Some covariance inequalities for non-monotonic functions with applications to mean-variance indifference curves and bank hedging

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Cogent Mathematics (2015), 2: 991082
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Abstract: In several problems of decision-making under uncertainty, it is necessary to study the sign of the covariance between marginal utilities. All of the works that study the covariance signs are based on Chebyshev’s integral inequality. However, this inequality requires that both functions be monotonic. There are many cases, originated basically by new alternative theories, which assume that the marginal utilities of interest are non-monotonic. Thus, we cannot use Chebyshev’s result as it relies on monotonic functions. In this article, I derive some new covariance inequalities for utility functions which have non-monotonic marginal utilities. I also apply the theoretical results to two problems in economics: First, I study some properties of the indifference curve in the mean-variance space for Prospect Theory and for Markowitz utility functions. Second, I analyze the asset hedging policies of a bank that behaves as predicted by Prospect Theory.

Keywords: Chebyshev’s inequality; decision-making under uncertainty; bank hedging
1. Introduction

Many problems of choice under uncertainty involve studying the sign of a covariance. In particular, many times it is necessary to determine the sign of the covariance of two real functions $\alpha$ and $\beta$ of a random variable $X$:

$$\text{Cov}[\alpha(X), \beta(X)].$$  

The sign of (1) is deduced with the following argument: If these two functions are increasing (or both are decreasing), its sign is non-negative, while if one function is increasing and the other is decreasing the sign is non-positive (cf. e.g. Gurland, 1967; Lehmann, 1966; McEntire, 1984; Schmidt, 2014). This argument relies on Chebyshev’s integral inequality (cf. e.g. Mitrinovic & Vasic, 1970; Simonovits, 1995).

We shall see that there are some economic problems where $\alpha$ or $\beta$ is a marginal utility function. For instance, suppose that $u$ is an increasing and concave utility function, then setting $\alpha(x) = x$ and $\beta(x) = u'(x)$, Equation (1) can be written as follows

$$\text{Cov}[X, u'(X)].$$  

In this particular case, as $x$ is an increasing function and $u'$ is a decreasing function (given the concavity of $u$), the sign of (2) is non-positive. This particular case of (1) has been used in many papers in economics. For instance, Sandmo (1971) studies the sign of covariance (2) to characterize the conditions under which a competitive firm, facing an uncertain price, would produce more or less than under certainty. Similarly, Batra and Russell (1974) use this tool to analyze the effect of international price uncertainty over the social welfare of a small country with two goods. While Mossin (1968) uses this covariance sign to show that full insurance is optimal at an actuarial fair price and partial insurance is optimal if the premium includes a positive loading.

Nevertheless, Chebyshev’s integral inequality crucially depends on the monotonic behavior of both functions. Sometimes this assumption does not hold. For instance, Wagener (2006) studies the sign of an expression similar to (2), which involves a non-monotonic function, to derive some results of comparative statics under uncertainty. Besides, other types utility functions have non-monotonic marginal utilities. For instance, prospect theory (Kahneman & Tversky, 1979) proposes an $S$-shaped utility function, which means the marginal utility is non-monotonic. On the other hand, Markowitz (1952) proposes a utility function that, in its simplest case, is reverse $S$-shaped ($RS$-shaped), implying that the marginal utility is also non-monotonic. Therefore, Chebyshev’s integral inequality does not work for marginal utilities of $S$-shaped or $RS$-shaped utility functions.

This paper contributes in the following. First, I derive some new covariance inequalities involving non-monotonic functions. In particular, I study the sign of covariance (2) for $S$-shaped and $RS$-shaped utility functions. Second, I apply these new results to two problems in economics. In the first application, I study the shape of the mean-variance indifference curves for $S$-shaped and $RS$-shaped utility functions. In this application, I study whether the monotonicity of the indifference curve in the $(\mu, \sigma)$ space continues to hold for these types of utility functions. Finally, I establish the optimal strategies for hedging asset price risk within prospect theory. Specifically, I examine the optimal strategy for an enterprise that behaves according to prospect theory.

The paper continues as follows. In the next section, I give a brief view of non-monotonic marginal utility functions. In Section 3, I present some previous covariance inequalities. In Section 4, I derive the main results of the paper. In Section 5, I present the applications noted above. Section 6 provides concluding remarks.
2. Non-monotonic marginal utility functions

In this section, I give a brief introduction to different types of non-monotonic marginal utility functions. As we have seen, a utility function, \( u \), can take on various shapes: concave, convex, \( S \)-shaped, and reversed \( S \)-shaped, among others. For a discussion of different forms of the utility function, I refer to Alghalith (2010) and Gillen and Markowitz (2009).

Friedman and Savage (1948) are among the first to propose an utility function different from the classical Bernoulli utility type. Why individuals buy lotteries (reflecting risk seeking) and purchase insurance (reflecting risk aversion) at the same time.

Markowitz (1952) criticizes Friedman and Savage’s proposal and posits an alternative model that modifies the shape of the utility function. In particular, he proposes a utility function in which the domain is all the real line. This utility function starts convex, then turns concave with an inflection point at the origin, becoming convex and finishing concave. For simplicity, many authors have used a reverse-shaped type utility function with only one inflection point at the origin, (cf. e.g. Egozcue, Fuentes García, Wong, & Zitikis, 2011; Levy, 2006). Although Markowitz’s proposal is appealing, there is mixed empirical evidence to support this theory (e.g. Hershey & Schoemaker, 1980; Loubergé & Outreville, 2001; Post & Levy, 2005; Reilly, 1986).

Egozcue et al. (2011) use a power function to represent an \( RS \)-shaped utility function given by

\[
\begin{align*}
    u(x) &= \begin{cases} 
    dx^3 & \text{when } x < 0, \\
    x^3 & \text{when } x \geq 0,
    \end{cases}
\end{align*}
\]

where \( d > 0 \).

The marginal utility is decreasing for negative values of \( x \) and increasing for positive values of \( x \). This corresponds with the assumption that the decision maker is risk averse in its negative domain and is risk seeker in its positive domain.

Prospect theory is one of the most famous alternative theories to expected utility theory (Kahneman & Tversky, 1979; Tversky & Kahneman, 1992). It serves to explain a wide range of phenomena that are not explained within the traditional expected utility framework. It is used in different fields such as: economics, finance, marketing, and psychology (cf. e.g. Barberis, Huang, & Santos, 2001; Dalal, 1983; Pennings & Smidts, 2003; and references therein).

