On certain categories of modules for affine Lie algebras

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Abstract

In this paper, we re-examine certain integrable modules of Chari-Pressley for an (untwisted) affine Lie algebra $\hat{g}$ by exploiting basic formal variable techniques. We define and study two categories $\mathcal{E}$ and $\mathcal{C}$ of $\hat{g}$-modules using generating functions, where $\mathcal{E}$ contains evaluation modules and $\mathcal{C}$ unifies highest weight modules, evaluation modules and their tensor product modules, and we classify integrable irreducible $\hat{g}$-modules in categories $\mathcal{E}$ and $\mathcal{C}$.

1 Introduction

Let $g$ be a finite-dimensional simple Lie algebra equipped with the Killing form $\langle \cdot, \cdot \rangle$ which is suitably normalized. Associated to the pair $(g, \langle \cdot, \cdot \rangle)$ we have the (untwisted) affine Lie algebra $\hat{g}$ (without the degree derivation added). For affine algebras, a very important class of modules is the class of highest weight modules (cf. [K1]) in the well known category $\mathcal{O}$, where highest weight integrable (irreducible) modules (of nonnegative integral levels) have been the main focus. We also have another class of modules, called evaluation modules (of level zero) associated with a finite number of $g$-modules and with the same number of nonzero complex numbers, studied by Chari and Pressley in [CP2] (cf. [CP1], [CP3]). Furthermore, Chari and Pressley in [CP2] studied the first time the tensor product module of an integrable highest weight $\hat{g}$-module with a (finite-dimensional) evaluation $\hat{g}$-module associated with finite-dimensional irreducible $g$-modules and distinct nonzero complex numbers. (Such a tensor product module is integrable as the tensor product module of any two integrable modules is integrable.) A surprising result, proved in [CP2], is that such a tensor product module is also irreducible. In this way, a new family of integrable $\hat{g}$-modules were constructed.

We know that integrable highest weight $\hat{g}$-modules are exactly the irreducible integrable modules in the well-known category $\mathcal{O}$ (see [K1]) and that irreducible integrable evaluation modules are exactly the finite-dimensional irreducible modules (see [C], [CP2]). In view of this, naturally one would want to find a canonical characterization for the new integrable modules, instead of presenting them as tensor product modules. This is part of our motivation for this paper. Part of our motivation is to look for canonical connections
of the new integrable modules with modules and fusion rules for affine vertex operator algebras.

In this paper, we give a canonical characterization of the new integrable modules using generating functions and formal calculus. Notice that highest weight \( \hat{\mathfrak{g}} \)-modules belong to a bigger class of modules called restricted modules, where a \( \hat{\mathfrak{g}} \)-module \( W \) is said to be restricted (cf. [K1]) if for any \( a \in \mathfrak{g} \), \( w \in W \), \( (a \otimes t^n)w = 0 \) for \( n \) sufficiently large. In terms of generating functions, a \( \hat{\mathfrak{g}} \)-module \( W \) is restricted if and only if for all \( a \in \mathfrak{g} \), \( w \in W \), \( a(x)w \in W((x)) \) for \( a \in \mathfrak{g} \), \( w \in W \), where \( a(x) = \sum_{n \in \mathbb{Z}} (a \otimes t^n)x^{-n-1} \) (the generating function).

For an evaluation module \( U \) (see [CP2]), we show that there is a nonzero polynomial \( p(x) \) such that \( p(x)a(x)u = 0 \) for \( a \in \mathfrak{g} \), \( u \in U \). Then \( p(x)a(x)v \in (W \otimes U)((x)) \) for \( a \in \mathfrak{g} \), \( v \in W \otimes U \). Motivated by these facts, we define a category \( \mathcal{E} \) to consist of \( \hat{\mathfrak{g}} \)-modules \( W \) such that there exists a nonzero polynomial \( p(x) \) such that \( p(x)a(x)w = 0 \) for \( a \in \mathfrak{g} \), \( w \in W \) and we define a category \( \mathcal{C} \) to consist of \( \hat{\mathfrak{g}} \)-modules \( W \) such that there exists a nonzero polynomial \( f(x) \) such that \( f(x)a(x)w \in W((x)) \) for \( a \in \mathfrak{g} \), \( w \in W \). Then category \( \mathcal{E} \) contains all the evaluation modules and category \( \mathcal{C} \) contains all the restricted modules, the evaluation modules and their tensor products, so that category \( \mathcal{C} \) unifies all the mentioned modules. In this paper we prove that the irreducible integrable \( \hat{\mathfrak{g}} \)-modules in the category \( \mathcal{E} \) are exactly the finite-dimensional irreducible evaluation modules up to isomorphism. (This result is analogous and closely related to a result of Chari-Pressley [C], [CP2].) It was proved in [DLM] that every restricted integrable \( \hat{\mathfrak{g}} \)-module is a direct sum of highest weight irreducible integrable \( \hat{\mathfrak{g}} \)-modules. As our main result of this paper we prove that the irreducible integrable \( \hat{\mathfrak{g}} \)-modules in the category \( \mathcal{C} \) up to isomorphism are exactly the tensor product modules of highest weight irreducible integrable \( \hat{\mathfrak{g}} \)-modules with finite-dimensional irreducible evaluation modules. The key to our main result is a factorization result which states that every irreducible representation of \( \hat{\mathfrak{g}} \) in the category \( \mathcal{C} \) can be factorized canonically as the product of two representations of \( \hat{\mathfrak{g}} \) such that the first representation defines a restricted module and the second one defines a module in the category \( \mathcal{E} \). The proof of this factorization uses formal calculus in an essential way.

It is well known (cf. [Li2], [LL]) that restricted \( \hat{\mathfrak{g}} \)-modules are closely related to affine vertex operator algebras and their modules. But the tensor product \( \hat{\mathfrak{g}} \)-modules in the category \( \mathcal{C} \) is not a module for the affine vertex operator algebra. In this paper, by using a result of [L3] we show that if \( W \) and \( W_1 \) are highest weight integrable irreducible \( \hat{\mathfrak{g}} \)-modules of the same level and \( U(z) \) is a finite-dimensional evaluation module, \( \text{Hom}_{\hat{\mathfrak{g}}}(W \otimes U(z), W_2) \) gives the fusion rule of a certain type as generally defined in [FHL] in terms of vertex operator algebras and their modules.

In this paper, most of the results are proved in the generality that \( \mathfrak{g} \) is only assumed to be of countable dimension, so those results in fact hold for toroidal Lie algebras.

This paper is organized as follows: In Section 2, in the first half we review the definitions and examples of restricted modules and evaluation modules for affine Lie algebras.
and we recall certain results of Chari-Pressley. In the second half we define categories $E$ and $C$ and we give slight generalizations of Chari-Pressley’s results. In Section 3, we classify the irreducible integrable modules in the categories $E$ and $C$. In Section 4, we give a connection between the tensor product module of a highest weight irreducible integrable module with an evaluation module and fusion rules of certain types.

2 Categories $R$, $E$ and $C$ of modules for affine Lie algebras

In this section we review the definitions and examples of restricted modules and evaluation modules for an affine Lie algebra $\hat{g}$. We define a category $E$ of $\hat{g}$-modules, including evaluation modules, and we define a category $C$ of $\hat{g}$-modules, including restricted modules, evaluation modules and their tensor product modules. We give a generalization of certain results of Chari and Pressley ([C], [CP2]) with a different proof using formal calculus.

First let us fix some formal variable notations (see [FLM], [FHL], [LL]). Throughout this paper, $x,x_1,x_2,\ldots$ are independent mutually commuting formal variables. We shall typically use $z,z_1,z_2,\ldots$ for complex numbers. For a vector space $U$, $U[[x_1^{\pm 1},\ldots,x_n^{\pm 1}]]$ denotes the space of all formal (possibly doubly infinite) series in $x_1,\ldots,x_n$ with coefficients in $U$, $U((x_1,\ldots,x_n))$ denotes the space of all formal (lower truncated) Laurent series in $x_1,\ldots,x_n$ with coefficients in $U$ and $U[[x_1,\ldots,x_n]]$ denotes the space of all formal (nonnegative) powers series in $x_1,\ldots,x_n$ with coefficients in $U$.

Remark 2.1. As it was pointed out in [FLM] (cf. [LL]), in formal calculus, associativity law and cancellation law for products of formal series do not hold in general, but they do hold if all the involved (sub)products exist. For example, if $a(x) \in U((x))$, $f(x),g(x) \in \mathbb{C}((x))$, we have

$$f(x)(g(x)a(x)) = (f(x)g(x))a(x).$$

For $a(x),b(x) \in U((x))$, if $h(x)a(x) = h(x)b(x)$ for some $h(x) \in \mathbb{C}((x))$, then $a(x) = b(x)$.

We shall use the traditional binomial expansion convention: For $m \in \mathbb{Z}$,

$$(x_1 \pm x_2)^m = \sum_{i \geq 0} \binom{m}{i} (\pm 1)^i x_1^{m-i} x_2^i \in \mathbb{C}[x_1,x_1^{-1}][[x_2]]. \quad (2.1)$$

Recall from [FLM] the formal delta function

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n \in \mathbb{C}[[x,x^{-1}]]. \quad (2.2)$$

Its fundamental property is that

$$f(x)\delta(x) = f(1)\delta(x) \quad \text{for } f(x) \in \mathbb{C}[x,x^{-1}]. \quad (2.3)$$
For any nonzero complex number \( z \),
\[
\delta \left( \frac{z}{x} \right) = \sum_{n \in \mathbb{Z}} z^n x^{-n} \in \mathbb{C}[x, x^{-1}] 
\]  
(2.4)

and we have
\[
f(x)\delta \left( \frac{z}{x} \right) = f(z)\delta \left( \frac{z}{x} \right) \quad \text{for } f(x) \in \mathbb{C}[x, x^{-1}].
\]  
(2.5)

In particular,
\[
(x - z)\delta \left( \frac{z}{x} \right) = 0.
\]  
(2.6)

Let \( \mathfrak{g} \) be a Lie algebra (not necessarily finite-dimensional) equipped with a nondegenerate symmetric invariant bilinear form \( \langle \cdot, \cdot \rangle \), fixed throughout this section. Let \( \hat{\mathfrak{g}} \) be the corresponding (untwisted) affine Lie algebra, i.e.,
\[
\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k
\]  
(2.7)

with the defining commutator relations
\[
[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\langle a, b \rangle \delta_{m+n,0}k \quad \text{for } a, b \in \mathfrak{g}, \ m, n \in \mathbb{Z}
\]  
(2.8)

and with \( k \) as a nonzero central element. A \( \hat{\mathfrak{g}} \)-module \( W \) is said to be of level \( \ell \) in \( \mathbb{C} \) if the central element \( k \) acts on \( W \) as the scalar \( \ell \). By the standard untwisted affine algebra \( \hat{\mathfrak{g}} \) we mean the affine Lie algebra \( \hat{\mathfrak{g}} \) with \( \mathfrak{g} \) a finite-dimensional simple Lie algebra and with \( \langle \cdot, \cdot \rangle \) the normalized Killing form so that the squared length of the longest roots is 2.

For \( a \in \mathfrak{g} \), form the generating function
\[
a(x) = \sum_{n \in \mathbb{Z}} (a \otimes t^n)x^{-n-1} \in \hat{\mathfrak{g}}[[x, x^{-1}]].
\]  
(2.9)

In terms of generating functions the defining relations (2.8) exactly amount to
\[
[a(x_1), b(x_2)] = [a, b](x_2)x_2^{-1}\delta \left( \frac{x_1}{x_2} \right) + \langle a, b \rangle k \frac{\partial}{\partial x_2}x_2^{-1}\delta \left( \frac{x_1}{x_2} \right).
\]  
(2.10)

Following the tradition (cf. [FLM], [LL]), for \( a \in \mathfrak{g}, \ n \in \mathbb{Z} \) we shall use \( a(n) \) for the corresponding operator associated to \( a \otimes t^n \) on \( \hat{\mathfrak{g}} \)-modules. We now introduce the category \( \mathcal{R} \) of the so-called restricted modules for the affine algebra \( \hat{\mathfrak{g}} \). A \( \hat{\mathfrak{g}} \)-module \( W \) is said to be restricted (cf. [K1]) if for any \( w \in W, \ a \in \mathfrak{g}, \)
\[
a(n)w = 0 \quad \text{for } n \text{ sufficiently large.}
\]  
(2.11)
We define a $\mathbb{Z}$-grading $\hat{g} = \bigsqcup_{n \in \mathbb{Z}} \hat{g}(n)$ by

$$
\hat{g}(0) = g \oplus \mathbb{C}k \quad \text{and} \quad \hat{g}(n) = g \otimes t^{-n} \quad \text{for} \quad n \neq 0,
$$

(2.12)

making $\hat{g}$ a $\mathbb{Z}$-graded Lie algebra. It is clear that any $\mathbb{N}$-graded $\hat{g}$-module is automatically a restricted module.

