Prolongations of Convenient Lie Algebroids

Patrick Cabau · Fernand Pelletier

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Abstract
We first define the concept of Lie algebroid in the convenient setting. In reference to the finite dimensional context, we adapt the notion of prolongation of a Lie algebroid over a fibred manifold to a convenient Lie algebroid over a fibred manifold. Then we show that this construction is stable under projective and direct limits under adequate assumptions.

Keywords Convenient Lie algebroid · Prolongation of a convenient Lie algebroid · Projective and direct limits of sequences of Banach Lie algebroids

Mathematics Subject Classification 53D35 · 55P35

1 introduction

In classical mechanics, the configuration space is a finite dimensional smooth manifold \( M \) where the tangent bundle \( p_M : TM \to M \) corresponds to the velocity space. This geometrical object plays a relevant role in the Lagrangian formalism between tangent and cotangent bundles (cf. [12]). In [21], Weinstein develops a generalized theory of Lagrangian Mechanics on Lie algebroids. In [14], Libermann shows that such a formalism is not possible, in general, if we consider the tangent bundle of a Lie algebroid. The notion of prolongation of a Lie algebroid introduced by Higgins and Mackenzie [11] offers a nice context in which such a formalism was generalized by Martínez (cf. [15, 16]).
The notion of Lie algebroid in the Banach setting was simultaneously introduced in [3, 17]. Unfortunately, in this setting, there exist many problems to generalizing all canonical Lie structures on a finite dimensional Lie algebroid and, at first, a nice definition of a Lie bracket (cf. [6]). In this paper, we consider the more general convenient setting (cf. [13]) in which we give a definition of a convenient Lie algebroid (and more generally a partial convenient Lie algebroid) based on a precise notion of sheaf of Lie brackets\(^1\) on a convenient anchored bundle (cf. Sect. 2.4).

As in finite dimension, given a convenient anchored bundle \((\mathcal{A}, \pi, M, \rho)\),\(^2\) the total space of the prolongation \(\hat{p} : T\mathcal{M} \to \mathcal{M}\) of \((\mathcal{A}, \pi, M, \rho)\) over a fibred manifold \(p : M \to M\), is the pullback over \(\rho\) of the bundle \(Tp : T\mathcal{M} \to TM\). Moreover, we have an anchor \(\hat{\rho} : T\mathcal{M} \to TM\).

In finite dimension, the Lie bracket on \(\mathcal{A}\) gives rise to a Lie bracket on \(T\mathcal{M}\). Unfortunately, this is not any more true in infinite dimension. If \(\hat{\pi} : \hat{\mathcal{A}} \to \mathcal{M}\) is the pullback of \(\pi : \mathcal{A} \to M\) over \(p\), then the module of local sections of \(\hat{\mathcal{A}}\) is no longer finitely generated by local sections along \(p\). For this reason, the way to define the prolongation of the bracket does not work as in finite dimension. Thus the prolongation of such a Lie algebroid is not again a Lie algebroid but only a strong partial Lie algebroid (cf. Sect. 2.5). We also define (non linear) connections on such a prolongation. As for the tangent bundle of a Banach vector bundle (cf. [2]), we then show that the kernel bundle of \(\hat{p} : T\mathcal{M} \to \mathcal{M}\) is split if and only if there exists a linear connection on \(\mathcal{M}\).

In the Banach setting, it is proved that if the kernel of the anchor \(\rho\) of a Banach Lie algebroid \((\mathcal{A}, \pi, M, \rho, [, , ]_A)\) is split and its range is closed, then the associated distribution on \(M\) defines a (singular) foliation (cf. [17]). Under these assumptions, we show that the prolongation \((T\mathcal{A}, \hat{p}, \mathcal{A}, \hat{\rho})\) has the same properties, and the foliation defined by \(\hat{\rho}(T\mathcal{A})\) on \(\mathcal{A}\) is exactly the set \(\{A_L, L \text{ leaf of } \rho(\mathcal{A})\}\) (cf. Theorem 47).

As an illustration of these notions, in the convenient setting, and not only in a Banach one, we end this work by the prolongation of projective (resp. direct) limits of sequences of projective (resp. ascending) sequences of fibred Banach Lie algebroids with finite or infinite dimensional fibres.

This work can be understood as the basis for further studies on how the Lagrangian formalism on finite dimensional Lie algebroid (cf. [16] for instance) can be generalized in this convenient framework.

Section 2 is devoted to the presentation of the prerequisities needed in this paper about (partial) convenient Lie algebroids. After some preliminaries on notations, we introduce the notion of convenient anchored bundle. Then we define a notion of almost bracket on such a vector bundle. The definition of a convenient Lie algebroid and some of its properties are given in Sect. 2.4. The next subsection exposes the concept of partial convenient Lie algebroid. The following subsection is devoted to the definitions of some derivative operators (Lie derivative with respect to some sheaf of \(k\)-forms sections and exterior derivative), in particular of strong partial Lie algebroids. The last subsection recall some results about integrability of Banach Lie algebroids when the Banach Lie algebroid is split and the range of its anchor is closed (cf. [18]).

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\(^1\) cf. Remark 4.

\(^2\) The anchor \(\rho\) is a vector bundle morphism \(\rho : \mathcal{A} \to TM\).
The important part of this work is contained in Sect. 3. In the first subsection, we build the prolongation of a convenient anchored bundle. Then, in Sect. 3.3, we can define a Lie bracket on local projectable sections of the total space of the prolongation. We then explain why this bracket cannot be extended to the whole set of local sections if the typical fibre of the anchored bundle is not finite dimensional, which is the essential difference with the finite dimensional setting. We end this section by showing that if a Banach Lie algebroid is split and the range of its anchor is closed, the same is true for its prolongation and the range of the anchor of the prolongation defines also a foliation even if its Lie bracket is not defined on the set of all local sections.

The two last sections show that, under adequate assumptions, the prolongation of a projective (resp. direct) limit of a projective sequence (resp. ascending sequence) of Banach Lie algebroids is exactly the prolongation of the projective (resp. direct) limit of this sequence.

In order to make the two last sections more affordable for non-informed readers, we have added two appendices recalling the needed concepts and results on projective and direct limits.

2 Convenient Lie Algebroid

2.1 Local Identifications and Expressions in a Convenient Bundle

In all this paper, we work in the convenient setting and we refer to [13].

Consider a convenient vector bundle $\pi : A \to M$ where the typical fibre is a convenient linear space $A$. For any open subset $U \subset M$, we denote by $\mathcal{C}^\infty(U)$ the ring of smooth functions on $U$ and by $\Gamma(A_U)$ the $\mathcal{C}^\infty(U)$-module of smooth sections of the restriction $A_U$ of $A$ over $U$, simply $\Gamma(A)$ when $U = M$.

Consider a chart $(U, \phi)$ on $M$ such that we have a trivialization $\tau : A_U \to \psi(U) \times A$ Then $T\phi$ is a trivialization of $TM|_U$ on $\phi(U) \times M$ and $T\tau$ is a trivialization of $T(A|_U)$ on $\phi(U) \times \psi(U) \times \mathcal{M} \times A$.

For the sake of simplicity, we will denote these trivializations:

- $A|_U \equiv \phi(U) \times A$;
- $TM|_U \equiv \phi(U) \times M$;
- $T(A|_U) \equiv (\phi(U) \times \mathcal{A}) \times (M \times \mathcal{A})$;
- $T(A|_U^*) \equiv \phi(U) \times \mathcal{A}^* \times \mathcal{M} \times \mathcal{A}^*$.

where $U \subset M$ is identified with $\psi(U)$.

We will also use the following associated local coordinates where $\equiv$ stands for the representation in the corresponding trivialization.

\[
\begin{align*}
(a, a) & \equiv (x, a) \in U \times A \\
(x, v) & \equiv (x, v) \in U \times M \\
(a, b) & \equiv (x, a, v, b) \in U \times A \times M \times A \\
(\sigma, w\eta) & \equiv (x, \xi, w_1, \eta_2) \in U \times A^* \times \mathcal{M} \times A^*
\end{align*}
\]
2.2 Convenient Anchored Bundle

Let \( \pi : \mathcal{A} \to M \) be a convenient vector bundle whose fibre is a convenient linear space \( \mathbb{A} \).

**Definition 1** A morphism of vector bundles \( \rho : \mathcal{A} \to TM \) is called an anchor and the quadruple \((A, \pi, M, \rho)\) is called a convenient anchored bundle.

**Notations 2**
1. In this section, if there is no ambiguity, the anchored bundle \((A, \pi, M, \rho)\) is fixed and, in all this work, the Lie bracket of vector fields on a convenient manifold will be simply denoted \[..\].
2. For any open set \( U \subseteq M \), the morphism \( \rho \) gives rise to a \( C^\infty(U) \)-morphism of modules \( \rho_U : \Gamma(A_U) \to \mathfrak{X}(U) \) defined, for any \( x \in M \) and any smooth section \( a \) of \( A_U \), by:

\[
(\rho_U(a))(x) = \rho(a(x))
\]

and still denoted by \( \rho \).
3. For any convenient spaces \( E \) and \( F \), we denote by \( L(E, F) \), the convenient space of bounded linear operators from \( E \) to \( F \) and for \( E = F \), we set \( L(E) := L(E, E) \); \( GL(E) \) is the group of bounded automorphisms of \( E \).
4. In local coordinates in a chart \((U, \phi)\), the restriction of \( \rho \) to \( U \) will give rise to a smooth field \( x \mapsto \rho_x \) from \( \phi(U) \equiv U \) to \( \mathcal{A}, \mathbb{M} \).

2.3 Almost Lie Bracket

**Definition 3** An almost Lie bracket on an anchored bundle \( \mathcal{A} \) is a sheaf of skew-symmetric bilinear maps

\[
[\cdot, \cdot]_{A_U} : \Gamma(A_U) \times \Gamma(A_U) \to \Gamma(A_U)
\]

for any open set \( U \subseteq M \) which satisfies the following properties:

**\( AL 1 \)** The Leibniz identity

\[
\forall (a_1, a_2) \in \Gamma(A_U)^2, \forall f \in C^\infty(M), \ [a_1, f a_2]_A = f. [a_1, a_2]_A + df(\rho(a_1)).a_2.
\]

**\( AL 2 \)** For any open set \( U \subseteq M \) the map

\[
(a_1, a_2) \mapsto [a_1, a_2]_{A_U}
\]

only depends on the 1-jets of \( a_1 \) and \( a_2 \) of sections of \( A_U \).

By abuse of notation, such a sheaf of almost Lie brackets will be denoted \([\cdot, \cdot]_A\).
Remark 4 In finite dimension, the bracket is defined on global sections and induces a Lie bracket on local sections which depend on the 1-jets of sections. In the convenient setting (as in the Banach one), if \( M \) is not smoothly regular, the set of restrictions to some open set \( U \) of global sections of \( \mathcal{A} \) could be different from \( \Gamma(\mathcal{A}_U) \) but, unfortunately, we have no example of such a situation. Thus, any bracket defined on the whole space \( \Gamma(\mathcal{A}) \times \Gamma(\mathcal{A}) \) will not give rise to a bracket on local sections of \( \mathcal{A} \) and, even if it is true, the condition (AL 2) will not be true in general.

In the context of local trivializations (Sect. 2.1), if \( L^2_{alt}(\mathbb{A}; \mathbb{A}) \) is the convenient space of bounded skew-symmetric operators on \( \mathbb{A} \) with values in \( \mathbb{A} \), there exists a smooth field

\[
C : U \to L^2_{alt}(\mathbb{A}, \mathbb{A})
\]

\[
x \mapsto C_x
\]
such that, for \( a_1(x) \equiv (x, a_1(x)) \) and \( a_2(x) \equiv (x, a_2(x)) \), we have:

\[
[a_1, a_2]_U(x) \equiv (x, C_x(a_1(x), a_2(x)) + d\alpha_2(\rho_x(a_1(x))) - d\alpha_1(\rho_x(a_2(x))).
\] (1)

2.4 Almost Lie Algebroid and Lie Algebroid

Definition 5 The quintuple \((\mathcal{A}, \pi, M, \rho, [,.,.]_A)\) where \((\mathcal{A}, \pi, M, \rho)\) is an anchored bundle and \([.,.,.]_A\) an almost Lie bracket is called a convenient almost Lie algebroid—convenient almost Lie algebroid almost Lie algebroid!convenient.

In this way, the Jacobiator is the \( \mathbb{R} \)-trilinear map \( J_{\mathcal{A}_U} : \Gamma(\mathcal{A}_U)^3 \to \Gamma(\mathcal{A}_U) \) defined, for any open set \( U \) in \( M \) and any section \((a_1, a_2, a_3) \in \Gamma(\mathcal{A}_U)^3\) by

\[
J_{\mathcal{A}_U}(a_1, a_2, a_3) = [a_1, [a_2, a_3]]_A + [a_2, [a_3, a_1]]_A + [a_3, [a_1, a_2]]_A.
\]

Definition 6 A convenient Lie algebroid!convenient Lie algebroid almost Lie algebroid almost Lie algebroid!convenient is a convenient almost Lie algebroid \((\mathcal{A}, \pi, M, \rho, [,.,.]_A)\) such that the associated jacobiator \( J_{\mathcal{A}_U} \) vanishes identically on each module \( \Gamma(\mathcal{A}_U) \) for all open sets \( U \) in \( M \).

We then have the following result (cf. [4, Chap. 3]):

Proposition 7 Consider an almost convenient Lie algebroid \((\mathcal{A}, \pi, M, \rho, [,.,.]_A)\).

1. For any open set \( U \subseteq M \) and for all \((a_1, a_2) \in \Gamma(\mathcal{A}_U)^2\), the map

\[
(a_1, a_2) \mapsto \rho ([a_1, a_2]_A) - [\rho(a_1), \rho(a_2)]
\]

only depends on the 1-jet of \( \rho \) at any \( x \in U \) and the values of \( a_1 \) and \( a_2 \) at \( x \).

2. If the jacobiator \( J_{\mathcal{A}_U} \) vanishes identically, then we have:

\[
\forall (a_1, a_2) \in \Gamma(\mathcal{A}_U)^2, \rho ([a_1, a_2]_A) = [\rho(a_1), \rho(a_2)].
\] (2)
If the property (2) is true, then $\mathcal{J}_{A_U}$ is a bounded trilinear $C^\infty(U)$-morphism from $\Gamma(A_U)^3$ to $\Gamma(A_U)$ which takes values in $\ker \rho$ over $U$.

If for each open set $U$, the assumption (2) of Proposition 7 is satisfied, (3) implies that the family $\{\mathcal{J}_{A_U}, U \text{ open in } M\}$ defines a sheaf of trilinear morphisms from the sheaf $\{(\Gamma(A_U))^3, U \text{ open in } M\}$ into the sheaf $\{(\Gamma(A_U), U \text{ open in } M\}$. This sheaf will be denoted $\mathcal{J}_A$.

**Corollary 8** If $(\mathcal{A}, \pi, M, \rho, [\, , \, ]_A)$ is a convenient Lie algebroid, then $\rho$ induces a morphism of Lie algebras from $\Gamma(A_U)$ into $\mathfrak{X}(U)$ for any open sets $U$ in $M$.

**Definition 9** A convenient Lie algebroid $(\mathcal{A}, \pi, M, \rho, [\, , \, ]_A)$ will be called split if, for each $x \in M$, the kernel of $\rho_x = \rho|_{\pi^{-1}(x)}$ is supplemented in $\pi^{-1}(x)$.

For example, if $\ker \rho_x$ is finite dimensional or finite codimensional for all $x \in M$ or if $\mathcal{A}$ is a Hilbert space, then $(\mathcal{A}, \pi, M, \rho, [\, , \, ]_A)$ is split. Another particular situation is the case where the anchor $\rho \equiv 0$ and then $(\mathcal{A}, \pi, M, \rho, [\, , \, ]_A)$ is a Lie algebra Banach bundle.

### 2.5 Structure of Partial Lie Algebroid

We have the following generalization of the notion of convenient Lie algebroid:

**Definition 10** Let $(\mathcal{A}, \pi, M, \rho)$ be a convenient anchored bundle. Consider a sub-sheaf $\mathfrak{P}_M$ of the sheaf $\Gamma(A)_M$ of sections of $\mathcal{A}$. Assume that $\mathfrak{P}_M$ can be provided with a structure of Lie algebras sheaf which satisfies, for any open set $U$ in $M$:

(i) for any $(a_1, a_2) \in (\mathfrak{P}(U))^2$ and any $f \in C^\infty(U)$, we have the Leibniz conditions

$$[a_1, f a_2]_{\mathfrak{P}(U)} = df(\rho(a_1))a_2 + f[a_1, a_2]_{\mathfrak{P}(U)} \quad (3)$$

(ii) the Lie bracket $[\, , \, ]_{\mathfrak{P}(U)}$ on $\mathfrak{P}(U)$ only depends on the 1-jets of sections of $\mathfrak{P}(U)$;

(iii) $\rho$ induces a Lie algebra morphism from $\mathfrak{P}(U)$ to $\mathfrak{X}(U)$.

Then $(\mathcal{A}, \pi, M, \rho, \mathfrak{P}_M)$ is called a convenient partial Lie algebroid. The family $\{[\, , \, ]_{\mathfrak{P}(U)}, U \text{ open set in } M\}$ is called a sheaf bracketsheaf bracket and is denoted $[\, , \, ]_A$.

A partial convenient Lie algebroid $(\mathcal{A}, \pi, M, \rho, \mathfrak{P}_M)$ is called strong if for any $x \in M$, the stalk

$$\mathfrak{P}_x = \lim_{\to} \mathfrak{P}(U), \quad \varphi^U_V : V \to U, \quad U, V \text{ open neighbourhoods of } x : U \supset V$$

is equal to $\pi^{-1}(x)$ for any $x \in M$.

