The existence of simple choreographies for the $N$-body problem—a computer-assisted proof

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Abstract

We consider the question of finding a periodic solution for the planar Newtonian $N$-body problem with equal masses, where each body is travelling along the same closed path. We provide a computer-assisted proof for the following facts: the local uniqueness and the convexity of the Chenciner and Montgomery Eight, the existence (and the local uniqueness) for Gerver’s Super-Eight for 4-bodies and a doubly symmetric linear chain for 6-bodies.

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1. Introduction

In this paper we consider the problem of finding a periodic solution to the $N$-body problem in which all $N$ masses travel along a fixed curve in the plane. The $N$-body problem with $N$ equal unit masses is given by a differential equation (the gravitational constant is taken equal to 1)

$$\ddot{q}_i = \sum_{j \neq i} \frac{q_j - q_i}{r_{ij}^3},$$

where $q_i \in \mathbb{R}^n$, $i = 1, \ldots, N$, $r_{ij} = \|q_i - q_j\|$.

We consider the planar case ($n = 2$) only, we set $q_i = (x_i, y_i)$, $\dot{q}_i = (v_i, u_i)$. Using this we can express (1.1) by:

$$\dot{v}_i = \sum_{j \neq i} \frac{x_j - x_i}{r_{ij}^3},$$

$$\dot{u}_i = \sum_{j \neq i} \frac{y_j - y_i}{r_{ij}^3},$$

$$\ddot{x}_i = v_i,$$

$$\ddot{y}_i = u_i.$$
Recently, this problem received a lot of attention in the literature; see [M, CM, CGMS, S1, S2, S3, MR2] and papers cited therein.

By a simple choreography [S1, S2] we mean a collision-free solution of the $N$-body problem in which all masses move on the same curve with a constant phase shift. This means that there exists $q : \mathbb{R} \to \mathbb{R}^2$ a $T$-periodic function of time, such that the position of the $k$th body ($k = 0, \ldots, N - 1$) is given by $q_k(t) = q(t + kT/N)$ and $(q_0, q_1, \ldots, q_{N-1})$ is a solution of the $N$-body problem. The simplest choreographies are the Langrange solutions in which the bodies are located at the vertices of a regular $N$-gon and move with a constant angular velocity. Another simple choreography, a figure eight curve (see figure 1), was found numerically by Moore [M]. Chenciner and Montgomery [CM] gave a rigorous existence proof of the Eight in 2000. In December 1999, Gerver found an orbit for $N = 4$ called the ‘Super-Eight’ (figure 6). After that Simó found many more simple choreographies of different shape, and with the number of bodies ranging from 4 to several hundreds (see [S, S1, CGMS] for pictures, animations and more details).

Up to now the only choreographies whose existence has been established rigorously [MR2] are the Lagrange solutions and the Eight solution. While the Lagrange solution is given analytically, the existence of the Eight was proven in [CM] using variational arguments and there are still a lot of open questions about it [Ch, MR2], for example the uniqueness (up to obvious symmetries and rescaling) and the convexity of the lobes in the Eight. In section 3 we give a computer-assisted proof of the existence of the Eight, its local uniqueness and the convexity of the lobes.

In sections 4–6 we concentrate on choreographies called doubly symmetric linear chains. They are symmetric with respect to both coordinate axes and all self-intersection points of the curve are on the $x$-axis. We describe a computer-assisted proof of the existence (and the local uniqueness) of doubly symmetric linear chains for 4- (the Gerver Super-Eight) and 6-bodies.

Proofs given in this paper are computer assisted. By this we mean that we use a computer program to provide rigorous bounds for solutions of (1.1). Our method can be described as a variant of an interval shooting method. This approach has a long history in the interval analysis literature, where it was used in the more general context of the boundary value problem for ODEs (see, e.g. [Ke, Lo, Lo1, Sc]).

In this paper the problem of proving the existence of a choreography is reduced to finding a zero for a suitable function. For this purpose we use the interval Newton method and the Krawczyk method [A, K, KB, Mo, N] (see section 2). To integrate equations (1.1) we use a $C^1$-Lohner algorithm [ZLo]. All computations were performed on an AMD Athlon 1700XP with 256 MB DRAM memory, with the Windows 98SE operating system. We used the CAPD package [Capd] and Borland C++ 5.02 compiler. The source code of our program is available on the first author’s web page [Ka]. The total computation time for 6-bodies was under 90 s and was considerably smaller for the Eight and the Super-Eight (see section 7 for more details).

Using an approach described in section 4 we also proved the existence of doubly symmetric linear chains for 10- and 12-bodies. Numerical data from these proofs are available on the first author’s web page [Ka]. Our approach failed for doubly symmetric linear chain of 15-bodies (see section 8 for more details).

2. Two zero finding methods

The main technical tool used in this paper in order to establish the existence of solutions of equations of the form $f(x) = 0$ is the interval Newton method [A, Mo, N] and the Krawczyk method [A, K, N]. The interval Newton method was used to prove the existence of the Eight
Simple choreographies in $N$-body problem

Figure 1. The Eight—the initial position.

Figure 2. The Eight—the final position.

(figure 1) (also see figure 2) and the Super-Eight orbit (figure 6). The Krawczyk method has been used for the Super-Eight orbit (for comparison) and for an orbit with 6-bodies in a linear chain (figure 8).

2.1. Notation

In the application of interval arithmetics to the rigorous verification of theorems, single-valued objects, such as numbers, vectors, matrices, etc are in the formulae replaced by sets containing sure bounds for them. In the following, we will not use any special notation to distinguish between single-valued objects and sets. For a set $S$ by $[S]$ we denote the interval hull of $S$, i.e. the smallest product of intervals containing $S$. For a set which is an interval set (i.e. can be represented as a product of intervals) we may sometimes use square brackets to stress its interval nature. For any interval set $[S]$ by mid($[S]$) we denote a centre point of $[S]$. For any interval $[a, b]$ we define a diameter by $\text{diam}[a, b] := b - a$. For an interval vector (matrix) $S = [S]$ by $\text{diam}S$ we denote a vector (matrix) of diameters of each components.

2.2. The interval Newton method

**Theorem 2.1 ([A, N]).** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ function. Let $[X] = \prod_{i=1}^n [a_i, b_i], a_i < b_i$. Assume that, the interval hull of $DF([X])$, (denoted by $[DF([X])]$) is invertible. Let $\tilde{x} \in X$ and we define

$$N(\tilde{x}, [X]) = -[DF([X])]^{-1}F(\tilde{x}) + \tilde{x}. \quad (2.1)$$

Then

(1) if $x_1, x_2 \in [X]$ and $F(x_1) = F(x_2)$, then $x_1 = x_2$,

(2) if $N(\tilde{x}, [X]) \subset [X]$, then $\exists x^* \in [X]$ such that $F(x^*) = 0$. 

(3) if \( x_1 \in [X] \) and \( F(x_1) = 0 \), then \( x_1 \in N(\bar{x}, [X]) \).

