Random Walks on Comb-Type Subsets of $\mathbb{Z}^2$

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Abstract
We study the path behavior of the simple symmetric walk on some comb-type subsets of $\mathbb{Z}^2$ which are obtained from $\mathbb{Z}^2$ by removing all horizontal edges belonging to certain sets of values on the $y$-axis. We obtain some strong approximation results and discuss their consequences.

Keywords Random walk · 2-dimensional comb · Strong approximation · 2-dimensional Wiener process · Oscillating Brownian motion · Laws of the iterated logarithm · Iterated Brownian motion

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1 Introduction

An anisotropic walk is defined as a nearest-neighbor random walk on the square lattice $\mathbb{Z}^2$ of the plane with possibly unequal symmetric horizontal and vertical step probabilities, so that these probabilities depend only on the value of the vertical coordinate. More formally, consider the random walk \{$C(N) = (C_1(N), C_2(N)) ; N = 0, 1, 2, \ldots$\} on $\mathbb{Z}^2$ with the transition probabilities

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\[ P(C(N + 1) = (k + 1, j)|C(N) = (k, j)) \]
\[ = P(C(N + 1) = (k - 1, j)|C(N) = (k, j)) = \frac{1}{2} - p_j, \]
\[ P(C(N + 1) = (k, j + 1)|C(N) = (k, j)) \]
\[ = P(C(N + 1) = (k, j - 1)|C(N) = (k, j)) = p_j, \quad (1.1) \]

for \((k, j) \in \mathbb{Z}^2, N = 0, 1, 2, \ldots\) with \(0 < p_j \leq 1/2\) and \(\min_{j \in \mathbb{Z}} p_j < 1/2\). Unless otherwise stated, we assume also that \(C(0) = (0, 0)\).

In the present paper, we are interested in a special type of this anisotropic walk. We only want to consider walks for which \(p_j\) in (1.1) is either \(1/2\) or \(1/4\). In particular, for such walks we consider an arbitrary subset \(B\) of the integers on the \(y\)-axis and remove from the two-dimensional integer lattice all the horizontal lines which do not belong to the \(y\)-levels in \(B\). Denote this lattice by \(\mathbb{C}^2 = \mathbb{C}^2(B)\). The transition probabilities throughout this paper are

\[ p_y = P(C(N + 1) = (x \pm 1, y) \mid C(N) = (x, y)) \]
\[ = P(C(N + 1) = (x, y \pm 1) \mid C(N) = (x, y)) = \frac{1}{4}, \quad \text{if } y \in B \]
\[ p_y = P(C(N + 1) = (x, y \pm 1) \mid C(N) = (x, y)) = \frac{1}{2}, \quad \text{if } y \notin B, \quad (1.2) \]

A compact way of describing the just introduced transition probabilities for this simple random walk \(C(N)\) on \(\mathbb{C}^2(B)\) is via defining

\[ p(\mathbf{u}, \mathbf{v}) := P(C(N + 1) = \mathbf{v} \mid C(N) = \mathbf{u}) = \frac{1}{\deg(\mathbf{u})}, \quad (1.3) \]

for locations \(\mathbf{u}\) and \(\mathbf{v}\) that are neighbors on \(\mathbb{C}^2(B)\), where \(\deg(\mathbf{u})\) is the number of neighbors of \(\mathbf{u}\), otherwise \(p(\mathbf{u}, \mathbf{v}) := 0\).

Clearly when \(B = \{0\}\), we get the two-dimensional comb which inspired our choice for the name of these particular anisotropic walks. We are interested in the case when \(B_n := B \cap [-n, n]\), and \(|B_n| \sim cn^\beta\) with some \(0 \leq \beta \leq 1\). Here, and in the sequel, \(|B_n|\) stands for the (finite) number of elements in the set \(B_n\). We are to discuss these comb-type walks at first in general and then spell out three important cases, namely the case \(\beta = 1, 0 < \beta < 1\), and \(\beta = 0\). In each of these cases, we prove strong approximation results for both components of the walks, by (time-changed) Wiener processes. Our primary motivation was to connect and generalize some existing results on anisotropic walks. In each of these cases, we use the fact that the number of the horizontal steps can be approximated via the occupation time of the set \(B\) by the simple symmetric random walk on the \(y\)-axis defined by our approximation. It is important to emphasize that the scaling of the vertical component is always \(N^{1/2}\), but the scaling of the horizontal component is \(N^{1/2}\) only in the case of \(\beta = 1\). When \(0 < \beta < 1\), then the scaling depends on \(\beta\). Finally in the case of \(\beta = 0\), the horizontal component is approximated by a time-changed (by the local time of zero) Brownian motion resulting in scaling by \(N^{1/4}\).
In what follows, we give some historical introduction and try to place our present paper in the context of some of the existing body of work on anisotropic walks. Initial studies of anisotropic walks are due to Silver et al. [39], Seshadri et al. [37], Shuler [38], and Wescott [42], who were motivated by the so-called transport phenomena of statistical physics. Some of the most important contributions to the general anisotropic walk as in (1.1) are due to Heyde [25] and [26], and Heyde et al. [27].

As in Heyde et al. [27], let

\[
k^{-1} \sum_{j=1}^{k} p_j^{-1} = 2\gamma_1 + \varepsilon_k, \quad k^{-1} \sum_{j=-k}^{-1} p_j^{-1} = 2\gamma_2 + \varepsilon_k^*.
\]

(1.4)

**Theorem A** ([27]) *For the anisotropic random walk under condition (1.1), suppose that in (1.4) \(\varepsilon_k\) and \(\varepsilon_k^*\) are \(o(1)\) as \(k \to \infty\). Then

\[
\sup_{0 \leq t \leq T} |n^{-1/2}C_2([nt]) - Y(t)| \to 0 \text{ a.s.}
\]

as \(n \to \infty\), for all \(T > 0\), where \([Y(t), t \geq 0]\) is a diffusion process on the same probability space as \([C_2(n)]\) whose distribution is defined by

\[
Y(t) = W(A^{-1}(t)), \quad t \geq 0,
\]

where \([W(t), t \geq 0]\) is a standard Brownian motion (or standard Wiener process) and

\[
A(t) = \int_0^t \sigma^{-2}(W(s)) \, ds
\]

and

\[
\sigma^2(y) = \begin{cases} 
\frac{1}{\gamma_1} & \text{for } y \geq 0, \\
\frac{1}{\gamma_2} & \text{for } y < 0.
\end{cases}
\]

Here \(A^{-1}(\cdot)\) is the inverse of \(A(\cdot)\). The process \(Y(t)\) is called oscillating Brownian motion if \(\gamma_1 \neq \gamma_2\), that is a diffusion with speed measure \(m(dy) = 2\sigma^{-2}(y)dy\).

**Remark 1.1** Observe that \(A(t)\) in the above theorem is equal to

\[
A(t) = \gamma_1 \int_0^t I(W(s) \geq 0) \, ds + \gamma_2 \int_0^t I(W(s) < 0) \, ds.
\]

(1.5)

In an earlier paper, Heyde [25] used the following somewhat more restrictive asymptotic version of (1.4), when \(\gamma_1 = \gamma_2 = \gamma\):
\[
    n^{-1} \sum_{j=1}^{n} p_j^{-1} = 2\gamma + o(n^{-\tau}), \quad n^{-1} \sum_{j=1}^{n} p_{-j}^{-1} = 2\gamma + o(n^{-\tau}), \quad (1.6)
\]

as \( n \to \infty \), for some constants \( \gamma \), \( 1 < \gamma < \infty \) and \( 1/2 < \tau < \infty \). Under (1.6), he proved a strong approximation result for the second coordinate by a rescaled Brownian motion. Under the same condition, the following simultaneous strong approximation result was proved for \( (C_1(\cdot), C_2(\cdot)) \).

**Theorem B** ([15]) Under conditions (1.1) and (1.6) with \( 1/2 < \tau \leq 1 \), on an appropriate probability space for the random walk

\[
    \{C(N) = (C_1(N), C_2(N)); \ N = 0, 1, 2, \ldots\},
\]

one can construct two independent standard Wiener processes \( \{W_1(t); \ t \geq 0\}, \ \{W_2(t); \ t \geq 0\} \) so that, as \( N \to \infty \), we have with any \( \varepsilon > 0 \)

\[
    \left| C_1(N) - W_1 \left( \frac{\gamma - 1}{\gamma} N \right) \right| + \left| C_2(N) - W_2 \left( \frac{1}{\gamma} N \right) \right| = O(N^{5/8-\tau/4+\varepsilon}) \ a.s. \quad (1.7)
\]

We note that Theorems A and B are true for \( p_j \), more general than given in (1.2).

