FUNCTIONS DEFINABLE IN DEFINABLY COMPLETE
UNIFORMLY LOCALLY O-MINIMAL STRUCTURE OF THE
SECOND KIND

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Abstract. We consider a definably complete uniformly locally o-minimal expansion of the second kind of a densely linearly ordered abelian group (DCULOAS structure) in this paper. The first main theorem is the following monotonicity theorem. For a definable function \( f \) on an interval \( I \), the interval \( I \) is decomposed into four definable sets. Three sets are open definable sets on which \( f \) is locally constant, locally strictly increasing and continuous, and locally strictly decreasing and continuous, respectively. The last definable set is discrete and closed.

We also investigate uniform continuous definable functions and derive Arzela-Ascoli-type theorem for definable functions. Consider the parameterized function \( f : C \times P \to M \) which is equi-continuous with respect to \( P \). The projection image of the set at which \( f \) is discontinuous to the parameter space \( P \) is of dimension smaller than \( \dim P \) when \( C \) is closed and bounded.

Finally, we demonstrate that an archimedean DCULOAS structure which enjoys definable Tietze extension property is o-minimal.

1. Introduction

An o-minimal structure enjoys tame properties such as monotonicity and definable cell decomposition \([3, 7, 9]\). Toffalori and Vozoris first introduced locally o-minimal structures in \([11]\). Roughly speaking, a locally o-minimal structure is defined by simply localizing the definition of an o-minimal structure. See their paper \([11]\) for the precise definition of locally o-minimal structures. In spite of its similarity to the definition of o-minimal structures, a locally o-minimal structure does not enjoy the localized counterparts such as local monotonicity theorem and local definable cell decomposition theorem.

A uniformly locally o-minimal structure of the second kind was first introduced in \([4]\) as a structure which enjoys a local monotonicity theorem \([4, \text{ Theorem } 3.2]\) and a local definable cell decomposition theorem \([4, \text{ Theorem } 4.2]\) with the extra cost of definably completeness \([8]\). If it is also an expansion of a densely linearly ordered abelian group, it also enjoys more tame properties such as good dimension theory and a decomposition into special submanifolds \([4, 5, 6]\). We consider a definably complete uniformly locally o-minimal expansion of the second kind of a densely linearly ordered abelian group in this paper. We call it a DCULOAS structure for short.

We investigate functions definable in a DCULOAS structure in this paper. In a DCULOAS structure, we get a non-local monotonicity theorem which is stronger

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than the conventional local monotonicity theorem \cite[Theorem 3.2]{4}. For a definable function $f$ on an interval $I$, the interval $I$ is decomposed into four definable sets. Three sets are open definable sets on which $f$ is locally constant, locally strictly increasing and continuous, and locally strictly decreasing and continuous, respectively. The last definable set is a discrete and closed.

The new monotonicity theorem together with good dimension theory enables us to investigate definable continuous functions. For instance, a definable continuous function on a definable closed bounded set is uniformly continuous. We also demonstrate an Arzela-Ascoli-type theorem for definable functions. Consider the parameterized function $f : C \times P \to M$ which is equi-continuous with respect to $P$. One of our main theorems is that the projection image of the set at which $f$ is discontinuous to the parameter space $P$ is of dimension smaller than $\dim P$ when $C$ is closed and bounded.

Functions definable in a DCULOAS structure enjoy several tame properties as above. Definable Tietze extension theorem is a convenient tool for topological studies of structures such as \cite{10}. It is available in a definably complete expansion of an ordered field \cite[Lemma 6.6]{1}. Unfortunately, a uniformly locally o-minimal expansion of the second kind of an ordered field is o-minimal \cite[Proposition 2.1]{4}. The author’s concern is whether a DCULOAS structure enjoys definable Tietze extension property. We obtain a negative partial result on this conjecture. An archimedean DCULOAS structure which enjoys definable Tietze extension property is o-minimal.

We introduce the terms and notations used in this paper. The term ‘definable’ means ‘definable in the given structure with parameters’ in this paper. For a linearly ordered structure $\mathcal{M} = (M, <, \ldots)$, an open interval is a definable set of the form $\{x \in \mathbb{R} \mid a < x < b\}$ for some $a, b \in M$. It is denoted by $[a, b]$ in this paper. We define a closed interval similarly. It is denoted by $[a, b]$. An open box in $M^n$ is the direct product of $n$ open intervals. When the structure $\mathcal{M}$ is an expansion of an abelian group. The notation $M_{>r}$ denotes the set $\{x \in M \mid x > r\}$ for any $r \in M$. We set $|x| := \max_{1 \leq i \leq n} |x_i|$ for any vector $x = (x_1, \ldots, x_n) \in M^n$. The function $|x - y|$ defines a distance in $M^n$. Let $A$ be a subset of a topological space. The notations $\text{int}(A)$ and $\overline{A}$ denote the interior and the closure of the set $A$, respectively.

This paper is organized as follows: We first review the previous works in Section 2. The dimension theory of sets definable in a definably complete uniformly locally o-minimal structure of the second kind is reviewed in this section. The monotonicity theorem is proved in Section 3. Definable choice lemma for a DCULOAS structure is necessary for our study. Section 4 is devoted to the lemma and its corollaries. We prove the main theorems other than the monotonicity theorem and the theorem on Tietze extension in Section 5 using the assertions in the previous sections. We consider definable Tietze extension property in Section 6.

2. Review of previous works

We first review the dimension theory of sets definable in a definably complete uniformly locally o-minimal structure of the second kind. A definably complete uniformly locally o-minimal structure of the second kind admits local definable cell decomposition by \cite[Theorem 4.2]{3}. We review the definitions of cells and local definable cell decomposition in \cite[Definition 4.1]{4}.
Definition 2.1 (Definable cell decomposition). Consider a densely linearly ordered structure \( M = (M, <, \ldots) \). Let \((i_1, \ldots, i_n)\) be a sequence of zeros and ones of length \( n \). \((i_1, \ldots, i_n)\)-cells are definable subsets of \( M^n \) defined inductively as follows:

- A \((0)\)-cell is a point in \( M \) and a \((1)\)-cell is an open interval in \( M \).
- An \((i_1, \ldots, i_n, 0)\)-cell is the graph of a continuous definable function defined on an \((i_1, \ldots, i_n)\)-cell. An \((i_1, \ldots, i_n, 1)\)-cell is a definable set of the form \( \{(x, y) \in C \times M \mid f(x) < y < g(x)\} \), where \( C \) is an \((i_1, \ldots, i_n)\)-cell and \( f \) and \( g \) are definable continuous functions defined on \( C \) with \( f < g \).

A cell is an \((i_1, \ldots, i_n)\)-cell for some sequence \((i_1, \ldots, i_n)\) of zeros and ones. An open cell is a \((1, 1, \ldots, 1)\)-cell.

We inductively define a definable cell decomposition of an open box \( B \subset M^n \). For \( n = 1 \), a definable cell decomposition of \( B \) is a partition \( B = \bigcup_{i=1}^m C_i \) into finite cells. For \( n > 1 \), a definable cell decomposition of \( B \) is a partition \( B = \bigcup_{i=1}^m C_i \) into finite cells such that \( \pi(B) = \bigcup_{i=1}^m \pi(C_i) \) is a definable cell decomposition of \( \pi(B) \), where \( \pi : M^n \to M^{n-1} \) is the projection forgetting the last coordinate. Consider a finite family \( \{A_\lambda\}_{\lambda \in \Lambda} \) of definable subsets of \( B \). A definable cell decomposition of \( B \) partitioning \( \{A_\lambda\}_{\lambda \in \Lambda} \) is a definable cell decomposition of \( B \) such that the definable sets \( A_\lambda \) are unions of cells for all \( \lambda \in \Lambda \).

When a locally o-minimal structure admits local definable cell decomposition, we can define the dimension of definable sets and it enjoys several good properties. Three equivalent definitions of dimension are given in \cite{HH} Definition 5.1, Corollary 5.3. We only review the assertions on dimension in \cite{HH} \cite{HH5} which are necessary for this study.

