\section*{Introduction}

In the context of Lie groupoid theory, there are some geometric structures of wide interest (e.g. symplectic and Poisson structures). The compatibility of these structures with the groupoid multiplication can be described infinitesimally as morphisms between special kinds of Lie algebroids called $\mathcal{VB}$-algebroids \cite{5,14}. In this work, we build on \cite{8} to study $\mathcal{VB}$-algebroid morphisms within the framework of representations up to homotopy \cite{11}.

Roughly, a $\mathcal{VB}$-algebroid is a Lie algebroid object in the category of vector bundles. One particularly important example of a $\mathcal{VB}$-algebroid is the tangent prolongation $(TA \to TM; A \to M)$ of a Lie algebroid $A \to M$.

It was shown in \cite{8} that isomorphism classes of $\mathcal{VB}$-algebroids correspond to isomorphism classes of representations up to homotopy on 2-term graded vector bundles \cite{11}. Under this correspondence, the tangent prolongation corresponds to the adjoint representation. In
this work, we refine [8] by showing that morphisms between \( \mathcal{VB} \)-algebroids correspond to morphisms between the associated representations (see Theorem 4.11). It turns out that, the correspondence in [8] is an equivalence of categories. As a direct consequence, one can re-interpret IM-2-forms [6] and Lie bialgebroids [14] as morphisms between the adjoint and the coadjoint representations. Our main motivation comes from the study of multiplicative foliations and distributions on Lie groupoids [7, 9, 10]. These are structures which have drawn some attention in connection to geometric quantization of Poisson manifolds [9] and also in a modern approach to Cartan’s work on Lie pseudogroups [7]. For a Lie groupoid \( \mathcal{G} \), the infinitesimal counterpart of a multiplicative foliation on \( \mathcal{G} \) is an ideal object in its Lie algebroid \( A \) (see [9, 10]), which can be seen as a \( \mathcal{VB} \)-subalgebroid of both the tangent prolongation and the tangent double \( (TA \to A; TM \to M) \). One can expect that, with the proper notion of adjoint representation for Lie algebroids provided by [1], multiplicative foliations will correspond to subrepresentations of the adjoint representation. This is true when \( \mathcal{G} \) is a Lie group and \( A \) is its Lie algebra since a multiplicative foliation on \( \mathcal{G} \) is in this case the same as an ideal in the Lie algebra, and hence a subrepresentation of the adjoint representation [13]. In general, this is only one part of the picture. This phenomenon is due to the presence of some additional structure coming from the tangent double of the manifold of units. The precise correspondence between ideal objects on a Lie algebroid and subrepresentations is given by Theorem 5.8.

We point out that our results on multiplicative foliations belong to the more general context of sub-objects of double vector bundles. In this context, Theorem 3.13 gives a classification of double vector subbundles of a general double vector bundle using the affine structure on the space of its decompositions. Applied to the tangent double, this result recovers the correspondence between linear distributions on a vector bundle and the so-called Spencer operators defined in [7] (see Proposition 5.9). The general result relating \( \mathcal{VB} \)-subalgebroids and subrepresentations is given by Proposition 5.6. This result is used in \( \S \)5.2.1 and \( \S \)5.2.2 to study linear foliations on a vector bundle and distributions compatible with the Lie bracket on a Lie algebroid (the infinitesimal counterpart of a multiplicative distribution), respectively, leading to an extension of some of the results in [7, 10].

This paper is organized as follows. In section 2 we present the background material on representations up to homotopy, including the main definitions and examples. In section 3 we discuss double vector bundles and their duals as well as double vector subbundles. In section 4 we state and prove the main result of this work. This theorem establishes a correspondence between morphisms between decomposed \( \mathcal{VB} \)-algebroids and morphisms between the corresponding 2-terms representations up to homotopy. As a result, the category of 2-term representations up to homotopy of a Lie algebroid \( A \) is equivalent to the category of \( \mathcal{VB} \)-algebroids with side algebroid \( A \). In section 5, we study \( \mathcal{VB} \)-subalgebroids within the framework of representations up to homotopy. In order to have a self-contained presentation, we have included one appendix, where we show that the dictionary between \( \mathcal{VB} \)-algebroids and 2-term representations up to homotopy is compatible with dualizations in each category.

Acknowledgements: The authors would like to thank Henrique Bursztyn and Benoit Dherin for useful comments that have improved the presentation of this work. Drummond acknowledges support of CAPES-FCT at IST-Lisboa, where part of this work was developed. Jotz was supported by the Dorothea-Schlözer program of the University of Göttingen, and a fellowship for prospective researchers of the Swiss NSF (PBELP2_137534) for research conducted at UC Berkeley, the hospitality of which she is thankful for. Ortiz would like to thank IMPA (Rio de Janeiro) for a 2012-Summer Postdoctoral Fellowship and its hospitality while part of this work was carried out.
2. Background: Representations up to homotopy of Lie algebroids

We recall here some background material on representations up to homotopy. We follow mostly [1].

2.1. Definition and examples. Let \( E \to M \) be a vector bundle and \( V = \bigoplus_{k \in \mathbb{Z}} V_k \) a graded vector bundle. The space of \( V \)-valued \( E \)-differential forms, \( \Omega(E;V) := \Gamma(\wedge^* E^* \otimes V) \), has a grading given by

\[
\Omega(E;V)_k = \bigoplus_{i+j=k} \Gamma(\wedge^i E^* \otimes V_j)
\]

and a natural (graded-)module structure over the algebra \( \Omega(E) := \Gamma(\wedge^* E^*) \).

If \( W = \bigoplus_{k \in \mathbb{Z}} W_k \) is a second graded vector bundle, \( \text{Hom}(V,W) \) is the graded vector bundle whose degree \( k \) part is

\[
\text{Hom}(V,W)_k = \bigoplus_{i \in \mathbb{Z}} \text{Hom}(V_i, W_{i+k}).
\]

Let now \((A, [\cdot, \cdot], \rho_A)\) be a Lie algebroid over \( M \).

Definition 2.1. A homogeneous \( A \)-connection \( \nabla \) on the graded vector bundle \( V = \bigoplus_{k \in \mathbb{Z}} V_k \) is an \( A \)-connection \( \nabla : \Gamma(A) \times \Gamma(V) \to \Gamma(V) \) such that \( \nabla_a \) preserves \( \Gamma(V_k) \), for all \( k \in \mathbb{Z} \) and \( a \in \Gamma(A) \). Equivalently, an \( A \)-connection on \( V \) is given by a family \( \{\nabla^k\}_{k \in \mathbb{Z}} \), where each \( \nabla^k \) is an \( A \)-connection on \( V_k \).

From now on, we assume that all connections on graded vector bundles are homogeneous.

Definition 2.2. Let \( V \) be a graded vector bundle. A representation up to homotopy of \( A \) on \( V \) is a degree one map \( D : \Omega(A;V)_* \to \Omega(A;V)_{*+1} \) such that \( D^2 = 0 \) and

\[
D(\alpha \wedge \omega) = d_A \alpha \wedge \omega + (-1)^{[\alpha]} \alpha \wedge D(\omega), \quad \text{for } \alpha \in \Omega(A), \omega \in \Omega(A;V),
\]

where \( d_A : \Omega(A) \to \Omega(A) \) is the Lie algebroid differential.

Definition 2.3. A morphism between two representations up to homotopy of \( A \) is a degree zero \( \Omega(A) \)-linear map

\[
\Omega(A;V) \to \Omega(A;W)
\]

which intertwines the differentials \( D_V \) and \( D_W \). We denote it by \((A,V) \to (A,W)\).

In this paper we are mostly concerned with representations up to homotopy on g.v.b's \( V \) concentrated in degree 0 and 1. These are called 2-term g.v.b's and the representations up to homotopy of \( A \) on 2-term g.v.b's form a category which we denote by \( \text{Rep}^2(A) \). We denote by \( \text{Rep}^2(A) \) the set of isomorphism classes of objects of \( \text{Rep}^2(A) \).

For a 2-term representation up to homotopy \( V \in \text{Rep}^2(A) \), the derivation property \( \text{[2.1]} \) implies that the differential \( D : \Omega(A;V) \to \Omega(A;V) \) is determined by

1. a bundle map \( \vartheta : V_0 \to V_1 \);
2. an \( A \)-connection \( \nabla \) on \( V \) compatible with \( \vartheta \) (i.e. \( \vartheta \circ \nabla^0 = \nabla^1 \circ \vartheta \));
3. an element \( K \in \Omega^2(A, \text{End}(V)_{-1}) = \Omega^2(A, \text{Hom}(V_1, V_0)) \) such that \( d_{\nabla,\text{End}} K = 0 \) and the diagram below commutes

\[
\begin{array}{ccc}
V_0 & \xrightarrow{\vartheta} & V_1 \\
\bigcap & \searrow & \nearrow \\
V_0 & \xrightarrow{\nabla^0} & V_1 \\
R_{\nabla^0} & \uparrow -K & R_{\nabla^1} \\
V_0 & \xrightarrow{\vartheta} & V_1 \\
\end{array}
\]

where \( R_{\nabla^i} \) is the curvature of \( \nabla^i \), for \( i = 0, 1 \). We say that \((\vartheta, \nabla, K)\) are the structure operators for \( V \in \text{Rep}^2(A) \).
We refer to [1] for a detailed exposition of the correspondence \( \mathcal{D} \mapsto (\partial, \nabla, K) \) (pointing out that our sign convention for \( K \) is different from the one in [1]).

For \( V, W \in \mathbb{R}^\mathbb{P}(A) \), a morphism \( (A, V) \to (A, W) \) is determined by a triple \( (\phi_0, \phi_1, \Phi) \), where \( \phi_0 : V_0 \to W_0, \phi_1 : V_1 \to W_1 \) are bundle maps and \( \Phi \in \Omega^1(A; \text{Hom}(V_1, W_0)) \), satisfying
\[
(2.2) \quad \phi_1 \circ \partial_V = \partial_W \circ \phi_0,
\]
(2.3)

\[
\nabla^\text{Hom}_a(\phi_0, \phi_1) = (\Phi_a \circ \partial_V, \partial_W \circ \Phi_a) \quad \text{for all } a \in \Gamma(A)
\]
and
\[
(2.4) \quad d_{\nabla^\text{Hom}} \Phi = \phi_0 \circ K_V - K_W \circ \phi_1,
\]
where \( \nabla^\text{Hom} \) is the \( A \)-connection on \( \text{Hom}(V, W) \) (see [1] for more details).

In the following, given \( E, E' \) vector bundles over \( M \), we denote by \( E_{[0]} \oplus E'_{[1]} \) the graded vector bundle having \( E \) in degree 0 and \( E' \) in degree 1.

**Example 2.4** (Double representation). Let \( B \to M \) be a vector bundle and consider the graded vector bundle \( V = B_{[0]} \oplus B_{[1]} \). Any connection \( \nabla : \Gamma(TM) \times \Gamma(B) \to \Gamma(B) \) induces a representation up to homotopy of \( TM \) on \( V \) by taking \( \partial = \text{id}_B \), \( \nabla^0 = \nabla^1 = \nabla \) and \( K = -R_\nabla \), the curvature of \( \nabla \). The isomorphism class of this representation does not depend on the choice of \( \nabla \) and is called the double representation of \( TM \) on \( B \). We denote it by \( \mathcal{D}(B) \in \mathbb{R}^2(TM) \) and the representation itself by \( \mathcal{D}_\nabla(B) \in \mathbb{R}^2(A) \).

**Example 2.5** (Adjoint representation). Let \( (A, [-.,.]_A, \rho_A) \) be a Lie algebroid over \( M \). Any connection \( \nabla : \Gamma(TM) \times \Gamma(A) \to \Gamma(A) \) on \( A \) induces a representation up to homotopy of \( A \) on \( V = A_{[0]} \oplus TM_{[1]} \) in the following manner. The map \( \partial \) is just the anchor \( \rho_A : A \to TM \). The \( A \)-connection \( \nabla^\text{bas} \) on \( V \) (called the basic connection) has degree zero and degree one parts given by
\[
\nabla^\text{bas} : \quad \Gamma(A) \times \Gamma(A) & \longrightarrow \Gamma(A) \\
(a, b) & \longmapsto [a, b]_A + \nabla_{\rho_A(b)} a.
\]
and
\[
\nabla^\text{bas} : \quad \Gamma(A) \times \Gamma(TM) & \longrightarrow \Gamma(TM) \\
(a, X) & \longmapsto [\rho_A(a), X]_A + \rho_A(\nabla_X a),
\]
respectively. The element \( K \) is the basic curvature \( R^\text{bas}_\nabla \in \Omega^2(A, \text{Hom}(TM, A)) \) defined by
\[
R^\text{bas}_\nabla(a, b)(X) = \nabla_X [a, b] - [\nabla X a, b] - [a, \nabla_X b] + \nabla^\text{bas}_{aX} b - \nabla^\text{bas}_{bX} a.
\]
As before, the isomorphism class of this representation does not depend on the choice of \( \nabla \) and it is called the adjoint representation of \( A \). We denote it by \( \text{ad} \in \mathbb{R}^2(A) \) and the representation itself by \( \text{ad}_\nabla \in \mathbb{R}^2(A) \).

