A LOWER BOUND FOR DEPTHS OF POWERS OF EDGE IDEALS

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Abstract. Let $G$ be a graph and let $I$ be the edge ideal of $G$. Our main results in this article provide lower bounds for the depth of the first three powers of $I$ in terms of the diameter of $G$. More precisely, we show that

$$\text{depth } R/I^t \geq \left\lceil \frac{d-t+5}{3} \right\rceil + p - 1,$$

where $d$ is the diameter of $G$, $p$ is the number of connected components of $G$ and $1 \leq t \leq 3$. As an application of our result we obtain the corresponding lower bounds for the Stanley depth of the first three powers of $I$.

1. Introduction

Let $R$ be either a Noetherian local ring or a standard graded $k$-algebra, where $k$ is a field. Let $I$ be an ideal in $R$ and when $R$ is graded assume that $I$ is a graded ideal. Let $d = \dim R$. A classical result by Burch [4], which was improved by Broadmann [1], states that

$$\lim_{t \to \infty} \text{depth } R/I^t \leq d - \ell(I),$$

where $\ell(I)$ is the analytic spread of $I$. Eisenbud and Huneke showed that the equality holds if the associated graded ring $\text{gr}_R(I) = \bigoplus_{i=0}^{\infty} I^i/I^{i+1}$ of $I$ is Cohen–Macaulay [9]. Therefore the limiting behavior of the depth is well understood. However the initial behavior of the depth of powers is still mysterious. Thus it is natural to investigate lower bounds for depth $R/I^t$.

In the case of monomial ideals, lower bounds for the depth of the first power, depth $R/I$, have been studied extensively [13, 14, 19, 21]. Herzog and Hibi determined that depth $R/I^t$ is a non–increasing function if all the powers of $I$ have a linear resolution [18]. They also obtained lower bounds for depth $R/I^t$ if all the powers of $I$ have linear quotients, a condition that implies that all the powers of $I$ have linear resolutions [18]. In particular, they showed that all edge ideals associated to a finite graph whose complementary graph is chordal have linear quotients. Also, if $I$ is a square-free Veronese ideal (which includes the class of complete graphs) then all powers of $I$ have linear quotients. However, in general edge ideals and their powers do not have linear resolutions. It is known that depth $R/I^t$ will not necessarily be a non–increasing function for square-free monomial ideals (see [20, Theorem 13]), but the question is still open for edge ideals of graphs.

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Another motivation for studying lower bounds for depth $R/I^t$ is the fact that these lower bounds provide upper bounds for projdim$_R R/I^t$, the projective dimension of $R/I^t$. When $I$ is the edge ideal of a graph then an upper bound for the projective dimension of a graph’s edge ideal provides a lower bound for the first non-zero homology group of the graph’s independence complex [24, Observation 1.2]. Moreover, when $I$ is square-free monomial, its cohomological dimension and projective dimension are equal, [10, Theorem 0.2] or [31, Corollary 4.2]. Many researchers have studied the question of finding upper bounds for the projective dimension of $R/I$ and upper bounds for the cohomological dimension, see for example [11, 12, 17, 22, 26, 27].

We now describe our setup. Let $V = \{x_1, \ldots, x_n\}$ be a set of $n$ vertices and let $G$ be a simple graph (no multiple edges, no loops) on $V$. Let $I$ be the edge ideal of $G$ in the ring $R = k[x_1, \ldots, x_n]$, where $k$ is a field. By depth $R/I^t$ we mean the maximum length of an $R/I^t$-regular sequence in $m = (x_1, \ldots, x_n)$. When $I$ is the edge ideal of a bipartite graph then depth $R/I^t \geq 1$, since $m \notin \text{Ass}(R/I^t)$, by [30, Theorem 5.9]. In a recent article, Morey gives lower bounds for the depths of all powers when $I$ is the edge ideal of a forest, [24]. We focus our interest in studying lower bounds for the depth of powers of edge ideals of graphs without any restrictions on the shape of the graph.

The article is organized as follows. In Section 2 we give the necessary definitions and relevant background. In Sections 3 and 4 we establish the main results of this article. One can improve this bound by considering the diameters of each connected component of the graph. We show that when $G$ has $p$ connected components then depth $R/I^t \geq \sum_{i=1}^{p} \left\lceil \frac{d_i + 1}{3} \right\rceil$, where $d_i$ is the diameter of the $i$-th connected component of $G$, Corollary 3.3.

We develop a series of lemmas that leads us to prove lower bounds for the second and third powers of the edge ideal of a graph. We prove that depth $R/I^2 \geq \left\lceil \frac{d-3}{3} \right\rceil + p - 1$ and depth $R/I^3 \geq \left\lceil \frac{d-7}{3} \right\rceil + p - 1$, where $I$ is the edge ideal of a graph $G$, $d$ is the diameter of $G$ and $p$ is the number of connected components of $G$, Theorems 4.3, 4.13. It is worth noting here that in order to establish the bounds for the second and third powers we need to deal with the depth of the edge ideal of a graph that potentially has loops. We provide a lower bound on the depth of the edge ideal of a graph with loops based on knowledge of the position of the loops. More precisely, we prove that when $I$ is the edge ideal of a graph with loops and $\ell$ is an integer such that there exists a vertex $u$ with $d(u, x) \geq \ell$ for all vertices $x$ for which there is a loop on $x$, then depth $R/I \geq \left\lceil \frac{\ell - 1}{3} \right\rceil$, Proposition 3.5. This result for the depth of the edge ideal of a graph with loops is of independent interest.

In Section 5 we show that by using [3, Proposition 2.6] or [29, Lemma 2.2] in place of the Depth Lemma, the results from earlier sections can be extended to provide lower bounds on the Stanley depth of the powers of $I$. In particular, in Theorem 5.3 we show that sdepth $R/I^t \geq \left\lceil \frac{d-4t+2}{4} \right\rceil + p - 1$ for $1 \leq t \leq 3$, where sdepth denotes the Stanley depth. We also make explicit the consequences of our bounds on the depths of the low powers of $I$ to other invariants, such as the projective dimension and the regularity.
2. Background

Let $V = \{x_1, \ldots, x_n\}$ be a set of $n$ vertices and let $G$ be a graph on $V = V(G)$. Let $E = E(G)$ denote the set of edges of $G$. Unless otherwise stated we will assume that $G$ is a simple graph, that is, without loops and without multiple edges. Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring, where $k$ is a field. Note that we will not distinguish between the vertices of a graph and the variables in the corresponding polynomial ring. The edge ideal $I(G)$ of a graph $G$ is defined to be the monomial ideal in the ring $R$ generated by the monomials $x_ix_j$, where $\{x_i, x_j\} \in E$. Similarly, if $I$ is a square-free monomial ideal generated in degree two, $G(I)$ is the graph associated to $I$. That is, $\{x_i, x_j\} \in E(G(I))$ if and only if $x_ix_j$ is a generator of $I$.

We now collect some useful definitions from graph theory. For algebraic definitions and background material, see [23] or [34].

**Definition 2.1.** Let $G$ be a graph, let $V = V(G) = \{x_1, \ldots, x_n\}$ and let $E = E(G)$. Then

(a) A **path** of length $r - 1$ is a set of $r$ distinct vertices $x_{i_1}, \ldots, x_{i_r}$ together with $r - 1$ edges $x_{i_j}x_{i_{j+1}}$, where $x_{i_j} \in \{x_1, \ldots, x_n\}$ and $1 \leq j \leq r - 1$.

(b) The **distance** between two vertices $u$ and $v$ is the length of the shortest path between $u$ and $v$ and is denoted $d(u, v)$.

