Formalism for stochastic perturbations and analysis in relativistic stars

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Abstract
Perturbed Einstein’s equations with a linear response relation and a stochastic source, applicable to a relativistic star model are worked out. These perturbations which are stochastic in nature, are of significance for building a non-equilibrium statistical mechanics theory in connections with relativistic astrophysics. A fluctuation dissipation relation for a spherically symmetric star in its simplest form is obtained. The FD relation shows how the random velocity fluctuations in the background of the unperturbed star can dissipate into Lagrangian displacement of fluid trajectories of the dense matter. Interestingly in a simple way, a constant (in time) coefficient of dissipation is obtained without a delta correlated noise. This formalism is also extended for perturbed TOV equations which have a stochastic contribution, and show up in terms of the effective or root mean square pressure perturbations. Such contributions can shed light on new ways of analysing the equation of state for dense matter. One may obtain contributions of first and second order in the equation of state using this stochastic approach.

Keywords Einstein-Langevin equation · Stochastic perturbations of relativistic stars

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1 Introduction

Perturbations in relativistic stars have been of significance towards asteroseismology, and stability criteria in massive stars [1–3]. The mode analysis of perturbations plays a central role in gravitational wave detection [4–6]. The theoretical developments for this are based on the perturbed Einstein’s equations. In this article a part of the new basic formalism for relativistic stars and their perturbative analysis using a stochastic approach, is given. This is in continuation with an upcoming theme of research [7–9], using the framework of Langevin formalism in general relativity. The open directions in this regard deal with formulations and solutions for a classical Einstein–Langevin equation, as well as there is scope for modelling fluctuations in other ways which are not limited to Langevin type, for various astrophysical models. However our first approach is to frame the Langevin formalism for a spacetime structure. For considerations in general relativity, the Langevin noise has to be obtained in an elaborate form and is model dependent. Similar developments in the semiclassical theory are well established [10] for semiclassical Einstein–Langevin equation. However the semiclassical developments follow differently in terms of formulations and applications due to the quantum fields and quantum stress tensors that define the Langevin noise. The applications of the semiclassical stochastic gravity is towards early universe cosmology [11] and black hole physics [12]. We do not follow those lines, though the idea of a Langevin noise defined due to fluctuations in the matter fields is borrowed from the semiclassical counterpart. In many other areas of physics it is easy to model the system by assuming gaussian white noise or some coloured form of the noise term in the Langevin equation [13, 14] in a handwaving way. One cannot do this for the framework which we are developing here. It is the spacetime metric and the matter fields, that decide the form of the Langevin term for the Einstein’s equations. This has to be obtained elaborately, using the generalized fluctuations of the respective stress-energy tensors. The generalized fluctuations as defined in [15] are an extension of the random fluctuations in time to that of random fluctuations on a spacetime, w.r.t. the spatial as well as temporal coordinates. We use the terminology, generalized fluctuations, in our theme of research for the background noise in the unperturbed system. The term perturbations is specifically used for the induced shift in the trajectories of the fluid and the metric potentials. The background fluctuations in the fluid matter can be due to many reasons, they can be thermal or non-thermal in nature. For example in a perfect fluid having no, or negligible thermal effects, such generalized stochastic fluctuations can arise either due to dynamical effects of fluid on small scale, which are yet unexplored in the dense matter fluids, and lie in the mesoscopic range of scales above the quantum microscopic but below the macroscopic hydro-scales. Such effects can also be remnants (coarse grained effects) of quantum mechanical fluctuations in the bulk.

In this article, we assume the radial velocity fluctuations in the background space-time as noise, with a near-equilibrium (dynamical) configuration of the relativistic star. It is shown how these may induce random perturbations in the system. The radial perturbations are of significance towards stability properites of the relativistic star [1, 5]. In this article we also work out a fluctuation dissipation relation for a spherically symmetric configuarion in a simple way. The perturbed TOV relation obtained in the
later section is due to contributions from the generalised stochastic pressure perturbations which are induced due to the Langevin noise. In the concluding section we give the importance of developing such a framework and further directions for theoretical formulations and investigations.

We begin by reviewing the Einstein–Langevin equation based on a linear response theory \cite{16} in the section below.

