Generalization of the Kolmogorov-Sinai entropy: Logistic- and periodic-like dissipative maps at the chaos threshold

Ugur Tirnakli$^{1,2}$, Garin F.J. Ananos$^1$ and Constantino Tsallis$^{1,3,4}$

$^1$Centro Brasileiro de Pesquisas Fisicas, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro-RJ, Brazil
$^2$Department of Physics, Faculty of Science, Ege University, 35100 Izmir, Turkey
$^3$Department of Physics, University of North Texas, P.O. Box 311427, Denton, Texas 76203, USA
$^4$Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico 87501, USA

tirnakli@sci.ege.edu.tr, fedorja@cbpf.br, tsallis@cbpf.br

We numerically calculate, at the edge of chaos, the time evolution of the nonextensive entropic form $S_q \equiv [1 - \sum_{i=1}^{W} F_i^q]/[q - 1]$ for two families of one-dimensional dissipative maps, namely a logistic- and a periodic-like with arbitrary inflexion $z$ at their maximum. At $t = 0$ we choose $N$ initial conditions inside one of the $W$ small windows in which the accessible phase space is partitioned; to neutralize large fluctuations we conveniently average over a large amount of initial windows. We verify that one and only one value $q^* < 1$ exists such that the $\lim_{t \to \infty} \lim_{W \to \infty} \lim_{N \to \infty} S_q(t)/t$ is finite, thus generalizing the (ensemble version of) Kolmogorov-Sinai entropy (which corresponds to $q^* = 1$ in the present formalism). This special, $z$-dependent, value $q^*$ numerically coincides, for both families of maps and all $z$, with the one previously found through two other independent procedures (sensitivity to the initial conditions and multifractal $f(\alpha)$ function).

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I. INTRODUCTION

In the area of nonlinear dynamical systems, a number of studies have recently addressed the sensitivity to initial conditions, multifractality and the behavior of the Kolmogorov-Sinai (KS) entropy of these systems. In this context, it is worth mentioning that these attempts include both dissipative (low-dimensional maps, symbolic sequences) and conservative (long-ranged many-body Hamiltonians, conservative maps) systems. Before describing the purpose of the present work, we need to briefly review the sensitivity to the initial conditions and multifractality of the chaotic attractor.

Firstly, let us focus on the concept of the sensitivity to initial conditions. As already well studied in the literature, for one-dimensional systems it is convenient to introduce $\xi(t) \equiv \lim_{\Delta x(0) \to 0} \Delta x(t)/\Delta x(0)$, where $\Delta x(0)$ is the discrepancy of the initial conditions at time $t = 0$, and $\Delta x(t)$ its time dependence. It can be shown that $\xi$ satisfies the differential equation $d\xi/dt = \lambda_1 \xi$, where $\lambda_1$ is the Lyapunov exponent, thus $\xi(t) = \exp(\lambda_1 t)$. Consequently, if $\lambda_1 < 0$ ($\lambda_1 > 0$) the system is said to be strongly insensitive (sensitive) to the initial conditions (and intermediate rounding). On the other hand, if $\lambda_1 = 0$, then the function $\xi$ is expected to satisfy the differential equation $d\xi/dt = \lambda_q \xi^q$, hence

$$\xi(t) = [1 + (1 - q)\lambda_q t]^1/(1-q) \quad (q \in \mathbb{R}),$$

(1)

