On Local Type I Singularities of the Navier–Stokes Equations and Liouville Theorems

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Abstract. We prove that suitable weak solutions of the Navier–Stokes equations exhibit Type I singularities if and only if there exists a non-trivial mild bounded ancient solution satisfying a Type I decay condition. The main novelty is in the reverse direction, which is based on the idea of zooming out on a regular solution to generate a singularity. By similar methods, we prove a Liouville theorem for ancient solutions of the Navier–Stokes equations bounded in $L^3$ along a backward sequence of times.

1. Introduction

In this paper, we consider potential singularities of the Navier–Stokes equations from the perspective of Liouville theorems. The main idea is to “zoom in” on the singularity and classify the limiting objects. This approach is highly effective in the regularity theory of minimal surfaces [10], semilinear heat equations [14], harmonic maps [27], and many other PDEs.

Unlike the above examples, the three-dimensional Navier–Stokes equations have no known critical conserved quantities or monotonicity formulae. Because of this issue, Type I conditions are typically imposed on the solutions; that is, we often ask that a critical quantity is finite near the singularity. For example, in the famous paper [12], Escauriaza, Seregin, and Šverák demonstrated, via Liouville theorems, that $L^\infty_tL^3_x$ solutions do not form singularities. The axisymmetric case is exceptional because $rv_\theta$ satisfies a maximum principle, and in this case, Seregin and Šverák proved that interior Type I blow-up does not occur [29]. Liouville theorems were also used by Tsai in [39] and other authors (see [9,15,23]) to exclude self-similar singularities in quite general situations. However, many questions concerning feasible Type I scenarios, e.g., discretely self-similar blow-up, remain completely open. We refer to [34] for a recent survey of regularity results based on Liouville theorems.

A central object in the Liouville theory is the class of mild bounded ancient solutions, which arise naturally as “blow-up limits” of singular solutions (see [18]). These are defined to be solutions which satisfy the integral equation formulation of the Navier–Stokes equations and are bounded (in fact, smooth) for all backward times. The assumption that the solution is mild simply excludes the “parasitic solutions” $v = \tilde{c}(t)$, $q = -\tilde{r}'(t) \cdot x$. At a conceptual level, classifying mild bounded ancient solutions serves to determine the possible model solutions on which a Navier–Stokes singularity must be based.

Koch et al. [18] conjectured that mild bounded ancient solutions are constant. Remarkably, the same authors proved that this is true in two dimensions and in the axisymmetric case without swirl (see Theorems 5.1–5.2 therein). A special case of the conjecture was recently verified by Lei et al. [21] when the solution is axisymmetric and periodic in the $z$-variable (see also [8]). If true, the conjecture excludes Type I singularities and implies that $D$-solutions of the steady Navier–Stokes equations are constant.1

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1This is in contrast to the focusing semilinear heat equation $\partial_t u - \Delta u = |u|^{p-1}u$, for which there is a non-trivial ground state whenever $p \geq p_\text{c} := \frac{n+2}{n-2}$. 
there is a counterexample to the conjecture, it is conceivable that it already occurs in the axisymmetric class.

We are interested in a weak version of the above conjecture obtained by restricting to mild bounded ancient solutions having Type I decay in backward time. With this modification, we can clarify the relationship between these solutions and Type I singularities:

**Theorem 1.1.** The following are equivalent:

- There exists a suitable weak solution with Type I singular point.
- There exists a non-trivial mild bounded ancient solution with \( I < \infty \).

The relevant terminology will be defined below, as there is some subletty in the formulation of Type I, see (1.6). The quantity \( I \) is defined in (1.5). For suitable weak solutions, see Definition 2.1.

The main novelty of Theorem 1.1 is in the reverse direction. Our idea is to zoom out on an ancient (but regular) solution to generate a singular solution. This is known as the “blow-down limit” in free boundary problems, and it has not yet been exploited in the Navier–Stokes literature. Our primary tools are known and consist of estimates in Morrey spaces and the persistence of singularities introduced by Rusin and Šverák in [26]. In principle, constructing ancient solutions with Type I decay is a (difficult) route to obtaining Navier–Stokes singularities.

We will use a rather weak notion of Type I in terms of the rescaled energy. Let \( z = (x, t) \in \mathbb{R}^{3+1} \), \( Q(z, r) = B(x, r) \times [t - r^2, t] \) be a parabolic ball, \( Q' = Q(z, r) \), and

\[
A(Q') = \text{ess sup} \frac{1}{t - r^2 < t' < t} \int_{B(x, r)} |v(x', t')|^2 \, dx',
\]

\[
C(Q') = \frac{1}{r^2} \int_{Q(z, r)} |v|^3 \, dz',
\]

\[
D(Q') = \frac{1}{r^2} \int_{Q(z, r)} |q - [q]_{x, r}(t')|^{\frac{3}{2}} \, dz',
\]

\[
E(Q') = \frac{1}{r} \int_{Q(z, r)} |\nabla v|^2 \, dz'.
\]

If \( \omega \subset \mathbb{R}^{3+1} \) is open and \((v, q)\) is defined on \( \omega \), then

\[
I(\omega) = \sup_{Q' \subset \omega} A(Q') + C(Q') + D(Q') + E(Q')
\]

If \( \omega \) is unspecified, we use \( \omega = \mathbb{R}^3 \times \mathbb{R}_- \). Together, \( v \equiv \text{const.} \) and \( I < \infty \) imply \( v \equiv 0 \).

