Nonlinear Luenberger-like observers for some anthracnose models

David Jaures FOTSA MBOGNE\textsuperscript{a}, Duplex Elvis HOUPA DANGA\textsuperscript{b}, David BEKOLLE\textsuperscript{b}\textsuperscript{†}

\textsuperscript{a}Department of Mathematics and Computer Science, ENSAI, The University of Ngaoundere
\textsuperscript{b}Department of Mathematics and Computer Science, Fac. Sci., The University of Ngaoundere

Abstract

In this paper, we propose observers for the dynamics of anthracnose disease described in \cite{12}. Spatial and non spatial versions of the observers are given following an approach similar to \cite{12}. The models given in \cite{12} are improved in order to be more realistic. The conditions on parameters are more general. There are also changes in the equations modelling the dynamics of the berry volume ($v$) and the rot volume ($v_r$). We study theoretically the proposed observers in terms of well-posedness and convergence. We make several simulations to assess the effectiveness of observers which definitely display fairly good behaviour.

KeyWords— Anthracnose modelling, Nonlinear observer, Stability.

AMS Classification— 93C15, 93C20, 93B07, 93D15.

1 Introduction

Anthracnose is a phytopathology which occurs on several commercial tropical crops. The coffee is concerned by that disease due to the \textit{Colletotrichum kahawae} which is an ascomycete fungus \cite{2, 5, 6, 14, 17, 21, 24}. Several models of the anthracnose dynamics have been proposed in the literature \cite{9, 10, 11, 16, 17, 18, 19, 20, 24} in order to better understand and control it. Recently, in \cite{12, 13} the authors studied the optimal control of a diffusion model of anthracnose with continuous and impulsive strategies. Optimal controls were computed with respect to given cost functionals. The general model proposed in \cite{12} was given by the following equations

\begin{align}
\partial_t \theta &= \alpha (t, x) (1 - w(t, x) \theta) + \text{div} (A(t, x) \nabla \theta), \quad \text{on } \mathbb{R}_+^* \times U \\
\Delta \theta (t, x) &= 0, \quad \text{on } \mathbb{R}_+^* \times \partial U
\end{align}

where $n(x)$ denotes the normal vector on the boundary at $x$ and

\begin{align}
\theta (0, x) &= \rho (x) \in [0, 1], \quad x \in \overline{U} \subseteq \mathbb{R}^3 \\
\partial_t v &= \beta (t, x, \theta) (1 - v/((1 - \theta) \eta (t, x) v_{\text{max}})) \\
\partial_t v_r &= \gamma (t, x, \theta) (1 - v_r/v) \\
(v(0, x), v_r(0, x)) &\in [0, v_{\text{max}}] \times [0, v_{\text{max}}], \quad x \in \overline{U} \subseteq \mathbb{R}^3
\end{align}

where $w(t, x) = 1/ (1 - \sigma u(t, x))$ and $v(0) \geq v_r(0)$.

In the model above, $\theta$ is the inhibition rate, $v$ and $v_r$ are respectively the fruit volume density and the rot volume density. The function $v$ is upper bounded by a value $v_{\text{max}}$ which models the natural fact that the fruit growth is limited. Nonnegative functions $\alpha, \beta, \gamma : \mathbb{R}_+ \times U \times \mathbb{R} \rightarrow \mathbb{R}_+$ characterize the effects of environmental and climatic conditions on the rate of change of inhibition rate, fruit volume, and infected

\textsuperscript{a}Corresponding author Address: email: mjdavidfotsa@gmail.com, P.O. Box 455, ENSAI, The University of Ngaoundere
\textsuperscript{†}Co-authors emails: e_houpa@yahoo.com, bekolle@yahoo.fr
fruit volume respectively \cite{9, 10, 11}. There is a control $u$ representing the chemical strategy consisting on
the effects after application of fungicides. The positive term $1 - \sigma \in ]0, 1[$ models the inhibition rate correponding to
epidermis penetration by hyphae. Once the epidermis has been penetrated, the inhibition rate cannot fall below $1 - \sigma$, even under the maximum control effort ($u = 1$). In the absence of control effort ($u = 0$), the inhibition rate should increase towards 1. The effects of environmental and climatic conditions
on the maximum fruit volume are represented by the parameter $\eta$ which is a $[0, 1[$ valued function. There is
a spatial spread of the disease in the open domain $U \subset \mathbb{R}^3$ of class $C^1$ through the diffusion term $\text{div}(A \nabla \theta)$. The boundary condition $\langle A \nabla \theta, n \rangle = 0$ where $A$ is a $3 \times 3$-matrix ($a_{ij}$) could be viewed as the law steering
migration of the disease between $U$ and its exterior. For instance, if $A$ is reduced to the identity matrix $I$
then $\langle \nabla \theta, n \rangle = 0$ means that the domain $U$ does not have any exchange with its exterior.

Also in \cite{12} a within-host model has been studied and given by the following equations :

\begin{align}
d_t \theta &= \alpha (t) (1 - w(t) \theta) \\
d_t v &= \beta (t, \theta) (1 - v/\eta (t) v_{\text{max}} (1 - \theta)) \\
d_t v_r &= \gamma (t, \theta) (1 - v_r/v)
\end{align}

\begin{equation}
(\theta (0), v (0), v_r (0)) \in [0, 1[ \times [0, v_{\text{max}}] \times [0, v_{\text{max}}]
\end{equation}

where $w(t) = 1/(1 - \sigma u(t))$ and $v(0) \geq v_r (0)$. The parameters in (7) - (10) still have the same interpretations
given in the general model (1) - (6).

In several cases, especially for the results in \cite{12, 13} on anthracnose disease, an optimal control
strategy is given as a feedback and needs to know the state of the system and parameters values. It could
be difficult to know exactly the dynamics of the inhibition rate while it is easier to observe volumes $v$
and $v_r$. The aim of this paper is to provide some observers for the state $\theta$ in the models (1) - (6) and
(7) - (10) assuming that volumes $v$ and $v_r$ are known. An observer is a function $\hat{\theta}$ having its own dynamics
depending on the available observations ($v$ and $v_r$) and converging towards a neighborhood of $\theta$. Let us
recall the following useful adapted definitions about the stability of an observer. The authors refers to
\cite{8, 15, 22, 23} and references therein for more general and classical definitions.

**Definition 1.1** Assume that $\theta$ is $S$-valued. An observer $\hat{\theta}$ is said to be

(i) (Locally) stable if there are two neighborhoods $U, V \subseteq S$ of 0 and a time $t_0 > 0$ such that if $\theta (0) - \hat{\theta} (0) \in U$ then $\forall t > t_0, \theta (t) - \hat{\theta} (t) \in V$;

(ii) (Locally) asymptotically stable if there is a neighborhood $U \subseteq S$ of 0 such that if $\theta (0) - \hat{\theta} (0) \in U$ then

\[ \lim_{t \to \infty} \theta (t) - \hat{\theta} (t) = 0; \]

(iii) (Locally) exponentially asymptotically stable if there are a neighborhood $U \subseteq S$ of 0, a positive real
constant $k$ and a time $t_0 > 0$ such $\forall t > t_0$, \[ \left\| \theta (t) - \hat{\theta} (t) \right\| < \exp (-kt). \]

Those local stability properties are said to be global if one can extend the set $U$ to the whole set $S$.

The remainder of the work is organized as it follows. In section 2 we present some new realistic
considerations made for the models. We also show that the new model is well-posed. We design observers
for the non spatial model in section 3. A theoretical study is done in subsection 3.1 and we make numerical
evaluations in subsection 3.2. We also design observers for the spatial model in section 4 and we study them
theoretically in subsection 4.1. A numerical evaluation is carried out in subsection 4.2. Finally, there is a
conclusion in section 5.

\footnote{See \cite{22} for more details on the topic.}
2 New modelling of anthracnose dynamics

In this section we introduce some changes in the models given in [12] based on some remarks in order to become more realistic. Our modifications are especially made on the description of the dynamics of $v$ and $v_r$.

Looking at equation (4) we can see that when $\theta$ tends to the value 1, $v$ tends with an infinite speed to zero. However, we think that either there is a disease or not the berries will reach and remain greater than a minimal volume. Indeed, the disease can not be triggered when the berry is less than a minimal size as for instance the bud size. Even if the fruit totally rots it keeps a minimal size. We then suggest to add a very small nonegative term $\varepsilon \ll 1$ in such way that equation (4) becomes

$$\partial_t v = \beta (t, x, \theta) (1 - v/((1 + \varepsilon - \theta) \eta (t, x) v_{\max}))$$  \hspace{1cm} (11)$$

With the same arguments equation (8) becomes

$$d_t v = \beta (t, \theta) (1 - v/\eta (t) v_{\max} (1 + \varepsilon - \theta))$$  \hspace{1cm} (12)$$

The use of the term $\varepsilon$ generalizes the model in such way that we can recover the original model given in [12] by setting $\varepsilon = 0$. With a positive $\varepsilon$ even if $\theta = 1$, $v$ might be also positive. Indeed, the berries reach a minimal volume in the neighborhood of $\varepsilon v_{\text{max}} \min \{\eta (t)\}$.

