On solving nonlinear differential-functional equations of mechanics and other applications

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Abstract: The paper presents the results of a study aimed at creating a method for solving the Cauchy problem and the stationary problem for equations containing a linear differential operator with constant coefficients and a nonlinear analytical functional. The method of Volterra-Frechet series is used to obtain a stationary solution of a second-order differential equation with an analytical functional describing the oscillations of a nonlinear vibrator with hereditary behavior. The behavior of the solution and the dependence of energy losses on physical parameters are studied.

1. Introduction

When designing modern technical systems in electronics, radio engineering, and materials science, models describing the state of the system in the form of nonlinear functionals that determine the relationship between input and output variables are widely used to study their behavior over time and when the load changes

\[ y(r_2, t) = \hat{F}[x(r_1, r), t] + \hat{L}x, \quad r \in (-\infty, t), \]  

(1)

where \( \hat{L} \) is a differential operator, the functional \( \hat{F} \) uniquely maps the set of input variables \( x \) to the set of output variables \( y \), and these variables can be scalar, vector, or tensor in nature, \( r_1 \) and \( r_2 \) are parameters. The functional defining this dependence must satisfy the conditions of causality and decaying memory. For stationary systems, there is no differential operator on the right side of (1). Relation (1) indicates that the value of the output variable at a time \( t \) is determined by the behavior of the input variable at all previous moments up to the time point \( t \).

Examples of functional devices used in electronics are piezoelectric resonators, Hall sensors, negatrons, and other devices where the set of discrete components of the device can be replaced by a medium that performs the functions of this set [1]. Similarly, states and processes in radio engineering systems are described using nonlinear functionals, for example, antennas with a nonlinear load [2]. As is known [3], the linear relationship between the polarization vectors of the medium and the electric field strength is not suitable for describing the results of observation in condensed media, when the strength reaches the order of interatomic fields, which can be obtained in modern power plants by focusing laser beams. To describe optical effects with high field strengths, the material equation is represented by the sum of Volterra-type multilinear functionals. That is, the defining functional was approximated by the analytic functional.
The analysis of experiments on the creep of fiberglass, a number of polymers, metals and alloys at elevated temperatures shows that the creep curves are not described by the relations of the linear theory of viscoelasticity, but are rather well approximated by the nonlinear-hereditary dependence between stresses and deformations [4]. In the one-dimensional case, such a dependence is represented by a decomposition of the analytical functional into a Volterra-Frechet series

$$\sigma(t) = \sum_{n=1}^{\infty} \int_{0}^{t} V_n(t_{1}, \ldots, t_{n}) \prod_{m=1}^{n} \varepsilon(t-t_{m}) dt_{m},$$

where $\sigma(t)$, $\varepsilon(t)$ – are stress and strain, $V_n(t_1, \ldots, t_n)$ are n-th order relaxation kernels satisfying the conditions of causality, damped memory, and symmetry by permutations of arguments [4]. In the case of a steady-state process, the upper limit of the integrals must be equal to infinity. It is shown in [5] that if the kernel $V_n(t)$ the additive contains an $\delta$ -function, then the relation (2) can be reversed

$$\varepsilon(t) = \sum_{n=1}^{\infty} \hat{W}_n \sigma, \quad \hat{W}_n \sigma = \int_{0}^{t} W_n(t_{1}, \ldots, t_{n}) \prod_{m=1}^{n} \sigma(t-t_{m}) dt_{m},$$

where the creep kernels $W_n(t_1, \ldots, t_n)$ are uniquely defined in terms of relaxation kernels $V_k(t_1, \ldots, t_k)$. The problem of constructing the inverse relation for (2) is completed by the system of recurrent equations obtained in [6], which allows us to determine the Laplace images of creep nuclei $W^*_n(p_1, \ldots, p_n)$ through the Laplace images of relaxation nuclei $V_k^*(p_1, \ldots, p_k)$:

