On solutions of singular differential equations of the second order

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June 30, 2020

Abstract

We study the behaviour of solutions of ordinary differential equations of the second order with singular points, where the coefficients of the second-order derivative vanishes. In particular, we consider solutions entering a singular point without definite tangential direction. Great attention is paid to second-order equations, whose right-hand sides is a cubic polynomial by the first-order derivative.

Key words: vector fields, singular point, normal form, resonance, oscillation

1 Introduction

A. F. Filippov showed [4] that systems of ordinary differential equations

\[ F(x, y, p) = 0, \quad p = dy/dx, \] (1)

where \( x \in \mathbb{R}^1, y \in \mathbb{R}^n \) and \( F : \mathbb{R}^{2n+1} \to \mathbb{R}^n \) is a smooth vector function, may have solutions that enter a point without definite tangential direction.

A solution \( y(x) \) of system (1) is called oscillating at a point \((x_0, y_0)\), if \( y(x) \) is a vector function differentiable on the interval \((x_0, x_0 + \delta)\) or \((x_0 - \delta, x_0)\), \( \delta > 0 \), such that \( y(x) \to y_0 \) but \( y'(x) \) has neither finite nor infinite limit as \( x \to x_0 \). A solution \( y(x) \) is called proper at a point \((x_0, y_0)\), if \( y'(x) \) has limit (finite or infinite) as \( x \to x_0 \).

Let \( J^1 \) be the 1-jet space of smooth functions \( y(x) \), with the coordinates \((x, y, p)\). Filippov showed that if for a point \( T_0 \in J^1 \) there exists \( T'_0 \in J^1 \) such that \( x_0 = x'_0, y_0 = y'_0, p_0 \neq p'_0 \), and the matrix \( F_p \) degenerates at \( T'_0 \), then besides a unique proper solution passing through \((x_0, y_0)\) with the tangential direction \( p_0 \), system (1) may have oscillating solutions, which pass through \((x_0, y_0)\) and even have definite tangential direction \( p_0 \) at \((x_0, y_0)\).

For instance, consider solutions of the system

\[ p_1(1 - p_1^2 - p_2^2) + 8xy_1 + 4y_2 = 0, \quad p_2(1 - p_1^2 - p_2^2) + 8xy_2 - 4y_1 = 0, \] (2)

that pass through the origin with the tangential direction \( p_1 = p_2 = 0 \). The matrix \( F_p \) is non-degenerate at the point \( T_0 = 0 \), and system (2) has a unique proper solution with the initial condition \( y_1(x) = y_2(x) = 0 \). Moreover, this system has an infinite number of oscillating solutions given by the formula

\[ y_1(x) = x^2 \cos(x^{-1} + c), \quad y_2(x) = x^2 \sin(x^{-1} + c), \quad c = \text{const}, \]

0 for \( x \neq 0 \), and \( y_i(0) = 0 \). The both derivatives \( y'_i(x) \) are zero at \( x = 0 \), but the limits \( y'_i(x) \) as \( x \to 0 \) do not exists.

However, he shows that such systems are a sort of exception. Given a point \( q_0 = (x_0, y_0) \in \mathbb{R}^{n+1} \) define the set

\[ Q(q_0) = \{ T = (x_0, y_0, p) \in J^1 : F(T) = 0, \quad \det(F_p(T)) = 0 \}. \]

1 By smooth we mean \( C^\infty \), if the otherwise is not stated.
Theorem 1 ([4]) Assume that at least one of the following conditions holds true:
1. \( p_0 \notin \text{co} \, Q(q_0) \), where \( \text{co} \) denotes the convex hull,
2. the set \( Q(q_0) \) is at most countable (i.e., countable, finite, or empty).

Then system (1) has no solution oscillating at \( q_0 \) with tangential direction \( p_0 \) at \( q_0 \). Generically, the condition 2 holds true at all points, and system (1) has no oscillating solutions.

Although Theorem 1 has a very general character, it gives only a trivial result for systems
\[
A(x, y) \frac{d p}{d x} = b(x, y), \quad p = dy/dx,
\]
where \( A \) is an \( n \times n \) matrix, \( b \) is a vector function. Namely, it guarantees the absence of oscillating solution at those points where the matrix is non-degenerate.

In the present paper, we consider a partial case of system (3). Namely, we consider the second-order equations
\[
\Delta(x, y) \frac{d p}{d x} = M(x, y, p), \quad p = dy/dx,
\]
which are equivalent to systems (3) of a special type.

