An accurate predictor-corrector HOC solver for the two dimensional Riemann problem of gas dynamics

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Abstract. The work in the present manuscript is concerned with the simulation of two-dimensional (2D) Riemann problem of gas dynamics. We extend our recently developed higher order compact (HOC) method from one-dimensional (1D) to 2D solver and simulate the problem on a square geometry with different initial conditions. The method is fourth order accurate in space and second order accurate in time. We then compare our results with the available benchmark results. The comparison shows an excellent agreement of our results with the existing ones in the literature. Being a finite difference solver, it is quite straightforward and simple.

1. Introduction
Over the past few decades, simulation of the Riemann problem of gas dynamics has been an interesting and challenging task amongst the researchers and there have been several attempts on developing numerical schemes for the Euler equations of gas dynamics. A close look at the literature reveal that there are plenty of numerical methods for the one-dimensional (1D) Riemann problem of gas dynamics. Most of these numerical methods are lower order accurate in space, for example, the AUSM, HLLC solver, Godunov, Rusanov, Lax-Friedrich, Lax-Wendroff, MacCormack methods etc. but recently Euler/Navier-Stokes equations have also been solved by compact schemes with near spectral accuracy [1, 2]. Very recently, we have developed a class of fourth order compact schemes [3] for the one-dimensional euler equations of gas dynamics. These schemes were based on finite difference formulation and were spatially fourth order accurate and temporally second order accurate or first order accurate depending on a weighted average parameter. These schemes were seen to capture shock fronts and the contact discontinuities with very correct locations and highly satisfactory sharpness. Unlike the earlier proposed HOC schemes, the numerical results obtained by the present HOC schemes show almost no smearing across all the discontinuities. In the present work, we have extended this 1D solver to a 2D solver which is then used to simulate the 2D Riemann problem of gas dynamics. Several researchers [4–9] worked on the 2-D Riemann problem with varied numerical approaches. However, in majority of these methods, upwind schemes have been devised, which use Riemann problems [9, 10] to construct numerical flux in the finite volume setting. Higher order version of these schemes give spurious wiggles at discontinuities like shocks [8, 10]. Compared to the existing methods for the 2D Riemann problem, our proposed method is quite straightforward and simple.
The paper has been arranged in four sections. Section 2 deals with the problem description, basic formulation and numerical procedure, Section 3 the numerical test cases and finally, Section 4 summarizes the whole work.

2. The problem and discretization

Fig. 1 shows the geometrical set-up of the 2D Riemann problem. The four quadrant has four different initial conditions. At time $t = 0$, the four separations are removed and gas is allowed to flow.

![Figure 1. Problem Geometry](image)

The flow is governed by the 2D Euler equations, which can be written as

$$U_t + F(U)_x + G(U)_y = 0, \quad (1)$$

where

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \quad F(U) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E + p) \end{bmatrix}, \quad G(U) = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E + p) \end{bmatrix} \quad (2)$$

Here $\rho$ is the density, $u$ and $v$ are the $x$- and $y$-component of velocities, $p$ is the pressure and $E$ is the total energy. The system is closed by specifying an equation of state

$$E = \frac{p}{(\gamma - 1)} + \frac{1}{2}\rho(u^2 + v^2), \quad (3)$$

The above set of equations together with the following four initial conditions

$$(p, \rho, u, v)(t = 0) = \begin{cases} (p, \rho, u, v)(1) & \text{if } x > 0.5 \text{ and } y > 0.5 \\ (p, \rho, u, v)(2) & \text{if } x < 0.5 \text{ and } y > 0.5 \\ (p, \rho, u, v)(3) & \text{if } x < 0.5 \text{ and } y < 0.5 \\ (p, \rho, u, v)(4) & \text{if } x > 0.5 \text{ and } y < 0.5 \end{cases}$$

is the Riemann problem of gas dynamics in two space dimension.

