Uniformly rotating neutron stars in the global and local charge neutrality cases

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Abstract

In our previous treatment of neutron stars, we have developed the model fulfilling global and not local charge neutrality. In order to implement such a model, we have shown the essential role by the Thomas-Fermi equations, duly generalized to the case of electromagnetic field equations in a general relativistic framework, forming a coupled system of equations that we have denominated Einstein-Maxwell-Thomas-Fermi (EMTF) equations. From the microphysical point of view, the weak interactions are accounted for by requesting the beta stability of the system, and the strong interactions by using the $\sigma$-$\omega$-$\rho$ nuclear model, where $\sigma$, $\omega$ and $\rho$ are the mediator massive vector mesons. Here we examine the equilibrium configurations of slowly rotating neutron stars by using the Hartle formalism in the case of the EMTF equations indicated above. We integrate these equations of equilibrium for different central densities $\rho_c$ and circular angular velocities $\Omega$ and compute the mass $M$, polar $R_p$ and equatorial $R_{eq}$ radii, angular momentum $J$, eccentricity $e$, moment of inertia $I$, as well as quadrupole moment $Q$ of the configurations. Both the Keplerian mass-shedding limit and the axisymmetric secular instability are used to construct the new mass-radius relation. We compute the maximum and minimum masses and rotation frequencies of neutron stars. We compare and contrast all the results for the global and local charge neutrality cases.

1. Introduction

We have recently shown (Rotondo et al., 2011; Rueda et al., 2011; Belvedere et al., 2012) that the equations of Tolman-Oppenheimer-Volkoff (TOV) (Tolman, 1939; Oppenheimer and Volkoff, 1939), traditionally used to describe the neutron star equilibrium configurations, are superseded once the strong, weak, electromagnetic and gravitational interactions are taken into account. Instead, the Einstein-Maxwell system of equations coupled with the general relativistic Thomas-Fermi equations of equilibrium have to be used; what we called the Einstein-Maxwell-Thomas-Fermi (EMTF) system of equations. While in the TOV approach the condition of local charge neutrality, $n_e(r) = n_p(r)$ is imposed (see e.g. Haensel et al., 2007, and references therein), the EMTF approach requests the less stringent condition of global charge neutrality, namely

$$\int \rho_{eh}d^3r = \int e[n_p(r) - n_e(r)]d^3r = 0, \quad (1)$$

where $\rho_{eh}$ is the charge density, $e$ is the fundamental electric charge, and the integral is carried out on the entire volume of the system.

The Lagrangian density taking into account all the interactions include the free-fields terms $\mathcal{L}_\gamma$, $\mathcal{L}_\sigma$, $\mathcal{L}_\omega$, $\mathcal{L}_\rho$ (respectively for the gravitational, the electromagnetic, and the mesonic fields), the three fermion species (electrons, protons and neutrons) term $\mathcal{L}_f$ and the interacting part in the minimal coupling assumption, $\mathcal{L}_{\text{int}}$ (Rueda et al., 2011; Belvedere et al., 2012):

$$\mathcal{L} = \mathcal{L}_y + \mathcal{L}_f + \mathcal{L}_\sigma + \mathcal{L}_\omega + \mathcal{L}_\rho + \mathcal{L}_\gamma + \mathcal{L}_{\text{int}}, \quad (2)$$

where

$$\mathcal{L}_y = -\frac{R}{16\pi}, \quad \mathcal{L}_f = \sum_{i=e,N} \bar{\psi}_i (i\gamma^\mu D_\mu - m_i) \psi_i,$$

$$\mathcal{L}_\sigma = \frac{\nabla_\mu \nabla^\mu \sigma - U(\sigma)}{2}, \quad \mathcal{L}_\omega = -\frac{\Omega_{\mu \nu} \Omega^{\mu \nu}}{4} + \frac{m_\omega^2 \omega_{\mu \nu} \omega^{\mu \nu}}{2},$$

$$\mathcal{L}_\rho = \frac{R_{\mu \nu} R^{\mu \nu}}{4} + \frac{m_\rho^2 \rho_{\mu \nu} \rho^{\mu \nu}}{2}, \quad \mathcal{L}_\gamma = \frac{F_{\mu \nu} F^{\mu \nu}}{16\pi},$$

$$\mathcal{L}_{\text{int}} = -g_\sigma \bar{\psi}_N \sigma \psi_N - g_\omega \omega_{\mu \nu} J_{\mu \nu} - g_\rho \rho_{\mu \nu} J^{\mu \nu} + eA_\mu J_{\mu e}^\gamma,$$

where the description of the strong interactions between the nucleons is made through the $\sigma$-$\omega$-$\rho$ nuclear model in

\textsuperscript{1}We use spacetime metric signature (+,−,−,−) and geometric units $G = c = 1$ unless otherwise specified.
the version of [Boguta and Bodmer (1977)]. Thus \( \Omega_{\mu
u} \equiv \partial_\mu \omega_\nu - \partial_\nu \omega_\mu, \) \( \mathcal{R}_{\mu
u} \equiv \partial_\mu \rho_\nu - \partial_\nu \rho_\mu, \) \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \) are the field strength tensors for the \( \omega^\mu \) fields, \( \rho \) and \( A^\mu \) fields respectively, \( \nabla_\mu \) stands for covariant derivative and \( R \) is the Ricci scalar. We adopt the Lorenz gauge for the fields \( A_\mu, \omega_\mu, \) and \( \rho_\mu \). The self-interaction scalar field potential is \( U(\sigma), \) \( \psi_N \) is the nucleon isospin doublet, \( \psi_e \) is the electronic singlet, \( m_i \) states for the mass of each particle-specie and \( D_\mu = \partial_\mu + \Gamma_\mu, \) being \( \Gamma_\mu \) the Dirac spin connections. The conserved currents are \( J_\mu^\rho = \bar{\psi}_N \gamma^\rho \psi_N, \) \( J_\rho^\rho = \bar{\psi}_N \tau_3 \gamma^\rho \psi_N, \) \( J_\mu^\gamma_{\tau_3} = \bar{\psi}_e \gamma^\mu \psi_e, \) and \( J_\nu^\gamma_{\tau_3} = \bar{\psi}_N (1/2)(1 + \tau_3) \gamma^\nu \psi_N, \) being \( \tau_3 \) the particle isospin.