(c) The **diameter** of a connected graph is $d(G) = \max\{d(u, v) \mid u, v \in V\}$. Therefore, if $d = d(G)$ then there exist vertices $u, v$ of $G$ with $d(u, v) = d$. In this case we say that a path of length $d$ with endpoints $u$ and $v$ realizes the **diameter** of $G$. Although technically the diameter of a disconnected graph is infinite, we will find it useful to refer to the maximum diameter of a connected component of $G$ as the **diameter** of $G$ when $G$ is disconnected.

(d) Let $u \in V$. The **neighbor set** of $u$ is the set $N(u) = \{v \in V(G) \mid \{u, v\} \in E\}$. When $N(u) = \emptyset$ then $u$ is called an **isolated vertex** and when the cardinality of $N(u)$ is one then $u$ is called a **leaf**.

(e) A **loop** in a graph $G$ is an edge both of whose endpoints are equal, that is, an edge $\{x, x\} \in E$. A loop on $x$ corresponds to a generator $x^2$ in the edge ideal, so the edge ideal of a graph with loops is no longer square-free. Note that if loops are added to a graph, the distance between two vertices is unchanged.

When dealing with general graphs, it is helpful to consider a construction that is commonly used to produce a spanning tree. Although the spanning tree produced will not be used here, nonetheless the construction yields a partition of the vertices that we will exploit.

**Notation 2.2.** Suppose $G$ is a connected graph and $u \in V(G)$. Let $X^i_G(u) = \{x \in V(G) \mid d(u, x) = i\}$. Note that $X^0_G(u) = \{u\}$ and that $i$ runs from 0 to $d$, where $d = \max\{d(u, x) \mid x \in V(G)\}$. The sets $X^i_G(u)$ form a partition of $V(G)$. Once $u$ has been fixed, we will omit $G$ and $u$ from the notation when they are clear from context. We will frequently choose $u$ to be an endpoint of a path realizing the diameter, in which case $d$ will be the diameter of $G$. When a graph $G$ is not connected, this construction can be applied to the connected component of $G$ containing $u$.

When a vertex $u$ has been fixed in $G$, we will denote the connected component of $G$ that contains $u$ by $uG$. Thus if $I$ is an edge ideal and $u$ has been fixed, then $d(uG(I))$ denotes the diameter of the connected component of $G(I)$ containing $u$. 
There are two basic facts about these sets that will prove useful in the sequel. Fix $u$ and form $X^i = X^i_2(u)$. First note that if $x \in X^i$ for $i \geq 1$, then $N(x) \cap X^{i-1}$ is nonempty since there is a path from $u$ to $x$ of length precisely $i$ by the definition of $X^i$. Also, if $u$ and $v$ are the endpoints of a path realizing the diameter, then $v \in X^d$ and if $y \in N(v)$, then $y$ is not a leaf. If $y$ were a leaf, $d(u, y) = d + 1$, a contradiction.

The next lemma is well known, see for example [24] Lemma 2.2.

**Lemma 2.3.** Let $I$ be an ideal in a polynomial ring $R$, let $x$ be an indeterminate over $R$, and let $S = R[x]$. Then depth $S/IS = \text{depth } R/I + 1$.

If $x_1$ is an isolated vertex of a graph $G$, define $R' = k[x_2, \ldots, x_n]$. Notice that all generators of $I = I(G)$ lie in $R'$ and so by abuse of notation we can consider an ideal $I' = IR'$ in the ring $R'$ generated by the edges of the graph $G$. Then by Lemma 2.3 depth $R/I = \text{depth } R'/I' + 1$. Thus we will assume graphs are initially free of isolated vertices and that all variables of $R$ divide at least one generator of $I$.

Throughout the paper we will perform operations on the ideal that correspond to the graph minors of contraction and deletion. A deletion minor is formed by removing a vertex $x$ of $G$ and deleting any edge of $G$ containing $x$. This corresponds to the ideal $(I, x)$, or more precisely the quotient ring $R/(I, x)$. The process can result in isolated vertices, which will increase the depth of the quotient ring as in Lemma 2.3. To provide clarity we will count isolated vertices separately and will require connected components of a graph to have at least two vertices. A contraction minor is formed by removing a vertex $x$ from any edge containing $x$. This corresponds to forming the ideal $(I : x)$. Note that $N(x) \subseteq (I : x)$ and so such an ideal may have variables as generators. However, if $K = (J, x_1)$ is a minimal generating set of an ideal, then $R/K \cong k[x_2, \ldots, x_n]/J$. Thus we will refer to $K$ as an edge ideal if $J$ is an edge ideal.

For clarity and ease of reference, we now state several previously known results.

**Lemma 2.4.** Let $I$ be a monomial ideal in a polynomial ring $R$ and let $M$ be a monomial in $R$. If $y$ is a variable such that $y$ does not divide $M$ and $K$ is the extension in $R$ of the image of $I$ in $R/y$, then $((I : M), y) = ((K : M), y)$.

**Proof.** See the proof of [16] Theorem 3.5. \hfill $\Box$

**Lemma 2.5.** [24] Lemma 2.10 Suppose $G$ is a graph, $I = I(G)$, $x$ is a leaf of $G$, and $y$ is the unique neighbor of $x$. Then $(I^t : xy) = I^{t-1}$ for any $t \geq 2$.

We conclude this section with an extension of the preceding lemma when $t = 2$ that will allow us to use any edge of the graph. In Section 4 we will provide a general extension of Lemma 2.5 to allow for arbitrary edges for any $t$.

**Lemma 2.6.** Let $G$ be a graph, $I = I(G)$ and $\{x, y\} \in E(G)$. Then $(I^2 : xy) = (I, E)$, where $E = \langle x_i y_j | x_i \in N(x), y_i \in N(y) \rangle$.

**Proof.** Suppose first that $a$ is a minimal generator of $(I, E)$. If $a \in I$, then $a \in (I^2 : xy)$ since $xy \in I$. Else $a = x_i y_j \in E$ and $axy = x_i x_j y_j y \in I^2$. Thus $(I, E) \subseteq (I^2 : xy)$.

Conversely, suppose $b \in (I^2 : xy)$ but $b \notin I$. Since $(I^2 : xy)$ is a monomial ideal, we may assume that $b$ is a monomial. Then $bxy \in I^2$, so $bxy = e_1 e_2 h$, where $e_i$ are
degree two monomials corresponding to edges of $G$. Since $b \notin I$, $e_2$ does not divide $b$ for $i = 1, 2$, and so without loss of generality, $x$ divides $e_1$ and $y$ divides $e_2$. Thus $e_1 = xx_i$ and $e_2 = yy_j$ for some $x_i \in N(x)$ and $y_j \in N(y)$. Thus $x_i y_j$ divides $b$ and so $b \in E \subset (I, E)$. □

Note that the ideal $(I, E)$ in Lemma 2.6 is no longer guaranteed to be square-free. If $z \in N(x) \cap N(y)$, then $z^2 \in E$. However, $(I, E)$ is still a monomial ideal, and if $z^2$ and $w^2$ are both generators of $E$, then $zw \in (I, E)$. This follows easily since $z \in N(x)$ and $w \in N(y)$.

3. The first power

As a first step toward determining the depths of $R/I^d$ for arbitrary graphs, a lower bound, similar to the one given in [24] for trees, is needed for depth $R/I$. This lower bound is generally far from sharp, however it is of a form that generalizes to higher powers. Alternate bounds for this depth, or equivalently for the projective dimension of $R/I$, exist in the literature, [6, 7, 8, 18]. However the focus here is on providing a bound that will serve as the basis for bounds on the depths of higher powers, using techniques that will extend to higher powers. We first present the main result of this section. An alternate proof has been communicated to us by Russ Woodroofe.