We now focus on the equations (12) and (19). In [12] it was established that $v_r$ is $[0, v_{\text{max}}]$ valued. However, that behaviour is not sufficient. Indeed, $v_r$ should remain less than $v$. On the other hand $v_r$ should remain equal to zero while $v$ is less than the minimal value corresponding to the threshold volume where the disease can be triggered. We then suggest that $v_r$ remains equal to zero when $v$ is not greater than 0. Instead of studying directly $v_r$, we introduce a proportion function $\rho$ depending on time and space variables, and given such as

$$v_r = \rho v$$  \hspace{1cm} (13)$$

We propose the following dynamics for the rot proportion $\rho$: $\forall x \in U$, $\rho (0, x) \in [0, 1]$ and

$$\partial_t \rho (t, x) = \gamma (t, \theta (t, x), v (t, x), \rho (t, x)) (1 - \rho (t, x)), \forall (t, x) \in \mathbb{R}_+^* \times U,$$  \hspace{1cm} (14)$$

where $\gamma : \mathbb{R}_+ \times U \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is a function satisfying $\forall (t, x) \in \mathbb{R}_+ \times U$,

**Assumption 2.1** $\gamma$ is a measurable with respect to the two first parameters and locally Lipschitz continuous with respect to the third parameter.

**Assumption 2.2** $\gamma(t, x, .., 0)$ is nonnegative and such that

$$\gamma(t, x, 0, .., 0) = 0 = \gamma(t, x, 0, ..).$$  \hspace{1cm} (15)$$

**Assumption 2.3** $\gamma(t, x, .., .)$ is increasing with respect to the first parameter and $\gamma(t, x, 0, ..)$ is decreasing with respect to the last parameter.

For instance $\gamma$ could be chosen with the simple but general form $\gamma(t, x, y_1, y_2, y_3) = (\gamma_1 (t, x) y_1 - \gamma_2 (t, x) y_3) y_2$, with $\gamma_1, \gamma_2 : \mathbb{R}_+ \times U \times \mathbb{R} \rightarrow \mathbb{R}$. If we omit the dependence with the space variable we get the corresponding equation for the non spatial model: $\rho (0) \in [0, 1]$ and

$$d_t \rho (t) = \gamma (t, \theta (t), v (t), \rho (t)) (1 - \rho (t)), \forall t \in \mathbb{R}_+^*.$$

As we will check in the sequel, the function $\rho$ is $[0,1]$-valued. The assumption (2.2) guarantees that while the berry has a null volume (without berry) or the disease has not started the rot volume remains null. The assumption (2.3) means that the rot volume increases with the inhibition rate. When there is a not inhibition the volume of rot does not increase while the fruit grows better and therefore the proportion $\rho$ decreases.
2.1 Well-posedness of the problem for the non spatial model

This subsection is devoted to the proof of the well-posedness of the non spatial model. We make the following useful assumptions.

Assumption 2.4 $\alpha \in L^\infty (\mathbb{R}^+_+; [0,1])$.

Assumption 2.5 $u, \eta \in L^\infty ([0,1]; [0,1/((1+\varepsilon)])$ and $\forall t \geq 0, \eta (t) \geq \eta^* \in [0,1]$.

Assumption 2.6 $\beta : \mathbb{R}^+_+ \times U \times \mathbb{R} \rightarrow \mathbb{R}^+_+$ is nonincreasing function with respect to the third parameter. $\forall z \in \mathbb{R}, \beta (.,.,z)$ is a measurable function and belongs to the space $L^\infty_{loc} (\mathbb{R}^+_+ \times \mathbb{R}; \mathbb{R})$.

Assumption 2.7 $\forall t \geq 0$, the function $\beta (t,.)$ is differentiable and $\partial_z \beta (t,.) \in L^\infty (\mathbb{R}; \mathbb{R})$.

The assumptions (2.4)-(2.6) are more general than those given in [12] in the sense that all the parameters of the model were assumed continuous with respect to the state variable while here they are just measurable and essentially bounded. The following existence and uniqueness proposition holds:

Proposition 2.1 If $(\theta (0), v (0), \rho (0)) \in [0,1] \times [0,v_{\text{max}}] \times [0,1]$ then the problem (7), (12) and (16) has a unique absolute continuous solution $(\theta, v, \rho)$ valued in the set $[0,1] \times [0,v_{\text{max}}] \times [0,1]$.

Proof. Using the Carathéodory theorem (see [7]) we get easily the existence of an absolutely continuous solution. The logistic form of the equations (7) and (12) shows clearly that for a maximal solution $(\theta, v, \rho)$ is necessarily valued in the set $[0,1] \times [0,v_{\text{max}}]$. On the other hand from the equation (16), $\rho$ is valued in $[0,1]$. To show it let set $\rho_1 = \max \{0,-\rho\}$ and $\rho_2 = \max \{0,\rho-1\}$. The functions $\rho_1$ and $\rho_2$ are continuous as $\rho$ is continuous. Without loss of generality we will assume that $\rho_1$ and $\rho_2$ are positive on an open interval of time $]0,T[$. Then $\forall t \in ]0,T[$,

$$\rho_1 (t) = -\int_0^t \gamma (s, \theta (s), v (s), -\rho_1 (s)) (1 + \rho_1 (s)) ds \leq 0 \quad (17)$$

$$\rho_2 (t) = -\int_0^t \gamma (s, \theta (s), v (s), \rho_2 (s) + 1) \rho_2 (s) ds \leq 0 \quad (18)$$

There is a contradiction meaning that $\rho_1$ and $\rho_2$ are identically null. Therefore, $\rho$ is $[0,1]$-valued. \[\blacksquare\]

2.2 Well-posedness of the problem for the spatial model

In this subsection the well-posedness of the spatial model is proved. To that aim, we make some assumptions.

Assumption 2.8 $\alpha \in L^\infty_{loc} (\mathbb{R}^+_+; L^\infty (U; \mathbb{R}^+_+))$.

Assumption 2.9 $u, \eta \in L^\infty ([0,1]; [0,1])$ and $\forall t \geq 0, 1/(1+\varepsilon) \geq \eta (t,.) \geq \eta^* \in L^\infty (U; [0,1])$ with $0 < \inf_U (\eta^*) = \eta_m$.

Assumption 2.10 $\beta : \mathbb{R}^+_+ \times U \times \mathbb{R} \rightarrow \mathbb{R}^+_+$ is nonincreasing function with respect to the third parameter. $\forall z \in \mathbb{R}, \beta (.,.,z)$ is a measurable function and belongs to the space $L^\infty_{loc} (\mathbb{R}^+_+ \times \mathbb{R}; \mathbb{R}^+_+)$.

Assumption 2.11 $\forall i, j \in \{1,2,3\}, a_{ij} \in L^\infty_{loc} (\mathbb{R}^+_+; W^{1,\infty} (U; \mathbb{R})).$
**Assumption 2.12** \( \exists \zeta \in \mathbb{R}_+^* \) such that \( \forall t \in \mathbb{R}_+, \forall w \in H^1(U; \mathbb{R}), \)

\[
\int_U \langle A(t, x) \nabla w(x), \nabla w(x) \rangle \, dx \geq \zeta \int_U \langle \nabla w(x), \nabla w(x) \rangle \, dx.
\]

**Assumption 2.13** \( \forall t \geq 0, \forall x \in U, \) the function \( \beta(t, x, \cdot) \) is differentiable and \( \partial_\gamma \beta(t, x, \cdot) \in L^\infty_{\text{loc}}(\mathbb{R}; \mathbb{R}). \)

Assumptions (2.8)-(2.10) are also more general than those given in [12] in the sense that all the parameters of the model were assumed continuous with respect to the state variable while here they are just measurable and essentially bounded. From assumptions (2.11)-(2.12) the following problem has a unique solution in \( H^1(U) \) for an arbitrary but fixed time \( t > 0. \)

\[
\left\{ \begin{array}{l}
\text{div} (A(t, x) \nabla \varphi_n(t, x)) = f(x), \quad \forall x \in U \\
\langle A(t, x) \nabla \varphi_n(t, x), n(x) \rangle = 0, \quad \forall x \in \partial U
\end{array} \right.
\]

where \( f \in L^2(U) \). Following [3] in Theorems 3.6.1 and 3.6.2, there is an orthonormal complete system \( \{\varphi_n(t, \cdot)\}_{n \in \mathbb{N}} \subset L^2(U) \) of eigenfunctions and eigenvalues \( \{\lambda_n(t)\} \) such that \( \forall n \in \mathbb{N}, \)

\[
\left\{ \begin{array}{l}
\text{div} (A(t, x) \nabla \varphi_n(t, x)) = \lambda_n(t) \varphi_n(t, x), \quad \forall x \in U \\
\langle A(t, x) \nabla \varphi_n(t, x), n(x) \rangle = 0, \quad \forall x \in \partial U
\end{array} \right.
\]

Moreover, \( \{\varphi_n(t, \cdot)\}_{n \in \mathbb{N}} \subset H^1(U) \) and if \( \partial U \) is of class \( C^2 \) then \( \{\varphi_n(t, \cdot)\}_{n \in \mathbb{N}} \) is \( H^2(U) \) valued.

Now we make the following additional assumption.

**Assumption 2.14** The sequence \( \{\varphi_n\} \) does not depend on the time (ie \( \varphi_n(t, \cdot) = \varphi_n(\cdot), \forall t > 0 \)).

