EMBEDDED LENSING TIME DELAYS, THE FERMAT POTENTIAL, AND THE INTEGRATED SACHS–WOLFE EFFECT

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ABSTRACT

We derive the Fermat potential for a spherically symmetric lens embedded in a Friedman–Lemaître–Robertson–Walker cosmology and use it to investigate the late-time integrated Sachs–Wolfe (ISW) effect, i.e., secondary temperature fluctuations in the cosmic microwave background (CMB) caused by individual large-scale clusters and voids. We present a simple analytical expression for the temperature fluctuation in the CMB across such a lens as a derivative of the lenses’ Fermat potential. This formalism is applicable to both linear and nonlinear density evolution scenarios, to arbitrarily large density contrasts, and to all open and closed background cosmologies. It is much simpler to use and makes the same predictions as conventional approaches. In this approach the total temperature fluctuation can be split into a time-delay part and an evolutionary part. Both parts must be included for cosmic structures that evolve and both can be equally important. We present very simple ISW models for cosmic voids and galaxy clusters to illustrate the ease of use of our formalism. We use the Fermat potentials of simple cosmic void models to compare predicted ISW effects with those recently extracted from WMAP and Planck data by stacking large cosmic voids using the aperture photometry method. If voids in the local universe with large density contrasts are no longer evolving we find that the time delay contribution alone predicts values consistent with the measurements. However, we find that for voids still evolving linearly, the evolutionary contribution cancels a significant part of the time delay contribution and results in predicted signals that are much smaller than recently observed.

Key words: cosmic background radiation – cosmology: theory – gravitation – gravitational lensing: strong – large-scale structure of universe

1. INTRODUCTION

The cosmic microwave background (CMB) exhibits an intrinsic temperature anisotropy caused by the primordial density perturbations. Determining the spectrum of this primary anisotropy is of fundamental importance to modern cosmology. The accurate evaluation of this spectrum is complicated by the presence of secondary anisotropies caused by the thermo/kinetic Sunyaev–Zeldovich effect (Zeldovich & Sunyaev 1969; Sunyaev & Zeldovich 1980; Birkshaw 1999), the integrated Sachs–Wolfe (ISW) effect (Sachs & Wolfe 1967), gravitational lensing (Cole & Efstathiou 1989; Seljak 1996a), and reionization (Vishniac 1987; Hu 2000; Miralda-Escudé et al. 2000). In this paper we are mainly interested in the ISW effect caused by inhomogeneities that exist along the line of sight from observation back to the CMB’s decoupling at around redshift \( z \approx 1100 \) (the last scattering surface). ISW fluctuations caused by nonlinear growths of density perturbations at low redshifts were investigated in Rees & Sciaima (1968) using Swiss cheese models (Einstein & Straus 1945; Schücking 1954) and their presence is referred to as the Rees–Sciaima (RS) effect. The importance of the ISW/RS effect to modern cosmology is multifold. First, studying the effect probes the evolution history of cosmic structures and the dynamics of dark energy. For example, in a flat ΛCDM universe, a nonzero cosmological constant causes the expansion of the universe to accelerate, resulting in a decay of the gravitational potentials associated with cosmic voids, and thus leaving its evolutionary imprint on the observed CMB anisotropies. Detecting this effect is considered a crucial test for the presence of dark energy, since it directly probes the negative pressure nature of dark energy and complements other methods that are geometrical in nature, e.g., using SNe Ia as standard candles (Riess et al. 1998; Perlmutter et al. 1999) or using baryonic acoustic oscillations as standard rulers (Eisenstein et al. 2005). Second, the ISW/RS effect caused by local inhomogeneities, in particular, cosmic voids, has been used extensively to explain several large angle anomalies in CMB anisotropies observed by the WMAP (Bennett et al. 2003). For example, the octopole planarity (a preferred axis along which the octopole power is suppressed), the alignment between the quadrupole and the octopole moments, the anomalously cold spot on the south hemisphere, and the temperature asymmetry in the large angle power between opposite hemispheres all have possible explanations via the ISW/RS effect (Tegmark et al. 2003; Vielva et al. 2004; Cooray & Seto 2005; Inoue & Silk 2006; Rudnick et al. 2007; Finelli et al. 2014; Nadathur et al. 2014; Szapudi et al. 2014, 2015). Accurately measuring the ISW/RS effect is also important for detecting possible non-Gaussian signatures in the primordial density spectrum. To measure the primordial non-Gaussianity of the initial density spectrum, secondary non-Gaussianities caused by the ISW/RS effect have to be systematically investigated and removed (Kim et al. 2013). For example, the ISW-lensing bispectrum has been recently detected by Planck (Collaboration 2014a).

Even though Rees & Sciaima (1968) pointed out that the time delay experienced by a CMB photon crossing a mass overdensity can cause significant temperature fluctuations, the RS effect today usually refers only to those fluctuations caused by time-dependent potentials. Detailed theoretical investigation...
the ISW effect continues to date (Dyer 1976; Nottale 1984; Martínez-González et al. 1990; Panek 1992; Seljak 1996b; Cooray 2002a, 2002b; Sakai & Inoue 2008; Smith et al. 2009; Valkenburg 2009). Observational detection of the ISW effect includes the two following approaches. First, by constructing the correlation function between the CMB temperature sky map and tracers of large scale structures in different wavelengths (Boughn & Crittenden 2004; Ho et al. 2008; Dupé et al. 2011), hot and cold spot directions are compared with galaxy overdense and under-densities. This correlation has been tentatively found by the Planck Collaboration (2014b). The second approach is based on aperture photometry, i.e., patches of CMB maps around known large cosmic voids and clusters are stacked and averaged in order to minimize contamination from primordial anisotropies and to detect cold or hot spots due to the ISW effect. Recently several groups claimed detection of the ISW effect using this second method (Granett et al. 2008a, 2008b; Planck Collaboration 2014b), although doubts exist (Hernández-Monteagudo 2010; Nadathur et al. 2012; Ilić et al. 2013) based on linear structure growth rates in a ΛCDM universe.