Great attention is paid to the case that the right-hand side of equation (4) is a cubic polynomial in \( p \), that is,
\[
M(x, y, p) = \sum_{i=0}^{3} \mu_i(x, y) p^i.
\]

This interest to equations (4), (5) is motivated by their role in various geometric structures, for instance, the affine connection and the projective connection. Equations if this class were studies by Sophus Lie, A. Tresse, J. Liouville, E. Cartan, to name a few. (See, e.g., [3, 15, 1, 16] and the references therein.) For instance, the equation of geodesics generated by the metric tensor
\[
ds^2 = a(x, y) \, dx^2 + 2b(x, y) \, dxdy + c(x, y) \, dy^2,
\]
has the form (4), (5), where \( \Delta(x, y) = ac - b^2 \) \( \mu_i \) are expressed through \( a, b, c \) as follows:
\[
\begin{align*}
\mu_3 &= c(2b_y - c_x) - bc_y, \\
\mu_2 &= b(2b_y - 3c_x) + 2a_y c - ac_y, \\
\mu_1 &= b(3a_y - 2b_x) + a_x c - 2ac_x, \\
\mu_0 &= a(a_y - 2b_x) + a_x b.
\end{align*}
\]

Here the coefficient \( \Delta \) vanishes at point that the quadratic form (6) degenerates. In Riemannian geometry, such is not the case, and \( \Delta > 0 \) everywhere (except for irregular points of surfaces). However, degenerate quadratic forms generically appear on surfaces embedded into pseudo-Euclidean spaces: the metric (6) induced on a surface into pseudo-Euclidean space degenerates at those points where the surface tangents the light cone of the ambient space. Singularities of the geodesic equation appearing at degenerate points of the metric are studied in a recent series of papers. See, e.g., [10, 8, 11] and the survey [7].

It should be remarked that in all papers mentioned above (as well as in other works known to us) only proper geodesics were considered, while the possibility of oscillating geodesics was not studied. (By proper/oscillating geodesics we mean geodesics that are proper/oscillating solutions of the corresponding geodesic equation. In the present paper, we fill this gap.

In the next section, we prove that equation (4) with arbitrary cubic polynomial (5) has no oscillating solutions at \( q \), if the functions \( \Delta, \mu_0, \ldots, \mu_3 \) do not simultaneously vanish at \( q \).

The closing section of the paper is devoted to the behaviour of proper solutions of equation (4) with a cubic polynomial (5) at its generic singular points.
2 Oscillating solutions

Consider the differential equation

\[ \Delta(x, y) \frac{dp}{dx} = M(x, y, p), \quad p = dy/dx, \]  

(8)

where \( \Delta, M \) are smooth functions and \( M \) is analytic in \( p \). A point \( q \) such that \( \Delta(q) = 0 \) is called a singular point of equation (8). We shall always assume that the equation \( \Delta(x, y) = 0 \) defines a regular curve \( \Gamma \) on the \((x, y)\)-plane, i.e., \( d\Delta(q) \neq 0 \) at all points \( q \in \Gamma \). Then, in a neighborhood of every point singular \( q \), the curve \( \Gamma \) locally splits the \((x, y)\)-plane into two domains \( D^+ \) and \( D^- \) in accordance with the inequalities \( \Delta > 0 \) and \( \Delta < 0 \).

When dealing with singular points, we should refine the notion of solutions. Let \( q_0 = (x_0, y_0) \in \Gamma \). A solution of equation (8) entering the point \( q_0 \) or (equivalently) issuing from \( q_0 \) is a function \( y(x) \) that is continuous on the segment with endpoints \( x_0, x_0 + \varepsilon \) with some \( \varepsilon \neq 0 \), differentiable and satisfying (8) at all inner points of this segment and such that \( y(x_0) = y_0 \) is the only point of the intersection \( y(x) \cap \Gamma \). If in addition to the above conditions, the derivative \( y'(x) \) has a (finite or infinite) limit as \( x \to x_0 \), the solution \( y(x) \) is called proper. Otherwise the solution \( y(x) \) is called oscillating at \( q_0 \). See Fig. 1.

Moreover, a solution \( y(x) \) is called passing through the point \( q_0 \) if the function \( y(x) \) is differentiable and satisfying (8) at all points of the open interval with the endpoints \( x_0 - \varepsilon, x_0 + \varepsilon \) with some \( \varepsilon \neq 0 \), such that \( y(x_0) = y_0 \) is the only point of the intersection \( y(x) \cap \Gamma \). Every solutions passing through \( q_0 \) is the union of two solutions entering \( q_0 \).

The advantages of the given definitions become clear from the following example.

Example 1 Integrating the equation \( 2ydp/dx = p^2 \), we obtain the family of parabolas \( x = \alpha y^2, \alpha = \text{const} \), which intersect the curve \( \Gamma(x) \) at the origin 0. According to the given definitions, every parabola of this family is a solution of the equation passing through 0, but in addition to them there exists an infinite number of solution that consist of the branches of two parabolas \( x = \alpha y^2 \) with different values of \( \alpha \). To avoid such ambiguity, it was proposed to deal with solutions entering 0 (or, equivalently, issuing from 0), the branches of the parabolas \( x = \alpha y^2 \).

