Loop homotopy algebras in closed string field theory

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Abstract. Barton Zwiebach constructed [20] ‘string products’ on the Hilbert space of a combined conformal field theory of matter and ghosts, satisfying the ‘main identity.’ It has been well known that the ‘tree level’ of the theory gives an example of a strongly homotopy Lie algebra (though, as we will see later, this is not the whole truth).

Strongly homotopy Lie algebras are now well-understood objects. On the one hand, strongly homotopy Lie algebra is given by a square zero coderivation on the cofree cocommutative connected coalgebra [4, 13]; on the other hand, strongly homotopy Lie algebras are algebras over the cobar dual of the operad $\text{Com}$ for commutative algebras [9].

As far as we know, no such characterization of the structure of string products for arbitrary genera has been available, though there are two series of papers directly pointing towards the requisite characterization.

As far as the characterization in terms of (co)derivations is concerned, we need the concept of higher order (co)derivations, which has been developed, for example, in [2, 3]. These higher order derivations were used in the analysis of the ‘master identity.’ For our characterization we need to understand the behavior of these higher (co)derivations on (co)free (co)algebras.

The necessary machinery for the operadic approach is that of modular operads, anticipated in [5] and introduced in [8]. We believe that the modular operad structure on the compactified moduli space of Riemann surfaces of arbitrary genera implies the existence of the structure we are interested in the same manner as was explained for the tree level in [11].

We also indicate how to adapt the loop homotopy structure to the case of open string field theory [19].

Plan of the paper: Section 1 – Introduction
Section 2 – Sign interlude and the definition
Section 3 – Higher order (co)derivations
Section 4 – Loop homotopy Lie algebras - 1st description
Section 5 – Loop homotopy Lie algebras - operadic approach
Section 6 – Possible generalizations (open strings)

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1. Introduction.

Let $\mathcal{H}$ be the Hilbert space of a combined conformal field theory of matter and ghosts and let $\mathcal{H}_{\text{rel}} \subset \mathcal{H}$ be the subspace of elements annihilated by $b^- := b_0 - \bar{b}_0$ and $L^- := L_0 - \bar{L}_0$ (see, for example, [11, Section 4]). Barton Zwiebach constructed in [20], for each ‘genus’ $g \geq 0$ and for each $n \geq 0$, multilinear ‘string products’

$$\mathcal{H}_{\text{rel}}^\otimes n \ni B_1, \ldots, B_n \mapsto [B_1, \ldots, B_n]_g \in \mathcal{H}_{\text{rel}}.$$ 

Recall the basic properties of these products. If $\text{gh}(\cdot)$ denotes the ghost number, then [20, (4.8)]

$$\text{gh}([B_1, \ldots, B_n]_g) = 3 - 2n + \sum_{i=1}^n \text{gh}(B_i).$$

The string products are graded (super) commutative [20, (4.4)]:

$$[B_1, \ldots, B_i, B_{i+1}, \ldots, B_n]_g = (-1)^{\text{gh}(B_i)\text{gh}(B_{i+1})}[B_1, \ldots, B_i, B_{i+1}, \ldots, B_n]_g.$$

Here we used the notation

$$(-1)^{\text{gh}(B_i)\text{gh}(B_{i+1})} := (-1)^{\text{gh}(B_i)\text{gh}(B_{i+1})}.$$

For $n = 0$ and $g \geq 0$, $[\cdot]_g \in \mathcal{H}_{\text{rel}}$ is just a constant, and the products are constructed in such a way that $[\cdot]_0 = 0$ [20, (4.6)]. The linear operation $[B]_0 =: QB$ is identified with the BRST differential of the theory. These product satisfy, for all $n, g$, the main identity [20, (4.13)]:

$$0 = \sum \sigma(i_1, j_k) [B_{i_1}, \ldots, B_{i_l}, [B_{j_1}, \ldots, B_{j_k}]_{g_2}, g_1] + \frac{1}{2} \sum_s (-1)^{\Phi_s} [\Phi_s, \Phi^s, B_1, \ldots, B_n]_{g-1}.$$

Here the first sum runs over all $g_1 + g_2 = g$, $k + l = n$, and all sequences $i_1 < \cdots < i_l$, $j_1 < \cdots < j_k$ such that $\{i_1, \ldots, i_l, j_1, \ldots, j_k\} = \{1, \ldots, n\}$. Such sequences are called unshuffles (see the terminology introduced at the beginning of Section [2]). The sign $\sigma(i_1, j_k)$ is picked up by rearranging the sequence $(Q, B_1, \ldots, B_n)$ into the order $(B_{i_1}, \ldots, B_{i_l}, Q, B_{j_1}, \ldots, B_{j_k})$. In the second sum, $\{\Phi_s\}$ is a basis of $\mathcal{H}_{\text{rel}}$ and $\{\Phi^s\}$ its dual basis in the sense that

$$(-1)^{\Phi_s} \langle \Phi_r, \Phi^s \rangle = \delta^s_r \text{ (Kronecker delta)}.$$
where \((-,-)\) denotes the bilinear inner product on \(\mathcal{H}\) \([20, (2.44)]\). Let us remark that, in the original formulation of \([20]\), \(\{\Phi_s\}\) was a basis of the whole \(\mathcal{H}\), but the sum in (2) was restricted to \(\mathcal{H}_{rel}\). The product satisfies \([20, (2.62)]\):

\[
(3) \quad \langle A, B \rangle = (-1)^{(A+1)(B+1)} \langle B, A \rangle
\]

and it is nontrivial only for elements whose ghost numbers add up to five:

\[
(4) \quad \text{if } \langle A, B \rangle \neq 0, \text{ then gh}(A) + \text{gh}(B) = 5.
\]

The above two conditions in fact imply that \(\langle A, B \rangle = \langle B, A \rangle\). Moreover, the product \((-,-)\) is \(Q\)-invariant \([20, 2.63]\):

\[
(5) \quad \langle QA, B \rangle = (-1)^A \langle A, QB \rangle.
\]

Conditions (3) and (4) also imply that the element \(\Phi := (-1)^b \Phi_s \otimes \Phi^s \in \mathcal{H}_{rel}^\otimes 2\) is symmetric in the sense that

\[
(6) \quad (-1)^b \Phi_s \otimes \Phi^s = (-1)^b \Phi^s \otimes \Phi_s = -(-1)^b \Phi^s \otimes \Phi_s.
\]

We use, in the previous formula as well as at many places in the rest of the paper, the Einstein convention of summing over repeated indices.

The last important property of string products is that the element

\[
(7) \quad \Phi_s \otimes [\Phi^s, B_1, \ldots, B_{n-1}]_g \in \mathcal{H}_{rel}^\otimes 2
\]

is antisymmetric. This property is not explicitly stated in \([20]\), though it is used in the proof of the identity \([20, (4.28)]\):

\[
\sum_s [B_1, \ldots, B_l, \Phi_s, [\Phi^s, A_1, \ldots, A_k]_g]_g = 0, \text{ for arbitrary } l \geq 0, k \geq 0,
\]

which then immediately follows from the antisymmetry (7) by the graded commutativity (1) of string products. Equation (7) is a consequence of the important fact that the string products are defined with the aid of the multilinear string functions \([20, (7.72)]\)

\[
\mathcal{H}_{rel}^{\otimes (n+1)} \ni B_0, \ldots, B_n \mapsto \{B_0, \ldots, B_n\}_g \in \mathbb{C}
\]

by \([20, (4.33)]\)

