ABSTRACT. We review the current state of the homogeneous Banach space problem. We then formulate several questions which arise naturally from this problem, some of which seem to be fundamental but new. We give many examples defining the bounds on the problem. We end with a simple construction showing that every infinite dimensional Banach space contains a subspace on which weak properties have become stable (under passing to further subspaces). Implications of this construction are considered.

1. Introduction

A Banach space is said to be homogeneous if it is isomorphic to all of its infinite dimensional subspaces.

The Homogeneous Banach Space (HBS) Problem

Is every homogeneous Banach space isomorphic to a Hilbert space?

This problem has frequently been referred to as “Banach’s problem” since it is stated in Banach’s book [B]. However, it was pointed out in [MTJ2] that in the original polish version of his book, Banach attributes this problem to Mazur. The objective of this paper is to generate renewed interest in the HBS problem and a host of other interesting problems surrounding it.

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Essentially no progress was made on this problem of mazur until recently when J. Bourgain [B∗] solved the finite dimensional version and W. B. Johnson [J1] solved a special case of the general problem. Both papers rely on advances made in the local theory of Banach spaces in the 1980’s. These results are discussed in section 3.

In section 2, we consider the long list of immediate questions we can also ask about a homogeneous Banach space, most of which are still unanswered. In section 4, we will look at a series of problems which are stronger than the homogeneous Banach space problem, in the sense that a positive answer to one of these would yield a positive answer to the HBS problem. In section 5 we look at some fundamental questions which arise from the HBS problem, some of which do not seem to have been asked before. Finally, in section 6, we give a simple construction that can be carried out in any Banach space to yield a subspace which is stable for weak properties, for passing to further subspaces. We will also show that this is the strongest subspace that we can expect to find inside of every Banach space.

The author expresses his deepest gratitude to W. B. Johnson for many interesting discussions related to the material in this paper.

2. Some Immediate Questions

There are several immediate questions which we can ask about a homogeneous Banach space $X$: (1) Is $X$ reflexive?; (2) Is $X$ suprreflexive?; (3) Does $X$ have a separable dual space? None of these questions has been answered yet. However, we do not believe that any of these questions will be important for the solution of the problem. But these questions do give rise to some interesting problems in the general theory of Banach spaces which are discussed in section 5. Even the isometric version of the HBS problem is still open.

The Homogeneous Banach Space (HBS) Problem (Isometric Version)

If $X$ is isometric to all of its infinite dimensional subspaces, is $X$ isometric (or even isomorphic) to a Hilbert space?

With so little progress having been made on this problem, we might hope to get some movement on it by strengthening our hypotheses. Unfortunately, assuming that $X$ is homogeneous and has a unconditional basis, or even a symmetric basis, does not
seem to help. These hypotheses do yield that $X$ is superreflexive and that $X$ is weak cotype 2 [J1] but do not seem to lead to any serious breakthrough on the problem.

Another obvious question is: Is every homogeneous Banach space $X$ isomorphic to a square? (i.e. Is $X \approx X \oplus X$?) Even such an elementary question was unresolved until recently and requires some heavy machinery of W. T. Gowers and B. Maurey [GoM]. Recall that a projection $P$ on a Banach space is said to be nontrivial if $\text{rk} P = \text{rk}(I - P) = \infty$.

**Theorem 2.1 (GoM).** If an infinite dimensional Banach space $X$ has no nontrivial projections on any infinite dimensional subspace, then every bounded operator on $X$ is a strictly singular perturbation of the identity.

This means that every bounded operator $T$ on such a space can be written as: $T = aI + S$, where $S$ is a strictly singular operator (i.e. $S$ is not an isomorphism when restricted to any infinite dimensional subspace of $X$.) We also need the notion of Fredholm index. If $T : X \to Y$ is a bounded operator with closed range, put $\alpha(T) = \dim \ker T$, $\beta(T) = \dim Y / TX$. If either $\alpha(T) < \infty$ or $\beta(T) < \infty$, we define the Fredholm index $i(T)$ by: $i(T) = \alpha(T) - \beta(T)$. If $i(T)$ is defined and is finite, then $T$ is called a Fredholm operator. (See chapter 2.c of [LT1], for basic information on $i(T)$). We are now ready for:

**Result 2.2.** If $X$ is homogeneous then $X \approx X \oplus X$.

**Proof.** If not, then $X$ has no non–trivial projection on any infinite dimensional subspace. Let $T$ be an isomorphism of $X$ onto a hyperplane in $X$. By Theorem 2.1, $T = aI + S$. But, $T$ has Fredholm index $-1$ while $aI + S$ has Fredholm index 0 by Proposition 2.c.10 of [LT1]. This contradiction completes the proof. □

There are some immediate questions which would be useful for the solution to the HBS problem. One such question is,

**Question 2.3.** If $X$ is homogeneous, is $X^*$ homogeneous?

In section 3, we will see a case where W. B. Johnson [J1] made use of such a hypothesis. If $X$ is homogeneous then, since $X$ has a subspace with a basis, every subspace of $X$ has a basis. In particular, $X$ is separable and every subspace of $X$ has the approximation property. It follows ([LT2], Theorem 1.g.6) that;
\[ \sup \{ p \mid X \text{ is type } p \} = 2 = \inf \{ q \mid X \text{ is cotype } q \}. \]

This fact gives rise to some interesting open questions in section 4.

We also feel that the following open question will be important to the solution of the HBS problem.

**Question 2.4.** If \( X \) is homogeneous, is \( X \) uniformly isomorphic to all of its infinite dimensional subspaces?

That is, does there exist a \( K \geq 1 \) so that \( X \) is \( K \)-isomorphic to all of its infinite dimensional subspaces? One reason for the importance of question 2.4 is that, perhaps the HBS problem has a positive answer in the uniform case but a negative answer in the general case. We can prove, with some relatively soft infinite dimensional theory, the following result:

**Proposition 2.5.** If \( X \) is a homogeneous Banach space, then there is a constant \( K \geq 1 \) so that \( X \) \( K \)-embeds into every infinite dimensional subspace of \( X \).

Proposition 2.5 will follow from a general result on minimal Banach spaces. A Banach space is **minimal** if it embeds into every one of its infinite dimensional subspaces. We discuss such spaces in more detail in section 6.

**Proposition 2.6.** If \( X \) is a minimal Banach space then there is a \( K \geq 1 \) so that \( X \) \( K \)-embeds into every infinite dimensional subspace of \( X \).

