IDEAL characterization of higher dimensional spherically symmetric black holes

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Abstract
In general relativity, an IDEAL (intrinsic, deductive, explicit, algorithmic) characterization of a reference spacetime metric $g_0$ consists of a set of tensorial equations $T[g] = 0$, constructed covariantly out of the metric $g$, its Riemann curvature and their derivatives, that are satisfied if and only if $g$ is locally isometric to the reference spacetime metric $g_0$. We give the first IDEAL characterization of generalized Schwarzschild–Tangherlini spacetimes, which consist of $\Lambda$-vacuum extensions of higher dimensional spherically symmetric black holes, as well as their versions where spheres are replaced by flat or hyperbolic spaces. The standard Schwarzschild black hole has been previously characterized in the work of Ferrando and Sáez, but using methods highly specific to 4 dimensions. Specialized to 4 dimensions, our result provides an independent, alternative characterization. We also give a proof of a version of Birkhoff’s theorem that is applicable also on neighborhoods of horizon and horizon bifurcation points, which is necessary for our arguments.

Keywords: IDEAL characterization, Schwarzschild, Tangherlini, warped product, Birkhoff’s theorem

1. Introduction

In this work, we are interested in an intrinsic characterization of higher dimensional generalizations of the Schwarzschild black hole. These spherically symmetric, asymptotically flat vacuum spacetimes were first studied by Tangherlini [34]. By a spacetime $(\mathcal{M}, g)$, we mean a smooth manifold $\mathcal{M}$ with a Lorentzian metric $g$, though a similar discussion can be carried out for any pseudo-Riemannian geometry. While ‘intrinsic’ generally does preclude direct reference to the form of the spacetime metric in a special coordinate system, it is a vague enough
term to have multiple interpretations. To be specific, we refer to an IDEAL\(^1\) or Rainich-type characterization that has been used, for instance, in the works [2, 3, 7, 8, 10, 15, 16, 21, 28, 33]. It consists of a list of tensorial equations \((T_k[g] = 0, a = 1, 2, \ldots, N)\), constructed covariantly out of the metric \((g)\) and its derivatives (concomitants of the Riemann tensor) that are satisfied if and only if the given spacetime locally belongs to the desired class, possibly narrow enough to be the isometry class of a single reference spacetime geometry. This notion has a natural generalization \((T_k[g, \phi] = 0)\) to spacetimes equipped with scalar or tensor fields \((\phi)\), with equivalence still given by isometric diffeomorphisms that also transform the additional scalars or tensors into each other, though we will not make use of this generalization here. A nice historical survey of this and other local characterization results can be found in [24].

An IDEAL characterization requires neither the existence of any extra geometric structures, nor the translation of the metric and of the curvature into a frame formalism. Thus, it is an alternative to the Cartan–Karlhede characterization [31, chapter 9], which is based on Cartan’s moving frame formalism. Intrinsic characterizations, of various types, have been of long standing and independent interest in geometry and general relativity. But, in addition, they can be helpful in deciding when a metric, given for instance by some complicated coordinate formulas, corresponds to one that is already known. In this regard, an IDEAL characterization is especially helpful if one would like to find an algorithmic solution to this recognition problem. In numerical relativity, the near-satisfaction of the tensor equations \(T_k[g] = 0\) may signal the local proximity of a numerical spacetime to a desired reference geometry. In addition, the approach to zero \(T_k[g] \to 0\) could be used to study either linear or nonlinear stability of reference geometries, in an unambiguous and gauge independent way.

The following additional application should be noted. By the Stewart–Walker lemma [32, lemma 2.2], the vanishing of a tensor concomitant \(T_k[g] = 0\) for a metric \(g\) implies that its linearization \(\tilde{T}_k[h]\) \((T_k[g + \epsilon h] = T_k[g] + \epsilon \tilde{T}_k[g] + O(\epsilon^2))\) is invariant under linearized diffeomorphisms. Thus, any quantity of the form \(\tilde{T}_k[h]\) defines a gauge-invariant observable in linearized gravity, when Einstein or Einstein-matter equations are linearly perturbed about a background solution \(g\). The linearized Einstein tensor \(\tilde{G}[h]\) itself is of course an example of a linear gauge-invariant, when linearized about a vacuum solution, \(G[g] = 0\). In the presence of matter fields \(\phi\), it is the whole linearised Einstein equation \(\tilde{G}[h] - 8\pi \tilde{T}[h, \psi] = 0\) that is linearly gauge-invariant, when linearized about a joint solution of \(\tilde{G}[g] = 8\pi \tilde{T}[g, \phi]\) and the matter equations of motion. And by invariant, of course, we mean with respect to the action of linearized diffeomorphisms on both the metric \(g\) and the matter fields \(\phi\). A straightforward (though heuristic) argument shows that an IDEAL characterization of a local isometry class provides a list \(\tilde{T}_k[h], k = 1, \ldots, N\), of gauge-invariant observables that should also complete: the joint kernel of \(\tilde{T}_k\) should coincide with the tangent space to the isometry orbit (to make this argument completely rigorous, it suffices to check that \(T_k[g + \epsilon h]\) do not approach zero at \(O(\epsilon^2)\) or higher order). That is, the joint kernel of \(\tilde{T}_k[h] = 0\) locally consists only of pure gauge modes \((h = L_{\psi} g\) for some vector field \(\psi\)). The use of such local observables (given by differential operators) can be advantageous both in theoretical and practical investigations of classical and quantum field theoretical models because they cleanly separate the local (or ultraviolet) and global (or infrared) aspects of the theory. This may be of interest in the problem of reconstructing the metric of a linear gravitational wave from its complete set of gauge-invariant observables [26], or in the problem of determining the decay properties of linear gravitational waves in a gauge-independent way [4].

\(^{1}\)The acronym, explained in [10] (footnote, p 2), stands for intrinsic, deductive, explicit and algorithmic.
In this work, we add the family of generalized Schwarzschild–Tangherlini geometries to the (unfortunately still small, but slowly growing) literature concerning IDEAL characterizations of isometry classes of individual reference geometries. That family consists of all Λ-vacuum $2 + m$-warped products, where the warped $m$-dimensional factor is maximally symmetric. When the latter factor is a round sphere, we recover the asymptotically flat or (anti-)de Sitter generalization of the Schwarzschild–Tangherlini black holes [20, 34]. Replacing the sphere by flat Euclidean space, we get the higher dimensional generalizations of Taub’s plane symmetric spacetimes [35]. Replacing the sphere by hyperbolic space, we get so-called pseudo-Schwarzschild wormhole spacetimes [23]. Specializing to $2 + m = 4$ and $Λ = 0$, we get the so-called family of $A$-metrics of Ehlers and Kundt [5]. Other IDEAL characterizations for geometries of interest in general relativity include (4-dimensional) Schwarzschild [7, 16], Reissner–Nordström [6], Kerr [8, 15], Lemaître–Tolman–Bondi [10], Stephani universes [11] (see references for complete lists and details), and most recently FLRW and inflationary spacetimes (in any dimension) [2]. Of course, for completeness, we have to mention the classic cases of constant curvature spaces (see (14)), which are known to be fully characterized by the structure of the Riemann tensor (by theorems of Riemann and Killing–Hopf [37]).

For definiteness, let us state what we mean by a local isometry and local isometry class.

**Definition 1 (Locally isometric).** A pseudo-Riemannian geometry $(\mathcal{M}_1, g_1)$ is locally isometric at $x_1 \in \mathcal{M}_1$ to a pseudo-Riemannian geometry $(\mathcal{M}_2, g_2)$ at $x_2 \in \mathcal{M}_2$ if there exist open neighbourhoods $U_1 \ni x_1$, $U_2 \ni x_2$ and a diffeomorphism $χ: U_1 \to U_2$ such that $χ(x_1) = x_2$ and $χ^* g_2 = g_1$. If we can choose $U_1 = \mathcal{M}_1$ and $U_2 = \mathcal{M}_2$ then they are (globally) isometric. If for every $x_1 \in \mathcal{M}_1$ there is $x_2 \in \mathcal{M}_2$ such that $(\mathcal{M}_1, g_1)$ at $x_1$ is locally isometric to $(\mathcal{M}_2, g_2)$ at $x_2$, we simply say that $(\mathcal{M}_1, g_1)$ is locally isometric to $(\mathcal{M}_2, g_2)$ (note the asymmetry in the definition). If $(\mathcal{M}_1, g_1)$ is locally isometric to $(\mathcal{M}_2, g_2)$, as well as vice versa, we say that they are locally isometric to each other (which constitutes an equivalence relation). All pseudo-Riemannian geometries that are locally isometric to a reference $(\mathcal{M}, g)$ constitute its local isometry class.

