POT-flavored estimator of Pickands dependence function

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Abstract

This work proposes an estimator with both Peak-Over-Threshold and Block-Maxima flavors, uses it to estimate the Pickands dependence function of bivariate time series, and illustrates how it brings down the asymptotic bias and the overall mean squared error.

Keywords: Extreme value copula, Pickands dependence function, peak-over-threshold, madogram.

1 Introduction

Extreme value statistics is witnessing an intensive horse racing [1] between two fundamental methods: the Block Maxima (BM) method and the Peak-Over-Threshold (POT) method. Intuitively, The BM method partitions the observations into blocks and view the max of each block to be extreme, while the POT method sets a threshold and considers the observations above this threshold to be extreme. The BM method and the POT method are connected but not identical to each other.

In addition to asking which of the BM and POT methods prevails over the other, one may also wonder if these two methods could be mixed to obtain an even better performance. This manuscript aims to propose a estimator that have the flavor of both BM and POT methods in the bivariate time series setting. Consider strictly stationary bivariate time series $X_t = \{X_{t,1}, X_{t,2}\}_{t \in \mathbb{Z}}$ with continuous univariate stationary margins. First we refer to the BM method. For $j = 1, 2$, let

$$M_{m,1,j} = \max\{X_{t,j} : t = 1, \ldots, m\}$$ (1.1)

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be the coordinate-wise block maxima, \( M_{m,1} = (M_{m,1,1}, M_{m,1,2}) \) be the block maxima vector, and \( C_m \) be the copula of \( M_{m,1} \). Assume there exists an extreme copula \( C_\infty \) such that for all \((u,v) \in [0,1]^2\),

\[
\lim_{m \to \infty} C_m(u,v) = C_\infty(u,v).
\]

Indeed, under (1.2), \( C_\infty \) can be represented with some function \( A_\infty : [0,1] \to [0,1] \); for \((u,v) \in [0,1]^2\),

\[
C_\infty(u,v) = \exp \left\{ \log(uv)A_\infty \left( \frac{\log v}{\log(uv)} \right) \right\}.
\]

Hence, the inference of the extreme copula \( C_\infty \) boils down to the inference of \( A_\infty \), which is called the Pickands dependence function of \( C_\infty \). The estimation of the Pickand function \( A_\infty \) has drawn a considerable attention in the literature, e.g., [2]–[11]; for an overview, see [12], [13]. In particular, [14]–[17] develop fast-to-compute, easy-to-interpret, madogram-type estimators.

Most of the literature above postulate that \( C_m = C_\infty \) instead of (1.2) and as a result does not include the bias, \( C_m - C_\infty \), in their asymptotic analyses. To cope with this bias term, one way is to push the block size \( m \) to infinity; alternatively, one could, as in the POT method, potentially consider pushing the threshold to infinity. In this work, instead of pushing the threshold to infinity and only considering the observations above the threshold as in the POT method, we propose to put more “weight” on the larger observations and develop a POT-flavored, madogram-type estimator for the Pickand dependent function \( A_\infty \) in (1.3) in the bivariate time series setting. With asymptotic analyses and simulation on the bias and the variance under (1.2), we find that in some scenarios assigning more weight on larger observations, although increases the variance, can reduce the bias and can moreover bring down the overall mean squared error (MSE).

The remaining parts of this paper proceed as follows. Section 2 details the construction of this POT-flavored Pickands dependence function estimator. Section 3 analyzes the asymptotic property of this estimator for the Pickands dependence function. Section 4 discusses the choice of the copula estimators. Section 5 presents the simulation results. The Appendix includes all the proofs.
2 POT-flavored Pickands Dependence Function Estimator

Let

\[ S(t) = 1 - \int_0^1 C_\infty(y^{1-t}, y^t) \, dy. \]  

(2.1)

When designing madogram-type estimators for the Pickands dependence function \( A_\infty \) in (1.3), [14]–[17] leverage the fact that (2.1) gives

\[ A_\infty(t) = \frac{1}{c} \left( \frac{1}{1 - S(t)} - 1 \right). \]