Assumption (2.14) could happen if \( A(t, \cdot) \) has the form \( \mu(t) B(\cdot) \) with \( \mu(t) \in \mathbb{R}, \forall t \geq 0. \) It will be the case in particular if \( A(t, \cdot) = \mu(t) I \) and therefore div \( (A(t, \cdot) \nabla w) = \mu(t) \Delta w. \) Here \( I \) denotes the identity matrix of \( \mathbb{R}^3. \) Whether (2.14) is satisfied a weak solution \( \theta \) of (11)–(12) can be written as the sum \( \sum_{n=0}^\infty \theta_n \varphi_n \) where each \( \theta_n \) is an absolutely continuous function of the time and satisfies \( \forall t \geq 0, \)

\[
\theta_n(t) = \sum_{m=0}^\infty \theta_m \int_U \varphi_m(x) \varphi_n(x) \, dx
\]

\[
= \theta_n(0) + \int_0^t \lambda_n(s) \theta_n(s) \, ds + \int_0^t \int_U \alpha(s, x) \varphi_n(x) \, dx \, ds
\]

\[
- \int_0^t \int_U \alpha(s, x) w(s, x) \varphi(s, x) \varphi_n(x) \, dx \, ds.
\]

**Proposition 2.2** The problem (11)–(12) has a unique solution \( \theta \in C(\mathbb{R}_+; H^1(U; [0, 1])) \) which is absolutely continuous with respect to the time.

**Proof.** It is sufficient here to establish that if there is a maximal solution then that solution is valued in the set \([0, 1]. \) Indeed, conditions (2.8)-(2.12) are sufficient to use the Carathéodory theorem (see in [1] the methods used in the proof of the Lemma A 2.7, p187-191). Let \( \forall t \geq 0, \) \( \theta_1 = \max \{0, -\theta\} \) and \( \theta_2 = \max \{0, \theta - 1\}. \) We have \( \theta_1(0, \cdot) = \theta_2(0, \cdot) = 0. \) Let \( S_i \) be a convex open subset of \( \mathbb{R}_+ \) where functions \( \theta_i(t, \cdot) (i \in \{1, 2\}) \) are positive on subsets of \( U \) with a positive measure. Then \( \forall t \in S_i \) we have

\[
\partial_t \|f_1(t, \cdot)\|_{L^2(U; \mathbb{R})}^2/2 = \int_U f_1(t, x) \partial_t f_1(t, x) \, dx
\]

\[
= -\int_U \alpha(t, x) (1 + w(t, x) f_1(t, x)) f_1(t, x) \, dx
\]

\[
\leq 0
\]
\[ \partial_t \| f_2 (t, \cdot) \|^2_{L^2(U; \mathbb{R})} / 2 = \int_U f_2 (t, x) \partial_t f_2 (t, x) \, dx \]
\[ = \int_U \alpha (t, x) (1 - w(t, x)) f_2 (t, x) \, dx - \int_U \alpha (t, x) w(t, x) f_2 (t, x) \, dx \]
\[ \leq 0 \]

Using the above inequalities we get that functions \( \theta_i (t, .) \) are nonpositive on the sets \( S_i \) which are necessary empty. We conclude that \( \forall t > 0, \theta (t, .) \) is valued in \([0, 1]\).

**Proposition 2.3** The problem \((11), (14)\) has a unique solution \((v, \rho) \in C ([\mathbb{R}_+; L^\infty (U; [0, v_{\text{max}}] \times [0, 1])] \) which is absolutely continuous with respect to the time.

**Proof.** Since \( \theta \) is valued in the set \([0, 1]\) and there is not diffusion in the equations \((11)\) and \((14)\), we can fix the space variable and use a proof similar to the one given for proposition 2.1.

### 3 Observation for the within-host model

#### 3.1 Theoretical design of observers

The main objective of this subsection is to design observers for the problem \((7), (12)\) and \((16)\) in order to estimate the inhibition rate \( \theta \). Let us consider the following system: \( \forall t \geq 0, \)

\[ d_t \hat{\theta} (t) = \alpha (t) \left( 1 - w(t) \hat{\theta} (t) \right) + k_1 (t) \phi_1 \left( t, \hat{\theta} (t), \hat{v} (t) \right) + k_2 (t) \phi_2 \left( t, \hat{\theta} (t) \right) \] (20)

and

\[ d_t \hat{v} (t) = \beta \left( t, \hat{\theta} (t) \right) \phi_3 \left( t, \hat{\theta} (t), \hat{v} (t) \right) \] (21)

where

\[ \phi_1 (t, x, y) = \begin{cases} (1 - v(t) / y) (1 + \varepsilon - x), & \text{if } v(t) \leq y \text{ and } (x, y) \in [0, 1] \times \mathbb{R}_+^* , \\ 0, & \text{otherwise} \end{cases} \]

\[ \phi_2 (t, x) = \begin{cases} d_t \rho(t) - \mathcal{F}(t, x, \nu(t), \rho(t)) (1 - \rho(t)), & \text{if } x \in [0, 1] \\ 0, & \text{otherwise} \end{cases} \]

and

\[ \phi_3 (t, x, y) = 1 - \frac{y}{(1 + \varepsilon - x) \eta(t) v_{\text{max}}} \]

We make the following additional hypothesis:

**Assumption 3.1** \( k_1, k_2 \in L^\infty_{\text{loc}} ([\mathbb{R}_+; \mathbb{R}_+]). \)

By a solution of the system \((20) - (21)\) we mean an absolutely continuous function of the time \((\hat{\theta}, \hat{v})\) which satisfies \( \forall t > 0, \)

\[ \hat{\theta} (t) = \hat{\theta} (0) + \int_0^t \alpha (s) \left( 1 - w(s) \hat{\theta} (s) \right) ds + \int_0^t k_1 (s) \phi_1 \left( s, \hat{\theta} (s), \hat{v} (s) \right) ds \]

and

\[ \hat{v} (t) = \hat{v} (0) + \int_0^t \beta \left( s, \hat{\theta} (s) \right) \phi_3 \left( s, \hat{\theta} (s), \hat{v} (s) \right) ds. \]

We have the existence and uniqueness.
Proposition 3.1 If \( \left( \hat{\theta}(0), \hat{v}(0) \right) \in [0,1] \times [0,v_{\max}] \) then the problem (20) – (21) has a unique solution valued in \([0,1] \times [0,v_{\max}]\).

**Proof.** We first show that if \( \left( \hat{\theta}, \hat{v} \right) \) is a local solution of the system (20) – (21) then \( \left( \hat{\theta}', \hat{v}' \right) \) is valued in \([0,1] \times [0,v_{\max}]\). Let \( \forall t \geq 0, \ \theta_1 = \max \left\{ 0, -\hat{\theta} \right\}, \ \theta_2 = \max \left\{ 0, \hat{\theta} - 1 \right\}, v_1 = \max \left\{ 0, -\hat{v} \right\} \) and \( v_2 = \max \left\{ 0, \hat{v} - v_{\max} \right\} \). We have \( \theta_i(0) = v_i(0) = 0, \ \forall i \in \{1, 2\} \). Without loss of generality we can assume that the functions \( \theta_i \) and \( v_i \ (i \in \{1, 2\}) \) are positive on an open interval \([0,T[\). Then we have

\[
\theta_1(t) = - \int_{0}^{t} \alpha(s) \left( 1 + w(s) \theta_1(s) \right) ds - \int_{0}^{t} k_1(s) \phi_1(s, -\theta_1(s), \hat{v}(s)) ds \\
- \int_{0}^{t} k_2(s) \phi_2(s, -\theta_1(s)) ds \\
\leq 0
\]

and

\[
\theta_2(t) = \int_{0}^{t} \alpha(s) \left( 1 - w(s) - w(s) \theta_2(s) \right) ds + \int_{0}^{t} k_1(s) \phi_1(s, 1 + \theta_2(s), \hat{v}(s)) ds \\
+ \int_{0}^{t} k_2(s) \phi_2(s, 1 + \theta_1(s)) ds \\
\leq 0.
\]

That is \( \theta_1 \) and \( \theta_2 \) are identically null and \( \hat{\theta} \) is \([0,1]-valued \). In the same manner one can show that \( \hat{\theta} \) is valued in \([0, v_{\max}]\).

Now let show the existence and the uniqueness of the solution of (20) – (21). It suffices to establish existence and uniqueness of a local solution and use the fact every local solution of (20) – (21) is bounded to conclude using Theorem 5.7 in [4].