Prospect theory proposes an \( S \)-shaped utility function. A formal definition of this type of function is found in Neilson (2002):

**Definition 1**  A continuous strictly non-decreasing function \( u : R \rightarrow R \) is called \( S \)-shaped if there is a point \( x_0 \) such that the function is non-positive and convex to the left of \( x_0 \) and non-negative and concave to the right of \( x_0 \). The point \( x_0 \), that separates losses from gains, is frequently called the reference point, or the status quo. (Throughout the paper, I set \( x_0 = 0 \)).

Specifically, Kahneman and Tversky (1979) propose the following power function

\[
\begin{align*}
    u(x) &= \begin{cases} 
    -\lambda (-x)^\gamma_G & \text{when } x < 0, \\
    x^\gamma_L & \text{when } x \geq 0,
    \end{cases}
\end{align*}
\]

where \( \lambda > 0 \) is the degree of loss aversion, and \( \gamma_G \) and \( \gamma_L \in (0, 1) \) are degrees of diminishing sensitivity.

al-Nowaihi, Bradley, and Dhami (2008) prove that (4) with \( \gamma_G = \gamma_L \) is a proper \( S \)-shaped function that accounts for preference homogeneity and loss aversion. Nonetheless, this utility function has a
mathematical tractability limitation, which is that its first derivative does not exist at $x = 0$ (cf. e.g. Wakker, 2010). I shall consider this limitation in the main result.

Nevertheless, other different types of S-shaped utility functions have been proposed. For instance, De Giorgi and Hens (2006) suggest using the following S-shaped function:

$$ u(x) = \begin{cases} \lambda_L (e^{\gamma_L x} - 1) & \text{when } x < 0, \\ \lambda_G (1 - e^{-\gamma_G x}) & \text{when } x \geq 0, \end{cases} $$

with parameters $\gamma_L, \gamma_G \in [0,1]$ and $\lambda_L, \lambda_G \in (0, \infty)$.

Since we are interested in analyzing the marginal utility of an S-shaped utility function, then we can write:

(i) For Kahneman and Tversky (1979)

$$ u'(x) = \begin{cases} \lambda_L \gamma_L x^{\gamma_L - 1} & \text{when } x < 0, \\ \gamma_G x^{\gamma_G - 1} & \text{when } x \geq 0, \end{cases} $$

and (ii) for De Giorgi and Hens (2006) as follows

$$ u'(x) = \begin{cases} \lambda_L \gamma_L e^{\gamma_L x} & \text{when } x < 0, \\ \lambda_G \gamma_G e^{-\gamma_G x} & \text{when } x \geq 0. \end{cases} $$

Naturally, the marginal utility function $u'$ is non-negative because it is generally assumed that the underlying utility function $u$ is not a decreasing one. Furthermore, $u'$ in many situations is non-monotonic on the entire real line.

Nevertheless, other types of S-shaped utility functions have been little explored in the literature. For instance, Berhold (1973) and LiCalzi (2000) propose the use of cumulative distribution functions to represent S-shaped utility functions. In particular, Broll, Egozcue, Wong and Zitikis (2010) and LiCalzi (2000) use an S-shaped utility function of the form,

$$ u(x) = F(x), $$

where $F(x)$ is the cumulative distribution function of a symmetric random variable. For example, setting $u(x) = \Phi(x) - 1/2$, where $\Phi$ is the standard normal distribution function, then $u(x)$ has an S-shaped form, with a reference point at the origin.

One of the innovative features of prospect theory is loss aversion. The basic idea of loss aversion is that losses loom larger than gains of the same quantity. It can be defined in different ways (cf., Köbberling & Wakker, 2005; Neilson, 2002). I present a brief summary of the most well known definitions of loss aversion.

**Definition 2** Let $u$ be an S-shaped utility function with $u(0) = 0$. Then $u$ exhibits loss aversion if it fulfills one of the following conditions:

1. $-u(-x) \geq u(x)$ for all $x > 0$.
2. $\frac{u(y)}{y} \leq \frac{u(z)}{z}$ for all $z < 0 < y$ (weak loss aversion).
3. $u'(y) \leq u'(z)$ for all $z < 0 < y$ (strong loss aversion).
4. If $u$ is defined as in Definition (1) and $u'(0^-) > u'(0^+)$. 
5. If $u$ is defined as in Equation (4) with $\gamma_G = \gamma_L$ and $\lambda > 1$. 

Some authors also define loss aversion as $u'(x) \leq u'(-x)$ for all $x \geq 0$, which is a particular case of the third condition in Definition 2. Hereafter, I shall consider utility functions that possess this last particular characterization of loss aversion, and I will also use those utility functions with loss aversion as defined in condition (5).

Nonetheless, evidence of the presence of loss aversion has received mixed empirical support (cf. e.g. Harinck, Van Dijk, Van Beest, & Mersmann, 2007; McGraw, Larsen, Kahneman, & Schkade, 2010; Rozin & Royzman, 2001, and the references therein).

Indeed, although the idea of loss aversion is appealing, recent studies have found evidence of opposite effects. For instance, Harinck et al. (2007) and McGraw et al. (2010) find evidence that for small outcomes, loss aversion is reversed, and individuals weigh more heavily gains than losses, which is referred as reverse loss aversion. In order to make the analysis as general as possible, I will also consider the case of reversed loss aversion (i.e. gains loom larger than losses). I shall call reverse loss aversion utility functions, those with the condition $u'(x) \geq u'(-x)$ for all $x \geq 0$. Note that for power utility functions as presented in Definition 2 condition (4), reversed loss aversion implies that $\lambda < 1$.

For further features of prospect theory functionals, the interested reader is referred to Wakker (2010).

3. Literature review

In this section, I present a brief review of some well-known covariance inequalities. I begin with the celebrated Chebyshev's integral inequality, which can be stated in its integral (original) form as follows:

**Theorem 1**  Let $\alpha, \beta$ be two real functions from $[a, b] \rightarrow \mathbb{R}$ and $f(x): [a, b] \rightarrow \mathbb{R}^+$. Then we have:

1. If $\alpha$ and $\beta$ are both increasing (or both decreasing), then

$$
\int_a^b f(x) \alpha(x) \beta(x) dx \geq \int_a^b \alpha(x) f(x) dx \int_a^b \beta(x) f(x) dx.
$$

2. If one of the functions is increasing and the other is decreasing, then the inequality is reversed.

It is common to see this inequality in its probabilistic notation. This can easily be done, assuming $f(x)$ as a probability density function. Then, Chebyshev’s integral inequality in Equation (10) can be written as follows

$$
\text{Cov}[\alpha(X), \beta(X)] \geq 0.
$$

Therefore, Theorem 1, can be expressed in its most usual probabilistic form as follows.

