A physical field has an infinite number of degrees of freedom, as it has a field value at each location of a continuous space. Knowing a field exactly from finite measurements alone is therefore impossible. Prior information on the field is essential for field inference, but will not specify the field entirely. An information theory for fields is needed to join the measurement and prior information into probabilistic statements on field configurations. Such an information field theory (IFT) is built on the language of mathematical physics, in particular on field theory and statistical mechanics. IFT permits the mathematical derivation of optimal imaging algorithms, data analysis methods, and even computer simulation schemes. The application of such IFT algorithms to astronomical datasets provides high fidelity images of the Universe and facilitates the search for subtle statistical signals from the Big Bang. The concepts of IFT might even pave the road to novel computer simulations that are aware of their own uncertainties.

1 Information field theory

1.1 Physical fields

Physical fields like the electromagnetic field, the gravitational field, or even macroscopic ones like the atmospheric temperature generally exhibit different values at different locations in space and time. Knowing the configuration of a field, meaning knowing the field values at all locations, can be of tremendous scientific, technological or economical value. Unfortunately, this is impossible in general, as there are more locations to be known, often infinitely many, than available measurement data that provide mathematical constraints on the possible field configurations. Thus, inferring a field is in general an under-constrained problem. For a given dataset obtained by an instrument probing the field, there can be an infinite number of potential field configurations that are fully consistent with the data. How to choose among them?

Many of those configurations will be implausible. For example, imagine a radio telescope observing the cosmic microwave background (CMB). The telescope is a measurement device that integrates the field over some sub-volume of space and provides this integral. Then the integrated field in this volume is known, but how this amount of integrated field is distributed among the different locations is unclear. Among all configurations that obey this measurement constraint, very rough ones (i.e. ones that exhibit completely different values at the different locations) are possible and outnumber the also possible smooth ones largely. There are just many more ways for a field to be jagged than ways to be smooth. Nevertheless, for most physical fields, the smooth field configurations would be regarded as far more plausible. Rapid spatial changes of a field are less likely as they either cost energy (think of an electrical potential field) or are erased rapidly by the field dynamics (think of temperature jumps in a turbulent or diffusive atmosphere). Thus, among all configurations consistent with the data, smooth ones should be favored. But how to smooth? And how should this extra information be inserted into the field inference?

The first question requires a specification of the concept of smoothness. A field can be regarded to be smooth, if field values at nearby locations are similar. Thus, knowing the field at one location implies some constraints about the possible values at nearby positions. Such knowledge is best represented in terms of correlations between the different locations. Smoothness of a field $\phi$ is therefore well characterized by the two point correlation

$$\Phi(x, y) = \langle \phi(x) \phi(y) \rangle_{\langle \phi \rangle}$$

of locations $x$ and $y$, assuming for simplicity here that the field is fluctuating around zero such that its expectation value vanishes, $\langle \phi \rangle_{\langle \phi \rangle} = 0$. For fields that have a spatially

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1 We denote with $\langle f(a, b) \rangle_{\langle a | b \rangle} = \int d a f(a, b) \mathcal{P}(a | b)$ the expectation of $f(a, b)$ averaged over the conditional probability
homogeneous statistics, this correlation function is only a function of the difference of the locations, and for additionally isotropic fields, it depends only on the absolute distance, \( \Phi(x, y) = C_\varphi(|x - y|) \), with \( C_\varphi(r) \) the radial two-point correlation function of the field \( \varphi \). For example, the density field of matter in the cosmos on large scales is assumed to be the result of a statistically isotropic and homogeneous random process. For this reason, the CMB shows temperature fluctuations with isotropic correlation statistics (see Fig. 1).

The knowledge and usage of such correlations is essential for IFT, as it provides an infinite number of statements on the field, which can help to tame the infinite number of degrees of freedom of an unknown field. Traditional smoothness regularization, like the suppression of strong gradients or curvature of the field, are naturally included in this language, as we will see in Sect. 1.5.

Assuming that we have some usable knowledge on the field smoothness, the second question remains, how to incorporate this into the inference?

1.2 Bayes and statistical physics

Bayes’ theorem conveniently answers this question. Let us assume we are interested in a field \( \varphi : \Omega \to \mathbb{R} \), which lives over some volume \( \Omega \subset \mathbb{R}^u \) and we have obtained some measurement data

\[
d = R(\varphi) + n
\]

on it, where \( R(\varphi) = \langle d \rangle_{d(\varphi)} \) is the deterministic response of the instrument to the field, and \( n = d - R(\varphi) \) denotes any stochastic contribution to the outcome, the measurement noise. For example, a linearly integrating instrument is described by

\[
R_i(\varphi) = \int_{\Omega} dx r_i(x) \varphi(x),
\]

where \( i \) indexes the different measurements and \( r_i(x) \) how those respond to the field values at different locations \( x \).

Bayes’ theorem states that the posterior probability density \( \mathcal{P}(\varphi|d) \), which summarizes our knowledge on the field after the data is taken, is proportional to the product of the data likelihood \( \mathcal{P}(d|\varphi) \) for a given field configuration times its prior probability \( \mathcal{P}(\varphi) \), which summarizes our a priori knowledge on the field, like that smooth configurations are more plausible than rough ones. Bayes’ theorem reads

\[
\mathcal{P}(\varphi|d) = \frac{\mathcal{P}(d|\varphi) \mathcal{P}(\varphi)}{\mathcal{P}(d)},
\]
where the normalization of the posterior is provided by the so called evidence
\[ P(d) = \int \mathcal{D} \varphi P(d | \varphi) P(\varphi). \] (5)

Here, an integration over all field configurations is necessary, a technicality we postpone until Sect. 1.4.

The posterior is just the joint probability of data and field \( P(d, \varphi) = P(d | \varphi) P(\varphi) \) evaluated for the observed data \( d = d_{\text{obs}} \), and normalized such that \( \int \mathcal{D} \varphi P(\varphi | d = d_{\text{obs}}) = 1 \). Formally, it can be brought into a form resembling the Boltzmann distribution of statistical mechanics,\(^2\)
\[ P(\varphi | d) = \frac{\exp(-\mathcal{H}(d, \varphi))}{\mathcal{Z}(d)}, \] (6)
where we introduced the information Hamiltonian
\[ \mathcal{H}(d, \varphi) = -\ln P(d, \varphi), \] (7)
the negative logarithm of the joint probability of data and field and the partition function
\[ \mathcal{Z}(d) = \int \mathcal{D} \varphi \exp(-\mathcal{H}(d, \varphi)), \] (8)
which is just another name for the evidence \( P(d) \).