$$W^*_n(pV_1^*(p)) = 1,$$

$$W^*_n(p_1, \ldots, p_n) V_1^\prime \left( \sum_{m=1}^{n} p_m \right) = -\frac{1}{n} \sum_{n} \sum_{m=1}^{n} W^*_j(p_1, \ldots, p_j) \cdots W^*_m(p_{[m-1]+1}, \ldots, p_n) \times$$

$$\times V_m^\prime \left( \sum_{k=1}^{p_m} p_{[k]} \cdots \sum_{k-\varphi=\cdot \cdot \cdot}^{n} \right),$$

where $|J, m| = j_1 + \ldots + j_m$, the internal sum is taken from the natural solutions of the equation $|J, m| = n$, the outer sum is taken by a cyclic permutation of the indices.

2. Construction of solutions of differential equations with analytical operators

Nonlinear integro-differential equation describing the change in the state of a functional system

$$(\hat{L}_n + \hat{F}^\prime) x(t) = y(t),$$

$$\hat{L}_n = \sum_{k=0}^{N} c_{n-k} \frac{d^{n-k}}{dt^{n-k}}, \quad c_k = \text{const},$$

$\hat{F}^\prime$ – nonlinear analytical operator represented by the Volterra-Frechet expansion into a series of multilinear integral operators of the form (2).

Solution of equation (5) with initial conditions

$$x(0) = x_0, \quad x'(0) = x_1, \ldots, x^{(n-1)}(0) = x_{n-1}, \quad x_k = \text{const}.$$
\[ x(t) = \sum_{m=1}^{\infty} \int_{0}^{\infty} \prod_{k=1}^{m} W_n(t - t_1, \ldots, t - t_m) f(t_k) dt_k, \]

where the kernels of integral operators are determined from a system of recurrent equations of the form (4)

\[ f(t) = y(t) + \sum_{l=1}^{n} x_{l-1} \delta_{(l)}^{(n-k)}(t), \]

\( \delta_{(l)}^{(n-k)}(t) \) – derivatives of the Dirac function.

As an example of an application, consider the problem of the dynamic behavior of a nonlinear vibrator that has heredity. In most works, the solution of this problem was constructed using averaging methods, which allows the solution to take into account only the first harmonic of vibrations. The application of the solution method based on its representation by an analytical functional makes it possible to obtain a solution in any approximation and take into account the influence of the entire spectrum of harmonics.

In dimensionless variables and parameters \( t' = \beta \cdot t, \ \tilde{\omega}_0^2 = \frac{E}{m \beta^2}, \ \tilde{P} = \frac{P_0}{m \beta^2} \), where \( \beta^{-1} \) is the relaxation time of the spring material, \( m \) is the mass of the oscillator, \( E \) is the spring stiffness coefficient, \( P_0 \) is the amplitude of the external periodic force, the equation of motion takes the form

\[ \ddot{x}(t) + \tilde{\omega}_0^2 \sum_{n=1}^{\infty} \prod_{k=1}^{n} V_n(t_1, \ldots, t_n) \prod_{k=1}^{m} x(t - t_k) dt_k = \tilde{P} f(t), \]

the stroke of the time variable is omitted, \( x(t) \) – displacement, the nuclei of heredity \( V_n \) satisfy the condition of causality, the nucleus \( V_1 \) contains an additive \( \delta \)-function, and the spring material does not change properties over time. If the behavior of the spring is the same under tension and compression, then \( V_{2n} = 0, \ n = 1, 2, \ldots \).

