Spacing statistics in two-mode random lasing

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The distribution of spacings between the lasing frequencies for an ensemble of random lasers in the two-mode regime was computed. The random lasers are implemented as open chaotic cavities filled with an active medium. The spectral properties of the passive cavities are modeled with non-Hermitian random matrices. The spacing distribution is found to depend on the relation between the gain-profile width and the mean spacing of the passive-cavity modes. The distribution displays mode repulsion and, under certain conditions, agrees with the Wigner surmise. The role of mode competition is discussed.

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I. INTRODUCTION

The term “random laser” usually refers to lasing systems based on disordered materials or substantially open wave-chaotic resonators [1]. They are distinguished from conventional lasers by leaky modes (because of the absence or openness of the resonator) and almost random field distributions (due to disorder or chaotic shape). Coherent lasing in disordered materials has been observed via emission spectra [2] and photon-count measurements [3]. The role of strong localization of light in disordered lasers remains an open theoretical question [4, 5]. In the recent work on the chaotic-laser theory [6, 7], the standard laser models [8, 9] were extended to cavities with spectrally overlapping modes and equipped with the ideas of quantum chaos [10]. The spatial structure of lasing modes was also considered [11].

A substantial aspect of the probabilistic description of random lasers is the mode statistics. For example, the mean number of lasing modes in weakly [12] and strongly [13] open resonators, as well as the variance [14], were calculated with the help of random-matrix theory [15]. The present study was partly motivated by recent experiments in a porous-GaP laser [16]. The observed distribution of spectral mode spacings could be well fitted with the Wigner surmise for the Gaussian orthogonal ensemble (GOE) of random matrices [10]. As a first step toward understanding of the multimode regime, here I propose a theory of spacing statistics for two-mode lasing. The calculation is based on the random-matrix approach of Refs. [13, 14]. The results point to the spectral repulsion of lasing modes on the scale inherited from the passive cavity. A different repulsion mechanism, requiring inhomogeneous broadening, was suggested in Ref. [17].

II. THEORY

A. Laser equations

As a model of a random laser we adopt an open resonator of irregular shape, having almost random eigenfunctions [18]. The lasing takes place due to a pumped active medium inside the resonator. Both the field and the active atoms interact with the environment. The environment variables can be eliminated from the equations of motion by inclusion of the effective damping and noise terms [8]. The number of photons in a mode $k$ is controlled via the bosonic operators $a_k$ and $\bar{a}_k$. The occupation of the two active levels of an atom $p$ will be described by the pseudospin-$1/2$ operators $\sigma_p$ and $s_p = s_p^\dagger$ satisfying the commutation relation $[\sigma_p, s_p] = \sigma_p$. The operator $\sigma_p$ transfers the electron from the upper to the lower level and $s_p$ yields the population difference between the levels. We will work in the classical approximation, replacing operators with $c$ numbers and neglecting the noise. Then $I_k \equiv |a_k|^2$ will be proportional to the mode intensity and $\sigma_p$ will characterize the atom’s polarization. The coupled classical equations of motion (equivalent to the Maxwell-Bloch equations) take the form

\begin{align}
\dot{a}_k &= - (i\omega_k + \kappa_k) a_k + g \sum_{p'} \phi^*_k (r_{p'}) \sigma_{p'}, \\
\dot{\sigma}_p &= - (i\nu + \gamma_\perp) \sigma_p + 2g \sum_{k'} \phi_{k'} (r_p) a_{k'} s_p, \\
\dot{s}_p &= \gamma_\parallel (S - s_p) - g \sum_{k'} [\phi_{k'} (r_p) a_{k'} \sigma_p^* + \text{c.c.}] .
\end{align}

Here $\omega_k - i\kappa_k$ are the complex frequencies of passive modes in the cavity, $\nu$ is the atomic transition frequency, $\gamma_\perp$ and $\gamma_\parallel$ are the polarization and inversion decay rates, and $S$ specifies the pump strength. The atom-field coupling is measured by the parameter $g \simeq d \sqrt{2\pi\nu/\hbar}$, where $d$ is the dipole moment for the atomic transition. All modes $k$ are assumed to be linearly polarized in the same direction. The field amplitude, evaluated at the atom position $r_p$, is described by the normalized wavefunction $\phi_k (r)$. As an eigenfunction of a non-Hermitian oper-
ator, it has an adjoint eigenfunction \( \tilde{\phi}_k(r) \) (Cf. Ref. [19]).

