Local cosmological effects of the order of $H$ in the orbital motion of a binary system

L. Iorio★

1 Ministero dell’Istruzione, dell’Università e della Ricerca (M.I.U.R.), Viale Unità di Italia 68 Bari, (BA) 70125, Italy

1 INTRODUCTION

In this paper, we first deal with a certain hypothetical anomalous radial acceleration proportional to the radial velocity of the orbital motion of a two-body system through a coefficient $H$ having dimensions of $T^{-1}$. Some intuitive, Newtonian-like guesses about the possibility that such a putative acceleration may exist as a local manifestation of the cosmic expansion in the case of non-circular motions are offered in Section 2. They are motivated by the known fact that, within certain limits, several well established key features of a homogeneous and isotropic expanding universe, and also of its influence on local gravitationally bound systems, can be practically inferred within a classical framework (Harrison 1965; Tipler 1996; Carrera & Giuliani 2010; Bertelli 2012; Fabris & Velten 2012).

Such a putative extra-acceleration may, in principle, have interesting phenomenological consequences since it would induce peculiar orbital signatures which would not be mimicked by any other known competing dynamical effect. Moreover, by assuming for $H$ a value equal to that of the Hubble parameter at present epoch, the magnitude of these exotic effects for the planets of our Solar System would be close to the current level of accuracy in determining their orbits. Tensions might even occur between data and predictions in the case of Mercury and Mars. These topics are treated in Section 3.

In Section 4, we try unsuccessfully to find a theoretical justification for the guesses of Section 2 rooted in a full general relativistic treatment of the orbital dynamics of a local system embedded in an expanding homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime metric. Nonetheless, our results remain valid from a phenomenological point of view because of their actual independence of any specific theoretical scheme, and can be viewed as observational constraints on such a putative exotic force, whatever the physical mechanism yielding it (if any) may be. Moreover, we feel that the numerical values coming out from our analysis are interesting if compared with the observations, and may pursue further investigations to find a possible physical origin, cosmological or not, for it. Cosmological effects linear in $H$ were recently derived by Kopeikin (Kopeikin 2012) for the propagation of electromagnetic waves between atomic clocks in geodesic motion in a FLRW background.

In Section 5, we build on the certain aspects discussed in pre-
vious Sections and provide some very general viability criteria that must be met by modified models of gravity (Clifton et al. 2012) in order not to give rise to unphysical observable effects. In particular, it is important to check the behaviour of their detectable predictions in the limits $G \to 0$, $M \to 0$, where $G$ is the Newtonian constant of gravitation and $M$ is the mass of the central body acting as localized source of the gravitational field.

Section 6 provides an overview of the results obtained.

2 A LOCAL COSMOLOGICAL EFFECT LINEAR IN $H$ FOR TWO-BODY ORBITAL DYNAMICS?

Let us define the Hubble parameter $H$ and the deceleration parameter $q$ in the usual way as

$$ H = \frac{S}{\dot{S}} , $$

$$ q = -\frac{1}{H^2} \left( \frac{\dot{S}}{S} \right) , $$

where $S(t)$ is the cosmological scale factor. From an observational point of view, the value of the Hubble parameter at present epoch is $H_0 \doteq 73.8 \pm 2.4$ km s$^{-1}$ Mpc$^{-1}$ (Riess et al. 2011),

$$ H_0 \doteq 100 \text{ km s}^{-1} \text{ Mpc}^{-1} , $$

so that

$$ h = 0.738 \pm 0.024 . $$

The deceleration parameter can be connected to directly determined quantities from observations by means of $\Omega_m$ (Serjeant 2010)

$$ q_0 = \frac{\Omega_{m,0}}{2} - \Omega_{\Lambda,0} , $$

where $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ are the current matter and dark energy densities, respectively, in units of the critical density (Serjeant 2010)

$$ \rho_0^{m,0} = \frac{3H_0^2}{8\pi G} . $$

From (Jarosik et al. 2011)

$$ \Omega_{m,0} h^2 = 0.1334^{+0.0056}_{-0.0035} , $$

$$ \Omega_{\Lambda,0} = 0.728^{+0.015}_{-0.016} , $$

and eq. (4), it turns out

$$ q_0 \sim -0.7 . $$

Let us now consider a two-body system composed by a central object of mass $M$ and a test particle gravitationally bound to it. Their proper motion is determined by their mutual gravitational interaction according to the Newtonian/Einsteinian laws. It superimposes on the global Hubble flow in such a way that the radial velocity of the test particle is the sum of such two contributions

$$ v_r = \dot{r} = v_{orb} + Hr , $$

In it, $r$ is their relative distance and

$$ v_{orb} = \frac{n_a e \sin f}{\sqrt{1 - e^2}} , $$

is the radial component of the velocity vector $v$ of the test particle along its standard two-body Keplerian ellipse where $a$ is the semimajor axis, $e$ is the eccentricity, $f$ is the true anomaly, and $n_a = \sqrt{GM/a^3}$ is the Keplerian mean motion. As shown by Table 1 the contribution of the Hubble flow to the planetary radial velocities is quite negligible in the Solar System.