Theorem 3.1. Let $G$ be a connected graph and let $I = I(G)$. If there exist $u, v \in V(G)$ with $d(u, v) = d$, then depth $R/I \geq \left\lceil \frac{d+1}{3} \right\rceil$.

Proof. We proceed by induction on $d$ and on $n$, the number of vertices. Notice that for any fixed $d$, we have that $n \geq d + 1$. Since $m \notin \text{Ass}(R/I)$, then depth $R/I \geq 1$.

Note that if $d \leq 2$, then $\left\lceil \frac{d+1}{3} \right\rceil = 1$ and so the result holds. If $n = d + 1$, the graph is a path and thus the result holds by [24], Lemma 2.8. Hence we may assume $n - 1 > d \geq 3$.

Let $X^i = X^i_G(u)$ be as in Notation 2.2 and let $w \in N(v) \cap X^{d-1}$. Consider first $(I : w) = (J, N(w))$, where $J$ is the ideal corresponding to the minor $G'$ of $G$ formed by deleting the variables in $N(w)$. Since $d \geq 3$ then $X^{d-3}_{G'}(u) \neq \emptyset$. Let $z \in X^{d-3}_{G'}(u)$ and notice that $d(u, z) = d - 3$. Moreover, $w$ does not divide any generator of $(J, N(w))$. Thus $(J, N(w)) \subset R'[N(w)]$, where $R'$ is the polynomial ring formed by deleting $w \cup N(w)$. Then we have

$$\text{depth } R/(I : w) = \text{depth } R'[w, N(w)]/(J, N(w)) = \text{depth } R'[w]/J = \text{depth } R'/J + 1 \geq \left\lceil \frac{d-3+1}{3} \right\rceil + 1 = \left\lceil \frac{d+1}{3} \right\rceil$$

by induction on $n$. Next we consider $(I, w) = (K, w)$, where $K$ is the ideal of the minor $G''$ of $G$ formed by deleting $w$. If $G''$ is connected, then $d(u, v) = d$ in $G''$ and therefore depth $R/(I, w) = \text{depth } R/(K, w) \geq \left\lceil \frac{d+1}{3} \right\rceil$ by induction on $n$. If $G''$ is not connected, then there is a vertex $z \in \text{v}G''$ with $d(u, z) \geq d - 2$ and $v \notin \text{v}G''$. If $v$ is an isolated vertex, then by Lemma 2.3, we obtain depth $R/(K, w) \geq \left\lceil \frac{d-2+1}{3} \right\rceil + 1 \geq \left\lceil \frac{d+1}{3} \right\rceil$. Otherwise, $v$ is in a connected component of $G''$ that has depth at least one, so by [34], Lemma 6.2.7, we have depth $R/(K, w) \geq \left\lceil \frac{d-2+1}{3} \right\rceil + 1 \geq \left\lceil \frac{d+1}{3} \right\rceil$. In either case, depth $R/(I, w) \geq \left\lceil \frac{d+1}{3} \right\rceil$. 


Applying the Depth Lemma [2, Proposition 1.2.9] to the short exact sequence
\[ 0 \to R/(I : w) \to R/I \to R/(I, w) \to 0 \]
yields depth \( R/I \geq \left\lceil \frac{d+1}{3} \right\rceil \), as desired.

By selecting a pair of vertices \( u \) and \( v \) whose distance is maximal, we immediately obtain the following corollary.

**Corollary 3.2.** Let \( G \) be a connected graph of diameter \( d \geq 1 \) and let \( I = I(G) \). Then depth \( R/I \geq \left\lceil \frac{d+1}{3} \right\rceil \).

As an immediate corollary we extend Theorem 3.1 to graphs that are not necessarily connected.

**Corollary 3.3.** Let \( G \) be a graph with \( p \) connected components, \( I = I(G) \), and let \( d_i \) be the diameter of the \( i \)th connected component. Then depth \( R/I \geq \sum_{i=1}^{p} \left\lceil \frac{d_i+1}{3} \right\rceil \).

In particular, depth \( R/I \geq \left\lceil \frac{d+1}{3} \right\rceil + p - 1 \).

**Proof.** This follows directly from Theorem 3.1 and [34, Lemma 6.2.7].

The next corollary is an interesting result that follows from the proof of Theorem 3.1. Although the result could be used to prove the theorem above, it is difficult to obtain independently. However, it can be useful in bounding the depths of higher powers.

**Corollary 3.4.** Let \( G \) be a graph, let \( I = I(G) \), and fix \( u \in V(G) \). Let \( w \in X^\ell = X^\ell_G(u) \) for some \( 0 \leq \ell \). Then depth \( R/(I : w) \geq \left\lceil \frac{\ell}{3} \right\rceil \).

**Proof.** Let \( w \in X^\ell \). Notice that \((I : w) = (J, N(w))\), where \( J \) is the ideal corresponding to the minor \( G' \) of \( G \) formed by deleting the variables in \( N(w) \). Let \( R' \) be the polynomial ring formed by deleting \( w \) and the variables in \( N(w) \). As before we have
\[
\text{depth } R/(I : w) = \text{depth } R'[w, N(w)]/(J, N(w)) = \text{depth } R'[w]/J = \text{depth } R'/J + 1.
\]
Since the diameter of \( G' \) is at least \( \ell - 2 \), applying Theorem 3.1 yields
\[
\text{depth } R'/J + 1 \geq \left\lceil \frac{\ell - 2 + 1}{3} \right\rceil + 1 = \left\lceil \frac{\ell + 2}{3} \right\rceil
\]
and the result follows.

We conclude this section with an extension of Theorem 3.1 that gives a bound for the depth of the first power of the edge ideal of a graph with loops. This result is also of independent interest.

**Proposition 3.5.** Let \( G \) be a connected graph with loops and let \( I = I(G) \). If there exists \( u \in V(G) \) with \( d(u, x) \geq \ell \) for all \( x \) such that \( \{x, x\} \in E(G) \), then depth \( R/I \geq \left\lceil \frac{\ell+1}{2} \right\rceil \).

**Proof.** We induct on the number of loops. Let \( x \) be a variable corresponding to a vertex with a loop. Notice that \((I : x) = (I, N(x)) = (J, N(x))\), where \( J \) is the minor formed by deleting all vertices in \( N(x) \). Since \( x \in N(x) \), the number of loops
of $G(J)$ is less than the number of loops of $G$. Notice that since all deleted vertices are at least distance $\ell - 1$ from $u$, $d(u,G(J)) \geq \ell - 2$ and $d(u,z) \geq \ell$ for all loops $z$. If $uG(J)$ has no loops, then depth $R/(I:x) \geq \lceil \frac{\ell-1}{3} \rceil$, by Theorem 3.1 since $\left\lceil \frac{d(G(J))+1}{3} \right\rceil \geq \left\lceil \frac{\ell-2+1}{3} \right\rceil$. If $uG(J)$ contains a loop $y$, then $d(u,y) \geq \ell$ and hence depth $R/(I:x) \geq \lceil \frac{\ell-1}{3} \rceil$, by induction.

Now consider $(I,x) = (K,x)$, where $K$ is the minor formed by deleting $x$. Then $d(J(K)) \geq \ell - 1$ and $G(K)$ has fewer loops than $G$, so depth $R/(I,x) \geq \lceil \frac{\ell-1}{3} \rceil$, by either Theorem 3.1 or induction as above.

Applying the Depth Lemma [2, Proposition 1.2.9] to the short exact sequence

$$0 \to R/(I:x) \to R/I \to R/(I,x) \to 0,$$

completes the proof.

