Imprint of spatial curvature on inflation power spectrum

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Abstract

If the universe had a large curvature before inflation there is a deviation from the scale invariant perturbations of the inflaton at the beginning of inflation. This may have some effect on the CMB anisotropy at large angular scales. We calculate the density perturbations for both open and closed universe cases using the Bunch-Davies vacuum condition on the initial state. We use our power spectrum to calculate the temperature anisotropy spectrum and compare the results with the WMAP three year data. We find that our power spectrum gives a lower quadrupole anisotropy when $\Omega - 1 > 0$, but matches the temperature anisotropy calculated from the standard Ratra-Peebles power spectrum at large $l$. The determination of spatial curvature from temperature anisotropy data is not much affected by the different power spectra which arise from the choice of different boundary conditions for the inflaton perturbation.

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I. INTRODUCTION

The curvature of the universe goes down exponentially after the start of inflation [1]. If there is a residual curvature still present by the time the scales which are entering our horizon at present were leaving the inflationary horizon there will be deviation from the scale invariant perturbations due to non-zero curvature. The corrections to the power spectrum at horizon scales are multiplicative powers of \((1 \pm K/\beta^2)\), where the curvature \(K = (\Omega_0 - 1)(a_0 H_0)^2\) and \(\beta\) is the comoving canonical wavenumber. We calculate the primordial power spectrum for the case closed and open universe at the time of inflation. We choose the Bunch-Davies boundary condition to normalize the wave-functions. For the case of closed universe we obtain the following expression for the power spectrum

\[
P_R(\beta) = \frac{H^4}{2\pi^2 \phi^2} \left(1 + \frac{K}{\beta^2}\right)^2, \quad \frac{\beta}{\sqrt{K}} = 3, 4, 5 \cdots \text{ (for } K > 0) \tag{1}\]

and for the case of inflation in an open universe

\[
P_R(\beta) = \frac{H^4}{2\pi^2 \phi^2} \left(1 + \frac{|K|}{\beta^2}\right)^2, \quad \frac{\beta}{\sqrt{|K|}} > 1, \text{ (for } K < 0) \tag{2}\]

where \(\phi\) is the inflaton field. In the case of closed universe \(\beta\) takes discrete values in units of \(\sqrt{K} = R_c^{-1}\) \((R_c\text{ being the curvature radius})\), the modes corresponding to \(\beta/\sqrt{K} = 1, 2\) can be eliminated by gauge transformations [2] so there is a large-wavelength cutoff at \(\beta_c^{-1} = R_c/3\). This large wavelength cut-off in a closed universe has been used to explain the observed low CMB anisotropy at low multi-poles [3] and [4]. In the case of open universe only modes with \(\beta > \sqrt{|K|}\) cross the inflationary horizon.

Our result for the power spectrum in the closed and open universe cases differs from the phenomenological power spectrum [5],

\[
P_R(\beta) = \frac{H^4}{2\pi^2 \phi^2} \frac{1}{1 + \frac{K}{\beta^2}} \tag{3}\]

used in the calculation of CMB anisotropies in both the closed and open cases. Our results agree qualitatively with [3] in that the power at small \(\beta\) is suppressed in the closed universe inflation [1] and enhanced in the open universe [2].

According to inflation [1], the curvature of the present universe \(\Omega_0 - 1\) is related to the curvature at any time during inflation \(\Omega_i - 1\) as

\[
\frac{\Omega_0 - 1}{\Omega_i - 1} = \left(\frac{a_i H_i}{a_0 H_0}\right)^2. \tag{4}\]
If $a_i$ is the scale factor at the time during inflation when scales of the size of our present horizon were exiting the inflationary horizon then $a_0H_0 = a_iH_i$ and $\Omega_0 = \Omega_i$. If at the beginning of inflation $(\Omega_{\text{start}} - 1) = O(1)$ then in order to have a deviation of say one-percent from unity in the present curvature, the number of e-foldings prior to the $a_i$ must be small. Putting an upper bound on the present curvature $(\Omega_0 - 1)$ from observations also puts a lower bound on the number of extra e-foldings necessary in inflation in addition to the minimum number needed to solve the horizon problem [6].

