Numerical Computation of Delay Differential Equation using Laplace Transform and Lambert W Function

S. Sathiya Sujitha, D. Piriadarshani

Abstract: In this paper a novel approach using Laplace transform for the solution of delay differential equation with a single delay based on Lambert W function has been investigated. An obtained result is extended to the n° order DDEs. Numerical examples have been provided to illustrate the obtained result.

Keywords: Delay Differential Equation, Lambert W function, Laplace transform

I. INTRODUCTION

An appearance of time delays in a system is unavoidable. It will appear frequently in the field of engineering and science and deteriorate the main system performances. Time delay systems belong to the class of infinite dimensional systems. It could be expressed by delay differential equations. DDEs are generally solved by numerical methods such as the Least Squares Method, Pade Approximation Method, Adomian Decomposition Method, Homotopy Perturbation Method, Laplace Transform Method. In 1997, W.H. Enright et.al., found a novel approach for solving neutral delay differential equations by continuous Runge-Kutta formula [1]. In 2003, an analytical method on the basis of Lambert W function was developed to find the solution of DDEs by Ulsoy et.al.,[2]. F. Karako et.al., implemented Differential Transform Method (DTM) to obtain an exact, analytical, and numerical solutions of both linear and nonlinear equations in 2009[3]. The numerical solution of delay differential equation can be found using Coupled Block Method by Hue Chi San in 2011 [4]. Adomian decomposition method (ADM) was presented to solve both linear and nonlinear delay differential equation by Ogunfiditimi, F.O. in 2015 [5]. An optimal perturbation iteration method, was developed to find an approximate solutions of delay differential equations by Necdet Bildik et.al., in 2017 [6]. An analytical approach using Laplace transform for solving linear systems of DDEs was investigated based on the matrix Lambert W function method by sun Yi et.al., in 2007[7].

In this paper an analytical solution of linear system of delay differential equation with single delay is discussed using Laplace transform based on Lambert function.

II. PROPOSED METHODOLOGY

A. First order delay differential equation

Consider the following first order delay differential equation
\[ \dot{x} + ax(t) - Kx(t - \tau) = 0, \quad t > 0 \]
\[ x(t) = g(t), \quad x(t) = 0 \]
where a and K are real constants, g(t) is an initial function and x0 is an initial value. Now,
\[ L[x(t - \tau)] = \int_{0}^{\infty} e^{-st}x(t - \tau)dt \]
\[ = \int_{0}^{T} e^{-st}x(t - \tau)dt + \int_{T}^{\infty} e^{-st}x(t - \tau)dt \]
\[ = \int_{0}^{T} e^{-st}g(t)dt + \int_{T}^{\infty} e^{-st}e^{sK\tau}x(t)dt \]
\[ = G(s) + e^{-sT}X(s) \]
Taking Laplace Transform on (1),
\[ L[\dot{x}] + aL[x(t)] - KL[x(t - \tau)] = 0 \]
\[ [s - Ke^{-\tau st} + a]X(s) - x_0 - KG(s) = 0 \]
\[ X(s) = \frac{x_0 + KG(s)}{s - Ke^{-\tau st} + a} \]
The solution of eqn(1) with respect to Lambert W function
\[ x(t) = \sum_{k=0}^{\infty} e^{sT}C_k, \quad \text{when} \quad k \]
\[ \text{To find the co-efficient} \ C_k, \quad \text{L}[x(t)] = L[\sum_{k=0}^{\infty} e^{sT}C_k] \]
\[ X(s) = \frac{\sum_{k=0}^{\infty} e^{sT}C_k n_k(s)}{d(s)} \]
Where
\[ d(s) = \prod_{k=0}^{s} (s - S_k) \]
\[ n_k(s) = \frac{d(s)}{s - S_k} = \cdots (s - S_{k-2}) (s - S_{k-1}) (s - S_k + 1) (s - S_{k+2}) \]
From equations (2) and (3),
\[ d(s) = \prod_{k=0}^{s} (s - S_k) = J(s)(s - Ke^{-\tau st} + a) \]
\[ \sum_{k=0}^{\infty} C_k n_k(s) = J(s)(x_0 + KG(s)) \]
Here J(s) represents a polynomial in s,
\[ n_k(s = S_1) = 0, \quad \text{when} \quad k \neq l \]
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We compute $C_k$ from (5).