(4) if \( N(\bar{x}, [X]) \cap [X] = \emptyset \), then \( F(x) \neq 0 \) for all \( x \in [X] \).

The main problem with an application of the interval Newton method is that of the invertibility of \( [\frac{\partial \Phi}{\partial x}([X])] \). One often can overcome this difficulty with the Krawczyk method.

2.3. The Krawczyk method

We assume that:

- \( F : \mathbb{R}^n \to \mathbb{R}^n \) is a \( C^1 \) function,
- \( [X] \subset \mathbb{R}^n \) is an interval set,
- \( \bar{x} \in [X] \),
- \( C \in \mathbb{R}^{n \times n} \) is a linear isomorphism.

The Krawczyk operator \([A, K, N]\) is given by

\[
K(\bar{x}, [X], F) := \bar{x} - CF(\bar{x}) + (\text{Id} - C[DF([X])])([X] - \bar{x}).
\]  

(2.2)

Theorem 2.2.

(1) If \( x^* \in [X] \) and \( F(x^*) = 0 \), then \( x^* \in K(\bar{x}, [X], F) \).

(2) If \( K(\bar{x}, [X], F) \subset \text{int}[X] \), then there exists in \( [X] \) exactly one solution of equation \( F(x) = 0 \).

(3) If \( K(\bar{x}, [X], F) \cap [X] = \emptyset \), then \( F(x) \neq 0 \) for all \( x \in [X] \).

2.4. The zero-finding algorithm based on Newton or Krawczyk methods

Theorems 2.1 and 2.2 can be used as a basis for an algorithm finding rigorous bounds for a solution of equation \( F(x) = 0 \). Let \( T(\bar{x}, [X]) = N(\bar{x}, [X]) \) if we are using the interval Newton method and \( T(\bar{x}, [X]) = K(\bar{x}, [X], F) \) for the Krawczyk-method-based algorithm.

First, we need to have a good guess for \( x^* \in \mathbb{R}^n \). For this purpose we use a nonrigorous Newton method to obtain \( \bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \). Then we choose interval set \([X]\) which contains \( \bar{x} \) and perform the following steps:

Step 1. Compute \( T(\bar{x}, [X]) \).

Step 2. If \( T(\bar{x}, [X]) \subset [X] \), then return success.

Step 3. If \( X \cap T(\bar{x}, [X]) = \emptyset \), then return fail. There are no zeros of \( F \) in \([X]\).

Step 4. If \([X] \subset T(\bar{x}, [X]) \), then modify computation parameters (e.g. a time step, the order of Taylor method, size of \([X]\)). Go to Step 1.

Step 5. Define a new \([X]\) by \([X] := [X] \cap T(\bar{x}, [X]) \) and a new \( \bar{x} \) by \( \bar{x} := \text{mid}([X]) \), then go to Step 1.

In practical computation it is convenient to define the maximum number of iterations allowed and return failed if the actual iteration count is larger.

Observe that the third assertion in both theorems 2.1 and 2.2 can be used to exclude the existence of zero of \( F \). This has been used by Galias [G1, G2] to find all periodic orbits up to a given period for the Hénon map and the Ikeda map.
3. The Eight—the existence, the local uniqueness and the convexity

The existence of the Eight has been shown in [CM] by using a mixture of symmetry and variational arguments. Here we give another existence proof and in addition we obtain the local uniqueness and the convexity of each lobe of the Eight. We follow [CM] in the use of the symmetry, but the other component of the proof is different—we use the interval Newton method discussed in section 2.

In notation we follow [CM].

Let $T$ be any positive real number. We define the action of the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathbb{R}/T\mathbb{Z}$ and on $\mathbb{R}^2$ as follows: if $\sigma$ and $\tau$ are generators, then we set

$$
\sigma(t) = t + \frac{T}{2}, \quad \tau(t) = -t + \frac{T}{2}, \quad \sigma(x, y) = (-x, y), \quad \tau(x, y) = (x, -y).
$$

(3.1)

For loop $q : (\mathbb{R}/T\mathbb{Z}) \to \mathbb{R}^2$ and $i = 1, 2, 3$ we define the position for $i$th body by

$$
q_i(t) = q \left( t + (3 - i) \cdot \frac{T}{3} \right).
$$

(3.2)

The following theorem without the uniqueness part was proved in [CM].

**Theorem 3.1.** There exists an ‘eight’-shaped planar loop $q : (\mathbb{R}/T\mathbb{Z}) \to \mathbb{R}^2$ with the following properties:

1. for each $t$,

$$
q_1(t) + q_2(t) + q_3(t) = 0;
$$

2. $q$ is invariant with respect to the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathbb{R}/T\mathbb{Z}$ and on $\mathbb{R}^2$:

$$
q \circ \sigma(t) = \sigma \circ q(t) \quad \text{and} \quad q \circ \tau(t) = \tau \circ q(t);
$$

3. the loop $x : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}^6$ defined by

$$
x(t) = (q_1(t), q_2(t), q_3(t))
$$

is a $T$-periodic solution of the planar 3-body problem with equal masses.

Moreover, $q$ is locally unique (up to obvious spatial symmetries and rescaling).

As was mentioned in the introduction the proof of theorem 3.1 is computer assisted. The goal of the next few lemmas is to transform it to the problem of solving the equation $F(x) = 0$ for a suitable $F$.

**Remark 3.2.** If conditions (1)–(3) are satisfied, then

(a) $q_1(t) + q_2(t) + q_3(t) = 0$ for each $t$,

(b) $q \circ \sigma(t) = \sigma \circ q(t)$ and $q \circ \tau(t) = \sigma \circ q(t)$ for each $t$,

(c) $q_3(0) = (0, 0)$ and $q_3(0) = -2q_1(0)$,

(d) $q_1(0) = -q_2(0)$ and $q_1(0) = \dot{q}_2(0)$,

(e) $q_1(T/12)$ is on the $x$-axis and $q_1(T/12)$ is orthogonal to the $x$-axis,

(f) $q_3(T/12) = \tau \circ q_3(T/12)$ and $q_3(T/12) = \sigma \circ q_3(T/12)$.

The following lemma describes the symmetry reduction for the Eight.

**Lemma 3.3.** Assume that $\tilde{q} : [0, \tilde{T}] \to \mathbb{R}^6$ is a solution of the 3-body problem, such that $\tilde{q} = (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)$ satisfies conditions (c), (d), (e) and (f) in remark 3.2, then exists $q : (\mathbb{R}/T\mathbb{Z}) \to \mathbb{R}^2$ satisfying conditions (1), (2) and (3) in theorem 3.1 with $T = 12\tilde{T}$. 

