ALMOST CRITICAL REGULARITY OF NON-ABELIAN CHERN-SIMONS-HIGGS SYSTEM IN THE LORENZ GAUGE

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Abstract. In this paper we consider a Cauchy problem on the self-dual relativistic non-abelian Chern-Simons-Higgs model, which is the system of equations of $\text{su}(n)$ ($n \geq 2$)-valued matter field $\phi$ and gauge field $A$. Since the system is non-abelian, it is extremely hard to reveal all the null structure. Instead we use angular Whitney decomposition of $[23]$ based on the frequency localization as well as the null structure. By a technical reason we take $\ell^1$ summation over the localized frequencies, which results in the local well-posedness in Besov space $B_{2,1}^{3/4} \times B_{2,1}^{1/2}$. We also prove that the solution flow map $(\phi(0), A(0)) \mapsto (\phi(t), A(t))$ fails to be $C^2$ at the origin of $H^s \times H^\sigma$ when $\sigma < \frac{1}{4}$ regardless of $s \in \mathbb{R}$. This means the regularity $B_{2,1}^{3/4}$ is almost critical.

1. Introduction

The Chern-Simons theory effectively describes $1+2$ dimensional physical phenomena especially in condensed matter physics. In general, particles interacting via the abelian Chern-Simons gauge acquire fractional statistics, which play a role in the fractional quantum Hall effect and also in high temperature superconductivity [1]. After Chern and Simons first introduced geometric invariants [6], Chern-Simons gauge theory gets a lot of interest in physicists and mathematicians. For instance see [4, 9, 10] and references therein.

Recently, many mathematicians have studied various dispersive partial differential equations coupled with Chern-Simons gauges, especially, Chern-Simons-Dirac system (CSD) and Chern-Simons-Higgs system (CSH) under several gauges; temporal gauge $A_0 = 0$, Coulomb gauge $\partial^j A_j = 0$, and Lorenz gauge $\partial^\mu A_\mu = 0$. It is well-known that in Lorenz gauge, (CSD) is rewritten as a system of nonlinear wave equation coupled with Dirac equation. The authors of [13, 14, 15, 22] studied low regularity solutions to (CSD) under the Lorenz gauge condition. In fact, Sobolev space $H^{3/4}$ is expected to be the critical space in the sense of well-posedness [7, 21]. On the other hand, (CSD) becomes a cubic Dirac equation with an elliptic structure in the Coulomb gauge. A number of results on low regularity solutions to (CSD) in the Coulomb gauge have appeared in [3, 21]. Furthermore, the presence of elliptic structure in the Coulomb gauge allows the almost critical regularity in view of scaling invariance [12].

At the same time, the abelian Higgs model in $1+2$ dimensions has attracted attention by mathematicians as well as physicists and then a lot of works on local well-posedness of abelian (CSH) for low regularity initial data under the Lorenz gauge condition are given by [2, 13, 15, 20, 30]. On the other hand, the non-abelian Chern-Simons-Higgs system has also gained attention in physical research [16, 18], however, the mathematical analysis on non-abelian case is not sufficient yet. (See [5, 31] for the previous work.) Inspired by the antecedent works, in this paper we study and provide a mathematically rigorous and almost optimal

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local well-posedness theory on the self-dual relativistic non-abelian Chern-Simons-Higgs system in the Lorenz gauge.

Let us begin with the mathematical setup for the non-abelian Chern-Simons-Higgs system. Let $G$ be a compact Lie group and $g$ its Lie algebra. For the sake of simplicity, we shall assume $G = SU(n, \mathbb{C})$, $n \geq 2$ (the group of unitary matrices of determinant one). Then $g = su(n, \mathbb{C})$ is the algebra of trace-free skew-Hermitian matrices whose infinitesimal generators are denoted by $T^a$ $(a = 1, 2, \cdots, n^2 - 1)$, and are traceless Hermitian matrices. (For example, $g = su(2, \mathbb{C})$, $T^a$ is chosen to be Pauli matrices; $i\sigma^a$, $a = 1, 2, 3$.)

For a given $g$-valued gauge field $A$, the component $A_\mu$ is written as $A_\mu(t, x) = A_{\mu, a}(t, x)T^a$ $(a = 1, \cdots, n^2 - 1)$. We then define the curvature $F = dA + [A, A]$. More explicitly, given $A_\mu : \mathbb{R}^{1+2} \to g$, we define $F_{\mu\nu}$ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

The associated covariant derivative is denoted by $D_\mu = \partial_\mu + [A_\mu, \cdot]$. The matter field $\phi$ is also assumed to be $su(n)$-valued function and thus $\phi = \phi_a T^a$ because the most natural and interesting physical case seems to be with the matter fields and gauge fields in the same Lie algebra representation $[9, 10]$. Throughout this paper, we adopt the Einstein summation convention, where Greek indices refer to 0,1,2 and Latin indices $j, k$ refer to 1,2. Indices are raised or lowered with respect to the Minkowski metric $\eta$ with signature $(+, -, -)$.

The Lagrangian density of the 1 + 2 dimensional non-abelian relativistic Chern-Simons-Higgs system is defined by

$$\mathcal{L} = -\frac{\kappa}{2} \epsilon^{\mu\nu\alpha} \text{Tr}(\partial_\mu A_\nu A_\alpha + \frac{2}{3} A_\mu A_\nu A_\alpha) + \text{Tr}((D_\mu \phi)^\dagger (D^\mu \phi)) - V(\phi, \phi^\dagger),$$

where $V(\phi, \phi^\dagger)$ is the Higgs potential given by

$$V(\phi, \phi^\dagger) = \frac{1}{\kappa^2} \text{Tr}([[\phi, \phi^\dagger], \phi] - v^2 \phi^\dagger([[\phi, \phi^\dagger], \phi] - v^2 \phi)).$$

Higgs potential is of sixth order and self-dual form, which is designed for a lower bound of energy. The constant $v > 0$ measures either the scale of the broken symmetry or the subcritical temperature of the system $[29]$. The $\epsilon^{\mu\nu\alpha}$ is the totally skew-symmetric tensor with $\epsilon^{012} = 1$. $\text{Tr}A$ and $A^\dagger$ denote the trace and the complex conjugate transpose of a matrix $A$, respectively. $[A, B] = AB - BA$ is the matrix commutator.

The Euler-Lagrange equation of the above Lagrangian density is

$$\begin{cases} 
D_\mu D^\mu \phi = -V(\phi, \phi^\dagger), \\
F_{\mu\nu} = \epsilon_{\mu\nu\alpha} J^\alpha,
\end{cases}$$

(1.1)

where $J^\mu$ is defined by $J^\mu = [\phi^\dagger, D^\mu \phi] - [(D^\mu \phi)^\dagger, \phi]$ and $V(\phi, \phi^\dagger)$ is derived by $V(\phi, \phi^\dagger) = \frac{\partial V(\phi, \phi^\dagger)}{\partial \phi^\dagger}$. For simplicity, we assumed that the coupling constant $\kappa$ in front of $F_{\mu\nu}$ is 1 in this paper. The potential $V$ consists of linear, cubic, and quintic terms of $\phi$ and $\phi^\dagger$. (For details, see Appendix below.) In particular, it contains linear term $-2v^4 \phi$ which contributes as a Higgs mass $m = \sqrt{2} v^2$ and gives a relativistic nature to $[11]$. The initial data set for the system comprises $(f, g, a_0, a_1, a_2)$, where $(\phi, \partial_t \phi)(0, x) = (f(x), g(x))$ and $A_\mu(0) = a_\mu$. 

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Now we take $\partial^\mu$ of the second equation in (1.1) and use the Lorenz gauge $\partial^\mu A_\mu = 0$. Then we arrive at the following system of wave equations which describes the time evolution of the fields $A_\mu, \phi$.

$$
\begin{align*}
\Box \phi &= -2 [A^\mu, \partial_\mu \phi] - [A_\mu, [A^\mu, \phi]] - \mathcal{V}(\phi, \phi^\dagger), \\
\Box A_\mu &= [\partial^\nu A^\mu, \partial_\nu A_\mu] - \epsilon_{\mu\nu\alpha} (Q^\nu(\partial^\alpha, \phi) + Q^\rho(\partial^\alpha, \phi^\dagger)) - \epsilon_{\mu\nu\alpha} \partial^\nu \left(\left[\phi^\dagger, [A^\alpha, \phi]\right] - \left[[A^\alpha, \phi]^\dagger, \phi\right]\right) \\
(\phi, \partial_t \phi)(0) &= (f, g), \quad A_\mu(0) = \delta_\mu, \\
(\partial_\alpha A_\mu)(0) &= -\partial_\alpha a_j, \quad (\partial_\alpha A_j)(0) = \partial_\alpha A_0 - [a_0, a_j] + \epsilon_{0jk}([f^\dagger, \partial^k f] - [(\partial^k f)^\dagger, f]),
\end{align*}
$$

(1.2)

where $Q_{\alpha\beta}(u, v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v$. Note that due to the Lorenz gauge condition the initial data should satisfy the following constraint equation:

$$
\partial_1 a_2 - \partial_2 a_1 + [a_1, a_2] = [f^\dagger, gf + [a_0, f]] - [(gf)^\dagger + [a_0, f]^\dagger, f].
$$

(1.3)

We state our result on the local well-posedness:

**Theorem 1.1.** Suppose that $(f, g) \in B^{\frac{4}{3}}_{2,1} \times B^{\frac{4}{3}}_{2,1}, a_\mu \in B^{\frac{4}{3}}_{2,1}$ and they satisfy (1.3). Then the (1.2) is locally well-posed in $B^{\frac{4}{3}}_{2,1} \times B^{\frac{4}{3}}_{2,1} \times B^{\frac{1}{2}}_{2,1}$. That is, there exists $T = T(f, g, a_\mu, m) > 0$ such that there exist unique solution $(\phi, \partial_t \phi, A_\mu) \in C((0, T); B^{\frac{4}{3}}_{2,1} \times B^{\frac{1}{2}}_{2,1} \times B^{\frac{1}{2}}_{2,1})$ of (1.2), which depends continuously on the initial data.

Here, $B^{s}_{2,1}$ is the usual Besov space whose norm is given by $\|f\|_{B^{s}_{2,1}} = \sum_{N} \|P_{\xi} f\|_{L^2}$, where $P_{\xi}$ is the Littlewood-Paley projection on $\{\xi \in \mathbb{R}^2 : |\xi| \sim N\}$. Similarly, Sobolev space $H^s$ is defined by $\ell^2$-sum. Instead of applying global estimates by [8], we make fully use of localization of space-time Fourier side and adopt the Besov type $X^{s,b}$ space. Thanks to the dyadic decomposition of space-time frequencies, we gain a well-posedness of lower regularity compared to [31]. Also, we exploit the null structure hidden in (1.2) as [31]. In fact, this null structure has a similar form as Yang-Mills equation in $1 + 3$ dimensions introduced in [25, 27].

The essence of proof of Theorem 1.1 heavily depends on the argument by [7] and [12]. To treat bilinear estimates, as the present authors have seen in [7], we exclude the low-low-high modulation with high-high-low and low-high-high frequency cases and then apply angular Whitney decomposition and null form estimates by [23].

When we are concerned with trilinear estimates, however, we encounter different situations other than [12], where one can enjoy the null structure in cubic Dirac equations. Since we cannot reveal the null structure in cubic terms of $A^\mu, \phi$, we do not expect to use null form estimates. Instead, by angular Whitney decomposition with well-separation of angular sectors and bilinear estimates [3, 6], we obtain even better estimates than the required estimates. We also remark that in dealing with Higgs potential, direct use of 2D bilinear estimates of wave type is sufficient to obtain the required estimates.

Let us now deal with the smoothness of the flow map $(\phi(0), A(0)) \to (\phi(t), A(t))$. Since the nonlinearity is algebraic, one may expect the flow will be smooth in local time if the problem is well-posed. However, such smoothness can be shown to fail when the initial data of $\phi, A$ are rougher than in $H^\frac{2}{3}, H^\frac{4}{3}$, respectively, which can be stated as follows.

**Theorem 1.2.** Let $s \in \mathbb{R}, \sigma < \frac{1}{4}$, and $T > 0$. Then the flow map of $(\phi(0), A(0)) \to (\phi(t), A(t))$ from $H^s \times H^\sigma$ to $C([-T, T]; H^s \times H^\sigma)$ cannot be $C^2$ at the origin. Furthermore, if $s < \frac{1}{4}, \sigma \in \mathbb{R}$, then the flow map cannot be $C^3$ at the origin.
We proceed the proof by the argument of Knapp type example as in [19]. We investigate carefully $su(n)$-valued initial data which guarantee that a resonance of phases occurs and hence the Fourier transforms of matter field and gauge field have significant lower bounds, which enable us to get the necessary condition $s \geq \frac{1}{2}$ or $\sigma \geq \frac{1}{4}$ for the smoothness. In view of Theorem 1.2, the LWP of Theorem 1.1 is very sharp since $B_{2,1}^\frac{1}{4}$ is a proper subspace of $H^\frac{1}{4}$. For the present we could not have filled the regularity or failure of smoothness fully in $H^\frac{1}{4}$ by a technical reason. However, the problem will be hopefully resolved in the near future.

We end this section with the introduction of notations and organization of this paper.

**Notations.** Here we give some notations used throughout this paper. Since we only use $L^1$ means that $N$ dyadic numbers such as $N = N\{1,2,\ldots\}$, the symbol is $\mu \in \mathbb{R}$.

The spatial Fourier transform and space-time Fourier transform on $\mathbb{R}^2$ are defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x) dx,$$

$$\tilde{u}(X) = \int_{\mathbb{R}^1+2} e^{-i(t \tau + x \cdot \xi)} u(t,x) dtdx,$$

where $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^2$, and $X = (\tau, \xi) \in \mathbb{R}^{1+2}$. Also we denote $\mathcal{F}(u) = \tilde{u}$. Then we define space-time Fourier projection operator $P_E$ by $P_E \tilde{u}(\tau, \xi) = \chi_E \tilde{u}(\tau, \xi)$, for $E \subset \mathbb{R}^{1+2}$. We define spatial Fourier projection operator, similarly. For example, $P_{|\xi|\sim N}$ is the Littlewood-Paley projection on $|\xi| \in \mathbb{R}^2 : |\xi| \sim N$.

Since we prefer to use the differential operator $|\nabla|$ rather than $-i\nabla$, for the sake of simplicity, we put $D := |\nabla|$ whose symbol is $|\xi|$.

For brevity, we denote the maximum, median, and minimum of $N_0$, $N_1$, $N_2$ by

$$N_{\text{max}}^{012} = \max(N_0, N_1, N_2), \quad N_{\text{med}}^{012} = \text{med}(N_0, N_1, N_2), \quad N_{\text{min}}^{012} = \min(N_0, N_1, N_2).$$

**Organization.** In Section 2, we introduce the decomposition of d’Alembertian and Besov type $X^{s,b}$ space. Section 3 is devoted to the description on our main techniques; 2D wave type bilinear estimates and angular Whitney decomposition and on null structure. In Section 4, we give main estimates to prove the local well-posedness of (1.2). Here, we observe that the estimates of commutator of $su(n)$-valued functions are reduced to the nonlinear estimates of C-valued functions. Then Section 5,6,7 are on the proof of bilinear estimates, trilinear estimates, and estimates of Higgs potential, respectively. In Section 8 we show the failure of smoothness.

**2. Preliminaries**

2.1. **Decomposition of d’Alembertian.** We use the standard transform given by $(\phi, \partial_t \phi) \to (\phi_+, \phi_-)$ and $(A_\mu, \partial_t A_\mu) \to (A_{\mu,+}, A_{\mu,-})$ with

$$\phi_{\pm} = \frac{1}{2} \left( \phi \pm \frac{1}{iD} \partial_t \phi \right), \quad A_{\mu,\pm} = \frac{1}{2} \left( A_\mu \pm \frac{1}{iD} \partial_t A_\mu \right).$$

Then the system (1.2) transforms to

$$\left\{ \begin{aligned}
(i\partial_t \pm D)\phi_{\pm} &= \pm \frac{1}{2D} \left( -2 [A_\mu, \partial_\mu \phi] - [A_\mu, [A_\mu, \phi]] - \nabla(\phi, \phi^\dagger) \right), \\
(i\partial_t \pm D)A_{\mu,\pm} &= \pm \frac{1}{2D} \left( \partial^\nu [A_\mu, A_\nu] - \epsilon_{\mu\nu\sigma} (Q^\sigma (\phi^\dagger, \phi) + Q^\sigma (\phi, \phi^\dagger)) \right) \\
&\quad + \frac{1}{2D} \partial^\nu \left( [\phi^\dagger, [A_\nu, \phi]] - [A_\nu, \phi^\dagger, \phi] \right).
\end{aligned} \right. \tag{2.1}$$
2. Function spaces. As we advert in Section 1, we adopt the Besov type $X^{s,b}$ space. The $\ell^1$ summation in Besov space enables us to focus on lower regularity and uniqueness of solutions to (1.2) instead of using $\ell^2$ summation in Sobolev space $H^s$. In this section, we also introduce the energy estimate lemma in our function space. We define the function spaces as follows.

**Definition 2.1.** Let $s, b \in \mathbb{R}$. Let $N, L \geq 1$ be dyadic number. We define Besov type $X^{s,b}$ space by

$$B^{s,b}_{\pm,1} = \left\{ u \in L^2 : \|u\|_{B^{s,b}_{\pm,1}} = \sum_{N,L \geq 1} N^s L^b \|P_{K_{N,L}}^- u\| < \infty \right\}$$

and

$$B^{s,b;\infty} = \left\{ u \in L^2 : \|u\|_{B^{s,b;\infty}} = \sup_{N,L \geq 1} N^s L^b \|P_{K_{N,L}}^- u\| < \infty \right\},$$

where

$$K_{N,L}^\pm = \{(\tau, \xi) \in \mathbb{R}^{1+2} : |\xi| \sim N, |\tau \pm |\xi|| \sim L \}.$$  

Since we are only concerned with local time existence $T \leq 1$ throughout this paper, it is convenient to utilize our function space in the local time setting. Hence we introduce the restriction space. The time-slab which is the subset of $\mathbb{R}^{1+2}$ is given by

$$S_T = (-T, T) \times \mathbb{R}^2.$$  

Then we define the restriction norm $B^{s,b;1}_{\pm}(S_T)$ for a function $u$ on the time slab $S_T$ by

$$\|u\|_{B^{s,b;1}_{\pm}(S_T)} = \inf_{v = u \text{ on } S_T} \|v\|_{B^{s,b;1}_{\pm}},$$

and $B^{s,b;\infty}(S_T)$ is defined in the similar way. This becomes a semi-norm on $B^{s,b;1}_{\pm}$, but is a norm if we identify elements which agree on $S_T$ and the resulting space is denoted $B^{s,b;1}_{\pm}(S_T)$. In other words, $B^{s,b;1}_{\pm}(S_T)$ is the quotient space $B^{s,b;1}_{\pm}/X$, where $X = \{ v \in B^{s,b;1}_{\pm} : v = 0 \text{ on } S_T \}$.

Now we introduce the energy estimate lemma.

**Lemma 2.2.** Let us consider the integral equation:

$$v(t) = e^{\mp \tau d} f + \int_0^t e^{\mp (t-t')d} F(t') dt'.$$

with sufficiently smooth $f$ and $F$. If $T \leq 1$, then for any $s \in \mathbb{R}$ we have

$$\|v\|_{B^{s,1}_{\pm} (S_T)} \lesssim \|f\|_{B^{s,1}_{\pm}} + \|F\|_{B^{s,1}_{\mp} (S_T)}.$$  

**Proof.** Appendix in [2].

If integral equation has a special form, then we obtain the following lemma.

**Lemma 2.3.** Let us consider the integral equation:

$$w(t) = \int_0^t \frac{e^{\mp (t-t')D}}{D} F(t') dt'.$$

with sufficiently smooth $f$ and $F$. If $T \leq 1$, then for any $s \in \mathbb{R}$ and some $\epsilon > 0$ we have

$$\|w\|_{B^{s,1}_{\pm} (S_T)} \lesssim T^\epsilon \|F\|_{B^{s,1}_{\mp} (S_T)}.$$  

Proof. The proof is easily followed by Appendix in [7]. Indeed, we let $G \in B^{s,b}_{\pm} (S_T)$ be any representative of $F \in B^{s,b}_{\pm} (S_T)$. Taking space-time Fourier transform we have

$$
\tilde{w}(\tau, \xi) = \frac{\hat{G}(\tau, \xi)}{i(\tau \pm |\xi|) |\xi|} - \delta(\tau \pm |\xi|) \int \frac{\hat{G}(\lambda, \xi)}{i(\lambda \pm |\xi|) |\xi|} d\lambda.
$$

We split $G = G_1 + G_2$ corresponding to the Fourier domain $|\tau \pm |\xi|| \ll \frac{1}{T}$ and $|\tau \pm |\xi|| \gg \frac{1}{T}$. We write $w = w_1 + w_2$ accordingly. By Taylor’s expansion, we write $w_1(t) = \sum_{n \geq 1} \frac{\tau^n}{n!} e^{i\tau t} f_n$, where $f_n(\xi) = \int (i(\lambda \pm |\xi|))^{n-1} |\xi|^{-1} \chi_{|\lambda| \leq |\xi|} \hat{G}(\lambda, \xi) d\lambda$. Then $\|f_n\|_{B^{s,b}_{\pm}} \lesssim T^{-n + \frac{1}{2} + b} \|G\|_{B^{s,b}_{\pm}}$ for $b < \frac{1}{2}$. From this we obtain $\|w_1\|_{B^{s,b}_{\pm}} \lesssim T^\frac{1}{2} + b \|G\|_{B^{s,b}_{\pm}}$, where the last inequality is followed by the definition of $B^{s,b}_{\pm}$.

