BEYOND SUPERSYMMETRY AND QUANTUM SYMMETRY
(AN INTRODUCTION TO BRAIDED-GROUPS AND
BRAIDED-MATRICES)

SHAHN MAJID
Department of Applied Mathematics & Theoretical Physics
University of Cambridge, Cambridge CB3 9EW, U.K.

ABSTRACT
This is a systematic introduction for physicists to the theory of algebras and
groups with braid statistics, as developed over the last three years by the au-
thor. There are braided lines, braided planes, braided matrices and braided
groups all in analogy with superlines, superplanes etc. The main idea is that
the bose-fermi $\pm 1$ statistics between Grassmannn coordinates is now replaced
by a general braid statistics $\Psi$, typically given by a Yang-Baxter matrix $R$. Most
of the algebraic proofs are best done by drawing knot and tangle diagrams, yet
most constructions in supersymmetry appear to generalise well. Particles of
braid statistics exist and can be expected to be described in this way. At the
same time, we find many applications to ordinary quantum group theory: how
to make quantum-group covariant (braided) tensor products and spin chains,
action-angle variables for quantum groups, vector addition on $q$-Minkowski
space and a semidirect product $q$-Poincaré group are among the main applica-
tions so far. Every quantum group can be viewed as a braided group, so the
theory contains quantum group theory as well as supersymmetry. There also
appears to be a rich theory of braided geometry, more general than super-
geometry and including aspects of quantum geometry. Braided-derivations
obey a braided-Leibniz rule and recover the usual Jackson $q$-derivative as the
1-dimensional case.

1 Introduction

There are certain physical situations in which it is known that particles of braid-
statistics exist. For example, the soliton sector in Yang-Mills theory in 2+1 dimen-
sions with topological mass term [2] and (related to this) the theory of semions [58].
Models with various anyonic statistics are also known [15] [60] and there is a general
theory that any quantum field theory in two or three spacetime dimensions obey-
ing reasonable axioms with regard to locality, causality and structure of the vac-
uum, has particles of braid statistics [13] [21]. Finally, it has become apparent that

This paper is in final form and no version of it will be submitted for publication elsewhere

1991 Mathematics Subject Classification 18D10, 18D35, 16W30, 57M25, 81R50, 17B37

SERC Fellow and Fellow of Pembroke College, Cambridge
q-deformation of physics (as in q-regularization\cite{29}) in any dimension makes the particle statistics naturally braided\cite{10}. The usual spin-statistics theorem fails in the lower-dimensions for familiar reasons (in 1+1 dimensions because the complement of a light-cone is not connected) and in the q-deformed case because the space-time itself is non-commutative.

The existence of these new particles with braid statistics then motivates the development of an algebraic framework with braided-commuting variables, just as the existence of particles with bose-fermi statistics in 3+1 dimensions motivated the development of the algebraic framework of super-algebras based on Grassmann variables. Here I want to give a systematic account of such an algebraic framework for braided particles, *braided algebra*, introduced over a series of papers spanning the last three years. The main works are \cite{35} \cite{36} \cite{45} \cite{44} (for physicists) and \cite{40} \cite{41} \cite{42} \cite{43} \cite{39} (for mathematicians), while the remaining 20 or so are applications (many of them with collaborators). Since the subject is now beginning to be appreciated by physicists, perhaps the time is right for such a review. It will be aimed at theoretical physicists, although Chapter 4 should be of use to mathematicians also. An earlier review for physicists appeared in \cite{37}.

We will describe a variety of braided lines, braided planes, braided matrices etc, as well as braided-differential calculus. The basis of this generalization is to replace the \(\pm 1\) phases encountered for Grassmann variables by a more general phase\cite{38} \cite{44} or more generally by an \(R\)-matrix (a matrix obeying the celebrated quantum Yang-Baxter equations). These are the generalised braid statistics \(\Psi\) and play the role of usual transposition or super-transposition. Our work differs in one fundamental respect from previous attempts to generalise super-symmetry in this way (notably the theory of colour-Lie algebras \cite{24} where there is a phase given by a skew bicharacter on an group, and the theory of S-Lie algebras \cite{16} \cite{51} and others, where the phase is generalised by a triangular \(R\)-matrix.) In all these works the \(R\)-matrix or colouring bicharacter is such that the resulting \(\Psi\) was symmetric in the sense \(\Psi^2 = \text{id}\). This is a natural condition for any analogue of transposition or super-transposition but meant the resulting theory was too close to the usual theory of supersymmetry. In particular, there is no braiding in the picture at all. The main task of our braided theory was to show how to relax this symmetry condition and hence reach the case where \(\Psi \neq \Psi^{-1}\), and it is this aspect which leads to many unusual features in the theory to be described. As a result, \(\Psi\) and \(\Psi^{-1}\) are more properly represented by mutually inverse braid crossings and indeed, most of the algebraic manipulations are now best done by means of braid and tangle diagrams. The general setting is explained in Chapter 3.

In principle, this line of development leading to braided geometry has nothing whatever to do with quantum inverse scattering (QISM) and quantum groups\cite{11} \cite{18} \cite{12} even though these are also obtained from the same data. This is because the same phase factors or \(R\)-matrices can be used on the one hand to define associative non-commutative algebras (the philosophy of non-commutative geometry\cite{13}) or on the other hand to define non-commutative statistics (the philosophy of super-geometry and its generalizations). This is depicted in Figure 1. Not only are the mathematical ideas quite orthogonal but the physical distinction is also clear. Thus, in quantum
field theories which are characterised by a ‘functor’ of some form (in which the super-selections sectors are mapped to Hilbert spaces), such as any conformal field theory or topological quantum field theory, there are general arguments[34] that there is always some kind of weak quantum group of internal symmetries. By contrast, in a general non-topological 1+1 or 2+1 quantum field theory there is no such functor to Hilbert spaces, hence there is no reason to find any kind of quantum group. Instead, we proved a generalised reconstruction theorem and introduced the notion of braided group for the resulting structure[32][35]. We proved that if $C$ is any braided collection of objects (braided category) then there is an underlying braided group $\text{Aut} (C)$.

The same distinction holds mathematically. Thus, any $R$-matrix generates a braided category $C$. There is a functor and using it one reconstructs by standard arguments the quantum matrices $A(R)$ of [12]. By contrast, proceeding without the functor one reconstructs $\text{Aut} (C) = B(R)$, the braided-matrices. Thus quantum and braided matrices are cousins but have quite different origins and philosophy. Chapter 2 begins with an introduction to braided matrices with emphasis on this contrast. We explain what exactly is a braided matrix and in what sense it generalises a super-matrix. The chapter does not assume a background in quantum groups.

Although having a very different origin, it is natural to compare and contrast braided groups and quantum groups. Just as simple Lie groups have quantum group analogs, they also have braided-group analogues. We have already mentioned that every $R$-matrix leads to a braided matrix group just as it leads to a quantum matrix group. As a matter of fact, every strict quantum group (equipped with a universal $R$-matrix or its dual) has a braided-group analogue. This process of conversion of
a strict quantum group into a braided group is a process we call transmutation\[12\]. There is also an ‘adjoint’ process that converts any braided-group of a certain type into an ordinary (bosonic) quantum group\[13\]. These form the topic of Chapter 5. These mathematical theorems mean that braided groups, while more general, also provide a tool for working with quantum groups. Many quantum group constructions are much simpler when viewed as corresponding braided group constructions. The general reason is that braided groups, like super-groups, are in some sense classical (not quantum) objects, albeit in a modified sense. Hence super-geometry and braided geometry are much closer to classical geometry than the quantum case. Some applications to quantum groups are

- **the braided tensor product** – how to tensor product quantum-group covariant systems to obtain another such. The construction is by analogy with the super or $Z_2$-graded tensor product (The notions of statistics and group-covariance are unified in the notion of quantum symmetry). This is fundamental to the very definition of braided matrices and braided groups (and we introduced it for this purpose).

- quantum-covariant spin chains – obtained by iterated braided tensor products. Examples include a system of $n$-braided harmonic oscillators \[1\][3] (w/ W.K. Baskerville) and exchange algebras in 2-D quantum gravity \[18\].

- **the quantum Fourier transform** on factorizable quantum groups \[23\][24\] (w/ Lyubashenko). When quantum groups are viewed in our braided setting they appear more like $\mathbb{R}^n$ than anything else.

- Using braided matrices one can construct a mutually commuting set of ‘angle’ variables $\alpha_k$ for any quantum group $A(R)$. One also recovers Casimirs etc as bosonic elements from the point of view of the braided group.

- **quantum differential calculus** – a general construction for bicovariant quantum differential calculus that includes all known examples \[6\] (w/ T. Brzezinski). Any function of the $\alpha_k$ defines a calculus, so we have a new function or field in q-deformed physics corresponding to the choice of differential calculus.

- **semidirect product structure of Drinfeld’s quantum double $D(H)$** – which then leads to its interpretation as a quantum algebra of observables on a braided group quantized by Mackey’s method\[17\]. In the factorizable case we also have $D(H) \simeq H \bowtie H$ by the quantum adjoint action\[16\]. For example, $D(U_q(g))$ is generated by two copies of $U_q(g)$ with relations given by the quantum adjoint action (also computed in \[16\]) and leading to a kind of quantum Lie bracket.

- **$U_q(g)$ as a kind of deformation-quantization** – \[15\] (w/ D. Gurevich).

- Spectrum generating quantum groups – some braided groups, including all super groups, can be bosonized to equivalent quantum groups. Thus statistical non-commutativity can be swapped for quantum non-commutativity. This is physically more appropriate in some situations \[24\][25\] (w/ A.J. Macfarlane).
• Superization and Anyonization – some quantum groups are more naturally super quantum groups or anyonic-quantum groups. For example, the super-like aspects of the non-standard quantum group of Wu and Jing, are understood by showing that when superized it is the super-quantum group $GL_q(1|1)$. The usual and graded FRT constructions are related like this[50] (w/ M-J. Rodriguez-Plaza).

Some of these are outlined in Chapter 6. Thus braided groups and braided matrices are a useful tool for obtaining results about quantum groups. What about situations where we genuinely need braided groups and braided matrices? Some results at the time of writing are

• group-like structure on the degenerate Sklyanin algebra – many authors tried to find a quantum group here and failed. It turns out to be the braided $2 \times 2$ matrices $BM_q(2)$ [34]. This is important because it means that one can tensor product representations of the degenerate Sklyanin algebra in a way familiar for groups and quantum groups (but remembering the braid statistics).

• group-like structure on quantum homogeneous spaces $G/H$ – even in the normal case, if $H \subset G$ are two quantum groups, the quotient is not a group or quantum group but a braided group. This supplies examples of quantum group principal bundles [33, 34] (w/ T. Brzezinski).

• group-like structure for $q$-Minkowski space – many authors tried to find a quantum group here, but failed. It turns out to be a braided group $\mathbb{R}^{1,3}_q$ (of vector addition). As a consequence, we also obtain the $q$-Poincare quantum group of Schlieker et al as the semidirect product $SO_q(1,3) \ltimes \mathbb{R}^{1,3}_q$ [49].

• braided-differential calculus – operators $\partial^i$ generate the braided addition law on $\mathbb{R}^{1,3}_q$ and obey a braided-Leibniz rule. The usual Jackson derivative familiar in $q$-analysis is the 1-dimensional case and is thereby generalized to $n$-dimensions by means of an $R$-matrix.

Some of these are outlined in Chapter 7. This completes our outline of the theory and of the paper. Perhaps the simplest examples other than the general braided matrices (which we begin with in the next chapter) are the braided-line, either anyonic as in [38] or C-statistical as in [44] and as outlined at the start of Chapter 7.3.

**Historical Note** The reconstruction theorem leading to braided groups Aut($\mathcal{C}$) was circulated in preprint form in 1989 and was finally published (after two rejections) in [32]. The resulting notion of braided groups was presented in May 1990 at the conference on *Common Trends in Mathematics and Quantum Field Theory*, Kyoto. The result was also presented at the *XIX DGM*, Rapallo, June 1990 [33] and at the *Euler Institute Programme on Quantum Groups*, Leningrad, October 1990. The explicit formulae for the braided matrices $B(R)$ (3)-(7) and the example $BSL_q(2)$ were obtained at the end of 1990 and appeared in the proceedings of the Leningrad conference[41].
edited by P.P. Kulish (whom I thank for proof-reading the manuscript). At this time the equations were written with indices and the $R$’s on one side rather than in the compact form. Further examples were presented in December 1990 at the conference on Quantum Probability and Related Topics, Delhi, including the braided matrices $BM_q(1|1)$ associated to the $R$-matrix for the Alexander-Conway knot polynomial, and published in [37]. The main mathematical works [41] [42] [43] were circulated in preprint form in late 1990 – early 1991, with the key part presented at the Bilingual Meeting of the American Maths Society, San Francisco, January 1991. Thus the main part of the work presented in this review dates from the period 1989 – early 1991. The latter half of 1991 and the subsequent time has been devoted to developing applications of the theory, some of it with collaborators. For example, the work with V. Lyubashenko [23] made contact with some of his independent constructions for braided categories with left and right duals [22], while the work with D. Gurevich [17] generalised some of his work on $S$-quantization in [16] to the braided case. The compact notation (3)-(7) for the braided matrices was explicitly introduced in [40] [43] circulated in January and February 1992 respectively, where we recognised the first of these as equations of interest in another context in QISM, as for example in [52]. The braided-matrix property and braid-statistics $R_{12}^{-1}u_1'R_{12}u_2 = u_2R_{12}^{-1}u_1'R_{12}$ however, arose only in the theory of braided matrices and not in QISM.

It is hoped by means of this chronology to make clear the historical origin of the theory of braided groups and braided matrices. In particular, the braided matrix property expressed in (3)-(7) did not arise in any way in connection with the reflection equations in [57] as sometimes suggested. Perhaps a significant connection can be established in the future but so far no results about braided matrices have been obtained from this point of view beyond those already published a year ago in [35] [36]. Covariance (89) of the braided-commutativity equation (3), its braid-diagrammatic form (24), the study of the resulting quadratic algebras for non-standard $R$-matrices and finally the braided-matrix property were some of the first main results in the theory of braided groups and appeared in some form in [33] [36] [40] and numerous preprints and other publications dating from 1989-1990.

The braided-matrix results in the compact notation (and applications) have been presented at various conferences, most recently in a series of four talks here at the Nankai Maths Institute. I would like to thank the organisers, especially M-L. Ge for inviting me give two of these talks (at this Workshop and at the XXI DGM immediately preceding it) and P.P. Kulish and R. Sasaki for encouraging me to continue my physicists account of braided groups and braided matrices in two further unofficial seminars here in the Nankai Institute. I would like to thank all concerned for a very enjoyable visit to the Nankai Institute.

2 Braided Linear Algebra

We begin by explaining what is a matrix from the point of view of algebraic geometry. In fact, this point of view is entirely familiar to physicists in the context of classical mechanics. The main idea is that instead of working directly with matrices, we can
work with the algebra of functions $C(M_n)$ on the matrices. This is generated by the
co-ordinate functions $t^i_j \in C(M_n)$, thought of as abstract ‘matrix-entry observables’
with value $t^i_j(M) = M^i_j$ on any matrix $M$. Because they commute, they can all
simultaneously have values of this form. Indeed, any actual matrix $M$ determines
a state (a linear functional) on the algebra $C(M_n)$ viewed as a classical algebra of
observables of a classical system. In the corresponding representation, the mutually
commuting observables $t^i_j$ have the simultaneous values $M^i_j$.

In this slightly unfamiliar language, the fact that we can multiply matrices
expresses itself as follows. If $t$ and $t'$ are the generators of two independent (commuting)
copies of $C(M_n)$, then $tt'$ is a realisation of $C(M_n)$ in $C(M_n) \otimes C(M_n)$. It is the
collection of observables on the joint system whose value is the matrix product of the
values in each system, $t^n_j(M \otimes N) = t^i_k(M)t^k_j(N) = M^i_k N^k_j$. In terms of the
algebraic properties of the $t^i_j$, $t^i_j$ alone, the key property is

$$[t^i_j, t^k_l] = 0, \quad t^n_j = t^i_k t^k_j \Rightarrow [t^n_j, t^m_k] = 0$$

so that $t''$ realises the same commutative algebra $C(M_n)$ in the joint system.

There are two quite distinct directions in which to generalise this notion, in both
of which we drop the actual matrices $M, N$ and work with the algebraic structure
of the $t^i_j$ directly and in a generalized form. The first (and more topical) direction
is that of a quantum matrix. Here we allow the algebra $C(M_n)$ to be replaced by a
non-commutative algebra $A(R)$ (say), still with generators $t^i_j$. Now these no-longer
commute. Typically, their quantum-commutation relations are of the form \[[72x18]

$$R^i_{\, m} t^m_j t^n_l = t^k_l t^i_k t^m_n, \quad \text{ i.e. } \quad R t_1 t_2 = t_2 t_1 R$$

where the second expression is a compact notation where the numerical suffixes refer
to the position in the matrix tensor product, and $R \in M_n \otimes M_n$ obeys the QYBE
$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$ in the same compact notation. Thus $A(R)$ is some
non-commutative ‘quantized’ version of $C(M_n)$ but the generators $t^i_j$ can still be multi-
plied as before, in a way that would correspond (if the $t^i_j, t^i_i$ had definite values) to
matrix multiplication. Thus the analog of \[[72x18]\]

$$[t^i_j, t^k_l] = 0, \quad t^n_j = t^i_k t^k_j \Rightarrow R t^n_j t^m_k = t^m_k t^n_j R.$$ \[[72x28]

This says that the matrix product $t''$ obeys the same quantum-commutation relations
and so realises a copy of $A(R)$ in the joint system $A(R) \otimes A(R)$ generated by the two
independent (commuting) quantum matrices $t, t'$.

A second (and even more well-known) direction is that of a super-matrix. Here the
philosophy is quite different in that we keep the algebra ‘commutative’ (not quantized)
but in the generalised sense of being super-commutative. Thus the generators have a
grading such as $|t^i_j| = i + j \mod 2$, and obey the relations

$$t^i_j t^k_l = (-1)^{|t^i_j||t^k_l|} t^k_l t^i_j.$$ \[[72x38]

The ability to multiply super-matrices appears now as the following: if $t^{k_i}$ is another
independent copy of the super-matrix, independent now meaning that it has super-
statistics with $t^i_j$, then the product is also a super-matrix. Thus

$$t^{n_i} t^{k_l} = (-1)^{|t^{n_i}||t^{k_l}|} t^{k_l} t^{n_i}, \quad t^{m_j} = t^i_k t^k_j \Rightarrow t^{m_j} t^{n_k} = (-1)^{|t^{m_j}||t^{n_k}|} t^{n_k} t^{m_j}.$$ \[[72x48]
Thus the idea of a super-matrix is somewhat different from that of a quantum matrix. In the super case, the algebra remains ‘commutative’ and hence classical but in a modified sense appropriate to a modified exchange law or statistics between independent copies.