**Proof.** Assume, for the sake of contradiction, that \( X \) is not uniformly embeddable into all of its infinite dimensional subspaces. Then no subspace of \( X \) has this property either. So there are infinite dimensional subspaces \( Y_1 \supset Y_2 \supset \cdots \) so that \( X \) does not \( 2^{f(n)} \)-embed into \( Y_n \), where \( f(n) = n(1 + \sqrt{n})^2 \). Choose \( y_n \in Y_n \) so that \( (y_n) \) is a basic sequence in \( X \) and let \( E_n = \text{span}_{n \leq i} y_i \). By assumption, there is a subspace \( Z \subset E_1 \) and a \( K \geq 1 \) so that \( Z \) is \( K \)-isomorphic to \( X \). Without loss of generality, we may assume that \( \dim E_1/Z = \infty \). For each \( n \) let \( H_n = Z \cap E_n \) and choose \( F_n \subset E_n \) with \( H_n \subset F_n \) and \( k = \dim E_n/F_n = \dim Z/H_n \leq n \). There are projections \( P : E_n \to F_n \) and \( Q : Z \to H_n \) with \( \|P\| \leq 1 + \sqrt{n} \) and \( \|Q\| \leq 1 + \sqrt{n} \). Hence, there are \( k \)-dimensional spaces \( W \subset E_n \) and \( W' \subset Z \) so that \( W' \oplus_1 H_n \) is \( (1 + \sqrt{n}) \)-isomorphic to \( Z \) while \( \text{span}(W, H_n) \) is \( (1 + \sqrt{n}) \)-isomorphic to \( W \oplus_1 H_n \). Since \( W \) and \( W' \) are \( n \)-isomorphic, it follows that \( V = \text{span}(W, H_n) \) is \( n(1 + \sqrt{n})^2 \)-isomorphic to \( Z \) and hence \( V \) is \( K n(1 + \sqrt{n})^2 \)-isomorphic to \( X \). But \( V \subset E_n \) implies \( X \) is
$Kn(1 + \sqrt{n})^2$-embeddable into $E_n$. For large $n$, this contradicts our assumption that $X$ is not $2^{f(n)}$-embeddable into $E_n$, for $f(n) = n(1 + \sqrt{n})^2$. \(\square\)

Proposition 2.6 easily gives a subspace $Y \subset X$ with a finite dimensional decomposition $\Sigma \oplus E_n$ so that for every $k$, $(E_n)_{n=k}^\infty$ is dense in the family of all finite dimensional subspaces of $X$. Hence, there is a constant $K \geq 1$ so that every finite dimensional subspace of $X$ is $K$–isomorphic to a $K$–complemented subspace of every finite codimensional subspace of $X$.

Another consequence of a positive answer to 2.4 comes from Krivine’s Theorem [Kr], [MS]. If $X$ is homogeneous, a positive answer to 2.4, combined with 2.1 and Krivine’s Theorem implies:

**Theorem 2.7.** If $X$ is $K$–isomorphic to all of its infinite dimensional subspaces, then for every $n$, there is a basis $(x_i)$ of $X$ with basis constant $\leq K$ and for every finite set of natural numbers $I$ with $|I| = n$, $(x_i)_{i \in I} \approx_K (e^2_i)_{i \in I}$, where $(e^2_i)$ is the unit vector basis of $l_2$.

This says that $X$ has a sequence of bases $(x^k_n)_{n=1}^\infty$ with uniformly bounded basis constants so that every $k$ elements of $(x^k_n)_{n=1}^\infty$ are uniformly the $l_2^k$ unit vector basis. This property itself is very strong and we might hope that it already characterizes a Hilbert space. Although, as we will see, it does not characterize Hilbert space, it does give rise to the notion of sequences of successively better bases for a Banach space.

**Definition 2.8.** We say that a Banach space $X$ has basis property $p \ (1 \leq p \leq \infty)$ if there is a $K \geq 1$ so that for every $n$, there is a basis $(x_i)$ of $X$ with basis constant $\leq K$ and for every $n$–element subset $I$ of the natural numbers, $(x_i)_{i \in I} \approx_K (e^p_i)_{i \in I}$, where $(e^p_i)$ is the unit vector basis of $l_p$ (or $c_0$ if $p = \infty$). To simplify notation, we write $(x_n) \leq (y_n)$, for two sequences in Banach spaces $X, Y$ respectively, if there is a constant $K \geq 1$ so that for every sequence of scalars $(a_n)$,

$$\left\| \sum_{n=1}^\infty a_n x_n \right\| \leq K \left\| \sum_{n=1}^\infty a_n y_n \right\| .$$

We now have,

**Proposition 2.9.** Let $X$ be a Banach space with a normalized basis $(x_n)$ satisfying $(x_n) < (e^p_n)$. Then $X \oplus l_p$ has basis property $p$. 

Proof. Let \((x_n)\) be the normalized basis of \(X\) satisfying \((x_n) < (e_n^p)\). Fix a natural number \(m\). Define \(y_n \in X \oplus l_p\) by:

\[
y_{k2m+(k+1)} = \left( x_{k+1}, \frac{1}{(2m)^{1/p}} \sum_{i=k2m+1}^{(k+1)2m} e_i^p \right)
\]

\[
y_{k2m+(k+1)} + j = \left( 0, e_j^{p}\right), \quad \text{for } 1 \leq j \leq 2m,
\]

and \(k = 0, 1, 2, \ldots\). For any sequence of scalars \((a_i)\) and any \(t < s\), choose \(k_1, k_2, j_1, j_2\) so that \(t = k_1 2m + (k_1 + 1) + j_1\) and \(s = k_2 2m + (k_2 + 1) + j_2\). Then, treating \(X \oplus l_p\) as an \(l_p\)-sum and letting \(b\) be the basis constant of \((x_n)\) we have,

\[
\left\| \sum_{i=1}^{t} a_i y_i \right\|^p = \\
\left[ \left\| \sum_{i=0}^{k_1} a_{i2m+(i+1)} x_{i+1} \right\|^p + \\
\sum_{i=0}^{k_1} \sum_{j=1}^{2m} \left| a_{i2m+(i+1)+j} - \frac{1}{(2m)^{1/p}} a_{i2m+(i+1)} \right|^p \\
+ \sum_{j=1}^{j_1} \left| a_{i2m+(i+1)+j} - \frac{1}{(2m)^{1/p}} a_{i2m+(i+1)} \right|^p \right]^{1/p}
\]

\[
\leq b \left[ \sum_{i=0}^{k_1} a_{i2m+(i+1)} x_{i+1} \right]^p + \\
\sum_{i=0}^{k_1} \sum_{j=1}^{2m} \left| a_{i2m+(i+1)+j} - \frac{1}{(2m)^{1/p}} a_{i2m+(i+1)} \right|^p \\
+ \sum_{j=1}^{j_1} \left| a_{i2m+(i+1)+j} - \frac{1}{(2m)^{1/p}} a_{i2m+(i+1)} \right|^p \right]^{1/p} \\
\leq b^2 \left[ \sum_{i=0}^{k_1} a_{i2m+(i+1)} x_{i+1} \right]^p + \\
\sum_{i=0}^{k_2} \sum_{j=1}^{2m} \left| a_{i2m+(i+1)+j} - \frac{1}{(2m)^{1/p}} a_{i2m+(i+1)} \right|^p
\]
\[
+ \sum_{j=1}^{j_2} \left| a_{2m+(i+1)+j} - \frac{1}{(2m)^{1/p}} a_{2m+(i+1)} \right|^p + \frac{2m - j_2}{2m} \left| a_{k_2 2m+(k_2+1)} \right|^p \right]^{1/p} \\
\leq b^2 \left\| \sum_{i=1}^{s} a_i y_i \right\|.
\]

So the basis constant of \((y_n)\) is \(b^2\) where \(b\) is the basis constant of \((x_n)\).