Given a 2-term representation \( V \in \mathbb{R}^2(A) \) with structure operators \( (\partial, \nabla, K) \) of \( A \) on \( V = V_0 \oplus V_1 \), its dual is the representation \( V^\top \in \mathbb{R}^2(A) \), where \( V_0^\top = V_1^*, V_1^\top = V_0^* \), with structure operators given by
\[
(2.5) \quad \partial_{V^\top} = \partial^*, \nabla^{V^\top} = \nabla^* \quad \text{and} \quad K_{V^\top} = -K^*
\]
where \( \nabla^* \) is the \( A \)-connection dual to \( \nabla \), given by
\[
(2.6) \quad \langle \nabla_a^* \xi, v \rangle + \langle \xi, \nabla_a v \rangle = L_{\rho_A(a)}(\xi, v), \quad \forall v \in \Gamma(V), \xi \in \Gamma(V^\top).
\]

**Example 2.6** (Coadjoint representation). The coadjoint representation is the representation of \( A \) on \( T^* M_{[0]} \oplus A^*_{[1]} \) dual to the adjoint representation. It is denoted by \( \text{ad}^\top(A) \in \mathbb{R}^2(A) \).
2.2. Pullbacks. We define here morphisms between 2-term representations up to homotopy of different Lie algebroids. Let \((A', \iota_{A'}, \rho_{A'})\) be another Lie algebroid over \(M\) and \(T : A \to A'\) a Lie algebroid morphism over \(\text{id}_M\). Choose \(W \in \text{Rep}^2(A')\) with structure operators \((\partial, \nabla, K)\).

Define \(\nabla^T : \Gamma(A) \times \Gamma(W) \to \Gamma(W)\) to be the \(A\)-connection given by

\[
\nabla^T_{a} w := \nabla_{T(a)} w
\]

for \(a \in \Gamma(A)\) and \(w \in \Gamma(W)\) and \(T^*K \in \Omega^2(A; \text{Hom}(W_1, W_0))\) by

\[
T^*K(a_1, a_2) = K(T(a_1), T(a_2)), \quad (a_1, a_2) \in A \times_M A.
\]

**Lemma 2.7.** The triple \((\partial, \nabla^T, T^*K)\) defines structure operators for a representation up to homotopy of \(A\) on \(W\) which is called the pullback of \(W\) by \(T\) and it is denoted by \(T^*W \in \text{Rep}^2(A)\).

**Proof.** We leave the details to the reader. \(\square\)

**Example 2.8.** If \(T\) is the inclusion of a Lie subalgebroid \(A \to A'\), the pullback \(T^*W\) is just the restriction of the representation to \(A\).

The usefulness of taking pullbacks is that it allows one to define morphisms between representations up to homotopy of different algebroids.

**Definition 2.9.** Let \(W \in \text{Rep}^2(A')\) and \(V \in \text{Rep}^2(A)\) be representations up to homotopy. We define a morphism \((A, V) \Rightarrow (A', W)\) over a Lie algebroid morphism \(T : A \to A'\) to be an usual morphism \((A, V) \Rightarrow (A, T^*W)\) (as given in Definition 2.3).

**Remark 2.10.** The pullback operation can be defined for arbitrary representations up to homotopy. It was already defined in this generality for representations up to homotopy of Lie groupoids in \([8]\). Also, the pullback can be extended to morphisms and we get a functor \(T^1 : \text{Rep}^2(A') \to \text{Rep}^2(A)\).

3. DOUBLE VECTOR BUNDLES.

We briefly recall some definitions regarding double vector bundles, their sections and morphisms. We refer to \([11]\) for a more detailed treatment (see also \([8]\) for a treatment closer to ours). We also present a result classifying subbundles of double vector bundles.

3.1. Preliminaries.

**Definition 3.1.** A double vector bundle (DVB) is a commutative square

\[
\begin{array}{ccc}
D & \xrightarrow{q_B^D} & B \\
\downarrow{q_A^D} & & \downarrow{q_B} \\
A & \xrightarrow{q_A} & M
\end{array}
\]

satisfying the following three conditions:

DV1. all four sides are vector bundles;
DV2. \(q_B^D\) is a vector bundle morphism over \(q_A\);
DV3. \(+ : D \times_B D \to D\) is a vector bundle morphism over \(+ : A \times_M A \to A\), where \(+\) is the addition map for the vector bundle \(D \to B\).

Given a DVB \((D; A, B; M)\), the vector bundles \(A\) and \(B\) are called the side bundles. The zero sections are denoted by \(0^A : M \to A\), \(0^B : M \to B\), \(^a0 : A \to D\) and \(^b0 : B \to D\). Elements of \(D\) are written \((d; a, b; m)\), where \(d \in D\), \(m \in M\) and \(a = q_A^D(d) \in A_m\), \(b = q_B^D(d) \in B_m\).
The core $C$ of a DVB is the intersection of the kernels of $q_A^D$ and $q_B^D$. It has a natural vector bundle structure over $M$, the projection of which we call $q_C : C \to M$. The inclusion $C \to D$ is usually denoted by

$$C_m \ni c \mapsto \tau \in (q_A^D)^{-1}(0_m^A) \cap (q_B^D)^{-1}(0_m^B).$$

**Definition 3.2.** Let $(D; A, B; M)$ and $(D'; A', B'; M)$ be two double vector bundles. A DVB morphism $(F; F_{\text{ver}}, F_{\text{hor}}; f)$ from $D$ to $D'$ is a commutative cube

![Diagram](https://via.placeholder.com/150)

where all the faces are vector bundle morphisms.

Given a DVB morphism $(F; F_{\text{ver}}, F_{\text{hor}}; f)$, its restriction to the core bundles induces a vector bundle morphism $F_c : C \to C'$. In the following, we are mainly interested in DVB morphisms where $f = \text{id}_M : M \to M$. In this case, we omit the reference to $f$ and denote a DVB morphism by $(F; F_{\text{ver}}, F_{\text{hor}})$.

Given a DVB $(D; A, B; M)$, the space of sections $\Gamma(B, D)$ is generated as a $C^\infty(B)$-module by two distinguished classes of sections (see [12]), the linear and the core sections which we now describe.

**Definition 3.3.** For a section $c : M \to C$, the corresponding core section $\hat{c} : B \to D$ is defined as

$$\hat{c}(b_m) = a_0 b_m + c(m), \ m \in M, \ b_m \in B_m.$$

We denote the space of core sections by $\Gamma_c(B, D)$.

**Definition 3.4.** A section $\mathcal{X} \in \Gamma(B, D)$ is called linear if $\mathcal{X} : B \to D$ is a bundle morphism from $B \to M$ to $D \to A$. The space of linear sections is denoted by $\Gamma_\ell(B, D)$.

The space of linear sections is a locally free $C^\infty(M)$-module (see e.g. [3]). Hence, there is a vector bundle $\hat{A}$ over $M$ such that $\Gamma_\ell(B, D)$ is isomorphic to $\Gamma(\hat{A})$ as $C^\infty(M)$-modules. Note that for a linear section $\mathcal{X}$, there exists a section $\mathcal{X}_0 : M \to A$ such that $q_B^D \circ \mathcal{X} = \mathcal{X}_0 \circ q_B$. The map $\mathcal{X} \mapsto \mathcal{X}_0$ induces a short exact sequence of vector bundles

$$0 \to B^* \otimes C \to \hat{A} \to A \to 0,$$

where for $T \in \Gamma(B^* \otimes C)$, the corresponding section $\hat{T} \in \Gamma(\hat{A}, D)$ is given by

$$\hat{T}(b_m) = a_0 b_m + T(b_m).$$

We call splittings $h : A \to \hat{A}$ of the short exact sequence (3.2) horizontal lifts.

**Example 3.5.** Let $A, B, C$ be vector bundles over $M$ and consider $D = A \oplus B \oplus C$. With the vector bundle structures $D = q_A^D(B \oplus C) \to A$ and $D = q_B^D(A \oplus C) \to B$, one has that $(D; A, B; M)$ is a DVB called the trivial DVB with core $C$. The core sections are given by

$$b_m \mapsto (0_m^A, b_m, c(m)), \ m \in M, \ b_m \in B_m, \ c \in \Gamma(C).$$

The space of linear sections $\Gamma_\ell(B, D)$ is naturally identified with $\Gamma(A) \oplus \Gamma(B^* \otimes C)$ via

$$(a, T) : b_m \mapsto (a(m), b_m, T(b_m)), \ m \in M, \ b_m \in B_m, \ T \in \Gamma(B^* \otimes C), \ a \in \Gamma(A).$$
The canonical inclusion $\Gamma(A) \hookrightarrow \Gamma(A,B,D)$ is a horizontal lift.

Let $A',B',C'$ be another triple of vector bundles over $M$ and consider the corresponding trivial DVB with core $C'$, $D' = A' \oplus B' \oplus C'$. Any DVB morphism $(F; F_{\text{ver}}, F_{\text{hor}})$ from $D$ to $D'$ is given by

$$(a,b,c) \mapsto (F_{\text{ver}}(a), F_{\text{hor}}(b), F_c(c) + \Phi_a(b))$$

where $F_c : C \to C'$ is a vector bundle morphism and $\Phi \in \Gamma(A^* \otimes B^* \otimes C')$.

A decomposition for a DVB $(D; A, B; M)$ is an isomorphism $\sigma$ of DVBs from the trivial DVB with core $C$ to $D$ covering the identities on the side bundles $A, B$ and inducing the identity on the core $C$. The space of decompositions for $D$ will be denoted by $\text{Dec}(D)$. We recall now how this is an affine space over $\Gamma(A^* \otimes B^* \otimes C)$. Given an element $\Phi \in \Gamma(A^* \otimes B^* \otimes C)$, consider the DVB morphism

$$I_{\Phi} : A \oplus B \oplus C \longrightarrow A \oplus B \oplus C$$

$$(a,b,c) \mapsto (a,b,c + \Phi_a(b))$$

obtained from (3.5) by taking $F_{\text{ver}}, F_{\text{hor}}, F_c$ to be the identity morphisms. For a decomposition $\sigma$,

$$\Phi \cdot \sigma := \sigma \circ I_{\Phi}$$

defines the affine structure on $\text{Dec}(D)$.

**Remark 3.6.** The space of horizontal lifts as well is affine over $\Gamma(A^* \otimes B^* \otimes C)$ (this follows directly from the definition of horizontal lifts). There is a natural one-to-one correspondence between decompositions and horizontal lifts for $D$. Concretely, given a horizontal lift $h$, the decomposition $\sigma_h : A \oplus B \oplus C \to D$ is given by

$$\sigma_h(a_m, b_m, c_m) = h(a)(b_m) + (a^0)_h + (c^m)_h,$$

where $m \in M$ and $a \in \Gamma(A)$ is any section with $a(m) = a_m$. Conversely, given a decomposition $\sigma : A \oplus B \oplus C \to D$, the map $h_\sigma : \Gamma(A) \to \Gamma(\ell(B,D))$,

$$h_\sigma(a)(b_m) = \sigma(a(m), b_m, 0^C_m), \quad m \in M,$$

is a horizontal lift. The map $h \mapsto \sigma_h$ and its inverse $\sigma \mapsto h_\sigma$ are affine.

