CLOSED ORBITS AND INTEGRABILITY FOR SINGULARITIES OF COMPLEX VECTOR FIELDS IN DIMENSION THREE

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Abstract. This paper is about the integrability of complex vector fields in dimension three in a neighborhood of a singular point. More precisely, we study the existence of holomorphic first integrals for isolated singularities of holomorphic vector fields in complex dimension three, pursuing the discussion started in [10]. Under generic conditions, we prove a topological criteria for the existence of a holomorphic first integral. Our result may be seen as a kind of Reeb stability result for the framework of vector fields singularities in complex dimension three. As a consequence, we prove that, for the class of singularities we consider, the existence of a holomorphic first integral is invariant under topological equivalence.

1. Introduction: Integrability, first integrals and closed orbits

The problem of deciding whether a vector field or, more generally, an ordinary differential equation can be integrated by studying its number of non-transcendent solutions goes back to H. Poincaré, Dulac ([14]) and other authors. More recently the classical theorem of G. Darboux ([18]) states that a polynomial vector field in the complex plane admits a rational first integral provided that if, and only if, it admits infinitely many algebraic solutions. Of course the class of analytic equations is the one where the above problem makes more sense. Moreover, with the arrival of the Theory of foliations the use of geometrical/topological methods has given an important contribution to the comprehension of the problem as well as some important results. Indeed, a holomorphic vector field $X$ defined in a neighborhood $U \subset \mathbb{C}^n$, $n \geq 2$ of the origin $0 \in \mathbb{C}^n$, with an isolated singularity at the origin, defines a germ of a one-dimensional holomorphic foliation with a singularity at the origin in a natural way. Conversely, any germ of a holomorphic foliation with a singularity at the origin is defined in a small enough open neighborhood of the origin by holomorphic vector field with an isolated singularity at the origin. This is a consequence of Hartogs’ extension theorem ([16]).

The local framework is not less important than the global (algebraic) case. In this sense we have the important theorem of Mattei-Moussu ([21]) that states that a germ of a holomorphic vector field at the origin of $\mathbb{C}^2$ admits a holomorphic first integral if, and only if, the following conditions are satisfied. (i) the leaves are closed off the origin and (ii) only finitely many of them are separatrices, i.e., adhere to the origin. Condition (ii) is usually known as non-dicriticality of the (germ of a) foliation induced by the (germ of a) vector field ([9]). A foliation germ admitting a pure meromorphic first integral is necessarily dicritical. An example of Suzuki shows then that there is no such a topological criteria for existence of a meromorphic first integral ([24], [19]). Also interesting is the point of view adopted in [2] where the authors prove, for a germ of a holomorphic vector field singularity in dimension $n \geq 2$, the existence of a holomorphic first integral, under the hypothesis of existence of an uniform bound for the volume of the orbits of the vector field, and some additional condition that restricts the “dicritical case”.

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Our goal is to investigate topological conditions assuring the existence of holomorphic first integrals for vector field germs in dimension 3. This is done in Theorem 3. In few words, our result shows, for a generic class of singularities, an equivalence between the existence of a holomorphic first integral and the existence of a suitable stable separatrix, and also with the existence of a suitable flag, i.e., a codimension one foliation containing the orbits of the vector field. Our result may be seen as a kind of Reeb stability theorem for singularities of complex vector fields.

According to the above, we shall only consider the holomorphic, i.e., non-dicritical case. Let us then introduce the notation we use, already used in [10]. Denote the ring of germs of holomorphic functions on \((\mathbb{C}^n, 0)\) by \(O_n\) and its maximal ideal by \(\mathcal{M}_n\). Given a germ of a holomorphic vector field \(X \in \mathfrak{X}(\mathbb{C}^n, 0)\) we shall denote by \(\mathcal{F}(X)\) the germ of a one-dimensional holomorphic foliation on \((\mathbb{C}^n, 0)\) induced by \(X\).

**Definition 1** (holomorphic first integral). We say that a germ of a holomorphic foliation \(\mathcal{F}(X)\) has a holomorphic first integral, if there is a germ of a holomorphic map \(F: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{n-1}, 0)\) such that:

(a): \(F\) is a submersion almost everywhere, i.e., if we write \(F = (f_1, \cdots, f_{n-1})\) in coordinate functions, then the \((n-1)\)-form \(df_1 \wedge \cdots \wedge df_{n-1}\) is non-identically zero, equivalently, it has maximal rank except for a proper analytic subset;

(b): The leaves of \(\mathcal{F}(X)\) are contained in level curves of \(F\).

Further, a germ \(f\) of a meromorphic function at the origin \(0 \in \mathbb{C}^n\) is called \(\mathcal{F}(X)\)-invariant if the leaves of \(\mathcal{F}(X)\) are contained in the level sets of \(f\). This can be precisely stated in terms of representatives for \(\mathcal{F}(X)\) and \(f\), but can also be written as \(i_X(df) = X(f) \equiv 0\).

Next we pass to describe the class of vector field germs we shall work with. A germ of a holomorphic vector field \(X\) on \((\mathbb{C}^n, 0)\) is non-degenerate if its linear part \(DX(0)\) is non-singular. As a linear map, generically \(DX(0)\) has three distinct eigenvalues, thus is diagonalizable and \(X\) has an isolated singularity at the origin. From Poincaré-Dulac, Siegel and Brjuno linearization theorems and from [7], generically (i.e., for a full measure subset of the set of the set of germs of holomorphic vector fields), up to a change of coordinates, the vector field \(X\) leaves invariant the coordinate hyperplanes \(x_1 \cdots x_n = 0\). This motivates the following definition:

**Definition 2** (generic germs). We shall say that \(\mathcal{F}(X)\) is non-degenerate generic if \(DX(0)\) is non-singular, diagonalizable and, after some suitable change of coordinates, \(X\) leaves invariant the coordinate planes.