2 The linear response relation for the perturbed Einstein’s equation with a source term

A linear response relation for perturbations of relativistic stars has been introduced in \cite{9}, which gives the following form of the Einstein–Langevin equation.

\[
\delta G_{ab}[h; x] - 8\pi \delta T_{ab}[h; x] - 8\pi \int K(x - x') \delta T_{ab}[\xi]; x' = \tau_{ab}[g; x] \tag{1}
\]

where \( h \) denotes the metric perturbations and \( \xi \), the fluid perturbations. The factor \( K(x - x') \) is the response kernel and connects the metric and fluid perturbations. In the above equation \( \tau_{ab}[g; x] \) denotes the stochastic source inside the dense matter of the relativistic star. We define \( \tau_{ab}(x) = \delta_s T_{ab}(x) \), in terms of the generalized stochastic fluctuations of matter fields of the massive astrophysical object. The linearized perturbed Einstein’s equations are covariantly conserved w.r.t the background metric \( g_{ab} \), which also implies \( \nabla_a \tau^{ab}(x) = 0 \). In case the perturbations in the gravitating system are due to external sources, then we assume \( \tau_{ab}(x) = 0 \) in Eq. (1), and \( K(x - x') = \delta^4(x - x') \). In this case one obtains the conventional perturbed Einstein’s equation given by \( \delta G_{ab}(x) = 8\pi \delta T_{ab}(x) \), which conventionally forms the base of asteroseismology. One can also consider a configuration where few components of \( \tau_{ab}(x) \) but not all, may be zero. We will see this more clearly in the example that we solve in this article. We model with \( \tau_{ab}(x) \) only the internal sources which may perturb the astrophysical body.

In the next two sections we solve the classical Einstein–Langevin equation for a spherically symmetric model of the star with non-thermal background velocity fluctuations in the fluid. The configuration is assumed to be near a static equilibrium state by the end of the collapse, and has matter fields described by a perfect fluid.

3 The spherically symmetric model of relativistic star and a stochastic source of induced perturbations

For the spherically symmetric star in Schwarzschild coordinates,

\[
ds^2 = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 d\Omega^2 \tag{2}
\]
while the matter fields for perfect fluid (describing cold dense matter in hydrodynamic approximation) are given by,

\[ T_{ab} = (\epsilon + p)u_a u_b + g_{ab} p \]  

the four-velocity is given by,

\[ u^a = (e^{-v}, 0, 0, 0). \]  

At a later time a radial velocity which maintains spherical symmetry can be introduced in the fluid given by, \( v = e^{\lambda - v} \dot{r} \), such that the four-velocity has components

\[ u^a = \frac{1}{\sqrt{1 - v^2}}(e^{-v}, e^{-\lambda v}, 0, 0) \]  

Accordingly, the components of field equation are given as

\[ G^t_t = 8\pi T^t_t : \]
\[ e^{-2\lambda} \left( \frac{1}{r^2} - \frac{2}{r} \lambda' \right) - \frac{1}{r^2} = -8\pi \epsilon \frac{1}{1 - v^2} \]  

\[ G^r_r = 8\pi T^r_r : \]
\[ e^{-2\lambda} \left( \frac{1}{r^2} + \frac{2}{r} \nu' \right) - \frac{1}{r^2} = 8\pi \left( \epsilon \frac{v^2}{1 - v^2} + p \right) \]  

\[ G^r_t = 8\pi T^t_r : \]
\[ -2 \frac{e^{-2\nu} \dot{\lambda}}{r} = 8\pi e^{\nu - v} (\epsilon + p) \frac{v}{1 - v^2} \]  

\[ e^{2\lambda} G^\theta_\theta = 8\pi e^{2\lambda} T^\theta_\theta : \]
\[ v'' + v^2 - v' \lambda' + \frac{1}{r} (v' - \lambda') = 8\pi e^{2\lambda} p \]