**Theorem 2**  Let $X$ be a continuous random variable defined on $[a, b] \subset \mathbb{R}$, with well defined expectations. Consider two real functions $\alpha$ and $\beta$ then:

1. If $\alpha$ and $\beta$ are both increasing or both decreasing, then $\text{Cov}[\alpha(X), \beta(X)] \geq 0$.

2. If one function is non-decreasing and the other one is non-increasing, then we have $\text{Cov}[\alpha(X), \beta(X)] \leq 0$.

We note that if the random variable is non-degenerate and both functions are strictly monotonic then the inequalities in Theorem 2 are strict.
The following Lemma plays an important role in proving the Chebyshev’s inequality.

**Lemma 1** Let α and β be two continuous real functions and X be a continuous random variable defined on \([a, b] \subset \mathbb{R}\). Then

\[
\text{Cov} \left[ \alpha(X), \beta(X) \right] = \mathbb{E} \left[ (\alpha(X) - \alpha(c)) (\beta(X) - \beta(c)) \right],
\]

where \(c \in [a, b]\) is such that \(\alpha(c) = \mathbb{E}[\alpha(X)]\).

**Remark 1** This result follows directly from applying the Second Mean Value Theorem for integrals (cf. e.g. Sahoo & Riedel, 1998), and using the definition of the covariance (see e.g. Gurland, 1967; Schmidt, 2014). Notice that Theorem 2 can be proved invoking Lemma 1. For instance, assume both functions are increasing, then (i) If \(X > c\), then as \(\alpha\) and \(\beta\) are increasing then \((\alpha(X) - \alpha(c)) (\beta(X) - \beta(c))\) is non-negative. (ii) On the other hand, if \(X < c\), then \(\alpha(X) \leq \alpha(c)\) and \(\beta(X) \leq \beta(c)\), which yields the same result.

As I have noted earlier, there is an important limitation of the Chebyshev’s integral inequality. It requires that both functions must be monotonic. This strong assumption might be violated on several occasions. Hence, studying the sign of (1) by relaxing the monotonicity assumption is not only a problem of pure mathematical interest, but it is also of interest in applied mathematics.

Some results, relaxing the monotonicity assumption, have already been proved. For example, Steffensen (1925) proposes a non-monotonic version of Chebyshev integral inequality. However, one of the functions must hold a special condition. For future reference, since the proof is difficult to find in the literature, we present a simple proof expressed with probabilistic notation. Additionally, we extend Steffensen’s Theorem considering some other cases.

**Theorem 3** Let \(\alpha\) and \(\beta : [a, b] \to \mathbb{R}\), be differentiable real functions. Consider a random variable \(X\) with the density function \(f\), the cumulative density function \(F\) and support on \([a, b]\). Assume the expectations exist. If

\[
\mathbb{E}[\beta(X) | X \leq x] \leq (\geq) \mathbb{E}[\beta(X)] \quad \text{for all } x \in [a, b],
\]

where \(\mathbb{E}[\cdot|\cdot]\) is the conditional expectation operator, then:

1. If \(\alpha\) is non-decreasing then \(\text{Cov}[\alpha(X), \beta(X)] \geq (\leq) 0\),

2. If \(\alpha\) is non-increasing then \(\text{Cov}[\alpha(X), \beta(X)] \leq (\geq) 0\).

**Proof** We prove only the first case, the other cases can be proved with the same argument. Using the weighted average mean value theorem for integrals (Sahoo & Riedel, 1998), there is \(c \in [a, b]\) such that \(\beta(c) = \mathbb{E}[\beta(X)]\). The condition \(\mathbb{E}[\beta(X) | X \leq x] \leq \mathbb{E}[\beta(X)]\) for all \(x \in [a, b]\), is equivalent to

\[
T(x) = \mathbb{E}\left[(\beta(X) - \beta(c)) \cdot 1_{x \leq X} \right] \leq 0,
\]

where \(1_{A}, X \to \{0, 1\}\) is the indicator function, which is equal to 1 whenever \(x \in A\) and equal to 0 otherwise. First, notice that \(T(a) = T(b) = 0\). Now, we can express

\[
\text{Cov}[\alpha(X), \beta(X)] = \mathbb{E}[\alpha(X) (\beta(X) - \beta(c))] = \int_{a}^{b} \alpha(x) (\beta(x) - \beta(c)) f(x) \, dx.
\]

Integrating by parts in Equation (12), we obtain

\[
\text{Cov}[\alpha(X), \beta(X)] = \int_{a}^{b} \alpha'(x) T(x) f(x) \, dx = -\mathbb{E}[\alpha'(X) T(X)].
\]
Since \( a'(x) \geq 0 \) and \( T(x) \leq 0 \) for all \( x \in [a, b] \), we get \( \text{Cov}[a(X), \beta(X)] \geq 0 \).

We note that if \( \beta \) is increasing then inequality (11) holds. However, the opposite is not necessarily true. Nevertheless, the non-monotonic function must hold a special condition, which is expressed with a conditional expectation inequality.

Egozcue, Fuentes García, & Wong (2009), Egozcue et al. (2011) derive some new covariance inequalities relaxing the assumption of monotonicity, but their works result only for symmetric random variables. Actually, some of their results are a special case of Steffensen’s theorem, which we proceed to prove after presenting its statement.

**Theorem 4** Let \( X \) be a random variable symmetric about zero with support on \([-b, b]\) and with a density function \( f \). Consider two continuous real functions \( \alpha \) and \( \beta \). Assume that \( \beta \) is an odd function with \( \beta(x) \geq 0 \) for all \( x \geq 0 \). We have

1. if \( \alpha(x) \) is increasing, then \( \text{Cov}[\alpha(X), \beta(X)] \geq 0 \); and
2. if \( \alpha(x) \) is decreasing, then \( \text{Cov}[\alpha(X), \beta(X)] \leq 0 \).

