Blow-up Criteria of Classical Solutions of Three-Dimensional Compressible Magnetohydrodynamic Equations

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Abstract In this paper we consider the isentropic compressible magnetohydrodynamic equations in three space dimensions, and establish a blow-up criterion of classical solutions, which depends on the gradient of the velocity and magnetic field.

Keywords Magnetohydrodynamic (MHD) · Blow up · Isentropic

Mathematics Subject Classification (2010) 35B45 · 35L65 · 35Q60 · 76N10

1 Introduction

In this paper we consider the following three-dimensional isentropic compressible magnetohydrodynamic (MHD) equations:

\[\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - Lu + \nabla p &= \text{rot}H \times H, \quad p = A\rho^\gamma, \\
H_t - \text{rot}(u \times H) &= \nu\Delta H, \quad \text{div}H = 0,
\end{align*}\]

with initial data

\[\rho(0, x) = \rho_0(x), \quad u(0, x) = u_0(x), \quad H(0, x) = H_0(x),\]

where \(\rho, u = (u^1, u^2, u^3), \quad H = (H^1, H^2, H^3)\) and \(P\) are functions of \(x \in \mathbb{R}^3\) and \(t \geq 0\) representing density, velocity, magnetic field and pressure, respectively. The
Lamé operator $L$ is defined by

$$Lu = \mu \Delta u + (\lambda + \mu) \nabla \text{div} u.$$ 

The parameters $\mu$, $\lambda$, $A$ and $\gamma$ are constants which satisfy

$$A > 0, \quad \gamma > 1, \quad \mu > 0, \quad 2\mu + 3\lambda \geq 0,$$

(1.5)

where $\mu$, $\lambda$ are the shear viscosity coefficient and bulk viscosity coefficient, respectively.

Magnetohydrodynamic equations describe the motion of electrically conducting fluids in the presence of the magnetic field, which essentially needs to consider the interaction between the fluid velocity and the magnetic field. Before starting and proving our main results, let us first briefly recall the related results in the literature. For compressible Navier-Stokes equations, Xin [11], Rozanova [8] showed the blow up results of global smooth solutions when the initial density is compactly supported, or decreases to zero rapidly. Cho and Jin [4] generalized the results in [11] to the coefficient of heat conduction $\kappa > 0$, and give a sufficient condition for blow up result provided that the initial density is positive but decays at infinity. Recently, Lai [6] established the blow up results under the assumption that the gradient velocity satisfies some decay constraint and the initial total momentum does not vanish. For incompressible MHD system, Caflisch, Klapper and Steele [2] extended the well-known result of Beal, Kato and Majda [1] for incompressible Euler equations to the case of the 3D ideal MHD equaitons. Wu [10], He and Wang [5] and Y. Zhou [12] obtained some regularity criteria for incompressible MHD equations under different conditions respectively, and Chen, Miao and Zhang [3] extended these results to Besov spaces. For compressible MHD equations, Lu, Du and Yao [7] obtained a blow up criterion for the local strong solutions just in terms of the gradient of the velocity. Wang and Li [9] established a blow up criterion for global strong solutions depend only on density and magnetic field.

The purpose of this paper is to derive the corresponding blow-up criteria for the isentropic compressible MHD equations in the three-dimensional space. Moreover, we shall study the blow-up criteria when the gradient of velocity and magnetic field satisfies some decay constraint and the initial momentum and magnetic field does not vanish.

Now we are in this position to state our main results.

**Definition 1.1** (Classical Solutions). Let $T$ be positive. Then $(\rho(t, x), u(t, x), H(t, x))$ is called a classical solution to the compressible magnetohydrodynamic equations (1.1)–(1.4) on $(0, T) \times \mathbb{R}^3$ if $\rho(t, x) \in C^1([0, T) \times \mathbb{R}^3)$, $u, H \in C^1([0, T), C^2(\mathbb{R}^3))$, and satisfies the system (1.1)–(1.4) pointwisely on $(0, T) \times \mathbb{R}^3$.