Also we can easily obtain the relation,

\[ v' + \lambda' = 4\pi (\epsilon + p) e^{2\lambda} p \]  

from the above Einstein’s equations. Perturbations of the above equations can be carried out in a heuristic way, which suffice out requirements. The perturbations in the fluid can be introduced using the radial velocity such that,

\[ \delta v = e^{\lambda - v} \dot{\xi} \]  

where \( \xi \equiv \xi_r \) is the only non-zero fluid displacement vector and “.” denotes derivative w.r.t time. The terminology that we use here should be clarified as “perturbations” meaning shift in trajectory of the fluid and related physical quantities in the system, while the generalized “fluctuations” is used for the noise in the unperturbed background which we define in the subsection below.
3.1 The source term with four-velocity fluctuations

We consider a model of the source term or generalized noise \( \tau_{ab}(x) \) with randomness over the spatial as well as temporal coordinates. We will show that the perturbations are induced in the star due to the cumulative effect of these generalized fluctuations.

The specific model of the noise or source that we consider can be given by the single non-zero term of \( \tau_{ab}(x) \).

\[
\tau_t^r(r, t) = 8\pi e^{(\lambda-\nu)}(\epsilon + p)\delta_s v(t, r)
\]

where \( \delta_s v(t, r) \) denotes the generalized fluctuations in the radial velocity of the fluid. The 's' in \( \delta_s v(t, r) \) denotes “source”. The probability distribution of \( P(\delta_s v(r, t)) \) is crucial to the nature of solutions of Eq. (1). The source is Langevin if the distribution is gaussian, such that \( \langle \tau_{ab}(x) \rangle = 0 \), where \( \langle .... \rangle \) denotes the statistical average. The two point correlations \( \langle \tau_{ab}(x)\tau_{cd}(x') \rangle_s = N_{abcd}(x, x') \) define the point separated noise kernel. For the Langevin noise then, all the higher order correlations can be described in terms of the two point noise kernel or correlations.

4 Solution of the Einstein Langevin equation

Given the model of noise in the previous subsection and Eqs. (6) (7) and (8) for the spherically symmetric relativistic star, the perturbed equations with the source described by (1) have \( t-t, r-r, t-r, \theta-\theta, \phi-\phi \) as the non-zero components.

We require the following perturbed components of the E-L equation for a complete solution of the configuration discussed here,

\[
\delta G_t^t[h; x] - 8\pi \delta T_t^t[h; x] - 8\pi \delta T_t^t[\xi; x] = 0
\]

\[
\delta G_t^r[h; x] - 8\pi \delta T_t^r[h; x] - 8\pi \delta T_t^r[\xi; x] = 0
\]

\[
\delta G_r^t[h; x] - 8\pi \delta T_r^t[h; x] - 8\pi \int K(x - x')\delta T_r^t[\xi; x']dx' = \tau_r^t(g, x)
\]

In the above, the components \( t-t \) and \( r-r \) do not contain a source term, since \( \tau_t^t \) and \( \tau_r^r \) are zero for the model of noise that we use here. Thus, as discussed earlier, we assume a delta correlated response kernel \( K(x - x') = \delta^4(x - x') \) for these two components of the E-L equation. To solve the above equations we will also need to specify the form of the Lagrangian displacement vector \( \xi \), we assume it to be of the form \( \xi(r, t) = \tilde{\xi}(r)e^{\gamma r t} \), where \( \gamma r \) is complex valued. We choose a simple form of the response kernel \( K(x - x') = K_1(t - t')\delta(r - r') \) for Eq. (15) which gives,

\[
\dot{\delta}\lambda(r, t) + (v' + \lambda')\int K(t - t')\tilde{\xi}(r, t')dt' = -e^{\lambda-\nu}(\lambda' + v')\delta_s v(r, t)
\]
Taking $t - t' = T$, we can bring the above in the form

$$\delta \lambda(r, t) = (v' + \lambda') K_1(\gamma_r) \xi(r, t) - e^{(v - \lambda)}(v' + \lambda') \int \delta_s v(r, t) dt$$  \hspace{1cm} (17)$$

where $K_1(\gamma_r)$ is the Laplace transform of $K_1(t - t')$ and gives susceptibility of perturbations in the astrophysical configuration. With this we will show a relation between the $\xi(r, t)$ and the noise term. Using (13) one can get the following relation

$$-\frac{2}{r} \delta \lambda' + \delta \lambda \left( \frac{3\lambda'}{r} - \frac{2}{r^2} - \frac{v'}{r} \right) = \frac{(v' + \lambda')}{r} \xi'$$

$$+ \left( \frac{v' + \lambda'}{2} \right) \left( \frac{r}{2} + \frac{\lambda'}{r} \right) \xi + 8\pi \epsilon' e^{2\lambda}$$  \hspace{1cm} (18)$$

