EXISTENCE OF COMPATIBLE CONTACT STRUCTURES 
ON $G_2$–MANIFOLDS

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ABSTRACT. In this paper, we show the existence of (co-oriented) contact structures on certain classes of $G_2$-manifolds, and that these two structures are compatible in certain ways. Moreover, we prove that any seven-manifold with a spin structure (and so any manifold with $G_2$-structure) admits an almost contact structure. We also construct explicit almost contact metric structures on manifolds with $G_2$-structures.

1. Introduction

Let $(M, g)$ be a Riemannian 7-manifold whose holonomy group $Hol(g)$ is the exceptional Lie group $G_2$ (or, more generally, a subgroup of $G_2$). Then $M$ is naturally equipped with a covariantly constant 3-form $\varphi$ and 4-form $\ast \varphi$. We can define $(M, \varphi, g)$ as the $G_2$-manifold with $G_2$ structure $\varphi$.

We can also define a (co-oriented) contact manifold as a pair $(N, \xi)$ where $N$ is an odd dimensional manifold and $\xi$, called a (co-oriented) contact structure, is a totally non-integrable (co-oriented) hyperplane distribution on $N$.

In dimension 7, so far contact geometry and $G_2$ geometry have been studied independently and each geometry has very distinguished characteristics which are rather different than those in the other. A basic example of such differences is the following: In contact geometry there are no local invariants, in other words, every contact 7-manifold is locally contactomorphic to $\mathbb{R}^7$ equipped with the standard contact structure. On the other hand, in $G_2$ geometry it is the $G_2$ structure itself that determines how local neighborhoods of points look like, and as a result, manifolds with $G_2$ structures can look the same only at a point, [7], [9].

The aim of this paper is to initiate a new interdisciplinary research area between contact and $G_2$ geometries. More precisely, we study the existence of (almost) contact structures on 7-dimensional manifolds with (torsion free) $G_2$-structures.

2000 Mathematics Subject Classification. 53C38, 53D10, 53D15, 57R17.
Key words and phrases. (Almost) contact structures, $G_2$ structures.

The first named author is partially supported by NSF FRG grant DMS-1065910.
The third named author is partially supported by NSF grant DMS-1105663.
The paper is organized as follows: After the preliminaries (Section 2), we show the existence of almost contact structures on 7-manifolds with spin structures in Section 3. In particular, we prove the following theorem:

**Theorem:** Every manifold with $G_2$-structure admits an almost contact structure.

In Section 4, we define $A$- and $B$-compatibility between contact and $G_2$ structures, and also present the motivating example for $\mathbb{R}^7$. We also prove the nonexistence result:

**Theorem:** Let $(M, \varphi)$ be a manifold with $G_2$-structure such that $d\varphi = 0$. If $M$ is closed (i.e., compact and $\partial M = \emptyset$), then there is no contact structure on $M$ which is $A$-compatible with $\varphi$.

In Section 5, for any non-vanishing vector field $R$ on a manifold $M$ with $G_2$-structure $\varphi$, we construct explicit almost contact structure, denoted by $(J_R, R, \alpha_R, g_\varphi)$, and indeed prove the following theorems:

**Theorem:** Let $(M, \varphi)$ be a manifold with $G_2$-structure. Then the quadruple $(J_R, R, \alpha_R, g_\varphi)$ defines an almost contact metric structure on $M$ for any non-vanishing vector field $R$ on $M$. Moreover, such a structure exists on any manifold with $G_2$-structure.

**Theorem:** Let $(M, \varphi)$ be a manifold with $G_2$-structure. Suppose that $\xi$ is a contact structure on $M$ such that $(J_R, R, \alpha_R, g_\varphi)$ is an associated almost contact metric structure for $\xi$. Then $\xi$ is $A$-compatible.

In Section 6, we define contact--$G_2$--structures on 7-manifolds and analyze their relations with $A$-compatible contact structures, the main results of that section are:

**Theorem:** Let $(M, \varphi)$ be a manifold with $G_2$-structure. Assume that there are nowhere-zero vector fields $X$, $Y$ and $Z$ on $M$ satisfying $\iota_Z \varphi = Y^0 \wedge X^0$ where $X^0$ (resp. $Y^0$) is the covariant 1-form of $X$ (resp. $Y$) with respect to the $G_2$-metric $g_\varphi$. Also suppose that $d(\iota_X \iota_Y \varphi) = \iota_X \iota_Y \star \varphi$. Then the 1-form $\alpha := Z^0 = g_\varphi(Z, \cdot)$ is a contact form on $M$ and it defines an $A$-compatible contact structure $\text{Ker}(\alpha)$ on $(M, \varphi)$.

**Theorem:** Let $(\varphi, R, \alpha, f, g)$ be a contact--$G_2$--structure on a smooth manifold $M^7$. Then $\alpha$ is a contact form on $M$. Moreover, $\xi = \text{Ker}(\alpha)$ is an $A$-compatible contact structure on $(M, \varphi)$. In particular, if $M$ is closed, then it does not admit a contact--$G_2$--structure with $d\varphi = 0$.

**Theorem:** Let $(M, \varphi)$ be any manifold with $G_2$-structure. Then every $A$-compatible contact structure on $(M, \varphi)$ determines a contact--$G_2$--structure on $M$. 

Finally, in Section 7, we present some examples of $A$-compatible structures and contact $- G_2$-structures.