4. DEPTHS OF HIGHER POWERS OF EDGE IDEALS

Our main results in this section focus primarily on $I^2$ and $I^3$. Selected results are stated for all powers since our methods can extend to higher powers, particularly when one has some control over the structure of the underlying graph. The central idea of the proofs will be to apply the Depth Lemma [2, Proposition 1.2.9] to families of short exact sequences. We begin the section by introducing some notation.

We will frequently use deletion minors in the proofs, and often the minors will be formed using a collection of vertices. Let $G$ be a graph and let $I = I(G)$. For $a \in V(G)$ we let $I_a$ represent the edge ideal of the minor of $G$ formed by deleting $a$. We will refer to $I_a$ as a minor of $I$. Given a collection of vertices $y_1, \ldots, y_s$, define $I_0 = I$ and for $1 \leq i \leq s$ define $I_i$ to be the minor of $I$ formed by deleting $y_1, \ldots, y_i$. Define $R_i$ to be the corresponding polynomial ring, namely $R_i = R/(y_1, \ldots, y_i)$.

Recall that an induced graph on a subset $\{x_1, \ldots, x_r\}$ of vertices of a graph $G$ is a graph $G'$ with $V(G') = \{x_1, \ldots, x_r\}$ and $E(G') = \{\{x_i, x_j\} \in E(G) \mid x_i, x_j \in V(G')\}$.

**Lemma 4.1.** Let $G$ be a graph, with $V = V(G)$ and $I = I(G)$. Let $x_1, \ldots, x_r \in V$ be such that the induced graph on $x_1, \ldots, x_r$ is connected and fix a vertex $u$ in the connected component of $G$ containing $x_1, \ldots, x_r$. Let $\{y_1, \ldots, y_s\} \subset \bigcup_{i=1}^{s} N(x_i) \setminus \{x_1, \ldots, x_r\}$. Then there exists an ordering of the vertices $y_1, \ldots, y_s$ such that for all $i < s$, $x_1, \ldots, x_r \in_u G(I_i)$, where $I_i$ is obtained by deleting $y_1, \ldots, y_i$.

**Proof.** Using the fixed vertex $u$, form $X^i = X^i_G(u)$. Since $x_1, \ldots, x_r \in_u G$, then for each $i$, $x_i \in X^i$ for some $t$. Let $k$ be the least positive integer for which $x_i \in X^k$ for some $i$. Fix $x_q \in X^k$. Then there is a path from $u$ to $x_q$ containing precisely one vertex in $X^j$ for each $j \leq k$. Since for every $i$, $y_i \in N(x_i)$ for some $\ell$, then $y_i \in \bigcup_{j=k-1}^{d} X^j$ for all $i$. Thus at most one $y_i$ lies on the chosen path. We may reorder the variables so that $y_s$ is this vertex (if any). Then for all $i < s$, there is a path in $I_i$ from $u$ to $x_q$ and there is a path from $x_q$ to $x_i$ for all other $i$. 

Once we have ordered a collection of neighboring vertices as in Lemma 4.1 deleting the vertices in order will result in a series of graphs for which $u$ and $x_1, \ldots, x_r$ are in the same connected component, followed by a graph for which $u$
and $x_1$ might be disconnected. When $r = 1$ and $\{y_1, \ldots, y_s\} = N(x_1)$, deleting all vertices except $y_s$ will result in a graph for which $x_1$ is a leaf. The next lemma formalizes how this can be used to estimate depths. Although it will generally be used when $M = x_1$ is a single vertex or $M = x_1 \cdots x_r$ is the product of connected vertices and $\{y_1, \ldots, y_s\} = N(x_r) \setminus \{x_1, \ldots, x_{r-1}\}$, the result holds in the more general situation described here.

**Lemma 4.2.** Let $R$ be a polynomial ring over a field, $I$ an ideal, and let $M$ be a monomial in $R$. Let $\{y_1, \ldots, y_s\}$ be variables such that for all $i$, $y_i$ does not divide $M$. Let $a, b$ be two nonnegative integers. If depth $R_i/(I_{i-1}^i : My_i) \geq a$ for all $i \geq 1$ and depth $R_s/(I_s^1 : M) \geq b$, then depth $R_i/(I_i^1 : M) \geq \min\{a, b\}$ for each $i \geq 0$. In particular, depth $R/(I^1 : M) \geq \min\{a, b\}$.

**Proof.** Consider the family of short exact sequences

$$0 \to R/(I^i : My_1) \to R/(I^i : M) \to R/(I^i : M), y_1 \to 0$$

$$0 \to R_1/(I_1^1 : My_2) \to R_1/(I_1^1 : M) \to R_1/(I_1^1 : M), y_2 \to 0$$

$$0 \to R_2/(I_2^1 : My_3) \to R_2/(I_2^1 : M) \to R_2/(I_2^1 : M), y_3 \to 0$$

$$\vdots$$

$$0 \to R_{s-1}/(I_{s-1}^1 : My_s) \to R_{s-1}/(I_{s-1}^1 : M) \to R_{s-1}/(I_{s-1}^1 : M), y_s \to 0.$$

Notice that by Lemma 2.3, the right hand term of sequence $i$ is isomorphic to $R_i/(I_i^1 : M)$, which is the center term of sequence $i + 1$. Now depth $(R_i/(I_i^1 : My_i)) \geq a$ by hypothesis and $R_{s-1}/(I_{s-1}^1 : M), y_s \cong R_s/(I_s^1 : M)$, so by hypothesis, depth $(R_{s-1}/(I_{s-1}^1 : M), y_s) \geq b$. By applying the Depth Lemma [2] Proposition 1.2.9 repeatedly starting with the final sequence and working our way up we see that depth $R_i/(I_i^1 : M) \geq \min\{a, b\}$ for each $i$ from $i = s - 1$ to $i = 0$. Since depth $R_s/(I_s^1 : M) \geq b$, the result holds for all $i$.

The next theorem establishes a lower bound for the depth of the second power of an edge ideal.

**Theorem 4.3.** Let $G$ be a graph with $p$ connected components, $I = I(G)$, and let $d = d(G)$ be the diameter of $G$. Then

$$\text{depth } R/I^2 \geq \left\lfloor \frac{d - 3}{3} \right\rfloor + p - 1.$$

**Proof.** We proceed by induction on $d$ and on $n$, the number of vertices in $G$. Suppose $n \leq 4$. Then $d \leq 3$ and $p \leq 2$ since the number of connected components does not include isolated vertices. If $p = 1$ the bound is trivial. If $p = 2$, for $n \leq 4$ the graph must be a forest consisting of two disconnected edges and the result follows from [24] Theorem 3.4. Note that in general, if $p \geq 2$, then depth $R/I^2 \geq 1$ by [2] Lemma 2.1 since each component of $I$ is square-free.