To assign more weights to those \( C_\infty(u, v) \) evaluated at larger values of \( u, v \in [0,1]^2 \) in (2.1), for \( t \in [0,1] \) and \( c > 0 \), we let \( \hat{S}(t, c) \), a generalization of \( S(t) \), be defined by

\[ \hat{S}(t, c) = 1 - \int_0^1 C_\infty(y^{c(1-t)}, y^{ct}) \, dy. \]  

(2.2)

Plugging (1.3) into (2.2) gives that for all \( c > 0 \),

\[ A_\infty(t) = \frac{1}{c} \left( \frac{1}{1 - \hat{S}(t, c)} - 1 \right). \]  

(2.3)

As an empirical counterpart of (2.2) and (2.3), let \( \hat{S}(t, c) \), the estimator for \( S(t, c) \), be defined by

\[ \hat{S}(t, c) = 1 - \int_0^1 \hat{C}_\infty(y^{c(1-t)}, y^{ct}) \, dy, \]  

and let \( \hat{A}_\infty(t) \), the estimator for \( A_\infty(t) \), be defined by

\[ \hat{A}_\infty(t) = \frac{1}{c} \left( \frac{1}{1 - \hat{S}(t, c)} - 1 \right). \]

Remark 2.1. Notice that \( \hat{S}(t, c) \) connects closely to the mean of the maximum of variables and the mean of the absolute difference of variables. Indeed, for \( j = 1, 2 \), let

\[ F_{m,j}(x) = \mathbb{P}(M_{m,1,j} \leq x), \quad U_{m,1,j} = F_{m,j}(M_{m,1,j}), \]  

(2.5)
where $M_{m,1,j}$ is defined in (1.1). Then, by (1.2), (2.2), and the tail sum formula for expectation,

$$S(t,c) = \lim_{m \to \infty} \mathbb{E} \left[ \max \left( U_{m,1,1}^{\frac{1}{m}}, U_{m,1,2}^{\frac{1}{m}} \right) \right] = \lim_{m \to \infty} \frac{1}{2} \mathbb{E} \left[ U_{m,1,1}^{\frac{1}{m}} + U_{m,1,2}^{\frac{1}{m}} + U_{m,1,1}^{\frac{1}{m}} - U_{m,1,2}^{\frac{1}{m}} \right].$$

**Remark 2.2.** By a change of variables, (2.4) results in

$$\hat{S}(t,c) = 1 - \int_0^1 \hat{C}_\infty(y^{(1-t)}, y^t) \frac{1}{c} y^{1/c-1} \, dy.$$

Hence, when $c$ gets smaller, the integral in the expression of $\hat{S}(t,c)$ put more weights on those $\hat{C}_\infty(y^{(1-t)}, y^t)$ with larger value of $y$. Hence, when $c$ gets smaller, the values of $\hat{C}_\infty(u,v)$ at larger values of $u$ and $v$ will take more weight in the construction of $\hat{A}_\infty$. As a result, $\hat{A}_\infty$ has some flavor of the POT.

### 3 Asymptotic Properties

By the Continuous Mapping Theorem, the definition of equicontinuity, a Taylor expansion, and the Slutsky’s Theorem, for fixed $c > 0$, $\hat{S}(\cdot,c)$ and $\hat{A}_\infty(\cdot)$ will be consistent and asymptotically Gaussian if the copula estimator $\hat{C}_\infty(\cdot)$ is consistent and asymptotically Gaussian. The asymptotic bias and variance of $\hat{A}_\infty$ depend on the specific choice of $\hat{C}_\infty$. For simplicity, we choose $\hat{C}_\infty$ to be the disjoint-block copula estimator $\hat{C}_m^\circ$ in, e.g., [18]. More specifically, recall $F_{m,j}$, $j = 1, 2$, defined in (2.5). Let

$$\tilde{M}_{m,i,j} = \max \{ X_{t,j} : t \in [(i-1)m + 1, im] \cap \mathbb{Z} \} \quad U_{m,i,j} = F_{m,j}(\tilde{M}_{m,i,j})$$

$$\tilde{b} = \lfloor n/m \rfloor \quad \tilde{C}_m^\circ(u,v) = \frac{1}{\tilde{b}} \sum_{i=1}^{\tilde{b}} \mathbb{1}(U_{m,i,1} \leq u, U_{m,i,2} \leq v).$$