In order to develop the higher order compact (HOC) scheme for the 2D Riemann problem (extending our recently developed 1D scheme [3]), let us consider the unsteady 2-D purely convection equation for a transport variable $\phi$ in some continuous domain with suitable boundary conditions.
\[
\frac{\partial \phi}{\partial t} + c(x, y, t) \frac{\partial \phi}{\partial x} + d(x, y, t) \frac{\partial \phi}{\partial y} = f(x, y, t) \tag{4}
\]

where \(c\) and \(d\) are convection coefficients and \(f\) is the forcing function.

The steady-state form of equation (4) may be written as

\[
c(x, y) \frac{\partial \phi}{\partial x} + d(x, y) \frac{\partial \phi}{\partial y} = f(x, y) \tag{5}
\]

Considering a uniform mesh of step sizes \(h\) and \(k\) along \(x\) and \(y\) directions respectively, a standard central difference approximation to equation (5) at the \((i, j)\)-th node is given by

\[
c \delta_x \phi_{i,j} + d \delta_y \phi_{i,j} - \tau_{i,j} = f_{i,j} \tag{6}
\]

where \(\phi_{i,j}\) denotes \(\phi(x_i, y_j)\); while \(\delta_x\) and \(\delta_y\) denotes first order central difference operators along \(x\)- and \(y\)-directions respectively.

The truncation error \(\tau_{i,j}\) in (6) is of the form

\[
\tau_{i,j} = \left[ \frac{h^2}{6} \frac{\partial^3 \phi}{\partial x^3} + \frac{d}{6} \frac{\partial^3 \phi}{\partial y^3} \right]_{i,j} + O(h^4, k^4) \tag{7}
\]

Now, to derive a fourth order compact finite difference formulation of (5), we differentiate it twice w.r.t. \(x\) and then w.r.t. \(y\), which results in the following equations

\[
\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 c \partial \phi}{\partial x^2 \partial x} - 2 \frac{\partial c \partial^2 \phi}{\partial x \partial x^2} - \frac{\partial^2 d \partial \phi}{\partial x \partial y} - 2 \frac{\partial d \partial^2 \phi}{\partial x \partial y} - \frac{d}{2} \frac{\partial^3 \phi}{\partial x^2 \partial y} \tag{8}
\]

\[
\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 d \partial \phi}{\partial y^2 \partial y} - 2 \frac{\partial d \partial^2 \phi}{\partial y \partial y^2} - \frac{\partial^2 c \partial \phi}{\partial x \partial y} - 2 \frac{\partial c \partial^2 \phi}{\partial x \partial y} - \frac{c}{2} \frac{\partial^3 \phi}{\partial x \partial y} \tag{9}
\]

Putting these expressions for higher derivatives in (7) and then putting them back in (6) results in the following HOC scheme for (5).

\[
\left[ \alpha_{i,j} \delta_x^3 + \beta_{i,j} \delta_y^3 + A_{i,j} \delta_x + B_{i,j} \delta_y + C_{i,j} \delta_x \delta_y + D_{i,j} \delta_x \delta_y + E_{i,j} \delta_x^2 \delta_y \right] \phi_{i,j} = F_{i,j} \tag{10}
\]

The coefficients appearing in (10) are given by:

\[
\alpha_{i,j} = \frac{h^2}{3} \delta_x c_{i,j}; \quad \beta_{i,j} = \frac{k^2}{3} \delta_y d_{i,j}; \quad A_{i,j} = \left[ 1 + \frac{h^2}{6} \delta_x^2 + \frac{k^2}{6} \delta_y^2 \right] c_{i,j};
\]

\[
B_{i,j} = \left[ 1 + \frac{h^2}{6} \delta_x^2 + \frac{k^2}{6} \delta_y^2 \right] d_{i,j}; \quad C_{i,j} = \frac{1}{3} \left( h^2 \delta_x d_{i,j} + k^2 \delta_y c_{i,j} \right);
\]

\[
D_{i,j} = \frac{c_{i,j} k^2}{6}; \quad E_{i,j} = \frac{d_{i,j} k^2}{6}; \quad F_{i,j} = \left[ 1 + \frac{h^2}{6} \delta_x^2 + \frac{k^2}{6} \delta_y^2 \right] f_{i,j}
\]

