Stabilization to trajectories for parabolic equations

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Abstract
Both internal and boundary feedback exponential stabilization to trajectories for semi-linear parabolic equations in a given bounded domain are addressed. The values of the controls are linear combinations of a finite number of actuators which are supported in a small region. A condition on the family of actuators is given which guarantees the local stabilizability of the control system. It is shown that a linearization-based Riccati feedback stabilizing controller can be constructed. The results of numerical simulations are presented and discussed.

Keywords Feedback stabilization to trajectories · Semilinear parabolic equations

Mathematics Subject Classification 93D15 · 93B52 · 35K58

1 Introduction
We consider controlled parabolic equations, for time $t \geq 0$, in a $C^\infty$-smooth domain $\Omega \subset \mathbb{R}^d$ located locally on one side of its boundary $\Gamma = \partial \Omega$, with $d$ a positive integer. We will consider both the case of internal controls
\[
\frac{\partial}{\partial t} y - \nu \Delta y + f(y, \nabla y) + f_0 = \sum_{i=1}^{M} u_i \vartheta \Phi_i, \quad y|_{\Gamma} = g,
\]
and the case of boundary controls

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\[
\frac{\partial}{\partial t} y - v \Delta y + f(y, \nabla y) + f_0 = 0, \quad y|_\Gamma = g + \sum_{i=1}^{M} u_i \vartheta \Psi_i. \quad (2)
\]

In the variables \((t, x, \bar{x}) \in (0, +\infty) \times \Omega \times \Gamma\), the unknown in the equation is the function \(y = y(t, x) \in \mathbb{R}\). The diffusion coefficient \(v > 0\) is a positive constant; the functions \(g = g(t, \bar{x}) \in \mathbb{R}\), \(f: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\), and \(f_0(t, x) \in \mathbb{R}\) are fixed. The function \(\vartheta\) will be used to localize the support of the actuators \(\vartheta \Phi_i = \vartheta(x) \Phi_i(x)\), respectively \(\vartheta \Psi_i = \vartheta(\bar{x}) \Psi_i(\bar{x})\). For system (1), we take \(\vartheta \in C^2(\overline{\Omega})\) and for system (2) we take \(\vartheta \in C^2(\Gamma)\). \(M\) is a positive integer, and \(u = u(t) \in \mathbb{R}^M\) is a vector function at our disposal.

The problem we address here is the \textit{local} exponential stabilization to trajectories for systems (1) and (2). That is, given a positive constant \(\lambda > 0\) and a solution \(\hat{y}(t) = \hat{y}(t, \cdot)\) of the (uncontrolled) system with \(u = 0\), we want to find a control function \(u\) such that the solution \(y(t) := y(t, \cdot)\) of the system, suplemented with the initial condition

\[y(0) := y(0, x) = y_0(x),\]

is defined on \([0, +\infty)\) and approaches \(\hat{y}(t)\) exponentially with rate \(\lambda\), provided \(y(0) - \hat{y}(0)\) is \textit{small enough}. In other words, for a suitable Banach space \(X\) and positive constants \(C\) and \(\epsilon\), we want to have that

\[|y(t) - \hat{y}(t)|_X^2 \leq Ce^{-\lambda t} |y(0) - \hat{y}(0)|_X^2, \quad \text{provided} \quad |y(0) - \hat{y}(0)|_X < \epsilon, \quad (3)\]

with \(\epsilon\) \textit{small enough}. Notice that, the constants \(C\) and \(\epsilon\) may depend on \(\lambda\), but not on \(y(0) - \hat{y}(0)\).

We are particularly interested in actuators which are supported in a small domain: for example for system (1) we will assume that \(\text{supp}(\vartheta \Phi_i) \subseteq \overline{\omega} \subseteq \overline{\Omega}\). We also suppose that there is \(\omega_1 \subseteq \omega\) such that \(\vartheta(x) = 1\) for all \(x \in \omega_1\). That is, for (1) we assume that \(\omega_1\) and \(\omega\) are open subsets such that

\[\omega_1 \subseteq \omega \subseteq \Omega, \quad \text{supp}(\vartheta) \subseteq \overline{\omega}, \quad \text{and} \quad \vartheta|_{\omega_1} = 1. \quad (4a)\]

Similarly, for (2) we assume that \(\Gamma_1\) and \(\Gamma_\epsilon\) are open subsets such that

\[\Gamma_1 \subseteq \Gamma_\epsilon \subseteq \Gamma, \quad \text{supp}(\vartheta) \subseteq \overline{\Gamma_\epsilon}, \quad \text{and} \quad \vartheta|_{\Gamma_1} = 1. \quad (4b)\]

Internal and boundary actuators are taken from \(L^2(\Omega)\) and \(H^\frac{3}{2}(\Gamma)\), respectively. We denote the following linear spaces:

\[S_\Phi := \text{span}\{\Phi_i \mid i \in \{1, 2, \ldots, M\}\} \subseteq L^2(\Omega), \quad (4c)\]

\[S_\Psi := \text{span}\{\Psi_i \mid i \in \{1, 2, \ldots, M\}\} \subseteq H^\frac{3}{2}(\Gamma), \quad \text{with} \quad \vartheta \Psi_i \in H^\frac{3}{2}(\Gamma). \quad (4d)\]
The orthogonal projections onto the above subspaces will be denoted by $P_M$

$$P_M: L^2(\Omega) \mapsto S_\Phi, \quad P_M: L^2(\Gamma) \mapsto S_\Psi.$$  

**Remark 1.1** Using the same notation $P_M$ for both internal and boundary cases will lead to no ambiguity. Of course, if we take the orthogonal projections $\hat{P}_M: L^2(\Omega) \rightarrow \partial S_\Phi$ and $\tilde{P}_M: L^2(\Omega) \rightarrow \partial S_\Psi$ instead, then we observe that the range of $\partial P_M$ coincides with that of $\hat{P}_M$. This means that the results in this paper will follow if we replace $(\partial, P_M)$ by $(1, \hat{P}_M)$. However, it will be convenient to have a stability condition involving $P_M$ rather than involving $\hat{P}_M$, namely in Sect. 2.5 we will present an example of boundary actuators where we exploit properties of particular actuators $\Psi_i \in H^\frac{3}{2}(\Gamma) \backslash H^\frac{3}{2}(\Gamma)$ (still, $\partial \Psi_i \in H^\frac{3}{2}(\Gamma)$).

In order to state the main results, let us denote the indicator operators:

$$1_{\omega_0}: L^2(\Omega) \mapsto L^2(\Omega), \quad 1_{\omega_0}f(x) = f(x), \quad \text{if} \ x \in \omega_0,$$

$$1_{\Gamma_0}: L^2(\Gamma) \mapsto L^2(\Gamma), \quad 1_{\Gamma_0}g(\bar{x}) = g(\bar{x}), \quad \text{if} \ \bar{x} \in \Gamma_0,$$

for given open subsets $\omega_0 \subseteq \Omega$ and $\Gamma_0 \subseteq \Gamma$.

Without lack of generality, in either case, we suppose that the families of actuators $\{\partial \Phi_i \mid i \in \{1, 2, \ldots, M\}\}$ and $\{\partial \Psi_i \mid i \in \{1, 2, \ldots, M\}\}$ are linearly independent. So, we consider the bijections $B_\Phi: \mathbb{R}^M \rightarrow \partial S_\Phi$ and $B_\Psi: \mathbb{R}^M \rightarrow \partial S_\Psi$,

$$B_\Phi u := \partial \sum_{i=1}^M u_i \Phi_i, \quad B_\Psi u := \partial \sum_{i=1}^M u_i \Psi_i.$$

We will prove the stabilization results under a general condition on the pair $(\hat{\psi}, f)$, say $(\hat{\psi}, f) \in \mathcal{C}$ for a suitable class $\mathcal{C}$ to be precised hereafter.

The space of continuous linear mappings from a Banach space $X$ into a Banach space $Y$ will be denoted $\mathcal{L}(X, Y)$. When $X = Y$ we write simply $\mathcal{L}(X) := \mathcal{L}(X, X)$.

The usual Lebesgue and Sobolev spaces $L^p(\Omega)^m = L^p(\Omega, \mathbb{R}^m)$, with $p \in [1, +\infty]$, and $H^s(\Omega)^m = H^s(\Omega, \mathbb{R}^m)$, $s \geq 0$, will be denoted by simply $L^p(\Omega)$ and $H^s(\Omega)$, respectively, whenever there is no ambiguity concerning the superscript $m \in \mathbb{N}_0$. Sometimes to shorten the formulas we will write simply $L^p$ and $H^s$, if there is no ambiguity concerning the domain $\Omega$. Same notation for the spaces $L^p(\Gamma, \mathbb{R}^m) = L^p(\Gamma) = L^p$, and $H^s(\Gamma, \mathbb{R}^m) = H^s(\Gamma) = H^s$.

**1.1 Review on the general procedure and main tools**

To derive the local Riccati-based feedback stabilization results, we follow the following sequence of steps:

1. linearize the system around the reference trajectory.
2. find an appropriate set of actuators and construct a globally stabilizing open-loop control for the linear system.
3. use the dynamical programming principle and Karush–Kuhn–Tucker Theorem to find a time-dependent feedback control operator.
4. observe that the feedback operator satisfies a differential Riccati equation.
5. use a fixed point argument to prove that the same feedback operator also locally stabilizes the full nonlinear system.

The most difficult step is to construct an open-loop stabilizing control for the linearized system by means of a finite number of actuators supported in small regions. This has been done in previous works in the case of internal actuators, and a condition for stabilizability has been given in terms of the orthogonal projection $1 - P_M$. The main novelty of this paper is the construction of such open-loop stabilizing control with boundary actuators for parabolic equations, with a corresponding stabilizability condition in terms of the orthogonal projection $1 - P_M$. The second novelty is the consideration of a general class of nonlinearities $f$, for both internal and boundary cases. The third novelty is the presentation of numerical simulations for the boundary feedback control, confirming the theoretical results.

Below we give further comments on the general steps above. **Linearization of the system around the reference trajectory**. We present the computations here because they will be useful to write down the conditions we ask for the targeted trajectory $\hat{y}$ and for the nonlinear function $f$ in an easier way. Namely, as $(\hat{y}, f) \in \mathcal{C}$, with $\mathcal{C}$ defined in (9), hereafter.

We want the solution $y(t)$ to go to the reference trajectory $\hat{y}(t)$ exponentially. By direct computations, we find that $z := y - \hat{y}$ solves
\begin{equation}
\frac{\partial}{\partial t} z - \nu \Delta z + f(y, \nabla y) - f(\hat{y}, \nabla \hat{y}) = tib B_{\phi} u, \quad z|_{\Gamma} = (1 - tib) B_{\psi} u, \quad \text{(5)}
\end{equation}
with $tib = 1$ for (1) and $tib = 0$ for (2).

Writing $(\xi^1, \xi^2) \in \mathbb{R} \times \mathbb{R}^d$ we denote $\partial_1 f := \frac{\partial f}{\partial \xi^1}$ and $\partial_2 f := \frac{\partial f}{\partial \xi^2}$. Formally,
\begin{equation}
f(y, \nabla y) - f(\hat{y}, \nabla \hat{y}) =: \left[ \partial_1 f|_{(\hat{y}, \nabla \hat{y})} \partial_2 f|_{(\hat{y}, \nabla \hat{y})} \right] \begin{bmatrix} z \\ \nabla z \end{bmatrix} + F_\hat{y}(z) = \hat{L} z - \hat{N}(z), \quad \text{(6a)}
\end{equation}
with $\hat{L} := \hat{a} z + \nabla \cdot (\hat{b} z)$ and
\begin{equation}
\hat{a} := \partial_1 f|_{(\hat{y}, \nabla \hat{y})} - \nabla \cdot \partial_2 f|_{(\hat{y}, \nabla \hat{y})}, \quad \hat{b} := \partial_2 f|_{(\hat{y}, \nabla \hat{y})}, \quad \text{and} \quad \hat{N}(z) = - F_\hat{y}(z), \quad \text{(6b)}
\end{equation}
where the remainder $\hat{N}(\cdot) : \mathbb{R} \to \mathbb{R}$ either vanishes or is a nonlinear function.

We will be able to prove the local stabilization to the trajectory $\hat{y}$ provided the triple $(\hat{a}, \hat{b}, \hat{N})$, defined by $(\hat{y}, f)$, satisfies
\begin{equation}
\hat{a} \in L^\infty(\mathbb{R}_0, L^d(\Omega, \mathbb{R})), \quad \text{(7a)}
\end{equation}
\begin{equation}
\hat{b} \in L^\infty_w(\mathbb{R}_0, L^\infty(\Omega, \mathbb{R}^d)), \quad \nabla \cdot \hat{b} \in L^\infty_w(\mathbb{R}_0, L^r(\Omega, \mathbb{R})), \quad \text{(7b)}
\end{equation}
with \( r = 2 \) if \( d \in \{1, 2, 3\} \), \( r = \infty \) if \( d \geq 4 \), and for a suitable constant \( \hat{C} > 0 \),

\[
|\hat{N}(z) - \hat{N}(\hat{z})|^2_{L^2} \leq \hat{C}|z - \hat{z}|^2_{H^1}(1 + |z|^2_{H^1} + |\hat{z}|^2_{H^1})(|z|^2_{H^2} + |\hat{z}|^2_{H^2}) + \hat{C}|z - \hat{z}|^2_{H^2}(|z|^2_{H^1} + |\hat{z}|^2_{H^1}),
\]

(8a)

and

\[
(\hat{N}(z) - \hat{N}(\hat{z}), z - \hat{z})_{L^2} \leq \hat{C} \left( 1 + |z|^2_{H^1} + |\hat{z}|^2_{H^1} \right) \frac{1}{2} \left( 1 + |z|^2_{H^2} + |\hat{z}|^2_{H^2} \right) |z - \hat{z}|_{H^1} |z - \hat{z}|_{L^2} + \hat{C}(1 + |z|^2_{H^1} + |\hat{z}|^2_{H^1}) \left( 1 + |z|^2_{H^2} + |\hat{z}|^2_{H^2} \right) |z - \hat{z}|^2_{L^2}
\]

(8b)

with \( \{\varepsilon_1, \varepsilon_2\} \in [0, +\infty) \) and \( \{\varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\} \in [2, +\infty) \).

That is, for our results to hold, the property asked for the pair \((\hat{y}, f)\) is that it belongs to the class \( \mathcal{C} \) defined as follows

\[
\mathcal{C} := \{(\hat{y}, f) \mid (\hat{a}, \hat{b}, \hat{N}) \text{ is defined by (6)} \text{ and satisfies (7) and (8)}\}.
\]

(9)

We see that our goal (3) is to find the control \( u \), in system (5), such that

\[
|z(t)|^2_X \leq C e^{-\lambda t} |z(0)|^2_X,
\]

provided \( |z(0)|_X < \epsilon \).

for suitable positive constants \( C = C_\lambda \) and \( \epsilon = \epsilon_\lambda \).

Construction of a globally stabilizing control for the linear system. We consider system (5) without the nonlinearity \( \hat{N} \),

\[
\frac{\partial}{\partial t} z - \nu \Delta z + \hat{a}z + \nabla \cdot (\hat{b}z) = \iota_{ib} B\Phi u, \quad z|_{\Gamma} = (1 - \iota_{ib}) B\psi u.
\]

Observe that \( z(t) \) goes exponentially to zero with rate \( \frac{\lambda}{2} \) if, and only if, \( e^{\frac{\lambda}{2} t} z(t) \) remains bounded. So, we consider the shifted system solved by \( e^{\frac{\lambda}{2} t} z(t) \):

\[
\frac{\partial}{\partial t} z - \nu \Delta z + (\hat{a} + \frac{\lambda}{2}) z + \nabla \cdot (\hat{b}z) = \iota_{ib} B\Phi \tilde{u}, \quad z|_{\Gamma} = (1 - \iota_{ib}) B\psi \tilde{u}.
\]

\* For internal actuators, \( \iota_{ib} = 1 \), we rewrite the system as

\[
\frac{\partial}{\partial t} z - \nu \Delta z + (\hat{a} + \frac{\lambda}{2}) z + \nabla \cdot (\hat{b}z) = \partial P_M \eta, \quad z|_{\Gamma} = 0,
\]

(10)

where now we look for a stabilizing control \( \eta \in L^2((0, +\infty), L^2(\Omega)) \), taking its values in \( L^2(\Omega) \) and such that \( z \in L^2((0, +\infty), H^1(\Omega)) \). The key tool used to find \( \eta \) is the null controllability of the system

\[
\frac{\partial}{\partial t} z - \nu \Delta z + (\hat{a} + \frac{\lambda}{2}) z + \nabla \cdot (\hat{b}z) = 1_{\omega_1} \eta_1, \quad z|_{\Gamma} = 0,
\]

(11)
at final time \( t = s_1 \), for \( t \in (s_0, s_1) \), \( 0 < s_0 < s_1 < +\infty \), with controls \( \eta_1 \in L^2((0, +\infty), L^2(\Omega)) \). Note that we cannot guarantee that \( \{1_{\omega_1} \eta_1(t) \in L^2(\omega_1) \mid t \in (s_0, s_1)\} \) is a subset of a finite-dimensional space, that is, \( 1_{\omega_1} \eta_1 \) is an infinite-dimensional control, in general. To find suitable actuators and a finite-dimensional control, suitable truncated observability inequalities can be used as in [13] (together with the smoothing property of the parabolic system). Another option is to use a suitable boundedness/smallerness condition on the operator \( 1 - P_M \) as in [17,29]. The latter approach led to some estimates on the number \( M \) of actuators that allow us to stabilize the system, for example for piecewise-constant actuators. The main idea is to construct a control recursively in each interval \( J^i := (iT_*, (i + 1)T_*) \), \( i \in \mathbb{N} \) such that \( |z((i + 1)T_*)|_{L^2(\Omega)} \leq \rho |z(iT_*)|_{L^2(\Omega)} \), where \( \rho < 1 \). Then, we just take the concatenation of such controls. The time-length \( T_* \) is at our disposal, but it will be chosen to somehow minimize “the” cost of null controllability. The control in the interval \( J^i \) is constructed as follows: firstly from the null controllability of (11) we can take a control \( \eta_1 \) driving (11) from \( z(iT_*) \), at time \( t = iT_* \), to \( z((i + 1)T_*) = 0 \), at time \( t = iT_* + T_* \), such that \( |\eta^2|_{L^2(J^i, L^2(\Omega))} \leq C_{nc}(T_*) |z(iT_*)|_{L^2(\Omega)}^2 \), secondly we observe that, by setting \( \eta_2 := 1_{\omega_1} \eta_1 \), then the difference \( d \) between the solution of (11) with \( \eta_1 = \eta_2 \), and the solution of (10) with \( \eta = \eta_2 \), satisfies, since \( \theta |_{\omega_1} = 1 \),

\[
\begin{align*}
\partial_t d - \nu \Delta d + (\hat{a} + \frac{i}{2}) d + \nabla \cdot (\hat{b} d) = \partial (1 - P_M) \theta \eta_2, \quad z|_{\Gamma} = 0, \\
\end{align*}
\]

and \( |d(s)|^2_{L^2(\Omega)} \leq \hat{T}(\Gamma_*) |\theta(1 - P_M)\theta|^2_{L^2(L^2(\Omega), H^{-1}(\Omega))} |z(iT_*)|^2_{L^2(\Omega)} \), where for suitable constants \( C_1 \) and \( C_2 \), \( \hat{T}(\Gamma_*) = C_2 e^{C_1 T_*} C_{nc}(T_*) \). Therefore, if we have that \( |\theta(1 - P_M)\theta|^2_{L^2(L^2(\Omega), H^{-1}(\Omega))} \leq \rho \hat{T}(\Gamma_*)^{-1} \), with \( 0 < \rho < 1 \), then the solution of system (10), issued from \( z(iT_*) \) at time \( t = iT_* \), satisfies \( |z((i + 1)T_*)|^2_{L^2(\Omega)} = |d((i + 1)T_*)|^2_{L^2(\Omega)} \leq \rho |z(iT_*)|^2_{L^2(\Omega)} \).

The constants \( \hat{T}(\Gamma_*) \) and \( M \) may be taken the same in each interval \( J_i \), \( i \in \mathbb{N} \), due to the conditions (7). For further details, we refer to [17,29,30].

- For boundary actuators, \( \eta_{ib} = 0 \), we will find suitable actuators and construct a stabilizing control for the system

\[
\begin{align*}
\frac{\partial}{\partial t} z - \nu \Delta z + (\hat{a} + \frac{i}{2}) z + \nabla \cdot (\hat{b} z) = 0, \quad z|_{\Gamma} = B_{\theta} u, \quad (12)
\end{align*}
\]

so that \( z \in L^2((0, +\infty), H^1(\Omega)) \). Again the smoothing property and the null controllability of the system

\[
\begin{align*}
\frac{\partial}{\partial t} \hat{z} - \nu \Delta \hat{z} + (\hat{a} + \frac{i}{2}) \hat{z} + \nabla \cdot (\hat{b} \hat{z}) = 0, \quad z|_{\Gamma} = 1_{\Gamma_1} \xi_1 \quad (13)
\end{align*}
\]

at final time \( t = s_1 \), for \( t \in (s_0, s_1) \), \( 0 < s_0 < s_1 < +\infty \), will play a key role, where at each time \( 1_{\Gamma_1} \xi_1(t) \) takes values in a suitable infinite-dimensional subspace of \( H_0^2(\Gamma) \).

The open-loop stabilizing control will again be constructed recursively in the time intervals \( J^i = (iT_*, (i + 1)T_*) \), for a suitable \( T_* \).

The constructed control \( u \) in each interval \( J^i = (iT_*, (i + 1)T_*) \) will belong to \( H^1(J^i, \mathbb{R}^M) \). More comments on the derivation of such time regularity for the control

\( \hat{z} \) Springer
are given below. In this case, it follows that \( \kappa := \frac{\partial}{\partial t} u - \frac{\lambda}{2} u + \varsigma u \in L^2(J^i, \mathbb{R}^M) \). That is, we have the control dynamics

\[
\frac{\partial}{\partial t} u = -\varsigma u + \frac{\lambda}{2} u + \kappa, \quad \kappa \in L^2(J^i, \mathbb{R}^M).
\]

The advantage of having such a dynamical control is that we will be able to rewrite (12) in a canonical extended form, where the control operator is bounded.