The synopsis of the paper is the following: in section 2 we define and exhibit the main geometric features of $2 + m$-warped product geometries. Proposition 3 gives a geometric characterization of $2 + m$-warped products in terms of a symmetric projector whose covariant derivative satisfies a special constraint. In section 2.1 we introduce the family of generalized Schwarzschild–Tangherlini (gST) geometries, with special attention to the structure of their Riemann curvature. Section 2.2 states and proves a version of Birkhoff’s theorem, according to which a locally maximally symmetric $2 + m$-warped product that is also a Λ-vacuum must locally coincide with one of the gST geometries. The main reason to include a proof is to pay special attention to the applicability of this result to points lying on a (Killing) horizon. Finally, theorem 7 in section 3 puts all the pieces together to give an IDEAL characterization of the local isometry classes of the gST geometries. Due to a quirk of the structure of the gST Riemann curvature in $n = 4$ dimensions, the final result looks slightly different in $n = 4$ and $n \geq 5$ dimensions. This difference is accounted for by theorem 8. In the case of spherical symmetry in $n = 4$ dimensions with $Λ = 0$, our results provide an independent alternative characterization of the standard Schwarzschild spacetime, which was first characterized in [7]. All other instances of the results from section 3 are new. Finally, in section 4, we conclude with a discussion of our results and of directions for future work.

Throughout the paper we follow the conventions of [36]: $(-+\cdots +)$ for Lorentzian signature, and $2\nabla_\mu \nabla_\nu \omega_\lambda = R_{\mu\nu\lambda}^\nu \omega_\lambda$ for curvature. Unless otherwise specified, all functions will be considered $C^\infty$ smooth.
2. 2 + \textit{m}-warped products

Below, we consider 2 + \textit{m}-warped product geometries. That is, pseudo-Riemannian geometries on an \( n \)-dimensional manifold, which can be represented as a warped product of a 2-dimensional and an \( m \)-dimensional geometry. Our main family of examples consists of the generalized Schwarzschild–Tangherlini spacetimes (section 2.1), which includes the spherically symmetric black holes in four and higher dimensions. We will discuss the structure of Riemann curvature tensor of the warped product and the consequences for the 2-dimensional base factor when the product satisfies the Einstein equation (Birkhoff’s theorem, section 2.2).

\textbf{Definition 2 (Warped product).} A pseudo-Riemannian geometry \((\mathcal{M}, \bar{g}) \cong (\mathcal{M}, g) \times_r (S, \Omega)\) is a warped product with warping function \( r \) when \( \mathcal{M} \cong \mathcal{M} \times S \) and the metric can be written as

\[
\bar{g} = g + r^2 \Omega,
\]

where the metric tensors \( g \) and \( \Omega \) are lifted to the product space by pulling back along the projections \( \mathcal{M} \rightarrow \mathcal{M} \) and \( \mathcal{M} \rightarrow S \), while \( r \) is the pullback of a nowhere vanishing function on \( \mathcal{M} \). We call \((S, \Omega)\) the \textit{warped factor} and \((\mathcal{M}, g)\) the \textit{base factor}.

Let us now introduce some notational conventions that will simplify subsequent discussions. Denote by \( \nabla \), \( \bar{\nabla} \) and \( D \) the canonical Levi-Civita connections on \((\mathcal{M}, g) \cong (\mathcal{M}, g) \times_r (S, \Omega)\), \((\mathcal{M}, g)\) and \((S, \Omega)\), respectively. We will use Greek indices \((\alpha \beta \cdots)\) for tensors on \( \mathcal{M} \), lower case Latin indices \((ab \cdots)\) for tensors on \( \mathcal{M} \), and upper case Latin indices \((AB \cdots)\) on \( S \). Using the product structure \( \mathcal{M} \cong \mathcal{M} \times S \), any tensor or differential operator on \( \mathcal{M} \) or \( S \) can be canonically transferred to \( \mathcal{M} \). We will do so for the usual Riemann and Ricci curvature tensors and the derivative of the warping function:

\[
\begin{align*}
R_{abcd}[g], & \quad R_{ab}[g], \quad R[g] \rightarrow R_{\mu\nu\lambda\kappa}, \quad R_{\mu\nu}, \quad R, \\
R_{\mu\nu\lambda\kappa}[\Omega], & \quad R_{\alpha\beta}[\Omega], \quad R[\Omega] \rightarrow S_{\mu\nu\lambda\kappa}, \quad S_{\mu\nu}, \quad S,
\end{align*}
\]

\[
r_{\alpha} = \nabla_{\alpha} r \rightarrow r_{\mu} = \bar{\nabla}_{\mu} r.
\]

We will also use the obvious notation \( R_{\mu\nu\lambda\kappa} = R_{\mu\nu\lambda\kappa}[\bar{g}] \), \( R_{\mu\nu} = R_{\mu\nu}[\bar{g}] \), \( R = R[\bar{g}] \). We will use the following self-explanatory convention when raising and lowering indices on tensors transferred to \( \mathcal{M} \) from one of the factors: \( g^{\mu\nu} g_{\nu\lambda} = g^{\mu\lambda} \), but \( g^{\mu\nu} r^2 \Omega_{\nu\lambda} g^{\lambda\kappa} = r^{-2} \Omega^{\mu\kappa} \).

The covariant derivative on the warped product geometry acts as

\[
\nabla_{\alpha} X_{\beta} = \nabla_{\alpha} X_{\beta} + D_{\alpha} X_{\beta} - 2X^{\lambda} (r^2 \Omega)_{\lambda(\nu} \nabla_{\alpha}) \log r + X^{\lambda} (\nabla_{\lambda} \log r)(r^2 \Omega)_{\mu\nu} r_{\nu} \quad \text{(3)}
\]

where the action of \( \nabla \) on scalars is just through the exterior derivative. In particular, we get

\[
\nabla_{\alpha} g_{\nu\lambda} = 2 \frac{r^2 (\Omega_{\mu(\nu} r_{\lambda)}}{r}, \quad \text{and} \quad \nabla_{\alpha} (r^2 \Omega)_{\nu\lambda} = -2 \frac{r^2 \Omega_{\mu(\nu} r_{\lambda)}}{r}. \quad \text{(4)}
\]

The curvature tensors are given by

\[
\begin{align*}
\bar{R}_{\mu\nu\lambda\kappa} &= r^2 S_{\mu\nu\lambda\kappa} + R_{\mu\nu\lambda\kappa} - \left( \frac{\nabla \nabla r}{r} \otimes r^2 \Omega \right)_{\mu\nu\lambda\kappa} - \frac{r_{\sigma} r^{\sigma}}{2r^2}(r^2 \Omega \otimes r^2 \Omega)_{\mu\nu\lambda\kappa}, \\
\bar{R}_{\mu\nu} &= S_{\mu\nu} + R_{\mu\nu} - \frac{\nabla_{\mu} \nabla_{\nu} r}{r} - \frac{\Box r}{r} r^2 \Omega_{\mu\nu} - (m - 1) \frac{r_{\sigma} r^{\sigma}}{r^2} r^2 \Omega_{\mu\nu},
\end{align*}
\]

\[
\text{(5)} \quad \text{(6)}
\]
\[
R = \frac{S}{r^2} + R - 2m \frac{\Box r}{r} - m(m - 1) \frac{r s}{r^2},
\]
where for convenience we have introduced the Kulkarni–Nomizu product of symmetric tensors:
\[
(A \odot B)_{\mu\nu\lambda\kappa} = A_{\mu\lambda} B_{\nu\kappa} - A_{\nu\lambda} B_{\mu\kappa} - A_{\mu\nu} B_{\lambda\kappa} + A_{\lambda\kappa} B_{\mu\nu}.
\]
Note that we could rewrite the \(\nabla\)-derivatives using \(\hat{\nabla}\)-derivatives in the above formulas with the help of the identity
\[
\frac{\nabla_{\mu} \nabla_{\nu} r}{r} = \frac{\hat{\nabla}_{\mu} \hat{\nabla}_{\nu} r}{r} - \frac{r_{\lambda} r_{\nu} (r^2 \Omega)_{\mu\lambda}}{r^2} = \frac{\hat{\nabla}_{\mu} \hat{\nabla}_{\nu} r}{r} + \frac{r_{\mu} r_{\nu}}{r^2} - \frac{r_{\nu} r_{\lambda} (r^2 \Omega)_{\mu\lambda}}{r^2}.
\]
These formulas can be extracted from [27, proposition 7.43]. But the quickest way check them is to notice that \(g_{\mu\nu} = r^2 (g_{\mu\nu} + \Omega_{\mu\nu})\), in which a pair of nested conformal transformation relates \(g\) and a product metric, and use the standard formulas for the conformal transformations of covariant derivatives and curvatures [36, appendix D].