Proof. We define
\[ \dot{q}(t) = \begin{cases} \dot{q}_3(t) & \text{for } t \in [0, T], \\ \tau \circ \dot{q}_2(2T - t) & \text{for } t \in [T, 2T], \\ \sigma \circ \dot{q}_1(t - 2T) & \text{for } t \in [2T, 3T] \end{cases} \] (3.3)
and
\[ q(t) = \begin{cases} \dot{q}(t) & \text{for } t \in [0, 3T], \\ \tau \circ \dot{q}(\tau^{-1}(t)) & \text{for } t \in [3T, 6T], \\ \sigma \circ \dot{q}(\sigma^{-1}(t)) & \text{for } t \in [6T, 9T], \\ \sigma \circ \tau \circ \dot{q}(\tau^{-1} \circ \sigma^{-1}(t)) & \text{for } t \in [9T, 12T]. \end{cases} \] (3.4)

Let \( f_1 \) and \( f_2 \) be two solutions of (1.1) on intervals \([t_1, t_2]\) and \([t_2, t_3]\), respectively. If \( f_1(t_2) = f_2(t_2) \) and \( f_1'(t_2) = f_2'(t_2) \) then \( f = \{ f_1, f_2 \} \) is a solution on interval \([t_1, t_2]\). To show that \( q(t) \) is a solution it is enough to show that ‘the pieces fit together smoothly’.

For example for \( t = T \) from remark 3.2(f) we have
\[ \dot{q}_3(T) = \dot{q}_3 \left( \frac{T}{12} \right) = \tau \circ \dot{q}_2 \left( \frac{T}{12} \right) = \tau \circ \dot{q}_2(2T - T), \] (3.5)
\[ \dot{q}_3(T) = \sigma \circ \dot{q}_3(\hat{T}). \] (3.6)

Other cases are left to the reader.

Condition (2) in theorem 3.1 follows easily from the definition of \( \dot{q}(t) \) and the properties of \( \sigma \) and \( \tau \) (for \( T \)-periodic orbit \( \sigma^{-1}(t) = \sigma(t) \) and \( \tau^{-1}(t) = \tau(t) \)). For example, for \( t \in [0, 3T] \) we have
\[ q(\sigma(t)) = \sigma \circ \dot{q}(\sigma^{-1} \circ \sigma(t)) = \sigma \circ \dot{q}(t) = \sigma \circ q(t), \] (3.7)
\[ q(\tau(t)) = \tau \circ \dot{q}(\tau^{-1} \circ \tau(t)) = \tau \circ \dot{q}(t) = \tau \circ q(t). \] (3.8)

We omit other cases, because the proof is very similar.

To prove condition (1) observe that, from remark 3.2(c) and (d) we obtain that \( q_1(0) + q_2(0) + q_3(0) = 0 \) and \( \dot{q}_1(0) + \dot{q}_2(0) + \dot{q}_3(0) = 0 \). This and the conservation of linear momentum implies condition (1). \( \square \)

Hence to prove theorem 3.1 it is enough to show that there exists a locally unique (up to obvious degeneracies) function satisfying assumptions of lemma 3.3. To this end we rewrite these assumptions as a zero-finding problem to which we apply the interval Newton method in the reduced space.

Our original phase space is 12-dimensional; the state of bodies is given by \( (q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) \). The centre of mass is fixed at the origin. Hence one body’s position and velocity is determined by other 2-bodies. We start from a collinear position with the third body at the origin and with equal velocities of the first and the second body (see section 3). Hence it is enough to know the position and the velocity of the first body to reconstruct initial condition of other bodies. Moreover, if we have one solution we could get another solution by a suitable rotation (both have the same shape). To remove this degeneracy we place the first body on the \( x \)-axis (see figure 3) (also see figure 4). In addition we fix the size of trajectory by setting \( q_1(0) = (1, 0) \). This also fixes the period of the solution, but from Kepler’s third law we can obtain a solution of any period just by rescaling. Hence the initial conditions are defined by the velocity of one body.

The reduced space for the Eight is two-dimensional and is parametrized by the velocity of the first body. We define a map from the reduced space to the full phase space \( E : \mathbb{R}^2 \rightarrow \mathbb{R}^{12}, \)
which expands velocity of the first body, given by \((v, u)\), to the initial conditions for the 3-body problem \((x_1, y_1, x_2, y_2, x_3, y_3, v_1, u_1, v_2, u_2, v_3, u_3)\) for equation (1.1)

\[
E(v, u) = (1, 0, -1, 0, 0, v, u, u, -2v, -2u).
\]

For each such initial condition there exists a solution of the 3-body problem defined on some interval. To each initial configuration, following that solution, we associate, if it exists, a configuration in which for the first time the position vector of the first body is orthogonal to its velocity vector. This defines the Poincaré map

\[
P : \mathbb{R}^{12} \supset \Omega \rightarrow \mathbb{R}^{12}.
\]

Now, we define map \(R : \mathbb{R}^{12} \rightarrow \mathbb{R}^2\), by

\[
R(q_1, q_2, q_3, \dot{q_1}, \dot{q_2}, \dot{q_3}) = (\|q_2 - q_1\|^2 - \|q_3 - q_1\|^2, (\dot{q}_2 - \dot{q}_3) \times q_1),
\]

where by \(\times\) we denote vector product, and map \(\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) by

\[
\Phi = R \circ P \circ E.
\]

**Remark 3.4.** If \(R(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) = 0\), then \(\|q_2 - q_1\| = \|q_3 - q_1\|\) and if in addition \(y_1 = 0\) but \(x_1 \neq 0\), then \(y_2 = y_3\). The bodies are then located as in figure 4 and conditions (e) and (f) in remark 3.2 are satisfied in a suitably rotated coordinate frame.

The following lemma, which is crucial for the proof of theorem 3.1, was proved with computer assistance.

**Lemma 3.5.** There exists a locally unique point \((v, u) \in \mathbb{R}^2\), such that \(\Phi(v, u) = (0, 0)\).

**Proof:** We use the interval Newton method (theorem 2.1). First we come close to zero of \(\Phi\), starting with some rough initial condition (e.g. from [S1]) using the nonrigorous Newton
method. Once we have a good candidate \( x_0 = (v_0, u_0) \), we set \( [X] = [v_0 - \delta, v_0 + \delta] \times [u_0 - \delta, u_0 + \delta] \) and compute rigorously \( \Phi(x_0) \) and \( \partial \Phi([X])/\partial x \). For this purpose we use the \( C^1 \)-Lohner algorithm described in [ZLo]. In this computation we used the following settings: the time step \( h = 0.01 \) and the order \( r = 7 \).

It turns out that the assumption of assertion 1 in theorem 2.1 holds for \( x_0 = (0.347 116 768 716, 0.532 724 944 657) \) and \( \delta = 10^{-6} \). Numerical data from this computation are listed in table 1.

