Negative discriminant states in $\mathcal{N} = 4$ supersymmetric string theories

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Abstract

Single centered BPS black hole solutions exist only when the charge carried by the black hole has positive discriminant. On the other hand the exact dyon spectrum in heterotic string theory compactified on $T^6$ is known to contain states with negative discriminant. We show that all of these negative discriminant states can be accounted for as two centered black holes. Thus after the contribution to the index from the two centered black holes is subtracted from the total microscopic index, the index for states with negative discriminant vanishes even for finite values of charges, in agreement with the results from the black hole side. Bound state metamorphosis – which requires us to identify certain apparently different two centered configurations according to a specific set of rules – plays a crucial role in this analysis. We also generalize these results to a class of CHL string theories.
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# 1 Introduction and Summary

We now have exact results for the appropriate supersymmetric index carried by a class of dyons in a class of $\mathcal{N} = 4$ supersymmetric string theories in four dimensions 1–15. Furthermore the dependence of the index on the asymptotic values of the moduli fields has also been completely understood in these theories 18–24. These results have been used to test the correspondence between the microscopic results and the macroscopic results based on the analysis of quantum gravity in the near horizon geometry of the black hole, both at perturbative 1, 2, 9, 11, 25, 27 and non-perturbative 28, 30 level. The analysis has also been extended to compare the prediction for the sign of the index 31, weighted index 32, 33 etc. with complete success. In particular the black hole prediction for the sign of the index was tested against

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\(^1\)A review of these results with all the sign factors corrected can be found in 16, 17.
Table 1: Some results for the index $d(Q, P)$ in heterotic string theory on $T^6$ in the chamber $\mathbf{R}$ for different values of $Q^2$, $P^2$ and $Q \cdot P$. The boldfaced entries are for charges for which the discriminant is negative.

| $(Q^2, P^2) \setminus Q \cdot P$ | 2   | 3   | 4   | 5   | 6   | 7   |
|-------------------------------|-----|-----|-----|-----|-----|-----|
| $(2, 2)$                      | 648 | 327 | 0   | 0   | 0   | 0   |
| $(2, 4)$                      | 50064 | 8376 | -648 | 0   | 0   | 0   |
| $(2, 6)$                      | 1127472 | 130329 | -15600 | 972 | 0   | 0   |
| $(4, 4)$                      | 3859456 | 561576 | 12800 | 3272 | 0   | 0   |
| $(4, 6)$                      | 110910300 | 18458000 | 1127472 | 85176 | -6404 | 0   |
| $(6, 6)$                      | 4173501828 | 920577636 | 110910300 | 8533821 | 153900 | 26622 |

The microscopic prediction for finite values of the charges, indicating that the macroscopic description holds beyond the large charge limit.

In this paper we shall carry out yet another comparison of the results from the microscopic and the macroscopic sides at finite values of the charges. Let us for definiteness consider the particular $\mathcal{N} = 4$ supersymmetric string theory obtained by compactifying heterotic string theory on $T^6$ and denote by $(Q, P)$ the (electric, magnetic) charge vectors carried by a state in this theory. We can then define moduli independent T-duality invariant inner products $Q^2$, $P^2$ and $Q \cdot P$. Let us further focus on charge vectors satisfying $I \equiv \gcd\{Q_i P_j - Q_j P_i\} = 1$ \cite{19} for which $Q^2$, $P^2$ and $Q \cdot P$ are known to be the complete set of T-duality invariants describing a charge vector \cite{34,35}. Single centered black hole solutions \cite{36,37} are known to exist only when the discriminant $Q^2 P^2 - (Q \cdot P)^2$ is positive. On the other hand when we examine the microscopic result for the index, we often find non-zero results for the index for many charge vectors with negative discriminant. Table 1 shows examples of such negative discriminant states (denoted by boldfaced entries) computed in a specific chamber $\mathbf{R}$ in the moduli space defined in \S 2. Thus the question that arises is: what is the macroscopic interpretation of these negative discriminant states?

\footnote{The microscopic index has been computed from first principles only for these states. For states with $I > 1$ the result for the index was guessed in \cite{14} and some justification for this result based on an effective string picture of the states was proposed in \cite{15}. In principle our analysis of negative discriminant states can also be extended to these states, but in this paper we shall restrict our analysis to $I = 1$ states.}
Table 2: This table shows, for various choices of \((Q, P)\), the contribution to the index in the chamber \(R\) from two centered black holes. The entries in the last column, giving the total index from all the two centered configurations, agree with the entries in table 1 for the corresponding values of \((Q^2, P^2, Q.P)\), showing that all the negative discriminant states in table 1 can be accounted for as two centered configurations. This is a general result that we shall prove in this paper.
One point of view one might take is that for finite values of the charges for which the results are given in table \textbf{1} the description of the system as black holes breaks down and so one should not try to compare the results in table \textbf{1} with the black hole results. However, the point of view that we would like to advocate is that the black hole description of the system continues to hold even for finite charges although the correction to the Bekenstein-Hawking formula due to $\alpha'$ and string loop effects may become significant. In that case the absence of black holes with negative discriminant is in apparent conflict with the presence of the negative discriminant states in the microscopic spectrum. One could still argue that the absence of negative discriminant black hole solutions was derived in the classical supergravity theory, and perhaps this is modified in string theory. To explore this we shall use the description of this system as a collection of wrapped D5, D3 and D1-branes in type IIB string theory on $K3 \times T^2$ and assume that the entropy of a supersymmetric black hole is independent of the asymptotic values of the moduli. In this case, one can show, using a simple scaling argument along the line of \cite{38}, that the classical Wald entropy, if non-zero, will grow quadratically when we scale all the charges together. On the other hand, the logarithm of the microscopic index grows at most linearly with the charges. Thus having a regular black hole solution after inclusion of $\alpha'$ corrections will be in conflict with the microscopic results in the large charge limit. One might still wonder if quantum corrections could change the result, but as in \cite{38} we shall call a configuration a black hole only if it exists as a solution to the classical equations of motion. Thus the absence of black holes with negative discriminant in classical string theory would imply that the contribution to the index from single centered black holes vanishes even for finite charges, and we must look for different macroscopic configurations to account for these states in the microscopic theory.

For a special class of charge vectors, carrying $Q^2 = P^2 = -2$ and arbitrary values of $Q.P$, this question was analyzed by Dabholkar, Gaiotto and Nampuri \cite{19}, where they found that in the domain in the moduli space where these states exist, there are two centered configurations – with individual centers carrying charges $(Q, 0)$ and $(0, P)$ – having the same index. This allowed them to identify these negative discriminant states as these specific two centered bound states, resolving the apparent contradiction arising due to the absence of single centered black holes carrying these charges. However, the analysis of \cite{19} also raises some questions:

1. Are there other two centered configurations carrying the same total charge in the same chamber of the moduli space? If so then they would also contribute to the index and spoil the agreement. We shall indeed find that there are apparently a series of
other 2-centered configurations which contribute to the index in the same chamber of the moduli space. Some examples are bound states of centers carrying charges \((Q + (Q.P)P, 0)\) and \((-(Q.P)P, P)\), bound states of centers carrying charges \((0, P + (Q.P)Q)\) and \((Q, -(Q.P)Q)\) etc.

2. The full spectrum computed in the microscopic theory contains many other negative discriminant states carrying different charge vectors. Are all the negative discriminant states in the microscopic spectrum accounted for by the bound states of multi-centered black holes? This was indeed the motivation of [19] for studying the special case described above.

In this paper we shall analyze these questions in detail. We shall find that indeed all the negative discriminant states in the microscopic spectrum can be accounted for in the macroscopic description precisely as due to two centered bound states.\(^3\) The contribution of the latter to the index can be calculated exactly and matched with the microscopic results. When the centers carry charge \(Q^2 \geq 0\) the analysis is straightforward. However we find that when one or both of the centers carry charge \(Q^2 = -2\), as in the case of [19], there is a subtlety, – two or more bound states which have apparently different constituents but the same total charge and the same index, must be identified according to a precise set of rules. We call this bound state metamorphosis – a phenomenon similar to but not the same as the one observed in [48]. An analogous phenomenon in the context of \(\mathcal{N} = 4\) supersymmetric gauge theories was discussed in [49]. This identification is justified since for a center with charge \(Q^2 = -2\) there is no genuine black hole solution, – the metric produced by it agrees with that of a black hole only far away from the core [50]. Thus at finite separation the identity of individual centers is expected to be lost. We also show that in the special limit in which some of the dyons may be interpreted as gauge theory dyons, the prescription we use for identifying bound states reproduces the known spectrum of quarter BPS dyons in gauge theories, in agreement with the results found in [49].

Our result thus shows that a specific prediction of the black hole description of quarter BPS states in \(\mathcal{N} = 4\) supersymmetric string theory – absence of single centered black holes with negative discriminant – is borne out by the microscopic spectrum, not only in the large charge limit but also for finite charges. This shows that the description of the system as a

\(^3\)Here the individual centers, being half BPS states, are in the S-duality orbit of elementary heterotic string states [39], and could appear either as small black holes [40,41] or smooth solutions [42-47] depending on the duality frame used for their description [38]. See footnote [6] for an extended discussion on this.
black hole can give us useful information far beyond the leading semi-classical approximation. In this context we would like to note that in $\mathcal{N} = 2$ supersymmetric theories the absence of negative discriminant states in certain region of the moduli space can be argued independently by showing that such states, if present, would become massless at certain points in the interior of the moduli space and hence would produce additional singularities in the moduli space which are known to be absent $^5$. However in $\mathcal{N} = 4$ supersymmetric string theories negative discriminant states never become massless $^4$ in the interior of the moduli space and hence there does not seem to be an independent argument for their absence.