If \( v \) is not essentially bounded in any parabolic ball centered at \( z \), we say that \( z \) is a singular point. Finally, if there exists a parabolic ball \( Q' \) centered at the singular point \( z \) and

\[
I(Q') < \infty,
\]

then we say that \( z \) is a Type I singularity.

Observe that (1.6) is adapted to the minimal requirements needed to make sense of the local energy inequality and partial regularity theory. In particular, \( I(Q') \ll 1 \) implies regularity. Our notion is also natural because it follows from boundedness of a variety of quantities considered to be Type I in the literature, e.g.,

\[
\sup_{x, t} \left( |x^* - x| + \sqrt{T^* - t} \right) |v(x, t)|,
\]

\[
\sup_t \|v(\cdot, t)\|_{L^{3, \infty}},
\]

\[
\sup_t (T^* - t)^{\frac{1}{2} - \frac{3}{2p}} \|v(\cdot, t)\|_{L^p},
\]

\[
\sup_t \sqrt{T^* - t} \|v(\cdot, t)\|_{L^\infty},
\]

\[
\sup_{t} \sqrt{T^* - t} \|v(\cdot, t)\|_{L^\infty},
\]

\[
\sup_{t} \sqrt{T^* - t} \|v(\cdot, t)\|_{L^\infty}.
\]
in the class of suitable weak solutions, see Lemma 2.5. In Theorem 3.1, we prove a version of Theorem 1.1 in the context of \((a)−(c_p)\) \((3 < p < \infty)\) using Calderón-type energy estimates, introduced in [7]. Historically, \((c_\infty)\) has been considered important, in part due to its success in the work of Giga and Kohn (see [14]). However, an important distinction is that \((c_\infty)\) is not well suited to the reverse direction, see Remark 3.2. Note that boundedness of one of \((a)−(c_\infty)\) is not known to imply boundedness of the other quantities.

In this paper, we also prove a Liouville theorem for ancient solutions with Type I decay along a backward sequence of times. The Liouville theorem of Escauriaza, Seregin, and Šverák in [12] states that an ancient suitable weak solutions in \(L_t^\infty L_x^3\) vanishing identically at time \(t = 0\) must be trivial. It is natural to ask whether the condition on vanishing can be removed; is a mild ancient solution in \(L_t^\infty L_x^3\) necessarily zero? Yes, since estimates of the form

\[
\|v\|_{L^\infty(Q(R/2))} \leq R^{-1} f(\|v\|_{L_t^\infty L_x^2(Q(R))})
\]  

(1.7)

were considered by Dong and Du in [11], where \(f > 0\) is an increasing function. Hence, one may simply allow \(R \to \infty\) in (1.7). On the other hand, the analogous result along a sequence of times is less obvious, and we prove it in the sequel:

**Theorem 1.2.** If \(v\) is a mild ancient solution satisfying

\[
\sup_{k \in \mathbb{N}} \|v(\cdot, t_k)\|_{L^3} < \infty
\]  

(1.8)

for a sequence of times \(t_k \downarrow -\infty\), then

\[
v \equiv 0.
\]  

(1.9)

The proof relies essentially on zooming out and the persistence of singularities, as in Theorem 1.1. In this case, to control the solution, we use the theory of weak \(L^{3,\infty}\) solutions developed in [6,30], where \(L^{3,\infty} = L^{3,\infty}_{\text{weak}}\) is the Lorentz space/weak Lebesgue space. We prove a more quantitative version in Theorem 4.1.

Without Type I assumptions, it is unclear what the existence of non-constant mild bounded ancient solutions says about the regularity theory. For example, the one-dimensional viscous Burgers equation is easily seen to be regular, but it admits non-constant traveling wave solutions \(f(x - ct)\). These solutions are easily upgraded to higher dimensions by writing \(u(x,t) = f(x \cdot n - ct)\). Regarding Navier–Stokes solutions, as there are no non-constant mild bounded ancient solutions in two dimensions [18], no such “upgrade” is possible. The analogous results in the half-space remain open.

### 2. Preliminaries

In this section, we recall some known facts about suitable weak solutions. We refer to [12,29,35,37] for a review of the partial regularity theory; in particular, [29,37] contain many excellent heuristics.