The idea of a braided matrix as introduced in [36][40] is to use the same data $R$ obeying the QYBE but to use it according to the philosophy of the super case rather than the more conventional quantum one. Let us call the braided-matrix generators $u^i_j$ to distinguish them from the above. They generate an algebra $B(R)$ with braided-commutativity relations

$$R^k_a b u^b_c R^a_j d u^d_j = u^b_c R^a_j d u^d_j R^k_a b \quad \text{i.e.} \quad R_{21} u_1 R_{12} u_2 = u_2 R_{21} u_1 R_{12}$$

(6)

and if $u'$ is another independent braided matrix obeying the same relations and having certain braid-statistics with $u$ then the product $uu'$ is also a braided-matrix,

$$R^{-1}_{12} u'_1 R_{12} u_2 = u_2 R^{-1}_{12} u'_1 R_{12}, \quad u''_k = u^i_k u^{jk} \Rightarrow R_{21} u''_1 R_{12} u''_2 = u''_2 R_{21} u''_1 R_{12}.$$  

(7)

The relations between $u, u'$ are the braid-statistics relations between two independent identical braided matrices[36]. Given these, the proof of (7) in the present compact notation[43] is $R_{21} u'_1 R_{12} u'_2 = R_{21} u_1 u'_1 R u_2 u'_2 = R_{21} u_1 R(R^{-1} u'_1 R u_2) u'_2 = \left( R_{21} u_1 R u_2 \right) R^{-1} R_{21} (R_{21} u'_1 R u'_2) = u_2 R_{21} (u_1 R^{-1} u'_1 R) u_1 R = u''_2 R_{21} u''_1 R$ as required. In each expression, the brackets indicate how to apply the relevant relation to obtain the next expression.

Note that if $R_{21} = R^{-1}_{12}$ (the so-called triangular or unbraided case) the braid-statistics relations and the braided-commutativity relations (5) coincide. This is the case for super-matrices (which fit into this framework for suitable $R$) but in the general braided case, the notion of braided-commutativity and braid-statistics are slightly different. This was one of the key obstacles to the braided case that was solved in [36].

For one of the simplest non-trivial examples we take $R$ to be the standard $SL_q(2)$ R-matrix, giving the braided matrices $BM_q(2)[36][40]$. Denoting the matrix entries as $u = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$, the braided-commutativity relations become

$$ba = q^2 ab, \quad ca = q^2 ac, \quad da = ad, \quad bc = cb + (1 - q^{-2})a(d - a)$$  

(8)

$$db = bd + (1 - q^{-2})ab, \quad cd = dc + (1 - q^{-2})ca$$  

(9)

while the braid-statistics relations between two independent copies comes out as

$$a'a = aa' + (1 - q^2)bc' \quad a'b = ba' \quad a'c = ca' + (1 - q^2)(d - a)c' \quad a'd = da' + (1 - q^{-2})bc' \quad b'a = ab' + (1 - q^2)b(d' - a') \quad b'b = q^2bb'$$

$$b'c = q^{-2}cb' + (1 + q^2)(1 - q^{-2})^2bc' - (1 - q^{-2})(d - a)(d' - a') \quad b'd = db' + (1 - q^{-2})(d - a)(d' - a') \quad c'a = ac' \quad c'b = q^{-2}bc'$$

$$c'e = q^2ce' \quad c'd = dc' \quad d'a = ad' + (1 - q^{-2})bc' \quad d'b = bd' \quad d'c = cd' + (1 - q^{-2})(d - a)c' \quad d'd = dd' - q^{-2}(1 - q^{-2})bc'.$$  

(10)
In this example, as $q \to 1$ the algebra becomes commutative and the statistics also become commutative, so we return to the case (1).

For another choice of $R$-matrix, for example the non-standard $R$-matrix associated to the Alexander-Conway knot polynomial, we arrive at the braided matrices $BM_q(1|1)$ as computed in [34, Sec. 3]. Its braided-commutativity relations are

$$b^2 = 0, \quad c^2 = 0, \quad d - a \text{ central,} \quad (11)$$

$$ab = q^{-2}ba, \quad ac = q^2ca, \quad bc = -q^2cb + (1 - q^2)(d - a)a$$

and its braid-statistics relations are $d - a$ bosonic and

$$a'a = aa' + (1 - q^2)bc' \quad b'b = -bb'$$

$$a'b = ba'$$

$$b'a = ab' + (1 - q^2)b(d' - a')$$

$$a'c = ca' + (1 - q^2)(d - a)c'$$

$$c'a = ac'. \quad (13)$$

Here, in the limit $q \to 1$ the commutativity and statistics relations become exactly the ones for the super-matrices $M(1|1)$ (with $b, c$ odd and $a, d$ even) as in the (3)-(5). Thus the notion of braided matrices really generalises both ordinary and super-matrices. In the general case the statistics are not merely a phase as in the super-case, but given by a linear combination via an $R$-matrix. The braided-matrix property means that

$$(a'' \ b'') = (\begin{array}{cc} a & b \\ c & d \end{array}) \ (a' \ b') (\begin{array}{cc} c' & d' \end{array})$$

also obeys the braided-matrix relations provided we remember the braid-statistics.

There are also braided-traces and braided-determinants in analogy with the super-case. Let $\tilde{R} = ((R^2)^{-1})^t_2$ where $t_2$ is transposition in the second matrix factor of $M_n \otimes M_n$ and we assume that the relevant inverse here exists. Let $\vartheta_{ij} = \tilde{R}_{kj}^i$ then

$$c_k = Tr \vartheta^k \Rightarrow u'c_k = c_k u', \quad uc_k = c_k u \quad (15)$$

so that the $c_k$ are bosonic and central elements of the braided matrix algebra $B(R)$. The element $c_1$ is the braided-trace while products of the $c_k$ can be used to define the braided-determinant. The braided-determinant for the $SL_q(2)$ $R$-matrix is

$$BDET(u) = ad - q^2cb. \quad (16)$$

Setting this to 1 in $BM_q(2)$ gives the braided group $BSL_q(2)$ [36].

In super-symmetry of course, we have not only super-matrices but super-vectors and super-covectors with a linear addition law. Thus if we denote as usual the Grassmann variables by $\theta_i$ with degree $|\theta_i| = i - 1 \mod 2$ say, we have super-commutativity

$$\theta_i \theta_j = (-1)^{|\theta_i||\theta_j|} \theta_j \theta_i \quad (17)$$

and if $\theta'$ is another copy of the super-plane with super-statistics relative to $\theta$, we can add them. Thus

$$\theta' \theta_j = (-1)^{|\theta'||\theta|} \theta_j \theta'_i, \quad \theta'' = \theta_i + \theta'_i \Rightarrow \theta'' \theta'' = (-1)^{|\theta'||\theta|} \theta'' \theta'_i. \quad (18)$$
In the usual case these \( \theta_i \) would be the commuting co-ordinate functions generating \( C(\mathbb{R}^n) \) and their linear addition would express linear addition on \( \mathbb{R}^n \). In the super-case we work abstractly with the generators \( \theta_i \), just as we did for the \( t_j \) above, there being no underlying actual points any more.

Here our notion of braided linear algebra strikes an important success over its rival notion of quantum-linear algebra: quantum planes (such as the much-discussed quantum planes \( \mathbb{C}_q^{20}, \mathbb{C}_q^{02} \)) can be defined with non-commutation relations among their generators, but these are not preserved under linear addition. Hence, there is no real quantum linear algebra (with linear addition of quantum vectors). By contrast, generalising the super-case, we do have braided-vectors and braided-covectors. Denoting the covector generators \( x_i \) we use them to generate a braided-commutative algebra \( V^\ast(R') \) of ‘co-ordinate’ functions with braided-commutation relations

\[
x_j x_l = x_n x_m R^{m n}_{j l}, \quad \text{i.e.} \quad x_1 x_2 = x_2 x_1 R'_{12}.
\]

(19)

Here \( R' \) is built from \( P \) (the usual permutation matrix) and \( R \), and is characterised by the equation

\[
(PR + 1)(PR' - 1) = 0.
\]

(20)

For example, there is an \( R' \) for each non-zero eigenvalue \( \lambda_i \) in the functional equation \( \prod_i (PR - \lambda_i) = 0 \). First one should rescale \( R \) by dividing by \( -\lambda_i \), and then \( R' = P + \mu P \prod_{j \neq i} (PR + (\lambda_j/\lambda_i)) \). Here \( \mu \) is an arbitrary non-zero constant which does not change the algebra and which can be chosen so that \( R' \) is invertible. The linear addition of braided-covectors means that if \( x' \) is another copy with braided-statistics relative to \( x \) then \( x + x' \) is also a realization of the same algebra,

\[
x_1' x_2 = x_2 x_1' R_{12}, \quad x'' = x + x' \Rightarrow x_1'' x_2'' = x_2'' x_1'' R'_{12}.
\]

(21)

The proof is \( (x_1 + x_1')(x_2 + x_2') = x_2 x_2' + x_1 x_2' + x_1' x_2' = x_2 x_1^2 + x_1 x_1' (PR_1^2 + 1) + x_2 x_2' R_{12} \) while \( (x_2 + x_2')(x_1 + x_1') R'_{12} \) has the same outer terms and the cross terms \( x_2 x_1 R_{12}' + x_2 x_1' R_{12} = x_1 x_2' (P R_1 + P) R' \). These are equal since \( PR + 1 = PR PR' + PR' \) from (20).

Note that if \( R \) is a Hecke symmetry in the sense \( (PR + 1)(PR - q^2) = 0 \) (such as for all the \( SL_q(n) \) \( R \)-matrices) we can take \( R' = q^{-2} R \). In this case, our braided-covectors reduce to the usual Zamolodchikov algebra. For example, the \( SL_q(2) \) \( R \)-matrix gives for \( V^\ast(R) \) the usual quantum planes \( \mathbb{C}_q^{20} \) or \( \mathbb{C}_q^{02} \) according to the chosen eigenvalue. Thus \( \mathbb{C}_q^{20} \) has generators \( x, y \) and braided-commutativity relations \( xy = q^{-1}yx \). The corresponding braiding-statistics relations are

\[
x' x = q^2 xx', \quad x'y = qyx', \quad y' y = q^2 yy', \quad y' x = qxy' + (q^2 - 1)yx'.
\]

(22)

Similarly, \( \mathbb{C}_q^{02} \) has generators \( \theta, \eta \) say, and braided-commutativity relations \( \theta^2 = 0, \eta^2 = 0, \theta \eta = -q \eta \theta \) and braiding-statistics relations

\[
\theta' \theta = -\theta \theta', \quad \theta' \eta = -q^{-1} \eta \theta', \quad \eta' \eta = -\eta \eta', \quad \eta' \theta = -q^{-1} \theta \eta' + (q^2 - 1) \eta \theta'.
\]

(23)

In the limit \( q \to 1 \) we obtain exactly the usual \( \mathbb{C}_q^{02} \) plane with \( \theta, \eta \) fermionic.
By contrast, the Alexander-Conway $R$-matrix mentioned above gives two algebras $\mathbb{C}_q^{[1]}$ according to the eigenvalue. One has generators $x, \theta$ say, with braided-commutativity relations $x\theta = q^{-1}\theta x, \theta^2 = 0$ and braided-statistics relations

\[ x'x = q^2xx', \quad x'\theta = q\theta x', \quad \theta'\theta = -\theta\theta, \quad \theta'x = qx\theta' + (q^2 - 1)\theta x'. \tag{24} \]

In the limit $q \to 1$ we obtain exactly the $\mathbb{C}_1^{[1]}$ super-plane with $\theta$ fermionic and $x$ bosonic. The other eigenvalue is similar with the first co-ordinate fermionic. These examples are all of Hecke type.

Other $R$-matrices give more complicated algebras, for example the $SO_q(1,3)$ $R$-matrix gives the $q$-Minkowski space algebra as in \[8\] but now equipped with a braided addition law. The braided-vectors $V(R')$ are similar, with generators $v^i$ and relations $v_1v_2 = R'v_2v_1$ and a braided-addition law. Thus,

\[ v'_1v_2 = R_{12}v_2v'_1, \quad v'' = v + v' \Rightarrow v''v'' = R_{12}v''v'_1. \tag{25} \]

The proof is similar to that above. There are also other variants of $V(R'), V^*(R'), B(R)$ according to other conventions (our present conventions are right-handed as we will see later).

This describes braided-matrices, braided-vectors and braided-covectors in isolation. However, they are all part of a single unified braided-linear algebra. Thus just as in super-symmetry, where all algebras have super-statistics relative to each other, there are braided-statistics between the braided matrices, braided-vectors and covectors\[13\, Lemma 3.4]

\[ x'_1Rv_2 = v_2x'_1, \quad v'_1x_2 = x_2R^{-1}v'_1 \tag{26} \]

\[ u'_1x_2 = x_2R^{-1}u'_1, \quad x'_1Ru_2R^{-1} = u_2x'_1 \tag{27} \]

\[ v'_1u_2 = Ru_2R^{-1}v'_1, \quad R^{-1}u'_1Rv_2 = v_2u'_1. \tag{28} \]

For example, remembering such braided-statistics, we can act on a braided covector $x$ by a braided-matrix $u$ in the sense that $x' = xu'$ obey the same relations $x'_1x'_2 = x_1u'_1x_2u'_2 = x_1x_2R^{-1}u'_1Ru'_2 = x_2x_1R'R^{-1}u'_1Ru'_2$ as required. Here we used that $R'$ is some function $R' = Pf(PR)$ (see above) and

\[ (PR)R^{-1}u_1Ru_2 = R^{-1}u_1Ru_2(PR) \tag{29} \]

from (8) so that $f(PR)$ commutes past while the remaining $P$ in $R'$ interchanges 1, 2. For example,

\[ (x' \quad y') = (x \quad y) \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \tag{30} \]

obeys the quantum plane relations if $x, y$ do and $a', b', c', d'$ are a copy of the braided-matrices, and provided we remember the braid-statistics computed from (27). We proved the result in \[13\] for the Hecke case where $R' \propto R$, but the general proof given here is almost the same.
Likewise, remembering the braid-statistics the tensor product of a braided-vector and a braided-covector is a braided-matrix, at least in the Hecke case\([45]\). Thus

\[
\mathbf{v}\mathbf{x} = \begin{pmatrix}
v_1 x_1 & \cdots & v_1 x_n \\
\vdots & & \vdots \\
v_n x_1 & \cdots & v_n x_n
\end{pmatrix}
\] (31)

obeys the \(B(R)\) relations (it is a rank-one braided matrix)\([45]\).

This completes our summary of braided linear algebra in the most direct language. In the following two chapters our goal is to introduce a slightly more abstract way of working with and thinking about these braided objects. Our first goal is to find a systematic way of obtaining these various braid-statistics relations as in (27)-(28) etc. Since all braided objects will enjoy some braid-statistics with all other braided objects, we obviously need a general way of working these out. We also have to be sure that everything is consistent (for example, one can show that the composite object \(u'' = uu'\) has the same braid-statistics as \(u\) or \(u'\) in (4) so that we can make higher products). We will formulate these braid statistics by means of a braided-transposition operator \(\Psi\) and explain how this is obtained in a systematic way, often in terms of \(R\)-matrices. This is our goal in the next chapter. Secondly, it is useful to distinguish carefully between the original matrix algebra \(B(R)\) and the copies of it that are realised in terms of products of generators in \(B(R) \otimes B(R)\) etc. Mathematically, this realization forms the braided-coproduct map \(\Delta : B(R) \to B(R) \otimes B(R)\). It takes the matrix form

\[
\Delta u^i_j = u^i_k \otimes u^k_j
\] (32)

equivalent to the matrix multiplication in (4). The underlines are to remind us of the non-commuting braid statistics. Similar remarks apply for the braided-vectors and braided-covectors. Thus the realization corresponding to covector addition is the map \(\Delta : V^*(R') \to V^*(R') \otimes V^*(R')\) given by

\[
\Delta x_i = x_i \otimes 1 + 1 \otimes x_i.
\] (33)

This leads us into a study of abstract braided-bialgebras and braided-Hopf algebras in Chapter 4.

We see from this outline that our use of \(R\)-matrices in defining braided-linear algebra really is quite different from their traditional use in quantum inverse scattering (where they lead to quantum groups \(A(R)\) etc). Thus, it is not the case that all results concerning \(R\)-matrices arise in inverse scattering. Quite simply, in QISM the QYBE leads to an associative quantum-commutativity, while in our work the QYBE leads to an associative braided-tensor product. Of course, the QYBE are well known to lead to braid relations: our idea is to build these into the algebra from the start in the form of braid statistics.
3 Braided Categories and the Unification of Covariance and Statistics

In this chapter we formalise the notion of braid-statistics evident in the examples of the braided-matrices above, by means of the notion of braided tensor categories. Basically, this just means a collection of objects with a braided-transposition law $\Psi$ obeying a number of consistency conditions. By formalising it in this way, we keep track of those consistency conditions and at the same time learn how to solve them. They need not come only from an $R$-matrix as in the last chapter.

Let us recall that our goal is to generalise the notion of supersymmetry to the braided case. In this chapter we concentrate on the first step, which is to generalise the notion of vector-spaces and super-vector spaces. This is what a braided category is from our point of view. Mathematically, it means that everything is $\mathbb{Z}_2$-graded. On the other hand, in physics there is a slightly different notion of covariance under a group. We shall see that when these notions are generalised, which we do by means of quantum groups, they become the same: the notion of quantum-group graded and quantum group-covariant mathematically coincide. Hence we shall see that the notion of $\mathbb{Z}_2$-grading (bose-fermi statistics) and the notion of group covariance are unified by the notion of quantum symmetry. This is an important unification in physics made possible by quantum group technology. This point of view has been developed in [BG, Sec. 6] and is the main physical lesson of the present chapter.

Braided tensor categories have been formalised by category theorists in [19] as well as being known in other contexts such as in the theory of knot invariants and in connection with quantum groups. Firstly, a category $\mathcal{C} = \{V, W, Z, \cdots\}$ just means a collection of objects $V, W, Z$ etc and a specification of what are the allowed morphisms $\phi : V \to Z$ etc between them (in concrete situations they are maps between objects of some specified form). One should be able to compose morphisms in an obvious way. So, categories are nothing to be afraid of: they are just a specification of what kind of objects and maps we intend to deal with. A braided tensor category is $(\mathcal{C}, \otimes, \Phi, \Psi)$ where $\mathcal{C}$ is a category, $\otimes$ is a tensor product between any two objects (and between any corresponding morphisms), $\Phi_{V,W,Z} : V \otimes (W \otimes Z) \to (V \otimes W) \otimes Z$ is a collection of isomorphisms expressing associativity of the tensor product between any three objects, and $\Psi_{V,W} : V \otimes W \to W \otimes V$ (the braided transposition) is a collection of isomorphisms expressing commutativity of the tensor product between any two objects. In addition, there is a unit object $1$ with $V \otimes 1 \cong V \cong 1 \otimes V$. In our examples below, $\Phi$, the unit object and its associated maps are typically the obvious ones, so we will not write them too explicitly. However, they should be understood in all formulae. By contrast, $\Psi$ will typically be non-trivial and so we emphasise this.