In [15,17], related issues of recurrence, local time and range are discussed. Theorem 1.1 in [17] implies that all comb-type walks are recurrent.

Now we mention some particular cases of the random walk \( C(N) \) as in (1.1).

The case \( p_j = 1/4, \ j = 0, \pm 1, \pm 2 \ldots \) is the simple symmetric walk on the plane for which we refer to Erdős and Taylor [24], Dvoretzky and Erdős [23] and Révész [35].

As we mentioned earlier, the case when \( p_0 = 1/4 \) and \( p_j = 1/2 \) with \( j = \pm 1, \pm 2 \cdots \) means that all horizontal lines except the \( x \)-axis are missing. This is the so-called random walk on the two-dimensional comb, for which \( \gamma_1 = \gamma_2 = 1 \) and hence is excluded from Theorem B. This model and some similar ones have many applications in physics, so a number of these early results are in the physics literature [1–3,6,21,22,36,43,44]. A present study of these models is provided in a new book by Iomin et al. [29]. From the many papers about the exact comb model as above, we only wish to mention the following few. Weiss and Havlin [41] derived an asymptotic formula for the \( n \)-step transition probability \( P(C(N) = (k, j)) \), Bertacchi [5] was the first who noted that while a Brownian motion is the right object to approximate \( C_2(\cdot) \), for the first component \( C_1(\cdot) \), the right approximation is by a Brownian motion time changed by the local time of the second component, and she proved a simultaneous weak convergence result for the two components. Bertacchi and Zucca [6] investigated the ratio of the \( n \)-step vertical and horizontal transition probabilities, suggesting that this walk spends most of its time on some teeth of the comb. In Csáki et al. [12], we established a simultaneous strong approximation for the two coordinates of the random walk \( C(N) = (C_1(N), C_2(N)) \) that reads as follows.

**Theorem C** ([12]) On an appropriate probability space for the simple random walk

\[
    \{C(N) = (C_1(N), C_2(N)); \ N = 0, 1, 2, \ldots\} \text{ on the two-dimensional comb lattice}
\]
$\mathbb{C}^2$, one can construct two independent standard Wiener processes \{W_1(t); \ t \geq 0\}, \{W_2(t); \ t \geq 0\} so that, as \( N \to \infty \), we have with any \( \varepsilon > 0 \)

\[
N^{-1/4}|C_1(N) - W_1(\eta_2(0, N))| + N^{-1/2}|C_2(N) - W_2(N)| = O(N^{-1/8+\varepsilon}) \ a.s.,
\]

where \( \eta_2(0, \cdot) \) is the local time process at zero of \( W_2(\cdot) \).

Some Strassen-type theorems are also given in the same paper. Furthermore, the local time of the comb walk is investigated in Csáki et al. [13]. In Csáki et al. [14], we investigated another special case, when \( p_j = 1/4, \ j = 0, 1, 2, \ldots, p_j = 1/2, \ j = -1, -2, \ldots \), namely all horizontal lines under the \( x \)-axis are deleted; hence, it is a simple random walk on the half-plane half-comb (HPHC) structure. In this case \( \gamma_1 = 2 \) and \( \gamma_2 = 1 \); thus, Theorem B is not applicable. However, this model satisfies the conditions of Theorem A, and hence, the second coordinate can be approximated by an oscillating Brownian motion. Our main result therein reads as follows.

**Theorem D** ([14]) On an appropriate probability space for the HPHC random walk \( \{C(N) = (C_1(N), C_2(N)); N = 0, 1, 2, \ldots \} \) with \( p_j = 1/4, \ j = 0, 1, 2, \ldots, p_j = 1/2, \ j = -1, -2, \ldots \) one can construct two independent standard Wiener processes \{W_1(t); \ t \geq 0\}, \{W_2(t); \ t \geq 0\} such that, as \( N \to \infty \), we have with any \( \varepsilon > 0 \)

\[
|C_1(N) - W_1(N - A_2^{-1}(N))| + |C_2(N) - W_2((A_2^{-1}(N)))| = O(N^{3/8+\varepsilon}) \ a.s.,
\]

where \( A_2(t) = 2 \int_0^t I(W_2(s) \geq 0) \, ds + \int_0^t I(W_2(s) < 0) \, ds \) as in (1.5).

In what follows, we are to generalize Theorems C and D under a somewhat relaxed version of condition (1.6), when the \( \gamma \) values in the two sums are different, discuss its consequences and consider some other interesting choices of the set \( B \).

The structure of this paper from now on is as follows: In Sect. 2, we give preliminary facts and results. In Sect. 3, first we redefine the walk on \( \mathbb{C}^2(B) \) in terms of two independent simple symmetric walks. Then, we list some facts which do not depend on the choice of \( B \) and prove some results which we will need for the rest of the paper. Section 4 contains our main results. In Sect. 5 some further questions and problems are discussed.

### 2 Preliminaries

In this section, we list some well-known results, and some new ones which will be used in the rest of the paper. In case of the known ones, we will not give the most general form of the results, just as much as we intend to use, while the exact reference will also be provided for the interested reader.

Let \( \{X_i\}_{i \geq 1} \) be a sequence of independent i.i.d. random variables, with \( P(X_i = \pm 1) = 1/2 \). Then the simple symmetric random walk on the line is defined as \( S(n) = \sum_{i=1}^n X_i \), and its local time is \( \xi(j, n) = \#\{k : 0 < k \leq n, S(k) = j\}, \ n = 1, 2, \ldots, \) for any integer \( j \).
Define $M_n = \max_{0 \leq k \leq n} |S(k)|$. Then we have the usual law of the iterated logarithm (LIL) and Chung’s LIL [9].

Lemma A

$$\limsup_{n \to \infty} \frac{M_n}{(2n \log \log n)^{1/2}} = 1, \quad \liminf_{n \to \infty} \left( \frac{\log \log n}{n} \right)^{1/2} M_n = \frac{\pi}{\sqrt{8}} \quad a.s.$$ 

For $\xi(n) = \sup_x \xi(x, n)$, we have Kesten’s LIL for local time.

Lemma B (Kesten [31]) For the maximal local time, we have

$$\limsup_{n \to \infty} \frac{\xi(n)}{(2n \log \log n)^{1/2}} = 1 \quad a.s.$$ 

According to the lower lower class (LLC) result for the local time (see, e.g., Révész [35], page 119), the following holds true.

Lemma C ([35]) For the local time of the simple symmetric walk we have for any $\epsilon > 0$ and large enough $n$

$$\xi(0, n) \geq \frac{\sqrt{n}}{(\log n)^{1+\epsilon}} \quad a.s.$$ 

In Csáki and Földes [16], the following stability result was concluded for local time.

Lemma D ([16]) For the local time of the simple symmetric random walk we have that with any $\epsilon > 0$ and $h(n) = \frac{\sqrt{n}}{(\log n)^{1+\epsilon}}$,

$$\lim_{n \to \infty} \sup_{|x| \leq h(n)} \left| \frac{\xi(x, n) - \xi(0, n)}{\xi(0, n)} - 1 \right| = 0 \quad a.s.$$ 

In Csáki and Révész [19], the following result was given about the uniformity of the local time.

Lemma E ([19]) For the simple symmetric random walk for any $\epsilon > 0$, we have

$$\lim_{n \to \infty} \sup_{x \in \mathbb{Z}} \left| \frac{\xi(x + 1, n) - \xi(x, n)}{n^{1/4+\epsilon}} \right| = 0 \quad a.s.$$ 

Remark 2.1 In fact, Lemma E deals with more general random walks, but we only need it for a simple symmetric random walk.

The following result is a version of Hoeffding’s inequality, which is explicitly stated in Tóth [40].
Lemma F ([40]) Let $G_i, i = 1, 2, \ldots$ be i.i.d. random variables with the common geometric distribution $P(G_i = k) = 2^{-k-1}, \ k = 0, 1, 2 \ldots$ Then,

$$P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} (G_i - 1) \right| > \lambda \right) \leq 2 \exp(-\lambda^2/8n)$$

for $0 < \lambda < na$ with some $a > 0$.

Let $\{W(t), \ t \geq 0\}$ be a standard Wiener process. Its local time $\{\eta(x, t), x \in \mathbb{R}, \ t \geq 0\}$ is defined as

$$\eta(x, t) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{0}^{t} I\{W(s) \in (x - \varepsilon, x + \varepsilon)\} \, ds,$$

where $I\{\cdot\}$ denotes the indicator function.

Concerning the increments of the Wiener process, we quote the following result from Csörgő and Révész [20], page 69.