This rewriting of Bayes’ theorem for fields in the language of statistical mechanics does not solve the problem to extract useful statements from the field posterior \( P(\varphi | d) \). However, it permits the usage of the many methods theoretical physics has developed to treat statistical and quantum field theories. Consequently, the treatment of field inference problems in this language is naturally called information field theory (IFT) [1–4].\(^3\)

1.3 Expectation values

One of the strengths of the field theoretical language is that it provides tools to calculate expectation values. For example, the partition function can be turned into a moment generating functional by extending it to
\[ \mathcal{Z}(d, j) = \int \mathcal{D} \varphi \exp(-\mathcal{H}(d, \varphi) + j^\dagger \varphi), \] (9)
where \( j \) is a moment generating source term and
\[ j^\dagger \varphi = \int_\Omega dx j^*(x) \varphi(x) \] (10)

\(^2\) The temperature usually appearing in thermodynamics has been set to \( T = 1 \) here.

\(^3\) The term Bayesian field theory was proposed originally by Lemm [5] for field inference. This term, however, does not follow the convention to name field theories after subjects and not people. We do not talk about Maxwell, Einstein, or Feynman field theory, but about electromagnetic, gravitational, and quantum field theories.
is the usual scalar product for functions. Any posterior expectation value of (connected) moments of field values can be obtained from the partition function via

\[ \langle \varphi(x_1) \ldots \varphi(x_n) \rangle_{(\varphi | \varphi)}^{(C)} = \left. \frac{\partial^n \ln \mathcal{Z}(d,j)}{\partial j^{*}(x_1) \ldots \partial j^{*}(x_n)} \right|_{j=0}. \tag{11} \]

The logarithmic partition function \( \ln \mathcal{Z}(d,j) \) is known to be given by the sum of all singly connected Feynman diagrams without free vertices (see Fig. 1 for such diagrams).

Of particular interest in practical applications are the first and second posterior moments of the field,

\[ m(x) = \langle \varphi(x) \rangle_{(\varphi | \varphi)}^{(C)} = \left. \frac{\partial \ln \mathcal{Z}(d,j)}{\partial j^{*}(x)} \right|_{j=0} \]

\[ = \int \mathcal{D} \varphi \varphi(x) \mathcal{P}(\varphi | d) \]  

\[ D(x,y) = \langle \varphi(x) \varphi(y) \rangle_{(\varphi | \varphi)}^{(C)} = \left. \frac{\partial^2 \ln \mathcal{Z}(d,j)}{\partial j^{*}(x) \partial j^{*}(y)} \right|_{j=0} \]

\[ = \int \mathcal{D} \varphi \varphi(x) - m(x))((\varphi(x) - m(y)) \mathcal{P}(\varphi | d), \tag{12} \]

that represent the posterior mean field \( m \) and the uncertainty covariance \( D \) around this mean. The mean is the best guess for the field under the \( \mathcal{L}^2 \)-error loss function \( \mathcal{L}^2 = \langle |\varphi - m|^2 \rangle_{(\varphi | \varphi)}^{(C)} \).

Let us assume, for a moment, that the mean \( m = \langle \varphi \rangle_{(\varphi | \varphi)} \) and the covariance \( D = \langle \varphi \varphi^{\dagger} \rangle_{(\varphi | \varphi)} = \langle (\varphi - m)(\varphi - m)^{\dagger} \rangle_{(\varphi | \varphi)} \) is all what is known about a field, i.e. the data are this mean and covariance, \( d = (m, D) \). Which probability distribution should then be used? According to the maximum entropy principle [6], the knowledge provided by this data is best summarized by a Gaussian probability distribution with this mean and covariance:

\[ \mathcal{P}(\varphi | d) = \mathcal{G}(\varphi - m, D) \]

\[ = \frac{1}{\sqrt{|2\pi D|}} \exp \left[ -\frac{1}{2} (\varphi - m)^{\dagger} D^{-1}(\varphi - m) \right]. \tag{14} \]

1.4 Infinites

So far we have operated with fields, operators on fields, and integrals over field configurations as if they were finite dimensional vectors, matrices, and integrals. One can imagine to perform such operations with pixelized versions of fields, operators, and the like, with the pixelization taken so fine that further refinements would not make any difference to calculated expectation values anymore. In order to approach the realm of fields, the limit of an infinite number of pixels has to be taken. Clearly, some quantities used above become infinite or vanish in this limit, like the determinant of a covariance \( D \) in Eq. 14. However, these exploding terms are usually auxiliary expressions, ensuring the normalization of a probability distribution (with respect to a given pixelization) and not of fundamental interest. What needs to stay finite, in this limit are observables, the expectation values of field properties \( \langle f(\varphi) \rangle_{(\varphi | \varphi)} \). Here, \( f(\varphi) = r^{\dagger} \varphi \) might be the integration of the field over some probing kernel \( r(x) \), or the like.

1.5 Free fields

Let us discuss a concrete, most simple IFT scenario first. Imagine we observe relative temperature fluctuations

\[ \varphi(x,t) = \frac{\delta T(x,t)}{T} \ll 1 \tag{15} \]

that live in one spatial and one temporal dimension, \( \Omega = \mathbb{R}^2 \). These fluctuations are excited by some external Gaussian excitations \( \xi \sim \mathcal{G}(\xi, \Xi) \) and decay by diffusion, so that their dynamics should be

\[ \partial_t \varphi(x,t) = \kappa \partial_k^2 \varphi(x,t) + \nu \frac{1}{2} \xi(x,t), \tag{16} \]

with \( \kappa \) the spatial diffusion coefficient and \( \nu \) being a rate.

Fourier transforming this equation with respect to space and time yields

\[ \varphi(k,\omega) = \frac{\nu^{1/2} \hat{\xi}(k,\omega)}{i \omega + \kappa k^2}. \tag{17} \]

For simplicity, we assume the excitation to be white and of unit variance, \( \Xi_{(x,t)(x',t')} = \delta(x-x') \delta(t-t') \) or shortly \( \Xi = 1 \). This implies that in Fourier space the excitation is white as well, \( \Xi_{(k,\omega)(k',\omega')} = (2\pi)^2 \delta(k - k') \delta(\omega - \omega') \). The field covariance in Fourier space is then

\[ \Phi_{(k,\omega)(k',\omega')} = \langle \varphi(k,\omega) \varphi^{*}(k',\omega') \rangle_{\varphi} \]

\[ = \frac{(2\pi)^2 \nu}{\omega^2 + \kappa^2 k^2} \delta(k - k') \delta(\omega - \omega'). \tag{18} \]

---

4 The external excitation fluctuations should add up under time integration in quadrature, and if \( \xi \) is as dimensionless as \( \varphi \), then \( [\nu] = \sec^{-1} \).