For the unsteady Equation (4), HOC formulation is similar to (10), but coefficients \(c\) and \(d\) are now functions of \(x, y\) and \(t\) and the R.H.S. is of the form \(f(x, y, t) - (\partial \phi)/(\partial t)\). Taking \(\Delta t\) as the time-step and using (10) and applying forward differencing for \((\partial \phi)/(\partial t)\) we have the following HOC approximation for the unsteady Equation (4).

\[
\left[ 1 + \frac{h^2}{6} \delta_x^2 + \frac{k^2}{6} \delta_y^2 \right] \delta^n_x + \left[ \alpha_{i,j} \delta_x^2 + \beta_{i,j} \delta_y^2 + A_{i,j} \delta_x + B_{i,j} \delta_y + C_{i,j} \delta_x \delta_y + D_{i,j} \delta_x \delta_y + E_{i,j} \delta_x^2 \delta_y \right] \phi^n_{i,j} = F_{i,j} \tag{11}
\]
where $\delta^t_i$ is the forward difference operator and the superscript $n$ represents the $n$-th time level.

To simulate the 2D Riemann problem, we convert each equation in (1) to purely convection equations as

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = f_1$$  \hspace{1cm} (12)

$$\frac{\partial m}{\partial t} + u \frac{\partial m}{\partial x} + v \frac{\partial m}{\partial y} = f_2$$  \hspace{1cm} (13)

$$\frac{\partial n}{\partial t} + u \frac{\partial n}{\partial x} + v \frac{\partial n}{\partial y} = f_3$$  \hspace{1cm} (14)

$$\frac{\partial E}{\partial t} + u \frac{\partial E}{\partial x} + v \frac{\partial E}{\partial y} = f_4$$  \hspace{1cm} (15)

where

$$m = \rho u, \quad n = \rho v,$$

$$f_1 = -\rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right),$$

$$f_2 = -\frac{\partial p}{\partial x} - m \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right),$$

$$f_3 = -\frac{\partial p}{\partial y} - n \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right),$$

$$f_4 = -u \frac{\partial p}{\partial x} - \frac{\partial u}{\partial x} - v \frac{\partial p}{\partial y} - \frac{\partial v}{\partial y} - E \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right).$$

Then we use the following predictor-corrector strategy as summarized below

1. First solve (12) for $\rho$ using the developed HOC.
2. Replace $f_1$ by $f_1 = f_1 + \delta \left( \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} \right)$, and again solve (12) but with the updated $\rho$.
3. Solve equation (13), with $\rho$ obtained in 2, by putting $m = \rho u$.
4. Replace $f_2$ by $f_2 = f_2 + \delta \left( \frac{\partial^2 m}{\partial x^2} + \frac{\partial^2 m}{\partial y^2} \right)$, and again solve (13) but with the new $m$ obtained in step 3 and new $u$ obtained from new $m$ as $u = \frac{m}{\rho}$, where $\rho$ is the one obtained in step 2.
5. Solve now equation (14), with new $u$ (i.e. again do $u = \frac{m}{\rho}$ after step 4) and using $n = \rho v$, where $\rho$ is the one obtained in step 2.
6. Replace $f_3$ by $f_3 = f_3 + \delta \left( \frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2} \right)$, and again solve (14) but with the new $n$ obtained in step 5 and new $v$ obtained from new $n$ as $v = \frac{n}{\rho}$, where $\rho$ and $u$ are the ones obtained in step 2 and 4.
7. With latest values of $\rho$, $u$ and $v$ (here again do $v = \frac{n}{\rho}$ after step 6) write energy $E$ as $E = \frac{p}{(\gamma - 1)} + \frac{1}{2} \rho (u^2 + v^2)$. Solve equation (15) with all the new values.
8. Lastly, replace $f_4$ by $f_4 = f_4 + \delta \left( \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} \right)$, and again solve (15) with these new values along with the updated $p = (\gamma - 1)(E - \frac{1}{2} \rho (u^2 + v^2))$, obtained from step 7.
9. Repeat steps 1-8 until final time $T$ is reached. Note that $\delta$ should be a very small entity close to zero. We take it 0.01. Since it adopts a two step strategy where it predicts one value at one
step and again solve for it, we termed it as a predictor-corrector HOC.

