Entanglement of a qubit with a single oscillator mode

Gregory Levine
Department of Physics and Astronomy, Hofstra University, Hempstead, NY 11549

V. N. Muthukumar
Department of Physics, Princeton University, Princeton, NJ 08544
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We solve a model of a qubit strongly coupled to a massive environmental oscillator mode where the qubit backaction is treated exactly. Using a Ginzburg-Landau formalism, we derive an effective action for this well known localization transition. An entangled state emerges as an instanton in the collective qubit-environment degree of freedom and the resulting model is shown to be formally equivalent to a Fluctuating Gap Model (FGM) of a disordered Peierls chain. Below the transition, spectral weight is transferred to an exponentially small energy scale leaving the qubit coherent but damped. Unlike the spin-boson model, coherent and effectively localized behaviors may coexist.

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The subject of quantum computing has led to renewed interest in the theory of quantum decoherence and also exposed a practical side to the theory. Scalable, persistent current designs for qubits involve measurement devices that are permanently coupled, leading to a continuous dephasing of the qubit. Before a quantum measurement can be made, the qubit must become entangled with the measurement device. The conflicting demands of quantum measurement and minimal dephasing is an active topic of study within the theory of mesoscopic quantum detectors.

In this letter, we study the problem of how a qubit becomes entangled with a single environmental mode when the qubit backaction on the environment is not neglected. This is perhaps the simplest quantum environment, but already displays great complexity. Models of dissipative quantum environments such as the spin-boson model (SBM) are intentionally weakly coupled and therefore neglect backaction. However, environments imposed by quantum detectors may be strongly coupled and exhibit sharp spectral features. The entanglement described here might be realized in any environment (e.g. detector or qubit-qubit coupler) that exhibits a low energy monochromatic spectrum; as an example, we discuss the environment imposed by an underdamped DC-SQUID detector. Lastly, we point out the equivalence between the present model and a particular model of disordered fermions.

Consider a qubit coupled to a single harmonic oscillator described by the Hamiltonian

$$H = \Delta \sigma_x + \lambda \frac{1}{\sqrt{2m\omega}} (a + a^\dagger) \sigma_z + \omega a^\dagger a$$ (1)

If the oscillator is replaced by a classical, adiabatic oscillator (the limit $\omega \to 0$, $m\omega^2 \to \text{const}$), it is well known that exhibits a bifurcation at $\lambda^2 = \Delta m\omega^2$ in that the energy of the ground state is now minimized by assuming a nonzero value of the oscillator displacement, $x = \pm x_0$. In the doubly degenerate ground state, the qubit is localized with a nonzero expectation value, $\langle \sigma_z \rangle \neq 0$. The classical calculation begs two questions: 1) If the oscillator localizes the qubit, is all coherent behavior destroyed? 2) If the qubit and oscillator weakly interact ($\lambda^2 << \Delta m\omega^2$) aren’t they always entangled to some degree?

We show that the onset of entanglement with the oscillator displacement becomes discontinuous in the massive limit, with a component of the ground state wavefunction playing the role of an order parameter in a second order phase transition. Below the transition, damped coherent behavior and entangled, effectively localized behavior coexist.

Entanglement of the qubit in our case is distinct from that of weak-coupling theories (such as the SBM) in that the ergodicity of the environment is effectively broken and fluctuations become non-Gaussian. This latter effect, of course, cannot be seen by integrating out the environmental degrees of freedom to find the effective action of the qubit. In this respect, our calculation bears some relation to the description of quantum measurement by Single Electron Transistor (SET) and overdamped DC-SQUID detectors. In this scheme, the qubit degrees of freedom are integrated out to yield the response function of the detector.

To represent the qubit, we choose the finite temperature formalism of Popov and Fedotov in which the action is written in terms of spinors satisfying modified boundary conditions. The imaginary time action corresponding to is

$$S = \int d\tau \bar{\psi}(\tau) (\partial_\tau + \Delta \sigma_x + \lambda x(\tau) \sigma_z) \psi(\tau) + S_0$$ (2)

where $S_0 = \int d\tau \left( \frac{1}{2} m \dot{x}^2(\tau) + \frac{1}{2} k x^2(\tau) \right)$, $k \equiv m \omega^2$ and $\psi(\tau)$ is a spinor

$$\psi(\tau) = \begin{pmatrix} \psi_\uparrow(\tau) \\ \psi_\downarrow(\tau) \end{pmatrix}$$ (3)

The spin-1/2 is formally represented as a fermion spinor with an imaginary chemical potential. The latter
eliminates unphysical fermionic states from the Hilbert space. A gauge transformation removes the chemical potential and shifts the Matsubara frequencies to \( \omega_n = 2\pi(n+1/4)/\beta \). Conventional finite temperature field theory techniques may now be applied to the action \( \mathcal{S}_b \).

