Ergodicity, eigenstate thermalization, and the foundations of statistical mechanics
in quantum and classical systems

Lorenzo Campos Venuti$^1$ and Lawrence Liu$^1$

$^1$Department of Physics and Astronomy, University of Southern California, CA 90089, USA
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Boltzmann’s ergodic hypothesis furnishes a possible explanation for the emergence of statistical mechanics in the framework of classical physics. In quantum mechanics, the Eigenstate Thermalization Hypothesis (ETH) is instead generally considered as a possible route to thermalization. This is because the notion of ergodicity itself is vague in the quantum world and it is often simply taken as a synonym for thermalization. Here we show, in an elementary way, that when quantum ergodicity is properly defined, it is, in fact, equivalent to ETH. In turn, ergodicity is equivalent to thermalization, thus implying the equivalence of thermalization and ETH. This result previously appeared in [De Palma et al., Phys. Rev. Lett. 115, 220401 (2015)], but becomes particularly clear in the present context. We also show that it is possible to define a classical analogue of ETH which is implicitly assumed to be satisfied when constructing classical statistical mechanics. Classical and quantum statistical mechanics are built according to the familiar standard prescription. This prescription, however, is ontologically justified only in the quantum world.

Introduction. A possible mechanistic justification of classical statistical mechanics proceeds via the ergodic hypothesis of Boltzmann, i.e., the assumption—to be proven in the cases at hand—that a given classical Hamiltonian dynamical system is ergodic. As we will see, this is, in fact, not the whole story and more is needed. In any case, in the quantum world it is not entirely clear what constitutes a meaningful notion of ergodicity, let alone whether or not such a notion implies thermalization as it does classically. In fact often quantum ergodicity is not separately defined but simply taken as a synonym for thermalization (see e.g. [1] or footnote 1 in [2]). Recently, the Eigenstate Thermalization Hypothesis (ETH) has emerged as a promising hypothesis to explain thermalization in the framework of quantum mechanics as ETH trivially implies thermalization [3–6]. Quantum ergodicity, however, can and has been precisely defined in different settings (see e.g., [7–11]). In this paper we give a few characterizations of the notion of ergodicity in the quantum world. It is shown that ergodicity is indeed equivalent to thermalization. Moreover, ergodicity is also seen to be equivalent to ETH, thus implying at once an equivalence between ETH and thermalization. That ergodicity is indeed equivalent to thermalization, thus implying the equivalence of thermalization and ETH. This result is not entirely clear what constitutes a meaningful notion of ergodicity in the quantum world. As we will see, this is, in fact, not the whole story and more is needed. In any case, in the quantum world it is not entirely clear what constitutes a meaningful notion of ergodicity, let alone whether or not such a notion implies thermalization as it does classically. In fact often quantum ergodicity is not separately defined but simply taken as a synonym for thermalization (see e.g. [1] or footnote 1 in [2]). Recently, the Eigenstate Thermalization Hypothesis (ETH) has emerged as a promising hypothesis to explain thermalization in the framework of quantum mechanics as ETH trivially implies thermalization [3–6]. Quantum ergodicity, however, can and has been precisely defined in different settings (see e.g., [7–11]). In this paper we give a few characterizations of the notion of ergodicity in the quantum world. It is shown that ergodicity is indeed equivalent to thermalization. Moreover, ergodicity is also seen to be equivalent to ETH, thus implying at once an equivalence between ETH and thermalization. That ergodicity is indeed equivalent to thermalization, thus implying the equivalence of thermalization and ETH. This result previously appeared in [De Palma et al., Phys. Rev. Lett. 115, 220401 (2015)], but becomes particularly clear in the present context. We also show that it is possible to define a classical analogue of ETH which is implicitly assumed to be satisfied when constructing classical statistical mechanics. Classical and quantum statistical mechanics are built according to the familiar standard prescription. This prescription, however, is ontologically justified only in the quantum world.