Lemma G ([20]) Let $0 < \alpha T \leq T$ be a nondecreasing function of $T$. Then, as $T \to \infty$, we have

$$\sup_{0 \leq t \leq T - \alpha T} \sup_{s \leq \alpha T} |W(t + s) - W(t)| = O((\alpha T (\log(T/\alpha T) + \log \log T))^{1/2}) \ a.s.$$

The above statement is also true if $W(\cdot)$ replaced by the simple symmetric random walk $S(\cdot)$.

We quote the following simultaneous strong approximation result from Révész [34].

Lemma H ([34]) On an appropriate probability space for a simple symmetric random walk $\{S(n); n = 0, 1, 2, \ldots\}$ with local time $\{\xi(x, n); x = 0, \pm 1, \pm 2, \ldots; n = 0, 1, 2, \ldots\}$, one can construct a standard Wiener process $\{W(t); t \geq 0\}$ with local time process $\{\eta(x, t); x \in \mathbb{R}; t \geq 0\}$ such that, as $n \to \infty$, we have for any $\varepsilon > 0$

$$|S(n) - W(n)| = O(n^{1/4 + \varepsilon}) \ a.s.$$

and

$$\sup_{x \in \mathbb{Z}} |\xi(x, n) - \eta(x, n)| = O(n^{1/4 + \varepsilon}) \ a.s.,$$

simultaneously.

Finally, we recall the following lemma from Klass [32].

Lemma I ([32]) Let $\{E_n\}_{n \geq 1}$ be an arbitrary sequence of events such that $P(E_n \ i.o.) = 1$. Let $\{F_n\}_{n \geq 1}$ be another arbitrary sequence of events that is independent of $\{E_n\}_{n \geq 1}$, and assume that for some $n_0 > 0$, $P(F_n) \geq c > 0$, for $n > n_0$. Then, we have $P(E_n F_n \ i.o.) > 0$. 

\( \square \) Springer
3 The General Case

First we are to redefine our random walk \{C(N); N = 0, 1, 2, \ldots\}. It will be seen that the process described right below is equivalent to that given in Introduction.

To begin with, on a suitable probability space consider two independent simple symmetric (one-dimensional) random walks \(S_1(\cdot)\) and \(S_2(\cdot)\). We may assume that on the same probability space we have a sequence of i.i.d. geometric random variables \(\{G_i, i \geq 1\}\) which are independent from \(S_1(\cdot)\) and \(S_2(\cdot)\) with

\[
P(G_i = k) = \left(\frac{1}{2}\right)^{k+1}, \quad k = 0, 1, 2, \ldots
\]

We now construct our walk \(C(N)\) as follows. We will take all the horizontal steps consecutively from \(S_1(\cdot)\) and all the vertical steps consecutively from \(S_2(\cdot)\). First we will take some horizontal steps from \(S_1(\cdot)\), then as many vertical steps from \(S_2(\cdot)\), as needed to get to a level belonging to \(B\), then again some horizontal steps from \(S_1(\cdot)\) and so on. Now we explain how to get the number of horizontal steps on each occasion. Consider our walk starting from the origin. If the origin is in \(B\), then take \(G_1\) horizontal steps from \(S_1(\cdot)\). (Note that \(G_1 = 0\) is possible with probability 1/2).

If the origin does not belong to \(B\), then the walk moves vertically taking its steps from \(S_2(\cdot)\) until it hits a level belonging to \(B\). If this happens at the level \(j\), then it moves horizontally on level \(j\) taking \(G_1\) steps from \(S_1(\cdot)\). Then it takes some vertical steps from \(S_2(\cdot)\), to arrive again to a level belonging to \(B\), where the next \(G_2\) horizontal steps from \(S_1(\cdot)\) should be taken, and so on. In general, whenever the walk arrives at the level \(j \in B\) then take some horizontal steps, the number of which is given by the next in line (first unused) geometric random variables \(G_i\).

Let now \(H_N, V_N\) be the number of horizontal and vertical steps, respectively, from the first \(N\) steps of the just described process. Consequently, \(H_N + V_N = N\), and

\[
\{C(N); N = 0, 1, 2, \ldots\} = \{(C_1(N), C_2(N)); N = 0, 1, 2, \ldots\}
\]

\[
d \equiv \{(S_1(H_N), S_2(V_N)); N = 0, 1, 2, \ldots\}, \quad (3.1)
\]

where \(d\) stands for equality in distribution.

Now we introduce a few more notations. Let \(\xi_2(\cdot, \cdot)\) denote the local time of \(S_2(\cdot)\).

\[
M_1(N) = M_1(N, B) = \max_{0 \leq k \leq N} |C_1(N)|, \quad (3.2)
\]

\[
M_2(N) = M_2(N, B) = \max_{0 \leq k \leq N} |C_2(N)|, \quad (3.3)
\]

\[
H_N = H_N(B) = \#\{k : 1 < k \leq N, \quad C_1(k) \neq C_1(k - 1)\}, \quad (3.4)
\]

\[
V_N = V_N(B) = \#\{k : 1 < k \leq N, \quad C_2(k) \neq C_2(k - 1)\}, \quad (3.5)
\]

\[
D_2(V_N) = D_2(V_N, B) = \sum_{j \in B} \xi_2(j, V_N). \quad (3.6)
\]

Clearly \(M_1(N), M_2(N)\) are the absolute maxima of the first and second coordinates. \(H_N\) and \(V_N\) are the number of horizontal and vertical steps, respectively, in the first
$N$ steps of $C(\cdot)$. $D_2(V_N)$ is the occupation time of $B$ by $S_2(\cdot)$ in the first $N$ steps of $C(\cdot)$.

By Lemma B, we have for any $\varepsilon > 0$, any $y \in \mathbb{Z}$ and $N$ large enough:

- **Fact 1.** $\xi_2(y, V_N) \leq (1 + \varepsilon)(2V_N \log \log V_N)^{1/2} \leq (1 + \varepsilon)(2N \log \log N)^{1/2}$ a.s.

  Combining Lemmas C and D, we easily get

- **Fact 2.** $\min_{|y| \leq \left(\frac{V_N}{\log V_N}\right)^{1/\varepsilon}} \xi_2(y, V_N) \geq (1 - \varepsilon) \frac{V_N^{1/2}}{(\log V_N)^{1+\varepsilon}}$ a.s.

Lemma A gives the following two facts:

- **Fact 3.** $M_2(N) \leq (1 + \varepsilon)(2V_N \log \log V_N)^{1/2} \leq (1 + \varepsilon)(2N \log \log N)^{1/2}$ a.s.

- **Fact 4.** $M_2(N) \geq (1 - \varepsilon) \pi \left(\frac{V_N}{8 \log \log V_N}\right)^{1/2}$ a.s.

Lemmas A and D also imply

- **Fact 5.** $M_1(N) \leq (1 + \varepsilon)(2H_N \log \log H_N)^{1/2}$ a.s.

- **Fact 6.** $M_1(N) \geq (1 - \varepsilon) \pi \left(\frac{H_N}{8 \log \log H_N}\right)^{1/2}$ a.s.

- **Fact 7.** $H_N \geq \sum_{y \in B, |y| \leq \left(\frac{V_N}{\log V_N}\right)^{1/\varepsilon}} \xi_2(y, V_N)$ a.s.

Consider $A(t)$ defined by (1.5) and let $\alpha(t) = A(t) - t$.

**Lemma 3.1** Let $\gamma_1 > \gamma_2 \geq 1$. Then

- $A(t)$ and $\alpha(t)$ are nondecreasing
- $\gamma_2 t \leq A(t) \leq \gamma_1 t$ and $\frac{t}{\gamma_1} \leq A^{-1}(t) \leq \frac{t}{\gamma_2}$.

**Proof** Observe that

\begin{align*}
A(t) &= \gamma_2 t + (\gamma_1 - \gamma_2) \int_0^t I(W(s) \geq 0) \, ds, \quad (3.7) \\
\alpha(t) &= A(t) - t = (\gamma_2 - 1)t + (\gamma_1 - \gamma_2) \int_0^t I(W(s) \geq 0) \, ds. \quad (3.8)
\end{align*}

Then our first statement is obvious.

