Parahoric induction and chamber homology for

$SL_2$

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Abstract

We consider the special linear group $G = SL_2$ over a $p$-adic field, and its diagonal subgroup $M \cong GL_1$. Parabolic induction of representations from $M$ to $G$ induces a map in equivariant homology, from the Bruhat-Tits building of $M$ to that of $G$. We compute this map at the level of chain complexes, and show that it is given by parahoric induction (as defined by J.-F. Dat).

Introduction

Consider the special linear group $G = SL_2(F)$ over a $p$-adic field $F$. Parabolic induction is the functor $i^G_M$ which takes (smooth, complex) representations of the diagonal subgroup $M \subset G$, pulls them back to the upper-triangular subgroup $P$ along the quotient map $P \to M$, and then induces up to $G$. This construction is remarkably efficient: it generically preserves irreducibility, and the coincidences between the resulting representations of $G$ are few and (mostly) easily understood.

Now let $\mathcal{O}$ be the ring of integers of $F$. The functor producing representations of $K = SL_2(\mathcal{O})$ from representations of its diagonal subgroup $L$ according to the above recipe has fewer desirable properties: for example, the representations thus produced are infinite-dimensional, and therefore far from irreducible. Dat has proposed a replacement for $i^G_M$ in this context, called parahoric induction [10].

The representation theory of $K$ (and of other compact open subgroups of reductive $p$-adic groups) is of interest not just for its own sake, but also in relation to the representation theory of $G$: see [7], for example. An appealing feature of Dat’s construction is its compatibility with parabolic induction: there
is a commutative diagram

\[
\begin{array}{ccc}
\text{Mod}(L) & \xrightarrow{\text{parahoric induction}} & \text{Mod}(K) \\
\downarrow \text{compact induction} & & \downarrow \text{compact induction} \\
\text{Mod}(M) & \xrightarrow{\text{parabolic induction}} & \text{Mod}(G)
\end{array}
\]

of functors between categories of smooth representations. (Dat proves this for a general minimal Levi subgroup of a reductive group [10 (1.4)].)

The main result of this paper is a commutative diagram of a similar kind:

\[
\begin{array}{ccc}
\text{C}_*^M(X_M) & \xrightarrow{\text{parahoric induction}} & \text{C}_*^G(X_G) \\
\downarrow \text{compact induction} & & \downarrow \text{compact induction} \\
\text{C}_*(\text{Mod}_f(M)) & \xrightarrow{\text{parabolic induction}} & \text{C}_*(\text{Mod}_f(G))
\end{array}
\]

Here $X_G$ and $X_M$ denote the Bruhat-Tits buildings of $G$ and $M$. $C_*^G(X_G)$ is a complex of simplicial chains on $X_G/G$, the coefficients over a simplex $s$ being the representation ring of the isotropy group of $s$. (This is the canonical chain complex computing chamber homology for $G$; see [11]). $C_*^M(X_M)$ is the corresponding complex for the action of $M$ on $X_M$, whose isotropy groups are all equal to $L$. The map $C_*^M(X_M) \rightarrow C_*^G(X_G)$ combines the inclusion $X_M \hookrightarrow X_G$ with parahoric induction from $L$ to the isotropy subgroups of $G$.

In the bottom row of (\*), the subscripts $f$ indicate the subcategories of finitely generated representations. $C_*$ here denotes the Hochschild complexes associated to these categories by Keller ([14], cf. [19] and [18]), and the map $C_*(\text{Mod}_f(M)) \rightarrow C_*(\text{Mod}_f(G))$ is the one induced by the functor $i_M^G$. The vertical arrows are given, in degree zero, by inducing representations from the isotropy groups of vertices up to $G$ and $M$ respectively. In higher degree, these maps are defined only at the level of homology.

The commutativity of the diagram in degree zero is essentially Dat’s result, for which we do not offer a new proof. The point of (\*) is that Dat’s definition extends in a natural way to a map between chamber-homology complexes, which is still compatible with parabolic induction in higher degree. Since the homology groups for SL$_2$ vanish in degree $\geq 2$, our extension of Dat’s theorem is so far a modest one; partial results for SL$_n$, discussed at the end of the paper, point toward a more ambitious generalisation.

This paper has three sections. Section [1] reviews Dat’s construction, and presents a few new results, in a general setting; note, though, that unlike Dat we work only over $C$. Section [2] contains explicit calculations in the case of SL$_2(O)$. Section [3] contains the main result, Theorem [3.4], on the commutativity of (\*). The theorem also gives a realisation in chamber homology of the Jacquet
restriction functor $r_G^C_M$. This part is comparatively easy: the restriction map $C_G^H(X_G) \to C_M^H(X_M)$ is just the naive analogue of $r_G^G_M$ for compact subgroups.

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**Basic definitions, notation and conventions:**

All vector spaces are over $\mathbb{C}$. If $G$ is a locally compact, totally disconnected group, then $\mathcal{H}(G)$ denotes the Hecke algebra of $G$, and $\text{Mod}(G)$ is the category of (smooth) representations of $G$. The terminology is explained, for example, in [21]. For a closed subgroup $H$ of $G$, and a representation $\rho$ of $H$, $\text{ind}_H^G \rho$ is the space of locally constant, compactly supported functions $f : G \to \rho$ satisfying $f(hg) = hf(g)$ for all $g \in G$ and $h \in H$; $G$ acts on $\text{ind}_H^G \rho$ by right translation. When $G$ is compact, $R(G)$ denotes the complexified representation ring of $G$, and $\text{Cl}(G)$ the space of locally constant class functions on $G$; the latter two spaces are isomorphic via the map sending a representation $\pi$ to its character $\text{ch}_\pi$. We write $e_G$ for the function on $G$ with constant value $1/\text{vol}(G)$. We make frequent appeal to “the Mackey formula” for the composition of restriction and induction functors; the version of this formula proved by Kutzko in [15] covers all of the cases that arise here.

1 **Inflation for Groups with an Iwahori Decomposition**

**Definition and basic properties**

**Definition 1.1.** An *Iwahori decomposition* of a compact totally disconnected group $J$ is a triple $(U, L, U)$ of closed subgroups of $J$, such that

(1) $L$ normalises $U$ and $\overline{U}$, and

(2) The product map $U \times L \times \overline{U} \to J$ is a homeomorphism.

(Note that if (2) holds, then thanks to (1) the same is true for any ordering of the factors $U$, $L$, and $\overline{U}$.)

The motivation for this definition comes from reductive $p$-adic groups: every such group $G$ contains arbitrarily small compact open subgroups which admit Iwahori decompositions compatible with the Levi decompositions of the parabolic subgroups of $G$. See [21] V.5.2 for a precise statement; the primordial example is [13] §2.2.

The standard theory of invariant measures on homogeneous spaces (as in, e.g., [24]) shows that:
Lemma 1.2. Let \( J = ULU \) be a group with an Iwahori decomposition, and let \( du, dl \) and \( d\bar{w} \) be Haar measures on \( U, L \) and \( U \) respectively. The product measure \( du dl d\bar{w} \) is a Haar measure on \( J \).

Let \( J = ULU \) be a group with a fixed Iwahori decomposition. From now on we assume that the Haar measures on \( J, U, L, \) and \( U \) are all normalised to have total volume 1. The Hecke algebra \( \mathcal{H}(J) \) is both a left and a right module over \( \mathcal{H}(U), \mathcal{H}(L), \) and \( \mathcal{H}(\bar{U}) \). Since \( L \) normalises \( U \) and \( U \), the action of \( \mathcal{H}(L) \) commutes with the idempotents \( e_U \) and \( e_{\bar{U}} \).

