GLOBAL EXISTENCE AND GRADIENT BLOWUP OF SOLUTIONS FOR A SEMILINEAR PARABOLIC EQUATION WITH EXPONENTIAL SOURCE

ZHENGCE ZHANG AND YAN LI
School of Mathematics and Statistics, Xi’an Jiaotong University
Xi’an, 710049, China

(Communicated by Patricia Bauman)

ABSTRACT. Throughout this paper, we consider the equation
\[ u_t - \Delta u = e^{\|\nabla u\|}, \quad x \in \Omega, \quad t > 0, \]
with homogeneous Dirichlet boundary condition. One of our main goals is to show that the existence of global classical solution can derive the existence of classical stationary solution, and the global solution must converge to the stationary solution in \( C(\overline{\Omega}) \). On the contrary, the existence of the stationary solution also implies the global existence of the classical solution at least in the radial case. The other one is to show that finite time gradient blowup will occur for large initial data or domains with small measure.

1. Introduction. In this paper, we study the following Dirichlet problem
\[
\begin{aligned}
& u_t - \Delta u = e^{\|\nabla u\|}, \quad x \in \Omega, \quad t > 0, \\
& u = 0, \quad x \in \partial \Omega, \quad t > 0, \\
& u(x, 0) = u_0(x), \quad x \in \Omega,
\end{aligned}
\]  
(1)

where \( \Omega \) is a smoothly bounded domain in \( \mathbb{R}^N \), \( u_0 \in X := C^1_0(\overline{\Omega}) = \{ u_0 \in C^1(\overline{\Omega}); u_0 \geq 0, u_0 = 0 \text{ on } \partial \Omega \} \) and satisfies the second-order compatibility condition
\[ -\Delta u_0 = e^{\|\nabla u_0\|} \text{ on } \partial \Omega. \]

It’s well-known [9, Theorem 8.2, p.206] that there exists a unique, maximal-in-time classical solution of (1) with the maximal existence time \( T^* = T^*(u_0) \in (0, \infty) \) and the solution \( u \) satisfies
\[ u \in C^{2,1}(\overline{\Omega} \times (0, T^*)) \cap C(\overline{\Omega} \times [0, T^*]). \]

Then, the nonnegativity of \( u \) can be directly obtained by the maximum principle.

For the problem (1), there has some related results in the one-dimensional or radial case. In [17], Zhang and Hu considered the one-dimensional case, and they proved that \( \nabla u \) will blow up in a finite time, i.e.
\[ \|u(\cdot, t)\|_\infty < \infty, \quad \lim_{t \to T} \|\nabla u\|_\infty = \infty, \quad t \in (0, T) \]

2010 Mathematics Subject Classification. Primary: 35K58; Secondary: 35A01, 35B40, 35B44.

Key words and phrases. Global solution, stationary solution, gradient blowup.

This work was supported by the National Natural Science Foundation of China (No. 11371286) and by the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.
for some large $u_0 \in X$. The blowup rate was also established in that paper. In [18], Zhang and Li studied the radial case and obtained some similar results to [17]. In [3], the continuation after blowup and the rate of converging to a singular steady state were obtained in the one-dimensional case. Another important problem is the large time behavior for the global solution. For this problem, Zhu and Zhang gave some results on annular domains in [21]. Besides, the boundedness of the global solution is also an interesting problem. The one dimensional case had been solved by Zhang and Li in [19]. However, there has no result on general domains.

Similar to (1), the problem
\[
\begin{cases}
  u_t - \Delta u = |\nabla u|^p, & x \in \Omega, t > 0, \\
  u = 0, & x \in \partial \Omega, t > 0, \\
  u(x, 0) = u_0(x), & x \in \Omega
\end{cases}
\]

had been studied by many people so far. It’s well-known that the differential equation in (2) comes from the famous Kardar-Parisi-Zhang (KPZ) equation which was first mentioned in [6] and developed by Krug and Spohn aiming at studying the influence of the nonlinearity to the solution in [7]. When $p > 2$, it was shown in [5, 10, 12, 13, 14] that the solution of (2) cannot exist globally if the initial data is suitably large. While if there exists $\epsilon > 0$, such that the initial data satisfies: $\|u_0\|_{C^1(\Omega)} < \epsilon$, then it was shown in [12, Proposition 3.1] that the solution is global and bounded in $C^1(\Omega)$. If the solution of (2) exhibits gradient blowup phenomenon, we want to know where it occurs, what’s the exact gradient blowup rate and so on. In [8], Li and Souplet showed that the gradient blowup will occur at a single point of $\partial \Omega$ if $\Omega \subset \mathbb{R}^2$ and has some symmetric properties. For the blowup rate, it had been shown in [4] that the exact rate is $(T - t)^{-1/(p-2)}$ in the one-dimensional case, i.e. $C_1 (T - t)^{-1/(p-2)} \leq \|u(t)\|_\infty \leq C_2 (T - t)^{-1/(p-2)}$ for some $C_1, C_2 > 0$. The lower estimate was also extended to higher dimensional space by the rescaling methods, while the upper estimate in this case is still an open problem. Apart from the finite time gradient blowup, the properties of the global solution is also worth researching. This had been partially solved in [1, 14, 15] and the references therein.