These various collections of maps all fit together into a consistent framework. The consistency conditions for $\Psi$ are of two types and modeled on the idea that it behaves like usual transposition or super-transposition. Firstly, $\Psi_{V,W}$ should be well-behaved
under any morphisms of $V$ or $W$ to any other object,

$$
\Psi_{Z,W}(\phi \otimes \text{id}) = (\text{id} \otimes \phi)\Psi_{V,W} \quad \forall \phi \downarrow\quad \Psi_{V,Z}(\text{id} \otimes \phi) = (\phi \otimes \text{id})\Psi_{V,W} \quad \forall \phi \downarrow
$$

(34)

One says that the collection is \textit{functorial}. Secondly, it should be well-behaved under tensor products of objects,

$$
\Psi_{V \otimes W, Z} = \Psi_{V, Z} \Psi_{W, Z}, \quad \Psi_{V, W \otimes Z} = \Psi_{V, Z} \Psi_{V, W}.
$$

(35)

If we put in $\Phi$ and write these conditions as maps, they look like hexagons. They are enough to imply also that $\Psi_{1, V} = \text{id} = \Psi_{V, 1}$ for the trivial object. $\Phi$ is also functorial and obeys a famous pentagon condition.

These conditions (34)-(35) are just the obvious properties that we take for granted when transposing ordinary vector spaces or super-vector spaces. In these cases $\Psi$ is the twist map $\Psi_{V,W}(v \otimes w) = w \otimes v$ or the supertwist

$$
\Psi_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v
$$

(36)

on homogeneous elements of degree $|v|, |w|$. The form of $\Psi$ in these familiar cases does not depend directly on the spaces $V, W$ so we often forget this. But in principle there is a different map $\Psi_{V,W}$ for each $V, W$ and they all connect together as explained.

For example, the hexagons (34) say that if we transpose something in $V \otimes W$ past something in $Z$, we obtain the same result as first transposing the part in $W$ with that in $Z$, and then the part in $V$. Similarly on the other side. On the other hand, there is one crucial property familiar for transpositions or super-transpositions which we do not suppose. Usually, when transpositions or super-transpositions are applied twice, they give the identity. Such a category is a \textit{symmetric tensor category}. By contrast, in our case we do not suppose that $\Psi_{W,V} \Psi_{V,W} = \text{id}$ and so must distinguish carefully between $\Psi_{V,W}$ and $\Psi_{W,V}^{-1}$. They are both morphisms $V \otimes W \rightarrow W \otimes V$. A convenient shorthand for doing this is to write them as morphisms downwards and as braids rather than single arrows. Thus, we use the notation,

$$
\Psi_{V,W} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
V \\
W
\end{array}
\begin{array}{c}
\begin{array}{c}
W \\
V
\end{array}
\end{array}
\end{array}
\end{array}
\Psi_{W,V}^{-1} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
V \\
W
\end{array}
\begin{array}{c}
\begin{array}{c}
W \\
V
\end{array}
\end{array}
\end{array}
\end{array}
$$

(37)

Thus, $\Psi$ behaves more like a braid crossing (and $\Psi^{-1}$ an inverse braid crossing) than a usual transposition (hence the name). It gives an action of the braid group on tensor products of objects, rather than of the symmetric group. The coherence theorem for braided categories says that if we are given any two composites of the $\Psi, \Phi$ etc and write them as braids according to (37) (suppressing the $\Phi$) then the compositions are the same if the corresponding braids are topologically the same. This means that it does not matter too much in what order we make a series of braided-transpositions – the result is the same if the corresponding braids are the same. Thus, the order or
history behind a series of braided-transpositions does matter to some extent (unlike
the situation with usual permutations) but only up to topology. This is the novel
feature in the braided case. Much of algebra involves permuting objects past each
other and this aspect of algebra is now replaced by topology.

This diagrammatic notation is a powerful one. In terms of it, we can write the
hexagon conditions (33) as

\[
\begin{align*}
V & \quad W & \quad Z \\
\downarrow & & \downarrow \\
Z & \quad V & \quad W
\end{align*}
\]

\[
\begin{align*}
V & \quad W & \quad Z \\
\downarrow & & \downarrow \\
W & \quad Z & \quad V
\end{align*}
\]

\[
\begin{align*}
V & \quad W & \quad Z \\
\downarrow & & \downarrow \\
V & \quad W & \quad Z
\end{align*}
\]

where on the left of each equation we have extended our notation by writing the
strand for a composite object such as \(V \otimes W\) or \(W \otimes Z\) as a pair of strands, one for
each factor. The content of (33) is that we can do this and and thereby include (35)
in our rule that topologically identical diagrams correspond to the same resulting
morphism.

Finally, we can write any other morphisms such as \(\phi : V \to Z\) etc as nodes in
a strand connecting \(V\) to \(Z\). We write all morphisms downwards. A morphism to
or from tensor products will have multiple strands into or out of the node. In these
terms, the functoriality condition (34) comes out as

\[
\begin{align*}
\begin{array}{c}
V \\
\downarrow
\end{array} & \begin{array}{c}
W \\
\downarrow
\end{array} & \begin{array}{c}
Z
\end{array} \\
\begin{array}{c}
W \\
\downarrow
\end{array} & \begin{array}{c}
V \\
\downarrow
\end{array} & \begin{array}{c}
Z \\
\downarrow
\end{array}
\end{align*}
\]

Thus functoriality means in our topological notation just that a node (of any valency)
can be pulled through a braid crossing (similarly for \(\Psi^{-1}\) for inverse braid crossings).

Thus, the main operations of tensor products and transpositions for vector spaces
and super-vector spaces are generalised to the braided case. Unlike these cases, the
order now matters to some extent and we have to be careful that such braided-
transpositions don’t get tangled up. One other feature of vector spaces and super-
vector spaces that we will need is that of dual linear spaces. Thus for every object \(V\),
there should be a dual \(V^*\). In fact one must distinguish carefully between left-duals
and right-duals since they can only be related via some further structure involving
\(\Psi\). We concentrate on the left-duals (the right-handed ones are similar). Then the
properties we need are that for every object \(V\) there should be another object \(V^*\)
and morphisms \(\text{ev}_V : V^* \otimes V \to \mathbb{1}\), \(\text{coev}_V : \mathbb{1} \to V \otimes V^*\). For vector spaces and
super-vector spaces the trivial object \(\mathbb{1} = \mathbb{C}\) and these maps are given by

\[
ev(f \otimes v) = f(v), \quad \text{coev}(\lambda) = \lambda \sum_a e_a \otimes f^a
\]
where \( v \in V, f \in V^*, \{ e_a \} \) is a basis of \( V \) and \( \{ f^a \} \) is a dual basis. The abstract characteristic property of these maps is that

\[
(ev_V \otimes \text{id})(\text{id} \otimes \text{coev}_V) = \text{id}_{V^*}, \quad (\text{id} \otimes ev_V)(\text{coev}_V \otimes \text{id}) = \text{id}_V. \tag{41}
\]

These are then enough to imply the most important of the familiar properties of duals such as the ability to dualise morphisms. In the diagrammatic notation we suppress \( \perp \) entirely so that the maps \( ev, \text{coev} \) and the left-duality condition \((41)\) appear as

\[
\begin{array}{c}
\text{ev}_V = V^* \quad \text{coev}_V = V^* \quad V^* \quad V^* \quad V^* \quad V^* \quad V^* \quad V^* \quad V^*
\end{array}
\]

Thus the condition means in topological terms that an \( S \)-bend for \( V^* \) and a mirror-\( S \)-bend for \( V \) can be straightened out by pulling. Some care is needed however because the mirror images of these diagrams are not true (for them one would need right-duals). A braided tensor category with both left and right duals, suitably compatible, is often called a modular tensor category.

It is easy to see that these conditions hold for usual transpositions and super-transpositions. Another source is to take for \( \mathcal{C} \) the category of representations of a group. These have a commutative tensor product, with \( \Psi \) given by the usual vector-space transposition. On the other hand, a fundamental property of strict quantum groups (with universal \( R \)-matrix) is that they too can be used in this way. Thus, let us recall that a strict quantum group means for us a quasitriangular Hopf algebra. This is \((H, \Delta, \epsilon, S, \mathcal{R})\) where \( H \) is an algebra, \( \Delta : H \rightarrow H \otimes H \) the coproduct homomorphism, \( \epsilon : H \rightarrow \mathbb{C} \) the counit, \( S : H \rightarrow H \) the antipode and \( \mathcal{R} \) the quasitriangular structure or 'universal \( R \)-matrix' obeying \([11]\)

\[
(id \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}, \quad (\Delta^{op} \otimes \text{id})(\mathcal{R}) = \mathcal{R}_{23}\mathcal{R}_{13}, \quad \Delta^{op} = \mathcal{R}(\Delta ( ))\mathcal{R}^{-1} \tag{43}
\]

where \( \mathcal{R}_{12} = \mathcal{R} \otimes 1 \) etc, and \( \Delta^{op} \) is the opposite coproduct. We have written the middle axiom in a slightly unconventional form but one that generalises below. Here and throughout the paper \( \mathbb{C} \) can be replaced by a field of suitable characteristic or (with due care) a commutative ring.

Then a well-known theorem about strict quantum groups (and key to their applications in knot theory) is that the category \( \mathcal{C} = \text{Rep}(H) \) of \( H \)-representations is a braided tensor one. The objects are vector spaces on which \( H \) acts, the morphisms are the \( H \)-interwiners, the associativity and duals are the standard ones as for vector spaces. Here the tensor product representation \( V \otimes W \) is given by the action of \( \Delta(H) \subset H \otimes H \), the first factor acting on \( V \) and the second factor on \( W \). Finally, the braiding is given by

\[
\Psi_{V,W}(v \otimes w) = P(\mathcal{R} \triangleright (v \otimes w)) \tag{44}
\]
where $\triangleright$ is the action of $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$ with its first factor on $V$ and its second factor on $W$, and this is then followed by the usual vector-space transposition $P$. The reason we need to act first by $\mathcal{R}$ is that $P$ alone would not be an intertwiner. Thus $h \triangleright \Psi(v \otimes w) = (\Delta h) \triangleright P(\mathcal{R} \triangleright (v \otimes w)) = P((\Delta^\text{op} h) \mathcal{R} \triangleright (v \otimes w)) = P(\mathcal{R}(\Delta h) \triangleright (v \otimes w)) = \Psi(h \triangleright (v \otimes w))$ in virtue of the last of (43). It is easy to see that the first two of (43) likewise just correspond to the hexagons (33) or (38). Functoriality is also easily shown. For an early treatment of this topic see [38, Sec. 7]. Note that if $\mathcal{R}_{21} = \mathcal{R}^{-1}$ (the triangular rather than quasitriangular case) we have $\Psi$ symmetric rather than braided.

**Proposition 3.1** ([36], Sec. 6) Let $H = \mathbb{Z}_2'$ denote the quantum group consisting of the group Hopf algebra of $\mathbb{Z}_2$ (with generator $g$ and $g^2 = 1$) and a non-standard triangular structure,

$$\Delta g = g \otimes g, \quad \epsilon g = 1, \quad Sg = g, \quad \mathcal{R} = 2^{-1}(1 \otimes 1 + g \otimes 1 + 1 \otimes g - g \otimes g). \quad (45)$$

Then $\mathcal{C} = \text{Rep}(\mathbb{Z}_2') = \text{SuperVec}$ the category of super-vector spaces.

**Proof** One can easily check that this $\mathbb{Z}_2'$ is indeed a quasitriangular (in fact, triangular) Hopf algebra. Hence we have a (symmetric) tensor category of representations. Writing $p = \frac{1-2g}{2}$ we have $p^2 = p$ hence any representation $V$ splits into degree $V_0 \oplus V_1$ according to the eigenvalue of $p$. We can also write $\mathcal{R} = 1 - 2p \otimes p$ and hence from (44) we compute $\Psi(v \otimes w) = P(\mathcal{R} \triangleright (v \otimes w)) = (1 - 2p \otimes p)(w \otimes v) = (1 - 2|v||w|)w \otimes v = (-1)^{|v||w|}w \otimes v$ as in (43). $\square$

So this non-standard quantum group $\mathbb{Z}_2'$ (non-standard because of its non-trivial $\mathcal{R}$) recovers the category of super-spaces with its correct symmetry $\Psi$. On the other hand, there are plenty of other quasitriangular Hopf algebras $H$ we could use here. For example, in [38] we have introduced the non-standard quasitriangular Hopf algebra $\mathbb{Z}_n'$ with generator $g$ and relation $g^n = 1$ and

$$\Delta g = g \otimes g, \quad \epsilon g = 1, \quad Sg = g^{-1}, \quad \mathcal{R} = n^{-1} \sum_{a,b=0}^{n-1} e^{-\frac{2\pi i ab}{n}} g^a \otimes g^b. \quad (46)$$

The category $\mathcal{C}_n = \text{Rep}(\mathbb{Z}_n')$ consists of vector spaces that split as $V = \oplus_{a=0}^{n-1} V_a$ with the degree of an element defined by the action $g \triangleright v = e^{\frac{2\pi i ab}{n}} v$. From (44) and (10) we find

$$\Psi_{V,W}(v \otimes w) = e^{\frac{2\pi i ab}{n}|v||w|}w \otimes v. \quad (47)$$

Thus we call $\mathcal{C}_n$ the category of anyonic vector spaces of fractional statistics $\frac{1}{n}$, because just such a braiding is encountered in anyonic physics. The case $n = 2$ is that of superspaces. For $n > 2$ the category is strictly braided in the sense that $\Psi \neq \Psi^{-1}$. There are natural anyonic traces and anyonic dimensions generalizing the super-case [38]

$$\dim V = \sum_{a=0}^{n-1} e^{-\frac{2\pi i a^2}{n}} \dim V_a, \quad \text{Tr}(f) = \sum_{a=0}^{n-1} e^{-\frac{2\pi i a^2}{n}} \text{Tr} f|_{V_a}. \quad (48)$$
We see that the category is generated by the quantum group $\mathbb{Z}_n'$.

Note that these quantum groups are discrete and nothing whatever to do with usual $q$-deformations. Quite simply, this use of the mathematical structure of quasi-triangular Hopf algebras as generating statistics is different from how they arose in QISM. On the other hand, there is nothing stopping us going to the other extreme and taking $H = U_q(g)$. For example, if $H = U_q(sl_2)$ the role of $\mathbb{Z}_2$ or $\mathbb{Z}_n$-grading is now played by the spectral decomposition into irreducibles. Thus $V = \oplus_i V_i$ where the $V_i$ are the spin $i = 0, \frac{1}{2}, 1$ etc. representations. The braiding $\Psi$ from (44) is now given by the direct sum of the corresponding $R$-matrices for each spin, which in turn can be reduced to products of the fundamental $SL_q(2)$ $R$-matrix.

We call the quantum group $H$ used in this way the statistics generating quantum group [28, Sec. 6]. Each generates a braided category or ‘universe’ within which we can work. By this reasoning we can immediately generalise the notion of super-algebras. Thus, a braided-algebra means an algebra $B$ living in a braided tensor category. We mean by this that there is an object $B$ in the category and the product and unit maps $\cdot : B \otimes B \to B$, $\eta : 1 \to B$ (49)

are morphisms in the category.

Now we switch to another topic, that of covariance under a group. Recall that an algebra $B$ is $G$-covariant if the group $G$ acts on $B$ and $g \triangleright (bc) = (gb)(gc)$, $g \triangleright 1 = 1$ for all $g \in G$ and $b, c \in B$. Here $\triangleright$ denotes the action. The natural generalization of this to any Hopf algebra $H$ is that of an $H$-covariant algebra (or $H$-module algebra). This means an algebra $B$ on which $H$ acts according to

$$h \triangleright (b \cdot c) = \cdot ((\Delta h) \triangleright (b \otimes c)), \quad h \triangleright 1 = \epsilon(h)1$$

(50)

where $\Delta h$ in $H \otimes H$ acts with the first factor on $b$ and the second factor on $c$, and we have emphasised the product $\cdot$ in $B$. This notion also includes the notion of $g$-covariant where $g$ is a Lie algebra. There $\Delta \xi = \xi \otimes 1 + 1 \otimes \xi$ for $\xi \in g$ and so the covariance condition (50) becomes $\xi \triangleright (bc) = (\xi \triangleright b)c + b(\xi \triangleright c)$ as usual (groups and Lie algebras are unified when we work with Hopf algebras).

**Lemma 3.2** Let $(H, R)$ be a quasitriangular Hopf algebra and $C = \text{Rep}(H)$. An algebra $B$ lives in $C$ (is a braided algebra) iff it is $H$-covariant in the sense of (50).

**Proof** That $B$ lives in the category as in (49) means in the present setting that the product and unit maps are intertwiners for the action of $H$ (they are morphisms in $\text{Rep}(H)$). This means $h \triangleright (b \cdot c) = \cdot (h \triangleright (b \otimes c))$. But the action of $H$ on $B \otimes B$ is in the tensor product representation, so $h \triangleright (b \otimes c) = (\Delta h) \triangleright (b \otimes c)$. $\square$

This lemma in conjunction with the above analysis of super-spaces etc expresses the unification of two concepts made possible by quantum group theory. Thus

$$B \in \text{Rep}(H) \quad \text{means} \quad \begin{cases} \text{super – algebra} & H = \mathbb{Z}_2' \\ \text{G – covariant algebra} & H = CG \end{cases}$$

(51)

Moreover, there are plenty of other strict quantum groups (quasitriangular Hopf algebras) that one may take here, ranging from the anyonic statistics-generating Hopf algebra $\mathbb{Z}_n'$ to the more standard $U_q(g)$. Each can be interpreted either way, as covariance (a quantum symmetry) or as generating statistics.
4 Diagrammatic Methods for Braided-Hopf Algebras

The unification of statistics and covariance in the last chapter means that all results about braided algebra, developed by analogy with super-symmetry, can then be reinterpreted in terms of results about quantum-group covariant systems. We will make such applications in later chapters. Our goal in the present chapter is to develop this braided-algebra in analogy with the theory of super-algebras. This provides the mathematical underpinning of the examples such as the braided-matrices in Chapter 2, as well as providing some powerful diagrammatic tools for working with such objects.