5 We use the (one dimensional) Fourier convention \( \varphi_k = \int dx e^{ikx} \varphi_x, \varphi_x = \int dk/(2\pi) e^{-ikx} \varphi_k \), and therefore the identity operator in Fourier space reads \( \varphi_{kk} = 2\pi \delta(k - k') \).

Furthermore, we use arguments and indices interchangeably and to denote whether a quantity is in real space or Fourier space: \( \varphi_x = \varphi(x) \) and \( \varphi_k = \varphi(k) = \int dx e^{ikx} \varphi(x) \).
At time $t = 0$ we obtain some data on the field according to a linear measurement as described by Eqs. 2 and 3, with Gaussian, field independent measurement noise $n \sim \mathcal{G}(n, N)$ that is white, $N = \sigma_n^2 \mathbb{I}$, and has a variance $\sigma_n^2$. Thus the likelihood is

$$\mathcal{P}(d|\varphi) = \mathcal{G}(d - R\varphi, N).$$  \hfill (19)

To simplify the discussion further, we assume that the measurement probes every location, such that $r_i(x) = \delta(i - x)$ or $R = 1$. This is an idealization, as now an infinite number of locations are probed. Note, however, that this does not set us into a perfect knowledge about the field, as the data $d = \varphi + n$ are still corrupted by noise.

To infer the field configuration at $t = 0$ via Bayes’ theorem, Eq. 4, we need the prior. As the dynamics is linear and the excitation is Gaussian, the field statistics will be Gaussian, $\mathcal{P}(\varphi) = \mathcal{G}(\varphi, \Phi)$. This requires that we work out the field covariance for a specific time. Fourier back-transforming Eq. 18 with respect to time gives

$$\Phi(k, t, \omega) = \frac{\pi v}{k^2} \delta(k - k') \exp(-|t - t'|k^2).$$  \hfill (20)

This means that the equal time correlation is

$$\Phi(k, t, \omega) = \frac{\pi v}{k^2} \delta(k - k')$$  \hfill (21)

and we can ignore the time coordinate in the following, as we are only interested in $t = 0$. Thus, from now on $\Omega = \mathbb{R}$. A field realization $\varphi$ of a Gaussian process with this covariance $\Phi$ is shown in Fig. 3.

This covariance is actually equivalent to regularization of the solutions by an $L^2$-norm on the field gradient. The prior Hamiltonian,

$$\mathcal{H}(\varphi) = -\ln \mathcal{P}(\varphi) = -\ln \mathcal{P}(\varphi, \Phi)$$

$$= \frac{1}{2} \varphi^\dagger \Phi^{-1} \varphi + \frac{1}{2} \ln |2\pi\Phi|$$

$$= \frac{1}{2} \int \frac{dk}{2\pi} \frac{2k}{v} k^2 |\varphi_k|^2 + \text{const}$$

$$= \frac{k}{v} \int dx |\nabla \varphi|^2 + \text{const},$$  \hfill (22)

is regularizing the joint information Hamiltonian of data and field,

$$\mathcal{H}(d, \varphi) = \mathcal{H}(d|\varphi) + \mathcal{H}(\varphi),$$  \hfill (23)

with $\mathcal{H}(d|\varphi) = -\ln \mathcal{P}(d|\varphi)$ the likelihood information.

Since we started with a physical model, the strength of this regularization is specified by the ratio of the diffusion constant to the excitation rate $\kappa/v$, a physical quantity. This should be seen in contrast to the frequently used ad-hoc parameter put in front of an $L^2$-gradient regularization term $||\nabla \varphi||^2 = \int dx |\nabla \varphi|^2$. The physics provides here the necessary regularization to treat the otherwise ill-posed problem. No ad-hoceries are needed.

Now, we can work out the information Hamiltonian

$$\mathcal{H}(d, \varphi) = \frac{1}{2} \varphi^\dagger \Phi^{-1} \varphi + \frac{1}{2} (d - R\varphi)^\dagger N^{-1} (d - R\varphi)$$

$$+ \frac{1}{2} \ln |2\pi \Phi| + \frac{1}{2} \ln |2\pi N|$$  \hfill (24)

as well as the logarithmic partition function

$$\ln Z(d, J) = \ln \int \mathcal{D}\varphi e^{-\mathcal{H}(d, \varphi) + J^\dagger \varphi}$$

$$= \frac{1}{2} (J + J^\dagger)^\dagger D (J + J^\dagger) - \frac{1}{2} d^\dagger N^{-1} d$$

$$+ \frac{1}{2} \ln |2\pi D| - \frac{1}{2} \ln |2\pi \Phi| - \frac{1}{2} \ln |2\pi N|$$  \hfill (25)

with $D = (\Phi^{-1} + R^\dagger N^{-1} R)^{-1}$

$$\text{and } J = R^\dagger N^{-1} d.$$  \hfill (26)

It turns out that a separate moment generating variable $J$ is not needed, as the variable $j$ can take over this role. We just set $J = 0$, take derivatives with respect to $j$, but do not set it to zero afterwards. The first and second moments of the posterior field are then

$$\langle \varphi(c) | \varphi(d) \rangle = \frac{\partial \ln Z(d, j)}{\partial j^\dagger} = D j = m$$  \hfill (28)

$$\langle \varphi^\dagger(c) | \varphi(d) \rangle = \frac{\partial^2 \ln Z(d, j)}{\partial j^\dagger \partial j} = D.$$  \hfill (29)

Since the posterior is Gaussian, its mean is $m = D j$, and its uncertainty is $D$, it has to be given by Eq. 14.

