Method of variable separation for investigating exact solutions and dynamical properties of the time-fractional Fokker-Planck equation∗†

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Abstract

Based on the idea of variable separation, the time-fractional Fokker-Planck equation with external force field is studied by using the property of Mittag-Leffler function and some special algorithm skills. In the cases of various external potential functions such as linear potential, harmonic potential, logarithmic potential, exponential potential, and quartic potential, exact solutions and dynamical properties of the above mentioned equation is investigated. The some interesting dynamical behaviors and phenomena are discovered. The profiles of some representative exact solutions are illustrated by 3D-graphs.

Keywords: Time-fractional Fokker-Planck equation; Variable separation method; Mittag-Leffler function; Exact solution and dynamical property; Anomalous diffusion.

1 Introduction

Since the original concept of fractional-order derivative appeared in a famous correspondence which L’Hôpital sent to Leibniz in 1695, many mathematicians have further developed this concept. Through more than 300 years of development, the research works on theory

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and applications of the fractional calculus and the fractional differential equations have become many hot topics in the scientific field. Especially in recent decades, fractional calculus theory and modeling methods have been well applied in the fields of high energy physics, statistical physics, anomalous diffusion, viscoelastic material mechanics, complex control, signal processing, biomedical engineering, geophysics, rheology and economics and so forth in other research areas, and their unique advantages can not be replaced by integer-order calculus and differential equations.

In many scientific research fields mentioned above, the research work on anomalous diffusion phenomenon is undoubtedly a very important research topic. Anomalous diffusion is a kind of common phenomenon in nature, it is present in many physical situations such as transport of fluid in porous media, surface growth, diffusion of plasma, diffusion on liquid surface, two-dimensional rotating flow and so on, the details of introduction can be seen in the references [1-4] and references cited therein. As an anomalous diffusion model in an external potential field, the following time-fractional Fokker-Planck equation [5-11]

$$\frac{\partial P(x,t)}{\partial t} = R_0 D_t^{1-\alpha} \left[ \frac{\partial}{\partial x} \frac{V'(x)}{m \eta_\alpha} + K_\alpha \frac{\partial^2}{\partial x^2} \right] P(x,t),$$ (1.1)

has a wide range of applications in various scientific areas such as biology, chemistry, physics and finance. As a fractional subdiffusion model of continuous-time random walk, the equation (1.1) is also rewritten as

$$\frac{\partial P(x,t)}{\partial t} = R_0 D_t^{1-\alpha} \left[ \frac{1}{m \eta_\alpha} \frac{\partial [V'(x)P(x,t)]}{\partial x} + K_\alpha \frac{\partial^2 P(x,t)}{\partial x^2} \right],$$ (1.2)

where the function $P(x,t)$ defines the probability density of finding the test particle at a certain position $x$ and a given time $t$ with $(t, x) \in [0, \infty) \times (-\infty, +\infty)$, the $R_0 D_t^{1-\alpha}$ is fractional differential operator of Riemann-Liouville type and $0 < \alpha < 1$, the function $V(x)$ indicates the potential of over-damped Brownian motion and $V'(x) = \frac{dV(x)}{dx}$, $m$ denotes the mass of the test particle, $\eta_\alpha$ is a constant which denotes the generalized friction coefficient with dimension $[\eta_\alpha] = s^{\alpha - 2}$, $K_\alpha$ is also a constant which denotes the generalized diffusion coefficient with physical dimension $[K_\alpha] = m^2 s^{-\alpha}$ and satisfies the Einstein-Stokes-Smoluchowski relation $K_\alpha = \frac{\eta_\alpha T}{m \eta_\alpha}$, both $\eta_\alpha$ and $K_\alpha$ have nothing to do with time. The equation (1.2) can be derived from a generalized master equation which follows from a non-homogeneous random walk model in the diffusion limit. If writing the right-hand side of the equation (1.1) as

$$- R_0 D_t^{1-\alpha} \frac{\partial S(x,t)}{\partial x}$$

and letting

$$S(x,t) = \left[ - \frac{V'(x)}{m \eta_\alpha} - K_\alpha \frac{\partial}{\partial x} \right] P(x,t),$$

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then the equation (1.1) can be easily reduced to the equation (1.2). For these details, please participate in the reference [8]. Especially, when \( \alpha \to 1 \), equation (1.1) becomes a classical Fokker-Planck equation [12-14] (a model in statistical physics) as follows:

\[
\frac{\partial P(x,t)}{\partial t} = \left[ \frac{\partial}{\partial x} \frac{V'(x)}{m \eta_1} + K_1 \frac{\partial^2}{\partial x^2} \right] P(x,t).
\]

(1.3)

The equation (1.3) is a normal diffusion model and the corresponding diffusion process is governed by Fick’s second law in the force-free limit \( V(x) = c \) (constant).

By employing the fractional differential operator \( R_0 D_t^{\alpha-1} \) to act on both sides of the equation (1.2), it results

\[
R_0 D_t^{\alpha} P(x,t) - P(x,0)t^{-\alpha} = \frac{1}{m \eta_\alpha} \frac{\partial [V'(x)P(x,t)]}{\partial x} + K_\alpha \frac{\partial^2 P(x,t)}{\partial x^2}.
\]

(1.4)

By using the following relation of the definitions of Riemann-Liouville fractional derivative and Caputo fractional derivative

\[
R_0 D_t^{\alpha} P(x,t) - P(x,0)t^{-\alpha} = \frac{R_0 D_t^{\alpha}[P(x,t) - P(x,0)]}{\Gamma(1-\alpha)} = C_0 D_t^{\alpha} P(x,t),
\]

the equation (1.4) can be reduced to a simple form in the definition of Caputo fractional operator as follows:

\[
C_0 D_t^{\alpha} P(x,t) = \frac{1}{m \eta_\alpha} \frac{\partial [V'(x)P(x,t)]}{\partial x} + K_\alpha \frac{\partial^2 P(x,t)}{\partial x^2}.
\]

(1.5)

For details of the above reduction process, see the reference [8,15]. From the above reduction process, the equation (1.5) is equivalent to the equation (1.2) although their fractional differential operators are different. As far as the current techniques and methods are concerned, it is obviously much simpler to solve the equation (1.5) than to solve the equations (1.1) and (1.2). Therefore, we will directly investigate exact solutions of the equation (1.5) in the next section.