From (3.7), we have that

\[ \gamma_2 t \leq A(t) \leq \gamma_1 t \]

and

\[ A^{-1}(\gamma_2 t) \leq A^{-1}(A(t)) = t \leq A^{-1}(\gamma_1 t), \]

that, in turn, implies

\[ \frac{t}{\gamma_1} \leq A^{-1}(t) \leq \frac{t}{\gamma_2}, \quad (3.9) \]
For a simple random walk with local time \( \xi(\cdot, \cdot) \), let

\[
\hat{A}(n) = \gamma_1 \sum_{j=0}^{\infty} \xi(j, n) + \gamma_2 \sum_{j=1}^{\infty} \xi(-j, n).
\]  

(3.10)

**Lemma 3.2**  On a probability space as in Lemma H

\[ |\hat{A}(n) - A(n)| = O(n^{3/4+\varepsilon}) \quad \text{a.s.} \]

**Proof**  According to Lemma 5.3 of Bass and Griffin [4], we have

\[
sup_{t \leq T} \sup_{|k| \leq t^{1/2+\varepsilon}} |\eta(z, t) - \eta(k, t)| = O(T^{1/4+\varepsilon}) \quad \text{a.s.,}
\]

where \( \eta(x, t) \) is the local time of a Wiener process. Using this fact, Lemmas A and H, we have for the difference

\[
|\hat{A}(n) - A(n)| \leq \gamma_1 \left( \sum_{j=0}^{n} |\xi(j, n) - \eta(j, n)| + \sum_{j=0}^{n} \int_{j}^{j+1} \eta(x, n) \, dx - \eta(j, n)| \right) \\
+ \gamma_2 \left( \sum_{j=1}^{n} |\xi(-j, n) - \eta(-j, n)| + \sum_{j=1}^{n} \int_{-j}^{-j+1} \eta(x, n) \, dx - \eta(-j, n)| \right) \\
= O(n^{3/4+\varepsilon}) \quad \text{a.s.}
\]  

(3.11)

Let \( D_2(V_N) \) be defined by (3.6).

**Lemma 3.3**  For any \( \varepsilon > 0 \), as \( N \to \infty \),

\[
\max_{1 \leq i \leq N} |H_i - D_2(V_i)| = O(|D_2(V_N)|^{1/2+\varepsilon}) \quad \text{a.s.}
\]

**Proof**  Recall that \( H_N \) is the number of horizontal steps in our random walk \( C(N) \), \( N = 0, 1, 2, \ldots \). As it was explained in the construction, horizontal steps only occur on levels belonging to \( B \). When the vertical walk arrives to such a level, it takes some horizontal steps, the number of which follows geometric distribution with expected value 1. Thus the total number of horizontal steps is the sum of \( D_2(V_N) \) i.i.d. geometric random variables, with expected value 1. However, this statement is slightly incorrect, as if the \( N \)-th step is a horizontal one, the corresponding last geometric random variable might remain truncated. Denote by \( H^+_N \) the number of horizontal steps which includes all the steps of this last geometric random variable. Then

\[
H^+_N - D_2(V_N) = \sum_{j=1}^{D_2(V_N)} (G_j - 1),
\]
where \( G_i, i = 1, 2 \ldots \) are i.i.d. geometric random variables as in Lemma F. According to this lemma

\[
P\left( \max_{1 \leq j \leq k} \left| \sum_{i=1}^{j} (G_i - 1) \right| > \lambda \right) \leq 2 \exp(-\lambda^2 / 8k).
\]

Selecting \( \lambda = k^{1/2+\varepsilon} \) we get by the Borel–Cantelli lemma that for all large \( k \)

\[
\max_{1 \leq j \leq k} \left| \sum_{i=1}^{j} (G_i - 1) \right| \leq k^{1/2+\varepsilon} \text{ a.s.} \quad (3.12)
\]

We have to apply this for \( k = |D_2(V_N)| \) and conclude that

\[
\max_{1 \leq i \leq N} |H_i^+ - D_2(V_i)| = O(|D_2(V_N)|^{1/2+\varepsilon}) \text{ a.s.},
\]

as \( N \to \infty \). Now to finish the proof, it is enough to observe that

\[
H_N^+ - H_N \leq \max_{i \leq N} G_j,
\]

where \( G_j \) are geometric random variables with parameter 1/2. Thus for any \( \varepsilon > 0 \)

\[
P\left( \max_{j \leq N} G_j > N^\varepsilon \right) \leq N \left( \frac{1}{2} \right)^{N^\varepsilon}
\]

and hence by the Borel–Cantelli lemma

\[
H_N^+ - H_N \leq N^\varepsilon \text{ a.s.}
\]

for all large \( N \).

\( \Box \)

4 Main Results

Assume that we have the construction described in Sect. 3 with independent simple symmetric random walks \( S_1(\cdot), S_2(\cdot) \) together with two independent standard Wiener processes \( W_1(\cdot), W_2(\cdot) \) satisfying Lemma H, i.e., for \( j = 1, 2 \)

\[
|S_j(n) - W_j(n)| = O(n^{1/4+\varepsilon}) \text{ a.s.}
\]

and

\[
\sup_{x \in \mathbb{Z}} |\xi_j(x, n) - \eta_j(x, n)| = O(n^{1/4+\varepsilon}) \text{ a.s.}
\]
Now we define $A_2(\cdot)$ as in (1.5), with $W$ replaced by $W_2$, $\alpha_2(t) = A_2(t) - t$, and also $\hat{A}_2(\cdot)$ as in (3.10), with $\xi$ replaced by $\xi_2$.

In what follows we assume that for $B_n := B \cap [-n, n]$ we have

$$|B_n| \sim cn^\beta$$  \hspace{1cm} (4.1)

with some $0 \leq \beta \leq 1$ and with some constant $c > 0$. As we mentioned in the Introduction, we will separately consider the cases $\beta = 1$, $0 < \beta < 1$ and $\beta = 0$.

4.1 The Case $\beta = 1$

To consider the case $\beta = 1$, we now suppose that

$$n^{-1} \sum_{j=1}^{n} p_j^{-1} = 2\gamma_1 + o(n^{-\tau}), \quad n^{-1} \sum_{j=1}^{n} p_{-j}^{-1} = 2\gamma_2 + o(n^{-\tau})$$  \hspace{1cm} (4.2)

as $n \to \infty$ with some $1/2 < \tau \leq 1$. Clearly $1 \leq \gamma_1 \leq 2$ and $1 \leq \gamma_2 \leq 2$.

**Theorem 4.1** Under conditions (1.2), (4.2) and $\max(\gamma_1, \gamma_2) > 1$, on an appropriate probability space for the random walk $\{C(N) = (C_1(N), C_2(N)); \ N = 0, 1, 2, \ldots\}$ one can construct two independent standard Wiener processes $\{W_1(t); \ t \geq 0\}$, $\{W_2(t); \ t \geq 0\}$ so that, as $N \to \infty$, we have with any $\varepsilon > 0$

$$\left|C_1(N) - W_1 \left(N - A_2^{-1}(N)\right)\right| + \left|C_2(N) - W_2 \left(A_2^{-1}(N)\right)\right| = O(N^{5/8-\tau/4+\varepsilon}) \ a.s.$$  \hspace{1cm} (4.3)

where $A_2^{-1}(\cdot)$ is the inverse of $A_2(\cdot)$.

**Proof** In what follows, we take some ideas from Heyde [25]. By our Lemma 3.3

$$H_N = D_2(V_N) + O(N^{1/2+\varepsilon}) = \sum_{j \in B} \xi_2(j, V_N) + O(N^{1/2+\varepsilon})$$

$$= \sum_{j} \xi_2(j, V_N) \frac{1 - 2p_j}{2p_j} + O(N^{1/2+\varepsilon})$$

$$= -V_N + \frac{1}{2} \sum_{j} \xi_2(j, V_N) \frac{1}{p_j} + O(N^{1/2+\varepsilon}) \ a.s.$$  \hspace{1cm} (4.4)

In the second line above, we changed the summation for all $j$, recalling the fact that $p_j = 1/2$ for $j \notin B$, while $p_j = 1/4$ for $j \in B$. 

\[\square\] Springer
To proceed, introduce the notation

\[
\frac{1}{j} \sum_{k=1}^{j} \frac{1}{p_k} = \kappa_j, \quad \frac{1}{j} \sum_{k=1}^{j} \frac{1}{p-k} = v_j.
\]

\[
N = H_N + V_N = \frac{1}{2} \sum_j \xi_2(j, V_N) \frac{1}{p_j} + O(N^{1/2+\varepsilon}) \quad a.s. \quad (4.5)
\]

