THE EIGENSPLITTING OF THE FIBER OF THE CYCLOTOMIC TRACE FOR THE SPHERE SPECTRUM

ANDREW J. BLUMBERG AND MICHAEL A. MANDELL

ABSTRACT. Let \( p \in \mathbb{Z} \) be an odd prime. We show that the fiber sequence for the cyclotomic trace of the sphere spectrum \( \mathcal{S} \) admits an “eigensplitting” that generalizes known splittings on \( K \)-theory and \( TC \). We identify the summands in the fiber as the covers of \( \mathbb{Z}_p \)-Anderson duals of summands in the \( K(1) \)-localized algebraic \( K \)-theory of \( \mathbb{Z} \). Analogous results hold for the ring \( \mathbb{Z} \) where we prove that the \( K(1) \)-localized fiber sequence is self-dual for \( \mathbb{Z}_p \)-Anderson duality, with the duality permuting the summands by \( i \mapsto p - i \) (indexed mod \( p - 1 \)). We explain an intrinsic characterization of the summand we call \( \mathbb{Z} \) in the splitting \( TC(\mathbb{Z})_p^h \cong j \lor \Sigma j' \lor \mathbb{Z} \) in terms of units in the \( p \)-cyclotomic tower of \( \mathbb{Q}_p \).

1. INTRODUCTION

The algebraic \( K \)-theory of the sphere spectrum, \( K(\mathcal{S}) \), is an object of basic and fundamental interest, relating geometric topology and arithmetic. Celebrated work of Waldhausen establishes a comparison between \( K(\mathcal{S}) \) and a stable space of \( h \)-cobordisms for the disk \( D^n \). On the other hand, \( K(\mathcal{S}) \) is intimately related to \( K(\mathbb{Z}) \), the algebraic \( K \)-theory of the integers, which encodes arithmetic invariants (e.g., Bernoulli numerators and denominators). For instance, the natural map \( K(\mathcal{S}) \rightarrow K(\mathbb{Z}) \) is a rational equivalence, and the latter is understood rationally by old work of Borel.

At a prime \( p \), our understanding of algebraic \( K \)-theory of ring spectra relies on trace methods. Bökstedt, Hsiang, and Madsen constructed a topological version of negative cyclic homology called topological cyclic homology \( (TC) \) and a Chern character \( K \rightarrow TC \), the cyclotomic trace. Following earlier work of Rognes, in a previous paper we studied the homotopy groups of \( K(\mathcal{S}) \) in terms of the cyclotomic trace and linearization maps: a basic theorem of Dundas (building on work of Goodwillie and McCarthy) provides a homotopy cartesian square

\[
\begin{array}{ccc}
K(\mathcal{S})_p^h & \rightarrow & K(\mathbb{Z})_p^h \\
\downarrow & & \downarrow \\
TC(\mathcal{S})_p^h & \rightarrow & TC(\mathbb{Z})_p^h,
\end{array}
\]

Date: December 14, 2021.
2020 Mathematics Subject Classification. Primary 19D10, 19D55, 19F05.
Key words and phrases. Adams operations, algebraic \( K \)-theory of spaces, cyclotomic trace, Tate-Poitou duality.

The first author was supported in part by NSF grants DMS-1812064, DMS-2104420.
The second author was supported in part by NSF grants DMS-1811820, DMS-2104348.
where the maps $K(S) \to K(Z)$ and $TC(S) \to TC(Z)$ are the linearization maps induced by the unit map of $E_\infty$ ring spectra $S \to Z$.

For the rest of the paper, we restrict to the case of $p$ an odd prime. In [3, 5.3], the authors showed that the fiber square above splits as a wedge of $p - 1$ fiber squares of the form:

\[
\begin{array}{ccc}
\epsilon_i K(S)_p & \longrightarrow & \epsilon_i K(Z)_p \\
\downarrow & & \downarrow \\
\epsilon_i TC(S)_p & \longrightarrow & \epsilon_i TC(Z)_p
\end{array}
\]

for $i = 0, \ldots, p - 2$ (or better, numbered modulo $(p - 1)$). We identify the spectra $\epsilon_i(-)$ more specifically below. These squares are exactly the summands that would result from an eigensplitting of the fiber square for an action of $F \times \mathbb{F}_p$ via the Teichmüller character $\omega$ for a conjectural action of $p$-adically interpolated Adams operations; see [ibid., §5]. We refer to the summands in the $i$th square above as the "$\omega^i$ eigenspectra" even though such Adams operations have not been constructed in this generality. (If they do exist, the $\omega^i$ eigenspectrum is the summand where the $F \times \mathbb{F}_p$-action in the stable category is given by the character $\omega^i: \mathbb{F}^\times_p \to \mathbb{Z}^\times_p$ and the action of $\mathbb{Z}^\times_p$.)

As a formal consequence, the fiber of the cyclotomic trace for $S$, or equivalently, for $Z$ also comes with an eigensplitting. In this paper, we identify the eigenspectra summands. In [4], the authors identified the fiber of the cyclotomic trace Fib($\tau$) in $K$-theoretic terms as the $(-3)$-connected cover of $\Sigma^{-1} I\mathbb{Z}_p(L K(1) K(Z))$, where $I\mathbb{Z}_p$ denotes the $\mathbb{Z}_p$-Anderson dual. Taking the idea of eigenspectra seriously, the natural conjecture is that the $\omega^i$ eigenspectra of Fib($\tau$) should be the corresponding eigenspectra of this Anderson dual. In the spirit of the identification in [ibid.], the $\omega^i$ eigenspectrum of Fib($\tau$) should correspond to the $\mathbb{Z}_p$-Anderson dual of the $\omega^{p-i}$ eigenspectrum of $L K(1) K(Z)$. Our main theorem establishes this conjecture.

**Theorem 1.1.** There is a canonical weak equivalence between the fiber of the map $\epsilon_i \tau : \epsilon_i K(S)_p \to \epsilon_i TC(S)_p$ and the $(-3)$-connected cover of $\Sigma^{-1} I\mathbb{Z}_p(L K(1) K(Z))_p$.

We also describe how the duality map of [4] interacts with the fiber sequence; we discuss these additional results in Section 5 after reviewing terminology and notation.

Theorem 1.1 is consistent with an expansive picture of the behavior of the conjectural $p$-adically interpolated Adams operations on $K(S)$. In particular, it is natural to conjecture compatibility with the (known) $p$-adic Adams operations on $TC(S)$ as well as multiplicative properties. Given such operations on the $\infty$-category level, Theorem 1.1 would follow. However, while the existence of such Adams operations on the stable category level is enough to obtain a splitting on Fib($\tau$), it would not be enough to deduce Theorem 1.1 without additional arguments like the ones below.

While this paper obviously builds on the authors’ previous work [2, 3, 4], we have tried to make it as self-contained as possible, with specific citations to any facts needed from those papers.

**Conventions.** We use the term “stable category” to refer to the homotopy category of spectra with its structure as a tensor triangulated category. The symbol $\wedge$ denotes the smash product in the stable category.
Some of the statements and results below involve precise accounting for signs. For this we use the following conventions: suspension is $(\cdot) \wedge S^1$ and cone is $(\cdot) \wedge I$, where in the latter case we use 1 as the basepoint. Cofiber sequences are sequences isomorphic (in the stable category) to Puppe sequences formed in the usual way using this suspension and cone. A cofiber sequence leads to a long exact sequence of homotopy groups; we use the sign convention that for a map $f : A \to B$, the connecting map $\pi_n Cf \to \pi_{n-1} A$ in the long exact sequence of homotopy groups is $(-1)^{n} \sigma^{-1}$ composed with the Puppe sequence map $\pi_n C f \to \pi_n \Sigma A$, where $\sigma$ denotes the suspension isomorphism $\pi_{n-1} A \to \pi_n \Sigma A$. (Consideration of the example of standard cells explains the desirability of the sign.)

We form the homotopy fiber $\text{Fib}(f)$ of a map $f$ using the space of paths starting from the basepoint; we then have a canonical map $\Sigma \text{Fib}(f) \to \text{Cofiber}(f)$ in the usual way (using the suspension coordinate to follow the path and then follow the cone). We switch between fiber sequences and cofiber sequences at will, using the convention that for the fiber sequence

$$\Omega B \xrightarrow{\delta} \text{Fib}(f) \xrightarrow{\phi} A \xrightarrow{f} B,$$

the sequence

$$\text{Fib}(f) \xrightarrow{\phi} A \xrightarrow{f} B \xrightarrow{-\Sigma \delta \circ -1} \Sigma \text{Fib}(f)$$

is a cofiber sequence where $\epsilon : \Sigma \Omega B \to B$ is the counit of the $\Sigma$, $\Omega$ adjunction.

For the long exact sequence of homotopy groups associated to a fiber sequence, we use the long exact sequence of homotopy groups of the associated cofiber sequence. In terms of the fiber sequence displayed above, the connecting map $\pi_{n+1} B \to \pi_n \text{Fib}(f)$ is the composite of the canonical isomorphism $\pi_{n+1} B \cong \pi_n \Omega B$ and the map $(-1)^n \pi_n \delta$.

For a cofiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

and a fixed spectrum $X$, the sequence

$$\Omega F(A, X) \xrightarrow{-h^*} F(C, X) \xrightarrow{g^*} F(B, X) \xrightarrow{f^*} F(A, X)$$

is a fiber sequence and

$$F(C, X) \xrightarrow{g^*} F(B, X) \xrightarrow{f^*} F(A, X) \xrightarrow{h^*} \Sigma F(C, X)$$

is a cofiber sequence.

**Acknowledgments.** The authors thank Adebisi Agboola, Brian Conrad, Mirela Ciperiani, Bill Dwyer, Samit Dasgupta, Mike Hopkins, Lars Hesselholt, Mahesh Kakde, Bjorn Poonen, John Rognes, and Matthias Strauch for helpful conversations or remarks. We are grateful to the referee for many small improvements and corrections.

2. A review of the eigensplitting of the cyclotomic trace

In this section, we review the splitting constructed in [3] of the cofiber sequence

$$(2.1) \quad \text{Fib}(\tau) \to K(\mathbb{Z})_p^\wedge \xrightarrow{\tau} TC(\mathbb{Z})_p^\wedge \to \Sigma \text{Fib}(\tau).$$

It can be useful to express this in purely $K$-theoretic terms, using the Hesselholt-Madsen results that $TC(\mathbb{Z})_p^\wedge \to TC(\mathbb{Z}_p)_p^\wedge$ is a weak equivalence [11, Add. 5.2] and that $K(\mathbb{Z}_p)_p^\wedge \to TC(\mathbb{Z}_p)_p^\wedge$ is a connective cover [11, Th. D]. Thus, we can just as well
identify the connective cover of Fib(τ) as the fiber of the map $K(Z)_{\tau}^p \to K(Z_p)_{\tau}^p$ induced by the completion map $\mathbb{Z} \to \mathbb{Z}_p$. As we will recall below, little information is lost by working in the $K(1)$-local category and hence by studying the cofiber sequence
\begin{equation}
\text{Fib}(\kappa) \to L_{K(1)}K(\mathbb{Z}) \xrightarrow{\kappa} L_{K(1)}K(\mathbb{Z}_p) \to \Sigma \text{Fib}(\kappa).
\end{equation}
We will switch back and forth between discussing the localized sequence (2.2) and non-localized sequence (2.1).

We begin by reviewing notation for some of the basic building blocks of the splitting. Let $KU_p^r$ denote $p$-completed complex periodic $K$-theory, and let $L$ denote the $p$-complete Adams summand:
\[KU_p^r \simeq L \vee \Sigma^2 L \vee \cdots \vee \Sigma^{2p-4} L.\]

Let $J = L_{K(1)}\mathbb{S}$. If we choose an integer $l$ which multiplicatively generates the units of $\mathbb{Z}/p^2$, then $J$ is weakly equivalent to the homotopy fiber of $1 - \psi^l : L \to J$ for the Adams operation $\psi^l$. In deducing $p$-complete results from $K(1)$-local results, we write $kU_p^r$, $\ell$, and $j$ for the connective covers respectively of $KU_p^r$, $L$, and $J$.