When the accelerations are computed, neglecting the proper motion yields the well known Hooke-like term quadratic in $H$. Indeed, starting from the Hubble law

$$ \dot{r} = Hr , $$

yields

$$ A_r = \dot{H} r + Hr . $$

By recalling that

$$ \dot{H} = \frac{\ddot{S}}{S} - \frac{\dot{S}^2}{2} , $$

and by using eq. (2) and eq. (12) one gets just

$$ A_r^{(H^2)} = -qH^2 r . $$

If we include $v_{orb}$ in the second term of eq. (13), we get an additional term linear in $H$

$$ A_r^{(HN)} = Hv_{orb} $$

at Newtonian level.

3 A COSMOLOGICAL TEST PARTICLE ACCELERATION OF ORDER $HV_0$: A PHENOMENOLOGICALLY APPEALING POSSIBILITY

Table 2 shows the values of $A_r^{(HN)}$ and $A_r^{(H^2)}$ compared with the Newtonian monopoles for the planets of the Solar System. It can be noticed that the orders of magnitude of $A_r^{(HN)}$ and $A_r^{(H^2)}$ are completely different, being the terms linear in $H$ of the order of $10^{-14} - 10^{-16}$ m s$^{-2}$. Interestingly, such values are neither too large to unrealistically compromise the agreement between theory and observations nor too small to be completely undetectable in any foreseeable future. Indeed, according to a recent analysis of various kind of planetary data by Folkner (Folkner 2009), the largest unmodelled radial acceleration in the Solar System allowed by observations is just of the order of $10^{-12}$ m s$^{-2}$. More precisely, from the current Mars radio range data set, Folkner (Folkner 2005) found upper bounds on a radial acceleration of Earth and Mars to be less than $3 \times 10^{-14}$ m s$^{-2}$ and $8 \times 10^{-14}$ m s$^{-2}$, respectively. From Cassini radiotechnical data Folkner (Folkner 2009) inferred an upper bound of $1 \times 10^{-14}$ m s$^{-2}$ for Saturn. Thus, according to Table 2, $A_r^{(HN)}$ seems to be not too far from the edge of the present-day detectability. Moreover, the maximum values for $A_r^{(HN)}$ quoted in Table 2 are not in contrast with the upper bounds by Folkner (Folkner 2005) for Earth, Mars and Saturn. On the other hand, it must be remarked that Folkner (Folkner 2005) did not release details about the time and/or spatial variability of the anomalous acceleration constrained. This fact generally does matter since $A_r^{(HN)}$ is not constant, so that the constraints by Folkner (Folkner 2005) may not be straightforwardly applicable to $A_r^{(HN)}$.

We now explicitly work out some dynamical orbital effects induced by eq. (15) on some quantities routinely determined from observations by astronomers. The standard Gauss equations (Bertotti et al. 2003) for the variations of the Keplerian orbital elements, applied to eq. (15) evaluated onto the unperturbed Keplerian
Local cosmological effects of the order of $H$ in orbits

The evaluated rates of change of the semimajor axis $a$ and the eccentricity $e$

\[ \dot{a}^{(\text{H})} = 2aH_0 \left(1 - \sqrt{1 - e^2}\right) = aH_0 e^2 + O(e^4), \]
\[ \dot{e}^{(\text{H})} = \frac{H_0}{e} \left(1 - \sqrt{1 - e^2}\right) = \frac{H_0 e}{2} \left(1 - \frac{3}{4}e^2\right) + O(e^4). \]

It is intended that eq. (17)–eq. (18) are averages over one full orbital revolution of the test particle. The other orbital elements are left unaffected. Note that, according to eq. (17)–eq. (18), both the semimajor axis and the eccentricity increase. This implies that the mean distance $d = a \left(1 + e^2/2\right)$ increases at a rate

\[ \ddot{d}^{(\text{H})} = 3aH_0 \left(1 - \sqrt{1 - e^2}\right) = \frac{3aH_0 e^2}{2} + O(e^4). \]

Rather surprisingly, eq. (17)–eq. (19) describe small changes occurring in the otherwise close orbit of a gravitationally bound system, do not depend on $M$; in the limit $M \rightarrow 0$, eq. (17)–eq. (19) do not vanish. The limit $G \rightarrow 0$ does not pose problems in the sense that eq. (17)–eq. (19) correctly vanish. Indeed, in a spatially flat FLRW universe, it is

\[ H^2 = \frac{8\pi G \rho}{3}, \]

where $\rho$ is the density of the cosmic fluid, inclusive of any dark energy component. In Table 1 we calculate eq. (17)–eq. (19) for the planets of the Solar System. It is interesting to compare the values in Table 1 with the experimental bounds in Table 2, preliminarily inferred from a multi-year fit by Pitjeva (Pitjeva 2007) for the EPM2006 ephemerides. In general, the predicted rates of Table 1 are compatible with the bounds of Table 2 although discrepancies occur for Mercury and Mars at $4 - \sigma$ level. It must be stressed that, so far, astronomers did not explicitly determine corrections to the standard Newtonian rates of change of $a$ and $e$ from observations: the figures in Table 1 were inferred rather naively by simply taking the ratios of the formal, statistical uncertainties in the orbital elements, rescaled by a factor 10, to the time span of the fit performed by Pitjeva (Pitjeva 2007). In view of future analyses, it is important to remark that eq. (17)–eq. (19), assumed as real physical effects, show very distinctive patterns since there are no other known competing orbital effects on $a$ and $e$, both Newtonian and Einsteinian. Indeed, general relativity does not predict any long-term rates of change for such orbital elements: the Schwarzschild