From (17), we get

$$\delta \lambda'(r, t) = [(v' + \lambda')K_1(\gamma_r)\xi(r, t)]' - e^{v - \lambda}[(v'^2 - \lambda^2)$$

$$+ (v'' + \lambda'')] \int \delta_s v(r, t) dt$$  \hspace{1cm} (19)$$

in the above we assume $\delta_s v(r, t) = 0$.

Substitute equation (17) and (19) in (18) in terms of $\xi(r, t)$ we get,

$$g(r) \xi'(r, t) + f(r) \xi(r, t) = j(r) \int \delta_s v(r, t) dt$$  \hspace{1cm} (20)$$

where

$$g(r) = -\frac{(\lambda' + v')}{r} (2K_1(\gamma_r) + 1)$$

$$f(r) = \left[ (v'' + \lambda'')K_1(\gamma_r) + (v' + \lambda') \left\{ K_1(\gamma_r) \frac{3\lambda'}{r} - \frac{2}{r^2} - \frac{v'}{r} \right\} + \lambda' + \frac{2}{r} \right]$$

$$j(r) = e^{v - \lambda} \left[ \lambda'' + v'' + (v' + \lambda') \left( \frac{3\lambda'}{r} - \frac{2}{r^2} - \frac{v'}{r} \right)$$

$$- (v'^2 - \lambda'^2) \right]$$

The solution of (20) gives,

$$\xi(r, t) = m_1(r) \int \left\{ m_2(r') \int \delta_s v(r', t') dt' \right\} dr'$$  \hspace{1cm} (21)$$
where

\[ m_1(r) = e^{-\int \frac{f(r')}{g(r')} dr'} \]
\[ m_2(r) = e^{\int \frac{f(r')}{g(r')} dr'} \frac{j(r)}{g(r)} \]

We can conclude from the above that \( \xi(r, t) \) is random in nature due to being induced by \( \delta_s v(r, t) \). With the form it has \( \hat{\xi}(r) e^{i\gamma r} \), which is random in \( \xi(r) \) as well as \( \gamma_r \) can be random. The complex nature of the frequency \( \gamma_r \) is important as, this will be used for mode analysis of the stochastic perturbations in future work. Using equation (21) for \( \xi(r, t) \) in equation (17) we get,

\[ \delta \lambda(r, t) = l_1(r) \int m_2(r') \int \delta_s v(r', t') dt' dr' - l_2(r) \int \delta_s v(r, t') dt' \]  

(22)

where

\[ l_1 = (v' + \lambda') K_1(\gamma_r) e^{-\int \frac{f(r')}{g(r')} dr'} \]
\[ l_2 = e^{v'-\lambda}(v' + \lambda') \]

From equation (14) similarly one can obtain,

\[ \delta v' = \delta \lambda \left[ 2e^{-2\lambda} \left( \frac{1}{r^2} + \frac{2}{r} v' \right) - 8\pi \Gamma_1 p \left( e^{-\lambda} r^2 \right)' - 8\pi \xi p' \right] \]  

(23)

which on substituting for \( \delta \lambda \) and \( \xi \) gives,

\[ \delta v(r, t) = \int \int a_1(r') m_2(r'') \delta_s v(r'', t') dt'' dr'' dr' + \int \int a_2(r') \delta v(r', t') dt' dr' \]  

(24)

where

\[ a_1 = (v' + \lambda') K_1(\gamma_r) \left[ 2e^{-2\lambda} \left( \frac{1}{r^2} + \frac{2}{r} v' \right) - \Gamma_1 p \right] - 8\pi \Gamma_1 p (\lambda' + \frac{2}{r}) + 8\pi \Gamma_1 p \frac{f(r)}{g(r)} \]
\[ a_2 = 8\pi \Gamma_1 p \frac{j(r)}{g(r)} - e^{v'-\lambda}(v' + \lambda') \left[ 2e^{-2\lambda} \left( \frac{1}{r^2} + \frac{2}{r} v' \right) - \Gamma_1 p \right] \]