The rest of the paper is organized as follows. In §2 we review the known microscopic results on the dyon spectrum in $\mathcal{N} = 4$ supersymmetric string theories and their moduli dependence. In §3 we discuss the phenomenon of bound state metamorphosis and our prescription for identifying bound states with different constituents when one or both the constituents carry charge $^2 = -2$. In §4 we show that once we implement the rules of bound state metamorphosis, all the negative discriminant states are accounted for precisely as two centered bound states. To illustrate this point we have displayed in table 2 the specific origin of all the negative discriminant states which arise in table 1. As can be seen, the entries for the total index given in the last column of table 2 match precisely the results for the total index given in table 1. In §5 we discuss generalization of our analysis to CHL models $^52$ $^55$. We conclude with some general remarks in §6.

## 2 The dyon spectrum

We consider a dyon carrying electric charge $Q$ and magnetic charge $P$ in heterotic string theory compactified on $T^6$. Here $Q$ and $P$ are each 28 dimensional vectors, normalized so that the

$$ m_{BPS}^2 = \frac{1}{\tau_2} \left[ (Q_R - \tau_1 P_R)^2 + \tau_2^2 P_R^2 + 2 \sqrt{Q_R^2 P_R^2 - (Q_R P_R)^2} \right] $$

where $\tau_1 + i\tau_2$ is the axion-dilaton modulus and the subscript $R$ denotes projection of the charge vectors along the six graviphoton directions. In order for $m_{BPS}$ to vanish for finite non-zero $\tau_2$, each of the three terms inside $[ ]$ must vanish. Since the inner product matrix between the the right handed components of charges is positive definite, this requires $Q_R$ and $P_R$ to vanish. In that case $Q^2 = Q_R^2 - Q_L^2 = -Q_L^2$, and similarly $P^2 = -P_L^2$ and $Q.P = -Q_L P_L$. Thus the discriminant is given by $Q^2 P^2 - (Q.P)^2 = Q_R^2 P_R^2 - (Q_L P_L)^2$, and this is manifestly non-negative. This shows that the negative discriminant states in $\mathcal{N} = 4$ supersymmetric string theories cannot become massless in the interior of the moduli space.
components $Q_i$ and $P_i$ are integers. Let

\[ \begin{align*}
Q^2, & \quad P^2, & \quad Q \cdot P, \\
\end{align*} \]

be the SO(6,22) invariant inner products of these vectors. Then, if

\[ I \equiv \text{gcd}\{Q_iP_j - Q_jP_i, \quad 1 \leq i, j \leq 28\} = 1, \]

then the index $d(Q, P) \equiv Tr'_{(Q,P)}(-1)^F$, computed using the microscopic description of the system, is given by

\[ d(Q, P) = g \left( \frac{P^2}{2}, \frac{Q^2}{2}, Q \cdot P \right), \]

where $g(m,n,p)$ are the Fourier expansion coefficients of the inverse of the Igusa cusp form of weight 10 [56,57]:

\[ \frac{1}{\Phi_{10}(\rho, \sigma, v)} = \sum_{m,n,p} (-1)^{p+1} g(m,n,p) e^{2\pi i (m\rho + n\sigma + pv)}. \]

Here $Tr'_{(Q,P)}$ denotes trace over BPS states carrying charges $(Q, P)$ and preserving four supersymmetries, after removing the trace over the fermion zero modes associated with broken supersymmetries.

$\Phi_{10}(\rho, \sigma, v)$ is a well defined function in the Siegel upper half plane:

\[ \rho_2, \sigma_2 > 0, \quad \rho_2 \sigma_2 - v_2^2 > 0, \quad (\rho_2, \sigma_2, v_2) \equiv Im(\rho, \sigma, v), \]

and can be expressed as an infinite product

\[ \frac{1}{\Phi_{10}(\rho, \sigma, v)} = e^{-2\pi i (\rho + \sigma + v)} \prod_{j,k,l \in \mathbb{Z}, 4kl - j^2 \geq -1} (1 - e^{2\pi i (kj + l\sigma + jv)})^{-c(4kl - j^2)} \]

where the coefficients $c(s)$ are defined via the equation

\[ 8 \sum_{i=2}^{4} \frac{\vartheta_i(t, z)^2}{\vartheta_i(t, 0)^2} = \sum_{n,j} c(4n - j^2) e^{2\pi i (nt + jz)}, \]

$\vartheta_i$'s being the Jacobi theta functions. It follows from (2.7) that

\[ c(s) = 0 \quad \text{for} \quad s < -1, \quad c(-1) = 2. \]

Alternatively we can describe it as the sixth helicity supertrace [58,59] appropriately normalized so that a single supermultiplet with helicities ranging from $-1$ to 1 gives a contribution of 1 to the index.
Figure 1: The different domains in the \((\rho_2, \sigma_2, v_2)\) space. Here the \(x\)-axis labels \((v_2/\sigma_2)\) and the \(y\)-axis labels \((\rho_2/\sigma_2)\). The dashed line is the boundary \(\rho_2\sigma_2 = v_2^2\) and the thick straight lines label the walls across which \(g(m,n,p)\) changes.

It turns out that \(g(m,n,p)\) defined through (2.4) are ambiguous since \(1/\Phi_{10}(\rho,\sigma,v)\) has poles in the \((\rho,\sigma,v)\) plane, and hence the \((\rho,\sigma,v)\) space is divided into different domains with each domain having its own Fourier expansion that converges in that domain. We shall always work in the region where \(\rho_2, \sigma_2\) and \(|v_2|\) are large. In this region a term of the form

\[ (1 - e^{2\pi i (k\rho + l\sigma + jv)})^{-\alpha}, \tag{2.9} \]

has a convergent expansion in a power series in \(e^{2\pi i (k\rho + l\sigma + jv)}\) as long as \((k\rho_2 + l\sigma_2 + jv_2) > 0\).

On the other hand if \((k\rho_2 + l\sigma_2 + jv_2) < 0\), then we must express this as

\[ (-1)^{-\alpha}e^{-2\pi i (k\rho + l\sigma + jv)} (1 - e^{-2\pi i (k\rho + l\sigma + jv)})^{-\alpha}, \tag{2.10} \]

and expand this in a power series in \(e^{-2\pi i (k\rho + l\sigma + jv)}\). These two different modes of expansion will in general generate different Fourier coefficients \(g(m,n,p)\), leading to a dependence of \(g(m,n,p)\) on the domain in \((\rho_2, \sigma_2, v_2)\) space where we carry out the expansion.

Thus we need to examine for each \(j, k, l\) in the product in (2.6) whether \((k\rho_2 + l\sigma_2 + jv_2)\) is positive or negative and carry out the Fourier expansion accordingly. It is straightforward to see that if \(4kl - j^2 \geq 0, k,l \geq 0\), then \((k\rho_2 + l\sigma_2 + jv_2)\) is always positive in the Siegel upper half plane described in (2.5), and hence there is no ambiguity in expanding the corresponding term. Thus the only ambiguities arise from the terms in the product for which \(4kl - j^2 = -1\).

For such terms the rules for the expansion changes as we cross the plane

\[ k\rho_2 + l\sigma_2 + jv_2 = 0, \quad j, k, l \in \mathbb{Z}, \quad 4kl - j^2 = -1, \tag{2.11} \]

in the \((\rho_2, \sigma_2, v_2)\) space. If we plot them in the \((v_2/\sigma_2, \rho_2/\sigma_2)\) plane, then (2.11) describes a set of straight lines that divides the allowed region bounded by the parabola \(\rho_2/\sigma_2 > (v_2/\sigma_2)^2\).
Figure 2: The different domains in the complex $\tau$ plane separated by walls of marginal stability. into infinite number of triangles. Inside each triangle we shall have a different set of $g(m, n, p)$. This has been illustrated in Fig. 1.

It turns out that there is a one to one map between these domains in the $(\rho_2, \sigma_2, v_2)$ space and the chambers in the moduli space of heterotic string theory on $T^6$, separated by the walls of marginal stability \cite{18-22}. To describe this we shall fix the moduli associated with the metric, 2-form field and components of gauge fields along $T^6$ – labelled by points on the coset space $SO(6, 22)/SO(6) \times SO(22)$ – to a fixed value and study the walls in the upper half plane parametrized by the axion-dilaton modulus $\tau$. The walls of marginal stability turn out to be circles connecting rational points $p/r$ and $q/s$, such that $p, q, r, s \in \mathbb{Z}$ and $ps - qr = 1$ \cite{18}. In the special case when $r$ (or $s$) vanishes, one of the points is at infinity and the wall becomes a straight line connecting an integer to $i\infty$. The precise shapes of the walls depend on the charges $(Q, P)$. As we cross a wall of marginal stability in the $\tau$ plane, the index jumps and this is accounted for by a jump in $g(m, n, p)$ across a wall of the form (2.11) in the $(\rho_2, \sigma_2, v_2)$ plane, – with each wall in the $(\rho_2, \sigma_2, v_2)$ space being in one to one correspondence to a wall in the $\tau$ plane \cite{18}. In particular the wall in the $\tau$ plane connecting the rational points $p/r$ and $q/s$ is mapped to the wall

$$pq\sigma_2 + rs\rho_2 + (ps + qr)v_2 = 0$$  \hspace{1cm} (2.12)

in the $(\rho_2, \sigma_2, v_2)$ space. This in particular means that the chamber in the $\tau$ plane, bounded by the walls connecting $(0, i\infty)$, $(0, 1)$ and $(1, i\infty)$ gets mapped to the domain in the $(\rho_2, \sigma_2, v_2)$ space bounded by the walls $v_2 = 0$, $v_2 = -\rho_2$ and $v_2 = -\sigma_2$. We shall denote this chamber / domain by $\mathbf{R}$. This has been marked in Figs. 1 and 2.