---

**Table 1.** Keplerian two-body and cosmological radial velocities of the planets of the Solar System, in km s$^{-1}$. As far as the Keplerian two-body velocities are concerned, the maximum values of eq. (11) (sin $f = 1$) were taken. The average planetary distances $d = a \left(1 + \sqrt{1 - e^2}\right)$ were used in the Hubble terms which were evaluated at the present epoch according to (Riess et al. 2011) $H_0 = 73.8$ km s$^{-1}$ Mpc$^{-1} = 2.39 \times 10^{-18}$ s$^{-1}$.

| Planet | $\psi_a^{(\text{orb})}$ (km s$^{-1}$) | $H_0d$ (km s$^{-1}$) |
|--------|---------------------------------|-------------------|
| Mercury | 10.05                           | $1 \times 10^{20}$ |
| Venus  | 0.23                            | $2 \times 10^{20}$ |
| Earth  | 0.49                            | $3 \times 10^{20}$ |
| Mars   | 2.26                            | $5 \times 10^{20}$ |
| Jupiter| 0.59                            | $2 \times 10^{20}$ |
| Saturn | 0.52                            | $3 \times 10^{20}$ |
| Uranus | 0.31                            | $7 \times 10^{20}$ |
| Neptune| 0.04                            | $1 \times 10^{20}$ |
| Pluto  | 1.22                            | $1 \times 10^{20}$ |

---

**Table 2.** Newtonian and cosmological accelerations of the planets of the Solar System, in m s$^{-2}$. As far as the Hubble terms of order $H$ are concerned, the maximum values of eq. (11) (sin $f = 1$) were taken. The average planetary distances $d = a \left(1 + \sqrt{1 - e^2}\right)$ were used in both the Newtonian and the Hubble terms of order $H^2$ which were evaluated at the present epoch according to (Riess et al. 2011) $H_0 = 73.8$ km s$^{-1}$ Mpc$^{-1} = 2.39 \times 10^{-18}$ s$^{-1}$ and (Riess et al. 2011) $q_0 = -0.7$.

| Planet | $GM/a^2$ (m s$^{-2}$) | $H_0\psi_a^{(\text{orb})}$ (m s$^{-2}$) | $-q_0H_0^2d$ (m s$^{-2}$) |
|--------|----------------------|-------------------------------------|-------------------------|
| Mercury| $4 \times 10^{-2}$   | $2.4 \times 10^{-14}$               | $2 \times 10^{-25}$     |
| Venus  | $1 \times 10^{-2}$   | $5 \times 10^{-16}$                 | $4 \times 10^{-25}$     |
| Earth  | $6 \times 10^{-3}$   | $1.2 \times 10^{-15}$               | $5 \times 10^{-25}$     |
| Mars   | $2 \times 10^{-3}$   | $5.4 \times 10^{-15}$               | $8 \times 10^{-25}$     |
| Jupiter| $2 \times 10^{-4}$   | $1.4 \times 10^{-15}$               | $2.6 \times 10^{-24}$   |
| Saturn | $6 \times 10^{-5}$   | $1.2 \times 10^{-15}$               | $4.9 \times 10^{-24}$   |
| Uranus | $2 \times 10^{-5}$   | $7 \times 10^{-16}$                 | $9.8 \times 10^{-24}$   |
| Neptune| $6 \times 10^{-6}$   | $1 \times 10^{-16}$                 | $1.54 \times 10^{-23}$  |
| Pluto  | $4 \times 10^{-6}$   | $2.9 \times 10^{-15}$               | $2.09 \times 10^{-23}$  |

© 0000 RAS, MNRAS 000, 000–000
field of a spherical static body affects just the pericenter and the mean anomaly of a test particle, while the Lense-Thirring effect due to the rotation of the central body causes secular precessions of the node and the pericenter. As far as Newtonian mechanics is concerned, the centrifugal oblateness of the primary does not influence the anomaly of a test particle, whereas the Lense-Thirring effect just the pericenter and the longitude of the ascending node.

\[ \Delta Y = \int_0^{P_0} dY = \int_0^{P_0} \left( \frac{dY}{dt} \right) dt = \int_0^{2\pi} \left( \frac{\partial Y}{\partial f} \frac{dM}{dt} + \sum_k \frac{\partial Y}{\partial k} \frac{dk}{dt} \right) df + \frac{\partial Y}{\partial \kappa} \frac{d\kappa}{dt} \int df, \quad \kappa = a, e, I, \omega, \Omega, \]