**Example 3.7.** For a vector bundle $B \to M$,

$$\begin{align*}
TB & \longrightarrow B \\
\downarrow & \quad \downarrow \\
TM & \longrightarrow M
\end{align*}$$

is a DVB with core bundle $B \to M$. The core section corresponding to $b \in \Gamma(B)$ is the vertical lift $b^\uparrow : B \to TB$. One has that

$$b^\uparrow(\ell_\psi) = \langle \psi, b \rangle \circ q_B \quad \text{and} \quad b^\uparrow(f \circ q_B) = 0,$$

where $\ell_\psi$, $f \circ q_B \in C^\infty(B)$ are the linear function and the pullback function corresponding to $\psi \in \Gamma(B^*)$ and $f \in C^\infty(M)$, respectively. An element of $\Gamma(\ell(B,TB))$ is called a linear vector field. It is well-known (see e.g. [11]) that a linear vector field $X : B \to TB$ covering $x : M \to TM$ corresponds to a derivation $L : \Gamma(B^*) \to \Gamma(B^*)$ having $x$ as its symbol. The precise correspondence is given by

$$X(\ell_\psi) = L_x(\psi) \quad \text{and} \quad X(f \circ q_B) = L_x(f) \circ q_B.$$
Example 3.8. Let \( A \to M \) be a vector bundle and consider \( TA \to TM \) as the horizontal side bundle of the tangent double,

\[
\begin{array}{ccc}
TA & \longrightarrow & TM \\
\downarrow & & \downarrow \\
A & \longrightarrow & M.
\end{array}
\]

For any \( a \in \Gamma(A) \), \( Ta : TM \to TA \) is a linear section covering \( a \) itself. Yet, the map \( a \mapsto Ta \) splits \eqref{3.2} only at the level of sections, as it fails to be \( C^\infty(M) \)-linear. The choice of a connection \( \nabla \) on \( A \) restores the \( C^\infty(M) \)-linearity and induces a horizontal lift by

\[
(3.9) \quad h(a)(x) = Ta(x) + (T0(x) - \nabla_x a), \quad x \in TM, \ a \in \Gamma(A).
\]

The associated decomposition \( \sigma_h \in \text{Dec}(TA) \) coincides with the one induced by \( \nabla \) as in Example 3.7.

3.2. Dualization of DVBs. Given a DVB \((D; A, B; M)\) with core \( C \), its horizontal dual is the DVB

\[
(3.10) \quad D^*_B \overset{p_B}{\longrightarrow} B \\
\downarrow \sigma_{hor}^* \quad \downarrow q_B \\
C^* \overset{q_{C^*}}{\longrightarrow} M,
\]

where \( p_B : D^*_B \to B \) is the dual of \( q_B^D : D \to B \) and, for \( \xi \in (p_B)^{-1}(b_m) \),

\[
(3.11) \quad \langle p_{C^*}^\text{hor}(\xi), c_m \rangle = \langle \xi, b_m + \frac{\partial}{\partial t} |_{t=0} c_m \rangle.
\]

The core bundle of \( D^*_B \) is \( A^* \to M \). Similarly, the vertical dual is the DVB

\[
(3.12) \quad D^*_A \overset{p_A^\text{ver}}{\longrightarrow} C^* \\
\downarrow \sigma_{C^*} \quad \downarrow q_{C^*} \\
A \overset{q_A}{\longrightarrow} M
\]

with core \( B^* \to M \).

In the following, we are mostly interested in the horizontal dual. For \( \psi \in \Gamma(A^*) \), the corresponding core section \( \hat{\psi} \in \Gamma_c(B, D^*_B) \) is just \( (q_B^D)^* \psi \). In particular,

\[
(3.13) \quad \langle \hat{\psi}, \hat{c} \rangle = 0
\]

for \( c \in \Gamma(C) \) and

\[
(3.14) \quad \langle \hat{\psi}, h(a) \rangle = \langle \psi, a \rangle \circ q_B
\]

for \( a \in \Gamma(A) \) and any horizontal lift \( h : A \to \hat{A} \).

Given a decomposition \( \sigma : A \oplus B \oplus C \to D \), the inverse of its dual over \( B \), \( (\sigma_B^*)^{-1} : B \oplus C^* \oplus A^* \to D^*_B \), is a decomposition for \( D^*_B \).

Example 3.9. Let \( B \to M \) be a vector bundle and consider its tangent double \((TB; TM, B; M)\). The projection of the cotangent bundle \( T^*B \) to \( B^* \) is given, for \( \xi \in T^*_B B \), by

\[
\langle p_B^\text{hor}(\xi), c_m \rangle = \left\langle \xi, \frac{d}{dt} \bigg|_{t=0} (b_m + tc_m) \right\rangle, \quad \text{for } c_m \in B_m, \ m \in M.
\]
Given a decomposition $\sigma : TM \oplus B \oplus B \to TB$, let $\nabla$ be the corresponding connection on $B$. The inverse of the dual of $\sigma$ over $B$ induces a horizontal lift $h : \Gamma(B^*) \to \Gamma(T(B^*)B)$ given by

$$h(\psi)(b_m) = (\sigma_B^*)^{-1}(b_m, \psi(m), 0^{TM}) = d\ell_\psi(b_m) - (\nabla_{\gamma_B} \psi, b_m) \in T^*_m B,$$

where $\ell_\psi \in \mathcal{A}^\infty(B)$ is the linear function corresponding to $\psi \in \Gamma(B^*)$.

### 3.3. Double vector subbundles.

**Definition 3.10.** Let $(D'; A', B'; M)$ be a DVB. We say that $(D; A, B; M)$ is a double vector subbundle of $D'$ if

1. $(D; A; B; M)$ is a DVB;
2. $D \subset D'$; $A \subset A'$ and $B \subset B'$ are subbundles;
3. the inclusion $i : D \hookrightarrow D'$ is a morphism of DVBs inducing the inclusions $i_A : A \hookrightarrow A'$ and $i_B : B \hookrightarrow B'$ on the side bundles.

Note that the core $C'$ of $D'$ is a subbundle of $C$ and the map $i : D \hookrightarrow D'$ induces the inclusion $i_C : C' \hookrightarrow C$ on the core bundles.

**Example 3.11.** Let $D' = A' \oplus B' \oplus C'$ be the trivial DVB with core $C'$. Given vector sub-bundles $A \subset A'$, $B \subset B'$ and $C \subset C'$, the trivial DVB $D = A \oplus B \oplus C$ with core $C$ is canonically a double vector subbundle of $D'$.

The inclusion of trivial DVBs of Example 3.11 should be seen as a model for general double vector subbundles. Let us be more precise.

**Definition 3.12.** Let $(D; A, B; M)$ be a double vector subbundle of $(D'; A', B'; M)$. We say that a decomposition $\sigma' : A' \oplus B' \oplus C' \to D'$ is adapted to $D$ if $\sigma'(A \oplus B \oplus C) = D$. In this case, the induced decomposition $\sigma := \sigma'|_{A'B'B'C'}$ of $D$ makes the diagram below commutative

$$\begin{array}{ccc}
A' \oplus B' \oplus C' & \xrightarrow{\sigma'} & D' \\
\downarrow & & \downarrow i \\
A \oplus B \oplus C & \xrightarrow{\sigma} & D
\end{array}$$

(3.15)

where the left vertical arrow is the canonical inclusion of Example 3.11.

A horizontal lift $h : A' \to \tilde{A}'$ is adapted to $D$ if its corresponding decomposition $\sigma_h \in \mathcal{A}^\infty(B)$ is adapted to $D$. Equivalently, $h$ is adapted to $D$ if and only if, for $a \in \Gamma(A)$, the linear section $h(a) : B \to D'$ satisfies $h(a)(B) \subset D$. In this case, $h|_A$ is a horizontal lift for $D$.

Recall that $\text{Dec}(D')$, the space of decompositions of $D'$, is affine modelled over $\Gamma(A^* \otimes B^* \otimes C^*)$. Define

$$\Gamma_{A,B,C} = \{ \Phi \in \Gamma(A^* \otimes B^* \otimes C^*) \mid \Phi_a(B) \subset C, \forall a \in A \}.$$  

(3.16)

**Theorem 3.13.** Let $(D'; A', B'; M)$ be a DVB and $A$ and $B$ and $C$ vector subbundles of $A'$, $B'$ and $C'$, respectively. There is a one-to-one correspondence

$$\begin{array}{c}
\left\{ \text{Double vector subbundles } (D; A; B; M) \text{ of } D' \right\} \\
\text{having } C \text{ as core bundle.}
\end{array} \xrightarrow{1-1} \Gamma_{A,B,C} \cap \text{Dec}(D').$$

More precisely, for a double vector subbundle $(D; A; B; M)$, the set of decompositions adapted to $D$ is an orbit for $\Gamma_{A,B,C}$. Reciprocally, given a decomposition $\sigma' \in \text{Dec}(D')$, the double vector subbundle $D = \sigma'(A \oplus B \oplus C)$ only depends on the $\Gamma_{A,B,C}$-orbit of $\sigma'$ and any decomposition in this orbit is adapted to $D$.

**Proof.** For a double vector subbundle $(D; A; B; M)$, we shall first prove that decompositions adapted to $D$ always exist and then that they form an orbit for $\Gamma_{A,B,C}$. Begin with two arbitrary decompositions $\sigma : A \oplus B \oplus C \to D$ and $\sigma' : A' \oplus B' \oplus C' \to D'$ and consider
that they lie in the same orbit if and only if $I$ implies that the decompositions adapted to $D$ also check in Appendix A that it behaves well under dualization.

Example 3.14. A decomposition of the trivial DVB $D' = A' \oplus B' \oplus C'$ is given by $I_{\Phi}$, for some $\Phi \in \Gamma(A'^* \otimes B'^* \otimes C')$. Hence,

$$\text{Dec}(D') \cong \Gamma(A'^* \otimes B'^* \otimes C').$$

In this case, double vector subbundles of $D'$ having side bundles $A \subset A'$, $B \subset B'$ and core $C \subset C'$ are in one-to-one correspondence with

$$\frac{\Gamma(A'^* \otimes B'^* \otimes C')}{\Gamma_{A,B,C}} \cong \Gamma\left(A^* \otimes B^* \otimes \left(\frac{C'}{C}\right)\right),$$

the trivial subbundle $A \oplus B \oplus C$ corresponding to $0 \in \Gamma\left(A^* \otimes B^* \otimes (C'/C)\right)$.

4. VB-algebroids and Morphisms.

It is shown in [8] that representations up to homotopy of a Lie algebroid on a 2-term graded vector bundle encode the Lie algebroid structures of VB-algebroids, i.e. DVBs with some additional Lie algebroid structure that is compatible with the DVB structure. In this section, we recall this correspondence and show how it can be extended to morphisms. We also check in Appendix A that it behaves well under dualization.

4.1. VB-algebroids. We begin with the definition of VB-algebroids. We follow [8] in our treatment of the subject.

Definition 4.1. Let $(D; A, B; M)$ be a DVB. We say that $(D \to B; A \to M)$ is a VB-algebroid if $D \to B$ is a Lie algebroid, the anchor $\rho_D : D \to TB$ is a bundle morphism over $\rho_A : A \to TM$ and the three Lie bracket conditions below are satisfied:

(i) $[\Gamma_{\ell}(B, D), \Gamma_{\ell}(B, D)]_D \subset \Gamma_{\ell}(B, D)$;
(ii) $[\Gamma_{\ell}(B, D), \Gamma_{c}(B, D)]_D \subset \Gamma_{c}(B, D)$;
(iii) $[\Gamma_{c}(B, D), \Gamma_{c}(B, D)]_D = 0$.

A VB-algebroid structure on $(D; A, B; M)$ naturally induces a Lie algebroid structure on $A$ by taking the anchor to be $\rho_A$ and the Lie bracket $[\cdot, \cdot]_A$ defined as follows: if $X, Y \in \Gamma_{\ell}(B, D)$ cover $X_0, Y_0 \in \Gamma(A)$ respectively, then $[X, Y]_D \in \Gamma_{\ell}(B, D)$ covers $[X_0, Y_0]_A \in \Gamma(A)$. We call $A$ the base Lie algebroid of $D$.

The next result from [8] relates VB-algebroid structures on trivial DVBs and representations up to homotopy (see also Proposition 3.9 in [H] where they prove a related result regarding the relationship between representations up to homotopy and Lie algebroid extensions).
Proposition 4.2. Let \((A, \rho_A, [\cdot, \cdot]_A)\) be a Lie algebroid over \(M\) and \(B \to M\) and \(C \to M\) be vector bundles. There is a one-to-one correspondence between VB-algebroid structures on the trivial DVB with core \(C\) having \(A\) as the base Lie algebroid and 2-term representations up to homotopy of \(A\) on \(V = C[0] \oplus B[1]\).