2. Preliminaries

2.1. $G_2$-structures and $G_2$-manifolds. A smooth 7-dimensional manifold $M$ has a $G_2$-structure, if the structure group of $TM$ can be reduced to $G_2$. The group $G_2$ is one of the five exceptional Lie groups which is the group of all linear automorphisms of the imaginary octonions $\text{im}O \cong \mathbb{R}^7$ preserving a certain cross product. Equivalently, it can be defined as the subgroup of $GL(7, \mathbb{R})$ which preserves the 3-form

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

where $(x_1, ..., x_7)$ are the coordinates on $\mathbb{R}^7$, and $e^{ijk} = dx^i \wedge dx^j \wedge dx^k$. As an equivalent definition, a manifold with a $G_2$-structure $\varphi$ is a pair $(M, \varphi)$, where $\varphi$ is a 3-form on $M$, such that $(T_pM, \varphi)$ is isomorphic to $((\mathbb{R}^7, \varphi_0)$ at every point $p$ in $M$. Such a $\varphi$ defines a Riemannian metric $g_\varphi$ on $M$. We say $\varphi$ is torsion-free if $\nabla \varphi = 0$ where $\nabla$ is the Levi-Civita connection of $g_\varphi$. A Riemannian manifold with a torsion free $G_2$-structure is called a $G_2$-manifold. Equivalently, the pair $(M, \varphi)$ is called a $G_2$-manifold if its holonomy group (with respect to $g_\varphi$) is a subgroup of $G_2$. As another characterization, one can show that $\varphi$ is torsion-free if and only if $d\varphi = d(*\varphi) = 0$ where “$*$” is the Hodge star operator defined by the metric $g_\varphi$.

The 3-form $\varphi$ also determines the cross product and the orientation top (volume) form $\text{Vol}$ on $M$. In fact, for any vector fields $u, v, w$ on $M$, we have

$$\varphi(u, v, w) = g_\varphi(u \times v, w),$$

$$\iota_u \varphi \wedge \iota_v \varphi \wedge \varphi = 6g_\varphi(u, v) \text{Vol}.$$  

Also we will make use of the following formula as well:

$$u \times (u \times v) = -||u||^2v + g_\varphi(u, v)u.$$
where “\(\iota\)” denotes the interior product.

Using \(R\), we can co-orient \(\xi\) and, as a result, the structure group of the tangent frame bundle can be reduced to \(U(n) \times 1\). Such a reduction of the structure group is called an almost contact structure on \(M\). Therefore, for the existence of a co-oriented contact structure on \(M\), one should first ask the existence of an almost contact structure. We refer the reader to [1] and [7] for more on contact geometry.

**Definition 2.1** ([8]). Let \(M^{2n+1}\) be a smooth manifold. If the structure group of its tangent bundles \(TM^{2n+1}\) reduces to \(U(n) \times 1\), then \(M^{2n+1}\) is said to have an almost contact structure.

### 3. Almost Contact Structures on 7-Manifolds with a Spin Structure

Although no explicit description is given, nevertheless the following result shows the existence of almost contact structures not only on manifolds with \(G_2\)-structures but also on a much wider family of 7-manifolds. Recall that if a manifold admits a spin structure, then its second Stiefel-Whitney class is zero.

**Theorem 3.1.** Every 7-manifold with a spin structure admits an almost contact structure.

**Proof.** Assume that \(M\) is a 7-manifold with spin structure. By definition, \(M\) admits an almost contact structure if and only if the structure group of \(TM\) can be reduced to \(U(3) \times 1\). Equivalently, the associated fiber bundle \(TM[SO(7)/U(3)]\) with fiber \(SO(7)/U(3)\) admits a cross-section [13]. If \(s\) is a cross section of fiber bundle over the the \((i - 1)\)-skeleton of \(M\), then the cohomology class

\[
\partial^i(s) \in H^i(M, \pi_{i-1}(SO(7)/U(3))).
\]

is the obstruction to extending \(s\) over the \(i\)-skeleton. Since we have

\[
\pi_i(SO(7)/U(3)) = 0
\]

unless \(i = 2, 6\), the only obstructions to the existence of such a cross section arise in \(H^i(M, \mathbb{Z})\) for \(i = 3, 7\). In [11], Massey shows that these obstructions are the integral Stiefel-Whitney classes of the associated dimensions. Recall that the integral Stiefel-Whitney classes are defined as the images \(\beta(w_i)\) of the Stiefel-Whitney classes under the Bockstein homomorphism. Here the Bockstein homomorphism is the connecting homomorphism \(\beta : H^i(M, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^{i+1}(M, \mathbb{Z})\) which arises from the short exact sequence

\[
0 \rightarrow \mathbb{Z} \rightarrow 2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0.
\]

Therefore, the obstructions \(\partial^3, \partial^7\) to the existence of an almost contact structures on 7-manifolds are 2-torsion classes.
Now we know that $w_2(M) = 0$ (since $M$ is spin), and hence the third integral Stiefel-Whitney class vanishes, i.e., $\sigma^3 = W_3(M) = \beta(w_2) = 0$. Therefore, the only obstruction lies in the cohomology group $H^7(M)$.

We consider the following cases: First, if $M$ is a closed manifold, then by Poincaré duality $H^7(M) \cong H_0(M) \cong \mathbb{Z}$ and hence the top dimensional obstruction $\sigma^7$ vanishes. Secondly, if $M$ has a boundary, then (again by the duality) we have $\sigma^7 \in H^7(M) \cong H_0(M, \partial M) \cong 0$. Now, if $M$ is non-compact without a boundary, then the cohomology group $H^7(M) \cong (H^0_{cs}(M))^*$ where $H_{cs}$ denotes the compactly supported cohomology. Hence, it is torsion-free.