The following definition is Dat’s [10].

Definition 1.3. Consider the tensor-product functors

\[
i_{\pi, \bar{\pi}} : \text{Mod}(L) \to \text{Mod}(J) \quad i_{\pi, \bar{\pi}} \rho = \mathcal{H}(J) \varphi \otimes_{\mathcal{H}(L)} \rho
\]

\[
r_{\pi, \bar{\pi}} : \text{Mod}(J) \to \text{Mod}(L) \quad r_{\pi, \bar{\pi}} \pi = e_U e_{\bar{U}} \mathcal{H}(J) \otimes_{\mathcal{H}(J)} \pi.
\]

The following concrete realisations of \( i_{\pi, \bar{\pi}} \) and \( r_{\pi, \bar{\pi}} \) are sometimes useful. Let \( i_U, i_{\bar{\pi}} : \text{Mod}(J) \to \text{Mod}(J) \) be the composite functors \( i_U = \text{ind}_{L}^{J} \text{infl}_{U}^{L} \) and \( i_{\bar{\pi}} = \text{ind}_{L}^{J} \text{infl}_{\bar{\pi}}^{L} \), where for example \( \text{infl}_{U}^{L} \) is the functor of inflation, i.e., pull-back along the quotient map \( LU \to L \). Then \( i_U \rho \cong \mathcal{H}(J) e_U \otimes_{\mathcal{H}(L)} \rho \), and likewise for \( i_{\bar{\pi}} \) and \( \mathcal{H}(J) e_{\bar{\pi}} \). Computing the map \( \mathcal{H}(J) e_{\bar{\pi}} \mapsto \mathcal{H}(J) e_U \) in this picture, one finds that

\[
i_{\pi, \bar{\pi}} \rho \cong \text{image} \left( i_{\bar{\pi}} \rho \xrightarrow{I_U} i_U \rho \right), \quad \text{where} \quad I_U(f)(j) = \int_U f(uj) \, du.
\]

Similarly, let \( r_U, r_{\bar{\pi}} : \text{Mod}(J) \to \text{Mod}(L) \) be the functors \( r_U \pi = \pi^U \) (the \( U \)-invariants in \( \pi \)), and \( r_{\bar{\pi}} \pi = \pi^{\bar{U}} \). Then

\[
r_{\pi, \bar{\pi}} \pi \cong \text{image} \left( r_{\bar{\pi}} \pi \xrightarrow{e_U} r_U \pi \right), \quad \text{where} \quad e_U(x) = \int_U ux \, du.
\]

We shall use another characterisation of \( i_{\pi, \bar{\pi}} \) and \( r_{\pi, \bar{\pi}} \), based on the following observation.

Lemma 1.4. The map \( \Phi : \mathfrak{Z}(L) \to \text{Hom}_{J \times L}(\mathcal{H}(J)e_{\bar{\pi}}, \mathcal{H}(J)e_U) \) defined by \( \Phi(z) : f \mapsto f e_U z \) is a \( \mathfrak{Z}(L) \)-linear isomorphism.

(Here \( \mathfrak{Z}(L) \cong \text{End}_{L \times L}(\mathcal{H}(L)) \) denotes the Bernstein centre of \( L \).)

Proof. Frobenius reciprocity gives \( \text{Hom}(i_{\bar{\pi}}, i_U) \cong \text{Hom}_{L}(r_U i_{\bar{\pi}}, id) \). Evaluation of functions at the identity in \( J \) gives a natural transformation \( r_U i_{\bar{\pi}} \to id_L \), which is an isomorphism because \( J = ULU \). Applying the Yoneda lemma, we obtain an isomorphism

\[
\mathfrak{Z}(L) \cong \text{End}_{L \times L}(\mathcal{H}(L)) \xrightarrow{\Phi} \text{Hom}_{J \times L}(\mathcal{H}(J)e_{\bar{\pi}}, \mathcal{H}(J)e_U),
\]

which is \( \mathfrak{Z}(L) \)-linear by the naturality of the construction. Computing the Frobenius reciprocity isomorphism explicitly, one finds that \( \Phi(1) : f \mapsto f e_U \).
**Lemma 1.5.** If $\rho$ is an irreducible representation of $L$, then $i_U \rho$ is the unique irreducible representation of $J$ common to both $i_U \rho$ and $\pi \rho$. Moreover, $i_U \rho$ has multiplicity one in both $i_U \rho$ and $\pi \rho$.

**Proof.** Lemma 1.4 implies that $\text{Hom}_J(\pi \rho, i_U \rho)$ is one-dimensional, spanned by $I_U$. The result now follows from Schur’s lemma. □

**Examples 1.6.** (1) When $U = \{1\}$ is trivial, so that $J \cong L \ltimes U$, one has $i_U \rho \cong i_U \rho$, the usual inflation functor.

(2) Let $\text{triv}_L$ be the trivial representation of $L$. Then $\text{triv}_J$ sits inside both $i_U \text{triv}_L$ and $\pi \text{triv}_L$, as the space of constant functions in each case. So $i_U \rho \cong i_U \rho$.

(3) Let $\tilde{\rho}$ denote the (smooth) contragredient of $\rho$. Then $i_U \tilde{\rho} \cong i_U \tilde{\rho}$, and likewise for $\pi \tilde{\rho}$, and so Lemma 1.5 implies that $i_U \tilde{\rho} \cong i_U \tilde{\rho}$.

**Definition/Lemma 1.7.** Let $\tilde{z}_{U, \pi} \in \mathfrak{g}(L)$ be the preimage of the map

$$
\mathcal{H}(J)_{e_U} \xrightarrow{f \mapsto f e_U e_U} \mathcal{H}(J)_{e_U}
$$

under the isomorphism $\Phi$ of Lemma 1.2. Let $z_{U, \pi}$ be the image of $\tilde{z}_{U, \pi}$ under the involution $l \mapsto l^{-1}$ on $\mathfrak{g}(L)$. Then $z_{U, \pi}$ is invertible.

**Proof.** We must show that $\tilde{z}_{U, \pi}$ acts as a nonzero scalar on each irreducible representation $\rho$ of $L$. Lemma 1.5 ensures that $i_U \rho \cong i_U \rho$, and so Lemma 1.5 implies that $i_U \rho \cong i_U \rho$.

Other descriptions of $z$ appear in Proposition 1.11 and Remark 1.12. Explicit formulae for the Iwahori subgroup in $SL_2(F)$ are given in [10, Section 2.4] and in Proposition 1.4.

**Proposition 1.8.** [10, Proposition 2.2] The operator $z_{U, \pi}^{-1} e_U e_U$ acts as an idempotent on each representation of $J$.

**Proof.** The definition of $\tilde{z}_{U, \pi}$ ensures that

$$
(f e_U e_U)^2 = f \tilde{z}_{U, \pi} e_U e_U
$$

for every $f \in \mathcal{H}(J)$. Applying the involution $j \mapsto j^{-1}$ on $\mathcal{H}(J)$ gives the desired result. □

The following basic properties of $i_{U, \pi}$ and $r_{U, \pi}$ follow easily from Proposition 1.8 as in [10, Lemme 2.8 and Corollaire 2.9]:
Proposition 1.9. (1) There are isomorphisms $i_{U,U} \cong i_{U,U}$ and $r_{U,U} \cong r_{U,U}$.
(2) $i_{U,U}$ and $r_{U,U}$ are mutual two-sided adjoints.
(3) $r_{U,U}i_{U,U} \cong \text{id}_L$.