Let $U$ be a $g$-module and let $\ell$ be any complex number. Let $k$ act on $U$ as the scalar $\ell$ and let $g \otimes t\mathbb{C}[t] \oplus \mathbb{C}k$ act trivially, making $U$ a $(g \otimes \mathbb{C}[t] \oplus \mathbb{C}k)$-module. Form the following induced $\hat{g}$-module

$$
M_{\hat{g}}(\ell, U) = U(\hat{g}) \otimes_U (g \otimes \mathbb{C}[t] \oplus \mathbb{C}k) U.
$$

(2.13)

Endow $U$ with zero degree, making $M_{\hat{g}}(\ell, U)$ an $\mathbb{N}$-graded $\hat{g}$-module. This in particular implies that $M_{\hat{g}}(\ell, U)$ is a restricted $\hat{g}$-module. This $\hat{g}$-module is commonly called the Weyl module or the generalized Verma module associated with $g$-module $U$. If $g$ is a finite-dimensional simple Lie algebra and if $U$ is a (highest weight) Verma $g$-module, then $M_{\hat{g}}(\ell, U)$ is isomorphic to a (highest weight) Verma $\hat{g}$-module (cf. [K1]). Furthermore, any (highest weight) Verma $\hat{g}$-module is isomorphic to a module of the form $M_{\hat{g}}(\ell, U)$.

A homomorphic image of a Verma $\hat{g}$-module is called a highest weight module. Then the category $\mathcal{R}$ contains all the highest weight modules for the standard affine Lie algebra $\hat{g}$.

For the affine Lie algebra $\hat{g}$, we also have another family of $\hat{g}$-modules, called the evaluation modules (see [CP2]). Let $U$ be a $g$-module and let $z$ be a nonzero complex number. Define an action of $\hat{g}$ on $U$ by

$$
a(n) \cdot u = z^n(au) \quad \text{for} \quad a \in g, \ n \in \mathbb{Z},
$$

(2.14)

$$
k \cdot U = 0.
$$

(2.15)

Then $U$ equipped with the defined action is a $\hat{g}$-module (of level zero) (see [CP2]), which is denoted by $U(z)$. If $U$ is an irreducible $g$-module, it is clear that $U(z)$ is an irreducible $\hat{g}$-module. More generally, let $U_1, \ldots, U_r$ be $g$-modules and let $z_1, \ldots, z_r$ be nonzero complex numbers. Then $U = U_1 \otimes \cdots \otimes U_r$ is a $\hat{g}$-module where $k$ acts as zero and

$$
a(n)(u_1 \otimes \cdots \otimes u_r) = \sum_{i=1}^{r} z_i^n(u_1 \otimes \cdots \otimes au_i \otimes \cdots \otimes u_r)
$$

(2.16)

for $a \in g, \ n \in \mathbb{Z}, \ u_i \in U_i$. This module is nothing but the tensor product $\hat{g}$-module $\otimes_{i=1}^{r} U_i(z_i)$. Such a $\hat{g}$-module is called an evaluation module. The following results are due to Chari and Pressley (see [C] and [CP2]):

**Theorem 2.2.** Let $g$ be a finite-dimensional simple Lie algebra. Let $U_1, \ldots, U_r$ be (finite-dimensional) irreducible $g$-modules and let $z_1, \ldots, z_r$ be distinct nonzero complex numbers. Then $\otimes_{i=1}^{r} U_i(z_i)$ is a (finite-dimensional) irreducible $\hat{g}$-module of level zero. Furthermore, every finite-dimensional irreducible $\hat{g}$-module is isomorphic to such a $\hat{g}$-module.
Remark 2.3. Let \( g \) be a finite-dimensional simple Lie algebra. We fix a Cartan subalgebra \( h \) and denote by \( \Delta \) the set of roots, so that \( g = h \oplus \sum_{\alpha \in \Delta} g_\alpha \). We also fix a choice of set \( \Delta_+ \) of positive roots and denote by \( \theta \) the highest long root. Let \( \langle \cdot, \cdot \rangle \) be the normalized Killing form on \( g \) such that \( \langle \theta, \theta \rangle = 2 \). A \( \hat{g} \)-module \( W \) is said to be integrable (see [K1], [K2]) if \( g_\alpha(n) \) acts locally nilpotently on \( W \) for \( \alpha \in \Delta \), \( n \in \mathbb{Z} \). Then (see [K2]) the subalgebra \( h \oplus \mathbb{C}k \) of \( \hat{g} \) acts semisimply on \( W \). If \( W \) is an irreducible integrable \( \hat{g} \)-module, the central element \( k \) acts on \( W \) as a scalar \( \ell \) in \( \mathbb{N} \).

A singular vector of a \( \hat{g} \)-module \( W \) of level \( \ell \) is a (nonzero) \( h \)-eigenvector \( u \) such that \( a(n)u = 0 \) for \( a \in g \), \( n > 0 \) and \( a(0)u = 0 \) for \( a \in g_+ \). A known fact is that the submodule of an integrable \( \hat{g} \)-module generated by a singular vector is irreducible (see [K1]).

The following result was established by Chari and Pressley in [CP2]:

Theorem 2.4. Let \( \hat{g} \) be a standard affine Lie algebra (with \( g \) a finite-dimensional simple Lie algebra and \( \langle \cdot, \cdot \rangle \) the normalized Killing form). Let \( W \) be an irreducible highest weight integrable \( \hat{g} \)-module, let \( U_1, \ldots, U_r \) be finite-dimensional irreducible \( g \)-modules and let \( z_1, \ldots, z_r \) be distinct nonzero complex numbers. Then the tensor product \( \hat{g} \)-module \( W \otimes U_1(z_1) \otimes \cdots \otimes U_r(z_r) \) is irreducible.

Note that a restricted \( \hat{g} \)-module is defined canonically by the property (2.11) while typical evaluation \( \hat{g} \)-modules are finite-dimensional. Then naturally one would want to find a canonical characterization for the new family of (tensor product) \( \hat{g} \)-modules \( W \otimes U_1(z_1) \otimes \cdots \otimes U_r(z_r) \). In the following we give a characterization in terms of generating functions.

First consider restricted \( \hat{g} \)-modules (in the category \( \mathcal{R} \)). Note that the condition (2.11) amounts to that

\[
a(x)w \in W((x)) \quad \text{for } a \in g, \, w \in W.
\]

(2.17)

That is, a \( \hat{g} \)-module \( W \) is restricted if and only if

\[
a(x) \in \text{Hom}(W, W((x))) \quad \text{for } a \in g.
\]

(2.18)

Then we consider evaluation \( \hat{g} \)-modules. Let \( U_1, \ldots, U_r \) be \( g \)-modules and \( z_1, \ldots, z_r \) nonzero complex numbers. For \( a \in g \), \( u_i \in U_i(z_i) = U_i \), writing (2.16) in terms of generating functions, we have

\[
a(x)(u_1 \otimes \cdots \otimes u_r) = \sum_{n \in \mathbb{Z}} \sum_{i=1}^{r} z_i^n x^{-n-1}(u_1 \otimes \cdots \otimes au_i \otimes \cdots \otimes u_r)
\]

\[
= \sum_{i=1}^{r} x^{-1} \delta \left( \frac{z_i}{x} \right) (u_1 \otimes \cdots \otimes au_i \otimes \cdots \otimes u_r).
\]

(2.19)

Since \( (x - z_i)\delta \left( \frac{z_i}{x} \right) = 0 \) for \( i = 1, \ldots, r \), we get \( (x - z_1) \cdots (x - z_r)a(x)(u_1 \otimes \cdots \otimes u_r) = 0 \).

In view of this and (2.17) we immediately have:
Lemma 2.5. Let $U_1, \ldots, U_r$ be $\mathfrak{g}$-modules and let $z_1, \ldots, z_r$ be nonzero complex numbers. Then on the tensor product $\hat{\mathfrak{g}}$-module $U_1(z_1) \otimes \cdots \otimes U_r(z_r)$,

$$(x - z_1) \cdots (x - z_r)a(x) = 0 \quad \text{for } a \in \mathfrak{g}. \quad (2.20)$$

Furthermore, for any restricted $\hat{\mathfrak{g}}$-module $W$, we have

$$(x - z_1) \cdots (x - z_r)a(x) \in \text{Hom}(M, M((x))) \quad \text{for } a \in \mathfrak{g}, \quad (2.21)$$

where $M$ denotes the tensor product $\hat{\mathfrak{g}}$-module $W \otimes U_1(z_1) \otimes \cdots \otimes U_r(z_r)$.

Motivated by Lemma 2.5 we define the following two categories:

Definition 2.6. We define a category $\mathcal{E}$ to consist of $\hat{\mathfrak{g}}$-modules $W$ for which there exists a nonzero polynomial $p(x) \in \mathbb{C}[x]$ such that

$$p(x)a(x)w = 0 \quad \text{for } a \in \mathfrak{g}, \ w \in W. \quad (2.22)$$

We define category $\mathcal{C}$ to consist of $\hat{\mathfrak{g}}$-modules $W$ such that there exists a nonzero polynomial $p(x)$ such that

$$p(x)a(x) \in \text{Hom}(W, W((x))) \quad \text{for } a \in \mathfrak{g}. \quad (2.23)$$

Remark 2.7. In view of Lemma 2.5, all the evaluation $\hat{\mathfrak{g}}$-modules belong to the category $\mathcal{E}$ and all the restricted $\hat{\mathfrak{g}}$-modules, evaluation $\hat{\mathfrak{g}}$-modules and tensor products of restricted $\hat{\mathfrak{g}}$-modules with evaluation $\hat{\mathfrak{g}}$-modules belong to the category $\mathcal{C}$.

Remark 2.8. In [C], Chari defined a category $\hat{\mathcal{O}}$ of $\hat{\mathfrak{g}}$-modules and classified all the irreducible modules and all the integrable modules in this category. Furthermore, Chari-Pressley proved in [CP2] that irreducible integrable modules in category $\hat{\mathcal{O}}$ are exactly the finite-dimensional evaluation modules up to isomorphism. The categories $\mathcal{E}$ and $\hat{\mathcal{O}}$ are closely related, but they are different.

Lemma 2.9. The central element $k$ acts as zero on any $\hat{\mathfrak{g}}$-module in the category $\mathcal{E}$.

Proof. Let $W$ be a $\hat{\mathfrak{g}}$-module in the category $\mathcal{E}$ with a nonzero polynomial $p(x)$ such that $p(x)u(x) = 0$ on $W$ for $u \in \mathfrak{g}$. If $p(x)$ is of degree zero, we have $u(x) = 0$ for $u \in \mathfrak{g}$, i.e., $u(n) = 0$ for $u \in \mathfrak{g}$, $n \in \mathbb{Z}$. In view of the commutator relation (2.8) we see that $k$ must be zero on $W$. Assume that $p(x)$ is not a constant, that is, $p'(x) \neq 0$. Let $a, b \in \mathfrak{g}$ be such that $\langle a, b \rangle = 1$. (Notice that $\langle \cdot, \cdot \rangle$ is assumed to be nondegenerate.) Using the commutator relations (2.10) we get

$$0 = p(x_1)p(x_2)[a(x_1), b(x_2)] = kp(x_1)p(x_2)\frac{\partial}{\partial x_2}x_2^{-1} \delta \left(\frac{x_1}{x_2}\right). \quad (2.24)$$
Noticing that
\[ \text{Res}_{x_2}p(x_1)p(x_2)\frac{\partial}{\partial x_2}x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = -\text{Res}_{x_2}p(x_1)p'(x_2)x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = -p(x_1)p'(x_1), \] (2.25)
we get \( kp(x_1)p'(x_1) = 0 \), which implies that \( k = 0 \) on \( W \).

In the following we give a slight generalization of Theorems 2.2 and 2.4. First recall from [Li2] (Lemmas 2.10 and 2.11) the following result (which might be well known, but we do not know any other reference):

**Lemma 2.10.** Let \( A_1 \) and \( A_2 \) be associative algebras (with identity) and let \( U_1 \) and \( U_2 \) be irreducible modules for \( A_1 \) and \( A_2 \), respectively. If either \( \text{End}_{A_1}U_1 = \mathbb{C} \), or \( A_1 \) is of countable dimension, then \( U_1 \otimes U_2 \) is an irreducible \( A_1 \otimes A_2 \)-module.

The following result slightly generalizes the first assertion of Theorem 2.2 of Chari and Pressley with a slightly different proof:

**Proposition 2.11.** Assume that \( \mathfrak{g} \) is of countable dimension. Let \( U_1, \ldots, U_r \) be irreducible \( \mathfrak{g} \)-modules and let \( z_1, \ldots, z_r \) be distinct nonzero complex numbers. Then \( U_1(z_1) \otimes \cdots \otimes U_r(z_r) \) is an irreducible \( \hat{\mathfrak{g}} \)-module.