Any convenient Lie algebroid is a partial Lie algebroid.

More generally, if $(\mathcal{A}, \pi, M, \rho)$ is a convenient anchored bundle, any convenient
subbundle \( B \) of \( \mathcal{A} \) such that \((B, \pi_B = \pi|_B, M, \rho_B = \rho|_B, [, .]|_B)\) is a convenient Lie algebroid provided with a structure of convenient partial Lie algebroid on \( \mathcal{A} \) which is not strong in general. Another type of example of convenient partial Lie algebroids will be described in the context of the prolongation of a convenient Lie algebroid in the next section. This convenient partial Lie algebroid will be a strong partial Lie algebroid.

**Remark 11** In local coordinates, the Lie bracket \([. .]|_{\mathcal{P}(U)} \) can be written as in (1).

### 2.6 Derivative Operators

#### 2.6.1 Preliminaries

If \( U \) is a \( c^\infty \)-open subset of a convenient space \( \mathbb{E} \), the space \( C^\infty(U, \mathbb{F}) \) of smooth maps from \( U \) to a convenient space \( \mathbb{F} \) is a convenient space (cf. [13, 3.7 and 3.11]).

The space \( L(\mathbb{E}, \mathbb{F}) \) of bounded linear maps from \( \mathbb{E} \) to \( \mathbb{F} \) endowed with the topology of uniform convergence on bounded subsets in \( \mathbb{E} \) is a closed subspace of \( C^\infty(\mathbb{E}, \mathbb{F}) \) and so is a convenient space.

More generally, the set \( L^k_{alt}(\mathbb{E}, \mathbb{F}) \) of all bounded \( k \)-linear alternating mappings from \( \mathbb{E}^k \) to \( \mathbb{F} \) endowed with the topology of uniform convergence of bounded sets is a closed subspace of \( C^\infty(\mathbb{E}^k, \mathbb{F}) \) (cf. [13, Corollary 5.13]) and so \( L^k_{alt}(\mathbb{E}, \mathbb{F}) \) is a convenient space.

On the other hand, if \( \bigwedge^k(\mathbb{E}) \) is the set of alternating \( k \)-tensors on \( \mathbb{E} \) then \( L^k_{alt}(\mathbb{E}) := L^k_{alt}(\mathbb{E}, \mathbb{R}) \) is isomorphic as a locally convex topological space to \( L^k_{alt}(\mathbb{E}) \) (cf. [13, Corollary 5.9]) and so has a natural structure of convenient space.

Recall that bounded linear maps are smooth (cf. [13, Corollary 5.5]).

Let us consider a convenient vector bundle \( \pi : \mathcal{A} \to M \) with typical fibre \( \mathbb{A} \). We study the bundle

\[
\pi^k : L^k_{alt}(\mathcal{A}) = \bigcup_{x \in M} L^k_{alt}(\mathcal{A}_x) \to M
\]

Using any atlas for the bundle structure of \( \pi : \mathcal{A} \to M \), it is easy to prove that \( \pi^k : L^k_{alt}(\mathcal{A}) \to M \) is a convenient vector bundle. The vector space of local sections of \( L^k_{alt}(\mathcal{A}_U) \) is denoted by \( \bigwedge^k \Gamma^*(\mathcal{A}_U) \) and is called the set of \( k \)-exterior differential forms on \( \mathcal{A}_U \). We denote by \( \bigwedge \Gamma^*(\mathcal{A}) \) the sheaf of sections of \( \pi^k : L^k_{alt}(\mathcal{A}) \to M \) and \( \bigwedge \Gamma^*(\mathcal{A}) = \bigcup_{k=0}^\infty \bigwedge^k \Gamma^*(\mathcal{A}) \) the sheaf of associated graded exterior algebras.

In this section, we assume that \((\mathcal{A}, \pi, M, \rho)\) is an anchored bundle and that \((\mathcal{A}, \pi, M, \rho, \mathcal{P}_M)\) is a fixed strong partial Lie algebroid.

This situation is always satisfied if \((\mathcal{A}, \pi, M, \rho, [, .]|_\mathcal{A})\) is a Lie algebroid and occurs for the prolongation of a convenient Lie algebroid (cf. Sect. 3). This context also occurs in the setting of partial Poisson manifolds (cf. [19]).
2.6.2 Insertion Operator

Let \( a \) be a local section of \( \mathcal{A} \) defined on an open set \( U \). As in [13, 33.10], we have

**Proposition 12** The insertion operator \( i_a \) is the graded endomorphism of degree \(-1\) defined by:

1. (i) For any function \( f \in C^\infty(U) \)
   \[
   i_a(f) = 0
   \]
2. (ii) For any \( k \)-form \( \omega \) (where \( k > 0 \)),
   \[
   (i_a \omega)(a_1, \ldots, a_k)(x) = \omega(a(x), a_1(x), \ldots, a_k(x)).
   \]

2.6.3 Lie Derivative

**Proposition 13** For \( k \geq 0 \), let \( \omega \) be a local \( k \)-form that is an element of \( \bigwedge^k \Gamma^*(\mathcal{A}_U) \) for some open set \( U \) of \( M \). Given any section \( \overline{a} \in \mathfrak{P}(U) \), the Lie derivative with respect to \( \overline{a} \) on sections of \( \mathfrak{P}(U) \), denoted by \( L_{\overline{a}}^\rho \), is the graded endomorphism with degree \( 0 \) defined in the following way:

1. (i) For any function \( f \in C^\infty(U) \),
   \[
   L_{\overline{a}}^\rho(f) = i_{\rho \circ \pi}(df)
   \]
   where \( L_X \) denote the usual Lie derivative with respect to the vector field \( X \) on \( M \).
2. (ii) For any \( k \)-form \( \omega \) (where \( k > 0 \)),
   \[
   (L_{\overline{a}}^\rho \omega)(a_1, \ldots, a_k)(x) = L_{\overline{a}}^\rho(\omega(a_1, \ldots, a_k))(x) - \sum_{i=1}^{k} \omega(a_1, \ldots, a_{i-1}, [\overline{a}, a_i], a_{i+1}, \ldots, a_k)(x)
   \]
   where \( \overline{a}_i \) is any section of \( \mathfrak{P}(U) \) such that \( \overline{a}_i(x) = a_i(x) \) for \( i \in \{1, \ldots, k\} \).

**Proof** Since the problem is local, we may assume that \( U \) is a \( C^\infty \)-open set in \( M \) over which \( \mathcal{A} \) is trivial. Fix some section \( a \) of \( \mathcal{A} \) on \( U \) and fix some \( x \in U \). After shrinking \( U \) if necessary, if \( f \) is a smooth function on \( U \), it is clear that (6) is well defined.

For \( k > 0 \), let \( \omega \in \bigwedge^k \Gamma^*(\mathcal{A}_U) \). Since we have a strong partial Lie algebroid, this implies that for any \( k \)-uple \((a_1, \ldots, a_k)\) of local sections on \( U \) of \( \mathcal{A} \), the value \( \omega(a_1(x), \ldots, a_k(x)) \) is well defined. Let \( \overline{a}_i \in \mathfrak{P}(U) \) such that \( a_i(x) = \overline{a}_i(x) \) for \( i \in \{1, \ldots, k\} \), we apply the formula (7) to \((\overline{a}, \overline{a}_1, \ldots, \overline{a}_k)\). In our context, \( \omega \) is a...
smooth field over $U$ with values in $L^k_{\text{alt}}(\mathbb{A})$ and each $\bar{a}_i$ is a smooth map from $U$ to $\mathbb{A}$. In this way, we have

$$L^\rho_{\bar{a}}\omega(\bar{a}_1, \ldots, \bar{a}_k)(x) = d_x \omega(\rho(\bar{a}); \bar{a}_1(x), \ldots, \bar{a}_k(x)) + \sum_{i=1}^k \omega(\bar{a}_1(x), \ldots, d_x \bar{a}_i(\rho(\bar{a})), \ldots, \bar{a}_k(x))$$

Since $[\ldots]_{\mathbb{P}(U)}$ only depends on the 1-jets of sections, as for an almost Lie bracket (cf. Remark 11), we have:

$$[\bar{a}, \bar{a}_i]_{\mathbb{A}}(x) = d_x \bar{a}_i(\rho(\bar{a})) - d_x \bar{a}(\rho(\bar{a})) + C_x(\bar{a}(x), \bar{a}_i(x)).$$

It follows that we have

$$(L^\rho_{\bar{a}}\omega)(a_1, \ldots, a_k)(x) = d_x \omega(\rho(\bar{a}); a_1(x), \ldots, a_k(x)) + \sum_{i=1}^k \omega(a_1(x), \ldots, a_{i-1}(x), d_x \bar{a}(\rho(\bar{a})), a_i(x))$$

which implies that $L^\rho_{\bar{a}}\omega$ is a well defined $k$-skew symmetric form on $\mathbb{A}_x$ on $U$ since its value only depends on the 1-jet of $\bar{a}$ and of $\omega$ and the values of $(a_1, \ldots, a_k)$ at $x$. Now, as $x \mapsto C_x$ is a smooth map from $U$ to $L^2_{\text{alt}}(\mathbb{A})$ and so is bounded, the differential of functions is a bounded morphism of convenient space (cf. [13, 3]), and $\rho$ is a bounded morphism of convenient spaces, this completes the proof according to the uniform boundedness principle given in [13, Proposition 30.1].

**Remark 14** From the relation (6), it is clear that the Lie derivative of a function is defined for any section of $\mathbb{A}_U$. Of course, this is also true for any $k$-form on a Lie algebroid. But, for a strong partial Lie algebroid, this is not true for any $k$-form with $k > 0$, since the last formula in the previous proof shows clearly that $L^\rho_{\bar{a}}\omega$ also depends on the 1-jet of $\bar{a}$.

**Remark 15** Assume that $(\mathbb{A}, \pi, M, \rho)$ is provided with an almost Lie bracket $[\ldots]_{\mathbb{A}}$. Then the Lie derivative $L^\rho_{\bar{a}}\omega$ is again well defined by an evident adaptation of formula (7) for any local section $a$ and $k$-form $\omega$ defined on some open set $U$. Moreover, if the Lie bracket on $(\mathbb{A}, \pi, M, \rho, \mathbb{P}_M)$ is induced by the almost Lie bracket $[\ldots]_{\mathbb{A}}$, then the Lie derivative defined in Proposition 13 and the previous global one are compatible.

### 2.6.4 Exterior Derivative

At first, for any function $f$, we can also define the 1-form $d_\rho f$, by

$$d_\rho f = \rho^i \circ df$$

(8)
where \( \rho^t : T'M \to A' \) is the transposed mapping of \( \rho \).

The Lie derivative with respect to any local section \( a \) of \( A \) commutes with \( d_\rho \).

The exterior differential exterior differential on \( \bigwedge \Gamma^*(A) \) is defined as follows:

**Proposition 16**

1. The exterior differential \( d_\rho \) is the graded endomorphism of degree 1 on \( \bigwedge \Gamma^*(A) \) defined in the following way:

   (a) For any function \( f \), \( d_\rho f \) is defined previously;

   (b) For \( k > 0 \) and any \( k \)-form \( \omega \), the exterior differential \( d_\rho \omega \) is the unique \((k+1)\)-form such that, for all \( a_0, \ldots, a_k \in \Gamma(A) \),

   \[
   (d_\rho \omega)(a_0, \ldots, a_q)(x) = \sum_{i=0}^q (-1)^i L^{\rho}_{\alpha_i} \left( \omega \left( a_0, \ldots, \widehat{a_i}, \ldots, a_q \right)(x) \right) \\
   + \sum_{0 \leq i < j \leq q} (-1)^{i+j} \times \left( \omega \left( [\alpha_i, \alpha_j]_A, a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_q \right) \right)(x)
   \]

   where \( \alpha_i \) is any section of \( \mathfrak{p}(U) \) such that \( \alpha_i(x) = a_i(x) \) for \( i \in \{0, \ldots, k\} \).

2. For any \( k \)-form \( \eta \), \( l \)-form \( \zeta \) where \((k, l)\) in \( \mathbb{N}^2 \), we then have the following property

   \[
   d_\rho(\eta \wedge \zeta) = d_\rho(\eta) \wedge \zeta + (-1)^k \eta \wedge d_\rho(\zeta). 
   \]

   \[
   d_\rho \circ d_\rho = d_\rho^2 = 0. 
   \]

**Proof** (1) Using the same context as in the proof of Proposition 13, on the one hand, in local coordinates, we have

\[
L^{\rho}_{\alpha_i} \left( \omega \left( a_0, \ldots, \widehat{a_i}, \ldots, a_q \right)(x) \right) = \frac{\partial}{\partial x_i} \omega \left( \rho(a_i); \alpha_1(x), \ldots, \widehat{\alpha_i}, \ldots, \alpha_k(x) \right) \\
+ \sum_{i=1}^k \omega \left( \alpha_1(x), \ldots, d_x \alpha_j(\rho(a_i)), \ldots, \alpha_k(x) \right).
\]

On the other hand, we have

\[
\omega([\alpha_i, \alpha_j]_A, a_0, \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, a_q)(x) \\
= \omega(d_x \alpha_i(\rho(\alpha_j)) - d_x \alpha_j(\rho(\alpha_i)) + C_x(\alpha_i(x), \alpha_j(x)), \alpha_i(x), \alpha_j, a_0, \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, a_k)(x).
\]
Finally, as $\rho(a_i(x)) = \rho(\bar{a}_i(x))$ for $i \in \{0, \ldots, k\}$, we obtain:

$$(d\rho_\omega)(a_0, \ldots, a_q)(x) = \sum_{i=0}^k (-1)^i d_x \omega(\rho(\bar{a}_i); \bar{a}_1(x), \ldots, \bar{a}_i, \ldots, \bar{a}_k(x))$$

$$+ \sum_{0\leq i < j \leq k} (-1)^{i+j} \times (\omega(C_x(\bar{a}_i(x), \bar{a}_j(x)), \bar{a}_0(x), \ldots, \bar{a}_i(x), \ldots, \bar{a}_j(x), \ldots, \bar{a}_k(x)))$$

Since this value only depends on the 1-jet of $\omega$ at $x$ and the value of each $\bar{a}_i(x)$ for $i \in \{0, \ldots k\}$, it follows that $d\rho_\omega$ is a well defined $(k+1)$-form by the same arguments as at the end of the proof of Proposition 13.

(2) According to the definition of the wedge product, the last formula in local coordinates for $(d\rho_\omega)(a_0, \ldots, a_q)$ clearly implies relation (10).

Since the Lie bracket on $\mathcal{P}(U)$ satisfies the Jacobi identity for any open set $U$, and since the differential $d\rho_\omega$ only depends on the 1-jet of $\omega$, as in finite dimension, it follows that $d\rho(d\rho_\omega) = 0$.  

$\Box$

### 2.6.5 Nijenhuis Endomorphism

In this subsection, we only consider the case of a convenient Lie algebroid $(A, \pi, M, \rho, [., .]_A)$.

Let $A$ be an endomorphism of $A$. The **Lie derivative of** $A$ with respect to a local section $a$ is defined by

$$L^\rho_a A(b) = [a, A(b)]_A - A([a, b]_A)$$

(12)

for all local or global sections $b$ with the same domain as $a$.

The **Nijenhuis tensor** $N^A$ of $A$ is the tensor of type $(2, 0)$ defined by:

$$N^A(a, b) = [Aa, Ab]_A - A[Aa, b]_A - A[a, Ab]_A - [a, b]_A$$

(13)

for all local or global sections $a$ and $b$ with the same domain.

**Remark 17** Consider a partial Lie algebroid $(A, \pi, M, \rho, \mathcal{P}_M)$. If $A$ is an endomorphism of sheaves of $\mathcal{P}_M$, the same formulae are well defined for any local section $\bar{a}$ of $\mathcal{P}_M$. In this way, we can also define the Nijenhuis tensor $N^A$ as an endomorphism of sheaves of $\mathcal{P}_M$.

### 2.7 Lie Morphisms and Lie Algebroid Morphisms

Let $(A_1, \pi_1, M_1, \rho_1, [., .]_{A_1})$ and $(A_2, \pi_2, M_2, \rho_2, [., .]_{A_2})$ be two convenient Lie algebroids.

We consider a bundle morphism $\Psi : A_1 \to A_2$ over $\psi : M_1 \to M_2$.

In the one hand, according to [11] in finite dimension, we can introduce:
Definition 18  Consider a section $a_1$ of $A_1$ over an open set $U_1$ and a section $a_2$ of $A_2$ over an open set $U_2$ which contains $\psi(U_1)$. We say that the pair of sections $(a_1, a_2)$ are $\psi$-related pairs of sections if we have
\[(RS) \quad \Psi \circ a_1 = a_2 \circ \psi.\]

Definition 19  $\Psi$ is called a Lie morphism over $\psi$ if it fulfills the following conditions:
\[(LM 1) \quad \rho_2 \circ \Psi = T \psi \circ \rho_1;\]
\[(LM 2) \quad \Psi \circ [a_1, a'_1]_{A_1} = [a_2, a'_2]_{A_2} \circ \psi \text{ for all } \psi\text{-related pairs of sections } (a_1, a_2) \text{ and } (a'_1, a'_2).\]

Remark 20  For $i \in \{1, 2\}$, let $(A_i, \pi_i, M_i, \rho_i, \mathfrak{P}_{M_i})$ be a partial Lie algebroid.