Since \((x_n) < (e_n^p)\), there is a constant \(K \geq 1\) so that for every sequence of scalars \((a_n)\) we have,

\[
\left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq K \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p},
\]

On the other hand, if \(I \subset \{ k2m+(k+1)+1, k2m+(k+1)+2, \ldots, (k+1)2m+(k+1) \} \) with \(|I| \leq m\),

\[
\sum_{j \in I} \left| a_{k2m+(k+1)+j} - \frac{1}{(2m)^{1/p}} a_{k2m+(k+1)} \right|^p + \frac{2m - |I|}{2m} \left| a_{k2m+(k+1)} \right|^p \right]^{1/p} \\
\geq \frac{1}{4} \left[ \sum_{j \in I} \left| a_{k2m+(k+1)+j} \right|^p + \left| a_{k2m+(k+1)} \right|^p \right]^{1/p}.
\]

Now, if \(I \subset \mathbb{N}, |I| = m\), for each \(k = 0, 1, \ldots\) let \(I_k = I \cap \{ k2m+(k+1), k2m+(k+1)+1, \ldots, (k+1)2m+(k+1) \}\) and let \(P\) be the natural projection of \(X \oplus l_p\) onto \(l_p\). From 2.4 we have,

\[
\left\| \sum_{i \in I} a_i y_i \right\| \geq \left( \sum_{k=0}^{\infty} \left\| \sum_{i \in I_k} a_i P y_i \right\| \right)^{1/p} \\
\geq \frac{1}{4} \left( \sum_{k=0}^{\infty} \sum_{i \in I_k} |a_i|^p \right)^{1/p}.
\]

By 2.2 and 2.6, we have that \((y_i)_{i \in I} \approx_{8K} (e_i^p)_{i \in I}\).
This completes the proof of the proposition. □

For $1 < p \leq 2$, $L_p[0,1] \approx L_p[0,1] \oplus l_2$, and the Walsh system $(W_n)$ is a basis of $L_p[0,1]$ satisfying $(W_n) < (e_n^2)$. So we have,

**Corollary 2.10.** For $1 < p \leq 2$, $L_p[0,1]$ has basis property 2.

Since any space with basis property $p$ contains uniformly complemented $l_p^n$'s, it follows that $L_1[0,1]$ fails basis property 2. It is immediate that every basis $(x_n)$ for a Banach space $X$ satisfies $(e_n^0) < (x_n) < (e_n^1)$. So for example, $L_1[0,1]$ has basis property 1. That is, $L_1[0,1]$ has bases with uniformly bounded basis constants every $n$-elements of which are the unit vector basis of $l_1^n$ (and hence are well unconditional) despite the fact that $L_1[0,1]$ has no unconditional basis. It can be shown that $T^{(2)}$–Tsirelson’s space [CS] has basis property 2 despite its not containing a subspace isomorphic to $c_0$ or $l_p$, for any $p$. Also, $(\sum \oplus T^{(2)})l_2$ has basis property 2.

We could strengthen this notion to “unconditional” basis property $p$ by requiring the unconditional basis constant to be $\leq K$ in definition 2.9. Although $l_p \oplus l_2$ has basis property $p$, for $2 < p < \infty$ (by proposition 2.10) it fails to have unconditional basis property $p$ by the quantitative version of a result of Edelstein and Wojtawsczyk [EW] (see also [W], [CKT]). Also, $T^{(2)}$ fails to have unconditional basis property 2 by the uniqueness, up to a permutation, of the unconditional basis for $T^{(2)}$ [BCLT]. We do not know of a non–Hilbert space with unconditional basis property 2. Probably, such examples exist in the class of Orlicz spaces (Probably even spaces with “symmetric” basis property 2). This whole idea could warrant further study if we could first find a good use for this concept.

Recently, V. Mascioni [Ma] has introduced an interesting variant of the HBS problem.

**Problem 2.11.** [Ma] If $X$ is an infinite dimensional Banach space, and every infinite dimensional subspace of $X$ is isomorphic to its dual space, is $X$ isomorphic to a Hilbert space?

We could also ask the isometric version of problem 2.11. Mascioni [Ma] then finite dimensionalizes the problem.

**Definition 2.12.** [Ma] A Banach space is locally self dual (LSD) if there is a constant $c$ such that every finite dimensional subspace of $X$ is $c$–isomorphic to its dual space.

**Problem 2.13.** [Ma] Are LSD spaces isomorphic to Hilbert spaces?
We will discuss these results further in section 4.

3. Some Positive Results

The first major advance in this area was due to J. Bourgain \([B^*]\), who solved the finite dimensional version of the HBS problem. Later, N. Tomczak–Jaegermann and P. Mankiewicz \([MTJ1]\) gave the best constants for the finite dimensional homogeneous Banach space problem. V. D. Milman first posed this problem in its finite dimensional form. In the finite dimensional setting, we cannot ask for \(X\) to be isomorphic to its subspaces. Instead, we assume that \(\dim X = n\) and for some \(1 \leq k \leq n\), we assume that all \(k\)-dimensional subspaces of \(X\) are \(K\)-isomorphic. Now we must ask for a quantitative answer: what is the smallest constant \(f(K)\) so that \(X\) is \(f(K)\)-isomorphic to a Hilbert space? Clearly, \(n = 1\) will give no information about the Banach space \(X\). Yet, we have a quite exact answer to this problem from \([B^*]\), and \([MJT1]\).

**Theorem 3.1.** \([MTJ1]\) If an \(n\)-dimensional Banach space \(X\) has the property that all its \([\alpha n]\)-dimensional subspaces are \(K\)-isomorphic, for some \(0 < \alpha < 1\), then \(X\) is \(f(\alpha, K)\)-isomorphic to a Hilbert space, where \(f(\alpha, K) = CK^{3/2}\), if \(0 < \alpha < 2/3\) and \(f(\alpha, K) = CK^2\), if \(2/3 < \alpha < 1\), and \(C = C(\alpha)\) depends on \(\alpha\) only.

We will come back to this theorem in a moment. In the meantime, we consider the only solution of a special case of the infinite dimensional problem due to W. B. Johnson \([J1]\). Recall that a Banach space \(X\) is said to have the GL–property (Gordon–Lewis property) if every absolutely summing operator from \(X\) to \(L_2\) factors through \(L_1\). Y. Gordon and D. Lewis \([GL]\) in a landmark paper showed that every Banach space with an unconditional basis (or even LUST) has the GL–property. W. B. Johnson \([J1]\) conjectured that every Banach space has a subspace with the GL–property. This conjecture does not seem to have been tested yet on the new examples of Banach spaces without unconditional bases due to W. T. Gowers and B. Maurey \([GoM]\). For a full understanding of Johnson’s result, we need to recall some definitions.

