Thin-plate stress analysis solved by combining FEM functions and approximations of class $C^0$

1 Hussain Abdulaziz Abraham,  
1 Enaam Obeid Hassoun  
1 Hind D. Salman,  
2 Serazutdinov M. N.

1Electromechanical Engineering Department, University of Technology – Iraq,  
50008@uotechnology.edu.iq, 50109@uotechnology.edu.iq,  
50260@uotechnology.edu.iq  
2Kazan National Research Technological University, Kazan, Russia,  
serazmn@mail.ru

Abstract. The method for constructing functions with finite support is used, which combines the properties of FEM functions and approximations in the form of series with unknown coefficients. To illustrate the properties of the approximating function constructed, we present the results of the solution of the 2D theory of elasticity for a thin plate, shown on figure 1, b. The thickness of the plate is $h$. One facet of the plate (at $x = -a$) is fixed and the other one (at $x = a$) unfixed; the remaining two are under shear stresses $\tau$ and an evenly distributed load $q$. The solution of the problem of determining the stress and strain in the plate can be obtained from the steady-state condition of the Lagrange functional by using the functions of class $C^0$ of the high-degree approximation for scientific calculations. The calculations show that the obtained solution has high accuracy, even in cases when the length of the rectangular domain is considerably greater than its width.

Keywords: theory of elasticity, FEM, high-degree approximation, degenerated domain solution.

1. Introduction

Variational methods are frequently used to solve problems in various fields of science and technology. To approximate the unknowns contained in the variational equations, finite functions of FEM may be used as well as functions in the form of series with unknown coefficients, imposed across the domain of definition of the solution. One of their advantages is that their usage allows achieving a relatively simple solution, and in order to increase the accuracy of the solution, a larger number of members in the series is used.

The advantages of FEM functions are determined by the fact that they are defined in subdomains; they are functions with finite support. While using FEM functions, it is possible to increase the accuracy of...
the solution by dividing the domain of definition of the solution into smaller elements as well as by increasing the degree of approximation. A number of well-known experts in the FEM field prefer using approximations of a high degree [1, 2, 3].

The finite elements—during whose construction, the basic functions are associated with unknown parameters in the nodes located mainly at the boundaries of the elements—are widely used nowadays. For several reasons, the most commonly used finite elements are those that are based on the use of low-approximation degree polynomials.

However, the use of high-approximation degree functions may smoothen contradictions and eliminate unwanted effects that arise in some cases. In particular, it relates to the calculation of thin shells [4].

This article presents a method for constructing functions with finite support, which combines the properties of FEM functions and approximations in the form of series with unknown coefficients. Some of the parameters, which are used as unknowns in these functions, determine the value of the function at the boundaries of the rectangle; others are coefficients of series entered in the domain. They are defined in a way that allows connecting functions determined in different rectangular domains. The distinctive feature of these functions is that they combine the properties of finite element approximation functions in the form of polynomials, trigonometric series, and others. The degree of approximation inside and on the boundary of the domain can be defined. Further, polynomial, trigonometric, and other functions can be used as basis functions.

Using the isoparametric mapping [5,6] of rectilinear quadrangles on the curvilinear, these functions can be used for obtaining solutions of theory of elasticity and the theory of thin shell [7] problems in the domains of different shapes. A system has a lot of algebraic equations with band structure while solving problems and determining the unknown coefficients of the approximating function. The method described in this article was used to construct one-dimensional functions with finite support and to calculate beam systems [8–10]. The formulas, on the basis of which calculations of the stress-strain state of structural elements were presented in the article, are also given in [11]. Another way of constructing functions with a finite carrier to solve three-dimensional problems is presented in [12].

2. Constructing functions of class $C^0$

Let’s divide the domain of the solution $\Omega$ into rectangular subdomains $\Omega_i$ ($i = 1, I$), in each of which we introduce the dimensionless local coordinate system $O\xi\eta$ ($-1 \leq \xi \leq 1, -1 \leq \eta \leq 1$) with the origin (beginning) in the center $\Omega_j$ (Fig. 1). We define the sides of the rectangle and the nodes in its corners as $\Gamma_{ij}$ and $\Gamma_{ij}$ ($j = 1, 4$).