Proposition 2.2. Let \( \mathcal{M} = (M, <, \ldots) \) be a definably complete uniformly locally o-minimal structure of the second kind. The following assertions hold true:

1. Let \( X \subset Y \) be definable sets. Then, the inequality \( \dim(X) \leq \dim(Y) \) holds true.
2. Let \( X \) be a subset of \( M^n \). The set \( X \) is of dimension \( n \) if and only if \( X \) has a nonempty interior.
3. A definable subset of \( M^n \) is of dimension \( \geq d \) if it contains an \((i_1, \ldots, i_n)\)-cell with \( \sum_{j=1}^n i_j \geq d \).
4. Let \( X \) be a nonempty definable subset of \( M^n \). There exists a point \( x \in X \) such that \( \dim(X \cap U) = \dim X \) for any open box \( U \) containing the point \( x \).
5. Let \( X \) and \( Y \) be definable subsets of \( M^n \). We have \( \dim(X \cup Y) = \max\{\dim(X), \dim(Y)\} \).
6. Let \( X \) be a definable set. The frontier \( \overline{X} \setminus X \) is of dimension smaller than \( \dim X \).
7. Assume that \( \mathcal{M} \) is a DCULOAS structure. Let \( f : X \to R \) be a definable function. The set of the points at which \( f \) is discontinuous is of dimension smaller than \( \dim(X) \).

Proof. (1) \cite{HH} Lemma 5.1; (2) through (4) \cite{HH} Corollary 5.3; (5) \cite{HH} Corollary 5.4(ii); (6) \cite{HH} Theorem 5.6; (7) \cite{HH5} Corollary 1.2. \( \square \)

We also need the following lemma on the dimension of a definable subset in \( M \).

Lemma 2.3. Let \( \mathcal{M} = (M, <, \ldots) \) be a definably complete uniformly locally o-minimal structure of the second kind. Let \( X \) be a definable subset of \( M \). It is of
dimension zero if and only if it is discrete and closed. If $X$ is of dimension zero and bounded below, we have $\inf X \in X$.

Proof. The first assertion follows from [1] Lemma 5.2, Corollary 5.3. Let $X$ be a definable subset of $\mathcal{M}$ of dimension zero. Since $\mathcal{M}$ is definably complete, the infimum $\inf X$ is well-defined. It is finite because $X$ is bounded below. Since $X$ is closed, we have $\inf X \in X$. \qed

The following technical lemma is used in Section 5.

Lemma 2.4. Let $\mathcal{M} = (\mathbb{M}, <, \ldots)$ be a definably complete uniformly locally o-minimal structure of the second kind. Consider definable subsets $X_1$ and $X_2$ of $\mathbb{M}^m$ and $\mathbb{M}^n$, respectively. Let $\pi : \mathbb{M}^{m+n} \to \mathbb{M}^m$ be the projection onto the first $m$ coordinates. Let $W$ be an open box in $\mathbb{M}^{m+n}$. Consider a definable cell decomposition of $W$ partitioning $W \cap (X_1 \times X_2)$. Assume that a cell $E$ contained in $X_1 \times X_2$ satisfies the following conditions:

(a) $\dim \pi(E) = \dim X_1$;
(b) Any cell $E'$ such that $\dim E' > \dim E$, $\pi(E') = \pi(E)$ and $E' \cap E \neq \emptyset$ is not contained in $X_1 \times X_2$.

Then there exist a point $x \in E$ and an open box $U$ in $\mathbb{M}^{m+n}$ containing the point $x$ such that $(X_1 \times X_2) \cap U$ is contained in $E$.

Proof. Let $E$ be an $(i_1, \ldots, i_{m+n})$-cell. Consider the family $\mathcal{D}$ of the cells in $\pi(W)$ contained in $X_1$ which is not $\pi(E)$. Set $\tilde{D} = \pi(E) \setminus \bigcup_{D \in \mathcal{D}} D$. We have $\dim \tilde{D} = \dim X_1$ by Proposition 2.2(5) because $\dim \tilde{D} \cap \pi(E) \leq \dim \tilde{D} \setminus D < \dim D \leq \dim X_1$ for all $D \in \mathcal{D}$ by Proposition 2.2(1), (6). Let $D'$ be a cell contained in $\tilde{D}$ with $\dim D' = \dim X_1$, the intersection $\pi^{-1}(D') \cap E$ is obviously a cell of dimension $\dim D' + i_{m+1} + \cdots + i_{m+n} = \dim E$ by the definition of dimension. The definable set $F = \pi^{-1}(\tilde{D}) \cap E$ is of dimension $\dim E$ by Proposition 2.2(3).

Consider the family $\mathcal{E}$ of the cells $E'$ in $W$ contained in $X_1 \times X_2$ such that $\pi(E) = \pi(E')$ and $E' \cap E \neq \emptyset$. We have $\dim E' \leq \dim E$ by the assumption. Set $G = F \setminus \bigcup_{E' \in \mathcal{E}} E'$. We have $\dim G = \dim E$ for the same reason as above. In particular, the definable set $G$ is not an empty set. Take a point $x \in G$ and a sufficiently small open box $U$ containing $x$. It is obvious that $D \cap \pi(U) = \emptyset$ for all $D \in \mathcal{D}$ and $E' \cap U = \emptyset$ for all $E' \in \mathcal{E}$ by the definition. Take an arbitrary cell $E''$ contained in $X_1 \times X_2$ which is not a member of $\mathcal{E}$ and $\pi(E'') = \pi(E)$. We have $E'' \cap E = \emptyset$. Hence, we may assume that $E'' \cap U = \emptyset$ for such cells $E''$ shrinking $U$ if necessary. We have demonstrated that $(X_1 \times X_2) \cap U$ is contained in $E$ because $U$ has an empty intersection with the cells contained in $X_1 \times X_2$ other than $E$. \qed

3. Monotonicity

Parameterized local monotonicity theorem for uniformly locally o-minimal structure of the second kind is demonstrated in [1] Theorem 3.2. We derive another monotonicity theorem in this section when the structure is a DCULOAS structure. We first review the definition of local monotonicity.

Definition 3.1 (Local monotonicity). [1] Definition 3.1 A function $f$ defined on an open set $I$ is locally constant if, for any $x \in I$, there exists an open interval $J$ such that $x \in J \subseteq I$ and the restriction $f|_J$ of $f$ to $J$ is constant.
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A function $f$ defined on an open set $I$ is \textit{locally strictly increasing} if, for any $x \in I$, there exists an open interval $J$ such that $x \in J \subset I$ and $f$ is strictly increasing on the interval $J$. We define a \textit{locally strictly decreasing} function similarly. A \textit{locally strictly monotone} function is a locally strictly increasing function or a locally strictly decreasing function. A \textit{locally monotone} function is locally strictly monotone or locally constant.

We need the following lemmas which are given in [4].

\textbf{Lemma 3.2.} Let $\mathcal{M} = (M, <, \ldots)$ be a uniformly locally o-minimal structure of the second kind. Let $f : I \to M$ be a definable function on an open interval $I$. Assume that, for any $a \in I$, there exists an open interval $I_a$ such that $a \in I_a \subset I$, $f(x) < f(a)$ for all $x \in I_a$ with $x < a$ and $f(x) > f(a)$ for all $x \in I_a$ with $x > a$. Then, $f$ is locally strictly increasing.

\textit{Proof.} [4, Lemma 3.1] \hfill \Box

\textbf{Lemma 3.3.} Let $\mathcal{M} = (M, <, \ldots)$ be a uniformly locally o-minimal structure of the second kind. No injective definable functions defined on open intervals have the local minimum throughout the intervals.

\textit{Proof.} [4, Lemma 3.2] \hfill \Box

The following is one of the main theorems in this paper.

\textbf{Theorem 3.4 (Monotonicity theorem).} Let $\mathcal{M} = (M, <, +, 0, \ldots)$ be a DCULOAS structure. Let $I$ be an interval and $f : I \to M$ be a definable function. There exists a partition $I = X_d \cup X_c \cup X_+ \cup X_-$ of $I$ into definable sets satisfying the following conditions:

1. the definable set $X_d$ is discrete and closed;
2. the definable set $X_c$ is open and $f$ is locally constant on $X_c$;
3. the definable set $X_+$ is open and $f$ is locally strictly increasing and continuous on $X_+$;
4. the definable set $X_-$ is open and $f$ is locally strictly decreasing and continuous on $X_-$.

\textit{Proof.} We prove the theorem basically following the strategy of the proof of [4, Theorem 3.2]. We first demonstrate the following claim:

\textbf{Claim 1.} There exists a partition $I = X_d' \cup X_c \cup X_n$ such that

\begin{enumerate}
\item the definable set $X_d'$ is at most of dimension zero;
\item the definable set $X_c$ satisfies the condition (2) of the theorem;
\item the definable set $X_n$ is open and $f$ is locally injective on it.
\end{enumerate}

Here, a function $g : I \to M$ is called \textit{locally injective} if, for any $x \in I$, there exists an open interval $I'$ such that $x \in I' \subset I$ and the restriction of $g$ to $I'$ is injective.

