Comment on “Can we measure structures to a precision better than the Planck length?”, by Sabine Hossenfelder

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Abstract

First principles do imply a non-zero minimal distance between events in spacetime, but no positive lower bound to the precision of the measurement of a single coordinate.

Part of the literature on Noncommutative Spacetime is based on a misunderstanding; namely, it takes as identical the following two distinct statements:

1. The localisation of a single space-time coordinate of an event cannot be performed with precision higher than the Planck Length;
2. The Euclidean distance between two events in space-time cannot be smaller than the Planck Length.

The paper [1] in question does not seem to clarify this confusion.

We believe, and stressed in several occasions, that the first statement is incorrect, while the second is correct; in a sense made precise in what follows.

Of course, nobody can say the final word based on a reliable theory of Quantum Gravity; however, we can investigate what presently known and accepted first principles allow, and what they do not.

We do not know any basic principle which characterises the concurrence of General Relativity and Quantum Mechanics, in a similar way as the locality principle does for Special Relativity and Quantum Mechanics [2].

But the concurrence of Classical General Relativity and Quantum Mechanics points to a principle of stability of spacetime against localisation of events, formulated in [3]. There Space Time Uncertainty Relations (STUR) for the...
Minkowski space (Cartesian) coordinates were deduced from that principle; those STUR suggested a model of Quantum Spacetime (QST) exactly implementing the STUR; this model is fully Poincaré covariant.

More precisely, the starting point was the following

**Principle of Gravitational Stability against localisation of events:**

*The gravitational field generated by the concentration of energy required by the Heisenberg uncertainty principle to localise an event in spacetime should not be so strong to hide the event itself to any distant observer - distant compared to the Planck scale.*

This principle cannot prevent arbitrarily accurate measurements of a single coordinate, or even of two. For, if $a$ is the common value of two small but fixed space uncertainties, $b$ the value of the third, and we are clever enough to arrange that in our measurement the energy packed in that region, due to the Heisenberg principle, is uniformly spread, than a freshman computation shows that the associated Newton potential tends to zero everywhere as $b$ grows to infinity and $a$ is kept fixed, no matter how small $a$ is.

The quantum gravitational corrections can hardly be relevant in that limit.

The example of exact “cosmic string” solutions of the Einstein equations discussed in [1] is not strictly pertinent, as it postulates from the start, in the form of the metric, that the thickness of the string, and hence some space uncertainty, is exactly zero; but even if we took such a solution as an indication of what happens, in the example discussed above, where $a$ is (not zero, but) very small, if $a$ were kept fixed but $b$ would become very large the energy density per unit length of the string would cease to be Planckian and would actually tend to zero, and we would be brought back to the conclusion suggested by the Newtonian approximation, in agreement with our STUR.

Before continuing our discussion, some words are in order on what is meant here by “localising an event”.

In Classical General Relativity we deal with a four dimensional manifold (spacetime) equipped with a Lorentzian metric. All points of the spacetime can be labeled by four real numbers, namely their coordinates.

These coordinates describe the localisation of events. Clearly, this makes sense even if the coordinates themselves are not observable quantities.

But even if they were classical observables, specialists of Quantum Mechanics would like to see a quantum mechanical observable which measures a single coordinate. Quantum Field Theorists, however, are well acquainted, since the work by Bohr and Rosenfeld [4], and especially thanks to the ideas of Rudolf Haag [5], with the idea that observables can only be derived from fields, and fields at a point are meaningless. Special combinations of field operators, all smeared with test functions with support in the space–time region $\mathcal{O}$, describe “local observables which can be measured within $\mathcal{O}$”.

If we could replace the region $\mathcal{O}$ by a point, its coordinates would not be observables, of course, but mere labels to specify a point in the underlying
classical manifolds. But measuring the corresponding local observable, i.e. as above some combination of field operators, would be a way to localise some event in that point.

Bohr and Rosenfeld, as we recalled, taught us that this is not possible, and we know since years that locality combined with special relativity forbids that, on precise mathematical grounds, based on first principles.

The localisation region for an event, in other words, might be as tiny as we like but with non zero sizes.

That is, as long as General Relativity is not taken in account, i.e. we disregard gravitational forces between elementary particles as exceedingly weak, we can still talk of localising events in arbitrarily small regions $\mathcal{O}$, by means of measurements of local observables, along the above lines. This gives a meaning to “points” in a classical manifold as idealised localisation of events.

These very well known and very elementary considerations allow us to talk of Minkowski space (or, in presence of an external background gravitational field, of an Einstein manifold), as a classical background where Quantum Field Theory is formulated.

We must repeat: the points of such a manifold are not described by some universal Quantum Field Theoretic observables called “coordinates”, but they are accessible to observation, with arbitrarily high precision, as attributes of local observables (localisation), in the above sense. This will be more and more accurate the better the state we produce with such a measurement is localised in the desired region. Thus the Quantum Field Theoretic concept of localised state [5] is crucial.

For instance, the state describing the result of a measurement localizing an event around some point $x$ in Minkowski space with uncertainties in the coordinates $\Delta x_\mu, \mu = 0, 1, 2, 3$, will be represented by a vector obtained by the action on the vacuum of some appropriate combinations of field operators, smeared out with test functions with support—say—in a region of rectangular shape in Minkowski space, centered around the point $x$, with sides of size $\Delta x_\mu$.

With all this in mind, the core of our message is:

If the gravitational forces are not disregarded, the localisation of an event, in the sense described here above, cannot be realised unless the uncertainties in the coordinates of the event, described as in the above example, obey some specific uncertainty relations, derived from the concurrence of Quantum Mechanics and Classical General Relativity.

