Score test for missing at random or not

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Abstract

Missing data are frequently encountered in various disciplines and can be divided into three categories: missing completely at random (MCAR), missing at random (MAR) and missing not at random (MNAR). Valid statistical approaches to missing data depend crucially on correct identification of the underlying missingness mechanism. Although the problem of testing whether this mechanism is MCAR or MAR has been extensively studied, there has been very little research on testing MAR versus MNAR. A critical challenge that is faced when dealing with this problem is the issue of model identification under MNAR. In this paper, under a logistic model for the missing probability, we develop two score tests for the problem of whether the missingness mechanism is MAR or MNAR under a parametric model and a semiparametric location model on the regression function. The score tests require only parameter estimation under the null MAR assumption, which completely circumvents the identification issue. Our simulations and analysis of human immunodeficiency virus data show that the score tests have well-controlled type I errors and desirable powers.

Keywords: Missing at Random; Missing Not at Random; Score Test.

1 Introduction

Missing data are frequently encountered in economic, medical and social science disciplines. Valid statistical inferences for missing data depend crucially on correct identification of the underlying missingness mechanism, which was divided by Rubin (1976) into three categories. The missingness is called missing at random (MAR) or ignorable if it does not depend on the missing values themselves conditioning on the observed data, and it is called missing not at random (MNAR) or nonignorable otherwise. A degenerate case of MAR is missing completely at random (MCAR), where the missingness does not depend on either the observed or the missing data.

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Under the MAR assumption, both the propensity score and outcome regression models are non-parametrically identifiable, and it is therefore always tractable to conduct valid inferences. A wide range of statistical approaches have been developed for MAR data analysis, including likelihood-based approaches (Dempster et al., 1977; Horton and Laird, 2001; Ibrahim, 1990), multiple imputation (Rubin and Schenker, 1986; Vach and Schumacher, 1993; Rubin, 2004), semiparametric methods (Zhao et al., 1996; Rubins et al., 1994) and the inverse probability weighting method (Rosenbaum and Rubin, 1983; Rubins et al., 1994, 1995). To alleviate the risk of possible mis-specifications of propensity score or outcome regression models, estimators of double or multiple robustness have been proposed and have attracted much attention (Laird and Pauler, 1999; Kang and Schafer, 2007; Han and Wang, 2013; Chan and Yam, 2014; Han, 2014a,b, 2016a,b; Chen and Haziza, 2017; Han et al., 2019). For a more comprehensive literature review on the analysis of MAR data, see, for example, Little and Rubin (2002); Tsiatis (2006); Kim and Shao (2013) and references therein.

Things become much more challenging when the MAR mechanism is violated or data are MNAR. The foremost challenge is parameter identifiability: the underlying generating model is often not identifiable based on the observed data. For MNAR data, even completely parametric models for the data-generating model may be not identifiable (Heckman, 1979; Qin et al., 2002; Greenlees et al., 1982; Miao et al., 2016), let alone semiparametric models (Qin et al., 2002; Tang et al., 2003; Kim and Yu, 2011; Shao and Wang, 2016; Liu et al., 2021) or completely nonparametric models (Robins and Ritov, 1997). When no general identification results are available for MNAR data, the joint distribution of the full data can only be identified under specific model assumptions. A popular condition for model identifiability with MNAR data is the existence of an ‘instrumental variable’ (Wang et al., 2014) or ‘ancillary variable’ (Miao and Tchetgen Tchetgen, 2016), which does not affect the missingness but may affect the conditional distribution of the response variable. Quite a few studies of MNAR data under the instrumental variable condition have been conducted. Wang et al. (2014) found that the identification issue for MNAR data can be overcome with the help of instrumental variables, and they then proposed the use of the generalized method of moments to estimate model parameters. This approach was extended by Zhao and Shao (2015) to generalized linear models under a parametric propensity score model, and this was further extended by Shao and Wang (2016) to allow for the semiparametric propensity model of Kim and Yu (2011). In the presence of an instrumental variable or ancillary variable, double robust estimation and semiparametric efficient estimation under MNAR data have also been investigated (see, e.g., Morikawa and Kim, 2016; Miao and Tchetgen Tchetgen, 2016; Ai et al., 2020; Liu et al., 2021). However, an instrumental variable may not be readily available or may not be straightforward to find in practice, which complicates the identifiability and inferences of the existing statistical approaches. Tang and Ju (2018) and Wang and Kim (2021) provide more comprehensive reviews of statistical
inferences for nonignorable missing-data problems.

These discussions arguably reveal that methods for handling MAR data and MNAR data are totally different: the former are relatively easy whereas the latter are much more difficult. Correctly determining which mechanism is responsible for data being missing is crucial to the subsequent development of valid inference methods. This raises the hypothesis testing problem of whether the missingness mechanism is MAR or MNAR.

A relatively simple counterpart of this hypothesis testing problem is whether the missingness mechanism is MCAR or MAR. Many tests for MCAR have been developed in recent decades, since the MCAR category is at the centre of interest of many behavioural and social scientists confronted with missing data (Simon and Simonoff, 1986). Little (1988) constructed a test by comparing the means of recorded values of each variable between groups of different missingness patterns. Chen and Little (1999) extended Little’s test to longitudinal data by comparing the means of the general estimating equations across different missingness patterns with zero, with any departure from zero then indicating rejection of the MCAR hypothesis. More extensions of Little’s idea to comparisons of the means, the covariance matrices and/or the distributions across different missingness patterns have also been investigated (see, e.g., Jamshidian and Jalal, 2010; Kim and Bentler, 2002; Li and Yu, 2015). Recently, Zhang et al. (2019) proposed a nonparametric approach for testing MCAR based on empirical likelihood (Owen, 1988, 1990, 2001). Their approach also provides a unified procedure for estimation after the MCAR hypothesis has been rejected.

In contrast to MCAR, testing for MAR has not received much attention so far. The first contribution in this direction was the nonparametric test proposed by Breunig (2019), which was based on an integrated squared distance. Under a generalized linear regression model, Duan et al. (2020) proposed to test MAR by a quadratic form of the difference of the estimators of the regression coefficient under the MAR and MNAR assumptions, respectively. To the best of our knowledge, these are the only two formal tests for MAR. They both require the existence of an instrumental variable to guarantee identifiability, because their test statistics depend on consistent estimates under MNAR. However, as mentioned previously, the identification of an instrumental variable is usually not straightforward, and, even worse, it may not exist. Also, consistent parameter estimation itself under MNAR is rather challenging.

In this paper, we propose two score tests for MAR under a linear logistic model when a completely parametric model and a semiparametric location model, respectively, are imposed on the outcome regression model, respectively. Compared with the tests proposed by Breunig (2019) and Duan et al. (2020), the new tests have at least three advantages. The first remarkable advantage is that no identification condition is required under MNAR, which implies that no instrumental variable is needed. However, without an instrumental variable, the tests of Breunig (2019) and Duan et al. (2020) may fail to work. The second advantage is that the new tests involve much easier
calculations, because the underlying unknown parameters need only be estimated under MAR. As we have pointed out, identifiability is not an issue for MAR data and parameter estimation has been well studied. Third, our simulation results indicate that the two new tests are often more powerful than or at least comparable to that proposed by Duan et al. (2020).

