Abstract. We examine a number of results of infinite combinatorics using the techniques of reverse mathematics. Our results are inspired by similar results in recursive combinatorics. Theorems included concern colorings of graphs and bounded graphs, Euler paths, and Hamilton paths.

Reverse mathematics provides powerful techniques for analyzing the logical content of theorems. By contrast, recursive mathematics analyzes the effective content of theorems. In many cases, theorems of reverse mathematics have recursion theoretic corollaries. Conversely, theorems and techniques of recursive mathematics can often inspire related results in reverse mathematics, as demonstrated by the research presented here. In Section 1, a brief description of reverse mathematics is given. Sections 2 and 3 analyze theorems on graph colorings. Section 4 considers graphs with Euler paths. Stronger axiom systems are introduced in Section 5 and applied to the study of Hamilton paths in Section 6.

1. Reverse mathematics.

In [4], Friedman defined subsystems of second-order arithmetic useful in determining the proof-theoretic and recursion-theoretic strength of theorems. The language of

1991 Mathematics Subject Classification. 03F35, 03D45.
Key words and phrases. recursion theory, proof theory, graph theory.
Gasarch’s research was partially supported by NSF grant CCR9020079.
second-order arithmetic contains two types of variables, lower case variables representing elements of \( \mathbb{N} \), the natural numbers, and upper case variables representing subsets of \( \mathbb{N} \). Consequently, a model for a subsystem consists of a number universe and a collection of subsets of the number universe.

**RCA\(_0\)** is the weak base system used in reverse mathematics. It consists of the axioms of first order Peano arithmetic with induction restricted to \( \Sigma^0_1 \) formulas, and the recursive comprehension axiom, which states that any set definable by both a \( \Sigma^0_1 \) formula and a \( \Pi^0_1 \) formula exists. **RCA\(_0\)** suffices to prove fundamental facts about pairing functions, finite sequences, and other tools used to encode theorems as statements of second-order arithmetic. In this paper, much of the coding has been suppressed. Details on encoding techniques can be found in [16].

Stronger axiom systems can be constructed by adding additional set existence axioms to **RCA\(_0\)**. For example, the subsystem **WKL\(_0\)** consists of the axioms of **RCA\(_0\)** together with a weak version of König’s Lemma asserting that every infinite 0–1 tree contains an infinite path. **WKL\(_0\)** is strictly stronger than **RCA\(_0\)**. Often it is possible to show that a theorem is equivalent to a set comprehension axiom over the weak base system **RCA\(_0\)**. Results of this sort, called reverse mathematics, leave no doubt as to what set existence axioms are necessary in a proof. The following theorem of Simpson [14] illustrates this process, and is used in later sections. The notation (**RCA\(_0\)**) in the proclamation of a theorem or definition signifies that the theorem can be proved in **RCA\(_0\)**, or that the definition can be expressed in the language of **RCA\(_0\)** using coding techniques.

**Theorem 1 (RCA\(_0\)).** The following are equivalent:

1. **WKL\(_0\)**.

2. If \( f : \mathbb{N} \to \mathbb{N} \) and \( g : \mathbb{N} \to \mathbb{N} \) are injections such that for all \( j, k \in \mathbb{N}, f(j) \neq g(k) \), then there is a set \( X \) that separates the ranges of \( f \) and \( g \), formally,

\[
\forall j \forall n((f(j) = n \to n \in X) \land (g(j) = n \to n \notin X)).
\]
Adopting a model theoretic viewpoint can clarify the content of Theorem 1. In part, the theorem asserts that if \( f \) and \( g \) are injections (encoded) in a model of \( \mathbf{WKL}_0 \), then a separating set for \( f \) and \( g \) is also (encoded) in the model. In some sense, this implicitly restricts the choices of \( f \) and \( g \).

The axiom system \( \mathbf{ACA}_0 \) consists of \( \mathbf{RCA}_0 \) together with the arithmetical comprehension scheme. This scheme asserts that any set definable by a formula containing no set quantifiers exists. \( \mathbf{ACA}_0 \) is strictly stronger than \( \mathbf{WKL}_0 \). A proof of the following characterization of \( \mathbf{ACA}_0 \) can be found in Simpson [14].

**Theorem 2 (\( \mathbf{RCA}_0 \)).** The following are equivalent:

1. \( \mathbf{ACA}_0 \).
2. If \( f : \mathbb{N} \to \mathbb{N} \) is an injection, then the range of \( f \) exists.

Additional axiom systems are briefly described in section 5. For more detailed information on subsystems of second-order arithmetic and reverse mathematics, see [15] or [16].

2. **Graph Colorings.**

In this section we will consider theorems on node colorings of countable graphs. A (countable) graph \( G \) consists of a set of vertices \( V \subseteq \mathbb{N} \) and a set of edges \( E \subseteq [\mathbb{N}]^2 \). We will abuse notation by denoting an edge by \( (x, y) \) rather than \( \{x, y\} \). For \( k \in \mathbb{N} \), we say that \( \chi : V \to k \) is a \( k \)-coloring of \( G \) if \( \chi \) always assigns different colors to neighboring vertices. That is, \( \chi \) is a \( k \)-coloring if \( \chi : V \to k \) and \( (x, y) \in E \) implies \( \chi(x) \neq \chi(y) \). If \( G \) has a \( k \)-coloring, we say that \( G \) is \( k \)-chromatic. Using an appropriate axiom system, it is possible to prove that a graph is \( k \)-chromatic if it satisfies the following local condition.

**Definition 3 (\( \mathbf{RCA}_0 \)).** A graph \( G \) is *locally \( k \)-chromatic* if every finite subgraph of \( G \) is \( k \)-chromatic.

The following theorem is the simplest result concerning graph colorings. To prove that (1) implies (2), a tree is constructed in which every infinite path encodes a \( k \)-coloring. The
proof of the reversal uses a graph whose \( k \)-colorings encode separating sets for a pair of injections. Theorem 1 is then applied to finish the proof. For a detailed proof, see Theorem 3.4 in [9].

**Theorem 4 (RCA\(_0\)).** For every \( k \geq 2 \), the following are equivalent:

1. \( \text{WKL}_0 \).
2. If \( G \) is locally \( k \)-chromatic, then \( G \) is \( k \)-chromatic.