One major controversy surrounding the void-stacking method is the large value of the observed fluctuations compared to theoretical predictions and numerical simulations (Maturi et al. 2007; Cai et al. 2010, 2014; Nadathur et al. 2012; Ilić et al. 2013; Hernández-Monteagudo & Smith 2013; Flender et al. 2013; Watson et al. 2014; Hotchkiss et al. 2015) Another surprising result of this method is the strange shape of the photometric profile, including the hot ring in the outer part of the profile (Ilić et al. 2013; Planck Collaboration 2014b). This structure is also hard to explain within the framework of the linear ISW effect in a standard ΛCDM cosmology (see Figure 9 of Planck Collaboration 2014b).

Here we present a theory that is easy to use and that may help to resolve these differences. We investigate the effect of lensing by individual large-scale structures on observed CMB temperature fluctuations using our recently developed embedded lens theory (Kantowski et al. 2010, 2012, 2013, 2015; Chen et al. 2010, 2011, 2015). Our lens is the same as a general Swiss cheese condensation but so named to emphasize the consequences of embedding, i.e., the consequences of making the lens a contributor to the mean mass density of the universe. An embedded lens is constructed by first removing a comoving sphere from a homogeneous Friedmann–Lemaître–Robertson–Walker (FLRW) universe, producing a vacuole, or a Swiss cheese void, then deforming that mass into an arbitrary spherical shape, and placing it back into the void in such a way as to keep Einstein’s equations satisfied throughout (Einstein & Straus 1945; Schücking 1954). Deformed mass densities can be over-dense in the central region and represent galaxy clusters or be under-dense in the central region and represent cosmic voids. Because of general relativity’s (GR) boundary conditions the average density throughout the refilled Swiss cheese void is essentially the same as the background cosmology. Consequently the dense central region of an embedded cluster lens model must be surrounded by an under-dense (void-like) region (see Figure 1).

For the same reason an embedded lens model for a cosmic void, which is low in central mass density, must be surrounded by an overdense region (often a shell of mass). By using the embedded lens model and satisfying Einstein’s equations we are assured that gravitational approximation errors are not inherent at any order. Until now most analytical expressions for temperature fluctuations in the CMB caused by crossing individual cosmic structures have been complicated and were best presented in graphical form. By keeping only the lowest-order lensing terms and using the Fermat potential we are able to give a simple analytical expression for such fluctuations.

In Section 2 we present the fundamental theoretical result of this paper, Equation (5), a simple relation between the Fermat potential and the ISW effect. In Section 3 we apply this theory to two simple embedded lens models, one for linearly growing structures and one for co-expanding structures. For simplicity we assume the clusters have stopped evolving relative to the background cosmology but for the voids we additionally compute an evolutionary correction by assuming the voids are still evolving linearly. Conclusions are drawn in Section 4. In Appendix A we present a subtle derivation of Equation (5). In Appendix B we check the validity of Equation (5) by comparing its prediction for the embedded point mass lens with the prediction made by the conventional approach.

2. THE FERMAT POTENTIAL OF EMBEDDED SPHERICAL LENSES AND THE ISW EFFECT

The geometry of a CMB photon crossing an embedded lens of angular radius \( \theta_{\text{fl}} \) is sketched in Figure 1. We assume a standard FLRW background universe with radius \( R(t) \) in which last scattering occurs well before the CMB photons encounter the cluster or void of interest. In the homogeneous universe a CMB photon’s frequency evolves with time according to \( \nu_T(t) = \nu_T R(t_0)/R(t) \), reaching us at time \( t_0 \) with frequency \( \nu_T \). If this CMB photon is gravitationally lensed by an inhomogeneity at redshift \( z_d \) before reaching us, its frequency evolution is altered in its passage through the lens between points 1 and 2 (see Figure 1). It enters the inhomogeneity at cosmic time \( t_1 \) with CMB frequency \( \nu_T(t_1) \) and exits at time \( t_2 \), but with a frequency \( \nu_T(t_2) \), not equal to the CMB frequency \( \nu_T(t_2) \), and then evolves again in the homogeneous FLRW background according to \( \nu_T(t) = \nu_T R(t_0)/R(t) \), reaching us with frequency \( \nu_T = \nu_T^\prime \). Because all frequencies are similarly affected, the temperature of the deflected photons will differ.
It is straightforward to check that variation of Equation (4) with respect to the image angle $\theta_i$ gives the lens Equation (2). Equation (4) is essentially the same as the Fermat potential for a spherical lens under the thin lens approximation in the standard linearized lens theory (Schneider et al. 1992) except for the $f_{\text{RW}}$ term which accounts for the (lowest order) effects of embedding (Kantowski et al. 2013). The term “Fermat potential” can refer to the two terms within the square brackets with or without the angular distances factor and/or the $1 + z_d$ factor. All factors are required for how we use Fermat potential in this paper. The geometrical and potential parts of the time delay, i.e., $T_g$ and $T_p$, are respectively the first and the second terms in Equation (4). The key theoretical result in this paper is the relation of the secondary CMB temperature anisotropy (Sachs & Wolfe 1967; Rees & Sciama 1968) to a derivative of the potential part of the embedded lensing time delay,

$$\frac{\Delta T}{T} = H_d \frac{\partial T_p}{\partial z_d},$$

(5)

where $H_d$ is the Hubble parameter at the deflector’s redshift $z_d$, and from Equation (4)

$$c T_p (\theta_i, z_d) = 2 (1 + z_d) \int x f(x', z_d) - f_{\text{RW}} (x') dx'. (6)$$

$T_p$ in Equation (5) can be replaced by the full time delay $T$ (hence the Fermat potential) because the geometrical part $T_g$ does not affect the CMB temperature, i.e., $\partial T_g/\partial z_d = 0$ (the source angle $\theta_i$ as well as the image angle $\theta_i$ are held fixed during the differentiation). We first investigate the consequences of Equation (5) and postpone its subtler derivation to Appendix A. A check of Equation (5) for an embedded static lens appears in Appendix B.