Ergodicity in quantum physics. Our starting point is to give a meaningful definition of ergodicity in the quantum setting. It will be useful first to recall various equivalent characterizations of ergodicity in the classical setting (see [13]).

Theorem 1. (Characterizations of ergodicity) Let $(M, g^t, \mu)$ be a measure-preserving system. $M$ is a measure space, $g^t$ a flow, and $\mu$ a normalized, $g^t$-invariant measure on $M$. Denoting $\langle f \rangle_\mu = \int_M f(x) \, d\mu(x)$, the following are equivalent:

1. Any (Borel) set $X \subseteq M$ which is almost invariant ($g^t(X)$ differs from $X$ by a null set for all $t$) has either full measure or zero measure.

2. For any $f, g \in L^\infty(M, \mu)$,
   \[
   \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \langle f(t)g(t) \rangle_\mu = \langle f \rangle_\mu \langle g \rangle_\mu.
   \]

3. For any $f \in L^1(M, \mu)$, the averages $T^{-1} \int_0^T dt \circ g^t$ converge pointwise almost everywhere to $\langle f \rangle_\mu$.

Let us now switch to quantum mechanics. The analogue of the triple $(M, g^t, \mu)$ (also valid in infinite dimension), is, not surprisingly, given by a quantum dynamical system comprising a $C^*$-algebra, a dynamical evolution, and a quantum state. In this paper we use the standard point of view that the approach to thermodynamic equilibrium can be understood by studying increasingly large systems of finite size. Hence we make the assumption that the total Hilbert space $\mathcal{H}$ is finite-dimensional. The system’s Hamiltonian acting on $\mathcal{H}$ can then be written as $H = \sum E_n \Pi_n$ ($E_n$ eigenenergies, $\Pi_n$ possibly degenerate eigen-projectors). Since we are dealing with an isolated system we consider, as usual, a collection $\mathbb{A}$ of energy eigenvalues (usually called a shell, and typically but not necessarily of the form $\mathbb{A} = \{E | E \leq E_n \leq E + \Delta\}$). Let $\mathcal{H}_\mathbb{A}$ be the corresponding Hilbert space, $\Pi_{\mathbb{A}}$ the orthogonal projector onto it, $\Pi_{\mathbb{A}} = \sum_{n \in \mathbb{A}} \Pi_n$, and $\mathcal{S}_\mathbb{A}$ the set of quantum states with support on $\mathcal{H}_\mathbb{A}$. The Schrödinger dynamics is $\mathcal{S}_\mathbb{A}(\cdot) = U_t \cdot \Pi_{\mathbb{A}} U^\dagger_t$ with $U_t = e^{-iHt}$. The role played by the measure $\mu$ is now taken by a quantum state $\rho_\mathbb{A} \in \mathcal{S}_\mathbb{A}$ invariant under the dynamics (whence $[\rho_\mathbb{A}, \Pi_n] = 0$ for all $n$ and $\rho_\mathbb{A} = \Pi_{\mathbb{A}} \rho_\mathbb{A} = \rho_\mathbb{A} \Pi_{\mathbb{A}}$). The equivalent of the “phase space average” is $\langle A \rangle_{\mathbb{A}} = tr(\mathbb{A} \rho_{\mathbb{A}})$ where $A$ is an observable. In principle $A$ is defined only on $\mathcal{H}_\mathbb{A}$ but it is useful to consider observables defined on the whole space: $A \in \mathbb{B}(\mathcal{H})$. Henceforth we write $\mathcal{X}(t) = \lim_{T \to \infty} T^{-1} \int_0^T dt \mathcal{X}(t)$.

We now define thermalization for a specific observable. This definition is essentially the same as in the classical case but we single out a particular observable to leave open the possibility that only some (but not all) observables thermalize. Moreover, for clarity of exposition and in analogy with the
classical case, we first consider exact ergodicity. We will relax this condition later.