which recovers the standard exponential case for $q = 1$, whereas $q \neq 1$ yields a power-law behaviour. If $q > 1$ ($q < 1$) and $\lambda_q < 0$ ($\lambda_q > 0$) the system is said to be weakly insensitive (sensitive) to the initial conditions. Although asymptotic power-law sensitivity to the initial conditions has since long been observed, eq.(1) (which in fact corresponds to the power-law growth of the upper-bound of $\xi(t)$) provides in principle a more complete description since $t$ does not need to satisfy $t >> 1$. At the onset of chaos (where it is $\lambda_1 = 0$), this upper bound ($\xi \propto t^{1/(1-q)}$) allows us to estimate the value $q^*$ of index $q$ for the map under consideration. This method has been successfully used for a variety of maps such as logistic, $z$-logistic, circle and $z$-circular maps.
Secondly, we consider another interesting property of dynamical systems: the geometrical aspects of the attractor at the onset of chaos. In order to describe the scaling behavior of the critical dynamical attractor it is convenient to introduce a multifractal formalism \[\text{[13,14]}\]. In this formalism, it is possible to introduce a partition function \(\chi_Q(N) = \sum_{i=1}^{N} p_i^Q\), where \(p_i\) represents the probability on the \(i\)th box among the \(N\) boxes of the measure (it is necessary to warn the reader that we use here \(Q\) instead of the standard notation \(q\) of multifractal literature in order to avoid confusion with the present index \(q\)). In the \(N \rightarrow \infty\) limit, the contribution to this partition function is proportional to \(N^{-\alpha(Q)}\), which comes from a subset of all possible boxes whose number scales as \(N_Q \propto N^{f(Q)}\), where \(f(Q)\) is the fractal dimension of the subset. The content on each contributing box scales as \(P \propto N^{-\alpha}\) and all these exponents are related by a Legendre transformation \(\tau(Q) = Q\alpha(Q) - f(Q)\). The multifractal measure is then characterized by a multifractal function \(f(\alpha)\), which reflects the fractal dimension of the subset with singularity strength \(\alpha\). At the end points of the \(f(\alpha)\) curve, this singularity strength is associated with the most concentrated \(\alpha_{min} = \lim_{Q \to \infty} \alpha(Q)\) and the most rarefied \(\alpha_{max} = \lim_{Q \to -\infty} \alpha(Q)\) regions on the attractor. Recently, the scaling behaviour of these regions has been used to estimate the power-law divergence of nearby trajectories and a new scaling relation has been proposed \[\text{[3]}\] as

\[\frac{1}{1-q^*} = \frac{1}{\alpha_{min}} - \frac{1}{\alpha_{max}}.\]  

(2)

This relation constitutes a completely different method for the calculation of the index \(q^*\). Previous works \[\text{[1]}\] have shown, for various map families, that the results of these two abovementioned methods of calculating the \(q^*\) values are the same within a good precision.

We are now prepared to describe the purpose of the present work, i.e., a specific generalization of the KS entropy \(K_1\). For a chaotic dynamical system, one can define this entropy as the increase, per unit time, of the standard Boltzmann-Gibbs entropy \(S_1 = -\sum_{i=1}^{W} p_i \ln p_i\). Moreover, it is well-known that the KS entropy is deeply related to the Lyapunov exponents since the Pesin equality \[\text{[15]}\] states that, for vast classes of nonlinear dynamical systems, \(K_1 = \lambda_1\) if \(\lambda_1 > 0\) and \(K_1 = 0\) otherwise. Strictly speaking, as we shall detail later on, the KS entropy is defined in terms of a single trajectory in phase space, using a symbolic representation of the regions of a partitioned phase space. However, it appears that, in almost all cases, this definition can be equivalently replaced by one based on an ensemble of initial conditions. This is the version we use herein.

The marginal cases, i.e., those for which \(\lambda_1 = 0\), include period doubling and tangent bifurcations as well as the onset of chaos. For these cases, a generalized version of the KS entropy \(K_q\) has been introduced \[\text{[1]}\] as the increase rate of a proper nonextensive entropy

\[S_q(t) = \frac{1 - \sum_{i=1}^{W} [p_i(t)]^q}{q - 1}.\]  

(3)

This nonextensive entropy enables a generalization of Boltzmann-Gibbs statistical mechanics \[\text{[16]}\]; it recovers the standard Boltzmann-Gibbs entropy \(S_1 = -\sum_{i=1}^{W} p_i \ln p_i\) in the \(q \rightarrow 1\) limit. A general review of the properties of this entropy and related subjects can be found in \[\text{[12]}\]. So, for the generalized version of the KS entropy, it has been proposed \[\text{[1]}\]

\[K_q \equiv \lim_{t \to \infty} \lim_{W \to \infty} \lim_{N \to \infty} \frac{S_q(t)}{t}\]  

(4)

where \(t\) is the time steps, \(W\) is the number of regions in the partition of the phase space and \(N\) is the number of points that are evolving with time. The Pesin equality itself is expected to be generalizable as follows: \(K_q = \lambda_q\) (or some appropriate average) if \(\lambda_q > 0\) and \(K_q = 0\) otherwise.

