Self-Similar Algebras with connections to Run-length Encoding and Rational Languages

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Abstract. A self-similar algebra $(A, \psi)$ is an associative algebra $A$ with a morphism of algebras $\psi : A \rightarrow M_d(A)$, where $M_d(A)$ is the set of $d \times d$ matrices with coefficients from $A$. We study the connection between self-similar algebras with run-length encoding and rational languages. In particular, we provide a curious relationship between the eigenvalues of a sequence of matrices related to a specific self-similar algebra and the smooth words over a 2-letter alphabet. We also consider the language $L(s)$ of words $u$ in $(\Sigma \times \Sigma)^*$ where $\Sigma = \{0, 1\}$ such that $s \cdot u$ is a unit in $A$. We prove that $L(s)$ is rational and provide an asymptotic formula for the number of words of a given length in $L(s)$.

Keywords: Self-similar algebras, rational languages, matrix algebra, eigenvalues.

1 Introduction

"Each portion of matter can be conceived as like a garden full of plants, or like a pond full of fish. But each branch of a plant, each organ of an animal, each drop of its bodily fluids is also a similar garden or a similar pond"

Gottfried Leibniz
(La Monadologie, 1714)

Self-similar scaling is a fundamental property of non-computable solutions of the homogeneous Euler equation for an incompressible fluid (see Scheffer[9], Shnirelman[10]). Classical (finite dimensional) matrix algebra is the framework for classical geometry and elementary physical applications. Similarly, a kind of self-similar matrix algebra should be (see Villani[11]) the framework for the self-similar phenomena arising from contemporary research in fluid dynamics. The following definition from Bartholdi [12] could be considered as an attempt to formalize self-similarity\footnote{For a general introduction to the subject of self-similarity from the algebraic point of view, see [2] and [7].}.

A self-similar algebra $(A, \psi)$ is an associative algebra...
endowed with a morphism of algebras \( \psi : \mathfrak{A} \to M_d(\mathfrak{A}) \), where \( M_d(\mathfrak{A}) \) is the set of \( d \times d \) matrices with coefficients from \( \mathfrak{A} \). Throughout this paper, we consider the case \( d = 2 \). Given \( s \in \mathfrak{A} \) and integers \( a \geq 0 \) and \( b \geq 0 \), the \( 2 \times 2 \) matrix \( \psi_{a,b}(s) \) is obtained using the mapping

\[
\begin{pmatrix}
0 & x^a \\
y^a & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & x^b \\
y^b & 0
\end{pmatrix}.
\]

We simplify the notation of \( \psi_{a,b}(s) \) as \( \psi(s) \) for the case \( a = 1 \) and \( b = 0 \).

Example 1.

\[
\psi_{1,2}(x y^3) = \begin{pmatrix}
x^1 y^2 x^2 y^1 x^1 y^1 x^2 y^2 \\
0
\end{pmatrix}
\]

which corresponds to the following run-length encoding

\[
\begin{array}{cccccccc}
1 & 2 & 2 & 1 & 1 & 1 & 2 & 2
\end{array},
\]

A similar phenomenon occurs in the setting of smooth words over 2-letter alphabets \( \{a, b\} \), where \( a \) and \( b \) are positive integers. Brlek et al. were able to compute the asymptotic density\(^2\) of a given letter in the extremal words, w.r.t. lexicographic order in the space of smooth words over 2-letter alphabets \( \{a, b\} \) provided that \( a \equiv b \mod 2 \) (see [3], [4], [5]). The analogous result for \( a \not\equiv b \mod 2 \) still remains an open problem.

Given a self-similar algebra \( (\mathfrak{A}, \psi) \), with \( \psi : \mathfrak{A} \to M_2(\mathfrak{A}) \), we define a right action\(^3\) \( \mathfrak{A} \times (\Sigma \times \Sigma)^* \to \mathfrak{A} \) by

\[
s \cdot (i, j) := \psi(s)[i, j],
\]

where \( \Sigma = \{0, 1\} \) and \( \psi(s)[i, j] \) is the notation for the entry \( (i, j) \) in the matrix \( \psi(s) \) (we begin to count the rows and the columns by 0). We define the language of units\(^4\) of \( s \in \mathfrak{A} \) as follows

\[
L(s) := \{ w \in (\Sigma \times \Sigma)^* : s \cdot w \text{ is a unit in } \mathfrak{A} \}.
\]

We use the notation \( \mathbb{F}_q \) for the Galois field with exactly \( q \) elements, where \( q \) is a prime power.

