CLASSIFICATION OF MULTIVECTOR THEORIES AND THE MODIFICATION OF THE POSTULATES OF PHYSICS

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Abstract.
We propose a graded classification of the entire field of multivector physics, including all alternative points of view. The (often tacit) postulates of different types of formulations are contrasted, summarizing their consequences. Specifically, spin-gauge formulations of gravitation and GUT which assume standard column spinors will require unnecessarily large matrix algebras. An extreme generalization is introduced, where wavefunctions are assumed to be multivectors, in which multiple generations of particle families naturally appear without increasing the size of the algebra. Further, this allows for two-sided (bilateral) operators, which can accommodate in excess of 10 times more gauge fields without increasing the algebraic representation. As this generalization encompasses all the essential features of the other categories, it is proposed to be the best path to new physics.

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1. INTRODUCTION
Collectively, all work in this field can be divided into two broad categories: Mathematics and Physics (and the relationships between these disciplines). Before we concentrate on the latter, we note that the former can be further subdivided into two areas, Algebra and Analysis. The structure of Clifford algebra, its relation to other systems (e.g. Grassmann, Lie or Cayley algebras) is the concern of an algebraist with an ultimate goal of creating a single unified math language based on geometry. The analyst is concerned with the calculus of functions of multivectors, with the potential of providing coordinate-free formulations of fields in curved spacetime.

In contrast, only a small minority of published papers actually address the role of using Clifford Algebras in physical models. The category of multivector physics can be subdivided into two areas, Philosophy and Formulation. Generically, the former is concerned with guiding principles, which has been ignored by all but a few authors. This is unfortunate because basic postulates can not be deduced from logical reasoning or pure math (e.g. from the Clifford algebra). There has been more attention to formulation, i.e. the association of physical phenomena with mathematical symbols, however no "new" physics will be obtained by applying Clifford algebra.
while tacitly assuming the old postulates.

The intent of this paper is to focus more upon the different underlying philosophies involved in how multivector physics are formulated. We introduce two broad classifications for comparison and contrast, organized along classic greek lines, the Platonistic and Aristotelian.

2. Platonistic

Generally, standard theory has the classical Platonistic view that abstract mathematical entities exist. We are forbidden (due to the imperfection of physical existence) to directly experience (i.e. "measure") these ideal forms (e.g. "quantum phase"). They are only knowable through mathematical formulation. The physical structure of space is hence really mathematical structure in disguise.

2.1. Standard Theory

Specifically, in quantum mechanics the fundamental structure of "real" particles is represented by abstract multicomponent spinors which exist in transcendental complex spin space, which has no concrete physical analog. The unobservable nature of spin space is built into the postulates of quantum mechanics, wherein the wavefunction's "phase" angle (in complex phasor space) is ordained to be unmeasurable. Hence the theory (e.g. the Lagrangian) must be formally invariant with respect to changes in phase, which are generated by the $U(1)$ Lie group. This symmetry gauge transformation induces (via Noether's theorem) bilinear covariant forms independent of the unobservable phase. Obeying a conservation law, these forms are associated via the correspondence principle with concrete measurable quantities (e.g. $\psi^\dagger \psi$ is the observable probability density).

Requiring the formulation to be invariant under local changes in phase (different at each point in space) is called the principle of local gauge invariance. Through an inductive generalization called minimal coupling, a gauge covariant derivative is induced which contains a connection function interpreted to be the electromagnetic 4-vector potential. The curvature of this connection is the electromagnetic field strength tensor, whose source is found to be the bilinear covariant current induced by the gauge transformation on the Lagrangian.

Requiring invariance with respect to larger non-abelian symmetry groups induces other gauge fields e.g. "gluons" from $SU(3)$. The emphasis of modern physics has shifted to choosing the right "magical" symmetry group of the universe which will correctly describe the known four fundamental forces in one "unified theory". Generally the group is expressed as a complex matrix which operates on the column spinor wavefunction. The size of the representation induces the number of degrees of freedom in the "spinor-space", which in order to incorporate all the known fermions must unfortunately be in excess of 90.

2.2. Operationalist

The most conservative use of Clifford Algebra in quantum physics is to view its elements only as abstract operators on a spinor space. Only those symmetry groups
which are embedded within a global Clifford algebra are to be considered for gauge
field generators. Unfortunately this restricts one to groups such as $SU(2^d)$, where $d$
is an integer, eliminating the standard $SU(3)$ and $SU(5)$ [except as subgroups of a
larger algebra]. Further, since the representation size is restricted to $2d$, this forces
global multispinors of either probably two few (64) or too many (128) components
for the estimated 90 or 96 known degrees of freedom needed to represent all the
fermions of the standard model.