We also obtain a counterpart to Lemma 1.5 for $r_{U,U}$:

Lemma 1.10. Let $\pi$ be an irreducible representation of $J$. Then

$$\dim \text{Hom}_L(r_{U,U}\pi, r_{U,U}\pi) = 0 \text{ or } 1.$$ If the dimension is zero, then $r_{U,U}\pi = 0$. If the dimension is 1, then $r_{U,U}\pi \cong r_{U,U}\pi \cong r_{U,U}\pi$.

Proof. First note that if $\text{Hom}_L(r_{U,U}\pi, r_{U,U}\pi) = 0$, then in particular the map $e_U : r_{U,U}\pi \to r_{U,U}\pi$ is zero, and so its image $r_{U,U}\pi$ is zero.

Now suppose that the intertwining space is nonzero. There exists a non-zero irreducible representation $\rho$ of $L$ common to both $r_{U,U}\pi$ and $r_{U,U}\pi$, which by Frobenius reciprocity implies that $\pi$ is a common irreducible component of $i_{U,U}\rho$ and $i_{U,U}\rho$. So by Lemma 1.5, $\pi \cong i_{U,U}\rho$. Thus $r_{U,U}\pi$ is a nonzero quotient of the irreducible representation $r_{U,U}\rho \cong \rho$, so $r_{U,U}\pi \cong \rho$. Similarly, $r_{U,U}\pi \cong \rho$, and so $\text{Hom}_L(r_{U,U}\pi, r_{U,U}\pi) \cong \text{End}_L(\rho)$ is one-dimensional.

Character formulae

Let $J = ULU$ be a compact totally disconnected group with an Iwahori decomposition. All the groups in question will be fixed throughout this section, and we write $i = i_{U,U}$ and $r = r_{U,U}$. Passing from representations $\pi$ to their characters $\text{ch}_\pi$, we may view $i$ and $r$ as maps between the spaces $\text{Cl}^X(J)$ and $\text{Cl}^X(L)$ of class functions on $J$ and $L$.

For example, suppose that $J = UL$ (i.e., $U = \{1\}$), so that $i$ is the usual inflation of representations, while $r$ is the functor $\pi \mapsto \pi^U$. The action on characters is easily computed: $i$ is given by pulling functions back along the quotient map $J \to L$, while $r$ is given by integration along the fibres of this map.

Returning to the general case, consider the map $\lambda = \lambda_{U,U} : J \to L$ defined by $\lambda(u\bar{u}) = l$. Then define $\lambda^* : \text{Cl}^X(J) \to \text{Cl}^X(L)$ and $\lambda^* : \text{Cl}^X(L) \to \text{Cl}^X(J)$ by

$$(\lambda^* \varphi)(l) = \int_J \varphi(\lambda^{-1}k) dk, \quad (\lambda^* \psi)(j) = \int_J \psi(\lambda(jk^{-1}l)) dk,$$

for $\varphi \in \text{Cl}^X(J)$ and $\psi \in \text{Cl}^X(L)$.

Proposition 1.11. Let $J = ULU$ be a group with Iwahori decomposition, and let $z = z_{U,U} \in \mathfrak{Z}(L)$ be as in Proposition 1.8

(1) The maps $\text{Cl}^X(J) \xrightarrow{r} \text{Cl}^X(L)$ are given by $r = z^{-1}\lambda^*$ and $i = \lambda^*z^{-1}$.
(2) For each irreducible representation \( \rho \) of \( L \), one has \( z(\rho) = \frac{\dim \rho}{\dim(i \rho)} \).

Proof. We first consider \( r \). For each irreducible \( \pi \) of \( J \), and each \( l \in L \),

\[ \lambda_\pi(ch_\pi(l)) = \int_U \int_\mathbb{C} \text{Trace}(\tau(lu\overline{u})) \, d\tau \, du = \text{Trace}(\tau(l)\pi(\tau_\pi)) \]

If \( r = 0 \), then \( \pi(e_U)\pi(\tau_\pi) = 0 \), and so \( r(ch_\pi) = z^{-1}\lambda_\pi(\pi) = 0 \). On the other hand, suppose that \( r \pi = \rho \). Then

\[ \lambda_\pi(ch_\pi(l)) = \text{Trace}(\tau(l)\pi(\tau_\pi)) = z(\rho) \text{Trace}(\tau(l)z(\rho)\pi(\tau_\pi)) \]

and \( z(\rho)^{-1}\pi(e_U)\pi(\tau_\pi) \) is a projection of \( \pi \) onto \( r \pi \) (Proposition 1.8). So

\[ \lambda_\pi(ch_\pi(l)) = z(\rho) \text{Trace}(\tau(l)|_{r \pi}) = z(r \pi) \, ch_\pi(l), \]

giving \( r = z^{-1}\lambda_\pi \).

Now turn to the map \( i \). We consider the usual inner products on \( Cl^L(L) \) and \( Cl^J(J) \):

\[ \langle \psi_1, \psi_2 \rangle_L = \int_L \psi_1(l)\overline{\psi_2(l)} \, dl \]

for \( \psi_1, \psi_2 \in Cl^L(L) \), and similarly for \( J \). The characters of irreducible representations constitute orthonormal bases for \( Cl^L(J) \) and \( Cl^J(L) \).

A straightforward computation with Lemma 1.2 shows that for each \( \psi \in Cl^L(L) \) and \( \varphi \in Cl^J(J) \), one has \( \langle \lambda^*\psi, \varphi \rangle_J = \langle \psi, \lambda_\pi \varphi \rangle_L \). Also, \( \langle i \psi, \varphi \rangle_J = \langle \psi, r \varphi \rangle_L \), because the functors \( i \) and \( r \) are adjoints. Thus the formula \( r = z^{-1}\lambda_\pi \) gives, upon taking adjoints, \( i = \lambda^*z^{-1} \), where \( z^{-1} \) denotes the complex conjugate of \( z^{-1} \). Noting that \( \langle \lambda^*\psi \rangle(1) = \psi(1) \) for all \( \psi \in Cl^L(L) \), we find

\[ \dim(i \rho) = i(ch_\rho)(1) = \lambda^*z^{-1} \dim(ch_\rho)(1) = z^{-1}(\rho)\lambda^* \dim(ch_\rho)(1) = \overline{z^{-1}} \dim(\rho). \]

Therefore \( \overline{z^{-1}}(\rho) = \dim(\rho)/\dim(i \rho) \), which is real, and (2) follows. Putting \( \tau = z \) into \( i = \lambda^*z^{-1} \) completes the proof of (1). \Box

Remark 1.12. The number \( z(\rho) \) may be interpreted as measuring the relative position of the idempotents \( e_U \) and \( e_\tau \) in the representation \( i \rho \), as we shall now explain.

Let \( \pi \) be an irreducible representation of \( J \), and choose a \( J \)-invariant inner product on \( \pi \). The self-adjoint idempotents \( P = \pi(e_U) \) and \( Q = \pi(2q) \) determine a finite-dimensional unitary representation of the infinite dihedral group \( \Gamma = (\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/2\mathbb{Z}) \): the generating involutions \( s_1, s_2 \in \Gamma \) map to the self-adjoint unitary operators \( 2P - 1 \) and \( 2Q - 1 \), respectively. This representation \( \pi \) of \( \Gamma \) decomposes into a direct sum of isotypical components, and each isotypical component is stable under the action of \( L \).