The geometry of the universe can be determined from the CMB anisotropy from the angular size of the acoustic peak. However the constraints on the density of the universe $\Omega$ depend upon priors like the value $H_0$ and $\Omega_\Lambda$. For example the combination of WMAP [7], LSS [8] and HST [9] supernovae observations gives a constraint on the density of the universe as $(\Omega - 1) = 0.06^{+0.02}_{-0.02}$ [10], which means that the curvature at one-σ could be as large as $K/(a_0H_0)^2 = 0.08$. In the case of the closed universe the power spectrum $P_R \propto (1 + K/\beta^2)^{-2}$ at the scale of the horizon $\beta = a_0H_0$ would be suppressed by about 16% compared to the power for the flat universe.

We use our power spectrum to calculate the temperature anisotropy spectrum and compare the results with the WMAP three year data. We find that our power spectrum gives a lower quadrupole anisotropy when $\Omega - 1 > 0$, but matches the temperature anisotropy calculated from the standard Ratra-Peebles power spectrum at large $l$. We also find that using the closed universe power spectrum [42] for larger values of $\Omega_0$ the quadrupole anisotropy is suppressed more and fits the WMAP data better. This supports the idea proposed in [3] that a positive spatial curvature should suppress the power at low $l$.

II. SCALAR POWER SPECTRUM

We expand the inflaton field $\phi(x, t) \equiv \phi(t) + \delta \phi(x, t)$, where the perturbations $\delta \phi$ around the constant background $\phi(t)$ obey the minimally coupled KG equation

$$\ddot{\delta \phi} + 3\frac{\dot{a}}{a}\dot{\delta \phi} - \frac{1}{a^2}\nabla^2 \delta \phi = 0.$$  \hspace{1cm} (5)

With the separation of variables

$$\delta \phi(x, t) = \sum_k \delta \phi_k(t)Q(x, k)$$ \hspace{1cm} (6)
the KG equation can be split as

$$
\ddot{\delta \phi}_k + \frac{3}{a} \dot{a} \dot{\delta \phi}_k + \frac{k^2}{a^2} \delta \phi_k = 0 \quad (7)
$$

$$
\nabla^2 Q(x, k) = -k^2 Q(x, k) \quad (8)
$$

where $\nabla^2$ is the Laplacian operator for the spatial part. Making the transformation $d\eta = dt/a$ and $\sigma(\eta, k) = a(\eta) \delta \phi_k(\eta)$ we get the KG equation for $\sigma(\eta, k)$

$$
\sigma'' + (k^2 - \frac{a''}{a}) \sigma = 0 \quad (9)
$$

where primes denote derivatives w.r.t conformal time $\eta$.

The Friedman equations in conformal time are,

$$
\left( \frac{a'}{a} \right)^2 = \frac{8\pi G}{3} \rho a^2 - K \quad (10)
$$

$$
\left( \frac{a'}{a} \right)' = -\frac{4\pi G}{3} (\rho + 3p) a^2. \quad (11)
$$

Consider the universe with cosmological constant and curvature, then $\rho = \rho_\Lambda$ and $p = -\rho_\Lambda$ and we get using the Friedman equations,

$$
\frac{a''}{a} = \frac{16\pi G}{3} \rho_\Lambda a^2 - K \equiv 2a^2 H_\Lambda^2 - K \quad (12)
$$

where $H_\Lambda = (\frac{8\pi G}{3} \rho_\Lambda)^{1/2}$ is the Hubble parameter during pure inflation. Substituting (12) in the KG equation (9) we obtain,

$$
\sigma'' + (k^2 - 2a^2 H_\Lambda^2 + K) \sigma = 0. \quad (13)
$$

The curvature affects the wave equation of $\sigma(\eta)$ in the explicit dependence $K$ and also in the changed dynamics of $\eta$—dependence of the scale factor $a$ which is important in the early stages of inflation.

