Experimental and numerical identification of the spiral wave in a wide-gap spherical Couette flow

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Spherical Couette flow experiments were conducted according to the work of Egbers and Rath [Acta Mech. 111 pp.125–140 (1995)]. The flow was visualized using aluminum flakes drifted on a horizontal plane illuminated by a laser sheet. A comparison between the numerically calculated phase velocity and that calculated from the visualized flow images indicates the robust formation of a spiral wave with azimuthal wavenumber $m = 3$. By solving the equation of motion for the infinitesimal surface elements advecting in the flow field obtained numerically, this study obtained a visual distribution of reflected light, which is in good agreement with the picture obtained experimentally.

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INTRODUCTION

The Newtonian fluid flow between two concentric spherical boundaries is a model in and on astronomical bodies [1, 2]. Mechanical factors such as the Coriolis force, thermal instability due to gravity toward the center of the system [3, 4], or Lorentz force via the electromagnetic field [5, 6] have been incorporated into the governing equations. This paper considers the flow transition, called the spherical Couette flow (SCF), triggered by angular differential velocities between the inner and outer spheres, when the inner sphere rotates at a constant angular velocity and the outer sphere is at rest. This flow could be a model of the liquid outer core of planetary and satellite bodies [8] with different rotations and mode changes.

Early experimental studies using several different combinations of spherical boundaries with various radii [3, 10] suggest that SCF is slightly more complicated than cylindrical Taylor-Couette flow despite the apparent similarities. Belyaev et al. [10, 11] estimated the first transitional Reynolds number for the case where the relatively wide spherical radius ratio $\eta = r_{in}/r_{out}$ is 1/2 using the power spectra obtained from laser Doppler velocimetry. Using flow visualization techniques, Egbers and Rath [12] revealed a qualitative phase diagram of flow in SCF for the cases of $\eta = 2/3$ and 3/4. The first transition from the laminar state for these radius ratios is triggered by a travelling sinusoidal disturbance at mid-latitudes propagating at a significantly low angular phase velocity in the azimuthal direction, as observed in the flow on a rotating planar disk in a stationary casing [13]. The disturbance, called “spiral wave” [12], can be visualized as a spiral pattern with $m$ equally spaced arms extending from the poles to the equatorial zone in each hemisphere, displaced in an alternatively staggered way with respect to the equator. The spiral wave occurring due to cross-flow instability for a relatively wide-gap SCF [13] is associated with the inertial wave generated by the instability of the axisymmetric flows via Coriolis effect from a geophysical viewpoint. This could also generate polygonal patterns formed in the polar jet stream on Jupiter and Saturn that have been observed in recent spacecraft missions [14].

On the other hand, improvements in computational capabilities have made it possible to numerically reproduce non-axisymmetric flows such as spiral waves and turbulent transitions [13–17]. The numerical linear stability analysis of the axisymmetric basic flow in the SCF specifies the value of the critical Reynolds number $Re_{crit}(m)$ for some representative values of $\eta$, over which a spiral wave with wave number $m$ supercritically bifurcates from the basic flow. Additionally, competition between spiral waves with different wave numbers was observed on the route to turbulence [11, 18, 19]. Using laser Doppler velocimetry, Wulf et al. [20] described flow transition over a critical $Re$ in a reconstructed phase space. The transition process is accompanied by a locked torus with modulation typical in the route to chaos via Hopf bifurcations and mode changes.

While the values of the critical Reynolds number are in good agreement with the experimental and numerical results [21], there remains a need to confirm the transition of SCF more clearly. Therefore, classical flow visualization techniques are still an effective way to identify the phases of flow in SCF. For example, the measurement of flow in an SCF using laser Doppler velocimetry [10, 20], which can provide a continuous time series of flow velocity at an observation point located in the domain, does not necessarily allow us to identify the phase of flow describing the entire domain of the SCF. Particle image velocimetry, which was applied to experiments verifying the asymmetry of flow in either positive or negative Rossby numbers [22], may enable the identification of subtle differences among similar phases of SCF. Con-
Considering that the axisymmetric stream in the background is relatively strong, the appearance of any disturbance in the equilibrium state that emerges over the critical Reynolds number should appear circular or spiral-shaped when visualized. The sole but distinct characteristic to distinguish analogous patterns in the transition in SCF is probably the wave number. In principle, it is very difficult to distinguish the different phases of flow emerging in the transition of SCF, except for recognizing the difference in the wave number. This can probably account for previous studies introducing confusing terms such as “inclined vortex,” “spiral vortex,” “wavy vortex,” and “ring vortex,” all of which specify different phases of flow, respectively, and all of which are intrinsically different from the spiral wave as well [12, 13, 18, 20]. In reality, a robust procedure to clearly identify a phase among similar circular or spiral-shaped phases that emerged in the transition of SCF is yet to be established.