Using L'Hospital's rule, $J(s) = \lim_{s \to S_0} \frac{1}{s - K e^{-s t}}$. Consequently, $C_0 = \frac{x_0 + KG(s)}{1 + Ke^{-S_0 t}}$, $C_1 = \frac{x_1 + KG(s)}{1 + Ke^{-S_0 t}}$ hence $C_k = \frac{x_k + KG(s)}{1 + Ke^{-S_0 t}}$.

**B. Second order delay differential equation**

Consider the following second order delay differential equation

\[ x'(t) + ax'(t) + bx(t) - Kx(t - r) = 0, \quad t > 0 \]

Taking Laplace transform on (6),

\[ X(s) = \frac{s x_0 + x'(0) + KG(s)}{s^2 + as + b + Ke^{-s t}} \]  

The solution of eqn(6) in respect of Lambert W function is

\[ x(t) = \sum_{n=0}^{\infty} e^{St} C_k \text{ where } S_k = \frac{1}{2} W_k \left( \frac{1}{e} \right) \left( \frac{1}{e} \right) \]  

Then the Laplace transform of $x(t)$ is given by

\[ X(s) = \sum_{n=0}^{\infty} \frac{C_n KG(s)}{s} \]  

From equations (7) and (8),

\[ d(s) = \prod_{n=0}^{\infty} (s - S_n) = J(s) (s^2 + as + b + Ke^{-st}) \]

\[ \sum_{n=0}^{\infty} C_n KG(s) = J(s) (s^2 + as + b) x_0 + (s + a) x'_0 + x''(0) + KG(s) \]

From (9),

\[ C_0 = \frac{J(s)}{n_0(s)} \]  

\[ C_1 = \frac{J(s)}{n_1(s)} \]  

From (9),

\[ J(s) = \lim_{S_0 \to S_0} \frac{\prod_{n=0}^{\infty} (r - s_n)}{s^2 + as + b + Ke^{-st}} \]  

So that $C_0 = \frac{(s_0 + a)x_0 + x'(0) + KG(S_0)}{2s_0 + a + Ke^{-S_0 t}}$, $C_1 = \frac{(s_1 + a)x_0 + x'(0) + KG(S_1)}{2s_1 + a + Ke^{-S_1 t}}$ hence $C_k = \frac{(s_k + a)x_0 + x'(0) + KG(S_k)}{2s_k + a + Ke^{-S_k t}}$.

**C. Third order delay differential equation**

Consider the following third order delay differential equation

\[ x'''(t) + ax''(t) + bx'(t) + cx(t) - Kx(t - r) = 0, \quad t > 0 \]

\[ x(t) = g(t), \quad x'(t) = f(t) \quad \text{and} \quad x''(t) = h(t), \quad (t, c(t), 0) \]

\[ x(t) = x_0, \quad x'(t) = 0 \quad \text{and} \quad x''(t) = 0, \quad t = 0 \]

Taking Laplace transform on (11),

\[ X(s) = \frac{1}{s^3 + as^2 + bs + c + Ke^{-st}} \]

The solution of eqn.(11) in relation to Lambert W function is

\[ x(t) = \sum_{n=0}^{\infty} e^{St} C_k \text{ where } S_k = \frac{1}{2} W_k \left( \frac{1}{e} \right) \left( \frac{1}{e} \right) \]  

Using Laplace Transform

\[ X(s) = \sum_{n=0}^{\infty} \frac{C_n KG(s)}{d(s)} \]

From (12) and (13),

\[ d(s) = \prod_{n=0}^{\infty} (s - S_n) = J(s) (s^3 + as^2 + bs + c + Ke^{-st}) \]

\[ \sum_{n=0}^{\infty} C_n KG(s) = J(s) (s^3 + as + b) x_0 + (s + a) x'_0 + x''(0) + KG(s) \]

From (14),

\[ J(s) = \lim_{S_0 \to S_0} \frac{\prod_{n=0}^{\infty} (s - S_n)}{s^3 + as^2 + bs + c + Ke^{-st}} \]

Therefore $C_0 = \frac{(s_0 + a)x_0 + x'(0) + KG(S_0)}{3s_0^2 + 2as + b + Ke^{-s_0 t}}$, $C_1 = \frac{(s_1 + a)x_0 + x'(0) + KG(S_1)}{3s_1^2 + 2as + b + Ke^{-s_1 t}}$ hence

\[ C_k = \frac{(s_k + a)x_0 + x'(0) + KG(S_k)}{3s_k^2 + 2as + b + Ke^{-s_k t}} \]
Hence in general we conclude that for $n^{th}$ order,
\[ C_k = \frac{a_0 x^n(0) + a_1 x^n(0) + \cdots + x^n(0) + \cos(\alpha x(t - \tau))G(S_k)}{n!} \]
\[ + (n - 1)\cos(\alpha x(t - \tau))G(S_k^2) + \cdots + \pi \cos(\alpha x(t - \tau))e^{-\pi x(t)} \]