\[
(8) \quad [B_1, \ldots, B_n]_g := \sum_t (-1)^{b_t} \Phi^t \cdot \{\Phi_t, B_1, \ldots, B_n\}_g
\]
Let us show that the graded commutativity \[ (4.36) \]
\[ \{ B_0, \ldots, B_i, B_{i+1}, \ldots, B_n \} = (-1)^{B_i B_{i+1}} \{ B_0, \ldots, B_{i+1}, B_i, \ldots, B_n \} \]
of the string multilinear functions implies the antisymmetry of the element in (7). Indeed, because of (6), we may write (8) as
\[ [B_1, \ldots, B_n] = \sum_t (-1)^\Phi_t \Phi_t \cdot \{ \Phi_t, B_1, \ldots, B_n \} \]
thus the element in (7) takes the form
\[ \sum_{s,t} (-1)^\Phi_t (\Phi_s \otimes \Phi_t) \cdot \{ \Phi_t, \Phi_s, B_1, \ldots, B_{n-1} \} \]
The antisymmetry we are proving means that
\[ \sum_{s,t} (-1)^\Phi_t (\Phi_s \otimes \Phi_t) \cdot \{ \Phi_t, \Phi_s, B_1, \ldots, B_{n-1} \} = \]
\[ - \sum_{s,t} (-1)^{\Phi_t + \Phi_s} \Phi_t \otimes \Phi_s \cdot \{ \Phi_s, \Phi_t, B_1, \ldots, B_{n-1} \} \]
The replacement \( t \leftrightarrow s \) in the right-hand side of the above equation gives
\[ - \sum_{s,t} (-1)^{\Phi_s + \Phi_t} \Phi_s \otimes \Phi_t \cdot \{ \Phi_s, \Phi_t, B_1, \ldots, B_{n-1} \} \]
which can be further rewritten, using the graded commutativity of string functions, as
\[ - \sum_{s,t} (-1)^{\Phi_s + \Phi_t} \Phi_s \otimes \Phi_t \cdot \{ \Phi^t, \Phi^s, B_1, \ldots, B_{n-1} \} \]
Since \( gh(\Phi^s) \equiv gh(\Phi_s) + 1 \pmod{2} \) and \( gh(\Phi^t) \equiv gh(\Phi_t) + 1 \pmod{2} \),
\[ gh(\Phi^s)gh(\Phi^t) \equiv gh(\Phi_s)gh(\Phi_t) + gh(\Phi_s) + gh(\Phi_t) + 1 \pmod{2} \]
therefore the sign factor in (8) is \((-1)^{\Phi_t}\). This proves the claim.

2. Sign interlude and the definition.

In this brief section we rewrite the axioms of string products into a more usual and convenient formalism. All algebraic objects will be considered over a fixed field \( k \) of characteristic zero. This, of course, includes the case \( k = \mathbb{C} \) of the previous section. We will systematically use the Koszul sign convention meaning that whenever we commute two ‘things’
of degrees \( p \) and \( q \), respectively, we multiply by the sign factor \((-1)^{pq}\). Our conventions concerning graded vector spaces, permutations, shuffles, etc., will follow closely those of [15]. For graded indeterminates \( x_1, \ldots, x_n \) and a permutation \( \sigma \in \Sigma_n \) define the Koszul sign \( \epsilon(\sigma) = \epsilon(\sigma; x_1, \ldots, x_n) \) by

\[
x_1 \wedge \ldots \wedge x_n = \epsilon(\sigma; x_1, \ldots, x_n) \cdot x_{\sigma(1)} \wedge \ldots \wedge x_{\sigma(n)},
\]

which is to be satisfied in the free graded commutative algebra \( \wedge (x_1, \ldots, x_n) \). Define also

\[
\chi(\sigma) := \chi(\sigma; x_1, \ldots, x_n) := sgn(\sigma) \cdot \epsilon(\sigma; x_1, \ldots, x_n).
\]

We say that \( \sigma \in \Sigma_n \) is an \((i, j)\)-unshuffle, \( i + j = n \), if \( \sigma(1) < \cdots < \sigma(i) \) and \( \sigma(i + 1) < \cdots < \sigma(n) \). In this case we write \( \sigma \in \text{unsh}(i, j) \). In the obvious similar manner one may introduce \((i, j, k)\)-unshuffles, etc.

Let us denote, for a graded vector space \( U \), by \( \uparrow U \) (resp. \( \downarrow U \)) the suspension (resp. the desuspension) of \( U \), i.e. the graded vector space defined by \( (\uparrow U)_p := U_{p-1} \) (resp. \( (\downarrow U)_p := U_{p+1} \)). We have the obvious natural maps \( \uparrow: U \to \uparrow U \) and \( \downarrow: U \to \downarrow U \).

For a graded vector space \( U \), let its reflection \( r(U) \) be the graded vector space defined by \( r(U)_p := U_{-p} \). There is an obvious natural map \( r: U \to r(U) \). Observe that \( r^2 = 1 \), \( r \circ \uparrow = \downarrow \circ r \) and \( r \circ \downarrow = \uparrow \circ r \).

Take now \( V := r(\downarrow \mathcal{H}_{rel}) \). Define, for each \( g \geq 0 \) and \( n \geq 0 \), multilinear maps \( l^g_n : V^\otimes n \to V \) by

\[
l^g_n(v_1, \ldots, v_n) := (-1)^{(n-1)v_1+(n-2)v_2+\cdots+v_{n-1}} \downarrow [\uparrow r(v_1), \ldots, \uparrow r(v_n)]_g, \quad \text{for } v_1, \ldots, v_n \in V^\otimes n.
\]

Define also the bilinear form \( B : V \otimes V \to \mathbb{C} \) by

\[
B(u, v) := \langle \uparrow r(u), \uparrow r(v) \rangle
\]

and, finally, the element \( h = h_s \otimes h^s \) by \( h_s := (-1)^{\Phi_s} r(\downarrow \Phi_s) \), \( h^s := r(\downarrow \Phi^s) \), which means that \( h_s \otimes h^s := (-1)^{\Phi_s} r(\downarrow \Phi_s) \otimes r(\downarrow \Phi^s) \) (Einstein summation convention).

A technical, but absolutely straightforward, calculation shows that the above structure is an example of a loop homotopy Lie algebra in the sense of the following definition.

**Definition 2.1.** A loop homotopy Lie algebra is a triple \( V = (V, B, \{l^g_n \}) \) consisting of
(i) a \(\mathbb{Z}\)-graded vector space \(V\), \(V_s = \bigoplus V_i\),
(ii) a graded symmetric nondegenerate bilinear degree +3 form \(B : V \otimes V \to k\), and
(iii) the set \(\{l^g_n\}_{n,g \geq 0}\) of degree \(n-2\) multilinear antisymmetric operations \(l^g_n : V^\otimes n \to V\).

These data are supposed to satisfy the following two axioms.

(A1) For any \(n, g \geq 0\) and \(v_1, \ldots, v_n \in V\), the following ‘main identity’
\[
0 = \sum_{k+l=n+1 \atop g_1+g_2=g} \sum_{\sigma \in \text{unsh}(l,n-l)} \chi(\sigma)(-1)^{(k-1)l^g_1} l^g_{l_1} (l^g_{l_2} (v_{\sigma(1)}, \ldots, v_{\sigma(l)}), v_{\sigma(l+1)}, \ldots, v_{\sigma(n)})
\]
\[
+ \frac{1}{2} \sum_s (-1)^{h_s+n} l^g_{n+2}(h_s, h^s, v_1, \ldots, v_n)
\]
holds. In the second term, \(\{h_s\}\) and \(\{h^s\}\) are bases of the vector space \(V\) dual to each other in the sense that
\[
B(h_s, h_t) = \delta_{ts}.
\]

(A2) The element
\[
(-1)^{(n+1)h_s} h_s \otimes l^g_n (h^s, v_1, \ldots, v_{n-1}) \in V \otimes V
\]
is symmetric, for all \(g \geq 0, n \geq 0\), and \(v_1, \ldots, v_{n-1} \in V\).

Remarks 2.2. To give a reasonable meaning to the ‘basis \(\{h_s\}\) of \(V\),’ we must suppose either that \(V\) is finite dimensional, or that it has a suitable topology, as in the case of string products. We will always tacitly assume that assumptions of this form have been made. In the ‘main identity’ for \(g = 0\) we put, by definition, \(l^{-1}_n = 0\).

Because \(\deg(h_s) + \deg(h^s) = -3\), \(\deg(h_s) \deg(h^s)\) is even. The graded symmetry of \(B\) then implies that, besides of (12), also \(B(h_s, h^t) = \delta_{st}\). The element \(h = h_s \otimes h^s\) is easily seen to be symmetric, \(h_s \otimes h^s = (-1)^{h_s h^s} h^s \otimes h_s = h^s \otimes h_s\).