Let \(z = (x, t) \in \mathbb{R}^{3+1}, r > 0\), and \(Q' = Q(z, r)\) a parabolic ball. We also write \(Q(r) = Q(0, r)\) and \(Q = Q(1)\).

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1. Moreover, it does not appear to hold for other equations, e.g., the harmonic map heat flow or parabolic-elliptic Keller-Segel system in two dimensions. However, in the context of mild solutions, one may say that \((c_p)\) for \(p_1\) implies \((c_p)\) for \(p_2 \geq p_1\) (in a slightly smaller time interval), and in particular, implies \((c_\infty)\). Clearly, \((a)\) implies \((b)\). Of course, many more quantities are possible, e.g., space-time Lorentz norms, quantities involving the vorticity, quantities involving Besov spaces (see [32]), etc.
2. We thank Hongjie Dong for informing us of this proof. It is possible to prove (1.7) using a compactness argument, persistence of singularities, and the local regularity result for \(L_t^\infty L_x^2(Q)\) solutions in [12]. A similar Liouville theorem was proven in [28] for ancient solutions in \(L_t^p L_x^q(\mathbb{R}^4 \setminus [0,0])\) by duality methods.
3. One may obtain other mild bounded ancient solutions of 1d viscous Burgers by solving the backward heat equation using a superposition of solutions \(f(x_0) \exp[(x-x_0) + t]\) and applying the Cole–Hopf transformation.
4. The relevant literature includes [4,13,28,31]. Since the writing of this paper, Seregin has shown an analogue of Theorem 1.1 in the half-space [36]. We remark that the relationship between various formulations of Type I is less clear in the half-space.
**Definition 2.1** (Suitable weak solution). We say that \((v, q)\) is a suitable weak solution in \(Q'\) if
\[
v \in L^\infty_t L^2_x \cap L^2_t H^1_x(Q') \quad \text{and} \quad q \in L^{3/2}(Q'),
\]
(2.1)
\((v, q)\) satisfies the Navier–Stokes equations on \(Q'\) in the sense of distributions,
\[
\begin{align*}
\partial_t v - \Delta v + v \cdot \nabla v + \nabla q &= 0, \\
\text{div } v &= 0
\end{align*}
\]
(2.2)
and \((v, q)\) satisfies the local energy inequality,
\[
\int_{B(x, r)} \zeta |v(y, t')|^2 dy + 2 \int_{t-r^2}^{t'} \int_{B(x, r)} \zeta |\nabla v|^2 dy ds \\
\leq \int_{t-r^2}^{t'} \int_{B(x, r)} |v|^2 (\partial_t + \Delta) \zeta + (|v|^2 + 2q) v \cdot \nabla \zeta dy ds,
\]
(2.3)
for all non-negative \(\zeta \in C^\infty_0(B(x, r) \times [t-r^2, t])\) and almost every \(t' \in [t-r^2, t]\).
\(^6\)

Finally, we say that \(v\) is a suitable weak solution in \(Q'\) (without reference to the pressure) if there exists \(q \in L^{3/2}(Q')\) such that \((v, q)\) is suitable in \(Q'\).

The following proposition is our primary tool. It is contained in Lemma 2.1 and Lemma 2.2 of [26]. However, as the statement therein is slightly different, we include a proof for completeness.

**Proposition 2.3** (Persistence of singularities). Let \((v^{(k)}, q^{(k)})_{k \in \mathbb{N}}\) be a sequence of suitable weak solutions on \(Q\) satisfying
\[
\sup_{k \in \mathbb{N}} \|v^{(k)}\|_{L^3(Q)} + \|q^{(k)}\|_{L^{3/2}(Q)} < \infty.
\]
Then there exists a suitable weak solution \((u, p)\) on \(Q(R)\) for all \(0 < R < 1\) such that
\[
v^{(k)} \rightharpoonup u \text{ in } L^3_{\text{loc}}(B \times [1-\varepsilon, 1]), \quad q^{(k)} \rightharpoonup p \text{ in } L^{3/2}_{\text{loc}}(B \times [1-\varepsilon, 1]),
\]
(2.5)
along a subsequence.

The next proposition is our primary tool. It is contained in Lemma 2.1 and Lemma 2.2 of [26]. However, as the statement therein is slightly different, we include a proof for completeness.