Let us give an explicit description of the VB-algebroid structure on \(D = A \oplus B \oplus C\) corresponding to a 2-term representation \((\partial, \nabla, K)\) of \(A\) on \(C[0] \oplus B[1]\). For \(a \in \Gamma(A)\), let \(h : \Gamma(A) \to \Gamma_B(B, D)\) be the canonical inclusion of Example 4.3. Define the anchor of \(D\), \(\rho_D : D \to B\), on linear and core sections as follows:

\[
\rho_D(h(a)) = X^1 \cdot a, \quad \rho_D(c) = \partial(c),
\]

where \(X^1, \partial(c) \in \mathcal{X}(B)\) are, respectively, the linear vector field corresponding to the derivation \(\nabla^1_a : \Gamma(B^*) \to \Gamma(B^*)\) and the vertical vector field corresponding to \(\partial(c) \in \Gamma(B)\) (see Example 4.4). The Lie bracket \([\cdot, \cdot]_D\) on \(\Gamma(D)\) is given by the formulas below:

\[
[h(a), c]_D = \nabla^0_a c,
\]

and

\[
[h(a_1), h(a_2)]_D = h([a_1, a_2]_A),
\]

where \(a, a_1, a_2 \in \Gamma(A)\) and \(c, c_1, c_2 \in \Gamma(C)\) and \(K(a_1, a_2) \in \Gamma_B(B, D)\) is the linear section given by (4.3).

Remark 4.3. A VB-algebroid structure on a general double vector bundle \((D; A, B; M)\) induces a representation up to homotopy of the base Lie algebroid \(A\) on \(C[0] \oplus B[1]\) once a decomposition \(\sigma : A \oplus B \oplus C \to D\) is chosen. The structure operators \((\partial, \nabla, K)\) are obtained from exactly the same formulas (4.1), (4.2) and (4.3) by taking \(h : \Gamma(A) \to \Gamma_B(B, D)\) as the horizontal lift corresponding to \(\sigma\). The isomorphism class of this representation does not depend on the choice of the decomposition. More precisely, if \(\tilde{\sigma} = \Phi \cdot \sigma\), for some \(\Phi \in \Gamma(A^* \otimes B^* \otimes C)\) and the structure operators \((\partial, \nabla, \tilde{K})\) corresponding to \(\tilde{\sigma}\) are given by

\[
\tilde{\partial} = \partial;
\]

\[
\nabla^0_a = \nabla^0_a - \Phi_a \circ \partial \quad \text{and} \quad \nabla^1_a = \nabla^1_a - \nabla^1_a \circ \Phi_a;
\]

\[
\tilde{K}(a, b) = K(a, b) + d_{\mathcal{V}}(\Phi(a, b) + \Phi_b \circ \partial \circ \Phi_a - \Phi_a \circ \partial \circ \Phi_b),
\]

for \(a, b \in \Gamma(A)\). Moreover, \((id_C, id_B, \Phi)\) are the components of an \(\Omega(A)\)-linear isomorphism \(\Omega(A, C[0] \oplus B[1]) \to \Omega(A, C[0] \oplus B[1])\) which intertwines \(D\) and \(\tilde{D}\) (see [8] for more details).

The next two Examples recall how the double and the adjoint representation arise in this way from VB-algebroids.

Example 4.4. The tangent double \((TB; TM; B; M)\) of a vector bundle \(B \to M\) is canonically endowed with a VB-algebroid structure \((TB \to B; TM \to M)\) with \(\rho_{TB} = id_B, \rho_{TM} = id_M\) and \([\cdot, \cdot]_B\) given by the Lie bracket of vector fields. A horizontal lift \(h : TM \to TM\) is equivalent to a connection \(\nabla : \Gamma(TM) \times \Gamma(B) \to \Gamma(B)\) (see Example 3.7). So, (4.1) and (4.2) implies that \(\partial = id_B\) and the connection on \(B[0] \oplus B[1]\) is given by \(\nabla\) in degree 0 and 1. Also, it follows from (4.3) that \(K = -R\nabla\), the curvature of \(\nabla\). Hence, the element on \(\text{Rep}^2(TM)\) associated to \((TB \to B; TM \to M)\) is the isomorphism class of the double representation of \(TM\) on \(B \oplus B\) (see Example 2.4).
Example 4.5. Let \((A, \rho_A, [\cdot, \cdot]_A)\) be a Lie algebroid over \(M\). The tangent prolongation \((TA; A, TM; M)\) of \(A\) has a VB-algebroid structure \((TA \to TM; A \to M)\). We refer to [11] for more details about this. It is shown in [8] that the element on \(\text{Rep}^2(A)\) associated to such a VB-algebroid structure is exactly the adjoint representation of \(A\).

Given a VB-algebroid \((D \to B; A \to M)\), one can prove (see [12]) that the vertical dual \((D^*_A \to \mathcal{C}^*; A \to M)\) is a VB-algebroid. By choosing a decomposition \(\sigma \in \text{Dec}(D)\), the inverse of its dual over \(A\), \((\sigma^*_A)^{-1}\), is a decomposition for \(D^*_A\). In Appendix A we prove that the representations up to homotopy associated to \(\sigma\) and \((\sigma^*_A)^{-1}\) are dual to each other.

Example 4.6. Let \((A, [\cdot, \cdot]_A, \rho_A)\) be a Lie algebroid over \(M\). By Proposition A.1, the VB-algebroid structure of \((T^*A \to A^*; A \to M)\) obtained from taking the vertical dual of the tangent prolongation \((TA \to TM; A \to M)\) gives rise to the coadjoint representation \(\text{ad}^\top \in \text{Rep}^2(A)\), the isomorphism class of the representation up to homotopy dual to the adjoint representation of \(A\). We refer to [11] for more details concerning the cotangent Lie algebroid \(T^*A \to A^*\).

4.2. Lie algebroid differential. Let \((A, \rho_A, [\cdot, \cdot]_A)\) be a Lie algebroid over \(M\). Given a 2-term representation up to homotopy of \(A\) on \(V = C[0] \oplus B[1]\), we investigate how the Lie algebroid differential \(d_D\) of \(D \to B\), where \(D = A \oplus B \oplus C\), is related to the structure operators \((\partial, \nabla, K)\). Let \(h : \Gamma(A) \hookrightarrow \Gamma(B, D)\) be the natural inclusion considered in Example 3.3.

First of all, it is straightforward to check that, for \(f \in C^\infty(M)\),
\begin{equation}
(4.7) \quad d_D(f \circ q_B) = (\rho_A \circ q_A^D)^*(df) = q_A^D(d_Af),
\end{equation}
where \(d_A\) is the Lie algebroid differential of \(A\).

In the following, recall the identification \(\Gamma(B, D^*_B) = \Gamma(A^* \otimes B^*) \oplus \Gamma(C^*)\) (see Example 3.3).

Lemma 4.7. Let \(\ell_\psi \in C^\infty(B)\) be the linear function associated to \(\psi \in \Gamma(B^*)\). The map \(d_D(\ell_\psi) : B \longrightarrow D^*_B\) is a linear section given by
\begin{equation}
(4.8) \quad d_D(\ell_\psi) = (d\nabla^*, \psi, \partial^* \psi)
\end{equation}
where \(\nabla^*\) is the \(A\)-connection on \(V^* = B^*_0 \oplus C^*_1\) dual to \(\nabla\) and \(d\nabla^* : \Omega(A; V^*) \to \Omega(A; V^*)\) is the Koszul differential.

Proof. The result that \(d_D(\ell_\psi)\) is a linear section follows from the fact that \(d\psi : B \longrightarrow T^*B\) is a linear section of the cotangent bundle (covering \(\psi\) itself) and \(\rho_D : D \to TB\) is a DVB morphism. Also, for \(b_m \in B\), one has
\begin{equation}
\langle \delta^\text{hor}(d_D(\ell_\psi)), c_m \rangle = d\psi(\rho_D(0_{b_m} + \mathfrak{c}_m)) = \frac{d}{dt}
\end{equation}
\begin{equation}
(4.9) \quad |_{t=0} \langle \psi(m), b_m + t\partial(c_m) \rangle = \langle \partial^* \psi(m), c_m \rangle.
\end{equation}
Finally, since \(\rho_D(h(a))\) is the linear vector field on \(B\) which corresponds to the derivation \(\nabla_a\) on \(\Gamma(B)\), it follows that
\begin{equation}
(4.10) \quad \langle d_D(\ell_\psi), h(a) \rangle = \mathcal{L}_{\rho_D(h(a))} \ell_\psi = \ell_{\nabla^* \psi}.
\end{equation}
Formula (4.8) follows immediately. \(\square\)

Now let us consider the degree one part of \(d_D\) (i.e. \(d_D : \Gamma(D_B^*) \to \Gamma(\wedge^2 D_B^*)\)). As \(D = q_B^1(A \oplus C)\) as a vector bundle over \(B\), one has
\begin{equation}
(4.9) \quad \wedge^2 D_B^* = q_B^1(\wedge^2 A^* \oplus (A^* \otimes C^*) \oplus \wedge^2 C^*).
\end{equation}

Lemma 4.8. Choose \(\psi \in \Gamma(A^*)\) and consider the corresponding core section \(\hat{\psi} \in \Gamma_c(B, D)\). With respect to the decomposition (4.10), one has
\begin{equation}
(4.10) \quad d_D \hat{\psi} : b_m \longmapsto (d_A \psi(m), 0_{m}^A \otimes C^*, 0_{m}^{\wedge^2 C^*}).
\end{equation}
Proof. Choose \( a_1, a_2 \in \Gamma(A) \). As \( \rho_D : D \to TB \) is a vector bundle morphism over \( \rho_A : A \to TM \), one has that \( Tq_B \circ \rho_D(h(a_i)) = \rho_A \circ q_A^D(h(a_i)) = \rho_A(a_i) \), for \( i = 1, 2 \). Also, it follows from (3.14) that \( \langle \hat{\psi}, h(a_i) \rangle = \langle \psi, a_i \rangle \circ q_B \), for \( i = 1, 2 \). It is now straightforward to check that

\[
d_D \hat{\psi}(h(a_1), h(a_2)) = (d_A \psi(a_1, a_2)) \circ q_B - \langle \hat{\psi}, \hat{\mathcal{K}}(a_1, a_2) \rangle = (d_A \psi(a_1, a_2)) \circ q_B.
\]

As for the component on \( A^* \otimes C^* \), we have

\[
d_D \hat{\psi}(h(a_1), c) = \mathcal{L}_{\rho_D(h(a_1))}(\hat{\psi}, c) - \mathcal{L}_{\rho_D(c)}(\hat{\psi}, h(a_1)) - (\hat{\psi}, [h(a_1), c]_D)_D = 0.
\]

The first and the last term on the right hand side vanish because of (3.13). Also, since \( \rho_D(c) \) is a vertical vector field, it follows with (3.14) that the second term on the right hand side vanishes. One can prove in a similar manner that the component on \( \wedge^2 C^* \) is also zero. \( \square \)

Lemma 4.9. Let \( Q \in \Gamma(A^* \otimes B^*) \) and \( \gamma \in \Gamma(C^*) \). With respect to the decomposition (4.8), we have

\[
d_D(Q, 0) : b_m \mapsto (\langle d_V \gamma, b_m \rangle, -(id_{A^*} \otimes \partial^*)(Q), 0_m^{\wedge 2} C^*),
\]

and

\[
d_D(0, \gamma) : b_m \mapsto (-\langle K^* \gamma, b_m \rangle, d_V \gamma, 0_m^{\wedge 2} C^*),
\]

Proof. Fix \( a_1, a_2 \in \Gamma(A) \) and \( c \in \Gamma(C) \). We have

\[
d_D(Q, 0)(h(a_1), h(a_2)) = \mathcal{L}_{\rho_D(h(a_1))} \ell_Q(a_2) - \mathcal{L}_{\rho_D(h(a_2))} \ell_Q(a_1) - \langle (Q, 0), [h(a_1), h(a_2)]_D \rangle.
\]

As \( \rho_D(h(a_1)) \in \mathfrak{X}(B) \) is the linear vector field corresponding to \( \nabla_{a_1} \), we have

\[
\mathcal{L}_{\rho_D(h(a_1))} \ell_Q(a_1) = \ell_{\nabla_{a_1} Q}(a_1), \quad \text{for } 1 \leq i \neq j \leq 2.
\]

Also, by (3.13) and (4.3), we get

\[
\langle (Q, 0), [h(a_1), h(a_2)]_D \rangle = \langle (Q, 0), h([a_1, a_2]_A) \rangle = \ell_Q([a_1, a_2]_A).
\]

The formula for the component on \( \wedge^2 A^* \) now follows by assembling the terms. Similarly, using that \( \langle (Q, 0), c \rangle = \langle (Q, 0), [h(a_1), c]_D \rangle = 0 \), we get

\[
d_D(Q, 0)(h(a_1), c) = -\mathcal{L}_{\rho_D(c)} \ell_Q(a_1) = \langle Q(a_1), -\partial(c) \rangle \circ q_B.
\]

It is straightforward to check now that the component in \( \wedge^2 C^* \) is zero. The proof of (4.12) is a similar computation that we leave to the reader. \( \square \)

4.3. Morphisms.

Definition 4.10. Let \( (D \to B; A \to M) \) and \( (D' \to B'; A' \to M) \) be VB-algebroids. A VB-algebroid morphism from \( D \) to \( D' \) is a DVB morphism \( (F; F_{ver}; F_{hor}) \) from \( D \) to \( D' \) such that \( F \) is a Lie algebroid morphism.