To estimate $w_2$, we split $w_2 = a - b$ where

$$
\tilde{a}(\tau, \xi) = \frac{\chi_{|\tau \pm |\xi||\gg 1/T} \hat{G}(\tau, \xi)}{i(\tau \pm |\xi|) |\xi|}, \quad \tilde{b}(\tau, \xi) = \delta(\tau \pm |\xi|) \int \frac{\chi_{|\tau \pm |\xi||\gg 1/T} \hat{G}(\lambda, \xi)}{i(\lambda \pm |\xi|) |\xi|} d\lambda.
$$

Then we have $\|a\|_{B^{s,b}_{\pm}} \lesssim T^\frac{1}{2} + b \|G\|_{B^{s,b}_{\pm}} \lesssim T^\frac{1}{2} + b \|G\|_{B^{s,b}_{\pm}}$. Here we put $\epsilon = \frac{1}{2} + b > 0$. This completes the proof of Lemma 2.3.

3. Bilinear estimates and Null structure

3.1. Bilinear estimates. For dyadic $N, L \geq 1$, let us invoke that

$$K_{N,L}^{\pm} = \{ (\tau, \xi) \in \mathbb{R}^{1+2} : |\xi| \sim N, |\tau| \sim |\xi| \sim L \}.$$

Now we introduce the key ingredient to handle the nonlinear terms in (1.2).

**Theorem 3.1** (Theorem 2.1 of [23]). For all $u_1, u_2 \in L^2_t \mathbb{R}^{1+2}$ such that $\tilde{u}_j$ is supported in $K_{N_j,L_j}^{\pm}$, the estimate

$$\|P_{K_{N_0,L_0}^{\pm}} (u_1 \cdot \nabla u_2)\| \leq C \|u_1\| \|u_2\|$$

holds with

(3.1) $C \sim (N^{012}_{\text{min}} L^{12}_{\text{min}})^{\frac{1}{4}} (N^{12}_{\text{min}} L^{12}_{\text{max}})^{\frac{1}{4}},$

(3.2) $C \sim (N^{012}_{\text{min}} L^{0j}_{\text{min}})^{\frac{1}{4}} (N^{0j}_{\text{min}} L^{0j}_{\text{max}})^{\frac{1}{4}}, \quad j = 1, 2,$

(3.3) $C \sim (N^{012}_{\text{min}} L^{012}_{\text{min}})^{\frac{1}{4}}$

regardless of the choices of signs $\pm j$.

3.2. Angular Whitney decomposition. While proceeding the proof we encounter the most serious case; low-low-high modulation with high-high-low frequency, low-high-low frequency and high-low-high frequency. Fortunately, we can exclude them by the support condition among space-time localized waves (see Section 5.1). Then the remaining case in the summation is the low-low-high modulation with all input and output frequencies compatible, for which we cannot expect further cancellation. To deal with it, we first apply the angular Whitney decomposition of [23] as follows: For $\gamma, r > 0$ and $\omega \in \mathbb{S}^1$, where $\mathbb{S}^1 \subset \mathbb{R}^2$ is the unit circle, we define

$$\Gamma_\gamma(\omega) = \{ \xi \in \mathbb{R}^2 : \angle (\xi, \omega) \leq \gamma \},$$

$$T_\gamma(\omega) = \{ \xi \in \mathbb{R}^2 : |P_{\omega} \xi| \leq r \},$$

and denote by $\mathcal{A}$ the bilinear form with

$$\mathcal{A}(u_1, u_2) = \int_{\mathbb{R}^2} \int \int \int \mathcal{A}(u_1, u_2)(\xi, \eta, \xi', \eta') \hat{u}_1(\xi) \hat{u}_2(\eta) d\xi d\eta d\xi' d\eta'.$$
where $P_{\omega^\perp}$ is the projection onto the orthogonal complement $\omega^\perp$ of $\omega$ in $\mathbb{R}^2$. In other words, the set $T_r(\omega)$ is a strip of radius compatible to $r$ about $R\omega$.

By $\Omega(\gamma)$ we denote a maximal $\gamma$-separated subset of the unit circle. Then for $\xi_1, \xi_2 \in \mathbb{R}^2 \setminus \{0\}$ with $\angle(\omega_1, \omega_2) > 0$, we have

$$1 \sim \sum_{\begin{smallmatrix} 0 < \gamma < 1 \\ \gamma: \text{dyadic} \end{smallmatrix}} \sum_{\omega_1, \omega_2 \in \Omega(\gamma)} \chi_{\Gamma_\gamma(\omega_1)}(\xi_1) \chi_{\Gamma_\gamma(\omega_2)}(\xi_2). \quad (3.4)$$

Note that the lower bound $\angle(\omega_1, \omega_2) \geq 3\gamma$ says that the sectors $\Gamma_\gamma(\omega_1)$ and $\Gamma_\gamma(\omega_2)$ are well-separated. If this separation is not required, we then have the following modification. For $0 < \gamma < 1$ and $k \in \mathbb{N}$, we have

$$\chi_{\angle(\xi_1, \xi_2) \leq k\gamma} \lesssim \sum_{\omega_1, \omega_2 \in \Omega(\gamma)} \chi_{\Gamma_\gamma(\omega_1)}(\xi_1) \chi_{\Gamma_\gamma(\omega_2)}(\xi_2), \quad (3.5)$$

for all $\xi_1, \xi_2 \in \mathbb{R}^2 \setminus \{0\}$ with $\angle(\omega_1, \omega_2) > 0$.

Second, we apply the following null form estimate.

**Theorem 3.2** (Theorem 2.3 of [23]). Let $r > 0$ and $\omega \in S^1$. Then for all $u_1, u_2 \in L^2_{t,x}(\mathbb{R}^{1+2})$ such that $\vec{u}_j$ is supported in $K^{\pm j}_{N_j, L_j}$, we have

$$\|B_{\theta_{12}}(P_{T_r(\omega)} u_1, u_2)\| \lesssim (rL_1L_2)^\frac{1}{2} \|u_1\| \|u_2\|. \quad (3.6)$$

Here, the bilinear form $B_{\theta_{12}}(u_1, u_2)$ is defined on the Fourier side by inserting the angle $\theta_{12} = \angle(\pm_1 \xi_1, \pm_2 \xi_2)$ in the convolution of $u_1$ and $u_2$; that is,

$$\mathcal{F}B_{\theta_{12}}(u_1, u_2)(X_0) = \int_{X_0 = X_1 + X_2} \angle(\pm_1 \xi_1, \pm_2 \xi_2) \vec{u}_1(X_1) \vec{u}_2(X_2) \, dX_1 \, dX_2. \quad (3.7)$$

In the proof of Theorem 1.1 we need to treat trilinear estimates but we cannot use null structure and hence we cannot invoke null form estimates. However, by appealing the separation between angular sectors, we get the similar estimates as follows (See [23, 24]):

$$\|P_{K^{\pm 0}_{N_0, L_0}} (u_1^{\gamma_{\omega_1}} u_2^{\gamma_{\omega_2}})\| \lesssim \left(\frac{rL_1L_2}{\gamma}\right)^{\frac{1}{2}} \|u_1^{\gamma_{\omega_1}}\| \|u_2^{\gamma_{\omega_2}}\|. \quad (3.8)$$

Here, $r > 0$ is the radius of a strip in which $P_{K^{\pm 0}_{N_0, L_0}} (u_1^{\gamma_{\omega_1}} u_2^{\gamma_{\omega_2}})$ is contained and $u_j^{\gamma_{\omega_j}} = P_{\pm_1 \xi_j \in \Gamma_\gamma(\omega_j)} u_j$, where $\vec{u}_j$ is supported in $K^{\pm j}_{N_j, L_j}$. The angle between $\omega_1$ and $\omega_2$ is well-separated; $\angle(\omega_1, \omega_2) \geq 3\gamma$.

### 3.3. Bilinear interaction

The space-time Fourier transform of the product $\phi_2^\dagger \phi_1$ of two $\mathfrak{g}$-valued fields $\phi_1$ and $\phi_2$ is written as

$$\widehat{\phi_2^\dagger \phi_1}(X_0) = \int_{X_0 = X_1 - X_2} \widehat{\phi_2^\dagger}(X_2) \widehat{\phi_1}(X_1) \, dX_1 \, dX_2,$$

where $\phi^\dagger$ is the transpose of complex conjugate of $\phi$. Here the relation between $X_1$ and $X_2$ in the convolution integral of fields is given by $X_0 = X_1 - X_2$ so called bilinear interaction. This is also the case for the product of two complex scalar fields.

The following lemma is on the bilinear interaction.

**Lemma 3.3** (Lemma 2.2 of [23]). Given a bilinear interaction $(X_0, X_1, X_2)$ with $\xi_j \neq 0$, and signs $(\pm_0, \pm_1, \pm_2)$, let $h_j = \tau_j \pm_0 |\xi_j| \, |\theta_{12} = |\angle(\pm_1 \xi_1, \pm_2 \xi_2)|$. Then we have

$$\max(|h_0|, |h_1|, |h_2|) \gtrsim \min(|\xi_1|, |\xi_2|) \theta_{12}^2.$$
Moreover, we either have
\[ |\xi_0| \ll |\xi_1| \sim |\xi_2|, \quad \text{and} \quad \pm_1 \neq \pm_2, \]
in which case
\[ \theta_{12} \sim 1 \quad \text{and} \quad \max(|h_0|, |h_1|, |h_2|) \gtrsim \min(|\xi_1|, |\xi_2|), \]
or else we have
\[ \max(|h_0|, |h_1|, |h_2|) \gtrsim \frac{|\xi_1||\xi_2|}{|\xi_0|} \theta_{12}. \]

3.4. Null structure. While proving the local well-posedness of (1.2), we must encounter multilinear estimates such as bilinear and trilinear estimates. Since we make use of duality argument and Cauchy-Schwarz inequality, we essentially treat only bilinear forms of wave type. Then the most serious case resulting in resonance interaction occurs when two input-waves are collinear. However, once this bilinear form possesses cancellation property so called null structure, we can expect better estimates [17].

Before discussing the null structure, we note that the spatial part of vector potential \( A = (A_1, A_2) \) can be split into divergence-free and curl-free parts:
\[ A = A^{df} + A^{cf}, \]
where
\[ A^{df}_j = (-\Delta)^{-1} \epsilon_{ijk} \partial^k (\epsilon^{ilm} \partial_i A_m) \]
\[ A^{cf}_j = (-\Delta)^{-1} \partial_j \partial^k A_k. \]

Also we define the Riesz transform given by
\[ R_j = D^{-1} \partial_j = \frac{\partial_j}{D}. \]

Now we introduce the standard null forms:
\[ Q_0(u, v) = \partial_a u \partial^a v \]
\[ Q_{\alpha\beta}(u, v) = \partial_a u \partial_{\alpha} v - \partial_{\beta} u \partial_a v. \]

Then we define a commutator version of null forms by
\[ Q_0[u, v] = [\partial_a u, \partial^a v] \]
\[ Q_{\alpha\beta}[u, v] = [\partial_a u, \partial_{\beta} v] - [\partial_{\beta} u, \partial_a v]. \]
Here we give some remark on commutator version of null forms. For \( \mathfrak{su}(n) \)-valued functions \( u \) and \( v \), we write \( u = u_a T^a \) and \( v = v_b T^b \), where \( a, b = 1, 2, \ldots, n^2 - 1 \) and \( u_a, v_b \) are smooth scalar functions. Then there holds
\[ Q_{\alpha\beta}[u, v] = [\partial_a u, \partial_{\beta} v] - [\partial_{\beta} u, \partial_a v] \]
\[ = [\partial_a u_a T^a, \partial_{\beta} v_b T^b] - [\partial_{\beta} u_a T^a, \partial_a v_b T^b] \]
\[ = \partial_a u_a \partial_{\beta} v_b [T^a, T^b] - \partial_{\beta} u_a \partial_a v_b [T^a, T^b] \]
\[ = (\partial_a u_a \partial_{\beta} v_b - \partial_{\beta} u_a \partial_a v_b) [T^a, T^b] \]
\[ = Q_{\alpha\beta}(u_a, v_b) [T^a, T^b] \]
\[ = Q_{\alpha\beta}(u_a, v_b) i f^{ab}_{\epsilon} T^\epsilon. \]
For the last equality see Appendix below. Then for a function space $\mathcal{X}(\mathfrak{su}(n))$ defined by the functions with value in $\mathfrak{su}(n)$, we observe that
\[
\|Q_{\alpha\beta}[u,v]\|_{\mathcal{X}(\mathfrak{su}(n))} = \sum c \|Q_{\alpha\beta}(u_a,v_b)\|_{\mathcal{X}(\mathbb{C})} f^{ab}_c.
\]
Hence we conclude that the $\mathcal{X}(\mathfrak{su}(n))$ norm of commutator version of null forms is reduced to the $\mathcal{X}(\mathbb{C})$ norm of null forms of scalar functions.

The following lemma is on null structure hidden in (1.2).

**Lemma 3.4.** In the Lorenz gauge, we have the following identity:
\[
[A^\mu, \partial_\mu \phi] = \frac{1}{2} \epsilon_{0jk} \epsilon_{0lm} Q_{jk}[D^{-1} R^l A^m, \phi] - Q_{j0}[R^l(D^{-1}A_0), \phi].
\]

**Proof.** First, we note that
\[
A^\mu \partial_\mu \phi = A_0 \partial_t \phi + \mathbf{A}^{ef} \cdot \nabla \phi + \mathbf{A}^{df} \cdot \nabla \phi.
\]
By Lorenz gauge condition: $\partial^k A_k = \partial_t A_0$, we get
\[
\mathbf{A}^{ef} \cdot \nabla \phi = -(-\Delta)^{-1} \partial_j \partial^k A_k \partial_j \phi
\]
\[
= D^{-2} \partial_j (\partial_t A_0) \partial_j \phi
\]
\[
= \partial_t R^j(D^{-1}A_0) \partial_j \phi.
\]
Also we have
\[
A_0 \partial_t \phi = -D^{-2} \partial_j \partial^j A_0 \partial_t \phi
\]
\[
= -\partial_t R^j(D^{-1}A_0) \partial_t \phi,
\]
and hence we have
\[
A_0 \partial_t \phi + \mathbf{A}^{ef} \cdot \nabla \phi = -Q_{j0}(R^l D^{-1}A_0, \phi).
\]
Next, we see that
\[
\mathbf{A}^{df} \cdot \nabla \phi = (-\Delta)^{-1} \epsilon_{0jk} \epsilon_{0lm} \partial_k \partial^l A^m \partial_j \phi
\]
\[
= -\epsilon_{0jk} \epsilon_{0lm} \partial_k \partial_l (D^{-1} R^l A_m) \partial_j \phi
\]
\[
= \frac{1}{2} \epsilon_{0jk} \epsilon_{0lm} Q_{jk}(D^{-1} R^l A_m, \phi).
\]
We can treat $\partial_\mu A^\mu$ similarly and hence completes the proof. \(\Box\)

We have the following corollary by Lemma 3.4.

**Corollary 3.5.** In the Lorenz gauge, we have the following identity.
\[
[\partial^\nu A_\mu, A_\nu] = -\frac{1}{2} \epsilon_{0jk} \epsilon_{0lm} Q_{jk}(D^{-1} R^l A_m, A_\mu) + Q_{j0}[R^l(D^{-1}A_0), A_\mu].
\]
4. Proof of Local Well-Posedness

4.1. Picard’s iterates. To prove (LWP) of \( \|u\| \), we construct Picard’s iterates and follow the contraction principle. Indeed, by (2.1), we obtain the following integral equation for \( \phi_{\pm} \) and \( A_{\mu, \pm} \) respectively:

\[
(4.1) \quad \phi_{\pm}(t) = \phi_{\pm}^{\text{hom}}(t) + i \int_{0}^{t} \frac{e^{\pm i(t-t')D}}{2iD} \left( -2[A^{\mu}, \partial_\mu \phi] - [A_{\mu}, [A^{\mu}, \phi]] - V(\phi, \phi^\dagger) \right) dt'
\]

\[
(4.2) \quad A_{\mu, \pm}(t) = A_{\mu, \pm}^{\text{hom}}(t) + i \int_{0}^{t} \frac{e^{\pm i(t-t')D}}{2iD} \left( [\partial^\nu A_{\mu}, A_{\nu}] - \epsilon_{\mu\nu\alpha} \left( Q^{\nu\alpha}(\phi^\dagger, \phi) + Q^{\nu\alpha}(\phi, \phi^\dagger) \right) \right) dt' + i \int_{0}^{t} \frac{e^{\pm i(t-t')D}}{2iD} \epsilon_{\mu\nu\alpha} \partial^\nu \left( [\phi^\dagger, [A^{\alpha}, \phi]] - [A^{\alpha}, [\phi, \phi]] \right) dt',
\]

where

\[
\psi_{\pm}^{\text{hom}}(t) = \frac{1}{2} e^{\pm iD} \left( \psi(0, x) \mp \frac{1}{iD} \partial_0 \psi(0, x) \right).
\]

To prove that our Picard’s iteration converges, it suffices to show the following estimates:

\[
\| \phi_{\pm}^{\text{hom}} \|_{B^{1/2}_{2, 1} (S_T)} \lesssim \| f \|_{B^{1/2}_{2, 1}} + \| g \|_{B^{1/2}_{2, 1}},
\]

\[
\| A_{\mu, \pm}^{\text{hom}} \|_{B^{1/2}_{2, 1} (S_T)} \lesssim \sum_{\nu=0}^{2} \| a_{\nu} \|_{B^{1/2}_{2, 1}} + \| a_{0} \|_{B^{1/2}_{2, 1}} \sum_{\nu=0}^{2} \| a_{\nu} \|_{B^{1/2}_{2, 1}} + \| f \|_{B^{1/2}_{2, 1}},
\]

\[
\| [A_{\pm}^{\mu}, \partial_\mu \phi_{\pm}] \|_{B^{1/2}_{2, 1} (S_T)} \lesssim \| A_{\mu, \pm} \|_{B^{1/2}_{2, 1}} \| \phi_{\pm} \|_{B^{3/2}_{2, 1}},
\]

\[
\| [\partial^\nu A_{\mu, \pm}, A_{\nu, \pm}] \|_{B^{1/2}_{2, 1} (S_T)} \lesssim \| A_{\mu, \pm} \|_{B^{1/2}_{2, 1}} \| A_{\nu, \pm} \|_{B^{1/2}_{2, 1}},
\]

\[
\| \epsilon_{\mu\nu\alpha} Q^{\nu\alpha}(\phi_{\pm}^\dagger, \phi_{\pm}) \|_{B^{1/2}_{2, 1} (S_T)} \lesssim \| \phi_{\pm} \|_{B^{3/2}_{2, 1}} \| \phi_{\pm} \|_{B^{3/2}_{2, 1}},
\]

\[
\| [A_{\pm}^{\mu}, [A_{\mu, \pm}, \phi_{\pm}]] \|_{B^{1/2}_{2, 1} (S_T)} \lesssim \| A_{\pm}^{\mu} \|_{B^{1/2}_{2, 1}} \| A_{\mu, \pm} \|_{B^{1/2}_{2, 1}} \| \phi_{\pm} \|_{B^{3/2}_{2, 1}},
\]

\[
\| [\phi_{\pm}^\dagger, [A_{\mu, \pm}, \phi_{\pm}]] \|_{B^{1/2}_{2, 1} (S_T)} \lesssim \| \phi_{\pm} \|_{B^{3/2}_{2, 1}} \| A_{\mu, \pm} \|_{B^{1/2}_{2, 1}} \| \phi_{\pm} \|_{B^{3/2}_{2, 1}},
\]

\[
\| [A_{\pm}^{\mu}, \phi_{\pm}] \|_{B^{1/2}_{2, 1} (S_T)} \lesssim \| A_{\pm}^{\mu} \|_{B^{1/2}_{2, 1}} \| \phi_{\pm} \|_{B^{3/2}_{2, 1}},
\]

\[
\| [A_{\pm}^{\mu}, \phi_{\pm}] \|_{B^{1/2}_{2, 1} (S_T)} \lesssim \| A_{\pm}^{\mu} \|_{B^{1/2}_{2, 1}} \| \phi_{\pm} \|_{B^{3/2}_{2, 1}},
\]

\[
\int_{0}^{t} \frac{e^{\pm i(t-t')D}}{2iD} V(\phi, \phi^\dagger)(t') dt' \lesssim T^r \| \phi_{\pm} \|_{B^{3/2}_{2, 1}} + \sum_{j=1}^{3} \| \phi_{\pm} \|_{B^{3/2}_{2, 1}} + \prod_{k=1}^{5} \| \phi_{\pm} \|_{B^{3/2}_{2, 1}}.
\]
4.2. Estimates of $\phi^\text{hom}_{\mu, \pm}$ and $A^\text{hom}_{\mu, \pm}$. First, we note that $\|e^{itD}f\|_{B^\frac{1}{2} (-\infty, \infty)} \leq \|\rho e^{itD}f\|_{B^\frac{1}{2} (-\infty, \infty)} \leq \|\rho\|_{L^\infty} \|f\|_{B^\frac{1}{2} (-\infty, \infty)}$, where $\rho \in C^\infty_0 (\mathbb{R})$ with value 1 on $(-T, T)$. (See Appendix in [7].) Now $\phi^\text{hom}_{\mu, \pm}$ is given by $\phi^\text{hom}_{\mu, \pm} = \frac{1}{2} e^{itD} (\phi (0, x) \mp \frac{1}{i \partial_1} \partial_1 (\phi (0, x))) = \frac{1}{2} e^{itD} (f (x) \mp \frac{1}{i \partial_1} g (x))$. Since $f \in B^\frac{1}{2} (-\infty, \infty)$ and $g \in B^{-\frac{1}{2}} (-\infty, \infty)$, we see that

$$
\|\phi_{\mu, \pm}^\text{hom}\|_{B^\frac{1}{2} (-\infty, \infty)} \lesssim \|f \mp \frac{1}{i \partial_1} g\|_{B^\frac{1}{2} (-\infty, \infty)} \lesssim \|f\|_{B^\frac{1}{2} (-\infty, \infty)} \|\mp \frac{1}{i \partial_1} g\|_{B^{-\frac{1}{2}} (-\infty, \infty)} \lesssim \|f\|_{B^{\frac{1}{2}} (-\infty, \infty)} + \|g\|_{B^{-\frac{1}{2}} (-\infty, \infty)}.
$$