Due to the above equations have very good physical meaningful and very practical applications in diverse scientific fields, all of them have been studied by many researchers in recent years. Especially, in various external potential function \( V(x) \), many authors present different kinds of significative research works. For examples, in the constant potential \( V(x) = c \) (constant), the equations (1.1) and (1.2) become the following classical fractional diffusion equation

\[
\frac{\partial P(x,t)}{\partial t} = R_0 D_t^{1-\alpha} K_\alpha \frac{\partial^2 P(x,t)}{\partial x^2},
\]

(1.6)
its analytic solutions in closed form in terms of the Fox $H$–function were obtained by Wyss [16], Hilfer [17], Metzler and Klafter [8,18], respectively. In the harmonic potential (Ornstein-Uhlenbeck potential) $V(x) = \frac{1}{2}m\omega^2x^2$, the equation (1.1) becomes

$$\frac{\partial P(x,t)}{\partial t} = R_0D_1^{1-a}\left[\frac{\partial}{\partial x} \frac{\omega^2 x}{\eta_\alpha} + K_\alpha \frac{\partial^2}{\partial x^2}\right]P(x,t), \quad (1.7)$$

which studied by Metzler et al [5,8] and its series solution was obtained in [5]. In the quartic double-well potential $V(x) = ax^4+bx^2$ and the sextic double-well potential $V(x) = ax^6+bx^4$, by using the method of numerical calculation, the authors investigated the propagators, the auto-correlation function, the survival probability and the lifetime distribution of the equation (1.1), see the reference [19]. From the findings in the above references, we can see that the diffusion mode and the diffusion velocity of the particles are different under different external force fields. Usually, the external force fields have electric field, magnetic field, thermal field and etcetera. Under different kinds of external force fields, the Fokker-Planck equation has different application backgrounds. For examples, under contour-clamped homogeneous electric fields, the separation problem of large DNA molecules in the gel medium can be modeled by time-dependent time fractional Fokker-Planck equation on finite interval [20,21]. Under magnetic field, the study of plasma transport phenomena (diffusion of plasma) can be modeled by classical Fokker-Planck equation on infinite interval [22]. Under the action of various external force fields, the particle motion described by the fractional Fokker-Planck models often no longer satisfies the Brownian motion. The corresponding probability distribution of the particles no longer satisfies single normal distribution. The above fractional models contain a series of new type of distributions such as heavy-tailed distribution, long-tailed distribution, Lévy-like distribution, Boltzmann distribution and so forth under different external fields.

The research works in the references mentioned above mainly focuses on the numerical solutions and anomalous diffusion phenomena of these equations, but little is done on their exact solutions and approximate analytical solutions. Compared with the numerical solutions, the exact solutions of the model can more accurately describe and explain the inherent laws and dynamic phenomena entrusted by the model, and can systematically reveal how the solution and its dynamic properties of the model are affected by the changes of the parameters of the equation. Of course, the analytical solution of series type is also an exact solution. However, it is very difficult to obtain the exact solution of fractional partial differential equations. Many effective methods in the field of integer order can not be directly applied to solve partial differential equations in the field of fractional order. It is
often necessary for us to design a suitable method for solving a specific equation. It is worth mentioning that with the development of the research, on the solving methods of fractional differential equations, people have developed some effective methods to search the exact solutions and the approximate analytical solutions of the fractional differential equation in recent years, these methods contain Adomian decomposition method [23,24], homotopy analysis method [25,26], first integral method [27], invariant analysis method [28,29], fractional variational iteration method [30-32], invariant subspace method [33-36], homogenous balanced principle combined with integral bifurcation method [37-40], method of separating variables [41-43], and so forth. Similar to the methods in references [37-43], in this paper, by using method of separating variables together with Mittag-Leffler function, we will investigate exact solutions and dynamical properties of the time-fractional Fokker-Planck equation (1.5) in various potential functions such as linear potential, harmonic potential, logarithmic potential, exponential potential and quartic potential.

The organization of this paper is as follows: By using method of separating variables together with Mittag-Leffler function, in various external potential function $V(x)$; we will investigate different kinds of exact solutions of the time-fractional Fokker-Planck equation (1.5) defined by Caputo differential operator. Further, we will discuss the dynamical properties of these exact solutions.

2 Exact solutions and dynamical properties of the fractional model (1.5) in various external potential fields

In this section, based on the idea of separating variables, by using the property of Mittag-Leffler function and some algorithm skills, we will investigate exact solutions of the equation (1.5) and discuss their dynamical properties. The equation (1.5) can be rewritten as the following form

$$
\mathcal{D}_t^\alpha P(x,t) = \frac{1}{m\eta_s} \left[ \frac{d^2V(x)}{dx^2} P(x,t) + \frac{dV(x)}{dx} \frac{\partial P(x,t)}{\partial x} \right] + K_\alpha \frac{\partial^2 P(x,t)}{\partial x^2}. \tag{2.1}
$$

Obviously, the traditional method is to make Laplace transform for $t$ and Fourier transform for $x$ on both sides of equation (2.1), and then use the corresponding inverse transformation to obtain the solution of equation (2.1). Indeed, anyone who studied fractional differential equations knows that the expression of the exact solution of the equation (2.1) can not be obtained by the two transform because the number of terms of the equation is
too many (more than two terms). However, we can use a special variable separation method to obtain the exact solution of the equation. According to the idea of separating variables, similar to the methods in reference [37-43], we suppose that equation (2.1) has solutions as form in the below

\[ P(x,t) = W(x)E_{\alpha}(\lambda t^{\alpha}), \quad (2.2) \]

where \( W(x) \) is an undetermined function which can be determined later. By substituting (2.2) into (2.1), it yields

\[ \lambda W E_{\alpha}(\lambda t^{\alpha}) = \frac{1}{m\eta_{\alpha}} \left[ \frac{d^2 V}{dx^2} W E_{\alpha}(\lambda t^{\alpha}) + \frac{dV}{dx} \frac{dW}{dx} E_{\alpha}(\lambda t^{\alpha}) \right] + K_{\alpha} \frac{d^2 W}{dx^2} E_{\alpha}(\lambda t^{\alpha}), \quad (2.3) \]

where \( W = W(x), \ V = V(x) \). In equation (2.3), dividing out the Mittag-Leffler function \( E_{\alpha}(\lambda t^{\alpha}) \), we obtain an ODE as follows:

\[ K_{\alpha} \frac{d^2 W}{dx^2} + \left( \frac{1}{m\eta_{\alpha}} \frac{dV}{dx} \right) \frac{dW}{dx} + \left( \frac{1}{m\eta_{\alpha}} \frac{d^2 V}{dx^2} - \lambda \right) W = 0. \quad (2.4) \]

By the way, by using the classical variable separation method appeared in [9] and reference cited therein, the form of solutions of equation (2.1) can be supposed as

\[ P_{n}(x,t) = W_{n}(x)T_{n}(t) \quad (2.5) \]

for a given eigenvalue \( \lambda_{n,\alpha} \), where \( n \in \mathbb{N} \). Under this hypothesis, equation (2.1) can be reduced the following two eigenequations

\[ \frac{C_{0}}{D_{t}} D_{t}^{\alpha} T_{n}(t) = -\lambda_{n,\alpha} T_{n}(t), \quad (2.6) \]

\[ \frac{1}{m\eta_{\alpha}} \left[ \frac{d^2 V(x)}{dx^2} + \frac{dV(x)}{dx} \frac{d}{dx} \right] W_{n}(x) + K_{\alpha} \frac{d^2 W_{n}(x)}{dx^2} = -\lambda_{n,\alpha} W_{n}(x). \quad (2.7) \]

It is easy to obtain the solution of equation (2.6) as follows:

\[ T_{n}(t) = E_{\alpha}(-\lambda_{n,\alpha} t^{\alpha}). \quad (2.8) \]

Thus, the solution of equation (2.1) can be expressed as

\[ P(x,t) = \sum_{n=0}^{\infty} [W_{n}(x)E_{\alpha}(\lambda_{n,\alpha} t^{\alpha})], \quad (2.9) \]

the function \( W_{n}(x) \) can be obtained by equation (2.7) under different eigenvalue \( \lambda_{n,\alpha} \). In fact, we can improve this method further. According to the theory of Fourier series, the series in
can be regarded as the Fourier expansion of the function in (2.4). In other words, the series in (2.9) converges to the function in (2.4) theoretically. Thus, using equations (2.2) and (2.4) directly, we are sufficient to obtain all the solutions for equation (2.1). Obviously, the amount of computation will be greatly reduced if we do so.

