From braids to transverse slices in reductive groups

Wicher Malten

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Abstract

In 1965, Steinberg’s study of conjugacy classes in connected reductive groups led him to construct an affine subspace parametrising regular conjugacy classes, which he noticed is also a cross section for the conjugation action by the unipotent radical of a Borel subgroup on another affine subspace. Recently, generalisations of this slice and its cross section property have been obtained by Sevostyanov in the context of quantum group analogues of $W$-algebras and by He-Lusztig in the context of Deligne-Lusztig varieties. Such slices are often thought of as group analogues of Slodowy slices.

In this paper we explain their relationship via common generalisations associated to Weyl group elements and provide a simple criterion for cross sections in terms of roots. In the most important class of examples this criterion is equivalent to a statement about the Deligne-Garside factors of their powers in the braid monoid being maximal in some sense. Moreover, we show that these subvarieties transversely intersect conjugacy classes and determine for a large class of factorisable $r$-matrices when the Semenov-Tian-Shansky bracket reduces to a Poisson structure on these slices.

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1 Introduction

In 1965, Steinberg’s study of conjugacy classes in a connected reductive group $G$ led him to analyse its regular elements, characterising them in various ways: they can be defined as the elements whose conjugacy class has minimal codimension (which is equal to the rank of $G$), and together these elements form a dense open
subset. Fixing a Borel subgroup $B$ and maximal torus $T$, denoting by $N_+ = [B, B]$ the unipotent radical of $B$, by $N_-$ its opposite, by $W = N_G(T)/T$ the Weyl group, writing for $w$ in $W$ then $N_w := N_+ \cap w^{-1} N_- w$ and $\dot{w}$ for arbitrary lifts to the normaliser $N_G(T)$ of $T$ in $G$ and finally fixing $w$ to be a Coxeter element of minimal length, he proves that the Steinberg slice $\dot{w}N_w$ yields a cross section of these regular conjugacy classes [Ste65, §1.4].

Example 1.1 ([Ste65, §7.4]). Let $G = SL_{rk+1}$ over a scheme $S$ and consider the Coxeter element $w = s_{rk} \cdots s_1$. The usual lift $\dot{w}$ yields the space of Frobenius companion matrices

$$\dot{w}N_w = \begin{bmatrix}
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 1 \\
(-1)^{rk} c_{rk} & \cdots & c_2 & c_1
\end{bmatrix} : c_1, \ldots, c_{rk} \in \mathcal{O}_S.$$ 

Furthermore, Steinberg settles the existence of regular unipotent elements by locating some of them inside of the Bruhat cell $BwB$ [Ste65, §4]. He subsequently remarks that actually $BwB$ consists entirely of regular elements, which he notes follows from the same property of $\dot{w}N_w$ in combination with the isomorphism

$$N_+ \times \dot{w}N_w \cong N_+ \dot{w}N_+, \quad (n, g) \mapsto n^{-1}gn,$$

but the proof of that isomorphism is missing [Ste65, §8.9]. In the late 1970s this map was investigated by Spaltenstein, who found an example in type $A_5$ (see Example 1.15) showing that it is not necessarily an isomorphism when $w$ is replaced by a Coxeter element which is not of minimal length [HL12, §0.4]. Very recently, generalisations of this cross section appeared in two different settings:

Perhaps the most natural way to construct $W$-algebras is by applying quantum Hamiltonian reduction to a certain character of the Lie algebra $\mathfrak{n}$ of a subgroup $N$ of $N_+$ generated by root subgroups [dBT93, Pre02, GG02]. For example, Kostant’s study in 1978 of Whittaker representations for the Langlands program led him to construct the principal finite $W$-algebra [Kos78], which can be interpreted as applying this procedure to a regular character of the Lie algebra $\mathfrak{n}_+$ of $N_+$. The character of $\mathfrak{n}$ dequantises to a symplectic point of $\mathfrak{n}^*$ via Dixmier’s map [Dix63] and the semiclassical limit of the corresponding finite $W$-algebra is then obtained by reducing the inverse image of this point under the momentum map for the restriction of the coadjoint action of $G$ on the dual $\mathfrak{g}^*$ of its Lie algebra to $N$. The quantum group analogue of that reduces inverse images of symplectic points under momentum maps for the restriction of the dressing action of a factorisable Poisson-Lie group $G$ (with the Drinfeld-Sklyanin bracket) on its dual Poisson-Lie group $G^*$ (which is quantised by the Drinfeld-Jimbo quantum group $U_q(G)$) to coisotropic subgroups $N$ of $N_+$ generated by root subgroups. One can exponentiate the symplectic point in $\mathfrak{n}_+^*$ to a point in the dual $N_+^*$ of $N_+$, and show directly that it is symplectic if and only if the standard $r$-matrix on $G$ is modified by the Cayley transform of a Coxeter element of minimal length [Mal]. Denoting by $G_*$ the variety $G$ equipped with the corresponding Semenov-Tian-Shansky bracket (which is quantised by the integrable part $U^\text{int} G$ of $U_q G$), the resulting reduction in $G^*$ covers a reduction in $G_*$ along the $G$-equivariant factorisation mapping $G^* \to G_*$. More precisely, this is the Poisson reduction of $N_+ \dot{w}N_+ \subset G_*$ by the conjugation action of $N_+ \subset G$, so the cross section suggests that a certain reduction of $U^\text{int}_q G$ should quantise the Steinberg slice $\dot{w}N_w$. The differential graded algebra corresponding to the principal case was thoroughly studied in work of Sevostyanov [Sev00], the author [Mal], and Grojnowski [Gro]. Whilst searching for quantum group analogues of non-principal $W$-algebras, Sevostyanov generalised the Steinberg slice to some elements in every conjugacy class of the Weyl group, and proved that for any those elements a conjugation map similar to (1.1) is an isomorphism [Sev11].

All infinite families of finite simple groups, except those of the alternating groups, consist of finite reductive groups. In 1976, Deligne and Lusztig proved that all of their characters can be obtained from the compactly
supported étale cohomology of certain algebraic varieties $X_w^G$ constructed out of such groups [DL76], using a twist $F$ (a Frobenius morphism) and an element $w$ of the Weyl group. They showed that these virtual representations do not depend on the twisted conjugacy class of $w$ [DL76, Theorem 1.6], reducing much of the study of these Deligne-Lusztig varieties to minimal length elements. Any minimal length element is elliptic in a standard parabolic corresponding to a Levi subgroup $L$, and one can show that the orbit space satisfies $G^F \backslash X_w^G \simeq L^F \backslash X_w^L$, thus reducing to the case where $w$ is elliptic and has minimal length. Lusztig and his former student He generalised the cross section (1.4) to these elements, and deduced from this that $G^F \backslash X_w^G$ is universally homeomorphic to the quotient of $\mathcal{H}^{(w)}$ by a finite torus [HL12] (which implies a cohomology vanishing theorem [OR08], simplifying Lusztig’s classification of the representations of finite reductive groups [Lus84]). This statement was later generalised by He to obtain a dimension formula for affine Deligne-Lusztig varieties [He14], which are closely related to the reduction of integral models of Shimura varieties in arithmetic geometry. He and Lusztig mention that it is not clear to us what is the relation of the Weyl group elements considered [by Sevostyanov] with those considered in this paper (HL12, §0.3)) and the original aim of this paper and its prequel [Mal21] was to provide a common generalisation explaining this.

Following work by Springer [Spr69], Grothendieck around 1969 obtained a simultaneous resulution of the singularities of the fibres of $G \to G/\!/G \simeq T/\!/W$ and conjectured that a transverse slice to a subregular element of $G$ would yield a universal deformation of the corresponding Du Val-Klein singularity, and similarly for its Lie algebra. This was proven by Brieskorn [Bri71], and Slodowy thoroughly studied these deformations through suitable slices in the Lie algebra [Slo80]. Already appearing in the work of Harish-Chandra on invariant distributions on Lie algebras [HC64], Slodowy slices play a crucial rôle in the classification of Whittaker representations as the semiclassical limits of $W$-algebras [Kos78, Pre02], have recently been applied to reconstruct Khovanov homology [SS06, AS19] and numerous physicists are currently using them in their study of supersymmetric gauge theories (e.g. [GW09a, GW09b]).

Meanwhile, Drinfeld showed that every reductive Lie algebra (interpreted as a Casimir Lie algebra with the Killing form) admits a unique quantisation, up to gauge transformations [Dri89]; semiclassically, this corresponds to a choice of factorisable $r$-matrix, and those were classified by Belavin-Drinfeld [BD82]. In his study of zero curvature conditions for soliton equations, Semenov-Tian-Shansky discovered a natural Poisson bracket on $G$ associated with these $r$-matrices, which is compatible with conjugation [STS85].

By using the Kazhdan-Lusztig map [KL88], Sevostyanov associated certain Weyl group elements to the subregular unipotent classes and determined that his corresponding slices have the correct dimension, yielding analogous results. He subsequently showed that after modifying the standard $r$-matrix, the Semenov-Tian-Shansky bracket on $G$ reduces to his slices and lifts to Poisson structures on these minimal surface singularities [Sev11]. Very recently, it was shown that the analogue of the Grothendieck-Springer resolution for $G$-bundles on elliptic curves [BZN15, Dav19] yields del Pezzo surfaces [GSB21, Dav20], whilst slices in the affine Grassmannian were quantised [KWWY14] and immediately applied in the study of Coulomb branches in physics (e.g. [BDG17, BFN19]).

Employing Lusztig’s recent stratification of $G$ by unions of sheets of conjugacy classes [Lus15], Sevostyanov verified that any conjugacy class of $G$ is strictly transversally intersected by one of his slices [Sev19]. This forms the main ingredient in Sevostyanov’s approach [Sev21] to the long-standing De Concini-Kac-Procesi conjecture on the dimensions of irreducible representations of quantum groups at roots of unity [DCKP92, §6.8] (the analogous Kac-Weisfeiler conjecture for Lie algebras [VK71] was proven with Slodowy slices [Pre95]). Furthermore, Sevostyanov’s slices have been used in an attempt to obtain a group analogue [CE15] of Katsylo’s theorem [Kat82], which itself has been applied e.g. to analyse the space of one-dimensional representations of finite $W$-algebras and to the theory of primitive ideals [PT14].

It is natural in the theory of reductive Poisson-Lie groups and Drinfeld-Jimbo quantum groups to fix a torus and Borel subgroup as before. Recall from the introduction of [Mal21] the notion and notation for twisted finite Coxeter groups, (firmly) convex elements, reduced braids, braid power bounds, roots, stable...
roots, convex sets of roots, inversion sets, weak left Bruhat-Chevalley orders, right Deligne-Garside factors and normal forms and the braid equation

\[ \text{DG}(b_w^d) = \text{pb}(w). \] (1.2)

The relevance of this equation to the main theorem is through its close relationship with

**Definition 1.2.** Let \( w \) be an element of a twisted Weyl group. For any positive root \( \beta \) of its root system, we construct a subset of positive roots

\[ \text{cross}_w(\beta) := \{ w(\beta + \sum_{i=1}^m \beta_i) \in \mathcal{R} : \beta_1, \ldots, \beta_m \in \mathcal{R}_w, m \geq 0 \} \cap \mathcal{R}_+. \]

This extends to a map

\[ \text{cross}_w : \{ \text{subsets of } \mathcal{R}_+ \} \rightarrow \{ \text{subsets of } \mathcal{R}_+ \}, \quad \mathcal{N} \mapsto \bigcup_{\beta \in \mathcal{N}} \text{cross}_w(\beta), \]

and we let \( \text{cross}_w^d(\cdot) \) denote its \( d \)-th iterate for integers \( d \geq 0 \).

A more group-theoretic interpretation is given in Corollary 3.5, where it is explained that \( \text{cross}_w(\cdot) \) is a (simplification of a) first-order approximation to the polynomial equations appearing in our generalisation of the cross section (1.1). In Lemma 2.6 we prove that \( \text{cross}_w^d(\cdot) \) naturally extends to twisted braid monoids, which is one of the ingredients in the proof of the following

**Lemma.** Let \( w \) be an element of a twisted Weyl group \( W \).

(i) For any simple root \( \alpha \) not in the inversion set \( \mathcal{R}_w \), the set \( \text{cross}_w(\alpha) \) contains simple roots.

(ii) For any other element \( w' \) of \( W \) and integer \( d \geq 0 \), we have

\[ \text{DG}(b_w^d) \geq w' \quad \text{if and only if} \quad \text{cross}_w^d(\mathcal{R}_{w'}) = \emptyset, \]

if and only if \( \text{cross}_w^d(\mathcal{R}_{w'}) \) does not contain any simple roots.

(iii) Moreover, for any \( d > |\mathcal{R}_+ \setminus \mathcal{R}_w| - \ell(w) \) we have

\[ w \text{ is convex and satisfies (1.2) if and only if} \quad \text{cross}_w^d(\mathcal{R}_+ \setminus \mathcal{R}_w) = \emptyset, \]

if and only if \( \text{cross}_w^d(\mathcal{R}_+ \setminus \mathcal{R}_w) \) does not contain any simple roots.

**Example 1.3.** Consider \( w = s_2 \) in type \( B_2 \), then the only simple root in \( \text{cross}_w(\alpha_1) \) is \( \alpha_1 = w(\alpha_1 + 2\alpha_2) \).

**Example 1.4.** Consider \( w = s_1 s_3 \) in type \( A_3 \), then the only simple root in \( \text{cross}_w(\alpha_2) \) is \( \alpha_2 = w(\alpha_2 + \alpha_1 + \alpha_3) \).

Our proof of transversality involves root combinatorics which through this lemma is closely intertwined with

**Corollary A.** Let \( w \) be a convex element of a twisted Weyl group. Then

\[ \text{DG}(b_w^d) = \text{pb}(w) \quad \text{if and only if} \quad \text{DG}(b_{w-1}^d) = \text{pb}(w). \]

Typically however, these Deligne-Garside factors are not very similar:

**Example 1.5.** Consider \( w = s_1 s_2 s_3 s_2 s_1 s_2 \) in type \( B_3 \); it is convex and for any integer \( d > 1 \) we have

\[ \text{DGN}(b_w^d) = b_w^d \quad \text{and} \quad \text{DGN}(b_{w-1}^d) = b_{w-1 s_1} b_{w-1 s_2} b_{w-1 s_3} b_{w-1}. \]
The perspective furnished by Definition 1.2 allows us to construct cross sections out of quite general data:

**Definition 1.6.** Let \( w \) be an element of a twisted Weyl group and let \( \mathcal{R} \subseteq \mathcal{R}^+_w \) be a convex subset of positive roots. Then we will say that \( \mathcal{R} \) is \((w-)nimble\) if it contains \( \mathcal{R}_w \) and furthermore

\[
w(\mathcal{R}\setminus\mathcal{R}_w) \subseteq \mathcal{R}.
\]

**Example 1.7.** From [Mal21, Proposition 4.21] it follows that the set \( \mathcal{R}_+ \setminus \mathcal{R}_{st}^w \) is nimble itself if and only if the element \( w \) is convex.

**Notation 1.8.** Given two subsets of roots \( \mathcal{R}, \mathcal{L} \) of a root system \( \mathcal{R} \), we write

\[
\mathcal{R} + \mathcal{L} := \{ c_0 \beta_0 + c_1 \beta_1 \in \mathcal{R} : \beta_0 \in \mathcal{R}, \beta_1 \in \mathcal{L}, c_0, c_1 \in \mathbb{R}_{>0} \}.
\]

**Definition 1.9.** Let \( w \) be an element of a twisted Weyl group and let \( \mathcal{R} \) a \( w \)-nimble set. We will say that a convex subset of roots \( \mathcal{L} \subseteq \mathcal{R} \setminus \mathcal{R}_w \) is a \((\mathcal{R},-)leavener\) when

(i) \( w(\mathcal{L}) = \mathcal{L} = -\mathcal{L} \),

(ii) the set \( \mathcal{R}_w \cup \mathcal{L} \) is convex, and

(iii) the set \( \mathcal{R} \cup \mathcal{L} \) is convex.

We will then refer to \((\mathcal{R}, \mathcal{L})\) as a \((w-)crossing pair\). Note that (i) implies \( \mathcal{R}_{w-1} \cap \mathcal{L} = \emptyset \). If furthermore \( \mathcal{R}_+ \subseteq \mathcal{R} \cup \mathcal{L} \) (forcing \( \mathcal{L} \) to be a standard parabolic subsystem), then we will say that it is a \((w-)slicing pair\).

Proposition 2.21 explains that crossing pairs arise quite naturally. In particular:

**Proposition.** Let \( w \) be an element of a twisted Weyl group. If \( w \) is convex, then for any convex subset \( \mathcal{L} \subseteq \mathcal{R}^w \) satisfying \( w(\mathcal{L}) = -\mathcal{L} \), both

\[
\mathcal{R}_+ \setminus \mathcal{R}_{st}^w, \mathcal{L} \quad \text{and} \quad \mathcal{R}_{DG(\mathcal{L})}, \mathcal{L}
\]

for any \( d \geq 1 \) are crossing pairs.

I expect that the most important class of cross sections will be the one arising from firmly convex elements with crossing pair \((\mathcal{R}_+ \setminus \mathcal{R}_{st}^w, \mathcal{R}_w)\). In order to extend our results to twisted conjugacy classes and make the proofs slightly cleaner, we will employ the language of twisted reductive groups:

**Definition 1.10.** Let \( \tilde{G} \) be a split reductive group over a scheme with chosen Borel subgroup, and let \( \Omega \) be a group of twists of its Weyl group \( W \); these extend to automorphisms of \( \tilde{G} \). Also let \( \phi \) be an automorphism of the base scheme. We will then call \( G := \phi \Omega \ltimes \tilde{G} \) a twisted split reductive group, and \( W := \Omega \ltimes \tilde{W} \) its Weyl group.

**Notation 1.11.** Let \( w \) be an element of its Weyl group and let \((\mathcal{R}, \mathcal{L})\) be a \( w \)-crossing pair. Then we let \( N \subseteq N_w \) denote the unipotent subgroup corresponding to the roots in \( \mathcal{R} \), and let \( L \) denote the reductive subgroup of \( G \) generated by a chosen subgroup \( T' \) of the torus \( T \), and by the root subgroups corresponding to \( \mathcal{L} \).

In the firmly convex case, one typically sets \( T' \) to be the set of fixed points \( T^w \) under the action of \( w \) on the torus \( T \); we denote its Lie algebra by \( t^w \).

**Notation 1.12.** Factorisable \( r \)-matrices are parametrised by a Belavin-Drinfeld triple \( \mathfrak{T} = (\mathfrak{T}_0, \mathfrak{T}_1, \tau) \) defining a nilpotent isomorphism \( \tau : \mathfrak{T}_0 \to \mathfrak{T}_1 \) between two subsets of simple roots, and an element \( r_0 \) in \( t^w \cap \mathfrak{T} \), where \( \mathfrak{T} \) is a certain subspace of the Lie algebra of the maximal torus \( T \). Given such a triple \( \mathfrak{T} \) and a set of roots \( \mathcal{L} \), by \( \mathfrak{T} \subseteq \mathcal{L} \) we mean that \( \mathfrak{T}_0 \) and \( \mathfrak{T}_1 \) are both contained in \( \mathcal{L} \); thus this is always satisfied for the empty triple.
The main aim of this paper is to generalise He-Lusztig’s, Sevostyanov’s and Steinberg’s slices to the following.

**Theorem.** Let $G$ be a twisted split reductive group over a scheme with chosen Borel subgroup containing a maximal torus $T$. Pick any element $w$ in its Weyl group and let $(\mathfrak{N}, \mathfrak{L})$ be a $w$-crossing pair such that for some integer $d \geq 0$, or equivalently for any integer $d > |\mathfrak{N}| - \ell(w)$, there is an identity

$$\text{cross}^d_w(\mathfrak{N}) = \emptyset. \quad (1.3)$$

(i) Then the (right) conjugation map

$$N \times \check{\mathfrak{w}} LN_w \rightarrow N \check{\mathfrak{w}} LN, \quad (n, g) \mapsto n^{-1} gn \quad (1.4)$$

is an isomorphism.