On the other hand, $A_{\mu, \pm}^\text{hom}$ and $A_{\nu, \pm}^\text{hom}$ are given by

$$
A_{\mu, \pm}^\text{hom} = \sum_{\pm} \frac{1}{2} e^{itD} \left( a_{\mu, \pm} \mp \frac{1}{i \partial_1} a_j \right),
$$

$$
A_{\nu, \pm}^\text{hom} = \sum_{\pm} \frac{1}{2} e^{itD} \left( a_j \mp \frac{1}{i \partial_1} (\partial_1 a_{\nu, \pm} - [a_0, a_j] + \epsilon_{\nu, \pm} ([f_1, \partial_1 f] - ([\partial_1 f], f))) \right).
$$

Since Riesz transform is a bounded operator in $L^2$,

$$
\|A_{\mu, \pm}^\text{hom}\|_{B^\frac{1}{2} (-\infty, \infty)} \lesssim \|a_{\mu, \pm}\|_{B^\frac{1}{2} (-\infty, \infty)} + \|\mp \frac{1}{i \partial_1} a_j\|_{B^{\frac{1}{2}} (-\infty, \infty)} \lesssim \|a_{\mu, \pm}\|_{B^\frac{1}{2} (-\infty, \infty)} + \sum_{j=1}^{2} \|a_j\|_{B^{\frac{1}{2}} (-\infty, \infty)}.
$$

To treat $A_{j}^\text{hom}$ we recall Bernstein’s inequality,

$$
\|P_{[\xi] \sim N} f\|_{L^p (\mathbb{R}^n)} \lesssim N^{d(\frac{1}{q} - \frac{1}{p})} \|P_{[\xi] \sim N} f\|_{L^q (\mathbb{R}^n)},
$$

for $q < p \leq \infty$. Then we have

$$
\left\| \frac{1}{i \partial_1} (f \partial_k f) \right\|_{B_{2, 1}^{\frac{1}{2}}} \lesssim \left\| f \partial_k f \right\|_{B_{2, 1}^{\frac{3}{2}}} = \sum_{N \text{dyadic}} N^{-\frac{2}{3}} \|P_N (f \partial_k f)\|
$$

$$
\lesssim \sum_{N} N^{-\frac{2}{3}} \|P_N f\|_{L^4} \|P_N \partial_k f\|_{L^4}
$$

$$
\lesssim \sum_{N} N^{-\frac{2}{3}} N^\frac{1}{2} N^\frac{1}{4} \|P_N f\| \|P_N \partial_k f\|
$$

$$
\lesssim \left( \sum_{N} (N^\frac{1}{2} \|P_N f\|)^2 \right)^{\frac{1}{2}} \left( \sum_{N} (N^\frac{1}{4} \|P_N f\|)^2 \right)^{\frac{1}{2}}
$$

$$
\lesssim \|f\|_{B_{2, 1}^{\frac{1}{2}}},
$$

and similarly,

$$
\left\| \frac{1}{i \partial_1} (a_0 a_j) \right\|_{B_{2, 1}^{\frac{1}{2}}} \lesssim \left\| a_0 a_j \right\|_{B_{2, 1}^{\frac{3}{2}}} \lesssim \left\| a_0 \right\|_{B_{2, 1}^{\frac{1}{2}}} \left\| a_j \right\|_{B_{2, 1}^{\frac{1}{2}}}.
$$

Therefore,

$$
\left\| A_{j}^\text{hom}\right\|_{B^\frac{1}{2} (-\infty, \infty)} \lesssim \left\| a_{0} \right\|_{B^\frac{1}{2} (-\infty, \infty)} + \left\| a_{j} \right\|_{B^\frac{1}{2} (-\infty, \infty)} + \left\| a_{0} \right\|_{B^\frac{1}{2} (-\infty, \infty)} \left\| a_{j} \right\|_{B^\frac{1}{2} (-\infty, \infty)} + \|f\|_{B_{2, 1}^{\frac{1}{2}}}.
$$

4.3. Proof of Theorem 11.1. We give a brief proof of local well-posedness of (1.2) using the above estimates. Our nonlinear wave equations are written as

$$(i \partial_t \mp D)^{\phi_{\pm}} = N_{\phi_{\pm}} (\phi_{\pm} A_{\pm}),$$

$$(i \partial_t \mp D)A_{\mu, \pm} = N_{A_{\mu, \pm}} (\phi_{\pm} A_{\pm}),$$

where $N_{\phi}$ and $N_A$ are the right-handside of (2.1).
To construct Picard’s iterates $\phi_{\pm}^{(n)}$ and $A_{\mu,\pm}^{(n)}$, we set $\phi_{\pm}^{(-1)}$ and $A_{\mu,\pm}^{(-1)}$ to be identically zero, and $\phi_{\pm}^{(0)}$ and $A_{\mu,\pm}^{(0)}$ are defined by $\phi_{\pm}^{\text{hom}}$ and $A_{\mu,\pm}^{\text{hom}}$, respectively. Then the inductive step $\phi_{\pm}^{(n)}$ and $A_{\mu,\pm}^{(n)}$ are given by solving equations 2.1 on $S_T$ with the previous step $\phi_{\pm}^{(n-1)}$ and $A_{\mu,\pm}^{(n-1)}$ inserted on the right-hand side, respectively. To be precise, we write

$$\phi_{\pm}^{(n)}(t) = \phi_{\pm}^{(0)}(t) + i \int_0^t e^{\mp i(t-t')D}N_{\phi}(\phi_{\pm}^{(n-1)}, A_{\pm}^{(n-1)})(t')dt'$$

and

$$A_{\mu,\pm}^{(n)}(t) = A_{\mu,\pm}^{(0)}(t) + i \int_0^t e^{\mp i(t-t')D}N_A(\phi_{\pm}^{(n-1)}, A_{\pm}^{(n-1)})(t')dt'.$$

Also we set

$$p_{n,\phi}(T) = \sum_{\pm} \|\phi_{\pm}^{(n)}\|_{E_{T,\mu,\pm}^{1/2}(S_T)}, \quad p_{n,A}(T) = \sum_{\pm} \|A_{\mu,\pm}^{(n)}\|_{E_{T,\mu,\pm}^{1/2}(S_T)}$$

and

$$q_{n,\phi}(T) = \sum_{\pm} \|\phi_{\pm}^{(n)} - \phi_{\pm}^{(n-1)}\|_{E_{T,\mu,\pm}^{1/2}(S_T)}, \quad q_{n,A}(T) = \sum_{\pm} \|A_{\mu,\pm}^{(n)} - A_{\mu,\pm}^{(n-1)}\|_{E_{T,\mu,\pm}^{1/2}(S_T)}$$

Recall Lemma 2.2 and use the above estimates to get

$$p_{n+1,\phi}(T) \leq c_{\phi}(T) + m^2 T^s p_{n,\phi}(T) + p_{n,A}(T)p_{n,\phi}(T) + p_{n,A}(T)^2 p_{n,\phi}(T) + p_{n,\phi}(T)^3 + p_{n,\phi}(T)^5$$

and

$$q_{n+1,\phi}(T) \leq m^2 T^s q_{n,\phi}(T) + p_{n,A}(T)q_{n,\phi}(T) + p_{n,A}(T)^2 q_{n,\phi}(T) + p_{n,\phi}(T)^2 q_{n,\phi}(T) + p_{n,\phi}(T)^4 q_{n,\phi}(T),$$

where $c_{\phi}(T) = \sum_{\pm} \|\phi_{\pm}^{\text{hom}}\|_{E_{T,\mu,\pm}^{1/2}(S_T)}$. Note that the constant $m^2$ is from linear terms of $V$. We also have

$$p_{n+1,A}(T) \leq c_A(T) + p_{n,A}(T)^2 + p_{n,\phi}(T)^2 + p_{n,\phi}(T)^2 p_{n,A}(T)$$

and

$$q_{n+1,A}(T) \leq p_{n,A}(T)q_{n,A}(T) + p_{n,\phi}(T)q_{n,\phi}(T) + p_{n,\phi}(T)^2 q_{n,A}(T),$$

where $c_A(T) = \|A_{\mu,\pm}^{\text{hom}}\|_{E_{T,\mu,\pm}^{1/2}(S_T)}$. By absolute continuity, for any $\delta > 0$ we can take $T$ so small that $c_{\phi}(T), c_A(T) \leq \delta$. Thus if $4\delta + 16\delta^3 + 32\delta^4 \leq 1$ and $2m^2 T^s \leq \frac{1}{7}$, by induction $p_{n+1,\phi}(T), p_{n+1,A}(T) \leq 2\delta$ for $n \geq 1$ and $q_{n,\phi}(T) \leq \frac{1}{7} q_{n-1,\phi}(T), q_{n,A}(T) \leq \frac{1}{7} q_{n-1,A}(T)$. This proves the local existence and uniqueness of solutions to (1.2).

**Remark 4.1.** One can observe in the proof of Theorem 1.4 that the existence time $T$ depends on the profiles of initial data since (and also in explicitly) we control the iterates with $c_{\phi}(T)$ and $c_A(T)$.

4.4. Reduction to nonlinear estimates. Now we reduce the above estimates to nonlinear estimates. We have already observed in Section 3 that the matrix structure in null forms plays no crucial role in the estimates. Also, we claim that estimates of cubic and quintic terms of $A_{\mu,\phi}$ are reduced to the nonlinear estimates of scalar functions. For example, we consider the cubic terms $[A_{\mu,\pm}^\mu, [A_{\mu,\pm}^\mu, A_{\mu,\pm}^\phi]]$. We write $A_{\mu,\pm}^\mu = A_{\mu,\pm}^{\mu,a} T^a, A_{\mu,\pm}^\phi = A_{\mu,\pm}^{\mu,\phi,b} T^b$, and $\phi_{\pm,\phi} = \phi_{\pm,\phi} c T^c$. Then for a function space $X(\mathfrak{su}(n))$, the norm of $[A_{\mu,\pm}^\mu, [A_{\mu,\pm}^\mu, A_{\mu,\pm}^\phi]]$ is given by

$$\|[A_{\mu,\pm}^\mu, [A_{\mu,\pm}^\mu, A_{\mu,\pm}^\phi]]||_{X(\mathfrak{su}(n))} = \sum_{c} \|[A_{\mu,\pm}^\mu, A_{\mu,\pm}^\mu, A_{\mu,\pm,\phi,c}}||_{X(\mathfrak{C})}^{|f_{abe}^c|} ||f_{ade}^c||,$$

and thus the norm of cubic terms of $\mathfrak{su}(n)$-valued fields is reduced to the norm of cubic terms of scalar fields.
In Appendix, we shall see that $a$-th component of $\mathcal{V}(\phi, \phi^\dagger)$ is given by
\[
\frac{\partial \mathcal{V}(\phi, \phi^\dagger)}{\partial \phi_a^e} = 2 \sum_e (f^{ab}_d f^{d e'_c} \phi^e_c + v^2 \delta^a_{e'_c}) (f^{a'b'}_d f^{d' e'_c} \phi_{a'b'} \phi_{e'} + v^2 \phi_e).
\]
Then the $\mathcal{X}(\mathfrak{su}(n))$ norm of the quintic term in $\mathcal{V}$ is bounded by
\[
2 \sum a, e \| \phi^a_e \phi^e_c \phi_{a'b'} \phi_{e'} \|_{\mathcal{X}(\mathbb{C})} | f^{ab}_d f^{d e'_c} f^{a'b'}_d f^{d' e'_c} |,
\]
and hence the norm of quintic terms of $\phi, \phi^\dagger$ is reduced to the $\mathcal{X}(\mathbb{C})$ norm of quintic terms of scalar fields. Therefore, from now on, we consider the $\mathfrak{su}(n)$-valued functions $A^a, \phi$ as $\mathbb{C}$-valued functions.

Through these observations together with Lemma 3.3 and Corollary 3.5, we need to treat the following nonlinear estimates:

\[
\|Q_j(k) (R^1 (D^{-1} A^k_{\pm, 1}), \phi_{\pm, 2}) \|_{B^{2 - \frac{3}{4} - \frac{1}{4}}_\pm (S_T)} \lesssim \| A^k_{\pm, 1} \|_{B^{3 + 1}_\pm} \| \phi_{\pm, 2} \|_{B^{3 + 1}_\pm},
\]

\[
\|Q_j(k) (R^1 (D^{-1} A^k_{\pm, 1}), \phi_{\pm, 2}) \|_{B^{2 - \frac{3}{4} - \frac{1}{4}}_\pm (S_T)} \lesssim \| A^k_{\pm, 1} \|_{B^{3 + 1}_\pm} \| \phi_{\pm, 2} \|_{B^{3 + 1}_\pm},
\]

\[
\|Q_j(R^1 (D^{-1} A_{0, \pm, 1}), \phi_{\pm, 2}) \|_{B^{2 - \frac{3}{4} - \frac{1}{4}}_\pm (S_T)} \lesssim \| A_{0, \pm, 1} \|_{B^{3 + 1}_\pm} \| \phi_{\pm, 2} \|_{B^{3 + 1}_\pm},
\]

\[
\|Q_j(k) (R^1 (D^{-1} A^k_{\pm, 1}), A_{\mu, \pm, 2}) \|_{B^{2 - \frac{3}{4} - \frac{1}{4}}_\pm (S_T)} \lesssim \| A^k_{\pm, 1} \|_{B^{3 + 1}_\pm} \| A_{\mu, \pm, 2} \|_{B^{3 + 1}_\pm},
\]

\[
\|Q_j(R^1 (D^{-1} A_{0, \pm, 1}), A_{\mu, \pm, 2}) \|_{B^{2 - \frac{3}{4} - \frac{1}{4}}_\pm (S_T)} \lesssim \| A_{0, \pm, 1} \|_{B^{3 + 1}_\pm} \| A_{\mu, \pm, 2} \|_{B^{3 + 1}_\pm},
\]

\[
\|Q_j(k) (\phi_{\pm, 1}, \phi_{\pm, 2}) \|_{B^{2 - \frac{3}{4} - \frac{1}{4}}_\pm (S_T)} \lesssim \| \phi_{\pm, 1} \|_{B^{3 + 1}_\pm} \| \phi_{\pm, 2} \|_{B^{3 + 1}_\pm},
\]

\[
\|Q_j(R^1 (\phi_{\pm, 1}, \phi_{\pm, 2}) \|_{B^{2 - \frac{3}{4} - \frac{1}{4}}_\pm (S_T)} \lesssim \| \phi_{\pm, 1} \|_{B^{3 + 1}_\pm} \| \phi_{\pm, 2} \|_{B^{3 + 1}_\pm},
\]

\[
\| A_{\mu, \pm, 2} \|_{B^{2 - \frac{3}{4} - \frac{1}{4}}_\pm (S_T)} \lesssim \| A_{\mu, \pm, 2} \|_{B^{3 + 1}_\pm} \| \phi_{\pm, 2} \|_{B^{3 + 1}_\pm},
\]

\[
\| A_{\mu, \pm, 2} \|_{B^{2 - \frac{3}{4} - \frac{1}{4}}_\pm (S_T)} \lesssim \| A_{\mu, \pm, 2} \|_{B^{3 + 1}_\pm} \| \phi_{\pm, 2} \|_{B^{3 + 1}_\pm},
\]

\[
\| \phi_{\pm, 2} \|_{B^{2 - \frac{3}{4} - \frac{1}{4}}_\pm (S_T)} \lesssim \prod_{j=1}^3 \| \phi_{\pm, 3} \|_{B^{3 + 1}_\pm},
\]

\[ \|\phi_{\pm 1}\phi_{\pm 2}\phi_{\pm 3}\phi_{\pm 4}\phi_{\pm 5}\|_{B_{-}^{\frac{1}{2},-\frac{1}{4}}(S_{\Gamma})} \lesssim \prod_{j=1}^{5} \|\phi_{\pm j}\|_{B_{-}^{\frac{1}{2},\frac{1}{4}}}, \]

where we used \( \epsilon^{ijk}\epsilon_{ilm} = \delta_{im}^{l} \delta_{jm}^{k} - \delta_{im}^{k} \delta_{jm}^{l} \). Recall that the last two estimates are from Higgs potential \( V(\phi, \phi^{1}) \).

The estimates of linear terms in \( V(\phi, \phi^{1}) \) is followed trivially by Lemma 2.3 and definition of \( B_{-}^{\frac{1}{2},\frac{1}{4}} \), so we omit it. In the following three sections, we will focus on the proof of the above nonlinear estimates.

5. Bilinear estimates

This section is devoted to the proof of the bilinear estimates appearing in Section 4. Since the Riesz transforms \( R_{i} \) are bounded in the spaces under our consideration, these bilinear estimates can be reduced to the following:

\[ \|Q_{jk}(D^{-1}A_{\pm 1}A_{\pm 2})\|_{B_{-}^{\frac{1}{2},-\frac{1}{4}}(S_{\Gamma})} \lesssim \|A_{\pm 1}\|_{B_{-}^{\frac{1}{2},\frac{1}{4}}}, \]

\[ \|Q_{j0}(D^{-1}A_{0,\pm 1}\phi_{\pm 2})\|_{B_{-}^{\frac{1}{2},-\frac{1}{4}}(S_{\Gamma})} \lesssim \|A_{0,\pm 1}\|_{B_{-}^{\frac{1}{2},\frac{1}{4}}}, \]

\[ \|Q_{jk}(D^{-1}A_{\pm 1},A_{\mu,\pm 2})\|_{B_{-}^{\frac{1}{2},-\frac{1}{4}}(S_{\Gamma})} \lesssim \|A_{\pm 1}\|_{B_{-}^{\frac{1}{2},\frac{1}{4}}}, \]

\[ \|Q_{j0}(D^{-1}A_{0,\pm 1},A_{\mu,\pm 2})\|_{B_{-}^{\frac{1}{2},-\frac{1}{4}}(S_{\Gamma})} \lesssim \|A_{0,\pm 1}\|_{B_{-}^{\frac{1}{2},\frac{1}{4}}}, \]

\[ \|Q_{jk}(\phi_{\pm 1},\phi_{\pm 2})\|_{B_{-}^{\frac{1}{2},-\frac{1}{4}}(S_{\Gamma})} \lesssim \|\phi_{\pm 1}\|_{B_{-}^{\frac{1}{2},\frac{1}{4}}}, \]

\[ \|Q_{j0}(\phi_{\pm 1},\phi_{\pm 2})\|_{B_{-}^{\frac{1}{2},-\frac{1}{4}}(S_{\Gamma})} \lesssim \|\phi_{\pm 1}\|_{B_{-}^{\frac{1}{2},\frac{1}{4}}}. \]

To prove the above bilinear estimates via null forms for functions \( u \) and \( v \), we recall the substitution:

\[ u = u_{+} + u_{-}, \quad \partial_{i}u = iD(u_{+} - u_{-}), \]

\[ v = v_{+} + v_{-}, \quad \partial_{i}v = iD(v_{+} - v_{-}). \]

Then we have

\[ Q_{j0}(u,v) = \sum_{\pm 1, \pm 2} (\pm 1)(\pm 2) ((\pm 1\partial_{j}u_{\pm 1})(\pm 2Dv_{\pm 2}) - (\pm 1Dv_{\pm 1})(\pm 2\partial_{j}v_{\pm 2})), \]

\[ Q_{jk}(u,v) = \sum_{\pm 1, \pm 2} (\pm 1)(\pm 2) ((\pm 1\partial_{j}u_{\pm 1})(\pm 2\partial_{k}v_{\pm 2}) - (\pm 1\partial_{k}u_{\pm 1})(\pm 2\partial_{j}v_{\pm 2})), \]

and their symbols are given by

\[ q_{j0}(\xi_{1},\xi_{2}) = -\xi_{1,j}\xi_{2} + |\xi_{1}||\xi_{2,j}|, \]

\[ q_{jk}(\xi_{1},\xi_{2}) = -\xi_{1,j}\xi_{2,k} + \xi_{1,k}\xi_{2,j}. \]

For these symbols, we have the following estimates.