Let consider the function \( F : (t, x, y) \in \mathbb{R}_+ \times [0,1] \times [0,v_{\max}] \rightarrow \mathbb{R}^2 \) defined by

\[
F^1(t, x, y, z) = \alpha(t) \left( 1 - xw(t) \right) + k_1 \phi_1(t, x, y) + k_2 \phi_2(t, x),
\]

and

\[
F^2(t, x, y) = \beta(t, x) \phi_3(t, x, y).
\]

The function \( F \) is measurable with respect to the time \( t \) and continuous with respect to \( (x, y) \). Using the Carathéodory theorem there is a local solution of (20) – (21). Moreover, \( F \) is Lipschitz continuous with respect to \( (x, y, z) \). Therefore, the solution is unique and global. Note that the hypothesis (2.7) implies that \( \forall t \geq 0, \ \text{the function} \ \beta(t, .) \ \text{is Lipschitz continuous and bounded on the set} \ [0,1]. \]

Now we state the main results of this subsection. Let define the function \( \delta : \mathbb{R} \rightarrow \{0, 1\} \) such as

\[
\delta(x) = \begin{cases} 
1, & x \in [0, 1] \\
0, & \text{otherwise}
\end{cases}
\]

**Theorem 3.2** Let consider the system (20) – (21) and assume that \( \hat{\theta}(0) = v(0) \).

(i) If the functions \( k_1 \) and \( k_2 \) are identically null and

\[
\inf \{ \alpha(t) : t > 0 \} > 0
\]

then \( \hat{\theta} \) is a globally exponentially asymptotically stable observer for \( \theta \).
(ii) If the function $k_1$ is identically null and $k_2$ is not identically null then $\hat{\theta}$ is a a locally asymptotically stable observers for $\theta$. Moreover, if there is a positive function $C_\gamma \in L_{loc}^\infty (\mathbb{R}_+; \mathbb{R}_+)$ such that
\[
|\gamma(t, y_1, y_2, y_3) - \gamma(t, z_1, z_2, z_3)| \geq C_\gamma(t) \|y - z\| \tag{23}
\]
then $\hat{\theta}$ is a a globally exponentially asymptotically stable observers for $\theta$.

(iii) If the function $k_2$ is identically null and $\forall t > 0$,
\[
\alpha w + k_1 \delta \left(\hat{\theta}\right) + \frac{k_1 \delta \left(\hat{\theta}\right) \beta(t, \theta) (\eta \nu_{max} (1 + \varepsilon - \theta) - v)}{\alpha \eta \nu_{max} (1 - \theta w)} > 0. \tag{24}
\]
then $\hat{\theta}$ is a a locally exponentially asymptotically stable observers for $\theta$.

(iv) If the functions $k_1$ and $k_2$ are not identically null, and $\forall t > 0$,
\[
\alpha w + k_1 \delta \left(\hat{\theta}\right) + k_2 (t) \phi_2 (t, \hat{\theta}) + \frac{k_1 \delta \left(\hat{\theta}\right) \beta(t, \theta) (\eta \nu_{max} (1 + \varepsilon - \theta) - v)}{\alpha \eta \nu_{max} (1 - w \theta)} > 0 \tag{25}
\]
then $\hat{\theta}$ is a a locally exponentially asymptotically stable observers for $\theta$. Moreover, if the condition (23) is satisfied and
\[
k_2 \left|\phi_2 (t, \hat{\theta})\right| > k_1 \phi_1 \left(t, \hat{\theta}, \varepsilon\right) \tag{26}
\]
then $\hat{\theta}$ is a a globally exponentially asymptotically stable observers for $\theta$.

**Proof.** Let $\hat{\varepsilon} = \theta - \hat{\theta}$ be the error of estimation.

(i) The error $\hat{\varepsilon}$ satisfies
\[
d_t \hat{\varepsilon} = -\alpha(t) w(t) \hat{\varepsilon} \tag{27}
\]
and
\[
\hat{\varepsilon}(t) = \exp \left(- \int_0^t \alpha(s) w(s) ds\right) \hat{\varepsilon}(0). \\
\]
The result follows.

(ii) The error $\hat{\varepsilon}$ satisfies
\[
d_t \hat{\varepsilon}^2 = -2 \alpha(t) w(t) \hat{\varepsilon}^2(t) - 2 k_2(t) \phi_2 \left(t, \hat{\theta}(t)\right) \hat{\varepsilon}(t) \tag{28}
\]
\[
= -\alpha(t) w(t) \hat{\varepsilon}^2(t) - 2 k_2(t) (1 - \rho(t)) \left(\gamma(t, \theta(t), v(t), \rho(t)) - \gamma(t, \hat{\theta}(t), v(t), \rho(t))\right) \hat{\varepsilon}(t)
\]
\[
\leq -\alpha(t) w(t) \hat{\varepsilon}^2(t)
\]
and
\[
\hat{\varepsilon}^2(t) \leq \exp \left(- \int_0^t \alpha(s) w(s) ds\right) \hat{\varepsilon}^2(0). \\
\]
Moreover, if the condition (23) is satisfied then
\[
d_t \hat{\varepsilon}^2 \leq -\alpha(t) w(t) \hat{\varepsilon}^2(t) - 2 k_2(t) (1 - \rho(t)) C_\gamma(t) \hat{\varepsilon}^2(t)
\]
and
\[
\hat{\varepsilon}^2(t) \leq \exp \left(- \int_0^t \left(\alpha(s) w(s) + 2 k_2(s) (1 - \rho(s)) C_\gamma(s)\right) ds\right) \hat{\varepsilon}^2(0). \
\]

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(iii) We have
\[
\frac{\partial_t \beta(s, \theta)}{\beta(s, \theta) - d_t v} = \frac{d_t v \partial_\theta \beta(s, \theta) - \beta(s, \theta) \partial_\theta d_t v}{(\beta(s, \theta) - d_t v)^2} = \frac{\beta^2(s, \theta)((1 + \varepsilon - \theta) \partial_\theta v + v)}{\eta v_{\max} (1 + \varepsilon - \theta)^2 (\beta(s, \theta) - d_t v)^2}
\]
and at the neighborhood of 0,
\[
\frac{\beta(s, \hat{\theta})}{\beta(s, \hat{\theta}) - d_t \hat{v}} = \left(\frac{\beta(s, \theta)}{\beta(s, \theta) - d_t v}\right) - \frac{\beta^2(s, \theta)((1 + \varepsilon - \theta) \partial_\theta v + v)}{\eta v_{\max} (1 + \varepsilon - \theta)^2 (\beta(s, \theta) - d_t v)^2} \hat{\theta} + \mathcal{O}(\hat{\varepsilon})
\]
It follows that
\[
\frac{v}{\hat{v}} = \frac{v}{\eta v_{\max} (1 + \varepsilon - \theta)} \frac{\beta(s, \hat{\theta})}{\beta(s, \hat{\theta}) - d_t \hat{v}} = 1 - \left(\frac{1}{1 + \varepsilon - \theta} + \frac{\eta v_{\max} \partial_\theta v}{\eta v_{\max} (1 + \varepsilon - \theta)} \mathcal{O}(\hat{\varepsilon})\right) \hat{\varepsilon} - \frac{v \hat{\varepsilon}}{\eta v_{\max} (1 + \varepsilon - \theta)} \mathcal{O}(\hat{\varepsilon}).
\]
finally,
\[
d_t \hat{\varepsilon} = -\alpha(t) w(t) \hat{\varepsilon} - k_1 \phi_1(t, \hat{\theta}, \hat{v}) = -\frac{k_1(t) \delta(\hat{\theta})(1 + \varepsilon - \theta) \partial_\theta v}{v} \hat{\varepsilon} - \left(\alpha(t) w(t) - k_1(t) \delta(\hat{\theta})\right) \hat{\varepsilon}
\]
\[
\approx -\frac{k_1(t) \delta(\hat{\theta})(1 + \varepsilon - \theta) \partial_\theta v + \alpha \eta v_{\max} (1 - \theta w)}{\alpha \eta v_{\max} (1 - \theta w)} \hat{\varepsilon} - \left(\alpha(t) w(t) + k_1(t) \delta(\hat{\theta})\right) \hat{\varepsilon}
\]
The given approximation on the dynamics of \( \hat{\varepsilon} \) shows clearly that \( \hat{\theta} \) is a locally exponentially asymptotically stable observers for \( \theta \) if the condition (24) is satisfied.

(iv) With the same arguments developped in (ii) and (iii) we easily get that \( \hat{\theta} \) is a locally exponentially asymptotically stable observers for \( \theta \). Using the conditions (23) and (26) we are sure that even the behaviour of the term with \( \phi_1 \) tends to introduce unsteady, the stability is maintained by the term with \( \phi_2 \).

### 3.2 Numerical evaluation of the observers

This section is devoted to numerical simulations in order to check the effectiveness of the observers we studied theoretically in the subsection (3.1). The Theorem 3.2 guarantees asymptotical stability of the observers, but practically the convergence is needed in a finite time. Since the coffee cultivation campaign is a cycle of one year, our simulations will cover the year. Parameters are taken following [12]. The control strategy is
taken with many variations in order to study their impact on the performance of the observer. For every time \( t \geq 0 \), \( u \) is given by

\[
u(t) = \sin^2 \left( \omega_1 (t - \varphi_1)^2 \right) \exp \left( -\omega_2 (t - \varphi_2)^2 \right).
\]

The functions \( \alpha, \beta \) and \( \gamma \) are taken with the following form

\[
\alpha(t) = p_1(t) + b_1 (1 - \cos(c_1 t)) (t - d_1)^2, \quad \forall t \in \mathbb{R}_+,
\]

\[
\beta(t, x) = b_2 (1 - \cos(c_2 t)) (t - d_2)^2 p_2(x), \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R},
\]

and

\[
\gamma(t, x_1, x_2, x_3) = b_3 (1 - \cos(c_3 t)) (t - d_3)^2 (x_1 - \kappa x_3) x_2, \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3.
\]

\( p_1 \) is a nonnegative function of the time \( t \) and \( p_2 \) is a real positive function of \( x \). \( \forall i \in \{1, 2, 3\}, b_i, c_i, \) and \( d_i \) are positive coefficients corresponding respectively to the maximal amplitude, the pulsation and the global maximum of \( \alpha, \beta \) and \( \gamma \). \( \kappa \) is a positive constant regulating the evolution of the rot volume with respect to the inhibition rate. The terms \( 1 - \cos((c_i t)) \) represent the seasonality probably due to climatic and environmental variations.