Spiral waves visualized in SCF experiments have not yet been compared to those identified by numerical investigations. Therefore, the primary objective of this study is to propose an analysis procedure to clearly identify spiral waves via experimental and numerical methods. For the future scope of the study, the proposed procedure may be followed by the identification of a specific phase among analogous phases of flows successively emerging in the transition of SCF.

The remainder of this paper is organized as follows. The next section describes the SCF experimental apparatus used in this study with $\eta = 1/2$ and a visualization configuration. In Section 3, the dimensionless frequency is experimentally obtained and compared to that obtained from numerical calculations. In Section 4, the equations of both translational and rotational motions of the infinitesimal planar element drifting in flow are introduced, and subsequently, the visualized image expected from the numerical calculations is present. The latter part of the section contains brief remarks on the difference between the experimentally and numerically visualized images. The remainder of this paper is summarized in Section 5.

**EXPERIMENTAL SETUP**

A gap between concentric double spherical boundaries having diameters of 85 mm and 170 mm was filled with a typical 22~36 wt\% aqueous glycerol solution mixed with a small amount of aluminum flakes (Daiwa Metal Powder Co., No.1112, average particle diameter of 23 [$\mu$m]) and surfactant. The outer sphere (container) was made of acrylic glass with optical access, except for the equatorial zone, which allows visualization of the flow in the gap. The inner sphere, made of black anodized aluminum with its center fixed at the origin, was suspended by the lower end of a stainless shaft located along the vertical $z$-axis, and the angular velocity of the shaft was electrically controlled by a brushless DC motor (Oriental Motor, GFS2G5, BXM230-GFS, BXSD30-A). The diameter of the shaft was 6 mm to facilitate flow in the polar region as much as possible.

A sphere rotating at a constant angular velocity $\Omega_\text{in} = 2\pi/T_\text{in}$ produces a basic Stokesian laminar flow along spherical surfaces involving secondary circulation, i.e., Eckman downwellings at poles. Interpreted in terms of mechanics, the secondary flow streaming from the inner to outer sphere in the meridian plane (or from the outer to inner sphere near the poles) is caused by the centrifugal force at the equator around the inner sphere under non-slip conditions owing to viscosity.

This axisymmetric basic flow becomes stronger as the Reynolds number $Re = r_\text{in}^2\Omega_\text{in}/\nu$ increases. For $\eta = 1/2$, the basic flow becomes unstable against infinitesimal sinusoidal disturbances with azimuthal wave number $m = 4$, at $Re(m = 4) = 489$ [13, 21]. However, another infinitesimal disturbance with wave number $m = 3$, which is more dominant at further higher Reynolds numbers [15], may cause the $m = 3$ spiral wave, which is often observed in experiments. Ref. [10, 11] reported hysteresis loops related to the spiral waves with different wave number in multiple ranges of $Re$. Such hysteresis among the spiral states with $m = 5, 4, 3$ has also been reported in another numerical study [13] that examined flow at $\eta = 2/3$. Ref. [12] suggested that the hysteresis stems from the acceleration rate in prehistory of the flow development until the initial condition, but we shall not delve into the depths of that in the present study.

Our experiments are designed as iterations of typical routines, described as follows. First, the fluid temperature was measured when the inner sphere was at rest, and then, the inner sphere was rotated at a constant angular velocity by initiating an abrupt change from the state of rest. The sphere was rotated for more than 40 min, following which the fluid flow and its visualized appearance was recorded by the camera for approximately 5 min. Keeping the inner sphere at rest for 5 min, we measured the fluid temperature again to ensure that the temperature difference from the initial value was within 0.5 [°C]. From the viscosity of the aqueous glycerol solution obtained using the fluid temperature [23], the Reynolds number in the present experiments varied in the range of $500 \lesssim Re \lesssim 750$. Here, note that if SCF is described by the Navier-Stokes equations for incompressible flow with only the two control parameter, $Re$ and $\eta$, the state of SCF is determined by the initial condition from the viewpoint of the deterministic dynamical system.