Once $T_p (\theta_i, z_d)$ is known for a given lens, e.g., via integrating Equation (6), temperature fluctuations $\Delta T$ in the CMB caused by that lens can easily be found by taking the single derivative in Equation (5). For an arbitrary lens we can understand the source of $\Delta T$ by differentiating Equation (6) without performing the integration. Because $z_d$ appears in two places in Equation (6), two terms contribute to the derivative, i.e., $\Delta T = \Delta T_g + \Delta T_e$, where one term is proportional to the potential part of the time delay and is called the time-delay part

$$\frac{\Delta T_g}{T} = \frac{H_d T_p}{1 + z_d},$$

(7)

and a second term called the evolutionary part

$$\frac{\Delta T_e}{T} = \frac{2 \pi H_d}{c} \int x f(x', z_d) dx'. (8)$$

Time delays, $T_p$ values in Equation (7), produced by individual galaxies are too small to produce detectable effects on the CMB, but time delays produced by large superclusters and huge voids are (see Figure 2 and Table 1). The evolutionary part is only present when the perturbation evolves at a different rate than the background cosmology. We can relate these two parts to terms that appear when making a perturbative calculation of the ISW effect. If $\chi$ is the comoving radial
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Figure 2. Secondary anisotropies in the CMB across an embedded super cluster lens of angular size \( \theta_M \approx 1.2^\circ \) and a void lens of size \( \theta_M \approx 2^\circ \) (respectively left and right panels) both at \( z_d = 0.5 \). Three background FLRW cosmologies are shown in each panel: the Einstein–de Sitter universe (left curves), the concordance ΛCDM universe (middle curves), and an open baryonic-matter only universe (right curves). The concordance cluster and void lens masses are about \( 10^{16} M_\odot \) and \( 4.8 \times 10^{14} M_\odot \), respectively. For the cluster lens \( \Delta T \) (in μK) is shown for four models: three uniformly distributed mass condensations with central over-densities \( \rho/\bar{\rho} = 8, 125, \) and 1000, all co-expanding with the background cosmologies (the respective solid red, blue and cyan curves), and for a point mass condensation (the dotted curves). In the right panel cosmic voids are modeled as uniformly under-dense spherical regions of densities \( 0.1\bar{\rho}, 0.5\bar{\rho}, \) and \( 0.9\bar{\rho} \) bounded by thin walls containing the remaining mass of the removed homogeneous sphere (the cyan, blue, and red curves, respectively). We assume the void to be either co-expanding, the dashed curves (\( \Delta T \) is produced entirely by the time-delay contribution of Equation (13)) or evolving according to linear perturbation theory, the solid curves (\( \Delta T \) is produced by the sum of Equations (13) and (14) with appropriate values for \( d\log\delta/dz \); see Figure 3).

Table 1

| Amplitudes of ISW Cold Spots Produced by Compensated Voids in the Concordance ΛCDM Cosmology |
|----------------------------------|-----------------|-------------------|------------------|
|                                | \( z_d = 0.5 \) | \( z_d = 1.0 \)    | \( z_d = 1.0 \)   |
| \( r_d = 5 \text{ Mpc} \)       | \( M_\ell = 7.2 \times 10^{13} M_\odot \) | \( 1.1 \times 10^{16} M_\odot \) | \( 1.7 \times 10^{16} M_\odot \) |
| \( \theta_M = 0^\circ.23 \)     | \( \theta_M = 0^\circ.17 \) | \( \theta_M = 0^\circ.12 \) | \( \theta_M = 0^\circ.09 \) |
| \( \delta = -0.1 \)             | \( -0.0001, -0.00038 \) | \( -0.01, -0.078 \) | \( -0.09, -0.70 \) |
| \( \delta = -0.5 \)             | \( -0.0005, -0.0019 \) | \( -0.05, -0.39 \) | \( -0.09, -0.70 \) |
| \( \delta = -0.9 \)             | \( -0.0009, -0.0034 \) | \( -0.055, -0.22 \) | \( -0.09, -0.70 \) |

Note. CMB temperature reductions in μK are given for Swiss cheese voids of radii \( r = 5, 20, 50, \) and 100 Mpc at redshifts \( z = 0.5 \) and 1.0 with density contrasts \( \delta = (\rho - \bar{\rho})/\bar{\rho} = -0.1, -0.5, \) and \( -0.9 \). The comoving mass \( M_\ell \) contained in the void radius \( r_d \) is also given as well as its angular size \( \theta_M \).

