A classification of $SU(d)$-type $C^*$-tensor categories

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Abstract

Kazhdan and Wenzl classified all rigid tensor categories with fusion ring isomorphic to the fusion ring of the group $SU(d)$. In this paper we consider the $C^*$-analogue of this problem. Given a rigid $C^*$-tensor category $C$ with fusion ring isomorphic to the fusion ring of the group $SU(d)$, we can extract a constant $q$ from $C$ such that there exists a $*$-representation of the Hecke algebra $H_\alpha(q)$ into $C$. The categorical trace on $C$ induces a Markov trace on $H_\alpha(q)$. Using this Markov trace and a representation of $H_\alpha(q)$ in $\text{Rep}(SU_d(q))$ we show that $C$ is equivalent to a twist of the category $\text{Rep}(SU_d(q))$. Furthermore a sufficient condition on a $C^*$-tensor category $C$ is given for existence of an embedding of a twist of $\text{Rep}(SU_d(q))$ in $C$.

1 Introduction

Tannaka-type reconstruction theorems allow one to reconstruct an algebraic object (for example a group) from its category of representations. There are numerous of these theorems, the classical Tannaka-Krein duality \cite{14}, the Doplicher-Roberts theorem \cite{3}, Deligne’s theorem \cite{2}, Woronowicz’s duality for compact matrix pseudogroups \cite{10} and many more. Despite these theorems it is still very difficult (if not impossible) to give a complete list of all quantum groups which satisfy the fusion rules of a certain group. However, if one instead tries to classify all ($C^*$-) tensor categories which have a fusion ring isomorphic to the fusion ring of a certain group $G$, this problem becomes easier to solve. Kazhdan and Wenzl \cite{17} gave such a classification in the case of tensor categories with fusion ring isomorphic to the fusion ring $K[\text{Rep}(SU(d))]$. They showed that if $C$ is a tensor category with fusion ring isomorphic to $K[\text{Rep}(SU(d))]$, then there exists a constant $\mu \in C^*$ not a non-trivial root of unity such that $C$ is (monoidally) equivalent to a “twist” of $\text{Rep}(SU_\mu(d))$, the representation category of the quantum group $SU_\mu(d)$. These twists are determined by a $d$-th root of unity.

This paper contains two main results. The first one (cf. Theorem \ref{thm:main}) is the $C^*$-analogue of the result by Kazhdan and Wenzl. We will show that all $C^*$-tensor categories which satisfy the same fusion rules as $\text{Rep}(SU(d))$, the so-called $SU(d)$-type categories, can be classified by pairs $(\mu, \omega)$ where $\mu \in (0,1]$ and $\omega$ is a $d$-th root of unity. Namely given a $SU(d)$-type category $C$ we can extract constants $\mu$ and $\omega$ from $C$ such that $C$ is equivalent to a “twist” by $\omega$ of the category $\text{Rep}(SU_\mu(d))$. The other main result is inspired on the paper by Pinzari \cite{13}. In this paper she gives a sufficient condition when it is possible to embed $\text{Rep}(SU_\mu(d))$ in a given braided $C^*$-tensor category. We generalize this result to conditions on $C^*$-tensor categories which are sufficient to construct an embedding of a “twist” of $\text{Rep}(SU_\mu(d))$ in a given category (see Theorem \ref{thm:main}). These two main results are independent of each other, but the proofs of both of theorems are related. They are both based on Theorem \ref{thm:main} which gives some technical conditions when a given category is equivalent to a “twist” of $\text{Rep}(SU_\mu(d))$. In this paper Hecke algebras play a key-role, because the representation category $\text{Rep}(SU_\mu(d))$ has a natural representation of the Hecke algebra. Following Kazhdan and Wenzl we construct a representation of the Hecke algebra in the the endomorphisms of a $SU(d)$-type category. These representations allow us eventually to recover the category from its fusion ring. In these categories we need to make some explicit calculations. However, in general categories this is often very difficult. Therefore we use the categorical trace \cite{8} to show that the representation of the Hecke algebra is independent of the category $C$. This
result allows us to make computations in $\text{Rep}(SU_\mu(d))$ in which everything is more explicit.

This paper is organized as follows. In Section 2 we start by recalling the main definitions and properties of $C^*$-categories and specialize to $SU(d)$-type categories. We continue with the necessary results on Hecke algebras. Section 3 will be devoted to making the necessary computations in $\text{Rep}(SU_\mu(d))$ allowing us later to compute the twist invariant of a general $SU(d)$-type category. In section 5 we construct the representation of the Hecke algebra into $\text{End}(X \otimes n)$ and we establish that this representation is independent of the category $C$. In the next section we consider a specific class of $C^*$-tensor categories and we prove a technical theorem showing that all these categories of this specific type are equivalent to a "twist" of $\text{Rep}(SU_\mu(d))$. Section 7 contains the two main theorems of this paper.

2 Preliminaries on $C^*$-tensor categories

In this section we will introduce the specific class of $C^*$-tensor categories we are interested in, namely the $SU(d)$-type categories. We will also discuss "twists" of such categories. We will not give the full definitions of $C^*$-tensor categories and functors of these categories. Precise definitions of $C^*$-tensor categories and functors thereof can be found in e.g., [11, §2.1]. The essential element one has to keep in mind is that in a $C^*$-tensor category one is able to take "tensor products" of objects and morphisms and that there exists a conjugation operation. We will define the associativity morphisms and conjugate objects in $C^*$-tensor categories, because they will play a key-role later on.

**Definition 2.1.** A $C^*$-tensor category is a $C^*$-category equipped with a bilinear bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $(U,V) \mapsto U \otimes V$, which will be called the tensor product and it is required that there exist natural unitary isomorphisms

$$\alpha_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W),$$

the associativity morphisms such that the pentagonal diagram

$$
\begin{array}{ccc}
(U \otimes V) \otimes X & \xrightarrow{\alpha_{1,2,3,4}} & U \otimes (V \otimes (W \otimes X)) \\
(U \otimes (V \otimes W)) \otimes X & \xrightarrow{\alpha_{1,2,3,4}} & (U \otimes V) \otimes (W \otimes X) \\
U \otimes (V \otimes W) \otimes X & \xrightarrow{\epsilon \otimes \alpha} & U \otimes (V \otimes (W \otimes X))
\end{array}
$$

commutes. Here the convention of leg-numbering is used (e.g., $\alpha_{12,3,4} := \alpha_{U \otimes V, W, X}$). Furthermore, it is assumed that there exists an object 1 (the unit) and natural unitary isomorphisms

$$\lambda_U : 1 \otimes U \to U, \quad \rho_U : U \otimes 1 \to U$$

such that $\lambda_1 = \rho_1$ and the triangle diagram

$$
\begin{array}{ccc}
(U \otimes 1) \otimes V & \xrightarrow{\alpha} & U \otimes (1 \otimes V) \\
& \xrightarrow{\rho \otimes 1} & U \otimes V \\
& \xrightarrow{\epsilon \otimes \lambda} & U \otimes V
\end{array}
$$

commutes. A category will be called strict if

$$(U \otimes V) \otimes W = U \otimes (V \otimes W), \quad 1 \otimes U = U = U \otimes 1$$

and the associativity morphisms $\alpha$ and the morphisms $\lambda$ and $\rho$ are the identity morphisms. We assume that $C^*$-tensor categories are closed under subobjects and direct sums and that the unit object 1 is simple.
Remark 2.2. Sometimes we will use the terminology that an object $U$ is a subobject of an object $V$, or simply $U \subseteq V$. This means by this is that there exists a projection $p \in \text{End}(V)$ and a morphism $v \in \text{Hom}(U, V)$ such that $v^*v = id_U$ and $vv^* = p$. Via this $v$ we can restrict morphisms $T \in \text{End}(V)$ to the object $U$, we write $T|_U := v^*Tv \in \text{End}(U)$.

Definition 2.3. Let $\mathcal{C}$ and $\mathcal{D}$ be $C^*$-tensor categories. A functor $F : \mathcal{C} \to \mathcal{D}$ together with an isomorphism $F_0 : 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$ and natural isomorphisms $F_2 : F(U) \otimes F(V) \to F(U \otimes V)$ and $F_0$ are unitary. $F$ is called fully faithful if $F : \text{Hom}(U, V) \to \text{Hom}(F(U), F(V))$ is an isomorphism. $F$ is called essentially surjective if for every object $U \in \text{Ob}(\mathcal{D})$ there exists an object $V \in \text{Ob}(\mathcal{C})$ such that $U$ is isomorphic to $F(V)$. $F$ is called a monoidal equivalence if $F$ is fully faithful and essentially surjective. Two $C^*$-tensor categories $\mathcal{C}$ and $\mathcal{D}$ are monoidally equivalent if there exists a monoidal equivalence $F : \mathcal{C} \to \mathcal{D}$.

Remark 2.4. Any $C^*$-tensor category can be stricified [11] [XLI]. This means that if $\mathcal{C}$ is a (non-strict) $C^*$-tensor category, then there exists a strict $C^*$-tensor category $\mathcal{D}$ such that $\mathcal{C}$ and $\mathcal{D}$ are unitarily monoidally equivalent. So unless stated otherwise we deal with strict categories.

If $\mathcal{C}$ is a category which satisfies all the requirements of a $C^*$-tensor category except from the existence of direct sums and subobjects, then $\mathcal{C}$ can be completed to a new category which is a $C^*$-tensor category (see for example [11] [2.5]). For this define $\mathcal{C}'$ with $\text{Ob}(\mathcal{C}') := \{(U_1, \ldots, U_n) : n \geq 1, U_i \in \text{Ob}(\mathcal{C})\}$ and $\text{Hom}_{\mathcal{C}'}(U_1, \ldots, U_m), (V_1, \ldots, V_n)) := \bigoplus_{i,j} \text{Hom}_{\mathcal{C}}(U_i, V_j)$. Now $(U_i) \oplus (V_j)$ is given by the sum of the lexicographical ordering of $U_i \otimes V_j$. Let $\mathcal{C}''$ be the category with $\text{Ob}(\mathcal{C}'') := \{(U, p) : U \in \text{Ob}(\mathcal{C}), p \in \text{End}_{\mathcal{C}}(U) \text{ projection}\}$ and $\text{Hom}_{\mathcal{C}''}(U, p), (V, q) := q \text{Hom}_{\mathcal{C}}(U, V)p$. The tensor product of objects is given by $(U, p) \otimes (V, q) := (U \otimes V, p \otimes q)$. The involution, direct sums and tensor products of morphisms on $\mathcal{C}'$ and $\mathcal{C}''$ are defined in the obvious way. Then $\mathcal{C}''$ is a $C^*$-tensor category. It is clear that there exists a unitary tensor functor $i : \mathcal{C} \to \mathcal{C}''$.

The completion $\mathcal{C}''$ is universal in the following sense: if $\mathcal{D}$ is a $C^*$-tensor category and $F : \mathcal{C} \to \mathcal{D}$ is a unitary tensor functor, then $F$ extends uniquely (up to unitary monoidal equivalence) to a unitary tensor functor $F'' : \mathcal{C}'' \to \mathcal{D}$. To construct this functor define $F''(U_1, \ldots, U_n) := F(U_1) \oplus \cdots \oplus F(U_n)$ and on morphisms $F''(\iota_{ij}) := (F(\iota_{ij}))$. If $(U, p) \in \text{Ob}(\mathcal{C}'')$, then $F''(p)$ is a projection in $\text{End}_{\mathcal{D}}(F(U))$, so there exists $V \in \text{Ob}(\mathcal{D})$ and an isometry $v \in \text{Hom}_{\mathcal{D}}(V, F(U))$ such that $vv^* = F(p)$ and $v^*v = \iota_V$. Define $F''((U, p)) := V$ and for $pTp \in \text{Hom}_{\mathcal{C}''}(F(U), p)$ let $F''(pTp) := v^*F'(pTp)v = v^*F'(T)v$. The tensor and involutive structure are again defined in the obvious way.

Note that in both steps of this extension of $F$ one has to make a choice of objects, a different choice leads to an equivalent functor. Furthermore if both $\mathcal{C}$ and $\mathcal{D}$ are not necessarily closed under direct sums and subobjects and $F : \mathcal{C} \to \mathcal{D}$ is a unitary tensor functor, then $F$ extends to a functor $i : \mathcal{C}'' \to \mathcal{D}''$. This extension is constructed by applying the universal property to $i \circ F$, where $i : \mathcal{D} \to \mathcal{D}''$ is the inclusion. The properties “fully faithful” and “essentially surjective” are preserved under this extension of tensor functors.
**Definition 2.5.** Let \( \mathcal{C} \) be a strict \( C^* \)-tensor category and \( U \in \text{Ob}(\mathcal{C}) \). Then \( \overline{U} \in \text{Ob}(\mathcal{C}) \) is called *conjugate* to \( U \) if there exist \( R \in \text{Hom}(\mathbbm{1}, U \otimes U) \) and \( \overline{R} \in \text{Hom}(\mathbbm{1}, U \otimes U) \) such that the compositions

\[
U \overline{\otimes} R \mathrm{U} \otimes \mathrm{U} \overline{R} \otimes \mathrm{U} \ ; \\
\overline{U} \overline{\otimes} \overline{R} \mathrm{U} \otimes \mathrm{U} \overline{R}^{*} \otimes \mathrm{U}
\]

are the identity morphisms. If every object in \( \mathcal{C} \) has a conjugate object then \( \mathcal{C} \) is *rigid*. We say that the pair \((R, \overline{R})\) solves the conjugate equations for \( U \). If \((R, \overline{R})\) is of the form

\[
R = \sum_k (w_k \otimes w_k^*) R_k, \quad \overline{R} = \sum_k (w_k \otimes \overline{w_k}) \overline{R_k},
\]

where for all \( k \) the objects \( U_k \in \text{Ob}(\mathcal{C}) \) are simple, \( \|R_k\| = \|\overline{R_k}\| \) and \( w_k \in \text{Hom}(U_k, U) \) are isometries such that \( \sum_k w_k w^*_k = \iota_U \), then \((R, \overline{R})\) is called a *standard solution* of the conjugate equations.

If an object has a conjugate it also admits a standard solution of the conjugate equations \([11] \S 2\). Furthermore, if \((R, \overline{R})\) and \((R', \overline{R}')\) are both standard solutions for \((U, \overline{U})\) respectively \((U', \overline{U'})\), then there exists a unitary \( T \in \text{Hom}(\overline{U}, \overline{U'}) \) such that \( R' = (T \otimes \iota) R \) and \( \overline{R} = (\iota \otimes T) \overline{R} \) \([11] \text{ Prop. 2.2.13}\). We will only deal with rigid categories.

**Definition 2.6.** Suppose that \( U \in \text{Ob}(\mathcal{C}) \) and \((R, \overline{R})\) is a standard solution of the conjugate equations for \( U \). For \( T \in \text{End}(U) \) let \( \text{Tr}_U(T) \) be the composition

\[
1 \xrightarrow{R} U \otimes U \xrightarrow{\iota \otimes T} U \otimes U \xrightarrow{R^*} 1.
\]

This functional \( \text{Tr}_U : \text{End}(U) \to \mathbb{C} \) is called the *categorical trace* of \( U \). Note that from the remark above it is immediate that the categorical trace is independent on the choice of the standard solution.

**Proposition 2.7** \([11] \text{ Thm. 2.2.16}\). Let \( U \in \text{Ob}(\mathcal{C}) \), then \( \text{Tr}_U : \text{End}(U) \to \mathbb{C} \) is a tracial, positive and faithful functional. Furthermore \( \text{Tr}_U(T) = \overline{R}(T \otimes \iota) \overline{R} \) for any standard solution \((R, \overline{R})\) of the conjugate equations for \( U \).

**Definition 2.8.** Two objects \( U, V \in \text{Ob}(\mathcal{C}) \) are *isomorphic* if there exists an isomorphism in \( \text{Hom}(U, V) \). We write \([U] \) for the equivalence class of objects isomorphic to \( U \). Denote by \( K^+[\mathcal{C}] \) the *fusion semiring* of \( \mathcal{C} \), it is the universal semiring ring generated by the equivalence classes \([U] \) of objects \( U \in \text{Ob}(\mathcal{C}) \) with sum and product given by

\[
[U] + [V] := [V \oplus V], \quad [U][V] := [U \otimes V].
\]

Note that there is no need to define a subtraction as we define a semiring.

Before we define a \( SU(d) \)-type category let us first say something about the representations of the special unitary group. Details can be found in lots of books, e.g., \([4]\). To avoid trivialities we will always assume that \( d \geq 2 \). We have the fundamental (or defining) representation of \( SU(d) \) on \( V := \mathbb{C}^d \) by letting the group elements act on vectors of \( V \) in the straightforward way. By the highest weight classification of irreducible representations of \( SU(d) \), we can classify the irreducible representations by the tuples

\[
\Lambda_d := \{ \lambda = (\lambda_1, \ldots, \lambda_{d-1}) \in \mathbb{N}^{d-1} : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{d-1} \}.
\]

In this paper we use the convention \( \mathbb{N} := \{0, 1, 2, \ldots\} \). We denote \( V_\lambda \) for the irreducible representation corresponding to \( \lambda \). For \( \lambda \in \mathbb{N}^{d-1} \) let \( |\lambda| := \lambda_1 + \ldots + \lambda_{d-1} \). It can be shown that any irreducible representation \( V_\lambda \) is contained in the tensor product \( V^{\otimes |\lambda|} \). Another special fact for \( SU(d) \) is that the \( d \)-th anti-symmetric tensor power \( \Lambda_d V \) is isomorphic to the trivial representation. Thus there exists a non-zero map \( \mathbb{C} \to V^{\otimes d} \) intertwining the trivial representation and
the $d$-th tensor power of the defining representation. Given two irreducible representation $V_\lambda$ and $V_\mu$, we can decompose their tensor product representation into irreducible representations. So we have

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu \in \Lambda_d} m_{\lambda,\mu,\nu} V_\nu,$$

for some multiplicities $m_{\lambda,\mu,\nu} := \dim \text{Hom}(V_\nu, V_\lambda \otimes V_\mu)$. Here $m_{\lambda,\mu,\nu} V_\nu := V_\nu \oplus \cdots \oplus V_\nu$, with $m_{\lambda,\mu,\nu}$ copies.