Fig. 1 illustrates the setup of a light source and a camera in the present experiment. To illuminate the flow, a laser beam of H1.6 mm × W0.9 mm in diameter was emitted parallel to the $y$-axis from the light source (Integrated optics, 0520 L-11A, CW 520[nm] 200[mW]). The beam was spread uniformly on a horizontal plane at $z =$
domain approach approximately parallel to the refraction point and angle. The uniformity of light internal reflection, was simulated by extracting the cor-
spectively. The ray tracing used in this study is a simple lens, for the present light sheet by adjusting the distance to the container against both ambient air and the working fluid. However, fortunately, the container with spherical inner container refracts the laser sheet at the surface of the con-
tainer, and the fluid as well as the periodicity with wave number $m$ in the azimuthal direction $u_i(r, \phi + 2\pi/m) = u_i(r, \phi)$ (i = $r, \theta, \phi$). The symmetry requires that the flow patterns appearing in both hemispheres shift toward each other by half the azimuthal wavelength with the reflection at the equatorial plane. High-shear regions localized around the arms of the spiral wave rotate around the $z$ axis so that the light intensity in a narrow area in the mid-latitude zone, which is captured by the camera fixed at $z \gg r_{out}$, may oscillate periodically at intervals when an arm passes through the area.

In the actual experiment, we recorded the time series of the light intensity in a narrow area, which is surrounded by thick solid lines in Fig. 2 on the illuminated horizontal

$r_{in}$ through a Powell lens located at $(x, y, z) = (0, y_0, r_{in})$. If most of the aluminum flakes, which drift at a location on the plane illuminated in the flow, orient in a specific direction owing to the shear of the flow, the reflection from the location toward the camera may intensify or diminish compared to the average. The contrast pattern in the image captured by the camera reflects the shear structure of the flow experienced by the flakes. Moreover, if the flow structure is axisymmetric, the light intensity at a fixed observation point will not vary with time. However, if the flow structure is not axisymmetric, it will vary with time.

It is to be noted that the inner surface of the acrylic container is spherical and has a diameter of 170 mm centered at the origin, and the outer surface of the container is cylindrical having a diameter of 181 mm with the $z$ axis. Without refractive index matching, the acrylic container refracts the laser sheet at the surface of the container against both ambient air and the working fluid. However, fortunately, the container with spherical inner and circular outer surfaces partly acts as a collimator lens for the present light sheet by adjusting the distance to the lens, $y_0$. By some adjustment, the rays across the fluid domain approach approximately parallel to the $y$-axis, and the spatial distribution of the light intensity in the domain may be expected to be relatively uniform. Fig. 2 illustrates the light rays scattered by the lens and refracted at the boundaries, which were obtained from the calculation based on the refractive indices of the air, the container, and the fluid as $n = 1.00, 1.49, \text{and } 1.33$, respectively. The ray tracing used in this study is a simple geometrical mapping, where refraction, including total internal reflection, was simulated by extracting the correct refraction point and angle. The uniformity of light intensity in an image of flow visualization may be ensured if reflective flakes are uniformly distributed in a domain with isotropic orientation[24].

RESULTS

Suppose that a spiral wave with $m$ arms extending equally from the poles to the equator continues to rotate around the $z$-axis at a constant angular velocity $\omega$ without changing its shape. The flow structure of the spiral wave satisfies the following $m$-fold symmetry:

$$u_r(r, \pi - \theta, \phi + \pi/m) = u_r(r, \theta, \phi),$$

$$u_{\theta}(r, \pi - \theta, \phi + \pi/m) = -u_{\theta}(r, \theta, \phi),$$

$$u_{\phi}(r, \pi - \theta, \phi + \pi/m) = u_{\phi}(r, \theta, \phi),$$

as well as the periodicity with wave number $m$ in the azimuthal direction $u_i(r, \theta, \phi + 2\pi/m) = u_i(r, \theta, \phi)$ (i = $r, \theta, \phi$).
plane. The area shaped as a sector centered on the y-axis spans the range $|\phi - \pi/2| < \pi/6$ in the azimuthal direction. Fig. 4 shows an example of the oscillation of the recorded light intensity, which was used to calculate the period $T$, via Fourier transform. This period is related to the wave number $m$ and angular velocity of the spiral state $\omega$ as $m\omega = 2\pi/T$. Note that the values of $m$ and $\omega$ cannot be estimated in principle until the spiral wave is acquired in the entire domain; however, the product can be obtained directly from the time series.