Indeed, for each actuator \( \vartheta \Psi_i \in H^{\frac{3}{2}}(\Gamma) \), we will take the extension \( \tilde{\Psi}_i \in H^2(\Omega) \), which solves the elliptic system

\[
-\nu \Delta \tilde{\Psi}_i + \varsigma \tilde{\Psi}_i = 0, \quad \tilde{\Psi}_i |_{\Gamma} = \vartheta \Psi_i,
\]

and set the bijection \( B_{\tilde{\Psi}} : \mathbb{R}^M \to S_{\tilde{\Psi}} \), with \( S_{\tilde{\Psi}} := \text{span}\{\tilde{\Psi}_i | i \in \{1, \ldots, M\}\} \),

\[
B_{\tilde{\Psi}} \kappa := \sum_{i=1}^{M} \kappa_i \tilde{\Psi}_i.
\]

Now, we can consider the extended system (cf. [5,42]) for the new variables \( (v, \kappa) = (z - B_{\tilde{\Psi}} u, u) \in H^1_0(\Omega) \times \mathbb{R}^M \):

\[
\frac{\partial}{\partial t} \begin{bmatrix} v \\ \kappa \end{bmatrix} + \begin{bmatrix} -\nu \Delta \tilde{L} & -\frac{\lambda}{2} \\ \varsigma - \frac{\lambda}{2} & 0 \end{bmatrix} \begin{bmatrix} v \\ \kappa \end{bmatrix} = \begin{bmatrix} -B_{\tilde{\Psi}} \\ 1 \end{bmatrix} \kappa,
\]

with \( \tilde{L} w := \dot{\omega} + \nabla \cdot (\dot{b} w) \) and \( \tilde{L}_{\varsigma} w := \dot{L} w - 2 \varsigma w \). Our (new) control function is \( \kappa \).

In particular, the control operator is bounded, \( [-B_{\tilde{\Psi}} \ 1] \in \mathcal{L}(\mathbb{R}^M, L^2(\Omega) \times \mathbb{R}^M) \).

**Deriving the desired time regularity for the control.** Proceeding as in the internal case we may look for the control in (12) in the form \( P_M \zeta \) where \( \zeta \) drives (12) to zero at final time. As we will see this standard procedure will give us a control \( u \in H^{\frac{3}{2}}(J^i, \mathbb{R}^M) \). To obtain the desired extra regularity \( u \in H^1(J^i, \mathbb{R}^M) \), we will use an extra suitable projection in \( L^2(J^i) \) with range contained in \( H^1_0(J^i) \), together with a suitable density argument. The fact that (after concatenation) we still have \( u \in H^1([0, +\infty), \mathbb{R}^M) \) will follow from suitable uniform properties on \( i \). For example, notice that the length \( T_* \) of \( J^i \) is independent of \( i \).

**Remark 1.2** Usually, the variable \( u \) stands for control. This is why we renamed \( \kappa := u \) to underline that in the extended system \( \kappa \) is not a control (it is part of the state). Of course we could simply take \( \zeta = 0 \), however, taking \( \zeta \geq 0 \) will not bring additional difficulties and, as we observed in numerical simulations, the value of \( \zeta \) may play an important role [35, section 9.5].

**Remark 1.3** In [42], the analogous result concerning the existence of an open-loop boundary stabilizing control for the linearized Navier–Stokes equations is proven, using the corresponding null controllability result [40] and suitable boundary observability inequalities [41]. The procedure we follow here is different, instead of deriving
the appropriate truncated observability inequalities for parabolic equations, we give
a condition for stabilizability depending on the operator $1 - P_M$. Then, we find a
set of actuators satisfying the condition. Further, such condition allows us to derive
estimates on the number $M$ of actuators that will allow us to stabilize the system (for
actuators taken from a suitable class of functions).

**Finding a feedback rule.** Once we have the existence of a control stabilizing an evo-
lutionary linear system $\dot{v} = -A_\lambda(t)v + Bu$ in a Hilbert space $H$, with bounded control
operator $B \in L(\mathbb{R}^M, H)$, that is,

$$
|v|^2_{L^2(\mathbb{R}_{s_0}^+, H)} + |u|^2_{L^2(\mathbb{R}_{s_0}^+, \mathbb{R}^M)} \leq C |v(s_0)|^2_H, \quad \text{for all} \quad s_0 \geq 0,
$$

we can look for the optimal control minimizing a suitable linear quadratic cost. Then,
through the dynamical programming principle and the Karush–Kuhn–Tucker Theo-
rem, we will conclude that the optimal control is given in feedback form $u = -B^*\Pi v$
where $(\Pi(s_0)w, w)_H$ is the optimal “cost to go” with initial condition $v(s_0) = w$. For
more details, see [13,30].

In both internal and boundary cases, we can write our linearized system as

$$
\dot{v} = -A_\lambda(t)v + Bu \quad (15)
$$

where, in the internal case

$$
H = L^2(\Omega), \quad A_\lambda = -\nu\Delta + \widehat{L} - \frac{\lambda}{2}, \quad \text{and} \quad B = \partial P_M,
$$

and, in the boundary case

$$
H = L^2(\Omega) \times \mathbb{R}^M, \quad A_\lambda = \begin{bmatrix}
-\nu\Delta + \widehat{L} - \frac{\lambda}{2} & \widehat{L}_\xi B \Phi \\
0 & \zeta - \frac{\lambda}{2}
\end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix}
-B \Phi \\
1
\end{bmatrix}.
$$

**Differential Riccati equation. Computation of the feedback rule.** In case our linear
quadratic running cost reads $\int_{s}^{+\infty} |Mv(\tau)|^2_H + |u(\tau)|^2_{\mathbb{R}^M} \, d\tau$, then the symmetric
bounded feedback operator $\Pi = \Pi^*: H \rightarrow H$ will satisfy the differential equation

$$
\frac{d}{dt}\Pi - \Pi A_\lambda - A_\lambda^*\Pi - \Pi B B^*\Pi + M^*M = 0, \quad (16)
$$

For simplicity, let us denote

$$
V = H_0^1(\Omega) \quad \text{and} \quad D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)
$$
in the internal case, and

$$
V = H_0^1(\Omega) \times \mathbb{R}^M \quad \text{and} \quad D(\Delta) = (H^2(\Omega) \cap H_0^1(\Omega)) \times \mathbb{R}^M
$$
in the boundary case.
Formally, equality (16) may be understood in the weak sense
\[(\frac{d}{dt} \Pi z, w)_{L^2(\Omega)} = (\Pi \overline{\mathcal{A}} z, w)_{L^2(\Omega)} - (\mathcal{A}^* z, w)_{L^2(\Omega)}
\]
\[= (\Pi \mathcal{B}^* \Pi z, w)_{L^2(\Omega)} - (\mathcal{M}^* \mathcal{M} z, w)_{L^2(\Omega)}, \quad \text{for all } (z, w) \in \mathcal{D}(\Delta)^2,
\]
see [13]. More precisely, we can say that \(\Pi\) satisfies (16), in the time interval \((0, T)\), \(T > 0\), if for all \(z \in \mathcal{D}(\Delta)\), see [20,30],
\[\Pi(t)z = U^*_{(t,s)} \Pi(s) U_{(t,s)}z + \int_s^t U^*_{(r,s)} (\Pi(r) \mathcal{B}^* \Pi(r) - \mathcal{M}^* \mathcal{M}) U_{(r,s)} dr,
\]
where \(U^*_{(t,s)} w,\) with \(0 < s < t < T,\) stands for the solution of
\[\dot{y} = -\mathcal{A}_\lambda y, \quad y(s) = w, \quad \text{with } w \in \mathcal{H}.
\]

**Remark 1.4** Once \(\Pi(t)z \in \mathcal{H}\) is defined by (17) for all \(z \in \mathcal{D}(\Delta)\), then by a density argument it can be extended for all \(z \in \mathcal{H}\). Note, in particular, that (17) makes sense because \(\mathcal{B}: \mathcal{H} \to \mathcal{H}\) and \(-\mathcal{A}_\lambda: \mathcal{D}(\Delta) \to \mathcal{H}\) are bounded. The boundedness of \(-\mathcal{A}_\lambda: \mathcal{D}(\Delta) \to \mathcal{H}\) will follow from (7) (see (25) hereafter).

Notice that, \(\Pi = \Pi_\lambda\) has been obtained prior to its dynamics. Therefore, at this point the existence of a solution for the Riccati equation is known. The uniqueness can be guaranteed in the class of families \(P \in L^\infty(\mathbb{R}_0, \mathcal{L}(\mathcal{H}))\) such that \(P(t)\) is self-adjoint and positive definite for all \(t > 0\), and the family \(\{P(t) | t > 0\}\) is continuous in the weak operator topology. See [30]. For further references concerning the Riccati equations, we refer to [19,47].

**Stabilization of the nonlinear system.** We consider the unshifted system (15) with the feedback control, and perturbed with the nonlinearity,
\[\frac{\partial}{\partial t} v + \mathcal{A}_0 v + \mathbb{B}^* \Pi v = \mathcal{N} v, \quad v(0) = v_0 \in \mathcal{H}, \quad (18)
\]
where \(\mathcal{N} = \mathcal{N}\) in the internal case, and \(\mathcal{N} = \left[\begin{array}{c} \mathcal{N} \\ 0 \end{array}\right]\) in the boundary case. At this point, the linear system \(\frac{\partial}{\partial t} v - \mathcal{A}_0 v + \mathbb{B}^* \Pi v = 0\) is known to be globally exponentially stable, with rate \(\frac{\lambda}{2}\). That is, \(|v(t)|^2_{\mathcal{H}} \leq C e^{-\lambda t} |v(0)|^2_{\mathcal{H}}\).

The space \(\mathcal{H}\) is considered as a pivot space, \(\mathcal{H} = \mathcal{H}^\prime\). Since the Cauchy problem for the nonlinear system is in general not well-posed for so-called weak solutions \(v \in \{L^2_{\text{loc}}(\mathbb{R}_0, \mathcal{V}) | \frac{\partial}{\partial t} v \in L^2_{\text{loc}}(\mathbb{R}_0, \mathcal{V}^\prime)\}\), we need to guarantee firstly that the solution for the linear system is strong, \(v \in \{L^2_{\text{loc}}(\mathbb{R}_0, \mathcal{D}(\Delta)) | \frac{\partial}{\partial t} v \in L^2_{\text{loc}}(\mathbb{R}_0, \mathcal{H})\}\). This is possible form conditions (7), and from the compatibility condition \(v(0) \in \mathcal{V}\).

Secondly, we will conclude the local exponential stability of the nonlinear system, with the same rate \(\frac{\lambda}{2}\), by following a standard fixed point argument. We look for the fixed point in a subset
\[\mathcal{Z}_\phi^\lambda := \left\{ v \in L^2_{\text{loc}}(\mathbb{R}_0, \mathcal{H}) \mid \sup_{r \geq 0} \left| e^{\frac{\lambda r}{2}} v \right|_{W((r,r+1), \mathcal{D}(\Delta), \mathcal{H})} \leq \phi |v(0)|^2_{\mathcal{V}} \right\},
\]
for an appropriate $\varrho > 0$. In particular, we need the strong solutions of the linearized closed-loop system to go exponentially to zero in the $H^1(\Omega)$-norm, $|v(t)|^2_{H^1} \leq C_1 e^{-\lambda t} |v(0)|^2_{H^1}$, which will follow from the smoothing property for parabolic equations, $|v(s + 1)|^2_{H^1} \leq C_2 |v(s)|^2_{H^1}$ (with $C_2$ independent of $s \geq 0$, due to (7)). Then from standard estimates on the linear parabolic systems, it also follow that, indeed the strong solutions $v$ are in $L^2_{\text{loc}}(\mathbb{R}, D(\Delta))$ and $|v|^2_{W((r,r+1),D(\Delta),H)} \leq C_3 |v(r)|^2_{H^1} \leq C_4 e^{-\lambda r} |v(0)|^2_{H^1}$.

The fixed point argument is based on the mapping $\bar{v} \mapsto v$ where $v$ solves
\[ \frac{\partial}{\partial t} v - A_0 v + BB^* \Pi_\lambda v = \mathcal{N} \bar{v}, \quad v(0) = v_0 \in V. \]

See (18). Such mapping will be a contraction provided $|v_0|_V$ is small enough.

Though the fixed point argument above is standard, we would like to mention that we will consider a general class of nonlinearities.

**Remark 1.5** We look for, and find, the fixed point in $\mathbb{Z}_{\varrho,\lambda}$. Using this set, we will be able to use some results in previous works (e.g., as in Step 1 in the proof of Theorem 3.1). It seems possible that by looking for the fixed point in a subset of the more classical space $e^{\frac{1}{2} r} W(\mathbb{R}, D(\Delta), H)$ (with $\hat{\lambda} < \lambda$) we would be able to find it as well. However, the details should be checked.

### 1.2 The main results

Here, we state the main results of the paper. Recall the class $\mathcal{C}$ defined in (9).

**Main Theorem 1.1** (Internal case) *Let $\hat{y}$ solve system (1) with $u = 0$, and let $(\hat{y}, f) \in \mathcal{C}$. Then for any given $\lambda > 0$, there exists a constant $\hat{\varGamma} > 0$ such that if

\[ |\vartheta (1 - P_M)\vartheta|^2_{L^2(L^2(\Omega), H^{-1}(\Omega))} < \hat{\varGamma}^{-1}, \]

then there exists a family of linear operators $\{\Pi_\lambda(t) \in L(L^2(\Omega)) | t > 0\}$ such that the following properties hold true:

(i) the mapping $t \mapsto \Pi_\lambda(t)$ is continuous in the weak operator topology;

(ii) there exists $\epsilon > 0$ such that if $y_0 - \hat{y}_0 \in H^1_0(\Omega)$ and $|y_0 - \hat{y}_0|_{H^1(\Omega)} < \epsilon$,

then the solution $y$ of the system
\[ \frac{\partial}{\partial t} y - \nu \Delta y + f(y, \nabla y) + f_0 = -\vartheta P_M \vartheta \Pi_\lambda (y - \hat{y}), \quad y|_{\Gamma} = g, \quad y(0) = y_0, \]

exists, and is unique, in the affine space
\[ \hat{y} + L^2_{\text{loc}}(\mathbb{R}, H^2(\Omega)) \cap C([0, +\infty), H^1_0(\Omega)), \]
and satisfies
\[ |y(t) - \hat{y}(t)|^2_{H^1(\Omega)} \leq Ce^{-\lambda t}|y_0 - \hat{y}_0|^2_{H^1(\Omega)}, \text{ for all } t \geq 0, \]
for a suitable constant C independent of \((\epsilon, y_0 - \hat{y}_0)\).

Now we introduce the space
\[ H(\Gamma) := L^2(H^{3/2}(\Gamma), H^{1/2}(\Gamma)) \cap L^2(L^2(\Gamma), H^{-1/2}(\Gamma)). \quad (19) \]

**Main Theorem 1.2** (Boundary case) Let \( \hat{y} \) solve system (2) with \( u = 0 \), and let \((\hat{y}, f) \in \mathcal{C}\). Then for any given \( \lambda > 0 \), there exists a constant \( \hat{\Upsilon} > 0 \) such that: if
\[ |\vartheta(\hat{y} - \hat{y})|_{H(\Gamma)} < \hat{\Upsilon} - 1, \]
then there exists a family of linear operators \( \{\Pi_\lambda(t) \in L^2(\mathbb{R}^M) \mid t > 0\} \) such that the following properties hold true:

(i) the mapping \( t \mapsto \Pi_\lambda(t) \) is continuous in the weak operator topology;

(ii) there exists \( \epsilon > 0 \) such that: if
\[ y_0 - \hat{y}_0 \in H^1(\Omega), \quad (y_0 - \hat{y}_0) \mid \Gamma \in \partial S\Psi, \quad \text{and} \quad |y_0 - \hat{y}_0|_{H^1(\Omega)} < \epsilon, \]
then the solution \( y \) of the system
\[ \begin{align*}
\frac{\partial}{\partial t} y - v \Delta y + f(y, \nabla y) + f_0 &= 0, \quad y \mid \Gamma = g + B\Psi \kappa, \quad y(0) = y_0, \\
\frac{\partial}{\partial t} \kappa + \varsigma \kappa &= \begin{bmatrix} B_\Psi & -1 \end{bmatrix} \Pi_\lambda \begin{bmatrix} y - \hat{y} - B\hat{\Psi} \kappa \\ \kappa \end{bmatrix}, \quad \kappa(0) = (B\Psi)^{-1}(y_0 - \hat{y}_0) \mid \Gamma, 
\end{align*} \]
exists, and is unique, in the affine space
\[ \hat{y} + L^2_{\text{loc}}(\mathbb{R}_0, H^2(\Omega)) \cap C([0, +\infty), H^1(\Omega)), \]
and satisfies
\[ |y(t) - \hat{y}(t)|^2_{H^1(\Omega)} \leq Ce^{-\lambda t}|y_0 - \hat{y}_0|^2_{H^1(\Omega)}, \text{ for all } t \geq 0, \]
for a suitable constant C independent of \((\epsilon, y_0 - \hat{y}_0)\).

**Remark 1.6** In the Main Theorems 1.1 and 1.2, we have the compatibility conditions \((y_0 - \hat{y}_0) \mid \Gamma = 0\) and \((y_0 - \hat{y}_0) \mid \Gamma \in \partial S\Psi\), respectively. These conditions are necessary to have strong solutions in \( W(\mathbb{R}_0, H^2(\Omega), L^2(\Omega)) \) for the system (5) solved by \( y - \hat{y} \). In fact, to have strong solutions \( v \in W(\mathbb{R}_0, \mathbf{D}(\Delta), \mathbf{H}) \) for the linearized system (15), we will need that \( v(0) \in \mathbf{V} \). Strong solutions are needed to deal with the closed-loop nonlinear systems.
In terms of the difference to the target \( y - \hat{y} \), in the internal case the feedback control is given in linear form: 
\[
y - \hat{y} \mapsto K_{\text{int}}(y - \hat{y})
\]
with 
\[
K_{\text{int}} = \vartheta P M \vartheta \Pi \lambda \in L(L^2(\Omega), \vartheta S_\varphi).
\]
In the boundary case, the control function \( u = \kappa \) is given in dynamic form. In terms of the difference to the target \( y - \hat{y} \), the boundary feedback is given in integral linear form 
\[
y|_\Gamma = g + e^{-\varsigma \cdot (\cdot - \tau)} K_{\text{bdry}}(y(\tau) - \hat{y}(\tau)) d\tau
\]
with 
\[
K_{\text{bdry}} = [((B\Psi)^* - 1)] \Pi \lambda \left[ 1 - ((B\Psi)^{-1} \circ (\cdot |_\Gamma)) \right] \in L(H^0_0(\Omega) + S\bar{\varphi}, \mathbb{R}^M).
\]
The operators \( \Pi \lambda \) may be taken as the solution of the corresponding Riccati equation (16).

The constant \( \Gamma = \hat{\Gamma}(T_s) \) will depend on (and increase with) the cost of null controllability of systems (11) and (13) in intervals \((jT_s, (j+1)T_s), j \in \mathbb{N}\), which are used to construct recursively an open-loop stabilizing control.

Since a key tool for the procedure is the null controllability of the linearized system, we would like to refer to a short list of works related with null controllability, observability inequalities, and exact controllability to trajectories. Namely to [6, 26–28, 40, 51], see also references therein.

Though the details must be checked, it is plausible that the entire procedure can be followed (adapted) for systems of several coupled parabolic equations, provided we have the null controllability of the linearized systems [1, 2].

We are particularly interested in stabilization to time-dependent trajectories, which are important for applications where external forces depend on time, \( f_0 = f_0(t) \). Notice that in such cases, the free-dynamics (uncontrolled) trajectories are necessarily time-dependent. That is, the uncontrolled system has no equilibria (steady states). Of course, when time-independent solutions do exist, then it makes sense to consider the problem of stabilization to an equilibrium, which has been studied for the last years by many authors for several systems and is by now quite well understood, we refer to [7–12, 23, 24, 34, 36, 37, 39, 48] and references therein. At this point, we must say that the spectral approach used in the case of a targeted time-independent solution is not (or, seems not to be) appropriate to deal with the case of time-dependent targeted solutions, as the examples in [50] do suggest.

### 1.3 Contents and notation

The paper mainly focuses on the case of boundary actuators. Section 2 concerns the boundary stabilization to zero of the linearized system (12), provided the pair \( (\hat{a}, \hat{b}) \) satisfies (7). We prove that there exists a family of actuators \( \{\vartheta \Psi_i, i \in [1, 2, \ldots, M]\} \) (satisfying the conditions in Main Theorem 1.2) and \( \kappa \in L^2(\mathbb{R}_0, \mathbb{R}^M) \) such that the solution of the system (14) satisfies, in particular, \((v, \kappa) \in L^2(\mathbb{R}_0, (H^1_0(\Omega) \cap H^2(\Omega)) \times \mathbb{R}^M), \) if \((v(0), \kappa(0)) \in H^1_0(\Omega) \times \mathbb{R}^M). \) In Sect. 3 we deal with the feedback boundary stabilization to zero of the nonlinear (full) system.
provided the nonlinearity \( \hat{N} \) satisfies (8). Recall that if \( (v, \kappa) \) satisfies the dynamics of the latter system, then \( (y, \kappa) \), with \( y = v + \hat{y} + B\hat{\varphi}\kappa \), satisfies the dynamics of the system in the Main Theorem 1.2. The case of internal controls is briefly revisited in Sects. 2.3 and 2.4, we construct a stabilizing control provided a suitable stabilizability condition depending on 1 

\[ \sum_{j=1}^{n} a_j \] 

denotes a function of nonnegative variables \( a_j \) that increases in each of its arguments, and \( C, C_i, i = 1, 2, \ldots, \) stand for positive constants.

## 2 Boundary stabilization of the linearized system

We start by briefly recalling some classical results in Sects. 2.1 and 2.2. Then, in Sects. 2.3 and 2.4, we construct a stabilizing control provided a suitable stabilizability condition depending on \( 1 - P_M \) is satisfied (cf. Main Theorem 1.2 and Theorem 2.4). This stabilizability condition is one of the main results of the paper. In Sect. 2.5, we present a family of actuators satisfying the stabilizability condition. Finally, in Sect. 4, where we present the proof of Main Theorems 1.1 and 1.2. In Sect. 5, we show that our condition \( (\hat{y}, f) \in \mathcal{C} \), that is, (7) and (8), is satisfied for regular enough \( \hat{y} \) and for some polynomial nonlinearities, which appear in several models of real world evolution processes. Finally, Sect. 6 contains the results of some numerical simulations, for both internal and boundary actuators, showing that the feedback control can stabilize systems whose free dynamics is unstable. Appendix gathers the proofs of auxiliary results needed in the main text.

### Notation

We write \( \mathbb{R} \) and \( \mathbb{N} \) for the sets of real and nonnegative integer numbers, respectively, and we define \( \mathbb{R}_a := (a, +\infty) \) for all \( a \in \mathbb{R} \), and \( \mathbb{N}_0 := \mathbb{N} \setminus \{0\} \).

We denote by \( \Omega \subset \mathbb{R}^n, n \in \mathbb{N}_0 \), a bounded \( C^\infty \)-smooth domain with boundary \( \Gamma = \partial \Omega \).