**Definition 2.9.** A $C^*$-tensor category of $SU(d)$-type, or simply a category of $SU(d)$-type, is a rigid $C^*$-tensor category $\mathcal{C}$ such that the semirings $K^+[\mathcal{C}]$ and $K^+[\text{Rep}(SU(d))]$ are isomorphic. In particular since simple objects can not be further decomposed, this isomorphism maps simple objects onto simple objects. Therefore we can index the equivalence classes of simple objects of a $SU(d)$-type category by the set $\Lambda_d$. An object $X \in \mathcal{C}$ which corresponds to the fundamental representation $[C^d]$ of $SU(d)$ will be called the fundamental object of $\mathcal{C}$. From now on we fix a $SU(d)$-type category with fundamental object $X$ and for every $\lambda \in \Lambda_d$ we fix a simple object $X_\lambda$ corresponding to $\lambda$.

**Example 2.10.** The object $X_{\{k\}} := X_{(k,0,\ldots,0)}$ corresponds to $S^k(V)$, the $k$-th symmetric tensor power of the fundamental representation of $SU(d)$ on $V$. For $1 \leq k \leq d-1$, the object $X_{\{k^+\}} := X_{(1^+,\ldots,1^+,0,\ldots,0)}$ corresponds to $\Lambda^k(V)$, the $k$-th antisymmetric tensor power of the fundamental representation.

**Example 2.11.** The conjugate object $\overline{X}$ is isomorphic to $X_{\{1^\perp\}}$. Indeed, by the fusion rules of $SU(d)$ it follows that $X \otimes X_{\{1^\perp\}} \cong 1 \oplus X_{\{2^\perp\}}$. Therefore we obtain that $\text{Hom}(1, X \otimes X_{\{1^\perp\}}) \neq \{0\}$, from which the claim follows.

**Notation 2.12.** In a not necessarily strict $SU(d)$-type category denote the objects $X^{\otimes n} := X$ and $X^{\otimes n} := X \otimes X^{\otimes n-1}$ for $n \geq 2$. Unwrapping this recursive definition gives $X^{\otimes n} = X \otimes (X \otimes (\cdots (X \otimes X)\cdots))$ with $n$ factors of $X$.

**Lemma 2.13.** Let $X$ be a fundamental object of a $SU(d)$-type category $\mathcal{C}$. Then

$$X^{\otimes n} = \bigoplus_{\lambda \in \Lambda_d} m_{\lambda,n} X_\lambda \tag{2.1}$$

and the multiplicities satisfy $m_{\lambda,n} = 0$ if $|\lambda| \neq n \pmod{d}$. In particular if $m \neq n \pmod{d}$, then $\text{Hom}(X^{\otimes m}, X^{\otimes n}) = \{0\}$.

**Proof.** For $\text{Rep}(SU(d))$ the identity (2.1) follows for $X = V$ and $X_\lambda = V_\lambda$ from [4] Prop. 15.25. Since $\mathcal{C}$ is a $SU(d)$-type category it satisfies the same fusion rules as $SU(d)$. It is possible to obtain a new $C^*$-tensor category from an existing one by changing the associativity morphisms. This can be done using twists. We will only define twists in the case of a special type of categories, but twists can be defined in other settings as well, for example for representation categories of compact quantum groups see e.g., [11][12].

**Definition 2.14.** Suppose that $\mathcal{C}$ is a strict $C^*$-tensor category and $X$ is an object of $\mathcal{C}$. Let $\rho$ be a $d$-th root of unity and assume that $\text{Hom}_\mathcal{C}(X^{\otimes m}, X^{\otimes n}) = \{0\}$ if $m \neq n \pmod{d}$. Let $\mathcal{C}$ be the category with objects $\{1, X, X^{\otimes 2}, \ldots\}$ and morphisms $\text{Hom}_\mathcal{C}(X^{\otimes m}, X^{\otimes n}) := \text{Hom}_\mathcal{C}(X^{\otimes m}, X^{\otimes n})$. For $a, b, c \in \mathbb{N} = \{0, 1, 2, \ldots\}$ put $\omega(a,b) := \lfloor \frac{a+b}{d} \rceil - \lfloor \frac{a}{d} \rceil - \lfloor \frac{b}{d} \rceil$. Define the morphisms

$$\alpha_{X^{\otimes a},X^{\otimes b},X^{\otimes c}}^\rho := \rho^{\omega(a,b)c} \cdot \alpha_{X^{\otimes a},X^{\otimes b},X^{\otimes c}} : (X^{\otimes a} \otimes X^{\otimes b}) \otimes X^{\otimes c} \rightarrow X^{\otimes a} \otimes (X^{\otimes b} \otimes X^{\otimes c}) \tag{2.2}$$

It can be checked (see Lemma 2.16 below) that the morphisms $\alpha^\rho$ satisfy the pentagon axiom. As $\text{Hom}_\mathcal{C}(X^{\otimes m}, X^{\otimes n}) = \{0\}$ if $m \neq n \pmod{d}$, we have naturality of $\alpha^\rho$. Therefore $\alpha^\rho$ define new associativity morphisms on $\mathcal{C}$. Completing $\mathcal{C}$ with respect to subobjects and direct sums and extending $\alpha^\rho$ to this completion gives new associativity morphisms for the $C^*$-tensor category generated by $\mathcal{C}$. We denote this category by $\mathcal{C}^\rho$. If $\mathcal{C}$ is generated by $X$ (that is $\mathcal{C}$ is the direct sum and subobject completion of the full subcategory with objects $\{1, X, X^{\otimes 2}, \ldots\}$), we denote the category we obtain in this way by $\mathcal{C}^\rho$.  

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These associativity morphisms might seem a bit artificial, but one can prove that the functionals \( \rho^{\omega(a,b)c} \) as defined above represent all classes in \( H^3(\mathbb{Z}/\mathbb{Z}, T) \), see e.g., [12 Prop. A.3]. This cocycle property is exactly needed to make the pentagonal diagram commutative. Note further that in general twisting does not preserve the existence a braiding.

**Remark 2.15.** Note that \( \rho^{\omega(a,b)c} \rho^{\omega(a,c)b} = (\rho \rho')^{\omega(a,b)c} \) for all \( a, b \) and \( c \). So if \( C \) is generated by \( X \), we immediately obtain that \( (C')^{\rho'} \cong (C, \rho'). \)

**Lemma 2.16.** The morphisms \( \alpha^\rho \) defined in (2.2) satisfy the pentagon axiom.

**Proof.** Since \( \alpha \) are associativity morphisms, commutativity of the diagram

\[
\begin{array}{ccc}
(X^{\otimes a} \otimes X^{\otimes b} \otimes X^{\otimes c}) \otimes X^{\otimes e} & \xrightarrow{\alpha^\rho_{1,2,3,4}} & (X^{\otimes a} \otimes (X^{\otimes b} \otimes X^{\otimes c})) \otimes X^{\otimes e} \\
& \xrightarrow{\alpha^\rho_{2,3,4}} & (X^{\otimes a} \otimes X^{\otimes b} \otimes X^{\otimes c}) \otimes X^{\otimes e} \\
& \xrightarrow{\alpha^\rho_{1,2,34}} & (X^{\otimes a} \otimes ((X^{\otimes b} \otimes X^{\otimes c}) \otimes X^{\otimes e})) \otimes X^{\otimes e}
\end{array}
\]

is equivalent to

\[\rho^{\omega(b,c)e} \rho^{\omega(a,b+c)e} \rho^{\omega(a,b)c} = \rho^{\omega(a,b)(c+e)} \rho^{\omega(a+b,c)e}.\]

For which in turn it is sufficient to prove that

\[\omega(b,c)e + \omega(a,b + c)e + \omega(a,b)c - \omega(a,b)(c + e) - \omega(a + b, c)e \equiv 0 \quad (\text{mod } d).\]

One can verify directly that this is in fact an equality and not only a congruency.

**Lemma 2.17.** Suppose that \( C \) is a strict \( C^* \)-tensor category generated by an object \( X \) and \( \rho \) is a \( d \)-th root of unity for some \( d \geq 2 \). Let \( \alpha \) and \( \alpha^\rho \) be the associativity morphisms in \( C \) respectively in \( C^\rho \). Consider for \( m, n \geq 1 \) the associativity morphism \( \alpha_{m,n} : X^{\otimes m} \otimes X^{\otimes n} \to X^{\otimes m+n} \) in \( C \), defined by the following inductive relations

\[
\alpha_{m,n} := \begin{cases} 
(l_X \otimes \alpha_{m-1,1}) \alpha_X, & \text{if } m = n = 1; \\
(l \otimes \alpha_{m-1,1}) \alpha_{X^{\otimes m-1}, X} & \text{if } m \geq 2, n = 1; \\
\alpha_{m+1,n-1} \circ (\alpha_{m,1} \otimes \iota^{n-1}) \circ \alpha^{-1}_{X^{\otimes m}, X} & \text{if } m \geq 1, n \geq 2.
\end{cases}
\]

Define similarly the morphisms \( \alpha^\rho_{m,n} \) in \( C^\rho \). Then it holds that

\[\alpha^\rho_{m,n} = \rho^n |\frac{m}{d}| \alpha_{m,n}.
\]

**Proof.** Let us prove this lemma by induction on \( m \) and \( n \). If \( m = n = 1 \), the lemma is trivial. Suppose that \( n = 1 \). Note that because \( d \geq 2 \) it holds that \( |\frac{1}{d}| = 0 \). So by definition of the twist we obtain

\[
\alpha^\rho_{X^{\otimes m-1}, X} = \rho^{\frac{m}{d} \cdot |\frac{m-1}{d}|} \alpha_{X^{\otimes m-1}, X}
\]

as a map \( (X^{\otimes m}) \otimes X \to X \otimes ((X^{\otimes m-1}) \otimes X) \). Proceeding by induction on \( m \) it follows that

\[
\alpha^\rho_{m,1} = (l \otimes \alpha^\rho_{m-1,1}) \alpha^\rho_{X^{\otimes m-1}, X}
\]

\[
= \rho^{\frac{m-1}{d} \cdot |\frac{m-1}{d}|} \rho^{\frac{m}{d} \cdot |\frac{m-1}{d}|} (l \otimes \alpha_{m-1,1}) \alpha_{X^{\otimes m-1}, X}
\]

\[
= \rho^{\frac{m}{d} \cdot |\frac{m}{d}|} \alpha_{m,1}
\]

and the lemma is proved for \( n = 1 \). Now suppose that \( n > 1 \). By the definition and induction hypothesis, it holds that

\[
\alpha^\rho_{m,n} = (\rho^{(n-1)\cdot |\frac{m}{d}|} \alpha_{m+1,n-1}) (\rho^{\frac{m}{d} \cdot |\frac{m-n+1}{d}|} \alpha_{m,1} \otimes \iota^{n-1}) (\rho^{-(\frac{m}{d} \cdot |\frac{m-n+1}{d}|) \cdot |\frac{n-1}{d}|} \alpha^{-1}_{X^{\otimes m}, X} \otimes \iota^{n-1})
\]

as desired. \( \Box \)
3 Hecke algebras

In this section we will briefly recall some results about Hecke algebras which will be used later when considering SU(d)-type categories. More about Hecke algebras can be found in e.g., [15].

Definition 3.1. Given $n \in \mathbb{N}$ and $q \in \mathbb{C}$, define the Hecke algebra $H_n(q)$ to be the unital algebra generated by the $n - 1$ elements $g_1, \ldots, g_{n-1}$ which satisfy the following three relations

\begin{align*}
  g_i g_j &= g_j g_i & \text{if } |i - j| \geq 2; \\
  g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} & \text{for } i = 1, \ldots, n - 2; \\
  g_i^2 &= (q - 1)g_i + q & \text{for } i = 1, \ldots, n - 1.
\end{align*}

Note that if $q \neq 0$ we have

\[ g_i \left( \frac{1 - q}{q} + \frac{1 - q}{q} g_i \right) = \frac{1 - q}{q} g_i + \frac{1}{q} ((q - 1)g_i + q) = 1. \]

So for $q \neq 0$ the elements $g_i$ have inverses. We will denote these by $g_i^{-1} := \frac{1 - q}{q} + \frac{1 - q}{q} g_i$. Observe that if $q = 1$ relation (3.3) reads as $g_i^2 = 1$ hence $H_n(1) = \mathbb{C}[S_n]$, the group algebra of the symmetric group on $n$ elements. So for $q = 1$ we obtain a map $S_n \to H_n(q)$, but also for general $q \in \mathbb{C}$ we can define such a map.

Definition 3.2. An elementary transposition of $S_n$ is an element of the form $\sigma_i := (i, i + 1)$. Any element $\pi \in S_n$ can be written as a product of elementary transpositions $\pi = \sigma_{i_1} \cdots \sigma_{i_k}$. For a permutation $\pi$ choose such a product of shortest length. The corresponding $k$ will be referred to as the length of $\pi$, we put $l(\pi) := k$. A product of shortest length will be referred to as a reduced expression for $\pi$. If $e$ is the identity element of $S_n$ we put $g_e := 1 \in H_n(q)$. If $\pi \in S_n$ and $\pi \neq e$, we define $g_\pi := g_{i_1} \cdots g_{i_k} \in H_n(q)$. From the the lemma below it follows that the element $g_\pi$ is well-defined.

Lemma 3.3 ([15] §1.1]). Let $\pi \in S_n$, define $d_\pi(i) := \#\{1 \leq j < i : \pi(j) > \pi(i)\}$. Then $l(\pi) = \sum_{i=1}^{n} d_\pi(i)$. Put

\[ C_{i,j} := \begin{cases} 1, & \text{if } i \geq j; \\
                    \sigma_i \cdots \sigma_{j-1}, & \text{if } i < j,
\end{cases} \]

Then $C_{n-d_\pi(n),n} \cdots C_{3-d_\pi(3),3} C_{2-d_\pi(2),2}$ is a reduced expression for $\pi$. Any two reduced expressions for $\pi$ can be transformed in one another by only using the transformations

\begin{align*}
  \sigma_i \sigma_j &= \sigma_j \sigma_i & \text{if } |i - j| \geq 1; \\
  \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } i = 1, \ldots, n - 2.
\end{align*}

We can embed $H_n(q)$ into $H_{n+1}(q)$ via the homomorphism $H_n(q), g_i \mapsto g_i \in H_{n+1}(q)$. Iterating this procedure we obtain embeddings $i_{m,n} : H_m(q) \to H_n(q)$ for $m \leq n$. The inductive limit of $(H_n(q), i_{m,n})$ is denoted by $H_\infty(q)$. Similarly the shift map $\Sigma : H_n(q) \to H_{n+1}(q), g_i \mapsto g_{i+1}$ yields another embedding. Unless stated otherwise we will use the first embedding. Note that (3.3) can be rewritten as $(g_i + 1)(g_i - q) = 0$. So $g_i$ has exactly two spectral values: $-1$ and $q$. In accordance with Kazhdan and Wenzl we define the idempotents$^2$

\[ e_i := \frac{q - q_i}{q + 1}. \]

Lemma 3.4. Let $\sigma_{m,n} \in S_{m+n}$ be the permutation defined by

\[ \sigma_{m,n}(i) := \begin{cases} i + n, & \text{if } 1 \leq i \leq m; \\
                        i - m, & \text{if } m + 1 \leq i \leq m + n.
\end{cases} \]

Then

\[ g_{\sigma_{m,n}} g_i = \begin{cases} g_i + g_{\sigma_{m,n}} g_i & \text{if } 1 \leq i \leq m - 1; \\
                                      g_{i-m} g_{\sigma_{m,n}} & \text{if } m + 1 \leq i \leq m + n.
\end{cases} \]

$^2$Note that the other choice of idempotents $e_i' := \frac{q + q_i}{q + 1}$ is also used in the literature.
Explicitly
\[ g_{\sigma,m,n} = (g_n g_{n-1} \cdots g_1)(g_{n+1} g_n \cdots g_2) \cdots (g_{n+m-1} g_{n+m} \cdots g_{n-2} g_{n-1} g_n) \]
\[ = (g_n g_{n+1} \cdots g_{n+m-1})(g_{n-1} g_{n-2} \cdots g_{n+m-2})(g_{n-2} g_{n-1} g_n) \]

Proof. The explicit formulas for \( g_{\sigma,m,n} \) follow from the reduced expression of \( \sigma_{m,n} \) as stated in Lemma 3.3. To prove Lemma 3.3, suppose that \( m = 1 \) and \( i > 1 \), then
\[ (g_n \cdots g_1) g_i = g_n \cdots (g_i g_{i-1} g_i) g_{i-2} \cdots g_1 = g_n \cdots (g_{i-1} g_i g_{i-1}) g_{i-2} \cdots g_1 = g_{i-1} (g_n \cdots g_1) \]
Now for \( m > 1 \) and \( i > m \) the statement follows from the case \( m = 1 \), induction on \( m \) and the explicit formula of \( g_{\sigma,m,n} \). For \( i < m \) we use the other expression of \( g_{\sigma,m,n} \). Assume \( n = 1 \), we obtain
\[ (g_1 \cdots g_m) g_i = g_1 \cdots (g_i g_{i-1} g_i) g_{i-2} \cdots g_m = g_1 \cdots (g_{i-1} g_i g_{i-1}) g_{i-2} \cdots g_m = g_{i-1} (g_1 \cdots g_m) \]
Again the general case follows from this case, induction on \( n \) and the explicit formula of \( g_{\sigma,m,n} \).

\[ \square \]

Notation 3.5. For \( q \neq 1 \) put \([n]_q := \frac{1-q^n}{1-q} = (1+q+\ldots+q^{n-1}) \) and \([n]_1 := n\), it is called the \( q \)-analog, \( q \)-bracket or \( q \)-number. Define the \( q \)-factorial
\[ [1]_q! := 1, \quad [n]_q! := [n]_q [n-1]_q! \]

Definition 3.6. Suppose that \( q > 0 \) or \( |q| = 1 \). Define an involution on \( H_n(q) \) by \( e_i^* := e_i \) and by antilinear extension. This involution will be called the standard involution of \( H_n(q) \). From now on we will assume that for these values of \( q \) the Hecke algebra \( H_n(q) \) is equipped with this standard involution. Note that the idempotents \( e_i \) become the spectral projections corresponding to the spectral value \( -1 \) of \( g_i \). Furthermore if \( q > 0 \) the elements \( g_i \) become self-adjoint.