Denote the set of germs of non-degenerate generic vector fields on \((\mathbb{C}^n, 0)\) by \(\text{Gen}(\mathfrak{X}(\mathbb{C}^n, 0))\). Let \(X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^n, 0))\), \(S\) a smooth integral curve of \(\mathcal{F}(X)\) through the origin, and \(f\) a germ of an \(\mathcal{F}(X)\)-invariant meromorphic function. Then we denote by \(\text{Hol}(\mathcal{F}(X), S, \Sigma)\) the holonomy of \(\mathcal{F}(X)\) with respect to \(S\) evaluated at a section \(\Sigma\) transverse to \(S\), with \(\Sigma \cap S = \{q_\Sigma\}\) a single point. Notice that we can choose \(\Sigma\) to be biholomorphic to a disc in \(\mathbb{C}^{n-1}\) with center corresponding to \(q_\Sigma\). With this identification the group \(\text{Hol}(\mathcal{F}(X), S, \Sigma)\) is conjugate to a subgroup of the group \(\text{Diff}(\mathbb{C}^{n-1}, 0)\) of germs of complex diffeomorphisms fixing the origin in \(\mathbb{C}^{n-1}\). A germ \(f\) of a meromorphic function at the origin \(0 \in \mathbb{C}^n\) is called \(\mathcal{F}(X)\)-adapted to \((\mathcal{F}(X), S)\) if it can be written locally in the form \(f = g/h\) where \(g, h \in O_n\) are relatively prime, \(S \subset Z(g) \cap Z(h)\), where \(Z(g)\) and \(Z(h)\) denote the zero sets of \(g\) and \(h\) respectively, and \(f|_\Sigma\) is pure meromorphic for a generic transverse section \(\Sigma\) to \(S\). Given vector field germs \(X, Y \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))\) we have \(\mathcal{F}(X) = \mathcal{F}(Y)\) if and only if for some nonvanishing holomorphic function germ \(u\) we have \(Y = uX\). We shall then say that \(X\) and \(Y\) are tangent. Any vector field germ \(X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))\) admitting a holomorphic first integral must satisfy the following condition (cf. [10]):
Definition 3 (condition (⋆)). Let \( X \in \text{Gen}(\mathcal{X}(\mathbb{C}^3,0)) \). We say that \( X \) satisfies condition (⋆) if there is a real line \( L \subset \mathbb{C} \) through the origin, containing all the eigenvalues of \( X \) and such that not all the eigenvalues belong to the same connected component of \( L \setminus \{0\} \).

There is therefore one isolated eigenvalue of \( X \). The above condition holds for \( X \) if and only if it holds for any vector field \( Y \) such that \( X \) and \( Y \) are tangent. Condition (⋆) implies that \( X \) is in the Siegel domain, but is stronger than this last. Denote by \( \lambda(X) \) the isolated eigenvalue of \( X \) and by \( S_X \) its corresponding invariant manifold (the existence is granted by the classical invariant manifold theorem). We call \( S_X \) the distinguished axis of \( X \). We shall say that \( X \) is transversely stable with respect to \( S_X \) if for any representative \( X_U \) of the germ \( X \), defined in an open neighborhood \( U \) of the origin, any open section \( \Sigma \subset U \) transverse to \( S_X \) with \( \Sigma \cap S_X = \{ q_\Sigma \} \), and any open set \( q_\Sigma \in V \subset \Sigma \) there is an open subset \( q_\Sigma \in W \subset V \) such that all orbits of \( X_U \) through \( W \) intersect \( \Sigma \) only in \( V \).

In this paper we prove the following topological criterion for the integrability of a germ of a complex vector field singularity in dimension three:

Theorem 1. Suppose that \( X \in \text{Gen}(\mathcal{X}(\mathbb{C}^3,0)) \) satisfies condition (⋆) and let \( S_X \) be the distinguished axis of \( X \). Then \( F(X) \) has a holomorphic first integral if, and only if, the leaves of \( F(X) \) are closed off the singular set \( \text{Sing}(F(X)) \) and transversely stable with respect to \( S_X \).

From this result we conclude the invariance of the existence of a holomorphic first integral for generic germs in dimension three, under topological equivalence:

Corollary 1. Let \( X,Y \in \text{Gen}(\mathcal{X}(\mathbb{C}^3,0)) \) be generic germs of holomorphic vector fields, both satisfying condition (⋆). Assume that \( X \) and \( Y \) are topologically equivalent. Then \( X \) has a holomorphic first integral if and only if \( Y \) admits a holomorphic first integral.

Theorem 1 above can be completed (cf. Theorem 3), by weakening the topological hypothesis on the orbits, replacing the transverse stability by the existence of a suitable flag, i.e., a codimension one foliation, tangent to \( F(X) \).

2. Finite orbits and periodic maps

We determine the necessary conditions on the vector field \( X \in \text{Gen}(\mathcal{X}(\mathbb{C}^3,0)) \) in order that \( F(X) \) has a first integral. Let \( G \in \text{Diff}(\mathbb{C}^2,0) \) and \( V \) a neighborhood of the origin where a representative (also denoted by \( G \)) of the germ \( G \) is defined. Then we denote by

\[
\mathcal{O}_V^+(G,x) = \left\{ G^n(x) : G^j(x) \in V, j = 0, \ldots, n \right\}
\]

the so called positive semiorbit of \( x \in V \) by \( G \). Analogously, the negative semiorbit of \( x \in V \) by \( G \) is the set \( \mathcal{O}_V^-(G,x) := \mathcal{O}_V^+(G^{-1},x) \). The orbit of \( x \in V \) by \( G \) is the set \( \mathcal{O}_V(G,x) = \mathcal{O}_V^+(G,x) \cup \mathcal{O}_V^-(G,x) \). The cardinality of \( \mathcal{O}_V(G,x) \) is denoted by \( |\mathcal{O}_V(G,x)| \).

