A POINTWISE ERGODIC THEOREM FOR IMPRECISE MARKOV CHAINS

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ABSTRACT. We prove a game-theoretic version of the strong law of large numbers for submartingale differences, and use this to derive a pointwise ergodic theorem for discrete-time Markov chains with finite state sets, when the transition probabilities are imprecise, in the sense that they are only known to belong to some convex closed set of probability measures.

1. INTRODUCTION

In Ref. [2], De Cooman and Hermans made a first attempt at laying the foundations for a theory of discrete-event (and discrete-time) stochastic processes that are governed by sets of, rather than single, probability measures. They showed how this could be done by connecting Walley’s [1991] theory of coherent lower previsions with ideas and results from Shafer and Vovk’s [2001] game-theoretic approach to probability theory. In later papers, De Cooman et al. [5] applied these ideas to finite-state discrete-time Markov chains, inspired by the work of Hartfiel [6]. They showed how to do efficient inferences in, and proved a Perron–Frobenius-like theorem for, so-called imprecise Markov chains, which are finite-state discrete-time Markov chains whose transition probabilities are imprecise, in the sense that they are only known to belong to a convex closed set of probability measures—typically due to partial assessments involving probabilistic inequalities. This work was later refined and extended by Hermans and De Cooman [7] and Škulj and Hable [15].

The Perron–Frobenius-like theorems in these papers give equivalent necessary and sufficient conditions for the uncertainty model—a set of probabilities—about the state $X_n$ to converge, for $n \rightarrow +\infty$, to an uncertainty model that is independent of the uncertainty model for the initial state $X_1$.

In Markov chains with ‘precise’ transition probabilities, this convergence behaviour is sufficient for a pointwise ergodic theorem to hold, namely that:

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) = E_\infty(f) \text{ almost surely}$$

for all real functions $f$ on the finite state set $\mathcal{X}$, where $E_\infty$ is the limit expectation operator that the expectation operators $E_n$ for the state $X_n$ at time $n$ converge to pointwise, independently of the initial model $E_1$ for $X_1$, according to the classical Perron–Frobenius Theorem.\(^1\)

The aim of the present paper is to extend this result to a version for imprecise Markov chains; see Theorem 13.

How do we mean to go about this? In Section 2, we explain what we mean by imprecise probability models: we extend the notion of an expectation operator to so-called lower (and upper) expectation operators, and explain how these can be associated with (convex and closed) sets of expectation operators.

\(^1\)Actually, much more general results can be proved, for functions $f$ that do not depend on a single state only, but on the entire sequence of states; see for instance Ref. [8, Chapter 20]. In this paper, we will focus on the simpler version.
In Section 3, we explain how these generalised uncertainty models can be combined with event trees to form so-called imprecise probability trees, to produce a simple theory of discrete-time stochastic processes. We show in particular how to combine local uncertainty models associated with the nodes in the tree into global uncertainty models about the paths in the tree, and how this procedure is related to sub- and supermartingales. We also indicate how it extends and subsumes the (precise-)probabilistic approach.

In Section 4 we prove a very general strong law of large numbers for submartingale differences in our imprecise probability trees. Our pointwise ergodic theorem will turn out to be a consequence of this in the particular context of imprecise Markov chains. We briefly explain what imprecise Markov chains are in Section 5: how they are special cases of imprecise probability trees, how to do efficient inference for them, and how to define Perron–Frobenius-like behaviour.

In Section 6 we show that there is an interesting identity between the time averages that appear in our strong law of large numbers, and the ones that appear in the pointwise ergodic theorem. The discussion in Section 7 first focusses on a number of terms in this identity, and investigates the convergence of these terms for Perron–Frobenius-like imprecise Markov chains. This allows us to use the identity to prove our version of the pointwise ergodic theorem, whose significance we discuss briefly in Section 8.

2. BASIC NOTIONS FROM IMPRECISE PROBABILITIES

Let us begin with a brief sketch of a few basic definitions and results about imprecise probabilities. For more details, we refer to Walley’s [16] seminal book, as well as more recent textbooks [1, 13].

Suppose a subject is uncertain about the value that a variable \( Y \) assumes in a non-empty set of possible values \( \mathcal{Y} \). He is therefore also uncertain about the value \( f(Y) \) a so-called gamble—a bounded real-valued function—\( f : \mathcal{Y} \to \mathbb{R} \) on the set \( \mathcal{Y} \) assumes in \( \mathbb{R} \). We will also call such an \( f \) a gamble on \( Y \) when we want to make explicit what variable \( Y \) the gamble \( f \) is intended to depend on. The subject’s uncertainty is modelled by a lower expectation\(^2 \tilde{E} \), which is a real functional defined on the set \( \mathcal{G}(\mathcal{Y}) \) of all gambles on the set \( \mathcal{Y} \), satisfying the following basic so-called coherence axioms:

LE1. \( \tilde{E}(f) \geq \inf f \) for all \( f \in \mathcal{G}(\mathcal{Y}) \); \hspace{1cm} [bounds]

LE2. \( \tilde{E}(f+g) \geq \tilde{E}(f) + \tilde{E}(g) \) for all \( f, g \in \mathcal{G}(\mathcal{Y}) \); \hspace{1cm} [superadditivity]

LE3. \( \tilde{E}(\lambda f) = \lambda \tilde{E}(f) \) for all \( f \in \mathcal{G}(\mathcal{Y}) \) and real \( \lambda \geq 0 \). \hspace{1cm} [non-negative homogeneity]

One—but by no means the only\(^3 \)—way to interpret \( \tilde{E}(f) \) is as a lower bound on the expectation \( E(f) \) of the gamble \( f(Y) \). The corresponding upper bounds are given by the conjugate upper expectation \( \bar{E} \), defined by \( \bar{E}(f) := -\tilde{E}(-f) \) for all \( f \in \mathcal{G}(\mathcal{Y}) \). It follows from the coherence axioms LE1–LE3 that

LE4. \( \inf f \leq \bar{E}(f) \leq \tilde{E}(f) \leq \sup f \) for all \( f \in \mathcal{G}(\mathcal{Y}) \);

LE5. \( \tilde{E}(f) \leq \tilde{E}(g) \) and \( \bar{E}(f) \leq \bar{E}(g) \) for all \( f, g \in \mathcal{G}(\mathcal{Y}) \) with \( f \leq g \);

LE6. \( \bar{E}(f+\mu) = \bar{E}(f) + \mu \) and \( \tilde{E}(f+\mu) = \tilde{E}(f) + \mu \) for all \( f \in \mathcal{G}(\mathcal{Y}) \) and real \( \mu \).

Lower and upper expectations will be the basic uncertainty models we consider in this paper.

The indicator \( I_A \) of an event \( A \)—a subset of \( \mathcal{Y} \)—is the gamble on \( Y \) that assumes the value 1 on \( A \) and 0 outside \( A \). It allows us to introduce the lower and upper probabilities of \( A \) as \( \mathcal{P}(A) := \tilde{E}(I_A) \) and \( \mathcal{P}(A) := \bar{E}(I_A) \), respectively. They can be seen as lower and upper bounds on the probability \( P(A) \) of \( A \), and satisfy the conjugacy relation \( \mathcal{P}(A) = 1 - \mathcal{P}(\mathcal{Y} \setminus A) \).

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\(^2\)In the literature [1, 13, 16], other names, such as coherent lower expectation, or coherent lower prevision, have also been given to this concept.

\(^3\)See Refs. [10, 13, 16] for other interpretations.
When the lower bound $\underline{E}$ coincides with the upper bound $\overline{E}$, the resulting functional $E := \underline{E} = \overline{E}$ satisfies the defining axioms of an expectation:

E1. $E(f) \geq \inf f$ for all $f \in \mathcal{F}(\mathcal{Y})$; \hfill [bounds]
E2. $E(f + g) = E(f) + E(g)$ for all $f, g \in \mathcal{F}(\mathcal{Y})$; \hfill [additivity]
E3. $E(\lambda f) = \lambda E(f)$ for all $f \in \mathcal{F}(\mathcal{Y})$ and real $\lambda$. \hfill [homogeneity]

When $\mathcal{Y}$ is finite, $E$ is trivially the expectation associated with a (probability) mass function $p$ defined by $p(y) := \underline{E}\{\{y\}\} = \overline{E}\{\{y\}\}$ for all $y \in \mathcal{Y}$, because it follows from the expectation axioms that then $E(f) = \sum_{y \in \mathcal{Y}} f(y) p(y)$; see for instance also the detailed discussion in Ref. [13].

With any lower expectation $\underline{E}$, we can always associate the following convex and closed set of compatible expectations:

$$\mathcal{M}(\underline{E}) := \{E \text{ expectation}: (\forall f \in \mathcal{F}(\mathcal{Y})) \underline{E}(f) \leq E(f) \leq \overline{E}(f)\},$$

and the properties LE1–LE3 then guarantee that

$$\underline{E}(f) = \min\{E(f): E \in \mathcal{M}(\underline{E})\} \quad \text{and} \quad \overline{E}(f) = \max\{E(f): E \in \mathcal{M}(\underline{E})\} \quad \text{for all } f \in \mathcal{F}(\mathcal{Y}).$$

In this sense, an imprecise probability model $\underline{E}$ can always be identified with a closed convex set $\mathcal{M}(\underline{E})$ of compatible ‘precise’ probability models $E$.

3. DISCRETE-TIME FINITE-STATE IMPRECISE STOCHASTIC PROCESSES

We consider a discrete-time process as a sequence of variables, henceforth called states, $X_1, X_2, \ldots, X_n, \ldots$, where each state $X_k$ is assumed to take values in a non-empty finite set $\mathcal{X}_k$.

3.1. Event trees, situations, paths and cuts. We will use, for any natural $k \leq \ell$, the notation $X_{k,\ell}$ for the tuple $(X_k, \ldots, X_\ell)$, which can be seen as a variable assumed to take values in the product set $\mathcal{X}_{k,\ell} := \times_{r=k}^{\ell} \mathcal{X}_r$. We denote the set of all natural numbers (without 0) by $\mathbb{N}$, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

We call any $x_{1,n} \in \mathcal{X}_{1,n}$ for $n \in \mathbb{N}_0$ a situation and we denote the set of all situations by $\Omega^\mathcal{X}$. So any situation is a finite string of possible values for the consecutive states, and if we denote the empty string by $\emptyset$, then in particular, $\mathcal{X}_{1,0} = \{\emptyset\}$. $\emptyset$ is called the initial situation. We also use the generic notations $s, t$ or $u$ for situations.

