GLOBAL EMPIRICAL RISK MINIMIZERS WITH “SHAPE CONSTRAINTS” ARE RATE OPTIMAL IN GENERAL DIMENSIONS

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ABSTRACT. Entropy integrals are widely used as a powerful tool to obtain upper bounds for the rates of convergence of global empirical risk minimizers (ERMs), in standard settings such as density estimation and regression. The upper bound for the convergence rates thus obtained typically matches the minimax lower bound when the entropy integral converges, but admits a strict gap compared with the lower bound when it diverges. Birgé and Massart [BM93] provided a striking example showing that such a gap is real with the entropy structure alone: for a variant of the natural Hölder class with low regularity, the global ERM actually converges at the rate predicted by the entropy integral that substantially deviates from the lower bound. The counter-example has spawned a long-standing negative position on the use of global ERMs in the regime where the entropy integral diverges, as they are heuristically believed to converge at a sub-optimal rate in a variety of models.

The present paper demonstrates that this gap can be closed if the models admit certain degree of ‘shape constraints’ in addition to the entropy structure. In other words, the global ERMs in such ‘shape-constrained’ models will indeed be rate-optimal, matching the lower bound even when the entropy integral diverges. The models with ‘shape constraints’ we investigate include (i) edge estimation with additive and multiplicative errors, (ii) binary classification, (iii) multiple isotonic regression, (iv) $s$-concave density estimation, all in general dimensions when the entropy integral diverges. Here ‘shape constraints’ are interpreted broadly in the sense that the complexity of the underlying models can be essentially captured by the size of the empirical process over certain class of measurable sets, for which matching upper and lower bounds are obtained to facilitate the derivation of sharp convergence rates for the associated global ERMs.

1. INTRODUCTION

1.1. Overview. Empirical risk minimization (ERM) is one of the most widely used statistical procedures for the purpose of estimation and inference. Theoretical properties for various ERMs, in particular in terms of rates of convergence, have been intensively investigated by various authors.
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[Bir83, BM93, BBM99, vdG87, vdG90, vdG93, vdG95, WS95, vdVW96, vdG00, Kol06], in a number of by-now standard settings. To motivate our discussion, let us focus on the standard Gaussian regression setting: Let \( X_1, \ldots, X_n \) be i.i.d. covariates taking value in \((\mathcal{X}, \mathcal{A})\) with law \( P \), and the responses \( Y_i \)'s are given by

\[
Y_i = f_0(X_i) + \xi_i, \quad i = 1, \ldots, n, \tag{1.1}
\]

where \( \xi_i \)'s are i.i.d. \( N(0, 1) \), and \( f_0 \) belongs to a uniformly bounded class \( F \subset L_\infty(1) \). One canonical global ERM in the regression model (1.1) is the least squares estimator (LSE):

\[
\hat{f}_n \in \arg\min_{f \in F} \sum_{i=1}^{n} (Y_i - f(X_i))^2.
\]

The performance of \( \hat{f}_n \) is usually evaluated through the risk under squared \( L_2 \) loss \( \mathbb{E}_{f_0} \| \hat{f}_n - f_0 \|^2_{L_2(P)} \), or its ‘probability’ version.

The seminal work of Birgé and Massart [BM93] (and other references cited above) shows that an upper bound \( \bar{r}_n^2 \) for the risk \( \mathbb{E}_{f_0} \| \hat{f}_n - f_0 \|^2_{L_2(P)} \) can be obtained by solving

\[
\int_{c\sqrt{n}}^{\bar{r}_n} \sqrt{\log \mathcal{N}((\varepsilon, F, L_2(P)))} \, d\varepsilon \asymp \sqrt{n} \cdot \bar{r}_n^2. \tag{1.2}
\]

Here \( \mathcal{N}((\varepsilon, F, L_2(P))) \) is the \( \varepsilon \)-bracketing number of \( F \) under \( L_2(P) \). On the other hand, a lower bound \( \underline{r}_n^2 \) for the risk \( \mathbb{E}_{f_0} \| \hat{f}_n - f_0 \|^2_{L_2(P)} \geq \underline{r}_n^2 \), can be obtained (cf. [Bir83, YB99]) via a different equation

\[
\underline{r}_n \sqrt{\log \mathcal{N}((\underline{r}_n, F, L_2(P)))} \asymp \sqrt{n} \cdot \underline{r}_n^2, \tag{1.3}
\]

where \( \mathcal{N}((\varepsilon, F, L_2(P))) \) is the \( \varepsilon \)-covering number of \( F \) under \( L_2(P) \). Note that the left hand side of (1.2) is no smaller than the left hand side of (1.3), so we always have \( \underline{r}_n \lesssim \bar{r}_n \). Suppose for now that the difference in the covering and bracketing entropy can be ignored, and it holds for some \( \alpha > 0 \) that

\[
\log \mathcal{N}(\varepsilon, F, L_2(P)) \asymp \log \mathcal{N}((\varepsilon, F, L_2(P))) \asymp \varepsilon^{-2\alpha}.
\]

The parameter \( \alpha > 0 \) measures the complexity of \( F \), and is closely related to the ‘smoothness’ of \( F \), cf. [vdV96, vdG00, GN15]. Solving the equations (1.2) and (1.3) yields that

\[
\underline{r}_n \asymp n^{-\frac{1}{2(1+\alpha)}}, \quad \bar{r}_n \asymp (n^{-\frac{1}{2(1+\alpha)}} \vee n^{-\frac{1}{4\alpha}}) \sqrt{\log^{1/2}(\alpha=1)} n.
\]

Modulo the logarithmic factor in the boundary case \( \alpha = 1 \), we see a somewhat strange phase-transition phenomenon:

- If \( \alpha \in (0, 1) \), \( 0 < \lim \inf \underline{r}_n / \log n \leq \lim \sup \bar{r}_n / \log n < \infty \). In this regime, \( F \) is Donsker since a central limit theorem in \( \ell_\infty(F) \) holds for the empirical process.
• If $\alpha > 1$, $\liminf_n \frac{\bar{r}_n}{\log n} = \infty$. In this regime, $\mathcal{F}$ is non-Donsker since there does not exist a central limit theorem in $\ell^\infty(\mathcal{F})$ for the empirical process as the limiting Brownian bridge process indexed by $\mathcal{F}$ is not sample bounded.

Although at this point (1.2) only gives an upper bound for $\bar{r}_n$, Birgé and Massart [BM93] showed by a stunning example that in the regime $\alpha > 1$, $\bar{r}_n$ can actually be attained (up to logarithmic factors) for the global ERM (called ‘minimum contrast estimators’ therein) over a slightly constructed $\mathcal{F}$ based on Hölder classes on $[0, 1]$ with smoothness less than $1/2$. Consequently, there is a genuine gap between the upper bound $\bar{r}_n$ and the lower bound $r_n$ obtained from general techniques based on entropy structures (1.2) and (1.3) alone, in the regime $\alpha > 1$.

The counter-example in [BM93] results in a long-standing negative position on the use of global ERMs in the regime $\alpha > 1$, as they are heuristically believed to be rate-suboptimal in various problems falling into the non-Donsker regime, beyond the natural setting of Hölder-type smoothness classes, cf. [vdG00, SW10, Gun12, KS16], just to name a few references. A common (but perhaps vague) heuristic is that when $\alpha > 1$, the class $\mathcal{F}$ is too ‘massive’ for global ERMs to achieve the optimal rate.

It should be mentioned that the rate sub-optimality phenomenon is due to the global nature of ERM that searches over the entire parameter space, since it is easy to construct a ‘theoretical’ rate-optimal estimator by searching over certain maximal packing sets of $\mathcal{F}$ even in the regime $\alpha > 1$ (usually known as the ‘sieve’ estimator [Gre81, LC73]). Such a theoretical construction often occurs in a minimax approach for a given statistical model, cf. [MT95, Gun12, Bru13].

At a deeper level from the perspective of empirical process theory, the upper bound (1.2) comes from the Dudley’s entropy integral, and the lower bound (1.3) is inherited with Sudakov minorization. From the recent work [Cha14, vdGW17, HW18a], it is now understood that the risk $r^2_n \equiv \mathbb{E}_{f_0} \| \hat{f}_n - f_0 \|_{L_2(P)}^2$ can be completely characterized (at least in the simple Gaussian regression model with uniformly bounded $\mathcal{F}$), by the following (not fully rigorous but essential\footnote{Rigorously, $r_n$ is determined by the location of the maxima of the map $r \mapsto \sup_{f \in \mathcal{F} - f_0} \| f \|_{L_2(P)} \leq r_n, G_n(f - f_0) = \sqrt{nr^2}$ provided it exists uniquely, cf. [vdGW17, HW18a]. For Gaussian errors $\xi_i$’s and uniformly bounded $\mathcal{F}$, the order of $r_n$ can typically be obtained by matching upper and lower moment estimates for the LHS of (1.5).} equation:

\begin{equation}
\mathbb{E} \sup_{f \in \mathcal{F} - f_0} |G_n(f)| \asymp \sqrt{n} \cdot r^2_n.
\end{equation}

Here $G_n \equiv \sqrt{n}(\mathbb{P}_n - P)$ is the empirical process. Since Dudley’s entropy integral provides an upper bound, while the Sudakov minorization gives a lower bound, for the empirical process in (1.5) as soon as it enters the ‘Gaussian domain’ (= for $n$ large in our case), the only possibility for which
\( \bar{r}_n \) and \( r_n \) do not match lies in situations where the entropy integral bound deviates substantially from the Sudakov minorization. In other words:

- In the Donsker regime \( 0 < \alpha < 1 \), the Dudley’s entropy integral (typically) matches the Sudakov minorization at the desired rate, so the upper and lower bounds in (1.2) and (1.3), and the actual rate in (1.5) match each other: \( \bar{r}_n \asymp r_n \asymp r_n \).

- In the non-Donsker regime \( \alpha > 1 \), the Dudley’s entropy integral yields a strictly larger estimate than the Sudakov minorization, so \( \bar{r}_n \gg r_n \). The actual rate \( r_n \), determined by the size of the localized empirical process (1.5), can be anywhere in between \( r_n \ll \bar{r}_n \), but the counter-example in [BM93] shows that for the natural situation of Hölder smoothness classes, \( r_n \asymp \bar{r}_n \) (up to logarithmic factors). This also suggests that the Dudley’s entropy integral (1.2) cannot be improved in general in the regime \( \alpha > 1 \).

Despite the strong suspicion in the literature (cited above) that the actual rate \( r_n \) of global ERMs will likely match \( \bar{r}_n \), which has a strict gap compared with the minimax lower bound \( r_n \), there appears recently some surprising special cases in which global ERMs are proved to be rate optimal even in the regime \( \alpha > 1 \). One example is given by the multiple isotonic regression model studied by the author in [HWCS18]. When \( d \geq 3 \), by the entropy estimate in [GW07], the class of multiple isotonic functions is in the non-Donsker regime \( \alpha > 1 \), but interestingly [HWCS18] proved that the natural LSE (= global ERM) is still minimax rate-optimal (up to logarithmic factors) in \( L_2 \) loss. The proof techniques in [HWCS18] are rather intricate and somewhat indirect, so they unfortunately do not shed light on why the LSE must be rate-optimal (see Remark 3.12 for more technical details).

The purpose of the present paper is to demonstrate a general underlying mechanism for the rate-optimality phenomenon for global ERMs beyond the isotonic LSE in general dimensions as mentioned above. Indeed, we will show that in models with certain “shape constraints”, the global ERMs will achieve the minimax optimal rates of convergence, i.e. \( \bar{r}_n \asymp r_n \asymp r_n \) (up to logarithmic factors) even in the domain \( \alpha > 1 \) when the entropy integral diverges. Here “shape constraints” are interpreted broadly in the sense that the size of the underlying empirical process (1.5) indexed by \( \mathcal{F} \) can be characterized by an empirical process indexed by certain class of measurable sets. This concept will be illustrated throughout a detailed study on the risk behavior (or rates of convergence) for the natural global ERMs in the following models:

1. Edge estimation with additive and multiplicative errors;
2. Binary classification;
3. Multiple isotonic regression (revisited);
4. \( s \)-concave density estimation,

all of which will be considered in general dimensions, where the problems necessarily fall into the non-Donsker regime \( \alpha > 1 \).
The key to proving rate minimaxity of the above global ERMs rests in new matching upper and lower bounds for empirical processes over certain special function classes, including the class of measurable sets, even in the non-Donsker regime $\alpha > 1$. More specifically, for the empirical process indexed by a class of measurable sets $\mathcal{C}$, suppose the $L^2$-size of $\mathcal{C}$ is not too small, and $\log \mathcal{N}(\varepsilon, \mathcal{C}, P) \asymp \varepsilon^{-\alpha}$ for some appropriate entropy notion $\mathcal{N}$ that will be detailed in Section 2. Then for $\alpha \neq 1$,

$$E \sup_{C \in \mathcal{C}(\sigma)} |G_n(C)| \asymp \max\{\sigma^{1-\alpha}, n^{(\alpha-1)/(\alpha+1)}\}.$$  

(1.6)

Here $\mathcal{C}(\sigma) \equiv \{C \in \mathcal{C} : P(C) \leq \sigma^2\}$, and for a measurable set $C$, $\mathbb{G}_n(C) \equiv \mathbb{G}_n(1_C)$. For $\alpha > 1$, the empirical process (1.6) is in the non-Donsker regime, and our estimate (1.6) is still sharp in this challenging regime.

The estimate in (1.6) bypasses the problem of directly using Dudley’s entropy integral in (1.2) that creates a strict gap with the lower bound (1.3) in the regime $\alpha > 1$, thereby giving sharp rates for the global ERM procedures in the above models in view of (1.5). As will be clear below, the lower bound in (1.6) agrees with that of the Sudakov minorization, and therefore the convergence rates of these global ERMs, solved using (1.6), match the minimax lower bound, i.e. $\bar{r}_n \asymp r_n \asymp \underline{r}_n$, potentially up to logarithmic factors, in the previously unknown regime $\alpha > 1$. In the special case of the multiple isotonic regression model, our new techniques present a much easier and intuitive proof (compared with the previous work [HWCS18]) that explains the reason why the natural least squares estimator is indeed rate minimax (up to logarithmic factors) for $d \geq 3$—the complexity of the isotonic LSE is captured by that of the class of upper and lower sets.

The rest of the paper is organized as follows. Section 2 is devoted to the sharp upper and lower bounds for the size of the empirical process indexed by a class of functions satisfying certain special entropy conditions that include the class of measurable sets. The behavior of various ratio-type empirical processes are also investigated. Applications of these sharp bounds to the models mentioned above are detailed in Section 3. For clarity of presentation, proofs for some of the results in Section 2 are deferred to the Appendix.

### 1.2. Notation.

For a real-valued random variable $\xi$ and $1 \leq p < \infty$, let $\|\xi\|_p := (E|\xi|^p)^{1/p}$ denote the ordinary $p$-norm.

For a real-valued measurable function $f$ defined on $(\mathcal{X}, \mathcal{A}, P)$, $\|f\|_{L_p(P)} \equiv \|f\|_{P,p} \equiv (|P|f|^p|)^{1/p}$ denotes the usual $L_p$-norm under $P$, and $\|f\|_{\infty} \equiv \sup_{x \in \mathcal{X}} |f(x)|$. $f$ is said to be $P$-centered if $Pf = 0$. $L_p(g, B)$ denotes the $L_p(P)$-ball centered at $g$ with radius $B$. For simplicity we write $L_p(B) \equiv L_p(0, B)$.