We define the function $u(\xi, \eta)$ in each of the subdomains $\Omega_i$ as

$$u(\xi, \eta) = u'(\xi, \eta) = \sum_{n=1}^{M} u_{\infty} F_{\infty}(\xi, \eta) + \sum_{n=1}^{N_1} u_{\in} F_{1n}(\xi, \eta) + \sum_{n=1}^{N_2} u_{2n} F_{2n}(\xi, \eta) +$$

$$+ \sum_{n=1}^{N_3} u_{3n} F_{3n}(\xi, \eta) + \sum_{n=1}^{N_4} u_{4n} F_{4n}(\xi, \eta) + \sum_{j=1}^{4} u_{j} f_j(\xi, \eta).$$

(1)
Here, $u_{0n}, u_{jn}^i$ – unknown coefficients; $u_j^i$ – values $u(\xi, \eta)$ at the nodes $\omega_{ij}$; $\varphi_{om}(\xi, \eta)$, $F_{jn}(\xi, \eta)$, $f_{j}(\xi, \eta)$ – basis functions, which satisfy certain conditions. Thus, functions $\varphi_{om}(\xi, \eta)$ must vanish at $\Gamma_j$ ($j = \bar{1,4}$); $F_{jn}(\xi, \eta)$ – equal to zero at $\Gamma_{ij}$ if $k \neq j$, ($k = \bar{1,4}$, $j = \bar{1,4}$); $f_j(\xi, \eta) = 1$ at the nodes $\omega_{ij}$ and equal to zero on the sides of the square $\Omega_i$ opposite to $\omega_{ij}$.

If these conditions are met, placing the values $\eta = \pm 1$, $\xi = \pm 1$ into the formula (1), we get

$$ u(\xi, -1) = \sum_{n=1}^{N_1} u_{1n} F_{1n}(\xi, -1) + u_{1}^{i} f_{1}(\xi, -1) + u_{2}^{i} f_{2}(\xi, -1), $$

$$ u(1, \eta) = \sum_{n=1}^{N_1} u_{2n} F_{2n}(1, \eta) + u_{2}^{i} f_{2}(1, \eta) + u_{3}^{i} f_{3}(1, \eta), $$

$$ u(\xi, 1) = \sum_{n=1}^{N_1} u_{3n} F_{3n}(\xi, 1) + u_{3}^{i} f_{3}(\xi, 1) + u_{4}^{i} f_{4}(\xi, 1), $$

$$ u(-1, \eta) = \sum_{n=1}^{N_1} u_{4n} F_{4n}(-1, \eta) + u_{4}^{i} f_{4}(-1, \eta) + u_{1}^{i} f_{1}(-1, \eta). $$

(2)

As seen from relations (1), (2), coefficients $u_{om}$ determine values $u(\xi, \eta)$ only inside subdomains $\Omega_i$, while on the boundaries of $\Gamma_j$, values $u(\xi, \eta)$ depend on $u_{jn}^i F_{jn}(\xi, \eta)$ and $u_{jn}^i f_{j}(\xi, \eta)$, $j = \bar{1,4}$. If we define $u(\xi, \eta)$ in subdomains $\Omega_i$ ($i = \bar{1, I}$) in such a way that at the boundaries of adjacent subdomain, it is expressed through the same values $u_{jn}^i F_{jn}(\xi, \eta)$, $u_{jn}^i f_{j}(\xi, \eta)$, then we get a function $u(\xi, \eta)$ continuous in the transition from one subdomain $\Omega_i$ to another. Consequently, $u(\xi, \eta)$ is a function of class $C^0$ with finite support.

Fig. 1

$\omega_{14}$ $\Gamma_{13}$ $\eta$ $\omega_3$ $\omega_{12}$ $\omega_{11}$ $\Gamma_{12}$

$\omega_{44}$ $\Gamma_{13}$ $\eta$ $\omega_3$ $\omega_{12}$ $\omega_{11}$ $\Gamma_{12}$

$2a$ $2b$

$y$ $\tau$

$O$
Please note that basis functions \( \varphi_{om}(\xi, \eta) \), \( F_{jm}(\xi, \eta) \), \( f_j(\xi, \eta) \) must satisfy the conditions of completeness. Let’s have a look at one of the possible ways to select functions \( \varphi_{om}(\xi, \eta) \), \( F_{jm}(\xi, \eta) \), \( f_j(\xi, \eta) \):