**Assumption 3.1.** We assume there exists a positive function $a(\cdot)$ with $\lim_{m \to \infty} a(m) = 0$ and a non-null function $S$ on $[0,1]^2$ such that

$$\lim_{m \to \infty} \frac{C_m(u,v) - C_\infty(u,v)}{a(m)} = S(u,v) \quad (3.2)$$

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uniformly in \((u,v) \in [0,1]^2\). Subsequently, we assume that \(a(\cdot)\) is regularly varying of order \(\rho < 0\), that is, \(\lim_{m \to \infty} a(mx)/a(m) = x^\rho\) for all \(x > 0\).

**Assumption 3.2.** Assume \(\{X_t\}_{t \in \mathbb{Z}}\) to be \(\alpha\)-mixing with coefficient \(\alpha(k), k = 1, 2, \ldots\). Further, assume that, as \(n \to \infty\), there exists a positive integer sequence \(\ell = \ell_n\) such that

1. \(m \to \infty, m = o(n)\)
2. \(\ell \to \infty, \ell = o(m)\)
3. \((n/m)\alpha(\ell) = o(1), (m/\ell)\alpha(\ell) = o(1)\)
4. \(\alpha(k) = O(k^{-1+\rho})\) for some \(\rho > 0\).

**Proposition 3.1.** Under Assumption 3.1, Assumption 3.2, and (3.1),

1. \(|\mathbb{E}[\hat{A}_\infty(t) - A_\infty(t)]| = a(m)S\left(e^{-t}, e^{-t}\right)e^{A_\infty(t)}\Gamma(2-\rho)\left(\frac{cA_\infty(t) + 1}{c}\right)^\rho + o(a(m))\);
2. \(\text{Var} \left(\hat{A}_\infty(t)\right) = (m/n)\left(\frac{cA_\infty(t) + 1}{c}\right)^2 (cA_\infty(t) + 2) + o(m/n),\)

where \(\Gamma(\cdot)\) is the Gamma function.

**Remark 3.1.** Since \(\rho < 0\), by analyzing the derivatives, the dominating terms of \(|\mathbb{E}[\hat{A}_\infty(t) - A_\infty(t)]|\) and \(\text{Var} \left(\hat{A}_\infty(t)\right)\) turn out to be an increasing and a decreasing function, respectively, with respect to the constant \(c\). Recall that by Remark 2.2, a smaller constant \(c\) leads to a larger weights for higher values, or intuitively, a higher “threshold”. Hence, Proposition 3.1 indicates that when this “threshold” gets higher, the absolute value of bias of the estimator will be smaller while the variance will become larger.

**Remark 3.2.** In light of the POT method, we can set \(c = c_n \to 0\), namely, we can let the “threshold” goes to infinity as \(n \to \infty\). In this case, both the orders of \(|\mathbb{E}[\hat{A}_\infty(t) - A_\infty(t)]|\) and \(\text{Var} \left(\hat{A}_\infty(t)\right)\) depend on the ratio \(m/c\). Specifically, the absolute value of the bias will have an order of \(a(m/c)\) and the variance will have an order of \((m/c)/n\).
4 Choice of Copula Estimator