In the previous example, all solutions were proper. Now we give examples of equations (8) that possess oscillating solutions.

Example 2 The equation \( x^4dp/dx = 2x^3p - (2x^2 + 1)y \) has the family of oscillating solutions \( y = x^2(\alpha \cos x^{-1} + \beta \sin x^{-1}), \alpha, \beta = \text{const} \), which enter the origin 0 without definite tangential directions, although at the point 0 they have the tangential direction \( p = 0 \).

Example 3 The equation \( x^2dp/dx = xp - 2y \) has the family of oscillating solutions

\[ y = x(\alpha \cos \ln |x| + \beta \sin \ln |x|), \quad \alpha, \beta = \text{const}, \]  

(9)

which enter 0 without definite tangential directions and have no definite tangential directions at 0. Using formula (9), one can construct an equation (8) with oscillating solution, whose

\footnote{Here we use the terms “solution” instead of “the graph of solution”. We do not distinguish between the cases \( \varepsilon > 0, \varepsilon < 0 \). This reflect the point of view that \( x \) is considered geometrically, as a coordinate on the plane or another surface, whence the increasing or decreasing of \( x \) has no geometric sense (in contrast to the case that \( x \) is time, which cannot decrease).}
right-hand side is a polynomial of arbitrary degree in $p$. For instance, one can see that the function $\xi$ with $\alpha = \beta = 1$ is a solution of the first-order equations $F_i(x, y, p) = 0$, where
\[
F_2 = (xp)^2 - 2xp + 2(y^2 - x^2),
F_3 = (xp)^3 + y(xp)^2 - 2xp(x^2 + 2y^2) + 6y(y^2 - x^2).
\]
Therefore, it is a solution of the equations $x^2 dp/dx = xp - 2y + F_i(x, y, p)$.

**Theorem 2** Assume that at a point $q_0 \in \Gamma$ the conditions $d\Delta(q_0) \neq 0$ and $M(q_0, p) \neq 0$ hold true. Then equation (8) has no oscillating solutions that enter $q_0$.

**Proof.** Assume that $d\Delta(q_0) \neq 0$ and equation (8) has an oscillating solution $y(x)$, $x_0 < x < x_0 + \varepsilon$, that enter $q_0$. Without loss of generality, one can assume that $y(x)$ belongs to the domain $D^+$ except for the point $q_0$. From the absence of the limit $y'(x)$ as $x \to x_0$ it follows that there exist two sequences $x_n' \to x_0 + 0$ and $x_n'' \to x_0 + 0$ such that $p(x_n') \to p'$ and $p(x_n'') \to p''$, $p' \neq p''$. For definiteness, assume that $p' < p''$. Since the function $y'(x)$ is continuous on the interval $x_0 < x < x_0 + \varepsilon$, for every $p_* \in (p', p'')$ there exist two sequences $\xi_n \to x_0 + 0$ and $\xi_n' \to x_0 + 0$ such that
\[
\lim_{n \to \infty} p(\xi_n) = \lim_{n \to \infty} p(\xi_n') = p_*, \quad \frac{dp}{dx}(\xi_n) > 0, \quad \frac{dp}{dx}(\xi_n') < 0 \quad (\forall n).
\]
See Fig. 1 (right).

Substituting the solution $y(x)$ into (8), at the points $x = \xi_n$ and $x = \xi_n'$ we have
\[
\Delta(\xi_n, y_n) \frac{dp}{dx}(\xi_n) = M(\xi_n, y_n, p_n), \quad \Delta(\xi_n', y_n') \frac{dp}{dx}(\xi_n') = M(\xi_n', y_n', p_n'),
\]
where $y_n = y(\xi_n)$, $p_n = p(\xi_n)$, and similarly for $\xi_n'$. By our assumption, the right-hand sides of both equalities in (11) have the finite limit $M(x_0, y_0, p_*)$, whence their left-hand sides have the same limit. Since $\Delta(\xi_n, y_n) > 0$ and $\Delta(\xi_n', y_n') > 0$ from (10) it follows that left-hand sides of both equalities in (11) have different signs. Therefore, $M(x_0, y_0, p_*) = 0$.

Thus, we proved that $M(q_0, p_*) = 0$ for every $p_* \in (p', p'')$. Since the function $M$ is analytic in $p$, this implies $M(q_0, p) = 0$ for every $p$. This completes the proof.

