DIMENSION DEPENDENCE
OF FACTORIZATION PROBLEMS:
HARDY SPACES AND $SL^\infty$

BY

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ABSTRACT

Given $1 \leq p < \infty$, let $V_n$ denote the finite-dimensional dyadic Hardy space $H_p^n$, its dual or $SL^\infty_n$. We prove the following quantitative result: The identity operator on $V_n$ factors through any operator $T : V_N \to V_N$ which has large diagonal with respect to the Haar system, where $N$ depends linearly on $n$.

1. Introduction

Local theory of Banach spaces is concerned with the quantitative study of finite-dimensional Banach spaces and their relation to infinite-dimensional spaces and operators. To illustrate, we give the following example.

Suppose that for each $n \in \mathbb{N}$, the $n$-dimensional Banach space $X_n$ has a normalized 1-unconditional basis $e_j$, $1 \leq j \leq n$, and let $e_j^* \in X_n^*$, $1 \leq j \leq n$ denote the associated coordinate functionals.

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Question 1.1: Given \( n \in \mathbb{N} \) and \( \delta, \Gamma, \eta > 0 \), what is the smallest integer \( N = N(n, \delta, \Gamma, \eta) \), such that for any operator \( T : X_N \to X_N \) satisfying
\[
\|T\| \leq \Gamma \quad \text{and} \quad |\langle e_j^*, Te_j \rangle| \geq \delta, \quad 1 \leq j \leq N,
\]
there are operators \( E : X_n \to X_N \) and \( F : X_N \to X_n \) such that the diagram
\[
\begin{array}{c}
\xymatrix{ X_n \ar[r]^{|Id_{X_n}|} & X_n \\
X_N \ar[u]_{E} \ar[r]_{T} & X_N \ar[u]_{F} }
\end{array}
\]
is commutative?

Note that the diagonal operator \( D : X_n \to X_n \) given by \( D = \delta |Id_{X_n}| \), where \( |Id_{X_n}| \) denotes the identity operator on \( X_n \), shows that for every choice for \( E \) and \( F \) we have \( \|E\|\|F\| \geq \frac{1}{\delta} \).

Naturally, we are interested in estimates for \( N = N(n, \delta, \Gamma, \eta) \), especially in the relation between \( N \) and \( n \). For many Banach spaces, we have quantitative estimates for \( N \) (see, e.g., \([2, 3, 14, 1, 16, 15, 12, 10, 9]\)). One would hope to obtain linear estimates for \( N \) in \( n \), which, for example, has been achieved by J. Bourgain and L. Tzafriri in \([3]\) for \( X_n = \ell^p_n \), \( 1 \leq p \leq \infty \). However, for many other Banach spaces, the best known estimates are often super-exponential.

For instance, P. F. X. Müller showed that for \( X_{d_n} = H^1_n \), \( X_{d_n} = (H^1_n)^* \) (see \([14]\)) and \( X_{d_n} = L^p, 1 < p < \infty \) (see \([15]\)), where \( d_n = 2^{n+1} - 1 \), the estimate for \( N \) is a nested exponential, e.g.,
\[
N \leq 2^{8n2^8n - 1}2^{8n - 2}2^{8n - 3}2^{n - 3} \cdots
\]

Another example where \( N \) is estimated by a nested exponential in \( n \), is the one-parameter space \( X_{d_n} = SL^\infty_n \) (see \([9]\)); a similar statement is true for the bi-parameter mixed norm Hardy spaces \( H^p_n(H^q_n), 1 \leq p, q < \infty \) and their duals (see \([10]\)).

The cause for the super-exponential growth in the previous three examples can be pinpointed exactly: the use of combinatorics. In this work, we introduce a new method, which replaces these combinatorics with an entirely probabilistic approach. Consequently, we obtain for \( X_{d_n} = H^p_n \), \( X_{d_n} = (H^p_n)^* \), \( 1 \leq p < \infty \) and \( X_{d_n} = SL^\infty_n \) (see Theorem 3.1) the estimate
\[
N \leq cn, \quad \text{where} \quad c = c(\delta, \Gamma, \eta) > 0.
\]
2. Notation

The collection of dyadic intervals $\mathcal{D}$ contained in the unit interval $[0, 1)$ is given by

$$\mathcal{D} = \{(k-1)2^{-n}, k2^{-n}) : n \in \mathbb{N}_0, 1 \leq k \leq 2^n\}.$$ 

Let $|\cdot|$ denote the Lebesgue measure. For any $N \in \mathbb{N}_0$ we put

$$\mathcal{D}_N = \{I \in \mathcal{D} : |I| = 2^{-N}\} \quad \text{and} \quad \mathcal{D}_{\leq N} = \bigcup_{n=0}^{N} \mathcal{D}_n.$$ 

Given $n \in \mathbb{N}_0$ and a dyadic interval $I \in \mathcal{D}_n$, we define $I^r, I^l \in \mathcal{D}_{n+1}$ by

$$I^l \cup I^r = I \quad \text{and} \quad \inf I^l < \inf I^r.$$ 

The $L^\infty$-normalized Haar system $h_I$, $I \in \mathcal{D}$ is given by

$$h_I = \chi_{I^l} - \chi_{I^r}, \quad I \in \mathcal{D},$$ 

where $\chi_A$ denotes the characteristic function of a set $A \subset [0, 1)$.