Lemma 3.7. Denote \( A_n := \sum_{\sigma \in S_n} g_\sigma \). If \( q^m \neq 1 \) for \( m = 1, \ldots, n \) let \( E_n := ([n]_q)!^{-1} A_n \). With this notation the following holds:

(i) \( A_n = (1 + g_{n-1} + g_{n-2} g_{n-1} + \ldots + g_{1} \cdots g_{n-1}) A_{n-1} \)
\[ = A_{n-1}(1 + g_{n-1} + g_{n-1} g_{n-2} + \ldots + g_{1} \cdots g_{1}) \]
\[ = (1 + g_1 + g_2 g_1 + \ldots + g_{n-1} \cdots g_1) \Sigma(A_{n-1}) \]
\[ = \Sigma(A_{n-1})(1 + g_1 + g_2 g_1 + \ldots + g_{n-1} \cdots g_{n-1}); \]

(ii) \( A_n g_i = g_i A_n = q A_n \), for \( i = 1, \ldots, n-1 \);

(iii) \( E_n \) is a minimal idempotent in \( H_n(q) \). If \( q \in \mathbb{R} \) or \( |q| = 1 \) and \( q^m \neq 1 \) for \( m = 1, \ldots, n \), it is a projection.

Proof. (i) From Lemma 3.3 it follows that every element in \( \pi \in S_n \) can uniquely be written as \( \pi = \sigma_j \sigma_{j+1} \cdots \sigma_{n-1} \pi' \) for some \( j \in \{1, \ldots, n \} \) and \( \pi' \in S_{n-1} \). For a permutation \( \sigma \) denote \( \sigma S_n := \{ \sigma \sigma' : \sigma' \in S_n \} \). By Lemma 3.3
\[ S_{n+1} = S_n \cup \sigma_n S_n \cup \ldots \cup (\sigma_1 \cdots \sigma_n) S_n. \]  

Hence by induction the first equality follows. Also \( \pi^{-1} = \pi'^{-1} \sigma_{n-1} \cdots \sigma_j \), therefore
\[ \sum_{\pi \in S_n} \pi = \sum_{\pi \in S_n} \pi^{-1} = \sum_{\pi' \in S_{n-1}} \pi' \left( \sum_{j=1}^{n} \sigma_{n-1} \cdots \sigma_{j+1} \sigma_j \right) \]

Now by induction the second equality in (i) follows. The third and fourth equalities can be proved similarly (one can use the map \( \sigma_i \mapsto \sigma_{i-1} \)).

Assertions (ii) and (iii) can be found in [13, §2]. But (ii) can also quickly be derived from (i) and induction and statement (iii) follows again from (ii).

\[ \square \]
Notation 3.8. Define the maps $\alpha$ and $\beta$ on the generators by
\[
\begin{align*}
\alpha &: H_n(1) \to H_n(q), \quad g_i \mapsto q - g_i; \\
\beta &: H_n(1) \to H_n(q^{-1}), \quad g_i \mapsto -q^{-1}g_i.
\end{align*}
\]

A simple computation shows that $\alpha$ and $\beta$ respect the defining relations of the Hecke algebras (cf. (3.1) - (3.3)) and thus that $\alpha$ and $\beta$ are Hecke algebra morphisms. Furthermore $\alpha \circ \alpha = id$ and $\beta \circ \beta = id$. Note also that $\alpha(e_i) = e_i' = \frac{1 + q}{q + 1}$, where $e_i'$ is the other choice of idempotents, as discussed in §.

Lemma 3.9. Suppose that $q \neq 0$. Denote $B_n := \sum_{\sigma \in S_n} (-q)^{-\ell(\sigma)}g_\sigma$. If $q^m \neq 1$ for $m = 1, \ldots, n$, let $F_n := ([n]_q^{-1})^{-1}B_n$. With this notation the following holds:

(i) $B_n = \begin{pmatrix} 1 - q^{-1}g_{n-1} + q^{-2}g_{n-2}g_{n-1} + \ldots + (-q)^{-(n-1)}g_1 \cdots g_{n-1} \\ B_{n-1} = 1 - q^{-1}g_{n-1} + q^{-2}g_{n-1}g_{n-2} + \ldots + (-q)^{-(n-1)}g_1 \cdots g_1 \\ \vdots \\ B_1 = 1 - q^{-1}g_1 + q^{-2}g_1g_2 + \ldots + (-q)^{-(n-1)}g_1 \cdots g_1 \\ B_0 = 1 \\ \end{pmatrix}B_n - 1$  

(ii) $B_ng_i = g_iB_n$, for $i = 1, \ldots, n - 1$;  

(iii) $F_n$ is a minimal idempotent in $H_n(q)$. If $q \in \mathbb{R} \setminus \{0\}$, or $|q| = 1$ and $q^m \neq 1$ for $m = 1, \ldots, n$ it is a projection;  

(iv) $\alpha(B_n) = \mu^{-n(1-n)}A_n$.

Proof. It is immediate that $\beta(A_n) = B_n$. Thus all assertions except for the last equality in item (i) and item (iv) follow from the previous lemma. To prove (i), first note that if $i = 1, \ldots, n - 2$, then $g_i^{-1}B_{n-1} = \frac{1 - q}{q}g_i^{-1}B_{n-1}$.

Hence
\[
\begin{align*}
\sum_{i=0}^{n-1} (-q)^{-(n-i)}g_1 \cdots g_{n-1} &+ \sum_{i=0}^{n-2} (-q)^{-(n-i-1)}g_1 \cdots g_{n-1} + \sum_{i=0}^{n-1} (-q)^{-(n-i-2)}g_1 \cdots g_{n-1} + \sum_{i=0}^{n-3} (-q)^{-(n-i-3)}g_1 \cdots g_{n-1}
\end{align*}
\]

Gathering all terms $g_i \cdots g_{n-1}$, the constant in front of $g_i \cdots g_{n-1}$ becomes
\[
\frac{1 - q}{q} \left( (-1)^i q^{-(n-i)} + (-q)^{-1}(-1)^i q^{-(n-i)} + \ldots + (-q)^{(i-2)} q^{-(n-i)} \right) + (-q)^{-(i-1)} q^{-(n-i)}
\]

\[
= (-1)^i (-q)^{-(n+1-i)} \cdots (-q)^{-(n-i)}
\]

\[
= (-1)^{i+1} q^{-(n-i)}.
\]

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4 Computations in \( \text{Rep}(\text{SU}_\mu(D)) \)

In the second last equality above we use the fact that we have an alternating sum. We thus obtain that (3.6) equals
\[
\begin{align*}
((−1)^{1+1}q^{−(n−1)}g_1 \cdots g_{n−1} + (−1)^{2+1}q^{−(n−2)}g_2 \cdots g_{n−1} + \ldots + (−1)^{n+1}q^{−(n−n)})B_{n−1} \\
= (−1)^{n−1}((−q)^{−(n−1)}g_1 \cdots g_{n−1} + (−q)^{−(n−2)}g_2 \cdots g_{n−1} + \ldots + 1)B_{n−1},
\end{align*}
\]

which by the first equality of item (i) in this lemma gives the desired result.

We prove (iv) by induction. The case \( n = 2 \) is easy, as
\[
\alpha(B_2) = \alpha(1 − q^{-1}g_1) = 1 − q^{-1}(q − 1 − g_1) = q^{-1}(1 + g_1) = \mu^{-2(2−1)}A_2.
\]

To prove the induction step, first note that \( \alpha(−qg_i^{−1}) = \alpha(−1 + q − g_i) = −1 + q − 1 + g_i = g_i \).

Therefore using part (ii) of this lemma, the induction hypothesis and Lemma 3.7, we get
\[
\alpha(B_{n+1}) = \alpha(q^{−n}(1 − qg_1^{−1} \cdots qg_n^{−1}))B_n
\]
\[
= q^{−n}(1 + \alpha(−qg_1^{−1}) + \ldots + \alpha((−q)^nq_1^{−1} \cdots q_n^{−1}))μ^{−n(n−1)}A_n
\]
\[
= μ^{−(n+1)n}(1 + g_1 + \ldots + g_1 \cdots g_n)A_n
\]
\[
= μ^{−(n+1)n}A_{n+1},
\]

as desired.

**Definition 3.10.** A trace \( \text{tr} \) on the Hecke algebra \( H_\infty(q) \) is a linear functional \( \text{tr}: H_\infty(q) \to \mathbb{C} \) such that \( \text{tr}(ab) = \text{tr}(ba) \) for all \( a, b \in H_\infty(q) \) and \( \text{tr}(1) = 1 \). The trace is called a Markov trace if there exists an \( \eta \in \mathbb{C} \) such that for all \( n \in \mathbb{N} \) and \( x, y \in H_n(q) \subset H_\infty(q) \), the equality \( \text{tr}(xe_ny) = \eta \text{tr}(xy) \) holds. We will refer to this identity as the Markov property. It is known that for each \( \eta \in \mathbb{C} \) there exists a Markov trace with \( \text{tr}(e_1) = \eta \), for a proof of this fact see [10, Thm. 5.1].

**Lemma 3.11.** Let \( \text{tr} \) be a Markov trace on \( H_\infty(q) \) and \( \varphi: H_\infty(q) \to \mathbb{C} \) be a functional with the Markov property such that \( \text{tr}(e_1) = \varphi(e_1) \), then \( \text{tr} = \varphi \). In particular \( \varphi \) is tracial.

**Proof.** Any element \( x \in H_n(q) \subset H_\infty(q) \) can be written as \( x = x_1 + x_2e_{n−1}x_3 \) for some \( x_1, x_2, x_3 \in H_{n−1}(q) \). Now for \( \psi = \text{tr} \) and \( \psi = \varphi \), it holds
\[
\psi(x) = \psi(x_1) + \psi(x_2e_{n−1}x_3) = \psi(x_1) + \psi(e_1)\psi(x_2x_3)
\]
and the lemma follows by induction to \( n \) and the fact that \( H_\infty(q) = \bigcup_n H_n(q) \).

4 Computations in \( \text{Rep}(\text{SU}_\mu(d)) \)

In this section we will make some computations in the category \( \text{Rep}(\text{SU}_\mu(d)) \) (for \( \mu \in (0, 1] \)) which will be needed later on. The results are analogous to [13], but in that paper a different representation of the Hecke algebra in \( \text{End}_{\text{Rep}(\text{SU}_\mu(d))} (\mathcal{H}^\otimes n) \) is used. See Remark 4.3 for a short discussion on these two different representations.

Since the representation category of a \( q \)-deformed Lie group is very similar to the representation category of the Lie group itself (cf. [10, §10.1]), it is immediate that \( \text{Rep}(\text{SU}_\mu(d)) \) is a \( SU(d) \)-type category.

**Notation 4.1.** Consider the C*-tensor category \( \text{Hilb}_\mathbb{C} \), with objects all finite dimensional Hilbert spaces and the collection of morphisms between two objects is given by all linear maps between the corresponding Hilbert spaces. Let \( \mathcal{H} := \mathbb{C}^d \in \text{Ob}(\text{Hilb}_\mathbb{C}) \) and let \( \{\psi_i\}_{i=1}^d \) be an orthonormal basis in \( \mathcal{H} \). Jimbo and Woronowicz defined the following representation of the Hecke algebra \( H_n(q) \). Let \( q := \mu^2 \). Define the map \( T \in \text{End}(\mathcal{H} \otimes \mathcal{H}) \) by
\[
T(\psi_i \otimes \psi_j) := \begin{cases}
(q − 1)\psi_i \otimes \psi_j + \mu\psi_j \otimes \psi_i, & \text{if } i < j; \\
q\psi_i \otimes \psi_j, & \text{if } i = j; \\
\mu\psi_j \otimes \psi_i, & \text{if } i > j.
\end{cases}
\]
Then a straightforward computation shows that

\[ \eta: H_n(q) \to \text{End}(H^{\otimes n}), \quad g_i \mapsto \epsilon^i \otimes T \otimes \epsilon^{n-i-1} \]

defines a representation of \( H_n(q) \). If it is necessary to keep track of \( n \) we write \( \eta_n \) for this representation. The action of the idempotents \( e_i \) corresponds to the linear map

\[
\frac{q-T}{q+1} (\psi_i \otimes \psi_j) = \begin{cases} 
\frac{1}{q+1} (\psi_i \otimes \psi_j - \mu \psi_j \otimes \psi_i), & \text{if } i < j; \\
0, & \text{if } i = j; \\
\frac{1}{q+1} (q \psi_i \otimes \psi_j - \mu \psi_j \otimes \psi_i), & \text{if } i > j.
\end{cases}
\]  

(4.1)

To define the category \( \text{Rep}(SU_d) \) we also need an embedding \( \mathbb{C} \to H^{\otimes d} \), corresponding to the morphism intertwining the trivial representation of \( SU(d) \) on \( \mathbb{C} \) with the \( d \)-th tensor power of the standard representation on \( H^{\otimes d} \). Up to a normalization the following element in \( H^{\otimes d} \) plays the role of this embedding \( \mathbb{C} \to H^d \):

\[ S := \sum_{\sigma \in S_d} (-\mu)^{-l(\sigma)} \psi_{\sigma(d)} \otimes \cdots \otimes \psi_{\sigma(1)}. \]  

(4.2)

Here \( l(\sigma) \) denotes the length of \( \sigma \), see Definition 3.2. We write \( S \) both for the element defined in \( \text{(4.2)} \) and for the map \( \mathbb{C} \to H^{\otimes d}, c \mapsto cS \). This element \( S \) can be considered as the \( q \)-deformed determinant.

The representation category \( \text{Rep}(SU_d) \) can be described as being the smallest \( C^* \)-tensor category in \( \text{Hilb} \) which contains the object \( H \) and the morphisms \( S \in \text{Hom}(C,H^{\otimes d}) \) and \( T \in \text{End}(H^{\otimes 2}) \).

Let us compute \( \|S\| \). As \( \{\psi_i\}_{i=1,\ldots,d} \) is a basis for \( H \) for \( \sigma,\sigma' \in S_d \) it follows that \( \langle \psi_{\sigma(d)} \otimes \cdots \otimes \psi_{\sigma(1)}, \psi_{\sigma'(d)} \otimes \cdots \otimes \psi_{\sigma'(1)} \rangle = \delta_{\sigma,\sigma'} \). So \( \|S\|^2 = \langle S, S \rangle = \sum_{\sigma \in S_d} (-\mu)^{-2l(\sigma)} \). By induction, \( \text{(3.5)} \) and the fact \( l(\sigma_1 \cdots \sigma_n \sigma) = l(\sigma) + n - i + 1 \) for \( \sigma \in S_n \), it follows that

\[
\sum_{\pi \in S_{n+1}} q^{l(\pi)} = (1 + q + \ldots + q^n) \sum_{\pi \in S_n} q^{l(\pi)} = [n+1]_q [n]_q = [n+1]_q!
\]

and thus \( \|S\| = |d|_q^{1/4} \).

Recall the labelling of the simple objects as introduced in Definition 2.9. The representation \( \eta \) acts as follows.

**Lemma 4.2.** For the representation \( \eta: H_n(q) \to \text{End}(H^{\otimes n}) \) the morphism \( \eta(e_1) \in \text{End}(H^{\otimes 2}) \) is the projection onto \( H_{(1)} \).

**Proof.** Using \( \text{(4.1)} \) we obtain for \( i < j \) and a constant \( a \in \mathbb{C} \)

\[
\eta(e_1)(\psi_i \otimes \psi_j + a \psi_j \otimes \psi_i) = \frac{1 - \mu a}{q+1} (\psi_i \otimes \psi_j - \mu \psi_j \otimes \psi_i);
\]

\[
\eta(e_1)(\psi_i \otimes \psi_i) = 0.
\]

In particular putting \( a = -\mu \) respectively \( a = \frac{1}{\mu} \), shows that

\[
\eta(e_1)(\psi_i \otimes \psi_j - \mu \psi_j \otimes \psi_i) = \psi_i \otimes \psi_j - \mu \psi_j \otimes \psi_i;
\]

\[
\eta(e_1)(\psi_i \otimes \psi_j + \frac{1}{\mu} \psi_j \otimes \psi_i) = 0,
\]

which means that \( \eta(e_1) \) is the orthogonal projection onto

\[ U := \text{span} \{ \psi_i \otimes \psi_j - \mu \psi_j \otimes \psi_i : 1 \leq i < j \leq d \}. \]
Since \( g_1e_1 = e_1g_1 \), we have \( \eta(g_1)U \subset U \). Thus \( U \) is a subobject of \( \mathcal{H}^{\otimes 2} \) in \( \text{Rep}(SU_\mu(d)) \). Now note that \( \mathcal{H}^{\otimes 2} = \mathcal{H}_{(1^2)} \otimes \mathcal{H}_{(2)} \). Recall that \( V_\lambda \) was defined to be the irreducible representation of \( SU(d) \) corresponding to \( \lambda \). By \( \text{[13] \S 10.1} \) the dimensions of \( \mathcal{H}_\lambda \) are the same as the dimensions of \( V_\lambda \). Therefore \( \dim(\mathcal{H}_{(1^2)}) = \frac{1}{2}d(d-1) \) and \( \dim(\mathcal{H}_{(2)}) = \frac{1}{2}d(d+1) \). Note that \( \dim(U) = \frac{1}{2}d(d-1) \), therefore \( U = \mathcal{H}_{(1^2)} \). \( \-boxed{} \)

**Remark 4.3.** Recall the Hecke algebra morphism \( \alpha \) of Notation \( \text{[3.8]} \). It is immediate that \( \eta \circ \alpha \) is also a representation of \( H_n(q) \) on \( \mathcal{H}^{\otimes n} \). This is exactly the representation which Pinzari considers in \( \text{[13] \S 4} \). Explicitly \( \eta \circ \alpha(g_i) = \varepsilon^{2i-1} \otimes T' \otimes \varepsilon^{n-i-1} \), where

\[
T'(\psi_i \otimes \psi_j) := ((q - 1)t - T)(\psi_i \otimes \psi_j) = \begin{cases} 
-\mu \psi_j \otimes \psi_i, & \text{if } i < j; \\
-\psi_i \otimes \psi_j, & \text{if } i = j; \\
(q - 1)\psi_i \otimes \psi_j - \mu \psi_j \otimes \psi_i, & \text{if } i > j.
\end{cases}
\]

For later use we prove the following identities in \( \text{Rep}(SU_\mu(d)) \).