![Figure 1: An oscillating solution issuing from the origin (on the left) and its derivative (on the right).](image)

Applying Theorem 2 to equations (4), (5), we conclude that for the existence of a solution oscillating at $q_0$, it is necessary that all coefficients $\mu_0, \ldots, \mu_3$ simultaneously vanish at $q_0$. Moreover, in the case that equation (4), (5) is the geodesic equation in metric (6), we have the following:
Corollary 1 If the metric tensor (6) degenerates on a regular curve Γ, i.e., \( d\Delta(q) \neq 0 \) for all \( q \in \Gamma \), then the corresponding geodesic equation has no oscillating solutions.

Proof. The geodesic equation in metric (6) has the form (4), (5), where \( \Delta = ac - b^2 \) and \( \mu_i \) are expressed via \( a, b, c \) by formula (7). By Theorem 1 there are no oscillating solutions at point \( q_0 \notin \Gamma \). By Theorem 2, for the non-existence of solutions oscillating at a point \( q_0 \in \Gamma \), it is sufficient to prove that if all coefficients \( \mu_0, \ldots, \mu_3 \) simultaneously vanish, then the condition \( d\Delta \neq 0 \) fails.

To simplify the calculations, we choose local coordinates centered at \( q_0 \) such that \( b(q_0) = 0 \). Then form \( \Delta(q_0) = 0, d\Delta(q_0) \neq 0 \) it follows that only one of the coefficients \( a, c \) vanishes at \( q_0 \). Without loss of generality one can assume that \( a(q_0) \neq 0 \) and \( c(q_0) = 0 \). Then from the equalities \( \mu_i(q_0) = 0 \) and (7) we get \( c_x(q_0) = c_y(q_0) = 0 \). On the other hand, at the point \( q_0 \) we have the relations \( \Delta_x = ac_x, \Delta_y = ac_y \), which imply \( d\Delta(q_0) = 0 \). The obtained contradiction completes the proof.

3 Singularities of second-order equations cubic in the first-order derivative

In this section, we study generic singularities of equation (4), whose right-hand side has the form (5). Here we assume that \( \Delta \) and \( \mu_i \) are smooth functions not connected with each other, in contrast with the geodesic equation. As before, we assume that the equation \( \Delta = 0 \) defines a regular curve \( \Gamma \), which locally separates the \((x, y)\)-plane into domains \( D^+ \) and \( D^- \).

With the above assumptions, equation (4), (5) has no oscillating solution, and our purpose is to study its proper solutions issuing from its generic singular points. For brevity, we shall omit the adjective proper. We start with a brief survey of the basic facts established in the papers [7] – [11], for geodesic equations that are valid for all equations of the form (4), (5).

Solutions of equations (4), (5) are the projections of integral curves of the vector field

\[
\dot{x} = \Delta(x, y), \quad \dot{y} = p\Delta(x, y), \quad \dot{p} = M(x, y, p)
\]

(12)

from the space \( J^1 \) to the \((x, y)\)-plane along to the \( p\)-direction, which we call vertical. Denote by \( \pi \) the projection along the vertical direction. The \( \pi \)-projection of any integral curve of the field (12) different from a straight vertical lines, is a solution of equation (4). Conversely, the Legendrian lift of every solution of (12) gives an integral curve of the field (12).

Lemma 1 Assume that \( q_0 \in \Gamma \). Then the vertical line \( \pi^{-1}(q_0) \) is an integral curve of the field (12). Moreover, if \( p_0 \) satisfies the condition \( M(q_0, p_0) \neq 0 \), then \( \pi^{-1}(q_0) \) is the only integral curve of the field (12) passing through the point \((q_0, p_0)\).

From Lemma 1 it follows that solutions of equation (4), (5) can issue from a singular point \( q_0 \in \Gamma \) only in so-called admissible tangential directions \( p \), which correspond real roots of the cubic polynomial \( M(q_0, p) \). We shall understand the term cubic polynomial in the broad sense: the higher coefficients \( \mu_3 \) can vanish at some points or even identically. Under such a convention, the class of equations (4), (5) is invariant with respect to changes of the variables \( x, y \) and it includes second-order equations linear in \( y' \), for instance, the Bessel equation and Gaussian Hypergeometric Equation.

The tangential direction \( p = \infty \) is admissible at \( q_0 \), iff \( \mu_3(q_0) = 0 \). Indeed, interchanging the variables \( x, y \), we obtain the equation \( \Delta(y, x)dp/dx = -M^*(y, x, p) \), where \( M^* \) is the
reciprocal polynomial, and the direction becomes \( p = 0 \). The equality \( \mu_3(q_0) = 0 \) is equivalent to the condition that \( p = 0 \) is a root of the reciprocal polynomial \( M^*(q_0, p) \).

