Coarse cohomology theories

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Abstract

We propose the notion of a coarse cohomology theory and study the examples of coarse ordinary cohomology, coarse stable cohomotopy and of coarse cohomology theories obtained by dualizing coarse homology theories.

We show that the dualizing spectrum of a finitely generated torsion-free group only depends on the coarse motivic spectrum represented by the underlying bornological coarse space of the group. This in particular implies a conjecture of J. R. Klein that the dualizing spectrum of a group is a coarse invariant.

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1 Introduction

In [BE20b] we have introduced the category $\text{BornCoarse}$ of bornological coarse spaces and the notion of a $C$-valued coarse homology theory $E: \text{BornCoarse} \to C$, where $C$ is a stable $\infty$-category. In the present paper we place the notion of a coarse cohomology theory into the same framework.

Let $D$ be a stable $\infty$-category and $E: \text{BornCoarse}^{\text{op}} \to D$ be a functor.

**Definition 1.1.** We call $E$ a coarse cohomology theory if $E^{\text{op}}: \text{BornCoarse} \to D^{\text{op}}$ is a coarse homology theory.

The main purpose of the present paper is to present the construction of the following three examples of coarse cohomology theories:

1. To every abelian group $A$ we associate the coarse ordinary cohomology theory $HAX: \text{BornCoarse}^{\text{op}} \to \text{Ch}_\infty$.

   Our Definition 3.1 extends the original definition of Roe [Roe93] to the context of bornological coarse spaces, see Lemma 3.12.

2. If $C$ is a bicomplete stable $\infty$-category tensored over $\text{Sp}$, then in Definition 4.1 we associate to every object $C$ of $C$ a $C$-valued coarse cohomology theory $QC$. For the category of spectra $C = \text{Sp}$ and for the sphere spectrum $C = S$ we obtain a coarse version $QS$ of stable cohomotopy.

3. If $E$ is a $C$-valued coarse homology theory and $C$ is an object of $C$, then in the Definition 2.11 we define the dual $\text{Sp}$-valued coarse cohomology theory $DC(E)$ by forming the mapping spectrum with target $C$. We also consider versions of this construction where we replace the mapping space functor by some internal mapping object functor or a suitable power functor.

Note that there are also other approaches to a general framework for coarse cohomology theories. Let us mention exemplary the work of Schmidt [Sch99], Hartmann [Har20], and Wulff [Wul22].

In Section 5 we provide an application of the coarse cohomology $QS$ discussed above in Item 2. In order to formulate our result we recall the following:
1. Let $G$ be a discrete group and define the dualizing spectrum $D_G := S[G]^G$. Here $S[G]$ is the suspension spectrum associated to the underlying discrete space of $G$. The right-action of $G$ on itself induces an action on $S[G]$, and $S[G]^G$ denotes the spectrum of fixed points.

2. In [BE20b, Def. 4.3] we constructed a universal coarse homology theory $\text{Yo}^* : \text{BornCoarse} \to \text{Sp} \mathcal{X}$ with values in the stable $\infty$-category of coarse motivic spectra.

3. By [BE20b, Ex. 2.21] the group $G$ gives naturally rise to a bornological coarse space $G_{\text{can}, \text{min}}$ in $\text{BornCoarse}$ and therefore to a coarse motivic spectrum $\text{Yo}^*(G_{\text{can}, \text{min}})$ in $\text{Sp} \mathcal{X}$. Note that $\text{Yo}^*(G_{\text{can}, \text{min}})$ is in particular an invariant of the quasi-isometry class of $G$.

**Theorem 1.2** (Theorem 5.2). If $G$ is finitely generated and torsion-free, then we have an equivalence $D_G \simeq Q_S(G_{\text{can}, \text{min}})$ in $\text{Sp}$.

The following corollary settles a generalization of a conjecture stated by Klein [Kle01, Conj. on Page 455].

**Corollary 1.3** (Corollary 5.3). For a finitely generated and torsion-free group $G$ the spectrum $D_G$ only depends on the coarse motivic spectrum $\text{Yo}^*(G_{\text{can}, \text{min}})$. In particular, it is an invariant of the quasi-isometry class.

**Remark 1.4.** The $\mathbb{Z}$-linear analogues of the above results have been settled by Roe as we explain in the following.

Recall that a group $G$ is of the type $FP$ if the trivial $G$-module $\mathbb{Z}$ admits a finite-length projective resolution by finitely generated $\mathbb{Z}[G]$-modules. We let $n$ be the cohomological dimension of $G$ and define the dualizing $G$-module by $D_{\mathbb{Z}}^G := H^n(G; \mathbb{Z}[G])$. Here we consider $\mathbb{Z}[G]$ as a $G$-module with the action induced by the right multiplication on $G$, and the $G$-action on the cohomology is induced from the left $G$-action on $\mathbb{Z}[G]$.

By $D_{\mathbb{Z}}^G := H(G; \mathbb{Z}[G])$ we denote the cohomology complex of $G$ with coefficients in the module $\mathbb{Z}[G]$. We consider it as an object of the stable $\infty$-category $\text{Ch}_{\infty}$ of chain complexes in $\text{Ab}$ with quasi-isomorphisms inverted. The chain complex $D_{\mathbb{Z}}^G$ is the $\mathbb{Z}$-linear analogue of $D_G$ defined above.

The group $G$ is called a Bieri–Eckmann duality group, if it is of type $FP$ and $D_{\mathbb{Z}}^G \simeq D_{\mathbb{Z}}^G[-n]$ (with $G$-action forgotten), i.e., $D_{\mathbb{Z}}^G$ has only one non-trivial cohomology group which sits in degree $n$ and is isomorphic to $D_{\mathbb{Z}}^G$. If in addition the underlying abelian group of $D_{\mathbb{Z}}^G$ is torsion-free, then the four assertions stated in [Bro82, Thm. VIII.10.1] are satisfied. In particular, the group $G$ satisfies a version of Poincaré duality in the sense that there are natural isomorphisms $H^i(G; -) \cong H_{n-i}(G; D_{\mathbb{Z}}^G \otimes -)$ of functors on the category of $G$-modules.

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A Bieri–Eckmann duality group with the property that $D^Z_{G,n}$ is torsion-free is called a Poincaré duality group.

It is known from the work of Roe [Roe96, Prop. 2.6] (see also [Ger95, Thm. 8]) that the cohomology groups $H^*(G; Z[G])$ are invariants of $Yo^*(G_{can, min})$ because they coincide with the coarse ordinary cohomology groups of $G_{can, min}$. Thus if $G$ is of type FP, then the property of being a Bieri–Eckmann or being a Poincaré duality group only depends on the equivalence class of $Yo^*(G_{can, min})$.

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2 Coarse cohomology theories

In Section 2.1 we give an axiomatic definition of the notion of a coarse cohomology theory which by the Corollary 2.7 it is eventually equivalent to Definition 1.1. In Section 2.2 we construct first examples by dualizing coarse homology theories. In Section 2.3 we discuss coassembly maps for coarse cohomology theories.

2.1 Definition and basic properties

In this section we define coarse cohomology theories by dualizing the axioms for coarse homology theories given in [BE20b, Sec. 4.4].

We will use notions for bornological coarse spaces and coarse homology theories introduced in [BE20b]. In particular, we assume familiarity of the reader with the material in [BE20b, Sec. 2–4].

Let $C$ be a complete stable $\infty$-category.

We equip the two-point set $\{0, 1\}$ with the maximal bornological and coarse structures. The category $\text{BornCoarse}$ has a symmetric monoidal structure $\otimes$ introduced in [BE20b, Ex. 2.32]. Since $\{0, 1\}$ is bounded, for every bornological coarse space $X$ the projection $\{0, 1\} \otimes X \to X$ is a morphism of bornological coarse spaces.

We consider a functor $E: \text{BornCoarse}^{op} \to C$.

**Definition 2.1.** $E$ is called coarsely invariant if for every bornological coarse space $X$ the projection $\{0, 1\} \otimes X \to X$ induces an equivalence $E(X) \to E(\{0, 1\} \otimes X)$. 
Let $\mathcal{Y} := (Y_i)_{i \in I}$ be a big family on a bornological coarse space $X$ [BE20b, Def. 3.2]. Set

$$E(\mathcal{Y}) := \lim_{i \in I} E(Y_i)$$

and note that have a natural morphism

$$E(X) \to E(\mathcal{Y}).$$

The definition of excisiveness involves the notion of complementary pairs [BE20b, Def. 3.5].

**Definition 2.2.** $E$ is called excisive if for any complementary pair $(\mathcal{Y}, Z)$ on a bornological coarse space $X$ the square

$$\begin{array}{ccc}
E(X) & \longrightarrow & E(Z) \\
\downarrow & & \downarrow \\
E(\mathcal{Y}) & \longrightarrow & E(Z \cap \mathcal{Y})
\end{array}$$

is cartesian.

Recall the notion of a flasque bornological coarse space [BE20b, Def. 3.21].

**Definition 2.3.** $E$ vanishes on flasques if $E(X) \simeq 0$ for every flasque bornological coarse space $X$.

In the following let $\mathcal{C}$ denote the coarse structure of a bornological coarse space $X$. For an entourage $U$ in $\mathcal{C}$ we let $X_U$ be the bornological coarse space obtained from $X$ by replacing the coarse structure $\mathcal{C}$ by the coarse structure $\mathcal{C}(U)$ generated by $U$. The identity map of the underlying set of $X$ is a morphism $X_U \to X$ of bornological coarse spaces. We get a natural morphism

$$E(X) \to \lim_{U \in \mathcal{C}} E(X_U).$$

**Definition 2.4.** $E$ is $u$-continuous if for every bornological coarse space $X$ the natural morphism

$$E(X) \to \lim_{U \in \mathcal{C}} E(X_U)$$

is an equivalence.

Let $\mathcal{C}$ be a complete stable $\infty$-category and consider a functor $E : \text{BornCoarse}^{\text{op}} \to \mathcal{C}$.

**Definition 2.5.** $E$ is a coarse cohomology theory if it has the following properties:

1. $E$ is coarsely invariant.
2. $E$ is excisive.
3. $E$ vanishes on flasques.
4. $E$ is $u$-continuous.

**Remark 2.6.** In this remark we compare Definition 2.5 with Fukaya and Oguni’s definition of a coarse cohomology theory [FO16, Def. 3.3].

As usual in the current coarse geometry literature they only consider proper metric spaces. In order to encode coarse invariance they introduce the coarse category: It is obtained from the full subcategory of BornCoarse of proper metric spaces (where the coarse and bornological structures are induced from the metric) by identifying morphisms which are close to each other. Then a coarse cohomology theory in the sense of Fukaya and Oguni is a contravariant, $\mathbb{Z}$-graded, group-valued functor on the coarse category which vanishes on all spaces of the form $X \otimes N_{can}$ (where $N_{can}$ has the canonical metric structures) and satisfies a Mayer–Vietoris sequence for coarsely excisive decompositions.

If $E$ is an $\text{Sp}$-valued coarse cohomology theory as in Definition 2.5 then we can derive a coarse cohomology theory in the sense of Fukaya–Oguni by taking homotopy groups and restricting to proper metric spaces. Condition 2.5.1 ensures that the resulting $\mathbb{Z}$-graded group-valued functor factorizes over the coarse category. The excisiveness Condition 2.5.2 is stronger than satisfying a Mayer–Vietoris sequence for coarsely excisive decompositions, cf. [BE20b, Lem. 3.38]. Finally, the Condition 2.5.4 is actually equivalent to the requirement that $E(X \otimes N_{can}) \cong 0$, and $u$-continuity is not part of Fukaya and Oguni’s axioms.

If the $\infty$-category $C$ is complete and stable, then the opposite $\infty$-category $C^{\text{op}}$ is cocomplete and stable. If $E$: BornCoarse$^{\text{op}} \rightarrow C$ is a functor, then we let $E^{\text{op}}$: BornCoarse $\rightarrow C^{\text{op}}$ denote the induced functor between the opposite categories.