Where $a_0 = \left( S_k^{-1} + \cos(\alpha x(t - \tau))S_k^{-2} + \cdots + \cos(\alpha x(t - \tau))e^{-\pi x(t)} \right)$

\[ \alpha = \left( S_k^{-1} + \cos(\alpha x(t - \tau))S_k^{-2} + \cdots + \cos(\alpha x(t - \tau))e^{-\pi x(t)} \right) \]

III. NUMERICAL EXAMPLES

A. Example

Consider the following DDE
\[ x''(t) + 3x'(t) + 2x(t) = 5x(t - \tau) \]
with initial point $x'(0) = 1$ and initial function $\phi(t) = e^{3t}$.

Now, $G(S_k) = \int_0^\tau e^{-S_k\tau} d\tau$
\[ G(S_k) = e^{-S_k\tau} \int_0^\tau e^{S_k\tau} d\tau = e^{-S_k\tau} \left[ (e^{S_k\tau} - 1) / S_k \right] \]

The complete solution of the above DDE in terms of Lambert function is
\[ x(t) = \cdots + (0.2572 - 0.2737i)e^{(-0.1704 - 5.2099i)t} + (0.4856)e^{(0.3889)t} + (0.2572 + 0.2737i)e^{(-0.1704 + 5.2099i)t} + \cdots \]

B. Example

Consider the second order DDE
\[ x''(t) + 2x'(t) + x(t) = 0.2x(t - 1) \]
with an initial points $x(0) = 1, x'(0) = 1$ and an initial function $\phi(t) = te^{-t}$.

Here $G(S_k) = \frac{e^t}{S_k + 1} - \frac{e^{-(S_k+1)t}}{S_k+1} + \frac{1}{S_k+1}$

Where $S_k = \sqrt{2} \int e^{\left(\left(t/2\right)\alpha \right) \sqrt{2} \alpha} - \alpha$

Using the above, $G(S_0) = -1.1380; G(S_1) = -4.8657 + 0.2612i; G(S_1) = -4.8657 - 0.2612i.$

And $C_k = \frac{(S_k + \alpha x(t) + \alpha G(S_k))}{\alpha}$. Here $C_0 = 1.633; C_1 = -0.0477 - 0.0960i; C_{i+1} = -0.0477 + 0.0960i$

The solution is
\[ x(t) = (0.0214 - 0.0074i)e^{(-6.1720 - 8.3115\sqrt{2})t} + (1.633)e^{(-0.4421)t} - (0.0477 - 0.0960i)e^{(-6.1720 + 8.3115\sqrt{2})t} \]

The complete solution of the above DDE in terms of Lambert function is
\[ x(t) = (0.0214 - 0.0074i)e^{(-6.1720 - 8.3115\sqrt{2})t} + (0.0428)e^{(-6.1720 + 8.3115\sqrt{2})t} \]

Fig.2. Solutions of second order DDE using Lambert W Function and Laplace Transform combined with Lambert W Function. Fig.2 shows the solution of second order DDE obtained by Lambert W function and Laplace transform combined with the Lambert W function is same after the certain stage.

C. Example

Consider the third order DDE
\[ x'''(t) - 9x''(t) + 15x'(t) + 25x(t) = 2x(t - \tau) \]
with initial values $x(0) = 1, x'(0) = 1, x''(0) = 1$ and the initial function $\phi(t) = Ae^{-t} + (Bt + C)e^{2t}$, which is the preshape function for the given DDE.

Here
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\[ G(S_k) = \frac{1.476^r}{\bar{\lambda}^{(r-1)}} \left( e^{-(S_{k+1})} - 1 \right) + \frac{4.826^{-5r}}{S_k^{(-5)} - 1} \]

\[ S_k = \frac{1}{\lambda} \left[ \frac{0.4^{e^{-5r}}}{(S_k-5)^{r}} \left( e^{-(S_{k-5})} - 1 \right) \right] \]

Here \( S_0 = 3.4050, S_1 = -7.6238 + 11.9580i, S_{-1} = -7.6238 - 11.9580i \) and \( G(S_0) = 0.8585, G(S_1) = -1.6704e + 01 + 1.3674e + 02i \).

\[ C_0 = 0.6056, C_1 = -0.0131 + 0.0049i, C_{-1} = -0.0131 - 0.0049i \]

The solution is

\[ x(t) = -(0.0131 - 0.0049i)e^{-(-7.6238 + 11.9580)i} + (0.6056)e^{(3.4050)i} + (-0.0131 + 0.0049i)e^{(-7.6238 + 11.9580)i} \]

IV. RESULT ANALYSIS

In the current investigation first order, second order and third order differential equations with delayed argument are solved and extended to the nth order DDEs as well, that is done by using Laplace Transform Method combined with Lambert W function. The characteristic equation of linear delay differential equation is transcendental and has infinite number of roots. This may lead to a major challenge in solving DDEs. Here we apply the Laplace transform connected with Lambert function. First order, second order and third order linear DDEs have been solved using the above procedure.