**Proposition 2.3** (Persistence of singularities). Let \((v^{(k)}, q^{(k)})_{k \in \mathbb{N}}\) be a sequence of suitable weak solutions on \(Q\) satisfying (2.5). If
\[
\limsup_{k \to \infty} \|v^{(k)}\|_{L^\infty(Q(R))} = \infty \text{ for all } 0 < R < 1,
\]
(2.6)
then
\[
u \text{ has a singularity at the space-time origin.}
\]
(2.7)

**Proof.** We prove the contrapositive. Suppose that \(u \in L^\infty(Q(R))\) for some \(0 < R < 1\). Let \(\varepsilon > 0\) (to be fixed later). Then there exists \(0 < R_0 < R\) (depending also on \(\varepsilon\)) satisfying, for all \(0 < r \leq R_0\),
\[
\frac{1}{r^2} \int_{Q(r)} |u|^3 dx dt \leq \varepsilon.
\]
(2.8)
This is because \(u \in L^\infty(Q(R))\) is a subcritical assumption. Rescaling, we may set \(R_0 = 1\). By the strong convergence in (2.5), for \(k\) sufficiently large (depending on \(0 < r \leq 1\)),
\[
\frac{1}{r^2} \int_{Q(r)} |v^{(k)}|^3 dx dt \leq 2\varepsilon.
\]
(2.9)
We decompose the pressure as \(q^{(k)} = \tilde{q}^{(k)} + h^{(k)}\), where
\[
\tilde{q}^{(k)} = (-\Delta)^{-1} \text{div} \, \text{div} (\varphi v^{(k)} \otimes v^{(k)}),
\]
(2.10)
\(^6\)By weak continuity in time, one may remove the “almost every” restriction.
\[ \varphi \in C_0^\infty(B) \ (0 \leq \varphi \leq 1) \text{ satisfies } \varphi \equiv 1 \text{ on } B(3/4), \text{ and } h^{(k)}(\cdot, t) \text{ is harmonic in } B(3/4). \] By (2.10) and Calderón-Zygmund estimates,
\[ \frac{1}{r^2} \int_{Q(r)} |\bar{q}^{(k)}|^{\frac{3}{2}} \, dx \, dt \leq C \frac{1}{r^2} \int_{Q(r)} |v^{(k)}|^{\frac{3}{2}} \, dx \, dt. \quad (2.11) \]

By the triangle inequality and (2.11) (with \( r = 1 \)),
\[ \int_{Q} |h^{(k)}|^{\frac{3}{2}} \, dx \, dt \leq C \sup_{l \in \mathbb{N}} \left( \int_{Q} |v^{(l)}|^{3} + |q^{(l)}|^{\frac{3}{2}} \, dx \, dt \right) \leq CM. \quad (2.12) \]

\((M > 0 \text{ depends on } \varepsilon \text{ through } R_0,\) By Hölder’s inequality and interior regularity for harmonic functions, whenever \( 0 < r \leq 1/2, \)
\[ \frac{1}{r^2} \int_{Q(r)} |h^{(k)}|^{\frac{3}{2}} \, dx \, dt \leq C r \int_{0}^{1/4} \|h^{(k)}(\cdot, t)\|_{L^{\frac{3}{2}}(B(1/2))}^{\frac{3}{2}} \, dt \leq CMr, \quad (2.13) \]

Finally, one may combine (2.9), (2.11), and (2.13) and fix \( \varepsilon \) and \( r \) sufficiently small to obtain
\[ \limsup_{k \to \infty} \frac{1}{r^2} \int_{Q(r)} |v^{(k)}|^{3} + |q^{(k)}|^{\frac{3}{2}} \leq \varepsilon_{\text{CKN}}, \quad (2.14) \]

where \( \varepsilon_{\text{CKN}} > 0 \) is the constant in the \( \varepsilon \)-regularity criterion. This ensures
\[ \limsup_{k \to \infty} \sup_{Q(r/2)} |v^{(k)}| \leq \frac{C_{\text{CKN}}}{r}, \quad (2.15) \]
as desired.

Since the forward direction of Theorem 1.1 deals with local solutions, it is useful to locally mimic the situation of the “first singular time” in the Cauchy problem. The following proposition follows from partial regularity, see [20, Lemma 3.2] and [37, Theorem 3].

**Proposition 2.4** (Regular cylinder lemma). If \( v \) is a suitable weak solution in \( Q \) with singular point at the space-time origin, then there exist \( z^* \in B(1/2) \times ] - 1/4, 0] \) and \( 0 < R < 1/2 \) satisfying
\[ v \in L^{\infty}(Q(z^*, R) \setminus Q(z^*, r)) \text{ for all } 0 < r < R. \quad (2.16) \]

It is possible to combine Proposition 2.4 and Bogovskii’s operator to truncate the solution, see [24], [38, Remark 12.3], and [1]. We will not require this here.

As discussed in the introduction, boundedness of other widely considered critical quantities is known to imply \( I(Q') < \infty. \) For example, this is true of the weak Lebesgue spaces:

**Lemma 2.5** (Weak Serrin implies Type I). If \( v \) is a suitable weak solution on \( Q \) with
\[ v \in L^q_{t} L^p_{x}(Q), \quad (2.17) \]
where \( 3 \leq p \leq \infty \) and \( 2 \leq q \leq \infty \) satisfy the Ladyzhenskaya-Prodi-Serrin condition
\[ \frac{3}{p} + \frac{2}{q} = 1, \quad (2.18) \]
then, for all \( Q' = Q(R) \) with \( 0 < R < 1, \)
\[ I(Q') < \infty. \quad (2.19) \]

Notice that having one of (a)–(c) bounded is enough to apply Lemma 2.5 (for suitable weak solutions). It is already known that absolute smallness in the above \( L^q_{t} L^p_{x} \) spaces (with the exception of the case \( q = 2) \) implies regularity, see [17].