The fermions are now integrated out to yield an effective boson action:

\[
Z = \int D\bar{\psi}D\psi Dxe^{-S} \propto \int Dxe^{-S_{\text{eff}}} \tag{4}
\]

\[
S_{\text{eff}} = -\text{tr} \log (-G_0^{-1} + \frac{\lambda}{\beta}x(\omega_n - \omega_m)\sigma_z) + S_0 \tag{5}
\]

where \( G_0 = (i\omega_n - \Delta\sigma_z)^{-1} \) is the free fermion Green’s function. The coordinate \( x(\tau) \) now represents a collective coordinate for \( \sigma_z \) and the oscillator displacement.

Expanding the action to quartic order in \( \lambda \) and re-grouping into dynamic and static terms, respectively, we obtain:

\[
S_{\text{eff}} = \frac{1}{2}\sum (mp_n^2 + \pi(p_n) - \pi(0))|x(p_n)|^2
\]

\[
+ \frac{1}{2}\sum (k + \pi(0))|x(p_n)|^2 + \sum D(0)|x(p_n)|^4 \tag{6}
\]

where the coefficients are given by

\[
\pi(p_n) = \frac{\lambda^2}{\beta} \text{tr} \sum G_0(p_n + \omega_m)\sigma_z G_0(\omega_m)\sigma_z
\]

\[
- \lambda^2 \tanh \beta \Delta \frac{4\Delta}{p_n^2 + 4\Delta^2}
\]

\[
D(0) = \frac{\lambda^4}{\beta^2} \text{tr} \sum |G_0(\omega_m)\sigma_z|^4 = \frac{\lambda^4}{8\beta^3} \tanh \beta \Delta
\]

Owing to the modified Matsubara frequencies, these sums are computed with a modified Fermi distribution, \( f(\omega) = (e^{\beta\omega} + 1)^{-1} \). Defining the dimensionless control parameter \( \alpha = \frac{\lambda^2}{\Delta^2} \), the effective action is transformed back to imaginary time at zero temperature.

\[
S_{\text{eff}} = \int d\tau \left( \frac{1}{2} (m + \frac{\lambda^2}{4\Delta^2})x^2 + \frac{1}{2}k(1 - \alpha)x^2 + \frac{\lambda^4}{8\Delta^3}x^4 \right)
\]

The mass is enhanced by a dynamical quantity and, most significantly, the action exhibits an instability at \( \alpha = 1 \). When \( \alpha > 1 \) and \( \omega = 0 \), the action is minimized for \( x_0 = \pm \Delta \left( 2\alpha-2 \right)^{1/2} \). For \( \omega \neq 0 \), there is no true broken symmetry, however an approximate calculation of the instanton action

\[
S_I = \sqrt{2\Delta \omega} \left( \alpha - 1 \right)^{3/2} \frac{\omega^2}{\alpha^2} \sqrt{1 + \frac{\omega^2}{\Delta^2}} \tag{9}
\]

shows that, although the symmetry changes at \( \alpha > 1 \), the instanton is only stabilized for \( \Delta/\omega \gg (\alpha - 1)^{-3/2} \). This defines the entangled regime of the qubit and oscillator in that the ground state is a coherent superposition of two states of the collective coordinate, \( x(\tau) = \pm x_0 \).

Now we must find the corresponding dynamics of the qubit, in particular, the spin-spin correlation function. It may be shown that

\[
\langle T_\tau \sigma_3(\tau)\sigma_3(0) \rangle = \frac{1}{Z} \int Dx \langle T_\tau \bar{\psi}\sigma_3\psi\sigma_3 \rangle S_I[x] e^{-S_{\text{eff}}} \tag{10}
\]