Moreover, from theorem 2.1 we know that this zero is unique in the set \( X \).

\[ \square \]

**Proof of theorem 3.1.** From lemma 3.5 it follows that there exists \( (v, u) \in \mathbb{R}^2 \) such that \( \Phi(v, u) = (0, 0) \). Hence there exists a solution \( \tilde{q}(t) \) of the 3-body problem defined in the interval \( [0, \tilde{T}] \), such that for \( t = 0 \) all bodies are in the collinear configuration with the third body at the origin and for \( t = \tilde{T} \) the bodies form an isosceles triangle (because \( r_{12} = r_{13} \)). By a suitable rotation of the coordinate frame we obtain a solution \( q(t) \), such that for \( t = \tilde{T} \) the first body is on the \( x \)-axis. From remark 3.4 it follows that \( q(t) \) satisfies all conditions in lemma 3.3 and hence there exists \( q(t) \) satisfying conditions (1)–(3).

The local uniqueness follows from the local uniqueness in lemma 3.5.

\[ \square \]

### 3.1. Convexity of the Eight

**Theorem 3.6.** Each lobe of the Eight is convex.

**Proof.** For the proof it is enough to show that the only inflection point on the curve \( q(t) \) is the origin.

From lemma 3.5 we obtain the set \( [X] \) which includes the initial condition for the Eight in the reduced space. To expand \( [X] \) to the full space we set \( \bar{X} = E([X]) \). In the coordinate frame in which the Eight looks as in figure 1 and symmetries \( \sigma \) and \( \tau \) are the reflections with respect to coordinate axes (see section 3.1) we see immediately that the symmetry properties of the Eight imply that at the origin \( \partial^2 y_i/\partial x_i^2 = 0 \) and \( \partial^2 x_i/\partial y_i^2 = 0 \). To prove the convexity of the Eight we follow rigorously the trajectory of set \( \bar{X} \) and show that the only point in which \( \partial^2 y_i/\partial x_i^2 = 0 \) and \( \partial^2 x_i/\partial y_i^2 = 0 \) is the origin. The same is true if we rotate the Eight (as in figure 3) to the coordinate system in which we performed the actual computations and in which we will work for the remainder of this proof.

Let \( q_i(t) = (x_i(t), y_i(t)) \) be the position of \( i \)th body. If \( 0 \notin (\partial x_i/\partial t)[t_{k-1}, t_k] \) then we can write \( y_i \) as a function of \( x_i \) in the interval \( [x_i(t_{k-1}), x_i(t_k)] \). Otherwise we try to represent \( x_i \) as a function of \( y_i \). For small enough time steps at least one of these representations is always possible for the Eight (this is verified during rigorous computations). Below we list

| Table 1. Data from the proof of lemma 3.5. |
|-----------------------------------------|
| \( \Phi(x_0) \) | \([-2.107 029e-06, -2.106 467e-06], [2.974 991e-06, 2.976 034e-06]\) |
| \( \text{diam } \Phi(x_0) \) | (5.625 889e-10, 1.042 962e-09) |
| \( \partial \Phi([X]) \) | \( \begin{bmatrix} 17.622 624, 17.643 043 \end{bmatrix}, [1.809 772, 1.827 325] \) |
| \( \text{diam } \Phi([X]) \) | \( \begin{bmatrix} [−24.868 548, −24.848 432] \end{bmatrix}, [−10.056 629, −10.039 221] \) |
| \( N(x_0, [X]) \) | \( (0.347 116 886 243 943, 0.347 116 889 993 313), [0.532 724 941 587 373, 0.532 724 949 187 495]) \) |
| \( \text{diam } N(x_0, [X]) \) | (3.749 369e-09, 7.600 121e-09) |
Simple choreographies in N-body problem

formulae for the derivatives of \( y_i(t(x_i)) \) with respect to \( x_i \) in terms of derivatives of \( x_j \) and \( y_i \) with respect to the time variable.

\[
\frac{\partial y_i(t(x_i))}{\partial x_i} = \frac{\partial y_i}{\partial t} \left( \frac{\partial x_i}{\partial t} \right)^{-1},
\]

\[
\frac{\partial^2 y_i(t(x_i))}{\partial x_i^2} = \left( \frac{\partial^2 y_i}{\partial t^2} - \frac{\partial^2 x_i}{\partial t \partial x_i} \frac{\partial y_i}{\partial t} \right)^{-2},
\]

\[
\frac{\partial^3 y_i(t(x_i))}{\partial x_i^3} = \left( \frac{\partial^3 y_i}{\partial t^3} \frac{\partial x_i}{\partial t} - \frac{\partial^3 x_i}{\partial t^3} \frac{\partial y_i}{\partial t} + 2 \left( \frac{\partial^2 x_i}{\partial t^2} \right)^2 \frac{\partial y_i}{\partial x_i} - 2 \frac{\partial^2 x_i}{\partial t^2} \frac{\partial^2 y_i}{\partial t^2} \right) \left( \frac{\partial x_i}{\partial t} \right)^{-1}
\]

\[
- \frac{(\partial^2 x_i/\partial t^2)(\partial^2 y_i/\partial x_i^2)}{(\partial x_i/\partial t)^2}.
\]

To obtain derivatives of \( x_i \) with respect to \( y_i \) it is enough to exchange \( x_i \) and \( y_i \) variables in the above formulae.

The time derivatives of \( x_i \) and \( y_i \) for each time step are computed during the execution of the \( C^1 \)-Lohner algorithm.

Before we state explicitly the conditions we check we need to introduce some notation.

Let \( x_0 \in \mathbb{R}^{12} \) represent initial conditions (for \( t = 0 \)) for (1.2) then by \( \varphi(t,x_0) \) we denote the state of bodies (positions and velocities) at time \( t \). Let \( h_k \) be length of \( k \)th time step, \( t_k = h_1 + \cdots + h_k \) — the total time elapsed after \( k \) steps, \( [q_k] \subset \mathbb{R}^{12} \) be an interval set, such that \( \varphi(t_k,\bar{X}) \subset [q_k] \) and \( [Q_k] \subset \mathbb{R}^{12} \) be an interval set, such that \( \varphi([t_{k-1},t_k],\bar{X}) \subset [Q_k] \). Let us stress here, that both \( [q_k] \) and \( [Q_k] \) are computed during \( k \)th step of \( C^1 \)-Lohner algorithm.

For \( k \)th step and \( i \)th body (except the first step and the third body (\( i = 3 \)), which starts at the origin) we check if at least one of the following conditions is true:

\[
0 \notin \frac{\partial y_i}{\partial t}[X^i_k] \quad \text{and} \quad 0 \notin \frac{\partial^2 y_i}{\partial y_j^3}[X^i_k], \quad (3.9)
\]

\[
0 \notin \frac{\partial x_i}{\partial t}[Y^i_k] \quad \text{and} \quad 0 \notin \frac{\partial^2 y_i}{\partial x_j^3}[Y^i_k]. \quad (3.10)
\]

For the first step and the third body we check if one of the following conditions is satisfied:

\[
0 \notin \frac{\partial y_3}{\partial t}[X^3_1], \quad 0 \in \frac{\partial^2 x_3}{\partial y_3^2}[X^3_1] \quad \text{and} \quad 0 \notin \frac{\partial^3 x_3}{\partial y_3^3}[X^3_1], \quad (3.11)
\]

\[
0 \notin \frac{\partial x_3}{\partial t}[Y^3_1], \quad 0 \in \frac{\partial^2 y_3}{\partial x_3^2}[Y^3_1] \quad \text{and} \quad 0 \notin \frac{\partial^3 y_3}{\partial x_3^3}[Y^3_1]. \quad (3.12)
\]

To verify the above conditions we follow the trajectory of set \( \bar{X} \) using the \( C^1 \)-Lohner algorithm until we reach the section described in proof of lemma 3.5 and for each time step and each body we verify a suitable condition. This completes the proof.

\[\square\]

3.2. Some numerical data from the proof of the convexity of the Eight

Parameter settings for the \( C^1 \)-Lohner algorithm: the time step \( h = 0.01 \), the order \( r = 7, 53 \) time steps were needed to cross the section.