The nuclear model is fixed once the values of the coupling constants and the masses of the three mesons are fixed: for instance in the NL3 parameter set Lalazissis et al. (1997) used in Belvedere et al. (2012) and in this work we have \( m_\pi = 508.194 \text{ MeV}, m_\omega = 782.501 \text{ MeV}, \) \( m_\rho = 763.000 \text{ MeV}, \) \( g_\sigma = 10.2170, \) \( g_\omega = 12.8680, \) \( g_\rho = 4.4740, \) plus two constants that give the strength of the self-interactions, \( g_2 = -10.4310 \text{ fm}^{-1} \) and \( g_3 = -28.8850. \)

From the equations of motion of the above Lagrangian we obtain the EMTF equations (see Rueda et al. 2011; Belvedere et al. 2012 for details). The solution of the EMTF coupled differential equations leads to a new structure of the star, as shown in Fig 1: a positively charged core at supranuclear densities, \( \rho \) and \( A^\mu \) fields respectively, \( \nabla_\mu \) stands for covariant derivative and \( R \) is the Ricci scalar. We adopt the Lorenz gauge for the fields \( A_\mu, \omega_\mu, \) and \( \rho_\mu \). The self-interaction scalar field potential is \( U(\sigma), \) \( \psi_N \) is the nucleon isospin doublet, \( \psi_e \) is the electronic singlet, \( m_i \) states for the mass of each particle-specie and \( D_\mu = \partial_\mu + \Gamma_\mu, \) being \( \Gamma_\mu \) the Dirac spin connections. The conserved currents are \( J_\mu^\rho = \bar{\psi}_N \gamma^\rho \psi_N, \) \( J_\rho^\rho = \bar{\psi}_N \tau_3 \gamma^\rho \psi_N, \) \( J_\mu^\gamma_{\tau_3} = \bar{\psi}_e \gamma^\mu \psi_e, \) and \( J_\nu^\gamma_{\tau_3} = \bar{\psi}_N (1/2)(1 + \tau_3) \gamma^\nu \psi_N, \) being \( \tau_3 \) the particle isospin.

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From the equations of motion of the above Lagrangian we obtain the EMTF equations (see Rueda et al. 2011; Belvedere et al. 2012 for details). The solution of the EMTF coupled differential equations leads to a new structure of the star, as shown in Fig 1: a positively charged core at supranuclear densities, \( \rho > \rho_{\text{nuc}} \sim 2.7 \times 10^{34} \text{ g cm}^{-3} \), surrounded by an electron distribution of thickness \( \sim h/(m_e c^2) \) and, at lower densities \( \rho < \rho_{\text{nuc}}, \) a neutral ordinary crust.

![Figure 1: In the top and center panels we show the neutron, proton, electron densities and the electric field in units of the critical electric field \( E_c \) in the core-crust transition layer, whereas in the bottom panel we show a specific example of a density profile inside a neutron star. In this plot we have used for the globally neutral case a density at the edge of the crust equal to the neutron drip density, \( \rho_{\text{drip}} \sim 4.3 \times 10^{11} \text{ g cm}^{-3}. \) The thermodynamic equilibrium is ensured by the constancy of the particle Klein potentials \( \epsilon_{\text{Klein}} \) (1949) generalized to the presence of electrostatic and strong fields (Rotondo et al. 2011; Rueda et al. 2011; Belvedere et al., 2012).

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![Figure 2: Neutron star mass-radius relation in the static (non-rotating) case for both global and local charge neutrality configurations (see Belvedere et al. 2012 for details). In this plot we have used for the globally neutral case a density at the edge of the crust equal to the neutron drip density, \( \rho_{\text{drip}} \sim 4.3 \times 10^{11} \text{ g cm}^{-3}. \) We extend in this work the previous results to the case when the neutron star is rotating as a rigid body.]

\[ \frac{1}{u^i} \left[ \mu_i + \left( q_i A_0 + g_\rho \omega_\rho + g_\rho \tau_3 \rho_0 \right) u^0 \right] = \text{constant}, \quad (3) \]

where the subscript \( i \) stands for each kind of particle, \( \mu_i \) is the particle chemical potential, and \( q_i \) is the particle electric charge. In the static case only the time components of the vector fields, \( A_0, \omega_0, \rho_0 \) are present. In the above equation \( u^i = (g_{0i})^{-1/2} \) is the time component of the fluid four-velocity which satisfies \( u^i u_i = 1; \) \( g_{0i} \) is the \( \tau-t \) component of the spherically symmetric metric
this end we use the Hartle approach (Hartle 1967) which solves the Einstein equations accurately up to second order approximation in the angular velocity of the star, \( \Omega \) (see next section 2 for details).

In this rotating case, the condition of the constancy of the particle Klein potential has the same form as Eq. (3), but the fluid inside the star now moves with a four-velocity of a rigid rotating body, \( u^\alpha = (\bar{\nu} \bar{t}, 0, 0, \bar{u}^\phi) \), with (see Hartle and Sharp (1967) and Appendix A for details)

\[
u^t = (g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi})^{-1/2}, \quad u^\phi = \Omega u^t, \quad (5)
\]

where \( \phi \) is the azimuthal angular coordinate with respect to which the metric is symmetric, namely the metric is independent of \( \phi \) (axial symmetry). The metric functions \( g_{\alpha\beta} \) are now given by Eq. (3) below. It is then clear that in a frame comoving with the rotating star, \( u^\alpha = (\bar{g}_{tt})^{-1/2} \), and the Klein equilibrium condition becomes the same as Eq. (3), as expected.

We applied the Hartle formalism to the seed static solution obtained from the integration of the EMTF equations (Belvedere et al. 2012). For the construction of the new mass-radius relation we take into account the Keplerian mass-shedding limit and the secular axisymmetric instability (see section 3). We compute in section 4 the mass \( M \), polar \( R_p \) and equatorial \( R_q \) radii, and angular momentum \( J \), as a function of the central density and total mass of the neutron star mass and from the gravitational binding energy of equilibrium we calculate the maximum stable neutron star mass as

\[
\delta M = \frac{1}{2} \frac{\delta J}{\delta M},
\]

where \( \delta M \) is the quadrupole moment as

\[
M = M_0 + \delta M, \quad \delta M = m_0 (\bar{R}) + \frac{J^2}{\bar{R}^3}, \quad (13)
\]

The formalism of General Relativity. The solutions of the Einstein equations of slowly rotating stars in the context of General Relativity. The solutions of the Einstein equations are obtained through a perturbative method, expanding the metric functions up to the second order in the angular velocity \( \Omega \). Under this assumption the structure of compact objects can be approximately described by the total mass \( M \), angular momentum \( J \) and quadrupole moment \( Q \). The slow rotation regime implies that the perturbations owing to the rotation are relatively small with respect to the known non-rotating geometry. The interior solution is derived by solving numerically a system of ordinary differential equations for the perturbation functions. The exterior solution for the vacuum surrounding the star, can be written analytically in terms of \( M, J, \) and \( Q \) (see Hartle 1967, Hartle and Thorne 1968 for details). The numerical values for all the physical quantities are derived by matching the interior and the exterior solution on the border of the star.