Given $1 \leq p < \infty$, the Hardy space $H^p$ is the completion of $\text{span}\{h_I : I \in \mathcal{D}\}$ under the square function norm

$$\left\| \sum_{I \in \mathcal{D}} a_I h_I \right\|_{H^p} = \left( \int_0^1 \left( \sum_{I \in \mathcal{D}} a_I^2 h_I^2(x) \right)^{p/2} \, dx \right)^{1/p}.$$

For each $n \in \mathbb{N}_0$, we define the finite-dimensional space

$$H^p_n = \text{span}\{h_I : I \in \mathcal{D}_{\leq n}\} \subset H^p.$$

The non-separable Banach space $SL^\infty$ (see [7]) is given by

$$SL^\infty = \left\{ f \in L^2 : \int_0^1 f(x) \, dx = 0, \|f\|_{SL^\infty} < \infty \right\},$$ 

equipped with the norm

$$\left\| \sum_{I \in \mathcal{D}} a_I h_I \right\|_{SL^\infty} = \left( \left( \sum_{I \in \mathcal{D}} a_I^2 h_I^2 \right)^{1/2} \right)_{L^\infty}.$$ 

For all $n \in \mathbb{N}_0$, we define the finite-dimensional space

$$SL_n^\infty = \text{span}\{h_I : I \in \mathcal{D}_{\leq n}\} \subset SL^\infty.$$
We define the duality pairing $\langle \cdot, \cdot \rangle : SL^\infty \times H^1 \rightarrow \mathbb{R}$ by
\[(2.9) \quad \langle f, g \rangle = \int_0^1 f(x)g(x) \, dx, \quad f \in SL^\infty, \ g \in H^1.\]
An elementary computation (see, e.g., [5]) shows that
\[(2.10) \quad |\langle f, g \rangle| \leq \|f\|_{SL^\infty}\|g\|_{H^1}, \quad f \in SL^\infty, \ g \in H^1.\]

3. Main result

Let $1 \leq p < \infty$ and recall that we put $d_n = 2^{n+1} - 1$, $n \in \mathbb{N}$. Our main result (Theorem 3.1) gives a quantitative estimate for $N = N(n, \delta, \Gamma, \eta)$ in Question 1.1 for the spaces $X_{d_n} = H^p_n$, $X_{d_n} = (H^p_n)^*$ and $X_{d_n} = SL^\infty_n$.

Theorem 3.1: Let $1 \leq p < \infty$, and let $(V_k : k \in \mathbb{N})$ denote one of the following three sequences of spaces:
\[(3.1) \quad (H^p_k : k \in \mathbb{N}), \quad ((H^p_k)^* : k \in \mathbb{N}), \quad (SL^\infty_k : k \in \mathbb{N}).\]
Let $n \in \mathbb{N}$ and $\delta > 0, \Gamma, \eta > 0$. Define the integer $N = N(n, \delta, \Gamma, \eta)$ by the formula
\[(3.2) \quad N = 19(n + 2) + \lfloor 4\log_2(\Gamma/\delta) + 4\log_2(1 + \eta^{-1}) \rfloor.\]

Then for any operator $T : V_N \rightarrow V_N$ satisfying
\[(3.3) \quad \|T\| \leq \Gamma \quad \text{and} \quad \|\langle Th_K, h_K \rangle\| \geq \delta |K|, \quad K \in D_{\leq N},\]
there exist bounded linear operators $E : V_n \rightarrow V_N$ and $F : V_N \rightarrow V_n$, such that the diagram
\[(3.4) \quad \begin{array}{ccc}
V_n & \xrightarrow{\text{Id}_{V_n}} & V_n \\
\downarrow E & & \downarrow F \\
V_N & \xrightarrow{T} & V_N
\end{array} \quad \|E\| \|F\| \leq \frac{1 + \eta}{\delta}
\]
is commutative.

Firstly, we remark that the linear dependence of $N$ on $n$ amounts to a polynomial dependence of the dimensions of the respective spaces; i.e. $\dim V_N$ is a polynomial in $\dim V_n$. 
Secondly, although very similar in spirit, since the results of [3] concern operators with large diagonal with respect to the standard unit vector basis in $\ell_p^n$, the results in [3] are not applicable in the context of Theorem 3.1, which is concerned with operators having large diagonal with respect to the Haar system.

Thirdly, the novelty of Theorem 3.1 is the above formula (3.2) for $N$, specifically the linear relation between $N$ and $n$. Indeed, we point out that for the previous results

$\triangledown$ $(V_k : k \in \mathbb{N}) = (H^1_k : k \in \mathbb{N})$ and $(V_k : k \in \mathbb{N}) = ((H^1_k)^* : k \in \mathbb{N})$ in [14],

$\triangledown$ $(V_k : k \in \mathbb{N}) = (H^p_k : k \in \mathbb{N}), 1 < p < \infty$ in [15],

$\triangledown$ $(V_k : k \in \mathbb{N}) = (SL^\infty_k : k \in \mathbb{N})$ in [9],

the relation between $N$ and $n$ is super-exponential (see (1.3)). The cause for this growth is the use of combinatorics. In a first step, these combinatorial methods are used to almost diagonalize the operator $T$, and then, in a second step, probabilistic arguments are employed to preserve the large diagonal of $T$.

By contrast, our new and entirely probabilistic approach almost diagonalizes $T$ and preserves its large diagonal in a single step (see Section 4).

4. Random block bases

Given $1 \leq p < \infty$, let $V_N$ denote either $V_N = H^p_N$, $V_N = (H^p_N)^*$ or $V_N = SL^\infty_N$. In this section, we will show that every operator $T : V_N \rightarrow V_N$ is almost diagonalized by random block bases $\theta \mapsto b^{(\theta)}_I \subset V_N, I \in D_{\leq n}$.

To this end, let $\mathbb{P}$ denote the uniform measure on $\{\pm 1\}^D$, and let $\mathbb{E}$ denote the expectation with respect to the probability measures $\mathbb{P}$. Given $n, N \in \mathbb{N}$ and pairwise disjoint sets $B_I \subset D_{\leq N}, I \in D_{\leq n}$, we define the random block basis

$$b^{(\theta)}_I = \sum_{K \in B_I} \theta_K h_K, \quad I \in D_{\leq n}, \theta \in \{\pm 1\}^D. \tag{4.1}$$

Given a linear operator $T : V_N \rightarrow V_N$, we define the random variables $Y_{I, I'}, Z_I$ by putting

$$Y_{I, I'}(\theta) = \langle Tb^{(\theta)}_I, b^{(\theta)}_{I'} \rangle, \quad I, I' \in D_{\leq n}, I \neq I', \theta \in \{\pm 1\}^D, \tag{4.2a}$$

$$Z_I(\theta) = \langle Tb^{(\theta)}_I, b^{(\theta)}_I \rangle - \sum_{K \in B_I} \langle Th_K, h_K \rangle, \quad I \in D_{\leq n}, \theta \in \{\pm 1\}^D. \tag{4.2b}$$
The following Theorem 4.1 asserts that the matrix-valued random variable
\[ \theta \mapsto \langle T b^{(\theta)}_I, b^{(\theta)}_{I'} \rangle \]
is for the most part (depending on the collections \( B_I, I \in \mathcal{D}_{\leq n} \)) centered around the diagonal matrix \( \text{diag} \left( \sum_{K \in B_I} \langle T h_K, h_K \rangle \right) \).