For \(n = 0\) axiom (2) gives
\[
0 = \sum_{g_1+g_2=g} l^g_1 (l^g_0(v)) + \frac{1}{2} \sum_s (-1)^{h_s} l^g_{-1}(h_s, h^s),
\]
while for \(n = 1\) it gives
\[
0 = \sum_{g_1+g_2=g} (l^g_1 (l^g_0(v)) + l^g_0 (l^g_0(v), v)) - \frac{1}{2} \sum_s (-1)^{h_s} l^g_{-1}(h_s, h^s, v), \ v \in V.
\]
From this moment on, we will assume that \(l^g_0 = 0\), for all \(g \geq 0\), that is, the theory has ‘no constants.’ This assumption is not really necessary, but it will considerably simplify our exposition.

- August 30, 1999 -
Exercise 2.3. Let us denote $\partial := l_0^0$. Equation (14) implies that $\partial^2 = 0$ (recall our assumption $l_0^0 = 0$!). Thus $\partial$ is a degree $-1$ differential on the space $V$. The symmetry of $h_s \otimes \partial(h^s)$ (axiom (A2) with $n = 1$ and $g = 1$) is equivalent to the $d$-invariance of the form $B$, $B(\partial u, v) + (-1)^n B(u, \partial v) = 0$, for $u, v \in V$.

The tree level. Let us discuss the ‘tree level’ ($g = 0$) specialization of the above structure. The only nontrivial $l_n^g$’s are $l_n := l_n^0$, $n \geq 1$. The main identity (11) for $g = 0$ reduces to

$$0 = \sum_{k+l=n+1} \sum_{\sigma \in \text{unsh}(l,n-l)} \chi(\sigma)(-1)^{l(k-1)} l_k(l_1(v_{\sigma(1)}), \ldots, v_{\sigma(l)}), v_{\sigma(l+1)}, \ldots, v_{\sigma(n)})$$

while, for $g = 1$ it gives (after forgetting the overall factor $\frac{(-1)^n}{2}$)

$$0 = \sum_s (-1)^h s l_{n+2}(h_s, h^s, v_1, \ldots, v_n).$$

Axiom (A2) says that the elements

$$(-1)^{(n+1)h} h_s \otimes l_n(h^s, v_1, \ldots, v_n)$$

are symmetric. We immediately recognize (13) as the defining equation for strongly homotopy Lie algebras [13, Definition 2.1]. Thus the tree level loop homotopy Lie algebra is a strongly homotopy Lie algebra $(V, \{l_n\})$ with an additional structure given by a bilinear form $B$ such that the element $h = h_s \otimes h^s$, uniquely determined by $B$, satisfies (16) and (17). We see that the ‘tree-level’ specialization is a richer structure than just a strongly homotopy Lie algebra as it is usually understood. A proper name for such a structure would be a cyclic strongly homotopy Lie algebra.

3. Higher order (co)derivations.

In this section we investigate properties of higher order coderivations of cofree cocommutative coalgebras. Because this paper is meant for humans, not for robots, we derive necessary properties for derivations on free commutative algebras, and then simply dualize the results. This is an absolutely correct procedure, except one fine point related to the cofreeness, see Remark 3.6. The following definitions were taken from [1, 3].
Let $A$ be a graded (super) commutative algebra and $\nabla : A \to A$ a homogeneous degree $k$ linear map. We define inductively, for each $n \geq 1$, degree $k$ linear deviations $\Phi^n_\nabla : A^{\otimes n} \to A$ by

\[
\begin{align*}
\Phi^1_\nabla(a) & := \nabla(a), \\
\Phi^2_\nabla(a, b) & := \nabla(ab) - \nabla(a)b - (-1)^{ka}a\nabla(b), \\
\Phi^3_\nabla(a, b, c) & := \nabla(abc) - \nabla(ab)c - (-1)^{a(b+c)}\nabla(bc)a - (-1)^{c(a+b)}\nabla(ca)b \\
\quad & \quad + \nabla(a)bc + (-1)^{a(b+c)}\nabla(b)ca + (-1)^{c(a+b)}\nabla(c)ab,
\end{align*}
\]

\[
\vdots
\]

\[
\Phi^{n+1}_\nabla(a_1, \ldots, a_{n+1}) := \Phi^n_\nabla(a_1, \ldots, a_n a_{n+1}) - \Phi^n_\nabla(a_1, \ldots, a_n) a_{n+1}
- (-1)^{a_{n+1}a_n} \Phi^n_\nabla(a_1, \ldots, a_{n-1}, a_{n+1}) a_n.
\]

As a matter of fact, it is possible to give a non-inductive formula for $\Phi^n_\nabla$, namely

\[
(18) \quad \Phi^n_\nabla(a_1, \ldots, a_n) = \sum_{1 \leq i \leq n} (-1)^{n-i} \epsilon\sigma(\nabla(x_{\sigma(1)} \cdots x_{\sigma(i)}) x_{\sigma(i+1)} \cdots x_{\sigma(n)}).
\]

We say that $\nabla$ is a derivation of order $r$ if $\Phi^{r+1}_\nabla$ is identically zero. In this case we write $\nabla \in \text{Der}_k^r(A)$, where $k = \deg(\nabla)$. In the following proposition, which was stated in [1], $[-,-]$ denotes the graded anticommutator of endomorphisms.

**Proposition 3.1.** The subspaces $\text{Der}_k^r(A)$ satisfy:

(i) $\text{Der}_k^1(A) \subset \text{Der}_k^2(A) \subset \text{Der}_k^3(A) \subset \cdots$

(ii) $\text{Der}_k^r(A) \circ \text{Der}_k^s(A) \subset \text{Der}_k^{r+s}(A)$, and

(iii) $[\text{Der}_k^r(A), \text{Der}_k^s(A)] \subset \text{Der}_k^{r+s-1}(A)$.

Let now $A = \wedge X$ be the free graded commutative algebra on the graded vector space $X$. Let us prove the following useful proposition.

**Proposition 3.2.** Let $\nabla \in \text{Der}_k^r(\wedge X)$. Then $\nabla$ is uniquely determined by its values on the products $x_1 \cdots x_s$, $s \leq r$, $x_i \in X$ for $1 \leq i \leq s$. In particular,

$\nabla = 0$ if and only if $\nabla(x_1 \cdots x_s) = 0$, for $x_1 \cdots x_s$ as above.
Proof. Since $\nabla \in \text{Der}^r_k(\wedge X)$ is linear, it is enough to prove that $\nabla(x_1 \cdots x_s) = 0$ for all $s \leq r$ implies that $\nabla(x_1 \cdots x_n) = 0$ for each $n$. This we prove inductively. Suppose we already know $\nabla(x_1 \cdots x_k) = 0$, for each $k \leq n$, $n \geq r$, and consider $\nabla(x_1 \cdots x_{n+1})$. We compute from (18) that
\[
\Phi^n_{\nabla}(x_1, \ldots, x_{n+1}) = \nabla(x_1 \cdots x_{n+1}) + \sum_{\sigma \in \text{unsh}(s,n-i+1)} (-1)^{n-i+1} \epsilon(\sigma) \nabla(x_{\sigma(1)} \cdots x_{\sigma(i)})x_{\sigma(i+1)} \cdots x_{\sigma(n+1)},
\]
Since $\nabla \in \text{Der}^r_k(\wedge X)$ and $n \geq r$, $\Phi^n_{\nabla}(x_1, \ldots, x_{n+1}) = 0$, while the terms in the sum are zero by the inductive assumption. Thus $\nabla(x_1 \cdots x_{n+1}) = 0$ and the induction may go on. □

Remark 3.3. 1-derivations are ordinary derivations, $\text{Der}^1_k(A) = \text{Der}_k(A)$. Proposition 3.2 then states the standard fact that derivations on free algebras are given by their restrictions to the space of generators.