Theorem 2 (Brochero Martínez [4]). Let \( G \in \text{Diff}(\mathbb{C}^2,0) \), then the group generated by \( G \) is finite if and only if there exists a neighborhood \( V \) of the origin such that \( |\mathcal{O}_V(G,x)| < \infty \) for all \( x \in V \) and \( G \) preserves infinitely many analytic invariant curves at 0.

Using the same arguments as in the one-dimensional case (cf. [21], Proposition 1.1, p. 475-476), one can prove that a finite abelian (e.g., cyclic) subgroup of \( \text{Diff}(\mathbb{C}^n,0) \) is always periodic, i.e., it is generated by a periodic (and linearizable) element. Contrasting with the one-dimensional case, in greater dimensions the finiteness of the orbits in not enough to ensure the periodicity of the group (cf. [21], Theorem 2, p. 477).

Example 1. Consider the map \( G(x,y) = (x + y^2, y) \). The orbits of \( G \) are confined in the level set of \( f(x,y) = y \) and are clearly finite. Notice that \( |\mathcal{O}_V(G,(x,y))| \to \infty \) as \( y \to 0 \), thus \( G \) is not periodic nor linearizable. Furthermore, the orbits \( \mathcal{O}_V(G,(x,y)) \) are far from being stable, since in each line \( (y = c) \) the map \( G \) acts as a translation.
We say that two germs of holomorphic functions \( f, g \in \mathcal{O}_2 \) are generically transverse if \( df \wedge dg \) is not identically zero.

**Proposition 1.** Let \( f, g \in \mathcal{O}_2 \) be generically transverse germs and \( G \in \text{Diff}(\mathbb{C}^2, 0) \) be a complex map germ having finite orbits and preserving the level sets of both \( f \) and \( g \). Then \( G \) is periodic.

**Proof.** The idea of the proof is the following: Since \( f \) and \( g \) are generically transverse, then one can find a pure meromorphic function \( h_o = f o / g o \) whose level sets are preserved by \( G \). Hence the infinitely many curves \( f o(x, y) - c \cdot g o(x, y) = 0 \) with \( c \in (\mathbb{C}, 0) \) pass through the origin and are invariant by \( G \). Thus Theorem 2 ensures that \( G \) is periodic.

Now let us construct \( h_o \). If \( f / g \) is already pure meromorphic, then it is enough to pick \( h_o := f / g \). Otherwise one has \( f = h \cdot g^k \), where \( k \in \mathbb{Z}_+ \), and \( h \) is a germ of a holomorphic function not divisible by \( g \). Clearly, \( h \) is \( G \)-invariant, thus if it has an irreducible component distinct from the irreducible components of \( g \), then \( h / g \) must be a \( G \)-invariant pure meromorphic function.

Suppose that the decomposition in irreducible components of \( g \) and \( h \) are of the form \( g = g_1^{p_1} \cdots g_n^{p_n} \) and \( h = h_1^{q_1} \cdots h_n^{q_n} \). Since \( h \) is not divisible by \( g \), then there must be \( j_0 \in \{1, \ldots, n\} \) such that \( q_{j_0} < p_{j_0} \). If there is also \( j_1 \in \{1, \ldots, n\} \) such that \( q_{j_1} > p_{j_1} \), then \( h / g \) is a pure meromorphic \( G \)-invariant function.

From now on we suppose that \( q_j \leq p_j \) for all \( j = 1, \ldots, n \) with at least one \( j_0 \in \{1, \ldots, n\} \) such that \( q_{j_0} < p_{j_0} \). If there is \( j_1 \in \{1, \ldots, n\} \) such that \( q_{j_1} = p_{j_1} \), then after reordering the indexes (if necessary) we may suppose that: (i) \( q_i < p_i \) for all \( i = 1, \ldots, n; \) (ii) \( q_i = p_i \) for all \( i = n_0 + 1, \ldots, n \); for some \( n_0 \in \{1, \ldots, n - 1\} \). Then \( \bar{h} := g / h = g_0^{p_0 - q_0} \cdots g_{n_0}^{p_{n_0} - q_{n_0}} \) is a \( G \)-invariant germ of a holomorphic function. Now, let \( s_1 := [p_1 / (p_1 - q_1)] + 1 \) (where \( [x] \) denotes the integer part of \( x \in \mathbb{R} \)), then a straightforward calculation shows that \( g / \bar{h} \) is a pure meromorphic function.

Hereafter we suppose that \( q_j < p_j \) for all \( j = 1, \ldots, n \). Recall that the Euclid’s algorithm of a pair of positive integers \( (p, q) \), \( p > q \), is the sequence of pairs of positive integers \( \{(p_j, q_j)\}_{j=1}^{n+1} \) given by: (1) \( (p_{j+1}, q_{j+1}) := (p_j, q_j) \); (2) \( p_j = q_j \cdot r_j + s_j \), where \( r_j := [p_j / q_j] \) and \( s_j < q_j \); (3) \( (p_{j+1}, q_{j+1}) := (q_j, r_j) \); and (4) \( s_n > 0 \) and \( s_{n+1} = 0 \). This is called the Euclid’s sequence of the pair \( (p, q) \). For simplicity, suppose that \( g \) and \( h \) have only two irreducible components, say \( g = f^p(\bar{f})^q \) and \( h = f^s(\bar{f})^r \), and let \( \{(p_j, q_j)\}_{j=1}^{n+1} \) and \( \{(\bar{p}_j, \bar{q}_j)\}_{j=1}^{n+1} \) be the Euclid’s sequence of \( (p, q) \) and \( (\bar{p}, \bar{q}) \), respectively. If \( r_1 = [p_1 / q_1] < [\bar{p}_1 / \bar{q}_1] = \overline{r}_1 \), then \( p_1 - (r_1 + 1) q_1 < 0 \) and \( \overline{r}_1 = [\overline{p}_1 / \overline{q}_1] \geq 0 \). If \( \overline{r}_1 = [\overline{p}_1 / \overline{q}_1] \neq 0 \), then \( g / h^{r_1+1} \) is a \( G \)-invariant germ of a pure meromorphic function, otherwise \( g / h^{r_1+1} = 1 / f^{(r_1+1)q_1 - p_1} \) and \( g \cdot (g / h^{r_1+1})^{p_1} \) is a \( G \)-invariant germ of a pure meromorphic function. Arguing inductively along the Euclid’s sequences of \( (p, q) \) and \( (\bar{p}, \bar{q}) \) one can always construct a \( G \)-invariant pure meromorphic function unless \( r_j = \overline{r}_j \) for all \( j = 1, \ldots, n + 1 \). But this means that \( (p, q) = (\alpha s_n, \beta s_n) \) and \( (\bar{p}, \bar{q}) = (\alpha \overline{s}_n, \beta \overline{s}_n) \) for some \( \alpha, \beta \in \mathbb{Z}_+ \). Therefore \( g, h, \) and \( f \) are powers of the same holomorphic function \( f^{(s_n)}(\bar{f})^{\overline{s}_n} \), thus \( f \) and \( g \) cannot be generically transverse. A contradiction! The reasoning in the case of many irreducible factors is analogous, being in fact a consequence of the above reasoning. \( \Box \)