Theoretically the functions \( k_1 \) and \( k_2 \) might take arbitrary positive numbers and we can set them constant without loss of generality. However, for the stability of the numerical scheme we were constrained to set them less than \( 1/10 \Delta t \). Here \( \Delta t \) denotes the constant time step. The initial conditions are taken such as

\[
theta(0) = 0, \quad v(0) = \hat{v}(0) = v(0).
\]

Indeed, at the beginning the fruits are small and the inhibition rate is relatively small.

The following table gives the assumed parameters values.

| Parameters | Values | Source | Parameters | Values | Source |
|------------|--------|--------|------------|--------|--------|
| \( b_1 \)  | \( 5 \ln(10) \) | Assumed | \( p_1(t) \) | \( 0 \) | Assumed |
| \( b_2 \)  | \( v_{\text{max}} \ln(10^5 v_{\text{max}} (1 - \eta^*) / 2 \) | Assumed | \( p_2(x) \) | \( 2 - x \) or \( (2 - x)^2 \) | Assumed |
| \( b_3 \)  | \( 7.5 \times 10^{-1} \) | Assumed | \( \kappa \) | \( 1 \) | Assumed |
| \( d_1 \)  | \( 7.5 \times 10^{-1} \) | Assumed | \( v_{\text{max}} \) | \( 1 \text{ cm}^2 \) | Assumed |
| \( d_2 \)  | \( 7.5 \times 10^{-1} \) | Assumed | \( \varepsilon \) | \( 10^{-4} \) | Assumed |
| \( d_3 \)  | \( 7.5 \times 10^{-1} \) | Assumed | \( \eta(t) \) | \( 1/(1 + \varepsilon) \text{ cm}^2 \) | Assumed |
| \( c_1 \)  | \( 10\pi \) | Assumed | \( \sigma \) | \( 0.9 \) | Assumed |
| \( c_2 \)  | \( 10\pi \) | Assumed | \( k_1 \) | \( 0 \text{ or } 10^3 \) | Assumed |
| \( c_3 \)  | \( 10\pi \) | Assumed | \( k_2 \) | \( 0 \text{ or } 10^3 \) | Assumed |
| \( \omega_1 \) | \( 25\pi \) | Assumed | \( \Delta t \) | \( 10^{-4} \) | Assumed |
| \( \omega_2 \) | \( 10 \) | Assumed | \( \varphi_1 \) | \( 0.6 \) | Assumed |

| \( \varphi_1 \) | \( 0.4 \) | Assumed | | | |

Table 1: Simulation parameters for the non-spatial model

In the remainder of the subsection we present simulations of the observers and their estimation relative errors dynamics. The relative error seems to be a good mean to evaluate the performance. For each scenario we display two groups figures. The first one represents the dynamics both of the inhibition rate and observers corresponding to each values of \( \rho(0) \leq \theta(0) \). The second group of figures shows relative errors of observers corresponding to each values of \( \rho(0) \leq \theta(0) \).
3.2.1 Simulations with $\theta(0) = v(0) = 0.05$

The figures 1 and 2 show the behaviour of the observers when the observation is started early. As we can see the global performance of each observer is relatively good but the case $k_1 = 0$ and $k_2 = 10^3$ seems to be the better one.

$$k_1 = k_2 = 0 \quad \quad k_1 = 0 \text{ and } k_2 = 10^3$$

$$k_1 = 10^3 \text{ and } k_2 = 0 \quad \quad k_1 = 10^3 \text{ and } k_2 = 10^3$$

Figure 1: Inhibition rate and observers for $\theta(0) = v(0) = 0.05$. 
\[ k_1 = k_2 = 0 \quad k_1 = 0 \text{ and } k_2 = 10^3 \]

\[ k_1 = 10^3 \text{ and } k_2 = 0 \quad k_1 = 10^3 \text{ and } k_2 = 10^3 \]

Figure 2: Relative absolute estimation error for \( \theta(0) = v(0) = 0.05 \).
3.2.2 Simulations with $\theta(0) = 0.05$ and $v(0) = 0.5$

The figures 3 and 4 correspond to an observation starting at the beginning of the disease but with a well developed berry. We have better performances when $k_2 = 10^3$. With $k_1 = 10^3$ and $k_2 = 0$ we notice an instability behaviour in figure 4 although the error takes the null value. That may be due to the constant sign of the function $\phi_1$.

\[
\begin{align*}
k_1 &= k_2 = 0 & k_1 &= 0 \text{ and } k_2 = 10^3 \\
k_1 &= 10^3 \text{ and } k_2 = 0 & k_1 &= 10^3 \text{ and } k_2 = 10^3
\end{align*}
\]

Figure 3: Inhibition rate and observers for $\theta(0) = 0.05$ and $v(0) = 0.5$. 


\begin{align*}
k_1 &= k_2 = 0 \\
k_1 &= 0 \text{ and } k_2 = 10^3
\end{align*}

Figure 4: Relative absolute estimation error for \( \theta (0) = 0.05 \) and \( v (0) = 0.5 \).
3.2.3 Simulations with \( \theta(0) = 0.75 \) and \( v(0) = 0.05 \)

In the figures 5 and 6 the inhibition rate is already high while the berry is little. As expected the worst estimation corresponds to \( k_1 = k_2 = 0 \). The best estimations are still made when \( k_2 = 10^3 \). However, when \( k_1 = 10^3 \), the same unstability behaviour in figure 4 appears. Note that the performances of the observers are better with \( \rho(0) \) small.

\[
\begin{align*}
    k_1 = k_2 &= 0 \\
    k_1 = 0 \text{ and } k_2 &= 10^3
\end{align*}
\]

\[
\begin{align*}
    k_1 = 10^3 \text{ and } k_2 &= 0 \\
    k_1 = 10^3 \text{ and } k_2 &= 10^3
\end{align*}
\]

Figure 5: Inhibition rate and observers for \( \theta(0) = 0.75 \) and \( v(0) = 0.05 \).
Figure 6: Relative absolute estimation error for $\theta(0) = 0.75$ and $v(0) = 0.05$. 

\[ k_1 = k_2 = 0 \quad \text{and} \quad k_1 = 0 \text{ and } k_2 = 10^3 \]

\[ k_1 = 10^3 \text{ and } k_2 = 0 \quad \text{and} \quad k_1 = 10^3 \text{ and } k_2 = 10^3 \]
3.2.4 Simulations with $\theta(0) = 0.75$ and $v(0) = 0.5$

For the figures 7 and 8 the inhibition rate is already high and the fruit is relatively mature when the observation starts. The worst case are still when $k_2 = 0$ and the best are with $k_2 = 10^3$. Again the performances of the observers are better with $\rho(0)$ small. It seems that starting observation with $\theta(0)$ small and $v(0)$ big guarantees better results.

\begin{align*}
k_1 = k_2 &= 0 \\
k_1 = 0 \text{ and } k_2 &= 10^3
\end{align*}

\begin{align*}
k_1 = 10^3 \text{ and } k_2 &= 0 \\
k_1 = 10^3 \text{ and } k_2 &= 10^3
\end{align*}

Figure 7: Inhibition rate and observers for $\theta(0) = 0.75$ and $v(0) = 0.5$. 17
\[ k_1 = k_2 = 0 \quad \text{and} \quad k_1 = 0 \text{ and } k_2 = 10^3 \]

\[ k_1 = 10^3 \text{ and } k_2 = 0 \quad \text{and} \quad k_1 = 10^3 \text{ and } k_2 = 10^3 \]

Figure 8: Relative absolute estimation error for \( \theta (0) = 0.75 \) and \( v (0) = 0.5 \).
4 Observation for the spatial model

The aim of this section is similar to the previous section. We design observers for the spatial dynamical system \(1\) − \(3\), \(6\), \(11\) and \(14\) in order to estimate the inhibition rate \(\theta\) and we improve them using computer simulations.