**Proof.** Notice that the universal enveloping algebra \( U(\hat{\mathfrak{g}}) \) is of countable dimension. It follows from Lemma 2.10 (and induction) that \( U_1(z_1) \otimes \cdots \otimes U_r(z_r) \) is an irreducible module for the product Lie algebra \( \hat{\mathfrak{g}} \oplus \cdots \oplus \hat{\mathfrak{g}} \) (\( r \) copies). Denote by \( \pi \) the representation homomorphism map. For \( 1 \leq i \leq r \), denote by \( \psi_i \) the \( i \)-th embedding of \( \hat{\mathfrak{g}} \) into \( \hat{\mathfrak{g}} \oplus \cdots \oplus \hat{\mathfrak{g}} \) (\( r \) copies) and denote by \( \psi \) the diagonal map from \( \hat{\mathfrak{g}} \) to \( \hat{\mathfrak{g}} \oplus \cdots \oplus \hat{\mathfrak{g}} \) (\( r \) copies). Then \( \psi = \psi_1 + \cdots + \psi_r \). We also extend the linear maps \( \psi \) and \( \psi_1, \ldots, \psi_r \) on \( \mathfrak{g}[x, x^{-1}] \) canonically.

For \( 1 \leq i \leq r \), set \( p_i(x) = \prod_{j \neq i}(x - z_j)/(z_i - z_j) \). Then
\[ p_i(x)\delta\left(\frac{z_j}{x}\right) = p_i(z_j)\delta\left(\frac{z_j}{x}\right) = \delta_{i,j}\delta\left(\frac{z_j}{x}\right) \] (2.26)
for \( i, j = 1, \ldots, r \). Using (2.19) we have that on \( U_1(z_1) \otimes \cdots \otimes U_r(z_r) \),
\[ p_i(x)\pi\psi_j(a(x)) = \delta_{i,j}\pi\psi_j(a(x)) \quad \text{for } 1 \leq i, j \leq r, \ a \in \mathfrak{g}. \] (2.27)
Thus on \( U_1(z_1) \otimes \cdots \otimes U_r(z_r) \),
\[ p_i(x)\pi\psi(a(x)) = \pi\psi_i(a(x)) \quad \text{for } 1 \leq i \leq r, \ a \in \mathfrak{g}, \] (2.28)
which implies that
\[ \pi\psi_i(\hat{\mathfrak{g}}) \subset \pi\psi(\hat{\mathfrak{g}}) \quad \text{for } i = 1, \ldots, r. \] (2.29)
From this we have
\[ \pi\psi(\hat{\mathfrak{g}}) = \pi\psi_1(\hat{\mathfrak{g}}) + \cdots + \pi\psi_r(\hat{\mathfrak{g}}). \] (2.30)
It follows that \( U_1(z_1) \otimes \cdots \otimes U_r(z_r) \) is an irreducible \( \hat{\mathfrak{g}} \)-module. \( \square \)
We also have the following result which generalizes Theorem 2.4 of Chari and Pressley with a different proof:

**Proposition 2.12.** Assume that \( g \) is of countable dimension. Let \( W \) be an irreducible restricted \( \hat{g} \)-module (in the category \( \mathcal{R} \)) and let \( U \) be an irreducible \( \hat{g} \)-module in the category \( \mathcal{E} \). Then the tensor product module \( W \otimes U \) is irreducible.

**Proof.** Let \( M \) be any nonzero submodule of the tensor product \( \hat{g} \)-module \( W \otimes U \). We must prove that \( M = W \otimes U \). Since \( W \) and \( U \) are irreducible \( \hat{g} \)-modules and \( U(\hat{g}) \) is of countable dimension, by Lemma 2.10 \( W \otimes U \) is an irreducible \( \hat{g} \oplus \hat{g} \)-module. Now, it suffices to prove that \( M \) is a \( \hat{g} \oplus \hat{g} \)-submodule of \( W \otimes U \) and furthermore it suffices to prove that

\[
(a(n) \otimes 1)M \subset M \quad \text{for } a \in \mathfrak{g}, \ n \in \mathbb{Z}. \tag{2.31}
\]

(Notice that \((1 \otimes a(n))w = a(n)w - (a(n) \otimes 1)w\) for \( w \in M \).)

By definition there exists a nonzero polynomial \( p(x) \) such that \( p(x)a(x) = 0 \) on \( U \) for all \( a \in \mathfrak{g} \), so that

\[
p(x)(a(x) \otimes 1 + 1 \otimes a(x)) = p(x)a(x) \otimes 1 \quad \text{on } W \otimes U. \tag{2.32}
\]

With \( M \) being a \( \hat{g} \)-submodule of the tensor product module and with \( W \) being a restricted module we have

\[
p(x)(a(x) \otimes 1 + 1 \otimes a(x))M \subset M[[x, x^{-1}]], \quad p(x)(a(x) \otimes 1)M \subset (W \otimes U)((x)). \tag{2.33}
\]

From this, using (2.32) we have

\[
p(x)(a(x) \otimes 1)M \subset M((x)) \quad \text{for } a \in \mathfrak{g}. \tag{2.34}
\]

Let \( f(x) \) be the formal Laurent series of rational function \( 1/p(x) \) at zero, so that \( f(x) \in \mathbb{C}((x)) \). Then we have

\[
a(x) \otimes 1 = (f(x)p(x))(a(x) \otimes 1) = f(x)(p(x)(a(x) \otimes 1))
\]

on \( M \). Consequently,

\[
(a(x) \otimes 1)M \subset M((x)) \quad \text{for } a \in \mathfrak{g}. \tag{2.35}
\]

This proves (2.31), completing the proof. \( \square \)

The following result tells us when two \( \hat{g} \)-modules of the form \( W \otimes U \) obtained in Proposition 2.12 are isomorphic:
**Proposition 2.13.** Let $W_1, W_2$ be irreducible $\mathfrak{g}$-modules in category $\mathcal{R}$ and let $U_1$ and $U_2$ be irreducible $\mathfrak{g}$-modules in category $\mathcal{E}$. Then the tensor product $\mathfrak{g}$-modules $W_1 \otimes U_1$ and $W_2 \otimes U_2$ are isomorphic if and only if $W_1$ and $U_1$ are isomorphic to $W_2$ and $U_2$, respectively.

**Proof.** We only need to prove the “only if” part. Let $f$ be a $\mathfrak{g}$-module isomorphism from $W_1 \otimes U_1$ onto $W_2 \otimes U_2$. We have

$$f(a(x) \otimes 1 + 1 \otimes a(x))v = (a(x) \otimes 1 + 1 \otimes a(x))f(v) \quad \text{for } a \in \mathfrak{g}, \ v \in W_1 \otimes U_1. \quad (2.36)$$

Let $p(x)$ be a nonzero polynomial such that

$$p(x)a(x)U_1 = 0 \quad \text{and} \quad p(x)a(x)U_2 = 0 \quad \text{for } a \in \mathfrak{g}.$$

Using this and (2.36) we get

$$p(x)f((a(x) \otimes 1)v) = p(x)(a(x) \otimes 1)f(v) \quad \text{for } a \in \mathfrak{g}, \ v \in W_1 \otimes U_1. \quad (2.37)$$

In view of Remark 2.1, we have

$$f((a(x) \otimes 1)v) = (a(x) \otimes 1)f(v) \quad \text{for } a \in \mathfrak{g}, \ v \in W_1 \otimes U_1. \quad (2.38)$$

Let $u_i$ for $i \in S$ be a basis of $U_2$. Then $W_2 \otimes U_2 = \bigoplus_{i \in S} W_2 \otimes \mathbb{C}u_i$. Denote by $\phi_i$ the projection of $W_2 \otimes U_2$ onto $W_2 \otimes \mathbb{C}u_i$. We have

$$\phi_i((a(x) \otimes 1)v) = (a(x) \otimes 1)\phi_i(v) \quad \text{for } a \in \mathfrak{g}, \ v \in W_2 \otimes U_2,$$

so that

$$\phi_i((a(x) \otimes 1)v) = (a(x) \otimes 1)\phi_i(v) \quad \text{for } a \in \mathfrak{g}, \ v \in W_1 \otimes U_1.$$

Let $0 \neq u_1 \in U_1$. There exists an $i \in S$ such that $\phi_i f \neq 0$ on $W_1 \otimes \mathbb{C}u_1$. We see that the map $\phi_i f$ gives rise to a nonzero $\mathfrak{g}$-module homomorphism from $W_1 (= W_1 \otimes \mathbb{C}u_1)$ onto $W_2 (= W_2 \otimes \mathbb{C}u_i)$. Because $W_1$ and $W_2$ are irreducible, this nonzero homomorphism is an isomorphism. This proves that $W_1$ is isomorphic to $W_2$.

From (2.36) and (2.38) we have

$$f((1 \otimes a(x))v) = (1 \otimes a(x))f(v) \quad \text{for } a \in \mathfrak{g}, \ v \in W_1 \otimes U_1. \quad (2.39)$$

Then using the same strategy, we see that $U_1$ is isomorphic to $U_2$. \hfill \Box

Furthermore, the following result, which is a version of a result of Chari in [C], gives the equivalence on evaluation $\mathfrak{g}$-modules (in category $\mathcal{E}$):

**Proposition 2.14.** Let $U_1, \ldots, U_r, V_1, \ldots, V_s$ be nontrivial irreducible $\mathfrak{g}$-modules and let $z_1, \ldots, z_r$ and $\xi_1, \ldots, \xi_s$ be two groups of distinct nonzero complex numbers. Then the $\mathfrak{g}$-module $U_1(z_1) \otimes \cdots \otimes U_r(z_r)$ is isomorphic to $V_1(\xi_1) \otimes \cdots \otimes V_s(\xi_s)$ if and only if $r = s$, $z_i = \xi_i$ and $U_i \cong V_i$ up to a permutation.
Proof. We only need to prove the “only if” part. Let \( U \) be any \( \mathfrak{g} \)-module in category \( \mathcal{E} \). There exists a (unique nonzero) monic polynomial \( p(x) \) of least degree such that \( p(x)a(x)U = 0 \) for \( a \in \mathfrak{g} \). Clearly, isomorphic \( \hat{\mathfrak{g}} \)-modules in category \( \mathcal{E} \) have the same monic polynomial. If \( U = U_1(z_1) \otimes \cdots \otimes U_r(z_r) \), we are going to show that \( p(x) = (x - z_1) \cdots (x - z_r) \) is the associated monic polynomial. First, by Lemma 2.5 we have that \( p(x)a(x) = 0 \) on \( U_1(z_1) \otimes \cdots \otimes U_r(z_r) \) for \( a \in \mathfrak{g} \). Let \( q(x) \) be any polynomial such that \( q(x)a(x) = 0 \) on \( U_1(z_1) \otimes \cdots \otimes U_r(z_r) \) for \( a \in \mathfrak{g} \). Set \( p_i(x) = \prod_{j \not= i}(x - z_j)/(z_i - z_j) \) for \( i = 1, \ldots, r \) as in the proof of Proposition 2.11. For \( a \in \mathfrak{g} \), \( u_i \in U_i \) with \( i = 1, \ldots, r \), we have

\[
0 = q(x)p_i(x)a(x)(u_1 \otimes \cdots \otimes u_r) = q(x)x^{-1}\delta \left( \frac{z_i}{x} \right) (u_1 \otimes \cdots \otimes au_i \otimes \cdots \otimes u_r) = q(z_i)x^{-1}\delta \left( \frac{z_i}{x} \right) (u_1 \otimes \cdots \otimes au_i \otimes \cdots \otimes u_r).
\]

Since each \( U_i \) is a nontrivial \( \mathfrak{g} \)-module, we must have \( q(z_i) = 0 \) for \( i = 1, \ldots, r \). Thus \( p(x) \) divides \( q(x) \). This proves that \( p(x) \) is the associated monic polynomial.