We show Claim 1. Set

$$X_c = \{ x \in I \mid \exists x_1, x_2 \in I \text{ such that } x_1 < x < x_2 \text{ and } f(x) = f(x') \text{ for all } x' \text{ with } x_1 < x' < x_2 \}. $$

The set $X_c$ clearly satisfies the condition (2) of the theorem. Let $E$ be the boundary of the definable set $X_c$. It is at most of dimension zero by Proposition 2.2(6). Set $Y = I \setminus \overline{X_c}$. 

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The set $X_c$ clearly satisfies the condition (2) of the theorem. Let $E$ be the boundary of the definable set $X_c$. It is at most of dimension zero by Proposition 2.2(6). Set $Y = I \setminus \overline{X_c}$.
Consider the definable set
\[ X_n = \{ x \in Y \mid \exists x_1 < x, \exists x_2 > x, f \text{ is injective on the interval } ]x_1, x_2[ \}. \]
It is obviously open. It satisfies the condition (c) of Claim 1. We show that \( X_n \) is dense in \( Y \). Fix an arbitrary point \( x' \in Y \). There exists \( x_2 \in Y \) such that the open interval \( J := ]x', x_2[ \) is contained in \( Y \). For any \( y \in f(J) \), the definable set \( \{ x \in J \mid f(x) = y \} \) does not contain an open interval by the definition of \( X_n \) and \( f \).

It is of dimension zero by Proposition 2.2(2). It has the smallest element by Lemma 2.3. Consider a definable map \( g : f(J) \to J \) given by \( g(y) = \inf \{ x \in J \mid f(x) = y \} \). We have \( f(g(y)) = y \) for all \( y \in f(J) \). The image of \( g \) contains an open interval one of whose endpoints is \( x' \). Otherwise, there exists an open interval \( ]x', x_2[ \subset J \) with \( ]x', x_2[ \cap g(f(J)) = \emptyset \) because \( M \) is locally o-minimal. Take a point \( u \in ]x', x_2[ \). We have \( g(f(u)) > u \). However, \( g(f(u)) \) is the smallest element in \( \{ x \in J \mid f(x) = f(u) \} \) by the definition of \( g \). It is a contradiction because \( u \) is smaller than \( g(f(u)) \).

We have shown that there exists \( x_2' \) with \( ]x', x_2'[ \subset g(f(J)) \). It means that \( f \) is injective on the interval \( ]x', x_2'[ \). We have shown that \( X_n \) is dense in \( Y \).

Since \( X_n \) is dense in \( Y \), the set \( Y \setminus X_n \) does not contain an open interval. We have \( \dim(Y \setminus X_n) \leq 0 \) by Proposition 2.2(2). The definable set \( X_n \) satisfies the condition (c) of Claim 1. Set \( X'_n = E \cup (Y \setminus X_n) \), then the set \( X'_n \) satisfies the condition (a) of Claim 1 by Proposition 2.2(5). We have finished the proof of Claim 1.

**Claim 2.** There exists a partition \( X_n = X''_n \cup X'_n \cup X'_n \) such that
(a) the definable set \( X''_n \) is at most of dimension zero;
(b) the definable set \( X'_n \) is open and \( f \) is locally strictly decreasing on it;
(c) the definable set \( X'_n \) is open and \( f \) is locally strictly increasing on it.

We demonstrate Claim 2. Define the definable subsets \( X''_n, X' \), \( X_{\text{max}} \) and \( X_{\text{min}} \) of \( X_n \) as follows:
\( X''_n = \{ x \in X_n \mid \exists x_1 < x, \exists x_2 > x \text{ with } x_1, x_2 \in X_n, \)
\( \forall x'((x_1 < x' < x) \rightarrow (f(x') > f(x)) \wedge x' \in X_n) \} \]
\( X'_n = \{ x \in X_n \mid \exists x_1 < x, \exists x_2 > x \text{ with } x_1, x_2 \in X_n, \)
\( \forall x'((x_1 < x' < x) \rightarrow (f(x') < f(x)) \wedge x' \in X_n) \} \]
\( X_{\text{max}} = \{ x \in X_n \mid \exists x_1 < x, \exists x_2 > x \text{ with } x_1, x_2 \in X_n, \)
\( \forall x'((x_1 < x' < x) \rightarrow (f(x') < f(x)) \wedge x' \in X_n) \} \]
\( X_{\text{min}} = \{ x \in X_n \mid \exists x_1 < x, \exists x_2 > x \text{ with } x_1, x_2 \in X_n, \)
\( \forall x'((x_1 < x' < x) \rightarrow (f(x') > f(x)) \wedge x' \in X_n) \} \]
Since \( M \) is locally o-minimal and \( f \) is locally injective on \( X_n \), we have
\( X_n = X''_n \cup X'_n \cup X_{\text{max}} \cup X_{\text{min}} \).

Let \( E' \) be the union of boundaries of \( X''_n \) and \( X'_n \). The definable set \( X'_n = X''_n \cup E' \) satisfies the condition (c) of Claim 2 by Lemma 3.2. In the same way, the definable
set $X' = X'' \setminus E'$ satisfies the condition (b) of Claim 2. The definable set $X_{\min}$ is at most of dimension zero by Lemma 3.3 and Proposition 2.2(2). In the same way, $X_{\max}$ is at most of dimension zero. The definable set $X''_g = E' \cup X_{\min} \cup X_{\max}$ satisfies the condition (a) of Claim 2 by Proposition 2.2(5). We have proven Claim 2.

We are ready to finish the proof of the monotonicity theorem. Set $D = \{ x \in X'_+ \cup X'_- \mid f \text{ is discontinuous at } x \}$. It is at most of dimension zero by Proposition 2.2(7). Set $X_+ = X'_+ \setminus D$, $X_- = X'_- \setminus D$ and $X_d = X''_d \cup X''_g \cup D$. Since $D$ is closed by Lemma 2.3, $X_+$ and $X_-$ are open. The definable set $X_d$ is discrete and closed by Lemma 2.3 and Proposition 2.2(5). The definable sets $X_c$, $X_+$, $X_-$ and $X_d$ satisfy the conditions of the theorem.

We also need the following lemma in [4].

**Lemma 3.5.** Let $\mathcal{M} = (M, <, \ldots)$ be a definably complete local o-minimal structure. A locally strictly monotone definable function defined on an open interval is strictly monotone.

**Proof.** [4 Proposition 3.1] □

The following corollary guarantees the existence of the limit.

**Corollary 3.6.** Let $\mathcal{M} = (M, <, +, 0, \ldots)$ be a DCULOAS structure. Let $s > 0$ and $f : ]0, s[ \to M^n$ be a bounded definable map. There exists a unique point $x \in M^n$ satisfying the following condition:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall t, 0 < t < \delta \Rightarrow |x - f(t)| < \varepsilon.$$  

The notation $\lim_{t \to 0^+} f(t)$ denotes the point $x$.

**Proof.** We first reduce to the case in which $n = 1$. Assume that the corollary holds true for $n = 1$. Let $\pi_i$ be the projection onto the $i$-th coordinate for all $1 \leq i \leq n$. Apply the corollary to the composition $\pi_i \circ f$. Set $x_i = \lim_{t \to 0^+} \pi_i \circ f(t)$ for all $1 \leq i \leq n$. It is obvious that $x = (x_1, \ldots, x_n)$ is the unique point satisfying the condition in the corollary. We have succeeded in reducing to the case in which $n = 1$.

Set $I = ]0, s[$. Applying Theorem 3.4 to $f$, we get a partition $I = X_d \cup X_c \cup X_+ \cup X_-$ into definable sets satisfying the conditions in Theorem 3.4. Since $X_d$ is discrete and closed, we have $\inf X_d \in X_d$ by Lemma 2.3. Shrinking the interval $I$ if necessary, we may assume that $X_d$ is an empty set. Since the interval $I$ is definably connected by [8 Proposition 1.4], we have $I = X_c$, $I = X_+$ or $I = X_-$. We only consider the case in which $I = X_-$. We can prove the corollary similarly in the other cases.

The function $f$ is strictly decreasing by Lemma 3.5 because $I = X_-$. Set $x = \inf_{t < t < s} f(t)$, which exists because $f$ is bounded. It is obvious the point $x$ satisfies the required condition because $f$ is strictly decreasing. Let $x'$ be another point satisfying the condition. We fix an arbitrary $\varepsilon > 0$. There exists $\delta > 0$ with $|x - f(t)| < \varepsilon$ whenever $0 < t < \delta$. There exists $\delta' > 0$ with $|x' - f(t)| < \varepsilon$ whenever $0 < t < \delta'$. Set $\delta'' = \min(\delta, \delta')$. We have $|x - x'| \leq |x - f(t)| + |x' - f(t)| < 2\varepsilon$ whenever $0 < t < \delta''$. We get $x = x'$ because $\varepsilon$ is an arbitrary positive element. □
4. Definable choice

We review the following definable choice lemma and its applications.