For other similar problems which are related to the gradient blowup, we refer the readers to [2, 10, 16, 20] and the references therein. We are also eager to know whether the results obtained for (2) are still valid for the problem (1). An obvious fact is that (2) is invariant under the self-similar transformation while (1) is not. Hence, we conjecture that the properties for the self-similar solutions of (2) cannot extend to (1). Another important fact is that the differential equations in (1) and (2) are two specific examples of the more general equation $u_t - \Delta u = F(x, t, u, \nabla u)$ for some $F \in C^1(\Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^N)$ satisfying suitable growth conditions. Some typical results on the general case when $F = F(\nabla u)$ had been studied by Fila and Lieberman in the one-dimensional case and by Souplet in the high-dimensional case, see [2, 12]. In this paper, we are to consider the specific case (1), and to study the large time behavior of the global solution and the finite time gradient blowup.

For the large time behavior, we will consider the relation between the global solution and the stationary solution of the following stationary problem.
\[
\begin{cases}
  -\Delta v = e^{|\nabla v|}, & x \in \Omega, \\
  v = 0, & x \in \partial \Omega.
\end{cases}
\]

For the solution of (3), we will use the following definition.
Definition 1.1. A solution $v$ of (3) is a function $v \in C^2(\Omega) \cap C_0(\overline{\Omega})$ which solves the differential equation in (3) in the classical sense in $\Omega$.

Now, let us introduce our main results for the global solution of (1).

**Theorem 1.2.** Assume that the solution $u$ of (1) is global in time for some $u_0 \in X$. Then there exists a solution $v$ of (3) and $u$ must converge uniformly to $v$ in $C(\overline{\Omega})$, as $t \to \infty$.

**Remark 1.** As a direct consequence of the maximum principle, one can easily verify that the solution of (3) must be unique if it exists. Souplet and Zhang showed in [15] that the solution of $-\Delta u = |\nabla u|^p$ with homogeneous Dirichlet boundary condition can only be 0. However, this is incorrect for our problem as any constant can’t be the solution of (3). But, if we rewrite the differential equation in (3) as the inhomogeneous one $-\Delta v = e^{\nabla v^p} - 1$, then the only solution must be 0 provided that $v = 0$ on $\partial \Omega$.

**Theorem 1.3.** Assume that $N \geq 2$ and $\Omega$ is a ball $B_R$. If there exists a solution $v$ of (3), then $u_0 \leq v$ implies the global existence of the solution of (1), i.e. $T^* = \infty$.

**Remark 2.** The conclusion of Theorem 1.3 is also correct when $N = 1$.

Different from the global existence and convergence, the following two theorems give another behavior of the solution of (1).

**Theorem 1.4.** For the problem (1), if $u_0$ is large enough, $\nabla u$ must blow up in a finite time $T$: $$\lim_{t \to T^-} \|\nabla u\|_{L^\infty(\Omega)} = \infty.$$ 

**Remark 3.** As was shown in Theorem 1.4, the large initial data can imply the finite time gradient blowup, we are also naturally eager to know whether small initial can imply the global existence and boundedness or not. For example, if there exists some $\epsilon > 0$, such that $\|u_0\|_{C^1} < \epsilon$, then the solution of (1) is global and bounded in $C^1(\overline{\Omega})$. Our main thought is to construct a supersolution and a subsolution. The subsolution can be obtained by reforming the solution of $-\Delta \chi = 1$ with $\chi = 0$ on $\partial \Omega$. However, due to the technical difficulty, we are unable to construct a supersolution.

Unlike the condition that $u_0(x)$ is large enough in certain $\Omega$, the following theorem shows that whether gradient blowup can occur or not depends on the measure of $\Omega$. In [2], Fila and Lieberman construct an example saying: gradient blowup may occur only at $x = 0$ for any $u_0(x) \in C^1[0, L]$, where $L$ must be larger than a given $L_0$. Here, we develop their result to higher dimensional spaces for some given initial data. More precisely, our result implies that gradient blowup can occur if the measure of $\Omega$ is small enough.

**Theorem 1.5.** Let $\phi_1$ be the solution of the following eigenvalue problem

$$\begin{cases}
-\Delta \phi = \lambda_1 \phi, & x \in \Omega, \\
\phi = 0, & x \in \partial \Omega, \\
\int_{\Omega} \phi(x) \, dx = 1,
\end{cases}$$

where $\lambda_1$ is given by $\lambda_1 = \inf\{\|\nabla u\|_{L^2(\Omega)}; u \in K\}$, $K = \{u \in H_0^1(\Omega); \|u\|_{L^2(\Omega)} = 1\}$. Let $u_0(x) = \phi_1$ and $N \geq 2$. Then there exists a small constant $M > 0$ such that gradient blowup occurs in finite time if $|\Omega| < M$. 

Remark 4. We also note that the initial data used in Theorem 1.5 can be replaced by some smooth function $\phi(x)$ which satisfies: $\int_\Omega \phi(x) \, dx = 1$.

2. Preliminaries. At the beginning of this section, we bring in the following notations.

- $Q_T := \Omega \times (0, T)$, $S_T := (\overline{\Omega} \times \{0\}) \cup (\partial \Omega \times (0, T))$.
- $\vec{n}$: the unit exterior normal vector on $\partial \Omega$.
- $\delta(x) = \text{dist}(x, \partial \Omega)$: the distance function to $\partial \Omega$.
- $\Omega_\epsilon := \{x \in \Omega; \delta(x) > \epsilon > 0\}$.