The potentials \( \delta \lambda(r, t) \) and \( \delta v(r, t) \) also have a random nature due to being induced by \( \delta_s v(r, t) \). All the unperturbed variables in the above expressions are functions of \( r \) only. Equations (21), (22) and (24) give the main expressions for \( \delta \lambda(r, t), \xi(r, t) \) and \( \delta v(r, t) \). These expressions show that the perturbations are induced by the cumulative effect of the radial velocity fluctuations which are the background noise. These
expressions are meaningful only as statistical results, hence we need to take statistical averages for \( \delta \lambda(t, r), \delta \nu(t, r) \) and \( \xi(t, r) \), and appropriately write them as

\[
\langle \xi(r, t) \rangle = m_1(r) \int \{ m_2(r') \int \langle \delta_x v(r', t') \rangle dt' \} dr' = 0 \tag{25}
\]

\[
\langle \delta \lambda(r, t) \rangle = l_1(r) \int \{ m_2(r') \int \langle \delta_x v(r', t') \rangle dt' \} dr' - l_2(r) \int \langle \delta_x v(r, t') \rangle dt' = 0 \tag{26}
\]

\[
\langle \delta v(r, t) \rangle = \int a_1(r') \{ m_2(r'') \langle \delta_x v(r'', t') \rangle dt'' dr' \} + \int a_2(r') \langle \delta v(r', t') \rangle dt' dr' = 0 \tag{27}
\]

We see that these are vanishing due to the Langevin property of the noise. However the two point correlations of these perturbations are non-vanishing and read,

\[
\langle \xi(r_1, t_1) \xi(r_2, t_2) \rangle = m_1(r_1) m_1(r_2) \int \{ m_2(r_1') m_2(r_2') \}
\]

\[
\int \langle \delta_x v(r_1', t_1') \delta_x v(r_2', t_2') \rangle dt_1' dt_2' dr_1' dr_2' \tag{28}
\]

\[
\langle \delta \lambda(r_1, t_1) \delta \lambda(r_2, t_2) \rangle = l_1(r_1) l_1(r_2) \int \{ m_2(r_1') m_2(r_2') \}
\]

\[
\int \langle \delta_x v(r_1', t_1') \delta_x v(r_2', t_2') \rangle dt_1' dt_2' dr_1' dr_2' \tag{29}
\]

\[
\langle \delta v(r_1, t_1) \delta v(r_2, t_2) \rangle = \int a_1(r_1') a_1(r_2') \{ m_2(r_1'') m_2(r_2'') \}
\]

\[
\int \langle \delta_x v(r_1'', t_1') \delta_x v(r_2'', t_2') \rangle dt_1'' dt_2'' dr_1'' dr_2'' \tag{30}
\]

From the above one can easily take the coincidence of points and obtain root mean square value for the generalized stochastic perturbations. It is the rms value that gives the strength of these perturbations and fluctuations for a localized point in the spacetime. The point separated or two point correlations are important to probe the non-local and extended properties of the matter fields. These nonlocal or point separated correlations of the induced perturbations then carry the signature of the inherent nature of
state of matter at mesoscopic scales. In the mode analysis of these perturbations, the essential features would be reflected in the randomness of physical variables, which may characterize the state of matter and its composition at the new scales. Such modes which we will work out in another article with a detailed analysis, can be appropriately called "stochastic modes" of the induced perturbations.