**Lemma 4.1** (Definable choice). Consider a definably complete expansion of a densely linearly ordered abelian group $\mathcal{M} = (M, <, +, 0, \ldots)$. Let $X$ be a definable subset of $M^{m+n}$. The notation $\pi : M^{m+n} \to M^n$ denotes the projection onto the last $n$ coordinates. There exists a definable map $\varphi : \pi(X) \to X$ such that the composition $\pi \circ \varphi$ is the identity map on $\pi(X)$.

**Proof.** Let $\mathcal{M}$ be a definable subset of $M^n$ which is not closed. Take a point $a \in X \setminus \pi$. There exist a small positive $\varepsilon$ and a definable continuous map $\gamma : ]0, \varepsilon[ \to X$ such that $\lim_{t \to 0^+} \gamma(t) = a$.

The following curve selection lemma is worth to be mentioned.

**Corollary 4.2.** Consider a DCULOAS structure $\mathcal{M} = (M, <, +, 0, \ldots)$. Let $X$ be a definable subset of $M^n$ which is not closed. Take a point $a \in X \setminus \pi$. There exist a small positive $\varepsilon$ and a definable continuous map $\gamma : ]0, \varepsilon[ \to X$ such that $\lim_{t \to 0^+} \gamma(t) = a$.

**Proof.** Let $\pi : M^{n+1} \to M$ be the projection onto the last coordinate. Set $Y = \{(x, t) | \in X \times M | |a - x| = t\}$. Since $\mathcal{M}$ is locally o-minimal, the intersection $]-\delta, \delta[\cap Y)$ is a finite union of points and open intervals for a sufficiently small $\delta > 0$. Since the point $a$ belongs to the closure of $X$, the intersection $]-\delta, \delta[\cap Y)$ contains an open interval of the form $]0, \varepsilon[$ for some $\varepsilon > 0$. There exists a definable map $\gamma : ]0, \varepsilon[ \to X$ with $\gamma(t), t \in Y$ for all $0 < t < \varepsilon$ by Lemma 4.1. The set $D$ of points at which the definable function $\gamma$ is discontinuous is of dimension zero by Proposition 2.2(7). We have $\inf D \in D$ by Lemma 4.3. In particular, we get $\inf D > 0$. Taking a smaller $\varepsilon > 0$ if necessary, we may assume that $\gamma$ is continuous.

The following two lemmas are used in the subsequent section. They can be proved by using the definable choice lemma.

**Lemma 4.3.** Consider a DCULOAS structure $\mathcal{M} = (M, <, +, 0, \ldots)$. Let $C$ and $P$ be definable subsets of $M^m$ and $M^n$, respectively. Let $X$ be a definable subset of $X \times P$. The notation $\pi : M^{m+n} \to M^n$ denotes the projection onto the last $n$ coordinates. Assume that $\dim \pi(X) = \dim P$. Then there exists a point $(c, p) \in X$ such that $\dim \pi(X \cap W) = \dim P$ for all open boxes $W$ in $M^{m+n}$ containing the point $(c, p)$.

**Proof.** We can find a definable map $\tau : \pi(X) \to X$ such that the composition $\pi \circ \tau$ is the identity map on $\pi(X)$ by Lemma 4.1. Let $D$ be the closure of the set of points at which $\tau$ is discontinuous. We have $\dim D < \dim \pi(X) = \dim P$ by Proposition 2.2(5), (6), (7). Set $E = \pi(X) \setminus D$. We obtain $\dim E = \dim P$ by Proposition 2.2(5). Therefore there exists a point $p \in E$ with $\dim(E \cap U) = \dim P$ for all open box $U$ in $M^n$ containing the point $p \in X$ by Proposition 2.2(4). Set $(c, p) = \pi(p)$.

We demonstrate that the point $(c, p)$ satisfies the condition in the lemma. Take an arbitrary sufficiently small open box $W$ in $M^{m+n}$ containing the point $(c, p)$. We may assume that $D \cap \pi(W) = \emptyset$ because $p \notin D$ and $D$ is closed. Since $\tau$ is continuous on $E$, the set $\tau^{-1}(W) = \pi(\tau(E) \cap W)$ is open in $E$. There exists an open box $U$ in $R^n$ such that $p \in U$ and $E \cap U \subset \pi(\tau(E) \cap W)$. Shrinking $U$ if necessary, we may assume that $U$ is contained in $\pi(W)$. We have $\dim P = \dim E \cap U$ by the definition of the point $p$. We then get $\dim P = \dim E \cap U \leq \dim \pi(\tau(E) \cap W) \leq \dim \pi(X \cap W) \leq \dim P$ by Proposition 2.2(1). We have demonstrated the lemma. □
Lemma 4.4. Consider a DCULOAS structure $\mathcal{M} = (M, <, +, 0, \ldots)$. Let $C$ be a definable closed and bounded subset of $\mathbb{R}^m$. Let $\varphi, \psi : C \to M_{\geq 0}$ be two definable functions. Assume that the following condition is satisfied:

$$\forall x \in C, \exists \delta > 0, \forall x' \in C, \ |x' - x| < \delta \Rightarrow \varphi(x') \geq \psi(x).$$

Then we have $\inf \varphi(C) > 0$.

Proof. Set $l = \inf \varphi(C) \geq 0$, which exists by the definably completeness of $\mathcal{M}$. We have only to show that $l > 0$. Since $\mathcal{M}$ is locally o-minimal, we have $l \in \varphi(C)$ or there exists $u \in M$ with $l < u$ and $\{l, u\} \subset \varphi(C)$. It is obvious that $l > 0$ in the former case. We consider the latter case in the rest of the proof.

Let $\Gamma$ be the graph of the function $\varphi$. Let $\pi_1 : M^{m+1} \to M^m$ and $\pi_2 : M^{m+1} \to M$ be the projections onto the first $m$ coordinates and onto the last coordinate, respectively. We can take a definable map $\eta : [l, u] \to \Gamma$ such that the composition $\pi_2 \circ \eta$ is the identity map on $[l, u]$ by Lemma 4.1. Note that the map $\eta$ is bounded because the domain of definition $C$ of $\varphi$ is bounded and the interval $[l, u]$ is bounded. Since the set of points at which $\eta$ is discontinuous is at most of dimension zero by Proposition 2.2(7), we may assume that $\eta$ is continuous by taking a smaller $u$ if necessary.

Set $z = \lim_{t \to l^+} \eta(t)$, which uniquely exists by Corollary 3.6. We have $\pi_2(z) = l$ by the definition of $\eta$. Set $c = \pi_1(z)$. It belongs to $C$ because $C$ is bounded and closed. For any $t > l$ sufficiently close to $l$, $\pi_1(\eta(t)) \in C$ is close to the point $c$. We have $\pi_2(\eta(t)) \geq \psi(c)$ for such $t$ by the assumption. We finally obtain $l = \lim_{t \to l^+} \pi_2(\eta(t))) \geq \psi(c) > 0$. \hfill $\square$

5. Properties of definable functions

We investigate the properties of functions definable in a DCULOAS structure.

Definition 5.1. Consider an expansion of a densely linearly ordered abelian group $\mathcal{M} = (M, <, +, 0, \ldots)$. Let $C$ and $P$ be definable sets. Let $f : C \times P \to M$ be a definable function. The function $f$ is equi-continuous with respect to $P$ if the following condition is satisfied:

$$\forall \varepsilon > 0, \ \forall x \in C, \ \exists \delta > 0, \ \forall p \in P, \ \forall x' \in C, \ |x - x'| < \delta \Rightarrow |f(x, p) - f(x', p)| < \varepsilon.$$  

The function $f$ is uniformly equi-continuous with respect to $P$ if the following condition is satisfied:

$$\forall \varepsilon > 0, \ \exists \delta > 0, \ \forall p \in P, \ \forall x, x' \in C, \ |x - x'| < \delta \Rightarrow |f(x, p) - f(x', p)| < \varepsilon.$$  

The function $f$ is pointwise bounded with respect to $P$ if the following condition is satisfied:

$$\forall x \in C, \ \exists N > 0, \ \forall p \in P, \ |f(x, p)| < N.$$  

Proposition 5.2. Consider a DCULOAS structure $\mathcal{M} = (M, <, +, 0, \ldots)$. Let $C$ and $P$ be definable sets. Let $f : C \times P \to M$ be a definable function. Assume that $C$ is closed and bounded. Then $f$ is equi-continuous with respect to $P$ if and only if it is uniformly equi-continuous with respect to $P$.