The stationary solution of equation (6) is sought in the form of the Volterra integral series

\[ x(t) = \sum_{n=1}^{\infty} P^n \prod_{k=1}^{m} W_n(t_1, \ldots, t_n) \prod_{k=1}^{m} x(t - t_k) dt_k. \]

Thus, the problem is reduced to determining the sequence of nuclei \( W_n(t_1, \ldots, t_n) \). To construct the steady-state reaction of the vibrator, we obtain a recurrent relation for the Fourier transformant of nuclei of the form

\[ W_n^*(t_1, \ldots, t_n) \cdot V_1^{0*} \left( \sum_{k=1}^{n} \omega_k \right) + \tilde{\omega}_0^2 \sum_{n} \sum_{c,i=1}^{n} \sum_{m=1}^{n} W_j^*(\omega_1, \ldots, \omega_j) \ldots W_{jm}^*(\omega_{j,m-1}, \ldots, \omega_n) \times \]

\[ \times V_m^*(\sum_{k=1}^{j} \omega_k, \ldots, \sum_{k=1}^{n} \omega_k) = \delta_l^{(n)}, \]

\[ V_1^{0*}(\omega) = -\omega^2 + \omega_0^2 \cdot V_1^*(\omega). \]

Here \( [J, n] = j_1 + \ldots + j_n \) – the length of the multi-index, \( j_k \) – natural numbers, an asterisk above the letters \( V_k^*, W_k^* \) denotes the Fourier image of the corresponding function. The inner sum is taken
from the natural solutions of the equation $|J, m| = n$, the outer sum is taken from the cyclic permutation of the indices $1, ..., n$ and is symmetrizing. The latter is due to the requirement of symmetry of functions $W_m(t_1, ..., t_n)$ with respect to the permutation of arguments.

We will further assume, that the function of dimensionless time $f(t)$ is $\frac{2\pi}{\omega}$ periodic and satisfies the condition of Fourier series decomposability

$$f(t) = \sum_{n=-\infty}^{\infty} f_n e^{in\omega t}, \quad f(t) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} f(t)e^{-in\omega t} dt, \quad f_{-n} = \bar{f}_n,$$

where the dash above the letter denotes a complex conjugation.

Changing the order of summation and product operations in expression (8) and performing integration over all variables, we obtain the following form of solving the differential equation (6)

$$x(t) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} (\sqrt{2\pi} P) \sum_{l=0}^{\infty} \psi_{n,l}(\omega) e^{il\omega t},$$

$$\psi_{n,l}(\omega) = \frac{1}{|m_1 \omega|} \prod_{k=1}^{n} f_{m_k},$$

where the "length" of the multi-index $|M, n|$ in sum both positive and negative values. Expression (9) together with the recurrent relation (8) represents the solution of the problem of constructing the response of a nonlinear hereditary - elastic system to periodic excitation. From the type of solution follows, that the response to the perturbation is a periodic function $t$ with a perturbation period. The coefficients of the constructed series are analytical functions of the excitation amplitude.

In the case, when the input signal $f(t)$ is approximated by a trigonometric polynomial

$$f(t) = \sum_{m=-N}^{N} f_m e^{im\omega t},$$

the solution of the considered differential equation has the form

$$x(t) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} (\sqrt{2\pi} P) \sum_{l=-nN}^{nN} \psi_{n,l}(\omega) e^{il\omega t}.$$

### 3. Dissipative characteristic of a nonlinear oscillator

Based on the results obtained above, we construct a solution to the equation of stationary oscillations of a nonlinear hereditary elastic oscillator with a symmetric nonlinear elastic characteristic under the action of a monoharmonic perturbation. As a measure of energy dissipation, we use internal friction [7]. This measure is widely used to determine the parameters of inheritance functions based on experimentally measured excitations and responses of the studied system.

The equation of motion of the considered oscillator, taking into account the requirement of symmetry of its behavior, takes the form

$$\ddot{x}(t) + \omega_0^2 \sum_{n=0}^{\infty} V_n(t_1, ..., t_n) \prod_{k=1}^{2n+1} x(t - t_k) dt_k = P \sin \omega t,$$

(10)
where $\omega = \Omega / \beta$ – dimensionless excitation frequency, $t$ – dimensionless time, $(n) = 2n + 1$, $P, \beta, w_0$ – are defined in formula (6), $V_{(n)}$ – weight functions of heredity, symmetric with respect to permutations of arguments, satisfying the conditions of causality and not changing the form over time, $V_1(t)$ – additively contains the delta-function.