\( \square \)

From Theorem 4, we can deduce two recursion theoretic results due to Bean [2]. The first result can be proved directly by imitating the construction used in the proof of the reversal of Theorem 4, using a pair of recursive functions with no recursive separating set. We will provide an alternative model theoretic argument based on the following observation. By Theorem 1 and the existence of a pair of recursive functions with no recursive separating set, every \( \omega \)-model of \( \text{WKL}_0 \) must contain a non-recursive set.

**Corollary 5 (Bean [2]).** For every \( k \geq 2 \), there is a recursive \( k \)-chromatic graph which has no recursive \( k \)-coloring.

*Proof.* Suppose, by way of contradiction, that for some \( k \in \omega \) every recursive \( k \) chromatic graph has a recursive \( k \)-coloring. Then, by Theorem 4, \( \omega \) together with the recursive sets is a model of \( \text{WKL}_0 \), contradicting the fact that every \( \omega \)-model of \( \text{WKL}_0 \) contains a non-recursive set. \( \square \)

Our model theoretic proof of the next recursion theoretic corollary relies on the fact that the set universe of any \( \omega \)-model of \( \text{WKL}_0 \) is a Scott system [12]. Such a model will include the recursive sets, and additional sets which can be bounded in complexity. By the Shoenfield-Kreisel low basis theorem [13] there is an \( \omega \)-model of \( \text{WKL}_0 \) such that for each set \( X \) in the model, if \( a \) is the Turing degree of \( X \), then \( a' \leq 0' \). That is, every set in such a model of \( \text{WKL}_0 \) is of low degree.

**Corollary 6 (Bean [2]).** For every \( k \geq 2 \), every recursive \( k \)-chromatic graph has a \( k \)-coloring of low degree.
Proof. Let $M$ be an $\omega$-model of $\text{WKL}_0$ in which every set is of low degree. Let $G$ be a recursive $k$-chromatic graph. Then $G$ is (encoded) in $M$, and by Theorem 4, a $k$-coloring of $G$ is also (encoded) in $M$. Thus $G$ has a $k$-coloring of low degree. □

The number of colors allowed in a coloring of a locally $k$-chromatic graph can be substantially increased without weakening the logical strength of the resulting theorem. This contrast sharply with the situation for bounded graphs which is discussed in the next section.

**Theorem 7 (RCA₀).** For each $k \geq 2$, the following are equivalent:

1. $\text{WKL}_0$.
2. If $G$ is locally $k$-chromatic, then $G$ is $(2k - 1)$-chromatic.

Proof. Since RCA₀ proves that every $k$-chromatic graph is $(2k - 1)$-chromatic, (1) implies (2) follows immediately from Theorem 4.

We will now prove that (2) implies (1) when $k = 2$, and then indicate how the argument can be generalized to any $k \in \mathbb{N}$. By Theorem 1, $\text{WKL}_0$ can be proved by showing that the ranges of an arbitrary pair of disjoint injections can be separated. Let $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be injections such that for all $m, n \in \mathbb{N}$, $f(n) \neq g(m)$. We will construct a 2-chromatic graph with the property that any 3-coloring of $G$ encodes a set $S$ such that $y \in \text{Range}(f)$ implies $y \in S$, and $y \in S$ implies $y \notin \text{Range}(g)$.

The graph $G$ contains an infinite complete bipartite subgraph consisting of upper vertices $\{b^u_n : n \in \mathbb{N}\}$, lower vertices $\{b^l_n : n \in \mathbb{N}\}$, and connecting edges $\{(b^u_n, b^l_m) : n, m \in \mathbb{N}\}$. Also, $G$ contains an infinite collection of pairs of vertices, denoted by $n^u$ and $n^l$ for $n \in \mathbb{N}$. Each such pair is connected, so the edges $\{(n^u, n^l) : n \in \mathbb{N}\}$ are included in $G$. Additional connections depend on the injections $f$ and $g$. If $f(i) = n$, add the edges $(b^u_n, n^l)$ and $(b^l_m, n^u)$ for all $m \geq i$. If $g(i) = n$, add the edges $(b^l_m, n^u)$ and $(b^u_m, n^l)$ for all $m \geq i$. Naively, if $n$ is in the range of $f$ or $g$, then the pair $(n^u, n^l)$ is connected to the complete bipartite subgraph. If $n$ is in the range of $G$, the pair is “flipped” before it is connected.
The reader can verify that $G$ is $\Delta^0_1$ definable in $f$ and $g$, and thus exists by the recursive comprehension axiom. Every finite subgraph of $G$ is clearly bipartite, so $G$ is locally 2-chromatic. Thus, by (2), $G$ has a 3-coloring; denote it by $\chi : G \to 3$.

If $\chi$ is a 2-coloring, we can define the separating set, $S$, by

$$S = \{y \in \mathbb{N} : \chi(y^u) = \chi(b^u_0) \lor \chi(y^l) = \chi(b^l_0)\}.$$ 

When $\chi$ uses all 3 colors, we must modify the construction of $S$. In particular, we must find a $j \in \mathbb{N}$ such that

(a) $\forall y (\exists n(n \geq j \land f(n) = y) \rightarrow (\chi(y^u) = \chi(b^u_j) \lor \chi(y^l) = \chi(b^l_j)))$, and

(b) $\forall y (\exists n(n \geq j \land g(n) = y) \rightarrow (\chi(y^l) \neq \chi(b^l_j) \land \chi(y^u) \neq \chi(b^u_j))).$

Suppose, by way of contradiction, that no such $j$ exists. The for some $m$ and $y$, either $f(m) = y \land \chi(y^u) \neq \chi(b^u_0) \land \chi(y^l) \neq \chi(b^l_0)$ or $g(m) = y \land (\chi(y^l) = \chi(b^l_0) \lor \chi(y^u) = \chi(b^u_0))$.