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\(^2\) This computation is related to Keane conjecture (see [6]) about the Oldenburger trajectory in generating symbols 1, 2 (see [6]).

\(^3\) Here \( * \) stands for the Kleene closure.

\(^4\) An element \( s \in \mathfrak{A} \) is a unit if and only if \( s r = r s = 1 \) for some \( r \in \mathfrak{A} \).
Example 2. Consider $\mathfrak{A} = \mathbb{F}_2\langle x, y \rangle$ and $s = 1 + x^2yx^2 + yx^2y$

$$\psi(s) = \begin{pmatrix} 1 + xy & xy^2x \\ yx^2y & 1 + xy \end{pmatrix};$$

$$\psi(1 + xy) = \begin{pmatrix} 1 + y & 0 \\ 0 & 1 + x \end{pmatrix}; \quad \psi(1 + xy) = \begin{pmatrix} 1 + x & 0 \\ 0 & 1 + y \end{pmatrix};$$

$$\psi(xy^2x) = \begin{pmatrix} xy & 0 \\ 0 & yx \end{pmatrix}; \quad \psi(yx^2y) = \begin{pmatrix} yx & 0 \\ 0 & xy \end{pmatrix};$$

$$\psi(xy) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}; \quad \psi(yx) = \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}; \quad \psi(1 + x) = \begin{pmatrix} 1 & x \\ y & 1 \end{pmatrix}; \quad \psi(1 + y) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix};$$

In the above example, since $s \cdot (0, 0) \cdot (1, 1) \cdot (0, 1) = x$ and since $x$ is a unit in $\mathbb{F}_2\langle x, y \rangle$, the word $(0, 0)(1, 1)(0, 1) \in L(s)$.

In this paper, we study self-similar algebras $(\mathfrak{A}, \psi)$ in the above context, and obtain the following results:

- We prove some results about iterated matrices where $A = \mathbb{Q}\langle x, y \rangle$; in particular, we provide a relationship between the eigenvalues of a sequence of matrices and the smooth words over a 2-letter alphabet.

- We prove that $L(s)$ is rational and provide an asymptotic formula

$$\# \left( (\Sigma \times \Sigma)^k \cap L(s) \right) = 2^k \mu(s) - 2 \nu(s), \quad (k \to +\infty),$$

for the number of words of a given length in $L(s)$, where $\mathfrak{A} = \mathbb{F}_q\langle x, y \rangle$.

- We prove that the range of $\mu(s)$ is dense in the ray of nonnegative real numbers.

2 Eigenvalues

Given a self-similar algebra $(\mathfrak{A}, \psi)$, with $\psi : \mathfrak{A} \to M_d(\mathfrak{A})$, for any integer $k \geq 0$ we define a new self-similar algebra $(\mathfrak{A}, \psi^{(k)})$, with $\psi^{(k)} : \mathfrak{A} \to M_{dk}(\mathfrak{A})$ by

$$\psi^{(0)}(s) := s,$$

$$\psi^{(k+1)}(s) := \left( \psi^{(k)}(s_{i,j}) \right)_{0 \leq i,j \leq d-1},$$

where $\psi(s) = (s_{i,j})_{0 \leq i,j \leq d-1}$.

Consider the self-similar algebra $(\mathfrak{A}, \psi)$, where $\mathfrak{A} = \mathbb{Q}\langle x, y \rangle$ and $\psi_{a,b} : \mathfrak{A} \to M_2(\mathfrak{A})$ is given by

$$x \mapsto \begin{pmatrix} 0 & xa \\ y^a & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & xb \\ y^b & 0 \end{pmatrix},$$

$\footnote{We begin to count the rows and the columns of the matrices by 0.}$
Consider the following doubly stochastic matrix

\[ M_k(a, b) := \psi^{(k)}_{a,b} \left( \frac{1}{2} x + \frac{1}{2} y \right) \bigg|_{(x,y) \rightarrow (1,1)} \in M_{2^k}(\mathbb{Q}). \]  

(6)

where the vertical line at the right with the equality \((x,y) \rightarrow (1,1)\) means that after the computation of the iterated matrix \(\psi^{(k)}_{a,b} \left( \frac{1}{2} x + \frac{1}{2} y \right)\), we should evaluate \((x,y)\) at \((1,1)\).