The spacetime metric for the rotating configuration up to the second order of \( \Omega \) is given by Hartle (1967)

\[
ds^2 = e^{\nu} (1 + 2h) dt^2 - e^{\lambda} \left[ 1 + \frac{2m}{r - 2M_0} \right] dr^2 - r^2 (1 + 2k) \left[ dt^2 + \sin^2 \theta (d\phi - \omega dt)^2 \right]. \quad (6)
\]

where \( \nu = \nu(r) \), \( \lambda = \lambda(r) \), and \( M_0 = M^{J=0} \) are the metric functions and mass profiles of the corresponding seed non-rotating star with the same central density as the rotating one; see Eq. (4). The functions \( h = h(r, \theta) \), \( m = m(r, \theta) \), \( k = k(r, \theta) \) and the fluid angular velocity in the local inertial frame, \( \omega = \omega(r) \), have to be calculated from the Einstein equations. Expanding up to the second order the metric in spherical harmonics we have

\[
h(r, \theta) = \bar{h}(r) + h_2(r) P_2(\cos \theta), \quad (7)
\]

\[
m(r, \theta) = \bar{m}(r) + m_2(r) P_2(\cos \theta), \quad (8)
\]

\[
k(r, \theta) = \bar{k}(r) + k_2(r) P_2(\cos \theta), \quad (9)
\]

where \( P_2(\cos \theta) \) is the Legendre polynomial of second order. Because the metric does not change under transformations of the type \( r \to f(r) \), we can assume \( k_0(r) = 0 \).

The functions \( h = h(r, \theta) \), \( m = m(r, \theta) \), \( k = k(r, \theta) \) have analytic form in the exterior (vacuum) spacetime and they can be found in Appendix A. The mass, angular momentum, and quadrupole moment are computed from the matching condition between the interior and exterior metrics.

First the angular momentum is computed. It is introduced the angular velocity of the fluid relative to the local inertial frame, \( \bar{\omega}(r) = \Omega - \omega(r) \). It can be shown from the Einstein equations at first order in \( \Omega \) that \( \bar{\omega} \) satisfies the differential equation

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^4 \frac{d\bar{\omega}}{dr} \right) + \frac{4}{r^2} \frac{dj}{dr} \bar{\omega} = 0, \quad (10)
\]

where \( j(r) = e^{-(\nu + \lambda)/2} \) with \( \nu \) and \( \lambda \) the metric functions of the seed non-rotating solution (4).

From the matching equations, the angular momentum of the star results to be given by

\[
J = \frac{1}{6} R^4 \left( \frac{d\bar{\omega}}{dr} \right)_{r=R}, \quad (11)
\]

so the angular velocity \( \Omega \) is related to the angular momentum as

\[
\Omega = \bar{\omega}(R) + \frac{2J}{R^3}. \quad (12)
\]

The total mass of the rotating star, \( M \), is given by

\[
M = M_0 + \delta M, \quad \delta M = m_0 (\bar{R}) + \frac{J^2}{\bar{R}^3}, \quad (13)
\]
where $\delta M$ is the contribution to the mass owing to rotation. The second order functions $m_0$ and $p_0$ (related to the pressure perturbation) are computed from the solution of the differential equation

$$\frac{dn_0}{dr} = 4\pi r^2 \frac{dP}{dP}(E + P)p_0 + \frac{1}{12} r^2 r^4 \left( \frac{d\omega}{dr} \right)^2 - \frac{1}{3} \frac{d^2}{dr^2} r^3 \omega^2,$$

$$\frac{dp_0}{dr} = -\frac{m_0(1 + 8\pi r^2 P)}{(r - 2M_0)^2} - \frac{4\pi r^2 (E + P)}{(r - 2M_0)^2} p_0^* + \frac{1}{12} \frac{d^2}{dr^2} \left( \frac{d\omega}{dr} \right)^2 + \frac{1}{3} \frac{d}{dr} \left( \frac{r^3 j^2 \omega^2}{r - 2M_0} \right),$$

where $E$ and $P$ are the total energy-density and pressure.

Turning to the quadrupole moment of the neutron star, it is given by

$$Q = \frac{j^2}{M_0} + \frac{8}{5} KM_0^3,$$

where $K$ is a constant of integration. This constant is fixed from the matching of the second order function $h_2$ obtained in the interior from

$$\frac{dk_2}{dr} = -\frac{dh_2}{dr} + h_2 \frac{d\nu}{dr} + \left( \frac{1}{r} + \frac{1}{2} \frac{d\nu}{dr} \right) \left[ -\frac{1}{3} r^3 \omega^2 \frac{d^2}{dr^2} \right],$$

$$\frac{dh_2}{dr} = h_2 \left[ -\frac{dv}{dr} + \frac{r}{r - 2M_0} \left( \frac{dv}{dr} \right)^{-1} \right] \left[ 8\pi (E + P) - \frac{4M_0}{r^3} \right] - \frac{4(k_2 + h_2)}{r(r - 2M_0)} \left( \frac{dv}{dr} \right)^{-1} + \frac{1}{6} \left[ \frac{r}{2} \frac{dv}{dr} - \frac{1}{r - 2M_0} \left( \frac{dv}{dr} \right)^{-1} \right] r^3 \omega^2 \left( \frac{d\omega}{dr} \right)^2 - \frac{1}{3} \frac{r}{2} \frac{dv}{dr} + \frac{1}{r - 2M_0} \left( \frac{dv}{dr} \right)^{-1} r^2 \omega^2 \frac{d^2}{dr^2} \right],$$

with its exterior counterpart (see Hartle (1967) and Appendix A).

It is worth to underline that the influence of the induced magnetic field owing to the rotation of the charged core of the neutron star in the globally neutral case is negligible (Boshkayev et al., 2012). In fact, for a rotating neutron star of period $P = 10$ ms and radius $R \sim 10$ km, the radial component of the magnetic field $B_r$ in the core interior reaches its maximum at the poles with a value $B_r \sim 2.9 \times 10^{-16} B_c$, where $B_c = m_e c^2 / (e \hbar) \approx 4.4 \times 10^{13}$ G is the critical magnetic field for vacuum polarization. The angular component of the magnetic field $B_\theta$, instead, has its maximum value at the equator and, as for the radial component, it is very low in the interior of the neutron star core, i.e. $|B_\theta| \sim 2.9 \times 10^{-10} B_c$. In the case of a sharp core-crust transition as the one studied by Belvedere et al. (2012) and shown in Fig. 1, this component will grow in the transition layer to values of the order of $|B_\theta| \sim 10^2 B_c$ (see Boshkayev et al., 20121, for further details). However, since we are here interested in the macroscopic properties of the neutron star, we can ignore at first approximation the presence of electromagnetic fields in the macroscopic regions where they are indeed very small, and safely apply the original Hartle formulation without any generalization.