The experiments were conducted using two inner spheres with different radii, $r_{in} = 42.5$ mm and $r_{in} = 38.0$ mm, which correspond to $\eta = 1/2$ and $\eta = 0.447$, respectively, for the present container with radius $r_{out} = 85.0$ mm. Here, we define the product of $m$ and $\omega$, nondimensionalized by $\Omega_{in}$, as the *dimensionless frequency* in accordance with Ref. [11]. Fig. 4 shows the dimensionless frequency against $Re$ obtained from the present experiment with $\eta = 1/2$ and $\eta = 0.447$ against $Re$. The range of the Reynolds numbers was $500 \leq Re \leq 750$ for each $\eta$.

Direct numerical simulations of SCF satisfying the Navier-Stokes equations were performed recently [25]. Using the equilibrium states obtained from the simulations as seeds, the spiral states with $m = 4, 3$, and $2$ were solved using the Newton-Raphson algorithm, and hence, so that the angular phase velocity was also specified numerically [19]. The flow field was expanded into a series of spherical harmonics and modified Chebyshev polynomials, as used in Ref. [26–28], and the Helmholtz equation equivalent to the Navier-Stokes equation was solved numerically with the aid of LAPACK libraries [29]. The dimensionless frequencies calculated from the spiral states with $m = 3$ solved for $\eta = 1/2$ and 0.447 are plotted as dashed curves in Fig. 4 for reference. Although not shown in the figure, those of the spiral waves with $m = 4$ and 2 were in the range of $0.6 \pm 0.03$ and $0.28 \pm 0.02$, respectively, for $Re \leq 700$. For reference, Ref. [11] reported 0.614 as a typical value of the dimensionless frequency of a sinusoidal perturbation with the azimuthal wave number as $m = 4$. A comparison of $m = 3$ and $m = 4, 2$ in dimensionless frequency suggests that the present state that was realized experimentally corresponds to the spiral state with $m = 3$, which remains within a relative error of 8% of its value for various Reynolds numbers.

A spiral wave with $\eta = 0.447$ emerges over $Re_{cr} \approx 410$ smaller than that with $\eta = 1/2$. The fact that $Re_{cr}$ increases with increasing $\eta$ is in qualitative agreement with the phase diagram reported in Ref. [12]. The dimensionless frequency for both cases is kept almost constant but decreases slightly with an increase in $Re$, which is also in good agreement qualitatively with Fig. 1 in Ref. [11].

In addition, the figure shows that the dimensionless frequency for $\eta = 0.447$ is smaller than that for $\eta = 1/2$. An idealized spiral wave may be regarded as an instability of Stewartson shear layer compensating for the angular velocity difference between the inner and outer boundaries [22], which implies that as $\eta$ decreases (as the gap becomes wider), the critical layer leaves the inner boundary for the outer boundary.

Fig. 5 shows four sequential snapshots of the flow visualized by the aluminum flakes that reflect light rays on the illuminated approximately horizontal halfplane, $z \approx r_0$ and $y \gtrsim 0$, as shown in Fig. 2. The zenith of the inner sphere and shaft are seen at the center of the bottom of these snapshots, and the sphere rotates counterclockwise around the axis. The wave number of the phase of the flow cannot be estimated based on a quick glance at these snapshots. Thus, the above discussion on dimensionless frequency may ensure that the phase of the flow is in a spiral state with wave number $m = 3$. The intervals of the sequential snapshots were adjusted only to $T/9$, such that four snapshots can describe only a single period. Here, assuming $m = 3$, the

\[\text{FIG. 3. Time series of light intensity in an area in the mid-latitude zone.} \quad \eta = 1/2, \quad T_{in} = 12 \text{[sec]}, \quad Re = 677.\]