Our aim is to relate VB-algebroid morphisms with morphisms of representations up to homotopy. Using decompositions, it suffices to consider morphisms \( F \) between trivial DVBs \( D = A \oplus B \oplus C \) and \( D' = A' \oplus B' \oplus C' \). From Example 3.7, we know that a DVB morphism \( F : D \to D' \) is determined by vector bundle morphisms \( F_{ver} : A \to A' \), \( F_{hor} : B \to B' \), \( F_c : C \to C' \) and \( \Phi \in \Omega^1(A, \text{Hom}(B, C')) \).

Theorem 4.11. \( F : D \to D' \) is a VB-morphism if and only if \( F_{ver} : A \to A' \) is a Lie algebroid morphism and \( (F_c, F_{hor}, \Phi) \) are the components of a morphism \( (A, V) \Rightarrow (A', W) \) over \( F_{ver} \) between the associated representations up to homotopy \( V = C_{[0]} \oplus B_{[1]} \in \text{Rep}^2(A) \) and \( W = C'_{[0]} \oplus B'_{[1]} \in \text{Rep}^2(A') \).

Remark 4.12. Combining the results in [8] with Theorem 4.11 we conclude that the category of 2-term representations up to homotopy of a Lie algebroid \( A \) is equivalent to the category of VB-algebroids with side algebroid \( A \).
In the following, let $d_D$ and $d_{D'}$ be the Lie algebroid differentials of $D \to B$ and $D' \to B'$, respectively. Recall that $F: D \to D'$ is a Lie algebroid morphism if and only if the associated exterior algebra morphism, $F^*: \Gamma(\wedge^* D_B^*) \to \Gamma(\wedge^* D_{B'}^*)$, intertwines $d_D$ and $d_{D'}$. Theorem \ref{thm:main} will follow from the thorough study of the relation $F^* \circ d_{D'} = d_D \circ F^*$, which we carry on in Lemmas \ref{lem:core} and \ref{lem:core2} below. First, we shall need a Lemma which gives useful formulas for $F^*$ in degree $1$, $F^*: \Gamma(D_{B'}^*) \to \Gamma(D_B^*)$.

**Lemma 4.13.** Choose $Q \in \Gamma(B)^{\wedge} \otimes A^{\ast}$, $\gamma \in \Gamma(C)^{\ast}$ and $\psi \in \Gamma(A^{\ast})$. Then $F^* \hat{\psi} = F^*_{\text{ver}} \psi$ and $F^*(Q, \gamma) = (F^*_{\text{ver}} \otimes F^*_{\text{hor}})(Q) + (\Phi, \psi) \otimes \gamma$.

**Proof.** The result follows directly from Example \ref{eg:skew}. We leave the details to the reader. \hfill $\Box$

Let now $(\partial_W, \nabla_W, K_W)$ and $(\partial_V, \nabla_V, K_V)$ be the structure operators of the representations up to homotopy of $A$ on $V = C_{[0]} \oplus B_{[1]}$ and of $A'$ on $W = C'_{[0]} \oplus B'_{[1]}$, respectively, and let $\nabla^\text{Hom}$ be the $A$-connection on $\text{Hom}(V, W)$ obtained from $\nabla_V$ and $(\nabla_W)^{\text{Hom}}$ (see \ref{eg:hor}).

**Lemma 4.14.** $F^* \circ d_{D'} = d_{D} \circ F^*$ holds on $\Gamma(\wedge^1 D_B^*) = C^\infty(B)$ if and only if
\begin{equation}
\rho_{\beta'} \circ F_{\text{ver}} = \rho_\beta,
\end{equation}
\begin{equation}
F_{\text{hor}} \circ \partial_V = \partial_W \circ F_c
\end{equation}
and
\begin{equation}
\nabla^\text{Hor}_a F_{\text{hor}} = \partial_W \circ F_a, \ \forall a \in \Gamma(A).
\end{equation}

**Proof.** It suffices to consider $f \circ q_{B'}$, for $f \in C^\infty(M)$ and linear functions $\ell_\beta$, for $\beta \in \Gamma(B)$. Now, \ref{eq:12} follows directly from \ref{eg:hor}. For the other two equations, first observe that $F^*(\ell_\beta) = \ell_{F^* \beta}$. The identity
\begin{equation}
d_D (F_{\text{ver}}) = (d_{\nabla_V} F_{\text{ver}} \otimes \beta, (F_{\text{hor}} \circ \partial_V)^{\ast} \beta) \in \Gamma(A^* \otimes B^*) \oplus \Gamma(C^*)
\end{equation}
follows from \ref{eq:12}. On the other hand, due to \ref{eq:12} and Lemma \ref{lem:core} we have $F^*(d_D (\ell_\beta)) = (Q, \gamma)$, where
\begin{equation}
Q = F_{\text{ver}} \otimes F_{\text{hor}} (d_{\nabla_W} \beta + (\Phi, \partial_W \beta)) \quad \gamma = (\partial_W \circ F_c)^{\ast} \beta.
\end{equation}

By comparing the components in $\Gamma(C^*)$ and $\Gamma(A^* \otimes B^*)$, one finds equations which are dual to \ref{eq:12} and \ref{eq:12}, respectively. This proves the lemma. \hfill $\Box$

**Lemma 4.15.** $F^* \circ d_{D'} = d_D \circ F^*$ holds on $\Gamma(\wedge^1 D_B^*)$ if and only if
\begin{equation}
d_A \circ F_{\text{ver}} = F_{\text{ver}} \circ d_A \text{ in } \Gamma(A^*)
\end{equation}
\begin{equation}
\nabla^\text{Hor}_a F_c = \Phi_a \circ \partial_V, \ \forall a \in \Gamma(A)
\end{equation}
and
\begin{equation}
d_{\nabla_W} g_{\Phi} = F_c \circ K_V - (F_{\text{ver}} K_W) \circ F_{\text{hor}}.
\end{equation}

**Proof.** It suffices to consider core sections $\hat{\psi}$ and linear sections of the type $(0, \gamma)$, where $\gamma \in \Gamma(C'^\ast)$ and $\psi \in \Gamma(A'^\ast)$. Equation \ref{eq:13} is equivalent to $F^* \circ d_{D'}(\hat{\psi}) = d_{D} \circ F^*(\hat{\psi})$. Now, according to the decomposition \ref{eq:14}, we find $F^* \circ d_{D'}(0, \gamma) = (\Lambda_1, \Lambda_2, \Lambda_3)$, where $\Lambda_3$ is the zero section of $\wedge^2 q_{B'^\ast} C^\ast$,
\begin{equation}
\Lambda_1(b_m) = -((F_{\text{ver}} K_W)^{\ast} \gamma, F_{\text{hor}}(b_m)) + (F_{\text{ver}} \wedge \Phi^*(b_m))(d_{\nabla_W} \gamma(m)),
\end{equation}
and
\begin{equation}
\Lambda_2(b_m) = F_{\text{ver}} \otimes F_{\text{hor}} (d_{\nabla_W} \gamma(m)),
\end{equation}
where \( m \in M, b_m \in B_m \) and \( \Phi(b_m') \) is seen as a map from \( A \) to \( C' \) with dual \( \Phi^*(b_m) : C'^* \rightarrow A^* \). Similarly, by Lemma 1.12 and formulas (1.11), (1.12), it follows that \( d_D(F^*(0, \gamma)) = d_D(\langle \Phi, \gamma \rangle, F^*_c \gamma) = (\Theta_1, \Theta_2, \Theta_3) \), where \( \Theta_3 \) is again the zero section of \( \lambda^2 q^*_B C^* \),

\[
\Theta_1(b_m) = \langle d_{\nabla^*} \langle \Phi, \gamma \rangle, b_m \rangle - \langle K(F_c^*_\gamma), b_m \rangle.
\]

and

\[
\Theta_2(b_m) = -(\text{id}^* \otimes \delta^*_c) \langle \Phi, \gamma(m) \rangle + d_{\nabla^*} F_c^*_\gamma(m).
\]

The equalities \( \Lambda = \text{id} \) and \( \Lambda_1 = \Theta_1 \) are equivalent to the equations dual to (4.17) and (4.18), respectively.

Proof of Theorem 4.11. Equations (4.15), (4.17) are equivalent to \( F_{\text{ver}} \) being a Lie algebroid morphism. Similarly, equations (4.11), (4.12), (4.17) and (4.18) are equivalent to \( (F_c, F_{\text{hor}}, \Phi) \) being the components of a morphism \( (A, V) \Rightarrow (A', W) \). This proves the Theorem.

Example 4.16. Let \( (A, [\cdot, \cdot]_A, \rho_A) \) be a Lie algebroid over \( M \). An IM-2-form \( \Omega \) on \( A \) is a pair \( (\mu, \nu) \) where \( \mu : A \rightarrow T^* M \) and \( \nu : A \rightarrow \lambda^2 T^* M \) such that

1. \( \langle \mu(a), \rho_A(b) \rangle = -\langle \mu(b), \rho_A(a) \rangle \);
2. \( \mu([a, b]) = \mathcal{L}_{\rho_A(a)} \mu(b) - i_{\rho_A(b)} (d\mu(a) + \nu(a)) \);
3. \( \nu([a, b]) = \mathcal{L}_{\rho_A(a)} \nu(b) - i_{\rho_A(b)} d\nu(a) \),

for \( a, b \in \Gamma(A) \). In [3] (see also [5] for the case where \( \nu = 0 \)), it is shown that every IM-2-form is associated to a 2-form \( \Lambda \in \Omega^2(A) \) whose associated map \( \Lambda_1 : TA \rightarrow T^* A \) is a VB-algebroid morphism from \( (TA \rightarrow TM; A \rightarrow M) \) to \( (T^* A \rightarrow A^*; A \rightarrow M) \) inducing \( \mu : A \rightarrow T^* M \) on the core bundles and \( -\mu^* : TM \rightarrow A^* \) on the side bundles. Let \( \sigma \in \text{Dec}(TA) \) and \( \sigma_A^* \) be its dual over the side bundle \( A \). From [4] (see Lemma 3.6 there), it follows that \( F = \sigma_A^* \circ \Lambda_1 \circ \sigma : A \oplus TM \oplus A \rightarrow A \oplus A^* \oplus T^* M \) has components given by \( F_{\text{ver}} = \text{id}_A, F_{\text{hor}} = -\mu^*, F_c = \mu \) and

\[
\Phi = \nu + d_{\nabla^*} \mu^* \in \Omega^1(A; \text{Hom}(TM, T^* M)).
\]

where \( \nabla \) is the connection on \( A \) associated to \( \sigma \) and \( d_{\nabla^*} : \Omega(TM; A^*) \rightarrow \Omega(TM; A^*) \) the Koszul differential associated to the dual connection. Note that we are identifying \( \Omega^2(TM, A^*) \) with \( \Omega^1(A; \lambda^2 T^* M) \) and seeing \( \lambda^2 T^* M \) as a subset of \( \text{Hom}(TM, T^* M) \). So, as a result of Theorem 4.11, one has that \( (\mu, \nu) \) is an IM-2-form if and only if \( (\mu, -\mu^*, \nu + d_{\nabla^*} \mu^*) \) are the components of a morphism from the adjoint representation \( ad_{\nabla^*}(A) \) to the coadjoint representation \( ad_{\nabla^*(A)} \).