Just as two super algebras $B, C$ have a super-tensor-product superalgebra, $B \otimes C$, containing $B, C$ as mutually super-commuting sub-superalgebras, so we have now the fundamental lemma:

**Lemma 4.1** Let $B, C$ be two algebras living in a braided category (two braided algebras as in (49)). There is a braided tensor product algebra $B \otimes C$, also living in the braided category. It is built on the object $B \otimes C$ but with the product law

$$(a \otimes c)(b \otimes d) = a \Psi_{C,B}(c \otimes b)d \quad \forall a, b \in B, c, d \in C$$

where the first is a concrete description and the second an abstract one in terms of morphisms.

**Proof** The best proof that this product law is associative is the diagrammatic one

$$\begin{align*}
\begin{array}{c}
\begin{array}{c}
\includegraphics{fig1} \\
\end{array}
\end{array}
\end{align*} = \begin{array}{c}
\begin{array}{c}
\includegraphics{fig2} \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\includegraphics{fig3} \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\includegraphics{fig4} \\
\end{array}
\end{array}
\end{align*}$$

The first step uses functoriality as in (39) to pull the product morphism through the braid crossing. The second equality uses associativity of the products in $B, C$ and the third equality uses functoriality again in reverse. The product is manifestly a morphism in the category (is covariant) because it is built out of morphisms (covariant maps). Finally, the unit is the tensor product one because the braiding is trivial on $1$. □

In the concrete case the braided tensor product is generated by $B = B \otimes 1$ and $C = 1 \otimes C$ and an exchange law between the two factors given by $\Psi$. This is because
\( (b \otimes 1)(1 \otimes c) = (b \otimes c) \) while \( (1 \otimes c)(b \otimes 1) = \Psi(c \otimes b) \). Another notation is to label the elements of the second copy in the braided tensor product by 't. Thus \( b \equiv (b \otimes 1) \) and \( c \equiv (1 \otimes c) \). Then if \( \Psi(c \otimes b) = \sum b_k \otimes c_k \) say, we have the braided-tensor product relations

\[
c' \equiv (1 \otimes c)(b \otimes 1) = \Psi(c \otimes b) = \sum b_k \otimes c_k \equiv \sum b_k c'_k
\]

which is the notation used in Chapter 2. This also makes clear why we call \( \Psi \) the braid-statistics and why the lemma generalizes the notion of super-tensor product. From the unification of statistics and covariance in Lemma 3.2, we have equally well,

**Corollary 4.2** Let \( H \) be a strict quantum group. Given two \( H \)-covariant algebras \( B, C \), we have another \( H \)-covariant algebra \( B \otimes C \).

We note that there is an equally good **opposite braided tensor product** with the inverse braid crossing in Lemma 4.1. For any braided category \( C \) there is another mirror-reversed braided category \( \overline{C} \) with

\[
\overline{\Psi}_{V,W} = \Psi_{W,V}^{-1}
\]

in place of \( \Psi_{V,W} \) (braids and inverse braided interchanged), and the opposite braided tensor product algebra is simply the braided tensor product algebra in \( \overline{C} \).

This lemma and its corollary have many applications, as we shall see in Chapter 6. It is also the fundamental lemma for us now because it enables us to define the notion of braided group as a generalization of supergroups and super-enveloping algebras. Let us recall that in place of working with groups and Lie algebras we can work equivalently with the group Hopf algebras and cocommutative enveloping Hopf algebras that they generate (cocommutative means that \( \Delta^\text{op} = \Delta \)). The same holds in the super-case where one can work with super-cocommutative enveloping super-Hopf algebras. Thus, we will introduce braided groups formally as (in a certain sense braided-cocommutative) braided-Hopf algebras.

Recall that the key feature of a Hopf algebra is that it has a realization in its own tensor products. In the braided case we require a realization in its own braided tensor products. Thus, a braided-Hopf algebra (a Hopf algebra living in a braided category \( C \)) is \( (B, \Delta, \epsilon, S) \) where \( B \) is a braided algebra as in (49) and \( \Delta : B \to B \otimes B, \epsilon : B \to 1 \) are algebra homomorphisms where \( B \otimes B \) has the braided tensor product algebra structure. In addition \( \Delta, \epsilon \) are coassociative as usual, and \( S : B \to B \) obeys the usual axioms of an antipode. In diagrammatic form, the axioms are

\[
\text{(55)}
\]

If there is no antipode then we speak of a braided-bialgebra or bialgebra in a braided category.
This gives us the notion of a group-like or Hopf algebra-like object in a braided category. In what follows we shall see that these braided objects really behave just like usual groups or Hopf algebras (or their super-versions). For example, recall that the antipode behaves like a group inverse (for a group algebra $Sg = g^{-1}$) with the result of course that $S$ is an anti-algebra homomorphism.

**Lemma 4.3**: For a braided-Hopf algebra $B$, the braided-antipode obeys $S(b \cdot c) = \cdot \Psi(Sb \otimes Sc)$ and $S(1) = 1$, or more abstractly, $S \circ \cdot = \cdot \circ \Psi_{B,B} \circ (S \otimes S)$ and $S \circ \eta = \eta$. Also, $\Delta \circ S = (S \otimes S) \circ \Psi_{B,B} \circ \Delta$ and $\epsilon \circ S = \epsilon$.

**Proof**: In diagrammatic form the proof is

In the first two equalities we have grafted on some circles containing the antipode, knowing they are trivial from (55). We then use the coherence theorem to lift the second $S$ over to the left, and associativity and coassociativity to reorganise the branches. The fifth equality uses the axioms (55) for $\Delta$. For the second part of the lemma, turn the diagram-proof upside-down and read it again. \[ \square \]

To see that this is a useful notion, recall that any group acts on itself by the adjoint action $Ad_h(g) = hgh^{-1}$. Remembering that $\Delta h = h \otimes h$ for a group Hopf algebra, the steps here are to split $h$ using the coproduct $\Delta$, apply $S$ to the second factor and multiply up with $g$ in the middle. Writing this as maps in our braided case we have:

**Proposition 4.4**: Every braided-Hopf algebra $B$ acts in itself by a braided-adjoint action $Ad = (\cdot \otimes \cdot)(\text{id} \otimes \Psi_{B,B})(\text{id} \otimes S \otimes \text{id})(\Delta \otimes \text{id})$. 

21
Proof  The diagrammatic representation and proof is

Here the first step is functoriality while second equality uses the result in the preceding lemma. We then use the bialgebra axiom for $\Delta$ in (55). We have adopted the notation of combining repeated products into a single node, knowing (from associativity) that only the order of inputs into the combined product matters, not its original tree structure. □

We can continue to prove familiar properties for the braided-adjoint action. For example, in the group case it respects its own algebra structure. The corresponding condition for an algebra $C$ in a braided-category to be braided-covariant under $B$ (a braided $B$-module algebra) is

This is the braided analogue of the group covariance $g \triangleright (bc) = (g \triangleright b)(g \triangleright c)$ or the quantum covariance as in (50). For the record, we can also ask that a coalgebra $(C, \Delta, \epsilon)$ in the category is acted upon by $B$. The corresponding condition is

Proposition 4.5  The braided-adjoint action of $B$ on itself given in Proposition 4.4 respects its product in the sense of (57).
Proof

Here we used functoriality in the first step and grafted on a trivial circle involving $S$ in the second step. We then used associativity and coassociativity to organise the result. We see that (57) is obeyed with $\alpha = \text{Ad}$. □

Next, we come to the question of what we mean by a cocommutative (cf commutative) Hopf algebra in a braided category. The most naive idea would be to define $\Delta^\text{op} = \Psi_{B,B}^1 \circ \Delta$ or $\Delta^\text{op} = \Psi_{B,B}^{-1} \circ \Delta$ as an opposite Hopf algebra, and say that $B$ is cocommutative if $\Delta^\text{op} = \Delta$. Unfortunately, in the truly braided case the first does not give a Hopf algebra at all (things get too tangled up) while the second gives →

Lemma 4.6 $B^\text{cop}$ defined as the same algebra $B$ but with braided-coproduct $\Delta^\text{op} = \Psi_{B,B}^{-1} \circ \Delta$ defines a bialgebra in $\bar{C}$ (C with the opposite braiding). It is a Hopf algebra in $\bar{C}$ with antipode $S^\text{op} = S^{-1}$ if $S$ is invertible.

Proof We use the definitions and the axioms (55) for $B$,

Because of this, there is no intrinsic notion of opposite coproduct lying in our original category, hence no intrinsic notion of cocommutativity in the braided case. Our way out in the theory of braided groups is to realise that we dont usually need cocommutativity in an intrinsic sense but only when the Hopf algebra acts on things, i.e. relative to representations. The notion of a braided-representation or braided-module $V$ is just the obvious one ($V$ is an object in the category and $B$ acts on it by a morphism $B \otimes V \rightarrow V$). Thus, we can turn things around and (for any braided-Hopf algebra $B$) we say that $B$ behaves braided-cocommutatively with respect to a braided-representation $V$ if

Braided Cocommutativity (59)
In the symmetric (unbraided) case this reduces to

\[
\begin{array}{c}
\Delta \otimes B \otimes V \\
\alpha \otimes V \\
B \otimes V
\end{array}
\] =
\[
\begin{array}{c}
\Delta \otimes B \\
\alpha \otimes V \\
= \\
B \\
B \otimes V
\end{array}
\]

but in general we cannot untangle \( V \) from \( B \) and must work with the weak notion.

**Lemma 4.7** If \( B \) is braided-cocommutative with respect to the braided-adjoint action (for example, this is true for the examples in Chapter 5) then the braided-adjoint action respects the coproduct of \( B \) in the sense of (60).

**Proof**

Here the first and second equalities both use the bialgebra axiom for \( \Delta \) in (55). The third uses Lemma 4.3. The fourth uses functoriality to drag the second \( S \) up and over to the left, so that we can recognise that it forms Ad in the fifth. The seventh uses the braided-cocommutativity assumption (59) for Ad. □

This is a result usually reserved for honest groups and supergroups (or cocommutative (super)-Hopf algebras) and its failure for general Hopf algebras severely limits the use of the quantum-adjoint action. Here we see that the requirement of cocommutativity with respect to Ad plays an analogous role. This kind of cocommutativity then tells us that we have a braided group rather than merely a braided-Hopf algebra. Formally then, we define a braided group as \((B, O)\) where \( B \) is a braided-Hopf algebra and \( O \) is a useful class of \( B \)-modules with respect to which \( B \) behaves cocommutatively.

24
Next, let us recall that for any group or Hopf algebra we can tensor product representations, and also dualise them using the group inverse. The same is true in the braided case.

**Proposition 4.8** Let $B$ be a braided-Hopf algebra. The braided tensor product of two $B$-modules $V, W$ is given by the action of $B$ on $V \otimes W$ via the action of $\Delta(B)$ and remembering the braid-statistics $\Psi$. In concrete terms this is

$$b \triangleright (v \otimes w) = \sum b_{(1)} \triangleright \Psi(b_{(2)} \otimes v) \triangleright w$$

where $\triangleright$ denotes the relevant actions and $\Delta b = \sum b_{(1)} \otimes b_{(2)}$ is an explicit notation in the concrete case. As morphisms this is $\alpha_{V \otimes W} = (\alpha_V \otimes \alpha_W) \Psi_{B,V}(\Delta \otimes \text{id} \otimes \text{id})$.

**Proof**

![Diagrams](#)

□

**Lemma 4.9** If $V$ is a left braided $B$-module, then its left dual $V^*$ is a right braided $B$-module by dualising.

**Proof** The dualization uses the ev map in (11). In diagrammatic form the resulting action $\alpha^*$ on $V^*$ is

![Diagrams](#)

□

**Lemma 4.10** If $V$ is a right $B$-module then it is also a left $B$-module via the antipode $S$. 

25
Proof. We use the antipode to convert a right action to a left action (recall from Lemma 4.3 that $S$ is some kind of anti algebra homomorphism). If $\alpha^R$ is our initial right action on $V$, the required left action is

$$B \xrightarrow{\alpha^R} V \quad \xrightarrow{S} \quad V \xrightarrow{\alpha^L} B$$

The Proposition 4.8 says that we can tensor product braided-representations, while using the lemmas, we see that any braided-representation $(V, \alpha_V)$ of a braided-Hopf algebra (a left $B$-module) has a contragradient or dual one $(V^*, \alpha_{V^*})$ where $\alpha_{V^*}$ is given by feeding the result of Lemma 4.9 into Lemma 4.10. This is just as for usual groups or Hopf algebras where the inverse or antipode allows one to construct contragradient representations.

On the other hand, a feature of true groups (as against general Hopf algebras) is that the tensor product of their representations is symmetric under the usual transposition of the underlying vector space. In the braided case we have the analogous result with $\Psi$ in the role of the usual transposition. Thus, if $B$ is braided-cocommutative with respect to $V$ then

$$B \xrightarrow{\Delta} V \xrightarrow{\alpha_V} W \quad \text{and} \quad V \xrightarrow{\alpha_V} W \xrightarrow{\Delta} B$$

from (59) (by adding an action on $W$ to both sides in (59)) and is largely equivalent to it. This is how (59) was derived in [5]. These results thus demonstrate that our weak braided-cocommutativity condition (59) is an appropriate and useful one.

We now turn to another topic, namely that of dual braided-Hopf algebras. Recall that a novel feature of Hopf algebras is that the dual of a Hopf algebra is also a Hopf algebra. For example, the dual of a group Hopf algebra is the commutative algebra of functions on a group. Similarly in the super case. When we work with matrix super-groups we are making use of this duality by describing the super-commutative Hopf algebra of ‘functions’ dual to the corresponding enveloping super-Hopf algebra itself.

**Proposition 4.11** If $B$ is a braided-Hopf algebra, then its left-dual $B^*$ is also a
braided-Hopf algebra with product, coproduct, antipode, counit and unit given by

\[
\begin{align*}
\eta & = 1_B, \\
\Delta & = \Delta^*, \\
S & = S^*, \\
\varepsilon & = \varepsilon^*.
\end{align*}
\]

(65)

**Proof**  
Associativity and coassociativity follow at once from coassociativity and associativity of \( B \). Their crucial compatibility property comes out as

The antipode property comes out just as easily. \( \square \)

Clearly, one can go on and develop all the usual theory of Hopf algebras in this braided-diagrammatic way. We will content ourselves here with another canonical construction, namely the braided version of the coregular action of a Hopf algebra on its dual. This is needed in Chapter 7 in order to define braided-vector fields (just as a group or Lie algebra acts on its own function algebra by derivations).

The first step is to recall that an action of a Hopf algebra corresponds to a coaction of its dual. A coaction is like an action but with the arrows reversed. Thus, a left \( B \)-comodule is a map \( \beta : V \to B \otimes V \).

**Proposition 4.12** If \( V \) is a left \( B \)-comodule then it becomes a left \( B^* \)-module by dualising.

**Proof**  
Here we use ev in \((1)\). The diagrammatic form and proof is

\[
\begin{align*}
\beta^* & = \beta^*, \\
\beta^* & = \beta^*.
\end{align*}
\]

(66)
Next, if \( B \) coacts on some other algebra we can ask that \( \beta_C : C \rightarrow B \otimes C \) is an algebra homomorphism to (a realisation in) the braided tensor product as in Lemma 4.1. This is a \textit{braided-comodule algebra},

\[
\begin{array}{c}
\beta_C \colon C \rightarrow B \otimes C \\
\end{array}
\]

For example, the coproduct \( \Delta : B \rightarrow B \otimes B \) makes \( B \) automatically a left (and also a right) comodule algebra under itself. For the record, we can also ask that a coalgebra \((C, \Delta, \epsilon)\) is preserved under a coaction of \( B \). The corresponding condition is

\[
\begin{array}{c}
\Delta_C \colon B \rightarrow B \otimes B \\
\end{array}
\]

Lemma 4.13 \textit{Let} \( C \) \textit{be a braided left} \( B \)-\textit{comodule algebra as in (67). Then} \( C \) \textit{is also a left} \( B^\text{cop} \)-\textit{module algebra in} \( \overline{\mathcal{C}} \) \textit{by dualising.}

\textbf{Proof} \quad \text{We have seen that} \( B^* \) \textit{is a braided Hopf algebra in} \( \mathcal{C} \) \textit{and hence by Lemma 4.6 applied to} \( B^* \), \textit{we know that} \( B^\text{cop} \) \textit{is a Hopf algebra in the category} \( \overline{\mathcal{C}} \) \textit{with reversed braiding. Also, we know from Proposition 4.12 that a coaction} \( \beta \) \textit{of} \( B \) \textit{gives an action} \( \beta^* \) \textit{of} \( B^* \). \textit{We compute}

\[
\begin{array}{c}
\beta_C^* \colon B^* \rightarrow B^* \otimes C \\
\end{array}
\]

\begin{align*}
\beta_C^* & = \beta_C^* \\
\beta_C^* & = \beta_C^* \\
\beta_C^* & = \beta_C^* \\
\beta_C^* & = \beta_C^* \\
\beta_C^* & = \beta_C^* \\
\end{align*}

The first equality is the definition of \( \beta^* \) and the second uses our assumption (67) for \( \beta \). We then use the definition of the coproduct for \( B^* \) in (68) and recognise \( \beta^* \) again. The right hand side is just like the right-hand side of (67) except that we have the opposite coproduct of \( B^* \) and we have an inverse braid crossing as \( B^* \) passes \( C \), rather than the crossing in (67). Hence we have a \( B^\text{cop} \)-module algebra living in the opposite category \( \overline{\mathcal{C}} \). \( \Box \)

Lemma 4.14 \textit{Let} \( C \) \textit{be a braided left} \( B^\text{cop} \)-\textit{module algebra in} \( \overline{\mathcal{C}} \). \textit{Then} \( C \) \textit{is also a right} \( B \)-\textit{module algebra in} \( \mathcal{C} \) \textit{by the use of} \( S \).
Proof We convert the left $B$-module $\alpha$ to a right one $\alpha_R$ in a way similar to Lemma 4.10. The proof is also similar, namely take the diagram proof there, reflect it in a mirror about a vertical axis and put the braid crossings that are reversed by this process back to their original form. We show that the resulting right action $\alpha_R$ defines a right module algebra (acts covariantly on the algebra $C$ from the right). We have

\[ \alpha_R C \]

The first equality is functoriality while the second is our assumption that $C$ is a $B^{\text{cop}}$-module algebra in $\mathcal{C}$ in the sense explained in the preceding lemma (like (57) but with $\Delta^{\text{op}}$ rather than $\Delta$ and a reversed braid crossing when $B$ passes $C$). We then use Lemma 4.3 and recognise the result. The result is that $\alpha_R$ acts on $C$ according to a right-handed version of (57). This right handed version consists of (57) reflected in a mirror about a vertical axis and with its reversed braid crossing then restored. \(\square\)

For example, if we apply Lemma 4.13 to the left-regular coaction of $B$ in itself given by $\Delta$, we see that $B^{\text{cop}}$ acts on $B$ from the left, but in the opposite category. Also, feeding the result of this into Lemma 4.14 applied to $B^{\text{cop}}$, we see that every braided-Hopf algebra $B$ becomes a right $B^{\text{cop}}$-module algebra in our original category by

\[ B \quad B^{*} \]

In this chapter we have defined only the elementary theory of braided groups. Other results (in our diagrammatic form) appeared in [43] and [42] and we recall them now without proof. In the first we show that if $C$ is a left $B$-module algebra as in (57) then there is a cross product braided-algebra $C\triangleright B$. It is built on $C \otimes B$ with product given by the top left of

\[ C \quad B \quad C \quad B \]

\[ C \quad B \quad C \quad B \quad C \quad B \quad C \quad B \]

\[ C \quad B \quad C \quad B \quad C \quad B \quad C \quad B \]

Cross Product by right action

Cross Coproduct by left coaction

Cross Product by right coaction

(72)
We proved associativity etc. As an application we showed that if the $B$-module algebra $C$ is also a braided-Hopf algebra and $B$ is cocommutative with respect to $C$ then $C \triangleright \triangleleft B$ is a braided-Hopf algebra with the braided-tensor-coproduct. Again, this is a result usually formulated only for group actions or actions of cocommutative Hopf algebras. It leads to the bosonization result in Chapter 5.