An infinite sequence of state values is called a path, and we denote the set of all paths—also called the sample space—by $\Omega$. Hence

$$\Omega^\mathcal{X} := \bigcup_{n \in \mathbb{N}_0} \mathcal{X}_{1,n} \quad \text{and} \quad \Omega := \times_{n=1}^{\infty} \mathcal{X}_k.$$ 

We will denote generic paths by $\omega$. For any path $\omega \in \Omega$, the initial sequence that consists of its first $n$ elements is a situation in $\mathcal{X}_{1,n}$ that is denoted by $\omega^n$. Its $n$-th element belongs to $\mathcal{X}_n$ and is denoted by $\omega_n$. As a convention, we let its 0-th element be the initial situation $\omega^0 = \omega_0 = \emptyset$. The possible realisations $\omega$ of a process can be represented graphically as paths in a so-called event tree, where each node is a situation; see Figure 1.

We write that $s \sqsubseteq t$ and say that $s$ precedes $t$ or that $t$ follows $s$, when every path that goes through $t$ also goes through $s$. The binary relation $\sqsubseteq$ is a partial order, and we write $s \sqsubset t$ whenever $s \sqsubseteq t$ but not $s = t$. We say that $s$ and $t$ are incomparable when neither $s \sqsubseteq t$ nor $t \sqsubseteq s$.

A (partial) cut $U$ is a collection of mutually incomparable situations, and represents a stopping time. For any two cuts $U$ and $V$, we define the following sets of situations:

$$[U, V] := \{s \in \Omega^\mathcal{X} : (\exists u \in U)(\exists v \in V)u \sqsubseteq s \sqsubseteq v\}$$

$$[U, V] := \{s \in \Omega^\mathcal{X} : (\exists u \in U)(\exists v \in V)u \sqsubseteq s \sqsubseteq v\}$$

4The ‘closedness’ is associated with the weak* topology of pointwise convergence [16, Section 3.6].
When a cut implies that it is indeed a gamble.

We will denote this more succinctly by

\[ G(n) = (\mathcal{B} \cap \mathcal{A}^c) \]

Observe that in that case a slight abuse of notation will for instance allow us to write

\[ x \in \mathcal{A} \]

Conversely, with a gamble process \( \mathcal{F} \) is a map defined on \( \Omega \) such that

\[ \mathcal{F}(\omega) = (\exists u \in U)(\exists v \in V)u \sqsubset s \sqsubseteq v \]

When a cut \( U \) consists of a single element \( a \) then we will identify \( U = \{ a \} \) and \( a \). This slight abuse of notation will for instance allow us to write \([a, v] = \{ s \in \Omega : u \sqsubset s \subseteq v \} \) and also \([U, v] = \{ s \in \Omega : (\exists u \in U)(\exists v \in V)u \sqsubset s \subseteq v \} \). We also write \([U, V]\) if \((\forall v \in V)(\exists u \in U)u \sqsubset s \subseteq v \). Observe that in that case \( U \cap V = \emptyset \). In particular, \( s \sqsubseteq U \) when there is some \( u \in U \) such that \( s \sqsubseteq u \), or in other words if \([U, s] \neq \emptyset \).

A process \( \mathcal{F} \) is a map defined on \( \Omega \). A real process is a real-valued process: it associates a real number \( \mathcal{F}(x_1) \in \mathbb{R} \) with any situation \( x_1 \). It is called bounded below if there is some real \( B \) such that \( \mathcal{F}(x) \geq B \) for all situations \( x \in \Omega \).

A gamble process \( \mathcal{F} \) is a process that associates with any situation \( x_1 \) a gamble \( \mathcal{G}(x_1) \in \mathcal{G}(\mathcal{F}(x_{n+1})) \) on \( X_{n+1} \). It is called uniformly bounded if there is some real \( B \) such that \(|\mathcal{F}(x)| \leq B \) for all situations \( x \in \Omega \). With any real process \( \mathcal{F} \), we can always associate a gamble process \( \Delta \mathcal{F} \), called the process difference. For every situation \( x_1 \), the gamble \( \Delta \mathcal{F}(x_1) \in \mathcal{G}(\mathcal{F}(x_{n+1})) \) is defined by \( \Delta \mathcal{F}(x_1)(x_{n+1}) := \mathcal{F}(x_{n+1}) - \mathcal{F}(x_1) \) for all \( x_{n+1} \in \mathcal{F}_{n+1} \).

We will denote this more succinctly by \( \Delta \mathcal{F}(x_1) = \mathcal{F}(x_{n+1}) - \mathcal{F}(x_1) \), where the ‘.’ represents the generic value of the next state \( X_{n+1} \).

Conversely, with a gamble process \( \mathcal{G} \), we can associate a real process \( \mathcal{F} \), defined by

\[ \mathcal{F}(x_1) := \sum_{k=0}^{n-1} \mathcal{G}(x_{k+1})(x_{k+1}) \quad \text{for all } n \in \mathbb{N}_0 \text{ and } x_1 \in \mathcal{F}_{1,n} \]

Clearly, \( \Delta \mathcal{F} = \mathcal{G} \) and \( \mathcal{F} = \mathcal{F}(\square) + \mathcal{G} \Delta \mathcal{F} \).

Also, with any real process \( \mathcal{F} \) we can associate the path-averaged process \( \langle \mathcal{F} \rangle \), which is the real process defined by:

\[ \langle \mathcal{F} \rangle(x_1) := \begin{cases} 0 & \text{if } n = 0 \\ \frac{1}{n} \mathcal{F}(x_1) & \text{if } n > 0 \end{cases} \quad \text{for all } n \in \mathbb{N}_0 \text{ and } x_1 \in \mathcal{F}_{1,n} \].

\(^5\)Our assumption that \( \mathcal{F}_{n+1} \) is finite is crucial here because it guarantees that \( \Delta \mathcal{F}(x_1) \) is bounded, which in turn implies that it is indeed a gamble.
3.2. Imprecise probability trees, submartingales and supermartingales. The standard way to turn an event tree into a probability tree is to attach to each of its nodes, or situations $x_{1:n}$, a local probability model $Q([x_{1:n}])$ for what will happen immediately afterwards, i.e. for the value that the next state $X_{n+1}$ will assume in $\mathcal{X}_{n+1}$. This local model $Q([x_{1:n}])$ is then an expectation operator on the set $\mathcal{G}(\mathcal{X}_{n+1})$ of all gambles $g(X_{n+1})$ on the next state $X_{n+1}$, conditional on observing $X_{1:n} = x_{1:n}$.

In a completely similar way, we can turn an event tree into an imprecise probability tree by attaching to each of its situations $x_{1:n}$ a local imprecise probability model $\overline{Q}([x_{1:n}])$ for what will happen immediately afterwards, i.e. for the value that the next state $X_{n+1}$ will assume in $\mathcal{X}_{n+1}$. This local model $\overline{Q}([x_{1:n}])$ is then a lower expectation operator on the set $\mathcal{G}(\mathcal{X}_{n+1})$ of all gambles $g(X_{n+1})$ on the next state $X_{n+1}$, conditional on observing $X_{1:n} = x_{1:n}$. This is represented graphically in Figure 2.

![Figure 2](image)

In a given imprecise probability tree, a submartingale $\mathcal{M}$ is a real process such that $\overline{Q}(\Delta \mathcal{M}(x_{1:n}) | x_{1:n}) \geq 0$ for all $n \in \mathbb{N}_0$ and $x_{1:n} \in \mathcal{X}_{1:n}$: all submartingale differences have non-negative lower expectation. A real process $\mathcal{M}$ is a supermartingale if $-\mathcal{M}$ is a submartingale, meaning that $\overline{Q}(\Delta \mathcal{M}(x_{1:n}) | x_{1:n}) \leq 0$ for all $n \in \mathbb{N}_0$ and $x_{1:n} \in \mathcal{X}_{1:n}$: all supermartingale differences have non-positive upper expectation. We denote the set of all submartingales for a given imprecise probability tree by $\overline{\mathcal{M}}$—whether a real process is a submartingale depends of course on the local uncertainty models. Similarly, the set $\overline{\mathcal{M}} := -\overline{\mathcal{M}}$ is the set of all supermartingales.

In the present context of probability trees, we will also call variable any function defined on the so-called sample space—the set $\Omega$ of all paths. When this variable is real-valued and bounded, we will also call it a gamble on $\Omega$. An event $A$ in this context is a subset of $\Omega$, and its indicator $1_A$ is a gamble on $\Omega$ assuming the value 1 on $A$ and 0 elsewhere. With any situation $x_{1:n}$, we can associated the so-called exact event $\Gamma(x_{1:n})$ that $X_{1:n} = x_{1:n}$, which is the set of all paths $\omega \in \Omega$ that go through $x_{1:n}$:

$$\Gamma(x_{1:n}) := \{ \omega \in \Omega : \omega^n = x_{1:n} \}.$$  

For a given $n \in \mathbb{N}_0$, we call a variable $\xi$ $n$-measurable if it is constant on the exact events $\Gamma(x_{1:n})$ for all $x_{1:n} \in \mathcal{X}_{1:n}$, or in other words, if it only depends on the values of the first $n$ states $X_{1:n}$. We then use the obvious notation $\xi(x_{1:n})$ for its constant value $\xi(\omega)$ on all paths $\omega$ in $\Gamma(x_{1:n})$.