(21)

where \( P_0 = 2\pi/n_0 \) is the orbital period along the Keplerian ellipse, and \( M, I, \omega, \Omega \) are the mean anomaly, the inclination, the argument of pericenter and the longitude of the ascending node, respectively, of the orbit of the test particle. \( Y \) appearing in eq. (21) is the analytical expression for the observable \( Y \) evaluated onto the unperturbed Keplerian orbit. \( dM/dt \) and \( dk/dt, \kappa = a, e, I, \omega, \Omega \) in eq. (21) are the instantaneous rates of change of the Keplerian orbital elements given by the right-hand-sides of the Gauss equations evaluated onto the unperturbed Keplerian ellipse. Expressions for \( df/dM, dt/df \) can be found in standard textbooks.

The shift per orbit of the radial distance

\[ r = \frac{a(1-e^2)}{1+e \cos f} \]

(22)

\footnote{A time-dependent \( G(t) \) would change \( a \) as well, but not as predicted by eq. (19).}

\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
Planet & \( \dot{a}^{(HN)} \) (m cty\(^{-1} \)) & \( \dot{e}^{(HN)} \) (cty\(^{-1} \)) \\
\hline
Mercury & 18.68 ± 0.61 & (7.5 ± 0.2) \times 10^{-10} \\
Venus & 0.03 ± 0.001 & (2 ± 0.08) \times 10^{-11} \\
Earth & 0.31 ± 0.01 & (6 ± 0.2) \times 10^{-11} \\
Mars & 15.04 ± 0.49 & (3.5 ± 0.1) \times 10^{-10} \\
Jupiter & 11.94 ± 0.38 & (1.7 ± 0.05) \times 10^{-10} \\
Saturn & 31.26 ± 1.01 & (2.0 ± 0.06) \times 10^{-10} \\
Uranus & 46.56 ± 1.51 & (1.7 ± 0.05) \times 10^{-10} \\
Neptune & 2.47 ± 0.08 & (3 ± 0.1) \times 10^{-11} \\
Pluto & 2849.75 ± 92.67 & (9.0 ± 0.3) \times 10^{-10} \\
\hline
\end{tabular}
\caption{Predicted cosmological rates of change of the semimajor axis \( a \) and the eccentricity \( e \) of the planets of the Solar System for \( \dot{a}^{(HN)} = \dot{a}^{(MNRAS)} \) and \( \dot{e}^{(HN)} = \dot{e}^{(MNRAS)} \).} \label{tab:cosmological_rates}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
Planet & \( \sigma_a \) (m cty\(^{-1} \)) & \( \sigma_e \) (cty\(^{-1} \)) \\
\hline
Mercury & 3.6 & 4 \times 10^{-9} \\
Venus & 2.3 & 2 \times 10^{-10} \\
Earth & 1.5 & 5 \times 10^{-11} \\
Mars & 2.8 & 5 \times 10^{-11} \\
Jupiter & 6612.9 & 2 \times 10^{-8} \\
Saturn & 45763.4 & 1 \times 10^{-7} \\
Uranus & 433269.0 & 3 \times 10^{-7} \\
Neptune & 4.9818 \times 10^6 & 8 \times 10^{-7} \\
Pluto & 3.66961 \times 10^7 & 3 \times 10^{-6} \\
\hline
\end{tabular}
\caption{Uncertainties in the rates of change of the semimajor axis \( a \) and the eccentricity \( e \) of the planets of the Solar System. They were inferred by taking the ratios of the formal errors in Table 3 of (Pitjeva 2007), all rescaled by a factor 10, to the data time span \( \Delta T = 93 \text{ yr} (1913-2006) \) of the EPM2006 ephemerides used by Pitjeva (Pitjeva 2007). The results for Saturn are relatively inaccurate with respect to those of the inner planets since radiotechnical data from Cassini were not yet processed when Table 3 of (Pitjeva 2007) was produced. Here cty stands for century.} \label{tab:uncertainties}
\end{table}
turns out to be
\[ \Delta \rho^{(\text{HN})} = -\frac{3\pi a H_0}{n_b e^2} (1 - e^2) (2 + e^2 \sqrt{1 - e^2}) = -\frac{3\pi a H_0 e^2}{4n_b} + O(e^4) \] (23)

The radial velocity is left unaffected.

The shift per orbit of the line-of-sight projection of the orbit of a binary system in the sky
\[ \rho = r \sin I \sin(\omega + f), \] (24)
which is practically determined from timing measurements of compact objects, is
\[ \Delta \rho^{(\text{HN})} = -\frac{3\pi a H_0}{n_b} \sin I \frac{\omega}{\sqrt{1 - e^2}} \frac{1}{2} + O(e^4). \] (25)

where, in this case, \( I \) is the inclination of the orbital plane to the plane of the sky. Curiously, both eq. (23) and eq. (25) are independent of \( G \) because of eq. (20), while they depend on \( M \) in such a way that they diverge in the limit \( M \to 0 \). Such a singularity might be explained by noticing that both eq. (23) and eq. (25) are the outcome of a perturbative calculation in which a Keplerian ellipse was assumed as unperturbed, reference orbit. Taking the limit \( M \to 0 \) implies a breakdown of the validity of such an approximation since \( A^{(\text{HN})} \propto v_s \propto \sqrt{M} \), and, for \( M \to 0 \), the Newtonian monopole \( A^{(\text{HN})} = GM/r^2 \) becomes smaller than \( A^{(\text{HN})} \).