To prove Lemma 2.5, we use the critical Morrey-type quantities
\[ M^{s;l}(Q') = \frac{1}{R^c} \int_{t-r^2}^{t} \left( \int_{B(x,r)} |v|^s \, dx' \right)^{\frac{l}{s}} \, dt', \quad (2.20) \]
where $\kappa = l(2/l + 3/s - 1)$, defined for $1 \leq s, l \leq \infty$ (with the obvious modification when $l = \infty$). The next lemma asserts that finiteness of rescaled energies $A, C, E$ (see [33]) or critical Morrey-type quantities $M^{s,l}$ (see [37, Theorem 6] and [40]) implies Type I bounds for suitable weak solutions.

**Lemma 2.6** (Morrey-type estimates). Suppose $(v, q)$ is a suitable weak solution in $Q$ with

$$
\min_{s,l} \left\{ \sup_{Q'} A(Q'), \sup_{Q'} C(Q'), \sup_{Q'} E(Q'), \sup_{Q'} M^{s,l}(Q') \right\} < \infty,
$$

where $s > 3/2, l > 1$ are finite and required to satisfy

$$
\frac{3}{s} + \frac{2}{l} < 2.
$$

Then, for all $Q' = Q(R)$ with $0 < R < 1$,

$$
I(Q') < \infty.
$$

For the above result to hold, it is crucial that $(v, q)$ is already assumed to be suitable, since the proof relies on the local energy inequality. Indeed, the estimate which gives (2.23) depends on the background quantities $C(1)$ and $D(1)$.

**Proof of Lemma 2.5.** Let $\delta > 0$ sufficiently small, so that $s = p - \delta$ and $l = q - \delta$ satisfy the requirements of Lemma 2.6. Then the embedding properties of Lorentz spaces imply

$$
M^{s,l}(Q')^{1/2} \leq C \|v\|_{L^p_t L^q_x(Q)} \leq C \|v\|_{L^\infty_t L^\infty_x(Q')},
$$

for all parabolic balls $Q' \subset Q$.

\[\square\]

### 3. Proof of Theorem 1.1

We now prove Theorem 1.1. As the forward direction is essentially known, we focus on the reverse direction. The forward direction is also valid in the local setting with curved boundary without Type I assumptions, see [1].

**Proof.** **Forward direction.** Suppose that $v$ is a suitable weak solution in $Q$ with singularity at the space-time origin and $I(Q) < \infty$. By Proposition 2.4, we may assume that $v \in L^\infty(Q', Q(r))$ for all $0 < r \leq 1$. This may require considering an earlier singularity than the original. It is proven in [29, Theorem 2.8] and [37, Section 5] that, under an appropriate rescaling procedure, such a solution (even without the Type I assumption) gives rise to a non-trivial mild bounded ancient solution $u$. It is clear from the rescaling procedure in [29,37] that $u$ will satisfy $I < \infty$.

**Reverse direction.** Suppose that $v$ is a non-trivial mild bounded ancient solution satisfying $I < \infty$. By translating in space-time as necessary, we have

$$
\|v\|_{L^\infty(Q)} = N > 0.
$$

Consider the sequence $(v^{(k)})_{k \in \mathbb{N}}$ of suitable weak solutions

$$
v^{(k)}(x,t) = kv(kx,k^2t), \quad (x,t) \in Q(2).
$$

By the uniform estimate

$$
\sup_{k \in \mathbb{N}} I(v^{(k)}, Q(2)) < \infty
$$

and Lemma 2.2, there exists a subsequence and a suitable weak solution $(u, p)$ with

$$
v^{(k)} \to u \text{ in } L^3(Q) \text{ and } q^{(k)} \rightharpoonup p \text{ in } L^{3/2}(Q).
$$

\[\text{Remark:} \quad 3/\max\{2/l, 1/2 - 1/l\} \leq \kappa \leq l(2/l + 3/s - 1)
\]

The statement in [37] also contains the requirement $3/s + 2/l - 3/2 > \max\{2/l, 1/2 - 1/l\}$. However, this requirement can be avoided by decreasing $s$ and/or $l$ using embeddings of Morrey spaces.
Moreover, (3.1) and (3.2) give
\[ \|v^{(k)\text{}}\|_{L^\infty(Q(1/k))} = kN \to \infty. \]  
(3.5)

Hence, Proposition 2.3 implies that \( u \) is singular at the space-time origin. Finally,
\[ I(u) < \infty \]  
(3.6)
follows from (3.3). That is, the singularity is Type I.