With a real process $\mathcal{F}$, we can associate in particular the following extended real variables $\liminf \mathcal{F}$ and $\limsup \mathcal{F}$, defined for all $\omega \in \Omega$ by:

$$\liminf \mathcal{F}(\omega) := \liminf_{n \to \infty} \mathcal{F}(\omega^n) \quad \text{and} \quad \limsup \mathcal{F}(\omega) := \limsup_{n \to \infty} \mathcal{F}(\omega^n).$$

If $\liminf \mathcal{F}(\omega) = \limsup \mathcal{F}(\omega)$ on some path $\omega$, then we also denote the common value there by $\lim \mathcal{F}(\omega) = \lim_{n \to \infty} \mathcal{F}(\omega^n)$. 

In a completely similar way, we can turn an event tree into an imprecise probability tree by attaching to each of its situations $x_{1:n}$ a local imprecise probability model $\overline{Q}([x_{1:n}])$ for what will happen immediately afterwards, i.e. for the value that the next state $X_{n+1}$ will assume in $\mathcal{X}_{n+1}$. This local model $\overline{Q}([x_{1:n}])$ is then a lower expectation operator on the set $\mathcal{G}(\mathcal{X}_{n+1})$ of all gambles $g(X_{n+1})$ on the next state $X_{n+1}$, conditional on observing $X_{1:n} = x_{1:n}$. This is represented graphically in Figure 2.
The following useful result is a variation on a result proved in Ref. [12, Lemma 1], and is similar in spirit to a result proved earlier in Ref. [2, Lemma 2].

**Lemma 1.** Consider any submartingale \( \mathcal{M} \) and any situation \( s \in \Omega^\diamond \), then:

\[
\mathcal{M}(s) \leq \sup_{\omega \in \Gamma(s)} \liminf \mathcal{M}(\omega) \leq \sup_{\omega \in \Gamma(s)} \limsup \mathcal{M}(\omega).
\]

**Proof.** Consider any real \( \alpha \), and assume that \( \mathcal{M}(s) > \alpha \). Assume that \( s = x_{1:n} \) with \( n \in \mathbb{N}_0 \). Since \( \mathcal{M} \) is a submartingale, we know that \( \mathcal{Q}(\mathcal{M}(x_{1:n-1})) - \mathcal{M}(x_{1:n}) \geq 0 \), and therefore, by coherence [LE4 and LE6] and the assumption, that

\[
\max \mathcal{M}(x_{1:n-1}) \geq \mathcal{Q}(\mathcal{M}(x_{1:n-1})) \geq \mathcal{M}(x_{1:n}) > \alpha,
\]

implying that there is some \( x_{n+1} \in \mathcal{P}_{n+1} \) such that \( \mathcal{M}(x_{1:n+1}) > \alpha \). Repeating the same argument over and over again, this leads to the conclusion that there is some \( \omega \in \Gamma(x_{1:n}) \) such that \( \mathcal{M}(\omega^{n+k}) > \alpha \) for all \( k \in \mathbb{N}_0 \), whence \( \liminf \mathcal{M}(\omega) \geq \alpha \), and therefore also \( \sup_{\omega \in \Gamma(x_{1:n})} \liminf \mathcal{M}(\omega) \geq \alpha \). The rest of the proof is now immediate. \( \square \)

### 3.3. Going from local to global belief models

So far, we have associated local uncertainty models with an imprecise probability tree. These represent, in any situation \( x_{1:n} \), beliefs about what will happen immediately afterwards, or in other words about the step from \( x_{1:n} \) to \( x_{1:n}X_{n+1} \).

We now want to turn these local models into global ones: uncertainty models about which entire path \( \omega \) is taken in the event tree, rather than which local steps are taken from one situation to the next. We shall use the following expression for the global lower expectation conditional on the situation \( s \):

\[
\mathbf{E}(f|s) := \sup \{ \mathcal{M}(s) : \mathcal{M} \in \mathcal{M} \text{ and } \limsup \mathcal{M}(\omega) \leq f(\omega) \text{ for all } \omega \in \Gamma(s) \},
\]

and for the conjugate global upper expectation conditional on the situation \( s \):

\[
\overline{E}(f|s) := \inf \{ \mathcal{M}(s) : \mathcal{M} \in \mathcal{M} \text{ and } \liminf \mathcal{M}(\omega) \geq f(\omega) \text{ for all } \omega \in \Gamma(s) \},
\]

where \( f \) is any gamble on \( \Omega \), and \( s \in \Omega^\diamond \) any situation. We use the simplified notations \( \mathbf{E} = \mathbf{E}(\cdot|\square) \) and \( \overline{E} = \overline{E}(\cdot|\square) \) for the (unconditional) global models, associated with the initial situation \( \square \).

Our reasons for using these so-called Shafer–Vovk–Ville formulae\(^6\) are fourfold.

First of all, they are formally very closely related to the expressions for lower and upper prices in Shafer and Vovk’s game-theoretic approach to probabilities, see for instance Refs. [11, Chapter 8.3], [12, Section 2] and [14, Section 6.3]. This allows us to import and adapt, with the necessary care, quite a number of powerful convergence results from that theory, as we shall see in Section 4. Moreover, Shafer and Vovk (see for instance Refs. [11, Proposition 8.8] and [14, Section 6.3]) have shown that they satisfy our defining properties for lower and upper expectations in Section 2, which is why we are calling them lower and upper expectations.

Secondly, as we gather from Proposition 2 below, the expressions (3) and (4) coincide for \( n \)-measurable gambles on \( \Omega \) with the formulae derived in Ref. [2] as the most conservative\(^7\) global lower and upper expectations that extend the local models—see Corollary 3 below.\(^8\)

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\(^6\)We give this name to these formulae because Glenn Shafer and Vladimir Vovk first suggested them, based on the ideas of Jean Ville; see the discussion of Ville’s Theorem in Ref. [11, Appendix 8.5].

\(^7\)By more conservative, we mean associated with a larger set of precise models, so pointwise smaller for lower expectations, and pointwise larger for upper expectations.

\(^8\)We have also shown in recent, still unpublished work that in a more general context—where \( X_n \) takes values in a possibly infinite set \( X \)—for arbitrary gambles on \( \Omega \) they are the most conservative global models that extend the local ones and satisfy additional conglomerability and continuity properties.
Proposition 2. For any situation $x_{1:n} \in \Omega^\omega$ and any $n$-measurable gamble $f$ on $\Omega$, with $n, m \in \mathbb{N}_0$ such that $n \geq m$:
\[
E(f|\Omega^\omega) = \sup\{E(x_{1:n}): M \in \mathbb{M} \text{ and } (\forall x_{m+1:n} \in \mathbb{M}_{m+1}^n). M(x_{1:n}) \leq f(x_{1:n})\}
\]
\[
E(f|\Omega^\omega) = \inf\{E(x_{1:n}): M \in \mathbb{M} \text{ and } (\forall x_{m+1:n} \in \mathbb{M}_{m+1}^n). M(x_{1:n}) \geq f(x_{1:n})\}.
\]
Proof. We sketch the idea of the proof for the lower expectations; the proof for the upper expectations is completely similar.

First, consider any submartingale $M$ such that $M(x_{1:n}) \leq f(x_{1:n})$ for all $x_{m+1:n} \in \mathbb{M}_{m+1}^n$. Consider the submartingale $M'$ derived from $M$ by keeping it constant as soon as any situation in $\{x_{1:m}\} \times \mathbb{M}_{m+1}^n$ is reached, then clearly $\limsup M'(\omega) < f(\omega)$ for all $\omega \in \Gamma(x_{1:n})$ and $M(x_{1:n}) = M'(x_{1:n})$.

Next, consider any submartingale $M$ such that $\limsup M(\omega) \leq f(\omega)$ for all $\omega \in \Gamma(x_{1:n})$. Fix any $x_{m+1:n}$, then it follows from the $n$-measurability of $f$ that $\limsup M(\omega) \leq f(x_{1:n})$ for all $\omega \in \Gamma(x_{1:n})$, whence indeed

\[
M(x_{1:n}) \leq \sup_{\Omega \in \Gamma(x_{1:n})} \limsup \omega M(\omega) \leq f(x_{1:n}),
\]
where the first inequality follows from Lemma 1 with $s = x_{1:n}$.

Corollary 3. Consider any $n \in \mathbb{N}_0$. For any situation $x_{1:n} \in \Omega^\omega$ and any $(n+1)$-measurable gamble $f$ on $\Omega$, $E(f|\Omega^\omega) = Q(f(x_{1:n})|x_{1:n})$ and $E(f|\Omega^\omega) = Q(f(x_{1:n})|x_{1:n})$.

Proof. We give the proof for the lower expectations; the proof for the upper expectations is completely similar.

First, consider any $M \in \mathbb{M}$ such that $M(x_{1:n}) \leq f(x_{1:n})$, then it follows from coherence [LE5 and LE6] and the submartingale character of $M$ that

\[
Q(f(x_{1:n})|x_{1:n}) \geq Q(M(x_{1:n})|x_{1:n}) \geq M(x_{1:n}),
\]
so Proposition 2 guarantees that $E(f|\Omega^\omega) \leq Q(f(x_{1:n})|x_{1:n})$.

To show that the inequality is actually an equality, consider any submartingale $M$ such that $M(x_{1:n}) = Q(f(x_{1:n})|x_{1:n})$ and $M(x_{1:n}) = f(x_{1:n})$.

Thirdly, it is (essentially) the expressions in Proposition 2 that we have used in Ref. [5, 7, 15] for our studies of imprecise Markov chains, which we report in Section 5. The main result of the present paper, Theorem 13 in Section 7, will build on the ergodicity results proved in those papers.

Fourthly, it was also shown in Ref. [2] that the expressions in Proposition 2 have an interesting interpretation in terms of (precise) probability trees. Indeed, we can associate with an imprecise probability tree a (usually infinite) collection of (so-called compatible) precise probability trees with the same event tree, by associating with each situation $s$ in the event tree some arbitrarily chosen precise local expectation $Q(.|s)$ that belongs to the convex closed set $\mathcal{Q}(\Omega^\omega|\omega)$ of expectations that are compatible with the local lower expectation $\|\cdot\|$. For any $n$-measurable gamble $f$ on $\Omega$, the global precise expectations in the compatible precise probability trees will then range over a closed interval whose lower and upper bounds are given by the expressions in Proposition 2.