A straightforward consequence is the following:

**Corollary 2.** Let \( X \in \text{Gen}(\mathcal{X}(\mathbb{C}^3, 0)) \) and \( S_X \) be the distinguished axis of \( X \). Suppose that \( F(X) \) admits a meromorphic first integral, then the holonomy group \( \text{Hol}(F(X), S_X, \Sigma) \) is periodic.
Example 2. Blowing up the diffeomorphism (cf. [5]) \( G = (g_1, g_2) = (x + y^2, y) \) at the origin one has

\[
\tilde{G}(t, x) = \left( \frac{g_2(x, tx)}{g_1(x, tx)}, g_1(x, tx) \right) = \left( \frac{t}{1 + t^2}, x + tx \right) = (t(1 - t^2x + t^4x^2 - t^6x^3 + \cdots), x(1 + t))
\]

whose orbits are finite and confined in the level sets of \( \tilde{f}(t, x) = tx \). Further, \( G \) acts in these level sets of \( \tilde{f} \) in some sort of translation whose orbits increase in cardinality as \( \tilde{f}(t, x) \to 0 \). In particular, Proposition \( \text{II} \) ensures that \( G \) does not preserve the level sets of a couple of generically transverse functions \( f, g \in O_2 \).

3. Closed leaves versus first integrals

Now we construct an example showing that the closing of the leaves is not sufficient to ensure the existence of first integrals for \( F(X) \) with \( X \in \text{Gen}(\mathcal{X}(\mathbb{C}^3, 0)) \). The first thing to be remarked is that the linear part of a generic vector field germ having a first integral is determined by Proposition 1 in [10]. As a consequence (cf. 2.3 in [10]) \( \text{Hol}(F(X), S_X, \Sigma) \) must be a (cyclic) group generated by \( \lambda \) a resonant map preserving two smooth curves crossing transversely. In particular, one cannot expect a map like the one in Example \( \text{II} \) appearing as the (generator of the) holonomy of some \( X \in \text{Gen}(\mathcal{X}(\mathbb{C}^3, 0)) \) with respect to \( S_X \). Thus we blow up such map and look to a neighborhood of the point determined by the exceptional divisor and the strict transform of \( (y = 0) \). Let \( X \in \mathcal{X}(\mathbb{C}^3, 0) \) be given by

\[
X(x) = -m_1[x_1(1 + a_1(x)) + x_2b_1(x)] \frac{\partial}{\partial x_1} - m_2x_2(1 + a_2(x)) \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}
\]

where \( m_1, m_2, k \in \mathbb{Z}_+ \), \( S := (x_1 = x_2 = 0) \) and \( \Sigma := (x_3 = 1) \). Now consider the closed loop \( \gamma : [0, 1] \to S \) given by \( \gamma(t) = (0, 0, e^{2\pi i}t) \) and let \( \Gamma_{(x_1, x_2)}(t) = (\Gamma_1(t, x_1, x_2), \Gamma_2(t, x_1, x_2), \gamma(t)) \) be its lifting along the leaves of \( F(X) \) starting at \( (x_1, x_2, 1) \in \Sigma \). In particular, the map \( h \in \text{Diff}(\mathbb{C}^2, 0) \) given by \( \Gamma_{(x_1, x_2)}(1) = (h(x_1, x_2), 1) \) is a generator of \( \text{Hol}(F(X), S, \Sigma) \). Since \( \Gamma_{(x_1, x_2)}(t) \) belongs to a leaf of \( F(X) \), then

\[
\frac{\partial}{\partial t} \Gamma_{(x_1, x_2)}(t) = \alpha X(\Gamma_1(t, x_1, x_2), \Gamma_2(t, x_1, x_2), \gamma(t)).
\]

From this vector equation one has \( \gamma'(t) = \alpha \gamma(t) \), thus \( \alpha = 2\pi i \). Furthermore

(1) \[ \frac{\partial}{\partial t} \Gamma_1 = -2m_1\pi i[\Gamma_1 \cdot (1 + a_1(\Gamma_1, \Gamma_2, \gamma))] + \Gamma_2 \cdot b_1(\Gamma_1, \Gamma_2, \gamma)], \]

(2) \[ \frac{\partial}{\partial t} \Gamma_2 = -2m_2\pi i\Gamma_2 \cdot (1 + a_2(\Gamma_1, \Gamma_2, \gamma))). \]

Example 3. Let \( X(x) = -[x_1 + x_2b(x_3)] \frac{\partial}{\partial x_1} - 3x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \), then \( S := \{ x_1 = x_2 = 0 \} \) is invariant by \( X \) and the holonomy of \( F(X) \) with respect to \( S \) evaluated at \( \Sigma = (x_3 = 1) \) has the form \( h = (h_1, h_2) \) with \( h_j(x_1, x_2) = \Gamma_j(1, x_1, x_2) \), where \( \Gamma_1 \) and \( \Gamma_2 \) satisfy respectively equations (1) and (2) above. Now if we let \( \Gamma_n(t, x_1, x_2) = \sum_{i+j \geq 1} c_{i,j}(t)x_1^i x_2^j \), then (2) is written in the form

\[
\frac{\partial}{\partial t} \Gamma_2 = -6\pi i \Gamma_2.
\]