We now address other formulations of Type I.

**Theorem 3.1.** Let \( 3 \leq p < \infty \). The following are equivalent:

- There exists a suitable weak solution in \( Q \) with singularity at the space-time origin and
  \[ \text{ess sup}_{-1 < t < 0} (-t)^{\frac{2}{p} - \frac{2}{q}} \|v\|_{L^p,\infty} < \infty. \]  
(3.7)

- There exists a mild bounded ancient solution satisfying
  \[ \text{ess sup}_{t < 0} (-t)^{\frac{2}{p} - \frac{2}{q}} \|v\|_{L^p,\infty} < \infty. \]  
(3.8)

**Remark 3.2.** It is noteworthy that \( p = \infty \) is omitted despite being a popular formulation of Type I. This is because \( \sup_{t < 0} \sqrt{-t} \|v(\cdot, t)\|_{L^\infty} < \infty \) alone does not appear to guarantee \( I < \infty \), or even that the local energy is finite up to (and including) the blow-up time. This is related to the fact that no global-in-time weak solution theory is known for \( L^\infty \) initial data. However, the forward direction remains valid because Lemma 2.5 implies \( I(Q(1/2)) < \infty \) (with an estimate depending on the quantities \( C(1) \) and \( D(1) \) for suitable weak solutions).

When \( p > 3 \), it is possible to prove Theorem 3.1 with mild solutions replacing suitable weak solutions. One could also consider \( \sup_{x,t} (|x| + \sqrt{-t}) \|v\|_{L^p} \leq \infty \) alone does not appear to guarantee \( I < \infty \), or even that the local energy is finite up to (and including) the blow-up time. This is related to the fact that no global-in-time weak solution theory is known for \( L^\infty \) initial data. However, the forward direction remains valid because Lemma 2.5 implies \( I(Q(1/2)) < \infty \) (with an estimate depending on the quantities \( C(1) \) and \( D(1) \) for suitable weak solutions).

**Proof of Theorem 3.1 (Reverse direction).** Let \( 3 \leq p < \infty \). We allow the constants below to depend implicitly on \( p \). It suffices to prove that a mild bounded ancient solution satisfying (3.8) also satisfies \( I < \infty \). By translating in space-time and rescaling, we only need to demonstrate
\[ A(1/2) + C(1/2) + D(1/2) + E(1/2) \leq C(M), \]  
(3.9)
where
\[ \sup_{-1 < t < 0} \left( -t \right)^{\frac{2}{p} + \frac{2}{q}} \|v(\cdot, t)\|_{L^p,\infty} \leq M. \]  
(3.10)
We utilize a Calderón-type splitting, see [2,7,16]. Decompose \( a := v(\cdot, -1) = \bar{u}_0 + \bar{u}_1 \), where
\[ \bar{u}_0 = \mathbb{P} \left( 1_{\{|a| > \lambda M\}} a \right), \]  
(3.11)
and \( \lambda > 0 \) will be determined later. This gives
\[ \|\bar{u}_0\|_{L^2} \leq C(\lambda, M) \quad \text{and} \quad \|\bar{u}_0\|_{L^{2p}} \leq C_0(\lambda, M), \]  
(3.12)
where \( C_0(\lambda, M) \to 0 \) as \( \lambda \to 0^+ \). We decompose the solution as
\[ v = V + U, \]  
(3.13)
where \( V \in C([-1,0]; L^{2p}) \) is the mild solution of the Navier–Stokes equations on \( \mathbb{R}^3 \times ]-1,0[ \) with initial data \( \bar{u}_0 \). When \( 0 < \lambda \ll 1 \) (depending on \( M \)), \( V \) is guaranteed to exist on \( \mathbb{R}^3 \times ]-1,0[ \), and
\[ \|V\|_{L^\infty_x L^{2p}(\mathbb{R}^3 \times ]-1,0[)} + \|(t + 1)^{\frac{1}{2}} \nabla V\|_{L^\infty_x L^{2p}(\mathbb{R}^3 \times ]-1,0[)} \leq 1. \]  
(3.14)
By the Calderón-Zygmund estimates and pressure representation \( Q = (-\Delta)^{-1} \text{div} \text{div} V \otimes V \),
\[ \|Q\|_{L^\infty_x L^p(\mathbb{R}^3 \times ]-1,0[)} \leq C. \]  
(3.15)
The correction \( U \) solves a perturbed Navier–Stokes equations with initial data \( \bar{u}_0 \) and zero forcing term. It is possible to show that \( U \) (which belongs to subcritical spaces) belongs to the energy space on \( \mathbb{R}^3 \times ]-1,0[ \).
and satisfies the energy inequality. (There is standard perturbation theory involved, using that \(v\) and \(V\) are mild solutions, see [2] for details.) A Gronwall-type argument implies
\[
\|U\|_{L_t^\infty L_x^2([\mathbb{R}^3 \times] -1, 0])} + \|\nabla U\|_{L_t^2 L_x^2([\mathbb{R}^3 \times] -1, 0])} + \|U\|_{L_t^{10} L_x^3([\mathbb{R}^3 \times] -1, 0])} \leq C(\lambda, M). \tag{3.16}
\]
Using \(P = (-\Delta)^{-1} \text{div} \text{div}(U \otimes U + V \otimes U + U \otimes V)\), Calderón-Zygmund estimates, and Hölder’s inequality, we obtain
\[
\|P\|_{L_t^3 L_x^2(Q)} \leq C(\lambda, M). \tag{3.17}
\]
Combining (3.13) with (3.14)–(3.17) completes the proof of the reverse direction. We omit the proof of the forward direction. \(\Box\)