4.1 Theoretical design of the spatial model observers

Let consider the following systems:

\[
\hat{\theta}(t, x) = \alpha(t, x) \left(1 - w(t, x) \hat{\theta}\right) + K_1(t, x) \Phi_1\left(t, x, \hat{\theta}, \hat{v}\right)
+ K_2(t, x) \Phi_2\left(t, x, \hat{\theta}\right) + \text{div}\left(A(t, x) \nabla \hat{\theta}\right), \quad \text{on } \mathbb{R}_+^* \times U
\]

\[
\left\langle A(t, x) \nabla \hat{\theta}_1(t, x), n(x) \right\rangle = 0, \quad \text{on } \mathbb{R}_+^* \times \partial U
\]

\[
\hat{v}(t, x) = \beta(t, x, \hat{\theta}) \Phi_3\left(t, x, \hat{\theta}, \hat{v}\right)
\]

where

\[
\Phi_1(t, x, y, z) = \begin{cases} 
(1 - v(t, x)/z)(1 + \varepsilon - y), & \text{if } v(t, x) \leq y \text{ and } (x, y, z) \in \overline{U} \times ]0, 1[ \times \mathbb{R}_+^*, \\
0, & \text{otherwise}
\end{cases}
\]

\[
\Phi_2(t, x, y) = \begin{cases} 
\partial_t \rho(t, x) - \tau(t, y, v(t, x), \rho(t, x))(1 - \rho(t, x)), & \text{if } (x, y) \in \overline{U} \times ]0, 1[ \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
\Phi_3(t, x, y, z) = 1 - \frac{z}{(1 + \varepsilon - y) \eta(t, x) v_{\text{max}}}
\]

We have the following assumption:

**Assumption 4.1** \(K_1, K_2 \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^\infty(U; \mathbb{R}_+))\).

By a solution of the system \(33\) − \(35\), we mean an absolutely continuous function with respect to the time \((\hat{\theta}, \hat{v})\) which satisfies \(\forall t > 0, \forall x \in U\),

\[
\hat{\theta}(t, x) = \hat{\theta}(0, x) + \int_0^t \alpha(s, x) \left(1 - w(s, x) \hat{\theta}(s, x)\right) ds + \text{div}\left(A(s, x) \nabla \hat{\theta}_1\right) ds
+ \int_0^t K_1(s, x) \Phi_1\left(s, x, \hat{\theta}, \hat{v}\right) + \int_0^t K_2(s, x) \Phi_2\left(s, x, \hat{\theta}\right)
\]

\[
\hat{v}(t, x) = \hat{v}(0, x) + \int_0^t \beta(s, x, \hat{\theta}) \Phi_3\left(s, x, \hat{\theta}, \hat{v}\right) ds
\]

We have the following

**Proposition 4.1** If \((\hat{\theta}(0, .), \hat{v}(0, .)) \in L^2(U; [0, 1[ \times [0, v_{\text{max}}])\) then the system \(33\) − \(35\) has a unique weak solution \((\hat{\theta}, \hat{v}) \in C(\mathbb{R}_+; H^1(U; [0, 1])) \times C(\mathbb{R}_+; L^2(U; [0, v_{\text{max}}])))\).
Proof. We first show that if \( \left( \hat{\theta}, \hat{\nu} \right) \) is a local solution of the system (33 – 35) then \( \forall t > 0, \left( \hat{\theta}(t, \cdot), \hat{\nu}(t, \cdot) \right) \) is valued in \([0,1] \times [0, v_{\text{max}}] \). Let \( \forall t \geq 0, \hat{\theta}_1 = \max \{0, -\hat{\theta} \}, \hat{\nu}_2 = \max \{0, -\hat{\nu} \}, \hat{\nu}_3 = \max \{0, \hat{\nu} - v_{\text{max}} \} \). We have \( \hat{\theta}_1 (0, \cdot) = \hat{\nu}_2 (0, \cdot) = \hat{\nu}_3 (0, \cdot) = 0 \). Let \( S_\theta^t \) and \( S_\nu^t \) be convex open subsets of \( \mathbb{R}_+ \) where respectively functions \( \hat{\theta}_i (t, \cdot) \) and \( \hat{\nu}_i (t, \cdot) \) are positive on subsets of \( U \) with a positive measure. Then \( \forall t \in S_\theta^t \) (respectively \( S_\nu^t \)) we have

\[
\frac{d_t}{2} \left\| \hat{\theta}_1 (t, \cdot) \right\|_{L^2(U; \mathbb{R})}^2 = \int_U \hat{\theta}_1 (t, x) \partial_t f_1 (t, x) \, dx = - \int_U \alpha (t, x) \left(1 + w(t,x) \hat{\theta}_1 (t, x)\right) \hat{\theta}_1 (t, x) \, dx - \int_U K_1 (t, x) \Phi_1 (t,x, -\hat{\theta}_1(t,x), \hat{\nu}) \, dx \\
\quad - \int_U K_2 (t, x) \Phi_2 (t,x, -\hat{\theta}_1(t,x)) \, dx \leq 0
\]

\[
\frac{d_t}{2} \left\| \hat{\theta}_2 (t, \cdot) \right\|_{L^2(U; \mathbb{R})}^2 = \int_U \hat{\theta}_2 (t, x) \partial_t \hat{\theta}_2 (t, x) \, dx = \int_U \alpha (t, x) \left(1 - w(t,x) \hat{\theta}_2 (t, x)\right) \hat{\theta}_2 (t, x) \, dx - \int_U \alpha (t, x) w(t,x) \hat{\theta}_2^2 (t, x) \, dx \leq 0
\]

\[
\frac{d_t}{2} \left\| \hat{\nu}_1 (t, \cdot) \right\|_{L^2(U; \mathbb{R})}^2 = \int_U \hat{\nu}_1 (t, x) \partial_t \hat{\nu}_1 (t, x) \, dx = - \int_U \beta (t, x, -\hat{\nu}_1 (t, x)) \Phi_3 (t,x, \hat{\theta}_1, -\hat{\nu}_1(t,x)) \hat{\nu}_1 (t, x) \, dx \leq 0
\]

and

\[
\frac{d_t}{2} \left\| \hat{\nu}_2 (t, \cdot) \right\|_{L^2(U; \mathbb{R})}^2 = \int_U \hat{\nu}_2 (t, x) \partial_t \hat{\nu}_2 (t, x) \, dx \leq \int_U \beta (t, x, \hat{\nu}_2 (t, x) + v_{\text{max}}) \Phi_3 (t,x, \hat{\theta}_1, \hat{\nu}_2 (t,x) + v_{\text{max}}) \hat{\nu}_2 (t, x) \, dx \leq 0
\]

Using the above inequalities we get that functions \( \hat{\theta}_i (t, \cdot) \) and \( \hat{\nu}_i (t, \cdot) \) are nonpositive on the sets \( S_\theta^t \) and \( S_\nu^t \) which are necessary empty. We conclude that \( \forall t > 0, \left( \hat{\theta}(t, \cdot), \hat{\nu}(t, \cdot) \right) \) is valued in \([0,1] \times [0, v_{\text{max}}] \).

Now let show the existence and the uniqueness of the solution of (33 – 35). It suffices to establish existence of a local solution to conclude the result. Just as decomposition (19) a solution to (33 – 34) can be written as the sum \( \sum_{n=0}^{\infty} \hat{\theta}_n \phi_n \). Each \( \hat{\theta}_n \) satisfies the realtion

\[
\hat{\theta}_n (t) = \sum_{m=0}^{\infty} \hat{\theta}_m \int_U \phi_m (x) \phi_n (x) \, dx = \hat{\theta}_n (0) + \int_0^t \lambda_n (s) \hat{\theta}_n (s) \, ds + \int_0^t \int_U \phi_n (x) \alpha (s,x) \left(1 - w(s,x) \hat{\theta}(s,x)\right) \, dx \, ds + \int_0^t \int_U \phi_n (x) K_1 (s,x) \Phi_1 (s,x, \hat{\theta}(s,x), \hat{\nu}(s,x)) \, dx \, ds + \int_0^t \int_U \phi_n (x) K_2 (s,x) \Phi_2 (s,x, \hat{\theta}(s,x)) \, dx \, ds
\]

