excluded volume effects and other interactions among monopoles were introduced as usual, by coupling them to external fields integrated with appropriate Gaussian weights. The same steps could be done in Eq. (79), however, cubic terms would be missing in this formulation (see the discussion in Ref. [53]). They will be relevant to drive SU($N$) $\rightarrow$ Z($N$) and to describe the observed first order confining/deconfining phase transition when $N \geq 3$. In this section, they will be generated as a consequence of monopole fusion rules.

Initially, we shall replace $Z_{\text{loops}}[\Lambda]$ in Eq. (60) by a general monopole sector $Z_{\text{mix}}[\Lambda] = Z_{\text{loops}}[\Lambda] Z_{\text{lines}}[\Lambda]$. The first factor involves adjoint Wilson loops $W_{\text{Ad}}[\Lambda]$, giving rise to a power of the functional determinant in Eq. (77), originated from loop copies needed to accommodate the matching rules (see section VIC). The second is constructed in terms of holonomies $\text{Ad}(\Gamma[\Lambda])$ computed along open lines, forming (connected and disconnected) closed one-dimensional arrays. For a correct matching, they must be combined in a gauge invariant way. In this manner, when integrated over link variables, the lattice formulation of $Z_{\text{mix}}[s_{\mu\nu}]$ will receive contributions from
plaqettes distributed on: i) closed center vortex worldsurfaces generated by the dual YM term, ii) center vortices attached to loops, and iii) center vortices attached to one-dimensional arrays.

In the ensemble, the lines $\gamma$ between given initial and final points $x_0, x$ will be weighted and integrated, as we did with the Wilson loops in Eq. (63),

$$\int dL \, du \, du_0 \int [Dx]_{uv}^L e^{-\int_0^L ds \left[ \frac{1}{2} \partial_{\mu} \bar{\partial}_{\mu} + \mu \right] \text{Ad}(\Gamma[\Lambda])}_{AA'}.$$  

The path-integral over shapes with fixed length $L$ gives the factor $Q(x, u, x_0, u_0, L)$ treated in section V (cf. Eq. (64)). In the flexible limit, using Eqs. (73)-(76), we obtain

$$\int_0^\infty dL \, du \, du_0 \, Q(x, u, x_0, u_0, L) \sim G(x, x_0) \quad , \quad O \, G(x, x_0) = \delta(x - x_0) \, I_{DA} ,$$  

that is, a ($\Lambda$-dependent) Green’s function $G(x, x_0)$ for every adjoint line. As a result, each array yields an effective Feynman diagram. By including coupling constants to measure the arrays’ relative importance, the effective diagrams can be associated with a perturbative expansion of $Z_{\text{lines}}[\Lambda] = 1 + C_{\text{lines}} [\Lambda]$. In 4d, the relevant possibilities correspond to three and four fused lines. Therefore, we are interested in modeling contributions to $C_{\text{lines}} [\Lambda]$ that involve blocks of the form,

$$C_n \propto \int d^4 x \, d^4 x_0 \prod_{j=1}^n dL_j \, du_j \, du_0 \int [Dx_j] \, \left[ - f_{ij} \, ds_j \, e^{-\int_0^{\nu_j} ds \left[ \frac{1}{2} \partial_{\mu} \bar{\partial}_{\mu} + \mu \right] \text{Ad}(\Gamma[\Lambda])}_{AA'} \right] D_n ,$$  

originated from all shapes and lengths of $n$ lines $\gamma_j$ ($n = 3, 4$) with common endpoints $x_0, x$. For $n = 3$, we could take,

$$D_3 = f_{ABC} \, f_{AD'B'C'} \, \text{Ad}(\Gamma_1[\Lambda])_{AA'} \, \text{Ad}(\Gamma_2[\Lambda])_{BB'} \, \text{Ad}(\Gamma_3[\Lambda])_{CC'} ,$$  

or replace $f_{ABC}$ by a combination of symmetric and antisymmetric structure constants. To gain some insight about the possibilities, let us consider the gauge invariant object

$$D_n = \int d\mu(g) d\mu(g_0) \langle g, \varepsilon_1 | \text{Ad}(\Gamma_1[\Lambda]) | g_0, \varepsilon'_1 \rangle \cdots \langle g, \varepsilon_n | \text{Ad}(\Gamma_n[\Lambda]) | g_0, \varepsilon'_n \rangle ,$$  

where $|\varepsilon_j\rangle, |\varepsilon'_j\rangle$ denote coherent reference states chosen as rotated root vectors (see Appendix A). This choice allows to make contact with the monopole worldline interpretation, as we did for loops in section V A. In this regard, when $|\varepsilon'_j\rangle = |\varepsilon_j\rangle$, we can write (cf. Eq. (A20))

$$D_n = \int d\mu(g) d\mu(g_0) \int \prod_j [dg_j(s_j)] \, e^{i \sum_j \int ds_j (g_j^A \, g_j^B + ig_j^A \, \bar{g}_j^B \cdot X_j)} , \quad X_j = [E_j, E_j^\dagger] .$$  