\( ^a \) The number pairs are for linearly evolving voids (left) and co-expanding voids (right), respectively. The value for the linearly evolving case is about a factor of 0.25 and 0.13 of the value for the co-expanding case for \( z_d = 0.5 \) and 1.0, respectively (see Figure 3).

coordinate measured from the center of the embedded lens, if \( \tilde{x} \equiv x/x_b \) is the normalized comoving distance, and if \( \rho(\tilde{x}, z_d) \) and \( \rho(\tilde{z}_d) \) are the matter densities of the embedded spherical lens and the homogeneous FLRW background universe, then the comoving density contrast of the lens is \( \delta(\tilde{x}, z_d) \equiv (\rho - \bar{\rho})/\bar{\rho} \). Using Equation (6) and the relation between projected and volume mass densities, the projected mass fraction can be written as

\[
\frac{\Delta T}{T} = \frac{6c}{cH_d} \int_{\tilde{x}}^{\infty} \frac{dx'}{x'} \int_0^{R_d} dy \left[ \int_0^{1-\tilde{x}^2} d\xi \delta \left( \sqrt{\tilde{y}^2 + \tilde{z}^2} , z_d \right) \right],
\]

and the total temperature fluctuation can be written in terms of the lens density contrast \( \delta \) as

\[
\frac{\Delta T}{T} = \frac{6c}{cH_d} \int_{\tilde{x}}^{\infty} \frac{dx'}{x'} \int_0^{R_d} dy \times \left[ \int_0^{1-\tilde{x}^2} d\xi \delta \left( \sqrt{\tilde{y}^2 + \tilde{z}^2} , R_d \right) \right] \times 1 - \frac{R_d}{\delta} \left[ \frac{\partial \delta}{\partial R_d} \right].
\]
the CMB’s temperature fluctuations, we first construct simple embedded lens models for galaxy clusters and cosmic voids and then compute the late-time ISW effect by applying Equation (5) to these models. We consider three centrally over-dense lenses (galaxy clusters) and three centrally under-dense lenses (cosmic voids). We assume the central mass densities of the embedded cluster lenses are homogeneous and expanding with the background, i.e., have constant angular radii $\theta_m < \theta_D$ (see the central shaded region of Figure 1). The projected mass fraction for this lens is $f_{\text{lens}}(x) = f_{\text{RW}}(x/\bar{x})$ for $x \leq \bar{x}$ and equals 1 for $\bar{x} < x \leq 1$, where $\bar{x} \equiv \theta_m/\theta_D$.

From Equations (6) and (7) we obtain the temperature anisotropy for this simple cluster lens model

$$\frac{\Delta T}{T} = \frac{H_d}{1 + z_d} \left\{ \begin{array}{ll} T_p^\text{PM}(x) - T_p^\text{PM}(x/\bar{x}), & \text{if } x \leq \bar{x}, \\ T_p^\text{PM}(x), & \text{if } \bar{x} < x \leq 1, \end{array} \right.$$  \hspace{1cm} (11)

where $T_p^\text{PM}(x)$ is the point mass time delay given in Equation (7) of Kantowski et al. (2013)

$$T_p^\text{PM}(x) = \left( 1 + z_d \right) \frac{2\bar{r}_c}{c} \left( \log \left[ \frac{1 + \sqrt{1 - x^2}}{x} \right] - \frac{4 - x^2}{3} \sqrt{1 - x^2} \right).$$  \hspace{1cm} (12)

In the left panel of Figure 2 we give $\Delta T$ for $\bar{x} = 0.5, 0.2$, and 0.1, corresponding to respective density contrasts $\rho/\bar{\rho}$ of 8, 125, and 1000.

For voids the lensing picture is similar to Figure 1 except the central region is an under-density and the deflection angle $\alpha$ is outward, i.e., a void is defocusing rather than focusing. To be an exact solution of GR, Swiss cheese embedding requires the under-dense void interior to be surrounded by an over-dense bounding ridge. For simplicity we model a cosmic void as a uniform under-dense spherical region of density $\rho/\bar{\rho} = 1 + \delta = 0.1, 0.5,$ or 0.9 (i.e., having density contrasts $\delta = -0.9, -0.5$ or -0.1, see the right panel of Figure 2) bounded by an over-dense thin shell that contains the mass removed from the homogeneous sphere (i.e., $f_{\text{shell}} = -\delta [1 - (1 - x^2)^{1/2}]$). If the voids are co-expanding $\delta$ remains constant; however, if they evolve $\delta$ changes with the deflector’s redshift $z_d$. For this void model

$$\frac{\Delta T_v}{T} = \delta \frac{2\bar{r}_c}{3c/H_d} \left( 1 - x^2 \right)^{3/2},$$  \hspace{1cm} (13)

and

$$\frac{\Delta T_v}{T} = \left( 1 + z_d \right) \frac{d \log \delta}{dz_d} \times \frac{\Delta T_v}{T}.$$  \hspace{1cm} (14)

For simplicity we consider only two possibilities: the co-expanding case where $\delta'(z_d) \equiv d\delta/dz_d = 0$ and only $\Delta T_v$ contributes, and a linear evolving case where $\delta(z_d)$ is known from standard perturbation theory. Despite the extreme simplicity of this result, the model is useful for studying lensing by voids (Chen et al. 2015). Extension to other cluster or void models is straightforward. For example, if the

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To illustrate the simplicity of the embedded theory’s use, as well as to give an order of magnitude estimate of the effect on