where \( S_I[x] \) denotes averaging over the fermionic action for a particular realization of the boson field \( x(\tau) \). By Wick contractions, this average is related to the single fermion Green’s function, in the presence of an inhomogeneous field \( x(\tau) \), which solves the equation of motion: \( (\partial_\tau + \Delta\sigma_x + \lambda x(\tau)\sigma_z)G(\tau\tau'; x(\tau)) = \delta(\tau-\tau') \). This approach has been used to implement bosonization in 1-d fermion systems. In the present case, the Green’s function must satisfy the boundary conditions implied by the shifted Matsubara frequencies: \( G(\tau + \beta) = -iG(\tau) \). Furthermore, the boson field \( x \) must be written as the sum of an instanton trajectory and a small oscillation: \( x(\tau) = x_i(\tau) + r(\tau) \). We now make the non-Abelian Schwinger ansatz

\[
G(\tau\tau'; x(\tau)) = U(\tau)g(\tau\tau'; x_i(\tau))U^{-1}(\tau'), \tag{11}
\]

where \( g \) satisfies the equation of motion but with the field \( x(\tau) \) restricted to the instanton trajectory: \( (\partial_\tau + \Delta\sigma_x + \lambda x_i(\tau)\sigma_z)g(\tau\tau'; x_i(\tau)) = \delta(\tau-\tau') \), and \( x_i(\tau) = \pm x_0 \). Now \( U(\tau) \) must satisfy the auxiliary condition

\[
\partial_\tau U + [\bar{H}, U] = -\lambda x_i(\tau)\sigma_z U \tag{12}
\]

where \( \bar{H} \equiv \Delta\sigma_x + \lambda x_i(\tau)\sigma_z \).

A solution of the equation of motion for \( g \) which satisfies the boundary conditions is obtained for all \( \tau \) away from the instanton kinks (consistent with the dilute gas approximation); for \( \tau < \tau' \)

\[
g(\tau\tau'; x_i(\tau)) = \frac{1}{2} f(\Delta) \left[ 1 + \frac{x_i(\tau)}{\Delta} \right] e^{-\Delta(\tau-\tau')}
\]

\[
+ \frac{1}{2} f(-\Delta) \left[ 1 - \frac{x_i(\tau)}{\Delta} \right] e^{\Delta(\tau-\tau')}
\]

where \( \Delta \) is the Rabi energy \( \Delta = \sqrt{\Delta^2 + \lambda^2 x_0^2} \). The Green’s function \( g(\tau > \tau') \) is obtained by the replacement \( f(z) \rightarrow -f(z) \) where \( f(z) \equiv f^*(-z) \). The auxiliary condition \( \bar{H} \) leads to a set of Riccati equations for an appropriate parameterization of \( U \) which are difficult to solve generally. In the strong coupling limit (i.e. to lowest order in \( \Delta/\Delta \)) the interacting Green’s function is found to be \( G(\tau\tau'; x(\tau)) = g(\tau\tau'; x_i(\tau)) \exp (-\lambda x_i \int_{\tau'}^\tau r(\tau) d\tau) \).

Within this scheme we need to evaluate \( c_{zz} (\tau - \tau') = \langle T_\tau \sigma_3(\tau)\sigma_3(\tau') \rangle_{S_I[x]} \); other correlation functions are obtained similarly. \( c_{zz} (\tau - \tau') \) involves two contractions yielding a DC part of \( c_{zz} \) and a part at energy \( 2\Delta \), respectively. These correlation functions must in turn be averaged over the effective boson action, \( S_{\text{eff}} \).

\[
\langle T_\tau \sigma_3(\tau)\sigma_3(\tau') \rangle = \lim_{\tau_i \rightarrow \tau^+} \text{tr} \sigma_3 G(\tau_i \tau; x(\tau_i))
\]
The first contraction in (13) will involve products of $x_i(\tau)$ at different times, giving the proper long time behavior; the second contraction will involve the average over boson fluctuations $r(\tau)$ that dress the qubit oscillations at energy $\Delta$. To evaluate the average over $S_{eff}$ we need both boson correlation function. Within the instanton approximation, the first contraction in (13) involves
\[ (T_{\tau}x_i(\tau)x_i(\tau')) S_{eff} = x_0^2 e^{-|\Gamma| \tau - \tau'} \] (14)