In practice, we can substitute the disjoint-block (denoted by D) estimator of [18] and the overlapping-block (denoted by O) estimator of [19] for $\hat{C}_\infty$ in (2.4). Specifically, recall $\tilde{b}$ and $\tilde{M}_{m,i,j}$ in (3.1). Let $b = n - m + 1$, $M_{m,i,j} = \max\{X_{t,j} : t \in [i, i + m - 1] \cap \mathbb{Z}\}$, and $\hat{C}_D^m$ and $\hat{C}_O^m$ be the disjoint-block and overlapping-block estimator, respectively, defined by:

$$\hat{F}_{m,j}(x) = \frac{1}{\tilde{b}} \sum_{i=1}^{\tilde{b}} \mathbb{I}(\tilde{M}_{m,i,j} \leq x)$$

$$\hat{U}_{m,i,j} = \hat{F}_{m,j}(M_{m,i,j})$$

$$\hat{C}_D^m(u,v) = \frac{1}{\tilde{b}} \sum_{i=1}^{\tilde{b}} \mathbb{I}(\hat{U}_{m,i,1} \leq u, \hat{U}_{m,i,2} \leq v)$$

By plugging $\hat{C}_D^m$ and $\hat{C}_O^m$ back to (2.4), for $\Psi = D, O$, we can define estimators for the Pickands dependence function by

$$\hat{S}_\Psi^m(t,c) = 1 - \int_0^1 \hat{C}_\Psi^m(y^{(1-t)},y^c) \, dy.$$ 

**Remark 4.1.** Similar to Remark 2.1,

$$\hat{S}_D^m(t,c) = \frac{1}{b} \sum_{i=1}^{b} \max(\hat{U}_{m,i,1},\hat{U}_{m,i,2})$$

$$\hat{S}_O^m(t,c) = \frac{1}{b} \sum_{i=1}^{b} \max(\hat{U}_{m,i,1},\hat{U}_{m,i,2}).$$

5 Simulation

5.1 Data Generating Process

Let $n = 1000$ be the sample size. We consider the moving maximum processes in the setup section of Chapter 5 of [18]. In particular, we let

$$X_{t,1} = \max(W_{t,1}^{1/a}, W_{t-1,1}^{1/(1-a)}), X_{t,2} = \max(W_{t,2}^{1/b}, W_{t-1,2}^{1/(1-b)}),$$

where we set $a = 0.25$, $b = 0.5$, and let $\{W_{t,1}, W_{t,2}\}_{t \in \mathbb{Z}}$ be a bivariate i.i.d sequence with uniform marginal distributions on $[0,1]$ and a joint cumulative distribution function $D$ specified below.
5.1.1 Outer-power Transformation of Clayton Copula

The outer-power transformation of a Clayton copula is defined, for \((u, v) \in [0, 1]^2\), by

\[
D(u, v) = \left[ 1 + \{(u^{-\theta} - 1)^\beta + (v^{-\theta} - 1)^\beta\}^{1/\beta}\right]^{-1/\theta},
\]

where we set \(\theta = 1\) and \(\beta = \log(2) / \log(2 - 0.25)\).

5.1.2 t-Copula

The t-copula is defined, for \((u, v) \in [0, 1]^2\), as

\[
D(u, v) = \int_{-\infty}^{t_{\nu}^{-1}(u)} \int_{-\infty}^{t_{\nu}^{-1}(v)} \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\pi \nu |P|^{1/2}} \left(1 + \frac{x'P^{-1}x}{\nu}\right)^{-\frac{\nu+2}{2}} dx_2 dx_1,
\]

where \(x = (x_1, x_2)', P\) is a \(2 \times 2\) correlation matrix with off-diagonal element \(\theta\), and \(t_{\nu}\) is the cumulative distribution function of a standard univariate t-distribution with degrees of freedom \(\nu\). We set \(\nu = 4\) and \(\theta = 0.494217\) so that the coefficient of upper tail dependence of \(D\) matches the coefficient in the outer-power transformation of Clayton Copula in Section 5.1.1.