Example 4.17. Let \( (A, [\cdot, \cdot]_A, \rho_A) \) be a Lie algebroid such that its dual \( A^* \) has also a Lie algebroid structure \( (A^*, [\cdot, \cdot]_{A^*}, \rho_{A^*}) \). It is shown in [14] that \( (A, A^*) \) is a Lie bialgebroid if and only if \( \pi^*_A : T^* A \rightarrow TA \) is a VB-algebroid morphism from the cotangent Lie algebroid \( (T^* A \rightarrow A^*, A \rightarrow M) \) to the tangent prolongation \( (TA \rightarrow TM; A \rightarrow M) \), where \( \pi_A \in \Gamma(\lambda^2 TA) \) is the linear Poisson bivector corresponding to the Lie algebroid \( A^* \). For any decomposition \( \sigma \in \text{Dec}(TA) \), it follows from [14] (see Corollary 6.5 there) that \( \sigma^{-1} \circ \pi_A^* \circ (\sigma_A^*)^{-1} : A \oplus A^* \oplus T^* M \rightarrow A \oplus TM \oplus A \) has components given by \( F_{\text{ver}} = \text{id}_A, F_{\text{hor}} = \rho_{A^*}, F_c = -\rho_A^*, \) and \( \Phi = \sigma_A^* \in \Omega^1(A; \text{Hom}(A^*, A)) \) defined by

\[
\langle \Phi_\sigma(a, \beta), \beta \rangle = -d_{\rho_A^*} a(a, \beta) + (\beta, \nabla_{\rho_{A^*}}(\alpha) \alpha) - (\alpha, \nabla_{\rho_{A^*}}(\beta) \alpha), (\alpha, \beta) \in A^* \times_M A^*.
\]

where \( d_{\rho_A^*} : \Gamma(\Lambda A) \rightarrow \Gamma(\Lambda A) \) is the Lie algebroid differential of \( A^* \), \( \nabla : \Gamma(TM) \times A \rightarrow \Gamma(A) \) is the connection corresponding to \( \sigma \) and \( \sigma_A^* \) is the dual of \( \sigma \) over \( A \). Note that

\[
\langle \Phi_\sigma(a, \beta), \beta \rangle = \langle a, T^\text{bas}(a, \beta) \rangle,
\]

where \( T^\text{bas} \) is the torsion of the basic connection

\[
\Gamma(A') \times \Gamma(A^*) \rightarrow \Gamma(A^*)
\]

\[
(a, \beta) \mapsto [a, \beta]_A + \nabla_{\rho_{A^*}(\beta)} \alpha.
\]
As a result of Theorem 4.11, we have that \((A, A^\ast)\) is a Lie bialgebroid if and only if \((-\rho_A^\ast, \rho_{A^\ast}, T^{\text{bas}})\) are the components of a morphism from the coadjoint representation \(\text{ad}_{\nabla}\) to the adjoint representation \(\text{ad}_\nabla\).

5. VB-subalgebroids and subrepresentations.

5.1. VB-subalgebroids.

Definition 5.1. Let \((D' \to B'; A' \to M)\) be a VB-algebroid. We say that a double vector subbundle \((D; A, B; M)\) is a VB-subalgebroid of \(D'\) if \(D \to B\) is a Lie subalgebroid of \(D' \to B'\).

Proposition 5.2. A double vector subbundle \((D; A, B; M)\) is a VB-subalgebroid of \(D'\) if and only if

1. \((D \to B; A \to M)\) is a VB-algebroid;
2. the inclusion map \(i : (D \to B; A \to M) \hookrightarrow (D' \to B'; A' \to M)\) is a VB-algebroid morphism.

Proof. It is straightforward to see that conditions (1) and (2) imply that \(D\) is a VB-subalgebroid of \(D'\). Conversely, assume that \(D \to B\) is a Lie subalgebroid of \(D' \to B'\). The fact that the inclusion \(i : D \to D'\) is a bundle morphism over \(i_A : A \to A'\) implies that the anchor of \(D\), \(\rho_D = \rho_{D'} \circ i\), is a bundle morphism over \(\rho_A = \rho_{A'} \circ i_A\). To prove that \((D \to B; A \to M)\) is a VB-algebroid, we still have to check conditions (i), (ii) and (iii) of Definition 4.1. These will follow from exactly the same conditions on \(D'\) if we prove that core (respectively linear) sections of \(D\) can be extended to core (respectively linear) sections of \(D'\). Now, (5.1) implies that

\[ \Gamma_c(B, D) = \{ \tilde{c}|_B \mid c \in \Gamma(C)\} \text{ and } \tilde{c} \in \Gamma_c(B', D') \} \]

Also, for \(\mathcal{X} : B \to D\), a linear section of \(D\) covering \(a \in \Gamma(A)\), choose any horizontal lift \(h' : A' \to \hat{A}'\) adapted to \(D\) (the existence of which is guaranteed by Proposition 3.13). For \(b \in B\), \n
\[ \mathcal{X}(b) \circ h(a)(b) = \tilde{h}^B + \Phi(a)(b) \]

for some \(\Phi \in \Omega^1(A, \text{Hom}(B, C))\). Extend \(\Phi\) to \(\Phi' \in \Omega(A', \text{Hom}(B', C'))\). The horizontal lift \(h'' = h' + \Phi'\) is still adapted to \(D\) and \(\mathcal{X} = h''(a)|_B\).

Definition 5.3. Let \(W \in \text{Rep}^2(A)\) be a 2-term representation of a Lie algebroid \(A\) and let \((\partial, \nabla, K)\) be its structure operators. We say that a (graded) subbundle \(V \subset W\) is a subrepresentation if \(V \in \text{Rep}^2(A)\) and \((i_{V_0}, i_{V_1}, 0)\) are the components of a morphism \((A, V) \Rightarrow (A, W)\), where \(i_{V_0} : V_0 \hookrightarrow W_0\) and \(i_{V_1} : V_1 \hookrightarrow W_1\) are the inclusions and \(0 \in \Gamma(A^* \otimes V_1^\ast \otimes W_0)\).

Remark 5.4. If \((\partial, \nabla, K)\) are the structure operators for \(W \in \text{Rep}^2(A)\), then \(V \subset W\) is a subrepresentation if and only if

\[
\begin{align*}
&\text{(5.1) } \partial(V_0) \subset V_1; \\
&\text{(5.2) } \nabla_a \text{ preserves } V, \forall a \in \Gamma(A); \\
&\text{(5.3) } K(a_1, a_2)(V_1) \subset V_0, \forall a_1, a_2 \in A.
\end{align*}
\]

In this case, the restrictions \((\partial|_{V_0}, \nabla|_V, K|_{V_1})\) are the structure operators for the representation up to homotopy of \(A\) on \(V\). This follows directly from equations (2.22), (2.23) and (2.24).

Let us give an example.
Example 5.5. Let \( \nabla : \Gamma(TM) \times \Gamma(B) \to \Gamma(B) \) be a connection on \( B \) and consider the double representation \( D_\nabla(B) \in \text{Rep}^2(TM) \) (see Example 2.4). For a given vector subbundle \( C \subset B \), the graded vector bundle \( C_{[0]} \oplus B_{[1]} \) is a subrepresentation if and only if \( \nabla \) preserves \( C \) and the induced connection on the quotient \( \nabla : \Gamma(TM) \times \Gamma(B/C) \to \Gamma(B/C) \) is flat.

The following result shows how \( \mathcal{VB} \)-subalgebroids and subrepresentations are related.

Proposition 5.6. Let \( W = C'_0 \oplus B'_1 \in \text{Rep}^2(A') \) be the representation up to homotopy corresponding to a \( \mathcal{VB} \)-algebraic structure on \( (A' \oplus B' \oplus C' \to B'; A' \to M) \). Then \( (A \oplus B \oplus C \to B; A \to M) \) is a \( \mathcal{VB} \)-subalgebroid if and only if \( A \subset A' \) is a subalgebroid and \( C_{[0]} \oplus B_{[1]} \) is a subrepresentation of \( i^1_A W \in \text{Rep}^2(A) \), where \( i_A : A \hookrightarrow A' \) is the inclusion.

Proof. Assume \( A \oplus B \oplus C \) is a \( \mathcal{VB} \)-subalgebroid of \( D' \). By Proposition (5.2), it follows that \( (A \oplus B \oplus C \to B; A \to M) \) is a \( \mathcal{VB} \)-algebroid, so there is a corresponding Lie algebroid structure on \( A \) and \( V = C_{[0]} \oplus B_{[1]} \in \text{Rep}^2(A) \). As the inclusion \( i : A \oplus B \oplus C \to A' \oplus B' \oplus C' \) is a \( \mathcal{VB} \)-morphism, it follows from Theorem 4.11 that the inclusion \( i_A : A \to A' \) is a Lie algebroid morphism and \( (i_C, i_B, 0) \) are the components of a morphism \( (A, V) \Rightarrow (A', W) \) over the inclusion \( i_A \). Conversely, assume that \( A \subset A' \) is a subalgebroid and that \( V \) is a subrepresentation of \( i^1_A W \). The representation up to homotopy of \( A \) on \( V \) give \( (A \oplus B \oplus C \to B; A \to M) \) a \( \mathcal{VB} \)-algebraic structure and one can use Theorem 4.11 once again to prove that the inclusion \( i : A \oplus B \oplus C \to A' \oplus B' \oplus C' \) is a \( \mathcal{VB} \)-morphism. This concludes the proof.

Although Proposition 5.6 refers to a specific double vector subbundle of \( A' \oplus B' \oplus C' \) being a \( \mathcal{VB} \)-subalgebroid, it can be applied to general double vector subbundles by means of adapted splittings (see Remark 4.3) as diagram \((5.15)\) indicates.

5.2. Applications. Let \( q_B : B \to M \) be a vector bundle and \( \Delta \subset TB \) be a distribution on \( B \). We say that \( \Delta \) is linear if there exists a distribution \( \Delta_M \subset TM \) such that

\[
\begin{array}{ccc}
\Delta & \longrightarrow & B \\
\downarrow & & \downarrow q_B \\
\Delta_M & \longrightarrow & M
\end{array}
\]

is a double vector subbundle of \( TB \) \cite{10}. We denote by \( C \subset B \) the core bundle of \( \Delta \). A section \( b : M \to B \) belongs to \( \Gamma(C) \) if and only if its vertical lift \( b^v \in \mathfrak{X}(B) \) belongs to \( \Delta \). In case \( \Delta = TF \), we say that \( F \) is a linear foliation on \( B \).

Remark 5.7. If one considers the Lie groupoid \( B \rightrightarrows M \), where the source and the target are the projection \( q_B : B \to M \) and the multiplication is addition on the fibers, then a linear distribution is just a multiplicative distribution in the sense of \cite{10}.

In \cite{10} (see also \cite{9}), it is shown that the infinitesimal version of ideal systems \cite{11} on a Lie algebroid \( A \) (called IM-foliations) are equivalent to linear distributions on \( A \) which are \( \mathcal{VB} \)-subalgebroids of both \( (TA \to A; TM \to M) \) and \( (TA \to TM; A \to M) \). As a consequence of Proposition 5.6, we have the following result relating IM-foliations on a Lie algebroid and the adjoint representation.

Theorem 5.8. Let \( A \) be a Lie algebroid over \( M \). A linear distribution \( (\Delta; A, \Delta_M; M) \) on \( A \) is a \( \mathcal{VB} \)-subalgebroid of both \( (TA \to A; A \to M) \) and \( (TA \to TM; A \to M) \) if and only if

1. \( \Delta_M \subset TM \) is integrable;
2. \( C_{[0]} \oplus \Delta_M_{[1]} \) is a subrepresentation of \( \text{ad}_\nabla(A) \) and
3. \( C_{[0]} \oplus A_{[1]} \) is a subrepresentation of \( i^1_A \Delta_M^1 \nabla(A) \).
where $C$ is the core bundle of $\Delta$, $i_{\Delta_M} : \Delta_M \hookrightarrow TM$ is the inclusion and $\nabla$ is any connection adapted to $\Delta$. Moreover, if $\widetilde{\nabla} : \Gamma(D_M) \times \Gamma(A/C) \to \Gamma(A/C)$ is the connection given by
\[
\widetilde{\nabla}_x \pi(a) = \pi(\nabla_x a), \quad a \in \Gamma(A), \quad x \in \Gamma(D_M),
\]
where $\pi : A \to A/C$ the quotient projection, then $(A, \Delta_M, C, \widetilde{\nabla})$ is an infinitesimal ideal system in $A$.

In the remainder of this paper, we investigate separately the $\mathcal{VB}$-subalgebroids of $(TA \to A; TM \to M)$ and $(TA \to TM; A \to M)$.