The scalar field perturbation can be written as

$$
\delta \phi(x, \eta) = \sum_k \frac{\sigma(\eta, k)}{a(\eta)} Q(x, k) \quad (14)
$$

where $\sigma(\eta)$ is the solution of equation (13) and the spatial harmonics $Q(x, k)$ are solutions of equation (8). One can separate the radial and angular modes of $Q^{lm}_\beta(r, \theta, \phi)$ as

$$
Q^{lm}_\beta(r, \theta, \phi) = \Phi^{l}_\beta(r) Y^m_1(\theta, \phi) \quad (15)
$$
where $\beta = (k^2 + K)^{1/2}$ are the eigenvalues of the radial-part of the Laplacian with eigenfunctions given by the hyperspherical Bessel functions $\Phi_l^j(r)$ which are listed in [2]. In the limit $K \to 0$, the radial eigenfunctions $\Phi_\beta(r) \to j_l(kr)$. The main properties that are needed for the calculation of the power spectrum are orthogonality

$$\int \gamma r^2 dr d\Omega Q_{\beta}^{lm}(r, \theta, \phi) Q_{\beta'}^{*lm'}(r, \theta, \phi) = \frac{1}{\beta^2} \delta_{\beta \beta'} \delta_{l l'} \delta_{m m'}$$

(16)

where $\gamma = (1 + \frac{K r^2}{4})^{-3}$ is the determinant of the spatial metric, and completeness

$$\sum_{l,m} \int \beta^2 d\beta Q_{\beta}^{lm}(r, \theta, \phi) Q_{\beta}^{*lm}(r', \theta', \phi) = \gamma^{-1} \frac{1}{r^2} \delta(r - r') \delta(\theta - \theta') \delta(\phi - \phi').$$

(17)

In case of closed universe the integral over $\beta$ is replaced by sum over the integers $\beta/\sqrt{K} = 3, 4, 5,...$ For open and flat universes $\beta$ is a real non-negative variable.

The gauge invariant perturbations are a combination of metric and inflaton perturbations. The curvature perturbations are gauge invariant and at super-horizon scales are related to the inflaton perturbations as

$$\mathcal{R}(x, \eta) = \frac{H}{\dot{\phi}} \delta \phi(x, \eta).$$

(18)

Curvature perturbations generated during inflations are frozen outside the horizon till they re-enter in the radiation or matter era. CMB anisotropies at large angles are caused by curvature perturbations in the surface of last scattering which enter in the matter era. The Sachs-Wolfe effect at large angles, relates the temperature perturbation in the direction $\hat{n}$ observed by the observer located at the point $(x_0, \eta_0)$ to the curvature perturbation at the point $(x_{LS}, \eta_{LS})$ in the LSS,

$$\frac{\delta T(x_0, \hat{n}, \eta_0)}{T} = \frac{1}{5} \mathcal{R}(x_{LS}, \eta_{LS})$$

(19)

where $x_{LS} = \hat{n}(\eta_{LS} - \eta_0)$. Using the completeness of $Q_{\beta}^{lm}(r, \theta, \phi)$ we can expand $\mathcal{R}$ as a sum-over the eigenmodes,

$$\mathcal{R}(x_{LS}, \eta_{LS}) = \sum_{l,m} \int \beta^2 d\beta \left[ \frac{H}{\dot{\phi}} \delta \phi(\eta) \right]_{\eta = \eta_*} Q_{\beta}^{lm}(x_{LS}).$$

(20)

Here we have used the fact that $\mathcal{R}$ does not change after exiting the horizon during inflation (at a conformal time which we shall denote by $\eta_*$) till it re-enters the horizon close to the LS era. Using the Sachs-Wolfe relation (19) and the mode expansion of the curvature perturbation (20) and using the orthogonality (16) of $Q_{\beta}^{lm}$, we obtain

$$\left\langle \frac{\delta T(\hat{n}_1)}{T} \frac{\delta T(\hat{n}_2)}{T} \right\rangle = \sum_l \frac{2l + 1}{4\pi} P_l(\hat{n}_1 \cdot \hat{n}_2) \int \beta^2 d\beta \frac{1}{25} |\mathcal{R}(\beta, \eta_*)|^2 |\Phi_l^j(\eta_0 - \eta_{LS})|^2.$$  

(21)
The angular spectrum $C_l$ of temperature anisotropy defined by
\[
\left\langle \frac{\delta T(\hat{n}_1) \delta T(\hat{n}_2)}{T} \right\rangle = \sum_l \frac{2l + 1}{4\pi} P_l(\hat{n}_1 \cdot \hat{n}_2) C_l
\] (22)
can be written in terms of the power spectrum of curvature perturbations by comparing (22) with (21),
\[
C_l = 4\pi \int \frac{d\beta}{\beta^3} \frac{1}{25} |P_R(\beta)|^2 |\Phi(\eta_0 - \eta_{LS})|^2
\] (23)
where the power spectrum of curvature perturbations is defined as
\[
P_R(\beta) = \frac{\beta^3}{2\pi^2} \left[ \left( \frac{H}{\dot{\phi}} \right)^2 |\delta \phi(\eta)|^2 \right]_{\eta = \eta_*}.
\] (24)

We shall now derive the power spectrum for the open and closed inflation universes.