The functions \( \phi_k \) and \( \tilde{\phi}_{k'} \) are orthogonal for \( k \neq k' \) (biorthogonality). Equations (1)-(3) are written in the rotating-wave approximation. It neglects antiresonant products of the type \( a_k \sigma_p \), oscillating with a double optical frequency.

**B. Lasing frequencies**

To solve Eqs. (1)-(3), we assume that each lasing mode \( \alpha_k(t) = \alpha_k(t) \exp(-i \Omega_k t) \) oscillates with a constant frequency \( \Omega_k \) and a slowly varying amplitude \( \alpha_k(t) \). Following a standard procedure [9] we expand \( \sigma_p(t) = \sum_k \sigma_{pk}(t) \exp(-i \Omega_k t) \) and \( s_p(t) = \sum_{kk'} \{s_{pkk'}(t) \exp(-i (\Omega_k - \Omega_{k'}) t) + c.c. \} \) and neglect all other possible oscillating terms. Furthermore, only contributions up to the order of \( \alpha_k^2 \) in \( \sigma_p \) and \( \alpha_k^2 \) in \( s_p \) are kept. Under the assumption \( |\alpha_k|/\alpha_k < \gamma_{\|}, \gamma_{\perp} \), the atomic variables can be eliminated from the equations of motion. The resulting equations for \( \alpha_k \),

\[
\ddot{\alpha}_k = \alpha_k \left( p_k - q_k I_k - \sum_{k' \neq k} r_{kk'} I_{k'} \right),
\]

contain linear and nonlinear (intensity-dependent) terms. The coefficients are

\[
p_k = i (\Omega_k - \omega_k) - \kappa_k + G D_k, \tag{5}
\]

\[
q_k = 2 c B_{kkk'} D_k \mathcal{L}_{k}, \tag{6}
\]

\[
r_{kk'} = c D_k \left[ 2 B_{kkk'} \mathcal{L}_{k'} + B_{kk'k} D_{kk'} (D_{k'} + D_k) \right], \tag{7}
\]

where \( B_{kkmm} \simeq V \int dr \phi_k^*(r) \phi_m(r) \phi_m(r) \phi_n(r), \mathcal{D}_k = [1 - i (\Omega_k - \nu) / \gamma_{\perp}]^{-1}, \mathcal{D}_{kk'} = [1 - i (\Omega_k - \Omega_{k'}) / \gamma_{\perp}]^{-1}, \mathcal{L}_k = \text{Re} D_k, G = 2 g^2 S N / \gamma_{\perp} V, c = 4 g^2 S N / \gamma_{\perp}^2 V^2, N \) is the number of atoms, and \( V \) is the cavity volume. The linear absorption \( \kappa_k \) is counteracted by the linear gain \( G \mathcal{L}_k \), which has a Lorentzian profile of halfwidth \( \gamma_{\perp} \). The chaotic wavefunctions for different modes behave like independent Gaussian random functions, on scales larger than the wavelength [18]. This allows one to approximate the correlation parameters \( B_{kkk'} \simeq B_{kk'k} \simeq 1 + 2 \kappa_k \) [12, 13].