(ii) If furthermore $L$ contains $T^w$ and the pair is slicing, then the conjugation map

$$G \times \check{\mathfrak{w}} LN_w \rightarrow G, \quad (n, g) \mapsto n^{-1} gn \quad (1.5)$$

is smooth; in other words, the subspace $\check{\mathfrak{w}} LN_w$ of $G$ transversely intersects the conjugation orbits of $G$.

(iii) If $w$ is firmly closed and $(\mathfrak{N}, \mathfrak{L}) = (\mathfrak{N}_+, \mathfrak{L}_+, \mathfrak{N}_w, \mathfrak{L}_w)$ and $L \cap T \subseteq T^w$, then the Semenov-Tian-Shansky bracket associated to a factorisable $r$-matrix with Belavin-Drinfeld triple $\mathfrak{L}$ satisfying $\mathfrak{L}_+ \subseteq \mathfrak{L}$ reduces to the subspace $\check{\mathfrak{w}} LN_w$ if and only if

(a) $L \cap T = T^w$,

(b) $r_0$ preserves $t^w$, and

(c) $r_0$ acts on its orthogonal complement as the Cayley transform

$$\frac{1 + w}{1 - w}.$$

My reasons for choosing this formulation are given at the start of section §3.

**Example 1.13.** Let $G = \text{SL}_3$, and pick the Borel subgroup and maximal torus and of upper triangular matrices and diagonal matrices. Set $w := s_1s_2s_1$ and lift it to

$$\check{w} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in G.$$

As $w = w_o$, part (i) of the main Theorem implies that the conjugation map sending

$$\left( \begin{array}{ccc} 1 & n_1 & n_{12} \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{array} \right) \times \left( \begin{array}{cc} 0 & 0 \\ 0 & -t^{-2} \\ t & x_1 \end{array} \right) \in N_+ \times \check{w}^T N_+$$

to

$$\begin{bmatrix} (n_1n_2 - n_{12})t & n_1t^{-2} + (n_1n_2 - n_{12})(n_1t + x_1) \\ -n_2t & -t^{-2} - n_1n_2t - n_2x_1 \\ t & x_1 + n_1t \end{bmatrix}$$

is an isomorphism onto $N \check{w}^T N$. Over a field the subregular conjugacy classes of $G$ can be parametrised by

$$\left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & -c_2^{-2} & 0 \\ c_2 & 0 & c_1 \end{bmatrix} : c_1, c_2 \in \mathcal{O}_S \right\} \subset \check{w}^T N_+,$$

and from part (ii) of the main Theorem and a dimension count it now follows that they are strictly transversally intersected by the subspace $\check{w}^T N_+$. 

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Example 1.14. Consider \( w = s_1 s_2 s_3 s_1 s_2 \) in type \( B_3 \) again and let \( \mathfrak{H} := \mathfrak{R}_w \cup \{ \alpha_{1233} \} \). This set is convex and although \( \text{DG}(b^i_w) = b^i_w \), we have \( w(\alpha_{1233}) \in \mathfrak{R}_w \) so \( \mathfrak{H} \) is nimble and hence the cross section holds.

Example 1.15. Consider Spaltenstein’s example of the (inverse of the) element \( w = s_1 s_2 s_3 s_4 s_5 s_4 s_3 s_1 s_2 \) in type \( A_5 \) with the conjugation map corresponding to the crossing pair \( (\mathfrak{R}, \mathcal{L}) = (\mathfrak{R}_+, \emptyset) \); by induction on \( i \geq 0 \) one can show that

\[
\text{DGN}(b^{i+2}_w) = b_{w^{s_5}} b^{i+2}_{451234312} b_{s_5 w},
\]

so as \( w \) is elliptic the inequality \( s_5 w \neq w_0 = \text{pb}(w) \) and main Lemma imply that we cannot invoke the main Theorem here. In fact, (some of) such Coxeter elements of length 9 in \( A_5 \) are the “smallest” elliptic examples in type \( A \) such that the braid equation (1.2) does not hold. One computes that

\[
\text{cross}^{i+2}_w(\mathfrak{R}_+) = \text{cross}^2_w(\mathfrak{R}_+) = \{ \alpha_{23}, \alpha_{2345}, \alpha_3, \alpha_{345}, \alpha_5 \}
\]

and as \( \text{cross}_w(\cdot) \) can be interpreted as a first-order approximation to the equations of (1.4) (after parametrising both sides into products of root subgroups, see Corollary 3.5), this suggests that there might be a linear relationship. Indeed, employing the usual lift

\[
\hat{w} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

and studying the resulting equations for the root subgroups of the roots in (1.6), one obtains a family of first-order counterexamples which exponentiates to (a slight generalisation of) Spaltenstein’s counterexample

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -s & 0 & -s \\
0 & 0 & 1 & s & 0 & st^{-1} \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \in N_+ \times \hat{w} N_w : s, t \in \mathcal{O}_S
\]

Nevertheless, the converse to (i) does not hold, as linear combinations of the polynomials in the first-order approximation can sometimes cancel each other out:

Example 1.16. Consider the Coxeter element \( w = s_2 s_3 s_4 s_5 s_6 s_1 s_2 s_3 s_4 s_5 s_7 s_3 s_2 \) in type \( A_6 \). Then \( b^i_w \) is in Deligne-Garside normal form for any \( i \geq 0 \) so from part (iii) of the main Lemma it again follows that equation (1.3) is never satisfied, but a (rather lengthy) calculation shows that the conjugation map (1.4) corresponding to the slicing pair \( (\mathfrak{R}, \mathcal{L}) = (\mathfrak{R}_+, \emptyset) \) is an isomorphism.

Finally, Sevostyanov and He-Lusztig proved the cross section isomorphism (1.4) for certain firmly convex elements \( w \) with slicing pair \( (\mathfrak{R}, \mathcal{L}) = (\mathfrak{R}_+ \setminus \mathfrak{R}_w, \mathfrak{R}_w) \), under some extra conditions on the base ring. The relationship between the elements they consider is explained in the prequel [Mal21, §2.1], and it is proven there that all of these indeed satisfy the braid equation (1.2) when \( d > |\mathfrak{R}_+ \setminus \mathfrak{R}_w| - \ell(w) \) [Mal21, Theorem B]; there are many convex (and firmly convex) elements satisfying (1.2) which are contained are in neither, yielding more transverse slices. The main conclusion of these two papers is now obtained by combining their main theorems with Sevostyanov’s dimension calculations [Sev19] to

Corollary B. The Weyl group elements considered by He-Lusztig and the elements Sevostyanov uses to construct strictly transverse slices (for connected reductive groups over algebraically closed fields) are all minimally dominant, and conversely all minimally dominant elements of conjugacy classes appearing in Lusztig’s partition furnish strictly transverse slices with natural Poisson brackets.
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2 Crossing roots

A common strategy, dating back to Killing’s work around 1889, is to study reductive groups and Weyl groups through their root systems; we follow this perspective by developing properties of $\text{cross}_w(\cdot)$. Throughout this paper, root systems will always be assumed to be crystallographic; we begin by proving a lemma on decomposing sums of roots, that we will employ several times:

**Definition 2.1.** We will say that a sequence $(\beta_1, \ldots, \beta_m)$ of roots in a crystallographic root system is a **summing sequence** if each of the partial sums $\sum_{i=1}^j \beta_i$ is a root for $1 \leq j \leq m$, and if their total sum is denoted by $\gamma := \sum_{i=1}^m \beta_i$ then we will denote this sequence as $\sum (\beta_1, \ldots, \beta_m) = \gamma$.

If furthermore each of these roots $\beta_i$ lies in a subset of roots $\mathcal{R}$, then we will call this a summing sequence in $\mathcal{R}$.

The following lemma shows how summing sequences may be constructed:

**Lemma 2.2.** Let $\beta_0, \ldots, \beta_m$ be roots in a crystallographic root system such that their sum $\sum_{i=0}^m \beta_i$ is also a root.

(i) [Bou68, Proposition VI.1.19] Amongst these roots there exists a root $\beta_j$ such that the difference $\sum_{i=0}^m \beta_i - \beta_j$ is either a root or is zero; it is strictly positive when $m > 0$ and each root $\beta_i$ is positive.

(ii) Suppose briefly that $m = 3$ and that $\beta_i + \beta_j \neq 0$ for each distinct pair $i, j \in \{1, 2, 3\}$. Then at least two of the three sums $\beta_1 + \beta_2, \beta_1 + \beta_3, \beta_2 + \beta_3$ are also roots.

(iii) Hence we may obtain from these roots $\beta_1, \ldots, \beta_m$ by reordering and deleting a summing sequence $\sum (\beta_{i_1}, \ldots, \beta_{i_m}) = \sum_{i=1}^m \beta_i$.

If each of the roots $\beta_i$ is positive, then the resulting sequence $\beta_{i_1}, \ldots, \beta_{i_m}$ is simply a reordering and we may choose the sequence to start with any of the $\beta_i$.

In particular, if each of the $\beta_i$ lie in a convex subset $\mathcal{R}$, then $\sum_{i=0}^m \beta_i$ also lies in $\mathcal{R}$.

**Proof.** (i): We follow the proof of [Bou68, Proposition VI.1.19]: let’s write $\gamma := \sum_{i=1}^m \beta_i$. If $(\beta_i, \gamma) \leq 0$ for each root $\beta_i$, then by linearity also $(\gamma, \gamma) = (\sum_{i=1}^m \beta_i, \gamma) \leq 0$ which contradicts that $\gamma$ is a root. Thus there must be an inequality $(\beta_j, \gamma) > 0$ for some $j$, which in a crystallographic root system implies that $\gamma - \beta_j$ is either a root or is zero.
(ii): By (i), we may assume after relabelling that $\beta_1 + \beta_2$ is a root (as it is not zero by assumption). If $(\beta_i, \beta_1 + \beta_2) \leq 0$ for both $i \in \{1, 2\}$ then we derive the same contradiction

$$(\beta_1 + \beta_2, \beta_1 + \beta_2) \leq 0$$

as before, so we may assume that say $(\beta_1, \beta_1 + \beta_2) > 0$. If $\beta_1 + \beta_3$ is a root the claim follows and if it is not then $(\beta_1, \beta_3) \geq 0$. This would yield

$$(\beta_1, \beta_1 + \beta_2 + \beta_3) = (\beta_1, \beta_1 + \beta_2) + (\beta_1, \beta_3) > 0,$$

which implies that $(\beta_1 + \beta_2 + \beta_3) - \beta_1 = \beta_2 + \beta_3$ is a root.

(iii): The first claim of (iii) follows from (i) by induction on $m$. As a nontrivial sum of positive roots is never zero the sequence must be a reordering if each of the roots $\beta_i$ are positive. For the final claim we induct on $m$, so we may assume that $\beta_{im}$ is the chosen root we want the sequence to start with. Writing $\gamma_{m-1} := \sum_{j=1}^{m-2} \beta_{ij}$, part (ii) yields that at least one of

$$\gamma_{m-1} + \beta_{im} \quad \text{or} \quad \beta_{im-1} + \beta_{im}$$

is a root. In the former case we apply the induction hypothesis to that sum to conclude, and in the latter case we apply the induction hypothesis to the set of roots $\beta_{i1}, \ldots, \beta_{im-2}, \beta_{im-1} + \beta_{im}$ to find a summing sequence beginning with $\beta_{im-1} + \beta_{im}$, which immediately yields one starting with $\beta_{im}$.

**Proof.** We induct on $m$ and within that we induct on $2 \leq l \leq m$, which is the first time $\tilde{\beta}_i$ lies in $\mathfrak{N}$ but $\tilde{\beta}_{l-1}$ lies in $\mathfrak{N}'$; if there is no such $l$ we are already done. Set $\gamma_{l-1} := \beta + \sum_{i=1}^{l-2} \tilde{\beta}_i$ and consider the sum

$$\gamma_{l-1} + \tilde{\beta}_{l-1} + \tilde{\beta}_l.$$
If either $\gamma_{<l-1} = 0$ or $\tilde{\beta}_{l-1} + \tilde{\beta}_l = 0$ then we may shorten the sequence and conclude from the induction hypothesis on $m$. Moreover if
\[
\beta + \sum_{i=1}^{l-2} \tilde{\beta}_i + \tilde{\beta}_{l-1} = \gamma_{<l-1} + \tilde{\beta}_{l-1} = 0
\]
then rewriting yields $-\beta = \sum_{i=1}^{l-2} \tilde{\beta}_i + \tilde{\beta}_{l-1}$. From the last statement in part (iii) it would follow that $-\beta$ lies in the convex set $\mathcal{R} \cup \mathcal{R}'$. The case $\gamma_{<l-1} + \tilde{\beta}_l = 0$ similarly yields a contradiction. Hence by part (ii) at least one of $\gamma_{<l-1} + \tilde{\beta}_l$ or $\tilde{\beta}_{l-1} + \tilde{\beta}_l$ is a root; in the latter case it lies in $\mathcal{R} \cup \mathcal{R}'$ and we may shorten the sequence, whereas in the former case we may now swap the roots $\tilde{\beta}_{l-1}$ and $\tilde{\beta}_l$ in the sequence to lower $l$ and apply the induction hypothesis on $l$.

**Definition 2.5.** Let $b := b_w := b_{w_d} \cdots b_{w_1}$ be a braid in a braid monoid, constructed out of a sequence of elements $w = (w_d, \ldots, w_1)$ lying in the corresponding twisted Weyl group. Given a positive root $\beta$ or a subset of positive roots $\mathcal{R} \subseteq \mathcal{R}_+$, construct the set of roots

$$\text{cross}_b(\beta) := \text{cross}_{w_d} \cdots \text{cross}_{w_2} \text{cross}_{w_1}(\beta)$$

and extend it to a map

$$\text{cross}_b : \{\text{subsets of } \mathcal{R}_+\} \rightarrow \{\text{subsets of } \mathcal{R}_+\}, \quad \mathcal{R} \mapsto \bigcup_{\beta \in \mathcal{R}} \text{cross}_b(\beta).$$

We let $\text{cross}_w(\gamma, \beta)$ denote the set of sequences of elements in the $\mathcal{R}_w$, “confirming” that $\gamma$ lies in $\text{cross}_b(\beta)$; more precisely, it is the set of subsets of roots

$$\mathcal{R}_m \times \cdots \times \mathcal{R}_2 \times \mathcal{R}_1 \subseteq \mathcal{R}_{w_m} \times \cdots \times \mathcal{R}_{w_2} \times \mathcal{R}_{w_1}$$

with the property that the corresponding sequence $\beta = \beta_0', \ldots, \beta_d'$ inductively constructed via

$$\beta_j' := w_j(\beta_{j-1}' + \sum_{\beta \in \mathcal{R}_j} \tilde{\beta})$$

for $1 \leq j \leq d$, consists solely of roots which are all positive, and satisfies $\beta_d' = \gamma$.

Analysing these sequences shows that the set of roots $\text{cross}_b(\cdot)$ is well-defined:

**Lemma 2.6.** Let $w, w'$ be two sequences of elements in a twisted Weyl group such that there is an equality $b_w = b_{w'}$ in the associated braid monoid, and let $\beta, \gamma$ be positive roots in its root system. One can non-canonically construct “transfer maps”

$$\text{cross}_{w}(\gamma, \beta) \rightleftharpoons \text{cross}_{w'}(\gamma, \beta),$$

mapping nontrivial sequences to nontrivial sequences.

In particular, for any subset of positive roots $\mathcal{R}$ and any braid $b$ in the corresponding braid monoid, the set of roots $\text{cross}_b(\mathcal{R})$ does not depend on the chosen decomposition of $b$ into reduced braids.

**Proof.** We first show the claim for a reduced decomposition $w = xy$ of an arbitrary element $w$ in the twisted Weyl group. A (nontrivial) sequence in $\text{cross}_{(x,y)}(\gamma, \beta)$ rewrites as

$$\gamma = x(y(\beta + \sum_{\beta' \in \mathcal{R}_y} \beta') + \sum_{\beta'' \in \mathcal{R}_x} \beta'') = w(\beta + \sum_{\beta' \in \mathcal{R}_w} \beta' + y^{-1}(\sum_{\beta'' \in \mathcal{R}_y} \beta'')),$$

which through the identity

$$\mathcal{R}_w = y^{-1}(\mathcal{R}_x) \sqcup \mathcal{R}_y$$

(2.2)
immediately yields a (nontrivial) sequence in \( \alpha_{w}(\gamma, \beta) \). Conversely, given roots \( \beta_1, \ldots, \beta_m \) in \( \mathfrak{R}_w \) such that
\[
\beta + \sum_{i=1}^{m} \beta_i = w^{-1}(\gamma),
\]
onobtain through part (iii) of the previous lemma a summing sequence starting with \( \beta \), and according to equation (2.2) each of the subsequent roots lies in one of the convex sets \( v^{-1}(\mathfrak{R}_u) \) or \( \mathfrak{R}_v \). By the previous corollary we may modify the summing sequence to one which starts with \( \beta \), is then followed by roots in \( \mathfrak{R}_u \), and in \( v^{-1}(\mathfrak{R}_u) \). In the process, roots are only reordered or summed, so since both \( v^{-1}(\mathfrak{R}_u) \) and \( \mathfrak{R}_v \) consist of positive roots and the sum of positive roots is positive, it follows through (2.1) that a nontrivial sequence in \( \alpha_{w}(\gamma, \beta) \) yields another nontrivial sequence in \( \alpha_{w}(\gamma, \beta) \).

Now let \( w \) and \( w' \) be as in the statement of this lemma. We may decompose both \( b_w \) and \( b_{w'} \) into a product of elementary braids \( b_i \) of length one and twists; one first verifies that
\[
\text{cross}_{\delta, x} = \text{cross}_{\delta} \text{cross}_{x} = \text{cross}_{\delta, x, \delta^{-1}, x} \text{cross}_{\delta}
\]
for any twist \( \delta \) and element \( x \) in the underlying untwisted Weyl group, so we may move all of the twists to the left and combine them. The equality \( b_w \cdot b_{w'} \) then implies that these twists agree and can be safely ignored. This braid identity on the untwisted part implies that the first sequence transforms into the second one, through a finite sequence of braid moves \( s_i, s_j, s_i \cdots = s_j, s_i, s_j \cdots \). Now applying the result in the first paragraph several times for each such braid move, we obtain transfer maps
\[
\text{cross}_{s_1, s_2, \ldots}(\gamma, \beta) \Leftrightarrow \text{cross}_{s_1, s_2, \ldots}(\gamma, \beta) = \text{cross}_{s_1, s_2, \ldots}(\gamma, \beta) \Leftrightarrow \text{cross}_{s_1, s_2, \ldots}(\gamma, \beta)
\]
for any positive root \( \beta \). Hence we by induction on the number of braid moves to be made, we obtain transfer maps \( \text{cross}_{w}(\gamma, \beta) \Leftrightarrow \text{cross}_{w}(\gamma, \beta) \). \( \square \)

Transferring nontrivial sequences will play a key rôl in the proof of Lemma 2.19.

### 2.2 Crossing simple roots

In this subsection we prove part (i) of the main Lemma. It is independent of the previous subsection and in contrast, I do not know of a simple interpretation or proof in terms of root subgroups.

**Proposition 2.7.** Let \( \beta_0, \beta_1, \gamma_0, \gamma_1 \) be positive roots (or zero) in a crystallographic root system, such that their sums yield an equality
\[
\beta_0 + \beta_1 = \gamma_0 + \gamma_1
\]
between two positive roots. Then there exists a pair of indices \( i, j \in \{0, 1\} \) such that
\[
\beta_i - \gamma_j
\]
is either a positive root or is zero.