And finally, Shafer and Vovk have shown [11, Chapter 8] that when the local models are precise probability models, these formulae (3) and (4) lead to global models that coincide with the ones found in measure-theoretic probability theory. This implies that the results we shall prove below, subsume, as special cases, the classical results of measure-theoretic probability theory.

4. A STRONG LAW OF LARGE NUMBERS FOR SUBMARTINGALE DIFFERENCES

We now prove two powerful convergence results for the processes we have defined in the previous section.
We call an event $A$ null if $\mathcal{P}(A) = \mathcal{E}(I_A) = 0$, and strictly null if there is some test supermartingale $\mathcal{T}$ that converges to $+\infty$ on $A$, meaning that:

$$\lim \mathcal{T}(\omega) = +\infty \quad \text{for all } \omega \in A.$$  

Here, a test supermartingale is a supermartingale with $\mathcal{T}(\square) = 1$ that is moreover non-negative in the sense that $\mathcal{T}(s) \geq 0$ for all situations $s \in \Omega^\omega$. Any strictly null event is null, but null events need not be strictly null [14].

**Proposition 4.** Any strictly null event is null, but not vice versa.$^9$

**Proof.** Consider any event $A$. Recall the following expression for $\mathcal{P}(A)$:

$$\mathcal{P}(A) = \mathcal{E}(I_A) = \inf \{ \mathcal{M}(\square) : \mathcal{M} \in \mathcal{M} \text{ and } \liminf_{\omega \in \Omega} \mathcal{M} \geq I_A \}. \quad (5)$$

Also, consider any supermartingale $\mathcal{M}$ such that $\liminf_{\omega \in \Omega} \mathcal{M} \geq I_A$, then it follows from Lemma 1 and the fact that $-\mathcal{M}$ is a submartingale that

$$\mathcal{M}(\square) \geq \inf_{\omega \in \Omega} \liminf_{n} \mathcal{M}(\omega) \geq \inf_{\omega \in \Omega} I_A(\omega) \geq 0. \quad (6)$$

Combined with Equation (5), this implies that $\mathcal{P}(A) \geq 0$.

We are now ready for the proof. Assume that $A$ is strictly null, so there is some test supermartingale $\mathcal{T}$ that converges to $+\infty$ on $A$. Then for any $\alpha > 0$, $\alpha \mathcal{T}$ is a supermartingale such that $\liminf_{\omega \in \Omega} (\alpha \mathcal{T}) \geq I_A$, and therefore we infer that $0 \leq \mathcal{P}(A) \leq \alpha \mathcal{T} (\square) = \alpha$, where the second inequality follows from Equation (5). Since this holds for all $\alpha > 0$, we find that $\mathcal{P}(A) = 0$.

To show that not every null event is strictly null, we show that while an exact event may be null, it can never be strictly null.

First, we show that exact events may be null. Consider any situation $x_{1:n+1}$, with $n \in \mathbb{N}_0$, such that $\mathcal{Q}(1_{\{x_{n+1}\}} | x_{1:n}) = 0$, then we show that $\mathcal{P}(\Gamma(x_{1:n+1})) = 0$. Indeed, consider the real process $\mathcal{M}$ that assumes the value 1 on all situations that follow (or coincide with) $x_{1:n+1}$, and 0 elsewhere. Then clearly $\mathcal{M}(\square) = 0$, $\liminf_{\omega \in \Omega} \mathcal{M} = I_{\Gamma(x_{1:n+1})}$ and $\mathcal{M}$ is a supermartingale because $\mathcal{Q}(\mathcal{M}(x_{1:n+1}) | x_{1:n}) = \mathcal{Q}(1_{\{x_{n+1}\}} | x_{1:n}) = 0 = \mathcal{M}(x_{1:n})$. Equation (5) now implies that $\mathcal{P}(\Gamma(x_{1:n+1})) \leq 0$ and therefore—since we already know that $\mathcal{P}(\Gamma(x_{1:n+1})) \geq 0$—that $\mathcal{P}(\Gamma(x_{1:n+1})) = 0$.

Next, if $\Gamma(s)$ were strictly null, there would be a test supermartingale $\mathcal{M}$ that converges to $+\infty$ on $\Gamma(x)$, and therefore Lemma 1 and the fact that $-\mathcal{M}$ is a submartingale would imply that $\mathcal{M}(s) \geq \inf_{\omega \in \Omega} \liminf_{n} \mathcal{M}(\omega) = +\infty$, which is impossible for the real process $\mathcal{M}$.

In this paper, we shall use the ‘strict’ approach, and prove that events are strictly null—and therefore also null—by actually showing that there is a test supermartingale that converges to $+\infty$ there.

As usual, an inequality or equality between two variables is said to hold (strictly) almost surely when the event that it does not hold is (strictly) null. Shafer and Vovk [11, 14] have proved the following interesting result, which we shall have occasion to use a few times further on. It can be seen as a generalisation of Doob’s supermartingale convergence theorem [19, Sections 11.5–7] to imprecise probability trees. We provide its proof, adapted from Ref. [12] to our specific definitions and assumptions, with corrections for a few tiny glitches, for the sake of completeness.

**Theorem 5** ([14, Section 6.5] Supermartingale convergence theorem). Let $\mathcal{M}$ be a supermartingale that is bounded below. Then $\mathcal{M}$ converges strictly almost surely to a real variable.

$^9$We infer from the proof that for the null and strictly null events to be the same, it is necessary to consider supermartingales that may assume extended real values, as is done in Refs. [12, 14]. We see no need for doing so in the context of the present paper.
We conclude that when which liminf situations supermartingale. Also, because $\mathcal{M}$ is bounded below, it cannot converge to $-\infty$ on any path. Let $A$ be the event where $\mathcal{M}$ converges to $+\infty$, and let $B$ be the event where it diverges. We have to show that there is a test supermartingale that converges to $+\infty$ on $A \cup B$.

Associate with any couple of rational numbers $0 < a < b$ the following recursively defined sequences of cuts $U_k^{a,b}$ and $V_k^{a,b}$. Let $V_0^{a,b} := \{ \square \}$, and for $k \in \mathbb{N}$:

$$U_k^{a,b} := \{ s \sqsubset V_{k-1}^{a,b} : \mathcal{M}(s) > b \text{ and } (\forall t \in (V_{k-1}^{a,b}, s), \mathcal{M}(t) \leq b) \} \quad (7)$$

$$V_k^{a,b} := \{ s \sqsubset U_k^{a,b} : \mathcal{M}(s) < a \text{ and } (\forall t \in (U_k^{a,b}, s), \mathcal{M}(t) \geq a) \}. \quad (8)$$

Consider the real process $\mathcal{T}^{a,b}$ with the following recursive definition:

$$\mathcal{T}^{a,b}(\square) := 1 \text{ and } \mathcal{T}^{a,b}(s \cdot) := \begin{cases} \mathcal{T}^{a,b}(s) + \Delta \mathcal{M}(s) & \text{if } s \in \bigcup_{k \in \mathbb{N}} [V_k^{a,b}, U_k^{a,b}] \\ \mathcal{T}^{a,b}(s) & \text{otherwise.} \end{cases} \quad (9)$$

We now show that $\mathcal{T}^{a,b}$ is a test supermartingale that converges to $+\infty$ on any path $\omega$ for which liminf $\mathcal{M}(\omega) < a < b < \limsup \mathcal{M}(\omega)$.

In what follows, for any situation $s$ and for any $k \in \mathbb{N}$, when $s \sqsubset U_k^{a,b}$, we denote by $u_k^s$ the (necessarily unique) situation in $U_k^{a,b}$ such that $u_k^s \sqsubset s$. Similarly, for any $k \in \mathbb{N}_0$, when $s \sqsubset V_k^{a,b}$, we denote by $v_k^s$ the (necessarily unique) situation in $V_k^{a,b}$ such that $v_k^s \sqsubset s$; observe that $v_0^\square = \square$. Recall from Equations (7) and (8) that, for all $k \in \mathbb{N}$, $\mathcal{M}(u_k^s) > b$ and $\mathcal{M}(v_k^s) < a$.

Since it follows from Equation (9) that $\Delta \mathcal{T}^{a,b}(s)$ is zero or equal to $\Delta \mathcal{M}(s)$, it follows from coherence [LE4] and $\bar{\mathcal{D}}(\mathcal{A}, \mathcal{M})(s) \leq 0$ that $\bar{\mathcal{D}}(\Delta \mathcal{T}^{a,b}(s)) \leq 0$ for all situations $s$, so $\mathcal{T}^{a,b}$ is indeed a supermartingale.

To prove that $\mathcal{T}^{a,b}$ is non-negative, we recall from Equation (9) that $\mathcal{T}^{a,b}$ can only change in situations $s \in [V_k^{a,b}, U_k^{a,b}]$, with $k \in \mathbb{N}$. Since $\mathcal{T}^{a,b}(\square) = 1$, taking into account Lemma 6, this means that we only have to prove that $\mathcal{T}^{a,b}(c) \geq 0$ for the children $c$ of the situations $s \in [V_k^{a,b}, U_k^{a,b}]$, with $k \in \mathbb{N}$. There are two possible cases to consider: The first case (a) is that $s \in [\square, U_1^{a,b})$. Since $\mathcal{T}^{a,b}(\square) = \mathcal{M}(\square) = 1$, we gather from Equation (9) for $k = 1$ that then $\mathcal{T}^{a,b}(c) = \mathcal{M}(c) \geq 0$ for all children $c$ of $s$. The second case (b) is that $s \in [V_k^{a,b}, U_{k+1}^{a,b})$ for some $k \in \mathbb{N}$. We then gather from Equation (9) and Lemma 6 that for all children $c$ of $s$

$$\mathcal{T}^{a,b}(c) = \mathcal{T}^{a,b}(\square) + [\mathcal{M}(u^s_k) - \mathcal{M}(\square)] + \sum_{t=2}^{k} [\mathcal{M}(u^s_t) - \mathcal{M}(v^s_{t-1})] + [\mathcal{M}(c) - \mathcal{M}(v^s_k)]$$

$$\geq b + (k-1)(b-a) + \mathcal{M}(c) - \mathcal{M}(v^s_k) \geq k(b-a) + \mathcal{M}(c) \geq k(b-a) \geq 0.$$
And the third possible case is that \( s \in (V_a^b, U_a^{b+1}) \) for some \( k \in \mathbb{N} \). Then we gather from the discussion of case (b) above that \( \mathcal{T}^a(s) \geq k(b - a) \). Since \( b > a \), we conclude that indeed \( \lim \mathcal{M}(\omega) = +\infty \).