This is related to a monopole current,

$$K_\mu = 2\pi \, 2N \sum_j \bar{U} \, X_j \, \bar{U}^{-1} \int d\mu(\delta(s_j)) \delta(x - y) \quad , \quad \sum_j X_j = 0 ,$$  

with the identification $g_j(s_j) = \bar{U}(x_j(s_j))$, and $g_0, g$ given by the value of $\bar{U}$ at the line endpoints. The last condition is a requirement for the covariant conservation of $K_\mu$ that we shall impose at each fusion point, thus generalizing the matching rule in the Cartan subalgebra $X_j = \bar{\delta}_j \cdot \bar{T}_j$, $\sum_j \bar{\delta}_j = 0$, discussed in Eq. (31). In the flexible limit, path-integrating Eq. (85) over $\gamma_j$, we obtain,

$$C_n \propto \int d^4 x \, d^4 x_0 \, F_{A_1 \cdots A_n}^{\varepsilon_1 \cdots \varepsilon_n} \, G(x, x_0) \, G(x_0) |_{A_1 A_1' \cdots A_n A_n} ,$$  

$$F_{A_1 \cdots A_n}^{\varepsilon_1 \cdots \varepsilon_n} = \int d\mu(g) \, | g, \varepsilon_1 \rangle |_{A_1} \cdots | g, \varepsilon_n \rangle |_{A_n} .$$  

A. Fusion of three monopoles

For three open worldlines, we have to compute,
\[ P_{ABC;A'B'C'} = \int d\mu(g) |g, \varepsilon_1\rangle |A |g, \varepsilon_2\rangle |B |g, \varepsilon_3\rangle |C , \quad |g_0, \varepsilon\rangle = R(g_0) |\varepsilon\rangle , \quad (91) \]
\[ P_{ABC;A'B'C'} = \int d\mu(g) R(g) |A\rangle A'R(g) |B\rangle B'R(g) |C\rangle C' , \quad R(g) = \text{Ad}(g) . \quad (92) \]

The factor \( R(g) |A\rangle A'R(g) |B\rangle B' \) acts on a tensor product space carrying a reducible representation. Its decomposition into irreps, complemented with the orthogonality relations (B1), leads to the desired integral. For \( N \geq 3 \) there are seven irreps, projected by \( P_J \) [87]-[89],
\[ \sum_J P^{AB,A'B'}_{[J]} = \delta_{AA'} \delta_{BB'} , \quad P^{CD,A'B'}_{[J]} = \delta_{JK} P^{AB,A'B'}_{[J]} . \quad (93) \]

They include a singlet \( P_{[1]} \), and two projectors onto the adjoint \( P_{[a]} , P_{[\bar{a}]} \), with components
\[ \frac{1}{N^2 - 1} \delta AB \delta A'B' , \quad f_{ABC} f_{A'B'C} , \quad \frac{N^2}{N^2 - 4} d_{ABC} d_{A'B'C} , \quad (94) \]
respectively. Note that in our conventions,
\[ \{T_A, T_B\} = \frac{1}{N^2} \delta AB I + d_{ABC} T_C , \quad d_{ABC} d_{DBC} = \frac{N^2 - 4}{N^2} \delta_{AD} . \quad (95) \]

Hence, we can write
\[ R(g) |A\rangle A'R(g) |B\rangle B' = R(g) |A\rangle A'R(g) |B\rangle B' \delta_{AA'} \delta_{BB'} \]
\[ = \frac{1}{N^2 - 1} \delta AB \delta A'B' + R(g) |A\rangle A'R(g) |B\rangle B' P_{AB,A'B'} + \ldots , \quad (96) \]
\[ P_{AB,A'B'} = f_{ABC} f_{A'B'C} + \frac{N^2}{N^2 - 4} d_{ABC} d_{A'B'C} , \quad (97) \]
where the dots involve other irreps. Finally, the invariance property of the structure constants \( f_{ABC} \) and \( d_{ABC} \) yields
\[ P_{ABC;A'B'C'} = f_{ABC} f_{A'B'C'} + \frac{N^2}{N^2 - 4} d_{ABC} d_{A'B'C'} , \quad (98) \]
where \( E = E |A T_A \) is the Lie algebra element associated with \( |\varepsilon\rangle \). When this is replaced in Eq. (89), the cross terms with mixed symmetric and antisymmetric constants do not contribute, due to the symmetry of the product of Green’s functions under the interchange \( A \leftrightarrow B , A' \leftrightarrow B' \).

Furthermore, if \( |\varepsilon_j'\rangle \) is an even (+) or an odd (−) permutation of \( |\varepsilon_j\rangle \), we get,
\[ C_3^± \propto \int d^4 x d^4 x_0 G(x, x_0) |A\rangle A' G(x, x_0) |B\rangle B' G(x, x_0) |C\rangle C' \times \]
\[ \pm f_{ABC} f_{A'B'C'} (-i[E_1, E_2], E_3)^2 + \frac{N^4}{(N^2 - 4)^2} d_{ABC} d_{A'B'C'} (\{E_1, E_2\}, E_3)^2 . \quad (99) \]

The three-line Cartan matching only exists for \( N \geq 3 \),
\[ X_j = \delta_j \cdot \mathbf{T} , \quad j = 1, 2, 3 , \quad \delta_1 + \delta_2 + \delta_3 = 0 . \quad (100) \]
In this case, \( E_j = E_{\delta_j} \) thus implying
\[
([E_{\delta_1}, E_{\delta_2}], E_{\delta_3})^2 = N_{\delta_1 \delta_2}^2 (E_{\delta_1 + \delta_2}, E_{\delta_3})^2 = N_{\delta_1 \delta_2}^2,
\]
and \( \{E_{\delta_1}, E_{\delta_2}, E_{\delta_3}\}^2 = N_{\delta_1 \delta_2}^2 \). Now, the Weyl group for \( \mathfrak{su}(N) \) acts as \( S_N \), permuting the weights of the fundamental irrep \([90]\). This produces even but not odd permutations of three different roots. This is the property underlying the two different contributions \( C^\pm_3 \). In Eq. (86), it is not possible to change variables in the group integral over \( g_0 \) to undo odd permutations. In general, there are two independent combinations: the antisymmetric \( (C^+_3 - C^-_3) \),
\[
C^{[a]}_{3-\text{Cartan}} \propto \int d^4 x d^4 x_0 G(x, x_0)|_{AA'} G(x, x_0)|_{BB'} G(x, x_0)|_{CC'} N_{\delta_1 \delta_2} f_{ABC} f_{A'B'C'} ,
\]
and the symmetric one, with \(-if_{ABC} \to d_{ABC}\).