One argument for using Clifford algebra is that it naturally appears in relativistic
quantum mechanics. Four mutually anticommuting algebraic entities were required
to factor the Klein-Gordon equation to the first order Dirac equation. These four
elements generate the 16 element Majorana algebra, which has the lowest order
representation of 4 by 4 matrices, inducing 4 component bispinor wavefunctions.
These bispinors successfully correspond to a Dirac fermion (e.g. electron) with
"half-integral spin". In order for the formulation to be generally covariant, the 4
basis generators obey the defining condition of a Clifford algebra, $\frac{1}{2} \{\gamma_\mu, \gamma_\nu\} = g_{\mu\nu} =
e_\mu \bullet e_\nu$ where $e_\mu$ is a coordinate basis vector (not part of a Clifford algebra). Hence
the $\gamma_\mu$ can be determined from the Riemann space metric $g_{\mu\nu}$ up to a similarity
transformation, $\gamma'_\mu = S \gamma_\mu S^{-1}$, i.e. a change in matrix representation, called a spin
transformation.

Spin gauge theories invoke the principle of local representation invariance,
i.e. require the theory to be invariant under similarity transformations which are
a function of position. This induces a spin covariant derivative $\nabla_\mu$ where the spin
connection: $\Omega_\mu = \Omega_\mu^i \Gamma_i$ (Fock-Ivanenko coefficient) is interpreted to be a Clifford
aggregate ($\Gamma_i$ is an element of the algebra) of new spin gauge fields,

\[
\psi' = S \psi,
\]

\[
\nabla_\mu \psi'^\alpha = \partial_\mu \psi'^\alpha + \Omega_\mu^{\alpha\beta} \psi'^\beta,
\]

\[
\Omega'_\mu = S \Omega_\mu S^{-1} + (\partial_\mu S) S^{-1},
\]

\[
K_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = \Omega_{\mu;\nu} - \Omega_{\nu;\mu} + [\Omega_\mu, \Omega_\nu] = K_{\mu\nu}^i \Gamma_i,
\]

\[
\gamma'^\nu_{\mu\nu} - [\Omega_\mu, \gamma'^\nu] = 0
\]

The last equation, requiring the spin covariant derivative of any element of the
Clifford algebra to vanish is arbitrarily imposed. It is a sufficient (but not necessary)
condition to insure $g_{\mu\nu;\alpha} = 0$, i.e. the coordinate covariant derivative of the metric
to vanish defining a Riemann space. If it is desired that $\gamma'^\nu$ is grade preserving, then
the spin connection, and the resulting field strength tensor $K_{\mu\nu}$ must be a bivector
(or the commuting center of the algebra).

2.3. Structuralist

The connection between abstract spin space and tangible coordinate space can be
made more clear if we propose a geometric structure to spin a space. A spin transfor-
mation can now be associated with a change of spin basis. A spin vector $\Psi = \psi^\alpha \xi_\alpha$
will be manifestly invariant, where the basis spin vectors $\xi_\alpha$ and spinor components
$\psi^\alpha$ will be modified by the transformation. The Fock-Ivanenko coefficients $\Omega_\mu^{\alpha\beta}$, are
now interpretable as the spin space analogy of the Christoffel symbols, describing
how the spin vectors change with position in coordinate space, \( \partial_\mu \xi_\alpha = \xi_\delta \Omega_\mu^{\beta \alpha} \). A
non-vanishing field strength tensor \( K_{\mu \nu} \) (eq. 1d) is now interpretable as intrinsic
spin space curvature. If one demands that the spin metric \( \eta_{\alpha \beta} = n^*_{\alpha \beta} = \xi_\alpha^\dagger \xi_\beta \) be
coordinate independent then it follows that the spin connection must be restricted
\( \Omega^1_\mu = -\Omega_\mu \). In the particular case of Minkowski spacetime with the associated Majorana
Clifford algebra the spin connection would be restricted to Dirac bar-negative
elements, the vectors and bivectors, which are generators of the Poincare group.
Hence it is tempting to pursue this as a gauge derivation of gravity [2].

