ON TAU-FUNCTIONS FOR THE TODA LATTICE HIERARCHY

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Dedicated to the memory of Boris Anatol’evich Dubrovin, with gratitude and admiration

Abstract. We extend a recent result of [13] for the KdV hierarchy to the Toda lattice hierarchy. Namely, for an arbitrary solution to the Toda lattice hierarchy, we define a pair of wave functions, and use them to give explicit formulae for the generating series of $k$-point correlation functions of the solution. Applications to computing GUE correlators and Gromov–Witten invariants of the Riemann sphere are under consideration.

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1. Introduction

The Toda lattice hierarchy, which contains the Toda lattice equation

$$\ddot{\sigma}(n) = e^{\sigma(n-1)-\sigma(n)} - e^{\sigma(n)-\sigma(n+1)},$$

is an important integrable hierarchy of nonlinear differential-difference equations [18, 19, 22, 27]. In this paper, following the idea of [13] we derive new formulae for generating series of $k$-point correlation functions for the Toda lattice hierarchy by using the matrix resolvent approach [10] and by introducing a pair of wave functions.

1.1. Toda lattice hierarchy and tau-function. Let

$$\mathcal{A} := \mathbb{Z}[v_0, w_0, v_{±1}, w_{±1}, v_{±2}, w_{±2}, \cdots]$$

be the polynomial ring. Define the shift operator $\Lambda : \mathcal{A} \to \mathcal{A}$ via

$$\Lambda(1) = 1, \quad \Lambda(v_i) = v_{i+1}, \quad \Lambda(w_i) = w_{i+1}, \quad \Lambda(fg) = \Lambda(f) \Lambda(g)$$

for all $i \in \mathbb{Z}$ and $f, g \in \mathcal{A}$. Denote by $\Lambda^{-1}$ the inverse of $\Lambda$ satisfying $\Lambda^{-1}(v_i) = v_{i-1}$, $\Lambda^{-1}(w_i) = w_{i-1}$, and $\Lambda^{-1}(fg) = \Lambda^{-1}(f)\Lambda^{-1}(g)$. For a difference operator $P$ on $\mathcal{A}$, we mean an operator of the form $P = \sum_{m \in \mathbb{Z}} P_m \Lambda^m$, where $P_m \in \mathcal{A}$. Denote $P_+ := \Lambda$.
Lemma 1. The operators $A$ we associate with $B$ be a difference operator, and define a sequence of difference operators differential-difference equations, called the Toda lattice hierarchy, given by

$$
L := \Lambda + v_0 + w_0 \Lambda^{-1}
$$

be a difference operator, and define a sequence of difference operators $A_k$, $k \geq 0$ by

$$
A_k := (L^{k+1})_+.
$$

We associate with $A_k$ a sequence of admissible derivations $D_k : A \to A$ defined via

$$
D_k(v_0) := \text{Coef}([A_k, L], 0), \quad D_k(w_0) := \text{Coef}([A_k, L], -1), \quad k \geq 0.
$$

The first few $D_k(v_0)$ and $D_k(w_0)$ are $D_0(v_0) = w_1 - w_0$, $D_0(w_0) = w_0(v_0 - v_1)$; $D_1(v_0) = w_1(v_1 + v_0) - w_0(v_0 + v_1)$, $D_1(w_0) = w_0(w_1 - w_1 + v_0^2 - v_1^2)$; etc.

**Lemma 1.** The operators $D_k$, $k \geq 0$ pairwise commute.

This lemma was known. We call $D_k$ the Toda lattice derivations, and $A$ the abstract Toda lattice hierarchy.

A tau-structure associated to the derivations $(D_k)_{k \geq 0}$ is a collection of polynomials $(\Omega_{p,q}, S_p)_{p,q \geq 0}$ in $A$ satisfying

$$
\Omega_{p,q} = \Omega_{q,p}, \quad D_r(\Omega_{p,q}) = D_q(\Omega_{p,r}),
$$

$$
(\Lambda - 1)(\Omega_{p,q}) = D_q(S_p),
$$

$$
w_0(1 - \Lambda^{-1})(S_p) = D_p(w_0)
$$

for all $p, q, r \geq 0$. It can be shown (e.g. [10]) that the tau-structure exists and is unique up to replacing $\Omega_{p,q}, S_p$ by $\Omega_{p,q} + c_{p,q}$ and $S_p + a_p$ respectively, where $c_{p,q} = c_{q,p}$ and $a_p$ are arbitrary constants. The tau-structure $\Omega_{p,q}, S_p$ is called canonical if

$$
\Omega_{p,q}|_{v_i=0, w_i=0, i \in \mathbb{Z}} = 0, \quad S_p|_{v_i=0, w_i=0, i \in \mathbb{Z}} = 0.
$$

Let us take $\Omega_{p,q}, S_p$ the canonical tau-structure. For $m \geq 3$, define

$$
\Omega_{p_1, \ldots, p_m} := D_{p_1} \cdots D_{p_{m-2}}(\Omega_{p_{m-1}, p_m}) \in A, \quad p_1, \ldots, p_m \geq 0.
$$

By [8] we know that the $\Omega_{p_1, \ldots, p_m}$, $m \geq 2$ are totally symmetric with respect to permutations of the indices $p_1, \ldots, p_m$. The first few of these polynomials are

$$
S_0 = v_0, \quad S_1 = w_1 + w_0 + v_0^2,
$$

$$
\Omega_{0,0} = w_0, \quad \Omega_{0,1} = \Omega_{1,0} = w_1(v_1 + v_0).
$$

If we think of $v_0, w_0$ as two functions $v(n), w(n)$ of $n$, respectively, and $v_i, w_i$ as $v(n+i), w(n+i)$, then the Toda lattice derivations $D_k$ lead to a hierarchy of evolutionary differential-difference equations, called the Toda lattice hierarchy, given by

$$
\frac{\partial v(n)}{\partial t_k} = D_k(v_0)(n), \quad \frac{\partial w(n)}{\partial t_k} = D_k(w_0)(n),
$$

where $k \geq 0$.
where $k \geq 0$, and the $D_k(v_0)(n), D_k(w_0)(n)$ are defined as $D_k(v_0), D_k(w_0)$ with $v_i, w_i$ replaced by $v(n+i), w(n+i)$, respectively. Lemma 1 implies that the flows (12) all commute. So we can solve the whole Toda lattice hierarchy (12) together, which yields solutions of the form $(v = v(n, t), w = w(n, t))$. Here $t := (t_0, t_1, \ldots)$ denotes the infinite time vector. Note that the $k$ equations read

$$
\dot{v}(n) = w(n+1) - w(n), \quad \dot{w}(n) = w(n) \left(v(n) - v(n-1)\right),
$$

which are equivalent to equation (1) via the transformation

$$w(n) = e^{\sigma(n-1) - \sigma(n)}, \quad v(n) = -\sigma(n).$$

Here, dot, “”, is identified with $\partial/\partial t_0$.

Let $V$ be a ring of functions of $n$ closed under shifting by $\pm 1$. For two given $f(n), g(n) \in V$, consider the initial value problem for (12) with the initial condition:

$$v(n, 0) = f(n), \quad w(n, 0) = g(n).$$

The solution $(v(n, t), w(n, t)) \in V[[t]]^2$ exists and is unique, which gives the following 1-1 correspondence:

$$\{\text{solution } (v, w) \text{ of } (12) \text{ in } V[[t]]^2\} \longleftrightarrow \{\text{initial data } (f, g)\}. \quad (15)$$

**Example 1.** $f(n) = 0, g(n) = n$. (For this case, one can take $V = Q[n]$.) The corresponding unique solution governs the enumerations of ribbon graphs in all genera.

**Example 2.** $f(n) = (n + 1/2)\epsilon, g(n) = 1$. (For this case, one can take $V = Q[\epsilon][n]$.) The corresponding unique solution governs the Gromov-Witten invariants of $\mathbb{P}^1$ in the stationary sector in all genera and all degrees.

Let $(v, w) \in V[[t]]^2$ be an arbitrary solution to the Toda lattice hierarchy (12). Write $\Omega_{p,q}(n, t)$ and $S_p(n, t)$ as the images of $\Omega_{p,q}$ and $S_p$ under the substitutions

$$v_i \mapsto v(n+i, t), \quad w_i \mapsto w(n+i, t), \quad i \in \mathbb{Z},$$

respectively. (Similar notations will be used for other elements of $A$.) Equalities (9) then imply the existence of a function $\tau = \tau(n, t)$ such that for $p, q \geq 0$,

$$\Omega_{p,q}(n, t) = \frac{\partial^2 \log \tau(n, t)}{\partial t_p \partial t_q}, \quad (17)$$

$$S_p(n, t) = \frac{\partial}{\partial t_p} \log \frac{\tau(n+1, t)}{\tau(n, t)}, \quad (18)$$

$$w(n, t) = \frac{\tau(n+1, t) \tau(n-1, t)}{\tau(n, t)^2}. \quad (19)$$

We call $\tau(n, t)$ the Dubrovin–Zhang (DZ) type tau-function of the solution $(v, w)$, for short the tau-function of the solution. The symmetry in $\Omega_p$ is more obvious: the image $\Omega_{p_1,\ldots,p_m}(n, t)$ of $\Omega_{p_1,\ldots,p_m}$ under (16) satisfies

$$\Omega_{p_1,\ldots,p_m}(n, t) = \frac{\partial^m \log \tau(n, t)}{\partial t_{p_1} \cdots \partial t_{p_m}}, \quad m \geq 2, p_1, \ldots, p_m \geq 0. \quad (20)$$
Define $\Omega_p(n, t) = \partial_p \log \tau(n, t)$, $p \geq 0$. These logarithmic derivatives of $\tau(n, t)$ are called correlation functions of the solution $(v, w)$. The specializations $\Omega_{p_1,\ldots,p_m}(n, 0)$ are called $m$-point partial correlation functions of $(v, w)$.

**Remark 1.** The tau-function $\tau(n, t)$ of the solution $(v, w)$ is unique up to multiplying it by the exponential of a linear function of $n, t_0, t_1, t_2, \ldots$.

1.2. **Matrix resolvent.** The matrix resolvent (MR) method for computing correlation functions for integrable hierarchies was introduced in [1, 2, 3], and was extended to the discrete case in [10] (in particular to the Toda lattice hierarchy). Denote

$$U(\lambda) := \begin{pmatrix} v_0 - \lambda & w_0 \\ -1 & 0 \end{pmatrix}.$$ 

The following lemma for the Toda lattice hierarchy was proven in [10].