Let $u, v$ be the endpoints of a path that realizes the diameter and let $X^i = X^i_G(u)$. Let $w \in N(v)$ and let $\{y_1, \ldots, y_s\} = N(w)$ be ordered as in Lemma 2.1 so that $d(u, w)$ is finite in $I_i$ for $i < s$. Recall that $I_0 = I$. Then for each $1 \leq i \leq s$ we have $(I_{i-1}^2 : wy_i) = (I_{i-1}, E_{i-1})$, where $E_{i-1}$ is as in Lemma 2.0. Now $(I_{i-1}, E_{i-1})$ is the edge ideal of a graph, possibly with loops, of diameter at least $d - 1$ since $d(u, w) \geq d - 1$ even with the additional edges. Thus if $(I_{i-1}, E_{i-1})$ is square-free, depth $R_{i-1}/(I_{i-1}, E_{i-1}) \geq \left\lfloor \frac{d - 1 + 1}{3} \right\rfloor + p - 1$ by Corollary 3.3. If $(I_{i-1}, E_{i-1})$ is not
square-free, then there exists $x \in V(G)$ such that $x^2 \in E_{i-1}$. Now $x \in N(w)$, and so $d(u, x) \geq d - 2$. Note that each connected component of $G((I_{i-1}, E_{i-1}))$ other than $w, G((I_{i-1}, E_{i-1}))$ will be square-free, and so have depth at least one. Thus combining [34, Lemma 6.2.7] with Proposition 3.5 yields depth $R_{i-1}/(I_{i-1}, E_{i-1}) \geq \left\lceil \frac{d - 2}{3} \right\rceil + p - 1$ for $i \leq s$.

Now $w$ is isolated in $I_s$, so $(I^2_w : w) = I^2_s$ and $w$ is a free variable in $R_s/I^2_s$. Since $d_w G((I_s)) \geq d - 3$, then by induction and Lemma 2.3 we have depth $R_s/(I^2_s : w) \geq \left\lceil \frac{d - 3}{3} \right\rceil + p - 1 + 1 = \left\lceil \frac{d - 2}{3} \right\rceil + p - 1$. Hence by Lemma 4.2 we obtain depth $R/(I^2 : w) \geq \left\lceil \frac{d - 2}{3} \right\rceil + p - 1$.

Finally, consider $(I^2, w) = (I^2_s, w)$. If $v \in G(I_s)$, then $d(v, G(I_s)) \geq d$ and depth $R/(I^2, w) = depth R_s/I^2_s \geq \left\lceil \frac{d - 3}{3} \right\rceil + p - 1$ by induction on $n$. Otherwise $d(v, G(I_s)) \geq d - 2$ and $G(I_s)$ contains an additional connected component or an isolated vertex, so

$$\text{depth } R/(I^2, w) = \text{depth } R/(I^2_s, w) \geq \left\lceil \frac{d - 2 - 3}{3} \right\rceil + (p + 1) = \left\lceil \frac{d - 2}{3} \right\rceil + p - 1.$$

By applying the Depth Lemma [2, Proposition 1.2.9] to the following exact sequence

$$0 \to R/(I^2 : w) \to R/I^2 \to R/(I^2, w) \to 0$$

we see that depth $R/I^2 \geq \left\lceil \frac{d - 2}{3} \right\rceil + p - 1$ as desired.

\[\square\]

**Remark 4.4.** Notice that in the proof of Theorem 4.3 we required that $u$ and $v$ were endpoints of a path that realized the diameter. This was done in order to obtain the best possible lower bound for the depth of $R/I^2$. However, one may take $u$ and $v$ to be endpoints of any path of length $\ell = d(u, v)$. Then continuing as in the proof of Theorem 4.3 we would obtain that depth $R/I^2 \geq \left\lceil \frac{\ell - 3}{3} \right\rceil + p - 1$. Although this is a weaker lower bound, it can be useful in a more general setting.

As with the proof of Theorem 3.1 the proof of Theorem 4.3 yields the following interesting corollary.

**Corollary 4.5.** Let $G$ be a graph and let $I = I(G)$. Fix $u \in V(G)$ and let $w \in X^\ell = X^\ell_G(u)$ for some $0 \leq \ell$. Then depth $R/(I^2 : w) \geq \left\lceil \frac{\ell - 2}{3} \right\rceil$.

**Proof.** Let $\{y_1, \ldots, y_s\} = N(w)$ be ordered as in Lemma 4.1. As in the proof of Theorem 4.3 for each $1 \leq i \leq s$ we have $(I^2_{i-1} : wy_i) = (I_{i-1}, E_{i-1})$ as in Lemma 2.6 and $(I_{i-1}, E_{i-1})$ is the edge ideal of a graph of diameter at least $\ell$ since $d(u, w) = \ell$. Thus if $(I_{i-1}, E_{i-1})$ is square-free, depth $R_{i-1}/(I_{i-1}, E_{i-1}) \geq \left\lceil \frac{\ell - 1}{3} \right\rceil + p - 1$ by Corollary 4.3. If $x^2 \in E_{i-1}$, then $d(u, x) \geq \ell - 1$ so combining [34, Lemma 6.2.7] with Proposition 3.5 yields depth $R_{i-1}/(I_{i-1}, E_{i-1}) \geq \left\lceil \frac{\ell - 2}{3} \right\rceil$ for $i \leq s$.

Now $w$ is isolated in $I_s$, and $d_w G(I_s) \geq \ell - 2$, so by Lemma 2.3 and Theorem 4.3

$$\text{depth } R_s/(I^2_s : w) = \text{depth } R_s/I^2_s + 1 \geq \left\lceil \frac{\ell - 2 - 3}{3} \right\rceil + 1 = \left\lceil \frac{\ell - 2}{3} \right\rceil.$$

Hence by Lemma 4.2 we have depth $R/I^2 : w) \geq \left\lceil \frac{\ell - 2}{3} \right\rceil$.

\[\square\]

When exhausting the neighbors as in Lemma 4.2 we might end up with disconnected graphs. If the vertex $w$ is not in the connected component containing $u$, and
thus is not in \( X^i \) for any \( i \), the bound above needs to be modified, but can still be found using only the diameter of \( sG(I) \).

**Lemma 4.6.** Let \( G \) be a graph and let \( I = I(G) \). Fix \( u \in V(G) \) and let \( w \in V(G) \) such that \( w \notin uG \). Then depth \( R/(I^2 : w) \geq \left\lceil \frac{d}{2} \right\rceil \), where \( \ell = d(uG) \).

**Proof.** Suppose \( I = (J, K) \), where \( K = I(uG) \). Let \( \{z_1, \ldots, z_s\} \) be the neighbors of \( w \) ordered as in Lemma 4.1. Note that \( I_i = (J_i, K_i) \) and \( J_i = J \) for all \( i \). As in Lemma 2.6 we have \( (I_{i-1}^2 : w_{z_i}) = (I_{i-1}, E_{i-1}) \), where all the edges in \( E_{i-1} \) have endpoints in \( V(uG) \). Recall that \( R_{i-1} \) is the polynomial ring corresponding to \( I_{i-1} \) and let \( R'_{i-1} \) be the polynomial ring with variables corresponding to \( V(G(I_{i-1})) \). Then depth \( R_{i-1}/(I_{i-1}, E_{i-1}) \) \( \geq \) depth \( R'_{i-1}/J_{i-1} \) \( \geq \left\lceil \frac{d_{i-1}}{2} \right\rceil \), by \[34\] Lemma 6.2.7 and Theorem 3.1. Finally, \( w \) is an isolated vertex in \( I_i \), so \( (I^2 : w) = I_s^2 \) and \( w \) is a free variable. Thus depth \( R_s/(I^2 : w) \) \( \geq \) depth \( R_s/I_s^2 + 1 = \left\lceil \frac{d-3}{2} \right\rceil + 1 = \left\lceil \frac{d}{2} \right\rceil \). The result then follows from Lemma 4.2.

The lower bound for the depth of the first power of edge ideals that we obtained in Theorem 3.1 is realized by edge ideals of paths as was shown in \[24\] Lemma 2.8. Therefore, one cannot hope for any improvement of this bound for a general graph in terms of the invariants used. However, the lower bound for the depth of higher powers of edge ideals of paths given in \[24\] Proposition 3.2 is too high for general graphs. The next example shows that the bound we established in Theorem 4.3 is indeed attained, thus establishing that one cannot improve this bound in terms of the invariants used.