For some complete system of functions \( \psi_m(\xi, \eta) \), \( m = 1, 2, 3, ... \), using expressions

\[
F_0(\xi, \eta) = (1 - \xi^2)(1 - \eta^2), \\
F_1(\xi, \eta) = (1 - \xi^2)(1 - \eta)/2, \\
F_2(\xi, \eta) = (1 - \xi^2)(1 + \eta)/2, \\
F_3(\xi, \eta) = (1 - \xi^2)(1 - \eta^2)/2, \\
f_1(\xi, \eta) = (1 - \xi)(1 - \eta)/4, \\
f_2(\xi, \eta) = (1 + \xi)(1 - \eta)/4, \\
f_3(\xi, \eta) = (1 + \xi)(1 + \eta)/4. 
\]  

(3)

We can write down the system of functions (1) in the following way:

\[
\varphi_{om}(\xi, \eta) = F_0(\xi, \eta) \psi_m(\xi, \eta), \quad m = 1, 2, 3, ...; \\
F_{jm}(\xi, \eta) = F_j(\xi, \eta) \psi_m(\xi, \eta), \quad n = 1, 2, 3, ...; k(n) = 1, k_{j1}, k_{j2}, ... . 
\]  

(4)

It is easy to confirm that \( \varphi_{om}(\xi, \eta) \), \( F_{jm}(\xi, \eta) \) in the form (4) satisfy the required conditions. Have a look at the graphs representing the changes in some of the functions (3) in Fig. 2.

While selecting the value of indexes \( k_{j} \) in (4), it’s important to note that on the boundaries \( \Gamma_{jl} \), the function \( F_{jm}(\xi, \eta) \) must be linearly independent.

Using power functions, we get the following:

\[
\psi_m(\xi, \eta) = \xi^k \eta^l, \quad k = 0, 1, 2, ...; \quad l = 0, 1, 2, ...; \\
F_{1m}(\xi, \eta) = F_1(\xi, \eta) \xi^{m-1}; \quad F_{2m}(\xi, \eta) = F_2(\xi, \eta) \xi^{m-1}; \\
F_{3m}(\xi, \eta) = F_3(\xi, \eta) \eta^{m-1}; \quad F_{4m}(\xi, \eta) = F_4(\xi, \eta) \eta^{m-1}. 
\]  

(5)
In relations (5), exponents \( k \), \( l \) are selected in a way for functions \( \psi_m(\xi, \eta) \) form a complete polynomial of degree \( K \).

While using (3) – (5), expression (1) becomes a polynomial, degree and properties of which depend on the values \( M, N \).

We believe \( N_1 = N_2 = N_3 = N_4 \). Opening brackets in (1) and grouping terms with same exponents, we present expression (1) in the form of a polynomial with unknown coefficients. Let the total number of these coefficients be \( K_e \). Some of the members with a thus obtained function will form the complete polynomial of degree \( K_0 \), and the rest, power functions with unknown coefficients, with a degree greater than \( K_0 \). Let their total number be \( K_i \).

It is obvious that \( K_0, K_e, K_i \) depend on \( K, N_1 \). Table 1 illustrates this dependence. You can see that with the increase of values \( K \) and \( N_1 \), \( K_0 \) increases as well.

Thus, based on the relations (3) – (5), functions with class \( C_1 \) finite support can be obtained in the form of a complete polynomial of degree \( K_0 \), to which a number of power functions with an exponent greater than \( K_0 \) is added.

### Table 1

| \( K \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| \( N_1 \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| \( K_e \) | 9 | 13 | 17 | 21 | 25 | 30 | 35 |
| \( K_0 \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| \( K_i \) | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

### 3. Results of calculations.

To illustrate the properties of the approximating function constructed, we present the results of the solution of the 2D theory of elasticity for a thin plate, shown in figure 1,b. the thickness of the plate is \( h \). One facet of the plate (at \( x = -a \)) is fixed, the other one (at \( x = a \)) unfixed; the remaining two are under shear stresses \( \tau \) and an evenly distributed load \( q \).

