Distinct scalings for mean first-passage time of random walks on scale-free networks with the same degree sequence

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(Dated: August 7, 2009)

In general, the power-law degree distribution has profound influence on various dynamical processes defined on scale-free networks. In this paper, we will show that power-law degree distribution alone does not suffice to characterize the behavior of trapping problem on scale-free networks, which is an integral major theme of interest for random walks in the presence of an immobile perfect absorber. In order to achieve this goal, we study random walks on a family of one-parameter (denoted by $q$) scale-free networks with identical degree sequence for the full range of parameter $q$, in which a trap is located at a fixed site. We obtain analytically or numerically the mean first-passage time (MFPT) for the trapping issue. In the limit of large network order (number of nodes), for the whole class of networks, the MFPT increases asymptotically as a power-law function of network order with the exponent obviously different for different parameter $q$, which suggests that power-law degree distribution itself is not sufficient to characterize the scaling behavior of MFPT for random walks, at least trapping problem, performed on scale-free networks.

PACS numbers: 05.40.Fb, 89.75.Hc, 05.60.Cd, 89.75.Da

I. INTRODUCTION

As a fundamental stochastic process, random walks have received considerable attention from the scientific society, since they found a wide range of distinct applications in various theoretical and applied fields, such as physics, chemistry, biology, computer science, among others [1, 2, 3]. Among a plethora of interesting issues of random walks, trapping is an integral major one, which plays an important role in an increasing number of disciplines. The so-called trapping issue that was first discussed in [4], is a random-walk problem, where a trap is positioned at a fixed location, absorbing all particles that visit it. The highly desirable quantity closely related to the trapping issue is the first-passage time (FPT), also called trapping time (TT). The FPT for a given site (node, vertex) is the time spent by the walker starting from the site to hit the trap for the first time. The average of first-passage times over all nodes is referred to as the mean first-passage time (MFPT), or mean trapping time (MTT), which is frequently used to measure the efficiency of the trapping problem.

One of the most important questions in the research of trapping is determining its efficiency, namely, showing the dependence relation of MFPT on the size of the system where the random walks are performed. Previous studies have provided the answers to the corresponding problems in some particular graphs with simple structure, such as regular lattices [5, 6], Sierpinski fractals [7, 8], T-fractal [9, 10], and so forth. However, recent empirical studies [11, 12, 13] uncovered that many (perhaps most) real networks are scale-free characterized by a power-law degree distribution [14, 15], which cannot be described by above simple graphs. Thus, it appears quite natural and important to explore trapping issues on scale-free networks. In recent work [16, 17, 18], we have shown that scale-free properties can substantially improve the efficiency of the trapping problem: the MFPT behaves linearly or sublinearly with the order (number of nodes) of the scale-free networks, which is in sharp contrast to the superlinear scaling obtained for above-mentioned simple graphs [19, 20, 21]. It was also demonstrated [22] that the high efficiency of trapping on scale-free networks is attributed to their power-law degree distribution. Although this feature can strongly affect the various dynamics occurring on networks, it was shown that the scale-free pattern itself does not suffice to characterize some dynamical processes on networks, e.g. synchronization [23, 24, 25], disease spreading [26, 27], and the like. These facts are still unknown whether the power-law degree distribution is sufficient to characterize the behavior of trapping problem on scale-free networks.
In this paper, we study the trapping problem on the a class of scale-free networks with the same degree se-
quence, which are dominated by a tunable parameter $q$. We determine separately the explicit formulas of the mean first-passage time for the two limiting cases of $q = 1$ and $q = 0$. We show that in both cases the MFPT increases as a power-law function of the network order, with the exponent less than 1 for $q = 1$ and equal to 1 for $q = 0$. We also study numerically the MFPT for the case of $0 < q < 1$, finding that it is also a power-