**Proposition 4.4.** In \( \text{Rep}(SU_\mu(d)) \) the following relations hold:

\[
S = \eta(B_d)(\psi_d \otimes \cdots \otimes \psi_1); 
\]

(4.3)

\[
S^*S = [d\frac{1}{4}] \otimes \nu; 
\]

(4.4)

\[
SS^* = \eta(B_d); 
\]

(4.5)

\[
(S^* \otimes \iota)(i \otimes S) = (-\mu)^{-(d-1)}[d-1] \frac{1}{4} \nu; 
\]

(4.6)

\[
(S^* \otimes \iota)_{\otimes(d-1)}(i \otimes S) = (-\mu)^{-(d-1)}\eta(B_{d-1}); 
\]

(4.7)

\[
\eta(g_1 \cdots g_d)(S \otimes \iota) = \mu^{d+1}(i \otimes S). 
\]

(4.8)

Here \( B_n \in H_n(q) \) is as in Lemma \( \text{[3.8]} \).

**Proof.** As stated before, these identities are closely related to the identities proved by Pinzari in \( \text{[13] \S 5} \), in fact one can deduce the relations above to the identities of \( \text{[13]} \). We will do this first and then we will also show how one can compute everything directly. We denote Pinzari’s \( q \)-deformed determinant by \( \tilde{S} := \sum_{\sigma \in S_d} (-\mu)^{l(\sigma)}\psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(d)} \). Let \( r : S_d \to S_d \) be defined by \( r(\sigma)(i) := \sigma(d + 1 - i) \). Then by Lemma \( \text{[3.8]} \)

\[
l(r(\sigma)) = \# \{(i, j) : i < j, r(\sigma)(i) > r(\sigma)(j)\} 
\]

\[
= \# \{(i, j) : i < j, \sigma(d + 1 - i) > \sigma(d + 1 - j)\} 
\]

\[
= \# \{(i, j) : i < j, \sigma(i) < \sigma(j)\}
\]

and thus \( l(\sigma) + l(r(\sigma)) = d(d-1)/2 \). Therefore we obtain

\[
\tilde{S} = \sum_{\sigma \in S_d} (-\mu)^{l(\sigma)}\psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(d)} 
\]

\[
= \sum_{\sigma \in S_d} (-\mu)^{d(d-1)/2 - l(r(\sigma))} \psi_{r(\sigma)(d)} \otimes \cdots \otimes \psi_{r(\sigma)(1)} 
\]

\[
= (-\mu)^{d(d-1)/2} \sum_{\sigma \in S_d} (-\mu)^{-l(\sigma)}\psi_{\sigma(d)} \otimes \cdots \otimes \psi_{\sigma(1)} 
\]

\[
= (-\mu)^{d(d-1)/2} S.
\]

With this identity and the properties of \( \alpha \) (see Notation \( \text{[3.8]} \)), we can derive equations \( \text{[13] \S 5} \) from the results in \( \text{[13] \S 5} \). For example using \( \text{[13] \text{Lemma 5.1 b}} \) gives

\[
\eta(B_d)\psi_d \otimes \cdots \otimes \psi_1 = \mu^{-d(d-1)}(\eta \circ \alpha(A_d))\psi_d \otimes \cdots \otimes \psi_1 
\]

\[
= \mu^{-d(d-1)}(-\mu)^{d(d-1)/2} \tilde{S} 
\]

\[
= \mu^{-d(d-1)}(-\mu)^{d(d-1)/2}(-\mu)^{d(d-1)/2}S = S.
\]
Or by [13] Lemma 5.4

\[(S^* \otimes \iota)(\iota \otimes S) = (-\mu)^{-(d-1)}(\hat{S}^* \otimes \iota)(\iota \otimes \hat{S})\]
\[= (-\mu)^{-(d-1)} \mu^{d-1}[d-1]q \mu.\]

The other identities can be verified in a similar way, the details are left to the reader. To compute everything directly we start with a general identity. Suppose that \(1 \leq i_1 < i_2 < \ldots < i_n \leq d\) and \(1 \leq j \leq n-1\), then

\[\eta(g_{n \cdots g_j})(\psi_{i_n} \otimes \cdots \otimes \psi_{i_1}) = \mu^{n+1-j}(\psi_{i_n} \otimes \cdots \otimes \psi_{i_{n+2-j}} \otimes \psi_{i_{n-j}} \otimes \cdots \otimes \psi_{i_1} \otimes \psi_{i_{n+1-j}}).\quad (4.9)\]

Now suppose that \(\theta \in S_n\). From Lemma 3.3 we have the reduced expression \(\theta = (\theta^{-1})^{-1} = (C_{c_{\mu,n}} \cdots C_{c_{\mu,2}})^{-1}\), where \(c_i = i - d_{\theta^{-1}(i)}\). This gives in combination with \(4.9\) and the fact \(l(\theta) = l(\theta^{-1})\), that the following identity holds

\[\eta(g_{\theta})(\psi_{i_n} \otimes \cdots \otimes \psi_{i_1}) = \mu^{l(\theta)}(\psi_{i_{\theta^{-1}(n)}} \otimes \cdots \otimes \psi_{i_{\theta^{-1}(1)}}).\quad (4.10)\]

Suppose again \(1 \leq i_1 < i_2 < \ldots < i_n \leq d\), define \(S_{i_n, \ldots, i_1} := \sum_{\sigma \in S_n} (-\mu)^{-l(\sigma)} \psi_{i_{\sigma(n)}} \otimes \cdots \otimes \psi_{i_{\sigma(1)}}\). By \(4.10\) and Lemma 3.3 we get

\[\eta(B_n)(\psi_{i_{\theta^{-1}(n)}} \otimes \cdots \otimes \psi_{i_{\theta^{-1}(1)}}) = \mu^{-l(\theta)} \eta(B_n)\eta(g_{\theta})(\psi_{i_n} \otimes \cdots \otimes \psi_{i_1})\]
\[= (-\mu)^{-l(\theta)} \sum_{\sigma \in S_n} (-q)^{-l(\sigma)} \eta(g_{\sigma})(\psi_{i_n} \otimes \cdots \otimes \psi_{i_1})\]
\[= (-\mu)^{-l(\theta)} \sum_{\sigma \in S_n} (-\mu)^{-l(\sigma)} \psi_{i_{\sigma^{-1}(n)}} \otimes \cdots \otimes \psi_{i_{\sigma^{-1}(1)}}\]
\[= (-\mu)^{-l(\theta)} \sum_{\sigma \in S_n} (-\mu)^{-l(\sigma)} \psi_{i_{\sigma(n)}} \otimes \cdots \otimes \psi_{i_{\sigma(1)}}\]
\[= (-\mu)^{-l(\theta)} S_{i_n, \ldots, i_1}.\quad (4.11)\]

Setting \(n = d, (i_1, \ldots, i_d) = (1, \ldots d)\) and \(\theta = \text{id}\) gives \(S_{i_d, \ldots, i_1} = S\) and proves 4.3.

Equation \(4.4\) is immediate from the norm of \(S\).

Instead of proving \(4.5\), we will prove a stronger statement which we will use later in the proof of this proposition. Using the notation introduced above, we will show that

\[\sum_{d \geq i_n > \cdots > i_1 \geq 1} S_{i_n, \ldots, i_1} = \eta(B_n).\quad (4.12)\]

Suppose that \(j_1, \ldots, j_n \in \{i_1, \ldots, i_n\}\). Order the tuple \((j_n, \ldots, j_1)\) in decreasing order so we obtain \(k_n \geq \cdots \geq k_2 \geq k_1\). Then let \(p\) be minimal such that \(k_p = j_1\). Then

\[\eta(g_{p-1} \cdots g_{n-1})(\psi_{k_n} \otimes \cdots \otimes \psi_{k_1}) = \mu^{p-1} \psi_{k_n} \otimes \cdots \otimes \psi_{k_{p-1}} \otimes \psi_{k_{p-1}} \otimes \cdots \otimes \psi_{k_1} \otimes \psi_{j_1}.\]

Iterating this procedure, it follows that there exists a \(\sigma \in S_n\) and \(c \in \mathbb{R} \setminus \{0\}\) such that \(\psi_{j_n} \otimes \cdots \otimes \psi_{j_1} = c\eta(g_{\sigma})(\psi_{k_n} \otimes \cdots \otimes \psi_{k_1})\). Suppose that \(j_{l'} = j_{l''}\) for some \(l' \neq l''\), then \(k_l = k_{l+1}\) for some \(l\).

We thus have

\[\eta(B_n)(\psi_{k_n} \otimes \cdots \otimes \psi_{k_1}) = -\eta(B_n)\eta(g_{n-1})(\psi_{k_n} \otimes \cdots \otimes \psi_{k_1}) = -q\eta(B_n)(\psi_{k_n} \otimes \cdots \otimes \psi_{k_1}),\]

where the first equality follows by Lemma 3.3 and the second from the action of \(\eta(g_{n-1})\) on \((\psi_{i_n} \otimes \cdots \otimes \psi_{i_1})\). Recall \(q > 0\), so in particular \(q \neq -1\). Therefore \(\eta(B_n)(\psi_{k_n} \otimes \cdots \otimes \psi_{k_1}) = 0\).

Now

\[\eta(B_n)(\psi_{j_n} \otimes \cdots \otimes \psi_{j_1}) = c\eta(B_n)\eta(g_{\sigma})(\psi_{k_n} \otimes \cdots \otimes \psi_{k_1}) = c(-1)^{l(\sigma)}\eta(B_n)(\psi_{k_n} \otimes \cdots \otimes \psi_{k_1}) = 0.\]
Hence (4.12) holds. The choice \( l = 1 \) and (4.11) we conclude

\[
\ker \left( \sum_{i_n > \ldots > i_1} S_{i_n, \ldots, i_1} \right) \subset \text{span}(\{\psi_{\sigma(i_n)} \otimes \cdots \otimes \psi_{\sigma(i_1)} : \sigma \in S_n, \ d \geq i_n > \ldots > i_1 \geq 1\}).
\]

Now by the fact that \( l(\sigma) = l(\sigma^{-1}) \) and (4.11) we conclude

\[
\sum_{i_n > \ldots > i_1} S_{i_n, \ldots, i_1} \sum_{i_n > \ldots > i_1} \left( \psi_{\sigma(i_n)} \otimes \cdots \otimes \psi_{\sigma(i_1)} \right) = (-\mu)^{-l(\sigma)} S_{j_n, \ldots, j_1} = \eta(B_n)(\psi_{\sigma(i_n)} \otimes \cdots \otimes \psi_{\sigma(i_1)}).
\]

Since the vectors \( S_{i_n, \ldots, i_1} \) and \( S_{j_n, \ldots, j_1} \) are orthogonal if \( (i_n, \ldots, i_1) \neq (j_n, \ldots, j_1) \), it follows that \( \sum_{i_n > \ldots > i_1} S_{i_n, \ldots, i_1} \sum_{i_n > \ldots > i_1} \) and \( \eta(B_n) \) act the same on the space

\[
\text{span}(\{\psi_{\sigma(i_n)} \otimes \cdots \otimes \psi_{\sigma(i_1)} : \sigma \in S_n, \ d \geq i_n > \ldots > i_1 \geq 1\}).
\]

Hence (4.12) holds. The choice \( n = d \) and \( (i_d, \ldots, i_1) = (d, \ldots, 1) \) gives (4.13).

For the proof of (4.6) and (4.7) we introduce the following tensors

\[
S_j^{(1)} := \sum_{\sigma \in S_d, \sigma(d) = j} (-\mu)^{-l(\sigma)} \psi_{\sigma(d-1)} \otimes \cdots \otimes \psi_{\sigma(1)};
\]

\[
S_j^{(2)} := \sum_{\sigma \in S_d, \sigma(1) = j} (-\mu)^{-l(\sigma)} \psi_{\sigma(d)} \otimes \cdots \otimes \psi_{\sigma(2)}.
\]

Note that it is immediate that

\[
S = \sum_{j=1}^d S_j^{(1)} = \sum_{j=1}^d S_j^{(2)} \otimes \psi_j.
\]

For \( \sigma \in S_{d-1} \) and \( j \leq d \) define \( p(\sigma) \in S_d \) by

\[
p(\sigma)(i) := \begin{cases} j & \text{if } i = 1; \\ \sigma(i-1) & \text{if } \sigma(i-1) < j; \\ \sigma(i-1) + 1 & \text{if } \sigma(i-1) > j. \end{cases}
\]

Then \( l(p(\sigma)) = l(\sigma) + j - 1 \) and \( p: S_{d-1} \to \{\theta \in S_d : \theta(1) = j\} \) is a bijection. For the tuple \( (i_{d-1}, \ldots, i_1) := (d, \ldots, j + 1, j - 1, \ldots, 1) \) we then obtain that

\[
S_{i_{d-1}, \ldots, i_1} = \sum_{\sigma \in S_{d-1}} (-\mu)^{-l(\sigma)} \psi_{\sigma(d-1)} \otimes \cdots \otimes \psi_{\sigma(1)}
\]

\[
= \sum_{\sigma \in S_d, \sigma(1) = j} (-\mu)^{-l(\sigma) + j - 1} \psi_{\sigma(d)} \otimes \cdots \otimes \psi_{\sigma(2)}
\]

\[
= (-\mu)^{-j-1} S_j^{(2)}. \tag{4.13}
\]

Furthermore we have that the map

\[
s: \{\sigma \in S_d : \sigma(d) = j\} \to \{\sigma \in S_d : \sigma(1) = j\}; \quad s(\sigma)(i) := \begin{cases} j & \text{if } i = 1; \\ \sigma(i-1) & \text{if } i > 1, \end{cases}
\]

is a bijection and one easily checks that \( l(s(\sigma)) = l(\sigma) - (d + 1) + 2j \). It follows that

\[
S_j^{(1)} = (-\mu)^{-(d+1)+2j} S_j^{(2)}. \tag{4.14}
\]
Since $\mathcal{H}$ is an irreducible object in $\text{Rep}(SU_\mu(d))$, the morphism $(S^* \otimes \iota)(t \otimes S)$ acts as a scalar. Suppose that $\{\psi_i\}_{i=1}^d$ is an orthonormal basis with respect to the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$. We obtain

$$\langle \psi_i, (S^* \otimes \iota)(t \otimes S)\psi_j \rangle_{\mathcal{H}} = \left( \sum_{k=1}^d \psi_k \otimes S_k^{(1)} \otimes \psi_i, \sum_{k=1}^d \psi_j \otimes S_k^{(2)} \otimes \psi_k \right)_{\mathcal{H}^{d+1}} = \langle S_j^{(1)}, S_j^{(2)} \rangle_{\mathcal{H}^{d+1}}.$$  

To compute this scalar $(S^* \otimes \iota)(t \otimes S)$ it thus suffices to compute $\langle S_j^{(1)}, S_j^{(2)} \rangle$. For this we have

$$\langle S_j^{(1)}, S_j^{(2)} \rangle = (-\mu)^{d+1-2d} \langle S_j^{(1)}, S_j^{(1)} \rangle = \left( \sum_{\sigma, \theta \in S_d, \sigma(d) = \theta(d) = d} (-\mu)^{-l(\sigma)-l(\theta)} \langle \psi_{\sigma(d-1)}, \psi_{\theta(d-1)} \rangle \cdots \langle \psi_{\sigma(1)}, \psi_{\theta(1)} \rangle \right)$$

$$= (-\mu)^{d-1} \sum_{\sigma \in S_d, \sigma(d) = d} \left( -\mu \right)^{-2l(\sigma)} \sum_{j=1}^d q^{-2l(\sigma)} = (-\mu)^{-(d-1)} [d-1]!_q,$$

which establishes (4.6).

Suppose that $\xi_i \in \mathcal{H}$, then

$$(S^* \otimes \iota^{d-1})(t \otimes S)(\xi_1 \otimes \cdots \otimes \xi_{d-1}) = \sum_{i,j=1}^d (S_j^{(2)*} \otimes \psi_j^* \otimes \iota^{d-1})(\xi_1 \otimes \cdots \otimes \xi_{d-1} \otimes \psi_i \otimes S_i^{(1)})$$

$$= \sum_{j=1}^d S_j^{(2)*} (\xi_1 \otimes \cdots \otimes \xi_{d-1}) \cdot S_j^{(1)}.$$  

Thus $(S^* \otimes \iota^{d-1})(t \otimes S) = \sum_{j=1}^d S_j^{(1)} S_j^{(2)*}$, which equals $\sum_{j=1}^d (-\mu)^{-(d-1)+2j} S_j^{(2)*} S_j^{(2)}$ by (4.19). Using (4.19) this can be written as $\sum_{j=1}^d (-\mu)^{-(d-1)} S_{d-j+1,d-j-1,\ldots} S_{d-j,\ldots} S_{d-j,\ldots} \otimes \xi_{d-1} \otimes \xi_{d-1} \otimes S_{d-1}$. Now we invoke (4.12) to obtain

$$(S^* \otimes \iota^{d-1})(t \otimes S) = (-\mu)^{-(d-1)} \eta(B_{d-1}).$$  

Thus (4.7) holds.

