General scheme for construction of scalar separability criteria from positive maps.

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We present a general scheme allowing for construction of scalar separability criteria from positive but not completely positive maps. The concept is based on a decomposition of every positive map \( \Lambda \) acting on \( M_d(\mathbb{C}) \) into a difference of two completely positive maps \( \Lambda_1, \Lambda_2 \), i.e., \( \Lambda = \Lambda_1 - \Lambda_2 \). The scheme may also be treated as a generalization of the known entropic inequalities, which are obtained from the reduction map. Analyses performed on a few classes of states show that the new scalar criteria are stronger than the entropic inequalities and when derived from indecomposable maps allow for detection of bound entanglement.

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Introduction.- Though the theory of entanglement [1] has been developed for many years, it is still an open problem how to unambiguously determine whether a given quantum state is separable or not. Recall that we call a given bipartite state separable iff it can be written as a convex combination of product states [2], i.e.,

\[ \rho_{\text{sep}} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}, \quad \sum_i p_i = 1, \]

otherwise it is called entangled. Among the known criteria allowing for the detection of entanglement in bipartite quantum systems, one of the most common is based on positive but not completely positive (CP) maps [3]. It states that a given density matrix \( \rho \) is separable if and only if the operator \( X_\Lambda(\rho) = [I \otimes \Lambda](\rho) \) is positive for all positive maps \( \Lambda \) (here, \( I \) denotes an identity map). An important example of such a map is the transposition map [3, 4]. Among others, the so-called indecomposable [5] maps are of special interest, since they can detect bound entanglement [6]. However, so far only few examples of indecomposable maps are known (see, e.g., [7, 8, 9, 10, 11, 12, 13, 14, 15]). The positive maps criterion is a structural one. To apply it one must know the positive of a state.

Of interest are the criteria that can be directly applied in experiment. One of the best known are the entropic inequalities [16, 17, 18] saying that the entropy of the subsystems of a separable state cannot exceed the entropy of the full system. Mathematically it can be expressed as

\[ S_\alpha(\rho_A) \leq S_\alpha(\rho) \quad \text{and} \quad S_\alpha(\rho_B) \leq S_\alpha(\rho) \quad (\alpha \in [0, \infty]), \]

where \( \rho_{A(B)} = Tr_{B(A)}(\rho) \) and as \( S_\alpha \) one may choose the Renyi entropy \( S_\alpha(\rho) = \ln Tr\rho^\alpha/(1 - \alpha) \). Simple calculations lead to a short form of the above inequalities, i.e.,

\[ Tr\rho^\alpha(\rho_{\text{A(B)}}) \geq Tr\rho^\alpha \quad (\alpha \in (1, \infty)) \quad \text{and} \quad Tr\rho^\alpha(\rho_{\text{A(B)}}) \leq Tr\rho^\alpha \quad (\alpha \in [0, 1]). \]

These do not constitute strong separability criteria, for instance cannot detect bound entanglement, since they are implied by the reduction criterion [17]. Still, their experimental realization for \( \alpha = 2 \) is within reach of the existing technology [19].

Interesting questions arise here. Is it possible to derive inequalities such as entropic ones from any positive map not only from the reduction one? Moreover, would such inequalities detect entanglement more efficiently and in particular could detect bound entanglement? Recently, it was shown in Ref. [20] that imposing some conditions on a density matrix and utilizing an extended reduction criterion [14, 15] one may derive some entropic-like inequalities, stronger than the entropic ones. In particular, the inequalities can detect bound entanglement.

Here we provide alternative simple construction allowing for derivation of scalar separability criteria from any positive, not CP map. Surprisingly, for many positive maps the inequalities can be derived, unlike in [20], for all states, without any assumptions on commutativity.