2 Score test

Let \( Y \) denote an outcome that is subject to missingness and let \( X \) be a fully observed covariate vector whose first component is 1. We denote by \( D \) the missingness indicator of the outcome, with \( D = 1 \) if \( Y \) is observed and 0 otherwise. We wish to test whether the missingness mechanism is MAR or MNAR, namely \( H_0 : \Pr(D = 1 | Y, X) = \Pr(D = 1 | X) \). Suppose that the missingness probability or the propensity score follows a linear logistic model

\[
\Pr(D = 1 | X = x, Y = y) = \pi(x^\top \beta + \gamma y)
\]

with \( \pi(t) = e^t/(1 + e^t) \). Under the model (2.1), the testing problem of interest is equivalent to \( H_0 : \gamma = 0 \), because the missingness mechanism is MAR if \( \gamma = 0 \) and MNAR otherwise.

Suppose that \( \{(d_i, d_iy_i, x_i), i = 1, 2, \ldots, n\} \) are \( n \) independent and identically distributed observations from \((D, DY, X)\). Let \( f(y|x) \) denote the conditional density function of \( Y \) given \( X = x \). The loglikelihood based on the observed data is

\[
\ell(\gamma, \beta, f) = \sum_{i=1}^{n} \left[ d_i \{ \log \pi(x_i^\top \beta + \gamma y_i) + \log f(y_i|x_i) \} + (1 - d_i) \log \int \{1 - \pi(x_i^\top \beta + \gamma y)\} f(y|x_i)dy \right].
\]

The likelihood ratio test is the most natural and preferable for testing \( \gamma = 0 \). Unfortunately, Miao et al. (2016) showed that parameter identifiability is not guaranteed even when a parametric model is postulated for \( f(y|x) \). Without parameter identifiability, consistent parameter estimation is not feasible, and therefore nor is the likelihood ratio test, under general parametric assumptions because these require consistent parameter estimation under the null and alternative hypotheses. The Wald test has the same problem.

The score test was introduced by Rao (1948) as an alternative to the likelihood ratio test and Wald test. The most significant advantage of the score statistic is that it depends only on estimates of parameters under \( H_0 \); in other words, it automatically circumvents the notorious identifiability issue under MNAR. This motivates us to consider testing \( \gamma = 0 \) by a score test. Let \( \nabla_{\gamma} \) denote the partial differential operator with respect to \( \gamma \). The score function with respect to \( \gamma \) is

\[
\nabla_{\gamma} \ell(\gamma, \beta, f)|_{\gamma=0} = \sum_{i=1}^{n} \left[ d_i \{1 - \pi(x_i^\top \beta)\} y_i - (1 - d_i) \pi(x_i^\top \beta) \mu(x_i) \right],
\]
which depends on the unknown parameters $\beta$ and $\mu(x) = \int y f(y|x) dy$.

The score test statistic is constructed by replacing $\beta$ and $\mu(\cdot)$ with their estimators under the null hypothesis $H_0 : \gamma = 0$. The likelihood function under $H_0$ becomes

$$
\ell_0(\beta, f) = \sum_{i=1}^n \left[ d_i \log \pi(x_i^\top \beta) + d_i \log f(y_i|x_i) + (1 - d_i) \log \left(1 - \pi(x_i^\top \beta)\right)\right].
$$

In this situation, a natural estimator for $\beta$ is the maximum likelihood estimator $\hat{\beta} = \text{arg max}_\beta \ell_1(\beta)$, where

$$
\ell_1(\beta) = \sum_{i=1}^n \left[ d_i \log \pi(x_i^\top \beta) + (1 - d_i) \log \left(1 - \pi(x_i^\top \beta)\right)\right]
$$

is the likelihood function of $\beta$ under the null hypothesis. Estimation of $\mu(\cdot)$ depends on model assumptions on $f(y|x) = \text{pr}(Y = y|X = x)$. To finish the construction of the score test, we consider postulating either a fully parametric or semiparametric model on $f(y|x)$.

### 2.1 Score test under a parametric model $f(y|x, \xi)$

When $H_0$ or equivalently the missingness mechanism is MAR, $f(y|x) = \text{pr}(Y = y|X = x) = \text{pr}(Y = y|X = x, D = 1)$. Under a fully parametric model $f(y|x, \xi)$ on $f(y|x)$, this motivates us to estimate $\xi$ by $\hat{\xi} = \text{arg max}_\xi \ell_2(\xi)$, where $\ell_2(\xi) = d_i \log f(y_i|x_i, \xi)$ is the likelihood function of $\xi$ under $H_0$. The score test statistic is then

$$
S_1(\hat{\beta}, \hat{\xi}) = \sum_{i=1}^n \left[ d_i \{1 - \pi(x_i^\top \hat{\beta})\} y_i - (1 - d_i) \pi(x_i^\top \hat{\beta}) \int y f(y|x_i, \hat{\xi}) dy \right].
$$

To calculate the $p$-value of a score test statistic, we need to determine the sampling distribution of $S_1(\hat{\beta}, \hat{\xi})$ under $H_0$. The exact form of this sampling distribution is in general intractable. A more practical solution is to approximate it by its null limiting distribution under $H_0$ or MAR.

Let $\beta_0$ and $\xi_0$ be the true values of $\beta$ and $\xi$, respectively. Our theoretical results on $S_1(\hat{\beta}, \hat{\xi})$ are built on the following regularity conditions on $X$ and $f(y|x, \xi)$:

(C1) $\mathbb{E}\|X\|^2 < \infty$ and $A = \mathbb{E}[\pi(X^\top \beta_0)\{1 - \pi(X^\top \beta_0)\}XX^\top]$ is of full rank.

(C2) (i) The parameter space $\Omega$ of $\xi$ is independent of $(y, x)$ and compact. (ii) The true value $\xi_0$ of $\xi$ is an interior point of $\Omega$. (iii) $\xi$ is identifiable, i.e. $\mathbb{E}\{\int |f(y|X, \xi) - f(y|X, \xi')|dy\} > 0$ for any different elements $\xi$ and $\xi'$ in $\Omega$. (iv) $\mathbb{E}\{\sup_{\xi \in \Omega} |\log f(Y|X, \xi)|\} < \infty$. (v) $f(y|x, \xi)$ is continuous in $\xi$ for almost all $(y, x)$.

(C3) (i) $f(y|x, \xi)$ is twice differentiable with respect to $\xi$ for almost all $(y, x)$, and $\nabla_{\xi^\xi} f(y|x, \xi)$ is continuous at $\xi_0$. (ii) $B = \mathbb{E}[\pi(X^\top \beta_0)\{\nabla_{\xi} \log f(Y|X, \xi_0)\}^{\otimes 2}]$ is positive definite. (iii) There exist a $\delta > 0$ and positive functions $M_1(x)$ and $M_2(y, x)$ such that $\mathbb{E}\{M_1(X)\} < \infty$ and $\mathbb{E}\{M_2(y, x)\} < \infty$, and

$$
\|x\| \int |t| \{f(t|x, \xi) + \|\nabla_{\xi} f(t|x, \xi)\|\} dt \leq M_1(x) \quad \text{and} \quad \|\nabla_{\xi^\top} \log f(y|x, \xi)\| \leq M_2(y, x)
$$
for all \( \xi \) satisfying \( \| \xi - \xi_0 \| \leq \delta \).