□

Since every manifold with $G_2$-structure is spin, we get

**Corollary 3.2.** Every manifold with $G_2$-structure admits an almost contact structure. □

### 4. Compatibility and the Motivating Example

Assuming the existence of a contact structure on a manifold with a $G_2$-structure, we can also ask if and how these two structures are related. We define two different notions of compatibility between them as follows:

**Definition 4.1.** A (co-oriented) contact structure $\xi$ on $(M, \varphi)$ is said to be $A$-compatible with the $G_2$-structure $\varphi$ if there exist a vector field $R$ on $M$ and a nonzero function $f : M \to \mathbb{R}$ such that $d\alpha = \iota_R \varphi$ for some contact form $\alpha$ for $\xi$ and $fR$ is the Reeb vector field of a contact form for $\xi$.

**Definition 4.2.** A (co-oriented) contact structure $\xi$ on $(M, \varphi)$ is said to be $B$-compatible with the $G_2$-structure $\varphi$ if there are (global) vector fields $X, Y$ on $M$ such that $\alpha = \iota_Y \iota_X \varphi$ is a contact form for $\xi$.

In this paper, we will mainly consider $A$-compatible contact structures. We remark that if $\varphi$ is torsion-free or at least $d\varphi = 0$, then Definition 4.1 makes sense only if $M$ is noncompact or compact with boundary. Indeed, we can easily prove the following:

**Theorem 4.3.** Let $(M, \varphi)$ be a manifold with $G_2$-structure such that $d\varphi = 0$. If $M$ is closed (i.e., compact and $\partial M = \emptyset$), then there is no contact structure on $M$ which is $A$-compatible with $\varphi$.

**Proof.** Suppose $\xi$ is an $A$-compatible contact structure on $(M, \varphi)$. Therefore, $d\alpha = \iota_R \varphi$ for some contact form $\alpha$ for $\xi$ and some nonvanishing vector field $R$. Using the equation (2), we have

$$d\alpha \wedge d\alpha \wedge \varphi = (\iota_R \varphi) \wedge (\iota_R \varphi) \wedge \varphi = 6\|R\|^2 \text{ Vol.}$$
Since $d\varphi = 0$, we have $d\alpha \wedge d\alpha \wedge \varphi = d(\alpha \wedge d\alpha \wedge \varphi)$. Now by Stoke’s Theorem,

$$0 \leq \int_M 6\|R\|^2 \text{Vol} = \int_M d(\alpha \wedge d\alpha \wedge \varphi) = \int_{\partial M} \alpha \wedge d\alpha \wedge \varphi = 0$$

(as $\partial M = \emptyset$). This gives a contradiction.

\[\square\]

For another application of this argument on specific vector fields on manifolds with $G_2$ structures, see [5].

We now explore the relation between the standard contact structure $\xi_0$ and the standard $G_2$-structure $\varphi_0$ on $\mathbb{R}^7$. Indeed, the notion of A- and B-compatibility relies on this motivating example.

Fix the coordinates $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ on $\mathbb{R}^7$. In these coordinates,

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

where $e^{ijk}$ denotes the 3-form $dx_i \wedge dx_j \wedge dx_k$. Consider the standard contact structure $\xi_0$ on $\mathbb{R}^7$ as the kernel of the 1-form

$$\alpha_0 = dx_1 - x_3dx_2 - x_5dx_4 - x_7dx_6.$$ 

For simplicity, throughout the paper we will denote $\partial/\partial x_i$ by $\partial x_i$ (so we have $dx_i(\partial x_j) = \delta_{ij}$). Consider the vector fields

$$R = \partial x_1, \ X = \partial x_7 \text{ and } Y = -x_7\partial x_1 + x_5\partial x_3 - x_3\partial x_5 - \partial x_6 + f\partial x_7$$

where $f : \mathbb{R}^7 \to \mathbb{R}$ is any smooth function (in fact, it is enough to take $f \equiv 0$ for our purpose). By a straightforward computation, we see that

$$d\alpha_0 = \iota_R(\varphi_0), \quad \alpha_0 = \iota_Y \iota_X(\varphi_0).$$

Also observe that $R$ is the Reeb vector field of $\alpha_0$. Note that this contact structure is not unique A-compatible with $\varphi_0$. In fact we have various ways of choosing the contact structures by rotating indexes and signs. For example, the contact structure $\alpha = dx_2 + x_3dx_1 - x_6dx_4 + x_7dx_5$ with $R = \partial x_2$ is another A-compatible contact structure with $\varphi_0$ and by choosing two vectors $X = \partial x_7, Y = \partial x_5 - x_3\partial x_6 + x_6\partial x_3 - x_7\partial x_2 + f\partial x_7$ it is easily seen as being B-compatible with $\varphi_0$. Therefore, we have proved:

**Theorem 4.4.** There are contact structures $\xi$ on $\mathbb{R}^7$ which are both A- and B-compatible with the standard $G_2$-structure $\varphi_0$. \[\square\]

5. **An explicit almost contact metric structure**

We first give an alternative definition of an almost contact structure, and then construct an explicit almost contact structure on a manifold with $G_2$-structure. The reader is referred to [1] for the equivalence between the previous definition (Definition 2.1) and this new one.
**Definition 5.1 (12).** An almost contact structure on a differentiable manifold $M^{2n+1}$ is a triple $(J, R, \alpha)$ consists of a field $J$ of endomorphisms of the tangent spaces, a vector field $R$, and a 1-form $\alpha$ satisfying

(i) $\alpha(R) = 1$,
(ii) $J^2 = -I + \alpha \otimes R$

where $I$ denotes the identity transformation.

For completeness, we provide the proof of the following lemma.

