The dynamic wave coefficient

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Abstract. Seismic calculations nowadays are carried out without accounting for the wave nature of the problem. All the seismic wave energy is assumed to be transmitted to the structure, but the wave part is reflected. The oscillating structure, in turn, affects the soil and gives back some of the energy to the soil. Thus failure to the wave nature of the problem results in to the excess of the calculated seismic loads over the real loads. Moreover, the magnitude of this excess is unknown.

The paper presents a solution of the wave effects determining problem. To account for the wave effects the system consisting of the soil seismic vibrations wave equation and the structure vibrations equation is solved. Thus, the task seems to be very difficult.

The purpose of the work is to develop and analyze compact exact solutions of the above task, since compact exact solutions can be put into practice of engineering calculations.

In the works of well-known authors an exact solution of the system of equations consisting of a wave equation describing the joint longitudinal seismic vibrations of the earth's crust and the equation of oscillation of the structure in the form of a point insert was obtained.

But the numerical calculations for the exact solution were not carried out.

The abstract solution analytical calculation from the work [1] for a concrete initial form of a seismic wave was carried out in this paper. The initial seismic wave is selected in the truncated harmonic function form. This choice allowed to derive a new coefficient – a dynamic wave coefficient. Numerical calculation of the wave dynamic coefficient showed that the classical non-wave methods for calculating seismic forces give overstated assessment. The dynamic coefficient depends on the properties of the soil.

In general, it also depends on the material of which the structure is made. For example, if the soil passes only high frequency seismic vibrations, it may be useful to select new materials to create a low frequency structure.

But the exact solution of the wave seismic stability problem is difficult even in the initial formulation. Therefore, in the present work a detailed analysis of the obtained exact solution was carried out.
Introduction

The purpose of the work is to demonstrate the wave approach advantages to the problems of the structures seismic stability of. The soil longitudinal seismic vibrations equations system solution with the construction in the form of a concentrated mass with the wave formulation of the problem allowed to derive a new coefficient – the dynamic wave coefficient.

The dynamic wave coefficient allows to calculate the share of the reflected energy of the seismic wave and the share of the energy of the seismic wave transmitted to the structure. In view of the wave coefficient importance, four different forms of the wave coefficient were presented in the work.

The work also investigated the dependence of the dynamic wave coefficient on various system parameters: on the frequency of the wave falling on structures, on the mass of the structure, on the soil material, that is on the phase speed of the seismic wave, on the dimensions of the foundation. The dependence of the dynamic coefficient on the material properties of the structure was not considered in this model.

Problem statement

Seismic longitudinal oscillations of the soil are defined as solutions of a hyperbolic partial differential equation with known initial data. The oscillations of a structure in the form of a concentrated mass are set with an ordinary differential equation. The conditions for the interface between the vibrations of the soil and the vibrations of the structure are specified. As a result, we get the following mathematical model.

The earth’s crust is modeled by an Infinite Elastic Rod (refer with figure 1).

![Figure 1. One-dimensional scheme ground - structure](image)

The origin is a structure of mass M in the form of a point insert. Joint longitudinal vibrations of the earth’s crust and structures are studied. The data for the numerical calculation are taken from the work [3]. The seismic stability of a three-story ferroconcrete house with a total mass of \( M = 1200000 \text{ kg} = 1200 \text{ t} \) is calculated. The soils are loams with a module of elasticity and density, respectively:

\[
E = 7 \times 10^4 \times 10^3 \frac{H}{m^2}, \rho = 1.8 \times 10^3 \frac{kg}{m^2}
\]

Ideally, the phase speed of the wave is calculated by the formula \( a = \frac{E}{\sqrt{\rho}} = 197.2 \).