\[\text{FIG. 4. The dimensionless frequencies against } Re \text{ for } \eta = 1/2 (\triangle) \text{ and } \eta = 0.447 (\circ). \quad \text{The dimensionless frequency is equivalent to } T_{in}/T, \text{ where } T_{in} \text{ is the rotation period of the inner sphere and the period } T \text{ is calculated from the time series of reflection light intensity in an area in the mid-latitude zone. Dashed curves are based on the phase angular velocities } \omega_1 \text{ of } m = 3 \text{-fold spiral states numerically solved for } \eta = 1/2 \text{ (upper) and } 0.447 \text{ (lower).}]

\[\text{FIG. 5. Four sequential snapshots of the flow visualized by the aluminum flakes that reflect light rays on the illuminated approximately horizontal halfplane, } z \approx r_0 \text{ and } y \gtrsim 0, \text{ as shown in Fig. 2. The zenith of the inner sphere and shaft are seen at the center of the bottom of these snapshots, and the sphere rotates counterclockwise around the axis. The wave number of the phase of the flow cannot be estimated based on a quick glance at these snapshots. Thus, the above discussion on dimensionless frequency may ensure that the phase of the flow is in a spiral state with wave number } m = 3. \]
FIG. 5. Sequential snapshots of the flow at \( \eta = 1/2 \) visualized by aluminum flakes drifting on the approximately horizontal plane \( z = r_0 \) illuminated as shown in Fig. 2. The Reynolds number \( Re = 604 \) was estimated using the concentration of the glycerol solution and the fluid temperature. The practically obtained interval of snapshots was 10 s, which is close to the numerically deduced value, \( T/9 = 9.6 \) s, from dimensionless frequency at \( Re = 604 \). The top of the inner sphere and the shaft along \( z \)-axis are observed at the center of these snapshots, and the sphere rotates toward the counter-clockwise around the axis.

In Fig. 5, the practically obtained interval of snapshots selected to complete a single period of recorded time series was 10 s. With some exceptions, the fourth snapshot is similar to the first, which satisfies the periodic condition in the azimuthal direction of the spiral state with wave number \( m = 3 \). The difference between them might be associated with symmetry breaking, which would occur with successive phase transitions above the critical Reynolds numbers.

A bright pattern shaped as the constricted neck of a crane extending from the shaft was observed along the \( y \)-axis in the second snapshot of Fig. 5. The exact spiral state is frozen in the rotation with a constant angular velocity around the \( z \)-axis, i.e., rotating without any change in its flow structure over time; thus, one would expect that the neck captured at the sequential snapshot should rotate forward to the azimuthal direction by \( 2\pi/9 \) without any change in pattern. However, the neck appeared to be more constricted at the second snapshot than at the third snapshot. Similarly, in the first quadrant, any copy of the neck appears neither in the first nor in the fourth snapshot, but instead, blurred shadow regions as a horn extends in the direction \( \phi = \pi/4 \) in the first quadrant in both snapshots. This was incorporated with other shadow fragments into a shadow region adjacent to the crane’s neck in the second snapshot. Even if the flow structure and orientations of the drifting flakes are frozen in rotation by just \( 2\pi/9 \), the direction of a light ray reflected by flakes drifting on the plane is not kept constant over time. This does not occur when the optical axes of both the light source and the camera accord by means of a half-mirror, which was employed in Ref. [12].

Fig. 6 shows the similar sequential snapshots of the flow realized at \( Re = 751 \). We adopted 4.75 s as the interval of snapshots in the figure, whereas numerical studies predicted the interval to be \( T/9 = 4.5 \) s. In the first snapshot, blurred shadow regions as a horn extended in the \( \phi = \pi/4 \) direction in the first quadrant, which was incorporated with shadow fragments into a shadow region shaped as a circle on the \( y \)-axis in the second snapshot, and then developed into a larger shadow region in the second quadrant in the third snapshot. Note that this circular shadow should not be regarded as a swirling fluid motion until the flow field numerically obtained is compared to the snapshot. The fourth snapshot is quite similar to the first snapshot, which suggests that the realized state satisfies the periodic condition of the spiral state with wave number \( m = 3 \). A flow state practically consists of combinations of a small set of incommensurable frequencies, and the spiral state obtained at \( Re = 751 \) was complete as compared to that obtained at \( Re = 604 \), which contained a certain modulation with lower frequencies [11].