Inequalities.- Let \( \Lambda \) be a positive but not completely positive map. A corresponding necessary separability criterion states that if a given density matrix is separable, i.e., of the form \( \rho = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \), then \( [I \otimes \Lambda]\rho_{\text{sep}} \geq 0 \). Applying the fact that every positive but not CP map \( \Lambda \) defined on \( M_d(\mathbb{C}) \) can be written as a difference of two CP maps \( \Lambda_1 \) and \( \Lambda_2 \) (see, e.g., Refs. [21, 22], i.e.,

\[ \Lambda = \Lambda_1 - \Lambda_2, \]

we can rewrite the separability criterion in the form of the following operator inequality

\[ [I \otimes \Lambda_1](\rho_{\text{sep}}) \geq [I \otimes \Lambda_2](\rho_{\text{sep}}). \]

Now, we are prepared to show the aforementioned inequalities, which may be treated as a generalization of the standard entropic inequalities.

Theorem. Let \( \rho \) be a separable state and \( \Lambda \) a positive but not completely positive map that can be written as in Eq. (2). Then the following implications hold:

(i) If \( [I \otimes \Lambda_2](\rho) = 0 \) and \( \alpha \geq 0, \beta > 1 \) then

\[ Tr\rho^\alpha([I \otimes \Lambda_1](\rho))^\beta \geq Tr\rho^\alpha([I \otimes \Lambda_2](\rho))^\beta. \]

(ii) If \( \alpha \geq 0 \) and \( 0 \leq \beta \leq 1 \) then

\[ Tr\rho^\alpha([I \otimes \Lambda_1](\rho))^\beta \geq Tr\rho^\alpha([I \otimes \Lambda_2](\rho))^\beta. \]

(iii) If \( \alpha \geq 0 \) and \( \beta \in [-1, 0) \) then

\[ Tr\rho^\alpha([I \otimes \Lambda_1](\rho))^\beta \leq Tr\rho^\alpha([I \otimes \Lambda_2](\rho))^\beta. \]
(iv) If \( \alpha, \beta \geq 0 \) then
\[
\text{Tr} \rho^\alpha \left( \left[ I \otimes \Lambda_1 \right](\rho) \right)^\beta \geq \text{Tr} \left( \Sigma_1(\rho)^\alpha \Sigma_1(\left[ I \otimes \Lambda_2 \right](\rho)) \right)^\beta.
\] (7)

**Proof.** (i) Assuming that a given \( \rho \) is separable, the inequality \((9)\) is satisfied. Following Ref. [17] we have
\[
\text{Tr} \rho^\alpha \left( \left[ I \otimes \Lambda_1 \right](\rho) \right)^\beta = \text{Tr} \left[ e^{\ln e^\rho + \ln(I \otimes \Lambda_1)(\rho)} \right]^\beta
\geq \text{Tr} \left[ e^{\ln e^\rho + \ln(I \otimes \Lambda_1)(\rho)} \right]^\beta
\geq \text{Tr} \rho^\alpha \left( \left[ I \otimes \Lambda_2 \right](\rho) \right)^\beta.
\] (8)

Then, using the commutativity assumption we obtain
\[
\text{Tr} \rho^\alpha \left( \left[ I \otimes \Lambda_1 \right](\rho) \right)^\beta \geq \text{Tr} \rho^\alpha \left( \left[ I \otimes \Lambda_2 \right](\rho) \right)^\beta,
\] (9)

which finishes the first part of the proof.

(ii)-(iii) It suffices to use the fact that the operator function \( f(A) = A^r \) is monotonically increasing for \( r \in [0,1] \), and monotonically decreasing for \( r \in [-1,0] \) (cf. [23]).