The plan’s size of the building is 18×18. We assume that the height of the basement is 3 meters. Then the side wall area affected by the seismic wave is: \( S = 3 \times 18 = 54 \text{ m}^2 \). Let us set the frequency of the seismic wave, suggesting that:

\( n = 4 \text{ Hz}, \ \omega = 2\pi n = 25.13. \)

We introduce some necessary notations:

- \( U(x, t) \) – longitudinal deviations of the point \( x \) at the time \( t \),
- \( f(x) \) – initial seismic wave where \( a = \frac{2ES}{a^2M} = \frac{2\rho S}{M} \) – exponential coefficient
It is assumed that a seismic harmonic wave of the following form moves to the structure from left to right:

\[ U(x, t) = \sin(kx - \omega t) = \sin\left(\frac{\omega}{a} x - \omega t\right), \quad k = \frac{\omega}{a} = 0.127. \]

The parameter \( k \) is the wave number. The initial conditions are:

\[ f(x) = U(x, 0) = \sin(kx) = \sin\left(\frac{\omega}{a} x\right) \]

We construct a truncated sinusoidal harmonic. To this end, we set negative integers: \( d = -10; b = -2 \). To this end, we set negative integers:

We define the interval of localization of the initial seismic wave, outside of which it is equal to zero with the segment: \([ad; ab]\). That is, the function \( f(x) \) is

\[ f(x) = \begin{cases} f_0 \sin(kx); & x \in [ad; ab]; \\ 0; & x \not\in [ad; ab] \end{cases} \]

The function \( f(x) \) is defined on the negative part of the real line. The equation of the longitudinal oscillations of the elastic rod has the form:

\[ \rho(x)S(x)\frac{\partial^2 U(x, t)}{\partial t^2} = \frac{\partial}{\partial x}\left[ E(x)S(x)\frac{\partial U(x, t)}{\partial x}\right] \quad (1) \]

where:
- \( t \) – time,
- \( \rho \) – volume density of soil,
- \( S \) – bar cross-section in the point \( x \),
- \( E \) – elastic modulus of the soil in the point \( x \),
- \( \mu(t) \) – longitudinal displacements in the form of the material particle.

If bar cross-section, volume density of the soil, and elastic modulus of the bar are constant, the equation (1) of the longitudinal oscillations of the bar will have a form:

\[ U''(x, t) = a^2 U''_{xx}(x, t), \quad a = \sqrt{\frac{E}{\rho}} \]

The longitudinal force arising in the cross section is:

\[ F = ESU'_x(x, t). \]

The longitudinal displacements of the soil are expressed by the following form:

\[ U(x, t) = \begin{cases} U_1(x, t), & x < 0, \\ U_2(x, t), & x > 0 \end{cases} \]

The equations described the longitudinal oscillations of the soil are:

\[ \begin{cases} U'_{1t} = a^2 U''_{1xx}, & x < 0, \quad 0 \leq t < \infty \\ U'_{2n} = a^2 U''_{2xx}, & x > 0, \quad 0 \leq t < \infty \end{cases} \quad (2) \]

The equations described the longitudinal oscillations of the construction in the form of the material particle will have the following form:
\[ M \mu'_t = ES \left( U_{2r}(0,t) - U_{1r}(0,t) \right) \]  

(3)

The interface condition has a form: \( U_1(0,t) = U_2(0,t) \), and means that the seismic wave is a continuing function at any time. That is, it is not torn at the moments of hitting the building. The initial conditions are:

\[
\begin{cases} 
U_1(x,0) = f(x), & x < 0 \\
U_2(x,0) = 0, & x > 0 \\
U'_r(x,0) = -af'(x), & x < 0 
\end{cases}
\]

(4)

In the work [1] the following theorem was derived:

Theorem 1. System (2-4) has the following solution:

\[
\begin{cases} 
U_1(x,t) = f(x-at) + \psi(-x-at), & x < 0 \\
U_2(x,t) = \varphi(x-at), & x > 0 
\end{cases}
\]

where the function \( \varphi(z) \) is a solution of the following differential equation:

\[
\varphi^*(z) - \alpha \varphi'(z) = -af'(z), \quad \varphi(0) = 0, \varphi'(0), z \leq 0, 
\]

(5)

the function \( \psi(z) \), and the equation of the structure oscillations a equal to:

\[
\psi(z) = \varphi(z) - f(z), z \leq 0; \quad \mu(t) = \varphi(-at) 
\]

Integrating the equation with a truncated harmonic initial wave we obtain the following solution formulated as a theorem.