Under Condition (C1), the limit function of \( \ell_1(\beta)/n \) is well defined. Conditions (i) and (ii) in Condition (C2) are trivial. Condition (iii) guarantees that \( \xi_0 \) is a unique maximizer of the likelihood \( \ell_2(\xi) \). Condition (iv) provides an envelope for \( \{ \log f(y|x, \xi) : \xi \in \Omega \} \). These conditions plus the continuity condition of Condition (C2)(v) are sufficient for the consistency of \( \hat{\xi} \). Under Condition (C3), the loglikelihood \( \ell_2(\xi) \) can be approximated by a quadratic form of \( \xi \). The asymptotic normality of \( \hat{\xi} \) follows immediately. Condition (C3) also guarantees that the matrices defined in Theorem 1 are well defined.

**Theorem 1** Assume Conditions (C1)–(C3) and that \( H_0 : \gamma = 0 \) is true. As \( n \) goes to infinity, \( n^{-1/2}S_1(\hat{\beta}, \hat{\xi}) \xrightarrow{d} N(0, \sigma^2_0) \), where \( \sigma^2_0 = A_2 + B_2 - A_1^\top A_1 - B_1^\top B_1 \), and

\[
A_2 = \mathbb{E}\{\pi(X^\top \beta_0)\{1 - \pi(X^\top \beta_0)\}^2Y^2\}, \\
B_2 = \mathbb{E}\{(1 - \pi(X^\top \beta_0)\{\pi(X^\top \beta_0)\}^2\{\int yf(y|X, \xi_0)dy\}^2\}, \\
A_1 = \mathbb{E}[\pi(X^\top \beta_0)\{1 - \pi(X^\top \beta_0)\}XY], \\
B_1 = \mathbb{E}\left\{\{1 - \pi(X^\top \beta_0)\}\pi(X^\top \beta_0)\int y\nabla f(y|X, \xi_0)dy\right\}.
\]

An estimator for the asymptotic variance \( \sigma^2_0 \) is \( \hat{\sigma}^2_0 = \hat{A}_2 - \hat{B}_2 - \hat{A}_1^\top \hat{A}_1 - \hat{B}_1^\top \hat{B}_1 \), where

\[
\hat{A} = \frac{1}{n} \sum_{i=1}^{n} \pi(x_i^\top \hat{\beta})\{1 - \pi(x_i^\top \hat{\beta})\}x_i^\top x_i, \\
\hat{B} = -\frac{1}{n} \sum_{i=1}^{n} \pi(x_i^\top \hat{\beta}) \int \{\nabla \xi^\top \log f(y|x_i, \hat{\xi})\}f(y|x_i, \hat{\xi})dy, \\
\hat{B}_2 = \frac{1}{n} \sum_{i=1}^{n} \pi^2(x_i^\top \hat{\beta})\{1 - \pi(x_i^\top \hat{\beta})\}\{\int yf(y|x_i, \hat{\xi})dy\}^2, \\
\hat{A}_1 = \frac{1}{n} \sum_{i=1}^{n} \pi(x_i^\top \hat{\beta})\{1 - \pi(x_i^\top \hat{\beta})\}x_i \int yf(y|x_i, \hat{\xi})dy, \\
\hat{B}_1 = \frac{1}{n} \sum_{i=1}^{n} \pi(x_i^\top \hat{\beta})\{1 - \pi(x_i^\top \hat{\beta})\} \int y\nabla f(y|x_i, \hat{\xi})dy, \\
\hat{A}_2 = \frac{1}{n} \sum_{i=1}^{n} \pi(x_i^\top \hat{\beta})\{1 - \pi(x_i^\top \hat{\beta})\}^2 \int y^2 f(y|x_i, \hat{\xi})dy.
\]

The consistency of \( \hat{\sigma}_1^2 \) follows from the consistency of \( (\hat{\beta}, \hat{\xi}) \) and the continuity of \( \pi(x^\top \beta) \) and of \( f(y|x, \xi) \). Formally, the proposed score test rejects \( H_0 \) if \( |S_1(\hat{\beta}, \hat{\xi})|/(\sqrt{n}\hat{\sigma}_1) \) is too large, and its \( p \)-value is approximately \( 2 - 2\Phi\{|S_1(\hat{\beta}, \hat{\xi})|/(\sqrt{n}\hat{\sigma}_1)\} \), where \( \Phi(\cdot) \) is the standard normal distribution function.
2.2 Score test under a semiparametric location model on \( f(y|x) \)

The score function depends on \( f(y|x) \) through the conditional mean \( \mu(x) = \int f(y|x)dy \), which is equal to \( \mathbb{E}\{Y|X = x, D = 1\} \) under \( H_0 \). Instead of imposing a fully parametric conditional density model, it is sufficient to assume a parametric model \( \mu(x, \theta) \) for \( \mu(x) \), where \( \theta \) is an unknown parameter. Under \( H_0 \), this model assumption is equivalent to a location model on the completely observed data \( \{(x_i, y_i) : d_i = 1\} \), namely \( y_i = \mu(x_i, \theta) + \varepsilon_i \), where \( \varepsilon_i \) satisfies \( \mathbb{E}(\varepsilon_i|X_i = x_i, D_i = 1) = 0 \). We estimate \( \theta \) by the least square estimator

\[
\hat{\theta} = \arg\min \sum_{i=1}^{n} d_i \{y_i - \mu(x_i, \theta)\}^2.
\]

Given the estimators \( \hat{\beta} \) and \( \mu(x, \hat{\theta}) \) of \( \beta \) and \( \mu(x) \), the score test statistic under the location model on \( f(y|x) \) is

\[
S_2(\hat{\beta}, \hat{\theta}) = \sum_{i=1}^{n} \left[ d_i \{1 - \pi(x_i^\top \hat{\beta})\} y_i - (1 - d_i) \pi(x_i^\top \hat{\beta}) \mu(x_i, \hat{\theta}) \right].
\]

Let \( \theta_0 \) be the true value of \( \theta \). To establish the limiting distribution of \( S_2(\hat{\beta}, \hat{\theta}) \), we impose the following regularity conditions \( \mu(x, \theta) \):

**(D1)** (i) \( \mathbb{E}\{Y|X\} = \mu(X, \theta_0) \) holds for all \( X \). (ii) The parameter space \( \Theta \) of \( \theta \) is compact, and \( \theta_0 \) is an interior in \( \Theta \). (iii) \( \theta_0 \) is the unique minimizer of \( L_\mu(\theta) = \mathbb{E}[\pi(X^\top \beta_0)\{Y - \mu(X, \theta)\}^2] \).

**(D2)** (i) \( \mu(x, \theta) \) is twice differentiable with respect to \( \theta \). (ii) There exists \( M_3(X,Y) \) such that \( \mathbb{E}\{M_3(X,Y)\} < \infty \) and \( Y^2 + \{\mu(X, \theta)\}^2 \leq M_3(X,Y) \) holds for all \( \theta \). (iii) There exists \( M_4(X,Y) \) such that \( \mathbb{E}\{M_4(X,Y)\} < \infty \) and

\[
\{\|Y| + |\mu(X, \theta)|\| \nabla_{\theta\theta^\top} \mu(X, \theta)\| + \|\nabla_{\theta\mu} (X, \theta)\|\}^2 \leq M_4(X,Y)
\]

holds for all \( \theta \). (iv) \( C_1 = \mathbb{E}[\{\nabla_{\theta\mu} (X, \theta_0)\} \otimes^2 \pi(X^\top \beta_0)] \) is positive definite and \( C_2 = \mathbb{E}[\{Y - \mu(X, \theta_0)\}^2 \{\nabla_{\theta\mu} (X, \theta_0)\} \otimes^2 \pi(X^\top \beta_0)] \) is well defined.