Proof. A uniformly equi-continuous definable function is always equi-continuous. We prove the opposite implication.
Take a positive $c \in M$. Consider the definable function $\varphi : C \times M_{>0} \to M_{>0}$ given by

$$\varphi(x, \varepsilon) = \sup\{0 < \delta < c \mid \forall p \in P, \forall x' \in C, \ |x - x'| < \delta \Rightarrow |f(x, p) - f(x', p)| < \varepsilon\}.$$  

Since $f$ is equi-continuous with respect to $P$, we have $\varphi(x, \varepsilon) > 0$ for all $x \in C$ and $\varepsilon > 0$. Fix arbitrary $x \in C$ and $\varepsilon > 0$. We also fix an arbitrary point $x' \in C$ with $|x' - x| < \frac{\varepsilon}{2}\varphi(x, \frac{\varepsilon}{2})$. We have $|f(x', p) - f(x, p)| < \frac{\varepsilon}{2}$ by the definition of $\varphi$.

For all $y \in C$ with $|x' - y| < \frac{1}{2}\varphi(x, \frac{\varepsilon}{2})$, we have $|x - y| \leq |x - x'| + |x' - y| < \varphi(x, \frac{\varepsilon}{2})$. We get $|f(y, p) - f(x, p)| < \frac{\varepsilon}{2}$ by the definition of $\varphi$. We finally obtain $|f(y, p) - f(x', p)| \leq |f(x', p) - f(x, p)| + |f(y, p) - f(x, p)| < \varepsilon$. It means that $\varphi(x', \varepsilon) \geq \frac{1}{2}\varphi(x, \frac{\varepsilon}{2})$ whenever $|x' - x| < \frac{\varepsilon}{2}\varphi(x, \frac{\varepsilon}{2})$. Apply Lemma 5.4 to the definable functions $\varphi(\cdot, \varepsilon)$ and $\frac{1}{2}\varphi(\cdot, \frac{\varepsilon}{2})$ for a fixed $\varepsilon > 0$. We have inf $\varphi(C, \varepsilon) > 0$.

For any $\varepsilon > 0$, set $\delta = \inf \varphi(C, \varepsilon)$. For any $p \in P$ and $x, x' \in C$, we have $|f(x, p) - f(x', p)| < \varepsilon$ whenever $|x - x'| < \delta$ by the definition of $\varphi$. It means that $f$ is uniformly equi-continuous.

It is well known that a continuous function defined on a compact set is uniformly continuous. The following corollary claims that a similar assertion holds true for a definable function defined on a definable closed bounded set.

**Corollary 5.3.** Consider a DCULIOAS structure $\mathcal{M} = (M, <, +, 0, \ldots)$. Let $C$ be a definable closed and bounded set. A definable continuous function $f : C \to M$ is uniformly continuous.

**Proof.** Let $P$ be a singleton. Apply Proposition 5.2 to the function $g : C \times P \to M$ defined by $g(x, p) = f(x)$.

We define a definable family of functions and investigate its properties. Equi-continuity, convergence and uniform convergence are defined for sequences of functions in classical analysis. We consider similar notions for a definable family of functions.

**Definition 5.4.** Consider an expansion of a densely linearly ordered abelian group $\mathcal{M} = (M, <, +, 0, \ldots)$. Let $C$ be a definable set and $s$ be a positive element in $M$. A family $\{f_t : C \to M\}_{0 < t < s}$ of functions with the parameter variable $t$ is a definable family of functions if there exists a definable function $F : C \times [0, s] \to M$ such that $f_t(x) = F(x, t)$ for all $x \in C$ and $0 < t < s$. We call it a definable family of continuous functions if every function $f_t$ is continuous.

Consider a definable family of functions $\{f_t : C \to M\}_{0 < t < s}$. Set $I = [0, s]$. The map $F : C \times I \to M$ given by $F(x, t) = f_t(x)$ is a definable function by the definition. The family is a definable family of equi-continuous functions if $F$ is equi-continuous with respect to $I$. It is a definable family of pointwise bounded functions if $F$ is pointwise bounded with respect to $I$.

A definable family of functions $\{f_t : C \to M\}_{0 < t < s}$ is pointwise convergent if for all positive $\varepsilon > 0$ and for all $x \in C$, there exists $s' > 0$ such that $|f_t(x) - f_t'(x)| < \varepsilon$ for all $t, t' \in [0, s']$.

The following lemma is proved following a typical argument in classical analysis.

**Lemma 5.5.** Consider an expansion of a densely linearly ordered abelian group $\mathcal{M} = (M, <, +, 0, \ldots)$. Let $C$ be a definable set and $s$ be a positive element in $M$. Consider a pointwise convergent definable family of functions $\{f_t : C \to M\}_{0 < t < s}$. For any $x \in C$, there exists $s' > 0$ such that the set $\{f_t(x) \mid 0 < t < s'\}$ is bounded.
Proof. Fix \( x \in C \). Take a positive \( \varepsilon > 0 \). There exists \( s' > 0 \) such that \(|f_t(x) - f_{t'}(x)| < \varepsilon \) for all \( t, t' \in ]0, s'[\). Fix \( u \in ]0, s'[\). For any \( t \in ]0, s'[\), we have \(|f_t(x)| \leq |f_u(x)| + |f_u(x) - f_t(x)| < |f_u(x)| + \varepsilon \). It means that the set \( \{f_t(x) \mid 0 < t < s'\} \) is bounded. □

We also get the following converse when \( \mathcal{M} \) is a DCULOAS structure.

**Lemma 5.6.** Let \( \mathcal{M} = (M, <, +, 0, \ldots) \) be a DCULOAS structure. Let \( C \) be a definable set and \( s \) be a positive element in \( M \). A definable family of pointwise bounded functions \( \{f_t : C \to M\}_{0 < t < s} \) is pointwise convergent.

Proof. Fix \( x \in C \). Set \( I = ]0, s[\). Consider the definable function \( g : I \to M \) given by \( g(t) = f_t(x) \). It is bounded. There exists a limit \( y = \lim_{t \to 0^+} g(t) \) by Corollary 5.5.

Take a positive \( \varepsilon > 0 \). There exists \( s' > 0 \) such that \(|y - g(t)| < \varepsilon / 2\) for all \( t \in ]0, s'[\). We have \(|f_t(x) - f_{t'}(x)| \leq |f_t(x) - y| + |y - f_{t'}(x)| < \varepsilon \) whenever \( t, t' \in ]0, s'[\). It means that the family \( \{f_t : C \to M\}_{0 < t < s} \) is pointwise convergent. □

We define the limit of a pointwise convergent definable family of functions.

**Definition 5.7.** Consider a DCULOAS structure \( \mathcal{M} = (M, <, +, 0, \ldots) \). Let \( C \) be a definable set and \( s \) be a positive element in \( M \). Consider a pointwise convergent definable family of functions \( \{f_t : C \to M\}_{0 < t < s} \). For any \( x \in C \), consider the function \( g_x : ]0, s[\to M \to M \) given by \( g_x(t) = f_t(x) \). Taking a smaller \( s > 0 \) if necessary, we may assume that \( g_x \) is bounded by Lemma 5.5. There exists a unique limit \( \lim_{t \to 0^+} f_t(x) \) exists by Corollary 5.5. The limit \( \lim_{t \to 0^+} f_t : C \to M \) of the family \( \{f_t : C \to M\}_{0 < t < s} \) is defined by \( (\lim_{t \to 0^+} f_t)(x) = \lim_{t \to 0^+} g_x(t) \).

**Definition 5.8.** Consider an expansion of a densely linearly ordered abelian group \( \mathcal{M} = (M, <, +, 0, \ldots) \). Let \( C \) be a definable set and \( s \) be a positive element in \( M \). A definable family of functions \( \{f_t : C \to M\}_{0 < t < s} \) is uniformly convergent if for all positive \( \varepsilon > 0 \), there exists \( s' > 0 \) such that \(|f_t(x) - f_{t'}(x)| < \varepsilon \) for all \( x \in C \) and \( t, t' \in ]0, s'[\).

The following proposition and its proof is almost the same as the counterparts in classical analysis.

**Proposition 5.9.** Consider a DCULOAS structure \( \mathcal{M} = (M, <, +, 0, \ldots) \). Let \( C \) be a definable set and \( s \) be a positive element in \( M \). Consider a uniformly convergent definable family of continuous functions \( \{f_t : C \to M\}_{0 < t < s} \). The limit \( \lim_{t \to 0^+} f_t : C \to M \) is continuous.