The shift per orbit of the radial velocity
\[ v_r = \frac{n a \sin I}{\sqrt{1 - e^2}} \left[ e \cos \omega + \cos(\omega + f) \right], \] (26)
which is a typical spectroscopic observable in binaries studies, turns out to be
\[ \Delta v_r^{(\text{HN})} = -\frac{2\pi a H_0}{n_b} \sin I \frac{\omega}{\sqrt{1 - e^2}} \frac{1}{2} + O(e^4) \]
\[ = \frac{5\pi a H_0}{2} \sin I \frac{\omega}{\sqrt{1 - e^2}} \frac{7}{10} + O(e^4). \] (27)

Note that eq. (27) is independent of \( M \); in the limit \( M \to 0 \) eq. (27) does not vanish. Instead, in the limit \( G \to 0 \) eq. (27) correctly vanishes because of eq. (20).

4 THEORETICAL MOTIVATIONS AGAINST THE EXISTENCE OF A COSMOLOGICAL \( \Lambda \) TERM IN THE EQUATIONS OF MOTION OF TEST PARTICLES

Actually, although appealing, the existence of eq. (19) does not seem to be justified by a theoretical analysis of particle dynamics rooted in general relativity.

The influence of the cosmic expansion on the gravitation fields surrounding individual objects was the subject of several investigations since earlier times (M. H. McVittie 1933, 1945, 1946; Schücking 1954; Bonnor 2000, Adkins et al. 2007, Mashhoon et al. 2007, Sereno & Jetzer 2007; Kopeikin 2012, Nandra et al. 2012); for a recent review covering several aspects, see Carrera and Giulini (2010). Connections with possible local variations of cosmologically varying constants were elucidated by Shaw and Barrow in Shaw & Barrow (2006a, 2006b).}

Local cosmological effects of the order of \( H \) in orbits 5

\[ g_{00} = 1 - 2\mu - \hbar^2, \] (28)
\[ g_{01} = \frac{\hbar}{\sqrt{1 - 2\mu}}, \] (29)
\[ g_{02} = g_{03} = 0, \] (30)
\[ g_{11} = \frac{1}{1 - 2\mu}, \] (31)
\[ g_{22} = -r^2, \] (32)
\[ g_{33} = -r^2 \sin^2 \theta, \] (33)
\[ g_{12} = g_{13} = g_{23} = 0, \] (34)

where
\[ \hbar = \frac{H(t)r}{c}, \] (36)

The equations of motion can be obtained from the Lagrangian
\[ \mathcal{L} = -\frac{1}{2} g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu, \] (37)
where the dot denotes derivation with respect to the proper time \( \tau \), as
\[ \frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\nu} - \frac{\partial \mathcal{L}}{\partial x^\nu} = 0. \] (38)

As far as the radial equation of motion is concerned, the terms independent of \( c \) in \( \partial \mathcal{L}/\partial r \) and \( \partial \mathcal{L}/\partial \dot{r} \)
\[ \frac{\partial \mathcal{L}}{\partial r} = r \dot{\theta}^2 + \left( \frac{GM + H^2 r^2}{r^2} \right) \dot{\theta}^2 - H r (1 - 3\mu)/(1 - 2\mu)^{3/2} + O(c^{-2}) \] (39).

The generalized momentum for \( r \) is
\[ p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\dot{r}}{1 - 2\mu} - \frac{H r (1 - 3\mu)}{\sqrt{1 - 2\mu}}. \] (40)

As usual, we posed \( \theta = \pi/2 \).
In the weak-field and slow-motion approximation \(1 \gg \mu, c \to \infty, i \to 1\), eq. \((39)\) reduces to
\[
\frac{\partial L}{\partial r} \to r \ddot{r} - \frac{GM}{r^2} + H^2 r - Hr, \tag{41}
\]
while eq. \((40)\) becomes
\[
\frac{\partial L}{\partial r} \to \dot{r} - Hr. \tag{42}
\]
Thus, eq. \((42)\) yields
\[
d \left( \frac{\partial L}{\partial r} \right) = \dot{r} + qH^2 r + H^2 r - Hr. \tag{43}
\]
By equating eq. \((43)\) and eq. \((41)\) yields
\[
\dot{r} - r \ddot{r} = -\frac{GM}{c^2} - qH^2 r, \tag{44}
\]
the left-hand-side is nothing but the radial acceleration in polar coordinates, while both the terms \(-Hr\) appearing in eq. \((43)\) and eq. \((41)\) canceled each other in eq. \((44)\).