The main theorem of Dwyer-Mitchell \cite{Dwyer-Mitchell} (as reinterpreted in \cite{Blumberg-Mandell} §2) produces a canonical splitting of $L_{K(1)}K(\mathbb{Z})$ as a certain wedge of $K(1)$-local spectra
\begin{equation}
L_{K(1)}K(\mathbb{Z}) \simeq J \vee Y_0 \vee \cdots \vee Y_{p-2},
\end{equation}
where $Y_i$ is characterized by the property that $L^*(Y_i)$ is concentrated in degrees congruent to $2i - 1 \bmod 2(p - 1)$ with $L^{2i-1}(Y_i)$ defined as an $L^0L$-module in terms of a certain abelian Galois group. Because $L^{2i-1}(Y_i)$ is a finitely generated $L^0L$-module of projective dimension 1, $Y_i$ is the homotopy fiber of a map between wedges of copies of $\Sigma^{2i-1}L$ (which on $L^{2i-1}(-)$ give a projective $L^0L$-resolution of $L^{2i-1}(Y_i)$). From this it follows that $\pi_i Y_i$ is concentrated in degrees congruent to $2i - 1$ and $2i - 2 \bmod 2(p - 1)$; moreover, it is free (as a $\mathbb{Z}_p$-module) in odd degrees. As $\pi_* J$ is concentrated in degrees congruent to $-1 \equiv 2p - 3 \bmod 2(p - 1)$ and degree 0, any particular homotopy group of $L_{K(1)}K(\mathbb{Z})$ involves only at most a single $Y_i$ and possibly $J$. The following two results from \cite{Blumberg-Mandell} (q.v. (2.8) and the preceding paragraph) simplify certain arguments.

**Proposition 2.4.** $Y_0 \simeq *$.

**Proposition 2.5.** $L^1 Y_1$ is a free $L^0L$-module of rank 1, and so $Y_1$ is (non-canonically) weakly equivalent to $\Sigma L$.

We say more about the relationship of $Y_1$ and $\Sigma L$ in Remark $\mathcal{L}_7$ in Section $\mathcal{F}$.

Let $y_i$ be the 1-connected cover of $Y_i$ for all $i$; then
\[K(\mathbb{Z})_{\tau}^p \simeq j \vee y_0 \vee \cdots \vee y_{p-2}.\]
We have a similar canonical splitting of $L_{K(1)}K(\mathbb{Z}_p)$ that takes the form
\[L_{K(1)}K(\mathbb{Z}_p) \simeq J \vee \Sigma J' \vee Z_0 \vee \cdots \vee Z_{p-2}\]
where $Z_i$ is non-canonically weakly equivalent to $\Sigma^{2i-1}L$ and $J'$ is the $K(1)$-localization of the Moore spectrum $M_{\mathbb{Z}_p^r}$ for the units of $\mathbb{Z}_p$, $J':= L_{K(1)}M_{\mathbb{Z}_p^r}$. Alternatively, $J'$ is canonically weakly equivalent to $(J \wedge M_{\mathbb{Z}_p^r})_{\tau}^p$; it is non-canonically weakly equivalent to $J$ since $(M_{\mathbb{Z}_p^r})_{\tau}^p$ is non-canonically equivalent to $S_{\mathbb{Z}_p^r}$. The spectra $Z_i$ were denoted $\Sigma^{-1} L_{TC}(i)$ in \cite{Blumberg-Mandell} for the non-canonical weak equivalence.
$Z_i \simeq \Sigma^{-1}L(i) = \Sigma^{2i-1}L$. The $Z_i$ admit a canonical description in terms of the units of cyclotomic extensions of $\mathbb{Q}_p$, which we review in Section 7.

Let $j'$ be the connective cover of $J'$, and for $i \neq 0$, let $z_i$ be the 1-connected cover of $Z_i$; let $z_0$ be the $(-2)$-connected cover of $Z_0$. Then

$$TC(\mathbb{Z})_p \simeq j \vee \Sigma j' \vee z_0 \vee \cdots \vee z_{p-2}.$$ 

A key result proved in [3, 3.1] is that the cyclotomic trace and the completion map are diagonal on the corresponding pieces.

**Theorem 2.6 ([3, 3.1]).** In the notation above, the cyclotomic trace $\tau : K(\mathbb{Z})_p^\wedge \to TC(\mathbb{Z})_p^\wedge$ decomposes as the wedge of the identity map $j \to j$ and maps $y_i \to z_i$ for $i = 0, \ldots, p-2$; the completion map $\kappa : L_{K(1)}K(\mathbb{Z}) \to L_{K(1)}K(\mathbb{Z}_p)$ decomposes as the wedge of the identity map $J \to J$ and maps $Y_i \to Z_i$ for $i = 0, \ldots, p-2$. The composite with the projection to the summands $\Sigma j'$ and $\Sigma J'$ is the trivial map for $\tau$ and $\kappa$, respectively.

It follows that the cofiber sequences of equations (2.1) and (2.2) decompose into wedges of cofiber sequences. To explain this, we need to introduce some notation.

**Definition 2.7.** Let $X_i$ and $x_i$ denote the homotopy fiber of the maps $Y_i \to Z_i$ and $y_i \to z_i$, respectively.

We have the following analogues of Propositions 2.4 and 2.5 for $X_1$ and $X_0$. The first is an immediate consequence of Proposition 2.4, the second is a restatement of [3, 4.4], which asserts (in the notation here) that $Y_1 \to Z_1$ is a weak equivalence.

**Proposition 2.8.** The connecting map $Z_0 \to \Sigma X_0$ is a weak equivalence; in particular, $X_0$ is (non-canonically) weakly equivalent to $\Sigma^{-2}L$.

**Proposition 2.9.** $X_1 \simeq *$

As a consequence of Definition 2.7 we have that

$$\text{Fib}(\tau) \simeq j' \vee x_0 \vee \cdots \vee x_{p-2} \quad \text{and} \quad \text{Fib}(\kappa) \simeq J' \vee X_0 \vee \cdots \vee X_{p-2}.$$ 

Because the maps in the definition are only defined in the stable category, the splitting of the fibers do not automatically give a canonical splitting of the fiber sequences. However, looking at the cofiber sequence

$$\Sigma^{-1}Z_i \longrightarrow X_i \longrightarrow Y_i \longrightarrow Z_i$$

we see that $X_i$ can have $L$-cohomology only in degrees congruent to $2i - 1$ and $2i - 2 \mod 2(p - 1)$, or equivalently:

**Proposition 2.10.** For $i = 0, \ldots, p-2$, $[X_i, \Sigma^j L] = 0$ unless $j \equiv 2i - 1$ or $j \equiv 2i - 2 \mod 2(p - 1)$.

From this we see that $[X_i, \Sigma^{-1}Z_j] = 0$ for $j \neq i$ and $[x_i, \Sigma^{-1}z_j] = 0$ for $j \neq i$. The latter is clear from Proposition 2.8 in the case $i = 0$ and in the remaining cases follows from the isomorphisms

$$[x_i, \Sigma^{-1}z_j] \cong [x_i, \Sigma^{-1}Z_j] \cong [X_i, \Sigma^{-1}Z_j].$$

The first isomorphism holds since $x_i$ is 0-connected for $i = 1, \ldots, p - 2$ while $\Sigma^{-1}z_j \to \Sigma^{-1}Z_j$ is a weak equivalence on 0-connected covers for all $j$. The second isomorphism holds because $X_i$ is the $K(1)$-localization of $x_i$ and $\Sigma^{-1}Z_j$ is $K(1)$-local. We also note that $[J', \Sigma^{-1}Z_j] = 0$ and $[j', \Sigma^{-1}Z_j] = 0$ for all $j \neq 1$. 
since $\pi_0\Sigma^{-1}Z_j = 0$. We have that $[J',\Sigma^{-1}Z_1] \cong \mathbb{Z}_p$ (non-canonically) and so $[J',\Sigma^{-1}Z_1[0,\infty]] \cong \mathbb{Z}_p$ with the isomorphism induced by $\pi_0$:

$$[J',\Sigma^{-1}Z_1[0,\infty]] \to \text{Hom}(\pi_0J',\pi_0\Sigma^{-1}Z_1) = \text{Hom}((\mathbb{Z}_p^\vee)_p,\mathbb{Z}_p) \cong \mathbb{Z}_p.$$ 

Since $\Sigma^{-1}Z_1$ is the fiber of the map $\Sigma^{-1}Z_1[0,\infty] \to H\mathbb{Z}_p$ that induces the identification $\pi_1Z_1 = \mathbb{Z}_p$, we see that $[J',\Sigma^{-1}Z_1] = 0$. In particular, we have shown that all of the indeterminacy in the map of fiber sequences is zero (in the non-localized case), and we get the following consequence.

**Corollary 2.11.** The cofiber sequence (2.1) splits canonically as a wedge of the cofiber sequences

$$\begin{array}{cccccccc}
& * & \to & j & \to & j & \to & * \\
& j' & \to & * & \to & \Sigma j' & \to & \Sigma j' \\
x_0 & \to & y_0 & \to & z_0 & \to & \Sigma x_0 \\
& : & : & : & : & : & : & : \\
x_{p-2} & \to & y_{p-2} & \to & z_{p-2} & \to & \Sigma x_{p-2}.
\end{array}$$

The cofiber sequence (2.2) splits canonically as an analogous wedge in terms of $J$, $J'$, $X_i$, $Y_i$, and $Z_i$.

3. Review of arithmetic duality in algebraic $K$-theory

In [4], we identified the fiber of the cyclotomic trace using a spectral lift of global arithmetic duality. Relating this identification to the splittings described above requires some of the details of the global duality map and a closely related local duality map. We give a short and mostly self-contained review of the construction in this section.

In arithmetic, local duality is an isomorphism

$$H_{et}^i(k;M) \cong (H_{et}^{2-i}(k;M^*(1)))^*$$

where $k$ is the field of fractions of a complete discrete valuation ring whose residue field is finite (e.g., a finite extension of $\mathbb{Q}_p$), $M$ is a finite Galois module, and $(-)^*$ denotes the Pontryagin dual. A version of this duality holds in algebraic $K$-theory where it takes the following form. (For a proof, see [4, 1.4].)

**Theorem 3.1** ($K$-Theoretic Local Duality). Let $k$ be the field of fractions of a complete discrete valuation ring whose residue field is finite. The map

$$L_{K(1)}K(k) \to I_{\mathbb{Q}/\mathbb{Z}}(L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \cong I_{\mathbb{Z}_p}L_{K(1)}K(k)$$

adjoint to the composite map

$$L_{K(1)}K(k) \wedge L_{K(1)}K(k) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p} \to I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}$$

described below is a weak equivalence.

We have stated the theorem in a way that emphasizes the parallel with the algebraic result. The functor $I_{\mathbb{Q}/\mathbb{Z}}(-)$ denotes Brown-Comenetz duality, the spectral analogue of Pontryagin duality; it can be constructed in terms of the Brown-Comenetz dual of the sphere spectrum $I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}$ as the derived function spectrum
F(−, I\textsubscript{Z}ZSQL). The functor $I_{Z_p}(-)$ denotes $Z_p$-Anderson duality; it can be constructed as

$$I_{Z_p}(-) = I_{Z/Q}(- \wedge M_{Z_p/Z_p}) \cong F(-, F(M_{Z_p/Z_p}, I_{Z/Q}S)) \cong F(-, I_{Z_p}S).$$

The duality map in Theorem\textsuperscript{[3.1]} is constructed as follows. The $E_\infty$ multiplication on $K(k)$ induces a map

$$L_{K(1)}K(k) \wedge L_{K(1)}K(k) \wedge M_{Q_p/Z_p} \to L_{K(1)}K(k) \wedge M_{Q_p/Z_p}$$

and we have a canonical map

$$L_{K(1)}K(k) \wedge M_{Q_p/Z_p} \to I_{Q/Z}S$$

(3.2)

essentially induced by the Hasse invariant (see [4] (1.2) for details).