Recall the definition of a $C^{\text{op}}$-valued coarse homology theory from [BE20b, Def. 4.22]. The following corollary immediately follows from a comparison of the definitions.

**Corollary 2.7.** $E$ is a $C$-valued coarse cohomology theory if and only if $E^{\text{op}}$ is a $C^{\text{op}}$-valued coarse homology theory.

Using the correspondence between coarse homology theories and coarse cohomology theories we can transfer the results and definitions concerning coarse homology theories shown or stated in, e.g. [BE20b] and [BE20a] to the case of coarse cohomology theories. Here are two examples of such a transfer of definitions.

Recall the notion of a weakly flasque bornological coarse space [BEKW20, Def. 4.18].

**Definition 2.8.** A coarse cohomology theory $E$ is called strong if $E(X) \cong 0$ for every weakly flasque bornological coarse space $X$.

Thus a coarse cohomology theory $E$ is strong if and only if $E^{\text{op}}$ is a strong coarse homology theory.
Recall from [BE20b, Def. 6.3] that a coarse homology theory $F$ is called strongly additive if for every family $(X_i)_{i \in I}$ of bornological coarse spaces we have an equivalence

$$F(\bigcup_{i \in I} X_i) \simeq \prod_{i \in I} F(X_i)$$

(induced by the collection of projection maps which exist by excision). Hence for coarse cohomology theories we get the following definition of (strong) additivity:

**Definition 2.9.** A coarse cohomology theory $E$ is called strongly additive if for every family $(X_i)_{i \in I}$ of bornological coarse spaces we have an equivalence

$$\bigoplus_{i \in I} E(X_i) \simeq E(\bigcup_{i \in I} X_i)$$

induced by the natural map.

We say that $E$ is additive, if the equivalence above is satisfied for all families of one-point spaces.

In [BE20b, Def. 4.3] we have introduced the stable $\infty$-category of coarse motivic spectra $\text{Sp}_\mathcal{X}$ and the universal classical coarse homology theory

$$\text{Yo}^s : \text{BornCoarse} \to \text{Sp}_\mathcal{X}.$$ For a complete stable $\infty$-category $\mathcal{C}$ precomposition with $\text{Yo}^s$ induces by [BE20b, Cor. 4.6] an equivalence between the $\infty$-categories of colimit preserving functors from $\text{Sp}_\mathcal{X}$ to $\mathcal{C}^{\text{op}}$ and $\mathcal{C}^{\text{op}}$-valued coarse homology theories. By Corollary 2.7 we have the analogous statement for coarse cohomology theories, as follows.

Let $\mathcal{C}$ be a complete stable $\infty$-category.

**Corollary 2.10.** Precomposition by $\text{Yo}^s^{\text{op}}$ induces an equivalence between the $\infty$-categories of limit-preserving functors $\text{Sp}_\mathcal{X}^{\text{op}} \to \mathcal{C}$ and $\mathcal{C}$-valued coarse cohomology theories.

### 2.2 Coarse cohomology theories by duality

In this section we provide a simple construction of coarse cohomology theories by dualizing coarse homology theories.

Let $\mathcal{C}$ be a stable $\infty$-category and $C$ be an object of $\mathcal{C}$. We assume one of the following:

1. $\mathcal{C}$ is cocomplete and $F : \text{BornCoarse} \to \mathcal{C}$ is a coarse homology theory. Then we use the notation

   $$C(-) := \text{map}(-, C) : \mathcal{C}^{\text{op}} \to \text{Sp}$$

   for the mapping spectrum functor and set $\mathcal{D} := \text{Sp}$. 

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2. $F: \text{BornCoarse} \to \text{Sp}$ is a coarse homology theory. In this case we assume that $C$ is complete and powered over $\text{Sp}$. We write

$$C^(-): \text{Sp}^{\text{op}} \to C, \quad A \mapsto C^A$$

for the power functor and set $D := C$.

3. $C$ is complete and cocomplete and $F: \text{BornCoarse} \to C$ is a coarse homology theory. Furthermore, $C$ is closed symmetric monoidal and in particular admits a limit-preserving internal mapping object functor

$$\text{map}(-,C): C^{\text{op}} \to C.$$

In this case we write $C^(-) := \text{map}(-,C)$ and set $D := C$.

**Definition 2.11.** We define the functor

$$D_C(F): \text{BornCoarse}^{\text{op}} \to D, \quad D_C(F) := C^(-) \circ F^{\text{op}}.$$

We consider $D_C(F)$ as the dual of $F$ with respect to $C$.

**Theorem 2.12.** $D_C(F)$ is a $D$-valued coarse cohomology theory.

**Proof.** The functor $C^(-)^{\text{op}}$ is colimit-preserving. The composition

$$D_C(F)^{\text{op}} = C^(-)^{\text{op}} \circ F: \text{BornCoarse} \to D^{\text{op}}$$

is therefore a $D^{\text{op}}$-valued coarse homology theory. Hence $D_C(F)$ is a $D$-valued coarse cohomology theory by Corollary 2.7.

The proof of the following lemma is straightforward:

**Lemma 2.13.** If $F$ is strong, then so is $D_C(F)$.

In general $D_C(F)$ is not additive even if $F$ is additive, as the next example shows.

**Example 2.14.** We consider the additive, $\text{Ch}_{\infty}$-valued coarse ordinary homology theory $H_{QX}$ with rational coefficients (see [BE20b, Def. 6.18] for the definition and [BE20b, Cor. 6.24] for additivity) and the object $Q[0]$ in $\text{Ch}_{\infty}$. Then, for every infinite set $I$,

$$D_{Q[0]}(H_{QX})(\biguplus_I^\text{free} *) \simeq \text{map}(H_{QX}^{\text{op}}(\biguplus_I^\text{free} *), Q[0]) \simeq \text{map}(\prod_I Q[0], Q[0]) \not\simeq \bigoplus_I \text{map}(Q[0], Q[0]) \simeq \bigoplus_I D_{Q[0]}(H_{QX})(*)$$
showing that \( D_{\Q(0)}(H\Q X) \) is not additive. Here map is the internal mapping object of \( \Ch_{\infty} \).

We let \( E: \BornCoarse^{\text{op}} \to D \) be a coarse cohomology theory.

**Definition 2.15.** A pairing between \( E \) and \( F \) with values in \( C \) is a morphism of coarse cohomology theories \( p: E \to D_C(F) \).

In the examples above the identity provides a \( C \)-valued pairing between \( D_C(F) \) and \( F \). The examples \( H\AX \) from Theorem 3.3 and \( Q_C \) from Theorem 4.2 further below come with a natural pairing with a corresponding homology theory.

### 2.3 Coassembly maps

In this section we translate some of the result from [BE20a] from coarse homology to coarse cohomology theories. A coarse homology theory gives rise to a local homology theory on the category of uniform bornological coarse spaces by pulling back along the cone functor. On the other hand, a local homology theory can be coarsified to a coarse homology theory by composing with the Rips complex construction. Applying these two constructions in a row we get a new coarse homology theory which is connected with the original one by a coarse assembly map. These constructions can be analogously performed with coarse cohomology theories and yield coarse coassembly maps. In the following we describe the details.

In [BE20a, Sec. 2] we introduced the category of uniform bornological coarse spaces \( \UBC \). The notion of a local homology theory is defined in [BE20a, Def. 3.11]. The universal local homology theory \( \Yo^s B: \UBC \to \Sp B \) is constructed in [BE20a, Sec. 4]. Furthermore, in [BEKW20, Sec. 4.4] we introduced the universal strong coarse homology theory \( \Yo^s wfl: \BornCoarse \to \Sp X_{wfl} \).

We let \( \O^\infty: \UBC \to \Sp X \) denote the germs-at-\( \infty \) of the cone functor ([BE20a, Sec. 8] and [BEKW20, Sec. 9.6]). By [BE20a, Lem. 9.5] the composition

\[
\O^\infty_{wfl}: \UBC \xrightarrow{\O^\infty} \Sp X \to \Sp X_{wfl}
\]

is a local homology theory and therefore extends essentially uniquely along \( \Yo^s B \) to a colimit-preserving functor (denoted by the same symbol)

\[
\O^\infty_{wfl}: \Sp B \to \Sp X_{wfl}.
\]
By [BE20a, Prop. 5.2] the Rips complex construction yields a coarse homology theory

\[ P : \text{BornCoarse} \to \text{SpB} . \]

Therefore the composition

\[ \text{BornCoarse} \xrightarrow{P} \text{SpB} \xrightarrow{\mathcal{O}_\text{wfl}^\infty} \text{Sp} \mathcal{X}_\text{wfl} \]

is a coarse homology theory. We assume now that \( \mathcal{E} : \text{BornCoarse}^{\text{op}} \to \text{C} \) is a strong coarse cohomology theory. Then we can interpret \( \mathcal{E} \) as a limit-preserving functor defined on \( \text{Sp} \mathcal{X}_\text{wfl}^{\text{op}} \). The composition

\[ \mathcal{E} \mathcal{O}_\text{wfl}^\infty P := F \circ \mathcal{O}_\text{wfl}^\infty \circ P^{\text{op}} : \text{BornCoarse}^{\text{op}} \to \text{C} \]

is a new coarse cohomology theory with the same target as \( \mathcal{E} \). It is related with \( \mathcal{E} \) by the coarse coassembly map, a natural transformation of functors

\[ \mu^\mathcal{E} : \Sigma^{-1} E \to \mathcal{E} \mathcal{O}_\text{wfl}^\infty P . \]

The construction of the coarse coassembly map is dual to the construction of the coarse assembly map in the homological case [BE20a, Def. 9.7]. In detail \( \mu^\mathcal{E} \) is given by composing \( E \) with the transformation

\[ \mathcal{O}_\text{wfl}^\infty P \xrightarrow{\partial \text{Cone}} \Sigma F P \xrightarrow{\sim} \Sigma \]

of functors \( \text{Sp} \mathcal{X} \to \text{Sp} \mathcal{X}_\text{wfl} \). Here \( \partial \text{Cone} : \mathcal{O}_\text{wfl}^\infty \to \Sigma F \) is the cone boundary (see [BE20a, Def. (9.1)]), \( F : \text{SpB} \to \text{Sp} \mathcal{X}_\text{wfl} \) (see [BE20a, Sec. 6]) is induced by forgetting the uniform structure, and we used [BE20a, Prop. 6.2] for the second equivalence.

In [BE20a Thm’s. 1.3, 1.4 & 1.5] we discussed conditions implying that the coarse assembly map is an equivalence. Using the relation between coarse cohomology theories and coarse homology theories these results yield conditions on the bornological coarse space \( X \) and the coarse cohomology theory \( E \) which imply that the coarse coassembly map

\[ \mu^{\mathcal{E},X} : \Sigma^{-1} E(X) \to \mathcal{E} \mathcal{O}_\text{wfl}^\infty P(X) \]

is an equivalence. We will only spell out the cohomological version of [BE20a, Thm. 1.3] and leave the translation of the other two theorems to the interested reader.

**Theorem 2.16.** If \( X \) admits a cofinal set of entourages \( U \) such that \( X_U \) has finite asymptotic dimension, then the coarse coassembly map \( \mu^{\mathcal{E},X} \) is an equivalence.

### 3 Coarse ordinary cohomology

#### 3.1 Construction of \( HAX \)

In this section we construct coarse cohomology

\[ HAX : \text{BornCoarse} \to \text{Ch}_\infty \]
with coefficients in an abelian group \( A \). Its target is the presentable stable \( \infty \)-category of chain complexes defined as a Dwyer–Kan localization
\[
\iota: \text{Ch} \to \text{Ch}_\infty
\]
of the category \( \text{Ch} \) of chain complexes of abelian groups at quasi-isomorphisms.

To a set \( X \) we can functorially associate a simplicial set \( \hat{X} \), the Čech nerve of the projection \( X \to * \). For \( n \) in \( \mathbb{N} \) its set of \( n \)-simplices is given by \( \hat{X}[n] := X^{x(n+1)} \).