Another natural matching type can be proposed for \( N \geq 2 \), in the \( \mathfrak{su}(2) \) subalgebras generated by \( \vec{a}, \vec{T}, \frac{T}{\sqrt{\alpha^2}}, \frac{T}{\sqrt{\alpha^2}}. \) As the directions \( X_j \) have the same length, the solutions to
\[
X_1^\alpha + X_2^\alpha + X_3^\alpha = 0 ,
\]
must be on the same plane and at angles of \( 2\pi/3 \). Note that there is no common adjoint group action that can transform \( X_2^\alpha \) into \( \delta_j \cdot \vec{T} \), for \( j = 1, 2, 3 \). Then, the former rule is physically inequivalent to the Cartan fusion type. The elements \( X_\alpha^j = X_\alpha^\delta_j \),
\[
X_\alpha^0 = \vec{a} \cdot \vec{T} \cos \theta + \sqrt{\alpha^2} T_\alpha \sin \theta = g(\theta) \vec{a} \cdot \vec{T} g(\theta)^{-1},
\]
associated with \( \theta_1 = 0, \theta_2 = \frac{2\pi}{3}, \) and \( \theta_3 = -\frac{2\pi}{3} \), satisfy Eq. (102). In this case, the rotated root vectors are given by \( E_j = E_\alpha^\theta \),
\[
\{E_0^\alpha, E_\alpha^\theta\} = X_\alpha^0 , \quad E_{\pm \alpha}^\theta = g(\theta) E_{\pm \alpha} g(\theta)^{-1},
\]
\[
(-i[E_0^\alpha, E_\alpha^\theta], E_\alpha^\theta) = \frac{3\sqrt{3} i}{4\sqrt{2}} \sqrt{\alpha^2} , \quad \{E_0^\alpha, E_\alpha^\theta\}, E_\alpha^\theta) = 0 .
\]
This only leaves the antisymmetric part in Eqs. (98) and (99), \( F_{ABC} = \frac{3\sqrt{3}}{4\sqrt{2}} i \sqrt{\alpha^2} f_{ABC} \), and the corresponding contribution,
\[
C_{3-\text{su}(2)} \propto \int d^4 x d^4 x_0 G(x, x_0)|_{AA'} G(x, x_0)|_{BB'} G(x, x_0)|_{CC'} \alpha^2 f_{ABC} f_{A'B'C'} .
\]

**B. Fusion of four monopoles**

For \( n = 4 \), we obtain,
\[
F_{A\ldots D}^{d_1\ldots d_4} = \frac{1}{N^2 - 1} \delta_{AB} \delta_{CD} (E_1, E_2)(E_3, E_4) + (-if_{ABE})(-if_{CDE})([E_1, E_2], [E_3, E_4]) + \ldots ,
\]
where the dots involve \( d_{ABE}\{E_1, E_2\}, \) \( d_{CDE}\{E_3, E_4\} \), and contributions due to other irreps. Using references \( |\varepsilon_j\rangle = |\varepsilon_{\delta_j}\rangle \), associated with the matching rules in the Cartan sector (cf. Eq. (31)), we obtain the following pair of terms for the antisymmetric combination,
\[
C^{[a]}_{4-\text{Cartan}} \propto \int d^4 x d^4 x_0 G(x, x_0)|_{AA'} G(x, x_0)|_{BB'} G(x, x_0)|_{CC'} G(x, x_0)|_{DD'} \times V_{\delta_1 \delta_2, \delta_3 \delta_4} f_{ABC} f_{CDE} f_{A'B'C'} f_{D'E'D'} ,
\]
\[
V_{\delta_1 \delta_2, \delta_3 \delta_4} = \left\{ \begin{array}{ll} N_{\delta_1 \delta_2} N_{\delta_3 \delta_4} , & \vec{\delta}_1 + \vec{\delta}_2 \neq 0 \\ \vec{\delta}_1 \cdot \vec{\delta}_3 , & \vec{\delta}_1 + \vec{\delta}_2 = 0 . \end{array} \right. \]
Then, in this case, Eq. (60) becomes,

\[
\text{(Det } O)^{-1} = \int [\mathcal{D}\zeta][\mathcal{D}\zeta^\dagger] e^{- \int d^4x \langle \zeta | e^{-1} O | \zeta \rangle}. \tag{108}
\]

Then, in this case, Eq. (60) becomes,

\[
Z_{\text{mix}}[s_{\mu\nu}] = \int [\mathcal{D}A_{\mu}][\mathcal{D}\zeta][\mathcal{D}\zeta^\dagger] e^{- \int d^4x \frac{1}{8\pi} (F_{\mu\nu}(\Lambda)-2\pi s_{\mu\nu}\beta_{\alpha}\tilde{T})^2} e^{- \int d^4x ((D_{\mu}\zeta^\dagger,D_{\mu}\zeta)+m^2(\zeta^\dagger,\zeta))},
\]

\[
m^2 = (12/\pi) \mu \kappa, \quad D_{\mu}(\Lambda) \zeta = \partial_{\mu} \zeta - i [A_{\mu}, \zeta]. \tag{109}
\]