5 Fluctuation-dissipation relation

The fluctuation dissipation relation is valid near the static equilibrium state of the relativistic star. For this we consider the perturbed Euler equation for the system

$$\delta \nabla_a T^a_b (x) = 0$$  \hspace{1cm} (31)

For the perfect fluid stress tensor, consider the component

$$\delta \nabla_a T^a_1 (x) = 0$$  \hspace{1cm} (32)

which gives, for the spherically symmetric configuration of the star,

$$\ddot{\xi} - \nu' \dot{\xi} + \frac{e^{2(v'-\lambda)}}{(\epsilon + p)} \delta p' - \frac{p}{(\epsilon + p)} e^{2(v'-\lambda)} \delta \nu' - \nu' \frac{e^{2(v'-\lambda)}}{(\epsilon + p)} \delta p = 0$$  \hspace{1cm} (33)

On substituting values of $\delta p$, $\delta \lambda$ and $\delta \nu'$ from the previous section, one can obtain,

$$\ddot{\xi} - \nu' \dot{\xi} - B[\xi] = b_1(r) \int \delta_s v(r, t') dt'$$  \hspace{1cm} (34)

where,

$$B[\xi] = \left[ 4 \pi r K_1(\gamma_r) e^{2\lambda} p (2 e^{-2\lambda} \left( \frac{1}{r^2} + \frac{2 \nu'}{r} \right) - 8 \pi \Gamma_1 p \right]$$

$$- \frac{8 \pi p \nu' + \frac{e^{2(v'-\lambda)}}{(\epsilon + p)} \Gamma_1 (2 \nu' + \lambda') K_1(\gamma_r)}{\left( \frac{p}{(\epsilon + p)} \right)^2} e^{2(v'-\lambda)} \left( \frac{4 \pi r e^{2\lambda} + \frac{(\nu' + \lambda')}{4} r (\lambda' + \frac{2}{r})}{(\epsilon + p)} - 1 \right)$$

$$- \frac{8 \pi p e^{2(v'-\lambda)} - e^{3(\nu' - \lambda)}/(\epsilon + p)}{\left( \frac{e^{2(v'-\lambda)}}{(\epsilon + p)} \right)^2} \left[ p' + p \left( 3 \lambda' - \frac{2}{r} - \nu' \right) \right]$$

$$b_1(r) = 8 \pi p e^{2\lambda} \left[ e^{-2\lambda} \left( \frac{1}{r^2} + \frac{2 \nu'}{r} \right) - 8 \pi \Gamma_1 p \right] - e^{3(v'-\lambda)}/\Gamma_1 (\nu' + \lambda') \left[ p' + p \left( 3 \lambda' - \frac{2}{r} - \nu' \right) \right]$$  \hspace{1cm} (35)

The analysis at near-static-equilibrium state using adiabatic stochastic perturbations can be of interest for characterizing the dense matter at sub-hydro mesoscopic scales.
Here we use our formalism, to define a non-thermal adiabatic fluctuation-dissipation relation. From (34) which is of the form of equation of motion, one can see the dissipative term with $\dot{\xi}$, and the fluctuations or noise term on the rhs of the equation. Assuming a particular form for $\delta_s v(r, t) = \delta_s v(r)e^{i\omega_s r t}$, where $\delta_s v(r)$ the amplitude, as well as $\omega_s r$ are both random in nature, such that they constitute a stochastic noise, in the form of stochastic oscillations. The superscript $s$ in $\omega_s r$ stands for stochasticity or randomness in the phase of oscillations, due to uncertainty of the exact value of the frequency at each shell denoted by radius $r$. Thus we see here how a generalized stochastic behaviour in w.r.t the spatial variable, here, play a role in the physical picture.

Using this form of the generalized fluctuations for velocity, we can write (34) in the form

$$\ddot{\xi} - v' \dot{\xi} - B[\xi] = -i b_1(r) \frac{\delta_s v(r)}{\omega_s r} \delta_s v(r)e^{i\omega_s r t}$$  \hspace{1cm} (37)

The above equation stands for an equation of motion, with a stochastic source. The noise and dissipation can be compared with the the rhs and second term on the lhs, respectively. Thus we can write a fluctuation -dissipation relation in a conventional way, relating the two by

$$\frac{b_1(r)^2}{\langle |\omega_s r|^2 \rangle} \langle \delta_s v(r)^2 \rangle = v'(r)$$  \hspace{1cm} (38)

where for the mean square of the noise, we have taken $\langle \delta_s v(r, t)\delta_s v(r, t)^* \rangle$. Thus for the form of the equation (34) we can see that $v'$ can be considered as a coefficient of dissipation in the system. While (38) relates this to the noise in the system through a fluctuation dissipation relation, which can be written in the form

$$\langle \delta_s v(r)^2 \rangle = \frac{v'(r)}{b_1(r)^2} \frac{\langle |\omega_s r|^2 \rangle}{\langle |\omega_s r|^2 \rangle}$$ \hspace{1cm} (39)