4. Proof of Theorem 1.2

We will now prove the Liouville theorem. In fact, we will prove the following, more quantitative generalization to the Lorentz space \(L^{3,\infty}\). Let \(B\) denote the subspace of \(\dot{B}^{-1}_{\infty,\infty}\) whose functions \(f\) satisfy
\[
f(\lambda \cdot) \to 0 \text{ in the sense of distributions as } \lambda \to \infty. \tag{4.1}
\]

**Theorem 4.1** (Liouville theorem). For all \(M > 0\), there exists a constant \(\varepsilon = \varepsilon(M) > 0\) satisfying the following property. Suppose that \(v\) is a mild ancient solution\(^8\) such that
\[
\|v(\cdot, t_k)\|_{L_t^3 L_x^\infty([\mathbb{R}^3 \times] -1, 0])} \leq M \tag{4.2}
\]
for a sequence \(t_k \downarrow -\infty\). If
\[
\text{dist}_{\dot{B}^{-1}_{\infty,\infty}}(v(\cdot, 0), B) \leq \varepsilon, \tag{4.3}
\]
then
\[
\limsup_{k \to \infty} \sqrt{|t_k|/2} \|v\|_{L_t^\infty(Q(\sqrt{|t_k|/2}))} < \infty. \tag{4.4}
\]
Hence,
\[
v \equiv 0. \tag{4.5}
\]

We will use the theory of weak \(L^{3,\infty}\) solutions developed in [6]. These are defined to be suitable weak solutions of the Navier–Stokes equations with initial data \(u_0 \in L^{3,\infty}\) that additionally satisfy a decomposition \(v = V + U\), where \(V(\cdot, t) = S(t)u_0\) is the Stokes evolution of the initial data and \(U\) belongs to the energy space with \(\|U(\cdot, t)\|_{L^2} \to 0\) as \(t \to 0^+\). We will also use the following proposition, which is proven in [3] by contradiction and backward uniqueness arguments.

**Proposition 4.2** (Auxiliary proposition). For all \(M > 0\), there exists a constant \(\varepsilon_0 = \varepsilon_0(M) > 0\) satisfying the following property. Suppose that \(v\) is a weak \(L^{3,\infty}\) solution on \(\mathbb{R}^3 \times [0, 1]\) satisfying
\[
\|v(\cdot, 0)\|_{L_t^3 L_x^\infty([\mathbb{R}^3 \times] 0, 1])} \leq M \tag{4.6}
\]
and
\[
\|v(\cdot, 1)\|_{\dot{B}^{-1}_{\infty,\infty}} \leq \varepsilon_0. \tag{4.7}
\]
Then
\[
v \text{ is essentially bounded in } \mathbb{R}^3 \times [1/4, 1]. \tag{4.8}
\]

In fact, one may give pointwise bounds for \(v\) on \(\mathbb{R}^3 \times [1/4, 1]\), but this will not be necessary.

\(^8\)In this section, we consider mild solutions belonging to the class \(L_t^{\infty} L_x^2(\mathbb{R}^3 \times] -\infty, 0]).\)
Proof of Theorem 4.1. Suppose otherwise. That is, there exists a mild ancient solution $v$ satisfying
\[ \|v(\cdot,t_k)\|_{L^3,\infty} \leq M \] (4.9)
for a sequence $t_k \downarrow -\infty$,
\[ \text{dist}_{B_{\infty,\infty}^{1,1}}(v(\cdot,0), \mathcal{B}) \leq \varepsilon_0/2, \] (4.10)
with $\varepsilon_0 = \varepsilon_0(M) > 0$ as in Proposition 4.2, and
\[ \limsup_{k \to \infty} \sqrt{\|t_k\|/2} \|v\|_{L^\infty(Q(\sqrt{\|t_k\|/2}))} = \infty. \] (4.11)
Regarding (4.10), we decompose $v(\cdot,0) = U + W$, where $U \in \mathcal{B}$ and $\|W\|_{B_{\infty,\infty}^{1,1}} \leq \varepsilon_0$.