More precisely \( \frac{\partial}{\partial t} c_{i,j}^2(t) = -6\pi i \cdot c_{i,j}^2(t) \), thus \( c_{i,j}^2(t) = \lambda_{i,j}^2 \exp(-6\pi i t) \) for some \( \lambda_{i,j}^2 \in \mathbb{C} \). Since \( \Gamma_2(0, x_1, x_2) = x_2 \), then \( \lambda_{0,1}^2 = 1 \) and \( \lambda_{2,0}^2 = 0 \) otherwise. Therefore \( \Gamma_2(t, x_1, x_2) = \exp(-6\pi i t) \cdot x_2 \)
and \( h_2(x_1, x_2) = x_2 \). On the other hand, (1) is written in the form
\[
\frac{\partial}{\partial t} \Gamma_1 = -2\pi i[\Gamma_1 + e^{-6\pi i t}x_2^2 b(\gamma(t))] = -2\pi i(\Gamma_1 + e^{-6\pi i t} b(e^{2\pi i t} \cdot x_2^2)).
\]

Analogously, \( \frac{d}{dt} c_{i,j}^1(t) = -2\pi i \cdot c_{i,j}^1(t) \) for all \((i, j) \neq (0, 2)\). Since \( \Gamma_1(0, x_1, x_2) = x_1 \), then \( c_{1,0}^1(t) = \exp(-2\pi it) \cdot x_1 \) and \( c_{i,j}^1(t) = 0 \) for all \((i, j) \notin \{(1, 0), (0, 2)\} \). Finally \( \frac{d}{dt} c_{0,2}^1(t) = -2\pi i(c_{0,2}^1(t) + e^{-6\pi it} b(e^{2\pi it})) \). Now recall that the solution to the Cauchy problem
\[
\alpha'(t) = -2\pi i \cdot \alpha(t) - 2\pi i e^{-6\pi it} b(e^{2\pi it}), \quad \alpha(0) = 0.
\]
is given by
\[
\alpha(t) = -2\pi i e^{-2\pi it} \int_0^t e^{2\pi is} e^{-6\pi is} b(e^{2\pi is}) ds = -e^{-2\pi it} \int_0^t e^{-6\pi is} b(e^{2\pi is}) 2\pi i e^{2\pi is} ds
\]
In particular, \( \alpha(1) = -e^{-2\pi i} \int_0^1 \frac{b(z)}{z^2} dz \). Thus, if we let \( b(z) = -z^2/2\pi i \), then \( \alpha(1) = 1 \), and \( h(x_1, x_2) = (x_1 + x_2^2, x_2) \).

Completing the above example we obtain:

**Example 4.** Consider the vector field \( X(x, y, z) = -[x - \frac{1}{2\pi i} y^2 z^2] \frac{\partial}{\partial x} - 3y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \), after one blow up along the z-axis one has
\[
\pi^*X(t, x, z) = -(x - \frac{1}{2\pi i} t^2 x^2 z^2) \frac{\partial}{\partial x} + \frac{1}{x}(-3tx - t(-x - t^2 x^2 z^2)) \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}
\]
which has an isolated singularity at the origin, and whose holonomy with respect to the z-axis is precisely the map \( \tilde{G} \) in Example 2. Thus it satisfies condition (a) and has all leaves closed but does not admit a first integral in the sense of [10] (or Definition 1).

4. The main result: Stability, flags and first integrals.

Example 4 shows that the statements of Theorems 1.2 and 1.3 in [10] are incomplete. The correct statements are thus suggested by Proposition 1. In one word, we need to consider stability.

**Definition 4** (stability). A germ of a holomorphic vector field \( X \in \text{Gen}(\mathcal{X}(\mathbb{C}^3, 0)) \) is said to be *transversely stable* with respect to \( S_X \) if for any representative \( X_U \) of the germ \( X \), defined in an open neighborhood \( U \) of the origin, any open section \( \Sigma \subset U \) transverse to \( S_X \) with \( \Sigma \cap S_X = \{q_x\} \), and any open set \( q_x \in V \subset \Sigma \) there is an open subset \( q_x \in W \subset V \) such that all orbits of \( X_U \) through \( W \) intersect \( \Sigma \) only in \( V \).

A germ of a map \( G \in \text{Diff}(\mathbb{C}^2, 0) \) is said to be (topologically) *stable* if for any representative \( G: U \to G(U) \), where \( U \) is an open neighborhood of the origin, and any open set \( V \subset U \) there is an open subset \( W \subset V \) such that \( G^{\circ n}(W) \subset V \) for all \( n \in \mathbb{Z} \), i.e., all iterates of \( G \) starting in \( W \) remain in \( V \).

**Lemma 1.** Let the germ \( G \in \text{Diff}(\mathbb{C}^2, 0) \) be represented by the map \( G: W \to V \), where \( W \subset V \) are open neighborhoods of the origin with compact closure. Suppose \( G \) has finite orbits with stable positive semi-orbit, i.e., there are \( W \) and \( V \) as above with \( W \subset V \) and satisfying \( G^{\circ n}(x) \subset V \) for all \( x \in W \) and \( n \in \mathbb{Z}^+ \). Then \( G \) is periodic, i.e., there is \( p \in \mathbb{Z}^+ \) such that \( G^{\circ p} = \text{id} \).