**Proof.** Since \( (\beta_0 + \beta_1, \gamma_0 + \gamma_1) > 0 \), we must have \( (\beta_0 + \beta_1, \gamma_j) > 0 \) for some \( j \in \{0, 1\} \), and thus \( (\beta_i, \gamma_j) > 0 \) for some \( i \in \{0, 1\} \). In a crystallographic root system it follows that \( \beta_i - \gamma_j \) must be either a root or zero, so if it is not a negative root we are done. If it is negative, then the complementary pair of indices \( i' = 1 - i \) and \( j' = 1 - j \) yields a positive root
\[
\beta_{i'} - \gamma_{j'} = - (\beta_i - \gamma_j).
\]
\( \square \)

**Corollary 2.8.** Let \( \beta_0, \ldots, \beta_m \) for \( m \geq 0 \) and \( \gamma_0, \gamma_1 \) be positive roots in a crystallographic root system \( \mathfrak{R} \), such that their sums yield an equality
\[
\sum_{i=0}^{m} \beta_i = \gamma_0 + \gamma_1
\]
between two positive roots. Then for some \(0 \leq j \leq m\) the smaller sum \(\sum_{i=0,i\neq j}^{m} \beta_i\) is also a positive root (or zero if \(m = 0\)) and for some \(t \in \{0,1\}\) either
\[
\sum_{i=0,i\neq j}^{m} \beta_i - \gamma_t \quad \text{or} \quad \beta_j - \gamma_t
\]
lies in \(\mathfrak{A}_+ \sqcup \{0\}\).

**Proof.** This now follows by combining the previous proposition with Lemma 2.2(i). \(\square\)

This is close to what we need; the next lemma refines this statement to show that if \(\beta_0\) is a simple root, then we can ensure that this root does not appear in the conclusion.

**Lemma 2.9.** Suppose that \(\alpha\) is a simple root and \(\beta_1, \ldots, \beta_m\) for \(m \geq 1\) and \(\gamma\) are positive roots in a crystallographic root system, such that
\[
\alpha + \sum_{i=1}^{m} \beta_i \quad \text{and} \quad \alpha + \sum_{i=1}^{m} \beta_i - \gamma
\]
are both positive roots.

Furthermore suppose that for any subset \(\{i_1, \ldots, i_k\}\) of \(\{1, \ldots, m\}\), the expression
\[
\alpha + \sum_{j=1}^{k} \beta_{i_j} - \gamma
\]
is neither a negative root nor zero. \hfill (2.3)

Then there exists a subset \(\{i_1, \ldots, i_k\}\) of \(\{1, \ldots, m\}\) with the property that
\[
\sum_{j=1}^{k} \beta_{i_j} - \gamma
\]
is a positive root or is zero. \hfill (2.4)

**Proof.** We induct on \(m\), so we presume that the claim is true \(< m\). Let \(0 \leq k \leq m\) be the largest integer such that both \(\gamma - \sum_{j=1}^{k} \beta_{i_j}\) and \(\alpha + \sum_{j=k+1}^{m} \beta_{i_j}\) are positive roots, for some partition
\[
\{i_1, \ldots, i_k\}, \quad \{i_{k+1}, \ldots, i_m\}
\]
of \(\{1, \ldots, m\}\) into two subsets. If \(k = m\) then \(\gamma - \sum_{j=1}^{m} \beta_{i_j} \in \mathfrak{A}_+\), but this combines with the initial assumption \(\alpha + \sum_{i=1}^{m} \beta_i \in \mathfrak{A}_+\) and simpleness of \(\alpha\) to a contradiction, so \(k < m\). By Lemma 2.2(iii) there then exists an integer \(k + 1 \leq t \leq m\) such that
\[
\alpha + \sum_{j=k+1,j\neq t}^{m} \beta_{i_j}
\]
is a positive root. \(\square\)

Now applying Proposition 2.7 to the equality of roots
\[
(\alpha + \sum_{j=k+1,j\neq t}^{m} \beta_{i_j}) + \beta_{i_t} = (\alpha + \sum_{i=j}^{m} \beta_{j} - \gamma) + (\gamma - \sum_{j=1}^{k} \beta_{i_j})
\]
yields that at least one of the expressions
\[
\gamma - \beta_{i_t} - \sum_{j=1}^{k} \beta_{i_j}, \quad \beta_{i_t} + \sum_{j=1}^{k} \beta_{i_j} - \gamma, \quad \alpha + \sum_{j=1,j\neq t}^{m} \beta_{i_j} - \gamma, \quad \gamma - \alpha - \sum_{j=1,j\neq t}^{m} \beta_{i_j}
\]
lies in \(\mathfrak{A}_+ \sqcup \{0\}\). In the first case, if this expression equals zero we’re done and if it’s a positive root then combining this with equation (2.4) implies that \(k\) was not maximal. The second case would yield the claim immediately. For the third case, the assumption of (2.3) implies that this expression can’t be zero and then the claim would follow from the induction hypothesis on \(m\). The fourth case is excluded by the same assumption. \(\square\)
Corollary 2.10. Suppose that $\alpha$ is a simple root and $\beta_1, \ldots, \beta_m$ for $m \geq 1$ and $\gamma_0, \gamma_1$ are positive roots in a crystallographic root system, such that there is an equality

$$\alpha + \sum_{i=1}^{m} \beta_i = \gamma_0 + \gamma_1$$

of positive roots. Then for some subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, m\}$ and some $t \in \{0, 1\}$, the expression

$$\sum_{j=1}^{k} \beta_{i_j} - \gamma_t$$

is a positive root or is zero.

Proof. We may assume that $\gamma_t \neq \alpha$ for both $t \in \{0, 1\}$ (otherwise the claim is immediate), and we first invoke Corollary 2.8 with $\beta_0 = \alpha$. If in its conclusion $j = 0$, then the claim follows immediately as $\alpha - \gamma_t$ in $\mathfrak{R}_+ \cup \{0\}$ would imply that $\gamma_t = \alpha$. Suppose on the other hand that $j \neq 0$ and that moreover $\beta_j - \gamma_t$ is not in $\mathfrak{R}_+ \cup \{0\}$ (otherwise the claim follows). Then this corollary states that for certain $\gamma_t$, the expression

$$\alpha + \sum_{i=1, i \neq j}^{m} \beta_i - \gamma_t$$

lies in $\mathfrak{R}_+ \cup \{0\}$.

If this expression is zero then $\beta_j = \gamma_{t'}$, where $t' := 1 - t$, and the claim follows for $\gamma_{t'}$ so we may assume that this expression is a positive root. In fact, from the equation

$$\sum_{k+1}^{m} \beta_{i_k} - \gamma_{t'} = - (\alpha + \sum_{i=1}^{k} \beta_{i_j} - \gamma_t)$$

it follows may furthermore assume that condition (2.3) holds, as again the claim would otherwise follow for $\gamma_{t'}$; then we may invoke the previous lemma, which yields the claim. \hfill \square

Notation 2.11. Given two positive roots $\gamma$ and $\gamma'$ in some root system, we write $\gamma' < \gamma$ if their difference $\gamma - \gamma'$ lies in the convex cone of positive roots.

Finally, we prove part (i) of the main Lemma:

Proof. As the simple root $\alpha$ is not in $\mathfrak{R}_w$ by assumption, the root $w(\alpha)$ is positive. Given an integer $m \geq 0$ and some roots $\beta_1, \ldots, \beta_m \in \mathfrak{R}_w$ such that $w(\alpha + \sum_{i=1}^{m} \beta_i)$ is a positive root but is not simple, we will find roots $\beta'_1, \ldots, \beta'_m \in \mathfrak{R}_w$ (with $m' \leq m + 1$) such that $w(\alpha + \sum_{i=1}^{m'} \beta'_i)$ is still a positive root, but is smaller in the sense that

$$w(\alpha + \sum_{i=1}^{m'} \beta'_i) < w(\alpha + \sum_{i=1}^{m} \beta_i) := \tilde{\gamma}.$$

By downwards induction on height, the claim then follows. Since $\tilde{\gamma}$ is not simple, we may split $\tilde{\gamma} = \tilde{\gamma}_0 + \tilde{\gamma}_1$ into some other positive roots $\tilde{\gamma}_0, \tilde{\gamma}_1$.

If both roots $w^{-1}(\tilde{\gamma}_0)$ and $w^{-1}(\tilde{\gamma}_1)$ are positive then by the previous corollary (setting $\gamma_i := w^{-1}(\tilde{\gamma}_i)$), for at least one $t \in \{0, 1\}$ there is a subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, m\}$ such that $\sum_{i=1}^{k} \beta_{i_j} - w^{-1}(\tilde{\gamma}_t)$ is a positive root or is zero; in the former case, it moreover lies in $\mathfrak{R}_w$ as

$$w(\sum_{i=1}^{k} \beta_{i_j} - w^{-1}(\tilde{\gamma}_t)) = \sum_{i=1}^{k} w(\beta_{i_j}) - \tilde{\gamma}_t$$

is a sum of negative roots.
If on the other hand some \( w^{-1}(\gamma_i) \) is negative, then \(-w^{-1}(\gamma_i)\) lies in \( R_w \) so the same conclusion holds with \( k = 0 \). Hence in either case
\[
w(\alpha + \sum_{i=k+1}^{m} \beta_i) + \left( \sum_{i=1}^{k} \beta_i - w^{-1}(\gamma_i) \right) = w(\alpha + \sum_{i=1}^{m} \beta_i - w^{-1}(\gamma_i)) = \tilde{\gamma}' < \tilde{\gamma},
\]
where \( t' := 1 - t \).

**Example 2.12.** Consider \( w = s_1s_2 \) in type \( B_2 \). Then \( \alpha_1 \notin R_w \) and as \( \text{cross}_w(\alpha_1) = \{\alpha_{122}, \alpha_2\} \) it follows that there is no “path of simple roots” within \( \text{cross}_w(\alpha_1) \) from \( w(\alpha_1) = \alpha_{122} \) down to a simple root.

**Remark 2.13.** I am not sure whether these results naturally generalise to the noncrystallographic case; consider for any positive root \( \beta \) the set
\[
\text{cross}_w(\beta) := \{w(c_0\beta + \sum_{i=1}^{m} c_i \beta_i) \in R : c_0, \ldots, c_m \in \mathbb{R}_{>0}, \beta_1, \ldots, \beta_m \in R_w, m \geq 0\} \cap R_+,
\]
where one might want to put some restrictions on the \( c_i \). For \( H_3 \) and \( H_4 \) there is a well-known “folding” argument showing that one can realise their root systems inside those of \( D_6 \) and \( E_8 \), preserving simple roots and embedding the corresponding reflection groups; part (i) of the main Lemma then holds for \( c_0 = 1 \) and \( c_i \in \{1, \varphi^{\pm 1}\} \) for \( i > 0 \), where \( \varphi \) denotes the golden ratio. On the other hand, if we fix \( c_0 = 1 \) then part (i) of the main Lemma fails for say Coxeter elements of minimal length in type \( I_2(7) \). For dihedral groups there are only two simple roots and in nontrivial cases the other simple root lies in \( R_w \); thus for (i) a positive linear combination of the simple roots can be used if there are no restrictions on \( c_0 \). However, the main point of (i) was to obtain (ii) by combining it with braid invariance, but braid invariance still fails in \( H_3 \).

**Example 2.14.** Consider \( w = s_2s_1 \) in type \( H_3 \) and denote the golden ratio by \( \varphi \). Then \( R_w = \{\alpha_1, \varphi\alpha_1 + \alpha_2\} \) and the only simple root in \( \text{cross}_w(\alpha_2) \) is
\[
\alpha_1 = w(\varphi\alpha_1 + \varphi\alpha_2) = w(\alpha_2 + \varphi^{-1}\alpha_1 + \varphi^{-1}(\varphi\alpha_1 + \alpha_2)).
\]

### 2.3 Crossing for the Deligne-Garside normal form

In this subsection we prove the remainder of the main Lemma, Corollary A and the main Proposition.

Recall (e.g. from [Mal21, §4.1]) that two elements \( w_2, w_1 \) of a twisted Weyl group are in right Deligne-Garside normal form (modulo moving twists) if and only if for any simple reflection \( s_i \) with \( \ell(w_2s_i) < \ell(w_2) \) we also have \( \ell(s_iw_2) < \ell(w_2) \). Furthermore, we will repeatedly use the following property from [Mal21, Proposition C(ii)]: for any element \( w \) of a twisted Weyl group and integer \( d \geq 0 \), we have
\[
R_{DG(w^d)} \cap R_{st} = \emptyset.
\]

From part (i) of the main Lemma we deduce

**Corollary 2.15.** Let \( w_m, \ldots, w_1 \) be elements of a twisted Weyl group \( W \) such that their product \( b_{w_m} \cdots b_{w_1} \) is in Deligne-Garside normal form, after moving twists. Then for any \( y \) in \( W \) such that \( w_1 \geq y \) and any simple root \( \alpha \) such that \( \beta := y^{-1}(\alpha) \) lies in \( R_+ \setminus R_{w_1} \), the set
\[
\text{cross}_{w_{m-1}} \cdots \text{cross}_{w_1}(\beta)
\]
contains simple roots not lying in \( R_{w_m} \).

**Proof.** By induction on \( m \), it suffices to show that \( \text{cross}_{w_1}(\beta) \) contains a simple root not in \( R_{w_2} \). By assumption we have a reduced decomposition \( w_1 = xy \), and the root \( w_1(\beta) = x(\alpha) \) is positive so \( \alpha \) is not in \( R_x \). By part (i) of the main Lemma there then exists a simple root \( \alpha' \in \text{cross}_x(\alpha) \), which means that
\[
x(\alpha + \sum_{i=1}^{m} \beta_i) = \alpha'.
\]
for some $\beta_1, \ldots, \beta_m \in R_x$. As $x = w_1 y^{-1}$, this yields the equation

$$w_1 (\beta + \sum_{i=1}^{m} y^{-1}(\beta_i)) = \alpha'$$

with each $y^{-1}(\beta_i) \in y^{-1}(R_x) \subseteq R_w$, which then implies that $\alpha' \in \text{cross}_{w_1}(\beta)$. The normal form condition on the pair $w_2, w_1$ is equivalent to requiring that $w_1^{-1}(\alpha)$ is a negative root for any simple root $\alpha$ lying in $R_w$. As $w_1^{-1}(\alpha') = \beta + \sum_{i=1}^{m} y^{-1}(\beta_i)$ is a sum of positive roots and is therefore positive, it now follows that $\alpha'$ does not lie in $R_w$. \hfill \Box

When $\alpha$ is not simple, there may not be simple roots in these sets:

**Example 2.16.** Consider again $w = s_1 s_2 s_3 s_1 s_2$ in type $B_3$. Then $\text{DGN}(b^i_w) = b^i_w$ and the root $\alpha_{1233}$ does not lie in $R_w \sqcup R^w$, yet

$$\text{cross}_w(\alpha_{1233}) = \{w(\alpha_{1233})\} = \{\alpha_{233}\} \in R_w.$$  

**Notation 2.17.** Recall that we write $\text{DG}_{\geq 1}(b^d_w) := b^d_w \text{DG}(b^d_w)$ for the left complement to $\text{DG}(b^d_w)$ in $b^d_w$.

**Corollary 2.18.** Let $w$ be an element of a twisted Weyl group $W$, let $\mathfrak{M} \subseteq R_x$ be a subset of positive roots and pick an integer $d \geq 0$.

(i) If $\mathfrak{M} \subseteq \mathfrak{R}_{\text{DG}(b^d_w)}$ then $\text{cross}^d_w(\mathfrak{M}) = \emptyset$.

(ii) If there exist a simple root $\alpha$ in $\mathfrak{M}$ and element $y$ in $W$ satisfying

$$\text{DG}(b^d_w) \geq y \quad \text{and} \quad y^{-1}(\alpha) \in R_+ \setminus \mathfrak{R}_{\text{DG}(b^d_w)},$$

then $\text{cross}^d_w(\mathfrak{M})$ contains simple roots.

**Proof.** (i): If $\mathfrak{M} \subseteq \mathfrak{R}_{\text{DG}(b^d_w)}$ then by Lemma 2.6 we have

$$\text{cross}^d_w(\mathfrak{M}) = \text{cross}_{\text{DG}_{\geq 1}(b^d_w)} \text{cross}_{\text{DG}(b^d_w)}(\mathfrak{M}) = \emptyset.$$  

(ii): Corollary 2.15 yields that there are simple roots lying in

$$\text{cross}_{\text{DG}_{\geq 1}(b^d_w)} \text{cross}_{\text{DG}(b^d_w)}(\mathfrak{M}) = \text{cross}^d_w(\mathfrak{M}).$$

From this corollary we can deduce part (ii) of the main Lemma:

**Proof.** The implication $\Rightarrow$ follows immediately from the first part of the corollary. For $\Leftarrow$, suppose that the inequality $\text{DG}(b^d_w) \geq w'$ does not hold. So we may suppose that there exists a simple reflection $s_i$ and an element $y \in W$ such that

$$\text{DG}(b^d_w) \geq y, \quad w' \geq s_i y, \quad \text{DG}(b^d_w) \not\geq s_i y.$$  

Then $y^{-1}(\alpha_i)$ is a positive root lying in $R_{w'} \setminus \mathfrak{R}_{\text{DG}(b^d_w)}$, so from the second part of the corollary it follows that $\text{cross}^d_w(\mathfrak{M})$ contains a simple root. \hfill \Box

**Lemma 2.19.** Let $w$ be an element of a twisted Weyl group and pick an integer $d \geq 0$. Then for any simple root $\alpha$ whose orbit under $w$ consists solely of other simple roots, the root $\text{DG}(b^d_w)(\alpha)$ is again simple.

**Proof.** From equation (2.5) it follows that $\alpha$ is not in $\mathfrak{R}_{\text{DG}(b^d_w)}$, so part (i) of the main Lemma implies that there has to exist a simple root $\alpha''$ in $\text{cross}_{\text{DG}(b^d_w)}(\alpha)$, which means that there exist roots $\beta_1, \ldots, \beta_m \in \mathfrak{R}_{\text{DG}(b^d_w)}$ such that

$$\text{DG}(b^d_w)(\alpha + \sum_{i=1}^{m} \beta_i) = \alpha''.$$
In particular, if $\text{DG}(\beta^d_{w})(\alpha)$ is not simple then $m > 0$. From Corollary 2.15 we similarly obtain a sequence in $\text{cross}_{\text{DG}_{+}}(\beta^d_{w})(\alpha', \alpha'')$ for some simple root $\alpha'$, which we may concatenate with the $(\beta_{i,1})$ to a sequence $(\beta_{i,j}) \in \text{cross}_{\text{DG}_{N}(\beta^d_{w})}(\alpha', \alpha)$. Since the $(\beta_{i,1})$ part of this sequence is nontrivial, Lemma 2.6 implies that we can transfer $(\beta_{i,j})$ to a nontrivial sequence $(\beta'_{i,j})$ in $\text{cross}_{\text{DG}}(\alpha', \alpha)$. This gives positive roots $\alpha'_{j}$ inductively defined for $1 \leq j \leq d$ as

$$\alpha'_{j} := w(\alpha'_{j-1} + \sum_{i} \beta'_{i,j}), \quad \alpha'_{0} := \alpha,$$

with each $\beta'_{i,j} \in \mathcal{R}_{w}$. A priori the roots $\alpha'_{j}$ are not necessarily simple, but we now inductively prove that they are all simple roots lying in the $w$-orbit of $\alpha$, and that the elements $\beta'_{i,j}$ are all zero: if it’s true $< j$ then the induction hypothesis yields that

$$\alpha'_{j} = w(\alpha'_{j-1} + \sum_{i} w(\beta'_{i,j}) \in \mathcal{R}_{+}$$

but as $w(\alpha'_{j-1})$ is simple by assumption and each nontrivial root $\beta'_{i,j}$ lies in $\mathcal{R}_{w}$ this implies that each $\beta'_{i,j} = 0$ and then $\alpha'_{j} = w(\alpha'_{j-1})$. But that is a contradiction as the sequence $(\beta'_{i,j})$ was constructed to be nontrivial, and therefore $\text{DG}(\beta^d_{w})(\alpha)$ must be a simple root.