To transform the solution of the equation, we use the relation

$$\hat{W}_n \sin \omega t = \frac{(\sqrt{2\pi})^{n-1}}{(2i)^n} \sum_{m=0}^{n} (-1)^m \hat{C}_{m}^n W_n(t_1, \ldots, t_n) \exp[i\omega(n-2m)t],$$

where the integral operator $\hat{W}_n$ is represented by a common term of the series in (7), the operator $\hat{C}_{m}^n$ is defined by the rule

$$\hat{C}_{m}^n \Phi(t_1, \ldots, t_n) = \left(\sqrt{2\pi}\right)^{n-1} \sum_{m=0}^{n} \Phi^*(\omega_1, \ldots, \omega_m, -\omega, \ldots, -\omega),$$

$\Phi^*(\omega_1, \ldots, \omega_m)$ – Fourier transform of a function $\Phi(t_1, \ldots, t_n)$, the sum is taken by permutations of the arguments of the group $(\omega)$ with $m$ the arguments of the group $(-\omega)$. The number of terms in the sum is equal to the binomial coefficient $\binom{n}{m}$. The given relations allow us to present the solution of the oscillator oscillation problem in the form of

$$x(t) = \frac{P}{T} \int_0^T f(t) \dot{x}(t) dt,$$

where $T$ is the period of the exciting oscillation force. Using the form of the solution and the expression for the function $f(t)$ and, changing the summation order, we get

$$\Delta N = \frac{\pi P^2}{2i} \sum_{n=1}^{\infty} \left(\frac{\pi P^2}{2}\right)^n \sum_{m=0}^{n+1} (-1)^{n+m} (2n - 2m + 1)(\delta_{n+1,m} - \delta_{n,m}) \hat{C}_{2n+1}^m W_n(t_1, \ldots, t_n),$$

where $\delta_{i,j}$ is the Kronecker symbol. By doing the summation on the index $m$ and, given the equality

$$\hat{C}_{2n+1}^m W_n(t_1, \ldots, t_n) = \hat{C}_{2n+1}^n W(n),$$

where the trait denotes a complex conjugation, we have

$$\Delta N = \pi P^2 \sum_{n=0}^{\infty} \left(\frac{\pi P^2}{2}\right)^n \text{Im}\left[\hat{C}_{2n+1}^n W_n(t_1, \ldots, t_n)\right].$$

(12)

$\text{Im}[\ldots]$ denotes the imaginary part of the expression in parentheses.
We calculate approximately the energy loss of the oscillator per oscillation cycle, taking into account only two terms in the solution. Assuming in formula (11) \( n = 0 \) and \( n = 1 \), we obtain

\[
x(t) = P \sum_{n=0}^{1} \left[ A_{2n+1} \sin(2n+1)\omega t + B_{2n+1} \cos(2n+1)\omega t \right],
\]

\[
A_{1}(\omega) = \text{Re}\left(W_{1}^{*}(\omega) + \frac{3\pi P^2}{2} \text{Re}\left(W_{3}^{*}(-\omega, \omega, \omega)\right)\right),
\]

\[
B_{1}(\omega) = -\text{Im}\left(W_{1}^{*}(\omega)\right) - \frac{3\pi P^2}{2} \text{Im}\left(W_{3}^{*}(-\omega, \omega, \omega)\right),
\]

\[
A_{3}(\omega) = -\frac{\pi P^2}{2} \text{Re}\left(W_{3}^{*}(\omega, \omega, \omega)\right), \quad B_{3}(\omega) = \frac{\pi P^2}{2} \text{Im}\left(W_{3}^{*}(\omega, \omega, \omega)\right),
\]

(13)

where \( W_{n}^{*}(\omega_{1}, \ldots, \omega_{n}) = \text{Re}\left(W_{n}^{*}(\omega_{1}, \ldots, \omega_{n})\right) - i \text{Im}\left(W_{n}^{*}(\omega_{1}, \ldots, \omega_{n})\right) \).