2.1. Comparison principle and applications. Here, we will give the following useful lemmas which are similar to the ones in [15, Propositions 2.1-2.4, Lemma 2.1]. The proofs there are also valid to ours with a little modification, here we omit it.

Lemma 2.1. Let $0 < T < \infty$ and $\Omega \subset \mathbb{R}^N$. Assume that $u_1, u_2 \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$, satisfy

$$\partial_t u_1 - \Delta u_1 - e^{\|\nabla u_1\|} \leq \partial_t u_2 - \Delta u_2 - e^{\|\nabla u_2\|}, \quad \text{in } Q_T.$$  

If $\Omega$ is unbounded, assume additionally that $\sup_{Q_T} (u_1 - u_2) < \infty$ and that either $\nabla u_1 \in L^\infty(Q_T)$ or $\nabla u_2 \in L^\infty(Q_T)$. Then

$$\sup_{Q_T} (u_1 - u_2) \leq \sup_{S_T} (u_1 - u_2).$$

Lemma 2.2. Let $0 < T < \infty$ and $K \geq 0$. Let $\Omega$ be any (possibly unbounded) domain of $\mathbb{R}^N$, and let $f = f(x, t)$ be such that $f(\cdot, t) \in L^2_{\text{loc}}(\Omega)$ for a.e. $t \in (0, T)$. Assume that $w \in C([0, T]; L^2_{\text{loc}}(\Omega))$ satisfies $\sup_{Q_T} w < \infty, w_i, \nabla w, D^2 w \in L^2_{\text{loc}}(\Omega \times (0, T))$, and

$$w_i - \Delta w \leq K|\nabla w| + f \quad \text{a.e. in } Q_T.$$  

Finally assume that, for some $A, B \in \mathbb{R}, f \leq 0$ a.e. on $\{(x, t) \in Q_T; w(x, t) \geq A\}$ and $w \leq B$ on $S_T$. Then

$$\sup_{Q_T} w \leq \max(A, B).$$

Lemma 2.3. (i) Any solution of (1) satisfies

$$\|\partial_x u(t)\|_{\infty} \leq \max \left( \|\partial_x u_0\|_{\infty}, \sup_{\partial \Omega \times (0, t)} \left| \frac{\partial u}{\partial \vec{n}} \right| \right), \quad 0 < t < T^*.$$  

(ii) Let $u$ be a solution of (1) and $v$ a solution of (3). Then

$$\|u(t) - v\|_{\infty} \leq \|u_0 - v\|_{\infty}, \quad 0 < t < T^*(u_0).$$

Lemma 2.4. Assume that $\Delta u_0 \in L^\infty(\Omega)$, then the solution of (1) satisfies

$$\min \left\{ 0, \essinf_{\Omega} \left( \Delta u_0 + e^{\|\nabla u_0\|} \right) \right\} \leq u_t \leq \max \left\{ 0, \esssup_{\Omega} \left( \Delta u_0 + e^{\|\nabla u_0\|} \right) \right\}$$  

for $x \in \Omega, 0 < t < T^*$.

Next, we will introduce the following lemma as an application of the comparison principles above. This result will play an important role in the proof of Theorem 1.3.
Lemma 2.5. Assume that there exists a \( \bar{u} \) satisfying
\[
\begin{cases}
\bar{u}_t - \Delta \bar{u} \geq e^{\langle \nabla \bar{u} \rangle}, & x \in \Omega, 0 < t < \infty, \\
\bar{u} = 0, & x \in \partial \Omega, 0 < t < \infty, \\
\bar{u}(x, 0) \geq u_0(x), & x \in \Omega
\end{cases}
\]
for some \( u_0 \in X \). In additional, \( \bar{u} \in C^{2,1}(\Omega \times (0, \infty)), \) and \( \bar{u}, \nabla \bar{u} \in C(\overline{\Omega} \times [0, \infty)) \).
Then \( T^* (u_0) = \infty \) and there exists \( \tilde{u}_0 \in X \) satisfying: \( T^* (\tilde{u}_0) = \infty \) and the corresponding solution is increasing in time.

2.2. Gradient estimates. In this section, we will give two different gradient estimates which are based on the Bernstein-type arguments.

Theorem 2.6. Let \( x_0 \in \mathbb{R}^N, R, T > 0, Q_{T,R} = B(x_0, R) \times (0, T) \). Assume that \( u \in C^{2,1}(Q_{T,R}) \) is a classical solution of (1) and that \( u \leq M \) in \( Q_{T,R} \). Then
\[
|\nabla u| \leq C(1 + R^{-1} + R^{-2} + t^{-1})(M + 2 - u)
\]
in \( Q_{T,R}/2 \) for some \( C > 0 \).