**Corollary 2.20.** If $w$ is convex (resp. firmly convex), then each of the elements in the sequence

$$w, \quad \text{DG}(\beta^d_{w}) w \text{DG}(\beta^2_{w})^{-1}, \quad \text{DG}(\beta^3_{w}) w \text{DG}(\beta^3_{w})^{-1}, \quad \ldots,$$

of cyclic shifts is convex (resp. firmly convex).

**Proof.** It was proven in [Mal21, Proposition 4.39] that conjugation by $\text{DG}(\beta^d_{w})$ induces a sequence of cyclic shifts, so from [Mal21, Proposition A(i)] we then deduce that

$$\mathcal{R}^{\text{DG}(\beta^d_{w}) \text{DG}(\beta^d_{w})^{-1}}_{\text{st}} = \text{DG}(\beta^d_{w})(\mathcal{R}^w_{\text{st}}).$$

If $w$ is convex then by height considerations it follows that it must map any of the simple roots in $\mathcal{R}^w_{\text{st}} \cap \mathcal{R}_{+}$ to other simple roots. The lemma now implies that $\text{DG}(\beta^d_{w})(\mathcal{R}^w_{\text{st}})$ is also a standard parabolic subsystem. \hfill \square

The main Proposition follows from

**Proposition 2.21.** Let $w$ be an element of a twisted Weyl group.

(i) Let $\mathcal{R} = \mathcal{R}_{y}$ for some $y \in W$. Then $\mathcal{R}$ is $w$-nimble if and only if $y \geq w$ and $y \geq yw^{-1}$ in the weak left Bruhat-Chevalley order.

In particular, the inversion sets associated to the elements in the sequence

$$w = \text{DG}(\beta^d_{w}), \quad \text{DG}(\beta^2_{w}), \quad \text{DG}(\beta^3_{w}), \quad \ldots,$$

yield $w$-nimble sets. On the other hand, $\mathcal{R}_{+} \mathcal{R}^w_{\text{st}}$ is $w$-nimble if and only if $w$ is convex.

(ii) If indeed $w$ is convex, then

$$\mathcal{R}_{+} \mathcal{R}^w_{\text{st}} \subseteq \mathcal{R}^w_{\text{st}} \quad \text{and} \quad \mathcal{R}_{\text{DG}(\beta^d_{w})} \mathcal{R}^w_{\text{st}} \subseteq \mathcal{R}_{\text{DG}(\beta^d_{w})} \mathcal{R}^w_{\text{st}}$$

for any natural number $d \geq 1$.

**Proof.** (i): The inclusion $\mathcal{R}_{w} \subseteq \mathcal{R}_{y}$ is equivalent to $y \geq w$. Under this assumption, there is a reduced decomposition $y = (yw^{-1})w$ which means

$$\mathcal{R}_{y} = w^{-1}(\mathcal{R}_{yw^{-1}} \cup \mathcal{R}_{w}).$$
and applying $w$ to this identity then yields
\[
 w(R_y) \cap R_+ = R_{yw-1} \cup (w(R_w) \cap R_+) = R_{yw-1},
\]
which reduces the nimbleness condition to the inclusion $R_{yw-1} = w(R_y) \cap R_+ \subseteq R_y$; this yields the first two claims.

From $R_w \cap R_{st}^w = \emptyset$ and $w(R_{st}^w) = R_{st}^w$ we obtain the inclusions
\[
 R_w \subseteq R_+ \setminus R_{st}^w \quad \text{and} \quad w(R_+ \setminus R_{st}^w) \cap R_+ \subseteq R_+ \setminus R_{st}^w.
\]
Thus $R_+ \setminus R_{st}^w$ is $w$-nimble if and only if $R_+ \setminus R_{st}^w$ is convex, so the final claim follows from [Mal21, Proposition 4.21].

(ii): The first identity follows from the assumption that $R_{st}^w$ forms a standard parabolic subsystem. For the final one, let $\beta \in R_{DG(b'_w)}$ and $\gamma \in R_{st}^w$ and suppose that $c_0 \beta + c_1 \gamma$ is a root for some $c_0, c_1 \in \mathbb{R}_{>0}$. As $R_{st}^w$ is a standard parabolic subroot system and $\beta$ is positive, equation (2.5) implies that $c_0 \beta + c_1 \gamma$ must be a positive root. If $\gamma$ is negative then the same equation implies that $DG(b'_w)(\gamma)$ is still negative and therefore so is
\[
 DG(b'_w)(c_0 \beta + c_1 \gamma) = c_0 DG(b'_w)(\beta) + c_1 DG(b'_w)(\gamma).
\]
On the other hand, if $\gamma$ is positive then the previous lemma implies that $DG(b'_w)(\gamma)$ lies in the positive half of the standard parabolic subsystem $DG(b'_w)(R_{st}^w)$. If furthermore $DG(b'_w)(c_0 \beta + c_1 \gamma)$ is positive then as $DG(b'_w)(\beta)$ is negative it must lie in the negative half of $DG(b'_w)(R_{st}^w)$, but then $\beta$ lies in $R_{st}^w$ which contradicts equation (2.5) again. \qed

By [Mal21, Proposition C(ii)], the sequence (2.7) (and hence also (2.6)) stabilises after $|R_+ \setminus R^w| - \ell(w)$ terms; we will reprove this at the end of this subsection.

**Example 2.22.** Consider $w = s_3s_2s_1$ and $w' = s_2s_1$ in type $A_3$. Then $w$ is elliptic so it is convex and
\[
 w = DG(b'_w) > w' > w,
\]
but as $w(\alpha_3) = \alpha_2$ we have
\[
 w(R_{w'}) \cap R_+ \nsubseteq R_w,
\]
so $R_w$ is not nimble.

Typically however, the sets $R_w \cup (R^w \cap R_+)$ and $R_w \cup (R_{st}^w \cap R_+)$ are not convex:

**Example 2.23.** Consider $s_2$ in type $B_2$. It is not firmly convex, and
\[
 R_w = \{\alpha_2\}, \quad R^w \cap R_+ = \{\alpha_{12}\}, \quad R_w + (R^w \cap R_+) = \{\alpha_{122}\}.
\]

**Example 2.24.** Consider $s_3s_1s_2s_1$ in type $B_3$. It is not firmly convex, and
\[
 R_w = \{\alpha_{23}\}, \quad R_{st}^w \cap R_+ = \{\alpha_1, \alpha_{12}, \alpha_2, \alpha_{123}\}, \quad R_w + (R_{st}^w \cap R_+) = \{\alpha_{12233}\}.
\]

**Lemma 2.25.** Let $w$ be an element of a twisted Weyl group and pick a natural number $d \geq 0$. Then the following are equivalent:

(i) The set $R_+ \setminus (R_{st}^w \cup R_{DG(b'_d)})$ is empty,

(ii) The set $R_+ \setminus (R_{st}^w \cup R_{DG(b'_d)})$ does not contain any simple roots,

(iii) The set $\text{cross}^d_w(R_+ \setminus R_{st}^w)$ is empty,

(iv) The set $\text{cross}^d_w(R_+ \setminus R_{st}^w)$ does not contain any simple roots.
The claim follows from the previous lemma.

\[ \text{Lemma 2.28.} \quad \text{Let } w \text{ be an element of a twisted Weyl group and let } \mathcal{R} \text{ a } w\text{-nimble set also containing } \mathcal{R}_{w^{-1}}. \text{ Then } \mathcal{R} \text{ is also nimble for the element } w^{-1}, \text{ and for any integer } d \geq 0 \text{ we have } \]

\[ \text{cross}^d_{w^{-1}}(\mathcal{R}) = \emptyset \quad \text{if and only if} \quad \text{cross}^d_{w^{-1}}(\mathcal{R}) = \emptyset. \]

Proof. Pick a root } \beta \text{ in } \mathcal{R}\setminus\mathcal{R}_{w^{-1}}. \text{ If } \beta \text{ also lies in } \mathcal{R}_{w^{-1}} = \mathcal{R}_{w^{-1}} \text{ then by nimbleness of } w \text{ and positivity the root } w^{-1}(\beta) = w_{\text{ord}(w)^{-1}}(\beta) \text{ also lies in } \mathcal{R}. \text{ If on the other hand it does not lie in } \mathcal{R}_{w^{-1}}, \text{ then } w^{-1}(\beta) \in \mathcal{R}_{w^{-1}} \subseteq \mathcal{R} \text{ for some } i > 0. \text{ Applying } w^{-1}, \text{ nimbleness yields that } w^{-1}(\beta) \text{ also lies in } \mathcal{R}. \text{ If say } \text{cross}^d_{w^{-1}}(\mathcal{R}) \neq \emptyset \text{ then there exists a sequence of roots } (\beta_{i,j}) \in \text{cross}^d_{w^{-1}}(\beta', \beta) \text{ for some roots } \beta, \beta' \text{ in } \mathcal{R}. \text{ As } -w(\mathcal{R}_{w}) = \mathcal{R}_{w^{-1}}, \text{ this yields another sequence } -w(\beta_{d+1-i,j}) \in \text{cross}^d_{w^{-1}}(\beta', \beta'), \text{ implying that } \text{cross}^d_{w^{-1}}(\mathcal{R}) \neq \emptyset. \]


Example 2.26. Let } w \text{ be reflection in a non-simple root in type } B_2, \text{ then } \mathcal{R}_w \cup (\mathcal{R}_w \cap \mathcal{R}_+) = \mathcal{R}_+.


Example 2.27. Consider the elements } w = s_2s_3s_2s_1 \text{ and } w' = s_2s_1w \text{ in type } B_3. \text{ Then } w_2s_3 = DG(b_2^2) > w' > w, \text{ but } \mathcal{R}_+ \setminus (\mathcal{R}_w \cup \mathcal{R}_w) = \{\alpha_{23}, \alpha_{233}\}.

We deduce part (iii) of the main Lemma:

Proof. If } w \text{ is convex then the claim follows from part (ii) of the main Lemma, combined with the fact that } pb(w) \text{ is an upper bound for } DG(b_2^i). \text{ If on the other hand it is not convex, then } \mathcal{R}_+ \setminus \mathcal{R}_{w} \text{ is not convex by } [Mal21, Proposition 4.21] \text{ again so } \mathcal{R}_+ \setminus (\mathcal{R}_{w} \cup \mathcal{R}_{DG(b_2^i)}) \text{ must be nonempty (as } \mathcal{R}_{DG(b_2^i)} \text{ is convex), and then the claim follows from the previous lemma.}


Corollary A follows from

\[ \text{Corollary 2.30.} \quad \text{Let } w \text{ be an element of a twisted Weyl group } W \text{ and let } y \text{ be an element of } W \text{ such that } y \geq w^{-1}. \text{ Then for any integer } d \geq 0 \text{ we have } \]

\[ DG(b_2^d) \geq y \quad \text{if and only if} \quad DG(b_{w^{-1}}^d) \geq y. \]

Proof. This now follows by combining the previous lemma with part (ii) of the main Lemma.

In the remainder of this subsection we reprove the bounds of [Mal21, Proposition C], in the case } i = 1:\n
\[ \text{Proposition 2.31.} \quad \text{Let } w \text{ be an element of a twisted Weyl group and let } \mathcal{R} \text{ be a nimble set of roots. Then } \text{we have a sequence of inclusions } \]

\[ \mathcal{R} \supseteq \text{cross}_w(\mathcal{R}) \supseteq \text{cross}_w^2(\mathcal{R}) \supseteq \cdots, \]

\[ \text{which stabilises after } |\mathcal{R}| - \ell(w) \text{ terms.} \]

\[ \text{In particular we have } \text{cross}_w^d(\mathcal{R}) = \emptyset \text{ for some integer } d \geq 0, \text{ if and only if this holds for all } d > |\mathcal{R}| - \ell(w). \]

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Proof. For any $i \geq 0$ let $w_i := (w, \ldots, w, w)$ denote the sequence of $i$ copies of $w$. If $\gamma$ lies in cross$^d_w(\mathfrak{H})$, then there exists a root $\beta$ in $\mathfrak{H}$ such that cross$^d_w(\gamma, \beta)$ is nonempty. In other words, there exists roots $\gamma = \beta_1', \ldots, \beta_0' = \beta$ inductively constructed via

$$\beta_j' := w(\beta_{j-1}' + \sum_{\beta \in \mathfrak{H}_w} \tilde{\beta})$$

By assumption we have $\mathfrak{H} \supseteq \text{cross}_w^d(\mathfrak{H})$, so $\beta_1'$ lies in $\mathfrak{H}$. But then cross$^{-1}_w(\gamma, \beta_1')$ is nonempty, which means that $\gamma$ also lies in cross$^{d-1}_w(\mathfrak{H})$. \hfill $\Box$

Although the sets $\mathfrak{H}$ and cross$^d_w(\mathfrak{H})$ are convex, this does not necessarily hold for the other sets appearing in such a sequence:

**Example 2.32.** Consider $w = s_1 s_2 s_3 s_1$ and let $v = w_c s_2 s_1$ in type $B_3$. Then $\mathfrak{H} := \mathfrak{H}_w$ is $w$-nimble and this sequence is

$$\mathfrak{H} \supseteq \{\alpha_3, \alpha_{23}, \alpha_{233}\} \supseteq \{\alpha_3, \alpha_{23}\} \supseteq \{\alpha_3, \alpha_{23}\} \supseteq \cdots$$

**Corollary 2.33.** Let $w$ be an element of a twisted Weyl group. Then for any $d \geq 1$ we have

(i) an inclusion

$$\mathfrak{H}_w \subseteq \mathfrak{H}_{DG(b^d_w)} \subseteq \mathfrak{H}_+ \setminus \mathfrak{H}_{st}$$

(ii) and if $d > |\mathfrak{H}_+ \setminus \mathfrak{H}_{st}| - \ell(w)$ then for any $d' \geq d$ we have

$$\text{DG}(b^d_w) = \text{DG}(b^{d'}_w)$$

**Proof.** (i): If $\beta$ lies in $\mathfrak{H}_{st}$ then $w^d(\beta)$ lies in cross$^d_w(\beta)$. By part (ii) of the main Lemma we have cross$^d_w(\mathfrak{H}_{DG(b^d_w)}) = \emptyset$, so $\beta$ does not lie in $\mathfrak{H}_{DG(b^d_w)}$.

(ii): The previous proposition yields

$$\text{cross}_w^d(\mathfrak{H}_{DG(b^d_w)}) = \text{cross}_w^d(\mathfrak{H}_{DG(b^{d'}_w)}) = \emptyset$$

and then we conclude from part (ii) of the main Lemma that $\text{DG}(b^d_w) \supseteq \text{DG}(b^{d'}_w) \supseteq \text{DG}(b^d_w)$. \hfill $\Box$

### 2.4 From roots to root subgroups

The theory of reductive groups over schemes was originally developed by Demazure and Grothendieck [SGA3-III]; some simplifications were recently made in an exposition by Conrad [Con14]. The main property that we will use is

**Theorem 2.34.** Consider a split reductive group $G$ over a scheme. Trivialisations of the root spaces of its Lie algebra exponentiate to parametrisations $p_\beta : \mathbb{G}_a \rightarrow N_\beta$ of its root subgroups.

(i) [Che55, p. 27] For any pair of roots $\beta, \gamma$ with $\beta \neq -\gamma$, there is the Chevalley commutator formula

$$[p_\beta(c_\beta), p_\gamma(c_\gamma)] := p_\beta(c_\beta)p_\gamma(c_\gamma)p_\beta(-c_\beta)p_\gamma(-c_\gamma) = \prod_{i,j > 0} p_{i\beta + j\gamma}(c_{\beta,\gamma}^{ij}c_{\beta,\gamma}^{ij})$$

for some global functions $c_{\beta,\gamma}^{ij}$ on the underlying scheme and ordering on the roots. In particular,

$$N_\beta N_\gamma = \left( \prod_{i \geq 0, j > 0} N_{i\beta + j\gamma} \right) N_\beta$$

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Let \( \dot{w} \) denote a lift of a Weyl group element \( w \) to \( G \). Then there is the Steinberg relation

\[
\dot{w}N_{\beta} = N_{w(\beta)}\dot{w}.
\]

In order to analyse such commutators, we will focus on the roots appearing in the product on the right-hand-side and hence define

**Definition 2.35.** Let \( w \) be an element of a twisted Weyl group. Given a convex set \( \mathcal{R} \) of positive roots containing \( \mathcal{R}_w \), we set

\[
\text{Cross}_w(\mathcal{R}) := w(\mathcal{R} + \mathcal{R}_w) \cap \mathcal{R}_+, \quad \text{and we let } \text{Cross}^d_w(\mathcal{R}) \text{ denote its } d \text{-th iterate.}
\]

We won’t be using

**Proposition 2.36.** Let \( w \) be an element of a twisted Weyl group and let \( \mathcal{R} \) be such a set.

(i) The sets \( \text{cross}_w(\mathcal{R}) \) and \( \text{Cross}_w(\mathcal{R}) \) are also convex.

(ii) Let \( \mathcal{L} \subseteq \mathcal{R} \setminus \mathcal{R} \) be a subset satisfying \( w(\mathcal{L}) = \mathcal{L} = -\mathcal{L} \) and such that \( \mathcal{L} \cup \mathcal{R}_w \) is convex. Then

\[
\text{cross}_w(\mathcal{R}) \cap \mathcal{L} = \text{Cross}_w(\mathcal{R}) \cap \mathcal{L} = \emptyset.
\]

**Proof.** (i): Let \( \gamma_0 \) and \( \gamma_1 \) be elements of \( \text{cross}_w(\mathcal{R}) \), so \( \gamma_i \in \text{cross}_w(\beta'_i) \) for some \( \beta'_i \) in \( \mathcal{R} \). If \( \gamma_0 + \gamma_1 \) is a root, then there is a sum

\[
w^{-1}(\gamma_0 + \gamma_1) = w^{-1}(\gamma_0) + w^{-1}(\gamma_1) = \beta'_0 + \beta'_1 + \sum_{i=1}^{m} \beta_i, \quad \beta_i \in \mathcal{R}_w,
\]

with \( \beta'_0 \) and \( \beta'_1 \) in \( \mathcal{R} \). Since \( \mathcal{R} \) is convex and contains \( \mathcal{R}_w \), it follows from Lemma 2.2(iii) that it contains the right-hand-side, so that \( \text{cross}_w(\mathcal{R}) \) contains \( \gamma_0 + \gamma_1 \). The second case is analogous.