Step 1. We start in the collinear position with the third body at the origin (at the inflection point). Numerical data for this case are given in table 2. We see that for the third body (\( i = 3 \)
variable parameterizing curves as follows (see figure 5).

By \( S_x \) we denote the reflection against the origin. These spatial symmetries act also on time variable parameterizing curves as follows (see figure 5).

we have 0 \( \in (\partial^3 x_i/\partial y_i^3)[X_i^1] \), but this derivative is monotonic (because \( (\partial^3 x_i/\partial y_i^3)[X_i^1] < 0 \)), hence there can be only one zero in the interval \( [X_i^1] \). But from the symmetry we know that this zero is at \( x_3 = 0 \in [X_i^1] \).

Steps 2–36 and 38–53. In this case all second derivatives do not contain 0. In table 3 we list as an example data obtained in the second step.

Step 37. The first body is in the rightmost position. In this case we cannot represent \( y_1 \) as a function of \( x_1 \), hence we represent \( x_1 \) as a function of \( y_1 \) and we check condition (3.10) instead of (3.9). Table 4 contains the derivatives for this case.

| Table 2. Data from the proof of theorem 3.6, for step 1, neighbourhood of the deflection point. |
| --- |
| \( i \) & \( \frac{\partial x_i}{\partial t}[X_i^1] \) & \( \frac{\partial^2 x_i}{\partial y_i^2}[X_i^1] \) & \( \frac{\partial^3 x_i}{\partial y_i^3}[X_i^1] \) |
| 1 & [0.334 402, 0.347 118] & [15.3592, 17.9897] & [136.616, 219.114] |
| 2 & [0.347 116, 0.360 049] & [−16.5013, −14.111] & [119.951, 192.562] |
| 3 & [−0.695 034, −0.693 85] & [−0.068 271 3, 0.269 952] & [−30.969, −26.3718] |

| Table 3. Data from the proof of theorem 3.6, for step 2. |
| --- |
| \( i \) & \( \frac{\partial x_i}{\partial t}[X_i^2] \) & \( \frac{\partial^2 x_i}{\partial y_i^2}[X_i^2] \) & \( \frac{\partial^3 x_i}{\partial y_i^3}[X_i^2] \) |
| 1 & [0.322 22, 0.334 722] & [16.7085, 19.6225] & [155.007, 250.021] |
| 2 & [0.359 72, 0.372 882] & [−15.2203, −13.0177] & [105.778, 171.191] |
| 3 & [−0.695 669, −0.694 44] & [0.126 359, 0.472 046] & [−30.9533, −26.1203] |

| Table 4. Data from the proof of theorem 3.6, for step 37. |
| --- |
| \( i \) & \( \frac{\partial x_i}{\partial t}[X_i^3] \) & \( \frac{\partial^2 x_i}{\partial y_i^2}[X_i^3] \) & \( \frac{\partial^3 x_i}{\partial y_i^3}[X_i^3] \) |
| 1 & [−0.002 872 09, 0.004 684 03] & [−] & [−] |
| 2 & [0.904 939, 0.922 079] & [−2.557 15, −2.168 31] & [4.351 97, 10.6201] |
| 3 & [−0.919 428, −0.909 617] & [2.563 71, 3.032 59] & [−3.512 03, 3.485 64] |
| \( \frac{\partial y_i}{\partial t}[Y_i^3] \) & \( \frac{\partial^2 y_i}{\partial x_i^2}[Y_i^3] \) & \( \frac{\partial^3 y_i}{\partial x_i^3}[Y_i^3] \) |
| 1 & [0.480 975, 0.482 888] & [−3.248 24, −3.117 37] & [−2.984 53, −0.860 616] |

4. Doubly symmetric choreographies with even number of bodies

4.1. Symmetries

Many of the choreographies found by Simo [S1] have at least one symmetry. In this section we introduce a notation for symmetries which will be used till the end of this paper. By \( S_x \), \( S_y \) we denote the reflection with symmetry with respect to the \( x \)-axis and the \( y \)-axis, respectively. By \( S_0 \) we denote the reflection against the origin. These spatial symmetries act also on time variable parameterizing curves as follows (see figure 5).
Let $T$ be any positive real number. We define actions of $S_x$, $S_y$ and $S_0$ on $\mathbb{R}/T\mathbb{Z}$ and on $\mathbb{R}^2$ by

\[
S_x(t) = -t + \frac{T}{2}, \quad S_y(x, y) = (-x, y), \quad S_y(t) = -t, \quad S_0(t) = t + \frac{T}{2}, \quad S_0(x, y) = (-x, -y).
\]

It follows from these definitions that $S_0 = S_x \circ S_y = S_y \circ S_x$.

Let $q(t) : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}^2$ be a $C^2$ function. We say that $q(t)$ is invariant (equivariant) with respect to the action of $S$ if $S(q(t)) = q(S(t))$ for all $t$. If $q(t)$ is invariant with respect to $S_x$ (respectively $S_y$) then $\dot{q}(S_x(t)) = S_y(\dot{q}(t))$ (respectively $\dot{q}(S_y(t)) = S_x(\dot{q}(t))$). Hence the $S_0$-invariance implies that $\dot{q}(S_0(t)) = S_0(q(t))$.

From now on we will enumerate bodies starting from 0. We set $q_i(t) = q(t + \frac{T}{N} \cdot i)$ for $i = 0, \ldots, N - 1$.

4.2. Doubly symmetric choreographies with even number of bodies

We will consider only cases with even number of bodies. Let $T = N \cdot \bar{T}$ > 0. We search for a function $q(t) : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}^2$ which has the following properties:

(P1) for each $t$ the origin is the centre of mass,

\[
\sum_{i=0}^{N-1} q(t + \bar{T} \cdot i) = 0, \quad (4.1)
\]

(P2) $q(t)$ is invariant with respect to

(a) $S_x$ i.e. $q(S_x(t)) = S_x(q(t))$,
(b) $S_0$ i.e. $q(S_0(t)) = S_0(q(t))$,

(P3) the function $x = (q_0(t), q_2(t), \ldots, q_{N-1}(t))$, where $q_i(t) = q(t + \bar{T} \cdot i)$ for $i = 0, \ldots, N - 1$, is a $T$-periodic solution of the $N$-body problem (1.1).