For our specific problem we have $R = \mathbb{1}$, $N = \sigma_n^2 \mathbb{I}$, and

$$\Phi_{kk'} = \frac{2\pi \delta(k - k')}{2k v^{-1} k^2}$$  \hfill (30)

and therefore

$$D_{kk'} = 2\pi \delta(k - k') \left[2k v^{-1} k^2 + \sigma_n^{-2}\right]^{-1}.$$  \hfill (31)

$$j_k = \sigma_k^{-2} d_k,$$  \hfill (32)

$$m_k = \frac{1}{D_{kk'}} \frac{dk'}{2\pi} D_{kk'} j_{k'} = \frac{d_k}{1 + \lambda^2 k^2} \text{ with}$$  \hfill (33)

$$\lambda^2 = \frac{2k v}{\sigma_n^2}.$$  \hfill (34)

This means that the optimal field reconstruction $m$ follows the data on large spatial scales $1/k \gg \lambda$ (as $m_k \approx d_k$ there) and should be a strongly suppressed version of the (Fourier space) data for small spatial scales with $1/k \ll \lambda$ (as $m_k \approx d_k (\lambda k)^{-2} \ll d_k$ there). The optimal reconstruction is the result of a Fourier space filter operation applied to the data. This result was first found by Wiener [7] and the resulting filter is therefore called the Wiener filter.
It should be noted that in ad-hoc regularization, the parameter $\lambda$, which controls the threshold wavelength of this low-pass filter, would be chosen out of the blue, whereas here the knowledge about the statistics of the signal is determining by physical quantities, as $\lambda = \sqrt{2k/\nu \sigma_n}$.

We made a number of assumptions in this derivation, namely that the noise is white, the field has a specific covariance, and the instrument response is the identity. None of these were essential, and assuming colored noise with covariance $N$, a general field covariance $\Phi$, and an arbitrary (but linear) response $R$ would have provided us with the solution given by Eqs. 14, 26, 27, 28, and 29. Solely the nice property of this generalized Wiener filter being a Fourier space only operation would have been lost.

1.6 Information propagation

Before we treat more complicated measurement situations, let us try to understand what the (generalized) Wiener filter does. It turns linear data $d = R\varphi + n$ on a field into an optimal reconstruction $m$ (see Fig. 3) with

$$m = D j,$$

$$j = R^\dagger N^{-1} d,$$  \hspace{1cm} (35)  \hspace{1cm} (36)

$$D = \left(\Phi^{-1} + R^\dagger N^{-1} R\right)^{-1}.$$  \hspace{1cm} (37)

First, we inspect $j$. This is the inversely noise weighted data $N^{-1}d$ back projected into the field domain $\Omega$. Every piece of data is thrown onto all locations that have influenced it in the measurement process, with exactly the strength of this influence, as encoded in the adjoint (or transposed) response $R^\dagger$. Since $j$ carries the essential information of the data and sources our posterior knowledge on the field, the name *information source* seems appropriate.

Now, let’s see what $D$ does to $j$ in $m = D j$. In components, this is

$$m(x) = \int_{\Omega} dx \, D(x, y) \, j(y).$$  \hspace{1cm} (38)

Thus, $D(x, y)$ transports (and weights) the information source $j$ from all locations $y$ to location $x$, where they are added up to provide the posterior mean field. For this reason the term *information propagator* is natural for $D$. This nomenclature is very much in accordance with conventions in quantum field theory and also serves IFT well.

For our simplistic example, the information propagator in position space

$$D(x, y) = \frac{\sigma^2_n}{2\lambda} e^{-\frac{|x-y|}{\lambda}}$$  \hspace{1cm} (39)

is a translation invariant convolution kernel, since neither the prior nor measurement singles out any location. The mean field $m$ perceives most strongly the local information source $j_x = d_x/\sigma^2_n$ at every location $x$, but also

**Figure 3** Wiener filter reconstruction $m$ (Eq. 35) of the unknown field $\varphi$ from noisy and incomplete data $d$ as discussed in the text. The 1-$\sigma$ posterior uncertainty is shown as well (Eqs. 41 and 37). Note the enlarged uncertainty in regions without data.

**Figure 4** Classical field reconstruction $\varphi_{cl}$ (Eq. 48) of the unknown field $\varphi$ from Poisson counts $d$ subject to a non-linear, exponential response. The 1-$\sigma$ posterior uncertainty is shown as well (Eqs. 41 and 49). Note the enlarged uncertainty in regions without data. The field realization $\varphi$ is the same as in Fig. 3.
more distant contributions with exponentially decaying weights:
\[
m(x) = \int \frac{dy}{2\lambda} e^{-\frac{|x-y|}{\lambda}} d(y).
\]

This operation averages down the noise in an optimal fashion.

The information propagator \(D\) is informed about how the measurement was performed as it incorporates \(R\) and \(N\). Furthermore, \(D\) knows about the correlations of field values at different locations through \(\Phi\). It uses the first to properly weight the information source and the second to extrapolate it to locations not or only poorly probed by the instrument.

Furthermore, since \(D\) is also the posterior uncertainty covariance in \(\mathcal{P}(\varphi|d) = \mathcal{G}(\varphi - m, D)\), it also encodes how certain we can be about the reconstructed field via
\[
\sigma^2_{\varphi_x} \equiv \langle (\varphi_x - m_x)^2 \rangle_{\varphi_x|d} = D_{xx}
\]

and how much the uncertainty is correlated between different locations (via \(D_{xy}\)). Eq. 39 illustrates that in case of larger noise (as controlled by \(\sigma_n\)) this uncertainty is larger, since \(\sigma^2_{\varphi_x} = D_{xx} = \sigma_n^2/(2\lambda) = \sigma_n/\sqrt{2}\kappa/\nu \propto \sigma_n\). It is correlated over larger distances, since \(\lambda = \sqrt{2\kappa/\nu}\sigma_n \propto \sigma_n\). Fig. 3 also shows the \(\sigma_{\varphi_x}\)-uncertainty range around the Wiener filter reconstruction \(m\).

**1.7 Critical filter**

So far, the prior signal covariance \(\Phi\) was assumed to be known. If it is unknown, it has to be inferred as well. A hyperprior, a prior on a prior quantity, is necessary for this. In case statistical homogeneity can be assumed, \(\Phi\) is diagonal in Fourier space, \(\Phi_{kq} = (2\pi)^d \delta(k-q) |P_{\varphi}(k)|\), with \(P_{\varphi}(k)\) the spectral power density and \(k\) a Fourier space vector, denoted here in bold face to distinguish it from its length \(k = |k|\). In case statistical isotropy can be assumed, the spectral power density is isotropic \(P_{\varphi}(k) = P_{\varphi}(k)\), and all Fourier modes with the same length \(k = |k|\) have the same power.