Throughout the article $\varepsilon_1, \ldots, \varepsilon_n$ will be i.i.d. Rademacher random variables independent of all other random variables. $C_x$ will denote a generic constant that depends only on $x$, whose numeric value may change from
line to line unless otherwise specified. \( a \preceq_x b \) and \( a \succeq_x b \) mean \( a \leq C_x b \) and \( a \geq C_x b \) respectively, and \( a \asymp_x b \) means \( a \preceq_x b \) and \( a \succeq_x b \) \( [a \preceq b \leq Cb \text{ for some absolute constant } C] \). For two real numbers \( a, b \), \( a \lor b \equiv \max\{a, b\} \) and \( a \land b \equiv \min\{a, b\} \). Slightly abusing notation, we write \( \log(x) \equiv \log(e \lor x) \), and \( \log \log(x) \equiv \log(e \lor \log(e \lor x)) \).

2. Empirical processes indexed by sets

2.1. Setup and assumptions. Let \( X_1, \ldots, X_n \) be i.i.d. random variables with distribution \( P \) on a sample space \( (\mathcal{X}, \mathcal{A}) \), and \( \mathcal{C} \) be a collection of measurable sets contained in \( \mathcal{X} \). To avoid measurability digressions, we assume that \( \mathcal{C} \) is countable throughout the article. For any \( \sigma > 0 \), let \( \mathcal{C}(\sigma) \equiv \{C \in \mathcal{C} : P(C) \leq \sigma^2\} \). Following the notation in [Dud14] (page 270, (7.4)), let \( N_1(\varepsilon, \mathcal{C}, P) \) be the \( \varepsilon \)-bracketing number for \( \mathcal{C} \) under \( P \), i.e. the smallest integer \( m \) such that there exist \( \{C_i \subset D_i\}_{i=1}^m \subset \mathcal{A} \) with the following property: for any \( C \in \mathcal{C} \), there exists some \( i \in \{1, \ldots, m\} \) such that \( C_i \subset C \subset D_i \), and \( P(D_i \setminus C_i) \leq \varepsilon \). \( N(\varepsilon, \mathcal{C}, P) \) will be used for the standard \( \varepsilon \)-covering number for \( \mathcal{C} \) under \( P \).

Assumption A. Let \( \alpha > 0 \).

(E1) \( \log N_1(\varepsilon, \mathcal{C}, P) \leq L\varepsilon^{-\alpha} \).

(E2) \( \log N(\varepsilon/\sqrt{\alpha}, \mathcal{C}(\sqrt{\alpha}), P) \geq L^{-1}\varepsilon^{-\alpha} \).

For examples satisfying the above entropy conditions, see [Dud14]. \( L \) will be a large enough absolute constant throughout the article, the dependence on which will not be explicitly stated in the theorems.

For \( 0 < \alpha < 1 \), the bracketing condition in (E1) can also be replaced by a uniform entropy condition \( \sup_Q \log N(\varepsilon, \mathcal{C}, Q) \leq L\varepsilon^{-\alpha} \), where the supremum is taken over all finitely discrete probability measures \( Q \). This case is essentially covered in [GK06]. Our proof techniques also apply to this case; see remarks after the proof of Theorem 2.1.

2.2. Upper and lower bounds.

Theorem 2.1. Suppose (E1) holds and \( \sigma^2 \geq n^{-1/(\alpha+1)} \), \( \alpha \neq 1 \). Then
\[
\mathbb{E} \sup_{C \in \mathcal{C}(\sigma)} |G_n(C)| \preceq_{\alpha} \max \{\sigma^{1-\alpha}, n^{(\alpha-1)/2(\alpha+1)}\}.
\]
If furthermore (E2) holds, then
\[
\mathbb{E} \sup_{C \in \mathcal{C}(\sigma)} |G_n(C)| \succeq_{\alpha} \max \{\sigma^{1-\alpha}, n^{(\alpha-1)/2(\alpha+1)}\}.
\]

Theorem 2.1 follows from matching upper and lower bounds for empirical processes indexed by more general function classes satisfying certain entropy conditions. First we state the general upper bound.

Theorem 2.2. Suppose that \( \mathcal{F} \subset L_\infty(1) \) is a class of measurable functions and that the following entropy estimate holds for some \( \alpha > 0 \):
\[
(2.1) \quad \log N_1(\varepsilon, \mathcal{F}, L_1(P)\mathbf{1}_{\alpha > 1} + L_2^2(P)) \leq L\varepsilon^{-2\alpha}.
\]
Then for $\sigma^2 \gtrsim n^{-1/(\alpha+1)}$, $\alpha \neq 1$, we have:

$$E \sup_{f \in \mathcal{F}(\sigma)} |G_n(f)| \lesssim_\alpha \max \left\{ \sigma^{1-\alpha}, n^{(\alpha-1)/(2(\alpha+1))} \right\}.$$ 

Here $\mathcal{F}(\sigma) \equiv \{ f \in \mathcal{F} : Pf^2 \leq \sigma^2 \}$.

The precise meaning of (2.1) is:

- In the regime $0 < \alpha < 1$, we require the usual bracketing entropy condition $\log N_1[\varepsilon, \mathcal{F}, L_2^2(P)] = \log N_1[\varepsilon, \mathcal{F}, L_2(P)] \leq L \varepsilon^{-2\alpha}$.
- In the regime $\alpha > 1$, we require that the minimal number of brackets $[f, \tilde{f}](f \leq \tilde{f})$ needed for $\mathcal{F}$ so that $P(\tilde{f} - f) + P(\tilde{f} - f)^2 \leq \varepsilon$, is no more than $\exp(L \varepsilon^{-2\alpha})$.

Next we state the general lower bound.

**Theorem 2.3.** Suppose that $\mathcal{F} \subset L_\infty(1)$ is a class of measurable functions, and that the following entropy estimate holds for some $\alpha > 0$:

$$\log N_1[\varepsilon, \mathcal{F}, L_2^2(P)] \leq L \varepsilon^{-2\alpha}, \quad \log N(\varepsilon/2, \mathcal{F}, L_2(P)) \geq L^{-1} \varepsilon^{-2\alpha}.$$ 

Then for $\sigma^2 \gtrsim n^{-1/(\alpha+1)}$, $\alpha \neq 1$, we have:

$$E \sup_{f \in \mathcal{F}(\sigma)} |G_n(f)| \gtrsim_\alpha \max \left\{ \sigma^{1-\alpha}, n^{(\alpha-1)/(2(\alpha+1))} \right\}.$$ 

The proofs for Theorems 2.2 and 2.3 will be deferred to later subsections.

**Proof of Theorem 2.1.** (2.1) is verified using the fact that for any measurable set $C$, with $f \equiv 1_C$ we have $Pf = Pf^2$. The entropy conditions can be verified by noting that $N_1[\varepsilon/4, \mathcal{C}(\sqrt{\varepsilon}), P] = N_1[\varepsilon/2, \mathcal{F}(\sqrt{\varepsilon}), L_2(P)]$ holds with $\mathcal{F} \equiv \{ 1_C : C \in \mathcal{C} \}$ and any $\varepsilon, \varepsilon' > 0$. (and similarly for the covering number). \(\Box\)

**Remark 2.4.** Some technical remarks:

1. Upper bounds in Theorem 2.2 in the regime $0 < \alpha < 1$ follow from standard techniques. Upper bounds for $\alpha > 1$ in probability for the class of measurable sets are obtained in [Dud82, Ale84]. Here we generalize their results both to function classes satisfying (2.1) and in expectation.

2. The condition (2.1) requires an entropy in $L_1(P) + L_2^2(P)$ rather than the usual $L_2^2(P)$ in the regime $\alpha > 1$. This is of crucial importance in obtaining a sharp estimate that matches the lower bound in Theorem 2.3. Indeed, if we only assume the entropy condition in $L_2^2(P)$, then by the standard local maximal inequality for the empirical processes (cf. Lemma 2.14.3 of [vdVW96]), we have

$$E \sup_{f \in \mathcal{F}(\sigma)} |G_n(f)| \lesssim \inf_{\gamma > 0} \left\{ \sqrt{n} \gamma + \int_0^\sigma \sqrt{\log N_1[\varepsilon, \mathcal{F}, L_2(P)]} \, d\varepsilon \right\}.$$
For $\sigma^2 = c^2 n^{-1/(\alpha+1)}$ and the entropy condition of Theorem 2.1 with $\alpha > 1$, the above bound reduces to
\[
\mathbb{E} \sup_{f \in F(\mathcal{F}_n)} |\mathcal{G}_n(f)| \lesssim \inf_{\gamma > 0} \left\{ \sqrt{n\gamma} + \gamma^{1-\alpha} \right\} - \sigma^{1-\alpha} \asymp n^{(\alpha - 1)/2\alpha}.
\]

Compared with the bounds obtained in Theorem 2.2, we see that general empirical process tools lead to strictly sub-optimal bounds for non-Donsker classes of functions. This discrepancy here underlies the rate sup-optimality of global ERM procedures observed in [BM93].

(3) The upper bounds derived in [GK06] take into account the size of the envelope function of the localized function class. Such a consideration is especially suited to the study of VC-type classes, where the entropy bound naturally incorporates the information of the size of the envelope function. We refer the reader to the discussion in [GK06] page 1171 for more details. Since we are not aware of any specific example of a class of measurable sets that has a bracketing entropy estimate taking into account the size of the envelope function, we will henceforth not address this point.

(4) The proof techniques for the lower bound in Theorem 2.3 are very different from [GK06]:
- The approach of [GK06] crucially relies on a lower bound for the Rademacher processes (cf. [LT11]), for which a uniform entropy condition in the Donsker regime $0 < \alpha < 1$ is indispensable.
- Our proof techniques are based on Gaussian randomization followed by an application of the multiplier inequality derived in the author’s previous work [H18a] that removes the effect of Gaussianization. This only requires some sharp upper bounds for the localized empirical processes. The bracketing entropy upper bound in (2.2) serves as a sufficient condition, and applies to set examples (cf. [Dud14]) in the non-Donsker regime $\alpha > 1$. In the Donsker regime $0 < \alpha < 1$, one may use instead a uniform entropy upper bound condition in (2.2).

As described above, our approach for the lower bound seems to offer some more generality than the techniques in [GK06]. We note that the Gaussianization idea is exploited in [GZ84, GZ91] for asymptotic purposes, while our techniques here are non-asymptotic.

(5) It is also possible to consider the boundary case $\alpha = 1$ in Theorem 2.1. The downside is that the upper bound deviates from the lower bound by a logarithmic factor.

(6) We derive in Theorems 2.2 and 2.3 upper and lower bounds for the first moment of the empirical process. Higher moment and concentration inequalities follow easily from Talagrand’s concentration inequality (cf. Appendix I).
2.3. Ratio limit theorems. As a direct application of Theorem 2.1, we consider limit theorems for ratio-type empirical processes. Such limit theorems are initiated in [Wel78, SW82, Stu82, MSW83, Stu84] for uniform empirical processes on (subsets of) \( \mathbb{R} \) (or \( \mathbb{R}^d \)), and are further investigated in [Ale87] for VC classes of sets, and extended by [GKW03, GK06] who studied more general VC-subgraph classes. These authors work with classes satisfying uniform entropy conditions, and the class of sets (or functions) need to be Donsker apriori. The lack of corresponding results for non-Donsker class of sets are mainly due to the lack of sharp upper and lower bounds for the behavior of the empirical process. Here we fill in this gap by using Theorem 2.1.

**Theorem 2.5.** Let \( r_n^2 \gtrsim n^{-1/(\alpha+1)} \) and \( \gamma_n \equiv n^{1/2} r_n \left( r_n^{\alpha-1} \wedge n^{-\frac{\alpha-1}{2(\alpha+1)}} \right) \). Then we have the following:

1. If (E1) holds and \( r_n^{2\alpha} \log \log n \to 0 \),

\[
\limsup_{n \to \infty} \gamma_n \sup_{C \in \mathcal{C} : r_n^2 \leq P(C) \leq 1} \frac{|P_n(C) - P(C)|}{\sqrt{P(C)}} < \infty \quad \text{a.s.}
\]

2. If (E1)-(E2) hold,

\[
\liminf_{n \to \infty} \gamma_n \sup_{C \in \mathcal{C} : r_n^2 \leq P(C) \leq 1} \frac{|P_n(C) - P(C)|}{\sqrt{P(C)}} > 0 \quad \text{a.s.}
\]

**Theorem 2.6.** Let \( r_n^2 \gtrsim n^{-1/(\alpha+1)} \) and there exists some large constant \( K_\alpha > 0 \) such that:

1. If (E1) holds and \( \liminf_{n \to \infty} r_n^2 \cdot n^{1/(\alpha+1)} \geq \rho \) for some \( \rho \in (K_\alpha, \infty] \),

\[
\limsup_{n \to \infty} \sup_{C \in \mathcal{C} : r_n^2 \leq P(C) \leq 1} \left| \frac{P_n(C)}{P(C)} - 1 \right| \leq O \left( \rho^{-(1\wedge\frac{1+\alpha}{2})} \right) \quad \text{a.s.}
\]

2. If furthermore (E2) holds and \( \limsup_{n \to \infty} r_n^2 \cdot n^{1/(\alpha+1)} \leq \bar{\rho} \) for some \( \bar{\rho} \in (K_\alpha, \infty] \),

\[
\liminf_{n \to \infty} \sup_{C \in \mathcal{C} : r_n^2 \leq P(C) \leq 1} \left| \frac{P_n(C)}{P(C)} - 1 \right| \geq O \left( \bar{\rho}^{-(1\wedge\frac{1+\alpha}{2})} \right) \quad \text{a.s.}
\]

Proofs of Theorems 2.5 and 2.6 combine Theorem 2.1 and the general strategy in [GK06, GKW03]; details are deferred to the Appendix.

**Remark 2.7.** Some technical remarks:

1. An interesting corollary of Theorem 2.6 is that under entropy conditions (E1)-(E2), the sequence in the theorem converges to 0 as \( n \to \infty \) almost surely if and only if \( r_n^2 \cdot n^{1/(\alpha+1)} \to \infty \).

2. Theorems 2.5 and 2.6 are also valid in their \( L_p \) (1 \leq p < \infty) versions (which can be seen by integrating the tail estimates in the proofs).
For instance, if (E1) holds, then
\[
\limsup_{n \to \infty} \left\| \gamma_n \sup_{C \in \mathcal{E}, r_n^2 \leq P(C) \leq 1} \left| \frac{\mathbb{P}_n(C) - P(C)}{\sqrt{P(C)}} \right| \right\|_{L_p(P^{\otimes n})} < \infty,
\]
and
\[
\limsup_{n \to \infty} \left\| \sup_{C \in \mathcal{E}, r_n^2 \leq P(C) \leq 1} \left| \frac{\mathbb{P}_n(C)}{P(C)} - 1 \right| \right\|_{L_p(P^{\otimes n})} \leq O \left( n \right).
\]

(3) We may consider more general weighting functions of form \( \phi(\sqrt{P(C)}) \) as in [GK06] rather than the special cases \( \phi_1(t) = t \) in Theorem 2.2 and \( \phi_2(t) = t^2 \) in Theorem 2.3. Here we make these choices mainly due to the fact that \( \phi_1, \phi_2 \) are of special interest in the history of empirical process theory [Wel78, SW82, Stu82, MSW83, Stu84, Ale87], and the corresponding results for more general cases follow from minor modifications of the proofs.

(4) It is also straightforward to consider corresponding ratio limit theorems for function classes satisfying the conditions of Theorems 2.2 and 2.3; we omit these digressions.