Reflection of this left semidirect or cross product construction in the vertical axis takes us into an analogous construction for a right $B$-module algebra living in $\bar{C}$, and restoration of the reversed braid-crossings gives us the analogous construction for a right $B$-module algebra in our original category (such as obtained in the last lemma), as displayed at the top right in (72). For example, we have by this construction the braided Weyl algebra $B^\ast \triangleleft B$ using (71). This is the semidirect product of $B$ regarded as ‘position observables’ by $B^\ast$ acting from the right as ‘braided vector-fields’.

As well as the reflection principle about a vertical axis, we also encountered above a reflection principle about a horizontal axis. This converts the theory with left modules into one with left comodules living in the opposite category. Restoring the reversed braid crossings gives us the theory for left comodules in our original category. Since this applies to all the axioms, it applies to all our results above obtained by diagram-proofs. For example, we have a semidirect product coalgebra for a coalgebra $C$ crossed by a left coaction obeying (58), displayed bottom left in (72). Finally, by making both reflections (turning the page up-side-down) we have the theory for right-comodules in our original category. Again, it is not necessary to repeat all the diagram proofs: All the results above about left modules hold for right comodules (and vice versa) by turning the page upside down and interchanging products and coproducts, evaluations and coevaluations etc. The duality and Hopf algebra axioms (41) and (55) are unchanged by this. The notion of braided-cocommutativity with respect to a module in (59) now becomes the notion of braided-commutativity with respect to a comodule. The left adjoint action becomes the right adjoint coaction etc. This is the form of the theory actually used in Chapter 2. See [45] for the details of the derivation.

Finally, in [42], we study quantum-braided-groups, by which we mean a braided-Hopf algebra equipped with a braided-universal $\mathcal{R}$-matrix. The latter is a morphism $\mathcal{R} : 1 \rightarrow B \otimes B$ obeying

$$
\begin{align*}
\Delta = & \quad R \\
\Delta^\text{op} = & \quad R \\
\Delta^\text{op} = & \quad R
\end{align*}
$$

(73)

where $\Delta^\text{op}$ is any second Hopf algebra structure on $B$. Again, its usefulness is limited to $B$-modules $V$ obeying the condition (59) with $\Delta^\text{op}$ in place of the $\Delta$ on the left (representations in which $\Delta^\text{op}$ behaves like an opposite coproduct). We show, for example, that the class $\mathcal{O}(B, \Delta^\text{op})$ of all such $B$-modules is closed under tensor products and itself a braided category in the quantum-braided-group case. If our initial braiding is equivalent to some kind of quantization [44], this is equivalent to second-quantization. Again, there is a dual version of the theory with comodules, which is the original setting in [41].
5 Transmutation and Bosonization

In the above we have given a full account of the elementary part of the theory of braided groups and braided matrices. In what remains we shall content ourselves with the briefest of outlines, without proof, of some of the more advanced theorems and their applications. In the present chapter we shall state the main results about transmutation and bosonization, and something of the general principles involved. The full details are in [40][41][42][43]. In the next two chapters we will conclude with a few illuminating examples and applications of interest in physics. Readers who have had enough of abstract mathematics should proceed directly to these chapters and skip the present one.

Recall that in quantum theory, and also in our algebraic work in Chapter 2, there are really two sources of non-commutativity. The first is non-commutativity of the algebra of observables resulting from quantization and expressed in algebraic terms in the language of non-commutative geometry (or as non-co-commutativity in the case of quantum enveloping algebras), while the second is statistical non-commutativity expressed as non-commutation relations between independent copies of the algebra. Thus super-groups and braided-groups (of function algebra type) are viewed not as quantum objects (they are super- or braided-commutative) but rather, the non-commutativity is of this second type and expressed in Chapter 2 as braid statistics. Thus the two kinds of non-commutativity are conceptually quite different.

The idea behind this chapter is that sometimes one can systematically trade one of these kinds of non-commutativity for the other. Thus, we can exchange some of the quantum non-commutativity for statistical non-commutativity, and vice-versa. This means that how we view an object, as a quantum group, super-group or braided-group is to some extent a matter of choice. The category we work in is a kind of ‘co-ordinate system’ and one system may be better for some purposes than another.

The precise formulation is as follows. Let $H \xrightarrow{f} H_1$ be a pair of quantum groups with a Hopf algebra map between them (for example, $H_1 \subseteq H$). At least $H_1$ should have a universal R-matrix. As explained in Chapter 3, it generates a braided category $\text{Rep}(H_1)$.

**Theorem 5.1** [40][42] $H$ can be viewed equivalently as a braided-Hopf algebra $B(H_1, H)$ living in the braided category $\text{Rep}(H_1)$ (so with braid statistics induced by $H_1$). Here

$$B(H_1, H) = \left\{ \begin{array}{l} H \\ \Delta, S \end{array} \right. \sum \text{modified coproduct and antipode}$$

If $H$ has a universal R-matrix then $B(H_1, H)$ has a braided-universal R-matrix as in (73), given by the ratio of the universal R-matrices of $H, H_1$.

The explicit formulae are as follows. Firstly, the action of $H_1$ is the induced quantum adjoint action $h \triangleright b = \text{Ad}_{f(h)}(b) = \sum f(h_{(1)})b(Sf(h_{(2)}))$ where $\Delta h = \sum h_{(1)} \otimes h_{(2)}$ is an explicit standard notation for the coproduct. Using this, the braided coproduct, braided-antipode and braided quasitriangular structure (or braided-universal R-matrix) are

$$\Delta b = \sum b_{(1)}f(SR_1(2)) \otimes R_1(1) \triangleright b_{(2)}, \quad Sb = \sum f(R_1(2))S(R_1(1) \triangleright b)$$

(75)
\[ \mathcal{R} = \sum \rho^{(1)} f(S\mathcal{R}_1^{(2)}) \otimes \mathcal{R}_1^{(1)} \triangleright \rho^{(2)} \]  

(76)

where \( \rho = f(\mathcal{R}_1^{-1}) \mathcal{R} \) is the ratio as promised and \( ^{(1)}, (2) \) refer to the component factors in a tensor product. There is also a specific opposite coproduct characterised by

\[ \sum \Psi(b_{(1)} \otimes Q_1^{(1)} \triangleright b_{(2)}) f(Q_1^{(2)}) = \sum b_{(1)} \otimes b_{(2)} \]  

(77)

where \( Q_1 = (\mathcal{R}_1)_{21} (\mathcal{R}_1)_{12} \) and \( f(Q_1^{(2)}) \) multiplies the result of \( \Psi \) from the right as \( (1 \otimes f(Q_1^{(2)})) \). The underlines in the superscripts are to remind us that we intend here the braided-coproducts \( \Delta \) and \( \Delta^{op} \). The equation can also be inverted to give an explicit formula for \( \Delta^{op} \). That these formulae obey the axioms (55) and (73) of Chapter 4 is verified explicitly in [42].

**Corollary 5.2** [38] Let \( H \) be a quantum group containing a group-like element \( g \) of order \( n \) (so \( g^n = 1 \) and \( \Delta g = g \otimes g \)). Then \( H \) has a corresponding anyonic version \( B \). It has the same algebra and

\[ \Delta b = \sum b_{(1)} g^{-|b_{(2)}|} \otimes b_{(2)}, \quad eb = eb, \quad Sb = g^{b} Sb \]  

(78)

\[ \Delta^{op} b = \sum b_{(2)} g^{-2|b_{(1)}|} \otimes g^{-|b_{(2)}|} b_{(1)}, \quad \mathcal{R} = \mathcal{R}_n^{-1} \sum \mathcal{R}^{(1)} g^{-|\mathcal{R}^{(2)}|} \otimes \mathcal{R}^{(2)} \]  

(79)

**Proof** We apply the transmutation theorem, Theorem 5.1 and compute the form of \( B = B(\mathbb{Z}_n', H) \). Here \( \mathbb{Z}_n' \) is the non-standard quantum group in (16) with universal R-matrix \( \mathcal{R}_n' \). The action of \( g \) on \( H \) is in the adjoint representation \( g \triangleright b = g b g^{-1} \) for \( b \in H \) and defines the degree of homogeneous elements by \( g \triangleright b = e^{\frac{2\pi i |b|}{n}} b \). \( \square \)

**Corollary 5.3** Let \( H \) be a quantum group containing a group-like element \( g \) of order 2 (so \( g^2 = 1 \) and \( \Delta g = g \otimes g \)). Then \( H \) has a corresponding super-version \( B \).

**Proof** The formulae are as in (78) and (79) with \( n = 2 \). \( \square \)

The first corollary was applied, for example to \( H = u_q(g) \) at a root of unity to simplify its structure. It leads to a new simpler form for its universal R-matrix (by finding its anyonic universal R-matrix and working back) [38]. The second corollary was usefully applied in [50] to superise the non-standard quantum group associated to the Alexander-Conway polynomial. In these examples, a sub-quantum group is used to generate the braid statistics (braided category) in which the entire quantum group is then viewed by transmutation. In the process its quasitriangular structure or universal R-matrix becomes reduced because the part from the subgroup is divided out (see Theorem 5.1). This means that the part corresponding to the subgroup is made in some sense cocommutative.

**Corollary 5.4** [33] Every strict quantum group (with universal R-matrix) has a braided-group analogue \( B(H, H) \) which is braided-cocommutative in the sense that \( \mathcal{R} = 1 \otimes 1 \) and \( \Delta^{op} = \Delta \). The latter is

\[ \sum \Psi(b_{(1)} \otimes Q_{1}^{(1)} \triangleright b_{(2)}) Q_{1}^{(2)} = \sum b_{(1)} \otimes b_{(2)}. \]  

(80)

We call \( B(H, H) \) the braided group of enveloping algebra type associated to \( H \). It is also denoted by \( H_3 \). [33]
**Proof**  Here we take the transmutation principle to its logical extreme and view any quantum group $H$ in its own braided category $\text{Rep}(H)$, by $H \subseteq H$. This is a bit like using a metric to determine geodesic co-ordinates. In that co-ordinate system the metric looks locally linear. Likewise, in its own category (as a braided group) our original quantum group looks braided-cocommutative. □

This completely shifts then from one point of view (quantum=non-cocommutative and bosonic object) to the other (classical=cocommutative but braided object), and means that ordinary quantum group theory is contained in the theory of braided-groups. On the other hand, not all braided-Hopf algebras are obtained in this way. We shall encounter some that do not appear to come from quantum groups in Chapter 7.

We have said in Theorem 5.1 that the resulting braided-Hopf algebra $B$ is equivalent to the original one. The sense in which this is true is that spaces and algebras etc on which $H$ act also become transmuted to corresponding ones for $B$. Partly, this is obvious since $B = H$ as an algebra, so any representation $V$ of $H$ is also a representation of $B$. The key point is that $V$ is also acted upon by $H_1$ through the mapping $H_1 \to H$. So the action of $H$ is used in two ways, both to define the corresponding action of $B$ and to define the ‘grading’ of $V$ as an object in a braided category $\text{Rep}(H_1)$ (as explained at the end of Chapter 3). This extends the process of transmutation to view a representation of $H$ also as a braided-representation of $B$.

**Proposition 5.5** [43, Prop 3.2] If $C$ is an $H$-covariant algebra ($H$-module algebra) in the sense of (50) then its transmutation is a $B$-module algebra in the sense of (57). Here the transmutation does not change the action, but simply views it in the braided category.

Thus covariant algebras for quantum groups as in (50) become braided-covariant algebras for their transmuted (super, anyonic or generally braided) versions. For example, the adjoint action of $H$ on itself transmutes to the braided-adjoint action of $B = B(H, H)$ on itself as studied in Chapter 4. Moreover, it means that $B(H, H)$ is braided-cocommutative with respect to $\text{Ad}$ as promised there. Indeed

**Proposition 5.6** [53, 42] For $B(H_1, H)$ the $\Delta^{op}$ behaves like an opposite coproduct on all braided-representations that arise from transmutation. In particular, $B(H, H)$ is cocommutative in the sense of (59) with respect to all such braided-representations that arise from transmutation.

**Proof**  Writing the braids in (59) in terms of the universal R-matrix $R$ as explained in (44) we see that the condition for all $V$ is implied by (and essentially equivalent to) the intrinsic braided-cocommutativity formula (80) in Corollary 5.4. □

To conclude this topic, we make some general remarks about how the formulae in Theorem 5.1 were obtained. The idea is that not only do quantum groups generate braided categories as in Chapter 3, but conversely there are classical theorems that any braided category for which the objects can be strongly identified with vector spaces, is essentially of the form $\text{Rep}(H)$ for some quantum group $H$. Strongly
identified means mathematically a monoidal functor $C \to \text{VectorSpaces}$. (If we are given only a weaker identification, we may obtain only some kind of weaker quantum group as actually happens in realistic topological quantum field theories). The idea behind transmutation is that the category $C$ could be targeted wherever we like, for example to super-vector spaces or to another braided category $V$. Then the generalised reconstruction theorem would give a quantum group $\text{Aut}(C, V)$ living not in the usual category of VectorSpaces, but in our chosen $V$. Thus is how we can shift or transmute the category in which an algebraic structure lives, by targeting its category of representations elsewhere. Some of the philosophy behind this is developed in \cite{31}. Moreover, the braided-quasitriangular structure is given by the ratio of the braidings in $C$ and $V$ so that the case $\text{Aut}(C) = \text{Aut}(C, C)$ is braided-cocommutative.

For example, given $H_1 \to H$, we have such a functor $\text{Rep}(H) \to \text{Rep}(H_1)$ and can reconstruct a braided-quantum group living in $\text{Rep}(H_1)$ and in the case $H_1 = H$ it is braided-cocommutative. The detailed structure is computed in \cite{35,42} with results as above.

What about going the other way, from braided groups to quantum groups? Not all braided groups are of the type coming from transmutation, so we cannot simply apply the above formulae in reverse. Here the unification of statistics and covariance in Lemma 3.2 comes to our aid. For if $B$ is a braided-Hopf algebra in a category of the form $\text{Rep}(H)$, then to say that $H$ lives in the category is to say equivalently that $B$ is fully $H$-covariant (its product and coproduct, antipode etc all commute with the action of $H$). Now, when a structure such as a quantum group acts covariantly on some other structure (algebra, coalgebra etc), we can make a semidirect product.

**Theorem 5.7** [43, Thm 4.1] Suppose that $B$ is a braided-Hopf algebra living in a braided category of the form $\text{Rep}(H)$ with action $\rhd$ of $H$. Then there is also an induced coaction $\beta : B \to H \otimes B$ of $H$ on the coalgebra of $B$ given by $\beta(b) = R_{21} \rhd b$. Here $R$ is the universal $R$-matrix of $H$ and acts here with its first factor on $b$. The cross product algebra by $\rhd$ and cross coproduct coalgebra by $\beta$ fit together to form an ordinary Hopf algebra $\text{bos}(B) = B \triangleright \triangleleft H$.

**Proof** Perhaps the simplest proof is by direct computation using elementary Hopf algebra techniques and the axioms in \cite{13}. For example, the conversion of modules to comodules by $R$ is in \cite{30}. The original conceptual proof in \cite{33} can be sketched as follows. Because $B$ is acted upon by $H$, it becomes automatically (by transmutation) a $B(H, H)$-module algebra in the braided category. Hence we can make a braided cross-product using the formulae as shown on the top left in (72) with $B$ in the role of $C$ and $B(H, H)$ in the role of $B$ there. Because $B(H, H)$ is cocommutative with respect to $B$, the result is a braided-Hopf algebra $B \triangleright \triangleleft B(H, H)$ with the braided tensor product coalgebra structure (in analogy with the standard situation for usual group or Lie algebra cross products). We can then recognise the result as that obtained by transmutation from some ordinary quantum group $\text{bos}(B)$ characterised by $B(H, \text{bos}(B)) = B \triangleright \triangleleft B(H, H)$. □

The resulting bosonised Hopf algebra can be built on $B \otimes H$ equipped with a cross product algebra and coalgebra. The algebra is generated by $B, H$ with cross relations
and crossed coproduct
\[ \sum h_{(1)}bSh_{(2)} = h \triangleright b, \quad \Delta b = \sum b_{(1)} \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} \triangleright b_{(2)} \] (81)

where \( b \equiv (b \otimes 1 \text{ and } h \equiv (1 \otimes h) \) and \( H \) is a sub-Hopf algebra. This \( \text{bos}(B) \) is equivalent to the original \( B \) in the sense that its ordinary representations correspond to the braided-representations of \( B \).

**Corollary 5.8** Every super-group (or super-quantum group) can be bosonised to an equivalent ordinary quantum group. It consists of adjoining an element \( g \) with relations \( g^2 = 1, \ gb = (-1)^{|b|}bg \) and
\[ \Delta g = g \otimes g, \ \Delta b = \sum b_{(1)} g^{b_{(2)}} \otimes b_{(2)}, \quad Sb = g^{-|b|} S b, \quad \mathcal{R} = \mathcal{R}_{\mathbb{Z}^2} \sum \mathcal{R}^{(1)} g^{\mathcal{R}^{(2)}} \otimes \mathcal{R}^{(2)}. \] (82)

**Proof** We have seen in Proposition 3.1 that the category of super-vector spaces is of the required form, with \( H = \mathbb{Z}^2 \). Here the super-representations of the original super-(quantum)-group are in one-to-one correspondence with the usual representations of the bosonised algebra. We have written the formulae in a way that works also in the anyonic case with 2 replaced by \( n \) and \( (-1)^{|b|} \) by \( e^{2\pi i n/2} \) for the Hopf algebra structure.

