

A NOTE ON COMMUTATIVE AND NONCOMMUTATIVE GURARIY SPACES

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Dedicated to the memory of Professor Sudipta Dutta!

Abstract. In this short note, we answer two questions about Gurariy spaces asked in the literature in the affirmative. We also prove the analogue of one of the results for the noncommutative Gurariy space.

1. Introduction

In this short note, we show that if a Banach space $X$ is almost isometric to a Gurariy space, then $X$ is also a Gurariy space. This answers a question posed in [16] in the affirmative. We prove an analogue of this result for the noncommutative Gurariy space (Definition 3.1) discussed in [12], as well as for $\mathcal{E}$-Gurariy property (Definition 3.3), and $\mathcal{E}$-injective property (Definition 3.5) discussed in [13].

We also show that a Banach space $X$ is a Gurariy space if and only if it is an almost isometric ideal in every superspace in which it embeds as a hyperplane, thereby answering a question posed in [15] in the affirmative.

We end this note with some more observations about Gurariy spaces.

2. Almost isometry

Definition 2.1. Let $X$ and $Y$ be Banach spaces.

(a) Let $\varepsilon > 0$. A linear operator $T : X \to Y$ is an $\varepsilon$-isometry if

\[(1 + \varepsilon)^{-1}\|x\| \leq \|T(x)\| \leq (1 + \varepsilon)\|x\| \text{ for all } x \in X.\]

(b) We say that $X$ and $Y$ are almost isometric if for every $\varepsilon > 0$, there is an onto $\varepsilon$-isometry $T : X \to Y$.

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(c) [6] A Banach space $X$ is called a *Gurariy space* if for every $\varepsilon > 0$, and every pair of finite dimensional spaces $E \subseteq F$, every isometry $T : E \to X$ extends to an $\varepsilon$-isometry $\tilde{T} : F \to X$.

In [6], Gurariy proved the existence of a separable Gurariy space. We refer to [3, 5] and [7, Chapter 4] for more information on Gurariy spaces and to [4, 9] for some recent results. For example, the separable Gurariy space is unique up to isometry. Throughout this note, this space will be denoted by $G$.

Here is our first result. This is related to [11, Problem 9.2].

**Theorem 2.2.** If a Banach space $X$ is almost isometric to a Gurariy space $G$, then $X$ is also a Gurariy space.

**Proof.** Let $\varepsilon > 0$. Choose $\delta > 0$ such that $(1 + \delta)^3 \leq 1 + \varepsilon$. Since $X$ is almost isometric to $G$, there exists an onto $\delta$-isometry $T_1 : X \to G$.

Let $E \subseteq F$ be finite dimensional spaces and let $T : E \to X$ be an isometry. Then $S = T_1 \circ T : E \to G$ is a $\delta$-isometry.

Define a new norm on $E$ by $\|e\|_1 = \|S(e)\|$. Clearly, $\| \cdot \|_1$ is $\delta$-equivalent to $\| \cdot \|$ on $E$. By [5, Lemma 1.1], we can extend this norm to $F$, still denoted by $\| \cdot \|_1$, so that it is $\delta$-equivalent to $\| \cdot \|$ on $F$.

Now, $S : (E, \| \cdot \|_1) \to G$ is an isometry. Since $G$ is a Gurariy space, $S$ can be extended to a $\delta$-isometry $\tilde{S} : (F, \| \cdot \|_1) \to G$, that is,

$$(1 + \delta)^{-1} \|z\|_1 \leq \|\tilde{S}(z)\| \leq (1 + \delta)\|z\|_1 \quad \text{for all } z \in F.$$ 

Now coming back to $\| \cdot \|$ we see that

$$(1 + \delta)^{-2} \|z\| \leq \|\tilde{S}(z)\| \leq (1 + \delta)^2 \|z\| \quad \text{for all } z \in F.$$ 

Now consider the map $\tilde{T} = T_1^{-1} \circ \tilde{S} : F \to X$. Clearly, $\tilde{T}$ extends $T$ and

$$(1 + \varepsilon)^{-1} \|z\| \leq (1 + \delta)^{-3} \|z\| \leq \|\tilde{T}(z)\| \leq (1 + \delta)^3 \|z\| \leq (1 + \varepsilon)\|z\|$$

for all $z \in F$. Hence $X$ is a Gurariy space. \hfill $\Box$

Since the separable Gurariy space is unique up to isometry, this answers the question posed in [116, Remark 6] in the affirmative.