Given an open interval \( I \subseteq \mathbb{R} \), and Banach spaces \( X \) and \( Y \), we write \( W(I, X, Y) := \{ f \in L^2(I, X) \mid \frac{d}{dt} f \in L^2(I, Y) \} \), where the derivative \( \frac{d}{dt} f \) is taken in the sense of distributions. This space is endowed with the natural norm \( |f|_{W(I, X, Y)} := (|f|^2_{L^2(I, X)} + |\frac{d}{dt} f|^2_{L^2(I, Y)})^{1/2} \). If the inclusion \( X \subseteq Y \) is continuous, we write \( X \rightarrow Y \); we write \( X \hookrightarrow Y \), respectively \( X \hookleftarrow Y \), if the inclusion is also dense, respectively, compact. If \( X \hookrightarrow \mathcal{H} \) and \( Y \hookrightarrow \mathcal{H} \) for a Hausdorff topological space \( \mathcal{H} \), then \( X \cap Y \) is a Banach space, with \( \| \cdot \|_{X \cap Y} := (\| \cdot \|_X^2 + \| \cdot \|_Y^2)^{1/2} \).

Given a Hilbert space \( H \), with scalar product \( \langle \cdot, \cdot \rangle_H \) and a subset \( S \subseteq H \), the subspace orthogonal to \( S \) will be denoted \( S^\perp := \{ h \in H \mid \langle h, s \rangle_H = 0 \text{ for all } s \in S \} \), as usual.

\( \overline{C[a_1, \ldots, a_k]} \) denotes a function of nonnegative variables \( a_j \) that increases in each of its arguments, and \( C, C_1, i = 1, 2, \ldots, \) stand for positive constants.
\[(v, w)_V := (\nabla v, \nabla w)_{L^2(\Omega, \mathbb{R}^d)} \quad \text{and} \quad (v, w)_{D(\Delta)} := (\Delta v, \Delta w)_H,\]

and corresponding norms \(|v|_V := (v, v)_{V}^{\frac{1}{2}}\) and \(|v|_{D(\Delta)} := (v, v)_{D(\Delta)}^{\frac{1}{2}}\). We have

\[D(\Delta) \xrightarrow{d.c.} V \xrightarrow{d.c.} H \xrightarrow{d.c.} V' \xrightarrow{d.c.} D(\Delta)',\]

and the sequence of repeated eigenvalues \(\alpha_i, i = 1, 2, \ldots\), of \(-\Delta\) satisfies

\[0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \ldots, \quad \lim_{i \to +\infty} \alpha_i = +\infty.\]

Furthermore, \((v, w)_V \cdot V = (v, w)_H\), for all \((v, w) \in H \times V\).

**Boundedness assumption.** For \(m \in \mathbb{N}_0\), for simplicity we denote

\[\mathcal{W} := L^\infty(\mathbb{R}_0, L^d(\Omega, \mathbb{R})) \times L^\infty_w(\mathbb{R}_0, L^\infty(\Omega, \mathbb{R}^d)).\]  \(\text{(20)}\)

We fix \(\hat{a}\) and \(\hat{b}\), and a constant \(C_{\mathcal{W}} \geq 0\), satisfying

\[\left|\begin{split} \left|\hat{a}\right|_{\mathcal{W}}^2 := \left|\hat{a}\right|_{L^\infty(\mathbb{R}_0, L^d(\Omega, \mathbb{R}))}^2 + \left|\hat{b}\right|_{L^\infty_w(\mathbb{R}_0, L^\infty(\Omega, \mathbb{R}^d))}^2 \leq C_{\mathcal{W}}. \end{split}\]  \(\text{(21)}\)

**Remark 2.1** Notice that (21) is weaker than (7). Condition (21) is sufficient for the existence and uniqueness of weak solutions (for the linearized system). We use (7) to derive the existence and uniqueness of strong solutions.

Throughout this paper \(I\) stands for the bounded time interval \(I = (s_0, s_1)\), with \(0 \leq s_0 < s_1 < +\infty\), whose length is denoted by \(|I| := s_1 - s_0\).

We will look for weak solutions in \(W(I, H^1(\Omega), V')\) and strong solutions in \(W(I, H^2(\Omega), H)\). The corresponding traces on the boundary are denoted

\[G^1(I, \Gamma) := W(I, H^1(\Omega), V')|_{\Gamma} \quad \text{and} \quad G^2(I, \Gamma) := W(I, H^2(\Omega), H)|_{\Gamma},\]

respectively. As usual, we endow the trace spaces with the norms

\[|\gamma|_{G^1(I, \Gamma)} := \inf_{\gamma = v|_{\Gamma}} |v|_{W(I, H^1(\Omega), V')}, \quad |\gamma|_{G^2(I, \Gamma)} := \inf_{\gamma = v|_{\Gamma}} |v|_{W(I, H^2(\Omega), H)}.\]

### 2.1 Weak solutions

Here, we recall some regularity results for the weak solutions for systems as (5). We start considering the more general system

\[\begin{align*}
\frac{\partial}{\partial t} z - v \Delta z + \hat{a}z + \nabla \cdot (\hat{b}z) + h &= 0, \quad \text{(22a)} \\
z|_{\Gamma} &= \gamma, \quad z(s_0) = z_0, \quad \text{(22b)}
\end{align*}\]

where the control is replaced by a general external force.
Existence and uniqueness of weak solutions can be derived by standard arguments, by using the estimates in the following Lemma, whose proof is also standard and is omitted.

**Lemma 2.1** We have, for \( z \in H^1(\Omega) \) and \( y \in V \),

\[
\langle \hat{a}z, y \rangle_{V', V} \leq C \left| \hat{a} \right|_{L^d} \left| z \right|_{H^1(\Omega)}^{\frac{1}{2}} \left| y \right|_{H^1}^{\frac{1}{2}} \left| y \right|_V, \quad \text{for } d \in \{1, 2\},
\]

\[
\langle \hat{a}z, y \rangle_{V', V} \leq C \left| \hat{a} \right|_{L^d} \left| z \right|_{H^1} \left| y \right|_V, \quad \text{for } d \geq 3.
\]

\[
\langle \nabla \cdot (\hat{b}z), y \rangle_{V', V} \leq C \left| \hat{b} \right|_{L^\infty} \left| z \right|_{H^1} \left| y \right|_V, \quad \text{for } d \geq 1.
\]

where \( C \geq 0 \) is a suitable constant depending only on \((\Omega, d)\).

**Lemma 2.2** Given \((\hat{a}, \hat{b}) \in W\) satisfying \( (21) \), \( h \in L^2(I, V') \), \( \gamma = 0 \), and \( z_0 \in H \), there is a weak solution \( z \in W(I, V, V') \) for \( (22) \), which is unique and depends continuously on the data:

\[
|z|^2_{W(I, V, V')} \leq \bar{C} \left[ |I|, C_W, \frac{1}{r} \right] \left( |z_0|^2_H + |h|^2_{L^2(I, V')} \right).
\]

Furthermore, \( |z(s)|^2_H \leq e^{\frac{C}{r} C_W (s - s_0)} \left( |z_0|^2_H + \frac{1}{r} |h|^2_{L^2((s_0, s), V')} \right) \), for \( s \in I \).

**Lemma 2.3** With \((\hat{a}, \hat{b}), h, \) and \( z_0 \) as in Lemma 2.2, and \( \gamma \in G^1(I, \Gamma) \), there is a weak solution \( z \in W(I, H^1(\Omega, \mathbb{R}), V') \) for \( (22) \), which is unique and depends continuously on the data:

\[
|z|^2_{W(I, H^1(\Omega, \mathbb{R}), V')} \leq \bar{C} \left[ |I|, C_W, \frac{1}{r} \right] \left( |z_0|^2_H + |h|^2_{L^2(I, V')} + |\gamma|^2_{G^1(I, \Gamma)} \right).
\]

In Appendix, Sect. A.1 we present the proof of Lemma 2.2. For the nonhomogeneous boundary case \( \gamma \neq 0 \), we recall that we can define weak solutions by a standard lifting argument (e.g., see [40]).

### 2.2 Strong solutions

For strong solutions, we need further regularity for \((\hat{a}, \hat{b})\). Roughly, we will need further regularity for the reference trajectory \( \hat{y} \) (cf. system \( (6) \)). We denote

\[
W_{st} := \left\{ (a, b) \in W \mid \nabla \cdot b \in L^\infty_w(\mathbb{R}_0, L^r(\Omega, \mathbb{R})) \right\},
\]

\[
\text{with } \quad r = 2 \quad \text{if } d \in \{1, 2, 3\}, \quad \text{and } \quad r = \infty \quad \text{if } d \geq 4,
\]

(cf. \( (20) \) and \( (7) \)). We fix \( \hat{a} \) and \( \hat{b} \), and a constant \( C_{W_{st}} \geq 0 \), satisfying

\[
|\hat{a}, \hat{b}|_{W_{st}}^2 := \left| \hat{a}, \hat{b} \right|_W^2 + \left| \nabla \cdot \hat{b} \right|_{L^\infty_w(\mathbb{R}_0, L^r(\Omega, \mathbb{R}))} \leq C_{W_{st}}.
\]
Now, we have the following estimates for the convection term

\[ |\nabla \cdot ( \hat{b} z) |_{L^2} = |(\nabla \cdot \hat{b}) z + \hat{b} \cdot \nabla z|_{L^2} \leq |(\nabla \cdot \hat{b})|_{L^2} |z|_{L^\infty} + |\hat{b}|_{L^\infty} |z|_{H^1} \]

and, by using the Agmon inequalities, we find

\[ |\nabla \cdot ( \hat{b} z) |_{L^2} \leq C |(\nabla \cdot \hat{b})|_{L^2} |z|_{L^2}^{1/2} |z|_{H^1}^{1/2} + |\hat{b}|_{L^\infty} |z|_{H^1}, \quad \text{for } d = 1. \]  
\[ |\nabla \cdot ( \hat{b} z) |_{L^2} \leq C |(\nabla \cdot \hat{b})|_{L^2} |z|_{L^2}^{1/2} |z|_{H^2}^{1/2} + |\hat{b}|_{L^\infty} |z|_{H^1}, \quad \text{for } d = 2. \]  
\[ |\nabla \cdot ( \hat{b} z) |_{L^2} \leq C |(\nabla \cdot \hat{b})|_{L^2} |z|_{L^2}^{1/2} |z|_{H^2}^{1/2} + |\hat{b}|_{L^\infty} |z|_{H^1}, \quad \text{for } d = 3. \]  

(25a) (25b) (25c)

The Agmon inequalities can be found in [45, chapter II, Section 1.4] for \( d \geq 2 \). For \( d = 1 \) and \( \Omega = (l, r) \) with \( l < r \), the inequality reads

\[ |z|_{L^\infty} \leq 2^{1/2} |z|_{H^1}^{1/2} |z|_{V}^{1/2} \]

and follows from the fact that, for all \( x_1 \in \Omega \),

\[ |z(x_1)|_V^2 = |z(x_1)|_W^2 - |z(l)|_W^2 = 2 \int_l^{x_1} z(\rho) \frac{d}{d\rho} z(\rho) \, d\rho \leq 2 |z|_{L^2(\Omega)} \left| \frac{d}{d\rho} z \right|_{L^2(\Omega)} . \]

For \( d \geq 4 \), the Agmon inequality does not allow us to bound the \( L^\infty \)-norm by the \( H^2 \)-norm. This is the reason we (need to) take different spaces in (23). Notice that

\[ |\nabla \cdot (bz) |_{L^2} \leq |\nabla \cdot b|_{L^\infty} |z|_{L^2} + |b|_{L^\infty} |z|_{H^1}, \quad \text{for all } d \geq 1, \]  

(25d)

We have the following results concerning the existence of strong solutions, whose proofs are standard.

**Lemma 2.4** Given \((\hat{a}, \hat{b}) \in \mathcal{W}_{\text{ad}}\) satisfying (24), \( h \in L^2(I, H) \), \( \gamma = 0 \), and \( z_0 \in V \), then there is a strong solution \( z \in W(I, D(\Delta), H) \) for (22), which is unique and depends continuously on the data

\[ |z|^2_{W(I, D(\Delta), H)} \leq \overline{C} \left[ |I|, C_{\mathcal{W}_{\text{ad}}}, \frac{1}{\nu} \right] \left( |z_0|^2_V + |h|^2_{L^2(I, H)} \right) . \]

**Lemma 2.5** With \((\hat{a}, \hat{b}) \) and \( h \) as in Lemma 2.4, \( \gamma \in G^2(I, \Gamma) \), and \( z_0 \in H^1(\Omega) \) with \( z_0|_{\Gamma} = \gamma(0) \), there is a strong solution \( z \in W(I, H^2(\Omega), H) \) for (22), which is unique and depends continuously on the data

\[ |z|^2_{W(I, H^2(\Omega), H)} \leq \overline{C} \left[ |I|, C_{\mathcal{W}_{\text{ad}}}, \frac{1}{\nu} \right] \left( |z_0|^2_{H^1(\Omega)} + |h|^2_{L^2(I, H)} + |\gamma|^2_{G^2(I, \Gamma)} \right) . \]

We also have a smoothing property as follows.

**Lemma 2.6** Let \((\hat{a}, \hat{b}), h, \) and \( \gamma \) be as in Lemma 2.5, and let \( z_0 \in H \). Then, the weak solution \( z \) of system (22) satisfies \((\cdot - s_0)z \in W(I, H^2(\Omega), H)\), and

\[ |((\cdot - s_0)z|^2_{W(I, H^2(\Omega), H)} \leq \overline{C} \left[ |I|, C_{\mathcal{W}_{\text{ad}}}, \frac{1}{\nu} \right] \left( |z_0|^2_{H^1} + |h|^2_{L^2(I, H)} + |\gamma|^2_{G^2(I, \Gamma)} \right) . \]
Proof Since \( z \) solves (22), then also \( w = (\cdot - s_0)z \) does, with different data:

\[
\frac{\partial}{\partial t} w - v \Delta w + \hat{a} w + \nabla \cdot (\hat{b} w) + (\cdot - s_0)h - z = 0,
\]

\[
w|_{\Gamma} = (\cdot - s_0)\gamma, \quad w(0) = 0.
\]

From Lemma 2.5, we can derive that, with \( \tilde{h} := (\cdot - s_0)h \) and \( \tilde{\gamma} := (\cdot - s_0)\gamma \),

\[
|w|^2_{W(I, H^2(\Omega), H)} \leq C \left( |I|, C_{W^1_{st}, \nu} \right) \left( |\tilde{h}|^2_{L^2(I, H)}, |z|^2_{W(I, H^1(\Omega), H^{-1}(\Omega))} + |\tilde{\gamma}|^2_{G^2(I, \Gamma)} \right).
\]

The result follows from \( |z|^2_{W(I, H^1(\Omega), H^{-1}(\Omega))} \leq |z|^2_{W(I, H^1(\Omega), H^{-1}(\Omega))} \) and Lemma 2.2. \( \square \)

2.3 Controls supported in a subset

Consider, in the cylinder \( I \times \Omega \), the controlled system (22)

\[
\frac{\partial}{\partial t} z - v \Delta z + \hat{a} z + \nabla \cdot (\hat{b} z) = 0, \tag{26a}
\]

\[
z|_{\Gamma} = B\zeta, \quad z(s_0) = z_0, \tag{26b}
\]

where \( B \in \mathcal{L}(\mathcal{Z}, G^1(I, \Gamma)) \) is to be seen as a control operator, and \( \mathcal{Z} \) is a given Hilbert space.

Following a standard argument, see [40, Section 4] and also [23] and references therein, we can construct open subsets \( \tilde{\omega} \) with

\[
\Omega \cap \tilde{\omega} = \emptyset, \quad \Gamma \cap \partial \tilde{\omega} \subseteq \tilde{T}_1, \quad \text{and} \quad \tilde{\Omega} := \Omega \cup \tilde{\omega} \cup (\Gamma \cap \partial \tilde{\omega}),
\]

such that \( \tilde{\Omega} \) is still a smooth domain.

Let us be given \( (\hat{a}, \hat{b}) \in \mathcal{W} \), and let \( \tilde{a} \) and \( \tilde{b} \) be, respectively, the extensions of \( \hat{a} \) and \( \hat{b} \) by zero outside \( \Omega \). Notice that we still have \( (\tilde{a}, \tilde{b}) \in \tilde{\mathcal{W}} \) (cf. (20), (21))

\[
\tilde{\mathcal{W}} := L^\infty(\mathbb{R}_0, L^d(\tilde{\Omega}, \mathbb{R})) \times L^\infty_w(\mathbb{R}_0, L^\infty(\tilde{\Omega}, \mathbb{R}^d)), \quad \left| (\tilde{a}, \tilde{b}) \right|^2_{\tilde{\mathcal{W}}} \leq C_\mathcal{W}.
\]

However, for given \( (\hat{a}, \hat{b}) \in \mathcal{W}_{st} \), see (23), we cannot guarantee that we still have \( (\tilde{a}, \tilde{b}) \in \tilde{\mathcal{W}}_{st} \),

\[
\tilde{\mathcal{W}}_{st} := \{(a, b) \in \tilde{\mathcal{W}} \mid \nabla \cdot b \in L^\infty_w(\mathbb{R}_0, L^r(\tilde{\Omega}, \mathbb{R}))\},
\]

with \( r \) as in (23). We need a smoother extension for the vector field \( \hat{b} \), given by the following proposition whose proof is given in Appendix, Sect. A.2.

Proposition 2.1 There exists \( \tilde{\omega} \) satisfying (27) and there exists an extension \( \tilde{\mathcal{W}} \) of \( \tilde{b} \) such that the linear mapping \( (\tilde{a}, \tilde{b}) \mapsto (\tilde{a}, \tilde{b}) \) is continuous.
Remark 2.2 In the literature, we may find some results allowing us to construct/extend vectors fields satisfying some divergence constraint. See the results in [3,4,14], for \( d \in \{2,3\} \). See also [46, chapter 1, Theorem 2.4]. We were not able to use those results to “construct” an extension \( \mathcal{W}_{\text{st}} \to \tilde{\mathcal{W}}_{\text{st}} \). Proposition 2.1, in some sense, generalizes some results in [14, Section 3] to higher dimensions and for essentially bounded vectors \( \hat{b} \). Furthermore, the extension \( \tilde{b} \) constructed in Appendix will be divergence free if so is \( \hat{b} \), see (A.4) in Appendix. Thus, Proposition 2.1 also generalizes to higher dimensions the result in Proposition 4.2 presented in [40], for \( d = 3 \).

It is known, see for example [21], that we can find a family of internal controls \( \{ \hat{\eta}(w) \mid w \in L^2(\hat{\Omega}) \} \), with \( \hat{\eta} \in \mathcal{L}(H, L^2(1, L^2(\hat{\Omega}))) \), such that the solution \( z^e \) of the system

\[
\begin{align*}
\frac{\partial}{\partial t} z^e - v \Delta z^e + \tilde{\alpha} z^e + \nabla \cdot (\tilde{b} z^e) &= 1_{\tilde{\omega}} \hat{\eta}(z^e_0), \\
z^e|_{\tilde{T}} &= 0, \quad z^e(s_0) = z^e_0,
\end{align*}
\]

(30a)

where \( z^e_0 \) is the extension of \( z_0 \) by zero outside \( \Omega \), satisfies \( z^e(s_1) = 0 \) and

\[
|\hat{\eta}(z^e_0)|^2_{L^2(1, L^2(\hat{\Omega}))} \leq e^{C_{\tilde{\omega}, \Omega} \Theta^I_v} |z^e_0|^2_{L^2(\hat{\Omega})} = e^{C_{\tilde{\omega}, \Omega} \Theta^I_v} |z_0|^2_H,
\]

(31)

with \( \Theta^I_v := \Theta \left( |v|, |\tilde{v}|_{L^\infty(I, L^d(\hat{\Omega}))), |\tilde{b}|_{L^\infty(I, L^\infty(\hat{\Omega}))), d \right) \), and

\[
\Theta(r, \theta_1, \theta_2, d) := 1 + \theta_1^2 + d \theta_2^2 + \frac{1}{r} + r \left( \theta_1 + d \theta_2^2 \right),
\]

(32)

and where \( C_{\tilde{\omega}, \Omega} \) is a constant depending on \( \tilde{\omega} \) and \( \Omega \).

Moreover, if \( z_0 \in V \) then \( z^e_0 \in H^1_0(\hat{\Omega}) \), and we have that \( z^e \) is a strong solution, which implies that \( z := z^e|_{\Omega} \) solves (26), with \( Z = G^2(1, \Gamma) \), \( \zeta = z^e|_{\Gamma} \), and the operator

\[
\mathcal{B} \in \mathcal{L}(\mathcal{C}^2(1, \Gamma), \mathcal{C}^2(1, \Gamma)), \quad \mathcal{B} \zeta := \partial \zeta.
\]

Recall that \( \partial \in \mathcal{C}^2(\Gamma) \) and \( \partial|_{\Gamma_1} = 1 \), that is, \( \partial \zeta = \zeta \) since \( \text{supp}(z^e|_{\Gamma}) \subseteq \overline{T_1} \).

Furthermore, we find that

\[
|\partial \zeta|^2_{G^2(1, \Gamma)} \leq |z|_{W(I, H^2(\hat{\Omega}), L^2(\hat{\Omega}))}^2 \leq \overline{C} \left[ |I|, C_W_{\text{st}}, \frac{1}{\Gamma} \right] \left( 1 + e^{C_{\tilde{\omega}, \Omega} \Theta^I_v} \right) |z^e(s_0)|_{H^1_0(\hat{\Omega})}^2
\]

\[
\leq 2 \overline{C} \left[ |I|, C_W_{\text{st}}, \frac{1}{\Gamma} \right] e^{C_{\tilde{\omega}, \Omega} \Theta^I_v} |z(s_0)|^2_{\tilde{V}}
\]

and, since the choice of such subset \( \tilde{\omega} \) is at our disposal, we can conclude that there exists a constant \( C_{\Gamma_1, \Omega} > 0 \) depending on \( \Gamma_1 \) and \( \Omega \), such that

\[
|\partial \zeta|^2_{G^2(1, \Gamma)} \leq \overline{C} \left[ |I|, C_W_{\text{st}}, \frac{1}{\Gamma} \right] e^{C_{\Gamma_1, \Omega} \Theta^I_v} |z(s_0)|^2_{\tilde{V}}.
\]
Therefore, we have the following.