This means that the theory of super-Lie algebras (and likewise for colour-Lie algebras, anyonic quantum groups etc) is in a certain sense redundant – we could have worked with their bosonized ordinary quantum groups. This is especially true in the super or colour case where there is no braiding to complicate the picture. An application to physics is in [26] where we observe that the spectrum generating algebra of the harmonic oscillator is more naturally a quantum group than the usual \( \text{osp}(1|2) \).

This completes our lightning survey of transmutation and bosonization. As we noted at the end of Chapter 4, all our constructions have left-right reflected versions (like the above but with right-modules rather than left-modules) and dual versions (with left-comodules or after reflection, right comodules). The right-comodule theory is obtained by obtaining all diagram proofs above (and the diagram-proofs in a diagrammatic form of the transmutation theorem) up-side-down. We conclude the chapter by summarising the formulae above in this dual form.

Firstly, instead of working with quasitriangular Hopf algebras \( H \) to generate the braid-statistics as in Chapter 3, we must work with dual-quasitriangular Hopf algebras \((A, \mathcal{R})\) where \( \mathcal{R} : A \otimes A \to \mathbb{C} \) is the dual-quasitriangular structure and obeys
\[ \mathcal{R}(ab \otimes c) = \sum \mathcal{R}(a \otimes c_{(1)}) \mathcal{R}(b \otimes c_{(2)}), \quad \mathcal{R}(a \otimes bc) = \sum \mathcal{R}(a_{(1)} \otimes c) \mathcal{R}(a_{(2)} \otimes b) \] (83)
\[ \sum b_{(1)} a_{(1)} \mathcal{R}(a_{(2)} \otimes b_{(2)}) = \sum \mathcal{R}(a_{(1)} \otimes b_{(1)}) a_{(2)} b_{(2)}. \] (84)

and is invertible in a certain convolution algebra. This is just the dual of (43). Now it is the category of \( A \)-comodules rather than \( H \)-modules that is braided. So we
will work in the braided category $\mathcal{C} = \text{CoRep}(A)$. The braiding $\Psi_{V,W}$ is given by applying the coactions to $V, W$ and evaluating with $\mathcal{R}$ on the $A \otimes A$ that results. As the same time one makes the usual permutation $P$ on the remaining $V \otimes W$. This is the analogue of (44). The unification of statistics and covariance in Lemma 3.2 now reads that the notions of and algebra in $\text{CoRep}(A)$, and of an $A$-comodule algebra, coincide.

The transmutation theory also has an analogue for $A \rightarrow A_1$ a Hopf algebra map where $A_1$ is dual quasitriangular. Then $A$ has the additional structure of a braided-group $B(A, A_1)$ with

$$B(A, A_1) = \begin{cases} A & \text{as a coalgebra} \\ \triangleright, \triangleleft & \text{modified product and antipode} \end{cases} \quad \text{(85)}$$

For brevity we focus on the identity map (the general case is strictly analogous), then the modified structures take the explicit form

$$a \cdot b = \sum a_{(2)} b_{(2)} \mathcal{R}((Sa_{(1)})a_{(3)} \otimes Sb_{(1)}), \quad S a = \sum S a_{(2)} \mathcal{R}((S^2 a_{(3)}) Sa_{(1)} \otimes a_{(4)}) \quad \text{(86)}$$

and live in the braided category $\text{CoRep}(A)$ by the right quantum-adjoint coaction $\beta(a) = \text{Ad}_R(a) = \sum a_{(2)} \otimes (Sa_{(1)})a_{(3)}$. This time $B(A, A)$ is braided-commutative in the sense of (59) turned up-side-down, for all comodules $V$ that come from transmutation of comodules of $A$. This reduces to an intrinsic form of commutativity dual to (80), and from what we have said so far, it can be computed explicitly as

$$b \cdot a = \sum a_{(3)} b_{(3)} \mathcal{R}(Sa_{(2)} \otimes b_{(1)}) \mathcal{R}(a_{(4)} \otimes b_{(2)}) \mathcal{R}(b_{(5)} \otimes Sa_{(1)}) \mathcal{R}(b_{(4)} \otimes a_{(5)}). \quad \text{(87)}$$

We call $A = B(A, A)$ the braided group of function algebra type associated to $A$. A direct proof that these formulae define a braided-Hopf algebra as in (55) appears in [41, Appendix].

**Proposition 5.9** If $A$ is dual to $H$ then the corresponding braided groups $B(A, A)$ and $B(H, H)$ are dual in the sense $B(A, A)^* = B(H, H)$ in (65).

**Proof** Here we view $B(A, A)$ as living in $\text{Rep}(H)$ by the quantum-coadjoint action defined by $\langle h \triangleright a, g \rangle = \sum (Sh_{(1)})g h_{(2)}$ on test-elements $g \in H$. Roughly speaking, a right $A$-comodule is the same thing (at least in the finite-dimensional case) as a left $H$-module by evaluation. Thus we have two braided-Hopf algebras $B(H, H)$ and $B(A, A)$ in $\text{Rep}(H)$. From the duality principle above they must be dual in the category. Explicitly, the duality is given by $b \in B(H, H)$ mapping to a linear functional $\langle S b, ( ) \rangle$ on $B(A, A)$, where $S$ is the usual antipode of $H$. See [47] for full details. The reader should be warned that the categorical dual as in (65) reduces in the bosonic case not to the usual dual but to the usual dual with opposite product and coproduct (which is isomorphic to the usual dual via the antipode). \(\square\)

It is these dual formulae for the braided-commutative braided-Hopf algebra $B(A, A)$ that leads to the braided matrix examples that we began with in Chapter 2. Thus if $A$ is a matrix quantum group obtained as quotient of the FRT bialgebra $A(R)$ [12] we
know that $A(R)$ and (we suppose) $A$ is dual-quasitriangular. This is true of course for the standard $R$-matrices (since $U_q(g)$ has a universal R-matrix\[11\]) but is also true for general non-standard $R$-matrices as proven in \[28\] Sec. 3. The dual-quasitriangular structure consists of

\[ R(t_1 \otimes t_2) = R_{12} \] (88)

extended according to \[83\]. Also, in Theorem 5.1 and its dual version one can transmute not only Hopf algebras but bialgebras. In particular, we can have $B(R) = B(A(R), A)$ as a braided-bialgebra in the category of $A$-comodules. The general formulae above become in this case that $B(R)$ coincides with $A(R)$ as a coalgebra (so has generators $u$ say with the matrix braided-coproduct \[32\]) and is covariant under the right quantum-adjoint coaction

\[ \beta(u^i_j) = u^m_n \otimes (St^i_m) t^a_j, \quad \text{i.e.,} \quad u \rightarrow t^{-1}u t. \] (89)

Meanwhile, the braided-commutativity relations \[87\] reduce using \[88\] to a matrix equation of the form $uu = uuR^{-1}RR$ with appropriate indices, see \[10\] \[36\]. Putting two of the $R$‘s to the left (or rearranging \[87\]) gives the braided-commutativity relations \[6\] in the compact form used in Chapter 2 \[15\]. Finally, $\Psi$ computed from \[89\] and \[88\] gives the braid-statistics \[7\] used there.

This is how the braided-matrices in Chapter 2 were first introduced. We obtain not only the results in Chapter 2 but also that they are related to usual quantum matrices by transmutation. The formula for the modified product comes out from \[86\] as \[45\]

\[ u^i_j = t^i_j \]

\[ u^i_j u^k_l = t^a_d t^i_d R^{i c}_{a} R^{k c}_{d} \]

\[ u^i_j u^k_l u^m_n = t^b_s t^a_d t^c_z n R^{i d}_{a} p R^{a w}_{d} y \tilde{R}^{b v}_{c} \tilde{R}^{k v}_{j} p R^{q y}_{s} \tilde{R}^{c v}_{j} \] (90)

etc. Here the products on the left are in $B(R)$ and are related by transmutation to the products on the right which are in $A(R)$. If we write some or all of the $R$-matrices over to the left hand side we have equally well the compact matrix form \[45\],

\[ u = t \]

\[ R_{12}^{-1} u_1 R_{12} u_2 = t_1 t_2 \]

\[ R_{23}^{-1} R_{13}^{-1} R_{12}^{-1} u_1 R_{12} u_2 R_{13} R_{23} u_3 = t_1 t_2 t_3 \] (91)

etc. This is just a rearrangement of \[90\] or our universal formula \[86\]. For the transmuted product of multiple strings the universal formula from \[86\] involves a kind of partition function made from products of $R$ to transmute the bosonic $A(R)$ to the braided $B(R)$ \[14\].

Because of these transmutation formulae, braided groups are a useful tool (a kind of conceptual ‘co-ordinates’) for doing ordinary quantum group computations. On the braided side they can be much simpler or more like the group case because of the braided-commutativity of $B(R)$ (which make it behave more like a classical group as we have seen in Chapter 4), after which formulae can be converted back by applying the transmutation formulae \[90\]-(\[91\]) in the reverse point of view. This was the reason given for introducing the transmutation process in \[10\] \[11\].
6 Applications to Quantum Groups

In this chapter we mention a few selected applications of the above theory to quantum groups. The first makes use of the unification of statistics and covariance in Chapter 3 to study quantum-group covariant systems. The other two make use of the transmutation theory in the last chapter. There are many more applications of braided groups as mentioned in the introduction.

6.1 Quantum-Covariant Spin Chains and Exchange Algebras

The braided-tensor product construction in Chapter 4 in the form of Corollary 4.2 has many applications in physics and $q$-deformed physics, for it tells us precisely how to combine two quantum group covariant systems in a quantum-group covariant way. One has to use the braiding $\Psi$ given by the action of the universal R-matrix of the quantum group.

In particular, given one quantum-group covariant algebra $C$ we can repeatedly take the braided-tensor product $C \otimes^N$ for the corresponding $N$-body system. Here we regard each $C$ as the quantum algebra of observables of a single particle or site and $C \otimes^N$ as the system for a chain of $N$ such particles with a natural interaction or statistics designed to maintain quantum-group covariance. Building up quantum-spin chains in this way ensures that they are manifestly quantum group covariant. The importance of quantum covariance for quantum spin chains is well-known in physics, see for example [20].

As an example, one can take for $C$ the braided-Heisenberg group corresponding to the quantum Heisenberg group $U_q(h)$ of [9]. As an algebra it is basically the usual harmonic oscillator with $\hbar$ viewed as a central operator rather than a constant, and has braid statistics $\Psi$ which tend to trivial bosonic statistics as $q \to 1$. This model has been worked out in complete detail by W.K. Baskerville in [4]. At first sight it might seem that the system of $N$ braided-harmonic oscillators with braid-statistics between them, would become horribly tangled. However, it is found instead that the system is in fact non-trivially isomorphic as an algebra to the unbraided system of $N$ commuting copies of $U_q(h)$. This allows constructions on the braided-side, which are manifestly covariant, to be mapped over to the unbraided side. In particular, $U_q(h)$ acts covariantly on the system and the action of one of its generators (the number operator) recovers the usual free particle evolution. One can also add covariant interaction terms to the Hamiltonian using these methods. We refer to [4] for all these results.

The general principle here is that in general $B(H, H) \otimes^n \cong H \otimes^n$ as an algebra, using the isomorphism that connects $\Delta$ to $\Delta$ or $\rho$ to $R$ in the transmutation formulae (75)-(76). Recall that the universal R-matrix is needed here in a non-trivial way. One can apply it also, for example to $BM_q(1, 1) \otimes^N$ in (11)-(13). This is a system with bose-fermi like statistics (and in the limit $q \to 1$ becomes an N-fold product of supermatrices). The right hand side on the other hand is the usual tensor product of a bosonic quantum group associated to the relevant $R$-matrix.
We can also take N-fold or infinite braided-tensor products such as \( V^*(R')^{\otimes^\infty} \) of the braided-(co)vectors \( V^*(R') \) and \( V(R') \) (which do not come from transmutation). As well as being thought of a quantum-spin chains, we can also think of these in terms of braided-geometry. There is a braided vector space at each site, and the independent sites have braid-statistics. Thus they should be thought of precisely as braided-vector-fields (or sections of a braided-vector bundle) in one space dimension (in a lattice approximation). For example, the structure of \( V^*(R')^{\otimes^\infty} \) comes out as follows. Let \( x_i(m) \) where \( m \in \mathbb{N} \) denote the generators of the copy of \( V^*(R') \) in the \( m \)’th position in the infinite (or finite) braided tensor product and 1 elsewhere. If \( m < n \) then \( x_i(m)x_j(n) = \cdots \otimes x_i \otimes \cdots x_j \cdots \) while if \( m > n \) we must use \( \Psi \) to take \( \cdots \otimes x_i \otimes \cdots \) past \( \cdots \otimes x_j \otimes \cdots \) when we multiply as in \((72)\). Here the braiding is \( \Psi(x_i \otimes x_j) = x_i \otimes x_k R_{ij}^{kl} \) as in \((25)\) so we obtain \( \cdots \otimes x_i \otimes \cdots x_k \otimes \cdots R_{ij}^{kl} = x_i(n)x_k(m)R_{ij}^{kl} \). Thus the braided tensor product is a kind of ‘ordered product’ of the \( x_i \). From this we can write \( V^*(R')^{\otimes^\infty} \) as generated by \( \{x_i(m)\} \) with relations

\[
x_1(m)x_2(n) = x_2(n)x_1(m)R(n - m)
\]  
where

\[
R(n - m) = \begin{cases} 
R & \text{if } m > n \\
R' & \text{if } m = n \\
R_{21}^{-1} & \text{if } m < n
\end{cases}
\]

obeys the parametrized Yang-Baxter equations

\[
R_{12}(m - n)R_{13}(m)R_{23}(n) = R_{23}(n)R_{13}(m)R_{12}(m - n)
\]

with discrete spectral parameter in the sense \( m, n \in \mathbb{N} \), cf[27].

These are relations of precisely the form that are noted for the exchange algebra in conformal field theory and in 2D quantum gravity. See for example[14] and later works. In this context there are fields \( \xi(\sigma) \) obeying \( \xi_1(\sigma)\xi_2(\sigma') = \xi_2(\sigma')\xi_2(\sigma)R(\sigma' - \sigma) \) where \( R(\sigma' - \sigma) = \begin{cases} 
R & \text{if } \sigma > \sigma' \\
R_{12}^{-1} & \text{if } \sigma < \sigma'
\end{cases} \). Here \( \sigma, \sigma' \in [0,1] \) are continuous and one does not worry about the case of equality. We see then that this exchange algebra has the mathematical structure of a continuous braided-tensor product of braided-covectors or braided-vectors. This is striking for several reasons. Firstly, we see at once, by construction, that the fields \( \xi_i \) generate a quantum-group-covariant algebra (because they are built up using the braided tensor product). The action or coaction is pointwise. Secondly, because \( V^*(R') \) has a braided Hopf algebra structure corresponding to braided-covector addition as explained in Chapter 2, we have in the braided tensor product or (by iterating) in any braided tensor power, a realization \( \Delta^N : V^*(R') \rightarrow V^*(R')^{\otimes^N} \). In the present case this comes out from \((33)\) as the realization

\[
x_i = \sum_m x_i(m), \quad \text{i.e.,} \quad \xi_i = \int d\sigma \xi_i(\sigma).
\]

Moreover, there are all sorts of other braided-linear algebra constructions that one can make. For example, the pointwise-version of \((31)\) means that using \( \xi \) and its conjugate fields one should be able to realise rank-one braided matrices. As we shall
see in Chapter 7.1 this is not a quantum group but, in the $SL_q(2)$ case, the degenerate Sklyanin algebra. Indeed, because the relations of $V^*(R')$ are some kind of (braided)-commutativity relation, what we gain by our approach is a picture of this version of the exchange algebra as describing a classical but braid-statistical (like fermionic) vector-field, and yet equivalent to its usual picture as describing a quantum field. This is like the transmutation principle above and is a step to a formulation of the geometrical structure of such quantum field theories.

Note that one can do the same thing with $V^*(\lambda R)$ where $\lambda$ is a suitable constant and this is the usual Zamolodchikov algebra. This is also a covariant algebra which can be tensor-producted (this was the actual point of view in [48]) but because $V^*(\lambda R)$ is not in general a braided-Hopf algebra, we do not have the full picture as above. It appears to coincide only in the Hecke case where $R' = \lambda R$. Thus it appears that the right generalization of the Zamolodchikov and exchange algebras to general non-Hecke $R$-matrices (such as the $SO_q(1,3)$ $R$-matrix) is with $R'$ and not $\lambda R$ if we wish to have a full picture.

6.2 Self-Duality and Factorizable Quantum Groups

In Chapter 5 we computed the braided groups and braided-matrices arising from transmutation of quantum function algebras such as $A(R)$. We can also compute the braided groups arising from the standard quantum groups $U_q(g)$ in Theorem 5.1. This was done in [10] and the result is as follows. For $U_q(g)$ we take the generators in ‘FRT’ form[12] as $I = \{l_{\pm i}j\}$, and write $L = 1^+ S L^-$. Then the braided-coproduct for $B(U_q(g), U_q(g))$ in Theorem 5.1 comes out as

$$\Delta L = L \otimes L. \quad (96)$$

Although the generators $L$ for $U_q(g)$ have been useful in the past, this [46] was not discussed or used because it does not form an ordinary Hopf algebra (the ordinary coproduct is different). One needs the theory of braided-Hopf algebras to appreciate it.

Next, a quantum group $H$ is called factorizable[53] if $Q = R_{21} R_{12}$ is non-degenerate in a certain sense. We call $Q$ the quantum Killing form. The condition is that the map $Q : A \rightarrow H$ given by $a \rightarrow (a \otimes \text{id})(Q)$ is a linear isomorphism, where $A$ is the dual of $H$. The usual quantum groups $U_q(g)$ with their suitable quantum function algebras $G_q$ as dual are factorizable (at least up to suitable formulation of generators). So we have two quantum groups related by $Q$ as linear spaces. Each has a braided version as in Proposition 5.9. Remarkably, one can show by explicit computation that [10]

$$Q : B(A, A) \rightarrow B(H, H) \quad (97)$$

as a homomorphism of braided-Hopf algebras, and an isomorphism in the factorizable case. See also [23]. That $Q$ is a morphism says that it is quantum-Ad invariant, which is indeed a property of the quantum Killing form, in analogy with the usual (inverse) Killing form in $g \otimes g$

$$K^{-1} : \ g^* \rightarrow g. \quad (98)$$
What the braided theory tells us is much more than this semiclassical limit: it tells us that the homomorphism or isomorphism is not only as linear spaces (as in the semiclassical case) but as entire braided-groups, mapping their algebras and braided-coproducts etc on onto the other. This, the semiclassical notion of semisimplicity of $g$ is a remnant of something deeper: the self-duality of the braided-groups $B(A, A) \cong B(H, H)$. Although conceptually very different (one is of function algebra type, and the other of enveloping algebra type) they become the same in the factorizable case. For $U_q(g)$ one has $Q = 1 + 2\hbar K^{-1} + O(\hbar^2)$. Here $\hbar$ is a deformation parameter rather than physical Planck’s constant.

