AN APPLICATION OF NONLINEAR SUPRATRANSMISSION TO
THE PROPAGATION OF BINARY SIGNALS IN WEAKLY
DAMPED, MECHANICAL SYSTEMS OF COUPLED
OSCILLATORS

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Abstract. In the present article, we simulate the propagation of binary sig-
nals in semi-infinite, mechanical chains of coupled oscillators harmonically
driven at the end, by making use of the recently discovered process of nonlinear
supratransmission. Our numerical results — which are based on a brand-new
computational technique with energy-invariant properties — show an efficient
and reliable transmission of information.

The process of nonlinear supratransmission consists of a sudden increase in the
amplitude of wave signals transmitted into a nonlinear chain by a harmonic dis-
turbance at the end, irradiating at a frequency in the forbidden band gap. The
phenomenon was first discovered in mechanical chains of oscillators described by
coupled sine-Gordon and Klein-Gordon equations [1], and it was quickly studied in
other nonlinear models [2, 3, 4, 5]. Several applications of nonlinear supratransmis-
sion have been realized so far [6, 7], and more physical applications have been given
a mathematical foundation to its realization [8, 9] have been suggested; however,
the problem of transmitting coded information into nonlinear chain systems has
not been attacked at all.

In this paper, we develop an application of nonlinear supratransmission to the
propagation of binary signals in weakly damped semi-infinite mechanical chains of
coupled oscillators, by modulating the amplitude of the driving signal at the end.
The first section introduces the mathematical model under study and the energy
expressions to be used. A numerical study of the idealized problem is carried
out in the following section; here we provide a numerical analysis of the breather
propagation, based on the method proposed in [10], which in turn is based on a
numerical scheme to approximate radially symmetric solutions of modified Klein-
Gordon equations [11]. The next section presents a simulation of the propagation
of a binary signal in a weakly damped system. Finally, we present a section of
concluding remarks and propose complementary directions of research.

1. Preliminaries

1.1. Mathematical model. Throughout this paper, we assume that $\alpha$, $\beta$ and $\gamma$
are nonnegative real numbers, and that $c \gg 1$ (see [1]). Likewise, we consider a
system $(u_n)_{n=1}^{\infty}$ of oscillators satisfying the mixed-value problem studied in [10],

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Figure 1. Bifurcation diagram of critical amplitude vs. driving frequency for system (1) for various values of $\gamma$: 0 (solid), 0.1 (dashed), 0.2 (dash-dotted), 0.3 (dotted).

namely,

$$\frac{d^2 u_n}{dt^2} - \left( c^2 + \alpha \frac{d}{dt} \right) \Delta^2 u_n + \beta \frac{du_n}{dt} + V'(u_n) = 0,$$

subject to:

$$\begin{align*}
    u_n(0) &= 0, \quad n \in \mathbb{Z}^+, \\
    \frac{du_n}{dt}(0) &= 0, \quad n \in \mathbb{Z}^+, \\
    u_0(t) &= \psi(t), \quad t \geq 0,
\end{align*}$$

where $c$ is the coupling coefficient, and $\alpha$ and $\beta$ evidently play the roles of internal and external damping coefficients, respectively. Here, $\Delta^2 u_n$ is used to denote the spatial second-difference $u_{n+1} - 2u_n + u_{n-1}$ for every $n \in \mathbb{Z}^+$, the boundary-driving function is given by $\psi(t) = A(t) \sin(\Omega t)$ for every $t \in (0, +\infty)$, and $V(u_n) = 1 - \cos(u_n) - \gamma u_n$ where, due to the analogy with the Josephson model [12], $\gamma$ will be called here the normalized current. Notice that the Hamiltonian of the $n$-th lattice site is given by

$$H_n = \frac{1}{2} \left[ u_n^2 + c^2 (u_{n+1} - u_n)^2 \right] + V(u_n),$$

for any differentiable function $V$. After including the potential energy from the coupling between the first two oscillators, the total energy of the system becomes

$$E = \sum_{n=1}^\infty H_n + \frac{c^2}{2} (u_1 - u_0)^2.$$
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Figure 2. Local energies $H_n(t)$ of (1) vs. lattice site $n$ and time $t$, corresponding to the transmission of binary signal ‘1111’.

\[
\frac{\delta^2 u_n^k}{(\Delta t)^2} - \left( c^2 + \frac{\alpha}{2\Delta t} \delta_t \right) \frac{\delta^2 u_n^k}{\delta t^2} + \frac{\beta'}{2\Delta t} \delta_t u_n^k + \frac{V(u_{n+1}^k) - V(u_{n-1}^k)}{u_{n+1}^k - u_{n-1}^k} = 0,
\]

where $\beta'$ includes both the effect of external damping and a simulation of an absorbing boundary slowly increasing in magnitude on the last $N - N_0$ oscillators. More concretely, we let $u_{N+1}(t)$ be equal to zero at all time $t$, and let $\beta'$ be the sum of external damping and the function

\[
\beta''(n) = 0.5 \left[ 1 + \tanh \left( \frac{2n - N_0 + N}{6} \right) \right],
\]

where usually $N_0 = 50$ and $N \geq 200$.