To finish, use the countable set of rational couples \( K := \{(a, b) \in \mathbb{Q}^2 : 0 < a < b\} \) to define the process \( \mathcal{T} \) by letting \( \mathcal{T}(\square) := 1 \) and, for all \( s \in \Omega^C \), \( \mathcal{T}(s) := \sum_{(a, b) \in K} w_{a, b} \mathcal{T}^{a, b}(s) \), a countable convex combination of the real numbers \( \mathcal{T}^{a, b}(s) \), with coefficients \( w_{a, b} > 0 \) that sum to 1. Observe that

\[
\mathcal{T}(s) = \sum_{(a, b) \in K} w_{a, b} \Delta \mathcal{T}^{a, b}(s) = \gamma(s) \Delta \mathcal{M}(s) \in \mathbb{R},
\]

where \( \gamma(s) \in [0, 1] \), because it follows from Equation (9) that for any \( (a, b) \in K \), \( \mathcal{T}^{a, b}(s) \) is equal to \( \Delta \mathcal{M}(s) \) or zero. As an immediate consequence, \( \mathcal{T} \) is a real process and \( \mathcal{T} = \sum_{(a, b) \in K} w_{a, b} \mathcal{T}^{a, b} \). Clearly, \( \mathcal{T}(\square) = 1 \), is non-negative and converges to \( +\infty \) on \( B \). Moreover, since \( \Delta \mathcal{T}(s) = \gamma(s) \Delta \mathcal{M}(s) \), it follows from coherence [LE3] that \( \mathcal{T}(\Delta \mathcal{T}(s)) = \gamma(s) \mathcal{T}(\Delta \mathcal{M}(s)) \leq 0 \) for all \( s \in \Omega^C \), so \( \mathcal{T} \) is a test supermartingale.

Since coherence [LE2 and LE3] implies that a convex combination of two test supermartingales is again a test supermartingale, we conclude from all these considerations that the process \( \frac{1}{2}(\mathcal{M} + \mathcal{T}) \) is a test supermartingale that converges to \( +\infty \) on \( A \cup B \). \( \square \)

**Lemma 6.** \( V_{k-1}^a \cap U_k^a \cap V_k^a \) for all \( k \in \mathbb{N} \).

**Proof.** The statement follows immediately from Equations (7) and (8). The case \( V_{k-1}^a = \emptyset \) presents no problem, because Equation (7) tells us that then \( U_k^a = 0 \) as well. Neither does the case \( U_k^a = 0 \), because Equation (8) tells us that then \( V_k^a = 0 \) as well. \( \square \)

We now turn to a very general version of the strong law of large numbers. Weak (as well as less general) versions of this law were proven by one of us in Refs. [2, 3]. It is this law that will, in Section 7, be used to derive our version of the pointwise ergodic theorem. Its proof is based on a tried-and-tested method for constructing test supermartingales that goes back to an idea in Ref. [11].

**Theorem 7** ([Strong law of large numbers for submartingale differences]. Let \( \mathcal{M} \) be a submartingale such that \( \Delta \mathcal{M} \) is uniformly bounded. Then \( \liminf \langle \mathcal{M} \rangle \geq 0 \) strictly almost surely.

**Proof.** We have to show that there is some test supermartingale \( \mathcal{T} \) that converges to \( +\infty \) on the set \( A := \{ \omega \in \Omega : \liminf \langle \mathcal{M} \rangle(\omega) < 0 \} \). Let \( B > 0 \) be any uniform real bound on \( \Delta \mathcal{M} \), meaning that \( \| \Delta \mathcal{M}(s) \| \leq B \) for all situations \( s \in \Omega^C \). We can always assume that \( B > 1 \).

For any \( r \in \mathbb{N} \), let \( A_r := \{ \omega \in \Omega : \liminf \langle \mathcal{M} \rangle(\omega) < -\frac{1}{2^r} \} \), and let \( A := \bigcup_{r \in \mathbb{N}} A_r \). So fix any \( r \in \mathbb{N} \) and consider any \( \omega \in A_r \), then

\[
\liminf_{n \to +\infty} \langle \mathcal{M} \rangle(\omega^n) < -\frac{1}{2^r}
\]

and therefore

\[
(\forall m \in \mathbb{N})(\exists n_m \geq m) \langle \mathcal{M} \rangle(\omega^{n_m}) < -\frac{1}{2^r} = -\varepsilon,
\]

with \( \varepsilon := \frac{1}{2^r} > 0 \). Consider now the positive supermartingale of Lemma 8, with in particular \( \xi := \frac{\varepsilon}{2^r} = \frac{1}{2^{r+1}} \).\(^{10}\) Denote this test supermartingale by \( \mathcal{T}^{(r)} \). It follows from Lemma 8 that

\[
\mathcal{T}^{(r)}(\omega^{n_m}) \geq \exp \left( n_m \frac{\varepsilon^2}{4B^2} \right) = \exp \left( n_m \frac{1}{2^{2r+2}B^2} \right) \quad \text{for all } m \in \mathbb{N},
\]

\(^{10}\) One of the requirements in Lemma 8 is that \( 0 < \varepsilon < B \), and this is satisfied because we made sure that \( B > 1 \).
where $c$ which, since $n_m \geq m$, implies that $\limsup \mathcal{F}(x) = +\infty$. Observe that for this test supermartingale, $\mathcal{F}(x) \leq \left(\frac{3}{7}\right)^n$ for all $n \in \mathbb{N}$ and $x \in \mathcal{F}_{1,n}$.

Now define the process $\mathcal{F}_{\mathcal{M}} := \sum_{r \in \mathbb{N}} w(r) \mathcal{F}(x)$ as a countable convex combination of the $\mathcal{F}_{\mathcal{M}}$ constructed above, with positive weights $w(r) > 0$ that sum to one. This is a real process, because each term in the series $\mathcal{F}_{\mathcal{M}}(x)$ is non-negative, and moreover

$$\mathcal{F}_{\mathcal{M}}(x) \leq \sum_{r \in \mathbb{N}} w(r) \mathcal{F}(x) \leq \sum_{r \in \mathbb{N}} w(r) \left(\frac{3}{7}\right)^n$$

for all $n \in \mathbb{N}$ and $x \in \mathcal{F}_{1,n}$.

This process is also positive, has $\mathcal{F}_{\mathcal{M}}(\emptyset) = 1$, and, for any $\omega \in \mathcal{A}$, it follows from the argumentation above that there is some $r \in \mathbb{N}$ such that $\omega \in \mathcal{A}_r$, and therefore

$$\limsup \mathcal{F}_{\mathcal{M}}(\omega) \geq w(r) \limsup \mathcal{F}(x) = +\infty,$$

so $\limsup \mathcal{F}_{\mathcal{M}}(\omega) = +\infty$.

We now prove that $\mathcal{F}_{\mathcal{M}}$ is a supermartingale. Consider any $n \in \mathbb{N}$ and any $x \in \mathcal{F}_{1,n}$, then we have to prove that $\mathcal{F}(x) \mathcal{M}(x) \geq 0$. Since it follows from the argumentation in the proof of Lemma 8 that

$$\mathcal{F}(x) = \frac{1}{2^{r+1}B^2} \mathcal{F}(x) \mathcal{M}(x)$$

we see that

$$\mathcal{F}(x) = \sum_{r \in \mathbb{N}} w(r) \mathcal{F}(x) \mathcal{M}(x) = \mathcal{M}(x) \sum_{r \in \mathbb{N}} \frac{w(r)}{2^{r+1}B^2} \mathcal{F}(x),$$

where $c(x) \geq 0$ must be a real number, because, using a similar argument as before

$$c(x) = \sum_{r \in \mathbb{N}} \frac{w(r)}{2^{r+1}B^2} \mathcal{F}(x) \leq L \sum_{r \in \mathbb{N}} w(r) \mathcal{F}(x) \leq L \left(\frac{3}{7}\right)^n$$

for some real $L > 0$. Therefore indeed, using the non-negative homogeneity of lower expectations [LE3]:

$$\mathcal{F}(x) \mathcal{M}(x) = \mathcal{M}(x) \mathcal{F}(x) = c(x) \mathcal{F}(x) \mathcal{M}(x) \geq 0,$$

because $\mathcal{M}$ is a submartingale.

Since we now know that $\mathcal{F}_{\mathcal{M}}$ is a supermartingale that is furthermore bounded below (by 0) it follows from the supermartingale convergence theorem (Theorem 5) that there is some test supermartingale $\mathcal{F}_{\mathcal{M}}$ that converges to $+\infty$ on all paths where $\mathcal{F}_{\mathcal{M}}$ does not converge to a real number, and therefore in particular on all paths in $\mathcal{A}$. Hence $\mathcal{A}$ is indeed strictly null.