**Lemma 5.1.** For \( \xi_{1}, \xi_{2} \in \mathbb{R}^{2} \) with \( \xi_{1}, \xi_{2} \neq 0 \),

\[ |q_{j0}(\xi_{1},\xi_{2})|, |q_{jk}(\xi_{1},\xi_{2})| \lesssim |\xi_{1}||\xi_{2}||\xi(\xi_{1},\xi_{2})|. \]

**Proof.** See Lemma 5. of [25]. \( \square \)

By Lemma 5.1, it suffices to prove (5.4), (5.5), and (5.6). Furthermore, the proof of (5.4) and (5.5) is essentially same. So we focus on the proof of (5.1) and (5.2).
5.1. Exclusion of low-low-high modulation. In advance of proving bilinear estimates \[5.1\], \[5.5\], we give some remark on the low-low-high modulation, i.e., \(L_{\text{max}}^{12} \ll L_0 \ll N_{\text{min}}^{012}\). Via duality, we essentially need to treat the integral \(I\) of the form:

\[
I = \int P_{K_{N_1}, L_1} \varphi_1 P_{K_{N_2}, L_2} \varphi_2 P_{K_{N_0}, L_0} \varphi \, dt \, dx.
\]

By Plancherel’s theorem, we get

\[
I = \int \mathcal{F}[P_{K_{N_1}, L_1}] \mathcal{F}[P_{K_{N_2}, L_2}] \varphi_1 \mathcal{F}[P_{K_{N_0}, L_0}] \varphi \mathcal{F}(-X_0) \, dX_0,
\]

where we have the bilinear interaction \(X_0 = X_1 + X_2\). We note that by the standard Littlewood-Paley trichotomy, spatial frequency \(N_0, N_1, N_2\) should satisfy the following relation:

\[
N_{\text{min}}^{012} \sim N_{\text{med}}^{012} \sim N_{\text{max}}^{012},
\]

and more explicitly, satisfy one of the following:

\[
N_0 \ll N_1 \sim N_2 \quad \text{(High-High-Low)},
\]

\[
N_1 \lesssim N_0 \sim N_2 \quad \text{(Low-High-High)},
\]

\[
N_2 \lesssim N_0 \sim N_1 \quad \text{(High-Low-High)},
\]

with \(N_{\text{min}}^{012} \sim N_{\text{med}}^{012} N_{\text{max}}^{012}\).

Also, our integral \(I\) can be written as

\[
I = \int \mathcal{F}[P_{K_{N_1}, L_1}] \mathcal{F}(-X_1) \mathcal{F}[P_{K_{N_2}, L_2}] \varphi_2 \mathcal{F}[P_{K_{N_0}, L_0}] \varphi (X_1) \, dX_1,
\]

since we have \(X_1 = X_0 - X_2\). Now suppose that \(L_{\text{max}}^{12} \ll L_0 \ll N_0 \ll N_1 \sim N_2\). Then in view of first representation of \(I\), the case \(\pm_1 \neq \pm_2\) is excluded, since \(\varphi_1\) and \(\varphi_2\) have up and down cones or down and up cones, respectively. We also exclude the case \(\pm_1 = \pm_2\) from second representation of \(I\). This concludes that the low-low-high modulation with high-high-low frequency case does not appear in \(I\).

On the other hand, suppose that \(N_1 \ll N_0 \sim N_2\). The support condition excludes the case \(\pm_0 = \pm_2\) since \(\varphi\) and \(\varphi_2\) have up and down cones or down and up cones, respectively from first representation. The case \(\pm_0 \neq \pm_2\) is also excluded by second representation, because \(\varphi\) and \(\varphi_2\) have then up and up, or down and down cones, respectively. We can argue similarly to exclude the case \(N_2 \ll N_0 \sim N_1\). Consequently, the low-low-high modulation with high-low-high and low-high-low frequency does not occur in \(I\). Hence in the low-low-high modulation case, we only consider the case \(L_0 \ll N_0 \sim N_1 \sim N_2\).

After the low-low-high modulation with all input and out frequency compatible, we consider the case \(L_0 \gtrsim N_{\text{min}}^{012}\). From the relation on spatial frequency \[5.7\], we are concerned with \(N_{\text{min}}^{012} \lesssim L_0 \lesssim N_{\text{med}}^{012} \sim N_{\text{max}}^{012}\). Appealing the similar argument which we have seen above, we exclude the case \(L_0 \ll N_{\text{med}}^{012} \sim N_{\text{max}}^{012}\) and \(N_{\text{min}}^{012} \ll L_0\). Hence the remaining case is \(L_0 \sim N_0 \sim N_1 \sim N_2\).

In summary, for the low-low-high modulation case, we only need to treat the following cases:

\[
L_{\text{max}}^{12} \ll L_0 \ll N_0 \sim N_1 \sim N_2,
\]

\[
L_{\text{max}}^{12} \ll L_0 \sim N_0 \sim N_1 \sim N_2.
\]
5.2. Proof of \( (5.1) \). By definition of Besov type \( X^{s,b} \) space,

\[
\|Q_{jk}(D^{-1}A^k_{\pm 1}, \phi_{\pm 2})\|_{B_{p,q}^{s,b}} \leq \sum_{N_0, L_0 \geq 1} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} \sum_{N_0, L_0 \geq 1} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} \sup_{\|\phi\|=1} \left| \int Q_{jk}(D^{-1}A^k_{\pm 1}, \phi_{\pm 2}) P_{K_{N_0, L_0}} ^{-1} \phi \, dt \, dx \right| \equiv J^1.
\]

We also apply a dyadic decomposition of space-time Fourier side to \( A^k \) and \( \phi \) and it yields

\[
|J^1| \leq \sum_{N, L} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} |J^1_{N, L}|,
\]

where

\[
J^1_{N, L} = \left| \int Q_{jk}(D^{-1}P_{K_{N_1, L_1}} ^{-1} A^k_{\pm 1}, P_{K_{N_2, L_2}} ^{-1} \phi_{\pm 2}) P_{K_{N_0, L_0}} ^{-1} \phi \, dt \, dx \right|,
\]

and

\[
N = (N_0, N_1, N_2), \quad L = (L_0, L_1, L_2).
\]

We can decompose the integrand of \( J^1 \) suitably and assume that \( \mathcal{F}(P_{K_{N_0, L_0}} ^{-1} \phi), \mathcal{F}(P_{K_{N_1, L_1}} ^{-1} A^k_{\pm 1}), \) and
\( \mathcal{F}(P_{K_{N_2, L_2}} ^{-1} \phi_{\pm 2}) \) are nonnegative real-valued functions. Then Lemma 5.1 and Cauchy-Schwarz inequality give rise to

\[
|J^1| \lesssim \sum_{N, L} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} \left( \frac{L_0^{12}}{N_0^{12}} \right)^{\frac{1}{2}} N_2 \|P_{K_{N_0, L_0}} ^{-1} (P_{K_{N_1, L_1}} ^{-1} A^k_{\pm 1} P_{K_{N_2, L_2}} ^{-1} \phi_{\pm 2})\|.
\]

5.2.1. Case 1: \( L_0 \leq L_1 \leq L_2 \).

If \( N_0 \ll N_1 \sim N_2 \), then by \( (5.2) \) with \( j = 1 \), we have

\[
|J^1| \lesssim \sum_{N, L} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} \left( \frac{L_2}{N_1} \right)^{\frac{1}{2}} N_0(N_0L_0)^{\frac{1}{2}} (N_0L_1)^{\frac{1}{2}} \|P_{K_{N_1, L_1}} ^{-1} A^k_{\pm 1} \| P_{K_{N_2, L_2}} ^{-1} \phi_{\pm 2}\|
\]

\[
= \sum_{N, L} \left( \frac{N_2}{N_1} \right)^{\frac{1}{2}} N_0^{\frac{1}{2}} N_2^{\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|P_{K_{N_1, L_1}} ^{-1} A^k_{\pm 1} \| P_{K_{N_2, L_2}} ^{-1} \phi_{\pm 2}\|
\]

\[
\lesssim \|A^k_{\pm 1}\|_{B_{p,q}^{s,b}} \|\phi_{\pm 2}\|_{B_{p,q}^{s,b}} \left( \sum_{L_0: L_0 \leq L_1} 1 \lesssim L_1^{\frac{1}{2}} \right).
\]

For \( N_1 \sim N_0 \sim N_2 \), using \( (5.2) \) with \( j = 1 \) we get

\[
|J^1| \lesssim \sum_{N, L} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} \left( \frac{L_2}{N_1} \right)^{\frac{1}{2}} N_0(N_0L_0)^{\frac{1}{2}} (N_0L_1)^{\frac{1}{2}} \|P_{K_{N_1, L_1}} ^{-1} A^k_{\pm 1} \| P_{K_{N_2, L_2}} ^{-1} \phi_{\pm 2}\|
\]

\[
= \sum_{N, L} N_0^{-\frac{1}{2}} N_1^{\frac{1}{2}} N_0^{\frac{1}{2}} N_2^{\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|P_{K_{N_1, L_1}} ^{-1} A^k_{\pm 1} \| P_{K_{N_2, L_2}} ^{-1} \phi_{\pm 2}\|
\]

\[
\lesssim \|A^k_{\pm 1}\|_{B_{p,q}^{s,b}} \|\phi_{\pm 2}\|_{B_{p,q}^{s,b}} \left( \sum_{L_0: L_0 \leq L_1} 1 \lesssim L_1^{\frac{1}{2}} \right).
\]
For $N_2 \lesssim N_0 \sim N_1$, then

\[
|J^1| \lesssim \sum_{N,L} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} \left( \frac{L_2}{N_2} \right)^{\frac{1}{2}} N_2 (N_2 L_0)^{\frac{1}{2}} \langle N_0 L_1 \rangle^{\frac{1}{2}} \| P_{K^{\pm}_{N_1,L_1}}^{\pm_1} A^{\pm}_{\pm_1} \| \| P_{K^{\mp}_{N_2,L_2}}^{\pm_2} \phi^{\pm_2} \|
\]

\[
= \sum_{N,L} N_2 L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \| P_{K^{\pm}_{N_1,L_1}}^{\pm_1} A^{\pm}_{\pm_1} \| \| P_{K^{\mp}_{N_2,L_2}}^{\pm_2} \phi^{\pm_2} \|
\]

\[
\lesssim \| A^{\pm}_{\pm_1} \|_{B^{\frac{1}{2}}_{\frac{1}{2}}} \| \phi^{\pm_2} \|_{B^{\frac{1}{2}}_{\frac{1}{2}}}.
\]

5.2.2. Case 2: $L_1 \leq L_0 \leq L_2$.
If $N_0 \ll N_1 \sim N_2$, then

\[
|J^1| \lesssim \sum_{N,L} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} \left( \frac{L_2}{N_1} \right)^{\frac{1}{2}} N_2 (N_0 L_1)^{\frac{1}{2}} \langle N_0 L_0 \rangle^{\frac{1}{2}} \| P_{K^{\pm}_{N_1,L_1}}^{\pm_1} A^{\pm}_{\pm_1} \| \| P_{K^{\mp}_{N_2,L_2}}^{\pm_2} \phi^{\pm_2} \|
\]

\[
= \sum_{N,L} L_0^{-\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \left( \frac{N_0}{N_1} \right)^{\frac{1}{2}} N_2 \| P_{K^{\pm}_{N_1,L_1}}^{\pm_1} A^{\pm}_{\pm_1} \| \| P_{K^{\mp}_{N_2,L_2}}^{\pm_2} \phi^{\pm_2} \|
\]

\[
\lesssim \sum_{N,L} N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \| P_{K^{\pm}_{N_1,L_1}}^{\pm_1} A^{\pm}_{\pm_1} \| \| P_{K^{\mp}_{N_2,L_2}}^{\pm_2} \phi^{\pm_2} \|
\]

\[
\lesssim \| A^{\pm}_{\pm_1} \|_{B^{\frac{1}{2}}_{\frac{1}{2}}} \| \phi^{\pm_2} \|_{B^{\frac{1}{2}}_{\frac{1}{2}}}.
\]

For $N_1 \lesssim N_0 \sim N_2$, we have

\[
|J^1| \lesssim \sum_{N,L} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} \left( \frac{L_2}{N_1} \right)^{\frac{1}{2}} N_2 (N_0 L_1)^{\frac{1}{2}} \langle N_0 L_0 \rangle^{\frac{1}{2}} \| P_{K^{\pm}_{N_1,L_1}}^{\pm_1} A^{\pm}_{\pm_1} \| \| P_{K^{\mp}_{N_2,L_2}}^{\pm_2} \phi^{\pm_2} \|
\]

\[
= \sum_{N,L} L_0^{-\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \left( \frac{N_2}{N_1} \right)^{\frac{1}{2}} N_1 \| P_{K^{\pm}_{N_1,L_1}}^{\pm_1} A^{\pm}_{\pm_1} \| \| P_{K^{\mp}_{N_2,L_2}}^{\pm_2} \phi^{\pm_2} \|
\]

\[
\lesssim \| A^{\pm}_{\pm_1} \|_{B^{\frac{1}{2}}_{\frac{1}{2}}} \| \phi^{\pm_2} \|_{B^{\frac{1}{2}}_{\frac{1}{2}}}.
\]

If $N_2 \lesssim N_0 \sim N_1$, then by [3.3] with $j = 2$, we get

\[
|J^1| \lesssim \sum_{N,L} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} \left( \frac{L_2}{N_2} \right)^{\frac{1}{2}} N_2 (N_2 L_1)^{\frac{1}{2}} \langle N_0 L_0 \rangle^{\frac{1}{2}} \| P_{K^{\pm}_{N_1,L_1}}^{\pm_1} A^{\pm}_{\pm_1} \| \| P_{K^{\mp}_{N_2,L_2}}^{\pm_2} \phi^{\pm_2} \|
\]

\[
= \sum_{N,L} L_0^{-\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \left( \frac{N_2}{N_0} \right)^{\frac{1}{2}} N_0 \| P_{K^{\pm}_{N_1,L_1}}^{\pm_1} A^{\pm}_{\pm_1} \| \| P_{K^{\mp}_{N_2,L_2}}^{\pm_2} \phi^{\pm_2} \|
\]

\[
\lesssim \| A^{\pm}_{\pm_1} \|_{B^{\frac{1}{2}}_{\frac{1}{2}}} \| \phi^{\pm_2} \|_{B^{\frac{1}{2}}_{\frac{1}{2}}}.
\]

5.2.3. Case 3: $L_1 \leq L_2 \leq L_0$.
We first consider the case $L^{12}_{\text{max}} \ll L_0 \ll N_0 \sim N_1 \sim N_2$. To treat this case, we use [3.3] to get

\[
|J^1_{N,L}| \lesssim \sum_{\omega_1, \omega_2} \left| \mathcal{F} [Q_{jk} (D^{-1} P_{K^{\pm}_{N_1,L_1}}^{\pm_1} A^{\pm_{\omega_1}}_{\pm_1}, P_{K^{\pm}_{N_2,L_2}}^{\pm_2} \phi^{\pm_{\omega_2}}_{\pm_2})] \mathcal{F} [P_{K^{\pm}_{N_0,L_0}}^{\pm_0} \varphi] dX_0 \right|.
\]
Then applying Cauchy-Schwarz inequality and Theorem \ref{cor:3.2}, we obtain
\[
|J|^1 \lesssim \sum_{N, L} N_0^{-\frac{2}{3}} L_0^{-\frac{2}{3}} N_2 \|B_{\theta_2}(D^{-1} P_{K_{N, L}}^{1}, A_{\pm, 1}^{\pm}, \phi_{\pm, 2})\|
\]
\[
\lesssim \sum_{N, L} N_0^{-\frac{2}{3}} L_0^{-\frac{2}{3}} N_2 \left( \frac{L_0}{N_0} \right)^{\frac{1}{2}} \|P_{K_{N, L}}^{1} \| \|P_{K_{N, L}}^{2} \phi_{\pm, 2}\|
\]
\[
\lesssim \sum_{N, L} N_2 L_0^{-\frac{2}{3}} (L_1 L_2)^{\frac{1}{2}} \|P_{K_{N, L}}^{1} \| \|P_{K_{N, L}}^{2} \phi_{\pm, 2}\|
\]
\[
\lesssim \|A_{\pm, 1}^k \| \|\phi_{\pm, 2}\| \|B_{\pm, 1}^k \|.
\]
For the case \(L_0 \sim N_0 \sim N_1 \sim N_2\), as before, using \textit{3.5} we see that
\[
|J|^1 \lesssim \sum_{N, L} N_0^{-\frac{2}{3}} L_0^{-\frac{2}{3}} N_2 \|B_{\theta_2}(D^{-1} P_{K_{N, L}}^{1}, A_{\pm, 1}^{\pm}, \phi_{\pm, 2})\|
\]
\[
\lesssim \sum_{N, L} N_2 L_0^{-\frac{2}{3}} (L_1 L_2)^{\frac{1}{2}} \|P_{K_{N, L}}^{1} \| \|P_{K_{N, L}}^{2} \phi_{\pm, 2}\|
\]
\[
\lesssim \|A_{\pm, 1}^k \| \|\phi_{\pm, 2}\| \|B_{\pm, 1}^k \|.
\]
This completes the low-low-high modulation case.

We turn our attention to the case \(L_2 \sim L_0\). This is very straightforward. For \(N_0 \ll N_1 \sim N_2\), we use \textit{3.2} with \(j = 1\) to get
\[
|J|^1 \lesssim \sum_{N, L} N_0^{-\frac{2}{3}} L_0^{-\frac{2}{3}} \left( \frac{L_0}{N_0} \right)^{\frac{1}{2}} N_2 (N_0 L_1)^{\frac{1}{2}} (N_0 L_0)^{\frac{1}{2}} \|P_{K_{N, L}}^{1} \| \|P_{K_{N, L}}^{2} \phi_{\pm, 2}\|
\]
\[
= \sum_{N, L} \left( \frac{N_0}{N_1} \right)^{\frac{2}{3}} N_2 L_0^{-\frac{2}{3}} L_1^{-\frac{2}{3}} \|P_{K_{N, L}}^{1} \| \|P_{K_{N, L}}^{2} \phi_{\pm, 2}\|
\]
\[
\lesssim \|A_{\pm, 1}^k \| \|\phi_{\pm, 2}\| \|B_{\pm, 1}^k \| \left( \sum_{L_0 \sim L_2} L_0^{-\frac{2}{3}} \lesssim L_2^{-\frac{2}{3}} \right)
\]
If \(N_1 \lesssim N_0 \sim N_2\), using \textit{3.1}, we have
\[
|J|^1 \lesssim \sum_{N, L} N_0^{-\frac{2}{3}} L_0^{-\frac{2}{3}} \left( \frac{L_0}{N_0} \right)^{\frac{1}{2}} N_1 (N_1 L_1)^{\frac{1}{2}} (N_1 L_2)^{\frac{1}{2}} \|P_{K_{N, L}}^{1} \| \|P_{K_{N, L}}^{2} \phi_{\pm, 2}\|
\]
\[
= \sum_{N, L} L_1^{-\frac{2}{3}} L_2^{-\frac{2}{3}} \left( \frac{N_2}{N_0} \right)^{\frac{2}{3}} N_1 \|P_{K_{N, L}}^{1} \| \|P_{K_{N, L}}^{2} \phi_{\pm, 2}\|
\]
\[
\lesssim \|A_{\pm, 1}^k \| \|\phi_{\pm, 2}\| \|B_{\pm, 1}^k \|.
\]
If \(N_2 \ll N_0 \sim N_1\), by \textit{3.1},
\[
|J|^1 \lesssim \sum_{N, L} N_0^{-\frac{2}{3}} L_0^{-\frac{2}{3}} \left( \frac{L_0}{N_0} \right)^{\frac{1}{2}} N_2 (N_2 L_1)^{\frac{1}{2}} (N_2 L_0)^{\frac{1}{2}} \|P_{K_{N, L}}^{1} \| \|P_{K_{N, L}}^{2} \phi_{\pm, 2}\|
\]
\[
= \sum_{N, L} L_1^{-\frac{2}{3}} L_2^{-\frac{2}{3}} N_2 \|P_{K_{N, L}}^{1} \| \|P_{K_{N, L}}^{2} \phi_{\pm, 2}\|
\]
\[
\lesssim \|A_{\pm, 1}^k \| \|\phi_{\pm, 2}\| \|B_{\pm, 1}^k \|.
\]
This completes the proof of \textit{3.1}.
5.3. Proof of (5.3). Recalling the definition of $B^r_{\pm}$, we have

\[
\|Q_j(k(\phi_{\pm1}, \phi_{\pm2}))\|_{B^r_{\pm,1}} = \sum_{N_0, L_0 \geq 1} N_0^{-\frac{r}{2}} L_0^{-\frac{r}{2}} \|P_{K_{N_0, L_0}}^{-\frac{1}{2}} Q_j(k(\phi_{\pm1}, \phi_{\pm2}))\|
\]

\[
= \sum_{N_0, L_0 \geq 1} N_0^{-\frac{r}{2}} L_0^{-\frac{r}{2}} \sup_{\|\varphi\| = 1} \left| \int Q_j(k(\phi_{\pm1}, \phi_{\pm2})) P_{K_{N_0, L_0}}^{-\frac{1}{2}} \phi \, dt \, dx \right|
\]

\[
:= J^3.
\]