It is then convenient to introduce the spectral band projector \((P_{\varphi})_{qq} \equiv (2\pi)^d \delta(q - q') \delta(|q| - k)\). With this the maximum a posteriori estimator for the power spectrum\(^6\) is
\[
P_{\varphi}(k) = \frac{\text{Tr} [(m m^\dagger + D) P_k]}{\text{Tr} [P_k]},
\]

where a flat prior on the spectral power density entries was assumed \(^8\). A compact derivation of this result can be found on Wikipedia \cite{Wikipedia}. Eq. 42 and 28 have to be solved jointly, as \(m = D j\) depends on the power spectrum \(P_{\varphi}(k)\) through \(D\), and the power spectrum itself also depends on \(m\). This so called critical filter is already an example of an interacting field theory, where the field solution depends in a non-linear way on the data.

The critical filter, its variations and extensions\(^7\) play an important role for real world applications of IFT, since very often a sufficiently well understood theory that predicts the statistical properties of a field is not available. With the critical filter, a field and its power spectrum can be estimated simultaneously from the same data.

**1.8 Interacting fields**

The (generalized) Wiener filter of the free IFT appeared under a number of assumptions. These were Gaussianity and mutual independence of signal and noise, linearity of the measurement and the precise knowledge of all involved operators (\(\Phi\), \(R\), and \(N\)).

In case these conditions are not given, the situation becomes more complicated. This is expressed by the information Hamiltonian \(\mathcal{H}(d, q) = -\ln \mathcal{P}(d, q)\) getting terms of higher than second order in the field \(q\) (after marginalization of all other unknowns), so called interaction terms. These lead – in general – to non-linear relations between the data and mean field as well as to non-Gaussian posteriors.

Let us stay concrete and turn our simplistic example into a non-linear problem. To do so we modify Eq. 15 that it can cope with large temperature fluctuations without becoming unphysical (= allowing negative temperatures). We achieve this by setting now
\[
q(x) = \ln \left(1 + \frac{\delta T(x)}{T} \right)
\]
such that
\[
T(x) = \frac{\overline{T}}{\exp(q(x))}
\]

\(\overline{T}\) For informative spectral prior \([8]\), spectral smoothness hyperpriors \([9]\), unknown noise \([10]\), non-linear measurement \([9, 11]\) and others.

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\(^6\) For this \(\mathcal{P}(P_{\varphi}|d) = \int \mathcal{D}q \mathcal{P}(P_{\varphi}, q|d)\) has to be maximized, the power spectrum posterior, which is marginalized over the field.
is now a strictly positive temperature, irrespective of the
value of \( \varphi(x) \), which we still assume to be drawn from
a zero centered Gaussian, \( \varphi \sim \mathcal{N}(\varphi, \Phi) \), with covariance
given by Eq. 21.

Let us assume that we are able to observe the thermal
photons emitted from our system, such that the expected
number of received photons by a detector in an observa-
tional period is

\[
\mu_i(\varphi) = \int dx r_i'(x) T^3(x) = \int dx r_i(x) \exp[3\varphi(x)]
= r_i^\dagger e^{3\varphi},
\]

(45)

where with \( i \) we label the detector and the observational
period, we have absorbed physical constants into the re-
sponses \( r_i \), have set \( r = r^\dagger T^3 \), and introduced the conven-
tion to apply functions position wise to fields, \( e^{\varphi(x)} \equiv e^{\varphi_x} \).

Observing photons is a Poisson process, and each data
bin has an independent distribution function

\[
\mathcal{P}(d_i|\mu_i) = \frac{\mu_i^{d_i}}{d_i!} e^{-\mu_i}
\]

(46)

for the number of photons \( d_i \) received by the individual
detectors' observations as labeled by \( i \). For the full dataset
we write \( \mu = R \exp(3\varphi) \), where we define the linear part of
the response operator as \( R_{ix} = r_i(x) \).

Thus, all elements to construct the information Hamilton-
ian are available, and we get

\[
\mathcal{H}(d, \varphi) = \mathcal{H}(\varphi) + \sum_i \mathcal{H}(d_i|\mu_i(\varphi))
\equiv \frac{1}{2} \varphi^\dagger \Phi^{-1} \varphi + 1^\dagger R e^{3\varphi} - d^\dagger \ln(R e^{3\varphi}),
\]

(47)

where \( \equiv \) denotes that we dropped irrelevant (since \( \varphi \)-
dependent) constants and \( 1^\dagger \) is a row vector in data
space with all entries being 1. This is clearly an informa-
tion Hamiltonian with strong self-interaction terms that
are provided by the an-harmonic orders of the exponen-
tial function and the logarithm.

To treat this, we will use two approximative ap-
proaches from theoretical physics, by calculating the clas-
sical and the mean field solution.

1.8.1 Classical field approximation

The classical field \( \varphi_{cl} \) is simply given by the minimum of
the Hamiltonian:

\[
0 = \frac{\partial \mathcal{H}(d, \varphi)}{\partial \varphi} = \Phi^{-1} \varphi + e^{3\varphi} R^3 - \frac{e^{3\varphi} R^3}{R e^{3\varphi}}
= \Phi^{-1} \varphi + 3 e^{3\varphi} R^3 \left( 1 - \frac{d}{R e^{3\varphi}} \right) \Rightarrow
\]

\[
\varphi_{cl} = 3 \Phi e^{3\varphi_{cl}} R^3 \left( \frac{d - R e^{3\varphi_{cl}}}{R e^{3\varphi_{cl}}} \right)
\]

(48)

A solution to this non-linear integral equation should ex-
hibit \( d \approx R e^{3\varphi_{cl}} \) as this satisfies the likelihood, but will
also be smoothed due to the application of the smoothing
action of the covariance kernel \( \Phi \).

The classical field minimizes the Hamiltonian and, ac-
cording to Eq. 6, maximizes the posterior. Thus, the clas-
sical field is the maximum a posteriori (MAP) field estimate.
In general the classical or MAP solution is not identical to
the posterior mean. This requires special conditions, like
that the posterior is point symmetric about \( \varphi_{cl} \), as it is the
case of the free theory discussed in Sect. 1.5.