Let \( X \) be a bornological coarse space, \( U \) be a coarse entourage of \( X \), and \( B \) be a bounded subset of \( X \). An \( n \)-simplex \((x_0, \ldots, x_n)\) in \( \hat{X} \) is called \( U \)-controlled if \((x_i, x_j) \in U\) for all \( i, j \) in \([n]\). We say that this simplex is contained in \( B \) if \( x_i \in B \) for all \( i \) in \([n]\).

If the entourage \( U \) contains the diagonal, then the \( U \)-controlled simplices form a simplicial subset \( \hat{X}_U \) of \( \hat{X} \). For simplicity, we assume that all entourages appearing below contain the diagonal.

To any simplicial set \( S \) and abelian group \( A \) we can functorially associate a chain complex \( C(S; A) \) in \( \text{Ch} \). It is defined as the chain complex associated to the cosimplicial abelian group \( A^S \). For \( n \) in \( \mathbb{Z} \) the group of \( n \)-chains is given by \( C^n(S; A) := A^{S[n]} \), and the boundary operator \( d: C^n(S; A) \to C^{n+1}(S; A) \) is given by \( \sum_{i=0}^{n+1} (-1)^i \partial_i \), where \( \partial_i \) is induced by the \( i \)th face map \( \partial_i: S[n + 1] \to S[n] \). For example, \( \partial_0(x_0, \ldots, x_{n+1}) := (x_1, \ldots, x_{n+1}) \).

We let \( C_U(X; B; A) \) be the \( \mathbb{Z} \)-graded subgroup of \( C(\hat{X}_U; A) \) of functions which vanish on all simplices which are not contained in \( B \). Observe that the \( \mathbb{Z} \)-graded subgroup \( C_U(X; B; A) \) is not a subcomplex. Indeed, the differential of \( C(\hat{X}_U; A) \) restricts to maps
\[
d: C_U(X, B; A) \to C_U(X, U[B]; A)
\]
for all \( B \) in the bornology \( B \) of \( X \). Hence, if we form the union of these \( \mathbb{Z} \)-graded subgroups over the bounded subsets \( B \) of \( X \), then we obtain the following subcomplex of \( C(\hat{X}_U; A) \):
\[
C_U(X; A) := \colim_{B \in B} C_U(X, B; A).
\]

Note that we consider \( C_U(X; A) \) as an object of \( \text{Ch} \).

Let \( f: X' \to X \) be a morphism between bornological coarse spaces. The map \( f \) induces a map of simplicial sets \( \hat{f}: \hat{X}' \to \hat{X} \). Assume that \( U' \) is an entourage of \( X' \), \( U \) is an entourage of \( X \), and that \( f(U') \subseteq U \). Then \( \hat{f} \) restricts to a map of simplicial sets \( \hat{X}_U' \to \hat{X}_U \). Since \( f \) is proper, pull-back along this map induces a morphism of chain complexes
\[
f^*: C_U(X; A) \to C_U(X'; A).
\]

We now want to perform the derived limit of \( C_U(X; A) \) over the entourages \( U \) in the coarse structure \( C \) of \( X \) (see the Remark 3.7 for the reason why we want the limit to be derived). In order to produce a functor on the level of \( \infty \)-categories, technically we will work with right Kan extensions. To this end we consider the following category \( \text{BornCoarse}^C \):
1. The objects of $\text{BornCoarse}^\mathcal{C}$ are pairs $(X,U)$ of a bornological coarse space $X$ and a coarse entourage $U$ on $X$.

2. The morphisms $(X',U') \to (X,U)$ in $\text{BornCoarse}^\mathcal{C}$ are morphisms of bornological coarse spaces $f : X' \to X$ such that $f(U') \subseteq U$.

We have functors

$$p : \text{BornCoarse}^\mathcal{C} \to \text{BornCoarse}, \quad (X,U) \mapsto X \quad (3.4)$$

and, using $\iota$ from (3.1),

$$\iota C(A) : (\text{BornCoarse}^\mathcal{C})^{\text{op}} \to \text{Ch}_{\infty}, \quad (X,U) \mapsto \iota C_U(X; A).$$

**Definition 3.1.** We define the functor $H\mathcal{A}\mathcal{X} : \text{BornCoarse}^{\text{op}} \to \text{Ch}_{\infty}$ as the right Kan extension

$$\begin{array}{ccc}
(\text{BornCoarse}^\mathcal{C})^{\text{op}} & \xrightarrow{\iota C(A)} & \text{Ch}_{\infty} \\
p^{\text{op}} & \downarrow & \downarrow H\mathcal{A}\mathcal{X} \\
\text{BornCoarse}^{\text{op}} & & \text{Ch}_{\infty}
\end{array}$$

of $\iota C(A)$ along $p^{\text{op}}$.

**Remark 3.2.** The point-wise formula for the right Kan extension provides the formula

$$H\mathcal{A}\mathcal{X}(X) \simeq \lim_{U \in \mathcal{C}} \iota C_U(X; A) \quad (3.5)$$

for the evaluation of the functor $H\mathcal{A}\mathcal{X}$ on the bornological coarse space $X$. It is crucial to apply $\iota$ before taking the limit. This ensures that the limit is derived.

**Theorem 3.3.** The functor $H\mathcal{A}\mathcal{X} : \text{BornCoarse}^{\text{op}} \to \text{Ch}_{\infty}$ is a coarse cohomology theory.

**Proof.** The axioms given in Definition 2.5 for a coarse cohomology theory will be verified in the following four Lemmas 3.4, 3.5, 3.6 and 3.8.

**Lemma 3.4.** $H\mathcal{A}\mathcal{X}$ is $u$-continuous

**Proof.** Let $X$ be a bornological coarse space with coarse structure $\mathcal{C}$. Using (3.5) and a cofinality consideration we get the chain of canonical equivalences

$$H\mathcal{A}\mathcal{X}(X) \simeq \lim_{U \in \mathcal{C}} \iota C_U(X; A) \simeq \lim_{U \in \mathcal{C}} \lim_{V \in (\mathcal{C}(U))} \iota C_V(X; A) \simeq \lim_{U \in \mathcal{C}} H\mathcal{A}\mathcal{X}(X_U). \quad \square$$

**Lemma 3.5.** $H\mathcal{A}\mathcal{X}$ is coarsely invariant.

**Proof.**
Proof. Let \( f, g : X \to X' \) be two maps of bornological coarse spaces which are close to each other. If \( U \) is an entourage of \( X \), then we choose an entourage \( U' \) of \( X' \) so large that \( f(U) \subseteq U' \), \( g(U) \subseteq U' \) and \( (f, g)(\text{diag}(X)) \subseteq U' \). Then for all \( n \in \mathbb{N} \) and \( i \) in \( [n] \) we have the maps \( h_i : X_U[n] \to X_U'[n + 1] \) given by
\[
(x_0, \ldots, x_n) \mapsto (f(x_0), \ldots, f(x_i), g(x_i), \ldots, g(x_n)).
\]
Pull-back along \( h_i \) induces a map
\[
h_i^* : C^{m+1}_U(X'; A) \to C^m_U(X; A).
\]
We form \( h^n := \sum_{i=0}^{n} (-1)^i h_i^{n*} \) and the map \( h := \bigoplus_{n \in \mathbb{Z}} h^n \) of degree \(-1\). Then one checks directly that
\[
d \circ h + h \circ d = g^* - f^* : C_U'(X'; A) \to C_U(X; A).
\]
Let \( X \) be a bornological coarse space. We consider the maps
\[
p : \{0, 1\} \otimes X \to X, \quad i : X \to \{0, 1\} \otimes X
\]
given by the projection and the inclusion of the point 0, respectively. Then \( p \circ i = \text{id}_X \) and \( i \circ p \) is close to the identity of \( \{0, 1\} \otimes X \). For an entourage \( U \) of \( X \) let \( \tilde{U} := \{0, 1\}^2 \times U \) be the corresponding entourage of \( \{0, 1\} \otimes X \). The above construction then shows that \((i \circ p)^* \) is chain homotopic to the identity on \( C(\{0, 1\} \otimes X; A) \). This implies that
\[
p^* : C_U(X; A) \to C_{\tilde{U}}(\{0, 1\} \otimes X; A)
\]
is an equivalence for every entourage \( U \) of \( X \). We conclude that
\[
HAX(p) : HAX(X) \to HAX(\{0, 1\} \otimes X)
\]
is an equivalence. \(\square\)

Lemma 3.6. \( HAX \) is excisive.

Proof. Let \( X \) be a bornological coarse space \( X \). For an entourage \( U \) and a subset \( Y \) of \( X \) we write \( U_Y := U \cap (Y \times Y) \). We have a surjective restriction
\[
C_U(X; A) \to C_{U_Y}(Y; A).
\]
We denote its kernel by \( C_U(X, Y; A) \).

Let \((Z, Y)\) be a complementary pair on \( X \) with \( Y = (Y_i)_{i \in I} \). For every \( i \) in \( I \) we consider the map of exact sequences
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & C_U(X, Y; A) & \longrightarrow & C_U(X; A) & \longrightarrow & C_{U_Y}(Y_i; A) & \longrightarrow & 0 \\
\downarrow r_i & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{U_Z}(Z, Z \cap Y_i; A) & \longrightarrow & C_{U_Z}(Z; A) & \longrightarrow & C_{U_{Z \cap Y_i}}(Z \cap Y_i; A) & \longrightarrow & 0
\end{array}
\]

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where the vertical morphisms are the corresponding restriction maps.

We claim that the morphism \((r_i)_i\) is an isomorphism of pro-systems indexed by \(I\). Indeed, we can define a map of complexes \(s_i: C_U(Z, Z \cap Y; A) \to C_U(X, Y; A)\) by extension by zero as long as \(j\) in \(I\) satisfies \(U[Y_i] \subseteq Y_j\). Since \(Y\) is a big family, for every \(i\) in \(I\) we can choose such an index \(j\) in \(I\). Then the resulting family \((s_i)_i\) is an inverse of \((r_i)_i\).

The localization \(\iota\) sends short exact sequences of chain complexes to fibre sequences. We apply \(\iota\) and \(\lim_{i \in I}\) and \(\lim_{U \in C}\) in order to get the morphism between fibre sequences

\[
\begin{array}{ccc}
\lim_{U \in C} \lim_{i \in I} (C_U(Z, Z \cap Y; A) \to C_U(X, Y; A)) & \to & \lim_{U \in C} (C_U(Z, Z \cap Y; A) \to C_U(X, Y; A)) \\
\approx & & \approx \\
\end{array}
\]

In view of the left vertical equivalence the right square is cartesian.

**Remark 3.7.** It is important for the proof of Lemma 3.6 that we consider the limits after applying \(\iota\). Limits in \(\text{Ch}_\infty\) preserve fibre sequences. In contrast, limits in \(\text{Ch}\) in general do not preserve short exact sequences. ♦

**Lemma 3.8.** \(HAX\) vanishes on flasques.

**Proof.** Let \(X\) be a flasque bornological coarse space with flasqueness implemented by the morphism \(f: X \to X\). We define a map of chain complexes

\[
S: C_V(X; A) \to C_U(X; A)
\]

by

\[
S(\phi) := \sum_{n=0}^{\infty} f^* n^* \phi,
\]

where \(U\) is an entourage of \(X\) and \(V := \bigcup_{n \in \mathbb{N}} f^n(U)\). Since \(\phi\) is supported on some bounded subset of \(X\) almost all summands vanish and the sum has a well-defined interpretation. One furthermore checks that

\[
f^* \circ S + r = S,
\]

where \(r: C_V(X; A) \to C_U(X; A)\) is the restriction. Applying \(\iota\) and \(\lim_{U \in C}\), the morphisms \(S\) for various \(U\) induce a morphism of chain complexes

\[
\tilde{S}: HAX(X) \to HAX(X)
\]

satisfying

\[
HAX(f) \circ \tilde{S} + \text{id}_{HAX(X)} \simeq \tilde{S}. \tag{3.6}
\]

Since \(HAX\) is coarsely invariant by Lemma 3.5 we get the equivalence

\[
\tilde{S} + \text{id}_{HAX(X)} \simeq \tilde{S}. \tag{3.7}
\]

It implies \(HAX(X) \simeq 0\). □
We have thus verified all four axioms for coarse cohomology theories and hence finished the proof of Theorem 3.3.