Theorem 2. The equation of the structure oscillation – the function \( \mu(t) \) with a falling truncated harmonic wave has the form:

\[
\mu(t) = \begin{cases} 
0 \text{ with } t \ll -b = 2; \\
\alpha f_0 \frac{k \cos(\alpha t) - \alpha \sin(\alpha t)}{k^2 + \alpha^2} - \frac{k \cos(abk) e^{-\alpha a(t+b)}}{k^2 + \alpha^2} \text{ with } (-b < t < -d); \\
\alpha f_0 k \cos(abk) e^{-\alpha a(t+d)} - \cos(abk) e^{-\alpha a(t+b)} \text{ with } t \gg -d = 10.
\end{cases}
\]

(6)

In work [2] only the harmonic component of the obtained solution (6) without exponential components was considered. In the first approximation this is permissible, since the exponential components are small. By rejecting the exponential components, we get an approximate solution:

\[
p(t) = \begin{cases} 
0 \text{ with } t \ll -b = 2; \\
\alpha f_0 \frac{k \cos(\alpha t) - \alpha \sin(\alpha t)}{k^2 + \alpha^2} \text{ with } (-b < t < -d); \\
0 \text{ with } t \gg -d = 10.
\end{cases}
\]
Considering this function only on the interval \( 2 = -b < t < -d = 10 \), we get:

\[
p(t) = \frac{\alpha f_0}{\sqrt{k^2 + \alpha^2}} \sin(\omega t - \varphi), \varphi = \arcsin\left(\frac{k}{\sqrt{k^2 + \alpha^2}}\right), (2 = -b < t < -d = 10)
\]

Thus, for the approximate solution in the form of a harmonic seismic wave passing through the structure, the amplitude decreases and the oscillation phase changes.

**The calculation of the dynamic wave coefficient. Wave effect №1**

Definition. The dynamic wave coefficient \( \beta \) is called the amplitudes ratio of the falling and transmitted approximate waves. Calculate the dynamic wave coefficient:

\[
\beta = \frac{\alpha f_0}{f_0} = \frac{\alpha}{\sqrt{k^2 + \alpha^2}} = \frac{0.162}{\sqrt{0.127^2 + 0.162^2}} = 0.786 \quad (7)
\]

The fact that the dynamic wave coefficient has been strictly less than unity is the manifestation of the wave effect of the problem. The result obtained (7) means that the seismic load calculated with consideration of wave factors is 21.4% less than the seismic load calculated by classical non-wave methods. We study the cases when the dynamic wave coefficient will be significantly less than unity.

Transform (7):

\[
\beta = \frac{\alpha}{\sqrt{k^2 + \alpha^2}} = \frac{\alpha}{\sqrt{\frac{\omega}{\alpha} + \alpha^2}} \quad (8)
\]

From the form (8) of the dynamic wave coefficient we derive its new property formulated as the following theorem:

**Theorem 3.** The greater the frequency of the falling harmonic wave, the lower the dynamic wave coefficient. Carry out calculations of the dynamic wave coefficient depending on the frequency falling on the construction of the wave (refer with Table 1):

**Table 1.** The dynamic wave coefficient dependence on the seismic frequency harmonic wave

| Frequency [Hz] | The dynamic wave coefficient |
|----------------|-----------------------------|
| 1              | 0.98                        |
| 3              | 0.86                        |
| 5              | 0.71                        |
| 7              | 0.59                        |
| 9              | 0.49                        |
| 11             | 0.42                        |
| 13             | 0.36                        |
| 15             | 0.32                        |

Obviously, with a high-frequency earthquake, the dynamic wave coefficient decreases to 0.32. That meant the wave approach to seismic problems is justified in this case.