period \( T = 2\pi/\omega(Re = 604) \) was estimated from the numerically obtained value of the dimensionless frequency at \( Re \) calculated from the fluid temperature and glycerol solution concentration used in the experiment. The numerically estimated dimensionless frequency at \( Re = 604 \) was 0.452, and thus, \( T/9 \) equals to 9.64 s for the rotation rate 4.61 rpm adopted in the present experiment.
uniform accumulation. The motion of the flakes in the flow can be assumed to be the rotation of infinitesimal planar elements without inertia advecting passively in the flow. We solved the simultaneous equations of both the translational and rotational motions of the element drifting in a spiral wave flow corresponding to \( m = 3 \). The temporal evolution of an element located at \( x(\tau) \) with orientation \( n(\tau) \) at time \( \tau \) is as follows:

Suppose that the elements are uniformly distributed on the illuminated plane \( z = r_0 \) at time \( t \) between the spherical boundaries, where the \( i \)-th element is located at \( x_i(t) \). With the aid of utilities in the numerical libraries of spherical harmonics \([26, 27]\), the position of the \( i \)-th element at \( t - T_0 \) can be solved by integrating the following translational motion equation backward in time:

\[
\frac{dx_i}{dt} = u_i(x_i, \tau). \tag{1}
\]

Thus, the map from a slightly complicated curved surface consisting of a set of \( x_i(t - T_0) \) to \( z = r_0 \) is the translation of elements by a time advance from \( t - T_0 \) to \( t \).

Next, each element at the instance of \( t - T_0 \) was assumed to face the orientations of the 12 vertices of the icosahedron with respect to its center with an equal probability. The governing equation for the orientation of each element \([24]\) is

\[
\frac{dn_i}{dt} = n_i \times u_i \times \nabla (n_i \cdot u), \tag{2}
\]

where the unit normal vector of the face at instance \( \tau \) is denoted by \( n_i(\tau) \) and the velocity field at \( x = x_i(\tau) \) is applied to the value of \( u \) in the equation. Note that the divergence of the right-hand side of the equation by \( n \) does not necessarily vanish, and hence, the probability of the orientation on the unit sphere may either accumulate or diffuse with time. To obtain the velocity field at \( \tau \), the velocity field solved using the Newton–Raphson method was rotated numerically by \( \omega \tau \) around the \( z \)-axis. We determine the orientation of the \( i \)-th element at the original instance \( t \) by integrating the governing equation of the forward orientation until each element arrives at the illuminated plane \( z = r_0 \). If the elements are strongly sheared during the integrated period, they can be aligned in a particular direction at \( t \), even if the plane elements are oriented isotropically at \( t - T_0 \). Since \( n_i(\tau) \) moves on the unit sphere with time, the governing equation of the element rotation was converted to the equivalent first-order Euler method of a quaternion algorithm, which was numerically integrated with a fine time step. Here, we adopted \( 2.8T_{in} \) as \( T_0 \) under the expectation that the order of \( T_{in} \) is sufficiently long for elements oriented isotropically at the initial stage to experience the shear region of the spiral state. For reference, \( 2.8T_{in} \) is 36 s in the case of the rotation rate 4.61 rpm for Fig. 5.

The direction of the incident rays \( I \) to an element is assumed to be \( I = -e_y \), independent of the position of the element. If the orientation of an element \( n \) is in the bisectonal direction between \(-I\) and the optical axis of