**Theorem 4.1**: Let \( n, N \in \mathbb{N} \), and let \( B_I \subset \mathcal{D}_{\leq N}, I \in \mathcal{D}_{\leq n} \) denote non-empty collections of dyadic intervals satisfying

\[ (4.3a) \quad B_I \cap B_{I'} = \emptyset, \quad I, I' \in \mathcal{D}_{\leq n}, \quad I \neq I', \]

\[ (4.3b) \quad K \cap K' = \emptyset, \quad K, K' \in B_I, \quad K \neq K', \quad I \in \mathcal{D}_{\leq n}. \]

Define \( \alpha \) by putting

\[ (4.4) \quad \alpha = \max \{|K| : K \in B_I, \ I \in \mathcal{D}_{\leq n} \}. \]

Given \( 1 \leq p < \infty \), let \( V_N \) denote either \( V_N = H^p_N \), \( V_N = (H^p_N)^* \) or \( V_N = SL^\infty_N \). Then for any operator \( T : V_N \to V_N \), we have that

\[ (4.5) \quad \mathbb{E} Y_{I, I'} = \mathbb{E} Z_I = 0, \quad I, I' \in \mathcal{D}, \quad I \neq I', \]

and the random variables \( Y_{I, I'}, Z_I \) satisfy the estimates

\[ (4.6) \quad \mathbb{E} Y_{I, I'}^2 \leq \| T \|^2 \alpha^{1/2}, \quad \mathbb{E} Z_I^2 \leq 2\| T \|^2 \alpha^{1/2}, \]

for all \( I, I' \in \mathcal{D}, \ I \neq I' \).

Before we proceed to the proof of Theorem 4.1, we record the following elementary facts.

**Lemma 4.2**: Let \( \mathcal{B} \) be a non-empty, finite collection of pairwise disjoint dyadic intervals, and define

\[ (4.7) \quad b^{(\theta)} = \sum_{K \in \mathcal{B}} \theta_K h_K, \quad \theta \in \{ \pm 1 \}^\mathcal{D}. \]

Then for all \( 1 \leq p < \infty \), \( 1 < p' \leq \infty \) with \( \frac{1}{p} + \frac{1}{p'} = 1 \), we have

\[ (4.8) \quad \| b^{(\theta)} \|_{H^p} = \left( \bigcup_{I} |B|^{1/p} \right)^{1/p}, \quad \| b^{(\theta)} \|_{(H^p)^*} = \left( \bigcup_{I} |B|^{1/p'} \right)^{1/p'} \quad \text{and} \quad \| b^{(\theta)} \|_{SL^\infty} = 1. \]

**Proof of Lemma 4.2**. The proof is simple and straightforward, and therefore omitted. \( \blacksquare \)
Proof of Theorem 4.1. Clearly, $\mathbb{E} Y_{I,I'} = \mathbb{E} Z_I = 0$.

Note that for $K_0, K_1, K_0', K_1' \in \mathcal{D}$, we have $
abla \theta_{K_0} \theta_{K_1} \theta_{K_0'} \theta_{K_1'} = 1$ if and only if one of the following conditions is satisfied:

(K1) $K_0 = K_1 = K_0' = K_1'$;
(K2) $K_0 = K_1 \neq K_0' = K_1'$;
(K3) $K_0 = K_0' \neq K_1 = K_1'$;
(K4) $K_0 = K_1' \neq K_1 = K_0'$.

Estimates for $Y_{I,I'}$, if $V_N = H_N^p$. Note that

\begin{equation}
\mathbb{E} Y_{I,I'}^2(\theta) = \sum_{K_0, K_1, K_0', K_1' \in \mathcal{B}_I} \mathbb{E} \theta_{K_0} \theta_{K_1} \theta_{K_0'} \theta_{K_1'} \langle Th_{K_0}, h_{K_0'} \rangle \langle Th_{K_1}, h_{K_1'} \rangle.
\end{equation}

In view of (4.3) and (K1)–(K4), the cases (K1), (K3), (K4) are eliminated from the sum in (4.9). Thus, with only (K2) terms left, (4.9) reads as follows:

\begin{equation}
\mathbb{E} Y_{I,I'}^2 = \sum_{K_0, K_0' \in \mathcal{B}_I} \langle Th_{K_0}, h_{K_0'} \rangle^2.
\end{equation}

Put $a_{K_0, K_0'} = \langle Th_{K_0}, h_{K_0'} \rangle$ and note that

\begin{equation}
|a_{K_0, K_0'}| \leq \| T \| \| K_0 \|^{1/p} |K_0'|^{1/p'}.
\end{equation}

We will now estimate (4.10) in two different ways.

Firstly, we rewrite (4.10) and then use duality to obtain

\begin{align*}
\mathbb{E} Y_{I,I'}^2 &= \sum_{K_0, K_0' \in \mathcal{B}_I} \langle Th_{K_0}, \sum_{K_0' \in \mathcal{B}_I} a_{K_0, K_0'} h_{K_0'} \rangle
\leq \sum_{K_0 \in \mathcal{B}_I} \| T \| \| K_0 \|^{1/p} \sum_{K_0' \in \mathcal{B}_I} a_{K_0, K_0'} h_{K_0'} \| (H^p)^*.
\end{align*}

System (4.3), Lemma 4.2 and (4.11) give us

\begin{align*}
\mathbb{E} Y_{I,I'}^2 &\leq \sum_{K_0 \in \mathcal{B}_I} \| T \| \| K_0 \|^{1/p} \max_{K_0' \in \mathcal{B}_I} |a_{K_0, K_0'}| \| h_{K_0'} \|
\leq \sum_{K_0 \in \mathcal{B}_I} \| T \| \| K_0 \|^{2/p} \max_{K_0' \in \mathcal{B}_I} |K_0'|^{1/p'}.
\end{align*}

Applying Hölder’s inequality yields

\begin{equation}
\mathbb{E} Y_{I,I'}^2 \leq \| T \|^{2/p} \max_{K_0 \in \mathcal{B}_I, K_0' \in \mathcal{B}_I'} \max_{K_0 \in \mathcal{B}_I, K_0' \in \mathcal{B}_I'} |K_0|^{2/p-1} |K_0'|^{1/p'}.
\end{equation}
Using (4.4) gives us the estimate
\begin{equation}
\mathbb{E} Y_{I,I'}^2 \leq \|T\|^2 \alpha^{1/p},
\end{equation}
for all $1 \leq p \leq 2$.