When monopole fusion is included, the partition function has the general form,

\[
Z_{\text{mix}}[s_{\mu\nu}] = \int [\mathcal{D}A_{\mu}][\mathcal{D}\zeta][\mathcal{D}\zeta^\dagger] e^{- \int d^4x (F_{\mu\nu}(\Lambda)-2\pi s_{\mu\nu}\beta_{\alpha}\tilde{T})^2} Z_m[\Lambda], \quad Z_m[\Lambda] = Z_{\text{loops}}[\Lambda] Z_{\text{lines}}[\Lambda]. \tag{110}
\]

Relying on a single complex field \(\zeta\), although we can write

\[
G(x,x_0)_{AA'} \propto \int [\mathcal{D}\zeta][\mathcal{D}\zeta^\dagger] \zeta^\dagger(x)|A\zeta(x_0)|A' e^{- \int d^4x ((D_{\mu}\zeta^\dagger,D_{\mu}\zeta)+m^2(\zeta^\dagger,\zeta))}, \tag{111}
\]

the correlator in Eq. (101) cannot be reproduced. In fact, as there is no common group element that can orient \(\tilde{\delta}_j \cdot \tilde{T}\), \(j = 1, 2, 3\) along the same Cartan direction, each monopole line entering a fusion point must be associated with different internal degrees \(\delta_j\). Accordingly, the loop types must also be expanded, which in turn allows capturing the desired one-dimensional arrays by using

\[
Z_m[\Lambda] = \int [\mathcal{D}\zeta][\mathcal{D}\zeta^\dagger] e^{- \int d^4x [(D_{\mu}\zeta^\dagger,D_{\mu}\zeta)+V_H(\zeta)]}, \tag{112}
\]

\[
V_H(\zeta) = m^2(\zeta_{\alpha\dagger}, \zeta_{\alpha}) + \gamma_c N_{\delta_1\delta_2}(\zeta_{\delta_1}, \zeta_{\delta_1} \wedge \zeta_{\delta_2}) + \text{c.c.} + \ldots, \quad X \wedge Y \equiv -i [X, Y], \tag{113}
\]

with the fields summed over positive roots \(\alpha\) and over roots \(\delta_j (\delta_1 + \delta_2 + \delta_3 = 0)\). For negative root indices \(-\alpha\), the notation \(\zeta_{-\alpha} \equiv \zeta^\dagger_{\alpha}\) is understood. The dots involve the symmetric product \(\{X, Y\}\) and constants \(d_{ABC}\). If only fusion types with \(|\xi_1\rangle = |\xi_j\rangle\) were considered in Eq. (86), then the precise combination of vertices would be fixed by Eq. (98).

Expanding in \(\gamma_c\), we get a factor \(Z_{\text{loops}}[\Lambda] = \prod_{\alpha} Z_\alpha[\Lambda] = (\text{Det } O)^{-N(N-1)/2}\) times effective Feynman diagrams associated with three-line fusion. For instance, Eq. (101) is obtained from the average of the second order term

\[
\int d^4x d^4x_0 \gamma_c^2 N_{\delta_1\delta_2}^2 (\zeta_{\delta_3}^\dagger(x), \zeta_{\delta_3}^\dagger(x) \wedge \zeta_{\delta_2}^\dagger(x)) (\zeta_{\delta_3}(x_0), \zeta_{\delta_1}(x_0) \wedge \zeta_{\delta_2}(x_0)).
\]

Now, let us include three-line fusion in the \(su(2)\) subalgebras. To accomodate the matching condition (102), which involves generalized directions \(X_{\alpha}^\dagger\), one possibility is to further expand the loop types, labeling them with the different global orientations \(X_\xi = \xi \tilde{\alpha} \cdot \tilde{T} \xi^{-1}\). The loop contribution gets replaced by,

\[
Z_{\text{loops}}[\Lambda] = e^{\sum_{\alpha} \ln Z_\alpha} \rightarrow e^\int d\mu(\xi) \ln Z_\xi, \tag{114}
\]
where, to avoid overcounting, the integral over the coset must be restricted. For every $\xi$, there is a $\xi'$ such that $X_{\xi'} = -X_{\xi}$. On the other hand, opposite points are already included in the loop orientations, so the integral is in fact over half the coset, 

$$Z_{\text{loops}}[\Lambda] = e^{2\lambda_{\text{Ad}} / 2i \ln Z_0} = (\text{Det} O)^{-2} \frac{d\lambda_{\text{Ad}}}{2} ,$$

(115)

thus leading to $D_{\text{Ad}}$ real adjoint fields $\psi_A \in \mathfrak{su}(N)$. Like in the Cartan decomposition of a Lie basis (cf. (A15)), the $N^2 - 1$ fields may be also organized as $\psi_\alpha, \psi_\beta$, labeled by the positive roots $\alpha$, plus a sector $\psi_q, q = 1, \ldots, N - 1$. In this manner, both fusion types are accommodated by the kinetic and potential terms,

$$\frac{1}{2} (D_\mu \psi_A, D_\mu \psi_A) = (D_\mu \zeta_A^\dagger, D_\mu \zeta_A) + \frac{1}{2} (D_\mu \psi_q, D_\mu \psi_q)$$

$$V_H(\psi) = V_H(\zeta) + \frac{m^2}{2} (\psi_\gamma, \psi_\gamma) + \gamma_2 (\psi_q, \zeta_\beta \zeta_\alpha \zeta_\gamma) + \ldots ,$$

(116)

where the complex fields are understood as $\zeta_{\pm \alpha} = (\psi_\alpha \pm i \psi_\beta) / \sqrt{2}$. When $\gamma_c = \gamma_2$, the Ad($SU(N)$)-flavor symmetry of the loop sector is extended to the interactions, in which case,

$$V_H(\psi) = \frac{m^2}{2} (\psi_A, \psi_A) + \gamma f_{ABC} (\psi_A, \psi_B \wedge \psi_C) + \ldots$$