For a fixed $n$, we denote by $\wedge^n X$ the subspace of $\wedge X$ spanned by the products $x_1 \cdots x_n$, $x_i \in X, 1 \leq i \leq n$; we put, by definition, $\wedge^0 X := k$. Let $\iota_n : \wedge^n X \hookrightarrow \wedge X$ be the inclusion. The following proposition says that $r$-derivations of the free algebra $\wedge X$ are in one-to-one correspondence with $r$-tuples of linear maps, $\{f_s : \wedge^s X \rightarrow \wedge X\}_{1 \leq s \leq r}$.

**Proposition 3.4.** Suppose we are given homogeneous degree $k$ linear maps $f_s : \wedge^s X \rightarrow \wedge X$, for $1 \leq s \leq r$. Then there exists a unique order $r$ derivation $\nabla \in \text{Der}^r_k(\wedge X)$ such that
\[
(19) \quad \nabla \circ \iota_s = f_s, \text{ for } 1 \leq s \leq r.
\]

**Proof.** The uniqueness follows immediately from Proposition 3.2. To prove the existence, observe first that, given degree $k$ linear maps $g_s : \wedge^s X \rightarrow \wedge X$, $1 \leq s \leq r$, the formula
\[
\nabla(x_1 \cdots x_n) := \sum_{\begin{subarray}{c}1 \leq s \leq \min(r,n) \\ \sigma \in \text{unsh}(s,n-s) \end{subarray}} \epsilon(\sigma) g_s(x_{\sigma(1)} \cdots x_{\sigma(s)})x_{\sigma(s+1)} \cdots x_{\sigma(n)},
\]
defines an order $k$ derivation. Condition (19) then leads to the following system of equations:
\[
\begin{align*}
    f_1(x_1) &= g_1(x_1), \\
    f_2(x_1 x_2) &= g_2(x_1 x_2) + g_1(x_1)x_2 + (-1)^{x_1 x_2} g_1(x_2)x_1, \\
                  & \vdots \\
    f_r(x_1 \cdots x_r) &= \sum_{\begin{subarray}{c}1 \leq s \leq r \\ \sigma \in \text{unsh}(s,n-s) \end{subarray}} \epsilon(\sigma) g_s(x_{\sigma(1)} \cdots x_{\sigma(s)})x_{\sigma(s+1)} \cdots x_{\sigma(r)}. \\
\end{align*}
\]
This system can obviously be solved for $g_s$, $1 \leq s \leq r$.

Let us turn our attention to coalgebras. Suppose that $C = (C, \Delta)$ is a cocommutative coassociative coalgebra. To define higher-order coderivations of $C$, we need analogs of the deviations $\Phi^r$ introduced above. By duality, we define, for any homogeneous degree $k$ linear endomorphism $\Omega$ of $C$, degree $k$ multilinear maps $\Psi^n_\Omega : C \to C^{\otimes n}$ inductively as

\[
\Psi^1_\Omega := \Omega,
\Psi^2_\Omega := \Delta \circ \Omega - (\Omega \otimes \mathbb{I}) \circ \Delta - (\mathbb{I} \otimes \Omega) \circ \Delta,
\Psi^3_\Omega := \Delta^{[3]} \circ \Omega - (\Delta \otimes \mathbb{I}) \circ (\Omega \otimes \mathbb{I}) \circ \Delta - T_{312} \circ (\Delta \otimes \mathbb{I}) \circ (\Omega \otimes \mathbb{I}) \circ \Delta - T_{231} \circ (\Delta \otimes \mathbb{I}) \circ (\Omega \otimes \mathbb{I}) \circ \Delta
\]

\[
+ (\Omega \otimes \mathbb{I}^2) \circ \Delta^{[3]} + T_{312} \circ (\Omega \otimes \mathbb{I}^2) \circ \Delta^{[3]} + T_{231} \circ (\Omega \otimes \mathbb{I}^2) \circ \Delta^{[3]}
\]

\[
\vdots
\]

\[
\Psi^{n+1}_\Omega := (\mathbb{I}^{n-1} \otimes \Delta) \circ \Psi^n_\Omega - (\Psi^n_\Omega \otimes \mathbb{I}) \circ \Delta - T_{1,2,...,n-1,n+1,n} \circ (\Psi^n_\Omega \otimes \mathbb{I}) \circ \Delta,
\]

where

\[
\Delta^{[3]} := (\Delta \otimes \mathbb{I}) \Delta = (\mathbb{I} \otimes \Delta) \Delta \text{ by the coassociativity}
\]

and, for $\sigma \in \Sigma_n$, $T_{\sigma(1)...\sigma(n)} : C^{\otimes n} \to C^{\otimes n}$ is defined by

\[
T_{\sigma(1)...\sigma(n)}(x_1 \otimes \cdots \otimes x_n) := \epsilon(\sigma)(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}).
\]

We say that a linear map $\Omega : C \to C$ is an order $r$ coderivation, if $\Psi^{r+1}_\Omega$ is identically zero. Let $\text{coDer}^r_k(C)$ be the space of all such maps. The following proposition is an exact dual of Proposition 3.1.

**Proposition 3.5.** The subspaces $\text{coDer}^r_k(C)$ satisfy:

(i) $\text{coDer}^1_k(C) \subset \text{coDer}^2_k(C) \subset \text{coDer}^3_k(C) \subset \cdots$

(ii) $\text{coDer}^r_k(C) \circ \text{coDer}^s_k(C) \subset \text{coDer}^{r+s}_{k+l}(C)$, and

(iii) $[\text{coDer}^r_k(C), \text{coDer}^s_l(C)] \subset \text{coDer}^{r+s-1}_{k+l}(C)$.

Let $W$ be a graded vector space and consider again the free graded commutative algebra $\wedge W$ on $W$. We introduce on $\wedge W$ a cocommutative coassociative comultiplication $\Delta = \mathbb{I} \otimes \mathbb{I} + \Delta + \mathbb{I} \otimes \mathbb{I}$ by defining the reduced diagonal $\overline{\Delta}$ as

\[
\overline{\Delta}(w_1 \cdots w_n) = \sum_{1 \leq i \leq n-1} \sum_{\sigma} \epsilon(\sigma)(w_{\sigma(1)} \cdots w_{\sigma(i)}) \otimes (w_{\sigma(i+1)} \cdots w_{\sigma(n)}), \quad w_1 \cdots w_n \in \wedge^n W,
\]

where $\sigma$ runs through all $(i, n-i)$ unshuffles. We denote the coalgebra $(\wedge W, \Delta)$ by $\wedge^\epsilon W$. 

- August 30, 1999 -
Remark 3.6. Here it must be pointed out that $\wp W$ is not the cofree cocommutative coassociative coalgebra cogenerated by $W$, as it is generally supposed to be. It is the cofree coalgebra in the category of connected coalgebras, see the discussion in [13, page 2150].

Denote by $\pi_n : \wp W \to \wedge^n W$ the natural projection of vector spaces. The following theorem is the exact dual of Proposition 3.4.

Proposition 3.7. For each $r$-tuple $u_s : \wp W \to \wedge^s W$, $1 \leq s \leq r$, of homogeneous degree $k$ linear maps there exists a unique order $r$ coderivation $\Omega \in \text{coDer}_k^r(\wp W)$ such that

$$\pi_s \circ \Omega = u_s, \text{ for } 1 \leq s \leq r.$$  

(20)

4. Loop homotopy Lie algebras - 1st description.

We already observed at the end of Section 2 that strongly homotopy Lie algebras are closely related to the ‘tree level’ specializations of loop homotopy Lie algebras. Recall [13, Theorem 2.3] that strongly homotopy Lie algebras have the following characterization.

Proposition 4.1. There exists a one-to-one correspondence between strongly homotopy Lie algebra structures on a graded vector space $V$ and degree $-1$ coderivations $\delta \in \text{coDer}_{-1}(\wp W)$, $W := \uparrow V$, with the property $\delta^2 = 0$.

In this section we give a similar characterization for loop homotopy Lie algebras. Suppose that the vector space $V$ and the bilinear form $B$ is the same as in Definition 2.1. Let $h = h_s \otimes h^s \in (V \otimes V)_{-3}$ be as in (22) (of course, $h$ is uniquely determined by the nondegenerate form $B$).