In terms of Anderson duality, the weak equivalence

$$L_{K(1)}K(k) \overset{\cong}{\to} I_{Z_p}(L_{K(1)}K(k))$$

in Theorem\textsuperscript{[3.1]} is adjoint to the map

$$L_{K(1)}K(k) \wedge L_{K(1)}K(k) \to I_{Z_p}S$$

induced by the multiplication and the map

$$v_k : L_{K(1)}K(k) \to I_{Z_p}S.$$ (3.3)

adjoint to (3.2).

The local duality theorem relates to our work in this paper when we consider the case $k = Q_p$. In this case, Quillen’s localization theorem and Quillen’s calculation of the $K$-theory of $F_p$ together say that the fiber of the map $K(Z_p)_{\infty} \to K(Q_p)_{\infty}$ is $HZ_p$ and it follows that the map $L_{K(1)}K(Z_p) \to L_{K(1)}L(Q_p)$ is a weak equivalence. We then have the following corollary.

**Corollary 3.4** (Local duality for $Z_p$). Let $v_{Z_p} : L_{K(1)}K(Z_p) \to I_{Z_p}S$ be the composite of the map $L_{K(1)}K(Z_p) \overset{\cong}{\to} L_{K(1)}K(Q_p)$ and the map $v_{Q_p} : L_{K(1)}K(Q_p) \to I_{Z_p}S$ of (3.3). The map $L_{K(1)}K(Z_p) \to I_{Z_p}(L_{K(1)}K(Z_p))$ adjoint to the composite of multiplication and $v_{Z_p}$

$$L_{K(1)}K(Z_p) \wedge L_{K(1)}K(Z_p) \to L_{K(1)}K(Z_p) \overset{v_{Z_p}}{\to} I_{Z_p}S$$

is a weak equivalence.

The $K$-theoretic analogue of global duality identifies Fib(κ) in (2.2) in terms of the $Z_p$-Anderson dual of $L_{K(1)}K(Z)$. Rather than stating it in the full generality proved in [4], we state it just in this case. In [4] (1.7), we construct a map

$$u_Q : \text{Fib}(\kappa) \to \Sigma^{-1}I_{Z_p}S.$$ (3.5)

from the Albert-Brauer-Hasse-Noether sequence for $Q$. It is compatible with the map $u_{Q_p}$ above in the sense that $v_{Q_p}$ is the composite

$$L_{K(1)}K(Q_p) \simeq L_{K(1)}K(Z_p) \overset{\Sigma \text{Fib}(\kappa)}{\to} \Sigma \text{Fib}(\kappa) \overset{\Sigma u_Q}{\to} \Sigma \Sigma^{-1}I_{Z_p}S \cong I_{Z_p}S$$

where $L_{K(1)}K(Z_p) \to \Sigma \text{Fib}(\kappa)$ is the connecting map in the fiber sequence (2.2). The map (3.5) induces the following $K$-theoretic global duality theorem.
Theorem 3.6 (K-Theoretic Tate-Poitou Duality for \( \mathbb{Z} \)). The map

\[
\text{Fib}(\kappa) \to \Sigma^{-1}I_{\mathbb{Z}_p}L_{K(1)}K(\mathbb{Z})
\]

adjoint to the map

\[
L_{K(1)}K(\mathbb{Z}) \wedge \text{Fib}(\kappa) \to \Sigma^{-1}I_{\mathbb{Z}_p}\mathcal{S}
\]

induced by the \( L_{K(1)}K(\mathbb{Z}) \)-module structure map \( L_{K(1)}K(\mathbb{Z}) \wedge \text{Fib}(\kappa) \to \text{Fib}(\kappa) \) and the map \( u_\mathbb{Q} : \text{Fib}(\kappa) \to \Sigma^{-1}I_{\mathbb{Z}_p}\mathcal{S} \) is a weak equivalence.

4. THE EIGENSPLITTING OF THE FIBER OF THE CYCLOTOMIC TRACE

Combining Theorem 3.6 with the canonical wedge decomposition of \( K(\mathbb{Z}) \) described above in equation (2.3), we obtain the following decomposition of \( \text{Fib}(\kappa) \).

\[
\text{Fib}(\kappa) \simeq \Sigma^{-1}I_{\mathbb{Z}_p}(L_{K(1)}K(\mathbb{Z})) \simeq \Sigma^{-1}I_{\mathbb{Z}_p}(J \vee Y_0 \vee \cdots \vee Y_{p-2})
\]

\[
\simeq \Sigma^{-1}I_{\mathbb{Z}_p}J \vee \Sigma^{-1}I_{\mathbb{Z}_p}Y_0 \vee \cdots \vee \Sigma^{-1}I_{\mathbb{Z}_p}Y_{p-2}.
\]

Our goal in this section is to identify this wedge decomposition with (a permutation of) the wedge decomposition

\[
\text{Fib}(\kappa) \simeq J' \vee X_0 \vee \cdots \vee X_{p-2}
\]

constructed above in Corollary 2.11. This is accomplished in the following theorem together with an observation on the \( J' \) summand stated in Theorem 4.3 below.

Theorem 4.1. The canonical isomorphism in the stable category

\[
\text{Fib}(\kappa) \simeq \Sigma^{-1}I_{\mathbb{Z}_p}J \vee \Sigma^{-1}I_{\mathbb{Z}_p}Y_0 \vee \cdots \vee \Sigma^{-1}I_{\mathbb{Z}_p}Y_{p-2}.
\]

identifies \( J' \) as \( \Sigma^{-1}I_{\mathbb{Z}_p}J \) and \( X_i \) as \( \Sigma^{-1}I_{\mathbb{Z}_p}Y_{p-i} \) (for \( i \neq 0, 1 \)) or \( \Sigma^{-1}I_{\mathbb{Z}_p}Y_{1-i} \) (for \( i = 0, 1 \)). Moreover:

(i) \( [X_i, \Sigma^{-1}I_{\mathbb{Z}_p}Y_j] = 0 \) unless \( i + j \equiv 1 \text{ mod } (p-1) \).

(ii) \( [X_i, \Sigma^{-1}I_{\mathbb{Z}_p}J] = 0 \) for all \( i \).

(iii) \( [J', \Sigma^{-1}I_{\mathbb{Z}_p}Y_j] = 0 \) for all \( j \).

In later formulas we will simply write \( X_i \simeq \Sigma^{-1}I_{\mathbb{Z}_p}Y_{p-i} \) and understand the indexing to be mod \( (p-1) \).

Proof. To simplify notation, we write \( D \) for \( \Sigma^{-1}I_{\mathbb{Z}_p} \) inside this proof. The multiplication \( L \wedge L \to L \) together with the canonical identification of \( \pi_0L \) as \( \mathbb{Z}_p \) induces a map \( L \to I_{\mathbb{Z}_p}L \) that is easily seen to be an isomorphism in the stable category, q.v. [13, 2.6] (this is essentially due to Anderson [1]). This gives us a canonical identification of \( DL \) as \( \Sigma^{-1}L \), which is the main tool we use.

All statements follow from verification of (i), (ii), and (iii). As discussed above, \( Y_j \) fits in a cofiber sequence of the form

\[
\bigvee \Sigma^{2j-2}L \to Y_j \to \bigvee \Sigma^{2j-1}L \to \bigvee \Sigma^{2j-1}L
\]

(for some finite wedges of copies of \( \Sigma^nL \)); it follows that \( DY_j \) fits into a cofiber sequence of the form

\[
\bigvee \Sigma^{-2j}L \to \bigvee \Sigma^{-2j}L \to DY_j \to \bigvee \Sigma^{-2j+1}L.
\]

Applying Proposition 2.10 we see that \( [X_i, DY_j] = 0 \) unless \( -2j \equiv 2i - 2 \text{ mod } 2(p-1) \), or equivalently \( i + j \equiv 1 \text{ mod } (p-1) \). This proves (i).
Writing $J$ as the fiber of a self-map of $L$, $DJ$ fits then into a cofiber sequence of the form
\[ \Sigma^{-1}L \to DJ \to L \to L \]
and again applying Proposition 2.10, it follows that $[X_i, DJ] = 0$ unless $2i - 2 \equiv 0$ or $2i - 1 \equiv -1 \pmod{2(p - 1)}$. In the first case, $X_1 \simeq *$ by Proposition 2.9. In the second case $i = 0$; by Proposition 2.8 $X_0$ is non-canonically weakly equivalent to $\Sigma^{-2}L$, and $[\Sigma^{-2}L, DJ] = 0$. This proves (ii).

Finally, to prove (iii), we note that $DY_j$ is $K(1)$-local. Since $J'$ is non-canonically weakly equivalent to $J \simeq L_{K(1)} S$, to see that $[J', DY_j] = 0$ it suffices to see that $\pi_0 DY_j = 0$. Since $\pi_0 Y_j$ is concentrated in degrees congruent to $2j - 1$ and $2j - 2 \pmod{2(p - 1)}$, we have that $\pi_0 DY_j$ can only possibly be non-zero for $j = 0$ but by Proposition 2.9 $Y_0 \simeq *$. $\square$

While we have defined the $X_i$ solely in terms of the fiber sequence, we have defined $J'$ intrinsically, and so the equivalence of $J'$ with $\Sigma^{-1}I_{\mathbb{Z}_p} J$ under the isomorphism in the stable category $\text{Fib}(\kappa) \simeq \Sigma^{-1}I_{\mathbb{Z}_p} K(\mathbb{Z})$ constitutes additional information. In fact, we have a canonical weak equivalence
\[ J' \to \Sigma^{-1}I_{\mathbb{Z}_p} J \]
that we call the \textit{the standard weak equivalence}, constructed as follows. Since $J'$ is defined as the $K(1)$-localization of the Moore spectrum $M_{\mathbb{Z}_p}$, and $(\mathbb{Z}_p^\times)^\wedge$ is a projective $\mathbb{Z}_p$-module, maps in the stable category from $J'$ into $K(1)$-local spectra are in canonical one-to-one correspondence with homomorphisms from $(\mathbb{Z}_p^\times)^\wedge$ into $\pi_0$. We note that $\Sigma^{-1}I_{\mathbb{Z}_p} J$ is $K(1)$-local, and to calculate $\pi_0 \Sigma^{-1}I_{\mathbb{Z}_p} J$, we use the fundamental short exact sequence for the $\mathbb{Z}_p$-Anderson dual: For any spectrum $X$, there is a canonical natural short exact sequence
\[ 0 \to \text{Ext}(\pi_{-n-1} X, \mathbb{Z}_p) \to \pi_n I_{\mathbb{Z}_p} X \to \text{Hom}(\pi_{-n} X, \mathbb{Z}_p) \to 0. \]
For finitely generated $\mathbb{Z}_p$-modules, $\text{Hom}(\pi_{-n} X, \mathbb{Z}_p)$ and $\text{Ext}(\pi_{-n} X, \mathbb{Z}_p)$ coincide with $\text{Hom}_{\mathbb{Z}_p}(\pi_{-n} X, \mathbb{Z}_p)$ and $\text{Ext}_{\mathbb{Z}_p}(\pi_{-n} X, \mathbb{Z}_p)$. In the case of $X = J$, since $\pi_{-2} J = 0$, we then have a canonical identification of $\pi_0 \Sigma^{-1}I_{\mathbb{Z}_p} J$ as $\text{Hom}(\pi_{-1} J, \mathbb{Z}_p)$. The Morava Change of Rings Theorem identifies $\pi_{-1} J$ canonically in terms of continuous group cohomology:
\[ \pi_{-1} J \cong H^1_{\mathbb{Z}_p}(\mathbb{Z}_p^\times; \mathbb{Z}_p) \cong \text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p) \cong \text{Hom}((\mathbb{Z}_p^\times)^\wedge, \mathbb{Z}_p), \]
q.v. [9, (1.1)], for the continuous action of $\mathbb{Z}_p^\times$ on $\pi_* KU^\wedge_p$ arising from the $p$-adic interpolation of the Adams operations. This then gives a canonical isomorphism
\[ \pi_0 \Sigma^{-1}I_{\mathbb{Z}_p} J \cong \text{Hom}(\text{Hom}((\mathbb{Z}_p^\times)^\wedge, \mathbb{Z}_p), \mathbb{Z}_p). \]
Since $(\mathbb{Z}_p^\times)^\wedge$ is projective of rank 1, the double dual map is an isomorphism, giving us a canonical isomorphism
\[ (\mathbb{Z}_p^\times)^\wedge \to \pi_0 \Sigma^{-1}I_{\mathbb{Z}_p} J \]
specifying the standard weak equivalence.