(ii): Suppose the first intersection is nonempty, so there exists a positive root \( \beta \in \mathcal{R} \) and a root \( \gamma \in \mathcal{L} \) such that

\[
\beta + \sum_{i=1}^{m} \beta_i = \gamma, \quad \beta_i \in \mathcal{R}_w,
\]

then as

\[
\sum_{i=1}^{m} \beta_i - \gamma = -\beta
\]

is a root and \( -\gamma \) lies in \( \mathcal{L} \) and \( \mathcal{R}_w \cup \mathcal{L} \) is convex it again follows from Lemma 2.2(iii) that \( -\beta \) lies in \( \mathcal{R}_w \cup \mathcal{L} \), which implies that \( \beta \) lies in \( \mathcal{L} \). The second case is analogous. \( \square \)

These statements do not extend to higher iterates because these sets might not contain all of \( \mathcal{R}_w \). The remainder of this subsection is devoted to proving

**Lemma 2.37.** Let \( w \) be an element of a twisted finite Weyl group and let \( \mathcal{R} \) be a subset of positive roots. Then

\[
\text{cross}_w^d(\mathcal{R}) = \emptyset \quad \text{if and only if} \quad \text{Cross}_w^d(\mathcal{R}) = \emptyset.
\]

**Lemma 2.38.** Let \( \sum(\gamma_1, \ldots, \gamma_m) \) be a summing sequence of positive roots in a crystallographic root system, pick any \( 1 \leq k \leq m \) and write \( \gamma_{<k} := \sum_{j=1}^{k-1} \gamma_j \). Then we may partition the set \( \{k+1, \ldots, m\} \) into two subsets

\[
\{i_1, \ldots, i_{k'}\}, \quad \{i_1, \ldots, i_{m-k}\},
\]

such that

\[
\gamma_{<k} + \sum_{i=1}^{k'} \gamma_{i_j} \quad \text{and} \quad \gamma_k + \sum_{j=k'+1}^{m-k} \gamma_{i_j}
\]

are both positive roots.
Proof. We induct on \( m - k \): applying Lemma 2.2(ii) to the root

\[ \gamma_{<k} + \gamma_k + \gamma_{k+1}, \]

it follows that at least one of \( \gamma_{<k} + \gamma_{k+1} \) or \( \gamma_k + \gamma_{k+1} \) is a root. By replacing the corresponding pair with this sum, this also shortens the sequence and then the claim follows from the induction hypothesis. \( \square \)

Lemma 2.39. Let \( \beta_1, \ldots, \beta_k \) and \( \gamma_1, \ldots, \gamma_m \) be positive roots in a crystallographic root system such that their sum

\[ \sum_{i=1}^{k} \beta_i + \sum_{i=1}^{m} \gamma_i =: \delta \]

is a root. Then we may partition the set \( \{1, \ldots, m\} \) into \( k \) subsets \( \{i_{j,1}, \ldots, i_{j,m_j}\} \) with \( 1 \leq j \leq k \), such that for each \( j \) the sum

\[ \beta_j + \sum_{l=1}^{m_j} \gamma_{ij,l} =: \delta_j \]

is a root, and thus the sum over those roots yields

\[ \sum_{j=1}^{k} \delta_j = \delta. \]

Proof. We use Lemma 2.2(iii) to construct a summing sequence. Let \( \beta_{ik} \) denote the final root from the first set of roots that appears in there, and denote the sum of the elements before it by \( \delta_{<ik} \). Thus

\[ \sum_{(\ldots, \beta_{ik}, \gamma_{i_m}, \ldots, \gamma_{im})} = \delta. \]

The previous lemma now yields a partition

\[ \{i'_1, \ldots, i'_{m''}\} \prod \{i''_{m''+1}, \ldots, i''_{m-m'}\} = \{i_{m'}, \ldots, i_m\}, \]

such that

\[ \delta_{<ik} + \sum_{j=1}^{m''} \gamma_{ij} \quad \text{and} \quad \beta_{ik} + \sum_{j=m'+1}^{m-m'} \gamma_{ij} \]

are both roots and sum to \( \delta \). Applying the induction hypothesis on the first sum then furnishes the claim. \( \square \)

Lemma 2.40. For any root \( \beta' \in \text{Cross}_w^d(\beta) \), there exist roots \( \beta'_1, \ldots, \beta'_k \in \text{cross}_w^d(\beta) \) such that

\[ \sum_{i=1}^{k} \beta'_i = \beta'. \]

(2.10)

In particular, (2.9) holds.

Proof. We induct on \( d \), so we may assume that there exists an integer \( k \in \mathbb{N}_1 \) and a root \( \tilde{\beta}' \) in \( \text{Cross}_w^{d-1}(\beta) \) such that

\[ \beta' = w(\tilde{k}\tilde{\beta}' + \sum_{i=1}^{m} \gamma_i), \quad \gamma_i \in \Pi_w. \]

The induction hypothesis furnishes \( \tilde{\beta}_1, \ldots, \tilde{\beta}_k \in \text{cross}_w^d(\beta) \) such that \( \sum_{i=1}^{k} \tilde{\beta}_i = \tilde{\beta}' \). Thus

\[ w^{-1}(\beta') = \tilde{k} \sum_{i=1}^{k'} \tilde{\beta}_i + \sum_{i=1}^{m} \gamma_i. \]
The previous lemma now implies that we may rename the $\tilde{k}$-fold concatenation of $\tilde{\beta}_1, \ldots, \tilde{\beta}_k$ into $\beta_1, \ldots, \beta_k'$ and partition the $\gamma_1, \ldots, \gamma_m$ such that for $1 \leq j \leq kk'$ there are roots
\[
\tilde{\beta}_j' := \beta_j + \sum \gamma_{i,j}, \quad \gamma_{i,j} \in R_w
\]
not lying in $R_w$ (otherwise we add them to the list of $\gamma$'s and start over with a smaller list of $\beta$'s). Setting $\beta'_j := w(\tilde{\beta}_j')$ and $k := \tilde{k}k'$, we have
\[
\sum_{i=1}^k \beta'_j = \sum_{i=1}^{kk'} w(\beta_j + \sum \gamma_{i,j}) = w(\tilde{k} \sum_{i=1}^{k'} \tilde{\beta}_i + \sum_{i=1}^m \gamma_i) = \beta'
\]
so (2.10) is satisfied, and they lie in $\text{cross}^d_w(\beta)$.

\[\square\]

3 Cross sections and transversality

In this section we prove part (i) and (ii) of the main Theorem.

In the first subsection we prove part (i). The main part of He-Lusztig’s proof employs the existence of certain “good” elements in each elliptic conjugacy class of the Weyl group [GM97]. Combining this part with the geometric construction of such elements in [HN12, §5.2] and unravelling the resulting proof, one finds that it is very similar to Sevostyanov’s (in the elliptic case). As He-Lusztig’s techniques are neater and yield an explicit inverse map, the proof of this subsection is based upon their approach.

More specifically, He-Lusztig constructed a candidate inverse $\Psi'$ to the conjugation map $\Psi$ when the Weyl group element $w$ is elliptic, and proved that $\Psi' \circ \Psi$ is the identity when $b_w^d$ is divisible by $b_w$ for some natural number $d$ [HL12, §3.7]. The core of their argument states that a certain variety with a projection map constructed out of root subgroups and sequences of Weyl group elements only depends on the image of this sequence in the braid monoid [HL12, §2.9], and this argument can be generalised to work in the nonelliptic case when $L$ is nontrivial. More directly however, we observe that the rôle that these Weyl group elements play here is in asserting the identity
\[
\text{cross}^d_w(\mathcal{R}) = \emptyset,
\]
which through Lemma 2.28 is crucial in our approach to proving transversality. Rather surprisingly, the previous section demonstrated that such equations about roots are equivalent to similar identities about braids. Hence in the first subsection we’ve rewritten this part of their proof in terms of $w$-crossing pairs (for arbitrary $w$) satisfying this equation, yielding many new cross sections along the way.

Rather than following this up with a proof that $\Psi \circ \Psi'$ also equals the identity, He-Lusztig then appeal to Ax-Grothendieck type results about affine $n$-space to conclude that $\Psi'$ is indeed inverse to $\Psi$, under suitable conditions on the base ring and its ring automorphism. However, for nonelliptic $w$ the slices $\dot{w}LN_w$ are not isomorphic to affine $n$-space; the following proof shows directly that $\Psi \circ \Psi'$ is the identity, shedding any conditions on the base ring and its automorphism.

In the second subsection, we will also prove that part (ii) implies the following variant on part (i):

(i') The conjugation action (1.4) is an isomorphism when restricted to a first order infinitesimal neighborhood of the subscheme $\{\text{id}\} \times \dot{w}LN_w$.

A priori (i') is weaker; I have not studied whether they might be equivalent.

3.1 Crossing root subgroups

The following construction was inspired by [HL12, §2.7]:
Definition 3.1. Fix an integer \( d \geq 1 \). We consider the set of orbits in the \( d \)-fold Cartesian product
\[
N(\ddot{w}L) := N\dot{w}LN \times \cdots \times N\dot{w}LN
\]
for the \( N^{d-1} \)-action given by
\[
(n_{d-1}, \ldots, n_1) \cdot (g_d, \ldots, g_1) = (g_d n_{d-1}, n_{d-1}^{-1} g_d n_{d-2}, \ldots, n_2^{-1} g_2 n_1, n_1^{-1} g_1).
\]
We denote the (naive) orbit space by \( N[\ddot{w}L] \) and the quotient map by
\[
N(\ddot{w}L) \longrightarrow N[\ddot{w}L], \quad (g_d, \ldots, g_1) \longmapsto [g_d, \ldots, g_1]. \tag{3.1}
\]
This map is equivariant with respect to the \( N \times N \)-actions on \( N(\ddot{w}L) \) and \( N[\ddot{w}L] \) coming from left and right multiplication on the outer factors:
\[
n'(g_d, \ldots, g_1)n := (n' g_d, \ldots, g_1 n), \quad n'[g_d, \ldots, g_1]n := [n' g_d, \ldots, g_1 n].
\]
Given an element \( g \in N\dot{w}LN \), we shall write
\[
[g] := [g, g, \ldots, g, g] \in N[\ddot{w}L]
\]
for the image of \((g, g, \ldots, g, g) \in N(\ddot{w}L)\) under the quotient map (3.1).

We can describe \( N[\ddot{w}L] \) more explicitly: first consider the \( d \)-fold product
\[
N^+_{\ddot{w}L} := \dot{w}LNw \times \cdots \times \dot{w}LNw
\]
and enlarge it to \( N^+_{\ddot{w}L} := N \times N^+_{\ddot{w}L} \). By multiplying the first two components of its \( d+1 \)-Cartesian product, we obtain a natural inclusion
\[
N^+_{\ddot{w}L} = N \times \dot{w}LNw \times \dot{w}LNw \times \cdots \times \dot{w}LNw \longrightarrow \dot{w}LNw \times \dot{w}LNw \times \cdots \times \dot{w}LNw \longrightarrow N(\ddot{w}L). \tag{3.2}
\]

Notation 3.2. We write \( N^w := w^{-1}Nw \cap N \) for the product of root subgroups corresponding to the roots in \( \mathfrak{R}_w \), and given elements \( x, g \in G \) we abbreviate left conjugation by \( xg := xgx^{-1} \).

Lemma 3.3. (i) There is a natural factorisation
\[
L \times N \longrightarrow LN = L \longrightarrow N^w \times L \times N_w, \quad (l, n) \longmapsto (n_1, l', n_2), \tag{3.3}
\]
implying \( N\dot{w}LN = N\dot{w}LN_w \); if \( (\mathfrak{R}_w + \mathfrak{L}) \cap \mathfrak{L} = \emptyset \) (e.g. \( \mathfrak{L} \) is a standard parabolic subsystem) then \( l' = l \).

(ii) Hence the inclusion (3.2) yields an algebraic cross section
\[
\begin{array}{ccc}
\overset{N(\ddot{w}L)}{\downarrow} & & \\
N^+_{\ddot{w}L} & \sim & N[\ddot{w}L]
\end{array}
\]
of the quotient map (3.1).

(iii) Assume that \( L \subseteq T \), pick a root \( \beta \in \mathfrak{R}_w \), elements \( n \in LN_w \) and \( m \in N_\beta \), and use (3.3) to factorise \( nm \in LN \) into a pair of elements \((m_1, n_1)\) in \( N^w \times LN_w \). Then
\[
\dot{w}m_1 \in \prod_{\gamma \in \text{Cross}_w(\beta)} N_\gamma \subseteq N.
\]
Proof. (i): Since $R \sqcup L$ is convex and $L = -L$ and $L \cap R = \emptyset$, it follows that
\[(\mathfrak{H} \setminus R_w) \cap R = \emptyset,\]
which through convexity of $R \sqcup L$ yields $L N^w = N^w L$, and the first claim follows. Nimbleness then implies that
\[N \dot{w} L N = N \dot{w} N^w L N = N \dot{w} L N_w.\]

(ii) The case of $d = 2$ now follows from
\[N \dot{w} L N \times N \dot{w} L N = N \dot{w} L N \times \dot{w} L N = N \dot{w} L N_w \times \dot{w} L N_w \times \dot{w} L N_w \simeq N^+ \dot{w} L.\]

(iii): Follows similarly. □

Notation 3.4. We now denote by $\text{Cross}^d_w L$ the composition
\[N[\dot{w} L] \sim \rightarrow N^+ \dot{w} L = N \times N^+ \dot{w} L \rightarrow N\]
of the inverse of this isomorphism with projection onto the first component of the Cartesian product.

Corollary 3.5. Assume that $L \subseteq T$, pick an integer $d > 0$ and a positive root $\beta$ in $\mathfrak{H}$, fix an element $h \in N[\dot{w} L]$ and consider the morphism of schemes
\[G_a \simeq N \beta \rightarrow N, \quad m \mapsto \text{Cross}^d_w L(hm).\]

(i) Then
\[\text{Cross}^d_w L(hm) \in \text{Cross}^d_w L(h) \prod_{\gamma \in \text{Cross}_w(\beta)} N_\gamma\]

(ii) If $\text{Cross}^d_w L(h)$ is the identity, then the derivative of this map at the identity of $N \beta$
\[n_\beta \rightarrow n, \quad x \mapsto \text{cross}^d_w L(hx)\]
satisfies
\[\text{cross}^d_w L(hx) \in \bigoplus_{\gamma \in \text{Cross}_w(\beta)} n_\gamma.\]

Proof. (i): Denote the inverse of the element $h$ under the isomorphism
\[N \times \dot{w} L N_w \times \cdots \times \dot{w} L N_w = N^+ \dot{w} L \sim \rightarrow N[\dot{w} L]\]
by $(m', \dot{w} m_2, \ldots, \dot{w} m_1)$, so $m'$ lies in $N$ and each $n_i$ lies in $L N_w$. Let $(m_1, n'_1) \in N^w \times L N_w$ be the factorisation of $n_1 m \in LN$ in (3.3) and inductively define for $1 < i \leq d$ elements $(m_i, n'_i) \in N^w \times L N_w$ as the factorisation of the element $n_{i-1}(\dot{w} m_{i-1}) \in LN$. By induction on $d$, the second part of the previous proposition implies that
\[\dot{w} m_i \in \prod_{\gamma \in \text{Cross}_w(\beta)} N_\gamma.\]
Then
\[ hm = [m', \dot{w}_0, \ldots, \dot{w}_2, \dot{w}_1 m] \]
\[ = [m', \dot{w}_0, \ldots, \dot{w}_2(\dot{w}_1 m_1), \dot{w}_1] \]
\[ = [m'(\dot{w}_d), \dot{w}_d', \ldots, \dot{w}_2, \dot{w}_1] \]
so that
\[ \text{Cross}_{w,L}^d(hm) = m'(\dot{w}_d) = \text{Cross}_{w,L}^d(h)(\dot{w}_d). \]

(ii): Taking derivatives with respect to the first subgroup in (2.8), the component on the right-hand-side with \(i > 1\) vanishes. This implies that for \(d = 1\), the image lands in
\[ \dot{w}(\bigoplus_{\gamma \in \{\beta + \sum_{i=1}^m \beta_i \in \mathcal{R}_+ : \beta_i \in \mathcal{R}_w\}} n_\gamma) \dot{w}^{-1} \cap n_+ = \bigoplus_{\gamma \in \text{cross}_w(\beta)} n_\gamma. \]
The claim then follows by induction. \(\square\)

**Proposition 3.6.** Both \(\text{cross}_w(\cdot)\) and \(\text{Cross}_w(\cdot)\) lift to the braid monoid.

**Proof.** For any reduced decomposition \(w = uv\) we have
\[ \dot{w}N_w = \dot{u}N_u \dot{v}N_v \]
regardless of characteristic, and hence
\[ \prod_{\beta' \in \text{Cross}_w(\beta)} N_{\beta'} \times \dot{w}N_w = \dot{w}N_wN_\beta = \dot{u}N_u \dot{v}N_vN_\beta = \prod_{\beta' \in \text{Cross}_u(\text{Cross}_w(\beta))} N_{\beta'} \times \dot{w}N_w, \]
so that
\[ \text{Cross}_w(\beta) = \text{Cross}_u(\text{Cross}_w(\beta)). \]
Taking derivatives as before then yields
\[ \text{cross}_w(\beta) = \text{cross}_u(\text{cross}_w(\beta)). \]
The rest of the proof is analogous to that of Lemma 2.6. \(\square\)

We now prove the crucial

**Lemma 3.7.** If
\[ \text{cross}_w^d(\beta) = \emptyset, \]
then for all \(h \in N[\dot{w}L]\) and \(n_\beta \in N_\beta\) we have
\[ \text{Cross}_{w,L}^d(hn_\beta) = \text{Cross}_{w,L}^d(h). \]

**Proof.** Consider the notions introduced in the first paragraph of Definition 3.1; we add a tilde to denote the orbits for \(NL\) instead. Then from \(\dot{w}L = Lu\) and \(LN_w = N_wL\) we similarly derive a factorisation
\[ N \times L \times \dot{w}N_w \times \cdots \times \dot{w}N_w = N \times L \times N_{\dot{w}} =: \tilde{N}_{\dot{w}/L} \]
yielding projection maps
\[ \text{Cross}_{\dot{w},L}^d : \tilde{N}_{\dot{w}/L} \overset{\sim}{\rightarrow} \tilde{N}_{\dot{w}/L}^+ = N \times L \times N_{\dot{w}} \rightarrow N, \]
\[ \text{proj}_L : N[\dot{w}L] \overset{\sim}{\rightarrow} \tilde{N}[\dot{w}/L] \overset{\sim}{\rightarrow} \tilde{N}_{\dot{w}/L}^+ = N \times L \times N_{\dot{w}} \rightarrow L. \]
\[ \text{proj}_{\dot{w}} : N[\dot{w}L] \overset{\sim}{\rightarrow} \tilde{N}[\dot{w}/L] \overset{\sim}{\rightarrow} \tilde{N}_{\dot{w}/L}^+ = N \times L \times N_{\dot{w}} \rightarrow N_\dot{w}, \]
\[ \text{proj}_{\dot{w},L} : N[\dot{w}L] \overset{\sim}{\rightarrow} \tilde{N}_{\dot{w}/L}^+ = N \times N_{\dot{w}/L} \rightarrow N_{\dot{w}/L}. \]
This gives a natural commutative diagram

\[
\begin{array}{ccc}
N(wL) & \longrightarrow & N(wL) \\
\downarrow & & \downarrow \\
N[wL] & \longrightarrow & N[wL] \\
\downarrow \text{Cross}^d_{wL} & & \downarrow \text{Cross}^d_{wL} \\
N & \longrightarrow & N
\end{array}
\]

Write \( h_L := \text{proj}_L(h) \), \( h_w := \text{proj}_{Nw}(h) \) and \( h_{wL} := \text{proj}_{wL}(h) \). Then

\[
\text{Cross}^d_{wL}(hn_\beta) = \text{Cross}^d_{wL}(h)\text{Cross}^d_{wL}(h_\beta n_\beta),
\]

\[
= \text{Cross}^d_{wL}(h)\text{Cross}_{wL}(h_L h_\beta n_\beta)
\]

\[
= \text{Cross}^d_{wL}(h) h_L^{-1} \text{Cross}_{wL}(h_L n_\beta) h_L
\]

so the claim follows from part (ii) of the previous statement.

\[
\square
\]

**Remark 3.8.** Instead of considering orbits for \( NL \), we could have also upgraded \( \text{Cross}_w(\cdot) \) to a version \( \text{Cross}_{wL}(\cdot) \) taking into account the roots in \( \mathfrak{L} \), and then proven that \( \text{cross}^d_{wL}(\beta) = \emptyset \) if and only if \( \text{Cross}_{wL}(\beta) = \emptyset \) purely by studying roots. By combining this with

\[
\text{Cross}^d_{wL}(hm) \in \text{Cross}^d_{wL}(h) \prod_{\gamma \in \text{Cross}_{wL}(\beta)} N_\gamma,
\]

for \( L \) arbitrarily, the previous lemma can be obtained almost entirely by analysing roots, but the arguments become a bit longer and perhaps less transparent, despite being essentially identical.