$\Phi_{10}(\rho, \sigma, v)$ remains invariant under the $SL(2, \mathbb{Z})$ S-duality group which has a natural
action on the charges, $\tau$ as well as on $(\rho, \sigma, v)$ space. It takes the form

\[
\begin{pmatrix}
Q \\
P
\end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\
P
\end{pmatrix}, \quad \tau \to \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),
\]

\[
\begin{pmatrix}
Q^2 \\
P^2
\end{pmatrix} \to \begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix} \begin{pmatrix} 2ab \\
2cd
\end{pmatrix} \begin{pmatrix} Q^2 \\
P^2
\end{pmatrix},
\]

\[
\begin{pmatrix}
\sigma \\
\rho
\end{pmatrix} \to \begin{pmatrix} ac & bd \\ ad + bc \end{pmatrix} \begin{pmatrix} \sigma \\
\rho
\end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).
\]

(2.13)

This $SL(2, \mathbb{Z})$ transformation maps the walls in the $\tau$ plane into each other and also the walls in the $(\rho_2, \sigma_2, v_2)$ space into each other. In fact just by knowing that the wall $v_2 = 0$ gets mapped to the wall connecting $0$ and $i\infty$ in the $\tau$ plane we can derive (2.12), since we can make use of the $SL(2, \mathbb{Z})$ transformation generated by the matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ to map the $v_2 = 0$ wall to the wall (2.12) in the $(\rho_2, \sigma_2, v_2)$ space and the wall connecting $0$ to $i\infty$ to the wall connecting $q/s$ to $p/r$ in the $\tau$ plane.

As we cross the walls of marginal stability in the $\tau$ plane shown in Fig. 2, certain bound states of two half BPS black holes either cease to exist or come into existence thereby causing a jump in the index \([60, 61]\). Let us for definiteness consider the wall connecting $0$ to $i\infty$ in the $\tau$ plane. If we take $Q.P > 0$, then on the left of this wall we have a bound state of half BPS black holes carrying charges $(Q, 0)$ and $(0, P)$ \([19–21]\), giving a total contribution to the index

\[
(-1)^{Q.P+1} |Q.P| f(Q^2/2) f(P^2/2),
\]

(2.14)

where $f(n)$ is defined through:

\[
q^{-1} \prod_{k=1}^{\infty} (1 - q^k)^{-24} = \sum_{n=-1}^{\infty} f(n) q^n.
\]

(2.15)

These two centered solutions cease to exist on the other side of the wall, thereby causing a jump in the index given by $(-1)^{Q.P} |Q.P| f(Q^2/2) f(P^2/2)$ as we cross the wall from the left to the right. This can be shown to be equal to the jump in $g(m, n, p)$ as we cross the corresponding wall in the $(\rho_2, \sigma_2, v_2)$ space \([19–21]\). For $Q.P < 0$ the situation is opposite, with the bound

\[\text{Since each center represents a half BPS state, it may appear either as a small black hole or as a smooth solution depending on the duality frame in which we describe this \([38]\). We shall call all of them black holes. Since near a wall of marginal stability the distance between the centers go to infinity, a substructure of the center will not affect the counting given in (2.14).}\]
states existing on the right of the wall. This can be seen by making an S-duality transformation by \[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\] which exchanges the two sides of the wall connecting 0 to \(i\infty\), and at the same time changes the sign of \(Q.P\).

By making an S-duality transformation of this result we can also figure out the physical significance of the other walls. In particular the S-duality transformation by \[
\begin{pmatrix}
p & q \\
r & s
\end{pmatrix}
\] takes the wall connecting 0 to \(i\infty\) to the wall connecting \(q/s\) to \(p/r\) preserving orientation, i.e. 0 gets mapped to \(q/s\) and \(i\infty\) gets mapped to \(p/r\). Let us denote by \((Q', P')\) the transformed charges:

\[
\begin{pmatrix}
Q' \\
P'
\end{pmatrix} = \begin{pmatrix}
p & q \\
r & s
\end{pmatrix} \begin{pmatrix}
Q \\
P
\end{pmatrix} \Rightarrow \begin{pmatrix}
Q \\
P
\end{pmatrix} = \begin{pmatrix}
s & -q \\
-r & p
\end{pmatrix} \begin{pmatrix}
Q' \\
P'
\end{pmatrix}.
\] (2.16)

It then follows from S-duality that if \(Q.P > 0\), i.e. \((sQ' - qP'),(-rQ' + pP') > 0\) then to the left of the wall connecting \(q/s\) to \(p/r\) there exists a bound state of two centers carrying charges

\[
\begin{pmatrix}
p & q \\
r & s
\end{pmatrix} \begin{pmatrix}
Q \\
P
\end{pmatrix} = \begin{pmatrix}
p(sQ' - qP') \\
r(sQ' - qP')
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
p & q \\
r & s
\end{pmatrix} \begin{pmatrix}
0 \\
P
\end{pmatrix} = \begin{pmatrix}
q(-rQ' + pP') \\
s(-rQ' + pP')
\end{pmatrix}.
\] (2.17)

On the other hand there are no such bound states to the right of this wall. If \((sQ' - qP'),(-rQ' + pP') < 0\) then the situation is reverse, with the bound state existing to the right of the wall.

Now in (2.17) \((Q, P)\) and hence \((Q', P')\) are arbitrary charge vectors. Renaming \((Q', P')\) as \((Q, P)\) we can state the above result as follows. Given a charge vector \((Q, P)\), and a point in the \(\tau\) plane, we have a bound state of charges \((p(sQ - qP), r(sQ - qP))\) and \((q(-rQ + pP), s(-rQ + pP))\) provided one of the following two conditions hold:

1. \((sQ - qP),(-rQ + pP) > 0\) and the point in the \(\tau\) plane lies to the left of the wall connecting \(q/s\) to \(p/r\).

2. \((sQ - qP),(-rQ + pP) < 0\) and the point in the \(\tau\) plane lies to the right of the wall connecting \(q/s\) to \(p/r\).

In either case the net contribution to the index from this bound state is given by

\[
(-1)^{Q,P+1} \left| (sQ - qP),(-rQ + pP) \right| f((sQ - qP)^2/2) f((-rQ + pP)^2/2),
\] (2.18)

where in computing the sign we have used the fact that \((-1)^{Q,P}\) remains invariant under S-duality transformation. This allows us the determine the list of possible two centered bound

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\(^7\)Throughout this paper whenever we refer to a wall connecting \(A\) to \(B\) we shall implicitly assign an orientation to the wall directed from \(A\) to \(B\), and the left or right of the wall is specified with respect to this orientation.
Figure 3: For $P^2 = -2$ and $u \equiv Q.P > 0$, the bound state of $(Q, 0)$ and $(0, P)$ exists to the left of the wall connecting 0 and $i\infty$, while the bound state of $(Q + uP, 0)$, $(-uP, P)$ exist to the right of the wall connecting $-u$ and $i\infty$. We propose that these two bound states are identical and exist in the region $Y$ between the two walls.

states which can contribute to the index at a given point in the moduli space and their contribution to the index. As we shall discuss in §3 and §4, this prescription is modified somewhat when $(sQ - qP)^2$ and/or $(-rQ + pP)^2$ takes the value $-2$.

3 Bound state metamorphosis

The prescription given at the end of §2 allows us to determine the total contribution to the index from two centered bound states at any point in the moduli space. We shall now argue that there are some exceptions to this prescription when $(sQ - qP)^2$ or $(-rQ + pP)^2$ or both are equal to $-2$ due to bound state metamorphosis, – a phenomenon first observed in [49] in the context of $\mathcal{N} = 4$ supersymmetric gauge theories.