III. CLOSED UNIVERSE INFLATION

For a closed universe, $K > 0$, from the Friedman equation (10), we get
\[
\dot{a} = H\lambda a \sqrt{1 - \frac{K}{H^2 a^2}}
\] (25)
which can be integrated to give
\[
a(t) = \frac{\sqrt{K}}{H\lambda} \cosh H\lambda t.
\] (26)

We consider as initial conditions the moment when the inflaton energy density starts to dominate over the curvature, i.e. for $t = 0$ we have $a(0) = \frac{\sqrt{K}}{H\lambda}$. The conformal time is then given by
\[
\eta(a) = \frac{-1}{\sqrt{K}} \arcsin \frac{\sqrt{K}}{aH\lambda}.
\] (27)

The conformal time spans the interval $\eta = (-\frac{\pi}{2\sqrt{K}}, 0)$ as the scale factor $a$ varies between $(\frac{\sqrt{K}}{H\lambda}, \infty)$, so for $K \neq 0$ our initial conditions are different from the standard inflation case. The dependence of the scale factor on the conformal time is obtained from (27)
\[
a(\eta) = -\frac{\sqrt{K}}{H\lambda} \frac{1}{\sin \sqrt{K}\eta}.
\] (28)

The conformal time KG equation (13) for the closed-inflationary universe is of the form
\[
\sigma''(\eta) + \left[ k^2 - K \left( 2\csc^2\sqrt{K}\eta - 1 \right) \right] \sigma(\eta) = 0.
\] (29)
This equation can be solved exactly and the solutions are

$$\sigma(\eta) = c_1 \left( -\sqrt{K} \cot \sqrt{K} \eta + i \sqrt{k^2 + K} \right) e^{i\sqrt{k^2 + K} \eta}$$

$$+ c_2 \left( -\sqrt{K} \cot \sqrt{K} \eta - i \sqrt{k^2 + K} \right) e^{-i\sqrt{k^2 + K} \eta}.$$  \hspace{1cm} (30)

The normalization constants $c_1$ and $c_2$ are determined by imposing the Bunch-Davies vacuum condition i.e the assumption that in the infinite past limit $\eta \to -\pi/(2\sqrt{K})$, $\sigma$ is a plane wave which obeys the canonical commutation relation

$$\sigma^* \sigma' - \sigma^* \sigma = i$$  \hspace{1cm} (31)

from which we obtain

$$|c_1| = \frac{1}{\sqrt{2(k^2 + K)^{3/4}}}, \quad c_2 = 0.$$  \hspace{1cm} (32)

Replacing these constants into (30) we find that in the limit $\eta \to -\pi/(2\sqrt{K})$,

$$\sigma(\eta \to -\pi/(2\sqrt{K})) \equiv \sigma_{BD} = \frac{1}{\sqrt{2\beta}} e^{i\beta \eta}$$  \hspace{1cm} (33)

where

$$\beta = (k^2 + K)^{1/2}.$$  \hspace{1cm} (34)

We will assume that the vacuum state of the universe $|0\rangle$ is the state in which there are no $\sigma_{BD}$ particles. The creation and annihilation operators are for the the BD vacuum can be written as

$$\sigma_{BD}(x, \eta) = \sum_{lm} \int \beta^2 d\beta \left( a_{\beta lm} Q_{\beta}^{lm}(x) \frac{e^{i\beta \eta}}{\sqrt{2\beta}} + a_{\beta lm}^\dagger Q_{\beta}^{lm}(x) \frac{e^{-i\beta \eta}}{\sqrt{2\beta}} \right).$$  \hspace{1cm} (35)