The following lemma, which is analogous to lemma 3.3, gives necessary and sufficient conditions for the existence of a choreography satisfying (P1), (P2) and (P3).

**Lemma 4.1.** Let $N = 2n$ be the number of bodies. There exists a function $q(t)$ with properties (P1), (P2) and (P3) if and only if there are functions $q_i : [0, \bar{T}/2] \rightarrow \mathbb{R}^2$ for $i = 0, 1, \ldots, N - 1$ such as:

1. $q_0(0) = (0, y_0)$ for some $y_0 \neq 0$ ($q_0(0)$ is on $y$-axis),
2. At time $t_0 = 0$ for $i = 0, 1, \ldots, N/2 - 1$ we have
   (a) $q_i(t_0) = S_x(q_{N/2-i}(t_0))$,
   (b) $\dot{q}_i(t_0) = S_y(q_{N/2-i}(t_0))$,
   (c) $q_i(t_0) = S_0(q_{N/2+i}(t_0))$,
   (d) $\dot{q}_i(t_0) = S_0(q_{N/2+i}(t_0))$. 

Figure 5. Linear chain with 8-bodies.
This defines the Poincaré map $E$ and $N$.

In both cases a choreography is equivalent to some boundary value problem for the $N$ variables to recover the rest of them. We may still obtain solutions of any period and determine it, hence we can fix the size of curve by fixing one variable. Hence our reduced space is $(N-1)$-dimensional. In the next paragraph we will be more specific.

We define map $E : \mathbb{R}^{N-1} \to \mathbb{R}^{4N}$, which expands, using symmetries from (2), initial conditions from the reduced space to the full phase space: $(x_0, y_0, \bar{x}_0, \bar{y}_0, x_1, y_1, \bar{x}_1, \bar{y}_1, \ldots, x_{N-1}, y_{N-1}, \bar{x}_{N-1}, \bar{y}_{N-1})$. We consider two cases: $N = 4k$ and $N = 4k + 2$.

For $N = 4k$ we set

$$E \left( x_0 \times \prod_{i=1}^{k-1} (x_i, y_i, \dot{x}_i, \dot{y}_i) \times (x_k, \dot{x}_k) \right) = (0, a, \dot{x}_0, 0) \times \prod_{i=1}^{k-1} (x_i, y_i, \dot{x}_i, \dot{y}_i)$$

$$\times (x_k, 0, 0, \dot{y}_k) \times \prod_{i=1}^{k-1} (x_{k-i}, -y_{k-i}, -\dot{x}_{k-i}, \dot{y}_{k-i})$$

$$\times (0, -a, -x_0, 0) \times \prod_{i=1}^{k-1} (-x_i, -y_i, -\dot{x}_i, -\dot{y}_i)$$

$$\times (-x_k, 0, 0, -\dot{y}_k) \times \prod_{i=1}^{k-1} (-x_{k-i}, y_{k-i}, \dot{x}_{k-i}, -\dot{y}_{k-i}).$$

For $N = 4k + 2$ we set

$$E \left( x_0 \times \prod_{i=1}^{k} (x_i, y_i, \dot{x}_i, \dot{y}_i) \right) = (0, a, \dot{x}_0, 0) \times \prod_{i=1}^{k} (x_i, y_i, \dot{x}_i, \dot{y}_i)$$

$$\times \prod_{i=0}^{k-1} (x_{k-i}, -y_{k-i}, -\dot{x}_{k-i}, -\dot{y}_{k-i}) \times (0, -a, -\dot{x}_0, 0)$$

$$\times \prod_{i=1}^{k} (-x_i, -y_i, -\dot{x}_i, -\dot{y}_i) \times \prod_{i=0}^{k-1} (-x_{k-i}, y_{k-i}, \dot{x}_{k-i}, -\dot{y}_{k-i}).$$

In both cases $a$ is a parameter fixing the size of the orbit.

We define the Poincaré map by requiring that

- for $N = 4k$: bodies $k$ and $k-1$ have equal $x$ coordinate ($x_k = x_{k-1}$),
- for $N = 4k + 2$: $k$th body is on the $x$-axis ($y_k = 0$).

This defines the Poincaré map $P : \mathbb{R}^{4N} \supset \Omega \to \mathbb{R}^{4N}$.
We define the reduction map $R : \mathbb{R}^{4N} \to \mathbb{R}^{N-1}$ in such way that $R$ has zeros in points satisfying conditions (3a) and (3b) in lemma 4.1 and only in such points. Observe that we do not have to worry about conditions (3c) and (3d) in lemma 4.1, because from the properties of (1.1) it follows that if (2c) and (2d) holds, then (3c) and (3d) are satisfied for any $t_1$.

For $N = 4k$ we set

$$R \left( \prod_{i=0}^{N-1} (x_i, y_i, \dot{x}_i, \dot{y}_i) \right) = (y_k + y_{k-1}, \dot{x}_k + \dot{x}_{k-1}, \dot{y}_k - \dot{y}_{k-1})$$

$$\times \prod_{i=0}^{k-2} (x_i - x_{2k-i-1}, y_i + y_{2k-i-1}, \dot{x}_i + \dot{x}_{2k-i-1}, \dot{y}_i - \dot{y}_{2k-i-1}).$$

For $N = 4k + 2$ we set

$$R \left( \prod_{i=0}^{N-1} (x_i, y_i, \dot{x}_i, \dot{y}_i) \right) = \{x_k\} \times \prod_{i=0}^{k-1} (x_i - x_{2k-i-1}, y_i + y_{2k-i-1}, \dot{x}_i + \dot{x}_{2k-i-1}, \dot{y}_i - \dot{y}_{2k-i-1}).$$

We define the map $\Phi : \mathbb{R}^{N-1} \supset E^{-1}(\Omega) \to \mathbb{R}^{N-1}$ by

$$\Phi = R \circ P \circ E.$$ 

We have the following easy theorem.

**Theorem 4.2.** If for some $x \in \mathbb{R}^{N-1}$ $\Phi(x) = 0$, then there exists a trajectory with properties (P1), (P2) and (P3).

5. Existence of the Super-Eight—the Gerver orbit

The Gerver orbit (figures 6 and 7) is a choreography with 4-bodies forming a linear chain. It is the simplest trajectory after the Eight.

With computer assistance we proved the following

**Theorem 5.1.** The Gerver Super-Eight exists and is locally unique (up to obvious symmetries and rescaling).

We show the existence of Super-Eight using an approach described in section 4, i.e. we verify assumptions of theorem 4.2. Below we give some details.

We set

$$E(x_1, \dot{x}_0, \dot{y}_1) = (0, a, \dot{x}_0, 0, x_1, 0, \dot{y}_1, 0, -a, -\dot{x}_0, 0, -x_1, 0, 0, -\dot{y}_1),$$

(5.1)

![Figure 6. The Gerver orbit—the initial position.](image-url)
Figure 7. The Gerver orbit—the final position.