**Theorem 2.1** Let \((\hat{a}, \hat{b}) \in \mathcal{W}_d\) and \(B\xi := \vartheta \xi\). Then, there is a family \(\{\xi(z_0) \mid z_0 \in V\}\), with \(\xi \in \mathcal{L}(H, G^2(I, \Gamma))\), such that the solutions \(z = z(z_0, \xi(z_0))\) to (26) satisfy \(z(z_0, \xi(z_0))(s_1) = 0\) and, for a constant \(\tilde{C}_0 = C(\Gamma_1, \Omega)\), we have

\[
\vartheta \xi(z_0) = \tilde{\xi}(z_0), \quad \text{and} \quad |\vartheta \xi(z_0)|^2_{G^2(I, \Gamma)} \leq \tilde{C} \left| |I|_1|, C_{W_d, \Gamma} \right| \tilde{C}_0 \left| z_0 \right|^2_V.
\]

**Theorem 2.2** Let \((\hat{a}, \hat{b}) \in \mathcal{W}_d\), \(B\xi := \vartheta \xi\), and \(s_{1/2} \in I\). Then, there is a family \(\{\xi_1(z_0) \mid z_0 \in H\}\), with \(\xi_1 \in \mathcal{L}(H, G^2(I, \Gamma))\), such that the solutions \(z = z(z_0, \xi(z_0))\) to (26) satisfy \(z(z_0, \xi(z_0))(s_1) = 0\) and, for a constant \(\tilde{C}_0 = C(\Gamma_1, \Omega)\), we have that

\[
\vartheta \xi_1(z_0) = \tilde{\xi}_1(z_0), \quad \text{and} \quad |\vartheta \xi_1(z_0)|^2_{G^2(I, \Gamma)} \leq |I_1|^{-2} \tilde{C} \left| |I_1|, C_{W_d, \Gamma} \right| \tilde{C}_0 \left| z_0 \right|^2_H,
\]

where \(I_1 = (s_0, s_{1/2})\) and \(I_2 = (s_{1/2}, s_1)\).

**Proof** Firstly we apply zero control for time \(t \in I_1\) in this way, see Lemma 2.6, we arrive at a vector \(z(s_{1/2}) = z(z_0, 0)(s_{1/2}) \in V\) and the mapping \(z(s_0) = z_0 \mapsto z(z_0, 0)(s_{1/2})\) is linear and continuous:

\[
|z(z_0, 0)(s_{1/2})|^2_V \leq |I_1|^{-2} \tilde{C} \left| |I_1|, C_{W_d, \Gamma} \right| |z_0|^2_H.
\]

Next we apply, in \(I_2\), the control \(\bar{\xi}(z(s_{1/2}))\) given by Theorem 2.1. Thus

\[
|\vartheta \bar{\xi}(z(s_{1/2}))|^2_{G^2(I_2, \Gamma)} \leq \tilde{C} \left| |I_2|, C_{W_d, \Gamma} \right| \tilde{C}_0 \left| s_{1/2} \right|^2_V \leq |I_1|^{-2} \tilde{C} \left| |I_1|, C_{W_d, \Gamma} \right| \tilde{C}_0 \left| s_{1/2} \right|^2_H.
\]

It remain to check that the concatenated control

\[
\vartheta \bar{\xi}_1(z_0) := \begin{cases} 0, & \text{if } t \in I_1 \\ \vartheta \bar{\xi}(z(z_0, 0)(s_{1/2})), & \text{if } t \in I_2 \end{cases}
\]

is in \(G^2(I, \Gamma)\). It is enough to check that the following weighted concatenation of the corresponding solutions

\[
\tilde{z}(t) := \begin{cases} \psi(t)z(z_0, 0)(t), & \text{if } t \in I_1 \\ z(z(s_{1/2}), \xi(z(s_{1/2}))) (t), & \text{if } t \in I_2 \end{cases}
\]

is in \(W(I, H^2(\Omega), L^2(\Omega))\), for some smooth function \(\psi\) vanishing for \(t \in [s_0, r_1]\) and taking the value 1 for \(t \in [r_2, s_{1/2}]\), with \(s_0 < r_1 < r_2 < s_{1/2}\). Notice that \(\psi\) does not change the trace on the boundary, \(\tilde{z}|_\Gamma = z|_\Gamma\). Since
\[ \tilde{z}|_{I_1} \in W(I_1, H^2(\Omega), L^2(\Omega)) \quad \text{and} \quad \tilde{z}|_{I_2} \in W(I_1, H^2(\Omega), L^2(\Omega)), \]

and recalling that (cf. [32, chapter 1, sections 3.2 and 9.3])

\[ \{ v(s_{1/2}) \mid v \in W(I_1, H^2(\Omega), L^2(\Omega)) \} = H^1(\Omega) = \{ v(s_{1/2}) \mid v \in W(I_2, H^2(\Omega), L^2(\Omega)) \}, \]

it follows that the concatenation is in \( W(I, H^2(\Omega), L^2(\Omega)) \), because by construction \((\tilde{z}|_{I_1})(s_{1/2}) = (\tilde{z}|_{I_2})(s_{1/2}) \in V \subset H^1(\Omega)\). \( \square \)

### 2.4 Stabilization to zero by finite-dimensional controls

Here \((\hat{a}, \hat{b}) \in \mathcal{W}_{st}\). We look for stabilizing controls, of the form \( B_\Psi u(t) = \partial \sum_{i=1}^{M} u_i(t) \Psi_i(x) \), with \( u \in H^1(\mathbb{R}_0, \mathbb{R}^M) \).

We will construct the finite-dimensional stabilizing control from the control \( \tilde{z}_1(z_0) \in G^2(I, \Gamma) \), given by Theorem 2.2. Notice that the range of \( \tilde{z}_1(z_0) \) is not necessarily finite-dimensional. Note also that, if we take \( \partial P_M \tilde{z}_1(z_0) \) instead, then such control takes values in the space \( \partial S_\Psi \) spanned by the actuators \( \partial \Psi_i \). Recall that \( P_M \) is the orthogonal projection in \( L^2(\Gamma, \mathbb{R}) \) onto \( S_\Psi = \text{span}\{ \Psi_i \mid i \in \{1, 2, \ldots, M\} \} \). Moreover, writing \( u = (B_\Psi)^{-1} \partial P_M \tilde{z}_1(z_0) \), that is, \( \partial P_M \tilde{z}_1(z_0) = B_\Psi u \), we do not necessarily have that \( u \in H^1(I, \mathbb{R}^M) \), as we show now by recalling the characterization of \( G^2(I, \Gamma) \) in terms of (fractional) Sobolev–Bochner spaces.

We consider (cf. [25, section 2.1] and [38, section 2.2], see also [33, chapter 4, section 2]) the following subspace of \( W(I, H^1(\Omega), H^{-1}(\Omega)) \) defined by

\[
W(I, H^1(\Omega), H^{-1}(\Omega)) := W(\mathbb{R}, H^1(\mathbb{R}^d), H^{-1}(\mathbb{R}^d))|_{I \times \Omega} \\
\hookrightarrow W(I, H^1(\Omega), H^{-1}(\Omega)),
\]

and the corresponding trace space

\[
G^1(I, \Gamma) := W(I, H^1(\Omega), H^{-1}(\Omega))|_{\Gamma} \hookrightarrow G^1(I, \Gamma).
\]

Analogously, we consider the space

\[
W(I, H^2(\Omega), L^2(\Omega)) := W(\mathbb{R}, H^2(\mathbb{R}^d), L^2(\mathbb{R}^d))|_{I \times \Omega} \\
\hookrightarrow W(I, H^2(\Omega), L^2(\Omega)),
\]

and the corresponding trace space

\[
G^2(I, \Gamma) := W(I, H^2(\Omega), L^2(\Omega))|_{\Gamma} \hookrightarrow G^2(I, \Gamma).
\]
Notice that (cf. [38, section 2.2]) for a general domain $\Omega \subset \mathbb{R}^d$, we have

$$W(I, H^2(\Omega), L^2(\Omega)) = W(I, H^2(\Omega), L^2(\Omega)), \quad G^2(I, \Gamma) = G^2(I, \Gamma),$$

$$G^1(I, \Gamma) \neq G^1(I, \Gamma).$$

We have the following characterizations in [25, Theorem 3.1],

$$G^1(I, \Gamma) = G^1(I, \Gamma) := L^2(I, H^\frac{1}{2}(\Gamma)) \cap H^\frac{1}{2}(I, H^{-\frac{1}{2}}(\Gamma)), \quad (33a)$$

$$G^2(I, \Gamma) = G^2(I, \Gamma) := L^2(I, H^\frac{1}{2}(\Gamma)) \cap H^\frac{1}{2}(I, L^2(\Gamma)). \quad (33b)$$

That is, $u$ defined by $B_{\Psi}u = \partial P_{M} \partial \bar{\xi}_1(z_0)$ belongs to $H^\frac{1}{2}(I, \mathbb{R}^M)$, but not necessarily to $H^1(I, \mathbb{R}^M)$. In order to obtain the desired regularity $H^1(I, \mathbb{R}^M)$ for the control, we will take controls of the form $Q_M \partial P_{M} \partial \bar{\xi}_1(z_0)$ where $Q_M$ is a suitable orthogonal projection in $L^2(I)$ with range contained in $H^1_0(I)$.

### 2.4.1 Further regularity in time variable for the control

The stabilizing control in $\mathbb{R}_{s_0} = (s_0, +\infty)$, $s_0 \geq 0$, will be constructed recursively in intervals of the same length, as $J^j := (s_0 + jT_s, s_0 + (j + 1)T_s)$, with $j \in \mathbb{N}$, where the length $T_s$ will be fixed.

Let $J := (0, T_s)$ and let $\xi_k \in H^1_0(J)$ be the orthonormalized eigenfunctions of the Dirichlet Laplacian $\Delta_J := -\frac{\partial^2}{\partial t^2}$ in $L^2(J)$, that is,

$$\xi_k(t) := \left(\frac{2}{T_s}\right)^{\frac{1}{2}} \sin\left(\frac{\pi t}{T_s}\right), \quad \Delta_J \xi_k = \beta_k \xi_k, \quad \text{with} \quad 0 < \beta_k := \left(\frac{\pi}{T_s}\right)^2 k^2 \rightarrow +\infty.$$

We define $Q_M$ as the orthogonal projection in $L^2(J)$ onto the finite-dimensional subspace span($\xi_k | k \in \{1, 2, \ldots, M\}$).

Then for each interval $J^j$, with $j \in \mathbb{N}$, we define the orthogonal projection $Q^j_M$ in $L^2(J^j)$, with range $Q^j_M(L^2(J^j)) \subset H^1_0(J^j)$, by

$$Q^j_M f := T_{s_0 + jT_s} Q_M T_{s_0 - jT_s} f \quad (34)$$

where $T_r$ is the translation operator $T_r f = f(\cdot - r)$.

Let us now fix $\lambda \geq 0$ and $j \in \mathbb{N}$, and consider, in $J^j \times \Omega$, the system:

$$\frac{\partial}{\partial t} z - v \Delta z + (\hat{a} - \frac{1}{2}) z + \nabla \cdot (\hat{b} z) = 0, \quad (35a)$$

$$z|_{\Gamma} = Q^j_M \partial P_{M} \partial \bar{\xi}_1(z_0), \quad z(s_0 + jT_s) = z^j_0, \quad (35b)$$

where $z^j_0 \in H$ and $\bar{\xi}_1(z^j_0) \in G^2(J^j, \Gamma)$ is given by Theorem 2.2, with $(\hat{a} - \frac{1}{2})$ in the place of $\hat{a}$. To fix ideas, we take the point $s_{1/2} = s_0 + (j + \frac{1}{2})T_s$ of $J^j$ in Theorem 2.2, $I_1 = (s_0 + jT_s, s_0 + (j + \frac{1}{2})T_s)$ and $I_2 = (s_0 + (j + \frac{1}{2})T_s, s_0 + (j + 1)T_s)$. 

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Proposition 2.2 Both $\vartheta P_M \vartheta$ and $Q \tilde{M} \vartheta P_M \vartheta$ are in $L(G^2(J, \Gamma))$.

The proof of Proposition 2.2 is given in Appendix, Sect. A.3.

Let $z$ solve (35) with $\vartheta$ in the place of $Q_j \tilde{M} \vartheta P_M \vartheta$, and let $z_M$ solve (35). Then, $d = z - z_M$ solves

$$\frac{\partial}{\partial t} d - \nu \Delta d + (\hat{a} - \frac{\lambda}{2}) d + \nabla \cdot (\hat{b} d) = 0,$$

$$ d|_{\Gamma} = \vartheta \left(1 - Q_j \tilde{M} P_M \vartheta \right) \vartheta \hat{\zeta}_1(z_j^0), \quad d(s_0) = 0. $$

Notice that $z_M(s_0 + (j + 1)T_*) = -d(s_0 + (j + 1)T_*)$. From Lemma 2.3 and Theorem 2.2, it follows, with $R := \vartheta(1 - Q_j \tilde{M} P_M) \vartheta$,

\begin{equation}
|\vartheta \tilde{\zeta}_1(z_j^0)|^2_{G^2(J_j, \Gamma)} \leq (\frac{T_*}{\tau})^{-2C} \left[ T_*, C_{Wst}^{1, \frac{1}{\tau}} \right] e^{C_0 \theta \tilde{\zeta}_1^2} \left| z_j^0 \right|_{H}^2, \tag{36a}
\end{equation}

\begin{equation}
|z_M(s_0 + (j + 1)T_*)|^2_{H} \leq \mathcal{E}(T_*) \left[ R \right]_{L(G^2(J_j, \Gamma), G^1(J_j, \Gamma))}^2 \left| z_j^0 \right|_{H}^2, \tag{36b}
\end{equation}

\begin{equation}
|z_M|^2_{L^\infty(J_j, H)} \leq \left( C + \mathcal{E}(T_*) \left[ R \right]_{L(G^2(J_j, \Gamma), G^1(J_j, \Gamma))}^2 \right) \left| z_j^0 \right|_{H}^2, \tag{36c}
\end{equation}

with $C = \overline{C} \left[ T_*, C_{Wst}^{1, \frac{1}{\tau}} \right]$ and where

$$\mathcal{E}(\tau) := 4r^{-2C} \left[ \tau, C_{Wst}^{1, \frac{1}{\tau}} \right] e^{D_{\vartheta} \left( \frac{r}{\tau} - \frac{1}{2} \right) \left| \frac{\hat{a}}{\vartheta} - \frac{\lambda}{2} \right|_{L^\infty(H_0, L^d(\Omega))} \left| \frac{\hat{b}}{\vartheta} \right|_{L^\infty(H_0, L^\infty(\Omega))}^2 d} \geq \left| \vartheta \tilde{\zeta}_1 \right|^2_{L(H, G^2(J_j, \Gamma))}, \quad |J_j| = \tau.$$

We can see that when $(\hat{a} - \frac{\lambda}{2}, \hat{b}) = (0, 0)$ then the system

$$\frac{\partial}{\partial t} z - \nu \Delta z + (\hat{a} - \frac{\lambda}{2}) z + \nabla \cdot (\hat{b} z) = 0, \quad z|_{\Gamma} = 0,$$

(cf. system (35)) is exponentially stable. Therefore, from now, we consider the case $(\hat{a} - \frac{\lambda}{2}, \hat{b}) \neq (0, 0)$ where we can see that it holds

$$\lim_{\tau \to +\infty} \mathcal{E}(\tau) = +\infty \quad \text{and} \quad \lim_{\tau \to 0} \mathcal{E}(\tau) = +\infty.$$

Hence, we can set $T_* > 0$ such that

$$\mathcal{E}(T_*) = \min_{\tau > 0} \mathcal{E}(\tau) =: \Upsilon. \tag{37}$$
2.4.2 The first stabilizability condition

We show that a stabilizing control can be constructed, under a boundedness condition on the operator \( Q_j^L \partial P_M \partial \). Let us consider, in \( \mathbb{R}^s_0 \times \Omega \), the system,

\[
\begin{align*}
\frac{\partial}{\partial t} z - v \Delta z + (\hat{a} - \frac{\hat{b}}{2})z + \nabla \cdot (\hat{b}z) &= 0, \quad (38a) \\
|z|_r = \zeta_c(z_0), \quad z(s_0) = z_0, \quad (38b) \\
\end{align*}
\]

where the control \( \zeta_c(z_0) \) is defined recursively as follows.

1. In \( J^0 = (s_0, s_0 + T_*) \) we take the control as in system (35), with \( z_0^0 := z_0 \),

\[
\zeta_c(z_0)|_{J^0} = Q_j^L \partial P_M \partial \zeta_1^0(z_0). \quad (38c)
\]

2. Once the control has been constructed for time \( t \in (s_0, s_0 + jT_*) \), \( j \geq 1 \), we solve the system and take the final state \( z_j^0 := z(s_0 + jT_*) \). Then, we take again the control as in system (35), in the time interval \( J^j \),

\[
\zeta_c(z_0)|_{J^j} = Q_j^L \partial P_M \partial \zeta_1^j(z_0^j). \quad (38d)
\]

In Theorem 2.3, we will give a stabilizability condition in terms of the value \( \chi := |\partial (1 - Q_j^L P_M^L) \partial |_{L(G^2(j, \Gamma), G^1(j, \Gamma))}^2 \). Denoting also, for each \( j \in \mathbb{N} \), \( \chi^j := |\partial (1 - Q_j^L P_M^L) \partial |_{L(G^2(j, \Gamma), G^1(j, \Gamma))}^2 \), we observe that the dependence of \( \chi^j \) on \( j \) is only terms of the length \( |J^j| \) of \( J^j \). Since \( |J^j| = |J| = T_* \), then \( \chi^j = \chi \).

**Theorem 2.3** The system (38) is exponentially stable with rate \( \delta T_* > 0 \), if

\[
|\partial (1 - Q_j^L P_M^L) \partial |_{L(G^2(j, \Gamma), G^1(j, \Gamma))}^2 \leq e^{-\delta T_*} \chi^{-1}. \quad (39)
\]

Furthermore, \( |\zeta_c(z_0)|_{G^2(\mathbb{R}^s_0, \Gamma)}^2 \leq \frac{C}{1 - e^{-\delta T_*}} |z_0|^2_H \), for a suitable constant \( C > 0 \).

**Proof** From (36) and (39), we find that the solution of (38) satisfies

\[
|z(s_0 + (j + 1)T_*)|^2_H \leq e^{-\delta T_*} |z(s_0 + jT_*)|^2_H \leq e^{-(j+1)\delta T_*} |z_0|^2_H, \\
|z|^2_{L^\infty(J^j, H)} \leq \left( \overline{C} + e^{-\delta T_*} \right) e^{-j\delta T_*} |z_0|^2_H.
\]

Since for \( t \in J^j \) we have that \( t = s_0 + jT_* + rT_* \) with \( r \in (0, 1) \), it follows that \( jT_* = t - s_0 - rT_* \), and

\[
|z(t)|^2_H \leq \left( \overline{C} + e^{-\delta T_*} \right) e^{rT_*} e^{-\delta(t-s_0)} |z(s_0)|^2_H, \quad t \geq s_0.
\]
Finally, we observe that  
\[ \left| \tilde{\zeta}(z_0) \right|^2_{G^2(\mathbb{R}_r, \Gamma)} \leq C \sum_{j \in \mathbb{N}} \left| \tilde{\zeta}(z_0 + jT_s) \right|^2_{G^2(j, \Gamma)}, \]  
that is,  
\[ \left| \tilde{\zeta}(z_0) \right|^2_{G^2(\mathbb{R}_r, \Gamma)} \leq C \sum_{j \in \mathbb{N}} e^{-j\delta T_s} |z_0|^2_{H^s} \leq \frac{C}{1-e^{-\delta T_s}} |z_0|^2_{H^s}. \]  
\( \square \)

2.4.3 The main stabilizability condition

Notice that (39) involves spaces of functions defined in the cylinder \( J \times \Gamma \). Here, we present the main stabilizability condition in terms of spaces of functions defined on \( \Gamma \) only.

The norms of the spaces \( G^i(J, \Gamma) \) have been introduced as the trace norm in Sect. 2. Here, the (fractional) Sobolev–Bochner spaces \( G^1(J, \Gamma) \) and \( G^2(J, \Gamma) \), in (33), are supposed to be endowed with the usual norms (based on the Fourier Transform). See [41, Section A.2] and references therein. That is, the norms may not coincide, but they are equivalent

\[ D_1^b |.|_{G^1(J, \Gamma)} \leq |.|_{G^1(J, \Gamma)}^2 \leq D_1^a |.|_{G^1(J, \Gamma)}, \quad (40a) \]
\[ D_2^b |.|_{G^2(J, \Gamma)} \leq |.|_{G^2(J, \Gamma)}^2 \leq D_2^a |.|_{G^2(J, \Gamma)}, \quad (40b) \]

The constants \( D_i^b, D_i^a, i \in \{1, 2\} \), depend on the length \( |J| \) of the interval \( J \).

Recall the space \( \mathcal{H}(\Gamma) \) defined in (19). We have the following result.

**Theorem 2.4** The system (38) is stabilizable to zero with rate \( \frac{\delta}{2} > 0 \), if

\[ |\partial (1 - P_M)\vartheta|^2_{\mathcal{H}(\Gamma)} \leq \frac{D_2^b e^{-\delta T_s}}{4D_1^a} \gamma^{-1}. \quad (41) \]

Furthermore, there exists \( \tilde{M} = \mathcal{C}[M] \) so that, with \( u\tilde{\zeta}(z_0) = (B_\psi)^{-1}\tilde{\zeta}(z_0) \), where \( \tilde{\zeta}(z_0) \) is as in Theorem 2.3, we have the estimate

\[ \left| z(z_0, B_\psi u\tilde{\zeta}(z_0)) \right|^2_{L^2(\mathbb{R}_r, H)} + \left| u\tilde{\zeta}(z_0) \right|^2_{H^1(\mathbb{R}_r, \mathbb{R}^M)} \leq \mathcal{C}[\mathcal{C}_\text{Vg}, \lambda, \frac{1}{2}, \frac{2}{3}, \delta, \gamma] |z_0|^2_{H^s}. \]

For the proof of Theorem 2.4, we will need the following auxiliary results.

**Proposition 2.3** Let us be given Banach spaces \( X_1, Y_1, X_2, Y_2 \) and Hausdorff topological spaces \( Z_1, Z_2 \) such that \( X_i \hookrightarrow Z_i \) and \( Y_i \hookrightarrow Z_i, i \in \{1, 2\} \). Then for any given \( A \in \mathcal{L}(X_1, X_2) \cap \mathcal{L}(Y_1, Y_2) \), we have that \( A \in \mathcal{L}(X_1 \cap Y_1, X_2 \cap Y_2) \) and \( |A|_{\mathcal{L}(X_1 \cap Y_1, X_2 \cap Y_2)} \leq |A|_{\mathcal{L}(X_1, X_2)} + |A|_{\mathcal{L}(Y_1, Y_2)}. \)

The proof is straightforward and is omitted.