**Theorem 1.1** Let (1.5) be satisfied, and assume that $(\rho, u, H)$ is a finite energy classic solution of the compressible magnetohydrodynamic equations (1.1)–(1.4) in the sense that

$$E(t) + \int_0^t \int_{\mathbb{R}^3} \left( \mu |\nabla u|^2 + (\lambda + \mu) |\text{div} u|^2 + \nu |\nabla H|^2 \right) (s, x) dx ds \leq E(0) < \infty, \quad \forall t \geq 0,$$

(1.6)
where
\[ E(t) = \int_{\mathbb{R}^3} \left( \frac{1}{2} (\rho|u|^2 + |H|^2) + \frac{1}{\gamma - 1} A\rho^\gamma \right) (t, x) dx \]

Moreover, we suppose that \((\rho, u, H)\) satisfies
\[ \int_{\mathbb{R}^3} (\rho(t, x) + |H(t, x)|) \, dx < \infty, \quad \forall t \geq 0. \quad (1.7) \]

and
\[ \int_{\mathbb{R}^3} \left( |\nabla u(t, x)| + |\nabla H(t, x)| \right) \, dx < \infty, \quad \forall t \geq 0. \quad (1.8) \]

If the initial data satisfy
\[ \left| \int_{\mathbb{R}^3} \rho_0(x)u_0(x) \, dx \right| \neq 0, \quad (1.9) \]

then \((\rho, u, H)\) must develop a finite time singularity in the sense that the equations in (1.1)–(1.4) are not valid for \(t \geq T^*_n\) for some \(T^*_n > 0\).

2 Proof of Theorem 1.1

The steps of the proof are: (a) proof of conservation of the total mass, the total momentum and the magnetic field; (b) proof of blow-up of classical solutions.

(a) proof of conservation of the total mass, the total momentum and the magnetic field

Lemma 2.1 Let (1.5) and (1.7) hold, then the total mass, the total momentum and the magnetic field of the isentropic compressible magnetohydrodynamic equations (1.1)–(1.4) are conserved in the sense that
\[ \int_{\mathbb{R}^3} \rho(t, x) \, dx = \int_{\mathbb{R}^3} \rho_0(x) \, dx, \quad t > 0, \quad (2.1) \]
\[ \int_{\mathbb{R}^3} \rho(t, x)u(t, x) \, dx = \int_{\mathbb{R}^3} \rho_0(x)u_0(x) \, dx, \quad t > 0, \quad (2.2) \]
\[ \int_{\mathbb{R}^3} H(t, x) \, dx = \int_{\mathbb{R}^3} H_0(x) \, dx, \quad t > 0. \quad (2.3) \]

Proof By combing (1.6)–(1.9), it is easy to see that \(\forall t \geq 0\), there exists \(R_n(t) \to \infty\) such that
\[ \lim_{n \to \infty} \int_{|x| = R_n} \left( \frac{1}{2} (\rho|u|^2 + |H|^2) + \rho + |H| + \frac{1}{\gamma - 1} A\rho^\gamma \right) \, d\sigma = 0, \quad (2.4) \]
and
\[ \lim_{n \to \infty} \int_0^t \int_{|x| = R_n} \left( \mu|\nabla u|^2 + (\lambda + \mu)|\text{div} u|^2 + \nu|\nabla H|^2 \right) \, d\sigma ds = 0. \quad (2.5) \]
Moreover, the total momentum is bounded in the sense that
\[
\left| \int_{\mathbb{R}^3} \rho u(t,x) \, dx \right| \leq \left( \int_{\mathbb{R}^3} \rho(t,x) \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} \rho |u|^2(t,x) \, dx \right)^{\frac{1}{2}} < \infty. \tag{2.6}
\]

Integrating (1.1) over the ball \( B_R := \{ x \in \mathbb{R}^3 : |x| < R \} \) with radius \( R \), we obtain
\[
\int_{|x| \leq R} \rho_t(t,x) \, dx = - \int_{|x| \leq R} \text{div}(\rho u)(t,x) \, dx. \tag{2.7}
\]