On the other hand, we have a canonical map $J' \to \text{Fib}(\kappa)$ arising from the fiber sequence (2.2) and the $\Sigma J'$ summand of $L_{K(1)} K(\mathbb{Z}_p) \simeq L_{K(1)} K(\mathbb{Z}_p)$. In terms of maps from $(\mathbb{Z}_p^\times)^\wedge$ into $\pi_0 \text{Fib}(\kappa)$, we can therefore identify this map $J' \to \text{Fib}(\kappa)$ as coming from the canonical identification of the cokernel of
\[ \pi_1 L_{K(1)} K(\mathbb{Z}_p) \to \pi_1 L_{K(1)} K(\mathbb{Z}_p) \]
as \((\mathbb{Z}_p^\times)^\wedge_p\) (the \(p\)-completion of the cokernel of the map \((\mathbb{Z}[1/p])^\times \to \mathbb{Q}_p^\times\)). The following theorem compares the two maps.

**Theorem 4.3.** The composite map \(J' \to \text{Fib}(\kappa) \simeq \Sigma^{-1} I_p K(Z) \to \Sigma^{-1} I_p J\) is the standard weak equivalence.

We postpone the proof to Section 6.

5. **Self-duality of the fiber sequence of the cyclotomic trace**

In this section, we extend the analysis from the previous section by showing that the fiber sequence defining \(\text{Fib}(\kappa)\) is self-dual. This requires the compatibility of \(K\)-theoretic local and global duality discussed in Section 3.

**Theorem 5.1.** The following diagram commutes up to the indicated sign

\[
\begin{array}{ccc}
\text{Fib}(\kappa) & \xrightarrow{\rho} & L_{K(1)} K(Z) \\
\downarrow & & \downarrow \\
\Sigma^{-1} I_p (L_{K(1)} K(Z)) & \xrightarrow{\Sigma^{-1} I_p \text{(Fib}(\kappa))) & I_p L_{K(1)} K(Z) \\
\end{array}
\]

where the top sequence is the cofiber sequence (associated to the fiber sequence) defining \(\text{Fib}(\kappa)\), the bottom sequence is the \(\mathbb{Z}_p\)-Anderson dual of its rotation, and the vertical maps are induced by the \(K\)-theoretic Tate-Poitou duality theorem for \(\mathbb{Z}\) (Theorem 3.6) and the \(K\)-theoretic local duality theorem for \(\mathbb{Z}_p\) (Theorem 3.4).

**Proof.** The assertion is that \(\Sigma \rho\) is \(\mathbb{Z}_p\)-Anderson dual to \(\rho\) and \(\kappa\) is \(\mathbb{Z}_p\)-Anderson dual to \(\partial\). Given pairings

\[
\epsilon_1 : A_1 \wedge B_1 \longrightarrow I_p S
\]

whose adjoints \(\eta_1 : B_1 \to F(A_1, I_p S)\) are weak equivalences, then for maps \(f : A_1 \to A_2\) and \(g : B_2 \to B_1\), \(\eta_1 \circ g \circ \eta_2^{-1}\) is \(\mathbb{Z}_p\)-Anderson dual to \(f\) exactly when the diagram

\[
\begin{array}{ccc}
A_1 \wedge B_2 & \xrightarrow{id \wedge g} & A_1 \wedge B_1 \\
\downarrow f \wedge id & & \downarrow \epsilon_1 \\
A_2 \wedge B_2 & \xrightarrow{\epsilon_2} & I_p S
\end{array}
\]

commutes. In this case, when the weak equivalences \(\eta_1, \eta_2\) are fixed and understood, we say that \(g\) is \(\mathbb{Z}_p\)-Anderson dual to \(f\). By construction, the following diagram commutes

\[
\begin{array}{ccc}
L_{K(1)} K(Z) \wedge L_{K(1)} K(p) & \xrightarrow{id \wedge \partial} & L_{K(1)} K(Z) \wedge \Sigma \text{Fib}(\kappa) \\
\kappa \wedge id & & \downarrow \xi \\
L_{K(1)} K(Z) \wedge L_{K(1)} K(p) & \xrightarrow{\mu} & L_{K(1)} K(Z) \\
\downarrow & & \downarrow \Sigma u \xi \\
& & I_p S
\end{array}
\]

where \(\mu\) denotes the multiplication, \(\xi\) denotes the \(L_{K(1)} K(Z)\)-module action map, and \(u\) and \(v\) are the maps in the global and local duality theorems, respectively.
This gives the duality between $\partial$ and $\kappa$. To compare $\rho$ and $\Sigma \rho$, consider the diagram

$$
\begin{array}{c}
\text{Fib}(\kappa) \wedge \Sigma \text{Fib}(\kappa) \xrightarrow{\text{id} \wedge \Sigma \rho} \text{Fib}(\kappa) \wedge \Sigma L_{K(1)} K(\mathbb{Z}) \\
\rho \wedge \text{id} \\
L_{K(1)} K(\mathbb{Z}) \wedge \Sigma \text{Fib}(\kappa) \xrightarrow{} \mathbb{I}_{\mathbb{Z}} \mathbb{S}
\end{array}
$$

where the unlabeled maps are induced by the duality pairing. The down-then-right composite is $u_2$ composed with the suspension of the non-unital multiplication on $\text{Fib}(\kappa)$, whereas the right-then-down composite is $u_2$ composed with the suspension of the opposite of the non-unital multiplication on $\text{Fib}(\kappa)$. Since the non-unital multiplication on $\text{Fib}(\kappa)$ is $E_{\infty}$ and in particular commutative in the stable category, the diagram commutes.

As an immediate consequence, we get duality between the cofiber sequences in Corollary 5.2. Theorem 1.3 indicates the relationship between the $j$ and $j'$ sequences. The relationship between the remaining ones is summarized in the following corollary.

**Corollary 5.2.** For each $i$ in $\mathbb{Z}/(p-1)$, the following diagram commutes up to the indicated sign

$$
\begin{array}{c}
\Sigma^{-1} I_{\mathbb{Z}}(Y_{p-i}) \xrightarrow{\rho_i} \Sigma^{-1} I_{\mathbb{Z}}(X_{p-i}) \xrightarrow{\kappa_i} \Sigma I_{\mathbb{Z}}(Z_{p-i}) \xrightarrow{\partial_i} \Sigma X_i \\
\Sigma^{-1} I_{\mathbb{Z}}(Y_{p-i}) \xrightarrow{\rho_i} \Sigma^{-1} I_{\mathbb{Z}}(X_{p-i}) \xrightarrow{\kappa_i} \Sigma I_{\mathbb{Z}}(Z_{p-i}) \xrightarrow{\partial_i} \Sigma X_i
\end{array}
$$

where the top sequence is the cofiber sequence (associated to the fiber sequence) defining $X_i$, the bottom sequence is the $\mathbb{Z}_p$-Anderson dual of its rotation, and the vertical maps are induced by the maps

$$
\begin{align*}
X_i &\rightarrow \text{Fib}(\kappa) \rightarrow \Sigma^{-1} I_{\mathbb{Z}}(L_{K(1)} K(\mathbb{Z})) \rightarrow \Sigma^{-1} I_{\mathbb{Z}}(Y_{p-i}) \\
Y_i &\rightarrow L_{K(1)} K(\mathbb{Z}) \rightarrow \Sigma^{-1} I_{\mathbb{Z}}(\text{Fib}(\kappa)) \rightarrow \Sigma^{-1} I_{\mathbb{Z}}(X_{p-i}) \\
Z_i &\rightarrow L_{K(1)} K(\mathbb{Z}) \rightarrow I_{\mathbb{Z}}(L_{K(1)} K(\mathbb{Z})) \rightarrow I_{\mathbb{Z}}(Z_{p-i})
\end{align*}
$$

arising from local and global $K$-theoretic duality.

In the case of primes that satisfy the Kummer-Vandiver condition, we can be a bit more specific. A prime $p$ satisfies the Kummer-Vandiver condition when $p$ does not divide the order of the class group of $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$ for $\zeta_p = e^{2\pi i/p}$. In this case, Dwyer-Mitchell [10, 12.2] identifies the homotopy type of the spectra $Y_i$ in terms of the Kubota-Leopoldt $p$-adic $L$-function: Given any power series $f$ in the $p$-adic integers, there is a unique self-map $\phi_f$ on $L$ in the stable category such that $\text{on } \pi_{2(p-1)n}$, $\phi_f$ is multiplication by $f((1 + p)^n - 1)$ (cf. [10, 2.4]). A celebrated theorem of Iwasawa [12] implies in this context that for $i = 2, 4, \ldots, p-3$, there exists a self-map of $\Sigma^{2i-1} L$ which on $\pi_{2n+1}$ is multiplication by the value of the Kubota-Leopoldt $p$-adic $L$-function $L_p(-n, \omega^i)$. The spectrum $Y_i$ is non-canonically weakly equivalent to the homotopy fiber of this map. The $\mathbb{Z}_p$-Anderson self-duality of $L$ then identifies $I_{\mathbb{Z}} Y_i$ as (non-canonically) weakly equivalent to the fiber of the self-map of $\Sigma^{2-2i} L$ that on $\pi_{2n}$ is multiplication by $L_p(n, \omega^i)$. In particular, for
$j = 3, 5, \ldots, p - 2$, $X_j \simeq \Sigma^{-1}I_{Z_p}Y_{p-j}$ is then non-canonically weakly equivalent to the homotopy fiber of the self-map of $\Sigma^{2j-1}L$ that on $\pi_{2n-1}$ is multiplication by $L_p(n, \omega^{-j})$, or equivalently, on $\pi_{2n+1}$ is multiplication by $L_p(n + 1, \omega^{-j})$. For $i$ odd and for $j$ even, $Y_i$ and $X_i$ are non-canonically weakly equivalent to $\Sigma^{2i-1}L$ and $\Sigma^{2j}L$, respectively. (Independently of the Kummer-Vandiver condition $Y_0 \simeq *$ and $X_1 \simeq *$ by Propositions 2.4 and 2.9)

In the case of an odd regular prime, the relevant values of the Kubota-Leopoldt $p$-adic $L$-functions are units, and the spectra $X_j$ and $x_j$ are trivial for $j$ odd. This is consistent with Rognes’ computation [19, 3.3] of the homotopy fiber of the cyclotomic trace as (non-canonically) weakly equivalent to $j \vee \Sigma^{-2}k_{p^i}$ in this case. More generally, we have the following corollary.

**Corollary 5.3.** Let $p$ be an odd prime that satisfies the Kummer-Vandiver condition. The cofiber sequence

$$\text{Fib}(\tau) \to K(Z)_p^\wedge \to TC(Z)_p^\wedge \to \Sigma \text{Fib}(\tau)$$

is (non-canonically) weakly equivalent to the wedge of the cofiber sequences

\[
\begin{array}{cccc}
* & \to & j & = \to & j & \to & * \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma j' & \to & \Sigma j' & = \to & \Sigma j' & & \\
\Sigma^2 i & \to & \Sigma^2 i & = \to & \Sigma^2 i & & \\
\Sigma x_i & \to & \Sigma x_i & = \to & \Sigma x_i & & \\
\Sigma^2 \ell & \to & \Sigma^2 \ell & = \to & \Sigma^2 \ell & & \\
\Sigma^2 i & \to & \Sigma\ell & = \to & \Sigma \ell & & \\
\end{array}
\]

where $\lambda_i^p$ is the unique self-map of $\Sigma^{2i-1}\ell$ that on $\pi_{2n+1}$ is multiplication by the value $L_p(n + 1, \omega^{-i})$, $\lambda_i^p$ is the unique self-map of $\Sigma^{2i-1}\ell$ that on $\pi_{2n+1}$ is multiplication by the value $L_p(-n, \omega^i)$, and $L_p$ denotes the Kubota-Leopoldt $p$-adic $L$-function.