We represent the wave dynamic coefficient by the following form:
\[ \beta = \frac{\alpha}{\sqrt{k^2 + \alpha^2}} = \frac{1}{\sqrt{\frac{k}{\alpha}} + 1} = \frac{2ES}{\sqrt{(\omega M)^2 + (2ES)^2}} \tag{9} \]

The form (9) of the dynamic wave coefficient allows to make the following conclusion:

**Theorem 4.** The larger the mass of the structure, the lower the dynamic wave coefficient.

We give another form of the dynamic wave coefficient.

**Definition.** The given mass \( M_1 = \rho Sa \) is called the mass of the wave passing through the building foundation per unit of time.

Then the dynamic wave coefficient can be represented by the following form:

\[ \beta = \frac{\alpha}{\sqrt{k^2 + \alpha^2}} = \frac{1}{\left( \sqrt{\frac{\omega}{2M_1/M}} \right)^2 + 1} \tag{10} \]

From the last representation of the dynamic wave coefficient (10) it follows that the dynamic wave coefficient first of all depends on the ratio of the reduced mass and the present mass of the structure – the value \( 2M_1/M \). The ratio has the frequency dimension.

And, secondly, the dynamic wave coefficient depends on the relationship between the ratios. That is, the dynamic wave coefficient depends on the ratio of the frequency of the wave falling on the structure and the ratio of the reduced mass and the present mass of the structure.

All four forms of the dynamic wave coefficient (7-10) are important, since each of these forms expresses one or another physical regularity of the studied problem.

**Wave effect № 2.** From the equation of the structure oscillation (6) it follows that the harmonic component acts only on the time interval \( [2t = -b < t < -d = 10] \) and then disappears. Then at \( t = 10 = -d \leq t < \infty \), the exponential component acts. This means that after passing through the harmonic component the structure returns to a stationary state indefinitely. At first glance, it seems that the system of equations has a non-zero friction coefficient, although there is no friction in the model. There is a wave effect.

**Wave effect № 3.** Let’s study the function \( \mu(t) \) at the time \( t = 2 \), that is, at the time when the structure only begins to accelerate. We expand the function \( \mu(t) \) into a Taylor series in a proximity of the point \( t = -b = 2 \). We get:

\[ \mu(t) \approx -\frac{\alpha k f_0 \left( \frac{a^2}{2} + \omega^2 \right)}{k^2 + \alpha^2} (t-2)^2 = -2.54 (t-2)^2 \tag{11} \]

At low values an approximate equality takes place: \( \sin(\omega \tau) \approx \omega \tau \). It means that the falling truncated harmonic wave accelerates from zero with values in direct proportion to time. From (11) it follows that the structure accelerates from scratch according to the square law, that is, slower. It means that an inertial lag due to the large mass of the structure takes place. Thus, there is another wave effect. The Tashkent earthquake of 1966 lasted only 12 seconds. The initial phase of acceleration of the structure oscillation is a split second. But we are interested in something else. We want to know whether the wave model reflects correctly the physics of phenomena.
We note that at the initial moment of acceleration of the structure oscillation, the function $\mu(t)$ has a negative value. It means that at the moment of seismic wave hit, the structure did not swing to the right, but to the left. The thing is, what form to set the initial seismic wave.

We have it truncated by a whole number of periods, a sinusoidal seismic wave. This wave moves from left to right, and therefore the negative part of the sine, which is a rarefaction wave, hits the structure at the time of the collision. The rarefaction wave does not push, but draws the structure to itself. Therefore, at the moment of impact, the structure will swing to the left, not to the right.

**Comparison of the approximate and exact values of the solution of the problem**

Approximate formula of the structure oscillation – the function $p(t)$ is a truncated harmonic function, extended in both directions by the numerical straight line by zero. It has gaps in two points: $t_1 = -b = 2, t_2 = -d = 10$ (refer with figure 2).