**Example 4.7.** Let \( R = k[x_1, \ldots, x_5] \) and let \( I \) be the edge ideal of the graph \( G \) below

![Graph](image)

Then \( d(G) = 3 \) and using Macaulay 2 \[15\] we have that depth \( R/I^2 = \left\lceil \frac{d-3}{2} \right\rceil = 0 \). Therefore, the bound in Theorem 4.3 is sharp.

We now prove a series of lemmas that will allow us to establish a bound for the depth of the third power.

**Lemma 4.8.** Let \( G \) be a graph and let \( I = I(G) \). Suppose \( I = (J, y_1y_2, y_2y_3, y_1y_3) \), where \( J \subset k[x_1, \ldots, x_n] \). Then depth \( R/(I^3 : y_1y_2y_3) \) \( \geq \left\lceil \frac{d(J)-3}{3} \right\rceil \), where \( d(J) = d(G(J)) \).

**Proof.** Let \( M = y_1y_2y_3 \) and consider the family of short exact sequences

\[
\begin{align*}
0 \to & R/(I^3 : My_1) \to R/(I^3 : M) \to R/(I^3 : My_1) \to 0 \\
0 \to & R/(I^3 : M, y_1) : y_2) \to R/(I^3 : My_1) \to R/(I^3 : My_1, y_2) \to 0 \\
0 \to & R/(I^3 : M, y_1, y_2) : y_3) \to R/(I^3 : M, y_1, y_2) \to R/(I^3 : My_1, y_2, y_3) \to 0.
\end{align*}
\]
We now explore the various terms of these sequences. Notice that \((I^3 : M y_1) = (I, y_2^2, y_3^2) = (J, y_1 y_2, y_2 y_3, y_1 y_3, y_2^2, y_3^2)\), and so by \([34] \text{Lemma 6.2.7}\),
\[
\begin{align*}
depth R/(I^3 : M y_1) &= \depth k[x_1, \ldots, x_n]/J + \depth k[y_1, y_2, y_3]/(y_1 y_2, y_2 y_3, y_1 y_3, y_2^2, y_3^2) \\
&= \depth k[x_1, \ldots, x_n]/J,
\end{align*}
\]
where the last equality follows since \(m \in \operatorname{Ass}(k[y_1, y_2, y_3]/(y_1 y_2, y_2 y_3, y_1 y_3, y_2^2, y_3^2))\) (see for instance \([25] \text{Corollary 4.14}\)). By Theorem \([34] \text{Theorem 3.4}\),
\[
\depth k[x_1, \ldots, x_n]/J \geq \left\lfloor \frac{d(J) + 1}{3} \right\rfloor.
\]

By Lemma \([24] \text{Lemma 2.3}\) and some straightforward computations we have
\[(I^3 : M, y_1) : y_2) = ((I^3 : M y_2), y_1) = (I, y_1, y_3^2) = (J, y_2 y_3, y_1, y_3^2) .
\]

As before,
\[
\depth R/((I^3 : M, y_1) : y_2) = \depth k[x_1, \ldots, x_n]/J \geq \left\lfloor \frac{d(J) + 1}{3} \right\rfloor.
\]

Similarly, \(((I^3 : M), y_1, y_2) : y_3) = ((I^3 : M y_1), y_1, y_2) = (I, y_1, y_2) = (J, y_1, y_2)\) and thus
\[
\begin{align*}
\depth R/(((I^3 : M), y_1, y_2) : y_3) &= \depth k[x_1, \ldots, x_n]/J + 1 \\
&> \depth k[x_1, \ldots, x_n]/J \geq \left\lfloor \frac{d(J) + 1}{3} \right\rfloor .
\end{align*}
\]

Finally, \(((I^3 : M), y_1, y_2, y_3) = (J^2, y_1, y_2, y_3)\), and thus
\[
\depth R/((I^3 : M), y_1, y_2, y_3) \geq \left\lfloor \frac{d(J) - 3}{3} \right\rfloor .
\]

by Theorem \([43] \text{Theorem 4.3}\). The result now follows from repeated applications of the Depth Lemma \([2] \text{Proposition 1.2.9}\). \(\square\)

**Lemma 4.9.** Let \(G\) be a graph and let \(I = I(G)\). Let \(u, x_1, \ldots, x_4 \in V(G)\) and suppose that \(x_1 x_2 x_3 x_4 \in I^2\) and that for some \(0 \leq \ell \leq d\) we have \(x_i \in \bigcup_{j=\ell}^d X^j\) for all \(i\), where \(X^j = X^j_G(u)\). Then \(\depth R/(I^3 : x_1 x_2 x_3 x_4) \geq \left\lfloor \frac{d + 2}{3} \right\rfloor \).

**Proof.** Notice that \((I^3 : x_1 x_2 x_3 x_4) = (I, E)\), where \(E\) is the ideal generated by all degree two monomials \(y_1 y_2\) supported on \(\bigcup_{i=1}^d N(x_i)\) satisfying \(y_1 y_2 x_1 x_2 x_3 x_4 \in I^3\).

To see this, suppose \(a \in (I^3 : x_1 x_2 x_3 x_4)\) is a monomial such that \(a \notin I\). Then \(ax_1 x_2 x_3 x_4 = e_1 e_2 e_3 h\) for some generators \(e_i\) of \(I\) and some monomial \(h\). Since \(a \notin I\), without loss of generality \(e_1 = x_1 a_1, e_2 = x_2 a_2, e_3 = x_3 a_3\), where \(a_i \in N(x_i)\).

It may happen that \(a_i = x_4\) for some \(i\), say \(i = 3\). But then \(a_1 a_2\) divides \(a\) and so \(a \in E \subseteq (I, E)\). The other inclusion is clear, so \((I^3 : x_1 x_2 x_3 x_4) = (I, E)\).

Let \(G'\) be the graph, possibly with loops, associated to \((I, E)\). Notice that \(X^j_G(u) = X^j_{G'}(u)\) for \(i \leq \ell - 1\) since both endpoints of any generator of \(E\) lie in \(\bigcup_{i=\ell-1}^d X^j\). This also implies that all loops of \(G'\) are contained in \(\bigcup_{i=\ell-1}^d X^j\). So by
Proposition 3.5 we have depth $R/(I, E) \geq \left\lceil \frac{t-2}{3} \right\rceil$.

The lemma above is an extension of Lemma 2.6. This can be further extended to allow for arbitrary powers of $I$.

**Remark 4.10.** Let $G$ be a graph and let $I = I(G)$. Let $t \geq 1$ be an integer and let $u, x_1, \ldots, x_{2t} \in V(G)$ with $x_1 \cdots x_{2t} \in I^t$. Then $I^{t+1} : x_1 \cdots x_{2t} = (I, E)$, where $E$ is the ideal generated by all degree two monomials $y_1y_2$ supported on $2t$ $N(x_i)$ satisfying $y_1y_2x_1 \cdots x_{2t} \in I^{t+1}$. The proof of this follows as in Lemma 2.6 and Lemma 4.9. Furthermore, as in Lemma 4.9 if $x_i \in \bigcup_{j=\ell}^d X^j$ for all $i$, where $X^j = X_G^j(u)$ then depth $R/(I, E) \geq \left\lceil \frac{t-2}{3} \right\rceil$.

We now return to our computations concerning the depths of various ideals involving the third power of an edge ideal.