Then

\[
\sum_j \xi_2(j, V_N) \frac{1}{p_j} = \sum_{j=1}^{\infty} \xi_2(j, V_N)(j\kappa - (j - 1)\kappa_{j-1})
\]

\[+ \sum_{j=1}^{\infty} \xi_2(-j, V_N)(jv_j - (j - 1)v_{j-1}) + \xi_2(0, V_N) \frac{1}{p_0}
\]

\[= \sum_{j=1}^{\infty} j\kappa_j (\xi_2(j, V_N) - \xi_2(j + 1, V_N))
\]

\[+ \sum_{j=1}^{\infty} jv_j (\xi_2(-j, V_N) - \xi_2(-j - 1, V_N)) + \xi_2(0, V_N) \frac{1}{p_0}
\]

\[= \sum_{j=1}^{\infty} j(\kappa_j - 2\gamma_1) (\xi_2(j, V_N) - \xi_2(j + 1, V_N))
\]

\[+ 2\gamma_1 \sum_{j=1}^{\infty} j (\xi_2(j, V_N) - \xi_2(j + 1, V_N))
\]

\[+ \sum_{j=1}^{\infty} j(v_j - 2\gamma_2)(\xi_2(-j, V_N) - \xi_2(-j - 1, V_N))
\]

\[+ 2\gamma_2 \sum_{j=1}^{\infty} j(\xi_2(-j, V_N) - \xi_2(-j - 1, V_N)) + \xi_2(0, V_N) \frac{1}{p_0}
\]

\[= 2\gamma_1 \sum_{j=0}^{\infty} \xi_2(j, V_N) + 2\gamma_2 \sum_{j=1}^{\infty} \xi_2(-j, V_N)
\]

\[+ \sum_{j=1}^{\infty} j(\kappa_j - 2\gamma_1)(\xi_2(j, V_N) - \xi_2(j + 1, V_N))
\]

\[+ \sum_{j=1}^{\infty} j(v_j - 2\gamma_2)(\xi_2(-j, V_N) - \xi_2(-j - 1, V_N))
\]

\[+ \xi_2(0, V_N) \left( \frac{1}{p_0} - 2\gamma_1 \right).
\]
Observe that from (4.2) we have that
\[ |j(\kappa_j - 2\gamma_1)| \leq cj^{1-\tau}, \quad |j(v_j - 2\gamma_2)| \leq c|j|^{1-\tau} \]
for some \(c > 0\). Now applying Lemma A for \(S_2(\cdot)\), and Lemma E, we get that
\[
\sum_{j=1}^{\infty} j(\kappa_j - 2\gamma_1)(\xi_2(j, V_N) - \xi_2(j + 1, V_N)) \\
+ \sum_{j=1}^{\infty} j(v_j - 2\gamma_2)(\xi_2(-j, V_N) - \xi_2(-j - 1, V_N)) \\
= O(N^{1/4+\epsilon}) \sum_{j=1}^{\max_{k\leq N}|S_2(k)|} j^{1-\tau} \\
= O(N^{1/4+\epsilon})O(N^{1-\tau/2+\epsilon}) = O(N^{5/4-\tau/2+\epsilon}) \quad a.s.,
\]
where here and throughout the paper the value of \(\epsilon\) might change from line to line. So we conclude, using Lemma 3.2, that
\[
\frac{1}{2} \sum_j \xi_2(j, V_N) \frac{1}{p_j} = \gamma_1 \sum_{j=0}^{\infty} \xi_2(j, V_N) + \gamma_2 \sum_{j=1}^{\infty} \xi_2(-j, V_N) + O(N^{5/4-\tau/2+\epsilon}) \\
= \hat{A}_2(V_N) + O(N^{5/4-\tau/2+\epsilon}) = A_2(V_N) + O(N^{5/4-\tau/2+\epsilon}) + O(N^{3/4+\epsilon}) \\
= A_2(V_N) + O(N^{5/4-\tau/2+\epsilon}) \quad a.s.
\]
since \(1/2 < \tau \leq 1\). Consequently from (4.5)
\[ N = A_2(V_N) + O(N^{5/4-\tau/2+\epsilon}) \quad a.s. \]
and
\[ V_N = A_2^{-1}(N) + O(N^{5/4-\tau/2+\epsilon}) \quad a.s. \]

**Remark 4.1** In the previous line, we used the fact that \(A_2^{-1}(u + v) - A_2^{-1}(u) \leq v\). To see this, first recall from Lemma 3.1 that \(A_2(t)\), \(A_2^{-1}(t)\) and \(\alpha(t) = A_2(t) - t\) are all nondecreasing. Then
\[
v = A_2(A_2^{-1}(u + v)) - A_2(A_2^{-1}(u)) \\
= \alpha(A_2^{-1}(u + v)) + A_2^{-1}(u + v) - \alpha(A_2^{-1}(u)) - A_2^{-1}(u) \\
\geq A_2^{-1}(u + v) - A_2^{-1}(u).
\]
So we can conclude, using Lemmas H and G that
\[
C_2(N) = S_2(V_N) = W_2(V_N) + O(N^{1/4+\varepsilon}) = W_2((A_2^{-1}(N) + O(N^{5/4-\tau/2+\varepsilon}))
\]
\[
= W_2((A_2^{-1}(N)) + O(N^{5/8-\tau/4+\varepsilon}) \quad a.s.
\]

Furthermore, by Lemmas H and G again
\[
C_1(N) = S_1(H_N) = S_1(N - V_N) = W_1(N - V_N) + O(N^{1/4+\varepsilon})
\]
\[
= W_1(N - A_2^{-1}(N)) + O(N^{5/8-\tau/4+\varepsilon}) \quad a.s.,
\]
proving our theorem. \(\Box\)

**Remark 4.2** Here we would like to mention that Theorem 4.1 is a generalization of Theorem D and Corollary 4.1 contains Corollary 4.4 of [14].

**Corollary 4.1** Suppose that \(\gamma_1 > \gamma_2 \geq 1\). The following laws of the iterated logarithm hold.

- (i)
  \[
  \limsup_{t \to \infty} \frac{W_1(t - A_2^{-1}(t))}{(t \log \log t)^{1/2}} = \limsup_{N \to \infty} \frac{C_1(N)}{(N \log \log N)^{1/2}} = \sqrt{2 \left( 1 - \frac{1}{\gamma_1} \right)} \quad a.s.,
  \]

- (ii)
  \[
  \liminf_{t \to \infty} \frac{W_1(t - A_2^{-1}(t))}{(t \log \log t)^{1/2}} = \liminf_{N \to \infty} \frac{C_1(N)}{(N \log \log N)^{1/2}} = -\sqrt{2 \left( 1 - \frac{1}{\gamma_1} \right)} \quad a.s.,
  \]

- (iii)
  \[
  \limsup_{t \to \infty} \frac{W_2(A_2^{-1}(t))}{(t \log \log t)^{1/2}} = \limsup_{N \to \infty} \frac{C_2(N)}{(N \log \log N)^{1/2}} = \sqrt{\frac{2}{\gamma_1}} \quad a.s.,
  \]

- (iv)
  \[
  \liminf_{t \to \infty} \frac{W_2(A_2^{-1}(t))}{(t \log \log t)^{1/2}} = \liminf_{N \to \infty} \frac{C_2(N)}{(N \log \log N)^{1/2}} = -\sqrt{\frac{2}{\gamma_2}} \quad a.s.
  \]

**Proof of (i) and (ii).** By the law of the iterated logarithm for \(W_1\), and (3.9) we have for all large enough \(t\)
\[
W_1(t - A_2^{-1}(t)) \leq (1 + \varepsilon)(2(t - A_2^{-1}(t)) \log \log(t - A_2^{-1}(t)))^{1/2}
\]
\[
\leq (1 + \varepsilon) \left( 2t \left( 1 - \frac{1}{\gamma_1} \right) \log \log t \right)^{1/2} \quad a.s.,
\]
which gives an upper bound in (i).

To give a lower bound in (i), for any sufficiently small $\delta > 0$ define the events

$$A^*_n = \{ W_1(u_n) \geq (1 - \delta)(2u_n \log \log u_n)^{1/2} \},$$
$$B^*_n = \left\{ \int_0^{u_n(1+\delta)/(\gamma_1-1)} I(W_2(s) \geq 0) \, ds > \frac{u_n}{\gamma_1 - 1} \right\},$$

$n = 1, 2, \ldots$ Then, with some sequence $\{u_n\}$ ($u_n = a^n$ with sufficiently large $a$ will do), we have

$$P(A^*_n \ i.o.) = 1, \quad P(B^*_n) > c > 0.$$ 

It follows from Lemma I that

$$P(A^*_n B^*_n \ i.o.) \geq c > 0.$$ 

By the 0-1 law, this probability is equal to 1. Recall that $\alpha_2(t) = A_2(t) - t$. We claim that if $B^*_n$ occurs then

$$\alpha_2 \left( \frac{u_n(1 + \delta)}{\gamma_1 - 1} \right) \geq u_n. \quad (4.6)$$

Now if $B^*_n$ occurs then by (3.8)

$$\alpha_2 \left( \frac{u_n(1 + \delta)}{\gamma_1 - 1} \right) = \frac{(\gamma_2 - 1)u_n(1 + \delta)}{\gamma_1 - 1} + (\gamma_1 - \gamma_2) \int_0^{u_n(1+\delta)/(\gamma_1-1)} I(W_2(s) \geq 0) \, ds$$
$$> \frac{(\gamma_2 - 1)u_n(1 + \delta)}{\gamma_1 - 1} + \frac{(\gamma_1 - \gamma_2)u_n}{\gamma_1 - 1} \geq u_n.$$