Secondly, we rewrite (4.10) as follows:
\begin{equation}
\mathbb{E} Y_{I,I'}^2 = \sum_{K_0 \in \mathcal{B}_{I'}} \left( \sum_{K_0 \in \mathcal{B}_I} a_{K_0, K_0'} h_{K_0} h_{K_0'} \right).
\end{equation}
The analogous computation to the one above shows that
\begin{equation}
\mathbb{E} Y_{I,I'}^2 \leq \|T\|^2 \alpha^{1/p'},
\end{equation}
whenever $2 \leq p < \infty$.

Finally, combining (4.12) and (4.13) yields
\begin{equation}
\mathbb{E} Y_{I,I'}^2 \leq \|T\|^2 \alpha^{1/2}.
\end{equation}

ESTIMATES FOR $Z_I$, IF $V_N = H_p^N$.
In the following sums, the variables $K_0, K_0', K_1, K_1'$ will always be summed over the collection $\mathcal{B}_I$. Note that
\begin{equation}
\mathbb{E} Z_I^2(\theta) = \sum_{K_0 \neq K_0'} \mathbb{E} \theta_{K_0} \theta_{K_0'} \theta_{K_1} \theta_{K_1'} \langle Th_{K_0}, h_{K_0'} \rangle \langle Th_{K_1}, h_{K_1'} \rangle.
\end{equation}

In view of (4.3) and (K1)–(K4), the cases (K1) and (K3) are eliminated from the sum in (4.15).

If we restrict the sum in (4.15) to Case (K2), (4.15) reads
\begin{equation}
\mathbb{E} Z_I^2(\theta) = \sum_{K_0 \neq K_0'} \langle Th_{K_0}, h_{K_0'} \rangle^2.
\end{equation}

Note that the expressions (4.10) and (4.16) are algebraically the same, except for the conditions $I \neq I'$ in (4.10) and $I = I'$ in (4.16). Hence, we can repeat the proof for $Y_{I,I'}$, which yields
\begin{equation}
\mathbb{E} Z_I^2 \leq \|T\|^2 \alpha^{1/2}.
\end{equation}

Restricting the sum in (4.15) to Case (K4) gives us
\begin{equation}
\mathbb{E} Z_I^2(\theta) = \sum_{K_0 \neq K_1} \langle Th_{K_0}, h_{K_1} \rangle \langle Th_{K_1}, h_{K_0} \rangle.
\end{equation}

Put $a_{K_0, K_1} = \langle Th_{K_0}, h_{K_1} \rangle$ and note that
\begin{equation}
|a_{K_0, K_1}| \leq \|T\| |K_0|^{1/p} |K_1|^{1/p'}.
\end{equation}
We will now estimate (4.18) in two different ways. Firstly, rewriting (4.18) and then using duality yields

\[ \mathbb{E} Z_I^2 = \sum_{K_0} \left< T \sum_{K_1} a_{K_0,K_1} h_{K_1}, h_{K_0} \right> \leq \sum_{K_0} \|T\| \left\| \sum_{K_1} a_{K_0,K_1} h_{K_1} \right\|_{H^p} |K_0|^{1/p'}. \]

System (4.3), Lemma 4.2 and (4.19) give us

\[ \mathbb{E} Z_I^2 \leq \sum_{K_0} \|T\| \max_{K_1} |a_{K_0,K_1}| \left\| \sum_{K_1} h_{K_1} \right\|_{H^p} |K_0|^{1/p'} \]

\[ \leq \sum_{K_0} \|T\|^2 |K_0| \max_{K_1} |K_1|^{1/p'} = \|T\|^2 \max_{K_1} |K_1|^{1/p'}. \]

Using (4.4), we obtain the estimate

(4.20) \[ \mathbb{E} Z_I^2 \leq \|T\|^2 \alpha^{1/p'}. \]

Secondly, we rewrite (4.18) as follows:

\[ \mathbb{E} Z_I^2 = \sum_{K_1} \left< T h_{K_1}, \sum_{K_0} a_{K_0,K_1} h_{K_0} \right> . \]

The analogous computation to the one above shows that

(4.21) \[ \mathbb{E} Z_I^2 \leq \|T\|^2 \alpha^{1/p}. \]

Finally, combining (4.20) with (4.21) gives us

(4.22) \[ \mathbb{E} Z_I^2 \leq \|T\|^2 \alpha^{1/2} \]

in Case (K4).

Adding (4.17) and (4.22) yields

(4.23) \[ \mathbb{E} Z_I^2 \leq 2\|T\|^2 \alpha^{1/2}. \]

Estimates for \( V_N = (H^p_N)^* \) and \( V_N = SL^\infty_N \). If \( V_N = (H^p_N)^* \), we repeat the above proof, but with the roles of \( H^p_N \) and \( (H^p_N)^* \) reversed.

If \( V_N = SL^\infty_N \), we only need to repeat half of the above proof (only the parts where the inner sum is on the \( SL^\infty \) side of the duality pairing). To be more precise, we repeat the proof for estimate (4.13) for \( Y_{I,I'} \), and the proof for the estimates (4.17) (which is actually repeating the proof for \( Y_{I,I'} \), again) and (4.20) for \( Z_I \). This way, we obtain the estimates

(4.24) \[ \mathbb{E} Y_{I,I'}^2 \leq \|T\|^2 \alpha \quad \text{and} \quad \mathbb{E} Z_I^2 \leq 2\|T\|^2 \alpha. \]
5. Embeddings, projections and factorization

First, we record essential facts about embeddings and projections in $H^p$, $(H^p)^*$, $1 \leq p < \infty$ and $SL^\infty$, and then we prove the main result Theorem 3.1.