A dyadic decomposition on \(\phi_{\pm1}\) and \(\phi_{\pm2}\) gives us \(|J^3| \leq \sum_{N, L} N_0^{-\frac{r}{2}} L_0^{-\frac{r}{2}} |J^3_{N, L}|\), where

\[
J^3_{N, L} = \left| \int Q_j(k(P_{K_{N, L}^1}^{-\frac{1}{2}} \phi_{\pm1}, P_{K_{N, L}^2}^{-\frac{1}{2}} \phi_{\pm2}) P_{K_{N_0, L_0}}^{-\frac{1}{2}} \phi \, dt \, dx \right|
\]

and

\[
N = (N_0, N_1, N_2), \quad L = (L_0, L_1, L_2).
\]

As we have done in the proof of (5.1), we assume that \(F(P_{K_{N_1, L_1}^1}^{-\frac{1}{2}} \phi_{\pm1}), F(P_{K_{N_2, L_2}^2}^{-\frac{1}{2}} \phi_{\pm2})\), and \(F(P_{K_{N_0, L_0}^2}^{-\frac{1}{2}} \phi)\) are nonnegative real-valued functions. Then Lemma 5.3 and Cauchy-Schwarz inequality give us

\[
|J^3| \lesssim \sum_{N, L} N_0^{-\frac{r}{2}} L_0^{-\frac{r}{2}} \theta_{12} N_1 N_2 \|P_{K_{N_0, L_0}^{-\frac{1}{2}}}(P_{K_{N_1, L_1}^{-\frac{1}{2}}}(\phi_{\pm1}) P_{K_{N_2, L_2}^{-\frac{1}{2}}}(\phi_{\pm2}))\|
\]

and by Lemma 5.3

\[
\theta_{12} \lesssim \min \left( \gamma', \left( \frac{N_0 \theta_{\text{max}}}{N_1 N_2} \right)^{\frac{1}{2}} \right), \quad \text{for some } 0 < \gamma' \ll 1.
\]

5.3.1. Case 1: \(L_0 \leq L_1 \leq L_2\).

If \(N_0 \ll N_1 \sim N_2\), then

\[
|J^3| \lesssim \sum_{N, L} N_0^{-\frac{r}{2}} L_0^{-\frac{r}{2}} \left( \frac{N_0 L_0}{N_1 N_2} \right)^{\frac{1}{2}} (N_1 N_2)(N_0 L_0)^{\frac{1}{2}} (N_0 L_1)^{\frac{1}{2}} \|P_{K_{N_1, L_1}^{-\frac{1}{2}}}(\phi_{\pm1}) \| \|P_{K_{N_2, L_2}^{-\frac{1}{2}}}(\phi_{\pm2})\|
\]

\[
\lesssim \sum_{N, L} \left( \frac{N_0}{N_1} \right)^{\frac{1}{2}} L_1^{\frac{1}{2}} N_1 N_2^{\frac{1}{2}} L_2^{\frac{1}{2}} \|P_{K_{N_1, L_1}^{-\frac{1}{2}}}(\phi_{\pm1}) \| \|P_{K_{N_2, L_2}^{-\frac{1}{2}}}(\phi_{\pm2})\|
\]

\[
\lesssim \|\phi_{\pm1}\|_{B^{\frac{1}{2}}_{2,1}} \|\phi_{\pm2}\|_{B^{\frac{1}{2}}_{2,1}}.
\]

If \(N_1 \ll N_0 \sim N_2\), then

\[
|J^3| \lesssim \sum_{N, L} N_0^{-\frac{r}{2}} L_0^{-\frac{r}{2}} \left( \frac{L_0}{N_1} \right)^{\frac{1}{2}} (N_1 N_2)(N_1 L_0)^{\frac{1}{2}} (N_1 L_1)^{\frac{1}{2}} \|P_{K_{N_1, L_1}^{-\frac{1}{2}}}(\phi_{\pm1}) \| \|P_{K_{N_2, L_2}^{-\frac{1}{2}}}(\phi_{\pm2})\|
\]

\[
\lesssim \sum_{N, L} \left( \frac{N_1}{N_0} \right)^{\frac{1}{2}} \left( \frac{N_2}{N_0} \right)^{\frac{1}{2}} N_1 N_2^{\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|P_{K_{N_1, L_1}^{-\frac{1}{2}}}(\phi_{\pm1}) \| \|P_{K_{N_2, L_2}^{-\frac{1}{2}}}(\phi_{\pm2})\|
\]

\[
\lesssim \|\phi_{\pm1}\|_{B^{\frac{1}{2}}_{2,1}} \|\phi_{\pm2}\|_{B^{\frac{1}{2}}_{2,1}}.
\]
5.3.2. Case 2: $L_1 \leq L_0 \leq L_2$.

For $N_0 \ll N_1 \sim N_2$, we have

$$|J^3| \lesssim \sum_{N,L} N_0^{-\frac{3}{2}} N_2^{-\frac{1}{2}} \left( \frac{N_0 L_2}{N_1 N_2} \right)^{\frac{1}{2}} (N_1 N_2) (N_0 L_1)^{\frac{1}{2}} (N_0 L_0)^{\frac{1}{2}} \|P_{K_{N_1 L_1}} \phi_{\pm} \| \|P_{K_{N_2 L_2}} \phi_{\pm} \|$$

$$\lesssim \sum_{N,L} L_0^{-\frac{1}{2}} \left( \frac{N_0}{N_1} \right)^{\frac{1}{2}} \left( \frac{N_2}{N_0} \right)^{\frac{1}{2}} (N_1 N_2)^{\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|P_{K_{N_1 L_1}} \phi_{\pm} \| \|P_{K_{N_2 L_2}} \phi_{\pm} \|$$

$$\lesssim \|\phi_{\pm}\|_{L^\infty_{\pm\frac{1}{2}}} \|\phi_{\pm}\|_{L^\infty_{\pm\frac{1}{2}}}.$$ 

If $N_1 \lesssim N_0 \sim N_2$, then

$$|J^3| \lesssim \sum_{N,L} N_0^{-\frac{3}{2}} N_2^{-\frac{1}{2}} \left( \frac{L_2}{N_1} \right)^{\frac{1}{2}} (N_1 N_2) (N_1 L_1)^{\frac{1}{2}} (N_1 L_0)^{\frac{1}{2}} \|P_{K_{N_1 L_1}} \phi_{\pm} \| \|P_{K_{N_2 L_2}} \phi_{\pm} \|$$

$$\lesssim \sum_{N,L} L_0^{-\frac{1}{2}} \left( \frac{N_1}{N_0} \right)^{\frac{1}{2}} \left( \frac{N_2}{N_0} \right)^{\frac{1}{2}} (N_1 N_2)^{\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|P_{K_{N_1 L_1}} \phi_{\pm} \| \|P_{K_{N_2 L_2}} \phi_{\pm} \|$$

$$\lesssim \|\phi_{\pm}\|_{B^{\frac{1}{2}}_{\pm\frac{1}{2}}} \|\phi_{\pm}\|_{B^{\frac{1}{2}}_{\pm\frac{1}{2}}}.$$ 

5.3.3. Case 3: $L_1 \leq L_2 \leq L_0$.

We only consider the case $L_{12}^{12} \ll N_0 \sim N_1 \sim N_2$ for low-low-high modulation. By (3.5), we get

$$|J^3_{NL}| \lesssim \sum_{\omega_1, \omega_2} \int |\mathcal{F}(Q_{jk}(P_{K_{N_1 L_1}} \phi_{\pm}, P_{K_{N_2 L_2}} \phi_{\pm}, \phi_{\pm}))| dX_0.$$ 

Then applying Cauchy-Schwarz inequality and null form estimates, we have

$$|J^3| \lesssim \sum_{N,L} N_0^{-\frac{3}{2}} L_0^{-\frac{1}{2}} (N_1 N_2) \left( \frac{L_0}{N_1} \right)^{\frac{1}{2}} \left( \frac{L_1 L_2}{N_1 L_1} \right)^{\frac{1}{2}} \|P_{K_{N_1 L_1}} \phi_{\pm} \| \|P_{K_{N_2 L_2}} \phi_{\pm} \|$$

$$\lesssim \sum_{N,L} N_0^{-\frac{3}{2}} L_0^{-\frac{1}{2}} N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} \|P_{K_{N_1 L_1}} \phi_{\pm} \| \|P_{K_{N_2 L_2}} \phi_{\pm} \|$$

$$\lesssim \|\phi_{\pm}\|_{B^{\frac{1}{2}}_{\pm\frac{1}{2}}} \|\phi_{\pm}\|_{B^{\frac{1}{2}}_{\pm\frac{1}{2}}}.$$ 

The case $L_0 \sim N_0 \sim N_1 \sim N_2$ is readily followed by angular Whitney decomposition (3.5) and null form estimates (3.2). Then the remaining case $L_2 \sim L_0$ is straightforward. We omit the details. This completes the proof of (5.5) and hence the bilinear estimates.

6. Trilinear estimates

In this section, we give the proof of the trilinear estimates in Section 4. Even though we cannot reveal null forms which we enjoy in the previous section, by the well-separation of angular sectors and bilinear estimates (5.6), we obtain the required estimates. We shall show the following estimates.

$$\|A_{\pm}^{\mu} A_{\pm}^{\pm} \phi_{\pm} \|_{B^{\frac{1}{2}}_{\pm\frac{1}{2}}} \lesssim \|A_{\pm}^{\mu} \|_{B^{\frac{1}{2}}_{\pm\frac{1}{2}}} \|A_{\pm}^{\pm} \|_{B^{\frac{1}{2}}_{\pm\frac{1}{2}}} \|\phi_{\pm} \|_{B^{\frac{1}{2}}_{\pm\frac{1}{2}}},$$

$$\|\phi_{\pm} A_{\pm}^{\pm} \phi_{\pm} \|_{B^{\frac{1}{2}}_{\pm\frac{1}{2}}} \lesssim \|\phi_{\pm} \|_{B^{\frac{1}{2}}_{\pm\frac{1}{2}}} \|A_{\pm}^{\pm} \|_{B^{\frac{1}{2}}_{\pm\frac{1}{2}}} \|\phi_{\pm} \|_{B^{\frac{1}{2}}_{\pm\frac{1}{2}}}.$$ 

The proof of (6.1) and (6.2) is essentially same. We only prove the estimate (6.1).
6.1. Exclusion of low-low-high modulation. As in Section 5, we give a remark on the low-low-high modulation. Since we now deal with trilinear estimates, we consider the integral $J$ of the form:

$$J = \int P_{K_{N_0, L_0}} \varphi_1 P_{K_{N_2, L_2}} \varphi_2 P_{K_{N_3, L_3}} \varphi_3 P_{K_{N_4, L_4}} \varphi \, dt \, dx$$

with assumption on the modulation: $L_{12}^{12} \ll L_0 \ll N_{12}^{12}$ and $L_{max}^{34} \ll L_0 \ll N_{max}^{34}$.

We put $\psi = P_{K_{N_0, L_0}} \varphi_1 P_{K_{N_2, L_2}} \varphi_2 P_{K_{N_3, L_3}} \varphi_3 P_{K_{N_4, L_4}} \varphi$. Since the space-time Fourier support of $\psi$ is contained in $K_{N_0, L_0}$, with no harm we write $\psi = P_{K_{N_0, L_0}} \varphi$. Then $J$ can be rewritten as

$$J = \int F[P_{K_{N_0, L_0}} \varphi_1 P_{K_{N_2, L_2}} \varphi_2](X_0) F[P_{K_{N_3, L_3}} \varphi_3 P_{K_{N_4, L_4}} \varphi](X_0)^{-1} \, dX_0.$$

Here, from the relation on frequencies $\xi_0 = \xi_1 + \xi_2$ and $\xi_0 = \xi_4 - \xi_3$, we have the bilinear interaction:

$X_0 = X_1 + X_2$ and $X_0 = X_4 - X_3$. Keeping in mind this relation, $J$ is also represented as

$$J = \int F[P_{K_{N_0, L_0}} \varphi_1](X_1) F[P_{K_{N_2, L_2}} \varphi_2](X_2) \, dX_0.$$

Then we can exclude $L_{12}^{12} \ll L_0 \ll N_0 \ll N_1 \sim N_2$ and $L_{max}^{34} \ll L_0 \ll N_{max}^{34} \ll N_0 \sim N_{max}^{34}$. On the other hand, we put $\psi = P_{K_{N_0, L_0}} \varphi_1 P_{K_{N_2, L_2}} \varphi_2$. Also, we can rewrite $\psi$ by $\psi = P_{K_{N_0, L_0}} \varphi$. Then our integral $J$ is rewritten as

$$J = \int F[P_{K_{N_0, L_0}} \varphi](X_0) F[P_{K_{N_2, L_2}} \varphi_3 P_{K_{N_4, L_4}} \varphi](X_0) \, dX_0,$$

and

$$J = \int F[P_{K_{N_0, L_0}} \varphi P_{K_{N_3, L_3}} \varphi_3](X_4) F[P_{K_{N_4, L_4}} \varphi](X_4) \, dX_4.$$

Then we see that the cases $L_{max}^{34} \ll L_0 \ll N_0 \sim N_3 \sim N_4$ and $L_{max}^{34} \ll L_0 \ll N_{max}^{34} \ll N_0 \sim N_{max}^{34}$ do not appear. Hence for $L_{12}^{12} \ll L_0$ and $L_{max}^{34} \ll L_0$, we only need to consider the following cases:

$\begin{align*}
L_{12}^{12}, L_{max}^{34} \ll L_0 \ll N_0 \sim N_1 \sim N_2 \sim N_3 \sim N_4,
L_{12}^{12}, L_{max}^{34} \ll L_0 \ll N_0 \sim N_1 \sim N_2 \sim N_3 \sim N_4.
\end{align*}$

Furthermore, we can exclude the case $L_3 \ll L_4 \ll L_0$.

6.2. Proof of (6.1). Recall the definition of $B_{\pm, k, l}^s$ to get

$$\| A_{\pm, k, l}^s \|_{B_{\pm, k, l}^s} \leq \sum_{N_4, L_4 \geq 1} N_4^{-\frac{1}{2}} L_4^{-\frac{1}{2}} \| P_{K_{N_4, L_4}} (A_{\pm, k, l}^s \phi) \|$$

$$= \sum_{N_4, L_4 \geq 1} N_4^{-\frac{1}{2}} L_4^{-\frac{1}{2}} \sup_{\| \varphi \|=1} \left| \int (A_{\pm, k, l}^s \phi) \varphi \, dt \, dx \right|$$

$$:= K_{N, L}^1.$$

A dyadic decomposition of space-time Fourier side on $A_{\pm, k, l}^s \phi$ gives us

$$|K_{N, L}^1| \leq \sum_{N, L} N_4^{-\frac{1}{2}} L_4^{-\frac{1}{2}} |K_{N,L}^1|,$$

where

$$K_{N,L}^1 = \left| \int P_{K_{N_1, L_1}} A_{\pm, k, l}^s P_{K_{N_2, L_2}} \phi \, dt \, dx \right|$$

and

$$N = (N_0, N_1, N_2, N_3, N_4), \quad L = (L_0, L_1, L_2, L_3, L_4).$$
Now we can write the integrand of $K^1$ with the combination of positive and negative parts of every component of $\mathcal{F}(P_{K_{N_1}^{\pm_1} A_{\mu}^{\pm_1}}), \mathcal{F}(P_{K_{N_2}^{\pm_2} A_{\mu}^{\pm_2}}), \mathcal{F}(P_{K_{N_3}^{\pm_3} \phi_{\pm_3}}), \mathcal{F}(P_{K_{N_4}^{\pm_4} \varphi})$. Then without loss of generality, we may assume that $\mathcal{F}(P_{K_{N_1}^{\pm_1} A_{\mu}^{\pm_1}}), \mathcal{F}(P_{K_{N_2}^{\pm_2} A_{\mu}^{\pm_2}}), \mathcal{F}(P_{K_{N_3}^{\pm_3} \phi_{\pm_3}}), \mathcal{F}(P_{K_{N_4}^{\pm_4} \varphi})$ are nonnegative positive real-valued functions.

6.2.1. Case 1: $L_0 \leq L_1 \leq L_2$, $L_0 \leq L_3 \leq L_4$.

This case is treated by only using bilinear estimates (3.1) and (3.2). Indeed, if $N_0 \ll N_1 \sim N_2$ and $N_0 \ll N_3 \sim N_4$, then

\[
|K^1| \lesssim \sum_{N,L} N_4^{-\frac{3}{4}} L_4^{-\frac{1}{4}} \|P_{K_{N_0}^{\pm_0}} (P_{K_{N_1}^{\pm_1} A_{\mu}^{\pm_1}} P_{K_{N_2}^{\pm_2} A_{\mu}^{\pm_2}}) \| \|P_{K_{N_0}^{\pm_0}} (P_{K_{N_3}^{\pm_3} \phi_{\pm_3}} P_{K_{N_4}^{\pm_4} \varphi})
\]

\[
\lesssim \sum_{N,L} N_4^{-\frac{3}{4}} L_4^{-\frac{1}{4}} (N_0 L_0)^{\frac{1}{2}} (N_0 L_1)^{\frac{1}{2}} (N_0 L_3)^{\frac{1}{2}} (N_0 L_4)^{\frac{1}{2}}
\]

\[
\times \|P_{K_{N_1}^{\pm_1}} A_{\mu}^{\pm_1} \| \|P_{K_{N_2}^{\pm_2}} A_{\mu}^{\pm_2} \| \|P_{K_{N_3}^{\pm_3} \phi_{\pm_3}} \| \|P_{K_{N_4}^{\pm_4} \varphi} \|
\]

\[
\lesssim \sum_{N,L} N_4^{-\frac{3}{4}} N_0^{-\frac{1}{2}} N_0^{\frac{1}{4}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_4^{-\frac{1}{2}} \|P_{K_{N_1}^{\pm_1}} A_{\mu}^{\pm_1} \| \|P_{K_{N_2}^{\pm_2}} A_{\mu}^{\pm_2} \| \|P_{K_{N_3}^{\pm_3} \phi_{\pm_3}} \| \|P_{K_{N_4}^{\pm_4} \varphi} \|
\]

\[
\lesssim \|A_{\mu}^{\pm_1} \|_{B_{L_1}^{1,2}} \|A_{\mu}^{\pm_2} \|_{B_{L_2}^{1,2}} \|\phi_{\pm_3} \|_{B_{L_3}^{1,2}} \|\varphi \|_{B_{L_4}^{1,2}}.
\]

For $N_0 \ll N_1 \sim N_2$ and $N_3 \ll N_0 \sim N_4$,

\[
|K^1| \lesssim \sum_{N,L} N_4^{-\frac{3}{4}} L_4^{-\frac{1}{4}} (N_0 L_0)^{\frac{1}{2}} (N_0 L_1)^{\frac{1}{2}} (N_0 L_3)^{\frac{1}{2}} (N_0 L_4)^{\frac{1}{2}}
\]

\[
\times \|P_{K_{N_1}^{\pm_1}} A_{\mu}^{\pm_1} \| \|P_{K_{N_2}^{\pm_2}} A_{\mu}^{\pm_2} \| \|P_{K_{N_3}^{\pm_3} \phi_{\pm_3}} \| \|P_{K_{N_4}^{\pm_4} \varphi} \|
\]

\[
\lesssim \sum_{N,L} N_4^{-\frac{3}{4}} N_0^{-\frac{1}{2}} N_0^{\frac{1}{4}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_4^{-\frac{1}{2}} \|P_{K_{N_1}^{\pm_1}} A_{\mu}^{\pm_1} \| \|P_{K_{N_2}^{\pm_2}} A_{\mu}^{\pm_2} \| \|P_{K_{N_3}^{\pm_3} \phi_{\pm_3}} \| \|P_{K_{N_4}^{\pm_4} \varphi} \|
\]

\[
\lesssim \|A_{\mu}^{\pm_1} \|_{B_{L_1}^{1,2}} \|A_{\mu}^{\pm_2} \|_{B_{L_2}^{1,2}} \|\phi_{\pm_3} \|_{B_{L_3}^{1,2}} \|\varphi \|_{B_{L_4}^{1,2}}.
\]

If $N_0 \ll N_1 \sim N_2$ and $N_4 \ll N_0 \sim N_3$,

\[
|K^1| \lesssim \sum_{N,L} N_4^{-\frac{3}{4}} L_4^{-\frac{1}{4}} (N_0 L_0)^{\frac{1}{2}} (N_0 L_1)^{\frac{1}{2}} (N_0 L_4)^{\frac{1}{2}} (N_0 L_3)^{\frac{1}{2}}
\]

\[
\times \|P_{K_{N_1}^{\pm_1}} A_{\mu}^{\pm_1} \| \|P_{K_{N_2}^{\pm_2}} A_{\mu}^{\pm_2} \| \|P_{K_{N_3}^{\pm_3} \phi_{\pm_3}} \| \|P_{K_{N_4}^{\pm_4} \varphi} \|
\]

\[
\lesssim \sum_{N,L} N_4^{-\frac{3}{4}} N_0^{-\frac{1}{2}} N_0^{\frac{1}{4}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_4^{-\frac{1}{2}} \|P_{K_{N_1}^{\pm_1}} A_{\mu}^{\pm_1} \| \|P_{K_{N_2}^{\pm_2}} A_{\mu}^{\pm_2} \| \|P_{K_{N_3}^{\pm_3} \phi_{\pm_3}} \| \|P_{K_{N_4}^{\pm_4} \varphi} \|
\]

\[
\lesssim \|A_{\mu}^{\pm_1} \|_{B_{L_1}^{1,2}} \|A_{\mu}^{\pm_2} \|_{B_{L_2}^{1,2}} \|\phi_{\pm_3} \|_{B_{L_3}^{1,2}} \|\varphi \|_{B_{L_4}^{1,2}}.
\]