(117)

The remaining four-line fusion rules in Eq. (107) are obtained from,

$$N_{\delta_1 \delta_2} N_{\delta_3 \delta_4} (\zeta_{\delta_1} \wedge \zeta_{\delta_2} \wedge \zeta_{\delta_3} \wedge \zeta_{\delta_4}) + \text{c.c.} \quad \bar{\alpha} \cdot \bar{\sigma} (\zeta_\alpha \wedge \zeta_\beta^\dagger, \zeta_\gamma \wedge \zeta_\delta),$$

$$\bar{\alpha} \cdot \bar{\sigma} (\zeta_\alpha \wedge \zeta_\beta^\dagger, \zeta_\gamma \wedge \zeta_\delta) \quad \bar{\alpha} = \gamma_2 (\psi_q, \zeta_\beta \zeta_\alpha \zeta_\gamma \zeta_\delta) \quad \bar{\alpha} \cdot \bar{\sigma} (\zeta_\alpha \wedge \zeta_\beta^\dagger, \zeta_\gamma \wedge \zeta_\delta) .$$

(118)

The first (second) term contributes to the case $\delta_1 + \delta_2 \neq 0 (\delta_1 + \delta_2 = 0)$, $\delta_1 + \ldots + \delta_4 = 0$, while the third contributes to the matching type $\delta_1 = -\delta_2 = \delta_3 = -\delta_4 = \bar{\alpha}$.

As discussed throughout this work, the lattice version of

$$Z_{\text{mix}}[s_{\mu \nu}] = \int [D\Lambda_{\mu}][D\psi] e^{-\int \frac{d^4 x}{16} \left( F_{\mu \nu}(\Lambda) - 2 \pi s_{\mu \nu} \mathcal{T} \right)^2 + \frac{1}{2} (D_\mu \psi_A, D_\mu \psi_A) + V_H(\psi) / 2} ,$$

(119)

normalized by $Z_{\text{mix}}[0]$, is an average of center elements over percolating surfaces, generated by the dual gauge sector $\Lambda_{\mu}$, that may be attached to loops and one-dimensional arrays, generated by the $\psi$-sector. The various couplings measure the abundances of each fusion type. A reduced model, without the Ad($SU(N)$)-flavor symmetry of the loop sector, is achieved in the Cartan matching rules that involve different root $s$, may have the form,

$$V_H(\psi) = (\zeta_\alpha \wedge \zeta_\beta^\dagger - m \bar{\alpha} \cdot \bar{\psi})^2 + (\bar{\alpha} \cdot \bar{\psi} \wedge \zeta_\alpha - m \zeta_\alpha, \zeta_\alpha^\dagger \wedge \bar{\alpha} \cdot \bar{\psi} - m \zeta_\alpha^\dagger) .$$

(120)

More generally, we could expand the squares and assign different couplings to the interaction terms. The Higgs potential may also involve the symmetric product $\{X, Y\}$ and terms originated from other irreps, such as the singlet $(\zeta_\alpha^\dagger, \zeta_\alpha)(\zeta_\beta^\dagger, \zeta_\beta)$ (cf. Eqs. (98), (106)). Among the alternatives, there is a natural Ad($SU(N)$) flavor-symmetric one that encompasses all the couplings in Eqs. (113), (116), (118),

$$V_H(\psi) = \frac{m^2}{2} (\psi_A, \psi_A) + \frac{\gamma}{3} f_{ABC} (\psi_A, \psi_B \wedge \psi_C) + \frac{\lambda}{4} f_{ABC} f_{ADE} (\psi_B \wedge \psi_C)(\psi_D \wedge \psi_E) .$$

(121)

This model is analogous to that introduced in Ref. [52], with the difference that in that work the quartic term was taken as $\lambda (\psi_A \wedge \psi_B)^2$. For a given parameter choice, $V_H(\psi)$ can also be written as a perfect square $V_H(\psi) = \frac{1}{2} (m \psi_A - f_{ABC} \psi_B \wedge \psi_C)^2$. In this case, as well as in Eq. (120), the structure is similar to that present in models motivated by $N = 1^*$ supersymmetric theories, based on three complex adjoint Higgs fields [45].
D. Physical consequences

The obtained models have several common features that can be highlighted. The parameters can be chosen in order for the vacua manifolds to be given by

\[ \zeta_\alpha \wedge \zeta_\alpha^\dagger = v \bar{\psi} \cdot \psi \quad \bar{\alpha} \cdot \bar{\psi} \wedge \zeta_\alpha = v \zeta_\alpha \quad \text{and} \quad f_{ABC} \psi_B \wedge \psi_C = v \psi_A , \]  

(122)