6. **Proof of Theorem 4.3**

As discussed above the statement of Theorem 4.3 maps from $J'$ to $\text{Fib}(\kappa)$ are determined by maps from $(Z_p^\times)_p$ to $\pi_0 \text{Fib}(\kappa)$; we have two isomorphisms of $(Z_p^\times)_p$ with $\pi_0 \text{Fib}(\kappa)$ and we need to show that they are the same. It is slightly easier to work with the $Z_p$-duals instead. We can canonically identify the $Z_p$-dual of $\pi_0 \text{Fib}(\kappa)$ as $\pi_{-1}L_{K(1)}K(Z)$: the fundamental short exact sequence 1.2 for $\pi_0 \Sigma^{-1}I_{Z_p}$ $K(Z)$ gives an isomorphism

$$\pi_0 \Sigma^{-1}I_{Z_p}K(Z) \cong \text{Hom}(\pi_{-1}L_{K(1)}K(Z), Z_p)$$

since $\pi_{-2}L_{K(1)}K(Z) = 0$ (as $Y_0 \simeq *$). The identification of $\pi_{-1}J$ as $\text{Hom}(Z_p^\times, Z_p)$ above and the canonical map $J \to L_{K(1)}K(Z)$ gives one isomorphism of $\pi_{-1}L_{K(1)}K(Z)$ with $\text{Hom}(Z_p^\times, Z_p)$, which we denote as $\alpha$. The isomorphism of $\pi_0 \text{Fib}(\tau)$ with $(Z_p^\times)_p$ as the quotient of $\pi_1L_{K(1)}K(Q_p) \cong (Q_p^\times)_p$ gives another isomorphism of $\pi_{-1}L_{K(1)}K(Z)$ with $\text{Hom}(Z_p^\times, Z_p)$, which we denote as $\eta$. We need to prove that $\eta = \alpha$. 
We have an intrinsic identification of $\pi_{-1}L_{K(1)}K(\mathbb{Z})$ coming from Thomason’s descent spectral sequence \cite[4.1]{Thomason}, which in this case canonically identifies

$$\pi_{-1}L_{K(1)}K(\mathbb{Z}) \cong \pi_{-1}L_{K(1)}K(\mathbb{Z}[1/p])$$

as $H^0_S(\text{Spec} \mathbb{Z}[1/p]; \mathbb{Z}_p)$ (continuous étale cohomology). Let $\mathbb{Q}_S$ denote the maximal algebraic extension of $\mathbb{Q}$ that is unramified except over $S = \{p\}$, and let $G_S = \text{Gal}(\mathbb{Q}_S/\mathbb{Q})$. The abelianization $G^{ab}_S$ of $G_S$ corresponds to the maximal abelian extension of $\mathbb{Q}$ that is unramified except over $p$, which is $\mathbb{Q}(\mu_{p^n})$ (where $\mu_{p^n}$ denotes the group of $p^n$th roots unity for all $n$). We have the standard identification of the continuous étale cohomology $H^0_S(\text{Spec} \mathbb{Z}[1/p]; \mathbb{Z}_p)$ as the Galois cohomology $H^0(\text{Gal}(\mathbb{Q}_S/\mathbb{Q}); \mathbb{Z}_p)$ \cite[II.2.9]{Milne}, which we can identify as the abelian group of continuous homomorphisms

$$\text{Hom}_c(G_S, \mathbb{Z}_p) \cong \text{Hom}_c(G^{ab}_S, \mathbb{Z}_p).$$

We have a further isomorphism

$$G^{ab}_S = \text{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q}) \cong \lim_n \text{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q}) \cong \lim_n (\mathbb{Z}/p^n)^\times \cong \mathbb{Z}_p^\times,$$

where the first isomorphism is inverse to the isomorphism $(\mathbb{Z}/p^n)^\times \cong \text{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$ taking an element $u \in (\mathbb{Z}/p^n)^\times$ to the automorphism of $\mathbb{Q}(\mu_{p^n})$ induced by the automorphism $\zeta \mapsto \zeta^n$ on $\mu_{p^n}$. This then constructs an isomorphism

$$\gamma: \pi_{-1}L_{K(1)}K(\mathbb{Z}) \longrightarrow \text{Hom}_c(\mathbb{Z}_p^\times, \mathbb{Z}_p) \cong \text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p).$$

First we show $\gamma = \alpha$. Choose a prime number $l$ such that $l$ is a topological generator of $\mathbb{Z}_p^\times$, or equivalently, a generator of $\mathbb{Z}/p^2$, and consider the quotient map $\mathbb{Z}[1/p] \to \mathbb{Z}/l = \mathbb{F}_l$. By celebrated work of Quillen \cite{Quillen}, the composite map

$$j \longrightarrow K(\mathbb{Z})_p \longrightarrow K(\mathbb{F}_l)_p$$

is a weak equivalence and an embedding of $\widehat{\mathbb{F}_l^\times}$ in $\mathbb{C}^\times$ induces a weak equivalence $K(\widehat{\mathbb{F}_l^\times}) \to ku^\wedge_p$ with the automorphism $\Phi$ on $K(\widehat{\mathbb{F}_l^\times})$ induced by the Frobenius automorphism $\Phi$ on $ku^\wedge_p$ (independently of the choice of embedding). We will also write $\Phi$ for the corresponding automorphism of $L_{K(1)}K(\widehat{\mathbb{F}_l})$. For any functorial model of $L_{K(1)}K(-)$, the induced map from $L_{K(1)}K(\mathbb{F}_l)$ into the homotopy fixed points of $\Phi$ (the homotopy equalizer of $\Phi$ and the identity on $L_{K(1)}K(\widehat{\mathbb{F}_l})$) is a weak equivalence. Writing $L_{K(1)}K(\mathbb{F}_l)^{h\Phi}$ for the homotopy fixed points of $\Phi$, the map

$$L_{K(1)}K(\mathbb{F}_l) \longrightarrow L_{K(1)}K(\mathbb{F}_l)^{h\Phi}$$

is the unique map that takes the unit element of $\pi_0(L_{K(1)}K(\mathbb{F}_l))$ to the unique element of $\pi_0(L_{K(1)}K(\mathbb{F}_l)^{h\Phi})$ that maps to the unit element of $\pi_0(L_{K(1)}K(\mathbb{F}_l))$. This gives a canonical identification of $\pi_{-1}(L_{K(1)}K(\mathbb{F}_l))$ as $H^1(\langle \Phi \rangle; \mathbb{Z}_p)$, where $\langle \Phi \rangle$ denotes the cyclic group generated by $\Phi$; we have used the canonical isomorphism $\pi_0(L_{K(1)}K(\mathbb{F}_l)) \cong \mathbb{Z}_p$ induced by the unit and we note that this isomorphism is consistent with the canonical isomorphism $\pi_0(KU^\wedge_p) \cong \mathbb{Z}_p$ under the weak equivalence $L_{K(1)}K(\mathbb{F}_l) \to KU^\wedge_p$ (independently of the choice of the embedding $\mathbb{F}_l^\times \to \mathbb{C}^\times$).

Under the identification of $\pi_{-1}J$ as $H^1_c(\mathbb{Z}_p^\times; \mathbb{Z}_p)$ above, the composite map

$$J \longrightarrow L_{K(1)}K(\mathbb{Z}) \longrightarrow L_{K(1)}K(\mathbb{F}_l)$$

induces on $\pi_{-1}$ the map

$$H^1_c(\mathbb{Z}_p^\times; \mathbb{Z}_p) \longrightarrow H^1(\langle \Phi \rangle; \mathbb{Z}_p).$$
induced by the inclusion of \( l \) in \( \mathbb{Z}_p^\times \) (the inclusion of \( \Psi^l \) in the group of \( p \)-adically interpolated Adams operations). This gives us information about \( \alpha \). In terms of the identification of \( \pi_1(L_{K(1)}K(\mathbb{F}_l)) \) as \( H^1_{\text{et}}(\mathbb{F}_l; \mathbb{Z}_p) \) from Thomason’s descent spectral sequence, the map

\[
H^1_{\text{et}}(\mathbb{F}_l; \mathbb{Z}_p) \cong H^1_{\text{Gal}}(\mathbb{F}_l/\mathbb{F}_l; \mathbb{Z}_p) \rightarrow H^1(\langle \Phi \rangle; \mathbb{Z}_p)
\]

is induced by the inclusion of the Frobenius in \( \text{Gal}(\mathbb{F}_l/\mathbb{F}_l) \). By naturality, the composite map

\[
H^1_{\text{et}}(\mathbb{Z}^{[\frac{1}{p}]}_p; \mathbb{Z}_p) \cong H^1(G_S; \mathbb{Z}_p) \rightarrow H^1(\text{Gal}(\mathbb{F}_l; \mathbb{Z}_p)) \rightarrow H^1(\langle \Phi \rangle; \mathbb{Z}_p)
\]

is induced by the inclusion of \( \Phi \) in \( G_S \) as the automorphism of (the \( p \)-integers in) \( \mathbb{Q}_S \) that does the automorphism \( \zeta \mapsto \zeta^l \) on \( \mu_{p^\infty} \). This then shows that \( \gamma = \alpha \).

We now compare \( \gamma \) and \( \eta \). Here it is easiest to work first in terms of \( L_{K(1)}K(\mathbb{Q}_p) \).

Using the standard identification of \( \pi_1(L_{K(1)}K(\mathbb{Q}_p)) \) as the \( p \)-completion of the units, we have a \( \mathbb{Q}_p \)-analogue of \( \eta \) using local duality: Let

\[
\eta_p: \pi_1L_{K(1)}K(\mathbb{Q}_p) \rightarrow \text{Hom}(\mathbb{Z}_p^\times; \mathbb{Z}_p) \cong \text{Hom}(\mathbb{Q}_p^\times; \mathbb{Z}_p)
\]

be the isomorphism derived from the isomorphism \( \pi_1L_{K(1)}K(\mathbb{Q}_p) \cong (\mathbb{Q}_p^\times)^\wedge \) by Anderson duality. We then have a commutative diagram

\[
\begin{array}{ccc}
\pi_1L_{K(1)}K(\mathbb{Z}) & \cong & \pi_1L_{K(1)}K(\mathbb{Z}^{[\frac{1}{p}]}_p) \\
\eta & \cong & \eta_p \\
\text{Hom}(\mathbb{Z}_p^\times; \mathbb{Z}_p) & \xrightarrow{q^*} & \text{Hom}(\mathbb{Q}_p^\times; \mathbb{Z}_p)
\end{array}
\]

by the compatibility of local and global duality. (Here \( q^* \) is the map induced by the standard quotient isomorphism \( \mathbb{Q}_p^\times/(p) \cong \mathbb{Z}_p^\times \).) To produce a local analogue of \( \gamma \), we use the Artin symbol \( \theta \) in local class field theory [21, §3.1]. The Artin symbol gives an isomorphism between the finite completion of the units of \( \mathbb{Q}_p \) and the Galois group of the maximal abelian extension \( \mathbb{Q}_p^{ab} \) of \( \mathbb{Q}_p \): For \( x \in \mathbb{Q}_p^\times \), writing \( x = ap^m \) for \( a \in \mathbb{Z}_p^\times \), the Artin symbol takes \( x \) to the unique element \( \theta(x) \) of \( \text{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p) \) that acts on the \( p^m \)th roots of unity \( \mu_{p^m} \) by raising to the \( 1/a \) power and acts on the maximal unramified extension \( (\mathbb{Q}_p)^{nr} \) of \( \mathbb{Q}_p \) by the \( m \)th power of a lift of the Frobenius. Using the isomorphism

\[
\pi_1L_{K(1)}K(\mathbb{Q}_p) \cong H^1_{\text{et}}(\mathbb{Q}_p; \mathbb{Z}_p)
\]

from Thomason’s descent spectral sequence and the canonical isomorphism

\[
H^1_{\text{et}}(\mathbb{Q}_p; \mathbb{Z}_p) \cong H^1_{\text{Gal}}(\mathbb{Q}_p/\mathbb{Q}_p; \mathbb{Z}_p) \cong \text{Hom}_c(\text{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p), \mathbb{Z}_p)
\]