If $f(m) = y$, since $\chi$ is a 3-coloring, either $\chi(y^u) = \chi(b^u_0)$ or $\chi(y^l) = \chi(b^l_0)$. By the construction of $G$, for every $n > m$, $\chi(b^u_m) = \chi(b^u_n)$ and $\chi(b^l_m) = \chi(b^l_n)$. Similarly, the case $g(m) = y$ also yields a point beyond which the complete bipartite subgraph of $G$ is 2-colored. By the negation of (a) and (b), there is an $m' > m$ and a $z \in \mathbb{N}$ such that either $f(m') = z \land \chi(z^u) \neq \chi(b^u_m) \land \chi(z^l) \neq \chi(b^l_m)$ or $g(m') = z \land (\chi(z^l) = \chi(b^l_m) \lor \chi(z^u) = \chi(b^u_m))$.

If $f(m') = z$, then since $\chi$ is a 3-coloring, either $\chi(z^u) = \chi(b^u_m)$ or $\chi(z^l) = \chi(b^l_m)$. Since $m' > m$, $\chi(b^l_m) = \chi(b^l_{m'})$ and $\chi(b^u_m) = \chi(b^u_{m'})$, so either $\chi(z^l) = \chi(b^l_{m'})$ or $\chi(z^u) = \chi(b^u_{m'})$.

But $(z^l, b^u_{m'})$ and $(z^u, b^l_{m'})$ are edges of $G$, so $\chi$ is not a 3-coloring. Assuming $g(m') = z$ yields a similar contradiction. Thus, a $j$ satisfying (a) and (b) exists.

Given an integer $j$ satisfying (a) and (b), the separating set $S$ may be defined as the union of $\{y \in \mathbb{N} : \exists n < j f(n) = y\}$ and

$$\{y \in \mathbb{N} : (\forall n < j g(n) \neq y) \land (\chi(y^u) = \chi(b^u_j) \lor \chi(y^l) = \chi(b^l_j))\}$$

$S$ is $\Delta^0_1$ definable in $\chi$ and $j$, so the recursive comprehension axiom assures the existence of $S$. If $f(n) = y$ and $n < j$, then $y \in S$. If $f(n) = y$ and $n \geq j$, then by (a) and the fact
that \( f \) and \( g \) have disjoint ranges, \( y \in S \). Thus \( \text{Range}(f) \subseteq S \). If \( g(n) = y \), and \( n < j \), then since the ranges of \( f \) and \( g \) are disjoint we have \( y \notin S \). If \( g(n) = y \) and \( n \geq j \), by (b) \( y \notin S \). Thus \( S \) is the desired separating set. This completes the proof for \( k = 2 \).

For \( k > 2 \), the preceding proof requires the following modifications. Replace the complete bipartite subgraph of \( G \) by a complete \( k \)-partite subgraph with vertices \( \{b^p_m : p < k \land m \in \mathbb{N}\} \). Each pair \((n^u, n^l)\) is replaced by a complete graph on the vertices \( \{n^p : p < k\} \). If \( f(i) = n \), add the edges \((b^p_m, n^{p'})\) for all \( m \geq i \) and all \( p \neq p' \) less than \( k \). If \( g(i) = n \), twist the subgraph before attaching it. That is, add the edges \((b^p_m, n^{p'})\) for all \( m \geq i \) and all \( p \) and \( p' \) less than \( k \) such that \( p \equiv p' + 1 \pmod{k} \). The argument locating the integer \( j \) is similar, except that \( m \) and \( m' \) must be replaced by a sequence \( m_1, \ldots, m_k \). Beyond the point \( m_{k-1} \), the complete \( k \)-partite subgraph of \( G \) is \( k \)-colored by \( \chi \). The definition of \( S \) is very similar, except that a bounded quantifier should be used to avoid the \( k \)-fold conjunction. \( \square \)

The following recursion theoretic consequence of Theorem 7 is a special case of a result due to Bean.

**Corollary 8 (Bean [2]).** For every \( k \geq 2 \), there is a recursive graph \( G \) which has no recursive \((2k-1)\)-coloring.

**Proof.** Imitate the reversal of Theorem 7, using disjoint recursive injections with no recursive separating set. \( \square \)

Bean [2] showed that Corollary 8 holds with \( 2k - 1 \) replaced by any value larger than \( k \). In light of this, the following conjecture seems reasonable. Unfortunately, even the case where \( k = 2 \) and \( m = 4 \) remains open.

**Conjecture 9 (RCA\(_0\)).** For each \( k \geq 2 \) and each \( m \geq k \) the following are equivalent:

1. \( \text{WKL}_0 \).
2. If \( G \) is locally \( k \)-chromatic, then \( G \) is \( m \)-chromatic.
Remark. Note that (1) implies (2) follows from Theorem 7. Also, the full reversal is an easy corollary of the reversal for $k = 2$ and arbitrary $m$. To see this, note that if $G$ is the graph used to prove the reversal for $k$ and $m$, the graph resulting from adding one vertex to $G$ and attaching it to every existing vertex will provide a proof of the reversal for $k + 1$ and $m + 1$.

3. Bounded graphs and sequences of graphs.

As noted above, a locally $k$-chromatic recursive graph may not have a recursive coloring, regardless of the number of colors used. By contrast, highly recursive graphs always have recursive colorings. A proof theoretic analog of a highly recursive graph is a bounded graph.

**Definition 10 (RCA$_0$).** A graph $G = \langle V, E \rangle$ is bounded if there is a function $h : V \to \mathbb{N}$ such that for all $x, y \in V$, $(x, y) \in E$ implies $h(x) \geq y$.

Using Definition 10, we can state a proof theoretic version of a theorem on highly recursive graphs proved by Schmerl [11] and independently rediscovered by Carstens and Pappinghaus [3].

**Theorem 11 (RCA$_0$).** For $k \in \mathbb{N}$, if $G$ is a bounded locally $k$-chromatic graph, then $G$ is $(2^k - 1)$-chromatic.

**Proof.** The proof of Theorem 1 of Schmerl [11] can be carried out in RCA$_0$. □

**Corollary 12 (Schmerl [11]).** For every $k$, every highly recursive $k$-chromatic graph has a recursive $(2^k - 1)$-coloring.

If the number of colors allowed is less than $2^k - 1$, a $k$-chromatic highly recursive graph may not have a recursive coloring.