Recall the list of irreducible unitary representations of \( \Gamma \): for each angle \( \alpha \in [0, \pi/2] \) one forms the two-dimensional representation \( \tau_\alpha \) in which \( P \) and \( Q \) are represented by the matrices

\[ \tau_\alpha(P) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tau_\alpha(Q) = \begin{bmatrix} \cos^2 \alpha & \cos \alpha \sin \alpha \\ \cos \alpha \sin \alpha & \sin^2 \alpha \end{bmatrix}. \]
For $\alpha \in (0, \pi/2)$, the $\tau_\alpha$ are irreducible and mutually inequivalent. The representations $\tau_0$ and $\tau_\pi$ each decompose into one-dimensional summands: $\tau_0 = \tau_0' \oplus \tau_0''$ and $\tau_\pi = \tau_\pi' \oplus \tau_\pi''$. These four one-dimensional representations, together with the irreducible $\tau_\alpha$, form a complete list of the irreducible unitary representations of $\Gamma$. (The list is obtained by expressing $\Gamma$ as a semidirect product $(\mathbb{Z}/2\mathbb{Z}) \ltimes \mathbb{Z}$, and applying Mackey theory [17, Section 14].)

Now, $r \pi$ is the range of $PQ$, and $PQ$ is nonzero only in $\tau_0'$ and in the $\tau_\alpha$ components for $\alpha \in (0, \pi/2)$. So

$$r \pi \neq 0 \iff \pi|_\Gamma \text{ contains } \tau_0' \text{ or } \tau_\alpha \text{ for some } \alpha \in (0, \pi/2).$$

Suppose $r \pi \neq 0$, so that $\pi = i\rho$ for some irreducible $\rho$ of $L$. Since $r \pi$ is an irreducible representation of $L$, and $L$ preserves the isotypical decomposition of $\pi|_\Gamma$, it follows that $\pi|_\Gamma$ contains exactly one of the representations $\tau_0'$ or $\tau_\alpha$ (possibly with multiplicity $> 1$). We then have $PQP = \cos^2(\alpha)P$ (setting $\alpha = 0$ if $\pi|_\Gamma$ contains $\tau_0'$), which by the definition of $z$ implies that

$$z(\rho) = \cos^2(\alpha).$$

Thus the formula $z(\rho) = \dim \rho / \dim(i\rho)$ imposes a restriction on the irreducible representations of $\Gamma$ that may occur in irreducible representations of $J$. For example, if $L$ is commutative, so that $\dim \rho = 1$ for every irreducible $\rho$, then the representation $\tau_\alpha$ of $\Gamma$ may occur only in those irreducibles $\pi$ of $J$ having $\dim \pi = 1 / \cos^2(\alpha)$.

## 2 The Iwahori Subgroup of $\text{SL}_2(F)$

Let $F$ be a $p$-adic field, with ring of integers $\mathcal{O}$ and maximal ideal $\mathfrak{p}$. Choose a generator $\varpi$ for $\mathfrak{p}$. We write $\mathfrak{f}$ for the residue field $\mathcal{O}/\mathfrak{p}$, and $q$ for the cardinality of $\mathfrak{f}$.

Let $G = \text{SL}_2(F)$, and consider the standard Iwahori subgroup [13, §2.2]

$$J = \left[ \begin{array}{cc} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{array} \right].$$

The notation means that $J$ is the group of determinant-one matrices whose bottom-left entry lies in $\mathfrak{p}$, and whose other entries lie in $\mathcal{O}$. (Similar notation will be used throughout the paper.) $J$ admits an Iwahori decomposition $J = ULU$, where

$$U = \left[ \begin{array}{cc} 1 & \mathcal{O} \\ 0 & 1 \end{array} \right], \quad L = \left[ \begin{array}{cc} \mathcal{O}^\times & 0 \\ 0 & \mathcal{O}^\times \end{array} \right], \quad U = \left[ \begin{array}{cc} 1 & 0 \\ \mathfrak{p} & 1 \end{array} \right].$$

Throughout this section we write $i$, $r$ and $z$ for $i_{U,V}$, $r_{U,V}$ and $z_{U,V}$.
Computations of $i$, $r$, and $z$

Let $\rho$ be an irreducible representation of $L$; identifying $L$ with $O^\times$ via $[\alpha_{-1}] \mapsto a$, we view $\rho$ as a smooth homomorphism $O^\times \to \mathbb{C}^\times$. If $\rho$ is trivial, then $i\rho$ is the trivial representation of $J$. Assume that $\rho$ is nontrivial, and let $c$ denote the conductor of $\rho$:

$$c = \min\{n \geq 1 \mid \rho \text{ is trivial on } 1 + p^n\}.$$

Then define $J_\varepsilon = \left[\frac{O}{p^\varepsilon} \frac{O}{p^\varepsilon}\right]$, and let $\rho : J_\varepsilon \to \mathbb{C}^\times$ be the homomorphism $\rho[\frac{a}{c \delta}] = \rho(a)$.

**Proposition 2.1.** (1) $i\rho \cong \text{ind}_{J_\varepsilon}^J \rho$.

(2) $z(\rho) = \begin{cases} 1 & \text{if } \rho \text{ is trivial,} \\ q^{1-c} & \text{if } \rho \text{ is nontrivial with conductor } c. \end{cases}$

**Proof.** A short computation shows that the image of $i_U : i_U \rho \to i_U \rho$ lies in the subspace $\text{ind}_{J_\varepsilon}^J \rho$, and so $i\rho \subseteq \text{ind}_{J_\varepsilon}^J \rho$. Using the Mackey formula, and the minimality of $c$, one can show that $\text{ind}_{J_\varepsilon}^J \rho$ is irreducible: see [2, Lemma 9.2]. This proves part (1).

For part (2), Proposition 1.11 gives $z(\rho) = \dim(i\rho)^{-1}$. For nontrivial $\rho$, part (1) implies that

$$\dim(i\rho) = [J : J_\varepsilon] = [p : p^\varepsilon] = q^{-1}.$$  

We now turn to the functor $r$. Let $t = \left[\begin{smallmatrix} \pi & \rho \\ 0 & \pi \end{smallmatrix}\right] \in G$. If $\pi$ is a representation of a subgroup $H$ of a group $G$, and if $g \in G$, then $\pi^g$ denotes the representation $x \mapsto \pi(gxg^{-1})$ of the group $H^g = g^{-1}Hg$.

**Lemma 2.2.** Let $\pi$ be a smooth, finite-dimensional representation of $J$. Then

$$\text{Hom}_L(r\pi, i_U \pi) \cong \text{Hom}_{J_\varepsilon \cap J^n}(\pi, \pi^n)$$

for all sufficiently large $n$. If $\pi$ is irreducible, then

$$r \pi = 0 \iff \text{Hom}_{J_\varepsilon \cap J^n}(\pi, \pi^n) = 0 \text{ for all } n >> 0.$$

**Proof.** To compactify the notation, let $J^n = J \cap J^n$. Explicitly, $J^n = \left[\frac{O}{p^n} \frac{O}{p^n}\right]$. This group has an Iwahori decomposition $J^n = U^n \overline{U}$, where $U^n = U^{t^n}$.