5.1. Jones’ compatibility condition. Given $B_I \subset \mathcal{D}$, $I \in \mathcal{D}$, we put

$$B_I = \bigcup B_I.$$

We say that the collections $B_I$, $I \in \mathcal{D}$ satisfy Jones’ compatibility condition (C) (see [8]; see also [13]) with constant $\kappa \geq 1$, if the following four conditions are satisfied:

(C1) For each $I \in \mathcal{D}$, the collection $B_I$ consists of finitely many pairwise disjoint dyadic intervals; moreover, $B_I \cap B_{I'} = \emptyset$, whenever $I, I' \in \mathcal{D}$, $I \neq I'$.

(C2) For every $I \in \mathcal{D}$, we have that $B_{I'} \cup B_{I'} \subset B_I$ and $B_{I'} \cap B_{II} = \emptyset$.

(C3) $\kappa^{-1}|I| \leq |B_I| \leq \kappa|I|$, for all $I \in \mathcal{D}$.

(C4) For all $I_0, I \in \mathcal{D}$ with $I_0 \subset I$ and $K \in B_I$, we have

$$\frac{|K \cap B_{I_0}|}{|K|} \geq \kappa^{-1} \frac{|B_{I_0}|}{|B_I|}.$$

Theorem 5.1: Let $B_I \subset \mathcal{D}$, $I \in \mathcal{D}$ satisfy Jones’ compatibility condition (C) with constant $\kappa = 1$. Let $\theta \in \{\pm 1\}^\mathcal{D}$ and define

$$b_I^{(\theta)} = \sum_{K \in B_I} \theta_K h_K, \quad I \in \mathcal{D}.$$

Given $1 \leq p < \infty$, let $V$ denote either $H^p$, $(H^p)^*$ or $SL^\infty$. Then the operators $B^{(\theta)}, A^{(\theta)} : V \to V$ given by

$$B^{(\theta)} f = \sum_{I \in \mathcal{D}} \frac{\langle f, h_I \rangle}{\|h_I\|^2_2} b_I^{(\theta)}$$

and

$$A^{(\theta)} f = \sum_{I \in \mathcal{D}} \frac{\langle f, b_I^{(\theta)} \rangle}{\|b_I^{(\theta)}\|^2_2} h_I$$

satisfy the estimates

$$\|B^{(\theta)} f\|_V \leq \|f\|_V, \quad f \in V,$$

$$\|A^{(\theta)} f\|_V \leq \|f\|_V, \quad f \in V.$$

Moreover, the diagram

$$\begin{array}{cc}
V & \xrightarrow{\operatorname{Id}_V} & V \\
\downarrow & & \downarrow \\
V & \xleftarrow{B^{(\theta)}} & V \\
\uparrow & & \uparrow \\
V & \xrightarrow{A^{(\theta)}} & V
\end{array}$$
is commutative and the composition $P^{(\theta)} = B^{(\theta)} A^{(\theta)}$ is the norm 1 projection $P^{(\theta)} : V \to V$ given by

$$P^{(\theta)}(f) = \sum_{I \in \mathcal{D}} \frac{\langle f, b_{I}^{(\theta)} \rangle}{\|b_{I}\|_{2}^{2}} b_{I}^{(\theta)}. \tag{5.5}$$

Consequently, the range of $B^{(\theta)}$ is complemented (by $P^{(\theta)}$), and $B^{(\theta)}$ is an isometric isomorphism onto its range.

Remark 5.2: In [4], Gamlen and Gaudet showed a similar version of Theorem 5.1 for $V = L^p$, $1 < p < \infty$. Let us point out two major aspects of their method: Firstly, they are using functions $(d_{i})_{i=1}^{\infty}$, which are not adapted to any dyadic filtration, therefore, their method is not applicable in $H^p$, $1 < p < \infty$. Secondly, condition (C4) is not part of their hypothesis. Instead, the collections $\mathcal{B}_{I}$, $I \in \mathcal{D}$ and the sets $\{b_{I}^{(\theta)} = \pm 1\}$, $I \in \mathcal{D}$ are linked, so that their projection $P$ can be viewed as a conditional expectation. Hence, $P$ is bounded in $L^1$ and their result can be extended to $L^1$.

In [6, Proposition 9.6], Johnson, Maurey, Schechtman and Tzafriri specify conditions for a block basis of the Haar system, so that the conclusion of Theorem 5.1 is true for $V = H^p$, $1 < p < \infty$. Since the proof relies on Stein’s martingale inequality, their result does not extend to $V = H^1$ or $V = (H^1)^\ast$. If Jones’ compatibility condition (C) is satisfied, the operator $B^{(\theta)}$ and the projection $P^{(\theta)}$ in Theorem 5.1 are the same as the respective operators occurring in [6, Proposition 9.6].

In [8], Jones showed Theorem 5.1 for $V = H^1$ and $V = (H^1)^\ast$. In order to achieve this, it was crucial to have condition (C4) in place.

The case $V = SL^\infty$ is proved in [11], even without requiring (C3).

In this work, we use Theorem 5.1 in a very particular case: Jones’ compatibility condition (C) is satisfied with constant $\kappa = 1$, and the collections $\mathcal{B}_{I}$, $I \in \mathcal{D}$ have a very special structure (see (5.14) and (5.15)).