**Theorem 13 (RCA$_0$).** For every $k \geq 2$, the following are equivalent:

1. WKL$_0$.

2. If $G$ is a bounded locally $k$-chromatic graph, then $G$ is $(2k - 2)$-chromatic.
Proof. Since $\text{RCA}_0$ proves that every $k$-chromatic graph is $(2k - 2)$-chromatic, (1) implies (2) follows immediately from Theorem 4. For the case $k = 2$, the statement that (2) implies (1) is included in Theorem 3.4 of [9]. Alternately, this case could be proved by formalizing the proof of Theorem 2 of Schmerl [11]. We will adopt this approach for the case $k > 2$.

Let $k > 2$, and assume that (2) holds. As in Theorem 7, we will prove $\text{WKL}_0$ by finding a separating set for the ranges of two disjoint injections, $f$ and $g$. The separating set must be encoded in a $(2k - 2)$-coloring of a bounded locally $k$-chromatic graph $G$.

$G$ will be constructed from subgraphs called blocks. A block $B$ consists of $k^2$ vertices $\{v_{ij} : i < k \land j < k\}$ connected by the edges $(v_{ij}, v_{rs})$ for $i \neq r$ and $j \neq s$. A block can be viewed as a $k \times k$ matrix where each vertex is connected to all the elements of its associated cofactor matrix. We can link two blocks $B$ and $B'$ by adding the edges $(v_{ij}, v'_{rs})$ for $i \neq r$ and $j \neq s$.

Given a coloring $\chi$ of a block $B$, we say that $B$ has a colorful row if for some $i$, whenever $j \neq r$, $\chi(v_{ij}) \neq \chi(v_{ir})$. Similarly, $B$ has a colorful column if all the elements in some column have distinct colors. $\text{RCA}_0$ proves that if $\chi$ is a $(2k - 2)$-coloring of a block $B$, then $B$ has either a colorful row or a colorful column, but not both. (To prove this, formalize the proof of Lemma 2.1 in [11] or Lemma 5.25 in [6].) Furthermore, $\text{RCA}_0$ proves that if $\chi$ is a $(2k - 2)$-coloring of two linked blocks $B$ and $B'$, then $B$ has a colorful row (column) if and only if $B'$ has a colorful column (row). (To prove this, formalize the proof of Lemma 2.2 in [11] or Lemma 5.27 in [6]).

The graph $G$ is constructed from two sets of blocks, $\{B_j : j \in \mathbb{N}\}$ and $\{B_{ij} : i, j \in \mathbb{N}\}$. For each $i$ and $j$, $B_j$ is linked to $B_{j+1}$, and $B_{ij}$ is linked to $B_{i(j+1)}$. Additional links depend on the injections $f$ and $g$. If $f(m) = n$, link $B_{n(2m)}$ to $B_{(2m)}$. If $g(m) = n$, link $B_{n(2m)}$ to $B_{(2m+1)}$. The reader may verify that $G$ is $\Delta^0_1$ definable in $f$ and $g$, bounded, and locally $k$-chromatic. Applying (2), $G$ has a $(2k - 2)$-coloring, $\chi$. By the recursive comprehension axiom, the set

$$S = \{n : B_{n0} \text{ has a colorful column}\}$$
exists. We will show that $S$ is the desired separating set. Suppose first that $\chi$ induces a colorful row in $B_0$. If $f(m) = n$, since $B_0$ and $B_{n_0}$ are connected by a sequence of linked blocks of even length, $B_{n_0}$ has a colorful column, and $n \in S$. Also, if $g(m) = n$, $B_0$ and $B_{n_0}$ are linked by an odd length sequence, so $n \notin S$. Thus, $\text{Range}(f) \subseteq S$ and $\text{Range}(g) \cap S = \emptyset$. Similarly, if $\chi$ induces a colorful column in $B_0$, then $S$ is a separating set containing $\text{Range}(g)$. $\square$

Imitating the reversal of Theorem 13 using disjoint recursive injections with recursively inseparable ranges yields Schmerl’s proof of the following result.

**Corollary 14 (Schmerl [11]).** For each $k \geq 2$ there is a highly recursive $k$-chromatic graph which has no recursive $2k - 2$ coloring.

We will close this section with a theorem concerning sequences of graphs and its recursion theoretic corollary. We say that a graph $G$ is **colorable** if there exists an integer $k$ such that $G$ is $k$-chromatic.

**Theorem 15 (RCA$_0$).** The following are equivalent:

1. ACA$_0$.
2. Given a countable sequence of graphs, $\langle G_i : i \in \mathbb{N} \rangle$, there is a function $f : \mathbb{N} \to 2$ such that $f(i) = 1$ if $G_i$ is colorable and $f(i) = 0$ otherwise.

**Proof.** To prove that (1) implies (2), assume ACA$_0$ and let $\langle G_i : i \in \mathbb{N} \rangle$ be a sequence of graphs. Define $f : \mathbb{N} \to \mathbb{N}$ by setting $f(i) = 1$ if there exists a $k \in \mathbb{N}$ such that $G_i$ is locally $k$-chromatic, and setting $f(i) = 0$ otherwise. Since “$G_i$ is locally $k$-chromatic” is an arithmetical sentence with parameter $G_i$, $f$ exists by the arithmetical comprehension axiom. Since ACA$_0$ implies WKL$_0$, we may apply Theorem 4 to show that $f(i) = 1$ if and only if $G_i$ is colorable.

To prove the converse, assume RCA$_0$ and (2). By Theorem 2, to prove ACA$_0$ it suffices to show that for every injection $g$, $\text{Range}(g)$ exists. Define the sequence of graphs $\langle G_i : i \in \mathbb{N} \rangle$ as follows. Let $\{v_j : j \in \mathbb{N}\}$ be the vertices of $G_i$. If $j < k$ and $\forall m \leq k(g(m) \neq i)$,
add the edge \((v_j, v_k)\) to \(G_i\). \(\text{RCA}_0\) can prove that \(\langle G_i : i \in \mathbb{N} \rangle\) exists, and \(G_i\) is colorable if and only if \(i \in \text{Range}(g)\). Thus, the function \(f\) supplied by (2) is the characteristic function for \(\text{Range}(g)\). By the recursive comprehension axiom, \(\text{Range}(g)\) exists. □

**Corollary 16.** There is a recursive sequence of recursive graphs \(\langle G_i : i \in \mathbb{N} \rangle\) such that \(0'\) is recursive in \(\{i \in \mathbb{N} : G_i\text{ is colorable}\}\).