Assume that \( U_1(z_1) \otimes \cdots \otimes U_r(z_r) \) is isomorphic to \( V_1(\xi_1) \otimes \cdots \otimes V_s(\xi_s) \) with \( f \) a \( \hat{\mathfrak{g}} \)-module isomorphism map. Then the two tensor product modules must have the same associated monic polynomial. That is, \( (x - z_1) \cdots (x - z_r) = (x - \xi_1) \cdots (x - \xi_s) \). Thus \( r = s \) and up to a permutation \( z_i = \xi_i \) for \( i = 1, \ldots, r \). Assume that \( z_i = \xi_i \) for \( i = 1, \ldots, r \). For \( 1 \leq i \leq r \), \( a \in \mathfrak{g} \) and for \( u_j \in U_j \), \( v_j \in V_j \) with \( j = 1, \ldots, r \), we have

\[
p_i(x)a(x)(u_1 \otimes \cdots \otimes u_r) = x^{-1}\delta \left( \frac{z_i}{x} \right) (u_1 \otimes \cdots \otimes au_i \otimes \cdots \otimes u_r), \quad (2.40)
\]

\[
p_i(x)a(x)(v_1 \otimes \cdots \otimes v_r) = x^{-1}\delta \left( \frac{z_i}{x} \right) (v_1 \otimes \cdots \otimes av_i \otimes \cdots \otimes v_r). \quad (2.41)
\]

Then

\[
f(u_1 \otimes \cdots \otimes au_i \otimes \cdots \otimes u_r) = \text{Res}_xF_x^{-1}\delta \left( \frac{z_i}{x} \right) f(u_1 \otimes \cdots \otimes au_i \otimes \cdots \otimes u_r)
= \text{Res}_x f(p_i(x)a(x)(u_1 \otimes \cdots \otimes u_r))
= \text{Res}_s p_i(x)a(x) f(u_1 \otimes \cdots \otimes u_r)
= \sigma_i(a) f(u_1 \otimes \cdots \otimes u_r), \quad (2.42)
\]

where for \( a \in \mathfrak{g} \), \( v_1 \in V_1, \ldots, v_r \in V_r \),

\[
\sigma_i(a)(v_1 \otimes \cdots \otimes v_r) = (v_1 \otimes \cdots \otimes av_i \otimes \cdots \otimes v_r).
\]

Now, from the proof of Proposition 2.13 we see that \( U_i \) is isomorphic to \( V_i \). \( \square \)

In view of Remark 2.4 and Proposition 2.12 naturally one wants to know whether irreducible \( \hat{\mathfrak{g}} \)-modules of the form \( W \otimes U \) as in Proposition 2.12 exhaust the irreducible
\[ \hat{g} \]-modules in the category \( \mathcal{C} \) up to isomorphism. In the next section we shall prove that this is true if we restrict ourselves to integrable module for a standard affine Lie algebra \( \hat{g} \).

### 3 Classification of irreducible integrable \( \hat{g} \)-modules in the categories \( \mathcal{R}, \mathcal{E} \) and \( \mathcal{C} \)

In this section we classify irreducible integrable \( \hat{g} \)-modules in the categories \( \mathcal{E} \) and \( \mathcal{C} \) for a standard affine Lie algebra \( \hat{g} \) (with \( g \) a finite-dimensional simple Lie algebra and with \( \langle \cdot, \cdot \rangle \) the normalized Killing form). It has been proved in [DLM] that every irreducible integrable \( \hat{g} \)-module in the category \( \mathcal{R} \) is a highest weight module and every integrable \( \hat{g} \)-module in the category \( \mathcal{R} \) is completely reducible. We here show that every irreducible integrable \( \hat{g} \)-module in the category \( \mathcal{E} \) is isomorphic to a finite-dimensional evaluation module and that every irreducible integrable \( \hat{g} \)-module in the category \( \mathcal{C} \) is isomorphic to a tensor product of a highest weight integrable module with a finite-dimensional evaluation module, constructed by Chari and Pressley.

We start with some formal calculus. First we have ([Li1], [LL])

\[
(x_1 - x_2)^m \left( \frac{\partial}{\partial x_2} \right)^n x_2^{-1} \delta \left( \frac{x_1}{x_2} \right) = 0
\]

for \( m > n \geq 0 \), and we have

\[
(x_1 - x_2)^m \frac{1}{n!} \left( \frac{\partial}{\partial x_2} \right)^n x_2^{-1} \delta \left( \frac{x_1}{x_2} \right) = \frac{1}{(n-m)!} \left( \frac{\partial}{\partial x_2} \right)^{n-m} x_2^{-1} \delta \left( \frac{x_1}{x_2} \right)
\]

for \( 0 \leq m \leq n \).

**Definition 3.1.** Let \( W \) be any vector space. Following [LL] (cf. [Li1]) we set

\[
\mathcal{E}(W) = \text{Hom} \left( W, W((x)) \right).
\]

We define \( \hat{\mathcal{E}}(W) \) to be the subspace of \( (\text{End} W)[[x, x^{-1}]] \), consisting of formal series \( a(x) \) such that \( p(x)a(x) \in \text{Hom} (W, W((x))) \) for some nonzero polynomial \( p(x) \). Define \( \hat{\mathcal{E}}_0(W) \) to be the subspace of \( \hat{\mathcal{E}}(W) \) consisting of the formal series \( a(x) \) such that \( p(x)a(x) = 0 \) for some nonzero polynomial \( p(x) \).

**Remark 3.2.** If \( a(x) \in \hat{\mathcal{E}}(W) \) and if \( x^m f(x)a(x) \in \text{Hom} (W, W((x))) \) for some integer \( m \) and for some polynomial \( f(x) \), then \( f(x)a(x) \in \text{Hom} (W, W((x))) \). In view of this, if we need, we may assume that \( p(0) \neq 0 \) for our nonzero polynomial \( p(x) \).
Let \( \mathbb{C}(x) \) denote the algebra of rational functions of \( x \). We define \( \iota_{x:0} \) to be the linear map from \( \mathbb{C}(x) \) to \( \mathbb{C}((x)) \) such that for \( f(x) \in \mathbb{C}(x) \), \( \iota_{x:0}(f(x)) \) is the formal Laurent series of \( f(x) \) at 0. Notice that both \( \mathbb{C}(x) \) and \( \mathbb{C}((x)) \) are (commutative) fields. The linear map \( \iota_{x:0} \) is a field embedding. If \( p(x) \) is a polynomial with \( p(0) \neq 0 \), then \( \iota_{x:0}(p(x)) \in \mathbb{C}[[x]] \).

**Definition 3.3.** Let \( W \) be a vector space. Define a linear map

\[
\psi_R : \mathcal{E}(W) \to \mathcal{E}(W) (= \text{Hom}(W, W((x))))
\]

by

\[
\psi_R(a(x))w = \iota_{x:0}(f(x)^{-1})(f(x)a(x)w) \quad \text{for } a(x) \in \mathcal{E}(W), \ w \in W,
\]

where \( f(x) \) is any nonzero polynomial such that \( f(x)a(x) \in \text{Hom}(W, W((x))) \).

First of all, the map \( \psi_R \) is well defined; the expression on the right-hand side of (3.4) makes sense (which is clear) and does not depend on the choice of \( f(x) \). Indeed, let \( 0 \neq f, g \in \mathbb{C}[x] \) be such that \( f(x)a(x), g(x)a(x) \in \text{Hom}(W, W((x))) \).

Set \( h(x) = f(x)g(x) \). Then \( h(x)a(x) \in \text{Hom}(W, W((x))) \). For \( w \in W \), we have

\[
\iota_{x:0}(h(x)^{-1})(h(x)a(x)w) = \iota_{x:0}(h(x)^{-1})f(x)(g(x)a(x)w) = \iota_{x:0}(g(x)^{-1})(g(x)a(x)w).
\]

Similarly, we have

\[
\iota_{x:0}(h(x)^{-1})(h(x)a(x)w) = \iota_{x:0}(f(x)^{-1})(f(x)a(x)w).
\]

**Remark 3.4.** Note that the expression \( \iota_{x:0}(f(x)^{-1})a(x)w \) may not exist in \( W[[x, x^{-1}]] \). Thus, in (3.4), it is necessary to use all the parenthesis.

The following is an immediate consequence of (3.4) and the associativity law (recall Remark 2.1):

**Lemma 3.5.** For \( a(x) \in \mathcal{E}(W) \), we have

\[
f(x)\psi_R(a(x)) = f(x)a(x),
\]

where \( f(x) \) is any nonzero polynomial such that \( f(x)a(x) \in \text{Hom}(W, W((x))) \).

Furthermore we have the following result:
Proposition 3.6. Let $W$ be any vector space. We have

$$\mathcal{E}(W) = \mathcal{E}(W) \oplus \mathcal{E}_0(W).$$

(3.6)

Furthermore, the linear map $\psi_\mathcal{E}$ from $\mathcal{E}(W)$ to $\mathcal{E}(W)$, defined in Definition 3.3, is the projection map of $\mathcal{E}(W)$ onto $\mathcal{E}(W)$, i.e.,

$$\psi_\mathcal{E}|_{\mathcal{E}(W)} = 1 \quad \text{and} \quad \psi_\mathcal{E}|_{\mathcal{E}_0(W)} = 0.$$  

(3.7)

Proof. Let $a(x) \in \mathcal{E}(W) = \text{Hom}(W,W((x)))$. In Definition 3.3 we can take $f(x) = 1$, so that $\psi_\mathcal{E}(a(x))w = a(x)w$ for $w \in W$. Thus $\psi_\mathcal{E}(a(x)) = a(x)$. Now, let $a(x) \in \mathcal{E}_0(W)$. By definition there is a nonzero polynomial $p(x)$ such that $p(x)a(x) = 0$ on $W$, so that $p(x)a(x) \in \text{Hom}(W,W((x)))$. From definition we have

$$\psi_\mathcal{E}(a(x))w = t_xo(p(x)^{-1})(p(x)a(x)w) = 0 \quad \text{for} \quad w \in W.$$  

This proves the property (3.7) and it follows immediately that the sum $\mathcal{E}(W) + \mathcal{E}_0(W)$ is a direct sum.

Let $a(x) \in \mathcal{E}(W)$ and let $0 \neq f(x) \in \mathbb{C}[x]$ be such that $f(x)a(x) \in \text{Hom}(W,W((x)))$. In view of Lemma 3.3 we have $f(x)\psi_\mathcal{E}(a(x)) = f(x)a(x)$. Then $f(x)(a(x) - \psi_\mathcal{E}(a(x))) = 0$, which implies that $a(x) - \psi_\mathcal{E}(a(x)) \in \mathcal{E}_0(W)$. Thus, $a(x) \in \mathcal{E}(W) \oplus \mathcal{E}_0(W)$. This proves that $\mathcal{E}(W) \subset \mathcal{E}(W) \oplus \mathcal{E}_0(W)$, from which we have (3.6).

Definition 3.7. Let $W$ be a vector space. Denote by $\psi_\mathcal{E}$ the projection map of $\mathcal{E}(W)$ onto $\mathcal{E}_0(W)$ with respect to the decomposition (3.6). For $a(x) \in \mathcal{E}(W)$ we set

$$\tilde{a}(x) = \psi_\mathcal{E}(a(x)), \quad (3.8)$$

$$\tilde{a}(x) = \psi_\mathcal{E}(a(x)) = a(x) - \psi_\mathcal{E}(a(x)) = a(x) - \tilde{a}(x). \quad (3.9)$$

From Lemma 3.3 we have

$$f(x)\tilde{a}(x) = f(x)a(x), \quad (3.10)$$

$$f(x)\tilde{a}(x) = 0 \quad (3.11)$$

for any nonzero $f(x) \in \mathbb{C}[x]$ such that $f(x)a(x) \in \text{Hom}(W,W((x)))$.

The following result relates the actions of $\psi_\mathcal{E}(a(x))$ and $a(x)$ on $W$:

Lemma 3.8. For $a(x) \in \tilde{\mathcal{E}}(W)$, $n \in \mathbb{Z}$, $w \in W$, we have

$$\psi_\mathcal{E}(a(x))(n)w = \sum_{i=0}^r \beta_i a(n + i)w$$

(3.12)

for some $r \in \mathbb{N}$, $\beta_1, \ldots, \beta_r \in \mathbb{C}$, depending on $a(x), w$ and $n$. 

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Proof. Let $p(x)$ be a polynomial with $p(0) \neq 0$ such that $p(x)a(x) \in \text{Hom}(W,W((x)))$. Then $x^kp(x)a(x)w \in W[[x]]$ for some nonnegative integer $k$. Assume that

$$
\iota_{x,0}(1/p(x)) = \sum_{i \geq 0} \alpha_i x^i \in \mathbb{C}[x].
$$

(3.13)

Noticing that $\text{Res}_x x^{k+m}p(x)a(x)w = 0$ for $m \geq 0$, we have

$$
\psi_R(a(x))(n)w = \text{Res}_x x^n\psi_R(a(x))w = \text{Res}_x x^n t_{x,0}(1/p(x))(p(x)a(x)w)
$$

$$
= \text{Res}_x \sum_{0 \leq i \leq k-n-1} \alpha_i x^{n+i}(p(x)a(x)w)
$$

$$
= \text{Res}_x \left( \sum_{0 \leq i \leq k-n-1} \alpha_i x^{n+i}p(x)a(x) \right) w.
$$

(3.14)

Then it follows immediately. \hfill \Box

We also have:

Lemma 3.9. Let

$$a(x), b(x) \in \mathcal{E}(W), \ c_0(x), \ldots, c_r(x) \in (\text{End } W)[[x, x^{-1}]]$$

be such that on $W$,

$$[a(x_1), b(x_2)] = \sum_{i=0}^r \frac{1}{i!} c_i(x_2) \left( \frac{\partial}{\partial x_2} \right)^i x_1^{-1} \delta \left( \frac{x_2}{x_1} \right).$$

(3.15)

Then $c_0(x), \ldots, c_r(x) \in \mathcal{E}(W)$ and

$$[	ilde{a}(x_1), \tilde{b}(x_2)] = \sum_{i=0}^r \frac{1}{i!} \tilde{c}_i(x_2) \left( \frac{\partial}{\partial x_2} \right)^i x_1^{-1} \delta \left( \frac{x_2}{x_1} \right).$$

(3.16)

Proof. Using (3.11) and (3.12), and noticing that

$$\text{Res}_{x_1} c_j(x_2) \left( \frac{\partial}{\partial x_2} \right)^r x_1^{-1} \delta \left( \frac{x_2}{x_1} \right) = (-1)^r \text{Res}_{x_1} c_j(x_2) \left( \frac{\partial}{\partial x_1} \right)^r x_1^{-1} \delta \left( \frac{x_2}{x_1} \right) = 0$$

for $r \geq 1$, we get

$$c_i(x_2) = \text{Res}_{x_1} (x_1 - x_2)^i [a(x_1), b(x_2)].$$

(3.17)

Then it is clear that $c_i(x) \in \mathcal{E}(W)$ for $i = 0, \ldots, r$, since $b(x) \in \mathcal{E}(W)$.