An estimate of the uncertainty covariance is given by
the inverse Hessian of the Hamiltonian at its minimum

\[
(D^{-1})_{xy} \approx \left( \frac{\partial^2 \mathcal{H}(d, \varphi_{cl})}{\partial \varphi_{clx} \partial \varphi_{cly}} \right)^{-1}
= \Phi^{-1}_{xy} + 9 \delta_{xy} e^{3\varphi_{cl}} R^3 \left( 1 - \frac{d}{R e^{3\varphi_{cl}}} \right)
+ 9 e^{3\varphi_{clx}} + 3 \varphi_{cly} \sum_i \frac{R_{ix} R_{iy} d_i}{(R e^{3\varphi_{cl}})^2}.
\]

(49)

The uncertainty in this Laplace approximation (treat-
ning Posterior as a Gaussian by expanding it around its max-
imum) clearly depends on the classical filed, and there-
fore on the data. For a different data realization, the un-
certainty would differ. Both, a field estimate and its un-
certainty for this measurement situation, can be found in
Fig. 4.

1.8.2 Mean field approximation

The mean field approximation \( m' \) should not be confused
with the a posteriori mean field \( m \equiv \langle \varphi \rangle_{\langle \varphi \rangle_d} \) it tries to
approximate. Mean fields in statistical field theory are constructed by minimization of the Gibbs free energy,
\[ G(m', D') = U(m', D') - TS(D'). \] (50)

In IFT the internal energy \( U \) is defined as
\[ U(m', D') = \langle \mathcal{H}(d, \phi) \rangle_{\mathcal{P}(\phi|\tau = m', D')}, \] (51)

the information Hamiltonian averaged over an approximative Gaussian knowledge state \( \mathcal{P}(\phi|\tau = m', D') \) with mean \( m' \) and variance \( D' \). The inference temperature \( T = 1 \) was kept in the formula to highlight the formal similarity to thermodynamics. Finally,
\[ S(D') = - \int D\phi \mathcal{G}(\phi - m', D') \ln \left( \mathcal{G}(\phi - m', D') \right) \]
\[ = \frac{1}{2} \ln |2\pi e D'| \] (52)
is the Boltzmann-Shannon entropy of this Gaussian state. It measures the phase space volume of the uncertainty of the approximative knowledge state.

The Gibbs free energy can be rewritten as
\[ G(m', D') = -\int D\phi \mathcal{G}(\phi - m', D') \ln \left( \frac{\mathcal{G}(\phi - m', D')}{\exp(-\mathcal{H}(d, \phi))} \right) \]
\[ = -\int D\phi \mathcal{P}(\phi'|m', D') \ln \left( \frac{\mathcal{P}(\phi'|m', D')}{\mathcal{P}(d)} \right) \]
\[ = -\int D\phi \mathcal{P}(\phi'|m', D') \ln \left( \frac{\mathcal{P}(\phi'|m', D')}{\mathcal{P}(\phi|d) \mathcal{P}(d)} \right) \]
\[ = \text{KL}[\mathcal{P}|\mathcal{P}] - \ln \mathcal{Z}(d), \] (53)
where \( \text{KL}[\mathcal{P}|\mathcal{P}] \) denotes the Kullback-Leibler (KL) divergence of \( \mathcal{P}(\phi'|m', D') \) and \( \mathcal{P}(\phi|d) \). Thus, the Gibbs free energy is (up to an irrelevant constant) the information distance between the correct and approximate probability distributions.\(^9\)

In case of our toy example, the Gibbs free energy reads
\[ G(m', D') \equiv \frac{1}{2} \text{tr} \left[ \Phi^{-1} \left( m' m'^{\dagger} + D' \right) \right] + 1^\dagger R e^{3m'^{\dagger} + \frac{3}{2} D'} \]
\[ - d^\dagger \ln \left( R e^{3m'^{\dagger} + \frac{3}{2} D'} \right) - \frac{1}{2} \ln |2\pi e D'|, \] (54)
since \( \langle \phi \phi' \rangle_{\mathcal{P}} = m' m'^{\dagger} + D' \) and \( \langle e^{\phi} \rangle_{\mathcal{P}} = e^{m' + \frac{3}{2} D'}. \) Here \( D'_{xx} \) denotes the diagonal of \( D' \).

The mean field \( m' \) is the minimum of the Gibbs free energy. A short calculation analogously to Eq. 48 yields
\[ m' = 3 \Phi e^{3m''} R \left( \frac{d - R e^{3m''}}{R e^{3m''}} \right), \] (55)
with \( m'' = m' + \frac{3}{2} D' \). The mean field \( m' \) therefore depends on the uncertainty \( D' \), which itself has to be determined, either by minimizing Eq. 54, or by using equivalently the thermodynamical relation [13]
\[ D' = \left( \frac{\partial^2 G(m', D')}{\partial m' \partial m'^{\dagger}} \right)^{-1} \] (56)
The latter yields
\[ (D')_{xy} = \Phi_{xy}^{-1} + 9 \kappa_{xy} e^{3m''} R_{xx} \left( 1 - \frac{d}{R e^{3m''}} \right) + 9 e^{3m''} \sum_{i} \frac{R_{xx} R_{yy} d_i}{(R e^{3m''})^2}, \] (57)
a rather complex expression, which depends on \( m' \) and \( D' \) itself. Thus the mean field and its uncertainty co-variance have to be solved for jointly, as they are interdependent.

A comparison of the mean field and Laplace approximations shows that the estimated uncertainty of the mean field is larger and this estimate is therefore more conservative. The Laplace approximation usually underestimates uncertainties. This larger uncertainty also provides corrections to the (mean) field (with respect to the Laplace/MAP field).

2 Applications

Current applications of IFT are mostly in astrophysics and cosmology. The elements of the theory presented here already cover many typical measurement situations in these areas: Gaussian and Poissonian noise, Gaussian and log-Gaussian fields, linear instrument responses, known and unknown field correlation structures, etc.

2.1 Numerical information field theory

The implementation of algorithms derived within of IFT is facilitated by the NIFTy library\(^10\) for numerical information field theory [14, 15]. For example, the calculations

\(^9\) The Gibbs free energy is equivalent to \( \text{KL}[\mathcal{P}|\mathcal{P}] \), the amount of information needed (as measured in nits) to change the correct posterior \( \mathcal{P} \) to the approximate posterior \( \mathcal{P}' \). Information theoretically, the reverse would be more appropriate, as \( \text{KL}[\mathcal{P}|\mathcal{P}] \) would measure how much information has to be added to the approximate posterior \( \mathcal{P}' \) to restore the actual \( \mathcal{P} \) [12]. This, however, would involve the calculation of non-Gaussian path integrals and is therefore often not an option.