In this paper, we consider generic singular points: in addition to the condition that the curve \( \Gamma \) is regular, we shall assume that all roots of the cubic polynomial \( M(q_0, p) \) are prime and the corresponding tangential directions are transversal to \( \Gamma \). Then there exist local coordinates such that \( \mu_3(q_0) \neq 0 \), and consequently, all admissible directions are finite. For a generic equation of the form (4), (5), the above conditions hold true at almost all points of \( \Gamma \) and fail at isolated points only.

By Lemma 1 all solutions of equations (4), (5) that issue from a point \( q_0 \in \Gamma \) are \( \pi \)-projections of integral curves of the field (12) that enter its singular point \((q_0, p)\) as time tends to infinity (positive or negative). Singular points of the field (12) are given by the equations \( \Delta = 0 \) and \( M = 0 \), they fill a curve (or several curves) in the space \( J^1 \). The spectrum of the linear part of (12) at its singular point has the form

\[
\Sigma = (0, \lambda_1, \lambda_2), \quad \lambda_1 = \Delta_x + p \Delta_y, \quad \lambda_2 = M_p.
\] (13)

By the above assumptions, \( \lambda_{1,2} \neq 0 \). It is worth observing that \( \partial_p \) is the eigenvector with the eigenvalue \( \lambda_2 \).

**Lemma 2** The germ of vector field (12) at its singular point is orbitally topologically equivalent to

\[
\dot{\xi} = \xi, \quad \dot{\eta} = \text{sgn}(\lambda) \eta, \quad \dot{\zeta} = 0, \quad \lambda = \lambda_2/\lambda_1.
\] (14)

Lemma 2 is a trivial corollary of the reduction principle. See, e.g., [2, 6].

Thus, the polynomial \( M(q_0, p) \) has a unique real \( p_0 \) or three different real roots \( p_0, p_1, p_2 \), and the set of singular points of the field (12) consists respectively of a single curve \( W^c_0 \) or three non-intersecting curves \( W^c_i \) passing through the points \((q_0, p_i)\). From the condition \( \lambda_{1,2} \neq 0 \) it follows that \( W^c_i \) are center manifolds of the field (12), and the slow dynamics on the center manifold (non-trivial for generic vector fields) is identically zero. By the reduction principle, this yields topological orbital normal form (14). Moreover, due to this circumstance, the normal form of the field (12) in the smooth category is much simpler than those for generic vector fields with an isolated singular point with the same spectrum \( \Sigma \).

**Lemma 3** If \( \lambda \) defined in (14) is positive or negative irrational, then the germ of (12) at its singular point is \( C^k \)-smoothly equivalent to the germ

\[
\dot{\xi} = \xi v_1 + \eta w_2, \quad \dot{\eta} = \xi w_1 + \eta v_2, \quad \dot{\zeta} = 0,
\] (15)

where \( v_i \) and \( w_i \) are smooth functions on \( \xi, \eta, \zeta \). If \( \lambda > 0 \), then the exponent of smoothness \( k = \infty \). If \( \lambda < 0 \), then \( k < \infty \) is an arbitrary integer number.

For \( k < \infty \), the statement of the lemma follows from general results obtained in \[13, 14\] taking into account that the field (12) has a one-dimensional center manifold filled with singular points of the field. If \( \lambda \) is positive, the exponent of smoothness \( k \) can be improved to infinity using the homotopy method. See [5] (Sections 1.2–1.5), where a similar result is proved. However, if \( \lambda \) is negative, \( k \) can be improved to infinity only in exceptionally rare case: if the number \( \lambda \) is the same at all singular points in a neighborhood of the given point. See, e.g., [12, 9].

We remark that if \( \lambda > 0 \), then there are only two possible types of resonances between the eigenvalues \( \lambda_{1,2} \), namely: \( \lambda_1 = n \lambda_2 \) or \( \lambda_2 = n \lambda_1 \) with integer \( n \geq 1 \).
Lemma 4 Assume that the condition of Lemma 3 holds true and $\lambda \neq 1$. Then the germ of vector field (12) at its singular point is $C^k$-smoothly equivalent to the germ

$$\dot{\xi} = a_1(\xi)\xi, \quad \dot{\eta} = a_2(\eta)\eta, \quad \dot{\zeta} = 0, \quad \text{if} \quad \lambda \notin \mathbb{N} \cup \mathbb{N}^{-1} \lor \lambda_1 = n\lambda_2, \quad (16)$$

$$\dot{\xi} = a_1(\xi)\xi, \quad \dot{\eta} = a_2(\eta)\eta + \rho(\xi)^n\xi^n, \quad \dot{\zeta} = 0, \quad \text{if} \quad \lambda_2 = n\lambda_1, \quad (17)$$

where $n > 1$ is an integer, $a_i, \rho$ are smooth functions such that $a_i(0) = \lambda_i$. Here the exponent of smoothness $k$ is the same as in Lemma 3.