Noting that
\[
\int_0^\infty \left| \int_{|x| \leq R} \text{div} (\rho u) \, dx \right| \, dR = \int_0^\infty \left| \int_{|x| = R} x \cdot (\rho u) / R \, d\sigma \right| \, dR \leq \left\| \sqrt{\rho} \right\|_{L^2(\partial B_R)} \left\| \sqrt{\rho u} \right\|_{L^2(\partial B_R)} < \infty,
\]
then by integrating (2.7) over \([0,t]\), we get
\[
\int_0^\infty \left| \int_{|x| \leq R} (\rho(t,x) - \rho_0(x)) \, dx \right| \, dR \leq t \left\| \sqrt{\rho} \right\|_{L^2(\mathbb{R}^3)} \left\| \sqrt{\rho u} \right\|_{L^2(\mathbb{R}^3)} < \infty. \tag{2.9}
\]

Hence, for each \( t \geq 0 \), there exist \( R_n \to \infty \) such that
\[
\lim_{n \to \infty} \int_{|x| \leq R_n} (\rho(t,x) - \rho_0(x)) \, dx = 0, \tag{2.10}
\]
which, together with (1.7), yields the mass conservation (2.1).

Similarly, the conservation of momentum and magnetic field can be proved. Indeed, integrating (1.2)–(1.3) over the ball \( B_R := \{ x \in \mathbb{R}^3 : |x| < R \} \) with radius \( R \) respectively, we have
\[
\int_{|x| \leq R} (\rho u)_t(t,x) \, dx = - \int_{|x| \leq R} \left( \text{div}(\rho u \otimes u - \mu \nabla u - H \otimes H + \frac{1}{2} |H|^2 I) \right.

+ \nabla (p - (\lambda + \mu) \text{div} u) \left( t,x \right) \, dx, \tag{2.11}
\]
\[
\int_{|x| \leq R} (H)_t(t,x) \, dx = \int_{|x| \leq R} (\text{div}(v \nabla H) + \nabla \times (u \times H)) \, dx

= \int_{|x| \leq R} (\text{div}(v \nabla H) + H \cdot \nabla u - u \cdot \nabla H - H \text{div} u) \, dx, \tag{2.12}
\]
where we have used the following conclusions:
\[
\nabla \times (u \times v) = (v \cdot \nabla) u - (u \cdot \nabla) v + u \text{div} v - v \text{div} u,

(\nabla \times w) \times w = \text{div}(w \otimes w - \frac{1}{2} |w|^2 I), \quad \text{provided that} \quad \text{div} w = 0.
\]

Using (1.6)–(1.8), we obtain
\[
\int_0^\infty \left| \int_{|x| \leq R} \left( \text{div}(\rho u \otimes u - \mu \nabla u - H \otimes H + \frac{1}{2} |H|^2 I) + \nabla (p - (\lambda + \mu) \text{div} u) \right) \, dx \right| \, dR

\leq \left\| \rho |u|^2 \right\|_{L^1(\mathbb{R}^3)} + \| p \|_{L^1(\mathbb{R}^3)} + (\lambda + 2\mu) \| \nabla u \|_{L^1(\mathbb{R}^3)} + \frac{3}{2} \| |H|^2 \|_{L^1(\mathbb{R}^3)} < \infty, \tag{2.13}
\]

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\[
\int_0^{\infty} \left| \int_{|x| \leq R} (\text{div}(\nu \nabla H) + H \cdot \nabla u - u \cdot \nabla H - H \text{div} u) \, dx \right| \, dR \\
\leq \| \nabla H \|_{L^1(\mathbb{R}^3)} + \| H \|_{L^2(\mathbb{R}^3)} + \| \nabla u \|_{L^2(\mathbb{R}^3)} < \infty. \tag{2.14}
\]