![Figure 2. Graphs of approximate and exact formulas for the structure oscillation](image)

The reason for the appearance of gaps in the approximate solution is evident. The initial seismic wave falling on the structure – the function $f(x)$ is specified in the form of a truncated sinusoidal harmonic wave, extended to both directions of a numerical straight line by zero. "Sine clipping" was performed at those points where the sine is zero. Therefore, with the continuation of the truncated sine to the numerical straight line the function $f(x)$ has been continuous.

But when this wave passes the structure, the oscillation phase changes and the sine function shifts along the numerical axis to the formed phase. Therefore, by truncating of the shifted sine and its continuing by zero to the entire numerical straight line, result function became discontinuous at two points.

Despite the two discontinuity points and the fact that the function $p(t)$ is only an approximate solution of the system of equations, it turns out that the function $p(t)$ has the following good property: the approximate function $p(t)$ and the corresponding approximate solutions of the system – the functions $\hat{U}_1(x,t), \hat{U}_2(x,t)$, satisfy the system of equations (2-4) at all points except two points.

Therefore, because of the importance of the discontinuity points, it is necessary to substitute the functions $p(t), \hat{U}_1(x,t), \hat{U}_2(x,t)$ into the system (2-4) and organize the calculation in generalized functions. The first derivative of the function $p(t)$ at the points of discontinuity will be the delta function – $\delta(t)$, and the second derivative of the function will be the derivative of the delta function – $\delta'(t)$ with some numerical factors.

Therefore, when we substitute approximate solutions in the wave equations, we have:

$$\hat{U}_{1n} = a^2 \hat{U}_{1xx}, \hat{U}_{2n} = a^2 \hat{U}_{2xx},$$
delta functions will appear on both sides of the equality. But the specificity of an approximate solution is such that the same generalized functions of the left and right equalities appear at the same points.

The only equation that does not satisfy the approximate solution is equation (5). Let's check this fact. We solve the inverse problem. We find the function \( \varphi(z) \) from the equation: \( p(t) = \varphi(-at) \). It follows that the function \( \varphi(z) \) must have the form of a truncated harmonic function with two discontinuity points. Then in equation (5) on the right of the equality there will be the usual discontinuous function, and on the left there will be the derivative of the delta function. We get a contradiction and we conclude: the approximate solution \( p(t) \) does not satisfy equation (5). It means that it is not an exact solution, but it is an approximate solution of the system (2-4).

**Summary**
The main result of the article is the fact that with the wave formulation of the seismic problem we get another dynamic coefficient called the dynamic wave coefficient. The dynamic wave coefficient shows that the classical non-wave methods for solving seismic problems can lead to the seismic forces overstated assessment acting on the structure. And the greater the purity of the seismic impact, the greater can be this error.

Another disadvantage of classical seismic calculation formula is the fact that the parameter \( T \) — the earthquake duration is not included in this case. To put the \( T \) parameter in the mathematical model, the initial seismic wave must have a compact limited vector, it is zero outside of this vector. To put the dynamic wave coefficient, the initial wave is to be harmonic. Therefore, the initial seismic wave must have a truncated sinusoid shape.

The truncated sinusoid can be extended by zero to the whole numerical axis, so that it becomes a continuous function. But it is impossible to continue so that it also becomes differentiable function at each point of the line. Then numerical effects arise due to the non-smoothness of the initial seismic wave. But wave effects No.2 and No.3 are not related to the non-smoothness of the initial truncated wave as it would seem at first glance.

It is possible to set the initial seismic wave in the finite function form:

\[
\varphi(x) = \begin{cases} 
0, & |x| \geq a \\
\exp\left(-\frac{a^2}{a^2-x^2}\right), & |x| < a
\end{cases}
\]

But in this case the wave effects No.1, No.2, No.3 will take place.

**References**
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