**Proposition 3 ([9], [14, theorem 16(2)]).** A pseudo-Riemannian geometry \((\mathcal{M}, \bar{g})\) can be locally put into the form of a \(2 + m\)-warped product
\[
\bar{g}_{\mu\nu} = g_{\mu\nu} + r^2 \Omega_{\mu\nu}
\]
iff there exist a 1-form \(\ell_\mu\) and a symmetric tensor \(\Omega_{\mu\nu} = \Omega_{(\mu\nu)}\) that together satisfy the following conditions
\[
\nabla_{[\mu} \ell_{\nu]} = 0, \quad \ell^\mu \hat{\nabla}_{\mu\nu} = 0, \quad \Omega_{\mu} \nabla_{\nu} = \Omega_{\nu} \nabla_{\mu}, \quad \hat{\nabla}_{\mu} \Omega_{\nu\lambda} = -2 \Omega_{\mu(\nu} \ell_{\lambda)}. \tag{11}
\]
Then we can choose \(g_{\mu\nu} = \bar{g}_{\mu\nu} - \Omega_{\mu\nu}\) and \(\hat{\nabla}_{\mu} \Omega_{\nu\lambda} = r^{-2} \Omega_{\nu\lambda}\), with \(r\) satisfying \(\nabla_{\mu} \log |r| = \ell_{\mu}\).

**Proof.** In one direction, starting with the definitions of \(\hat{\nabla}_{\mu} \ell_{\nu}\) and \(\ell_{\mu}\) in terms of \(r\) and \(\Omega_{\mu\nu}\), verifying the above identities is a matter of direct calculation (see equation (4)).

In the other direction, the key observation is that a warped product metric is conformal to a direct product metric, namely \(g_{\mu\nu} = r^2 \bar{g}_{\mu\nu}\), with \(\hat{g}_{\mu\nu} = r^2 g_{\mu\nu} + \Omega_{\mu\nu}\), where the warping function \(r^2\) plays the role of the conformal factor. Given our first condition on \(\ell_{\mu}\), and since we are working locally, we can always find a smooth function \(r\) satisfying \(\nabla_{\mu} \log |r| = \nabla_{\mu} r / r = \ell_{\mu}\). Again locally, we can choose \(r\) to be nowhere vanishing. The choice is unique, up to a multiplicative constant.

Define \(\bar{g}_{\mu\nu} = r^2 \bar{g}_{\mu\nu}\), \(\hat{g}^\mu_{\nu}\) its inverse, \(\Omega_{\mu\nu} = r^{-2} \Omega_{\mu\nu}\), and let \(\hat{\nabla}\) be the \(\hat{g}\)-compatible Levi-Civita connection. A straightforward calculation shows that our conditions on \(\Omega\) translate to
\[
\Omega_{\mu\nu} \hat{g}^\mu_{\lambda} \hat{g}^\nu_{\kappa} = \Omega_{\mu\kappa}, \quad \hat{g}^\mu_{\nu} \Omega_{\mu\nu} = m, \quad \text{and} \quad \nabla_{\mu} \Omega_{\nu\lambda} = 0.
\]
It is well known [9] that the existence of such a rank-\(m\) \(\hat{\nabla}\)-covariantly constant symmetric projector \(\Omega_{\mu\nu}\) implies that \(\bar{g}_{\mu\nu}\) can be locally put into \(2 + m\)-product form \(\bar{g}_{\mu\nu} = (\Omega_{\mu\nu} - \Omega_{\mu\nu}) + \Omega_{\mu\nu}\). Our second condition on \(\ell_{\mu}\) implies that \(r\) does not depend on the \(\Omega\)-factor. Thus, without loss of generality, we can define \(g_{\mu\nu} = r^2 (\bar{g}_{\mu\nu} - \Omega_{\mu\nu})\) and write the product form as \(\bar{g}_{\mu\nu} = r^{-2} \bar{g}_{\mu\nu} + \Omega_{\mu\nu}\).

Undoing the conformal transformation, we end up with the desired local \(2 + m\)-warped product form \(g_{\mu\nu} = g_{\mu\nu} + r^2 \Omega_{\mu\nu}\).
2.1. Generalized Schwarzschild–Tangherlini geometries

Consider an integer $n \geq 4$ and a triple of real numbers $(\alpha, M, \Lambda)$, where $M \neq 0$. The 2-dimensional metric

$$g_{ab} = -f dt^a dt^b + \frac{1}{f} dr^a dr^b, \quad f(r) = \alpha - \frac{2M}{r^{n-3}} - \frac{2\Lambda}{(n-1)(n-2)} r^2, \quad (13)$$

is well-defined and Lorentzian in the interiors of the $r$-intervals separated by $r = 0$ and the roots of $f(r) = 0$. It is well-known that each one of these regions has a unique maximal analytic, connected and simply-connected extension [22, 29]. Each region with $r > 0$ generates the same extension (topologically $\mathbb{R}^2$), and similarly for each region with $r < 0$. When $n$ is even, the extensions with $r > 0$ and $r < 0$ are distinct. However, the $r < 0$ extension is isometric to the $r > 0$ extension with $M$ replaced by $-M$, by sending $r \mapsto -r$. When $n$ is odd, the extensions with $r > 0$ and $r < 0$ are isometric, again by sending $r \mapsto -r$, but the geometry obtained by replacing $M$ by $-M$ is different. Thus, for book keeping convenience, let us denote by $(M, g)_{\alpha, M, \Lambda}$ the disjoint union of the $r > 0$ and $r < 0$ extensions with the same $M$ parameter when $n$ is even, and the disjoint union of the $r > 0$ extension with parameter $M$ and the $r < 0$ extension with parameter $-M$ when $n$ is odd. In either case, $M \cong \mathbb{R}^2 \sqcup \mathbb{R}^2$.

Naturally, by our construction, each of these maximally extended geometries is accompanied by the distinguished scalar function $r$, taking on all non-zero real values, which was analytically extended along with the metric.

The precise way in which the $(t, r)$ charts are glued together along horizons and horizon bifurcation points to form the analytic extension can be glimpsed from proposition 6, where (a) corresponds to a generic points covered by an $(t, r)$ chart, (b) corresponds to a horizon points, and (c) corresponds to a horizon bifurcation point of the extension. The gluing is done with the help of the tortoise coordinate $r$, from (49). Penrose conformal diagrams for the extensions can be found in [22, 29].

Recall that $(\mathcal{S}, \Omega)$ is of constant curvature [37], with sectional curvature $\alpha$, if its Riemann curvature tensor is

$$R_{ABCD}[\Omega] = \frac{\alpha}{2} (\Omega \odot \Omega)_{ABCD}. \quad (14)$$

When an $m$-dimensional Riemannian geometry $(\mathcal{S}, \Omega)$ is simply connected, geodesically complete and of constant curvature it can only be one of the following [37, section 2.4]: Euclidean $m$-space, round $m$-sphere, hyperbolic $m$-space. These are called maximally symmetric spaces. Let us denote the corresponding maximally symmetric space with sectional curvature $\alpha$ by $(\mathcal{S}, \Omega)_{m, \alpha}$.

**Definition 4 (Generalized Schwarzschild–Tangherlini spacetime).** Fix a dimension $m \geq 2$ and a triple of real numbers $(\alpha, M, \Lambda)$, with $M \neq 0$. Set $n = 2 + m$ and denote the warped product $(\mathcal{M}, \tilde{g})_{\alpha, M, \Lambda} \cong (\mathcal{M}, \tilde{g})_{(\alpha, M, \Lambda)} \times_r (\mathcal{S}, \Omega)_{\text{m,} \alpha, \text{m,} \alpha}$, where the base factor and the warping function $r$ are defined in the discussion following (13), and the warped factor is the $m$-dimensional maximally symmetric space of sectional curvature $\alpha$. We call $(\mathcal{M}, \tilde{g})_{\alpha, M, \Lambda}$ a $n$-dimensional generalized Schwarzschild–Tangherlini (gST) spacetime.