Let $0 \neq f(x) \in \mathbb{C}[x]$ be such that

$$f(x)a(x) = f(x)\tilde{a}(x), \ f(x)b(x) = f(x)\tilde{b}(x), \ f(x)c_i(x) = f(x)\tilde{c}_i(x)$$
for \(i = 0, \ldots, r\). Then by multiplying both sides of (3.15) by \(f(x_1)f(x_2)\) we obtain
\[
f(x_1)f(x_2)[\tilde{a}(x_1), \tilde{b}(x_2)] = \sum_{i=0}^{r} \frac{1}{i!} f(x_1)f(x_2) \tilde{c}_i(x_2) \left( \frac{\partial}{\partial x_2} \right)^i x_1^{-1} \delta \left( \frac{x_2}{x_1} \right). \tag{3.18}
\]

Then we may multiply both sides by \(\iota_{x_1;0}(f(x_1)^{-1})\iota_{x_2;0}(g(x_2)^{-1})\) to get (3.16).

The following is the key factorization result:

**Theorem 3.10.** Let \(\pi\) be a representation of \(\hat{g}\) on module \(W\) in the category \(C\). Define linear maps \(\pi_R\) and \(\pi_E\) from \(\hat{g}\) to \(\text{End} W\) in terms of generating functions by
\[
\begin{align*}
\pi_R(a(x) + \alpha k) &= \psi_R(\pi(a(x))) + \alpha \pi(k), \tag{3.19} \\
\pi_E(a(x) + \beta k) &= \psi_E(\pi(a(x))) \tag{3.20}
\end{align*}
\]
for \(a \in g\), \(\alpha, \beta \in \mathbb{C}\), where we extend \(\pi\) to \(\hat{g}\) canonically. Then
\[
\pi = \pi_R + \pi_E \tag{3.21}
\]
and the linear map
\[
\hat{g} \oplus \hat{g} \rightarrow \text{End} W \\
(u, v) \mapsto \pi_R(u) + \pi_E(v) \tag{3.22}
\]
defines a representation of \(\hat{g} \oplus \hat{g}\) on \(W\). If \((W, \pi)\) is irreducible, \(W\) is an irreducible \(\hat{g} \oplus \hat{g}\)-module. Furthermore, \((W, \pi_R)\) is a restricted \(\hat{g}\)-module (in the category \(R\)) and \((W, \pi_E)\) is a \(\hat{g}\)-module in the category \(E\).

**Proof.** The relation (3.21) follows from Proposition 3.6. It follows immediately from the defining commutator relations (2.10) and Lemma 3.9 that \((W, \pi_R)\) is a \(\hat{g}\)-module and it is clear that it is restricted. (We view \(k\) as an element of \(\hat{E}(W)\).) Consequently, \((W, \pi_E)\) is a \(\hat{g}\)-module, since \(\pi_E = \pi - \pi_R\).

Let \(0 \neq p(x) \in \mathbb{C}[x]\) be such that \(p(x)\pi(a(x)) \in \text{Hom}(W, W((x)))\) for all \(a \in g\). Then
\[
p(x)\pi_R(a(x)) = p(x)\pi(a(x)), \quad p(x)\pi_E(a(x)) = 0,
\]
so that
\[
\begin{align*}
p(x)\pi_R(a(x)) &= p(x)\psi_R \pi(a(x)) = p(x)\pi(a(x)) \tag{3.23} \\
p(x)\pi_E(a(x)) &= 0 \tag{3.24}
\end{align*}
\]
for \(a \in g\). From this we have that \((W, \pi_E)\) belongs to the category \(E\).
For $a, b \in \mathfrak{g}$, using the commutator relations (2.10) and the basic delta-function property we have

\[
p(x_1)\pi_R(a(x_1)), \pi_\xi(b(x_2)) = p(x_1)\pi_R(a(x_1)), \pi_\xi(b(x_2)) - p(x_1)\pi_R(a(x_1)), \pi_R(b(x_2)) = p(x_1)\pi(a(x_1)), \pi(b(x_2)) - p(x_1)\pi_R(a(x_1)), \pi_\xi(b(x_2)) = p(x_1)\pi([a, b](x_2))x_1^{-1}\delta \left(\frac{x_2}{x_1}\right) + \langle a, b \rangle \pi(k)p(x_1)\frac{\partial}{\partial x_2}x_1^{-1}\delta \left(\frac{x_2}{x_1}\right) - p(x_1)\pi_R([a, b](x_2))x_1^{-1}\delta \left(\frac{x_2}{x_1}\right) - \langle a, b \rangle \pi_R(k)p(x_1)\frac{\partial}{\partial x_2}x_1^{-1}\delta \left(\frac{x_2}{x_1}\right) = 0.
\]

(3.25)

Since $\pi_R(a(x_1)) \in \text{Hom}(W, W((x_1)))$, we can multiply both sides by $\psi_{x_1, 0, 1}/p(x_1)$ and use associativity to get

\[
[\pi_R(a(x_1)), \pi_\xi(b(x_2))] = 0.
\]

(3.26)

It follows that $\langle u, v \rangle \mapsto \pi_R(u) + \pi_\xi(v)$ defines a representation of $\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$ on $W$. With $\pi = \pi_R + \pi_\xi$, it is clear that if $(W, \pi)$ is irreducible, $W$ is an irreducible $\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$-module. □

Furthermore, we have:

**Proposition 3.11.** Let $(W_1, \pi_1)$ and $(W_2, \pi_2)$ be $\hat{\mathfrak{g}}$-modules in the category $\mathcal{C}$ and let $f$ be a $\hat{\mathfrak{g}}$-module homomorphism (isomorphism) from $(W_1, \pi_1)$ to $(W_2, \pi_2)$. Then $f$ is a $\hat{\mathfrak{g}}$-module homomorphism (isomorphism) from $(W_1, (\pi_1)_\mathcal{R})$ to $(W_2, (\pi_2)_\mathcal{R})$ and a $\hat{\mathfrak{g}}$-module homomorphism (isomorphism) from $(W_1, (\pi_1)_\xi)$ to $(W_2, (\pi_2)_\xi)$.

**Proof.** Let $p(x)$ be a nonzero polynomial such that for every $a \in \mathfrak{g}$,

\[
p(x)\pi_1(a(x)) \in \text{Hom}(W_1, W_1((x))) , \quad p(x)\pi_2(a(x)) \in \text{Hom}(W_2, W_2((x))).
\]

Then we have

\[
p(x)\psi_\mathcal{R}(\pi_1(a(x))) = p(x)\pi_1(a(x)), \quad p(x)\psi_\mathcal{R}(\pi_2(a(x))) = p(x)\pi_2(a(x)),
\]

so that

\[
p(x)(\pi_1)_\mathcal{R}(a(x)) = p(x)\psi_\mathcal{R}(\pi_1(a(x))) = p(x)\pi_1(a(x)) \quad \text{and} \quad p(x)(\pi_2)_\mathcal{R}(a(x)) = p(x)\psi_\mathcal{R}(\pi_2(a(x))) = p(x)\pi_2(a(x)).
\]

(3.27)

(3.28)
For $a \in \mathfrak{g}$, $w_1 \in W_1$, we have
\[
p(x)f((\pi_1)_R(a(x))w_1) = p(x)f(\pi_1(a(x))w_1) = p(x)f(\pi_2(a(x))f(w_1)) = p(x)(\pi_2)_R(a(x))f(w_1).
\] (3.29)
Since $f((\pi_1)_R(a(x))w_1), (\pi_2)_R(a(x))f(w_1) \in W_2((x))$, in view of Remark 2.1 we have
\[
f((\pi_1)_R(a(x))w_1) = (\pi_2)_R(a(x))f(w_1) \quad \text{for } a \in \mathfrak{g}.
\] (3.30)
This proves that $f$ is a $\hat{\mathfrak{g}}$-module homomorphism from $(W_1, (\pi_1)_R)$ to $(W_2, (\pi_2)_R)$. (Notice that $a \otimes t^n$ for $a \in \mathfrak{g}$, $n \in \mathbb{Z}$ generates $\hat{\mathfrak{g}}$.) Because $(\pi_i)_\mathcal{E} = \pi_i - (\pi_i)_R$ for $i = 1, 2$, it follows that $f$ is also a $\hat{\mathfrak{g}}$-module homomorphism from $(W_1, (\pi_1)_\mathcal{E})$ to $(W_2, (\pi_2)_\mathcal{E})$. \qed

In view of Theorem 3.10 we classify irreducible representations of $\hat{\mathfrak{g}}$ in the category $\mathcal{C}$ we need to classify irreducible representations of $\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$ which are composed of a representation of $\hat{\mathfrak{g}}$ in the category $\mathcal{R}$ and a representation of $\hat{\mathfrak{g}}$ in the category $\mathcal{E}$. Motivated by this, we next present some elementary results (or facts) about modules for a tensor product associative algebra $A_1 \otimes A_2$.

Remark 3.12. We here collect some facts for general (maybe infinite-dimensional) associative algebras, which follow from the proofs for the finite-dimensional case. The first fact is that if $A$ is an associative algebra (with identity), $U$ a finitely generated $A$-module and $W = \bigsqcup_{i \in I} W_i$ a direct sum of $A$-modules, then $\text{Hom}_A(U, W) \cong \bigsqcup_{i \in I} \text{Hom}_A(U, W_i)$. With this fact, using the usual proof one can prove the second fact: Let $A_1$ and $A_2$ be associative algebras (with identity), let $W$ be an $A_1 \otimes A_2$-module such that $W$ viewed as an $A_1$-module is completely reducible and let $\{U_i \mid i \in I\}$ be a complete set of representatives of equivalence classes of irreducible $A_1$-submodules of $W$. Assume that $\text{End}_{A_i} U_i = \mathbb{C}$ for $i \in I$. Then $W \cong \bigsqcup_{i \in I} U_i \otimes \text{Hom}_{A_i}(U_i, W)$, as an $A_1 \otimes A_2$-module. A version of Schur lemma (cf. [Di]) is that if $A$ is an associative algebra (with identity) of countable dimension, then $\text{End}_{A} U = \mathbb{C}$ for any irreducible $A$-module $U$. In view of this, for the second fact, the condition that $\text{End}_{A_i} U_i = \mathbb{C}$ can be replaced by that condition that $A_i$ is of countable dimension.

The following two lemmas are very useful in the proof of our main theorems later:

Lemma 3.13. Let $A_1$ and $A_2$ be associative algebras (with identity) and let $U$ be an irreducible $A_1 \otimes A_2$-module. Suppose that $A_1$ is of countable dimension and that $U$ as an $A_1$-module has an irreducible submodule. Then $U$ is isomorphic to an $A_1 \otimes A_2$-module of the form $U_1 \otimes U_2$ as in Lemma 2.14.

Proof. Let $U_1$ be an irreducible $A_1$-submodule of $U$. Since $U$ is an irreducible $A_1 \otimes A_2$-module, we have $U = (A_1 \otimes A_2)U_1 = A_2U_1$. For any $a \in A_2$, $u \mapsto au$ is an $A_1$-homomorphism from $U_1$ to $U$. Consequently, for $a \in A_2$, either $aU_1 = 0$ or $aU_1$ is an
irreducible $A_1$-submodule isomorphic to $U_1$. It follows that $U$ as an $A_1$-module is a direct sum of irreducible submodules isomorphic to $U_1$. Furthermore, since $A_1$ is of countable dimension, from Remark 3.12 we have $W \cong U_1 \otimes \text{Hom}_{A_1}(U_1, U)$, where $\text{Hom}_{A_1}(U_1, U)$ is a natural $A_2$-module which is necessarily irreducible.

