On the Reversible Geodesics for a Finsler space with Randers change of Quartic metric

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Abstract. In this paper, we consider a Finsler space with a Randers change of Quartic metric $F = \sqrt[4]{\alpha^4 + \beta^4}$. The conditions for this space to be with reversible geodesics are obtained. Further, we study some geometrical properties of $F$ with reversible geodesics and prove that the Finsler metric $F$ induces a generalized weighted quasi-distance $d_F$ on $M$.

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1 Introduction

An interesting topic in Finsler geometry is to study the reversible geodesics of a Finsler metric. Recall that, a Finsler space is said to have reversible geodesics if for every one of its oriented geodesic paths, the same path traversed in the opposite sense is also a geodesic. In the last decade many interesting and applicable results have been obtained on the theory of Finsler spaces with reversible geodesics. In [1], Crampin discussed Randers space with reversible geodesics. In ( [4], [5]), Masca, Sabau and Shimada have discussed reversible geodesics with $(\alpha, \beta)$-metric and two dimensional Finsler space with $(\alpha, \beta)$-metric to be with reversible geodesic, respectively. In [6], Sabau and Shimada have given some important results on reversible geodesics. In [10], Shanker and Baby have discussed reversible geodesics for generalized $(\alpha, \beta)$-metric. Recently, Shanker and Rani [11] have studied weighted quasi metric associated with Finsler spaces with reversible geodesics.
In this paper, we find conditions for a Finsler space \((M, F)\) with Randers change of Quartic metric \(F = \sqrt[4]{a^4 + b^4 + \beta}\) to be with reversible geodesics. The main results of this paper lies in theorem (3.1), (4.1), (4.2), (5.1) and (5.2).

2 Preliminaries

Let \(F^n = (M, F)\) be a connected \(n\)-dimensional Finsler manifold and let \(TM = \bigcup_{x \in M} T_x M\) denotes the tangent bundle of \(M\) with local coordinates \(u = (x, y) = (x^i, y^i) \in T_x M\), where \(i = 1, \ldots, n\), \(y^i = \frac{\partial}{\partial x^i}\).

If \(\gamma : [0, 1] \rightarrow M\) is a piecewise \(C^\infty\) curve on \(M\), then its Finslerian length is defined as

\[
L_F(\gamma) = \int_0^1 F(\gamma(t), \dot{\gamma}(t)) dt,
\]

and the Finslerian distance function \(d_F : M \times M \rightarrow [0, \infty)\) is defined by \(d_F(p, q) = \inf_{\gamma} L\), where infimum is taken over all piecewise \(C^\infty\) curves \(\gamma\) on \(M\) joining the points \(p, q \in M\). In general, this is not symmetric.

A curve \(\gamma : [0, 1] \rightarrow M\) is called a geodesic of \((M, F)\) if it minimizes the Finslerian length for all piecewise \(C^\infty\) curves that keep their endpoints fixed. We denote the reverse Finsler metric of \(F\) as \(\tilde{F} : TM \rightarrow (0, \infty)\), given by \(\tilde{F}(x, y) = F(x, -y)\). One can easily see that \(\tilde{F}\) is also a Finsler metric.

**Lemma 2.1** A Finsler metric is with a reversible geodesic if and only if for any geodesic \(\gamma(t)\) of \(F\), the reverse curve \(\tilde{\gamma}(t) = \gamma(1-t)\) is also a geodesic of \(F\).

**Lemma 2.2** Let \((M, F)\) be a connected, complete Finsler manifold with associated distance function \(d_F : M \times M \rightarrow [0, \infty)\). Then, \(d_F\) is a symmetric distance function on \(M \times M\) if and only if \(F\) is a reversible Finsler metric, i.e., \(F(x, y) = F(x, -y)\).

**Lemma 2.3** A smooth curve \(\gamma : [0, 1] \rightarrow M\) is a constant Finslerian speed geodesic of \((M, F)\) if and only if it satisfies \(\ddot{\gamma} + 2 G^i(\gamma(t), \dot{\gamma}(t)) = 0\), \(i = 1, \ldots, n\), where the functions \(G^i : TM \rightarrow \mathbb{R}\) are given by

\[
G^i(x, y) = \Gamma^i_{jk}(x, y)y^jy^k,
\]

with \(\Gamma^i_{jk}(x, y) = \frac{g^{is}}{2} \left( \frac{\partial g_{sj}}{\partial x^k} + \frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s} \right)\).
Remark 1. It is well known [7] that the vector field $\Gamma = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$.
is a vector field on $TM$, whose integral lines are the canonical lifts $\tilde{\gamma}(t) = (\gamma(t), \dot{\gamma}(t))$ of the geodesics of $\gamma$. This vector field $\Gamma$ is called the canonical geodesics spray of the Finsler space $(M, F)$ and $G^i$ are called the coefficients of the geodesics spray $\Gamma$.