**Corollary 2.3.** If a Banach space $X$ is almost isometric to $G$, then it is isometric to $G$. 
Combining [4, Theorem 2.6] and Corollary 2.3 we get an improvement of [4, Theorem 2.6].

**Theorem 2.4.** A (non-separable) Banach space $X$ is a Gurariy space if and only if every separable a.i.-ideal in $X$ is almost isometric to $G$.

### 3. The Noncommutative Analogue

Noncommutative Gurariy space was introduced by Oikhberg [12] within the framework of operator spaces. Recently Pisier [13] has done an extensive study on the Gurariy property in noncommutative set up.

**Definition 3.1.**

(a) If $V$ and $W$ are operator spaces, we define the completely bounded (cb) Banach-Mazur distance $d_{cb}(V, W)$ by

$$d_{cb}(V, W) = \inf \{ \| \phi \| \| \phi^{-1} \| : \phi : V \to W \text{ is a cb linear isomorphism} \}.$$  

(b) An operator space $X$ is called $c$-exact if, for every finite-dimensional subspace $E$ of $X$, and for every $\varepsilon > 0$ there exists a subspace $F$ of $M_n(\mathbb{C})$ satisfying $d_{cb}(E, F) < c + \varepsilon$.

(c) [12] A separable 1-exact space $X$ is a non-commutative Gurariy space if it satisfies the following condition:

- suppose a finite dimensional operator space $F$ is 1-exact, $E \subseteq F$, $E' \subseteq X$ and $u : E \to E'$ is a cb-isomorphism.
- Then for any $\delta > 0$ there exists a subspace $F' \subseteq X$ containing $E'$ and a linear map $\tilde{u} : F \to F'$ such that $\tilde{u}|_{E} = u$ and $\| \tilde{u} \|_{cb} \| \tilde{u}^{-1} \|_{cb} < (1 + \delta) \| u \|_{cb} \| u^{-1} \|_{cb}$.

In [10], the authors prove the main properties of the noncommutative Gurariy space. For example, the noncommutative Gurariy space is unique up to completely isometric isomorphism, and this space is universal for separable 1-exact operator space. Throughout this note, this space will be denoted by NG. NG can also be thought as the operator space analogue of the Cuntz algebra $O_2$. NG is the first example of a separable 1-exact operator space that is not an $M_n$-space for any $n \in \mathbb{N}$.

Now we prove the noncommutative analogue of Corollary 2.3. This is also a contribution to [11, Problem 9.2]. We believe that this result also is of independent interest and will prove useful in future investigations.

**Theorem 3.2.** If an operator space $X$ satisfies $d_{cb}(X, \text{NG}) = 1$, then $X$ is completely isometrically isomorphic to NG.
Proof. Let $\varepsilon > 0$. Choose $\eta > 0$ such that $(1 + \eta)^2 < 1 + \varepsilon$. Let $E \subseteq X$ be a finite-dimensional subspace. Since $d_{cb}(X, \mathbb{N}G) = 1$, there exists a finite dimensional subspace $E' \subseteq \mathbb{N}G$ such that $d_{cb}(E, E') \leq 1 + \eta$. As $\mathbb{N}G$ is 1-exact, there exists $F \subseteq M_n(\mathbb{C})$ such that $d_{cb}(E', F) < 1 + \eta$. Therefore $d_{cb}(E, F) \leq d_{cb}(E, E')d_{cb}(E', F) < (1 + \eta)^2 < 1 + \varepsilon$. So, $X$ is 1-exact.