**Definition 1.** We say that \( A \) thermalizes on \( \mathcal{H}_V \) (with equilibrium state \( \rho_V \)) if \( \text{tr}(A(t)\rho_0) = \langle A \rangle_V \) for all \( \rho_0 \in \mathcal{S}_V \).

As we have seen there are several equivalent characterization of ergodicity in classical dynamical systems. We first consider characterization 2 of Theorem 1 which can be trivially reformulated quantum mechanically. Once again we retain the possibility of ergodicity only for some specific observables.

**Definition 2.** We say that an observable \( A \) is ergodic on the energy shell \( V \) (shell-ergodic) if \( \langle A(t)A \rangle_V = (\langle A \rangle_V)^2 \).

This definition appears in [8, 9]. We now give an alternative characterization of ergodicity which may help clarify its meaning.

**Proposition 1.** The observable \( A \) is shell-ergodic if and only if \( A(t)\Pi_V = \langle A \rangle_V \Pi_V \).

**Proof.** Note that \( [\mathcal{A}(t), \Pi_V] = 0 \) so \( \mathcal{A}(t)\Pi_V = \Pi_V \mathcal{A}(t) = \Pi_V \mathcal{A}(t) \Pi_V \). The \( \Leftarrow \) direction is clear: multiply by \( A \rho_V \) since \( \rho_V \in \mathcal{S}_V \). \( \rho_V = \Pi_V \rho_V \Pi_V \) and take the trace. For the other direction, we use the auxiliary result \( \mathcal{A}(t)\Pi_V = \langle A \rangle_V \Pi_V \) (also valid classically). We define the dephasing operator \( \mathcal{D}(A) = \sum_n \Pi_n A \Pi_n = A(t) \) and its complement \( Q = I - \mathcal{D} \). Both \( \mathcal{D} \) and \( Q \) are orthogonal projectors with respect to the Hilbert–Schmidt scalar product \((X, Y)_{HS} = \text{tr}(X^\dagger Y)\). Since \( \rho_V \) is diagonal in the energy eigenbasis \( \rho_V = \mathcal{D}(\rho_V) \). So \( \text{tr}(A \rho_V) = \langle A \mathcal{D}(\rho_V) \rangle_{HS} = \langle \mathcal{D}(A) \rho(V) \rangle_{HS} = \langle A(t) \rangle_V \). In a similar way \( \langle A(t) \rangle_V = \text{tr}(A \rho_V) = \text{tr}(A \mathcal{D}(\rho_V) \Pi_V) = \langle A \rangle_V \langle A \rangle_V \Pi_V \). With these results \( \langle A(t) \rangle_V = (\langle A \rangle_V)^2 \) can be written as \( (\langle A \rangle_V - \langle A \rangle_V)^2 \Pi_V = 0 \). Finally, by Lemma 1 (see Appendix A), \( \langle A(t) - \langle A \rangle_V \rangle \Pi_V = 0 \); that is, \( A \) is shell-ergodic. \( \square \)

It should be clear that this is the quantum mechanical equivalent to the standard ergodicity statement (characterization 3), according to which time averages of functions are the constant functions a.e. with values given by the equilibrium averages. Indeed, the analogue of a constant function in quantum mechanics is a projector. Moreover, since invariant spaces in quantum mechanics are linear subspaces, the usual restriction “for almost any initial state” loses its meaning. We will see later how a similar condition can be re-introduced in quantum mechanics. Note that this is essentially the definition of ergodicity given for abstract C*-algebras [10, 11]. There, however, the statement is taken for all observables in the algebra and here the shell Hilbert space appears.

The following result illustrates the connection between thermalization and (shell-)ergodicity.

**Proposition 2.** An observable \( A \) thermalizes if and only if it is shell-ergodic.

**Proof.** The \( \Leftarrow \) direction is obvious. For the other direction, thermalization means \( \text{tr}(A(t) - \langle A \rangle_V)\rho_0 = 0 \), \( \forall \rho_0 \in \mathcal{S}_V \). By Lemma 2 (see Appendix A), \( \Pi_V (A(t) - \langle A \rangle_V) \Pi_V = 0 \). Noting that \( A(t) \) commutes with \( \Pi_V \), we obtain \( \langle A(t) - \langle A \rangle_V \rangle \Pi_V = 0 \).