These ideas have been used very recently by Latora et al \[\text{[18]}\] to construct a third method for the calculation of the \(q^*\) values. They conjectured that (i) a special \(q^*\) value exists such that \(K_q\) is finite for \(q = q^*\), vanishes for \(q > q^*\) and diverges for \(q < q^*\), (ii) this value of \(q^*\) coincides with that coming from other two methods of finding \(q^*\) (namely, from eq.(1) and eq.(2)). Latora et al have checked these conjectures with numerical calculations for the standard logistic map and found that the growth of \(S_q(t)\) is linear when the value of index \(q\) equals \(q^* \simeq 0.2445\), which strongly supports the point that all three methods yield one and the same special value of the index \(q^*\). Although the results of Latora et al provide strong evidence in favor of this scenario, it is no doubt necessary to study more general maps along these lines in order to see whether the results of this method reproduce those of the previous two methods. The aim of this work is to address this question by studying the \(z\)-logistic and \(z\)-periodic maps.
II. NUMERICAL RESULTS

We first recall the z-logistic map

\[ x_{t+1} = 1 - a|x_t|^z , \tag{5} \]

where \( 1 < z, 0 < a \leq 2, -1 \leq x_t \leq 1 \), and the z-periodic map

\[ x_{t+1} = d \cos \left( \pi \left| x_t - \frac{1}{2} \right| z/2 \right) , \tag{6} \]

where \( 1 < z, 0 < d < \infty, -d \leq x_t \leq d \). These maps display a period-doubling route to chaos and the parameter \( z \) is the inflexion of the map at its extremal point. Both maps belong to the same universality class, namely they share the same value \( q^* \) for a given \( z \). The values of \( a_c \) and \( d_c \) at the onset of chaos, as well as the \( q^* \) values calculated from eq.(1) and eq.(2) are indicated in the Table for three representative values of \( z \).

Now we can describe the numerical procedure that we used as the third method for calculating \( q^* \). This method was first introduced in [8] for conservative systems and then used by Latora et al [18] for the standard \((z = 2)\) logistic map. We partition the interval \(-1 \leq x \leq 1 (-d \leq x \leq d)\) into \( W \) equal windows for the z-logistic map \((z\text{-periodic map})\). Then we choose (randomly or not) one of these windows and select (randomly or uniformly) \( N \) initial values of \( x \) (all inside the chosen window) for the z-logistic \((z\text{-periodic map})\) at a given value of \( a \ (d) \). As \( t \) evolves, these \( N \) points typically spread within the interval of phase space and this gives us a set \( \{N_i(t)\} \) with \( \sum_{i=1}^{W} N_i(t) = N, \forall t \), which consequently yields a set of probabilities \( \{p_i(t) \equiv N_i(t)/N\} \). Since for \( t = 0 \) all \( N \) points are inside the chosen window, \( S_q(0) = 0 \) and, as time evolves, \( S_q(t) \) starts increasing, although bounded by the equiprobability value (i.e., by \((W^{1-q} - 1)/(1-q)\) for \( q \neq 1 \), and \( \ln W \) for \( q = 1 \)). Let us anticipate our main result. For \( N \) and \( W \) increasingly large (and always satisfying \( N >> W \)) we observe, in all cases, that a linear increase of \( S_q \) with time emerges only for a special value of \( q \), namely \( q^*(z) \). We define the corresponding slope as the generalized Kolmogorov-Sinai entropy \( K_q \). For values of \( a \) and \( d \) for which \( \lambda_1 > 0 \) we verify that \( q^* = 1 \). But at the edge of chaos, we obtain \( q^* \leq 1 \). In contrast with the cases where strong chaos exists, considerably large fluctuations appear in \( S_q(t) \) at the chaos threshold, due to the fact that the attractor occupies only a tiny part of the available phase space. These fluctuations make uneasy the task of determining the value of \( q \) for which linearity is present. In order to neutralize these fluctuations, we adopt a procedure of averaging over the “efficient” initial conditions as introduced in [18]. The procedure is as follows: we choose the initial distribution of \( N \) points in one of the \( W \) cells of the partition and count the total number of occupied cells as time evolves between say \( t = 1 \) and \( t = 50t \); this number is a measure of the efficiency of that particular window in spreading itself. After studying each one of the \( W/2 \) cells in the interval \( 0 \leq x \leq 1 \) \((0 \leq x \leq d)\) for the z-logistic \((z\text{-periodic map})\), the averaging is done over the cells for which the total number of occupied cells is larger than a fixed threshold, say 5000 (see Fig. 1).