3.1 Prescription

We begin with the case when $P^2 = -2$, $Q.P > 0$. In this case a two centered bound state of $(Q, 0)$ and $(0, P)$ exists to the left of the wall connecting 0 to $i\infty$, with a net contribution to the index given by (2.14). Now consider the bound state of charges $(Q + uP, 0)$ and $(-uP, P)$ for $u \equiv Q.P$. In the convention described at the end of §2 this corresponds to the choice

\[
\begin{pmatrix}
p \\
r \\
q \\
s
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
-u \\
1
\end{pmatrix}
\]

and according to the prescription given there, this bound state exists to the right of the wall connecting $-u$ to $i\infty$ (see Fig. 3) since $(Q + uP).P = -Q.P < 0$. Furthermore (2.18) shows that the contribution to the index from this bound state is given by

\[
(-1)^{Q.P+1}|Q.P + uP^2|f((Q + uP)^2/2)f(P^2/2) = (-1)^{Q.P+1}|Q.P|f(Q^2/2)f(P^2/2),
\]
Figure 4: For $Q^2 = -2$ and $u \equiv Q.P > 0$, the bound state of $(Q,0)$ and $(0,P)$ exists to the left of the wall connecting 0 and $i\infty$, while the bound state of $(0,uQ + P)$, $(Q,-uQ)$ exists to the right of the wall connecting 0 and $-1/u$. We propose that these two bound states are identical and exists in the region $X$ between the two walls.

where we have used the fact that $Q.P + uP^2 = -Q.P$ and $(Q + uP)^2 = Q^2$. Thus (3.1) coincides with the (2.14), i.e. the index carried by the bound state of $(Q,0)$ and $(0,P)$ and that carried by the bound state of $(Q + uP,0)$ and $(-uP,P)$ coincide. We propose that these two configurations describe the same physical states, and hence should be counted only once. Furthermore the bound state exists only in the region $Y$ between the walls connecting 0 and $i\infty$ and $-u$ and $i\infty$ (see Fig. 3). Had we not identified these bound states then both two centered bound states would exist in the region between the walls connecting 0 to $i\infty$ and $-u$ and $i\infty$ and one of the two centered bound states will exist in each of the two regions to the right of the wall from 0 to $i\infty$ and to the left of the wall from $-u$ to $i\infty$.

If we have $Q.P < 0$ and $P^2 = -2$ then the analysis remains more or less unchanged with left and right exchanged. Thus in this case there will be a bound state of $(Q,0)$ and $(0,P)$ to the right of the wall connecting 0 to $i\infty$ and there will be a bound state of charges $(Q + uP,0)$ and $(-uP,P)$ to the left of the wall connecting $-u$ to $i\infty$. Note however that now $-u = -Q.P$ is positive. Our prescription is to identify these two bound states and postulate that it exists in the region between the walls connecting 0 to $i\infty$ and $-u$ to $i\infty$. Finally if $Q^2 = -2$ and $P^2 \geq 0$ then the whole analysis can be repeated by exchanging $Q$ and $P$, and we shall identify the bound state of $(Q,0)$ and $(0,P)$ with the bound state of $(Q,-uQ)$ and $(0,P+uQ)$, existing in the region between the walls connecting 0 to $i\infty$ and 0 to $-1/u$ in the upper half $\tau$ plane (see Fig. 4). In the convention described at the end of §2, the second bound state corresponds to the choice

$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}$.

If we have $P^2 = Q^2 = -2$ then we have to identify the bound states of $(Q,0)$ and $(0,P)$,
Figure 5: For $Q^2 = -2$ and $P^2 = -2$, there are a series of choices for $(p, q, r, s)$ for which the associated bound states all have the same index and are identified. Thus these bound state exist in a region $V$ bounded by the walls corresponding to all of these matrices. 

$$(Q + uP, 0) \text{ and } (-uP, P) \text{ and of } (0, uQ + P) \text{ and } (Q, -uQ),$$ all of which carry the same index. Furthermore it exists in the common region lying between the walls connecting $0$ to $i\infty$, $-u$ to $i\infty$ and $0$ to $-1/u$ (see Fig. 5). However unless $u = \pm 1$ the identification does not stop here since we can construct an infinite set of matrices $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ by taking alternate products of $\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ or of $\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, and the bound states associated with all of these matrices have the same index and should be identified. Furthermore they exist in the region bounded by the walls associated with these matrices, as shown by the region $V$ in Fig. 5.

Finally note that the analysis given above can be easily extended to the case when neither $Q^2$ nor $P^2$ is equal to $-2$, but for some given $SL(2, \mathbb{Z})$ matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, either $(sQ - qP)^2 = -2$ or $(-rQ + pP)^2 = -2$ or both are equal to $-2$. This is related to the cases discussed above via a simple $SL(2, \mathbb{Z})$ transformation by the matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, and the region of the moduli space where these bound states exist will be related to the regions depicted in Figs. 3, 4 and 5 by this $SL(2, \mathbb{Z})$ transformation. We shall elaborate on this in §4.

3.2 Justification

Even though we have given a prescription for identifying certain bound states which carry the same index, we have not proved the result. We shall now try to provide some justification for this prescription. This stems from the observation that a center with charge $Q^2 = -2$ does not
correspond to a conventional black hole in any duality frame. In particular the central charge associated to such a charge vector has a zero on a subspace of the moduli space where the corresponding state becomes massless, and the attractor flow cannot be continued past this point to generate a horizon \[51,60\]. Beyond this point the solution is described by a core in which the moduli are frozen at a constant value at which the gauge symmetry is enhanced, and the charge carried by the black hole is spread over the surface of this core \[50\]. Thus the description of this state as a black hole is valid only outside the core. Consequently a two centered configuration where one or both the centers carry charge \(2 = -2\) can be regarded as a genuine two centered solution only when the centers do not come too close to each other. Given that in the interior of the moduli space away from the walls of marginal stability the centers do come close to each other, it is not inconceivable that a two centered solution of this type can change its description as we move from one wall of marginal stability to another. Since the index of a BPS state cannot jump except at the walls of marginal stability, such a change in the description is possible only if the new configuration carries the same index as the old one.

While the above argument provides a possible reason for the metamorphosis, it clearly does not prove the validity of our prescription. Presumably more insight into this can be obtained by a detailed study of the two centered solution, but we shall not do this here. We shall see in \[4\] that this identification is necessary for a consistent description of the negative discriminant states. In the remaining of this section we shall show that in the special case of \(Q^2 = P^2 = -1\) and \(Q.P = \pm 1\) this identification is necessary for getting agreement with the spectrum of quarter BPS states in gauge theory. Let us choose \(Q.P = 1\) for definiteness. In this case our prescription requires us to identify the bound states of \((Q,0)\) and \((0,P)\), \((Q+P,0)\) and \((-P,P)\), and \((0,P+Q)\) and \((Q,-Q)\) in the chamber \(W\) bounded by the walls connecting 0 to \(i\infty\), \(-1\) to \(i\infty\) and 0 to \(-1\) (see Fig. \[6\]). The index associated with each of these bound states according to eq.(2.18) is 1. Thus if we follow our prescription then the contribution to the index from these bound states will be 1 in the chamber \(W\) in Fig. \[6\] and will vanish outside this chamber. On the other hand if we do not make this identification then the index will be 3 in the chamber \(W\) and 2 outside this chamber (since only 2 of the 3 bound states will exist in each of these regions). The correct answer can be found by going near an appropriate region of the moduli space where we can regard this state as a quarter BPS state in an \(\mathcal{N} = 4\) supersymmetric SU(3) gauge theory \[62,63\]. The state is known to have index 1 and to exist only inside the chamber bounded by the walls connecting 0 to \(i\infty\), \(-1\) to \(i\infty\) and 0 to \(-1\) \[64,68\]. This will
Figure 6: For $Q^2 = -2$, $P^2 = -2$ and $Q.P = 1$ the bound states of centers $(Q, 0)$ and $(0, P)$, $(Q + P, 0)$ and $(-P, P)$ and of $(0, Q + P)$ and $(Q, -Q)$ all have the same index (=1) and are identified. They exist in the region $W$ bounded by the walls from 0 to $i\infty$, from $-1$ to $i\infty$ and from 0 to $-1$. This is consistent with the gauge theory limit in which the BPS state carrying charge $(Q, P)$ with $Q^2 = P^2 = -2$ and $Q.P = 1$ is known to have index 1 and to exist precisely in the chamber bounded by these three walls.

be consistent with the result obtained from the spectrum of two centered bound states only if we identify the different two centered bound states as prescribed above. Thus at least the results in gauge theory are in agreement with our general prescription.

### 3.3 Uniqueness

In order to further justify that the bound state metamorphosis is associated with centers carrying negative charge$^2$, we shall now show that in order that two different bound states with the same total charge have identical index and exist in the same region of the moduli space, at least one of the centers must carry negative charge$^2$. Let us consider two possible bound states with total charge $(Q, P)$. Using S-duality we can take one of them to be a bound state of charges $(Q, 0)$ and $(0, P)$; let the other have centers carrying charges $(p(sQ - qP), r(sQ - qP))$ and $(q(-rQ + pP), s(-rQ + pP))$ for some $SL(2, \mathbb{Z})$ matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Using the freedom of changing the sign of $p, q, r, s$ without changing the configuration, we shall choose $p$ to be non-negative. These two bound states will manifestly carry the same index if and only if$^8$

$$Q^2 = (sQ - qP)^2, \quad P^2 = (-rQ + pP)^2, \quad Q.P = \mp(sQ - qP).(-rQ + pP).$$

$^8$I wish to thank the referee for drawing my attention to this issue.

$^9$We are not including the accidental cases where the factors in (2.18) are different for the two bound states but the product is identical. Also we ignore the other possibility where we equate $Q^2$ with $(-rQ + pP)^2$ and $P^2$ with $(sQ - qP)^2$ since this just corresponds to a redefinition of $p, q, r, s$. 

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Leaving out the trivial case where \( Q.P = 0 \) and hence the bound state index vanishes, and the cases \( \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) or \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) for which the two bound states are manifestly identical, one can show as a consequence of (3.2) that

\[ p = s \quad 2p Q.P = rQ^2 + qP^2, \tag{3.3} \]

or

\[ rQ^2 + qP^2 = 0, \quad 2q Q.P = (s - p)Q^2, \quad 2r Q.P = (p - s)P^2. \tag{3.4} \]

The solutions given in (3.3) and (3.4) are related to the \(-\) and \(+\) signs in (3.2). For (3.3) if \( Q.P \) is positive (negative) then the first bound state exists to the left (right) of the wall connecting 0 to \( i\infty \) and the second bound state exists to the right (left) of the wall connecting \( q/s \) to \( p/r \).