Choose $\delta > 0$ such that $(1 + \delta)^5 < 1 + \varepsilon$. Since $d_{cb}(X, \mathbb{N}G) = 1$, there exists an onto $\delta$-cb isomorphism $v : X \to \mathbb{N}G$, that is, $v_n : M_n(X) \to M_n(\mathbb{N}G)$ is a $\delta$-isomorphism for all $n \geq 1$ and $\|v\| = \sup_{n \geq 1} \|v_n\| < \infty$. Let $F$ be 1-exact finite dimensional operator space. Consider $E \subseteq F$. Let $u : E \to E' \subseteq X$ is a cb isomorphism from $\mathbb{N}G$ is non-commutative Gurariy space, there exists $\tilde{w} : F \to \mathbb{N}G$ linear map such that $\tilde{w}|_E = w$, and $\|\tilde{w}|_E\|\|\tilde{w}^{-1}\|_{cb} < (1 + \delta)\|w\|_{cb}\|w^{-1}\|_{cb}$. Define $\tilde{u} = v^{-1} \circ \tilde{w}$. Now $\tilde{u}|_E = v^{-1} \circ \tilde{w}|_E = v^{-1} \circ w = v^{-1} \circ v \circ u = u$. Here $\tilde{u}$ is a cb-isomorphism from $F$ to $\tilde{u}(F)$. Now,

$$\|\tilde{u}\|_{cb}\|\tilde{u}^{-1}\|_{cb} = \|v^{-1} \circ \tilde{w}\|_{cb}\|\tilde{w}^{-1} \circ v\|_{cb} \leq \|v\|_{cb}\|v^{-1}\|_{cb}\|\tilde{w}\|_{cb}\|\tilde{w}^{-1}\|_{cb} \leq (1 + \delta)^2\|\tilde{w}\|_{cb}\|\tilde{w}^{-1}\|_{cb}.$$ 

Since $\mathbb{N}G$ is non-commutative Gurariy space,

$$(1 + \delta)^2\|\tilde{w}\|_{cb}\|\tilde{w}^{-1}\|_{cb} < (1 + \delta)^3\|w\|_{cb}\|w^{-1}\|_{cb}.$$ 

So,

$$(1 + \delta)^3\|w\|_{cb}\|w^{-1}\|_{cb} = (1 + \delta)^3\|v \circ u\|_{cb}\|u^{-1} \circ v^{-1}\|_{cb} \leq (1 + \delta)^3\|v\|_{cb}\|v^{-1}\|_{cb}\|u\|_{cb}\|u^{-1}\|_{cb} \leq (1 + \delta)^5\|u\|_{cb}\|u^{-1}\|_{cb} < (1 + \varepsilon)\|u\|_{cb}\|u^{-1}\|_{cb}.$$ 

Since non-commutative Gurariy space is unique upto complete isomtery [10, Theorem 1.1], $X$ is completely isometrically isomorphic to $\mathbb{N}G$.

This completes the proof. \hfill \Box

**Definition 3.3.** [13] Definition 4.4] Let $E$ be a class of finite dimensional operator spaces. An operator space $X$ has the $E$-Gurariy property if for any $\varepsilon > 0$ and for any pair of spaces $E \subseteq F$ with $F \in E$ the following holds:

for any injective linear map $u : E \to X$ there exists an injective extension $\tilde{u} : F \to X$ such that $\|\tilde{u}\|_{cb}\|\tilde{u}^{-1}\|_{cb} \leq (1 + \varepsilon)\|u\|_{cb}\|u^{-1}\|_{cb}$. 


Using the same argument of Theorem 3.2 we get the following

**Corollary 3.4.** Let $X$ be an operator space has the $\mathcal{E}$-Gurariy property. If an operator space $Y$ satisfies $d_{cb}(Y, X) = 1$, then $Y$ also has the $\mathcal{E}$-Gurariy property.

**Definition 3.5.** [13, Definition 4.5] Let $\mathcal{E}$ be a class of finite dimensional operator spaces. An operator space $X$ is called $\mathcal{E}$-injective if for any $\varepsilon > 0$, any $E \in \mathcal{E}$ and any $S \subseteq E$, any map $u : S \to X$ admits an extension $\tilde{u} : E \to X$ with $\|\tilde{u}\|_{cb} \leq (1 + \varepsilon)\|u\|_{cb}$.