**Proof.** First notice that the topological hypothesis on the orbits of \( G \) ensures that these orbits are all periodic. Now consider the analytic set \( C_q := \{x \in W : G^{\circ q} = x\} \), where \( q \in \mathbb{Z}^+ \). Then \( C_q \) is a closed set without interior points. Suppose that there is no \( p \in \mathbb{Z}^+ \) as stated, then
Recall that a germ of a foliation $F$ in $(\mathbb{C}^2, 0)$ has a dicritical component if there appears a dicritical singularity along its resolution process or, equivalently, there are infinitely many leaves adhering only at the origin ([9]). By a flag containing $F(X)$, we mean a germ of a codimension one holomorphic foliation $F$ at $0 \in \mathbb{C}^2$ with the property that (for some representatives of each foliation defined in a common domain containing the origin) each leaf of $F(X)$ is contained in some leaf of $F$. The notion of flag is detailed in [22]. As for our purposes, assume that $F$ is defined by a germ of an integrable holomorphic one-form $\omega = Adx + Bdy + Cdz$ with a singularity at the origin. In this case one writes $F = F_\omega$ and then the flag condition can be stated as $i_X \omega \equiv 0$. Given such a germ $F_\omega$, the singular set $\text{Sing}(F_\omega)$ is a codimension $\geq 2$ germ of an analytic subset at the origin. A codimension two irreducible component $K \subset \text{Sing}(F) \setminus \{0\}$ is a Kupka type component if $d\omega$ does not vanish along $K$. According to Kupka's theorem ([6, 20]), for a representative $F_U$ of $F$ in an open neighborhood $0 \in U$, where $F$ is given by an integrable holomorphic one-form $\omega_U$, and a representative $K_U \subset \text{Sing}(F_U)$ of the component $K \subset \text{Sing}(F_\omega)$, there is a plane foliation $\eta(K)$ in a neighborhood of the origin $0 \in \mathbb{C}^2$ such that for each point $q \in K_U$ there is a holomorphic submersion $\varphi_q : V_q \rightarrow \mathbb{C}^2$, with the property that $q \in V_q \subset U, \varphi_q(q) = 0$ and $\varphi_q^* (\eta(K)) = F_U|_{V_q}$. The foliation $\eta(K)$ is then called the Kupka transverse type of $F$ along the Kupka component $K$. One says that the Kupka component $K$ is dicritical if the corresponding transverse type $\eta(K)$ has a dicritical singularity at the origin $0 \in \mathbb{C}^2$. A particular case of a dicritical Kupka component is the one given by $F(X) \times \mathbb{C}$, where $F(X)$ is a germ of a foliation in $(\mathbb{C}^2, 0)$ defined by a resonant linearizable foliation in the Poincaré domain, i.e., in some appropriate coordinate system $F(X)$ is given by a vector field $X$ on $(\mathbb{C}^2, 0)$ of the form $X = mx \frac{\partial}{\partial x} + ny \frac{\partial}{\partial y}$, where $m, n \in \mathbb{Z}$. Such kind of dicritical Kupka component shall be called of radial type.

Given a flag $F_\omega$ containing the foliation $F(X)$, consider its restriction $F_\omega|_\Sigma$ to a transverse section $\Sigma$ as above. Since $\Sigma$ is transverse to $F(X)$, it is also transverse to $F_\omega$ off the singular set $\text{Sing}(F(X))$ and therefore one may identify the germ of $F_\omega$ at the point $q_0 = \Sigma \cap S(X)$ with the germ of a foliation at the origin $0 \in \mathbb{C}^2$. One says then that $F_\omega|_\Sigma$ is dicritical if this corresponding germ in dimension two is dicritical.

Let $\mathcal{G}$ be a germ of a foliation in $(\mathbb{C}^2, 0)$ having a dicritical component and $\phi \in \text{Diff}_{id}(\mathbb{C}^2, 0)$ be given by $\phi(x, y) = \exp[1] \hat{X}(x, y)$ for a (unique) formal vector field $\hat{X}$ of order at least two (cf. [5, 12]) called the infinitesimal generator of $\phi$. Then one says that $\mathcal{G}$ is adapted to $\phi$ if there is a resolution $\pi : (\mathcal{M}, D) \rightarrow (\mathbb{C}^2, 0)$ of $\hat{X}$ such that $\pi^* (\mathcal{G})$ has infinitely many curves transversal to $D$ (this happens precisely when we blow-up a dicritical component of $\mathcal{G}$ along the resolution of $\hat{X}$). In particular, any dicritical $\mathcal{G}$ is automatically adapted to $\phi$.

Now recall that $X$ has a linear part given by $J^1(X) = mx \frac{\partial}{\partial x} + ny \frac{\partial}{\partial y} - k z \frac{\partial}{\partial z}$. Thus the holonomy of $F$ with respect to the distinguished axis $z$ is periodic with linear part given by $J^1(h)(x, y) = (\exp(-\frac{2\pi i}{k}) x, \exp(-\frac{2\pi i}{k}) y)$; in particular $\phi := h^{\omega(k)}$ is tangent to the identity. Therefore, this map can be written locally in the form $\phi(x, y) = \exp[1] \hat{X}(x, y)$, where $\hat{X}$ is its infinitesimal generator. Then one says that $(F(X), F_\omega)$ is an adapted flag if $F(X) \subset F_\omega$ is a flag such that $F_\omega|_\Sigma$ is a germ of a foliation having a dicritical component and adapted to $\phi = h^{\omega(k)}$. In particular, if $F_\omega|_\Sigma$ is dicritical, then $(F(X), F_\omega)$ is automatically an adapted flag.

Notice that the last definitions are of finite determinacy character. Using this terminology, one may correctly restate the main result of [10] as follows.

**Theorem 3.** Suppose that $X \in \text{Gen}(\mathfrak{X}(\mathbb{C}^3, 0))$ satisfies condition $(\ast)$ and let $S_X$ be the distinguished axis of $X$. Then the following conditions are equivalent:
The leaves of $\mathcal{F}(X)$ are closed off the singular set $\text{Sing}(\mathcal{F}(X))$ and transversely stable with respect to $S_X$;

(2): $\text{Hol}(\mathcal{F}(X), S_X, \Sigma)$ has finite orbits and is (topologically) stable;

(3): $\text{Hol}(\mathcal{F}(X), S_X, \Sigma)$ is periodic (in particular linearizable and finite);

(4): $\mathcal{F}(X)$ has a holomorphic first integral.