Definition 2.1 If $F$ and $\tilde{F}$ are two different fundamental Finsler functions on the same manifold $M$, then they are said to be projectively equivalent if their geodesics coincide as set points.

Lemma 2.4 A Finsler structure $(M, F)$ is with a reversible geodesic if and only if $F$ and its reverse function $\tilde{F}$ are projectively equivalent.

3 Reversible Geodesics for a Finsler space with Randers change of Quartic metric.

Consider a Finsler space $(M, F)$ with a special $(\alpha, \beta)$-metric $F = \sqrt[4]{\alpha^4 + \beta^4}$. Here, $F$ can be treated as the Randers change of Quartic-metric $\tilde{F} = \sqrt[4]{\alpha^4 + \beta^4}$. One can easily see that $\tilde{F}(x, -y) = \tilde{F}(x, y)$.

As we know that [6] if $(M, F)$ is a non-Riemannian $n(n \geq 2)$-dimensional Finsler space with $(\alpha, \beta)$-metric, which is not absolute homogeneous, then $F$ is with reversible geodesics if and only if $F(\alpha, \beta) = F_0(\alpha, \beta) + \epsilon \beta$, where $F_0$ is absolute homogeneous $(\alpha, \beta)$-metric, $\epsilon$ is a non-zero constant and $\beta$ is a closed 1-form on the Manifold $M$.

In our case, $F_0 = \tilde{F}$, which is absolute homogeneous. If $\beta$ is a closed 1-form, then $F$ is with reversible geodesics. Further, a necessary and sufficient condition for $F$ to have reversible geodesics is that [6]

$$\tilde{\Gamma}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = 0, \quad (3.1)$$

where $\tilde{\Gamma}$ is the reverse of $\Gamma$, the geodesic spray of $F$; moreover $\tilde{\Gamma}$ is geodesic spray of $\tilde{F}$. We have $F = \tilde{F} + \beta$, where, $\tilde{F} = \sqrt[4]{\alpha^4 + \beta^4}$.
Therefore,

\[
\hat{\Gamma}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = \hat{\Gamma}\left(F_\alpha \frac{\partial \alpha}{\partial y^i} + F_\beta \frac{\partial \beta}{\partial y^i}\right) - F_\alpha \frac{\partial \alpha}{\partial x^i} - F_\beta \frac{\partial \beta}{\partial x^i} = \hat{\Gamma}(F_\alpha) \frac{\partial \alpha}{\partial y^i} + F_\alpha \left[\hat{\Gamma}\left(\frac{\partial \beta}{\partial y^i}\right) - \frac{\partial \beta}{\partial x^i}\right] + \hat{\Gamma}(F_\beta) \frac{\partial \beta}{\partial y^i} + F_\beta \left[\hat{\Gamma}\left(\frac{\partial \beta}{\partial y^i}\right) - \frac{\partial \beta}{\partial x^i}\right].
\]

(3.2)

For the Riemannian metric \(\alpha\), the Euler-Lagrange equation gives \(\hat{\Gamma}\left(\frac{\partial \alpha}{\partial y^i}\right) = 0\). Also, one knows [6] that if \((M, F(\alpha, \beta))\) is a Finsler space with \((\alpha, \beta)\)-metric, then \(f(x, y)\frac{\partial \alpha}{\partial y^i} + g(x, y)b_i = 0\), \(\forall i = 1, 2, \ldots, n\), implies that \(f = g = 0\), for any smooth functions \(f\) and \(g\) on \(TM\). It is known that, if \(\beta\) is closed and \(F\) is projectively equivalent to the Riemannian metric \(\alpha\), then

\[
\frac{\partial \alpha}{\partial x^i} = 0
\]

and hence by using lemma 2.7 of [6], we find that \(\hat{\Gamma}(F_\alpha) = \hat{\Gamma}(F_\beta)b_i = 0\).

Again, since \(F = 4\sqrt{\alpha^4 + \beta^4} + \beta\), therefore \(F_\beta = 1 + \frac{\beta^3}{(\alpha^4 + \beta^4)^{\frac{3}{4}}}\).