respectively. The nontrivial solutions contain tuples \((\psi_1, \ldots, \psi_{N^2-1})\), \(\psi_A = v ST_A S^{-1}\), identified with points in \(\text{Ad}(\text{SU}(N))\). When \(V_H(\psi)\) is a perfect square \((v = m)\), the nontrivial vacua are degenerate with the trivial point \(\psi_A = 0\). However, for appropriate parameters, the degeneracy can be lifted. This triggers a phase where the dual gauge group \(\text{SU}(N)\) is broken to \(\mathbb{Z}(N)\), which allows to compute \(Z_{\text{mix}}[s_{\mu\nu}]\) by means of a saddle point and collective modes. Therefore, in the presence of the source \(2\pi s_{\mu\nu}\vec{\beta} \cdot \vec{T}\), a flux tube with \(N\)-ality is induced. These models are also well-known to possess flux tube solutions with confined dual monopoles [41]-[56]. In particular, as the distance between a pair of adjoint quarks is increased, the saddle point will eventually favor string-breaking by screening the external sources with induced dual monopoles, which get identified with valence gluons. Hence, gluon confinement follows from the fact that the second homotopy group of \(\text{Ad}(\text{SU}(N))\) (a compact group) is trivial. The difference-in-areas law for doubled pairs of \(\text{SU}(2)\) fundamental quarks can be similarly understood [53]. Furthermore, we could consider an observable formed by one adjoint and two fundamental holonomies with common endpoints, combined in a (chromoelectric) gauge invariant way. This object could be used to calculate the hybrid potential for a quark-gluon-antiquark state in pure YM theory. The associated source in \(Z_{\text{mix}}[s_{\mu\nu}]\) contains a pair of surfaces carrying two different fundamental weights. They are spanned between the adjoint and the fundamental lines. In accordance with the gluon interpretation, the induced saddle point will be a flux tube, running between the fundamental sources, with an induced dual monopole localized at the adjoint line.

With respect to the Lüscher corrections, in a flavor non-symmetric model the soft modes will be given by the transverse fluctuations. This is welcomed, since the presence of additional gapless modes would modify [91] the correction observed in lattice simulations up to \(N = 6\) [92]. YMH models that support flux tubes with \(N\)-ality and non-Abelian internal collective modes were constructed in Refs. [45] and [46]. They display \(SO(3)_{C-F}\) and \(SU(N)_{C-F}\) color-flavor locking, respectively. The phenomenological effective models we derived may display a tensor product of \(SO(3)_{C-F}\) symmetries, one for each root, or \(\text{Ad}(\text{SU}(N))\) \(C-F\) symmetry. Nevertheless, in a YM context, the parameters would be related with a single scale, implying that possible non-Abelian degrees on the flux tube worldsheet are in fact frozen [91]. For this reason, these phases would also be compatible with the observed universal corrections.

VII. CONCLUSIONS

In this work, we initially considered a gauge fixing in the continuum that induces a partition of the \(\text{SU}(N)\) YM path-integral into sectors with center vortex worldsheets and monopole worldlines. They are not only labeled by the location of defects but also by non-Abelian magnetic degrees of freedom. The average of an observable involves two steps: a path-integral over general fluctuations in each sector, followed by an ensemble integration.

In the continuum, thin configurations amount to gauge fields \(A_\mu\) such that the field strength is localized on closed surfaces. In this case, neither monopoles nor non-Abelian degrees affect the quark Wilson loop \(W_e[4]\). However, there are many possibilities for the ensemble measure, which dictates how to weight configurations when computing center element averages. This measure should be obtained by taking the first step with the YM action, which is a difficult task. Instead,
we analyzed possible effects by considering a simple example based on a smoothed Gaussian version of the Wilson loop. In doing so, we observed that monopoles with non-Abelian d.o.f. get coupled to a dual non-Abelian gauge field $\Lambda_\mu$, in much the same way as in compact QED(4). In turn, $\Lambda_\mu$ gets coupled to an external source originated from the linking numbers of magnetic defects. In this case, while the treatment of fluctuations is simple, the ensemble integration is not. Motivated by the above example, and guided by effective theories in 3d, we proposed a measure to compute center element averages in 4d ensembles of percolating center vortices and chains.

In four dimensions, as center vortices are two-dimensional, the obtention of an effective theory is a hard problem. However, according to the Julia-Toulouse mechanism, when closed surfaces form a condensate, they can be described by a gauge field representing the Goldstone modes. As a synthesis of the above physical inputs, we associated center vortices with a Wilson action for a non-Abelian gauge field $\Lambda_\mu$. This was implemented in a manner such that the lattice path-integral receives contributions from plaquettes distributed on closed surfaces. Moreover, they are accompanied by the center element that would be generated in the Wilson loop for quarks in representation D. For stronger chromoelectric coupling, larger and multiple surfaces are favored.

In the next stage, we included effects that might be originated from the topologically trivial sector on top of which magnetic defects are created. Monopoles were introduced by products of adjoint magnetic Wilson loop variables $W_{\text{Ad}}[A]$, which single out plaquette configurations distributed on surfaces attached in pairs to these loops. Using the Petrov-Diakonov representation, they were interpreted as monopole worldlines with non-Abelian d.o.f. Likewise, monopole fusion rules were introduced by means of gauge invariant combinations of magnetic holonomies, involving three and four fused monopole lines.

Finally, we integrated the monopole sector and showed that the large distance behavior is given by a dual SU($N$) YMH model with emergent adjoint Higgs fields. The field content depends on the physically inequivalent monopole loop types. Fusion rules in the Cartan and su(2) subalgebras can be accommodated in models with $N^2 - 1$ real fields. When monopoles condense, the gauge group undergoes dual SU($N$) $\rightarrow$ Z($N$) SSB, which allows capturing the ensemble by means of a saddle point formed by flux tubes with $N$-ality and confined dual monopoles. If the parameters correspond to a flavor non-symmetric model, the soft modes are only given by flux tube transverse fluctuations. In fact, this also occurs in the color-flavor locking phase, as the phenomenological parameters are expected to be originated from a single scale, leaving no window for gapless non-Abelian modes [91]. From this point of view, both possible scenarios are equally interesting, as they lead to the correct Lütscher term observed up to $N = 6$ in lattice simulations. In order to narrow down the possibilities, the various implied observables will be compared with lattice calculations in a future contribution.

Thus, following the path proposed, we showed a possible mechanism to explain confining flux tubes and confined gluons as emergent objects in mixed ensembles of percolating center vortices and chains.