Conditions (D1) and (D2) are the analogues of Conditions (C2) and (C3) for the conditional mean model \( \mu(x, \theta) \).

**Theorem 2** Assume Conditions (C1), (D1) and (D2) and that \( H_0 : \gamma = 0 \) is true. As \( n \) goes to infinity, \( n^{-1/2} S_2(\hat{\beta}, \hat{\theta}) \xrightarrow{d} N(0, \sigma^2_2) \), where \( \sigma^2_2 = A_2 + B_4 - A_1^1 A_1^{-1} A_1 + B_3^1 C_1^{-1} C_2 C_1^{-1} B_3 - 2B_3^1 C_1^{-1} C_3 \).
and

\[ B_3 = \mathbb{E}\{1 - \pi(X^\top \beta_0)\} \pi(X^\top \beta_0) \nabla_{\theta} \mu(X, \theta_0), \]
\[ B_4 = \mathbb{E}\{1 - \pi(X^\top \beta_0)\} \{\pi(X^\top \beta_0)\}^2 \{\mu(X, \theta_0)\}^2, \]
\[ C_1 = \mathbb{E}\{\nabla_{\theta} \mu(X, \theta_0)\} \otimes \cdot \pi(X^\top \beta_0), \]
\[ C_2 = \mathbb{E}\{(Y - \mu(X, \theta_0))^{\otimes 2} \pi(X^\top \beta_0)\}, \]
\[ C_3 = \mathbb{E}\{1 - \pi(X^\top \beta_0)\} \pi(X^\top \beta_0) \{Y - \mu(X, \theta_0)\}^2 \nabla_{\theta} \mu(X, \theta_0). \]

A consistent estimator for the asymptotic variance \( \sigma_2^2 \) is

\[ \hat{\sigma}_2^2 = \hat{A}_2 + \hat{B}_4 - \hat{A}_1 \hat{A}^{-1} \hat{A}_1 + \hat{B}_3 \hat{C}_1^{-1} \hat{C}_2 \hat{C}_1^{-1} \hat{B}_3 - 2 \hat{B}_3 \hat{C}_1^{-1} \hat{C}_3, \]

where

\[ \hat{A} = \frac{1}{n} \sum_{i=1}^{n} \pi(x_i^\top \hat{\beta}) \{1 - \pi(x_i^\top \hat{\beta})\} x_i x_i^\top, \]
\[ \hat{A}_1 = \frac{1}{n} \sum_{i=1}^{n} \pi(x_i^\top \hat{\beta}) \{1 - \pi(x_i^\top \hat{\beta})\} x_i \mu(x_i, \hat{\theta}), \]
\[ \hat{A}_2 = \frac{1}{n} \sum_{i=1}^{n} \{1 - \pi(x_i^\top \hat{\beta})\}^2 [d_i \{y_i - \mu(x_i, \hat{\theta})\}^2 + \pi(x_i^\top \hat{\beta}) \mu^2(\pi(x_i^\top \hat{\beta})], \]
\[ \hat{B}_3 = \frac{1}{n} \sum_{i=1}^{n} \pi(x_i^\top \hat{\beta}) \{1 - \pi(x_i^\top \hat{\beta})\} \nabla_{\theta} \mu(x_i, \hat{\theta}), \]
\[ \hat{B}_4 = \frac{1}{n} \sum_{i=1}^{n} \pi^2(\pi(x_i^\top \hat{\beta})) \{1 - \pi(x_i^\top \hat{\beta})\} \{\mu(x_i, \hat{\theta})\}^2, \]
\[ \hat{C}_1 = \frac{1}{n} \sum_{i=1}^{n} (\nabla_{\theta} \mu(x_i, \hat{\theta}) \otimes \pi(x_i^\top \hat{\beta})), \]
\[ \hat{C}_2 = \frac{1}{n} \sum_{i=1}^{n} [d_i \{y_i - \mu(x_i, \hat{\theta})\}^2 \nabla_{\theta} \mu(x, \hat{\theta}) \otimes \pi(x_i^\top \hat{\beta})], \]
\[ \hat{C}_3 = \frac{1}{n} \sum_{i=1}^{n} \{1 - \pi(x_i^\top \hat{\beta})\} d_i \{y_i - \mu(x_i, \hat{\theta})\}^2 \nabla_{\theta} \mu(x_i, \hat{\theta}). \]

We reject \( H_0 \) if \(|S_2(\hat{\beta}, \hat{\theta})|/(\sqrt{n} \hat{\sigma}_2)\) is large enough. Its \( p \)-value is approximately \( 2 - 2\Phi(|S_2(\hat{\beta}, \hat{\theta})|/(\sqrt{n} \hat{\sigma}_2)) \).

The result in Theorem 2 and the variance estimator \( \hat{\sigma}_2^2 \) allow the error \( \varepsilon_i \) given \( D_i = 1 \) and \( X_i = x \) to depend on \( x \), or to have a heterogeneous variance. If we assume that \( x_i \) and \( \varepsilon_i \) are conditionally independent given \( D_i = 1 \), then \( C_3 = B_3 \times \text{Var}(\varepsilon_i|D_i = 1) \) and \( C_2 = C_1 \times \text{Var}(\varepsilon_i|D_i = 1) \) and \( \sigma_2^2 \) reduces to \( \sigma_2^2 = A_2 + B_4 - A_1 A^{-1} A_1 - B_3 C_1^{-1} B_3 \times \text{Var}(\varepsilon_i|D_i = 1) \).
2.3 Local power

To study the asymptotic power of the proposed score tests, we consider the following local alternative:

\[ H_a : \gamma = n^{-1/2}\gamma_0, \tag{2.2} \]

where \( \gamma_0 \) is fixed. The local alternative (2.2) tends to the null hypothesis at a root-\( n \) rate as \( n \) goes to infinity. A test for \( H_0 \) is root-\( n \) consistent if it can detect the local alternative (2.2) as \( n \) goes to infinity for any fixed \( \gamma_0 \). We expect that both of the proposed score tests have root-\( n \) consistency, which is a desirable property of a nice test for MAR.

Theorem 3 Assume Condition (C1) and that the alternative \( H_a \) is true. Let \( \beta_0, \xi_0 \) and \( \theta_0 \) be the true values of \( \beta \), \( \xi \) and \( \theta \), respectively. (i) If Conditions (C2) and (C3) are satisfied, then, as \( n \) goes to infinity, \( n^{-1/2}S_1(\hat{\beta}, \hat{\xi}) \xrightarrow{d} N(\gamma_0\sigma_1^2, \sigma_1^2) \), where \( \sigma_1^2 \) is defined in Theorem 1. (ii) If conditions (D1) and (D2) are satisfied, then, as \( n \) goes to infinity, \( n^{-1/2}S_2(\hat{\beta}, \hat{\theta}) \xrightarrow{d} N(\delta, \sigma_2^2) \), where \( \delta = \gamma_0( A_2 + B_4 - A_1^TA^{-1}A_1 - B_3C_1^{-1}C_3 ) \) and \( \sigma_2^2 \) is defined in Theorem 2.