2.4. Proof of Theorem 2.2

Proposition 2.8. Suppose there exists some \( \alpha > 1 \) such that for all \( \varepsilon > 0, \log \mathcal{N}[\varepsilon^2, \mathcal{F}, L_1(P) + L_2^2(P)] \leq L \varepsilon^{-2\alpha} \). Then
\[
\mathbb{E} \sup_{f \in \mathcal{F}} |G_n(f)| \leq C_{L, \alpha} \cdot n^{(\alpha-1)/2(\alpha+1)}.
\]

Proof. Let \( k_n \in \mathbb{N} \) be such that \( 2^{k_n} = n^{1/(\alpha+1)} \) (by slightly ignoring the rounding issue for notational convenience). For any \( 1 \leq k \leq k_n \), let \( N_k = \mathcal{N}[1(2^{-k}, \mathcal{F}, L_1(P) + L_2^2(P))] \leq \exp(L \cdot 2^{k_n}) \), and let \( \{f_{k,i}, \bar{f}_{k,i}\}_{i=1}^{N_k} \) denote a minimal bracketing set such that \( P(\bar{f}_{k,i} - f_{k,i})^2 \vee P(f_{k,i} - \bar{f}_{k,i}) \leq 2^{-k} \). Let \( N_0 = 1 \) and \( \bar{f}_{0,1} = 1, f_{0,1} = -1 \). Then for any \( f \in \mathcal{F} \), there exists some \( i = i(k,f) \in [1:N_k] \) such that \( f_{k,i} \leq f \leq \bar{f}_{k,i} \). Note that for any \( k \geq 1 \),
\[
P \left| f_{k,i(k,f)} - f_{k-1,i(k-1,f)} \right|^2 \leq 2P \left| f_{k,i(k,f)} - f_{k-1,i(k-1,f)} \right|^2 + 2P \left| f_{k,i(k,f)} - f_{k-1,i(k-1,f)} \right|^2 \leq 2^{-k+1} + 2^{-k+2} \leq 2^{-k+3}.
\]
Consider the set
\[
Q_k \equiv \{ f \in \{ f_{k,i} - f_{k-1,i}, \bar{f}_{k,i} - f_{k,i} \}_{1 \leq i \leq N_k, 1 \leq j \leq N_{k-1}, Pj^2 \leq 2^{-k+3}} \}.
\]
Then \( |Q_k| \leq N_{k-1}N_k + N_k \leq 2 \exp(2L \cdot 2^{k_n}) \). By Bernstein’s inequality (cf. page 36 of [BLM13]), for any \( 1 \leq k \leq k_n \),
\[
\mathbb{E} \max_{f \in Q_k} |G_n(f)| \lesssim \frac{\log |Q_k|}{\sqrt{n}} + 2^{-k/2} \sqrt{\log |Q_k|} \lesssim L \frac{2^{k_n}}{\sqrt{n}} + 2^{k(\alpha-1)/2}.
\]
For any $f \in \mathcal{F}$, consider the following chaining:

$$f_{k_n,i(k_n,f)} = \sum_{r=2}^{k_n} \left( f_{r,i(r,f)} - f_{r-1,i(r-1,f)} \right) + f_{1,i(1,f)}. $$

Then,

$$\mathbb{E} \sup_{f \in \mathcal{F}} |G_n(f_{k_n,i(k_n,f)})| \leq \sum_{r=2}^{k_n} \mathbb{E} \sup_{f \in \mathcal{F}} |G_n(f_{r,i(r,f)} - f_{r-1,i(r-1,f)})| + \mathbb{E} \sup_{f \in \mathcal{F}} |G_n(f_{1,i(1,f)})|
\leq \sum_{r=2}^{k_n} \mathbb{E} \max_{f \in \mathcal{Q}_r} |G_n(f)| + \mathbb{E} \sup_{f \in \mathcal{F}} |G_n(f_{1,i(1,f)})|
\lesssim \sum_{r=1}^{k_n} \left( \frac{2^{\alpha r} \sqrt{\alpha}}{\sqrt{n}} + 2^{r(\alpha - 1)/2} \right) \lesssim L_\alpha \frac{2^{k_n \alpha}}{\sqrt{n}} + 2^{k_n (\alpha - 1)/2} \leq C_{L,\alpha} \cdot n^{(\alpha - 1)/2(\alpha + 1)}
$$

by our choice of $k_n$ such that $2^{k_n} = n^{1/(\alpha + 1)}$. On the other hand,

$$\mathbb{E} \sup_{f \in \mathcal{F}} |G_n(f - f_{k_n,i(k_n,f)})|\n\leq \mathbb{E} \sup_{f \in \mathcal{F}} \sqrt{n} P\left( f_{k_n,i(k_n,f)} - f_{k_n,i(k_n,f)} \right) + \sup_{f \in \mathcal{F}} \sqrt{n} P\left( f_{k_n,i(k_n,f)} - f_{k_n,i(k_n,f)} \right)
\leq \mathbb{E} \sup_{f \in \mathcal{Q}_{k_n}} |G_n(f)| + 2 \sup_{f \in \mathcal{F}} \sqrt{n} P\left( f_{k_n,i(k_n,f)} - f_{k_n,i(k_n,f)} \right)
\lesssim L_\alpha \frac{2^{k_n \alpha}}{\sqrt{n}} + 2^{k_n (\alpha - 1)/2} + \sqrt{n} 2^{-k_n} \lesssim n^{(\alpha - 1)/2(\alpha + 1)}.
$$

This entails that

$$\mathbb{E} \sup_{f \in \mathcal{F}} |G_n(f)| \leq \mathbb{E} \sup_{f \in \mathcal{F}} |G_n(f - f_{k_n,i(k_n,f)})| + \mathbb{E} \sup_{f \in \mathcal{F}} |G_n(f_{k_n,i(k_n,f)})|
\lesssim L_\alpha n^{(\alpha - 1)/2(\alpha + 1)},$$

as desired. \hfill \square

The following local maximal inequality for empirical processes will be needed to complete the proof for the upper bound of Theorem \ref{thm:upper_bound}.

**Lemma 2.9** (Lemma 3.4.2 of [vdVW96]). Suppose that $\mathcal{F} \subset L_\infty(1)$, and $X_1, \ldots, X_n$’s are i.i.d. random variables with law $P$. Then with $\mathcal{F}(\delta) \equiv \{f \in \mathcal{F} : Pf^2 < \delta^2 \}$. Then

$$\mathbb{E} \sup_{f \in \mathcal{F}(\delta)} \left| \sum_{i=1}^n \varepsilon_i f(X_i) \right| \lesssim \sqrt{n} J_{11}(\delta, \mathcal{F}, L_2(P)) \left( 1 + \frac{J_{11}(\delta, \mathcal{F}, L_2(P))}{\sqrt{n} \delta^2} \right).
$$

Here $J_{11}(\delta, \mathcal{F}, \|\cdot\|) \equiv \int_0^\delta \sqrt{1 + \log \mathcal{N}(\varepsilon, \mathcal{F}, \|\cdot\|)} \ d\varepsilon$.

**Proof of Theorem 2.2.** First consider $0 < \alpha < 1$. Upper bounds in this regime follow from the local maximal inequality as in Lemma \ref{lem:local_maximal} by noting that $J_{11}(\sigma, \mathcal{F}, L_2(P)) \lesssim \int_0^\sigma \varepsilon^{-\alpha} \ d\varepsilon \lesssim \sigma^{1-\alpha}$. For $\sigma^2 \gtrsim n^{1/(\alpha + 1)}$, we
then have $\mathbb{E} \sup_{f \in \mathcal{F}(\sigma)} |G_n(f)| \lesssim_\sigma \sigma^{1-\alpha}$. Next consider $\alpha > 1$. A global upper bound follows from Proposition 2.8 $\mathbb{E} \sup_{f \in \mathcal{F}} |G_n(f)| \lesssim_\sigma n^{(\alpha-1)/2(\alpha+1)}$, completing the proof.

**Remark 2.10.** From the proof we see that for $0 < \alpha < 1$, the bracketing entropy condition in $L^2_2$ can be replaced by the uniform entropy condition in $L^2_2$—then we only need to use the corresponding local maximal inequality for empirical processes instead of Lemma 2.9 used here, cf. [vdVW11] or Section 3 of [GK06].

2.5 **Proof of Theorem 2.3.** We first prove the lower bound for Gaussian-ized empirical process.

**Proposition 2.11.** Let $\mathcal{F} \subset L_\infty(1)$. For any $\sigma \geq 50n^{-1/2}$ such that $\log \mathcal{N}(\sigma/4, \mathcal{F}(\sigma), L_2(P)) \leq n \sigma^2/4000$, we have

$$\mathbb{E} \sup_{f \in \mathcal{F}(\sigma)} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_i f(X_i) \right| \gtrsim \sigma \sqrt{\log \mathcal{N}(\sigma/2, \mathcal{F}(\sigma), L_2(P))}.$$ 

Here $g_1, \ldots, g_n$ are i.i.d. $\mathcal{N}(0, 1)$.

**Lemma 2.12** (Sudakov minorization [Sud69]). Let $(X_t)_{t \in T}$ be a centered separable Gaussian process, and $\|t - s\|_2 := \mathbb{E}(X_t - X_s)^2$. Then

$$\mathbb{E} \sup_{t \in T} X_t \geq C^{-1} \sup_{\epsilon > 0} \epsilon \sqrt{\log \mathcal{N}(\epsilon, T, \|\cdot\|)}.$$ 

Here $C$ is a universal constant.

**Proof of Proposition 2.11.** By Sudakov minorization (cf. Lemma 2.12), for any $\sigma > 0$,

$$(2.4) \quad \mathbb{E} \sup_{f \in \mathcal{F}(\sigma)} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_i f(X_i) \right| \gtrsim \mathbb{E} \sigma \sqrt{\log \mathcal{N}(\sigma/10, \mathcal{F}(\sigma), L_2(\mathbb{P}_n))}.$$ 

We claim that for any $\sigma > 0$ such that $\log \mathcal{N}(\sigma/10, \mathcal{F}(\sigma), L_2(P)) \leq n \sigma^2/4000$,

$$(2.5) \quad \mathbb{P}(\mathcal{N}(\sigma/10, \mathcal{F}(\sigma), L_2(\mathbb{P}_n))) \geq \mathcal{N}(\sigma/2, \mathcal{F}(\sigma), L_2(P))) \geq 1 - e^{-n \sigma^2/20000}.$$ 

To see this, let $f_1, \ldots, f_N$ be a maximal $\sigma/2$-packing set of $\mathcal{F}(\sigma)$ in the $L_2(P)$ metric, i.e. for $i \neq j$, $P(f_i - f_j)^2 \geq \sigma^2/4$. Since $P(f_i - f_j)^4 \leq 4P(f_i - f_j)^2 \leq 16\sigma^2$, we apply Bernstein’s inequality followed by a union bound to see that with probability at least $1 - N^2 \exp(-t)$,

$$\max_{1 \leq i \neq j \leq N} \left( nP(f_i - f_j)^2 - \sum_{k=1}^{n} (f_i - f_j)^2(X_k) \right) \leq \frac{2t}{3} + \sqrt{32t n \sigma^2}.$$ 

With $t = c n \sigma^2$ for a constant $c > 0$ to be specified below, we obtain

$$\mathbb{P} \left( \min_{1 \leq i \neq j \leq n} \frac{1}{n} \sum_{k=1}^{n} (f_i - f_j)^2(X_k) \geq \sigma^2 \left( \frac{1}{4} - 2c/3 - \sqrt{32c} \right) \right) \geq 1 - e^{2 \log \mathcal{D}(\sigma/2, \mathcal{F}(\sigma), L_2(P)) - c n \sigma^2} \geq 1 - e^{2 \log \mathcal{N}(\sigma/4, \mathcal{F}(\sigma), L_2(P)) - c n \sigma^2},$$
where \( D(\cdot, \cdot) \) stands for the packing number. By choosing \( c = 1/10^3 \) and
\[
\log \mathcal{N}(\sigma/4, \mathcal{F}(\sigma), L_2(P)) \leq n\sigma^2/4000,
\]
we have
\[
\mathbb{P}\left( \min_{1 \leq i \neq j \leq N} \frac{1}{n} \sum_{k=1}^{n} (f_i - f_j)^2(X_k) \geq 0.04\sigma^2 \right) \geq 1 - \exp(-n\sigma^2/2000).
\]
This entails that \( D(\sigma/5, \mathcal{F}(\sigma), L_2(\mathbb{P}_n)) \geq N \equiv D(\sigma/2, \mathcal{F}(\sigma), L_2(P)) \) with the
above probability. Hence for any \( \sigma > 0 \) such that \( \log \mathcal{N}(\sigma/4, \mathcal{F}(\sigma), L_2(P)) \leq n\sigma^2/4000 \),
with probability at least \( 1 - e^{-n\sigma^2/2000} \),
\[
\mathcal{N}(\sigma/10, \mathcal{F}(\sigma), L_2(\mathbb{P}_n)) \geq D(\sigma/5, \mathcal{F}(\sigma), L_2(\mathbb{P}_n)) \geq D(\sigma/2, \mathcal{F}(\sigma), L_2(P)) \geq \mathcal{N}(\sigma/2, \mathcal{F}(\sigma), L_2(P)),
\]
completing the proof of (2.5). Hence for any \( \sigma \geq 50n^{-1/2} \) such that the
entropy \( \log \mathcal{N}(\sigma/4, \mathcal{F}(\sigma), L_2(P)) \leq n\sigma^2/4000 \), the claim of the proposition
follows from (2.4) and (2.5).

Next we eliminate the effect of the Gaussian multiplier. We need the
following form of a multiplier inequality due to [HW18a].

Lemma 2.13 (Theorem 1 in [HW18a]). Suppose that \( \xi_1, \ldots, \xi_n \) are i.i.d.
mean-zero random variables independent of i.i.d. \( X_1, \ldots, X_n \). Let \( \{\mathcal{F}_k\}_{k=1}^n \) be
a sequence of function classes such that \( \mathcal{F}_k \subseteq \mathcal{F}_n \) for all \( 1 \leq k \leq n \).
Assume further that there exist non-decreasing concave functions \( \{\psi_n\} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) with \( \psi_n(0) = 0 \) such that
\[
\mathbb{E} \sup_{f \in \mathcal{F}_k} \left| \sum_{i=1}^{k} \xi_i f(X_i) \right| \leq \psi_n(k)
\]
holds for all \( 1 \leq k \leq n \). Then
\[
\mathbb{E} \sup_{f \in \mathcal{F}_n} \left| \sum_{i=1}^{n} \xi_i f(X_i) \right| \leq 4 \int_{0}^{\infty} \psi_n(n \cdot \mathbb{P}(|\xi_1| > t)) \, dt.
\]

The following alternative formulation of the multiplier inequality, proved
in Proposition 1 of [HW18a], will also be useful.

Lemma 2.14. Let \( \xi_1, \ldots, \xi_n \) be i.i.d. symmetric mean-zero multipliers in-
dependent of i.i.d. samples \( X_1, \ldots, X_n \). For any function class \( \mathcal{F} \),
\[
\mathbb{E} \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^{n} \xi_i f(X_i) \right| \leq \mathbb{E} \left[ \sum_{k=1}^{n} (|\xi(k)| - |\xi(k+1)|) \mathbb{E} \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^{k} \xi_i f(X_i) \right| \right]
\]
where \( |\xi(1)| \geq \cdots \geq |\xi(n)| \geq |\xi(n+1)| \equiv 0 \) are the reversed order statistics for
\( \{\xi_i\}_{i=1}^n \).

We need a further technical lemma.

Lemma 2.15. Let \( g_1, \ldots, g_n \) be i.i.d. \( \mathcal{N}(0, 1) \), and \( |g(n)| \leq \cdots \leq |g(1)| \) be
reversed order statistics of \( \{|g_1|, \ldots, |g_n|\} \). Then there exists an absolute
constant $K > 0$ such that for any $c \in (0, K^{-1})$, and $0 \leq t \leq K^{-1} \sqrt{\log(1/c)}$, we have $\mathbb{P}(\{|g_{(cn)}| \leq t\}) \leq e^{-c^2 n/K}$.