An element \( \gamma^\mu_{\alpha \beta} \) of the matrix Clifford algebra is viewed as a second rank spinor.
The manifestly spin representation independent bilinear form \( e^\mu = \xi_\alpha \gamma^\mu_{\alpha \beta} \xi_\beta \) can be
concretely interpreted as the observable basis vector of Riemann space. This forces
a relationship between spin and coordinate space, in particular eq. (1e) is built into
this definition. More importantly, this will force a decomposition of the spin gauge
field when the Clifford gradient replaces the standard in Lagrangians and equations
of motion. For example, consider the non-relativistic Pauli-Schrodinger equation
for a two-component spinor. The associated Clifford geometry of three dimensional
space is the Pauli algebra. The generators of the unitary spin transformations form
the \( U(1) \otimes SU(2) \) group with elements \( \Gamma^j = \{ i, i\sigma_1, i\sigma_2, i\sigma_3 \} \). The induced spin
connection \( \Omega_\mu = \Omega_\mu^k \Gamma_k = iA_\mu + i\sigma_k Z^k_\mu \) apparently contains the usual electromagnetic
vector field \( A_\mu \) plus three other vectors [3]. However, it is \( \sigma^k \nabla_k \) that appears in the
Lagrangian (and hence equation of motion) which forces a decomposition of the
gauge fields,

\[
-\imath \sigma^k \nabla_k = -\imath \nabla + A + \phi + iC,
\]

(2a)

\[
A = A^k \sigma_k,
\]

(2b)

\[
\phi = Z^k_k,
\]

(2c)

\[
C^m = \epsilon^{jkn} Z^j_k.
\]

(2d)

Apparently the symmetric part of the spin field \( Z^{jk} \) does not directly couple to the
particle! The remaining components enter as scalar and pseudovector interactions
(e.g. analogous to \( f_0 \) and \( f_1 \) mesons respectively). The sources of these fields will
be spin currents; in particular it is interesting to note that the scalar field \( \phi \) couples
to the helicity density: \( J_\phi = i\psi^\dagger \sigma^k \partial_k \psi \).

3. ARISTOTELIAN

In contrast, consider an Aristotelian view that abstract entities (e.g. spin space) do
not exist, rather the structure of particle fields is (represented by) tangible physical
geometry. Both operators and structure will be described by a single unifying,
geometrically interpretable Clifford Algebra. We make a distinction as to the degree
by which the theory is committed to the full exploitation of the geometric algebra.

3.1. PRAGMATISM

The pragmatist notes that restricted combinations of the algebra called minimal
ideals can be used to replace the column spinor. For example, in the Pauli equation
the basis spin vectors of a two component spin space could be replaced by \( \xi_2 = \sigma_1 \xi_1 \) and \( \xi_1 = \frac{1}{2}(1 + \sigma_3) \). The quadratic form of the wavefunction \( \psi^\dagger \psi \) may no longer be a real scalar, but have other geometric pieces, however the pragmatist does not feel obligated to utilize these extra pieces. In fact, he may restrict the geometric content of the wavefunction such that the non scalar portions disappear. For example, consider a Dirac-Hestenes equation of the form \( \gamma^\mu \partial_\mu \psi = m \psi \Gamma \), where \( \Gamma \), is an element of the algebra such that \( \Gamma^2 = +1 \) if in the \((-+++)\) metric, else \( \Gamma^2 = -1 \) for the \((+---)\) metric. For \( \bar{\Gamma} = -\Gamma \), (the tilda represents the reversion anti-involution) the multivector current equation is,

\[
\partial_\mu = m[\bar{\psi} \psi, \Gamma],
\]

\[
j^\mu = \bar{\psi} \gamma^\mu \psi = s^\mu + T^{\mu \nu} \gamma_\nu + \rho^\mu \gamma_0 \gamma_1 \gamma_2 \gamma_3.
\] (3a)

For a unrestricted multivector wavefunction \( \psi \) only the scalar current \( s^\mu \) is conserved, so the pragmatist could interpret it as an observable and ignore the parts of eq. (3b) by taking the scalar part of eq. (3a). Alternatively, he would restrict the form of the wavefunction such that the commutator of eq. (3a) vanishes, hence all three of the currents of eq. (3b) are conserved and hence interpretable as observables.