Let $W := \uparrow V$ and $y = y_s \otimes y^s := \uparrow h_s \otimes \uparrow h^s \in (W \otimes W)_{-1}$. Because $h$ is symmetric, $y$ is symmetric as well, thus, in fact, $y = y_s y^s \in \Lambda^2 W_{-1}$. Let us consider the extension $\wp W[t]$ of $\wp W$ over the polynomial ring $k[t]$, $\wp W[t] := \wp W \otimes_k k[t]$. By Proposition 3.7, there exist a unique coderivation $\theta \in \text{coDer}_{-1}^2(\wp W[t])$ such that

$$\pi_1(\theta) = 0 \text{ and } \pi_2(\theta)(w) = \begin{cases} 0, & w \in \Lambda^n W[t], \ n > 0, \\ \frac{1}{2} ty, & w = 1 \in \Lambda^0 W \cdot t^0 \cong k. \end{cases}$$

(21)

The rôle of $\theta$ is to incorporate the form $B$ into our theory. In the rest of this section we prove the following theorem.
Theorem 4.2. Under the above notation, there is a one-to-one correspondence between loop homotopy Lie algebra structures on the graded vector space \( V \) and degree \(-1\) coderivations \( \delta \in \text{coDer}^1_{-1}(\bigwedge W[t]) \) such that
\[
(\delta + \theta)^2 = 0.
\]

Let us analyze equation (22). It is, of course, equivalent to
\[
\delta^2 + \theta \delta + \delta \theta + \theta^2 = 0.
\]

Sublemma 4.3. Under the above notation, \( \theta^2 = 0 \), \( \delta^2 \in \text{coDer}^1_{-2}(\bigwedge W[t]) \), and \( (\theta \delta + \delta \theta) \in \text{coDer}^2_{-2}(\bigwedge W[t]) \).

**Proof.** For \( w_1 \cdots w_n \in \bigwedge^n W \) obviously
\[
\theta(w_1 \cdots w_n) = \frac{1}{2} t y_s y^s w_1 \cdots w_n,
\]
thus
\[
\theta^2(w_1 \cdots w_n) = \frac{1}{4} t^2 y_s y^s y_t y^t w_1 \cdots w_n.
\]
The graded commutativity implies that
\[
y_s y^s y_t y^t = (-1)^{(y_s + y^s)(y_t + y^t)} y_t y^t y_s y^s = -y_t y^t y_s y^s.
\]
On the other hand, the substitution \( s \leftrightarrow t \) gives \( y_s y^s y_t y^t = y_t y^t y_s y^s \), therefore \( y_t y^t y_s y^s = 0 \), and \( \theta^2 = 0 \) by (23).

The remaining two statements follow from Proposition 3.3(iii) and the observation that \( \delta^2 = \frac{1}{2}[\delta, \delta] \) and \( \theta \delta + \delta \theta = [\delta, \theta] \).

By Sublemma 4.3, (23) reduces to
\[
\delta^2 + \theta \delta + \delta \theta = 0.
\]

By the same sublemma and Proposition 3.1(i), \( \delta^2 + \theta \delta + \delta \theta \) is an order 2 coderivation. Thus (28) is, by Proposition 3.7, equivalent to
\[
\pi_1(\delta^2 + \theta \delta + \delta \theta) = 0, \text{ and}
\]
\[
\pi_2(\delta^2 + \theta \delta + \delta \theta) = 0.
\]
Because, by (21), \( \pi_1(\theta) = 0 \), equation (27) further reduces to
\[
\pi_1(\delta^2 + \delta \theta) = 0.
\]

To understand better the meaning of this equation, let us introduce, for any \( g \geq 0 \) and \( n \geq 0 \), linear maps \( \delta^g_n : \wedge^n W \to W \) by
\[
\delta^g_n(w_1 \cdots w_n) := \text{Coef}_g(\pi_1(\delta(w_1 \cdots w_n))), \quad w_1 \cdots w_n \in \wedge^n W,
\]
where \( \text{Coef}_g(\cdot) \) is the coefficient at \( t^g \). By Proposition 3.7, the set \( \{ \delta^g_n \}_{n,g \geq 0} \) uniquely determines the coderivation \( \delta \). The explicit formula is (compare explicit formulas for coderivations acting on coalgebras in [14]):
\[
\delta(w_1 \cdots w_n) = \sum_{0 \leq i \leq n} \epsilon(\sigma) t^g \delta^g_1(w_{\sigma(1)} \cdots w_{\sigma(i)}) w_{\sigma(i+1)} \cdots w_{\sigma(n)},
\]
where the summation is taken over all \( g \geq 0 \) and all \( \sigma \in \text{unsh}(i,n-i) \). From this and (24) we obtain
\[
\pi_1(\delta^2 + \delta \theta)(w_1 \cdots w_n) = \sum_{k+l=n+1, g_1+g_2=g} \epsilon(\sigma) t^{g_1} \delta^g_1(w_{\sigma(1)} \cdots w_{\sigma(l)}) w_{\sigma(l+1)} \cdots w_{\sigma(n)}
\]
\[+ \frac{1}{2} \sum_{s,g \geq 0} \delta^{g+1}_{n+2}(y_s, y^s, w_1, \ldots, w_n).
\]

We formulate the result as:

**Sublemma 4.4.** Equation (29) means that, for all \( n \geq 0 \), \( w_1 \cdots w_n \in \wedge^n W \) and \( g \geq 0 \),
\[
0 = \sum_{k+l=n+1, g_1+g_2=g} \epsilon(\sigma) \delta^{g_1}_k(\delta^{g_2}_l(w_{\sigma(1)} \cdots w_{\sigma(l)}) w_{\sigma(l+1)} \cdots w_{\sigma(n)})
\]
\[+ \frac{1}{2} \sum_{s} \delta^{g-1}_{n+2}(y_s, y^s, w_1, \ldots, w_n).
\]

We will see that equation (33) will correspond to the ‘main identity’ (11). Let us make a similar analysis of equation (28). Because clearly \( \pi_2(\theta \delta) = 0 \), it reduces to
\[
\pi_2(\delta^2 + \delta \theta) = 0.
\]
Using the similar arguments as above, we obtain, for any \( g \geq 0 \) and \( w_1 \cdots w_n \in \wedge^n W \),

\[
(35) \quad \text{Coef}_g(\pi_2(\delta^2)(w_1 \cdots w_n)) = \\
= \frac{1}{2} \sum_{1 \leq i \leq n} (-1)^{w_i(w_{i+1} + \cdots + w_n)} \delta^n_{n+1}(y_s y^i w_1 \cdots w_{i-1} w_{i+1} \cdots w_n) w_i \\
+ \frac{1}{2} \sum_s (-1)^{y_s(y_s + w_1 + \cdots + w_n)} \delta^n_{n+1}(y_s w_1 \cdots w_n) y_s \\
+ \frac{1}{2} \sum_s (-1)^{y_s(y_1 + \cdots + w_n)} \delta^n_{n+1}(y_s w_1 \cdots w_n) y_s.
\]

Now, assuming \( (33) \), it is immediate to see that the first term at the right hand side of \( (35) \) is minus the first term at the right hand side of \( (36) \). The symmetry \( y_s y^i = (-1)^{y_s y^i} y^i y_s \) implies that the second and third terms at the left hand side of \( (33) \) are the same, both equal to \( 1/2 \sum_s (-1)^{y_s y_s} \delta^n_{n+1}(y_s w_1 \cdots w_n) \). We formulate these observations as

\textbf{Sublemma 4.5.} Assuming \( (33) \), equation \( (34) \) is equivalent to

\[
(37) \quad \frac{1}{2} \sum_s (-1)^{y_s y_s} \delta^n_{n+1}(y_s w_1 \cdots w_n) = 0,
\]

Since we work in the free commutative algebra, \( (33) \) is equivalent to the antisymmetry of

\[
(38) \quad \frac{1}{2} \sum_s (-1)^{y_s y_s} \delta^n_{n+1}(y_s w_1 \cdots w_n) \in W \otimes W.
\]

\textbf{Proof of Theorem 4.2.} Recall that \( W = \uparrow V \). The correspondence between the structure operations \( \{l^g_n\}_{g,n \geq 0} \) of a loop homotopy Lie algebra and coderivations \( \delta \) of Theorem 4.2 is given by

\[
l^g_n(v_1, \ldots, v_n) = (-1)^{(n-1)v_1+\cdots+v_n-1} \downarrow \delta^n_{n+1}(\uparrow v_1 \cdots \uparrow v_n), \quad v_1, \ldots, v_n \in V,
\]

with the inverse formula

\[
\delta^n_{n+1}(w_1 \cdots w_n) = (-1)^{n(n-1)/2}(-1)^{(n-1)w_1+\cdots+w_n-1} \uparrow l^g_n(\downarrow w_1, \ldots, \downarrow w_n), \quad w_1 \cdots w_n \in \wedge^n W.
\]
where the multilinear maps \( \{ \delta^n \} \) were introduced in (30). Observe the sign \((-1)^{n(n-1)/2}\) in the second formula; it is typical for formulas of this type, see [15, Example 1.6]. A routine
calculation shows that the substitution \( l^n \leftrightarrow \delta^n \) converts (33) to (11) and that the symmetry
of the element in (13) is equivalent to the antisymmetry of the element of (38).