At this point we are ready to recall the definition of ETH. There are two points to note. First, the ETH is never supposed to be valid for the entire spectrum but only for the levels in some shell, here \( V \). The other point is that ETH is naturally made up of two statements, a diagonal one and an off-diagonal one. It will be useful to separate them.

**Definition 3.** ETH-D. An observable \( A \) satisfies the diagonal eigenstate thermalization hypothesis with respect to \( V \) and \( \rho_V \), if

\[
\Pi_n A \Pi_n = \langle A \rangle_V \Pi_n, \quad \forall E_n \in V.
\]

Of course if the eigen-projectors \( \Pi_n \) are one-dimensional this reduces to the standard, diagonal part of the ETH given in many references.

**Definition 4.** ETH-O. An observable \( A \) satisfies the off-diagonal eigenstate thermalization hypothesis with respect to \( V \) and \( \rho_V \), if \( \Pi_n A \Pi_m = 0, \forall E_n, E_m \in V, n \neq m \).

If an observable satisfies both ETH-D and ETH-O we will simply talk of ETH. ETH-O can have an important impact on the relaxation time to the equilibrium state but not on the nature of the equilibrium state itself [14].

We now come to one of the main results of this paper.

**Proposition 3.** An observable \( A \) is shell-ergodic if and only if it satisfies ETH-D.

**Proof.** It is possible to give a proof of this fact using Definition 2. However using the characterization of Proposition 1 the proof is particularly elementary. Consider first the (standard) \( \Rightarrow \) direction. Simply sum Eq. (1) for all \( n \) such that \( E_n \in V \). Using the fact that \( \mathcal{A}(t) = \sum_n \Pi_n A \Pi_n \) we obtain

\[
\sum_{E_k \in V} \Pi_k A \Pi_k = \mathcal{A}(t) \Pi_V = \langle A \rangle_V \Pi_V,
\]

that is, shell-ergodicity. The proof of the other implication is equally trivial. Simply multiply both sides by \( \Pi_n \) with \( E_n \in V \) and we obtain Eq. (1). \( \square \)

Recalling Proposition 2 we obtain the following.

**Corollary.** An observable \( A \) thermalizes if and only if it satisfies ETH-D.

The standard notion of ergodicity, however, is a property of a dynamical systems, and not of single observables. This is particularly evident in the characterization 1 of Theorem 1. We call this property metric indecomposability to avoid confusion. Roughly speaking, shell-ergodicity for a sufficiently large class of observables should become equivalent to metric indecomposability. Moreover, in the classical setting, it is usually believed (see e.g. [13]) that metric indecomposability implies that the only equilibrium state (i.e., invariant measure) is the microcanonical one [14].

In any case we will give a characterization of metric indecomposability as it arises in quantum mechanics.

Let \( \mathcal{E}_t^X \) (the star indicates Hilbert–Schmidt adjoint) denote the Heisenberg evolution operator. It is easy to see that shell-ergodicity means that

\[
T(X) = \mathcal{E}_t^X(\Pi_V X \Pi_V) = \Pi_V \langle X \rangle_V,
\]
where $\mathcal{T} : \mathcal{B}(\mathcal{H}_Y) \rightarrow \mathcal{B}(\mathcal{H}_Y)$ is the restriction of $\overline{\mathcal{E}}_V$ to $\mathcal{B}(\mathcal{H}_Y)$. If the above were true for all $X \in \mathcal{B}(\mathcal{H})$ then $\mathcal{T}$ would be equal to $T_{MC} \equiv |\Pi V|_{HS} (\rho V|_{HS}$ where we used Hilbert–Schmidt (HS) notation. Incidentally, this is the definition of ergodicity in the theory of quantum semi-groups. Moreover, this would imply at once that the invariant state $\rho V$ is the microcanonical one. In fact the superoperator $\overline{\mathcal{E}}_V$ as well as $\mathcal{T}$ are HS self-adjoint (this is a statement of the von Neumann ergodic theorem) and $T = |\Pi V|_{HS} (\rho V|_{HS}$ implies $\Pi V = \Pi V/\text{tr} \Pi V \equiv \rho_{MC}$.