Using the commutation relation (31) of $\sigma_{BD}$ and the orthogonality of $Q_{\beta}^{lm}$ (16) we see that the creation and annihilation operators obey the canonical commutation relations

$$[a_{\beta lm}, a_{\beta' lm'}^\dagger] = \frac{1}{\beta^2} \delta(\beta - \beta') \delta_{ll'} \delta_{mm'}.$$  \hspace{1cm} (36)

From the foregoing discussion it is clear that $\beta$ is the radial canonical momentum. The quantum fluctuations become classical when $\beta = aH$. We shall evaluate the power spectrum at horizon crossing, as the modes do not change after exiting the inflation horizon till they re-enter the horizon in the radiation or matter era.
Substituting the constants $c_1$ and $c_2$ in the general solution (30) and going back to the $\delta\phi$, we find that

$$\langle 0|\delta\phi(\eta)^2|0\rangle = \frac{1}{a(\eta)^2} \left[ \frac{\beta^2 + K \cot^2 \sqrt{K\eta}}{2\beta^3} \right].$$  \hspace{1cm} (37)

We want to evaluate the spectrum of perturbations at horizon crossing. The horizon crossing condition is given by

$$\beta = a_* H(a_*) = a_* \left( H^2 - \frac{K}{a_*^2} \right)^{1/2}$$  \hspace{1cm} (38)

from which we obtain the values of the scale factor

$$a_* = \frac{(\beta^2 + K)^{1/2}}{H_\lambda}$$  \hspace{1cm} (39)

and of the conformal time

$$\eta_* = -\frac{1}{\sqrt{|K|}} \arctan \frac{\sqrt{K}}{\beta}$$  \hspace{1cm} (40)

at horizon crossing. The corresponding value of the Hubble parameter is

$$H(a_*) = H_\lambda \frac{\beta}{(\beta^2 + K)^{1/2}}.$$  \hspace{1cm} (41)

The power spectrum $P(\beta)$ in this case is given by

$$P_R(\beta) = \frac{H^4}{2\pi^2 \phi^2} \frac{1}{(1 + \frac{K}{\beta^2})^2}.$$  \hspace{1cm} (42)

**IV. OPEN UNIVERSE INFLATION**

Now we consider the case of an open universe with $K < 0$. From the Friedman equation (10) we have

$$\dot{a} = H_\lambda a \sqrt{1 + \frac{|K|}{H^2_\lambda a^2}}$$  \hspace{1cm} (43)

where we work with the absolute value of the curvature, taking into account that $|K| = -K$ in this case. The above expression can be integrated to give

$$a(t) = \frac{\sqrt{|K|}}{H_\lambda} \sinh H_\lambda t$$  \hspace{1cm} (44)

with initial condition $a(t = 0) = 0$. The conformal time is

$$\eta(a) = -\frac{1}{\sqrt{|K|}} \arcsinh \frac{\sqrt{|K|}}{H_\lambda a}.$$  \hspace{1cm} (45)
The conformal time spans the interval $\eta = (-\infty, 0)$ as the scale factor varies in the interval $a = (0, \infty)$. We can solve for $a(\eta)$ and obtain

$$a(\eta) = -\frac{\sqrt{|K|}}{H_\lambda} \frac{1}{\sinh \sqrt{|K|} \eta}.$$ (46)

The conformal time KG equation for the open-inflationary universe is of the form

$$\sigma''(\eta) + \left[ k^2 - |K| \left( 2 \text{cosech}^2 \sqrt{|K|} \eta + 1 \right) \right] \sigma(\eta) = 0.$$ (47)

This equation has exact solutions

$$\sigma(\eta) = c_1 \left( -\sqrt{|K|} \coth \sqrt{|K|} \eta + i \sqrt{k^2 - |K|} \right) e^{i \sqrt{k^2 - |K|} \eta}$$
$$+ c_2 \left( \sqrt{|K|} \coth \sqrt{|K|} \eta + i \sqrt{k^2 - |K|} \right) e^{-i \sqrt{k^2 - |K|} \eta}.$$ (48)

The normalization constants $c_1$ and $c_2$ are chosen so that in the infinite past $\eta \to -\infty$ limit one gets plane waves satisfying the following relation

$$\sigma^* \sigma' - \sigma^* \sigma = i$$ (49)

and obtain,

$$|c_1| = \frac{1}{\sqrt{2k(k^2 - |K|)^{1/4}}}, \quad c_2 = 0.$$ (50)