Table 5. Data from the proof of the existence of Gerver Super-Eight: initial values.

| Initial values |
|----------------|
| $\vec{x}$      | (1.382 857, 1.871 935 108 24, 0.584 872 579 881) |
| $a$            | 0.157 029 944 461 |
| $[X]$          | $\vec{x} + [-10^{-7}, 10^{-7}]^3$ |

where $a$ is a parameter fixing the size of the orbit.

$$R(x_0, y_0, x_1, y_1, x_2, y_2, x_3, y_3, \dot{x}_1, \dot{y}_1, \dot{x}_2, \dot{y}_2, \dot{x}_3, \dot{y}_3) = (y_1 + y_0, \dot{x}_1 + x_0, \dot{y}_1 + y_0). \quad (5.2)$$

The Poincaré section $S$ is defined by

$$S(x_0, y_0, x_1, y_1, x_2, y_2, x_3, y_3) = x_1 - x_0 = 0. \quad (5.3)$$

At first we proved the existence of a zero of $\Phi(x)$ using the interval Newton method, but here we present data from the proof based on the Krawczyk method. In case of the Gerver orbit the choice of the method was not important, because if we take a time step small enough or smaller set $[X]$, then the computed matrix $[\partial \Phi/\partial x([X])]$ becomes invertible and the proof based on the interval Newton method goes through.

We used the $C^1$-Lohner algorithm with the order $r = 6$ and the time step was set to $h = 0.002$. As matrix $C$ we used the value of $(\partial \Phi(\vec{x})/\partial x)^{-1}$ computed using a nonrigorous algorithm.

Tables 5 and 6 contain numerical data from this proof.

6. Existence of the ‘Linear chain’ orbit for 6-bodies

Figure 8 displays a linear chain choreography with 6-bodies.

In this section we report about the computer assisted proof of the following theorem.

**Theorem 6.1.** The linear chain for 6-bodies exists and is locally unique (up to obvious symmetries and rescaling).

We prove the theorem above using an approach described in section 4 with some minor changes. To speed up the calculation and to increase the accuracy we take into account that for all time $q_3(t) = -q_0(t)$, $q_4(t) = -q_1(t)$ and $q_5(t) = -q_2(t)$. Hence, our phase space become 12-dimensional. We also use a different time parametrization (we use a time shift of $\frac{1}{4}$ of the period). After this time shift in order to use the approach described in section 4 we also need to interchange axes (see figures 8 and 9). From lemma 4.1 we obtain, in this coordinate frame,
Table 6. Data from the proof of the existence of the Gerver Super-Eight: matrix $C$ and results of computation.

| Computed values |        |
|-----------------|--------|
| $C$             |        |
| $\begin{bmatrix} -2.154 & 0.257 & 0.786 \\ -0.081 & 0.293 & 0.043 \\ 0.939 & 0.100 & 0.158 \end{bmatrix}$ |        |
| $\Phi(\vec{x})$ | $\begin{bmatrix} -2.870 \times 10^{-9} \\ -1.211 \times 10^{-8} \\ -5.455 \times 10^{-8} \end{bmatrix}$ |
| $\text{diam } \Phi(\vec{x})$ | $\begin{bmatrix} 6.040 \times 10^{-6} \\ 1.434 \times 10^{-9} \\ 3.552 \times 10^{-9} \end{bmatrix}$ |
| $\frac{\partial \Phi}{\partial x} ([X])$ | $\begin{bmatrix} -0.1664 \\ -0.1764 \\ 0.871 \end{bmatrix}$ |
| $K(\vec{x}, [X], \Phi)$ | $\begin{bmatrix} 1.382 \\ 1.871 \end{bmatrix}$ |
| $\text{diam } K(\vec{x}, [X], \Phi)$ | $\begin{bmatrix} 5.386 \times 10^{-6} \\ 7.520 \times 10^{-6} \\ 1.542 \times 10^{-5} \end{bmatrix}$ |

Figure 8. ‘Linear chain’ orbit for 6-bodies—the initial position.

Figure 9. ‘Linear chain’ orbit for 6-bodies—the final position.
**Table 7.** Data from the proof of the existence of linear chain for 6-bodies: initial values.

| Initial value | \(-0.635\ 277\ 524\ 319^\circ\) |
|---------------|----------------------------------|
| \(\bar{x}\)   | \(0.140\ 342\ 838\ 651\)       |
| \(\bar{y}\)   | \(0.100\ 637\ 737\ 317\)       |
| \(a\)         | \(1.887\ 041\ 548\ 253\ 914\) |

\[
\begin{bmatrix}
-0.635\ 277\ 525\ 319, \\
0.140\ 342\ 837\ 651, \\
0.100\ 637\ 736\ 317, \\
-2.031\ 522\ 279\ 64
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.140\ 342\ 837\ 651, \\
0.100\ 637\ 738\ 317, \\
-2.031\ 522\ 277\ 64
\end{bmatrix}
\]

\[
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2.0e-09
\]

a doubly symmetric periodic solution \(q(t)\). Then it is easy to see that \(\bar{q} = q(t - T/4)\) is a solution with required symmetries in the original coordinate frame.

We set

\[
E(\dot{x}_0, x_1, y_1, \dot{y}_1) = (0, a, \dot{x}_0, 0, x_1, \dot{y}_1, x_1, \dot{x}_1, \dot{y}_1, y_1, -\dot{x}_1, \dot{y}_1),
\] (6.1)

where \(a\) is a parameter fixing the size of the orbit.

\[
R(x_0, y_0, \dot{x}_0, x_1, y_1, \dot{x}_1, \dot{y}_1, x_2, y_2, \dot{x}_2, \dot{y}_2) = (\dot{x}_1, x_0 - x_2, y_0 + y_2, \dot{x}_0 + \dot{x}_2, \dot{y}_0 - \dot{y}_2).
\] (6.2)

The Poincaré section \(S\) is defined by

\[
S(x_0, y_0, \dot{x}_0, x_1, y_1, \dot{x}_1, \dot{y}_1, x_2, y_2, \dot{x}_2, \dot{y}_2) = y_1 = 0.
\] (6.3)

To find a zero of \(\Phi(x)\) we use the Krawczyk method. To compute the Poincaré map we use the \(C^1\)-Lohner algorithm [ZLo] of the order \(r = 9\) and the time step \(h = 0.0025\) for the computation in point \(\bar{x}\) and \(h = 0.001\) for the computation on the set \([X]\). As matrix \(C\) we used a nonrigorously computed point matrix \((\partial \Phi(\bar{x})/\partial x)^{-1}\). In tables 7 and 8 we give data from these computations.

### 7. Some technical data

All computations were performed on an AMD Athlon 1700XP with 256 MB DRAM memory, with Windows 98SE or Linux Red Hat 8.0 operating system. We used the CAPD package [Capd] and Borland C++ 5.02 compiler (under Linux we used g++ 3.02). Our interval arithmetics (part of CAPD package) was based on the double precision reals. The source code of our program is available at the first author’s web page [Ka].