Then by integrating (2.11)–(2.12) over \([0, t]\) respectively, we have
\[
\int_0^{\infty} \left| \int_{|x| \leq R} (\rho u(t, x) - \rho_0 u_0(x)) \, dx \right| \, dR < \infty, \tag{2.15}
\]
\[
\int_0^{\infty} \left| \int_{|x| \leq R} (H(t, x) - H_0(x)) \, dx \right| \, dR < \infty, \tag{2.16}
\]
and hence, for each \(t \geq 0\), there exist \(R_n \to \infty\) such that
\[
\lim_{n \to \infty} \int_{|x| \leq R_n} (\rho u(t, x) - \rho_0 u_0(x)) \, dx = 0, \tag{2.17}
\]
\[
\lim_{n \to \infty} \int_{|x| \leq R_n} (H(t, x) - H_0(x)) \, dx = 0, \tag{2.18}
\]
which, together with (1.7) and (2.6), yields the (2.2)–(2.3).

(b) proof of blow-up of classical solutions. We split the proof into two cases:
\[\gamma \geq \frac{6}{5}\] and \(1 < \gamma < \frac{6}{5}\).

For the case \(\gamma \geq \frac{6}{5}\), by interpolation inequality we have
\[
\left| \int_{\mathbb{R}^3} \rho u \, dx \right| \leq \| \rho \|_{L^6(\mathbb{R}^3)} \| u \|_{L^6(\mathbb{R}^3)} \\
\leq C \| \rho \|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \| u \|_{L^{\frac{12}{5}}(\mathbb{R}^3)} \| \nabla u \|_{L^2(\mathbb{R}^3)}, \tag{2.19}
\]
where \(\theta = \frac{5\gamma - 6}{6\gamma - 6}\) satisfies \(0 < \theta < 1\) and the constant \(C > 0\).

For the case \(1 < \gamma < \frac{6}{5}\), we have
\[
\left| \int_{\mathbb{R}^3} \rho u \, dx \right| \leq \| \sqrt{\rho} u \|_{L^2(\mathbb{R}^3)} \| u \|_{L^{\frac{12}{5}}(\mathbb{R}^3)} \| \rho \|_{L^\gamma(\mathbb{R}^3)} \| \nabla u \|_{L^2(\mathbb{R}^3)}, \tag{2.20}
\]
where \(\alpha = \frac{6 - 5\gamma}{3 - 2\gamma}\) satisfies \(0 < \alpha < 1\) and the constant \(C > 0\). On the other hand, by interpolation inequality, we get
\[
\| H \|_{L^2(\mathbb{R}^3)} \leq C \| \nabla H \|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \| H \|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \| \nabla H \|_{L^2(\mathbb{R}^3)}. \tag{2.21}
\]
Combining (1.6)–(1.9), Lemma 2.1 with (2.19)–(2.20), one has for \(\gamma > 1\)
\[
\left( \| \nabla u \|_{L^2(\mathbb{R}^3)}^2 + \| \nabla H \|_{L^2(\mathbb{R}^3)}^2 \right) \geq C_0 > 0, \tag{2.22}
\]
where $C_0$ denotes a constant depending on the following initial data
\begin{align}
\int_{\mathbb{R}^3} (\rho_0 u_0 + \rho_0 + H_0) (x) dx, \\
\int_{\mathbb{R}^3} \left( \frac{1}{2} (\rho_0 |u_0|^2 + |H_0|^2) + \frac{1}{\gamma - 1} A \rho_0^\gamma \right) (x) dx.
\end{align}
(2.23) (2.24)

Inserting (2.22) into (1.6), we have
\begin{align}
\int_{\mathbb{R}^3} \left( \frac{1}{2} (\rho |u|^2 + |H|^2) + \frac{1}{\gamma - 1} A \rho^\gamma \right) (t, x) dx + C_1 t \\
\leq \int_{\mathbb{R}^3} \left( \frac{1}{2} (\rho_0 |u|^2 + |H_0|^2) + \frac{1}{\gamma - 1} A \rho_0^\gamma \right) (x) dx,
\end{align}
(2.25)

where constant $C_1$ depending on $C_0$, $\lambda$, $\mu$, and $\nu$, so there exists $t_\ast > 0$, s.t.
$E(t_\ast) = 0$, which contradicts to the fact that the total energy is conserved and positive, and we finish the proof of Theorem 1.1.

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