**Lemma 4.11.** Let $G$ be a graph and let $I = I(G)$. Let $u, x_1, x_2, x_3 \in V(G)$ and suppose that that $x_1, x_3 \in N(x_2)$ and $x_1, x_2, x_3 \in \bigcup_{i=1}^d X^i$, where $X^i = X_G^i(u)$ for some $0 \leq \ell \leq d$. Then depth $R/(I^3 : x_1x_2x_3) \geq \left\lceil \frac{t-3}{3} \right\rceil$.

**Proof.** We may assume $\ell \geq 6$ since otherwise the bound is trivial. First suppose $x_3$ is a leaf. Then $(I^3 : x_1x_2x_3) = (I^2 : x_1)$ and by Corollary 4.5 we have depth $R/(I^3 : x_1x_2x_3) = \text{depth } R/(I^2 : x_1) \geq \left\lceil \frac{t-2}{3} \right\rceil$.

Suppose $x_3$ is not a leaf. We consider two cases. If $x_1x_3$ is a generator of $I$, let $\{z_1, \ldots, z_s\} = N(x_1) \cup N(x_2) \cup N(x_3) \setminus \{x_1, x_2, x_3\}$. If $x_1x_3$ is not a generator of $I$, let $\{z_1, \ldots, z_s\} = N(x_3) \setminus \{x_2\}$.

In either case, order the vertices $z_1, \ldots, z_s$ as in Lemma 4.1. Then by considering $\nu G(I_{r-1})$, we have depth $R_{r-1}/(I^3_{r-1} : x_1x_2x_3z_i) \geq \left\lceil \frac{t-3}{3} \right\rceil$ by Lemma 4.9 since $z_i \in \bigcup_{i=\ell}^d X^i$. If $x_1x_3 \in I$, then by Lemma 4.8 we have that depth $R_{r}/(I^3_r : x_1x_2x_3) \geq \left\lceil \frac{d(I_r) - 3}{3} \right\rceil \geq \left\lceil \frac{t-3}{3} \right\rceil$, since $d(\nu G(I_r)) \geq \ell - 2$. When $x_1x_3 \not\in I$ then $x_3$ is a leaf in $I_s$, so as above, $(I^3_s : x_1x_2x_3) = (I^2_s : x_1)$. If $I_s$ is disconnected, then $d(\nu G(I_s)) \geq \ell - 2$. This is also true of $I_s$ when $\nu G(I_s)$ is connected, we obtain depth $R_s/(I^3_s : x_1x_2x_3) \geq \left\lceil \frac{t-3}{3} \right\rceil$. In either case, applying Lemma 4.2 yields depth $R/(I^3 : x_1x_2x_3) \geq \left\lceil \frac{t-3}{3} \right\rceil$.

**Lemma 4.12.** Let $G$ be a graph and let $I = I(G)$. Fix $u \in V(G)$ and suppose that $xy \in E(G)$ with $x \in X^\ell$, where $X^\ell = X_G^\ell(u)$ for some $0 \leq \ell \leq d$. Then depth $R/(I^3 : xy) \geq \left\lceil \frac{t-2}{3} \right\rceil$.

**Proof.** First suppose either $x$ or $y$ is a leaf of $G$. Then by Lemma 2.5 we have that $(I^3 : xy) = I^2$ and by Theorem 4.3 we obtain depth $R/(I^3 : xy) \geq \left\lceil \frac{d(I) - 3}{3} \right\rceil + p(I) - 1$. Since $d(I) \geq \ell$, the result follows.

Next we assume that neither $x$ nor $y$ is a leaf of $G$. Let $\{z_1, \ldots, z_s\} = N(x) \setminus \{y\}$ be ordered as in Lemma 4.1. Then depth $R_{r-1}/(I^3_{r-1} : xyz_i) \geq \left\lceil \frac{t-4}{3} \right\rceil$, by Lemma 4.11 since $x, y, z_i \in \bigcup_{j=\ell-1}^d X^j$. Now $x$ is a leaf of $I_s$, so $I^3_s : xy = I^2_s$, by
Lemma [2.5] Let \( d(I_s) = d(uG(I_s)) \). Then since \( z_i \in \bigcup_{j=1}^{d} X^j \), we have \( d(I_s) \geq \ell - 2 \).

Thus \( \text{depth } R_s/(I_s^3 : xy) = \text{depth } R_s/I_s^2 \geq \left\lceil \frac{d - 7}{3} \right\rceil \) by Theorem 4.3. Hence by Lemma 4.2 we have \( \text{depth } R/(I^3 : xy) \geq \left\lceil \frac{d - 7}{3} \right\rceil \). \( \boxdot \)

We are now ready to establish a bound for the depth of the third power of an edge ideal.

**Theorem 4.13.** Let \( G \) be a graph with \( p \) connected components, \( I = I(G) \), and let \( d = d(G) \) be the diameter of \( G \). Then \( \text{depth } R/I^3 \geq \left\lceil \frac{d - 7}{3} \right\rceil + p - 1 \).

**Proof.** We proceed by induction on \( d \) and on \( n \), the number of vertices. Notice that if \( d \leq 7 \), the result is trivial. So we may assume \( d \geq 8 \). For any fixed \( d \), we have \( n \geq d + 1 \). If \( n = d + 1 \), the graph is a path and the result follows from [24, Theorem 3.4].

Let \( u, v \) be the endpoints of a path that realizes the diameter of \( G \) and let \( X^i \) be as in Notation 4.2. Let \( w \in N(v) \cap X^{d-1} \).

Notice that \( (I^3, w) = (J^3, w) \), where \( J \) is the minor of \( I \) formed by deleting \( w \). We have two cases to consider. If \( u \) and \( v \) are in the same connected component of \( J \) then \( d(J) \geq d \) and \( p(J) \geq p \), where \( p(J) \) is the number of connected components of the graph associated to \( J \). Hence by induction on \( n \) we have

\[
\text{depth } R/(I^3, w) \geq \left\lceil \frac{d(J) - 7}{3} \right\rceil + p(J) - 1 \geq \left\lceil \frac{d - 7}{3} \right\rceil + p - 1.
\]

If \( u \) and \( v \) are not connected in \( J \), then \( d(J) \geq d(uG(J)) \geq d - 2 \) and \( p(J) \geq p + 1 \), or if \( v \) is isolated, Lemma 4.3 applies. Hence again by induction on \( n \) we have

\[
\text{depth } R/(I^3, w) \geq \left\lceil \frac{d - 9}{3} \right\rceil + p + 1 - 1 \geq \left\lceil \frac{d - 7}{3} \right\rceil + p - 1.
\]

Let \( \{z_1, \ldots, z_b\} = N(w) \) be ordered as in Lemma 4.1. Since \( w \in X^{d-1} \) then \( \text{depth } R_{d-1}/(I_{d-1}^3 : wz_i) \geq \left\lceil \frac{d - 7}{3} \right\rceil \), by Lemma 4.12.