Proof. For notational convenience, we assume that $cn \in \mathbb{N}$. Let $\phi(t) \equiv \mathbb{P}(\{|g_1| > t\})$. By (2.23) of [GN15], $\sqrt{2/\pi} \cdot \frac{1}{e} e^{-t^2/2} \leq \phi(t) \leq \min\{1, \sqrt{2/\pi} \cdot t^{-1}\} e^{-t^2/2}$. Let $t_c > 0$ be such that $\phi(t_c) = 2c$. Then $t_c \leq \sqrt{2 \log(1/2c)}$. By Bernstein’s inequality, for $0 \leq t \leq t_c$, $2c \leq \phi(t) \leq 1$, so

$$
\mathbb{P}(\{|g_{(cn)}| \leq t\}) \leq \mathbb{P}\left( \sum_{i=1}^{n} \mathbb{1}_{|g_i| > t} \leq cn \right) = \mathbb{P}\left( \sum_{i=1}^{n} (1_{|g_i| > t} - \phi(t)) \leq -\phi(t) - cn \right)
$$

$$
\leq \exp\left(-\frac{(\phi(t) - c)^2 n^2}{2n^3 (\phi(t) + 4(\phi(t) - c)n/3)}\right) \leq e^{-c^2 n/K},
$$

proving the claim. \(\square\)

**Proposition 2.16.** Let the conditions in Theorem 2.3 hold for some $\alpha \in (0, 1)$ and $L > 0$ large enough. Then for $\sigma_n^2 \geq cn^{-1/(\alpha+1)}$ with some constant $c > 0$, $\mathbb{E} \sup_{f \in \mathcal{F}(\sigma_n)} |\mathcal{G}_n(f)| \gtrsim \sigma_n^{1-\alpha}$.

**Proof.** By Proposition 2.11, the Gaussianized empirical process satisfies

$$
\mathbb{E} \sup_{f \in \mathcal{F}(\sigma_n)} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i f(X_i) \right| \gtrsim \sigma_n \sqrt{\log N(\sigma_n/2, \mathcal{F}(\sigma_n), L_2(P))} \geq C_1^{-1} \sigma_n^{1-\alpha}.
$$

Suppose that $\sigma_n^2 \leq c$. Without loss of generality, we assume that $\sigma_n^2 \equiv \sigma_n(\gamma)^2 = cn^{-\gamma}$ for some $0 \leq \gamma \leq 1/(\alpha + 1)$ and define $\sigma_k^2 \equiv ck^{-\gamma}$. We first prove the following claim: there exists some $c_1 \equiv c_1(c, \alpha) > 0$ such that for any $0 \leq \gamma \leq 1/(\alpha + 1)$,

$$
(2.9) \quad \mathbb{E} \sup_{f \in \mathcal{F}(\sigma_n)} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i f(X_i) \right| \geq c_1 \sigma_n^{1-\alpha}.
$$

To this end, let $a_{\alpha} \equiv (\sigma_n^{1-\alpha})^{-1} \mathbb{E} \sup_{f \in \mathcal{F}(\sigma_n)} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i f(X_i) \right|$. Then by the local maximal inequality (cf. Lemma 2.9), we see that $\sup_{k \in \mathbb{N}} a_k \leq C_2 \equiv C_2(\alpha)$. Since $\sqrt{n} \sigma_n^{1-\alpha} = c^{(1-\alpha)/2} n^\beta$ where $0 \equiv \beta = \beta(\alpha, \gamma) \equiv \frac{1}{2}(1 - (1 - \alpha) \gamma) \in [\alpha/(1 + \alpha), 1/2]$, we have by Lemma 2.14 that for a constant $c' > 0$ to be determined later,

$$
(2.10) \quad C_1^{-1} n^\beta \leq c'^{-(1-\alpha)/2} \mathbb{E} \sup_{f \in \mathcal{F}(\sigma_n)} \left| \sum_{i=1}^{n} g_i f(X_i) \right|
$$

$$
\leq c'^{-(1-\alpha)/2} \mathbb{E} \left[ \sum_{k=1}^{n} (|g_{(k)}| - |g_{(k+1)}|) \right] \mathbb{E} \sup_{f \in \mathcal{F}(\sigma_k)} \left| \sum_{i=1}^{k} \varepsilon_i f(X_i) \right|
$$

$$
\leq \mathbb{E} \left[ \sum_{k=1}^{c'n} (|g_{(k)}| - |g_{(k+1)}|) a_k k^\beta \right].
$$
ity (cf. Lemma 2.3.6 of [vdVW96]), we have that any variables have finite moments of any order, and the last inequality follows from (2.11). On the other hand, the claim is trivial for \( \alpha > n \). From the proof of Proposition 2.16, the bracketing entropy upper bound is only used to prove \( \sup_{k \leq n} a_k \geq 1/(2C_1 G_a) \). This proves our claim (2.9).

Now by de-symmetrization inequality (cf. Lemma 2.3.6 of [vdVW96]), we have that

\[
\sum_{k=1}^{n} (|g(k)| - |g(k+1)|) a_k k^\beta = (I) + (II).
\]

For \( (I) \) in (2.10), using the same notation as in the proof of Lemma 2.15,

\[
(I) \leq C_2 \cdot \mathbb{E} \left[ \sum_{k=1}^{[c'|n]} (|g(k)| - |g(k+1)|) k^\beta \right] \leq C_2 \cdot \mathbb{E} \left[ \sum_{k=1}^{[c'|n]} \int_{|g(k+1)|}^{[g(k)]} k^\beta \, dt \right]
\]

\[
\leq C_2 \cdot \mathbb{E} \int_0^{\infty} \left( \sum_{i=1}^{n} 1_{|g_i| \geq t} 1_{|g([c'|n])| \leq |g(i)|} \right)^{\beta/2} \, dt
\]

\[
\leq C_2 n^{\beta} \left( \int_0^{K^{-1} \sqrt{\log(1/c)}} \phi(t)^{\beta/2} \mathbb{P}(|g([c'|n])| \leq t) \, dt + \int_{K^{-1} \sqrt{\log(1/c)}}^{\infty} \phi(t)^{\beta/2} \, dt \right)
\]

\[
\leq C_3 n^{\beta} \left( e^{-\beta t^2/2} e^{-(c'|n)^2/2} \right)
\]

by choosing \( c' \equiv \exp(-C_3 \log(4C_1 C_3)) \) and \( n \geq C_3 \log(4C_1 C_3)/(c'|n)^2 \). On the other hand, for \( (II) \) in (2.10), we have

\[
(II) \leq \left( \max_{[c'|n] \leq k \leq n} a_k \right) \mathbb{E} \left[ \sum_{k=1}^{n} (|g(k)| - |g(k+1)|) k^\beta \right] \leq \left( \max_{[c'|n] \leq k \leq n} a_k \right) G_\alpha n^\beta,
\]

where \( G_\alpha = \int_0^{\infty} (\mathbb{P}(|g_1| > t))^\alpha/(1+\alpha) \, dt < \infty \) since Gaussian random variables have finite moments of any order, and the last inequality follows from Jensen’s inequality. Combining the above displays we see that \( \max_{[c'|n] \leq k \leq n} a_k \geq 1/(2C_1 G_a) \). This proves our claim (2.9).

Now by de-symmetrization inequality (cf. Lemma 2.3.6 of [vdVW96]), we have that

\[
\mathbb{E} \sup_{f \in \mathcal{F}(\sigma_n)} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i f(X_i) \leq 2 \mathbb{E} \sup_{f \in \mathcal{F}(\sigma_n)} |G_n(f)| + 2\sigma_n.
\]

For \( \sigma_n \leq (c_1/4)^{1/\alpha} \wedge c_1^{1/2} \), the claim of the proposition follows from (2.9) and (2.11). On the other hand, the claim is trivial for \( \sigma_n > (c_1/4)^{1/\alpha} \wedge c_1^{1/2} \).

**Remark 2.17.** From the proof of Proposition 2.16, the bracketing entropy upper bound is only used to prove \( \sup_{k \leq n} a_k \leq C_2 \equiv C_2(\alpha) \). This means that we may impose instead a uniform entropy upper bound condition as in [GK06] in the regime \( 0 < \alpha < 1 \).

**Proposition 2.18.** Let the conditions in Theorem 2.3 hold for some \( \alpha > 1 \) and \( L > 0 \) large enough. Then for \( \sigma_n \equiv cn^{-1/(\alpha+1)} \) with some constant \( c > 0 \), \( \mathbb{E} \sup_{f \in \mathcal{F}(\sigma_n)} |G_n(f)| \geq n^{(\alpha-1)/2(\alpha+1)} \).
Proof. Proposition 2.11 shows that
\[ \mathbb{E} \sup_{f \in \mathcal{F}(\sigma_n)} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_i f(X_i) \right| \gtrsim \sigma_n \sqrt{\log N(\sigma_n/2, \mathcal{F}(\sigma_n), L_2(P))} \gtrsim n^{(\alpha-1)/(\alpha+1)}. \]

Now applying Lemma 2.14 in the following form,
\[ \mathbb{E} \sup_{f \in \mathcal{F}(\sigma_n)} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_i f(X_i) \right| \lesssim \max_{1 \leq k \leq n} \mathbb{E} \sup_{f \in \mathcal{F}(\sigma_n)} \left| \frac{1}{\sqrt{k}} \sum_{i=1}^{k} \varepsilon_i f(X_i) \right|, \]
we see that for some \( K > 0, \)
\[ \max_{1 \leq k \leq n} \mathbb{E} \sup_{f \in \mathcal{F}(\sigma_n)} |G_k(f)| \gtrsim K^{-1} n^{(\alpha-1)/(\alpha+1)}. \]

On the other hand, by enlarging \( K \) if necessary, Proposition 2.8 entails that \( \mathbb{E} \sup_{f \in \mathcal{F}(\sigma_k)} |G_k(f)| \leq K \cdot k^{(\alpha-1)/(\alpha+1)}, \) and hence
\[ \max_{1 \leq k \leq n} \mathbb{E} \sup_{f \in \mathcal{F}(\sigma_k)} |G_k(f)| \leq Kn^{(\alpha-1)/(\alpha+1)} \]
by the assumption \( \alpha > 1. \) Combining the upper and lower estimates we see that
\[ K^{-1} n^{(\alpha-1)/(\alpha+1)} \leq \max_{1 \leq k \leq n} \mathbb{E} \sup_{f \in \mathcal{F}(\sigma_n)} |G_k(f)| \leq \max_{1 \leq k \leq n} \mathbb{E} \sup_{f \in \mathcal{F}(\sigma_k)} |G_k(f)| \leq Kn^{(\alpha-1)/(\alpha+1)}. \]

Now we will argue that the max operator can be ‘eliminated’. To this end, let \( a_k \equiv \mathbb{E} \sup_{f \in \mathcal{F}(\sigma_n)} |G_k(f)| \) and \( \beta \equiv (\alpha - 1)/(\alpha + 1) \) for notational convenience. Let \( k_n \equiv \arg \max_{1 \leq k \leq n} a_k. \) We claim that \( k_n \in [cn,n] \) where \( c = K^{-2/\beta} \in (0,1) \). To see this, we only need to note \( K^{-1} n^\beta \leq \max_{1 \leq k \leq n} a_k = a_{k_n} \leq K k_n^\beta, \) which entails \( k_n^\beta \geq K^{-2} n^\beta. \) Hence
\[ K^{-1} n^{(\alpha-1)/(\alpha+1)} \leq \mathbb{E} \sup_{f \in \mathcal{F}(\sigma_n)} |G_{k_n}(f)| \leq \frac{1}{\sqrt{c}} \mathbb{E} \sup_{f \in \mathcal{F}(\sigma_n)} |G_n(f)| \]
where the last inequality follows from Jensen’s inequality, proving the claim.

Proof of Theorem 2.3. The claims follow by combining Propositions 2.16 and 2.18.

3. Rate-optimal global ERMs

In this section, we apply the sharp bounds derived in the previous section to several models including (i) edge estimation problem in additive and multiplicative regression model, (ii) the binary classification problem, (iii) multiple isotonic regression, and (iv) s-concave density estimation, all in general dimensions. Global ERMs in these models are non-Donsker problems in general dimensions, but we will show that in each of these models, the underlying empirical process problem [1,5] can be essentially characterized
by an empirical process indexed by certain class of measurable sets. The sharp bounds in Theorem 2.1 can then be used to prove that these global ERMs converge at an optimal rate (up to logarithmic factors), rather than a strictly sub-optimal rate as predicted using the entropy integral (= (1.2)) in [BM93].

3.1. Edge estimation: additive errors. Let $X_1, \ldots, X_n$ be i.i.d. samples with law $P$ on a sample space ($\mathcal{X}, \mathcal{A}$). In this subsection we consider the regression model with additive errors:

$$Y_i = 1_{C_0}(X_i) + \xi_i, \quad i = 1, \ldots, n. \quad (3.1)$$

This model has been considered by [KT92, KT93] and more recently by [Bru13]. We assume for simplicity that the $\xi_i$'s are i.i.d. $\mathcal{N}(0,1)$ and are independent of $X_i$'s. Let $\mathcal{C}$ be a collection of measurable sets in $\mathcal{X}$, and we will fit the regression model by $\{1_C : C \in \mathcal{C}\}$. Our interest will be the behavior of the least squares estimator $\hat{C}_n$ defined by

$$\hat{C}_n \in \arg\min_{C \in \mathcal{C}} \sum_{i=1}^{n} (Y_i - 1_C(X_i))^2. \quad (3.2)$$

We assume that $\hat{C}_n$ is well-defined without loss of generality.

**Theorem 3.1.** Suppose that for some $\alpha \neq 1$, $\log N_I(\varepsilon, \mathcal{C}, P) \leq L\varepsilon^{-\alpha}$. Then

$$\sup_{C_0 \in \mathcal{C}} \mathbb{E}_{C_0}|\hat{C}_n \Delta C_0| \lesssim n^{-1/(\alpha+1)}. \quad (3.3)$$

By [YB99], the rate $n^{-1/(\alpha+1)}$ cannot be improved in a minimax sense if furthermore a lower bound on the metric entropy on the same order as that of the upper bound is available.

As a straightforward corollary of the above Theorem 3.1, let $\mathcal{C}_d$ be the collection of all convex bodies contained in the unit ball in $\mathbb{R}^d$ and $P$ the uniform distribution on the unit ball.

**Corollary 3.2.** Fix $d \geq 4$. Then

$$\sup_{C_0 \in \mathcal{C}_d} \mathbb{E}_{C_0}|\hat{C}_n \Delta C_0| \lesssim n^{-2/(d+1)}. \quad (3.4)$$

**Proof.** The claim essentially follows from Theorem 8.25 and Corollary 8.26 of [Dud14], asserting that we can take $\alpha = (d - 1)/2$ in Theorem 3.1. \qed

The corollary shows that we can use a global least squares estimator rather than a sieved least squares estimator (cf. [Bru13]) to achieve the optimal rate of convergence.

**Remark 3.3.** It is possible to impose certain tail conditions on the density of $P$ to extend the above corollary to a maximum risk bound over all convex sets in $\mathbb{R}^d$. A proof in this vein is carried out in the context of $s$-concave density estimation in $\mathbb{R}^d$ to be detailed below.