If we had included the $M = 0$ cases, then each such geometry would correspond to a particular representation of a subset of a maximally symmetric geometry (de Sitter or anti-de Sitter spacetime), as we will see shortly (equation (23)). Since this case has already been extensively studied (e.g. see our previous works [2, 18]), we exclude it from consideration.
For tensors with two or four indices, we define contractions

\[(A \cdot B)_{\mu\nu} = A_{\mu}{}^\lambda B_{\nu}{}^\lambda, \quad \text{and} \quad (R \cdot S)_{\mu\nu\lambda\kappa} = R_{\mu\nu}{}^{\sigma\tau}S_{\sigma\tau\lambda\kappa}.\]  

(15)

Recalling also the definition of the Kulkarni–Nomizu product (8), when \(A, B, C\) and \(D\) are symmetric, we have the useful identities

\[\[(A \odot B) \cdot (C \odot D)]_{\mu\nu\lambda\kappa} = 2[(A \cdot C) \odot (B \cdot D) + (A \cdot D) \odot (B \cdot C)]_{\mu\nu\lambda\kappa},\]  

(16)

\[(A \odot B)_{\mu\nu} = [A \cdot B - (\text{tr} A)B - A(\text{tr} B) + B \cdot A]_{\mu\nu}.\]  

(17)

Now, we compute the curvature of the gST geometries that we have defined above. Let us start with the 2-dimensional \((M, g)\) factor. We basically follow the presentation from [25]. Working in the \((t, r)\) chart, clearly \(t^a = (\partial_t)^a\) is a timelike Killing vector. For convenience, we also introduce the notation \(t_a = g_{ab}t^b = -f^b \varepsilon_{ab}\) and \(r_a = dr_a\). They are related as \(t_a = -\varepsilon_{ab} r_b\), where \(\varepsilon_{ab} = (dt \wedge dr)_{ab}\). Then, of course, \(r_a r^a = f\) and \(t_a t^a = -1/f\). The action of the covariant derivative is summarized by

\[\nabla_a t_b = f \frac{1}{2} r_{ab}\]  

(18)

For the record, let us write out in full the following identities for \((M, g)\):

\[\frac{r_a r^a}{r^2} = f \frac{\alpha}{r^2} - 2 \frac{M r}{r^2 - 1} - 2 \frac{\Lambda}{(n - 1)(n - 2)},\]  

(19)

\[\nabla_a \nabla_b r = \left( (n - 3) \frac{M}{r^{n - 1}} - 2 \frac{\Lambda}{(n - 1)(n - 2)} \right) g_{ab},\]  

(20)

\[R_{abcd} = \frac{R}{4} (g \odot g)_{abcd}, \quad R_{ab} = \frac{R}{2} g_{ab},\]  

(21)

\[R = \frac{4\Lambda}{(n - 1)(n - 2)} + 2(n - 2)(n - 3) \frac{M}{r^{n - 1}}.\]  

(22)

They can be directly plugged into (5), the formula for the Riemann tensor of a \(2 + m\)-warped product, to get the explicit expression for the Riemann tensor \(R_{\mu\nu\lambda\kappa}\) of a gST geometry \((\bar{M}, \bar{g})\):

\[R_{\mu\nu\lambda\kappa} = \frac{\Lambda}{(n - 1)(n - 2)} (g \odot \bar{g})_{\mu\nu\lambda\kappa} + \frac{M}{r^{n - 1}} \left( (n - 2)(n - 3) g \odot (r^2 \Omega \odot r^2 \Omega)_{\mu\nu\lambda\kappa} + (n - 3)(g \odot r^2 \Omega)_{\mu\nu\lambda\kappa} \right).\]  

(23)

Next, let us define several tensors and scalars built out of the Riemann tensor and its derivatives:

\[T_{\mu\nu\lambda\kappa}[\bar{g}] := R_{\mu\nu\lambda\kappa} - \frac{\Lambda}{(n - 1)(n - 2)} (g \odot \bar{g})_{\mu\nu\lambda\kappa},\]  

(24)
\[ \rho[g] := \left[ \frac{(\bar{T} \cdot \bar{T} \cdot \bar{T})_{\mu \nu}^{\mu \nu}}{8(n-1)(n-2)(n-3)(n-2)(n-3)(n-4) + 2} \right]^{\frac{1}{4}}, \]  
(25)

\[ \ell_\mu[g] := -\frac{1}{(n-1)} \frac{\nabla_\mu \rho}{\rho}, \]  
(26)

\[ A[g] := \ell_\mu \ell^\mu + 2 \rho + \frac{2 \Lambda}{(n-1)(n-2)}. \]  
(27)

For future reference, we also compute some algebraic combinations among these tensors (see [19, section 3.3] for more intermediate steps of the calculations):

\[ \bar{T} = \frac{M}{r^{n-1}} \left[ \frac{(n-2)(n-3)}{2} (g \circ g) + (r^2 \Omega \circ r^2 \Omega) - (n-3)(g \circ r^2 \Omega) \right], \]  
(28)

\[ \bar{T} \cdot \bar{T} = \left( \frac{M}{r^{n-1}} \right)^2 \left[ (n-2)^2(n-3)^2(g \circ g) + 4(r^2 \Omega \circ r^2 \Omega) + 2(n-3)^2(g \circ r^2 \Omega) \right], \]  
(29)

\[ \bar{T} \cdot \bar{T} \cdot \bar{T} = \left( \frac{M}{r^{n-1}} \right)^3 \left[ 2(n-2)^3(n-3)^3(g \circ g) + 16(r^2 \Omega \circ r^2 \Omega) - 4(n-3)^3(g \circ r^2 \Omega) \right], \]  
(30)

\[ (\bar{T} \cdot \bar{T})_{\mu \nu}^{\mu \nu} = -\left( \frac{M}{r^{n-1}} \right)^2 \left[ (n-1)(n-2)(n-3)^2 g_{\mu \nu} + 4(n-1)(n-3) r^2 \Omega_{\mu \nu} \right], \]  
(31)

\[ (\bar{T} \cdot \bar{T})_{\mu \nu}^{\mu \nu} = 4(n-1)(n-2)^2(n-3) \left( \frac{M}{r^{n-1}} \right)^2, \]  
(32)

\[ (\bar{T} \cdot \bar{T} \cdot \bar{T})_{\mu \nu}^{\mu \nu} = 8(n-1)(n-2)(n-3)(n-2)(n-3)(n-4) + 2 \left( \frac{M}{r^{n-1}} \right)^3. \]  
(33)

\[ \rho = \frac{M}{r^{n-1}}, \quad \ell_\mu = \frac{r_\mu}{r}, \quad A = \frac{\alpha}{r^2}. \]  
(34)

Given the last row of identities, it is clear that

\[ \text{sgn } A = \text{sgn } \alpha, \quad \text{and } A^{n-1} \rho^{-2} = \alpha^{n-1} M^{-2}. \]  
(35)

Next, we find a way to express the projector \( r^2 \Omega_{\mu \nu} \) onto the warped factor in terms of the curvature. Here we find a slight dimension dependence (as already noted in [19, section 3.3]). In dimension \( n \geq 5 \), one can find a formula that involves only products and contractions \( g \) of \( T \):

\[ (r^2 \Omega)_{\lambda \kappa} = \frac{2(n-2)^2}{(n-1)(n-4)} (\bar{T} \cdot \bar{T})_{\lambda \nu}^{\nu \kappa} + \frac{(n-2)(n-3)}{(n-1)(n-4)} g_{\lambda \kappa}. \]  
(36)

Obviously, the above formula has poles and hence fails when \( n = 4 \). On the other hand, the following slightly more complex formula works both in \( n = 4 \) as well as higher dimensions:
\[(r^2\Omega)_{\lambda\kappa} = -\frac{1}{(n-1)(n-3)}\rho \ell^2 \left( \bar{T}_{\mu\lambda\kappa} - \frac{(n-2)(n-3)}{2} \rho \left( \bar{\mathcal{g}} \otimes \bar{\mathcal{g}} \right)_{\mu\lambda\kappa} \right) \ell^\lambda \ell^\kappa. \tag{37} \]

The complexity of the second formula is due to the presence of the \(\ell_\mu\) vector, which is itself defined as the gradient of a scalar of a gradient constructed from \(T\). Thus formula (36) may be preferable in \(n \geq 5\) to (37), even if the latter also works in higher dimensions.

The following result simply identifies the invariant parameters that can be used to exhaustively label the distinct isometry classes of gST reference geometries as we have defined them earlier. Note that below we adopt the convention that the sign function satisfies \(\text{sgn } 0 = 0\).

**Proposition 5.** A gST geometry \((\mathcal{M}, \bar{\mathcal{g}})_{\alpha,\Lambda,\Lambda}\) is locally isometric at \(x \in \mathcal{M}\) to another gST geometry \((\mathcal{M}', \bar{\mathcal{g}}')_{\alpha',\Lambda',\Lambda'}\) at \(x' \in \mathcal{M}'\) iff

\[
\left(\text{sgn } \alpha \right) |\alpha|^{n-1} M^{-2}, \Lambda = \left(\text{sgn } \alpha' \right) |\alpha'|^{n-1} M'^{-2}, \Lambda'.
\]

and one of the following holds

(a) \(x\) is a generic point, \(\ell^2[\bar{\mathcal{g}}](x) \neq 0 \neq \ell^2[\bar{\mathcal{g}}'](x')\), or
(b) \(x\) is a horizon point, \(\ell_\mu[\bar{\mathcal{g}}](x) \neq 0 \neq \ell_\mu[\bar{\mathcal{g}}'](x')\) and \(\ell^2[\bar{\mathcal{g}}](x) = 0 = \ell^2[\bar{\mathcal{g}}'](x')\), or
(c) \(x\) is a horizon bifurcation point, \(\ell_\mu[\bar{\mathcal{g}}](x) = 0 = \ell_\mu[\bar{\mathcal{g}}'](x')\).