5. Loop homotopy Lie algebras - operadic approach.

In this section we give an operadic characterization of loop homotopy Lie algebras.

We will not repeat here all details of necessary definitions concerning operads, because it
would stretch the paper beyond any reasonable limit. Operads are introduced in the classical
book [17]. The (co)bar construction over a (co)operad is defined in [5], see also [6]. Cyclic
operads are introduced in [4] while modular operads and the corresponding modular (co)bar
construction (called the Feynman transform) in [8]. There is also a nice overview [10]. These
sources are easily available, we will thus rely on them and indicate only basic ideas.

Recall that a collection is a system \( E = \{ E(n) \}_{n \geq 1} \) of graded vector spaces such that each
\( E(n) \) possesses a right action of the symmetric group \( \Sigma_n \). Any collection \( E \) extends to a
functor (denoted by the same symbol) from the category of finite sets to the category of
graded vector spaces with the property that \( E(n) = E(\{1, \ldots, n\}) \) [6, 1.3].

Let \( T^r_n \) denote the set of rooted (= directed) trees with \( n \) labelled leaves. For a tree \( T \in T^r_n \)
and a collection \( E \), denote (\[9, 1.2.13\])
\[
E(T) := \bigotimes_{v \in \text{Vert}(T)} E(\text{In}(v)),
\]
where \( \text{Vert}(T) \) is the set of the vertices of \( T \) and \( \text{In}(v) \) the set of incoming edges of \( v \). The
free operad on \( E \) [4, 2.1.1] is then the collection
\[
\mathcal{F}(E)(n) := \bigotimes_{T \in T^r_n} E(T), \ n \geq 1,
\]
with the operadic structure induced by the grafting of underlying trees.

Let \( \mathcal{P} \) be an operad. Consider the free operad \( \mathcal{F}(\downarrow s\mathcal{P}^*) \) on the collection
\[
\downarrow s\mathcal{P}^*(n) := \uparrow^{n-2}\mathcal{P}^*(n), \ n \geq 1,
\]
where \((-)^*\) is the linear dual. As proved in [3, 3.2], structure operations of the operad \(\mathcal{P}\) induce a differential \(\partial_{\mathcal{P}}\) on \(\mathcal{F}(\downarrow s\mathcal{P}^*)\). The differential operad \(\mathcal{D}(\mathcal{P}) := (\mathcal{F}(\downarrow s\mathcal{P}^*), \partial_{\mathcal{P}})\) is called the (operadic) cobar dual of the operad \(\mathcal{P}\). It is well-known [9, 4.2.14] that ‘classical’ strongly homotopy Lie algebras are characterized as follows.

**Proposition 5.1.** Strongly homotopy Lie algebras are algebras over the cobar dual \(\mathcal{D}(\mathcal{C}\text{om})\) of the operad \(\mathcal{C}\text{om}\) for commutative algebras.

The above proposition means that a strongly homotopy Lie algebra structure on a differential graded vector space \(V = (V, \partial)\) is the same as a morphism \(a : \mathcal{D}(\mathcal{C}\text{om}) \to \mathcal{E}\text{nd}_V\) from the operad \(\mathcal{D}(\mathcal{C}\text{om})\) to the endomorphism operad \(\mathcal{E}\text{nd}_V\) of \(V\) [9, 1.2.9].

Our aim is to give a similar characterization of loop homotopy Lie algebras, based on a certain generalization of operads, called modular operads.

An intermediate step between ordinary operads and modular operads are cyclic operads whose definition we briefly recall. A **cyclic collection** is a system \(E = \{E((n))\}_{n \geq 1}\) of graded vector spaces such that each \(E((n))\) has a right \(\Sigma_{n+1}\)-action. Each cyclic collection \(E\) induces a functor from the category of finite sets into the category of graded vector spaces (denoted again by \(E\)) such that \(E((\{0, \ldots, n\})) = E((n))\). This notation differs from that of [7] and [5] where \(E((\{0, \ldots, n\})) = E((n + 1))\).

Let \(T_{\text{ur}}^n\) denote the set of **unrooted** trees \(T\) with leaves indexed by \(\{0, \ldots, n\}\). For a cyclic collection \(E\) and a tree \(T \in T_{\text{ur}}^n\), let

\[
E(T) := \bigotimes_{v \in \text{Vert}(T)} E((\text{Leg}(v))),
\]

where \(\text{Leg}(v)\) is the set of all edges of \(T\) adjacent to the vertex \(v\).

A **cyclic operad** is then a cyclic collection \(\mathcal{C} = \{\mathcal{C}((n))\}_{n \geq 1}\) together with a ‘coherent’ system of ‘contractions’

\[
(39) \quad \alpha_T : \mathcal{C}((T)) \to \mathcal{C}((n)), \quad T \in T_{\text{ur}}^n, \ n \geq 1,
\]

see [4, Definition 2.1]

Modular operads, anticipated in [6], were introduced by Getzler and Kapranov [8] for the study of moduli spaces of Riemann surfaces of arbitrary genera. Recall that a **modular collection** is a cyclic collection \(E\) with a second grading by the ‘genus’ \(g \geq 0\), \(E = \{E((g, n))\}_{n \geq 1}\). A modular operad \(\mathcal{A}\) is then modular collection which posses, besides a cyclic operadic structure, also operations \(\mathcal{A}((g, n + 2)) \to \mathcal{A}((g + 1, n))\). These operations are abstractions of the
Figure 1: An example of ‘self-gluing.’ The surface on the right has 2 punctures and genus 2. It is obtained from the surface on the left with 4 punctures and genus 1 by sewing along the punctures marked by 3 and 4.

‘self-gluing’ which produces, from a surface of genus \(g\) with \((n+2)\) punctures, a new surface of genus \(g+1\) with \(n\) punctures, as indicated in Figure 1.

As cyclic operads are characterized by a system of contractions \((\mathfrak{B})\) indexed by unrooted trees, there is a similar characterization of modular operads, but based on labelled (or ‘modular’) graphs rather than trees.

Following \([5, 12]\), by a graph \(\Gamma\) we mean a finite set \(\text{Flag}(\Gamma)\) (whose elements are called flags or half-edges) together with an involution \(\sigma\) and a partition \(\lambda\). The vertices \(\text{Vert}(\Gamma)\) of a graph \(\Gamma\) are the blocks of the partition \(\lambda\). The edges \(\text{Edg}(\Gamma)\) are pairs of flags forming a two-cycle of \(\sigma\) relative to the decomposition of a permutation into disjoint cycles. The legs \(\text{Leg}(\Gamma)\) are the fixed-points of \(\sigma\). We also denote by \(\text{Leg}(v)\) the flags belonging to the block \(v\) or, in common speech, half-edges adjacent to the vertex \(v\).

Each graph \(\Gamma\) has its geometric realization, a finite one-dimensional cell complex \(|\Gamma|\), obtained by taking one copy of \([0, \frac{1}{2}]\) for each flag and imposing the following equivalence relation: the points \(0 \in [0, \frac{1}{2}]\) are identified for all flags in a block of the partition \(\lambda\), and the points \(\frac{1}{2} \in [0, \frac{1}{2}]\) are identified for pairs of flags exchanged by the involution \(\sigma\). We will usually make no distinction between a graph and its geometric realization.