We then obtain for the magnitude of $\delta \phi_\beta(\eta) = \sigma(\eta)/a(\eta)$ the expression,

$$|\delta \phi_\beta(\eta)|^2 = \frac{1}{a(\eta)^2} \left[ \frac{\beta^2 + |K| \coth^2 \sqrt{|K|} \eta}{2(\beta^2 + |K|) \beta} \right].$$ (51)

where for the open universe,

$$\beta = (k^2 - |K|)^{1/2}.$$ (52)

The horizon crossing condition is given by

$$\beta = a_* H(a_*) = a_* \left( H_\lambda^2 + \frac{|K|}{a_*^2} \right)^{1/2}$$ (53)

and we obtain for the scale factor at Hubble crossing

$$a_* = \frac{(\beta^2 - |K|)^{1/2}}{H_\lambda}$$ (54)

and the corresponding conformal time is given by

$$\eta_* = -\frac{1}{\sqrt{|K|}} \arctanh \frac{\sqrt{|K|}}{\beta}.$$ (55)
The Hubble parameter at horizon crossing is

\[ H(a_*) = H_\Lambda \frac{\beta}{\sqrt{\beta^2 - |K|}}. \]  

(56)

We notice that in an open-universe stage of inflation, only the modes satisfying the condition \( \beta^2 > |K| \) will cross the Hubble radius.

With this, we obtain the following expression for the curvature power spectrum at Hubble crossing

\[ P_R(\beta) = \frac{H_\Lambda^4}{2\pi^2 \phi^2} \frac{1}{\left(1 - \frac{|K|}{\beta^2}\right)^2 \left(1 + \frac{|K|}{\beta^2}\right)}. \]  

(57)

V. EFFECT OF CURVATURE ON TEMPERATURE ANISOTROPY SPECTRUM

There are first principle calculations of power spectrum in a non-flat inflationary universe \([11], [12], [13], [14], [15], [16], [17]\) and \([18]\). Our results for the primordial power spectra for both the closed and open pre-inflation universe cases differ in some details from these earlier papers because of differences in the way we have implemented the initial conditions. Our results of the primordial power spectrum have been derived assuming that the vacuum state in the infinite past was the Bunch-Davies vacuum and we have evaluated the primordial power spectrum at horizon crossing of the perturbation modes.

The power spectrum obtained by Ratra and Peebles \([11]\) and Lyth and Stewart \([12]\) for the open universe case, obtained by assuming conformal boundary condition for the initial state at \( \eta \to -\infty \) is

\[ P_R(\beta) = \frac{H_\Lambda^4}{2\pi^2 \phi^2} \frac{1}{\left(1 + \frac{|K|}{\beta^2}\right)^2}. \]  

(58)

This is sometime written in the form

\[ \beta^{-3} P_R(\beta) \propto \frac{1}{\beta(\beta^2 + K)} \equiv \frac{1}{\beta(\beta^2 + 1)}. \]  

(59)

Bucher, Goldhaber and Turok \([17]\) consider an open universe with a tunneling solution and assume that the initial states annihilate the Bunch-Davies vacuum and obtain a power spectrum

\[ P_R(\beta) = \frac{H_\Lambda^4}{2\pi^2 \phi^2} \frac{1}{\left(1 + \frac{|K|}{\beta^2}\right) \coth \left(\frac{\pi \beta}{\sqrt{|K|}}\right)}. \]  

(60)

In our paper we also assume a Bunch-Davies vacuum but we consider the standard slow roll inflation model, where the expansion was dominated by the curvature term prior to inflation,
FIG. 1: Comparison of temperature anisotropy with the Ratra-Peebles power spectrum (59) and the power spectrum (42) derived assuming a Bunch-Davies vacuum. The temperature anisotropy has been calculated for a closed universe with $\Omega_0 = 1.06$

and evaluate the power spectrum at the horizon exit $a_* H(a_*) = \beta$. In our solution for the power spectrum of the open universe case (2) we have a factor of $1/(1 - |K|/\beta^2)$ instead of $\coth(\pi\beta/\sqrt{|K|})$ of (60). All three solutions for the power spectrum (2), (58) and (60) agree in the limit of small curvature $|K|/\beta^2 \to 0$. 