In a steady state \( \dot{\alpha}_k = 0 \), the bracketed portion of Eq. (4) must vanish for all \( k \), such that \( \alpha_k \neq 0 \). This makes a system of complex algebraic equations, from which the \( \Omega_k \)'s and \( I_k \)'s of the lasing modes can be determined. To use this procedure, however, it must be known a priori which modes have nonzero intensity. If this is not the case, one has to assume that certain modes are lasing, solve the system of equations, and then verify that all found intensities are positive and the solution is stable. In view of the complexity of the problem, I restricted the present study to the case of two-mode lasing.

If only one mode is lasing, its frequency is [9]

\[
\Omega_k^{(1)} = \frac{\omega_k + \nu \kappa_k / \gamma_{\perp}}{1 + \kappa_k / \gamma_{\perp}}. \tag{8}
\]

The positive-intensity condition requires that the linear gain for this mode exceeds the linear absorption. Hence, the actual first lasing mode, labeled \( k = 1 \), without loss of generality, is the one with the lowest threshold

\[
G_1 = \frac{\kappa_1}{\mathcal{L}_1^{(1)}} = \min_k \left( \frac{\kappa_k}{\mathcal{L}_k^{(1)}} \right), \quad \mathcal{L}_k^{(1)} = \mathcal{L}_k |_{\Omega_k = \Omega_k^{(1)}}. \tag{9}
\]

In other words, the first lasing mode \( k = 1 \) emerges at the pump level \( G = G_1 \). Unlike \( \Omega_1^{(1)} \), the frequency of the second lasing mode depends on the mode intensities. In the results for the spacing statistics below, this frequency is always taken at the threshold. If \( k \neq 1 \) is the second lasing mode, Eq. (4) provides a pair of complex equations, with an unknown \( I_1 \) and \( I_k \to 0^+ \). They yield a single nonlinear equation for the frequency \( \Omega_k^{(2)} \) at the threshold,

\[
\Omega_k^{(2)} - \Omega_k^{(1)} = R_k \frac{\text{Im} \left[ \mathcal{D}_k^\| (D_1^\| + D_k) \right]}{1 + \frac{2
\kappa_k}{\gamma_{\perp}}}, \tag{10}
\]

\[
R_k = \frac{\kappa_k - (\Omega_k - \omega_k) \Omega_k - \nu - G_1}{4 \mathcal{L}_k^{(1)} - \text{Re} \left[ \mathcal{D}_k^\| (D_1^\| + D_k) \right]} |_{\Omega_k = \Omega_k^{(1)}}. \tag{11}
\]

The stability of the two-mode solution can be checked as described in Ref. [8]. Of all stable candidates, the genuine second lasing mode, labeled \( k = 2 \), minimizes the threshold, i.e.,

\[
G_2 = \min_{\text{stable} k \neq 1} \left( G_1 + 6 \mathcal{L}_k^{(1)} R_k \right). \tag{12}
\]

The first-mode intensity at the second-mode threshold is \( I_1 = R_2 / c \). In order to estimate the importance of the nonlinear effects (mode competition), we can also define the would-be second mode neglecting the \( I_1 \) dependence in Eq. (4). This mode, labeled \( k = 2' \), has the threshold, simply, \( G_{2'} = \min_k (\kappa_k / \mathcal{L}_k^{(1)}) \) and the frequency \( \Omega_2^{(1)} \).