In the remaining part of this section we will verify that \( HAX \) is a strong coarse cohomology theory that satisfies additivity, and in the following section we compare our definition to the original one of Roe [Roe03] and describe a natural pairing with coarse homology.

**Lemma 3.9.** \( HAX \) is strong.

**Proof.** We assume that \( X \) is weakly flasque and we repeat the argument for Lemma 3.8. Because we already know that \( HAX \) is a coarse cohomology theory, in order to see that (3.6) implies (3.7) it suffices to have \( Yo^*(f) \simeq id_{Yo^*(X)} \).

Recall Definition 2.9 of (strong) additivity and the definition of a free union (see [BE20b, Def. 2.27]). The following result implies additivity of \( HAX \), but because of the additional assumption it is weaker than strong additivity.

Let \( (X_i)_{i \in I} \) be a family of bornological coarse spaces.

**Lemma 3.10.** If \( X_i \) has a maximal coarse entourage for every \( i \) in \( I \), then we have an equivalence

\[
\bigoplus_{i \in I} HAX(X_i) \simeq HAX\left(\bigsqcup_{i \in I} X_i\right).
\]

**Proof.** The set of entourages of \( X \) of the form \( \bigsqcup_{i \in I} U_i \) for families \( (U_i)_{i \in I} \) in \( \coprod_{i \in I} C_i \) (where \( C_i \) denotes the coarse structure of \( X_i \)) is cofinal in the set \( C \) of entourages of \( X \). Let \( B \) denote the bornology of \( X \) and let \( B_i \) denote the bornology of \( X_i \) for every \( i \) in \( I \). For any \( B \) in \( B \) we have \( B \cap X_i \in B_i \) for all \( i \) in \( I \), and \( B \cap X_i = \emptyset \) for all but finitely many \( i \) in \( I \). Consequently, we get the following chain of equivalences:

\[
\begin{align*}
HAX(X) &\simeq \lim_{U \in C} \colim_{B \in B} C_U(X, B; A) \\
&\simeq \lim_{(U_i)_{i \in I} \in \prod_{i \in I} C_i} \lim_{B \in B} \colim_{(U_i)_{i \in I} \in I} C_{(U_i)_{i \in I}}(X, B; A) \\
&\simeq \lim_{(U_i)_{i \in I} \in \prod_{i \in I} C_i} \colim_{B \in B_i} \bigoplus_{i \in I} C_{U_i}(X_i, B \cap X_i; A) \\
&\simeq \lim_{(U_i)_{i \in I} \in \prod_{i \in I} C_i} \colim_{B \in B_i} \bigoplus_{i \in I} C_{U_i}(X_i; B_i; A) \\
&\simeq \lim_{(U_i)_{i \in I} \in \prod_{i \in I} C_i} \bigoplus_{i \in I} C_{U_i}(X_i; A) \\
&\simeq \bigoplus_{i \in I} C_{U_i}(X_i; A) \\
&\simeq \bigoplus_{i \in I} HAX(X_i)
\end{align*}
\]
In addition to the properties of the bounded subsets of $X$ mentioned above, for the marked equivalence we also use the fact that the chain boundary operator does not mix the different coarse components of $X$. In order to see that the left-pointing arrow is an equivalence we use the additional assumption of the lemma to replace the limits by the evaluations at the corresponding maximal entourages.

**Remark 3.11.** We have written the argument for the Lemma 3.10 in greater detail than necessary in order to locate the problem that appears if one wants to show strong additivity, i.e., if one drops the additional assumption. Then in order to show that the left-pointing arrow is an equivalence one must distribute a cofiltered limit over an infinite sum \cite[Rem. 6.28]{BE20b}, see also the proof of Lemma 4.8 below. It is not clear that this is always possible in $\mathbf{Ch}_\infty$.

### 3.2 Further properties

If $X$ is a metric space, then Roe \cite{Roe93} has defined coarse cohomology groups $HAX_{Roe}(X; A)$. Roe’s coarse cohomology groups are defined as the cohomology groups of the complex $C\mathcal{X}_{Roe}(X; A)$ of locally bounded, $A$-valued Borel functions on the simplicial space $\hat{X}$ whose restrictions to $\hat{X}_{U_r}$ have bounded support for every entourage $U_r := \{x, y \in X \mid d(x, y) < r\}$. In our notation

$$C\mathcal{X}_{Roe}(X; A) := \lim_{U \in \mathcal{C}} C_{U, Roe}(X; A),$$

where $C_{U, Roe}(X; A)$ is the subcomplex of $C_U(X; A)$ of the locally bounded Borel functions. We have a natural morphism

$$\iota C\mathcal{X}_{Roe}(X; A) \to HAX(X). \quad (3.8)$$

**Lemma 3.12.** If $X$ is a proper metric space, then the morphism (3.8) is an equivalence.

**Proof.** It suffices to show that (3.8) induces a quasi-isomorphism. Since the domain and the target of this morphism are coarsely invariant functors we can replace $X$ by a locally finite, discrete subset which is coarsely equivalent to $X$. Furthermore, we can replace the limit over $U$ in $\mathcal{C}$ by the limit over the family of entourages $U_n := \{x, y \in X \mid d(x, y) < n\}$ indexed by $n \in \mathbb{N}$.

If $X$ is a locally finite, discrete metric space, then the conditions of being a Borel function and of being locally bounded are vacuous. In this case the only difference between Roe’s complex and our complex is the order of the limit $\lim_{U \in \mathcal{C}}$ and the localization $\iota$. In Roe’s case the limit is not derived.

We now observe that the restriction maps $C_{U_{n+1}}(X; A) \to C_{U_n}(X; A)$ are surjective for all $n$ in $\mathbb{N}$. As explained below this implies that one can interchange the order of taking the limit and the localization so that the assertion of the lemma follows.

In order to see that we can interchange the order of taking the limit and the localization we model $\mathbf{Ch}_\infty$ using the model category structure on $\mathbf{Ch}$ whose weak equivalences are
quasi-isomorphisms and whose fibrations are degree-wise epimorphisms. On \( \text{Fun}(N^{\text{op}}, \text{Ch}) \) we then have a Reedy model category structure in which the diagram \((C_{U_n}(X; A))_{n \in \mathbb{N}}\) is fibrant. This implies that the canonical morphism

\[
\lim_{\mathbb{N}^{\text{op}}} C_{U_n}(X; A) \xrightarrow{\sim} \lim_{\mathbb{N}^{\text{op}}} tU_n(X; A)
\]

is an equivalence.

Therefore our construction extends Roe’s coarse cohomology from proper metric spaces to all bornological coarse spaces.

We now describe a natural pairing with values in \( \iota A[0] \) in the sense of Definition \[ 2.15 \] between the coarse cohomology \( H\mathcal{A}\mathcal{X} \) and the coarse homology theory with \( \mathbb{Z} \)-coefficients \( H\mathcal{X}^{\text{hlg}} \) from \[ BE20b \] Def. 6.18\(^1\).

We use the closed symmetric monoidal structure of the \( \infty \)-category \( \text{Ch}_{\infty} \). The dualizing object (denoted by \( C \) in Section \[ 2.2 \]) is the object \( \iota A[0] \) in \( \text{Ch}_{\infty} \), where \( A[0] \) in \( \text{Ch} \) is the chain complex with \( A \) in degree zero.

Recall that the coarse homology of the bornological coarse space \( X \) is given by

\[
H\mathcal{X}^{\text{hlg}}(X) \cong \iota C\mathcal{X}^{\text{hlg}}(X),
\]

where \( C\mathcal{X}^{\text{hlg}}(X) \) is the complex of locally finite and controlled chains \[ BE20b \] Def. 6.17. Let \( U \) be a coarse entourage of \( X \). Then we let

\[
\overline{C\mathcal{X}}^{\text{hlg}}(X, U) := C\mathcal{X}^{\text{hlg}}(X)
\]

be the subcomplex of \( C\mathcal{X}^{\text{hlg}}(X) \) of the locally finite, \( U \)-controlled chains. We thus get a functor

\[
C(A): (\text{BornCoarse}^C)^{\text{op}} \to \text{Ch}, \quad (X, U) \mapsto \overline{C\mathcal{X}}^{\text{hlg}}(X, U).
\]

Furthermore, recall the functor

\[
C(A): (\text{BornCoarse}^C)^{\text{op}} \to \text{Ch}, \quad (X, U) \mapsto C(A)(X, U) := C_U(X; A).
\]

We first define a natural transformation of \( \text{Ch} \)-valued functors \( (\text{BornCoarse}^C)^{\text{op}} \to \text{Ch} \)

\[
\tilde{p}: C(A) \to \text{Hom}((\overline{C\mathcal{X}}^{\text{hlg}, \text{op}}), A[0])
\]

as follows. Let \((X, U)\) be an object of \( \text{BornCoarse}^C \), fix an \( n \) in \( \mathbb{N} \), and let \( \phi \) be an element of \( C_A(X, U)^n \). Then we define the homomorphism \( \tilde{p}_{(X, U)}(\phi) \) in \( \text{Hom}((\overline{C\mathcal{X}}^{\text{hlg}, \text{op}}), A[0]) \) as the \( \mathbb{Z} \)-linear extension of the map which sends the simplex \((x_0, \ldots, x_n)\) in \( X_U[n] \) to \( \phi((x_0, \ldots, x_n)) \) and vanishes on simplices of dimensions different from \( n \). One easily checks that \( \tilde{p}_{(X, U)} \) is a map of chain complexes, and that the collection of maps \( \tilde{p}_{(X, U)} \) for all \((X, U)\) in \( \text{BornCoarse}^C \) defines a natural transformation of functors \( \tilde{p} \).

\(^1\)In the reference \( H\mathcal{X}^{\text{hlg}} \) is denoted by \( H\mathcal{X} \).
The natural transformation $\hat{p}$ induces a morphism

$$\iota \hat{p}: \iota C(A) \to \iota \text{Hom}(\bar{C}A^{\text{hlg}, \text{op}}, A[0])$$

between functors from $\text{BornCoarse}^C \text{op}$ to $\text{Ch}_\infty$. We derive the desired pairing

$$p: HA X \to D_{\iota A[0]}(HA^{\text{hlg}})$$

by a right Kan extension of $\iota \hat{p}$ along the forgetful functor (3.4) from $\text{BornCoarse}^C \text{op}$ to $\text{BornCoarse}^\text{op}$. To this end we must check that the domain and target of this extension are the correct functors.

By Definition 3.1 the domain of the Kan extension of $\iota \hat{p}$ is $HA X$. We now evaluate the target on $X$ in $\text{BornCoarse}$. Using the objectwise formula for the Kan extension this evaluation is given by

$$\lim_{U \in C} \iota \text{Hom}(\bar{C}X^{\text{hlg}}(X, U), A[0]) \cong \lim_{U \in C} \text{map}(\iota \bar{C}X^{\text{hlg}}(X, U), \iota A[0]) \cong \text{map}(\text{colim} \iota \bar{C}X^{\text{hlg}}(X, U), \iota A[0]).$$

We now use the chain of equivalences

$$\text{colim}_{U \in C} \iota \bar{C}X^{\text{hlg}}(X, U) \cong \iota \text{colim}_{U \in C} \bar{C}X^{\text{hlg}}(X, U) \cong \iota C \bar{X}^{\text{hlg}}(X) \cong HA^{\text{hlg}}(X),$$

where the first equivalence follows from the fact that the poset $C$ of entourages of $X$ is filtered and $\iota$ commutes with filtered colimits. We therefore obtain the following formula for the target:

$$\text{map}(HA^{\text{hlg}}(X), \iota A[0]) \cong D_{\iota A[0]}(HA^{\text{hlg}}(X)).$$

The right Kan extension of $\iota \hat{p}$ therefore is a morphism as in (3.9) and this is the desired pairing.