Under the propensity model (2.1), a nonzero \( \gamma \) characterizes the departure of the true missingness mechanism from the null hypothesis. Theorem 3 indicates that if for some fixed \( \gamma_0 \) the alternative \( H_a \) is true, then both of the score test statistics converge in distribution to nondegenerate distributions with nonzero location parameters. Because the absolute values of the location parameters are increasing functions of \( |\gamma_0| \), the powers of both score tests tend to 1 as \( \gamma_0 \) goes to infinity, which means that both of them are root-\( n \) consistent.

3 Simulation

We conduct simulations to evaluate the finite-sample performance of the proposed score tests. Specifically, we compare the following three tests: (1) S1, the proposed score test under a parametric model on \( \text{pr}(y|x) \), (2) S2, the proposed score test under a semiparametric location model on \( \text{pr}(y|x) \) and (3) DUAN, the test proposed by Duan et al. (2020). We generate data from two examples. In Example 1, which comes from Duan et al. (2020), an instrumental variable is present, whereas there is no instrumental variable in Example 2. All our simulation results are calculated based on 5000 simulated samples and the significance level is set to 5%.

Example 1 Let \( (Y, U, Z) \) follow a multivariate normal distribution such that \( (Y|U, Z) \sim N(1 + U + b_zZ, 1) \), \( (U|Z) \sim N(1 - Z, 1) \) and \( Z \sim N(0, 1) \). The missingness indicator \( D \) for \( Y \) follows a Bernoulli distribution with success probability \( \text{pr}(D = 1 | Y, U, Z) \sim \Phi(c_0 + c_1w(Y) + c_2U) \), where \( \Phi(\cdot) \) is the standard normal distribution function. We consider three choices for \( w(y) \), namely
$y, 0.4y^2$ and $2.5I(y > 1)$, two choices for $b_z$, namely 0.5 and 1, four choices for $c_2$, namely 0, 0.25, 0.5, and 0.75, and 11 choices for $c_1$, namely $0.05 \times k$, for $k = 0, 1, \ldots, 10$.

The DUAN test requires the existence of an instrumental variable. Under the settings of Example 1, the variable $Z$ is an instrumental variable and therefore the DUAN test is applicable. For data generated from this example, we model $pr(Y = y | X = x)$ by $f(y|x, \xi) = (2\pi)^{-1/2} \exp\{-(x^\top \xi)^2/2\}$ in the construction of the score test $S_1$, and we model $E(Y | X = x)$ by $\mu(x, \theta) = x^\top \theta$ for $S_2$. Tables 1 and 2 present the simulated rejection rates of $S_1$, $S_2$, and DUAN when the sample size $n = 1000$. The rejection rates corresponding to the DUAN test are directly copied from Tables 3 and 4 in Duan et al. (2020), which were calculated based on 1000 simulated samples. Although the true missingness indicator is generated from a probit model, we model it by the logistic model (2.1) in the constructions of the proposed two score tests.

The coefficient $c_1$ quantifies the departure of the true missingness mechanism from the null hypothesis. When $c_1 = 0$, the null hypothesis holds and the results reported are all type I errors. We see that all three tests have desirable controls on their type I errors. As $c_1$ increases, the true missingness mechanism departs more and more from the null hypothesis and, as expected, all tests have increasing powers. The proposed two score tests are more powerful than DUAN in most situations. When $b_z = 0.5$, their power gains against DUAN can be greater than 25%; see the case with $c_1 = 0.4, c_2 = 0.75$ and $w(y) = y$. As $b_z$ increases from 0.5 to 1, the power gain can be as large as 41%; see the case with $b_z = 1, c_1 = 0.25, c_2 = 0.75$ and $w(y) = 0.4y^2$. These observations show that the proposed score tests have obvious advantages over the DUAN test. Meanwhile, the two score tests have almost the same powers in all cases, although the $S_2$ test requires much weaker model assumptions. We also conduct simulations for $n = 2000$, and the simulation results, provided in the Supplementary Material, are similar.

For better comparison, we display the power (versus $c_1$) lines of the $S_2$ and DUAN tests in Figs. 1 and 2, corresponding to $b_z = 0.5$ and 1, respectively. The power lines of the $S_1$ test coincide with those of the $S_2$ test and hence are omitted. It is clear that the power lines of the $S_2$ test always lie above those of the DUAN test, or $S_2$ is uniformly more powerful, except for two scenarios where $b_z = 0.5, c_2 = 0$, and $w(y) = 0.4y^2$ or $2.5I(y > 1)$. In the two exceptional cases, compared with DUAN, $S_2$ is more powerful for small $c_1$ and becomes less powerful for large $c_1$. As $c_1$ quantifies the departure of the true missingness mechanism from the null hypothesis, a possible explanation for this phenomenon is that the score test is usually most powerful for ‘local’ alternatives, but may be suboptimal when the alternative is not very local.

**Example 2** Let $X$ denote a univariate covariate following $\mathcal{N}(0, 1)$. The conditional distribution of $Y$ given $X = x$ is $\mathcal{N}(\xi_1 x + \xi_2 x^2, \exp(\xi_3 + \xi_4 x))$, with $\xi = (\xi_1, \ldots, x_4) = (-1, 1, 0.5, 0)$ or $(1, 1, 0.5, 1)$. The missingness indicator $D$ of $Y$ conditional on $(Y = y, X = x)$ follows a logistic model, $pr(D =
Table 1: Empirical rejection rates (%) of the S1, S2, and DUAN tests based on 5000 simulated samples from Example 1. The sample size is $n = 1000$ and $b_z = 0.5$.