Moreover, in terms of flags of foliations, the above conditions are also equivalent to each of the following conditions:

(5): The leaves of $\mathcal{F}(X)$ are closed off $\text{Sing}(\mathcal{F}(X))$ and there is an adapted flag $(\mathcal{F}(X), \mathcal{F}_\omega)$;

(6): The leaves of $\mathcal{F}(X)$ are closed off $\text{Sing}(\mathcal{F}(X))$ and there is a flag $\mathcal{F}(X) \subset \mathcal{F}_\omega$ such that $\mathcal{F}_\omega$ is a Kupka component of radial type.

Proof of the first part of Theorem 3. It follows immediately from the definition of transverse stability of germs of vector fields and from (topological) stability of maps that (1) implies (2). It comes from Lemma 1 that (2) implies (3). Now let us prove that (3) implies (4). Since $X$ satisfies condition $(\ast)$ and $\text{Hol}(\mathcal{F}(X), S_X, \Sigma)$ is linearizable, then [15] ensures that $\mathcal{F}(X)$ is linearizable. Therefore one may suppose without loss of generality that $X(x) = \lambda x_1 \frac{\partial}{\partial x_1} + \mu x_2 \frac{\partial}{\partial x_2} - \kappa x_3 \frac{\partial}{\partial x_3}$, where $\lambda, \mu, \kappa \in \mathbb{R}_+$. Since $\text{Hol}(\mathcal{F}(X), S_X, \Sigma)$ is periodic one may suppose without loss of generality that $\lambda = m, \mu = n, \kappa = k \in \mathbb{Z}_+$. The result then follows from Lemma 2.3 in [10]. Finally let us verify that (4) implies (1). The existence of a first integral for $\mathcal{F}(X)$ ensures that the leaves of $\mathcal{F}(X)$ are closed off $\text{Sing}(\mathcal{F}(X))$. Furthermore, $\text{Hol}(\mathcal{F}(X), S_X, \Sigma) = (H)$ admits a couple of generically transverse $\mathcal{F}(X)$-invariant holomorphic functions whose restrictions to $\Sigma$ have the level sets preserved by $H$. Thus Proposition 1 ensures that $\text{Hol}(\mathcal{F}(X), S_X, \Sigma)$ is periodic and, in particular, topologically stable. Hence the leaves of $\mathcal{F}(X)$ are transversely stable with respect to $S_X$. This proves the first four equivalences in Theorem 3. □

As a straightforward consequence, one has the following topological criterion for the existence of invariant meromorphic functions for elements in $\text{Gen}(\mathfrak{x}^3, 0)$.

Theorem 4. Let $X \in \text{Gen}(\mathfrak{x}(\mathbb{C}^3, 0))$ satisfy condition $(\ast)$, and $S_X$ be its distinguished axis. Suppose that $\mathcal{F}(X)$ has closed leaves off $\text{Sing}(\mathcal{F}(X))$ and is transversely stable with respect to $S_X$, then there is an $\mathcal{F}(X)$-invariant meromorphic function adapted to $(\mathcal{F}(X), S_X)$.

Proof. The proof of Theorem 4 is now almost identical to the original version (Theorem 2 in [10]), only including the stability hypothesis in the proof to obtain then a holomorphic first integral and after the desired $\mathcal{F}(X)$-invariant meromorphic function adapted to $(\mathcal{F}(X), S_X)$. □

Now we study the topological invariance of the existence of a holomorphic first integral for a generic germ of a holomorphic vector field, as a consequence of our preceding results. We recall that two germs of holomorphic vector fields $X$ and $Y$ at the origin $0 \in \mathbb{C}^n$ are topologically equivalent if there is a homeomorphism $\psi: U \to V$ where $U, V$ are neighborhoods of the origin $0 \in \mathbb{C}^n$, where $X$ and $Y$ have representatives $X_U$ and $Y_V$ respectively, such that $\psi$ takes orbits of $X_U$ into orbits of $Y_V$. Such a map $\psi$ takes separatrices of $X_U$ into separatrices of $Y_V$: indeed, a separatrix of $X_U$ is an orbit which is closed off the origin, and the same holds for its image under $\psi$. Assume that the vector field $X$ is generic satisfying condition $(\ast)$ and admits a holomorphic first integral. In this case one has:

Claim 1. The vector field $X$ is analytically linearizable, say $X(x, y, z) = X_{n,m,k} := nx \frac{\partial}{\partial x} + my \frac{\partial}{\partial y} - k \frac{\partial}{\partial z}$ with $n, m, k \in \mathbb{Z}_+$ and suitable local coordinates $(x, y, z) \in (\mathbb{C}^3, 0)$. In particular, $X$ admits a unique separatrix off the dicritical plane, and this separatrix corresponds to the distinguished separatrix $S_X$.

Proof. Indeed, the analytic linearization of $X$ is a straightforward consequence of the first part of Theorem 3 (or, since by hypothesis there is a holomorphic first integral, in view of Proposition 1).
and [15]). In this normal form
\[ X(x, y, z) = X_{n,m,-k} := nx \frac{\partial}{\partial x} + my \frac{\partial}{\partial y} - k \frac{\partial}{\partial z} \]
the “dicritical plane” is the plane \( \{ z = 0 \} \) and the distinguished separatrix is the z-axis. The orbit \( \mathcal{O}_{(a,b,c)} \) of \( X \) through the point \((a, b, c)\) is given by
\[ \phi(t) = (x(t), y(t), z(t)) = (ae^{nt}, bxe^{mt}, c - kt), \quad t \in \mathbb{C}. \]
Thus, if \( \mathcal{O}_{(a,b,c)} \) accumulates at the origin, either \( c = 0 \) or \( c \neq 0 \) and \( a = b = 0 \). For instance, if \( c \neq 0 \neq a \), then the orbit is contained in the hypersurface \( x^kz^n = a^k c^n \neq 0 \), which does not accumulate at the origin. □

Lemma 2. A topological equivalence takes the distinguished axis of \( X \) into the distinguished axis of \( Y \).