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Let \( \ell^2(\mathbb{R}) \) denote the Hilbert space of real-valued sequences \( \{s_n\}_{n \in \mathbb{N}} \) such that \( \langle s, s \rangle = \sum_{n \in \mathbb{N}} s_n^2 < \infty \). We set
\[
S_\theta = \left\{ s \in \ell^2(\mathbb{R}) : 0 \leq \sum_{n \in \mathbb{N}} s_n \leq 1 \right\}
\]
and
\[
S_v = \left\{ w \in L^2(U; \mathbb{R}) : \forall x \in U, w(x) \in [0, v_{\text{max}}] \right\}
\]
Let consider the linear operator \( G : D(G) \to \ell^2(\mathbb{R}) \times L^2(U; \mathbb{R}) \) where \( D(G) \subseteq \mathbb{R}_+ \times \ell^2(\mathbb{R}) \times L^2(U; \mathbb{R}) \) defined by
\[
G_1(t, y, z) = \lambda_n(t) y_n - \int_U \alpha(t, x) w(t, x) \varphi_n(x) \langle y, \varphi(x) \rangle \, dx - \int_U K_1(t, x) \varphi_n(x) \delta \langle \langle y, \varphi(x) \rangle \rangle \langle y, \varphi(x) \rangle \, dx
\]
\[
G_2(t, y, z) = 0
\]
\( G \) is the infinitesimal generator of the evolution system \( E \) defined such as \( \forall n \in \mathbb{N}, \forall t_0^+ \leq s \leq t \leq t_1, \)
\[
(E^1(s, t)(y, z))_n = \int_U \exp \left( \int_s^t (\lambda_n(\tau) - \alpha(\tau, x) w(\tau, x) - K_1(\tau, x)) \, d\tau \right) \langle y, \varphi(x) \rangle \varphi(x) \, dx
\]
\[
E^2(s, t)(y, z) = z
\]
Let also consider the operators \( F : D(F) \cap (\mathbb{R}_+ \times S_\theta \times S_v) \to \ell^2(\mathbb{R}) \times L^2(U; \mathbb{R}) \) where \( D(F) \subseteq \mathbb{R}_+ \times \ell^2(\mathbb{R}) \times L^2(U; \mathbb{R}) \) defined by
\[
F_1(t, y, z) = \int_U \varphi_n(x) (\alpha(t, x) + K_2(t, x) \overline{\gamma}(t, x, \theta(t, x), v(t, x), \rho(t, x)) (1 - \rho(t, x))) \, dx
\]
\[
+ \int_U K_1(t, x) \delta \langle \langle y, \varphi(x) \rangle \rangle \frac{(1 + \varepsilon)(z - v) \varphi_n(x)}{z} \, dx
\]
\[
+ \int_U K_1(t, x) \delta \langle \langle y, \varphi(x) \rangle \rangle \frac{v \varphi_n(x) \langle y, \varphi(x) \rangle}{z} \, dx
\]
\[
- \int_U \varphi_n(x) K_2(t, x) \overline{\gamma}(t, x, y, v(t, x), \rho(t, x)) (1 - \rho(t, x)) \, dx
\]
\[
F_2(t, y, z) = \beta(t, x, \varphi(x)) \Phi_3(t, x, \varphi(x), z)
\]
The function \( F \) is measurable with respect to the time \( t \). We can show as in the proof of proposition 3.1 that \( F \) is Lipschitz continuous with respect to \( (y, z) \) and there is a global and unique solution for the system (33) – (35). Note that \( \forall (t, x) \in \mathbb{R}_+ \times U \) the function \( \beta(t, x, \cdot) \) is Lipschitz continuous and bounded on the set \([0, 1]\) using the assumption (2.13). ■

**Theorem 4.2** Let consider the system (33) – (35) and assume that \( \hat{\gamma}(0, \cdot) = v(0, \cdot) \).

(i) If the functions \( K_1 \) and \( K_2 \) are identically null almost everywhere on \( U \) and
\[
\inf \{ \alpha(t, x) ; t > 0, x \in U \} > 0
\]
then \( \hat{\theta} \) is a globally exponentially asymptotically stable observer for \( \theta \).

(ii) If the function \( K_1 \) is identically null almost everywhere on \( U \) and \( K_2 \) is not identically null at least on a subset of \( U \) with positive measure then \( \hat{\theta} \) is a locally asymptotically stable observers for \( \theta \). Moreover, if there is a positive function \( C_\gamma \in L^\infty_{\text{loc}}(\mathbb{R}_+ \times U; \mathbb{R}_+) \) such that
\[
|\overline{\gamma}(t, x, y_1, y_2, y_3) - \overline{\gamma}(t, x, z_1, z_2, z_3)| \geq C_\gamma(t, x) \|y - z\|
\]
then \( \hat{\theta} \) is a globally exponentially asymptotically stable observers for \( \theta \).
(iii) If the function $K_2$ is identically null and
\[
\inf \left\{ \frac{v + (1 + \varepsilon - \theta) \partial v}{v} K_1 \delta \left( \hat{\theta} \right) + \alpha (t, x) w (t, x); \ (t, x) \in \mathbb{R}_+ \times U \right\} > 0. \tag{38}
\]
then $\hat{\theta}$ is a a locally exponentially asymptotically stable observers for $\theta$.

(iv) If the functions $K_1$ and $K_2$ are not identically null, and
\[
\inf \left\{ K_1 \delta \left( \hat{\theta} \right) \frac{v + (1 + \varepsilon - \theta) \partial v}{v} + K_2 \Phi_2 \right\} (t, x, \hat{\theta}) + \alpha (t, x) w (t, x); \ (t, x) \in \mathbb{R}_+ \times U \right\} > 0. \tag{39}
\]
then $\hat{\theta}$ is a locally exponentially asymptotically stable observers for $\theta$. Moreover, if the condition (37) is satisfied and
\[
K_2 (t, x) \left| \Phi_2 \left( t, \hat{\theta} (t, x) \right) \right| > K_1 (t, x) \Phi_1 \left( t, \hat{\theta} (t, x), \hat{w} (t, x) \right), \ \forall (t, x) \in \mathbb{R}_+ \times U \tag{40}
\]
then $\hat{\theta}$ is a globally exponentially asymptotically stable observers for $\theta$.

**Proof.** Let $\hat{e} = \theta - \hat{\theta}$ be the error of estimation.

(i) The error $\hat{e}$ satisfies
\[
d_t \left\| \hat{e} (t,.) \right\|_{L^2 (U)}^2 / 2 = - \int_U \alpha (t, x) w (t, x) \hat{e}^2 (t, x) dx \leq - \left\| \hat{e} (t,.) \right\|_{L^2 (U)}^2 \inf \left\{ \alpha (s, y) \right\} \tag{41}
\]
and
\[
\left\| \hat{e} (t,.) \right\|_{L^2 (U)}^2 \leq \exp \left( -2t \inf \left\{ \alpha (s, y) \right\} \left\| \hat{e} (0,.) \right\|_{L^2 (U)}^2 \right). \tag{42}
\]
We deduce the results.

(ii) The error $\hat{e}$ satisfies
\[
d_t \left\| \hat{e} (t,.) \right\|_{L^2 (U)}^2 / 2 = - \int_U \alpha (t, x) w (t, x) \hat{e}^2 dx - \int_U (A (t, x) \nabla \hat{e}, \nabla \hat{e}) dx \leq - \int_U K_2 (t, x) \Phi_2 \left( t, x, \hat{\theta} \right) \hat{e}^2 (t, x) dx \leq - \int_U \frac{v + (1 + \varepsilon - \theta) \partial v}{v} K_1 (t, x) \delta \left( \hat{\theta} \right) \hat{e}^2 dx - \int_U \alpha (t, x) w (t, x) \hat{e}^2 dx \leq - \int_U K_2 (t, x) (1 - \rho) \left( \gamma (t, x, \theta, v, \rho) - \gamma (t, \hat{\theta}, \hat{v}, \rho) \right) \hat{e} (t, x) \tag{43}
\]
and
\[
\left\| \hat{e} (t, .) \right\|_{L^2 (U)}^2 \leq \exp \left( -2t \inf \left\{ \alpha (s, y) \right\} \left\| \hat{e} (0, .) \right\|_{L^2 (U)}^2 \right). \tag{44}
\]
Moreover, if the condition (37) is satisfied then
\[
d_t \left\| \hat{e} (t, .) \right\|_{L^2 (U)}^2 \leq - \int_U 2 \alpha (t, x) w (t, x) \hat{e}^2 (t, x) dx - \int_U 2 K_2 (t, x) (1 - \rho (t, x)) C \gamma (t) \hat{e}^2 (t, x) dx \leq -2 \left\| \hat{e} (t, .) \right\|_{L^2 (U)}^2 \inf \left\{ \alpha (s, y) w (s, y) + K_2 (s, y) (1 - \rho (s, y)) C \gamma (s, y) \right\} \tag{45}
\]
and
\[
\left\| \hat{e} (t, .) \right\|_{L^2 (U)}^2 \leq \exp \left( -2t \inf \left\{ \alpha (s, y) w (s, y) + K_2 (s, y) (1 - \rho (s, y)) C \gamma (s, y) \right\} \right) \left\| \hat{e} (0, .) \right\|_{L^2 (U)}^2 . \tag{46}
\]
Since the function $C \gamma$ is positive the results holds.
(iii) The error $\hat{e}$ satisfies
\[
\frac{d}{dt}\|\hat{e}(t, .)\|_{L^2(U)}^2 / 2 = - \int_U \alpha(t, x) w(t, x) \hat{e}^2(t, x) \, dx - \int_U \left\langle A(t, x) \nabla \hat{e}^2(t, x), \nabla \hat{e}^2(t, x) \right\rangle \, dx \\
- \int_U K_1(t, x) \Phi_1(t, x, \hat{\theta}_1(t, x), \hat{v}_1(t, x)) \hat{e}^2(t, x) \, dx \\
\leq - \int_U \left( \frac{v + (1 + \varepsilon - \theta)}{v} \right) \alpha(t, x) w(t, x) \hat{e}^2(t, x) \, dx \\
- \int_U K_1(t, x) \Phi_1(t, x, \hat{\theta}_1(t, x), \hat{v}_1(t, x)) \hat{e}^2(t, x) \, dx \\
\approx - \int_U \left( \frac{K_1(t, x) \delta(\hat{\theta}) \beta(t, \theta) (q \nu_{\text{max}} (1 + \varepsilon - \theta) - v)}{\alpha \eta \nu_{\text{max}} (1 - \theta w)} \right) \hat{e}^2(t, x) \, dx \\
- \int_U \left( \alpha(t) w(t) + K_1(t) \delta(\hat{\theta}) \right) \hat{e}^2(t, x) \, dx.
\]

Using the condition (38) we deduce that $\hat{\theta}$ is a locally exponentially asymptotically stable observer for $\theta$.