Arakida (Arakida 2011) recently studied the effects of the McVittie metric on the gravitational time delay.

Another way to realize that standard general relativity does not predict the existence of eq. \((16)\) for a two-body system immersed in a FLRW expanding universe consists of looking at the cosmological impact on the binary system through the generalized Jacobi equation as a tidal effect in the local Fermi frame. (Hodgkinson 1972; Mashhoon 1973; Chicone & Mashhoon 2002).

The generalized Jacobi equation is suitable for our purposes since it actually takes into account the relative velocity of the geodesics immersed in a FLRW expanding universe consists of looking at the cosmological impact on the binary system through the generalization approach, did not find eq. \((16)\) in the equations of motion \((40)\), which amounts to Serjeant (2010)
\[
\omega_{(40)} \approx -\frac{3\mu H_0^2 \sqrt{1 - \epsilon^2}}{2n_0}, \tag{48}
\]
gets singular in the limit \(G \to 0\) because of \(n_0^{-1}\) in eq. \((45)\); actually, it is not so since Sereno & Jetzer (2007) by assuming eq. \((15)\) much smaller than the Newtonian monopole \(A(0)\) which, instead, would become smaller than eq. \((15)\) for \(M \to 0\). The propagation of electromagnetic waves continuously exchanged between two atomic clocks geodesically moving in an expanding FLRW universe retains a cosmological imprint of order \(H_0^2\) large enough to allow for a possible detection in accurate range-rate Doppler experiments (Kopeikin 2012). According to Kopeikin (Kopeikin 2012), it may provide an explanation of the Pioneer anomaly (Turyshev & Toth 2010) as well.

5 GENERAL CONSIDERATIONS ON THE VIABILITY OF MODIFIED MODELS OF GRAVITY FROM THEIR OBSERVATIONAL CONSEQUENCES

The discussions of the previous sections allow us to provide preliminary general criteria of viability of non-standard models of gravity by inspecting the behaviour of their observable consequences.

Let us suppose to have a given modified model of gravity \(M\) yielding a test particle extra-acceleration \(A(0)\) that can be viewed as a small correction to \(A(0)\) in appropriate circumstances. Depending on the type of model, the parameter(s) entering \(A(0)\), collectively denoted as \(\alpha\), may or may not contain explicitly \(M\) and \(G\). Let us, now, suppose to work out a certain observable effect \(X(0)\), say, an orbital extra-precession which slowly alters the otherwise unperturbed Keplerian ellipse. It is clear that, if we ideally switch off gravity, \(X(0)\) must vanish in the limit \(G \to 0\), at least for a certain class of modified models. Clearly, the same should occur in the limit \(M \to 0\) as well, unless certain conditions pertaining how \(X(0)\) is calculated must be satisfied. That poses certain basic constraints on the viability of the model \(M\) and on the nature of its building blocks parameterized by \(\alpha\). It is important to note that it may happen that a well-behaved \(A(0)\) yields a \(X(0)\) which, instead, is not. As far as the condition on vanishing \(G\) is concerned, it must be remarked that, strictly speaking, it applies only to a particular class of modified gravity models where the perturbations introduced by...
them to general relativity scale with $G$ itself; it has not necessarily a general validity.

More precisely, let us consider the case of the Cosmological Constant $\Lambda$ and the orbital effects it causes on a gravitationally bound binary system. The resulting particle acceleration is $A^{(\Lambda)} = \frac{\Lambda c^2 r}{3}$, (50)

which yields the orbital precession $\dot{\omega}^{(\Lambda)} = \frac{\Lambda c^2 \sqrt{1 - e^2}}{2n_0}$. (51)

Contrary to what might seem at first sight, $\dot{\omega}^{(\Lambda)}$ is, actually, well-behaved with respect to $G$. Indeed, from $\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$ (52)

and eq. (6), it is possible to write $\dot{\omega}^2 = 3\Omega_\Lambda H^2$. (53)

Since, according to eq. (20), $H^2 \propto G$ in a spatially flat universe, we see from eq. (51) that $\dot{\omega}^{(\Lambda)} \propto \sqrt{G}$; thus, $\dot{\omega}^{(\Lambda)} \rightarrow 0$ in the limit $G \rightarrow 0$. The mass $M$ of the central body enters eq. (51) as $M^{3/2}$ through the Keplerian mean motion $n_0$. As discussed in the previous Sections, it is meaningless to consider the limit $M \rightarrow 0$ for the precession of eq. (51) since it was obtained perturbatively, while $A^{(\Lambda)}/A^{(\Lambda)} < 1$ for $M \rightarrow 0$.