Our results for typical values of \( z \) for the z-logistic and the z-periodic maps at the threshold of chaos are indicated in Figs. 2-4. We used \( N = 10 \times W = 10^5 \) for the z-logistic maps, and \( N = 10 \times W = 5 \times 10^5 \) for the z-periodic maps. The values we have used are large enough so that the time evolution of the entropy in the intermediate time region (after the initial transient and before the saturation) does not depend much on \( W \). This point has been illustrated in Fig. 2, where we plot the time evolution of \( S_q(t) \) for two values of \( W \) for the case of the \( z = 1.75 \) logistic map. It is clear from this figure that the starting points of saturation of the curves shift to larger time as \( W \) increases, but in the intermediate time region the curves are almost insensitive to \( W \).

In all cases, the growth of \( S_q(t) \) in this intermediate time region is found to be linear when \( q = q^* \), whereas it curves upwards for \( q < q^* \) and downwards for \( q > q^* \), i.e., \( K_q \) is finite for \( q = q^* \), diverges for \( q < q^* \) and vanishes for \( q > q^* \). In order to provide quantitative support to this behavior, we fit the curves with the polynomial \( S(t) = A + Bt + Ct^2 \) in the interval \([t_1, t_2]\) characterizing the intermediate region. We define the nonlinearity coefficient \( R \equiv C(t_1 + t_2)/B \) as a measure of the importance of the nonlinear term in the fit; \( R \) vanishes for a strictly linear fit (see the Insets of Figs. 2-4). The times \( t_1 \) and \( t_2 \) are respectively defined as the end of the initial transient (during which \( S_q \) is roughly constant) and the beginning of the saturation; \( R(q) \) is almost independent from \((t_1, t_2)\) for any set of specific values \((t_1, t_2)\) which satisfy the above characterizations. We also see in the insets that the signs of \( R \) on both sides of \( q^* \) are consistent with the curvatures of \( S_q(t) \) in the intermediate region.
III. CONCLUSIONS

Let us make a few general considerations in order to better expose the meaning of the present effort and its results. The Kolmogorov-Sinai entropy $K_1$ is an important concept in chaotic dynamical systems, either dissipative (like the one-dimensional maps considered here) or conservative (like classical many-body Hamiltonians, satisfying the Liouville theorem). Its definition is based on a partition of the accessible phase space in a set of $W$ subspaces that are visited, along time, in some complex order starting from a single initial point in phase space. If we associate to each subspace a symbol, we shall have $W^N$ possible words of length $N$. These $W^N$ words are visited along time with probabilities $\{\pi_l\}$ ($l = 1, 2, ..., W^N$). This set of probabilities enable the calculation of the Boltzmann-Gibbs-Shannon entropy $S_1(\{\pi_l\}) = -\sum_l \pi_l \ln \pi_l$. In the limit $N \to \infty$, $S_1(\{\pi_l\})$ is proportional to $N$ if the system mixes exponentially quickly (i.e., positive Lyapunov exponents), and $K_1$ is defined as the supremum of the $\lim_{N \to \infty} S_1(N)/N$. Although we are not aware of any general proof, a common belief exists that this computationally quite heavy definition can be conveniently replaced by the one we have used here, based on an ensemble of initial conditions instead of on single trajectories. This is to say, we choose $N$ initial conditions within one of the $W$ subspaces, and we follow along time the set of probabilities $\{p_i\}$ ($i = 1, 2, ..., W$) associated with the occupancies of those subspaces. The set $\{p_i\}$ enables the calculation of $S_1(\{p_i\}) = -\sum_i p_i \ln p_i$, and $K_1$ is expected to be the supremum of the $\lim_{t \to \infty} S_1(t)/t$.

The scenario we have just described is the standard one. What we have exhibited here is a step forward, although only for very simple cases, namely the maps herein considered. By analyzing the generalization of $K_1$ of the KS entropy based on the nonextensive entropy $S_q$, either in its trajectory formation or in its ensemble one, we exhibit a path that might be applicable to situations much more general than the maps focused on here. We have (numerically) shown that an unique value $q^*$ exists such that $K_q$ is finite, being zero for $q > q^*$ and infinite for $q < q^*$. If the nonlinear dynamical system is strongly chaotic (exponential mixing in phase space), then $q^* = 1$, thus recovering the usual scheme. But if the system is only weakly chaotic (power-law mixing, i.e., vanishing Lyapunov exponents, but positive generalized Lyapunov exponents, like $\lambda_q$), then one essentially expects $q^* < 1$. Let us emphasize that in the present work we have only addressed $q^*$ and not $K_q$ (nor $\lambda_q$). Indeed, the averaging procedure we have implemented is expected to preserve linearity with time, but certainly not the supremum, or whatever analogous to that. Further analysis is certainly welcome.