Lemma 3.14. Let $A_1$ and $A_2$ be associative algebras (with identity) and let $W$ be an $A_1 \otimes A_2$-module. Assume that $A_1$ is of countable dimension and assume that $W$ is a completely reducible $A_1$-module and a completely reducible $A_2$-module. Then $W$ is isomorphic to a direct sum of irreducible $A_1 \otimes A_2$-modules of the form $U \otimes V$ with $U$ an irreducible $A_1$-module and $V$ an irreducible $A_2$-module.

Proof. Let $\{U_1^{(i)} \mid i \in I\}$ be a complete set of representatives of equivalence classes of irreducible $A_1$-submodules of $W$. With $A_1$-being countable dimensional, from Remark 3.12 we have

$$W \cong \bigoplus_{i \in I} U_1^{(i)} \otimes \text{Hom}_{A_1}(U_1^{(i)}, W)$$

as an $A_1 \otimes A_2$-module. Since $W$ is a completely reducible $A_2$-module, $\text{Hom}_{A_1}(U_1^{(i)}, W)$ is a completely reducible $A_2$-module. Now it follows from Lemma 2.10 that $W$ is a completely reducible $A_1 \otimes A_2$-module. □

We now classify finite-dimensional irreducible $\mathfrak{g}$-modules in category $\mathcal{E}$. For $a \in \mathfrak{g}$, we have (cf. [HL])

$$a(x) = \sum_{n \in \mathbb{Z}} (a \otimes t^n)x^{-n-1} = a \otimes x^{-1}\delta\left(\frac{t}{x}\right).$$

(3.31)

For $f(x) \in \mathbb{C}[x]$, $m \in \mathbb{Z}$, $a \in \mathfrak{g}$, we have

$$x^m f(x)a(x) = a \otimes x^m f(x)x^{-1}\delta\left(\frac{t}{x}\right) = a \otimes t^m f(t)x^{-1}\delta\left(\frac{t}{x}\right),$$

(3.32)

so that

$$\text{Res}_x x^m f(x)a(x) = a \otimes t^m f(t).$$

(3.33)

It follows immediately that for any $\mathfrak{g}$-module $W$, $f(x)a(x)W = 0$ if and only if $(a \otimes f(t)\mathbb{C}[t,t^{-1}])W = 0$. For a nonzero polynomial $p(x)$, we define a subcategory $\mathcal{E}_p$ of $\mathcal{E}$, consisting of $\mathfrak{g}$-modules $W$ such that

$$p(x)a(x)w = 0 \quad \text{for } a \in \mathfrak{g}, \ w \in W.$$

(3.34)

Then a $\mathfrak{g}$-module in the category $\mathcal{E}_{p(x)}$ exactly amounts to a module for the Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]/p(t)\mathbb{C}[t,t^{-1}]$ (recall Lemma 2.9).
Lemma 3.15. Let \( p(x) = x^k(x - z_1) \cdots (x - z_r) \) with \( z_1, \ldots, z_r \) distinct nonzero complex numbers and with \( k \in \mathbb{N} \). Then any finite-dimensional irreducible \( \hat{g} \)-module \( W \) in the category \( \mathcal{E}_{p(x)} \) is isomorphic to a \( \hat{g} \)-module \( U_1(z_1) \otimes \cdots \otimes U_r(z_r) \) for some finite-dimensional irreducible \( g \)-modules \( U_1, \ldots, U_r \).

Proof. Noticing that \( \mathbb{C}[t, t^{-1}]/p(t)\mathbb{C}[t, t^{-1}] = \prod_{i=1}^r \mathbb{C}[t, t^{-1}]/(t - z_i)\mathbb{C}[t, t^{-1}] \), we have

\[
\hat{g} \otimes (\mathbb{C}[t, t^{-1}]/p(t)\mathbb{C}[t, t^{-1}]) = \prod_{i=1}^r \hat{g} \otimes \mathbb{C}[t, t^{-1}]/(t - z_i)\mathbb{C}[t, t^{-1}].
\]  

(3.35)

Notice that for any nonzero complex number \( z \), a \( \hat{g} \otimes \mathbb{C}[t, t^{-1}]/(t - z)\mathbb{C}[t, t^{-1}] \)-module exactly amounts to an evaluation \( \hat{g} \)-module \( U(z) \). Set

\[
A_i = U \left( \hat{g} \otimes \mathbb{C}[t, t^{-1}]/(t - z_i)\mathbb{C}[t, t^{-1}] \right)
\]

for \( i = 1, \ldots, r \). Since \( W \) is finite-dimensional, \( W \) viewed as an \( A_i \)-module contains an irreducible submodule. It now follows from Lemma 3.13 (and induction). \( \square \)

We also have the following result:

Proposition 3.16. Assume that \([\hat{g}, \hat{g}] = g\). Then any finite-dimensional irreducible \( \hat{g} \)-module \( W \) in the category \( \mathcal{E} \) is isomorphic to a \( \hat{g} \)-module \( U_1(z_1) \otimes \cdots \otimes U_r(z_r) \) for some finite-dimensional \( g \)-modules \( U_1, \ldots, U_r \) and for some distinct nonzero complex numbers \( z_1, \ldots, z_r \).

Proof. In view of Lemma 3.15 it suffices to prove that \( W \) is in the category \( \mathcal{E}_{p(x)} \) with \( p(x) \) a nonzero polynomial whose any nonzero root is multiplicity-free. In view of Remark 3.2 there exists a polynomial \( p(x) \) with \( p(0) \neq 0 \) such that \( p(x)a(x)W = 0 \) for \( a \in \hat{g} \). Let \( p(x) \) be such a monic polynomial with the least degree. Thus

\[
p(x) = (x - z_1)^{k_1} \cdots (x - z_r)^{k_r},
\]  

(3.36)

where \( z_1, \ldots, z_r \) are distinct nonzero complex numbers and \( k_1, \ldots, k_r \) are positive integers.

Let \( I \) be the annihilating ideal of \( W \) in \( \hat{g} \). Then \( (\hat{g} \otimes p(t)\mathbb{C}[t, t^{-1}]) \subset I \) and \( W \) is an irreducible faithful \( \hat{g}/I \)-module. Therefore (cf. [H]) \( \hat{g}/I \) is reductive (where we using the fact that \( W \) is finite-dimensional). Set \( f(x) = (x - z_1) \cdots (x - z_r) \) and let \( k \) be the largest one among \( k_1, \ldots, k_r \). We see that \( p(x) \) is a factor of \( f(x)^k \). It follows that the quotient space \( (\hat{g} \otimes f(t)\mathbb{C}[t, t^{-1}])/I \) is a solvable ideal of \( \hat{g}/I \). With \( \hat{g}/I \) being reductive, \( (\hat{g} \otimes f(t)\mathbb{C}[t, t^{-1}])/I \) must be in the center of \( \hat{g}/I \). From this we have that \([\hat{g}, \hat{g}] \otimes f(t)\mathbb{C}[t, t^{-1}] \subset I \), which implies that \( \hat{g} \otimes f(t)\mathbb{C}[t, t^{-1}] \subset I \), since \( \hat{g} = [\hat{g}, \hat{g}] \) by assumption. This proves that \( f(x)a(x)W = 0 \) for \( a \in \hat{g} \). Consequently, \( f(x) = p(x) \), that is, \( k_1 = \cdots = k_r = 1 \). \( \square \)
Remark 3.17. In Proposition 3.16, the condition $g = [g, g]$ is necessary. For example, let $g$ be an abelian Lie algebra. For any nonzero linear functional $\chi$ on $\hat{g}$ with $\psi(k) = 0$, we have a one-dimensional irreducible $\hat{g}$-module $C$ with $\hat{g}$ acting according to $\chi$. In general, such a module may not be in category $E$.

For the rest of this section we assume that $\hat{g}$ is a standard affine Lie algebra with $g$ a finite-dimensional simple Lie algebra and with $\langle \cdot, \cdot \rangle$ the normalized Killing form. We retain all the notations and definitions in Remark 2.3.

The following result is a refinement of Theorem 3.10:

Proposition 3.18. Let $\pi$ be a representation of $\hat{g}$ on integrable $\hat{g}$-module $W$ in the category of $C$. Then $(W, \pi_\mathcal{R})$ is a restricted integrable $\hat{g}$-module (in the category $\mathcal{R}$) and $(W, \pi_\mathcal{E})$ is an integrable $\hat{g}$-module in the category $\mathcal{E}$.

Proof. In view of Theorem 3.10, we only need to show that $(W, \pi_\mathcal{R})$ and $(W, \pi_\mathcal{E})$ are integrable $\hat{g}$-modules. We must prove that for $a \in g$, with $\alpha \in \Delta$ and for $n \in \mathbb{Z}$, $\tilde{a}(n)$ and $\tilde{a}(n)$ act locally nilpotently on $W$.

Let $a \in g$, with $\Delta$ and $n \in \mathbb{Z}$. Notice that $[a(r), a(s)] = 0$ for $r, s \in \mathbb{Z}$, since $[a, a] = 0$ and $\langle a, a \rangle = 0$. For $w \in W$, we have

$$a(r)\tilde{a}(x)w = a(r)\iota_{x:0}(1/p(x))(\iota_{x:0}(p)(a(x))a(r)w) = \iota_{x:0}(1/p(x))(p)(a(x))a(r)w = \tilde{a}(x)a(r)w.$$ 

Thus

$$a(r)\tilde{a}(s) = \tilde{a}(s)a(r) \quad \text{for } r, s \in \mathbb{Z}. \quad (3.37)$$

Let $w \in W$ be an arbitrarily fixed vector. By Lemma 3.8

$$\tilde{a}(n)w = \sum_{i=0}^{r} \beta_i a(n+i)w$$

for some positive integer $r$ and for some complex numbers $\beta_1, \ldots, \beta_r$. Using (3.37) we get

$$\tilde{a}(n)^p w = (\beta_0 a(n) + \cdots + \beta_r a(n+r))^p w \quad \text{for any } p \geq 0. \quad (3.38)$$

Since $(W, \pi)$ is an integrable $\hat{g}$-module, there is a positive integer $k$ such that

$$a(m)^k w = 0 \quad \text{for } m = n, n+1, \ldots, n+r.$$ 

Combining this with (3.38) we obtain $\tilde{a}(n)^{k(r+1)}w = 0$.

Since $\tilde{a}(n) = a(n) - \tilde{a}(n)$ and $[a(n), \tilde{a}(n)] = 0$, we get

$$\tilde{a}(n)^{k(r+2)}w = (a(n) - \tilde{a}(n))^{k(r+2)}w = \sum_{i \geq 0} \binom{k(r+2)}{i} (-1)^ia(n)^{k(r+2)-i}\tilde{a}(n)^iw = 0. \quad (3.39)$$

This proves that $\tilde{a}(n)$ and $\tilde{a}(n)$ act locally nilpotently on $W$, completing the proof. \hfill \Box
We shall need the following fact which is a reformulation of Lemma 3.6 of [DLM]:

**Lemma 3.19.** There is a basis \( \{a_1, \ldots, a_r\} \) of \( \mathfrak{g} \) such that

\[
[a_i(m), a_i(n)] = 0 \quad \text{for } 1 \leq i \leq r, \ m, n \in \mathbb{Z}
\] (3.40)

and such that for \( 1 \leq i \leq r \) and for any \( n \in \mathbb{Z} \), \( a_i(n) \) acts locally nilpotently on all integrable \( \hat{\mathfrak{g}} \)-modules.

**Proof.** For \( \alpha \in \Delta_+ \), choose nonzero vectors \( e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}, h_\alpha \in \mathfrak{h} \) such that \([h_\alpha, e_\alpha] = 2e_\alpha, [h_\alpha, f_\alpha] = -2f_\alpha \) and \([e_\alpha, f_\alpha] = h_\alpha \). Set \( \sigma_\alpha = e^{ad_{e_\alpha}} \), an inner automorphism of Lie algebra \( \mathfrak{g} \). Then \( \sigma_\alpha(f_\alpha) = f_\alpha + h_\alpha - e_\alpha \). Since \( \{e_\alpha, f_\alpha, h_\alpha \mid \alpha \in \Delta_+\} \) is a basis of \( \mathfrak{g} \), \( \{e_\alpha, f_\alpha, \sigma_\alpha(f_\alpha) \mid \alpha \in \Delta_+\} \) is also a basis of \( \mathfrak{g} \). On any integrable \( \hat{\mathfrak{g}} \)-module we have (cf. [H], [K1])

\[
\exp(e_\alpha(0))f_\alpha(n)\exp(-e_\alpha(0)) = \sigma_\alpha(f_\alpha)(n) \quad \text{for } n \in \mathbb{Z}.
\] (3.41)

Since for \( n \in \mathbb{Z} \), \( f_\alpha(n) \) acts locally nilpotently on any integrable \( \hat{\mathfrak{g}} \)-module, \( \sigma_\alpha(f_\alpha)(n) \) also acts locally nilpotently on any integrable \( \hat{\mathfrak{g}} \)-module. Then \( \{e_\alpha, f_\alpha, \sigma_\alpha(e_\alpha) \mid \alpha \in \Delta_+\} \), is a basis of \( \mathfrak{g} \), satisfying the desired property. \( \square \)

The following result is a reformulation of Theorem 3.7 (cf. Remark 3.9) of [DLM]:

**Theorem 3.20.** Every nonzero restricted integrable \( \hat{\mathfrak{g}} \)-module is a direct sum of (irreducible) highest weight integrable modules. In particular, every irreducible integrable \( \hat{\mathfrak{g}} \)-module \( W \) is a highest weight integrable module.