(iv) We start from the fact that for any \( A, B \in M_d(\mathbb{C}) \) one has \( e^{A+B} = \lim_{m \to \infty} (e^{A/2m})^m (e^{B/2m})^m \) (cf. [23]). This, due to the continuity of the trace, implies
\[
\text{Tr} \rho^\alpha \left( \left[ I \otimes \Lambda_2 \right](\rho) \right)^\beta = \lim_{m \to \infty} \text{Tr} \left[ \rho^{\alpha/2m} \left( \left[ I \otimes \Lambda_2 \right](\rho) \right)^{\beta/m} \right]^m
\] (10)

Now, we use the following inequality [24]:
\[
\text{Tr} \left[ B^r(\sqrt{BAV})^s \right] \geq \text{Tr} \left[ \left( \Sigma_1(A)^s \Sigma_1(B)^r \right)^{r+s} \right],
\] (11)
satisfied by positive matrices \( A \) and \( B \), and \( r, s \geq 0 \). Here \( \Sigma_1(X) = \text{diag}[\sigma_1, \ldots, \sigma_n] \) and \( \Sigma_1(X) = \text{diag}[\sigma_n, \ldots, \sigma_1] \), where \( \sigma_1 \geq \ldots \geq \sigma_n \) are singular values of \( X \), i.e., the eigenvalues of \( \sqrt{X} \). Putting \( r = 0, A = \left[ I \otimes \Lambda_2 \right](\rho)^{\beta/m} \), and \( B = \rho^{\alpha/m} \) we arrive at the claimed inequality. \( \square \)

Remarks. It should be emphasized that though parts (ii)-(iv) of the theorem are proved without any assumptions on commutativity, there is, to our knowledge, no direct method of measurement of such functions (except for the case \( \beta=1 \) and natural \( \alpha \) included in part (ii)). Meaning that they do not lead to many-copy entanglement witnesses, whereas the inequality from part (i) of the theorem taken with integer \( \alpha \) and \( \beta \) can be experimentally checked in a collective measurement (see [20] [26]). The main drawback, however, is that to apply the criterion we need to assume that \( \left[ I \otimes \Lambda_2 \right](\rho), \rho = 0 \). Remarkably, as we will see in the examples given below, for many of the known positive maps, \( \Lambda_2 \) could be taken as an identity map. Then the commutativity assumption is trivially satisfied by every state and the inequality \((1)\) becomes
\[
\text{Tr} \rho^\alpha \left( \left[ I \otimes \Lambda_1 \right](\rho) \right)^\beta \geq \text{Tr} \rho^{\alpha+\beta} \quad (\alpha \geq 0, \beta > 1).
\] (12)

Because of their structure we shall call the inequalities from parts (i)-(iii) of the Theorem the \( (\alpha, \beta) \)-inequalities.

**Examples.** Below we provide four examples of positive maps and show that most of them have identity map as \( \Lambda_2 \) in Eq. (2), which leads to entropic-like inequalities [12].

**Example 1.** First, we consider the reduction map \( \tau^{U} \) which acts on a \( d \times d \) matrix \( X \) as \( R(X) = \left( \text{Tr} \right) X \mathbb{1}_d - X \). It can be written as a difference of CP maps \( R_1, R_2 \) taken as \( R_1(X) = \left( \text{Tr} \right) X \mathbb{1}_d \) and \( R_2(X) = X \). The complete positivity of both is obvious. Moreover, \( R_2 \) is an identity map so by the remarks given above the inequality is suitable for all states. Putting \( \alpha = 1 \) and using the fact that \( \text{Tr} \left[ \rho \otimes \mathbb{1}_d \right]^{\beta} = \text{Tr} \rho^{\alpha+\beta} \), one obtains the standard entropic inequality \( \text{Tr} \rho^\alpha \geq \text{Tr} \rho^{\beta+1} \). On the other hand, as shown in Fig. 1, inequalities of the form \((12)\) derived from the reduction map for \( \alpha > 1 \) and \( \alpha > \beta \) are stronger than the entropic ones.