Next, by using equation (2.4) we will search exact solutions of the equation (1.5) and discuss their dynamical properties in various external potential fields.

### 2.1 Case of the linear potential field

When $V = m \omega^2 x$ is a linear potential function, equation (2.4) can be reduced to a linear and homogeneous ODE of constant coefficient as follows:

$$K_\alpha \frac{d^2 W}{dx^2} + \frac{\omega^2}{\eta_\alpha} \frac{dW}{dx} - \lambda W = 0.$$  \hspace{1cm} (2.10)

If $\lambda > -\frac{\omega^4}{4\eta_\alpha^2 K_\alpha}$, then equation (2.10) has a general solution,

$$W = C_1 \exp \left( \frac{-\omega^2 - \sqrt{\Delta}}{2\eta_\alpha K_\alpha} x \right) + C_2 \exp \left( \frac{-\omega^2 + \sqrt{\Delta}}{2\eta_\alpha K_\alpha} x \right),$$  \hspace{1cm} (2.11)

where $\Delta = \omega^4 + 4\lambda \eta_\alpha^2 K_\alpha$, $C_1$ and $C_2$ are two arbitrary constants. Plugging (2.11) into (2.2), we obtain a type of exact solution of the equation (1.5) as follows:

$$P(x, t) = \left[ C_1 \exp \left( \frac{-\omega^2 - \sqrt{\Delta}}{2\eta_\alpha K_\alpha} x \right) + C_2 \exp \left( \frac{-\omega^2 + \sqrt{\Delta}}{2\eta_\alpha K_\alpha} x \right) \right] E_\alpha(\lambda t^\alpha).$$  \hspace{1cm} (2.12)

In the range of $-\frac{\omega^4}{4\eta_\alpha^2 K_\alpha} < \lambda < 0$, the solution (2.12) has attenuation property because $E_\alpha(\lambda t^\alpha) \to 0$ as $t \to +\infty$.

If $\lambda = -\frac{\omega^4}{4\eta_\alpha^2 K_\alpha}$, then equation (2.10) has a general solution,

$$W = (C_1 + C_2 x) e^{-\frac{\omega^2}{2\eta_\alpha K_\alpha} x},$$  \hspace{1cm} (2.13)

where $C_1$, $C_2$ are two arbitrary constants. Plugging (2.13) and $\lambda = -\frac{\omega^4}{4\eta_\alpha^2 K_\alpha}$ into (2.2), we obtain a type of exact solution of the equation (1.5) as follows:

$$P(x, t) = \left[ (C_1 + C_2 x) e^{-\frac{\omega^2}{2\eta_\alpha K_\alpha} x} \right] E_\alpha \left( -\frac{\omega^4}{4\eta_\alpha^2 K_\alpha} t^\alpha \right).$$  \hspace{1cm} (2.14)

If $\lambda < -\frac{\omega^4}{4\eta_\alpha^2 K_\alpha}$, then equation (2.10) has a general solution,

$$W = e^{-\frac{\omega^2}{2\eta_\alpha K_\alpha} x} \left[ C_1 \cos \left( \frac{\sqrt{-\Delta}}{2\eta_\alpha K_\alpha} x \right) + C_2 \sin \left( \frac{\sqrt{-\Delta}}{2\eta_\alpha K_\alpha} x \right) \right],$$  \hspace{1cm} (2.15)
where \(C_1, C_2\) are two arbitrary constants. Plugging (2.15) into (2.2), we obtain an exact solution of the equation (1.5) formed as follows:

\[
P(x, t) = e^{-\frac{\omega^2}{2\eta_\alpha K_\alpha} x} \left[ C_1 \cos \left( \frac{\sqrt{-\Delta}}{2\eta_\alpha K_\alpha} x \right) + C_2 \sin \left( \frac{\sqrt{-\Delta}}{2\eta_\alpha K_\alpha} x \right) \right] E_\alpha(\lambda t^\alpha). \tag{2.16}
\]

Here, we naturally have to ask which of the above three solutions is the unique solution of satisfying physical problem described by model (1.5)? In fact, once the initial conditions and boundary conditions are given, we can uniquely determine the solution of the mixed problem. For examples, if the initial condition of equation (1.5) is given by

\[
P(x, 1) = \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{4n}}{2^n \eta_\alpha^n K_\alpha(n+1)} e^{-\frac{\omega^2}{2\eta_\alpha K_\alpha} x} \left( -\frac{\omega^4}{2\eta_\alpha K_\alpha} t^\alpha \right), \quad 0 \leq x \leq L,
\]
and the boundary condition is given by

\[
P(0, t) = P(L, t) = 0, \quad t > 0,
\]
then by using (2.16) we obtain an unique solution corresponding to the mixed problem,

\[
P(x, t) = e^{-\frac{\omega^2}{2\eta_\alpha K_\alpha} x} \sin \left( \frac{\omega^2 \sqrt{2\eta_\alpha - 1}}{2\eta_\alpha K_\alpha} x \right) E_\alpha \left( -\frac{\omega^4}{2\eta_\alpha K_\alpha} t^\alpha \right), \quad 0 \leq x \leq L, \tag{2.17}
\]
where \(L = \frac{2\eta_\alpha K_\alpha \pi}{\omega^2 \sqrt{2\eta_\alpha - 1}}\). Similarly, if the initial condition of equation (1.5) is given by

\[
P(x, 1) = \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{4n}}{4^n \eta_\alpha^n K_\alpha(n+1)} e^{-\frac{\omega^2}{2\eta_\alpha K_\alpha} x}, \quad 0 \leq x < +\infty,
\]
and the boundary condition is given by

\[
P(0, t) = P(+\infty, t) = 0, \quad t > 0,
\]
then by using (2.14) we obtain an unique solution corresponding to above mixed problem,

\[
P(x, t) = x e^{-\frac{\omega^2}{2\eta_\alpha K_\alpha} x} E_\alpha \left( -\frac{\omega^4}{4\eta_\alpha^2 K_\alpha} t^\alpha \right), \quad 0 \leq x < +\infty. \tag{2.18}
\]

It is easy to know that the solutions (2.17) and (2.18) have attenuating property (decay characteristic) according to time increase, which satisfy \(P(x, t) \to 0\) as time \(t \to +\infty\).