We construct a sequence $(v^{(k)})_{k \in \mathbb{N}}$ of mild solutions on $\mathbb{R}^3 \times ]-1,0[\$ by rescaling appropriately:
\[ v^{(k)}(x,t) = \sqrt{|t_k|}v(\sqrt{|t_k|}x, |t_k|t). \] (4.12)
Since $v$ is mild, it is not difficult to show that $v^{(k)}$ is a weak $L^{3,\infty}$ solution on $\mathbb{R}^3 \times ]-1,0[\$. Moreover,
\[ \|v^{(k)}(\cdot,-1)\|_{L^{3,\infty}} \leq M, \] (4.13)
\[ v^{(k)}(\cdot,0) = \sqrt{|t_k|}v^{(k)}(\sqrt{|t_k|},0) = U^{(k)} + W^{(k)}, \] (4.14)
where $U^{(k)}$ and $W^{(k)}$ correspond to $U$ and $W$, appropriately rescaled, and
\[ \|v^{(k)}\|_{L^\infty(Q(1/2))} \to \infty. \] (4.15)
Regarding (4.14), we find that $U^{(k)} \to 0$ in the sense of distributions and $\|W^{(k)}\|_{B_{\infty,\infty}^{1,1}} \leq \varepsilon_0$. Hence, there exists $W^\infty$ satisfying $\|W^\infty\|_{B_{\infty,\infty}^{1,1}} \leq \varepsilon_0$ and
\[ v^{(k)}(\cdot,0) \to W^\infty \text{ in the sense of distributions} \] (4.16)
along a subsequence.

Next, we recall a compactness result for the above sequence of weak $L^{3,\infty}$ solutions (see [3,6]). There exists a weak $L^{3,\infty}$ solution $v^\infty$ on $\mathbb{R}^3 \times ]-1,0[\$ and a subsequence such that
\[ v^{(k)}(\cdot,-1) \overset{\Delta}{\rightharpoonup} u(\cdot,-1) \text{ in } L^{3,\infty}, \] (4.17)
where $\|u(\cdot,-1)\|_{L^{3,\infty}} \leq M$,
\[ v^{(k)} \to v^\infty \text{ in } L^3_{\text{loc}}(\mathbb{R}^3 \times ]-1,0[), \] (4.18)
\[ q^{(k)} \to q^\infty \text{ in } L^2_{\text{loc}}(\mathbb{R}^3 \times ]-1,0[), \] (4.19)
and
\[ v^{(k)}(\cdot,0) \to v^\infty(\cdot,0) \text{ in the sense of distributions}. \] (4.20)
In particular,
\[ v^\infty(\cdot,0) = W^\infty. \] (4.21)
By Proposition 4.2, $v^\infty$ is essentially bounded in $\mathbb{R}^3 \times ]-3/4,0[\$. We claim that $v^\infty$ has a singular point $z^* \in \overline{Q(1/2)}$. Indeed, due to (4.15), we have
\[ \limsup_{k \to \infty} \|v^{(k)}\|_{L^\infty(Q(z^*,R))} = \infty \text{ for all } 0 < R < 1/4, \] (4.22)
for some $z^* \in \overline{Q(1/2)}$, and we may invoke Proposition 2.3. This contradicts that $v^\infty$ is essentially bounded in $\mathbb{R}^3 \times ]-3/4,0[\$ and completes the proof. 

We conclude with a few remarks:
Remark 4.3. The proof also implies that, if there exists a non-trivial mild ancient solution satisfying (4.2), then there exists a singular weak $L^{3,\infty}$ solution $v^\infty$ on $\mathbb{R}^3 \times [-1,0]$. By considering the energy-class correction $u^\infty(\cdot,t) = v^\infty - S(t)v(\cdot,-1)$ after the initial time, one obtains a singular weak Leray-Hopf solution with subcritical forcing term.

Using the theory of weak Besov solutions developed in [3], similar statements hold when $L^{3,\infty}$ is replaced by $B^{1+\frac{3}{p}}_{p,\infty}$ and when $L^3$ is replaced by $B^{-1+\frac{3}{p}}_{p,p}$ ($p > 3$). While similar results remain unknown in $\text{BMO}^{-1}$, a mild ancient solution satisfying $\|v(\cdot,t_k)\|_{\text{BMO}^{-1}} \to 0$ as $t_k \to -\infty$ must be identically zero. This follows from the perturbation theory in [19].

Similar statements seem to hold mutatis mutandis in the half-space with a different decomposition of the pressure, e.g., the one in [5] (see also the weak $L^3(\mathbb{R}^3)$ solution theory developed in [25]). It is interesting to note that, in the half-space case, one has the option to zoom out on an interior or boundary point.

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