Proof. Fix arbitrary \( \varepsilon > 0 \) and \( x \in C \). Since the family is uniformly convergent, we may assume that \(|f_t(x') - f_{t'}(x')| < \frac{\varepsilon}{2} \) for all \( x' \in C \) and \( t, t' \in ]0, s[\) by taking a smaller \( s > 0 \) if necessary. Fix \( t_0 \) with \( 0 < t_0 < s \). There exists \( \delta > 0 \) such that \(|f_{t_0}(x') - f_{t_0}(x)| < \frac{\varepsilon}{2} \) whenever \(|x - x'| < \delta \) because \( f_{t_0} \) is continuous. Fix a point \( x' \in C \) with \(|x - x'| < \delta \). We can take \( t_1, t_2 \in ]0, s[\) with \(|(\lim_{t \to 0^+} f_t)(x') - f_{t_1}(x')| < \frac{\varepsilon}{2} \) and \(|(\lim_{t \to 0^+} f_t)(x') - f_{t_2}(x')| < \frac{\varepsilon}{2} \) by the definition of the limit \( \lim_{t \to 0^+} f_t \). We finally have \(|(\lim_{t \to 0^+} f_t)(x') - (\lim_{t \to 0^+} f_t)(x)| \leq |(\lim_{t \to 0^+} f_t)(x') - f_{t_1}(x')| + |f_{t_2}(x') - f_{t_0}(x')| + |f_{t_0}(x') - f_{t_0}(x)| + |f_{t_0}(x) - f_{t_1}(x)| + |f_{t_1}(x) - (\lim_{t \to 0^+} f_t)(x)| < \varepsilon \).

We have proven that \( \lim_{t \to 0^+} f_t \) is continuous. □

The following Arzela-Ascoli-type theorem is a main theorem of this paper.
Theorem 5.10. Consider a DCULOAS structure \( \mathcal{M} = (M, <, +, 0, \ldots) \). Let \( C \) be a definable closed and bounded set. A pointwise convergent definable family of equi-continuous functions \( \{f_t : C \to M\}_{0 < t < s} \) is uniformly convergent.

Proof. Set \( I = [0, s] \). Consider the map \( F : C \times I \to M \) given by \( F(x, t) = f_t(x) \). It is an equi-continuous definable function with respect to \( I \) by the definition. Set \( g = \lim_{t \to 0^+} f_t \). It is well-defined because the family is pointwise convergent.

Take \( c > 0 \). Consider the definable function \( \varphi : C \times M_{>0} \to M_{>0} \) given by

\[
\varphi(x, \varepsilon) = \sup\{0 < \delta < c \mid \forall t, t' \in [0, \delta], |F(x, t) - F(x, t')| < \varepsilon\}.
\]

We first show that it is well-defined. Fix \( x \in C \) and \( \varepsilon > 0 \). There exists \( \delta > 0 \) such that \(|F(x, u) - g(x)| < \frac{\varepsilon}{3}\) for all \( u \in [0, \delta] \) by the definition of \( g \). For any \( t, t' \in [0, \delta] \), we have \(|F(x, t) - F(x, t')| \leq |F(x, t) - g(x)| + |g(x) - F(x, t')| < \varepsilon\). The definable set \( \{0 < c < c \mid \forall t, t' \in [0, \delta], |F(x, t) - F(x, t')| < \varepsilon\} \) is not empty and the function \( \varphi \) is well-defined.

We fix \( x \in C \) and \( \varepsilon > 0 \) again. Since \( F \) is equi-continuous with respect to \( I \), there exists \( \delta' > 0 \) such that

\[
\forall t \in [0, s], \forall x' \in C, |x - x'| < \delta' \Rightarrow |F(x, t) - F(x', t)| < \frac{\varepsilon}{3}.
\]

Fix arbitrary \( x' \in C \) with \(|x - x'| < \delta'\). For any \( t, t' \in [0, \varphi(x, \frac{\varepsilon}{3})] \), we have \(|F(x, t) - F(x, t')| < \frac{\varepsilon}{3}\) by the definition of \( \varphi \). We finally get

\[
|F(x', t) - F(x', t')| \leq |F(x', t) - F(x, t)| + |F(x, t) - F(x', t')| + |F(x', t') - F(x', t')| < \varepsilon
\]

whenever \( t, t' \in [0, \varphi(x, \frac{\varepsilon}{3})] \). It means that \( \varphi(x', \varepsilon) \geq \varphi(x, \frac{\varepsilon}{3}) \). Apply Lemma 4.4 to the definable functions \( \varphi(\cdot, \varepsilon) \) and \( \varphi(\cdot, \frac{\varepsilon}{3}) \) for a fixed \( \varepsilon > 0 \). We have \( \inf \varphi(C, \varepsilon) > 0 \) for all \( \varepsilon > 0 \).

Fix \( \varepsilon > 0 \). Set \( \delta = \inf \varphi(C, \varepsilon) > 0 \). We have \(|f_t(x) - f_{t'}(x)| = |F(t, x) - F(t', x)| < \varepsilon\) for all \( x \in C \) and \( t, t' \in [0, \delta] \). It means that the family \( \{f_t : C \to M\}_{0 < t < s} \) is uniformly convergent.

The above theorem together with the curve selection lemma yields the following corollary:

Corollary 5.11. Consider a DCULOAS structure \( \mathcal{M} = (M, <, +, 0, \ldots) \). Let \( C \) and \( P \) be definable sets. Assume that \( C \) is closed and bounded. Let \( f : C \times P \to M \) be a definable function which is equi-continuous and pointwise bounded with respect to \( P \). Take \( p \in \overline{P} \). There exists a definable continuous curve \( \gamma : [0, \varepsilon] \to P \) such that \( \lim_{t \to 0^+} \gamma(t) = p \) and the definable family of functions \( \{g_t : C \to M\}_{0 < t < \varepsilon} \) defined by \( g_t(x) = f(x, \gamma(t)) \) is uniformly convergent.

Proof. The corollary follows from Corollary 4.2, Lemma 5.6, and Theorem 5.10. \( \square \)

Consider a parameterized function \( f : C \times P \to M \) which is equi-continuous with respect to \( P \). The following theorem claims that the projection image of the set at which \( f \) is discontinuous onto the parameter space \( P \) is of dimension smaller than \( \dim P \) when \( C \) is closed and bounded.

Theorem 5.12. Consider a DCULOAS structure \( \mathcal{M} = (M, <, +, 0, \ldots) \). Let \( C \) be a definable closed and bounded set and \( P \) be a definable set. Let \( \pi : C \times P \to P \) be the projection. Consider a definable function \( f : C \times P \to M \) which is equi-continuous with respect to \( P \). Set \( D = \{(x, q) \in C \times P \mid f \text{ is discontinuous at } (x, q)\} \). We have \( \dim \pi(D) < \dim P \).
Proof. Let $C$ and $P$ be definable subsets of $M^n$ and $M^m$, respectively. We first consider the set

$$S = \{(x, p) \in D \mid \exists U \subset M^m : \text{open box with } x \in U \text{ and } C \cap U = D_p \cap U\},$$

where the notation $D_p$ denotes the fiber $\{x \in C \mid (x, p) \in D\}$. We first demonstrate that $\dim \pi(S) < \dim P$.

Assume the contrary. There exists a point $(c, p) \in S$ such that $\dim \pi(S \cap W) = \dim P$ for all open box $W$ in $M^{m+n}$ containing the point $(c, p)$ by Lemma 4.3. Fix a sufficiently small open box $W$ containing the point $(c, p)$. Let $\tau : C \times P \to P \times C$ be the map defined by $\tau(x, p) = (p, x)$ and $\pi' : M^{m+n} \to M^n$ be the projection onto the first $n$ coordinates. Shrinking $W$ if necessary, there exists a definable cell decomposition of $W' = \tau(W)$ partitioning the definable sets $W' \cap (P \times C)$ and $S' = \tau(S \cap W)$ by [3, Theorem 4.2]. There exists a cell $E$ contained in $S'$ with $\dim \pi'(E) = \dim P$ by the assumption. Let $E_{\text{max}}$ be a cell of the maximum dimension among such cells.