We can now prove (i) of the main Theorem:

**Proof.** Set \( d > 0 \) such that (1.3) holds, then we construct the (algebraic) inverse \( \Psi' \) to the conjugation map

\[
\Psi : N \times \dot{w}LN_w \longrightarrow N\dot{w}LN = N\dot{w}LN_w, \quad (n, \dot{w}\hat{n}) \mapsto n^{-1}\dot{w}\hat{n}
\]

as follows. For an element \( \tilde{g} \in N\dot{w}LN_w \) we set

\[
n_{\tilde{g}} := \text{Cross}^d_{wL}([\tilde{g}])^{-1} \in N. \quad \text{(3.4)}
\]

Denoting the image of \( n_{\tilde{g}} \hat{g} n_{\tilde{g}}^{-1} \in N\dot{w}LN_w \) under the inverse of the usual multiplication map

\[
N \times \dot{w}LN_w \longrightarrow N\dot{w}LN_w, \quad (\tilde{m}, \dot{w}\hat{n}) \mapsto \tilde{m}\tilde{n}\hat{n} \quad \text{(3.5)}
\]

by \((n', g_{\tilde{g}})\), now set \( \Psi'(\tilde{g}) := (n_{\tilde{g}}, g_{\tilde{g}}) \). We will calculate these elements more explicitly and see that equation (1.3) implies that \( n' = \text{id} \).

\( \Psi' \circ \Psi = \text{id} \): Pick \((n, g) \in N \times \dot{w}LN_w \) and set \( \tilde{g} := \Psi(n, g) := n^{-1}gn \in N\dot{w}LN_w \). Then

\[
[\tilde{g}] = [n^{-1}gn, n^{-1}gn, \ldots, n^{-1}gn] = n^{-1}[g]n,
\]

so as \( \text{Cross}^d_{wL}([\tilde{g}]) = e \), Lemma 3.7 implies that

\[
n_{\tilde{g}}^{-1} = \text{Cross}^d_{wL}([\tilde{g}]) = \text{Cross}^d_{wL}(n^{-1}[g]n) = \text{Cross}^d_{wL}(n^{-1}[g]) = \text{Cross}^d_{wL}([n^{-1}g, \ldots, g]) = n^{-1}.
\]

Hence \( n_{\tilde{g}} = n \), and thus

\[
n_{\tilde{g}}\hat{g} n_{\tilde{g}}^{-1} = n\hat{g} n^{-1} = g
\]

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which lies in \( \dot{w}LN \) by assumption, so \( g_\bar{g} = g \) and therefore \( (\Psi' \circ \Psi)(n, g) = (n_\bar{g}, g_\bar{g}) = (n, g) \).

\( \Psi \circ \Psi' = \text{id} \): Pick \( \bar{g} \in \dot{N} \dot{w}LN \) and use (3.5) to decompose \( \bar{g} = m \dot{w}m \) for some \( m \in N \) and \( n \in LN \).

Let \((m_1, n_1) \in N^w \times LN_n \) be the factorisation of \( m \dot{w}m \) \((\dot{N} \dot{w}LN) \) in (3.3) for \( i = 1 \) and inductively for \( i > 1 \) as the factorisation of \( n_{i-1} \dot{w}m_{i-1} \) \((\dot{N} \dot{w}LN) \) in (3.3).

These elements were constructed to obtain the inverse image of \( [\bar{g}] \) under the isomorphism \( N^w \overset{\sim}{\to} N[\dot{w}L] \), as

\[
[\bar{g}] = [m \dot{w}m, \ldots, m \dot{w}m, \dot{w}n]
\]

\[
= [\dot{w}nm, \ldots, \dot{w}nm, \dot{w}n]
\]

\[
= [\dot{w}(m_1 \dot{w}m_1) \dot{w}n_1, \ldots, (\dot{w}m_1) \dot{w}n_1, \dot{w}n]
\]

\[
= [m(\dot{w}(m_1 \cdots m_d)) \dot{w}n_{d-1}, \dot{w}n_{d-2}, \ldots, \dot{w}n_1, \dot{w}n].
\]

In particular, this yields

\[
n_\bar{g}^{-1} = \text{Cross}^d_{\dot{w}L}(\bar{g}) = m(\dot{w}(m_1 \cdots m_d)).
\]

A similar calculation furnishes that \( m_d = \text{id} \): briefly setting

\[
a := \text{Cross}^d_{\dot{w}L}([\bar{g}]^{-1} \dot{g}) = [\dot{w}n_{d-1}, \dot{w}n_{d-2}, \ldots, \dot{w}n_1, \dot{w}n] \in N[\dot{w}L],
\]

we have by construction

\[
\dot{w}mn = [\dot{w}n_{d-1}, \dot{w}n_{d-2}, \ldots, \dot{w}n_1, (\dot{w}m_1) \dot{w}n_1]
\]

\[
= [\dot{w}n_{d-1}, \dot{w}n_{d-2}, \ldots, (\dot{w}m_2) \dot{w}n_2, \dot{w}n_1]
\]

and then Lemma 3.7 implies that

\[
\dot{w}m_d = \text{Cross}^d_{\dot{w}L}(a) = \text{Cross}^d_{\dot{w}L}(am) = \dot{w}m_d.
\]

We now obtain an expression for \( \Psi'(\bar{g}) \) by computing

\[
n_\bar{g}^{-1} \dot{w}m_d = (\dot{w}(m_d^{-1} \cdots m_1^{-1})) m^{-1} m \dot{w}m mn (\dot{w}(m_1 \cdots m_d))
\]

\[
= (\dot{w}(m_d^{-1} \cdots m_1^{-1})) \dot{w}mn (\dot{w}(m_1 \cdots m_d))
\]

\[
= (\dot{w}(m_d^{-1} \cdots m_1^{-1})) \dot{w}n_{d-1} \dot{w}m_2 \cdots \dot{w}m_d
\]

\[
= \dot{w}n_{d-1} \dot{w}m_d = \dot{w}n_d
\]

which already lies in \( \dot{w}LN \). Thus \( \dot{g} = \dot{w}n_d = n_\bar{g} \dot{w}n_\bar{g}^{-1} \), and hence \( (\Psi \circ \Psi')(\bar{g}) = \Psi(n_\bar{g}, n_\bar{g} \dot{w}n_\bar{g}^{-1}) = \bar{g} \).

**Remark 3.9.**

(i) We could have written the same proof with \( \dot{N}(\dot{w}L) \) (as defined in the proof of Lemma 3.7) instead of \( N(\dot{w}L) \); nothing changes except for the final paragraph, where more factoring is required.

(ii) For convenience, let’s briefly add a \( d \) to the notions introduced in Definition 3.1, so \( N(\dot{w}L) := \dot{N}(\dot{w}L) \), etc. The previous proof easily generalises to show that if (1.3) holds for some \( d \), then

\[
N \times \dot{i}N_{\dot{w}L} \to \dot{i}N[\dot{w}L], \quad (n, [g_1, \ldots, g_1]) \mapsto [n^{-1}g_1, \ldots, g_1 n]
\]

is an isomorphism for any \( i \geq 1 \); this proof was just the case \( i = 1 \).

(iii) If \( \mathcal{L} \) is a standard parabolic subsystem then Lemma 3.3(i) implies that the \( L \)-component of image of \( \dot{w}n' \in N\dot{w}LN \) in \( \dot{w}LN \) under the inverse map \( \Psi' \) is again \( l \); this plays a rôle in [Sev19, Lemma 5.7].
3.2 Charts on the quotient stack \([G/G]\)

Sevostyanov deduced from the cross section isomorphism (1.4) that his slices transversely intersect the conjugacy classes of \(G\) \cite[Proposition 2.3]{Sev11}. In this subsection we adapt his approach to prove that (ii) still holds in our more general setting, and simultaneously refine it to show that (ii) \(\Rightarrow\) (i):

**Notation 3.10.** We denote by \(\text{Ad}_w(\cdot)\) the right adjoint action map of an element \(m\) of \(G\) on its Lie algebra \(\mathfrak{g}\). We let \(n_w, n^w, l, \pi, \pi^w, \pi^{-1}\) denote the free submodules of \(\mathfrak{g}\) corresponding to roots in \(\mathfrak{R}_w, \mathfrak{N}\setminus\mathfrak{R}_w, \mathfrak{L}, -\mathfrak{N}, -\mathfrak{N}\setminus\mathfrak{R}_w\) and \(w(\mathfrak{R}_w) = -\mathfrak{R}_{w^{-1}}\) respectively. (These are actually all Lie subalgebras, as convexity of \(\mathfrak{R}_+\setminus\mathfrak{R}_w\) and \(\mathfrak{N}\) implies that \(\mathfrak{N}\setminus\mathfrak{R}_w\) is convex.) We let \(t_w^\perp\) denote the orthogonal complement inside \(t\) to \(t\); since \(L\) contains \(T^w\) we have \(t_w^\perp \subseteq t_w = (t^w)^{-1}\), which is \(w\)-invariant as both \(T^w\) and \(\mathfrak{L}\) are. Finally, we denote by \(\mathfrak{N}\) the unipotent subgroup of \(G\) corresponding to \(\pi\) (and \(-\mathfrak{N}\)).

**Lemma 3.11.** The image of the differential of the conjugation map

\[
G \times \mathfrak{w}LN_w \rightarrow G, \quad (g, m) \mapsto g^{-1}mg
\]

at any point \((id, m)\) is given in the left trivialisation of the tangent bundle of \(G\) by

\[
(id - \text{Ad}_m)(n \oplus \pi) + t_w + l \oplus n_w \oplus \pi^{-1}
\]

Moreover, we have

\[
(id - \text{Ad}_m)(\pi) \subseteq \pi \oplus l \oplus n_w.
\]

**Proof.** The left trivialisation of the tangent bundle of \(G\) induces for all points \(m\) in \(\mathfrak{w}LN_w\) identifications of their tangent spaces \(T_m(\mathfrak{w}LN_w) \simeq \pi \oplus n_w\) with a free submodule of \(\mathfrak{g}\), and this differential is then given at a point \((id, m)\) by the linear map

\[
\mathfrak{g} \oplus (l \oplus n_w) \rightarrow \mathfrak{g}, \quad (x, l) \mapsto (id - \text{Ad}_m)(x) + l.
\]  

For any \(t \in \mathfrak{t}\) and \(m'\) in \(L\) we have \(\text{Ad}_{m'}(t) \in \mathfrak{t} + l\). As \(\mathfrak{R}_w \cup \mathfrak{L}\) is convex and \(\mathfrak{L} = -\mathfrak{L}\), it then follows for \(m'' \in N_w\) that \(\text{Ad}_{m''}(t + l) \in l \oplus n_w\), so that finally for \(m = \mathfrak{w}m''\) we have

\[
\text{Ad}_m(t) = \text{Ad}_{m''}\text{Ad}_{m'}\text{Ad}_w(t) \in \text{Ad}_w(t) + l \oplus n_w.
\]

By definition \(t_w\) the linear operator \(id - \mathfrak{w}\) restricts to an isomorphism, and as \(t_w^\perp\) is invariant under \(w\) it then further restricts to an isomorphism of \(t_w^\perp\). Hence the previous equation now yields

\[
(id - \text{Ad}_m)(t_w^\perp) + l \oplus n_w = (id - \text{Ad}_w)(t_w^\perp) + l \oplus n_w = t_w^\perp + l \oplus n_w.
\]

Similarly, from \(\text{Ad}_m(l) = \text{Ad}_{m''}(l) \subseteq l \oplus n_w\) it follows that

\[
(id - \text{Ad}_m)(l) + l \oplus n_w = l \oplus n_w.
\]

Finally, from \(\text{Ad}_m(\pi^{-1}) = \text{Ad}_{m''}\text{Ad}_{m'}(n_w) \subseteq l \oplus n_w\) we deduce that

\[
(id - \text{Ad}_m)(\pi^{-1}) + l \oplus n_w = l \oplus n_w \oplus \pi^{-1}.
\]

Since the pair \((\mathfrak{N}, \mathfrak{L})\) is slicing there is a decomposition \(\mathfrak{g} \simeq \mathfrak{n} \oplus \pi \oplus t_w^\perp \oplus l\), so the first claim now follows.

As \(\mathfrak{L} \cup \mathfrak{R}_w\) and \(\mathfrak{L} \cup -\mathfrak{R}_w\) are convex, it follows from \(\mathfrak{L} = -\mathfrak{L}\) that so is \(\mathfrak{L} \cup (\mathfrak{N}\setminus\mathfrak{R}_w)\). But then \(\mathfrak{L} \cup - (\mathfrak{N}\setminus\mathfrak{R}_w)\) is also convex, so that \(\text{Ad}_{m''}(\pi^{-1}) \subseteq \pi^{-1} \oplus l\) for any \(m''\) in \(L\). As

\[
- (\mathfrak{N}\setminus\mathfrak{R}_w) \cap - \mathfrak{R}_w = (\mathfrak{N}\setminus\mathfrak{R}_w) \cap \mathfrak{R}_w = \emptyset,
\]

for any \(m''\) in \(N_w\) the operator \(\text{Ad}_{m''}\) acts as the identity on \(\mathfrak{w}^\perp\). Then for \(m = \mathfrak{w}m''\) we have

\[
\text{Ad}_m(\pi) = \text{Ad}_{m''}\text{Ad}_{m'}\text{Ad}_w(\pi) = \text{Ad}_{m''}\text{Ad}_{m'}(\pi^{-1} \oplus n_w) \subseteq \text{Ad}_{m''}(\pi^{-1} \oplus l \oplus n_w) = \pi^{-1} \oplus l \oplus n_w,
\]

yielding the second claim. \(\square\)
We now prove (ii):

**Proof.** Since \( G \) and \( \hat{w}LN_w \) are smooth, the claim is equivalent to requiring that the image of the differential of (3.6) is surjective at each point of \( G \times \hat{w}LN_w \). By equivariance for the \( G \)-action on the first component by left translation, it suffices to prove this at each point of the form \((id, m)\). When we restrict (3.6) to \( N \), it yields the cross section morphism (1.4) which is an isomorphism by the previous subsection. In the left trivialisation we have

\[
\mathfrak{n} \subseteq \text{Ad}_m(\mathfrak{n}_{w^{-1}}) \oplus \mathfrak{l} \oplus \mathfrak{n} \simeq T_m(N_{w^{-1}} \hat{w}LN) = T_m(N\hat{w}LN),
\]

so by this isomorphism the image of the differential certainly contains \( \mathfrak{n} \).

Now consider the Chevalley anti-automorphism which switches positive and negative root vectors of a Chevalley basis. Expressing a lift of a simple reflection as a product of exponentials of such elements, an \( \text{SL}_2 \)-calculation shows that its image under this involution is again a lift of this simple reflection. Hence the involution maps a lift of \( w^{-1} \) to a lift of \( w \), but then the image of the slice for (a suitable lift of) \( w^{-1} \) is

\[
\overline{N}_{w^{-1}}\overline{L}\overline{w} = \overline{N}_{w^{-1}}L\overline{w} = \overline{N}_{w^{-1}}\overline{w}L = \hat{w}LNw.
\]

Since by assumption equation (1.3) holds and the pair is slicing, Lemma 2.28 implies that equation (1.3) also holds with \( w \) replaced by \( w^{-1} \). Thus by the previous subsection the cross section isomorphism (1.4) also holds for \( w^{-1} \). Hence from the involution we now obtain an isomorphism

\[
\overline{N} \times \overline{\hat{w}LN}_w \cong \overline{\hat{w}LN},
\]

so by the same reasoning as in the previous paragraph, the image of the differential (in the left trivialisation) also contains \( \overline{n} \). Combining this with the first part of the previous lemma, the claim again follows from the decomposition \( g \simeq \mathfrak{n} \oplus \overline{\mathfrak{n}} \oplus t_w' \oplus \mathfrak{l} \).

And finally, we prove (ii) \( \Rightarrow \) (i'):

**Proof.** Concretely, (i') says that image of the differential of the conjugation map

\[
N \times \hat{w}LN_w \longrightarrow N\hat{w}LN, \quad (g, m) \longmapsto g^{-1}mg
\]

at any point \((id, m)\) is an isomorphism. In the left trivialisation we have

\[
T_m(N\hat{w}LN) = T_m(N\hat{w}LN_w) \simeq \text{Ad}_m(\mathfrak{n}) \oplus \mathfrak{l} \oplus \mathfrak{n}_w
\]

and this differential is given by

\[
\mathfrak{n} \oplus (\mathfrak{l} \oplus \mathfrak{n}_w) \longrightarrow \text{Ad}_m(\mathfrak{n}) \oplus \mathfrak{l} \oplus \mathfrak{n}_w, \quad (x, l) \longmapsto (id - \text{Ad}_m)(x) + l.
\] (3.8)

By assumption the differential of (3.6) is surjective. Since \((\overline{\mathfrak{n}} \oplus \mathfrak{n}_w) \cap \mathfrak{n}^w = \emptyset\), the decomposition \( g = \mathfrak{n}^w \oplus \overline{\mathfrak{n}} \oplus \mathfrak{l} \oplus \mathfrak{n}_w \oplus t_w' \) and the statements of the previous lemma imply that \( \mathfrak{n}^w \subseteq (id - \text{Ad}_m)(\mathfrak{n}) + \mathfrak{l} \oplus \mathfrak{n}_w \). But then we have

\[
\mathfrak{n} \subseteq (id - \text{Ad}_m)(\mathfrak{n}) + \mathfrak{l} \oplus \mathfrak{n}_w \subseteq \text{Ad}_m(\mathfrak{n}) \oplus \mathfrak{l} \oplus \mathfrak{n}_w
\]

which implies that the second inclusion is an equality. Hence the differential (3.8) is surjective; as it is a morphism of (sheaves of) (locally) free modules of finite rank, it is thus an isomorphism. \( \square \)
4 Poisson reduction

Having obtained that the action of $N$ on $N\dot{w}LN$ is free (whilst working over $\mathbb{C}$, for his particular choice of firmly convex Weyl group element $w$ with the slicing pair $(\mathbb{R}_+ \backslash \mathbb{R}_w, \mathbb{R}_w)$), Sevostyanov proceeds to proving that the Semenov-Tian-Shansky bracket on $G$ reduces to the slice $\dot{w}LN_w$ when the $r$-matrix is changed from the standard one $r_{st}$ to

$$r = r_{st} + \frac{1+w}{1-w} \text{proj}_{t_w} \text{proj}_{n_w} - \frac{1+w}{1-w} \text{proj}_{n_w},$$

by employing a general Poisson reduction method for manifolds [MR86, §2]. This approach was based on earlier work he did with his advisor on their loop analogues [STSS98, Theorem 2.5]. We will continue to work in the algebraic setting:

**Definition 4.1.** A Poisson scheme is a scheme $X$ with a Poisson bracket on its sheaf of functions. On its smooth locus this bracket corresponds to a Poisson bivector field which we will denote by $\Pi$; there it induces a musical morphism $\Pi^# : T^*X \to TX$ from the cotangent sheaf to the tangent sheaf. A function $f$ then defines a Hamiltonian vector field $\text{Ham}_f = \Pi^#(df)$ on this locus.

We modify this reduction method in Proposition 4.25; one obtains a statement that is very similar to a standard characterisation of smooth Poisson subschemes (recalled in Proposition 4.6), which explains the focus on Hamiltonian vector fields in the final proof. By analysing tangent spaces with some new root combinatorics we can work with a larger class of factorisable $r$-matrices, and settle which of them yield reducible Poisson brackets.

As in the previous sections, we are implicitly working over a base scheme but will omit it from all notation.

4.1 Coisotropic subgroups

Motivated by work of physicists on integrable systems, Drinfeld initiated the study of Poisson-Lie groups (and their quantisations); they translate to the algebraic setting as

**Definition 4.2** ([Dri83, §3]). A group scheme $G$ equipped with a Poisson bracket is called a Poisson algebraic group if this bracket is multiplicative, i.e. if the multiplication map $G \times G \to G$ is a morphism of Poisson schemes.

The identity element of $G$ is then a symplectic point, so that the Poisson bracket

$$\mathcal{O}_{G,\text{id}} \otimes \mathcal{O}_{G,\text{id}} \longrightarrow \mathcal{O}_{G,\text{id}}, \quad f \otimes f' \longmapsto (df \otimes df')(\Pi)$$

induces on its tangent space the structure of its Lie bialgebra.