Proof. Let \( v = \ln (2 + M - u) \) and \( w = |\nabla v|^2 \). Then \( v \) satisfies
\[
v_t - \Delta v = -e^{v|\nabla v| - v} + |\nabla v|^2.
\]
Differentiating the equation (5) with respect to \( x_i \):
\[
\partial_i v_t - \Delta v_i = 2\nabla v \cdot \nabla v_i - e^{v|\nabla v| - v} \left( -v_t + e^v v_i |\nabla v| + e^v \nabla v \cdot \nabla v_i \right),
\]
where \( v_i := \frac{\partial v}{\partial x_i} \). Multiplying (6) by \( 2v_i \) and summing up, we have
\[
w_t - \Delta w = -2|D^2 v|^2 + \left( 2\nabla w \cdot \nabla v - e^v |\nabla v| - v e^v w^{-\frac{1}{2}} \nabla v \cdot \nabla w - e^{v|\nabla v| - v} \left( -2w + 2e^v w^2 \right) \right),
\]
where we used the fact that \( \nabla w = 2D^2 v \nabla v, \Delta w = 2\nabla (\Delta v) \cdot \nabla v + 2|D^2 v|^2 \) and \( |D^2 v|^2 = \sum_{i,j=1}^N (v_{ij})^2 \). Define
\[
\begin{align*}
\tilde{b} &= \left( e^v |\nabla v| w^{-\frac{1}{2}} - 2 \right) \nabla v, \\
Nw &= -e^{v|\nabla v| - v} \left( -2w + 2e^v w^2 \right), \\
\mathcal{L}w &= w_t - \Delta w + \tilde{b} \cdot \nabla w.
\end{align*}
\]
Then there holds \( \mathcal{L}w = -2|D^2 v|^2 + Nw \). Now, we need a cut-off function \( \eta \in C^2(\overline{B}(x_0, R')) \) which satisfies
\[
0 \leq \eta \leq 1 \text{ and } \eta = 0 \text{ for } |x - x_0| = R',
\]
and
\[
|\nabla \eta| \leq CR^{-1} \eta^a, |\Delta \eta| + \eta^{-1} |\nabla \eta|^2 \leq CR^{-2} \eta^a, \text{ for } |x - x_0| < R',
\]
where \( R' = 3R/4 \), and \( a > 0 \) will be determined below, \( C = C(a) > 0 \).

From now on, all computations will take place in \( Q_{T,R} \) without specific statement. Let \( z = \eta w \), then it satisfies
\[
\mathcal{L}z = \eta \mathcal{L}w + w \mathcal{L} \eta - 2\nabla \eta \cdot \nabla w.
\]
Using Young's inequality, we have
\[
2|\nabla \eta \cdot \nabla w| \leq 4\eta^{-1} |\nabla \eta|^2 w + \eta |D^2 v|^2,
\]
then
\[ \mathcal{L}z + \eta |D^2v| \leq \eta Nw + (\mathcal{L} \eta + 4\eta^{-1}|\nabla \eta|^2)w. \]  

(8)

Let us now estimate each term appearing in the inequality (8).

Since \( u \leq M \), we have \( 2 \leq e^u \leq 2 + M \) which implies that \( 2 < e^u|\nabla v|w^{-\frac{1}{2}} \), then
\[ |(\tilde{b} \cdot \nabla \eta)w| = \left| w \left( e^{\eta} |\nabla v|w^{-\frac{1}{2}} - 2 \right) \nabla v \cdot \nabla \eta \right| \]
\[ \leq w^{\frac{3}{2}} |\nabla \eta| e^{\eta} |\nabla v|w^{-\frac{1}{2}} - 2 \]
\[ \leq w|\nabla \eta|e^{\eta}w^2 \leq CR^{-2}\eta^a we^{\alpha}w^{\frac{1}{2}}. \]

By the second inequality in (7), we can get
\[ -w\Delta \eta + 4\eta^{-1}|\nabla \eta|^2w \leq C'R^{-2}\eta^a w, \]

(10)

where \( C' = 4C \). Besides, we have
\[ \eta Nw = -2\eta e^{\eta}|\nabla v|^{-\alpha} \left(-w + e^u w^\frac{3}{2}\right) \leq 2 \left( \eta w - \eta w^\frac{3}{2} \right) e^{\eta}|\nabla v|. \]

(11)

Combining inequality (8) with estimates (9)–(11), and using Young’s inequality and the fact that \( 0 \leq \eta \leq 1 \), we can get
\[ e^{-e^{\eta}|\nabla v|} \mathcal{L}z \leq -2\eta w^{\frac{3}{2}} + 2\eta w + CR^{-2}\eta^a w + CR^{-1}\eta^a w \leq -\frac{3}{2} \eta w^{\frac{3}{2}} + C_1 \left( 1 + R^{-6} + R^{-3} \right) \]
\[ \leq -\frac{3}{2} \eta w^{\frac{3}{2}} + A^2, \]

where \( a \geq \frac{2}{3}, A = C_1^2 (1 + R^{-2} + R^{-1})^2 \). Then we can deduce that
\[ \mathcal{L}z \leq -\frac{1}{2} \eta w^{\frac{3}{2}} \quad \text{in} \quad \{(x,t) \in QT,R'; z \geq A\}. \]

In order to obtain the estimate (4), it’s necessary to construct another auxiliary function \( \phi(t) = ct^{-2} \) which satisfies \( \phi(t) \geq -\frac{1}{2} \phi \) (t) with suitable \( c > 0 \).