### 3. Stability of uniformly rotating neutron stars

#### 3.1. Secular axisymmetric instability

In a sequence of increasing central density in the $M-\rho_c$ curve, $\rho_c \equiv \rho(0)$, the maximum mass of a non-rotating neutron star is defined as the first maximum of such a curve, namely the point where $\partial M / \partial \rho_c = 0$. This derivative defines the secular instability point, and, if the perturbation obeys the same equation of state (EOS) as the equilibrium configuration, it coincides also with the dynamical instability point (see e.g. Shapiro and Teukolsky, 1983). In the rotating case, the situation becomes more complicated and in order to find the axisymmetric dynamical instability points, the perturbed solutions with zero frequency modes (the so-called neutral frequency line) have to be calculated. Friedman et al. (1988) however, following the works of Sorkin (1981, 1982), described a turning-point method to obtain the points at which secular instability is reached by uniformly rotating stars. In a constant angular momentum sequence, the turning point is located in the maximum of the mass-central density relation, namely the onset of secular axisymmetric instability is given by

$$\left( \frac{\partial M}{\partial \rho_c} \right)_{J=\text{constant}} = 0,$$

and once the secular instability sets in, the star evolves quasi-stationarily until it reaches a point of dynamical instability where gravitational collapse sets in (Stergioulas, 2003).

The above equation defines an upper limit for the mass at a given $J$ for a uniformly rotating star, however this criterion is a sufficient but not necessary condition for the instability. This means that all the configurations with the given angular momentum $J$ on the right side of the turning point defined by Eq. (19) are secularly unstable, but it does not imply that the configurations on the left side of it are stable. An example of dynamically unstable configurations on the left side of the turning-point limiting boundary in neutron stars was recently shown in (Takami et al., 2011), for a specific EOS.

#### 3.2. Keplerian mass-shedding instability

The maximum velocity for a particle to remain in equilibrium on the equator of a star, kept bound by the balance between gravitational and centrifugal force, is the Keplerian velocity of a free particle computed at the same location. As shown, for instance in (Stergioulas, 2003), a star rotating at Keplerian rate becomes unstable due to
the loss of mass from its surface. The mass shedding limiting angular velocity of a rotating star is the Keplerian angular velocity evaluated at the equator, \( r = R_{\text{eq}} \), i.e. \( \Omega_K^{J=0} = \Omega_K(r = R_{\text{eq}}) \). Friedman et al. (1986) introduced a method to obtain the maximum possible angular velocity of the star before reaching the mass-shedding limit; however Torok et al. (2008) and Bini et al. (2013), showed a simpler way to compute the Keplerian angular velocity of a rotating star. They showed that the mass-shedding angular velocity, \( \Omega_K^{J=0} \), can be computed as the orbital angular velocity of a test particle in the external field of the star and corotating with it on its equatorial plane at the distance \( r = R_{\text{eq}} \). For the Hartle external solution, this is given by

\[
\Omega_K^{J=0}(r) = \sqrt{\frac{M}{r^3}} \left[ 1 - j F_1(r) + j^2 F_2(r) + q F_3(r) \right],
\]

where \( j = J/M^2 \) and \( q = Q/M^3 \) are the dimensionless angular momentum and quadrupole moment. Further details and the analytical expression of the functions \( F_i \) can be found in Appendix A.

3.3. Gravitational binding energy

Besides the above stability requirements, one should check if the neutron star is gravitationally bound. In the non-rotating case, the binding energy of the star can be computed as

\[
W_{J=0} = M_0 - M_{\text{rest}}^0, \quad M_{\text{rest}}^0 = m_b A_{J=0},
\]

where \( M_{\text{rest}}^0 \) is the rest-mass of the star, \( m_b \) is the rest-mass per baryon, and \( A_{J=0} \) is the total number of baryons inside the star. So the non-rotating star is considered bound if \( W_{J=0} < 0 \).

In the slow rotation approximation the total binding energy is given by (Hartle and Thorne, 1968)

\[
W_{J\neq0} = W_{J=0} + \delta W, \quad \delta W = \frac{J^2}{R^3} - \int_0^R 4\pi r^2 B(r) dr,
\]

where

\[
B(r) = (E + P) p_0 \left\{ \frac{dE}{dP} \left[ \left( 1 - \frac{2M}{r} \right)^{-1/2} - 1 \right] - \frac{dP}{dP} \left( 1 - \frac{2M}{r} \right)^{-1/2} + (E - u) \left( 1 - \frac{2M}{r} \right)^{-3/2} \left[ \frac{m_0}{r} \right] \right. + \left. \frac{1}{3} \right. \frac{J^2}{r^2 \omega^2} - \frac{1}{4\pi r^2} \left( \frac{d\omega}{dr} \right)^2 - \frac{1}{3} \frac{dj^2}{dr^2} \left[ \frac{3}{4} \right. \left. \omega^2 \right],
\]

where \( u = E - m_b n_b \) is the internal energy of the star, with \( n_b \) the baryon number density.

We will therefore request that the binding energy be negative, namely \( W_{J\neq0} < 0 \). As we will show below in section 4.2.2 this condition leads to a minimum mass for the neutron star under which the star becomes gravitationally unbound.

4. Structure of uniformly rotating neutron stars

We show now the results of the integration of the Hartle equations for the globally and locally charge neutrality neutron stars; see e.g. Fig. 1. Following Belvedere et al. (2012), we adopt, as an example, globally neutral neutron stars with a density at the edge of the crust equal to the neutron drip density, \( \rho_{\text{crust}} = \rho_{\text{drip}} \approx 4.3 \times 10^{11} \text{ g cm}^{-3} \).

4.1. Secular instability boundary

In Fig. 3 we show the mass-central density curve for globally neutral neutron stars in the region close to the axisymmetric stability boundaries. Specifically we show some J-constant sequences to show that indeed along each of these curves there exist a maximum mass point (turning point). The line joining all the turning points defines the secular instability limit. In Fig. 3 the axisymmetric stable zone is on the left side of the instability line.

![Figure 3: Total mass versus central density of globally neutral neutron stars. The solid line represents the configuration with Keplerian angular velocity, the dashed line represents the static configuration, the dotted-dashed lines represent the J-constant sequences (in units of 10^{11} cm^2). The gray line joins all the turning points of the J-constant sequences, so it defines the secular instability boundary.](image-url)

Clearly we can transform the mass-central density relation in a mass-radius relation. In Fig. 4 we show the mass versus the equatorial radius of the neutron star that correspond to the range of densities of Fig. 3. In this plot the stable zone is on the right side of the instability line.

We can construct a fitting curve joining the turning points of the J-constant sequences line which determines the secular axisymmetric instability boundary. Defining \( M_{\text{max,0}} \) as the maximum stable mass of the non-rotating neutron star constructed with the same EOS, we find that for globally neutral configurations the instability line is well fitted by the function

\[
\frac{M_{\text{GCN}}^{\text{sec}}}{M_{\odot}} = 21.22 - 6.68 \left( \frac{M_{\text{GCN}}^{\text{max,0}}}{M_{\odot}} \right) - \left( \frac{77.42 - 28 \frac{M_{\text{GCN}}^{\text{max,0}}}{M_{\odot}}}{10 \text{ km}} \right)^{-6.08},
\]

where 12.38 km \( \lesssim R_{\text{eq}} \lesssim 12.66 \) km, and \( M_{\text{GCN}}^{\text{max,0}} \approx 2.67 M_{\odot} \).
The turning points of locally neutral configurations in the mass-central density plane are shown in Fig. 5, the corresponding mass-equatorial radius plane is plotted in Fig. 6.