We consider a sheaf morphism $\Psi : \mathfrak{P}_{M_1} \to \mathfrak{P}_{M_2}$ over a smooth map $\psi : M_1 \to M_2$. Then the Definition 18 makes sense for pairs of sections $(a_1, a_2) \in \mathfrak{P}_{M_1} \times \mathfrak{P}_{M_2}$ which are then called $\psi$-related. If $[..]_{A_i}$ is the sheaf of Lie bracket defined on $\mathfrak{P}_{M_i}$, the assumption $(LM 2)$ in Definition 19 also makes sense for two pairs of such $\psi$-related sections.

Thus $\Psi$ will be called a Lie morphism of partial convenient Lie algebroids if it satisfies the assumptions $(LM 1)$, and $(LM 2)$ for two pairs of $\psi$-related sections of $\mathfrak{P}_{M_1} \times \mathfrak{P}_{M_2}$.

For any local $k$-form $\omega$ on $A_2$ defined on $U_2$, we denote by $\Psi^* \omega$ the local $k$-form on $A_1$ defined on $U_1 = \psi^{-1}(U_2)$ by:
\[(\Psi^* \omega)_{x}(a_1 \ldots a_k) = \omega_{\psi(x)}(\Psi(a_1), \ldots, \Psi(a_k)) \quad (14)\]
for all $x \in U_1$.

On the other hand, as classically in finite dimension, we can introduce:

Definition 21  $\Psi$ is a Lie algebroid morphism over $\psi$ if and only if we have
\[(LAM 1) \quad \Psi^*(d_{\rho_2} f) = d_{\rho_1} (f \circ \psi) \text{ for all } f \in C^\infty(U_2);\]
\[(LAM 2) \quad \Psi^*(d_{\rho_2} \omega) = d_{\rho_1} \Psi^*(\omega) \text{ for any } 1\text{-form } \omega \text{ on } \mathcal{A}_2|_{U_2}.\]

It is easy to see that condition $(LM 1)$ and $(LAM 1)$ are equivalent (cf. Proof of Proposition 22). Property $(LM 2)$ implies property $(LAM 2)$ for $\psi$-related sections but, in general, a pair of local sections $(a_1, a_2)$ of $A_1$ and $A_2$ are not $\psi$-related while each member $(14)$, for any two such pairs, is well defined. On the other hand, under the assumption of $(LM 2)$, we have
\[[\Psi(a_1), \Psi(a_2)]_{A_2}(\psi(x_1)) = ([a'_1, a'_2]_{A_2})(\psi(x_1)). \quad (15)\]
For any $x_1 \in U_1$. Therefore the relation $(LAM 1)$ is satisfied for any such pair $(a_1, a_2)$ of sections of $A_1$ which are $\psi$-related to a pair $(a'_1, a'_2)$ of sections of $A_2$. Of course, this property is no longer true for any pair $(a_1, a_2)$ of local sections of $A_2$ and so the bracket $"[\Psi(a_1), \Psi(a'_1)]_{A_2}(\psi(x_1))"$ is not defined. Thus, in general, both definitions are not comparable. However, if $\psi$ is a local diffeomorphism, we have:
Proposition 22 Let \((A_1, \pi_1, M_1, \rho_1, [\ldots]_{A_1})\) and \((A_2, \pi_2, M_2, \rho_2, [\ldots]_{A_2})\) be two convenient Lie algebroids. We consider a bundle morphism \(\Psi : A_1 \to A_2\) over a local diffeomorphisms \(\psi : M_1 \to M_2\). Then \(\Psi\) is a Lie algebroid morphism if and only if it is a Lie morphism.

For instance, given any convenient Lie algebroid \((A, \pi, M, \rho, [\ldots]_A)\), then \(\rho\) is Lie morphism and a Lie algebroid morphism from \((A, \pi, M, \rho, [\ldots]_A)\) to the convenient Lie algebroid \((T M, p_M, M, Id, [\ldots].)\).

Proof Since the set of differentials \(\{df, f\text{ smooth map around }x_2 \in M_2\}\) is a separating family on \(T_x M_2\), by an elementary calculation, we obtain the equivalence \((LM 1) \Leftrightarrow (LAM 1)\).

At first note that \((LM 2)\) and \((LAM 2)\) are properties of germs. Thus the equivalence is in fact a local problem. Fix some \(x_1^0 \in M_1\) and \(x_2^0 = \psi(x_1^0)\). Since \(\psi\) is a local diffeomorphism, the model of \(M_1\) and of \(M_2\) are the same convenient space \(M\) and we have charts \((U_1, \phi_1)\) and \((U_2, \phi_2)\) around \(x_1^0\) in \(M_1\) and \(x_2^0\) in \(M_2\) such that for \(i \in \{1, 2\}\):

\[
\begin{align*}
- \phi_i(x_i) &= 0 \in M; \\
- \phi_2 \circ \psi \circ \phi_1^{-1} &\text{ is a diffeomorphism between the } c^\infty-\text{open sets } U_1 := \phi_1(U_1) \text{ and } U_2 := \phi_2(U_2); \\
- \text{a trivialization } \tau_i [A_i] U_i = U_i \times A_i.
\end{align*}
\]

Thus, without loss of generality, we may assume that \(M_i\) is a \(c^\infty\)-open neighbourhood of \(0 \in M\), \(\psi\) is a diffeomorphism from \(M_1\) to \(M_2\) and \(A_i = M_i \times A_i\). By the way, the anchor \(\rho_i\) is a smooth map from \(M_i\) to \(L(M, A_i)\) and each section \(a_i\) of \(A_i\) is a smooth map from \(M_i\) to \(A_i\). Under this context, on the one hand, for all \(x_1 \in M_1\), we have

\[
\begin{align*}
\Psi^* d\rho_2 \omega(a, a') (x_1) &= d\psi(x_1) \left( (\omega(\Psi(a')) (\rho_2 \circ \Psi(a))) - d\psi(x_1) (\omega(\Psi(a)) (\rho_2 \circ \Psi(a'))) \right) \\
&= \omega(\Psi([a, a']_{A_2}) (\psi(x_1))).
\end{align*}
\]

On the other hand, we have

\[
\begin{align*}
d\rho_1 \Psi^* \omega(a, a')(x_1) &= d\psi(x_1) \omega((\Psi(a')) (T \psi \circ \rho(a)) \\
&= d\psi(x_1) \omega((\Psi(a)) (T \psi \circ \rho(a')) - \omega(\Psi([a, a']_{A_1}) (\psi(x_1))).
\end{align*}
\]

Note that for any two pairs of \(\psi\)-related sections \((a_1, a_1')\) and \((a_2, a_2')\), as \(\psi\) is a diffeomorphism, \((LM 2)\) is equivalent to

\[
[\Psi(a_1), \Psi(a_2)]_{A'} (\psi(x_1)) = \Psi([a_1, a_2]_A) (\psi(x_1)) \quad (16)
\]

for all \(x_1 \in M_1\).

Thus, if \((LM 1)\) and \((LM 2)\) are true, then, in the previous local context, \((LAM 2)\) is equivalent to

\[
\omega(\Psi([a_1, a_2]_{A_1}) (\psi(x_1)) = \omega([\Psi(a_1), \Psi(a_2)]_{A'}) (\psi(x_1)) \quad (17)
\]
for any 1-form $\omega$ on $\mathcal{A}_2$ and any $x_1 \in M_1$. As $\psi$ is a diffeomorphism for any pair of sections $(a_1, a_2)$ of $\mathcal{A}_1$ if we set $a'_1 = \Psi(a_1) \circ \psi^{-1}$ and $a'_2 = \Psi(a_2) \circ \psi^{-1}$, then $(a_1, a'_1)$ and $(a_2, a'_2)$ are $\psi$-related and it follows that (LM 1) and (LM 2) implies (LAM 1) and (LAM 2).

Conversely, assume that (LAM 1) and (LAM 2) are true. Consider any two pairs of $\psi$-related sections $(a_1, a'_1)$ and $(a_2, a'_2)$. In this case the relation (17) evaluated on $(a_1, a_2)$ is equivalent to (LAM 2) for any 1-form $\omega$ on $U_2$. Since around each point in $M_2$, the set of germs of 1-forms on $\mathcal{A}_2$ is separating for germs of sections of $\mathcal{A}_2$ and as $\psi$ is a diffeomorphism, this implies (17).

It follows that the relation (LM 2) evaluated on both pairs $(a_1, a'_1)$ and $(a_2, a'_2)$ is satisfied, which ends the proof. 

\[ \square \]

### 2.8 Foliations and Banach–Lie Algebroids

We first recall the classical notion of integrability of a distribution on a Banach manifold (cf. [17]).

Let $M$ be a Banach manifold.

1. A distribution $\Delta$ on $M$ is an assignment $\Delta : x \mapsto \Delta_x \subset T_x M$ on $M$ where $\Delta_x$ is a subspace of $T_x M$. The distribution $\Delta$ is called closed if $\Delta_x$ is closed in $T_x M$ for all $x \in M$.

2. A vector field $X$ on $M$, defined on an open set $\text{Dom}(X)$, is called tangent to a distribution $\Delta$ if $X(x)$ belongs to $\Delta_x$ for all $x \in \text{Dom}(X)$.

3. Let $X$ be a vector field tangent to a distribution $\Delta$ and $\text{Fl}^X_t$ its flow. We say that $\Delta$ is $X$-invariant if $T_x \text{Fl}^X_t(\Delta_x) = \Delta_{\text{Fl}^X_t(x)}$ for all $t$ for which $\text{Fl}^X_t(x)$ is defined.

4. A distribution $\Delta$ on $M$ is called integrable if, for all $x_0 \in M$, there exists a weak submanifold $(N, \phi)$ of $M$ such that $\phi(y_0) = x_0$ for some $y_0 \in N$ and $T\phi(T_x N) = \Delta_{\phi(y)}$ for all $y \in N$. In this case $(N, \phi)$ is called an integral manifold of $\Delta$ through $x$. A leaf $L$ is a weak submanifold which is a maximal integral manifold.

5. A distribution $\Delta$ is called involutive if for any vector fields $X$ and $Y$ on $M$ tangent to $\Delta$ the Lie bracket $[X, Y]$ defined on $\text{Dom}(X) \cap \text{Dom}(Y)$ is tangent to $\Delta$.

Classically, in the Banach context, when $\Delta$ is a supplemented subbundle of $TM$, according to the Frobenius Theorem, involutivity implies integrability.

In finite dimension, the famous results of H. Sussman and P. Stefan give necessary and sufficient conditions for the integrability of smooth distributions.

Few generalizations of these results in the framework of Banach manifolds can be found in [20].

In the context of this section, we have (cf. [17]):

**Theorem 23** Let $(\mathcal{A}, \pi, M, \rho, \ldots, \mathcal{A})$ be a split Banach–Lie algebroid.

If $\rho(\mathcal{A})$ is a closed distribution, then this distribution is integrable.

Note that if $\rho$ is a Fredholm morphism, the assumptions of Theorem 23 are always satisfied. In the Hilbert framework, only the closeness of $\rho$ is required.
3 Prolongation of a Convenient Lie Algebroid Along a Fibration

3.1 Prolongation of an Anchored Convenient Bundle

Let $p : E \to M$ be a convenient vector bundle with typical fibre $E$ and $M$ an open submanifold such that the restriction of $p$ to $M$ is a surjective fibration over $M$ of typical fibre $O$ (open subset of $E$). We consider some anchored convenient bundle $(A, \pi, M, \rho)$.

**Notations 24** If $(U, \phi)$ is a chart such that $E_U$ and $A_U$ are trivializable, then $TM_U = TM|_U$ and $T_M|_U$ are also trivializable. In this case, we have trivializations and local coordinates

- $E_U \equiv U \times E$ and $M_U \equiv U \times O$ with local coordinates $m = (x, e)$
- $T_M|_U \equiv (U \times O) \times M \times E$ with local coordinates $(m, v, z)$.

For $m \in p^{-1}(x)$ we set

$$T^A_m M = \{(a, \mu) \in A_x \times T_m M : \rho(a) = Tp(\mu)\}.$$

An element of $T^A_m M$ will be denoted $(m, a, \mu)$.

We set $T^A M = \bigcup_{m \in M} T^A_m M$ and we consider the projection $\hat{p} : T^A M \to M$

defined by $\hat{p}(m, a, \mu) = m$.

We introduce the following context:

1. Let $\tilde{\pi} : \tilde{A} \to M$ be the pull-back of the bundle $\pi : A \to M$ by $p : M \to M$.

We denote by $\tilde{p}$ the canonical vector bundle such that the following diagram is commutative:

```
    \tilde{A} \xrightarrow{\tilde{\pi}} A
       \downarrow \hat{p} \uparrow \pi
      M \xrightarrow{p} M
```

2. Consider the map

$$\hat{\rho} : T^A M \to TM (m, a, \mu) \mapsto (m, \mu)$$

and let $p_{\tilde{A}} : T^A M \mapsto \tilde{A}$ be the map defined by $p_{\tilde{A}}(m, a, v, z) = (m, a)$. Then the following diagrams are commutative

```
  T^A M \xrightarrow{p_{\tilde{A}}} \tilde{A} \xrightarrow{\hat{\rho}} TM
  \downarrow \hat{\pi} \downarrow \pi \uparrow Tp
  M \xrightarrow{\text{Id}} M \xrightarrow{\rho} TM
```
(3) If \( p : \mathcal{M} \mapsto \mathcal{M} \) is the tangent bundle, consider the associated vertical bundle \( p^V : T\mathcal{M} \mapsto \mathcal{M} \). Then there exists a canonical isomorphism bundle \( \nu \) from the pull-back \( \tilde{p} : \tilde{E} \mapsto \mathcal{M} \) of the the bundle \( p : E \mapsto \mathcal{M} \) over \( p : \mathcal{M} \mapsto \mathcal{M} \) to \( p^V : \mathcal{V} \mathcal{M} \mapsto \mathcal{M} \) so that the following diagram is commutative:

\[
\begin{array}{ccc}
\tilde{E} & \overset{\nu}{\longrightarrow} & \mathcal{V}\mathcal{M} \\
\downarrow{\tilde{p}} & & \downarrow{p^V} \\
\mathcal{M} & \overset{\text{Id}}{\longrightarrow} & \mathcal{M}
\end{array}
\] (18)

**Theorem 25**

(1) \( \hat{p} : T^A\mathcal{M} \mapsto \mathcal{M} \) is a convenient bundle with typical fibre \( \mathbb{A} \times \mathbb{E} \) and \( (T^A\mathcal{M}, \hat{p}, \mathcal{M}, \hat{\rho}) \) is an anchored bundle.

(2) \( p_A \) is a surjective bundle morphism whose kernel is a subbundle of \( T\mathcal{M} \). The restriction of \( \hat{\rho} \) to \( \ker p_A \) is a bundle isomorphism on \( \mathcal{V}\mathcal{M} \).

(3) Given an open subset \( \mathcal{V} \), then, for each section \( X \) of \( T\mathcal{M} \) defined on the open set \( \mathcal{V} = \hat{p}^{-1}(\mathcal{V}) \subset \mathcal{M} \), there exists a pair \((a, X)\) of a section \( a \) of \( \tilde{A} \) and a vector field \( X \) on \( \mathcal{V} \) such that

\[
\forall m \in \mathcal{V}, \ T\hat{p}(X(m)) = \rho \circ p_A(a(m)).
\] (19)

Conversely such a pair \((a, X)\) which satisfies (19) defines a unique section \( X \) on \( \mathcal{V} \), the associated pair of \( X \) is precisely \((a, X)\) and with these notations, we have

\[
\hat{\rho}(X) = X.
\]

**Proof**

(1) Let \((U, \phi)\) be a chart on \( \mathcal{M} \) such that we have a trivialization \( \tau : \mathcal{A} U \mapsto \phi(U) \times \mathbb{A} \) and \( \Phi : \mathcal{M} U \mapsto \phi(U) \times \mathbb{A} \subset \phi(U) \times \mathbb{E} \). Then \( T\Phi \) is a trivialization of \( T\mathcal{M} U \) on \( \phi(U) \times \mathbb{M} \) and \( T\Phi \) is a trivialization of \( T\mathcal{M} U \) on \( \phi(U) \times \mathbb{A} \times \mathbb{M} \times \mathbb{E} \).

To be very precise, according to the notations in Sect. 2.1, we have

\[
\begin{align*}
\phi(x) & \equiv x; \\
\tau(x, a) & \equiv (x, a); \\
\Phi(x, e) & \equiv (x, a, e) \text{ and for } m = (x, e), \Phi(m) \equiv m; \\
T\phi(x, v) & \equiv (x, v); \\
T\Phi(m, \mu) & = T\Phi(x, e, v, z) \equiv (x, e, v, z)
\end{align*}
\]

where \( \equiv \) stands for "denoted".

By the way, in this local context and with these notations, we have

\[
T^A\mathcal{M} U \equiv \{(x, e, a, \nu, z) \in \phi(U) \times \mathbb{A} \times \mathbb{M} \times \mathbb{E} : \nu = \rho_{\phi(a)}\}
\] (20)

where \( \rho \) corresponds to the local expression of the anchor. It follows that:

\[
T^A\mathcal{M} U \equiv \{(x, e, a, \rho_{\phi(b)} z) : (x, e, a, z) \in \phi(U) \times \mathbb{A} \times \mathbb{M} \times \mathbb{E}\},
\] (21)

and so the map \( T\Phi : T^A\mathcal{M} U \to \phi(U) \times \mathbb{A} \times \mathbb{M} \times \mathbb{E} \) defined by \( T\Phi(x, e, a, \nu, z) \equiv (x, e, a, z) \) is a smooth map which is bijective. Moreover, for each \( m = (x, e) \in \mathcal{M} U \) the
restriction $T\Phi_m$ to $T_m\mathcal{M}_U$ is clearly linear and we have the following commutative diagram

$$
\begin{array}{ccc}
T^A\mathcal{M}_U & \xrightarrow{T\Phi} & \Phi(U) \times \emptyset \times A \times E \\
\downarrow \hat{\rho} & & \downarrow \hat{\pi}_1 \\
\mathcal{M}_U & \xrightarrow{\Phi} & \phi(U) \times \emptyset
\end{array}
$$

This shows that $T\Phi$ is a local trivialization of $T^A\mathcal{M}_U$ modelled on $A \times E$.