Semenov-Tian-Shansky used the formalism of Poisson algebraic groups to study the “hidden symmetry groups” (dressing transformations) of certain integrable systems, as these don’t preserve Poisson structures; instead, they are Poisson actions:

**Definition 4.3** ([STS85, p. 1238]). A group action of a Poisson algebraic group $G$ on a Poisson scheme $X$ is called Poisson if the action map $G \times X \to X$ is a morphism of Poisson schemes.

Concretely, a point $x$ in $X$ then induces a map $x : G \to X$ via $g \mapsto gx$ and in terms of the Poisson brackets on $G$ and $X$, the Poisson condition can then be rephrased as

$$\{f, f'(gx)\} = \{x^* f, x^* f'\}(g) + \{g^* f, g^* f'\}(x) \quad (4.1)$$

for $f, f' \in \mathcal{O}_X$ and arbitrary points $g \in G, x \in X$.

As shown at the end of this subsection, in order to construct interesting quotients out of Poisson actions one sometimes uses subgroups of $G$ that are not necessarily Poisson themselves:
Definition 4.4. Let $X$ be a Poisson scheme. A smooth closed subscheme $Z \hookrightarrow X$ is called coisotropic (resp. Poisson) if
\[ \Pi_z \in T_z Z \cap T_z X \quad \text{(resp. } \Pi_z \in T_z Z \cap T_z \mathcal{O}_X) \]
for all points $z$ lying in $Z$.

Notation 4.5. Given a scheme $X$ we denote its sheaf of functions by $\mathcal{O}_X$. The inclusion of a closed subscheme $\iota : Z \hookrightarrow X$ induces a morphism $\iota^{-1}(\mathcal{O}_X) \to \mathcal{O}_Z$, and its ideal ideal sheaf is denoted by $\mathcal{I}_Z := \ker(\iota^\#: \iota^{-1}(\mathcal{O}_X) \to \mathcal{O}_Z)$. Given a function $f$ in $\iota^{-1}(\mathcal{O}_X)$ we denote its image under $\iota^\#$ by $f|_Z$.

The focus in the final proof of this section will be on Hamiltonian vector fields; heuristically, this is due to a group action analogue of the following

Proposition 4.6 ([Wei83, Lemma 1.1]). Let $X$ be a Poisson scheme and $\iota : Z \hookrightarrow X$ a smooth closed subscheme. Then the following are equivalent:

(i) $Z$ is Poisson.

(ii) $\ker(\iota^{-1}(\mathcal{O}_X) \to \mathcal{O}_Z)$ is a subsheaf of Poisson ideals; in other words, the Poisson bracket on $\mathcal{O}_X$ reduces to $\mathcal{O}_Z$.

(iii) For any function $f$ in $\iota^{-1}(\mathcal{O}_X)$, its Hamiltonian vector field lies in $T_Z \subseteq T_X|_Z$.

An algebraic proof of (ii) $\Leftrightarrow$ (iii) can be recovered as a special case of Lemma 4.25. One can characterise coisotropic smooth closed subschemes similarly [Wei88, Proposition 1.2.2].

Proposition 4.7. Let $G$ be a Poisson algebraic group and let $H$ be a closed algebraic subgroup.

(i) If $H$ is coisotropic, then the annihilator $\mathfrak{h}^0 \subseteq \mathfrak{g}^*$ of its Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}^*$.

(ii) [STS85, Proposition 2] If $H$ is Poisson, then this annihilator $\mathfrak{h}^0$ is an ideal of $\mathfrak{g}^*$.

If $H$ is connected, then the converse to (i) and (ii) holds as well.

Proof. (i) is well-known, it follows from: let $V$ be a locally free module over a ring, $\Pi$ an element of $V \wedge V$ and $U$ a locally free submodule of $V$; denote the annihilator of $U$ in the dual $V^*$ by $U^0$. Then $\Pi^\#(U^0) \subseteq U$ if and only if $\Pi \in U \wedge V$.

(ii): Similarly, this follows from $\Pi^\#(V^*) \subseteq U$ if and only if $\Pi \in U \wedge U$. \hfill $\Box$

Notation 4.8. If a group scheme $H$ acts on a scheme $X$, then we denote the resulting sheaf of $H$-invariant functions on $X$ by $\mathcal{O}_X^H$.

Proposition 4.9 ([STS85, Theorem 6]). Let $G$ be a Poisson algebraic group with a Poisson action on a Poisson scheme $X$, and let $H$ be a closed coisotropic subgroup of $G$. Furthermore assume that the restriction of the action on $X$ to $H$ preserves a closed subscheme $\iota : Z \hookrightarrow X$. Then $(\iota^\#)^{-1}(\mathcal{O}_X^H)$ is a sheaf of Poisson subalgebras of $\mathcal{O}_X$.

Proof. Let $f, f' \in (\iota^\#)^{-1}(\mathcal{O}_X^H)$, $h \in H$ and let $z \in Z$. Since $H$ is coisotropic we have
\[ \{z^* f, z^* g\}(h) = (z_* \Pi_h)(f, g) = 0, \]
so as the action is Poisson it then follows from (4.1) that
\[ \{f, f'\}(hz) = \{z^* f, z^* f'\}(h) + \{h^* f, h^* f'\}(z) = \{h^* f, h^* f'\}(z) = \{f, f'\}(z). \]
\hfill $\Box$

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4.2 Factorisable $r$-matrices

In order to obtain explicit coisotropic subgroups for reductive group schemes, we will now specialise to a particular class of Poisson structures.

**Notation 4.10.** Given an element $x = \sum_{i=1}^{n} a_i \otimes b_i$ in $\mathfrak{g} \otimes \mathfrak{g}$, we will write $x_{12} := \sum_{i=1}^{n} a_i \otimes 1 \otimes b_i$ and similarly $x_{23}$ for the usual elements in $\otimes^3 \mathfrak{g}$. Furthermore, we denote its “flip” by $x_{21} := \sum_{i=1}^{n} b_i \otimes a_i \in \mathfrak{g} \otimes \mathfrak{g}$.

The existence of a multiplicative Poisson structure can be rephrased cohomologically: using the left or right trivialisation, they are 1-cocycles on $G$ with values in $\mathfrak{g} \wedge \mathfrak{g}$. Subsequently employing e.g. Whitehead’s first lemma, there often exists an element $r \in \mathfrak{g} \wedge \mathfrak{g}$ such that the Lie cobracket $\delta$ of a Lie bialgebra $\mathfrak{g}$ equals the differential $\partial r$, i.e. such that

$$\delta(x) = \partial r(x) := \frac{1}{2} [x \otimes 1 + 1 \otimes x, r]$$

for all $x \in \mathfrak{g}$. Conversely, in order for an arbitrary element $r \in \mathfrak{g} \wedge \mathfrak{g}$ to define a compatible cobracket it is necessary and sufficient [Dri83, §6] that this $r$-matrix satisfies the generalised Yang-Baxter equation

$$[r, r] \in (\wedge^3 \mathfrak{g})^\mathfrak{g},$$

where for any element $r \in \mathfrak{g} \otimes \mathfrak{g}$ we denote by

$$[r, r] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in \otimes^3 \mathfrak{g}$$

its Drinfeld bracket; up to scalar, this coincides with the canonical Gerstenhaber (or Schouten-Nijenhuis) bracket when restricted to $\wedge^2 \mathfrak{g} \subset \wedge^* \mathfrak{g}$. Note that for any element $c \in (\mathfrak{g} \otimes \mathfrak{g})^\mathfrak{g}$, we have $[c, c] = [c_{12}, c_{23}]$.

**Theorem 4.11.** Let $\mathfrak{g}$ be a reductive Lie algebra over a field $\mathbb{k}$.

(i) [CE48, Theorem 19.1] There exists a natural quasi-isomorphism between the exterior algebra $(\wedge^* \mathfrak{g})^\mathfrak{g}$ and the Chevalley-Eilenberg complex computing Lie algebra homology, and hence

$$(\wedge^* \mathfrak{g})^\mathfrak{g} \simeq H_*(\mathfrak{g}).$$

(ii) [Kos50, §11] If the characteristic is 0 (or sufficiently large) then the Koszul map

$$\text{Sym}^2(\mathfrak{g})^\mathfrak{g} \longrightarrow (\wedge^3 \mathfrak{g})^\mathfrak{g}, \quad c \longmapsto [c_{12}, c_{13}]$$

is an isomorphism.

Throughout the rest of this section, we will implicitly make use of the identification

$$\mathfrak{g} \wedge \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g} \simeq \text{Hom}(\mathfrak{g}^*, \mathfrak{g}), \quad x \otimes y \longmapsto (\xi \mapsto \xi(x)y).$$

**Proposition 4.12 ([BD82]).** Let $\mathfrak{g}$ be a Lie algebra over a ring with 2 invertible, let $r \in \mathfrak{g} \wedge \mathfrak{g}$ and let $c \in \text{Sym}^2(\mathfrak{g})^\mathfrak{g}$ and consider the two maps

$$r_{\pm} : \mathfrak{g}^* \longrightarrow \mathfrak{g}, \quad \xi \longmapsto r_{\pm} c.$$ 

Then the following are equivalent:

(i) $[r, r] = -[c, c]$.

(ii) The map $r_{\pm}$ is a Lie algebra homomorphism.

(iii) The map $r_{\pm}$ is a Lie coalgebra antihomomorphism.
The equation \([r, r] = -[c, c]\) is called the \textit{modified classical Yang-Baxter equation}.

**Definition 4.13.** If \(G\) is a Poisson algebraic group whose Lie bialgebra \(\mathfrak{g}\) admits an \(r\)-matrix \(r\), then we will abbreviate this by writing \(G_r\) and \(\mathfrak{g}_r\). If \([r, r] = -[c, c]\) and \(c\) defines a perfect pairing, then we say that \(r\) is \textit{factorisable}, and we will say that the Lie bialgebra \(\mathfrak{g}_r\) and any corresponding Poisson algebraic group \(G_r\) are \textit{factorisable}.

**Notation 4.14.** We denote the torus component of such \(c\) by \(c_t\). Let \(L_g\) and \(R_g\) denote the translations on \(G\) of left and right multiplication by \(g\). Given an element \(r \in \mathfrak{g} \otimes \mathfrak{g}\), we then write \(r^R := r_{g, r}^R := (dR_g \otimes dR_g)(r)\), \(r_{R,L}^R := (dR_g \otimes dL_g)(r)\), etc. Then set \(r^\text{id} := r_{R,R} + r_{L,L} - r_{R,L} - r_{L,R}\).

Semenov-Tian-Shansky used Proposition 4.12 to geometrically prove over \(\mathbb{C}\) the following

**Theorem 4.15** ([STS85, p. 1247]). \textit{Given a factorisable Poisson algebraic group \(G_r\) over a scheme where 2 is invertible, let \(c\) in \(\text{Sym}^2(\mathfrak{g})^\text{g}\) be such that \([r, r] = -[c, c]\), let \(G_*\) denote the underlying scheme of \(G\) but now equipped with the bivector}

\[
\frac{1}{2} (r^\text{id} - c^L \otimes c^R) = (r_{+}^{R,R} - r_{+}^{L,R}) - (r_{-}^{R,L} - r_{-}^{L,L}).
\]

This yields a Poisson structure on \(G\), and the right conjugation map \(G_r \times G_* \to G_*\) is Poisson.

This can also be proven directly, without using factorisability.

**Definition 4.16.** This is called the \textit{(right) Semenov-Tian-Shansky bracket} on \(G\).

**Notation 4.17.** In order to obtain more such brackets in low characteristic, we slightly enlarge \(t\) to by adding the dual basis \(\{\hat{\omega}_i\}_{i=1}^{rk}\) to the simple roots and denote the result by \(t_{\text{sc}}\). We then assume that \(c_t\) lies in \(t_{\text{sc}} \otimes t\).

**Proposition 4.18.** If \(\mathfrak{g}\) is defined over an arbitrary ring then such \(c_t\) and \(r_{\pm}\) might not lie in \(\mathfrak{g}\), but the corresponding Semenov-Tian-Shansky bracket still yields integral formulas.

**Proof.** We only prove the second part. In the left-trivialisation of the tangent bundle of \(G\), the right-hand-side of equation (4.2) is given at any point \(g\) in \(G\) by

\[
((\text{Ad}_g - \text{id}) \otimes \text{Ad}_g)(r_+) - ((\text{Ad}_g - \text{id}) \otimes \text{id})(r_-),
\]

whose torus component is

\[
((\text{Ad}_g - \text{id}) \otimes (\text{Ad}_g + \text{id}))(c_t/2).
\]

We may decompose \(g\) into a product of root subgroups, and induct on the length of such an expression. By projecting \(x\) to root spaces, it suffices to prove the claim for elements of the form \((\hat{\omega}_i \otimes t_i/2)\). For \(g\) an element of a root subgroups, it follows from

\[
\text{Ad}_{\exp_\beta(g)}(x) = x + \beta(x)y,
\]

where \(\exp_\beta : \mathfrak{g}_\beta \to N_\beta\) is the exponential map and \(y\) lies in \(\mathfrak{g}_\beta\). The claim now follows by induction on the length of the decomposition of \(g\): if \(g = g'g''\) with \(g''\) lying in a root subgroup, then

\[
\text{Ad}_g(x) = \text{Ad}_{g'}(\text{Ad}_{g''}(x) \pm x) \in \text{Ad}_{g''}(\mathfrak{g}) + \mathfrak{g} = \mathfrak{g}.
\]

**Example 4.19.** Let \(\mathfrak{g}\) be a reductive Lie algebra over a field. Extend Chevalley generators to a basis \(\{e_\beta, f_\beta\}_{\beta \in \mathfrak{t}^+}\) and \(\{t_i\}_{i=1}^{rk}\) and then denote by \(\{\hat{\omega}_i\}_{i=1}^{rk}\) the dual basis in \(t\) to the simple roots. If we set

\[
r := \sum_{\beta \in \mathfrak{t}^+} d_\beta f_\beta \wedge e_\beta, \quad c := \sum_{\beta \in \mathfrak{t}^+} d_\beta e_\beta \otimes f_\beta + \sum_{\beta \in \mathfrak{t}^+} d_\beta f_\beta \otimes e_\beta + \sum_{i=1}^{rk} \hat{\omega}_i \otimes t_i,
\]

where \(d_\beta \in \{1, 2, 3\}\) are the usual symmetrisers with \(d_\beta = 1\) for long roots, then this Casimir element yields \([r, r] = -[c, c]\).
The claim on equality follows from nondegeneracy of \( c \).

**Proposition 4.21.** Let \( g \) be a factorisable Lie bialgebra.

(i) \([BD82]\) The annihilator of \( b_\pm \) in \( g^* \) is \( \ker r_\pm \).

(ii) Let \( h \) be a subspace of \( g \) containing \( b_\pm \). Then \( h^\perp = r_\pm(h^0) \subseteq m_\pm \), with equality if and only if \( h^0 = (b_\pm)^0 \).

In particular, we have
\[
\iota_r(h^0) = (h^\perp, 0) \quad \text{(resp. } \iota_r(h^0) = (0, h^\perp)) \text{.}
\]

**Proof.** (i): Any element of \( b_\pm \) is of the form \( r_x x \) for some \( x \in g^* \). For \( y \in g^* \) we then have
\[
\langle y, r_x x \rangle = \langle r_x^2 y, x \rangle = \langle -r_x y, x \rangle.
\]
Since \( x \) can be chosen arbitrarily, the claim follows.

(ii): From (i) it follows that \( h^0 \subseteq (b_\pm)^0 = \ker r_\pm \), so
\[
h^\perp = c(h^0) \subseteq (r_+ - r_-)(\ker r_\pm) = \pm r_\pm(\ker r_\pm) =: m_\pm.
\]
The claim on equality follows from nondegeneracy of \( c \). For \( h \) containing \( b_\pm \) we now find
\[
\iota_r(h^0) = (r_+, r_-)(h^0) = (r_+(h^0), 0) = (h^\perp, 0). \quad \square
\]

**Definition 4.22.** A Belavin-Drinfeld triple is a triple \( \mathfrak{T} = (\mathfrak{T}_0, \mathfrak{T}_1, \tau) \) where \( \mathfrak{T}_0, \mathfrak{T}_1 \) are sets of simple roots and \( \tau : \mathfrak{T}_0 \to \mathfrak{T}_1 \) is a bijection such that
- \( (\tau(\alpha), \tau(\tilde{\alpha})) = (\alpha, \tilde{\alpha}) \) for any pair of simple roots \( \alpha, \tilde{\alpha} \in \mathfrak{T}_0 \), and
- \( \tau \) is nilpotent: for any \( \alpha \in \mathfrak{T}_0 \) there exists \( m \in \mathbb{N}_1 \) such that \( \tau^m(\alpha) \in \mathfrak{T}_1 \setminus \mathfrak{T}_0 \).

Such a triple gives the set of positive roots a partial ordering: for positive roots \( \beta, \beta' \) we set \( \beta < \beta' \) if \( \beta \in \mathbb{N}_0 \mathfrak{T}_0, \beta' \in \mathbb{N}_0 \mathfrak{T}_1 \) and \( \tau^m(\beta) = \beta' \) for some \( m \in \mathbb{N}_1 \).

**Theorem 4.23** ([BD82]). Let \( g \) be a factorisable reductive Lie bialgebra over a field of characteristic 0 (or sufficiently large). Then there exists a Cartan decomposition and a Belavin-Drinfeld triple \( \mathfrak{T} = (\mathfrak{T}_0, \mathfrak{T}_1, \tau) \) such that
\[
r_+ = \frac{1}{2}(r_0 + c_1) + \sum_{\beta \in \mathfrak{T}_1} d_\beta f_\beta \otimes e_\beta + \sum_{\beta < \beta'} d_\beta f_\beta \wedge e_{\beta'},
\]
where \( f_\beta \) and \( e_\beta \) are root vectors with weight \( -\beta \) and \( \beta \) respectively, are normalised by \( d_\beta(f_\beta, e_\beta) = 1 \) for an invariant bilinear form corresponding to some element \( c \in \text{Sym}^2(g)^* \), whilst the element \( r_0 \in \mathfrak{t} \wedge \mathfrak{t} \) satisfies
\[
(\tau(\alpha) \otimes 1)(r_0 + c_1) + (1 \otimes \alpha)(r_0 + c_1) = 0, \quad \forall \alpha \in \mathfrak{T}_0,
\]
where \( c_1 \) is the image of the \( t \)-component of \( c \).

Furthermore, such solutions \( r_0 \) form a torsor for the \( k_\mathfrak{T}(k_\mathfrak{T} - 1)/2 \)-dimensional vector space \( \mathfrak{t}_\mathfrak{T} \wedge \mathfrak{t}_\mathfrak{T} \), where
\[
k_\mathfrak{T} := \text{rank}(g) - |\mathfrak{T}_0| \quad \text{and} \quad \mathfrak{t}_\mathfrak{T} := \{ t \in \mathfrak{t} : \beta(t) = \beta'(t) \text{ for all } \beta < \beta' \}.
\]

Explicitly, we then have
\[
r_- = \frac{1}{2}(r_0 - c_1) - \sum_{\beta \in \mathfrak{T}_1} d_\beta e_\beta \otimes f_\beta + \sum_{\beta < \beta'} d_\beta f_\beta \wedge e_{\beta'}.
\]

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Proposition 4.24. Let $G_r$ be a reductive factorisable Poisson algebraic group and let $H \subseteq G_r$ be a connected closed subgroup with Lie algebra $\mathfrak{h}$.

(i) If $\mathfrak{h}$ contains $\mathfrak{b}_+^*$ or $\mathfrak{b}_-^*$, then $H$ is Poisson.

(ii) Now suppose that the r-matrix comes from a Belavin-Drinfeld triple $\mathfrak{T}$, and that $H$ is a product of root subgroups corresponding to a subset of positive (resp. negative) roots $\mathfrak{R}$ of the form $\mathfrak{R}_w$. If $\mathfrak{T} \subseteq \mathfrak{R}^c$ then $H$ is coisotropic.