**Definition 3.2.** (1) A Banach space \(X\) is a weak cotype \(q\) space if there is a constant \(wc_q(X)\) so that for all \(n\) and all operators \(u : l^2_2 \rightarrow X\), we have

\[
\sup_{k \geq 1} k^{1/q} a_k(u) \leq wc_q(X) l(u),
\]

where

\[
l(u) = \left( \int_{\mathbb{R}} \|u(x)\|^2 \gamma_n(dx) \right)^{1/2}
\]
and $\gamma_n$ is the canonical Gaussian probability measure on $\mathbb{R}^n$, and

$$a_k(u) = \inf\{\|u - s\| | s : l_2^n \to X, \text{ rank } s \leq k\}.$$

(2) A Banach space $X$ is a weak type $p$ space if there is a constant $wt_p(X)$ so that for all $n$ and all operators $V : X \to l_2^n$ we have

$$\sup_{k \geq 1} k^{1/p} a_k(V) \leq wt_p(X) l^*(V)$$

where

$$l^*(V) = \sup\{tr(uv) | u : l_2^n \to X, l(u) \leq 1\}.$$ 

(3) A Banach space $X$ is a weak Hilbert space if it is weak type 2 and weak cotype 2 space.

(4) A Banach space is as–Hilbertian (asymptotically Hilbertian) if there is a constant $K \geq 1$ so that for every $n$, there is a finite codimensional subspace $H$ of $X$ with the property that every $n$–dimensional subspace of $H$ is $K$–isomorphic to $l_2^n$.

See G. Pisier [P] for a complete treatment of these notions. Johnson observed;

**Theorem 3.3.** [J1] A homogeneous weak Hilbert space is isomorphic to a Hilbert space.

The main result from [J1] is:

**Theorem 3.4.** [J1] If $X$ and $X^*$ are homogeneous and if $X$ has the GL–property, then $X$ is isomorphic to a Hilbert space.

Returning to Theorem 3.1, we might naturally formulate an alternative infinite dimensional HBS problem as:

If $X$ is $K$–isomorphic to all of its finite codimensional subspaces, is $X f(K)$–isomorphic to a Hilbert space?

It is easily seen that this property is so strong that even $c_0$, $l_p$, $1 \leq p < \infty$, fail it. In particular, a space $X$ with this property also satisfies the property that all finite dimensional subspaces of $X$ are $K$–isomorphic to $K$–complemented subspaces of $X$. Also, the proof of Theorem 3.4 shows that whenever $X$ and $X^*$ have this property and $X$ has the GL–property, then $X$ is isomorphic to a Hilbert space. However, this property does not characterize a Hilbert space as our next proposition shows.
Proposition 3.5.

(1) If $X$ is a Banach space with a separable dual $X^*$, and $\varepsilon > 0$, then there is a Banach space $Y$ with a separable dual containing $X$ and $Y$ is $1 + \varepsilon$–isomorphic to every subspace of $Y$ of finite codimension.

(2) If in (1), $X^{**}$ is separable, we may construct our $Y$ satisfying (1) and so that $Y^*$ is also $1 + \varepsilon$–isomorphic to every subspace of $Y^*$ of finite codimension.

Proof. Let $X_1 = (\sum_{k=1}^{\infty} \oplus X)_l^2$ and choose $f_n^1 \in X_1^*$ so that \{f_n^1\} is dense in $X_1^*$ and each $f_n^1$ appears infinitely many times in the sequence. Let $J$ be the family of all finite subsets of $\mathbb{N}$. For each $A \in J$ let $Z(A) = \text{span} \{f_n^1 \mid n \in A\}$ and let

(1) $X_2 = X_1 \oplus_2 (\sum_{A \in J} \oplus Z(A))_l^2$.
   It follows that

(2) $X_2 \approx (1) X_1 \oplus X_2$,
   and if $A \in J$, then

(3) $X_2 \approx (1) X_2 \oplus_2 Z(A)_l^2$.
   Now choose $(f_n^2)$ dense in $X_2^*$ with each $f_n^2$ appearing infinitely many times in the sequence and repeat the construction to get $X_3$. By induction, we construct $X_1, X_2, \ldots$. Now let

(4) $Y = (\sum_{n=1}^{\infty} \oplus X_n)_l^2$.
   If follows that

(5) $Y \approx (1) (\sum_{k=n}^{\infty} \oplus X_k)_l^2$, for every $n$, and

(6) if $H$ is a subspace of $(\sum_{i=1}^{n} \oplus X_i)_l^2$ of finite codimension, and $H = (\text{span}_{k \in A} f_k^1)_l^2$, for some $A \in J$, then

$$\left( \sum_{k=n+1}^{\infty} \oplus X_k \right)_l^2 \approx (1) \left( \sum_{k=n+1}^{\infty} \oplus X_k \right)_l^2 \oplus_2 H.$$ 

Now let $H \subset Y$ be a subspace of finite codimension and $H^\perp = \text{span}_{1 \leq i \leq n} f_i$, where $f_i = \sum_{j=1}^{\infty} f_{ij}, f_{ij} \in X_j^*$ and the basis constant of \{f_i\}_{i=1}^{n} is $\leq n^2$. Choose $\varepsilon_0 < 1 + \varepsilon/2\sqrt{mn^2}$ and an $m$ so that

(7) $\|f_i - \sum_{j=1}^{m} f_{ij}\| < \varepsilon_0$, for all $i = 1, 2, \ldots, n$. 

Let $g_i = \sum_{j=1}^{m} f_{ij}$, $1 \leq i \leq n$ and choose $f_{ki}^m \in \left( \sum_{j=1}^{m} \oplus X_j^* \right)_{l_2}$ so that

$$\text{(8)} \quad \| f_{ki}^m - g_i \| < \varepsilon_0.$$  

It follows that

$$\text{(9)} \quad \| f_{ki}^m - f_i \| < 2\varepsilon_0.$$  

Choose a $w^*$–continuous projection $P^*$ of $Y^*$ onto $\text{span}_{1 \leq i \leq n} f_{ki}^m$ with $\| P^* \| < 2\sqrt{n}$. Finally, define $T : Y^* \longrightarrow Y^*$ by: Given $f \in Y^*$, let $P^*(f) = \sum_{i=1}^{n} a_i f_{ki}^m$ and define

$$\text{(10)} \quad T(f) = \sum_{i=1}^{n} a_i f_i + (I - P^*)(f).$$  

We now have, for all $f \in Y^*$,

$$\text{(11)} \quad \|(I - T)(f)\| \leq \sum_{i=1}^{n} |a_i| \| f_i - f_{ki}^m \| < 2\sqrt{nn^2}\varepsilon_0 < 1 + \varepsilon.$$  

So $T$ is a $w^*$ continuous isomorphism on $Y^*$ with $\| T^* \| \| T^{-1} \| \leq (1 + \varepsilon)^2$. Let $S : T \longrightarrow Y$ be the isomorphism satisfying $T = S^*$. Finally, $T \left( \text{span}_{1 \leq i \leq n} f_{ki}^n \right) = H^\perp$ implies $S(H) \subset \left( \text{span}_{1 \leq i \leq n} f_{ki}^m \right)_{\perp}$, and dimension considerations yields that $S$ is a $(1 + \varepsilon)^2$ isomorphism of $H$ onto $\left( \text{span}_{1 \leq i \leq n} f_{ki}^n \right)_{\perp}$. Now, $H \approx_{(1+\varepsilon)^2} \left( \text{span}_{1 \leq i \leq n} f_{ki}^n \right)_{\perp}$ and so $H \approx_{(1+\varepsilon)^2} H_1 \oplus \left( \sum_{j=m+1}^{\infty} \oplus X_j \right)_{l_2}$ where $H_1$ has codimension $n$ in $\left( \sum_{j=1}^{m} \oplus X_j \right)_{l_2}$. So by ($5$) and ($6$),

$$H \approx_{(1+\varepsilon)^2} \left( \sum_{j=m+1}^{\infty} \oplus X_j \right)_{l_2} \approx (1) Y.$$  

For part (2) of the proposition, we alternate the above construction between $X_n$ and $X_n^*$. 