where $\Gamma = \omega \sqrt{S_i/2\pi} \exp S_i$. The second contraction in (13) involves the average over fluctuations of $r(\tau)$, $\langle \cosh \int d\tau r(\tau) \rangle S_{eff}$. Averages of this form and their Fourier transforms have been discussed in several places. The relationship to the FGM may now be seen: Replacing (in ref.10) the 1-d spatial degree of freedom in the FGM by imaginary time, eqn. 2 is the action for a fermion of frequency $\nu$ propagating in a Peierls chain with a gap function $\Delta x(\tau)$. Averaging (in eqn. 13) over the instanton fluctuations of $x(\tau)$ with correlator 14, is equivalent to the disorder average in ref.10 with a correlation length given by the instanton time, $\Gamma^{-1}$. In the FGM, the density of states at the Fermi surface is accessed by the zero frequency limit $\omega \rightarrow 0^+$ which corresponds to $\Delta \rightarrow 0^+$, the strong coupling limit in the present work. The two models differ in that the correlation length of the gap disorder in the FGM is an independently controlled parameter whereas the instanton fluctuations are generated spontaneously in the present model. In addition, the fermions in the present model satisfy twisted boundary conditions.

Continuing to real frequencies and denoting the imaginary part of the fourier transform of 13 by $s_{zz}(\nu)$, we obtain for $\alpha > 1$ a correlation function of the form
\[ s_{zz}(\nu) = \left( \frac{\lambda x_0}{\Delta} \right)^2 \delta(\nu - \Gamma) + \frac{1}{2} \left( 1 - \left( \frac{\lambda x_0}{\Delta} \right)^2 + \left( \frac{\Delta}{\omega} \right)^2 \right) x_0^2 \sum_{mn} \frac{p^{n+m} e^{-2p}}{m! n!} \delta(\nu - 2\Delta - (n-m)\omega) \] (15)

for $\nu > 0$ where $p \equiv \alpha \Delta$. This is our main result. Equation 15 shows that for the broken symmetry phase ($\alpha > 1$) two distinct spectral features are formed. The low energy feature with weight $w_{ent} = \frac{1}{2} (\lambda x_0/\Delta)^2 S_{eff} \propto \Delta^2$ (at energy $\Gamma$) corresponds to the qubit entangled with the oscillator and in a superposition of the form $|+; +x_0 \rangle \pm |--; -x_0 \rangle$, where $|+/-\rangle$ refer to states of the qubit with nonvanishing z-polarization. The high energy set of delta functions (at energy $\sim 2\Delta$), corresponds to the decoupled (unentangled) but dephased qubit with weight $w_{free} \simeq 1/\alpha$. Although the spectrum is discrete, dephasing may be considered by examining the envelope of the sidebands associated with the primary resonance. For small $\omega$ (large $p$) the Poisson distributions become approximately Gaussian and the dephasing rate is estimated to be $\omega p^{1/2}$; the fractional width of the resonance, $1/Q$, is then $(\alpha \Delta)^{1/2}$. The qubit becomes critically damped when this factor is unity. When $m \rightarrow \infty$, keeping $k$ and therefore $\alpha$ constant, the width of the resonance goes to zero as $O(m^{-1/4})$ (although the number of bosons diverges), in agreement with the classical calculation.

The spectral weights appearing in equation 15 satisfy the sum rule: $w_{ent} + w_{free} = 1$. At critical coupling, a redistribution of spectral weight is initiated and spectral weight flows from the coherent feature to the entangled one. Unlike the SBM where the system spin becomes overdamped before it becomes localized, coherent and effectively localized behavior coexist in the present model12. An approximate numerical diagonalization of the Hamiltonian 11 confirm these results. Figure 1 shows that a feature of energy $\Gamma$ appears in the correlation function $s_{zz}(\nu)$ for $\alpha \sim 1$, while the primary resonance remains distinct from the instanton feature. As expected from the effective action $S_{eff}$, $w_{ent}^{1/2}$ is the order parameter for a second order phase transition; in the strict adiabatic limit ($\Delta/\omega \rightarrow \infty$), the derivative $dw_{ent}^{1/2}/d\alpha$ would diverge. To illustrate the sharp onset of entanglement, we have computed the entanglement entropy $S_\rho = -\text{tr} \rho \log_2 \rho$, where $\rho$ is the ground state reduced density matrix of the qubit. The entropy, shown in fig. 2, is seen to sharply increase from zero at the onset of entanglement ($\alpha = 1$). Similarly to the order parameter, the slope of this graph appears to have a discontinuity at $\alpha = 1$ as $\Delta/\omega \rightarrow \infty$. This is not unexpected, noting that thermodynamic entropy is given in terms of the
Gibbs free energy by \( S_{\text{th}} = -\partial G / \partial T \). Taking \( G \) to be the static limit of \( S_{\text{eff}} \), \( -\partial G / \partial \alpha = \frac{1}{2} k x_0^2 \), which is proportional to the square of the order parameter. Thus, entanglement entropy appears to behave analogously to thermodynamic entropy in this second order phase transition.