**Lemma 5.2 (12).** Suppose that $(J, R, \alpha)$ is an almost contact structure on $M^{2n+1}$. Then $J(R) = 0$ and $\alpha \circ J = 0$

**Proof.** Since $J^2(R) = -R + \alpha(R)R = -R + 1 \cdot R = 0$, we have either $J(R) = 0$ or $J(R)$ is nonzero vector field whose image is 0. Suppose $J(R)$ is nonzero vector field which is mapped to 0 by $J$. Then from

$0 = J^2(J(R)) = -J(R) + \alpha(J(R)) \cdot R$

we get $J(R) = \alpha(J(R)) \cdot R$, and so $\alpha(J(R)) \neq 0$ (as $J(R) \neq 0$). But then

$J^2(R) = J(J(R)) = J(\alpha(J(R))R) = \alpha(J(R)) \cdot J(R) = [\alpha(J(R))]^2 \cdot R \neq 0$

which contradicts to assumption that $J^2(R) = J(J(R)) = 0$. Hence, we conclude that $J(R) = 0$ must be the case.

Now for any vector $X$, we see that

$J^3(X) = J(J^2(X)) = J((-X) + \alpha(X)R) = -J(X) + J(\alpha(X)R)$

and also we have

$J^3(X) = J^2(J(X)) = -J(X) + \alpha(J(X))R$.

So combining these we compute

$\alpha(J(X))R = J^3(X) + J(X)$

$= -J(X) + J(\alpha(X)R) + J(X) = J(\alpha(X)R)$.

But using the fact $J(R) = 0$ we have

$J(\alpha(X)R) = \alpha(X)J(R) = 0$.

Therefore, $\alpha(J(X)) = 0$ as $R \neq 0$. Hence, $\alpha \circ J = 0$ for any vector $X$. \qed

We can also introduce a Riemannian metric into the picture as suggested in the following definition.

**Definition 5.3 (12).** An almost contact metric structure on a differentiable manifold $M^{2n+1}$ is a quadruple $(J, R, \alpha, g)$ where $(J, R, \alpha)$ is an almost contact structure on $M$ and $g$ is a Riemannian metric on $M$ satisfying

(5) $g(Ju, Jv) = g(u, v) - \alpha(u) \alpha(v)$

for all vector fields $u, v$ in $TM$. Such a $g$ is called a compatible metric.
Remark 5.4. Every manifold with an almost contact structure admits a compatible metric (see [1], for a proof). Also setting $u = R$ in Equation (5) gives $g(JR, Jv) = g(R, v) - \alpha(R)\alpha(v)$. Since $J(R) = 0$, an immediate consequence is that $\alpha$ is the covariant form of $R$, that is, $\alpha(v) = g(R, v)$.

Definition 5.5 ([12]). Let $M$ be an odd dimensional manifold, and $\alpha$ be a contact form on $M$ with the Reeb vector field $R$. Therefore, $d\alpha$ is a symplectic form on the contact structure (or distribution) $\xi = \text{Ker}(\alpha)$. We say that the triple $(J, R, \alpha)$ is an associated almost contact structure for $\xi$ if $J$ is $d\alpha$-compatible almost complex structure on the complex bundle $\xi$, that is

$$d\alpha(JX, JY) = d\alpha(X, Y)$$

for all $X, Y \in \xi$. Furthermore, if $g$ is a metric on $M$, we consider two equations:

$$g(JX, JY) = g(X, Y) - \alpha(X)\alpha(Y) \tag{6}$$

$$d\alpha(X, Y) = g(JX, Y) \tag{7}$$

for all $X, Y \in TM$. We say that $(J, R, \alpha, g)$ is an associated almost contact metric structure if two equations (6) and (7) hold. In this case, $g$ is called an associated metric.

Suppose that $(M, \varphi)$ is a manifold with $G_2$-structure. There might be many ways to construct almost contact metric structures on $(M, \varphi)$. Here we give a particular way of constructing almost contact metric structures on $(M, \varphi)$. Denote the Riemannian metric and the cross product (determined by $\varphi$) by $g_\varphi = \langle \cdot, \cdot \rangle_\varphi$ and $\times_\varphi$, respectively. Suppose that $R$ is a nowhere vanishing vector field on $M$. By normalizing $R$ using $g_\varphi$, we may assume that $\|R\| = 1$. Then using the metric, we define the 1-form $\alpha_R$ as the metric dual of $R$, that is,

$$\alpha_R(u) = g_\varphi(R, u) = \langle R, u \rangle_\varphi.$$

Moreover, using the cross product and $R$, we can define an endomorphism $J_R : TM \to TM$ of the tangent spaces by

$$J_R(u) = R \times_\varphi u.$$

Note that $J_R(R) = 0$, and so $J_R$, indeed, defines a complex structure on the orthogonal complement $R^\perp$ of $R$ with respect to $g_\varphi$. With these, we have

**Theorem 5.6.** Let $(M, \varphi)$ be a manifold with $G_2$-structure. Then the quadruple $(J_R, R, \alpha_R, g_\varphi)$ defines an almost contact metric structure on $M$ for any non-vanishing vector field $R$ on $M$. Moreover, such a structure exists on any manifold with $G_2$-structure.