4 The coarse cohomology theory $Q_C$

4.1 The construction

In this section we introduce the $C$-valued coarse cohomology theory $Q_C$ for $C$ an object of a complete and cocomplete stable $\infty$-category $C$ which is tensored and cotensored over $\text{Spc}$. This coarse cohomology theory can be thought of as a generalized version of coarse stable cohomotopy which is the special case $C = \text{Sp}$ and the sphere spectrum $C := S$. We prove that $Q_C$ is a strong coarse cohomology theory and in Section 4.2 we discuss a natural pairing with coarse stable homotopy which was introduced in [BE20b, Def. 6.29].

If $X$ is a bornological coarse space and $U$ is a coarse entourage of $X$, then $P_U(X)$ denotes the space of probability measures on the discrete measurable space $X$ with finite, $U$-bounded support. This space has the structure of a simplicial complex, and it is a (quasi-)metric
space with the path (quasi-)metric induced by the spherical metric on the simplices. We actually have a functor

\[ \text{BornCoarse}^C \to \text{Top}, \quad (X,U) \mapsto P_U(X) \]

(see Section [3] for the definition of \( \text{BornCoarse}^C \)).

Let \( \iota: \text{Top} \to \text{Spc} \)

(4.1)

be the Dwyer–Kan localization of \( \text{Top} \) at the weak equivalences, where \( \text{Spc} \) denotes the \( \infty \)-category of spaces.

Assume that \( \mathbf{C} \) is a complete and cocomplete \( \infty \)-category which is tensored and cotensored over \( \text{Spc} \). In particular, any object \( C \) of \( \mathbf{C} \) gives rise to a functor

\[ C(-): \text{Spc}^{\text{op}} \to \mathbf{C}, \quad A \mapsto C^A. \]

(4.2)

We shall define a functor \( Q_C: \text{BornCoarse}^{\text{op}} \to \mathbf{C} \)

whose evaluation on objects is given by

\[ X \mapsto \lim_{U \in \mathbf{C}} \colim_{B \in B} \text{Fib}(C^u P_U(X) \to C^u P_U(X \setminus B)). \]

(4.3)

To this end we consider the category \( \text{BornCoarse}^{C,B} \).

1. An object of \( \text{BornCoarse}^{C,B} \) is a triple \((X,U,B)\) of a bornological coarse space \( X \),
a coarse entourage \( U \) of \( X \), and a bounded subset \( B \) of \( X \).

2. A morphism \( f: (X',U',B') \to (X,U,B) \) is a morphism of bornological coarse spaces \( f: X' \to X \) such that \( f(U') \subseteq U \) and \( f^{-1}(B) \subseteq B' \).

We have forgetful functors

\[ \text{BornCoarse}^{C,B} \xrightarrow{q} \text{BornCoarse}^C \xrightarrow{p} \text{BornCoarse}, \quad (X,U,B) \mapsto (X,U) \mapsto X. \]

We furthermore have a functor

\[ W: \text{BornCoarse}^{C,B} \to \text{Top}^{\Delta^1}, \quad W(X,U,B) := (P_U(X \setminus B) \to P_U(X)). \]

It induces the functor

\[ \tilde{Q}_C: (\text{BornCoarse}^{C,B})^{\text{op}} \to \mathbf{C}, \quad \tilde{Q}_C(X,U,B) := \text{Fib}(C^u W). \]

\(^2\)For a quasi-metric we allow infinite distances.
**Definition 4.1.** We define the functor $Q_C$ as the composition of a left and of a right Kan extension as follows:

$$
\begin{array}{ccc}
(\text{BornCoarse}_\mathcal{C})^\text{op} & \xrightarrow{Q_C} & \mathcal{C} \\
q \downarrow & & \downarrow \hat{Q}_C \\
(\text{BornCoarse}_\mathcal{C})^\text{op} & \xrightarrow{Q_C} & \text{BornCoarse}_\mathcal{C}^\text{op}
\end{array}
$$

**Theorem 4.2.** $Q_C$ is a $\mathcal{C}$-valued coarse cohomology theory.

**Proof.** In the following four Lemmas 4.3, 4.4, 4.5 and 4.6 we verify the four axioms from Definition 2.5 on coarse cohomology theories.

**Lemma 4.3.** $Q_C$ is $u$-continuous.

**Proof.** We have $\hat{Q}_C(X,U) \simeq \colim_{B \in \mathcal{B}} \tilde{Q}_C(X,U,B)$. Then, by (4.3),

$$Q_C(X) \simeq \lim_{U \in \mathcal{C}} \hat{Q}_C(X,U) \simeq \lim_{U \in \mathcal{C}} \lim_{n \in \mathbb{N}} \tilde{Q}_C(X,U^n) \simeq \lim_{U \in \mathcal{C}} Q_C(X_U).$$

**Lemma 4.4.** $Q_C$ is coarsely invariant.

**Proof.** For a coarse entourage $U$ of $X$ we form the entourage $\tilde{U} := \{0,1\}^2 \times U$ of $\{0,1\} \times X$. The projection $P_U(\{0,1\} \times Y) \to P_U(Y)$ is a homotopy equivalence for every subset $Y$ of $X$. For every bounded subset $B$ of $X$ we define the bounded subset $\tilde{B} := \{0,1\} \times B$ of $\{0,1\} \times X$. Then $\tilde{Q}_C(X,U,B) \to \tilde{Q}_C(\{0,1\} \times X, \tilde{U}, \tilde{B})$ is an equivalence for every $B$ in $\mathcal{B}$ and $U$ in $\mathcal{C}$. We get an equivalence after applying $\lim_{U \in \mathcal{C}} \colim_{B \in \mathcal{B}}$. Since the bounded subsets of the form $\tilde{B}$ for $B$ in $\mathcal{B}$ and the entourages of the form $\tilde{U}$ for $U$ in $\mathcal{C}$ are cofinal in the bounded subsets or entourages, respectively, of $\{0,1\} \times X$ we get the desired equivalence $Q_C(X) \to Q_C(\{0,1\} \times X)$.

**Lemma 4.5.** $Q_C$ is excisive.

**Proof.** Let $\mathcal{Y} := (Y_i)_{i \in I}$ be a big family on $X$ and let $(\mathcal{Y}, Z)$ be a complementary pair. Let $W$ be a subset of $X$. If $i$ is sufficiently large, then $(Y_i, Z)$ is a $U$-covering of $X$, i.e., every $U$-bounded subset of $X$ is contained in at least one of $Y_i$ or $Z$. In this case

$$
P_U(W \cap Z \cap Y_i) \xrightarrow{P_U(W \cap Z)} P_U(W \cap Z)$$

$$
P_U(W \cap Y_i) \xrightarrow{P_U(W)} P_U(W)$$
is a homotopy cocartesian diagram since it is cocartesian and all maps are inclusions of subcomplexes. It follows that

\[
\begin{array}{ccc}
C^i P_U(W) & \rightarrow & C^i P_U(W \cap Z) \\
\downarrow & & \downarrow \\
C^i P_U(W \cap Y_i) & \rightarrow & C^i P_U(W \cap Z \cap Y_i)
\end{array}
\]

is cartesian, from which we conclude that

\[
\begin{array}{ccc}
\hat{Q}_C(X, U, B) & \rightarrow & \hat{Q}_C(Z, U, B) \\
\downarrow & & \downarrow \\
\hat{Q}_C(Y_i, U, B) & \rightarrow & \hat{Q}_C(Z \cap Y_i, U, B)
\end{array}
\]

is cartesian. We apply \(\lim_{i \in I} \lim_{U \in C} \colim_{B \in B} \) and get a square

\[
\begin{array}{ccc}
Q_C(X) & \rightarrow & Q_C(Z) \\
\downarrow & & \downarrow \\
Q_C(Y) & \rightarrow & Q_C(Z \cap Y)
\end{array}
\]

in \(C\). We can interchange the order of taking the limits, i.e., apply \(\lim_{U \in C} \lim_{i \in I} \colim_{B \in B} \) without changing the result. For every \(U\) in \(C\) let \(I(U)\) be the subset of those \(i\) in \(I\) such that \((Y_i, Z)\) is a \(U\)-covering. By cofinality, we can restrict the limit to \(\lim_{U \in C} \lim_{i \in I(U)} \colim_{B \in B} \).

Then the square above is obtained by applying this operation to a diagram of cartesian squares and, by stability of \(C\) in order to deal with the colimit, is itself cartesian. \(\Box\)

**Lemma 4.6.** \(Q_C\) vanishes on flasques.

**Proof.** Let \(X\) be a flasque bornological coarse space with flasqueness implemented by the morphism \(f: X \rightarrow X\). We write

\[
F_U(B) := \hat{Q}_C(X, U, B).
\]

Note that by definition

\[
Q_C(X) \simeq \lim_{U \in C} \colim_{B \in B} F_U(B).
\]
For an entourage \( U \) of \( X \) we define \( \tilde{U} := \bigcup_{n \in \mathbb{N}} f^n(U) \). We then have the diagram

\[
\begin{array}{c}
\text{colim}_{B \in B, B \cap f^n(X) = \emptyset} F_U(B) \xrightarrow{f^0_*} \text{colim}_{B \in B} F_U(B) \\
\downarrow \\
\text{colim}_{B \in B, B \cap f^n(X) = \emptyset} F_U(B) \xrightarrow{f^0_* + f^1_*} \text{colim}_{B \in B} F_U(B) \\
\downarrow \\
\text{colim}_{B \in B, B \cap f^n(X) = \emptyset} F_U(B) \xrightarrow{f^0_* + f^1_* + f^2_*} \text{colim}_{B \in B} F_U(B) \\
\vdots \\
\downarrow \\
\text{colim}_{B \in B} F_U(B) \xrightarrow{s_U} \text{colim}_{B \in B} F_U(B)
\end{array}
\]

The squares commute since the composition

\[
\text{colim}_{B \in B, B \cap f^n(X) = \emptyset} F_U(B) \to \text{colim}_{B \in B, B \cap f^n(X) = \emptyset} F_U(B) \xrightarrow{f^n_*} \text{colim}_{B \in B} F_U(B)
\]

has a preferred equivalence to zero. The map \( s_U \) is induced. If \( U' \) is a second entourage of \( X \) such that \( U \subseteq U' \), then we have a natural commuting diagram

\[
\begin{array}{c}
\text{colim}_{B \in B} F_{U'}(B) \xrightarrow{s_{U'}} \text{colim}_{B \in B} F_{U'}(B) \\
\downarrow \\
\text{colim}_{B \in B} F_U(B) \xrightarrow{s} \text{colim}_{B \in B} F_U(B)
\end{array}
\]

More precisely, one can perform the construction above in diagrams indexed by the poset \( C \). The construction then yields an interpretation of the family of morphisms \( (s_U)_{U \in C} \) as a morphism between diagrams. By applying \( \lim_{U \in C} \) we get a morphism

\[
s : Q_C(X) \to Q_C(X).
\]

By construction it satisfies

\[
Q_C(f) \circ s + \text{id}_{Q_C(X)} \simeq s.
\]

Since \( Q_C \) is coarsely invariant we conclude that

\[
s + \text{id}_{Q_C(X)} \simeq s
\]

and therefore \( Q_C(X) \simeq 0. \)

4.2 Further properties of \( Q_C \)

In the next lemmas we establish that \( Q_C \) is strong and strongly additive. We furthermore describe the natural pairing with coarse stable homotopy.
Lemma 4.7. $Q_C$ is strong.

Proof. Let $X$ be weakly flasque. We repeat the argument for Lemma 4.6. Since we already know that $Q_C$ is a coarse cohomology theory, in order to see that (4.4) implies (4.5) we only need that $\text{Yo}^*(f) \cong \text{id}_{\text{Yo}^*(X)}$. \hfill \Box

Recall the Definition 2.9 of strong additivity.

Lemma 4.8. If $C$ has the property that cofiltered limits distribute over coproducts, then $Q_C$ is strongly additive.