Assuming separability of the nuclei of heredity

\[
V_{n}^{*}(\omega_{1}, \ldots, \omega_{n}) = a_{n} \prod_{k=1}^{n} V_{1}^{*}(\omega_{k})
\]

from the recurrent relation (8) for odd numbers \( n \), we define the functions included in relation (13) explicitly

\[
W_{3}^{*}(\omega, \omega, \omega) = -\frac{a_{3}}{2\pi} W_{1}^{*}(3\omega) \left| W_{1}^{0*}(\omega) \right|^3,
\]

\[
W_{3}^{*}(-\omega, \omega, \omega) = -\frac{a_{3}}{2\pi} W_{1}^{*}(\omega) W_{1}^{0*}(\omega) \left| W_{1}^{0*}(\omega) \right|^2,
\]

(14)

where \( W_{1}^{0*}(\omega) = W_{1}^{*}(\omega)V_{1}^{*}(\omega) \).

Substituting the relation (14) in (13), we obtain expressions for the coefficients of harmonics included in the approximate solution

\[
A_{1}(\omega) = \text{Re}\left[W_{1}^{*}(\omega) - \alpha_{3} W_{1}^{*}(\omega) W_{1}^{0*}(\omega) \left| W_{1}^{0*}(\omega) \right|^2\right],
\]

\[
B_{1}(\omega) = \text{Im}\left[W_{1}^{*}(\omega) - \alpha_{3} W_{1}^{*}(\omega) W_{1}^{0*}(\omega) \left| W_{1}^{0*}(\omega) \right|^2\right],
\]

\[
A_{3}(\omega) = \frac{\alpha_{3}}{3} \text{Re}\left[W_{1}^{*}(3\omega) \left| W_{1}^{0*}(\omega) \right|^3\right], \quad B_{3}(\omega) = \frac{\alpha_{3}}{3} \text{Im}\left[W_{1}^{*}(3\omega) \left| W_{1}^{0*}(\omega) \right|^3\right], \quad \alpha_{3} = \frac{3a_{3}P^2}{4}.
\]

(15)

Given the equalities (14) and (15), the expression for the dissipation energy, which includes the first two harmonics, under the condition of separability of the influence functions \( V_{n}(t_{1}, \ldots, t_{n}) \), takes the form

\[
\Delta N \approx -\pi P^2 B_{1}(\omega),
\]

(16)

where \( B_{1}(\omega) < 0 \).
The average accumulated energy can be calculated by averaging the sum of potential \( U \) and kinetic \( E_k \) energies over the period [8]

\[
N = \langle U + E_k \rangle,
\]

where

\[
U = \frac{x^2(t)}{2} + a_3 \frac{x^4(t)}{4} + \cdots, \quad E_k = \frac{\dot{x}^2(t)}{2}.
\]

For sufficiently small perturbation amplitudes, given \(|a_3| < 1\), the calculation \( U \) can be limited to the first term in the expansion.

By performing transformations similar to those used in obtaining the expression for energy losses and by performing an averaging over the oscillation period, we obtain

\[
\left\langle \frac{x^2(t)}{2} \right\rangle = \frac{P^2}{8} \sum_{n=1}^{\infty} \left( \frac{\pi P^2}{2} \right)^{n-1} \sum_{p_1+p_2=n-1} \sum_{m_1+m_2=n} 2 \hat{C}_{2p_1+1} W_{2p_1+1}, \quad (17)
\]

\[
\left\langle \frac{\dot{x}^2(t)}{2} \right\rangle = -\frac{(\omega P)^2}{8} \sum_{n=1}^{\infty} \left( \frac{\pi P^2}{2} \right)^{n-1} \sum_{p_1+p_2=n-1} \sum_{m_1+m_2=n} 2 \hat{C}_{2p_1+1} W_{2p_1+1}, \quad (18)
\]

where the internal sums are taken by the integer nonnegative solution of the diophantine equations.