On the other hand for (3.4) if \( Q.P \) is positive (negative) then the first bound state exists to the left (right) of the wall connecting 0 to \( i\infty \) and the second bound state exists to the left (right) of the wall connecting \( q/s \) to \( p/r \).

First consider the case given in (3.3). If \( p = s \geq 2 \) then \( qr = (ps - 1) \geq 3 \) and neither \( q \) nor \( r \) can vanish. We now see from (3.3) that for \( Q^2 \) and \( P^2 \geq 0 \), at least one of \( q/p = q/s \) and \( r/p \) must be positive (negative) for \( Q.P \) positive (negative). Since \( pqr \) is positive, \( ps \) and \( qr \) have the same sign and hence \( q/s \) and \( p/r \) also have the same sign. Thus both \( q/s \) and \( p/r \) must be positive (negative) for \( Q.P \) positive (negative). Furthermore since \( q \) and \( r \) have the same sign, \( rs \) has the same sign as \( q/s \) and hence \( \frac{r}{s} - \frac{q}{s} = \frac{1}{rs} \) has the same sign as \( q/s \). Thus for \( Q.P \) positive (negative) we have \( \frac{r}{s} > \frac{q}{s} \) \((\frac{r}{s} < \frac{q}{s}) \). Thus we see that for \( Q.P \) positive (negative), the first bound state which lies to the left (right) of the wall connecting 0 and \( i\infty \), and the second bound state which lies to the right (left) of the wall connecting \( q/s \) to \( p/r \), with \( 0 < \frac{q}{s} < \frac{r}{s} \) \((\frac{r}{s} < \frac{q}{s} < 0) \), exist in non-overlapping regions of the moduli space. Thus there is no sense in which we can identify them.

The special cases \( p = s = 1 \) and \( p = s = 0 \) can be treated easily. If \( p = s = 0 \) then we must have \( q = -r = \pm 1 \). In this case the second bound state is identical to the first one with the centers exchanged and hence we can ignore this case. For \( p = s = 1 \) we must have \( qr = 0 \). For \( q = 0 \) (3.3) gives \( 2Q.P = rQ^2 \). Thus if \( Q^2 > 0 \), then for \( Q.P \) positive (negative), we must have \( r \) positive (negative) and \( p/r \) positive (negative). Thus the first bound state exists to the left (right) of the line from 0 to \( i\infty \) and the second bound state exists to the right (left) of the line connecting 0 to \( p/r \), with no common region in which both bound states exist. The case \( r = 0 \) gives similar result. Thus we cannot identify these different bound states.
Finally we turn to the case (3.4). First consider the special case $Q^2 = P^2 = 0$. In this case if $Q, P \neq 0$ then we must have $q = 0$ and $r = 0$ and hence $ps = 1$, $p = s = 1$. Thus the two bound states are identical and we can skip this case. If $Q^2 = 0$, $P^2 > 0$ then we must have $q = 0$ and hence again $ps = 1$, $p = s = 1$. This in turn implies from (3.4) that $r$ must also vanish, and we again have that the two bound states are manifestly identical. Thus we need to consider the case $Q^2, P^2 > 0$. In this case either $q$ and $r$ both vanish with $p = s = 1$ in which case we again have identical bound states, or $q$ and $r$ are both non-vanishing and have opposite sign. Restricting our attention to the latter case, we now get from (3.4) that

$$2Q.P = \left(\frac{s}{q} - \frac{p}{q}\right)Q^2 = \left(\frac{p}{r} - \frac{s}{r}\right)P^2. \quad (3.5)$$

Using $ps - qr = 1$ and $qr \leq -1$ we see that we must have $ps \leq 0$. First consider the case $qr \leq -2$ so that $ps \leq -1$, and $p, q, r, s$ are all non-zero. (3.5) shows that for $Q.P$ positive (negative) $\frac{s}{q} - \frac{p}{q}$ is positive (negative) and also $\frac{p}{r} - \frac{s}{r}$ is positive (negative). But since $ps < 0$ we know that $\frac{s}{q}$ and $\frac{p}{q}$ have opposite sign and $\frac{s}{r}$ and $\frac{p}{r}$ have opposite sign. Thus we must have $\frac{s}{q}$ and hence $\frac{q}{s}$ positive (negative), $\frac{p}{r}$ positive (negative) and $\frac{s}{r}$ negative (positive). Furthermore we have $\frac{q}{s} - \frac{p}{r} = -\frac{1}{rs}$ positive (negative). Thus the first bound state exists to the left (right) of the wall connecting $0$ to $i\infty$ and the second bound state exists to the left (right) of the wall connecting $\frac{q}{s}$ to $0$, with $0 < \frac{p}{r} < \frac{q}{s} \left(\frac{q}{s} < \frac{p}{r} < 0\right)$. Thus there is no common region where both bound states exist.

Finally we need to consider the special case $qr = -1, ps = 0$. First suppose $p = 0$. In this case (3.5) shows that for $Q.P$ positive (negative) we must have $\frac{s}{q}$ positive (negative). Thus the first bound state exists to the left (right) of the wall connecting $0$ and $i\infty$ and the second bound state exists to the left (right) of the wall connecting $\frac{q}{s}$ to $0$, with $0 < \frac{p}{r}$ $\left(\frac{2}{s} < \frac{p}{r} < 0\right)$. Thus again we see that there is no common region where both bound state exists. For $s = 0$ we see from (3.5) that for $Q.P$ positive (negative) we have $\frac{p}{r}$ positive (negative). In this case the first bound state exists to the left (right) of the wall connecting $0$ and $i\infty$ and the second bound state exists to the left (right) of the wall connecting $i\infty$ to $\frac{p}{r}$, with $0 < \frac{p}{r} \left(\frac{p}{r} < 0\right)$. Again there is no common region where both bound state exists.

This shows that as long as $Q^2 \geq 0$ and $P^2 \geq 0$, a bound state of $(Q, 0)$ and $(0, P)$ cannot have the same index as another bound state with same total charge, with both the bound states existing in a common region in the moduli space. Thus the phenomenon of bound state metamorphosis takes place if and only if the at least one of the centers carry negative charge$^2$. This is consistent with the fact that precisely in this case the classical black hole solution
becomes singular at a finite distance away from the center, and the mechanism suggested in [50] is necessary to modify the solution. Thus while a multi-centered solution can be trusted for large separation between the centers, it cannot be trusted everywhere in the moduli space. As we shall see in §4, this is also consistent with the microscopic results.

4 Negative discriminant states

Given a charge vector \((Q, P)\) we denote the discriminant associated with the charge vector as

\[ D(Q, P) = Q^2P^2 - (Q.P)^2. \]  

(4.1)

The spectrum of heterotic string theory on \(T^6\), given by eqs.(2.3), (2.4), contains many states with negative discriminant. These arise from non-zero values of \(g(m, n, p)\) for \((m, n, p)\) satisfying \(4mn - p^2 < 0\), and in turn arise from the fact that the product over \(k, l, j\) in (2.6) contains terms with \(4kl - j^2 < 0\). Explicit examples of such negative discriminant states can be found in the boldfaced entries in table 1 where we have given the values of the index \(d(Q, P)\) for various combinations of \(Q^2, P^2\) and \(Q.P\) in the chamber \(R\) depicted in Figs. 1 and 2. On the other hand single centered black hole solutions [36, 37] always have positive discriminant. Our goal in this section will be to argue that all the negative discriminant states in heterotic string theory on \(T^6\) arise from bound states of two half BPS black holes. Thus after subtracting the contribution from the two centered configurations, we shall be left with only states with non-negative discriminant, in agreement with the prediction based on classical black hole solutions.

4.1 Duality transformation

In order to prove this result we shall use the fact that the spectrum is consistent with wall crossing, so that the jump in the spectrum as we cross a wall of marginal stability can be explained as due to (dis-)appearance of a particular 2-centered solution. Thus if we can establish the result in any one chamber, then this will prove the result in all other chambers. We shall also make use of the duality invariance of the spectrum, which tells us that if we can prove this for any charge vector, then it also holds for all other charge vectors related to the original charge vector by a duality transformation.

Let us begin by making use of the duality transformation to bring a charge vector \((Q, P)\) with \(Q^2P^2 < (Q.P)^2\) to a ‘standard form’. This is done using the following steps:
1. If either $Q^2$ or $P^2$ is $\leq 0$ we skip this step and proceed directly to step 2. Suppose however that both $P^2$ and $Q^2$ are positive. Let us take for definiteness $P^2 \geq Q^2 > 0$. In this case $|Q.P|$ must be bigger than $Q^2$. Now let us make a duality transformation $P \rightarrow P + aQ$, $Q \rightarrow Q$ for some integer $a$. Under this $Q^2 \rightarrow Q^2$ and $Q.P \rightarrow Q.P + aQ^2$. Thus by choosing a suitable $a$ we can ensure that $|Q.P| \leq Q^2$. Under this duality transformation the new $P^2$ must be less than $Q^2$ and $|Q.P|$ since otherwise the discriminant will not be negative. If $P^2 \leq 0$ we proceed to step 2. Otherwise we repeat the process with $Q$ and $P$ exchanged so that $|Q.P|$ becomes less than $P^2$. By continuing this process we can ensure that either $Q^2$ or $P^2$ eventually becomes less than or equal to zero.