In order to intuitively show the dynamical profiles (or properties) of above solutions obtained by us, as examples, the 3D-graphs and 2D-graphs of the solutions (2.17) and (2.18) are illustrated, which are shown in Fig. 1(a), (b), (c) and (d), respectively.
It can be seen from Fig. 1 that the probability density determined by solutions (2.17) and (2.18) does not conform to the normal distribution under the above conditions. In addition, although the graph of solution (2.17) is very similar to that of solution (2.18) in shape, they are essentially different; in the solution (2.17), the space variable satisfies the region $0 < x < L$, and in the solution (2.18), the space variable satisfies the region $0 < x < +\infty$. Thus, it can be seen that the continuous random variable defined by the solution (2.17) obeys short-tailed distribution and the random variable defined by the solution (2.18) obeys long-tailed distribution. Therefore, the solution (2.17) is well suited to explain the phenomenon of the separation of large DNA molecules from the gel medium in a finite region (interval); this indicates that the probability of the large DNA molecules separated from the gel medium in the left-central region of the separator is very high, while in the edge of the separator, the corresponding probability is very low. The solution (2.18) is well suited to explain the phenomenon of plasma transport in an infinite region; this shows that the probability of plasma acoustic particles being distributed at infinity is very low, that is to say, the more
distant the particle acoustic wave is transmitted, the weaker the signal is.

In the following discussion, we only discuss the solution of equation (1.5) (i.e. equation (2.1)) under free conditions, and no longer discuss the special solution under a certain boundary condition and an initial value condition in detail. Of course, as long as the certain boundary condition and the initial value condition are given, the special solution of the practical problem can always be obtained by the solution under the free condition.

### 2.2 Case of the harmonic potential field

When \( V = \frac{1}{2}m\omega^2x^2 \) is a harmonic potential function, equation (2.4) can be reduced to a linear and homogeneous ODE of variable coefficient as follows:

\[
K_\alpha \frac{d^2W}{dx^2} + \frac{\omega^2}{\eta_\alpha} x \frac{dW}{dx} + \left( \frac{\omega^2}{\eta_\alpha} - \lambda \right) W = 0. \tag{2.19}
\]

Especially, when \( \lambda = \frac{\omega^2}{\eta_\alpha} \), equation (2.19) becomes the following simple equation

\[
K_\alpha \frac{d^2W}{dx^2} + \frac{\omega^2}{\eta_\alpha} x \frac{dW}{dx} = 0. \tag{2.20}
\]

it has a general solution formed as follows:

\[
W = C_1 + C_2 \text{erf} \left( \frac{\omega}{\sqrt{2\eta_\alpha K_\alpha}} x \right), \tag{2.21}
\]

where

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\tau^2} d\tau = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}
\]

defines an error function and \( C_1, C_2 \) are two arbitrary constants. Plugging (2.21) and \( \lambda = \frac{\omega^2}{\eta_\alpha} \) into (2.2), we obtain a kind of solution of equation (1.5) formed as

\[
P(x,t) = \left[ C_1 + C_2 \text{erf} \left( \frac{\omega}{\sqrt{2\eta_\alpha K_\alpha}} x \right) \right] E_\alpha \left( \frac{\omega^2}{\eta_\alpha} \, t^\alpha \right) \tag{2.22}
\]

or

\[
P(x,t) = \left[ C_1 + \tilde{C}_2 \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n+1} x^{2n+1}}{n!(2n+1)2^{n+1} n! \eta_\alpha K_\alpha} \right] E_\alpha \left( \frac{\omega^2}{\eta_\alpha} \, t^\alpha \right), \tag{2.23}
\]

where \( \tilde{C}_2 = \frac{2C_2}{\sqrt{\pi}} \) is a new arbitrary constant, the (2.23) is a solution of infinite series type. The 3D and 2D graphs of the profiles for the solution (2.22) are shown in Fig.2(a) and (b), respectively.

It can be seen from the Fig.2 that the probability density function defined by the solution (2.22) does not satisfy the case of normal distribution. In the case of subdiffusion, this is

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a new type of distribution. Since the solution (2.22) is expressed by the error function, we call this distribution an error-type distribution. This phenomenon seems to be used to explain that the shape of sound wave of the plasma particle is kink-type. But what is being explained in other models is not known at the moment.

Fig. 2. The dynamical profiles of the solution (2.22); the parameters are taken as 
\( \eta_\alpha = 0.8, K_\alpha = 0.6, \alpha = 0.25, \omega = 0.3, C_1 = C_2 = 0.3 \) in drawing.

When \( \lambda \neq \frac{\omega^2}{\eta_\alpha} \), we make a transformation as follows:

\[
W = y(x)e^{-\frac{x^2}{2\eta_\alpha K_\alpha}}.
\]  

(2.24)

By substituting (2.24) into (2.19), it can be reduced to

\[
\frac{d^2 y}{dx^2} - \frac{\omega^2}{\eta_\alpha K_\alpha} \frac{dy}{dx} - \frac{\lambda}{K_\alpha} y = 0,
\]

(2.25)

where \( y = y(x) \). Letting \( \xi = \frac{\omega}{\sqrt{2\eta_\alpha K_\alpha}} x \) and writing \( \nu = -\frac{\lambda \eta_\alpha}{\omega^2} \), the equation (2.25) can be reduced to the following Hermite equation

\[
\frac{d^2 y}{d\xi^2} - 2\xi \frac{dy}{d\xi} + 2\nu y = 0,
\]

(2.26)

which is a special case of Kummer equation, its solution can be expressed by Hermite function or Kummer function. Supposing the equation (2.25) has solution of the series type as follows:

\[
y = \sum_{n=0}^{+\infty} a_n x^n.
\]

(2.27)

Letting \( y(0) = C_1 \) and \( \frac{dy(0)}{dx} = C_2 \) as initial conditions of the equation (2.25), then it is easily to obtain \( a_0 = C_1, a_1 = C_2 \) in (2.27). After substituting (2.27) into (2.25), comparing with
the same power coefficient of $x$ in the reduced equation, we obtain an iterative formula for coefficients as follows:

$$a_0 = C_1, \ a_1 = C_2, \ a_{n+2} = \frac{n \left( \frac{\omega^2}{\eta \kappa} \right) + \frac{\lambda}{K_\alpha}}{(n + 2)(n + 1)} a_n, \ (n = 0, 1, \cdots).$$  \hfill (2.28)

According to (2.28) and (2.27), we obtain a general solution of the series type of equation (2.25) as follows:

$$y = C_1 \left[ 1 + \sum_{n=1}^{\infty} \frac{1}{(2n)!} \prod_{k=1}^{n} \left( \frac{2(k - 1)\omega^2}{\eta \kappa} + \frac{\lambda}{K_\alpha} \right) x^{2n} \right] + C_2 \left[ 1 + \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} \prod_{k=1}^{n} \left( \frac{(2k - 1)\omega^2}{\eta \kappa} + \frac{\lambda}{K_\alpha} \right) x^{2n+1} \right]. \hfill (2.29)$$