**Lemma 2 ([10]).** There exists a unique series $R(\lambda) \in \text{Mat}(2, \mathcal{A}[[\lambda^{-1}]]$) satisfying

$$\Lambda(R(\lambda)) U(\lambda) - U(\lambda) R(\lambda) = 0,$$  \hspace{1cm} (21)

$$\text{Tr} R(\lambda) = 1, \quad \det R(\lambda) = 0,$$ \hspace{1cm} (22)

$$R(\lambda) - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \text{Mat}(2, \mathcal{A}[[\lambda^{-1}]]) \lambda^{-1}).$$ \hspace{1cm} (23)

The unique series $R(\lambda)$ in Lemma 2 is called the basic matrix resolvent. The first few terms of $R(\lambda)$ are given by

$$R(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -w_0 \\ 1 & 0 \end{pmatrix} \frac{1}{\lambda} + \begin{pmatrix} w_0 & -v_0 w_0 \\ v_{-1} & -w_0 \end{pmatrix} \frac{1}{\lambda^2}$$

$$+ \begin{pmatrix} w_0(v_0 + v_{-1}) & -w_0(w_0 + w_1 + v_0^2) \\ w_0 + w_{-1} + v_{-1}^2 & -w_0(v_0 + v_{-1}) \end{pmatrix} \frac{1}{\lambda^3} + \cdots.$$ \hspace{1cm} (24)

**Proposition 1 ([10]).** For any $k \geq 2$, the following formula holds true:

$$\sum_{i_1,\ldots,i_k \geq 0} \frac{\Omega_{i_1,\ldots,i_k}}{\lambda_{i_1}^{1+2} \cdots \lambda_{i_k}^{k+2}} = - \sum_{\pi \in S_k/C_k} \text{tr} \frac{\prod_{j=1}^k R(\lambda_{\pi(j)})}{\prod_{j=1}^k (\lambda_{\pi(j)} - \lambda_{\pi(j+1)})} - \frac{\delta_{k,2}}{(\lambda_1 - \lambda_2)^2},$$ \hspace{1cm} (25)

where $S_k$ denotes the symmetry group and $C_k$ the cyclic group, and $\pi(k + 1) := \pi(1)$.

The meaning of (25) is the following: For any fixed permutation $(j_1, \ldots, j_k)$ of $(1, \ldots, k)$, expanding the right-hand side with respect to $|\lambda_{j_1}| > \cdots > |\lambda_{j_k}| >> 0$ gives identical formal power series with the left-hand side. This is because, after the summation over the $S_k/C_k$ and subtracting $\frac{\delta_{k,2}}{(\lambda_1 - \lambda_2)^2}$, the poles in the diagonal cancel (cf. the Proposition 2 of [12] for a straightforward proof of this point). We note that, as formal power series, the coefficients of the both sides of (25) are in $\mathcal{A}$. We give in Section 2 a new proof of (25), where we keep all derivations with coefficients in $\mathcal{A}$. 


1.3. From wave functions to correlation functions. In [13] we introduced the notion of a tuple of wave functions (in many cases a pair) to the study of tau-function without using the Sato theory. Let us generalize it to the Toda lattice hierarchy. Our definition of a pair will be based on the standard construction of wave functions for the Toda lattice hierarchy [27, 6, 5]. For given \((f(n), g(n))\) a pair of arbitrary elements in \(V\), let \(L\) be the linear difference operator \(L = \Lambda + f(n) + g(n)\Lambda^{-1}\). Denote
\[
s(n) := -(1 - \Lambda^{-1})^{-1}(\log g(n)).
\] (26)
The function \(s(n)\) is in a certain extension \(\hat{V}\) of \(V\), and is uniquely determined by \(\log g(n)\) up to a constant. Below we fix a choice of \(s(n)\). An element \(\psi_A(\lambda, n) = (1 + O(\lambda^{-1}))\lambda^n\) in the module \(\hat{V}[\lambda^{-1}]\lambda^n\) is called a (formal) wave function of type A associated to \(f(n), g(n)\), if \(L(\psi_A(\lambda, n)) = \lambda\psi_A(\lambda, n)\). Here, \(\hat{V}\) is a ring of functions of \(n\) satisfying
\[
V \subset (\Lambda - 1)(\hat{V}) \subset \hat{V}.
\]
The existence of a pair of wave functions is proven in Section 3.

Denote by \((v(n, t), w(n, t))\) the unique solution in \(V[[t]]^2\) to the Toda lattice hierarchy with \((f(n), g(n))\) as its initial value, by \(\psi_A(\lambda, n)\) and \(\psi_B(\lambda, n)\) a pair of wave functions associated to \((f(n), g(n))\), and by \(\tau(n, t)\) the DZ type tau-function of \((v(n, t), w(n, t))\). Introduce
\[
D(\lambda, \mu, n) := \frac{\psi_A(\lambda, n) \psi_B(\mu, n) - \psi_A(\lambda, n - 1) \psi_B(\mu, n)}{\lambda - \mu}.
\] (29)

**Theorem 1.** Fix \(k \geq 2\) being an integer. The generating series of \(k\)-point partial correlation functions has the following expression:
\[
\sum_{i_1, \ldots, i_k \geq 0} \frac{\partial^k \log \tau(n, 0)}{\partial t_{i_1} \cdots \partial t_{i_k}} \frac{1}{\lambda_1^{i_1+2} \cdots \lambda_k^{i_k+2}} = (-1)^{k-1} \frac{e^{ks(n-1)}}{\prod_{j=1}^k \lambda_j} \sum_{\pi \in S_k/C_k} \prod_{j=1}^k D(\lambda_{\pi(j)}, \lambda_{\pi(j+1)}, n) - \frac{\delta_{k, 2}}{(\lambda_1 - \lambda_2)^2}.
\] (30)

Theorem 1 gives an algorithm with the initial value \((f(n), g(n))\) as the only input for computing the \(k\)-th-order logarithmic derivatives of the tau-function \(\tau(n, t)\) evaluated at \(t = 0\) for \(k \geq 2\). Indeed, by solving the spectral problem \(L(\psi) = \lambda\psi\) with \(L = \Lambda + f(n) + g(n)\Lambda^{-1}\) and with the normalization condition (28), one constructs a pair of wave functions; the coefficients in the \(t\)-expansion of \(\log \tau(n, t)\) are then obtained.
through algebraic manipulations by using (85). (Recall that in the inverse scattering method (cf. e.g. [18, 19]), an additional integral equation needs to be solved.) Two applications of Theorem 1 are given in Section 5. For a certain class of bispectral solutions (cf. [20]) it would be possible to give a canonical way of constructing a pair of wave functions, which was briefly mentioned in [13] for the KdV hierarchy; we plan to do this for KdV and for Toda lattice in a future publication.

**Organization of the paper.** In Section 2 we review the MR method of studying tau-structure for the Toda lattice hierarchy. In Section 3 we prove the existence of a pair of wave functions. In Section 4 we prove Theorem 1 and several other theorems. Applications to the computations of GUE correlators and Gromov–Witten invariants of $\mathbb{P}^1$ are given in Section 5. In Appendix A we give an extension of $A$, define a pair of abstract pre-wave functions, and prove an abstract version for Theorem 1.

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## 2. Matrix resolvent and tau-structure

We continue in this section with more details in reviewing the MR method [10] to the Toda lattice hierarchy. Denote by $\mathcal{L}$ the matrix Lax operator for the Toda lattice:

$$\mathcal{L} := \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} + \begin{pmatrix} v_0 - \lambda & w_0 \\ -1 & 0 \end{pmatrix} = \Lambda + U(\lambda).$$

Let $R(\lambda)$ be the basic matrix resolvent (of $\mathcal{L}$). Write

$$R(\lambda) = \begin{pmatrix} 1 + \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & -\alpha(\lambda) \end{pmatrix},$$

(31)

$$\alpha(\lambda) = \sum_{i \geq 0} a_i \frac{\lambda^i}{\lambda^{i+1}}, \quad \beta(\lambda) = \sum_{i \geq 0} b_i \frac{\lambda^{i+1}}{\lambda^{i+1}}, \quad \gamma(\lambda) = \sum_{i \geq 0} c_i \frac{\lambda^{i+1}}{\lambda^{i+1}},$$

(32)

where $a_i, b_i, c_i \in A$. From the defining equations (21)-(23), we see that the series $\alpha, \beta, \gamma$ satisfy the equations

$$\beta(\lambda) = -w_0 \Lambda(\gamma(\lambda)),$$

(33)

$$\gamma(\lambda) = \frac{1 + \alpha(\lambda) + \Lambda^{-1}(\alpha(\lambda))}{\lambda - v_1},$$

(34)

$$\left(\alpha(\lambda) - \Lambda(\alpha(\lambda))\right)(\lambda - v_0) - w_0 \frac{1 + \alpha(\lambda) + \Lambda^{-1}(\alpha(\lambda))}{\lambda - v_1}$$

$$+ w_1 \frac{1 + \Lambda(\alpha(\lambda)) + \Lambda^2(\alpha(\lambda))}{\lambda - v_1} = 0,$$

(35)

$$\alpha(\lambda) + \alpha(\lambda)^2 + \beta(\lambda)\gamma(\lambda) = 0.$$  

(36)
These equalities give rise to the following recursion relation for $a_i, b_i, c_i$:

$$b_j = -w_0 \Lambda(c_j), \quad c_{j+1} = v_{-1} c_j + (1 + \Lambda^{-1})(a_j), \quad (37)$$

$$\left(1 - \Lambda \right)(a_{j+1}) + v_0 \left(\Lambda - 1\right)(a_j) + w_1 \Lambda(c_j) - w_0 c_j = 0, \quad (38)$$

$$a_\ell = \sum_{i+j=\ell-1} \left(w_0 c_i \Lambda(c_j) - a_i a_j\right) \quad (39)$$

along with

$$a_0 = 0, \quad c_0 = 1. \quad (40)$$

Equations (37)–(40) are called the matrix-resolvent recursion relation.