Similar to [15], we can set \( \tilde{z}(t) = z(t + t_0) - \phi(t) \) for fixed \( t_0 \in (0,T) \). Then \( \tilde{z}(t) \) satisfies
\[ \mathcal{L} \tilde{z} \leq 0 \quad \text{in} \quad \{(x,t) \in QT_{-t_0,R'}; \tilde{z}(x,t) \geq A\}. \]

As \( \lim_{t \to 0} \tilde{z}(t) \leq 0 \), we can assert that \( \tilde{z} \leq A \) in \( QT_{-t_0,R'} \) by Lemma 2.2. Letting \( t_0 \to 0 \), using the fact that \( z = \eta |\nabla v|^2 \) and that \( 0 < \delta \leq \eta \leq 1 \) in \( QT_{R/2} \), we can deduce that
\[ |\nabla u| = |\nabla v|(M + 2 - u) \leq C (A + t^{-2})^\frac{\gamma}{2} (M + 2 - u) \]
\[ \leq C \left( 1 + R^{-2} + R^{-1} + t^{-1} \right) (M + 2 - u) \]
in \( QT_{R/2} \).

\[ \square \]

**Remark 5.** The cutoff function \( \eta \) had been constructed explicitly in [15]:
\[ \eta = \rho^k, \quad \rho(x) = 1 - \left( \frac{x - x_0}{R^2} \right)^2, \quad k > \frac{2}{1 - a}. \]

If we put \( v = u \) instead of \( v = \ln(M + 2 - u) \), then we have the following gradient estimate which is independent of \( u \) and the variable \( t \).
Corollary 1. The solution of (1) satisfies
\[ |\nabla u| \leq C_1 \ln \left( \frac{1}{\delta(x)} + C_2 \right) + C_3, \quad x \in \Omega, 0 \leq t < T^*, \] (12)
where \( C_1 = C_1(N), C_2 = C_2(\|u_0\|_{C^1}), C_3 = C_3(\|u_0\|_{C^1}) > 0. \)

Proof. For the proof of Theorem 2.7, we just need to modify the last step of the proof of [18, Theorem 3.1] as
\[ |\nabla u(x_0, t)| \leq \sup x \frac{1}{x} \leq C_1 \ln \left( \frac{1}{\delta(x)} + C_2 \right) + C_3. \]

\[ \square \]

Remark 6. One can also see from the estimate (12) that if \( \Omega = \mathbb{R}^N \), then \( |\nabla u| \leq C(\Omega, N) \). Moreover, Theorem 2.7 implies that gradient blowup, in finite or infinite time, can occur only on the boundary, and it gives an upper estimate of the blowup profile of \( \nabla u \). This estimate is sharp in view of [17, Lemma 2.1], where the one-dimensional case has been studied.

We can also derive the following estimate for \( u \) from (12).

Corollary 1. The solution of (1) satisfies
\[ |u| \leq C_1 \delta(x) \ln \left( \frac{1}{\delta(x)} + C_2 \right) + C_3 \delta(x) \text{ in } \Omega \times [0, T^*]. \] (13)

3. Proof of the results on stationary solution. In this section, we will give the simple proofs of Theorems 1.2 and 1.3 which are based on the parabolic regularity theory.

Proof of Theorem 1.2. By Lemma 2.5, we know that there exists a global solution \( \tilde{u} \) which is increasing in time for some \( \tilde{u}_0 \in X \) and satisfies (1). The estimate (13) implies that there exists a function \( v \in C_0(\Omega) \) such that \( \lim_{t \to \infty} \tilde{u}(x, t) = v(x) \) for all \( x \in \Omega \).

The next step is to show that \( v \in C^2(\Omega) \cap C_0(\Omega) \) and that \( v \) is a solution of (3) in the sense of distribution. Hence, \( v \) is a classical solution of (3). According to the gradient estimate (12), we assert that \( |\nabla \tilde{u}| \leq C(\epsilon) \) in \( \Omega_t \times [0, \infty) \). Thus, \( e^{\|\nabla u\|} \in L^\infty_{\text{loc}}(\Omega) \) for \( t > 0 \). Combining this with the fact that \( \tilde{u} \in L^\infty_{\text{loc}}(\Omega) \) and using parabolic regularity theory, we deduce that there exists \( 0 < \beta < 1 \) such that \( \|\nabla \tilde{u}\|_{C^{\beta, \beta/2}(\Omega_t \times [\tau, \tau + 1])} \leq C(\epsilon) \) for each \( \epsilon > 0 \) and \( \tau \geq 1 \). Then the interior Schauder estimates imply that
\[ \|\tilde{u}\|_{C^{2+\gamma, 1+\gamma/2}(\Omega_t \times [\tau, \tau + 1])} \leq C(\epsilon), \quad \tau \geq 2, \]
where \( 0 < \gamma < 1 \) and \( \epsilon > 0 \).

For each \( \epsilon > 0 \), let \( \tilde{u}_n := \tilde{u}(\cdot, t_n + \cdot) \) and \( t_n \to \infty \), by the diagonal procedure and Ascoli-Arzelà theorem, we can find a subsequence, still denoted by \( \tilde{u}_n \), such that \( \tilde{u}_n \) converges in \( C^{2,1}_{\text{loc}}(\Omega \times [0, 1]) \) to a function \( \tilde{v} \) which must coincide with \( v \). Thus, we can deduce that \( v \in C^2(\Omega) \cap C_0(\Omega) \).