To prove (4.8) we use (4.3) and Lemma 3.4. We obtain the following

$$\eta(g_1 \cdots g_d)(S \otimes \psi_i) = \eta(g_1 \cdots g_d)(\psi_d \otimes \cdots \otimes \psi_1 \otimes \psi_i)$$

$$= \eta(S(B_d)) \eta(g_1 \cdots g_d)(\psi_d \otimes \cdots \otimes \psi_1 \otimes \psi_i)$$

$$= \mu^{d-1} \eta(S(B_d)) (\psi_1 \otimes \psi_d \otimes \cdots \otimes \psi_1)$$

$$= \mu^{d+1} (\psi_1 \otimes S),$$

which concludes the proof of this proposition. \(\Box\)

Remark 4.5. From relations (4.6) and (4.7) it follows directly that $R := \mu^{(d-1)/2}([d-1]!_q)^{-1/2} S$ and $R := (-1)^{d+1} \mu^{(d-1)/2}([d-1]!_q)^{-1/2} S$ solve the conjugate equations for $\mathcal{H}$ in $\text{Rep}(SU_\mu(d))$.

5 Representations of Hecke algebras

Suppose that $\mathcal{C}$ is a strict $SU(d)$-type category, with fundamental object $X$. The aim of this section is to show that one can extract a constant $q$ from $\mathcal{C}$ such that there exists a representation of the Hecke algebra $H_n(q)$ into $\text{End}(X^{\otimes n})$. This section is closely related to [7] \S4. Once we established this representation, we will show that this representation essentially only depends on the constant $q$ and not on the other information of the category $\mathcal{C}$. To obtain this result the Markov traces will be used.
**Notation 5.1.** Recall that $V = \mathbb{C}^d$ is the fundamental representation of $SU(d)$, in $\text{Rep}(SU(d))$ the object $V_{[1^2]}$ is a subrepresentation of $V^{\otimes 2}$. Therefore if $C$ is a $SU(d)$-type category, there exists a projection $a \in \text{End}(V^{\otimes 2})$ and a morphism $v \in \text{Hom}(X_{[1^2]}, X^{\otimes 2})$ such that $v^*v = id_{X_{[1^2]}}$ and $vv^* = a$. We say that $a$ is the projection of $X^{\otimes 2}$ onto $X_{[1^2]}$. Define the elements $a_k := \gamma^{\otimes k-1} a \in \text{End}(X^{\otimes k+1})$. If $k < n$ we also write $a_k$ for the element $\gamma^{k-1} a \otimes a, \gamma^{n-k-1} \in \text{End}(X^{\otimes n})$.

Denote by $\Sigma$ the map $\Sigma(a_i) := a_{i+1}$.

**Lemma 5.2.** Let $a \in \text{End}(X^{\otimes 2})$ be the projection onto $X_{[1^2]} \subset X^{\otimes 2}$. Put $a_1 := a \otimes 1$ and $a_2 := 1 \otimes a$. Then there exists a constant $\gamma \in (0, 1]$ such that

$$a_1 a_2 a_1 - \gamma a_1 = a_2 a_1 a_2 - \gamma a_2.$$  

(5.1)

**Proof.** This is a slightly stronger statement than what it is proved in [7] Prop. 4.2] this is due to the fact that $a$ is a projection and not only an idempotent, we will follow the proof by Kazhdan and Wenzl. By the fusion rules of $SU(d)$ we have

$$X^{\otimes 3} \cong \begin{cases} X_{[2,1]} \oplus X_{[2,1]} \oplus X_{[3]} & \text{if } d = 2; \\ X_{[1^3]} \oplus X_{[2,1]} \oplus X_{[2,1]} \oplus X_{[3]} & \text{if } d \geq 3. \end{cases}$$

Therefore

$$\text{End}(X^{\otimes 3}) \cong \begin{cases} M_2(\mathbb{C}) \oplus \mathbb{C} & \text{if } d = 2; \\ \mathbb{C} \oplus M_2(\mathbb{C}) \oplus \mathbb{C} & \text{if } d \geq 3. \end{cases}$$  

(5.2)

We now only consider the case $d \geq 3$, the case $d = 2$ is similar. By the fusion rules of $SU(d)$ it follows that $X_{[1^3]}$ is a subobject of $X_{[1^2]} \otimes X$, so there exists a projection $p \in \text{End}(X_{[1^2]} \otimes X)$ and a morphism $v \in \text{Hom}(X_{[1^3]}, X_{[1^2]} \otimes X)$ such that $v^*v = id_{X_{[1^3]}}$ and $vv^* = p$. Similarly there exists $w \in \text{Hom}(X_{[1^2]}, X^{\otimes 2})$ such that $w^*w = id_{X_{[1^2]}}$ and $ww^* = a$. Then

$$a_1 |_{X_{[1^3]}} = v^* (w^* \otimes \iota) a_1 (w \otimes \iota) v = v^* (w^* \otimes \iota)(ww^* \otimes \iota)(w \otimes \iota) v = v^* id_{X_{[1^2]}} \otimes \iota v = id_{X_{[1^3]}}.$$  

So $a_1$ acts on $X_{[1^3]}$ as the identity. Similarly $a_2 |_{X_{[1^3]}} = id_{X_{[1^3]}}$ Using this terminology of subobjects, $X_{[3]}$ is not a subobject of $X_{[1^2]} \otimes X$ and $X \otimes X_{[1^2]}$, which implies $a_1 |_{X_{[3]}} = a_2 |_{X_{[3]}} = 0$. We have $\text{dim}(\text{Hom}(X_{[2,1]}, X_{[1^2]} \otimes X)) = \text{dim}(\text{Hom}(X_{[2,1]}, X \otimes X_{[1^2]})) = 1$, thus in $\text{End}(X_{[2,1]} \otimes X_{[1^2]})$ the morphisms $a_i$ act as rank 1 projections. So using the isomorphism [5.2] there exist rank 1 projections $f_i \in M_2(\mathbb{C})$ such that the projection $a_i \in \text{End}(X^{\otimes 3})$ corresponds to $(1, f_i, 0) \in \mathbb{C} \oplus M_2(\mathbb{C}) \oplus \mathbb{C}$. Since $\text{ran}(f_1 f_2 f_1) \subset \text{ran}(f_1)$ there exists a $\gamma_1 \in \mathbb{C}$ such that $f_1 f_2 f_1 = \gamma_1 f_1$. Similarly there exists $\gamma_2 \in \mathbb{C}$ such that $f_2 f_1 f_2 = \gamma_2 f_2$.

$$\gamma_1 f_1 f_2 = f_1 (f_2 f_1 f_2) = (f_1 f_2 f_1) f_2 = \gamma_2 f_2 f_1.$$  

(5.3)

Hence either $\gamma_1 = \gamma_2$ or $f_1 f_2 = 0$ in the latter case we can set $\gamma_1 = \gamma_2 = 0$. Put $\gamma_C := \gamma_1$. Because $f_i$ are projections and thus positive, it must hold that $\gamma_C \in [0, 1]$. Since $a_i$ corresponds to $(1, f_i, 0)$, [5.3] gives [5.1]. It remains to show that $\gamma \neq 0$, this is non-trivial and makes use of certain projections on objects in $X^{\otimes d}$, the proof can be found in [7] Prop. 4.2.

**Notation 5.3.** Put $\gamma_C$ to be the constant obtained from $C$ as in the previous lemma. Pick $q_C$ such that $\gamma_C = \frac{q_C}{(1+q_C^2)}$, i.e. such that $q_C + q_C^{-1} = \gamma_C^{-1} - 2$. From this it is clear that $q_C$ is uniquely determined up to $q_C \leftrightarrow q_C^{-1}$. Therefore to fix a unique $q_C$ we select $q_C \in \{ z \in \mathbb{C} : |z| \leq 1, \text{Im}(z) \geq 0 \} \cup \{ z \in \mathbb{C} : |z| < 1, \text{Im}(z) < 0 \}$. If it is clear which category $C$ is considered we will omit the subscript $C$ in $q_C$ and $\gamma_C$.

**Remark 5.4.** At this point it is not clear why $q_C$ is indeed an invariant of the category. A priori it might be dependent on the choice of $X$. However this constant is indeed independent, we will say more about this issue later (cf. Remark [7.3]).
Lemma 5.5. For a $SU(d)$-type category $\mathcal{C}$ we have $q_C \in (0,1] \cup \{e^{i\alpha} : 0 < \alpha < \frac{2\pi}{3}\}$.

Proof. The function $(0,1] \rightarrow [2,\infty)$, $q \mapsto q + q^{-1}$ is a bijection, so for $\gamma \in (0,1/4)$ it holds $q \in (0,1]$. If $\gamma \in (\frac{1}{4},1]$, then write $\gamma = \frac{1}{4} \cos^{-2}(\alpha/2)$ for a unique $\alpha \in (0,\frac{2\pi}{3}]$. We have

$$\gamma = (e^{i\alpha/2} + e^{-i\alpha/2})^{-2} = \frac{e^{i\alpha}}{(1 + e^{i\alpha})^2},$$

which implies that $q = e^{i\alpha}$.

Corollary 5.6. The map

$$H_n(q_C) \rightarrow \text{End}(X^{\otimes n}), \quad e_i \mapsto a_i$$

extends to a $*$-representation of the Hecke algebra $H_n(q_C)$.

Proof. Since $g_i = q - (q + 1)e_i$, in the Hecke algebra $H_n(q)$ the relations (3.1), (3.2) and (3.3) can equivalently be described in terms of the idempotents $e_i$ by

$$e_ie_j = e_j e_i, \quad \text{if } |i - j| \geq 2; \quad (5.5)$$

$$e_i e_{i+1} e_i - \frac{q}{(1 + q)^2} e_i e_{i+1} e_i e_{i-1} - \frac{q}{(1 + q)^2} e_{i+1}, \quad \text{for } i = 1, \ldots, n - 2; \quad (5.6)$$

$$e_i^2 = e_i, \quad \text{for } i = 1, \ldots, n - 1. \quad (5.7)$$

From this characterization, the fact that $a$ is a projection satisfying (5.1) and the choice of $q$ it is immediate that the map (5.4) extends to a representation of $H_n(q)$. Since $e_i$ is self-adjoint in $H_n(q)$ and $a_i$ is self-adjoint in $\text{End}(X^{\otimes n})$ the map is $*$-preserving.

Lemma 5.7. If $q_C = e^{i\alpha}$ for some $0 < \alpha < \pi$, then $q_C$ is a root of unity.

Proof. We can write $q = e^{2\pi i/3}$ for some $0 < \beta < \frac{1}{2}$. A representation of $H_n(q)$ into a C*-algebra is a C*-representation if the idempotents $e_i$ are mapped to projections. Such a representation is called trivial if it is a direct sum of representations $\pi_1$ and $\pi_0$ where $\pi_1 : e_i \mapsto id$ for all $i$ and $\pi_0 : e_i \mapsto 0$ for all $i$. If $q$ is not a root of unity, then there exists an $m \in \mathbb{N} \setminus \{0\}$ such that $m - 1 < \frac{1}{2} < m$. Now [15] Prop. 2.9 implies that there exist no non-trivial C*-representations of $H_n(q)$ for $n > ((m + 1)/2)^2$. However by Corollary 5.6 for each $n$ we do have a non-trivial C*-representation. Hence $q$ must be a root of unity.

Definition 5.8. Suppose that $\mathcal{C}$ is a strict $SU(d)$-type category with fundamental object $X$.

Consider for $m \leq n$ the map

$$i_{m,n} : \text{End}(X^{\otimes m}) \rightarrow \text{End}(X^{\otimes n}), \quad T \mapsto T \otimes i^{\otimes (n-m)}.$$ 

Clearly if $k \leq m \leq n$, then $i_{m,n} i_{k,m} = i_{k,n}$. Thus the algebraic inductive limit of $(\text{End}(X^{\otimes n}), i_{m,n})$ exists, denote this limit by $M_C$. The representations $\theta_n : H_n(q_C) \rightarrow \text{End}(X^{\otimes n})$ obtained from Corollary 5.6 satisfy $i_{m,n} \circ \theta_m(x) = \theta_n \circ i_{m,n}(x)$ for all $m, n$ and $x \in H_m(q_C)$. Thus the collection $\{\theta_n\}_n$ extends to a representation of the inductive limits $\theta_C : H_\infty(q_C) \rightarrow M_C$. We denote $\theta_C(x) = \theta_n(x) = i_{m,n}(\theta_m(x)) \in \text{End}(X^{\otimes m})$ for $x \in H_n(q) \subset H_\infty(q)$. Again we will write just $\theta$ if no confusion is possible.

Proposition 5.9. Let $R : 1 \rightarrow \overline{X} \otimes X$, $\overline{R} : 1 \rightarrow X \otimes \overline{X}$ be a standard solution of the conjugate equations. The categorical trace $\text{Tr}_C$ on $\mathcal{C}$ induces a Markov trace $\text{tr}_C$ on $H_\infty(q)$ via

$$\text{tr}_C(x) := \|R\|^{-2n} \text{Tr}_{X^{\otimes n}}(\theta_C(x)), \quad (x \in H_n(q) \subset H_\infty(q)). \quad (5.8)$$
Hence the Markov trace satisfies the Kazhdan-Wenzl condition, independent of the category $C$ in the following sense.

**Theorem 5.10 (Kazhdan–Wenzl).** If $C$ is a strict $\text{SU}(d)$-type category, then $q_C \in (0,1]$ and the Markov trace satisfies $\text{tr}_C(g_1) = \frac{q_C^d}{|\partial_C|^d}$. Therefore the kernel of the representation $\theta_C : H_n(q_C) \to \text{End}(X^\otimes n)$ depends only on $q_C$. Furthermore $\text{tr}_C(H_n(q_C)) = \text{End}(X^\otimes n)$.

*Proof.* Since $||\theta(x^*x)|| = ||\theta(x)||^2$, it holds that $\theta(x) = 0$ if and only if $\theta(x^*x) = 0$. Because the categorical trace $\text{Tr}_X^{\otimes n}$ is faithful, we obtain that

$$\ker(\theta : H_n(q) \to \text{End}(X^\otimes n)) = \{x \in H_n(q) : \text{tr}_C(x^*x) = 0\}.$$ 

To characterize the kernel of $\theta$ by Proposition 5.9 and Lemma 3.11 it suffices to show that $\text{tr}_C(g_1)$ can be computed in terms of $q$. This is non-trivial and has been done by Kazhdan and Wenzl (see [7] Thm. 4.1]), here they also prove surjectivity of $\theta$. The idea of their proof is to decompose $H_n(q)/I_n^\mu \cong \bigoplus M_i(C)$ as a direct sum of matrix algebras $M_i(C)$. Here $I_n^\mu := \{x \in H_n(q) : \text{tr}_\mu(xy) = 0 \text{ for all } y \in H_n(q)\}$ and $tr_\mu$ is the unique Markov trace such that $tr_\mu(g_1) = \mu$. The values $\mu = \frac{q_C^d}{|\partial_C|^d}$ for $m \in \mathbb{N}$ play a special role (see [7] Prop. 3.1]). These matrix blocks $M_i(C)$ are related to Young diagrams and thus to representations of $\text{SU}(n)$. This allows to compare the dimensions of $H_n(q)/I_n^\mu$ and $\text{End}(X^\otimes n)$. From these dimensions one can deduce that $\text{tr}(g_1) = \frac{q_C^d}{|\partial_C|^d}$ and that $q$ cannot be a non-trivial root of unity. Therefore by Lemmas 5.5 and 5.7 it follows that $q \in (0,1]$. Furthermore one can show that $I_n^{tr\theta(g_1)} = \ker(\theta : H_n(q) \to \text{End}(X^\otimes n))$ and by using another dimension argument one has $H_n(q)/\ker(\theta) \cong \text{End}(X^\otimes n)$, thus $\theta$ must be surjective.

**Remark 5.11.** Combining the above theorem, Remark 5.3 and 13 Prop. 4.1 it follows that for $n > d$ the kernel $\ker(\theta_C : H_n(q) \to \text{End}(X^\otimes n))$ equals the ideal generated by the element $B_{d+1} \in H_n(q)$. 

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6 Categories generated by Hecke algebras

In this section we will give a number of technical requirements on C*-tensor categories which allow us to prove that a category which satisfies these assumptions is in fact unitarily monoidally equivalent to a twist of \(\text{Rep}(SU_\mu(d))\). In the next section we will use this result to show that all \(SU(d)\)-type categories are equivalent to \(\text{Rep}(SU_\mu(d))\). Furthermore we will show that in a special case these categories admit a braiding.

**Assumption 6.1.** Assume \(C\) is a strict C*-tensor category generated by an object \(X\) which satisfies the following requirements:

(i) there exists a constant \(q_C \in (0, 1]\) and a projection \(a \in \text{End}(X^\otimes 2)\) such that

\[
(a \otimes \iota)(\iota \otimes a)(a \otimes \iota) - \frac{q_C}{(1 + q_C)} = (\iota \otimes a)(a \otimes \iota)(a \otimes \iota) - \frac{q_C}{(1 + q_C)^2}.
\]

This requirement defines a representation \(\theta_n: H_n(q_C) \to \text{End}(X^\otimes n), c_i \mapsto \iota^\otimes i \otimes a \otimes \iota^\otimes n - i - 1\).

(ii) \(\theta_n: H_n(q_C) \to \text{End}(X^\otimes n)\) is surjective;

(iii) \(\ker(\theta_n): H_n(q_C) \to \text{End}(X^\otimes n)\) = \(\ker(\eta_n): H_n(q_C) \to \text{End}(H^\otimes n)\), here \(\eta_n\) is as in Notation 4.1

(iv) there exists an integer \(d_C \geq 2\) and a morphism \(\nu \in \text{Hom}(1, X^\otimes d_C)\) such that \(\nu^\otimes \nu = \iota\) and \(\nu^{\otimes n} = \theta(F_{dC})\);

(v) there exists a \(d_C\)-th root of unity \(\omega_C\) such that \(\theta(g_{dC} \cdots g_1)(\iota \otimes \nu) = \omega_C q^{(d_C + 1)/2}(\nu \otimes \iota);\)

(vi) \(\text{Hom}(X^\otimes m, X^\otimes n) = \{0\}\), if \(m \neq n\) (mod \(d_C\)).