The solution of the problem of determining the stress and strain in the plate can be obtained from the steady-state condition of the Lagrange functional:

\[
\frac{1}{2} \delta \iint_{\Omega} \left\{ \frac{Eh}{1-\mu^2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 + Gh \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 - 4 \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right\} \, d\Omega =
\]

\[
= h \int_{-b}^{b} \left[ \varepsilon \delta u(-a, y) \right] dy + h \int_{-a}^{a} \left[ q \delta u(x, b) \right] dx.
\]

Here, \( u, v \) – displacements of the plate in the direction of axes \( Ox \) and \( Oy \) module, \( h \) – thickness of plate, \( E \) – Young's modulus, \( \mu \) – Poisson's ratio, \( G = E/2(1+\mu) \).

If the domain of the definition of the solution \( \Omega \) is not divided into the rectangular subdomain,
then \( l = 1 \), \( \Omega_i \) occupies the entire domain \( \Omega \). We introduce the dimensionless coordinates \( \xi = x/a \), \( \eta = y/b \) and believe that \( N_1 = N_2 = N_3 = N_4 \). In this case, \( u(x, y) \) and \( v(x, y) \), using (1) can be written down as follows:

\[
\begin{align*}
    u(\xi, \eta) &= \sum_{m=1}^{M} u_{om} \varphi_{om}(\xi, \eta) + \sum_{j=1}^{4} u_{j} f_{j}(\xi, \eta) + \sum_{k=1}^{N} \sum_{n=1}^{N} u_{kn} F_{kn}(\xi, \eta), \\
    v(\xi, \eta) &= \sum_{m=1}^{M} v_{om} \varphi_{om}(\xi, \eta) + \sum_{j=1}^{4} v_{j} f_{j}(\xi, \eta) + \sum_{k=1}^{N} \sum_{n=1}^{N} v_{kn} F_{kn}(\xi, \eta).
\end{align*}
\]  

(7)

We believe that the approximating functions are defined in the form of (5).

The terms of the plate fixation will be satisfied, if the following is true for (7) formulas:

\[
    u_{2n} = v_{2n} = 0, \quad u_2 = u_3 = v_2 = v_3 = 0.
\]

Placing expression (7) into the condition (3), we obtain a system of algebraic equations of \( N_c = 2K_c \) order for finding unknown coefficients in the series (7).

![Graph of stress changes](image)

**Fig. 3**

The calculations assumed that the lengths of the sides of the plate were \( a = b = 0.25 \, m \), \( E = 2 \cdot 10^5 \, \text{MPa} \), \( \mu = 0.3 \), \( h = 0.01 \, m \), \( q = 0 \); shear stresses affecting facets of plates at \( x = -a \) have constant thickness; height of the plates change in accordance to the law \( \tau = \tau_0 (b^2 - y^2) / b^2 \), \( \tau_0 = 10 \, \text{MPa} \). Fig. 3 shows graphs of the changes in stress \( \sigma_x \) at the point \( x = 0/125 \, m \), \( y = 0.25 \, m \), depending on the used approximation. The numbers in the circles by the lines of graphs show degrees of \( K \) of complete polynomials inside \( \Omega_i \) used in the calculations. The numbers on the lines around the selected points indicate under which degrees of \( K \) of approximating polynomials on the boundaries of domain \( \Omega_i \) data of
calculations were received ($K_f = N_1 - 1$).

Here are the results of the calculations of the plate: for domain $\Omega$ divided into a number of subdomain $\Omega_i$. Tables 2–4 present results obtained for $E = 2 \cdot 10^5$ MPa, $\mu = 0.3$, $a = 0.3$ m, $b = h = 0.01$ m, $q = 0.1$ MPa, $\tau = 0$.

In this case, $b/a = 0.033$; therefore, the sizes of the sides of the domain $\Omega$ differ considerably and the plate can be seen as a beam fixed on one end. For the beam, the maximum stresses occur at $x = -a$ and are calculated according to the formula $\sigma_{\text{max}}^b = M_z^\text{max} / W_z$. Taking into consideration that $M_z^\text{max} = 2qh2a^2 = 180$ Hm, $W_z = (2b)^2 / 6 = 0.67 \cdot 10^{-6}$ m$^3$, we obtain $\sigma_{\text{max}}^b = 270$ MPa.