II. THE SCALE-FREE NETWORKS WITH IDENTICAL DEGREE SEQUENCE

The networks considered exhibit some interesting topo-
logical properties. Their nodes have same degree se-
quence (thus the same degree distribution), independent
of the value of parameter $q$. Concretely, the networks have a power-law degree distribution with the exponent $\gamma = 3 \ [? \ ]$. On the other hand, since there is no triangle in the whole class of the networks, the clustering coefficient is zero. Although the degree distribution and clustering coefficient do not depend on the parameter $q$, other structural characteristics are closely related to $q$. For example, for $q = 1$, the network is reduced to the $(2, 2)$-flower introduced in $[? \ ]$. In this case, it is a small world, its average path length (APL), defined as the mean of shortest distances between all pairs of nodes, grows logarithmically with the network order $[? \ ]$; at the same time, it is a non-fractal network $[? \ ? \ ]$. While for $q = 0$, it is exactly the hierarchical lattice that was proposed by Berker and Ostlund $[? \ ]$ and was extensively studied by many authors $[? \ ? \ ? \ ? \ ? \ ? \ ]$. For this case, the network is not small-world with the APL increasing as a square power of the network order $[? \ ? \ ]$; moreover, it is fractal with the fractal dimension $d_B = 2$. When $q$ increases from 0 to 1, the networks undergo a transition from fractal to non-fractal scalings, and exhibit a crossover from ‘large’ to small worlds at the same time $[? \ ]$; these similar phenomena are also observed in a family of treelike networks $[? \ ]$.

The peculiar topological features make the networks unique within the category of scale-free networks, since these particular structures strongly affect the dynamical processes defined on the networks. For instance, different thresholds of bond percolation were recently observed in the networks, which implies that power-law degree distribution alone does not suffice to characterize the percolation threshold on scale-free networks under bond percolation $[? \ ? \ ]$. In what follows, we will study random walks with a single immobile trap on the networks. We will show that the degree distribution is not sufficient to determine the scalings for MFPT of trapping process occurring on the networks under consideration.

III. RANDOM WALKS WITH A FIXED TRAP

In this section, we study the so-called simple discrete-time random walks of a particle on network $H_n$. At each time step, the particle (walker) jumps from its current location to one of its neighbors with equal probability. In particular, we focus on the trapping problem, i.e., a special issue for random walks with a trap positioned at a given node. To this end, we first we distinguish different nodes in $H_n$, by labeling them in the following way. The two nodes in $H_2$ have labels 1 and 2. For each new generation, we only label the new nodes created at this generation, while we keep the labels of all pre-existing nodes unchanged. In other words, we label sequentially new nodes as $M + 1, M + 2, \ldots, M + \Delta M$, where $M$ is the number of the old nodes and $\Delta M$ the number of newly-created nodes. In this way, every node is labeled by a unique integer, at generation $n$ all nodes are labeled from 1 to $V_n = \frac{2}{3}(4^n + 2)$. Figures ?? and ?? show how the nodes are labeled for two special cases of $q = 1$ and $q = 0$.

We place the trap at node 1, denoted by $i_T$. At each time step, the particle, starting from any node except the trap $i_T$, moves uniformly to any of its nearest neighbors. It should be mentioned that, due to the symmetry, the trap can be also situated at nodes 2, 3, or 4, which has not any effect on MFPT. The special selection we made for the trap allows to address the issue conveniently. Particularly, this makes it possible to analytically compute the MFPT for the two deterministic networks corresponding to $q = 1$ and $q = 0$ (details will be discussed below), because of their special structures and the convenience of identifying the trap $i_T$ since the first generation.

As mentioned above, one of the most important quantity characterizing such a trapping problem is the FPT defined as the expected time a walker takes, starting from a source node, to first reach the trap node. The significance firstly originates from the fact that the first encounter properties are relevant to those in a plethora of real situations $[? \ ]$, including transport, disease spread-
aging is that the particular construction of the networks proposed to get around this problem. What is encouraging is that the particular construction of the networks and the special choice of the trap location allow to calculate analytically MFPT to obtain a closed-form formula, at least for the two special cases of $q = 1$ and $q = 0$. The computation details will be provided in the following text.