Since $t^n$ centralises $L$, we have an isomorphism $r\pi = \pi^U \cong (\pi^n)^U$ of representations of $L$, and so

$$\text{Hom}_L(\pi^U, \pi^n) \cong \text{Hom}_L(\pi^U, (\pi^n)^U).$$

Because $\pi$ is smooth and finite-dimensional, the kernel of $\pi$ contains some congruence subgroup $\left[\begin{smallmatrix} 1+p^n & p^n \\ p^n & 1+p^n \end{smallmatrix}\right]$. Clearly $U^n$ lies in this subgroup for sufficiently large $n$, as does $U^{t^n}$. So, for sufficiently large $n$, $\pi$ is trivial on $U^n$, while $\pi^{t^n}$ is trivial on $U$. We thus have for large $n$ that

$$\text{Hom}_L(\pi^U, (\pi^n)^U) \cong \text{Hom}_{U^n \overline{U}}(\pi, (\pi^n)^U) \cong \text{Hom}_{U^n \overline{U}}(\pi, \pi^n).$$

The second assertion follows immediately from the first and Lemma 1.10. 

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Proposition 2.3. Let $\pi$ be an irreducible representation of $J$. Then
$$r\pi = 0 \iff \dim \left( \operatorname{End}_G(\operatorname{ind}_J^G \pi) \right) < \infty.$$ 

Proof. The Mackey formula gives
$$\operatorname{End}_G(\operatorname{ind}_J^G \pi) \cong \bigoplus_{g \in J \backslash G / J} \operatorname{Hom}_{J \cap \mathcal{J}}(\pi, \pi^g).$$
Let $w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. According to the Bruhat decomposition, $\{t^n, t^nw \mid n \in \mathbb{Z}\}$ is a set of representatives for the double-coset space $J \backslash G / J$ [23, II.1.7].

If $r\pi \neq 0$, then Lemma 2.2 ensures that the space $\operatorname{Hom}_{J \cap \mathcal{J}}(\pi, \pi^{tn})$ is nonzero for all $n \gg 0$. Thus $\operatorname{End}_G(\operatorname{ind}_J^G \pi)$ is infinite-dimensional in this case.

For the converse, suppose that $r\pi = 0$. Lemma 2.2 implies that the cosets $Jt^nJ$, for $n \geq 0$, contribute only finitely many dimensions to $\operatorname{End}_G(\operatorname{ind}_J^G \pi)$. Since $\operatorname{Hom}_{J \cap \mathcal{J}}(\pi, \pi^{tn}) \cong \operatorname{Hom}_{J \cap \mathcal{J}}(\pi^{tn}, \pi)$, the same is true for $n < 0$. A small modification of Lemma 2.2 shows that the contribution of the double cosets $Jt^n\mathcal{J}J$ is likewise finite-dimensional.

Remark 2.4. The vanishing of $r\pi$ does not guarantee that $\operatorname{ind}_J^G \pi$ is a supercuspidal representation of $G$. For example, consider the groups $B(f) = \begin{bmatrix} f^* & f^* \\ f & 0 \end{bmatrix}$ and $N(f) = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix}$, and let $\psi$ be a nontrivial one-dimensional representation of $N(f)$. Since $B(f)$ is a quotient of $J$, the representation $\pi = \operatorname{ind}_{N(f)}^{B(f)} \psi$ may be inflated to a representation of $J$. A Mackey-formula computation shows that $\pi^\mathcal{J} = \pi^{N(f)} = 0$, and so $r\pi = 0$. Now, $\pi$ does contain a nonzero vector fixed by the diagonal subgroup $M(f)$: namely, the function $f \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \psi(xy)$. The quotient map $J \to B(f)$ sends $J \cap \mathcal{J}J$ onto $M(f)$, and so we have $\pi^{J \cap \mathcal{J}J} \neq 0$.

An application of the Mackey formula then gives $(\operatorname{ind}_J^G \pi)^J \neq 0$, and so $\operatorname{ind}_J^G \pi$ has a nonzero summand in the unramified principal series [10, Lemma 4.7].

It is true, on the other hand, that the cuspidality of $\operatorname{ind}_J^G \pi$ implies $r\pi = 0$: indeed, if $r\pi = \rho \neq 0$, then the pair $(J, \pi)$ is a type for the (non-cuspidal) Bernstein component $[M, \rho]_G$ of $G$ [11].

Parahoric Induction

We continue to consider the Iwahori subgroup $J \subset \text{SL}_2(F)$, with its decomposition $J = U \mathcal{L} \mathcal{U}$. Let $K = \text{SL}_2(O)$, and define a functor
$$i^K_{U, \mathcal{U}} : \text{Mod}(L) \to \text{Mod}(K), \quad i^K_{U, \mathcal{U}} = \operatorname{ind}_J^K i_{U, \mathcal{U}}.$$ 
This is an example of parahoric induction; see [10] for the general definition.

The family of representations $i^K_{U, \mathcal{U}} \rho$, as $\rho$ ranges over the irreducibles of $L$, may be considered a kind of principal series for $K$. We will show that the irreducibility and intertwining properties of these representations are exactly analogous to those of the principal series for $\text{SL}_2(F)$ (as explained in [11], for instance).
Lemma 2.5. Suppose that $I$ and $I'$ are closed subgroups of a compact totally disconnected group, having Iwahori decompositions $I = WMW$ and $I' = VMV$, where $V \subseteq W$ and $W \subseteq V$. Then $\text{Hom}_{I'\cap I'}(i_{W,WM} \rho, i_{W,VM} \tau) \cong \text{Hom}_M(\rho, \tau)$ for all $\rho, \tau \in \text{Mod}(M)$.

Proof. Let $H = I \cap I'$. This group has an Iwahori decomposition $H = YMY'$, where $Y = V$ and $Y' = W$. (We write $Y$ and $Y'$ in an attempt to avoid ambiguity in the notation; so, for example, $i_Y$ is a functor from $\text{Mod}(M)$ to $\text{Mod}(H)$, while $i_{Y'}$ is a functor from $\text{Mod}(M)$ to $\text{Mod}(I')$.)

Restriction of functions from $I$ to $H$ gives an $H$-equivariant isomorphism $i_W \rho \cong i_Y \rho$; similarly $i_{Y'} \tau \cong i_{Y'} \tau$. Embedding $i_W \rho \subseteq i_Y \rho$ and $i_{Y'} \tau \subseteq i_{Y'} \tau$, we obtain an injective map

\[(2.6) \quad \text{Hom}_H(i_W, i_Y \rho) \cong \text{Hom}_H(i_Y, i_{Y'} \rho) \cong \text{Hom}_M(\rho, \tau);\]
the last isomorphism holds by Frobenius reciprocity, as in Lemma 1.4.

On the other hand, restriction of functions from $I$ to $H$ gives a surjective, $H$-equivariant map $i_{Y'} \rho \to i_{Y'} \rho$ making the diagram

\[
\begin{array}{ccc}
& i_W \rho & \\
\text{restrict} & \downarrow & \text{restrict} \\
i_Y \rho & \quad & i_{Y'} \rho
\end{array}
\]
commute. This diagram exhibits $i_Y \rho$ as a quotient of $i_W \rho$; a similar argument shows that $i_{Y'} \tau$ is a quotient of $i_{Y'} \tau$. We therefore have an injective map

\[(2.7) \quad \text{Hom}_M(\rho, \tau) \cong \text{Hom}_H(i_Y, i_Y \rho) \cong \text{Hom}_H(i_{Y'} \rho, i_{Y'} \tau) \cong \text{Hom}_M(\rho, \tau);\]
the first isomorphism holds by Proposition 1.9.