Such formulations tend to be calculationally isomorphic to standard theory, hence one can argue that it is mere reformulation. One can adopt a "minimalistic" principle, where the formulation uses the least number of geometric degrees of freedom to represent the phenomena. Hence the postulates of quantum mechanics are mildly challenged in that for example there is no imaginary "i" present in the 4 dimensional real Clifford algebra. Specifically the standard bias that the "i" is required for charged fields is clearly not true.

3.2. Radicalism

In contrast, the Radicalist holds to the extreme position that in formulation we are ontologically committed to use and interpret EVERY geometric degree of freedom within the given dimension of the space. For example, consider the multivector Dirac equation \( \gamma^\mu \partial_\mu \psi = m \psi \) (only in the \(-+++)\) metric) which has Greider \( \bar{\psi} \psi \)

multivector current,

\[
\partial_\mu j^\mu = 0,
\]

\[
j^\mu = \bar{\psi} \gamma^\mu \psi = T^{\mu \nu} \gamma_\nu + M^{\mu \alpha \beta} \gamma_\alpha \wedge \gamma_\beta,
\] (4a)

where the Dirac "bar" operator is the main algebra anti-involution (reversion followed by inversion of the basis vector elements).

In contrast to eq. (3a), each part of the Greider current is conserved (and hence interpretable as an observable) without any restriction on the wavefunction \( \psi \). Hence the general multivector wavefunction is no longer constrained to be a minimal ideal describing one particle, but contains multiple generations of particles [5]. This general structure may violate the standard Fierz identities [6], and introduce new quantum (hidden?) parameters. A Lagrangian constructed of the general multivector wavefunction may have non-scalar parts, which could contribute in the path integral formulation of quantum mechanics.
Now allowable dextral (right-side applied) operators on the wavefunction must be included, which brings a challenge to the postulate of compatible observables. Two operators may not commute but will represent compatible observables if one is a dextral operator while the other a sinistral (left-side applied, e.g. a spin-transformation). Further, the same operator when applied to the right has an entirely different physical effect (observable) than when applied on the left. For example, in the Pauli algebra, the spin would be given by $S_j = SP(\psi^\dagger \sigma_j \psi)$ while the isospin by $T_j = SP(\psi^\dagger \psi \sigma_j) = SP(\psi \sigma_j \psi^\dagger)$. The SP notation means to take the scalar part (matrix trace) as the fundamental bilinear covariant form. Each piece of such a multigeometric current must be given interpretation; we cannot just pick the ones we like and exclude the rest. In particular, we have to consider new bilateral currents of the form $R_{jk} = SP(\psi^\dagger \sigma_j \psi \sigma_k) = SP(\psi \sigma_k \psi^\dagger \sigma_j)$.

Finally one needs to address a generalized principle of geometric covariance, which considers left, right and both-sided operations on the wavefunction. This will give interactions which couple to both the right and left side of the multivector wavefunction. The generalized bilateral multivector covariant derivative would have the form,

$$\nabla_{\mu}(\psi) = \partial_{\mu} \psi + \Omega^{\alpha \beta}_{\mu} \Gamma_{\alpha} \psi \Gamma_{\beta}, \quad (5a)$$

$$[\nabla_{\mu}, \nabla_{\nu}](\psi) = K^{\alpha \beta}_{\mu \nu} \Gamma_{\alpha} \psi \Gamma_{\beta}. \quad (5b)$$

This poses several difficulties in formulation which have yet to be fully addressed. For example, eq. (1d) will not give the correct field strength tensor of eq. (5b) because of the necessity of operating on both sides of the wavefunction. However, it is apparent that far more number of gauge fields can be represented in this way for a given dimension Clifford algebra than for the spin-gauge type theory. For example, in the particular case of the multivector Dirac equation in Majorana algebra, it was earlier argued that spin gauge theory would yield 10 fields (from the bar-negative vectors and bivectors of the Clifford group). The general bilateral connection of eq. (5a) implies $16 \times 16 = 256$ possible fields. It has been shown that 136 of these can be interpreted as describing all the known spin 0 and 1 light unflavored mesons. Geometric constraints between spin and isospin suppresses the remaining 120 couplings, which exactly corresponds to those mesons forbidden by the standard model (which must appeal to the requirement of antisymmetry of a composite quark wavefunction).

4. Summary

The last category is the most general, containing all the previous categories as subsets. It is hence a prototype for the full exploitation of Clifford algebra in a physical formulation. Further it allows for a broadening of the postulates of physics (n.b. quantum mechanics) beyond mere reformulation.

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