5.2.1. Linear foliations on a vector bundle. Let $(\Delta; \Delta_M, B; M)$ be a linear distribution on a vector bundle $B$. Let $C$ be the core bundle of $\Delta$ and $\pi : B \to B/C$ be the quotient projection. Define
\[
D^\Delta : \Gamma(D_M) \times \Gamma(B) \to \Gamma(B/C)
\]
where $X : B \to \Delta$ is any linear section covering $x$, $L_X : \Gamma(B) \to \Gamma(B)$ is the derivation defined by
\[
L_X(b) = [X, b], \quad b \in \Gamma(B).
\]
It is straightforward to check that $\pi \circ L_X$ depends only on $x$ and not on the particular choice of $\chi : B \to \Delta$. The map $D^\Delta$ is $C^\infty(M)$-linear in the first coordinate and satisfies
\[
D^\Delta_x (f b) = f D^\Delta_x (b) + (\mathcal{L}_x f) \pi(b), \quad \forall x \in \Gamma(D_M), \quad b \in \Gamma(B), \quad f \in C^\infty(M).
\]

**Proposition 5.9.** The map $\Delta \mapsto D^\Delta$ gives a one-to-one correspondence between linear distributions on $(\Delta; \Delta_M, B; M)$ on $B$ having $C$ as core bundle and maps $D : \Gamma(B) \to \Gamma(D_M \otimes (B/C))$ satisfying (5.6). Moreover, a connection $\nabla : \Gamma(TM) \times \Gamma(B) \to \Gamma(B)$ is adapted to $\Delta$ if and only if
\[
\pi \circ \nabla_x = D^\Delta_x, \quad \forall x \in \Gamma(D_M).
\]

**Proof.** Let us first prove the second assertion. On the one hand, if $\nabla$ is adapted to $\Delta$, then the linear vector field $X_\nabla : B \to TB$ corresponding to the derivation $\nabla_x : \Gamma(B) \to \Gamma(B)$ is a linear section of $\Delta$. Hence, it follows from the definition (5.4) that
\[
D^\Delta_x = \pi \circ L_{X_\nabla}, \quad \forall x \in \Gamma(D_M).
\]
for $x \in \Gamma(D_M)$. On the other hand, if (5.7) holds, then, for every $x \in \Gamma(D_M)$, there exists a linear section $X : B \to \Delta$ covering $x$ such that $\delta := \nabla_x - L_X \in \text{Hom}(B, C)$, where $L_X$ is the derivation defined by (5.5). In terms of sections,
\[
X_{\nabla} = X + \delta^\Delta.
\]
As $C$ is the core bundle of $\Delta$, one gets that $X_{\nabla}$ is a section of $\Delta$ and, therefore, $\nabla$ is adapted to $\Delta$.

Now, it remains to prove that given a map $D : \Gamma(B) \to \Gamma(D_M \otimes (B/C))$ satisfying (5.6), there exists a linear distribution $\Delta$ on $B$ such that $\Delta = D^\Delta$. The idea is that all connections $\nabla$ satisfying $D_x = \pi \circ \nabla_x, \ x \in \Delta_M$, belong to the same $\Gamma_{\Delta_M,B,C}$-orbit on Dec$(TB)$, which, in turn, correspond to a linear distribution on $B$ to which any connection on the orbit is adapted (as shown in Proposition 3.13). So, we are left to prove that such a connection always exists. For this, let $s : B/C \to B$ be a linear section for the quotient projection $\pi : B \to B/C$. In the rest of the proof, we identify $B$ with $(B/C) \oplus C$ using $s$. First, note that (5.7) implies that the map
\[
\Gamma(D_M) \times \Gamma(B/C) \to \Gamma(B/C)
\]
\[
(x, \gamma) \mapsto D_x(s(\gamma))
\]
is a $\Delta_M$-connection on $B/C$. Extend it to a $\Delta_M$-connection $\nabla^{\text{rest}}$ on $B$ by choosing an arbitrary $\Delta_M$-connection on $C$. Second, (5.6) implies that, for $x \in \Gamma(\Delta_M)$, $\nabla_x : \Gamma(B) \to \Gamma(B/C)$ is actually linear when restricted to $\Gamma(C)$. So, define $\Phi \in \Gamma(\Delta_M^* \otimes B^* \otimes B)$

$$\Phi_x(u) = \begin{cases} s \circ \tilde{D}_x(u), & \text{if } u \in C; \\ 0, & \text{if } u \in B/C. \end{cases}$$

It is now straightforward to check that any connection $\nabla : \Gamma(TM) \times \Gamma(B) \to \Gamma(B)$ such that $\nabla_x = \nabla_x^{\text{rest}} + \Phi_x$ for $x \in \Delta_M$ satisfies $\nabla_x = \pi \circ \nabla_x$, for $x \in \Delta_M$.

**Remark 5.10.** Proposition [5.9] can be put in the more general setting of multiplicative distributions on Lie groupoids considered in [7] (see Remark 5.7). There, the one-to-one correspondence above is shown in the case $\Delta_M = TM$. The map $\mathbb{D}^\Delta$ is called in this case the *Spencer operator associated to $\Delta$* by the authors.

The following result (see [7, 10]), can now be reobtained as a direct consequence of Proposition 5.11.

**Proposition 5.11.** A linear distribution $(\Delta; \Delta_M, B; M)$ on $B$ with core bundle $C$ is involutive if and only if $\Delta_M$ is involutive and the associated map $\mathbb{D}^\Delta : \Gamma(B) \to \Gamma(\Delta_M^* \otimes (B/C))$ satisfies

1. $\mathbb{D}^\Delta|_{\Gamma(C)} = 0$;
2. the map induced on the quotient $\Gamma(B/C) \to \Gamma(\Delta_M^* \otimes (B/C))$ is a flat $\Delta_M$-connection on $B/C$.

**Proof.** Choose any connection adapted to $\Delta$ and consider the double representation $\mathcal{D}_\nabla(B) \in \text{Rep}^2(TM)$. It is the representation up to homotopy of $TM$ associated to the VB-algebroid $(TB \to B; TM \to M)$ and the decomposition induced by $\nabla$. So, by Proposition [5.6] one has that $\Delta$ is involutive if and only if $\Delta_M$ is involutive and $C[0] \oplus B[1]$ is a subrepresentation of $i_{\Delta_M}^* \mathcal{D}_\nabla \in \text{Rep}^2(\Delta_M)$, where $i_{\Delta_M} : \Delta_M \hookrightarrow TM$ is the inclusion. The result now follows from Example 5.5 and the fact that $\mathbb{D}^\Delta_x = \pi(\nabla_x)$, for every $x \in \Gamma(\Delta_M)$.

5.2.2. *Subrepresentations of the adjoint.* Let $(A, [\cdot, \cdot]_A, \rho_A)$ be a Lie algebroid over $M$ and consider the associated VB-algebroid $(TA \to TM; A \to M)$.

**Definition 5.12.** A linear distribution $(\Delta; A, \Delta_M; M)$ is said to be *compatible with the Lie algebroid structure if $\Delta \to \Delta_M$ is a Lie subalgebroid of $TA \to TM$.*

**Remark 5.13.** Distributions on $A$ compatible with the Lie algebroid structure are the infinitesimal counterpart of multiplicative distributions on groupoids.

The following result gives a characterization of distributions $(\Delta; A, \Delta_M; M)$ with core $C$ which are compatible with the Lie algebroid structure in terms of their associated operators $\tilde{\mathbb{D}} : \Gamma(A) \to \Gamma(\Delta_M^* \otimes (A/C))$ (see (5.4)). We will make use of $\pi : A \to A/C$ as well as the quotient projection $\pi_{TM} : TM \to TM/\Delta_M$.

**Proposition 5.14.** Let $\Delta$ be a linear distribution and choose any connection $\nabla : \Gamma(TM) \times \Gamma(A) \to \Gamma(A)$ adapted to $\Delta$. One has that $\Delta$ is compatible with the Lie algebroid structure if and only if $\rho_A(C) \subset \Delta_M$,

$$\begin{cases} \tilde{\mathbb{D}}_{\rho_A(c)}(a) = -\pi([a, c]); \\ \tilde{\rho}_A(\tilde{\mathbb{D}}_x(a)) = -\pi(\rho_A([a, x])) \end{cases}$$

where $\tilde{\rho}_A : A/C \to TM/\Delta_M$ is the quotient map (i.e. $\tilde{\rho}_A \circ \pi = \pi_{TM} \circ \rho_A$) and

$$\mathbb{D}_x([a, b]_A) = \tilde{\nabla}_a^{\text{bas}} \mathbb{D}_x(b) - \tilde{\nabla}_b^{\text{bas}} \mathbb{D}_x(a) + \pi(\nabla_{[\rho_A(b), x]}a - \nabla_{[\rho_A(a), x]}b).$$
where $\nabla^{\text{bas}}$ is the $A$-connection on the quotient $A/C$ given by

\begin{equation}
\nabla^{\text{bas}}_a \pi(b) = \pi([a, b] + \nabla_{\rho(a)} b) = \pi(\nabla^{\text{bas}}_a b)
\end{equation}

for $a, b \in \Gamma(A)$, $c \in \Gamma(C)$ and $x \in \Gamma(\Delta_M)$.

**Proof.** As $\nabla$ is adapted to $\Delta$, Proposition 5.10 assures that $\Delta$ is compatible with the Lie algebroid structure if and only if $V = C_{[0]} \oplus \Delta_{M[1]}$ is a subrepresentation of $\text{adv} \in \mathfrak{h}^{\Lambda^2}(A)$. It is straightforward to see that (5.1) is equivalent to $\rho_A(C) \subset \Delta_M$. So, one is left to prove that (5.8) and (5.9) correspond to (5.2) and (5.3), respectively. Now, saying that $\nabla^{\text{bas}}$ preserves $V$ is equivalent to

\[
\left\{ \begin{array}{l}
[a, c] + \nabla_{\rho_A(c)} a \in \Gamma(C) \\
[\rho_A(a), x] + \rho_A(\nabla_a x) \in \Gamma(\Delta_M)
\end{array} \right. \iff \left\{ \begin{array}{l}
\pi(\nabla_{\rho_A(c)} a) = -\pi([a, c]) \\
\pi_T \rho_A(\nabla_a x) = -\pi_T \rho_A(x)
\end{array} \right.
\]

for every $c \in \Gamma(C)$, $a \in \Gamma(A)$ and $x \in \Gamma(\Delta_M)$. So, (5.8) now follows from the fact that, for any adapted connection $\nabla$, $\nabla_x = \pi \circ \nabla$, $\forall x \in \Gamma(\Delta_M)$ and $\pi_T \circ \rho_A = \rho_A \circ \pi$. In particular, as $\nabla^{\text{bas}}$ preserves $C$, it follows that $\nabla^{\text{bas}}$ is well-defined.

At last, (5.3) for the adjoint representation says that $R^{\text{bas}}(\Delta_M) \subset C$ which, in turn, holds if and only if

\begin{equation}
\pi(\nabla_a [a, b] - [\nabla_a a, b] - [\nabla_a b, x] - \nabla^{\text{bas}}_a a + \nabla^{\text{bas}}_b b) = 0,
\end{equation}

for $a, b \in \Gamma(A)$ and $x \in \Gamma(\Delta_M)$. Now, note that

\[
-\nabla_a [a, b] + \nabla^{\text{bas}}_a [a, b] = [b, \nabla_a a] + \nabla_{\rho_A(\nabla_a a)} b + \nabla_{[\rho_A(a), x]} b = \nabla^{\text{bas}}_a \nabla_a a + \nabla_{[\rho_A(a), x]} b
\]

and similarly

\[
[a, \nabla_a b] + \nabla^{\text{bas}}_a \nabla_a b = \nabla^{\text{bas}}_a \nabla_a b + \nabla_{[\rho_A(a), x]} b.
\]

So, by definition of $\nabla^{\text{bas}}$, one has that (5.11) is equivalent to

\[
\nabla_x ([a, b]) + \nabla^{\text{bas}}_a \nabla_a x = \nabla^{\text{bas}}_a \nabla_a b + \pi(\nabla_{[\rho_A(a), x]} b - \nabla_{[\rho_A(b), x]} a) = 0,
\]

as required. \hfill \Box

**Remark 5.15.** In the special case where $\Delta_M = TM$, one can get rid of the choice of an adapted connection. Indeed, note that $\nabla^{\text{bas}}$ can be alternatively given by

\[
\nabla^{\text{bas}}_a b = \pi([a, b] + \nabla_{\rho(b)} a)
\]

and (5.3) becomes

\[
\nabla_x ([a, b] + \rho(b)) = \nabla^{\text{bas}}_a \nabla_a x - \nabla^{\text{bas}}_b \nabla_b x - \nabla^{\text{bas}}_{[\rho_A(a), x]} a - \nabla^{\text{bas}}_{[\rho_A(b), x]} b.
\]

In this form, Proposition 5.14 gives the infinitesimal counterpart of a result from [7] characterizing (wide) multiplicative distributions in terms of their Spencer operators. In the general case, one seems to need the choice of an adapted connection to write individual terms on the right hand side of (5.9), although the whole expression on the right hand side does not depend on the choice of the connection.