As an example of a modified model, let us consider the Dvali-Gabadadze-Porrati (DGP) model (Dvali et al. 2000). It predicts an acceleration $A^{(DGP)} = \mp \left( \frac{c}{2r_0} \right) \sqrt{\frac{GM}{r}}$, (54)

which is well-behaved with respect to the limits $G \rightarrow 0, M \rightarrow 0$; $r_0$ is a free length scale determined by observations of cosmological nature, while the $\mp$ sign depends on the cosmological expansions phases. As far as the observational consequences of eq. (54) are concerned, it yields an orbital precession $\dot{\omega}^{(DGP)} \sim \pm \frac{3c}{8r_0} + O(c^{-2})$. (55)

which is pathologically independent of both $G$ and $M$. The same drawback is common to Galileon-based models as well (Nicolis et al. 2009; Burrage & Seery 2010) yielding orbital precession $\dot{\omega}^{(G)} \sim \frac{\psi}{c^2 n_0}$, (56)

where $\psi$ may or may not depend on $GM$ along with other parameters $\alpha$. It can be shown perturbatively that the resulting orbital precessions are $\dot{\omega}^{(A^{(\Lambda)})} \sim \frac{\psi}{n_0}$; (57)

if $\psi$ does not contain $G$, an unphysical singularity would appear in eq. (57) for $G \rightarrow 0$. From this point of view, the Randall-Sundrum model (Randall & Sundrum 1999) is well-behaved since $\psi \propto GM\ell^2$ in it, where $\ell$ is the anti-de Sitter (AdS) curvature scale.

The Kehagias-Sfetsos (KS) solution of the Hořava-Lifshitz (HL) modified gravity (Hořava 2009) predicts the existence of an additional radial acceleration $A^{(KS)} \sim \frac{3G^2 M^4}{\phi_0 c^5 r^2}$, (58)

where $\phi_0$ is a parameter of the HL model. The orbital precession induced by eq. (58) is $\dot{\omega}^{(KS)} \sim \frac{3G^2 M^{7/2}}{2\phi_0 c^5 r^{3/2}}$, (59)

and the lack of singularity is due to the fact that the Newtonian monopole remains always larger than $A^{(KS)}$ for vanishingly small values of $M$.

Clearly, if a model is satisfactory from the point of view illustrated here, this does not necessarily imply it is really valid as a good description of (a part of) the physical reality: it has to pass other independent tests. On the other hand, as already remarked, not all modified models induce perturbations scaling with $G$ itself.

6 CONCLUSIONS

We investigated the possibility that the cosmological expansion may impact the orbital motion of a localized two-body system at first order in the Hubble parameter $H$.

Reasoning classically, intuitive guesses lead us to postulate, at Newtonian level, the existence of an additional radial acceleration $A^{(R)}$ of order $H$ proportional to the radial velocity $v_{\phi}$ of the proper motion of the test particle with respect to the primary. By considering $A^{(R)}$ as a small correction to the Newtonian monopole $A^{(N)}$, we perturbatively worked out its long-term effects on the Keplerian orbital elements of the test particle and on its distance from the primary, and, in the case of a binary in the plane of sky, on the projection of its orbit onto the line-of-sight and on its radial velocity. It turned out that both the semimajor axis $a$ and the eccentricity $e$ of the test particle would secularly increase. The analytical expressions of their rates of change are independent of the mass $M$ of the primary, so that they do not vanish in the limit $M \rightarrow 0$, contrary to the formal expectations. Such a feature constitutes a general requisite to be satisfied in the search of potentially viable alternative, i.e. non-cosmological, physical mechanisms able to provide $A^{(R)}$.

Nonetheless, the consequences of $A^{(R)}$ would be interesting from a phenomenological point of view since their magnitude is close to the current level of accuracy in determining the planetary orbits in our Solar System.

Then, we looked for theoretical justifications of the existence of $A^{(R)}$ in three different frameworks within general relativity. None of them provided it at Newtonian level, contrary to the known Hooke-like term of order $H^2$. Instead, as recently pointed out in literature, a velocity-dependent acceleration $A^{(R)}_{v \propto v}$ of order $H$ and directed along the velocity $v$ of the test particle exists at post-Newtonian level. We perturbatively worked out its orbital effects by finding secular rates of change of $a$ and $e$ proportional to the Schwarzschild radius $r_s$ of the primary. They are well-behaved since they correctly vanish in the limit $M \rightarrow 0$. For a planet of our Sun such effects are negligibly small: suffice it to say that the semimajor axis increases at a rate of just 20 $\mu m$ per century. We discussed the limits of validity of the orbital precession due to the $H^2$ term. We noticed that the formal singularity occurring in it for $M \rightarrow 0$ actually has no physical meaning since such a limit is beyond the
regime of validity of the perturbative calculation yielding the precession itself.

Finally, we extended some points emerged in dealing with the previous cosmological issues by discussing certain basic criteria of viability that certain classes of modified models of gravity should generally meet in view of their predicted observable effects like, e.g., orbital precessions. Given that such modified gravities must reduce to small perturbations of standard Newtonian gravity in appropriate circumstances, their precessions, which slowly alter an otherwise fixed Keplerian ellipse, must necessarily vanish in the limit of no gravity, i.e. for $G \to 0$, at least as far as modified gravities whose perturbations to general relativity scale with $G$ are concerned; it is not necessarily valid in all cases. The same should occur also for $M \to 0$, unless such a limit violates the validity of the perturbative regime in which the precessions are calculated.