In the case of $U_q(g)$ we have seen that $B(A, A)$ is a quotient of $B(R)$ (by braided determinants etc) and has generators $u$ as in Chapter 2. The map $Q$ then comes out as

$$Q(u) = \langle R_{21} R_{12}, t \otimes \text{id} \rangle = I^+ S I^- = L.$$  \hspace{1cm} (99)

This then explains facts about $U_q(g)$ that have been found in other ways. Firstly, it is well-known that some of the relations of $U_q(g)$ take the form $R_{21} L_1 R_{12} = L_2 R_2 L_1 R_{12}$ and we see from our point of view that this is a consequence of the self-duality expressed in (99). Also, the bosonic elements $c_k$ from (13) recover the Casimirs of $U_q(g)$ first found in [12] as $Q(c_k)$. Note that (97) and (99) are a purely quantum phenomenon (with the above semiclassical remnant) in the sense that the map $Q$ is trivial (or $Q^{-1}$ is rather singular) at $q = 1$. Thus we are recovering results about $U_q(g)$ by working strictly at generic $q \neq 1$.

All this has many concrete applications. We mention here only one, where we showed that Drinfeld’s quantum double $D(U_q(g))$ (of some interest in physics for various reasons) is in fact isomorphic to a semidirect product $U_q(g) \rtimes U_q(g)$ by the quantum adjoint action [46]. In this new form of the quantum double, the generators are $l^\pm$ and $m^\pm$ say (two copies of $U_q(g)$) with cross relations [46, Corol 4.3]

$$R_{12} l^+_1 M_1 = M_1 R_{12} l^+_1, \quad R_{21}^{-1} l^- M_1 = M_1 R_{21}^{-1} l^-_1$$  \hspace{1cm} (100)

where $M = m^+ S m^-$. The coalgebra is also a semidirect coproduct. Mathematically, this form of the quantum double is obtained as nothing other than an example of the bosonization Theorem 5.7 applied to $B(U_q(g), U_q(g))$. The quantum adjoint action here can also be computed as [46, Prop 2.4]

$$L^d \triangleright L^b = \bar{R}^{a}_{\ j} \ k \ R^{-1 b}_{\ i} L^c_{\ a} L^d_{\ e}$$  \hspace{1cm} (101)

where $Q = R_{21} R_{12}$. This can be used to write the action of $L$ on $M$ for the cross relations in $U_q(g) \rtimes U_q(g)$ (in place of (100)), as well as to define some kind of quantum or braided Lie bracket [46]. We refer to [46] for the details of these results.

### 6.3 Action-Angle Variables for Quantum Groups

If we regard $A(R)$ and other quantum function algebras as analogous to some kind of quantum algebra of observables (this is not their actual role in physics in QISM), then it is natural to look for a complete commuting set of observables. The braided theory above does precisely this. Let $R$ be a matrix obeying the QYBE. It need not
be standard but should be generic enough that the second inverse \( \tilde{R} \) exists. Then we find
\[
\{ \alpha_k \in A(R) \}, \quad [\alpha_k, \alpha_l] = 0, \quad \text{Ad}_R(\alpha_k) = \alpha_k \otimes 1.
\]
(102)

In the standard case there are as many algebraically independent \( \alpha_k \) (roughly speaking) as the rank of the Lie algebra, so in some sense these \( \alpha_k \) are a ‘complete’ set.

They can perhaps be called ‘quantum angle variables’ in honour of the role of action-angle variables in the theory of classical inverse scattering; one may hope that they could prove correspondingly useful in quantum inverse scattering. So far, they have been used in [7] to define a ‘ring’ of bicovariant differential calculi for any matrix quantum group. Here any polynomial function in the \( \alpha_k \) defines some Ad-invariant element, and every Ad-invariant element defines a differential calculus[7]. This function is a new kind of ‘field’ in physics, one that governs the choice of differential structure. It is interesting to ask if there is some kind of action-principle for such fields leading to the usual differential structures assumed in physics as extrema.

This result follows very simply from the transmutation theory in Chapter 5. Recall that this is some kind of non-linear transformation that maps

\[
\text{qu. non – commutativity of } A(R) \rightarrow \left\{ \begin{array}{l}
\text{braided – commutativity of } B(R) \\
\text{& statistical non – commutativity}
\end{array} \right. \]

(103)

Under this mapping, the bosonic and central elements \( c_k = \text{Tr} \varphi^k \) in (103) are the image of some elements \( \alpha_k \) in \( A(R) \). From (100) one finds at once[4]
\[
\begin{align*}
\alpha_1 &= \tilde{\varphi}_{ij}^j t_i^j \\
\alpha_2 &= \tilde{\varphi}_{ij}^j R_k^l R_m^k \varphi_{lm}^n t_i^j t_m^n \\
\alpha_3 &= \tilde{\varphi}_{ij}^j R_k^l R_m^k \varphi_{lm}^n \varphi_{pq}^p R_{qs}^q R_{tv}^r \varphi_{vw}^v t_i^j t_m^n t_p^q t_s^r \\
\end{align*}
\]
(104)

etc. One can compute all the \( \alpha_k \) in a similar way from (100) or (86). Here \( \alpha_1 \) is the quantum trace (which is well-known to be Ad-invariant) but the higher \( \alpha_k \) are also useful and can be used, for example to obtain quantum determinants and other expressions.

The proof that these \( \alpha_k \) obey (102) is then easy. That the \( c_k \) are bosonic comes from the fact that they are Ad-invariant under the quantum-adjoint coaction. Here we identify the original dual quantum group \( A(R) \) with the braided group \( B(R) \) as linear spaces as in (85). The product is modified, but from (86) we see that if \( \text{Ad}_R(a) = a \otimes 1 \) (quantum-Ad invariant) then \( a \cdot b = ab \). Hence \( \alpha_k = c_k \) as elements of a linear space and \( \alpha_k \alpha_l = c_k c_l = c_l c_k = \alpha_l \alpha_k \). Here we used that the \( c_k \) are central in \( B(R) \). The \( \alpha_k \) themselves are not central but we see that they mutually commute as a remnant under the inverse of (103) of the fact that \( c_k \) are both bosonic and central.

7 Applications Beyond Quantum Groups

In this final chapter we announce from the preprints [46][49] a few selected applications of the above theory to problems where quantum-groups have been found to fail. When one begins to systematically \( q \)-deform everything in physics one soon finds...
that quantum groups are not enough, and that more naturally, everything acquires braid statistics. If the $q$ is physical then these braid statistics are physical. If the $q$ is only a regularization parameter before renormalization then the braid-statistics are an artifact of the regularization procedure. However, they can still have physical consequences after renormalisation (which can now be done elegantly using braided or $q$-analysis) and setting $q \to 1$. Roughly speaking the role of $q$ here is like a systematic variant of ‘point splitting’ (see (52) and (111)) with a choice of braid crossing or inverse-braid crossing reflecting topologically distinct ways to set $q \to 1$. This approach to regularization could be especially relevant to quantum gravity where some formulations already make use of knots and loops.

### 7.1 The Degenerate Sklyanin Algebra

An important algebra in quantum inverse scattering is the Sklyanin algebra. It is the homomorphic image of a bialgebra associated to the 8-vertex solution of the parametrized Yang-Baxter equations (and hence its representations lead to ones of this bialgebra). It has 3-parameters $J_{12}, J_{23}, J_{31}$ obeying one equation $J_{12} + J_{23} + J_{31} + J_{12}J_{23}J_{31} = 0$, and an elegant structure in terms of four generators $S_0, S_1, S_2, S_3$. On the other hand, in spite of many efforts, there has not been found any bialgebra structure on the Sklyanin algebra itself. Even in the degenerate case where one of the parameters vanishes, which is associated with the 6-vertex solution and the quantum group $U_q(sl_2)$, the degenerate Sklyanin algebra itself does not appear to be a usual bialgebra.

We have shown in [46] that the degenerate Sklyanin algebra, in a form with suitable generators, is a braided-bialgebra. It turns out to be isomorphic to the braided matrices $BM_q(2)$ in (31)-(44). This comes out as the identification

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = \begin{pmatrix}
K_+^2 & q^{-\frac{1}{2}}(q - q^{-1})K_+Y_- \\
q^{-\frac{1}{2}}(q - q^{-1})Y_+K_+ & K_-^2 + q^{-1}(q - q^{-1})^2Y_+Y_-
\end{pmatrix}
\]

(105)

where the four generators of the degenerate Sklyanin algebra are $Y_{\pm} = \frac{1}{2}\sqrt{1 - t^2}(S_1 \pm tS_2)$ and $K_{\pm} = S_0 \pm tS_3$. Here $t = \sqrt{J_{23}}$ (a fixed square root) is the remaining free parameter in the degenerate case and $q = \frac{1 + t}{1 - t}$. In making this identification we allow $a, d - ca^{-1}b$ of $BM_q(2)$ to be invertible and have square roots.

What this means is that the degenerate Sklyanin algebra lives in some braided category in the sense explained in Chapter 3. In our case the relevant category is the braided category of $U_q(sl_2)$-modules. If we denote by $X_{\pm}, q^{\pm\frac{1}{2}}$ the usual generators of $U_q(sl_2)$ in Jimbo’s conventions, then the action comes out explicitly as

\[
\begin{align*}
q^{\frac{1}{2}}b & K_+ = K_+ \\
X_+bK_+ = (1 - q^{\pm1})Y_+ & X_+bK_- = (q^{\pm1} - 1)K_+^{-1}Y_+K_- \\
X_+bY_- = (1 - q^{-1})Y_+K_+^{-1} & X_+bY_+ = K_+^{-1}(Y_+Y_- - qY_+Y_-) \\
X_-bY_- = (1 - q)Y_+K_+^{-1} & X_-bY_+ = K_+^{-1}(Y_+Y_- - q^{-1}Y_+Y_-).
\end{align*}
\]

(106)

The braid statistics and braided-coproduct for the degenerate Sklyanin algebra can likewise be worked out explicitly from (110) (or from (114)) and (32) respectively.
results on the individual generators themselves rather than products involve infinite power-series (because the universal R-matrix for $U_q(sl_2)$ is an infinite power-series).
An alternative is to realise that, more precisely, it is the subalgebra of the degenerate Sklyanin algebra generated by $a, b, c, d$ in (105) that is $BM_q(2)$ and has finite braiding and coproduct.

Either way (working with formal power-series or with the subalgebra) we have a braided-bialgebra. Hence, for example, we can make braided-tensor products of braided-representations as in (111). Moreover, we have the braided-cocommutativity properties as in (129) (for suitable representations) and (124) for their tensor product. It might be thought that this braided-bialgebra is not much different from $U_q(sl_2)$ (because it is known that adding one more relation to the degenerate Sklyanin algebra gives the algebra of $U_q(sl_2)$). But this is not so. For example, in our braided-matrix description this additional relation turns out to be $BDET(u) = 1$ as in (116) and gives $U_q(sl_2)$ via the self-duality (129) [46]. By contrast, the realization (111) of $BM_q(2)$ in terms of braided-vectors and braided-covectors, and hence the corresponding realization of the degenerate Sklyanin algebra, has $BDET(u) = 0$. So, such realizations of the degenerate Sklyanin algebra, and associated representations, are far removed from the usual realizations and representations of $U_q(sl_2)$. We refer to [46] for the details.

7.2 $q$-Homogeneous Spaces and $q$-Poincaré Group

Among the several general results about braided groups in [46] we note here one that says that quantum-homogeneous spaces are, in general, braided groups.

Recall that if we have Lie algebras $h \subset g$ in a strong sense then $g = m > h$ is a semidirect sum Lie algebra. This happens for example, for the Poincaré group Lie algebra $p \supset so(1,3)$. Likewise, if we have an inclusion of ordinary Hopf algebras $H \subset H_1$ which is a strong one in the sense that there is a Hopf algebra map $H_1 \to H$ covering the inclusion then a theorem in [46, Prop. A.2] asserts that $H_1 = B > H$ as a semidirect product and coproduct, where $B$ is a braided-Hopf algebra. It lives in the braided category of $D(H)$-modules where $D(H)$ is Drinfeld’s quantum double construction. If $H$ is a strict quantum group (with universal R-matrix) then the braided-category of $H$-modules is contained in that of $D(H)$-modules, and $B$ often lives in this smaller category. Either way, we see that quantum groups are not enough – even in nice cases the homogeneous spaces associated to inclusions of quantum groups force us to braided groups.

On these very general grounds, we know for example that if we can find some $q$-Poincaré group $U_q(p)$ strongly containing the known $q$-Lorentz group $U_q(so(1,3))$ dual to $SO_q(1,3)$ in [3], then $U_q(p) \cong B > sU_q(so(1,3))$ for some braided-group $B$ in the role of Minkowski space.

We can also use the theorem mentioned above for quantum function algebras rather than quantum enveloping algebras. Now the usual inclusion of the Lorentz group in the Poincaré group becomes a projection $P_q \to SO_q(1,3)$ where $P_q$ is our supposed quantum function algebra for our quantum Poincaré group. If we suppose that this is strong in the sense that it covers an inclusion $SO_q(1,3) \subset P_q$ of Hopf
algebras, then we again have on general grounds that \( P_q \cong B \rtimes SO_q(1,3) \). Here \( B \) is a braided group of ‘functions’ on an analogue of Minkowski space.

Armed with this general picture, we have been motivated in [49] to construct \( P_q \) by beginning with a suitable candidate for \( B \), since we know from the above that it should be a braided-group version of \( q \)-Minkowski space. Thus, we take \( B = V^\ast(R') \) where \( R \) is the \( q \)-Lorentz group \( R \)-matrix (such as the one for \( SO_q(1,3) \) in [8]). We have seen in Chapter 2 that this has a braided-covector addition law (21), so we can add \( q \)-Minkowski position vectors provided we treat them with braid statistics. We denote \( q \)-Minkowski space with this braided addition law by \( R_{1,3}^q \).

The first step is to identify more precisely the quantum group under which \( V^\ast(R') \) and \( V(R') \) are covariant. The simplest possibility would be to take for this the bialgebra \( A(R) \), which becomes dual-quasitriangular as explained in (88). Its category of comodules is braided and it indeed coacts from the right on \( V^\ast(R') \) by \( x \rightarrow xt \). The induced braiding is

\[
\Psi(x_1 \otimes x_2) = x_2 \otimes x_1 R(t_1 \otimes t_2) = x_2 \otimes x_1 R_{12}
\]

as used in (21) and Chapter 6.1.

While this works for \( A(R) \), this \( R \) defined by (88) needs \( R \) to be correctly normalised if it is to descend to the relevant quantum group \( A \) (such as \( SO_q(1,3) \)). On the other hand, the normalization of \( R \) in (21) is fixed by (20) and is not usually this quantum group normalization. This forces us to extend \( A \) as follows[49]. We normalise \( R \) according to (20) and let \( \lambda R \) be the quantum group normalization. The extension \( \tilde{A} \) is then given by adjoining to \( A \) a single invertible and commuting generator \( g \), with coproduct \( \Delta g = g \otimes g \) and dual-quasitriangular structure \( R(g \otimes g) = \lambda^{-1} \).

This \( \tilde{A} \) also coacts from the right on \( V^\ast(R') \) and \( V(R') \) in Chapter 2, by

\[
x \rightarrow xt g, \quad v \rightarrow g^{-1}t^{-1}v
\]

and gives the correct braiding \( \Psi \). In this way, these braided-vectors and covectors live in the braided category of \( \tilde{A} \)-comodules. We are using here (in a dual form) the unification of statistics and covariance in Chapter 3. In the present setting it means that \( V^\ast(R') \rightarrow V^\ast(R') \otimes \tilde{A} \) etc are comodule algebras (they are algebra homomorphisms, so the algebra \( V^\ast(R') \) is realised in \( V^\ast(R') \otimes \tilde{A} \) by (108) and similarly for \( V(R') \)). This means that

\[
x_1 x_2 \cdots x_n \rightarrow x_1 x_2 \cdots x_n t_1 \cdots t_n g^n
\]

so that we see that \( g \) measures the degree of a polynomial in the \( x_i \). It coacts according to the physical scaling dimension.

Not only is the algebra of \( V^\ast(R') \) respected by the coaction of \( \tilde{A} \), but so are all the other braided-group maps, such as the braided-coproduct corresponding to covector addition. This means that we can form a right-handed semidirect product and coproduct algebra \( A \ltimes V^\ast(R') \) by a dual version of the bosonization Theorem 5.7. For example, with an \( SO_q(1,3) \) \( R \)-matrix we have \( P_q = SO_q(1,3) \rtimes \mathbb{R}^{1,3}_q \) as an ordinary Hopf algebra. Explicitly, it consists of the generators of the \( q \)-Minkowski space \( \mathbb{R}^{1,3}_q \).
which we now denote by \( \{p_i\} \) (since we think of them as momentum) and the generators \( g, t \) of the extended quantum rotation group, and relations and coproducts:

\[
pg = \lambda^{-1} gp, \quad p_1 t_2 = \lambda t_2 p_1 R_{12}, \quad \Delta p = p \otimes tg + 1 \otimes p, \quad \Delta t = t \otimes t, \quad \Delta g = g \otimes g.
\]

Relations of this form were first proposed as a kind of \( q \)-Poincaré group in [55], but we see now their structure as a semidirect product of a momentum braided-covector space by quantum rotations. Its dual is a quantum enveloping algebra also of the semidirect product form \( U_q(p) = B \circledast \tilde{H} \).

Finally, because all our constructions are covariant, it is not hard to see that this \( q \)-Poincaré group (110) coacts from the right on another copy of \( \mathbb{R}^{1,3} \), which we think of as \( q \)-spacetime, by

\[
x \to xtg + p
\]

The \( q \)-Minkowski spacetime is fully covariant under this (the right hand side obeys the same algebra etc). We defer to [49] for the details.

### 7.3 Braided-Differential Calculus

As a final application of braided-groups, we mention some results about braided-differential calculus. The idea is to develop these strictly in analogy with super-differentials. Thus, they obey what we call a braided Leibniz rule and generate braided-translations as in (21).

We begin with the simplest case, the braided line (114). This has a single generator \( x \) with no relations. So the algebra \( B \) is just polynomials in one variable. To this, we add braid statistics and braided-coproduct

\[
\Psi(x \otimes x) = q^2 x \otimes x, \quad \Delta x = x \otimes 1 + 1 \otimes x
\]

In the language of Chapter 2 this means that an independent copy has braid statistics \( x' x = q^2 x x' \) and in this braided tensor product we realise \( x + x' \).