This appealing possibility, however, can never be realized in finite dimension. In fact in this case one can explicitly compute

$$\overline{\eta} \Pi V = \sum_{E_n \in V} \Pi_n A \Pi_n$$

(4)

and this expression cannot be proportional to $\Pi V$ for all $A$ unless $\Pi V$ is one-dimensional\[^{[17]}\], in which case we must have $\rho V = |E_n| \langle E_n |$ for some $E_n$.

The conclusion of this elementary argument is the following. In quantum mechanics all the eigenspaces are invariant subspaces. Each eigenspace with $\Pi \text{thermalizes to precision at most } 2 \epsilon$ is the one we prefer, e.g. the microcanonical one, and have

$$\langle \Pi V \rangle (\rho V) = T^* \text{ implies } \rho V = \Pi V/\text{tr} \Pi V \equiv \rho_{MC}.$$
for this to make sense one must require that for almost any $E \in I = [E, E + \Delta]$ the system is metrically indecomposable. Note that the quantum equivalent of this situation corresponds to a Hamiltonian whose spectrum $E_n$ in $[E, E + \Delta]$ is non-degenerate. In quantum mechanics this is certainly a very common property. For a given observable function $f$ (defined on $V_{E,\Delta}$) we say it is $M$-ergodic if $\overline{f}(x) = \langle f \rangle_E$ for almost all $x \in M_E$ and for almost all $E \in I$. In this setting phase space averages are defined as

$$\langle f \rangle_{V,\Delta} = \frac{\int_{V_{E,\Delta}} dx f(x)}{\int_{V_{E,\Delta}} dx} = \frac{\int dE \omega(E) \langle f \rangle_E}{\int dE \omega(E)},$$

whereas the entropy is defined as $S_V(E) = \ln \Omega = \ln \int_{E-E+\Delta} dE \omega(E)$.

Of course, we want averages computed with approach a) to be equal to those computed with b). Hence we require

$$\langle f \rangle_E = \langle f \rangle_{V,\Delta} \quad (5)$$

for almost all $E \in I$. This is clearly reminiscent of ETH-D. Truthfully, this is equivalent to ETH-D only if $\Pi_n$ are one-dimensional \[21\], so we call it ETH-C (classical). Its quantum mechanical version is $(A)_{\text{eq}} = (A)_V$.

Obviously if $f$ is $M$-ergodic and satisfies ETH-C then $f$ is shell-ergodic, namely, $\overline{f}(x) = \langle f \rangle_{V,\Delta}$ for $V$-almost any $x \in V_{E,\Delta}$, which is what we wanted. We see a clear parallel with the quantum world. It is also clear that Eq. 5, as well as shell-ergodicity, cannot be satisfied for all functions $f$ (simply take an $f$ which is not constant over different $M_E$) and in general can be valid only approximately. It is the introduction of the shell $V_{E,\Delta}$ that forces us to consider approximate thermalization or ergodicity.