(as above), the Artin symbol induces an isomorphism

\[
-\gamma_p: \pi_1L_{K(1)}K(\mathbb{Q}_p) \rightarrow \text{Hom}(\mathbb{Q}_p^\times; \mathbb{Z}_p).
\]
We have implicitly defined an isomorphism $\gamma_p$: The formula for the Artin symbol implies that the following diagram commutes

\[
\begin{array}{ccc}
\pi_{-1}L_{K(1)}K(\mathbb{Z}) & \xrightarrow{\cong} & \pi_{-1}L_{K(1)}K(\mathbb{Z}^{1/p}) \\
\gamma & \cong & \gamma_p \\
\Hom(\mathbb{Z}_p^\times, \mathbb{Z}_p) & \xrightarrow{q^*} & \Hom(\mathbb{Q}_p^\times, \mathbb{Z}_p) \\
\downarrow & & \downarrow \cong \\
\Hom(\mathbb{Z}_p^\times, \mathbb{Z}_p) & \xrightarrow{(id,0)} & \Hom(\mathbb{Z}_p^\times, \mathbb{Z}_p) \times \mathbb{Z}_p
\end{array}
\]

(where the bottom right vertical isomorphism is induced by the $ap^n$ decomposition of $\mathbb{Q}_p^\times$ as $\mathbb{Z}_p^\times \times \mathbb{Z}$). In other words, omitting notation for the isomorphism arising from Thomason’s descent spectral sequence and the usual isomorphism $H^1_{\text{ét}}(\mathbb{Q}_p; \mathbb{Z}_p) \cong \text{Hom}(\text{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p), \mathbb{Z}_p)$, $\gamma_p$ is the $\mathbb{Z}_p$-dual of $-\theta$. The Artin symbol has a cohomological characterization \[21\] §2.3, Prop. 1]: For a character $\rho$: $\text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p) \to \mathbb{Q}/\mathbb{Z}$ and $x \in \mathbb{Q}_p^\times$,

\[
\rho(\theta(x)) = \text{inv}(x \cup \rho)
\]

(cf. \[17\] p. 386) where on the right we interpret $x$ as an element of $H^1_{\text{ét}}(\mathbb{Q}_p; \mathbb{Z}_p)$ and $\rho$ as an element of $H^1_{\text{ét}}(\mathbb{Q}_p; \mathbb{Q}/\mathbb{Z})$, while the symbol $\cup$ denotes the cup product on étale cohomology

\[
H^1_{\text{ét}}(\mathbb{Q}_p; \mathbb{Z}_p(1)) \otimes H^1_{\text{ét}}(\mathbb{Q}_p; \mathbb{Q}/\mathbb{Z}) \to H^2_{\text{ét}}(\mathbb{Q}_p; \mathbb{Q}/\mathbb{Z}(1)),
\]

and $\text{inv}$ denotes the map induced by the Hasse invariant (q.v. \[22\] XIII§3, Prop. 6ff]). Because

\[
\text{inv}(x \cup y): H^1_{\text{ét}}(\mathbb{Q}_p; \mathbb{Z}_p(1)) \otimes H^1_{\text{ét}}(\mathbb{Q}_p; \mathbb{Q}_p/\mathbb{Z}_p) \to \mathbb{Q}/\mathbb{Z}
\]

is the local duality pairing, restricting to

\[
H^1_{\text{ét}}(\mathbb{Q}_p; \mathbb{Z}/p^n(1)) \otimes H^1_{\text{ét}}(\mathbb{Q}_p; \mathbb{Z}/p^n) \to \mathbb{Z}/p^n,
\]

taking the inverse limit, and applying the isomorphism from Thomason’s descent spectral sequence gives us the Anderson duality pairing

\[
\pi_{-1}L_{K(1)}K(\mathbb{Q}_p) \otimes \pi_{-1}L_{K(1)}K(\mathbb{Q}_p) \to \mathbb{Z}_p.
\]

We conclude that $\eta_p = \gamma_p$. Since the map $q^*: \text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p) \to \text{Hom}(\mathbb{Q}_p^\times, \mathbb{Z}_p)$ is an injection, we conclude that $\eta = \gamma$.

7. The eigenspectra of $TC(\mathbb{Z})_p^\times$

As shown in \[3\] (2.4)], $TC(\mathbb{Z})_p^\times$ admits a canonical splitting with summands $j$, $\Sigma_j'$, and the spectra denoted in Section 2 as $z_i$, for $i = 0, \ldots, p-2$. As indicated there, $z_i$ is non-canonically weakly equivalent to $\Sigma^{2i-1}L$ (for $i \neq 1$) or $\Sigma^{2p-1}L$ (for $i = 1$). The purpose of this section is to give an identification of these summands in intrinsic terms. We work in terms of the $K(1)$-localizations $Z_i$, and our main result is to explain the perspective that $Z_i$ is $\Sigma^{2i-1}L$ tensored over $\Lambda \cong [L, L]$ with a free $\Lambda$-module and to identify that $\Lambda$-module intrinsically in number theoretic terms; see Corollary \[26\] for a precise statement. The remainder of the section discusses the problem of finding a generator for this free module. The number
theory literature discusses several approaches, which we review. The content of this section is independent of the other sections.

Let

$$Z = Z_0 \lor \cdots \lor Z_{p-2} = L_{K(1)Z_0} \lor \cdots \lor L_{K(1)Z_{p-2}}.$$  

Since $z_i$ is the $1$-connected cover (for $i \neq 0$) or $(-2)$-connected cover (for $i = 0$) of $Z_i = L_{K(1)Z_i}$, to identify $z_i$, it suffices to identify $Z_i$. Since $Z_i$ is non-canonically weakly equivalent to $\Sigma^{2i-1}L$ (for $L$ the Adams summand of $KU_p^n$), we have that $[Z_i, Z_j] = 0$ for $i \neq j$. The decomposition of $Z$ into the summands $Z_i$ is therefore unique, and so it suffices to identify $Z$. It follows from Hesselholt-Madsen [11 Th. D, Add 6.2] that $TC$ of the $p$-completion map and the cyclotomic trace $\alpha$ are $K(1)$-equivalences. These maps induce a weak equivalence from $Z$ to a summand of $L_{K(1)K(Z_p)}$, which Dwyer-Mitchell [10, §13] identifies in terms of units of cyclotomic extensions of $Q_p$. We now review this identification.

Let $F_n = Q_p(\mu_{p^n+1})$ (with $F_{n-1} = Q_p$), and let $E_n = (F_n/\text{torsion})^\wedge_p$ (where $\mu_{p^n+1}$ denotes the $p^{n+1}$ roots of unity in some algebraic closure of $Q_p$). The norm (in Galois theory) gives maps $E_n \to E_{n-1}$; let $E_\infty$ be the inverse limit. The Galois group $\text{Gal}(F_n/Q_p)$ is canonically isomorphic to $(Z/\mathbb{p}^{n+1})^\times$ and this makes $E_n$ a $p$-complete $Z_p[(Z/\mathbb{p}^{n+1})^\times]$-module. The norm $E_n \to E_{n-1}$ is a $Z_p[(Z/\mathbb{p}^{n+1})^\times]$-module map and so in the inverse limit $E_\infty$ is a module over the Iwasawa algebra $\Lambda' = Z_p[[Z_p^\times]] = \text{lim} Z_p[(Z/\mathbb{p}^{n+1})^\times]$.

As a completed group ring, $\Lambda'$ comes with an (anti)involution $\chi$ that sends the group elements to their inverses; for a $\Lambda'$-module $M$, we denote by $M^\chi$ the $\Lambda'$-module obtained via this involution. Coincidentally, $\Lambda' \cong [KU_p^\wedge_p, KU_p^\wedge_p]$ via the map that takes an element $a$ of $Z_p^\times$ to the $(p$-adically interpolated) Adams operation $\psi^a$ (which exists for $p$-completed topological $K$-theory), and in particular, we have a canonical action of $\Lambda'$ on $KU_p^\wedge_p$ (in the stable category). Dwyer-Mitchell [10] §13 then shows that $E_\infty$ (denoted there as $E'_\infty$(red)) is free of rank 1 as a $\Lambda'$-module, observes that

$$[Z, KU_p^\wedge_p] = (KU_p^\wedge_p)^0(Z) = 0$$

(cf. [ibid., 6.11]), and constructs a canonical isomorphism

$$[\Sigma^{-1}Z, KU_p^\wedge_p] = (KU_p^\wedge_p)^1(Z) \cong \text{Hom}_{\Lambda'}(E_\infty, \Lambda')^\chi$$

of $\Lambda'$-modules (cf. [ibid., 8.10]). For formal reasons, this characterizes $Z$ in the stable category via the Yoneda Lemma (see [ibid., 4.12]).

**Theorem 7.2** (Dwyer-Mitchell [10] 9.1). For any spectrum $X$, the natural map

$$[X, \Sigma^{-1}Z] \to \text{Hom}_{\Lambda'}(KU_p^\wedge_p, [\Sigma^{-1}Z, KU_p^\wedge_p], [X, KU_p^\wedge_p])$$

$$\cong \text{Hom}_{\Lambda'}(E_\infty, \Lambda')^\chi, [X, KU_p^\wedge_p] \cong E_\infty^\chi \otimes_{\Lambda'} [X, KU_p^\wedge_p]$$

is an isomorphism.

The last isomorphism follows from the fact that $E_\infty$ is free of finite rank, using the isomorphism

$$\text{Hom}_{\Lambda'}(E_\infty, \Lambda')^\chi \cong \text{Hom}_{\Lambda'}(E_\infty^\chi, \Lambda')$$
adjoint to the $\chi$-twisted evaluation map
\begin{equation}
\text{Hom}_\Lambda(E_\infty, \Lambda')^\chi \otimes E_\infty^\chi \rightarrow (\Lambda')^\chi \xrightarrow{\cong} \Lambda'.
\end{equation}

Plugging $X = KU_p^\wedge$ into Theorem 7.2 we get an isomorphism of $\Lambda'$-modules
\[
[KU_p^\wedge, \Sigma^{-1}Z] \xrightarrow{\cong} E_\infty^\chi \otimes_{\Lambda'} [KU_p^\wedge, KU_p^\wedge] \cong E_\infty^\chi.
\]

Concisely, this isomorphism and the isomorphism of (7.1) identify the $\chi$-twisted evaluation map (7.3) with the composition in the stable category
\[
\Sigma^{-1}Z, KU_p^\wedge \otimes [KU_p^\wedge, \Sigma^{-1}Z] \rightarrow [KU_p^\wedge, KU_p^\wedge] \cong \Lambda'.
\]

We have a canonical identification of endomorphism rings $\Lambda' \cong [KU_p^\wedge, KU_p^\wedge]$ and $[\Sigma^{-1}Z, \Sigma^{-1}Z]$ induced by choosing any weak equivalence $KU_p^\wedge \simeq \Sigma^{-1}Z$: because $\Lambda'$ is commutative every choice induces the same isomorphism. Commutativity also implies that the $\Lambda'$-module structure on $[KU_p^\wedge, \Sigma^{-1}Z]$ from $[\Sigma^{-1}Z, \Sigma^{-1}Z]$ coincides with the $\Lambda'$-structure from $[KU_p^\wedge, KU_p^\wedge]$ and we see that the isomorphism in Theorem 7.2 is an isomorphism of $\Lambda'$-modules for the $\Lambda'$-module structure on $[X, \Sigma^{-1}Z]$ from $[\Sigma^{-1}Z, \Sigma^{-1}Z]$. Using the duality of the invertible $\Lambda'$-modules $[\Sigma^{-1}Z, KU_p^\wedge]$ and $[KU_p^\wedge, \Sigma^{-1}Z]$, Theorem 7.2 then implies the following slightly less complicated isomorphism.

**Corollary 7.4.** For any spectrum $X$, the natural map
\[
E_\infty^\chi \otimes_{\Lambda'} [X, KU_p^\wedge] \cong [KU_p^\wedge, \Sigma^{-1}Z] \otimes_{[KU_p^\wedge, KU_p^\wedge]} [X, KU_p^\wedge] \rightarrow [X, \Sigma^{-1}Z]
\]
is an isomorphism.