Since $\text{Hom}_M(\rho, \tau)$ is finite-dimensional when $\rho$ and $\tau$ are, the injective maps (2.6) and (2.7) are isomorphisms.

Applied to the Iwahori subgroup in $SL_2(F)$, Lemma 2.5 gives the following Mackey-type formula for parahoric induction and restriction. (The corresponding formula in the general case is the subject of ongoing work with Ehud Meir and Uri Onn.)

Lemma 2.8. Let $\rho$ and $\tau$ be representations of $L$. Then

\[\text{Hom}_K(i_U, i_U \rho, i_U \tau) \cong \text{Hom}_L(\rho, \tau) \oplus \text{Hom}_L(\rho, \tau^w).\]

Proof. Using the Mackey formula and the Bruhat decomposition $K = J \sqcup JwJ$, we find

\[\text{Hom}_K(i_U, i_U \rho, i_U \tau) \cong \text{Hom}_J(i_U, i_U \rho, i_U \tau) \oplus \text{Hom}_{JwJ}(i_U, i_U \rho, i_U \tau);\]
The first summand is isomorphic to $\text{Hom}_L(\rho, \tau)$, by Proposition 1.9. We have 
$(i_{U, \mathcal{T}} \tau)^w \cong i_{U, \mathcal{T}} \tau^w(\tau^w)$, and so Lemma 2.5 implies that the second summand is isomorphic to $\text{Hom}_L(\rho, \tau^w)$.

An application of Schur’s lemma then gives:

**Proposition 2.9.** Let $\rho$ and $\rho'$ be irreducible representations of $L$.

1. $i_K^U \rho$ is irreducible if and only if $\rho \neq \rho^w$.
2. If $\rho \cong \rho^w$, then $i_K^U \rho$ is a sum of two inequivalent irreducibles.
3. $i_K^U \rho \cong i_K^U \rho^w$.
4. $\text{Hom}_K(i_K^U \rho, i_K^U \rho') = 0$ if $\rho' \neq \rho$ or $\rho^w$.

## 3 Parahoric Induction and Chamber Homology for $\text{SL}_2(F)$

**Background**

Keep the notation $G$, $J$, $K$, $L$, $U$, $U'$, $F$, etc., from the previous section. We also set

$$M = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}, \quad N = \begin{bmatrix} 1 & F \\ 0 & 1 \end{bmatrix}, \quad \overline{N} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}, \quad K' = \begin{bmatrix} \mathcal{O} & p^{-1} \\ p & \mathcal{O} \end{bmatrix},$$

$$V = \begin{bmatrix} 1 & p^{-1} \\ 0 & 1 \end{bmatrix}, \quad w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad t = \begin{bmatrix} \varpi^{-1} & 0 \\ 0 & \varpi \end{bmatrix}, \quad \text{and } W = \{1, w\}/\pm 1$$

($p^{-1}$ means $\varpi^{-1} \mathcal{O}$). We consider the normalised Jacquet functors $i_M^G$ and $r_M^G$ of parabolic induction and Jacquet restriction along $P$ [21 VI.1].

Work of Bernstein [3] and Keller [14] implies that the Hochschild homology groups $\text{HH}_*(\mathcal{H}(G))$ and $\text{HH}_*(\mathcal{H}(M))$ may be defined in terms of the categories of finitely generated modules over $\mathcal{H}(G)$ and $\mathcal{H}(M)$, respectively: see [8]. The Jacquet functors preserve the subcategories of finitely generated modules in $\text{Mod}(G)$ and $\text{Mod}(M)$, and so they induce natural maps between $\text{HH}_*(\mathcal{H}(G))$ and $\text{HH}_*(\mathcal{H}(M))$.

We let $G_c$ denote the union of the compact subgroups of $G$. This set is open, closed, and conjugation-invariant in $G$, and so it determines a direct-summand $\text{HH}_*(\mathcal{H}(G))_c$ of $\text{HH}_*(\mathcal{H}(G))$: see [3]. The map $r_M^G$ sends $\text{HH}_*(\mathcal{H}(G))_c$ to $\text{HH}_*(\mathcal{H}(M))_c$, and the map $i_M^G$ sends $\text{HH}_*(\mathcal{H}(M))_c$ to $\text{HH}_*(\mathcal{H}(G))_c$ [8 Corollaries 3.12 and 3.19].

The Bruhat-Tits building $X$ of $G$ is an infinite, locally finite, connected tree, on which $G$ acts properly, simplicially, and without inversions [23 II.1]. The action is transitive on the set $X^1$ of edges, and has two orbits in the vertex-set $X^0$. The Iwahori subgroup $J$ is the isotropy group of an edge, whose vertices

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have isotropy groups $K$ and $K'$. The chamber homology $H^G_\ast(X)$ of $X$ is, by
definition, the homology of the following chain complex [2]:

$$C^G_\ast(X): \quad R(J) \xrightarrow{\partial_G} R(K) \oplus R(K') \quad \partial_G(\pi) = \text{ind}_J^K \pi \oplus -\text{ind}_J^{K'} \pi$$

The building $Y$ of $M$ identifies with an apartment (i.e., a line) in $X$. With
respect to the decomposition $M \cong L \times \langle t \rangle$, $L$ acts trivially on $Y$ while $t$ translates
the $i$th vertex to the $(i + 2)$nd. The chamber homology $H^M_\ast(Y)$ of $Y$ is the
homology of the chain complex

$$C^M_\ast(Y): \quad R(L) \oplus R(L) \xrightarrow{\partial_M} R(L) \oplus R(L) \quad \partial_M(\rho_0, \rho_1) = (\rho_0 + \rho_1, -\rho_0 - \rho_1)$$

In pictures, showing the (oriented) quotient complexes $Y/M$ and $X/G$ labelled
by their respective coefficient systems:

$$Y/M : \quad \begin{array}{c}
R(L) \circ \quad R(L) \\
\circ \quad R(L)
\end{array}$$

$$X/G : \quad \begin{array}{c}
R(K') \quad R(J) \\
\quad \circ \quad R(K)
\end{array}$$

There are canonical isomorphisms $H^G_\ast(X) \cong \text{HH}_\ast(\mathcal{H}(G))_c$ and $H^M_\ast(Y) \cong \text{HH}_\ast(\mathcal{H}(M))_c$: see [12] and [22]. (Part of the argument is also outlined below.)

The action of the Weyl group $W$ on $Y$ and $M$ induces an action on $C^M_\ast(Y)$
as follows: in degree zero, $w(\rho_0, \rho_1) = (\rho_0^w, \rho_1^w)$. In degree one, $w(\rho_0, \rho_1) = (\rho_1^w, \rho_0^w)$. The induced action on chamber homology agrees, under the embedding $H^M_\ast(Y) \hookrightarrow \text{HH}_\ast(\mathcal{H}(M))$, with the one given by the action of $W$ on $M$ by
conjugation.