It has been proven [10] that the abstract Toda lattice hierarchy (5) can be equivalently written as

$$D_j(v_0) = (\Lambda - 1)(a_{j+1}),$$

$$D_j(w_0) = w_0(\Lambda - 1)(c_{j+1}),$$

where $j \geq 0$. Define an operator $\nabla(\lambda)$ by

$$\nabla(\lambda) := \sum_{j \geq 0} \frac{D_j}{\lambda^{j+2}}. \quad (41)$$

We have

$$\nabla(\lambda)(v_0) = (\Lambda - 1)(\alpha(\lambda)), \quad (42)$$

$$\nabla(\lambda)(w_0) = w_0(\Lambda - 1)(\gamma(\lambda) - 1). \quad (43)$$

**Lemma 3.** There exists a unique element $W(\lambda, \mu)$ in $\mathcal{A} \otimes \text{sl}_2(\mathbb{C})[[\lambda^{-1}, \mu^{-1}]]\lambda^{-1}\mu^{-1}$ of the form

$$W(\lambda, \mu) = \begin{pmatrix} X(\lambda, \mu) & Y(\lambda, \mu) \\ Z(\lambda, \mu) & -X(\lambda, \mu) \end{pmatrix}$$

satisfying the following linear inhomogeneous equations for the entries of $W$:

$$\Lambda(W(\lambda, \mu))U(\lambda) - U(\lambda)W(\lambda, \mu) + \Lambda(R(\lambda))\nabla(\mu)(U(\lambda)) - \nabla(\mu)(U(\lambda))R(\lambda) = 0, \quad (44)$$

$$X(\lambda, \mu) + 2\alpha(\lambda)X(\lambda, \mu) + \gamma(\lambda)Y(\lambda, \mu) + \beta(\lambda)Z(\lambda, \mu) = 0. \quad (45)$$

**Proof.** The existence part of this lemma follows from Lemma [2]. Indeed, if we define

$$W(\lambda, \mu) := \nabla(\mu)(R(\lambda)), \quad (46)$$

then $W(\lambda, \mu)$ satisfies (44)–(45). To see the uniqueness part, we first note that the (1,2)-entry and the (2,1)-entry of the matrix equation (44) imply that $Y$ and $Z$ can be uniquely expressed in terms of $X$. Indeed, we have

$$Z(\lambda, \mu) = \frac{(1 + \Lambda^{-1})(X(\lambda, \mu))}{\lambda^{-v_1}} + \gamma(\lambda)\frac{\Lambda^{-1}\nabla(\mu)(v_0)}{\lambda^{-v_1}}, \quad (46)$$

$$Y(\lambda, \mu) = -\nabla(\mu)(w_0)\frac{1 + \alpha(\lambda) + \Lambda(\alpha(\lambda))}{\lambda^{-v_0}} - w_0\frac{(1 + \Lambda)(X(\lambda, \mu))}{\lambda^{-v_1}} + \beta(\lambda)\frac{\nabla(\mu)(v_0)}{\lambda^{-v_0}}. \quad (47)$$
Substituting these two expressions in (45) we obtain the following linear inhomogeneous difference equation for $X$:

$$
\left(1 + 2\alpha(\lambda) + \frac{\beta(\lambda)}{\lambda - v_{-1}} - \frac{\nu_0 \gamma(\lambda)}{\lambda - v_0}\right)X(\lambda, \mu) - \frac{\nu_0 \gamma(\lambda)}{\lambda - v_0} \Lambda \left(X(\lambda, \mu)\right) + \frac{\beta(\lambda)}{\lambda - v_{-1}} \Lambda^{-1} \left(X(\lambda, \mu)\right)
= \left(1 + \alpha(\lambda) + \Lambda(\alpha(\lambda))\right) \gamma(\lambda) \left(\nabla(\mu) \frac{v_0}{\lambda - v_0}\right) - \beta(\lambda) \gamma(\lambda) \left(1 + \Lambda^{-1} \left(\nabla(\mu) \frac{v_0}{\lambda - v_0}\right)\right).$$

(48)

Suppose this equation has two solutions $X_1, X_2$ in $A[[\lambda^{-1}, \mu^{-1}]] \lambda^{-1} \mu^{-1}$. Let $X_0 = X_1 - X_2$, then $X_0 \in A[[\lambda^{-1}, \mu^{-1}]] \lambda^{-1} \mu^{-1}$, and it satisfies the following equation:

$$
\left(1 + 2\alpha(\lambda) + \frac{\beta(\lambda)}{\lambda - v_{-1}} - \frac{\nu_0 \gamma(\lambda)}{\lambda - v_0}\right)X_0(\lambda, \mu) - \frac{\nu_0 \gamma(\lambda)}{\lambda - v_0} \Lambda \left(X_0(\lambda, \mu)\right) + \frac{\beta(\lambda)}{\lambda - v_{-1}} \Lambda^{-1} \left(X_0(\lambda, \mu)\right) = 0.
$$

(49)

It follows that $X_0$ vanishes. Indeed, write $X_0 = \sum_{j \geq 0} X_{0,j}(\mu) \lambda^{-j+1}$. Observe that

$$
\frac{1}{\lambda - v_{-n}} = \frac{1}{\lambda} + \frac{v_n}{\lambda^2} + \cdots \in A[[\lambda^{-1}]] \lambda^{-1}, \quad m = -1, 0,
$$

and recall that $\alpha(\lambda), \beta(\lambda), \gamma(\lambda) \in A[[\lambda^{-1}]] \lambda^{-1}$. Then by comparing the coefficients of powers of $\lambda^{-1}$ consecutively we find that $X_{0,0}(\mu) = 0$, $X_{0,1}(\mu) = 0$, $X_{0,2}(\mu) = 0, \cdots$. So $X_0 = 0$. Hence $X_1 = X_2$. The lemma is proved. □

Based on this lemma we now give a new proof for the following proposition.

**Proposition 2** (10). The following equation holds true:

$$
\nabla(\mu) R(\lambda) = \frac{1}{\mu - \lambda} \left[R(\mu), R(\lambda)\right] + \left[Q(\mu), R(\lambda)\right],
$$

(50)

where

$$
Q(\mu) := -\frac{id}{\mu} + \begin{pmatrix} 0 & 0 \\ 0 & \gamma(\mu) \end{pmatrix}.
$$

Proof. Define $W^*$ as the right-hand side of (50), i.e.,

$$
W^* := \frac{1}{\mu - \lambda} \left[R(\mu), R(\lambda)\right] + \left[Q(\mu), R(\lambda)\right].
$$

More precisely, the entries of $W^*$ have the expressions:

$$
X^* = \frac{\nu_0}{\mu - \lambda} \left(\frac{(\alpha(\lambda) + \Lambda(\alpha(\lambda)) + 1)(\Lambda^{-1} \alpha(\mu) + \alpha(\mu) + 1)}{\lambda - v_0 (\mu - v_{-1})} - \frac{(\Lambda^{-1} \alpha(\lambda) + \alpha(\lambda) + 1)(\alpha(\mu) + \alpha(\mu))}{\lambda - v_{-1} (\mu - v_0)}\right),
$$

(51)

$$
Y^* = \frac{\nu_0}{\lambda - \mu} \left(\frac{(\alpha(\lambda) + \Lambda(\alpha(\lambda)) + 1)(\Lambda^{-1} \alpha(\mu) + \alpha(\mu))}{\lambda - v_0 (\mu - v_{-1})} + \frac{(\alpha(\lambda) + \alpha(\lambda) + \Lambda(\alpha(\lambda)) + 1)}{\lambda - v_{-1}}\right),
$$

(52)

$$
Z^* = \frac{1}{\lambda - \mu} \left(\frac{(\Lambda^{-1} \alpha(\lambda) + \alpha(\lambda) + 1)(\Lambda^{-1} \alpha(\mu) - \alpha(\mu))}{\lambda - v_{-1}} + \frac{(\Lambda^{-1} \alpha(\lambda) - \alpha(\lambda))}{\lambda - \mu}\right)\right).
$$

(53)

We can then verify that $W^* \in A \otimes \text{sl}_2(\mathbb{C})[[\lambda^{-1}, \mu^{-1}]] \lambda^{-1} \mu^{-1}$, as well as that $W := W^*$ satisfies the two equations (44), (45). The latter is done by a lengthy but straightforward calculation. The proposition is proved due to Lemma □
If we define $\tilde{\Omega}_{i,j}, \tilde{S}_{i}$ by
\begin{equation}
\sum_{i,j \geq 0} \tilde{\Omega}_{i,j} \lambda^{i+2} \mu^{j+2} = \text{Tr} \left( R(\lambda)R(\mu) \right) - \frac{1}{(\lambda - \mu)^2},
\end{equation}
\begin{equation}
\Lambda(\gamma(\lambda)) = \lambda^{-1} + \sum_{i \geq 0} \tilde{S}_{i} \lambda^{-i-2},
\end{equation}
then according to [10], $\tilde{\Omega}_{i,j}, \tilde{S}_{i}$ gives the canonical tau-structure for the Toda lattice, i.e.,
\[ \tilde{\Omega}_{i,j} = \Omega_{i,j}, \quad \tilde{S}_{i} = S_{i}. \]
These equalities together with Proposition 2 lead to Proposition 1; see [10] for the detailed proof of Proposition 1.

Before ending this section, we will make two remarks. The first remark is that all the entries of $R(\lambda)$ can be expressed by the canonical tau-structure. Indeed, we have
\begin{equation}
\alpha(\lambda) = \sum_{p \geq 0} \Omega_{p,0} \lambda^{-p-2}, \quad \beta(\lambda) = -w_{0} \Lambda(\gamma(\lambda)),
\end{equation}
\begin{equation}
\Lambda(\gamma(\lambda)) = \lambda^{-1} + \sum_{p \geq 0} S_{p} \lambda^{-p-2}.
\end{equation}
The proof was in [10]. The second remark is that existence of a tau-structure in general implies Lemma 1 and note that the proof in [10] of the fact that $\tilde{\Omega}_{i,j}, \tilde{S}_{i}$ is a tau-structure does not use the commutativity of the abstract Toda lattice hierarchy, so, as a byproduct of the matrix resolvent method we get a new proof of Lemma 1 together with a simple construction of the Toda lattice hierarchy. Similar idea was in [3].

3. Pair of wave functions

As in the Introduction, we start with the linear operator $L(n) = \Lambda + f(n) + g(n) \Lambda^{-1}$, where $f(n)$ and $g(n)$ are two given arbitrary elements in $V$. We show in this section the existence of pairs of wave functions associated to $(f(n), g(n))$. Let us write
\begin{equation}
\psi_{A}(\lambda, n) = e^{(\Lambda^{-1})^{-1} y(\lambda, n) \lambda^{n}}, \quad y(\lambda, n) := \sum_{i \geq 1} \frac{y_{i}(n)}{\lambda^{i}},
\end{equation}
\begin{equation}
\psi_{B}(\lambda, n) = e^{(\Lambda^{-1})^{-1} z(\lambda, n) e^{-s(n)} \lambda^{-n}}, \quad z(\lambda, n) := \sum_{i \geq 1} \frac{z_{i}(n)}{\lambda^{i}}.
\end{equation}
Then the spectral problems $L(n)\left( \psi(\lambda, n) \right) = \lambda \psi(\lambda, n)$ for $\psi = \psi_{A}$ and for $\psi = \psi_{B}$ recast into the following equations:
\begin{equation}
\lambda e^{y(\lambda, n)} + f(n) - \lambda + g(n) \lambda^{-1} e^{-y(\lambda, n) - 1} = 0,
\end{equation}
\begin{equation}
\lambda e^{-z(\lambda, n) - 1} + f(n) - \lambda + g(n + 1) \lambda^{-1} e^{z(\lambda, n)} = 0,
\end{equation}
yielding recursions of the form (as equivalent conditions to (60)–(61))

\[ y_{k+1}(n) = - \sum_{m_1, \ldots, m_k \geq 0} \frac{\prod_{i=1}^{k} y_i(n) m_i}{\prod_{i=1}^{k} m_i!} - f(n) \delta_{k,0} \]

\[ - g(n) \sum_{m_1, \ldots, m_{k-1} \geq 0} \frac{\prod_{i=1}^{k-1} (-1)^m y_i(n-1) m_i}{\prod_{i=1}^{k-1} m_i!}, \] (62)

\[ z_{k+1}(n) = \sum_{m_1, \ldots, m_k \geq 0} \frac{\prod_{i=1}^{k} (-1)^{m_i} z_i(n) m_i}{\prod_{i=1}^{k} m_i!} + f(n + 1) \delta_{k,0} \]

\[ + g(n + 2) \sum_{m_1, \ldots, m_{k-1} \geq 0} \frac{\prod_{i=1}^{k-1} z_i(n) m_i}{\prod_{i=1}^{k-1} m_i!}, \] (63)

where \( k \geq 0 \). From these recursions, it easily follows that \( y_k, z_k \in V, k \geq 0 \). This proves the existence of wave functions of type A and of type B meeting the definitions in Section 1.3. Clearly, \( \psi_A \) and \( \psi_B \) are unique up to multiplying by arbitrary series \( G(\lambda) \) and \( E(\lambda) \) of \( \lambda^{-1} \) with constant coefficient of the form \( G(\lambda) \in 1 + \mathbb{C}[\lambda^{-1}] \lambda^{-1} \) and \( E(\lambda) \in 1 + \mathbb{C}[\lambda^{-1}] \lambda^{-1} \). Since \( \psi_A(\lambda, n) = (1 + O(\lambda^{-1})) \lambda^n \) and since \( \psi_B(\lambda, n) = (1 + O(\lambda^{-1})) e^{-s(n)} \lambda^{-n} \), we find that the \( d(\lambda, n) \) defined in (27) must have the form

\[ d(\lambda, n) = \lambda e^{-s(n-1)} e^{\sum_{k \geq 1} d_k(n) \lambda^{-k}}. \]