Now, in this local context, $\hat{\rho}(x, e, a, v, z) \equiv (x, e, a, v, z)$.

Consider two such chart domains $U$ and $U'$ in $M$ such that $U \cap U' \neq \emptyset$. Then we have:

- the transition maps associated to the trivializations $\tau$ and $\tau'$ in $\mathcal{A}$ are of type

$$
(x, a) \mapsto (t(x), G_x(a))
$$

where $x \mapsto G_x$ takes value in $\text{GL}(A)$, and is a smooth map from $U \cap U'$ into $L(A)$;

- the transition maps associated to the trivializations $\Phi$ and $\Phi'$ in $\mathcal{M}$ are of type

$$
(x, e, a, v) \mapsto (t(x), F_x(e))
$$

where $x \mapsto F_x$ takes value in $\text{GL}(E)$, and is a smooth map from $U \cap U'$ into $L(E)$;

- if $\tilde{\Phi}$ and $\tilde{\Phi}'$ are the trivializations of $\tilde{\mathcal{A}}$ associated to $\Phi$ and $\Phi'$, the transition maps associated to trivializations $\tilde{\Phi}$ and $\tilde{\Phi}'$ in $\tilde{\mathcal{A}}$ are of type

$$
(x, e, a) \mapsto (t(x), F_x(e), G_x(a)) ;
$$

- the transition maps associated to trivializations $T\Phi$ and $T\Phi'$ in $T\mathcal{M}$ are of type

$$
(x, e, v, z) \mapsto (t(x), F_x(e), d_{xt(y)}, H_{(x,e)}(v))
$$

where $(x, e) \mapsto H_{(x,e)}$ takes value in $\text{GL}(E)$ and is a smooth map from $U \cap U'$ to $L(E)$;

- the transition maps associated to trivializations $T\Phi$ and $T\Phi'$ in $T^A\mathcal{M}$ are of type

$$
(x, e, a, z) \mapsto (t(x), F_x(e), G_x(a), H_{(x,e)}(z)) .
$$

Clearly, this implies that $\hat{\rho} : T\mathcal{M} \rightarrow \mathcal{M}$ is a convenient bundle.

Now, in a trivialization $\tau$ and $T\phi$, we write $\rho(x, a) \equiv (x, a) \mapsto (x, \rho_x(a))$. If, in another trivialization $\tau'$ and $T\phi'$, we write $\rho(x, a) \equiv (x', a') \mapsto (x', \rho'_{x'}(a'))$, then for the associated transition maps, we have

$$
(x', \rho'_{x'}) = (x', dx \circ \rho_x \circ G_x).
$$
It follows easily that $\hat{\rho}$ is a bundle convenient morphism.

(2) By construction, the following diagram is commutative:

$$
\begin{array}{ccc}
T^A\mathcal{M} & \overset{\hat{\rho}}{\longrightarrow} & TM \\
\downarrow{p_\mathcal{A}} & & \downarrow{p_M} \\
\tilde{\mathcal{A}} & \overset{\tilde{\pi}}{\longrightarrow} & M
\end{array}
$$

In the trivialization $\tilde{\Phi} : \tilde{\mathcal{A}}_U \to \phi(U) \times \mathbb{O} \times \mathbb{A}$, using the same convention as previously, we have

$$
\{(x, e, a, v, z) \mapsto p_\mathcal{A}(x, e, a, v)\} \equiv \{(x, e, a) \mapsto (x, e, a)\}.
$$

Thus, by analog arguments as in the proof of (1), it is clear that $p_\mathcal{A}$ is compatible with the transition maps associated to the trivializations over the chart domains $U$ and $U'$ of $M$ for $T^A\mathcal{M}$ and for $\tilde{\mathcal{A}}$.

Thus $p_\mathcal{A} : T^A\mathcal{M} \to \tilde{\mathcal{A}}$ is a Banach bundle morphism which is surjective.

From the construction of $T^A\mathcal{M}$ we have

$$
\ker p_\mathcal{A} = \{(m, a, v) \in T^A_m\mathcal{M} : \rho(a) = 0\}.
$$

The definition of $\hat{\rho}$ implies that its restriction to $\ker p_\mathcal{A}$ is an isomorphism onto $V\mathcal{M}$, which ends the proof of (2).

(3) Let $X$ be a section of $T\mathcal{M}$ defined on $\mathcal{V} = \hat{p}^{-1}(V)$. According to (2), if we set $a = p_\mathcal{A} \circ X$ and $X = \hat{\rho}(X)$, the pair $(a, X)$ is well defined and from the definition of $T^A\mathcal{M}$ the relation (19) is satisfied. Conversely, if $a$ is a section $a$ of $\tilde{\mathcal{A}}$ and $X$ a vector field on $\mathcal{V}$, the relation (19) means exactly that $X(m) = (\hat{p}(a(m)), X(m))$ belongs to $T^A_m\mathcal{M}$; so we get a section $X$ of $T^A\mathcal{M}$. Now it is clear that $a = p_\mathcal{A} \circ X$ and $X = \hat{\rho}(X)$.

**Definition 26** The anchored bundle $(T^A\mathcal{M}, \hat{\rho}, M, \hat{\rho})$ is called the prolongation of $(\mathcal{A}, \pi, M, \rho)$ over $\mathcal{M}$. The subbundle $\ker p_\mathcal{A}$ will be denoted $V^A\mathcal{M}$ and is called the vertical subbundle.

**Remark 27** According to the proof of Theorem 25, if $V^A\mathcal{M}_U$ is the restriction of $V^A\mathcal{M}$ to $\mathcal{M}_U$, we have

$$
V^A\mathcal{M}_U \equiv \{(x, e, a, 0, z) : (x, e, a, z) \in \phi(U) \times \mathbb{O} \times \mathbb{A} \times \mathbb{E}\}.
$$

**Examples 28** For simplicity in these examples we assume that the model space $\mathbb{M}$ of $M$ and the typical fiber $\mathbb{A}$ of $\mathcal{A}$ are Banach spaces.

(1) If $\mathcal{A} = M = TM$ then we have $T\mathcal{A} = TTM$ and $T\mathcal{A}^* = TT^*M$ and the anchor $\hat{\rho}$ is the identity.

(2) If $\mathcal{A} = M$ then $T^A\mathcal{A}$ is simply denoted $T\mathcal{A}$ and the anchor $\hat{\rho}$ is the map $(x, a, b, c) \mapsto (x, a, \rho(b), \nu(c))$ from $T\mathcal{A}$ to $T\mathcal{A}$.
(3) If \( \mathcal{M} = \mathcal{A}^* \) then \( T^A\mathcal{A}^* \) is simply denoted \( T\mathcal{A}^* \) and the anchor \( \hat{\rho} \) is the map 
\[(x, \xi, a, \eta) \mapsto (x, \xi, \rho(a), v(\eta)) \) from \( T\mathcal{A}^* \) to \( T\mathcal{A}^* \).

(4) If \( \mathcal{M} = \mathcal{A} \times_M \mathcal{A}^* \) then \( T^A\mathcal{M} \) is simply denoted \( T(\mathcal{A} \times \mathcal{A}^*) \) and the anchor 
\( \hat{\rho} \) is the map 
\[(x, a, \xi, b, c, \eta) \mapsto (x, a, \xi, \rho(b), v(v)(\eta)) \) from \( T(\mathcal{A} \times \mathcal{A}^*) \) to 
\( T(\mathcal{A} \times \mathcal{A}^*) \).

(5) If \( \mathcal{M} \) is a conic submanifold of \( \mathcal{A} \) (cf. [18]), then \( T^A\mathcal{M} \) is simply denoted \( T\mathcal{M} \) and the anchor \( \hat{\rho} \) is the same expression as in (2).

**Remark 29** We come back to the previous general context: \( (\mathcal{A}, \pi, M, (...)_\mathcal{A}) \) is a 
convenient Lie algebroid and \( p : E \to M \) is a convenient vector bundle and \( \mathcal{M} \) is an 
open submanifold of \( E \) which is fibered on \( M \).

Let \((U, \phi)\) be a chart of \( M \) such that \( \mathcal{A}_U \) (resp. \( \mathcal{E}_U \)) is trivial and we have denoted 
\[ \tau : \mathcal{A}_U \to \phi(U) \times \mathbb{A} \] (resp. \( \Phi : \mathcal{E}_U \to \phi(U) \times \mathbb{E} \))
the associated trivialization (cf. notations in the proof of Theorem 25). Then we have 
a canonical anchored bundle 
\[ (\phi(U) \times \mathbb{A}, \pi_1, \phi(U), \tau = T\phi \circ \rho \circ \phi^{-1}) \]
where \( \pi_1 : \phi(U) \times \mathbb{A} \to \phi(U) \).

Therefore, the prolongation \( \phi(U) \times \mathbb{A} \) over \( \phi(U) \times \mathbb{E} \) is then 
\[ T^{\phi(U) \times \mathbb{A}}(\phi(U) \times \mathbb{E}) = \phi(U) \times \mathbb{E} \times \mathbb{A} \times \mathbb{E} \]
and the anchor \( \hat{\rho} \) is the map \((x_e, a, z) \mapsto (x_e, r(a), z)\).

Note that the bundle \( T^{\phi(U) \times \mathbb{A}}(\phi(U) \times \mathbb{E}) = \phi(U) \times \mathbb{E} \times \mathbb{A} \times \mathbb{E} \) can be identified 
with the subbundle 
\[ \{(x_e, a, 0, v) \in U \times \mathbb{E} \times \mathbb{A} \times \mathbb{M} \times \mathbb{E} \} \]
of \( T(\phi(U) \times \mathbb{E}) = \phi(U) \times \mathbb{E} \times \mathbb{A} \times \mathbb{M} \times \mathbb{E} \).

An analog description is true for any open set \( \mathcal{U} \) in \( \mathcal{E} \) such that \( p(\mathcal{U}) = U \) since \( \mathcal{U} \) is contained in \( \mathcal{M}_U \).

Given a fibred morphism \( \Psi : \mathcal{M} \to \mathcal{M}' \) between the fibre bundles \( p : \mathcal{M} \to M \) and \( p' : \mathcal{M}' \to M' \) over \( \psi : M \to M' \) and a morphism of anchored bundles \( \varphi \) between 
\( (\mathcal{A}, \pi, M, \rho) \) and \( (\mathcal{A}', \pi', M', \rho') \), over \( \psi : M \to M' \), we get a map
\[ T\Psi : T^A\mathcal{M} \to T^{A'}\mathcal{M}' \]
\[ (m, a, \mu) \mapsto (\Psi(m); \varphi(a), T_m\Psi(\mu)) \] (22)

**Remark 30** As in the proof of Theorem 25, consider a chart \((\mathcal{U}, \Phi)\) of \( \mathcal{M} \) where 
\( p(\mathcal{U}) = U \) and let \( \tau : \mathcal{A}_U \to U \times \mathbb{A} \) be an associated trivialization of \( \mathcal{A}_U \). According 
to Remark 29, since \( \Phi \) is a smooth diffeomorphism from \( \mathcal{U} \) onto its range \( \phi(U) \times \mathbb{E} \), then 
\( T\Phi : T^A\mathcal{M}|_{\mathcal{U}} \to T^{\phi(U) \times \mathbb{E}}\mathcal{U} = \Phi(\mathcal{U}) \times \mathbb{A} \times \mathbb{E} \) is a bundle 
isomorphism and so \( (T^A\mathcal{M}|_{\mathcal{U}}, T\Phi) \) is a chart for \( T^A\mathcal{M} \).
Notations 31 From now on, the anchored bundle \((\mathcal{A}, \pi, M, \rho)\) is fixed and, if no confusion is possible, we simply denote by \(T\mathcal{M}\) and \(V\mathcal{M}\) the sets \(T^A\mathcal{M}\) and \(V^A\mathcal{M}\) respectively. In particular when \(\mathcal{M} = A\), the prolongation \(TA\) will be simply called the prolongation of the the Lie algebroid \(A\) prolongation of a convenient Lie algebroid. The bundle \(V\mathcal{M}\) will be considered as a subbundle of \(T\mathcal{M}\) as well as of \(T\mathcal{M}\).

3.2 Connections on a Prolongation

Classically ([13, 37]), a connection on a convenient vector bundle \(p : \mathcal{E} \to M\) is a Whitney decomposition \(T\mathcal{E} = H\mathcal{E} \oplus V\mathcal{E}\). Now, as in finite dimension we introduce this notion on \(T\mathcal{M}\).

Definition 32 A connection on \(T\mathcal{M}\) is a decomposition of this bundle in a Whitney sum \(T\mathcal{M} = H\mathcal{M} \oplus V\mathcal{M}\).

Such a decomposition is equivalent to the datum of an endomorphism \(N\) of \(T\mathcal{M}\) such that \(N^2 = \text{Id}\) with \(T\mathcal{M} = \ker(\text{Id} + N)\) and \(H\mathcal{M} = \ker(\text{Id} - N)\) where \(\text{Id}\) is the identity morphism of \(T\mathcal{M}\). We naturally get two projections:

\[
\begin{align*}
    h_N &= \frac{1}{2}(\text{Id} + N) : T\mathcal{M} \to H\mathcal{M} \\
    v_N &= \frac{1}{2}(\text{Id} - N) : T\mathcal{M} \to V\mathcal{M}.
\end{align*}
\]

\(v_N\) and \(h_N\) are called respectively the vertical and horizontal projector of \(N\).

Using again the context of Remark 30, we have charts

\[
\Phi : \mathcal{M}_U \to \phi(U) \times O \\
T\Phi : T\mathcal{M}_U \to \phi(U) \times O \times A \times E.
\]

If \(N\) is a connection on \(T\mathcal{M}\), then \(N = T\Psi \circ N \circ \Psi^{-1}\) is a non linear connection on the trivial bundle \(\phi(U) \times O \times A \times E\). Thus \(N\) can be written as a matrix field of endomorphisms of \(A \times E\) of type

\[
\begin{pmatrix}
    \text{Id}_A & 0 \\
    -2\mathcal{F} & -\text{Id}_E
\end{pmatrix}
\]

and so the associated horizontal (resp. vertical) projector is given by

\[
\begin{align*}
    h_{Nm}(a,z) &= \frac{1}{2}(a,z) - \mathcal{F}(a) \\
    v_{Nm}(a,z) &= \frac{1}{2}(a,v) + \mathcal{F}(a).
\end{align*}
\]

The associated horizontal space in \(\{m\} \times A \times E\) is

\[
\left\{ \frac{1}{2}(a,z) - \mathcal{F}(a), (a,z) \in \{m\} \times A \times E \right\}
\]

and the associated vertical space in \(\{m\} \times A \times E\) is \(\{m\} \times \{0\} \times E\).

\[\text{In the finite dimensional context such a connection is sometimes called a nonlinear connection.}\]
$\omega$ is called the (local) Christoffel symbol of $N$.

Let $\tilde{\rho} : \tilde{A} \times \tilde{E} \to M$ the fibered product bundle over $\tilde{M}$ of $(\pi, \rho) : A \times \mathcal{E} \to M$.

We have natural inclusions $\iota_1 : \tilde{A} \to \tilde{A} \times \tilde{E}$ and $\iota_2 : \tilde{E} \to \tilde{A} \times \tilde{E}$, given respectively by $\iota_1(m, a) = (m, a, 0)$ and $\iota_1(m, z) = (m, 0, z)$, such that

$$\tilde{A} \times \tilde{E} = \iota_1(\tilde{A}) \oplus \iota_2(\tilde{E}).$$

(24)

With these notations, we have

**Proposition 33**

1. There exists a non-linear connection $N$ on $TM$ if and only if there exists a convenient bundle morphism $H$ from $\tilde{A}$ to $TM$ such that $TM = H(\tilde{A}) \oplus V_M$. In this case $TM$ is isomorphic to $\tilde{A} \times \tilde{E}$.

2. Assume that $N$ is a connection on $TM$. Let $\Upsilon$ be semi-basic vector valued $\Upsilon$, then $N + \Upsilon$ is a connection on $TM$. Conversely, given any nonlinear connection $N'$ on $TM$, there exists a unique semi-basic vector valued $\Upsilon$ such that $N' = N + \Upsilon$.

According to this Proposition we introduce:

**Definition 34**

A prolongation of $A$ over $\rho : M \to M$ is called a split prolongation if there exists a Whitney decomposition $TM = K_M \oplus V_M$.

The following result is a clear consequence of Proposition 33.

**Corollary 35**

Let $TM$ be a split prolongation. Then there exists a connection on $TM$ and $TM$ is isomorphic to $\tilde{A} \times \tilde{E}$.

**Proof of Proposition 33**

(1) Assume that we have a connection $N$ on $TM$ and let $TM = H_M \oplus V_M$ be the associated Whitney decomposition.

Let $H$ be the restriction of $p_{\tilde{A}}$ to $H_M$. Since $V_M$ is the kernel of the surjective morphism $p_{\tilde{A}}$, it follows that $H$ is an isomorphism. Since we have an isomorphism $\nu : \tilde{E} \to V_M$, according to (24), it follows easily that $\tilde{A} \times \tilde{E}$ is isomorphic to $TM$. The converse is clear.

(2) At first, if $\Upsilon$ is semi basic, then $\ker(\text{Id} + N + \Upsilon) = V_M$ and clearly the range of $\text{Id} + N + \Upsilon$ is a supplemented subbundle of $V_M$.