**Lemma 2.7** We have the continuous inclusions

\[ G^2(J, \Gamma) \hookrightarrow H^{\frac{9}{10}}(J, H^{\frac{3}{5}}(\Gamma)) \cap H^{\frac{3}{5}}(J, H^{\frac{3}{5}}(\Gamma)) \hookrightarrow G^1(J, \Gamma). \]
Proof The inclusion $H^{\frac{9}{2}}(J, H^{\frac{3}{2}}(\Gamma)) \cap H^{\frac{3}{2}}(J, H^{\frac{3}{2}}(\Gamma)) \hookrightarrow G^1(J, \Gamma)$ follows easily from (33). From (33) and [41, Lemma A.12], we find that $G^2(J, \Gamma) \hookrightarrow H^{\frac{3}{2}}(J, H^{\frac{3}{2}}(\Gamma))$. Then, using again [41, Lemma A.12], we also obtain the inclusion $H^{\frac{3}{2}}(J, L^2(\Gamma)) \cap H^{\frac{3}{2}}(J, H^{\frac{3}{2}}(\Gamma)) \hookrightarrow H^{\frac{9}{2}}(J, H^{\frac{3}{2}}(\Gamma))$. \hfill \Box

Lemma 2.8 We have that for big enough $\tilde{M} = \overline{C}_M$,

$$\|\vartheta |1 - Q_{\tilde{M}} P_M\vartheta\|^2_{L(G^2, G^1)} \leq 4 D_1^{\frac{5}{3}} |\vartheta |(1 - P_M\vartheta\|^2_{\mathcal{H}(\Gamma)},$$

where $D_1^{\frac{5}{3}}$ and $D_2^{\frac{5}{3}}$ are as in (33).

Proof From $\hat{R} := \vartheta |1 - Q_{\tilde{M}} P_M\vartheta = \vartheta |1 - P_M\vartheta + \vartheta |1 - Q_{\tilde{M}} P_M\vartheta$, we have

$$\|\hat{R}\|^2_{L(G^2, G^1)} \leq 2 \|\vartheta |1 - P_M\vartheta\|^2_{L(G^2, G^1)} + 2 \|\vartheta |1 - Q_{\tilde{M}} P_M\vartheta\|^2_{L(G^2, G^1)}.$$ (42)

Now recalling (33) and (19), we find

$$\|\vartheta |1 - P_M\vartheta\|^2_{L(G^2, G^1)} \leq \|\vartheta |L(G^2, G^1)\| \|\vartheta |1 - P_M\vartheta\|^2_{L(G^2, G^1)} \leq \frac{D_1^{\frac{5}{3}}}{D_2^{\frac{5}{3}}} \|\vartheta |1 - P_M\vartheta\|^2_{\mathcal{H}(\Gamma)},$$ (43)

because, for any $w \in G^2$, and denoting shortly $\mathcal{P} := \vartheta |1 - P_M\vartheta$,

$$\|\vartheta |1 - P_M\vartheta w\|^2_{G^1} = \|\mathcal{P}w\|^2_{L^2(J, H^{\frac{1}{2}}(\Gamma))} + \|\mathcal{P}w\|^2_{H^{\frac{3}{2}}(J, H^{-\frac{1}{2}}(\Gamma))} \leq \|\mathcal{P}\|^2_{L(H^{\frac{3}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma))} \|w\|^2_{L^2(J, H^{\frac{3}{2}}(\Gamma))} + \|\mathcal{P}\|^2_{L(H^2(\Gamma), H^{-\frac{1}{2}}(\Gamma))} \|w\|^2_{H^2(J, L^2(\Gamma))} \leq \|\mathcal{P}\|^2_{\mathcal{H}(\Gamma)} \|w\|^2_{G^2}.$$  

Now for simplicity we denote, see Lemma 2.7,

$$\mathcal{I}(\Gamma) := H^{\frac{9}{2}}(J, H^{\frac{3}{2}}(\Gamma)) \cap H^{\frac{3}{2}}(J, H^{\frac{3}{2}}(\Gamma)).$$

Recalling that by assumption $\mathcal{S}_\Psi \subset H^{\frac{3}{2}}(\Gamma)$, see (4), we observe that

$$\|\vartheta |1 - Q_{\tilde{M}} P_M\vartheta\|^2_{L(G^2, G^1)} \leq \|P_M\vartheta\|^2_{L(G^2, \mathcal{I}(\Gamma))} \|1 - Q_{\tilde{M}}\|^2_{L(P_M \mathcal{I}(\Gamma), G^1)} \|\vartheta\|^2_{L(G^1)}.$$  

\[ Springer\]
and, recalling Lemma 2.7 and Proposition 2.3,

\[
|1 - Q_{\tilde{M}}|_{L(P_{\tilde{M}}\mathcal{I}(\Gamma),\mathcal{G}^1)}^2 = |1 - Q_{\tilde{M}}|_{L(H^{\frac{3}{2}}(J,P_{M}H^{\frac{3}{2}}(\Gamma))),L^2(J,P_{M}H^{\frac{1}{2}}(\Gamma)))}^2 \\
+ |1 - Q_{\tilde{M}}|_{L(H^{\frac{3}{2}}(J,P_{M}H^{\frac{3}{2}}(\Gamma)),L^2(J,P_{M}H^{\frac{1}{2}}(\Gamma)))}^2 \\
\leq |1 - Q_{\tilde{M}}|_{L(H^{\frac{3}{2}}(J),L^2(J))}^2 + |1 - Q_{\tilde{M}}|_{L(H^{\frac{1}{2}}(J),H^{\frac{1}{2}}(J))}^2.
\]

Therefore, with \(\overline{C}_{[M,\|\cdot\|_{L(G^1)}]} = |P_{M}\vartheta|_{L(G^2,G^1)}^2 |\vartheta|_{L(G^1)}^2\), we arrive at,

\[
|\vartheta (1 - Q_{\tilde{M}}) P_{M}\vartheta|_{L(G^2,G^1)}^2 \\
\leq \overline{C}_{[M,\|\cdot\|_{L(G^1)}]} \left( |1 - Q_{\tilde{M}}|_{L(H^{\frac{3}{2}}(J),L^2(J))}^2 + |1 - Q_{\tilde{M}}|_{L(H^{\frac{1}{2}}(J),H^{\frac{1}{2}}(J))}^2 \right).
\]

Above we considered the fractional Sobolev spaces \(H^{s}(J)\), \(s \in [0,1]\), being endowed with the norm defined through the Fourier Transform. Recall that we can also see \(H^{s}(J)\) as being the domain \(H^{s}(J)\) of the fractional operator \((-\Delta_{J} + 1)^{\frac{s}{2}}\), where we recall \(-\Delta_{J} + 1: H^{2}(J) \cap H_{0}^{1}(J) \rightarrow L^{2}(J)\), with the equivalent norm

\[
|g|_{H^{s}(J)}^2 := \sum_{i=1}^{+\infty} (1 + \beta_{i})^s g_{i}^2, \quad \text{with} \quad g = \sum_{i=1}^{+\infty} g_{i} \xi_{i} \in H^{s}(J) = H_{0}^{s}(J).
\]

\[
C_{s}^{\beta_{\cdot}} |g|_{H^{s}(J)}^2 \leq |g|_{H_{0}^{s}(J)}^2 \leq C_{s}^{\beta_{\cdot}} |g|_{H^{s}(J)}^2.
\]

In particular, observe that for \(1 \geq r \geq s \geq 0\), and

\[
|g|_{H_{r}^{s}(J)}^2 = (1 + \beta_{\tilde{M}})^{s-r} \sum_{i=1}^{+\infty} (1 + \beta_{\tilde{M}})^{r-s} (1 + \beta_{i})^{s} g_{i}^2 \leq (1 + \beta_{\tilde{M}})^{s-r} |g|_{H_{r}^{s}(J)}^2.
\]

This allow us to conclude that

\[
|\vartheta (1 - Q_{\tilde{M}}) P_{M}\vartheta|_{L(G^2,G^1)}^2 \\
\leq \overline{C}_{[M,\|\cdot\|_{L(G^1)}]} \left( C_{1}(1 + \beta_{\tilde{M}})^{-\frac{s}{8}} + C_{2}(1 + \beta_{\tilde{M}})^{-\frac{1}{16}} \right) \leq C_{3}(1 + \beta_{\tilde{M}})^{-\frac{1}{16}},
\]

with \(C_{3} = 2 \max\{C_{1}, C_{2}\}\overline{C}_{[M,\|\cdot\|_{L(G^1)}]}\), where \(C_{1} = C_{3}^{\frac{s}{8}} (C_{0}^{\beta_{\cdot}})^{-1} = C_{3}^{\frac{s}{8}}, \) and \(C_{2} = C_{3}^{\frac{s}{16}} (C_{1}^{\beta_{\cdot}})^{-1} = C_{3}^{\frac{s}{16}}\).

Therefore, by setting \(\tilde{M} \in \mathbb{N}\) large enough so that

\[
1 + \beta_{\tilde{M}} \geq C_{3}^{16} \left( \frac{D_{1}}{D_{2}} \right)^{-16} |\vartheta (1 - P_{M})\vartheta|_{H^{0}(\Gamma)}^2,
\]

(44) Springer
we find \( |(1 - Q_{\tilde{M}})P_M \vartheta|^2_{L^2(G^2, G^1)} \leq \frac{D_2^2}{D_1^2} |(1 - P_M) \vartheta|^2_{\mathcal{H}(\Gamma)} \). Finally, by (42) and (43), we obtain \( |(1 - Q_{\tilde{M}})P_M \vartheta|^2_{L^2(G^2, G^1)} \leq 4 \frac{D_2^2}{D_1^2} |(1 - P_M) \vartheta|^2_{\mathcal{H}(\Gamma)}. \)

Now, Theorem 2.4 follows as a corollary of Theorem 2.3 and Lemma 2.8.

\textbf{Proof of Theorem 2.4} Let us set \( \tilde{M} \) as in Lemma 2.8. From (41), we find that the stability condition (39) is satisfied. Then by Theorem 2.3, we have that system (38) is exponentially stable with rate \( \delta \frac{\sigma}{2} \), and the control \( u \tilde{c}(z_0) := (B_{\psi})^{-1} \tilde{c}(z_0) \) (i.e., \( z|\Gamma = \tilde{c}(z_0) = B_{\psi} u \tilde{c}(z_0) \)) satisfies

\[
\left( B_{\psi} \right)^{-1} \tilde{c}(z_0) \leq \sum_{j \in \mathbb{N}} \left( B_{\psi} \right)^{-1} \tilde{c}(z_0) \leq \sup_{i \in \mathbb{N}} \left( B_{\psi} \right)^{-1} \left( L(\partial Q_{\tilde{M}} P_M \vartheta G^2(J^i, \Gamma), H^1(J^i, \mathbb{R}^M)) \right) \sum_{j \in \mathbb{N}} \tilde{c}(z_0) \leq \left[ G^2(J^i, \Gamma) \right]
\]

where \( \partial Q_{\tilde{M}} P_M \vartheta G^2(J^i, \Gamma) \) is supposed to be endowed with norm inherited from \( G^2(J^i, \Gamma) \). Therefore, we arrive at

\[
\left( B_{\psi} \right)^{-1} \tilde{c}(z_0) \leq C_1 \tilde{c}(z_0) \leq \left( B_{\psi} \right)^{-1} \left[ G^2(\mathbb{R}_0^\infty, \Gamma) \right] \leq \frac{C}{1 - e^{-\delta T_\Gamma}} |z_0|^2_{\mathcal{H}}.
\]

Note that \( \left| (B_{\psi})^{-1} \right|^2_{L(\partial Q_{\tilde{M}} P_M \vartheta G^2(J^i, \Gamma), H^1(J^i, \mathbb{R}^M))} \) is independent of \( i \), that is, the supremum coincides with \( C_1 := \left| (B_{\psi})^{-1} \right|^2_{L(\partial Q_{\tilde{M}} P_M \vartheta G^2(J, \Gamma), H^1(J, \mathbb{R}^M))}. \)

\section{2.5 An example of suitable actuators: estimate on the number of actuators}

Here, we present a set of actuators, supported in the subset \( \Gamma_c \subset \Gamma \), which allows us to stabilize the system, that is, such that the stabilizability condition (41) is satisfied.

We suppose that \( \vartheta \in C^2(\Gamma) \) and that the open subset \( \Gamma_c \subset \Gamma \) are as in (4). For simplicity we suppose that the boundary \( \partial \Gamma_c \) of \( \Gamma_c \) in \( \Gamma \) is either empty or \( C^\infty \)-smooth.

Let us take as actuators the functions (cf. [41,42]),

\[
\{ \partial \psi_i = \partial \bar{E}_{\Gamma_c}^0 \psi_i(\bar{x}) | 1 \leq i \leq M \}
\]

where the \( \psi_i \)'s are the first eigenfunctions of the Laplace (Laplace–de Rham) operator \( \Delta_{\Gamma_c} \) under homogeneous Dirichlet boundary conditions on \( \Gamma_c \),

\[
\Delta_{\Gamma_c} \psi_i = \sigma_i \psi_i, \quad \text{with} \quad 0 \leq \sigma_i \leq \sigma_{i+1} \to +\infty,
\]

and where \( \bar{E}_{\Gamma_c}^0 \) stands for the extension, to \( \Gamma \), by zero outside \( \Gamma_c \).
We see that
\[
|\vartheta (1 - P_M) \vartheta |^2_{\mathcal{L}^2(\Gamma)} \\
\leq |\vartheta |^2_{\mathcal{L}(H^\frac{\alpha}{2}(\Gamma))} (1 - P_M) |\mathcal{L}(H^\frac{\alpha}{2}(\Gamma))| |\vartheta |^2_{\mathcal{L}(H^\frac{\alpha}{2}(\Gamma))} \\
+ |\vartheta |^2_{\mathcal{L}(L^2(\Gamma))} (1 - P_M) |\mathcal{L}(L^2(\Gamma))| |\vartheta |^2_{\mathcal{L}(L^2(\Gamma))} \\
\leq \overline{C}_{\vartheta |^{2}c_{2}(\Gamma)} (1 + \sigma_M)^{-1} (1 + \sigma_M)^{-\frac{1}{2}} \leq 2\overline{C}_{\vartheta |^{2}c_{2}(\Gamma)} (1 + \sigma_M)^{-\frac{1}{2}}.
\]

Therefore, the stabilizability condition (41) is satisfied if
\[
1 + \sigma_M > \left( \frac{8\overline{C}_{\vartheta |^{2}c_{2}(\Gamma)} D_2^2 T e^{\delta T_n}}{C_d D_2^2} \right)^{d-1}.
\]

We may expect (as it is the case when \(\Gamma_1\) is flat, see [31]) that, \(\sigma_M \geq C_d M \overline{\varphi}^{-1}\), for a suitable constant \(C_d > 0\). Then, we would obtain a sufficient condition for stabilizability in terms of the number of actuators as \(1 + C_d M \overline{\varphi}^{-1} > \left( \frac{8\overline{C}_{\vartheta |^{2}c_{2}(\Gamma)} D_2^2 T e^{\delta T_n}}{C_d D_2^2} \right)^{d-1}\), which follows from \(M \geq \left( \frac{8\overline{C}_{\vartheta |^{2}c_{2}(\Gamma)} D_2^2 T e^{\delta T_n}}{C_d D_2^2} \right)^{d-1}\).

Remark 2.3 Note that the mapping \(\vartheta : f \mapsto \vartheta f\) is in \(\mathcal{L}(H^2(\Gamma)) \cap \mathcal{L}(L^2(\Gamma))\) with \(|\vartheta |_{\mathcal{L}(H^2(\Gamma))} \leq C |\vartheta |_{c_{2}(\Gamma)}\), for all \(s \in \{0, 2\}\). Then by an interpolation argument, we can also show that the inequality holds for all \(s \in \{0, 2\}\), and from \(\langle \vartheta f , g \rangle_{H^{\sigma}(\Gamma)} : = \langle f, \vartheta g \rangle_{H^{-\sigma}(\Gamma)}\), we can conclude that \(|\vartheta |_{\mathcal{L}(H^{\sigma}(\Gamma))} \leq C |\vartheta |_{c_{2}(\Gamma)}\) for all \(s \in [-2, 2]\).

Remark 2.4 Note that the functions \(\Psi_i = \mathbb{E}_{I_i}^{0}\psi_i\) as above are not in \(H^\frac{\alpha}{2}(\Gamma)\) in general, though we have \(\psi_i \in H^2(\Gamma_{\delta}) \cap H^1(\Gamma_{\delta})\) and \(\Psi_i \in H^s(\Gamma)\), for any \(s < \frac{3}{2}\).

2.6 Feedback stabilizing rule

We assume, in this section, that the actuators \(\vartheta \Psi_i \mid i \in \{1, 2, \ldots, M\}\) allow us to stabilize the system. That is we assume that there exists a family of controls \(u = u(z_0) \in H^1(\mathbb{R}_{x_0}, \mathbb{R}^M)\), so that the system
\[
\frac{\partial}{\partial t} z - v \Delta z - \frac{\lambda}{2} z + \tilde{L} z = 0, \quad z|_{\Gamma} = B_{\Psi} u(z_0), \quad z(s_0) = z_0
\]
is exponentially stable, with \(|z|^2_{L^2(\mathbb{R}_{x_0}, H^1(\Omega))} + |u|^2_{H^1(\mathbb{R}_{x_0}, \mathbb{R}^M)} \leq C_2 |z_0|^2_H\). We can rewrite (46) in the variables \((v, \kappa) = (z - B_{\Psi} u, u)\), in the form of the extended system (14). Then as explained in the Introduction, we can follow a standard procedure.
to find a stabilizing feedback control operator \( \mathcal{F}_0 = \mathcal{F}_0(t) \), in the form \( \mathcal{F}_0(t) \begin{bmatrix} v \\ \kappa \end{bmatrix} = - \begin{bmatrix} -B\tilde{\psi} \\ 1 \end{bmatrix} \kappa = - \begin{bmatrix} -B\tilde{\psi} \\ 1 \end{bmatrix} \begin{bmatrix} -B\tilde{\psi} \\ 1 \end{bmatrix} \Pi \begin{bmatrix} v \\ \kappa \end{bmatrix} \), where \( \Pi = \Pi(s) \) can be taken as the solution of a differential Riccati equation as (16). That is, the solution of

\[
\frac{\partial}{\partial t} \begin{bmatrix} v \\ \kappa \end{bmatrix} + \begin{bmatrix} -\nu \Delta + \hat{L} \end{bmatrix} \begin{bmatrix} v \\ \kappa \end{bmatrix} + \mathcal{F}_0(t) \begin{bmatrix} v \\ \kappa \end{bmatrix} = 0, \quad \begin{bmatrix} v(s_0) \\ \kappa(s_0) \end{bmatrix} = \begin{bmatrix} v_0 \\ \kappa_0 \end{bmatrix} \tag{47}
\]

satisfies, with \( C \) independent of \( (v_0, \kappa_0) \), the estimate

\[
\| (v, \kappa) (t) \|_{H_{loc}}^2 \leq C e^{-\lambda t} \| (v_0, \kappa_0) \|_{H_{loc}}^2, \quad \text{for all } t \geq 0. \tag{48}
\]

Recall the notations \( H = H \times \mathbb{R}^M, \quad V = V \times \mathbb{R}^M, \quad D(\Delta) = D(\Delta) \times \mathbb{R}^M \).

**Theorem 2.5** If \( (v_0, \kappa_0) \in V \), then the solution of (47) satisfies

\[
\sup_{t \geq 0} \| e^{\hat{\xi} t} (v, \kappa) (t) \|_{V}^2 \leq C \| (v_0, \kappa_0) \|_{V}^2, \quad \text{for all } t \geq 0,
\]

with \( C \) independent of \( (v_0, \kappa_0) \). The solution \( (v, \kappa) \) is, and is unique, in the space \( L^2_{loc}(H_0, D(\Delta)) \cap C([0, +\infty), V) \).

The proof is omitted. It follows from (48) and from the smoothing property for parabolic equations (cf. (2.6)). For further details, we refer to [42]. Here we just note that we can rewrite the system (47) with a general external forcing \( h \) in the place of the controller as

\[
\frac{\partial}{\partial t} \begin{bmatrix} v \\ \kappa \end{bmatrix} + A \begin{bmatrix} v \\ \kappa \end{bmatrix} + R \begin{bmatrix} v \\ \kappa \end{bmatrix} + C \begin{bmatrix} \bar{v} \\ \bar{\kappa} \end{bmatrix} + h = 0, \quad \begin{bmatrix} v_0 \\ \kappa_0 \end{bmatrix} = \begin{bmatrix} v_0 \\ \kappa_0 \end{bmatrix} \tag{49}
\]

with \( \begin{bmatrix} v \\ \kappa \end{bmatrix}, \quad \begin{bmatrix} v_0 \\ \kappa_0 \end{bmatrix} \), the diffusion term \( A \begin{bmatrix} v \\ \kappa \end{bmatrix} = \begin{bmatrix} -\nu \Delta v \\ 0 \end{bmatrix} \), the reaction term \( R \begin{bmatrix} v \\ \kappa \end{bmatrix} := \begin{bmatrix} \hat{a}(v + B\tilde{\psi}, \kappa) - 2\xi B\tilde{\psi} \kappa \\ 0 \end{bmatrix} \), and the convection term \( C \begin{bmatrix} \bar{v} \\ \bar{\kappa} \end{bmatrix} := \begin{bmatrix} \nabla \cdot (\bar{\hat{b}}(v + B\tilde{\psi}, \kappa)) \end{bmatrix} \). In this case, from (25) and Lemma 2.1, we can obtain the analogous estimates

\[
\langle Rz, \tilde{z} \rangle_{V^*, V} \leq C \| \hat{a} \|_{L^d} \| z \|_H^2 \| \tilde{z} \|_V^2, \quad \text{for } d \in \{1, 2\},
\]

\[
\langle Rz, \tilde{z} \rangle_{V^*, V} \leq C \| \hat{a} \|_{L^d} \| z \|_H \| \tilde{z} \|_V, \quad \text{for } d \geq 3.
\]

\[
\langle Cz, \tilde{z} \rangle_{V^*, V} \leq C \| \hat{b} \|_{L^d} \| z \|_H \| \tilde{z} \|_V, \quad \text{for } d \geq 1.
\]
For all \((z, \tilde{z}) \in V \times V\), and

\[
|Cz|_H \leq C|(|\nabla \cdot \hat{b}|)_L^2 |z|_V^\frac{1}{2} |\hat{b}|_L^\infty |z|_V, \quad \text{for } d = 1.
\]

\[
|Cz|_H \leq C|(|\nabla \cdot \hat{b}|)_L^2 |z|_V^\frac{1}{2} |\hat{b}|_L^\infty |z|_V, \quad \text{for } d = 2.
\]

\[
|Cz|_H \leq C|(|\nabla \cdot \hat{b}|)_L^2 |z|_V^\frac{1}{2} |\hat{b}|_L^\infty |z|_V, \quad \text{for } d = 3.
\]

\[
|Cz|_H \leq C|\nabla \cdot b|_L^\infty |z|_H + |b|_L^\infty |z|_V, \quad \text{for } d \geq 1,
\]

for all \((z, \tilde{z}) \in D(\Delta) \times D(\Delta)\) where \(C > 0\) is a positive constant. In particular, notice that \(D(\Delta) \hookrightarrow V \hookrightarrow H\). The estimates above allow us to derive the analogous regularity properties for system (49) as in Lemmas 2.2 and 2.4.