5.2. Proof of the main result Theorem 3.1. For convenience, we repeat Theorem 3.1 here.

**Theorem 5.3 (Main result Theorem 3.1):** Let $1 \leq p < \infty$, and let $(V_k : k \in \mathbb{N})$ denote one of the following three sequences of spaces:

$$H^p_k : k \in \mathbb{N}, \quad ((H^p_k)^\ast : k \in \mathbb{N}), \quad (SL^\infty_k : k \in \mathbb{N}). \tag{5.6}$$
Let $n \in \mathbb{N}$ and $\delta, \Gamma, \eta > 0$. Define the integer $N = N(n, \delta, \Gamma, \eta)$ by the formula

\begin{equation}
N = 19(n + 2) + \lfloor 4 \log_2(\Gamma / \delta) + 4 \log_2(1 + \eta^{-1}) \rfloor.
\end{equation}

Then for any operator $T : V_N \to V_N$ satisfying

\begin{equation}
\|T\| \leq \Gamma \quad \text{and} \quad |\langle Th_K, h_K \rangle| \geq \delta |K|, \quad K \in \mathcal{D}_{\leq N},
\end{equation}

there exist bounded linear operators $E : V_n \to V_N$ and $F : V_N \to V_n$, such that the diagram

\begin{equation}
\begin{array}{ccc}
V_n & \xrightarrow{\text{Id}_{V_n}} & V_n \\
\downarrow E & & \downarrow F \\
V_N & \xrightarrow{T} & V_N \\
\end{array}
\end{equation}

is commutative.

**Proof.** Define the norm 1 multiplication operator $M : V_N \to V_N$ as the linear extension of

\[ h_K \mapsto \text{sign}(\langle Th_K, h_K \rangle)h_K, \quad K \in \mathcal{D}_{\leq N}, \]

and observe that by (5.8), we obtain

\[ \langle TMh_K, h_K \rangle = |\langle Th_K, h_K \rangle| \geq \delta |K|, \quad K \in \mathcal{D}_{\leq N}. \]

Thus, we can assume that

\begin{equation}
\langle Th_K, h_K \rangle \geq \delta |K|, \quad K \in \mathcal{D}_{\leq N}.
\end{equation}

Before we proceed, we define the following two constants: Let $m_0 \in \mathbb{N}_0$ be the smallest integer for which

\begin{equation}
2^{m_0} > \frac{2^6(n+2)}{\eta_0^4} \frac{\Gamma^4}{\eta \delta}, \quad \text{where} \quad \eta_0 = \frac{\eta \delta}{(1 + \eta)2^3(n+2)}.
\end{equation}

**Step 1: Overview.** The operators $E$ and $F$ will be defined in terms of a block basis $b_I^{(\theta)}$, $I \in \mathcal{D}_{\leq n}$ of the Haar system $h_K$, $K \in \mathcal{D}_{\leq N}$ having the following form:

\begin{equation}
b_I^{(\theta)} = \sum_{K \in \mathcal{B}_I} \theta_K h_K, \quad I \in \mathcal{D}_{\leq n}, \quad \theta \in \{\pm 1\}^\mathcal{D}.
\end{equation}
Our goal is to find collections $B_I \subset \mathcal{D}_{\leq N}$, $I \in \mathcal{D}_{\leq n}$ satisfying Jones’ compatibility condition (C) with constant $\kappa = 1$, and signs $\theta \in \{\pm 1\}^D$ such that

\begin{align}
(5.13a) & \quad |\langle T_I^{(\theta)}, b_I^{(\theta)} \rangle| \leq \eta_0, \quad I, I' \in \mathcal{D}_{\leq n}, \ I \neq I', \\
(5.13b) & \quad \langle T_I^{(\theta)}, b_I^{(\theta)} \rangle \geq (\delta - 2^n \eta_0)\|b_I^{(\theta)}\|_2^2, \quad I \in \mathcal{D}_{\leq n}.
\end{align}

**Step 2: Constructing the random block basis $b_I^{(\theta)}$, $I \in \mathcal{D}_{\leq n}$.** First, we will use a minimalist Gamlen–Gaudet construction to define the collections $B_I$, $I \in \mathcal{D}_{\leq n}$, and then we will rely on Theorem 4.1 to find signs $\theta \in \{\pm 1\}^D$ such that (5.13) is satisfied.

We will now inductively define the collections $B_I$, $I \in \mathcal{D}_{\leq n}$. We begin by putting,

\begin{equation}
(5.14) \quad B_{[0,1)} = \mathcal{D}_{m_0}.
\end{equation}

Let $0 \leq k \leq n - 1$ and assume that we have already constructed the collections $B_I$, $I \in \mathcal{D}_{\leq k}$. Then, we define

\begin{equation}
(5.15) \quad B_I^+ = \{K^+ : K \in B_I\} \quad \text{and} \quad B_I^- = \{K^- : K \in B_I\}, \quad I \in \mathcal{D}_k.
\end{equation}

One can easily verify that the collections, $B_I$, $I \in \mathcal{D}_{\leq n}$ satisfy Jones’ compatibility condition (C) with constant $\kappa = 1$, and that

\begin{equation}
(5.16) \quad B_I \subset \mathcal{D}_{\leq m_0 + n}, \quad I \in \mathcal{D}_{\leq n}.
\end{equation}

Next, we will use a probabilistic argument to find $\theta \in \{\pm 1\}^D$ such that (5.13) is satisfied. To this end, let us now define the off-diagonal events

\begin{equation}
(5.17a) \quad O_{I, I'} = \{\theta \in \{\pm 1\}^D : |\langle T_I^{(\theta)}, b_I^{(\theta)} \rangle| > \eta_0\}, \quad I, I' \in \mathcal{D}_{\leq n}, \ I \neq I',
\end{equation}

and the diagonal events

\begin{equation}
(5.17b) \quad D_I = \left\{\theta \in \{\pm 1\}^D : \left|\langle T_I^{(\theta)}, b_I^{(\theta)} \rangle - \sum_{K \in B_I} \langle Th_K, h_K \rangle\right| > \eta_0\right\}, \quad I \in \mathcal{D}_{\leq n}.
\end{equation}