Now \( w \) is isolated in \( I_s \) and thus \( (I_s^3 : w) = I_s^3 \). Therefore by induction on \( n \) we have that

\[
\text{depth } R_s/(I_s^3 : w) = \text{depth } R_s/I_s^3 \geq \left\lceil \frac{d(I_s) - 7}{3} \right\rceil + p(I_s) - 1 + 1 \\
\geq \left\lceil \frac{d - 3 - 7}{3} \right\rceil + p - 1 + 1 = \left\lceil \frac{d - 7}{3} \right\rceil + p - 1,
\]

since \( d(I_s) \geq d(uG(I_s)) \geq d - 3 \) and \( w \) is an isolated vertex. Hence by Lemma 4.2 we have that \( \text{depth } R/(I^3 : w) \geq \left\lceil \frac{d - 7}{3} \right\rceil + p - 1. \)

By applying the Depth Lemma [2 Proposition 1.2.9] to the following exact sequence

\[
0 \rightarrow R/(I^3 : w) \rightarrow R/I^3 \rightarrow R/(I^3, w) \rightarrow 0
\]

we have that \( \text{depth } R/I^3 \geq \left\lceil \frac{d - 7}{3} \right\rceil + p - 1. \) \( \boxdot \)

As in Remark 4.4 one may take \( u \) and \( v \) in the proof of Theorem 4.13 to be endpoints of a path of length \( \ell = d(u, v) \) and obtain \( \text{depth } R/I^3 \geq \left\lceil \frac{\ell - 2}{3} \right\rceil + p - 1. \) The next corollary follows from the proof of Theorem 4.13.
Corollary 4.14. Let $G$ be a graph and let $I = I(G)$. Fix $u \in V(G)$ and let $w \in X^\ell$ for some $0 \leq \ell$, where $X^i = X^i_G(u)$. Then depth $R/(I^3 : w) \geq \lceil \frac{\ell - 6}{3} \rceil$.

Proof. Let $\{z_1, \ldots, z_s\} = N(w)$ be ordered as in Lemma 4.1. By Lemma 4.12 we have depth $R_{i-1}/(I_{i-1}^3 : wz_i) \geq \lceil \frac{\ell - 6}{3} \rceil$.

Now $w$ is isolated in $I_s$ and thus $(I_s^3 : w) = I_s^3$ and $d(I_s) \geq \ell - 2$. Therefore by Theorem 4.13, we obtain

$$\text{depth } R_s/(I_s^3 : w) \geq \lceil \frac{d(I_s) - 7}{3} \rceil + 1 \geq \lceil \frac{\ell - 9}{3} \rceil + 1 = \lceil \frac{\ell - 6}{3} \rceil.$$ 

Hence by Lemma 4.2, the result follows.

We conclude this section with an example that shows that the bound for the depth of the third power of an edge ideal given in Theorem 4.13 is attained. This example extends naturally, which suggests a lower bound for the depth of any power.

Example 4.15. Let $R = k[x_1, \ldots, x_{10}]$ and let $I$ be the edge ideal of the graph $G$ below

\[ x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \quad x_9 \quad x_{10} \]

Then $d(G) = 7$ and using Macaulay 2 [15] we have that depth $R/I^3 = \lceil \frac{d - 7}{3} \rceil = 0$. Therefore, the bound in Theorem 4.13 is sharp.

Notice that the graph in Example 4.15 is a graph with two copies of the graph in Example 4.7 attached by an additional edge. One may attach more copies of the graph in Example 4.7 to obtain examples of graphs where depth $R/I^t = \lceil \frac{d - 4t + 5}{3} \rceil + p - 1$ for any $t \geq 1$. A natural question then arises.

Question 4.16. Let $G$ be a graph with $p$ connected components, $I = I(G)$, and let $d = d(G) \geq 1$ be the diameter of $G$. Then is it true that for all $t \geq 1$ we have that depth $R/I^t \geq \lceil \frac{d - 4t + 5}{3} \rceil + p - 1$?

5. Extensions and Applications

In this last section we show how our results and our techniques can be used to obtain bounds on projective dimension, regularity and Stanley depth.

Remark 5.1. Let $R$ be a polynomial ring over a field and let $I$ be a square-free monomial ideal in $R$ generated in degree 2. Let $G = G(I)$. Using the Auslander-Buchsbaum Formula [21, Theorem 1.3.3] one can observe that the lower bounds we obtain for the depths of the first 3 powers of $I$ immediately give upper bounds for the corresponding projective dimensions as well, namely

$$\text{projdim}_R R/I^t \leq n - \left\lceil \frac{d - 4t + 5}{3} \right\rceil - p + 1,$$

where $n = \dim R$, $d = d(G)$ is the diameter of $G$, $p$ is the number of connected components of $G$ and $1 \leq t \leq 3$. 

When $I$ is a square free monomial ideal then $\text{projdim} R/I = \text{reg}(I^\vee)$, where $I^\vee$ is the Alexander dual of $I$, [33, Corollary 0.3]. Since $I^\vee = I$ then $\text{reg}(I) = \text{projdim}(R/I^\vee) = n - \text{depth} R/I^\vee$, where $n = \dim R$. Using our result for the depth of the first power of edge ideals we immediately obtain the following bound on the regularity of $I$.

**Observation 5.2.** Let $G$ be a graph on $n$ vertices and let $I = I(G)$. Let $I^\vee$ denote the Alexander dual of $I$. Then $\text{reg}(I^\vee) \leq n - \left\lceil \frac{d(G)+1}{3} \right\rceil - p + 1$, where $d(G)$ is the diameter of $G$ and $p$ is the number of connected components of $G$. Moreover, if $I$ is an unmixed ideal of height 2 then $\text{reg}(I) \leq n - \left\lceil \frac{d(I^\vee)+1}{3} \right\rceil - p' + 1$, where $d(I^\vee)$ is the diameter of $G(I^\vee)$ and $p'$ is the number of connected components of $G(I^\vee)$.

As a final application of our results we obtain lower bounds on the Stanley depth of the first three powers of edge ideals. Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$. Let $M$ be a nonzero finitely generated $\mathbb{Z}^n$-graded $R$-module, let $u \in M$ be a homogeneous element and let $Z \subset \{x_1, \ldots, x_n\}$. Then $uk[Z]$ is the $k$-subspace generated by all monomials $uv$, where $v$ is a monomial in $k[z]$. A presentation of $M$ as a finite direct sum of such spaces $D$: $M = \bigoplus_{i=1}^{r} u_i k[Z_i]$ is called a Stanley decomposition of $M$. Let $\text{sdepth} D = \min\{|Z_i| : i = 1, \ldots, r\}$ and let $\text{sdepth} M = \max\{\text{sdepth} D : D$ is a Stanley decomposition of $M\}$. Then $\text{sdepth} M$ is called Stanley depth of $M$. It was conjectured by Stanley in [32] that $\text{sdepth} M \geq \text{depth} M$ for all $\mathbb{Z}^n$-graded modules $M$.

There has been a lot of interest in the recent years concerning this conjecture by Stanley. For the case of edge ideals of graphs and their powers we are able to obtain lower bounds for the Stanley depth using our results from the previous sections as well as the following version of the Depth Lemma for Stanley depth.

**Lemma 5.3.** [3 Proposition 2.6] Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$. Let $0 \to M \to N \to L \to 0$ be a short exact sequence of finitely generated $\mathbb{Z}^n$-graded $R$-modules. Then $\text{sdepth} N \geq \min\{\text{sdepth} M, \text{sdepth} N\}$.

**Theorem 5.4.** Let $G$ be a graph with $p$ connected components, $I = I(G)$, and let $d = d(G)$ be the diameter of $G$. Then for $1 \leq t \leq 3$ we have

$$\text{sdepth} R/I^t \geq \left\lceil \frac{d - 4t + 5}{3} \right\rceil + p - 1.$$

**Proof.** The proof follows by induction on $d$ and $n$, the number of vertices of $G$. Given Lemma 5.3 we can proceed the same way as in the proofs of Theorems 5.1 5.10 4.13 as long as we can establish the bounds for the base case of the induction, that is when $n = d + 1$ and $G$ is the graph of a path. The required bounds are known to hold for the Stanley depth, see for example [28, Theorem 2.7].

One consequence of Theorem 5.4 is that any class of ideals for which at least one of the bounds in Theorems 5.1 5.3 4.13 is an equality will correspond to a class of modules that satisfy the Stanley conjecture. Thus discovering when the bounds are achieved is an area of further interest.
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