As a potential example of this transition, we now turn to experimental realizations of superconducting qubits. In, it was demonstrated that the dominant source of decoherence in a persistent current (phase) qubit was the electromagnetic environment of the DC-SQUID, characterized by an \( RL_1C \) impedance \( Z(\omega) \) (\( L_1 \) is the Josephson equivalent inductance.) Dephasing follows from the spin-boson model where \( J(\omega) \propto \frac{1}{\omega} \text{Re} Z(\omega) \); in particular, \( J \) is ohmic at low frequencies. However, \( J(\omega) \) is highly structured, dominated by the plasma mode at the frequency \( \omega = 1/\sqrt{L_0C} \) with a \( Q \approx 10 \) for typical parameters. The \( \sigma_z \) coupling to the qubit comes from the circulating component of current in the DC-SQUID. The effective coupling constant, \( \lambda \), reflecting the mutual inductance \( M \) between the persistent current in the qubit \( I_\phi \) and circulating current of the DC-SQUID is then \( \lambda = 2M I_\phi I_0 \sin \frac{\phi}{2} \sin \theta \), where \( I_0 \) is the critical current, \( \theta \) is the net Josephson phase and \( \phi \equiv 2\pi (\Phi/\Phi_0) \) the dimensionless flux at the operating point. Linearizing the effective potential of the SQUID about the operating point, the stiffness is found to be \( k = \frac{\Phi_0}{2\pi} \sqrt{4I_0^2 \cos^2 \frac{\phi}{2} - I_e^2} \), where \( I_e \) is the external current bias. Thus the effective potential softens and the dimensionless coupling constant \( \alpha = \frac{\lambda^2}{k^2} \) increases as \( I_e \) approaches the switching value.

To illustrate our point, we use \( M = 20 \mu \text{H} \), \( I_p = 900 \text{nA} \), \( I_0 = 150 \text{nA} \), and fix \( I_e \) at 0.95 of the critical switching value, \( 2I_0 \cos \frac{\phi}{2} \), to obtain a dimensionless coupling of \( \alpha \approx 1 \). Once entangled, two distinct coherent oscillations in \( \sigma_z \) would be observed: the primary oscillation at \( 2\Delta \) and the entangled one at \( \Gamma \). Within this model, the oscillation at \( \Gamma \) is undamped; however, external couplings would be expected to cause dephasing. The single mode contribution to the dephasing of the \( 2\Delta \) oscillation becomes significant when \( p(\equiv \alpha \bar{\phi}) > > 1 \) in which case \( Q = (\bar{\phi}/2\pi)^{1/2} \). Using \( \alpha = 1 \) and the adiabatic ratio \( \Delta/\omega = 10 \), yields a \( Q \) for the qubit of \( Q \approx 3.3 \). As in fig. 1, however, the spectrum is discrete consisting of several sidebands. In contrast, the ohmic spectrum, \( J(\omega) = \eta \omega \), for the same DC-SQUID parameters (and lead resistance \( R = 100 \text{ohm} \)) has a dimensionless coupling of \( \eta \approx 0.03 \). The ohmic contribution to the dephasing rate is \( 1/\tau \approx 2\pi \eta (\bar{\phi}/2\pi)^2 \), which vanishes for both zero bias and zero temperature (the present case). For \( \epsilon = \Delta \approx 1 \text{ GHz} \), the ohmic \( Q \) becomes comparable for a temperature of \( T \approx 10 \text{ mK} \) and diverges for \( T = 0 \). In contrast, the single mode contribution to dephasing remains constant at \( T = 0 \).

In conclusion, we have considered a model in which a qubit is strongly coupled to a single environmental mode and described the resulting dephasing and entanglement. We have solved the resulting model by appealing to a formal analogy with the Fluctuating Gap model, a model for disordered fermions. We applied these results to an undamped SQUID environment, however, this transition might be studied in a variety of existing qubit designs, the main attribute being a variable coupling between the qubit and an adiabatic (\( \Delta/\omega \approx 10 \)) oscillator mode.

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