Applying (2.29), (2.24) and (2.2), we get an analytic solution of equation (1.5) formed as

$$P(x, t) = C_1 e^{-\frac{\omega^2 x^2}{2\eta \kappa}} \left[ 1 + \sum_{n=1}^{\infty} \frac{1}{(2n)!} \prod_{k=1}^{n} \left( \frac{2(k - 1)\omega^2}{\eta \kappa} + \frac{\lambda}{K_\alpha} \right) x^{2n} \right] E_\alpha (\lambda t^\alpha) + C_2 e^{-\frac{\omega^2 x^2}{2\eta \kappa}} \left[ 1 + \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} \prod_{k=1}^{n} \left( \frac{(2k - 1)\omega^2}{\eta \kappa} + \frac{\lambda}{K_\alpha} \right) x^{2n+1} \right] E_\alpha (\lambda t^\alpha). \hfill (2.30)$$

Of course, directly solving the equation (2.26) and then changing the variables $\xi = \sqrt{\frac{\omega}{\sqrt{2\eta \kappa}}} x$ and $\nu = -\frac{\lambda \eta \omega}{2\kappa}$ back, the following result is obtained.

$$y = C_1 M \left( \frac{\lambda \eta \omega}{2\omega^2}, \frac{1}{2}, \frac{\omega^2 x^2}{2\eta \kappa} \right) + C_2 x M \left( \frac{\omega^2 + \lambda \eta \omega}{2\omega^2}, \frac{3}{2}, \frac{\omega^2 x^2}{2\eta \kappa} \right), \hfill (2.31)$$

where $C_1, \ C_2$ are two arbitrary constants, the function $M(\mu, \nu, x)$ is the first kind of Kummer function which defined by

$$M(\mu, \nu, x) = \sum_{n=0}^{\infty} \frac{\Gamma(\nu) \Gamma(\mu + n)}{n! \Gamma(\mu) \Gamma(\nu + n)} x^n,$$

and the Pochhammer symbol denotes $(\mu)_n = \frac{\Gamma(\mu + n)}{\Gamma(\mu)}$. In the program editing of mathematical software Maple, the function $M(\mu, \nu, x)$ is usually represented by notation $Kummer M(\mu, \nu, x)$. According to (2.31), (2.24) and (2.2), we also obtain an analytic solution of equation (1.5) as follows:

$$P(x, t) = e^{-\frac{\omega^2 x^2}{2\eta \kappa}} \left[ C_1 M \left( \frac{\lambda \eta \omega}{2\omega^2}, \frac{1}{2}, \frac{\omega^2 x^2}{2\eta \kappa} \right) + C_2 x M \left( \frac{\omega^2 + \lambda \eta \omega}{2\omega^2}, \frac{3}{2}, \frac{\omega^2 x^2}{2\eta \kappa} \right) \right] E_\alpha (\lambda t^\alpha). \hfill (2.32)$$
Although the solutions (2.30) and (2.32) are different in forms, they are in fact equivalent. As in Fig.1 and Fig.2, when the parametric values are properly selected, we plot the 3D and 2D graphs of the solution (2.32), which are shown in Fig.3 under $\lambda < 0$ and $\lambda > 0$, respectively. Obviously, no matter how the value of the parameter $\lambda$ changes, the probability density determined by the solution (2.32) always accords with the normal distribution, as can be seen from Fig. 3.

![3D-graph as $\lambda < 0$](a)

![2D-graph at $t = 2$ as $\lambda < 0$](b)

![3D-graph as $\lambda > 0$](c)

![2D-graph at $t = 2$ as $\lambda > 0$](d)

Fig. 3. The dynamical profiles of the solution (2.32) under the case $\lambda < 0$ and $\lambda > 0$.

From the Fig.3, we know that the random variable defined by the solution (2.32) obeys normal distribution. The corresponding particle motion belongs to Brownian motion. This indicates that in the case of subdiffusion, if the external field is adjusted properly, the particle motion also satisfies Brownian motion and corresponding random variable also obeys normal distribution. It is well known that this phenomenon will not occur in the simple field of subdiffusion. However, with the interference of the external field, that is, the interaction between the subdiffusion and the external field, everything will become possible. This phenomenon of normal distribution which appeared in the subdiffusion environment is the result of the interaction between the subdiffusion and the external field to achieve a new
equilibrium.

### 2.3 Case of the exponential potential field

When \( V(x) = m\eta_0 K_{\alpha} e^{\sqrt{\frac{K_{\alpha}}{K}} x} \) is an exponential potential function and \( \lambda > 0 \), equation (2.4) can be reduced to

\[
\frac{d^2 W}{dx^2} + \left( \sqrt{\frac{\lambda}{K_{\alpha}}} e^{\sqrt{\frac{K_{\alpha}}{K}} x} \right) \frac{dW}{dx} + \frac{\lambda}{K_{\alpha}} \left( e^{\sqrt{\frac{K_{\alpha}}{K}} x} - 1 \right) W = 0. \tag{2.33}
\]

Obviously, \( W_1 = e^{-\sqrt{\frac{K_{\alpha}}{K}} x} \) is a particular solution of the equation (2.33). Writing \( p(x) = \sqrt{\frac{\lambda}{K_{\alpha}}} e^{\sqrt{\frac{K_{\alpha}}{K}} x} \) and then using the following formula of general solution

\[
W = W_1 \left[ C_1 + C_2 \int \frac{1}{W_1^2} e^{-\int p(x)dx} dx \right], \tag{2.34}
\]

we get a general solution of (2.33) as follows:

\[
W = e^{-\sqrt{\frac{K_{\alpha}}{K}} x} \left[ C_1 + C_2 \left( \frac{e^{\sqrt{\frac{K_{\alpha}}{K}} x}}{e e^{\sqrt{\frac{K_{\alpha}}{K}} x}} + \frac{1}{e^{\sqrt{\frac{K_{\alpha}}{K}} x}} \right) \right], \tag{2.35}
\]

where \( C_1, C_2 \) are two arbitrary constants. Plugging (2.35) into (2.2), we obtain an exact solution of the equation (1.5) as follows:

\[
P(x, t) = e^{-\sqrt{\frac{K_{\alpha}}{K}} x} \left[ C_1 + C_2 \left( \frac{e^{\sqrt{\frac{K_{\alpha}}{K}} x}}{e e^{\sqrt{\frac{K_{\alpha}}{K}} x}} + \frac{1}{e^{\sqrt{\frac{K_{\alpha}}{K}} x}} \right) \right] E_{\alpha}(\lambda t^\alpha). \tag{2.36}
\]

When \( V(x) = m\eta_0 K_{\alpha} e^{-\sqrt{\frac{K_{\alpha}}{K}} x} \) is an exponential function and \( \lambda > 0 \), equation (2.4) can be reduced to

\[
\frac{d^2 W}{dx^2} - \left( \sqrt{\frac{\lambda}{K_{\alpha}}} e^{-\sqrt{\frac{K_{\alpha}}{K}} x} \right) \frac{dW}{dx} + \frac{\lambda}{K_{\alpha}} \left( e^{-\sqrt{\frac{K_{\alpha}}{K}} x} - 1 \right) W = 0. \tag{2.37}
\]