Multiplying (1) by \( \varphi(x) \in C^\infty_0(\Omega) \) and integrating by part, we have
\[ \int_0^1 \int_\Omega (\tilde{u}_n \Delta \varphi + e^{\|\nabla \tilde{u}_n\|} \varphi) \, dx \, dt = \left[ \int_\Omega \tilde{u}_n \varphi \, dx \right]_0^1. \]
Letting \( n \to \infty \), we can get

\[
\int_{\Omega} \left( v \Delta \varphi + e^{\|\nabla v\|_{\infty}} \varphi \right) \, dx = 0,
\]

which implies that \( v \) is a solution of (3) in \( \Omega \) in the distributional sense. Therefore, \( v \) is a classical solution of (3).

Let us now show that \( u \) converges to \( v \) in \( C(\Omega) \) as \( t \to \infty \). Put \( \phi(t) = \|u(t) - v\|_{\infty} \), then by Lemma 2.3 (ii), we know that \( \varphi(t) \) is nonincreasing in \( t \). Thus we can define: \( l = \lim_{t \to \infty} \varphi(t) \in [0, \infty) \). Our main goal is to proof that \( l \equiv 0 \). Similar to the proof of the existence of \( v \), we can set \( u_n(t, \cdot) := u(\cdot, t_n + \cdot) \) with \( t_n \to \infty \). Combining (12) with (13), and following the same procedure as above, we can obtain a function \( z \in C^{2,1}(\Omega \times (0, \infty)) \) which satisfies

\[
z_t - \Delta z = e^{\|\nabla z\|_{\infty}} \text{ in } \Omega \times (0, \infty).
\]

Using the estimates (12) and (13) again, we know that \( \{u(\tau); \tau \geq 0\} \) is relatively compact in \( C_0(\Omega) \). Then, by Ascoli-Arzela theorem, we can find a subsequence \( u_{n_k} \) (\( \equiv u(\cdot, t_{n_k}) \)) which converges to \( z(t) \) in \( C_0(\Omega) \). Hence, there holds

\[
z(t) \in C_0(\Omega), \quad \|z(t) - v\|_{\infty} = \lim_{k \to \infty} \|u(t_{n_k} + t) - v\|_{\infty} = l, \ t \geq 0.
\]

Define \( \bar{w}(t) := z(t) - v \), then it satisfies

\[
\bar{w}_t - \Delta \bar{w} = b(x,t) \cdot \nabla \bar{w}, \text{ in } \Omega \times (0, \infty),
\]

where

\[
b(x,t) = \int_0^1 e^{\|\nabla v + s\nabla \bar{w}\|_{\infty}} \frac{\nabla v + s\nabla \bar{w}}{|\nabla v + s\nabla \bar{w}|}(x,t) \, dx \in C(\Omega \times (0, \infty)).
\]

Assume that \( l > 0 \), then there exists \( x_0 \in \Omega \), such that \( |\bar{w}(x_0, 2)| = \|\bar{w}\|_{\infty} = l \). Applying the strong maximum principle in \( B(x_0, \rho) \times [1, 2] \) for \( \rho < \delta(x_0) \), we can deduce that \( |\bar{w}| \equiv l \) in \( B(x_0, \rho) \times [1, 2] \). Then we can obtain a contradiction by letting \( \rho \to \delta(x_0) \), since \( \bar{w} \in C_0(\Omega) \). Hence, \( l \equiv 0 \) as the sequence \( t_n \) is arbitrary. \( \square \)

**Remark 7.** If we assume that the initial data satisfies: \( \Delta u_0 + e^{\|\nabla u_0\|_{\infty}} \leq 0 \), then the solution is decreasing with respect to \( t \). Thus, we can apply Dini’s theorem [11, Theorem 7.13] on uniform convergence of monotone sequences to obtain that \( u \) converges to \( v \) uniformly in \( C(\Omega) \).

**Proof of Theorem 1.3.** If \( v \in C^1(\Omega) \), this is a direct consequence of Lemma 2.5. If \( v \notin C^1(\Omega) \), then there holds \( \lim_{r \to R} v'(r) = -\infty \) by the estimate (12). In order to obtain the global existence, we may assume that \( T^* < \infty \). Our next step is to show that

\[
\frac{\partial u}{\partial n} \geq C > -\infty, \text{ on } \partial \Omega \times [0, T^*).
\]

Combining this with the fact that

\[
\frac{\partial u}{\partial n} < 0, \text{ on } \partial \Omega \times [0, T^*),
\]

we can derive a contradiction which implies the global existence of the solution of (1). For (14), we just need to consider the annular domain \( B_{\rho,R} := \{ x \in \mathbb{R}^N; \rho < r < \rho \} \) for some \( \rho > 0 \). By the same arguments as in [21, Section 4], we can obtain (14). \( \square \)
Remark 8. It’s necessary to point out that the condition for \( v \) in [21] is \( v(\rho) = 0, v(R) = M \). However, the condition in Theorem 1.3 is opposite. Hence the monotonicity is also opposite. But the proof there is still valid for our problem with little modification, here we omit it.

Remark 9. We note that the gradient blowup may occur in infinite time. For example, when the stationary solution \( v \notin C^1(\Omega) \), Theorem 1.3 implies that gradient blowup will occur in infinite time.