If $N_1 \ll N_0 \sim N_2$ with $N_0 \ll N_3 \sim N_4$, we get

\[
|K^1| \lesssim \sum_{N,L} N_4^{-\frac{3}{4}} L_4^{-\frac{1}{4}} (N_1 L_0)^{\frac{1}{2}} (N_1 L_1)^{\frac{1}{2}} (N_0 L_4)^{\frac{1}{2}} (N_0 L_3)^{\frac{1}{2}}
\]

\[
\times \|P_{K_{N_1}^{\pm_1}} A_{\mu}^{\pm_1} \| \|P_{K_{N_2}^{\pm_2}} A_{\mu}^{\pm_2} \| \|P_{K_{N_3}^{\pm_3} \phi_{\pm_3}} \| \|P_{K_{N_4}^{\pm_4} \varphi} \|
\]

\[
\lesssim \sum_{N,L} N_4^{-\frac{3}{4}} N_1^{-\frac{1}{2}} N_0^{\frac{1}{4}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_4^{-\frac{1}{2}} \|P_{K_{N_1}^{\pm_1}} A_{\mu}^{\pm_1} \| \|P_{K_{N_2}^{\pm_2}} A_{\mu}^{\pm_2} \| \|P_{K_{N_3}^{\pm_3} \phi_{\pm_3}} \| \|P_{K_{N_4}^{\pm_4} \varphi} \|
\]

\[
\lesssim \|A_{\mu}^{\pm_1} \|_{B_{L_1}^{1,2}} \|A_{\mu}^{\pm_2} \|_{B_{L_2}^{1,2}} \|\phi_{\pm_3} \|_{B_{L_3}^{1,2}} \|\varphi \|_{B_{L_4}^{1,2}}.
\]
For $N_1 \lesssim N_0 \sim N_2$ with $N_3 \lesssim N_0 \sim N_4$, we have

$$|K^1| \lesssim \sum_{N,L} N_4^{-\frac{3}{4}} L_4^{-\frac{3}{4}} (N_1 L_0)^{\frac{3}{4}} (N_1 L_1)^{\frac{3}{4}} (N_3 L_0)^{\frac{3}{4}} (N_3 L_3)^{\frac{3}{4}}$$

$$\times \|P_{K_{N_1, L_1}} A_{\mu, \pm 2}^N \| \|P_{K_{N_2, L_2}}^0 A_{\mu, \pm 2}^N \| \|P_{K_{N_3, L_3}}^0 \phi_{\pm 3} \| \|P_{K_{N_4, L_4}}^0 \varphi \|$$

$$\lesssim \sum_{N,L} N_4^{-\frac{3}{4}} N_1^{-\frac{3}{4}} L_0 L_1 L_3^{-\frac{3}{4}} L_4^{-\frac{3}{4}} \|P_{K_{N_1, L_1}} A_{\mu, \pm 2}^N \| \|P_{K_{N_2, L_2}}^0 A_{\mu, \pm 2}^N \| \|P_{K_{N_3, L_3}}^0 \phi_{\pm 3} \|$$

$$\lesssim \|A_{\mu, \pm 2}^N\|_{B_{\frac{3}{4}}^{\frac{3}{4}}} \|A_{\mu, \pm 2}^N\|_{B_{\frac{3}{4}}^{\frac{3}{4}}} \|\phi_{\pm 3}\|_{B_{\frac{3}{4}}^{\frac{3}{4}}}.$$

If $N_1 \lesssim N_0 \sim N_2$ and $N_4 \lesssim N_0 \sim N_3$, then

$$|K^1| \lesssim \sum_{N,L} N_4^{-\frac{3}{4}} L_4^{-\frac{3}{4}} (N_1 L_0)^{\frac{3}{4}} (N_1 L_1)^{\frac{3}{4}} (N_3 L_0)^{\frac{3}{4}} (N_3 L_3)^{\frac{3}{4}}$$

$$\times \|P_{K_{N_1, L_1}} A_{\mu, \pm 2}^N \| \|P_{K_{N_2, L_2}}^0 A_{\mu, \pm 2}^N \| \|P_{K_{N_3, L_3}}^0 \phi_{\pm 3} \| \|P_{K_{N_4, L_4}}^0 \varphi \|$$

$$\lesssim \sum_{N,L} N_4^{-\frac{3}{4}} N_1^{-\frac{3}{4}} L_0 L_1 L_3^{-\frac{3}{4}} L_4^{-\frac{3}{4}} \|P_{K_{N_1, L_1}} A_{\mu, \pm 2}^N \| \|P_{K_{N_2, L_2}}^0 A_{\mu, \pm 2}^N \| \|P_{K_{N_3, L_3}}^0 \phi_{\pm 3} \|$$

$$\lesssim \|A_{\mu, \pm 2}^N\|_{B_{\frac{3}{4}}^{\frac{3}{4}}} \|A_{\mu, \pm 2}^N\|_{B_{\frac{3}{4}}^{\frac{3}{4}}} \|\phi_{\pm 3}\|_{B_{\frac{3}{4}}^{\frac{3}{4}}}.$$

6.2.2. Case 2: $L_1 \leq L_2 \leq L_0$, $L_3 \leq L_4 \leq L_0$. For the case $L_{12}^{12} \ll L_0$ and $L_3 \leq L_4 \ll L_0$, $(L_3 \ll L_4 \ll L_0$ is already excluded.) we are left to treat $L_0 \ll N_0 \sim N_j$, $j = 0, 1, 2, 3, 4$. To do this, we apply angular Whitney decomposition [3.4] to get

$$|K_{N,L}^1| \lesssim \sum_{\omega_1, \omega_2, \omega_3, \omega_4} \int \mathcal{F}[P_{K_{N_1, L_1}} A_{\mu, 2}^{2 \theta_{12} \omega_1} P_{K_{N_2, L_2}}^0 A_{\mu, 2}^{2 \theta_{12} \omega_2}](X_0)$$

$$\times \mathcal{F}[P_{K_{N_3, L_3}}^0 \phi_{\pm 3}^{2 \theta_{12} \omega_3} P_{K_{N_4, L_4}}^0 \varphi^{2 \theta_{12} \omega_4}](X_0) dX_0,$$

where $\omega_1, \omega_2 \in \Omega(\theta_{12})$ and $\omega_3, \omega_4 \in \Omega(\theta_{12})$ with $3 \theta_{12} \leq \angle(\omega_1, \omega_2), \angle(\omega_3, \omega_4) \leq 12 \theta_{12}$. By the separation of angular sectors, we recall Theorem [3.6] to obtain

$$|K^1| \lesssim \sum_{N,L} \sum_{\omega_1, \omega_2, \omega_3, \omega_4} N_4^{-\frac{3}{4}} L_4^{-\frac{3}{4}} \|P_{K_{N_0, L_0}}^{2 \theta_{12} \omega_1} P_{K_{N_1, L_1}} A_{\mu, 2}^{2 \theta_{12} \omega_2} P_{K_{N_2, L_2}} A_{\mu, 2}^{2 \theta_{12} \omega_2} \|$$

$$\times \|P_{K_{N_3, L_3}}^0 \phi_{\pm 3}^{2 \theta_{12} \omega_3} P_{K_{N_4, L_4}}^0 \varphi^{2 \theta_{12} \omega_4} \|$$

$$\lesssim \sum_{N,L} \sum_{\omega_1, \omega_2, \omega_3, \omega_4} \sum_{N_4^{-\frac{3}{4}} L_4^{-\frac{3}{4}} \max(L_1 L_2)} \|P_{K_{N_1, L_1}} A_{\mu, 2}^{2 \theta_{12} \omega_1} \| \|P_{K_{N_2, L_2}} A_{\mu, 2}^{2 \theta_{12} \omega_2} \|$$

$$\times \|P_{K_{N_3, L_3}}^0 \phi_{\pm 3}^{2 \theta_{12} \omega_3} \| \|P_{K_{N_4, L_4}}^0 \varphi^{2 \theta_{12} \omega_4} \|$$

$$\lesssim \sum_{N,L} \sum_{\omega_1, \omega_2, \omega_3, \omega_4} \sum_{N_4^{-\frac{3}{4}} L_4^{-\frac{3}{4}} \max(L_1 L_2)} \|P_{K_{N_1, L_1}} A_{\mu, 2}^{2 \theta_{12} \omega_1} \| \|P_{K_{N_2, L_2}} A_{\mu, 2}^{2 \theta_{12} \omega_2} \|$$

$$\times \|P_{K_{N_3, L_3}}^0 \phi_{\pm 3}^{2 \theta_{12} \omega_3} \| \|P_{K_{N_4, L_4}}^0 \varphi^{2 \theta_{12} \omega_4} \|$$

$$\lesssim \|A_{\mu, 2}^{2 \theta_{12} \omega_1} \|_{B_{\frac{3}{4}}^{\frac{3}{4}}} \|A_{\mu, 2}^{2 \theta_{12} \omega_2} \|_{B_{\frac{3}{4}}^{\frac{3}{4}}} \|\phi_{\pm 3}^{2 \theta_{12} \omega_3} \|_{B_{\frac{3}{4}}^{\frac{3}{4}}} \|\varphi^{2 \theta_{12} \omega_4} \|_{B_{\frac{3}{4}}^{\frac{3}{4}}}.$$
Finally, it remains to consider the case \( L_2 \sim L_0 \) and \( L_4 \sim L_0 \). Here, we only treat the case \( N_1 \lesssim N_0 \sim N_2 \) and \( N_{\text{min}}^{34} \lesssim N_0 \sim N_{\text{max}}^{34} \). If \( N_{\text{min}}^{34} = N_4 \), using (3.1), we get

\[
|K^1| \lesssim \sum_{N, L} N_4^\frac{1}{4} L_4^\frac{1}{4} (N_1 L_1)^\frac{1}{4} (N_2 L_2)^\frac{1}{4} (N_3 L_3)^\frac{1}{4} (N_4 L_4)^\frac{1}{4} \\
\times \|P_{K_{N_1, L_1}} A_{\pm}^\mu, \pm \| \|P_{K_{N_2, L_2}} A_{\mu, \pm} \| \|P_{K_{N_3, L_3}} \phi_{\pm} \| \|P_{K_{N_4, L_4}} \varphi\| \\
\lesssim \sum_{N, L} N_4^\frac{1}{4} L_4^\frac{1}{4} L_2^\frac{1}{4} L_4^\frac{1}{4} \|P_{K_{N_1, L_1}} A_{\pm}^\mu, \pm \| \|P_{K_{N_2, L_2}} A_{\mu, \pm} \| \|P_{K_{N_3, L_3}} \phi_{\pm} \| \\
\lesssim \|A_{\pm}^\mu \|_{B_{\pm}^\frac{1}{2} \frac{1}{2}} \|A_{\mu, \pm} \|_{B_{\pm}^\frac{1}{2} \frac{1}{2}} \| \phi_{\pm} \|_{B_{\pm}^\frac{3}{2} \frac{1}{2}}.
\]

For \( N_{\text{min}}^{34} = N_3 \), we have

\[
|K^1| \lesssim \sum_{N, L} N_4^\frac{1}{4} L_4^\frac{1}{4} (N_1 L_1)^\frac{1}{4} (N_2 L_2)^\frac{1}{4} (N_3 L_3)^\frac{1}{4} (N_4 L_4)^\frac{1}{4} \\
\times \|P_{K_{N_1, L_1}} A_{\pm}^\mu, \pm \| \|P_{K_{N_2, L_2}} A_{\mu, \pm} \| \|P_{K_{N_3, L_3}} \phi_{\pm} \| \|P_{K_{N_4, L_4}} \varphi\| \\
\lesssim \sum_{N, L} N_4^\frac{1}{4} N_3^\frac{1}{4} L_4^\frac{1}{4} L_2^\frac{1}{4} L_4^\frac{1}{4} \|P_{K_{N_1, L_1}} A_{\pm}^\mu, \pm \| \|P_{K_{N_2, L_2}} A_{\mu, \pm} \| \|P_{K_{N_3, L_3}} \phi_{\pm} \| \\
\lesssim \|A_{\pm}^\mu \|_{B_{\pm}^\frac{1}{2} \frac{1}{2}} \|A_{\mu, \pm} \|_{B_{\pm}^\frac{1}{2} \frac{1}{2}} \| \phi_{\pm} \|_{B_{\pm}^\frac{3}{2} \frac{1}{2}}.
\]

6.2.3. Case 3: \( L_1 \leq L_2 \leq L_0 \), \( L_0 \leq L_3 \leq L_4 \).

First, we consider the case \( L_{\text{max}}^{12} \ll L_4 \). As stated in Section 6.1, we only have to consider the case \( L_{\text{max}}^{12} \ll L_4 \ll N_3 \sim N_4 \). Also, we already have \( L_{\text{max}}^{12} \ll L_0 \ll N_0 \sim N_1 \sim N_2 \). To do this case, we invoke angular Whitney decomposition \([3,4]\) to get

\[
|K_{N, L}| \lesssim \sum_{\omega_1, \omega_2} \mathcal{F}[P_{K_{N_1, L_1}} A_{\pm}^{\theta_{12}, \omega_1} P_{K_{N_2, L_2}} A_{\mu, \pm}^{\theta_{12}, \omega_2}(X_0) \\
\times \mathcal{F}[P_{K_{N_3, L_3}} \phi_{\pm} P_{K_{N_4, L_4}} \varphi](X_0) \, dX_0,
\]

where \( \omega_1, \omega_2 \in \Omega(\theta_{12}) \) satisfy \( 3\theta_{12} \leq \angle(\omega_1, \omega_2) \leq 12\theta_{12} \). Then we use Cauchy-Schwarz inequality and Theorem (3.3) to obtain

\[
|K^1| \lesssim \sum_{N, L} \sum_{\omega_1, \omega_2} N_4^\frac{1}{4} L_4^\frac{1}{4} \|P_{K_{N_1, L_0}} (P_{K_{N_1, L_1}} A_{\pm}^{\theta_{12}, \omega_1} P_{K_{N_2, L_2}} A_{\mu, \pm}^{\theta_{12}, \omega_2})\| \\
\times \|P_{K_{N_0, L_0}} (P_{K_{N_3, L_3}} \phi_{\pm} P_{K_{N_4, L_4}} \varphi)\| \\
\lesssim \sum_{N, L} \sum_{\omega_1, \omega_2} N_4^\frac{1}{4} L_4^\frac{1}{4} (N_{\text{max}}^{12} L_1 L_2) \|P_{K_{N_1, L_1}} A_{\pm}^{\theta_{12}, \omega_1}\| \|P_{K_{N_2, L_2}} A_{\mu, \pm}^{\theta_{12}, \omega_2}\| \\
\times (N_0 L_0) \|P_{K_{N_3, L_3}} \phi_{\pm}\| \|P_{K_{N_4, L_4}} \varphi\| \\
\lesssim \sum_{N, L} (N_{\text{max}}^{12}) \|L_1 L_2\| L_4^\frac{1}{4} \|P_{K_{N_1, L_1}} A_{\pm}^{\theta_{12}, \omega_1}\| \|P_{K_{N_2, L_2}} A_{\mu, \pm}^{\theta_{12}, \omega_2}\| \\
\lesssim \|A_{\pm}^{\theta_{12}}\|_{B_{\pm}^\frac{1}{2} \frac{1}{2}} \|A_{\mu, \pm}^{\theta_{12}}\|_{B_{\pm}^\frac{1}{2} \frac{1}{2}} \| \phi_{\pm} \|_{B_{\pm}^\frac{3}{2} \frac{1}{2}},
\]

where we use the summation by \( L_4; L_0 \ll L_4 \) and \( L_0 \ll L_3 \). If \( L_0 \gtrsim N_{\text{min}}^{12} \), as before, we have \( L_0 \sim N_k \), \( k = 0, 1, 2 \) and hence \( L_0 \sim N_0 \sim L_4 \), which contradicts to the assumption \( L_{\text{max}}^{12} \ll L_4 \), so this case is excluded. Then we consider the case \( L_4 \gtrsim N_{\text{min}}^{34} \). This implies that \( L_4 \sim N_0 \sim N_j, j = 3, 4 \). It is also excluded since we have assumed \( L_{\text{max}}^{34} \ll L_4 \). This completes the case \( L_{\text{max}}^{12} \ll L_0 \) and \( L_{\text{max}}^{12} \ll L_4 \).
Now it remains to consider the case \( L_0 \sim L_2 \) and \( L_3 \sim L_4 \). This is very straightforward. Here, we only treat the case \( N_1 \lesssim N_0 \sim N_2 \) and \( N_4 \lesssim N_0 \sim N_3 \) as follows.

\[
|K^1| \lesssim \sum_{N,L} N_4^{-\frac{1}{4}} L_4 \left( N_1 L_1 \right)^{\frac{1}{4}} (N_1 L_2)^{\frac{1}{4}} (N_4 L_3)^{\frac{1}{4}} (N_4 L_4)^{\frac{1}{4}} \times \left\| P_{K_{N_1,L_1}} A_{\mu, \pm} \right\| \left\| P_{K_{N_2,L_2}} A_{\mu, \pm} \right\| \left\| P_{K_{N_3,L_3}} A_{\mu, \pm} \right\| \left\| P_{K_{N_4,L_4}} A_{\mu, \pm} \right\|
\]

\[
\lesssim \sum_{N,L} N_4^{-\frac{1}{4}} L_4 \left( N_1 L_1 \right)^{\frac{1}{4}} (N_1 L_2)^{\frac{1}{4}} (N_4 L_3)^{\frac{1}{4}} (N_4 L_4)^{\frac{1}{4}} \times \left\| P_{K_{N_1,L_1}} A_{\mu, \pm} \right\| \left\| P_{K_{N_2,L_2}} A_{\mu, \pm} \right\| \left\| P_{K_{N_3,L_3}} A_{\mu, \pm} \right\|
\]

\[
\lesssim \| A_{\mu, \pm} \|_{B^{0,\frac{1}{4},\frac{1}{4}}_{1,1}} \| \phi_{\pm, \pm} \|_{B^{0,\frac{1}{4},\frac{1}{4}}_{1,1}} \| \phi_{\pm, \pm} \|_{B^{0,\frac{1}{4},\frac{1}{4}}_{1,1}}
\]

6.2.4. Case 4: \( L_0 \leq L_1 \leq L_2, L_3 \leq L_4 \leq L_0 \).

As Case 3, first we consider \( L_0 \leq L_2 \leq N_{0, \min} \) and \( L_3 \leq L_4 \leq L_0 \sim N_k, k = 0, 3, 4 \). Similarly, we only need to consider \( L_2 \leq N_0 \sim N_1 \sim N_2 \). We deal with this case by angular Whitney decomposition. Indeed, we write

\[
|K^1_{N,L}| \lesssim \sum_{\nu, \omega} \int \mathcal{F} \left[ P_{K_{N_1,L_1}} A_{\pm, \pm} P_{K_{N_2,L_2}} A_{\mu, \pm} \right] (X_0) dX_0
\]

Then we have

\[
|K^1| \lesssim \sum_{N,L} \left\| P_{K_{N,L}} A_{\pm, \pm} \right\| \left\| P_{K_{N,L}} A_{\mu, \pm} \right\| \left\| P_{K_{N,L}} A_{\mu, \pm} \right\|
\]

\[
\lesssim \sum_{N,L} \left\| P_{K_{N,L}} A_{\pm, \pm} \right\| \left\| P_{K_{N,L}} A_{\mu, \pm} \right\| \left\| P_{K_{N,L}} A_{\mu, \pm} \right\|
\]

where we used the summation \( L_0 \leq L_2 \).

Now we deal with the case \( N_{0, \min} \sim L_2 \), which implies \( L_2 \sim N_j, j = 0, 1, 2, \). This is impossible, because we must have \( L_2 \sim L_0 \). Thus we are left to treat the case \( L_1 \sim L_2 \) and \( L_4 \sim L_3 \). This is treated by the similar way as in previous sections and so we omit details. There are several cases left, but the required estimates can be obtained by direct calculation. We omit the details. This completes the proof of (6.1).