REFERENCES

Adelberger E. G., Gundlach J. H., Heckel B. R., Hoedl S., Schlamminger S., 2009, Progress in Particle and Nuclear Physics, 62, 102

Adelberger E. G., Heckel B. R., Hoedl S., Hoyle C. D., Kapner D. J., Upadhye A., 2007, Physical Review Letters, 98, 131104

Adkins G. S., McDonnell J., 2007, Physical Review D, 75, 082001

Adkins G. S., McDonnell J., Fell R. N., 2007, Physical Review D, 75, 064011

Arakida H., 2011, General Relativity and Gravitation, 43, 2127

Barrow J. D., Shaw D. J., 2007, General Relativity and Gravitation, 39, 1235

Bertello U., 2012, Master Thesis. University of Helsinki. Department of Physics

Bertotti B., Farinella P., Vokrouhlický D., 2003, Physics of the Solar System. Kluwer Academic Press, Dordrecht

Bonnor W. B., 2000, Classical and Quantum Gravity, 17, 2739

Burrell C., Seery D., 2010, Journal of Cosmology and Astroparticle Physics, 8, 11

Carrera M., Giulini D., 2010, Reviews of Modern Physics, 82, 169

Chicone C., Mashhoon B., 2010, Classical and Quantum Gravity, 19, 4231

Clifton T., Ferreira P. G., Padilla A., Skordis C., 2012, Physics Reports, 513, 1

Cooperstock F. I., Faraoni V., Vollick D. N., 1998, The Astrophysical Journal, 503, 61

Dvali G., Gabadadze G., Porrati M., 2000, Physics Letters B, 485, 208

Einstein A., Straus E. G., 1945, Reviews of Modern Physics, 17, 120

Einstein A., Straus E. G., 1946, Reviews of Modern Physics, 18, 148

Fabris J. C., Velten H., 2012, arXiv:1207.0060 [physics.space-ph]

Folkner W., 2009, in Klioner S., Seidelman P., Soifer M., eds, Proceedings of the IAU Symposium, Relativistic aspects of the jpl planetary ephemeris. Cambridge University Press, Cambridge, pp 155–158

Harrison E. R., 1965, Annals of Physics, 35, 437

Hodgkinson D. E., 1972, General Relativity and Gravitation, 3, 351

Horava P., 2008, Journal of High Energy Physics, 3, 031

Iorio L., 2005, Classical and Quantum Gravity, 22, 5271

Iorio L., 2008, Advances in Astronomy, 2008

Iorio L., 2012a, Journal of Cosmology and Astroparticle Physics, 7, 1

Iorio L., 2012b, Annalen der Physik, 524, 371

Jarosik N., Bennett C., Dunkley J., Gold B., Greason M., Halpern M., Hill R., Hinshaw G., Kogut A., Komatsu E., Larson D., Limon M., Meyer S., Nolta M., Odegard N., Page L., Smith K., Spiegel D., Tucker G., Weiland J., Wollack E., Wright E., 2011, The Astrophysical Journal Supplement, 192, 14

Kaloper N., Kleban M., Martin D., 2010, Physical Review D, 81, 104044

Kehagias A., Sfetsos K., 2009, Physics Letters B, 678, 123

Kerr A. W., Hauck J. C., Mashhoon B., 2003, Classical and Quantum Gravity, 20, 2727

Kopeikin S., 2012, Physical Review D, 86, 064004

Lake K., Abdelqader M., 2011, Physical Review D, 84, 044045

Lue A., Starkman G., 2003, Physical Review D, 67, 064002

Mashhoon B., 1975, The Astrophysical Journal, 197, 705

Mashhoon B., Mobed N., Singh D., 2007, Classical and Quantum Gravity, 24, 5031

McVittie G. C., 1933, Monthly Notices of the Royal Astronomical Society, 93, 325

Nandra R., Lasenby A. N., Hobson M. P., 2012, Monthly Notices of the Royal Astronomical Society, 422, 2931

Nicolis A., Rattazzi R., Trincherini E., 2009, Physical Review D, 79, 064036

Pitjeva E., 2007, in Jiw W., Platais I., Perryman M., eds, A Giant Step: from Milli- to Micro-arcsecond Astrometry Vol. 248 of Proceedings IAU Symposium, Use of optical and radio astrometric observations of planets, satellites and spacecraft for ephemeris astronomy. Cambridge University Press, Cambridge, pp 20–22

Randall L., Sundrum R., 1999, Physical Review Letters, 83, 4690

Riess A. G., Macri L., Casertano S., Lampeitl H., Ferguson H. C., Filippenko A. V., Jha S. W., Li W., Hornoch R., 2011, The Astrophysical Journal, 730, 119

Schücking E., 1954, Zeitschrift fur Physik, 137, 595

Serenno M., Jetzer P., 2007, Physical Review D, 75, 064031

Serdjev S., 2010, Observational Cosmology. Cambridge University Press, Cambridge, pp 20–22