2. At the end of step 1, we shall have a charge vector for which either $Q^2 \leq 0$ or $P^2 \leq 0$. Using $Q \rightarrow P$, $P \rightarrow -Q$ symmetry we can ensure that $Q^2 \leq 0$. We now make a transformation $P \rightarrow P + bQ$ so that $P^2 \rightarrow P^2 + b^2Q^2 + 2bQ.P$. If $Q^2 < 0$, then by choosing $b$ to be sufficiently large we can make $P^2$ arbitrarily large and negative. If on the other hand $Q^2 = 0$ then by choosing $b$ to be sufficiently large in magnitude, and having a sign opposite to that of $Q.P$ we can again make $P^2$ sufficiently large and negative. We shall use this freedom to choose $P^2 \leq -4$ and call this the standard form.

Next we focus on the choice of the chamber in which we shall work. We shall work in the chamber $R$ shown in Figs. 1 and 2. In the $(\rho^2, \sigma^2, v^2)$ plane this corresponds to choosing $-\sigma^2, -\rho^2 < v^2 < 0$. In this case one can show that

$$k\rho^2 + l\sigma^2 + jv^2 > 0$$

(4.2)

for the range of values of $j,k,l$ over which the product runs in (2.6). Thus in each factor of the product we need to expand the $(1 - e^{2\pi i(k\rho + l\sigma + jv)})^{e^{2\pi i(4kl-j^2)}}$ factor in $1/\Phi_{10}$ in powers of $e^{2\pi i(k\rho + l\sigma + jv)}$ with $k,l \geq 0$ and $j < 0$ if $k = l = 0$. This in particular will mean that $g(m, n, p)$ vanishes for $m < -1$ or $n < -1$. Since the standard form in which we have brought $(Q, P)$, we have $P^2 \leq -4$, it follows that $d(Q, P) = g(P^2/2, Q^2/2, Q.P)$ vanishes in the chamber $R$. Thus if we can prove that in this chamber there are no two centered configurations contributing to the index, then we would have proved that in any chamber the index for negative discriminant states vanish after removing the contribution from the two centered configurations.

Thus our task now is to analyze the two centered bound states which could contribute to the index $d(Q, P)$ in the chamber $R$. As discussed at the end of §21 the possible charges carried
Figure 7: This figure displays two possibilities for how the chamber \( R \) is situated with respect to the wall connecting \( q/s \) to \( p/r \) in the \( \tau \) plane.

by the two centers are of the form

\[
(p\tilde{Q}, r\tilde{Q}), \quad (q\tilde{P}, s\tilde{P}), \quad \tilde{Q} = sQ - qP, \quad \tilde{P} = -rQ + pP, \quad \begin{pmatrix} p/q \\ r/s \end{pmatrix} \in SL(2, \mathbb{Z}), \quad s \geq 0,
\]

where the last condition is chosen using the freedom of changing the sign of \( \tilde{Q}, \tilde{P} \) and \( (p,q,r,s) \) simultaneously. It follows from (4.3) that

\[
P^2 = r^2\tilde{Q}^2 + s^2\tilde{P}^2 + 2rs\tilde{Q}\tilde{P}.
\]

The net index carried by such a bound state is given by

\[
(-1)^{\tilde{Q}\tilde{P} + 1} |\tilde{Q}\tilde{P}| f(\tilde{Q}^2)f(\tilde{P}^2).
\]

It was also shown at the end of §2 that in the \( \tau \) plane these bound states exist to the left of the wall connecting \( q/s \) to \( p/r \) if \( \tilde{Q}\tilde{P} > 0 \) and to the right of the same wall if \( \tilde{Q}\tilde{P} < 0 \). We shall now analyze various possibilities separately.

4.2 \( \tilde{Q}^2 \geq 0, \tilde{P}^2 \geq 0 \)

Since \( P^2 < 0 \), it follows from (4.4) that we must have \( rs < 0 \) for \( \tilde{Q}\tilde{P} > 0 \) and \( rs > 0 \) for \( \tilde{Q}\tilde{P} < 0 \). Now since \( ps - qr = 1 \) we have \( ps > qr \). Dividing both sides by \( rs \) we see that for \( \tilde{Q}\tilde{P} > 0 \) we have \( p/r < q/s \) and hence in the \( \tau \) plane the chamber \( R \) lies to the right of the wall connecting \( q/s \) to \( p/r \) (see Fig.7(a)), – precisely opposite to the side in which bound states exist. On the other hand for \( \tilde{Q}\tilde{P} < 0 \) we have \( rs > 0 \), \( p/r > q/s \), and hence the chamber \( R \)
Figure 8: The two walls of marginal stability corresponding to bound states of charges given in eqs. (4.3) and (4.6) for \( \tilde{P}^2 = -2, \tilde{Q}.\tilde{P} > 0 \). The bound states exists in the region between the two walls, and hence not in \( R \).

lies to the left of the wall connecting \( q/s \) to \( p/r \) (see Fig. 7(b)), – again on the side opposite to which the bound state exists. For \( \tilde{Q}.\tilde{P} = 0 \) there is no contribution to the index from such a bound state (see eq. (4.5)), so we need not consider this case. This shows that in the chamber \( R \) there are no bound states of charges of the form given in (4.3) as long as \( \tilde{Q}^2 \geq 0 \) and \( \tilde{P}^2 \geq 0 \).

4.3. \( \tilde{P}^2 = -2, \tilde{Q}^2 \geq 0 \) or \( \tilde{Q}^2 = -2, \tilde{P}^2 \geq 0 \)

Consider first the case when \( \tilde{P}^2 = -2, \tilde{Q}^2 \geq 0, \tilde{Q}.\tilde{P} > 0 \). In this case if \( rs < 0 \) then the argument of the previous paragraph establishes that there is no bound state of charges given in (4.3) in the chamber \( R \). Thus we need to analyze the case when \( rs \geq 0 \). First consider the case \( rs > 0 \) so that we have \( p/r > q/s \). In this case the chamber \( R \) lies to the left of the wall connecting \( q/s \) to \( p/r \), – the same side on which we have a two centered solution carrying the charges given in (4.3) (see Fig. 8). This would seem to indicate that in the chamber \( R \) there is a bound state of the charges given in (4.3) carrying total index given in (4.5). We shall however argue that we encounter another wall before reaching the chamber \( R \) across which this bound state ceases to exist. To see this let us examine another bound state carrying total charge \((Q, P)\) where the individual centers carry charges

\[
(p'\tilde{Q}', r'\tilde{Q}'), \quad (q'\tilde{P}', s'\tilde{P}'), \quad \tilde{Q}' = s'Q - q'P, \quad \tilde{P}' = -r'Q + p'P,
\]

\[
\begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix} = \begin{pmatrix} p & q - pu \\ r & s - ru \end{pmatrix}, \quad \left( \frac{\tilde{Q}'}{\tilde{P}'} \right) = \left( \frac{\tilde{Q} + u\tilde{P}}{\tilde{P}} \right), \quad u \equiv \tilde{Q}.\tilde{P}. \tag{4.6}
\]
This has the same index as (4.5), and according to the bound state metamorphosis proposal of §3 this bound state should be identified with the one given in (4.3). Now it follows from (4.4) and the $P_2 < 0$ condition that

$$s/r > \tilde{Q} \tilde{P}.$$  \hfill (4.7)

From (4.7), (4.6) we get

$$\frac{q'}{s'} = \frac{q - pu}{s - ru} < \frac{q}{s},$$  \hfill (4.8)

as shown in Fig. 8. Thus the wall connecting $q'/s'$ to $p'/r' = p/r$ lies in between the wall connecting $q/s$ to $p/r$ and the chamber $R$. Furthermore since $\tilde{Q}' \tilde{P}' = -\tilde{Q} \tilde{P} < 0$, the bound state of the charges given in (4.6) exists to the right of the wall connecting $q'/s'$ to $p/r$. Thus according to the prescription of §3 such a bound state will exist only in the region bounded by the walls connecting $q/s$ to $p/r$ and $q'/s'$ to $p/r$ (see Fig. 8). In particular it will not exist in the chamber $R$.

Next consider the case where $rs = 0$. It follows from (4.4) and that $\tilde{Q}^2 \geq 0, P^2 \leq -4$ that $s$ cannot vanish, hence $r$ must vanish. In this case we have $ps = 1$ and hence $p = s = \pm 1$. Eq. (4.4) now gives $P^2 = -2$ which contradicts the fact that $P^2$ has been chosen to be $\leq -4$. Thus we see that $rs$ cannot vanish.

The case when $\tilde{Q} \tilde{P} < 0$ can be dealt with in the same way as in the previous case with some changes in sign. In this case the undesirable situation arises when $rs < 0$ so that we have $p/r < q/s$, and the chamber $R$ lies to the right of the wall connecting $q/s$ to $p/r$, – the same side on which the bound state of the charges given in (4.3) exists. One can now repeat the argument given for $\tilde{Q} \tilde{P} > 0$ case to arrive at a diagram similar to that in Fig. 8 but with the end points of the walls arranged in the order $q'/s' > q/s > p/r$. The bound state exists in the region between the walls connecting $q/s$ to $p/r$ and $q'/s'$ to $p/r$, and not in the chamber $R$.