7. Estimates of \( \mathcal{V}(\phi, \phi^\dagger) \)

In this section we focus on the Higgs potential \( \mathcal{V}(\phi, \phi^\dagger) \). We only need to prove the following estimates:

\[
\| \phi_{\pm, \pm}, \phi_{\pm, \pm} \|_{B^{0,\frac{1}{4},\frac{1}{4}}_{1,1}(S_T)} \lesssim \prod_{j=1}^3 \| \phi_{\pm, \pm} \|_{B^{0,\frac{1}{4},\frac{1}{4}}_{1,1}(Y)}
\]

\[
\| \phi_{\pm, \pm}, \phi_{\pm, \pm} \|_{B^{0,\frac{1}{4},\frac{1}{4}}_{1,1}(S_T)} \lesssim \prod_{j=1}^5 \| \phi_{\pm, \pm} \|_{B^{0,\frac{1}{4},\frac{1}{4}}_{1,1}(Y)}
\]
7.1. Proof of (7.1). The proof of (7.1) is easier than Section 6. In fact, by definition of $B^{s,b}_j$, 
\[ \|\phi_{\pm_1} \phi_{\pm_2} \phi_{\pm_3}\|_{B^{s,b}_j} = \sum_{N_4, L_4 \geq 1} N_4^{-\frac{s}{2}} L_4^{-\frac{b}{2}} \|P_{K_{N_4}^{\frac{-s}{2},L_4}} \phi_{\pm_1} \phi_{\pm_2} \phi_{\pm_3}\| = \sum_{N_4, L_4 \geq 1} N_4^{-\frac{s}{2}} L_4^{-\frac{b}{2}} \sup_{\|\varphi\|=1} \left| \int (\phi_{\pm_1} \phi_{\pm_2} \phi_{\pm_3}) P_{K_{N_4}^{\frac{-s}{2},L_4}} \varphi \, dt \, dx \right| := \mathbf{H}^1. \]

By a dyadic decomposition on $\phi_{\pm_j}$, $j = 1, 2, 3$, we have $|\mathbf{H}^1| \leq \sum_{N,L} N_4^{-\frac{s}{2}} L_4^{-\frac{b}{2}} |\mathbf{H}^1_{N,L}|$, where 
\[ |\mathbf{H}^1_{N,L}| \leq \int P_{K_{N_4}^{\frac{-s}{2},L_4}} \phi_{\pm_1} P_{K_{N_2}^{\frac{-s}{2},L_2}} \phi_{\pm_2} P_{K_{N_3}^{\frac{-s}{2},L_3}} \phi_{\pm_3} \frac{P_{K_{N_4}^{\frac{-s}{2},L_4}} \varphi}{\|\varphi\|_{L^\infty}} \, dt \, dx \]
and 
\[ N = (N_0, N_1, N_2, N_3, N_4), \quad L = (L_0, L_1, L_2, L_3, L_4). \]

7.1.1. Case 1: $L_1 \leq L_2 \leq L_0$, $L_3 \leq L_4 \leq L_0$. 
If $N_0 \ll N_1 \sim N_2$ and $N_0 \ll N_3 \sim N_4$, then by (7.1) we get 
\[ |\mathbf{H}^1| \lesssim \sum_{N,L} N_4^{-\frac{s}{2}} L_4^{-\frac{b}{2}} (N_0 L_1)^{\frac{s}{2}} (N_1 L_2)^{\frac{s}{2}} (N_2 L_3)^{\frac{s}{2}} (N_3 L_4)^{\frac{s}{2}} \|P_{K_{N_4}^{\frac{-s}{2},L_4}} \phi_{\pm_1}\| \|P_{K_{N_2}^{\frac{-s}{2},L_2}} \phi_{\pm_2}\| \|P_{K_{N_3}^{\frac{-s}{2},L_3}} \phi_{\pm_3}\| \]
\[ \lesssim \sum_{N,L} N_0 N_1^3 (L_1 L_3)^{\frac{s}{2}} (L_2 L_4)^{\frac{s}{2}} \|P_{K_{N_4}^{\frac{-s}{2},L_4}} \phi_{\pm_1}\| \|P_{K_{N_2}^{\frac{-s}{2},L_2}} \phi_{\pm_2}\| \|P_{K_{N_3}^{\frac{-s}{2},L_3}} \phi_{\pm_3}\| \]
\[ \lesssim \prod_{j=1}^3 \|\phi_{\pm_j}\|_{B_{s,j}^{\frac{3}{2},1}}. \]

If $N_0 \ll N_1 \sim N_2$ and $N_3 \lesssim N_0 \ll N_4$, then 
\[ |\mathbf{H}^1| \lesssim \sum_{N,L} N_4^{-\frac{s}{2}} L_4^{-\frac{b}{2}} (N_0 L_1)^{\frac{s}{2}} (N_1 L_2)^{\frac{s}{2}} (N_2 L_3)^{\frac{s}{2}} (N_3 L_4)^{\frac{s}{2}} \|P_{K_{N_4}^{\frac{-s}{2},L_4}} \phi_{\pm_1}\| \|P_{K_{N_2}^{\frac{-s}{2},L_2}} \phi_{\pm_2}\| \|P_{K_{N_3}^{\frac{-s}{2},L_3}} \phi_{\pm_3}\| \]
\[ \lesssim \sum_{N,L} N_0^3 N_1^3 (L_1 L_3)^{\frac{s}{2}} (L_2 L_4)^{\frac{s}{2}} \|P_{K_{N_4}^{\frac{-s}{2},L_4}} \phi_{\pm_1}\| \|P_{K_{N_2}^{\frac{-s}{2},L_2}} \phi_{\pm_2}\| \|P_{K_{N_3}^{\frac{-s}{2},L_3}} \phi_{\pm_3}\| \]
\[ \lesssim \prod_{j=1}^3 \|\phi_{\pm_j}\|_{B_{s,j}^{\frac{3}{2},1}}. \]

For $N_0 \ll N_1 \sim N_2$ and $N_4 \lesssim N_0 \sim N_3$, 
\[ |\mathbf{H}^1| \lesssim \sum_{N,L} N_4^{-\frac{s}{2}} L_4^{-\frac{b}{2}} (N_0 L_1)^{\frac{s}{2}} (N_1 L_2)^{\frac{s}{2}} (N_2 L_3)^{\frac{s}{2}} (N_3 L_4)^{\frac{s}{2}} \|P_{K_{N_4}^{\frac{-s}{2},L_4}} \phi_{\pm_1}\| \|P_{K_{N_2}^{\frac{-s}{2},L_2}} \phi_{\pm_2}\| \|P_{K_{N_3}^{\frac{-s}{2},L_3}} \phi_{\pm_3}\| \]
\[ \lesssim \sum_{N,L} N_0^3 N_1^3 (L_1 L_3)^{\frac{s}{2}} (L_2 L_4)^{\frac{s}{2}} \|P_{K_{N_4}^{\frac{-s}{2},L_4}} \phi_{\pm_1}\| \|P_{K_{N_2}^{\frac{-s}{2},L_2}} \phi_{\pm_2}\| \|P_{K_{N_3}^{\frac{-s}{2},L_3}} \phi_{\pm_3}\| \]
\[ \lesssim \prod_{j=1}^3 \|\phi_{\pm_j}\|_{B_{s,j}^{\frac{3}{2},1}}. \]
If $N_1 \lesssim N_0 \sim N_2$ and $N_0 \ll N_3 \sim N_1$,

$$\left| H^1 \right| \lesssim \sum_{N,L} N_4^2 \frac{1}{5} L_4^2 \zeta \frac{1}{5} (N_1 L_1) \zeta \frac{1}{5} (N_1 L_2) \zeta \frac{1}{5} (N_2 L_3) \zeta \frac{1}{5} (N_3 L_4) \zeta \frac{1}{5} \left\| P_{K_{N_1},1} \phi_{\pm 1} \right\| \left\| P_{K_{N_2},L_2} \phi_{\pm 2} \right\| \left\| P_{K_{N_3},L_3} \phi_{\pm 3} \right\|$$

$$\lesssim \sum_{N,L} N_4^2 \frac{1}{3} N_3 \frac{1}{3} (L_1 L_3) \zeta \frac{1}{3} L_4^2 \zeta \frac{1}{3} \left\| P_{K_{N_1},1} \phi_{\pm 1} \right\| \left\| P_{K_{N_2},L_2} \phi_{\pm 2} \right\| \left\| P_{K_{N_3},L_3} \phi_{\pm 3} \right\|$$

$$\lesssim \prod_{j=1}^3 \left\| \phi_{\pm j} \right\|_{B^\frac{1}{2} \frac{1}{4} \cdot 1}.$$  

If $N_1 \lesssim N_0 \sim N_2$ and $N_3 \lesssim N_0 \sim N_4$ then,

$$\left| H^1 \right| \lesssim \sum_{N,L} N_4^2 \frac{1}{5} L_4^2 \zeta \frac{1}{5} (N_1 L_1) \zeta \frac{1}{5} (N_1 L_2) \zeta \frac{1}{5} (N_2 L_3) \zeta \frac{1}{5} (N_3 L_4) \zeta \frac{1}{5} \left\| P_{K_{N_1},1} \phi_{\pm 1} \right\| \left\| P_{K_{N_2},L_2} \phi_{\pm 2} \right\| \left\| P_{K_{N_3},L_3} \phi_{\pm 3} \right\|$$

$$\lesssim \sum_{N,L} N_4^2 \frac{1}{3} N_3 \frac{1}{3} (L_1 L_3) \zeta \frac{1}{3} L_4^2 \zeta \frac{1}{3} \left\| P_{K_{N_1},1} \phi_{\pm 1} \right\| \left\| P_{K_{N_2},L_2} \phi_{\pm 2} \right\| \left\| P_{K_{N_3},L_3} \phi_{\pm 3} \right\|$$

$$\lesssim \prod_{j=1}^3 \left\| \phi_{\pm j} \right\|_{B^\frac{1}{2} \frac{1}{4} \cdot 1}.$$  

For $N_1 \lesssim N_0 \sim N_2$ and $N_4 \lesssim N_0 \sim N_3$,

$$\left| H^1 \right| \lesssim \sum_{N,L} N_4^2 \frac{1}{5} L_4^2 \zeta \frac{1}{5} (N_1 L_1) \zeta \frac{1}{5} (N_1 L_2) \zeta \frac{1}{5} (N_4 L_3) \zeta \frac{1}{5} (N_3 L_4) \zeta \frac{1}{5} \left\| P_{K_{N_1},1} \phi_{\pm 1} \right\| \left\| P_{K_{N_2},L_2} \phi_{\pm 2} \right\| \left\| P_{K_{N_3},L_3} \phi_{\pm 3} \right\|$$

$$\lesssim \sum_{N,L} N_4^2 \frac{1}{3} N_3 \frac{1}{3} (L_1 L_3) \zeta \frac{1}{3} L_4^2 \zeta \frac{1}{3} \left\| P_{K_{N_1},1} \phi_{\pm 1} \right\| \left\| P_{K_{N_2},L_2} \phi_{\pm 2} \right\| \left\| P_{K_{N_3},L_3} \phi_{\pm 3} \right\|$$

$$\lesssim \prod_{j=1}^3 \left\| \phi_{\pm j} \right\|_{B^\frac{1}{2} \frac{1}{4} \cdot 1}.$$  

The remaining cases are treated similarly. We omit the details.

**Remark 7.1.** Here, we do not use the exclusion of low-low-high modulation. The fact $L^{12}_{\text{max}} \ll L_0 \ll N^{012}_{\text{min}}$ is not harmful in this case. The reason is that we get essential terms of $N^\frac{3}{2}$ bound in doing bilinear estimates. However, we need the bound $N^\frac{3}{2}$ to obtain the required estimates in view of the right-hand side of (7.1). Thus the bilinear estimates (3.1), (3.2) are sufficient for it.

### 7.2. Proof of (7.2).

By definition of $B^{s,b;1}$, we write

$$\left\| \phi_{\pm 1} \phi_{\pm 2} \phi_{\pm 3} \phi_{\pm 4} \right\|_{B^{s,b;1}} = \sum_{N_6,L_6 \geq 1} N_6^2 L_6^{-\frac{3}{2}} \left\| P_{K_{N_6},L_6} \phi_{\pm 1} \phi_{\pm 2} \phi_{\pm 3} \phi_{\pm 4} \phi_{\pm 5} \phi_{\pm 6} \right\|$$

$$= \sum_{N_6,L_6 \geq 1} N_6^2 L_6^{-\frac{3}{2}} \sup_{\| \psi \|_{L^6} = 1} \left| \int (\phi_{\pm 1} \phi_{\pm 2} \phi_{\pm 3} \phi_{\pm 4} \phi_{\pm 5} \phi_{\pm 6}) P_{K_{N_6},L_6} \psi \, dt \, dx \right|$$

$$:= H^2$$

We apply a dyadic decomposition on $\phi_{\pm j}$, $j = 1, \ldots, 5$ to get $|H^2| \leq \sum_{N',L'} N_6^2 L_6^{-\frac{3}{2}} |H^2_{N',L'}|$ where

$$H^2_{N,L} = \int P_{K_{N_1},L_1} \phi_{\pm 1} P_{K_{N_2},L_2} \phi_{\pm 2} P_{K_{N_3},L_3} \phi_{\pm 3} \times P_{K_{N_4},L_4} \phi_{\pm 4} P_{K_{N_5},L_5} \phi_{\pm 5} \frac{1}{P_{K_{N_6},L_6}} \psi \, dt \, dx,$$

and

$$N' = (N_0, N_1, N_2, N_3, N_4, N_5, N_6), \quad L' = (L_0, L_1, L_2, L_3, L_4, L_5, L_6).$$
But to treat this integral as previously, we need to work further. Using Cauchy-Schwarz inequality and duality, we obtain

$$|H_{N,L}^2| \leq \sup \int P_{K_{N_1,L_1}} \phi_{\pm 1} P_{K_{N_2,L_2}} \phi_{\pm 2} P_{K_{N_3,L_3}} \phi_{\pm 3} \psi' \, dt \, dx$$

$$\times \int P_{K_{N_4,L_4}} \phi_{\pm 4} P_{K_{N_5,L_5}} \phi_{\pm 5} P_{K_{N_6,L_6}} \psi'' \, dt \, dx,$$

where the supremum is taken over $\|\psi'\|, \|\psi''\| = 1$. Also, a dyadic decomposition on $\psi'$ and $\psi''$ gives us $|H^2| \leq \sum_{N,L} N_6^\pm L_6^\mp |H_{N,L}|$, where

$$|H_{N,L}| \leq \sup \int P_{K_{N_1,L_1}} \phi_{\pm 1} P_{K_{N_2,L_2}} \phi_{\pm 2} P_{K_{N_3,L_3}} \phi_{\pm 3} P_{K_{N_4,L_4}} \phi_{\pm 4} P_{K_{N_5,L_5}} \phi_{\pm 5} P_{K_{N_6,L_6}} \psi' \, dt \, dx$$

$$\times \int P_{K_{N_7,L_7}} \phi_{\pm 6} P_{K_{N_8,L_8}} \phi_{\pm 7} P_{K_{N_9,L_9}} \phi_{\pm 8} \psi'' \, dt \, dx,$$

and

$$N = (N', N'', L), \quad L = (L', L'', L''').$$

**Remark 7.2.** Here we give a remark that in proving (7.2), the required estimate can be obtain without exclusion of low-low-high modulation case. Indeed, in view of right-handside of (7.2), the summation of $H_{N,L}$ need essential terms of $N^\pm$ bound. However, bilinear estimates (8.1) and (8.2) yields essential terms of $N^\pm$ bound in each bilinear estimates. Since we treat four bilinear estimates and $\frac{3}{4} \times 4 - \frac{1}{4} = \frac{1}{2} < \frac{5}{4}$, we see that bilinear estimates give sharp estimate on this case.

**8. The failure of smoothness**

In this section we show the flow $(\phi(0), A_\mu(0)) \mapsto (\phi(t), A_\mu(t))$ is not $C^3$ near the origin. Especially, we show the gauge field is not smooth at the origin in $H^\pm$. For this purpose we consider the system given by

$$= \phi \cdot -2 [A^\mu, \partial_\mu] - [A_\mu, [A^\mu, \phi]] + V(\phi, \phi),$$

$$\Box A_\mu = \partial^\nu A_\mu A_\nu - \epsilon_{\mu\nu\alpha} Q^{\alpha} \phi^\delta \phi + Q^{\alpha} \phi^\delta \phi - \epsilon_{\mu\nu\alpha} \partial^\nu \left( [\phi^\alpha, [A^\alpha, \phi]] - [A^\alpha, \phi] \right),$$

$$(\phi, \partial_\phi(0)) = (\delta f, 0), \quad A_\phi(0) = \delta a_0, \quad A_\phi(0) = 0 \quad \partial_\phi A_\phi(0) = 0,$$

where $0 < \delta \ll 1$ and $f$ is a $g$-valued smooth function. We denote the local solution of (8.1) by $(\phi(\delta, t), A_\mu(\delta, t))$. If $f = f_1 T^1$ for a smooth scalar function $f_1$, then $\partial_\phi A_\phi(0) = \partial_\phi a_0$.

**8.1. Set up.** We prove by contradiction. Assume that the flow is $C^3$ at the origin in $H^\pm$ and $f = f_1 T^1$ and $a_0 = a_0, T^2$. Since $(\phi, \partial_\phi(0)) = (0, t = 0) = (0, 0)$ and $(A_\mu, \partial_\phi A_\mu)(\delta = 0, t = 0) = (0, 0)$, the solution $\phi(\delta = 0, t) = 0$ and $A_\mu(\delta = 0, t) = 0$. By taking derivative to (8.1) w.r.t. $\delta$ we see that since

$$A_\mu(\delta = 0, t) = 0, \quad \partial_\phi A_\mu(\delta = 0, t) = \delta a_0, \quad \partial_\phi A_\mu(\delta = 0, t) = \delta a_0,$$

the solution $\partial_\phi A_\mu(\delta = 0, t)$ is written as

$$\partial_\phi A_\mu(\delta = 0, t) = \sum_{\pm} \delta a_0 \pm \frac{1}{2} e^{\pm i D} \left( \frac{\delta a_0 a_0}{1 D} \partial_\phi a_0 \right) T^2.$$

On the other hand, $\partial_\phi \phi$ satisfies from the formula (8.1) below that

$$\Box \partial_\phi \phi = -m^2 \partial_\phi \phi = 0, \quad (\partial_\phi \phi(\delta = 0, t) = 0, \partial_\phi \phi(\delta = 0, t) = 0) = (f, 0),$$
where \( m^2 = 2v^4 \). Then the solution \( \partial_\delta \phi_0 (\delta = 0) = 0 \) for \( a > 1 \), and hence

\[
\partial_\delta \phi (\delta = 0) = \sum_{\pm} \frac{1}{2} e^{\mp it \sqrt{m^2 - \Delta}} f_i T^1.
\]

Let us consider the second derivatives.

\[
\partial^2_\delta [\partial^\nu A_\mu, A_\nu] (\delta = 0, t) = 2 \sum_{\pm_1, \pm_2} [\partial^\nu a_{\mu, \pm_1} (t), A_{\nu, \pm_2} (t)] = 2 \sum_{\pm_1, \pm_2} (\cdots) [T^2, T^2] = 0.
\]

\[
\partial^2_\delta (Q^{\nu a} (\phi^\dagger, \phi) + Q^{\nu a} (\phi, \phi^\dagger)) (\delta = 0, t)
= \frac{1}{2} \sum_{\pm_1, \pm_2} Q^{\nu a} \left( (e^{\mp \pm_1 it \sqrt{m^2 - \Delta}} f_1) T^1, e^{\mp \pm_2 it \sqrt{m^2 - \Delta}} f_1 T^1 \right)
+ \frac{1}{2} \sum_{\mp_1, \pm_2} Q^{\nu a} \left( (e^{\mp \pm_1 it \sqrt{m^2 - \Delta}} f_1 T^1, (e^{\mp \pm_2 it \sqrt{m^2 - \Delta}} f_1) T^1 \right)
= 0 \quad (\cdots; Q^{\nu a} (g_1 T^1, g_1 T^1) = -Q^{\nu a} (g_1 T^1, h_1 T^1)).
\]

Clearly the second derivative of cubic term of RHS of \( \Box A_\mu \) is 0 at \( \delta = 0 \). Hence \( \partial^2_\delta A_\mu (\delta = 0, t) = 0 \).