Before the proof of Theorem 3.1, we need following:
Lemma 3.4 (Proposition 2 of [HW18a]). Consider the regression model \((3.7)\) and the least squares estimator \(\hat{C}_n\) in \((3.2)\). Suppose that \(\xi_1, \ldots, \xi_n\) are mean-zero random variables independent of \(X_1, \ldots, X_n\). Further assume that

\[
\mathbb{E} \sup_{C \in \mathcal{C} : \|C\Delta C_0\| \leq \delta^2} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i (1_C - 1_{C_0})(X_i) \lesssim \phi_n(\delta),
\]

\[
\mathbb{E} \sup_{C \in \mathcal{C} : \|C\Delta C_0\| \leq \delta^2} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i (1_C - 1_{C_0})(X_i) \lesssim \phi_n(\delta),
\]

hold for some \(\phi_n\) such that \(\delta \rightarrow \phi_n(\delta)/\delta\) is non-increasing. Then \(\mathbb{E}|\hat{C}_n\Delta C_0| = O(\delta_n^2)\) holds for any \(\delta_n\) such that \(\phi_n(\delta_n) \leq \sqrt{n} \delta_n^2\), where the constant in \(O\) only depends on the constants in \((3.3)\).

Proof of Theorem 3.1. By Lemma 3.4 the risk of the least squares estimator

\[\delta_n^2 \equiv \sup_{C_0 \in \mathcal{C}} \mathbb{E}|\hat{C}_n\Delta C_0| = \sup_{C_0 \in \mathcal{C}} \mathbb{E}(1_{\hat{C}_n} - 1_{C_0})^2\]

can be solved by estimating the empirical processes in \((3.3)\). Since the global entropy estimate is translation invariant, by Theorem 2.1 we obtain an estimate for the Rademacher randomized empirical process:

\[\sup_{C_0 \in \mathcal{C}} \mathbb{E} \sup_{C \in \mathcal{C} : \|C\Delta C_0\| \leq \delta^2} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i (1_C - 1_{C_0})(X_i) \lesssim \max\{\delta_n^{1-\alpha}, n^{(\alpha-1)/2(\alpha+1)}\} \]

It is now easy to see that the choice \(\delta_n^2 \asymp n^{-1/(\alpha+1)}\) leads to an upper bound of the above display on the desired order \(\sqrt{n} \delta_n^2\). The Gaussian randomized empirical process can be handled via the multiplier inequality Lemma 2.13 by letting \(\psi_n(t) \equiv \psi(t) \equiv t^{\alpha/(\alpha+1)}\) and then use the fact that Gaussian random variables have infinitely many moments. We omit the details. \(\square\)

3.2. Edge estimation: multiplicative errors. In this subsection we consider the regression model with multiplicative errors as in [KT93, MT95]:

\[Y_i = f_{C_0}(X_i) \eta_i\]

where \(f_{C_0}(x) = 21_{C_0}(x) - 1\) and \(\eta_i\)'s are i.i.d. random variables such that \(\mathbb{P}(\eta_i = 1) = 1/2 + a\) and \(\mathbb{P}(\eta_i = -1) = 1/2 - a\) for some \(a \in (0, 1/2)\). Such a model is motivated by estimation of sets in multi-dimensional ‘black and white’ pictures, where \(Y_i = 1\) is interpreted as observing black, and \(Y_i = -1\) is white. We refer the reader to [MT95] for more motivation for this model. The model \((3.3)\) can be rewritten as

\[Y_i = 2af_{C_0}(X_i) + \xi_i\]

where \(\xi_i = f_{C_0}(X_i)(\eta_i - 2a)\)'s are bounded errors. An important property for these errors is that \(\mathbb{E}[\xi_i | X_i] = 0\) for all \(i = 1, \ldots, n\). Note here \(\xi_i\) is not
independent of $X_i$ and hence a different analysis is needed. Now consider
the least squares estimator

\begin{equation}
\hat{C}_n \equiv \arg \min_{C \in \mathcal{C}} \sum_{i=1}^{n} (Y_i - 2afC(X_i))^2.
\end{equation}

**Theorem 3.5.** Suppose that for some $\alpha \neq 1$, $\log N(\epsilon, \mathcal{C}, P) \leq L\epsilon^{-\alpha}$. Then

$$
\sup_{C_0 \in \mathcal{C}} \mathbb{E}[\hat{C}_n \Delta C_0] \lesssim n^{-1/(\alpha+1)}.
$$

Compared with Theorem 4.1 in [MT95], we use an unsieved least squares estimator to achieve the optimal rate, rather than their theoretical ‘sieved’ estimator. This provides another example for which the simple least squares estimator can be rate-optimal for non-Donsker function classes in a natural setting.

We need the following analogy of Lemma 3.4 before proving Theorem 3.5.

**Lemma 3.6.** Consider the regression model (3.3) and the least squares estimator $\hat{C}_n$ in (3.6). Further assume that

\[
\mathbb{E} \sup_{C \in \mathcal{C} : |C - C_0| \leq \delta^2} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \epsilon_i (1_C - 1_{C_0})(X_i) \right| \quad \vee \quad \mathbb{E} \sup_{C \in \mathcal{C} : |C - C_0| \leq \delta^2} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \epsilon_i (1_C - 1_{C_0})(X_i) \right| \lesssim \phi_n(\delta),
\]

holds for some $\phi_n$ such that $\delta \mapsto \phi_n(\delta)/\delta$ is non-increasing. Then $\mathbb{E}(|\hat{C}_n - C_0|) = \mathcal{O}(\delta^2)$ holds for any $\delta_n$ such that $\phi_n(\delta_n) \leq \sqrt{n}\delta_n^2$, where the constant in $\mathcal{O}$ only depends on the constants in the above inequality.

**Proof.** We sketch the proof. Let

$$
\mathbb{M}_n 1_C = \frac{2}{n} \sum_{i=1}^{n} (1_C - 1_{C_0})(X_i) \xi_i - \frac{1}{n} \sum_{i=1}^{n} (1_C - 1_{C_0})^2(X_i),
$$

$$
\mathbb{M}_n 1_{C^2} = -P(1_C - 1_{C_0})^2.
$$

Here we used the fact that $\mathbb{E}[\xi_i | X_i] = 0$. Then it is easy to see that

$$
|\mathbb{M}_n 1_C - \mathbb{M}_n 1_{C_0} - (\mathbb{M}_n 1_C - \mathbb{M}_n 1_{C_0})|
$$

\[
\leq \left| \frac{2}{n} \sum_{i=1}^{n} (1_C - 1_{C_0})(X_i) \xi_i \right| + |(\mathbb{P}_n - P)(1_C - 1_{C_0})^2| \equiv (I) + (II).
\]

By symmetrization (cf. Theorem 3.1.21 of [GN15]) and contraction principle (cf. Theorem 3.1.17 of [GN15]) for the empirical process, the moments of the first term $(I)$ can be handled by

\[
\mathbb{E} \sup_{C \in \mathcal{C} : |C - C_0| < \delta^2} \left| \sum_{i=1}^{n} (1_C - 1_{C_0})(X_i) \xi_i \right|^p
\]

\[
\lesssim \mathbb{E} \sup_{C \in \mathcal{C} : |C - C_0| < \delta^2} \left| \sum_{i=1}^{n} \epsilon_i (1_C - 1_{C_0})(X_i) f_{C_0}(X_i) (\eta_i - 2\alpha) \right|^p
\]
It is then natural to consider an estimator of \( g \) attained by a Bayes classifier. It is known that for a given law \( P \) the empirical training error:

\[
\sum_{i=1}^{n} \epsilon_i (1_C - 1_{C_0})(X_i) f_{C_0}(X_i)\]

\( \varepsilon \) \( \geq \mathbb{E} \sup_{C \in \mathcal{E} : |C \Delta C_0| < \delta^2} \left| \sum_{i=1}^{n} \epsilon_i (1_C - 1_{C_0})(X_i) f_{C_0}(X_i) \right|^p \)

\( \varepsilon \) \( \geq \mathbb{E} \sup_{C \in \mathcal{E} : |C \Delta C_0| < \delta^2} \left| \sum_{i=1}^{n} \epsilon_i (1_C - 1_{C_0})(X_i) \right|^p \)

\( \varepsilon \) \( \geq \mathbb{E} \sup_{C \in \mathcal{E} : |C \Delta C_0| < \delta^2} \left| \sum_{i=1}^{n} \epsilon_i (1_C - 1_{C_0})(X_i) \right|^p \)

where the last inequality follows by noting that

\( (1_C - 1_{C_0})f_{C_0} = (1_C - 1_{C_0})(21_{C_0} - 1) = 2(1_{C \cap C_0} - 1_{C_0}) - (1_C - 1_{C_0}) \)

and using triangle inequality. The second term (II) is standard, and then we use the proof strategy of Proposition 2 in [HW18a], by noting that the class of interest here is uniformly bounded. Details are omitted.

**Proof of Theorem 3.5.** The proof follows by Lemma 3.6 and similar arguments as the proof of Theorem 3.1. \( \square \)

### 3.3. Binary classification: excess risk bounds.

In this subsection we consider the binary classification problem in the learning theory, cf. [Tsy04, MN06]. Suppose one observes i.i.d. \((X_1,Y_1), \ldots, (X_n,Y_n)\) with law \(P\), where \(X_i\)’s take values in \(\mathcal{X}\), and the responses \(Y_i \in \{0,1\}\) over a class \(\mathcal{G}\) has a generalization error \(P(Y \neq g(X))\). The excess risk for a classifier \(g\) over \(\mathcal{G}\) under law \(P\) is given by

\[
\mathcal{E}_P(g) \equiv P(Y \neq g(X)) - \inf_{g' \in \mathcal{G}} P(Y \neq g'(X)).
\]

It is known that for a given law \(P\) on \((X,Y)\), the minimal generalized error is attained by a Bayes classifier \(g_0(x) \equiv 1_{\eta(x) \geq 1/2}\) where \(\eta(x) \equiv \mathbb{E}[Y|X = x]\), cf. [DGL96]. It is then natural to consider an estimator of \(g_0\) by minimizing the empirical training error:

\[
\hat{g}_n \equiv \arg\min_{g \in \mathcal{G}} \frac{1}{n} \sum_{i=1}^{n} 1_{Y_i \neq g(X_i)}.
\]

We assume \(g_0 \in \mathcal{G}\) for simplicity. The quality of the estimator \(\hat{g}_n\) is measured by the excess risk:

\[
\mathcal{E}_P(\hat{g}_n) \equiv P(Y \neq \hat{g}_n(X)) - P(Y \neq g_0(X)).
\]

Let \(\Pi\) be the marginal distribution of \(X\) under \(P\). We assume the following ‘Tsylvak’s margin(low noise) condition’ (cf. [MT99, Tsy04]): there exists some \(c > 0\) such that for all \(g \in \mathcal{G}\),

\[
\mathcal{E}_P(g) \geq c(\Pi(g(X) \neq g_0(X))) = c\|g - g_0\|^2_{L_2(\Pi)}.
\]

Here we have assumed that the margin condition holds with \(\kappa = 1\). Although faster rates are possible under more general margin condition \(\kappa \geq 1\) (cf. [MT99, Tsy04]), we do not go into this direction to avoid distraction from our main points.
Below is the main result in this subsection, the formulation of which follows that of [Kol06] [GK06].

**Theorem 3.7.** Suppose $G \equiv \{1_C : C \in \mathcal{C}\}$ satisfies the following entropy condition: there exists some $\alpha \neq 1$ such that for all $\varepsilon > 0$, $\log \mathcal{N}_f(\varepsilon, \mathcal{C}, P) \leq L\varepsilon^{-\alpha}$. If $r_n^2 \geq Kn^{-1/(\alpha+1)}$ for a large enough constant $K > 0$, then

$$
\mathbb{P}(\mathcal{E}_P(\hat{g}_n) \geq r_n^2) \leq K' \exp(-nr_n^2/K')
$$

holds for some constant $K' > 0$.

[Tsy04] considered the same problem under the working assumption $\alpha \in (0, 1)$ (cf. Assumption A2, page 140). [MN06] used ratio-type empirical process techniques to give a more unified treatment of deriving risk bounds for this problem, when the class of classifiers satisfies a Donsker bracketing entropy condition (i.e. $0 < \alpha < 1$), or a Donsker uniform entropy condition. [GK06] further improved the result of [MN06] in the Donsker regime under a uniform entropy condition, by taking into account the size of the localized envelopes. See also [Kol06] page 2618, [Lec07] page 1706 for similar Donsker conditions. To the best knowledge of the author, our Theorem 3.7 gives a first result for the global ERM $\hat{g}_n$ in (2.7) to be rate-optimal in the non-Donsker regime $\alpha > 1$ in the classification problem.

We need some further notations before the proof. For any $g \in \mathcal{G}$, write $f_g(x, y) \equiv 1_{y \neq g(x)}$. Let $\mathcal{G}(\delta) \equiv \{g \in \mathcal{G} : \mathcal{E}_P(g) \leq \delta\}$. Let $\ell$ be the smallest integer such that $r_n^2 2^\ell \geq 1$, and for any $1 \leq j \leq \ell$, let $\mathcal{F}_j \equiv \{f_{g_1} - f_{g_2} : g_1, g_2 \in \mathcal{G}(r_n^2 2^j)\}$.

**Lemma 3.8.** Suppose $G \equiv \{1_C : C \in \mathcal{C}\}$ satisfies the same entropy condition as in Theorem 3.7. Then

$$
\mathbb{P}\left(\max_{1 \leq j \leq \ell} \sup_{f \in \mathcal{F}_j} \frac{\mathbb{P}(f) - \mathbb{P}(f)}{r_n^2 2^{j+1}} \geq c \left(\frac{1}{4} + K \sqrt{\frac{s}{nr_n^2} + \frac{s}{nr_n^2}}\right)\right) \leq K' \exp(-s/K')
$$

holds for some constants $K, K' > 0$ provided $r_n^2 \cdot n^{1/(\alpha+1)} \geq K''$ for a large enough constant $K'' > 0$ depending on $c > 0$ in (2.7) only.