As in the case discussed by Latora et al.\cite{18}, we have verified that the value $q^*$ that has emerged is precisely the same previously obtained through two completely different procedures, namely the power-law sensitivity to the initial conditions (see eq. 1), and the multifractal structure of the chaotic attractor (see eq. 2). This uniqueness of the special value $q^*$ clearly provides to the whole scenario a very robust consistency.

Let us end with a speculative question quite analogous to the one we have addressed here. If we consider two systems $A$ and $B$ that are independent in the sense that $p_{ij}^{A+B} = p_i^A p_j^B$, then we straightforwardly verify that $S_q(A+B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B)$. In other words, we have extensivity, superextensivity and subextensivity for $q = 1$, $q < 1$ and $q > 1$ respectively. Let us now assume that we have a classical $d$-dimensional many-body Hamiltonian system whose particles interact through two-body interactions which are nowhere singular and which (attractively) decay like $r^{-\alpha}$ at long distances. It is known that such a system is thermodynamically extensive if $\alpha/d > 1$ (short-range interaction) and nonextensive if $0 \leq \alpha/d < 1$ (long-range interaction)\cite{17}. For $\alpha/d > 1$, if the system is thermodynamically large (say it contains $M$ particles with $M \sim 10^{12}$), any two, also thermodynamically large, of its subsystems can be considered as probabilistically independent in the above sense. Then, because of its well known extensivity, the special value of $q$ to be associated clearly is $q = 1$. Consequently the entropy $S_1$ is generically expected to yield a finite value for $\lim_{t \to \infty} \lim_{M \to \infty} S_1(t, M)/M$. The question we wish to leave open at the present stage is: for $0 \leq \alpha/d < 1$, is there a special (possibly unique) value of $q$ such that $\lim_{t \to \infty} \lim_{M \to \infty} S_q(t, M)/M$ also is finite?

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Figure and Table Captions

Figure 1 - Total number of occupied cells as function of the window rank (varying from rank 30000 to rank 60000, respectively corresponding to $x = 0$ and $x = 1$) for the $z = 3$ periodic map with $W = 6 \times 10^4$. The horizontal line indicates the cutoff at 5000. Average for $S_q(t)$ is done over the windows associated with values above the cutoff.

Figure 2 - Time evolution of $S_q(t)$ for two different values of $W$ for the $z = 1.75$ logistic map. Inset: The nonlinearity coefficient $R$ versus $q$ (see text for details).

Figure 3 - Time evolution of $S_q(t)$ for three different values of $q$ and $W = 10^5$ for the $z = 3$ logistic map. Inset: The nonlinearity coefficient $R$ versus $q$ (see text for details).

Figure 4 - Time evolution of $S_q(t)$ for three different values of $q$ and $W = 6 \times 10^4$ for the $z$-periodic maps: (a) $z = 1.75$; (b) $z = 2$; (c) $z = 3$. Insets: The nonlinearity coefficient $R$ versus $q$ (see text for details).

Table - The values of $a_c$, $d_c$ and $q^*$ at the chaos threshold for typical values of the inflexion parameter $z$.

| $z$   | $a_c$    | $d_c$    | $q^*$ |
|-------|----------|----------|-------|
| 1.75  | 1.355000075... | 0.77946432... | 0.100 |
| 2.0   | 1.40115518...  | 0.86557926... | 0.2445|
| 3.0   | 1.52187879...  | 1.07848805... | 0.472 |
Fig 2

\[ S_q(t) \]

\[ z=1.75 \]

\[ W=10^5 \]

\[ W=2 \times 10^4 \]

\[ q=-0.1 \]

\[ q^*=0.100 \]

\[ q=0.3 \]
Fig 3

![Graph showing the relationship between $S_q(t)$ and $q$ for different values of $q$. The inset shows the behavior of $R$ as a function of $q$ at $z=3$. The graph includes data points for $q=0.3$, $q^*=0.472$, and $q=0.75$.](image-url)
Fig 4a
Fig 4c

\[ S_q(t), R, t \]

- \( z = 3 \)
- \( q = 0.3 \)
- \( q^* = 0.472 \)
- \( q = 0.65 \)