Prove. Making use of Lemma 3 we consider the vector field (15), whose linear part at every singular point has the spectrum $\Sigma$ with one zero and two non-zero real eigenvalues $\lambda_{1,2}$. If $\lambda \notin \mathbb{N} \cup \mathbb{N}^{-1}$, then the germ of such a field is equivalent to the normal form (16). If $\lambda_2 = n\lambda_1$ or $\lambda_1 = n\lambda_2$ with $n > 1$, the normal form consists of the linear part (16) plus the resonance term $\rho(\xi)\xi^n\xi^n$ or $\rho(\xi)^n\xi^n\eta$, respectively. See, e.g., [11] (Appendix A). Integral curves of the field (12) and their $\pi$-projections are presented in Fig. 2 (a) and (b) for the cases $|\lambda_1| > |\lambda_2|$ and $|\lambda_1| < |\lambda_2|$, respectively.

The above results were obtained for arbitrary vector field (15), whose linear part at every singular point has one zero eigenvalue. Now we recall that we deal with a vector field equivalent to (12). Then, in the case of the resonance $\lambda_1 = n\lambda_2$, the coefficient $\rho(\xi)$ in normal form

$$\dot{\xi} = a_1(\xi)\xi + \rho(\xi)\eta^n, \quad \dot{\eta} = a_2(\eta)\eta, \quad \dot{\zeta} = 0 \quad (18)$$

is identically zero.

Indeed, the field (18) has the invariant foliation $\{\zeta = \text{const}\}$, which corresponds to a certain two-dimensional invariant foliation of the field (12), each leaf of this foliation contains a vertical curve $\pi^{-1}(q), q \in \Gamma$. The restriction of the field (18) to the invariant leaf $\{\zeta = \zeta_0\}$ is a node with the eigenvalues $a_{1,2}(\zeta_0)$. Since $\lambda_1 = n\lambda_2$ at the origin, the inequality $|a_1(\zeta_0)| > |a_2(\zeta_0)|$ holds true on all leaves $\{\zeta = \zeta_0\}$ sufficiently close to the origin. Therefore, almost all integral curves of the restriction of field (12) to $\{\zeta = \zeta_0\}$ tangent to the vertical direction; see Fig. 2 (a). Integrating the restriction of the filed (18) to the leaf $\{\zeta = \zeta_0\}$, one can find the exact formula for these integral curves:

$$\xi = \eta^n(c + \varepsilon \ln |\eta|), \quad \zeta = \zeta_0, \quad \varepsilon = \frac{\rho(\zeta_0)}{a_2(\zeta_0)}, \quad c = \text{const}. \quad (19)$$

If $\varepsilon = 0$, then all curves of the family (19) are $C^\infty$-smooth, while if $\varepsilon \neq 0$, they are only $C^{n-1}$-smooth (but not $C^n$ at the origin). On the other hand, it is easy to see that the family (19) contains at least one $C^\infty$-smooth curve: it is the straight integral curve $\pi^{-1}(q_0)$ of the field (12), see Lemma 1. Therefore, $\varepsilon = 0$, that is, $\rho(\zeta_0) = 0$ for every $\zeta_0$ sufficiently close to zero.

Solutions of equation (4), (5) passing through a generic singular point $q_0 \in \Gamma$ with the admissible tangential direction $p_i$ are described by the following theorem:

Theorem 3 If $\lambda < 0$, then there exists a unique solution that passes through $q_0$, it is $C^\infty$-smooth. If $\lambda > 0$, then there exist an infinite number of solutions issuing from $q_0$ in $\Gamma_+$ and an infinite number of solutions issuing from $q_0$ in $\Gamma_-$.

There exist local coordinates centered at $q_0$ such that the curve $\Gamma$ coincides with the axis $x = 0$ and the infinite family of solutions mentioned above has the form

$$y = cx|x|^\lambda \varphi(x, c|x|^\lambda), \quad c = \text{const}, \quad \text{if} \quad \lambda \notin \mathbb{N} \cup \mathbb{N}^{-1} \lor \lambda_1 = n\lambda_2, \quad (20)$$

$$x = t^n(c + \varepsilon \ln |t|), \quad y = \psi(t, x), \quad c = \text{const}, \quad \text{if} \quad \lambda_2 = n\lambda_1, \quad (21)$$
where $n > 1$ is an integer number; $\varphi, \psi$ are smooth functions, $\varphi(0) \neq 0$, $\psi(0) = 0$.