For locally neutral neutron stars, the secular instability line is fitted by

\[
\frac{M_{\text{sec}}^{\text{LCN}}}{M_\odot} = 20.51 - 6.35 \frac{M_{\text{max},0}^{\text{LCN}}}{M_\odot} - \left(80.98 - 29.02 \frac{M_{\text{max},0}^{\text{LCN}}}{M_\odot}\right) \left(\frac{R_{\text{eq}}}{10 \text{ km}}\right)^{-5.71},
\]

where \(12.71 \text{ km} \lesssim R_{\text{eq}} \lesssim 13.06 \text{ km}\), and \(M_{\text{max},0}^{\text{LCN}} \approx 2.70 M_\odot\).

4.2. Keplerian mass-shedding sequence

We turn now to analyze in detail the behavior of the different properties of the neutron star along the Keplerian mass-shedding sequence. For the sake of reference we have indicated in the following plots stars with the selected masses \(M \approx [1, 1.4, 2.04, 2.5] M_\odot\). The cyan star indicates the fastest observed pulsar, PSR J1748–2446ad (Hessels et al. 2006), with a rotation frequency of \(f \approx 716 \text{ Hz}\). The gray filled circles indicate the last stable configuration of the Keplerian sequence, namely the point where the Keplerian and the secular stability boundaries cross each other.

4.2.1. Maximum mass and rotation frequency

The total mass of the rotating star is computed from Eq. (13). In Fig. 7 is shown the total mass of the neutron star as a function of the rotation frequency for the Keplerian sequence. It is clear that for a given mass, the rotational frequency is higher for a globally neutral neutron star with respect to the locally neutral one.

The configuration of maximum mass, \(M_{\text{max}}^{J=0}\), occurs along the Keplerian sequence, and it is found before the secular instability line crosses the Keplerian curve. Thus,
the maximum mass configuration is secularly stable. This implies that the configuration with maximum rotation frequency, \( f_{\text{max}} \), is located beyond the maximum mass point, specifically at the crossing point between the secular instability and the Keplerian mass-shedding sequence. The results are summarized in Table 1.

It is important to discuss briefly the validity of the present perturbative solution for the computation of the properties of maximally rotating neutron stars. The expansion of the radial coordinate of a rotating configuration \( r(R, \theta) \) in powers of angular velocity is written as [Hartle, 1967]

\[
r = R + \xi(R, \theta) + O(\Omega^4),
\]

where \( \xi \) is the difference in the radial coordinate, \( r \), between a point located at the polar angle \( \theta \) on the surface of constant density \( \rho(R) \) in the rotating configuration, and the point located at the same polar angle on the same constant density surface in the non-rotating configuration. In the slow rotation regime, the fractional displacement of the surfaces of constant density due to the rotation have to be small, namely \( \xi(R, \theta)/R \ll 1 \), where \( \xi(R, \theta) = \xi_0(R) + \xi_2(R)P_2(\cos \theta) \) and \( \xi_0(R) \) and \( \xi_2(R) \) are function of \( R \), proportional to \( \Omega^2 \). From Table 1 we can see that the configuration with the maximum possible rotation frequency has a maximum fractional displacement \( \delta R_{\text{max}} = \xi(R, \pi/2)/R \) as low as \( \approx 2\% \) and \( \approx 3\% \), for the globally and locally neutral neutron stars respectively.

In this line, it is worth to quote the results of Benhar et al. [2005], who showed that the inclusion of a third-order expansion \( \Omega^3 \) in the Hartle’s method improves the value of the maximum rotation frequency by less than 1\% for different EOS. The reason for this is that as mentioned above, along the Keplerian sequence the deviations from sphericity decrease with density and frequency (see Figs. 15 and 17), which ensures the accuracy of the perturbative solution.

Turning to the increase of the maximum mass, Weber and Glendenning [1992] showed that the mass of maximally rotating neutron stars, computed with the Hartle’s second order approximation, is accurate within an error as low as \( \lesssim 4\% \).

### 4.2.2. Minimum mass and rotation frequency

We compute now the gravitational binding energy of the neutron star from Eq. (22) as a function of the central density and angular velocity. We make this for central densities higher than the nuclear density, thus we impose the neutron star to have a supranuclear hadronic core. In Fig. 8 we plot the binding energy \( W \) of the neutron star as a function of the neutron star mass along the Keplerian sequence. For the sake of comparison we show also the binding energy of the non-rotating configurations.

We found that the globally neutral neutron stars studied here are bound up to some minimum mass at which the gravitational binding energy vanishes. For the static and Keplerian configurations we find that \( W_{J=0} = 0 \), and

| Global Neutrality | Local Neutrality |
|-------------------|------------------|
| \( M_{\text{max}}^{J=0} (M_\odot) \) | 2.67 | 2.70 |
| \( R_{\text{max}}^{J=0} (\text{km}) \) | 12.38 | 12.71 |
| \( M_{\text{max}}^{J\neq0} (M_\odot) \) | 2.76 | 2.79 |
| \( R_{\text{max}}^{J\neq0} (\text{km}) \) | 12.66 | 13.06 |
| \( \delta M_{\text{max}} \) | 3.37% | 3.33% |
| \( \delta R_{\text{max}} \) | 2.26% | 2.75% |
| \( f_{\text{max}} \) (kHz) | 1.97 | 1.89 |
| \( P_{\text{min}} \) (ms) | 0.51 | 0.53 |

Table 1: \( M_{\text{max}}^{J=0} \) and \( R_{\text{max}}^{J=0} \): maximum mass and corresponding radius of non-rotating stars as computed in [Belvedere et al. 2012]; \( M_{\text{max}}^{J\neq0} \) and \( R_{\text{max}}^{J\neq0} \): maximum mass and corresponding radius of rotating stars; \( \delta M_{\text{max}} \) and \( \delta R_{\text{max}} \): increase in mass and radius of the maximum mass configuration with respect to its non-rotating counterpart; \( f_{\text{max}} \) and \( P_{\text{min}} \): maximum rotation frequency and associated minimum period.

![Figure 8: Neutron star binding energy versus total mass along the Keplerian sequence both for the global (red) and local (blue) charge neutrality.](image_url)
where with the superscript $K$ we indicate that this value corresponds to the minimum mass on the Keplerian sequence. Clearly this minimum mass value decreases with decreasing frequency until it reaches the above value $M_{\text{min}}^{J=0}$ of the non-rotating case.

We did not find any unbound configuration in the local charge neutrality case for the present EOS (see Fig. 8). The corresponding plot of $W$ as a function of the central density is shown in Fig. 9.