| $w(y)$ | $c_2$ | Test | 0  | 0.05 | 0.1  | 0.15 | 0.2  | 0.25 | 0.3  | 0.35 | 0.4  | 0.45 | 0.5   |
|--------|-------|------|----|------|------|------|------|------|------|------|------|------|------|
| $y$    | 0     | S1   | 5.0| 6.5  | 12.5 | 20.8 | 32.3 | 46.3 | 59.9 | 70.7 | 81.3 | 87.9 | 92.7  |
|        |       | S2   | 5.0| 6.5  | 12.5 | 20.8 | 32.4 | 46.3 | 59.9 | 70.8 | 81.3 | 87.9 | 92.8  |
|        |       | DUAN | 5.2| 7.6  | 12.4 | 17.0 | 27.8 | 37.0 | 55.3 | 60.8 | 72.3 | 84.2 | 88.2  |
|        | 0.25  | S1   | 5.2| 6.4  | 12.5 | 19.8 | 30.2 | 43.0 | 56.2 | 68.1 | 78.4 | 86.0 | 92.0  |
|        |       | S2   | 5.1| 6.4  | 12.4 | 19.8 | 30.4 | 43.0 | 56.5 | 68.2 | 78.6 | 86.2 | 92.1  |
|        |       | DUAN | 7.0| 7.2  | 11.4 | 16.0 | 23.0 | 36.4 | 49.8 | 52.6 | 70.7 | 76.4 | 84.4  |
|        | 0.50  | S1   | 5.0| 6.6  | 11.1 | 18.6 | 30.1 | 43.5 | 55.9 | 68.9 | 79.8 | 87.1 | 92.4  |
|        |       | S2   | 5.0| 6.6  | 11.0 | 18.6 | 30.0 | 43.8 | 56.2 | 69.1 | 80.2 | 87.3 | 92.7  |
|        |       | DUAN | 5.8| 7.6  | 10.4 | 14.0 | 24.4 | 29.0 | 38.6 | 57.8 | 66.8 | 78.2 | 85.6  |
|        | 0.75  | S1   | 5.2| 6.6  | 11.9 | 20.3 | 31.5 | 44.4 | 58.4 | 70.3 | 81.3 | 87.9 | 92.8  |
|        |       | S2   | 5.1| 6.4  | 11.8 | 20.0 | 31.2 | 44.8 | 59.0 | 70.7 | 81.4 | 88.1 | 93.1  |
|        |       | DUAN | 4.4| 6.4  | 9.4  | 11.6 | 20.6 | 28.6 | 39.4 | 49.0 | 54.6 | 68.6 | 82.4  |
| $0.4y^2$ | 0     | S1   | 4.9| 8.5  | 17.3 | 27.1 | 35.9 | 45.4 | 54.7 | 60.5 | 65.3 | 71.4 | 76.1  |
|        |       | S2   | 4.9| 8.5  | 17.3 | 27.1 | 35.8 | 45.4 | 54.7 | 60.3 | 65.3 | 71.3 | 75.9  |
|        |       | DUAN | 5.0| 7.8  | 11.4 | 19.0 | 27.8 | 35.8 | 54.8 | 63.9 | 65.2 | 69.6 | 72.6  |
|        | 0.25  | S1   | 4.9| 6.4  | 11.4 | 17.6 | 23.0 | 27.6 | 36.5 | 41.4 | 48.9 | 56.3 | 62.0  |
|        |       | S2   | 4.9| 6.4  | 11.4 | 17.4 | 22.7 | 27.2 | 35.9 | 41.0 | 48.3 | 55.3 | 61.1  |
|        |       | DUAN | 4.6| 6.0  | 9.6  | 13.6 | 19.8 | 29.0 | 35.9 | 45.6 | 45.6 | 55.9  |
|        | 0.50  | S1   | 5.1| 6.2  | 9.3  | 12.2 | 16.5 | 22.8 | 28.1 | 34.4 | 42.3 | 49.1 | 55.6  |
|        |       | S2   | 5.0| 6.2  | 9.1  | 11.9 | 16.2 | 22.0 | 27.0 | 33.2 | 40.7 | 47.5 | 53.4  |
|        |       | DUAN | 6.4| 6.0  | 8.6  | 10.8 | 16.2 | 19.0 | 23.6 | 33.4 | 30.5 | 41.6 | 39.9  |
|        | 0.75  | S1   | 5.0| 6.3  | 8.7  | 11.4 | 16.1 | 22.8 | 29.5 | 37.2 | 46.8 | 54.2 | 63.4  |
|        |       | S2   | 5.0| 6.1  | 8.0  | 10.7 | 15.0 | 21.2 | 27.5 | 34.8 | 44.4 | 51.6 | 60.5  |
|        |       | DUAN | 5.4| 6.4  | 5.2  | 8.4  | 11.4 | 19.2 | 15.6 | 22.6 | 33.4 | 31.0 | 32.2  |
| $2.5I(y > 1)$ | 0     | S1   | 5.2| 5.7  | 7.0  | 9.9  | 14.1 | 20.9 | 26.7 | 36.3 | 45.8 | 54.4 | 64.3  |
|        |       | S2   | 5.2| 5.7  | 7.0  | 9.9  | 14.1 | 20.9 | 26.7 | 36.3 | 45.9 | 54.5 | 64.3  |
|        |       | DUAN | 5.2| 7.6  | 8.6  | 9.6  | 16.4 | 21.4 | 22.8 | 33.0 | 45.8 | 56.8 | 72.0  |
|        | 0.25  | S1   | 5.0| 5.4  | 8.2  | 11.8 | 17.1 | 23.6 | 33.0 | 43.7 | 54.6 | 65.4 | 75.8  |
|        |       | S2   | 5.0| 5.4  | 8.1  | 11.8 | 17.0 | 23.6 | 33.0 | 43.7 | 54.3 | 65.2 | 75.6  |
|        |       | DUAN | 6.0| 9.2  | 7.8  | 11.2 | 11.4 | 16.4 | 35.0 | 37.0 | 52.5 | 60.4 | 65.4  |
|        | 0.50  | S1   | 4.6| 6.1  | 8.9  | 13.2 | 20.8 | 28.8 | 39.9 | 54.1 | 65.9 | 76.1 | 85.7  |
|        |       | S2   | 4.6| 6.1  | 8.8  | 13.1 | 20.5 | 28.5 | 39.6 | 53.7 | 65.4 | 75.8 | 85.4  |
|        |       | DUAN | 6.4| 6.6  | 6.4  | 11.6 | 16.2 | 23.6 | 30.8 | 40.2 | 46.2 | 56.2 | 68.2  |
|        | 0.75  | S1   | 4.8| 6.5  | 10.1 | 16.3 | 24.8 | 34.1 | 48.7 | 62.2 | 72.6 | 83.4 | 90.4  |
|        |       | S2   | 4.9| 6.4  | 9.9  | 16.1 | 24.5 | 33.8 | 48.1 | 61.6 | 72.3 | 82.8 | 90.2  |
|        |       | DUAN | 5.6| 5.4  | 8.0  | 10.0 | 13.6 | 18.8 | 36.6 | 37.2 | 49.4 | 53.4 | 62.8  |
Table 2: Empirical rejection rates (%) of the S1, S2, and DUAN tests based on 5000 simulated samples from Example 1. The sample size is $n = 1000$ and $b_z = 1.$