**Proposition 3.6.** Let $X$ be an $\mathcal{E}$-injective operator space. If an operator space $Y$ satisfies $d_{cb}(Y, X) = 1$, then $Y$ is also $\mathcal{E}$-injective.

**Proof.** Let $\varepsilon > 0$. Choose $\delta > 0$ such that $(1 + \delta)^3 < 1 + \varepsilon$.

Since $d_{cb}(Y, X) = 1$, there exists an onto $\delta$-cb isomorphism $v : Y \to X$, that is, $v_n : M_n(Y) \to M_n(X)$ is a $\delta$-isomorphism for all $n \geq 1$ and $\|v\| = \sup_{n \geq 1} \|v_n\| < \infty$. Let $E \in \mathcal{E}$ and $S \subseteq E$. Let $u : S \to Y$ be a cb linear map. Consider the cb linear map $w = v \circ u : S \to X$. Since $X$ is $\mathcal{E}$-injective, there exists an extension $\tilde{w} : E \to X$ with $\|\tilde{w}\|_{cb} \leq (1 + \delta)\|w\|_{cb}$.

Define $\tilde{u} = v^{-1} \circ \tilde{w} : E \to Y$ be a cb linear map. Now $\tilde{u}|_S = v^{-1} \circ \tilde{w}|_S = v^{-1} \circ v \circ u = u$. And

$$\|\tilde{u}\|_{cb} = \|v^{-1} \circ \tilde{w}\|_{cb} \leq \|v^{-1}\|_{cb}\|\tilde{w}\|_{cb}.$$ 

Since $X$ is $\mathcal{E}$-injective space,

$$\|v^{-1}\|_{cb}\|\tilde{w}\|_{cb} \leq (1 + \delta)\|v^{-1}\|_{cb}\|w\|_{cb} = (1 + \delta)\|v^{-1}\|_{cb}\|v \circ u\|_{cb}$$

$$\leq (1 + \delta)\|v\|_{cb}\|v^{-1}\|_{cb}\|u\|_{cb} \leq (1 + \delta)^3\|u\|_{cb} < (1 + \varepsilon)\|u\|_{cb}.$$ 

Hence, $Y$ is also $\mathcal{E}$-injective.

This completes the proof.

\[\square\]

### 4. Almost isometric ideals

Now we return to Banach spaces again.

**Definition 4.1.** [2] A (closed linear) subspace $Y$ of a Banach space $X$ is an almost isometric ideal (a.i.-ideal) in $X$ if for every $\varepsilon > 0$ and every finite dimensional subspace $E \subseteq X$ there exists $T : E \to Y$ such that
(i) $Te = e$ for all $e \in Y \cap E$, and
(ii) $T$ is an $\varepsilon$-isometry.

It is known \[2, \text{Theorem 4.3}\] that if $X$ is a Gurariy space and $Y \subseteq X$ is an a.i.-ideal in $X$, then $Y$ is a Gurariy space. Indeed, $X$ is a Gurariy space if and only if it is an a.i.-ideal in every superspace containing it. Here we prove:

**Proposition 4.2.** Let $X$ be a Gurariy space and $Y \subseteq X$ be an infinite dimensional subspace such that for every $x \in X \setminus Y$, $Y$ is an a.i.-ideal in span$\{x, Y\}$. Then $Y$ is a Gurariy space.

**Proof.** We first observe that to prove $Y$ is a Gurariy space, by \[7, \text{Lemma 4.3.1}\], it is enough to show that for every $\varepsilon > 0$, and every pair of finite dimensional spaces $E \subseteq F$ with $\dim(F/E) = 1$, every isometry $T : E \to Y$ extends to an $\varepsilon$-isometry $\tilde{T} : F \to Y$.

Let $\varepsilon > 0$. Choose $\delta > 0$ such that $(1 + \delta)^2 \leq 1 + \varepsilon$.

Let $E \subseteq F$ be finite dimensional subspaces with $\dim(F/E) = 1$, and $T : E \to Y \subseteq X$ be a linear isometry. Since $X$ is a Gurariy space, there exists a linear extension $\tilde{T} : F \to X$ of $T$ with

$$(1 + \delta)^{-1}\|f\| \leq \|\tilde{T}f\| \leq (1 + \delta)\|f\|$$

for all $f \in F$.