**Proposition 1.** For all \(a \equiv b \pmod{2}\), if \(\lambda \in \mathbb{C}\) is an eigenvalue of the matrix \(M_k(a, b)\) then either \(\lambda = -1\) or \(\lambda = 1\).

**Proof.** We shall consider the following cases.

(i) If \(a \equiv b \equiv 1 \pmod{2}\) then \(M_k(a, b)\) is the exchange matrix, i.e. the matrix \(J = (J_{i,j})_{0 \leq i,j \leq 2^k-1}\), where

\[ J_{i,j} = \begin{cases} 1 & \text{if } j = 2^k - 1 - i, \\ 0 & \text{if } j \neq 2^k - 1 - i. \end{cases} \]  

(7)

(ii) If \(a \equiv b \equiv 0 \pmod{2}\) then \(M_k(a, b)\) is the following block matrix

\[ \begin{pmatrix} I_{2^k-1} & 0_{2^k-1} \\ 0_{2^k-1} & I_{2^k-1} \end{pmatrix}, \]  

(8)

where \(I_n\) and \(0_n\) are the \(n \times n\) identity matrix and the \(n \times n\) zero matrix respectively.

In both cases, all the eigenvalues belong to the set \((-1,1)\).

The structure of \(M_k(1,0)\) is less trivial than in the previous examples, although it is not so complex as in the case \(a \not\equiv b \pmod{2}\).

**Proposition 2.** If \(\lambda \in \mathbb{C}\) is an eigenvalue of the matrix \(M_k(1, 0)\) then

\[ \lambda = \cos(\pi \theta) \]  

(9)

for some \(\theta \in \mathbb{Q}\).

**Proof.** Denoting

\[ A_k := \left. \psi^{(k)}(x) \right|_{(x,y) = (1,1)}, \]  

(10)

\[ B_k := \left. \psi^{(k)}(y) \right|_{(x,y) = (1,1)}, \]  

(11)
we have
\[
\left[ \frac{1}{2} (A_{2k} + B_{2k}) \right]^2 = \left( \frac{1}{2} (B_{2k-1} + I_{2k-1}) \right)^2 \]
\[
= \left( \frac{1}{2} (A_{2k-1} + I_{2k-1}) \right)^2 \]
\[
= \frac{1}{4} \left( (A_{2k-1} + I_{2k-1}) (B_{2k-1} + I_{2k-1}) \right) (A_{2k-1} + I_{2k-1}) \]
\[
= \frac{1}{4} \left( \begin{array}{cc}
A_{2k-1} + I_{2k-1} & 0_{2k-1} \\
B_{2k-2} + I_{2k-2} & 0_{2k-2}
\end{array} \right) \left( \begin{array}{cc}
A_{2k-2} + I_{2k-2} & 0_{2k-2} \\
B_{2k-2} + I_{2k-2} & 0_{2k-2}
\end{array} \right)
\]
\[
= \frac{1}{4} \left( \begin{array}{cc}
0_{2k-2} & 0_{2k-2} \\
0_{2k-2} & 0_{2k-2}
\end{array} \right) \left( \begin{array}{cc}
0_{2k-2} & 0_{2k-2} \\
B_{2k-2} + I_{2k-2} & 0_{2k-2}
\end{array} \right)
\]
So,
\[
\lambda I_{2k} - \frac{1}{2} (A_{2k} + B_{2k}) \]
\[
= \left( \lambda^2 \right)^{2^{k-2}} \left( \lambda^2 I_{2k-2} - \frac{1}{4} (A_{2k-2} + B_{2k-2} + 2I_{2k-2}) \right)^2 \]
\[
= \left( \lambda^{2^k-1} \right)^{2} \left( \frac{1}{2} \left( 2\lambda^2 I_{2k-2} - \frac{1}{2} (A_{2k-2} + B_{2k-2}) - I_{2k-2} \right) \right)^2 \]
\[
= \frac{\lambda^{2^k-1}}{2^{2^k}} \left( 2\lambda^2 - 1 \right) I_{2k-2} - \frac{1}{2} (A_{2k-2} + B_{2k-2}) \right)^2 \]
Hence, the characteristic polynomial of \( M_k(1, 0) \), denoted
\[
C_k(\lambda) := |\lambda I_{2k} - M_k(1, 0)|,
\]
satisfies the recurrence relations
\[
C_0(\lambda) = \lambda - 1,
\]
\[
C_1(\lambda) = \lambda^2 - 1,
\]
\[
C_k(\lambda) = \frac{\lambda^{2^k-1}}{2^{2^k}} C_{k-2} \left( 2\lambda^2 - 1 \right).
\]
It follows by induction on \( k \geq 1 \) that \( C_k(\lambda) \) is equal to \( \lambda^2 - 1 \) times the product of normalized irreducible Chebyshev polynomials of the first kind.\(^6\)

\(^6\) This property is false for \( k = 0 \).