$$R_d \equiv R(t_0)/(1 + z_d)$$ and is used as a “time” variable rather than $t_d$. There is no time delay or temperature fluctuation at the boundary of the Swiss cheese void where $x = 1$ (i.e., $\theta_d = \theta_D$) for any embedded lens. For linearly evolving structures in an Einstein–de Sitter (EdS) universe, $\delta \propto R_d$ and the two terms in Equation (10) exactly cancel, resulting in $\Delta T = 0$, consistent with the classical results of Sachs & Wolfe (1967). For linearly evolving structures in the standard $\Lambda$CDM universe, $|\delta|$ grows slower than $R_d$, and the evolutionary contribution only partially cancels the time-delay contribution, resulting in a non-zero ISW effect (see Figure 3 for an example). For highly nonlinear structures (Bertschinger 1985a, 1985b; Sheth & van de Weygaert 2004), $|\delta|$ might cease to grow or even start to decrease (e.g., a deep cosmic void with density contrast $\delta$ already approaching the lower bound $-1$), the evolutionary contribution can be of the same sign as the time-delay contribution. For such cases the time-delay contribution $\Delta T_D$ can be used as a conservative estimate of the magnitude of the ISW signal (see the examples in the next section). For embedded point mass lenses or lenses that expand with the background $f(x, z_d) = f(x)$ does not explicitly depend on $z_d$ and the temperature fluctuation is simply proportional to the time delay; see Equation (7). The time-delay contribution to the temperature fluctuation is of the order $H_d T_p \sim H_d (2\bar{r}_c/c) \sim 2\bar{r}_c H_d/c$, where $c\bar{r}_D \equiv H_d T_p$ is the expansion rate of the Swiss cheese void’s boundary relative to static observers and $r_D$ is the physical radius of the void at redshift $z_d$ (Kantowski et al. 2013). This is in line with estimates originally made in Rees & Sciama (1968) although the details differ. From Equation (8) the evolutionary contribution is expected to be the same order of magnitude as long as only moderate expansion/contraction occurs.
compensating ring of mass at the boundary of the Swiss cheese void in this simple model is moved inward it can produce a hot ring surrounding the void’s central cold spot.

In the left panel of Figure 2 we plot the radial profile of the ISW effect across a super cluster at redshift \( z_d = 0.5 \) with an angular radius \( \theta_d \approx 2.0 \). The cluster lens mass is \( \sim 10^{16} M_\odot \). We test for three background cosmologies: an EdS universe (\( \Omega_m = 1.0, \Omega_\Lambda = 0 \)), a concordance \( \Lambda \text{CDM} \) universe (\( \Omega_m = 0.3, \Omega_\Lambda = 0.7 \)), and a baryonic-matter only universe (\( \Omega_m = 0.05, \Omega_\Lambda = 0 \)), i.e., the left, middle, and right curves, respectively. From the left panel of Figure 2 we see that for a cluster lens of fixed mass, the amplitude of the perturbation increases with central concentration (approaching the limiting case of an embedded point mass lens; \( f_{\text{point}} = 1 \)).

The CMB temperature increases by \( \gtrsim 2 \mu K \) for a co-expanding cluster of \( \sim 10^{16} M_\odot \) with density contrast \( \delta \gtrsim 100 \). The ISW signal predicted by the simple cluster model scales with the cluster mass; see Equations (11) and (12). Outside of the over-dense region, i.e., the vacuum, the cluster behaves exactly the same as a (static) point mass lens; inside the over-dense region, the lens might be still evolving. Consequently, prediction based on the simple model might be biased (e.g., increases more slowly near the cluster center than predicted in Figure 2). However, since the over-dense region takes only a small fraction of the lens volume, our model should give a reasonable estimate for the averaged signal for rich galaxy clusters (\( \Delta T \)) about \( \sim 0.3 \mu K \) for a \( 10^{16} M_\odot \) cluster assuming \( \delta \gtrsim 100 \). A more physical embedded cluster model (e.g., non-uniform density profile with different evolution scenarios) will be pursued in future works (e.g., Kantowski et al. 2015). We also see that various background cosmologies influence \( \Delta T_F \) through the Hubble parameter \( H_\odot \); however, the differences are not very significant.

The right panel of Figure 2 and Table 1 show predictions for the two simple void lens models (co-expanding and linearly evolving) at redshift \( z_d = 0.5 \). The concordance void lens has an angular radius \( \theta_d \approx 2.0 \) with a physical radius \( r_0 \approx 40 \) Mpc and mass \( \sim 4.8 \times 10^{16} M_\odot \). The dashed and solid curves show the results for co-expanding and linearly evolving void models, respectively. The signal scales linearly with the lens mass \( M_\odot \), equivalently as the cube of the void size \( r_0 \) and emptier voids result in colder CMB spots, i.e., they evolve according to standard linear perturbation theory, then linear perturbation theory should give a better estimate of the ISW signal, whereas for large density contrast (e.g., \( \delta = -0.9 \)), the co-expanding model should give a better estimate. The two numbers can be used as approximate lower and upper bounds for the ISW signal caused by cosmic voids.

Figure 3 shows the redshift dependence of the amplitude of the ISW effect for the two simple void lens models presented in this paper. We plot the growth curves of the ISW effect with redshift for the time-delay contribution \( \Delta T_F \), evolutionary contribution \( \Delta T_F \), assuming linear perturbation theory, and the sum of the two contributions (the blue, red, and black curves respectively). To compute the ISW temperature profiles for a void at an arbitrary redshift \( z \), you simply scale the curves in the right panel of Figure 2 (\( z_d = 0.5 \)) using the corresponding curves in Figure 3, i.e., apply a multiplicative factor \( \text{ISW}_v(z)/\text{ISW}(0.5) \) and ISW(z)/ISW(0.5) for the co-expanding and linear evolving model respectively. The reader can observe from the left panel, as stated earlier, that linearly evolving perturbations in an EdS universe do not directly alter the CMB’s temperature, i.e., the net growth factor vanishes \( \text{ISW}(z) = 0 \) for all \( z \). What is clear for the other two cosmologies is that for the highest redshifts (\( z > 20 \)) the earlier in cosmic time they are encountered the less they perturb the CMB. However, somewhat closer voids (\( 20 > z > 0 \)) in an open baryonic matter universe (see the black curve in the third panel) will produce larger effects on CMB temperatures with increasing redshifts, whereas voids in a concordance universe (see the black curve in the middle panel) have a rapidly diminishing effect (with increasing redshift) on the CMB for all \( z > 0 \). Seeing a decrease in ISW temperatures with redshift is clearly expected if we do live in a concordance universe.