**Parameters used:** $\text{CFL} = 0.475$ and grid size $401 \times 401$. $C_i = \sqrt{(\gamma \cdot p_i/\rho_i)}$, where $i = 1, 2, 3, 4$ and $\gamma = 1.4$ and $C = \max(C_i)$ and $\Delta t = (\text{CFL} \cdot \min(h, k))/C$.

3. Results
We simulate 4 different configurations and compare our results with those of Kurganov and Tadmor [4], which gives us a good comparison with theirs.

We consider the following four initial datas (taken from Kurganov and Tadmor [4]):

**Configuration 3:**
$$(p, \rho, u, v)(1) = (1.500, 1.5000, 0.000, 0.000), \quad (p, \rho, u, v)(2) = (0.300, 0.5323, 1.206, 0.000)$$
$$(p, \rho, u, v)(3) = (0.029, 0.1380, 1.206, 1.206), \quad (p, \rho, u, v)(4) = (0.300, 0.5323, 0.000, 1.206).$$

**Configuration 5:**
$$(p, \rho, u, v)(1) = (1.0000, 1.0000, -0.7500, -0.5000), \quad (p, \rho, u, v)(2) = (1.0000, 2.0000, -0.7500, 0.0500)$$
$$(p, \rho, u, v)(3) = (1.0000, 1.0000, 0.7500, 0.5000), \quad (p, \rho, u, v)(4) = (1.0000, 3.0000, 0.7500, -0.5000).$$

![Figure 2. 2-D Riemann problem: Configuration 3 at $t = 0.23$ (left), Configuration 5 at $t = 0.3$ (right).](image)

**Configuration 11:**
$$(p, \rho, u, v)(1) = (1.0000, 1.0000, 0.1000, 0.0000), \quad (p, \rho, u, v)(2) = (0.400, 0.5313, 0.8276, 0.0000)$$
$$(p, \rho, u, v)(3) = (0.400, 0.8000, 0.1000, 0.0000), \quad (p, \rho, u, v)(4) = (0.400, 0.5313, 0.1000, 0.7276).$$

**Configuration 12:**
$$(p, \rho, u, v)(1) = (0.400, 0.5313, 0.0000, 0.0000), \quad (p, \rho, u, v)(2) = (1.000, 1.0000, 0.7276, 0.0000)$$
$$(p, \rho, u, v)(3) = (1.000, 0.8000, 0.0000, 0.0000), \quad (p, \rho, u, v)(4) = (1.000, 1.0000, 0.0000, 0.7276).$$

4. Conclusion
A new higher order compact (HOC) finite difference solver has been proposed for the two-dimensional Riemann problem of gas dynamics. The scheme is fourth order accurate in space.
Figure 3. 2-D Riemann problem: Configuration 11 at $t = 0.3$ (left), Configuration 12 at $t = 0.25$ (right).

The robustness of the scheme has been illustrated by its application to some of the test cases (with different initial conditions) of this problem on a $401 \times 401$ grid. In all the numerical cases, our computed numerical solutions are found to be in excellent match with those of the established results in the literature. Being simple and straight-forward, this method is quite easier to understand and implement. Moreover, the implicitness of the scheme allows us to choose a very larger time-step in contrast to the earlier existing finite difference methods for the Riemann problem. The proposed scheme thus has a very good potential for implementation in more intriguing problems of gas dynamics.

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