(iv) Using the same arguments developed in (ii) and (iii) we get that $\hat{\theta}$ is a locally exponentially asymptotically stable observer for $\theta$. With the conditions (37) and (40) we are sure that even if the term with $\Phi_1$ introduces an instability, the stability is maintained by the term with $\Phi_2$.

\section{4.2 Numerical evaluation of the spatial model’s observers}

In this section we present numerical simulations in order to check the effectiveness of the observers we studied theoretically in the subsection 4.1. The duration of the simulations is the year. Parameters are taken following [12] and generalize those in the subsection 3.2. We assume an anisotropic radial control strategy $u$ given for every $(t, x) \in \mathbb{R}_+ \times U$,
\[
u(t, x) = \sin^2 \left( \|M (x - x_0)\| \right) \sin^2 \left( \omega_1 (t - \varphi_1) \right) \exp \left( -\omega_2 (t - \varphi_2) \right).
\]

The matrix $M$ introduces a geometrical anisotropy in the distribution of $u$. The functions $\alpha$, $\beta$ and $\gamma$ are taken such as
\[
\alpha(t, x) = p_1(t, x) + b_1 q_1(x) (1 - \cos(c_1 t)) (t - d_1)^2, \quad (t, x) \in \mathbb{R}_+ \times U
\]
\[
\beta(t, x, y) = b_2 q_2(x) (1 - \cos(c_2 t)) (t - d_2)^2 p_2(y), \quad (t, x, y) \in \mathbb{R}_+ \times U \times \mathbb{R}
\]
and
\[
\gamma(t, x, y_1, y_2, y_3) = b_3 q_3(x) (1 - \cos(c_3 t)) (t - d_3)^2 (y_1 - \kappa y_3) y_2, \quad (t, x, y) \in \mathbb{R}_+ \times U \times \mathbb{R}^3
\]
p_1 is a nonnegative function of the time, $p_2$ is a positive function and for every $i \in \{1, 2, 3\}$, $q_i$ is a real anisotropic radial nonnegative function of the space variable as the control $u$. For every $i \in \{1, 2, 3\}$, $b_i, c_i,$ and $d_i$ are positive coefficients corresponding respectively to the maximal amplitude, the pulsation and the global maximum of $\alpha$, $\beta$ and $\gamma$. For every $i \in \{1, 2, 3\}$, $q_i$ is a $[0, 1]$-valued function of the space variable. The matrix $A$ in equation (11) is assumed to be equal to $10^{-2}I$ because the diffusion is relatively slow. The functions $K_1$ and $K_2$ are taken constant and are similar with $k_1$ and $k_2$ defined in subsection 3.2. The rest of the parameters are the same described in the table 1. The initial conditions are also taken following subsection 3.2 and are constant with respect to the space variable in order to easily fulfill the boundary conditions (2). The following table specifies some assumed values of additional parameters: For every $i \in \{1; 2; 3\}$,
\[ q_i(x) = \frac{\sin^2(\|M_i(x-x_i)\|^2) + 1}{2} \]

Table 2: Simulation parameters for the spatial model

In the remainder of the subsection we present simulations of the observers and their estimation absolute relative error dynamics. To keep the representation simple we just plot the spatial minimum values, the spatial average values and the spatial maximum values. Instead of the estimation relative error we consider the estimation absolute relative error to avoid compensations when averaging. The initial condition \( \rho(0,.) \) is always taken equal to \( \theta(0,.) \). Indeed, following simulations in subsection 3.2 the performances of the observers decrease with respect to \( \rho(0,.) \) and on the other hand, it seems realistic to take \( \rho(0,.) \) less than \( \theta(0,.) \).

4.2.1 Simulations with \( \theta(0,.) = v(0,.) = 0.05 \)

The figures 9 and 10 show the behaviour of the observers when the observation is started early. As we can see the global performance of each observer is relatively good but the case \( k_1 = 0 \) and \( k_2 = 10^3 \) seems to be the better one.

\[ k_1 = k_2 = 0 \quad k_1 = 0 \text{ and } k_2 = 10^3 \]

\[ k_1 = 10^3 \text{ and } k_2 = 0 \quad k_1 = 10^3 \text{ and } k_2 = 10^3 \]

Figure 9: Inhibition rate and observers for \( \theta(0,.) = v(0,.) = 0.05 \).
\[ k_1 = k_2 = 0 \] \hspace{2cm} \[ k_1 = 0 \text{ and } k_2 = 10^3 \]

\[ k_1 = 10^3 \text{ and } k_2 = 0 \] \hspace{2cm} \[ k_1 = 10^3 \text{ and } k_2 = 10^3 \]

Figure 10: Relative absolute estimation error for \( \theta (0,.) = v (0,.) = 0.05 \).
4.2.2 Simulations with $\theta(0,.) = 0.05$ and $v(0,.) = 0.5$

The figures [11] and [12] correspond to an observation starting at the beginning of the disease but with a well developed berry. We have better performances when $k_2 = 10^3$. With $k_1 = 10^3$ and $k_2 = 0$ we notice an instability behaviour in figure [12] although the error takes the null value. That may be due to the constant sign of the function $\phi_1$.

\[
\begin{align*}
  k_1 = k_2 &= 0 
  &\quad k_1 = 0 \text{ and } k_2 = 10^3 \\
  k_1 = 10^3 \text{ and } k_2 &= 0 
  &\quad k_1 = 10^3 \text{ and } k_2 = 10^3
\end{align*}
\]

Figure 11: Inhibition rate and observers for $\theta(0,.) = 0.05$ and $v(0,.) = 0.5$.  

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Figure 12: Relative absolute estimation error for $\theta(0, .) = 0.05$ and $v(0, .) = 0.5$. 

$k_1 = k_2 = 0$  \quad $k_1 = 0$ and $k_2 = 10^3$

$k_1 = 10^3$ and $k_2 = 0$  \quad $k_1 = 10^3$ and $k_2 = 10^3$
4.2.3 Simulations with $\theta (0, .) = 0.75$ and $v (0, .) = 0.05$

In the figures 13 and 14 the inhibition rate is already high while the berry is little. As expected the worst estimation corresponds to $k_1 = k_2 = 0$. The best estimations are still made when $k_2 = 10^3$. However, when $k_1 = 10^3$, the same unstability behaviour in figure 12 appears.

\[ k_1 = k_2 = 0 \quad \text{and} \quad k_1 = 0 \text{ and } k_2 = 10^3 \]

\[ k_1 = 10^3 \text{ and } k_2 = 0 \quad \text{and} \quad k_1 = 10^3 \text{ and } k_2 = 10^3 \]

Figure 13: Inhibition rate and observers for $\theta (0, .) = 0.75$ and $v (0, .) = 0.05$. 

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$k_1 = k_2 = 0$ \hspace{1cm} $k_1 = 0$ and $k_2 = 10^3$

$k_1 = 10^3$ and $k_2 = 0$ \hspace{1cm} $k_1 = 10^3$ and $k_2 = 10^3$

Figure 14: Relative absolute estimation error for $\theta(0, .) = 0.75$ and $v(0, .) = 0.05$. 
4.2.4 Simulations with $\theta(0,.) = 0.75$ and $v(0,.) = 0.5$

For the figures 15 and 16 the inhibition rate is already high and the fruit is relatively mature when the observation starts. The worst case are still when $k_2 = 0$ and the best are with $k_2 = 10^3$.

\[
\begin{align*}
k_1 &= k_2 = 0 & \quad k_1 &= 0 \text{ and } k_2 = 10^3 \\
k_1 &= 10^3 \text{ and } k_2 = 0 & \quad k_1 &= 10^3 \text{ and } k_2 = 10^3
\end{align*}
\]

Figure 15: Inhibition rate and observers for $\theta(0,.) = 0.75$ and $v(0,.) = 0.5$. 

$k_1 = k_2 = 0$ \hspace{1cm} $k_1 = 0$ and $k_2 = 10^3$

$k_1 = 10^3$ and $k_2 = 0$ \hspace{1cm} $k_1 = 10^3$ and $k_2 = 10^3$

Figure 16: Relative absolute estimation error for $\theta (0,.) = 0.75$ and $v (0,.) = 0.5$. 
5 Conclusion and prospects

We have studied four observers for the dynamics of anthracnose disease. Our constructions were based on the models given in [12]. We consider two models: a spatial version and a non spatial one stated respectively by the equations (1) – (6) and (7) – (10). Some changes have been made in the original form of those models in order to be more realistic. Precisely, we have changed the equations modelling the dynamics of the berry volume \((v)\) and the rot volume \((v_r)\). Indeed, before the launch of the disease, the berry has to reach a minimal volume. On the other hand the rot volume shall never be greater than total volume. We found that the system describing the disease dynamics can be viewed as the natural observer of the system even the performances are poor. We have studied theoretically two observers which gives fairly good results looking at the relative error. Several simulations have been carried out to assess the effectiveness of our proposed observers. That work has been done for the spatial and the non spatial models.

Although proposed observers seem to display fairly good results, we think that a stochastic approach could be introduced in the estimation process. Indeed, the parameters of the models are submitted to several stochastic variations. Moreover, the measures of \(v\) and \(v_r\) also contains errors. A mean to take into account those errors could consist in adding noises in our models and proceed to the stochastic filtering by different existing tools. That aspect is studied in on going works of some of the authors of the current paper.

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