Figure 4 shows an example of ISW-induced CMB temperature fluctuations (with and without linear evolution) using the recent void catalog from SDSS DR7 (Abazajian et al. 2009; Sutter et al. 2012) and the void models described above. For simplicity we have assumed the same density contrast \( \delta = -0.5 \) for all voids (density profiles for individual cosmic voids are not available, only mean profiles have been estimated by stacking/averaging a large number of cosmic voids). The majority of the voids have radii of the order of \( \sim 10 \) Mpc, and their evolution can be nonlinear. Consequently, the time-delay contribution \( \Delta T_F \) (the left panel) might give better estimates of the ISW signals. For those really large voids (with radius \( \sim 80 \) Mpc), if they are shallow as predicted by standard theory, then linear perturbation theory should give better estimates of the signal (the right panel). Because most of the voids are at low redshifts (\( z \lesssim 0.5 \)), the ISW signal of the linear evolution scenario is about one-third to one-fourth of the time-delay contribution because of the cancellation of \( \Delta T_F \) by \( \Delta T_F \) (see Figure 3). Consequently, the right panel looks very similar to the left panel except for the color-bar. Assuming a different void density contrast will also result in a similar Figure except for a pure rescaling of the color-bar. For example, if we chose \( \delta = -0.1 \) instead of \( \delta = -0.5 \), then the signal will be reduced to one-fifth of its current values for both scenarios.

## 4. DISCUSSIONS AND CONCLUSIONS

Granett et al. (2008a) claimed a detection of the ISW effect at \( -11.3 \pm 3.1 \mu K \) by stacking 50 of the largest voids (effective radius \( \gtrsim 100 \) Mpc) from their catalog using the WMAP 5 yr
data. Similar analyses have also been conducted for the same and other void catalogs (Pan et al. 2012; Sutter et al. 2012; Nadathur & Hotchkiss 2014) by several other groups using WMAP or Planck data (e.g., Ilić et al. 2013; Cai et al. 2014; Planck Collaboration 2014b; Hotchkiss et al. 2015). Analyses of the more recent catalogs have typically given signals of more moderate amplitude and lower significance (from negligible to \( \sim 2.5\sigma \); Ilić et al. 2013; Planck Collaboration 2014b). For example, Ilić et al. (2013) confirmed the Granett et al. (2008a) results (at slightly lower significance) using WMAP 7 yr data. They detected no signals using the Pan et al. (2012) catalog and a signal up to \( \sim 2.4\sigma \) using the Sutter et al. (2012) catalog (after rescaling of the voids to the same size on the stacked images). Recently the amplitude of the Granett et al. (2008a) detection has been re-confirmed by Planck by applying the aperture photometry method to Planck CMB temperature maps and the same void catalog (Planck Collaboration 2014b). Several groups found that the observed CMB fluctuations from stacking and averaging cosmic voids are significantly larger than expected from the linear ISW effect. For example, Nadathur et al. (2012) found a theoretical value from a linear perturbation analysis in \( \Lambda \)CDM cosmology \( \gtrsim 2\mu K \), inconsistent with the observed value from Granett et al. (2008a) at \( >3\sigma \). Numerical simulations have also predicted smaller effects (Maturi et al. 2007; Cai et al. 2010, 2014; Flender et al. 2013; Watson et al. 2014; Hotchkiss et al. 2015). For example, Cai et al. (2014) found a cold ISW imprint of amplitude between 2.6 and 2.9 \( \mu K \) and significance of \( \sim 2\sigma \) using voids selected in the SDSS DR7 spectroscopic redshift galaxy catalog calibrated by N-body simulations. A \( \Lambda \)CDM signal consistent with zero was predicted by Hotchkiss et al. (2015) using the Jubilee N-body simulation and confirmed using a catalog of voids and superclusters extracted from SDSS, thus challenging all positive ISW interpretations.

We tentatively compare predictions based on our simple embedded void model with recent observations based on void-stacking (Granett et al. 2008a; Planck Collaboration 2014b). The time delay of a deep cosmic void (\( \delta = -0.9 \)) of angular radius \( \sim 4° \) at \( z_d = 0.5 \) (about the mean radius and redshift for the 50 voids in Granett et al. 2008a) results in \( \Delta T \approx -18.8 \mu K \) toward the void center and an averaged cold spot of about \( -7.5 \mu K \) after convolving with a compensated top-hat filter of aperture radius \( \theta_p = 4° \). These temperature reductions are of the same order as observed from stacking and averaging large cosmic voids in the Granett et al. (2008a) catalog (Granett et al. 2008a; Planck Collaboration 2014b). However, to produce an effect of this order of magnitude requires those large voids to be very deep, \( \delta \sim -0.9 \) and/or possibly evolve in an unknown nonlinear way. Cosmic voids in galaxy surveys were selected as regions with low galaxy number densities; however, luminous matter as a tracer of dark matter might be biased. The dark matter density contrast within a cosmic void might be shallower than the \( \delta \sim -0.9 \) estimate based on galaxy number counts (Sutter et al. 2012, 2014). If the void is much shallower, as predicted by standard theory, then the ISW effect will not be able to produce signals of the order of the recent detections. For example, if we assume \( \delta = -0.5 \) (still a large density contrast), then estimates based on time delays alone are not sufficient because the void is probably still evolving and the evolutionary contribution needs to be included (see Figure 3). For linear evolution, our model predicts a signal of the order of \( \sim 2 \mu K \), similar to results of other predictions (Nadathur et al. 2012; Flender et al. 2013; Ilić et al. 2013; Cai et al. 2010, 2014). Calibrating the galaxy bias for void catalogs is an active field of research (Rassat et al. 2007; Szapudi