The configuration with the minimum mass, $M_{\text{min}}^{K} \approx 0.167 M_{\odot}$, has a rotation frequency

$$f_{\text{min}}^{K} = f(M_{\text{min}}^{K}) \approx 700.59 \text{ Hz} ,$$

that is the minimum rotation rate that globally neutral configurations can have along the Keplerian sequence in order to be gravitationally bound. Interestingly, the above value is slightly lower than the frequency of the fastest observed pulsar, PSR J1748–2446ad, which has a frequency of 716 Hz [Hessels et al. (2006)]. Further discussions on this issue are given below in section 8.

In Fig. 10 we show in detail the dependence of $W$ on the rotation frequency.

### 5. Neutron star mass-radius relation

We summarize now the above results in form of a new mass-radius relation of uniformly rotating neutron stars, including the Keplerian and secular instability boundary limits. In Fig. 11 we show a summary plot of the equilibrium configurations of rotating neutron stars. In particular we show the total mass versus the equatorial radius: the dashed lines represent the static (non-rotating, $J = 0$) sequences, while the solid lines represent the corresponding Keplerian mass-shedding sequences. The secular instability boundaries are plotted in pink-red and light blue color.
for the global and local charge neutrality cases, respectively.

It can be seen that due to the deformation for a given mass the radius of the rotating case is larger than the static one, and similarly the mass of the rotating star is larger than the corresponding static one. It can be also seen that the configurations obeying global charge neutrality are more compact with respect to the ones satisfying local charge neutrality.

6. Moment of inertia

The neutron star moment of inertia $I$ can be computed from the relation

$$ I = \frac{J}{\Omega}, \quad (29) $$

where $J$ is the angular momentum and $\Omega$ are related via Eq. (12). Since $J$ is a first-order quantity and so proportional to $\Omega$, the moment of inertia given by Eq. (29) does not depend on the angular velocity and does not take into account deviations from the spherical symmetry. This implies that Eq. (11) gives the moment of inertia of the non-rotating unperturbed seed object. In order to find the perturbation to $I$, say $\delta I$, the perturbative treatment has to be extended to the next order $\Omega^3$, in such a way that $I = I_0 + \delta I = (J_0 + \delta J)/\Omega$, becomes of order $\Omega^5$, with $\delta J$ of order $\Omega^3$ (see e.g. Hartle, 1973; Benhar et al., 2005). In this work we keep the solution up to second order and therefore we proceed to analyze the behavior of the moment of inertia for the non-rotating configurations. In any case, as we will show in section 6.1 even the fastest observed pulsars rotate at frequencies much lower than the Keplerian rate, and under such conditions we expect that the moment of inertia can be approximated with high accuracy by the one of the corresponding static configurations.

In Figs. 12 and 13 we show the behavior of the total momentum of inertia, i.e. $I = I_{\text{core}} + I_{\text{crust}}$, with respect to the total mass and central density for both globally and locally neutral non-rotating neutron stars.

![Figure 12: Total moment of inertia versus total mass both for globally (red) and locally (blue) neutral non-rotating neutron stars.](image)

![Figure 13: Total moment of inertia versus central density for globally (red) and locally (blue) neutral non-rotating neutron stars.](image)

6.1. Core and crust moment of inertia

In order to study the single contribution of the core and the crust to the moment of inertia of the neutron star, we shall use the integral expression for the moment of inertia. Multiplying Eq. (10) by $r^5$ and making the integral of it we obtain

$$ I(r) = -\frac{2}{3} \int_0^r r^3 \frac{d\bar{\omega}(r)}{dr} \frac{dr}{\Omega} = \frac{8\pi}{3} \int_0^r r^4 (E + P) e^{(\lambda - \nu)/2} \bar{\omega}(r) \frac{dr}{\Omega}, \quad (30) $$

where the integration is carried out in the region of interest. Thus, the contribution of the core, $I_{\text{core}}$, is obtained integrating from the origin up to the radius of the core, and the contribution of the crust, $I_{\text{crust}}$, integrating from the base of the crust to the total radius of the neutron star.

We show in Figs. 14 and 15 the ratio between the moment of inertia of the crust and the one of the core as a function of the total mass and central density, respectively, for both the globally and locally neutral configurations.

7. Deformation of the neutron star

In this section we explore the deformation properties of the neutron star. The behavior of the eccentricity, the rotational to gravitational energy ratio, as well as the quadrupole moment, are investigated as a function of the mass, density, and rotation frequency of the neutron star.

---

2 It is clear that this expression approaches, in the weak field limit, the classic Newtonian expression $I_{\text{Newtonian}} = (8\pi/3) \int r^4 \rho dr$ where $\rho$ is the mass-density (Hartle, 1967).
7.1. Eccentricity

A measurement of the level of deformation of the neutron star can be estimated with the eccentricity

$$\epsilon = \sqrt{1 - \left(\frac{R_p}{R_{eq}}\right)^2},$$

(31)

where $R_p$ and $R_{eq}$ are the polar and equatorial radii of the configuration. Thus, $\epsilon = 0$ defines the spherical limit and $0 < \epsilon < 1$ corresponds to oblate configurations.

In Fig. 16 we show the behavior of the total eccentricity (31), as a function of the neutron star frequency.

We can see that in general the globally neutral neutron star has an eccentricity larger than the one of the locally neutral configuration for almost the entire range of frequencies and the corresponding central densities, except for the low frequencies $f \lesssim 0.8$ kHz and central densities $\rho(0) \lesssim 1.3\rho_{\text{nuc}}$; see also Fig. 17. Starting from low values of the frequency $f$ and central density $\rho(0)$, the neutron stars increase their oblateness, and after reaching the maximum value of the eccentricity, the compactness increases and the configurations tend to a more spherical shape.

7.2. Rotational to gravitational energy ratio

Other property of the star related to the centrifugal deformation of the star is the ratio between the gravitational energy and the rotational energy of the star. The former is given by Eq. (22), whereas the latter is

$$T = \frac{1}{2} I \Omega^2,$$

(32)

We show in Fig. 18 the ratio $T/|W|$ as a function of the mass of the neutron stars along the Keplerian sequence. In Fig. 19 instead we plot the dependence of the ratio on the central density and in Fig. 20 on the Keplerian frequency.
7.3. Quadrupole moment

In Figs. 21 and 22 we show the quadrupole moment, \( Q \) given by Eq. (16), as a function of the total mass and central density for both globally and locally neutral neutron stars along the Keplerian sequence. The dependence of \( Q \) on the rotation frequency is shown in Fig. 23. We have normalized the quadrupole moment \( Q \) to the quantity \( MR^2 \) of the non-rotating configuration with the same central density.