**Proof.** In the proof of the reversal for Theorem 15, let \(g\) be a recursive function such that \(0'\) is recursive in \(\text{Range}(g)\). The sequence of graphs constructed in the proof has the desired properties. □

4. Euler paths.

Now, we will turn to the study of Euler paths. A *path* in a graph \(G\) is a sequence of vertices \(v_0, v_1, v_2, \ldots\) such that for every \(i \in \mathbb{N}\), \((v_i, v_{i+1})\) is an edge of \(G\). A path is called an *Euler path* if it uses every edge of \(G\) exactly once.

The following terminology is useful in determining when a graph has an Euler path. A graph \(G = \langle V, E \rangle\) is *locally finite* if for each vertex \(V\), the set \(\{u \in V : (v, u) \in E\}\) is finite. If \(H\) is a subgraph of \(G\), \(G - H\) denotes the graph obtained by deleting the edges of \(H\) from \(G\). Using this terminology, we can describe a condition which, from a naive viewpoint, is sufficient for the existence of an Euler path.

**Definition 17 (RCA\(_0\)).** A graph \(G\) is *pre-Eulerian* if it is

1. connected,
2. has at most one vertex of odd degree,
3. if it has no vertices of odd degree, then it has at least one vertex of infinite degree, and
4. if \(H\) is any finite subgraph of \(G\) then \(G - H\) has exactly one infinite connected component.

Note that the formula “\(G\) is pre-Eulerian” is arithmetical in the set parameter \(G\). \(\text{RCA}_0\) suffices to prove that every graph with an Euler path is pre-Eulerian. However, \(\text{RCA}_0\)
can only prove that bounded pre-Eulerian graphs have Euler paths. (Bounded graphs are defined in Section 3.) This result is just a formalization of Bean’s [1] proof that every highly recursive pre-Eulerian graph has a recursive Euler path.

**Theorem 18 (RCA₀).** If $G$ is a bounded pre-Eulerian graph, then $G$ has an Euler path.

*Proof.* The proof of this theorem is just a straightforward formalization of Theorem 2 of Bean [1]. The formalization requires verification that Euler’s Theorem for finite graphs (see [10]) can be proved using only RCA₀. □

If $G$ is not bounded, additional axiomatic strength is required to prove the existence of an Euler path.

**Theorem 19 (RCA₀).** The following are equivalent:

1. ACA₀
2. If $G$ is a pre-Eulerian graph, then $G$ has an Euler path.
3. If $G$ is a locally finite pre-Eulerian graph, then $G$ has an Euler path.

*Proof.* To prove that (1) implies (2), assume ACA₀ and let $G$ be a pre-Eulerian graph. Let $\langle E_i : i \in \mathbb{N} \rangle$ be an enumeration of the edges of $G$. Let $v_0$ be the vertex of $G$ of odd degree, or a vertex of infinite degree if no odd vertex exists. Imitating the proof of Theorem 3.2.1 of Ore [10], there is a finite path $P$ containing the edge $E_0$ such that

- $P$ starts at $v_0$,
- $G - P$ is connected, and
- $P$ ends at the odd vertex of $G - P$, or at an infinite vertex of $G - P$ if no odd vertex exists.

Furthermore, since the finite paths of $G$ can be encoded by integers, we can pick the unique path $P_0$ satisfying the conditions above and having the least code. Similarly, any path $P_i$ satisfying the three conditions can be extended to a unique path $P_{i+1}$ which contains the edge $E_{i+1}$, satisfies the three conditions, and has the least code among all paths with these
properties. Note the $P_{i+1}$ extends $P_i$ by including $P_i$ as an initial segment. The reader may verify that the sequence of paths $\langle P_i : i \in \mathbb{N} \rangle$ is arithmetically definable in $G$, and so exists by arithmetical comprehension. Let $v_i$ denote the $i^{th}$ vertex of $P_i$. Then the sequence $\langle v_i : i \in \mathbb{N} \rangle$ exists by recursive comprehension and includes each $P_i$ as an initial segment. Consequently, $\langle v_i : i \in \mathbb{N} \rangle$ defines an Euler path through $G$.

Since (3) is a special case of (2), showing that (3) implies (1) will complete the proof of the theorem. Assume $\text{RCA}_0$ and fix an injection $f : \mathbb{N} \to \mathbb{N}$. We will construct a locally finite pre-Eulerian graph $G$ such that every Euler path through $G$ encodes $\text{Range}(f)$. Define the vertices of $G$ by

$$V = \{ a_n, b_n, c_n : n \in \mathbb{N} \}.$$  

For each $n$, include the edges $(a_n, a_{n+1})$ and $(b_n, c_n)$ in $G$. Additionally, for each $i$ and $n$, if $f(i) = n$ then include the edges $(a_n, b_i)$ and $(c_i, a_n)$ in $G$. $\text{RCA}_0$ suffices to prove that $G$ exists, and is both locally finite and pre-Eulerian. By (3), $G$ has an Euler path. Note that $n \in \text{Range}(f)$ if and only if the first occurrence of $a_n$ in the Euler path is not followed immediately by $a_{n+1}$. By the recursive comprehension axiom, $\text{Range}(f)$ exists. Since $f$ was an arbitrary injection, by Theorem 2 this suffices to prove $\text{ACA}_0$. $\square$

**Corollary 20.** There is a recursive pre-Eulerian graph $G$ such that $0'$ is recursive in every Euler path through $G$.

**Proof.** Let $f$ be a recursive function such that $0'$ is recursive in $\text{Range}(f)$. Construct the graph $G$ as in the proof of the reversal in Theorem 19. Then $G$ is recursive, and $\text{Range}(f)$ is recursive in every Euler path through $G$. $\square$

$\text{ACA}_0$ also suffices to address the problem of determining which elements of a sequence of graphs have Euler paths. This contrasts sharply with the situation for Hamilton paths, as described in Theorem 30.