5.1.3 Gaussian Copula

The Gaussian copula is defined, for \((u, v) \in [0, 1]^2\), as

\[
D(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi |P|^{1/2}} \exp \left(-\frac{x'P^{-1}x}{2}\right) dx_2 dx_1,
\]

where \(x = (x_1, x_2)', P\) is a \(2 \times 2\) correlation matrix with off-diagonal element \(\theta\), and \(\Phi\) is the cumulative distribution function of a standard univariate normal distribution. We set \(\theta = 0.5\) to match the coefficient in the t-copula in Section 5.1.2.
5.2 Algorithm

Choose block size $m = 1, 2, \ldots, 30$. We estimate the Pickands Dependence Function $A_\infty(t)$ with the additive boundary corrected estimator $\hat{A}_m^\Psi(t)$. Specifically, for $\Psi = D, O$, define

$$\hat{A}_m^\Psi(t) = \hat{A}_m^\Psi(t) - (1 - t)\{\hat{A}_m^\Psi(0) - 1\} - t\{\hat{A}_m^\Psi(1) - 1\},$$

where

$$\hat{A}_m^\Psi(t) = \frac{1}{c}\left(\frac{1}{1 - \hat{S}_m^\Psi(t, c)} - 1\right),$$

where $\hat{S}_m^\Psi(t, c), \Psi = D, O,$ are generated by (4.1).

5.3 Result

We examine the criteria below by setting $T = 51$ below and by averaging out over $N = 1000$ iterations:

$$B^{(\text{sum})} = \sum_{t=0}^{T-1} \left\{\hat{E} \left[\hat{A}_m^\Psi\left(t/(T-1)\right) - A\left(t/(T-1)\right)\right]\right\}^2,$$

$$\text{Var}^{(\text{sum})} = \sum_{t=0}^{T-1} \text{Var} \left(\hat{A}_m^\Psi\left(t/(T-1)\right)\right),$$

$$\text{MSE}^{(\text{sum})} = B^{(\text{sum})} + \text{Var}^{(\text{sum})}.$$

Figures 1 to 3 include $B^{(\text{sum})}$, $\text{Var}^{(\text{sum})}$, and $\text{MSE}^{(\text{sum})}$ of the estimators in Table 1.

| Disjoint/Overlap | Value of $c$ | Shorthand |
|------------------|--------------|-----------|
| Disjoint         | 1            | D         |
| Overlap          | 0.25         | O_0.25    |
| Overlap          | 0.5          | O_0.5     |
| Overlap          | 1            | O_1       |
| Overlap          | 2            | O_2       |
| Overlap          | 4            | O_4       |

Table 1: Estimators for $A_\infty$ and their shorthands
Figures 1 to 3 show that having a smaller constant $c$, or equivalently, having a higher “threshold”, increases the variance, reduces the bias, and can potentially diminish the overall MSE. Particularly, the overlapping-block estimator with $c = 0.25$ corresponds to MSE curves with the smallest nadirs.
Data-adaptive selections of the combination of parameter $c$ and block size $m$ will be left to future works.

Appendix

Proof of Proposition 3.1(i). By [19, (B.1) and Lemma 3.4], (1.2), Assumption 3.1, and Assumption 3.2(iv), for all $s > 0$ and $(u, v) \in (0, 1]^2$,

$$S(u^s, v^s) = s^{1-p}S(u, v) \frac{C_\infty(u^s, v^s)}{C_\infty(u, v)} = s^{1-p}S(u, v) \left( C_\infty(u, v) \right)^{s^{-1}}.$$

By (1.3), $C_\infty(e^{-(1-t)}, e^{-t}) = e^{-A_\infty(t)}$. Hence, for $c > 0$ and $0 < y < 1$,

$$S\left(y^{e^{(1-t)}}, y^{e^{ct}}\right) = S\left(e^{-c \log(y)(1-t)}, e^{-c \log(y)(-t)}\right)$$
$$= \left(-c \log(y)\right)^{1-p}S\left(e^{-(1-t)}, e^{-t}\right) \left(C_\infty\left(e^{-(1-t)}, e^{-t}\right)\right)^{-c \log(y)^{-1}}$$
$$= \left(-c \log(y)\right)^{1-p}S\left(c^{e^{-(1-t)}}, e^{c \log(y)+1}A_\infty(t)\right).$$