8. Observational constraints

In Fig. 24 we show the above mass-radius relations together with the most recent and stringent constraints indicated by [Trümper (2011)]:

1) The largest mass. Until 2013 it was given by the mass of the 3.15 millisecond pulsar PSR J1614-2230 \( M = 1.97 \pm 0.04 M_\odot \) [Demorest et al. (2010)], however the recent reported mass 2.01\( \pm 0.04 M_\odot \) for the neutron star in the relativistic binary PSR J0348+0432 [Antoniadis et al. (2013)]
puts an even more stringent request to the nuclear EOS. Thus, the maximum mass of the neutron star has to be larger than the mass of PSR J0348+0432, this constraint is represented by the orange-color stars in Fig. 23.

2) The largest radius. It is given by the lower limit to the radius of RX J1856-3754. The lower limit to the radius as seen by an observer at infinity is $R_\infty = R[1 - 2GM/c^2R]^{-1/2} > 16.8$ km, as given by the fit of the optical and X-ray spectra of the source Trümper et al. (2004); so in the mass-radius relation this constraint reads $2GM/c^2 > R - R^3/(R_\infty^3)$, with $R_\infty = 16.8$ km. We represent this constraint with the dotted-dashed curve in Fig. 23.

3) The maximum surface gravity. Using a neutron star of $M = 1.4M_\odot$ to fit the Chandra data of the low-mass X-ray binary X7, it turns out that the radius of the star satisfies at 90% confidence level, $R = 14.5^{+1.8}_{-1.5}$ km, which gives $R_\infty = [15.64, 18.86]$ km, respectively Heinke et al. (2006). Using the same formula as before, $2GM/c^2 > R - R^3/(R_\infty^3)$, we obtain the dotted curves shown in Fig. 23.

4) The highest rotation frequency. The fastest observed pulsar is PSR J1748–2446ad with a frequency of 716 Hz Hessels et al. (2006). We show the constant rotation frequency sequence $f = 716$ Hz for both globally (dashed pink) and locally (dashed light blue) neutral neutron stars. We indicated with cyan-color stars the point where these curves cross the corresponding Keplerian sequences in the two cases (see Fig. 23).

Every $f$-constant sequence crosses the stability region of the objects in two points: these crossing points define the minimum and maximum possible mass that an object rotating with such a frequency may have in order to be stable. In the case of PSR J1748-2446ad, the cut of the $f = 716$ Hz constant sequence with the Keplerian curve establishes the minimum mass of this pulsar. We find that its minimum mass is $\approx 0.175 M_\odot$, which is slightly lower than the frequency of PSR J1748-2446ad. It would imply that PSR J1748-2446ad is very likely rotating at a rate much lower than the Keplerian one.

It is interesting that the above minimum mass, given by its constant rotation frequency sequence, is slightly larger than the minimum mass for bound configurations on the Keplerian sequence, $M_{\text{min}}^K \approx 0.167 M_\odot$; see Eq. (24). In fact, as we shown in Eq. (25) the minimum rotation frequency along the Keplerian sequence for bound configurations in the globally neutral case is, $f_{\text{min}}^K \approx 700.59$ Hz, which is slightly lower than the frequency of PSR J1748-2446ad. It would imply that PSR J1748-2446ad is very likely rotating at a rate much lower than the Keplerian one.

Similarly to what presented in Belvedere et al. (2012) for the static neutron stars and introduced by Trümper (2011), the above observational constraints show a prefer-
ence on stiff EOS that provide highest maximum masses for neutron stars. Taking into account the above constraints, the radius of a canonical neutron star of mass $M = 1.4M\odot$ is strongly constrained to $R \geq 12$ km, disfavoring at the same time strange quark matter stars. It is evident from Fig. 24 that mass-radius relations for both the static and the rotating case presented here, are consistent with all the observational constraints. In Table 2 we show the radii predicted by our mass-radius relation both for the static and the rotating case for a canonical neutron star as well as for the most massive neutron stars discovered, namely, the millisecond pulsar PSR J1614–2230 (Demorest et al. 2010), $M = 1.97 \pm 0.04M\odot$, and PSR J0348+0432 (Antoniadis et al. 2013), $M = 2.01 \pm 0.04M\odot$. These configurations are computed under the constraint of global charge neutrality and for a density at the edge of the crust equal to the neutron drip density. The nuclear parameterizations NL3 has been used.

| $M(M\odot)$ | $R_{J=0}^{\text{eq}}$ (km) | $R_{\text{eq}}^{J=0}$ (km) |
|------------|----------------|----------------|
| 1.4        | 12.313         | 13.943         |
| 1.97       | 12.991         | 14.104         |
| 2.01       | 13.020         | 14.097         |

Table 2: Radii for a canonical neutron star of $M = 1.4M\odot$ and for PSR J1614–2230 (Demorest et al. 2010), $M = 1.97 \pm 0.04M\odot$, and PSR J0348+0432 (Antoniadis et al. 2013), $M = 2.01 \pm 0.04M\odot$. These configurations are computed under the constraint of global charge neutrality and for a density at the edge of the crust equal to the neutron drip density.

9. Concluding remarks

We have constructed equilibrium configurations of uniformly rotating neutron stars in both the global charge neutrality and local charge neutrality cases, generalizing our previous work (Belvedere et al. 2012). To do this we have applied the Hartle method to the seed static solution obtained from the integration of the Einstein-Maxwell-Thomas-Fermi equations (Belvedere et al. 2012). We calculated the mass, polar and equatorial radii, angular momentum, moment of inertia, quadrupole moment, and eccentricity, as functions of the central density and the rotation angular velocity of the neutron star.

The Keplerian mass-shedding limit and the secular axisymmetric instability have been analyzed for the construction of the region of stability of rotating neutron stars. We have given fitting curves of the secular instability boundary in Eqs. (24) and (25) for global and local charge neutrality, respectively. With this analysis we have established in section 4.2.2 the maximum mass and maximum rotation frequency of the neutron star. We computed in section 4.2.2 the gravitational binding energy of the configurations as a function of the central density and rotation rate. We did this for central densities higher than the nuclear one, so imposing that the neutron star has a supranuclear hadronic core. We found that there is a minimum mass under which the neutron star becomes gravitationally unbound. Along the Keplerian sequence, to this minimum mass object we associate a minimum frequency under which an object rotating at the Keplerian rate becomes unbound; see Eq. (28). We found that locally neutral neutron stars with supranuclear cores remained always bound for the present EOS. In Table 3 we summarize all these results.

| $M_{\text{max}}(M\odot)$ | 2.67 | 2.70 |
| $M_{\text{max}}(M\odot)$ | 2.76 | 2.79 |
| $j_{\text{max}}$(kHz) | 1.97 | 1.89 |
| $P_{\text{min}}$(ms) | 0.51 | 0.53 |
| $M_{\text{max}}^{J=0}(M\odot)$ | 0.18 | – |
| $M_{\text{max}}^{J=0}(M\odot)$ | 0.17 | – |
| $f_{\text{K}}$(kHz) | 0.70 | – |

Table 3: Maximum mass, maximum frequency, minimum period, minimum mass of globally and locally neutral neutron stars.