Let $t_n$ be defined by

$$u_n = t_n - A_2^{-1}(t_n) = \alpha_2(A_2^{-1}(t_n)).$$

Since $B^*_n$ implies

$$\alpha_2 \left( \frac{u_n(1 + \delta)}{\gamma_1 - 1} \right) > u_n = \alpha_2(A_2^{-1}(t_n)),$$

and $\alpha_2(\cdot)$ is nondecreasing, we have that

$$\frac{u_n(1 + \delta)}{\gamma_1 - 1} > A_2^{-1}(t_n).$$
Thus, using (3.9)

\[ u_n > \frac{\gamma_1 - 1}{1 + \delta} A_2^{-1}(t_n) > \frac{\gamma_1 - 1 - t_n}{\gamma_1} = \left(1 - \frac{1}{\gamma_1}\right) \frac{t_n}{1 + \delta}. \]

Hence, \( A_n^* B_n^* \) implies

\[ W_1(t_n - A_2^{-1}(t_n)) \geq (1 - \delta) \left(\frac{2(1 - \frac{1}{\gamma_1}) t_n \log \log t_n}{1 + \delta}\right)^{1/2} \quad a.s. \]

Since \( \delta > 0 \) is arbitrary, this gives a lower bound in (i).

The proof of (ii) follows by symmetry.

**Proof of (iii).** We have infinitely often with probability 1

\[ W_2(A_2^{-1}(t)) \geq (1 - \epsilon)(2A_2^{-1}(t) \log \log t)^{1/2} \geq (1 - \epsilon) \sqrt{\frac{2}{\gamma_1}} (t \log \log t)^{1/2}, \]

where we used (3.9) to get the second inequality.

To give an upper bound, we use the formula for the distribution of the supremum of \( W_2(A_2^{-1}(t)) \) given in Corollary 2 of Keilson and Wellner [30], which in our case is equivalent to

\[
P\left( \sup_{0 \leq s \leq t} W_2(A_2^{-1}(s)) > y \right) \leq \frac{4}{\sqrt{\gamma_1}} \sum_{k=0}^{\infty} \left( \frac{\sqrt{\gamma_2} - \sqrt{\gamma_1}}{\sqrt{\gamma_1} + \sqrt{\gamma_2}} \right)^{k} \left(1 - \Phi \left( \frac{(2k + 1) \sqrt{\gamma_1}}{\sqrt{t} y} \right) \right) \quad y \geq 0.
\]

From this, it is easy to give the estimation

\[
P\left( \sup_{0 \leq s \leq t} W_2(A_2^{-1}(s)) > y \right) \leq c \exp \left( -\frac{\gamma_1 y^2}{2t} \right)
\]

with some constant \( c \), from which the upper estimation in (iii) follows by the usual procedure.

**Proof of (iv).** The lower estimation is easy. Namely, by (3.9) we have for \( t \) big enough that

\[ W_2(A_2^{-1}(t)) \geq -(1 + \epsilon)(2A_2^{-1}(t) \log \log A_2^{-1}(t))^{1/2} \]

\[ \geq -(1 + \epsilon) \left(2 \frac{t}{\gamma_2} \log \log t \right)^{1/2} \quad a.s. \]

It remains to prove an upper estimation in (iv). By the law of the iterated logarithm for \( W_2(\cdot) \)

\[ W_2(v) \leq -((2 - \epsilon)v \log \log v)^{1/2} \quad (4.7) \]
almost surely for infinitely many \( v \) tending to infinity. Define

\[
\mu(v) = \int_0^v I(W_2(s) \geq 0) \, ds. \tag{4.8}
\]

Let \( \zeta(v) \) be the last zero of \( W_2(\cdot) \) before \( v \), i.e.,

\[
\zeta(v) = \max\{u \leq v : W_2(u) = 0\},
\]

By Theorem 1 of Csáki and Grill [18] (or from the Strassen theorem), for large \( v \) satisfying (4.7), we have \( \zeta(v) \leq \varepsilon v \), and hence, also \( \mu(v) \leq \zeta(v) \leq \varepsilon v \). Now put \( v = A_2^{-1}(t) \), i.e., \( t = A_2(v) \leq \varepsilon \gamma_1 + \gamma_2 v = (\varepsilon \gamma_1 + \gamma_2)v \), from which \( v = A_2^{-1}(t) \geq t/\varepsilon (\gamma_1 + \gamma_2) \). Hence

\[
W_2(v) = W_2(A_2^{-1}(t)) \leq - \left(2 - \varepsilon \right) \frac{t}{\varepsilon \gamma_1 + \gamma_2} \log \log t \right)^{1/2}
\]

infinitely often with probability 1. Since \( \varepsilon > 0 \) is arbitrary, this gives an upper bound in (iv).

It is not hard to calculate the density function of \( A_2^{-1}(t) \) and \( t - A_2^{-1}(t) \).

Lemma 4.1

Suppose that \( \gamma_1 > \gamma_2 \geq 1 \).

\[
\begin{align*}
\mathbb{P}(A_2^{-1}(t) \in dv) &= \frac{t}{\pi v} \frac{1}{\sqrt{(v \gamma_1 - t)(t - \gamma_2 v)}} \, dv \quad \text{for} \quad \frac{t}{\gamma_1} < v < \frac{t}{\gamma_2}, \\
\mathbb{P}(t - A_2^{-1}(t) \in dv) &= \frac{t}{\pi(t - v)} \frac{1}{\sqrt{((\gamma_1 - 1)t - \gamma_1 v)(t(1 - \gamma_2) + \gamma_2 v)}} \, dv \\
&\quad \text{for} \quad \frac{1}{1 - \gamma_2} < v < t \left(1 - \frac{1}{\gamma_1}\right).
\end{align*}
\]

Proof

Recall the definition of \( \mu(v) \) from (4.8).

\[
\begin{align*}
\mathbb{P}(A_2^{-1}(t) < v) &= \mathbb{P}(t < A_2(v)) = \mathbb{P}(t < \gamma_2 v + (\gamma_1 - \gamma_2) \mu(v)) \\
&= P \left( \frac{\mu(v)}{v} > \frac{t - \gamma_2 v}{v(\gamma_1 - \gamma_2)} \right) \\
&= 1 - \frac{2}{\pi} \arcsin \left( \frac{t - \gamma_2 v}{v(\gamma_1 - \gamma_2)} \right). \tag{4.9}
\end{align*}
\]

By differentiation, we get the first statement and the second goes similarly. \( \square \)
4.2 The Case $0 < \beta < 1$

Now we want to consider the second case when we have $0 < \beta < 1$ in (4.1), which means that a considerable portion of the horizontal lines are missing. Recall that

$$D_2(V_N) = \sum_{j \in B} \xi(j, V_N).$$

**Theorem 4.2** Under the condition (4.1) with $0 < \beta < 1$, on an appropriate probability space for the random walk \{C(N) = (C_1(N), C_2(N)); N = 0, 1, 2, \ldots\} one can construct two independent standard Wiener processes \{W_1(t); t \geq 0\}, \{W_2(t); t \geq 0\} so that, as $N \to \infty$, we have with any $\varepsilon > 0$

$$|C_1(N) - W_1(D_2(V_N))| = O(N^{1/8+\beta/8+\varepsilon}) \text{ a.s.}$$
$$|C_2(N) - W_2(N)| = O(N^{1/4+\beta/4+\varepsilon}) \text{ a.s.}$$

**Proof** Using Lemmas A and B and (4.1), we conclude

$$D_2(V_N) = O(N^{1/2+\beta/2+\varepsilon}) \text{ a.s.}$$

Clearly from Lemma 3.3

$$H_N = D_2(V_N) + O(D_2(V_N))^{1/2+\varepsilon} = D_2(V_N) + O(N^{1/4+\beta/4+\varepsilon}) \text{ a.s.}$$

and by Lemmas H and G

$$C_1(N) = S_1(H_N) = W_1(H_N) + O(H_N^{1/4+\varepsilon}) = W_1(H_N) + O(N^{1/8+\beta/8+\varepsilon})$$
$$= W_1(D_2(V_N)) + O(N^{1/8+\beta/8+\varepsilon}) \text{ a.s.}$$

$$C_2(N) = S_2(V_N) = W_2(V_N) + O(N^{1/4+\varepsilon}) = W_2(N - H_N) + O(N^{1/4+\varepsilon})$$
$$= W_2(N) + O(N^{1/4+\beta/4+\varepsilon}) + O(N^{1/4+\varepsilon})$$
$$= W_2(N) + O(N^{1/4+\beta/4+\varepsilon}) \text{ a.s.}$$

\[\square\]