**Proof.** From Lemmas 2–4 it follows that integral curves of the field (12) that pass through its singular point $(q_0, p_i)$ and their $\pi$-projections are located as it presented in Fig. 2 (from the left to right: $\lambda < 0$, $0 < \lambda < 1$, $\lambda > 1$).

If $\lambda < 0$, then the field (12) has two integral curves that enter the point $(q_0, p_i)$, the stable and unstable one-dimensional manifolds of the field, both of the class $C^\infty$. One of them coincides with the vertical line $\pi^{-1}(q_0)$. Another one has a non-vertical tangential direction, and its $\pi$-projection is a unique solution that passes through $q_0$ with the tangential direction $p_i$. See Fig. 2 (a).

If $\lambda > 0$, then the field (12) has a one-parameter family of integral curves that enter the point $(q_0, p_i)$ as time tends to plus or minus infinity. The essential difference is between the cases $0 < \lambda < 1$ and $\lambda > 1$. In the first case, almost all (except one) integral curves of this family are tangent to the vertical direction, while in the second case, they have a non-vertical tangential direction. See Fig. 2 (b, c).

In the normal form (16) or (17), the above family of integral curves of (12) is given respectively by the formula

$$\eta = c|\xi|^\lambda \quad \text{or} \quad \xi = \eta^n(c + \varepsilon \ln |\eta|), \quad c \in \mathbb{R} \cup \infty, \quad (22)$$

where $c = \infty$ corresponds to the integral curves $\xi = 0$ or $\eta = 0$. In the initial coordinates, $\xi = 0$ corresponds to the vertical straight line $\pi^{-1}(q_0)$, while all the remaining curves of the family (22) correspond to integral curves of the field (12) whose $\pi$-projections consist an infinite family of solutions issuing from the point $q_0$ with the tangential direction $p_i$.

Now we consider other points $(q, p_i(q)) \in W^c_i$. By the conditions of generality, $p_i(q)$ is a smooth function. On $\Gamma$, it defines a field of directions transversal to $\Gamma$ for all $q$ close to $q_0$. Let us choose smooth local coordinates on the $(x, y)$-plane centered in $q_0$ such that $\Gamma$ coincides with the axis $x = 0$ and the direction filed $p_i(q) = 0$, that is, $W^c_i = \{x = p = 0\}$. In the new coordinates, the field (12) is parallel to the field

$$\dot{x} = \lambda_1 x, \quad \dot{y} = \lambda_1 xp, \quad \dot{p} = \lambda_2 p A(x, y, p) + x B(x, y), \quad (23)$$

where $A, B$ are smooth functions, $A$ is a quadratic polynomial in $p$, and $A(0) = 1$. 

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**Figure 2:** Integral curves of the field (12) and their $\pi$-projections passing through a fixed point $q \in \Gamma$. From left to right: $\lambda < 0$ (a), $0 < \lambda < 1$ (b), $\lambda > 1$ (c). The curve $\Gamma$ is depicted as a dotted line.
A conjugating diffeomorphism that brings the germ (23) to the normal form (16) or (17) can be chosen in the form

\[ x = \xi, \quad p = \eta F(\xi, \eta, \zeta) + \xi G(\xi, \eta, \zeta), \quad y = H(\xi, \eta, \zeta) \tag{24} \]

where \( F(0) = 1, \quad H(0) = 0 \). To obtain the family of solutions issuing from \( q_0 \) with the tangential direction \( p_i = 0 \), consider the images of integral curves of the field (16) or (17) lying on the leaf \( \{ \xi = 0 \} \) under the mapping (24). In the case of the normal form (17) this yields the second family from (22). Put \( \eta = t \), then from (24) it follows \( x = t^n(c + \varepsilon \ln |t|) \) and \( p = tF(x, t, 0) + xG(x, t, 0) \). Taking into account \( dy/dt = pdx/dt \), we obtain (21).

In the case of the normal form (16), there exists a one-parameter family of integral curves of the filed (12), each of which passes through the point \( q \) transversally to \( \Gamma \). For such a family \( \{ \gamma_q \}, \quad q \in \Gamma \), one can take the images of the curves \( \eta = c|\xi|^\lambda \) with \( c = 0 \), lying on various leaves \( \{ \xi = \text{const} \} \). Every leaf can be uniquely parametrized by the point of the intersection \( \gamma_q \) and \( \Gamma \), denoted by \( q \). Let us choose local coordinates on the \((x, y)\)-plane such that the family \( \{ \gamma_q \}, \quad q \in \Gamma \), becomes the family of parallel lines \( y = \text{const} \). In such coordinates, the function \( B \) in (23) is identically zero, and a conjugating diffeomorphism that brings the germ (23) to the normal form (16) can be chosen in the form (24) with \( G \equiv 0 \). This diffeomorphism sends the family of curves \( \eta = c|\xi|^\lambda \) lying on the leaf \( \{ \xi = 0 \} \) to \( p = \eta F(x, \eta, 0), \quad \eta = c|x|^\lambda \). Finally, integrating \( dy/dx = c|x|^\lambda F(x, c|x|^\lambda, 0) \), we obtain (20).