Now, using the above results, the equation (3.2) reduces to the form

\[
\hat{\Gamma}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = F_\beta \left[\hat{\Gamma}\left(\frac{\partial \beta}{\partial y^i}\right) - \frac{\partial \beta}{\partial x^i}\right] = \left(1 + \frac{\beta^3}{(\alpha^4 + \beta^4)^{\frac{3}{4}}}\right) \left[\hat{\Gamma}(b_i) - \frac{\partial b_j}{\partial x^i} y^j\right] = \left(1 + \frac{\beta^3}{(\alpha^4 + \beta^4)^{\frac{3}{4}}}\right) \left[\frac{\partial b_i}{\partial y^i} y^j - \frac{\partial b_j}{\partial x^i} y^j\right] = \left(1 + \frac{\beta^3}{(\alpha^4 + \beta^4)^{\frac{3}{4}}}\right) \left[\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}\right] y^j.
\]

(3.3)

Now, \(1 + \frac{\beta^3}{(\alpha^4 + \beta^4)^{\frac{3}{4}}}\) can not be zero. Therefore, from equation (3.1) and (3.3) we conclude that \(F\) is with reversible geodesics if and only if \(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}\) \(y^j = 0\).
i.e., \( \tilde{F} \) is with reversible geodesic if and only if \( \beta \) is closed 1-form. Hence, we have the following theorem:

**Theorem 3.1** A Finsler space \((M, F)\) with Randers change of Quartic metric \( F = \tilde{F} + \beta \), where, \( F = \sqrt[4]{\alpha^4 + \beta^4} \), is with reversible geodesics if and only if the differential 1-form \( \beta \) is closed on \( M \).

### 4 Projective Flatness of Randers change of Quartic metric

A Finsler space \((M, F)\) is called (locally) projectively flat if all its geodesics are straight lines \([8]\). An equivalent condition is that the spray coefficients \( G^i \) of \( F \) can be expressed as \( G^i = P(x, y)y^i \), where \( P(x, y) = \frac{1}{2F} \frac{\partial F}{\partial x^k} y^k \).

An equivalent characterization of projective flatness is the Hamel’s relation \([2]\)

\[
\frac{\partial^2 F}{\partial x^m \partial y^k} y^m - \frac{\partial F}{\partial x^k} = 0.
\]

Recall that \([10], [11]\) if \( F = F_0 + \epsilon \beta \) is a Finsler metric, where \( F_0 \) is an absolute homogeneous \((\alpha, \beta)\)-metric, then any two of the following properties imply the third one:

1. \( F \) is projectively flat;
2. \( F_0 \) is projectively flat;
3. \( \beta \) is closed.

In our case, \( F = \tilde{F} + \beta \), where \( \tilde{F} = \sqrt[4]{\alpha^4 + \beta^4} \), which is absolute homogeneous. Hence, we have the following:

**Theorem 4.1** Let \((M, F)\) be a Finsler space with Randers change of Quartic metric \( F = \sqrt[4]{\alpha^4 + \beta^4} + \beta \). Then, \( F \) is projectively flat if and only if \( \tilde{F} \) is projectively flat.

**Proof:** Let \((M, F)\) be projectively flat, then by Hamel’s relation for projective flatness, we have

\[
\frac{\partial^2 F}{\partial x^m \partial y^k} y^m - \frac{\partial F}{\partial x^k} = 0.
\]

The proof of the theorem directly follows from it.
**Theorem 4.2** Let \((M, F)\) be a Finsler space with Randers change of Quartic metric. If \(F\) is projectively flat, then it is with reversible geodesics.

**Proof.** Applying Hammel's equation, one can easily see that \(F\) is projectively flat if and only if \(\tilde{F}\) is projectively flat, which implies that both \(F\) and \(\tilde{F}\) are projectively equivalent to the standard Euclidean metric and therefore \(F\) must be projective to \(\tilde{F}\). Thus, \(F\) must be with a reversible geodesic.

## 5 Weighted quasi metric associated with Randers change of Quartic metric

It is well known that the Riemannian spaces can be represented as metric spaces. Indeed, for a Riemannian space \((M, \alpha)\), one can define the induced metric space \((M, d_\alpha)\) with the metric

\[
d_\alpha : M \times M \longrightarrow [0, \infty), d_\alpha(x, y) = \inf_{\gamma \in \Gamma_{xy}} \int_a^b \alpha(\gamma(t), \dot{\gamma}(t))dt,
\]

(5.1)

where \(\Gamma_{xy} = \{\gamma : [a, b] \longrightarrow M | \gamma\) is piecewise, \(\gamma(a) = x, \gamma(b) = y\}\) is the set of curves joining \(x\) and \(y\), \(\dot{\gamma}(t)\) is the tangent vector to \(\gamma\) at \(\gamma(t)\). Then \(d_\alpha\) is a metric on \(M\) satisfying the following conditions:

1. Positiveness : \(d_\alpha(x, y) > 0\), if \(x \neq y\), \(d_\alpha(x, x) = 0\), \(x, y \in X\).

2. Symmetry : \(d_\alpha(x, y) = d_\alpha(y, x)\), \(\forall x, y \in M\).

3. Triangle inequality: \(d_\alpha(x, y) \leq d_\alpha(x, z) + d_\alpha(z, y)\), \(\forall x, y, z \in M\).

Similar to the Riemannian space, one can induce the metric \(d_F\) to a Finsler space \((M, F)\), given by

\[
d_F : M \times M \longrightarrow [0, \infty), d_F(x, y) = \inf_{\gamma \in \Gamma_{xy}} \int_a^b F(\gamma(t), \dot{\gamma}(t))dt.
\]

(5.2)

But unlike the Riemannian case, here \(d_F\) lacks the symmetric condition. In fact, \(d_F\) is a special case of quasi metric defined below:

**Definition 5.1** A quasi metric \(d\) on a set \(X\) is a function \(d : X \times X \longrightarrow [0, \infty)\) that satisfies the following axioms:

1. Positiveness : \(d(x, y) > 0\), if \(x \neq y\), \(d(x, x) = 0\), \(x, y \in X\).
2. **Triangle inequality**: \( d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X. \)

3. **Separation axiom**: \( d(x, y) = d(y, x) = 0 \Rightarrow x = y, \forall x, y \in X. \)

One special class of quasi metric spaces are the so called weighted quasi metric spaces \((M, d, w)\), where \(d\) is a quasi-metric on \(M\) and for each \(d\), there exist a function \(w : M \rightarrow [0, \infty)\), called the *weight* of \(d\) that satisfies

4. **Weightability**: \( d(x, y) + w(x) = d(y, x) + w(y), \forall x, y \in M. \)

In this case, the weight function \(w\) is \(\mathbb{R}\)-valued, and is called generalized weight.

**Theorem 5.1** Let \((M, F)\) be an \(n\)-dimensional simply connected smooth Finsler manifold with \(F\) as Randers change of Quartic metric. Then, \(F\) induces generalized weighted quasi metric \(d_F\) on \(M\).

**Proof.** We consider that \((M, F)\) is a Finsler space with \(F = \beta + \sqrt{(\alpha^4 + \beta^4)}\), which can be written as \(F = \tilde{F} + \beta\), where \(\tilde{F} = \sqrt[4]{(\alpha^4 + \beta^4)}\) is an absolute homogeneous Finsler metric on \(M\) and \(\beta\) an exact 1-form.

Let \(\gamma_{xy} \in \Gamma_{xy}\) be an \(F\)-geodesic, which is at the same time \(\tilde{F}\)-geodesic, then from equation (5.2), we get

\[
d_F(x, y) = \int_a^b F(\gamma(t), \dot{\gamma}(t))dt
= \int_a^b \left(\beta + \sqrt[4]{(\alpha^4 + \beta^4)}\right)dt
= \int_a^b \left(\sqrt[4]{(\alpha^4 + \beta^4)}\right)dt + \int_a^b \beta dt
= \int_{\gamma_{xy}} \left(\sqrt[4]{(\alpha^4 + \beta^4)}\right) + \int_{\gamma_{xy}} \beta. \tag{5.3}
\]

Consider a fixed point \(a \in M\) and define the function \(w_a : M \rightarrow \mathbb{R}\) by \(w_a(x) := d_F(a, x) - d_F(x, a)\).

From the equation (5.3) it follows that

\[
w_a(x) = \int_{\gamma_{ax}} \beta - \int_{\gamma_{xa}} \beta = -2 \int_{\gamma_{xa}} \beta, \tag{5.4}
\]

where we have used the Stokes theorem for the 1-form \(\beta\) on the closed domain \(D\) with boundary \(\partial D := \gamma_{ax} \cup \gamma_{xa} \).
It can be easily seen that \( w_a \) is an anti-derivative of \( \beta \). This is well defined if and only if the integral in the R.H.S. of equation (5.4) is path independent, i.e., \( \beta \) must be exact.

Then \( d_F \) is a weighted quasi-metric with generalized weight \( w_a \). Next we have

\[
d_F(x, y) + w_a(x) = \int_{\gamma_{xy}} \left( \sqrt[4]{(\alpha^4 + \beta^4)} \right) - \int_{\gamma_{xa}} \beta - \int_{\gamma_{ya}} \beta, \tag{5.5}
\]

where we have again used the Stokes theorem for the one form \( \beta \) on the closed domain with boundary \( \gamma_{ax} \cup \gamma_{xy} \cup \gamma_{ya} \).