A modular or labelled graph is a connected graph \(\Gamma\) together with a map \(g : \text{Vert}(\Gamma) \to \{0, 1, 2, \ldots\}\). The genus \(g(\Gamma)\) of a modular graph \(\Gamma\) is the number

\[
g(\Gamma) := \dim H_1(|\Gamma|) + \sum_{v \in \text{Vert}(\Gamma)} g(v).
\]

Let \(\mathfrak{G}((g, S))\) be the category whose objects are pairs \((|\Gamma|, \rho)\) consisting of a modular graph \(\Gamma\) of genus \(g\) and an isomorphism \(\rho : \text{Leg}(\Gamma) \to S\) labeling the legs of \(\Gamma\) by elements of a finite set \(S\). As usual, we write \(\mathfrak{G}((g, n)) := \mathfrak{G}((g, \{0, \ldots, n\}))\).
For a modular collection \( \mathcal{A} = \{ \mathcal{A}(\langle g, n \rangle) \}_{n \geq 1} \) and a modular graph \( \Gamma \), let \( \mathcal{A}(\langle \Gamma \rangle) \) be the tensor product
\[
\mathcal{A}(\langle \Gamma \rangle) := \bigotimes_{v \in \text{Vert}(\Gamma)} \mathcal{A}(\langle g(v), \text{Leg}(v) \rangle).
\]
(40)

A modular operad structure on \( \mathcal{A} \) is then given by a coherent system of contractions [8, 2.10]
\[
\alpha_{\Gamma} : \mathcal{A}(\langle \Gamma \rangle) \to \mathcal{A}(\langle g, S \rangle),
\]
for any \( \Gamma \in \Gamma(\langle g, S \rangle) \), \( g \geq 0 \) and a finite set \( S \).

**Example 5.2.** Let \( V = (V, B) \) be a differential graded vector space with a graded symmetric inner product \( B : V \otimes V \to k \). Let us define, for each \( g \geq 0 \) and a finite set \( S \),
\[
\mathcal{E}nd_V(\langle g, S \rangle) := V^{\otimes S} \text{ (the tensor product of copies of } V \text{ indexed by } S).\]

It follows from definition that, for any graph \( \Gamma \in \Gamma(\langle g, S \rangle) \), \( \mathcal{E}nd_V(\langle \Gamma \rangle) = V^{\otimes \text{Flag}(\Gamma)} \).

Let \( B^{\otimes \text{Edg}(\Gamma)} : V^{\otimes \text{Flag}(\Gamma)} \to V^{\otimes \text{Leg}(\Gamma)} \) be the multilinear form which contracts the factors of \( V^{\otimes \text{Flag}(\Gamma)} \) corresponding to the flags which are paired up as edges of \( \Gamma \). Then we define \( \alpha_{\Gamma} : \mathcal{E}nd_V(\langle g, \Gamma \rangle) \to \mathcal{E}nd_V(\langle g, S \rangle) \) to be the map
\[
(41) \quad \alpha_{\Gamma} : \mathcal{E}nd_V(\langle \Gamma \rangle) \cong V^{\otimes \text{Flag}(\Gamma)} \xrightarrow{B^{\otimes \text{Edg}(\Gamma)}} V^{\otimes \text{Leg}(\Gamma)} \cong V^{\otimes S} = \mathcal{E}nd_V(\langle g, S \rangle).
\]

It is easy to show that the contractions \( \{ \alpha_{\Gamma} \mid \Gamma \in \Gamma(\langle g, S \rangle) \} \) define on \( \mathcal{E}nd_V \) the structure of a modular operad.

We would like to modify Example 5.2 to the situation when the degree of the form \( B \) is +3, as in the definition of a loop homotopy Lie algebra. Formula (41) does not work, among other things also because \( \alpha_{\Gamma} \) will not be of degree zero.

For this modification we need to introduce ‘twisted’ modular operads. If \( X \) is a finite set with \( \text{card}(X) = s \), let \( \text{Det}(X) := \Lambda^s(\downarrow k)^{\otimes X} \), the top dimensional piece of the \( s \)-fold exterior power of the direct sum of the copies of \( \downarrow k \) indexed by elements of \( X \). Clearly \( \text{Det}(X) \) is an one-dimensional vector space concentrated in degree \(-s\). The *determinant of a graph* \( \Gamma \in \Gamma(\langle g, S \rangle) \) is defined by \( \text{Det}(\Gamma) := \text{Det}(\text{Edg}(\Gamma)) \).

A *twisted* modular operad ([3], p. 293), also called a \( \mathfrak{K} \)-modular operad in [7]) is then a modular collection \( \mathcal{A} \) together with a coherent system of contractions
\[
\tilde{\alpha}_{\Gamma} : \mathcal{A}(\langle \Gamma \rangle) \otimes \text{Det}(\Gamma) \to \mathcal{A}(\langle g, S \rangle), \text{ for any } \Gamma \in \Gamma(\langle g, S \rangle), \ g \geq 0 \text{ and a finite set } S.
\]
**Example 5.3.** Let \( W = (W, H) \) be a graded vector space with a nondegenerate degree \(-1\) symmetric bilinear form \( H \). Define the modular collection \( \widetilde{\text{End}}_W \) by

\[
\widetilde{\text{End}}_W((g, S)) := W^\otimes S,
\]

for \( g \geq 0 \) and a finite set \( S \). For \( \Gamma \in \Gamma((g, S)) \), the twisted modular contraction

\[
\tilde{\alpha}_\Gamma : \widetilde{\text{End}}_W((\Gamma)) \otimes \text{Det}(\Gamma) \to \widetilde{\text{End}}_W((g, S))
\]

is defined as follows. Let us choose labels \( s_e, t_e \) such that \( e = \{s_e, t_e\} \) for each edge \( e \in \text{Edg}(\Gamma) \) and define \( \tilde{\alpha}_\Gamma \) to be the composition:

\[
\widetilde{\text{End}}_W((\Gamma)) \otimes \text{Det}(\Gamma) \cong W^\otimes \text{Flag}(\Gamma) \otimes \text{Det}(\Gamma) \cong W^\otimes \bigotimes_{e \in \text{Edg}(\Gamma)} \left(W^\otimes \{s_e, t_e\} \otimes \text{Span}(\downarrow e)\right)
\]

\[
\cong W^\otimes \bigotimes_{e \in \text{Edg}(\Gamma)} \left(W_{s_e} \otimes W_{t_e} \otimes \text{Span}(\downarrow e)\right)
\]

\[
\xymatrix{ \mathbf{1} \otimes \bigotimes_{e} H_e \ar[r]^{\cong} & W^\otimes \bigotimes_{e \in \text{Edg}(\Gamma)} \mathbf{k} \ar[r]^-{\cong} & \widetilde{\text{End}}_W((g, S)),}
\]

where \( H_e \) is the map that sends \( u \otimes v \otimes \downarrow e \in W_{s_e} \otimes W_{t_e} \otimes \text{Span}(\downarrow e) \) to \( H(u, v) \in \mathbf{k} \). The symmetry of \( H \) assures that the definition of \( \tilde{\alpha}_\Gamma \) does not depend on the choice of labels. The system \( \{\tilde{\alpha}_\Gamma | \Gamma \in \Gamma((g, S))\} \) induces on \( \widetilde{\text{End}}_W \) the structure of a twisted modular operad.

If \( V = (V, B) \) is a graded vector space with a nondegenerate degree \(+3\) symmetric bilinear form \( B \), then \( W = (W, H) \) with \( W := \uparrow^2 V \) and the form \( H \) defined by \( H(u, v) := B(\downarrow^2 u, \downarrow^2 v) \), \( u, v \in W \), form the data as in Example 5.3, so we may consider the twisted modular operad \( \widetilde{\text{End}}_{\uparrow^2 V} \).