### 3 The nonlinear systems

To derive the local stabilization result for the nonlinear system, we consider (47) with \(N\) as a perturbation:

\[
\frac{\partial}{\partial t} \begin{bmatrix} v \\ \kappa \end{bmatrix} + A_0 \begin{bmatrix} v \\ \kappa \end{bmatrix} + F_0 \begin{bmatrix} v \\ \kappa \end{bmatrix} = N \left( \begin{bmatrix} v \\ \kappa \end{bmatrix} \right), \quad \begin{bmatrix} v(0) \\ \kappa(0) \end{bmatrix} = \begin{bmatrix} v_0 \\ \kappa_0 \end{bmatrix},
\]

with

\[
A_0 := \begin{bmatrix} -\nu \Delta + \tilde{L} & \tilde{\Sigma} B \tilde{\psi} \\ 0 & \tilde{\Sigma} \end{bmatrix} \quad \text{and} \quad N \left( \begin{bmatrix} v \\ \kappa \end{bmatrix} \right) := \begin{bmatrix} \tilde{N}(v + B \tilde{\psi} \kappa) \\ 0 \end{bmatrix}.
\]

The procedure is analogous to the one in [13,30], however, since we are considering a general class of nonlinearities, we will recall the main steps.

Let us define \(z := \begin{bmatrix} v \\ \kappa \end{bmatrix}\). Note that we can identify \(v + B \tilde{\psi} \kappa \in V \oplus B \tilde{\psi} \mathbb{R}^M\), with its components \((v, \kappa)\).

It follows that \(N\) satisfy, for a suitable \(\tilde{C}_1 > 0\), estimates (8) in the form

\[
|N(z) - N(\tilde{z})|_H^2 \leq \tilde{C}_1|z - \tilde{z}|_V^2 \left( 1 + |z|_V^{\epsilon_1} + |\tilde{z}|_V^{\epsilon_2} \right) \left( |z|_{D(\Delta)}^2 + |\tilde{z}|_{D(\Delta)}^2 \right)
\]

\[
+ \tilde{C}_1|z - \tilde{z}|_{D(\Delta)}^2 \left( |z|_V^{\epsilon_3} + |\tilde{z}|_V^{\epsilon_4} \right),
\]

\begin{equation}
(51a)
\end{equation}

and

\[
(N(z) - N(\tilde{z}), z - \tilde{z})_H
\]

\[
\leq \tilde{C}_1(1 + |z|_V^{\epsilon_5} + |\tilde{z}|_V^{\epsilon_6}) \left( 1 + |z|_{D(\Delta)}^2 + |\tilde{z}|_{D(\Delta)}^2 \right)^{\frac{1}{2}} |z - \tilde{z}|_V |z - \tilde{z}|_H
\]

\[
+ \tilde{C}_1(1 + |z|_V^{\epsilon_3} + |\tilde{z}|_V^{\epsilon_4})(1 + |z|_{D(\Delta)}^2 + |\tilde{z}|_{D(\Delta)}^2) |z - \tilde{z}|_H^2
\]

\begin{equation}
(51b)
\end{equation}

with \(\{\epsilon_1, \epsilon_2\} \in [0, +\infty)\) and \(\{\epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6\} \in [2, +\infty)\).
With (51), we will be able to follow the argument used in the internal case in [13,30] to prove the following result, saying that the feedback control stabilizes the nonlinear system to zero locally. Notice that Theorem 2.5 says that the feedback controller stabilizes the linearized system to zero globally.

**Theorem 3.1** We assume that (47) is stable, with \( F_0 \in L(H) \), and that the estimates in Theorem 2.5 hold true. Then, there exists \( \epsilon > 0 \) with the following property: if we have that 
\[
|v_0, \kappa_0|_V \leq \epsilon,
\]
then there is a solution for system (50), in \( \mathbb{R}_0 \times \Omega \), which is in \( L^2_{\text{loc}}(\mathbb{R}_0, D(\Delta)) \cap C([0, +\infty), V) \), is unique, and satisfies
\[
|v, \kappa(t)|^2_V \leq C e^{-\lambda t} |v_0, \kappa_0|^2_V, \quad \text{for all} \ t \geq 0,
\]
for a suitable constant \( C \) independent of \( (\epsilon, (v_0, \kappa_0)) \).

**Proof** We sketch/recall the main steps. We define the Banach space
\[
Z^\lambda := \left\{ z \in L^2_{\text{loc}}(\mathbb{R}_0, H) \mid |z|_{Z^\lambda} < \infty \right\}
\]
endowed with the norm \( |z|_{Z^\lambda} := \sup_{r \geq 0} \left| e^{\lambda r} z \right|_{W((r, r+1), D(\Delta), H)} \). We also set
\[
Z^\lambda_{\text{loc}} := \left\{ z \in L^2_{\text{loc}}(\mathbb{R}_0, H) \mid \left| e^{\lambda r} z \right|_{W((r, r+1), D(\Delta), H)} < \infty, \ \text{for all} \ r \geq 0 \right\}.
\]

For a given constant \( \varrho > 0 \), we define the subset
\[
Z^\lambda_{\varrho} := \left\{ z \in Z^\lambda \mid |z|_{Z^\lambda}^2 \leq \varrho |z_0|^2_V \right\},
\]
with \( z_0 = \begin{bmatrix} v_0 \\ \kappa_0 \end{bmatrix} \in V \), and the mapping \( \Psi : Z^\lambda_{\varrho} \to Z^\lambda_{\text{loc}}, \tilde{z} \mapsto z \), taking a given vector \( \tilde{z} \) to the solution \( z = \begin{bmatrix} v \\ \kappa \end{bmatrix} \) of
\[
\frac{\partial}{\partial t} z + A_0 z + F_0 z = \mathcal{N}(\tilde{z}), \quad z(0) = z_0.
\]  

\( \S \) Step 1: a preliminary estimate. Proceeding as in [13], we can conclude that the solution of the system (53) with a general \( g \in L^2_{\text{loc}}(\mathbb{R}_0, H) \) in the place of \( \mathcal{N}(\tilde{z}) \) satisfies
\[
\sup_{r \geq 0} \left| e^{\lambda \tau} z(\cdot) \right|_{W((r, r+1), D(\Delta), H)}^2 \leq \overline{C} \left( |z_0|^2_V + \sup_{k \in \mathbb{N}} \int_{k}^{k+1} e^{2\lambda s} |g(s)|^2_H \, ds \right),
\]
with \( \overline{C} = \overline{C}_s \left[ C_{W(M, \lambda, \frac{1}{2}, \frac{r\kappa}{\delta}, \delta, \gamma) \right] \right)$. 

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Step 3: $\Psi$ maps $Z_0^\lambda$ into itself, if $|z_0|_V$ is small. Now we will replace $g$ by $\mathcal{N}(\tilde{z})$ in (54). From (51a), with $(z, \tilde{z}) = (\tilde{z}, 0)$, and from $e^{\frac{1}{2}s}\epsilon \geq 1$ since $\frac{1}{2}s\epsilon \geq 0$, we find that

$$\sup_{k \in \mathbb{N}} \int_k^{k+1} e^{2\lambda s}|\mathcal{N}(\tilde{z})(s)|^2_H ds \leq \sup_{k \in \mathbb{N}} \tilde{C}_k \int_k^{k+1} |e^{\frac{1}{2}r}\tilde{z}(r)|^2_{D(\Delta)} dr$$

with $\tilde{C}_k := \sup_{s \in [k, k+1]} \tilde{C}_1 \left( |e^{\frac{1}{2}s}\tilde{z}(s)|^2_V + |e^{\frac{1}{2}s}\tilde{z}(s)|^{2+\epsilon_1}_V + |e^{\frac{1}{2}s}\tilde{z}(s)|^{\epsilon_3}_V \right)$. Therefore

$$\sup_{k \in \mathbb{N}} \int_k^{k+1} e^{2\lambda s}|\mathcal{N}(\tilde{z})(s)|^2_H ds \leq \tilde{C}_1 \left( |\tilde{z}|^4_{Z_0^\lambda} + |\tilde{z}|^{4+\epsilon_1}_{Z_0^\lambda} + |\tilde{z}|^{\epsilon_3+2}_{Z_0^\lambda} \right),$$

because $W((k, k+1), D(\Delta), H) \rightarrow C([k, k+1], V)$ uniformly with respect to $k \in \mathbb{N}$. Thus, inequality (54) with $g = \mathcal{N}(\tilde{z})$ and $\tilde{z} \in Z_0^\lambda$ gives us

$$|\Psi(\tilde{z})|^2_{Z_0^\lambda} \leq \tilde{C} \left( |z_0|^2_V + \tilde{C}_1 \left( |\tilde{z}|^4_{Z_0^\lambda} + |\tilde{z}|^{4+\epsilon_1}_{Z_0^\lambda} + |\tilde{z}|^{\epsilon_3+2}_{Z_0^\lambda} \right) \right)$$

$$\leq C_2 \left( 1 + \rho^2 |z_0|^2_V + \rho^{4+\epsilon_1} |z_0|^{2+\epsilon_1}_V + \rho^{\epsilon_3+2} |z_0|^{\epsilon_3}_V \right) |z_0|^2_V$$

and if we set $\rho = 4C_2$ and $\epsilon < \min \left\{ \rho^{-1}, \rho^{-\frac{4+\epsilon_1}{2+\epsilon_1}}, \rho^{-\frac{\epsilon_3+2}{2\epsilon_3}} \right\}$, then we obtain

$$|\Psi(\tilde{z})|^2_{Z_0^\lambda} \leq C_2 \left( 1 + \rho^2 \epsilon^2 + \rho^{4+\epsilon_1} e^{-2+\epsilon_1} + \rho^{\epsilon_3+2} e^{-\epsilon_3} \right) |z_0|^2_V$$

$$\leq 4C_2 |z_0|^2_V = \rho |z_0|^2_V,$$

(55) if $|z_0|_V \leq \epsilon$, which means that $\Psi(\tilde{z}) \in Z_0^\lambda$.

Step 3: $\Psi$ is a contraction, if $|z_0|_V$ is smaller. Let us take two functions $\tilde{z}_1, \tilde{z}_2 \in Z_0^\lambda$ and let $\Psi(\tilde{z}_1)$ and $\Psi(\tilde{z}_2)$ be the corresponding solutions for (53). Set $e := \tilde{z}_1 - \tilde{z}_2$ and $d^\Psi := \Psi(\tilde{z}_1) - \Psi(\tilde{z}_2)$. Then $d^\Psi$ solves (53) with $d^\Psi(0) = 0$ and $g = \mathcal{N}(\tilde{z}_1) - \mathcal{N}(\tilde{z}_2)$ in the place of $\mathcal{N}(\tilde{z})$. Therefore, by (54), we have

$$|d^\Psi|^2_{Z_0^\lambda} \leq \tilde{C} \sup_{t \geq 0} \int_t^{t+1} e^{2\lambda s}|\mathcal{N}(\tilde{z}_1) - \mathcal{N}(\tilde{z}_2)|^2_H ds,$$

and from $e^{2\lambda s}|\mathcal{N}(\tilde{z}_1) - \mathcal{N}(\tilde{z}_2)|^2_H \leq |e^{\frac{1}{2}s}e(s)|^2_V \Xi_1 + |e^{\frac{1}{2}s}e(s)|^2_{D(\Delta)} \Xi_2$, we have

$$\Xi_1 = \left( 1 + |e^{\frac{1}{2}s}\tilde{z}_1(s)|^{\epsilon_1}_V + |e^{\frac{1}{2}s}\tilde{z}_2(s)|^{\epsilon_3}_V \right) \left( |e^{\frac{1}{2}s}\tilde{z}_1(s)|^2_{D(\Delta)} + |e^{\frac{1}{2}s}\tilde{z}_2(s)|^2_{D(\Delta)} \right),$$

$$\Xi_2 = \left( |e^{\frac{1}{2}s}\tilde{z}_1(s)|^{\epsilon_3}_V + |e^{\frac{1}{2}s}\tilde{z}_2(s)|^{\epsilon_4}_V \right),$$

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it follows that
\[ |d^\Psi|^2_{Z^\lambda} \leq C_3 |e|^2_{Z^\lambda} \left( 1 + |\bar{z}_1|^2_{Z^\lambda} + |\bar{z}_2|^2_{Z^\lambda} \right) \left( |\bar{z}_1|^2_{Z^\lambda} + |\bar{z}_2|^2_{Z^\lambda} + |\bar{z}_1|^2_{Z^\lambda} + |\bar{z}_2|^2_{Z^\lambda} \right), \]
and since \( \bar{z}_1 \) and \( \bar{z}_2 \) are both in \( Z^\lambda \), we arrive to
\[ |d^\Psi|^2_{Z^\lambda} \leq C_3 |e|^2_{Z^\lambda} \left( 1 + \frac{\varepsilon_1}{\varepsilon_1 + 3} |z_0|^{\varepsilon_1} + \frac{\varepsilon_2}{\varepsilon_2 + 3} |z_0|^{\varepsilon_2} \right) \left( 2\varepsilon \varepsilon_1 + \frac{\varepsilon_1}{\varepsilon_1 + 3} |z_0|^{\varepsilon_1} + \frac{\varepsilon_2}{\varepsilon_2 + 3} |z_0|^{\varepsilon_2} \right). \]
Choosing \( \varepsilon > 0 \), smaller than the one in Step 2, such that
\[ \varepsilon < \min \left\{ \varepsilon_1^{-1}, \varepsilon_2^{-1}, \varepsilon_3^{-1}, \left( \frac{\varepsilon_1}{\varepsilon_1 + 3} \right)^{\frac{1}{2}}, \left( \frac{\varepsilon_2}{\varepsilon_2 + 3} \right)^{\frac{1}{2}}, \left( \frac{\varepsilon_3}{\varepsilon_3 + 3} \right)^{\frac{1}{2}}, \left( \frac{\varepsilon_4}{\varepsilon_4 + 3} \right)^{\frac{1}{2}} \right\}, \]
than we have that \( \Psi \) maps \( Z^\lambda \) into itself and
\[ |d^\Psi|^2_{Z^\lambda} \leq C_3 |e|^2_{Z^\lambda} \left( 1 + \frac{\varepsilon_1}{\varepsilon_1 + 3} \varepsilon_1 + \frac{\varepsilon_2}{\varepsilon_2 + 3} \varepsilon_2 \right) \left( 2\varepsilon \varepsilon_1 + \frac{\varepsilon_1}{\varepsilon_1 + 3} \varepsilon_1 + \frac{\varepsilon_2}{\varepsilon_2 + 3} \varepsilon_2 \right), \]
provided \( |z_0|^2_{V} \leq \varepsilon \). That is \( |\Psi(\bar{z}_1) - \Psi(\bar{z}_2)|^2_{Z^\lambda} < \varepsilon^2 |\bar{z}_1 - \bar{z}_2|^2_{Z^\lambda} \). Furthermore, we can see that \( \varepsilon \) can be taken independent of \( (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_6) \in \mathbb{R}^6_0 \times \mathbb{R}^6_3 \), because the function \( t \mapsto \frac{c_1 t + c_2}{c_1 t + c_4}, t > 0 \) and \( (c_1, c_2, c_3, c_4) \in \mathbb{R}^4 \), is monotone if \( c_3 t + c_4 \neq 0 \) for all \( t > 0 \). Indeed, we can take \( \varepsilon = \min \left\{ \varepsilon_1^{-1}, \varepsilon_2^{-1}, \left( \frac{\varepsilon_1}{\varepsilon_1 + 3} \right)^{\frac{1}{2}}, \left( \frac{\varepsilon_2}{\varepsilon_2 + 3} \right)^{\frac{1}{2}}, \left( \frac{\varepsilon_3}{\varepsilon_3 + 3} \right)^{\frac{1}{2}}, \left( \frac{\varepsilon_4}{\varepsilon_4 + 3} \right)^{\frac{1}{2}} \right\} \). That is, \( \varepsilon \) depends essentially on the constant \( \bar{C} \) in (54) and on the constant \( \hat{C}_1 \) in (51). From [13, Section 4.2], we can see that the constant \( \bar{C} \) in (54) will essentially depend on the constant \( C \) in Theorem 2.5.

\( \vartriangle \) Step 4: Fixed point argument. We can conclude that if \( z_0 \in V \) is sufficiently small, \( |z_0|^2_{V} < \varepsilon \), then there exists a unique fixed point \( z = \Psi(\bar{z}) = \bar{z} \in Z^\varepsilon_0 \) for \( \Psi \). It follows from the definitions of \( \Psi \) and \( Z^\varepsilon_0 \) that \( z \) solves the system (53), with \( \bar{z} = z \). We can conclude that \( z \) solves (50). Further, inequality (52) can be concluded from (55).

\( \vartriangle \) Step 5: Uniqueness. Finally, we show the uniqueness of the solution for (50) in the space \( Z := L^2_{\text{loc}}(\mathbb{R}, D(\Delta)) \cap C([0, +\infty), V) \supseteq Z^\varepsilon_0 \). Let \( z_1 \) and \( z_2 \) be two solutions, in \( Z \), for (50). It turns out that \( e = z_1 - z_2 \) solves (53) with \( g = \mathcal{N}(z_1) - \mathcal{N}(z_2) \) in the place of \( \mathcal{N}(\bar{z}) \). Using (51b), and following standard arguments, we can obtain that
\[ \frac{d}{dt} |e|^2_H \leq C_4 \left( 1 + |z_1|^2_V + |z_2|^2_V \right) \left( 1 + |z_1|^2_{D(\Delta)} + |z_2|^2_{D(\Delta)} \right) |e|^2_H. \]
Notice that the function
\[ s \mapsto G(s) := C_4 \left( 1 + |z_1(s)|^2_V + |z_2(s)|^2_V \right) \left( 1 + |z_1(s)|^2_{D(\Delta)} + |z_2(s)|^2_{D(\Delta)} \right) \]
is locally integrable, which allows us to write

\[ |e(t)|_H^2 \leq e^{\int_0^t G(s) ds} |e(0)|_H^2 = 0, \quad \text{for all } t \geq 0. \]

That is, the uniqueness holds true: \( z_1 - z_2 = e = 0 \).

\[ \square \]

4 Proofs of Main Theorems 1.1 and 1.2

Let us be given a solution \( \hat{y} \) for the uncontrolled system (2), with \( u = 0 \). We suppose that \( (\hat{y}, f) \in \mathcal{C} \), with \( \mathcal{C} \) defined as in (9). That is, we suppose that \( \hat{N} \) and \( (\hat{a}, \hat{b}) \) defined in (6) satisfy (8) and (24), respectively, for suitable nonnegative constants \( \hat{C} \) and \( C_{W_\alpha} \).

By taking the Riccati feedback \( F_0(t) \begin{bmatrix} v \\ \kappa \end{bmatrix} = \begin{bmatrix} -B\hat{\psi} & -B\hat{\psi}^* \\ 1 \end{bmatrix} \begin{bmatrix} v \\ \kappa \end{bmatrix} \), a si n Sect. 2.6, we see that the Main Theorem 1.2, in the Introduction, is a corollary of Theorems 3.1, 2.5 and 2.4 (with \( s_0 = 0 \)).

Notice that the condition \((v_0, \kappa_0) \in V \times \mathbb{R}^M \) in Theorem 3.1 leads us to the compatibility condition \( 0 = v(0)|_F = (y(0) - \hat{y}(0) - B\psi\kappa(0))|_F \), that is, \( (y(0) - \hat{y}(0))|_F \in \mathcal{S}_\psi \).

The case of internal controls. The analogous of Theorems 2.4 and 2.5, for the case of internal controls can be found in \([17, \text{Theorem 2.10 and Corollary 2.15}]\), where the stabilizability condition reads (see also \([29]\)),

\[ |\vartheta (1 - P_M)\vartheta|^2_{L^2(H,V')} < Y^{-1}. \]  

Observe that the computations in Sect. 3 can be followed by just using (8) instead of (51) and replacing the triple \((H, V, D(\Delta))\) by \((H, V, D(\Delta))\). Therefore, we arrive at the analogous of Theorem 2.4 for the internal case. As a corollary, we obtain the Main Theorem 1.1 as in the Introduction.

5 Examples of covered nonlinearities

Many systems modeling real evolutions involve polynomial nonlinearities, for example Fisher-like equations \([22,49]\) modeling population dynamics, Burgers-like equations \([18,30]\) modeling fluid (e.g., traffic) flow, and the Schlögl equations \([43]\) modeling certain chemical reactions. Here, we check the property

\( (\hat{y}, f) \in \mathcal{C} \)

we ask/assume for the pair \((\hat{y}, f)\). See (9). That is, we investigate whether both (7) and (8) hold true, in case the function \( f(y, \nabla y) \) takes (or can be written in) the form

\[ f(y, \nabla y) = f_r(y) + f_c(y) \cdot \nabla y, \]
where \( f_r \) and \( f_c = [f_{c1} \ f_{c2} \ \ldots \ f_{cd}]^\top \), are polynomials:

\[
f_r(y) = \sum_{j=0}^{\hat{p}} \bar{r}_j y^j \quad \text{and} \quad f_{ck}(y) = \sum_{j=0}^{p_k} r_{kj} y^j,
\]

with \( \bar{r}_j \) and \( r_{kj} \) real numbers, and \( p_k \in \mathbb{N} \) for \( k \in \{1, \ldots, d\} \).

It is enough to analyze the case of monomials, with degree greater than or equal to 2:

\[
f(y) = y^n \quad \text{with} \quad n \geq 2
\]

and

\[
f(y, \nabla y) = y^n \partial_{x_k} y = \frac{1}{n+1} \partial_{x_k} y^{n+1}, \quad \text{with} \quad n \geq 1 \quad \text{for some} \quad \bar{k} \in \{1, \ldots, d\}.
\]

In this case, recalling the notation in Sect. 1.1, for a given trajectory \( \hat{y} \), we obtain, respectively,

\[
\hat{a} = \bar{n} \hat{y}^{\bar{n}-1}, \quad \hat{b} = 0,
\]

and

\[
\hat{a} = n \hat{y}^{n-1} \partial_{x_k} \hat{y} - \nabla \cdot \hat{b} = 0, \quad \hat{b} = [\hat{b}_1 \ \hat{b}_2 \ \ldots \ \hat{b}_d]^\top, \quad \text{with} \quad \hat{b}_k = \begin{cases} 0 & \text{if} \quad k \neq \bar{k}, \\ \hat{y}^n & \text{if} \quad k = \bar{k}. \end{cases}
\]

For illustration, we consider here the case \( d = 3 \). The following estimates will be also valid for \( d \in \{1, 2\} \), though in those cases better estimates may hold true. On the other hand, some of the following arguments will not work in dimension \( d \geq 4 \), in that case some changes are needed.