**Lemma 8.** Consider any real $B > 0$ and any $0 < \xi < \frac{1}{B}$. Let $\mathcal{M}$ be any submartingale such that $|\Delta \mathcal{M}| \leq B$ and $\mathcal{M}(\emptyset) = 0$. Then the process $\mathcal{F}_{\mathcal{M}}$ defined by:

$$\mathcal{F}_{\mathcal{M}}(x) := \prod_{k=0}^{n-1} \left[1 - \xi \Delta \mathcal{M}(x_{k+1}) \right] \text{ for all } n \in \mathbb{N} \text{ and } x_{k+1} \in \mathcal{F}_{1,n}$$

is a positive supermartingale with $\mathcal{F}_{\mathcal{M}}(\emptyset) = 1$, and therefore in particular a test supermartingale. Moreover, for $\xi := \frac{B}{2\epsilon}$, with $0 < \epsilon < B$, we have that

$$\langle \mathcal{M} \rangle(x_{n+1}) \leq -\epsilon \Rightarrow \mathcal{F}_{\mathcal{M}}(x_{n+1}) \geq \exp \left(\frac{2e^2}{B\epsilon}\right) \text{ for all } n \in \mathbb{N} \text{ and } x_{n+1} \in \mathcal{F}_{1,n}.$$  

**Proof.** $\mathcal{F}_{\mathcal{M}}(\emptyset) = 1$ trivially. For any $n \in \mathbb{N}$ and any $x_{n+1} \in \mathcal{F}_{1,n+1}$:

$$\Delta \mathcal{F}_{\mathcal{M}}(x_{n+1})(x_{n+1}) = \mathcal{F}_{\mathcal{M}}(x_{n+1}) - \mathcal{F}_{\mathcal{M}}(x_{n+1})$$
where the equality holds because we can show that

implying that \( -\Delta \mathcal{F}(x_{1:n}) = \xi \mathcal{F}(x_{1:n}) \Delta \mathcal{M}(x_{1:n}) \). It will therefore follow directly from \( \mathcal{O}(\Delta \mathcal{M}(x_{1:n})|x_{1:n}) \geq 0 \) and the non-negative homogeneity property [LE3] of a lower expectation that \( \mathcal{O}(\mathcal{F}(x_{1:n})|x_{1:n}) \geq 0 \)—implying that \( \mathcal{F} \) is a supermartingale—provided we can show that \( \mathcal{F}(x_{1:n}) > 0 \). Since it follows from \( 0 < \xi B < 1 \) and \( |\mathcal{M}| \leq B \) that

For the second statement, consider any \( 0 < \varepsilon < B \) and let \( \xi := \frac{\varepsilon}{2B} \). Then for any \( n \in \mathbb{N}_0 \) and \( x_{1:n} \in \mathcal{X}_{1:n} \) such that \( \langle \mathcal{M}(x_{1:n}) \rangle \leq -\varepsilon \), we have for all real \( K \):

Since \( |\mathcal{M}| \leq B \) and \( 0 < \varepsilon < B \), we know that \( -\xi \Delta \mathcal{M}(x_{1:n}) \geq -\xi B = -\frac{\varepsilon B}{2} > -\frac{\varepsilon}{2} \). As \( \ln(1 + x) \geq x - x^2 \) for \( x > -\frac{1}{2} \), this allows us to infer that

where the equality holds because \( \mathcal{M}(\Box) = 0 \). Now choose \( K := \frac{n\varepsilon^2}{4B^2} \) in Equation (10).

5. Imprecise Markov chains

We are now ready to apply what we have learned in the previous sections to the special case of (time-homogeneous) imprecise Markov chains. These are imprecise probability trees where (i) all states \( X_t \) assume values in the same finite set \( \mathcal{X}_t = \mathcal{F} \), called the state space, and (ii) all local uncertainty models satisfy the so-called Markov condition:

\[ \mathcal{O}(\cdot|x_{1:n}) = \mathcal{O}(\cdot|x_n) \text{ for all situations } x_{1:n} \in \Omega^0, \]

meaning that these local models only depend on the last observed state; see Figure 3.

We refer to Refs. [5, 7, 15] for detailed studies of the behaviour of these processes. We restrict ourselves here to a summary of the material that is relevant for the present discussion of ergodicity.

From now on, we shall start using a convenient notational device often encountered in texts on stochastic processes: when we want to indicate which states a process or variable depends on, we indicate them explicitly in the notation. Thus, we use for instance the notation \( \mathcal{F}(X_{1:n}) \) to indicate the ‘uncertain’ value of the process \( \mathcal{F} \) after the first \( n \) time steps, and write \( f(X_n) \) for a gamble that only depends on the value of the \( n \)-th state.
The conjugate transition operator $T^\ast$ is defined by $T^\ast f := \mathcal{Q}(-f)$ for all $f \in \mathcal{G}$. In particular, $T^\ast[1](x)$ is the lower probability to go from state $x$ to state $y$ in one time step, and $T^\ast[1](x)$ the conjugate upper probability. This seems to suggest that the lower/upper transition operators $T$ are generalisations of the concept of a Markov transition matrix for ordinary Markov chains. This is confirmed by the following general result, proved in Ref. [5, Corollary 3.3] as a special case of the so-called Law of Iterated (Lower) Expectations [2, 11]. If, for any $n \in \mathbb{N}$, we denote by $E_n(f)$ the value of the (global) lower expectation $\mathcal{E}(f(X_n))$ of a gamble $f(X_n)$ on the state at time $n$, then

$$E_n(f) = E_1(T^{n-1}f), \text{ with } T^{n-1}f := \underbrace{T \cdots T}_{n-\text{times}}f,$$

and where, of course, $E_1 = \mathcal{Q}(-\square)$ is the marginal local model for the state $X_1$ at time 1.

As a consequence, for any $n \in \mathbb{N}$, $T^n[1](x)$ is the lower probability to go from state $x$ to state $y$ in time steps, and $T^n[1](x)$ the conjugate upper probability.

We can formally call lower transition operator any transformation $T$ of $\mathcal{G}$ such that for any $x \in \mathcal{X}$, the real functional $T_x$ on $\mathcal{G}$, defined by $T_x(f) := T(f(x))$ for all $f \in \mathcal{G}$, is a lower expectation—satisfies the coherence axioms LE1–LE3. The composition of any two lower transition operators is again a lower transition operator. See Ref. [5] for more details on the definition and properties of such lower transition operators, and Ref. [4] for a mathematical discussion of the general role of these operators in imprecise probabilities.

We call an imprecise Markov chain with lower transition operator $T$ Perron–Frobenius-like if for all $f \in \mathcal{G}$, the sequence of gambles $T^n f$ converges pointwise to a constant real number, which we shall denote by $E_\infty(f)$.

The following result was proved in Ref. [5, Theorem 5.1], together with a simple sufficient (and quite weak) condition on $T$ for a Markov chain to be Perron–Frobenius-like: there is some $n \in \mathbb{N}$ such that $\min T^n[1](y) > 0$ for all $y \in \mathcal{X}$, or in other words, all state values can be reached from any state value with positive upper probability in (precisely) $n$ time steps. More involved necessary and sufficient conditions were given later in Refs. [7, 15]; see also Theorem 10(iv) further on.
Proposition 9 ([5]). If the imprecise Markov chain with lower transition operator $\mathbb{T}$ is Perron–Frobenius-like, then for any initial model $E_1$ and any $f \in \mathcal{G}(\mathcal{X})$, it holds that $E_n(f) = E_1(\mathbb{T}^{n-1}f) \to E_{\infty}(f)$. Moreover, the functional $E_{\infty}$ is then a lower expectation on $\mathcal{G}(\mathcal{X})$, called the stationary lower expectation, and it is the only lower expectation that is $\mathbb{T}$-invariant in the sense that $E_{\infty} \circ \mathbb{T} = E_{\infty}$.

6. AN INTERESTING EQUALITY IN IMPRECISE MARKOV CHAINS

We now prove an interesting equality for imprecise Markov chains, which will be instrumental in proving our pointwise ergodic theorem in the next section.

Consider, for any $f \in \mathcal{G}(\mathcal{X})$, the corresponding gain process $\mathcal{W}[f]$, defined by:

$$\mathcal{W}[f](X_{1:n}) := [f(X_1) - E_1(f)] + \sum_{k=2}^{n} [f(X_k) - \mathbb{T} f(X_{k-1})] \text{ for any } n \in \mathbb{N},$$

(11)

the corresponding average gain process $\langle \mathcal{W} \rangle[f]$, defined by:

$$\langle \mathcal{W} \rangle[f](X_{1:n}) := \frac{1}{n} \left[ f(X_1) - E_1(f) + \sum_{k=2}^{n} [f(X_k) - \mathbb{T} f(X_{k-1})] \right] \text{ for any } n \in \mathbb{N},$$

(12)

and the ergodic average process $\mathcal{W}^{\ast}[f]$, defined by:

$$\mathcal{W}^{\ast}[f](X_{1:n}) := \frac{1}{n} \sum_{k=1}^{n} \left[ f(X_k) - E_k(f) \right] \text{ for any } n \in \mathbb{N}. \quad (13)$$

We can let these processes be 0 in the initial situation $\square$—the choice is immaterial. Now observe that, for any $n \in \mathbb{N}$ and any $f \in \mathcal{G}(\mathcal{X})$:

$$\sum_{\ell=0}^{n-1} \langle \mathcal{W} \rangle[\mathbb{T}^\ell f](X_{1:n}) = \frac{1}{n} \sum_{\ell=0}^{n-1} \mathbb{T}^\ell f(X_1) - E_1(\mathbb{T}^\ell f) + \frac{1}{n} \sum_{\ell=0}^{n-1} \sum_{k=2}^{n} \mathbb{T}^\ell f(X_k) - \mathbb{T}^\ell+1 f(X_{k-1}),$$