Hence, \((\mathcal{M}, \bar{\mathcal{g}})_{\alpha,\Lambda,\Lambda}\) and \((\mathcal{M}', \bar{\mathcal{g}}')_{\alpha',\Lambda',\Lambda'}\) are isometric iff

\[
\left(\text{sgn } \alpha \right) |\alpha|^{n-1} M^{-2}, \Lambda = \left(\text{sgn } \alpha' \right) |\alpha'|^{n-1} M'^{-2}, \Lambda'.
\]

In other words, the pair or real numbers

\[
\left(\text{sgn } \alpha \right) |\alpha|^{n-1} M^{-2}, \Lambda \in \mathbb{R}^2
\]

uniquely and exhaustively identifies all isometry classes among the gST geometries (definition 4). Moreover, non-isometric gST geometries are not even locally isometric.

**Proof.** Many of the arguments below are based on the fact that the existence of a local isometry linking the points \(x\) and \(x'\) forces the pairwise equality of all curvature scalars respectively evaluated at these points. A slight generalization of this idea to covariant identities involving curvature tensors immediately establishes our claims (a), (b) and (c). Also, recall that, from our definition of a reference gST geometry, the coordinate transformation \(r \mapsto -r\) always corresponds to the parameter flip \(M \mapsto -M\), independent of the parity of the dimension \(n\). Finally, we will assume that the points \(x \in \mathcal{M}\) and \(x' = \mathcal{M}'\) belong to the regions where we can introduce the \((t, r)\) and \((t', r')\) coordinates, as in (13), on the base factors. Then, simple coordinate transformations on \((t, r)\) extend to globally defined diffeomorphisms of \(\mathcal{M}\) or \(\mathcal{M}'\) by analyticity. When such coordinates are ill-defined on neighborhoods of \(x\) or \(x'\), the same logic applies, but where we need to directly apply the diffeomorphisms defined by analytic extension.

First, note that \(\Lambda = \Lambda'\) is necessary for local isometry. Relying on (23), we can obtain this constant from the Ricci scalar of the geometry, \(\bar{R}[\bar{\mathcal{g}}] = \frac{2n\Lambda}{(n-2)}\) and \(\bar{R}[\bar{\mathcal{g}}'] = \frac{2n\Lambda'}{(n-2)}\). Let us assume the equality \(\Lambda = \Lambda'\) from now on.
Next, relying on (35), note that
\[ \text{sgn} \, A[g] = \text{sgn} \, \alpha, \quad A[g]^{\mu \nu - 1} \rho[g]^{-2} = \alpha^{n-1} M'^{-2} \]  
(41)
and
\[ \text{sgn} \, A[g'] = \text{sgn} \, \alpha', \quad A[g']^{\mu \nu - 1} \rho[g']^{-2} = \alpha'^{n-1} M'^{-2}. \]  
(42)
Hence, since knowledge of \( (\text{sgn} \, \alpha)|\alpha|^{n-1} M'^{-2} \) is equivalent to the knowledge of both \( \text{sgn} \, \alpha \) and \( \alpha^{n-1} M'^{-2} \), the equality \( (\text{sgn} \, \alpha)|\alpha|^{n-1} M'^{-2} = (\text{sgn} \, \alpha')|\alpha'|^{n-1} M'^{-2} \) is also necessary for local isometry. Let us assume this equality from now on. It remains only to check that both equalities guarantee the existence of an isometry.

When \( \text{sgn} \, \alpha = 0 = \text{sgn} \, \alpha' \), apply the coordinate transformations \( (1/t, r) \mapsto |M|^{\frac{1}{t^2}} (1/t, r) \) and \( (1/t', r') \mapsto |M'|^{\frac{1}{t'^2}} (1/t', r') \), together with a possible \( r \mapsto -r \) and/or \( r' \mapsto -r' \) flip, depending on the signs of \( M \) and \( M' \), to bring the base factors to isometric form with \( M = 1 = M' \). To keep \( r \) and \( r' \) as the warping functions, these transformations must be accompanied by the conformal rescalings \( \Omega \mapsto |M|^{\frac{1}{t^2}} \Omega \) and \( \Omega' \mapsto |M'|^{\frac{1}{t'^2}} \Omega' \). However, since \( \alpha = 0 = \alpha' \) and the two warped factors are flat, these rescalings do not affect their isometry class. Hence, the two gST geometries must be isometric since they have the same warped product structure.

Now, assume that \( \alpha \neq 0 \neq \alpha' \), while necessarily \( \text{sgn} \, \alpha = (-1)^k = \text{sgn} \, \alpha' \). Then the coordinate redefinitions \( (1/t, r) \mapsto |\alpha|^\frac{1}{2} (1/t, r) \) and \( (1/t', r') \mapsto |\alpha|^\frac{1}{2} (1/t', r') \) bring them both to \( \alpha = (-1)^k = \alpha' \). Let us assume this equality from now on.

The only possible remaining difference between the parameters is that \( \text{sgn} \, M \neq \text{sgn} \, M' \), while \( |M| = |M'| \). But then, applying \( r \mapsto -r \) or \( r' \mapsto -r' \) brings about the equality \( M = M' \) and hence the desired isometry.

Clearly, the above arguments apply both to local isometries as well as to global isometries. This concludes the proof.

\[ \square \]

2.2. Birkhoff’s theorem

It is well-known that being a \( 2 + m \)-warped product solution of cosmological Einstein’s equations is highly restrictive. In particular, the geometry of the base factor is restricted to locally coincide with one of the base factors of a gST geometry, whether the warped factor is spherically symmetric or not. This rigidity result (though usually stated with the spherical symmetry assumption) is known as Birkhoff’s theorem \([1, 22, 29, 30]\). Below, we state and prove a version that is convenient for our purposes. The main reason to include a proof is to make sure that we can cover the corner cases (when \( \nabla \tau \) becomes null or even vanishes) that are often skipped in the literature.

Recall that a metric \( g_{\mu \nu} \) is called a \( \Lambda \)-vacuum when it satisfies the source-free Einstein equations with cosmological constant \( \Lambda \):
\[ \bar{R}_{\mu \nu}[g] - \frac{1}{2} \bar{R}[g] g_{\mu \nu} + \Lambda g_{\mu \nu} = 0 \iff \bar{R}_{\mu \nu}[g] - \frac{2\Lambda}{(n-2)} g_{\mu \nu} = 0. \]  
(43)

**Proposition 6** \([1, 22, 29, 30]\). Consider a pseudo-Riemannian geometry \((\mathcal{M}, \bar{g})\) of dimension \( n = 2 + m \) that locally, say at \( x \in \mathcal{M} \), has the form \((\mathcal{M}, \bar{g}) \times_\tau (S, \Omega)\) of a \( 2 + m \)-warped product, with
\[ \bar{g}_{\mu \nu} = g_{\mu \nu} + \bar{r}^2 \Omega_{\mu \nu}, \]  
(44)
where \((\mathcal{M}, g)\) is Lorentzian and \(r\) is not locally constant at \(x \in \mathcal{M}\), the projection of \(\bar{x}\) to \(\mathcal{M}\). When \(g_{\mu\nu}\) is a \(\Lambda\)-vacuum that is not locally of constant curvature at \(\bar{x}\), the metric of the base factor can be locally put into one of the following forms at \(x\):

(a) when \(\nabla_a r \neq 0\), \((\nabla r)^2 \neq 0\) at \(x\), in local \((t, r)\) coordinates,
\[
    g_{ab} = -f \, dt dt + \frac{1}{f} \, dr dr,
\]
(b) when \(\nabla_a r \neq 0\), \((\nabla r)^2 = 0\), \(r = r_H\) at \(x\), in local \((v, r)\) coordinates,
\[
    g_{ab} = -f \, dv dv + 2 \, dv (dr),
\]
(c) when \(\nabla_a r = 0\), \(r = r_H\) at \(x\), in local \((U, V)\) coordinates,
\[
    r = r(UV) = r_H + r_H U V + O^3(U, V), \quad \text{with} \quad z r'(z) = \frac{f(r)}{f'(r_H)}, \quad \text{and}
\]
\[
    g_{ab} = \frac{-4 e^{-\Lambda}}{f'(r_H)^2 (1 - r/r_H)} \, dv dv, \quad \text{with} \quad h(r) = \int_{r_H}^r ds \left( \frac{f'(r_H)}{f(s)} - \frac{1}{s - r_H} \right),
\]

where in each case
\[
    f(r) = \alpha - \frac{2M}{r^{n-2}} - \frac{2\Lambda}{(n-1)(n-2)} r^2,
\]
for some constants \(\alpha\) and \(M \neq 0\). In cases (b) and (c), \(r = r_H\) is a root of \(f(r) = 0\); in case (c) the root is always simple.