**Example 4** Integrating the differential equations \( xdp/dx = \alpha p(p^2 - 1) \), we get two families

\[ p = \pm \frac{1}{\sqrt{1 + c|x|^\lambda}}, \quad c = \text{const}, \tag{25} \]

and the single solution \( p = 0 \), which can be considered as the limit of (25) as \( c = +\infty \).

The polynomial \( M(p) = \alpha p(p^2 - 1) \) has three roots: \( p_0 = 0, \quad p_{1,2} = \pm 1 \). If \( \alpha > 0 \), then for every point \( q_0 \in \Gamma = \{ x = 0 \} \) there exist two infinite families of solutions issuing from \( q_0 \) with tangential directions \( p_{1,2} \) (these solutions are obtained from (23) when the constant \( c \) runs through the real numbers), and the single solution \( y = y_0 \) issuing from \( q_0 \) with tangential directions \( p_0 \).

If \( \alpha < 0 \), then there exist two solutions issuing from \( q_0 \) with the admissible directions \( p_{1,2} \): \( y = y_0 \pm x \), which can be obtained from (25) with \( c = 0 \). Moreover, there exists a one-parameter family of solutions issuing from \( q_0 \) with the admissible direction \( p_0 \), which can be obtained from formula (25) with all possible \( c > 0 \), including the value \( c = +\infty \) (the solution \( y = y_0 \)).

**Acknowledgements.** This work is supported by the Ministry of Science and Higher Education of the Russian Federation (Goszadaniye N. 075-00337-20-03, project 0714-2020-0005).

**References**

[1] Aminova, A. V., Aminov, N. A.-M. Projective geometry of systems of second-order differential equations. Sb. Math. 197:7, 951–975 (2006).
[2] Arnol’d, V. I., Il’yashenko, Yu. S. Ordinary differential equations. Dynamical systems I. Encycl. Math. Sci. 1, 1–148 (1988).
[3] Cartan, E. Sur les varietes a connexion projective. Bull. Soc. Math. France, 52, 205–241 (1924).
[4] Filippov, A. F. Uniqueness of the solution of a system of differential equations unsolved for the derivatives. Differ. Equ. 41:1, 90–95 (2005).
[5] Ilyashenko, Yu. S., Yakovenko, S. Yu. Finitely-smooth normal forms of local families of diffeomorphisms and vector fields. Russian Math. Surveys, 46:1, 1–43 (1991).

[6] Hirsch, M. W., Pugh, C. C., Shub, M. Invariant manifolds. Lect. Notes Math. 583, Springer-Verlag, Berlin-New York, 1977.

[7] Pavlova, N. G., Remizov, A. O. A brief survey on singularities of geodesic flows in smooth signature changing metrics on 2-surfaces. Singularities and foliations. Geometry, topology and applications, Springer Proc. Math. Stat., 222, 135–155 (2018).

[8] Pavlova, N. G., Remizov, A. O. Completion of the classification of generic singularities of geodesic flows in two classes of metrics. Izv. Math. 83:1, 104–123 (2019).

[9] Pavlova, N. G., Remizov, A. O. Smooth local normal forms of hyperbolic Roussarie vector fields. Moscow Mathematical Journal (to appear).

[10] Remizov, A. O. Geodesics on 2-surfaces with pseudo-Riemannian metric: Singularities of changes of signature. Sb. Math. 200:3, 385–403 (2009).

[11] Remizov, A. O., Tari, F. Singularities of the geodesic flow on surfaces with pseudo-Riemannian metrics. Geom. Dedicata, 185:1, 131–153 (2016).

[12] Roussarie, R. Modèles locaux de champs et de formes. Asterisque, 30, 1–181 (1975).

[13] Samovol, V. S. Equivalence of systems of differential equations in a neighborhood of a singular point. Trans. Mosc. Math. Soc. 2, 217–237 (1983).

[14] Samovol, V. S. Normal form of autonomous systems with one zero eigenvalue. Math. Notes 75:5, 660–668 (2004).

[15] Tresse, A. Détermination des Invariants Ponctuels de l’Equation Differentielle Ordinaire du Second Ordre: $y'' = \omega(x, y, y')$. Hirzel: Leipzig, 1896.

[16] Yumaguzhin, V. A. Differential invariants of second order ODEs. I. Acta Appl. Math. 109:1, 283–313 (2010).

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