Similarly, we can obtain an exact solution of the equation (1.5) formed as

\[
P(x, t) = e^{\sqrt{\frac{K_{\alpha}}{K}} x} \left[ C_1 + C_2 \left( \frac{1}{e^{\sqrt{\frac{K_{\alpha}}{K}} x}} - \frac{1}{e e^{-\sqrt{\frac{K_{\alpha}}{K}} x}} \right) \right] E_{\alpha}(\lambda t^\alpha). \tag{2.38}
\]

In order to show profiles of the solutions (2.36) and (2.38) intuitively, we plot their graphs which are shown in Fig.4; under the values of the parameters \( \lambda = 0.2, K_{\alpha} = 0.9, \alpha = 0.6, C_1 = 1, C_2 = -1 \), we plot 3D and 2D graphs of the solution (2.36) which are shown in Fig.4 (a) and (b). Under the values of the parameters \( \lambda = 0.2, K_{\alpha} = 0.6, \alpha = 0.75, C_1 = \)
\( C_2 = 0.2 \), we plot 3D and 2D graphs of the solution (2.38) which are shown in Fig. 4 (c) and (d). Obviously, the probability density determined by the solution (2.36) accords with the normal distribution, but the probability density determined by the solution (2.38) does not accord with the normal distribution, as can be seen from Fig. 4(b) and (d), respectively.

![3D-graph of the solution (2.36)](image1)

![2D-graph of the solution (2.36) at \( t = 2 \)](image2)

![3D-graph of the solution (2.38)](image3)

![2D-graph of the solution (2.38) at \( t = 2 \)](image4)

Fig. 4. The dynamical profiles of the solutions (2.36) and (2.38).

From Fig. 4, we can see that the probability density function defined by solution (2.36) satisfies normal distribution, its dynamical property is similar to the property of solution (2.32). Similarly, the probability density function defined by solution (2.38) satisfies error-type distribution, its dynamical property is similar to that of solution (2.22).

### 2.4 Case of the quartic potential field

When \( V(x) = \frac{1}{4}m\omega^2 x^4 \) is a quartic potential and \( \lambda = -\frac{3\omega^2}{\eta_\alpha} \), equation (2.4) can be reduced to

\[
\frac{d^2 W}{dx^2} + \frac{\omega^2}{\eta_\alpha K_\alpha} x^3 \frac{dW}{dx} + \frac{3\omega^2}{\eta_\alpha K_\alpha} (x^2 + 1)W = 0. \tag{2.39}
\]
Under the transformation $W = y(x)e^{-\frac{\omega x^2}{\eta_\alpha K_\alpha}}$, the equation (2.39) can be reduced to

\[
\frac{d^2y}{dx^2} - \frac{\omega^2}{\eta_\alpha K_\alpha} x^3 \frac{dy}{dx} + \frac{3\omega^2}{\eta_\alpha K_\alpha} y = 0.
\]  

(2.40)

The general solution of equation (2.40) can be expressed by biconfluent Heun function as follows:

\[
y = C_1 H \left( -\frac{1}{2}, 0, \frac{3}{2}, \frac{3\omega}{\sqrt{\eta_\alpha K_\alpha}}, -\frac{\omega x^2}{2\sqrt{\eta_\alpha K_\alpha}} \right) + C_2 x H \left( \frac{1}{2}, 0, \frac{3}{2}, \frac{3\omega}{\sqrt{\eta_\alpha K_\alpha}}, -\frac{\omega x^2}{2\sqrt{\eta_\alpha K_\alpha}} \right), \]

(2.41)

where the function $H(a, b, \mu, \nu, x)$ is biconfluent Heun function and the $HeunB(a, b, \mu, \nu, x)$ is a special notation in the program of mathematical software Maple, the $C_1$, $C_2$ are two arbitrary constants. Thus, the general solution of equation (2.39) is given by

\[
W = C_1 e^{-\frac{\omega^2 x^4}{4\eta_\alpha K_\alpha}} H \left( -\frac{1}{2}, 0, \frac{3}{2}, \frac{3\omega}{\sqrt{\eta_\alpha K_\alpha}}, -\frac{\omega x^2}{2\sqrt{\eta_\alpha K_\alpha}} \right) +
C_2 x e^{-\frac{\omega^2 x^4}{4\eta_\alpha K_\alpha}} H \left( \frac{1}{2}, 0, \frac{3}{2}, \frac{3\omega}{\sqrt{\eta_\alpha K_\alpha}}, -\frac{\omega x^2}{2\sqrt{\eta_\alpha K_\alpha}} \right).
\]  

(2.42)

Plugging (2.42) and $\lambda = -\frac{3\omega^2}{\eta_\alpha}$ into (2.2), we obtain an analytical solution of the equation (1.5) as follows:

\[
P(x, t) = C_1 e^{-\frac{\omega^2 x^4}{4\eta_\alpha K_\alpha}} H \left( -\frac{1}{2}, 0, \frac{3}{2}, \frac{3\omega}{\sqrt{\eta_\alpha K_\alpha}}, -\frac{\omega x^2}{2\sqrt{\eta_\alpha K_\alpha}} \right) E_{\alpha} \left( -\frac{3\omega^2}{\eta_\alpha t^\alpha} \right) +
C_2 x e^{-\frac{\omega^2 x^4}{4\eta_\alpha K_\alpha}} H \left( \frac{1}{2}, 0, \frac{3}{2}, \frac{3\omega}{\sqrt{\eta_\alpha K_\alpha}}, -\frac{\omega x^2}{2\sqrt{\eta_\alpha K_\alpha}} \right) E_{\alpha} \left( -\frac{3\omega^2}{\eta_\alpha t^\alpha} \right).
\]  

(2.43)

When $V(x) = \frac{1}{4} m \omega^2 x^4 + \frac{1}{2} m \omega^2 x^2$ is a quartic double-well potential and $\lambda = -\frac{2\omega^2}{\eta_\alpha}$, equation (2.4) can be reduced to

\[
\frac{d^2W}{dx^2} + \frac{\omega^2}{\eta_\alpha K_\alpha} (x^3 + x) \frac{dW}{dx} + \frac{3\omega^2}{\eta_\alpha K_\alpha} (x^2 + 1) W = 0.
\]  

(2.44)

As in the above computational process, we can obtain an analytical solution of the equation (1.5) as follows:

\[
P(x, t) = C_1 e^{-\frac{\omega^2 x^4}{4\eta_\alpha K_\alpha}} H \left( -\frac{1}{2}, \frac{\omega}{\sqrt{\eta_\alpha K_\alpha}}, \frac{3}{2}, \frac{5\omega}{2\sqrt{\eta_\alpha K_\alpha}}, \frac{\omega x^2}{2\sqrt{\eta_\alpha K_\alpha}} \right) E_{\alpha} \left( -\frac{2\omega^2}{\eta_\alpha t^\alpha} \right) +
C_2 x e^{-\frac{\omega^2 x^4}{4\eta_\alpha K_\alpha}} H \left( \frac{1}{2}, \frac{\omega}{\sqrt{\eta_\alpha K_\alpha}}, \frac{3}{2}, \frac{5\omega}{2\sqrt{\eta_\alpha K_\alpha}}, \frac{\omega x^2}{2\sqrt{\eta_\alpha K_\alpha}} \right) E_{\alpha} \left( -\frac{2\omega^2}{\eta_\alpha t^\alpha} \right).
\]  