**Proof.** As before, we will assume that $R$ is already normalized using $g_\varphi$. First, note that $\alpha_R(R) = g_\varphi(R, R) = \|R\|^2 = 1$. Also we have

$$J_R^2(u) = J_R(R \times_\varphi u) = R \times_\varphi (R \times_\varphi u) = -\|R\|^2 u + g_\varphi(R, u)R = -u + \alpha(u)R.$$
where we made use of the identity (3). This shows that the endomorphism $J_R : TM \to TM$ satisfies the condition

$$J_R^2 = -I + \alpha \otimes R.$$

Therefore, the triple $(J_R, R, \alpha_R)$ is an almost contact structure on $M$. Next, we check $g_\varphi$ is a compatible metric with this structure. Using (1) and (3), we compute

$$g_\varphi(J_R u, J_R v) = g_\varphi(R \times_\varphi u, R \times_\varphi v) = \varphi(R, u, R \times_\varphi v, u)$$

$$= -g_\varphi(R \times_\varphi (R \times_\varphi v), u) = -g_\varphi(-\|R\|^2 v + g_\varphi(R, v)R, u)$$

$$= -g_\varphi(-v + g_\varphi(R, v)R, u) = g_\varphi(v, u) - g_\varphi(\alpha_R(v)R, u)$$

$$= g_\varphi(u, v) - \alpha_R(v) \left( g_\varphi(R, u) - g_\varphi(\alpha_R(u)R, v) \right)$$

which holds for all vector fields $u, v$ in $TM$. This proves that $g_\varphi$ satisfies (5). Hence, $(J_R, R, \alpha_R, g_\varphi)$ is an almost contact metric structure on $M$.

For the last statement, we know by [14] that there exists a nowhere vanishing vector field $R$ on any 7-dimensional manifold. In particular, $(J_R, R, \alpha_R, g_\varphi)$ can be constructed on any manifold $M$ with $G_2$-structure $\varphi$.

\[\Box\]

**Theorem 5.7.** Let $(M, \varphi)$ be a manifold with $G_2$-structure, and $(J_R, R, \alpha_R, g_\varphi)$ be an almost contact metric structure on $M$ constructed as above. Suppose that $\xi$ is a contact structure on $M$ such that $(J_R, R, \alpha_R, g_\varphi)$ is an associated almost contact metric structure for $\xi$. Then $\xi$ is $A$-compatible.

**Proof.** By assumption $(J_R, R, \alpha_R, g_\varphi)$ is an associated almost contact metric structure for $\xi$. Therefore, $g_\varphi$ is an associated metric and satisfies

$$d\alpha_R(u, v) = g_\varphi(J_R(u), v)$$

for all $u, v \in TM$.

But then using the equation defining $J_R$ and (1), we obtain

$$d\alpha_R(u, v) = g_\varphi(R \times_\varphi u, v) = \varphi(R, u, v) = i_R \varphi(u, v), \quad \forall u, v \in TM.$$

Therefore, we have $d\alpha_R = i_R \varphi$. Also $R$ is the Reeb vector field of $\alpha_R$ by assumption. Hence, $\xi$ is $A$-compatible by definition.

\[\Box\]

**Corollary 5.8.** Let $(M, \varphi)$ be a manifold with $G_2$-structure such that $d\varphi = 0$, and $(J_R, R, \alpha_R, g_\varphi)$ be an almost contact metric structure on $M$ constructed as above. If $M$ is closed, then there is no contact structure on $M$ whose associated almost contact metric structure is $(J_R, R, \alpha_R, g_\varphi)$.

**Proof.** On the contrary, suppose that $\xi = \text{Ker}(\alpha_R)$ is a contact structure on a closed manifold $M$ equipped with a $G_2$-structure $\varphi$ and $d\varphi = 0$, and also that $(J_R, R, \alpha_R, g_\varphi)$ is an associated almost contact metric structure. Then, by Theorem 5.7, $\xi$ is $A$-compatible, but this contradicts to Theorem 4.3.

\[\Box\]
6. Contact–$G_2$–structures on 7-manifolds

Suppose that $(M, \varphi)$ is a manifold with $G_2$-structure. Let us recall the decomposition of the space $\Lambda^2$ of 2-forms on $M$ obtained from $G_2$-representation and some other useful formulas which we will use. A good source for these is [2] and also [9]. According to irreducible $G_2$-representation, $\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}$, where

\[
\Lambda^2_7 = \{ \iota_v \varphi; v \in \Gamma(TM) \} = \{ \beta \in \Lambda^2; \ast(\varphi \wedge \beta) = -2\beta \}
\]

\[
(8)
\]

\[
\Lambda^2_{14} = \{ \beta \in \Lambda^2; \ast \varphi \wedge \beta = 0 \} = \{ \beta \in \Lambda^2; \ast(\varphi \wedge \beta) = \beta \}
\]

Also on any Riemannian $n$-manifold, for any $k$-form $\alpha$ and a vector field $v$, the following equalities hold:

\[
i_v \ast \alpha = (-1)^k \ast (v^b \wedge \alpha) \quad \text{and}
\]

\[
i_v \alpha = (-1)^{n+k+1} \ast (v^b \wedge \ast \alpha).
\]

As a last one we recall a very useful equality: For any $k$-form $\lambda$, and any $(n+1-k)$-form $\mu$ and any vector field $v$ on a smooth manifold of dimension $n$, we have

\[
(\iota_v \lambda) \wedge \mu = (-1)^{k+1} \lambda \wedge (\iota_v \mu).
\]

Now we are ready to prove:

**Theorem 6.1.** Let $(M, \varphi)$ be a manifold with $G_2$-structure. Assume that there are nowhere-zero vector fields $X$, $Y$ and $Z$ on $M$ satisfying

\[
i_Z \varphi = Y^b \wedge X^b
\]

where $X^b$ (resp. $Y^b$) is the covariant 1-form of $X$ (resp. $Y$) with respect to the $G_2$-metric $g_\varphi$. Also suppose that

\[
d(i_X i_Y \varphi) = i_X i_Y \ast \varphi.
\]

Then the 1-form $\alpha := Z^b = g_\varphi(Z, \cdot)$ is a contact form on $M$ and it defines an $A$-compatible contact structure $\text{Ker}(\alpha)$ on $(M, \varphi)$.