(14)

and moreover

$$\sum_{\ell=0}^{n-1} \sum_{k=2}^{n} \mathbb{T}^\ell f(X_k) - \mathbb{T}^\ell+1 f(X_{k-1})$$

$$= \sum_{\ell=0}^{n-1} \sum_{k=2}^{n} \mathbb{T}^\ell f(X_k) - \sum_{\ell=0}^{n-1} \sum_{k=2}^{n} \mathbb{T}^{\ell+1} f(X_{k-1}) = \sum_{\ell=0}^{n-1} \sum_{k=2}^{n} \mathbb{T}^\ell f(X_k) - \sum_{\ell=1}^{n-1} \sum_{k=1}^{n-1} \mathbb{T}^\ell f(X_k)$$

$$= \sum_{k=2}^{n} f(X_k) + \sum_{\ell=1}^{n-1} \left( \mathbb{T}^\ell f(X_{\ell+1}) + \sum_{k=2}^{n} \mathbb{T}^\ell f(X_k) \right)$$

$$- \sum_{k=1}^{n-1} \mathbb{T}^\ell f(X_k) - \sum_{\ell=1}^{n-1} \left( \mathbb{T}^\ell f(X_1) + \sum_{k=2}^{n} \mathbb{T}^\ell f(X_k) \right)$$

$$= \sum_{k=2}^{n} f(X_k) + \sum_{\ell=1}^{n-1} \mathbb{T}^\ell f(X_{\ell+1}) - \sum_{k=1}^{n-1} \mathbb{T}^{\ell+1} f(X_k) - \sum_{\ell=1}^{n-1} \mathbb{T}^{\ell+1} f(X_1)$$

and if we substitute this back into Equation (14), we find that, after getting rid of the cancelling terms, recalling that $E_1(\mathbb{T}^\ell f) = E_{\ell+1}(f)$, and reorganising a bit:

$$\sum_{\ell=0}^{n-1} \langle \mathcal{W} \rangle[\mathbb{T}^\ell f](X_{1:n}) = \frac{1}{n} \left[ - \sum_{\ell=0}^{n-1} E_1(\mathbb{T}^\ell f) + \sum_{k=1}^{n} f(X_k) + \sum_{\ell=1}^{n} \mathbb{T}^\ell f(X_{\ell+1}) - \sum_{k=1}^{n} \mathbb{T}^\ell f(X_k) \right]$$

$$= \frac{1}{n} \sum_{k=1}^{n} \left[ f(X_k) - E_k(f) \right] + \frac{1}{n} \sum_{\ell=1}^{n} \mathbb{T}^\ell f(X_{\ell+1}) - \frac{1}{n} \sum_{k=1}^{n} \mathbb{T}^\ell f(X_k)$$
or in other words:
\[
\mathcal{A}[f](X_{1:n}) = \sum_{i=0}^{n-1} \langle \mathcal{W}' \rangle [T\ell\ell f](X_{1:n}) + \frac{1}{n} \sum_{k=1}^{n} \mathcal{T}^n f(X_k) - \frac{1}{n} \sum_{k=1}^{n} \mathcal{T}^k f(X_k).
\]  

(15)

7. Consequences of the Perron–Frobenius-like Character

Let us associate with a lower transition operator \( \mathcal{T} \) the following (weak) coefficient of ergodicity [7, 15]:
\[
\rho(\mathcal{T}) := \max_{x \in X} \max_{h \in \mathcal{H}(X)} \| \mathcal{T}h(x) - \mathcal{T}h(y) \| = \max_{h \in \mathcal{H}(X)} \| \mathcal{T}h \|,
\]
where \( \mathcal{H}(X) := \{ h \in \mathcal{G}(X) : 0 \leq h \leq 1 \} \), and where for any \( h \in \mathcal{G}(X) \), its variation (semi)norm is given by \( \| h \| := \max h - \min h. \) If we define the following distance between two lower expectation operators \( \mathcal{E} \) and \( \mathcal{F} \) [15]:
\[
d(\mathcal{E}, \mathcal{F}) = \max_{h \in \mathcal{H}(X)} | \mathcal{E}(h) - \mathcal{F}(h) |,
\]
then it is not difficult to see [using LE3, LE4 and LE6] that \( 0 \leq d(\mathcal{E}, \mathcal{F}) \leq 1 \), and that for any \( f \in \mathcal{G}(X) \):
\[
| \mathcal{E}(f) - \mathcal{F}(f) | \leq d(\mathcal{E}, \mathcal{F}) \| f \|.
\]  

(16)

Škulj and Hable [15] prove the following results, which will turn out to be crucial to our argument.

Theorem 10 ([15]). Consider lower transition operators \( \mathcal{T} \) and \( \mathcal{S} \) on \( \mathcal{G}(X) \). Then the following statements hold:

(i) \( 0 \leq \rho(\mathcal{T}) \leq 1 \).

(ii) \( \rho(S\mathcal{T}) \leq \rho(S) \rho(\mathcal{T}) \) and therefore \( \rho(\mathcal{T}^n) \leq \rho(\mathcal{T})^n \) for all \( n \in \mathbb{N} \).

(iii) \( d(\mathcal{E}_\ell \mathcal{T}, \mathcal{E}_k \mathcal{T}) \leq d(\mathcal{E}_\ell, \mathcal{E}_k) \rho(\mathcal{T}) \).

(iv) The lower transition operator \( \mathcal{T} \) is Perron–Frobenius-like if and only if there is some \( r \in \mathbb{N} \) such that \( \rho(\mathcal{T}^r) < 1 \).

Indeed, they allow us to derive useful bounds for the various terms on the right-hand side of Equation (15). For any non-negative real number \( a \) we denote by \( |a| = \max\{n \in \mathbb{N}_0 : n \leq a \} \) the largest natural number that it still dominates—its integer part.

Lemma 11. Let \( \mathcal{T} \) be a Perron–Frobenius-like lower transition operator, with invariant lower expectation \( \mathcal{E}_\omega \), and let \( r \) be the smallest natural number such that \( \rho := \rho(\mathcal{T}^r) < 1 \). Let \( \mathcal{E}_\ell \) and \( \mathcal{E}_k \) be any two lower expectations on \( \mathcal{G}(X) \). Then for all \( f \in \mathcal{G}(X) \), \( \ell, \ell_2 \in \mathbb{N}_0 \):
\[
| \mathcal{E}_\ell(T^{\ell_1} f) - \mathcal{E}_k(T^{\ell_2} f) | \leq \| f \| \rho^{\min(\ell_1, \ell_2)}.
\]  

(17)

As a consequence, for all \( f \in \mathcal{G}(X) \), \( \ell, \ell_1, \ell_2 \in \mathbb{N}_0 \) and \( k, k_1, k_2 \in \mathbb{N} \):
\[
| \mathcal{T}^\ell f(X_k) - \mathcal{E}_\omega(f) | \leq \| f \| \rho^{\ell_1},
\]  

(18)

\[
| \mathcal{E}_\ell(T^{\ell_1} f) - \mathcal{E}_k(f) | \leq \| f \| \rho^{\ell_2},
\]  

(19)

\[
| \mathcal{T}^\ell f(X_{k_1}) - \mathcal{E}_\omega(T^{\ell_1} f) | \leq \| f \| \rho^{\ell_2},
\]  

(20)

\[
| \mathcal{T}^{\ell_1} f(X_{k_2}) - \mathcal{T}^{\ell_2} f(X_{k_2}) | \leq \| f \| \rho^{\min(\ell_1, \ell_2)}.
\]  

(21)

Proof. We may assume without loss of generality that \( \ell_1 \leq \ell_2 \). Using Equation (16), Theorem 10(iii) and the fact that we can consider \( \mathcal{T}^{-1} \) as a lower transition operator in its own right:
\[
| \mathcal{E}_\ell(T^{\ell_1} f) - \mathcal{E}_k(T^{\ell_2} f) | \leq d(\mathcal{E}_\ell, \mathcal{E}_k) \| f \| \rho^{\min(\ell_1, \ell_2)}.
\]

Our proof of the first inequality (17) is complete if we realise that \( 0 \leq d(\mathcal{E}_\ell, \mathcal{E}_k \mathcal{T}^{\ell_2-\ell_1}) \leq 1, \) and that \( \rho(\mathcal{T}^{-1}) \leq \rho(\mathcal{T}^{1/2}) \leq \rho(\mathcal{T}^{1/2}) \) by Theorem 10(ii)&(ii).
Denote, for any $x \in \mathcal{X}$, by $E_x$ the expectation operator that assigns all probability mass to $x$, meaning that $E_x(f) := f(x)$ for all $f \in \mathcal{B}(\mathcal{X})$. To prove the second inequality (18), consider any $x \in \mathcal{X}$ and let $E_x = E_b = E_w$ and $\ell_1 = \ell_2 = \ell$, then we infer from (17) that indeed:

$$|\mathbf{T}^\ell f(x) - E_w(f)| = |E_x(T^\ell f) - E_w(T^\ell f)| \leq \|f\|_\ell \rho^{|\ell|}.$$ 

To prove the third inequality (19), let $E_b = E_w$ and $\ell_1 = \ell_2 = \ell$, then we infer from (17) that indeed:

$$|E_w(T^\ell f) - E_w(f)| = |E_w(T^\ell f) - E_w(T^\ell f)| \leq \|f\|_\ell \rho^{|\ell|},$$

where we used that $E_w(f) = E_w(T^\ell f)$ for all $\ell \in \mathbb{N}_0$; see Proposition 9.

To prove the fourth inequality (20), consider any $x \in \mathcal{X}$ and let $E_a = E_x$ and $\ell_1 = \ell_2 = \ell$, then we infer from (17) that indeed:

$$|T^\ell f(x) - E_a(T^\ell f)| = |E_x(T^\ell f) - E_a(T^\ell f)| \leq \|f\|_\ell \rho^{|\ell|}.$$ 

To prove the fifth inequality (21), consider any $x, y \in \mathcal{X}$ and let $E_a = E_x$ and $E_b = E_y$. Then we infer from (17) that indeed:

$$|T^\ell f(x) - T^\ell f(y)| = |E_a(T^\ell f) - E_b(T^\ell f)| \leq \|f\|_\ell \rho^{\min(|\ell_1|, |\ell_2|)}.$$  

Lemma 12. Consider an imprecise Markov chain with initial—or marginal—model $E_x$ and lower transition operator $\mathbf{T}$. Assume that $\mathbf{T}$ is Perron–Frobenius-like, with invariant lower expectation $E_w$, and let $r$ be the smallest natural number such that $\rho := \rho(\mathbf{T}) < 1$. Then the following statements hold for all $f \in \mathcal{B}(\mathcal{X})$:

(i) $\|f\|_\ell \rho^{|\ell|} \leq \|f\|_\ell \rho^{|\ell|}$ for all $\ell \in \mathbb{N}_0$ and $n \in \mathbb{N}$.

(ii) $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n T^k f(X_k) = E_w(f)$.

(iii) $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n E_a(T^k f(X_k)) = E_w(f)$.