We can identify the spectra $Z_i$ using the eigensplitting of $E_\infty^\chi$. To explain this, let $\Lambda$ be the completed group ring
\[
\Lambda = \mathbb{Z}_p[[U^1]] = \lim \mathbb{Z}_p[U^1/U^n]
\]
where $U^n$ denotes the subgroup of $\mathbb{Z}_p^\times$ of elements congruent to 1 mod $p^n$. Then $\Lambda$ is a (topological) subring of $\Lambda'$ and because $\mathbb{Z}_p^\times \cong \mu_{p-1} \times U^1$, $\Lambda'$ is isomorphic to the group algebra $\Lambda[\mu_{p-1}]$. We prefer to write $\Lambda' \cong \Lambda[\mathbb{F}_p^\times]$, using the Teichmüller character $\omega: \mathbb{F}_p^\times \rightarrow \mathbb{Z}_p^\times$ for the isomorphism $\mathbb{F}_p^\times \rightarrow \mu_{p-1}$. Because $p-1$ is invertible in $\mathbb{Z}_p$ and $\mu_{p-1} \subset \mathbb{Z}_p$, we get orthogonal idempotent elements
\[
\epsilon_i = \frac{1}{p-1} \sum_{\alpha \in \mathbb{F}_p^\times} \omega(\alpha)^{-i} \psi(\alpha)
\]
in $\Lambda'$ for $i = 0, \ldots, p-2$ (where we have written the group elements (in $\mathbb{Z}_p^\times$) using Adams operation notation to distinguish them from the coefficient elements (in $\mathbb{Z}_p$) of the completed group algebra $\Lambda'$). These give a Cartesian product decomposition of $\Lambda'$,
\[
\Lambda' = \epsilon_0 \Lambda' \times \cdots \times \epsilon_{p-2} \Lambda'
\]
where the elements of $\epsilon_i \Lambda$ can be characterized as the elements of $\Lambda'$ on which $\mathbb{F}_p^\times$ acts by multiplication by $\omega^i$; we call this the $\omega^i$ eigenspace of $\Lambda'$. Since $\omega^i = \omega^i \cdot (p-1)$, it makes sense and can be convenient to index the eigenspaces on elements of $\mathbb{Z}/(p-1)$ rather than the specific representatives $0, \ldots, p-2$. The inclusion $\Lambda \rightarrow \Lambda'$ induces an isomorphism of (topological) $\mathbb{Z}_p$-algebras between $\Lambda$ and $\epsilon_i \Lambda'$ for each $i$. Every $\Lambda'$-module admits a corresponding decomposition into $\omega^i$ eigenspaces, which we can regard as $\Lambda$-modules.
The Cartesian product decomposition of $\Lambda'$ above corresponds exactly to the Cartesian product decomposition of $[KU_p^\wedge, KU_p^\wedge]$, $$[KU_p^\wedge, KU_p^\wedge] \cong [L, L] \times [\Sigma^2L, \Sigma^2L] \times \cdots \times [\Sigma^{2p-4}L, \Sigma^{2p-4}L]$$ induced by the Adams splitting $KU_p^\wedge \cong L \vee \Sigma^2L \vee \cdots \vee \Sigma^{2p-4}L$. The decomposition isomorphism takes $[\Sigma^2L, \Sigma^2L]$ to precisely the subset of $[KU_p^\wedge, KU_p^\wedge]$ of elements on which $\psi(\alpha)$ acts by multiplication by $\omega(\alpha)^i$ for all $\alpha \in \mathbb{F}_p^\times$. The isomorphism $\Lambda' \cong [KU_p^\wedge, KU_p^\wedge]$ induces an isomorphism $\Lambda \cong [L, L]$ for the inclusion of $[L, L]$ as the diagonal in $[\prod [L, L] \cong \prod [\Sigma^2L, \Sigma^2L]$. Since the identification of $[KU_p^\wedge, \Sigma^{-1}Z]$ as $E_\infty$ above is an isomorphism of $\Lambda'$-modules, we get an isomorphism of $\Lambda$-modules $$\epsilon_i[KU_p^\wedge, \Sigma^{-1}Z] \cong \epsilon_iE_\infty$$ on the $\omega^i$ eigenspaces for all $i$. The idempotent $\epsilon_i$ of $[KU_p^\wedge, KU_p^\wedge]$ is the composite of the projection and inclusion $$KU_p^\wedge \rightarrow \Sigma^2L \rightarrow KU_p^\wedge,$$ so we see that $$\epsilon_i[KU_p^\wedge, \Sigma^{-1}Z] \cong [\Sigma^2L, \Sigma^{-1}Z] \cong [\Sigma^2L, \Sigma^{-1}Z_{i+1}]$$ (numbering mod $p-1$) as $[\Sigma^2L, \Sigma^{-1}Z_j] = 0$ unless $j \equiv i+1 \pmod{p-1}$. We then get the following characterization of $Z_i$.

**Corollary 7.5.** Let $i \in \mathbb{Z}/(p-1)$. For any spectrum $X$, the natural map $$\epsilon_iE_\infty \otimes_{\Lambda} [X, \Sigma^2L] \cong [\Sigma^2L, \Sigma^{-1}Z_{i+1}] \otimes_{[\Sigma^2L, \Sigma^2L]} [X, L] \rightarrow [X, \Sigma^{-1}Z_{i+1}]$$ is an isomorphism.

More can be said about choosing weak equivalences $Z_i \simeq \Sigma^{2i-1}L$. From the discussion above, choosing such a weak equivalence is equivalent to choosing a generator of the free rank one $\Lambda$-module $\epsilon_{i-1}E_\infty$, and this is equivalent to choosing a generator of the free rank one $\Lambda$-module $\epsilon_{p-i}E_\infty$.

We treat the case of $\epsilon_0E_\infty$ separately, but all cases require a choice of a system of primitive $p$th roots of units $\zeta_n \in \mu_{p^{n+1}}$ with $\zeta_p = \zeta_{n-1}$, which we now fix. (For example, choosing an embedding of $\mathbb{Q}_p$ in $\mathbb{C}$, one could take $\zeta_n = e^{2\pi i/p^{n+1}}$.) Then $\zeta_{n-1}$ is a uniformizer for $\mathcal{O}_{F_n}$ for each $n$: we have $N_{F_{n+1}/F_n}(\zeta_{n+1}-1) = \zeta_n - 1$ and $N_{F_{n}/\mathcal{O}_p}(\zeta_0 - 1) = p$. This argument also shows that the system $\{\epsilon_0(\zeta_n - 1)\}$ specifies an element of $E_\infty$, and we can consider its $\omega^0$ eigenfactor $\epsilon_0(\zeta_n - 1)$.

**Proposition 7.6.** In the $\Lambda'$-module $E_\infty$, $\epsilon_0(\zeta_n - 1)$ generates $\epsilon_0E_\infty$.

**Proof.** The valuation $F_n^\times \rightarrow \mathbb{Z}$ is a homomorphism that sends roots of unity to zero and so extends to a homomorphism $E_n \rightarrow \mathbb{Z}_p$. It commutes with the norm and so assembles to a homomorphism $E_\infty \rightarrow \mathbb{Z}_p$. Giving $\mathbb{Z}_p$ the trivial $\Lambda'$-action, the homomorphism $E_\infty \rightarrow \mathbb{Z}_p$ is $\Lambda'$-linear and factors through $\epsilon_0E_\infty$. Because $\zeta_{n-1} \in F_n^\times$ has valuation 1, $\Lambda$ is a local ring, and $\epsilon_0E_\infty$ is a free rank one $\Lambda$-module, it follows by Nakayama's Lemma that $\epsilon_0(\zeta_n - 1)$ is a generator of $\epsilon_0E_\infty$. \qed

**Remark 7.7.** We note that the element $\epsilon_0(\zeta_n - 1)$ of $\epsilon_0E_\infty$ is in the image of the corresponding $\Lambda'$-module defined in terms of $\mathbb{Z}[1/p]$ in place of $\mathbb{Q}_p$, that is, the inverse limit over norm maps of the $p$-completion of the units of $\mathbb{Z}[\zeta_n, 1/p]$ modulo torsion. This element can be used to construct a weak equivalence $\Sigma^{-1}Y_1 \simeq L$ by an
argument analogous to the one for Theorem 7.2 (but for the $\epsilon_0$ piece only). Since the resulting weak equivalence is just the composite of the weak equivalence $\Sigma^{-1}Y_1 \to \Sigma^{-1}Z_1$ induced by the cyclotomic trace and the weak equivalence $\Sigma^{-1}Z_1 \simeq L$ coming from the previous proposition (and Corollary 7.5), we omit the details.

For other eigenspaces, $\epsilon_i((\zeta_n - 1))$ is generally not a generator. If $i \neq 0 \mod p - 1$, then $\epsilon_i((\zeta_n - 1)) \in F_n^\times$ is a cyclotomic unit [14 3§5]. In particular, it is a real multiple of a root of unity [14 p. 84], and as such, $\epsilon_i((\zeta_n - 1))$ becomes the identity element in $\epsilon_iE_n$ for $i$ odd. For $i$ even, $\epsilon_i((\zeta_n - 1))$ is a generator of $\epsilon_iE_\infty$ if and only if the Bernoulli number $B_i$ is relatively prime to $p$ (see the argument for Theorem 1.4 in Chapter 7 of Lang [14]).

The authors know of two (for $i = 1$) or three (for $i = 2, \ldots, p - 2$) distinct ways of producing generators for the other eigenspaces. For the constructions, let $U_n^1$, denote the subgroup of $F_n^\times$ congruent to 1 mod $\zeta_n - 1$ and let $U_n^1 = \lim U_n^1$ under the norm maps. Since $U_n^1$ is $p$-complete, the Galois action on $U_n^1$ makes it a $\Lambda'$-module. Let $T_p(\mu) = \lim \mu_{p^{n+1}}$, a $\Lambda'$-submodule of $U_n^1$. Since $\mu_{p^{n+1}}$ is the torsion of $U_n^1$ and $\mu_{(p-1)p^{n+1}}$ is the torsion in $F_n^\times$, we have an exact sequence of $\Lambda'$-modules

$$0 \to T_p(\mu) \to U_n^1 \to E_\infty \to Z_p$$

where $Z_p$ has the trivial Galois action and the map $E_\infty \to Z_p$ is induced by the valuation as in the proof of Proposition 7.6 above. In particular $U_n^1 \to E_\infty$ is an isomorphism on $\omega^i$ eigenspaces for $i \neq 0, 1$ and an epimorphism on $\omega^i$.

Given an element $(\langle u_n \rangle) \in U_n^1$, we can detect whether $\epsilon_i(\langle u_n \rangle)$ maps to a generator of $\epsilon_iE_\infty$ using the Kummer homomorphisms (see [14 7§1–2]) $\phi_i: U_n^1 \to F_p$. Let $D: Z_p[[X]] \to Z_p[[X]]$ be the homomorphism of $Z_p$-modules $DF = (1 + X)f'(X)$. Given $u \in U_n^1$, $u = f_u(\zeta_n - 1)$ for some (non-unique) $f_u \in Z_p[[X]]$ with leading coefficient congruent to 1 mod $p$; for $i = 1, \ldots, p - 2$, define

$$\phi_i(u) = D^i(\log(f_u))|_{X = 0} \pmod{p} \in F_p$$

where $\log(f) = (f - 1) - (f - 1)^2/2 + \cdots$ (or more generally, for any power series $f$ with constant term a unit in $Z_p$, we can equivalently interpret $D^i(\log(f)$ as $D^{i-1}(1 + X)f'/f$). The power series $f_u$ is well defined mod $(p, X^{p-1})$ since $O_{F_0}$ has ramification index $p - 1$ over $Z_p$. As a consequence, $f_u(X)f_v(X) \equiv f_{uv}(X) \mod (p, X^{p-1})$. The formal power series identity for $\log(1 + X)$ (or the product rule applied to $(gh)'/(gh)$) implies

$$D^i\log(f_u f_v) = D^i\log(f_u) + D^i\log(f_v)$$

and $\phi_i$ is a well-defined homomorphism. The (easily checked) formula

$$D((f + 1)^a - 1) = a(X + 1)^a f'_i((X + 1)^a - 1) = aDf_i|_{X + 1)^a - 1}$$

shows that $\phi_i(\psi^a) = a^\phi_i(u)$ for any $u \in U_n^1$, $a \in Z_p^\times$. It follows that $\phi_i(\psi u) = \phi_i(u)$. Since $\epsilon_iE_\infty$ is a free rank one $\Lambda$-module and $\Lambda$ is a local ring, if $\phi_i(\psi u) \neq 0$ for some $((u_n)) \in U_n^1$ and either $i \in \{2, \ldots, p - 2\}$ or $i = 1$ and $\epsilon_1 u_0 \notin \mu_p$, then the image of $\epsilon_i(\langle u_n \rangle)$ generates $\epsilon_iE_\infty$. For the first construction of generators, we use the following proposition from [14] 7§3.