Another example of a twisted modular operad is provided by the *Feynman transform* of a modular operad. Recall \([3, 4.2]\) that the *free twisted modular operad* \( \widetilde{M}(E) \) on a modular collection \( E \) is given by

\[
\widetilde{M}(E)((g, n)) := \text{colim}_{\Gamma \in \text{Iso}\Gamma((g, n))} E((\Gamma)) \otimes \text{Det}(\Gamma),
\]

where \( \text{Iso}\Gamma((g, n)) \) is the full subcategory of isomorphisms in \( \Gamma((g, n)) \). The twisted modular operad structure is induced by the ‘grafting’ of underlying graphs.

If \( \mathcal{A} \) is a modular operad, then \( \widetilde{M}(\mathcal{A})((g, n)) \) carries a natural differential \( \partial_\Gamma [4, \text{Theorem 4.4}] \). The twisted differential modular operad \( F(\mathcal{A}) := (\widetilde{M}(\mathcal{A}), \partial_\Gamma) \) is called the Feynman transform of the modular operad \( \mathcal{A} \).
Let us consider the ‘forgetful’ functor \( \square : \text{MOp} \to \text{COp} \) from the category of modular operads to the category of cyclic operads given by \( \square(\mathcal{A})(S) := \mathcal{A}(\langle 0, S \rangle) \), for any finite set \( S \). It is not difficult to show \[16\] that this functor has a left adjoint \( \text{Mod} : \text{COp} \to \text{MOp} \).

**Definition 5.4.** The modular operad \( \text{Mod}(\mathcal{P}) \) is called the modular operadic completion of the cyclic operad \( \mathcal{P} \).

An easy calculation shows that

\[
\text{Mod}(\text{Com})(\langle g, n \rangle) \cong k, \quad \text{for each } g \geq 0, n \geq 1,
\]

with the trivial action of the symmetric group \( \Sigma_{n+1} \).

The key role in our characterization is played by the Feynman transform \( F(\text{Mod}(\text{Com})) \) of the modular completion of the operad \( \text{Com} \). It follows from \[12\] that, as a nondifferential operad, \( F(\text{Mod}(\text{Com})) \) is the free twisted modular operad on the generators \( \omega_n^g \),

\[
\widetilde{M}(\text{Mod}(\text{Com})) \cong \widetilde{M}(\{\omega_n^g; n \geq 1, g \geq 0\}),
\]

where \( \omega_n^g \) corresponds to the dual of \( 1 \in k \cong \text{Mod}(\text{Com})(\langle g, n \rangle) \). The central result of this section reads as follows.

**Theorem 5.5.** There exists a natural one-to-one correspondence between twisted modular \( F(\text{Mod}(\text{Com})) \)-algebra structures on \( (\uparrow^2 V, B(\downarrow^2-, \downarrow^2-)) \), i.e. morphisms

\[
a : \left( F(\text{Mod}(\text{Com})), \partial_k \right) \to \left( \mathcal{E} \text{nd}_{\uparrow^2 V}, \partial = 0 \right)
\]

of differential twisted modular operads, and loop homotopy algebra structures on \( (V, B) \) in the sense of Definition \[2.1\].

**Sketch of proof.** Description \[13\] shows that a map \( a \) of \[44\] is determined by its values \( \xi_n^g := a(\omega_n^g) \in \mathcal{E} \text{nd}_{\uparrow^2 V}(\langle g, n \rangle) \) on the generators. Moreover, the map \( a \) ought to commute with the differentials, so the equation

\[
a(\partial_F(\omega_n^g)) = 0
\]

must be satisfied, for each \( g \geq 0 \) and \( n \geq 1 \). Observe that \( \xi_n^g \in \mathcal{E} \text{nd}_{\uparrow^2 V}(\langle g, n \rangle) \) can be interpreted as a degree \(-2(n+1)\)-element of the graded vector space \( V^{\otimes n+1} \). Let us introduce a map \( \Xi : V^{\otimes n+1} \to \text{Hom}(V^{\otimes n}, V) \) by

\[
\Xi(x_0 \otimes \cdots \otimes x_n)(v_1, \ldots, v_n) := (-1)^{n x_0 + (n-1) x_1 + \cdots + x_{n-1}} x_0 B(x_1, v_1) B(x_2, v_2) \cdots B(x_n, v_n),
\]
for $x_0 \otimes \cdots \otimes x_n \in V^\otimes n+1$ and $v_1, \ldots, v_n \in V$. The map $\Xi$ is clearly a degree $3n$ isomorphism of $V^\otimes n+1$ and $\text{Hom}(V^\otimes n, V)$. Finally, let $l^g_n : V^\otimes n \to V$ be a homogeneous degree $n-2$ map given by

$$l^g_n(v_1, \ldots, v_n) := (-1)^{\frac{n(n+1)}{2} + n(v_1 + \cdots + v_n)} \Xi(\omega^g_n)(v_1, \ldots, v_n), \text{ for } v_1, \ldots, v_n \in V.$$ 

A long but straightforward calculation shows that $l^g_n$ are antisymmetric operations satisfying (13) and that (45) translates to the main identity (11).

On the other hand, all steps above can clearly be reversed, thus a loop homotopy Lie algebra structure induces a map (44).

\textbf{Remark 5.6.} Observe that Theorem 5.5 is formulated in such a way that the differential $\partial$ on $V$ is a part of the structure, namely $\partial := a(\omega^0_1)$.

\section{Possible generalizations (open strings).}

Let $\mathcal{P}$ be an operad. It is now well-understood what a ‘strongly homotopy $\mathcal{P}$-algebra’ is. In case when $\mathcal{P}$ is Koszul, it is an algebra over the cobar construction on the quadratic dual $\mathcal{P}!$ of $\mathcal{P}$ [3, Definition 4.2.14].

An alternative characterization is that a homotopy $\mathcal{P}$-algebra is a square zero differential on the cofree connected $\mathcal{P}!$-coalgebra. The equivalence of these two characterizations follows for example from [3, Proposition 4.2.15].

The quadratic dual of the operad $\mathcal{L}ie$ for Lie algebras is $\mathcal{C}om$, the operad for commutative associative algebras, and the above characterization give Proposition 4.1 resp. Proposition 4.11. Another example is $\mathcal{P} = \mathcal{A}ss$, the operad for associative algebras. It is quadratic self-dual, $\mathcal{P}! = \mathcal{A}ss$, and the corresponding strongly homotopy algebras are called strongly homotopy associative or $A_\infty$-algebras [18, 15].

Let us look for possible generalizations to the loop case. If $\mathcal{P}$ is a cyclic operad (recall that both $\mathcal{L}ie$ and $\mathcal{A}ss$ are cyclic), the quadratic dual $\mathcal{P}!$ is again cyclic [3], so it makes sense to consider the modular completion $\text{Mod}(\mathcal{P}!)$ (Definition 5.4). We suggest the following definition.

\textbf{Definition 6.1.} Let $\mathcal{P}$ be a Koszul cyclic operad. A loop homotopy $\mathcal{P}$-algebra is then a modular algebra over the twisted differential modular operad $\mathcal{F}(\text{Mod}(\mathcal{P}!))$. 

- August 30, 1999 -
For $\mathcal{P} = \mathcal{L}ie$ we get Theorem 5.3. It would be interesting to write out explicitly axioms of loop homotopy associative algebras, because these structures should play an important rôle in the higher-genera open string field theory, as suggested by [19]. While in the Lie case we had, for each $n$ and $g$, only one antisymmetric operation $l^g_n : V^{\otimes n} \to V$, in the loop homotopy associative case we expect to have

\[
\frac{(n + 1)!}{2^g \cdot g! \cdot (n + 1 - 2g)!}
\]

operations $V^{\otimes n} \to V$, due to the dimension of Mod($\mathcal{A}ss$)(($g, n$)).

A seemingly easier approach would be the one based on coderivations. We would like to say that a loop homotopy $\mathcal{P}$-algebra is an order 2 coderivation of the cofree connected $\mathcal{P}$'-coalgebra, having properties analogous to (22). This works nicely for $\mathcal{P} = \mathcal{L}ie$, because we know what is a higher order coderivation of a cocommutative coalgebra. But we are not sure whether there exist a reasonable concept of higher-order coderivations without the cocommutativity, though the paper [4] seems to suggest this.

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