\(^7\) A normalized Chebyshev polynomial is a Chebyshev polynomial divided by the coefficient of its leading term. So, the leading term of a normalized Chebyshev polynomial is always 1.

\(^8\) The Chebyshev polynomials of the first kind are defined by \( T_n(x) := \cos (n \arccos(x)) \).
Therefore, if $\lambda \in \mathbb{C}$ is an eigenvalue of the matrix $M_k(1,0)$ then $\lambda = \cos(\pi \theta)$ for some $\theta \in \mathbb{Q}$.

**Example 3.** The matrices $M_1(1,0)$, $M_2(1,0)$, and $M_3(1,0)$ are

\[
M_1(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
M_2(1,0) = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix},
M_3(1,0) = \begin{pmatrix} 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \end{pmatrix}.
\]

**Example 4.** The first values of $C_k(\lambda)$ are

\[
C_2(\lambda) = (\lambda^2 - 1) \cdot \lambda^2,
\]
\[
C_3(\lambda) = (\lambda^2 - 1) \cdot \lambda^6,
\]
\[
C_4(\lambda) = (\lambda^2 - 1) \cdot \lambda^{10} \cdot \left(\lambda^2 - \frac{1}{2}\right)^2,
\]
\[
C_5(\lambda) = (\lambda^2 - 1) \cdot \lambda^{18} \cdot \left(\lambda^2 - \frac{1}{2}\right)^6,
\]
\[
C_6(\lambda) = (\lambda^2 - 1) \cdot \lambda^{34} \cdot \left(\lambda^2 - \frac{1}{2}\right)^{10} \cdot \left(\lambda^4 - x^2 + \frac{1}{8}\right)^2,
\]
\[
C_7(\lambda) = (\lambda^2 - 1) \cdot \lambda^{66} \cdot \left(\lambda^2 - \frac{1}{2}\right)^{18} \cdot \left(\lambda^4 - \lambda^2 + \frac{1}{8}\right)^6,
\]
\[
C_8(\lambda) = (\lambda^2 - 1) \cdot \lambda^{130} \cdot \left(\lambda^2 - \frac{1}{2}\right)^{34} \cdot \left(\lambda^4 - \lambda^2 + \frac{1}{8}\right)^{10} \cdot \left(\lambda^8 - 2\lambda^6 + \frac{5}{4}\lambda^4 - \frac{1}{4}\lambda^2 + \frac{1}{128}\right)^2.
\]

The following common property seem to be true because of the empirical evidences.

**Conjecture 1.** For all $a \geq 0$, $b \geq 0$, $k \geq 0$, the matrix $2M_k(a,b)$ is nilpotent mod 2, i.e. for some integer $N \geq 0$, all the entries of

\[
(2M_k(a,b))^N
\]

are even integers.

**Example 5.** The 15th power of $2M_{10}(a,b)$ for $(a,b) = (1,0)$ and $(a,b) = (1,2)$ are represented in Fig 1 and Fig 2 respectively. The odd entries correspond to the black points and the even entries, to the white points.

\footnote{These pictures were obtained in SageMath using a program created by the authors.}
Fig. 1. Representation of the matrix $(2 M_{10}(1, 0))^{15}$.

Fig. 2. Representation of the matrix $(2 M_{10}(1, 2))^{15}$.
3 Rational languages

Proposition 3. For all $s \in \mathbb{F}_q[x, y]$, the language $L(s)$ is rational.

Proof. We can construct a deterministic finite automata $\Gamma$ for $L(s)$ as follows:

Alphabet: $\Sigma \times \Sigma$.

States: $Q := \{0\} \cup \{v \in \mathbb{A}\{0\} : \deg v \leq \deg s\}$ (this set is finite because the algebra is of finite rank over a finite field).

Initial state: $s$.

Final states: $\mathbb{F}_q\{0\}$.