**Proof.** From (8) we know that $\iota_Z \varphi$ is an element of $\Lambda^2_7$. Set $\iota_Z \varphi = \beta \in \Lambda^2_7$, and so we have $\iota_Z \varphi = \beta = Y^b \wedge X^b$ by (12). Also applying (9) twice gives

\[
i_X i_Y \ast \varphi = -i_X (* (Y^b \wedge \varphi)) = -* (X^b \wedge X^b \wedge \varphi) = * (Y^b \wedge X^b \wedge \varphi) = * (\beta \wedge \varphi)
\]

from which we get

\[
i_X i_Y \varphi = -2\beta
\]
where we use the second line in (8). Moreover, by (10) followed by (9),

\[
(15) \quad i_X i_Y \varphi = i_X (* (Y^\flat \wedge * \varphi)) = - * (X^\flat \wedge Y^\flat \wedge * \varphi) = * (\beta \wedge * \varphi).
\]

Now putting (14) and (15) into (13) gives us

\[
(16) \quad d * (\beta \wedge * \varphi) = -2 \beta = -2 i_Z \varphi.
\]

Recall the formula \((i_v \varphi) \wedge * \varphi = 3 * v^\flat\) which is true for any vector field \(v\). By taking \(v = Z\), we compute the left-hand side in (16) as

\[
\frac{1}{3} Z \wedge \alpha \wedge (d\alpha) = -Z \wedge \alpha \wedge (d\alpha).
\]

Combining these together we obtain

\[
(17) \quad d\alpha = -\frac{2}{3} \iota_Z \varphi.
\]

Next, consider the identity (11) by taking \(\lambda = \varphi\), \(v = Z\) and \(\mu = \alpha \wedge (d\alpha)^2\): Using (17), we compute the left-hand side as

\[
(\iota_Z \varphi) \wedge \alpha \wedge (d\alpha)^2 = -\frac{3}{2} \alpha \wedge (d\alpha)^3,
\]

and the right-hand side as

\[
\varphi \wedge \iota_Z (\alpha \wedge (d\alpha)^2) = \alpha(Z) \varphi \wedge d\alpha \wedge d\alpha = \frac{4}{9} \|Z\|^2 \varphi \wedge (\iota_Z \varphi) \wedge (\iota_Z \varphi).
\]

Therefore, by using the identity (2) in the right-hand side, we obtain

\[
\alpha \wedge (d\alpha)^3 = -\frac{16}{9} \|Z\|^4 \text{Vol}.
\]

Hence, we conclude that \(\alpha \wedge (d\alpha)^3\) is a volume form on \(M\) (as being a nonzero function multiple of the volume form \(\text{Vol}\) on \(M\) determined by the metric \(g_\varphi\)). Equivalently, \(\alpha\) is a contact form on \(M\). Moreover, it follows from (17) that \((1/\|Z\|^2)Z\) is the Reeb vector field of \(\alpha\), i.e., it satisfies (4). Hence, \(\text{Ker}(\alpha)\) is an \(A\)-compatible contact structure on \((M, \varphi)\).

With the inspiration we get from the proof of Theorem 6.1, we define a new structure on 7-manifolds as follows:

**Definition 6.2.** Let \(M^7\) be a smooth manifold. A \textit{contact}–\(G_2\)–\textit{structure} on \(M\) is a quintuple \((\varphi, R, \alpha, f, g)\) where \(\varphi\) is a \(G_2\)-structure, \(R\) is a nowhere-zero vector field, \(\alpha\) is a 1-form on \(M\), and \(f, g : M \to \mathbb{R}\) are nowhere-zero smooth functions such that

(i) \(\alpha(R) = f\)

(ii) \(d(g \alpha) = \iota_R \varphi\).

Observe that we have already seen an example of a contact–\(G_2\)–structure in the above proof (of course under the assumptions of Theorem 6.1) with \(R = Z, \alpha = Z^\flat, f = \|Z\|^2, g \equiv -3/2\). The reason why we call the quintuple \((\varphi, R, \alpha, f, g)\) “contact–\(G_2\)–structure” is given by the following theorem.
Theorem 6.3. Let \((\varphi, R, \alpha, f, g)\) be a contact–\(G_2\)–structure on a smooth manifold \(M\). Then \(\alpha\) is a contact form on \(M\). Moreover, \(\xi = \text{Ker}(\alpha)\) is an \(A\)-compatible contact structure on \((M, \varphi)\). In particular, if \(M\) is closed, then it does not admit a contact–\(G_2\)–structure with \(d\varphi = 0\). 