5.1 Checking the conditions on the pair \((\hat{a}, \hat{b})\). Case \( d = 3 \)

Observe that in the case of a reaction nonlinearity \( f(y) = y^n \), we find that condition (7) is satisfied provided \( \hat{y} \in L^\infty(\mathbb{R}_0, L^3(\bar{n}-1)) \). In the case of a convection nonlinearity \( f(y, \nabla y) = y^n \partial_{x_k} y \), we find that conditions (7) is satisfied provided \( \hat{y} \in L_w^\infty(\mathbb{R}_0, L^\infty) \) and \( \partial_{x_k} \hat{y} \in L^\infty(\mathbb{R}_0, L^2) \).

5.2 Checking the conditions on the nonlinearity \( \mathcal{N} \). Case \( d = 3 \)

For simplicity, we restrict ourselves to the case \( \hat{y} \in L_w^\infty(\mathbb{R}_0, L^\infty) \).

Example 1 In case \( \mathcal{N}(z) = \hat{y}^m z^2 \), \( m \in \mathbb{N} \), conditions (8) hold true. We may write

\[
|\mathcal{N}(z) - \mathcal{N}(\bar{z})|_H^2 = |\hat{y}^m (z - \bar{z})(z + \bar{z})|_H^2 \leq |z - \bar{z}|_H^2 |\hat{y}^m|_{L^\infty}^2 |z + \bar{z}|_{L^\infty}^2,
\]

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and
\[ |N(z) - N(\bar{z})|_H^2 \leq C |\hat{y}^m|_{L^\infty}^2 |z - \bar{z}|_V^2 \left( |z|_{D(\Delta)}^2 + |\bar{z}|_{D(\Delta)}^2 \right), \]

which shows that (8a) holds true. Furthermore, we find
\[
(N(z) - N(\bar{z}), z - \bar{z})_H \leq |\hat{y}^m(z - \bar{z})^2(z + \bar{z})|_{L^1} \leq |\hat{y}^m|_{L^\infty} |z + \bar{z}|_{L^\infty} |z - \bar{z}|_{L^2}^2
\]
\[
\leq C_1(|z|_{L^\infty} + |\bar{z}|_{L^\infty})|z - \bar{z}|_{L^2}^2 \leq 2^{1/2} C_1(|z|_{L^2}^2 + |\bar{z}|_{L^2}^2)^{1/2} |z - \bar{z}|_{L^2}^2
\]
\[
\leq C_2(|z|_V |z|_{D(\Delta)} + |\bar{z}|_V |\bar{z}|_{D(\Delta)})^{1/2} |z - \bar{z}|_{L^2}^2
\]
\[
\leq C_3(|z|_V^2 + |\bar{z}|_V^2 + |z|_{D(\Delta)}^2 + |\bar{z}|_{D(\Delta)}^2)^{1/2} |z - \bar{z}|_{L^2}^2
\]
\[
\leq C_4 (1 + |z|_V^2 + |\bar{z}|_V^2)^{1/2} (1 + |z|_{D(\Delta)}^2 + |\bar{z}|_{D(\Delta)}^2)^{1/2} |z - \bar{z}|_H |z - \bar{z}|_V
\]

which shows that (8b) holds true.

**Example 2** In case $N(z) = \hat{y}^m z^n$, $m \in \mathbb{N}$ and $n = \{3, 4, 5\}$, (8) holds true. We may write, for suitable nonzero constants $r_j$,
\[
N(z) - N(\bar{z}) = \hat{y}^m (z - \bar{z}) \sum_{j=0}^{n-1} r_j z^j \bar{z}^{n-1-j},
\]
where in the sum we have monomials of degree $n - 1$. For example for $z^1 \bar{z}^{n-2}$, by standard (yet appropriate) Young, Hölder, Sobolev, and Agmon inequalities, we may write
\[
\left| (z - \bar{z}) z^1 \bar{z}^{n-2} \right|_H^2 = \left| (z - \bar{z})^2 z^2 \bar{z}^{2n-4} \right|_{L^1} \leq |z - \bar{z}|_{L^\infty}^2 |z|_{L^\infty} |\bar{z}|_{L^\infty} |z^{2n-5}|_{L^1}
\]
\[
\leq C_1 |z - \bar{z}|_V |z - \bar{z}|_{D(\Delta)} |\bar{z}|_{V}^{1/2} |z|_{D(\Delta)}^{1/2} |\bar{z}|_{D(\Delta)}^{1/2} |z|_{L^6}^{2n-5} |\bar{z}|_{V}^{2n-5} L_{6(2n-5)}^{-5},
\]
and, since $H^1(\Omega) \hookrightarrow L^6(\Omega) \hookrightarrow L^{6(2n-5)}(\Omega),$
\[
\left| (z - \bar{z}) z^1 \bar{z}^{n-2} \right|_H^2 \leq \frac{C_1}{\bar{V}} |z - \bar{z}|_V |z|_{D(\Delta)} |\bar{z}|_{D(\Delta)} + C_2 |z - \bar{z}|_{D(\Delta)}^2 |z|_V^{3/2} |\bar{z}|_{V}^{1/2} |z|_{D(\Delta)}^{1/2} |\bar{z}|_{D(\Delta)}^{1/2} |z|_{L^6}^{4n-6} |\bar{z}|_{V}^{4n-6}
\]
\[
\leq \frac{C_1}{\bar{V}} |z - \bar{z}|_V^2 \left( |z|_{D(\Delta)}^2 + |\bar{z}|_{D(\Delta)}^2 \right) + C_3 |z - \bar{z}|_{D(\Delta)}^2 \left( |z|_V^{4n-6} + |\bar{z}|_V^{4n-6} \right).
\]

Furthermore, we find
\[
((z - \bar{z}) z^1 \bar{z}^{n-2}, z - \bar{z})_H \leq |z^1 \bar{z}^{n-2}|_{L^\infty} |z - \bar{z}|_{L^2}^2 \leq C_4 (|z|_{L^\infty}^{n-1} + |\bar{z}|_{L^\infty}^{n-1}) |z - \bar{z}|_{L^2}^2
\]
\[
\leq C_5 (|z|_V^{n-1} |z|_{D(\Delta)}^{n-1} + |\bar{z}|_V^{n-1} |\bar{z}|_{D(\Delta)}^{n-1}) |z - \bar{z}|_{L^2}^2
\]
which implies the inequality

\[(z - \tilde{z}) z^{1} \tilde{z}^{n-2} \leq C_7 (|z|_V^{\delta_n} + |\tilde{z}|_V^{\delta_n}) (|z|_{D(\Delta)}^{2} + |\tilde{z}|_{D(\Delta)}^{2}) |z - \tilde{z}|_{L^2}^{2},\]

with \(\delta_n = 2\), for \(n = 5\), and \(\delta_n = \frac{2(n-1)}{5-n}\) for \(n \in \{3, 4\}\).

where we have used, for \(n \in \{3, 4\}\), the Young inequality

\[|z|_{V}^{\frac{n-1}{2}} |z|_{D(\Delta)}^{\frac{n}{4-n}} \leq C_6 \left( |z|_{V}^{\frac{n-1}{2}} + |z|_{D(\Delta)}^{\frac{n}{4-n}} \right).\]

For the other monomials, we can obtain analogous estimates, which give us

\[|\mathcal{N}(z) - \mathcal{N}(\tilde{z})|_{H}^{2} \leq C_8 \left( \tilde{y}^m |z|_{V}^{2} \left( |z|_{D(\Delta)}^{2} + |\tilde{z}|_{D(\Delta)}^{2} \right) + \tilde{y}^{m} |z|_{V}^{2} \left( |z|_{D(\Delta)}^{2} + |\tilde{z}|_{D(\Delta)}^{2} \right) \right),\]

\[(\mathcal{N}(z) - \mathcal{N}(\tilde{z}), z - \tilde{z})_{H} \leq C_9 \left( 1 + |z|_{V}^{6} + |\tilde{z}|_{V}^{6} \right) \left( |z|_{D(\Delta)}^{2} + |\tilde{z}|_{D(\Delta)}^{2} \right) |z - \tilde{z}|_{L^2}^{2},\]

which show that conditions (8) hold true.

**Example 3** In the case \(\mathcal{N}(z) = \tilde{y}^m z^6\), we were not able to derive (8a). Proceeding as above, for suitable nonzero constants \(r_j\),

\[\mathcal{N}(z) - \mathcal{N}(\tilde{z}) = \tilde{y}^m (z - \tilde{z}) \sum_{j=0}^{4} r_j z^j \tilde{z}^{5-j},\]

where in the sum we have now monomials of degree 5. If for example for \(z^1 \tilde{z}^4\), we proceed as above and write

\[|z - \tilde{z})z^1 \tilde{z}^4|_{L^1}^{2} = \left( |z - \tilde{z})z^{2} \tilde{z}^8|_{L^1} \leq |z - \tilde{z}|_{L^\infty}^{2} |z|_{L^\infty} |\tilde{z}|_{L^\infty} \frac{z \tilde{z}^7}{L^1} \leq C_1 |z - \tilde{z}|_{V} |z - \tilde{z}|_{D(\Delta)} |z|_{D(\Delta)}^{\frac{1}{2}} |\tilde{z}|_{D(\Delta)}^{\frac{1}{2}} |z \tilde{z}^7|_{L^1},\]

we cannot bound the term \(|z \tilde{z}^7|_{L^1}\) by the \(V\)-norms of \(z\) and \(\tilde{z}\) (for \(d = 3\)). Trying to use again the \(D(\Delta)\)-norms, we were not able to arrive to (8a) (the \(D(\Delta)\)-norms will appear with a power strictly greater than 2).

**Example 4** In the case \(\mathcal{N}(z) = \nabla \cdot (g(\tilde{y}) z^n)\), where \(n \in \{2, 3\}\) and \(g: \mathbb{R} \to \mathbb{R}^3\) is a smooth function, estimates (8) hold true provided \(g(\tilde{y}) \in W_{st}\). We will consider the cases \(n = 2\) and \(n = 3\) separately.

The case \(n = 3\). We write, for suitable nonzero constants \(r_j\),

\[\mathcal{N}(z) - \mathcal{N}(\tilde{z}) = \nabla \cdot \left( g(\tilde{y}) (z - \tilde{z}) \sum_{j=0}^{2} r_j z^j \tilde{z}^{2-j} \right)\]
where in the sum we have monomials of degree 2. For example for $z\ddot{z}$, we find

$$\left| \nabla \cdot (g(\dot{y})(z - \ddot{z})z\ddot{z}) \right|_H^2 \leq \left| (\nabla \cdot g(\dot{y})) (z - \ddot{z})^2 z\ddot{z}^2 \right|_{L^1} + \left| g(\dot{y}) \right|_{L^\infty} \left| (\nabla ((z - \ddot{z})z\ddot{z})) \right|_{L^1} \leq \left| (\nabla \cdot g(\dot{y})) \right|_{L^3} |z - \ddot{z}|_{L^6} |z^2\ddot{z}^2|_{L^\infty} + \left| g(\dot{y}) \right|_{L^\infty} \left| (\nabla ((z - \ddot{z})z\ddot{z})) \right|_{L^1} \leq C |z - \ddot{z}|_{V}^2 |z|_{L^\infty} |\ddot{z}|_{L^\infty} + C |z - \ddot{z}|_{V} |z - \ddot{z}|_{D(\Delta)} \left( |z|_{V}^2 |\ddot{z}|_{L^\infty} + |z|_{L^\infty} |\ddot{z}|_{V}^2 \right) \leq C_1 |z - \ddot{z}|_{V}^2 \left( |z|_{V}^2 + |\ddot{z}|_{V}^2 \right) \left( |z|_{D(\Delta)}^2 + |\ddot{z}|_{D(\Delta)}^2 \right) + C_1 |z - \ddot{z}|_{V} |z - \ddot{z}|_{D(\Delta)} \left( |z|_{V}^2 |\ddot{z}|_{V} |\ddot{z}|_{D(\Delta)} + |z|_{V} |z|_{D(\Delta)} |\ddot{z}|_{V}^2 \right) \leq C_2 |z - \ddot{z}|_{V}^2 \left( |z|_{V}^2 + |\ddot{z}|_{V}^2 + 1 \right) \left( |z|_{D(\Delta)}^2 + |\ddot{z}|_{D(\Delta)}^2 \right) + C_2 |z - \ddot{z}|_{D(\Delta)}^2 \left( |z|_{V}^6 + |\ddot{z}|_{V}^6 \right).$$

We can obtain analogous estimates for the other monomials, and obtain

$$|N(z) - N(\ddot{z})|_H \leq C_3 |z - \ddot{z}|_{V}^2 \left( |z|_{V}^2 + |\ddot{z}|_{V}^2 + 1 \right) \left( |z|_{D(\Delta)}^2 + |\ddot{z}|_{D(\Delta)}^2 \right) + C_3 |z - \ddot{z}|_{D(\Delta)}^2 \left( |z|_{V}^6 + |\ddot{z}|_{V}^6 \right)$$

which shows that (8a) holds true. Furthermore, we also obtain

$$(\nabla \cdot (g(\dot{y})(z - \ddot{z})z\ddot{z}), z - \ddot{z})_H = (g(\dot{y})z\ddot{z}(z - \ddot{z}), \nabla (z - \ddot{z}))_{L^2(\Omega, \mathbb{R}^d)} \leq |g(\dot{y})|_{L^\infty(\Omega, \mathbb{R}^d)} |z\ddot{z}|_{L^\infty(\Omega, \mathbb{R})} |z - \ddot{z}|_H |z - \ddot{z}|_V \leq C_4 (|z|_{V} |z|_{D(\Delta)} |\ddot{z}|_{V} |\ddot{z}|_{D(\Delta)}) \frac{1}{2} |z - \ddot{z}|_H |z - \ddot{z}|_V \leq C_5 (|z|_{V}^2 + |\ddot{z}|_{V}^2)^{\frac{1}{2}} (|z|_{D(\Delta)}^2 + |\ddot{z}|_{D(\Delta)}^2)^{\frac{1}{2}} |z - \ddot{z}|_H |z - \ddot{z}|_V$$

and

$$(N(z) - N(\ddot{z}), z - \ddot{z})_H \leq C_6 (|z|_{V}^2 + |\ddot{z}|_{V}^2)^{\frac{1}{2}} (|z|_{D(\Delta)}^2 + |\ddot{z}|_{D(\Delta)}^2)^{\frac{1}{2}} |z - \ddot{z}|_H |z - \ddot{z}|_V$$

which shows that (8b) holds true.

*The case n = 2.* We write, for suitable nonzero constants $r_j$,

$$N(z) - N(\ddot{z}) = \nabla \cdot \left( g(\dot{y})(z - \ddot{z})(r_0 z^1 + r_1 z^1) \right).$$
Again, we consider the monomials in the last sum separately, and find
\[ |\nabla \cdot (\hat{g}(\tilde{y})(z - \tilde{z})z)|^2_H \]
\[ \leq C |z - \tilde{z}|^2_V |\hat{z}|^2_{L^\infty} + C |z - \tilde{z}|_V |z - \hat{z}|_{D(\Delta)} |z|_V^2 \]
\[ \leq C_1 |z - \tilde{z}|_V |\hat{z}|_{D(\Delta)}^2 + C \frac{C}{2} (|z - \tilde{z}|_V^2 + |z - \hat{z}|_{D(\Delta)}^2) |z|_V^2 \]
\[ \leq C_2 |z - \tilde{z}|_V |\hat{z}|_{D(\Delta)}^2 + C \frac{C}{2} |z - \hat{z}|_{D(\Delta)}^2 |z|_V^2 . \]

We can obtain analogous estimates for the other monomial, and conclude that
\[ |\mathcal{N}(z) - \mathcal{N}(\tilde{z})|^2_H \leq C_3 |z - \tilde{z}|_V^2 (|\hat{z}|_{D(\Delta)}^2 + |\tilde{z}|_{D(\Delta)}^2) + C_3 |z - \hat{z}|_{D(\Delta)} (|\hat{z}|_V^2 + |\tilde{z}|_V^2) \]
which shows that (8a) holds true. Furthermore, we also obtain
\[
(\nabla \cdot (\hat{g}(\tilde{y})(z - \tilde{z})z), z - \tilde{z})_H = (\hat{g}(\tilde{y})z(z - \tilde{z}), \nabla (z - \tilde{z}))_{L^2(\Omega, \mathbb{R}^d)} \\
\leq C_4 |z|_{L^\infty} |z - \tilde{z}|_H |z - \tilde{z}|_V \leq C_5 |z|_{D(\Delta)} |z - \hat{z}|_H |z - \tilde{z}|_V
\]
and
\[
(\mathcal{N}(z) - \mathcal{N}(\tilde{z}), z - \tilde{z})_H \leq C_6 (|\hat{z}|_{D(\Delta)}^2 + |\tilde{z}|_{D(\Delta)}^2) |z - \tilde{z}|_H |z - \tilde{z}|_V
\]
which shows that (8b) holds true.

6 A numerical example

The simulations below have been done by considering a finite element approximation for the space variable based on the classical piecewise linear hat functions. For the time variable, we have used the Crank–Nicolson scheme. Since the manuscript is already long we skip the details on the discretization.

We consider the following nonlinear parabolic equation, for time \( t \in [0, T] \), with \( T = 8 \), in the unit ball \( \Omega = \mathbb{D} = \{ x = (x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 < 1 \} \).

\[
\frac{\partial}{\partial t} y - \nu \Delta y + c_3 y^3 + c_2 y^2 + c_1 y + \frac{1}{2} \nabla \cdot (y^2, y^2) + f_0 = 0,
\]
\[ y|_\Gamma = g, \quad y(0) = y_0, \tag{57} \]

where \( c_1, c_2, \) and \( c_3 \) are constants in \( \mathbb{R} \), and \( f_0 \) is a fixed appropriate function.

Notice that to make a given smooth function \( \hat{y}(t, x) \) a solution of (57), we have just to set the appropriate external forces \( f_0 \) and \( g \) as

\[
f_0 = f_0(\hat{y}) = -\left( \frac{\partial}{\partial t} \hat{y} - \nu \Delta \hat{y} + c_3 \hat{y}^3 + c_2 \hat{y}^2 + c_1 \hat{y} + \frac{1}{2} \nabla \cdot (\hat{y}^2, \hat{y}^2) \right),
\]
\[ g = g(\hat{y}) = \hat{y}|_\Gamma. \tag{58} \]
Hereafter, we will take the reference trajectory and parameters as
\[
\hat{y}(t) = (2x_1^3 + x_2^2) \sin(t),
\]
\[
v = 0.2, \quad (c_1, c_2, c_3) = (-2, -1, -3), \quad \text{and} \quad \lambda = 1,
\]
with the external forces as in (58).

Our internal actuators will be defined as follows. We define the rectangle
\[
\omega := (0, \frac{1}{2}) \times (0, \frac{1}{3}).
\]
Then, we take a regular partition of \( \omega \) into \( M = mn \) subrectangles
\[
\omega_{l_1,l_2} := \left( \frac{l_1-1}{2m}, \frac{l_1}{2m} \right) \times \left( \frac{l_2-1}{3n}, \frac{l_2}{3n} \right), \quad (l_1, l_2) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}.
\]
We take the \( M \) actuators \( 1_{\omega_{l_1,l_2}} \). Thus, at any given time instant, in each subrectangle \( \omega_{l_1,l_2} \) the control is constant. As an illustration, we plot a linear combination of 4 piecewise-constant actuators in Fig. 1a, corresponding to the arrangement \( (m, n) = (2, 2) \).

For the boundary control case, our boundary, once parameterized by arc length, is \( \Gamma = [0, 2\pi) \). We use \( M \) boundary actuators whose form is
\[
\Psi_i(\theta) = 1_{(\theta_0, \theta_1)} \sin \left( \frac{i(\theta - \theta_0)}{\theta_1 - \theta_0} \right), \quad i = 1, 2, \ldots, M,
\]
with \( \theta_0 = \pi \) and \( \theta_1 = \frac{5\pi}{4} \). As an illustration, the boundary actuator \( \Psi_2 \) is plotted in Fig. 1b.

The feedback operators have been computed by solving the Riccati equation 16, with \( \hat{a} \) and \( \hat{b} \) as in (6b), with \( M = 6 \), and with \( \lambda = 1 \). In the boundary case, we have taken \( \zeta = 8 \). In the internal case, the actuators are those corresponding to the arrangement \( (m, n) = (3, 2) \).
The initial condition has been taken in the form $y_0 = \hat{y}(0) + \epsilon v_0$, where $v_0$ is the (numerical) solution of the elliptic system

$$-0.5 \Delta v_0 + 0.1 v_0 = \cos(3x_2)^2 + \sin(x_1) + 2; \quad v_0|_\Gamma = \sum_{i=1}^{M} q_i \Psi_i,$$

and $q = (1, 1, 0, 0, 0, 0, 0, 0)$ in the boundary case and $q = (0, 0, 0, 0, 0, 0)$ in the internal case. The corresponding functions are shown in Fig. 2.

### 6.1 The case of boundary actuators

We will confirm that the feedback control is able to stabilize locally system (57) to the targeted trajectory $\hat{y}$ with exponential rate $\frac{\lambda}{2}$, see Main Theorem 1.2. That is, the solutions of the system

$$\begin{align*}
\frac{\partial}{\partial t} y - \nu \Delta y + f(y, \nabla y) + f_0 &= 0, \quad y|_\Gamma = g + B\hat{y}, \quad y(0) = y_0, \\
\frac{\partial}{\partial t} \kappa + \zeta \kappa &= \left[B_{\hat{y}}^* - 1\right] \Pi_h \left[ y - \hat{y} - B_{\hat{y}} \kappa \right], \quad \kappa(0) = \epsilon q
\end{align*}$$

(61a) (61b)

with $y_0 = \hat{y}(0) + \epsilon v_0$, go exponentially to $\hat{y}$, with rate $\frac{\lambda}{2}$, provided $\epsilon \|v_0\|_{H^1(\Omega)} = |y_0 - \hat{y}(0)|_{H^1(\Omega)}$ is small enough, that is, provided $\epsilon$ is small enough.

In Fig. 3a, we observe that under the boundary feedback control, the system is stable and the solution $y$ goes exponentially to $\hat{y}$ with rate $\frac{\lambda}{2}$, for small $|\epsilon|_R$. The feedback fails to stabilize the system to $\hat{y}$ for bigger magnitudes of $\epsilon$, as we see in Fig. 3b. In Fig. 3c, we see that the uncontrolled system is not stable, and the solution may explode even for small $\epsilon$.