Table 2 shows the results of calculations in the case where subdomain $\Omega_i$ feels the whole domain $\Omega$ ($I = 1$). We present the values of the maximum stress $\sigma_{\text{max}}^b$ obtained using the relations (6), (7), (5) at different values of the number of members $N_1, M$ in the series (7). In the table, $\Delta \sigma = \left| (\sigma_{\text{max}}^b - \sigma_{\text{max}}^a) / \sigma_{\text{max}}^a \right| \cdot 100\%$; the percentage difference between the value $\sigma_{\text{max}}^b$ is calculated by the theory of rods and $\sigma_{\text{max}}^a$ is found with the use of elasticity theory.

| $M$ | $N_1$ | $\sigma_{\text{max}}^b$, MPa | $\Delta \sigma$, % |
|-----|-------|-----------------------------|-----------------|
| 2   | 4     | 270,17                      | 0,063           |
| 4   | 6     | 270,51                      | 0,18            |
|     | 8     | 271,14                      | 0,42            |
|     | 10    | 272,22                      | 0,82            |

Table 3 shows the solution in the case when $I = 2$, domain $\Omega$ is divided into 2 parts $\Omega_1$ and $\Omega_2$ (Fig. 4,a). Domain $\Omega_1$ on the figure 4, a is shaded, the length of the vertical side $\Omega_1$ equals $b_1$. The length of the side $\Omega_2$ equals $b_2 = 2b - b_1$, the relative value $b_1$ calculated by the formula:
\[ \Delta b_1(b) = \frac{b_1}{2b} \cdot 100\% , \quad \Delta b_2(a) = \frac{b_2}{2a} \cdot 100\% . \]

The calculations assumed \( M = 4, \quad N_1 = 6 \).

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{Fig4}
\caption{Fig. 4}
\end{figure}

The results of calculations for domain \( \Omega \) divided into 3 subdomains \( \Omega_1, \Omega_2, \Omega_3 \) (Fig. 4, b) are shown in Table 4. Domain \( \Omega_1 \) in Fig. 4, b is shaded, the length of the vertical side \( \Omega_1 \) equals \( b_1 \). The length of the vertical side \( \Omega_2 \) equals \( b_2 = 5 \cdot 10^{-3} \text{ m} \). For \( \Omega_3 \), the length of the vertical side \( b_3 = 2b - b_1 - b_2 \). We assumed \( M = 4, \quad N_1 = 6 \).

| \( b_i, m \) | \( \Delta b_1(b), \% \) | \( \Delta b_2(a), \% \) | \( \sigma_{\text{max}}, \text{MPa} \) | \( \Delta, \% \) |
|---------------|----------------|----------------|----------------|---------|
| \( 2 \cdot 10^{-3} \) | 10 | 0.33 | 277,74 | 2.85 |
| \( 10^{-3} \) | 5 | 0.17 | 277,17 | 2.65 |
| \( 5 \cdot 10^{-4} \) | 2.5 | \( 8.3 \cdot 10^{-2} \) | 276,84 | 2.53 |
| \( 5 \cdot 10^{-5} \) | 0.25 | \( 8.3 \cdot 10^{-3} \) | 276,51 | 2.41 |
| \( 10^{-7} \) | \( 5 \cdot 10^{-4} \) | \( 1.7 \cdot 10^{-6} \) | 276,25 | 2.31 |

4. Discussions

As can be seen from the graphs shown in Fig. 3, for the given problem, starting from a certain value \( K_f \), the increase of the order of approximation for unknown functions on the boundaries \( \Omega_1 \) practically does not increase the accuracy of the solution. This conclusion is valid for the other case as well. Therefore, to construct the functions that have a high degree of approximation, it is important to use the degree of freedom inside the element as well as on its boundaries. It is possible to obtain a solution with high accuracy without breaking down the domain of the definition of the solution \( \Omega \) into several subdomains \( \Omega_i \).
In some cases, it may be required to break down the domain of the definition of the solution into a certain amount of subdomains. The advantage of functions described here is that they allow using $\Omega_i$ that have significantly different sizes of sides while breaking down the domain into subdomains. The data in Tables 2, 3 and 4 shows that by using the functions with the finite support described here, the solution can also be obtained with high accuracy in the case where the sizes of subdomains $\Omega_i$ differ significantly. As seen from Tables 3 and 4, the solution is obtained with good accuracy in the cases where the length $b_i$ of one side of the subdomain $\Omega_i$ is shorter than the length $2a$ of the other subdomain side by $1.7 \cdot 10^{-6}$%. Therefore, the solution is obtained when a rectangular domain almost degenerates into a line.