**FIG. 6.** Same as in Fig. 5 but \( Re = 751 (\eta = 1/2) \). The practical interval of snapshots was 4.75 s, which is close to the numerically deduced interval of \( T/9 = 4.5 \) s from dimensionless frequency at \( Re = 751 \).
the camera, reflected light is observable. In general, the
direction of reflected ray \( \mathbf{R} \) from the element is \( \mathbf{R} = I - 2(I \cdot \mathbf{n})\mathbf{n} \). The alignment of the reflected ray against
the optical axis of the camera must be correlated with the intensity distribution in the recorded images. Thus,
the degree of reflected light intensity captured by the
camera can be evaluated by the magnitude of \( (\mathbf{C} \cdot \mathbf{R}) \),
where \( \mathbf{C} \) is the unit vector along the optical axis of the
camera lens with respect to the reflecting element, and we assumed \( \mathbf{C} = \mathbf{e}_z \) for ease. Here, the square bracket
represents the ensemble average, such that the contrast
in light intensity at \( \mathbf{x} \), \( I(\mathbf{x}) \), is the average of \( \mathbf{C} \cdot \mathbf{R} \) for
all computed flakes positioned at \( \mathbf{x} \). If the orientations
of flakes at \( \mathbf{x} \) are distributed isotropically, the intensity vanishes; the orientation of the flakes at \( \mathbf{x} \) is aligned in
the direction, and the intensity is close to \( \pm 1 \).

Fig.7 shows a time series of the contrast pattern of light
intensity calculated visually, which is generated by the
reflection from infinitesimal planar elements advecting in
a spiral wave corresponding to \( m = 3 \) at \( Re = 560 \). The
interval between the figures was \( T/9 \) deduced from the
dimensionless frequency at the corresponding Reynolds
number. The range of \( I(\mathbf{x}) \) from 0–1 corresponds to the
gray-scale in Fig.7. It is worth noting that the value of
\( I(\mathbf{x}) \) is negative in most of the region \( y < 0 \) in the case
of \( i = -e_y \) (not shown in the figure), even though the
incident ray is assumed to be distributed as uniformly in
the region \( y < 0 \) as in the region \( y > 0 \). This was not
confirmed in our experiment because light emitted from
\( y > 0 \) is practically absorbed by flakes floating in the
region \( y > 0 \).

A shadow region as a circular spot centered at \( (x, y) =
(-0.5, 1) \) is identified in the third part of Fig.7 \((t =
2T/9)\). This shadow region is probably identical to that
located around \( (x, y) = (0.2, 1.3) \) in the second part of
Fig.7 and around \( (x, y) = (-1.2, 0.5) \) in the fourth Fig.7.
A bright pattern at the right hand side, adjacent to the
spot, is reminiscent of the pattern shaped as a crane’s
constricted neck extending from the shaft, which was ob-
served along the \( y \)-axis in the second snapshot of Fig.5.
Both, the shadow regions and the neck, appear to change
their shapes over time. Although the \( m = 3 \) spiral state
is principally a simple rotating wave solution without its
shape change, the contrast of the light intensity changes
over time, which is confirmed both experimentally and
numerically. As shown above, there are similarities be-
 tween Fig.5 experimentally captured at \( Re = 604 \) and
Fig.6 numerically calculated at \( Re = 560 \). From this simi-
arity, it seems possible to relate the experimental visual-
ization images to the numerically obtained velocity field
of the spiral wave.

FIG. 7. Time series of contrast in light intensity reflected
from infinitesimal planar element on the plane \( z = r_0 \) advect-
ing in the flow field of a spiral wave of wave number \( m = 3 \) at
(\( \eta, Re \) = (1/2, 560)) was calculated. The divisions on axes are
scaled according to the radius of the inner sphere, \( r_{in} \). The
images correspond to Fig.5 \( t = 0, T/9, 2T/9, 3T/9 \), from the
top to the bottom, respectively.

SUMMARY

This study conducted spherical Couette flow experi-
ments according to Ref.\[12\]. A comparison of the dimen-
sionless frequency between the experimental and numer-
ical results for \( \eta = 1/2 \) and \( \eta = 0.447 \) suggests that a
spiral wave with wave number \( m = 3 \) was realized in our
experiments. The spiral states were visualized using alu-
minum flakes drifted on a horizontal plane illuminated
by a laser sheet. Solving the equation of motion for the
infinitesimal planar elements advecting in the flow field
obtained numerically, we obtained the distribution of re-
lected light virtually, which was in agreement with the
experimentally obtained image. A robust procedure to
clearly identify a flow phase among similar circular or spiral-shaped phases that emerged in the transition of SCF has not been established yet, and we have not obtained the detail phase diagram of SCF. For future, the present procedure to compare experimental and numerical images may be developed so that we can distinguish the spiral wave from different but similar phase states, which were named as inclined vortex, wavy vortex, ring vortex, and so on, in previous observation on SCF.

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