**Appendix A. Duality of VB-algebroids.**

Let $(D; A, B; M)$ be a DVB and consider its horizontal (8.10) and vertical (8.12) duals. The vector bundles $p_{C^*}^*: D^*_A \to C^*$ and $q_{B^*}^*: D^*_B \to C^*$ are dual to each other via the nondegenerate pairing $\|\cdot, \cdot\|: D^*_A \times C^* \to D^*_B \to \mathbb{R}$ given by

\begin{equation}
\|\Theta, \Psi\| := \langle \Psi, d^*_A \Theta - \Theta, d^*_B \rangle
\end{equation}

where $d \in D$ is any element with $q^B_B(d) = p_A(\Theta)$ and $q^B_B(d) = p_B(\Psi)$. The pairings on the right-hand side of (A.1) are defined with respect to the fibers over $B$ and over $A$, respectively. Henceforth, we identify the dual of $D^*_B \to C^*$ with $D^*_A \to C^*$ via the pairing.
Poisson structure on $E$. Duals of $\ell T$.

Also, for $T \in \Gamma(B^* \otimes C)$,

$\ell_{\hat{T}}^{C^*} = -\ell_{\hat{T}},$

where $\hat{T} \in \Gamma_i(B, D)$ and $\hat{T}^{\hat{x}} \in \Gamma_i(C^*, D_A)$ are the linear sections corresponding to $T$ and its dual $T^* \in \Gamma(C^* \otimes B)$, respectively.

We shall need one more formula (which follows directly from (3.11)) for $\ell_{[x_1, x_2]}^{C^*}$ for $x \in \Gamma(C)$ and the corresponding core section $\hat{c} \in \Gamma(B, D)$, namely

$$\ell_{\hat{c}} = \ell_{\hat{c}} \circ p_{D}^{hor}.$$  

A.1. Duals of $\mathcal{V}B$-algebroids and representations up to homotopy. Assume now that $(\mathcal{D} \rightarrow B; A \rightarrow M)$ is a $\mathcal{V}B$-algebroid. Then $(D_A^* \rightarrow C^*; A \rightarrow M)$ has a natural $\mathcal{V}B$-algebroid structure. The Lie algebra structure on $D_A^* \rightarrow C^*$ is obtained by noticing that the linear Poisson structure on $D_B^* \rightarrow B$ associated to the Lie algebroid structure $D_B^* \rightarrow C^*$ is linear with respect to the vector bundle structure.

In particular, besides the usual formulas

$$\ell_{[x_1, x_2]}^{C^*} = \{\ell_{x_1}, \ell_{x_2}^D\} D_B^*, \quad \ell_{\rho_D^D A}^{C^*} \circ p_B = \{\ell_{\rho_D^D A}, f \circ p_B\} D_B^*,$$

defining the Lie bracket $[\cdot, \cdot]_D$ and the anchor $\rho_D$ on $D \rightarrow B$, for $\chi, \chi_1, \chi_2 \in \Gamma(B, D)$ and $f \in C^\infty(B)$, we have

$$\ell_{[x_1, x_2]}^{C^*} = \{\ell_{x_1}, \ell_{x_2}^D\} D_B^*, \quad \ell_{\rho_D^D A}^{C^*} \circ p_B = \{\ell_{\rho_D^D A}, g \circ p_B^{hor}\} D_B^*,$$

defining the Lie bracket $[\cdot, \cdot]_{D_B^*}$ and the anchor $\rho_{D_B^*}$ on $D_B^* \rightarrow C^*$, for $\Theta, \Theta_1, \Theta_2 \in \Gamma(C^*, D_A^*)$ and $g \in C^\infty(C^*)$. We refer to (3.11) (see also (3.8) for more details.

Our aim here is to prove that the representations up to homotopy associated to $\partial$, $\Theta_{\overline{\Gamma}(B, D)}$ be a horizontal lift for $D$. There exists a corresponding horizontal lift $h^\Gamma : \Gamma(A) \rightarrow \Gamma_i(C, D_A^*)$ given as follows: take the decomposition $\sigma_h \in \text{Dec}(D)$ associated to $h$ by (3.7) and consider the inverse of its dual over $A$, $(\sigma_h)^{-1} \in \text{Dec}(D_A^*)$. Set $h^\Gamma$ to be the horizontal lift corresponding to $(\sigma_h)^{-1}$. It is straightforward to check that

$$\ell_{h(a)} = \ell_{h^\Gamma(a)} \in C^\infty(D_B^*)$$

for every $a \in \Gamma(A)$.

We define operators of the representation up to homotopy $C_{(0)} \oplus B_{(1)} \in \text{Rep}^2(A)$ associated to $(D, h)$ and $(\partial, \nabla, K)$ be the structure operators of the representation up to homotopy $B_{(0)}^* \oplus C_{(1)} \in \text{Rep}^2(A)$ associated to $(D_A^*, h^\Gamma)$. The next result relates the two representations.

Proposition A.1. The structure operators $(\partial, \nabla, K)$ coincide with the structure operators of the representation $(C_{(0)} \oplus B_{(1)})^\Gamma \in \text{Rep}^2(A)$ dual to $(\partial, \nabla, K)$.
Proof. Let $c \in \Gamma(C)$ and $\psi \in \Gamma(B^*)$. By (4.1), we have
\begin{align*}
(\delta \vartheta^\psi)(c) \circ q_{C^*} \circ p_{C^*}^\text{hor} &= L_{p_D\psi}(c) \circ p_{C^*}^\text{hor} = \{c_{C^*}, \ell_c \circ p_{C^*}^\text{hor}\}_{D_B^*} = \{-\{\psi \circ p_B, \ell_c\}_{D_B^*} = L_{p_D\psi}(c) \circ p_B = (\psi, \partial(c)) \circ q_B \circ p_B.
\end{align*}
Since $q_{C^*}^\text{hor} \circ p_{C^*}^\text{hor} = q_B \circ p_B$, the equality $\delta \vartheta^\psi = \partial^* \psi$ follows.

Let us prove the relation between the $A$-connections. By (4.1) and (4.2) together with (A.2) and (A.3), one has that
\begin{align*}
\ell_{\nabla_{\vartheta^\psi}^\text{ver}} \circ p_B &= \ell_{\nabla_{\vartheta^\psi}^\text{ver}} = -\ell_{C^*} = -\{c_{C^*}, \ell_c \circ p_{C^*}^\text{ver}\}_{D_B^*} = \left\{\ell_{h(a)}, \ell_\psi \circ p_B\right\}_{D_B^*} = L_{p_D\psi}(c) \circ p_B = \ell_{\nabla_{\vartheta^\psi}^\text{ver}} \circ p_B.
\end{align*}

Similarly,
\begin{align*}
\ell_{\nabla_{\vartheta^\psi}^\text{ver}} \circ p_{C^*}^\text{hor} &= L_{p_D\psi}(c) \circ p_{C^*}^\text{hor} = \{c_{C^*}, \ell_c \circ p_{C^*}^\text{hor}\}_{D_B^*} = \left\{\ell_{h(a)}, \ell_\psi \circ p_{C^*}^\text{hor}\right\}_{D_B^*} = \left\{\ell_{h(a)}, \ell_\psi \circ p_{C^*}^\text{ver}\right\}_{D_B^*} = \ell_{|h(a),\ell_\psi\circ p_{C^*}^\text{ver}}.
\end{align*}

Hence, we have verified the equality $\nabla_{\vartheta^\psi}^\text{ver} = \nabla^*\psi$.

It remains to compare the curvatures. For that, choose $a_1, a_2 \in \Gamma(A)$ and let $\hat{K} \in \Gamma(B^* \otimes C)$ and $K_{\nabla\psi} \in \Gamma(C^*, D_B^*)$ be the linear sections corresponding to $K(a_1, a_2) \in \Gamma(B^* \otimes C)$ and $K_{\nabla\psi}(a_1, a_2) \in \Gamma(C^* \otimes B)$ respectively. First note that, by (A.3),
\begin{align*}
\ell_{c_{\psi}} \big|_{D_B^*} &= \left\{\ell_{h(a_1), h(a_2)}\right\}_{D_B^*} = \left\{\ell_{h(a_1)}, \ell_{h(a_2)}\right\}_{D_B^*} = \ell_{|h(a_1), h(a_2)|_D} = \ell_{\nabla_{\vartheta^\psi}^\text{ver}} = \ell_{\nabla^*\psi}.
\end{align*}

Therefore, by (4.3) and (A.3), we find
\begin{align*}
\ell_{K^{-\nabla\psi}} &= \ell_{C^*_{\psi}} = \ell_{C^*_{\psi}} = \ell_{h(a_1), h(a_2)} - h((a_1, a_2)_A) = \ell_{|h(a_1), h(a_2)|_D - h((a_1, a_2)_A)} = \ell_{\hat{K}} = -\ell_{C^*_{\psi}}\psi.
\end{align*}

This proves that $K_{\nabla\psi} = -K^*$. 

\begin{thebibliography}{99}

[1] C. Arias Abad, M. Crainic, Representations up to homotopy of Lie algebroids, J. Reine Angew. Math. 663, 91-126, 2012.
[2] C. Arias Abad, M. Crainic, The Weil algebra and the Van Est isomorphism, Ann. Inst. Fourier (Grenoble) 61(3), 927-970, 2011.
[3] C. Arias Abad, M. Crainic, Representations up to homotopy and Bott’s spectral sequence for Lie groupoids, preprint available at arXiv:0911.2859v1
[4] H. Bursztyn, A. Cabrera, Multiplicative forms at the infinitesimal level, preprint arXiv:1001.0534v2 [math.DG]
[5] H. Bursztyn; A. Cabrera; C.Ortiz, Linear and multiplicative 2-forms Lett. Math. Phys. 90, 59–83, 2009
[6] H. Bursztyn; M. Crainic; A. Weinstein; C. Zhu, Integration of twisted Dirac brackets Duke Math. J. 123(3), 549–607, 2004.
[7] M. Crainic, M. A. Salazar, I. Struchiner, Multiplicative forms and Spencer operators, preprint available at arXiv:1210.2977 [math.DG]
[8] A. Gracia-Saz, R. Mehta, Lie algebroid structures on double vector bundles and representation theory of Lie algebroids, Advances in Mathematics, 223, 1236–1275, 2010.
[9] E. Hawkins A groupoid approach to quantization, J. Symplectic Geom. 6(1), 61–125, 2008.
[10] M. Jotz, C. Ortiz, Foliated groupoids and their infinitesimal data, preprint available at arXiv: 1109.4515 [math.DG].
[11] K. Mackenzie, General theory of Lie groupoids and Lie algebroids, Lecture Notes London Math. Soc. 213 (2005)
[12] K. Mackenzie, Ehresmann doubles and Drinfel’d doubles for Lie algebroids and Lie bialgebroids, J. Reine Angew. Math. 658, 193-245, 2011.
[13] C. Ortiz, Multiplicative Dirac structures on Lie groups, C.R. Acad. Sci. Paris Ser. I. 346, (23-24) 1279-1282, 2008
[14] K.Mackenzie; P. Xu, Lie bialgebroids and Poisson groupoids, Duke Math. J. 73(2), 41542, 1994
\end{thebibliography}
Universidade Federal do Rio de Janeiro, Instituto de Matemática, 21945-970, Rio de Janeiro - Brazil. 
E-mail address: drummond@im.ufrj.br

Department of Mathematics, University of California, Berkeley, CA 94720. 
E-mail address: madeleine.jotz@gmail.com

Departamento de Matemática, Universidade Federal do Paraná, Setor de Ciências Exatas - Centro Politécnico 81531-990 Curitiba - Brasil. 
E-mail address: cristian.ortiz@ufpr.br