On the one hand, if $N'$ is a connection, we set $\Upsilon = N' - N$. Then $\Upsilon(Z) = 0$, for all local vertical sections. On the other hand $\Upsilon(X) = (\text{Id} + N')(X) - (\text{Id} + N(X)$ which belongs to $V_M$. \qed

A sufficient condition for the existence of a connection on $TM$ is given by the following result:

**Theorem 36**

Assume that there exists a linear connection on the bundle $\rho : E \to M$. Then there exists a connection $N$ on $TM$.

**Proof**

let $\rho^* TM$ (resp. $\rho^* E$) is the pull-back over $M$ of $TM \to M$ (resp. $E \to M$).

\footnote{That is a morphism from $TM$ to $V_M$ such that $\Upsilon(Z) = 0$ for any local vertical section $Z$.}
If there exists a linear connection $N$ on the bundle $E \to M$, there exists a convenient isomorphism bundle

$$\kappa = (\kappa_1, \kappa_2) : T E \to p^* TM \oplus p^* E.$$  

(cf. Theorem 3.1 [2, in Banach setting], [4, Chap. 6 in convenient setting]). Therefore, for an open fibred submanifold $\mathcal{M}$ of $E$, by restriction, we obtain an isomorphism (again denoted $\kappa$).

$$\kappa : T M \to p^* TM \oplus p^* E$$

Without loss of generality, we can identify $T M$ with $p^* TM \oplus p^* E$. For $m = (x, e)$, the fibre $T_m M$ is then

$$T_m M = \{(a, \mu) \in A_x \times T_x M \times V_m M : \rho(a) = T p(\mu)\}.$$  

But, under our identification of $T M$ with $p^* TM \oplus p^* E$, if $m = (x, e)$, this implies that $\mu \in T_m M$, can be written as a pair $(v, z) \in T_x M \times E_x$ and so we replace the condition $\rho(a) = T p(\mu)$ by $\rho_x(a) = v$.

Recall that, in the one hand, we have a chart (cf. proof of Theorem 25):

$$T \Phi : T M_U \to \Phi(M_U) \times A \times E.$$  

The value of $T \Phi(m, a, \mu)$ can be written $(\phi(x), \Phi_x(e), \tau_x(a), T_m \Phi(\mu))$ (value denoted $(x, e, a, z)$) with $T_m \Psi(T p(\mu)) = T_x \phi(\rho_x(a))$. But, under our assumption, we have $T_m \Phi(\mu) = (T_x \phi(\rho_x(a)), z) \in \{(x, e)\} \times \mathbb{M} \times E$ and so we obtain

$$T \Phi(m, a, \mu) \equiv (x, e, a, z). \quad (25)$$

On the other hand, we have a trivialization $\tau \times \Phi$ from $A \times E_{M_U}$ to $\Phi(M_U) \times A \times E$ over $\Phi$. In fact, we have $(\tau \times \Phi)(x, e, a, z) = (\phi(x), \Phi_x(e), \tau_x(a), \Phi_x(z))$.

According to our assumption, the map $\widetilde{\Psi} : A \times E \to T M$ is given by

$$\widetilde{\Psi}(m, a, z) = (m, a, \rho(a), z)$$

is well defined. In local coordinates, we have

$$\widetilde{\Psi}(e, x, a, \rho(a), z) \equiv (x, e, a, z).$$

Thus $\widetilde{\Psi}$ is the identity in local coordinates and so is a local bundle isomorphism. To complete the proof, we only have to show that under our assumption $\widetilde{\Psi}$ is a convenient bundle isomorphism.

By analogy with the notations used in the proof of Theorem 25 (1), let $(U', \phi')$ be another chart on $M$ and consider all the corresponding trivializations $T \phi'$, $\Phi'$, and $\Phi'$. We set $(x', e', v', z') = T \Phi'(e, z)$ (i.e. $(e, z) \equiv (x', e', v', z')$ following our convention), in
these new coordinates, we have \( \widetilde{\Psi}(e, z) \equiv (x', e', a', z') \). Assume that \( U \cap U' \neq \emptyset \). For the change of coordinates, we set \( \theta_x = \phi' \circ \phi^{-1}(x) \), and each associated transition map gives rise to a smooth field of isomorphisms of convenient spaces as follows:

\[
T_x \theta(v) = T_x(\phi' \circ \phi^{-1})(v)
\]

\[
\mathcal{T}_x(a) = \left( \tau' \circ \tau^{-1} \right)_x(a)
\]

\[
\Theta_x(e) = (\Phi' \circ \Phi^{-1})_x(e).
\]

Thus, under our assumption, according to (25), in fact we have

\[
T \Phi' \circ T \Phi^{-1}(x, e, a, z) = (\theta(x), \Theta_x(e), \mathcal{T}_x(a), \Theta_x(z)).
\]

Now with the previous notations we have

\[
(\tau' \times \Phi') \circ (\tau \times \Phi)^{-1}(x, e, a, z) = (\theta(x), \Theta_x(e), \mathcal{T}_x(a), \Theta_x(z)) = T \Phi' \circ T \Phi^{-1}(x, e, a, z).
\]

Since, in such local coordinates, \( \widetilde{\Psi} \) is the identity map, so under our assumption \( \widetilde{\Psi} \) is a convenient bundle isomorphism. \( \Box \)

3.3 Prolongation of the Lie Bracket

In the finite dimensional framework, a Lie bracket for smooth sections of \( \hat{\mathfrak{p}} : T\mathcal{M} \to \mathcal{M} \) is well defined. Unfortunately, we will see that it is not true in this general convenient context.

According to the Notations 24, for each open set \( U \) of \( \mathcal{M} \), we denote by \( \Gamma(T\mathcal{M}_U) \), \( \Gamma(V\mathcal{M}_U) \) and \( \Gamma(\bar{A}_U) \) the \( C^\infty(\mathcal{M}_U) \)-module of sections of \( T\mathcal{M}_U \), \( V\mathcal{M}_U \) and \( \bar{A}_U \) respectively. We also denote \( \Gamma(A_U) \) the \( C^\infty(U) \)-module of sections \( \pi : A_U \to U \).

**Definition 37** Let \( U \) be an open subset of \( \mathcal{M} \). A section \( X \) in \( \Gamma(T\mathcal{M}_U) \) is called projectable if there exists \( a \in \Gamma(A_U) \) such that

\[
p_A \circ X = a.
\]

Therefore \( X \) is projectable if and only if there exists a vector field \( X \) on \( \mathcal{M}_U \) and \( a \in \Gamma(A_U) \) such that (see Theorem 25)

\[
X = (a \circ p, X) \text{ with } Tp(X) = \rho \circ a.
\]

Assume now that \( (\mathcal{A}, \pi, \mathcal{M}, \rho, \ldots, \mathcal{A}) \) is a convenient Lie algebroid.

Let \( X_i = (a_i \circ p, X_i) \), \( i \in \{1, 2\} \) be two projectable sections defined on \( \mathcal{M}_U \). We set

\[
[X_1, X_2]_{T\mathcal{M}} = ([a_1, a_2]_\mathcal{A} \circ p, [X_1, X_2]) \tag{26}
\]
Since $\rho([a_1, a_2]_A) = [\rho(a_1), \rho(a_2)]$ and $T_p([X_1, X_2]) = [T_p(X_1), T_p(X_2)]$, it follows that $[X_1, X_2]_{T_M}$ is a projectable section which is well defined. Moreover, we have

$$\hat{\rho}([X_1, X_2]_{T_M}) = [\hat{\rho}(X_1), \hat{\rho}(X_2)].$$

(27)

**Comments 38** Now, in finite dimension, it is well known that the module of sections of $\tilde{A}_U \to M_U$ is the $C^\infty(M_U)$-module generated by the set of sections $a \circ p$ where $a$ is any section of $A_U \to U$. Therefore, according to Theorem 25, the module $\Gamma(T_M U)$ is generated, as $C^\infty(M_U)$-module, by the set of all projectable sections of $\Gamma(T_M U)$. This result is essentially a consequence of the fact that, in the local context used in the proof of Theorem 25, over such a chart domain $U$, the bundle $\tilde{A}_U$ is a finite dimensional bundle and so the module of sections of $\tilde{A}_U$ over $M_U$ is finitely generated as $C^\infty(M_U)$. Thus, if $A$ is a finite rank bundle over $M$, then the module $\Gamma(T_M U)$ is generated, as $C^\infty(M_U)$-module, by the set of all projectable sections of $\Gamma(T_M U)$.

Unfortunately, this is no longer true in the convenient context and even in the Banach setting in general.

Note that, under some type of approximation properties of $A$, we can show that the module generated by the set of all projectable sections of $\Gamma(A_U)$ as $C^\infty(M_U)$-module, is dense in $\Gamma(\tilde{A}_U)$ as a convenient space. In this case, the $C^\infty(M_U)$-module, generated by the set of all projectable sections of $\Gamma(T_M U)$ will be dense in $\Gamma(T_M U)$ (as a convenient space). We could hope that, in this context, the Lie bracket $[\cdot, \cdot]_{T_M U}$ can be extended to $\Gamma(T_M U)$.

**Definition 39** We denote by $\mathcal{P}(T_M U)$ the $C^\infty(M_U)$-module $\Gamma(T_M U)$ generated by the set of projectable sections defined on $U$ in the $C^\infty(M_U)$-module.

Each module $\mathcal{P}(T_M U)$ has the following properties:

**Lemma 40** (1) For any open subset $U$ in $M$, there exists a well defined Lie bracket $[\cdot, \cdot]_{T_M U}$ on $\mathcal{P}(T_M U)$ which satisfies the assumption of Definition 3 and whose restriction to projectable sections is given by the relation (26).

(2) For each $x \in M$, there exists a chart domain $U$ around $x$ in $M$ such that $T_M U$ is trivializable over $M_U$ and for each $(m, v) \in T_M M$ where $m = (x, e) \in M_U$, there exists a projectable section $X$ defined on $M_U$ such that $X(m) = (a, v)$.

(3) Assume that we have a Whitney decomposition $T_M = K_M \oplus V M$ and let $p_K$ be the associated projection on $K_M$. Then, for any section $X \in \mathcal{P}(T_M U)$, the induced section $X^K = p_K \circ X$ belongs to $\mathcal{P}(K_M U) = \Gamma(K_M U) \cap \mathcal{P}(T_M U)$. In particular, $X^K$ is projectable if and only if $X$ is so.

**Proof** (1) First of all, using the Leibniz formula, if $X_1$ and $X_2$ are projectable sections defined on $M_U$, we have

$$[X_1, f X_2]_{T_M U} = df(\hat{\rho}(X_1))X_2 + f[X_1, X_2]_{T_M U}.$$
for any \( f \in C^\infty(\mathcal{M}) \). Now any local section \( X \) of \( \mathfrak{P}(T\mathcal{M}_U) \) is a finite linear functional sum

\[
X = f_1 X_2 + \cdots + f_k X_k
\]

where, for \( i \in \{1, \ldots, k\} \), each \( X_i \) is projectable and \( f_i \) is a local smooth function on \( \mathcal{M}_U \). Therefore, such a decomposition allows to define the bracket \( [Y, X]_{T\mathcal{M}_U} \) for all projectable sections \( Y \) defined on the same open set as \( X \). Note that, from (26) and the Leibniz formula, the value \( [Y, X]_{T\mathcal{M}_U}(m) \) only depends on the 1-jet of \( Y \) and \( X \) at point \( m \) and so the value of \( [Y, X]_{T\mathcal{M}_U}(m) \) is well defined. Since such a value is independent on the expression of \( X \), by similar arguments, we can define the bracket \( [\hat{X}', X]_{T\mathcal{M}_U}(m) \), for any other local section \( \hat{X}' \) of \( \mathfrak{P}(T\mathcal{M}_U) \). Now since, by assumption, \( [\ldots, \ldots]_\mathfrak{L} \) and the Lie bracket of vector fields satisfies the assumption of Definition 3, the restriction of \( [\ldots, \ldots]_{T\mathcal{M}_U} \) to projectable sections satisfies also the assumption of Definition 3 and so is its extension to sections of \( \mathfrak{P}(T\mathcal{M}_U) \) for any open set \( U \) of \( M \) is an almost Lie bracket.

In this context, the jacobitor on \( T\mathcal{M}_U \) is defined by

\[
J_{T\mathcal{M}_U}(X_1, X_2, X_3) = [[X_1, X_2]_{T\mathcal{M}_U}, X_3]_{T\mathcal{M}_U} + [[X_2, X_3]_{T\mathcal{M}_U}, X_1]_{T\mathcal{M}_U} + [[X_3, X_1]_{T\mathcal{M}_U}, X_2]_{T\mathcal{M}_U}
\]

But according to relation (27) for projectable sections, using the Leibnitz property, it is easy to see that, for all \( X_1, X_2 \) in \( \mathfrak{P}(T\mathcal{M}_U) \)

\[
\hat{\rho}([X_1, X_2]_{T\mathcal{M}_U}) = [\hat{\rho}(X_1), \hat{\rho}(X_2)].
\]

On the other hand, from(26), for projectable sections \( X_i, i \in \{1, 2, 3\} \), it follows that

\[
J_{T\mathcal{M}_U}(X_1, X_2, X_3) = 0.
\]

Therefore, according to these properties, it follows that \( J_{T\mathcal{M}_U} \) vanishes identically on \( \mathfrak{P}(T\mathcal{M}_U) \), which ends the proof of (1).

(2) Choose \( x_0 \in M \). According to the proof of Theorem 25, there exists a chart domain \( U \) around \( x \) such that

\[
\begin{align*}
\mathcal{M}_U & \equiv U \times \mathbb{O} \\
\mathfrak{A}_U & \equiv U \times \hat{\mathbb{A}} \\
T\mathcal{M}_U & \equiv U \times \mathbb{O} \times \mathcal{M} \times \mathbb{E} \\
\hat{T}\mathcal{M}_U & \equiv U \times \mathbb{O} \times \mathbb{A} \times \mathbb{E}.
\end{align*}
\]

Consider \((a_0, v_0) \in T_{m_0}\mathcal{M}\) where \( m = (x_0, e_0) \in \mathcal{M}_U \). Using local coordinates, if \( m_0 \equiv (x_0, e_0) \) and \((a_0, v_0) \equiv (a_0, v_0)\) we consider the section

\[
X : U \times \mathbb{O} \to U \times \mathbb{O} \times \mathbb{A} \times \mathbb{E}
\]
given by $X(x,e) = (x,e, a_0, v_0)$. Then, by construction, the corresponding local section $X \equiv X$ is a projectable section defined on $\mathcal{M}_U$.

(3) Under the assumptions of (3), note that the restriction $p_{K,\mathcal{M}}$ of $p_{\hat{A}}$ to $K,\mathcal{M}$ is an isomorphism onto $\hat{A}$. Let $X$ be a section of $T\mathcal{M}_U$. Thus the difference $X - X^K$ is a vertical section. Since the sum of two projectable sections is a projectable section, it follows that $X$ is projectable if and only if $X^K$ is projectable. □

**Notations 41** (1) **Sheaf $\mathcal{P}_\mathcal{M}$ of sections of $T\mathcal{M}$**: On the one hand, for any open set $\mathcal{U}$ of $\mathcal{M}$, if $U = p(\mathcal{U})$, then $\mathcal{U} \subset \mathcal{M}_U$ and so $\mathcal{P}(\mathcal{U}) := \{(X|_\mathcal{U}, X \in \mathcal{P}(\mathcal{M}_U))\}$. On the other hand, since the set of smooth sections of a Banach bundle defines a sheaf over its basis, it follows that the set $\{\mathcal{P}(\mathcal{U}), \mathcal{U} \text{ open set in } \mathcal{M}\}$ defines a sub-sheaf of modules $\mathcal{P}_\mathcal{M}$ of the sheaf of modules $\Gamma_\mathcal{M}$ of sections of $T\mathcal{M}$. Thus $\{\mathcal{P}(\mathcal{M}_U), \mathcal{U} \text{ open set in } \mathcal{M}\}$ generates a sheaf of modules $\mathcal{P}_\mathcal{M}$ on $\mathcal{M}$. According to Definition 10, in this context, when no confusion is possible, we simply denote by $[\ldots]_{T\mathcal{M}}$ the sheaf of brackets generated by $\{[\ldots]_{T\mathcal{M}_U}, U \text{ open set in } \mathcal{M}\}$.