Thus, \((\mathcal{M}, g)\) is locally isometric at \(x\) to either (a) a generic point, (b) a horizon point, or (c) a horizon bifurcation point of a gST, as classified in proposition 5.

**Proof.** We address the last statement first. The transitions between the different charts in (a), (b) and (c) are effected with the help of the *tortoise coordinate*
\[
    r_* = \frac{1}{f'(r_H)} \log \left( \frac{r}{r_H} - 1 \right) + \frac{h(r)}{f'(r_H)}, \quad \text{which satisfies} \quad dr_* = \frac{dr}{f(r)},
\]
implicitly defining \(h(r)\) as in (47). The null coordinate from (b) has the form \(v = t + r_*\), while the double null coordinates from (c) have the form \(U = -e^{-f'(r_H)}(t + r_*)/2, V = e^{f'(r_H)}(t + r_*)/2\).

Direct calculation shows that the metrics \(g_{ab}\) expressed in these charts agree on overlaps. Hence, charts (b) and (c) constitute analytic extensions of the charts from (a), which when glued in a simply connected way, form the maximal analytic extension, of the gST geometry from definition 4. The correspondence between points (a), (b) and (c) from the current proposition with those from proposition 5 is obvious.

Before entering further specific arguments, we use formulas (6) and (9) to project the Einstein equation (43) onto the base factor of the warped product:
\[
    R_{ab} - (n - 2) \frac{\nabla_a \nabla_b r}{r} - \frac{2\Lambda}{(n-2)} g_{ab} = 0.
\]
Recalling that in 2 dimensions \(R_{ab} = \frac{1}{2} R g_{ab}\) the equation decomposes into its trace and traceless parts:
\[ R - (n - 2) \frac{\Box r}{r} - \frac{4A}{(n - 2)} = 0, \quad (n - 2) \frac{\nabla_a \nabla_b r}{r} - \frac{\Box r}{2r} g_{ab} = 0. \]  

(51)

Contracting the traceless part with \( \varepsilon_{ab} \) shows that \( t_0 = -\varepsilon_{ab} \rho^b \) is Killing, \( \nabla_{(\rho t_0)} = 0 \).

Suppose that \( x \) is a critical point of \( r \), that is, \( \nabla_a r(x) = 0 \), and hence also \( t_0(x) = 0 \). We will now argue that this critical point must be non-degenerate and hence isolated (see remark 2.9 in [1]). From the projected Einstein equations above, \( \nabla_a \nabla_b \rho^a \propto g_{ab} \), hence either \( \nabla_a \nabla_b r(x) = 0 \) or the critical point is non-degenerate. If indeed \( \nabla_a \nabla_b r(x) = 0 \), then the formula \( t_0 = -\varepsilon_{ab} \nabla^b \rho \) tells us that \( t_0(x) = 0 \) and \( \nabla_a t_0(x) = 0 \) as well. In turn, this implies that locally \( t_0 \equiv 0 \), and hence also \( \nabla_a r \equiv 0 \), which contradicts our hypothesis that \( r \) is not locally constant. The reason is that \( t_0 \) solves the Killing equation, which is an equation of finite type. In short, knowing \( t_0(x) \) and \( \nabla_a t_0(x) \) determines \( t_0 \) uniquely in a neighborhood of \( x \) [17, appendix B], which in this case gives \( t_0 \equiv 0 \).

Next, we address each of the possibilities stated in the theorem. The function \( f(r) \) from (48) always appears as the general solution, parametrized by constants \( \alpha \) and \( M \), of the differential equation

\[ r \left( r f' + (n - 3) f \right)' = -\frac{4A}{(n - 2)} r^2. \]  

(52)

\begin{itemize}
  \item[(a)] When \( \nabla_a r \neq 0 \) is non-null, we are free to pick orthogonal coordinates \( (t, r) \), with \( (\partial_t)^a \propto r^a \) Killing. Then, the most general ansatz for the metric is \( g_{ab} = -f(r) dt dt + 1/r h(r) dr dr \).
    The projected Einstein equations reduce to

\[ \frac{h'}{h} = \frac{f'}{f}, \quad r \left( r f' + (n - 3) f \right)' = -\frac{4A}{(n - 2)} r^2. \]  

(53)

Up to rescaling \( t \) by a constant, \( h(r) = f(r) \) and \( f(r) \) is as stipulated. The metric \( g_{ab} \) is singular only when \( \alpha = M = \Lambda = 0 \), meaning that all other values of the parameters are allowed.

\item[(b)] When \( \nabla_r r \neq 0 \) is null, we are free to pick coordinates \( (v, r) \), with \( \nabla_a v \) null everywhere, with \( (\partial_v)^a \propto r^a \) Killing. Then, the most general ansatz for the metric is \( g_{ab} = -f(r) dv dv + 2/h(r) dr dr \).
    The projected Einstein equations reduce to

\[ h' = 0, \quad r \left( r f' + (n - 3) f \right)' = -\frac{4A}{(n - 2)} r^2. \]  

(54)

Up to rescaling \( v \) by a constant, \( h(r) = 1 \) and \( f(r) \) is as stipulated. For \( \nabla_a r \) to be null at \( x \) as well, we must have \( f(r_H) = 0 \).

\item[(c)] When \( x \) is a non-degenerate critical point of \( r \), we are free to pick double null coordinates \( (U, V) \) centered at \( x \). Let \( r_H = r(x) \), which must be a non-zero constant. In 2 dimensions, double null coordinates are unique up to permutation and individual reparametrization of each coordinate. Then, the most general ansatz for the metric is

\[ g_{ab} = 2F(U, V) dU dV + O^2(U, V). \]  

(55)

Our hypotheses on \( r \) force its Taylor expansion to start, up to a constant rescaling, as

\[ r(U, V) = r_H + \frac{1}{2} L^2 (U \partial_U)^a (V \partial_V)^b + O^3(U, V) = r_H + \frac{F(0, 0)}{L} UV + O^3(U, V), \]  

(56)
for some constant \( L \neq 0 \), which will be constrained later on. The precise form of \( r = r(U, V) \) is to be determined from the equations. The traceless part of the projected Einstein equations reduces to

\[
\frac{\partial U}{F} \frac{\partial r}{F} = 0, \quad \frac{\partial U}{\xi(U)} = V \frac{F(U, V)}{L\xi(U)\eta(V)}, \quad \frac{\partial U}{\eta(V)} = U \frac{F(U, V)}{L\xi(U)\eta(V)},
\]

(57)

for some arbitrary \( \xi(U) \) and \( \eta(V) \), though with \( \xi(0) = 1 = \eta(0) \) as needed to maintain our hypotheses on the Taylor expansion of \( r \). We are free to change our ansatz by \( F(U, V) \mapsto F(U, V)\xi(U)\eta(V) \) and reparametrize the coordinates subject to \( \xi(U)\,dU \mapsto dU, \ \eta(V)\,dV \mapsto dV \), effectively setting \( \xi = 1 = \eta \). Then, two immediate integrability conditions follow:

\[
U(\partial_U r - VF) - V(\partial_V r - UF) = U\partial_U r - V\partial_V r = 0,
\]

(58)

\[
U\partial_U(U\partial_U r - UF) - V\partial_V(U\partial_V r - VF) = UV(U\partial_U F - V\partial_V F) = 0.
\]

(59)

Therefore, both \( F \) and \( r \) are constant along the flow lines of the vector field \( U\partial_U - V\partial_V \), which are the connected components of the level sets of \( UV \). Without loss of generality, we can restrict to a neighborhood where each level set of \( UV \) consists of exactly two connected components, exchanged by the flip \( (U, V) \mapsto (-U, -V) \).

At this point, we would like to conclude that \( r = r(UV) \) and \( F = F(UV) \) for some smooth functions \( r(z) \) and \( F(z) \), but this conclusion must be postponed due to the technical complication (not shared by polynomial or analytic functions) that a smooth function invariant under the flow of \( U\partial_U - V\partial_V \) takes such a form only on those open regions where the product \( UV \) may play the role of a simple coordinate (no critical points, connected level sets). For instance, \( F = F_{\{U>0\}}(UV) \) on \( U > 0 \) and \( F = F_{\{V>0\}}(UV) \) on \( V > 0 \), but \( F_{\{U>0\}}(z) \) and \( F_{\{V>0\}}(z) \) may be different smooth functions. Of course, these functions have to agree on overlapping regions, namely for \( z > 0 \), in this case. Below, we will presume that we are restricting to one of the open regions \( U > 0, U < 0, V > 0, \) or \( V < 0 \).