**Proof.** As in [DLM], in view of the complete reducibility theorem in [K1] we only need to show that every nonzero restricted integrable \( \mathfrak{g} \)-module \( W \) contains a highest weight integrable (irreducible) submodule. We now reformulate the proof of [DLM, Theorem 3.7] as follows:

**Claim 1:** There exists a nonzero \( u \in W \) such that \( (\mathfrak{g} \otimes t\mathbb{C}[t])u = 0 \). For \( n \in \mathbb{Z} \), set \( \mathfrak{g}(n) = \{a(n) \mid a \in \mathfrak{g}\} \). For any nonzero \( u \in W \), since \( W \) is restricted, \( \mathfrak{g}(n)u = 0 \) for \( n \) sufficiently large, so that \( \sum_{n \geq 1} \mathfrak{g}(n)u \) is finite-dimensional. For any \( u \in W \), we define \( d(u) = \dim \sum_{n \geq 1} \mathfrak{g}(n)u \). If there is a \( 0 \neq u \in W \) such that \( d(u) = 0 \), then \( (\mathfrak{g} \otimes t\mathbb{C}[t])u = 0 \).

Suppose that \( d(u) > 0 \) for any \( 0 \neq u \in W \). Take \( 0 \neq u \in W \) such that \( d(u) \) is minimal. By Lemma 3.19 there exists a basis \( \{a_1, \ldots, a_r\} \) of \( \mathfrak{g} \) such that \( a_i(n) \) locally nilpotently act on \( W \) for \( i = 1, \ldots, r \), \( n \in \mathbb{Z} \). Let \( k \) be the positive integer such that \( \mathfrak{g}(k)u \neq 0 \) and \( \mathfrak{g}(n)u = 0 \) whenever \( n > k \). By the definition of \( k \), \( a_i(k)u \neq 0 \) for some \( 1 \leq i \leq r \).

Notice that \( a_i(k)^su = 0 \) for some nonnegative integer \( s \). Let \( m \) be the nonnegative integer such that \( a_i(k)^mu \neq 0 \) and \( a_i(k)^{m+1}u = 0 \). Set \( v = a_i(k)^mu \). We will obtain a contradiction by showing that \( d(v) < d(u) \). First we prove that if \( a(n)u = 0 \) for some
$a \in \mathfrak{g}$, $n \geq 1$, then $a(n)v = 0$. In the following we will show by induction on $m$ that $a(n)a_i(k)^m u = 0$ for any $a \in \mathfrak{g}$ and $m \geq 0$. If $m = 0$ this is immediate by the choice of $u$. Now assume that the result holds for $m$. Since $[a, a_i](k + n)u = 0$ (from the definition of $k$) and $a(n)u = 0$, by the induction assumption that $a(n)a_i(k)^m u = 0$ we have

$$[a, a_i](k + n)a_i(k)^m u = 0, \ a(n)a_i(k)^m u = 0. \tag{3.42}$$

Thus

$$a(n)a_i(k)^{m+1} u = [a(n), a_i(k)]a_i(k)^m u + a_i(k)a(n)a_i(k)^m u$$

$$= [a, a_i](k + n)a_i(k)^m u + a_i(k)a(n)a_i(k)^m u$$

$$= 0,$$  \hspace{1cm} \tag{3.43}

as required. In particular, we see that $a(n)v = a(n)a_i(k)^r u = 0$. Therefore, $d(v) \leq d(u)$. Since $a_i(k)v = 0$ and $a_i(k)u \neq 0$, we have $d(v) < d(u)$, a contradiction.

Claim 2: $W$ contains an irreducible highest weight integrable submodule. Set

$$\Omega(W) = \{u \in W \mid (\mathfrak{g} \otimes t\mathbb{C}[t])u = 0\}. \tag{3.44}$$

Then $\Omega(W)$ is a $\mathfrak{g}$-submodule of $W$ and it is nonzero by Claim 1. Since $a_i(0)$ for $i = 1, \ldots, r$ act locally nilpotently on $\Omega(W)$, it follows from the PBW theorem that for any $u \in \Omega(W)$, $U(\mathfrak{g})u$ is finite-dimensional, so that $U(\mathfrak{g})u$ is a direct sum of finite-dimensional irreducible $\mathfrak{g}$-modules. Let $u \in \Omega(W)$ be a highest weight vector for $\mathfrak{g}$. It is clear that $u$ is a singular vector for $\hat{\mathfrak{g}}$. It follows from [K1] that $u$ generates an irreducible $\hat{\mathfrak{g}}$-module. \hfill \square

We also have the following result (cf. Theorem 2.2):

**Proposition 3.21.** The irreducible integrable $\hat{\mathfrak{g}}$-modules in the category $\mathcal{E}$ up to isomorphism are exactly those evaluation modules $U_1(z_1) \otimes \cdots \otimes U_r(z_r)$ where $U_i$ are finite-dimensional irreducible $\mathfrak{g}$-modules and $z_i$ are distinct nonzero complex numbers.

**Proof.** In view of Proposition 3.16 it suffices to prove that every irreducible integrable $\hat{\mathfrak{g}}$-module $W$ in the category $\mathcal{E}$ is finite-dimensional. Since $W$ is in the category $\mathcal{E}$, there is a nonzero polynomial $p(x)$ such that $(a \otimes p(t)\mathbb{C}[t, t^{-1}])W = 0$ for $a \in \mathfrak{g}$. Let $I$ be the annihilating ideal of $W$ in $\hat{\mathfrak{g}}$. Then $\hat{\mathfrak{g}}/I$ is finite-dimensional. Recall from Lemma 3.19 that there is a basis $\{a_1, \ldots, a_r\}$ of $\mathfrak{g}$ such that for any $1 \leq i \leq r$, $n \in \mathbb{Z}$, $a_i(n)$ acts locally nilpotently on $W$. Let $0 \neq w \in W$. Since $W$ is irreducible, we have $W = U(\hat{\mathfrak{g}})w = U(\hat{\mathfrak{g}}/I)w$. In view of the PBW theorem (for $\hat{\mathfrak{g}}/I$ using a basis consisting of the cosets of finitely many $a_i(n)$’s) we have that $W$ is finite-dimensional, completing the proof. \hfill \square

Now, we are in a position to prove our main result:
Theorem 3.22. Every irreducible integrable \( \hat{\mathfrak{g}} \)-module in the category \( C \) is isomorphic to a module of the form \( W \otimes U_1(z_1) \otimes \cdots \otimes U_r(z_r) \), where \( W \) is an irreducible integrable highest weight \( \hat{\mathfrak{g}} \)-module and \( U_1, \ldots, U_r \) are finite-dimensional irreducible \( \mathfrak{g} \)-modules with \( z_1, \ldots, z_r \) distinct nonzero complex numbers.

Proof. Let \( \pi \) be an irreducible integrable representation of \( \hat{\mathfrak{g}} \) on module \( W \) in the category \( C \). By Theorem 3.10, \( W \) is an irreducible \( \hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}} \)-module with \( (u, v) \) acting as \( \pi_R(u) + \pi_E(v) \) for \( u, v \in \hat{\mathfrak{g}} \) and we have \( \pi = \pi_R + \pi_E \). Furthermore, by Proposition 3.18, \( (W, \pi_R) \) is an integrable restricted \( \hat{\mathfrak{g}} \)-module and \( (W, \pi_E) \) is an integrable \( \hat{\mathfrak{g}} \)-module in the category \( E \).

In view of Theorem 3.20, \( (W, \pi_R) \) is a direct sum of integrable highest weight (irreducible) \( \hat{\mathfrak{g}} \)-modules. Now it follows immediately from Lemma 3.13 with \( A_1 = A_2 = U(\hat{\mathfrak{g}}) \) (which is of countable dimension) and Proposition 3.21.

Recall (Theorem 3.20) that every integrable \( \hat{\mathfrak{g}} \)-module in the category \( \mathcal{R} \) is completely reducible. But, an integrable \( \hat{\mathfrak{g}} \)-module in the category \( \mathcal{E} \) is not necessarily completely reducible. (Notice that any finite-dimensional \( \hat{\mathfrak{g}} \)-module in the category \( \mathcal{E} \) is integrable, but it is not necessarily completely reducible.) Nevertheless we have:

Proposition 3.23. Let \( p(x) \) be a nonzero polynomial such that all the nonzero roots are multiplicity-free. Then every integrable \( \hat{\mathfrak{g}} \)-module in the category \( \mathcal{E}_{p(x)} \) is semisimple and every integrable \( \hat{\mathfrak{g}} \)-module in the category \( \mathcal{C}_{p(x)} \) is semisimple.

Proof. Set \( p(x) = x^k(x - z_1) \cdots (x - z_r) \), where \( k \in \mathbb{N} \) and \( z_1, \ldots, z_r \) are distinct nonzero complex numbers. From the proof of Lemma 3.15 a \( \hat{\mathfrak{g}} \)-module in the category \( \mathcal{E}_{p(x)} \) amounts to a module for the product Lie algebra

\[
\prod_{i=1}^r (\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]/(t - z_i)\mathbb{C}[t, t^{-1}]).
\]

Let \( W \) be an integrable \( \hat{\mathfrak{g}} \)-module in the category \( \mathcal{E}_{p(x)} \). Using the basis of \( \mathfrak{g} \) as in the proof of Proposition 3.21 it follows from the PBW theorem that any vector in \( W \) generates a finite-dimensional \( \hat{\mathfrak{g}} \)-submodule. With \( \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]/(t - z_i)\mathbb{C}[t, t^{-1}] = \mathfrak{g} \), it follows that \( W \) as a module for each of the Lie algebras \( \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]/(t - z_i)\mathbb{C}[t, t^{-1}] \) is completely reducible. Now it follows from Lemma 3.14 that \( W \) is completely reducible.

Finally, with the first assertion and Theorem 3.20 it follows from Lemma 3.14 that every integrable \( \hat{\mathfrak{g}} \)-module in the category \( C \) is completely reducible.

4 A relation between tensor product module \( W \otimes U(z) \) and fusion rules

In this section we relate the tensor product module \( W \otimes U(z) \) in the category \( C \) with the fusion rules of certain type for the vertex operator algebra associated with the affine Lie algebra \( \hat{\mathfrak{g}} \) of level \( \ell \).
As in Section 2, let \( g \) be a (not necessarily finite-dimensional) Lie algebra equipped with a nondegenerate symmetric invariant bilinear form \( \langle \cdot, \cdot \rangle \) and let \( \hat{g} \) be the associated affine Lie algebra. Recall the extended affine Lie algebra (cf. \([K1]\))

\[
\tilde{g} = \hat{g} \oplus \mathbb{C} d = g \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} k \oplus \mathbb{C} d,
\]

where \([d, k] = 0\) and

\[
[d, a \otimes t^n] = n(a \otimes t^n) \quad \text{for } a \in g, \ n \in \mathbb{Z}.
\]

A \( \tilde{g} \)-module \( W \) is said to be upper truncated if \( W = \coprod_{\lambda \in \mathbb{C}} W(\lambda) \), where for \( \lambda \in \mathbb{C} \),

\[
W(\lambda) = \{ w \in W \mid dw = \lambda w \},
\]
such that for any \( \lambda \in \mathbb{C} \), \( W(\lambda + n) = 0 \) for \( n \in \mathbb{Z} \) sufficiently large. Clearly, we have

\[
a(n)W(\lambda) \subset W(\lambda + n) \quad \text{for } a \in g, \ n \in \mathbb{Z}, \ \lambda \in \mathbb{C}.
\]

Then every upper truncated \( \tilde{g} \)-module is a restricted \( \hat{g} \)-module. For an upper truncated \( \tilde{g} \)-module \( W = \coprod_{\lambda \in \mathbb{C}} W(\lambda) \), we set (cf. \([HL]\))

\[
\overline{W} = \prod_{\lambda \in \mathbb{C}} W(\lambda),
\]

the formal completion of \( W \). Then \( \overline{W} \) is again a \( \tilde{g} \)-module (but not a restricted module).