**Jacquet functors in chamber homology**

**Definition 3.1.** Let $i_c : H^M_\ast(Y) \to H^G_\ast(X)$ and $r_c : H^G_\ast(X) \to H^M_\ast(Y)$ be the
maps induced by restricting the Jacquet functors $i^G_M$ and $r^G_M$ to the compact
part of Hochschild homology. Thus $i_c$ and $r_c$ are the unique maps making the
diagrams

$$\begin{array}{ccc}
H^M_\ast(Y) & \xrightarrow{i_c} & H^G_\ast(X) \\
\cong & & \cong \\
\text{HH}_\ast(\mathcal{H}(M))_c & \xrightarrow{i^G_M} & \text{HH}_\ast(\mathcal{H}(G))_c
\end{array}$$

$$\begin{array}{ccc}
H^G_\ast(X) & \xrightarrow{r_c} & H^M_\ast(Y) \\
\cong & & \cong \\
\text{HH}_\ast(\mathcal{H}(G))_c & \xrightarrow{r^G_M} & \text{HH}_\ast(\mathcal{H}(M))_c
\end{array}$$

commute.
Recall that we have defined $i^K_{U,U} : \text{Mod}(L) \to \text{Mod}(K)$ as the composition
\[ \text{ind}^K_J i^K_{U,U} \]. We likewise define $i^{K'}_{U,U} \coloneqq \text{ind}^{K'}_{J'} i^{K'}_{U,U}$.

**Definition/Lemma 3.2.** The following diagrams commute, and therefore define maps of complexes $I : C^M(Y) \to C^G(X)$ and $R : C^G(X) \to C^M(Y)$ (in that order).

\[
\begin{array}{c}
\begin{array}{c}
\text{(3.3)}
\end{array}
\end{array}
\]

\[
\begin{array}{cccc}
\pi & \downarrow & & \\
R(J) & \xrightarrow{\hat{\delta}_G} & R(K) \oplus R(K') & \text{(3.3)} \\
\downarrow & & \downarrow & \\
R(L) \oplus R(L) & \xrightarrow{\hat{\delta}_M} & R(L) \oplus R(L) & \text{(3.3)}
\end{array}
\]

**Proof.** The first diagram commutes by virtue of the equality $i^K_{U,U} \rho \cong i^K_{U,U} \rho^w$ from Proposition 2.9, along with the analogous equality for $K'$.

In the second diagram we are asserting that for each representation $\pi$ of $J$,

\[
(\text{ind}^K_J \pi)^U \cong \pi^U \oplus (\pi^w)^U
\]

and similarly for induction to $K'$. An application of the Mackey formula gives

\[
(\text{ind}^K_J \pi)^U \cong \pi^U \oplus (\text{ind}^J_{J',J''} \pi^w)^U,
\]

and a character computation confirms that the second summand is isomorphic to $(\pi^w)^w$.

**Theorem 3.4.** $I = i_c$ and $R = r_c$ as maps on chamber homology.

The proof of Theorem 3.4 occupies most of the remainder of the paper.

**Proof that** $R = r_c$

An explicit formula for the map $r^G_M$ on Hochschild homology is given, for a general reductive group $G$ and Levi subgroup $M$, in [8]. The same map appeared earlier in [20], where Nistor computes the corresponding map on smooth group homology. Let us recall these results, in summary.
Let $\mathcal{H}(G_c)$ denote the space of locally constant, compactly supported functions on $G_c$, considered as a $G$-module under the adjoint action. As observed in [12] and [22], $C^G_\pi(Y)$ is isomorphic to the $G$-coinvariants of the following projective resolution of $\mathcal{H}(G_c)$:

$$C_\ast(X, G): \quad \bigoplus_{e \in X^1} \mathcal{H}(G_c) \xrightarrow{\partial} \bigoplus_{v \in X^0} \mathcal{H}(G_c)$$

(The boundary $\partial$ and the augmentation $C_0(X, G) \to \mathcal{H}(G_c)$ are given by extending functions by zero.) It follows that $H^G_\ast(X) \cong H_\ast(G, \mathcal{H}(G_c))$, the right-hand side being smooth group homology (the left-derived functor of $G$-coinvariants on $\text{Mod}(G)$). Blanc and Brylinski show in [5] that there is a canonical isomorphism $H_\ast(G, \mathcal{H}(G_c)) \cong \text{HH}_\ast(\mathcal{H}(G)_c)$, whence the identification of chamber homology with the compact part of Hochschild homology. Similar considerations apply to $M$: $C^M_\pi(Y)$ is the complex of $M$-coinvariants of the complex $C_\ast(Y, M)$ of simplicial chains on $Y$ with coefficients in $\mathcal{H}(L)$, giving $H^M_\ast(Y) \cong H_\ast(M, \mathcal{H}(L))$ (note that $L = M_c$).

Let $\delta$ be the modular function on $P$, characterised by $d(pq) = \delta(q)dp$ for any left Haar measure $dp$ on $P$. For each $\rho \in \text{Mod}(M)$, $\rho_{\delta/2} := \rho \otimes C \delta^{1/2}$ denotes the twisting of $\rho$ by the one-dimensional representation $\delta^{1/2}$. For each representation $\pi \in \text{Mod}(G)$, the idempotent $\pi(e_K) : \pi \to \pi$ descends to a well-defined map $\pi_G \to (r^G_M(\pi)_{\delta/2})_M$ between the $G$-coinvariants of $\pi$ and the $M$-coinvariants of $r^G_M(\pi)_{\delta/2}$. (Here one appeals to the Iwasawa decomposition $G = KMN$.) This map is natural in $\pi$, and so it lifts to a natural transformation of derived functors, $\kappa : H_\ast(G, \pi) \to H_\ast(M, r^G_M(\pi)_{\delta/2})$.

The “Harish-Chandra transform”

$$\Psi : \mathcal{H}(G_c) \to \mathcal{H}(L), \quad \Psi(f)(l) = \int_N f(nl) \, dn$$

descends to an $\text{Ad}_M$-equivariant map $r^G_M \mathcal{H}(G_c)_{\delta/2} \to \mathcal{H}(L)$. The Jacquet restriction $r_c : H^G_\ast(X) \to H^M_\ast(Y)$ is then equal to the composition

$$H^G_\ast(X) \xrightarrow{\kappa} H_\ast(G, \mathcal{H}(G_c)) \xrightarrow{\Psi} H_\ast(M, \mathcal{H}(L)) \xrightarrow{\kappa} H^M_\ast(Y).$$

See [20] and [8] for details.

**Proof that $R = r_c$ in Theorem 3.4.** The inclusion of $Y$ into $X$ gives an isomorphism $Y \cong X/N$. It follows that the image of the resolution $C_\ast(X, G)$ under the functor $r^G_M(\underline{\_})_{\delta/2}$ is isomorphic to

$$C_\ast(Y, rG): \quad \bigoplus_{e \in Y^1} \mathcal{H}(G_c)_{N_e} \to \bigoplus_{v \in Y^0} \mathcal{H}(G_v)_{N_v},$$

the subscripts $N_e$ and $N_v$ denoting coinvariants with respect to the adjoint action. The maps

$$\Psi_s : \mathcal{H}(G_s)_{N_s} \to \mathcal{H}(L), \quad \Psi_s(f)(l) = \int_{N_s} f(nl) \, dn,$$
where \( s \) ranges over the simplices in \( Y \), provide a lift of \( \Psi \) to a map of resolutions, \( C_s(Y, rG) \rightarrow C_s(Y, M) \). We claim that the composition

\[
(3.5) \quad C^G_s(X) \xrightarrow{\sim} C_s(X, G)_G \xrightarrow{\kappa} C_s(Y, rG)_M \xrightarrow{\Psi} C_s(Y, M)_M \xrightarrow{\sim} C^M_s(Y)
\]

is equal to \( R \).