Proof. Let $(X_i)_{i \in I}$ be a family of bornological coarse spaces and

$$U := \bigsqcup_{i \in I} U_i \quad (4.6)$$

be an entourage of the free union \cite[Def. 2.27]{BE20b}

$$X := \bigsqcup_{i \in I}^\text{free} X_i.$$

Then we have an isomorphism of topological spaces

$$P_U(X) \cong \bigsqcap_{i \in I} P_{U_i}(X_i).$$

A subset $B$ of $X$ is bounded if and only if $B_i := B \cap X_i$ is bounded for all $i$ in $I$ and empty for all but finitely many $i$ in $I$. We conclude that

$$\hat{Q}_C(X, U, B) \simeq \text{Fib}(C^{uP_v}(X) \to C^{uP_v(X \setminus B)}) \simeq \bigsqcup_{i \in I} \text{Fib}(C^{uP_v}(X_i) \to C^{uP_v(X_i \setminus B_i)}) \simeq \bigsqcup_{i \in I} \hat{Q}_C(X_i, U_i, B_i).$$

We get the equivalence

$$\hat{Q}_C(X, U) \simeq \text{colim}_{B \in \mathcal{B}} \hat{Q}_C(X, U, B) \simeq \text{colim}_{B \in \mathcal{B}} \bigsqcup_{i \in I} \hat{Q}_C(X_i, U_i, B_i) \simeq \bigsqcup_{i \in I} \text{colim}_{B_i \in \mathcal{B}_i} \hat{Q}_C(X_i, U_i, B_i) \simeq \bigsqcup_{i \in I} \hat{Q}_C(X_i, U_i),$$

where $\mathcal{B}$ is the poset of bounded subsets of $X$. The subset of entourages of the form (4.6) is cofinal in the coarse structure $C$ of $X$. In the definition of $Q_C(X)$ we can therefore restrict the limit over $\mathcal{C}$ to this set and get the equivalence

$$Q_C(X) \simeq \lim_{U \in \mathcal{C}} \hat{Q}_C(X, U) \simeq \lim_{(U_i)_{i \in I} \in \mathcal{C}} \bigsqcup_{i \in I} \hat{Q}_C(X_i, U_i) \simeq \bigsqcup_{i \in I} \lim_{U_i \in \mathcal{C}_i} \hat{Q}_C(X_i, U_i) \simeq \bigsqcup_{i \in I} Q_C(X_i).$$

In the marked equivalence we use the assumption on $C$. One checks that this equivalence is indeed induced by the collection of morphisms $Q_C(X_i) \to Q_C(X)$ for all $i$ in $I$ given by excision for the complementary pair $(X_i, \{X \setminus X_i\})$ on $X$. \hfill \Box

\footnote{see e.g. \cite[Rem. 6.28]{BE20b}}
We now describe a $C$-valued pairing
\[ p: Q_C \to D_C(Q^{\text{hlg}}) \]
in the sense of Definition 2.15, where $Q^{\text{hlg}}(X)$ is the coarse stable homotopy theory of $X$ [BE20b, Def. 6.29].

We first recall the definition of $Q^{\text{hlg}}$. We start with the functor
\[ \text{BornCoarse}^{C,B} \to \text{Top}^\Delta^1, \ (X,U,B) \mapsto (P_U(X \setminus B) \to P_U(X)) \].

We apply the localization functor $\iota: \text{Top} \to \text{Spc}$, the stabilization functor $\Sigma^\infty_+: \text{Spc} \to \text{Sp}$, and finally the cofibre functor in order to get the functor
\[ \tilde{Q}^{\text{hlg}}: \text{BornCoarse}^{C,B} \to \text{Sp}, \quad \tilde{Q}^{\text{hlg}}(X,U,B) \simeq \text{Cofib}(\Sigma^\infty_+ P_U(X \setminus B) \to \Sigma^\infty_+ P_U(X)) \].

Similarly as in Definition 4.1, the coarse homology theory $Q^{\text{hlg}}$ is obtained as the composition of a right and of a left Kan extension

\[ \begin{array}{ccc}
\text{BornCoarse}^{C,B} & \xrightarrow{\tilde{Q}^{\text{hlg}}} & \text{Sp} \\
\downarrow & & \downarrow \\
\text{BornCoarse}^{C} & \xrightarrow{Q^{\text{hlg}}} & \text{Sp} \\
\downarrow & & \downarrow \\
\text{BornCoarse} & & \\
\end{array} \]

Since $C$ is stable, the power structure of $C$ over $\text{Spc}$ extends to a power structure over $\text{Sp}$. If we fix the object $C$ in $C$, then in analogy with (4.2) we have a functor
\[ C^{(-)}: \text{Sp}^{\text{op}} \to C, \quad W \mapsto C^W. \quad (4.7) \]

For a space $A$ we have the natural equivalence $C^A \simeq C^{\Sigma^\infty_+ A}$.

We now construct the pairing. We first observe that we have an equivalence of functors
\[ \tilde{Q}_C \simeq C^\tilde{Q}^{\text{hlg}}: (\text{BornCoarse}^{C,B})^{\text{op}} \to C. \]

Indeed, for $(X,U,B)$ in $\text{BornCoarse}^{C,B}$ we have the natural equivalences
\[
\tilde{Q}_C(X,U,B) \simeq \text{Fib}(C^{P_U(X)} \to C^{P_U(X \setminus B)}) \\
\simeq \text{Fib}(C^{\Sigma^\infty_+ P_U(X)} \to C^{\Sigma^\infty_+ P_U(X \setminus B)}) \\
\simeq C^{\text{Cofib}(\Sigma^\infty_+ P_U(X \setminus B) \to \Sigma^\infty_+ P_U(X))}. 
\]

We now form the left Kan extension of this equivalence along the functor
\[ (\text{BornCoarse}^{C,B})^{\text{op}} \to (\text{BornCoarse}^{C})^{\text{op}} \]
and get the natural transformation

\[ \hat{Q}_C \simeq LK(C^\hat{Q}_{hlg}) \overset{1}{\to} C^{RK(\hat{Q}_{hlg})} \simeq C^\hat{Q}_{hlg}. \]

Here \( LK \) and \( RK \) stand for the left, resp. right Kan extension. In general, the marked transformation is not an equivalence since the functor \( (4.7) \) in general does not preserve colimits. We now form the right Kan extension of this morphism along the functor

\[ (\text{BornCoarse}^C)^{op} \to \text{BornCoarse}^{op} \]

and get the morphism

\[ p: Q_C \simeq RK(\hat{Q}_C) \to RK(C^\hat{Q}_{hlg}) \overset{1}{\to} C^{LK(\hat{Q}_{hlg})} \simeq C^\hat{Q}_{hlg} \simeq D_C(\hat{Q}_{hlg}) \]

which is the desired pairing. Note that here the marked morphism is an equivalence since \( (4.7) \) preserves limits.

5 The dualizing spectrum of a group

5.1 Poincaré duality groups

Let \( G \) be a group in \( \text{Spc} \), or equivalently, a group-like \( E_1 \)-monoid. Then we can form its classifying space \( BG \) which is a pointed object of \( \text{Spc} \). In fact, every connected pointed space \( X \) in \( \text{Spc} \) is equivalent to the classifying space \( B(\Omega X) \) of its loop space \( \Omega X \) considered as a group in \( \text{Spc} \).

Considering \( BG \) as an \( \infty \)-groupoid we define the category of \( G \)-spectra by

\[ G\text{Sp} := \text{Fun}(BG, \text{Sp}). \]  (5.1)

For \( E \) in \( G\text{Sp} \) we can form the fixed points and orbits

\[ E^G := \lim_{BG} E, \quad E^G := \colim_{BG} E \]

in \( \text{Sp} \).

We can form the spherical group ring \( S[G] := S \wedge G_+ \) in \( (G \times G)\text{Sp} \) using the left and right actions of \( G \) on itself. Following Klein [Kle01] we then define the dualizing spectrum of \( G \) by

\[ D_G := \lim_{BG} S[G] \]

in \( G\text{Sp} \), where we use the right action to form the fixed points and keep the residual left action. We will write \( D_G \) in \( \text{Sp} \) for the underlying spectrum of \( D_G \).

For any \( E \) in \( G\text{Sp} \) Klein [Kle01, Sec. 3] introduces a norm map in \( \text{Sp} \)

\[ N^E: D_G \wedge_G E \to E^G \]
as the composition

\[
\begin{align*}
D_G \wedge_G E &\overset{\text{def}}{=} \text{colim} \lim_{BG} S[G] \wedge E \\
\rightarrow &\text{colim} \lim_{BG} S[G] \wedge E \\
\rightarrow &\lim \text{colim}_{BG} S[G] \wedge E \\
\uparrow &\text{lim}_{BG} E \\
\cong &\lim_{BG} E \\
\cong &E^G.
\end{align*}
\]

The marked equivalence uses \(\text{colim}_{BG}(S[G] \wedge E) \cong E\) in \(G\text{Sp}\) which holds since \(S[G] \wedge E\) is freely induced. If \(BG\) is a compact object in \(\text{Spc}\), using that \(\text{Sp}\) is stable, the limit over \(BG\) commutes with \(- \wedge E\) and the colimit of \(BG\) so that the two arrows in (5.2) are equivalences. In this case the norm map is an equivalence, compare [Kle01, Thm. D].

Recall that homology of \(BG\) with coefficients in \(D_G \wedge E\) and the cohomology of \(BG\) with coefficients in \(E\) are defined by

\[
H_*(BG, D_G \wedge E) := \pi_*(D_G \wedge_G E) \quad \text{and} \quad H^*(BG, E) := \pi_{-*}(E^G).
\]

The norm map induces a map from the homology of \(BG\) with coefficients in \(D_G \wedge E\) to the cohomology with coefficients in \(E\). If the underlying spectrum \(D_G\) is equivalent to a shift \(S^{-n}\) of the sphere spectrum and \(BG\) is compact in \(\text{Spc}\), then the norm map induces a Poincaré duality isomorphism

\[
H_{n-*}(BG, L \wedge E) \overset{\cong}{\rightarrow} H^*(BG, E),
\]

where \(L := \Sigma^n D_G\). Indeed we have:

**Theorem 5.1** (Klein [Kle01 Thm. A]). Assume that \(BG\) is compact in \(\text{Spc}\). Then the following are equivalent:

1. \(BG\) is a Poincaré duality space.
2. \(D_G\) is a shift of the sphere spectrum.
3. \(D_G\) is a finite spectrum.

We now restrict to discrete groups \(G\). The importance of the homotopy type \(D_G\) indicated above raises the question how it can be calculated and how it depends on the group. Klein [Kle01 Conj. on Page 455] conjectured that \(D_G\) only depends on the quasi-isometry class of \(G\) with respect to the word metric.

Referring to Section 4.1, we consider the case \(C = \text{Sp}\) and the sphere spectrum \(C := S\).

By Definition 4.1 we get a coarse cohomology theory

\[
Q_S: \text{BornCoarse}^{op} \rightarrow \text{Sp}.
\]

By \(G_{\text{can,min}}\) (see [BE20b, Ex. 2.21]) we denote the object of \(\text{BornCoarse}\) obtained by equipping the group \(G\) with the bornological coarse structure given by the minimal bornology and the canonical coarse structure. Our main technical result is now:
Theorem 5.2. If $G$ is finitely generated and torsion-free, then we have an equivalence $D_G \simeq Q_S(G_{can,min})$ in $\text{Sp}$.

Thus $D_G$ is the value of a coarse cohomology theory on $G_{can,min}$ and therefore we get as an immediate consequence:

Corollary 5.3. For finitely generated and torsion-free groups $G$ the spectrum $D_G$ only depends on the coarse motivic spectrum $Yo^*(G_{can,min})$. In particular, it is an invariant of the quasi-isometry class.

Remark 5.4. If $BG$ is compact in $\text{Spc}$, then $G$ is finitely generated and torsion-free.