**Proof** The first case; the others can be proved similarly. Note that because \( \beta \) is an odd function and \( X \) is symmetric, then \( E[\beta(x)] = 0 \). Let \( T(x) = E[\beta(X)X \leq x] \), then \( T(b) = T(-b) = 0 \). Now, applying the Leibniz’s rule, we get \( T'(x) = \beta(x)f(x) \). We therefore deduce that \( T'(x) \leq 0 \) for \( x \leq 0 \) and \( T'(x) \geq 0 \) for \( x \geq 0 \). Thus, we have Steffensen’s theorem condition,

\[
T(x) = E[\beta(X)X \leq x] = E[\beta(X)X \leq x] - E[\beta(X)] \leq 0.
\]

Hence, as \( \alpha \) is increasing we apply Theorem 3.

This result relaxes the monotonicity assumption of one function, but it requires symmetry with the random variable. Naturally, as one assumption is relaxed it requires an additional assumption to get consistent results.

In the next theorem, Egozcue et al. (2009) relax the monotonicity assumption of both random variables.

**Theorem 5** Let \( X \) be a random variable symmetric about zero. Consider two real functions \( \alpha(x) \) and \( \beta(x) \). Let \( \beta(x) \) be an odd function of bounded variation with \( \beta(x) \geq 0 \) for all \( x \geq 0 \). We have that

1. if \( \alpha(x) \geq \alpha(-x) \) for all \( x \geq 0 \), then \( \text{Cov}[\alpha(X), \beta(X)] \geq 0 \); and
2. if \( \alpha(x) \leq \alpha(-x) \) for all \( x \geq 0 \), then \( \text{Cov}[\alpha(X), \beta(X)] \leq 0 \).

Broll et al. (2010) extend this result considering S-shaped utility function. They show that the mean has an important role to determine the covariance sign for a particular type of S-shaped utility functions, as we can see in the following theorem.

**Theorem 6** Let \( X \) be symmetric around its mean \( \mu = E[X] \). If \( u \) is an S-shaped function, with \( u'(x) = u'(-x) \) for all \( x \in \mathbb{R} \). Then, we have the following statements:

1. if \( \mu \geq 0 \) then \( \text{Cov}[X, u'(X)] \leq 0 \).
2. if \( \mu \leq 0 \) then \( \text{Cov}[X, u'(X)] \geq 0 \).

Theorem (6) characterizes the sign of the covariance (2) for a non-monotonic marginal utility. However, it works only for a utility function that does not consider strict loss aversion, as it is defined in Definition (2). In the next section, I present a general result of this theorem that includes discussion
for an $S$-shaped utility function with loss aversion and reversed loss aversion as well. I also extend this theorem considering reverse $S$-shaped utility functions.

4. Main results

In this section, I present the main results of this paper. We have seen that $S$-shaped utility functions have non-monotonic marginal utilities. We have also seen that for some $S$-shaped utility functions, (e.g. (4)), the marginal utility $u'$ does not exist at the reference point $0$. Nevertheless, there other $S$-shaped utility functions with more mathematically tractable behavior such as (5) or (9). First, I shall state a general theorem where the marginal utility exists in all the real line. Second, I shall relax the assumption of existence of the marginal utility at the origin.

First, in the next result, I extend Broll et al. (2010) findings considering general $S$-shaped and $RS$-shaped utility functions. The novelty of this result is that I shall consider loss aversion and reversed loss aversion as well.

**THEOREM 7** Let $X$ be a symmetric random variable about its mean $\mu$. Let $u$ be a differentiable utility function.

(1) If $u$ is an $S$-shaped utility function, then we have the following two statements:

(a) If $\mu \geq 0$ and $u'(x) \leq u'(-x)$ for all $x \geq 0$, then $\text{Cov}[X, u'(X)] \leq 0$.

(b) If $\mu \leq 0$ and $u'(x) \geq u'(-x)$ for all $x \geq 0$, then $\text{Cov}[X, u'(X)] \geq 0$.

(2) If $u$ is an $RS$-shaped utility function, then we have the following two statements:

(a) If $\mu \geq 0$ and $u'(x) \geq u'(-x)$ for all $x \geq 0$, then $\text{Cov}[X, u'(X)] \geq 0$.

(b) If $\mu \leq 0$ and $u'(x) \leq u'(-x)$ for all $x \geq 0$, then $\text{Cov}[X, u'(X)] \leq 0$.

**Proof** First, I prove case (1a). Define the random variable $Z = X - \mu$. Therefore, $Z$ is symmetric about zero with $E[Z] = 0$. Thus, we rewrite the covariance as follows

\[
\text{Cov}[X, u'(X)] = \text{Cov}[Z + \mu, u'(Z + \mu)]
\]

\[
= \text{Cov}[Z, u'(Z + \mu)] + E[Zu'(\mu + Z)]
\]

\[
= E[Zu'(\mu + Z) \cdot 1\{Z \geq 0\}] + E[Zu'(\mu + Z) \cdot 1\{Z < 0\}]
\]

\[
= E[Z \cdot (u'(\mu + Z) - u'(\mu - Z)) \cdot 1\{Z \geq 0\}].
\]

There are two cases to consider: (i) Let $\mu - z \geq 0$. Since $z \geq 0$ implies $\mu + z \geq \mu - z$ and $u'$ is non-increasing on $(0, \infty)$, we have that

\[
u'(\mu + z) - u'(\mu - z) \leq 0.
\]

(ii) Now, assume that $\mu - z \leq 0$. Since $\mu \geq 0$ and $z \geq 0$, we therefore have that $\mu - z \leq 0 \leq \mu + z$. Consequently, using the assumption of $u'(x) \leq u'(-x)$ for all $x \geq 0$, we have that

\[
u'(\mu + z) = u'(-(z - \mu)) \geq u'(\mu - z),
\]

and thus

\[
u'(\mu + z) - u'(\mu - z) \leq u'(\mu + z) - u'(\mu - z).
\]

We exploit the fact that the right-hand side of bound (15) is non-positive because $u'$ is non-increasing on $(0, \infty)$ and $0 \leq z - \mu \leq \mu + z$. Consequently, $u'(z - \mu) \geq u'(\mu + z)$, and thus

\[
u'(\mu + z) - u'(\mu - z) \leq u'(\mu + z) - u'(z - \mu) \leq 0.
\]

Therefore, together from (14) and (16), we conclude that

\[
u'(\mu + z) - u'(\mu - z) \leq 0
\]

for all $\mu \geq 0$ and $z \geq 0$. 

\[
\text{(17)}
\]
Multiplying in both sides of (17) by \(1\{Z \geq 0\}\), taking expectations in both sides and using equality (13), we obtain that \(\text{Cov}[X, u'(X)] \leq 0\). This finishes the proof of part 1(a).