Proof. We first show that \(\alpha\) is a contact form on \(M\). Consider the 1-form 
\[ \alpha' := g \alpha. \]
Note that \(\text{Ker}(\alpha) = \text{Ker}(\alpha')\) as \(g\) is a nowhere-zero function. Therefore, if we show that \(\alpha'\) is a contact form on \(M\), then it will imply that so is \(\alpha\). The conditions in Definition 6.2 translate into 
\[ \alpha'(R) = fg \quad \text{and} \quad d\alpha' = \imath_R \varphi. \]
Also from the equation (2) we get 
\[ (d\alpha')^2 \wedge \varphi = (\imath_R \varphi) \wedge (\imath_R \varphi) \wedge \varphi = 6 \|R\|^2 \text{Vol}. \]
Now if we write the equation (11) by taking \(\lambda = \varphi, \mu = \alpha' \wedge (d\alpha')^2\) and \(v = R\), then the left-hand side gives 
\[ (\imath_R \varphi) \wedge \alpha' \wedge (d\alpha')^2 = (d\alpha') \wedge \alpha' \wedge (d\alpha')^2 = \alpha' \wedge (d\alpha')^3; \]
and from the right-hand side we have 
\[ \varphi \wedge \imath_R (\alpha' \wedge (d\alpha')^2) = \alpha'(R) \varphi \wedge (d\alpha')^2 = fg \varphi \wedge (d\alpha')^2 = 6 fg \|R\|^2 \text{Vol}. \]
Therefore, we conclude 
\[ \alpha' \wedge (d\alpha')^3 = 6 fg \|R\|^2 \text{Vol} \]
which implies that \(\alpha'\) (and so \(\alpha\)) is a contact form on \(M\) as \(6 fg \|R\|^2\) is a nowhere-zero function on \(M\).

Next, we consider the vector field \(R' = (1/fg)R\). Clearly, \(\alpha'(R') = 1\). Also we compute 
\[ \imath_{R'} d\alpha' = (1/fg) \imath_R d\alpha' = (1/fg) \imath_R (\imath_R \varphi) = 0 \]
as \(\varphi\) is skew-symmetric. Therefore, \(R'\) is the Reeb vector field of \(\alpha'\), and so \(\xi = \text{Ker}(\alpha')\) is an \(A\)-compatible contact structure on \((M, \varphi)\) by definition. Finally, the last statement now follows from Theorem 4.3. \(\square\)

The next result shows that we can go also in the reverse direction.

Theorem 6.4. Let \((M, \varphi)\) be any manifold with \(G_2\)-structure. Then every \(A\)-compatible contact structure on \((M, \varphi)\) determines a contact–\(G_2\)–structure on \(M\).

Proof. Let \(\xi\) be a given \(A\)-compatible contact structure on \((M, \varphi)\). By definition, there exist a non-vanishing vector field \(R\) on \(M\), a contact form \(\alpha\) for \(\xi\) and a nowhere-zero function \(h : M \to \mathbb{R}\) such that \(d\alpha = \imath_R \varphi\) and \(hR\) is the Reeb vector field of some contact form (possibly different than \(\alpha\)) for \(\xi\). Being a Reeb vector field, \(hR\) is transverse to the contact distribution \(\xi\). Therefore,
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$R$ is also transverse to $\xi$ because $h$ is nowhere-zero on $M$. As a result, there must be a nowhere-zero function $f : M \to \mathbb{R}$ such that

$$\alpha(R) = f.$$  

To check this, assume, on the contrary, that the function $M \to \mathbb{R}$ given by $x \mapsto \alpha_x(R_x)$ has a zero, say at $p$. So, we have $\alpha_p(R_p) = 0$ which means that $R_p \in \ker(\alpha_p) = \xi_p$. But this contradicts to the fact that $R$ is everywhere transverse to $\xi$. Hence, we obtain a contact $-G_2$–structure $(\varphi, R, \alpha, f, 1)$. This finishes the proof. 

□

7. SOME EXAMPLES

In this final section, we give some examples of $G_2$–manifolds admitting $A$-compatible contact structures. In fact, by Theorem 6.4 in each example we will also have a corresponding contact $-G_2$–structure.

7.1. $CY \times S^1$ (or $CY \times \mathbb{R}$). Consider a well-known example of $G_2$–manifold $(CY \times S^1, \varphi)$ where we assume $CY(\Omega, \omega)$ is a 3-fold Calabi-Yau manifold which is either noncompact or compact with boundary. Assume Kähler form $\omega$ on $CY$ is exact, i.e. $\omega = d\lambda$ for some $\lambda \in \Omega^1(CY)$ and set $\alpha = dt + \lambda$ where $t$ is the coordinate on $S^1$. Then $\alpha \wedge (d\alpha)^3 = \omega^3 \wedge dt$ is a volume form, and so $\alpha$ is a contact 1-form on $CY \times S^1$. Moreover, $\partial t$ is the Reeb vector field of $\alpha$ as $i_{\partial t} \alpha = 1$ and $i_{\partial t} d\alpha = i_{\partial t} \omega = 0$. Also observe that since $\varphi = \text{Re}(\Omega) + \omega \wedge dt$ see [10], for instance), we compute $i_{\partial t} \varphi = i_{\partial t}(\text{Re}(\Omega) + \omega \wedge dt) = i_{\partial t} \text{Re}(\Omega) + i_{\partial t}(\omega \wedge dt) = \omega i_{\partial t} dt = \omega = d\lambda = d\alpha$.

Thus, $\xi = \ker(\alpha)$ is an A-compatible contact structure on $(CY \times S^1, \varphi)$, or in other words, $(\varphi, \partial t, \alpha, 1, 1)$ is a contact $-G_2$–structure on $CY \times S^1$. We note that, by considering $t$ as a coordinate on $\mathbb{R}$, the above argument also gives a contact $-G_2$–structure on $CY \times \mathbb{R}$.

7.2. $W \times S^1$ (or $W \times \mathbb{R}$). We now give a special case of the above example. First, we need some definitions: A Stein manifold of complex dimension $n$ is a triple $(W^{2n}, J, \psi)$ where $J$ is a complex structure on $W$ and $\psi : W \to \mathbb{R}$ is a smooth map such that the 2–form $\omega_\psi = -d(\psi \circ J)$ is non-degenerate (and so an exact symplectic form) on $W$. Indeed, $(W, J, \omega_\psi)$ is an exact Kähler manifold. We say that $(M^{2n-1}, \xi)$ is Stein fillable if there is a Stein manifold $(W^{2n}, J, \psi)$ such that $\psi$ is bounded from below, $M$ is a non-critical level of $\psi$, and $-d(\psi \circ J)$ is a contact form for $\xi$.