4. Proof of the results on finite time gradient blowup. In this section, we will first show that for some large initial data, \( \nabla u \) must blow up in a finite time. In \([17, 18]\), the results were established in one-dimensional and radial cases respectively by constructing self-similar subsolutions. Here, we give a more general subsolution. We refer the readers to \([12]\) for the results on finite time gradient blowup of the equations with generalized nonlinearities. We can also verify that the self-similar subsolution used in \([18]\) is available for our problem. Indeed, we can consider a ball \( B_R \subset \Omega \) which is internally tangent to \( \partial \Omega \) at some point \( x_0 \in \partial \Omega \). Then applying the same manner as \([18, \text{Theorem } 2.1]\) in \( B_R \), we can derive

\[
\lim_{t \to T'} \frac{\partial u}{\partial n}(x_0, t) = -\infty
\]

for some \( T' < \infty \) by which the conclusion follows.

The following local boundary control property for the gradient of the solution \( v \) of (15) below will be useful. Its proof can be found in \([8, \text{Lemma } 2.1]\).

Lemma 4.1. If there exist \( M, R > 0 \) such that

\[
\left| \frac{\partial v}{\partial n} \right| \leq M \quad \text{on} \quad (B_R(x_0) \cap \partial \Omega) \times [0, T'),
\]

then \( x_0 \) is not a gradient blowup point.

Proof of Theorem 1.4. Due to the gradient estimate (12), we know that the gradient blowup can only occur on the boundary of \( \Omega \). Let \( v \) denote the maximal, classical solution of

\[
\begin{align*}
  v_t - \Delta v &= \frac{1}{6} |\nabla v|^3, & x \in \Omega, t > 0, \\
  v &= 0, & x \in \partial \Omega, t > 0, \\
  v(x, 0) &= u_0(x), & x \in \Omega
\end{align*}
\]  

(15)

with maximal existence time \( T' \). Suppose that the solution of (1) exists globally in time, i.e. \( T^* = \infty \). We can see from \([4, 8, 10, 12, 20]\) that \( \nabla v \) will blow up in finite time on the boundary of \( \Omega \). Thus we can set \( 0 < T < T' < \infty \). Applying Lemma 2.1 and the fact that \( \frac{1}{6} |\nabla v|^3 \leq e^{\|\nabla v\|} \), we know that \( u \geq v \) in \( \overline{\Omega} \times (0, T] \). Moreover, we have

\[
\frac{\partial u}{\partial n} \leq \frac{\partial v}{\partial n} < 0, \quad \text{on} \ \partial \Omega.
\]

Then letting \( T \to T' \) and applying Lemma 4.1, we can deduce that

\[
\frac{\partial u}{\partial n} \leq \frac{\partial v}{\partial n} < -\infty
\]

at some point \( x_0 \in \partial \Omega \) satisfying \( \lim_{t \to T'} |\nabla u(x_0, t)| = \infty \). This contradicts with the assumption that \( T^* = \infty \). Hence, \( \nabla u \) must blow up in a finite time if \( u_0 \) is large enough. \( \square \)
Remark 10. From the proof of Theorem 1.4, we can also see that \( T^* \leq T' \). The subsolution used above is not unique, we can also construct other subsolutions by using the fact that \( e^x \geq \frac{1}{m}x^m \) for \( m \in \mathbb{Z}^+, m > 2 \).

Next, we will give a simple proof of Theorem 1.5.

**Proof of Theorem 1.5.** Here, we just need to show that the solution of problem (15) will have gradient blowup phenomenon.

Let \( k = 3 \) and \( F(t) = \frac{1}{k+1} \int_{\Omega} v^{k+1} \, dx \), then we have

\[
F'(t) = \frac{1}{6} \int_{\Omega} |\nabla v|^3 v^k \, dx - k \int_{\Omega} |\nabla v|^2 v^{k-1} \, dx. \tag{16}
\]

By H"older’s inequality, for any \( \epsilon \in (0, \frac{1}{12}) \), we have

\[
\int_{\Omega} |\nabla v|^2 v^{k-1} \, dx \leq \left( \int_{\Omega} |\nabla v|^3 v^k \, dx \right)^{\frac{2}{3}} |\Omega|^{\frac{1}{3}} \leq \frac{2}{3} \epsilon \int_{\Omega} |\nabla v|^3 v^k \, dx + C(\epsilon)|\Omega|. \tag{17}
\]

Using Poincaré’s inequality, we have

\[
\int_{\Omega} |\nabla v|^3 v^k \, dx = \left( \frac{3}{k+3} \right)^3 \int_{\Omega} \left| \nabla v \right|^{k+3} \, dx \geq C(k)O \left( |\Omega|^{-\frac{k+3}{k+1}} \right) \int_{\Omega} v^{k+3} \, dx. \tag{18}
\]