It is very natural to differentiate this braided-addition law to obtain the braided-differential

\[
\partial_q f(x) = (f(x + \epsilon') - f(x))\epsilon'|_{\epsilon' = 0}
\]

where we mean here the linear part in \( \epsilon' \) of \( f(x + \epsilon') - f(x) \) and where \( \epsilon' \) has the braid-statistics relative to \( x \). Using the \( q \)-binomial theorem one finds

\[
\partial_q x^m = [m]_q x^{m-1}, \quad [m]_q = \frac{q^{2m} - 1}{q^2 - 1}, \quad \partial_q (x^m x^n) = (\partial_q x^m) x^n + q^{2m} x^m (\partial_q x^n).
\]

We see that the generator of braided-addition on the braided line is the usual \( q \)-derivative well-known in \( q \)-analysis. Its well-known skew-Leibniz rule is seen now to be nothing other than the rule for a braided-differential where \( \partial_q \) gives a factor of \( q^{2m} \) as it goes past \( x^m \). This factor \( q^2 \) plays precisely the role of \( \pm 1 \) in the super case.

This differentiates the right regular coaction \( \Delta x = x \otimes 1 + 1 \otimes \epsilon \) and is in fact more properly viewed as a derivative acting from the right. Another option is to use
the left regular coaction $\Delta x = \epsilon \otimes 1 + 1 \otimes x$ with the same results in this simplest case.

At a primitive root of unity we are in the anyonic situation\[38\]. There we have defined the enveloping algebra of the one-dimensional anyonic Lie algebra $U_n(C)$ with one generator $\xi$ and

$$\xi^n = 0, \quad \Psi(\xi \otimes \xi) = e^{\frac{2\pi i}{n}} \xi \otimes \xi, \quad \Delta \xi = \xi \otimes 1 + 1 \otimes \xi. \quad (115)$$

This forms a braided-Hopf algebra (an anyonic one) and one can check that $(\xi + \xi')^n = 0$ if we remember the statistics $\xi'\xi = c^{\frac{2\pi i}{n}} \xi \xi'$ (the anyonic degree is $|\xi| = 1$). Differentiating the anyonic addition law gives $\partial_q$ similar to the formula above and obeying a braided-Leibniz rule with $\partial_q$ of anyonic degree -1. This is

$$\partial_q(ab) = (\partial_qa)b + \Psi^{-1}(\partial_q \otimes a)b = (\partial_qa)b + e^{\frac{2\pi i|a|}{n}}a(\partial_qb) \quad (116)$$

where the braiding is computed from $\langle 17 \rangle$ with $|\partial_q| = -1$. Note that we must take the degree of $\partial_q$ to be -1 so that the map $\partial_q \otimes a \to \partial_qa$ is degree-preserving (a morphism in the anyonic category). This then forced us to use $\Psi^{-1}$ rather than $\Psi$.

Mathematically, we can formalise this braided Leibniz rule as the statement that one copy of $U_n(C)$ (regarded as the anyonic enveloping algebra of the one-dimensional Lie algebra) and equipped with the opposite coproduct as in Lemma 4.6, acts on another copy of itself (regarded as an anyonic line co-ordinate algebra) by $\xi \to \partial_q$. The anyonic line becomes a left braided $U_n(C)^{\text{cop}}$-module algebra in the sense of $\langle 17 \rangle$ but living in $C$ (with the opposite braid crossing to the one shown in $\langle 57 \rangle$). Since 1 is always bosonic, the opposite coproduct $\Psi^{-1} \circ \Delta$ looks just the same as in $\langle 113 \rangle$ but now extended as a braided-Hopf algebra in $C$ (so using $\Psi^{-1}$ to compute the value on products). Putting this into $\langle 77 \rangle$ with the opposite braiding means that $\partial_q$ obeys the braided-Leibniz rule as described.

The generalization of these constructions to the higher-dimensional case is straightforward using the braided covectors and braided vectors in Chapter 2. We differentiate their braided-addition law $\langle 25 \rangle$ from the left. The result is

$$\partial^i x_j = \delta^i_j, \quad \partial^i(x_j x_k) = (\partial^i x_j) x_k + x_a R^a \cdot x_b \partial^b x_k, \quad \text{etc.} \quad (117)$$

One can check directly that the $\partial^i$ themselves obey the algebra relations of $V(R')$ and that we have a well-defined action $V(R')^{\text{cop}} \otimes V^*(R') \to V^*(R')$ making $V^*(R')$ into a left braided-module algebra in $C$, as in $\langle 57 \rangle$ but with the opposite braiding. Thus the general Leibniz rule is

$$\partial^i(ab) = (\partial^i a)b + \cdot \Psi^{-1}(\partial^i \otimes a)b \quad (118)$$

where the statistics between $\partial^i$ and $x_j$ is as between $v^i$ and $x_j$ in $\langle 29 \rangle$. Here the coproduct on $V(R')^{\text{cop}}$ is the linear one as usual but extended to products with the opposite braid statistics (this simply means with $R^{-1}$ in place of $R$ in $\langle 21 \rangle$). Iterating $\langle 118 \rangle$ along the lines of $\langle 117 \rangle$ leads to

$$\partial^i(x_1 \cdots x_n) = e^i_1 x_2 \cdots x_n(1 + (PR)_{12} + (PR)_{12}(PR)_{23} + \cdots + (PR)_{12} \cdots (PR)_{n-1n}) \quad (119)$$
where \( e^i = \{ \delta^i_j \} \) is a basis covector.

Thus, the vector fields on a braided-covector space (such as on \( q \)-Minkowski space in the last section) obtained from differentiating the braided addition law, obey the relations of the braided-vectors \( V(R^c) \). This completes the braided-geometrical picture of Chapter 7.2. There are several variants of these formulae, including ones where the \( \partial^k \) act from the right.

These constructions can also be understood as variants of some of our diagrammatic results in Chapter 4. For any braided-Hopf algebra we can use Lemma 4.13 applied to \( \Delta : B \to B \otimes B \) (the left regular coaction) to have a left action of \( B^{\text{coop}} \), by evaluating the left hand output of \( \Delta \) as proven in (64) and working in the category \( \tilde{C} \) with reversed braid crossings. The above construction is a variant of something like this and lives in the category with reversed \( \Psi \). Recall that we could also convert the diagrammatic action over to a right action of \( B^* \) in our original category using the braided-antipode, with resulting right action (71).

The abstract diagrammatic constructions are especially useful when we have good information about \( B^* \). Thus we can apply them to braided matrices and braided groups to obtain the corresponding left-invariant and right-invariant braided-vector fields. For example, the canonical right-action of \( B^* \) on \( B \) in (71) in the case when \( B = B(A, A) \) (the braided group of function algebra type in Chapter 5) comes out from Proposition 5.9 and (71) as

\[
a \triangleright b = \sum < R^{(2)} b, a(1) > a(2) < R^{(1)}, a(3) >, \quad a \in B(A, A), \ b \in B(H, H). \tag{120}
\]

This makes \( B(A, A) \) into a right braided \( B(H, H) \)-module algebra in our category. For example, on \( BSL_q(2) \) as in Chapter 2 there is an action of the braided-group associated to \( U_q(sl_2) \) (which has the same algebra but the matrix braided-coproduct as explained in Chapter 6.2). Similarly for any \( U_q(g) \) in FRT form. The action computed from (120) comes out on generators as

\[
u^i_j \triangleright k \partial^l = u^a_b \tilde{R}^k_m n_j Q^n p_a R^b m_p l \tag{121}
\]

and must now obey the right braided-module algebra axioms as in the proof of Lemma 4.14. Writing the action \( \triangleright L^i_j = \triangleright (1^+ S 1^-)^i_j = \partial^i_j \), this comes out as

\[
(ab) \partial^i_j = a \cdot \Psi(b \otimes \partial_k \partial_j)^i_k \partial_j \tag{122}
\]

using the braided-matrix coproduct on the braided-group version of \( U_q(g) \). For example, when we compute \( u_1 u_2 \triangleright \partial_3 \) (in the compact notation), we already know the braiding of \( u_2 \) with \( \partial_3 \) since it is the same as between \( u, u' \) in (7) as computed in 6410. In the compact notation this means

\[
R_{23}^{-1}(u_1 u_2)R_{23} \triangleright \partial_3 = (u_1 \triangleright \partial_3) R_{23}^{-1}(u_2 R_{23} \triangleright \partial_3). \tag{123}
\]

There are as usual plenty of variants of these results with left-right interchanged etc, as well the more standard regular actions dualised as in Lemma 4.13 and working
with the variant $B(H, H)^{\text{cop}}$ and the opposite braiding. These left or right regular actions are ‘matrix-differentials’ in the sense of (122) etc. and define the differential structure on the braided group (just as the left regular or right regular representation defines the differential structure on a usual Lie group). The semidirect product by such actions can be called the braided Weyl algebra.

Of course, every braided module algebra action induces some kind of braided vector field. For example, we have already seen in Chapter 4 that there is a braided-adjoint action, so $B(H, H)$ acts on itself from the left by this. As a linear map it coincides with the usual quantum adjoint action already given in (101), giving Ad-vector fields $\partial^i_j = L^i_j \triangleright$ obeying the braided left-module algebra axioms. On the generators this means

$$\partial^i_j(ab) = \partial^i_k \Psi(\partial^k_j \otimes a)b.$$  \hspace{1cm} (124)

Here the statistics between $\partial$ and $L$ are again as braided matrices. So, for example,

$$\partial_1 R_{12} (u_2 u_3) R^{-1}_{12} = (\partial_1 R_{12} u_2) R^{-1}_{12} (\partial_1 u_3)$$  \hspace{1cm} (125)

eq etc.

It would be interesting to compare and contrast these braided-group constructions to the physicists approach to quantum differentials in [59][61] and elsewhere.

References

[1] G.E. Andrews. $q$-series: Their development and application in analysis, number theory, combinatorics, physics and computer algebra. Technical Report 66, AMS, 1986.

[2] A.P. Balachandran et al. Mod. Phys. Lett. A, 3:1725, 1988.

[3] W.K. Baskerville and S. Majid. Braided and unbraided harmonic oscillators. To appear in Proc. of XIX GTMP, Salamanca, 1992.

[4] W.K. Baskerville and S. Majid. The braided heisenberg group. Preprint, Damtp/92-39, 1992.

[5] T. Brzeziński and S. Majid. Quantum group gauge theory on quantum spaces. Preprint, Damtp/92-27, 1992.

[6] T. Brzeziński and S. Majid. Quantum group gauge theory and $Q$-monopoles. To appear in Proc. of XIX GTMP, Salamanca, 1992.

[7] T. Brzeziński and S. Majid. A class of bicovariant differential calculi on Hopf algebras. Lett. Math. Phys., 26:67-78, 1992.

[8] U. Carow-Watamura, M. Schlieker, M. Scholl, and S. Watamura. A quantum Lorentz group. Int. J. Mod. Phys., 6:3081–3108, 1991.

[9] E. Celeghini, R. Giachetti, E. Sorace, and M. Tarlini. The quantum Heisenberg group $H(1)_q$. J. Math. Phys., 32:1155–1158, 1991.
[10] A. Connes. $C^*$ algebres et géométrie différentielle. *C.R. Acad. Sc. Paris*, 290:599–604, 1980.

[11] V.G. Drinfeld. Quantum groups. In A. Gleason, editor, *Proceedings of the ICM*, pages 798–820, Rhode Island, 1987. AMS.

[12] L.D. Faddeev, N.Yu. Reshetikhin, and L.A. Takhtajan. Quantization of Lie groups and Lie algebras. *Algebra i Analiz*, 1, 1989. In Russian. Transl. in *Leningrad Math. J.*, 1990.

[13] K. Fredenhagen, K.H. Rehren, and B. Schroer. Superselection sectors with braid statistics and exchange algebras. *Comm. Math. Phys.*, 125:201–226, 1989.

[14] J.-L. Gervais. The quantum group structure of 2d gravity and minimal models, I. *Comm. Math. Phys.*, 130:257, 1990.

[15] G.A. Goldin, R. Menikoff, and D.H. Sharp. *J. Math. Phys.*, 21:650, 1980.

[16] D.I. Gurevich. Algebraic aspects of the quantum Yang-Baxter equation. *Leningrad Math. J.*, 2:801–828, 1991.

[17] D.I. Gurevich and S. Majid. Braided groups of Hopf algebras obtained by twisting, 1991. To appear in *Pac. J. Math.*

[18] M. Jimbo. A $q$-difference analog of $U(g)$ and the Yang-Baxter equation. *Lett. Math. Phys.*, 10:63–69, 1985.

[19] A. Joyal and R. Street. Braided monoidal categories. Mathematics Reports 86008, Macquarie University, 1986.

[20] P.P. Kulish. Quantum groups and quantum algebras as symmetries of dynamical systems. Preprint, YITP/K-959, 1991.

[21] R. Longo. Index of subfactors and statistics of quantum fields. *Comm. Math. Phys.*, 126:217, 1989.

[22] V.V. Lyubashenko. Modular transformations for tensor categories. Preprint, 1991.

[23] V.V. Lyubashenko and S. Majid. Braided groups and quantum Fourier transform, 1991. To appear in *J. Algebra.*

[24] V.V. Lyubashenko and S. Majid. Fourier transform identities in quantum mechanics and the quantum line. *Phys. Lett. B*, 284:66–70, 1992.

[25] A. Macfarlane and S. Majid. Quantum group structure in a fermionic extension of the quantum harmonic oscillator. *Phys. Lett. B*, 268:71–74, 1991.

[26] A. Macfarlane and S. Majid. Spectrum generating quantum group of the harmonic oscillator. *Int. J. Mod. Phys.*, 7(18):4377–4393, 1992.

[27] S. Majid. More examples of bicrossproduct and double cross product Hopf algebras. *Izr. J. Math.*, 72:133–148, 1990.

[28] S. Majid. Quasitriangular Hopf algebras and Yang-Baxter equations. *Int. J. Modern Physics A*, 5(1):1–91, 1990.
[29] S. Majid. On q-regularization. *Int. J. Mod. Phys. A*, 5(24):4689–4696, 1990.

[30] S. Majid. Doubles of quasitriangular Hopf algebras. *Comm. Algebra*, 19(11):3061–3073, 1991.

[31] S. Majid. Principle of representation-theoretic self-duality. *Phys. Essays*, 4(3):395–405, 1991.

[32] S. Majid. Reconstruction theorems and rational conformal field theories. *Int. J. Mod. Phys. A*, 6(24):4359–4374, 1991.

[33] S. Majid. Some physical applications of category theory. In C. Bartocci, U. Bruzzo, and R. Cianci, editors, *XIXth DGM, Rapallo (1990)*, volume 375 of *Lec. Notes in Phys.*, pages 131–142. Springer.

[34] S. Majid. Quasi-quantum groups as internal symmetries of topological quantum field theories. *Lett. Math. Phys.*, 22:83–90, 1991.

[35] S. Majid. Braided groups and algebraic quantum field theories. *Lett. Math. Phys.*, 22:167–176, 1991.

[36] S. Majid. Examples of braided groups and braided matrices. *J. Math. Phys.*, 32:3246–3253, 1991.

[37] S. Majid. Braided groups and braid statistics. In *Quantum Probability and Related Topics VIII (Proc. Delhi, 1990)*. World Sci.

[38] S. Majid. Anyonic quantum groups, May 1991. To appear in *Proc. of 2nd Max Born Symposium, Wroclaw, Poland, 1992*, eds. A. Borowiec, B. Jancewicz and Z. Oziewicz. Kluwer.

[39] S. Majid. Braided groups and duals of monoidal categories. *Canad. Math. Soc. Conf. Proc.*, 13:329–343, 1992.

[40] S. Majid. Rank of quantum groups and braided groups in dual form. In *Proc. of the Euler Institute, St. Petersberg (1990)*, volume 1510 of *Lec. Notes in Math.*, pages 79–88. Springer.

[41] S. Majid. Braided groups, 1990. To appear in *J. Pure Applied Algebra*.

[42] S. Majid. Transmutation theory and rank for quantum braided groups, 1991. To appear in *Math. Proc. Camb. Phil. Soc*.

[43] S. Majid. Cross products by braided groups and bosonization, 1991. To appear in *J. Algebra*.

[44] S. Majid. C-statistical quantum groups and Weyl algebras. *J. Math. Phys.*, 33:3431–3444, 1992.

[45] S. Majid. Quantum and braided linear algebra, February 1992 (Damtp/92-12). To appear in *J. Math. Phys*.

[46] S. Majid. Braided matrix structure of the Sklyanin algebra and of the quantum Lorentz group, January 1992 (Damtp/92-10). To appear in *Comm. Math. Phys*.
[47] S. Majid. The quantum double as quantum mechanics. Preprint, Damtp/92-48, 1992.

[48] S. Majid. Infinite braided tensor products and 2D quantum gravity. To appear in Proc. of XXI DGM, Tianjin, China June, 1992. World Sci.

[49] S. Majid. Braided momentum in the q-Poincaré group, 1992 (Damtp/92-65). To appear in J. Math. Phys.

[50] S. Majid and M.J. Rodriguez-Plaza. Universal R-matrix for non-standard quantum group and superization. Preprint, Damtp/91-47, 1991.

[51] Yu.I. Manin. Quantum groups and non-commutative geometry. Technical report, Centre de Recherches Math, Montreal, 1988.

[52] N.Yu. Reshetikhin and M.A. Semenov-Tian-Shansky. Central extensions of quantum current groups. Lett. Math. Phys., 19:133–142, 1990.

[53] N.Yu. Reshetikhin and M.A. Semenov-Tian-Shansky. Quantum R-matrices and factorization problems. J. Geom. Phys., 1990.

[54] M. Scheunert. Generalized lie algebras. J. Math. Phys., 20:712–720, 1979.

[55] M. Schlieker, W. Weich, and R. Weixler. Inhomogeneous quantum groups. Zeit. fur. Phys. C, 53:79–82, 1992.

[56] E.K. Sklyanin. Some algebraic structures connected with the Yang-Baxter equations. Func. Anal. Appl., 16:263–270, 1982.

[57] E.K. Sklyanin. Boundary conditions for the integrable quantum systems. J.Phys. A, 21:2375–2389, 1988.

[58] D.P. Sorokin and D.V. Volkov. (Anti)commuting spinors and supersymmetric dynamics of semions. Trieste Preprint, IC/92/121, 1992.

[59] J. Wess and B. Zumino. Covariant differential calculus on the quantum hyperplane. Proc. Supl. Nucl. Phys. B, 18B:302, 1990.

[60] F. Wilczek. Phys. Rev. Lett., 48:1144, 1982.

[61] B. Zumino. Introduction to the differential geometry of quantum groups. In K. Schmüdgen, editor, Mathematical Physics X. Springer-Verlag, 1992.