Then by using the definitions of wave functions and of \( s(n) \) one easily derives that

\[ e^{s(n)} d(\lambda, n + 1) = e^{s(n)} d(\lambda, n). \] (64)

It follows that all \( d_k(n) \), \( k \geq 1 \) are constants. Therefore, for any fixed choice of \( \psi_A \), we can suitably choose the factor \( E(\lambda) \) for \( \psi_B \) such that \( \psi_A, \psi_B \) form a pair. This proves the existence of pair of wave functions associated to \( f(n), g(n) \).

We proceed with the time-dependence. Let \( (v(n, t), w(n, t)) \) be the unique solution in \( V[[t]]^2 \) to the Toda lattice hierarchy satisfying the initial condition \( v(n, 0) = f(n), \)

\( w(n, 0) = g(n) \). Let \( L(n, t) := \Lambda + v(n, t) + w(n, t) \Lambda^{-1}. \) Define \( \sigma(n, t) \) as the unique up to a constant function satisfying the following equations:

\[ w(n, t) = e^{\sigma(n-1, t) - \sigma(n, t)}, \]

\[ \frac{\partial \sigma(n, t)}{\partial t_p} = -S_p(n, t), \quad p \geq 0. \] (66)

An element \( \psi_A(n, t, \lambda) = (1 + O(\lambda^{-1})) \lambda^n e^{\sum_{k \geq 0} t_k \lambda^{k+1}} \) in \( \mathcal{V}[[t, \lambda^{-1}]] \) \( \lambda^n e^{\sum_{k \geq 0} t_k \lambda^{k+1}} \) is called a wave function of type A associated to \( (v(n, t), w(n, t)) \) if

\[ L(n, t) (\psi_A(\lambda, n, t)) = \lambda \psi_A(\lambda, n, t), \quad \frac{\partial \psi_A}{\partial t_k} = (L^{k+1})_+ (\psi_A). \] (67)
An element $\psi_B(n, t, \lambda) = (1 + O(\lambda^{-1}))\lambda^n e^{-\sum_{k \geq 0} t_k \lambda^{k+1}}$ in $\tilde{V}[\{t, \lambda^{-1}\}] e^{-\sigma(n, t)} \lambda^{-n} e^{-\sum_{k \geq 0} t_k \lambda^{k+1}}$ is called a wave function of type B associated to $(v(n, t), w(n, t))$ if

$$L(n, t) \left( \psi_B(\lambda, n, t) \right) = \lambda \psi_B(\lambda, n, t), \quad \frac{\partial \psi_B}{\partial t_k} = -(L^{k+1})^- \left( \psi_B \right).$$

(68)

The existence of wave functions $\psi_A$ and $\psi_B$ of type A and of type B associated to $(v(n, t), w(n, t))$ is a standard result in the theory of integrable systems (cf. [27, 6, 5, 13]); therefore we omit its details. Denote

$$d(\lambda, n, t) := \psi_A(\lambda, n, t) \psi_B(\lambda, n - 1, t) - \psi_B(\lambda, n, t) \psi_A(\lambda, n - 1, t),$$

and introduce

$$m(\mu, \lambda, n, t) := \frac{R(\mu, n, t)}{\mu - \lambda} + Q(\mu, n, t),$$

(70)

where $Q(\mu, n, t) := \frac{-id}{\mu} + \begin{pmatrix} 0 & 0 \\ 0 & \gamma(\mu, n, t) \end{pmatrix}$. We know from e.g. [10] that the wave function $\psi_A(\lambda, n, t)$ satisfies

$$\nabla(\mu) \begin{pmatrix} \psi_A(\lambda, n, t) \\ \psi_A(\lambda, n - 1, t) \end{pmatrix} = m(\mu, \lambda, n, t) \begin{pmatrix} \psi_A(\lambda, n, t) \\ \psi_A(\lambda, n - 1, t) \end{pmatrix}.$$  

(71)

Similarly, the wave function $\psi_B(\lambda, n, t)$ satisfies

$$\nabla(\mu) \begin{pmatrix} \psi_B(\lambda, n, t) \\ \psi_B(\lambda, n - 1, t) \end{pmatrix} = \left( m(\mu, \lambda, n, t) - \frac{\lambda}{\mu(\mu - \lambda)} I \right) \begin{pmatrix} \psi_B(\lambda, n, t) \\ \psi_B(\lambda, n - 1, t) \end{pmatrix}.$$  

(72)

Here, $I$ denotes the $2 \times 2$ identity matrix.

**Lemma 4.** The following formula holds true:

$$\nabla(\mu) \left( d(\lambda, n, t) \right) = \left( -\frac{1}{\mu} + \gamma(\mu, n, t) \right) d(\lambda, n, t).$$  

(73)

**Proof.** Recalling the definition (69) for $d$ and using (71) - (72) we find

$$\nabla(\mu) \left( d(\lambda, n, t) \right) = \left( \text{tr}(m(\mu, \lambda, n, t)) - \frac{\lambda}{\mu(\mu - \lambda)} \right) d(\lambda, n, t).$$  

(74)

The lemma is then proved via a straightforward computation. \[\square\]

**Definition 1.** We say $\psi_A, \psi_B$ form a pair if $e^{\sigma(n-1,t)} d(\lambda, n, t) = \lambda$.

The next lemma shows the existence of a pair.

**Lemma 5.** There exist a pair of wave functions $\psi_A, \psi_B$ associated to $(v(n, t), w(n, t))$. Moreover, the freedom of the pair is characterized by a factor $G(\lambda)$ via

$$\psi_A(\lambda, n, t) \mapsto G(\lambda) \psi_A(\lambda, n, t), \quad \psi_B(\lambda, n, t) \mapsto \frac{1}{G(\lambda)} \psi_B(\lambda, n, t),$$

(75)

$$G(\lambda) = \sum_{j \geq 0} G_j \lambda^{-j}, \quad G_0 = 1$$

(76)

with $G_j, j \geq 1$ being arbitrary constants.
Proof. Firstly, the freedom of a wave function \( \psi_A \) associated to \((v, w)\) is characterized by the multiplication by a factor \( G(\lambda) \) of the form (70). Fix an arbitrary choice of \( \psi_A \). For \( \psi_B \) being a wave function of type B associated to \((v, w)\), from (69) and the definitions of wave functions we know \( e^{\sigma(n-1, t)}d(\lambda, n, t) \) must have the form

\[
e^{\sigma(n-1, t)}d(\lambda, n, t) = \lambda e^{\sum_{k \geq 1} d_k(n, t) \lambda^{-k}}
\]

for some \( d_k(n, t) \), \( k \geq 1 \). By using (67), (68), (69) we find

\[
d(\lambda, n+1, t) = w(n, t)d(\lambda, n, t) = e^{\sigma(n-1, t)-\sigma(n, t)}d(\lambda, n, t),
\]
i.e.,

\[
e^{\sigma(n, t)}d(\lambda, n+1, t) = e^{\sigma(n-1, t)}d(\lambda, n, t),
\]

Using Lemma 4 and (66) we have

\[
\nabla(\mu)\left(e^{\sigma(n-1, t)}d(\lambda, n, t)\right)
= e^{\sigma(n-1, t)}\nabla(\mu)\left(\sigma(n-1, t)\right)d(\lambda, n, t) + e^{\sigma(n-1, t)}\nabla(\mu)\left(d(\lambda, n, t)\right)
= -e^{\sigma(n-1, t)} \sum_{p \geq 0} \frac{S_p(n-1, t)}{\mu^{p+2}}d(\lambda, n, t) + e^{\sigma(n-1, t)}d(\lambda, n, t)\left(-\frac{1}{\mu} + \gamma(\mu, n, t)\right) = 0.
\]

So we have

\[
\frac{\partial(e^{\sigma(n-1, t)}d(\lambda, n, t))}{\partial \mu} = 0, \quad \forall \mu \geq 0.
\]

We deduce from (77), (78), (79) that \( d_k(n, t), k \geq 1 \) are all constants. Therefore, there exists a unique choice of \( \psi_B \) such that \( \psi_A, \psi_B \) form a pair. The lemma is proved. \( \square \)

4. The k-point Generating Series

Let \((v, w) = (v(n, t), w(n, t)) \in V[[t]]^2\) be the unique solution to the Toda lattice hierarchy with the initial value \((v(n, 0), w(n, 0)) = (f(n), g(n))\), and \((\psi_A, \psi_B)\) a pair of wave functions associated to \((v, w)\). Define

\[
\Psi_{\text{pair}}(\lambda, n, t) = \begin{pmatrix}
\psi_A(\lambda, n, t) & \psi_B(\lambda, n, t) \\
\psi_A(\lambda, n-1, t) & \psi_B(\lambda, n-1, t)
\end{pmatrix}.
\]

Proposition 3. The following identity holds true:

\[
R(\lambda, n, t) \equiv \Psi_{\text{pair}}(\lambda, n, t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Psi_{\text{pair}}^{-1}(\lambda, n, t).
\]

Proof. Define

\[
M = M(\lambda, n, t) := \Psi_{\text{pair}}(\lambda, n, t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Psi_{\text{pair}}^{-1}(\lambda, n, t).
\]

It is easy to verify that \( M \) satisfies

\[
[\mathcal{L}, M] (\Psi_{\text{pair}}) = 0, \quad \det M = 0.
\]

The entries of \( M \) in terms of the pair of wave functions read

\[
M = \frac{1}{d(\lambda, n, t)} \begin{pmatrix}
\psi_A(\lambda, n, t) \psi_B(\lambda, n-1, t) & -\psi_A(\lambda, n, t) \psi_B(\lambda, n, t) \\
\psi_A(\lambda, n-1, t) \psi_B(\lambda, n-1, t) & -\psi_A(\lambda, n-1, t) \psi_B(\lambda, n, t)
\end{pmatrix},
\]

where \( \psi_A(\lambda, n, t) \) and \( \psi_B(\lambda, n, t) \) are the entries of the pair \((\psi_A, \psi_B)\).
where we recall that $d(\lambda, n, t) = \psi_A(\lambda, n, t) \psi_B(\lambda, n - 1, t) - \psi_B(\lambda, n, t) \psi_A(\lambda, n - 1, t)$, which coincides with the determinant of $\Psi(\lambda, n, t)$. It follows from (81) that

$$
M(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \text{Mat}(2, \mathbb{C}) = [t, \Lambda^{-1}]\Lambda^{-1}.
$$

(83)

The proposition then follows from the uniqueness theorem proven in Section 2.