Taking the second derivative to the equation of \( \phi \) and then removing the cubic terms, since the initial data are zero and the linear term plays a role of mass, we have

\[
\partial_\delta^2 \phi (\delta = 0, t) = - \sum_{\pm_1, \pm_2, \pm_3} \int_0^t e^{\pm_1 i(t - t') \sqrt{m^2 - \Delta}} (2iD)^{-1} [A_{\mu, \pm 2} (t') , \partial_\nu e^{\mp_3 it' \sqrt{m^2 - \Delta}} f] dt,'
\]

If the flow is \( C^2 \) in \( H^s \times H^s \), within a local existence time interval, the following inequality holds:

\[
\sup_{0 \leq t \leq T} \| \partial^2_\delta \phi (\delta = 0, t) \|_{H^s} \lesssim \| f_i \|_{H^s} \| a_0 \|_{H^s}.
\]

In order to treat the third derivative, let us first observe that

\[
Q^{\nu a} (\partial^2_\delta \phi (\delta = 0), \partial_\delta \phi (\delta = 0)) + Q^{\nu a} (\partial_\delta \phi (\delta = 0), \partial^2_\delta \phi (\delta = 0))
= (\cdots) (\cdots) (T^2, T^1) T^1 + [T^1, [T^2, T^1]] = (\cdots) (T^2, T^1) T^1 + [T^1 T^2, T^1] = (\cdots) [T^2, T^1 T^1].
\]

It is an easy exercise to show that \( [T^a, T^b] [T^b, T^b] = 0 \) for any generator \( T^a \) and \( T^b \). (For instance see the page 420 of [26].) Therefore, we obtain

\[
\partial^3_\delta A_2 (\delta = 0)
= \frac{3i}{2} \sum_{\pm_1, \pm_2, \pm_3, \pm_4} \mp_1 \int_0^t \frac{e^{\mp \pm_1 i(t - t') \Delta}}{2iD} e^{210} \partial^{10} \left( \left[ (e^{\mp_2 it' \sqrt{m^2 - \Delta}} f) \right], \left[ A_{\pm_3}^{\text{hom}, 1}, e^{\mp_4 it' \sqrt{m^2 - \Delta}} f \right] \right)
- \left[ A_{\pm_3}^{\text{hom}, 1}, e^{\mp_4 it' \sqrt{m^2 - \Delta}} f \right] \left[ e^{\mp_2 it' \sqrt{m^2 - \Delta}} f \right] \right) dt'
\]

\[
= I + II.
\]

If the flow is \( C^3 \) in \( H^s \times H^s \) within a local existence time interval, then we have

\[
\sup_{0 \leq t \leq T} \| \partial^3_0 A_0 (\delta = 0, t) \|_{H^s} \lesssim \| f_2 \|_{H^s} \| a_0 \|_{H^s}.
\]
8.2. Failure of \( \frac{\omega}{\lambda} \) when \( s < \frac{1}{2} \). Given \( \lambda \gg 1 + m^2 \), let us define \( W_{\lambda} \) by \( \{ \xi = (\xi_1, \xi_2) : |\xi_1 - \lambda| \leq 10^{-6}\lambda \}, \) \( |\xi_2| \leq 10^{-6}\lambda \), and \( -W_{\lambda} = \{ \xi : -\xi \in W_{\lambda} \} \). Then for any \( \xi \in 2W_{\lambda} \) there exists \( 0 < \rho < 1 \) such that \( 2(1 - \rho)\lambda \leq |\xi| \leq 2(1 + \rho)\lambda \).

We now take \( f_1 \) and \( a_{0,2} \) as

\[
\hat{f}_1(\xi) = \chi_{W_{\lambda}}, \quad \hat{a}_{0,2} = \chi_{2W_{\lambda}}(\xi).
\]

Using integration by parts and the fact that \( A_{\pm}^{\text{hom},1} = \pm e^{\mp \imath t \sqrt{m^2 - \omega^2}} f \), \( [A_{\pm}^{\text{hom},1}(t), e^{\mp \imath t \sqrt{m^2 - \omega^2}} f] \)

\[
I = \frac{3}{4} \sum_{\pm, \cdots, \pm} + \frac{1}{D} \left( [e^{\mp \imath t \sqrt{m^2 - \omega^2}} f, [A_{\pm}^{\text{hom},1}(t), e^{\mp \imath t \sqrt{m^2 - \omega^2}} f]]
\right.
\]

\[
\left. + [e^{\mp \imath t \sqrt{m^2 - \omega^2}} f, A_{\pm}^{\text{hom},1}(t), e^{\mp \imath t \sqrt{m^2 - \omega^2}} f] \right)
\]

\[
+ \frac{3}{4} \sum_{\pm, \cdots, \pm} \pm \int_0^t e^{\mp \imath t(t-t')D} \left( [e^{\mp \imath t \sqrt{m^2 - \omega^2}} f, [A_{\pm}^{\text{hom},1}(t'), e^{\mp \imath t \sqrt{m^2 - \omega^2}} f]]
\right.
\]

\[
\left. + [e^{\mp \imath t \sqrt{m^2 - \omega^2}} f, A_{\pm}^{\text{hom},1}(t'), e^{\mp \imath t \sqrt{m^2 - \omega^2}} f] \right) dt'
\]

\[
=: I_1 + I_2.
\]

Taking space Fourier transform \( \mathcal{F}_x \) for \( \xi \in 2W_{\lambda} \) gives

\[
|\mathcal{F}_x I_1(t, \xi)| \lesssim \frac{1}{|\xi|} \int_{R^2 \times R^2} \chi_{W_{\lambda}}(\zeta - \eta) \chi_{W_{\lambda}}(\xi - \eta) d\zeta d\eta \lesssim \lambda^2.
\]

Then the direct integration w.r.t. \( t' \) gives us

\[
\mathcal{F}_x I_2(t, \xi) = \frac{3i}{8} \sum_{\pm, \cdots, \pm} \pm e^{\mp \imath t |\xi|} \int_{R^2 \times R^2} \mathbf{m}_{1234}(\xi, \xi', \eta, \eta') \chi_{2W_{\lambda}}(\xi - \eta) \left( \pm \frac{\zeta_1}{|\zeta|} \right) \chi_{2W_{\lambda}}(\zeta - \eta) d\zeta d\eta d\xi d\eta d\xi' d\eta'.
\]

\[
=: (\mathcal{N}_{1234} + \tilde{\mathcal{N}}_{1234})[T^1, [T^2, T^1]],
\]

where

\[
\mathbf{m}_{1234}(t, \xi, \eta, \eta') = \frac{e^{it \omega_{1234}} - 1}{i \omega_{1234}}, \quad \tilde{\mathbf{m}}_{1234}(t, \xi, \eta, \eta') = \frac{e^{it \tilde{\omega}_{1234}} - 1}{i \tilde{\omega}_{1234}},
\]

\[
\omega_{1234} = \pm 1 |\xi| \pm 2 \sqrt{m^2 + |\zeta - \eta|^2} \pm 3 |\zeta| - \pm 4 \sqrt{m^2 + |\xi - \eta|^2},
\]

\[
\tilde{\omega}_{1234} = \pm 1 |\xi| \pm 2 \sqrt{m^2 + |\zeta - \eta|^2} \pm 3 |\zeta| - \pm 4 \sqrt{m^2 + |\xi - \eta|^2}.
\]

For \( \xi \in 2W_{\lambda} \) from the support conditions of the integrand of \( \mathcal{N}_{1234} \) that \( \eta \) is at least in \( W_{\lambda} \). On the other hand, the support condition of integrand in \( \tilde{\mathcal{N}}_{1234} \) enforces that

\[
\eta \in (-W_{\lambda} - 2W_{\lambda}) \cap (2W_{\lambda} - W_{\lambda}),
\]

which is impossible. Hence \( \tilde{\mathcal{N}}_{1234} \) is vanishing.

Now by Taylor expansion \( \omega_{1234} \) can be estimated as follows: if \( (\pm_1 = \pm_3) \& (\pm_2 = \pm_4) \), or \( (\pm_1 = \pm_3) \neq (\pm_2 = \pm_4) \), or \( (\pm_1, \pm_3) = (+, -) \& (\pm_2, \pm_4) = (-, +) \), or \( (-, +) \& (+, -) \) (these cases are called resonance ones denoted by \( \mathcal{R} \)), then

\[
(8.6) \quad |\omega_{1234}| \lesssim \lambda^{\frac{3}{2}}.
\]

Otherwise,

\[
(8.7) \quad |\omega_{1234}| \sim \lambda.
\]
If \( \pm_1, \cdots, \pm_4 \in \mathcal{R} \), then by taking \( t = \varepsilon \lambda^{-\frac{1}{2}} \) for some fixed \( \lambda^{-\frac{1}{2}} \ll \varepsilon \ll 1 \), from (8.6) we obtain
\[
m_{1234} = t(1 + O_{1234}(\varepsilon)),
\]
where \( O_{1234}(\varepsilon) \) is \( O(\varepsilon) \) depending on \( \pm_1, \cdots, \pm_4 \). If \( \pm_1, \cdots, \pm_4 \in \mathcal{R}^c \), then by (8.7)
\[
|m_{1234}| \lesssim \lambda^{-1}.
\]
Using these estimates we have
\[
|\mathcal{N}_{1234}| \geq \left| \sum_{\pm_1, \cdots, \pm_4 \in \mathcal{R}} (\cdots) \right| - \left| \sum_{\pm_1, \cdots, \pm_4 \in \mathcal{R}^c} (\cdots) \right|
\]
In the summation in \( \mathcal{R} \) there are four cases of \( \pm_1 = \pm_3 \) and two cases of \( \pm_1 \neq \pm_3 \), which results in
\[
\left| \sum_{\pm_1, \cdots, \pm_4 \in \mathcal{R}} (\cdots) \right| \geq \frac{3}{4} \left| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \cos t|s| \chi_{W, \lambda}(\zeta - \eta) \chi_{W, \lambda}(\zeta + \eta) \, d\zeta d\eta \right|
\]
\[
- C t \varepsilon \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\zeta_1}{|s|} \chi_{W, \lambda}(\zeta - \eta) \chi_{W, \lambda}(\zeta + \eta) \, d\zeta d\eta
\]
\[
\geq \frac{3}{4} t |s| \left| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\zeta_1}{|s|} \chi_{W, \lambda}(\zeta - \eta) \chi_{W, \lambda}(\zeta + \eta) \, d\zeta d\eta \right|
\]
\[
- C t \varepsilon \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\zeta_1}{|s|} \chi_{W, \lambda}(\zeta - \eta) \chi_{W, \lambda}(\zeta + \eta) \, d\zeta d\eta.
\]
Let us take \( \lambda = \lambda(\varepsilon, \rho, k) \) for any large integer \( k \) such that
\[
\frac{2k\pi - \varepsilon}{\varepsilon(1 - \rho)} \leq \lambda^{\frac{1}{2}} \leq \frac{2k\pi + \varepsilon}{\varepsilon(1 + \rho)}.
\]
Then
\[
\left| \sum_{\pm_1, \cdots, \pm_4 \in \mathcal{R}} (\cdots) \right| \gtrsim t(1 - C\varepsilon) \int_{\mathbb{R}^2 \times \mathbb{R}^2} \chi_{W, \lambda}(\zeta) \chi_{W, \lambda}(\eta) \, d\zeta d\eta \gtrsim t\lambda^3 = \varepsilon\lambda^\frac{3}{2}
\]
Since \( \varepsilon \gg \lambda^{-\frac{3}{2}} \), we have
(8.8) \[
|\mathcal{F}_x(t, \xi)| \gtrsim \varepsilon\lambda^\frac{3}{2}.
\]
Now let us consider \( II \). By the same argument as above we have
\[
\mathcal{F}_x(II)(t, \xi) = \frac{3i}{8} \sum_{\pm_1, \cdots, \pm_4} \mp t |s| \chi_{W, \lambda}(\zeta - \eta) \chi_{W, \lambda}(\zeta + \eta) \, d\zeta d\eta
\]
\[
= \frac{3i}{8} \left( \sum_{\mathcal{R}} (\cdots) + \sum_{\mathcal{R}^c} (\cdots) \right).
\]
The estimate (8.7) yields
\[
\left| \frac{3i}{8} \sum_{\mathcal{R}^c} (\cdots) \right| \lesssim \lambda^2.
\]
On the other hand, (8.6) leads us to
\[
\left| \sum_{\mathcal{R}} (\cdots) \right| \lesssim t(\sin(|s|) + \varepsilon) \int_{\mathbb{R}^2 \times \mathbb{R}^2} \chi_{W, \lambda}(\zeta - \eta) \chi_{W, \lambda}(\zeta + \eta) \, d\zeta d\eta.
\]
By the choice of $\lambda$, $|\sin(t|\xi|)| \lesssim \varepsilon$. This implies that
\[
\left| \sum_{\mathcal{R}} (\cdots) \right| \lesssim \varepsilon \lambda^{-\frac{1}{2}} \varepsilon \lambda^3 \lesssim \varepsilon^2 \lambda^{\frac{3}{2}}.
\]
Combining this with $\text{8.8}$ we conclude that for $\xi \in 2W_\lambda$
\[
|\mathcal{F}_x(\partial_t^2 A_2(\delta = 0))(t, \xi)| \gtrsim \varepsilon \lambda^{\frac{3}{2}}.
\]
Now let us invoke the necessary-sufficient condition of $\text{(8.5)}$ for the $C^3$ smoothness, from which we deduce that
\[
\varepsilon \lambda^{\frac{3}{2}} \lambda^{\sigma + \frac{1}{2}} \lesssim \lambda^{2\sigma + \frac{1}{2}} \lambda^{\sigma + \frac{1}{2}}
\]
and hence regardless of $\sigma$, $s$ should be greater than or equal to $\frac{1}{2}$ for the $C^3$ smoothness. This completes the proof of Theorem $1.2$.

8.3. Failure of $\text{8.3}$ when $\sigma < \frac{1}{2}$. Let us set $f = f_1 T_1$, $a_0 = a_{0,2}T_2$ and $\tilde{f}_1 = \tilde{a}_{0,2} = \chi W_\lambda$. By taking Fourier transform to $\text{8.2}$ we have
\[
\mathcal{F}_x(\partial_t^2 \phi(\delta = 0))(\xi)
\]
\[-\frac{1}{4} \sum_{\pm_1 = \pm_2 = \pm_3} \frac{e^{\mp i t \sqrt{m^2 + |\xi|^2}}}{|\xi|} \int \tilde{m}_{123} \chi W_\lambda(\xi - \eta) \left( \mp_3 \sqrt{m^2 + |\eta|^2} \pm \frac{\xi_1 - \eta_1}{|\xi - \eta|} \right) \chi W_\lambda(\eta) \, d\eta \, [T^2, T^1],
\]
where
\[
\tilde{m}_{123} = \frac{e^{it \omega_{123}} - 1}{i \omega_{123}}, \quad \omega_{123} = \pm_1 \sqrt{m^2 + |\xi|^2} \mp \pm_2 |\xi - \eta| \mp \pm_3 |\eta|.
\]
The resonance case $\mathcal{R}$ occurs only when $\pm_1 = \pm_2 = \pm_3$ for which $|\omega_{123}| \lesssim \lambda^{\frac{3}{2}}$. For other cases $|\omega_{123}| \sim \lambda$.

If we take $t = \varepsilon \lambda^{-\frac{1}{2}}$ and $\xi \in 2W_\lambda$, then
\[
\mathcal{F}_x(\partial_t^2 \phi(\delta = 0))(\xi)
\]
\[-\frac{1}{4} \sum_{\pm_1 = \pm_2 = \pm_3} \frac{e^{\mp i t \sqrt{m^2 + |\xi|^2}}}{|\xi|} \int t(1 + O_{123}(\varepsilon)) \chi W_\lambda(\xi - \eta) \left( \mp_3 \sqrt{m^2 + |\eta|^2} \pm \frac{\xi_1 - \eta_1}{|\xi - \eta|} \right) \chi W_\lambda(\eta) \, d\eta \, [T^2, T^1]
\]
\[-\frac{1}{4} \sum_{\mathcal{R}^c} \frac{e^{\mp i t \sqrt{m^2 + |\xi|^2}}}{|\xi|} \int \tilde{m}_{123} \chi W_\lambda(\xi - \eta) \left( \mp_3 \sqrt{m^2 + |\eta|^2} \pm \frac{\xi_1 - \eta_1}{|\xi - \eta|} \right) \chi W_\lambda(\eta) \, d\eta \, [T^2, T^1]
\]
\[-\frac{1}{4} \sum_{\pm_1 = \pm_2 = \pm_3} \frac{e^{\mp i t \sqrt{m^2 + |\xi|^2}}}{|\xi|} \int \chi W_\lambda(\xi - \eta) \left( \sqrt{m^2 + |\eta|^2} \right) \chi W_\lambda(\eta) \, d\eta \, [T^2, T^1]
\]
\[-\frac{1}{4} \sum_{\pm_1 = \pm_2 = \pm_3} \frac{e^{\mp i t \sqrt{m^2 + |\xi|^2}}}{|\xi|} \int \chi W_\lambda(\xi - \eta) \left( \frac{\xi_1 - \eta_1}{|\xi - \eta|} \right) \chi W_\lambda(\eta) \, d\eta \, [T^2, T^1]
\]
\[-\frac{1}{4} \sum_{\mathcal{R}^c} \frac{e^{\mp i t \sqrt{m^2 + |\xi|^2}}}{|\xi|} \int \tilde{m}_{123} \chi W_\lambda(\xi - \eta) \left( \mp_3 \sqrt{m^2 + |\eta|^2} \pm \frac{\xi_1 - \eta_1}{|\xi - \eta|} \right) \chi W_\lambda(\eta) \, d\eta \, [T^2, T^1]
\]
\[=: I + II + III + IV.
\]
A direct calculation gives
\[
|II| \lesssim t \lambda^{\frac{3}{2}} = \varepsilon, \quad |III| \lesssim \varepsilon^2 \lambda,
\]
and for the non-resonance case $\mathcal{R}^c$, $|\tilde{m}_{123}| \lesssim \lambda^{-1}$ holds and hence $|IV| \lesssim \lambda^{\frac{3}{2}}$. 

As for $I$ we further take $\lambda = \lambda(\varepsilon, \rho, k)$ such that
\[
\frac{k\pi + \frac{\pi}{2}}{\varepsilon(1 - \rho)} \leq \lambda^2 \leq \frac{k\pi + \frac{3\pi}{2}}{\varepsilon(1 + \rho)}
\]
for each integer $k$. Then for $\xi \in 2W_\lambda$
\[
|I| = \frac{t}{2} \sin t|\xi| \int \chi_{W_\lambda}(\xi - \eta) \left( \sqrt{m^2 + |\eta|^2} \right) \chi_{W_\lambda}(\eta) \, d\eta |||T^2, T^1|||
\]
\[
= \frac{t}{2} \sin t|\xi| \int \chi_{W_\lambda}(\xi - \eta) \left( \sqrt{m^2 + |\eta|^2} \right) \chi_{W_\lambda}(\eta) \, d\eta |||T^2, T^1|||
\]
\[
\geq t|\sin t||\xi||\lambda^2
\]
\[
\geq \varepsilon \lambda.
\]
Hence we get
\[
|F_x(\partial_x^2 \phi(\delta = 0)(\xi)| \geq \varepsilon \lambda.
\]
Suppose that (8.3) holds. Then
\[
\varepsilon \lambda \lambda^{s+\frac{1}{2}} \lesssim \lambda^{s+\frac{1}{2}} \lambda^{\sigma+\frac{1}{2}}.
\]
Therefore, $\sigma$ should be greater than equal to $\frac{1}{2}$.

9. Appendix

We introduce a set of infinitesimal generators $T^a$, $a = 1, 2, \cdots, n^2 - 1$, of Lie algebra $\mathfrak{g}$ which are traceless Hermitian matrices. These matrices obey the normalization condition $\text{Tr}(T^a T^b) = \delta^{ab}$, where $\delta^{ab}$ is Kronecker delta. Now we introduce so-called structure coefficients $f^{ab}_c$ of the Lie algebra $\mathfrak{g}$ given by $[T^a, T^b] = i f^{ab}_c T^c$. We write $\phi = \phi_a T^a$. Then $\frac{\partial V(\phi, \phi^\dagger)}{\partial \phi^\dagger}$ is a matrix in $\mathfrak{su}(n, \mathbb{C})$ and its $a$-th component (with respect to the basis $T^a$) is given by $\frac{\partial V(\phi, \phi^\dagger)}{\partial \phi^a}$. Since
\[
[\phi_a T^a, \phi_b T^b] = i \phi_a \phi_b f^{ab}_c T^c, \quad [T^d, T^c] = i f^{dc}_e T^e,
\]
we write
\[
[[\phi, \phi^\dagger], \phi] - v^2 \phi = [[[\phi_a T^a, \phi_b T^b], \phi_c T^c] - v^2 \phi_c T^c
\]
\[
= - (f^{ab}_d f^{dc}_e \phi_a \phi_b \phi_c + v^2 \phi_c) T^e.
\]
Then we have
\[
V(\phi, \phi^\dagger) = \text{Tr}(([[\phi, \phi^\dagger], \phi] - v^2 \phi) ([\phi, \phi^\dagger], \phi] - v^2 \phi))
\]
\[
= (f^{ab}_d f^{dc}_e \phi_a \phi_b \phi^*_c + v^2 \phi_c^*) (f^{a'b'}_d f^{d'c'}_e \phi_a \phi_{b'} \phi_{c'} + v^2 \phi_{c'}) \text{Tr}(T^{a'b'} T^{d'c'})
\]
\[
= 2 \sum_c (f^{ab}_d f^{dc}_e \phi_a \phi_b \phi^*_c + v^2 \phi_c^*) (f^{a'b'}_d f^{d'c'}_e \phi_a \phi_{b'} \phi_{c'} + v^2 \phi_{c'}) \text{Tr}(T^{a'b'} T^{d'c'})
\]
where we used $\text{Tr}(T^{a'b'} T^{d'c'}) = 2 \delta^{c'}. Taking the partial derivative with respect to $\phi^*_a$, we have
\[
(9.1) \quad \frac{\partial V(\phi, \phi^\dagger)}{\partial \phi^*_a} = 2 \sum_c (f^{ab}_d f^{dc}_e \phi_a \phi_b \phi^*_c + v^2 \delta^{a}_c) (f^{a'b'}_d f^{d'c'}_e \phi_a \phi_{b'} \phi_{c'} + v^2 \phi_{c'}),
\]
and hence we conclude that the essential terms in $V(\phi, \phi^\dagger)$ are linear, cubic and quintic terms of $\phi$ and $\phi^\dagger$.

The readers find more discussion on non-abelian gauge symmetry in [20, 28, 29].
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