| $w(y)$ | $c_2$ | Test | 0   | 0.05 | 0.1  | 0.15 | 0.2  | 0.25 | 0.3  | 0.35 | 0.4  | 0.45 | 0.5  |
|--------|-------|------|-----|------|------|------|------|------|------|------|------|------|------|
|        |       | S1   | 5.0 | 12.2 | 34.4 | 63.0 | 86.6 | 95.9 | 99.9 | 100.0 | 100.0 | 100.0 |
|        |       | S2   | 5.0 | 12.2 | 34.4 | 63.0 | 86.6 | 95.9 | 99.9 | 100.0 | 100.0 | 100.0 |
|        |       | DUAN | 5.8 | 11.0 | 20.0 | 40.0 | 70.6 | 86.4 | 97.4 | 99.0 | 100.0 | 100.0 | 100.0 |
|        |       | S1   | 5.7 | 12.4 | 31.8 | 60.2 | 82.9 | 94.5 | 98.7 | 99.8 | 100.0 | 100.0 | 100.0 |
|        |       | S2   | 5.7 | 12.4 | 31.8 | 60.2 | 82.9 | 94.5 | 98.7 | 99.8 | 100.0 | 100.0 | 100.0 |
|        |       | DUAN | 5.5 | 10.3 | 27.2 | 49.5 | 73.5 | 89.4 | 96.3 | 99.2 | 99.8 | 99.8 | 100.0 |
|        |       | S1   | 4.9 | 11.6 | 28.9 | 54.8 | 77.0 | 91.2 | 97.8 | 99.6 | 99.8 | 100.0 | 100.0 |
|        |       | S2   | 5.0 | 11.6 | 28.8 | 54.7 | 77.0 | 91.2 | 97.8 | 99.6 | 99.8 | 100.0 | 100.0 |
|        |       | DUAN | 6.6 | 10.2 | 21.6 | 36.8 | 63.4 | 79.4 | 96.6 | 99.2 | 99.8 | 99.8 | 100.0 |
| $y^2$  | 0     | S1   | 5.2 | 20.2 | 55.6 | 81.1 | 94.3 | 98.4 | 99.5 | 99.8 | 100.0 | 100.0 | 100.0 |
|        |       | S2   | 5.2 | 20.2 | 55.6 | 81.1 | 94.3 | 98.4 | 99.5 | 99.8 | 100.0 | 100.0 | 100.0 |
|        |       | DUAN | 7.0 | 13.0 | 34.0 | 60.6 | 82.6 | 95.4 | 96.6 | 98.4 | 99.2 | 99.2 | 100.0 |
| $0.4y$ | 0.25  | S1   | 4.5 | 16.6 | 43.2 | 70.7 | 87.3 | 95.1 | 98.2 | 99.5 | 99.9 | 100.0 | 100.0 |
|        |       | S2   | 4.5 | 16.6 | 43.2 | 70.6 | 87.2 | 95.0 | 98.2 | 99.5 | 99.9 | 100.0 | 100.0 |
|        |       | DUAN | 7.0 | 9.2  | 24.0 | 44.2 | 57.4 | 82.6 | 89.0 | 95.2 | 95.8 | 98.2 | 99.0 |
| $0.5y$ | 0.50  | S1   | 5.6 | 14.1 | 33.9 | 58.8 | 78.8 | 90.8 | 96.5 | 98.9 | 99.6 | 100.0 | 100.0 |
|        |       | S2   | 5.5 | 14.0 | 33.9 | 58.6 | 78.7 | 90.8 | 96.4 | 98.9 | 99.6 | 100.0 | 100.0 |
|        |       | DUAN | 4.6 | 06.6 | 17.0 | 35.2 | 50.4 | 56.8 | 77.6 | 87.0 | 90.8 | 95.2 | 97.2 |
| $y^2$  | 0.75  | S1   | 5.0 | 11.1 | 30.2 | 52.2 | 72.3 | 86.1 | 94.1 | 97.8 | 99.3 | 99.8 | 100.0 |
|        |       | S2   | 5.0 | 11.0 | 30.0 | 52.0 | 72.1 | 85.7 | 94.0 | 97.7 | 99.2 | 99.8 | 100.0 |
|        |       | DUAN | 6.8 | 8.4  | 16.4 | 36.2 | 45.6 | 62.8 | 78.0 | 86.8 | 86.8 | 91.6 | 100.0 |
| $2.5I(y > 1)$ | 0 | S1   | 5.3 | 7.5  | 14.7 | 27.7 | 45.9 | 66.7 | 82.1 | 92.0 | 97.2 | 99.2 | 99.8 |
|        |       | S2   | 5.3 | 7.5  | 14.7 | 27.7 | 45.9 | 66.7 | 82.1 | 92.0 | 97.2 | 99.2 | 99.8 |
|        |       | DUAN | 6.0 | 8.6  | 14.8 | 21.2 | 35.2 | 60.2 | 74.4 | 86.0 | 94.2 | 97.6 | 99.2 |
| $2.5I(y > 1)$ | 0.25 | S1   | 5.3 | 7.7  | 16.6 | 30.2 | 51.5 | 69.6 | 85.4 | 94.3 | 98.0 | 99.5 | 99.9 |
|        |       | S2   | 5.3 | 7.7  | 16.6 | 30.2 | 51.5 | 69.5 | 85.3 | 94.3 | 98.0 | 99.5 | 99.9 |
|        |       | DUAN | 8.0 | 10.4 | 16.0 | 28.8 | 39.6 | 54.8 | 70.8 | 94.6 | 95.6 | 97.8 | 99.8 |
| $2.5I(y > 1)$ | 0.50 | S1   | 5.2 | 7.9  | 16.2 | 32.3 | 52.4 | 72.7 | 86.8 | 94.6 | 98.3 | 99.7 | 99.9 |
|        |       | S2   | 5.2 | 7.9  | 16.2 | 32.3 | 52.4 | 72.8 | 86.7 | 94.6 | 98.3 | 99.7 | 99.9 |
|        |       | DUAN | 5.2 | 7.6  | 13.6 | 23.6 | 41.0 | 50.8 | 74.0 | 85.8 | 92.8 | 97.0 | 99.6 |
| $2.5I(y > 1)$ | 0.75 | S1   | 5.4 | 8.1  | 16.0 | 32.2 | 51.5 | 71.5 | 85.0 | 94.1 | 98.0 | 99.4 | 99.9 |
|        |       | S2   | 5.3 | 8.1  | 16.0 | 32.1 | 51.4 | 71.5 | 85.1 | 94.2 | 98.0 | 99.4 | 99.9 |
|        |       | DUAN | 7.0 | 7.2  | 11.6 | 20.6 | 31.6 | 41.8 | 66.8 | 83.6 | 92.0 | 97.2 | 99.2 |
Figure 1: Plots of rejection rates when $b_z = 0.5$ for the S2 test (solid lines) and the DUAN test (dotted lines): $w(y) = y$ (circles); $w(y) = 0.5y^2$ (triangles); $w(y) = 2.5I(y > 1)$ (squares).
Figure 2: Plots of rejection rates when $b_z = 1$ for the S2 test (solid lines) and the DUAN test (dotted lines): $w(y) = y$ (circles); $w(y) = 0.5y^2$ (triangles); $w(y) = 2.5I(y > 1)$ (squares).
\[ 1|x, y = \pi(\beta_0 + \beta_1 x + \gamma y). \] We consider six choices of \((\beta_0, \beta_1)\), namely \((0.85, 0), (0.6, 0.25), (0.4, 0.5)\) in the case of \(\xi = (-1, 1, 0.5, 0)\) and \((0.85, 0), (0.7, 0.25) (0.5, 0.5)\) in the case of \(\xi = (1, 1, 0.5, 1)\). These settings are chosen such that the missingness rates are about 20\%-30\%. The parameter \(\gamma\) is set to 0, 0.05, \ldots, and 0.25, respectively.

Example 2 is designed to represent the case where no instrument is present, and therefore the DUAN test is not applicable. The choices of \(\xi_4 = 0\) and 1 correspond to a homogeneous variance and a heterogeneous variance, respectively. We take \(f(y|x, \xi)\) and \(\mu(x, \theta)\) in the constructions of S1 and S2 to be the density functions of \(N(\xi_1 x + \xi_2 x^2, \exp(\xi_3 + \xi_4 x))\) and \(\theta_1 x + \theta_2 x^2\), respectively, where \(\theta = (\theta_1, \theta_2)^T\). Table 3 presents the simulated rejection rates of the S1 and S2 tests when data are generated from Example 2 and the sample size \(n = 1000\). The results corresponding to \(\gamma = 0\) are type I errors, and the type I errors of both S1 and S2 are under control. As \(\gamma\) increases from 0 to 0.25, both tests have desirable and increasing powers whether the variance is homogeneous or heterogeneous. Again, the results for both tests are all nearly equal to each other in all cases. As S1 requires stronger model assumptions, it may be more risky for model mis-specification than S2. Hence, we would recommend using S2 rather than S1 for testing whether the missingness mechanism is ignorable missing or nonignorable missing.