The case where $\tilde{Q}^2 = -2, \tilde{P}^2 \geq 0$ is related to the case discussed above by S-duality transformation $\tilde{Q} \rightarrow \tilde{P}, \tilde{P} \rightarrow -\tilde{Q}$, and so need not be analyzed separately.

4.4 $\tilde{Q}^2 = \tilde{P}^2 = -2$

First we note that in this case we cannot have $rs = 0$, since if $r = 0$ we have $p = s = \pm 1$ and if $s = 0$ we have $q = -r = \pm 1$. In both cases eq. (4.4) will give $P^2 = -2$ contradicting the assumption that $P^2 \leq -4$. Thus we choose $rs \neq 0$. First consider the case $\tilde{Q} \tilde{P} > 0$. In this case for $rs < 0$ we have $p/r < q/s$ and the chamber $R$ lies to the right of the wall connecting $q/s$ to $p/r$ as in Fig. 7(a) – on the opposite side of the domain in which the bound
state exists. Thus the problematic case, where the bound state exists on the same side of the wall of marginal stability as the chamber $\mathbf{R}$, arises for

$$\tilde{Q} \cdot \tilde{P} > 0, \quad rs > 0,$$

so that we have $q/s < p/r$ (see Fig. 9(a) and (b)). We now consider two other bound states carrying charges:

$$\left( p', r', \tilde{Q}' \right), \quad \left( q', s', \tilde{P}' \right), \quad \tilde{Q}' = s'Q - q'P, \quad \tilde{P}' = -r'Q + p'P,$$

$$\left( \frac{p'}{r'} \quad \frac{q'}{s'} \right) = \left( \frac{p - pu}{r - su} \quad \frac{q}{s} \right), \quad \left( \frac{\tilde{Q}'}{\tilde{P}'} \right) = \left( \frac{\tilde{Q} + u\tilde{P}}{\tilde{P}} \right), \quad u \equiv \tilde{Q}, \tilde{P}. \quad (4.10)$$

and

$$\left( p'', r'', \tilde{Q}'' \right), \quad \left( q'', s'', \tilde{P}'' \right), \quad \tilde{Q}'' = s''Q - q''P, \quad \tilde{P}'' = -r''Q + p''P,$$

$$\left( \frac{p''}{r''} \quad \frac{q''}{s''} \right) = \left( \frac{p - qu}{r - su} \quad \frac{q}{s} \right), \quad \left( \frac{\tilde{Q}''}{\tilde{P}''} \right) = \left( \frac{\tilde{Q} + u\tilde{P}}{\tilde{P}} \right), \quad u \equiv \tilde{Q}, \tilde{P}. \quad (4.11)$$

It can be checked that $\tilde{Q}'^2 = \tilde{Q}^2$, $\tilde{P}'^2 = \tilde{P}^2$ and $\tilde{Q}' \cdot \tilde{P}' = \tilde{Q}'' \cdot \tilde{P}'' = -\tilde{Q} \cdot \tilde{P}$. Thus these bound states carry the same index as the bound state of the charges given in (4.3). By applying the transformations (4.10) and (4.11) alternatively (with $u \rightarrow -u$ at every step to account for the change in the sign of $\tilde{Q} \cdot \tilde{P}$) we can generate a series of other bound states with the same index. According to the proposal of §3 all these bound states represent the same physical state.

Since $\tilde{Q}' \cdot \tilde{P}'$ and $\tilde{Q}'' \cdot \tilde{P}''$ are both negative, the bound state described in (4.10) exists to the right of the wall connecting $q'/s'$ to $p/r$ and the bound state given in (4.11) exists on the right of the wall connecting $q/s$ to $p''/r''$. Let us first consider the case $r > s$. In this case we must have $u < r/s + 1$ since if $u \geq 1 + r/s$, it follows from (4.4) and (4.9) that

$$P^2 \geq -2r^2 - 2s^2 + 2rs \left( 1 + \frac{r}{s} \right) \geq 0,$$

contradicting the fact that $P^2 \leq -4$. Furthermore it can be seen as follows that $u$ cannot be equal to $r/s$. Since $u \in \mathbb{Z}$, $u = r/s$ will imply that $r/s$ must be an integer. Since $r$ and $s$ are relatively prime this would imply $s = 1$, $r = u = \tilde{Q} \cdot \tilde{P}$, and from (4.4) we shall get $P^2 = -2$. This contradicts the fact that we have chosen $P^2 \leq -4$. Thus we are left with two possibilities: $u < r/s$ and $r/s < u < r/s + 1$. Let us first assume that $u < r/s$. Furthermore
since $u \geq 1$ and $s/r < 1$ we have $u > s/r$. In this case it follows from (4.10) and (4.11) that $p''/r'' > q'/s' > p/r > q/s$ and the different points are arranged on the real $\tau$ axis as shown in Fig. 9(a). Since the bound state exists only to the right of the wall connecting $q/s$ to $p''/r''$, we see that it does not exist in the chamber $\mathbb{R}$. The case $s > r$ with $u < s/r$ can be analyzed similarly and leads to the arrangement described in Fig. 9(b). Since the bound state exists to the right of the wall connecting $q'/s'$ to $p/r$, we again see that it does not exist in chamber $\mathbb{R}$. This leaves us with the cases $r/s < s/r < u < s/r + 1$ and $s/r < r/s < u < r/s + 1$.

Before we discuss these cases let us see what happens when $u = \tilde{Q} / \tilde{P} < 0$. Here the problematic case arises when $s/r < 0$ so that we have $q/s > p/r$ and the bound state exists to the right of the wall connecting $q/s$ to $p/r$. The same side on which the chamber $\mathbb{R}$ lies. Using logic similar to the one for the $\tilde{Q} / \tilde{P} > 0$ case one can argue that for $|s/r| < |u| < |s/r|$ case the arrangements of various points are opposite to that given in Fig. 9(a) and the bound state lies to the left of the wall connecting $q/s$ to $p''/r''$ so that there is no bound state in the chamber $\mathbb{R}$. Similarly for $|r/s| < |u| < |s/r|$ we can show that the arrangements of various points are opposite to that given in Fig. 9(b) and the bound state lies to the left of the wall connecting $q'/s'$ to $p/r$ so that there is no bound state in the chamber $\mathbb{R}$. Thus here again we are left to deal with the cases $|s/r| < |r/s| < |u| < |s/r| + 1$ and $|r/s| < |s/r| < |u| < |s/r| + 1$, which, taking into account the negative signs of $u$ and $sr$, can be expressed as $r/s - 1 < u < r/s < s/r$ and $s/r - 1 < u < s/r < r/s$.

Thus it remains to analyze the cases $r/s < s/r < u < s/r + 1$ and $s/r < r/s < u < r/s + 1$. 

Figure 9: The arrangement of different walls of marginal stability for bound states of charges given in (4.3), (4.10) and (4.11) for $\tilde{Q}^2 = \tilde{P}^2 = -2$ and $\tilde{Q} / \tilde{P} > 0$. Fig.(a) shows the arrangement for $r > s$ and Fig.(b) shows the arrangement for $r < s$. In both cases the bound states exist in the region between the walls and hence not in $\mathbb{R}$. 

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for \( sr > 0, u > 0 \) and \( r/s - 1 < u < r/s < s/r \) and \( s/r - 1 < u < s/r < r/s \) for \( sr < 0, u < 0 \).

It can be seen that

\[
\begin{align*}
sr > 0, \quad u > 0, \quad s/r < 1, \quad r/s < u < r/s + 1 & \Rightarrow r''s'' < 0, \quad |r''s''| < rs \\

sr > 0, \quad u > 0, \quad s/r > 1, \quad s/r < u < s/r + 1 & \Rightarrow r's' < 0, \quad |r's'| < rs \\

sr < 0, \quad u \leq 0, \quad |s/r| < 1, \quad r/s - 1 < u < r/s & \Rightarrow r''s'' > 0, \quad r''s'' < |rs| \\

sr < 0, \quad u \leq 0, \quad |s/r| > 1, \quad s/r - 1 < u < s/r & \Rightarrow r's' > 0, \quad r's' < |rs|.
\end{align*}
\]

(4.13)

The common feature of all these transformations is that the value of \(|rs|\) is reduced under this transformation. Given the new values of \((p, q, r, s)\) there are two possibilities: either \(|u|\) lies between \(|r/s|\) and \(|r/s| + 1\) for \(|r| > |s|\) and between \(|s/r|\) and \(|s/r| + 1\) for \(|s| > |r|\) or it lies outside this range. In the former case we can apply (4.13) again to further reduce the value of \(|rs|\). In the second case we can use the results of our previous analysis to conclude that at the next step either the wall connecting \(q'/s'\) to \(p/r\) or the wall connecting \(q/s\) to \(p''/r''\) will shield the chamber \(R\) from the region where the bound state exists. This process will have to stop eventually since \(|rs|\) has a lower bound of 1, and for \(|rs| = 1\) we have \(|r/s| = |s/r| = 1\), making it impossible to have \(|r/s| < |u| < |r/s| + 1\) or \(|s/r| < |u| < |s/r| + 1\) with non-zero integer \(u\). Thus we shall eventually produce a wall which will shield from \(R\) the region in which the bound state exists.

This establishes that in the standard form \(P^2 \leq -4\), there are no two centered bound states contributing to the index in the chamber \(R\). Since we have already seen that the microscopic index vanishes in this case, we conclude that the contribution to the index from single centered black holes must vanish in this case. This is in agreement with the macroscopic result that there are no single centered black holes for charge vectors with negative discriminant.