$\square$

A variation of this construction carried out transfinitely would yield an alternate proof of a result in the literature. (This result definitely exists in the literature despite our inability to locate it at this time). One can view this as a counterexample to the “non–separable” HBS problem.

**Theorem 3.6.** There is a non–separable Banach space $X$ which is not isomorphic to a Hilbert space, but such that $X$ is isometric to every subspace of the same density as $X$. 

At this time, weak Hilbert space theory seems to be closing in on the HBS problem. Several natural questions from that area would yield a positive solution to the HBS problem. Since every subspace of a homogeneous Banach space has a basis, a positive answer to the following question, combined with Theorem 3.3, would yield a positive answer to the HBS problem:

**Question 4.1.** If every subspace of $X$ has a basis, is $X$ a weak Hilbert space?

The converse of 4.1 is also an open problem. That is; Does every separable weak Hilbert space have a basis? A result of B. Maurey and G. Pisier (which appears for the first time in [Ma]) states that a separable weak Hilbert space has a finite dimensional decomposition. R. Komorowski [K] has constructed the first weak Hilbert spaces which fail to have unconditional bases (and they are even unconditional sums of two dimensional subspaces).

Johnson’s result 3.3 actually asserts that a homogeneous Banach space which is as–Hilbertian is isomorphic to a Hilbert space. And an earlier result of Johnson (see [P]) states that every weak Hilbert space is as–Hilbertian. So a weaker formulation of 4.1 would be:

**Question 4.2.** If every subspace of $X$ has the approximation property, is $X$ as–Hilbertian?

It is easily seen that there are as–Hilbertian spaces which fail the approximation property. Also, Johnson [J2] has exhibited a class of Banach spaces which are as–Hilbertian, every subspace has the approximation property (even the finite dimensional decomposition property) but they have subspaces without bases. There is a possible counterexample to question 4.2. That is, the symmetric convexified Tsirelson space [CS]. This is a Banach space with a symmetric basis for which all $n$–dimensional subspaces are within a fixed iterated logarithm of $l_2^n$. It is possible, however, that every non Hilbert space with a symmetric basis has a subspace which fails the approximation property.

N. Tomczak–Jaegermann and P. Mankiewicz [MTJ1] obtained the following corollary while proving some deep results in the local theory of Banach spaces:

**Theorem 4.3.** [MTJ1] Let $X$ be a Banach space for which there exists a constant $K$ such that every finite dimensional subspace $F$ of $X$ satisfies $bc(F) \leq K$. Then $X$ is of weak cotype 2.
This Theorem gives added importance to the following question which has been around for quite some time.

**Question 4.4.** Are the following equivalent for a Banach space $X$?

1. Every subspace of $X$ has a basis.
2. There is a constant $K \geq 1$ so that every finite dimensional subspace of $X$ has a basis with basis constant $\leq K$.

The importance of question 4.4 is that a positive solution to $(1) \implies (2)$ would yield that every homogeneous Banach space is a weak cotype 2 space. This might then be combined with a positive solution to question 2.3 to solve the whole problem.

The argument of Johnson [J2] is “local” and shows that convexified Tsirelson’s space has both properties (1) and (2) of question 4.4. N. Nielsen and N. Tomczak–Jaegermann [NTJ] have shown that very weak Hilbert space with LUST also has these properties.

Recently, P. Mankiewicz and N. Tomczak–Jaegermann [MTJ3] made an important step in resolving the above questions:

**Theorem 4.5.** [MTJ3] If a Banach space $X$ has the property that every subspace of every quotient space of $\ell_2(X)$ has a Schauder basis, then $X$ is isomorphic to a Hilbert space.

Neither implication in question 4.4 is known, which points out a serious gap in the available techniques in Banach space theory. Namely, we have no reasonable way of passing results back and forth between local theory and infinite dimensional theory. The paper [MTJ3] should be read not only for the main result, but also because it is the first serious integration of local theory and infinite dimensional theory.

Mascioni [Ma] has proved the corresponding result for LSD–spaces.

**Theorem 4.6.** [Ma] If $\ell_2(X)$ is LSD, then $X$ is isomorphic to a Hilbert space.

Another possible counterexample to question 4.2 is $X = \left( \sum \oplus T^{(2)} \right)_{\ell_2}$, where $T^{(2)}$ is convexified Tsirelson space (see [CS]). This space is of type 2 but fails to be weak cotype 2, while still satisfying:

$$\inf\{q : X \text{ is cotype } q\} = 2.$$
Results of N. Tomczak–Jaegermann and P. Mankiewicz [MTJ1] show that $X$ fails the finite dimensional basis property (i.e. (2) of question 4.4). Also, results from [MTJ3] show that this space has a subspace of a quotient space which fails to have a basis. It is possible, however, that every subspace of $X$ has the approximation property.

Finally, a positive answer to the following question (plus its dual formulation for weak cotype) would yield a positive solution to the HBS problem:

**Question 4.7.** If $\sup\{p|X \text{ is Type } p\} = p_0$, does $X$ contain a subspace of weak type $p_0$?

5. Some Basic Questions

Let us return to some of the open questions of section 2. It seems strange that we do not know if a homogeneous Banach space is reflexive, especially in light of formula (1) of section 2. R. C. James [Ja] has given us a class of Banach spaces with type which fail to be reflexive. But a positive answer to the following question would show that homogeneous Banach spaces are reflexive.

**Question 5.1.** If a Banach space $X$ is of type $p$, for some $1 < p \leq 2$, must $X$ contain a reflexive subspace?

If a Banach space has a subspace with LUST, the answer is “yes” (see [LT2]). Actually, a stronger conclusion could hold:

**Question 5.2.** Does every Banach space of type $p$ for some $1 < p \leq 2$, contain a superreflexive subspace?

The cotype version of question 5.1 also seems to be unasked and unanswered:

**Question 5.3.** If $X$ is of cotype $q$, for some $2 \leq q \leq \infty$, must $X$ contain a separable dual space?

An equivalent formulation of question 5.3 would be to ask if $X$ must contain a boundedly complete basic sequence. These questions are special cases of the question of H. P. Rosenthal: Does every Banach space contain either a reflexive subspace or a subspace isomorphic to $c_0$ or $l_1$? In fact, question 5.2 is the “uniform” or “local” version of Rosenthal’s question. That is, question 5.2 is equivalent to:

**Question 5.5.2’.** Does every Banach space contain either a superreflexive subspace or subspaces uniformly isomorphic to $l_1^n$ or $l_\infty^n$, for every $n = 1, 2, \ldots$?
In this setting, the “$l_\infty$” in the question is redundant.