Finite-difference scheme (2) (which is a modified version of the one proposed in [11]) is consistent with our mixed-value problem and conditionally stable, having the inequality $(c\Delta t)^2 < 1 + (\alpha + \beta'/4) \Delta t$ as a necessary condition for stability when $\beta'$ is assumed constant [10]. Moreover, if the energy of the system at the $k$-th time step is computed using the expression

\[
E_k = \frac{1}{2} \sum_{n=1}^{M} \left( \frac{u_{n+1}^k - u_n^k}{\Delta t} \right)^2 + \frac{c^2}{2} \sum_{n=1}^{M} (u_{n+1}^k - u_n^k)(u_{n+1}^{k+1} - u_n^{k+1}) + \sum_{n=1}^{M} \frac{V(u_{n+1}^k) + V(u_n^k)}{2} + \frac{c^2}{2} (u_1^k - u_0^k)(u_1^{k+1} - u_0^{k+1}),
\]

then the discrete rate of change of energy turns out to be a consistent approximation of order $O(\Delta t)^2$ for the corresponding instantaneous rate of change.

2. Numerical study
2.1. Bifurcation analysis. The existence of a bifurcation threshold of the energy administered into a semi-infinite chain of damped coupled oscillators described by (1) has been established and numerically predicted in [10] for a potential $V(u) = 1 - \cos u$. Numerical experiments on undamped mechanical chains with nonzero normalized bias currents have shown that the process of nonlinear supratransmission is likewise present in these models. In fact, Fig. 1 provides bifurcation diagrams of driving amplitude at which supratransmission first occurs vs. driving frequency $\Omega$ for a system of 200 undamped oscillators with coupling coefficient equal to 4 and several values of $\gamma$, over a time interval $[0, T(\Omega)]$ where $T(\Omega)$ is equal to 200 for all frequencies except for those satisfying $\Omega > 0.95$, in which case $T(\Omega)$ had to be increased up to 500.

2.2. Moving breather solutions. For the sake of simplification, let $\gamma = 0$ and consider a discrete system described by (1), harmonically driven at the boundary by a frequency $\Omega$. A binary bit $b$ will be transmitted into the medium during a fixed and sufficiently long period of signal generation $P$ equal to an integer multiple of the driving period, by defining

$$A(t) = \frac{8}{5} b C A_s \left( e^{-\Omega t/4.5} - e^{-\Omega t/0.45} \right)$$

for every $t \in [0, P]$, where $A_s$ represents the critical amplitude at which nonlinear supratransmission starts and $C > 0$ is an adjusting constant.

Fig. 2 shows the time evolution of the local energies $H_n$ corresponding to solutions of the undamped problem (1) with coupling coefficient equal to 4, driving frequency 0.9 (in which case $A_s$ is approximately equal to 1.79) and $C = A_s$. The period of signal generation is equal to 20 driving periods, and the binary code transmitted at the boundary is ‘1111’. The results show that a single moving breather is generated per period, and that the phase velocity $v_p$ through the medium is approximately constant. To verify this claim, we include in Fig. 3 the graphs of the positions of the four breathers generated by the binary signal vs. time. The
Figure 4. Time-dependent graphs of the maximum energy of the breathers generated in (1) by the binary signal ‘1111’.

fact that the breathers are transmitted at a constant velocity is now obvious, the common phase velocity being approximately 2.549.

In order to determine the strength of the emitted signals at sites located far away from the source, it is important to determine the time behavior of the maximum local energy attained by a discrete breather. Fig. 4 presents this behavior for the breather solutions obtained under the conditions above. Notice that the maximum local energy is well above a cutoff limit of 2.

3. Application

The discrete medium described by (1) with $\alpha = \beta = \gamma = 1 \times 10^{-3}$, and the rest of the parameters as in the previous section, will be our object of study in this section. Local energy-based reception devices will be placed on the 100-th and 300-th lattice sites, and the signal to be transmitted through the chain system is ‘10110010111011001’. Our system will consist of 600 sites, and a time step of 0.05 will be employed.

Assuming the general convention that site $n_0 \geq 100$ will start signal reception at time $t = (n_0 - 100)/v_p$, Fig. 5 shows the evolution of the local energy in the 100-th and 300-th sites in terms of the time normalized with respect to the period of signal generation. In either case, it is clear that the transmission of a bit equal to 1 in the $n$-th period is completely characterized in the graph by peak(s) of height greater than the cutoff limit 2 on the interval $[n-1, n]$.

4. Conclusions and perspectives

In this letter, we have proposed a simple mathematical model to transmit binary information in discrete, semi-infinite chains of coupled oscillators using the process of nonlinear supratransmission. In the absence of dispersive and dissipative effects, our model (which is based on the modulation of amplitudes of source signals with constant frequency) has shown to be highly reliable for sufficiently long periods of single-bit generation, independently of the distance between the source of transmission and the point of reception.
When weak damping is present the general picture does not change much. Stronger damping, however, manifests itself through a substantial decrease in the amplitude of the maximum local energy of the moving breathers with respect to the lattice site. In a forthcoming work, we will examine the possibility to overcome this problem via the concatenation of chain systems, where the driving at the beginning of each lattice will irradiate with an amplitude equal to a value just below its critical point multiplied by the amplitude of the last site in the previous lattice.

Finally, we wish to point out that the problem of determining whether it is possible to design a propagation system of binary signals in Josephson junction arrays still remains an open topic of research. Moreover, in view of the recently discovered phenomenon of nonlinear infratransmission (or lower-transmission, as named by the authors [3]), the problem of finding more efficient pathways to achieve signal transmission in other models is still an open question of general interest.

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