4 Application to human immunodeficiency virus data

For illustration, we analyse human immunodeficiency virus (HIV) data from AIDS Clinical Trials Group Protocol 175 (Hammer et al., 1996; Han et al., 2019; Liu et al., 2021). These data are available from the R package \texttt{speff2trial} and consist of various measurements of \(n = 2139\) HIV-infected patients. The patients were randomly divided into four arms according to the regimen of treatment they received: (I) zidovudine monotherapy, (II) zidovudine + didanosine, (III) zidovudine + zalcitabine and (IV) didanosine monotherapy. Important measurements from the patients include CD4 cell count at baseline (cd40), CD4 cell count at 20±5 weeks (cd420), CD4 cell count at 96±5 weeks (cd496), CD8 cell count at 20±5 weeks (cd820) and arm number (arms). The effectiveness of a HIV treatment can be assessed by monitoring the CD4 cell counts of HIV-positive patients: an increase in such counts is an indication of improvement in the patients’ health. The typical problem of interest is to estimate the mean of the CD4 cell counts in each arm after the patients were treated for about 96 weeks.

We take cd496 as a response variable \(Y\) and we take cd40, cd420 and cd820 as covariates \(X_1, X_2\) and \(X_3\), respectively. Owing to the end of the trial or loss to follow-up, 39.66\% of the patients’ responses were missing. The analyses of Hammer et al. (1996) and Han et al. (2019) were based on the MAR assumption, whereas Liu et al. (2021) and Zhang et al. (2020) assumed that the response was nonignorable missing. As correctly specifying the underlying missingness mechanism
Table 3: Empirical rejection rates (%) of the S1 and S2 tests based on 5000 simulated samples of size \( n = 1000 \) from Example 2.

| \( \xi \) | \( \beta_1 \) | Test | \( \gamma \) | 0  | 0.05 | 0.1 | 0.15 | 0.2  | 0.25 |
|----------|----------------|------|------------|----|-----|-----|------|------|------|
| \((-1, 1, 0.5, 0)\) | 0  | S1   | 4.7 | 17.4 | 50.9 | 81.7 | 95.6 | 99.2 |
|  |  | S2   | 4.8 | 17.3 | 50.8 | 81.7 | 95.5 | 99.2 |
|  | 0.25 | S1   | 4.9 | 17.4 | 50.6 | 81.7 | 96.4 | 99.4 |
|  |  | S2   | 4.9 | 17.2 | 50.4 | 81.6 | 96.4 | 99.4 |
|  | 0.5 | S1   | 5.4 | 16.9 | 47.6 | 79.3 | 94.8 | 99.1 |
|  |  | S2   | 5.4 | 16.8 | 47.3 | 79.0 | 94.7 | 99.1 |
|  | 1  | S1   | 4.7 | 13.0 | 35.8 | 66.0 | 86.3 | 97.3 |
|  |  | S2   | 5.1 | 12.8 | 35.3 | 65.4 | 86.1 | 97.1 |
| \((1, 1, 0.5, 1)\) | 0  | S1   | 4.6 | 14.4 | 37.4 | 60.9 | 77.7 | 87.4 |
|  |  | S2   | 5.0 | 13.0 | 36.3 | 60.4 | 77.5 | 88.4 |
|  | 0.25 | S1   | 5.2 | 13.6 | 35.1 | 57.4 | 75.9 | 86.6 |
|  |  | S2   | 5.3 | 13.4 | 34.7 | 57.6 | 76.5 | 87.4 |
|  | 0.5 | S1   | 4.7 | 14.2 | 33.6 | 55.3 | 73.7 | 86.4 |
|  |  | S2   | 4.7 | 14.1 | 34.3 | 56.2 | 74.6 | 87.0 |
|  | 1  | S1   | 4.7 | 11.0 | 28.1 | 47.2 | 66.0 | 79.9 |
|  |  | S2   | 4.6 | 11.1 | 27.5 | 46.8 | 65.2 | 79.4 |
is crucial to validation of the subsequent inference, it is also necessary to formally check whether the missingness is MAR or not.

Let $X = (1, X_1, X_2, X_3)$. We choose $f(y|x, \xi)$ to be the normal density with mean $\mu(x, \xi) = \xi_1 + \xi_2 x_1 + \xi_3 x_2 + \xi_4 x_3 + \xi_5 x_2^2$ and variance $\sigma(x, \xi) = \xi_6$, where $\xi = (\xi_1, \ldots, \xi_6)^\top$. After explorative analysis, we find that among the three covariates, only $x_2$ is significantly correlated with the missingness indicator. We assume that the missingness indicator in each regimen of treatment follows a linear logistic regression model with covariates $x_2$ and $y$ only. As no instrumental variable is present in this application, we apply only the proposed two scores to test whether the missingness of cd496 depends on itself.

The $p$-values of the proposed two score tests are reported in Table 4. None of the results are significant at the 5% level. In other words, they all support the MAR mechanism in the four regimens. At the same time, both the tests have very close $p$-values. Table 4 also presents their $p$-values if we remove the covariate $x_2$ from the propensity score model. The $p$-values for regimens II and IV are seemingly unchanged and insignificant. However, those for regimens I and III become much less, and much smaller than the 5% significance level. These results indicate that the MNAR mechanism seems more reasonable than MAR if the propensity score depends potentially on $y$. A possible explanation for the insignificant result in the presence of $x_2$ is that $x_2$ and $y$ stand for CD4 cell counts at $20 \pm 5$ weeks and at $96 \pm 5$ weeks, respectively, and they are highly correlated.

Table 4: P-values of the proposed score tests S1 and S2 under the four regimens of treatment based on the HIV data

| Treatment regimen | I   | II  | III | IV  |
|-------------------|-----|-----|-----|-----|
| $X_2$ appears in the logistic model |     |     |     |     |
| S1                | 0.3263 | 0.3730 | 0.4490 | 0.2081 |
| S2                | 0.1291 | 0.3388 | 0.3730 | 0.1584 |
| $X_2$ does not appear in the logistic model |     |     |     |     |
| S1                | 0.0065 | 0.3731 | 0.0104 | 0.2081 |
| S2                | 0.0003 | 0.3389 | 0.0006 | 0.1584 |

5 Discussion

Valid data analyses of missing data rely on a correctly specified missingness mechanism. The problem of testing whether the missingness mechanism is MCAR or MAR is relatively easy to solve and has been extensively studied. However, it is much more challenging to test whether the mechanism is MAR or not, because parameters may no longer be identifiable. We avoid this thorny issue by using a score test, which is constructed under the null hypothesis, namely the MAR
mechanism. The underlying parameters are usually identifiable based on MAR data. This is one of the nice properties of a score test (Rao, 2005). A score test is also invariant under transformation of parameters. Transformation of parameters may simplify parameter estimation without affecting the value of the statistic. We derive two score tests, S1 and S2, when the conditional density of \( Y \) given \( X \) is modelled by a completely parametric model and a semiparametric location model, respectively. Our numerical results indicate that these tests generally have nearly the same performance (type I error and power), but S2 is preferable because it requires weaker model assumptions.

ACKNOWLEDGEMENTS

This research was supported by the National Natural Science Foundation of China (11771144), the State Key Program of the National Natural Science Foundation of China (71931004 and 32030063), the Development Fund for Shanghai Talents, and the 111 Project (B14019). Drs Lu and Liu are the corresponding authors.

Supplementary Materials

The proofs of Theorems 1–3 and more simulation results are available with this paper at the Supplementary Materials.

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