**Theorem 21 (RCA$_0$).** The following are equivalent:

(1) $\text{ACA}_0$
(2) Given a countable sequence of graphs, \( \langle G_i : i \in \mathbb{N} \rangle \), there is a set \( Z \subseteq \mathbb{N} \) such that \( i \in Z \) if and only if \( G_i \) has an Euler path.

Proof. First assume \( \text{ACA}_0 \) and let \( \langle G_i : i \in \mathbb{N} \rangle \) be a sequence of graphs. Define the set \( Z \) by \( Z = \{ i \in \mathbb{N} : G_i \text{ is pre-Eulerian} \} \). Note that \( Z \) is arithmetically definable in \( \langle G_i : i \in \mathbb{N} \rangle \), so \( \text{ACA}_0 \) proves the existence of \( Z \). Since \( \text{RCA}_0 \) proves that every graph with an Euler path is pre-Eulerian, and Theorem 19 proves that every pre-Eulerian graph has an Euler path, \( i \in Z \) if and only if \( G_i \) has an Euler path.

To prove the converse, assume \( \text{RCA}_0 \) and (2). By Theorem 2, it suffices to prove that \( \text{Range}(f) \) exists for an arbitrary injection \( f : \mathbb{N} \to \mathbb{N} \). Let \( v_0, v_1, v_2, \ldots \) denote the vertices of \( G_i \). For each \( n \), if \( f(n) \neq i \), add the edge \((v_n, v_{n+1})\) to \( G_i \). By the recursive comprehension axiom, the sequence of graphs \( \langle G_i : i \in \mathbb{N} \rangle \) exists. Let \( Z \) be as in (2). Then \( \text{Range}(f) = \{ i \in \mathbb{N} : i \notin Z \} \), so \( \text{Range}(f) \) exists by the recursive comprehension axiom. Note that this proof actually shows that Theorem 21 holds with (2) restricted to sequences of bounded graphs. □

Theorem 21 can be used to establish rough upper and lower bounds for the complexity of the problem of determining which graphs in a sequence have Euler paths.

**Corollary 22 (Beigel and Gasarch [6]).** If \( \langle G_i : i \in \mathbb{N} \rangle \) is an arithmetical sequence of graphs, then the set \( \{ i \in \mathbb{N} : G_i \text{ has an Euler path} \} \) is arithmetical.

Proof. \( \omega \) together with the arithmetical sets is a model of \( \text{ACA}_0 \), and thus models (2) of Theorem 21. □

**Corollary 23 (Beigel and Gasarch [6]).** There is a recursive sequence of recursive graphs, \( \langle G_i : i \in \mathbb{N} \rangle \) such that \( 0' \) is recursive in the set \( \{ i \in \mathbb{N} : G_i \text{ has an Euler path} \} \).

Proof. Let \( f \) be a recursive function such that \( 0' \) is recursive in \( \text{Range}(f) \). The sequence of graphs constructed from \( f \) as in the proof of the reversal in Theorem 21 has the desired property. □
Remark. A two-way or endless Euler path is a bijection between the integers (both positive and negative) and the set of edges of $G$ such that each edge shares one vertex with its predecessor and its other vertex with its successor. Theorems 18, 19, and 21 can be modified to address the existence of two-way Euler paths.

5. Stronger axiom systems.

The discussion of Hamilton paths in the next section uses three axiom systems which are each strictly stronger than $\text{ACA}_0$. These axiom systems, in strictly increasing order of strength, are $\Sigma^1_1-\text{AC}_0$, $\text{ATR}_0$, and $\Pi^1_1-\text{CA}_0$.

The subsystem $\Sigma^1_1$-axiom of choice, denoted by $\Sigma^1_1-\text{AC}_0$, consists of $\text{RCA}_0$ together with the comprehension scheme

$$\left(\forall k(\exists X \Psi(k,X))\right) \rightarrow \left(\exists Y(\forall k(\Psi(k,(Y)_k)))\right)$$

where $\Psi$ is any $\Sigma^1_1$ formula and $(Y)_k = \{i : (i,k) \in Y\}$.

The subsystem $\text{ATR}_0$ consists of $\text{ACA}_0$ and an existence axiom for sets constructed by applying a form of arithmetical transfinite recursion. We will need the following notation. Let $\text{Seq}$ denote the set of (codes for) finite sequences of elements of $\mathbb{N}$. Given $T \subseteq \text{Seq}$, we say that $T$ is a tree if whenever $\tau \in T$ and $\sigma$ is an initial segment of $\tau$, $\sigma \in T$. In this way, we can encode infinitely splitting trees as subsets of $\text{Seq}$, which can in turn be encoded as subsets of $\mathbb{N}$. The following result (which is Theorem 5.2 of [16]) gives two combinatorial characterizations of $\text{ATR}_0$.

**Theorem 24 ($\text{RCA}_0$).** The following are equivalent

1. $\text{ATR}_0$.
2. The schema

$$\left(\forall i (\exists \text{ at most one } X) \Psi(i,X) \rightarrow (\exists Z)(\forall i)(i \in Z \leftrightarrow (\exists X)\Psi(i,X))\right),$$

where $\Psi(i,X)$ is any arithmetical formula in which $Z$ does not occur.
3. For any sequence of trees $(T_i : i \in \mathbb{N})$, if for each $i \in \mathbb{N}$, $T_i$ has at most one path, then $\exists Z \forall i (i \in Z \leftrightarrow T_i \text{ has a path})$.
The system \( \Pi^1_1 - \text{CA}_0 \) consists of \( \text{RCA}_0 \) plus a comprehension axiom asserting that the set \( \{ n \in \mathbb{N} : \Psi(n) \} \) exists for any \( \Pi^1_1 \) formula \( \Psi \). \( \Pi^1_1 - \text{CA}_0 \) is strictly weaker than \( \Pi^1_\infty - \text{CA}_0 \) (full second-order arithmetic.)

6. Hamilton paths.

Now we will consider theorems on the existence of Hamilton paths. A path through a graph \( G \) is called a (one way) Hamilton path if it uses every vertex of \( G \) exactly once. There is no known analog of the characterization “pre-Eulerian” for graphs containing Hamilton paths. Consequently, all the results of this section concern sequences of graphs.