Now, the proof of part (2b). Starting with equality (13), we have, again, two cases:

(i) Assume \(\mu + z \leq 0\), then (since \(\mu \leq 0\) and \(z \geq 0\), we have \(\mu - z \leq \mu + z \leq 0\). And, since we are in the negative domain of \(u\), it is concave. Therefore, we conclude that
\[ u'(\mu - z) \geq u'(\mu + z). \]

(ii) Now, assume that \(\mu + z \geq 0\). Since we have that \(u'(x) \leq u'(-x)\) for all \(x \geq 0\) then \(u'(z - \mu) \leq u'(\mu - z)\). So that,
\[ u'(\mu + z) - u'(z - \mu) \geq u'(\mu + z) - u'(\mu - z). \]
Notice that \(z - \mu \geq \mu + z \geq 0\), thus we are in the positive domain of \(u\), implying that it is convex. Thus, \(u'(\mu + z) \leq u'(z - \mu)\), therefore, we conclude that
\[ u'(\mu + z) - u'(\mu - z) \leq 0. \]
At the end, we conclude that for all \(z \geq 0\) and \(\mu \leq 0\) we have that
\[ u'(\mu + z) - u'(\mu - z) \leq 0. \]
Following the same steps as in the proof of part 1(a), we conclude that \(\text{Cov}[X, u'(X)] \leq 0\). This ends the proof of part (2b). The other cases could almost certainly be adapted in the same way.

The extension of Theorem 7 to other S-shaped such as (4) can be done considering certain types of symmetric random variables. For example, Theorem 7 could almost certainly be adapted to those random variables, \(X\), such as \(X \neq 0\) almost surely. I present this extension in the next theorem.

**Theorem 8** Let \(X\) be a random variable symmetric about its mean \(\mu\), and such that \(X \neq 0\) almost surely. Suppose \(u\) is an S-shaped utility function, as defined in (4), then we have the following two statements:

1. If \(\mu \geq 0\) and \(u'(x) \leq u'(-x)\) for all \(x > 0\), then \(\text{Cov}[X, u'(X)] \leq 0\).
2. If \(\mu \leq 0\) and \(u'(x) \geq u'(-x)\) for all \(x > 0\), then \(\text{Cov}[X, u'(X)] \geq 0\).

**Proof** I only prove the first case; the other case can be proved in the same way. The proof mimics the proof of Theorem 7. First, with the notation \(Z = X - \mu\), we rewrite the covariance \(\text{Cov}[X, u'(X)]\) as the expectation \(E[Zu'(\mu + Z)]\). Since \(Z \neq -\mu\) by assumption, we have also \(Z \neq \mu\), by the symmetry of \(X\). Consequently, \(\text{Cov}[X, u'(X)]\) is equal to \(E[Zu'(\mu + Z) \cdot 1\{Z \neq \pm \mu\}]\), and thus

\[
\text{Cov}[X, u'(X)] = E[Zu'(\mu + Z) \cdot 1\{Z \neq \pm \mu\} \cdot 1\{Z < 0\}] \\
+ E[Zu'(\mu + Z) \cdot 1\{Z = \pm \mu\} \cdot 1\{Z \leq 0\}] \\
= E[Z(u'(\mu + Z) - u'(\mu - Z))1\{Z \neq \pm \mu\}1\{Z < 0\}].
\]

I skip the rest of the proof, since it is similar to that of Theorem 7.

**5. Applications**

This section shows some applications of the main results of the paper. The range of applications of our findings is broad, but I restrict the analysis to two cases. First, I study the monotonicity condition of the mean-variance indifference curve for an S-shaped utility function and RS-shaped utility function. Second, I apply the findings to the hedging policies of an enterprise that behaves according to prospect theory.
5.1. Mean-variance indifference curves for S-shaped and RS-shaped utility functions

The expected utility approach and the mean-variance approach, which is known as \((\mu, \sigma)\) criterion, are in general two different approaches for decision-making under uncertainty. The expected utility approach says that choice \(X\) is preferred to choice \(Y\) if and only if

\[
E[\mathcal{U}(X)] < E[\mathcal{U}(Y)],
\]

where \(\mathcal{U}\) is a concave utility function. On the other hand, the mean-variance approach (sometimes also called mean-variance rule) was introduced by Markowitz (1952) and states that choice \(X\) is preferred over choice \(Y\) if

\[
\mu_X \geq \mu_Y \text{ and } \sigma_X \leq \sigma_Y,
\]

with at least one strict inequality. Here, \(\mu_x\) and \(\mu_y\) denote the mean of \(X\) and \(Y\), and \(\sigma_x\) and \(\sigma_y\) denote their respective standard deviations. The idea is that decision-makers use only the mean and variance to make decisions. This is a common tool used by practitioners in finance (Shefrin, 2008). However, it has strong theoretical limitations. For example, it does not satisfy the expected utility independence axiom (e.g. Hens & Rieger, 2010, p. 50).

Many scholars study when both approaches are equivalent. Tobin (1958) shows that the two approaches are compatible under normally distributed assets or quadratic utility functions. Moreover, under the normal distribution assumption, the mean-variance rule also coincides with the expected utility approach (Hanoch and Levy, 1969). Sinn (1983) and Meyer (1987) show the equivalence of these approaches when the distributions differ only by a location and scale parameters. That is, suppose that \(X\) has a distribution that belongs to a class \(\Omega\), then \(Y = \mu + \sigma X\) where, \(\mu \in \mathbb{R}\) and \(\sigma > 0\), also belongs to that class of distribution \(\Omega\). In other words, if the distribution of \(X\) is \(F(x)\), then the distribution of \(Y\) is equal to \(F(\mu + \sigma X)\). Some distributions that satisfy the location scale condition are, among others: the elliptical distributions; the normal distribution; the uniform distribution; the Cauchy distribution; and the Student’s t distribution.

Sinn (1983) and Meyer (1987) derive several properties of the indifference curve in a \((\mu, \sigma)\) space, generated by a general risk averse von Neumann-Morgenstern utility function. In particular, these studies prove that these indifference curves, represented as a function \(\sigma \mapsto \mu(\sigma)\), are increasing and convex. These conditions are useful when the indifference curve is maximized over convex feasible sets. They explained, among other things, issues such as the existence of the CAPM equilibrium, as elucidated by Ormiston and Schlee (2001).