5. Conclusions

Based on the results of the research presented, the following conclusions can be written:

1. The distinctive features of the approximating function $u(\xi, \eta)$ with finite support are that it is expressed through the four values at the nodes of the rectangular domain $\Omega_i$ and generalized parameters that determine the function inside and on the boundaries of $\Omega_i$. To change the order of the approximation of the function, you can just change the number of these generalized parameters.

2. Due to its structure, the boundaries of the subdomains $u(\xi, \eta)$ is presented by functions $F_{jn}(\xi, \eta)$, which are independent of each other. Further, the functions are defined inside $\Omega_i$. The required degree of approximation in each subdomain $\Omega_i$ can be reached by specifying a corresponding number of members of series $M, N_j (j = 1, 4)$. Moreover, in different subdomains, different $M$ as well as different functions $\phi_{om}(\xi, \eta)$ and even different $F_{jn}(\xi, \eta), f_j(\xi, \eta)$ can be used if certain conditions are met.

3. As can be seen from the structure $u(\xi, \eta)$, various systems of functions can be used as the basis function. If necessary, in order to improve approximation, inside one or more subdomains $\Omega_i$, various additional functions (e.g. singular)—depending on the properties of the chosen problem—can be introduced on the boundaries $\Gamma_y$ and at the nodes $\omega_y$.

4. While using the isoparametric mapping of rectilinear quadrangles $\Omega_i$ on curvilinear quadrangles, functions $u(\xi, \eta)$ can be used to obtain solutions in the domains other than rectangular.

5. Similar to the finite element method, to determine the unknown coefficients $u(\xi, \eta)$, a system of algebraic equations of the tape structure is obtained.
References

[1] G. Strang, G. Fix, *An Analysis of the Finite Element Method*, Wellesley-Cambridge Press, 2008.
[2] O.C. Zienkiewicz, K. Morgan, *Finite Elements and Approximation*, Courier Corporation, 2006.
[3] P. Solin, K. Segeth, I. Dolezel, *Higher-Order Finite Element Methods*, Chapman & Hall/CRC, 2004.
[4] K.-J. Bathe, E.L. Wilson, *Numerical Methods in Finite Element Analysis*, Prentice Hall, 1976.
[5] A.I. Golovanov, M.S. Kornishin, *Introduction to the Finite Element Method* in Statics of Thin Shells, Kazan, Fiz.-Tekh. Inst. KF AN SSSR, Kazan, 1989.
[6] K.-J. Bathe *Finite Element Procedures*, Prentice Hall, 1996.
[7] M.N. Serazutdinov, M.F. Garifullin, Shells of complex shape. *Soviet Applied Mechanics*, No. 27, 11, 1991, 1077–1082.
[8] M.N. Serazutdinov, M.N. Ubaidulloev, H.A. Abraham, Calculation of reinforced loaded structures by the variational method. *Scientific and Theoretical Journal “Proceedings of Higher Educational Institutions. Construction”*, No. 7, 2010, S.118–123. (Russian copy)
[9] M.N. Serazutdinov, M.N. Ubaidulloev, H.A. Abraham, Increasing the bearing capacity of reinforced loaded structures. *Journal "Structural Mechanics of Engineering Structures and Structures"*, No. 3, 2011, pp.23–30. (Russian copy)
[10] M.N. Serazutdinov, M.N. Ubaidulloev, H.A. Abraham, The influence of mounting forces on the bearing capacity of reinforced rod systems. *Bulletin of Kazan Technological University*, No. 10, 2011, pp.116–124. (Russian copy)
[11] M.N. Serazutdinov, A method for constructing a compactly supported function of a high degree of approximation of class C0. *Bulletin of Kazan Technological University*, T. 19, No. 11, 2016, S.162–165. (Russian copy)
[12] F.S. Khayrullin, O.M. Sakhbiev, Calculation of orthotropic constructions by a variation method on the basis of three-dimensional functions with final carriers. *PNRPU Mechanics Bulletin*, No. 2, 2017, pp.195–207. doi:10.15593/perm.mech/2017.2.11.