The equivalence between approach a) and approach b) is usually not discussed at length. A necessary condition is that $\langle f \rangle_E$ is a smooth function of $E$ and $\Delta$ sufficiently small. Assuming $M_E$ is sufficiently well behaved (a Lipschitz domain) and $\langle f \rangle_E$ is differentiable as a function of $E$, we have, as $\Delta \to 0$,

$$\langle f \rangle_{V,\Delta} = \langle f \rangle_E + \frac{\Delta}{2} \langle f \rangle_E'' + O(\Delta^2). \quad (6)$$

From the above, an estimate for the relative error is

$$\left| \frac{\langle f \rangle_{V,\Delta} - \langle f \rangle_E}{\langle f \rangle_E} \right| \leq \frac{\Delta}{2\epsilon_f}, \quad (7)$$

where the energy scale $\epsilon_f$ is $\epsilon_f = \langle f \rangle_E/|\langle f \rangle_{V,E}'|$. All in all, in order for $f$ to thermalize, we need metric indecomposability for almost all $E \in I$, and ETH-C, Eq. 5, for which a convenient proxy is given by $\Delta \ll \epsilon_f$. For the Hamiltonian function $\epsilon_H = E$ and we obtain the standard requirement $\Delta/E \ll 1$.

Let us now go back to the quantum realm. Now possibility a) is not allowed for at least two reasons. First we can argue (as in [13]) that the uncertainty in energy is a consequence of the system not being exactly isolated. In this case one cannot be in an exact eigenstate because of a time-energy uncertainty where $\Delta$ is the duration of the interaction process. Likewise, interactions with an environment would cause a broadening of levels. These arguments do not apply to a truly isolated system. For a truly isolated system however, we can say that we could not define a meaningful entropy function. So considering scenario b) becomes a necessity in quantum mechanics.

Reproducing the classical argument, we then need metric indecomposability for all the levels in a certain shell (now called $V$). As we have seen, in quantum mechanics, this is simply the requirement that the levels in $V$ are non-degenerate, a quite common property. After that we still demand equality of the two scenarios, i.e., $\langle A \rangle_{\text{eq}} = \langle A \rangle_V$ which, as we have seen, is equivalent to thermalization.

Note that any invariant state can be written as $\rho_V = \sum_n p_n \Pi_n/\text{tr} \Pi_n$. Then the phase space average can be written as

$$\langle A \rangle_V = \sum_{E_n \in I} p_n \langle A \rangle_{E_n} = \int dE \omega(E) \langle A \rangle_E \int dE \omega(E)$$

where we introduced the density of levels function

$$\omega(E) \equiv \sum_{E_n \in I} \delta(E - E_n) p_n$$

—in other words it is formally precisely the classical one, upon identifying $\omega = \omega$.

Conclusions. We show that a proper definition of ergodicity in the quantum framework is equivalent to the standard notion of thermalization for all initial states generally used in the literature. Moreover we prove that ergodicity is equivalent to the diagonal part of the eigenstate thermalization hypothesis (ETH) implying equivalence between ETH and thermalization, thus resolving a current conjecture. The arguments are elementary and in fact similar results also hold in the classical framework once suitably translated to the corresponding language. Indeed, we show that ETH is also present and implicitly assumed in the foundations of classical statistical mechanics. One point where the analogy breaks down is the conceptual impossibility in quantum mechanics of fixing the energy of an isolated statistical system to infinite precision.

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which implies $XP\sqrt{P}P = 0$. Inside the range of $P$, $\sqrt{P}$ is invertible and we can multiply by $P\sqrt{P}^{-1}P$ and get $XP = 0$. □

Lemma 2. If $\text{tr}(X\rho_0) = 0$, $\forall \rho_0 \in \mathcal{S}_V$ then $\Pi_V\Pi_V = 0$ (in fact the two statements are equivalent).

Proof. Although $\mathcal{S}_V$ is not a linear space it is possible to find $n^2$ linearly independent matrices in it, where $n = \dim \mathcal{H}_V$. The equation $\text{tr}(X\rho_0) = 0$ can be written as $\langle X\vert\rho_0\rangle_{\mathcal{H}_V} = 0$ using the Hilbert–Schmidt scalar product. Let $|e_i\rangle$, $i = 1, 2, \ldots, n$ be a basis of $\mathcal{H}_V$. Then the last equation means that $X = \Pi_VXQ + Q\Pi_VQ = Q\Pi_VQ$ with $Q = 1 - \Pi_V$. In other words $\Pi_V\Pi_V = 0$. □

Appendix B: Proofs of the $\epsilon$-relations

Proof of Proposition 4. The equivalence of ETH-D and strong shell-ergodicity to precision $\epsilon$ was established in the main text. Here we complete the proof by establishing an equivalence between thermalization and strong shell-ergodicity.