**Proposition 7.8.** Fix $i \in 1, \ldots, p - 2$. There exists $\lambda \in \mu_{p-1}, \lambda \neq 1$ such that $\phi_i(\omega(\lambda - 1)^{-1}(\lambda - \zeta_0)) \neq 0$. 


We then have
\[ \beta \in \mathbb{Z} \]

Since \( \lambda \) is a rational polynomial in \( \mathbb{Z} \), let \( \lambda \) be the indefiniteness of the unspecified choice \( \theta \) and let \( G \) be the unique strict isomorphism from the multiplicative formal group law \( \Gamma \). (For an introduction to this Lubin-Tate theory, see for example [14, §1], particularly Theorems 1.1 and 1.2.) Because \( \pi_\mathbb{Z} = \zeta_n - 1 \) satisfies
\[ [p]_{\mathbb{Z}}(\pi_{n+1}) = (\pi_{n+1} + 1)^p - 1 = \pi_n, \]
for \( n \geq 0 \), \( x_n = \theta(\pi_n) \) satisfies
\[ x_{n+1}^p + px_{n+1} = x_n \]
for \( n \geq 0 \), and \( x_0^p + px_0 = 0 \). Let \( u_n = \beta - x_n \). We have \( N_{F_{n+1}/F_n}(u_{n+1}) = u_n \), and it is easy to calculate \( \phi_i(u_0) \) as follows. Because \( \Gamma \) agrees with the additive formal

**Proposition 7.9.** Let \( \lambda \in \mu_{p-1} \setminus \{1\} \) and let \( u_0(\lambda) = \omega(\lambda - 1)^{-1}(\lambda - \zeta_0) \). Then \( \epsilon_1(u_0(\lambda)) \) is not a \( p \)th root of unity.

**Proof.** Let \( \pi = \zeta_0 - 1 \in \mathcal{O}_{F_0} \); we note \( (p) = (\pi^{p-1}) \). It suffices to check that
\[ \epsilon_1(u_0(\lambda))^p \neq 1 \pmod{\pi^{p+1}}. \]

Since \( \omega(\lambda - 1)^{-1}(\lambda - 1) \) is in \( \mathbb{Z}_p \), and is by construction congruent to 1 mod \( p \) \( (a \text{ fortiori} \text{ congruent to 1 mod } \pi \text{ in } \mathcal{O}_{F_0}) \), it is in \( \epsilon_0U_0^1 \) and \( \epsilon_1 \) takes it to the identity. We then have
\[ \epsilon_1(u_0(\lambda)) = \epsilon_1 \left( \frac{u_0(\lambda)}{\omega(\lambda - 1)^{-1}(\lambda - 1)} \right) = \epsilon_1 \left( 1 - \frac{\pi}{\lambda - 1} \right), \]
and
\[ \left( 1 - \frac{\pi}{\lambda - 1} \right)^p \equiv 1 - \frac{p\pi}{\lambda - 1} \pmod{\pi^{p+1}}. \]

Moreover, since
\[ \psi^a \left( 1 - \frac{p\pi}{\lambda - 1} \right) \equiv 1 - \frac{ap\pi}{\lambda - 1} \equiv \left( 1 - \frac{p\pi}{\lambda - 1} \right)^a \pmod{\pi^{p+1}}, \]
it follows that \( \epsilon_1(u_0(\lambda))^p \equiv 1 - p\pi/(1 - \lambda) \neq 1 \pmod{\pi^{p+1}}. \)

Taking \( \lambda \) as in Proposition [18] let \( u_n = \omega(\lambda - 1)^{-1}(\lambda - \zeta_0) \). Because \( \lambda \in \mu_{p-1} \), we have \( N_{F_{n+1}/F_n}(u_{n+1}) = u_n \), and the system \( \epsilon_1(u_n) \) maps to a generator of \( \epsilon_1E_\infty \).

**Proposition 7.10.** Let \( i \in 1, \ldots, p - 2 \), and choose \( \lambda \in \mu_{p-1} \setminus \{1\} \) such that \( \phi_i(\omega(\lambda - 1)^{-1}(\lambda - \zeta_0)) \neq 0 \). Then \( \epsilon_i(\omega(\lambda - 1)^{-1}(\lambda - \zeta_0)) \in \epsilon_1U_\infty^1 \) maps to a generator of \( \epsilon_iE_\infty \).

The next construction of generators, due to Coates-Wiles [6, Theorem 4], avoids the indefiniteness of the unspecified choice \( \lambda \) in the previous construction. Let \( \beta \in \mathbb{Z}_p \) denote the unique \((p - 1)\)th root of \( 1 - p \) which is congruent to 1 mod \( p \). Let \( \Gamma \) be the unique Lubin-Tate formal group law over \( \mathbb{Z}_p \) with \([p]_\Gamma[X] = X^p + pX\), and let \( \theta \) be the unique strict isomorphism from the multiplicative formal group law \( G_m \) to \( \Gamma \). (For an introduction to this Lubin-Tate theory, see for example [14, §1], particularly Theorems 1.1 and 1.2.) Because \( \pi_\mathbb{Z} = \zeta_n - 1 \) satisfies
\[ [p]_{G_m}(\pi_{n+1}) = (\pi_{n+1} + 1)^p - 1 = \pi_n, \]
for \( n \geq 0 \), \( x_n = \theta(\pi_n) \) satisfies
\[ x_{n+1}^p + px_{n+1} = x_n \]
for \( n \geq 0 \), and \( x_0^p + px_0 = 0 \). Let \( u_n = \beta - x_n \). We have \( N_{F_{n+1}/F_n}(u_{n+1}) = u_n \), and it is easy to calculate \( \phi_i(u_0) \) as follows. Because \( \Gamma \) agrees with the additive formal
group law to order \( p - 1 \), \( \theta(X) \equiv \log(1 + X) \pmod{X^p} \). Calculating

\[
D^i(\beta - \log(1 + X)) = \frac{-(i - 1)!}{(\beta - \log(1 + X))^i},
\]

we get \( \phi_1(u_0) = -(i - 1)! \neq 0 \in \mathbb{F}_p \). In the case \( i = 1 \), \( \epsilon_1u_0 \) is not a \( p \)-th root of unity (see [20, 2.5]). This proves that \( \epsilon_i((u_n)) \) maps to a generator of \( \epsilon_iE_\infty \) for all \( i = 1, \ldots, p - 2 \).

**Proposition 7.11.** Let \( i \in 1, \ldots, p - 2 \). In the notation of the previous paragraph, \( \epsilon_i((\beta - \theta(\zeta_n - 1))) \in \epsilon_iU_\infty \) maps to a generator of \( \epsilon_iE_\infty \).

Finally, Coleman [7, 8] constructs an extremely nice isomorphism \( \epsilon_iE_\infty \rightarrow \Lambda \) for \( i \neq 0, 1 \). The Coleman map \( \mathcal{L}: U_\infty^1 \rightarrow \Lambda' \) fits into an exact sequence of \( \Lambda' \)-modules

\[
0 \rightarrow T_p(\mu) \rightarrow U_\infty^1 \xrightarrow{\mathcal{L}} \Lambda' \rightarrow \mathbb{Z}_p(1) \rightarrow 0
\]

(see [5, 3.5.1] or [20, Theorem 2]), where \( \mathbb{Z}_p(1) \) is the \( \Lambda' \)-module with underlying \( \mathbb{Z}_p \)-module \( \mathbb{Z}_p \) where \( \mathbb{Z}_p \) acts by multiplication (it is isomorphic to \( T_p(\mu) \) under the isomorphism \( a \mapsto ((\zeta_p^n)) \)). In particular, the Coleman map is an isomorphism on \( \alpha^i \) eigenspaces for \( i \neq 1 \). Choosing a generator of \( \epsilon_i\Lambda' \) for \( i \neq 0, 1 \), \( \mathcal{L}^{-1} \) gives a generator of \( \epsilon_iU^1 \) which maps to a generator of \( \epsilon_iE_\infty \). (This generator has \( \phi_i \) equal to \( 1 \) by [5, 3.5.2].)

**References**

[1] D. W. Anderson. Universal coefficient theorems for \( K \)-theory. Mimeographed notes.

[2] Andrew J. Blumberg and Michael A. Mandell. The nilpotence theorem for the algebraic \( K \)-theory of the sphere spectrum. *Geom. Topol.*, 21(6):3453–3466, 2017.

[3] Andrew J. Blumberg and Michael A. Mandell. The homotopy groups of the algebraic \( K \)-theory of the sphere spectrum. *Geom. Topol.*, 23(1):101–134, 2019.

[4] Andrew J. Blumberg and Michael A. Mandell. \( K \)-theoretic Tate-Poitou duality and the fiber of the cyclotomic trace. *Invent. Math.*, 221(2):397–419, 2020.

[5] J. Coates and R. Sujatha. *Cyclotomic fields and zeta values*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2006.

[6] J. Coates and A. Wiles. On \( p \)-adic \( L \)-functions and elliptic units. *J. Austral. Math. Soc. Ser. A*, 26(1):1–25, 1978.

[7] Robert F. Coleman. Division values in local fields. *Invent. Math.*, 53(2):91–116, 1979.

[8] Robert F. Coleman. The arithmetic of Lubin-Tate division towers. *Duke Math. J.*, 48(2):449–466, 1981.

[9] Ethan S. Devinatz and Michael J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Topol. J.*, 43(1):1–47, 2004.

[10] W. G. Dwyer and S. A. Mitchell. On the \( K \)-theory spectrum of a ring of algebraic integers. *K-Theory*, 14(3):201–263, 1998.

[11] Lars Hesselholt and Ib Madsen. On the \( K \)-theory of finite algebras over Witt vectors of perfect fields. *Topology*, 36(1):29–101, 1997.

[12] Kenkichi Iwasawa. On \( p \)-adic \( L \)-functions. *Ann. of Math. (2)*, 89:198–205, 1969.

[13] Karlheinz Knapp. Anderson duality in \( K \)-theory and \( \text{Im}(J) \)-theory. *K-Theory*, 18(2):137–159, 1999.

[14] Serge Lang. *Cyclotomic fields I and II*, volume 121 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990. With an appendix by Karl Rubin.

[15] J. S. Milne. *Arithmetic duality theorems*. BookSurge, LLC, Charleston, SC, second edition, 2006.

[16] Stephen A. Mitchell. On \( p \)-adic topological \( K \)-theory. In *Algebraic \( K \)-theory and algebraic topology (Lake Louise, AB, 1991)*, volume 407 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*., pages 197–204. Kluwer Acad. Publ., Dordrecht, 1993.
[17] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. Cohomology of number fields, volume 323 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, second edition, 2008.

[18] Daniel Quillen. On the cohomology and $K$-theory of the general linear groups over a finite field. Ann. of Math. (2), 96:552–586, 1972.

[19] John Rognes. The smooth Whitehead spectrum of a point at odd regular primes. Geom. Topol., 7:155–184 (electronic), 2003.

[20] A. Saikia. A simple proof of a lemma of Coleman. Math. Proc. Cambridge Philos. Soc., 130(2):209–220, 2001.

[21] J.-P. Serre. Local class field theory. In Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), pages 128–161. Thompson, Washington, D.C., 1967.

[22] J.P. Serre. Local fields, volume 67 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1979.

[23] R. W. Thomason. Algebraic $K$-theory and étale cohomology. Ann. Sci. École Norm. Sup. (4), 18(3):437–552, 1985.