**Proof.** By Talagrand’s concentration inequality (cf. Appendix B), with $\sigma_j^2 \equiv \sup_{f \in \mathcal{F}_j} \|f\|_{L_2(P)}^2$

$$
\mathbb{P}\left[\sup_{f \in \mathcal{F}_j} |G_n(f)| \geq K \left(\mathbb{E}_{f \in \mathcal{F}_j} |G_n(f)| + \sqrt{\sigma_j^2 s_j + \frac{s_j}{\sqrt{n}}}\right)\right] \leq K \exp(-s_j/K).
$$

Let $\mathcal{S} \equiv \{S : f_g = 1_S, g \in \mathcal{G}\}$. Note that for $g_1 = 1_{C_1}, g_2 = 1_{C_2} \in \mathcal{G}$, where $C_1, C_2 \in \mathcal{C}$, we have $f_{g_1} = 1_{S_1}, f_{g_2} = 1_{S_2}$, and hence

$$
P(S_1 \Delta S_2) = P(f_{g_1} - f_{g_2})^2 \leq P(g_1 - g_2)^2 = P(C_1 \Delta C_2).
$$

This shows that $\mathcal{N}_f(\varepsilon, \mathcal{S}, P) \leq \mathcal{N}_f(\varepsilon, \mathcal{C}, P)$. Furthermore, for any $g \in \mathcal{G}(r_n^2 2^j)$, let $S \in \mathcal{S}$ be such that $f_g = 1_S$. Then similar to the above display, we have

$$
P(S \Delta S_0) \leq \|g - g_0\|_{L_2(P)}^2 \leq c^{-1} i_n^2 2^j,
$$
where the last inequality follows from the margin condition. Now by Theorem 2.1, we obtain
\[
\mathbb{E} \sup_{f \in \mathcal{F}_j} |\mathbb{G}_n(f)| \leq \mathbb{E} \sup_{g \in \mathcal{G}(\mathbb{r}_2^2)} |\mathbb{G}_n(f_g)| \leq \mathbb{E} \sup_{S \in \mathcal{P}(\mathcal{S}) \leq c^{-1}\mathbb{r}_2^2} |\mathbb{G}_n(S)| \leq \max\{ (\mathbb{r}_2^2)^{(1-\alpha)/2}, n^{(\alpha-1)/2(\alpha+1)} \}.
\]

On the other hand,
\[
\sigma_j^2 \equiv \sup_{f \in \mathcal{F}_j} \|f\|_{L_2}^2 = \sup_{g_1, g_2 \in \mathcal{G}(\mathbb{r}_2^2)} \|g_1 - g_2\|_{L_2}^2 \leq 4 \sup_{g \in \mathcal{G}(\mathbb{r}_2^2)} \|g - g_0\|_{L_2}^2 \leq 4c^{-1} \sup_{g \in \mathcal{G}(\mathbb{r}_2^2)} \mathcal{E}_P(g) \leq 4c^{-1}\mathbb{r}_2^2 j.
\]
This implies that with \(s_j = s2^j\),
\[
\mathbb{P} \left[ \frac{\sup_{f \in \mathcal{F}_j} |\mathbb{P}_n(f) - P(f)|}{\mathbb{r}_n^2 2^j} \right] \geq K_c \mathbb{r}_n^{-2-2j} \left( \max\{ n^{-1/2}(\mathbb{r}_2^2)^{(1-\alpha)/2}, n^{-1/(\alpha+1)} \} + n^{-1/2}(\mathbb{r}_2^2)^{1/2} \sqrt{s2^j} + n^{-1}s2^j \right) \leq K \exp(-s2^j/K).
\]

Note that
\[
\mathbb{r}_n^{-2-2j} \left( \max\{ n^{-1/2}(\mathbb{r}_2^2)^{(1-\alpha)/2}, n^{-1/(\alpha+1)} \} + n^{-1/2}(\mathbb{r}_2^2)^{1/2} \sqrt{s2^j} + n^{-1}s2^j \right)
\]
\[
\leq \max\left\{ \frac{1}{\sqrt{n}\mathbb{r}_n^{2(1-\alpha)}}, \frac{1}{\mathbb{r}_n^{1/(\alpha+1)}} \right\} + \frac{s}{n\mathbb{r}_n^2} + \frac{s}{\mathbb{r}_n^2} \leq \frac{c}{4K} + \frac{s}{n\mathbb{r}_n^2} + \frac{s}{\mathbb{r}_n^2}
\]
under the assumption. Now a union bound leads to the desired claim. \(\square\)

**Proof of Theorem 3.7.** Given the estimate in Lemma 3.8, the proof of the theorem closely follows that of Theorem 7.1 of [GK06]. We provide some details for the convenience of the reader. On the event
\[
E \equiv \left\{ \max_{1 \leq j \leq \ell} \sup_{f \in \mathcal{F}_j} |\mathbb{P}_n(f) - P(f)|^{2^j} \leq c \left( \frac{1}{4} + K \sqrt{\frac{s}{n\mathbb{r}_n^2}} + K \frac{s}{\mathbb{r}_n^2} \right) \right\},
\]
we have for any \(g \in \mathcal{G}(\mathbb{r}_2^2) \setminus \mathcal{G}(\mathbb{r}_2^2)^{-1}\) and \(g' \in \mathcal{G}(\sigma)\) for some \(0 < \sigma < \mathbb{r}_2^2 j\),
\[
\mathcal{E}_P(g) = P(f_g - f_{g'}) + [P(f_{g'}) - P(f_{g'})] \leq P(f_g - f_{g'}) + \sigma
\]
\[
\leq \mathbb{P}_n(f_g - f_{g'}) + \sigma + \sup_{f \in \mathcal{F}_j} |(\mathbb{P}_n - P)(f)|
\]
\[
\leq \mathcal{E}_\mathbb{P}_n(g) + \sigma + c \left( \frac{1}{4} + K \sqrt{\frac{s}{n\mathbb{r}_n^2}} + K \frac{s}{\mathbb{r}_n^2} \right)^{\mathbb{r}_n^2 2^j}
\]
\[
\leq \mathcal{E}_\mathbb{P}_n(g) + \sigma + \left( \frac{1}{4} + K \sqrt{\frac{s}{n\mathbb{r}_n^2}} + K \frac{s}{\mathbb{r}_n^2} \right) 2\mathcal{E}_P(g).
\]

Since \(\sigma > 0\) is taken arbitrarily, we see that on the event \(E\), it holds that
\[
\frac{\mathcal{E}_\mathbb{P}_n(g)}{\mathcal{E}_P(g)} \geq 1 - \left( \frac{1}{2} + 2K \sqrt{\frac{s}{n\mathbb{r}_n^2}} + 2K \frac{s}{\mathbb{r}_n^2} \right)
\]
for all $g \in \mathcal{G}$ such that $\mathcal{E}_P(g) \geq r_n^2$. Furthermore, the above display entails that on the event $E$, we necessarily have $\mathcal{E}_P(\hat{g}_n) < r_n^2$ for $n$ large enough. Hence for any $g \in \mathcal{G}(r_n^2 2^j) \setminus \mathcal{G}(r_n^2 2^{j-1})$, we have

$$\mathcal{E}_{\mathbb{P}_n}(g) = \mathbb{P}_n(f_g) - \mathbb{P}_n(f_{\hat{g}_n}) \leq P f_g - P f_{\hat{g}_n} + \sup_{f \in \mathcal{F}_j} |(\mathbb{P}_n - P)(f)|$$

$$\leq \mathcal{E}_P(g) + \left( \frac{1}{4} + K \sqrt{\frac{s}{nr_n^2}} + K \frac{s}{nr_n^2} \right) 2\mathcal{E}_P(g).$$

This entails that

$$\frac{\mathcal{E}_{\mathbb{P}_n}(g)}{\mathcal{E}_P(g)} \leq 1 + \left( \frac{1}{2} + 2K \sqrt{\frac{s}{nr_n^2}} + 2K \frac{s}{nr_n^2} \right).$$

The proof of the claim is complete by combining (3.9)-(3.10) along with Lemma 3.8. \qed

### 3.4. Multiple isotonic regression

Let $X_1, \ldots, X_n$ be i.i.d. with law $P$ on $[0, 1]^d$. For simplicity we assume that $P$ is the uniform distribution on $[0, 1]^d$. Consider the multiple isotonic regression model

$$(3.11) \hspace{1cm} Y_i = f_0(X_i) + \xi_i, \hspace{0.5cm} i = 1, \ldots, n,$$

where $\xi_i$'s are i.i.d. Gaussian errors $\mathcal{N}(0, 1)$, and $f_0 \in \mathcal{M}_d \equiv \{ f : [0, 1]^d \to \mathbb{R}, f(x) \leq f(y) \text{ for any } x \leq y \}$. Consider the isotonic least squares regression estimator $\hat{f}_n$ defined via:

$$\hat{f}_n \equiv \arg \min_{f \in \mathcal{M}_d} \sum_{i=1}^n (Y_i - f(X_i))^2.$$ 

The performance of $\hat{f}_n$ in the multivariate setting is examined by [CGS18] for $d = 2$ and [HWCS18] for $d \geq 3$. By the entropy estimate for uniformly bounded multiple isotonic functions in [GW07], $\mathcal{M}_d \cap L_\infty(1)$ is in the non-Donsker regime when $d \geq 3$, which is the main interesting case here.

**Theorem 3.9.** Let $d \geq 2$. Then with $\gamma^{iso}_d = (2)^{d=2} + 1_{d \geq 3}$,

$$\sup_{f_0 \in \mathcal{M}_d \cap L_\infty(1)} \mathbb{E}_{f_0} \|\hat{f}_n - f_0\|_{L_2(P)}^2 \lesssim_d n^{-1/d} \log^{\gamma^{iso}_d} n.$$

**Remark 3.10.** Compared with Theorem 4 of [HWCS18], the above result gives improvements over logarithmic factors. The logarithmic factors in $d \geq 3$ are due to boundary behavior of $\hat{f}_n$. For instance, if the errors are bounded, then we may remove these logarithmic factors to get a sharp rate $n^{-1/d}$ for $d \geq 3$. These logarithmic factors cannot be removed by the proof techniques in [HWCS18] even if the errors are bounded. The rate $n^{-1/d}$ is shown to be minimax optimal for squared $L_2$ loss in [HWCS18].

**Lemma 3.11.** It holds for $C > 0$ large enough that

$$\mathbb{E}_{f_0} \|\hat{f}_n - f_0\|_{L_2(P)}^2 \leq \mathbb{E}_{f_0} \|\hat{f}_n - f_0\|_{L_2(P)}^2 1_{\|\hat{f}_n - f_0\|_{\infty} \leq C \sqrt{\log n}} + O(n^{-1}).$$

The $O$ term is uniform in $f_0 \in \mathcal{M}_d \cap L_\infty(1)$. 
Proof. Fix \( f_0 \in \mathcal{M}_d \cap L_\infty (1) \). By Lemma 10 in the supplement of [HWCST18],
\[
\sup_{x \in [0,1]^d} (\hat{f}_n - f_0)(x) \leq \max_{1 \leq i \leq n} Y_i + \| f_0 \|_\infty \leq 2 + \max_{1 \leq i \leq n} \xi_i ,
\]
Hence with \( Z_n \equiv \| \hat{f}_n - f_0 \|_\infty \), for \( u \) large, \( \mathbb{P}( Z_n > u \sqrt{\log n} ) \leq e^{-cu^2 \log n} \) for some \( c > 0 \). In particular, \( \mathbb{E} Z_n^4 \leq \log^2 n \). Now the claim of the lemma follows by noting that
\[
\mathbb{E}_f \| \hat{f}_n - f_0 \|_{L_2 (P)}^2 \mathbf{1}_{\| \hat{f}_n - f_0 \|_\infty > C \sqrt{\log n}} \leq \mathbb{E} Z_n^2 \mathbf{1}_{Z_n > C \sqrt{\log n}}
\]
\[
\leq \sqrt{\mathbb{E} Z_n^4} \cdot \sqrt{\mathbb{P}( Z_n > C \sqrt{\log n} )} \leq \log n \cdot e^{-cC^2 \log n/2} = O(n^{-1})
\]
for \( C > 0 \) large.

Proof of Theorem 3.9. First consider \( d \geq 3 \). By Lemma 4.11 we only need to compute an upper bound for \( \mathbb{E}_f \| \hat{f}_n - f_0 \|_{L_2 (P)}^2 \mathbf{1}_{\| \hat{f}_n - f_0 \|_\infty \leq C \sqrt{\log n}} \leq r_n^2 \).
This can be done by evaluating the size of two empirical processes
\[
\mathbb{E}_f \sup_{f \in \mathcal{M}_d \cap L_\infty (C \sqrt{\log n})} | G_n (\xi (f - f_0)) | \leq \mathbb{E}_f \sup_{f \in \mathcal{M}_d \cap L_\infty (C \sqrt{\log n})} | G_n ((f - f_0)^2) | \lesssim \sqrt{nr_n^2}.
\]
Note that for any \( f \in \mathcal{M}_d \),
\[
| (\mathbb{P}_n - P) f | \leq | \mathbb{E}_{\mathbb{P}_n} f (X) - \mathbb{E}_P f (X) | \leq \int_0^\infty ( \mathbb{P}_{\mathbb{P}_n} (f_+ (X) > t) - \mathbb{P}_P (f_+ (X) > t) ) \, dt \leq 2 \| f \|_\infty \sup_{C \in \mathcal{L}_d} | (\mathbb{P}_n - P) (C) |.
\]
Here \( \mathcal{L}_d \) is the class of all upper and lower sets in \([0,1]^d\). The last inequality follows since for any \( f \in \mathcal{M}_d \), \( \{ f_+ (x) > t \} = \{ f (x) \vee 0 > t \} \in \mathcal{L}_d \) and \( \{ f_- (x) > t \} = \{ -f (x) \wedge 0 > t \} = \{ f (x) \wedge 0 < -t \} \in \mathcal{L}_d \). Hence by Theorem 8.22 of [Dud14], we may apply Theorem 2.1 with \( \alpha = d - 1 \) to see that
\[
\mathbb{E} \sup_{f \in \mathcal{M}_d \cap L_\infty (C \sqrt{\log n})} | G_n (f - f_0) | \lesssim \sqrt{\log n} \cdot \mathbb{E} \sup_{C \in \mathcal{L}_d} | G_n (C) | + 1 \lesssim \sqrt{\log n} \cdot n^{d-2/2d}.
\]
Using the multiplier inequality (cf. Lemma 2.13 and contraction principle for empirical processes (cf. Lemma 6 in the supplement of [HWCST18]), we may further bound the two empirical processes in (3.12) by
\[
\mathbb{E} \sup_{f \in \mathcal{M}_d \cap L_\infty (C \sqrt{\log n})} | G_n (\xi (f - f_0)) | \leq \sqrt{\log n} \cdot \mathbb{E} \sup_{C \in \mathcal{L}_d} | G_n (C) | + 1 \lesssim \sqrt{\log n} \cdot n^{d-2/2d}.
\]
SHAPE-CONSTRAINED ERMS ARE RATE OPTIMAL

\[
\forall \mathbb{E} \sup_{f \in \mathcal{M}_d \cap L_\infty(C\sqrt{\log n}): \|f - f_0\|_{L_2(P)} \leq \delta_n} |G_n((f - f_0)^2)| \lesssim n^{\frac{d-2}2 \log n}.
\]

Solving (3.12) using the above inequality we obtain the rate \( \bar{r}_n \) for \( d \geq 3 \).

For \( d = 2 \), we may estimate the empirical process with an additional \( \log n \):

\[
\mathbb{E} \sup_{f \in \mathcal{M}_d \cap L_\infty(C\sqrt{\log n})} |G_n(f - f_0)| \lesssim \log^{3/2} n,
\]

and the rate can be obtained similarly as above. \( \square \)

**Remark 3.12.** The proof for the analogue of Theorem 3.9 in [HWCS18], i.e. Theorem 4 of [HWCS18], uses a completely different strategy. A rough argument is as follows. [HWCS18] first consider the problem \( f_0 = 0 \), where it is shown in Proposition 9 therein that for \( \delta_n > 0 \) not too small,

\[
(3.13) \quad \mathbb{E} \sup_{f \in \mathcal{M}_d \cap L_\infty(C\sqrt{\log n})} |G_n(f)| \lesssim \delta_n \cdot n^{1/2 - 1/d} \log^\gamma n.
\]

Then by a simple triangle inequality, if \( d \geq 2 \),

\[
(3.14) \quad \mathbb{E} \sup_{f \in \mathcal{M}_d \cap L_\infty(C\sqrt{\log n})} |G_n(f - f_0)|
\]

\[
\lesssim \mathbb{E} \sup_{f \in \mathcal{M}_d \cap L_\infty(C\sqrt{\log n})} |G_n(f)| + \mathbb{E} |G_n(f_0)| \lesssim (\delta_n + \|f_0\|_{\infty})n^{1/2 - 1/d} \log^\gamma n.
\]

Using the above inequality and (3.12), we obtain \( \bar{r}_n^2 \lesssim n^{-1/d} \) up to logarithmic factors. It is clear from the sketch here that the property of isotonic regression functions is only used in (3.13) where the problem is \( f_0 = 0 \). The proof for general \( f_0 \in L_\infty(1) \) in (3.14) is not very informative in the sense that the method of (3.14) is valid for any problem as long as one could solve the risk problem (= empirical process problem (3.13)) for one particular \( f_0 \). In contrast, our proof here shows that it is the complexity of the class of upper and lower sets \( \mathcal{L}_d \) that leads to the minimax rate of convergence for the multiple isotonic LSE.