**Example 2.** The second example is concerned with the modified transposition map \( \tau^{U} \) acting on \( X \) as \( \tau^{U}(X) = UT(X)U^\dagger \) with \( T \) being the known transposition map and \( U \) denoting arbitrary \( d \times d \) unitary matrix. It may be written as a difference of \( \tau^{U}(X) = (1/2)\tau^{U} \otimes \mathbb{1}_d = (1/2)\tau^{U} \circ \mathbb{1}_d \), where \( \mathbb{1}_d \) is the identity map. Both maps may be easily shown to be completely positive [27]. To check which states fulfill the assumption of the theorem we may write \( \tau^{U}(X) = (1/2)[\left( \text{Tr} \right) X \mathbb{1}_d - \tau^{U}(X)] \) and then \( \left[ I \otimes \tau^{U}(X) \right] = (1/2)[\rho_A \otimes \mathbb{1}_d + \tau^{U}(\rho)] \), which means that one needs to assume that \( \rho_A \otimes \mathbb{1}_d, \rho = [\tau^{U}(\rho), \rho] \), where \( \tau^{U}(\rho) = I \otimes \tau^{U} \). Notice that this is the only map studied here for which one needs to impose some assumptions on \( \rho \) to derive inequalities \((1)\) from it.

**Example 3.** As the third example we consider the indecomposable map introduced in Refs. [14] [15], that is \( \Lambda^{U}(X) = \left( \text{Tr} \right) X \mathbb{1}_d - \tau^{U}(X) \) and its slight modification. Here \( \tau^{U} \) is defined as in Example 2 but with \( U \) being an antisymmetric matrix \( UT = -U^\dagger \) such that \( U^\dagger \mathbb{1}_d \leq \mathbb{1}_d \). This map may be expressed as a difference of the maps \( \Lambda^U(X) = (\text{Tr}X \mathbb{1}_d - \tau^{U}(X)) \) and \( \Lambda^U(X) = X \). Note that the map \( \Lambda^U \) is the same as \( \tau^{U} \) (up to one-half factor) and thus is completely positive. Note that combining maps \( \tau^{U} \) and \( \Lambda^U \) we obtain \( 2\tau^{U}(\rho_{\text{sep}}) \geq 2\tau^{U}(\rho_{\text{sep}}) \geq \Lambda^U(\rho_{\text{sep}}) \), which leads to another map \( \Lambda^{U}(X) = (\text{Tr}X \mathbb{1}_d + \tau^{U}(X)) \). It may be easily shown that \( \Lambda^{U} \) is positive but not CP. For both maps considered in this example \( \Lambda_2 \) from Eq. (9) is an identity map. Thus, due to previous remarks the resulting inequalities can be applied to arbitrary states.

**Example 4.** Here we discuss two families of indecomposable maps analyzed in [7] [8] [9] and in [10] [11] [12]. The first class of maps \([7] [8] [9]\) acts on a \( d \times d \) matrix \( X \) as
\[
\varphi_{d,k}(X) = (d-k)\epsilon(X) + \sum_{i=1}^{k} \epsilon(S^i X S^{i\dagger}) - X,
\] (13)

where \( S \) denotes a shift operator, i.e., \( S|i\rangle = |i+1\rangle \), where addition is understood mod \( d \) for \( i = 1, \ldots, d \). Here \( \epsilon \) stands for a CP map defined as \( \epsilon(X) = \sum_{i=1}^{d} |i\rangle \langle i| X |i\rangle \langle i| \). For \( k = 1, \ldots, d-2 \) the maps \( \varphi_{d,k} \) were shown to be indecomposable [9], for \( k = 0 \) one has a CP map, \( k = d-1 \) gives a reduction map \( R \), and \( \varphi_{3,1} \) is the Choi map.
Form (2) of \( \varphi_{d,k} \) follows directly from its definition and reads \( \varphi_{d,k} = \varphi^{(1)}_{d,k} - I \), where \( \varphi^{(1)}_{d,k} \) is CP for \( k = 0, \ldots, d - 1 \) due to complete positivity of \( \epsilon \) map.