Combining (16) with estimates (17) and (18), using H"older’s inequality, we get

\[
F'(t) \geq C(k, \epsilon) \int_{\Omega} |\nabla v|^3 v^k \, dx - C(\epsilon)|\Omega| \geq C_1 \left( \int_{\Omega} v^{k+1} \, dx \right)^{\frac{k+3}{k+1}} - C_2
\]

which is equivalent to

\[
F'(t) \geq C_1 F^{\frac{k+3}{k+1}}(t) - C_2,
\]

where \( C_1 = O(|\Omega|^{-\frac{k}{k+1}}) > 0, C_2 = O(|\Omega|) > 0 \). Using also the fact that

\[
(k+1)F(0) = \int_{\Omega} \phi_1^{k+1} \, dx \geq \left( \int_{\Omega} \phi_1 \, dx \right)^{k+1} |\Omega|^{-k} = |\Omega|^{-k},
\]

we can conclude that \( \nabla v \) must blow up in finite time if \( |\Omega| < M \) is small. Then, following the same procedure as in the proof of Theorem 1.4, we know that \( \nabla u \) must blow up in finite time. \( \square \)

Remark 11. Following the proof of Theorem 1.5, we can extend this result to the following problem

\[
\begin{cases}
    u_t - \Delta u = |\nabla u|^p, & x \in \Omega, t > 0, \\
    u = 0, & x \in \partial \Omega, t > 0, \\
    u(x, 0) = \phi_1(x), & x \in \Omega,
\end{cases}
\]

where \( p > 2 \).

Acknowledgments. We would like to thank the referees very much for their valuable comments and suggestions.
REFERENCES

[1] J. M. Arrieta, A. Rodríguez-Bernal and Ph. Souplet, Boundedness of global solutions for nonlinear parabolic equations involving gradient blow-up phenomena, *Ann. Scuola. Norm. Super. Pisa Cl. Sci.*, 3 (2004), 1–15.

[2] M. Fila and G. M. Lieberman, Derivative blow-up and beyond for quasilinear parabolic equations, *Differential Integral Equations*, 7 (1994), 811–821.

[3] M. Fila, J. Taskinen and M. Winkler, Convergence to a singular steady state of a parabolic equation with gradient blow-up, *Appl. Math. Lett.*, 20 (2007), 578–582.

[4] J.-S. Guo and B. Hu, Blowup rate estimates for the heat equation with a nonlinear gradient source term, *Discrete Contin. Dyn. Syst.*, 20 (2008), 927–937.

[5] M. Hesaaraki and A. Moameni, Blow-up of positive solutions for a family of nonlinear parabolic equations in general domain in $\mathbb{R}^N$, *Michigan Math. J.*, 52 (2004), 375–389.

[6] M. Kardar, G. Parisi and Y. C. Zhang, Dynamic scaling of growing interfaces, *Phys. Rev. Lett.*, 56 (1986), 889–892.

[7] J. Krug and H. Spohn, Universality classes for deterministic surface growth, *Phys. Rev. A.*, 38 (1988), 4271–4283.

[8] Y. X. Li and Ph. Souplet, Single-point gradient blow-up on the boundary for diffusive Hamilton-Jacobi equations in planar domains, *Commun. Math. Phys.*, 293 (2010), 499–517.

[9] G. M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific, Singapore, 2005.

[10] P. Quittner and Ph. Souplet, *Superlinear Parabolic Problems: Blow-up, Global Existence and Steady States*, Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser, Verlag, Basel, 2007.

[11] W. Rudin, *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill, 2007.

[12] Ph. Souplet, Gradient blow-up for multidimensional nonlinear parabolic equations with general boundary conditions, *Differential Integral Equations*, 15 (2002), 237–256.

[13] Ph. Souplet, Recent results and open problems on parabolic equations with gradient nonlinearities, *Electron. J. Differential Equations*, 2001 (2001), 1–19.

[14] Ph. Souplet and J. L. Vázquez, Stabilization towards a singular steady state with gradient blow-up for a diffusion-convection problem, *Discrete Contin. Dyn. Syst.*, 14 (2006), 221–234.

[15] Ph. Souplet and Q. S. Zhang, Global solutions of inhomogeneous Hamilton-Jacobi equations, *J. d’Analyse Math.*, 99 (2006), 335–396.

[16] Z. C. Zhang and B. Hu, Gradient blowup rate for a semilinear parabolic equation, *Discrete Contin. Dyn. Syst.*, 26 (2010), 767–779.

[17] Z. C. Zhang and B. Hu, Rate estimates of gradient blowup for a heat equation with exponential nonlinearity, *Nonlinear Anal.*, 72 (2010), 4594–4601.

[18] Z. C. Zhang and Y. Y. Li, Gradient blowup solutions of a semilinear parabolic equation with exponential source, *Comm. Pure Appl. Anal.*, 12 (2013), 269–280.

[19] Z. C. Zhang and Y. Y. Li, Boundedness of global solutions for a heat equation with exponential gradient source, *Abstr. Appl. Anal.*, 2012 (2012), 1–10.

[20] Z. C. Zhang and Z. J. Li, A note on gradient blowup rate of the inhomogeneous Hamilton-Jacobi equations, *Acta Math. Sci. Ser. B Eng. Ed.*, 33 (2013), 678–686.

[21] L. P. Zhu and Z. C. Zhang, Rate of approach to the steady state for a diffusion-convection equation on annular domains, *Electron. J. Qual. Theory Differ. Equ.*, 39 (2012), 1–10.

Received May 2013; revised September 2013.

E-mail address: zhangzc@mail.xjtu.edu.cn
E-mail address: liyan1989@stu.xjtu.edu.cn