Each reversal in this section will rely on the construction of a sequence of graphs from a sequence of trees, as in the following lemma.

**Lemma 25 (\( \text{RCA}_0 \)).** Given a sequence of trees \( \langle T_i : i \in \mathbb{N} \rangle \), there is a sequence of graphs \( \langle G_i : i \in \mathbb{N} \rangle \) such that

1. for each \( i \in \mathbb{N} \), \( T_i \) has a (unique) path if and only if \( G_i \) has a (unique) Hamiltonian path, and
2. if there is a sequence \( \langle P_i : i \in \mathbb{N} \rangle \) such that \( P_i \) is a Hamiltonian path through \( G_i \) for each \( i \in \mathbb{N} \), then there is a sequence \( \langle P'_i : i \in \mathbb{N} \rangle \) such that \( P'_i \) is a path through \( T_i \) for each \( i \in \mathbb{N} \).

**Proof.** For each \( T_i \), use the graph constructed in the proof of Theorem 1 of Harel [5]. \( \square \)

The next three theorems analyze the following tasks:

1. finding Hamilton paths through graphs known to have such paths,
2. determining whether graphs that have at most one Hamilton path have such a path, and
3. determining whether arbitrary graphs have Hamilton paths.

Using proof theoretic strength as a measure of difficulty, we shall see that these tasks are strictly increasing in order of difficulty.
Theorem 26 (RCA₀). The following are equivalent:

(1) \( \Sigma^1_1 \text{AC}_0 \).

(2) If \( \langle G_i : i \in \mathbb{N} \rangle \) is a sequence of graphs such that each \( G_i \) has a Hamilton path, then there is a sequence \( \langle P_i : i \in \mathbb{N} \rangle \) such that for each \( i \), \( P_i \) is a Hamilton path through \( G_i \).

Proof. To prove that (1) implies (2), assume \( \Sigma^1_1 \text{AC}_0 \) and let \( \langle G_i : i \in \mathbb{N} \rangle \) be a sequence of graphs, each with a Hamilton path. Let \( \Psi(k, P) \) be the arithmetical sentence formalizing “\( P \) is a Hamilton path through \( G_k \).” By \( \Sigma^1_1 \text{AC}_0 \), since \((\forall k)(\exists P)\Psi(k, P)\), there is a \( Y \) such that \((\forall k)\Psi(k, (Y)_k)\). Since the desired sequence of paths is \( \Delta^0_1 \)-definable in \( Y \), it exists by the recursive comprehension axiom.

The first step in proving that (2) implies (1) is to deduce \( \text{ACA}_0 \) from (2). Let \( f : \mathbb{N} \to \mathbb{N} \) be an injection. We will show that \( \text{Range}(f) \) exists. Construct a sequence of graphs as follows. Let \( v_0, v_1, v_2, \ldots \) be the vertices of \( G_n \). Include the edge \((v_0, v_1)\) in \( G_n \). For each \( j \in \mathbb{N} \), if \( f(j) \neq n \), add the edge \((v_{j+1}, v_{j+2})\) to \( G_n \). If \( f(j) = n \), add \((v_0, v_{j+2})\) to \( G_n \). By the recursive comprehension axiom, the sequence \( \langle G_i : i \in \mathbb{N} \rangle \) exists. Furthermore, for each \( n \), \( G_n \) has a Hamiltonian path. In particular, if \( n \notin \text{Range}(f) \) the only Hamiltonian path in \( G_n \) is \( v_0v_1v_2\ldots \), while if \( f(m) = n \), the only path is given by \( v_{m+1}v_m\ldots v_0v_{m+2}v_{m+3}\ldots \). Applying (2), we obtain a sequence of paths \( \langle P_i : i \in \mathbb{N} \rangle \), and by the recursive comprehension axiom, the set

\[
\text{Range}(f) = \{ n : v_0 \text{ is not the first vertex in } P_n \}
\]

exists. By Theorem 2, this suffices to prove \( \text{ACA}_0 \).

To complete the deduction of \( \Sigma^1_1 \text{AC}_0 \) from (2), suppose that \( \Psi \) is a \( \Sigma^1_1 \) formula and \((\forall k)(\exists X)\Psi(k, X)\). By Lemma 3.14 of [5], there is a sequence of trees \( \langle T_i : i \in \mathbb{N} \rangle \) such that for all \( k \in \mathbb{N} \), \( P \) is a path through \( T_k \) if and only if \( \Psi(k, X) \), where \( X \) is uniformly \( \Delta^0_1 \)-definable in \( P \). (Lemma 3.14 of [5] has \( \text{ACA}_0 \) as a hypothesis.) Let \( \langle G_i : i \in \mathbb{N} \rangle \) be the sequence of graphs obtained by applying Lemma 25 to \( \langle T_i : i \in \mathbb{N} \rangle \). Since \((\forall k)(\exists X)\Psi(k, X)\),
each of the trees has an infinite path, so each of the graphs has a Hamilton path. By (2), there is a sequence of paths \( \langle P_i : i \in \mathbb{N} \rangle \) for the graphs. By Lemma 25, there is a sequence of paths \( \langle P'_i : i \in \mathbb{N} \rangle \) for the trees. Using these paths as a parameter, arithmetic comprehension suffices to prove the existence of a set \( Y \) such that \((\forall k)\Psi(k, (Y)_k)\). Thus \( \Sigma_1^1 - \text{AC}_0 \) holds, as desired. \( \square \)

From Theorem 26, we can draw the following recursion theoretic conclusion.

**Corollary 27.** If \( \langle G_i : i \in \mathbb{N} \rangle \) is a hyperarithmetical sequence of graphs, each of which has a Hamilton path, then there is a hyperarithmetical sequence \( \langle P_i : i \in \mathbb{N} \rangle \) such that for each \( i \), \( P_i \) is a Hamilton path through \( G_i \).

**Proof.** \( \omega \) together with the hyperarithmetical sets is a model of \( \Sigma_1^1 - \text{AC}_0 \) [16]. \( \square \)

Using Theorem 24, it is easy to prove:

**Theorem 28 (RCA₀).** The following are equivalent:

1. \( \text{ATR}_0 \).