We let \(\mu_C \in (0, 1]\), \(\mu_C := q^{1/2}_C\). If it is clear which category is considered, the subscript \(C\) will be dropped.

**Remark 6.2.** The results in Proposition 4.4 show that \(\text{Rep}(SU_\mu(d))\) satisfies the conditions (i) and (iii)-(vi) of the assumption above. The fact that the representation \(\eta: H_n(q) \to \text{End}(H^\otimes n)\) is surjective follows from Theorem 5.10. So \(\text{Rep}(SU_\mu(d))\) satisfies Assumption 6.1.

**Notation 6.3.** If \(\mathcal{C}\) is strict, then \(\mathcal{C}^\rho\) (see Definition 2.11) is in general not strict. We define \(\theta_n(g_i) \in \text{End}_{\mathcal{C}^\rho}(X^\otimes n)\) to be the composition

\[
X^\otimes n \xrightarrow{\alpha} X^\otimes -1 \otimes (X^\otimes 2 \otimes X^\otimes n - i - 1) \xrightarrow{\beta} X^\otimes i - 1 \otimes (X^\otimes 2 \otimes X^\otimes n - i - 1) \xrightarrow{\alpha^{-1}} X^\otimes n.
\]

Here \(\alpha\) is the appropriate associativity morphism in \(\mathcal{C}^\rho\) and \(\beta := \iota^\otimes i - 1 \otimes (\theta_2(g_1) \otimes \iota^\otimes n - i - 1)\).

As shown in the next proposition the constant \(\omega\) behaves nicely with respect to twisting the associativity morphisms of a category \(\mathcal{C}\). This proposition will be of importance, because in some cases it implies that we can restrict ourselves to the case \(\omega = 1\).

**Proposition 6.4.** Suppose that \(\mathcal{C}\) satisfies the requirements of Assumption 6.1 and \(\rho\) is a root of unity of order \(d_C\), then in \(\mathcal{C}^\rho\) the equality \((\nu^\otimes \iota)\theta(g_{d_C}) \cdots \theta(g_1)(\iota \otimes \nu) = \rho^{-1} \omega_C \mu^{d_C + 1}_C\) holds. In particular if \(\overline{\mathcal{C}}\) is the strictification of \(\mathcal{C}^{\omega_C}\), then in \(\overline{\mathcal{C}}\) it holds that \((\nu^\otimes \iota)\theta(g_{d_C}) \cdots \theta(g_1)(\iota \otimes \nu) = \mu^{d_C + 1}_C\).

**Proof.** Since in general \(\mathcal{C}^\rho\) is not strict, consider \((\nu^\otimes \iota)\theta(g_{d_C}) \cdots \theta(g_1)(\iota \otimes \nu)\) which equals the composition

\[
X = X \otimes \iota \xrightarrow{\iota^\otimes \nu} X \otimes X^\otimes d \xrightarrow{\alpha^\rho_1} X^\otimes 2 \otimes X^\otimes d - 1 \xrightarrow{\theta(g_1) \otimes \iota^\otimes d - 1} X^\otimes 2 \otimes X^\otimes d - 1 \xrightarrow{\alpha^{-1}_2} X^\otimes d \otimes X \xrightarrow{\nu^\otimes \iota} \text{Id} \otimes X = X.
\]
Here \( \alpha^n \) are the associativity morphisms in \( C \). The composition of these morphisms \( \alpha^n \circ \cdots \circ \alpha^2 \circ \alpha^1 \) equals the associativity morphism \( \alpha^n : X \otimes X^{\otimes d} \rightarrow X^{\otimes d} \otimes X \), which by Lemma 4.17 acts as multiplication by \( \rho^{-1}(\frac{d}{2}) = \rho^{-1} \). In \( C \) the associativity morphisms are trivial. Thus if we replace \( \alpha^n \) by the associativity morphisms \( \alpha \) of \( C \), in \( C \) the composition \( \mu_{C}^{d+1} \omega_{C} \) by requirement (v) of Assumption 6.1. Hence in \( C \) the morphism \( \delta \) acts as \( \rho^{-1} \mu_{C}^{d+1} \omega_{C} \), as desired.

**Notation 6.5.** Let \( \delta_C := (\omega_{C} \mu_{C}^{d+1})^{-1} \frac{\partial}{\partial t} \). Denote

\[
T_{m,n} := \delta_{C}^{m,n} \omega_{C}(g_{\sigma_{m,n}}) \in \text{End}(X^{\otimes m+n}).
\]

Observe the crucial property \( T_{1,dc} = (\omega_{C} \mu_{C}^{d+1})^{-1} \theta(g_{dc} \cdots g_{1}) \), which implies that \( T_{1,dc}(\iota \otimes \nu) = \nu \otimes \iota \).

The following proposition is similar to [7 Prop. 2.2 a)].

**Proposition 6.6.** Suppose that \( C \) satisfies Assumption 6.1 and \( \omega_{C} = \pm 1 \), then the collection of morphisms \( \{ T_{m,n} \}_{m,n \in \mathbb{N}} \) defines a braiding on the category \( C \). Explicitly,

\[
T_{k,m+n} = (\iota^{m} \otimes T_{k,m})T_{k,m} \otimes \iota^{m};
\]

\[
T_{k+m,n} = (T_{k,n} \otimes \iota^{m})T_{k,m} \otimes \iota^{m};
\]

\[
(\beta \otimes \alpha)T_{k,m} = T_{1,n}(\alpha \otimes \beta),
\]

for all \( \alpha \in \text{Hom}(X^{\otimes k}, X^{\otimes l}), \beta \in \text{Hom}(X^{\otimes m}, X^{\otimes n}) \). (6.4)

Note that the case \( \omega = -1 \) can only occur when \( d \) is even, because \( \omega \) is a \( d \)-th root of unity.

**Proof.** From the explicit formulas in Lemma 3.4 we obtain the identities

\[
\Sigma^{m}(g_{\sigma_{m,n}})g_{\sigma_{k,m+n}} = g_{\sigma_{k,m+n}},
\]

\[
g_{\sigma_{k,m+n}} \Sigma^{k}(g_{\sigma_{m,n}}) = g_{\sigma_{k+m,n}},
\]

from which (6.2) and (6.3) immediately follow. Denote the morphism \( \nu_{m,n} := \iota^{m} \otimes \nu \otimes \iota^{n} \in \text{Hom}(X^{\otimes m+n}, X^{\otimes m+d+n}) \). The collection \( \{ T_{m,n} \}_{m,n} \) satisfies the following relations

\[
(T_{m,d} \otimes \iota^{n})\nu_{m,n} = \nu_{0,m+n};
\]

\[
(T_{d,m} \otimes \iota^{n})\nu_{0,m+n} = \nu_{m,n};
\]

\[
\nu_{m,n}^{*}(T_{d,m} \otimes \iota^{n}) = \nu_{0,m+n}^{*};
\]

\[
\nu_{0,m+n}^{*}(T_{m,d} \otimes \iota^{n}) = \nu_{m,n}^{*}.
\]

Indeed, the case \( m = 1 \) of (6.5) follows immediately from

\[
(\nu^{*} \otimes \iota)^{\Sigma_{m}(g_{\sigma_{m,n}})}(g_{d} \cdots g_{1})(\nu \otimes \iota) = \omega^{d+1} \iota
\]

and the definition of \( T_{1,d} \). The case \( m > 1 \) can be proved using induction and (6.3). For (6.6) observe that taking the adjoint of (6.9) gives

\[
(\nu^{*} \otimes \iota)^{\Sigma_{m}(g_{\sigma_{m,n}})}(g_{d} \cdots g_{1})(\iota \otimes \nu) = \omega^{d+1} \iota
\]

Here it is crucial that \( \omega = \pm 1 \), otherwise we would have the factor \( \overline{\omega} \). From this equation (6.6) follows for \( m = 1 \) and the general case can again be proved by induction. The identities (6.7) and (6.8) follow from respectively (6.5) and (6.6) by taking conjugates. Again the requirement \( \omega = \pm 1 \) is implicitly used.

By assumption on \( C \) the map \( \theta: H_{n}(q) \rightarrow \text{End}(X^{\otimes n}) \) is surjective. Combination with Lemma 5.4 gives immediately that for all \( \alpha \in \text{End}(X^{\otimes m}) \) and \( \beta \in \text{End}(X^{\otimes n}) \)

\[
T_{m,n}^{*}(\alpha \otimes \beta) = (\beta \otimes \alpha)T_{m,n}.
\]

(6.10)

It remains to show that (6.4) also holds for \( \alpha \in \text{Hom}(X^{\otimes k}, X^{\otimes l}) \) and \( \beta \in \text{Hom}(X^{\otimes m}, X^{\otimes n}) \). We may assume that \( k = l + pd \) and \( m = n + qd \) for some \( p, q \in \mathbb{Z} \). We will proceed by induction
on \( p \) and \( q \). The basis case \( p = q = 0 \) is exactly (6.10). So first suppose that \( p \geq 1, q = 0 \), \( \alpha \in \text{Hom}(X \otimes^k, X \otimes^l) \) and \( \beta \in \text{Hom}(X \otimes^m, X \otimes^n) \). Then \((\nu \otimes \alpha) \in \text{Hom}(X \otimes^k, X \otimes^{l+d})\). Using the induction hypothesis, (6.5) and (6.6) we have

\[
\nu_{m,l}(\beta \otimes \alpha)T_{k,m} = (\beta \otimes \nu \otimes \alpha)T_{k,m} = T_{l+d,m}(\nu \otimes \alpha \otimes \beta)
\]

\[
= (T_{d,m} \otimes \iota^{\otimes l})(\nu \otimes \alpha \otimes \beta) = (T_{d,m} \otimes \iota^{\otimes l})(T_{0,l+d,m})T_{l,m}(\alpha \otimes \beta)
\]

\[
= \nu_{m,l}T_{l,m}(\alpha \otimes \beta).
\]

Since the map \( \text{Hom}(X \otimes^r, X \otimes^{u+v}) \to \text{Hom}(X \otimes^r, X \otimes^{u+d+v}) \), \( \gamma \mapsto \nu_{u,v} \circ \gamma \) is injective, it follows by induction that \((\beta \otimes \alpha)T_{k,m} = T_{l,m}(\alpha \otimes \beta)\). Now suppose that \( p < 0 \), then \((\nu^* \otimes \alpha) \in \text{Hom}(X \otimes^{k+d}, X \otimes^l)\) with a similar argument as above involving the relations (6.2) and (6.7) one can show that

\[
(\beta \otimes \alpha)T_{k,m} \nu_{0,k+m}^* = T_{l,n}(\alpha \otimes \beta) \nu_{0,k+m}^*.
\]

Injectivity of the map \( \text{Hom}(X \otimes^{u+v}, X \otimes^s) \to \text{Hom}(X \otimes^{u+d+v}, X \otimes^s) \), \( \gamma \mapsto \gamma \circ \nu_{u,v} \) closes the induction on \( p \). Induction on \( q \) is similar and thus (6.3) holds. \( \Box \)

**Theorem 6.7.** Suppose that \( \mathcal{C} \) satisfies the requirements of Assumptions (6.1). Then \( \mathcal{C} \) is unitarily monoidally equivalent to \( \text{Rep}(SU\mu(d)) \).

This theorem uses the ideas of monoidal algebras as described by Kazhdan and Wenzl in [7, §2]. The proof of this theorem is very similar to the proof of [7, Proposition 2.2 b)] and therefore the computational details will be omitted.

**Proof.** From Proposition 6.4 and Remark 2.15 it follows that it suffices to consider the case \( \omega_C = 1 \). The idea of the proof of this theorem is to extend the isomorphisms \( \text{End}(X^{\otimes n}) \to \text{End}(\mathcal{H}^{\otimes n}) \) to \( \text{Hom}(X^{\otimes k}, X^{\otimes l}) \to \text{Hom}(\mathcal{H}^{\otimes k}, \mathcal{H}^{\otimes l}) \) by embedding \( \text{Hom}(X^{\otimes k}, X^{\otimes l}) \) into \( \text{End}(\mathcal{H}^{\otimes p}) \) for some large \( p \in \mathbb{N} \) using the maps \( \alpha \mapsto \alpha \otimes \nu \) and \( \alpha \mapsto \alpha \otimes \nu^* \). For this, suppose that \( k, l, m, n, p \in \mathbb{N} \) such that \( p = m + kd = n + ld \). We will define some subspaces and maps for \( \mathcal{C} \). Note that these constructions can of course also be performed in \( \text{Rep}(SU\mu(d)) \). Define the map

\[
H^{m,n}_p : \text{Hom}(X \otimes^m, X \otimes^n) \to \text{End}(X \otimes^p),
\]

\[
\alpha \mapsto (\nu^* \otimes \iota^{\otimes n}) \alpha((\nu^*)^{\otimes k} \otimes ^{\otimes m}) = \nu^{\otimes l} \otimes (\nu^*)^{\otimes k} \otimes \alpha.
\]

Then clearly \( H^{m,n}_p \) is linear. Define the subspace \( \Sigma^{m,n}_p \subset \text{End}(X \otimes^p) \) to be

\[
\Sigma^{m,n}_p := \{ \beta \in \text{End}(X \otimes^p) : ((\nu^* \otimes \iota^{\otimes n}) \beta = \beta((\nu^*)^{\otimes k} \otimes \iota^{\otimes m}) = \beta \}.
\]

The proof of the following lemmas is omitted, because it is very similar to [7, §2], the only additional requirement one has to check is compatibility of the *-structure, but this follows directly from the definitions.

**Lemma 6.8.** \( H^{m,n}_p \) is an isomorphism of \( \text{Hom}(X \otimes^m, X \otimes^n) \) onto \( \Sigma^{m,n}_p \). Furthermore for \( \alpha \in \text{Hom}(X \otimes^m, X \otimes^n) \) and \( \beta \in \text{Hom}(X \otimes^n, X \otimes^r) \) the following identities hold

\[
H^{m,n}_p(\alpha)^* = H^{n,m}_p(\alpha^*), \quad H^{n,r}_p(\beta) \circ H^{m,n}_p(\alpha) = H^{m,r}_p(\beta \circ \alpha).
\]

For each \( p \), let \( \psi_p : \text{End}(X \otimes^p) \to \text{End}(\mathcal{H} \otimes^p) \) be a *-isomorphism making the diagram

\[
\begin{array}{ccc}
H_p(q) & \xrightarrow{\theta_p} & \text{End}(X \otimes^p) \\
\downarrow{\eta_p} & & \downarrow{\psi_p} \\
\text{End}(\mathcal{H} \otimes^p)
\end{array}
\]

commute. Such an isomorphism exists, because by assumption and Theorem 6.10 \( \theta_p : H_p(q) \to \text{End}(X \otimes^p) \) and \( \eta_p : H_p(q) \to \text{End}(\mathcal{H} \otimes^p) \) are surjective and \( \ker(\theta) = \ker(\eta) \). Let us write \( \kappa := \)
unitary tensor functor. Taking the completions of \( \tilde{\alpha} \) (up to natural unitary isomorphism) to a unitary tensor functor \( C \) of subobjects gives us the categories

\[
\text{Rep}(SU_d) \quad \text{and} \quad \text{Rep}(\mu(d))
\]

\( \psi \) is again fully faithful and essentially surjective. So \( D \) and \( C \) are unitarily monoidally equivalent, in other words \( C \) is unitarily monoidally equivalent to \( \text{Rep}(SU_d) \).

\( \tau \) is given by six identities which basically state that if a category satisfies those requirements, there is a unitary tensor functor. It will be shown that all \( SU(d) \)-type categories can be classified by a pair \((g, \omega)\) where \( g \in \{0, 1\} \) and \( \omega \) is a \( d \)-th root of unity. The requirement for existence an embedding is given by six identities which basically state that if a category satisfies those requirements, there exist a representation of the Hecke algebra, and the twist and solutions of the conjugate equations can be explicitly computed. The proofs of both theorems consist of showing that in both cases the Assumptions \( A_1 \) are satisfied allowing to apply Theorem \( 6.7 \).

\[ \|S\|^{-1}S, \text{ where } S: 1 \to \mathcal{H}^{\otimes d} \text{ is the intertwiner defined in } (4.2). \]

Because \( \nu\nu^* = \theta(F_d) \) and \( \kappa\kappa^* = \|S\|^{-2}SS^* = \eta(F_d) \), we have \( \psi_d(\nu\nu^*) = \|S\|^{-2}SS^* = \kappa\kappa^* \). Define for \( m \equiv n \) (mod \( d \)) the map \( \psi_{m,n} \) which is the composition

\[
\text{Hom}_C(X^{\otimes m}, X^{\otimes n}) \xrightarrow{H^{\otimes m,n}_{p,c}} \sum_{p,c} \psi_{p,c} \xrightarrow{\sum_{p,c} \psi_{p,c}} \text{Hom}_{\text{Rep}(SU_d)}(H^{\otimes m,n}, H^{\otimes m,n}).
\]

**Lemma 6.9.** The morphisms \( \psi_{m,n} \) are well-defined (independent of \( p \)) isomorphisms of linear spaces and satisfy

\[
\psi_{m,n}(\alpha^*) = \psi_{n,m}(\alpha)^*, \quad \psi_{n,r}(\beta) \circ \psi_{m,n}(\alpha) = \psi_{m,r}(\beta \circ \alpha).
\]

With these isomorphisms \( (\psi_{m,n})_{m,n} \) we are able to define a unitary tensor functor from \( C \) to \( \text{Rep}(SU_d) \). For this consider the full subcategory \( \tilde{C} \) of \( C \) with objects \( \text{Ob}(\tilde{C}) := \{X^{\otimes n} : n \in \mathbb{N}\} \) and \( D \) the full subcategory of \( \text{Rep}(SU_d) \) with objects \( \text{Ob}(D) := \{H^{\otimes n} : n \in \mathbb{N}\} \). Then the completion of \( \tilde{C} \) and \( D \) with respect to direct sums and subobjects equal respectively \( C \) and \( \text{Rep}(SU_d) \). Define \( \tilde{F}: \tilde{C} \to D \) by \( X^{\otimes n} \mapsto H^{\otimes n} \) on objects, \( \tilde{F}(\alpha) := \psi_{m,n}(\alpha) \) for morphisms \( \alpha \in \text{Hom}(X^{\otimes m}, X^{\otimes n}) \) and \( \tilde{F}_0 = id, \tilde{F}_2 = id. \tilde{F}(\alpha) \) is well-defined, because by assumption \( m \equiv n \) (mod \( d \)) if \( \alpha \neq 0 \).