**Remark 4.3** It is very easy to see using Lemmas G and H that the first statement in Theorem 4.2 can be replaced by the following more natural one:

$$|C_1(N) - W_1(L_2(V_N))| = O(N^{1/8+\beta/4+\varepsilon}) \text{ a.s.}$$

where $L_2(V_N) := \sum_{j \in B} \eta(j, V_N)$, that is to say using the Wiener local time instead of the random walk local time. However, this change slightly weakens the rate of the approximation. Moreover, we can replace $L_2(V_N)$ in the argument of $W_1(\cdot)$ with $L_2(N)$ using the local time increment result in Csáki et al. [10] (see, e.g., [35], page 121, Theorem 11.9), to get the following weaker approximation:

$$|C_1(N) - W_1(L_2(N))| = O(N^{1/8+3\beta/8+\varepsilon}) \text{ a.s.}$$
Clearly, the weakness of Theorem 4.2 is that we do not know the limiting distribution and other limit theorems (lower and upper classes) for $D_2(V_N)$. So we will not be able to get LL-L-s for $C_1(N)$ as in the case of Theorem 4.1. In what follows we get some simple estimates instead. Namely, using Lemmas A and B and (4.1), we have that with some positive constant $c > 0$

$$D_2(V_N) \leq c(N \log \log N)^{1 + \beta} \ a.s.$$  

On the other hand, from Lemmas C and D and (4.1) we have for any $\varepsilon > 0$ that

$$D_2(V_N) \geq \frac{N^{1 + \beta}}{(\log N)^{1 + \beta + \varepsilon}} \ a.s.$$  

Using these bounds, we can conclude that with some $c_1 > 0$

$$\frac{N^{1 + \beta}}{(\log N)^{1 + \beta + \varepsilon}} \leq C_1(N) \leq c_1 N^{1 + \beta} (\log N)^{1 + \beta + \varepsilon} \ a.s.$$  

Furthermore, for the second coordinates of our walk we have

$$\limsup_{N \to \infty} \max_{0 \leq k \leq N} |C_2(k)| \leq \frac{1}{2(N \log \log N)^{1/2}} \ a.s. \quad (4.10)$$

From the so-called other law of the iterated logarithm due to Chung [9], we obtain for $C_2(N)$

$$\liminf_{N \to \infty} \left( \frac{8 \log \log N}{\pi^2 N} \right)^{1/2} \max_{0 \leq k \leq N} |C_2(k)| = 1 \ a.s. \quad (4.11)$$

### 4.3 The Case $\beta = 0$

Now we consider the case when $B$ is finite, that is to say, we only have finitely many horizontal lines (so in (4.1) $\beta = 0$).

**Theorem 4.4** Suppose that $B$ is finite. Then on an appropriate probability space for the random walk $\{C(N) = (C_1(N), C_2(N)); N = 0, 1, 2, \ldots\}$ one can construct two independent standard Wiener processes $\{W_1(t); t \geq 0\}, \{W_2(t); t \geq 0\}$ so that, as $N \to \infty$, we have with any $\varepsilon > 0$

$$N^{-1/4} |C_1(N) - W_1(|B|\eta_2(0, N))| + N^{-1/2} |C_2(N) - W_2(N)| = O(N^{-1/8 + \varepsilon}) \ a.s.$$  

**Proof** By Lemma 3.3 and Lemmas B, E and H

$$H_N = D_2(V_N) + (D_2(V_N))^{1/2 + \varepsilon} = |B|\xi_2(0, N) + O(N^{1/4 + \varepsilon})$$

$$= |B|\eta_2(0, N) + O(N^{1/4 + \varepsilon}) \ a.s.$$
Hence by Lemmas H and G, we have

\[
C_1(N) = S_1(H_N) = S_1(|B|\eta_2(0, N) + O(N^{1/4+\varepsilon})) = W_1(|B|\eta_2(0, N) + O(N^{1/4+\varepsilon})) + O(N^{1/8+\varepsilon}) = W_1(|B|\eta_2(0, N)) + O(N^{1/8+\varepsilon}) \text{ a.s.}
\]

(4.12)

Similarly,

\[
C_2(N) = S_2(V_N) = S_2(N) + O(H_N^{1/2+\varepsilon}) = W_2(N) + O(N^{1/4+\varepsilon}) \text{ a.s.}
\]

Observe that Theorem 4.4 is a generalization of Theorem C. Now we list some conclusions of Theorem 4.4.

Define the continuous version of the random walk process on our lattice, having horizontal lines only in \(B\) as follows:

\[
\{C(xN) = (C_1(xN), C_2(xN)) : 0 \leq x \leq 1\}.
\]

We have almost surely, as \(N \to \infty\),

\[
\sup_{0 \leq x \leq 1} \left\| \frac{C_1(xN) - W_1(|B|\eta_2(0, xN))}{N^{1/4}(\log N)^{3/4}}, \frac{C_2(xN) - W_2(xN)}{(N \log \log N)^{1/2}} \right\| \to 0,
\]

where \(\|.|\|\) means the Euclidean norm. We have the following laws of the iterated logarithm (for the first component see Theorem 2.2 in Csáki et al. [11]).

\[
\limsup_{n \to \infty} \frac{C_1(N)}{\sqrt{|B|N^{1/4}(\log N)^{3/4}}} = \frac{2^{5/4}}{3^{3/4}} \text{ a.s.}
\]

\[
\limsup_{N \to \infty} \frac{C_2(N)}{(2N \log \log N)^{1/2}} = 1 \text{ a.s.}
\]

As to the liminf behavior of the max functionals of the two components, we have the same results as for the two-dimensional comb lattice [12]. These results are based on the corresponding ones for Wiener process and the iterated process \(W_1(\eta_2(0, t))\) and the work of Chung [9], Hirsch [28], Bertoin [7], and Nane [33].

Based on [33], we get the following: Let \(\rho(n), n = 1, 2, \ldots\), be a nonincreasing sequence of positive numbers such that \(n^{1/4}\rho(n)\) is nondecreasing. Then, we have almost surely that

\[
\liminf_{N \to \infty} \frac{\max_{0 \leq k \leq N} C_1(k)}{N^{1/4}\rho(N)} = 0 \text{ or } \infty
\]

and

\[
\liminf_{n \to \infty} \frac{\max_{0 \leq k \leq N} C_2(k)}{N^{1/2}\rho(N)} = 0 \text{ or } \infty.
\]
according as to whether the series $\sum_1^{\infty} \rho(n)/n$ diverges or converges.

$$\liminf_{N \to \infty} \left( \frac{8 \log \log N}{\pi^2 N} \right)^{1/2} \max_{0 \leq k \leq N} |C_2(k)| = 1 \ a.s. \quad (4.13)$$

On the other hand, for the max functional of $|C_1(\cdot)|$ we obtain from [12] the following result.

Let $\rho(n), n = 1, 2, \ldots$ be a nonincreasing sequence of positive numbers such that $n^{1/4} \rho(n)$ is nondecreasing. Then, we have almost surely that

$$\liminf_{N \to \infty} \frac{\max_{0 \leq k \leq N} |C_1(k)|}{N^{1/4} \rho(N)} = 0 \ or \ \infty,$$

as to whether the series $\sum_1^{\infty} \rho^2(n)/n$ diverges or converges.

5 Final Remarks and Further Questions

As we mentioned in Introduction, the main advantage of using (1.2) instead of (1.1) in this paper is that the number of horizontal steps can be easily approximated by the total time spent by the vertical walk in the set $B$, as shown in Lemma 3.3. It would be interesting to get the results of this paper under the much less restrictive transition probabilities of (1.1). Some further areas of investigation would be to discuss the other usual questions on random walks on the plane. Namely, we would like to investigate the return time to zero, local times and the range. These issues were considered in [15] only for the so-called periodic case, when $p_j = p_{j+L}$ for each $j \in \mathbb{Z}$, where $L \geq 1$ is a positive integer.

To illustrate our results, we would like to mention some examples.

**Example 1** In [15], we discussed a special periodic case, the so-called uniform case $p_j = 1/4$ if $|j| = 0 \ (mod L)$ and $p_j = 1/2$ otherwise. Now our Theorem 4.1 contains the case where

$$p_j = 1/4 \ if \ j = 0 \ (mod L) \ for \ j \geq 0$$

$$p_j = 1/4 \ if \ j = 0 \ (mod K) \ for \ j < 0$$

$$p_j = 1/2 \ otherwise.$$

Then we get $\gamma_1 = \frac{L+1}{L}$ and $\gamma_2 = \frac{K+1}{K}$ for any pairs of integers $L$ and $K$ in (4.2) and Theorem 4.1 holds with $\tau = 1$. Clearly, many more elaborate patterns of keeping and discarding horizontal lines could be handled.