Define

$$
D(\lambda, \mu, n, t) := \frac{\psi_A(\lambda, n, t) \psi_B(\mu, n - 1, t) - \psi_A(\lambda, n - 1, t) \psi_B(\mu, n, t)}{\lambda - \mu}.
$$

(84)

**Theorem 2.** Fix $k \geq 2$ being an integer. The generating series of $k$-point correlation functions of the solution $(\nu(n, t), \omega(n, t))$ has the following expression:

$$
\sum_{i_1, \ldots, i_k \geq 0} \frac{\Omega_{i_1, \ldots, i_k}(n, t)}{\lambda_1^{i_1+2} \cdots \lambda_k^{i_k+2}} = (-1)^k e^{k \sigma(n-1, t)} \prod_{j=1}^k \sum_{\pi \in S_k/C_k} D(\lambda_{\pi(j)}, \lambda_{\pi(j+1)}, n, t) - \frac{\delta_{k, 2}}{(\lambda_1 - \lambda_2)^2}.
$$

(85)

**Proof.** It follows from (81) that

$$
R(\lambda, n, t) = \frac{r_1(\lambda, n, t)^T r_2(\lambda, n, t)}{d(\lambda, n, t)},
$$

(86)

where $r_1(\lambda, n, t) := (\psi_A(\lambda, n, t), \psi_A(\lambda, n - 1, t))$, $r_2(\lambda, n, t) := (\psi_B(\lambda, n - 1, t), -\psi_B(\lambda, n, t))$. Substituting this expression into the identity

$$
\sum_{i_1, i_2 \geq 0} \frac{\Omega_{i_1, i_2}(n, t)}{\lambda_1^{i_1+2} \lambda_2^{i_2+2}} = \frac{\text{Tr} \left( R_1(\lambda_1, n, t) R_2(\lambda_2, n, t) \right)}{(\lambda_1 - \lambda_2)^2} - \frac{1}{(\lambda_1 - \lambda_2)^2},
$$

(87)

we obtain

$$
\sum_{i_1, i_2 \geq 0} \frac{\Omega_{i_1, i_2}(n, t)}{\lambda_1^{i_1+2} \lambda_2^{i_2+2}} = \frac{\text{Tr} \left( r_1(\lambda_1, n, t)^T r_2(\lambda_1, n, t) r_1(\lambda_2, n, t)^T r_2(\lambda_2, n, t) \right)}{d(\lambda_1, n, t) d(\lambda_2, n, t) (\lambda_1 - \lambda_2)^2} - \frac{1}{(\lambda_1 - \lambda_2)^2}
$$

$$
= \frac{\left( r_2(\lambda_2, n, t) r_1(\lambda_1, n, t)^T \right) \left( r_2(\lambda_1, n, t) r_1(\lambda_2, n, t)^T \right)}{d(\lambda_1, n, t) d(\lambda_2, n, t) (\lambda_1 - \lambda_2)^2} - \frac{1}{(\lambda_1 - \lambda_2)^2}
$$

$$
= - \frac{D(\lambda_1, \lambda_2, n, t) D(\lambda_2, \lambda_1, n, t)}{\lambda_1 \lambda_2 e^{-2 \sigma(n-1, t)}} - \frac{1}{(\lambda_1 - \lambda_2)^2},
$$

(88)

where we used the definition (84) and

$$
\frac{\psi_A(\lambda, n, t) \psi_B(\mu, n - 1, t) - \psi_A(\lambda, n - 1, t) \psi_B(\mu, n, t)}{\lambda - \mu} = \frac{r_2(\mu, n, t) r_1(\lambda, n, t)^T}{\lambda - \mu}.
$$
In particular, let \( \phi_k \) be the wave functions introduced. In (85) or (30), the freedom (75) affects the \( G \) by a factor of the form \( \frac{G(\lambda)}{G(\mu)} \), but the product \( \prod_{j=1}^{\lambda} D(\lambda_{i(j)}, \lambda_{i(j+1)}) \) remains unchanged.

In Appendix A, the abstract form of (85) is obtained, where a pair of abstract pre-wave functions are introduced. Write

\[
\psi_A(\lambda, n, t) = \phi_A(\lambda, n, t) \lambda^n, \quad \psi_B(\lambda, n, t) = \phi_B(\lambda, n, t) e^{-\sigma(n,t)} \lambda^{-n}. \tag{90}
\]

Theorem 1 can then be alternatively written in terms of \( \phi_A, \phi_B \) by the following corollary.

**Corollary 1.** The following formula holds true for \( k \geq 2 \):

\[
\sum_{i_1, \ldots, i_k \geq 0} \frac{\Omega_{i_1, \ldots, i_k}(n, t)}{\lambda_{i_1+1}^{1+1} \cdots \lambda_{i_k+1}^{k+1}} = (-1)^{k-1} \sum_{\pi \in S_k/C_k} \prod_{j=1}^{\lambda} B(\lambda_{i(j)}, \lambda_{i(j+1)}; n, t) - \frac{\delta_{k,2}}{(\lambda_1 - \lambda_2)^2}, \tag{91}
\]

where \( B(\lambda, \mu, n, t) \) is defined by

\[
B(\lambda, \mu, n, t) := \frac{\phi_A(\lambda, n, t) \phi_B(\mu, n-1, t) \phi_A(\lambda, n-1, t) \phi_B(\mu, n) \phi_A(\mu, n-1) \phi_B(\mu, n)}{\lambda - \mu}. \tag{92}
\]

In particular, let \( \phi_A(\lambda, n) := e^{(\lambda-1)(y(\lambda,n))} \), \( \phi_B(\lambda, n) := e^{(\lambda-1)(z(\lambda,n))} \) (cf. (60) – (61)), and let \( B(\lambda, \mu, n) := \phi_A(\lambda, n) \phi_B(\mu, n-1) - \phi_B(\lambda, n) \phi_A(\mu, n) \), then we have

\[
\sum_{i_1, \ldots, i_k \geq 0} \frac{\Omega_{i_1, \ldots, i_k}(n, 0)}{\lambda_{i_1+1}^{1+1} \cdots \lambda_{i_k+1}^{k+1}} = (-1)^{k-1} \sum_{\pi \in S_k/C_k} \prod_{j=1}^{\lambda} B(\lambda_{i(j)}, \lambda_{i(j+1)}; n) - \frac{\delta_{k,2}}{(\lambda_1 - \lambda_2)^2}. \tag{93}
\]

For some particular examples related to matrix models, it turns out that the suitable chosen \( D \) coincides, possibly up to simple factors, with certain kernel of the matrix model. However, the \( D \) is not unique. We now introduce a formal series \( K(\lambda, \mu) \)
such that the generating series of multi-point correlation functions still has an explicit expression, but this time $K$ is \textit{local} and is therefore unique for the given solution. The series $K$ is defined by

$$K(\lambda, \mu) := \frac{(1 + \alpha(\lambda))(1 + \alpha(\mu)) - w_0 \gamma(\lambda) \Lambda(\gamma(\mu))}{\lambda - \mu}, \quad (94)$$

where $1 + \alpha(\lambda)$ is the (1,1)-entry of the basic matrix resolvent $R(\lambda)$, and $\gamma(\lambda)$ is the (2,1)-entry. The next theorem expresses the left-hand side of \textcolor{blue}{(85)} in terms of $K$.

\textbf{Theorem 3.} For any $k \geq 2$, the following formula holds true:

$$\sum_{i_1, \ldots, i_k \geq 0} \Omega_{i_1, \ldots, i_k}^{\lambda^{i_1} \cdots \lambda_k^{i_k}} = (-1)^{k-1} \sum_{\pi \in S_k/C_k} \prod_{j=1}^{k} K(\lambda_{\pi(j)}, \lambda_{\pi(j+1)}, n, t) - \frac{\delta_{k,2}}{\lambda_1 - \lambda_2}. \quad (95)$$

\textbf{Proof.} The identity \textcolor{blue}{(81)} gives

$$\psi_B(\lambda, n - 1, t) = \frac{(1 + \alpha(\lambda, n, t)) d(\lambda, n, t)}{\psi_A(\lambda, n, t)},$$

$$\psi_B(\lambda, n, t) = -\beta(\lambda, n, t) d(\lambda, n, t) = w_n \frac{\gamma(\lambda, n + 1, t) d(\lambda, n, t)}{\psi_A(\lambda, n, t)},$$

$$\psi_A(\lambda, n - 1, t) = \psi_A(\lambda, n, t) \frac{\gamma(\lambda, n, t)}{1 + \alpha(\lambda, n, t)}.$$

Substituting these expressions into \textcolor{blue}{(84)} we obtain

$$D(\lambda, \mu, n, t) = d(\mu, n, t) \frac{\psi_A(\lambda, n, t)}{\psi_A(\mu, n, t)} e(\lambda, n, t), \quad (96)$$

where

$$e(\lambda, n, t) := \frac{(1 + \alpha(\lambda, n, t))(1 + \alpha(\mu, n, t)) - w_n(t) \gamma(\lambda, n, t) \gamma(\mu, n + 1, t)}{(\lambda - \mu)(1 + \alpha(\lambda, n, t))}. \quad (97)$$

Combining with the definition of $K(\lambda, \mu, n, t)$ and Theorem \textcolor{blue}{[4]} we find

$$\sum_{i_1, \ldots, i_k \geq 0} \Omega_{i_1, \ldots, i_k}^{\lambda^{i_1} \cdots \lambda_k^{i_k}} = (-1)^{k-1} \sum_{\pi \in S_k/C_k} \prod_{j=1}^{k} K(\lambda_{\pi(j)}, \lambda_{\pi(j+1)}, n, t) - \frac{\delta_{k,2}}{\lambda_1 - \lambda_2}. \quad (98)$$