Proof. (i): Let’s assume that $\mathfrak{h}$ contains $\mathfrak{b}_+^*$, then by Proposition 4.21(ii) we have $\iota_r(\mathfrak{h}^0) = (\mathfrak{h}^+, 0)$. Thus, in order to prove that $\mathfrak{h}^0$ is an ideal of $\mathfrak{g}^*$, it suffices to show that $[\mathfrak{b}_+^*, \mathfrak{h}^+] \subseteq \mathfrak{h}^+$. Given $x \in \mathfrak{b}_+^* \subseteq \mathfrak{h}$ and $y \in \mathfrak{h}^+$ and $z \in \mathfrak{h}$, invariance of $(\cdot, \cdot)_r$ yields

$$([x, y], z)_r = (y, [z, x])_r = 0$$

so that $[x, y] \in \mathfrak{h}^-$.

(ii): Let $\{e_\beta, f_\beta\}_{\beta \in \mathfrak{R}_+}$ be the basis of $\mathfrak{n}_+^* \oplus \mathfrak{n}_-^* \subset \mathfrak{g}^*$ dual to the usual basis $\{e_\beta, f_\beta\}_{\beta \in \mathfrak{R}_+}$ of $\mathfrak{n}_+ \oplus \mathfrak{n}_-$, so $e_\beta^*$ is the element of $\mathfrak{g}^*$ vanishing on $\mathfrak{t}$ and all root spaces other than $\mathfrak{n}_\beta$, where it is given on its basis vector by $e_\beta^*(e_\beta) = 1$, etc. Then

$$\iota_r(e_\beta) = -d_\beta(\sum_{i>1} f_{\tau^i(\beta)} - \sum_{1 \geq i} f_{\tau^{-i}(\beta)}), \quad \iota_r(f_\beta) = d_\beta(\sum_{i>1} e_{\tau^i(\beta)} - \sum_{1 \geq i} e_{\tau^{-i}(\beta)})$$

(4.7)

where we set $e_{\tau^{i+1}(\beta)} = 0$ (resp. $f_{\tau^{i-1}(\beta)} = 0$) when $\tau^i(\beta)$ is not in the span of $\mathfrak{R}_0$ (resp. $\tau^{-i}(\beta)$ not in the span of $\mathfrak{R}_1$). In particular, if $\beta$ is not in $\mathfrak{R}^c$ then it is not in $\mathfrak{R}_0$ or $\mathfrak{R}_1$ and we find

$$\iota_r(e_\beta) = -d_\beta(0, f_\beta), \quad \iota_r(f_\beta) = d_\beta(e_\beta, 0).$$

(4.8)

Since by assumption $\mathfrak{h}$ is a sum of root subspaces, its dual is spanned by a subset of $\{e_\beta^*, f_\beta^*\}_{\beta \in \mathfrak{R}_+}$ plus a basis for $\mathfrak{t}^*$. The adjoin action of elements of $\iota_r(\mathfrak{t}^*)$ on elements in (4.7) is through scalars due to (4.5), so we can safely ignore them.

Let’s do the positive case: since $\mathfrak{T} \subseteq \mathfrak{R}^c$ and the set of roots $\mathfrak{R}^c$ is convex [Pap94], it follows from (4.7) that bracket of $\iota_r(e_\beta^*)$ and $\iota_r(e_{\beta'}^*)$ for $\beta \in \mathfrak{R}^c$ corresponds to root vectors lying in (4.7). As $\mathfrak{g}$ is closed under bracketing, (4.8) implies that it must be a linear combination of elements $\iota_r(e_{\beta''}^*)$ with $\beta''$ in $\mathfrak{R}^c$: there are simply no other elements in $\iota_r(\mathfrak{g}^*)$ projecting to the right weight spaces.

For $\iota_r(e_\beta^*)$ and $\iota_r(f_\beta^*)$ with $\beta \in \mathfrak{R}^c$ and $\beta' \in \mathfrak{R}_+$, note that only the parts lying in $\mathfrak{T}$ can bracket nontrivially. As $\mathfrak{T}$ forms a standard parabolic subsystem it similarly follows that the bracket is a linear combination of elements $\iota_r(e_{\beta''}^*)$ with $\beta''$ in $\mathfrak{R}^c$ and $\iota_r(f_{\beta''}^*)$ with $\beta''$ arbitrary.

Restricting the left or right multiplication of $G$ on itself to $H$ as in (i), we thus obtain Poisson structures on the GIT quotients $H \backslash G$ and $G//H$. In the particular case where $r$ is the standard r-matrix (so that $N_+ = N_\perp$ and $B_+^* = B_\perp$) and $H$ is a parabolic subgroup, the corresponding Poisson structure has been extensively studied (e.g. [GY09]).

4.3 Reducing the Semenov-Tian-Shansky bracket

Lemma 4.25. Let $G$ be a Poisson group scheme with an action on a Poisson scheme $X$. Let $H \subseteq G$ be a closed subgroup preserving a smooth closed subscheme $\iota: Z \hookrightarrow X$ such that $(\iota^*)^{-1}(\mathcal{O}_H^X)$ is a sheaf of Poisson subalgebras of $(\iota^*)^{-1}(\mathcal{O}_X)$. Then the following are equivalent:

(i) The ideal sheaf $\mathcal{I}_Z$ is a subsheaf of Poisson ideals of $(\iota^*)^{-1}(\mathcal{O}_H^X)$; in other words, the Poisson bracket on $(\iota^*)^{-1}(\mathcal{O}_H^X)$ reduces to $\mathcal{O}_Z^H$.

(ii) For any function $f$ in $(\iota^*)^{-1}(\mathcal{O}_H^X)$, its Hamiltonian vector field lies in $T_Z$. 

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Proof. (i) ⇒ (ii): Pick a function $f$ in $(i^t)^{-1}(O^H_Z)$ and a covector $\alpha$ at $z$ which is annihilated by all tangent vectors in $T_z Z$. Lifting $\alpha$ to a one-form in a neighbourhood of $z$, by the conormal exact sequence we can find a function $f'$ in $I_Z$ such that $df'|_z = \alpha$ and $f'|_Z = 0$. Then the hypothesis yields that

$$\alpha(\text{Ham}_f) = \{f, f'\}|_z = \{f|_Z, 0\} = 0,$$

which means that $\text{Ham}_f$ is annihilated by the covectors annihilated by $T_z Z$, which by smoothness implies that it lies in the tangent space $T_z Z$.

(ii) ⇒ (i): Let $f$ be a function in $(i^t)^{-1}(O^H_Z)$ and $f'$ in $I_Z$, then from the inclusion $\text{Ham}_f \in T_z Z$ it follows that

$$\{f, f'\}|_z = df'(\text{Ham}_f) = d(f'|_Z)(\text{Ham}_f) = 0,$$

implying that $\{f, f'\}$ lies in $I_Z$. \qed

**Corollary 4.26.** Equip $G$ with the right Semenov-Tian-Shansky bracket coming from a factorisable $r$-matrix with $\mathfrak{T} \subseteq \mathfrak{L}$. Consider the right conjugation action of $N \subset G$ on $G_*$ and its restriction to the closed subscheme $\iota : N\bar{w}LN \hookrightarrow G_*$. The subgroup $N$ is coisotropic and $(i^t)^{-1}(O^N_{\bar{w}LN})$ is a subsheaf of $i^{-1}(O_{G_*})$ of Poisson subalgebras.

*Proof.* This follows by combining Proposition 4.24 with Proposition 4.9. \qed

**Notation 4.27.** For any function $f$ in $O_G$ and element $g$ in $G$, we define $df^g \in \mathfrak{g}^*$ by evaluating on a tangent vector in $\mathfrak{g}$ as follows:

$$df^g(\cdot) := df(R_g(\cdot) - L_g(\cdot)) \in \mathfrak{g}^*,$$

so in the left trivialisation we have $df^g(\cdot) = df(\text{Ad}_g(\cdot) - \text{id}(\cdot))$. The tangent space to $N\bar{w}LN = NL\bar{w}N$ at one of its elements $g$ is given by

$$T_g := \text{Ad}_g(n) + l \oplus n = \text{Ad}_g(n \oplus l) + n \subseteq \mathfrak{g},$$

(4.9)

where $\text{Ad}_g(\cdot)$ still denotes the right adjoint action.

**Lemma 4.28.** Assume that $\mathfrak{L}$ is a standard parabolic subroot system. Then

$$T_g \cap t = l \cap t.$$ 

*Proof.* Let $g = n\bar{w}\bar{n}$. Since $\mathfrak{R}$ is convex, we have $\text{Ad}_n(n) \subseteq n$. From the assumption on $\mathfrak{L}$ it follows that

$$\text{Ad}_l(n) \subseteq \bigoplus_{\beta \in \mathfrak{R}_+ \setminus \mathfrak{L}} \mathfrak{g}_\beta.$$ 

As $\mathfrak{L}$ is preserved by $w$, it now follows that

$$\text{Ad}_{n\bar{w}}(n) \subseteq \text{Ad}_{\bar{w}}(\text{Ad}_l(n)) \subseteq \text{Ad}_{\bar{w}}\left( \bigoplus_{\beta \in \mathfrak{R}_+ \setminus \mathfrak{L}} \mathfrak{g}_\beta \right) \subseteq \bigoplus_{\beta \in \mathfrak{R}_+ \setminus \mathfrak{L}} \mathfrak{g}_\beta.$$ 

Now let $x$ be an arbitrary element in the right-hand-side. If $\beta$ is of minimal height among the roots in $\mathfrak{R} \setminus \mathfrak{L}$ such that the projection of $x$ to $\mathfrak{g}_\beta$ is nonzero, then the projection of $\text{Ad}_n(x)$ to $\mathfrak{g}_\beta$ is still nonzero. Thus if $\beta$ is negative, $\text{Ad}_n(x)$ does not lie in $l \oplus n$; if $\beta$ is positive then $\text{Ad}_n(x)$ still lies in $n_+$. Hence

$$T_g \cap t = (\text{Ad}_g(n) + l \oplus n) \cap t = (l \oplus n) \cap t = l \cap t.$$ \qed

**Proposition 4.29.** Fix an ordering $\beta_1, \ldots, \beta_l$ of the roots of $\mathfrak{R}_w$ by height, and fix the element $g' := p_{\beta_1}(1) \cdots p_{\beta_l}(1)$. Then

$$t \mapsto n_w, \quad t \mapsto \text{proj}_{n_w}(\text{Ad}_g(t) - t)$$

is injective.
Proof. By construction of $t_{sc}$, we can recover the coordinates of $t$ in the standard basis of $t_{sc}$ through studying $t \mapsto \alpha(t)$ as $\alpha$ ranges over the simple roots. By [Mal21, Corollary 3.13], these values can be obtained from the roots in $R_w$. For $\beta$ in $R_w$, let $e_\beta$ denote the element of $g_\beta$ exponentiating to $p_\beta(1)$. We write $O(\beta_{i+1})$ to mean a polynomial in the $e_\beta$ with $j \geq i + 1$. It follows from

$$\text{Ad}_{g'}(t) = \text{Ad}_{p_{\beta_1}(1) \cdots p_{\beta_l}(1)}(t + \beta_1(t)e_{\beta_1} + \cdots + \beta_l(t)e_{\beta_l} - t) = \sum_{i=1}^{l} (\beta_i(t)e_{\beta_i} + O(\beta_{i+1}))$$

that the values $\beta_i(t)$ can be recovered inductively. \hfill \square

**Corollary 4.30.** Now set $g := \dot{w}g'$. The map

$$(t^g)^{-1}(O_{N\dot{w}LN}^N) \longrightarrow t^*_{sc}, \quad f \mapsto (t \mapsto df^g(t))$$

is surjective.

**Proof.** By the previous statement, we can recover $w^{-1}(t)$ and hence $t$ from the projection of

$$\text{Ad}_g(t) - w^{-1}(t) = \text{Ad}_g'(w^{-1}(t)) - w^{-1}(t)$$

to $n_w$, and then the same is true for

$$t^g := (\text{Ad}_g - \text{id})(t) = (\text{Ad}_g(t) - w^{-1}(t)) + (w^{-1}(t) - t)$$

since we are adding an element of $t$. Thus

$$O_{\dot{w}Nw} \longrightarrow t^*_{sc}$$

is surjective. As $\dot{w}LN_w = L\dot{w}N_w \simeq L \times \dot{w}N_w$, the the cross section isomorphism

$$O_{N\dot{w}LN} \sim O_N \otimes O_{\dot{w}LN_w} \sim O_N \otimes O_L \otimes O_{\dot{w}N_w}$$

now yields the claim as elements of the form $1 \otimes 1 \otimes f'$ pull back to $N$-invariants in $O_{N\dot{w}LN}$ satisfying

$$df^g(t) = df(t^g) = d(1 \cdot f')(t^g).$$

\hfill \square

**Example 4.31.** Consider the usual group

$$G = \text{SL}_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1 \right\}$$

of type $A_1$ and let

$$\dot{w} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

be the usual lift of its Coxeter element. Let $t$ be the usual diagonal element of its Lie algebra and

$$g = \dot{w}p_\alpha(1) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix},$$

then

$$R_g t - L_g t = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$ 

The matrix coordinate $c$ is invariant under the conjugation action of $N$, and $dc^g(t) = 2$. 

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The following lemma was obtained by dissecting Sevostyanov’s proof [Sev11, Theorem 5.2]:

**Notation 4.32.** Given a function \( f \) in \( O_G \) defined at a closed point \( g \) of \( G \), we write
\[
d f^g \_\pm := r_\pm(d f^g) \in \mathfrak{g}
\]
and then decompose these elements along the Cartan decomposition \( \mathfrak{g} = \mathfrak{t} \oplus (\mathfrak{n}_+ \oplus \mathfrak{n}_-) \) as
\[
d f^g_\pm = df^g_{\pm 0} + df^g_{\pm 0}, \quad df^g_{\pm 0} \in \mathfrak{t}, \quad df^g_{\pm 0} \in \mathfrak{n}_+ \oplus \mathfrak{n}_-.
\]
From (4.3) and (4.6) one then sees that
\[
d f^g_{\pm 0} = d f^g(r_0 \pm c_t)
\]
and similarly for \( df^g_{\pm 0} \).

From the right-hand-side of equation (4.2), it follows that Hamiltonian vector field of a function \( f \) in \( O_G \) is given at a point \( g \) in this trivialisation by
\[
\text{Ham}_f(g) = \text{Ad}_g(r_+ (df^g)) - r_-(df^g).
\]

**Lemma 4.33.** For any function \( f \) in \((\mathbb{R})^{-1}(O^N_{N\hat{w}LN})\) defined at a point \( g \) of \( N\hat{w}LN \), we have
\[
\text{Ham}_f(g) \equiv \text{Ad}_\hat{w}(df^g_{\pm 0}) - df^g_{-0} \mod T_g.
\]

**Proof.** The slicing assumption \( \mathfrak{L} \sqcup \mathfrak{R} = \mathfrak{R}_+ \) implies that \( \mathfrak{t} + \mathfrak{t} \oplus \mathfrak{n}_+ = \mathfrak{t} + \mathfrak{t} \oplus \mathfrak{n} \), so from \( \mathfrak{T}_0 \subset \mathfrak{L} \) we deduce that
\[
\mathfrak{b}_+^r \subset \mathfrak{t} + \mathfrak{t} \oplus \mathfrak{n}_+ = \mathfrak{t} + \mathfrak{t} \oplus \mathfrak{n}.
\]

Thus \( df^g_{\pm 0} \in \mathfrak{t} \oplus \mathfrak{n} \) and hence
\[
\text{Ad}_g(df^g_{\pm 0}) \in \text{Ad}_g(\mathfrak{t} \oplus \mathfrak{n}) \subset T_g.
\]

Now pick \( n \in N \) and \( \tilde{n} \in LN \) such that \( g = n\hat{w}\tilde{n} \). As \( df^g_{\pm 0} \) lies in \( \mathfrak{t} \) by construction, the element \( \text{Ad}_n(df^g_{\pm 0}) - df^g_{\pm 0} \) lies in \( \mathfrak{n} \), so that
\[
\text{Ad}_{n\hat{w}\tilde{n}}(df^g_{\pm 0}) - \text{Ad}_{\hat{w}\tilde{n}}(df^g_{\pm 0}) = \text{Ad}_\tilde{n}(n) = \text{Ad}_g(n) \subset T_g.
\]

Furthermore as \( w(t) = t \) normalises \( \mathfrak{t} \oplus \mathfrak{n} \), we have
\[
\text{Ad}_{\hat{w}\tilde{n}}(df^g_{\pm 0}) - \text{Ad}_{\hat{w}}(df^g_{\pm 0}) \in \mathfrak{t} \oplus \mathfrak{n} \subset T_g
\]
so we conclude from the last three inclusions that
\[
\text{Ad}_g(df^g_{\pm 0}) = \text{Ad}_g(df^g_{\pm 0}) + \text{Ad}_g(df^g_{\pm 0}) = \text{Ad}_g(df^g_{\pm 0}) \mod T_g.
\]

As \( f \) is invariant under conjugation by \( N \), we have \( f(n g) - f(g n) = 0 \) for all \( n \) in \( N \) which implies that the element \( df^g \in \mathfrak{g}^* \) lies in the annihilator of \( \mathfrak{n} \). From (4.6) one sees that \( df^g_{\pm 0} \) lies in
\[
\oplus_{\beta \in \mathfrak{t}_+ \setminus \mathfrak{t}_-} \mathfrak{n}_{-\beta} + \oplus_{\beta \in \pi_+} \mathfrak{n}_{-\beta} + \oplus_{\beta \in \pi_-} \mathfrak{n}_\beta \subset (\mathfrak{l} \cap \mathfrak{n}_-) + \mathfrak{l} = \mathfrak{l} \subset T_g.
\]

Combined with (4.10), the claim follows.

Finally, we prove (iii):
Proof. Recall Corollary 4.26 and Proposition 4.25; they now imply that the bracket of \( O_G \) reduces to \( O_{N\tilde{w}LN} \) if and only if all Hamiltonian vector fields of functions in \((i^t)^{-1}(O_{N\tilde{w}LN})\) are tangent to \( N\tilde{w}LN \). From Corollary 4.30 we deduce that \( \text{Ham}_f(g) \in T_g \) for all \( f \in (i^t)^{-1}(O_{N\tilde{w}LN}) \) if and only if the image of

\[
\begin{align*}
t^*_{sc} \rightarrow t, \\
t^* \rightarrow w\left(\frac{r_0 + c_t}{2}(t^*)\right) - \left(\frac{r_0 - c_t}{2}\right)(t^*)
\end{align*}
\]

lands inside of \( T_g \cap t \), which by Lemma 4.28 equals \( t \cap t \subseteq t^w \). In other words, \( \text{Ham}_f(g) \in T_g \) if and only if the projection of

\[
\frac{1}{2}(w - 1)(r_0) + \frac{1}{2}(w + 1)(c_t)(t^*_{sc})
\]

to \( t_w \) is trivial in the orthogonal decomposition \( t = t^w \oplus t_w \). Since the right-hand-side would send \( t^w_{sc} \) to \( t^w \) and \( w - 1 \) acts nontrivially on anything outside of \( t^w \), it follows that \( r_0 \) must map \( t^w_{sc} \) to \( t^w \); for otherwise a nontrivial \( t_w \)-component would appear. By skew-symmetry, \( r_0 \) then also preserves \( t_w \).

By \( w \)-invariance we may decompose \( c_t = c_{t_w} + c_{t_w} \). On \( t_w \) the operator \( w - 1 \) then acts trivially, so it follows from nondegeneracy of \( c_{t_w} = \frac{1}{2}(w + 1)(c_{t_w}) \) that the image is all of \( t_w \), so \( t^w \subseteq I \).

Now consider \( r_0|_{t_w} \). It needs to satisfy

\[
(w - 1)(r_0|_{t_w}) + (w + 1)(c_t) = 0,
\]

which rewrites as

\[
r_0|_{t_w} = \frac{1 + w}{1 - w}.
\]

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Mathematical Institute, University of Oxford, Oxford OX2 6GG, United Kingdom
E-mail address: w.malten@gmail.com