Transition: If the machine, in the state $r \in Q$, reads $w \in \Sigma \times \Sigma$, then there is a transition to the state $r \cdot w \in Q$ (Recall equation 1).

Note that $\Gamma$ has this curious property that a state itself defines the subsequent transitions and states. For a given state $s \in \mathbb{F}_q(x, y)$, the non-zero locations in the matrix $\psi(s)$ dictate the transitions from $s$ and the corresponding non-zero elements define the respective destination states. Also note that $\psi_{a,b}(s)$ in general, do not preserve this property. For example, $\psi_{1,2}(s)$ does not generate an automaton.

Example 6. We construct an automaton using the same example we used for explaining $L(s)$ in the introduction, with $s = 1 + x^2yx^2 + yx^2y \in \mathbb{F}_2[x, y]$. 

![Diagram of automaton](image-url)
Proposition 4. For all \( s \in \mathbb{F}_q(x,y) \) and for all \( k \geq 0 \) large enough,
\[
\# \left( (\Sigma \times \Sigma)^k \cap L(s) \right) = 2^k \mu(s) - 2\nu(s),
\]  
where \( \mu(s) \in \mathbb{Q} \) and \( \nu(s) \in \mathbb{N} \), both independent of \( k \).

Proof. This result is trivially true provided that \( s = 0, s = \gamma \) or \( s = \gamma x \), for all \( \gamma \in \mathbb{F}_q \setminus \{0\} \). On the other hand, given \( s \in \mathbb{F}_q(x,y) \), there is \( k_s \geq 0 \) such that all the words \( w \in (\Sigma \times \Sigma)^{k_s} \) satisfy
\[
s \cdot w = 0, \quad s \cdot w = \gamma \quad \text{or} \quad s \cdot w = \gamma x.
\]
So, for all \( k \geq k_s \),
\[
(\Sigma \times \Sigma)^k \cap L(s) = \bigcup_{w \in (\Sigma \times \Sigma)^{k_s}} \{w\} \left( (\Sigma \times \Sigma)^{k-k_s} \cap L(s \cdot w) \right),
\]
where \( \{w\} \left( (\Sigma \times \Sigma)^{k-k_s} \cap L(s \cdot w) \right) \) is the concatenation of the languages \( \{w\} \) and \( (\Sigma \times \Sigma)^{k-k_s} \cap L(s \cdot w) \). It follows that for all \( k \) large enough,
\[
\# \left( (\Sigma \times \Sigma)^k \cap L(s) \right) = \sum_{w \in (\Sigma \times \Sigma)^{k_s}} \# \left( (\Sigma \times \Sigma)^{k-k_s} \cap L(s \cdot w) \right) = \sum_{w \in (\Sigma \times \Sigma)^{k_s}} \left( 2^{k-k_s} \mu(s \cdot w) - 2\nu(s \cdot w) \right) = 2^k \sum_{w \in (\Sigma \times \Sigma)^{k_s}} 2^{-k_s} \mu(s \cdot w) - 2 \sum_{w \in (\Sigma \times \Sigma)^{k_s}} \nu(s \cdot w).
\]
We conclude that, for
\[
\mu(s) := \frac{1}{2^{k_s}} \sum_{w \in (\Sigma \times \Sigma)^{k_s}} \mu(s \cdot w), \quad \nu(s) := \sum_{w \in (\Sigma \times \Sigma)^{k_s}} \nu(s \cdot w),
\]
the equality (32) holds for all \( k \) large enough. \( \Box \)

Proposition 5. The range of \( \mu \) is dense on the ray of nonnegative real numbers, i.e. given a real number \( \alpha \geq 0 \), for any real number \( \varepsilon > 0 \) there exists \( s \in \mathbb{F}_q(x,y) \) such that
\[
|\mu(s) - \alpha| < \varepsilon.
\]
Proof. For each \( s \in \mathbb{F}_q(x, y) \), we will use the notation \( s' \) for the substitution \( s' = s_{\substitute{(x,y)\rightarrow(y,x)}} \). We will divide the proof into 5 steps.