**C. Random-matrix model**

The working equations (8)-(12) use the eigenfrequencies \( \omega_k - i \kappa_k \) of the passive cavity as an input. In order to describe statistical characteristics of an ensemble of (nonidentical) chaotic lasers, the passive spectra can be modeled with the eigenvalues of non-Hermitian random matrices. Henceforth we will formally set \( \nu = 0 \), i.e., the frequencies will be measured with respect to the atomic frequency. Following Refs. [13–15], we associate with each cavity an \( L \times L \) matrix \( \hat{\omega} - i \gamma \). Here, the real
symmetric matrix $\tilde{\omega}$ is chosen from the GOE and $\tilde{\gamma}$ is a fixed diagonal matrix with $M \ll L$ positive and $L - M$ zero eigenvalues. The eigenvalue density of $\tilde{\omega}$ obeys the Wigner semicircle law $\rho(\omega) = \pi^{-1} \sqrt{1 - \omega^2/4}$ (in dimensionless units), which remains approximately valid for the real parts of the eigenvalues of $\omega - i\tilde{\gamma}$. The integer $M$ is interpreted as the number of spectral bands of the outside field, to which the cavity field is coupled [7]. The bands, called the coupling channels, are a consequence of the wave quantization due to the finite size of the cavity opening. They are similar to conductance channels of a quantum dot connected to ballistic leads. We will use the model of equivalent channels, when all nonzero eigenvalues of $\tilde{\gamma}$ are equal to some $\gamma > 0$. According to random-matrix theory, the coupling strength is characterized by the parameter $2\pi \rho(\omega)/\gamma$ [15]. Hence, $\gamma$ between 0 and 1 spans the whole coupling range from weak to strong. In order to exclude the effects of variable density $\rho(\omega)$, only the eigenvalues near the top of the Wigner semicircle were taken into account in the numerical simulations. That is, for each random matrix $\omega - i\tilde{\gamma}$, I used $L_0 \approx 0.36L$ eigenvalues with the smallest $|\omega_k|$, thereby allowing for a 4% variation of $\rho(\omega_k)$.

III. NUMERICAL RESULTS AND DISCUSSION

Numerical results for the mode-spacing statistics are shown in Figs. 1 and 2. The first two lasing frequencies $\Omega_1(1)$ and $\Omega_2(2)$, as well as the second frequency $\Omega_2(1)$ without mode competition, were computed for each random matrix in an ensemble. The plots show the spacing distributions $P(\Delta \Omega)$, $P'(\Delta \Omega')$ (the primed function is not a derivative), and $P(\Delta \omega)$, where $\Delta \Omega \equiv |\Omega_1(1) - \Omega_2(2)|$, $\Delta \Omega' \equiv |\Omega_1(1) - \Omega_2(1)|$, while $\Delta \omega$ are the nearest-neighbor spacings for the real parts $\omega_k$ of the passive eigenfrequencies. (The spacings in the figures are scaled with their respective mean values.) $P(\Delta \omega)$ for $\gamma \ll 1$ is close to the Wigner surmise

$$p_W(\Delta \omega) = \frac{\pi}{2} \Delta \omega \exp \left(-\frac{\pi}{4} \Delta \omega^2\right),$$

which displays eigenvalue repulsion in the GOE. For stronger coupling, the eigenvalues get spread in the complex plane, and the crossing probability for the real parts increases. According to the numerical data (Table I), the average spacings $\Delta \Omega$ and $\Delta \Omega'$ are of order of the gain-profile halfwidth $\gamma_\perp$. We will consider the three regimes of $\gamma_\perp$ being much greater, of order, or much smaller than the passive mean spacing $\Delta \omega \approx 1/L_0(0)$. The inversion decay rate $\gamma_\parallel$ is taken equal to $\gamma_\perp$.

1. $\gamma_\perp \gg \Delta \omega$ [Figs. 1(a) and 2(a)]. Although the distributions $P(\Delta \Omega)$ and $P'(\Delta \Omega')$ are quite similar, the mode competition still influences the lasing frequencies. In principle, this influence is twofold: (1) the indices $k = 2$ and $k' = 2'$ refer, in general, to two different passive modes and, (2) even if the modes are the same, the frequency $\Omega_2(2)$ is different from $\Omega_2(1)$ [Eq. (10)]. It can be deduced from Eq. (10) and was also checked numerically (not shown) that, in the present case, the second factor is insignificant. On the scale $\Delta \Omega \gtrsim \Delta \omega$, the passive distribution $P(\Delta \omega)$ is irrelevant for $P(\Delta \Omega)$, because the passive modes corresponding to $k = 1, 2$ are not the nearest neighbors. The passive modes with close $\omega_k$’s are less likely to become lasing modes, since they may lie far