Example 5.5. If $M$ is a complete, simply connected, negatively curved $n$-dimensional Riemannian manifold, then its Gromov boundary $\partial M$ is homeomorphic to $S^{n-1}$. By Higson–Roe [HR95, Sec. 8] we have a coarse homotopy equivalence between $M$ and the open cone over $\partial M$. By [BE20a, Sec. 6] we know the open cone and the Euclidean cone yield equivalent objects in $\text{Sp}X$ after applying Yo*. Since the Euclidean cone over $S^{n-1}$ is coarsely equivalent to $\mathbb{R}^n$ we get $Yo^*(M) \simeq Yo^*(\mathbb{R}^n)$.

If $G$ acts freely and cocompactly on $M$, then we have coarse equivalence $G_{can,min} \to M$ given by $g \mapsto gm_0$ for any choice of a point $m_0$ in $M$. So we get an equivalence

$$Yo^*(G_{can,min}) \simeq Yo^*(\mathbb{R}^n) \simeq \Sigma^n Yo^*(\ast),$$

where $!$ is a standard calculation in coarse homotopy theory (see e.g. [BE20a, Ex. 4.9]). Consequently,

$$Q_S(G_{can,min}) \simeq Q_S(\mathbb{R}^n) \simeq \Sigma^{-n} Q_S(\ast) \simeq S^{-n}$$

as expected since $BG \simeq M/G$ is an $n$-dimensional Poincaré duality space.

Example 5.6. In the following example we provide a pair of groups $G$ and $H$ which are not quasi-isometric, but for which $Yo^*(G_{can,min}) \simeq Yo^*(H_{can,min})$. This example shows that the first assertion of Corollary 5.3 is strictly stronger than the second. We consider torsion-free and cocompact lattices $G$ in $SO(2n, 1)$ and $H$ in $SU(n, 1)$. Such lattices exist by a result of Borel [Bor63]. Then $G$ is quasi-isometric to the hyperbolic space $\mathbb{H}^{2n}$ and $H$ is quasi-isometric to the complex hyperbolic space $\mathbb{H}C^n$ (of real dimension $2n$). By Mostow rigidity (Mostow [Mos73], Kleiner–Leeb [KL97, Cor. 1.1.4]) $\mathbb{H}^{2n}$ and $\mathbb{H}C^n$ are not quasi-isometric, and hence $G$ and $H$ are not quasi-isometric.

The boundaries of the negatively curved spaces $\mathbb{H}^{2n}$ and $\mathbb{H}C^n$ are both homeomorphic to $S^{2n-1}$. Hence

$$Yo^*(G_{can,min}) \simeq \Sigma^{2n-1} Yo^*(\ast) \simeq Yo^*(H_{can,min})$$

by Example 5.5.
5.2 Proof of Theorem 5.2

Let $\iota : \text{Top} \to \text{Spc}$ denote the canonical functor from topological spaces to the $\infty$-category of spaces. For a group $G$ we denote by $BG$ the category consisting of one object whose monoid of endomorphisms is given by the group $G$. Note that

$$\iota \vert \mathbb{N}(BG) \simeq BG,$$  \hfill (5.3)

where $\mathbb{N}(BG)$ is the geometric realization of the nerve of $BG$.

By $\text{Orb}(G)$ we denote the orbit category of $G$ which is the category of transitive $G$-sets and equivariant maps. We form the categories

$$G\text{Top} := \text{Fun}(BG, \text{Top})$$

of $G$-topological spaces (i.e., objects are topological spaces with an action of $G$, and morphisms are equivariant continuous maps) and

$$G\text{Spc} := \text{Fun}(BG, \text{Spc}), \quad G[\text{Spc}] := \text{Fun}(\text{Orb}(G)^{\text{op}}, \text{Spc})$$

of spaces with a $G$-action and of $G$-spaces. The category $G[\text{Spc}]$ models the $G$-equivariant homotopy theory and is the natural home for classifying spaces $E_{\mathcal{F}}G$ for families $\mathcal{F}$ of subgroups of $G$. We have a functor $\iota_G : G\text{Top} \to G[\text{Spc}]$ which sends the $G$-topological space $X$ to the functor

$$\text{Orb}(G)^{\text{op}} \ni O \mapsto \iota_G(X)(O) := \iota \text{Map}_G(O, X) \in \text{Spc},$$  \hfill (5.4)

where $\text{Map}_G(Y, X)$ denotes the topological space of $G$-equivariant maps from $Y$ to $X$ with the compact-open topology, and $O$ is considered as a discrete $G$-topological space.

The category $G\text{Spc}$ models the homotopy theory of topological spaces with $G$-action and equivariant maps, where weak equivalences are maps which are weak equivalences after forgetting the $G$-action. In contrast, by Elmendorf's theorem $\iota_G$ models the Dwyer–Kan localization of $G\text{Top}$ at the equivariant weak equivalences. We have an isomorphism of monoids

$$\text{End}_{\text{Orb}(G)}(G) \cong G^{\text{op}},$$

and therefore an inclusion

$$BG^{\text{op}} \hookrightarrow \text{Orb}(G).$$  \hfill (5.5)

This inclusion induces an adjunction

$$\text{Res} : G[\text{Spc}] \rightleftarrows G\text{Spc} : \text{Coind}$$  \hfill (5.6)

relating the two categories.

Furthermore, we let

$$G\text{Sp} := \text{Fun}(BG, \text{Sp}), \quad G[\text{Sp}] := \text{Fun}(\text{Orb}(G)^{\text{op}}, \text{Sp})$$
be the categories of spectra with a $G$-action and of naive $G$-spectra. Since $G$ is discrete and in view of (5.3) this new definition of $G\text{Sp}$ is equivalent to the former [5.1]. The inclusion (5.5) induces an adjunction

$$\text{Res} : G\text{Sp} \rightleftarrows \text{Coind} : G\text{Sp}.$$ (5.7)

An $\Omega$ spectrum is a spectrum $(E_n, \sigma_n)_{n \in \mathbb{N}}$ in topological spaces such that $\sigma_n : E_n \to \Omega E_{n+1}$ is a weak equivalence. A weak equivalence between $\Omega$-spectra is a morphism which is a level-wise weak equivalence. We denote by $\text{Sp}^\Omega$ the ordinary category of $\Omega$-spectra. The relative category $(\text{Sp}^\Omega, W)$, where $W$ denotes the class of weak equivalences, is a presentation of the category of spectra. In particular, we have a functor

$$\kappa : \text{Sp}^\Omega \to \text{Sp}^\Omega[W^{-1}] \simeq \text{Sp}.$$ (5.8)

We furthermore consider the category

$$G\text{Sp}^\Omega := \text{Fun}(BG, \text{Sp}^\Omega)$$

of $\Omega$-spectra with a $G$-action and use the symbol $\kappa$ also for the induced functor

$$\kappa : G\text{Sp}^\Omega \to G\text{Sp}.$$ We consider the $G$-spectrum $S[G]$ in $G\text{Sp}$, i.e., we only keep the right-action of $G$ on itself which is used to form the limit over $BG$ later.

**Remark 5.7.** In greater detail, $S[G]$ is given by $\coprod_{g \in G} S$, where the $G$-action is given by the action of $G$ on the index set by right multiplication. The technical description is

$$S[G] := \text{Ind}_G^G(S),$$

where $\text{Ind}_G^G : \text{Sp} \to G\text{Sp}$ is the left-adjoint of the forgetful functor $G\text{Sp} \to \text{Sp}$.

Equivalently, we can choose an $\Omega$-spectrum $QS$ in $\text{Sp}^\Omega$ with $\kappa(QS) \simeq S$. Then we form the $G$-$\Omega$-spectrum $\text{Map}_c(G, QS)$ of compactly supported maps from $G$ to $QS$ (see below for details), where $G$ is considered as a discrete $G$-space with the right action. Then we have an equivalence

$$S[G] \simeq \kappa\text{Map}_c(G, QS).$$ (5.8)

In order to see this equivalence we first observe that for a finite discrete space $F$ we have an equivalence

$$S[F] \simeq \prod_{F \subseteq G} \kappa(QS) \simeq \kappa\text{Map}(F, QS) \simeq \kappa\text{Map}_c(G, QS).$$

This equivalence is functorial for embeddings of finite sets where on the side of the mapping spaces we use extension of maps by zero. We then use that

$$S[G] \simeq \colim_{F \subseteq G} S[F] \simeq \colim_{F \subseteq G} \kappa\text{Map}(F, QS) \simeq \kappa\colim_{F \subseteq G} \text{Map}(F, QS) \simeq \kappa\text{Map}_c(G, QS),$$

where the colimit is taken over the finite subsets of $G$, and where for the last equivalence we use that extension by zero of functions is level-wise a closed embedding of $\Omega$-spectra.

---

$^4$We thank Thomas Nikolaus for pointing out that the equivalence [5.8] should be justified.
We have a functor \( \lim_{B G} : G \text{Sp} \to \text{Sp} \),
and, by definition, an equivalence
\[ D_G \simeq \lim_{B G} S[G]. \]

Let \( X \) be a \( G \)-topological space and \( Z \) a pointed topological space. For a subset \( K \) of \( X \) we let \( \text{Map}_K(X, Z) \) denote the subspace of \( \text{Map}(X, Z) \) of maps which send \( X \setminus K \) to the base point. We define the \( G \)-subset of \( \text{Map}(X, Z) \)
\[ \text{Map}_c(X, Z) := \bigcup_K \text{Map}_K(X, Z) \]
of compactly supported maps, where \( K \) runs over all compact subsets of \( X \). We equip \( \text{Map}_c(X, Z) \) with the inductive limit topology. Note that this topology is in general finer than the induced topology from \( \text{Map}(X, Z) \).

Let \( X,Y \) be \( G \)-topological spaces and \( Z \) be a pointed topological space.

**Lemma 5.8.** If \( G \) acts properly and cocompactly on \( X \) and \( Y \), then we have a homeomorphism
\[ \text{Map}_G(X, \text{Map}_c(Y, Z)) \cong \text{Map}_G(Y, \text{Map}_c(X, Z)). \] (5.9)

**Proof.** We define the \( G \)-space
\[ \text{Map}_d(X \times Y, Z) := \colim_{(K,L)} \text{Map}_G(K \times L)(X \times Y, Z) \]
equipped with the inductive limit topology, where \( K \) (or \( L \)) runs over the compact subsets of \( X \) (resp. \( Y \)) and \( G \) acts diagonally on \( X \times Y \). We compare both sides of (5.9) with \( \text{Map}_d(X \times Y, Z)^G \). We carry out the arguments only for the case of \( \text{Map}_G(X, \text{Map}_c(Y, Z)) \) since the other case is completely analogous.

Assume that \( f \) belongs to \( \text{Map}_G(X, \text{Map}_c(Y, Z)) \). By the exponential law for maps between sets it corresponds to a \( G \)-equivariant map \( \tilde{f} : X \times Y \to Z \) which by \( G \)-equivariance is determined by its restriction to \( K \times Y \) for any compact subset \( K \) of \( X \) with \( GK = X \). Since we equip \( \text{Map}_c(Y, Z) \) with the inductive limit topology, there exists a compact subset \( L \) of \( Y \) with \( \tilde{f}(k, y) = * \) for all \( k \in K \) and \( y \in Y \setminus L \). In other words, \( \tilde{f} \in \text{Map}_G(K \times L)(X \times Y, Z)^G \).