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The experimental bounds on the total density of the universe from a combination of WMAP, LSS and HST supernovae observations is $\Omega_0 = 1.06 \pm 0.02$ [10]. This implies that the curvature

$$K = (\Omega - 1) H_0^2 \sigma_0^2 = (0.06 \pm 0.02) H_0^2 \sigma_0^2.$$  

(61)
FIG. 3: Suppression of quadrupole temperature anisotropy with increasing spatial curvature from the power spectrum (42).

If one uses the Ratra-Peebles form of the power spectrum for the closed universe

\[ P_K(\beta) = \frac{H_0^4}{2\pi^2\beta^2 \left(1 - \frac{K}{\beta^2}\right)} \]  

we see that for perturbations of the horizon size \( \beta \simeq H_0 a_0 \), the power spectrum is suppressed by up to 8% (compared to the flat universe). On the other hand if one uses the power
spectrum (1) for the closed universe derived in this paper the suppression of large scale power can be as large as 16%.

In principle the choice of power spectrum used as an input (in numerical programmes like CAMB [19] and CMBFAST [20]) will affect the determination of cosmological parameters like $\Omega$, $H_0$, $n_s$, etc from the CMB data. In Fig. 1 we show the temperature anisotropy for a closed universe with $\Omega_0 = 1.06$ calculated using the power spectrum (42) (dashed line) and the temperature anisotropy calculated using the Ratra-Peebles power spectrum (59) (solid line). We modified the CAMB program to determine the temperature anisotropy spectrum and we have taken the best fit values of all other parameters like $n_s$, $h$, $\tau$ etc. We find that there is some difference between the two close to $l = 2$ but essentially no difference at large $l$. The difference at lower $l$ is highlighted in Fig. 2 where we have shown the same plot as in Fig. 1 but only for the low $l$ values. We see that the temperature anisotropy calculated using (42) fits the WMAP quadrupole anisotropy data slightly better than the one calculated using the Ratra-Peebles form.

In Fig. 3 we show that in the case of closed universe for larger values of $\Omega_0$ the quadrupole anisotropy is even more suppressed and fits the WMAP data better using the closed universe power spectrum (42). This supports the idea proposed in [3] that a positive spatial curvature should suppress the power at low $l$.

In Fig. 4 we show the allowed parameter space of the Hubble parameter and curvature from the WMAP data. We have used the power spectrum of this paper (42) and the Ratra-Peebles form (59) to calculate the theoretical prediction for the temperature anisotropy using CAMB. Marginalising all other parameters we plot the allowed values of $H_0$ and $\Omega_0$ at 90\%C.L. Since the theoretical prediction from the two power spectra match closely except at low $l$, the chi-square from the two differs only in the second decimal place and the allowed parameter space from the two power spectra are almost identical as shown in Fig. 4.

VI. CONCLUSION

At the beginning of inflation the curvature $\Omega - 1$ is expected to be of order one. By the time perturbations of our horizon size exit the inflation horizon, the curvature drops to $\Omega_0 - 1$ which is the present value. A non-zero observation of the curvature will tell us whether the universe prior to inflation was open or closed (even though it is almost flat
now) and put constraints on the number of extra e-foldings that must have occurred beyond
the minimum number needed to solve the horizon problem. Spatial curvature is a threshold
effect which can give us information on the pre-inflation universe from observations of the
CMB anisotropy at large angles, similar to the effect of a possible pre-inflation thermal era
[21, 22]. From the power spectrum of the closed (1) and open inflation (2) cases we see
that if $K > 0$, power is suppressed at large angles and if $K < 0$ power is enhanced at large
angles. The WMAP observation of a suppression of the quadrupole temperature anisotropy
supports inflation in a closed universe [3]. The spatial curvature is determined in the CMB
anisotropy mainly from the angular size of the acoustic horizon. The determination of the
spatial curvature from the WMAP data is not substantially affected by the choice of the
boundary condition used for the determination of the primordial power spectrum in a curved
inflationary universe.

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FIG. 4: The allowed parameter space for $\Omega_0$ and Hubble parameter $H_0$ (in units $Km/sec/Mpc$) at 90% C.L from WMAP3 data. There is no discernible difference in the parameter space when one assumes the Ratra-Peebles form of power spectrum (59) or the form (52) calculated in this paper.