Next, consider a parallelizable Stein manifold $(W, J, \psi)$ of complex dimension three. By a result of [8], we know that $c_1(W, J) = 0$, i.e., the first Chern class of $(W, J)$ vanishes. Therefore, $W$ admits a Calabi-Yau structure with associated
Kähler form $\omega_\varphi = -d(d\varphi \circ J)$. Let $\Omega$ be the non-vanishing holomorphic 3-form on $W$ corresponding to this Calabi-Yau structure. Then by the previous example, $(W \times S^1, \varphi)$ is a $G_2$-manifold with $\varphi = Re(\Omega) + \omega_\varphi \wedge d\theta$ (where $\theta$ is the coordinate on $S^1$), $\alpha = d\theta - (d\psi \circ J)$ is a contact 1-form on $W \times S^1$ with the Reeb vector field $\partial \theta$, and $\xi = \text{Ker}(\alpha)$ is an $A$-compatible contact structure on $(W \times S^1, \varphi)$. Again by considering $\theta$ as a coordinate on $\mathbb{R}$, we obtain an $A$-compatible contact structure on $(W \times \mathbb{R}, \varphi)$. Note that the corresponding contact-$G_2$-structure in both cases is $(\varphi, \partial \theta, \alpha, 1, 1)$.

Now consider the unit disk $\mathbb{D}^2 \subset \mathbb{C}$. Then $(W \times \mathbb{D}^2, J \times i, \psi + |z|^2)$ is a Stein manifold where $i$ is the usual complex structure and $z = re^{i\theta}$ is the coordinate on $\mathbb{C}$. Let $\eta$ be the induced contact structure on the boundary

$$\partial(W \times \mathbb{D}^2) = (\partial W \times \mathbb{D}^2) \cup (W \times S^1).$$

Then we remark that the restriction of the Stein fillable structure $\eta$ on $W \times S^1$ is the contact structure $\xi$ constructed above.

7.3. $\mathbb{R}^3 \times K^4$. Let $K$ be a Kähler manifold with an exact Kähler form $\omega$, i.e. $\omega = d\lambda$ for some $\lambda \in \Omega^1(K)$. Note that $K$ is either noncompact or compact with boundary. Consider the $G_2$-manifold $\mathbb{R}^3 \times K^4$ with the $G_2$-structure

$$\varphi = dx_1dx_2dx_3 + \omega \wedge dx_1 + Re(\Omega) \wedge dx_2 - Im(\Omega) \wedge dx_3$$

where $(x_1, x_2, x_3)$ are the coordinates on $\mathbb{R}^3$ (see [10]). Then $\alpha = dx_1 + x_2dx_3 + \lambda$ is a contact 1-form as $\alpha \wedge (d\alpha)^3 = dx_1dx_2dx_3 \wedge \omega^2$ is a volume form on $\mathbb{R}^3 \times K^4$.

One can easily check that $\partial x_1$ is the Reeb vector field of $\alpha$. Furthermore,

$$i_{\partial x_1}\varphi = i_{\partial x_1}(dx_1dx_2dx_3 + \omega \wedge dx_1 + \omega \wedge dx_2 + \omega \wedge dx_3)$$

$$= dx_2dx_3 + i_{\partial x_1}(\omega dx_1) = dx_2dx_3 + \omega = d(x_2dx_3 + \lambda) = d\alpha$$

Hence, $\xi = \text{Ker}(\alpha)$ is an $A$-compatible contact structure on $(\mathbb{R}^3 \times K^4, \varphi)$ and the corresponding contact-$G_2$-structure on $\mathbb{R}^3 \times K^4$ is $(\varphi, \partial x_1, \alpha, 1, 1)$.

7.4. $T^*M^3 \times \mathbb{R}$. Let $M$ be any oriented Riemannian 3-manifold and $T^*M$ denote the cotangent bundle of $M$. It is shown in [11] that $T^*M \times \mathbb{R}$ has a $G_2$-structure $\varphi$ with $d\varphi = 0$. To describe $\varphi$, let $(x_1, x_2, x_3)$ be local coordinates on $M$ around a given point, and consider the corresponding standard local coordinates $(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3)$ on the cotangent bundle $T^*M$. These define the standard symplectic structure $\omega = -d\lambda$ on $T^*M$ where $\lambda = \Sigma_{i=1}^3 \xi_i dx_i$ is the tautological 1-form on $T^*M$. Let $t$ denote the coordinate on $\mathbb{R}$. Then $\varphi = Re(\Omega) - \omega \wedge dt$ where $\Omega = (dx_1 + id\xi_1) \wedge (dx_2 + id\xi_2) \wedge (dx_3 + id\xi_3)$ is the complex-valued $(3,0)$-form on $M$. On the other hand, the 1-form $\alpha = dt + \lambda$ is a contact form on $T^*M \times \mathbb{R}$ with the Reeb vector field $\partial t$. Now it is straightforward to check that $\xi = \text{Ker}(\alpha)$ is an $A$-compatible contact structure on $(T^*M \times \mathbb{R}, \varphi)$ and also that $(\varphi, \partial t, \alpha, 1, 1)$ is the corresponding contact-$G_2$-structure on $T^*M \times \mathbb{R}$.
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