![Figure 4. Lambert equal-area projection of the sky map of CMB temperature fluctuations caused by passing through voids along the line of sight. We used the "central void catalog" (version 2014.06.18) of Sutter et al. (2012). The map is centered at R.A. = 180° and decl. = 30° with longitude and latitude lines separated by 30°. We use the same void model as in the right panel of Figure 2 (the blue curves in the middle panel). For each void, we estimate its mass (or Schwarzschild radius \( r_s \)) using its effective radius and redshift provided by Sutter et al. (2012) and, we then compute \( \Delta T \) across this void using Equation (13) assuming \( \delta = -0.5 \). For overlapping voids along a sight line, we add the \( \Delta T \) caused by each void. For the left panel, we assume the voids are co-expanding with the background universe and only the time delay contributes. For the right panel, we assume a linear perturbation theory (see Figure 3). Assuming a different void density contrast will also result in a similar Figure except for a pure rescaling of the color-bar. For example, if we chose \( \delta = -0.1 \) instead of \( \delta = -0.5 \), then the signal will be reduced to one fifth of its current values for both scenarios.](image-url)
A linear galaxy bias $b_g = 1.41 \pm 0.07$ was recently used by Szapudi et al. (2014, 2015) to model the CMB Cold Spot using the ISW effect caused by a supergiant void of radius $192 \pm 15$ h$^{-1}$ Mpc and density contrast $\delta = -0.13 \pm 0.03$ (see also Finelli et al. 2014). A similar estimate of $b_g$ was obtained earlier by Rassat et al. (2007) for galaxies selected from the 2MASS (Jarrett et al. 2000). The galaxy void profiles of Sutter et al. (2012) suggest very low density near the void center, $\delta \lesssim -0.8$ for voids of radii range from a few up to $\sim 100$ Mpc. For simplicity, we have assumed a uniform dark matter density contrast $\delta = -0.5$ for all voids (large and small) when simulating the CMB sky map modulated by ISW effect caused by cosmic voids (see Figure 4). This is equivalent to assuming a linear galaxy bias $b_g \approx 1.8$ for all voids (see Figure 9 of Sutter et al. 2012). For simplicity, we have also assumed all voids to be compensated (with zero net mass with respect to the homogeneous background). Numerical simulations show that small voids tend to be over-compensated, while large voids might be under-compensated (Sheth & van de Weygaert 2004; Cecarelli et al. 2013; Hamaus et al. 2014; Cai et al. 2014). More accurate modeling using more physical density profiles and better estimates of galaxy bias (Sutter et al. 2014; Hamaus et al. 2014; Leclercq et al. 2014) is beyond the scope of the paper but is important to pursue in further work.

The new results of this paper are Equations (4)–(6), which result in Equations (7)–(10), i.e., simple analytical expressions for secondary temperature anisotropies in the CMB caused by encountering mass inhomogeneities, i.e., embedded gravitational lenses. The temperature fluctuations are obtained from a simple derivative of the Fermat potential. That derivative is the same as a derivative of the potential part of the time delay, which is easy to compute for an arbitrary lens model via Equation (6). This new method of obtaining CMB temperature changes represents a significant improvement over previous methods. We find that the CMB temperature change, i.e., $\Delta T$ given in Equation (5), can be understood as coming from two sources. The first source of change is the time delay and occurs because the increase in the Swiss cheese void’s radius depends on how long the lens mass delays the passing photon. The longer it is delayed the larger the cosmic expansion and the more the unlensed CMB photons are redshifted. This term is given in Equation (7). Clusters tend to delay lensed photons, making them bluer, and voids shorten transit times, making them redder. A second source of temperature change is the evolution of the distribution of the embedded mass inside the Swiss cheese void relative to the background cosmology, i.e., an $f$ and $\delta$ dependence on cosmic time alters the frequency of the transiting photon. This temperature change can be positive or negative depending on lens details and is given in Equation (8). We have confirmed Equations (7) and (8) by comparing results with prior analytic examples, e.g., the embedded expanding homogeneous dust sphere model (Dyer 1976), and the compensated void/lump model under the potential approximation (Martínez-González et al. 1990). We have used Equation (8) in Figure 3 to estimate the effect of linear perturbation growth of the voids temperature profiles of Figure 2. Our theory is valid for arbitrarily large density contrast and is therefore very useful for studying CMB lensing in nonlinear regimes, e.g., lensing by cosmic voids where the density contrast can be large (Sutter et al. 2012, 2014) and by galaxy clusters where $\delta \gg 1$. For individual nonlinear cosmic structures, the temperature fluctuation can be easily computed provided that the expanding/collapsing histories of the structures are known from astrophysical theories or simulations (Press & Schechter 1974; Bertschinger 1985a, 1985b; Sheth & van de Weygaert 2004). This is in contrast to the standard linear ISW analysis where the Poisson Equation for the perturbation potential (or the density contrast) is solved in Fourier space with some assumed initial power spectrum. Such an approach is not convenient for modeling temperature fluctuations caused by individual structures with a fixed scale, whereas our embedded theory is. We found that to produce secondary anisotropies of the order of some recent measurements of the ISW effect made using the void-stacking method requires large cosmic voids to be more numerous, deeper, and evolving differently than predicted by linear perturbation theory.