Recently W. T. Gowers [Go] has constructed a Banach space not containing $c_0$, $l_1$ or any reflexive subspace. In fact, Gowers’ space has no subspace with a separable dual space. It is possible that refinements of this example will give counterexamples to the above problems.

6. A Construction

What is the “best” subspace we can find inside of every Banach space? The major problem here is not just to answer the question, but to formulate the question. We now know that almost every Banach space contains a subspace which fails the approximation property (see [LT2], Chapter 1g). Thanks to W. B. Johnson [J2] we also know of non–Hilbert spaces for which every subspace of every quotient space has a basis. We also know that every Banach space contains a basic sequence but may not contain an unconditional basic sequence [GoM]. The best subspace we could hope to find in a Banach space is a subspace isomorphic to $c_0$, or $l_p$, $1 \leq p < \infty$. Not just because we understand these spaces better than any others, but because they have the property that they embed (complementably) into every infinite dimensional subspace of themselves. That is, we can recover all of the properties of the whole space inside of every subspace. But in 1972, B. S. Tsirelson [T] (see also [CS]) showed that there are Banach spaces which do not contain copies of $c_0$ or $l_p$, $1 \leq p < \infty$. This example quickly blossomed into an “industry” [CS] and even today has its place in the recent exciting solutions to the unconditional basic sequence problem [GoM], the $c_0$ $l_1$, reflexive space counterexample [Go], and the distortion problem [S1], [OS], [MiTJ]. H. P. Rosenthal then raised the question whether every Banach space $X$ might contain a subspace $Y$ which embeds into every one of its infinite dimensional subspaces? Such a space $Y$ is called minimal. This was the “best” subspace we could hope for at the time. In 1982, P. G. Casazza and E. Odell [C0] (see also [CS]) showed that Tsirelson’s space contains no minimal subspaces. As of this writing, we know of only two new classes of minimal Banach spaces (besides subspaces of $c_0$, $l_p$, $1 \leq p < \infty$) [CJT], [S2], and only the second is complementably minimal. So, we now know that we cannot find, in every Banach space $X$, a subspace $Y$ so that infinite dimensional properties of $Y$ are invariant under passing to further subspaces.

Our next approach would be to look for a “locally best” subspace in every Banach space, i.e. a subspace $Y$ of $X$ so that $Y$ is crudely finitely representable in every one of its subspaces. This is not possible either as Tsirelson’s space again fails this property by the argument of [OS]. Since Tsirelson’s space seems to be blocking all our efforts, let’s see what property this space does have. The Tsirelson space $T_p$ enjoys
the property that there is a constant $K$ so that for every $n$ there is a subspace $H$ of $T_p$ of condimension $n$ and every $n$–dimensional subspace of $H$, $K$–embeds into every infinite dimensional subspace of $T_p$. From our earlier examples, this is the “best” we can hope for in an arbitrary Banach space. Our next theorem states that, indeed, every Banach space does contain such a subspace (and with $K$ arbitrarily close to 1).

Recall that a Banach space $Y$ (finite or infinite dimensional) almost isometrically embeds into a Banach space $X$ if for every $\varepsilon > 0$, there is a subspace $Z \subset X$ and an operator $T : Y \rightarrow Z$ so that $\|T\| \|T^{-1}\| \leq 1 + \varepsilon$. We can now state the theorem.

**Theorem 6.1.** For every Banach space $X$, for every $\varepsilon_n \downarrow 0$ and for every $f : N \rightarrow N$ there is a subspace $Y \subset X$ which the following properties:

1. $Y$ has a normalized basis $(y_n)$ with basis constant $\leq 1 + \varepsilon_0$,
2. Every $E \subset \text{span}_{n \leq i < \infty} y_i$, with $\dim E \leq f(n)$, is $1 + \varepsilon_n$–isomorphic to a Banach space $F$, and $F$ almost isometrically embeds into every infinite dimensional subspace of $Y$,
3. Every block basis $(z_i)_{i=1}^{f(n)}$ of $(y_i)_{i=n}^{\infty}$ is $1 + \varepsilon_n$–equivalent to a basis $\{w_i\}_{i=1}^{f(n)}$ of a Banach space $F$, and $\{w_i\}_{i=1}^{f(n)}$ is almost isometrically equivalent to a block basis of every basic sequence in $Y$.

Before we prove Theorem 6.1, let us consider some of its consequences. We have immediately from Definition 3.2 and Theorem 6.1:

**Corollary 6.2.** If $X$ is an infinite dimensional Banach space, there is an infinite dimensional subspace $Y$ of $X$ so that

1. $wt_p(Z) = wt_p(Y)$,
2. $wc_q(Z) = wc_q(Y)$,

for every infinite dimensional subspace $Z$ of $Y$ and every $1 \leq p \leq 2, 2 \leq q < \infty$.

It follows that $Y$ has type (respectively, cotype) if and only if $Y$ has a subspace with type (respectively, cotype). Also, $Y$ is a weak Hilbert space if and only if $Y$ contains a weak Hilbert space.

**Corollary 6.3.** For every Banach space $X$ there is a subspace $Y \subset X$ so that every spreading model of $Y$ (built from a weakly null sequence) is finitely representable in every (other) infinite dimensional subspace of $Y$. 

Proof. This is immediate since spreading models of subspaces of $Y$ are finitely representable in $\text{span}_{i\leq i<\infty} y_i$, for every $n = 1, 2, \ldots$. □

Thus, if $Y$ has $c_0$ as a spreading model (respectively $l_1$) then $Y$ has no subspaces with cotype (respectively, type). Recall that a basic sequence $(x_n)$ is block finitely representable in a basic sequence $(y_n)$ if for every $n$ and every $\varepsilon > 0$ there is a block basis $(z_i)_{i=1}^n$ of $(y_i)$ which is $1 + \varepsilon$–equivalent to $(x_i)_{i=1}^n$. In this notation, Krivine’s Theorem [Kr] (see also [MS]) says that for every basic sequence $(x_n)$ in a Banach space $X$ there is a $1 \leq p < \infty$ (or for $c_0$) so that the unit vector basis of $l_p$ (or $c_0$) is block finitely representable in $(x_n)$. Property (3) of Theorem 6.1 implies that whenever the unit vector basis of $l_p$ is block finitely representable on $(y_n)$ then it is block finitely representable on every basic sequence of $Y$. This result was first observed by H. P. Rosenthal [R].

Corollary 6.4. For every Banach space $X$, there is a subspace $Y$ and a $1 \leq p < \infty$ (or $c_0$) so that the unit vector basis of $l_p$ (or $c_0$) is block finitely representable in every basic sequence in $Y$. Moreover, if $l_p$ (or $c_0$) is block finitely representable in one basic sequence in $Y$, then it is block finitely representable in every basic sequence in $Y$.