**Remark 6.1** In the theoretical results, we have asked the boundary actuators to be in $H^{\frac{3}{2}}(\Gamma)$. The actuators in (60) are in $H^{s}(\Gamma)$ for all $s < \frac{3}{2}$, but not necessarily...
in $H^2(Γ)$ (cf [32, Chapter 1, Section 11.3, Theorem 11.4]). This lack of regularity was neglected for the simulations.

### 6.2 The case of internal actuators

We will confirm that, with $B_M = P_M$, the solutions of the system

$$\frac{d}{dt} y - v Δ y + f(y, \nabla y) + f_0 = - B_M B_M^* Π_λ(y - \hat{y}), \quad y|_{Γ} = g, \quad y(0) = \hat{y}(0) + ϵ v_0,  \tag{62}$$

go exponential to the targeted trajectory $\hat{y}$, with rate $\frac{1}{2}$, provided $ϵ |v_0|_{H^1(Ω)} = |y_0 - \hat{y}(0)|_{H^1(Ω)}$ is small enough.

In Fig. 4a, we observe that for small $|ϵ|_R$ the feedback control is able to stabilize the system to $\hat{y}$ with exponential rate $\frac{1}{2}$. Figure 4a shows that for bigger magnitudes of $ϵ$ the feedback controller is not able to stabilize the system to $\hat{y}$. In Fig. 4c, the uncontrolled system is unstable and exploding, even for small $ϵ$.
6.3 On the computation of the solution of the differential Riccati equations

We cannot solve numerically the differential Riccati equations backwards in the time interval \([0, +\infty)\). So we solve them in a finite interval \([0, T]\), and we refer to [30, sections 5.3.2 and 5.3.3] for a procedure to find an appropriate final condition \(\Pi(T)\). Here, we have followed essentially the same procedure wherein particular \(\Pi(T)\) will be the solution of a suitable algebraic Riccati equation. To solve such equations, we use the software in [15] (see also [16]). Though, the numerical issues are not the subject of this work we must, however, mention that it is well known that as our discretization is refined solving the Riccati equation becomes a very hard and challenging numerical problem.

Another remark is that the function \(f_0\) in the discretized systems has been taken as the function which makes \(\hat{y}\) a solution of the discrete system. This was done to somehow avoid the effects of the numerical error which is propagated over time. See the discussion on [30, Section 7.2].

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Appendix

A.1 Proof of Lemma 2.2

Observe that given \(z\) solving (22), with \(\gamma = 0\), by rescaling time \(t = \frac{z}{\nu}\) and defining \(\tilde{z}(\tau) = z(\frac{\tau}{\nu})\), \(\tilde{a}(\tau) = \tilde{a}(\frac{\tau}{\nu})\), \(\tilde{b}(\tau) = \tilde{b}(\frac{\tau}{\nu})\), and \(\tilde{h}(\tau) = h(\frac{\tau}{\nu})\), we find

\[
\partial_{\tau} \tilde{z} - \Delta \tilde{z} + \frac{\tilde{a}}{\nu} \tilde{z} + \nabla \cdot \left( \frac{\tilde{b}}{\nu} \tilde{z} \right) + \frac{\tilde{h}}{\nu} = 0, \quad \tilde{z}|_{\Gamma} = 0, \quad \tilde{z}(v_{s0}) = z_0.
\]

Then, by standard arguments, we can find

\[
|\tilde{z}(v_{s})|_{H}^2 \leq e^{\nu^2 C_{W}(v_{s}-v_{s0})} \left( |\tilde{z}(v_{s0})|_{H}^2 + \frac{1}{\nu^2} \left| \tilde{h} \right|_{L^2((v_{s0}, v_{s}), V')}^2 \right),
\]

\[
2\nu |\tilde{z}|_{L^2(v_{I}, V)}^2 = |\tilde{z}(v_{s0})|_{H}^2 - |\tilde{z}(v_{s1})|_{H}^2 + \int_{v_{s0}}^{v_{s1}} \left( \frac{\tilde{a}}{\nu} \tilde{z} + \nabla \cdot \left( \frac{\tilde{b}}{\nu} \tilde{z} \right) + \frac{\tilde{h}}{\nu}, \tilde{z} \right)_{V', V} d\tau
\]

\[
\leq C_1 |\tilde{z}|_{L^\infty(v_{I}, H)}^2 + 3 \left| \frac{\tilde{a}}{\nu} \right|_{L^2(v_{I}, V')}^2 + |\tilde{z}|_{L^2(v_{I}, V')},
\]

with \(C_1 := \left( 1 + 3C^2 \left| \frac{\tilde{a}}{\nu} \right|_{L^\infty(v_{I}, L^d)}^2 + 3C^2 \left| \frac{\tilde{b}}{\nu} \right|_{L^\infty(v_{I}, L^\infty)}^2 \right).\) Therefore,
\[ |z(s)|^2_H \leq e^{\frac{C}{2} C_W(s-s_0)} \left( |z(s_0)|^2_H + \frac{1}{\nu} |h|^2_{L^2((s_0,s),V')} \right), \]
\[ |z|^2_{L^2(I,V)} \leq \left( \frac{1}{2\nu} + \frac{3C^2}{2\nu^2} |(a, b)|^2_W \right) |z|^2_{L^2(I,V)} + \frac{3}{2\nu^2} |h|^2_{L^2(I,V')}, \]
\[ \left| \frac{\partial}{\partial w} z \right|_{L^2(I,V')} \leq \left( \nu + C |a|_{L^\infty(I,L^d)} + C |b|_{L^\infty(I,L^\infty)} \right) |z|_{L^2(I,V)} + |h|_{L^2(I,V')}, \]

which imply the statement of Lemma 2.2. \hfill \Box

\section*{A.2 Proof of Proposition 2.1}

We construct an extension for \( \hat{\theta} = \hat{\theta}(t, x) \) independent of \( t \). We consider a different local system of space coordinates \( w \), in order to flatten the boundary. We use some basic concepts from Riemannian manifolds, see [44, chapter 1, Section 13 and chapter 2, Section 2].

\textit{Change of coordinates.} Up to a translation and rotation, we may suppose that locally the boundary \( \Gamma = \partial \Omega \) is the graph of a smooth function \( \Lambda, \) with \( \bar{w} := (w_1, w_2, \ldots, w_d-1), \)

\[ \bar{w} \mapsto G_A(\bar{w}) := (\bar{w}, \Lambda(\bar{w})) \in \Gamma, \]
\[ \bar{w} \in \mathbb{D}_r^{d-1} := \{ \bar{w} \in \mathbb{R}^{d-1} | w_1^2 + w_2^2 + \cdots + w_{d-1}^2 \leq r \}, \]

with (small) \( r > 0 \). Locally, a tubular neighborhood is given by

\[ T = T_{r,l} := \left\{ x \in \mathbb{R}^d | x = G_A(\bar{w}) + w_d \mathbf{n}_{G_A(\bar{w})}, \ w := (\bar{w}, w_d) \in \mathbb{D}_r^{d-1} \times (-l, l) \right\}, \quad (A.1) \]

with (small) \( l > 0 \). Where \( \mathbf{n}_{G_A(\bar{w})} \) stands for the unit outward normal vector at \( G_A(\bar{w}) \in \Gamma \).

Let us denote \( \Omega^- = \mathbb{D}_r^{d-1} \times (-l, 0) \) and \( \Omega^+ = \mathbb{D}_r^{d-1} \times (0, l) \). Notice that the points outside \( \Omega \) correspond to those in \( \Omega^+ \). We may suppose that \( x_0 = (0, G_A(0)) \in \Gamma_1 \subset \Gamma \cap T \), that is, in Proposition 2.1 we may take \( \tilde{\omega} \) corresponding to a subset of \( \Omega^+ \).

The new coordinates \((w_1, w_2, \ldots, w_d)\) induce the vector fields

\[ \frac{\partial}{\partial w_i} = \sum_{i=1}^{d} \frac{\partial x_i}{\partial w_i} \frac{\partial}{\partial x_j}, \quad i = 1, 2, \ldots, d, \quad (A.2) \]

defined in \( \mathcal{O} := \mathbb{D}_r^{d-1} \times (-l, l) \). We can see the neighborhood \( T \) (endowed with the usual Euclidean scalar product) as the Riemannian manifold \((\mathcal{O}, g)\) by taking the metric tensor

\[ g = \sum_{i=1}^{d} \sum_{j=1}^{d} g_{ij} dw_j \otimes dw_j, \quad \text{with} \quad g_{ij} = \left( \frac{\partial}{\partial w_i}, \frac{\partial}{\partial w_j} \right)_{\mathbb{R}^d} \]

where \((\cdot, \cdot)_{\mathbb{R}^d}\) stands for the usual Euclidean scalar product in \( \mathbb{R}^d \).
The divergence of a vector field \( V = \sum_{i=1}^{d} V_i \frac{\partial}{\partial w_i} \) reads, in the new coordinates,

\[
\nabla_w \cdot V = (-1)^d \frac{1}{\sqrt{\bar{g}}} \sum_{j=1}^{d} \frac{\partial(V_j \sqrt{\bar{g}})}{\partial w_j},
\]

where \( \bar{g} := \det(g_{ij}) \) stands for the determinant of the matrix whose entries are the coefficients of the metric tensor. Recall that in our setting \( \sqrt{\bar{g}} \) coincides with the Jacobian \( \det \left( \frac{\partial x}{\partial w} \right) \) of the smooth diffeomorphism \( w \mapsto x \), because

\[
[g_{ij}] = \left[ \sum_{k=1}^{d} \frac{\partial x_k}{\partial w_i} \frac{\partial x_k}{\partial w_j} \right] = \left[ \frac{\partial x}{\partial w} \right] \left( \frac{\partial x}{\partial w} \right)^\top,
\]

where \( \left( \frac{\partial x}{\partial w} \right)^\top \) is the transpose matrix of \( \left( \frac{\partial x}{\partial w} \right) \).

**The extension.** For a given vector field \( V^- = \sum_{i=1}^{d} V_i^- \frac{\partial}{\partial w_i} \), defined in \( \mathcal{O}^- \), we consider the following vector field defined in \( \mathcal{O}^+ \):

\[
V^+ := \sum_{i=1}^{d} V_i^+ \frac{\partial}{\partial w_i}, \quad \text{with} \quad \begin{cases} V_i^+(\bar{w}, s) = -Q(w)V_i^-(\bar{w}, -s), & \text{if } i \neq d, \\ V_d^+(\bar{w}, s) = Q(w)V_d^-(\bar{w}, -s), & \end{cases}
\]

where \( s \in (0, 1) \) and

\[
Q(w) := \frac{\sqrt{\bar{g}}|_{(\bar{w}, -s)}}{\sqrt{\bar{g}}|_{(\bar{w}, s)}}.
\]

Then, denoting the mapping \( \sigma : \mathcal{O}^+ \to \mathcal{O}^-, (\bar{w}, s) \mapsto (\bar{w}, -s) \), we find

\[
\sqrt{\bar{g}}|_{(\bar{w}, s)} (-1)^d (\nabla_w \cdot V^+)|_{(\bar{w}, s)} = \sum_{j=1}^{d} \frac{\partial(V_j^+ \sqrt{\bar{g}})}{\partial w_j} \bigg|_{(\bar{w}, s)}
\]

\[
= \frac{\partial((V_j^- \sqrt{g})\sigma)}{\partial w_d} \bigg|_{(\bar{w}, s)} - \sum_{j=1}^{d-1} \frac{\partial((V_j^- \sqrt{g})\sigma)}{\partial w_j} \bigg|_{(\bar{w}, s)} = -\sum_{j=1}^{d} \frac{\partial(V_j^- \sqrt{g})}{\partial w_j} \bigg|_{(\bar{w}, -s)},
\]

which gives us

\[
(\nabla_w \cdot V^+)|_{(\bar{w}, s)} = -Q(w)(\nabla_w \cdot V^-)|_{(\bar{w}, -s)}. \tag{A.4}
\]

For \( S \subseteq \mathcal{O} \), let us denote \( \mathcal{W}_2(S) := \{ v \in L^\infty(S, \mathbb{R}^d) \mid (\nabla_w \cdot v) \in L^r_g(S, \mathbb{R}) \} \), with \( r \in [2, \infty) \) (cf. (29)). In particular, since \( Q(w) \) is a smooth function, we observe that \( V^+ \in \mathcal{W}_2(\mathcal{O}^+) \) if \( V^- \in \mathcal{W}_2(\mathcal{O}^-) \). Here, \( (f, h)_{L^2_g(S, \mathbb{R})} := \int_S Jh \, d\mathcal{O}, \quad d\mathcal{O} := \sqrt{g} \, dw_1 \wedge dw_2 \wedge \cdots \wedge dw_d \). It is also clear that the linear mapping \( V^- \mapsto V^+ \) is continuous. Finally, we prove that the function defined by
V \mapsto \nabla, \quad \text{with} \quad \nabla (w) := \begin{cases} V^- (w), & \text{if } w \in \mathcal{O}^-, \\ V^+ (w), & \text{if } w \in \mathcal{O}^+, \end{cases}

maps \mathcal{W}_2(\mathcal{O}^-) \text{ into } \mathcal{W}_2(\mathcal{O}). \text{ We need to prove that } \nabla_w \cdot \nabla \in L^2_g(\mathcal{O}, \mathbb{R}). \text{ For a smooth function } \phi \in \mathcal{D}(\mathcal{O}) := C_0^\infty(\mathcal{O}, \mathbb{R}) \text{ with support contained in } \mathcal{O} \text{ we find, in the distribution sense, with } \Gamma_0 := \{w \in \mathcal{O} \mid w_d = 0\},

\langle \nabla_w \cdot \nabla, \phi \rangle_{\mathcal{D}(\mathcal{O}), \mathcal{D}(\mathcal{O}')} := \langle \nabla, \nabla \phi \rangle_{\mathcal{D}(\mathcal{O}), \mathcal{D}(\mathcal{O}')} = -\langle \nabla, \nabla \phi \rangle_{L^2_g(\mathcal{O}, \mathbb{R})} = \langle \nabla_w \cdot \nabla, \phi \rangle_{L^2_g(\mathcal{O}^-, \mathbb{R})} + \langle \nabla_w \cdot \nabla, \phi \rangle_{L^2_g(\mathcal{O}^+, \mathbb{R})}

= \int_{\Gamma_0} \phi g \left( V^- - \frac{\partial}{\partial w_d} \right) - \phi g \left( V^+ - \frac{\partial}{\partial w_d} \right) \, d_g \Gamma_0.

Notice that \frac{\partial}{\partial w_d} is the unit outward normal at \Gamma_0 \subset \partial \mathcal{O}^-.

Now, since \( g \left( V^- - \frac{\partial}{\partial w_d} \right) - g \left( V^+ - \frac{\partial}{\partial w_d} \right) = V^-_d - V^+_d \), from \( V^+_d (w, s) = Q(w) V^-_d (w, -s) \) for all \( s > 0 \) and \( Q(\bar{w}, 0) = 1 \), we can conclude that \( V^-_d - V^+_d \) necessarily vanishes at \( \Gamma_0 \). Hence, the boundary term vanishes, and we can conclude that \( \nabla_w \cdot \nabla \in L^2_g(\mathcal{O}, \mathbb{R}) \). We may write

\langle \nabla_w \cdot \nabla, \phi \rangle_{\mathcal{D}(\mathcal{O}), \mathcal{D}(\mathcal{O}')} = \langle \nabla_w \cdot \nabla, \phi \rangle_{L^2_g(\mathcal{O}^-, \mathbb{R})} + \langle \nabla_w \cdot \nabla, \phi \rangle_{L^2_g(\mathcal{O}^+, \mathbb{R})}

= \langle \nabla_w \cdot \nabla, \phi \rangle_{L^2_g(\mathcal{O}, \mathbb{R})}.

Therefore, if in addition we have \( \nabla_w \cdot \nabla \in L^\infty_g(\mathcal{O}^-, \mathbb{R}) \) then from (A.4) it follows that \( \nabla_w \cdot \nabla \in L^\infty_g(\mathcal{O}, \mathbb{R}) \).

It is also clear that the mapping \( V^- \mapsto \nabla \) maps \( \mathcal{W}_2(\mathcal{O}^-) \) into \( \mathcal{W}_2(\mathcal{O}) \) continuously.

In the original coordinates, the extension above reads: given \( \hat{b} = \hat{b}_i \frac{\partial}{\partial y_i} \), we firstly rewrite \( \hat{b} \) in the new coordinates \( \hat{b} = \hat{b}_i \frac{\partial}{\partial y_i} =: V^- \), next we extend \( V^- \) to \( \nabla = \nabla_i \frac{\partial}{\partial y_i} \) (through \( V^+ \) as above), finally we rewrite \( \nabla \) in the original coordinates: \( \nabla = \nabla_i \frac{\partial}{\partial x_i} =: \tilde{b} \).

The continuity of \( (\tilde{a}, \tilde{b}) \mapsto (\hat{a}, \hat{b}) \) from \( \mathcal{W}_\text{st} \) into \( \tilde{\mathcal{W}}_\text{st} \) follows straightforwardly. \( \square \)

**Remark A.2** In [40, Proposition 4.2] we find, for \( d = 3 \), the result we present here in Proposition 2.1. Our proof borrows the idea from [40, Appendix]. We still present the proof in here because in [40], when computing the vector fields \( \frac{\partial}{\partial w_i} \) for \( i = 1, 2, \ldots, d-1 \), as in (A.2) above, the terms \( u_d \frac{\partial}{\partial w_i} n \cdot (\nabla \Lambda (w)) j \frac{\partial}{\partial x_j} \) have been missed, see [40, Eq. (A.2)].
A.3 Proof of Proposition 2.2

From [33, chapter 4, Section 2.5, Theorem 2.3] we know that \( u \mapsto (u_0, u|_\Gamma) \), maps the space \( W(\mathbb{R}_0, H^2(\Omega, \mathbb{R}), L^2(\Omega, \mathbb{R})) \) continuously onto the product space

\[
\left\{ (z, g) \in H^1(\Omega, \mathbb{R}) \times \left( L^2(\mathbb{R}_0, H^\frac{3}{2}(\Gamma, \mathbb{R})) \cap H^\frac{3}{2}(\mathbb{R}_0, L^2(\Gamma, \mathbb{R})) \right) \mid g(0) = z|_\gamma \right\}.
\]

In particular, this implies that \( G^2(J, \Gamma) = L^2(J, H^\frac{3}{2}(\Gamma, \mathbb{R})) \cap H^\frac{3}{2}(J, L^2(\Gamma, \mathbb{R})) \).

From the discussion in Remark 2.3, it follows that \( (\vartheta \cdot) \in \mathcal{L}(G^2(J, \Gamma)) \). It is also not difficult to check that

\[
\partial P_M \vartheta \in \mathcal{L}(L^2(J, H^\frac{3}{2}(\Gamma, \mathbb{R}))) \cap \mathcal{L}(H^1(J, L^2(\Gamma, \mathbb{R}))).
\]

Indeed, we may suppose, without loss of generality, that the family of functions \( \Psi_i \) is orthonormal in \( L^2(\Gamma, \mathbb{R}) \), and in that case, for a Hilbert space \( H^2(\Gamma, \mathbb{R}) \subseteq X \hookrightarrow L^2(\Gamma, \mathbb{R}) \),

\[
|\partial P_M \vartheta \xi|_{L^2(J, X)}^2 = \int J \left| \partial \sum_{i=1}^{M} (\partial \xi(s), \Psi_i)_{L^2(\Gamma, \mathbb{R})} \Psi_i \right|^2 ds 
\leq C_{\partial M} \sum_{i=1}^{M} \int J |\xi(s)|^2_{L^2(\Gamma, \mathbb{R})} |\Psi_i|^2_X ds 
\leq C_{\partial M} \max_{1 \leq i \leq M} |\Psi_i|_{L^2(\Gamma, \mathbb{R})}^2 |\xi|^2_{L^2(J, L^2(\Gamma, \mathbb{R}))} \leq C |\xi|^2_{L^2(J, X)}.
\]

Now, from \( \partial P_M \vartheta \in \mathcal{L}(L^2(J, L^2(\Gamma, \mathbb{R}))) \cap \mathcal{L}(H^1(J, L^2(\Gamma, \mathbb{R}))) \), by an interpolation argument, it follows that \( \partial P_M \vartheta \in \mathcal{L}(H^\frac{3}{2}(J, L^2(\Gamma, \mathbb{R}))) \). See [32, chapter 1, Section 5.1] and [33, chapter 4, Section 2.1]. Finally, \( \partial P_M \vartheta \in \mathcal{L}(L^2(J, H^\frac{1}{2}(\Gamma, \mathbb{R}))) \cap \mathcal{L}(H^\frac{3}{2}(J, L^2(\Gamma, \mathbb{R}))) \) implies, by Proposition 2.3, that \( \partial P_M \vartheta \in \mathcal{L}(G^2(J, \Gamma)) \).

Finally, we prove that \( \partial Q_{\tilde{M}} P_M \vartheta \in \mathcal{L}(G^2(J, \Gamma)) \). Since \( \partial P_M \vartheta \in \mathcal{L}(G^2(J, \Gamma)) \), it is enough to prove that \( Q_{\tilde{M}} \in \mathcal{L}(\partial P_M \vartheta G^2(J, \Gamma), G^2(J, \Gamma)) \). Notice that

\[
\partial P_M \vartheta G^2(J, \Gamma) \subseteq L^2(J, \partial P_M H^\frac{3}{2}(\Gamma)) \cap H^\frac{3}{2}(J, \partial P_M L^2(\Gamma)),
\]

and \( P_M H^\frac{3}{2}(\Gamma) = S_\Psi = P_M L^2(\Gamma) \). Since the space \( S_\Psi = \text{span}\{\Psi_i \mid i \in \{1, 2, \ldots, M\}\} \) is finite-dimensional, it remains to observe that \( Q_{\tilde{M}} \in \mathcal{L}\left( L^2(J, \mathbb{R}^M) \cap H^\frac{3}{2}(J, \mathbb{R}^M) \right) \), which follows from \( Q_{\tilde{M}} \in \mathcal{L}(L^2(J, \mathbb{R})) \cap \mathcal{L}(H^\frac{3}{2}(J, \mathbb{R})) \). Finally, observe that looking at \( H^\frac{3}{2}(J, \mathbb{R}) =: H^\frac{3}{2}_f(J, \mathbb{R}) \) as the domain of
\(-\Delta_j + 1\)^{\frac{3}{8}} \) (cf. proof of Lemma 2.8) we find the identities
\[ |Q^{\tilde{M}}|^2_{\mathcal{L}(L^2(J,\mathbb{R}))} = 1 = |Q^{\tilde{M}}|^2_{\mathcal{L}(H^1_0(J,\mathbb{R}))}. \]
\[ \square \]

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