It turns out that it is more convenient to write everything in terms of \( r, UV = UV(r) \) and \( F = F(r) \), which is always possible to do locally, away from \( (U, V) = (0, 0) \) and subject to the above caveats. Taking advantage of the usual identity \( (UV)' = 1/r' \), our previous integrating step (57) simply gives \( F = L/(UV)' \). Thus, the remaining trace part of the projected Einstein equations reduces to

\[
r\partial_U (r\partial_U + (n - 3)) \frac{2}{L} \frac{UV}{(UV)'} = - \frac{4A}{(n - 2)} r^2 \iff \frac{UV}{(UV)'} = \frac{L}{2} f(r),
\]

(60)

with \( f(r) \) as stipulated. At \( (U, V) = 0 \), the above left-hand side evaluates to 0. Hence, \( r = r_H \) must solve \( f(r) = 0 \). If it is a multiple root, that is \( f(r) = c(r - r_H)^k + o(r - r_H)^k\) for some constants \( c \) and \( k > 1 \), then asymptotically \( UV \sim e^{D/(r - r_H)^{k+1}} \) for some constant \( D \) as \( r \to r_H \), which is not compatible with our requirement that \( UV \) be a smooth function of \( r \) at \( r = r_H \) and vice versa. Hence, \( r = r_H \) must also be a simple root of \( f(r) = 0 \), meaning that \( f'(r_H) \neq 0 \).

Thus, given that \( f(r) = 0 \) has a simple root at \( r_H \), we can rewrite the equation for UV as

\[
\frac{(UV)'}{UV} = \frac{2}{Lf'(r_H)(r - r_H)} + h'(r) \iff UV = C(r/r_H - 1)^{\nu/r_H} e^{h(r)},
\]

(61)
for some constant $C$, with smooth
\[
h(r) = \frac{2}{f'(r_H)} \int_{r_H}^r ds \left( \frac{f'(r_H)}{f(s)} - \frac{1}{s - r_H} \right).
\] (62)

For this formula to be consistent with the Taylor expansion $r(U, V) = r_H + \frac{f(0,0)}{2} UV + O^3(U, V)$, we must have $L = 2/f'(r_H)$ and $C = r_H L F(0,0)$. The final form of the solution is then
\[
UV = \frac{2r_H}{f'(r_H)F(0,0)} \left( \frac{r}{r_H} - 1 \right) e^h, \quad F = L UV = \frac{f'(r_H)F(0,0)}{2r_H} - \frac{2f e^{-h}}{f'(r_H)^2(1 - r/r_H)}. \] (63)

To bring the metric into the desired form, it remains only to choose the value of $F(0, 0) = \frac{2r_H}{f'(r_H)}$, which could be done by appropriately rescaling $U$ or $V$ by a constant. This also finally fixes the initial coefficients in Taylor expansion $r = r_H + r_H UV + O^3(U, V)$.

Finally, recall that the above discussion, determining the precise form of $UV = UV(r)$ and $F = F(r)$, applies separately in each of the open regions $U > 0$, $U < 0$, $V > 0$ or $V < 0$, though that precise form of the functions $UV(r)$ and $F(r)$ may differ from region to region. It is obvious that the only differences may be in the values of the constants $\alpha$ and $M$, which a priori may be different in these different regions. However, $UV(r)$ and $F(r)$ must agree on the intersection whenever two of these regions overlap (e.g. $U > 0$ and $V > 0$), and this is only possible if the values of $\alpha$ and $M$ agree between the overlapping regions. Considering all possible overlaps, the values of $\alpha$ and $M$ must then agree in all these regions. In other words, the formulas in (63) actually hold on a whole neighborhood of $(U, V) = (0, 0)$ without any more reservations (the extension to the origin is by continuity). This concludes the proof.

\[\Box\]

3. Characterization

In this section, we state and prove our main result on the IDEAL characterization of local isometry classes (definition 1) of generalized Schwarzschild–Tangherlini (gST) geometries (definition 4). The result comes in two versions, one valid for any dimension $n \geq 5$ (theorem 7), and the other valid for $n = 4$ as well as higher dimensions (theorem 8). The only difference between them is the covariant formula for extracting the idempotent projector $\bar{\Omega}_{\mu \nu}$ from the curvature. In higher dimensions, $n \geq 5$, it can be obtained by a simpler formula than in $n = 4$. However, the more complicated formula that works in $n = 4$ also generalizes to higher dimensions. When restricted to the standard spherically symmetric, $\Lambda = 0$, $n = 4$ Schwarzschild solution, our theorem 8 provides an independent alternative IDEAL characterization compared to the one previously obtained in [7]. For other values of the dimension $n$, and the parameters $\alpha, M$ and $\Lambda$, the results of this section are new.

**Theorem 7.** Consider a Lorentzian geometry $(\mathcal{M}, \bar{g})$, with dim $\mathcal{M} = n \geq 5$. Given a constant $\Lambda$, define the following tensors and scalars from the metric and the curvature:
\[
T_{\mu \nu \lambda \kappa} := R_{\mu \nu \lambda \kappa} - \frac{\Lambda}{(n - 1)(n - 2)} (\bar{g} \odot \bar{g})_{\mu \nu \lambda \kappa}.
\] (64a)

[2] The notation in [7] is somewhat hard to follow. A transcription of the key formulas into more standard tensor notation can be found in [16].
\[\rho := \left[ \frac{(T \cdot T \cdot T)_{\mu\nu\mu\nu}}{8(n-1)(n-2)(n-3)(n-4)(n-5) + 2} \right]^{1/2}, \quad (64b)\]

\[\ell_{\mu} := -\frac{1}{\rho} \nabla_{\mu} \rho, \quad (64c)\]

\[A := \ell_{\mu} \ell^{\mu} + \frac{2\Lambda}{(n-1)(n-2)}, \quad (64d)\]

\[\Omega_{\mu\nu} := \frac{2(n-2)^2}{(n-1)(n-4)} \frac{(T \cdot T)_{\mu\nu\lambda\nu}}{(T \cdot T)_{\lambda\alpha\nu\nu} + (n-3)(g \circ T)_{\mu\nu\lambda\nu}} - \frac{1}{\rho} \left[ \frac{n-3}{2} (g \circ g)_{\mu\nu\lambda\nu} + (\Omega \circ \Omega)_{\mu\nu\lambda\nu} - (n-3)(g \circ \Omega)_{\mu\nu\lambda\nu} \right], \quad (64e)\]

\[g_{\mu\nu} := \bar{g}_{\mu\nu} - \Omega_{\mu\nu}, \quad (64f)\]

\[Z_{\mu\nu\lambda\kappa} := \bar{T}_{\mu\nu\lambda\kappa} - \rho \left[ \frac{n-2}{2} (g \circ g)_{\mu\nu\lambda\kappa} + (\Omega \circ \Omega)_{\mu\nu\lambda\kappa} - (n-3)(g \circ \Omega)_{\mu\nu\lambda\kappa} \right]. \quad (64g)\]

Then the geometry \((\bar{M}, \bar{g})\) is locally isometric at \(x \in \bar{M}\) to a gST geometry \((\bar{M}', \bar{g}'\rangle_{\alpha'} M', \Lambda')\) (definition 4) iff \(\Lambda = \Lambda'\) and the following conditions hold on some neighborhood of \(x\):

\[\rho \neq 0, \quad \ell_{\mu} \neq 0, \quad (T \cdot T)_{\lambda\alpha\nu\nu} \neq 0, \quad \Omega_{\mu\nu} \geq 0, \quad \Omega_{\mu\nu} = 0, \quad (65a)\]

\[\nabla_{[\mu} \ell_{\nu]} = 0, \quad \ell^{\alpha} \Omega_{\mu\nu} = 0, \quad (65b)\]

\[\Omega_{\mu}^{\nu} \Omega_{\nu\lambda} = \Omega_{\mu\lambda}, \quad \nabla_{\mu} \Omega_{\nu\lambda} = 2\Omega_{\mu(\nu} \ell_{\lambda)}, \quad (65c)\]

By the inequalities \(\rho \neq 0, \ell_{\mu} \neq 0\) and \((T \cdot T)_{\lambda\alpha\nu\nu} \neq 0\), we mean that these objects do not vanish at \(x \in \bar{M}\) and hence, by continuity, in some neighborhood of \(x\). By the inequality \(\Omega_{\mu\nu} \geq 0\) we mean that the quadratic form \(\Omega(\nu, \nu) = \Omega_{\mu\nu} \nu^{\mu} \nu^{\nu}\) is positive-semidefinite.