Recall that a Banach space $X$ is $K$–crudely finitely representable in a Banach space $Y$ if for every finite dimensional subspace $E \subset X$ there is a subspace $F \subset Y$ with $d(E, F) \leq K$. If for every $\varepsilon > 0$ and every finite dimensional subspace $E \subset X$ there is a subspace $F \subset Y$ with $d(E, F) \leq 1 + \varepsilon$, we say $X$ is finitely representable in $Y$.

Corollary 6.5. Suppose the separable Banach space $X$ is $K$–crudely finitely representable in every infinite dimensional subspace of $X$. Then there is an equivalent norm $1 \cdot 1$ on $X$ so that $(X, 1 \cdot 1)$ is finitely representable in every infinite dimensional subspace of $X$.

Proof. Choose the subspace $Y$ of $X$ with basis $(y_i)$ from Theorem 6.1. Choose finite dimensional subspaces $E_1 \subset E_2 \subset \cdots$ whose union is dense in $X$. For each $i, j = 1, 2, \ldots$, choose $F_{ij} \subset \text{span}_{i(j) \leq k<\infty} y_k$, where $l(j) \geq \dim F_{ij}$, and $d(E_i, F_{ij}) \leq K$. By switching to a subsequence, we may assume $\lim_{j \to \infty} F_{ij} = F_i$, for every $j = 1, 2, \ldots$ (the limit in Banach–Mazur distance) and $F_i$ is finitely representable in every infinite dimensional subspace of $X$, by Theorem 6.1 (2). For each $i = 1, 2, \ldots$ let $T_i : E_i \to F_i$ be a $K$–isomorphism. By switching to a subsequence again, we may assume that for every $x \in X$, $|x| = \lim_{i \to \infty} ||T_i x||$ exists. Then $1 \cdot 1$ is an equivalent norm on $X$ and clearly $(X, 1 \cdot 1)$ is finitely representable in $X$ (again by Theorem 6.1). □
If the above corollary had an infinite dimensional analogue, it could be quite useful for working in minimal Banach spaces. But our proof is local and we have not found a generalization of it for this case. Also, it would be much better if we could show that \((X, 1 \cdot 1)\) is finitely representable in every infinite dimensional subspace of itself.

Again, we might hope that such a strong property would characterize a Hilbert space. But, it is easily seen that \(c_0\) has the property that every finite dimensional subspace isometrically embeds into every infinite dimensional subspace.

We are now ready for the proof of Theorem 6.1.

**Proof of Theorem 6.1.** Fix \(\varepsilon > 0\) and \(n \in \mathbb{N}\). Choose \(n\)-dimensional Banach spaces \(H_1, H_2, \ldots, H_m\) which are \(1 + \varepsilon\)-dense in the set of all \(n\)-dimensional Banach spaces in the Banach–Mazur distance. We now ask: (+) Is \(H_1\), \(1 + \varepsilon\)-embeddable into every infinite dimensional subspace of \(X\)?

If the answer is “yes”, put \(H_1\) in the set \(A\) and go to \(H_2\). If the answer is “no”, put \(H_1\) in the set \(B\) and replace \(X\) by an infinite dimensional subspace \(X_1\) of \(X\) so that every \(H_i\), for \(i \in I\), is \(1 + \varepsilon\)-embeddable into every infinite dimensional subspace of \(Z\) while no \(H_i\), \(i \in J\), is \(1 + \varepsilon\)-embeddable into \(Z\).

It follows that for any \(E \subset Z\), \(\dim E = n\), \(E\) is \(1 + \varepsilon\)-isomorphic to \(H_i\), for some \(i \in I\), and hence \(E\) is \((1 + \varepsilon)^2\)-embeddable into every infinite dimensional subspace of \(Z\). Observe that this property is maintained if we switch to any infinite dimensional subspace of \(Z\). Hence, given \(f : N \to N\) and \(\varepsilon_n \downarrow 0\), we can inductively carry out this procedure to obtain infinite dimensional subspaces of \(X\), \(X \supset Z_1 \supset Z_2 \supset \cdots\) so that every \(f(n)\)-dimensional subspace of \(Z_n\) is \(1 + \varepsilon_n\)-embeddable into every infinite dimensional subspace of \(Z_n\). Now choose \(y_n \in Z_n\) so that \((y_n)\) is a \(1 + \varepsilon_0\)-basic sequence in \(X\). Now, if \(E \subset \text{span}_{i \leq n} y_i\), then there are subspaces \(F_k \subset \text{span}_{n+k} y_i\) so that \(d(E, F_k) \leq 1 + \varepsilon_n\). By switching to a subsequence, we may assume \(\lim_{k \to \infty} F_k = F\) (in Banach–Mazur distance). So \(d(E, F) \leq 1 + \varepsilon_n\) but \(F_k\) is \(1 + \varepsilon_{n+k}\) embeddable into every infinite dimensional subspace of \(Y\). Hence, \(F\) is finitely representable in every infinite dimensional subspace of \(Y\). This concludes the construction for part (1) of Theorem 6.1. Again, note that part (1) of the Theorem holds if \((y_n)\) is replaced by any block basis of \((y_n)\).

Part (2) of Theorem 6.1 is proved in a similar manner. Again, fixing \(n\) and \(\varepsilon > 0\), we list out \((x_{1i})_{i=1}^n, (x_{2i})_{i=1}^n, \ldots, (x_{mi})_{i=1}^n\) with the property: Every normalized basis
$(z_i)_{i=1}^{n}$, with basis constant $\leq 2$, for a Banach space $X$, is $1 + \varepsilon$–equivalent to one of the $(x_{ki})_{i=1}^{n}$. Using our basis $(y_i)$ from part (1), we ask: Is $(x_{1i})_{i=1}^{n}$, $1 + \varepsilon$–equivalent to a block basis of every infinite block basis of $Y$?

As before, if the answer is “yes”, put $(x_{1i})$ in the set $A$ and go on to $(x_{2i})$. If the answer is “no”, put $(x_{1i})$ in the set $B$ and replace $(y_i)$ by a block basis $(y_{i})$ of $(y_i)$ so that $(x_{1i})$ is not $1 + \varepsilon$–equivalent to any block basis of $(y_{i})$. Now go to $(x_{2i})$ and start over. After $m$ steps, we arrive at a block basis $(z_i)$ of $(y_i)$ so that every $(x_{ji})_{i=1}^{n}$ in $A$ is $1 + \varepsilon$–equivalent to a block basis of every block basis of $(z_i)$ and $(x_{ji})_{i=1}^{n}$ in $B$ is not $1 + \varepsilon$–equivalent to any block basis of $(z_i)$. As in part (1), we perform this construction inductively to produce successive block bases of block bases $(z_1^1), (z_1^2), \ldots$ with the above property for $1 + \varepsilon_n$. Then $(z_n^n)$ is the required block basis. Relabeling $(z_n^n)$ as $(y_n)$, we see that we now have property (3) of Theorem 6.1 while maintaining property (2). This completes the proof of the Theorem. □

Theorem 6.1 can be partially strengthened in several directions. For one, if we let $Y_n = \text{span}_{n \leq i} y_i$, then for every $E \subset Y_n^*$, $\dim E \leq f(n)$, and for every $m$, there is an $F \subset Y_m^*$ with $d(E, F) \leq 1 + \varepsilon$. It is easily seen using $X = l_1$ that we cannot expect to $1 + \varepsilon$ embed $E$ into every infinite dimensional subspace of $Y^*$. With significantly more effort, we can show that our blocks in Theorem 6.1 are constructable in a very strong way. That is, whenever $(z_i)_{i=1}^{n}$ is a block basis of $(y_i)_{i=n}^{\infty}$, then for every $m_1$, there is a $w_1 \in \text{span}_{m_1 \leq k} y_k$, so that for every $m_2$, there is a $w_2 \in \text{span}_{m_2 \leq k} y_k, \ldots$, so that for every $m_n$ there is a $w_n \in \text{span}_{m_n \leq k} y_k$ and $(w_i)_{i=1}^{n}$ is $1 + \varepsilon_n$–equivalent to $(z_i)_{i=1}^{n}$.