For example, let \( \pi \) be a representation of \( K \), viewed as a chain in \( C^G_0(X) \). The corresponding chain in \( C_0(X, G) \) is the function \( \text{ch}_\pi \in \mathcal{H}(K) \); recall that \( K \) is the isotropy group of a vertex in \( X \). This vertex lies in \( Y \), and so the map \( \kappa \) simply acts on \( \text{ch}_\pi \) by averaging over the adjoint action of \( K \); \( \text{ch}_\pi \) is already \( \text{Ad}_K \)-invariant, so \( \kappa(\text{ch}_\pi) = \text{ch}_\pi \in \mathcal{H}(K)_U \). The map \( \Psi : \mathcal{H}(K)_U \rightarrow \mathcal{H}(L) \) sends \( \text{ch}_\pi \) to \( \text{ch}_\pi^U \), and so \( \Psi \) equals \( R \) as maps \( R(K) \rightarrow R(L) \). The computations for \( R(K') \) and \( R(J) \) are only slightly more involved (because the cycles in question are not a priori \( K \)-invariant). We shall not present the details here. \( \Box \)

**Proof that** \( I = i_c \)

Unlike the preceding section, whose methods apply to general reductive \( G \) and Levi subgroup \( M \), our proof that \( I = i_c \) relies on some special features of \( \text{SL}_2 \): \( H^*_G(X) \) is nonzero only in degrees zero and one, and \( r_c : H^1_G(X) \rightarrow H^M_1(Y) \) is an isomorphism onto the space of \( W \)-invariants in \( H^1_M(Y) \); see [20] and [8].

**Theorem 3.6.** [4, Theorem 5.2] \( r_c i_c = 1 + w \) as endomorphisms of \( H^M_*(Y) \).

**Proof.** The cited result of Bernstein and Zelevinsky implies that the functor \( r^G \delta^G \) on \( \text{Mod}(M) \) has a natural filtration with quotients 1 and \( w \). This filtration becomes a sum in Hochschild homology [8]. \( \Box \)

**Proposition 3.7.** \( RI = 1 + w \) as endomorphisms of \( C^*_M(Y) \).

**Proof.** For each irreducible \( \rho \) of \( L \), one has

\[
\left( (i_{M}^{G} \rho)^{U} \right)^w \cong (i_{M}^{G} \rho)^{w} \oplus (i_{M}^{G} \rho)^{w} \cong \rho \oplus \rho^w ;
\]

the first isomorphism is \( \text{[3.3]} \), the second follows from Lemma [1.10]. This (and the corresponding computation for \( K' \)) shows that \( RI = 1 + w \) in degree zero.

In degree one, Lemma [1.10] gives \( RI = 1 + w \) immediately. \( \Box \)

**Proof that** \( I = i_c \) in **Theorem 3.4**. We have shown that \( r_c I = RI = 1 + w = r_c i_c \). Since \( r_c \) is one-to-one in degree one, this gives \( I = i_c \) as maps \( H^*_M(Y) \rightarrow H^G_1(X) \).

The equality in degree zero is deduced from a theorem of Dat, as follows. \( HH_0(\mathcal{H}(G)) \) is a quotient of the complex vector space \( \mathcal{V}_G \) with basis consisting of pairs \([\sigma, T]\), where \( \sigma \) is a finitely generated projective \( G \)-module, and \( T \in \text{End}_G(\sigma) \) ([3, 1.3], [8, Proposition 2.7]). The inclusion \( H^*_G(X) \rightarrow \text{HH}_0(\mathcal{H}(G)) \) is then the one induced in homology by

\[
R(K) \rightarrow \mathcal{V}_G, \quad \pi \mapsto [\text{ind}_K^G \pi, \text{id}]
\]
and by the corresponding map $R(K') \to V_G$. Similar considerations apply to $M$, and the map $i^G_M : HH_0(H(M)) \to HH_0(H(G))$ is the one induced by

$$V_M \to V_G \quad [\sigma, T] \mapsto [i^G_M \sigma, i^G_M T].$$

So the theorem in degree zero follows from the assertion that

$$i^G_M \text{ind}_L^M \rho \cong \text{ind}_K^G i^K_{U,G} \rho,$$

naturally with respect to $\rho \in \text{Mod}(L)$, and similarly for $K'$. This assertion is a special case of [10, (1.4)].

The following description of $H^G_1(X)$ follows immediately from Theorem 3.4. Together with Proposition 2.1(1), this gives a new proof of [2, Proposition 9.3], and also explains the resemblance with principal-series characters observed in [2, p.17].

**Corollary 3.8.** $H^G_1(X)$ has a basis consisting of cycles $i_{U,U}(\rho) - i_{U,U}(\rho^w) \in R(J)$, where $\rho$ ranges over a set of representatives for the two-element orbits of $W$ on the set of irreducible representations of $L$.

**Proof.** The map $r_c : H^G_1(X) \to H^M_1(Y)$ is injective, with range equal to the space of $W$-invariants in $H^M_1(Y)$. The cycles $c_\rho := (\rho - \rho^w, \rho^w - \rho) \in C^M_1(Y)$, for $\rho$ as in the statement of the corollary, constitute a basis for the latter space, and Proposition 3.7 shows that

$$R(i_{U,U}(\rho) - i_{U,U}(\rho^w)) = \frac{1}{2} RI(c_\rho) = c_\rho.$$  

**The case of $SL_n$**

The definitions of $R$ and $I$ make sense also for $G = SL_n(F)$, $\mathcal{M}$ the diagonal subgroup. For example, in degree $n - 1$ one sets

$$I : R(L)^n \to R(J), \quad (\rho_0, \ldots, \rho_{n-1}) \mapsto \sum_{w_i \in W} i_{U,U} \rho_i^{w_i},$$

where $J = ULU \subset G$ is the standard Iwahori subgroup, and $W = N_G(M)/M$ is the Weyl group, which acts simply transitively on the set of chambers in a fundamental domain for the action of $M$ on its apartment. In degree zero,

$$I : R(L)^n \to \bigoplus_{i=0}^{n-1} R(K_i), \quad (\rho_0, \ldots, \rho_{n-1}) \mapsto (i^i_{U,U} \rho_0, \ldots, i^i_{U,U} \rho_{n-1}),$$

where $K_0, \ldots, K_{n-1}$ are the isotropy groups of the vertices of the chamber stabilised by $J$, and $i^i_{U,U} = \text{ind}_J^i i_{U,U}$. The above proof carries over to give the following partial result:
Proposition 3.9. Let \( G = \text{SL}_n(F) \), and let \( M \subset G \) be the diagonal subgroup. Define maps \( R : C^G(X) \to C^M(Y) \) and \( I : C^M(Y) \to C^G(X) \) as above. Then \( R = r_c \) as maps \( H^0_{\text{an}}(X) \to H^0_{\text{an}}(Y) \), and \( I = i_c \) as maps \( H^M_0(Y) \to H^G_0(X) \) and \( H^M_{n-1}(Y) \to H^G_{n-1}(X) \).

Replacing the diagonal subgroup by a larger Levi subgroup, for example the \((2 \times 1)\)-block-diagonal subgroup of \( \text{SL}_3(F) \), one can still use parahoric induction to define a candidate for the map \( I \). It follows from our joint work (in progress) with Ehud Meir and Uri Onn that this map will no longer commute with the boundary maps; the issue is closely related to Dat’s question [10, Question 2.14]. It is likely that new tools will be needed in this situation.

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