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Appendix A: Group coherent states and holonomies

1. Group coherent states

Consider an irreducible $\mathcal{D}$-dimensional unitary representation over a vector space \{\langle \psi \rangle \}. The Lie algebra and group act according to,

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_\mathcal{D} \end{pmatrix}, \quad |\psi\rangle \rightarrow D(Y)|\psi\rangle, \quad |\psi\rangle \rightarrow D(U)|\psi\rangle.$$  \hspace{1cm} (A1)

Given a reference $|\phi\rangle$, $\langle \phi | \phi \rangle = 1$, the invariance subgroup $H_\phi \subset G$ is defined by,

$$D(h)|\phi\rangle = e^{ia(h)}|\phi\rangle, \quad h \in H_\phi.$$ \hspace{1cm} (A2)

Coherent states of type \{\mathcal{D}, |\phi\rangle\} are defined by $|\xi,\phi\rangle = D(\xi)|\phi\rangle$ [93], [94], after choosing a representative $\xi$ in the quotient $G/H_\phi$. Then, as for every group element there is a unique decomposition $g = \xi h$, the action of $g$ becomes,

$$D(g)|\phi\rangle = e^{ia(h)}|\xi,\phi\rangle.$$ \hspace{1cm} (A3)

Due to unitarity, the group invariance of the measure $d\mu(\xi)$ induced by the Haar measure $d\mu(g)$, and Schur’s Lemma [90], the operator $O = \int d\mu(\xi) |\xi,\phi\rangle \langle \xi,\phi| \ $ is proportional to $I_{\mathcal{D}}$, the $\mathcal{D} \times \mathcal{D}$ identity matrix,

$$\int d\mu(h) = 1, \quad \int d\mu(\xi) = \mathcal{D}, \quad \langle \phi | \phi \rangle = 1, \quad \int d\mu(\xi) |\xi,\phi\rangle \langle \xi,\phi| = I_{\mathcal{D}}.$$ \hspace{1cm} (A4)

That is, coherent states are overcomplete.

2. Holonomies

The overcompleteness property does not depend on the reference $|\phi\rangle$. However, in path-integral applications, some requirements must be considered. A reference state $|\phi\rangle$ such that the “dynamical” operator has a diagonal representation seems to be important to give meaning to the formal expressions [93], [95]. Some irreps have weight vectors that enable a classical description, that is, a symplectic structure on the coset space. In particular, the highest weight vectors are among the favorable states [93]-[96]. The coherent state representation of the holonomy

$$\Gamma[A] = P \left\{ e^{i \int ds A_\mu(x)} \right\},$$ \hspace{1cm} (A5)

is obtained by using the composition property with infinitesimal steps [97, 98],

$$D(\Gamma[A]) = (I_{\mathcal{D}} + i\epsilon D(A(s_{M-1})) \ldots (I_{\mathcal{D}} + i\epsilon D(A(s_0))) \quad , \quad A(s) = \frac{dx_\mu}{ds} A_\mu(x(s)),$$

and then taking the continuum limit. As usual, various completeness relations can be introduced. The reference $|\phi\rangle$ is chosen such that the order $\epsilon$ contribution is nonzero [95], with the second order providing a regularization [97]. In this case, the factors can approximated by,

$$1 + i\epsilon \langle \phi | D(X_n) |\phi\rangle \approx e^{i\epsilon \langle \phi | D(X_n) |\phi\rangle}, \quad X_n = \xi_n^A(s_n) \xi_n + i\xi_n^A \dot{\xi}_n$$ \hspace{1cm} (A6)
which leads to the representation
\[
\langle \xi, \phi | D(\Gamma[A]) | \xi_0, \phi \rangle = \int [d\xi(s)] e^{i \int ds \langle \phi | D(\xi^\dagger A \xi + i \xi^\dagger \xi) | \phi \rangle},
\]
(A7)

\[ [d\xi]\xi_0 = d\mu(\xi_1) d\mu(\xi_2) \ldots, \]
and the boundary conditions \( \xi(0) = \xi_0, \xi(L) = \xi. \) Note also that,
\[
\langle \phi | D(\xi^\dagger A \xi + i \xi^\dagger \xi) | \phi \rangle = D(A)|_{ct} z_d \bar{z}_c + \frac{i}{2} (\bar{z}_c \dot{z}_c - \dot{z}_c \bar{z}_c),
\]
(A8)

where \( a \) ranges from 1 to \( D \) and \( z_a(s) \) are the components of the coherent state \( | z(s) \rangle = | \xi(s), \phi \rangle. \) Following similar steps, using an identity based on the group, we obtain,
\[
\langle g, \phi | D(\Gamma[A]) | g_0, \phi \rangle = \int [dg(s)] e^{i \int ds \langle \phi | D(g^\dagger A g + i g^\dagger g) | \phi \rangle},
\]
(A9)

with \( g(0) = g_0, g(L) = g. \) The path \( g(s) \) can be uniquely decomposed in the form \( g(s) = \xi(s) h(s). \) Then, the left-hand side in Eq. (A9) becomes (cf. Eq. (A2))
\[
\langle g, \phi | D(\Gamma[A]) | g_0, \phi \rangle = e^{i(a(0) - a(L))} \langle \xi, \phi | D(\Gamma[A]) | \xi_0, \phi \rangle.
\]
(A10)

Of course, this can be checked on the right-hand side, by using
\[
g^\dagger A g + i g^\dagger g = h^\dagger (\xi^\dagger A \xi + i \xi^\dagger \xi) h + + i h^\dagger \dot{h}, \quad \langle \phi | D(h^\dagger h(s)) | \phi \rangle = i \dot{a}.
\]
(A11)