**Example 2** As we mentioned earlier, the case $\gamma_1 = \gamma_2$ in Theorem 4.1 gives our Theorem B. On the other hand, Theorem D about the HPHC walk means that in Theorem 4.1 we have $\gamma_1 = 2$, and $\gamma_2 = 1$. But Theorem 4.1 permits many HPHC-like structures as the following example shows. For any $\alpha_1 > 1, \alpha_2 > 1$ let
\[ B^{(1)} = \{ j : j = 0, 1, 2, \ldots \}, \quad B^{(2)} = \{ [k^{\alpha_1}], k = 1, 2, \ldots \}, \]
\[ B^{(3)} = \{ -[k^{\alpha_2}], k = 1, 2, \ldots \} \]

and

\[ B = (B^{(1)} \setminus B^{(2)}) \cup B^{(3)}, \]

namely above the \( x - \)axis, which is the plane part, we eliminate many horizontal lines, and under the \( x - \)axis, which is the comb part, we add many horizontal lines. Theorem 4.1 is saying that in this case we still have all the main features of the HPHC walk.

**Example 3** The easiest example for which our Theorem 4.2 can be applied is the increasing gap case, namely when

\[ B = B(\alpha) = \{ \pm [k^{\alpha}], k = 0, 1, 2, \ldots \quad \text{with} \quad \alpha > 1 \}. \]

Then, Theorem 4.2 holds true with \( \beta = \frac{1}{\alpha} \).

It would be interesting to generalize Theorem 4.2 by replacing condition (4.1) with

\[ |B_n| \sim cn^\beta L(n) \] (5.1)

where \( L(n) \) is a slowly varying function.

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**References**

1. Arkhincheev, V.E.: Anomalous diffusion and charge relaxation on comb model: exact solutions. Physica A **280**, 304–314 (2000)
2. Arkhincheev, V.E.: Random walks on the comb model and its generalizations. Chaos **17**, 043102–7 (2007)
3. Arkhincheev, V.E.: Unified continuum description for sub-diffusion random walks on multi-dimensional comb model. Physica A **389**, 1–6 (2010)
4. Bass, F., Griffin, P.S.: The most visited site of Brownian motion and simple random walk. Z. Wahrsch. verw. Gebiete **70**, 417–436 (1985)
5. Bertacchi, D.: Asymptotic behaviour of the simple random walk on the 2-dimensional comb. Electron. J. Probab. **11**, 1184–1203 (2006)
6. Bertacchi, D., Zucca, F.: Uniform asymptotic estimates of transition probabilities on combs. J. Aust. Math. Soc. **75**, 325–353 (2003)
7. Bertoin, J.: Iterated Brownian motion and stable (1/4) subordinator. Statist. Probab. Lett. **27**, 111–114 (1996)
8. Cassi, D., Regina, S.: Random walks on \( d \)-dimensional comb lattices. Modern Phys. Lett. B **6**, 1397–1403 (1992)
9. Chung, K.L.: On the maximum partial sums of sequences of independent random variables. Trans. Am. Math. Soc. 64, 205–233 (1948)
10. Csáki, E., Csörgő, M., Földes, A., Révész, P.: How big are the increments of the local time of a Wiener process? Ann. Probab. 11, 593–608 (1983)
11. Csáki, E., Csörgő, M., Földes, A., Révész, P.: Global Strassen-type theorems for iterated Brownian motions. Stoch. Process. Appl. 59, 321–341 (1995)
12. Csáki, E., Csörgő, M., Földes, A., Révész, P.: Strong limit theorems for a simple random walk on the 2-dimensional comb. Electron. J. Probab. 14, 2371–2390 (2009)
13. Csáki, E., Csörgő, M., Földes, A., Révész, P.: On the local time of random walk on the 2-dimensional comb. Stoch. Process. Appl. 121, 1290–1314 (2011)
14. Csáki, E., Csörgő, M., Földes, A., Révész, P.: Random walk on half-plane half-comb structure. Ann. Math. Inform. 39, 29–44 (2012)
15. Csáki, E., Csörgő, M., Földes, A., Révész, P.: Strong limit theorems for anisotropic random walks on \( \mathbb{Z}^2 \). Periodica Math. Hung. 67, 71–94 (2013)
16. Csáki, E., Földes, A.: A note on the stability of the local time of the Wiener process. Stoch. Process. Appl. 25, 203–213 (1987)
17. Csáki, E., Révész, P.: Some results and problems for anisotropic random walk on the plane. Asymptotic Laws and Methods in Stochastics. In: A Volume in Honour of Miklós Csörgő Fields Institute Communication, vol. 76, pp. 55–76 (2015)
18. Csáki, E., Grill, K.: On the large values of the Wiener process. Stoch. Process. Appl. 27, 43–56 (1988)
19. Csáki, E., Révész, P.: Strong invariance for local time. Z. Wahrsch. verw. Gebiete 50, 5–25 (1983)
20. Csörgő, M., Révész, P.: How big are the increments of a Wiener process? Ann. Probab. 7, 731–737 (1979)
21. Dean, D.D., Jansons, K.M.: Brownian excursions on combs. J. Stat. Phys. 70, 1313–1332 (1993)
22. Durhuus, B., Jonsson, T., Wheater, J.F.: Random walks on combs. J. Phys. A 39, 1009–1037 (2006)
23. Dvoretzky, A., Erdős, P.: Some problems on random walk in space. In: Proceedings of the Second Berkeley Symposium, pp. 353–367 (1951)
24. Erdős, P., Taylor, S.J.: Some problems concerning the structure of random walk paths. Acta Math. Acad. Sci. Hung. 11, 137–162 (1960)
25. Heyde, C.C.: On the asymptotic behaviour of random walks on an anisotropic lattice. J. Stat. Phys. 27, 721–730 (1982)
26. Heyde, C.C.: Asymptotics for two-dimensional anisotropic random walks. In: Stochastic Processes, pp. 125–130. Springer, New York (1993)
27. Heyde, C.C., Westcott, M., Williams, E.R.: The asymptotic behavior of a random walk on a dual-medium lattice. J. Stat. Phys. 28, 375–380 (1982)
28. Hirsch, W.M.: A strong law for the maximum cumulative sum of independent random variables. Commun. Pure Appl. Math. 18, 109–127 (1965)
29. Iomin, A., Méndez, V., Horsthemke, W.: Fractional Dynamics in Comb-like Structures. World Scientific, Singapore (2018)
30. Keilson, J., Wellner, J.A.: Oscillating Brownian motion. J. Appl. Probab. 15, 300–310 (1978)
31. Kesten, H.: An iterated logarithm law for the local time. Duke Math. J. 32, 447–456 (1965)
32. Klass, M.: Toward a universal law of the iterated logarithm. I. Z. Wahrsch. verw. Gebiete 36, 165–178 (1976)
33. Nane, E.: Laws of the iterated logarithm for a class of iterated processes. Stat. Probab. Lett. 79, 1744–1751 (2009)
34. Révész, P.: Local time and invariance. Lecture Notes in Math., vol. 861, pp. 128–145. Springer, New York (1981)
35. Révész, P.: Random Walk in Random and Non-Random Environments, 3rd edn. World Scientific, Singapore (2013)
36. Reynolds, A.M.: On anomalous transport on comb structures. Physica A 334, 39–45 (2004)
37. Seshadri, V., Lindenberg, K., Shuler, K.E.: Random walks on periodic and random lattices. II. Random walk properties via generating function techniques. J. Stat. Phys. 21, 517–548 (1979)
38. Shuler, K.E.: Random walks on sparsely periodic and random lattices I. Physica A 95, 12–34 (1979)
39. Silver, H., Shuler, K.E., Lindenberg, K.: Two-dimensional anisotropic random walks. In: Statistical Mechanics and Statistical Methods in Theory and Application (Proc. Sympos., Univ. Rochester, Rochester, N.Y., 1976), pp. 463–505. Plenum, New York (1977)
40. Tóth, B.: No more than three favorite sites for simple random walk. Ann. Probab. 29, 484–503 (2001)
41. Weiss, G.H., Havlin, S.: Some properties of a random walk on a comb structure. Physica A 134, 474–482 (1986)
42. Westcott, M.: Random walks on a lattice. J. Stat. Phys. 27, 75–82 (1982)
43. Zahran, Z.A.: 1/2-order fractional Fokker–Planck equation on comblike model. J. Stat. Phys. 109, 1005–1016 (2002)
44. Zahran, M.A., Abulwafa, E.M., Elwakil, S.A.: The fractional Fokker–Planck equation on comb-like model. Physica A 323, 237–248 (2003)

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