(2) **Local version of the Lie bracket $[\ldots]_{T\mathcal{M}}$**: Each section $X$ in $\mathcal{P}(T\mathcal{M}_U)$ has a decomposition:

$$X = \sum_{i=1}^{p} f_iX_i$$

with $f_i \in C^\infty(\mathcal{M}_U)$ and $X_i$ are projectable sections for all $i \in \{1, \ldots, p\}$. We consider a chart domain $U$ in $\mathcal{M}$ for which the situation considered in the proof of Lemma 40 (2) is valid. In the associated local coordinates, for any section $X$ of $T\mathcal{M}_U$, we have $X(x,e) \equiv (x,e, a(x,e), z(x,e))$. Now, $X$ is projectable if and only if $a$ only depends on $x$. Under these notations, if $X' \equiv (x,e, a'(e), z'(x,e))$ is another projectable section, we have (cf. (26) and Notations 26):

$$[X, X']_{T\mathcal{M}}(x,e) \equiv (x,e, C_{x(a,a')}, d\rho_{x(a)}(\rho_{x(a)}), dz'_{x(a),z'})).$$

(28)

Now consider two sections $X$ and $X'$ in $\mathcal{P}(T\mathcal{M}_U)$. We can write $X = \sum_{i=1}^{p} f_iX_i$ and $X' = \sum_{j=1}^{q} f'_jX'_j$ where $(f_i, f'_j) \in C^\infty(\mathcal{M}_U)^2$ and $X_i$ and $X'_j$ are projectable sections for all $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$. Since the value of the Lie bracket $[X, X']_{T\mathcal{M}}$ at $m$ only depends on the 1-jet of $X$ and $X'$ and $m$, so this value does not depend on the previous decompositions. Now $X$ and $X'$ can be also written as a pair $(a, X)$ and $(a', X')$ respectively. Of course, if $m = (x, e)$, we have

$$a(m) = \sum_{i=1}^{p} f_i(m)a_i(x) \text{ and } a'(m) = \sum_{j=1}^{q} f'_j(m)a'_j(x)$$

$$X = \sum_{i=1}^{p} f_iX_i \text{ and } X = \sum_{i=1}^{p} f_iX_i.$$
In local coordinates, we then have
\[ X \equiv (a,z) = \left( \sum_{i=1}^{p} f_i(m)i, \sum_{i=1}^{p} f_i z_i \right) \]
\[ X' \equiv (a',z') = \left( \sum_{j=1}^{q} f'_j a'_j, \sum_{j=1}^{q} f_j z_j \right). \]

Thus the Lie bracket \([X, X']_{T\mathcal{M}}\) has the following expression in local coordinates:
\[
[X, X']_{T\mathcal{M}} \equiv (C(a,a') + da'(\rho_{x(a')} - d\rho_{x(a)}, z - d\rho_{x(a)}, z'))
\]
where \(C : U \rightarrow L_{alt}^2(\mathbb{A})\) is defined in local coordinates by the Lie bracket \([., .]_{A}\) and where
\[
C(a,a')(m) = \sum_{i=1}^{p} \sum_{j=1}^{q} f_i(m)f'_j(m)C_x(a_i(x), a'_j(x)).
\]

**Remark 42** According to Comments 38, when \(A\) is a finite rank vector bundle, then \(\mathfrak{P}(\mathcal{M}_U) = \Gamma(T\mathcal{M}_U)\) and so \([., .]_{T\mathcal{M}_U}\) is defined for all sections of \(\Gamma(T\mathcal{M}_U)\).

Now, from Lemma 40, \((\mathfrak{P}(T\mathcal{M}_U), [., .]_{T\mathcal{M}})\) is a Lie algebra and \(\hat{\rho}\) induces a Lie algebra morphism from \((\mathfrak{P}(T\mathcal{M}_U), [., .]_{T\mathcal{M}})\) to the Lie algebra of vector fields on \(\mathcal{M}_U\). In this way, so we get:

**Theorem 43** The sheaf \(\mathfrak{P}_M\) on \(M\) which gives rise to a strong partial convenient Lie algebroid on the anchored bundle \((T\mathcal{M}, \hat{\rho}, \mathcal{M}, \hat{\rho})\). Moreover, the restriction of the bracket \([., .]_{T\mathcal{M}}\) to the module of vertical sections induces a convenient Lie algebroid structure on the anchored subbundle \((\mathcal{V}\mathcal{M}, \hat{\rho}_{|\mathcal{V}\mathcal{M}}, \mathcal{M}, \hat{\rho})\) which is independent on the bracket \([., .]_{A}\).

From Remark 42, we obtain:

**Corollary 44** If \(A\) is a finite dimensional fiber Banach bundle, then \((T\mathcal{M}, \hat{\rho}, \mathcal{M}, \hat{\rho}, [., .]_{T\mathcal{M}})\) is a convenient Lie algebroid.

**Proof of Theorem 43** From Lemma 40, \((\mathfrak{P}(T\mathcal{M}_U), [., .]_{T\mathcal{M}})\) is a Lie algebra and \(\hat{\rho}\) induces a Lie algebra morphism from \((\mathfrak{P}(T\mathcal{M}_U), [., .]_{T\mathcal{M}})\) to the Lie algebra of vector fields on \(\mathcal{M}_U\). This implies the same properties for \((\mathfrak{P}(U), [., .]_{T\mathcal{M}})\) for any open set \(U\) in \(\mathcal{M}\). Thus we obtain a sheaf \(\mathfrak{P}_M\) of Lie algebras and \(\mathfrak{C}_M^\infty\) modules on \(\mathcal{M}\), which implies that \((T\mathcal{M}, \hat{\rho}, \mathcal{M}, \hat{\rho}, [., .]_{T\mathcal{M}})\) is a partial convenient Lie algebroid which is strong, according to Lemma 40 (2).

It remains to prove the last property. From its definition, the restriction of \(\hat{\rho}\) to the vertical bundle \(\mathcal{V}\mathcal{M} \rightarrow \mathcal{M}\) is an isomorphism onto the vertical bundle of the tangent bundle \(T\mathcal{M} \rightarrow \mathcal{M}\). Now, from the definition of the bracket of projectable sections, it
is clear that the bracket of two (local) vertical sections of \( T\mathcal{M} \to \mathcal{M} \) is also a (local) vertical section which is independent on the choice of \([..].\)\( A \).

**Remark 45** Given a fibred morphism \( \Psi : \mathcal{M} \to \mathcal{M}' \) between fibred manifold \( p : \mathcal{M} \to M \) and \( p' : \mathcal{M}' \to M' \) over \( \psi : M \to M' \) and a morphism of anchored bundles \( \varphi \) between two Lie algebroids \( (\mathcal{A}, \pi, M, \rho, [..].)\) and \( (\mathcal{A}', \pi', M', \rho', [..].)\), over \( \psi : M \to M' \). Then, from Remark 20, the prolongation \( T\Psi : T\mathcal{A}\mathcal{M} \to T\mathcal{A}'\mathcal{M}' \), is a Lie morphism of partial convenient Lie algebroids from \( (T\mathcal{A}\mathcal{M}, \mathcal{M}, \hat{\rho}, \Psi_1) \) to \( (T\mathcal{A}'\mathcal{M}', \mathcal{M}', \hat{\rho}', \Psi_1) \).

### 3.4 Derivative Operator on \( T\mathcal{M} \)

Consider an open set \( \mathcal{U} \) in \( \mathcal{M} \). We simply denote \( T\mathcal{U} \) the restriction of \( T\mathcal{M} \) to \( \mathcal{U} \). Note that \( \mathcal{U} = p(\mathcal{U}) \) is an open set in \( M \) and of course \( \mathcal{U} \) is contained in the open set \( \mathcal{M}_U \); so we have a natural restriction map from \( \Gamma(T\mathcal{M}_U) \) (resp. \( \mathcal{P}(T\mathcal{M}_U) \)) into \( \Gamma(T\mathcal{U}) \) (resp. \( \mathcal{P}(T\mathcal{U}) \)).

Since \( T\mathcal{M} \) has only a strong partial convenient Lie algebroid structure, by application of results of Sect. 2.6, we then have the following result:

**Theorem 46** Fix some open set \( \mathcal{U} \) in \( \mathcal{M} \).

1. Fix some projectable section \( u \in \mathcal{P}(T\mathcal{U}) \). For any \( k \)-form \( \omega \) on \( \mathcal{U} \) the Lie derivative \( L^\omega u \) is a well defined \( k \)-form on \( \mathcal{U} \).
2. For any \( k \)-form \( \omega \) on \( T\mathcal{U} \) the exterior derivative \( d^\rho \omega \) of \( \omega \) is well defined \((k + 1)\)-form on \( \mathcal{U} \).

### 3.5 Prolongations and Foliations

We assume that \( (\mathcal{A}, M, \rho, [..].) \) is a split Banach–Lie algebroid.

Under the upper assumptions, by application of Theorem 25 and Theorem 2 in [17], we obtain the following link between the foliation on \( M \) and the foliation on \( T\mathcal{M} \):

**Theorem 47** Assume that \( (\mathcal{A}, \pi, M, \rho, [..].) \) is split and the distribution \( \rho(\mathcal{A}) \) is closed.

1. The distribution \( \hat{\rho}(T\mathcal{A}) \) is integrable on \( T\mathcal{M} \).
2. Assume that \( \mathcal{M} = \mathcal{A} \). Let \( L \) be a leaf of \( \rho(\mathcal{A}) \) and \( (\mathcal{A}_L, \pi|_L, L, \rho_L, [..].) \) be the Banach–Lie algebroid which is the restriction of \( (\mathcal{A}, \pi, M, \rho, [..].) \). Then \( \hat{L} = \mathcal{A}_L = \pi^{-1}(L) \) is a leaf of \( \hat{\rho}(T\mathcal{A}) \) such that \( \hat{p}(\hat{L}) = L \).
3. In the previous context, the partial Banach–Lie algebroid which is the prolongation of \( (\mathcal{A}_L, \pi|_L, L, \rho_L, \mathcal{P}_A)_L \) over \( \hat{L} \) is exactly the restriction to \( \hat{L} \) of the partial Banach–Lie algebroid \( (T\mathcal{A}, \hat{\rho}, \mathcal{A}, \hat{\rho}, \mathcal{P}_A) \).

**Remark 48** (1) If there exists a weak Riemannian metric on \( \mathcal{A} \), the distribution \( \rho(\mathcal{A}) \) is closed and the assumptions of Theorem 47 are satisfied. These assumptions are always satisfied if \( \rho \) is a Fredholm morphism.
(2) The set of leaves of the foliations defined by $\hat{\rho}(T\mathcal{A})$ is

$$\{A_L, \text{ $L$ leaf of } \rho(\mathcal{A}) \}.$$  

**Proof** From Theorem 25, if $m = (x, e) \in \mathcal{A}$, the fibre $T_m\mathcal{M}$ can be identified with $\mathcal{A}_x \times \tilde{E}_m \equiv \mathbb{A} \times \mathbb{E}$, so we have $\hat{\rho}_m(a, v) \equiv (\rho_x(a), v)$. It follows that ker$(\hat{\rho}_m)$ can be identified with ker $\rho_x \times \mathbb{V}_m \mathcal{M} \subset \mathcal{A}_x \times T_m\mathcal{A}$. Thus, ker $\hat{\rho}_m$ is split if and only if ker $\rho_x$ is split. Moreover $\hat{\rho}_m(T_m\mathcal{A}) = \rho_x(A_x) \times \mathbb{V}_m \mathcal{M}$ is closed in $T_m\mathcal{M} \equiv \mathbb{M} \times \mathbb{E}$ if and only if $\rho_x(A_x)$ is closed in $T_x\mathcal{M}$. Then (1) will be a consequence of [4, Theorem 8.39] if we show that, for any $(x, e) \in \mathcal{M}$, there exists an open neighbourhood $U$ of $x$ in $M$ such that the set $\mathfrak{F}(T\mathcal{M}_U)$ is a generating upper set for $\hat{\rho}(T\mathcal{M})$ around $(x, e)$ (cf. [4, Theorem 8.38]) and satisfied the condition (LB) given in [4, § Integrability and Lie invariance].

The point $m = (x, e) \in \mathcal{M}$ is fixed and we choose a chart domain $U$ of $x \in M$ such that $\mathcal{M}_U$ and $\mathcal{A}_U$ are trivializable. Without loss of generality, according to the notations in Sect. 2.1, we may assume that $U \subset \mathbb{M}$, $\mathcal{M}_U = U \times \mathbb{O} \subset \mathbb{M} \times \mathbb{E}$, $\mathcal{A}_U = U \times \mathbb{A}$. Thus, according to the proof of Theorem 25, we have $T\mathcal{M}_U = U \times \mathbb{O} \times \mathbb{A} \times \mathbb{E}$. In this context, if $(x) \times \mathbb{A} = \ker \rho_x \oplus \mathbb{S}$, then

$$(\{x\}, \mathbb{E}) \times \mathbb{A} \times \mathbb{E} = [(x, e)] \times (\ker \rho_x \oplus \mathbb{S}) \times \mathbb{E}.$$  

Now consider the upper trivialization $\rho : U \times \mathbb{A} \rightarrow U \times \mathbb{M}(= T\mathcal{M}_U)$ and the associated upper trivialization:

$$\hat{\rho} : U \times \mathbb{O} \times \mathbb{A} \times \mathbb{E} \rightarrow U \times \mathbb{O} \times \mathbb{A} \times \mathbb{M}(= T\mathcal{M}_U).$$  

Then, from the definition of $\hat{\rho}$, any upper vector field is of type

$$X_{(a, v)} = \hat{\rho}(a, v) = (\rho(a), v)$$  

for any $(a, v) \in \mathbb{A} \times \mathbb{E}$. From the proof of Lemma 40 (2), it follows that such a vector field is the range, under $\hat{\rho}$, of a projectable section. Moreover, the stability of projectable sections under the Lie bracket $[..]_{T\mathcal{M}_U}$ and the fact that $\hat{\rho}$ induces a Lie algebra morphism from $\mathfrak{F}(T\mathcal{M}_U)$ into the Lie algebra of vector fields on $\mathcal{M}_U$ implies that condition (LB) is satisfied for the set $\mathfrak{F}(T\mathcal{M}_U)$.

Assume that $\mathcal{M} = \mathcal{A}$. Fix some leaf $L$ in $M$. If $(A_L, \pi_L, L, \rho_L, [..]_{A_L})$ is the Banach–Lie algebroid which is the restriction of $(\mathcal{A}, \pi, M, \rho, [..]_{\mathcal{A}})$ then $\pi^{-1}(L) = A_L$ and $\rho_L(A_L) = TL$. Now by construction, the prolongation $T\mathcal{A}_L$ of $A_L$ relative to the Banach–Lie algebroid $(A_L, \pi_L, L, \rho_L, [..]_{A_L})$ is characterized by

$$T_{(x, a)}A_L = \{(b, (v, y)) \in A_x \times T_{(x, a)}A_L : \rho_x(b) = y\}.$$  

It follows that $T\mathcal{A}_L$ is the restriction of $T\mathcal{A}$ to $A_L$ and also $\hat{\rho}(T\mathcal{A}_L) = T\mathcal{A}_L$. Since $L$ is connected, so is $A_L$ and then $A_L$ is a leaf of $\hat{\rho}(T\mathcal{M})$.  \qed
4 Projective Limits of Prolongations of Banach Lie Algebroids

Definition 49  Consider a projective sequence of Banach–Lie algebroid bundles
\[
\left( (\mathcal{A}_i, \pi_i, M_i, \rho_i, [..,]) , \left( (\xi^j_i, \xi^j_i, \delta^j_i) \right) \right)_{(i, j) \in \mathbb{N}^2, j \geq i}
\]
(resp. of Banach bundles \(\left( (\mathcal{E}_i, \mathbf{B}_i, M_i, \rho_i, [..,]) , \left( (\xi^j_i, \delta^j_i) \right) \right)_{(i, j) \in \mathbb{N}^2, j \geq i}\)).

A sequence of open fibred manifolds \(M_i\) of \(\mathcal{E}_i\) is called compatible with algebroid prolongations, if, for all \((i, j) \in \mathbb{N}^2\) such that \(j \geq i\), we have

(PSPLAB 1) \(\xi^j_i(M_j) \subset M_i\);

(PSPLAB 2) \(p_i \circ \xi^j_i = \delta^j_i \circ p_j\).

Under the assumptions of Definition 66, for each \(i \in \mathbb{N}\), we denote by \(\left( T M_i , \hat{\rho}_i , M_i , \rho_i \right)\) the prolongation of \(A_i\) over \(M_i\) and \([..,]_{T M_i}\) the prolongation of the Lie bracket \([..,]_{A_i}\) on projectable sections of \(T M_i\).

We then have the following result.

Proposition 50  Consider a projective sequence of Banach–Lie algebroid bundles
\[
\left( (\mathcal{A}_i, \pi_i, M_i, \rho_i, [..,]) , \left( (\xi^j_i, \xi^j_i, \delta^j_i) \right) \right)_{(i, j) \in \mathbb{N}^2, j \geq i}
\]
(resp. Banach bundles \(\left( (\mathcal{E}_i, \mathbf{B}_i, M_i, \rho_i, [..,]) , \left( (\xi^j_i, \delta^j_i) \right) \right)_{(i, j) \in \mathbb{N}^2, j \geq i}\)) and a sequence of open fibred manifolds \(M_i\) of \(\mathcal{E}_i\) compatible with algebroid prolongations. Then

1. \(\left( \overset{\leftarrow}{\lim} T M_i , \overset{\leftarrow}{\lim} \hat{\rho}_i , \overset{\leftarrow}{\lim} M_i , \overset{\leftarrow}{\lim} \hat{\rho} = \overset{\leftarrow}{\lim} \hat{\rho}_i \right)\) is a Fréchet anchored bundle which is the prolongation of \(\left( A = \overset{\leftarrow}{\lim} A_i , \pi = \overset{\leftarrow}{\lim} \pi_i , M = \overset{\leftarrow}{\lim} M_i \right)\) over \(M\).

2. Consider any of open \(U\) in \(M\) and a sequence of open set \(U_i\) in \(M_i\) such that \(U = \overset{\leftarrow}{\lim} U_i\). We denote by \(\overset{\leftarrow}{\lim} p^l (T M_U)\) the \(C^\infty (U)\)-module generated by all projective limits \(X = \overset{\leftarrow}{\lim} X_i\) of projectable sections \(X_i\) of \(T M_i\) over \(\{M_i\}_{U_i}\).

Then there exists a Lie bracket \([..,]_{T M_U}\) defined on \(\overset{\leftarrow}{\lim} (T M_U)\) which satisfies the assumptions of Definition 3 characterized by
\[
[X, X']_{T M_U} = \overset{\leftarrow}{\lim} [X_i, X'_i]_{T M_i}
\]
where \(X = \overset{\leftarrow}{\lim} X_i\) and \(X' = \overset{\leftarrow}{\lim} X'_i\).