Similarly,

\[
d_F(y, x) + w_a(y) = \int_{\gamma_{xa}} \left( \sqrt[4]{(\alpha^4 + \beta^4)} \right) - \int_{\gamma_{ya}} \beta - \int_{\gamma_{xa}} \beta. \tag{5.6}
\]

From the equations (5.5) and (5.6) we conclude that \( d_F \) is weighted quasi-metric with generalized weight \( w_a \).

This completes the proof.

Next, recall the following:

**Lemma 5.1** ([3], [9]) Let \( (M, d) \) be any quasi-metric space. Then \( d \) is weightable if and only if there exists \( w : M \rightarrow [0, \infty) \) such that

\[
d(x, y) = \rho(x, y) + \frac{1}{2}[w(x) - w(y)], \forall x, y \in M, \tag{5.7}
\]

where \( \rho \) is the symmetrized distance function of \( d \). Moreover, we have

\[
\frac{1}{2}[w(x) - w(y)] \leq \rho(x, y), \forall x, y \in M. \tag{5.8}
\]

The proof is trivial from the definition of weighted quasi-metric.

**Remark 2.** If \( (M, F) \) is a Finsler space with a special \( (\alpha, \beta) \)-metric \( F = \sqrt[4]{(\alpha^4 + \beta^4)} + \beta \), then the induced quasi-metric \( d_F \) and the symmetrized metric \( \rho \) induce the same topology on \( M \). This follows immediately from ([7], [8]).

**Remark 3.** From Lemma 5.1, It can be seen that the assumption of \( w \) to be smooth is not essential.

Next, we discuss an interesting geometric property concerning the geodesic triangles.

**Theorem 5.2** Let \( (M, F) \) be a Finsler space with the Randers change of Quartic-metric \( F = \sqrt[4]{(\alpha^4 + \beta^4)} + \beta \). Then the parameteric length of any geodesic triangle on \( M \) does not depend on the orientation, that is,

\[
d_F(x, y) + d_F(y, z) + d_F(z, x) = d_F(x, z) + d_F(z, y) + d_F(y, x), \forall x, y, z \in M.
\]

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Proof. Since the Randers change of Quartic metric $F = \sqrt[4]{(\alpha^4 + \beta^4)} + \beta$ can be treated as the Randers change of absolute homogeneous Finsler metric $\tilde{F} = \sqrt[4]{(\alpha^4 + \beta^4)}$, i.e., $F = \tilde{F} + \beta$ with $d\beta = 0$, from theorem 5.1, it follows that the quasi-metric is weightable and therefore equation (5.7) holds good. By using the formula (5.7), a simple calculation gives the required result.

References

[1] M. Crampin, *Randers spaces with reversible geodesics*, Publ. Math. Debrecen, 67/3 (2005), 401-409.

[2] G. Hamel, *Über die Geomtrieen in denen die Geraden die Kürzeintensind*, Math. Ann., 57 (1903).

[3] H. P. A. Kunzi, V. Vajner, *Weighted quasi-metrics*, Annals New York Acad. Sci., 728 (1994), 64-77.

[4] I. Masca, S. V. Sabau, H. Shimada, *Reversible geodesics for $(\alpha, \beta)$-metrics*, Intl. J. Math., 21 (2010), 1071-1094.

[5] I. Masca, S. V. Sabau, H. Shimada, *Two dimensional $(\alpha, \beta)$-metrics with reversible geodesics*, Publ. Math. Debrecen, 82 (2013), 485-501.

[6] S. V. Sabau, H. Shimada *Finsler manifolds with reversible geodesics*, Rev. Roumaine Math. Pures Appl, 57 (2012), 91-103.

[7] Z. Shen, *Differential geometry of sprays and Finsler spaces*, Kluwer Academic Publishers, Dordrecht, (2001).

[8] Z. Shen, *On projectively flat $(\alpha, \beta)$-metrics*, Canad. Math. Bull., 52 (2009), 132-144.

[9] P. Vitolo, *A representation theorem for quasi-metric space*, Topology Appl., 65 (1995), 101-104.

[10] G. Shanker, S. A. Baby, *Reversible geodesics of Finsler spaces with a special $(\alpha, \beta)$-metric*, Bull. Cal. Math. Soc., 109 (2017), 183-188.

[11] G. Shanker, S. Rani, *Weighted quasi-metrics associated with Finsler metrics*, arXiv:1801.05636v2[math.DG].
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