These STUR lead to the Planck length as a lower bound to the Euclidean distance between two events, but the uncertainty of a single coordinate does not have a priori a nonzero lower bound.

This strongly suggests to abandon the concept of a classical Einstein Manifold as a geometric background, in favour of a non-commutative manifold, describing Quantum Spacetime. So far this procedure has been followed through only in the case of Quantum Minkowski Space [3, 6, 7].
But in general, we might expect that the form of Noncommutativity should not be a priori assigned, but part of the dynamics [8].

How do our comments here compare to the “generalised uncertainty principle”, is a point which also has been often emphasised (see e.g. [8]; or [9]), where you can read:

“There is no a priori lower limit on the precision in the measurement of any single coordinate (it is worthwhile to stress once more that the apparently opposite conclusions, still often reported in the literature, are drawn under the implicit assumption that all the space coordinates of the event are simultaneously sharply measured);”

or, more recently, [7], where you can read, about the same fact:

“This does not conflict with the famous Amati Ciafaloni Veneziano Generalized Uncertainty Relation: all the derivations we are aware of (see e.g [. . . ] ) implicitly assume that all space coordinates of the event are measured with uncertainties of the same order of magnitude; in which case they agree with our Spacetime Uncertainty Relations.”

For a more detailed discussion, see the review article [10].

That implicit hypothesis seems to have been noticed in [1], but the confusion minimal uncertainty in a coordinate / minimal length / minimal volume seems to persist.

Note that the STUR proposed in [3] \(^1\), namely, in absolute units,

\[
\Delta q_0 \cdot \sum_{j=1}^{3} \Delta q_j \gtrsim 1; \quad \sum_{1 \leq j < k \leq 3} \Delta q_j \Delta q_k \gtrsim 1. \quad (1)
\]

of course imply that each \(\Delta q_j\) is greater than one (i.e. of the Planck length) when they all coincide, for \(j = 1, 2, 3\).

Thus, the argument of Bronstein, and, if properly interpreted in view of the implicit hypothesis that all space uncertainties agree, the results of Mead, Amati–Ciafaloni–Veneziano, (referred to in [1, 7, 10]) are not at all contradicted by these STUR.

Moreover these STUR do lead to assertion (ii) (though assertion (i) is negated).

To make this point precise, note first that if one sticks to the uncertainty relations, no sharp assertion on the geometric operators, as distance, area, 3- or 4-volume, is possible; to that effect an explicit mathematical model of spacetime must be assumed (or derived from dynamics) where the corresponding analytical expressions (operators, in models of Quantum Spacetime) can be defined and analysed.

Within the “basic model” of QST, which exactly implements the above STUR, one can exactly analyse the operators describing distance between two

\(^1\)Strangely enough, the second relation is totally ignored in [1]
independent events [6], or, considering up to five independent events, area, three
volume or four volume operators [7].

The result is that the sum of the square moduli of all components, for each of
those operators, (the square Euclidean distance, . . . , the square modulus of the
pseudoscalar four volume operator) is bounded below by some positive constant
of order 1 in Planck units - in all Lorentz frames [7] (although the model is
fully Poincaré covariant; problems with Lorentz covariance arise only if dealing
with interactions between quantum fields on QST [3, 6]).

Note that obvious modifications of the computations in [3, Section 3] lead
to a similar lower bound (a factor 1/2 smaller, i.e. exactly 2 in Planck units)
for the square Euclidean spatial distance, i.e. for the sum of the square moduli
of the differences of the space coordinates of the two events.

The result on distances tells precisely that you cannot “measure structures
to a precision better that the Planck length”.

Are we asserting that we can measure a given coordinate of an event as
precisely as we wish?

Of course not: we are asserting that no presently known argument or prin-
ciple seems to forbid it.

The claim, repeatedly asserted, that we do not know whether there is a
positive lower bound on the accuracy with which a single coordinate can be
measured, apparently received very little attention; we would like to acknowl-
edge that the paper [1] is an exception.

Note that in all the quoted arguments it was important to talk of localisation
of events, not of particles: by Heisenberg’s principle, the uncontrolled amount
of energy transmitted to the system by such an accurate measurement would
create uncontrollable amounts of photons and particle-antiparticle pairs.

The well known fact we just mentioned takes a precise form in Quantum
Field Theory, in the light of the principle of locality.

Typically, localised states are obtained from the vacuum by acting with an
isometric operator, which lies in a local algebra of fields. Coherent states, with
components with arbitrarily many particles, are examples. But single particle
states are never of that form. 2

If one misses these points, one feeds confusions, as that between distance
resolution and uncertainty in a single coordinate.

The limitation 2. was deduced in [3, 6, 7] from the Basic Model of QST
based on the STUR; the weakness of the linear approximation (common to so
many other results; mitigated, however, by the consistency with exact solutions
as Schwarzschild’s or Kerr’s), made to derive the STUR in the starting point
of [3], was partly resolved in [11], based on the Hoop conjecture, and, with the

2Even if they are in the Newton-Wigner or in the Foldy Wouthuysen class, their localisation
will be merely the expression of a geometric relation between them, and not of the relativistic
localisation: the expectation value of a local observable in such a state will never have the
characteristic property, to become exactly the vacuum expectation value, when the observable
is translated away into the spacelike complement of some localisation region. It will merely
converge to it, for large spacelike translations, with a characteristic distance of order of the
Compton wave length of the particle.
limitation to spherical symmetry, but without use of the the Hoop conjecture, in [12].

Similarly, we are not asserting that the Euclidean distance between two events in spacetime can be as small as the Planck length; we are only saying that, under the specified hypothesis, it cannot be smaller.

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