Note that, in choosing the precise form of the conditions (65), we have aimed for clarity rather than any particular kind of minimality. So, for instance, the condition \(\Omega_{\mu}^{\nu} = (n-2)\) is automatically satisfied by virtue of our definition of \(\Omega_{\mu\nu}\), the same being true for the condition \(\nabla_{[\mu} \ell_{\nu]} = 0\).

**Proof.** In the easy direction, the direct calculations from section 2.1 show that all of the identities (65) hold for any gST geometry.

In the other direction, we first note that the conditions (65b) involving \(\ell_{\mu}\) and \(\Omega_{\mu\nu}\) are precisely needed by proposition 3 to locally put the geometry into \(2 + m\)-warped product form \((\bar{M}, \bar{g}) = (M, g) \times_r (S, \Omega)\), with \(m = n - 2\) and \(\bar{g}_{\mu\nu} = g_{\mu\nu} + r^{2}\Omega_{\mu\nu}\), where \(\Omega_{\mu\nu} = r^{-2}\Omega_{\mu\nu}\) and \(\ell_{\mu} = \nabla_{\mu} \log |r|\) and \(\nabla_{[\mu} \ell_{\nu]} \neq 0\). Since, \(\Omega_{\mu\nu} \geq 0\) and \(g_{\mu\nu}\) is Lorentzian, \(g_{\mu\nu}\) must be a Lorentzian metric when restricted to the base factor of the warped product.

Next, taking the trace of the \(Z_{\mu\nu\lambda\kappa}\) to be zero, we obtain precisely the \(\Lambda\)-vacuum Einstein equations, forcing the equality \(\Lambda = \Lambda'\). Appealing to Birkhoff’s theorem (proposition 6), we can conclude that the metric \(g_{ab}\) on the base of the warped product has the gST form \((M, g)_{a, a, d, a, M, 2}\) (section 2.1), for some values of the parameters \(\alpha\) and \(M\) (we have not yet drawn any conclusion about the warped factor).
cable as long as the warping function $r$ is not locally constant at $\bar{x} \in \bar{\mathcal{M}}$, without other restrictions on $\nabla_{\bar{\mu}} r(\bar{x})$, with the different possibilities listed as parts (a), (b) and (c) in proposition 6.

Two immediate consequences, again following the direct calculations in section 2.1, are the identities

$$\rho = \frac{M}{r^{n-1}} \quad \text{and} \quad A = \frac{\alpha}{r^2}. $$

Now, knowing that our geometry is locally of $2 + m$-warped product form, implies that its Riemann tensor takes the form (5), where we can replace $r_{\sigma r^r / r^2} = \ell^2$. Hence, projecting the identity $Z_{\mu \nu \lambda \kappa} = 0$ by $\bar{\Omega}_{\mu \nu}$ on each index, we obtain

$$\bar{\Omega}_{\mu \nu} \bar{\Omega}_{\lambda \kappa} \bar{\Omega}_{\rho \ell} \left( r^2 S_{\mu \nu \lambda \kappa} - \frac{A}{2} (\bar{\Omega} \odot \bar{\Omega})_{\mu \nu \lambda \kappa} \right) = 0. \quad (66)$$

Substituting in what we already know about $\bar{\Omega}_{\mu \nu}$ and $A$ into this formula, it reduces to the equality $S_{ABCD} = \frac{2}{3} (\bar{\Omega} \odot \bar{\Omega})_{ABCD}$ on the warped factor $(S, \bar{\Omega})$. In other words, the warped factor is locally of constant curvature, with sectional curvature $\alpha$. Hence, our geometry $(\bar{\mathcal{M}}, \bar{g})$ is indeed locally isometric to a $g$ST geometry $(\mathcal{M}', \bar{g}')_{\alpha, M, \Lambda}$ with parameters $\alpha, M$ and $\Lambda$ (definition 4).

Finally, referring again to the direct calculations from section 2.1, and in particular the identity (35), the last identity from (65) implies the equality

$$\left( \text{sgn } \alpha \right) |\alpha|^n - 1 M^{-2} = \left( \text{sgn } \alpha' \right) |\alpha'|^n - 1 M'^{-2}. \quad (67)$$

So, invoking proposition 5, we can at last conclude that the $g$ST geometry that we have identified locally about $\bar{x} \in \bar{\mathcal{M}}$ is indeed isometric to the desired reference $g$ST geometry, $(\mathcal{M}', \bar{g}')_{\alpha, M, \Lambda} \cong (\mathcal{M}', \bar{g}')_{\alpha', M', \Lambda'}$. $\square$

For a version of the above result that holds also when $n = 4$ we need only replace formula (64e) for $\bar{\Omega}_{\mu \nu}$ by formula (68) below (recall the discussion around equations (36) and (37) in section 2.1). Hence, the proof of the following result proceeds in an exactly analogous way.

**Theorem 8.** Consider a Lorentzian geometry $(\bar{\mathcal{M}}, \bar{g})$, with $\dim \bar{\mathcal{M}} = n \geq 4$. Then the same statement as in theorem 7 holds, with the exception that we must change the definition

$$\bar{\Omega}_{\mu \nu} := -\frac{1}{(n - 1)(n - 3) \rho \ell^2} \left( \bar{T}_{\rho \lambda \nu \kappa} - \frac{(n - 2)(n - 3)}{2} \rho (\bar{g} \odot \bar{g})_{\mu \lambda \nu \kappa} \right) \ell^\lambda \ell^\kappa. \quad (68)$$

while adding the hypotheses that $\ell^2 \neq 0$ on a neighborhood of $\bar{x} \in \bar{\mathcal{M}}$ and that $\bar{\Omega}_{\mu \nu}$ extends by continuity to a smooth tensor field on this neighborhood despite $\ell^2$ possibly vanishing at some points.

4. Discussion

We have given an IDEAL characterization (theorems 7 and 8) of each spacetime from the family of local isometry classes generalized Schwarzschild–Tangherlini ($g$ST) spacetimes (definition 4), which consists of maximally symmetric $\Lambda$-vacuum $2 + m$-warped products. In particular, this family includes the higher dimensional spherically symmetric black holes,
which generalize the 4-dimensional Schwarzschild solution and which were first investigated
by Tangherlini [34].

Our strategy, inspired by the related recent work on the characterization of cosmological
FLRW spacetimes [2], was to first identify a geometric characterization of the $2 + m$-warped
product structure in terms of a rank-$m$ symmetric projector $\hat{\Omega}$ (proposition 3) [9, 14] and
then to identify a covariant formula for $\hat{\Omega}$ in terms of the curvature of a given gST geometry.
The previously existing IDEAL characterization of the 4-dimensional Schwarzschild geom-
etry [7, 16] relied much more on an intricate algebraic classification of the Riemann tensor,
special to 4 dimensions. Unfortunately, we could not generalize the latter approach to higher
dimensions directly. On the other hand, our general strategy succeeds also in 4 dimensions
(theorem 8) and thus provides an alternative characterization of the Schwarzschild geometry,
which should be compared to that of [7]. We leave such a comparison to future work.

As discussed in [12], the linearization of the tensors of an IDEAL characterization of a given
reference geometry provides a set of gauge-invariants with respect to linearized gauge trans-
formations (diffeomorphisms) of linearized gravity on that geometry. Heuristically, this set of
invariants is a good candidate for being complete, but to be rigorous its completeness should be
proven separately. In the recent work [19], we have explicitly exhibited (by a different method)
complete sets of linear invariants for each geometry in the gST family. Relating these invariants
to the linearization of the IDEAL characterization tensors, as well as vice versa, can accomplish
two goals: give a geometric interpretation to the invariants of [19] and to prove the complete-
ess of the linearized invariants that can be obtained from the present work.

A natural direction for related future work is to extend it to an IDEAL characterization of
other black hole spacetimes. For instance, the generalization to charged spherical symmetric
black holes, the Reissner–Nordström geometry and its higher dimensional generalizations,
should be straightforward. A bigger challenge would be to generalize it to higher dimensional
rotating black holes, the Myers–Perry generalizations of the Kerr geometry, perhaps build-
ing on the existing characterization of the 4-dimensional Kerr spacetime [8]. Eventually, it
would be interesting to extend the characterization to the full Kerr–Taub–NUT–(A)dS family
[13] and higher dimensional versions.

An important future application of the above results could be an intrinsic and invariant
characterization of asymptotic flatness. Usually, asymptotic flatness is defined by an asymp-
totic condition on the metric in a special coordinate system. On the other hand, this definition
is supposed to capture the asymptotic approach to flatness or the asymptotic end of a black
hole spacetime. Thus, having an on hand an IDEAL characterization of these reference geom-
eties may give us a chance to intrinsically and invariantly define asymptotic approach to
them, providing an alternative definition of asymptotic flatness.

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