\(^10\)https://gitlab.mpcdf.mpg.de/ift/NIFTy/
for Figs. 3 and 4 were performed by NIFTy and the corresponding code\textsuperscript{11} became part of the current demo package delivered with the library.

NIFTy permits the abstract implementation of IFT formulas irrespective of the underlying domains the fields live over. The same algorithm implemented with NIFTy can reconstruct without change fields living over 1D, 2D, 3D Cartesian spaces, the sphere, or even product spaces built out of those spaces.

Different algorithmic choices for signal estimation are possible. Algorithms can be based on MAP or Gibbs-free energy minimization, or use perturbation series like expansion in Feynman diagrams. For the joint inference of a signal and its power spectrum the approach of [11] is performing well.

In the following we highlight three applications of IFT to illustrate the spectrum of potential usages: photon imaging, non-Gaussianity estimation, and simulation of field dynamics.

2.2 Photon imaging

Photon imaging is the reconstruction of the continuous spatial and/or spectral photon emission field from which a finite number of detected photons were emitted. The D\textsuperscript{3}PO and D\textsuperscript{4}PO algorithms [16, 17] address this task. The first one\textsuperscript{12}, D\textsuperscript{3}PO, denoises, deconvolves and decomposes photon observations (thus the acronym D\textsuperscript{3}PO) into a diffuse log-emission field plus a point-source log-emission field. For the diffuse field a power spectrum is inferred as well. For the point source field a virtual point source is assumed to reside in every image pixel, just most of them being insignificantly faint. The application of D\textsuperscript{3}PO to data of the Fermi gamma-ray satellite is shown in Fig. 2 [18].

The successor algorithm D\textsuperscript{4}PO does essentially the same, just with the difference that the fields can live over product spaces of spatial coordinates (the sky) and other domains (like the photon energy space). Furthermore, an arbitrary number of such fields can be reconstructed, e.g. to allow also for background counts living over the data space. The correlation structure of the fields is assumed to be the direct product over (a priori unknown and therefore simultaneously reconstructed) correlation functions in the different directions. A diffuse log-emission field $\varphi(x, y)$ over the product space of sky ($x \in S^2$) and log-energy ($y = \ln(E/E_0) \in \mathbb{R}$) therefore has the correlation structure $\Phi_{\varphi}(x, y) = \Phi_{\varphi}(S^2) \Phi_{\varphi}(E)$.

2.3 Non-Gaussianity

An important question of contemporary cosmology is how precisely the initial density fluctuations of the Universe were following Gaussian statistics. The different inflationary scenarios for the first fractions of a second of the cosmic history predict different amounts of deviations from Gaussianity. A simple parametrisation of non-Gaussianity is the so called $f_{\text{NL}}$ model, in which an initially Gaussian random field $\varphi \rightarrow \mathcal{G}(\varphi, \Phi)$ is non-linearly processed into the gravitational potential in the Early Universe observed at a later epoch:

$$\psi(x) = \psi(\varphi, f_{\text{nl}})(x) = \varphi(x) + f_{\text{nl}} \left[ \varphi^2(x) - \langle \varphi^2(x) \rangle_{\langle \varphi \rangle} \right]$$

(58)

If the measurement process is linear and Gaussian, with $d = R \psi + n$ and $n \sim \mathcal{N}(n, N)$, the information Hamiltonian

$$\mathcal{H}(d, \psi | f_{\text{nl}}) \equiv \frac{1}{2} \left( d - R \psi + f_{\text{nl}} \right) \dagger N^{-1} \left( d - R \psi + f_{\text{nl}} \right)$$

$$+ \frac{1}{2} \psi \dagger \Phi \psi$$

(59)

becomes fourth order in $\varphi$.

In order to decide which value $f_{\text{nl}}$ the data prefer, one needs to calculate the evidence $\mathcal{Z}(d | f_{\text{nl}})$. This is, however, also the partition function $\mathcal{Z}(d | f_{\text{nl}})$. Since $f_{\text{nl}}$ is a (comparably) small parameter, one can use that the logarithm of the partition function is given by the sum of all simply connected Feynman diagrams without external vertices. Sorting such diagrams by their order in $f_{\text{nl}}$ up to second order, one can construct the MAP estimator for $f_{\text{nl}}$ [1]. This is displayed in Fig. 1 superimposed to a map of the CMB, to which such a non-Gaussianity estimator can be applied. The Feynman diagrams containing $f_{\text{nl}}$ up to linear order are in the numerator and the ones up to quadratic order in the denominator of the fraction that comprises the $f_{\text{nl}}$-estimator (while $f_{\text{nl}}$ was removed from these terms as it is to be estimated). The numerator is identical to the well known Komatsu-Spergel-Wandelt (KSW) estimator [19] used in CMB studies, the denominator provides a Bayesian normalization, which depends on the particular data realization. Actually, the original KSW estimator comprised only the terms given by the first two diagrams of the numerator. The other terms were discovered later as correction terms for inhomogeneous noise [20]. The diagrammatic approach to IFT delivers all these terms naturally and simultaneously. Further details can be found in [1].

\textsuperscript{11}https://gitlab.mpcdf.mpg.de/ift/NIFTy/blob/NIFTy_4/demos/poisson_demo.py
\textsuperscript{12}http://ascl.net/1504.018
2.4 Information field dynamics

As a final application, we show how IFT can be used to build computer simulation schemes for partial differential equations (PDEs). A field \( \varphi(x, t) \) over space and time may follow a PDE of the form

\[
\partial_t \varphi = F(\varphi),
\]

(60)

For example \( F(\varphi) = \kappa \partial^2_x \varphi + v^{1/2} \xi \) reproduces Eq. 16.

In a computer, only a discretized version of the field can be stored in memory. In the framework of information field dynamics (IFD, [21–23]), the data in computer memory describing the field at an instant \( t \) is regarded as the result of a virtual measurement process of the form given by Eq. 2. A convenient linear response \( R \) might be given by the pixel window function, e.g. \( R_{ix} = P(x \in \Omega_i|x, \Omega_i) \in [0, 1] \) such that \( R_{ix} = 1 \) if \( x \) is within the volume of the \( i \)-th pixel \( \Omega_i \), otherwise \( R_{ix} = 0 \). In this case the data would contain the pixel-integrated field values. The measurement noise might be absent, \( n = 0 \), as the virtual measurements can be chosen to be noiseless.