It is important to study the monotonicity of function \(\mu(\sigma)\). An increasing function means that the investor is willing to take more risk in exchange for more expected return. This is a crucial assumption of portfolio theory, since larger returns are associated with higher risk. Therefore, as an application of the main results, I will study whether the monotonicity property still holds for S-shaped and RS-shaped utility functions.

To keep the analysis as simple as possible, I do not consider transformations of the distribution function as prospect theory suggests (Kahneman & Tversky, 1979; Tversky & Kahneman, 1992). Hereafter, I assume that the random return \(Y\) belongs to the location-scale family \(\{\mu + \sigma X : \mu \in \mathbb{R}, \sigma > 0 \}\), where \(X\) is a random variable with mean 0 and variance 1, and whose distribution function \(F\) does not depend on \(\mu\) and \(\sigma\). Hence, the expected utility \(E[\mathcal{U}(Y)]\) defines a two-argument function

\[
V(\mu, \sigma) = E[\mathcal{U}(\mu + \sigma X)] = \int u(\mu + \sigma x) dF(x).
\]
Various properties of $V(\mu, \sigma)$, its partial derivatives

$$V_\mu(\mu, \sigma) = \frac{\partial}{\partial \mu} V(\mu, \sigma),$$

$$\sigma(\mu, \sigma) = \frac{\partial}{\partial \sigma} V(\mu, \sigma),$$

and especially of

$$S(\mu, \sigma) = -\frac{V_\mu(\mu, \sigma)}{V_\sigma(\mu, \sigma)}.$$ 

have been extensively investigated in the literature (see, for example, Meyer, 1987; Sinn, 1983).

The quantity $S(\mu, \sigma)$ has played a particularly prominent role. For instance, it can be viewed as the derivative with respect to the standard deviation $\sigma$ of the indifference function $\sigma \mapsto \mu(\sigma)$, which, for a given constant $\alpha$, can be viewed as the curve $\{(\sigma, \mu) \mid V(\mu, \sigma) = \alpha\}$ drawn in the $(\sigma, \mu)$-plane.

Hence, if $S(\mu, \sigma)$ is positive, then the indifference function $\sigma \mapsto \mu(\sigma)$ is increasing, whereas if $S(\mu, \sigma)$ is negative, then the indifference function is decreasing. Assuming that the utility function $u$ is differentiable and some integrability conditions are satisfied, we have the equations

$$V_\mu(\mu, \sigma) = E[u'(Y)],$$

$$V_\sigma(\mu, \sigma) = \frac{1}{\sigma} \text{Cov}[Y, u'(Y)],$$

where $Y = \mu + \sigma X$. Since $S(\mu, \sigma) = -V_\sigma(\mu, \sigma)/V_\mu(\mu, \sigma)$, we therefore have

$$S(\mu, \sigma) = -\frac{1}{\sigma} \frac{\text{Cov}[Y, u'(Y)]}{E[u'(Y)]}.$$ 

(23)

We may view $V_\mu(\mu, \sigma)$ as the expected marginal utility or, in other words, the slope of the expected utility $V(\mu, \sigma)$ with respect to $\mu$. Likewise, we may view $V_\sigma(\mu, \sigma)$ as the expected marginal utility $V(\mu, \sigma)$ with respect to $\sigma$. Finally, we may view $S(\mu, \sigma)$ as the slope of the indifference function $\sigma \mapsto \mu(\sigma)$.

This indifference curve and its various properties (e.g. monotonicity, convexity, concavity, and so forth) have received considerable attention in the literature. As we have noted above, some of the properties follow from the corresponding ones of the indifference function $\sigma \mapsto S(\mu, \sigma)$. In particular, the following general property is well known (see, for example, Eichner, 2008; Eichner & Wagener, 2009; Meyer, 1987; and references therein).

**Theorem 9** If the distribution of $Y$ with mean $\mu$ and variance $\sigma^2$ belongs to a location and scale family, and the twice differentiable utility function $u$ is increasing on its domain of definition, then we have the following two statements:

1. If the utility function $u$ is concave then the indifference function $\sigma \mapsto \mu(\sigma)$ is increasing and convex.

2. If the utility function $u$ is convex then the indifference function $\sigma \mapsto \mu(\sigma)$ is decreasing and concave.

It is now natural to extend formulas (21)–(23) to the case of general marginal utility functions $u'$ and random variables $Y$. As before, I use the notation $\mu = E[Y]$ and $\sigma^2 = \text{Var}[Y]$. 
Determining the sign of (23) is obviously related to the sign of $\text{Cov}(Y, u'(Y))$. When the marginal utility is monotonic, then we know that $\text{Cov}(Y, u'(Y)) \geq 0$ for every non-decreasing $u'$ and $\text{Cov}(Y, u'(Y)) \leq 0$ for every non-increasing $u'$. However, the marginal utility may be non-monotonic, as noted earlier. To cover such functions, I establish the following theorem that studies the monotonicity of the indifference curve generated by $S$-shaped utility functions.

**Theorem 10** Suppose the utility $u$ is an $S$-shaped function. Let $Y = \mu + \sigma X$ be a random variable where $X$ is a symmetric random variable with zero mean and unit variance. Assume that the location-scale condition holds.

1. If $\mu \geq 0$ and $u'(x) \leq u'(-x)$ for any $x > 0$, then $V_\sigma(\mu, \sigma) \leq 0$ and thus the indifference function $\sigma \mapsto \mu(\sigma)$ is increasing.

2. If $\mu \leq 0$ and $u'(x) \geq u'(-x)$ for any $x > 0$, then $V_\sigma(\mu, \sigma) \geq 0$ and thus the indifference function $\sigma \mapsto \mu(\sigma)$ is decreasing.

**Proof** I only prove Part (1) of the theorem by considering the case $\mu \geq 0$. We have seen that the slope of the indifference function $\sigma \mapsto \mu(\sigma)$ is determined by the sign of $\text{Cov}(Y, u'(Y))$. Since $X$ is symmetric about zero, then $Y$ is also symmetric about $\mu$. Therefore, invoking the first part of Theorem 7, we deduce that $\text{Cov}(Y, u'(Y)) \leq 0$ and thus $S(\mu, \sigma) \geq 0$, which implies that the assertion in Part (1) of Theorem 10 holds. Part (2) can be proved in the same way.

Next, I study the monotonicity property of the indifference curve for $RS$-shaped utility functions.