By Theorem 4.1 and the definition of the random variables $Y_{I, I'}, Z_I$ (see (4.2)), we obtain

\begin{equation}
(5.18) \quad \mathbb{P}(O_{I, I'}) \leq \frac{\Gamma^2}{2^{m_0/2} \eta_0^2} \quad \text{and} \quad \mathbb{P}(D_I) \leq \frac{2 \Gamma^2}{2^{m_0/2} \eta_0^2}, \quad I, I' \in \mathcal{D}_{\leq n}, \ I \neq I.
\end{equation}
Using (5.18) and (5.11) gives us

\[ \mathbb{P} \left( \bigcup_{I, I' \in \mathcal{D} \leq n} O_{I, I'} \cup \bigcup_{I \in \mathcal{D} \leq n} D_I \right) \leq \sum_{I, I' \in \mathcal{D} \leq n} \mathbb{P}(O_{I, I'}) + \sum_{I \in \mathcal{D} \leq n} \mathbb{P}(D_I) \]

(5.19)

\[ \leq \frac{2^{3(n+2)} \Gamma^2}{2^{m_0/2} \eta_0^2} < 1. \]

By (5.19) and the definition of the events \( O_{I, I'}, D_I \) (see (5.17)), we can find at least one \( \theta \in \{ \pm 1 \}^\mathcal{D} \) such that

\[ |\langle Tb_I^{(\theta)}, b_I^{(\theta)} \rangle| \leq \eta_0, \quad I, I' \in \mathcal{D} \leq n, \quad I \neq I', \]

(5.20a)

\[ |\langle Tb_I^{(\theta)}, b_I^{(\theta)} \rangle - \sum_{K \in \mathcal{B}_I} \langle Th_K, h_K \rangle| \leq \eta_0, \quad I \in \mathcal{D} \leq n. \]

(5.20b)

Using (5.10), (C1), (C3) and that \( \kappa = 1 \) by (5.15), we obtain

\[ \sum_{K \in \mathcal{B}_I} \langle Th_K, h_K \rangle \geq \sum_{K \in \mathcal{B}_I} |K| = \delta \sum_{K \in \mathcal{B}_I} |I| = \delta \sum_{K \in \mathcal{B}_I} |I|, \quad I \in \mathcal{D} \leq n. \]

Combining this estimate with (5.20b) yields

\[ \langle Tb_I^{(\theta)}, b_I^{(\theta)} \rangle \geq \delta |I| - \eta_0, \quad I \in \mathcal{D} \leq n. \]

(5.21)

By Lemma 4.2, we have \( \|b_I^{(\theta)}\|_2^2 = |I| \), thus estimate (5.21) implies

\[ \langle Tb_I^{(\theta)}, b_I^{(\theta)} \rangle \geq (\delta - \eta_0 \eta_0^2) |b_I^{(\theta)}|_2^2, \quad I \in \mathcal{D} \leq n. \]

(5.22)

Together, the estimates (5.20a) and (5.22) give us (5.13), that is

\[ |\langle Tb_I^{(\theta)}, b_I^{(\theta)} \rangle| \leq \eta_0, \quad I, I' \in \mathcal{D} \leq n, \quad I \neq I', \]

(5.23a)

\[ \langle Tb_I^{(\theta)}, b_I^{(\theta)} \rangle \geq (\delta - 2^{n/2} \eta_0 |b_I^{(\theta)}|_2^2, \quad I \in \mathcal{D} \leq n. \]

(5.23b)

**Step 3: Conclusion of the Proof.** By (5.16) and (5.12) it is clear that \( b_I^{(\theta)} \in V_N, \quad I \in \mathcal{D}^n \), whenever \( N \geq m_0 + n \). By Theorem 5.1, the operators \( B^{(\theta)} : V_n \to V_N \) and \( A^{(\theta)} : V_N \to V_n \) given by

\[ B^{(\theta)} f = \sum_{I \in \mathcal{D} \leq n} \frac{\langle f, h_I \rangle}{\|h_I\|_2^2} b_I^{(\theta)}, \quad f \in V_n, \]

(5.24a)

\[ A^{(\theta)} f = \sum_{I \in \mathcal{D} \leq n} \frac{\langle f, b_I^{(\theta)} \rangle}{\|b_I^{(\theta)}\|_2^2} h_I, \quad f \in V_N \]

(5.24b)
satisfy the estimates

\[(5.25)\quad \|B^{(\theta)}\| \leq 1 \quad \text{and} \quad \|A^{(\theta)}\| \leq 1.\]

The operator \(P^{(\theta)} : V_N \to V_N\) defined by \(P^{(\theta)} = B^{(\theta)} A^{(\theta)}\) is a norm 1 projection given by

\[(5.26)\quad P^{(\theta)} f = \sum_{I \in D \leq n} \frac{\langle f, b^{(\theta)}_I \rangle}{\|b^{(\theta)}_I\|^2_2} b^{(\theta)}_I, \quad f \in V_N.\]

Now, we define the subspace \(Y\) by \(Y = P^{(\theta)}(V_N)\), and note that the following diagram is commutative:

\[(5.27)\quad \begin{array}{ccc}
V_n & \xrightarrow{\text{Id}_{V_n}} & V_n \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\text{Id}_Y} & Y \\
B^{(\theta)} & & A^{(\theta)}_Y \\
\end{array}\]

\[\|B^{(\theta)}\|, \|A^{(\theta)}_Y\| \leq 1.\]

Next, we define \(U^{(\theta)} : V_N \to Y\) by putting

\[(5.28)\quad U^{(\theta)} f = \sum_{I \in D \leq n} \frac{\langle f, b^{(\theta)}_I \rangle}{\langle Tb^{(\theta)}_I, b^{(\theta)}_I \rangle} b^{(\theta)}_I, \quad f \in V_N.\]

By the 1-unconditionality of the Haar system, the definition of \(P^{(\theta)}\) (see (5.26)) and the estimates (5.23), (5.25), we obtain

\[(5.29)\quad \|U^{(\theta)}\| \leq \frac{\|P^{(\theta)}\|}{\delta - \eta_0 2^n} \leq \frac{1}{\delta - \eta_0 2^n}.\]

Moreover, for all \(g = \sum_{I \in D \leq n} a_I b^{(\theta)}_I \in Y\), we have the following identity:

\[(5.30)\quad U^{(\theta)} T g - g = \sum_{I, I' \in D \leq n} a_{I'} \frac{\langle Tb^{(\theta)}_{I'}, b^{(\theta)}_I \rangle}{\langle Tb^{(\theta)}_I, b^{(\theta)}_I \rangle} b^{(\theta)}_I.\]