Figure 24: Observational constraints on the mass-radius relation given by Trümper (2011), and the theoretical mass-radius relation presented in this work in Fig. 14. The red lines represent the configuration with global charge neutrality, while the blue lines represent the configuration with local charge neutrality. The pink-red line and the light-blue line represent the secular axisymmetric stability boundaries for the globally neutral and the locally neutral case, respectively. The red and blue solid lines represent the Keplerian sequences and the red and blue dashed lines represent the static cases presented in Belvedere et al. (2012).
recent measurement of the mass PSR J0348+0432, $M = 2.01 \pm 0.04 M_\odot$ (Antoniadis et al., 2013), favors stiff nuclear EOS as the one used here.

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Benhar, O., Ferrari, V., Gualtieri, L., Marassi, S., Aug. 2005. Hartle, J. B., Sharp, D. H., and the functions $\Omega$, are derived from the Einstein equations (see Hartle, 1967; Hartle and Thorne, 1968, for details). Following this
prescriptions the eq. become:

\[ ds^2 = \left(1 - \frac{2M}{r} \right) \left[1 + 2k_1 P_2(\cos \theta) \right. \]
\[ + 2 \left(1 - \frac{2M}{r} \right)^{-1} \frac{J^2}{r^4} (2 \cos^2 \theta - 1) \right] dt^2 \]
\[ + \frac{4J}{r} \sin^2 \theta dtd\phi - \left(1 - \frac{2M}{r} \right)^{-1} \]
\[ \times \left[1 - 2 \left(1 - \frac{2M}{r} \right)^{-1} \frac{J^2}{r^4} \right] dr^2 \]
\[ - r^2[1 - 2k_2 P_2(\cos \theta)](d\theta^2 + \sin^2 \theta d\phi^2), \]  
\[ (A.1) \]

where

\[ k_1 = \frac{J^2}{M^3} \left(1 + \frac{M}{r} \right) + \frac{5Q - J^2/M}{8 M^3} Q^2_1(x), \]
\[ k_2 = k_1 + \frac{J^2}{4M^2r^3} \sqrt{1 - 2M/r} Q^2_1(x), \]

and

\[ Q^2_1(x) = (x^2 - 1)^{1/2} \left[ \frac{3x}{2} \ln \left(\frac{x + 1}{x - 1} \right) - \frac{3x^2 - 2}{x^2 - 1} \right], \]
\[ Q^2_2(x) = (x^2 - 1) \left[ \frac{3}{2} \ln \left(\frac{x + 1}{x - 1} \right) - \frac{3x^3 - 5x}{(x^2 - 1)^2} \right], \]

are the associated Legendre functions of the second kind, being \( P_2(\cos \theta) = (1/2)(3 \cos^2 \theta - 1) \) the Legendre polynomial, and where it has been effectuated the re-scaling \( x = r/M - 1 \). The constants \( M, J \) and \( Q \) are the total mass, angular momentum and mass quadrupole moment of the rotating object, respectively. This form of the metric corrects some misprints of the original paper by Hartle and Thorne [1968] (see also Berti et al. [2005]; Boskayev et al. [2012]). To obtain the exact numerical values of \( M, J \) and \( Q \), the exterior and interior metrics have to be matched at the surface of the star. It is worthy underline that in the terms involving \( J^2 \) and \( Q \), the total mass \( M \) can be substituted by \( M - \delta M \) since \( \delta M \) is already a second order term in the angular velocity.

**Appendix A.2. Angular velocity of equatorial circular orbits**

It is possible to obtain the analytical expression for the angular velocity \( \Omega \) given by Eq. (20) with respect to an observer at infinity, taking into account the parameterization of the four-velocity \( u \) of a test particle on a circular orbit in equatorial plane of axisymmetric stationary spacetime, regarding as parameter the angular velocity \( \Omega \) itself:

\[ u = \Gamma [\partial_t + \Omega \partial_\phi], \]  
\[ (A.2) \]

where \( \Gamma \) is a normalization factor such that \( u^a u_a = 1 \). Normalizing and applying the geodesics conditions we get the following expressions for \( \Gamma \) and \( \Omega = u^\phi/u^t \)

\[ \Gamma \pm (g_{\phi\phi} + 2 \Omega g_{\phi t} + \Omega^2 g_{tt})^{-1/2}, \]
\[ (A.3) \]
\[ g_{tt} = 2 \Omega g_{t\phi} + \Omega^2 g_{\phi\phi}, \]  
\[ (A.4) \]

Thus, the solution of Eqs. (A.3) and (A.4) can be written as

\[ \Omega_{\text{orb}}(r) = \frac{u^\phi}{u^t} = \frac{-g_{t\phi} \pm \sqrt{(g_{t\phi})^2 - g_{tt} g_{\phi\phi}}}{g_{\phi\phi}}, \]  
\[ (A.5) \]

where \( +/− \) stands for co-rotating/counter-rotating orbits, \( u^\phi \) and \( u^t \) are the angular and time components of the four-velocity respectively, and a colon stands for partial derivative with respect to the corresponding coordinate.

To determine the mass shedding angular velocity (the Keplerian angular velocity) of the neutron stars, we need to consider only the co-rotating orbit, so from here and thereafter we take into account only the plus sign in Eq. (A.3) and we write \( \Omega_{\text{orb}}(r) = \Omega_{\text{orb}}(r). \)

For the Hartle external solution given by Eq. (A.1) we obtain Eq. (20) with

\[ F_1 = \left( \frac{M}{r} \right)^{3/2}, \]
\[ F_2 = \frac{24M^7 - 80M^6r + 4M^5r^2 - 18M^4r^3}{16M^2r^4(r - 2M)} \]
\[ + \frac{40M^3r^4 + 10M^2r^5 + 15Mr^6 - 15r^7}{16M^2r^4(r - 2M)} + F, \]
\[ F_3 = \frac{6M^4 - 8Mr^3 - 2M^2r^2 - 3Mr^3 + 3r^4}{16M^2r(r - 2M)/5} - F, \]
\[ F = \frac{15(r^3 - 2M^3)}{32M^4} \ln \left(\frac{r}{r - 2M} \right). \]

The maximum angular velocity possible for a rotating star at the mass-shedding limit is the Keplerian angular velocity evaluated at the equator \( (r = R_{\text{eq}}) \), i.e.

\[ \Omega_J = \Omega_{\text{orb}}(r = R_{\text{eq}}). \]  
\[ (A.6) \]

In the static case i.e. when \( j = 0 \) hence \( q = 0 \) and \( \delta M = 0 \) we have the well-known Schwarzschild solution and the orbital angular velocity for a test particle \( \Omega_{K}^0 \) on the surface \( (r = R) \) of the neutron star is given by

\[ \Omega_{K}^0 = \sqrt{\frac{M_j}{R_{M}^j}}. \]  
\[ (A.7) \]