Let $T_i^{(n)}$ be the FPT for a walker initially placed at node $i$ to first reach the trap $i_T$ in $H_n$. This quantity can be expressed in terms of mean residence time (MRT) \[ T_i^{(n)} = \sum_{j=2}^{V_n} (b^{-1}_{n,i})_{ij}. \] (4)

Then, the mean first-passage time, $\langle T \rangle_n$, which is the average of $T_i^{(n)}$ over all initial nodes distributed uniformly over nodes in $H_n$ other than the trap, is given by

\[ \langle T \rangle_n = \frac{1}{V_n-1} \sum_{i=2}^{V_n} T_i^{(n)} = \frac{1}{V_n-1} \sum_{i=2}^{V_n} \sum_{j=2}^{V_n} (b^{-1}_{n,i})_{ij}. \] (5)

Equation (5) shows that the problem of determining $\langle T \rangle_n$ is reduced to computing the sum of all elements of the fundamental matrix $(B_n)^{-1}$. Although the expression of Eq. (5) seems compact, the complexity of inverting $L_n$ is $O(V_n^3)$. Since the network order increases exponentially with $n$, Eq. (5) becomes intractable for large $n$. Thus, restricted by time and computer memory, one can obtain $\langle T \rangle_n$ through direct calculation from Eq. (5) only for the first iterations. It would be satisfactory if good alternative computation methods could be proposed to get around this problem. What is encouraging is that the particular construction of the networks and the special choice of the trap location allow to calculate analytically MFPT to obtain a closed-form formula, at least for the two special cases of $q = 1$ and $q = 0$. The computation details will be provided in the following text.

A. Case of $q = 1$

We first establish the scaling relation governing the evolution for $T_i^{(n)}$ with generation $n$. In Table ??, we list the numerical values of $T_i^{(n)}$ for some nodes up to $n = 6$. From the numerical values, we can observe that for a given node $i$, the relation $T_i^{(n+1)} = 3 T_i^{(n)}$ holds. That is to say, upon growth of the network from generation $n$ to generation $n + 1$, the trapping time to first arrive at the trap increases by a factor 3. This is a basic characteristic of random walks on $H_n$ when $q = 1$, which can be established from the arguments below [??].

Consider an arbitrary node $i$ in $H_n$ of the $q = 1$ case, after $n$ generation evolution. From Eq. (5), we know that upon growth of the network to generation $n + 1$, the degree, $k_i$, of node $i$ doubles, namely, it increases from $k_i$ to $2k_i$. Among these $2k_i$ neighbors, one half are old neighbors, while the other half are new nodes created at generation $n + 1$, each of which has two connections, attached to node $i$ and another simultaneously emerging new node. We now examine the standard random walk in $H_{n+1}$. Let $X$ be the FPT for a particle going from node $i$ to any of its $k_i$ old neighbors; let $Y$ be the FPT for going from any of the $k_i$ new neighbors of $i$ to one of the $k_i$ old neighbors; and let $Z$ represent the FPT for starting from any of new neighbors (added to the network at generation $n + 1$) of an old neighbor of $i$ to this old neighbor. Then we can establish the following backward equations:

\[ \begin{cases} X = \frac{1}{2} + \frac{1}{2}(1 + Y), \\ Y = \frac{1}{2}(1 + X) + \frac{1}{2}(1 + Z), \\ Z = \frac{1}{2} + \frac{1}{2}(1 + Y). \end{cases} \] (6)

Equation (6) has a solution $X = 3$. Thus, upon the growth of the network from generation $n$ to generation $n + 1$, the first-passage time from any node $i$ to any node $j$ (both $i$ and $j$ belong to $H_n$) increases by a factor of 3. That is to say, $T_i^{(n+1)} = 3 T_i^{(n)}$, which will be useful for the derivation of the exact formula for the MFPT below.

After obtaining the scaling of first-passage time for old nodes, we now derive the analytical rigorous expression for the MFPT $\langle T \rangle_n$. Before proceeding further, we first introduce the notations that will be used in the rest of this section. Let $\Delta_n$ denote the set of nodes in $H_n$, and let $\overline{\Delta}_n$ stand for the set of those nodes entering the network at generation $n$. For the convenience of computation, we define the following quantities for $1 \leq m \leq n$:

\[ T_{m,\text{tot}}^{(n)} = \sum_{i \in \Delta_m} T_i^{(n)}, \] (7)

and

\[ T_{m,\text{tot}}^{(n)} = \sum_{i \in \overline{\Delta}_m} T_i^{(n)}. \] (8)

By definition, it follows that $\Delta_n = \overline{\Delta}_n \cup \Delta_{n-1}$. Thus,