Taking into account in the obtained expressions the contributions of only the first and third harmonics (13), as well as the separability of the nuclei \( V_a(t_1,\ldots,t_n) \) and the formula (14), we obtain expressions for the average accumulated energy

\[
N \approx \frac{P^2}{4} \left[ (\omega_0^2 + \omega^2)(A_1^2 + B_1^2 + A_3^2 + B_3^2) + 8\omega^2(A_3^2 + B_3^2) \right], \quad (18)
\]

where the amplitudes of harmonics are determined by expressions (15).

Choosing as a measure of internal friction the value [7],

\[
Q^{-1} = \frac{\Delta N}{2\pi N},
\]

we obtain in the considered approximation taking into account (16) and (18)

\[
Q^{-1} = -\frac{2B_i}{(\omega_0^2 + \omega^2)(A_1^2 + B_1^2 + A_3^2 + B_3^2) + 8\omega^2(A_3^2 + B_3^2)}.
\]

Note that the formula (19) determines the internal friction during the oscillations of the linear hereditary elastic oscillator at \( a_3 = 0 \).

The resulting expression for the dissipation measure determines not only the frequency, but also the amplitude dependence. Thus, the obtained results can be used for the phenomenological description of energy losses during the motion of point defects and dislocations in solids under the action of mechanical vibration [7], [8]. The possibility of applying the nonlinear theory of viscoelasticity to describe the amplitude dependence of internal friction was noted in the monograph [7]. The question of the frequency dependence of internal friction is very important in materials science. There are a significant number of experimental results for various materials, as well as theoretical developments on the relationship of the frequency dependence of internal friction with physical relaxation mechanisms. It is established that weakly singular nuclei of heredity are the most suitable for the description of experimental data.
Consider the frequency dependence of internal friction for the relaxation functions of Rzhanitsyn and Rabotnov. In dimensionless variables, they have the form

\[ V_1(t) = \delta(t) - vt^{-1} \frac{e^{-t}}{\Gamma(\gamma)}, \quad V_1(t) = \delta(t) - vt^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{\Gamma(\gamma(1+n))}, \]

where \(0 < \gamma \leq 1\), \(0 < v < 1\), \(\delta(t)\) is the delta-function, \(\Gamma(\ldots)\) is the gamma-function. Fourier images of the solution kernels have the form: for the Rzhanitsyn kernel

\[ W_1^*(\omega) = B_r^{-1}(\omega) \delta^\gamma \left[ (\omega_0^2 - \omega^2) - v\omega_0^2 \cos \gamma \varphi - iv\omega_0^2 \sin \gamma \varphi \right], \]

\[ \varphi = \arctan \omega, \quad \delta = \sqrt{1 + \omega^2}, \quad B_r(\omega) = \left[ \omega_0^2 - \omega_2^2 \right] \left[ -v^2 \omega_0^2 - 2v\omega_0^2 \delta^\gamma \cos \gamma \varphi \right]; \]

for the function Rabotnova

\[ W_1^*(\omega) = B_R^{-1}(\omega) \left[ \omega^2 (\omega_0^2 - \omega^2) + (v\omega_0^2 - \omega^2) + v^2 \left[ 2 - v \omega_0^2 - 2\omega_0^2 \right] \cos \frac{\omega}{2} - iv\omega^2 \sin \frac{\omega}{2} \right], \]

\[ B_R(\omega) = \omega^2 (\omega_0^2 - \omega^2) + ((1 - v)\omega_0^2 - \omega^2)^2 + 2v^2 (\omega_0^2 - \omega^2)((1 - v)\omega_0^2 - \omega^2) \cdot \cos \frac{\omega}{2}. \]

Figure 1 and figure 2 show the curves of the dependence of internal friction \(Q^{-1}\) on the dimensionless frequency \(\omega = \Omega\) for different values of the parameters \(v\) and \(\gamma\), fixed amplitudes of the external action and two relaxation nuclei of Rzhanitsyn and Rabotnov. Solid curves correspond to the internal friction calculated for the Rzhanitsyn core, dashed curves calculated for the Rabotnov core. In all calculations, the separability coefficient \(a_3 = 1\).