As the second class we consider the maps investigated in Ref. 12. Let \( a \) and \( c_1, \ldots, c_d \) be positive real numbers. Then the maps for \( d \times d \) matrices are defined as \( \Theta \equiv \Theta[a; c_1, \ldots, c_d] = \Theta_1[a; c_1, \ldots, c_d] - I \), where \( \Theta_1 \equiv \Theta_1[a; c_1, \ldots, c_d] \) is defined as

\[
\Theta_1(X) = a \varepsilon(X) + \text{diag}[c_d, c_1, \ldots, c_{d-1}] \varepsilon(SS\dagger).
\] (14)

For instance, in this notation \( \Theta[2; 1, 1, 1] \) is the Choi map and its further generalization studied in [11] is \( \Theta[a; c_1, c_2, c_3] \). The map \( \Theta \) is positive if \( (c_1 \ldots c_d)^{1/d} \geq d - a \) and \( a \geq d - 1 \), and indecomposable if additionally \( a < d \). Moreover, \( \Theta \) is completely positive whenever taken with nonnegative parameters \( a, c_1, \ldots, c_n \) [27].

As presented above both maps \( \varphi_{d,k} \) and \( \Theta \) have an identity map as the second CP map in Eq. (2). Therefore, again the resulting inequalities can serve as the separability criterion for all bipartite states.

Comparison. - Below we compare the inequalities of type (i) and (ii) from the Theorem, derived from some of the positive maps considered in the previous section. We use two classes of states, namely the SO(3)-invariant bipartite states (see, e.g., Refs. 28,29 and references therein for some results concerning separability properties of this class) and the class of states considered in Ref. 30.

Every SO(3)-invariant bipartite state is a convex combination of projections \( P_l \) onto common eigenspaces of the square of the total angular momentum and its \( z \) component. Here, \( J \) takes the values \( |J_A - J_B|, \ldots, J_A + J_B \), where \( j_{A(B)} \) denotes the angular momentum of the subsystem \( A(B) \). We focus on the case when \( J_A = J_B = 3/2 \).

Then an arbitrary state may be written as \( \rho(p, q, r) = p P_0 + q P_1 + r P_2 + s P_3 \), where \( p, q, r, s = 1 - p - q - r \in [0, 1] \). Entanglement for this states was characterized in Ref. 28 and can be fully described by the transposition map \( T \) and the map \( \Lambda^V \) (\( V \) is an antisymmetric and anti-diagonal unitary matrix with elements \( \pm 1 \)). Moreover, the states have maximally mixed subsystems and commute with their partial time reversal \( [I \otimes \tau^V](\rho) \). Therefore, they are appropriate to apply the inequalities derived from the reflection map \( \tau^V \).

In Fig. 1 we compare the \((\alpha, \beta)\)-inequalities of type (ii) derived from the reflection map \( \tau^V \), map \( \Lambda^V \) from Example 3, and reduction map \( R \) for integer \( \alpha \geq 1, \beta = 1 \), with the corresponding entropic inequalities (the \((\alpha = 1, \beta \geq 1)\)-inequalities derived from \( R \), with the same power \( \alpha + \beta \)). Note that here one may look at \( \alpha + \beta \) as the number of copies necessary to perform a collective measurement. As can be seen the region detected by each inequality (the region where the inequality is violated) becomes larger with the growth of parameter \( \alpha \).

In Fig. 2 we present the effectiveness of the inequalities of type (i) and (ii) derived from the map \( \Lambda^V \) for a few values of parameters \( \alpha \) and \( \beta \). One sees that the detected region becomes larger with the growth of parameter \( \alpha \). Moreover, for the same power \( \alpha + \beta \) the inequality with larger \( \alpha \) detects entanglement more effectively. It is interesting that even for small values of parameter \( \alpha + \beta \) \((\alpha = 3, \beta = 1)\) some PPT entanglement is detected.