V. CONCLUSION

In this paper the numerical solution of delay differential equation with a single delay is discussed with an approach using Laplace Transform based on Lambert function. This approach has been extended to the nth order system of delay differential equation. Numerical examples are given to support our result.

REFERENCES

1. W.H. Enright, H. Hayashi, A delay differential equation solver based on a continuous Runge- Kutta method with defect control, Numerical Algorithms, Vol.16, pp.349-364, 1997.
2. F.M. Asl, A.G. Ulsoy, Analysis of a System of Linear Delay Differential Equations, Journal of Dynamic Systems, Measurement, and Control, Vol.125, pp.215-223, 2003.
3. F. Karako, H. Bereketoglu, Solutions of delay differential equations by using differential transform method, International Journal of Computer Mathematics, Vol.86, pp.914-923, 2009.
4. Hue Chi San, Zarariah Abdul Majid and Mohamed Othman, Solving delay differential equations using coupled block method, Fourth International Conference on Modeling, Simulation and Applied Optimization, DOI:10.1109/RCSM2011.5775484, 2011.
5. F.O. Ogundiri, Numerical Solution Of Delay Differential Equations Using The Adomian Decomposition Method(ADM), The International Journal Of Engineering And Science(IJES), Vol.4, pp.18-23, 2015.
6. Necdet Bildik, Sinan Deniz, A new efficient method for solving delay differential equations and a comparison with other methods, The European Physical Journal Plus, DOI 10.1140/epjp/i2017-11344-9, 2017.
7. S. Yi, A.G. Ulsoy, and P.-W. Nelson, Solution of Systems of Linear Delay Differential Equations via Laplace Transformation, Proceedings of the IEEE Conference on Decision and Control, DOI: 10.1109/CDC.2006.377712, 2007.
8. Ljadovska, Application of Lambert W Function in Oscillation Theory, Acta Electrotechnicae Informatica, Vol.14, pp.9-17, 2014.
9. S. Sathiya Sujitha, D. Piriadarshani, A Lambert W Function Approach For Second Order Delay Differential Equation as a Special Case of the One-Mass System Controlled Over the Network, International Journal of Mechanical Engineering and Technology (IJMET), Vol.8, pp.502511, 2017.
10. S. Sathiya Sujitha, D. Piriadarshani, A Study on Stability Analysis of Dynamical Systems Modeled as Linear Time Invariant (LTI) Systems, Journal of Advanced Research in Dynamical and Control Systems, Vol.10, pp.1570-1575, 2018.
11. D. Piriadarshani and T. Sengadir, Asymptotic stability of differential equations with infinite delay, Journal of Applied Mathematics, Art. ID804509, 2012.

AUTHORS PROFILE

MS. SATHIYA SUJITHA. S was born in Tamil Nadu, India in 1987. She obtained B.Sc, M.Sc and M.Phil in Mathematics from Manonmanium Sundaranar University, Thirunelveli. She also obtained B.Ed in Mathematics from Manonmanium Sundaranar University. Now she is doing Ph.D in Mathematics in Hindustan Institute of Technology & Science (Deemed to be University), Chennai. Her research area is Stability Analysis of Networked Control System.

DR. PRIADARSHANI. D was born in Tamil Nadu, India in 1972. She obtained B.Sc, M.Sc and M.Phil in Mathematics from University of Madras. She also obtained her Doctorate in Mathematics from Anna University, Chennai. She started her teaching career as a Lecturer in September 1995 in Meenakshi Amman Engineering College, Kanchipuram District. Since September 1997 she has been working in Hindustan Institute of Technology & Science (Deemed to be University), Chennai and currently designated as Professor in Mathematics Department. She has published and presented papers in International Journals and Conferences. Her area of research interest is Applications of Delay Differential Equations in real time problems. She is an executive member of Association of Mathematics Teachers of India and a Life Member of the Ramanujan Mathematical Society.