Without loss of generality assume $||A|| = 1$. Consider $X \equiv (A(t) - \langle A\rangle_\nu)\Pi_V = \Pi_VX\Pi_V$. Note that $X$ is hermitian if $A$ is. The observable $A$ thermalizes if $|\text{tr}(X\rho_0)| \leq \epsilon$ for all $\rho_0 \in \mathcal{S}_V$. Since $\rho_0$ is a state we have $|\text{tr}(X\rho_0)| \leq ||X||$. Hence $A$ strongly shell-ergodic implies $A$-thermalizes. But we also have

$$||\Pi_V\Pi_V|| = \sup_{\rho \in \mathcal{S}_V} |\text{tr}(X\rho)|.$$  

(B1)

So $|\text{tr}(X\rho)| \leq \epsilon$ for all $\rho \in \mathcal{S}_V$ implies that the left hand side is also $\leq \epsilon$, i.e., we have the other direction with the same $\epsilon$. □

Proof of Proposition 5. Without loss of generality assume $||A|| = 1$. First the $\Rightarrow$ direction. With the same $X$ defined previously we have $X^2 = [A\hat{A} - 2\langle A\rangle_\nu + \langle A\rangle_\nu^2]\Pi_V$. Hence $\langle A(t)A\rangle - \langle A\rangle_\nu^2 = |\text{tr}(X^2\rho_0)| \leq ||X^2|| = ||X||^2 \leq \epsilon^2$. For the other direction note that for $Y$ a positive definite operator ($Y \geq 0$) we have $||\Pi_VY\Pi_V|| \leq |\text{tr}(Y\rho_0)||\rho_0^{-1}\Pi_V||$. In fact $\text{tr}(Y\rho_0) = \text{tr}(\Pi_VY\Pi_V\rho_0)$ where $\Pi_VY\Pi_V$ is again positive with spectral resolution $\Pi_VY\Pi_V = \sum_n \lambda_n |n\rangle\langle n|$. Hence $\text{tr}(Y\rho_0) = \sum_n \lambda_n n|n\rangle\langle n|$ with $\lambda_n \geq 0$ and $|n\rangle\langle n| \geq \min_j f_j$ where $f_j$ are the zero-non eigenvalues of $\rho_0$. Since $1/min_j f_j = max_j 1/f_j = ||\Pi_V\rho_0^{-1}\Pi_V||$ we obtain

$$||\Pi_VY\Pi_V|| \leq \text{tr}(\Pi_VY\Pi_V\rho_0)||\rho_0^{-1}\Pi_V||. \tag{B2}$$

Now use this with $Y = X^2$ to obtain

$$||\langle A(t) - \langle A\rangle_\nu\rangle\Pi_V||^2 = ||\Pi_VX\Pi_V||^2 \leq ||\langle A(t)A\rangle - \langle A\rangle_\nu^2||\rho_0^{-1}\Pi_V||. \tag{B3}$$

from which the result follows. □

Appendix A: Proofs of the lemmas

Lemma 1. If $\text{tr}(X^\dagger XP) = 0$ then $XP = 0$ (and $PX^\dagger = 0$) where $P$ is the orthogonal projector onto the support of $\rho$.

Proof. Because $\rho = P\sqrt{P}P\sqrt{P}$ (since $P$ and $\rho$ commute),

$$\text{tr}(X^\dagger XP) = \text{tr}[(XP\sqrt{P})^\dagger (XP\sqrt{P})] = ||XP\sqrt{P}||^2_{HS} = 0,$$

as required. □