**Lemma 6.10.** \( \tilde{F} \) is a unitary tensor functor. Clearly \( \tilde{F} \) is essentially surjective. Note that Lemmas \( 6.9 \) and \( 6.10 \) imply that \( \tilde{F} \) is a fully faithful unitary tensor functor. Taking the completions of \( \tilde{C} \) and \( D \) with respect to direct sums and subobjects gives us the categories \( C \) and \( \text{Rep}(SU_d) \). Under this completion \( \tilde{F} \) extends uniquely (up to natural unitary isomorphism) to a unitary tensor functor \( F: C \to \text{Rep}(SU_d) \). Then \( F \) is again fully faithful and essentially surjective. So \( F \) is a unitary monoidal equivalence, in other words \( C \) is unitarily monoidally equivalent to \( \text{Rep}(SU_d) \). \( \square \)

## 7 Two characterizations of \( SU(d) \)-type categories

The aim of this section is to prove the main results of this paper, namely to characterise all \( SU(d) \)-type categories and to give a condition when it is possible to embed \( \text{Rep}(SU_d) \) in a given \( C^* \)-tensor category. It will be shown that all \( SU(d) \)-type categories can be classified by a pair \((g, \omega)\) where \( g \in \{0, 1\} \) and \( \omega \) is a \( d \)-th root of unity. The requirement for existence an embedding is given by six identities which basically state that if a category satisfies those requirements, there exist a representation of the Hecke algebra, and the twist and solutions of the conjugate equations can be explicitly computed. The proofs of both theorems consist of showing that in both cases the Assumptions \( 6.1 \) are satisfied allowing to apply Theorem \( 6.7 \).

**Definition 7.1.** Let \( C \) be a strict \( SU(d) \)-type category. Since in \( \text{Rep}(SU(d)) \) the trivial representation \( \mathbb{T} \) is a subrepresentation of \( \rho^{\otimes d} \), there exist a morphism \( \nu: 1 \hookrightarrow \rho^{\otimes d} \), such that \( \nu
\nu^* \in \text{End}(\rho^{\otimes d}) \) is a projection. We define the **twist** \( \tau_C \) of \( C \) to be the number by which one multiplies in the following composition

\[
X = H^{\otimes d} \xrightarrow{\theta(g_1 \cdots g_1)} H^{\otimes d} \otimes X \xrightarrow{\nu^* \otimes 1} X \otimes X = X.
\]

Note that since \( X \) is simple, we obtain a scalar. Also \( \tau_C \) is clearly independent of the choice of \( \nu \). Again, a priori it is not clear why \( \tau_C \) is independent of the choice of \( X \). Fortunately this is the case as we will show later (cf. Remark 7.9).

**Lemma 7.2.** The following holds: \( \theta(g_1 \cdots g_1)(\nu \otimes 1) = \tau_C(\nu \otimes 1). \)

\(^3\text{Note that this twist differs a factor } (-1)^d \text{ from the twist defined in 4.}\)
Proof. Note that $\nu \nu^* \in \text{End}(X^d)$. From Theorem 5.10 we obtain that there exists an $x \in H_d(q)$ such that $\nu \nu^* = \theta(x)$. By Lemma 3.4 it therefore follows that $\theta(g_d \cdots g_1)(\nu \otimes \nu^*) = (\nu \nu^* \otimes \iota)\theta(g_d \cdots g_1)$. Since $\nu^* \nu = \iota$ we have

$$\theta(g_d \cdots g_1)(\nu \otimes \nu) = \theta(g_d \cdots g_1)(\nu \otimes \nu^*)(\nu \otimes \nu) = (\nu \nu^* \otimes \iota)\theta(g_d \cdots g_1)(\nu \otimes \nu)$$

and the result follows.

Observe that identity 4.18 of Proposition 4.3 implies that the twist of $\text{Rep}(SU_d(d))$ equals $\mu^d+1$.

**Notation 7.3.** Since for a strict $SU(d)$-type category $C$ the constant $q_C \in (0, 1]$, define $\mu_C \in (0, 1]$ to be the positive square root of $q_C$.

**Lemma 7.4.** Let $C$ be a strict $SU(d)$-type category. For $n \leq d$, the morphism $\theta(F_n) \in \text{End}(X^\otimes n)$ is the projection corresponding to the inclusion $X^\otimes n \subset X_{1^n}$. Here $X_{1^n} := 1$, to express the fact that there exists a non-zero morphism $\nu : 1 \to X^d$.

**Proof.** We proceed by induction. The case $n = 2$ is trivial. Suppose that $2 \leq n \leq d - 1$ and the result holds for $n$. Let $p_k \in \text{End}(X^\otimes k)$ be the projection corresponding to $X_{1^k} \subset X^\otimes k$. To prove the induction step we must show that $p_{n+1} = \theta(F_{n+1})$. By the fusion rules of $SU(d)$ we have $X_{1^n} \otimes X \cong X_{1^{n+1}} \oplus X_{21^{n-1}}$ and $X \otimes X_{1^n} X \cong X_{1^{n+1}} \oplus X_{21^{n-1}}$. So either $(p_n \otimes \iota)(\nu \otimes p_n) = p_{n+1}$ or $(p_n \otimes \iota) = (\nu \otimes p_n)$. Let us argue by contradiction and assume that the second case holds. Let for $i = 0, \ldots, n$, $r_i := \iota^{\otimes i} \otimes p_n \otimes \nu^{\otimes i} \in \text{End}(X^{\otimes 2n})$. From the assumption it follows that $r_i = r_{i+1}$ for all $i$ and therefore we have $r_0 = r_n$. In particular $r_0(1 - r_n) = r_0(1 - r_0) = 0$. On the other hand $r_0(1 - r_n)$ cannot be zero, because e.g., the non-zero object $X_{1^n} \otimes X_{1^n}$ lies in the range of $r_0(1 - r_n)$, which yields a contradiction. We conclude that $(p_n \otimes \iota)(\nu \otimes p_n) = p_{n+1}$.

By Lemma 3.9 we have $F_{n+1}F_n = F_{n+1} \Sigma(F_n) = F_{n+1}$, and thus by the induction hypothesis $\theta(F_{n+1})p_{n+1} = \theta(F_{n+1})(p_n \otimes \iota)(\nu \otimes p_n) = \theta(F_{n+1}F_n \Sigma(F_n)) = \theta(F_{n+1})$. Since $X_{1^n}$ is simple, $p_{n+1}$ is a minimal projection. By the previous calculation $\theta(F_{n+1})$ is a subprojection of $p_{n+1}$. To show that $\theta(F_{n+1})$ equals $p_{n+1}$ it thus suffices to show that $\theta(F_{n+1}) \neq 0$. For this we compute

$$(\iota \otimes \text{tr}_C)(F_{n+1}) = [n+1]^{-1}_{q^d} \left(1 + \frac{-1}{d} q^d \right) \frac{[1]}{[1]} [n]_q^{-1} \theta(F_n)$$

here we used Lemma 3.9. Since $\frac{[1]}{[1]} = \frac{1}{q}$ and $n < d$ it follows that

$$1 - \frac{q^d}{[d]_q} \frac{[n]_q}{[1]_q} = 1 - \frac{q^d}{[d]_q} \frac{[n]_q}{q^n} \neq 0.$$ 

By the induction hypothesis $\text{tr}_C(F_n) \neq 0$, thus $\text{tr}_C(F_{n+1}) \neq 0$ and hence $\theta(F_{n+1}) \neq 0$.

**Corollary 7.5.** Suppose that $C$ is a strict $SU(d)$-type category, then there exists a $d$-th root of unity $\omega_C$ such that $\tau_C = \omega_C \mu_C^{d+1}$.

**Proof.** First note that Lemma 3.4 implies that $g_{\sigma_k,d} = g_{\sigma_k-1,d} \Sigma^{k-1}(g_{\sigma_1,d})$. Combination with the identity $\theta(g_d \cdots g_1)(\nu \otimes \iota) = \tau_C(\nu \otimes \iota)$ gives

$$\theta(g_{\sigma_k,d})(\iota \otimes \nu) = \tau_C \theta(g_{\sigma_k-1,d})(\iota \otimes \nu \otimes \iota).$$

By induction we obtain for all $k \in \mathbb{N}$

$$\theta(g_{\sigma_k,d})(\iota \otimes \nu) = \tau_C^k(\nu \otimes \iota \otimes \kappa),$$
Thus in particular
\[ \theta(g_{a,d})(\nu \otimes \delta) = \tau^d_C(\nu \otimes \delta). \]
Multiplying both sides by \((\nu^* \otimes \delta^d)\) gives
\[ (\nu^* \otimes \delta^d)\theta(g_{a,d})(\nu \otimes \delta) = \tau^d_C(\nu \otimes \delta). \]
Combination with the above lemma gives that as a morphism in \(\text{End}(X^{\otimes 2d})\) we have
\[ (\theta(F_d) \otimes \theta(F_d))(\theta(g_{a,d})(\nu \otimes \delta)) = \tau^d_C(\theta(F_d) \otimes \theta(F_d)). \]
Remark 7.9. Theorem 7.7 shows that the representations \(\theta\) and \(\eta\) are equivalent. In particular this implies that \(\tau^d_C = \pi^d_C(\text{Rep}(SU_{\mu}(d))) = (\mu^{d+1})^d\), which proves the corollary.

\[ \square \]

Remark 7.6. In [12] Prop. 5.2 it is asserted that \(\tau_c = (-1)^d\omega\) for some \(d\)-th root of unity \(\omega\). This is not true as for example the explicit calculation for \(SU_{\mu}(d)\) shows (cf. (1.8)). The mistake in the proof, is that it is claimed that \(\theta(g_{a,d})\) acts as \((-1)^d\) on the object \(X_{\{a\}} \otimes X_{\{a\}}\).

Now all the technical work has been done to give a classification of \(SU(d)\)-type categories.

Theorem 7.7. If \(\mathcal{C}\) is a \((d)\)-type category with fundamental object \(X\). Then \((\text{Rep}(SU_{\mu}(d)))_{\mathcal{C}}\) is unitarily monoidal equivalent to \(\mathcal{C}\). Furthermore \(\mathcal{C}\) admits a braiding if \(\omega_c = \pm 1\).

Proof. By Corollary 5.6 we have a representation of the Hecke algebra \(H_n(q_c) \rightarrow \text{End}_\mathcal{C}(X^{\otimes n})\). By Theorem 6.10 this representation is surjective and depends only on \(q_c\). As the representation \(\eta: H_2(q_c) \rightarrow \text{End}_\text{Rep}(SU_{\mu}(d))((\mathcal{H})^{\otimes 2})\) satisfies \(\eta(\epsilon_1)\) is the projection onto \(H_{11}\) (cf. Lemma 2.12), we obtain that \(q_{\text{Rep}(SU_{\mu}(d))} = q_c\). Then again by Theorem 5.10 \(\ker(\eta) = \ker(\theta)\). Lemmas 2.13, 7.2, 7.4 and Corollary 7.5 show that the other requirements of Assumption 6.1 are satisfied.

Now Proposition 6.9 and Theorem 6.7 give the result.

\[ \square \]

Remark 7.8. It can be shown [12] Rem. 4.4 that in general a \((d)\)-type category is not braided; such a category \(\mathcal{C}\) admits a braiding if and only if \(\omega_c = \pm 1\).

Remark 7.9. Now we can also prove why the constants \(q_c\) and \(\tau_c\) are independent of the chosen generator \(X\) of the category \(\mathcal{C}\). By [10] all automorphisms of \(\text{Rep}(SU(d))\) are in 1-1 correspondence with symmetries of the Dynkin diagram of \(SU(d)\). This diagram, consisting of \(d - 1\) nodes \(\{1, 2, \ldots, d - 1\}\) where the nodes \(i\) and \(i + 1\) are connected by a single edge, has exactly two symmetries, namely the identity and the map given on the nodes by \(i \mapsto d - i\). So we only have to show that \(q_c\) and \(\tau_c\) are invariant under this second, non-trivial, map. This map induces an automorphism of \(U_{\mu}(SU(d))\), the quantum enveloping Hopf algebra of \(SU(d)\), given on the generators by \(E_i \mapsto E_{d-i}\), \(F_i \mapsto F_{d-i}\), \(K_i^\pm \mapsto K_{d-i}^\mp\). In \(\text{Rep}(SU_{\mu}(d))\) it thus maps every object to a conjugate object. Therefore it is sufficient to show that if we would have chosen \(\overline{X}\) instead of \(X\) generating object, the resulting constants \(q_c\) and \(\tau_c\) are the same. This is implicitly proved in [12] §4.2. The idea is the following, suppose that in \(\mathcal{C}\) the associativity morphisms are given by a cocycle \(\varphi \in H^3(\mathbb{Z}/d\mathbb{Z}, T)\), thus \(\alpha: (X^{\otimes d} \otimes X^{\otimes c}) \rightarrow X^{\otimes a} \otimes (X^{\otimes b} \otimes X^{\otimes c})\) acts as multiplication by \(\varphi(a, b, c)\) (in this case \(\varphi\) is of the form \(\varphi(a, b, c) = \omega_c^{\frac{a-b+c}{d}}\)). We write \(\text{Rep}(SU_{\mu}(d))^\varphi\) for the category \(\text{Rep}(SU_{\mu}(d))\) with these new associativity morphisms. Then \(\mathcal{C} \cong \text{Rep}(SU_{\mu}(d))^\varphi\). The map \(X \mapsto \overline{X}\) then corresponds to changing the cocycle \(\varphi\) to the new one given by \(\psi(a, b, c) := \varphi(-a, -b, -c)\). Then one obtains an isomorphism \(\theta: \text{Rep}(SU_{\mu}(d))^\varphi \rightarrow \mathcal{C} \rightarrow \text{Rep}(SU_{\mu}(d))^\psi\). The question is now whether this isomorphism acts trivially on \(H^3(\mathbb{Z}/d\mathbb{Z}, T)\). This is indeed the case, since \(\varphi = df, \psi = \partial g\), where \(f(a, b) = \omega_c^{\frac{-a+b}{d}}\) and \(g(a, b) = \omega_c^{\frac{a-b}{d}}\) are maps \(f, g: \mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} \rightarrow T\). Now a direct computation shows that \(fg^{-1}\) factors through \(\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}\) and thus \(\varphi\) and \(\psi\) are equivalent cocycles. Thus \(\theta\) acts trivially and \(\omega_c\) and \(q_c\) are invariant under \(X \mapsto \overline{X}\).

Another (more straightforward) method of proving that those constants are invariant is by explicitly computing everything. This can be done in the following way. We adopt the notation as in [11] §2.2 and denote \(F: \mathcal{C} \rightarrow \mathcal{C}\) for the contravariant tensor functor
\[
\text{Ob}\mathcal{C} \rightarrow \text{Ob}\mathcal{C}, \quad U \mapsto \overline{U}; \quad \text{Hom}(U, V) \rightarrow \text{Hom}(\overline{V}, \overline{U}), \quad T \mapsto T^\vee,
\]
where \( T^\nu := (\iota \otimes R_\nu)(\iota \otimes T \otimes \iota)(R_\nu \otimes \iota) \). Define \( F_2(U,V) : \nabla \otimes \nabla \to \widetilde{U} \otimes \nabla \) by the identity
\[
(F_2(U,V) \otimes \iota \otimes \iota)(\iota \otimes R_U \otimes \iota)R_V = R_U \otimes V.
\]
Put \( \alpha^\nu := F_2(X,X)\alpha^\nu F_2(X,X) \). Then it can be checked that the \( \alpha^\nu \) satisfy the relations of \( \epsilon_i \) in the Hecke algebra \( H_n(q_C) \) and thus we get a representation \( \theta^\nu : H_n(q) \to \text{End}_{C}(\mathbb{X}^\otimes n) \). Hence \( q_C \) is invariant. Now for \( \tau_C \) we define
\[
\nu^\nu := (F_2(X,X) \otimes \iota \otimes \iota)(F_2(X^\otimes d-2,X) \otimes \iota)(\nu^\nu) : \mathbb{1} \to \mathbb{X}^\otimes d.
\]
Then one can verify that \( \nu^\nu \) plays the role of \( \nu \) and
\[
(\nu^\nu \otimes \iota)\theta(g_d \cdots g_1)(\iota \otimes \nu^\nu) = \tau_C \iota,
\]
whence \( \tau_C \) is invariant under the transformation \( X \mapsto \mathbb{X} \).

**Remark 7.10.** The above theorem says that all \( SU(d) \)-type categories can be described by a pair \((q,\omega)\), where \( q \in \{0,1\} \) and \( \omega \) is a \( d \)-th root of unity. Namely we have shown that a \( SU(d) \)-type category \( C \) is isomorphic to \((\text{Rep}(SU(d))_\omega))^\circ \). Now one might wonder if each pair \((q,\omega)\) of this form can be realised by a compact quantum group. This is indeed the case, see \cite{12}.

Inspired by \cite{13} we have the following condition for the existence of an embedding of a twist of \( \text{Rep}(SU_\mu(d)) \) in a \( C^* \)-tensor category \( D \). We use the notation as introduced in Notation \ref{not2.1}.