However, even without measurement uncertainty, the field is not fully determined by the data. The remaining uncertainty is captured by the field posterior \( \mathcal{P}(\varphi|d) \), which can be specified via Bayes’ theorem (Eq. 6) in case a field prior \( \mathcal{P}(\varphi) \) is available.

The quintessence of IFD is to evolve the knowledge on the field as parametrized by the posterior, by evolving the data \( d \) in such a way that information on the field is conserved as much as possible. For this, the KL-divergence between the time evolved field posteriors and a posterior can be specified via Bayes’ theorem (Eq. 6) in case a field prior \( \mathcal{P}(\varphi) \) is available.

Let us regard the case in which the field prior is Gaussian, \( \mathcal{P}(\varphi) = \mathcal{N}(\varphi, \Phi) \), the measurement is linear and noise-free, \( d = R \varphi \), and thus the posterior is Gaussian as well \( \mathcal{P}(\varphi|d) = \mathcal{N}(\varphi - m, D) \), with \( m = m(d) = \Phi R^\dagger (R \Phi R^\dagger)^{-1} d \equiv W d \) is the noise-less Wiener filter reconstruction. In this case, the optimal temporal evolution of the data is given by [22]

\[
\partial_t d = R \langle F(\varphi) \rangle_{\varphi|d},
\]

(61)

Since the average is over a Gaussian knowledge state, this expression can be brought into the compact form

\[
\partial_t d = R F[O_m],
\]

(62)

where

\[
O_m = m + D \partial_m
\]

(63)

is the field operator [24]. This generates a field instance out of a Gaussian posterior knowledge state,

\[
O_m \mathcal{N}(\varphi - m, D) = \frac{m + D \partial_m e^{-\frac{1}{4}(\varphi - m)^2 D^{-1}(\varphi - m)}}{\sqrt{2\pi D}} = \mathcal{N}(\varphi - m, D)
\]

(64)

and permits to calculate Gaussian expectation values algebraically, since

\[
\langle F(\varphi) \rangle_{\varphi|d} = \int \mathcal{D} \varphi \mathcal{N}(\varphi| O_m) F(\varphi - m, D) = F[O_m] \langle \varphi|d \rangle = F[O_m] 1 \equiv F[O_m].
\]

(65)

Eq. 62 describes the data evolution that captures the (knowledge on the) field evolution best. For a linear PDE, like Eq. 16, the \( \delta_m \) terms make no difference, and the data equation becomes

\[
\partial_t d = R F[m] = R F \Phi R^\dagger (R \Phi R^\dagger)^{-1} d = \tilde{F} d,
\]

(66)

a linear ordinary differential equation (ODE). The data evolution operator \( \tilde{F} \) contains a weighting of the action of the PDE operator \( F \) with the statistics of the expected field fluctuations. These are encoded in \( \Phi \) and its projection into the data space via the measurement response \( R \). This weight \( R \Phi R^\dagger \) is also the ‘denominator’ that ensures the proper units of the data space operator. This way, the knowledge about the field modes that are not resolved by the data enters the data dynamics. For example, an IFD algorithm for a thermally excited Klein-Gordon field performs slightly better than a spectral scheme for the same problem, because the former exploits its knowledge about the correct statistics of the unresolved field fluctuations [21].

For a non-linear PDE, quadratic terms in \( F \) generate quadratic terms in the field operator,

\[
O_m^2 = (m + D \partial_m)(m + D \partial_m) = m^2 + D + \mathcal{O}(\delta_m)
\]

that contain non-vanishing contributions from the uncertainty dispersion \( D \). Thus, the data of a non-linearly evolving field not

13The constituents of the field operator \( O_m = m + D \partial_m \) have to be applied to the vacuum state \( 1 = 1_m \) at the end. This is constant in \( m \). Any differential \( \delta_m \) annihilates to zero on this state. One could also call \( D \partial_m \), the mean field annihilator, as it annihilates one power of the mean field \( m \) from expressions if applied to them. Consequently, the other part of \( O_m, m \), would be the mean field creator, as it creates one power of \( m \) in an expression.
only get a non-linear ODE, as \(m^2 = (W\,d)^2\) is quadratic in the data, they also get corrections that capture the effect of the uncertainty processed through the non-linearity, expressed by terms that contain the field uncertainty \(D\).

A naive discretization of a PDE, which just replaces the differential operators of the PDE with difference operators on the data, does not account for such non-linear sub-grid effects. For example, IFD schemes for the Burgers equation, which is a simplistic version of compressive hydrodynamics, handle the shock waves developing in the field dynamics better than (central) finite difference schemes [22], the latter representing the (most) naive discretization of the PDE.

Despite these encouraging results, it should be noted that IFD requires further theoretical investigation and algorithmic work before it can seriously compete with existing state-of-the-art simulation schemes in terms of usability and performance. However, as IFD is already formulated probabilistically, the assimilation of observational data into an IFD simulation should be relatively straight-forward.

## 3 Outlook

IFT, as the information theory for fields, has many potential scientific, technological, and economical applications. Two current development lines to bring IFT into broader usage should be presented as a closing outlook:

**Imaging.** The transformation of astronomical or medical data into visual informative images, is a central application area of IFT. The recipe to an IFT imaging algorithm is the construction of the information Hamiltonian and/or its Gibbs free energy, and the minimization of those with respect to the unknown fields. The joint Hamiltonian is comprised of an instrument description \(\mathcal{H}(\phi, \theta)\), a field prior \(\mathcal{H}(\phi|\theta)\), and hyper-prior \(\mathcal{H}(\theta)\) of all the remaining unknowns \(\theta\) (field or noise power spectra, calibration parameters of the response, etc.). Thus, imaging of very different instruments can be brought into a unified description, in which even the data of different instruments can be imaged jointly. To this end, a Universal Bayesian Imaging toolKit (UBIK) is under development, which will permit imaging based on multi-instrument data of fields living over multi-dimensional spaces with spatial, spectral, and/or temporal dimensions.

Besides the already described IFD development, the **inference of dynamical fields** including their unknown dynamics from data is a relevant research direction with promising first results [25]. Observations of an evolving field might be sufficient to determine the dynamical laws the field obeys, which then can be used to better estimate and predict the field from limited observational data.

To summarize, IFT seems to be applicable within all areas in which fields have to be handled under uncertainty. It is the appropriate language for field inference from imperfect data. Currently it is mostly used in astrophysics and cosmology, however, its potential for geophysics, medical imaging, and other areas of remote and non-invasive sensing should be obvious.

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