### 4.5 Explicit results

The above discussion has been somewhat abstract, but it allows us to identify precisely which two centered black holes contribute to the index in any given chamber. For this we make an appropriate duality transformation to bring the charge vector to the ‘standard form’ \(P^2 \leq -4\). This takes the original chamber to some other chamber \(C\). We can then find an appropriate path from \(R\) to \(C\), examine the walls crossed by this path and compute the contribution given in (2.18) from the 2-centered black hole solutions which appear as we move from the chamber \(R\) to \(C\). The net contribution to the index is obtained by adding up these contributions. To
determine the constituents in the original duality frame we need to make an inverse duality transformation to go back to the original frame. We have carried out this analysis for all the negative discriminant charge vectors appearing in table 1. The results are shown in table 2. We can see that the last column agrees with the entries for the index appearing in table 1, confirming that all the contributions to the index of negative discriminant states come from two centered black hole solutions.

5 Generalization to CHL models

In this section we shall discuss generalization of our analysis of §4 to CHL models. CHL models are obtained by taking a $\mathbb{Z}_N$ orbifold of heterotic string theory on $T^6$, where the $Z_N$ acts as a translation by $2\pi/N$ along one of the circles, and a $\mathbb{Z}_N$ rotation on the left-movers satisfying the level matching condition [52–55]. The index for a class of quarter BPS dyons in these theories is known exactly [6–9, 11, 12] and has a form similar to the one given in (2.3), (2.4) with $\Phi_{10}$ replaced by another modular form $\Phi$. For simplicity we shall restrict our analysis to the models with prime values of $N$: $N = 2, 3, 5$ and 7. In this case the main differences in the analysis arise due to the following reasons:

1. The new function $\Phi$ has period 1 in $\rho$ and $v$ but period $N$ in $\sigma$. As a result $Q^2$ is quantized in units of $1/N$.

2. The prefactor $e^{2\pi i(\rho+\sigma+v)}$ is replaced by $e^{2\pi i(\rho+\sigma/N+v)}$. As a result in the chamber $R$ the lowest values of $P^2$ and $Q^2$ for which $d(Q,P)$ is non-zero are $-2$ and $-2/N$ respectively. The chamber $R$ is defined as before as the chamber to the right of the wall connecting 0 to $i\infty$ in the $\tau$ plane. In the $(v_2/\sigma_2, \rho_2/\sigma_2)$ plane it represents the chamber to the left of the $v_2 = 0$ line.

3. The $SL(2, \mathbb{Z})$ S-duality group appearing in (2.13) is reduced to $\Gamma_1(N)$ described by the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1, \quad c = 0 \text{ mod } N, \quad a, d = 1 \text{ mod } N, \quad b \in \mathbb{Z}. \quad (5.1)$$

In fact it was shown in [9] that the modular form $\Phi$ is actually invariant under the $\Gamma_0(N)$ subgroup of $SL(2, \mathbb{Z})$ where we relax the conditions $a, d = 1 \text{ mod } N$ to $a, d \in \mathbb{Z}$. 

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4. The possible 2-centered bound states contributing to the index arise from configurations described at the end of §2 with the additional restriction that $r \in N \mathbb{Z}$ [18]. This implies that

$$\left( \begin{array}{cc} p & q \\ r & s \end{array} \right) \in \Gamma_0(N).$$

(5.2)

Furthermore eq. (2.18) for the index carried by the bound state is modified to a different formula

$$(-1)^{Q.P+1} |(sQ - qP)(-rQ + pP)| f_1((sQ - qP)^2/2) f_2((-rQ + pP)^2/2),$$

(5.3)

where $f_1(n)$ and $f_2(n)$ are defined via the equations:

$$q^{-1/N} \prod_{k=1}^{\infty} (1 - q^k)^{-24/(N+1)} (1 - q^{k/N})^{-24/(N+1)} = \sum_{n=-1/N}^{\infty} f_1(n)q^n,$$

$$q^{-1} \prod_{k=1}^{\infty} (1 - q^k)^{-24/(N+1)} (1 - q^{kN})^{-24/(N+1)} = \sum_{n=-1}^{\infty} f_2(n)q^n.$$

(5.4)

For $N = 1$, $f_1$ and $f_2$ both reduce to $f(n)$ defined in eq. (2.15) and we recover the result for heterotic string theory on $T^6$.

Let us now reexamine our analysis of §4 for these models. First of all we need to show that we can bring a charge vector $(Q, P)$ to the standard form where $P^2 \leq -4$ so that the index of the state vanishes in the chamber $R$. If $Q^2 < 0$, then by making a $P \rightarrow P + KNQ$, $Q \rightarrow Q$ transformation we get $P^2 \rightarrow P^2 + 2KNQ.P + K^2N^2Q^2$, and this can be made arbitrarily large negative for sufficiently large integer $K$. For $Q^2 = 0$ the same transformation works if we take $K$ to have the opposite sign of $Q \cdot P$. For $Q^2 > 0$ we consider a $\Gamma_1(N)$ matrix with the following values of $c, d$:

$$c = -Q.PNLK, \quad d = Q^2NLK + 1, \quad L = \text{l.c.m.}(Q,P,Q^2),$$

(5.5)

where $K$ is a large integer and l.c.m. stands for the lowest common multiple. In this case $c$ and $d$ cannot contain a common factor since all prime factors of $c$ are also prime factors of $(d - 1)$ by construction. Thus there exist $a$ and $b$ such that $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z})$. Furthermore by construction $c = 0 \mod N$ and $d = 1 \mod N$, and hence $ad - bc = 1$ gives $a = 1 \mod N$. Thus $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_1(N)$ and is an allowed duality transformation. It now follows from (2.13) and (5.5) that under this transformation

$$P^2 \rightarrow K^2 \left[ L^2N^2Q^2 (Q^2P^2 - (Q.P)^2) + \mathcal{O}(K^{-1}) \right].$$

(5.6)
Thus by taking $K$ to be sufficiently large and using the fact that $Q^2P^2 - (Q.P)^2 < 0$ we can make $P^2$ arbitrarily large and negative. This allows us to bring $(Q,P)$ to the standard form $P^2 \leq -4$.

Thus we now need to show following the analysis of §4 that for the new charge vector in the chamber $R$ there are no two centered bound states. We consider the possible bound states of the form (4.3) with \[
\left( \begin{array}{c}
p \\
r \\
q \\
q \\
s \\
\end{array} \right) \in \Gamma_0(N).
\] The case $\tilde{P}^2 \geq 0, \tilde{Q}^2 \geq 0$ proceeds as in the $T^6$ case. The case $P^2 = -2, \tilde{Q}^2 \geq 0$ is also identical, – the only point to note is that the transformation $\tilde{Q}' = \tilde{Q} + u\tilde{P}, \tilde{P}' = \tilde{P}$ with $u = Q.P$ correspond to multiplication by the matrix \[
\left( \begin{array}{cc}
1 & u \\
0 & 1 \\
\end{array} \right) \in \Gamma_1(N) \text{ and hence is an allowed duality transformation.}
\] The cases $\tilde{Q}^2 = -2/N$ needs a different analysis however. In this case a transformation $\tilde{Q} \rightarrow \tilde{Q}, \tilde{P} \rightarrow \tilde{P} + u\tilde{Q}$ appearing in (4.11) is neither a valid duality transformation, nor does it leave $\tilde{P}^2$ invariant. Instead we use the transformation is $\tilde{Q} \rightarrow \tilde{Q}, \tilde{P} \rightarrow \tilde{P} + Nu\tilde{Q}$. This is a valid duality transformation and has the effect of leaving $\tilde{Q}^2$ and $\tilde{P}^2$ invariant, and changing the sign of $\tilde{Q} \cdot \tilde{P}$. Thus we identify bound states of constituents related by this transformation. With this hypothesis we find, after a somewhat lengthy analysis along the lines of §4 that there are no two centered bound states in the chamber $R$ for $P^2 \leq -4$. This is turn establishes that the negative discriminant states in any chamber of the moduli space can be accounted for by the contribution to the index from multi-centered black holes.

### 6 Conclusion

Our analysis shows that the negative discriminant states in the microscopic spectrum of a class of $\mathcal{N} = 4$ supersymmetric string theories can be accounted for as bound states of two centered black holes. This is consistent with the fact that single centered black hole solutions exist only for charges with positive discriminant and shows that the description of the system as a black hole can capture exact information on the system even for finite charges. This in turn suggests that quantum gravity in the near horizon geometry may provide an exact dual description of the system instead of just being an emergent description that works only for large charges. The result of §31 showing that quantum gravity in the the near horizon geometry correctly predicts the sign of the index also points to the same conclusion.

It will clearly be interesting to extend this analysis to states carrying torsion ($I$ defined in eq.(2.2)) larger than 1. The bound state metamorphosis rules described in §3, §4 are likely to
be more complicated for these states. It will also be useful to understand the origin of these rules directly from the classical analysis of two centered solutions.

Another direction that should be explored is the macroscopic origin of the states with zero discriminant. There are plenty of examples (e.g. all states carrying $Q^2 = P^2 = Q \cdot P$ in table 1) for which the index does not vanish in the chamber $R$, but there are no two centered configurations contributing to this index. Smooth horizonless classical solutions constructed along the lines of [69] might play a role in providing the macroscopic description of these states.

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