N. Pagani, O. Tommasi

The orbifold cohomology of moduli of genus 3 curves

Received: 30 March 2011 / Revised: 14 November 2012
Published online: 2 February 2013

Abstract. In this work we study the additive orbifold cohomology of the moduli stack of smooth genus $g$ curves. We show that this problem reduces to investigating the rational cohomology of moduli spaces of cyclic covers of curves where the genus of the covering curve is $g$. Then we work out the case of genus $g = 3$. Furthermore, we determine the part of the orbifold cohomology of the Deligne–Mumford compactification of the moduli space of genus 3 curves that comes from the Zariski closure of the inertia stack of $\mathcal{M}_3$.

1. Introduction

It was a remarkable discovery of the beginning of this century, anticipated in physics in the nineties, that the degree zero small quantum cohomology of a smooth Deligne–Mumford stack $X$ does not in general coincide with the ordinary cup product, and that it is, in fact, a proper ring extension of the ordinary cohomology ring of $X$. More generally, the definition of quantum cohomology and Gromov–Witten invariants for orbifolds were given in symplectic geometry by Chen and Ruan in [9]. The algebraic counterparts of these theories were developed by Abramovich–Graber–Vistoli in [2,3].

The Chen–Ruan cohomology of a smooth Deligne–Mumford stack $X$ is, by definition, the degree zero part of the small quantum cohomology ring of $X$, and the orbifold cohomology of $X$ is the rationally graded vector space that underlies the Chen–Ruan cohomology algebra. The general idea, coming from stringy geometry, is that an important role in the study of $X$ is played by the so-called inertia stack of $X$. When $X$ is a moduli space for certain geometric objects, the inertia stack of $X$ parametrizes the same geometric objects, together with the choice of an automorphism on them. The stack $X$ itself appears as the connected component of its inertia stack associated with the trivial automorphism, but in general there are other connected components, usually called the twisted sectors of $X$, a terminology that originates from physics. Orbifold cohomology is simply the ordinary

N. Pagani, O. Tommasi (✉): Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany.
e-mail: tommasi@math.uni-hannover.de
N. Pagani: e-mail: npagani@math.uni-hannover.de

Mathematics Subject Classifications (2010): Primary: 14H10, 55N32. Secondary: 14N35, 14D23, 14H37, 32G15, 55P50.

DOI: 10.1007/s00229-013-0608-z
cohomology of the inertia stack, endowed with a different grading. Each twisted sector is assigned a rational number, called (depending on the author) degree shifting number, age or fermionic shift: this number depends on the action of the given automorphism on the normal bundle to the twisted sector in $X$. Then the degree of each cohomology class of the twisted sector is shifted by twice this rational number.

In this paper, we study the inertia stack of moduli spaces $\mathcal{M}_g$ of smooth genus $g$ curves. The starting point of our construction is that one can associate with each object $(C, \alpha)$ of the inertia stack the cover given by quotienting $C$ by the cyclic group generated by $\alpha$. Following an idea of Fantechi [13], we exploit this correspondence to tackle the problem of the identification of the twisted sectors of $\mathcal{M}_g$ by using the classical theory of cyclic (possibly ramified) covers of algebraic varieties, as developed in [24]. We identify some discrete data in order to separate the inertia stack of $\mathcal{M}_g$ in its connected components. The first data are the genus of the quotient curve and the order $N$ of the automorphism; the latter is a general invariant of twisted sectors as it appears already in the definition of the inertia stack. Finally, the branch locus of the cover can be split in $N - 1$ parts according to the local monodromy around each of its points. The last invariants are simply the degrees of each of these $N - 1$ divisors. It is a recent result of Catanese [8] that these numerical data single out a connected component of the moduli space of connected cyclic covers.

The twisted sectors of moduli of curves were studied with *ad hoc* methods in the case of genus 2 in [26], and in the case of pointed curves of genus 1 in [21]. The same approach explained in the above paragraph was used in [22] to identify the twisted sectors of $\mathcal{M}_{g,n}$ with $g = 2$ or $n \geq 1$, in this paper we complete the picture by analyzing the more delicate case when $n = 0$. A complete cohomological description of the twisted sectors of moduli of hyperelliptic curves for all genera is given in [23] following a similar approach.

After having determined the connected components of the inertia stack of $\mathcal{M}_g$ for general $g$, we study the topology of the twisted sectors in the case when $g$ equals 3. In most cases, the cohomology of the twisted sector is computed in a rather straightforward way. The main exceptions are the twisted sectors corresponding to bielliptic and to quadrielliptic genus 3 curves, which require a more detailed analysis. In particular, our computation of the cohomology of the moduli space of bielliptic genus 3 curves is achieved by using a combination of Vassiliev–Gorinov’s method for the computation of the cohomology of complements of discriminants with the study of certain Leray spectral sequences, following the approach of [27,28]. We expect that these techniques could be applied also in other cases of moduli spaces of cyclic covers, at least for small values of $g$.

Finally, we partially extend our investigation to the orbifold cohomology of the Deligne–Mumford compactification $\overline{\mathcal{M}}_3$ of $\mathcal{M}_3$. Specifically, we study the Zariski closure of $I(\mathcal{M}_3)$ inside the inertia stack $I(\overline{\mathcal{M}}_3)$. The connected components of this compactification are precisely the connected components of $I(\overline{\mathcal{M}}_3)$ whose general