**Theorem 7.11.** Suppose that \( D \) is a strict \( C^* \)-tensor category such that there exists an object \( X \in \text{Ob}(D) \), morphisms \( \nu \in \text{Hom}(1,X^\otimes d) \), \( a \in \text{End}(X^\otimes 2) \), a constant \( \mu \in \{0,1\} \) and a \( d \)-th root of unity \( \omega \) satisfying the following properties:
\[
a = a^* = a^2;
\]
\[
(a \otimes \iota)(a \otimes \iota) - \frac{q}{(1+q)^2}(a \otimes \iota) = (\iota \otimes a)(a \otimes \iota) - \frac{q}{(1+q)^2}(\iota \otimes a);
\]
\[
\nu^* \nu = \iota;
\]
\[
\nu \nu^* = \theta(F_d);
\]
\[
(\nu^* \otimes \iota)\iota \nu = \omega(-\mu)^{-(d-1)[d\frac{1}{q}]^1} \iota;
\]
\[
\theta(g_d \cdots g_1)(\iota \otimes \nu) = \omega^{d+1} (\nu \otimes \iota).
\]

Here \( q := \mu^2 \) and \( \theta : H_n(q) \to \text{End}_{D}(X^\otimes n) \) is the representation of the Hecke algebra as in Corollary \ref{cor2.6}. Let \( C \) be the sub \( C^* \)-tensor category of \( D \) generated by the object \( X \) and morphisms \( \nu \) and \( a \). Then \( C \) is a \( SU(d) \)-type category and there exists a unique (up to natural unitary isomorphism) unitary tensor functor \( F : (\text{Rep}(SU_\mu(d)))^\circ \to D \) such that \( F(H) = X \) and \( F(S) = (\{d\}_{q})^{1/2} \nu, F(T) = q - (q + 1)a \).

**Proof.** First note that equations \ref{7.1} and \ref{7.2} together with Corollary \ref{cor2.6} imply that we have a \( \star \)-representation \( \theta : H_n(q) \to \text{End}_{D}(X^\otimes n) \). Therefore the identities \ref{7.4} and \ref{7.6} make sense. We would like to use Theorem \ref{thm6.7} for this we only need to check three conditions: equality of the kernels of \( \theta \) and \( \eta \), surjectivity of the representation \( \theta : H_n(q) \to \text{End}_{C}(X^\otimes n) \) and \( \text{Hom}(X^\otimes m, X^\otimes n) = \{0\} \) if \( m \neq n \ (\text{mod} \ d) \). Let us start with the easiest one: the last one.

For this note that \( \text{Hom}_{C}(X^\otimes m, X^\otimes n) \) is generated by \( a \) and \( \nu \). So if \( a \in \text{Hom}_{C}(X^\otimes m, X^\otimes n) \), then \( \alpha = 0 \), or \( \alpha \) is a linear combination of words consisting of the letters \( i^{\otimes k} \otimes \nu \otimes i^{\otimes l}, i^{\otimes k} \otimes \nu^* \otimes i^{\otimes l} \) and \( \theta(x) \) for \( k,l \in N \) and \( x \in H_{\infty}(q) \). It is sufficient to consider individual words. If \( x \in H_p(q) \subset H_{\infty}(q) \), then \( \theta(x) \in \text{End}(X^\otimes p) \) and for \( k,l \in N \) we have \( i^{\otimes k} \otimes \nu \otimes i^{\otimes l} \in \text{Hom}(X^{k+l}, X^{k+l+i}), i^{\otimes k} \otimes \nu^* \otimes i^{\otimes l} \in \text{Hom}(X^{k+l+d}, X^{k+l+i}) \). Induction on the length of a word gives the result.

To be able to prove the other two remaining requirements we first compute \((\nu^* \otimes i^{\otimes k})(i^{\otimes k} \otimes \nu)\). In the upcoming computations we need the identity
\[
\theta(g_i)\nu = \theta(g_i)\nu(\nu^* \nu) = \theta(g_i)\nu = -\theta(F_d)\nu = -(\nu \nu^*)\nu = -\nu, \quad \text{for } i = 1, \ldots, d-1,
\]
which follows from \ref{eq6.3}, \ref{eq6.4} and Lemma \ref{lem6.9} This also implies that \( \nu^* \theta(g_i) = -\nu^* \).

[25]
Lemma 7.12. For \( k = 1, 2, \ldots, d - 1 \) the following equality holds
\[
(\nu^* \otimes \iota^\otimes k)(\iota^\otimes k \otimes \nu) = \omega^k \frac{[d - k + 1]^1_\frac{1}{d} [k - 1]^1_\frac{1}{d}}{[d]^1_\frac{1}{d}} (-\mu)^{-k(d-k)}\theta(F_k). \tag{7.8}
\]

Proof. We prove this by induction. The case \( k = 1 \) is exactly assumption (7.2) of the theorem, so we will prove the induction step. Consider the morphism \( T := (\nu^* \otimes \iota^\otimes k)(\iota^\otimes k \otimes \nu \otimes \iota)(\iota^\otimes k \otimes \nu) \). By the induction hypothesis and the assumption of this theorem, this morphism equals
\[
T = \omega^{k-1} \frac{[d - k + 1]^1_\frac{1}{d} [k - 1]^1_\frac{1}{d}}{[d]^1_\frac{1}{d}} (-\mu)^{-k(d-k+1)} \omega(-\mu)^{-d-1}(d-k)(\theta(F_{k-1}) \otimes \iota)
\]
\[
= \omega^k \frac{[d - k + 1]^1_\frac{1}{d} [k - 1]^1_\frac{1}{d}}{[d]^1_\frac{1}{d}} \quad (\text{7.9})
\]
On the other hand, as \( \nu^* = \theta(F_d), \theta(g_1) = -\nu \) and \( \nu^* \theta(g_1) = -\nu^* \) we have
\[
T = (\nu^* \otimes \iota^\otimes k)(\iota^\otimes k \otimes \theta(F_d) \otimes \iota)(\iota^\otimes k \otimes \nu)
\]
\[
= \omega^k \frac{[d - k + 1]^1_\frac{1}{d} [k - 1]^1_\frac{1}{d}}{[d]^1_\frac{1}{d}} \omega(-\mu)^{-d-1}(d-k)(\theta(F_{k-1}) \otimes \iota)
\]
\[
= \omega^k \frac{[d - k + 1]^1_\frac{1}{d} [k - 1]^1_\frac{1}{d}}{[d]^1_\frac{1}{d}} \quad (\text{7.10})
\]
Now note that by the assumptions and induction hypothesis
\[
(\nu^* \otimes \iota^\otimes k)(\theta(g_{d+1} \cdots g_k) \otimes \nu) = (\nu^* \otimes \iota^\otimes k)(\theta(g_{d+1} \cdots g_k) \otimes \nu)
\]
\[
= \theta(g_1 \cdots g_{d-1}) \otimes (\nu^* \otimes \iota^\otimes k) \otimes \omega^d \mu^d \omega^k \otimes (\theta(F_{d+1} \otimes \iota)
\]
\[
= \omega^d \mu^d \omega^k \frac{[d - k + 1]^1_\frac{1}{d} [k - 1]^1_\frac{1}{d}}{[d]^1_\frac{1}{d}} \quad (\text{7.11})
\]
Since
\[
(-\mu)^{-d}(\theta(F_{d+1}) \otimes \iota) = (-\mu)(\theta(F_{d+1}) \otimes \iota),
\]
identity (7.10) equals
\[
T = \omega^k \frac{[d - k + 1]^1_\frac{1}{d} [k - 1]^1_\frac{1}{d}}{[d]^1_\frac{1}{d}} \omega(-\mu)^{-d-1}(d-k)(\theta(F_{k-1}) \otimes \iota).
\]
If we now combine both expressions of \( T \), (7.9) and (7.11), we get
\[
[\frac{d^1_\frac{1}{d}}{d}]^1_\frac{1}{d} \omega^k \frac{[d - k + 1]^1_\frac{1}{d} [k - 1]^1_\frac{1}{d}}{[d]^1_\frac{1}{d}} \theta(g_1 \cdots g_{k-1} \cdots -q)^{-d+1}(-q)^{-d+1}(\theta(F_{k-1}) \otimes \iota).
\]
which by Lemma (3.3) equals
\[
(-1)^{k-1}(-1)^{d-k+1}(-1)^{d-k} \mu^k \frac{[d - k + 1]^1_\frac{1}{d} [k - 1]^1_\frac{1}{d}}{[d]^1_\frac{1}{d}} \theta(F_k).
\]
From this (7.8) follows immediately and the lemma is proved.
Lemma 7.13. The representation \( \theta \) satisfies \( \ker(\theta: H_n(q) \to \text{End}(X^\otimes n)) = \ker(\eta: H_n(q) \to \text{End}(H^\otimes n)) \).

Proof. From the above lemma it follows in particular that

\[
(\nu^* \otimes \iota^{\otimes d-1})(\iota^{\otimes d-1} \otimes \nu) = \overline{\omega}(-\mu)^{-(d-1)}[d]^{-1}_q \theta(F_{d-1})
\]

and thus that the morphisms

\[
R := \overline{\omega}(-1)^{d-1} [d]^{1/2}_q \mu^{(d-1)/2} \nu, \quad \overline{R} := [d]^{1/2}_q \mu^{(d-1)/2} \nu
\]

satisfy the conjugate equations for \( X \). Define a map

\[
\varphi^{(n)} : \text{End}(X^\otimes n) \to \text{End}(X^{\otimes n-1}), \quad \alpha \mapsto (\iota^{\otimes n-1} \otimes \nu^*)(\alpha \otimes \iota^{\otimes d-1})(\iota^{\otimes n-1} \otimes \nu)
\]

and the functional \( \varphi_n := \varphi^{(1)} \circ \cdots \circ \varphi^{(n-1)} \circ \varphi^{(n)} \). Now let \( (R', \overline{R}) \) be a standard solution of the conjugate equations of \( X \). The map

\[
\text{End}(X^\otimes n) \to \text{End}(X^\otimes n-1), \quad \alpha \mapsto (\iota^{\otimes n-1} \otimes \overline{R}^*)(\alpha \otimes \iota^{\otimes d-1})(\iota^{\otimes n-1} \otimes \overline{R})
\]

is a partial trace induced by a standard solution, so it is tracial and faithful. There exists an invertible morphism \( T \in \text{Hom}(X', X) \) such that \( R = (T^{-1} \otimes \iota)R' \) and \( \overline{R} = (\iota \otimes T')\overline{R'} \) [11, Prop. 2.2.4]. From this it is immediate that \( \varphi^{(n)} \) and thus \( \varphi_n \) are also faithful. Using the involution, equation (7.5) can be rewritten as

\[
(\iota \otimes \nu^*)(\nu \otimes \iota) = \overline{\omega}(-\mu)^{-(d-1)}[d]^{-1}_q \iota.
\]

Combination with (7.6) and (7.7) gives that

\[
\varphi^{(2)} \circ \theta(g_1) = (\iota \otimes \nu^*)\theta(g_1)(\iota \otimes \nu) = (1)^{d-1}(\iota \otimes \nu^*)\theta(g_d \cdots g_1)(\iota \otimes \nu) = (1)^{d-1}\omega \mu^{d+1} (\iota \otimes \nu^*)(\nu \otimes \iota) = (1)^{d-1} \omega \mu^{d+1} \overline{\omega}(-\mu)^{-(d-1)}[d]^{-1}_q \iota = q[d]^{-1}_q \iota = \frac{q^d}{[d]_q} \iota.
\]

Thus in particular \( \varphi^{(2)} \circ \theta(g_1) \) is a scalar in \( \text{End}(X) \) and thus \( \varphi^{(2)} \circ \theta(e_1) \) is a scalar as well. Therefore if \( x, y \in H_{n-1}(q) \)

\[
\varphi^{(n)}(\theta(xe_{n-1}y)) = (\iota^{\otimes n-1} \otimes \nu^*)\theta(x)(\iota^{\otimes n-2} \otimes \theta(e_1))\theta(y)(\iota^{\otimes n-1} \otimes \nu) = \theta(x)(\iota^{\otimes n-1} \otimes \nu^*)(\iota^{\otimes n-2} \otimes \theta(e_1))(\iota^{\otimes n-1} \otimes \nu)\theta(y) = \varphi_2(\theta(e_1)) \cdot \theta(xy).
\]

So \( \varphi_n \circ \theta \) defines a faithful functional with the Markov property on \( H_n(q) \). According to Lemma 8.11 this functional must be tracial and hence we obtain a Markov trace \( \text{tr}_C := \varphi_n \circ \theta \colon H_n(q) \to C \). Markov traces are characterized by their value on the generator \( g_1 \). Recall from Theorem 7.10 that \( \text{tr}_{\text{Rep}(SU_d(n))}(g_1) = \frac{q^d}{[d]_q} \). It follows that \( \text{tr}_C = \text{tr}_{\text{Rep}(SU_d(n))} \) and thus \( \ker(\theta) = \ker(\text{tr}_C) = \ker(\text{tr}_{\text{Rep}(SU_d(n))}) = \ker(\eta) \).

Now surjectivity of the map \( \theta: H_n(q) \to \text{End}(X^\otimes n) \). For this we need the following lemma. Recall the notation \( \nu_{k,l} := \iota^{\otimes k} \otimes \nu \otimes \iota^{\otimes l} \) and \( \nu^*_{k,l} := \iota^{\otimes k} \otimes \nu^* \otimes \iota^{\otimes l} \).

Lemma 7.14. Let \( x \in H_p(q) \) and \( k, l, m, n, k', l', m', n' \in \mathbb{N} \) be natural numbers satisfying the equality \( k + l + d = m + n + d = k' + l' = m' + n' = p \), then there exist \( x_1, x_2 \in H_\infty(q) \) such that

\[
\nu^*_{k,l} \theta(x) \nu^*_{m,n} = \theta(x_1), \quad \nu_{k',l'} \theta(x) \nu^*_{m',n'} = \theta(x_2).
\]
Proof. First we prove the second assertion. We write $k,l,m,n$ instead of $k',l',m',n'$. Note that by $(7.6)$ there exist $y_1 \in H_{k+d}(q)$ and $y_2 \in H_{m+d}(q)$ such that
\[ \nu_{k,l} = \theta(y_1)\nu_{0,k+l}, \quad \nu_{m,n} = \nu_{0,m+n}\theta(y_2). \] (7.12)
Then by $(\ref{eq:7.6})$,
\[ \nu_{k,l}\theta(x)\nu_{m,n} = \theta(y_1)\nu_{0,k+l}\theta(x)\nu_{0,m+n}\theta(y_2) = \theta(y_1)(\iota^{\otimes d} \otimes \theta(x))(\nu\nu^* \otimes \iota_{k+l})\theta(y_2) \]
\[ = \theta(y_1\Sigma^d(x)F_d y_2), \]
where we still use $\Sigma$ to denote the shift map.

Now the first case. Similar to $(7.12)$ there exist $y_1$ and $y_2$ such that
\[ \nu_{k,l}^*\theta(x)\nu_{m,n} = \nu_{0,k+l}\theta(y_1)\theta(x)\theta(y_2)\nu_{0,m+n}. \]
So we can assume to deal with the case $\nu_{0,k}^*\theta(x)\nu_{0,k}$ and $x \in H_{l+k}(q)$. Now observe that $\nu_{0,k}^*\theta(x)\nu_{0,k} = \nu_{0,k}^*\theta(F_d x F_d)\nu_{0,k}$. By surjectivity of the representation $\eta: H_k(q) \rightarrow \text{End}(H^k)$ there exists an $y \in H_k(q)$ such that $(S^* \otimes \iota^{\otimes k})\eta(F_d x F_d)(S \otimes \iota^{\otimes k}) = \eta(y)$, here $S$ and $\eta$ are as in Notation 4.1. This implies that
\[ \eta(F_d\Sigma^d(y)) = SS^* \otimes \eta(y) = (S \otimes \iota^{\otimes k})\eta(y)(S^* \otimes \iota^{\otimes k}) = (SS^* \otimes \iota^{\otimes k})\eta(F_d x F_d)(SS^* \otimes \iota^{\otimes k}) = \eta(F_d x F_d). \]

Because by Lemma 7.13 the representations $\eta$ and $\theta$ have the same kernel, it follows that $\theta(F_d x F_d) = \theta(F_d\Sigma^d(y))$. Combining all this gives
\[ \nu_{0,k}^*\theta(x)\nu_{0,k} = \nu_{0,k}^*\theta(F_d x F_d)\nu_{0,k} = \nu_{0,k}^*\theta(F_d\Sigma^d(y))\nu_{0,k} = \theta(y)\nu_{0,k}^*\nu_{k,0}\nu_{k,0}^*\nu_{0,k} = \theta(y) \]
and concludes the lemma. \(\Box\)

To prove that the representation $\theta: H_n(q) \rightarrow \text{End}_C(X^\otimes n)$ is surjective let $\alpha \in \text{End}_C(X^\otimes n)$. Then $\alpha$ is a linear combination of words consisting of the letters $\theta(x)$ for $x \in H_\infty(q)$ and $\nu_{k,l}, \nu_{m,n}$. Let $\beta = \beta_1 \cdots \beta_r$ be such a word and $\beta_i$ the letters. Then $\beta \in \text{End}_C(X^\otimes n)$ and thus
\[ \#\{i : \beta_i = \nu_{k,l} \text{ some } k,l\} = \#\{i : \beta_i = \nu_{k,l}^* \text{ some } k,l\}. \]

We now apply induction on $r$. If $r = 1$, then the above sums must be empty and thus $\beta = \theta(x)$ for some $x \in H_n(q)$. Suppose $r > 1$ and not all $\beta_i$ are of the form $\theta(x)$ for $x \in H_\infty(q)$, then there must exist $1 \leq i < j \leq r$ such that either $\beta_i = \nu_{k,l}, \beta_j = \nu_{m,n}$ for some $k,l, m, n$ and $\beta_s = \theta(x_s)$ for all $i < s < j, x_s \in H_\infty(q)$ or $\beta_i = \nu_{k,l}^*, \beta_j = \nu_{m,n}$ for some $k,l, m, n$ and $\beta_s = \theta(x_s)$ for all $i < s < j, x_s \in H_\infty(q)$. In both cases we can apply Lemma 7.14 to reduce $\beta_i\beta_{i+1} \cdots \beta_j$ to $\theta(x)$ for some $x \in H_\infty(q)$. In this way we obtain a word of length $r$ and by induction $\beta \in \theta(H_n(q))$. Hence $\theta: H_n(q) \rightarrow \text{End}_C(X^\otimes n)$ is surjective and the conclusion follows from Theorem 6.7. \(\Box\)

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