We put \(W_N = (H_N^P)^*\) if \(V_N = H_N^P\), \(W_N = H_N^P\) if \(V_N = (H_N^P)^*\), and \(W_N = H_N^1\) if \(V_N = SL_N^\infty\), and note that Lemma 4.2 yields

\[\|g\|_{V_N} \|b^{(\theta)}_{I'}\|_{W_N} \geq |\langle g, b^{(\theta)}_I \rangle| = |a_{I'}||I'| = |a_{I'}||b^{(\theta)}_I|_{V_N} \|b^{(\theta)}_{I'}\|_{W_N}.\]
Hence, $|a_{I'}| \leq \frac{\|g\|_{V_N}}{\|b_{I'}^{(\theta)}\|_{V_N}}$ and we obtain from (5.23) that

$$\|U^{(\theta)}Tg-g\|_{V_N} \leq \left\| \sum_{I, I' \in \mathcal{D}, I' \neq I} a_{I', I} \langle T_{b_{I'}^{(\theta)}}, b_{I}^{(\theta)} \rangle \right\|_{V_N} \leq \frac{\eta_0 2^{3(n+1)}}{\delta - \eta_0 2^n} \|g\|_{V_N}. \tag{5.31}$$

Now, let $I : Y \to V_N$ denote the operator given by $Iy = y$. By (5.11), we have that $\frac{\eta_0 2^{3(n+1)}}{\delta - \eta_0 2^n} < 1$; hence (5.31) yields

$$\|(U^{(\theta)}TI)^{-1}g\|_{V_N} \leq \frac{1}{1 - \frac{\eta_0 2^{3(n+1)}}{\delta - \eta_0 2^n}} \|g\|_{V_N}. \tag{5.32}$$

By (5.29), (5.32) and (5.11), the operator $V^{(\theta)} : V_N \to Y$ given by $V^{(\theta)} = (U^{(\theta)}TI)^{-1}U^{(\theta)}$ satisfies the estimate

$$\|V^{(\theta)}\| \leq \frac{1}{\delta - \eta_0 (2^n + 2^{3(n+1)})} \leq \frac{1 + \eta}{\delta},$$

and the following diagram is commutative:

$$\begin{array}{ccc}
Y & \xrightarrow{\text{Id}_Y} & Y \\
\downarrow_{U^{(\theta)}TI} & & \downarrow_{V^{(\theta)}} \\
Y & \xrightarrow{\|(U^{(\theta)}TI)^{-1}\|} & Y
\end{array} \quad \|I\| \|V^{(\theta)}\| \leq \frac{1 + \eta}{\delta}. \tag{5.33}
$$

Merging the diagrams (5.27) and (5.33) yields

$$\begin{array}{ccc}
V_n & \xrightarrow{I_{V_n}} & V_n \\
\downarrow_{B^{(\theta)}} & & \downarrow_{A^{(\theta)}_{Y}} \\
Y & \xrightarrow{\|(U^{(\theta)}TI)^{-1}\|} & Y \\
\downarrow_{(U^{(\theta)}TI)} & & \downarrow_{V^{(\theta)}} \\
V_n & \xrightarrow{U^{(\theta)}} & V_n
\end{array} \quad \|E\| \|F\| \leq \frac{1 + \eta}{\delta}. \tag{5.34}$$
Finally, reviewing the construction of the block basis $b^{(\theta)}_I, I \in D_{\leq n}$ (see (5.16)) and the definitions of the operators involved in diagram (5.34), $N$ must be at least $m_0 + n$; hence, considering the constants defined in (5.11) makes (5.7) an appropriate choice for $N$. ■

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References

[1] G. Blower, The Banach space $B(l^2)$ is primary, Bulletin of the London Mathematical Society 22 (1990), 176–182.
[2] J. Bourgain, On the primarity of $H^\infty$-spaces, Israel Journal of Mathematics 45 (1983), 329–336.
[3] J. Bourgain and L. Tzafriri, Invertibility of “large” submatrices with applications to the geometry of Banach spaces and harmonic analysis, Israel Journal of Mathematics 57 (1987), 137–224.
[4] J. L. B. Gamlen and R. J. Gaudet, On subsequences of the Haar system in $L_p[1, 1]$ $(1 \leq p \leq \infty)$, Israel Journal of Mathematics 15 (1973), 404–413.
[5] A. M. Garsia, Martingale Inequalities: Seminar Notes on Recent Progress, Mathematics Lecture Notes Series, W. A. Benjamin, Reading, MA–London–Amsterdam, 1973,
[6] W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri, Symmetric structures in Banach spaces, Memoirs of the American Mathematical Society 19 (1979).
[7] P. W. Jones and P. F. X. Müller, Conditioned Brownian motion and multipliers into $SL_\infty$, Geometric and Functional Analysis 14 (2004), 319–379.
[8] P. W. Jones, BMO and the Banach space approximation problem, American Journal of Mathematics 107 (1985), 853–893.
[9] R. Lechner, Direct sums of finite dimensional $SL_n^\infty$ spaces, https://arxiv.org/abs/1709.02297.
[10] R. Lechner, Factorization in mixed norm Hardy and BMO spaces, Studia Mathematica 242 (2018), 231–265.
[11] R. Lechner, Factorization in $SL^\infty$, Israel Journal of Mathematics 226 (2018), 957–991.
[12] R. Lechner and P. F. X. Müller, Localization and projections on bi-parameter BMO, Quarterly Journal of Mathematics 66 (2015), 1069–1101.
[13] P. F. X. Müller, Isomorphisms Between $H^1$ Spaces, Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne, Vol. 66, Birkhäuser, Basel, 2005.
[14] P.F. X. Müller, On projections in $H^1$ and BMO, Studia Mathematica 89 (1988), 145–158.
[15] P. F. X. Müller, Two remarks on primary spaces, Mathematical Proceedings of the Cambridge Philosophical Society 153 (2012), 505–523.
[16] H. M. Wark, A class of primary Banach spaces, Journal of Mathematical Analysis and Applications 326 (2007), 1427–1436.