(2.45)
where $C_1$, $C_2$ are two arbitrary constants. Under the values of the parameters $\lambda = 0.01$, $\eta = 0.6$, $K_\alpha = 0.9$, $\alpha = 0.25$, $\omega = 0.005$, $C_1 = 0.5$, $C_2 = 0$, we draw 3D and 2D graphs of profiles of the solutions (2.43) and (2.45), which are shown in Fig.5. It is not difficult to find from Fig. 5 that the probability density determined by solutions (2.43) and (2.45) always accords with normal distribution when $C_2 = 0$. When $C_2 \neq 0$, the function of density determined by solution (2.43) and (2.45) does not satisfy probability distribution because $C_2 x$ can be positive or negative in the two solutions, this implies the value of $P(x,t)$ can be positive or negative under this case. Therefore, when $C_2 \neq 0$, it is necessary to re-determine the boundary conditions in order to make the two solutions conform to the problem of probability distribution.

![3D-graph of the solution (2.43)](image1)

![2D-graph of the solution (2.43) at $t = 2$](image2)

![3D-graph of the solution (2.45)](image3)

![2D-graph of the solution (2.45) at $t = 2$](image4)

Fig. 5. The dynamical profiles of the solutions (2.43) and (2.45) under $C_2 = 0$.

Obviously, the probability density functions defined by the solutions (2.43) and (2.45) obey normal distribution, their dynamical properties are very similar to those of the solution (2.32).
2.5 Case of the logarithmic potential field

When $V(x) = m\omega^2 \ln |x|$ is a logarithmic potential function and $\lambda = -\frac{\omega^2}{\eta_\alpha}$, equation (2.4) can be reduced to a generalized Bessel equation

$$\frac{d^2W}{dx^2} + \frac{\omega^2}{\eta_\alpha K_\alpha} \frac{1}{x} \frac{dW}{dx} + \frac{\omega^2}{\eta_\alpha K_\alpha} \left(1 - \frac{1}{x^2}\right) W = 0. \quad (2.46)$$

Under the variable transformation $\xi = \sqrt{\frac{\omega}{\eta_\alpha K_\alpha}} x$, the equation (2.46) can be reduced to a normal Bessel equation of 1-order as follows:

$$\frac{d^2W}{d\xi^2} + \frac{1}{\xi} \frac{dW}{d\xi} + \left(1 - \frac{1}{\xi^2}\right) W = 0, \quad (2.47)$$

its solution is

$$W = C_1 J(1, \xi) + C_2 Y(1, \xi), \quad (2.48)$$

where the $J(1, \xi)$ is the first kind of Bessel function which is defined by

$$J(1, \xi) = BesselJ(1, \xi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} \left(\frac{\xi}{2}\right)^{2n+1},$$

the $Y(1, \xi)$ is the second kind of Bessel function which is defined by

$$Y(1, \xi) = \frac{2}{\pi} J(1, \xi) \ln \frac{\xi}{2} - \frac{2}{\pi \xi} - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} \left[2\psi(n+1) + \frac{1}{n+1}\right] \left(\frac{\xi}{2}\right)^{2n+1},$$

with

$$\psi(n+1) = -\gamma + \sum_{k=0}^{n-1} \frac{1}{k+1}, \quad \gamma = 0.57721 56649 01532 \cdots,$$

the $\gamma$ is Euler constant. By the way, in the program of mathematical software Maple, the function $J(1, x)$ is usually represented by notation $BesselJ(1, x)$ and the function $Y(1, x)$ is usually represented by notation $BesselY(1, x)$. Plugging (2.48), $\lambda = -\frac{\omega^2}{\eta_\alpha}$ and the variable transformation $\xi = \sqrt{\frac{\omega}{\eta_\alpha K_\alpha}} x$ into (2.2), we obtain an analytic solution of equation (1.5) as follows:

$$P(x, t) = \left[C_1 J\left(1, \frac{\omega}{\sqrt{\eta_\alpha K_\alpha}} x\right) + C_2 Y\left(1, \frac{\omega}{\sqrt{\eta_\alpha K_\alpha}} x\right)\right] E_\alpha \left(-\frac{\omega^2}{\eta_\alpha} t^\alpha\right). \quad (2.49)$$

When $V(x) = m\eta_\alpha K_\alpha \ln \left|\csc \left(\sqrt{-\frac{\lambda}{K_\alpha}} x\right) + \cot \left(\sqrt{-\frac{\lambda}{K_\alpha}} x\right)\right|$ is a logarithmic potential function and $\lambda < 0$, $x \in [0, \pi \sqrt{-\frac{K_\alpha}{\lambda}}]$, equation (2.4) can be reduced to a two-order ODE

$$\frac{d^2W}{dx^2} - \left(\sqrt{-\frac{\lambda}{K_\alpha}} \csc \sqrt{-\frac{\lambda}{K_\alpha}} x\right) \frac{dW}{dx} - \frac{\lambda}{K_\alpha} \left(1 + \csc \sqrt{-\frac{\lambda}{K_\alpha}} x \cot \sqrt{-\frac{\lambda}{K_\alpha}} x\right) W = 0. \quad (2.50)$$
It is easy to verify that \( W_1 = \sin \sqrt{-\frac{\lambda}{K_\alpha}} x \) is a particular solution of the equation (2.50). Writing \( p(x) = -\sqrt{-\frac{\lambda}{K_\alpha}} \csc \sqrt{-\frac{\lambda}{K_\alpha}} x \) and using the formula (2.34), we obtain a general solution of (2.50) as follows:

\[
W = \sin \sqrt{-\frac{\lambda}{K_\alpha}} x \left[ C_1 + C_2 \left( \frac{1}{1 + \cos \sqrt{-\frac{\lambda}{K_\alpha}} x} + \frac{1}{2} \ln \left( \frac{1 - \cos \sqrt{-\frac{\lambda}{K_\alpha}} x}{1 + \cos \sqrt{-\frac{\lambda}{K_\alpha}} x} \right) \right) \right],
\]  

(2.51)

where \( C_1, C_2 \) are two arbitrary constants. Plugging (2.51) into (2.2), we obtain an analytic solution of equation (1.5) as follows:

\[
P = \sin \sqrt{-\frac{\lambda}{K_\alpha}} x \left[ C_1 + C_2 \left( \frac{1}{1 + \cos \sqrt{-\frac{\lambda}{K_\alpha}} x} + \frac{1}{2} \ln \left( \frac{1 - \cos \sqrt{-\frac{\lambda}{K_\alpha}} x}{1 + \cos \sqrt{-\frac{\lambda}{K_\alpha}} x} \right) \right) \right] E_\alpha(\lambda t^n),
\]

(2.52)

where \( 0 < x < \pi \sqrt{-\frac{K_\alpha}{\lambda}} \) and let \( P = 0 \) under other regions of \( x \).