**Remark 3.13.** It is in principle possible to adapt the present approach to the problem of multivariate convex regression. The major difficulty here, however, is to understand the boundary behavior for the convex least squares estimator. In a recent paper of the author [HW18b], it is shown that for univariate convex regression, the boundary behavior of the LSE does not blow up; it remains a substantial challenge to understand the situation in a multivariate setting. One crucial difference here is that in the Gaussian multiple isotonic regression, it is easy to show that \( \|\hat{f}_n\|_{\infty} = \mathcal{O}_P(\sqrt{\log n}) \) with a Gaussian tail; while for the convex regression, even in \( d = 1 \), the tail of \( \|\hat{f}_n\|_{\infty} \) is so heavy that \( \mathbb{E} \|\hat{f}_0\|_{L_2(P)} = \infty \).
3.5. *s-concave density estimation in \( \mathbb{R}^d \).* We first introduce the class of *s-concave densities on \( \mathbb{R}^d \).* The exposition follows that of [HW16]. Let

\[
M_s(a, b; \theta) \equiv \begin{cases} 
(1 - \theta)a^s + \theta b^s \right)^{1/s}, & s \neq 0, a, b > 0, \\
0, & s < 0, ab = 0, \\
a^{1-\theta}b^\theta, & s = 0, \\
a \land b, & s = -\infty.
\end{cases}
\]

A density \( p \) on \( \mathbb{R}^d \) is called *s-concave*, i.e. \( p \in \mathcal{P}_s \) if and only if for all \( x_0, x_1 \in \mathbb{R}^d \) and \( \theta \in (0, 1) \), \( p((1 - \theta)x_0 + \theta x_1) \geq M_s(p(x_0), p(x_1); \theta) \). It is easy to see that the densities \( p \) have the form \( p = \varphi_{+}^{1/s} \) for some concave function \( \varphi \) if \( s > 0 \), \( p = \exp(\varphi) \) for some concave \( \varphi \) if \( s = 0 \), and \( p = \varphi_{+}^{1/s} \) for some convex \( \varphi \) if \( s < 0 \). The function classes \( \mathcal{P}_s \) are nested in \( s \) in that for every \( r > 0 > s \), we have \( \mathcal{P}_r \supset \mathcal{P}_0 \supset \mathcal{P}_s \supset \mathcal{P}_{-\infty} \).

Maximum likelihood estimation over \( \mathcal{P}_s \) is proposed in [SW10], where existence and consistency of the MLE \( \hat{p}_n \) is proved. Global rates of convergence of the MLE \( \hat{p}_n \) over \( \mathcal{P}_s \) is primarily studied in the special case \( s = 0 \), also known as the log-concave MLE, cf. [KS16]. For general *s-concave MLEs*, the only result concerning global convergence rates is due to [DW16], who studied the univariate case \( d = 1 \), \( s > -1 \), showing that \( h^2(\hat{p}_n, p_0) = \mathcal{O}_p(n^{-4/5}) \), where \( h(\cdot, \cdot) \) is the Hellinger distance. Here we will be interested in general *s-concave MLEs* in general dimensions.

**Theorem 3.14.** Suppose \( s > -1/d \) and \( d \geq 2 \). Then

\[
h^2(\hat{p}_n, p_0) = \mathcal{O}_p(n^{-2/(d+1)} \log^{\gamma_d} n),
\]

where \( \gamma_d = (2/3)1_{d=2} + (2)1_{d=3} + 1_{d \geq 4} \).

The most interesting regime here is \( d \geq 4 \) when the entropy integral for the class of *s-concave densities* diverges. Modulo logarithmic factors, the rates of convergence for the *s-concave MLE* \( \hat{p}_n \) in squared Hellinger distance is \( \mathcal{O}_p(n^{-2/(d+1)}) \), which matches the minmax lower bound for the smaller log-concave (= 0-concave) class, cf. [KS16].

The integrability restriction \( s > -1/d \) is very natural in this setting: if \( s < -1/d \), then there exists a family of *s-concave densities* with singularities so that the MLE does not exist. The following proposition makes this precise.

**Proposition 3.15.** The *s-concave MLE* does not exist for \( s < -1/d \).

**Proof.** For \( a \in \mathbb{R}^d, b > 0 \), let \( \varphi_{a,b}(x) \equiv \|x - a\| 1_{\|x-a\| \leq b} + \infty 1_{\|x-a\| > b} \). Since \( c_b \equiv \int \varphi_{a,b}^{1/s} = \int \varphi_{0,b}^{1/s} < \infty \) for \( s \neq -1/d \), \( p_{a,b} \equiv \varphi_{a,a}^{1/s} / c_b \) is an *s-concave density*. The log likelihood function for observed \( X_1, \ldots, X_n \) is \( \ell(a, b) \equiv \log \prod_{i=1}^{n} p_{a,b}(X_i) = \sum_{i=1}^{n} \frac{(1/s) \log(\|X_i - a\|)}{-\log c_b} \) for \( (a, b) \) such that \( \max_i \|X_i - a\| \leq b \) and \( X_i \neq a \) for \( i = 1, \ldots, n \). For \( b \) large enough and \( a \) approaches any of \( X_i \)'s, \( \ell(a, b) \nearrow \infty \), so the MLE does not exist. \( \square \)
The univariate case $d = 1$ for the above proposition can also be found in [DW16]. Now we prove Theorem 3.14.

**Proof of Theorem 3.14** We only provide the proof for the most difficult case $-1/d < s < 0$; the other cases are similar or simpler.

First consider $d \geq 4$. Using same arguments as in Step 1 of the proof of Theorem 4.3 in [DW16], we may assume without loss of generality that $p_0 \in \mathcal{P}_{s,M/2}$ and $\hat{p}_n$ belongs to

$$
\mathcal{P}_{s,M} \equiv \left\{ p \in \mathcal{P}_s : \sup_{x \in \mathbb{R}^d} p(x) \leq M, \inf_{x \in B(0,1)} p(x) \geq 1/M \right\}
$$

for some large $M$ with high probability. By the proof of Lemma F.7 of [HW16] (especially (F.3) therein),

$$
\sup_{p \in \mathcal{P}_{s,M}} p(x) \leq C_M \left( 1 + \|x\| \right)^{1/s} \leq C_{M,d} \left( 1 + \prod_{k=1}^{d} |x_k|^{1/d} \right)^{1/s}. \tag{3.15}
$$

Furthermore, it is not hard to see that $\hat{p}_n$ is supported in the convex hull of $X_1, \ldots, X_n$. As $1/p_0(X_1)$ admits $L_q$ moment for some $q > 0$ by (3.15), $
\log \max_i (1/p_0(X_i)) \leq C_1 \log n$ with high probability for large $C_1 > 0$. Hence with $c_n \equiv n^{-C_1}, X_1, \ldots, X_n \in \{ p_0 \geq c_n \}$ with high probability. Let $\tilde{p}_n \equiv (\hat{p}_n \lor c_n) 1_{p_0 \geq c_n} / \int (\hat{p}_n \lor c_n) 1_{p_0 \geq c_n}$. Then with $b_n \equiv \int (\hat{p}_n \lor c_n) 1_{p_0 \geq c_n}$, it follows that $b_n^{-1} = b_n^{-1} \int_{p_0 \geq c_n} \hat{p}_n \leq b_n^{-1} \int_{p_0 \geq c_n} (\hat{p}_n \lor c_n) = 1$, and $b_n - 1 = \int_{p_0 \geq c_n} (\hat{p}_n \lor c_n) - \hat{p}_n \leq c_n \{ p_0 \geq c_n \} \leq c_n^{1+sd}$ by (3.15). By the condition $s > -1/d$, by choosing $C_1 > 0$ large, we have $0 \leq b_n - 1 \leq O(n^{-1})$. This implies with high probability,

$$
h^2(\hat{p}_n, \tilde{p}_n) \leq \int |\hat{p}_n - \tilde{p}_n| \leq |1 - b_n^{-1}| \int \hat{p}_n + b_n^{-1}(b_n - 1) = O(n^{-1}).
$$

Hence by the integrability $\mathbb{E}_{P_0} \log^2 p_0 < \infty$, with $P_0$ denoting the distribution of $p_0$, it follows that with high probability,

$$
h^2(p_0, \hat{p}_n) \leq h^2(p_0, \tilde{p}_n) + h^2(\hat{p}_n, \tilde{p}_n) \leq \mathbb{E}_{P_0} \log(p_0/\tilde{p}_n) + O(n^{-1})
$$

$$
\leq \mathbb{E}_{P_n} \log p_0 - \mathbb{E}_{P_0} \log \tilde{p}_n + O(n^{-1/2}) \leq \mathbb{E}_{P_n} \log \hat{p}_n - \mathbb{E}_{P_0} \log \tilde{p}_n + O(n^{-1/2})
$$

$$
\leq \mathbb{E}_{P_n} \log \left[ b_n^{-1}(\hat{p}_n \lor c_n) 1_{p_0 \geq c_n} \right] - \mathbb{E}_{P_0} \log \hat{p}_n + \log b_n + O(n^{-1/2})
$$

$$
\leq \left| \mathbb{E}_{P_n} \left[ (\log \hat{p}_n)_{+} (X) > t \right] - \mathbb{E}_{P_0} \left[ (\log \hat{p}_n)_{+} (X) > t \right] \right| dt + \left| \mathbb{E}_{P_n} \left[ (\log \hat{p}_n)_{-} (X) \leq t \right] - \mathbb{E}_{P_0} \left[ (\log \hat{p}_n)_{-} (X) \leq t \right] \right| dt + O(n^{-1/2})
$$
\[
\leq \log n \cdot \mathbb{E} \sup_{C \in \mathcal{C}_d} |(\mathbb{P}_n - P_0)(C)| + \mathcal{O}(n^{-1/2}),
\]
where \(\mathcal{C}_d\) is the set of all convex bodies in \(\mathbb{R}^d\). The last inequality follows as for any \(s\)-concave density \(p\), \(\{\log p(x) > t\} = \{\log p(x) > 0 > t\} = \{p(x) \vee 1 > e^t\} = \{(\varphi(x) \wedge 1 < e^{st})\} = \{- (\log p(x) \wedge 0) \leq t\} = \{p(x) \wedge 1 \leq e^{-t}\} = \{\varphi(x) \vee 1 \leq e^{-ts}\}\) are convex sets, and 

\[-C_1 \log n \leq \log c_n \leq \log \tilde{p}_n \leq \log(M)\]

for \(n\) large.

Hence we only need to bound the entropy \(\mathcal{N}_I(\varepsilon, \mathcal{C}_d, P_0)\). To this end, for a multi-index \(I = (\ell_1, \ldots, \ell_d) \in \mathbb{Z}^d_{\geq 0}\), let \(I_\ell \equiv \prod_{k=1}^d [2^\ell_k - 1, 2^{\ell_k} - 1]\). Then \(|I_\ell| \asymp 2^{\sum \ell_k}\).

Let \(\{(A_{\ell,j}, B_{\ell,j}) : 1 \leq j \leq N_\ell\}\) be an \(\varepsilon_\ell\)-bracket for \(\{C|_{I_\ell} : C \in \mathcal{C}\}\) under the Lebesgue measure. By Theorem 8.25 of [DD14], we have \(\log N_{\ell} \leq_d (|I_\ell|^{-1} \varepsilon_\ell)^{(1-d)/2}\). Let \(\varepsilon_\ell = a_\ell \varepsilon\), where \(a_\ell \equiv |I_\ell|^{(1+\delta)}\) for some \(\delta > 0\) such that \(1 < 1+\delta < (-sd)^{-1} \in (1, \infty)\). Then \(\{(\sum_\ell 1_{A_{\ell,j}}, \sum_\ell 1_{B_{\ell,j}}) : 1 \leq j \leq N_\ell, \ell \in \mathcal{Z}^d_{\geq 0}\}\) forms a bracket for \(\mathcal{C} \cap \mathbb{R}^d_{\geq 0}\) with \(P_0\)-size

\[
\left| P_0 \left( \sum_\ell 1_{B_{\ell,j}} 1_{I_\ell} - \sum_\ell 1_{A_{\ell,j}} 1_{I_\ell} \right) \right| \leq \sum_\ell \varepsilon_\ell \sup_{x \in I_\ell} p_0(x) \leq \varepsilon \sum_\ell a_\ell |I_\ell|^{-(-sd)^{-1}} \lesssim \varepsilon.
\]

The logarithm of the number of the brackets can be bounded by

\[
C_d \sum_\ell |I_\ell|^{(d-1)/2} \varepsilon_\ell^{(1-d)/2} \lesssim \varepsilon^{(1-d)/2} \sum_\ell \left( a_\ell |I_\ell|^{-1} \right)^{(1-d)/2} \lesssim \varepsilon^{(1-d)/2}.
\]

Other quadrants can be handled similarly. This means that \(\log \mathcal{N}(\varepsilon, \mathcal{C}, P_0) \lesssim \varepsilon^{(1-d)/2}\), and hence Theorem 2.1 applies to (3.16).

The situation for \(d = 3\) is similar; but with an additional \(\log n\) term in the estimate for the empirical process \(\mathbb{E} \sup_{C \in \mathcal{C}_d} |G_n(C)| \lesssim \log n\), and therefore the rate in squared Hellinger comes with an additional \(\log n\).

Finally we consider \(d = 2\). Let \(\mathcal{P}_s(I, B) \equiv \{p \text{ is } s\text{-concave on } I \subset \mathbb{R}^2 : 0 \leq p(x) \leq B, \forall x \in I\}\). We write \(\mathcal{P}_s = \mathcal{P}_s([0,1]^2, 1)\) for simplicity. We claim that for \(s > 1\),

\[
\log \mathcal{N}_I(\varepsilon, \mathcal{P}_s, L_2) \lesssim_s \varepsilon^{-1} \log(1/\varepsilon).
\]

Fix \(\varepsilon > 0\). Let \(y_k \equiv 2^k, 1 \leq k \leq k_0\), where \(k_0\) is the smallest integer such that \(y_k = \varepsilon\). Let \(\{A_j, B_j\} : A_j \supseteq B_j\}_{j=1}^{N_1}\) be an \(\varepsilon_2\)-bracket for all convex sets in \([0,1]^2\) under the Lebesgue measure. By Theorem 8.25 and Corollary 8.26 of [DD14], we have \(\log N_1 \lesssim \varepsilon_0^{-1}\). By Proposition 4 in the supplement of [KSS16], for each \(j = 1, \ldots, N_1\), and \(k = 1, \ldots, k_0\), we may find a lower \(\varepsilon_{j,k}\)-bracket \(\{f_{j,k,m} : 1 \leq m \leq N_{j,k}\}\) in \(L_2\) (resp. upper \(\varepsilon_{j,k}\)-bracket \(\{f_{j,k,m} : 1 \leq m \leq N_{j,k}\}\) in \(L_2\)) for non-negative convex functions defined on \(B_j\) with an upper bound \(2^k\), such that \(\log(N_{j,k} \cup \sum_{j,k} \lesssim (2^k/\varepsilon_{j,k}) \log(2^k/\varepsilon_{j,k})\).