In this way we define a map
\[ \text{Map}_G(X, \text{Map}_c(Y, Z)) \to \text{Map}_d(X \times Y, Z)^G. \]

Assume now that \( \tilde{f} \) belongs to \( \text{Map}_d(X \times Y, Z)^G \). Then there is a pair \( (K, L) \) of compact subsets of \( X \) and \( Y \), respectively, such that \( \tilde{f} \) is supported on \( G(K \times L) \). Since \( G \) acts properly on \( X \), the set \( F := \{ g \in G \mid gK \cap K \neq \emptyset \} \) is finite. Then \( \tilde{L} := FL \) is compact and \( \tilde{f}(x, y) = * \) for \( x \) in \( K \) and \( y \in Y \setminus L \). Let \( f : X \to \text{Map}(Y, Z) \) be the adjoint of \( \tilde{f} \). Then \( f|_K \) takes values in \( \text{Map}_c(Y, Z) \). This shows that \( f \in \text{Map}_G(X, \text{Map}_c(Y, Z)) \). In this way we have constructed the inverse map
\[ \text{Map}_d(X \times Y, Z)^G \to \text{Map}_G(X, \text{Map}_c(Y, Z)). \]
If \( E = (E_n, \sigma_n)_{n \in \mathbb{N}} \) is a \( G\)-\( \Omega \)-spectrum, then for a \( G \)-topological space \( X \) we get a \( G\)-\( \Omega \)-spectrum
\[
\text{Map}(X, E) := (\text{Map}(X, E_n), \sigma^X_n)_{n \in \mathbb{N}},
\]
where
\[
\sigma^X_n : \text{Map}(X, E_n) \xrightarrow{\sigma_n} \text{Map}(X, \Omega E_{n+1}) \cong \Omega \text{Map}(X, E_{n+1}).
\]
If \( X \) is a CW-complex and \( E \) is an \( \Omega \)-spectrum, then we have the equivalence in \( \mathbf{Sp} \)
\[
\kappa \text{Map}(X, E) \simeq (\kappa E)^{iX}. \tag{5.10}
\]
In order to see this note that both sides are cohomology theories in the CW-complex argument and coincide for \( X = \ast \).

Furthermore, if \( X \) is a free \( G \)-CW-complex and \( E \) is a \( G\)-\( \Omega \)-spectrum, then we have the equivalence in \( \mathbf{Sp} \)
\[
\kappa \text{Map}_G(X, E) := \kappa \lim_{B_G} \text{Map}(X, E) \simeq \lim_{B_G} [(\kappa E)^{\text{Res}(\iota_G X)}], \tag{5.11}
\]
where \( \text{Res} \) is as in \((5.6)\) and \( \iota_G \) as in \((5.4)\). Again, both sides are cohomology theories on free \( G\)-CW-complexes and coincide on \( X = G \).

Similarly, we define the \( G\)-\( \Omega \)-spectrum \( \text{Map}_c(X, E) \) by \((\text{Map}_c(X, E_n), \sigma^X_{c,n})_{n \in \mathbb{N}}\), where
\[
\sigma^X_{c,n} : \text{Map}_c(X, E_n) \xrightarrow{\sigma_n} \text{Map}_c(X, \Omega E_{n+1}) \cong \Omega \text{Map}_c(X, E_{n+1}).
\]

For the last isomorphism we used that the circle \( S^1 \) is compact.

We now choose an \( \Omega \)-spectrum \( QS \) representing the sphere spectrum. We consider \( G \) as a discrete \( G \)-space and note that \( G \) acts properly and cocompactly on \( G \). By Remark \(5.7\) we have
\[
S[G] \simeq \kappa \text{Map}_c(G, QS).
\]
The Rips complex of a \( G \)-coarse space \( X \) with coarse structure \( C \) is defined by
\[
\text{Rips}(X) := \colim_{U \in \mathit{CG}} P_U(X),
\]
where the colimit is interpreted in the category \( G\text{Top} \) of \( G \)-topological spaces. Then by \[\{BEKW20\} \text{ Lemb. 11.4} \] we have an equivalence
\[
\iota_G \text{Rips}(G_{can}) \simeq E_{\text{Fin}G}
\]
in \( G[\mathbf{Sp}] \). Since we assume that \( G \) is torsion-free we have an equivalence \( E_{\text{Fin}G} \simeq EG \).

Since \( G \) is finitely generated the coarse structure \( C \) of \( G_{can} \) is generated by a single invariant entourage \( U_{\text{gen}} \). Hence we have an equivariant homeomorphism
\[
\text{Rips}(G_{can}) \cong \colim_{n \in \mathbb{N}} P_{U_{\text{gen}}^n}(G).
\]
We further observe that $P_{U_{gen}}^n(G)$ is a locally finite $G$-CW-complex, and that the morphisms $P_{U_{gen}}^n(G) \to P_{U_{gen}}^{n+1}(G)$ are inclusions of subcomplexes. It follows that the colimit over these inclusions is a homotopy colimit, i.e., that we have the equivalence

$$EG \cong \iota_G \text{Rips}(G_{can}) \cong \varprojlim n \in \mathbb{N} \iota_G P_{U_{gen}}^n(G)$$

in $G[Spc]$. Let $(-)^G : G[Sp] \to Sp$ be the evaluation functor at the one-point $G$-set. For a $G$-$\Omega$-spectrum $E$ we have the following chain of equivalences in $Sp$:

$$\lim_{BG} \kappa E \cong (\text{Coind } \kappa E)^G \cong ((\text{Coind } \kappa E)^{EG})^G \cong ((\text{Coind } \kappa E)^{\varprojlim n \in \mathbb{N} \iota_G P_{U_{gen}}^n(G)})^G \cong \lim_{n \in \mathbb{N}}((\text{Coind } \kappa E)^{\iota_G P_{U_{gen}}^n(G)})^G \cong \lim_{n \in \mathbb{N}} \lim_{BG} \kappa \text{Map}_G(P_{U_{gen}}^n(G), E).$$

(5.11)

For the first equivalence we interpret Coind in (5.7) as the right Kan extension functor along $BG \to G\text{Orb}^{op}$ and use the point-wise formula for the evaluation of the result at the initial object of $G\text{Orb}^{op}$. For the last equivalence we use the fact that $P_{U_{gen}}^n(G)$ is a free $G$-CW-complex since $G$ is torsion-free.

We obtain the equivalence

$$D_G \cong \lim_{n \in \mathbb{N}} \kappa \text{Map}_G(P_{U_{gen}}^n(G), \text{Map}_c(G, QS)).$$

Since $G$ acts properly and cocompactly on both $P_{U_{gen}}^n(G)$ and $G$, we get by Lemma 5.8

$$D_G \cong \lim_{n \in \mathbb{N}} \kappa \text{Map}_G(G, \text{Map}_c(P_{U_{gen}}^n(G), QS)) \cong \lim_{n \in \mathbb{N}} \kappa \text{Map}_c(P_{U_{gen}}^n(G), QS),$$

where the second equivalence is induced by the evaluation at the identity of $G$. We now observe that the subsets of the form $P_{U_{gen}}^n(G) \setminus P_{U_{gen}}^m(G \setminus B)$ for all bounded subsets $B$ of $G_{can,min}$ (i.e., finite subsets of $G$) are cofinal in the set of compact subsets of $P_{U_{gen}}^n(G)$. It follows that

$$\text{Map}_c(P_{U_{gen}}^n(G), QS) \cong \colim_{B \in \mathcal{B}} \text{Fib}(\text{Map}(P_{U_{gen}}^n(G), QS) \to \text{Map}(P_{U_{gen}}^n(G \setminus B), QS)),$$

where $\mathcal{B}$ denotes the bornology of $G_{can,min}$. Hence we get

$$D_G \cong \lim_{n \in \mathbb{N}} \kappa \colim_{B \in \mathcal{B}} \text{Fib}(\text{Map}(P_{U_{gen}}^n(G), QS) \to \text{Map}(P_{U_{gen}}^n(G \setminus B), QS)) .$$

Let $QS_k$ be the $k$'th space of the $\Omega$-spectrum $QS$. For $B$ in $\mathcal{B}$ we set

$$F(B)_k := \text{Fib}(\text{Map}(P_{U_{gen}}^n(G), QS_k) \to \text{Map}(P_{U_{gen}}^n(G \setminus B), QS_k)).$$
The structure maps of the spectrum $QS$ induce structure maps

$$\sigma(B)_n : \Sigma F(B)_k \to F(B)_{k+1}$$

which turn $F(B) := (F(B)_k, \sigma(B)_n)_{n \in \mathbb{N}}$ into an $\Omega$-spectrum. If $B'$ is a second element in $\mathcal{B}$ such that $B \subseteq B'$, then the natural morphism $F(B)_k \to F(B')_k$ is a closed embedding. Indeed, it is the embedding of the space of $QS_k$-valued functions on $P_{Ungen}(G)$ which are constant on the subset $P_{Ungen}(G \setminus B)$ into the space of such functions which are constant on the smaller subset $P_{Ungen}(G \setminus B')$. Since taking homotopy groups commutes with filtered colimits of systems of pointed spaces whose structure maps are closed embeddings we can conclude that $\text{colim}_{B \in \mathcal{B}} F(B)$ is again an $\Omega$-spectrum, and that this also represents the spectrum $\text{colim}_{B \in \mathcal{B}} \kappa F(B)$. Hence we can switch the order of $\kappa$ and taking the colimit. Furthermore, because $P_{Ungen}(G \setminus B) \to P_{Ungen}(G) \setminus B'$ is an inclusion of a locally finite subcomplex, the induced map between $\Omega$-spectra is a fibration between $\Omega$-spectra. Therefore $\text{Fib}$ in the formula above can be commuted with $\kappa$. Therefore

$$D_G \simeq \lim_{n \in \mathbb{N}} \text{colim}_{B \in \mathcal{B}} \text{Fib}(\kappa \text{Map}(P_{Ungen}(G), QS) \to \kappa \text{Map}(P_{Ungen}(G \setminus B), QS)).$$

We now use the relation (5.10) in order to get the equivalence

$$D_G \simeq \lim_{n \in \mathbb{N}} \text{colim}_{B \in \mathcal{B}} \text{Fib}((\kappa QS)^\ell P_{Ungen}(G) \to (\kappa QS)^\ell P_{Ungen}(G \setminus B)).$$

Finally, by cofinality we replace the limit over $\mathbb{N}$ by the limit over $\mathcal{C}$. In view of (4.3) we get the desired equivalence

$$D_G \simeq QS(G_{can, min}),$$

which finishes the proof of Theorem 5.2.

**Remark 5.9.** Let $\mathbf{Orb}_\mathcal{F}(G)$ denote the full subcategory of the orbit category of $G$ of transitive $G$-sets with stabilizers in the family of subgroups $\mathcal{F}$. We set

$$G_{\mathcal{F}}[\mathbf{Sp}] := \text{Fun}(\mathbf{Orb}_\mathcal{F}(G)^{\text{op}}, \mathbf{Sp}).$$

Then for every two families $\mathcal{F}$ and $\mathcal{F}'$ with $\mathcal{F} \subseteq \mathcal{F}'$ we have a corresponding pair of adjoint functors $(\text{Ind}_{\mathcal{F}}^\mathcal{F}', \text{Res}_{\mathcal{F}}^{\mathcal{F}})$, see [BEKW20, Sec. 10.3]. If $E \in G_{\mathbf{All}}[\mathbf{Sp}]$, then we define

$$E^{(h_{\mathcal{F}}G)} := \lim_{\text{Orb}_{\mathcal{F}}(G)} \text{Res}_{\mathcal{F}}^{\mathbf{All}} E.$$ 

If $H$ is a subgroup of $G$, then we have an induction functor

$$\text{Ind}_{H \cap \mathcal{F}}^G : H_{\mathcal{F} \cap \mathcal{H}}[\mathbf{Sp}] \to G_{\mathcal{F}}[\mathbf{Sp}].$$

We could consider $S[G]$ as an object $\text{Ind}_{(1)}^G G_{\mathbf{All}}(S)$ of $G_{\mathbf{All}}[\mathbf{Sp}]$. Then by construction

$$D_G \simeq S[G]^{(h_{(1)} G)}.$$

\[\checkmark\]
An appropriate modification (with $\text{BG} \simeq \text{Orb}_{\{1\}}(G)$ replaced by $\text{Orb}_{\text{Fin}}(G)$) of the proof of Proposition 5.2 shows the following more general statement:

**Proposition 5.10.** For every finitely generated group $G$ we have an equivalence

$$S[G]^{(\text{hFin}G)} \simeq Q_S(G_{\text{can,min}}).$$

If $G$ is torsion-free, then we have the equality of families $\text{Fin} = \{1\}$ and Proposition 5.2 follows from Proposition 5.10.

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