**Theorem 11** Consider the utility function as defined in (3), in which case, the utility is an $RS$-shaped function. Let $Y = \mu + \sigma X$ be a random variable where $X$ is a symmetric random variable with zero mean and unit variance.

1. If $\mu \geq 0$ and $u'(x) \geq u'(-x)$ for all $x > 0$, then $V_\sigma(\mu, \sigma) \geq 0$ and thus the indifference function $\sigma \mapsto \mu(\sigma)$ is decreasing.

2. If $\mu \leq 0$ and $u'(x) \leq u'(-x)$ for all $x > 0$, then $V_\sigma(\mu, \sigma) \leq 0$ and thus the indifference function $\sigma \mapsto \mu(\sigma)$ is increasing.

**Proof** The proof is analogous to the one in Theorem 10, but now invoking the results of the second part of Theorem 7.

### 5.2. Hedging policies within prospect theory

In this application, I follow the Broll and Wahl (2006) hedging model of a firm with one-period planning horizon.

The model setup is as follows. The enterprise that has risky assets with random returns, (future spot price) $r$. The assets are financed partially with external funds (deposits), denoted by $D$, which pays a certain return (price) $r_D > 0$. The enterprise also finances its assets with a fixed equity $K > 0$. Therefore, we can write the firm’s balance sheet constraint as follows,

$$A = D + K. \quad (24)$$

There are operational costs that depend on the deposits level. We represent these costs with a function $C(D)$, which we assume to be increasing and convex. In part of the uncertainty, the risky assets, $A$, can be hedged in the forward market at a certain price $r_F$. Let $H$ denote the amount of the hedged assets that is determined at the beginning of the period. When $H$ is positive means that the firm is selling assets in the futures market. On the other hand, if $H$ is negative, it means that firm is purchasing assets in the futures market. It is said that speculation is involved if $H \notin [0, A]$ otherwise, the assets are hedged without speculation. For instance, $H < 0$ it means that the firm is purchasing...
assets in the forward market, while \( H > A \) means that the firm selling an amount greater in the futures market than its current assets.

Since \( A = D + K \) is known, in this scenario, the next period enterprise’s profit function is given by

\[
\Pi(H) = r(A - H) - r_D D - C(D) + r_A H. \tag{25}
\]

The firm’s profit is uncertain and its mean is given by

\[
\mu(H) = E[\Pi(H)] = E[r(A - H) - r_D D - C(D) + r_A H]. \tag{26}
\]

As we shall see, the value of the mean, for the reasons studied in the previous section, has an important role in determining the optimal hedging decision.

Therefore, the firm manager’s problem is to find the optimal hedging that maximizes the expected utility of profits. However, instead of considering a traditional Bernoulli utility function, the firm uses an \( S \)-shaped utility function \( u \) as defined in Definition 1.

Thus, the firm wants to maximize its expected utility of profit

\[
\max_H E[u(\Pi(H))] = E[u(rA - r_D D - C(D) + H(r_A - r))]. \tag{27}
\]

In other words, we want to find the \( H \) that maximizes the expected utility of profits. I denote by \( H^* \) the solution of (27). Since \( u \) is an \( S \)-shaped function, then there is no guarantee that \( E[u(\Pi(H))] \) will be concave respect to \( H \). Therefore, I restrict the analysis to those cases where the first-order condition holds and there is a global solution of (27).

**Proposition 1** If the first-order condition of (27) holds then we have

\[
(r - E[r])(A - H^*) = \frac{\text{Cov}[\Pi(H^*), u'(\Pi(H^*))]}{E[u'(\Pi(H^*))]} \tag{28}
\]

**Proof** Taking the first-order condition of (27), evaluated at \( H^* \), we have

\[
E[(r_A - r)u'(\Pi(H^*))] = 0. \tag{29}
\]

The latter equation can be rewritten as follows:

\[
E\left[r u'(\Pi(H^*))\right] = r_A. \tag{30}
\]

Now, using the covariance function, we have

\[
r_A - E[r] = \frac{\text{Cov}[r, u'(\Pi(H^*))]}{E[u'(\Pi(H^*))]} \tag{31}
\]

After subtracting (25) from (26), we get

\[
(r - E[r])(A - H^*) = \Pi(H^*) - E[\Pi(H^*)], \tag{32}
\]

which implies that

\[
r = \frac{\Pi(H^*) - E[\Pi(H^*)]}{(A - H^*)} + E[r]. \tag{33}
\]

Substituting (30) in the covariance term of (25), we have

\[
\text{Cov}[r, u'(\Pi(H^*))] = \text{Cov}\left[\frac{\Pi(H^*) - E[\Pi(H^*)]}{(A - H^*)} + E[r], u'(\Pi(H^*))\right] \tag{34}
\]

\[
= \frac{1}{A - H^*} \text{Cov}[\Pi(H^*), u'(\Pi(H^*))]. \tag{35}
\]
Therefore, from Equation (29), we have

\[
(r_A - E[r]) (A-H^*) = \frac{\text{Cov}[\Pi(H^*), u'(\Pi(H^*))]}{E[u'(\Pi(H^*))]].
\]  

(31)

Finding \(H^*\) is generally a complex task. Nevertheless, Equation (31) has important derivations. Since \(E[u'(\Pi(H^*))]\) is strictly positive, the sign of the covariance \(\text{Cov}[\Pi(H^*), u'(\Pi(H^*))]\) determines the sign of the product \((r_A - E[r])(A-H^*)\). When \(u\) is more complexly shaped than being concave, then determining the sign of the covariance \(\text{Cov}[\Pi(H^*), u'(\Pi(H^*))]\) is a challenging task.

The relation between the expected spot price and the futures price will also determine the sign of \((A-H^*)\). When there are more hedgers taking short positions in the futures market than those that are going long then, to reach a balance, speculators must enter the market taking long positions. The speculators will do so, only if \(r_A < E[r]\) (a condition named normal backwardation). Conversely, if there are more hedgers taking long positions than those that are short, speculators will enter the market if \(r_A > E[r]\) (a condition named contango).

Notice that using the spot-futures parity relationship (which states that the ratio of return on perfectly hedged stocks equals the risk-free interest rate), we can write

\[
r_A = E[r] \left( \frac{1+r_f}{1+k} \right)^n
\]

where \(r_f\) is the risk free interest rate, \(k\) the required rate of return and \(n\) is the number of periods (Bodie, Kane, & Marcus, 1996, p. 708). Thus, \(r_A\) will be less than \(E[r]\) whenever \(k > r_f\) (i.e. whenever the asset has a positive beta). When the expected price equals the forward price, then the price is unbiased, which is the case when \(r_f = k\). Finally, \(r_A > E[r]\), whenever \(k < r_f\) (i.e. whenever the asset has a negative beta).