![FIG. 1: Distributions of spacings between the first two lasing modes for an ensemble of 6,000 non-Hermitian random matrices of size $L = 200$ with $M = 6$ coupling channels and coupling parameter $\gamma = 0.1$. In each matrix, $L_0 = 72$ eigenvalues closest to the top of the Wigner semicircle were taken into account. The average damping $k_\perp = 3.0 \times 10^{-3}$ was determined numerically. Shown are the distributions $P(\Delta \Omega)$ (bold solid line), $P'(\Delta \Omega')$ (circles) with the mode competition neglected, $p(\Delta \omega)$ (pluses) for the passive frequencies, the Wigner surmise $p_W(\Delta \omega)$ (thin solid line), and the approximation (16) for $P'(\Delta \Omega')$ (dashed line). The spacings are computed in units of the mean values for each distribution (Table I). The gain-profile halfwidth $\gamma_\perp$ is $10^{-1}$ (a), $10^{-2}$ (b), and $10^{-3}$ (c). The inversion decay rate $\gamma_\parallel = \gamma_\perp$.]

| $\gamma_\parallel$ | $\Delta \Omega$ | $\Delta \Omega'$ | $\Delta \omega$ |
|-------------------|----------------|-----------------|----------------|
| $10^{-3}$         | 0.0049         | 0.0043          |                |
| $10^{-2}$         | 0.017          | 0.015           | 0.016          |
| $10^{-1}$         | 0.102          | 0.079           |                |
| 1.0               | 0.0020         | 0.0016          |                |
| $10^{-2}$         | 0.017          | 0.014           | 0.015          |
| $10^{-1}$         | 0.11           | 0.087           |                |

TABLE I: Mean frequency spacings $\Delta \Omega$ between the lasing modes ($\Delta \Omega'$ neglecting the mode competition) and $\Delta \omega$ between the passive-cavity modes for different $\gamma$ and $\gamma_\perp = \gamma_\parallel$. |
from each other in the complex plane, i.e., one of them can be substantially damped. This explains the falloff of $P(\Delta \Omega)$ as $\Delta \Omega \to 0$. For small $\gamma$, the repulsion of lasing frequencies is guaranteed by the Wigner repulsion of the passive modes.

An approximate expression at $\Delta \Omega' \gg \overline{\Delta \omega}$ for the distribution $P'(\Delta \Omega')$ without mode competition can be derived from a simple probabilistic model as follows. Of all passive eigenvalues $\omega_k - i\kappa_k$, the two lasing modes have the smallest $\tilde{\kappa}_k \equiv \kappa_k \left(1 + \omega_k^2/\gamma_\perp^2\right)$ [here we can approximate $\Omega^{(1)}_k \approx \omega_k$, since only small $\kappa_k$ are relevant]. Let us divide the $\omega$ axis into $N_\omega \gg 1$ intervals, each containing $L/N_\omega \gg 1$ eigenvalues. These conditions ensure that the two lasing modes are unlikely to belong to the same interval and that the $\kappa_k$ distributions in different intervals are uncorrelated. The smallest $\kappa_k$ within an interval is distributed according to $P_{\text{gap}}(\kappa) = (dn/d\kappa) e^{-n(\kappa)}$ [assuming that the number of eigenvalues between $\kappa$ and $\kappa + dk$ obeys the Poisson distribution with the average $dn(\kappa)$]. Here $n(\kappa)$, the mean number of eigenvalues with $\kappa_k < \kappa$, has a small-$\kappa$ asymptotics $\kappa^{5/2}$ [15] (valid for arbitrary $\gamma$). The distribution of the smallest $\kappa_k$ is $P_{\text{gap}}(\tilde{\kappa}) = P_{\text{gap}}(\kappa) d\kappa/d\tilde{\kappa}$, where $\omega_i$ is the central frequency of the interval. The two lasing modes lie in the intervals $i$ and $j$ with the probability

$$P_{12}^{\omega_i, \omega_j} \propto \int d\tilde{\kappa}_i d\tilde{\kappa}_j \tilde{P}_{\text{gap}}(\tilde{\kappa}_i) \tilde{P}_{\text{gap}}(\tilde{\kappa}_j) \prod_{l \neq i,j} \left[1 - \int_0^{\max(\tilde{\kappa}_i, \tilde{\kappa}_j)} d\tilde{\kappa}'' P_{\text{gap}}(\tilde{\kappa}'')\right].$$