The theorem is proved. \hfill \square

It seems to be an interesting question to study the geometric and algebraic meaning of the kernel $K$ (as well as $D$). Below we give without proof some of their properties.
Proposition 4. The functions $K$ and $D$ are related by

\[
K(\lambda, \mu, n, t) = \frac{e^{\sigma(n-1,t)}}{\mu} \left(1 + \alpha(\lambda, n, t)\right) \frac{\psi_A(\mu, n, t)}{\psi_A(\lambda, n, t)} D(\lambda, \mu, n, t)
\]

\[
= \frac{e^{2\sigma(n-1,t)}}{\lambda \mu} \psi_A(\mu, n, t) \psi_B(\lambda, n - 1, t) D(\lambda, \mu, n, t)
\]

\[
= \frac{e^{\sigma(n-1,t)}}{\lambda} \left(1 + \alpha(\mu, n, t)\right) \frac{\psi_B(\lambda, n - 1, t)}{\psi_B(\mu, n - 1, t)} D(\lambda, \mu, n, t).
\]

We observe that the following three formal series

\[
K(\lambda, \mu) = \frac{1 + \alpha(\lambda)}{\lambda - \mu}, \quad K(\lambda, \mu) = \frac{1 + \alpha(\mu)}{\lambda - \mu}, \quad K(\lambda, \mu) = \frac{2 + \alpha(\lambda) + \alpha(\mu)}{2(\lambda - \mu)}
\]

all belong to $A[[\lambda^{-1}, \mu^{-1}]]$. Therefore, the following three formal series

\[
K(\lambda, \mu, n, t) = \frac{1 + \alpha(\lambda, n, t)}{\lambda - \mu}, \quad K(\lambda, \mu, n, t) = \frac{1 + \alpha(\mu, n, t)}{\lambda - \mu}, \quad K(\lambda, \mu, n, t) = \frac{2 + \alpha(\lambda, n, t) + \alpha(\mu, n, t)}{2(\lambda - \mu)}
\]

all belong to $V[[t]][[\lambda^{-1}, \mu^{-1}]]$. It follows from this observation and Proposition 4 that

\[
\frac{e^{\sigma(n-1,t)}}{\mu} D(\lambda, \mu, n, 0) \left(\frac{\mu}{\lambda}\right)^n - \frac{1}{\lambda - \mu} \in \tilde{V}[[\lambda^{-1}, \mu^{-1}]]. \tag{99}
\]

Remark 3. We could loosen both the conditions for wave functions and the pair-condition. Let us say $\psi_A$ and $\psi_B$ are pre-wave functions of type A and of type B, respectively, if they satisfy the first equations of \((67)\) and \((68)\). Define $d_{\text{pre}}(\lambda, \mu, n, t)$ and $D_{\text{pre}}(\lambda, \mu, n, t)$ by \((129)\) and \((140)\). Then the following formula holds true:

\[
\sum_{i_1, \ldots, i_k \geq 0} \frac{\Omega_{i_1, \ldots, i_k}(n, t)}{\lambda_{i_1+2} \cdots \lambda_{i_k+2}} \prod_{j=1}^k d_{\text{pre}}(\lambda_{\pi(j)}, \lambda_{\pi(j)+1}, n, t) - \frac{\delta_{k,2}}{(\lambda_1 - \lambda_2)^2}. \tag{100}
\]

Now $\psi_A$ and $\psi_B$ are determined by $(v(n, t), w(n, t))$ up to

\[
\psi_A(\lambda, n, t) \mapsto G(\lambda, t) \psi_A(\lambda, n, t), \quad \psi_B(\lambda, n, t) \mapsto E(\lambda, t) \psi_B(\lambda, n, t),
\]

where $G(\lambda, t) = 1 + \sum_{k \geq 1} G_k(t) \lambda^{-k}, E(\lambda, t) = 1 + \sum_{k \geq 1} E_k(t) \lambda^{-k}$ with $G_k(t), E_k(t) \in \mathbb{C}[[t]], k \geq 1$. This freedom affects $D_{\text{pre}}(\lambda, \mu, n, t)$ and $d_{\text{pre}}(\lambda, n, t)$ into

\[
D_{\text{pre}}(\lambda, \mu, n, t) \mapsto G(\lambda, t) E(\mu, t) D_{\text{pre}}(\lambda, \mu, n, t), \quad d_{\text{pre}}(\lambda, n, t) \mapsto G(\lambda, t) E(\lambda, t) d_{\text{pre}}(\lambda, n, t).
\]

Therefore, it gives rise to each summand of \((100)\) the factor

\[
\frac{\prod_{j=1}^k G(\lambda_{\pi(j)}, t) E(\lambda_{\pi(j)+1}, t)}{\prod_{j=1}^k G(\lambda_{j}, t) E(\lambda_{j}, t)},
\]

which is equal to 1. Hence the right-hand side of \((100)\) remains unchanged.
5. Applications

Partition functions in some matrix models and enumerative models are particular tau-functions for the Toda lattice hierarchy. Theorem 1 can then be used for computing their logarithmic derivatives. In this section we do two explicit computations.

5.1. Application I. Enumeration of ribbon graphs. The initial data of the GUE solution to the Toda lattice hierarchy is given by \( f(n) = 0 \) and \( g(n) = n \); see for example [10] for the proof. For this case, we can take \( V = Q[n] \) and \( \tilde{V} = V \). Substituting the initial data in (26) we find

\[
s(n) = -(1 - \Lambda^{-1})^{-1} \log g(n) = -(1 - \Lambda^{-1})^{-1} \log n = -\log \Gamma(n + 1) + C, \tag{101}
\]

where \( C \) is a constant. Below we fix this constant as 0.

**Proposition 5.** The \( \psi_A, \psi_B \) defined by

\[
\psi_A(\lambda, n) = \sum_{j \geq 0} (-1)^j \frac{(n - 2j + 1)2j}{2j! \lambda^{2j}} \lambda^n, \tag{102}
\]

\[
\psi_B(\lambda, n) = \Gamma(n + 1) \sum_{j \geq 0} \frac{(n + 1)2j}{2j! \lambda^{2j}} \lambda^{-n} \tag{103}
\]

form a particular pair of wave functions associated to \( f(n), g(n) \). Here and below \((a)_i\) denotes the increasing Pochhammer symbol defined by \((a)_i = a(a + 1) \cdots (a + i - 1)\).

**Proof.** It is straightforward to verify that both \( \psi_A \) and \( \psi_B \) satisfy the equation

\[
\psi(\lambda, n + 1) + n \psi(\lambda, n - 1) = \lambda \psi(\lambda, n). \tag{104}
\]

Moreover, from the definitions (102)–(103) we see that

\[
\psi_A \in \tilde{V}(\lambda^{-1}) \lambda^n, \quad \psi_B \in \tilde{V}(\lambda^{-1}) e^{-s(n)} \lambda^{-n}.
\]

We are left to show that

\[
\Gamma(n)^{-1} \left( \psi_A(\lambda, n) \psi_B(\lambda, n - 1) - \psi_B(\lambda, n) \psi_A(\lambda, n - 1) \right) = \lambda. \tag{105}
\]

Clearly, the meaning of this identity is the following: both sides of (105) are Laurent series of \( \lambda^{-1} \) with coefficients in \( \tilde{V} = V = Q[n] \), and the equality means all the coefficients should be equal. More precisely, the identity (105) can be equivalently written as the following sequence of identities:

\[
\frac{n}{j + 1} \sum_{j_1=0}^{j+1} \frac{(-1)^{j_1}}{2} \binom{j + 1}{j_1} \binom{n + 2j_1 - 1}{2j + 1} + \sum_{j_1=0}^{j} (-1)^{j_1} \binom{j}{j_1} \binom{n + 2j_1}{2j + 1} = 0, \quad j \geq 0. \tag{106}
\]

From (64) we know that the left-hand side of (106) as a polynomial of \( n \) is a constant for any \( j \geq 0 \). Note that the value of the left-hand side of (106) at \( n = 0 \) is obviously 0 for any \( j \geq 0 \). The proposition is proved. \( \square \)
It follows from the above proposition an explicit expression for the $D(\lambda, \mu, n, 0)$ (cf. equation (84)) associated to the pair (102)–(103):

$$\frac{e^{s(n-1)}}{\mu} D(\lambda, \mu, n, 0) \left(\frac{\mu}{\lambda}\right)^n = \frac{1}{\lambda - \mu} + A(\lambda, \mu, n),$$  \hspace{1cm} (107)

with $A(\lambda, \mu, n)$ given by

$$A(\lambda, \mu, n) = \sum_{k \geq 1} \frac{(2k-1)!!}{(2k)!} \sum_{p=0}^{2k-1} (-1)^{p+[(p+1)/2]} \left(\frac{k-1}{p/2}\right)^{2k-1-p} \prod_{j=-p}^{n+j} (n+j+1) \lambda^{-p-1} \mu^{-(2k-p)}.$$  \hspace{1cm} (108)

This explicit expression (108) first appeared in [31]. Denote

$$\hat{A}(\lambda, \mu, n) = \frac{1}{\lambda - \mu} + A(\lambda, \mu, n).$$  \hspace{1cm} (109)

As a corollary of Proposition 5, Theorem 1 and the above (107) we have achieved a new proof of the following theorem of Jian Zhou.

**Theorem 4** (Zhou, [31]). Fix $k \geq 2$ being an integer. The generating series of $k$-point connected GUE correlators has the following expression:

$$\sum_{i_1, \ldots, i_k \geq 1} \frac{\langle \text{tr} M_{i_1} \cdots \text{tr} M_{i_k} \rangle_c}{\lambda_{i_1+1} \cdots \lambda_{i_k+1}} = (-1)^{k-1} \sum_{\pi \in S_k/C_k} \prod_{j=1}^{k} \hat{A}(\lambda_{\pi(j)}, \lambda_{\pi(j+1)}, n) - \frac{\delta_{k,2}}{(\lambda_1 - \lambda_2)^2},$$  \hspace{1cm} (110)

where $\hat{A}$ is defined by (108)–(109). Here we recall that for any fixed $i_1, \ldots, i_k$, the connected GUE correlator $\langle \text{tr} M_{i_1} \cdots \text{tr} M_{i_k} \rangle_c$ is a polynomial of $n$ (cf. [4, 17, 21, 10]).

5.2. Application II. Gromov–Witten invariants of $\mathbb{P}^1$ in the stationary sector.

The initial data for the Gromov–Witten solution to the Toda lattice hierarchy was for example derived in [10, 12, 11]. It has the following explicit expression:

$$f(n) = nc + \frac{\epsilon}{2}, \hspace{0.5cm} g(n) = 1.$$  \hspace{1cm} (111)

We have

$$s(n) = -(1 - \Lambda^{-1})^{-1} \log 1 = C,$$

where $C$ is an arbitrary constant. Below we take $C = 0$.