The second class of states we consider was introduced.
TABLE I: Range of parameter $\gamma$ of the states given by Eq. (15), for which the $(\alpha, \beta)$-inequality of type (ii) is violated, versus parameters $\alpha$ and $\beta$.

| $\alpha$ | $\beta$ | Range of $\gamma$ |
|----------|---------|-------------------|
| 6        | 1       | $[3.016, 4.683]$  |
| 7        | 1       | $(3.191, 3.942)$  |
| 10       | 1       | $(3.016, 4.683)$  |
| 13       | 1       | $(3.002, 5.0)$    |
| $\infty$| 1       | $(3.0, 5.0)$      |

in Ref. [30] and is parameterized as follows

$$\sigma_\gamma = (1/7) \left[ |\psi_+^{(3)}\rangle\langle \psi_+^{(3)}| + \gamma \sigma_+ + (5 - \gamma)\sigma_- \right], \tag{15}$$

where $\gamma \in [2, 5]$, $|\psi_+^{(3)}\rangle = (1/\sqrt{3})\sum_{i=0}^{2} |ii\rangle$, $\sigma_+$ are defined as $\sigma_+ = (1/3) (|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|)$ and $\sigma_- = \gamma \sigma_+ + I$ with $I$ being a bipartite swap operator. The states are entangled for $\gamma \in (3, 5)$ and PPT for $\gamma \in [2, 4]$. They can be detected by the Choi map described in Example 4 either as $\varphi_{3,1}$, or $\Theta(2;1,1,1)$. The resulting $(\alpha, \beta)$-inequalities of type (ii) effectively detect both NPT and PPT entanglement in this class of states.

The range of parameter $\gamma$, for which the inequalities are violated, versus the values of $\alpha, \beta$ are given in Table I.

Conclusions.- In the paper we have provided a method for derivation of scalar separability criteria from positive but not CP maps. As a particular example, when one uses the reduction map [18, 25] and puts $\alpha = 1$, the construction leads to the common scalar separability criterion, known as entropic inequalities [16,17,18]. Therefore, it may be treated as their generalization. However, for the studied positive maps and states the $(\alpha, \beta)$-inequalities with $\alpha > 1$ are much stronger than the entropic ones. Moreover, as shown in the case of SO(3) invariant states and states considered in Ref. [30], the inequalities arising from indecomposable maps detect a large share of bound entangled states for sufficiently large $\alpha$ (see Fig. 2 and Table I), and in the limit $\alpha \to \infty$, the whole set.

In an attempt to explain how the obtained inequalities work we considered the limit $\alpha \to \infty$ of the inequalities of type (ii) with fixed $\beta = 1$. This leads to a condition $\text{Tr}[I \otimes A](\rho)P_{\text{max}} \geq 0$, where $P_{\text{max}}$ is such a projector $P$ that corresponds to a maximum eigenvalue of $\rho$ and for which $\text{Tr}[I \otimes A](\rho)P \neq 0$. Thus, whenever $P_{\text{max}}$ is an entangled state detected by the conjugate map $[I \otimes A](P_{\text{max}})$ it above condition can be treated as a mean value of a kind of state-dependent entanglement witness $W = [I \otimes A](P_{\text{max}})$. It seems that at least for some classes of states such witness can detect entanglement.

Another point that should be stressed here is that many positive maps give rise to inequalities that cannot be applied without the assumption of commutativity. It would be interesting to investigate which maps (except for those discussed in Examples 1-4) lead to inequalities applicable to all states. For instance, the maps considered in Example 4 belong to a general class analyzed in Ref. [12], namely maps that act as $\varphi(|ii\rangle\langle ii|) = \sum_j a_{ij} |jj\rangle\langle jj|$ and $\varphi(|jj\rangle\langle jj|) = -|ij\rangle\langle ij|$ for $i \neq j$. They can be written as $\varphi = \varphi_1 - I$, where $\varphi_1(|ii\rangle\langle ii|) = \sum_j (a_{ij} + \delta_{ij}) |jj\rangle\langle jj|$. One easily finds that $\varphi_1$ is completely positive if $a_{ij} + \delta_{ij} \geq 0$ for all $i,j$. However, exact conditions that should be imposed on $a_{ij}$ to obtain a positive but not CP map $\varphi$, while $\varphi_1$ remains CP, are not specified. This, in the context of the proposed inequalities, would be an interesting subject for further research.

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