Next, I present an application of Theorem 7 that studies the sign of the covariance in (28).

Proposition 2  Let the distribution of \(r\) be symmetric around its mean \(E[r]\). Let \(u\) be an \(S\)-shaped utility function, with loss aversion defined as \(u'(x) \leq u'(-x)\) for all \(x > 0\). If \(\mu(H) \geq 0\) then

\(\text{Cov}[\Pi(H), u'(\Pi(H))] \leq 0\).

Proof  The proof follows directly by invoking Theorem 7.

We see that \(\mu(H)\) plays a decisive role in determining the sign of the covariance \(\text{Cov}[\Pi(H), u'(\Pi(H))]\).

When \(H = H^*\), note that we have the following expressions for the mean:

\[
\mu(H^*) = (E[r] - r_g)(A-H^*) + r_f D - r_p D - C(D)
\]

\[
= -\frac{\text{Cov}[\Pi(H^*), u'(\Pi(H^*))]}{E[u'(\Pi(H^*))]} + r_f A - r_p D - C(D).
\]

Using Proposition 2, we obtain the following corollary, which guides the firm in deciding whether to speculate or not. More precisely, it will tell us whether \(H^*\) is smaller or greater than \(A\), depending on whether the expected price \(E[r]\) is smaller or greater than the forward rate \(r_A\).
Corollary 1  Let the distribution of $r$ be symmetric around its mean $E[r] > 0$. Let $u$ be an $S$-shaped utility function such that $u'(x) \leq u'(-x)$ for all $x > 0$. Assume that $H^*$ is a solution of (31) such that $\mu(H^*) = E[\Pi(H^*)] \geq 0$, then we have the following statements:

1) If $r_A < E[r]$, then $H^* \leq A$.

2) If $r_A > E[r]$, then $H^* \geq A$.

Proof  Now, I prove the first part. The third part can be proved similarly. From the first-order condition of (27) we have

$$ (r_A - E[r])(A - H^*) = \frac{\text{Cov}[\Pi(H^*), u'(\Pi(H^*))]}{E[u'(\Pi(H^*))]} $$

Since $\mu(H^*) \geq 0$ using Proposition 2 then $\text{Cov}[\Pi(H^*), u'(\Pi(H^*))] \leq 0$. Therefore, the sign of $(r_A - E[r])$ is the opposite to the sign of $(A - H^*)$. Therefore, since

$$ \frac{\text{Cov}[\Pi(H^*), u'(\Pi(H^*))]}{E[u'(\Pi(H^*))]} \leq 0, $$

and $r_A < E[r]$ we have $H^* \leq A$.

This result has the following intuition. In the first case, if the forward price is less than the expected spot price, then the firm could hedge an amount less than its current assets. However, if the gap between the forward price and the expected prices is large enough, then the firm could purchase assets in the futures market. The firm will expect to sell at a greater price in the future. In the second case, if the forward price is greater than the expected price, then the firm will speculate selling an amount greater than its assets, expecting to purchase the additional assets in the future at a lower price.

These are well known results for decision makers with strict risk aversion (Benninga, Eldor, & Zilcha, 1983; Feder, Just, & Schmitz, 1980; Holthausen, 1979; Meyer & Robison, 1988). At the end, under these conditions, the enterprise hedging policies with an $S$-shaped utility are similar to those if it has an increasing and concave utility function.

6. Concluding remarks

In this paper, I establish new covariance inequalities with non-monotone functions. In particular, I derive new results to study the sign of $\text{Cov}[\Pi, u'(\Pi)]$, when the marginal utility is non-monotonic. This is the case when the utility function is according to either utility, as presented by Markowitz, or as in prospect theory. I show that the sign depends on the mean of the random variable and on the degree of loss aversion.

Two applications illustrate the main results of this paper. First, I study the monotonicity properties of the indifference curves on the $(\sigma, \mu)$-plane for $S$-shaped and $RS$-shaped utility functions. My results show that the indifference curve of $S$-shaped utility is increasing when there is loss aversion and $\mu \geq 0$. Similar results are derived considering reverse loss aversion and using $RS$-shaped utility functions as well. Finally, I study hedging policies of a firm that uses a utility function as postulated by prospect theory. I examine the behavior of a firm whose utility function varies with gains and losses in firm’s profits. Even though the analysis with prospect theory is more complex than one that assumes risk aversion, I demonstrate that similar behavior hold for symmetric random variables.

This work can be extended in several directions. For instance, it would be interesting to generalize Theorems 7 and 8 considering skewed distributions and especially for skewed-normal distribution (Azzalini, 1985). This remains a task for future studies.
Acknowledgements
I am grateful to two anonymous reviewers for numerous insightful suggestions and critical remarks that have guided my work on revising the paper. Comments by Luis Fuentes Garcia, Hugh Schwartz and Ricardas Zitikis are much appreciated.

Funding
I gratefully acknowledge partial funding from Agencia Nacional de Investigación e Innovación (ANII).

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Notes
1. Throughout this work, I use $u$ to denote the first derivative of $u$ (when it exists), and the Radon-Nykodym derivative in the absolutely continuous case (when the derivative may not exist). For example, given the marginal utility $u(x)$ as in (4), the utility function $u(x)$ is given by the formula

$$u(x) = \begin{cases} -\int_0^x u'(t) \, dt & x < 0, \\ \int_0^x u'(t) \, dt & x > 0. \end{cases}$$

(6)

Notice, that $u(x)$ in (6) coincides with $u(x)$ in (4). This is in line with the most frequently used in the statistical notion of absolutely continuous distribution functions. For example, the uniform on $[0, 1]$, density function $f_U(x)$ is related to the uniform distribution $F_U(x)$ by the equation $F_U(x) = \int_0^x f_U(t) \, dt$, but $r_U(x)$ is not differentiable at the points 0 and 1.

2. Part of this proof was provided to me by Luis Fuentes Garcia.

3. I have left the case $r_\varepsilon = E[r]$ as a task for future research, because it is more involved. It is necessary to prove that $	ext{Cov}(u, u') = 0$ implies $a = 0$, where $a$ and $b$ are real numbers.

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