Now, a final word about Krivine’s Theorem. Until recently, there was the possibility for strengthening this powerful and useful result. First, given $\varepsilon > 0$, and $n \in N$, we might look in every Banach space $X$ for a $p$ and a basic sequence $(x_k)$ in $X$ so that every block basis $(y_i)_{i=1}^{n}$ of $(x_k)$ is $1 + \varepsilon$–equivalent to the unit vector basis of $l_p^n$. It can be shown that Tsirelson’s space $T(q)$, fails this property for every $1 \leq q < \infty$. But Tsirelson’s space has this property for $\varepsilon = 1/2$. Unfortunately, Schlumprecht’s space $[S2]$ fails this property for every $\varepsilon > 0$. Our final hope for a strengthening of Krivine’s Theorem has just fallen to the Gowers’–Maurey spaces [GoM]. That is, in general we cannot get basic sequences in a Banach space $X$ for which small numbers of blocks are even well unconditional. In particular, carrying out the construction of proposition 6.1 in the “unconditional” Gowers’ spaces [Go2] we obtain,

**Proposition 6.6.** For every $K \geq 1$, there is an $n$ and $n$–vectors $(x_i)_{i=1}^{n}$ in a Banach space $X$ and an infinite dimensional Banach space $Y$ with an unconditional basis so that $(x_i)$ is $1$–block finitely representable on every basic sequence in $Y$, yet $(x_i)$ is not $K$–unconditional.
References

[B] S. Banach, *Theorie des operations lineaires*, Warszawa, 1932.

[B*] J. Bourgain, *On finite dimensional homogeneous Banach spaces*, Geometric Aspects of Functional Analysis (J. Lindenstrauss and V. D. Milman, eds.), Springer Lecture Notes, 1317, pp. 232–239.

[BCLT] J. Bourgain, P. G. Casazza, J. Lindenstrauss, and L. Tzafriri, *Banach spaces with a unique unconditional basis, up to a permutation*, Memoirs, AMS, 322 (1985).

[CJT] P. G. Casazza, W. B. Johnson, and L. Tzafriri, *On Tsirelson’s space*, Israel J. Math. 47 (1984), 81–98.

[CKT] P. G. Casazza, N. J. Kalton, and L. Tzafriri, *Decompositions of Banach Lattices into direct sums*, Trans. AMS, no. 2, 304 (1987), 771–800.

[CO] P. G. Casazza and E. Odell, *Tsirelson’s space and minimal subspaces*, Longhorn Notes, University of Texas, 1982–83.

[CS] P. G. Casazza and T. Shura, *Tsirelson’s space*, Springer Lecture Notes, 1362 (1989).

[EW] I. S. Edelstein and P. Wojtaszczyk, *On projections and unconditional bases in direct sums of Banach spaces*, Studia Math. 56 (1976), 263–276.

[GL] Y. Gordon and D. Lewis, *Absolutely summing operators and local unconditional structures*, Acta Math. 133 (1974), 27–48.

[Go] W. T. Gowers, *A space not containing c₀, l₁ or a reflexive subspace*, (preprint).

[Go2] W. T. Gowers, *On the hyperplane problem*, (preprint).

[GoM] W. T. Gowers and B. Maurey, *A counterexample to the unconditional basic sequence problem*, (preprint).

[Ja] R. C. James, *A nonreflexive Banach space that is uniformly nonoctahedral*, Israel J. Math. 18 (1974), 145–155.

[J1] W. B. Johnson, *Homogeneous Banach spaces*, Geometric Aspects of Functional Analysis (J. Lindenstrauss and V. D. Milman, eds.), Springer Lecture Notes, 1317, 201–203.

[J2] W. B. Johnson, *Banach spaces all of whose subspaces have the approximation property*, Special Topics of Applied Math., functional Analysis, Numerical Analysis and Optimization, Proceedings Bonn 1979, (J. Frehse, D. Pallaschke, and V. Trottenberg, eds.), North Holland, 1980, pp. 15–26.

[K] Ryszard Komorowski, *Weak Hilbert spaces without unconditional bases*, (preprint).

[Kr] J. L. Krivine, *Sous–espaces de demension finie des espaces de Banach réticulés*, Ann. of Math. 104 (1976), 1–29.

[LT1] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I*, Berlin, Springer, 1977.

[LT2] , *Classical Banach spaces II*, Berlin, Springer, 1979.

[MTJ1] P. Mankiewicz and N. Tomczak–Jaegermann, *Random subspaces and quotients of finite dimensional Banach spaces*, Odense University, 1989, (preprint no. 8).

[MTJ2] , *A solution of the finite–dimensional homogeneous Banach space problem*, (preprint).

[MTJ3] , *Schauder bases in quotients of subspaces of l₂(X)*, (preprint).

[Ma] V. Mascioni, *On Banach spaces isomorphic to their duals*, (preprint).

[MS] V. D. Milman and G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, Springer Lecture Notes 1200 (1986).

[MiTJ] V. Milman and N. Tomczak–Jaegermann, (preprint).

[NTJ] N. Nielsen and N. Tomczak–Jaegermann, *On subspaces of Banach spaces with property (H) and weak Hilbert spaces*, (preprint).
[OS] E. Odell and T. Schlumprecht, \( l_p \) is distortable, \( 1 < p < \infty \), (preprint).

[P] G. Pisier, *The volume of convex bodies and Banach space geometry*, Cambridge University Press, 1989.

[R] H. P. Rosenthal, *On a Theorem of J. L. Krivine concerning block finite — representability of \( l_p \) in general Banach spaces*, J. Functional Anal. 28 (1978), 197–225.

[S1] T. Schlumprecht, *An arbitrarily distortable Banach space*, (preprint).

[S2] _____, *A complementably minimal Banach space not containing \( c_0 \) or \( l_p \)*, (preprint).

[T] B. S. Tsirelson, *Not every Banach space contains \( l_p \) or \( c_0 \)*, Functional Anal. & Appl. 8 (1974), 138–141.

[W] P. Wojtaszczyk, *On projections and unconditional bases in direct sums of Banach spaces II*, Studia math. 47 (1973), 197–206.

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