Proof. Recall from Equation (12) that:

$$n\langle \mathcal{W}, T^\ell f \rangle |X_{1:n}| = \left[ T^\ell f(X_1) - E_a(T^\ell f) \right] + \sum_{k=2}^n \left[ T^\ell f(X_k) - T^{\ell+1} f(X_{k-1}) \right].$$

If we also invoke Lemma 11, we find that:

$$n\langle \mathcal{W}, T^\ell f \rangle |X_{1:n}| \leq \left[ T^\ell f(X_1) - E_a(T^\ell f) \right] + \sum_{k=2}^n \left[ T^\ell f(X_k) - T^{\ell+1} f(X_{k-1}) \right]$$

$$\leq \|f\|_\ell \rho^{|\ell|} + \sum_{k=2}^n \|f\|_\ell \rho^{|\ell|} = n\|f\|_\ell \rho^{|\ell|},$$

which proves statement (i). Similarly, by Lemma 11:

$$\left| \frac{1}{n} \sum_{k=1}^n \left[ T^\ell f(X_k) - E_w(f) \right] \right| \leq \frac{1}{n} \left| \sum_{k=1}^n T^\ell f(X_k) - E_w(f) \right| \leq \frac{1}{n} \sum_{k=0}^n \|f\|_\ell \rho^{|\ell|} = \|f\|_\ell \rho^{|\ell|},$$

which proves statement (ii). Similarly, again by Lemma 11:

$$\left| \frac{1}{n} \sum_{k=1}^n \left[ T^\ell f(X_k) - E_w(f) \right] \right| \leq \frac{1}{n} \left| \sum_{k=1}^n T^\ell f(X_k) - E_w(f) \right| \leq \frac{1}{n} \sum_{k=1}^n \|f\|_\ell \rho^{|\ell|}$$

$$\leq \|f\|_\ell \rho^{|\ell|} \leq \frac{\|f\|_\ell}{n} \sum_{\ell=0}^n \rho^{|\ell|} \leq \frac{\|f\|_\ell}{n} \frac{\rho^{|\ell|}}{1 - \rho^{|\ell|}},$$

which proves statement (iii). Finally, by Lemma 11 and an argumentation similar to our proof for statement (iii):

$$\left| \frac{1}{n} \sum_{k=1}^n E_a(f) - E_w(f) \right| \leq \frac{1}{n} \left| \sum_{k=1}^n E_a(f) - E_w(f) \right| = \frac{1}{n} \left| \sum_{k=1}^n E_a(T^{k-1} f) - E_w(f) \right|$$

$$\leq \left| \frac{1}{n} \sum_{k=1}^n T^\ell f(X_k) - E_w(f) \right| \leq \|f\|_\ell \rho^{|\ell|}.$$
\[
\leq \frac{1}{n} \sum_{k=1}^{n} \| f \|_{\psi^\circ \rho^{k+1}} \leq \frac{\| f \|_{\psi}}{n} r \frac{1}{1 - \rho},
\]
which proves statement (iv).

We can now prove our main result.

**Theorem 13** (Pointwise ergodic theorem). Consider an imprecise Markov chain with initial—or marginal—model \( E_0 \) and lower transition operator \( \Upsilon \). Assume that \( \Upsilon \) is Perron–Frobenius-like, with invariant lower expectation \( E_0. \) Then for all \( f \in \mathcal{G}(\mathcal{H}) \):

\[
\liminf_{n \to \infty} \mathcal{A}[f] \geq 0 \text{ strictly almost surely,}
\]
and consequently,

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) \geq E_0(f) \text{ strictly almost surely.}
\]

**Proof.** We begin with the first inequality. Let \( r \) be the smallest natural number such that \( \rho := \rho(\Upsilon^r) < 1 \). Consider any \( q \in \mathbb{N} \), and let \( g_q := \sum_{t=0}^{rq-1} \Upsilon^t f \), then it follows from Equation (11) and LE2 that for all \( n \in \mathbb{N} \):

\[
\mathcal{W}[g_q](X_{1:n}) \leq \sum_{t=0}^{rq-1} \mathcal{W}[\Upsilon^t f](X_{1:n}) \text{ and therefore } \langle \mathcal{W} \rangle[g_q](X_{1:n}) \leq \sum_{t=0}^{rq-1} \langle \mathcal{W} \rangle[\Upsilon^t f](X_{1:n}).
\]

Hence, if we also take into account Equation (15) and Lemma 12, we find that:

\[
\liminf_{n \to \infty} \mathcal{A}[f] = \liminf_{n \to \infty} \sum_{t=0}^{n-1} \langle \mathcal{W} \rangle[\Upsilon^t f](X_{1:n})
\]
\[
\geq \liminf_{n \to \infty} \sum_{t=0}^{rq-1} \langle \mathcal{W} \rangle[\Upsilon^t f](X_{1:n}) + \liminf_{n \to \infty} \sum_{t=0}^{n-1} \langle \mathcal{W} \rangle[\Upsilon^t f](X_{1:n})
\]
\[
\geq \liminf \langle \mathcal{W} \rangle[g_q] - \| f \|_{\psi} \limsup_{n \to \infty} \sum_{t=0}^{n-1} \rho^t |f|
\]
\[
= \liminf \langle \mathcal{W} \rangle[g_q] - \| f \|_{\psi} \sum_{t=0}^{\infty} \rho^t |f| \geq \liminf \langle \mathcal{W} \rangle[g_q] - \| f \|_{\psi} r \frac{\rho^q}{1 - \rho}. \tag{22}
\]

By combining Equation (11) with the coherence [LE4 and LE6] of the local models of the Markov chain, we see that \( \mathcal{W}[g_q] \) is a submartingale for which \( \Delta \mathcal{W}[g_q] \) is uniformly bounded. It therefore follows from our strong law of large numbers for submartingale differences [Theorem 7] that \( \liminf \langle \mathcal{W} \rangle[g_q] \geq 0 \) strictly almost surely, meaning that there is some test supermartingale \( \mathcal{F}(q) \) that converges to \( +\infty \) on any path \( \omega \) for which \( \liminf \langle \mathcal{W} \rangle[g_q] < 0 \). Furthermore, by the argumentation in the proof of Theorem 7, we also know that \( \mathcal{F}(q)(x_{1:n}) \leq \langle \frac{1}{q} \rangle^n \) for all \( n \in \mathbb{N} \) and \( x_{1:n} \in \mathcal{F}_{1:n} \). If we now invoke Equation (22), we see that \( \mathcal{F}(q) \) converges to \( +\infty \) on any path \( \omega \) where \( \liminf_{n \to \infty} \mathcal{A}[f](\omega) < \| f \|_{\psi} r \frac{\rho^q}{1 - \rho} \).

Now consider any sequence of positive real numbers \( w(q) \) such that \( \sum_{q \in \mathbb{N}} w(q) = 1 \), then it follows from the considerations above that the sequence of non-negative real numbers \( a_i(x_{1:n}) := \sum_{q=1}^{n} w(q) \mathcal{F}(q)(x_{1:n}) \), \( i \in \mathbb{N} \) is non-decreasing and bounded above by \( \langle \frac{1}{q} \rangle^n \), and therefore converges to a non-negative real number, for all \( n \in \mathbb{N} \) and \( x_{1:n} \in \mathcal{F}_{1:n} \). Hence, we can define the real process \( \mathcal{F} := \sum_{q \in \mathbb{N}} w(q) \mathcal{F}(q) \), which clearly converges to \( +\infty \) on any path \( \omega \) where \( \liminf_{n \to \infty} \mathcal{A}[f](\omega) < 0 \). Moreover, \( \mathcal{F}(\emptyset) = 1 \) and \( \mathcal{F} \) is non-negative. So we are done with the first inequality if we can prove that \( \mathcal{F} \) is a supermartingale. Consider, therefore, any situation \( s \) and any \( Q(\cdot | s) \in \mathcal{M}(Q(\cdot | s)) \), then, if we denote its (probability) mass function by \( p(\cdot |s) \):

\[
Q(\Delta \mathcal{F}|s) = \sum_{x \in \mathcal{F}} p(x|s) \Delta \mathcal{F}(s)(x) = \sum_{x \in \mathcal{F}} p(x|s) \sum_{q \in \mathbb{N}} w(q) \Delta \mathcal{F}(q)(s)(x)
\]
\[
\sum_{q \in \mathbb{N}} w(q) \sum_{x \in X} p(x|s) \Delta \mathcal{F}(q)(s)(x) = \sum_{q \in \mathbb{N}} w(q) \bar{Q}(\Delta \mathcal{F}(q)(s)|s) \leq 0,
\]
where the inequality follows from \( \bar{Q}(\Delta \mathcal{F}(q)(s)|s) \leq \bar{Q}(\Delta \mathcal{F}(q)(s)|s) \leq 0; \) see Equation (1).

If we now recall Equation (2), we see that indeed \( \bar{Q}(\Delta \mathcal{F}(s)|s) \leq 0. \) The second inequality is equivalent with the first by Lemma 12(iv). \( \Box \)

8. CONCLUSIONS AND DISCUSSION

We have proved a version of the pointwise ergodic theorem for imprecise Markov chains involving functions of a single state. It is a subject of current research whether this result can be extended to gambles that depend on the entire state trajectory, and not just on a single state.

Our version subsumes the one for (precise) Markov chains, because there \( \overline{E}_\infty(f) = \overline{E}_\infty(f) = \overline{E}_\infty(f) \) and therefore

\[
\overline{E}_\infty(f) = \overline{E}_\infty(f) \geq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) \geq \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) \geq \overline{E}_\infty(f) = \overline{E}_\infty(f)
\]

strictly almost surely, implying that \( \frac{1}{n} \sum_{k=1}^{n} f(X_k) \) converges to \( \overline{E}_\infty(f) \) (strictly) almost surely. In our more general case, however, we cannot generally prove that there is almost sure convergence, and we retain only almost sure inequalities involving limits inferior and superior, as is also the case for our strong law of large numbers for submartingale differences. Indeed, that such convergence should not really be expected for imprecise probability models was already argued by Walley and Fine [17].

Ergodicity results for Markov chains are quite relevant for applications in queuing theory, where they are for instance used to prove Little’s Law [18], or ASTA (Arrivals See Time Averages) properties [9]. We believe the discussion in this paper could be instrumental in deriving similar properties for queues where the probability models for arrivals and departures are imprecise.

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