2. If \( \langle G_i : i \in \mathbb{N} \rangle \) is a sequence of graphs each of which has at most one Hamilton path, then there is a set \( Z \subseteq \mathbb{N} \) such that for all \( i \in \mathbb{N} \), \( i \in Z \) if and only if \( G_i \) has a Hamilton path.

**Proof.** To prove that (1) implies (2), apply the scheme in part (2) of Theorem 24, using “\( X \) is a Hamilton path through \( G_i \)” for \( \Psi(i, X) \).

To prove the converse, it suffices to deduce part (3) of Theorem 24 using (2). Let \( \langle T_i : i \in \mathbb{N} \rangle \) be a sequence of trees, each with at most one path. Lemma 25 yields a corresponding sequence of graphs, each with at most one Hamilton path. The set \( Z \) obtained by applying (2) satisfies part (3) of Theorem 24. \( \square \)

The following corollary is a recursion theoretic consequence of Theorem 28.

**Corollary 29.** There is a hyperarithmetical sequence of graphs \( \langle G_i : i \in \mathbb{N} \rangle \), each of which has at most one Hamilton path, such that the set \( \{ i \in \mathbb{N} : G_i \text{ has a Hamilton path} \} \) is not
hyperarithmetical.

Proof. $\omega$ together with the hyperarithmetical sets is not a model of $\mathbf{ATR}_0$ [16]. □

Now we will analyze the third and most difficult task. Theorem 30 is closely related to Harel’s proof [7] that the problem of finding a Hamiltonian path is $\Sigma^1_1$ complete.

**Theorem 30 (RCA$_0$).** The following are equivalent:

1. $\Pi^1_1$–$\mathbf{CA}_0$.
2. If $\langle G_i : i \in \mathbb{N} \rangle$ is a sequence of graphs, then there is a set $Z \subseteq \mathbb{N}$ such that $i \in Z$ if and only if $G_i$ has a Hamilton path.

Proof. To prove that (1) implies (2), assume (1) and let $\langle G_i : i \in \mathbb{N} \rangle$ be a sequence of graphs. By $\Pi^1_1$ comprehension, the set

$$C = \{i \in \mathbb{N} : G_i \text{ does not have a Hamilton path}\}$$

exists. By the recursive comprehension axiom, the desired set $Z$, which is the complement of $C$, also exists.

To prove the converse, we will use (2) to prove that $\{n \in \mathbb{N} : \Psi(n)\}$ exists, where $\Psi(n)$ is a $\Pi^1_1$ formula. Note that $\neg \Psi(n)$ is a $\Sigma^1_1$ formula. By Lemma 3.14 of [5], there is a sequence of trees $\langle T_i : i \in \mathbb{N} \rangle$ such that $T_i$ has a path if and only if $\neg \Psi(i)$. By Lemma 25, there is a sequence of associated graphs $\langle G_i : i \in \mathbb{N} \rangle$ such that $G_i$ has a Hamilton graph if and only if $\neg \Psi(i)$. Applying (2) yields the set $Z = \{n \in \mathbb{N} : \neg \Psi(n)\}$. By the recursive comprehension axiom, the complement of $Z$, $\{n \in \mathbb{N} : \Psi(n)\}$, also exists. □

Theorem 30 contrasts nicely with Theorem 21. Since $\Pi^1_1$–$\mathbf{CA}_0$ is a much stronger axiom system than $\mathbf{ACA}_0$, we can conclude that it is more difficult to determine if certain graphs have Hamilton paths than to determine if they have Euler paths. Determining which finite graphs have Hamilton paths is an NP-complete problem, while determining which finite graphs have Euler paths is polynomial time computable. It would be nice to know if this sort of parallel is common, and exactly what it signifies.
REFERENCES

1. D. R. Bean, Recursive Euler and Hamiltonian paths, Proc. Amer. Math. Soc. 55 (1976), 385–394.
2. D. R. Bean, Effective Coloration, J. Symb. Logic 41 (1976), 469–480.
3. H.-G. Carstens and P. Pappinghaus, Recursive coloration of countable graphs, Ann. Pure Appl. Logic 25 (1983), 19–45.
4. H. Friedman, Some systems of second order arithmetic and their use, Proceedings of the International Congress of Mathematicians, Vol. 1, (Vancouver, Canada, 1974), Canadian Mathematical Congress, 1975, pp. 235–242.
5. H. Friedman and J. Hirst, Weak comparability of well orderings and reverse mathematics, Ann. Pure Appl. Logic 47 (1990), 11–29.
6. W. Gasarch, A survey of recursive combinatorics, (in preparation).
7. D. Harel, Hamiltonian paths in infinite graphs, Israel J. of Math. 76 (1991), 317–336.
8. J. Hirst, Combinatorics in subsystems of second order arithmetic, Ph.D. Thesis, The Pennsylvania State University, 1987.
9. J. Hirst, Marriage theorems and reverse mathematics, Logic and Computation, Contemporary Math. 106 (W. Sieg, ed.), AMS, Providence, RI, 1990, pp. 181–195.
10. O. Ore, Theory of graphs, AMS Colloquium Publications Vol. 38, AMS, Providence, RI, 1962.
11. J. Schmerl, Recursive colorings of graphs, Can. J. Math. 32 (1980), 821–830.
12. D. Scott, Algebras of sets binumerable in complete extensions of arithmetic, Proc. Symp. Pure Math. 5 (1962), 117–121.
13. J. Shoenfield, Degrees of models, J. Symb. Logic 25 (1960), 233–237.
14. S. Simpson, Which set existence axioms are needed to prove the Cauchy/Peano theorem for ordinary differential equations?, J. Symb. Logic 49 (1984), 783–802.
15. ———, Subsystems of $Z_2$ and reverse mathematics, Proof Theory (G. Takeuti, ed.), North-Holland, Amsterdam New York, 1985.
16. ———, Subsystems of second order arithmetic, (in preparation).

DEPT. OF COMPUTER SCIENCE AND INSTITUTE FOR ADVANCED STUDIES, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742
E-mail address: gasarch@cs.umd.edu

DEPT. OF MATHEMATICAL SCIENCES, APPALACHIAN STATE UNIVERSITY, BOONE, NC 28608
E-mail address: jlh@math.appstate.edu