For any \( g \)-module \( U \), \( L(U) = U \otimes \mathbb{C}[t, t^{-1}] \) is naturally a \( \tilde{g} \)-module where

\[
a(m)(u \otimes t^n) = au \otimes t^{m+n},
\]

\[
d(u \otimes t^n) = (n + 1)(u \otimes t^n) \quad \text{for } a \in g, \ u \in U, \ m, n \in \mathbb{Z}
\]

and \( k \) acts as zero (cf. \([CP2], [K1]\)). Such a \( \tilde{g} \)-module is often called a loop module. We have \( L(U) = \coprod_{n \in \mathbb{Z}} L(U)(n) \), where \( L(U)(n) = (U \otimes \mathbb{C} t^{n-1}) \) for \( n \in \mathbb{Z} \).

**Remark 4.1.** Notice that if \( U \) is not a trivial \( g \)-module, i.e., \( gU = 0 \), then the action of \( \hat{g} \) on the evaluation \( \hat{g} \)-module \( U(z) \) cannot be extended to a module action for the extended affine Lie algebra for \( a \in g, \ m \in \mathbb{Z}, \ u \in U, \ \hat{g} \). Otherwise, we have

\[
0 = (da(m) - a(m)d - ma(m))u = z^m(da(u) - adu - ma)
\]

for \( a \in g, \ m \in \mathbb{Z}, \ u \in U \), which implies that \( au = 0 \) for \( a \in g, \ u \in U \), a contradiction.

Let \( W_1 \) and \( W \) be upper truncated \( \tilde{g} \)-modules of level \( \ell \) and \( U(z) \) be an evaluation \( \hat{g} \)-module (of level zero), where \( z \) is a fixed nonzero complex number. We have a (tensor product) \( \tilde{g} \)-module \( W_1 \otimes U(z) \) and a (tensor product) \( \tilde{g} \)-module \( W_1 \otimes L(U) \). For homogeneous vector \( w_1 \in W_1 \) and for \( u \in U, \ n \in \mathbb{Z} \), we have

\[
\deg(w_1 \otimes u \otimes t^n) = \deg w_1 + n + 1.
\]
We next show that there is a canonical linear isomorphism from $\text{Hom}_{\tilde{g}}(W_1 \otimes U \otimes \mathbb{C}[t, t^{-1}], W)$ to $\text{Hom}_{\hat{g}}(W_1 \otimes U(z), \overline{W})$.

Let $\psi$ be a $\tilde{g}$-module homomorphism from $W_1 \otimes U \otimes \mathbb{C}[t, t^{-1}]$ to $W$. We define a linear map

$$\hat{\psi} : W_1 \otimes U \rightarrow \overline{W} \quad w_1 \otimes u \mapsto \sum_{n \in \mathbb{Z}} z^{-n-1} \psi(w_1 \otimes u \otimes t^n). \quad (4.9)$$

We are going to show that $\hat{\psi}$ is in fact a $\hat{g}$-module homomorphism from the tensor product module $W_1 \otimes U(z)$ to $\overline{W}$.

Let $a \in g$, $m \in \mathbb{Z}$, $w_1 \in W_1$, $u \in U$. We have

$$\hat{\psi}(a(m))(w_1 \otimes u)) = \hat{\psi}(a(m)w_1 \otimes u + w_1 \otimes z^m au) = \sum_{n \in \mathbb{Z}} z^{-n-1} \psi(a(m)w_1 \otimes u \otimes t^n) + z^{m-n-1} \psi(w_1 \otimes au \otimes t^n) = \sum_{n \in \mathbb{Z}} z^{-n-1} (\psi(a(m)w_1 \otimes u \otimes t^n) + \psi(w_1 \otimes au \otimes t^{m+n})) = \sum_{n \in \mathbb{Z}} z^{-n-1} a(m) \psi(w_1 \otimes u \otimes t^n) = a(m) \hat{\psi}(w_1 \otimes u). \quad (4.10)$$

Since $a \otimes t^m$ for $a \in g$, $m \in \mathbb{Z}$ generate $\tilde{g}$, $\hat{\psi}$ is a $\tilde{g}$-module homomorphism. Clearly, $\hat{\psi} = 0$ implies $\psi = 0$. Then we obtain a one-to-one linear map from $\text{Hom}_{\tilde{g}}(W_1 \otimes U \otimes \mathbb{C}[t, t^{-1}], W)$ to $\text{Hom}_{\hat{g}}(W_1 \otimes U(z), \overline{W})$ sending $\psi$ to $\hat{\psi}$.

On the other hand, let $\phi$ be a $\hat{g}$-module homomorphism from $W_1 \otimes U(z)$ to $\overline{W}$. For any $\lambda \in \mathbb{C}$, denote by $p_\lambda^W$ the projection of $\overline{W}$ onto the homogeneous subspace $W(\lambda)$. We have

$$p_\lambda^W(a(m)\bar{w}) = a(m)p_{\lambda-m}^W(\bar{w}) \quad \text{for } a \in g, \lambda \in \mathbb{C}, m \in \mathbb{Z}, \bar{w} \in \overline{W}. \quad (4.11)$$

Define a linear map $\tilde{\phi}$ from $W_1 \otimes U \otimes \mathbb{C}[t, t^{-1}]$ to $W$ by

$$\tilde{\phi}(w_1 \otimes u \otimes t^n) = z^{n+1} p_{\deg w_1+n+1}^W(\phi(w_1 \otimes u)) \quad (4.12)$$

for homogeneous vector $w_1 \in W_1$ and for $u \in U$, $n \in \mathbb{Z}$. We now show that $\tilde{\phi}$ is a $\tilde{g}$-module homomorphism. Let $w_1 \in W_1$ be homogeneous and let $a \in g$, $u \in U$, $m, n \in \mathbb{Z}$.
Noticing that $\deg a(m)w_1 = \deg w_1 + m$, we have

\[
\tilde{\phi}(a(m)(w_1 \otimes u \otimes t^n)) \\
= \tilde{\phi}(a(m)w_1 \otimes u \otimes t^n + w_1 \otimes au \otimes t^{m+n}) \\
= z^{n+1}p_{\deg w_1+m+n+1}^W \phi(a(m)w_1 \otimes u) + z^{m+n+1}p_{\deg w_1+m+n+1}^W \phi(w_1 \otimes au) \\
= z^{n+1}p_{\deg w_1+m+n+1}^W \phi(a(m)w_1 \otimes u) \\
= z^{n+1}a(m)p_{\deg w_1+n+1}^W \phi(w_1 \otimes u) \\
= a(m)\tilde{\phi}(w_1 \otimes u \otimes t^n).
\]

This shows that $\tilde{\phi}$ is indeed a $\tilde{\mathfrak{g}}$-module homomorphism.

For $\phi \in \text{Hom}_{\tilde{\mathfrak{g}}}(W_1 \otimes U(z), W)$, set $\psi = \tilde{\phi}$. For homogeneous vector $w_1 \in W_1$ and for $u \in U$, we have

\[
\hat{\psi}(w_1 \otimes u) = \sum_{n \in \mathbb{Z}} z^{-n-1} \phi(w_1 \otimes u \otimes t^n) \\
= \sum_{n \in \mathbb{Z}} z^{-n-1} \tilde{\phi}(w_1 \otimes u \otimes t^n) \\
= \sum_{n \in \mathbb{Z}} p_{\deg w_1+n+1}^W \phi(w_1 \otimes u) \\
= \phi(w_1 \otimes u).
\]

This shows that the linear map $\psi \mapsto \hat{\psi}$ is also onto.

To summarize we have:

**Proposition 4.2.** Let $W_1, W$ be upper truncated $\tilde{\mathfrak{g}}$-modules of level $\ell$ and let $U$ be a $\mathfrak{g}$-module and $z$ a nonzero complex number. Then the map $\psi \mapsto \hat{\psi}$ from $\text{Hom}_{\tilde{\mathfrak{g}}}(W_1 \otimes U \otimes \mathbb{C}[t, t^{-1}], W)$ to $\text{Hom}_{\tilde{\mathfrak{g}}}(W_1 \otimes U(z), W)$ is a linear isomorphism. The inverse map is given by $\phi \mapsto \tilde{\phi}$.

Let $\ell$ be any complex number. Take $U$ to be the one-dimensional trivial $\mathfrak{g}$-module $\mathbb{C}$ in the (2.43) and set

\[
V_{\tilde{\mathfrak{g}}}(\ell, 0) = M_{\tilde{\mathfrak{g}}}(\ell, \mathbb{C}),
\]

which is usually called the vacuum $\tilde{\mathfrak{g}}$-module. It is well known ([FZ], [Lia], [Li2], [LL]) that $V_{\tilde{\mathfrak{g}}}(\ell, 0)$ has a natural vertex algebra structure. It is also known ([Li2], [LL], cf. [FZ]) that a module for $V_{\tilde{\mathfrak{g}}}(\ell, 0)$ (as a vertex algebra) exactly amounts to a restricted $\tilde{\mathfrak{g}}$-module of level $\ell$.

For the rest of this section we assume that $\tilde{\mathfrak{g}}$ is a standard affine Lie algebra (with $\mathfrak{g}$ a finite-dimensional simple Lie algebra and with $\langle \cdot, \cdot \rangle$ the normalized Killing form). For
any complex number $\ell$ not the negative dual Coxeter number of $\mathfrak{g}$, $V_{\mathfrak{g}}(\ell, 0)$ equipped with a canonical conformal vector is a vertex operator algebra (cf. [FZ]). For any restricted $\hat{\mathfrak{g}}$-module $W$ of level $\ell$, $W$ is naturally a module for $V_{\mathfrak{g}}(\ell, 0)$ viewed as a vertex algebra, then $W$ is naturally a $\hat{\mathfrak{g}}$-module with $\mathfrak{d}$ acting as $\alpha - L(0)$ where $\alpha$ is any complex number (cf. [LL]). Denote by $L_{\mathfrak{g}}(\ell, 0)$ the simple quotient vertex operator algebra of $V_{\mathfrak{g}}(\ell, 0)$. If $\ell$ is a positive integer, it was proved ([FZ], [DL], cf. [Li2]) that irreducible modules for $L_{\mathfrak{g}}(\ell, 0)$ viewed as a vertex operator algebra are exactly the irreducible highest weight integrable $\hat{\mathfrak{g}}$-modules of level $\ell$. Up to isomorphism, irreducible highest weight integrable $\hat{\mathfrak{g}}$-modules of level $\ell$ are $L(\ell, \lambda)$ where $\lambda$ is a dominant integrable weight for $\mathfrak{g}$ such that $\langle \lambda, \theta \rangle \leq \ell$ (see [K1]).

In [FHL], among other important results, for a general vertex operator algebra $V$ and for $V$-modules $W_1, W_2$ and $W_3$, a notion of fusion rule of type $(W_3^{W_1, W_2})$ was defined. Furthermore, in [FZ], a conceptual method for determining fusions was developed in terms of Zhu’s algebra and this method was applied to the case with $V = L_{\mathfrak{g}}(\ell, 0)$. In [Li3], a certain analogue of the classical hom-functor for vertex operator algebras was developed and by using this analogue it was proved ([Li3], Proposition 4.15) that the fusion rule of type $(L(\ell, \nu), L(\ell, \lambda), L(\ell, \mu))$ for the vertex operator algebra $L_{\mathfrak{g}}(\ell, 0)$ equals the dimension of $\text{Hom}_{\hat{\mathfrak{g}}}(L(\ell, \lambda) \otimes L(\mu) \otimes \mathbb{C}[t, t^{-1}], L(\ell, \nu))$, where $L(\mu)$ denotes the irreducible $\mathfrak{g}$-module of highest weight $\mu$. Combining this with Proposition 4.2 we immediately have:

**Proposition 4.3.** Let $\ell$ be a positive integer and let $L(\ell, \lambda), L(\ell, \mu)$ and $L(\ell, \nu)$ be highest weight irreducible $\hat{\mathfrak{g}}$-modules of level $\ell$. Then the fusion rule of type $(L(\ell, \nu), L(\ell, \lambda), L(\ell, \mu))$ for the vertex operator algebra $L_{\mathfrak{g}}(\ell, 0)$ equals the dimension of $\text{Hom}_{\hat{\mathfrak{g}}}(L(\ell, \lambda) \otimes L(\mu) \otimes \mathbb{C}[t, t^{-1}], L(\ell, \nu))$.

**Remark 4.4.** It was proved in [CP2] that for a highest weight irreducible integrable $\hat{\mathfrak{g}}$-module $W$ and a finite-dimensional irreducible $\mathfrak{g}$-module $U$, $W \otimes U \otimes \mathbb{C}[t, t^{-1}]$ is an irreducible $\hat{\mathfrak{g}}$-module if $W$ and $U$ satisfy certain conditions. In [A], the irreducibility of $\hat{\mathfrak{g}}$-modules $W \otimes U \otimes \mathbb{C}[t, t^{-1}]$ for certain nonintegrable $\hat{\mathfrak{g}}$-modules $W$ was studied in terms of vertex operator algebra $L_{\mathfrak{g}}(\ell, 0)$ and fusion rules, and certain interesting results were obtained in [A].

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