**Proposition 6.** The $\psi_1, \psi_2$ defined by

$$\psi_A(\lambda, n) = e^{\lambda - \frac{n}{2}} \Gamma\left(\frac{\lambda}{\epsilon} + \frac{1}{2}\right) J_{\frac{\lambda}{\epsilon} - n - \frac{1}{2}} \left(\frac{2}{\epsilon}\right),$$  \hspace{1cm} (112)

$$\psi_B(\lambda, n) = (-1)^{n+1} e^{-\lambda - \frac{n}{2}} \Gamma\left(\frac{-\lambda}{\epsilon} + \frac{1}{2}\right) J_{\frac{-\lambda}{\epsilon} + n + \frac{1}{2}} \left(\frac{2}{\epsilon}\right)$$  \hspace{1cm} (113)

form a particular pair of wave functions associated to $f(n) = nc + \frac{\epsilon}{2}, g(n) = 1$. Here, $J_\nu(y)$ denotes the Bessel function, and the right-hand sides of (112)–(113) are understood as the large $\lambda$ asymptotics of the corresponding analytic functions.
Proof. Firstly, using the properties of Bessel functions we can verify that $\psi_A(\lambda, n)$ and $\psi_B(\lambda, n)$ defined from the above asymptotics satisfy

$$
\psi_A(\lambda, n + 1) + \left( ne + \frac{\epsilon}{2} \right) \psi_A(\lambda, n) + \psi_A(\lambda, n - 1) = \lambda \psi_A(\lambda, n),
$$

$$
\psi_B(\lambda, n + 1) + \left( ne + \frac{\epsilon}{2} \right) \psi_B(\lambda, n) + \psi_B(\lambda, n - 1) = \lambda \psi_B(\lambda, n).
$$

Secondly, as $\lambda$ goes to $\infty$, the following asymptotics hold true:

$$
\epsilon^{\frac{\lambda}{2} - \frac{1}{2}} \Gamma\left(\frac{\lambda}{\epsilon} + \frac{1}{2}\right) J_{\lambda - n - \frac{1}{2}}\left(\frac{2}{\epsilon}\right) \sim \lambda^n \left(1 + O(\lambda^{-1})\right),
$$

$$
(-1)^n \epsilon^{\frac{\lambda}{2} - \frac{1}{2}} \lambda \Gamma\left(\frac{-\lambda}{\epsilon} + \frac{1}{2}\right) J_{-\lambda + n + \frac{1}{2}}\left(\frac{2}{\epsilon}\right) \sim \lambda^{-n} \left(1 + O(\lambda^{-1})\right).
$$

Thirdly, $\psi_A$ and $\psi_B$ also satisfy

$$
\psi_A(\lambda, n) \psi_B(\lambda, n - 1) - \psi_B(\lambda, n) \psi_A(\lambda, n - 1) = \lambda.
$$

We have verified all the defining properties for a pair of wave functions associated to $f(n) = ne + \frac{\epsilon}{2}, g(n) = 1$. The proposition is proved. \qed

Note that

$$
\psi_A(\lambda, n - 1) = \epsilon^{\frac{\lambda}{2} - \frac{1}{2}} \Gamma\left(\frac{\lambda}{\epsilon} + \frac{1}{2}\right) J_{\lambda - n - \frac{1}{2}}\left(\frac{2}{\epsilon}\right),
$$

$$
\psi_B(\lambda, n - 1) = (-1)^n \epsilon^{\frac{\lambda}{2} - \frac{1}{2}} \lambda \Gamma\left(\frac{-\lambda}{\epsilon} + \frac{1}{2}\right) J_{-\lambda + n + \frac{1}{2}}\left(\frac{2}{\epsilon}\right),
$$

and denote

$$
J_\nu(y) =: \frac{(y/2)^\nu}{\Gamma(\nu + 1)} J_{\nu + \frac{1}{2}}(y^2/4).
$$

It follows from (112)–(115), (84) that the $D(\lambda, \mu, 0, 0)$ associated to the pair (112)–(113) has the following explicit expression:

$$
\frac{1}{\mu} D(\lambda, \mu, 0, 0) = -\frac{1}{\epsilon} \left( \frac{\nu}{\lambda} J_{\lambda} \left( \frac{1}{\lambda^2} \right) + \frac{\epsilon^{-\nu}}{\lambda^{-\nu}} \frac{\nu}{\lambda} J_{-\lambda} \left( \frac{1}{\lambda^2} \right) \frac{\nu}{\lambda} J_{\lambda + \nu} \left( \frac{1}{\lambda^2} \right) \right). \mu / \epsilon - \lambda / \epsilon
$$

Then according to [12], the function $\frac{1}{\mu} D(\lambda, \mu, 0, 0)$ has the following expressions:

$$
\frac{1}{\mu} D(\lambda, \mu, 0, 0)
$$

$$
= -\frac{1}{\epsilon} \sum_{k=0}^{\infty} \frac{(a - b - 2k + 1)_{k-1}}{k! (-a + \frac{1}{2})_k (b + \frac{1}{2})_k} \epsilon^{-2k}
$$

$$
= \frac{-1}{\epsilon (a - b)} 2F_3\left(\frac{b-a}{2}, \frac{b-a+1}{2}, \frac{1}{2} - a, \frac{1}{2} + b, b - a + 1; -4\epsilon^{-2}\right)
$$

$$
\sim \frac{-1}{\epsilon (a - b)} - \sum_{p,q \geq 0} \frac{(-1)^{p+1}}{a^{p+1}b^{q+1}} \sum_{k \geq 1} \frac{\epsilon^{-2k-1}}{k!}
$$

$$
\sum_{1 \leq i, j \leq k} (-1)^{i+j} \frac{i+j}{(i-1)! (j-1)! (k-i)! (k-j)!} =: \hat{A}(\lambda, \mu),
$$
where \( a := \frac{\mu}{\epsilon}, \) \( b := \frac{1}{\epsilon}, \) the \((a - b + 1)_{-1}\) of \((116)\) is defined as \(1/(a - b)\), and \( \sim \) in \((118)\) is taken as \(a, b \to \infty\) away from the half integers. The explicit expression \((118)\) first appeared in \([12]\). So we have completed a new proof of the following theorem.

**Theorem 5** \((12)\). The generating series of \(k\)-point \((k \geq 2)\) Gromov–Witten invariants of \(\mathbb{P}^1\) in the stationary sector has the following explicit expression:

\[
\epsilon^k \sum_{i_1, \ldots, i_k \geq 0} \frac{(i_1 + 1)! \cdots (i_k + 1)!}{\lambda_1^{i_1+2} \cdots \lambda_k^{i_k+2}} \langle \tau_{i_1} (\omega) \cdots \tau_{i_k} (\omega) \rangle (\epsilon) = (-1)^{k-1} \sum_{\pi \in S_k / C_k} \prod_{i=1}^{k} \hat{A} (\lambda_{\pi(i)}, \lambda_{\pi(i+1)}) - \frac{\delta_{k,2}}{(\lambda_1 - \lambda_2)^2},
\]

(119)

where \(\hat{A} (\lambda, \mu)\) is explicitly defined in \((118)\), and

\[
\langle \tau_{i_1} (\omega) \cdots \tau_{i_k} (\omega) \rangle (\epsilon) := \sum_{g \geq 0} \epsilon^{2g-2} \sum_{d \geq 2} \int_{[M_{g,k}(\mathbb{P}^1, d)]^{\virt}} \text{ev}_1^* (\omega) \cdots \text{ev}_k^* (\omega) \psi_1 \cdots \psi_k.
\]

(120)

*(See for example [12] for the notation about the integral in the right-hand side of (120).)*

**Appendix A. Pair of abstract pre-wave functions**

Here we construct a ring that is suitable for defining abstract pre-wave functions. Recall that \(\mathcal{A}\) is the ring of polynomials of \(v_k, w_k, k \in \mathbb{Z}\). Instead of the \(\mathbb{Z}\)-coefficients, we will use in this appendix the \(\mathbb{Q}\)-coefficients, i.e., \(\mathcal{A} = \mathbb{Q}[v_k, w_k \mid k \in \mathbb{Z}]\) is now under consideration. For each monic monomial \(\alpha \in \mathcal{A}\setminus \mathbb{Q}\), we associate a symbol \(m_{\alpha}\). Denote by \(\mathcal{B}\) the polynomial ring

\[
\mathcal{B} := \mathbb{Q}[\{ m_{\alpha} \mid \alpha \text{ is a monic monomial in } \mathcal{A}\setminus \mathbb{Q} \}].
\]

(121)

Define the action of \(\Lambda^k\) on \(\mathcal{B}\) with \(k \in \mathbb{Z}\) by

\[
\Lambda^k(m_{\alpha_1} \cdots m_{\alpha_l}) = m_{\Lambda^k(\alpha_1)} \cdots m_{\Lambda^k(\alpha_l)}
\]

(122)

for \(\alpha_1, \ldots, \alpha_l\) being monic monomials in \(\mathcal{A}\setminus \mathbb{Q}\), as well as by linearly extending it to other elements of \(\mathcal{B}\). For a monic monomial \(\alpha = v_{i_1} \cdots v_{i_r} w_{j_1} \cdots w_{j_s} \in \mathcal{A}\setminus \mathbb{Q}\) with \(i_1 \leq \cdots \leq i_r, j_1 \leq \cdots \leq j_s\) and \(r + s \geq 1,\) let \(k_0 := -i_1\) (if \(r \geq 1\)), \(k_0 := -j_1\) (if \(r = 0\)); the monomial \(\Lambda^{k_0}(\alpha) \in \mathcal{A}\) is then called the (unique) reduced monomial (associated to \(\alpha\)). Denote by \(\mathcal{C}\) the polynomial ring generated by \(m_\beta, v_k, w_k\) with \(\mathbb{Q}\)-coefficients, where \(\beta\) are reduced monomials, and \(k \in \mathbb{Z}\). Let us also define an action of \(\Lambda^k\) on \(\mathcal{C}\), \(k \in \mathbb{Z}\). To this end, we introduce some notations: for \(\beta\) a reduced monomial of \(\mathcal{A}\), denote

\[
n_{\Lambda^k(\beta)} := \left\{ \begin{array}{ll}
m_\beta + \sum_{i=1}^{k-1} \Lambda^i(\beta), & k \geq 0, \\
m_\beta - \sum_{i=k}^{k-1} \Lambda^i(\beta), & k \leq -1.
\end{array} \right.
\]

(123)

Then for a monomial \(\alpha \cdot m_{\beta_1} \cdots m_{\beta_s}\) of \(\mathcal{C}\) with \(\alpha\) being a monomial in \(\mathcal{A}\), define

\[
\Lambda^k(\alpha \cdot m_{\beta_1} \cdots m_{\beta_s}) = \Lambda^k(\alpha) \cdot \prod_{j=1}^{s} n_{\Lambda^k(\beta_j)}, \quad k \in \mathbb{Z}.
\]

(124)

Define the action of \(\Lambda^k\) on other elements in \(\mathcal{C}\) by requiring it as a linear operator. Denote by \(p : \mathcal{B} \to \mathcal{C}\) the linear map which maps \(m_{\alpha_1} \cdots m_{\alpha_l} \in \mathcal{B}\) to \(n_{\alpha_1} \cdots n_{\alpha_l} \in \mathcal{C}\),
for \(\alpha_i, i = 1, \ldots, l\) being monic monomials in \(\mathcal{A} \setminus \mathbb{Q}\). Denote by \(\mathcal{B}^0\) the image of \(p\). Clearly, \(\mathcal{A} \subset \mathcal{B}^0\). Indeed, the element \((\Lambda - 1)(\sum_{i=1}^{l} \lambda_i m_{\alpha_i}) \in \mathcal{B}\) becomes \(\sum_{i=1}^{l} \lambda_i \alpha_i \in \mathcal{A}\) under the map \(p\). Here \(\alpha_1, \ldots, \alpha_l\) are distinct monic monomials in \(\mathcal{A} \setminus \mathbb{Q}\). Finally we define an operator \(S : \mathcal{A} \setminus \mathbb{Q} \to \mathcal{B}^0\) by

\[
S(\lambda_1 \alpha_1 + \cdots + \lambda_l \alpha_l) = \lambda_1 n_{\alpha_1} + \cdots + \lambda_l n_{\alpha_l}
\]

for \(\alpha_1, \ldots, \alpha_l\) being distinct monic monomials and \(\lambda_1, \ldots, \lambda_l \in \mathbb{Q}\).