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APPENDIX A

PROOF OF EQUATION (5)

Suppose two identical CMB photons with frequencies $\nu\gamma(t_1)$ reach the embedded lens at comoving point 1 in Figure 1 at cosmic time $t_1$. One photon is gravitationally deflected by the embedded lens and exits at comoving point 2 at time $t_2$ on a path differing in direction by an angle $\alpha$. The second photon is reflected by an angle $\alpha$ at point B as it travels from point 1 to point 2 on two straight line segments, just as if the embedded lens did not exist. The lensed photon exits comoving point 2 at cosmic time $t_2$ which is later than the reflected photon by an amount of time $T_p(t_2)$ which is by definition the potential part of the time delay (Chen et al. 2010). After leaving point 2, both photons travel the same comoving path but with the reflected photon leading the delayed gravitationally lensed photon in time by $T_p(t) = T_p(t_0)R(t)/R(t_0)$. Because both photons had the same frequency $\nu_1$ and period $\delta t_1$ at $t_1$ the ratio of the lensed photon’s frequency at $t_2$ to the reflected photon’s frequency at $t_2 - T_p(t_2)$ is given by the inverse ratio of their periods, $\delta(t_2 - T_p(t_2))/\delta t$. Because all photons redden equally after $t_2$, the ratio of the frequencies of lensed to unlensed CMB photons at equal $t$ is constant and is given by

$$\frac{\nu\gamma(t)}{\nu\gamma(t_1)} = \frac{R(t_0)}{R(t_2)} = \frac{R(t_2)}{R(t_2 - T_p(t_2))} \frac{\delta(t_2 - T_p(t_2))}{\delta t}. \tag{15}$$

4 This is an approximation that requires $H_0 T_p \ll 1$. 

8
This analytic expression can be evaluated at the cosmic time \( t_d \) equivalent to the redshift \( z_d \) of the deflector, giving

\[
\frac{\nu_{\ell c}}{\nu_{\ell}} = 1 + \frac{H_d}{T_p(1 + z_d)} \frac{\partial}{\partial z_d} \left[ (1 + z_d) T_p \right].
\]  

Equation (16)

The quantity \( T_p = (1 + z_d) T_p(z_d) \) is the potential part of the time delay as seen at the observer given in Equation (6), written as a function of the deflector’s redshift. When Equation (16) is used in Equation (1), Equation (5) results. Equation (15) can be interpreted as a product of a time-delay effect and a redshift effect associated with the density evolution. The evolutionary effect is produced by a time dependence in the projected mass fraction, see Equation (8), or equivalently in the relative mass density, see Equation (10), and is hence associated with a time dependence of the Newtonian potential of the relative perturbation.

**APPENDIX B**

**THE ISW/RS EFFECT FOR AN EMBEDDED POINT MASS LENS**

Because Equation (5) is new and its derivation in Appendix A is subtle we wish to confirm its validity by comparing its prediction for the late-time ISW/RS effect caused by an embedded point mass Kottler (1918) with the prediction made by the conventional approach. The conventional approach dates back to Rees & Sciama (1968) and requires a detailed calculation of the photon’s deflected orbit. Equation (2) of Dyer (1976) gives the needed ratio of observed frequencies of a lensed CMB photon \( \nu_{\ell c} \) to an identical unlensed CMB photon \( \nu_{\ell} \)

\[
\frac{\nu_{\ell c}}{\nu_{\ell}} = \frac{R(t_2)}{R(t_1)} \frac{\nu_{\ell c}(t_2)}{\nu_{\ell c}(t_1)}.
\]  

Equation (17)

where all frequencies are measured by observers comoving with the background FLRW cosmology. This equality is a result of both lensed and unlensed photons being identically redshifted after lensing (after the lensed photon exits at cosmic time \( t_2 \) with frequency \( \nu_{\ell c}(t_2) \), see point 2 of Figure 1) and both photons having the same frequencies prior to lensing which begins at \( t_1 \) with frequency \( \nu_{\ell}(t_1) = \nu_{\ell c}(t_1) \).

Using Equations (9) and (10) of Kantowski et al. (2010) and details of the photon’s orbit that follow, Equation (17) can be evaluated to give the ISW/RS effect for a flat background universe, but only after a significant amount of algebra. The result is

\[
\frac{T + \Delta T}{T} = \frac{R(t_2)}{R(t_1)} \frac{\nu_{\ell c}(t_2)}{\nu_{\ell c}(t_1)}
\]

Equation (18)

\[
= 1 - \frac{\beta_1 \Gamma}{6} \left[ 15 \cos \tilde{\phi}_1 + \cos 3\tilde{\phi}_1 - 2 \log \tan \frac{\tilde{\phi}_1}{2} \right],
\]

and results in Equations (18) and (19) giving identical results for \( \Delta T/T \) to order \( \delta^1 \).

To compare Equation (18) with Equation (19) the Kottler impact variables \( \beta_1, \tilde{\phi}_1 \), and \( r_0 \) in Equation (18) must be converted to image positions \( \theta_1 \), expansion velocities \( \beta_2 \), lens sizes \( r_d \), and maximum image positions \( \theta_M \) all evaluated at fixed cosmic time \( t_d \) defined by the deflector’s redshift \( z = (1 + z_d) \). The needed change of variables given in the next three equations can be found in Equation (3) of Kantowski et al. (2013), Equation (A1) of Kantowski et al. (2013), and Equations (2) and (15) of Chen et al. (2011)

\[
\tilde{\phi}_1 = \sin^{-1} \frac{\theta_1}{\theta_M} + \beta_2 \frac{\theta_1}{\theta_M} + \mathcal{O}(\delta^2),
\]

\[
r_0 = r_d \sin \tilde{\phi}_1 \left( 1 - \beta_2 \cos \tilde{\phi}_1 + \mathcal{O}(\delta^2) \right),
\]

\[
\beta_1 = \beta_2 + \delta \left( \frac{r_0}{2\tilde{\phi}_1} - \frac{\Lambda r_0^2}{3} \cos \tilde{\phi}_1 + \mathcal{O}(\delta^3) \right).
\]

Equation (19)

and results in Equations (18) and (19) giving identical results for \( \Delta T/T \) to order \( \delta^1 \).

The reader can easily see that because there is no time evolution of the projected mass fraction \( f(\theta_1/\theta_M) = 1 \) or equivalently in the density contrast \( \delta = -1 \) for an embedded point mass, only the time delay part, Equation (7), contributes to the ISW effect.

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