In the listing below \(r\) is an order and \(h\) is a time step used in the \(C^1\)-Lohner algorithm [ZLo].

- Computation times for the Eight, \(h = 0.01\), \(r = 7\)
  - in point \(\bar{x}\): 1.417 s,
  - for set \([X]\): 2.66 s,
  - convexity: 1.155 01 s.
Table 8. Data from the proof of the existence of linear chain for 6-bodies.

| Computed value | \[\Phi(\bar{x})\] | \[\text{diam } \Phi(\bar{x})\] | \[K(\bar{x}, [X], \Phi)\] | \[\text{diam } K(\bar{x}, [X], \Phi)\] |
|----------------|--------------------|-------------------------------|---------------------|-------------------------------|
|                | \[\begin{bmatrix} [-3.131 195 790 965 8e-11, 3.156 277 062 700 585e-11] \\ [-4.528 821 762 050 939e-12, 4.574 757 239 694 804e-12] \\ [-1.063 704 679 893 362e-11, 1.051 470 022 161 993e-11] \\ [-3.084 105 193 592e-11, 3.117 495 150 917 193e-11] \\ [-1.203 726 007 759 087e-11, 1.193 112 275 643 671e-11] \end{bmatrix}\] | \[\begin{bmatrix} 6.287 472 853 666 386e-11 \] \[9.103 579 001 745 743e-12 \] \[2.115 174 702 055 356e-11 \] \[6.201 600 344 368 785e-11 \] \[2.396 838 283 402 758e-11 \] \| \[\begin{bmatrix} [-0.635 277 524 361 667 955 7, -0.635 277 524 276 328 331 4] \\ [0.140 342 838 643 052 164 6, 0.140 342 838 659 099 994 3] \\ [0.797 833 001 999 263 769, 0.797 833 002 012 834 469] \\ [0.100 637 737 288 174 253 24, 0.100 637 737 345 775 718 9] \\ [-0.203 227 288 712 178 764, -0.203 227 287 575 771 612] \end{bmatrix}\] | \[\begin{bmatrix} 8.533 962 425 616 437 03e-11 \] \[1.604 782 973 174 678 733e-11 \] \[1.357 070 011 920 313 846e-11 \] \[5.760 096 566 387 318 262e-11 \] \[1.344 071 520 748 002 513e-10 \] \| | | |

For the proof of the existence of the Gerver solution with 4-bodies problem we used \(r = 6, h = 0.002\). The computation times for both \(\bar{x}\) and set \([X]\) were approximately equal to 30.5 s.

In the proof of the linear chain of 6-bodies
- the computation of the Poincaré map for set \([X]\) took 57.5 s with \(h = 0.001\) and \(r = 9\),
- the computation for \(\bar{x}\) took 23.8 s with \(h = 0.0025\) and \(r = 9\).

8. Conclusions and future directions

In principle there is no theoretical limit for the number of bodies in the doubly linear chain to which our method applies. By this we mean the following: if the choreography is isolated in the reduced space, then the computer-assisted proof of its existence is possible based on the approach and algorithms used in this paper. This may obviously require small time steps and/or higher order Taylor method and higher precision arithmetics.

We tried to see how far we can go with the number of bodies with double precision interval arithmetics and our code. Using the approach described in section 4 we proved the existence of doubly symmetric linear chains for 10- and 12-bodies. Data from these proofs are available on the first author’s web page [Ka]. Our approach failed for doubly symmetric linear chain of 16-bodies. We skipped the 14-bodies case, as we did not have good initial conditions. We believe that the reason for the failure for 16-bodies was the wrapping effect [Mo, Lo] for intervals sets arising from the round-off error of the double precision arithmetic.

We think that to obtain the proof for the existence of doubly symmetric linear chains with more bodies using our approach it should be enough to make one of the following modifications:
- use higher precision arithmetics,
- instead of simple shooting use multiple shooting [Lo, Lo1, Sc]. This technique corresponds to the intermediate section method used very effectively in the computer-assisted proofs of chaotic behaviour in the Lorenz system [GZ, MM, T].
One may also think of more radical changes (like the change of the algorithm for rigorous integration of ODEs or totally different approach), which may eventually result in improvements over the proposed method/algorithms. Below we list some possibilities:

- The Taylor model method for an integration of ODEs advocated by Berz and his coworkers (see [BMH] and references given therein). This method is very slow compared to the Lohner algorithm, but it still works in cases were the Lohner algorithm fails to produce reasonable bounds.
- The shadowing technique of Stoffer–Kichgraber [SK], which can be seen as an efficient mixture of hyperbolic shadowing and the intermediate section method. In [SK], in the context of the planar restricted 3-body problem this approach has been shown much more efficiently than the one based on the $C^1$-Lohner algorithm.
- Rigorous numerics for variational methods. We believe that it is an interesting problem in itself to ‘construct’ the rigorous numerics for the variational approach to the $N$-body problem, which will turn the numerical-variational work of Simó [S2] and Nauenberg [Na] into computer-assisted proofs.

Regarding the Eight: we believe that by combining the methods presented in this paper with some analytical estimates, which will make the reduced space to be searched for the choreography compact, one should be able to answer the following open questions [Ch]:

- the global uniqueness of the Eight in the class of doubly symmetric choreographies,
- the global uniqueness of the Eight in the class of choreographies with less symmetry (see [Ch, FT] for an explanation).

We hope to treat the above problems in the near future.

On the other hand the question of the (local) uniqueness of the Eight (up to obvious isometries and rescaling) in the class of choreographies with zero angular momentum appears to be more difficult. The problem is related to the geometric phase in the $N$-body problem (see [MR1] and the literature given there), which in our context can be represented as follows.

Assume that we have 3-bodies in the collinear configuration (with zero total linear and angular momenta). Let $\alpha_0$ be a line containing all the bodies, whose velocities and positions are given by $q_i'(0)$ and $q_i(0)$ for $i = 1, 2, 3$.

Assume now that after time $T$ the bodies are again in the collinear configuration (represented by the line $\alpha(T)$) and such that $q_i(T) = q_{\sigma(i)}(0)$, where $\sigma$ is a cyclic permutation. Can we then claim that $\alpha_0 = \alpha_T$ and $q_i(T) = q_{\sigma(i)}(0)$?

It turns out that a similar problem appears when one tries to use our approach to the case of the choreographies without any symmetry.

Another interesting question about the Eight is its stability. Numerical experiments of Simó [S3] show that the Eight is KAM-stable. Rigorous verification of this statement requires checking that the Eight is linearly stable (which is possible in principle using our algorithm) and that a twist condition is satisfied [SM]. The twist condition requires rigorous computations of higher derivatives of a Poincaré map, hence a development of the robust and efficient $C^k$-Lohner algorithm for $k > 1$ is desirable. We believe that a suitable generalization of the $C^1$-Lohner algorithm will work.

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Simple choreographies in \( N \)-body problem

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