For any \(p \in \mathcal{P}_s\), let \(\varphi = p^*\) be the underlying convex function. Let \(C_k \equiv \{\varphi \leq y_k\}\). Let \(A_{j,k} \subset B_{j,k}\) be a bracket for \(A_{j,k} \supseteq C_k \supseteq B_{j,k}\), and let \(\ell_{j,k,m}\) (resp. \(\tilde{f}_{j,k,m}\)) be a lower (resp. upper) bracket for \(\varphi|_{B_{j,k}}\).
Let $A_{j_0} \equiv \emptyset$ and $y_0 \equiv 1$. Consider an upper bracket for $p$ of form
\[
\sum_{k=1}^{k_0} \left[ (f_{jk,k,m} \land y_{k-1}) 1_{B_{jk} \setminus A_{jk-1}} \right]^{1/s} + \sum_{k=1}^{k_0} \left[ (y_{k-1})^{1/s} \land 1 \right] 1_{A_{jk} \setminus B_{jk}} + \varepsilon 1_{[0,1] \setminus A_{jk_0}},
\]
and a lower bracket of $p$ of form $\sum_{k=1}^{k_0} \left[ (f_{jk,k,m} \land y_{k}) 1_{B_{jk} \setminus A_{jk-1}} \right]^{1/s}$. For the choice $\varepsilon_0 \equiv \varepsilon$ and $\varepsilon_{j,k} \equiv \varepsilon \cdot 2^{2k}$, this bracket has squared $L_2$ size bounded, up to a constant depending only on $s$, by
\[
\sum_{k=1}^{k_0} \varepsilon_{jk,k}^{2} (2^{k})^{2(1/s-1)} + \varepsilon^2 \sum_{k=1}^{k_0} \left[ (y_{k-1})^{1/s} \land 1 \right] + \varepsilon^2 \lesssim \varepsilon^2.
\]
The logarithm of the total number of brackets can be bounded by
\[
\log \left[ \prod_{k=1}^{k_0} N_{jk,k} \bar{N}_{jk,k} \right] \lesssim \sum_{k=1}^{k_0} \left( \varepsilon_0^{-1} + \frac{2^{k}}{\varepsilon_{jk,k}} \log \left( \frac{2^{k}}{\varepsilon_{jk,k}} \right) \right) \lesssim \varepsilon^{-1} \log(1/\varepsilon),
\]
proving the claim (3.17). Let $I_\ell$ be the same as in the proof for $d \geq 4$. By rescaling, it follows that
\[
\log N_{\ell}(\varepsilon, \bar{P}_\ell(I_\ell, B), L_2) \lesssim \frac{(B^2 I_\ell[1])^{1/2}}{\varepsilon} \log \left( \frac{(B^2 I_\ell[1])^{1/2}}{\varepsilon} \right).
\]
By (3.15), on $I_\ell$, $\sup_{x \in I_\ell} \sup_{p \in \mathcal{P}_{s,M}} p(x) \leq |I_\ell|^{1/2s}$. Let $b_\ell = |I_\ell|^{-\delta'}$ for some $\delta' \in (0, (-1/s - 1)/2)$, and $\{f_{j,\ell}: 1 \leq j \leq N_\ell\}$ be a $b_\ell \varepsilon$-bracket for $\mathcal{P}_{s,M}|I_\ell$ under $L_2$. A global bracket for $\mathcal{P}_{s,M}$ can be obtained by assembling these local brackets for all (=four) quadrants, with squared $L_2$-size at most $\varepsilon^2 \sum_\ell b_\ell^2 \lesssim \varepsilon^2$, and the logarithm of the number of brackets is
\[
\sum_\ell \log N_\ell \lesssim \sum_\ell \frac{|I_\ell|^{(1+s)/2}}{b_\ell \varepsilon} \log \left( \frac{|I_\ell|^{(1+s)/2}}{b_\ell \varepsilon} \right) \lesssim \varepsilon^{-1} \log(1/\varepsilon).
\]
Hence for $s > -1/2$, $\log N_{\ell}(\varepsilon, \mathcal{P}_{s,M}, h) = \log N_{\ell}(\varepsilon, \mathcal{P}_{2s,M}, L_2) \lesssim \varepsilon^{-1} \log(1/\varepsilon)$. By the rest of the proof is a standard computation of the size of the localized empirical process via Hellinger bracketing numbers (cf. Theorem 3.4.4 of [vdVW96]), so we omit the details.

\textbf{Remark 3.16.} During the preparation of this paper, the author becomes aware of [DKT9] who derived global rates of convergence for the log-concave MLE (= 0-concave MLE) for $d \geq 4$, based on the previous results of [CDSS18]. Here we treat the general $s$-concave MLEs, showing that the natural boundary in this setting is $s > -1/d$.

We conclude this section with an open question.

\textbf{Question 3.17.} Examine if the $s$-concave MLEs have certain adaptation property as the log-concave (= 0-concave) MLE; see [KGS18] for related results in $d \leq 3$. The main interesting case would be $d \geq 4$. 


APPENDIX A. PROOFS OF THEOREMS 2.5 AND 2.6

We will investigate the behavior of ratio-type empirical processes in a more general setting as in [GK06]. Let \( \phi \) be a continuous and strictly increasing function with \( \phi(0) = 0 \). Let \( \mathcal{C}(r) \equiv \{ C \in \mathcal{C} : P(C) \leq r^2 \} \) and \( \mathcal{C}(r, s) \equiv \mathcal{C}(s) \setminus \mathcal{C}(r) \). Fix \( 0 < r < \delta \leq 1 \). For a real number \( 1 < q \leq 2 \), let \( \ell \equiv \ell(r, \delta, q) \) be the smallest integer no smaller than \( \log_q(\delta/r) \). For any \( s \equiv (s_1, \ldots, s_{\ell}) \in \mathbb{R}^\ell \), let

\[
\beta_{n,q}(r, \delta) \equiv \max_{1 \leq j \leq \ell} \frac{\mathbb{E}\sup_{C \in \mathcal{C}(r^q j \delta^2, r^q j)} |G_n(C)|}{\phi(r^q j)} \quad \text{and} \quad \tau_{n,q}(r, \delta, s) \equiv \max_{1 \leq j \leq \ell} \frac{r^q j \sqrt{\sigma_j} + s_j / \sqrt{n}}{\phi(r^q j)}.
\]

The following result is essentially due to [GK06]. We state a somewhat simplified and easier-to-use version.

**Proposition A.1.** Assume that \( \phi \) is continuous, strictly increasing and satisfies \( \sup_{r \leq x \leq 1} \phi(qx)/\phi(x) = \kappa_{r,q} < \infty \) for some \( 1 < q \leq 2 \). Then for any \( s \equiv (s_1, \ldots, s_{\ell}) \in \mathbb{R}^\ell \), both the probabilities

\[
\mathbb{P}\left[ \sup_{C \in \mathcal{C}(r^q j \delta^2, r^q j)} |G_n(C)| \geq K \kappa_{r,q} (\beta_{n,q}(r, \delta) + \tau_{n,q}(r, \delta, s)) \right]
\]

and

\[
\mathbb{P}\left[ \sup_{C \in \mathcal{C}(r^q j \delta^2, r^q j)} |G_n(C)| \leq K (\beta_{n,q}(r, \delta) - \tau_{n,q}(r, \delta, s)) \right]
\]

can be bounded by \( K \sum_{j=1}^{\ell} \exp\left( -s_j / K \right) \). Here \( K > 0 \) is a universal constant.

**Proof of Proposition A.1.** We only prove the first claim; the second follows from similar arguments. The proof is a simple application of Talagrand’s concentration inequality combined with a peeling device. Write \( \mathcal{C}_j \equiv \mathcal{C}(r^q j^{-1}, r^q j) \) and \( \phi_q(u) \equiv \phi(r^q j) \) if \( u \in (r^q j^{-1}, r^q j) \) for notational convenience. By Talagrand’s concentration inequality,

\[
\mathbb{P}\left[ \sup_{C \in \mathcal{C}_j} |G_n(C)| \geq K \left( \mathbb{E}\sup_{C \in \mathcal{C}_j} |G_n(C)| + \sqrt{\sigma_j^2 s_j + s_j / \sqrt{n}} \right) \right] \leq K \exp\left( -s_j / K \right)
\]

where \( \sigma_j^2 = \sup_{P \in \mathcal{C}_j} P(C) = r^q 2^j \). Hence by a union bound we see that with probability at least \( 1 - \sum_{j=1}^{\ell} K \exp(-s_j / K) \), it holds that

\[
\left( \mathbb{E}\sup_{C \in \mathcal{C}(r^q j \delta^2, r^q j)} |G_n(C)| / \phi_q(\sqrt{P(C)}) - K \beta_{n,q}(r, \delta) \right)_+ \leq \max_{1 \leq j \leq \ell} \left( \sup_{C \in \mathcal{C}_j} |G_n(C)| / \phi(r^q j) - K \mathbb{E}\sup_{C \in \mathcal{C}(r^q j^{-1}, r^q j)} |G_n(C)| / \phi(r^q j) \right)_+ \leq K \max_{1 \leq j \leq \ell} \frac{r^q j \sqrt{\sigma_j} + s_j / \sqrt{n}}{\phi(r^q j)}.
\]

Now the conclusion follows from \( \sup_{r \leq x \leq 1} \phi(qx)/\phi(x) < \infty \). ∎
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The next lemma, due to Lemma 7.2 of [Ale87], provides a convenient device to derive almost sure results for ratio-type empirical processes.

**Lemma A.2.** Let $c_n, u_n$ be such that $c_n/n \to \alpha$ and $u_n \to \alpha$, and assume that $r_n \to \alpha$ and $\sqrt{n}d_n \to \beta$. For a centered function class $F \subset L_2(P)$, let

$$A_n \equiv \{|G_n f| \geq c_n \phi(\sigma f) + u_n, f \in F, r_n \leq \sigma f \leq \delta_n\},$$

and

$$A_n^c \equiv \{|G_n f| \geq (1 - \varepsilon)(c_n \phi(\sigma f) + u_n), f \in F, r_n \leq \sigma f \leq \sqrt{1 + \varepsilon \delta_n}\}.$$

Assume that $\inf_{n \geq 1, t \in [n, \delta_n]} c_n \phi(t)/t > 0$. Then if $P(A_n) = O(1/(\log n)^{1+\theta})$ holds for some $\varepsilon, \theta > 0$, we have $P(A_n)$ i.o. = 0.

**Proof of Theorem 2.6.** Consider the first claim. Note that for $0 < \alpha < 1$, $\beta_{n, q} \approx \max_{1 \leq j \leq \ell} \frac{(r_n q_j)^{1-\alpha}}{r_n q_j^\alpha} \times r_n^{-\alpha}$, while for $\alpha > 1$, $\beta_{n, q} \approx \max_{1 \leq j \leq \ell} \frac{n^{(\alpha-1)/2(\alpha+1)}}{r_n q_j^\alpha} \times n^{2(\alpha+1)/r_n}$. For $s_j \equiv s + 2K \log j$, we have

$$\tau_{n, q} \leq \max_{1 \leq j \leq \ell} \left(\sqrt{s + 2K \log j} + \frac{s + 2K \log j}{\sqrt{r_n q_j^2}}\right) \leq \sqrt{s} \sqrt{\log \log (1/r_n)} + (s \vee 1)n^{-\frac{\alpha}{2(\alpha+1)}},$$

and the probability estimate $K \sum_{j=1}^\ell \exp(-s_j/K) = Ke^{-s} \sum_{j=1}^\ell j^{-2} \leq K' e^{-s}$. This proves that

$$P\left(\sup_{C \in C : r_n^2 \leq P(C) \leq 1} \left|\frac{G_n(C)}{P(C)}\right| \geq K \left(\beta_{n, q} + \sqrt{s} \sqrt{\log \log (1/r_n)} + (s \vee 1)n^{-\frac{\alpha}{2(\alpha+1)}}\right)\right) \leq K' e^{-s}.$$  

The first claim of (1) follows from Lemma A.2 by setting $s \asymp \log \log n$, and requiring $\beta_{n, q} \gg \sqrt{\log \log n} \log \log n \cdot n^{-\alpha/(2(\alpha+1))}$. The second claim follows from similar lines by observing that under (E2), Theorem 2.4 yields that for $0 < \alpha < 1$, $\beta_{n, q} \geq \max_{1 \leq j \leq \ell} \frac{(r_n q_j)^{1-\alpha}}{r_n q_j^\alpha} \times r_n^{-\alpha}$, while for $\alpha > 1$, $\beta_{n, q} \geq \max_{1 \leq j \leq \ell} \frac{n^{(\alpha-1)/2(\alpha+1)}}{r_n q_j^\alpha} \times n^{2(\alpha+1)/r_n}$, and $\tau_{n, q}$ can be estimated from above using the same arguments. \hfill $\Box$

**Proof of Theorem 2.6.** The proof of Theorem 2.6 uses a similar strategy as that of Theorem 2.5. For convenience of the reader we provide some details. Consider the first claim. Note that for $0 < \alpha < 1$, $\beta_{n, q} \approx \max_{1 \leq j \leq \ell} \frac{(r_n q_j)^{1-\alpha}}{r_n q_j^\alpha} \times r_n^{-\alpha}$, while for $\alpha > 1$, $\beta_{n, q} \approx \max_{1 \leq j \leq \ell} \frac{n^{(\alpha-1)/2(\alpha+1)}}{r_n q_j^\alpha} \times n^{-2(\alpha+1)/r_n}$. For $s_j \equiv s + 2K \log j$, we have

$$\tau_{n, q} \approx \max_{1 \leq j \leq \ell} \left(\frac{1}{r_n} \sqrt{s + 2K \log j} + \frac{s + 2K \log j}{\sqrt{r_n q_j^2}}\right).$$
\[ \leq \sqrt{r_n^{-2} (s \vee \log \log(1/r_n))} + (s \vee 1)(\sqrt{m r_n^2})^{-1}. \]

This shows that, for
\[ \bar{\gamma}_n \equiv (r_n^{-2}n^{-1+\alpha})^{1/\alpha}, \quad \alpha \in (0, 1); \]
\[ r_n^{-2}n^{-1+\alpha}, \quad \alpha > 1. \]

we have
\[
\mathbb{P} \left( \sup_{C \in \mathcal{F}, \|C\|^2 \leq P(C)} \frac{|P_n(C) - P(C)|}{P(C)} \right. \\
\left. \geq K \left( \bar{\gamma}_n + \sqrt{(nr_n^2)^{-1}(s \vee \log \log(1/r_n)) + (s \vee 1)(nr_n^2)^{-1}} \right) \right) \leq K'e^{-s}.
\]

The first claim of the theorem follows by taking \( s \approx \log \log n \), applying Lemma \textbf{[A.2]} and noting that \( \limsup_{n} \bar{\gamma}_n \leq \rho^{-1} \) by the assumption. The second claim follows similarly by estimating \( \beta_{n,q} \) from below, up to a multiplicative constant, by \( \bar{\gamma}_n \) and then repeat the arguments as above. \( \square \)

**Appendix B. Talagrand’s concentration inequality**

We frequently use Talagrand’s concentration inequality \textbf{[Tal96]} for the empirical process in the form given by Theorems 3.3.9 and 3.3.10 of \textbf{[GN15]} in the proofs. For sake of completeness, we record it as follows.

(Talagrand’s concentration inequality) Let \( \mathcal{F} \) be a countable class of real-valued measurable functions such that \( \sup_{f \in \mathcal{F}} \|f\|_{\infty} \leq b \). Then
\[
\mathbb{P} \left( \sup_{f \in \mathcal{F}} |G_n(f)| - \mathbb{E} \sup_{f \in \mathcal{F}} |G_n(f)| \geq \sqrt{2\sigma_n^2 x + 4b x \sqrt{n}} \right) \leq 2e^{-x},
\]
where \( \sigma_n^2 \equiv 2n^{-1/2} \mathbb{E} \sup_{f \in \mathcal{F}} |G_n(f)| + \sup_{f \in \mathcal{F}} \text{Var}_P(f) \).

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