Note that \(\overset{\leftarrow}{\lim} p^l (T M_U)\) is a submodule of the module \(\overset{\leftarrow}{\lim} p (T M_U)\) generated by projectable sections \(T M_U\). Therefore, by analog argument as used in the proof of Theorem 43, the set \(\{\overset{\leftarrow}{\lim} p^l (T M_U), U\\) open set in \(M\)\) generates a sheaf of \(\overset{\leftarrow}{\lim} p^l\) over \(M\). Moreover, for any open set \(U\) in \(M\), according to Proposition 50, the restriction of \(\hat{\rho}\) to each \(\overset{\leftarrow}{\lim} p^l (T M_U)\) is a Lie algebra morphism into the Lie algebra morphism into the Lie algebra of vector fields on \(U\). Thus we obtain:
Theorem 51 Consider a projective sequence of Banach–Lie algebroid bundles
\[
\left( (A_i, \pi_i, M_i, \rho_i, [\ldots]_{A_i}), \left( (\xi_i^j, \delta_i^j) \right) \right)_{(i,j) \in \mathbb{N}^2, j \geq i}
\]
(resp. Banach bundles \((\mathcal{E}_i, \mathcal{B}_i, M_i, \rho_i, [\ldots]_{\mathcal{E}_i}), \left( (\xi_i^j, \delta_i^j) \right) \) \((i,j) \in \mathbb{N}^2, j \geq i\)) and a sequence of open fibred manifolds \(\mathcal{M}_i\) of \(\mathcal{E}_i\) compatible with algebroid prolongations. Then
\[
\left( T\mathcal{M} = \lim_{\leftarrow} T\mathcal{M}_i, \hat{\mathcal{P}} = \lim_{\leftarrow} \hat{\mathcal{P}}_i, M = \lim_{\leftarrow} \mathcal{M}_i, \hat{\rho} = \lim_{\leftarrow} \hat{\rho}_i \right)\]

is a Fréchet anchored bundle which is the prolongation of \((A = \lim_{\leftarrow} A_i, \pi = \lim_{\leftarrow} \pi_i, M = \lim_{\leftarrow} M_i)\) over \(M\).
Moreover \((T\mathcal{M}, \hat{\mathcal{P}}, \hat{\mathcal{M}}, \hat{\rho}, \mathcal{P}^{pl}_{\mathcal{M}})\) is a strong partial Fréchet Lie algebroid.

**Remark 52** In general, since the projective limit of a projective sequence of Banach algebroids has only a structure of partial Fréchet Lie algebroid, it follows that \(\mathcal{P}^{pl}_{\mathcal{M}}\) is a subsheaf of modules of \(\mathcal{P}_{\mathcal{M}}\) and the inclusion is strict in the sense that for each open \(\mathcal{U}\) in \(\mathcal{M}\), the inclusion of \(\mathcal{P}^{pl}(\mathcal{U})\) in \(\mathcal{P}(\mathcal{U})\) is strict. Thus we do not have a structure of strong partial Fréchet Lie algebroid defined on \(\mathcal{P}_{\mathcal{M}}\) as in Theorem 43.

**Proof** (1) According to (PSPBLAB 1) and Theorem 67, \((\lim_{\leftarrow} A_i, \lim_{\leftarrow} \pi_i, \lim_{\leftarrow} M_i, \lim_{\leftarrow} \rho_i)\)

is a Fréchet anchored bundle.

From (PSPBLAB 2) and Proposition 65, we obtain a structure of Fréchet vector bundle on \((\lim_{\leftarrow} \mathcal{E}_i, \lim_{\leftarrow} \mathcal{P}_i, \lim_{\leftarrow} \mathcal{M}_i)\). Since each \(\mathcal{M}_i\) is an open manifold of \(\mathcal{E}_i\) such that the restriction of \(\mathcal{P}_i\) is a surjective fibration of \(\mathcal{M}_i\) over \(M_i\) and we have \(\xi_i^j (\mathcal{M}_j) \subset \mathcal{M}_i\), it follows that \((\mathcal{M}_i, \xi_i^j)_{(i,j) \in \mathbb{N}^2, j \geq i}\) is a projective sequence of Banach manifolds and so the restriction of \(\mathcal{P} = \lim_{\leftarrow} \mathcal{P}_i\) to \(\mathcal{M} = \lim_{\leftarrow} \mathcal{M}_i\) is a surjective fibration onto \(M\).

Recall that
\[
T\mathcal{M}_j = \{(a_j, \mu_j) \in \mathcal{A}_x \times T_{m_j} \mathcal{M}_j : \rho_j (a_j) = T_{m_j} \mathcal{P}_j (\mu_j)\}.
\]

Let \((a_j, \mu_j)\) be in \(T\mathcal{M}_j\) and consider \(a_i = \xi_i^j (a_j)\) and \(\mu_i = T \xi_i^j (\mu_j)\). We then have:
\[
\rho_i (a_i) = \rho_i \circ \xi_i^j (a_j) = T \delta_i^j \circ \rho_j (a_j)
\]

and also
\[
T_{m_j} \mathcal{P}_i (\mu_i) = T_{m_j} \mathcal{P}_i \circ T \xi_i^j (\mu_j) = T \delta_i^j \circ T_{m_j} \mathcal{P}_j (\mu_j).
\]

Since \(\rho_j (a_j) = T_{m_j} \mathcal{P}_j (\mu_j)\), we then obtain \(\rho_i (a_i) = T_{m_j} \mathcal{P}_i (\mu_i)\).
So $T\xi^j_i : TM_j \to TM_i$ is a morphism of Banach bundles and we have the following commutative diagram

\[
\begin{array}{c}
TM_i \\ \rho_i \downarrow \\
T\xi^j_i \\ \rho_j \downarrow \\
TM_j
\end{array}
\]

We deduce that $\left( (T^A_i M_i, \dot{p}_i, M_i, \dot{\rho}_i), \left( T\xi^j_i, \xi^j_i \right) \right)_{(i,j)\in \mathbb{N}^2, j \geq i}$ is a projective sequence of Banach anchored bundles. Applying again Theorem 67, we get a Fréchet anchored bundle structure on $(\lim_{i \to \infty} M_i, \lim_{i \to \infty} \dot{p}_i, \lim_{i \to \infty} \dot{\rho}_i)$ over $\lim_{i \to \infty} M_i$ and which appears as the prolongation of $\left( \lim_{i \to \infty} A_i, \lim_{i \to \infty} \pi_i, \lim_{i \to \infty} M_i, \lim_{i \to \infty} \rho_i \right)$ over $\lim_{i \to \infty} M_i$.

(2) Let $U$ be an open set in $M$. There exists $U_i$ in $M_i$ such that $\delta_i(U) \subset U_i$ for each $i \in \mathbb{N}$ and so that $U = \lim_{i \to \infty} U_i$. Now, from the definition of $\{M_i \}_U$, we must have $M_U = \lim_{i \to \infty} [M_i)_U$.

Recall that a projectable section $X$ on $\{M_i \}_U$ is characterized by a pair $(a_i, X_i)$ where $a_i$ is a section of $\{A_i \}_U$, and $X_i$ is a vector field on $\{M_i \}_U$ such that $\rho_i \circ a_i = Tp_i(X_i)$. Assume that $a = \lim a_i$ and $X = \lim X_i$, from the compatibility with bonding maps for sequences of sections $(a_i)$, anchors $(\rho_i)$ and vector fields $(X_i)$ we must have then $Tp(X) = \rho \circ a$. But from Theorem 25, it follows that $(a \circ p, X)$ defines a projectable section $X$ over $M_U$ and so $X = \lim X_i$.

On the other hand, recall that from (26), for each $i \in \mathbb{N}$, we have

\[
[X_i, X'_i]_{TM_i} = ([a_i, a'_i]_{A_i} \circ p, [X_i, X'_i])
\] (30)

Now since $\xi^j_i$ is a Lie algebroid morphism, over $\delta'_i$, according to Definition 19 we have

\[
\xi^j_i([a_j, a'_j]_{A_j})(x_j) = ([a_i, a_i]_{A_i}) (\delta^j_i(x_j)).
\] (31)

Since $\delta^j_i \circ p_j = p_i \circ \lambda^j_i$, we have:

\[
\xi^j_i([a_j, a'_j]_{A_j}) \circ p_j(m_j) = ([a_i, a'_i]_{A_i}) \circ p_i \circ \xi^j_i(m_j).
\] (32)

Naturally, since $X_i$ (resp. $X'_i$) and $X_j$ (resp. $X'_j$) are $\xi^j_i$ related, we also have

\[
[X_i, X'_i] \left( \xi^j_i(m_j) \right) = T\xi^j_i \left( [X_j, X_j] \right)(m_j).
\] (33)
From (30) and (22) we then obtain

\[ T \xi_j^i \left( [X_i, X'_j]_{T_M} \right) (m_j) = \left( [a_i, a'_j]_{A_i} \circ p_i(m_j), T m_j \xi_j^i([X_i, X'_j]) \right) \]

\[ = \left( [X_i, X'_j]_{T_M} \right) \circ \xi_j^i(m_j). \]  

(34)

It follows that we can define:

\[ [X, X']_{T_MU} = \lim_{\leftarrow} [X_i, X'_i]_{T_M}. \]  

(35)

Now, since each bracket \([..]_{T_M} \) satisfies the Jacobi identity, from (35), the same is true for \([..]_{T_M} \) on projective limit \( X \) and \( X' \) of projective sections \((X_i)\) and \((X'_i)\). Finally, as

\[ [\hat{\rho}_i(X_i), \hat{\rho}_i(X'_i)]_{T_M} = \hat{\rho}_i ([X_i, X_i]_{T_M}). \]  

(36)

from the compatibility with bonding maps for sequences of sections \((a_i)\), anchors \((\rho_i)\) vector fields \((X_i)\) and Lie brackets \([..]_{T_M} \) on projective sequence \((T_M)\), it follows that \(\hat{\rho}\) satisfies the same type of relation as (36).

Now using by same arguments as used in the proof of Lemma 40 we show that we can extend this bracket to a Lie bracket on the module \(P^{pl}(T_M U)\) so that we have a Lie algebra and the restriction of \(\hat{\rho}\) to this Lie algebra is a morphism of Lie algebra into the Lie algebra of vector fields on \(M U\).

5 Direct Limits of Prolongations of Banach Lie Algebroids

As in the previous section we introduce:

**Definition 53** Consider a direct sequence of Banach–Lie algebroid bundles

\[ \left( \left( A_i, \pi_i, M_i, \rho_i, [..]_{A_i}, \left( \eta_i^j, \chi_i^j, \epsilon_i^j \right) \right), \left( \pi_i^j, \chi_i^j, \delta_i^j \right) \right)_{(i,j) \in \mathbb{N}^2, i \leq j} \]

(resp. of Banach bundles \(\left( \left( E_i, \mathbf{B}_i, M_i \right), \left( \eta_i^j, \chi_i^j, \epsilon_i^j \right) \right)_{(i,j) \in \mathbb{N}^2, i \leq j}\).

A sequence of open fibred manifolds \(M_i\) of \(E_i\) is called compatible with algebroid prolongations, if for all \((i, j) \in \mathbb{N}^2\) such that \(i \leq j\), we have

(DSPBLAB 1) \(\chi_i^j(M_j) \subset M_i\);  
(DSPBLAB 2) \(\epsilon_i^j \circ p_i = p_j \circ \chi_i^j\).

**Remark 54** The context of direct limit in which we work concerns ascending sequences of Banach manifolds \((M_i)_{i \in \mathbb{N}}\) where \(M_i\) is a closed submanifold of \(M_{i+1}\). The reason of this assumption is essentially that their direct limit has a natural structure of \((n.n.H)\) convenient manifold.
Although each manifold $M_i$ is open in $E_i$, since $E_i$ is a closed subbundle of $E_j$, it follows that $(M_i, X^j_i)_{(i,j) \in \mathbb{N}^2, i \leq j}$ is an ascending sequence of convenient manifolds.

As in the previous section, for each $i \in \mathbb{N}$, we denote by $(\mathcal{A}_i, \hat{\pi}_i, \hat{A}_i, \hat{\rho}_i)$ the prolongation of $\mathcal{A}_i$ over $M_i = A_i$ and $[\ldots]_{\mathcal{T}\mathcal{A}_i}$ the prolongation of the Lie bracket $[\ldots]_{\mathcal{A}_i}$ on projectable section of $\mathcal{T}\mathcal{A}_i$.

Adapting the argument used in proof of Proposition 50 to this setting of strong ascending sequences and direct limits, we have the result below. Note that, in this context, the prolongation is not Hausdorff in general. However, all the arguments used in the proofs are local and so they still work in this context.

**Proposition 55** Consider a direct sequence of Banach–Lie algebroid bundles

$$(\mathcal{A}_i, \pi_i, M_i, \rho_i, [\ldots]_{\mathcal{A}_i}), (\eta^j_i, \xi^j_i, \varepsilon^j_i))_{(i,j) \in \mathbb{N}^2, i \leq j}$$

(resp. Banach bundles $((E_i, B_i, M_i, \rho_i, [\ldots])_i), (\xi^j_i, \varepsilon^j_i))_{(i,j) \in \mathbb{N}^2, i \leq j}$) and a sequence of open fibred manifolds $M_i$ of $E_i$ compatible with algebroid prolongations. Then

(1) $\left(\lim_{\to} \mathcal{T}M_i, \lim_{\to} \hat{\pi}_i, \lim_{\to} M_i, \lim_{\to} \hat{\rho}_i\right)$ is a convenient anchored bundle which is the prolongation of $\left(\lim_{\to} \mathcal{A}_i, \lim_{\to} \pi_i, M_i\right)$ over $\lim_{\to} M_i$.

(2) Consider any open set $U$ in $M$ and a sequence of open sets $U_i$ in $M_i$ such that $U = \lim_{\to} U_i$. We denote by $\mathfrak{D}^d(\mathcal{T}M_U)$ the $C^\infty(U)$-module generated by all direct limits $\mathfrak{X} = \lim_{\to} \mathfrak{X}_i$ of projectable sections $\mathfrak{X}_i$ of $\mathcal{T}M_i$ over $\{M_i\}_{U_i}$.

Then there exists a Lie bracket $[\ldots]_{\mathcal{T}M_U}$ defined on $\mathfrak{D}^p(\mathcal{T}M_U)$ which satisfies the assumptions of Definition 3 characterized by

$$[\mathfrak{X}, \mathfrak{X}']_{\mathcal{T}M_U} = \lim_{\to} [\mathfrak{X}_i, \mathfrak{X}'_i]_{\mathcal{T}M_i}$$

where $\mathfrak{X} = \lim_{\to} \mathfrak{X}_i$ and $\mathfrak{X}' = \lim_{\to} \mathfrak{X}'_i$.

As in the context of Projective sequences, $\mathfrak{D}^d(\mathcal{T}M_U)$ is a submodule of the module $\mathfrak{P}(\mathcal{T}M_U)$ generated by projectable sections $\mathcal{T}M_U$. Therefore, again by analog argument as used in the proof of Theorem 43, the set $\{\mathfrak{D}^d(\mathcal{T}M_U), U \text{ open set in } M\}$ generates a sheaf of $\mathfrak{D}^d_M$ over $\mathcal{M}$. Moreover, for any open set $\mathcal{U}$ in $\mathcal{M}$, according to Proposition 55, the restriction of $\hat{\rho}$ to each $\mathfrak{D}^d(\mathcal{T}\mathcal{U})$ is a Lie algebra morphism into the Lie algebra morphism into the Lie algebra of vector fields on $\mathcal{U}$. Thus we obtain:

**Theorem 56** Consider a direct sequence of Banach–Lie algebroid bundles

$$(\mathcal{A}_i, \pi_i, M_i, \rho_i, [\ldots]_{\mathcal{A}_i}), (\eta^j_i, \xi^j_i, \varepsilon^j_i))_{(i,j) \in \mathbb{N}^2, j \geq i}$$

(resp. Banach bundles $((E_i, B_i, M_i, \rho_i, [\ldots])_i), (\xi^j_i, \varepsilon^j_i))_{(i,j) \in \mathbb{N}^2, i \leq j}$) and a sequence of open fibred manifolds $M_i$ of $E_i$ compatible with algebroid prolongations. Then
\[(\lim T \mathcal{M}_i, \lim \hat{\mathcal{P}}_i, \lim \mathcal{M}_i, \lim \hat{\rho}_i)\] is a convenient anchored bundle which is the prolongation of \[(\lim \mathcal{A}_i, \lim \pi_i, \mathcal{M}_i)\] over \[\lim \mathcal{M}_i\]. Moreover \[(T \mathcal{M}, \hat{\mathcal{P}}, \mathcal{M}, \hat{\rho}, \mathcal{P}^d_M)\] is a strong partial convenient Lie algebroid.

**Appendix A: Projective Limits**

**Projective Limits of Topological Spaces**

**Definition 57** A projective sequence of topological spaces \( (\{(X_i, \delta^j_i)\}_{(i,j) \in \mathbb{N}^2, j \geq i}) \) is a sequence where

- (PSTS 1) For all \( i \in \mathbb{N} \), \( X_i \) is a topological space;
- (PSTS 2) For all \( (i, j) \in \mathbb{N}^2 \) such that \( j \geq i \), \( \delta^j_i : X_j \to X_i \) is a continuous map;
- (PSTS 3) For all \( i \in \mathbb{N} \), \( \delta^i_i = \text{Id}_{X_i} \);
- (PSTS 4) For all \( (i, j, k) \in \mathbb{N}^3 \) such that \( k \geq j \geq i \), \( \delta^j_i \circ \delta^k_j = \delta^k_i \).