Hence, by the uniform convergence in (3.2),

$$E\left[\tilde{S}(t, c) - S(t, c)\right] = -\int_0^1 C_m(y^{e^{(1-t)}}, y^{e^{ct}}) - C_\infty(y^{e^{(1-t)}}, y^{e^{ct}}) \, dy$$
$$= -a(m)\int_0^1 S\left(y^{e^{(1-t)}}, y^{e^{ct}}\right) \, dy + o(a(m))$$
$$= -a(m)S\left(e^{-(1-t)}, e^{-t}\right)e^{A_\infty(t)}c^{1-p} \int_0^1 \left(-c \log(y)\right)^{1-p}y^{e^{A_\infty(t)}} \, dy + o(a(m))$$
$$= -a(m)S\left(e^{-(1-t)}, e^{-t}\right)e^{A_\infty(t)}\Gamma(2 - \rho)c^{1-p} \left(cA_\infty(t) + 1\right)^{\rho-2} + o(a(m)),$$
where $\Gamma(\cdot)$ is the Gamma function. By a Taylor expansion, the Dominated Convergence Theorem, and (2.3),

$$
\left| \mathbb{E}[\hat{A}_\infty(t) - A_\infty(t)] \right| = \frac{1}{c} \mathbb{E} \left[ \left( \frac{1}{1 - \hat{S}(t,c)} - 1 \right) - \left( \frac{1}{1 - S(t,c)} - 1 \right) \right] = \frac{1}{c} \mathbb{E} \left[ \left( 1 - \hat{S}(t,c) \right)^{-2} + o(1) \right] \mathbb{E} \left[ \hat{S}(t,c) - S(t,c) \right] = a(m) S(e^{-1-t}, e^{-t}) e^{A_\infty(t) \Gamma(2 - \rho) \left( \frac{cA_\infty(t) + 1}{c} \right)} + o(a(m)).
$$

**Proof of Proposition 3.1(ii).** By Assumption 3.2(i)-(iii) and a proof similar to [18, Theorem 3.1],

$$
\text{Cov} \left( \hat{C}_m(u_1, v_1), \hat{C}_m(u_2, v_2) \right) = (m/n) \left( C_\infty(\min(u_1, v_1), \min(u_2, v_2)) - C_\infty(u_1, v_1) C_\infty(u_2, v_2) \right) + o(m/n).
$$

Hence, by (2.2),

$$
\text{Var} \left( \hat{S}(t,c) \right) = \int_0^1 \int_0^1 \text{Cov} \left( \hat{C}_m(y^{(1-t)}, y^{\epsilon t}), \hat{C}_m(z^{(1-t)}, z^{\epsilon t}) \right) \, dz \, dy = (m/n) \left( 2 \int_0^1 \int_y^1 C_m \left( y^{(1-t)}, y^{\epsilon t} \right) \, dz \, dy - \int_0^1 C_m \left( y^{(1-t)}, y^{\epsilon t} \right) \, dy \right)^2 + o(m/n) = (m/n) \frac{cA_\infty(t)}{(cA_\infty(t) + 1)^2 (cA_\infty(t) + 2)} + o(m/n).
$$

Hence, by a Taylor expansion, the Dominated Convergence Theorem, and (2.3),

$$
\text{Var} \left( \hat{A}_\infty(t) \right) = \frac{1}{c^2} \text{Var} \left[ \left( \frac{1}{1 - \hat{S}(t,c)} - 1 \right) - \left( \frac{1}{1 - S(t,c)} - 1 \right) \right] = \frac{1}{c^2} \left[ \left( 1 - \hat{S}(t,c) \right)^{-2} + o(1) \right]^2 \text{Var} \left( \hat{S}(t,c) - S(t,c) \right) = (m/n) \frac{(cA_\infty(t) + 1)^2 A_\infty(t)}{c(cA_\infty(t) + 2)} + o(m/n).
$$
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