The distributions for the amplitude $\delta_s v(r)$ and for the phase factor $\omega_s r$ of the random oscillations are independent of each other. In general we can consider $\omega_s r$ as complex. We also see the appearance of $'i'$ on the rhs of (34), which indicates an imaginary noise. This is to be compared with the fact that for the Langevin equations describing quantum systems coupled to a bath or in the theory of semiclassical stochastic gravity, the noise appears as the imaginary part of a kernel while dissipation appears as the real part of the same kernel, using influence action formalism. In the classical case as we present here, the action for obtaining the Einstein’s equations does not contain an imaginary contribution and is unlike the quantum case. This is the basic and fundamental difference between the classical and quantum action principles. However, this difference may show up as an explicit imaginary part appearing in the Euler-Langevin equation of the form (34). However here, this is attributed more to the form of the noise that we have considered with the part $e^{i\omega_s r t}$. In case a different form for $\delta_s v(r, t)$ is considered, we may not encounter the imaginary $i$ appearing of the rhs of the equation (34). However the FD relation remains similar and one sees a local dissipation coefficient associated with the system. This remains as $v'(r)$, as can be concluded.
from the second term of (34), which has a radial dependence. It is important to note that we have not put in by hand, the noise term in the perturbed Euler equation, but it follows naturally from the Einstein-Langevin equation that we have considered in this article.

Thus one can view the adiabatic non-thermal dissipative phenomena in the fluid as a result of $\delta_\alpha v(r, t)$ which also act as seeds for the induced random perturbations. In other words, we see the mechanical effect in terms of Lagrangian displacement vector $\xi$ of the fluid in which the energy of the velocity fluctuations is dissipated. These seeds in the form of the velocity fluctuations can be due to various reasons, including the end of dynamically collapsing phase or some other internal mechanical phenomena which leaves its footprints on the matter. In some way, the mechanical microscopic effects coarse grained as $\delta_\alpha v(r, t)$ can be seen to act as sources of perturbations in the system.

6 Perturbed TOV equations

The role of TOV equations for a relativistic star and its importance to study the structure is well established. In this section we it is shown how the induced stochastic perturbations affect the TOV equation. For this we assume, linear perturbations in teh fluid, while its impact of the metric can be ignored. We also recall that the E-L equation is covariantly conserved w.r.t the background metric $g_{ab}$. Thus we can write, $\nabla_a \delta T^a_b(x) = 0$, for which the component $\nabla_a \delta T^a_1 = 0$ gives,

$$\delta p' = -v'(\delta p + \delta \epsilon) - e^{2(\lambda - v)}(\epsilon + p)\dddot{\xi}(r, t)$$

(40)

The above equation can be written as

$$\delta p(r, t) = -\int \{v'(r')(\delta p(r') + \delta \epsilon(r')) + e^{2(\lambda(r') - v(r'))}((\epsilon(r') + p(r'))\dddot{\xi}(r', t))dr'

(41)

This is stochastic in nature due to the Langevin formalism. Moreover it is the root mean square of these perturbations which gives the effective value and can be added to the background unperturbed pressure for a new equilibrium state. Such an analysis can be suitable to study stability properties of the star, and also may also affect the equation of state, when used as an additive contribution to the regular TOV equation which is of the form,

$$p(r) = -\int v'(r')((\epsilon + p)dr'$$

(42)