Let $E'$ be a cell such that $\pi'(E') = \pi'(E_{\text{max}})$, $\dim E' > \dim E_{\text{max}}$ and $\overline{E'} \cap E_{\text{max}} \neq \emptyset$. We show that $E' \cap (P \times C)$ is an empty set. Assume the contrary. The cell $E'$ is contained in $P \times C$ because it is a cell of the cell decomposition partitioning the set $W' \cap (P \times C)$. Take $(p', c') \in \overline{E'} \cap E_{\text{max}}$. We obviously have $(p', c') \in P \times C$ because $E_{\text{max}} \subset S' \subset P \times C$. Since $(p', c')$ is an element of $S'$, there exists an open box $U'$ in $M^n$ containing the point $c'$ such that $C \cap U' = D_{p'} \cap U'$. We can take a point $d \in C \cap U'$ with $(p', d) \in E'$ because $(p', c') \in \overline{E'} \cap E_{\text{max}}$. Take an open box $V'$ in $M^m$ contained in $U'$ and containing the point $d$. We obviously have $C \cap V' = D_{p'} \cap V'$. It means that $(p', d) \in S'$. The cell $E'$ is contained in $S'$ because the cell decomposition partitions the set $S'$. We have $\dim E' > \dim E_{\text{max}}$, $\dim \pi'(E') = \dim \pi'(E_{\text{max}}) = \dim P$ and $E' \subset S'$. It is a contradiction to the definition of $E_{\text{max}}$. We have demonstrated that $E' \cap (P \times C)$ is an empty set.

We can take a point $(p_1, c_1) \in E_{\text{max}}$ such that intersection $(P \times C) \cap (V_1 \times U_1)$ is contained in $E_{\text{max}}$ for a sufficiently small open box $U_1$ in $M^m$ containing the point $c_1$ and a sufficiently small open box $V_1$ in $M^n$ containing the point $p_1$ by the previous claim and Lemma 2.4. It means that $(C \times P) \cap (U_1 \times V_1)$ is contained in $S$. Consider the restriction $g$ of $f$ to the set $(C \times P) \cap (U_1 \times V_1)$. The set of points at which $g$ is discontinuous is $D \cap (U_1 \times V_1)$, and $g$ is discontinuous everywhere because $S$ is contained in $D$. It contradicts to Proposition 2.2.7. We have demonstrated that $\dim \pi(S) < \dim P$.

We next demonstrate that $\dim \pi(D) < \dim P$. We lead to a contradiction assuming the contrary. Set $T = D \setminus \pi^{-1}(\pi(S))$. We have $\dim \pi(T) = \dim P$ by Proposition 2.2.5 because $\dim \pi(S) < \dim P$. There exists a point $(c, p) \in T$ such that $\dim \pi(T \cap W) = \dim P$ for all open box $W$ in $M^{m+n}$ containing the point $(c, p)$ by Lemma 4.3. Fix an arbitrary $\varepsilon > 0$. Since $f$ is uniformly equi-continuous with respect to $P$ by the assumption and Proposition 5.2, there exists $\delta > 0$ satisfying the following condition:

(1) $\forall q \in P, \forall x, x' \in C, |x - x'| < \delta \Rightarrow |f(x, q) - f(x', q)| < \varepsilon/3$.

Since $T \cap S = \emptyset$, there exists $c_1 \in C$ such that $|c - c_1| < \delta/2$ and $(c_1, p) \notin D$. There exists $\delta' > 0$ such that

(2) $\forall q \in P, |q - p| < \delta' \Rightarrow |f(c_1, q) - f(c_1, p)| < \varepsilon/3$.

because $f$ is continuous at $(c_1, p)$. 

Consider an arbitrary point \((c', p') \in C \times P\) with \(|c - c'| < \delta/2\) and \(|p - p'| < \delta'.\) We have \(|f(c_1, p) - f(c, p)| < \varepsilon/3\) by the inequality (1) because \(|c - c_1| < \delta/2.\) We also have \(|f(c', p') - f(c_1, p')| < \varepsilon/3\) by (1) because \(|c' - c_1| \leq |c' - c| + |c - c_1| < \delta.\) We get
\[
|f(c', p') - f(c, p)| \leq |f(c', p') - f(c_1, p')| + |f(c_1, p') - f(c_1, p)| + |f(c_1, p) - f(c, p)| \leq \varepsilon
\]
by the above inequalities together with the inequality (2). We have demonstrated that \(f\) is continuous at \((c, p)\). It is a contradiction to the condition that \((c, p) \in T \subset D.\) We have finished the proof of the theorem. □

6. Definable Tietze extension theorem and o-minimality

We treat the assertions satisfied in a DCULOAS structure in the previous sections. We consider a slightly different type of problem in this section. We consider whether a DCULOAS structure satisfying definable Tietze extension property is o-minimal or not.

**Definition 6.1.** A structure \(M = (M, \ldots)\) enjoys definable Tietze extension property if, for any positive integer \(n\), any definable closed subset \(A\) of \(M^n\) and any continuous definable function \(f : A \to M\), there exists a definable extension \(F : M^n \to M\) of \(f\).

We first prove the following lemma.

**Lemma 6.2.** Consider a definably complete expansion of a densely linearly ordered abelian group \(M = (M, <, +, 0, \ldots)\). If the structure \(M\) has a strictly monotone definable homeomorphism between a bounded open interval and an unbounded open interval, any two open intervals are definably homeomorphic and there exists a definable strictly increasing homeomorphism between them.

**Proof.** By the assumption, there exists a strictly monotone definable homeomorphism \(\varphi : I \to J\), where \(I\) is a bounded open interval and \(J\) is an unbounded open interval. We may assume that \(I = [0, u]\) for some \(u > 0.\) In fact, an open interval \([u_1, u_2]\) is obviously definably homeomorphic to \([0, u_2 - u_1]\). We may further assume that \(\varphi\) is strictly increasing because the map \(\tau : [0, u[ \to [0, u]\) defined by \(\tau(t) = u - t\) is a definable homeomorphism.

We next reduce to the case in which \(J = [0, \infty[.\) We have only three possibilities; that is \(J = [v, +\infty[, J = (-\infty, v[\) and \(J = M\) for some \(v \in M.\) In the first and second cases, we may assume that \(J = [0, \infty[\) because \(J = [v, +\infty[\) and \(J = (-\infty, v[\) are obviously definable homeomorphic to \([0, \infty[.\) In the last case, set \(u' = \varphi^{-1}(0)\). Then the restriction of \(\varphi\) to the open interval \([0, u'][\) is a definable homeomorphism between \([0, u'][\) and \([-\infty, 0[\). Hence, we can reduce to the second case. We have constructed a strictly increasing definable homeomorphism \(\varphi : [0, u[ \to [0, \infty[.\) We fix such a homeomorphism.

We next construct a definable strictly increasing homeomorphism between an arbitrary bounded open interval and \([0, \infty[.\) We may assume that the bounded interval is of the form \([0, v[\). We have nothing to do when \(v = u.\) When \(v < u,\) the map defined by \(\varphi(t + u - v) - \varphi(u - v)\) for all \(t \in [0, v[\) is a definable homeomorphism between \([0, v[\) and \([0, \infty[.\) When \(v > u,\) consider the map \(\psi : [0, v[ \to [0, \infty[\) given by \(\psi(t) = t\) for all \(t \leq v - u\) and \(\psi(t) = \varphi(t + u - v) + v - u\) for the other
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case. It is a desired definable homeomorphism. We have constructed a definable
homeomorphism between \([0, u]\) and all open intervals other than \(M\).

The remaining task is to construct a definable homeomorphism between \([0, u]\) and \(M\). There exists a strictly increasing definable homeomorphisms \(\psi_1 : [0, u/2] \to \] - \(\infty, 0]\) and \(\psi_2 : (u/2, 0, \infty\]. The definable map \(\psi : [0, u] \to M\) given by \(\psi(t) = \psi_1(t)\) for \(t < u/2\), \(\psi(t) = 0\) for \(t = u/2\) and \(\psi(t) = \psi_2(t)\) for \(t > u/2\) is a definable
homeomorphism. They are well-defined because \((M, +)\) is a divisible group by [8]
Proposition 2.2].

Lemma 6.3. Consider a uniformly locally o-minimal expansion of the second kind
of a densely linearly ordered abelian group \(\mathcal{M} = (M, <, +, 0, \ldots)\). If the structure \(\mathcal{M}\)
has a strictly monotone definable homeomorphism between a bounded open interval
and an unbounded open interval, it is o-minimal.

Proof. Since the map \(\tau : [0, u] \to [0, u]\) defined by \(\tau(t) = u - t\) is a definable
homeomorphism, we may assume that there exists a strictly decreasing definable homeomorphism \(\varphi : [0, u] \to [0, +\infty]\) for some \(u > 0\) by Lemma 6.2.

Let \(X\) be an arbitrary definable subset of \(M\). We show that it is a finite union
of points and open intervals.