Similarly, When \( V(x) = m n_\alpha K_\alpha \ln \left| \sec \left( \sqrt{-\frac{\lambda}{K_\alpha}} x \right) + \tan \left( \sqrt{-\frac{\lambda}{K_\alpha}} x \right) \right| \) is a logarithmic potential function and \( \lambda < 0, x \in \left[ -\frac{\pi}{2} \sqrt{-\frac{K_\alpha}{\lambda}}, \frac{\pi}{2} \sqrt{-\frac{K_\alpha}{\lambda}} \right] \), equation (2.4) can be reduced to

\[
\frac{d^2 W}{dx^2} + \left( \sqrt{-\frac{\lambda}{K_\alpha}} \sec \sqrt{-\frac{\lambda}{K_\alpha}} x \right) \frac{dW}{dx} - \frac{\lambda}{K_\alpha} \left[ 1 + \sec \sqrt{-\frac{\lambda}{K_\alpha}} x \tan \sqrt{-\frac{\lambda}{K_\alpha}} x \right] W = 0.
\]

(2.53)

Also, we can verify that \( W_1 = \cos \sqrt{-\frac{\lambda}{K_\alpha}} x \) is a particular solution of the equation (2.53). From (2.53), we know that \( p(x) = \sqrt{-\frac{\lambda}{K_\alpha}} \sec \sqrt{-\frac{\lambda}{K_\alpha}} x \). By using the formula (2.34), we obtain a general solution of (2.53) as follows:

\[
W = \cos \sqrt{-\frac{\lambda}{K_\alpha}} x \left[ C_1 + C_2 \left( \frac{1}{1 + \sin \sqrt{-\frac{\lambda}{K_\alpha}} x} + \frac{1}{2} \ln \left( \frac{1 - \sin \sqrt{-\frac{\lambda}{K_\alpha}} x}{1 + \sin \sqrt{-\frac{\lambda}{K_\alpha}} x} \right) \right) \right],
\]

(2.54)

where \( C_1, C_2 \) are two arbitrary constants. Thus, we obtain an analytic solution of equation (1.5) as follows:

\[
P = \cos \sqrt{-\frac{\lambda}{K_\alpha}} x \left[ C_1 + C_2 \left( \frac{1}{1 + \sin \sqrt{-\frac{\lambda}{K_\alpha}} x} + \frac{1}{2} \ln \left( \frac{1 - \sin \sqrt{-\frac{\lambda}{K_\alpha}} x}{1 + \sin \sqrt{-\frac{\lambda}{K_\alpha}} x} \right) \right) \right] E_\alpha(\lambda t^n),
\]

(2.55)

where \( -\frac{\pi}{2} \sqrt{-\frac{K_\alpha}{\lambda}} < x < \frac{\pi}{2} \sqrt{-\frac{K_\alpha}{\lambda}} \) and let \( P = 0 \) under other regions of \( x \).

In order to show profiles of above exact solutions, as example, under \( K_\alpha = 0.4, \lambda = -0.004, \alpha = 0.25, C_1 = 1, C_2 = 0.2 \), we plot 3D and 2D graphs of the solution (2.52) which are shown in Fig.6 (a) and (b), respectively.
The solution (2.52) implies compacton-type distribution. Obviously, the solution (2.52) is well suited to explain the phenomenon of the separation of large DNA molecules from the gel medium in a finite region, that is to say, the probability of the large DNA molecules separated from the gel medium in the central region of the separator is very high, but the probability near the edge of the separator becomes lower and lower.

3 Conclusions

In this work, based on the classical variable separation method, by using the properties of the Kittag-Leffler function, we improve the traditional practice by assuming the solution of equation (2.1) directly in the form of (2.4). Further, different kinds of exact solutions of the studied model (1.5) are obtained by solving the ODE (2.4) under various external force fields (potential functions). Most of these exact solutions are very new and most of them have attenuating property (decay characteristic) according to time increase, which satisfy $P(x,t) \to 0$ as time $t \to +\infty$. Some solutions (such as solutions (2.32), (2.36), (2.43) and (2.45)) imply normal distribution, other solutions don’t satisfy the case of normal distribution; the probability density (random variable) defined by solution (2.17) obeys sort-tailed distribution, the probability density defined by solution (2.18) obeys long-tailed distribution, the probability density defined by solutions (2.22) and (2.38) obeys error-type distribution, the probability density defined by solution (2.52) obeys compacton-type distribution.

According to symmetrical characteristic, this method also can be used to investigate exact solutions and dynamical property of a space-fractional PDE. Of cause, this new approach is
also suitable for solving high-dimensional time-fractional PDEs formed as

\[ C_0^\alpha D_t^\alpha P = a(x, y) \left( \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right) + b(x, y) \left( \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y} \right) + c(x, y)P, \]

(3.1)

where \( C_0^\alpha D_t^\alpha \) is time-fractional derivative of Caputo type and \( P = P(t, x, y) \). Obviously, equation (3.1) contains high-dimensional time-fractional Fokker-Planck equations. As in (2.2), if we suppose that the equation (3.1) has solution formed as

\[ P = W(x, y)E_\alpha(\lambda t^\alpha), \]

(3.2)

then the high dimensional equation (3.1) can be reduced to the following PDE

\[ \lambda W = a(x, y) \left[ \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} \right] + b(x, y) \left[ \frac{\partial W}{\partial x} + \frac{\partial W}{\partial y} \right] + c(x, y)W, \]

(3.3)

where \( W = W(x, y) \). Once the coefficients \( a(x, y), b(x, y) \) and \( c(x, y) \) are given (fixed), we can investigate the exact solution of equation (3.1) by solving the PDE (3.3). Of course, this is much more difficult than solving ODE (2.4), but in some special cases, the solution becomes easier. For example, if \( a(x, y) = a(x + ky), b(x, y) = b(x + ky) \) and \( c(x, y) = c(x + ky) \), that is to say, when the external force field is loaded on the plan \( x + ky \), then we can make a new transformation of variable as follows:

\[ W(x, y) = W(\xi) \quad \text{with} \quad \xi = x + ky. \]

(3.4)

Under the transformation (3.4), the PDE (3.3) always can be reduced to the following linear ODE of two order

\[ (1 + k)a(\xi) \frac{d^2 W(\xi)}{d\xi^2} + (1 + k)b(\xi) \frac{dW(\xi)}{d\xi} + [c(\xi) - \lambda]W(\xi) = 0. \]

(3.5)

As in equation (2.4), we can also solve equation (3.5). Thus the various exact solutions of equation (3.1) can be obtained similarly. Especially, when \( a(x, y) = a(\xi) = a_0, b(x, y) = b(\xi) = b_0 \) and \( c(x, y) = c(\xi) = c_0 \) are constants (i.e. constant potential), the equation (3.5) becomes a linear equation of constant coefficients, its exact solutions are very easy to obtained. Here we omit the detailed processes that are solved in the cases mentioned above.

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The authors declare that they have no conflict of interest.

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