**Notation 58** For the sake of simplicity, the projective sequence \( (\{(X_i, \delta^j_i)\}_{(i,j) \in \mathbb{N}^2, j \geq i}) \) will be denoted \( (X_i, \delta^j_i)_{j \geq i} \).

An element \( (x_i)_{i \in \mathbb{N}} \) of the product \( \prod_{i \in \mathbb{N}} X_i \) is called a thread if, for all \( j \geq i \), \( \delta^j_i (x_j) = x_i \).

**Definition 59** The set \( X = \lim X_i = \lim_{\to} X_i \) of all threads, endowed with the finest topology for which all the projections \( \delta^j_i : X \to X_i \) are continuous, is called the projective limit of the sequence \( (X_i, \delta^j_i)_{j \geq i} \).

A basis of a topology of the topology of \( X \) is constituted by the subsets \( (\delta^i_i)^{-1} (U_i) \) where \( U_i \) is an open subset of \( X_i \) (and so \( \delta^i_i \) is open whenever \( \delta^i_i \) is surjective).

**Definition 60** Let \( (X_i, \delta^j_i)_{j \geq i} \) and \( (Y_i, \gamma^j_i)_{j \geq i} \) be two projective sequences whose respective projective limits are \( X \) and \( Y \).

A sequence \( (f_i)_{i \in \mathbb{N}} \) of continuous mappings \( f_i : X_i \to Y_i \), satisfying, for all \( (i, j) \in \mathbb{N}^2, j \geq i \), the coherence condition

\[ \gamma^j_i \circ f_j = f_i \circ \delta^j_i \]

is called a projective sequence of mappings.
The projective limit of this sequence is the mapping

\[ f : X \rightarrow Y \]

\[
(x_i)_{i \in \mathbb{N}} \mapsto (f_i(x_i))_{i \in \mathbb{N}}
\]

The mapping \( f \) is continuous if all the \( f_i \) are continuous (cf. [1]).

**Projective Limits of Banach Spaces**

Consider a projective sequence \( \left( E_i, \delta^j_i \right)_{j \geq i} \) of Banach spaces.

**Remark 61** Since we have a countable sequence of Banach spaces, according to the properties of bonding maps, the sequence \( \left( \delta^j_i \right)_{(i,j) \in \mathbb{N}^2, j \geq i} \) is well defined by the sequence of bonding maps \( \left( \delta^j_i + 1 \right)_{i \in \mathbb{N}} \).

**Projective Limits of Differential Maps**

The following proposition (cf. [9, Lemma 1.2] and [4, Chap. 4]) is essential

**Proposition 62** Let \( \left( E_i, \delta^j_i \right)_{j \geq i} \) be a projective sequence of Banach spaces whose projective limit is the Fréchet space \( F = \lim \leftarrow E_i \) and \((f_i : E_i \rightarrow E_i)_{i \in \mathbb{N}}\) a projective sequence of differential maps whose projective limit is \( f = \lim \leftarrow f_i \). Then the following conditions hold:

1. \( f \) is smooth in the convenient sense (cf. [13])
2. For all \( x = (x_i)_{i \in \mathbb{N}} \), \( df_x = \lim \leftarrow (df_i)_{x_i} \).
3. \( df = \lim \leftarrow df_i \).

**Projective Limits of Banach Manifolds**

**Definition 63** The projective sequence \( \left( M_i, \delta^j_i \right)_{j \geq i} \) is called projective sequence of Banach manifolds if

1. **(PSBM 1)** \( M_i \) is a manifold modelled on the Banach space \( \mathbb{M}_i \);  
2. **(PSBM 2)** \( \left( \mathbb{M}_i, \delta^j_i \right)_{j \geq i} \) is a projective sequence of Banach spaces;  
3. **(PSBM 3)** For all \( x = (x_i) \in M = \lim \leftarrow M_i \), there exists a projective sequence of local charts \( (U_i, \xi_i)_{i \in \mathbb{N}} \) such that \( x_i \in U_i \) where one has the relation

\[
\xi_i \circ \delta^j_i = \delta^j_i \circ \varphi_j;
\]

4. **(PSBM 4)** \( U = \lim \leftarrow U_i \) is a non empty open set in \( M \).
Under the assumptions (PSBM 1) and (PSBM 2) in Definition 63, the assumptions (PSBM 3) and (PSBM 4) around \( x \in M \) is called the projective limit chart property around \( x \in M \) and \( (U = \lim U_i, \phi = \lim \phi_i) \) is called a projective limit chart.

The projective limit \( M = \lim M_i \) has a structure of Fréchet manifold modelled on the Fréchet space \( \mathbb{M} = \lim \mathbb{M}_i \) and is called a PLB-manifold. The differentiable structure is defined via the charts \( (U, \varphi) \) where \( \varphi = \lim \xi_i : U \to (\xi_i(U_i))_{i \in \mathbb{N}} \).

\( \varphi \) is a homeomorphism (projective limit of homeomorphisms) and the charts changings \( (\psi \circ \varphi^{-1})_{|\varphi(U)} = \lim \left( (\psi_i \circ (\xi_i)^{-1})_{|\xi_i(U_i)} \right) \) between open sets of Fréchet spaces are smooth in the sense of convenient spaces.

### Projective Limits of Banach Vector Bundles

Let \( \left( M_i, \delta_i^j \right)_{j \geq i} \) be a projective sequence of Banach manifolds where each manifold \( M_i \) is modelled on the Banach space \( \mathbb{M}_i \).

For any integer \( i \), let \( (E_i, \pi_i, M_i) \) be the Banach vector bundle whose type fibre is the Banach vector space \( \mathbb{E}_i \) where \( \left( \mathbb{E}_i, \lambda_i^j \right)_{j \geq i} \) is a projective sequence of Banach spaces.

**Definition 64** \( \left( (E_i, \pi_i, M_i), \left( \xi_i^j, \delta_i^j \right) \right)_{j \geq i} \), where \( \xi_i^j : E_j \to E_i \) is a morphism of vector bundles, is called a projective sequence of Banach vector bundles of projective sequence of Banach vector bundles if, for all \( (x_i) \), there exists a projective sequence of trivializations \( (U_i, \tau_i) \) of \( (E_i, \pi_i, M_i) \), where \( \tau_i : (\pi_i)^{-1}(U_i) \to U_i \times \mathbb{E}_i \) are local diffeomorphisms, such that \( x_i \in U_i \) (open in \( M_i \)) and where \( U = \lim U_i \) is a non empty open set in \( M \) where, for all \( (i, j) \in \mathbb{N}^2 \) such that \( j \geq i \), we have the compatibility condition

\[
(\text{PLVBV}) \left( \delta_i^j \times \lambda_i^j \right) \circ \tau_j = \tau_i \circ \xi_i^j.
\]

With the previous notations, \( (U = \lim U_i, \tau = \lim \tau_i) \) is called a projective bundle chart limit of projective bundle chart limit. The triple of projective limit \( (E = \lim E_i, \pi = \lim \pi_i, M = \lim M_i) \) is called a projective limit of Banach bundles or PLB-bundle for short.

The following proposition generalizes the result of [9] about the projective limit of tangent bundles to Banach manifolds (cf. [4, 8]).

**Proposition 65** Let \( \left( (E_i, \pi_i, M_i), \left( \xi_i^j, \delta_i^j \right) \right)_{j \geq i} \) be a projective sequence of Banach vector bundles. Then \( \left( \lim E_i, \lim \pi_i, \lim M_i \right) \) is a Fréchet vector bundle.

**Definition 66** \( \left( (E_i, \pi_i, M_i, \rho_i, [\ldots]), \left( \xi_i^j, \delta_i^j \right) \right)_{(i, j) \in \mathbb{N}^2 \!, \, j \geq i} \) is called a projective sequence of Lie algebroids of projective sequence of Lie algebroids if...
\( (\text{PSBLA} \ 1) \) \((E_i, \xi^i_j)_{j \geq i}\) \ is a projective sequence of Banach vector bundles \((\pi_i : E_i \to M_i)_{i \in \mathbb{N}}\) over the projective sequence of manifolds \((M_i, \delta^i_j)_{j \geq i}\).

\( (\text{PSBLA} \ 2) \) For all \((i, j) \in \mathbb{N}^2\) such that \(j \geq i\), one has
\[
\rho_i \circ \xi^i_j = T \delta^i_j \circ \rho_j
\]

\( (\text{PSBLA} \ 3) \) \(\xi^i_j : E_j \to E_i\) is a Lie morphism from \((E_j, \pi_j, M_j, \rho_j)\) to \((E_i, \pi_i, M_i, \rho_i)\).

We then have the following result (cf. [4]):

**Theorem 67** Let \(\left(\left(E_i, \pi_i, M_i, \rho_i, [\ldots, .]_i\right), (\xi^i_j, \delta^i_j)\right)_{(i,j) \in \mathbb{N}^2, j \geq i}\) be a projective sequence of Banach–Lie algebroids. If \((M_i, \delta^i_j)_{(i,j) \in \mathbb{N}^2, j \geq i}\) is a submersive projective sequence, then \((\lim_{\leftarrow} E_i, \lim_{\leftarrow} \pi_i, \lim_{\leftarrow} M_i, \lim_{\leftarrow} \rho_i)\) is a strong partial Fréchet Lie algebroid.

**Appendix B: Direct Limits**

**Direct Limits of Topological Spaces**

Let \(\left\{\left(Y_i, \varepsilon^i_j\right)\right\}_{(i,j) \in I^2, i \leq j}\) be a direct system of topological spaces and continuous maps. The direct limit \(\left\{(X_i, \varepsilon_i)\right\}_{i \in I}\) of the sets becomes the direct limit in the category \(\text{TOP}\) of topological spaces if \(X\) is endowed with the direct limit topology (DL-topology for short)DL-topology, i.e. the final topology with respect to the inclusion maps \(\varepsilon_i : X_i \to X\) which corresponds to the finest topology which makes the maps \(\varepsilon_i\) continuous. So \(O \subset X\) is open if and only if \(\varepsilon_i^{-1}(O)\) is open in \(X_i\) for each \(i \in I\).

**Definition 68** ascending sequence of topological spaces Let \(S = \left(\left(X_n, \varepsilon^m_n\right)\right)_{(m,n) \in \mathbb{N}^2, n \leq m}\) be a direct sequence of topological spaces such that each \(\varepsilon^m_n\) is injective. Without loss of generality, we may assume that we have
\[
X_1 \subset X_2 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots
\]
and \(\varepsilon^{n+1}_n\) becomes the natural inclusion.

**\(\text{ASTS}\) S** will be called an ascending sequence of topological spaces.

**\(\text{SASTS}\)** Moreover, if each \(\varepsilon^m_n\) is a topological embedding, then we will say that \(S\) is a strict ascending sequence of topological spaces (expanding sequence)

**Notation 69** The direct sequence \(\left(\left(X_n, \varepsilon^m_n\right)\right)_{(m,n) \in \mathbb{N}^2, n \leq m}\) will be denoted \(\left(X_n, \varepsilon^m_n\right)_{n \leq m}\).

If \(\left(X_n, \varepsilon^m_n\right)_{n \leq m}\) is a strict ascending sequence of topological spaces, each \(\varepsilon_n\) is a topological embedding from \(X_n\) into \(X = \lim_{\rightarrow} X_n\).
**Direct Limits of Ascending Sequences of Banach Manifolds**

**Definition 70.** Let $\mathcal{M} = (M_n, \epsilon_n^{n+1})_{n \in \mathbb{N}}$ be a sequence of Banach manifolds if, for any $n \in \mathbb{N}$, $(M_n, \epsilon_n^{n+1})$ is a submanifold of $M_{n+1}$.

**Proposition 71.** Let $\mathcal{M} = \lim_{\rightarrow} M_n$ be an ascending sequence of Banach manifolds. Assume that for $x \in M = \lim_{\rightarrow} M_n$, there exists a sequence of charts $((U_n, \phi_n))_{n \in \mathbb{N}}$ of $(M_n)_{n \in \mathbb{N}}$, such that:

1. $(U_n)_{n \in \mathbb{N}}$ is an ascending sequence of chart domains;
2. $\forall n \in \mathbb{N}$, $\phi_{n+1} \circ \epsilon_n^{n+1} = \iota_{n+1} \circ \phi_n$.

Then $U = \lim_{\leftarrow} U_n$ is an open set of $M$ endowed with the DL-topology and $\phi = \lim_{\rightarrow} \phi_n$ is a well defined map from $U$ to $\mathbb{M} = \lim_{\rightarrow} M_n$. Moreover, $\phi$ is a continuous homeomorphism from $U$ onto the open set $\phi(U)$ of $\mathbb{M}$.

**Definition 72.** We say that an ascending sequence $\mathcal{M} = (M_n, \epsilon_n^{n+1})_{n \in \mathbb{N}}$ of Banach manifolds has the direct limit chart property (DLCP) at $x$ if it satisfies both (ASC 1) and (ASC 2).

We then have the fundamental result (cf. [6]).

**Theorem 73.** Let $(M_n)_{n \in \mathbb{N}}$ be an ascending sequence of Banach $C^\infty$-manifolds, modelled on the Banach spaces $\mathbb{M}_n$. Assume that

1. $(M_n)_{n \in \mathbb{N}}$ has the direct limit chart property (DLCP) at each point $x \in M = \lim_{\rightarrow} M_n$;
2. $\mathbb{M} = \lim_{\rightarrow} \mathbb{M}_n$ is a convenient space.

Then there is a unique n.n.H. convenient manifold structure on $M = \lim_{\rightarrow} M_n$ modelled on the convenient space $\mathbb{M}$ such that the topology associated to this structure is the DL-topology on $M$.

In particular, for each $n \in \mathbb{N}$, the canonical injection $\epsilon_n : M_n \rightarrow M$ is an injective conveniently smooth map and $(M_n, \epsilon_n)$ is a submanifold of $M$.

Moreover, if each $M_n$ is locally compact or is open in $M_{n+1}$ or is a paracompact Banach manifold closed in $M_{n+1}$, then $M = \lim_{\rightarrow} M_n$ is provided with a Hausdorff convenient manifold structure.

**Direct Limits of Banach Vector Bundles**

**Definition 74.** Let $\mathcal{E} = (E_n, \pi_n, M_n), (\lambda_{n+1}, \iota_{n+1})_{n \in \mathbb{N}}$ be called a strong ascending sequence of Banach vector bundles if the following assumptions are satisfied:
\[
M = (M_n)_{n \in \mathbb{N}} \text{ is an ascending sequence of Banach } C^\infty\text{-manifolds, where } M_n \text{ is modelled on the Banach space } M_n \text{ such that } M_n \text{ is a supplemented Banach subspace of } M_{n+1} \text{ and the inclusion } \varepsilon_n^{n+1} : M_n \to M_{n+1} \text{ is a } C^\infty \text{ injective map such that } (M_n, \varepsilon_n^{n+1}) \text{ is a closed submanifold of } M_{n+1};
\]

\[\text{(ASBVB 2) The sequence } (E_n)_{n \in \mathbb{N}} \text{ is an ascending sequence such that the sequence of typical fibres } (E_n)_{n \in \mathbb{N}} \text{ is an ascending sequence of Banach spaces and } E_n \text{ is a supplemented Banach subspace of } E_{n+1};\]

\[\text{(ASBVB 3) For each } n \in \mathbb{N}, \pi_n^{n+1} \circ \lambda_n^{n+1} = \varepsilon_n^{n+1} \circ \pi_n \text{ where } \lambda_n^{n+1} : E_n \to E_{n+1} \text{ is the natural inclusion;}\]

\[\text{(ASBVB 4) Any } x \in M = \varinjlim M_n \text{ has the direct limit chart property } (\text{DLCP}) \text{ for } (U = \varinjlim U_n, \phi = \varinjlim \phi_n);\]

\[\text{(ASBVB 5) For each } n \in \mathbb{N}, \text{ there exists a trivialization } \Psi_n : (\pi_n)^{-1} (U_n) \to U_n \times E_n \text{ such that, for any } i \leq j, \text{ the following diagram is commutative:}\]

\[
\begin{array}{ccc}
\pi_i^{-1} (U_i) & \xrightarrow{\lambda_i^j} & \pi_j^{-1} (U_j) \\
\downarrow \Psi_i & & \downarrow \Psi_j \\
U_i \times E_i & \xrightarrow{\varepsilon_i^j \times \iota_i^j} & U_j \times E_j
\end{array}
\]

For example, the sequence \( ((TM_n, \pi_n, M_n), (T\varepsilon_n^{n+1}, \varepsilon_n^{n+1}))_{n \in \mathbb{N}} \) is a strong ascending sequence of Banach vector bundles whenever \((M_n)_{n \in \mathbb{N}} \) is an ascending sequence which has the direct limit chart property at each point of \( x \in M = \varinjlim M_n \) whose model \( M_n \) is supplemented in \( M_{n+1} \).

\textbf{Notation 75} From now on and for the sake of simplicity, the strong ascending sequence of vector bundles \( ((E_n, \pi_n, M_n), (\lambda_n^{n+1}, \varepsilon_n^{n+1}))_{n \in \mathbb{N}} \) will be denoted \( (E_n, \pi_n, M_n) \) _\rightarrow_.

We then have the following result given in [6].

\textbf{Proposition 76} Let \((E_n, \pi_n, M_n) \) _\rightarrow_ be a strong ascending sequence of Banach vector bundles. We have:

1. \( \varinjlim E_n \) has a structure of not necessarily Hausdorff convenient manifold modelled on the LB-space \( \varinjlim M_n \times \varinjlim E_n \) which has a Hausdorff convenient structure if and only if \( M \) is Hausdorff.
2. \( (\varinjlim E_n, \varinjlim \pi_n, \varinjlim M_n) \) can be endowed with a structure of convenient vector bundle whose typical fibre is \( \varinjlim E_n \).

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