Since $\dim(F/E) = 1$, there exists $f_0 \in F$ such that $F = \text{span}\{f_0, E\}$. Let $x_0 = \tilde{T}(f_0)$ and $H = \tilde{T}(F)$. Then $H \subseteq \text{span}\{x_0, Y\}$ is finite dimensional.

Since $Y$ is an a.i.-ideal in span$\{x_0, Y\}$, there exists $S : H \to Y$ such that

(a) $Sh = h$ for all $h \in H \cap Y$, and

(b) $(1 + \delta)^{-1}\|h\| \leq \|Sh\| \leq (1 + \delta)\|h\|$ for all $h \in H$.

It follows that $S \circ \tilde{T} : F \to Y$ is a linear extension of $T$ satisfying

$$(1 + \varepsilon)^{-1}\|f\| \leq \|S \circ \tilde{T}f\| \leq (1 + \varepsilon)\|f\|$$

for all $f \in F$.

Therefore $Y$ is a Gurariy space. \hfill $\square$

This answers the question posed in \[15, \text{Question 17}\] in the affirmative. If we replace a.i.-ideals by ideals in the above, this characterises $L_1$-predual spaces and was observed in \[14\].

**Corollary 4.3.** A Banach space $X$ is a Gurariy space if and only if it is an a.i.-ideal in every superspace in which it embeds as a hyperplane.
Proof. It is known that $X$ embeds isometrically into a Gurariy space $G$ of the same density [5]. By hypothesis, every $g \in G \setminus X$, $X$ is an a.i.-ideal in $\text{span}\{g, X\}$. By Proposition 4.2, $X$ is a Gurariy space. 

5. MISCELLANEOUS RESULTS

We end this note with few more properties of Gurariy spaces, which are of independent interest.

5.1. $M$-summands. The next observation is closely related to the work of [4].

**Remark 5.1.** Let $M$ be an $M$-summand of the Gurariy space $G$, that is, $G = M \oplus \infty N$, where $M$ and $N$ are both infinite dimensional. By [15, Theorem 3] we get $M \cong G$ and $N \cong G$. But we know that $G \oplus \infty G$ is not a Gurariy space (see e.g., [15, Remark 9]). Hence, there does not exist any $M$-summand in the Gurariy space $G$.

**Remark 5.2.** Every separable $L_1$-predual space is isometric to a 1-complemented subspace of $G$ [17]. Hence quotient of the Gurariy space $G$ need not be the Gurariy space $G$.

**Remark 5.3.** We know that $G \subset G \oplus \infty G \subset G$, where $G$ is the separable Gurariy space and $G$ is a non-separable Gurariy space. Here $G$ is an a.i-ideal in $G$ but $G \oplus \infty G$ is not an a.i-ideal in $G$ [15, Remark 9] [2, Theorem 4.3].

5.2. $M$-ideals.

**Definition 5.4.** A subspace $M$ of a Banach space $X$ is said to be an $M$-ideal in $X$ if there is a projection $P$ on $X^*$ with $\text{ker}(P) = M^\perp$ and for all $x^* \in X^*$, $\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$.

It is known [15, Theorem 3] that any $M$-ideal in $G$ is isometric to $G$, and hence, is an a.i.-ideal.

**Proposition 5.5.** Let $M$ be a separable $M$-ideal of a non-separable Gurariy space $G$. Then $M$ is the Gurariy space $G$ and hence an a.i.-ideal in $G$.

**Proof.** Let $M$ be a separable $M$-ideal of a non-separable Gurariy space $G$. So there exists a separable a.i-ideal $Z$ such that $M \subseteq Z \subseteq G$ [1]. Using [3, Theorem 2.6] we get $Z$ is isometric to $G$. By [8, Proposition 1.17] $M$
is an $M$-ideal in $Z$. Now using [15, Theorem 3] we conclude that $M$ is the Gurariy space $G$, hence it is an a.i-ideal in $G$.

This completes the proof. □

**Problem 5.6.** Let $M$ be a non-separable $M$-ideal of a non-separable Gurariy space $G$. Is $M$ a Gurariy space?

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