We first consider the case in which \(X\) is bounded. We may assume that \(x > 0\)
for all \(x \in X\) by shifting the definable set \(X\) if necessary. Take \(N > 0\) with \(x < N\)
for all \(x \in X\). Consider the map \(\psi : [0, \infty] \times [0, u] \to [0, u]\) defined by
\[
\psi(x, y) = \varphi^{-1}(x + \varphi(y)).
\]
Set \(Z = \{(z, y) \in [0, u] \times [0, u] \mid z = \psi(x, y)\text{ for some }x \in X\}\). The notation \(Z_y\)
denotes the set \(\{z \in M \mid (z, y) \in Z\}\) for all \(y \in M\). Since \(\mathcal{M}\) is a uniformly
locally o-minimal structure of the second kind, there exists \(c > 0\) and \(d > 0\) such
that, for any \(0 < y < d\), the intersection \(Z_y \cap [-c, c]\) is a finite union of points
and open intervals. We may assume that \(c < u\) taking a smaller \(c\) if necessary. Take
\(0 < y < c\). For all \(x \in X\), we have
\[
\psi(x, y) = \varphi^{-1}(x + \varphi(y)) < \varphi^{-1}(x + \varphi(c)) < \varphi^{-1}(\varphi(c)) = c
\]
because \(x > 0\) when \(x \in X\). It means that \(\psi(X, y)\) is contained in the open interval
\([0, c]\). Fix a sufficiently small \(y > 0\) with \(y < \min\{c, d\}\). We have \(\psi(X, y) =
Z_y \cap [-c, c]\), which is a finite union of points and open intervals. Since the map
\(\psi(\cdot, y)\) is a definable homeomorphism for the fixed \(y\), the set \(X\) itself is a finite
union of points and open intervals.

We next consider the case in which \(X\) is unbounded. Set \(X_+ = \{x \in X \mid x > 0\}\)
and \(X_- = \{x \in X \mid x < 0\}\). Consider the sets \(\varphi^{-1}(X_+)= \varphi^{-1}(-X_-)\). They
are bounded definable subsets of \(M\), and they are finite unions of points and open
intervals. Therefore, \(X\) itself is a finite union of points and open intervals because
\(\varphi\) is definable strictly decreasing homeomorphism. We have demonstrated that
\(\mathcal{M}\) is o-minimal.

Definition 6.4. Consider an expansion of a densely linearly ordered abelian group
\(\mathcal{M} = (M, <, +, 0, \ldots)\). It is called archimedean if, for any positive \(a, b \in M\), there
exists a positive integer \(n\) with \(na > b\). Here, \(na\) denotes the sum of \(n\) copies of \(a\).

The following theorem is the last main theorem of this paper. Its proof is inspired
by [2] Example 3.4].
Theorem 6.5. Consider an archimedean DCCUSOA structure \( \mathcal{M} = (M, <, +, 0, \ldots) \). If the structure \( \mathcal{M} \) enjoys definable Tietze extension property, the structure \( \mathcal{M} \) is \( \alpha \)-minimal and it has a strictly monotone definable homeomorphism between a bounded open interval and an unbounded open interval.

Proof. We have only to construct a strictly monotone definable homeomorphism between a bounded open interval and an unbounded open interval by Lemma 6.3. Fix \( c > 0 \). Set \( X = \{(x, y) \in M^2 \mid y < 0 \text{ or } x \geq c \} \). Consider the definable continuous map \( f : X \to M \) given by \( f(x) = -y \) if \( x \leq 0 \) and \( f(x, y) = y \) otherwise. Since the structure \( \mathcal{M} \) enjoys definable Tietze extension property by the assumption, the function \( f \) has a definable continuous extension \( F : M^2 \to M \).

Fix \( \varepsilon > 0 \). The map \( g_y : [0, c] \to M \) given by \( g_y(x) = F(x, y) \) are uniformly continuous for all \( y \in M \) by Corollary 5.3. Therefore there exists \( \delta_y > 0 \) such that the condition \( |x - x'| < \delta_y \) implies that \( |F(x, y) - F(x', y)| < \varepsilon \) for all \( x, x' \in [0, c] \).

It means that the definable function \( \varphi : M_{>0} \to M_{>0} \) defined by

\[
\varphi(y) = \sup \{0 < \delta \leq c \mid \forall x, x' \in [0, c], \ |x - x'| < \delta \Rightarrow |F(x, y) - F(x', y)| < \varepsilon \}
\]

is well-defined.

The infimum \( \inf \varphi(M_{>d}) \) always exists for any \( d > 0 \) because \( \mathcal{M} \) is definably complete. We prove that

\[
\inf \varphi(M_{>d}) = 0.
\]

We lead to a contradiction assuming that \( \inf \varphi(M_{>d}) > 0 \) for some \( d > 0 \). Take a positive \( \mu > 0 \) with \( \mu < \inf \varphi(M_{>d}) \). We have \( \varphi(y) > \mu \) for all \( y > d \). There exists a positive integer \( n \) with \( n\mu > c \) because \( \mathcal{M} \) is archimedean. Set \( x_i = \frac{i}{n}c \) for all \( 0 \leq i \leq n \). They are well-defined because \((M, +)\) is a divisible group by [8, Proposition 2.2]. We have \( |x_i - x_{i-1}| = \frac{1}{n} < \mu < \varphi(y) \) for all \( y > d \) and \( 1 \leq i \leq n \).

For any \( y > d \), we get

\[
2y = |F(c, y) - F(0, y)| \leq \sum_{i=1}^{n} |F(x_i, y) - F(x_{i-1}, y)| < n\varepsilon
\]

by the definition of \( \varphi(y) \). It is a contradiction because \( y \) is an arbitrary element with \( y > d \). We have demonstrated that \( \inf \varphi(M_{>d}) = 0 \).

Fix \( d > 0 \). Since \( \varphi(M_{>d}) \) is a set definable in a locally \( \alpha \)-minimal structure and \( \inf \varphi(M_{>d}) = 0 \), there exists \( u > 0 \) such that the open interval \([0, u]\) is contained in \( \varphi(M_{>d}) \). Consider the definable function \( \iota : [0, u] \to M_{>0} \) given by

\[
\iota(t) = \inf \{y \in M_{>d} \mid \varphi(y) = t\}.
\]

We define \( \psi : [0, u] \to M_{>0} \) as follows: Set \( \psi(t) = \iota(t) \) when \( t = \varphi(\iota(t)) \). Otherwise, the set \( T_i = \{y \in M_{>d} \mid y > \iota(t), \ \forall y', \ i(t) < y' < y \Rightarrow \varphi(y') = t\} \) is not empty because of local \( \alpha \)-minimality. The supremum \( e(t) = \sup T_i \in M \cup \{+\infty\} \) exists by definable completeness. Set \( \psi(t) = \frac{\iota(t) + e(t)}{2} \) when \( e(t) < \infty \), and set \( \psi(t) = \iota(t) + c \) otherwise. We have \( \varphi(\psi(t)) = t \) by the definition.

For any \( 0 < u' < u \), the restriction of \( \psi \) to the open interval \([0, u']\) is unbounded. Assume the contrary. There exists \( 0 < u' < u \) and \( v > 0 \) such that \( \psi([0, u'])\) is contained in \([d, v]\). Since the closed box \([0, c] \times [d, v]\) is bounded, there exists \( \delta > 0 \) such that the following condition holds true by Corollary 5.3.

\[
\forall (x, y), (x', y') \in [0, c] \times [d, v], \ |(x, y) - (x', y')| < \delta \Rightarrow |F(x, y) - F(x', y')| < \varepsilon.
\]
It implies that $\varphi(y) \geq \tilde{\delta}$ for all $d \leq y \leq v$. We may assume that $\tilde{\delta} < u'$ taking a smaller $\delta$ if necessary. Take $t > 0$ smaller than $\delta$. We have $d \leq \psi(t) \leq v$ and $\varphi(\psi(t)) = t < \tilde{\delta}$. Contradiction. We have proven that the restriction of $\psi$ to the open interval $]0, u']$ is unbounded for any $0 < u' < u$.

Taking a smaller $u > 0$ if necessary, we may assume that the function $\psi$ is continuous and monotone by Theorem 3.4 and Lemma 3.5. Since the restriction of $\psi$ to the open interval $]0, u']$ is unbounded for any $0 < u' < u$, it is strictly decreasing. The restriction $\psi$ to the open interval $]0, u/2]$ is a strictly monotone definable homeomorphism between the bounded open interval $]0, u/2]$ and the unbounded open interval $]\psi(u/2), \infty[$.

□

We have only proved that an archimedean DCULOAS structure which enjoys definable Tietze extension property is o-minimal in Theorem 6.5. The following conjecture is still open.

Conjecture. A DCULOAS structure is o-minimal when it enjoys definable Tietze extension property.

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