In particular, as the Wilson loop is related to periodic boundary conditions, the coset and the group path-integrals have no relative factor,
\[
\mathcal{W}_D[A] = \text{tr} D(\Gamma[A]) = \int [dg] p e^{i \int ds \langle \phi | D(g^\dagger A g + i g^\dagger g) | \phi \rangle}.
\]
(A12)

### 3. Maximal reference state

A general weight vector \( | \phi_\lambda \rangle \) (\( \langle \phi_\lambda | \phi_\lambda \rangle = 1 \)) satisfies,
\[
D(T_q)|\phi_\lambda \rangle = \bar{\lambda}|_q|\phi_\lambda \rangle,
\]
(A13)

where \( T_q, q = 1, \ldots, N-1, [T_q, T_p] = 0, \) are independent elements generating the Cartan subalgebra. To compute \( \langle \phi_\lambda | D(X) | \phi_\lambda \rangle \) for a general Lie algebra element \( X \in \text{su}(N) \), we can expand it in the basis, \( T_q, E_\alpha, E_-\alpha, \)
\[
[T_q, E_\alpha] = \bar{\alpha}|_q E_\alpha.
\]
(A14)

The step operators \( E_\alpha \) are labelled by the positive roots \( \bar{\alpha} \), which gives \( N(N-1)/2 \) possibilities\(^2\), while the hermitian generators can be identified with,
\[
\{T_\alpha\} = \{T_q, T_\alpha, T_\alpha\} \quad T_\alpha = \frac{1}{\sqrt{2}}(E_\alpha + E_-\alpha) \quad T_\bar{\alpha} = \frac{1}{\sqrt{2i}}(E_\alpha - E_-\alpha).
\]
(A15)

The remaining commutators are,
\[
[E_\alpha, E_-\alpha] = \bar{\alpha}|_q T_q \quad [E_\alpha, E_\gamma] = N_{\alpha\gamma} E_{\alpha + \gamma}, \quad \bar{\alpha} + \bar{\gamma} \neq 0,
\]
(A16)

where \( N_{\alpha\gamma} = 0, \) if \( \bar{\alpha} + \bar{\gamma} \) is not a root. If \( \bar{\lambda} \) is the highest weight, then \( | \phi_\lambda \rangle \) satisfies \( E_\alpha|\phi_\lambda \rangle = 0, \) \( \langle \phi_\lambda | E_-\alpha \rangle = 0. \) In this case, in terms of the Killing form, it is verified,
\[
\langle \phi_\lambda | D(X)|\phi_\lambda \rangle = X^q \bar{\lambda}|_q = (X, \bar{\lambda}|_q T_q),
\]
(A17)
\[
\langle \phi_\lambda | D(g^\dagger A g + i g^\dagger g)|\phi_\lambda \rangle = (g^\dagger A g + i g^\dagger g, \bar{\lambda}|_q T_q),
\]
(A18)

which leads to the Petrov-Diakonov representation of the Wilson loop in Eq. (A12) [78].

\(^2\) A weight is defined as positive if the last nonvanishing component is positive.
4. Adjoint representation

For the adjoint representation, we have,
\[
\text{Ad}(Y)_{AB} \zeta_B T_A = [Y, \zeta] , \quad \text{Ad}(U)_{AB} \zeta_B T_A = U \zeta U^{-1} .
\] (A19)

As the roots are formed by eigenvalues of the adjoint action of \( T_q \) (cf. Eq. (A14)), they are weights of the adjoint representation. In addition, the invariance subgroup, \( hE_{\alpha}h^{-1} = e^{ia(h)}E_{\alpha} \), is the Cartan subgroup \( h = e^{i \vec{\epsilon} \cdot \vec{T}} \), which gives \( a(h) = \vec{\epsilon} \cdot \vec{\alpha} \). Using the scalar product in Eq. (1), we get \( \langle \zeta | Y \rangle = \bar{\zeta} | A Y | A = (\zeta^\dagger, Y) \). Thus, for any reference \( |\epsilon\rangle = R(\xi) |\epsilon\rangle \) (i.e., \( E = \xi E_{\alpha} \xi^{-1} \)), the cyclicity of the Killing product yields,
\[
\langle \epsilon | \text{Ad}(Y) | \epsilon \rangle = (E^\dagger, [Y, E]) = (Y, [E, E^\dagger]) = (Y, X) , \quad X = [E, E^\dagger] = \xi \vec{\alpha} \cdot \vec{T} \xi^\dagger .
\]

In terms of the rotated reference, Eq. (A9) can be written in the form
\[
\langle g, \epsilon | \text{Ad}(\Gamma[A]) |g_0, \epsilon \rangle = \int [dg(s)] e^{i \int ds (g^\dagger \Lambda g + ig^\dagger g, X)} .
\] (A20)

On the other hand, a coherent state reference \( Z \) given by a combination of Cartan generators, \( [T_q, Z] = 0 \), cannot be used to derive a path-integral since,
\[
\langle z | \text{Ad}(Y) | z \rangle = (Z^\dagger, [Y, Z]) = (Y, [Z, Z^\dagger]) = 0 .
\]

Appendix B: Orthogonality Relations

If \( D^{(i)} \) and \( D^{(j)} \) are unitary irreps \( (i \neq j \) label inequivalent irreps), then \[99\]
\[
\int d\mu(g) D^{(i)}(g)_{ab} D^{(j)}(g^{-1})_{\beta\alpha} = \delta_{ij} \delta_{a\alpha} \delta_{b\beta} ,
\]
In particular, for the adjoint,
\[
\int d\mu(g) R(g)_{AB} R(g^{-1})_{B'A'} = \delta_{AA'} \delta_{BB'} .
\] (B1)

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