![Figure 1. Frequency dependence of internal friction for Rzhanitsyn and Rabotnov relaxation nuclei at \(v = 0.9\) and \(P = 10^{-2}\).](image)
Figure 2. Frequency dependence of internal friction for Rzhanitsyn and Rabotnov relaxation nuclei at $\nu = 0.1$ and $P = 10^{-7}$.

Figure 1 shows the values of the parameters $\nu = 0.9$ and $P = 10^{-2}$, figure 2 shows the values of the parameters $\nu = 0.1$ and $P = 10^{-7}$.

The figures show the characteristic bell-shaped behavior of a quantity $Q^{-1}$ with a well-defined maximum, the value of which depends on the singularity parameter of the nucleus $\gamma$. As the value $\gamma$ decreases, the losses decrease. For the Rabotnov core, the maximum of internal friction is shifted towards low frequencies relative to the maximum for the Rzhanitsyn core. With growth $\gamma$, the positions of these maxima converge and at $\gamma = 1$ the curves merge. Thus, the core of Rabotnov is more suitable for describing the peak of low-frequency losses than the core of Rzhanitsyn. The figures clearly show the presence of a region (small $\omega$) with almost linear dependence $Q^{-1}(\omega)$ and show small minima on the ascending and descending sections of the curves.

A large experimental data indicates the presence of a number of maxima and minima in the frequency dependence of internal friction, which are associated with various relaxation mechanisms.

Figure 3 and figure 4 illustrate the frequency dependence of the ascending and descending sections $Q^{-1}$ at different amplitudes of external influence. The nucleus of heredity is chosen in the form of Rzhanitsyn at $\nu = 0.1$ and $\gamma = 0.15$.

Figure 3. Frequency dependence of internal friction for Rzhanitsyn relaxation nuclei at $\nu = 0.1$ and $\gamma = 0.15$ (descending section).
Figure 4. Frequency dependence of internal friction for Rzhanitsyn relaxation nuclei at $\nu = 0.1$ and $\gamma = 0.15$ (ascending section).

The qualitative behavior of the curves is also preserved for the Rabotnov kernel. The dashed curve corresponds to the case of linear hereditary rheology.

The dip of the curve in figure 3 corresponds to the resonant frequency for the third harmonic, and in figure 4 – for the first harmonic. Calculations show, that the largest deviation from the curve for a linear hereditary body in the areas of failure is of the order of 3%. With the growth of $\gamma$ the failures are reduced, with large $\nu$ the dips of the curve virtually disappear.

Thus, the internal friction for a nonlinear hereditary-elastic body reveals both frequency and amplitude dependences and is characterized by the presence of a number of maxima and minima.

4. Conclusions

In this paper, we construct a method for solving ordinary differential equations containing nonlinear analytical functionals. Such functionals are used in modeling the state of a number of technical devices of modern electronics, in nonlinear optics to describe the behavior of dielectrics during the passage of intense electromagnetic waves, as well as in describing the creep of polymers and some alloys under significant mechanical loads. The proposed method is based on the representation of the solution of a functional equation in the form of a series of multilinear functionals of increasing order. Relations between the kernels of the functionals representing the solution and the kernels of a given functional are obtained.

The method developed in this paper is applied to the construction of a stationary solution of the equation of motion of a nonlinear hereditary-elastic oscillator, in which the functional describing the effects of heredity is represented by a Volterra series. An expression for the internal friction of the oscillator describing its frequency and amplitude dependences is obtained. The presence of peaks and dips in the frequency dependence of energy dissipation is established. These effects are not detected when averaging methods are applied to the solution of the equation.

It should be noted that the work on generalization of the presented method to the case of a system of differential equations with nonlinear analytical functionals is a promising area of research.

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