(14)

Using an argument similar to that of Ref. [10] (Sec. 5.6), the product in the second line can be estimated as $\exp\left\{-\mathcal{O}\left[L \max(\tilde{\kappa}, \tilde{\kappa}')^{M/2}\right]\right\}$. Thus, only $\tilde{\kappa} \ll L^{-2/M}$ is relevant. The spacing distribution is

$$P'\big(\Delta \Omega'\big) \propto \int d\omega \ P_{12}^{\omega + \Delta \Omega'} \propto \int d\omega \ (\mathcal{L}_\omega \mathcal{L}_{\omega + \Delta \Omega'})^{M/2},$$

(15)

where $\mathcal{L}_\omega \equiv \left(1 + \omega^2/\gamma_\perp^2\right)^{-1}$ and the integration was extended to infinity. For $M = 6$ (see the figures),

$$P'(\Delta \Omega') \approx \frac{32}{3\pi \gamma_\perp} \frac{x^4 + 24x^2 + 336}{(x^2 + 4)^5}, \quad x \equiv \frac{\Delta \Omega'}{\gamma_\perp}.$$  

(16)

2. $\gamma_\perp \lesssim \overline{\Delta \omega}$ [Figs. 1(b,c) and 2(b,c)]. The lasing modes are likely to emerge from the passive nearest neighbors. Thus, the spacing distribution approaches the Wigner surmise (13). At small $\gamma$, the mode competition is manifested mostly via the frequency shift (10), while $k = 2$ and $k = 2'$ is usually the same mode. This frequency shift is responsible for the vanishing $P(\Delta \Omega)$ at $\Delta \Omega \to 0$, even when $P'(\Delta \Omega')$ does not vanish [Figs. 1(c) and 2(c)]. In other words, when $\Omega^{(1)}_k$ gets close to $\Omega^{(1)}_k$, $\Omega^{(2)}_k$ is repelled from these two frequencies. It is expected that the distribution $P'(\Delta \Omega')$ without the mode competition is close to $p(\Delta \omega)$. In particular, when $\gamma \ll \tilde{\kappa}_k \sim \overline{\Delta \omega}$ [Fig. 2(c)], the frequencies (8) can be approximated as $\Omega^{(1)}_k \approx \omega_k \gamma_\perp / k_k$. This makes $\mathcal{L}'_k \approx \left[1 + (\omega_k/k_k)^2\right]^{-1}$, i.e., the gain profile becomes independent of $\gamma$. When, on the other hand, $\gamma \sim \tilde{\kappa}_k \ll \overline{\Delta \omega}$ [Fig. 1(c)], the two modes with the smallest $|\Omega_k|$, not the smallest $k$, are lasing. Since $k_k$’s for these modes can be substantially different, $\Delta \Omega'$ is not proportional to $\Delta \omega$ [Eq. (8)]. This explains the difference between $P'(\Delta \Omega')$ and $p(\Delta \omega)$ in Fig. 1(c).

IV. CONCLUSION

The spacing distribution for the first two lasing modes depends on the relation between the gain-profile width and the mean spacing of the passive-cavity modes. For a very wide gain profile, the passive-spacing statistics is irrelevant, except for very small spacings. When these parameters are of the same order, the spacing distribution is approximated by the Wigner surmise. The nonlinear frequency shift due to the mode competition provides for the mode repulsion, even when the unperturbed frequencies may approach each other.

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