Motivated by (62) and (63), define two families of elements \(y_i, z_i \in \mathcal{A}, i \geq 1\) by

\[
y_{k+1} = - \sum_{\sum_{i=1}^{m} i_{m_i} \geq 0} \prod_{i=1}^{k} y_i^{inom{m_i}{2}} - v_0 \delta_{k,0} - w_0 \sum_{\sum_{i=1}^{m} i_{m_i+1} \geq 0} \prod_{i=1}^{k} (-1)^{m_i} (\Lambda^{-1}(y_i))^{m_i},
\]

\[
z_{k+1} = \sum_{\sum_{i=1}^{m} i_{m_i} \geq 0} \prod_{i=1}^{k} (-1)^{m_i} z_i^{inom{m_i}{2}} + v_1 \delta_{k,0} + w_2 \sum_{\sum_{i=1}^{m} i_{m_i+1} \geq 0} \prod_{i=1}^{k} (\Lambda(z_i))^{m_i}.
\]

Equivalently, the generating series \(y(\lambda) := \sum_{i \geq 1} y_i / \lambda^i\), \(z(\lambda) := \sum_{i \geq 1} z_i / \lambda^i\) satisfy

\[
\lambda e^{y(\lambda)} + v_0 - \lambda + w_0 \lambda^{-1} \Lambda^{-1}(e^{-y(\lambda)}) = 0,
\]

\[
\lambda \Lambda^{-1}(e^{-z(\lambda)}) + v_0 - \lambda + w_1 \lambda^{-1} e^{z(\lambda)} = 0.
\]

Define

\[
\psi_A := e^{S(y(\lambda))} \otimes \lambda^n \otimes 1, \quad \psi_B := e^{S(z(\lambda))} \otimes \lambda^{-n} \otimes e^{-\sigma},
\]

where \(e^{-\sigma}\) is a formal element satisfying \(e^{(1-\Lambda^{-1})(-\sigma)} = w_0\), and \(\lambda^n, \lambda^{-n}\) are formal elements satisfying \(\Lambda^k(1 \otimes \lambda^n) = \lambda^k \otimes \lambda^n, \Lambda^k(1 \otimes \lambda^{-n}) = \lambda^{-k} \otimes \lambda^{-n}, k \in \mathbb{Z}\). We have

\[
L(\psi_A) = \lambda \psi_A, \quad L(\psi_B) = \lambda \psi_B,
\]

\[
\psi_A(\lambda) = (1 + O(\lambda^{-1})) \otimes \lambda^n \in C[[\lambda^{-1}]] \otimes \lambda^n,
\]

\[
\psi_B(\lambda) = (1 + O(\lambda^{-1})) \otimes \lambda^{-n} \otimes e^{-\sigma} \in C[[\lambda^{-1}]] \otimes \lambda^{-n} \otimes e^{-\sigma},
\]

where \(L = \Lambda + v_0 + w_0 \Lambda^{-1}\). We call \(\psi_A\) and \(\psi_B\) the abstract pre-wave functions of type A and of type B, respectively, associated to \(v_0, w_0\).

Denote

\[
d_{\text{pre}}(\lambda) := \psi_A(\lambda) \Lambda^{-1}(\psi_B(\lambda)) - \psi_B(\lambda) \Lambda^{-1}(\psi_A(\lambda))
\]

and

\[
\Psi(\lambda) := \begin{pmatrix} \psi_A(\lambda) & \psi_B(\lambda) \\ \Lambda^{-1}(\psi_A(\lambda)) & \Lambda^{-1}(\psi_B(\lambda)) \end{pmatrix}.
\]

Then we have the following identity:

\[
R(\lambda) = \Psi(\lambda) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Psi^{-1}(\lambda) =: M(\lambda).
\]

The proof is similar to that of Proposition 3. The main fact used in the proof is that from the definition the coefficients of entries of \(R(\lambda)\) are uniquely determined in an
algebraic way.) We omit its details here. However, let us explain the equality \((131)\) by an equivalent form. From definition we have

\[
M(\lambda) = \frac{1}{d_{\text{pre}}(\lambda)} \begin{pmatrix}
\psi_A(\lambda) \Lambda^{-1}(\psi_B(\lambda)) & -\psi_A(\lambda) \psi_B(\lambda) \\
\Lambda^{-1}(\psi_A(\lambda)) & -\Lambda^{-1}(\psi_A(\lambda)) \psi_B(\lambda)
\end{pmatrix}.
\]

Then from a straightforward calculation by using the definitions we find

\[
M_{11}(\lambda) = \frac{1}{1 - \frac{w_0}{\lambda^2} e^{\Lambda^{-1}(z(\lambda)-y(\lambda))}}, \quad (132)
\]

\[
M_{12}(\lambda) = \frac{1}{\lambda e^{\Lambda^{-1}(y(\lambda))} - \frac{w_0}{\lambda} e^{\Lambda^{-1}(z(\lambda))}}, \quad (133)
\]

\[
M_{21}(\lambda) = \frac{1}{\lambda e^{\Lambda^{-1}(y(\lambda))} - \frac{w_0}{\lambda} e^{\Lambda^{-1}(z(\lambda))}}, \quad (134)
\]

\[
M_{22}(\lambda) = \frac{1}{1 - \frac{\lambda^2}{w_0} e^{\Lambda^{-1}(y(\lambda)-z(\lambda))}}. \quad (135)
\]

Hence the equality \((131)\) means new expressions for the entries of the basic matrix resolvent \(R(\lambda)\) explicitly in terms of \(y(\lambda), z(\lambda)\). Substituting the following expansions

\[
y(\lambda) = -\frac{v_0}{\lambda} - \frac{1}{\lambda^2} v_0^2 + w_0 + \cdots, \quad z(\lambda) = \frac{v_1}{\lambda} + \frac{1}{\lambda^2} v_1^2 + w_2 + \cdots \quad (136)
\]

into \((132) - (135)\) we find that the new expressions agree with \((24)\). Combining with \((56) - (57)\) we obtain

\[
\sum_{p \geq 0} \Omega_{p,0} \lambda^{-p-2} = A \quad (137)
\]

\[
\sum_{p \geq 0} \Lambda^{-1}(S_p) \lambda^{-p-2} = B. \quad (138)
\]

We therefore arrive at the following formulae:

\[
e^{\Lambda^{-1}(y(\lambda))} = \frac{1}{\lambda} \frac{1 + A}{B}, \quad e^{\Lambda^{-1}(z(\lambda))} = \frac{\lambda}{w_0} \frac{A}{B}. \quad (139)
\]

Let us proceed to the generating series of multi-point correlation functions. Define

\[
D_{\text{pre}}(\lambda, \mu) := \frac{\psi_A(\lambda) \Lambda^{-1}(\psi_B(\mu)) - \Lambda^{-1}(\psi_A(\lambda)) \psi_B(\mu)}{\lambda - \mu}. \quad (140)
\]

Using \((131)\), Proposition \(1\) and a similar argument to the proof of Theorem \(2\) we obtain

\[
\sum_{i_1, \ldots, i_k \geq 0} \frac{\Omega_{i_1, \ldots, i_k}}{\lambda_{i_1}^{i_1+2} \cdots \lambda_k^{i_k+2}} = \frac{(-1)^{k-1}}{\prod_{j=1}^k d_{\text{pre}}(\lambda_j)} \sum_{\pi \in S_k / C_k} \prod_{j=1}^k D_{\text{pre}}(\lambda_{\pi(j)}, \lambda_{\pi(j+1)}) - \frac{\delta_{k,2}}{(\lambda_1 - \lambda_2)^2}. \quad (141)
\]
For the reader’s convenience, we give the first few terms of the abstract pre-wave functions \( \psi_A(\lambda) \) and \( \psi_B(\lambda) \) as follows:

\[
\psi_A = \left( 1 - \frac{m_{v_0}}{\lambda} + \frac{m_{v_0}^2 - m_{v_0}^2 - 2m_{w_0}}{2\lambda^2} \right. \\
- \frac{1}{6\lambda^3} \left( m_{v_0}^3 + 2m_{v_0}^3 - 3m_{v_0}m_{v_0}^2 + 6m_{v_0}w_0 + 6m_{v_0}w_1 \\
- 6m_{v_0}m_{w_0} - 6v_{-1}w_0 \right) + O\left( \frac{1}{\lambda^4} \right) \lambda^n,
\]

\[
\psi_B = \left( 1 + \frac{m_{v_0} + v_0}{\lambda} + \frac{m_{v_0}^2 + m_{v_0}^2 + 2v_0m_{v_0} + 2m_{w_0} + 2v_0^2 + 2w_0 + 2w_1}{2\lambda^2} \right. \\
+ \frac{1}{6\lambda^3} \left( m_{v_0}^3 + 6m_{v_0}m_{w_0} + 3m_{v_0}m_{v_0}^2 + 2m_{v_0}^3 + 6m_{v_0}w_0 + 6m_{v_0}w_0 \\
+ 3v_0m_{v_0}^2 + 6v_0^2m_{v_0} + 6w_0m_{v_0} + 6w_1m_{v_0} + 3v_0m_{v_0}^2 + 6v_0m_{w_0} \\
+ 6v_0^3 + 12v_0w_0 + 12v_0w_1 + 6v_1w_1 \right) + O\left( \frac{1}{\lambda^4} \right) \lambda^{-n} e^{-\sigma}.
\]

It turns out that the above abstract pre-wave functions form a pair. Namely, \( d_{\text{pre}}(\lambda) = \lambda e^{\lambda^{-1}(-\sigma)} \). Interestingly, for given arbitrary initial value \((f(n), g(n))\), based on this statement one obtains a constructive method for a pair of wave functions associated to \((f(n), g(n))\) (cf. (28) in Section 1.3 for the definition of a pair). This is important considering Theorem 1. We hope to confirm the statement on the pair property of the abstract pre-wave functions in another publication.

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