1) Let \( \sigma : \mathbb{F}_q(x, y) \times \mathbb{F}_q(x, y) \rightarrow \mathbb{F}_q(x, y) \) be the function
\[
\sigma(r, s) := (r + y s)_{\substitute{(x,y)\rightarrow(y,x)}},
\]
where \( (x, y) \rightarrow (y, x) \) is the notation for the simultaneous substitution of \( x \) and \( y \) by the products \( xy \) and \( yx \), respectively. Define
\[
\Omega := \bigcup_{n \geq 0} \Omega_n,
\]
where
\[
\Omega_0 := \left\{ 1 - (xy)^{2^k}, 1 - (yx)^{2^k} : k \geq 0 \right\} \cup \{0\},
\]
\[
\Omega_{n+1} := \{ \sigma(r, s), (\sigma(r, s))' : r, s \in \Omega_n \}.
\]

2) For all integers \( k \geq 0 \),
\[
\mu \left( 1 - (xy)^{2^k} \right) = \mu \left( 1 - (yx)^{2^k} \right) = \frac{1}{2^{k-1}} \in \mu(\Omega).
\]
It follows in a straightforward way by induction on \( k \) using the formulas
\[
\psi \left( 1 - x^{2^{k+1}} \right) = \left( \begin{array}{cc} 1 - (x y)^{2^k} & 0 \\ 0 & 1 - (y x)^{2^k} \end{array} \right),
\]
\[
\psi \left( 1 - (x y)^{2^k} \right) = \left( \begin{array}{cc} 1 - x^{2^k} & 0 \\ 0 & 1 - y^{2^k} \end{array} \right).
\]

3) For each \( s \in \Omega \), \( \psi(s) \) is a diagonal matrix and \( \mu(s) = \mu(s') \).

Indeed, this claim can be checked in a straightforward way for all \( s \in \Omega_0 \). Suppose that this claim is true for all \( s \in \Omega_n \), with \( n \geq 0 \). Let \( \sigma(r, s) \in \Omega_{n+1} \), with \( r, s \in \Omega_n \). We have,
\[
\psi(\sigma(r, s)) = \left( \begin{array}{cc} r + y s & 0 \\ 0 & (r + y s)' \end{array} \right),
\]
\[
\psi((\sigma(r, s))') = \left( \begin{array}{cc} (r + y s)' & 0 \\ 0 & r + y s \end{array} \right).
\]
Hence,
\[
\mu(\sigma(r, s)) = \frac{1}{2} \left( \mu(r + y s) + \mu(r' + x s') \right) = \mu((\sigma(r, s))').
\]
Therefore, the claim follows for all \( r, s \in \Omega \).
4) For all \( r, s \in \Omega \), we have \( \mu(\sigma(r, s)) = \mu(r) + \mu(s) \).

We know that both \( \psi(r) \) and \( \psi(s) \) are diagonal matrices. So,

\[
\mu(r + ys) = \mu(r) + \mu(s).
\]

We have already proved in step 3) that \( \mu(r) = \mu(r') \) and \( \mu(s) = \mu(s') \). It follows that

\[
\mu(r + ys) = \mu((r + ys)'),
\]

Using the equality

\[
\psi(\sigma(r, s)) = \begin{pmatrix} r + ys & 0 \\ 0 & (r + ys)' \end{pmatrix},
\]

we conclude that

\[
\mu(\sigma(r, s)) = \frac{1}{2} \left( \mu(r + ys) + \mu((r + ys)') \right) = \mu(r + ys) = \mu(r) + \mu(s).
\]

5) We conclude that \( \mu(\Omega) \) contains all the positive rational numbers with finite binary representation. Therefore, the set \( \mu(\Omega) \) is dense on the ray of nonnegative real numbers. \( \square \)

4 Final remarks

1. Many of the results proved in this paper hold with minor modifications for the following more general self-similar structure associated to the cyclic permutation of the variables \( x_0, x_1, x_2, \ldots, x_{d-1} \),

\[
x_0 \mapsto \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & x_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & x_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & x_{d-1} \\ x_0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \text{and} \quad x_r \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},
\]

for \( 1 \leq r \leq d - 1 \).

2. We have computed in a trivial way the eigenvalues of \( M_k(a, b) \) provided that \( a \equiv b \pmod{2} \). The determination of the eigenvalues of \( M_k(a, b) \), when \( a \not\equiv b \pmod{2} \), is a more difficult problem (the case \( a = 1, b = 0 \) is relatively easier in this category).

The possibility of a connection between the eigenvalues of \( M_k(a, b) \) and the extremal words, with respect to lexicographic order in the space of smooth words over 2-letter alphabets \( \{a, b\} \) is a question that deserves more attention for future research.
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