For equation (41), taking the two point correlation and the coincidence limit,

$$\lim_{r_1,t_1 \rightarrow r_2,t_2} <\delta p(r_1, t_1)\delta p(r_2, t_2)>= \lim_{r'_1,t'_1 \rightarrow r'_2,t'_2} \int \int [v'(r'_1)v'(r'_2) - (\delta p(r'_1, t_1)$$

$$+\delta \epsilon(r_1, t_1))(\delta p(r'_2, t_2) + \delta \epsilon(r'_2, t_2)) + e^{2(\lambda(r'_1) - v(r'_1)) - v(r'_2))(\epsilon(r'_1) + p(r'_1))}((\epsilon(r'_2) + p(r'_2))(\gamma_{r'_1}^r\gamma_{r'_2}^r)^2 <\dddot{\xi}(r'_1, t_1)\dddot{\xi}(r'_2, t_2)$$

$$v'(r'_1)v'(r'_2)e^{2(\lambda(r'_1) - v(r'_1)) - v(r'_2))) <\delta \epsilon(r'_2, t_2) + \delta p(r'_2, t_2))(\dddot{\xi}(r'_2, t_2) + \gamma_{r'_1}^r v'(r'_2)((\epsilon(r'_1) + p(r'_1))$$

$$<\dddot{\xi}(r'_1, t_1)(\delta p(r'_2, t_2) + \delta \epsilon(r'_2, t_2)) >dr'_1dr'_2$$

(43)
The root mean square then can be easily obtained and put in as an additive part in the regular TOV equation to get,

\[
p(r) + (\delta p(r, t))_{rms} = - \int v'(r') (\epsilon(r') + p(r')) dr' + \sqrt{\lim_{r_1, t_1 \to r_2, t_2} <\delta p(r_1, t_1) \delta p(r_2, t_2)>} \tag{44}
\]

where for the second term on the rhs of equation (44), one has to evaluate the rhs of equation (43). To solve this perturbed TOV equations one needs to consider an equations of state like \( p(r) = K_2\epsilon^\Gamma(r) \) and its perturbation \( \delta p(r, t) = K_2\Gamma(\delta\epsilon(r, t))^\Gamma(r, t) \). The quantity on the lhs of equation (44) gives the new near equilibrium dynamical pressure in the fluid. Note that on the rhs of equation (43) the correlations between the pressure and energy density perturbations with the Lagrangian displacement vector are non vanishing. One can also use this perturbed equation for finding out the first and second order corrections, to the equation of state using the rms value and coincidence of the two point correlation/variance, respectively.

### 7 Concluding remarks

In this article we have discussed a detailed form of the classical Einstein Langevin equation for induced perturbations (radial). These perturbations are expected to arise in the system due to non-thermal velocity fluctuations of the fluid elements as the background source. The results obtained are in closed analytical form. We have also addressed susceptibility of the spacetime to the fluid matter perturbations, and a linear response relation between fluid and metric perturbations. In this article, we have discussed the simple case of spherically symmetric spacetime in order to establish the main ingredients for the new theoretical base. These new theoretical foundations have to be developed rigorously from the basics in general relativity and the framework of Langevin approach. More realistic and wider range of configurations of the relativistic stars in a perturbative study for non-radial cases can be based on the framework presented here. Most of the realistic configurations will need numerical solutions for the Einstein-Langevin equations, which will be carried out later. The radial non-rotating case of the relativistic stars is just the beginning of the theme that is to follow. Our further goal in this theme of research as the next step, is to develop the same analysis for non-radial perturbations. We will also work on perturbations arising due to other matter fields like the electromagnetic field coupled to a relativistic star in near future. In this article, we have also discussed a linear response relation for perturbations in relativistic stars from the first principles. We show in a very simple way, how a perturbed TOV equation turns up with the stochastic formulation, and give the basic expressions for the same. This can be used to study new modes of random oscillations in the relativistic stars which may be named as “stochastic modes”. We are interested in characterizing such stochastic modes for non-radial random oscillations in future. It is through such formulations as presented here, that we intend to develop dynamical non-equilibrium and equilibrium theory at sub-hydro mesoscopic scales in the dense...
matter stars. One can analyse extended and non local properties of the dense matter, through two point or higher correlations of the induced perturbations. Hence this is the first step as the theoretical base ( along with work done in and developments in [7–9, 15, 17] ) into investigations that are to follow for the dense matter in gravitating bodies. The main interest in this direction of research lies in analysis of properties of the dense matter at mesoscopic sub-hydro scales for local as well as extended structure in the relativistic stars.

Author Contributions SS is the solo author of this article, and has given full contribution to all the parts and work done.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no conflict of interest.

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