Proof. Indeed, as we have seen above, the image \( \psi(S_X) \) is some separatrix of \( Y \). If this is not the distinguished axis of \( Y \), then the distinguished axis of \( Y \) is taken by \( \psi^{-1} \) into a separatrix other than the distinguished axis of \( X \). Therefore, according to Claim 1, \( \psi^{-1}(S_Y) \) must be a separatrix of the “dicritical” part of \( X \), i.e., in the coordinates \((x, y, z)\) above, where \( X(x, y, z) = X_{n,m,-k} \), we have \( \psi^{-1}(S_Y) \subset \{ z = 0 \} \). Nevertheless, any invariant neighborhood of a leaf contained in a dicritical separatrix of \( X \) off the origin intersects infinitely many separatrices (namely, those contained in the intersection of this neighborhood with the dicritical plane \( \{ z = 0 \} \)). On the other hand, this same phenomena does not occur for arbitrarily small invariant neighborhoods of a leaf contained in the distinguished axis \( S_Y \) of \( Y \). Therefore, necessarily \( \psi(S_X) \) is the distinguished axes of \( Y \). □

From the above considerations we immediately obtain Corollary 1 from Theorem 3.

5. Flags and integrability

In this section, the second part of the of the main result is proved, i.e., the equivalence of the first four with the final two equivalences in Theorem 3.

In fact, our main concern here is the following: given a germ of a foliation by curves \( \mathcal{F} \) induced by a germ of a vector field of the form
\[ X = mx(1 + a(x, y, z)) \frac{\partial}{\partial x} + ny(1 + b(x, y, z)) \frac{\partial}{\partial y} - kz(1 + c(x, y, z)) \frac{\partial}{\partial z} \]
with \( a, b, c \in \mathcal{M}_3 \), study the consequences of the existence of a codimension 1 germ of a holomorphic foliation tangent to \( X \), which is transversely dicritical with respect to \( S \).

5.1. Transverse structure. We begin by studying the consequences of the existence of a flag foliation with a dicritical transverse type for a vector field \( X \in \text{Gen}(X(C^3, 0)) \).

Lemma 3. Let \( \mathcal{F} \) be a germ of a foliation by curves on \((C^3, 0)\), \( S \) an invariant curve of \( \mathcal{F} \) through the origin and \( \mathcal{G} \) a codimension one foliation satisfying the following conditions:

(i): \( \mathcal{G} \) is tangent to \( X \);
(ii): There is a section \( \Sigma \) transverse to \( S \) such that \( \mathcal{G}|_{\Sigma} \) is dicritical.

Then \( \mathcal{G} \) is transversely dicritical with respect to \( S \).

Proof. Since the orbits of \( X \) are contained in the leaves of \( \mathcal{G} \), then they are invariant by the flow of \( X \). Therefore if \( \Sigma' \) is another section transversal to \( S \) and \( \phi : \Sigma \longrightarrow \Sigma' \) is an element of the holonomy pseudogroup of \( X \) with respect to \( S \), then it is a diffeomorphism taking the leaves of \( \mathcal{G}|_{\Sigma} \) onto the leaves of \( \mathcal{G}|_{\Sigma'} \). □
The following result leads to deep implications in the transverse dynamics of the foliation \( \mathcal{F}(X) \) in the presence of a dicritical flag \( \mathcal{F}(X) \subset \mathcal{F}_\omega \). First recall some notation. Let \( \text{Diff}_{id}(\mathbb{C}^2,0) \subset \text{Diff}(\mathbb{C}^2,0) \) denote the group of germs of diffeomorphism tangent to the identity. Further, let \( \mathcal{F}_\omega \) be a germ of a foliation in \((\mathbb{C}^2,0)\) given by \( \omega = 0 \), then we denote by \( \text{Aut}(\mathcal{F}_\omega) \) the subgroup of \( \text{Diff}(\mathbb{C}^2,0) \) given by those \( \phi \in \text{Diff}(\mathbb{C}^2,0) \) preserving \( \mathcal{F}_\omega \), i.e., such that \( \phi^* \omega \wedge \omega = 0 \).

**Lemma 4.** Let \( \mathcal{F}_\omega \) be a germ of a foliation in \((\mathbb{C}^2,0)\) having a dicritical component and adapted to \( \phi \in \text{Diff}_{id}(\mathbb{C}^2,0) \). Thus \( \phi \in \text{Aut}(\mathcal{F}_\omega) \) and has finite orbits if and only if \( \phi \) is the identity.

**Proof.** Let \( \pi : (\mathcal{M}, D) \to (\mathbb{C}^2,0) \) be the resolution of \( \phi \) introduced in [1], \( \tilde{\mathcal{F}} := \pi^* \mathcal{F} \) the strict transform of \( \mathcal{F} \) via \( \pi \), and \( \tilde{\phi} \) the lifting of \( \phi \). Since \( \phi \in \text{Diff}_{id}(\mathbb{C}^2,0) \), then \( \tilde{\phi}|_D = \text{id}|_D \). If \( \tilde{\mathcal{F}} \subset \mathcal{F} \) is a dicritical component of \( \mathcal{F} \) defined in a neighborhood of the irreducible component \( D_j \subset D \), then it is given in appropriate coordinate systems by a fibration transversal to \( D_j \), up to a finite number of singular leaves or smooths leaves tangent to \( D_j \). More precisely, there is an open set \( U_j := D_j \setminus \{p_1, \ldots, p_r\} \) such that \( \tilde{\phi}|_{U_j} \) can be seen as a family of germs of automorphisms of \((\mathbb{C},0)\) with parameters in \( U_j \subset D_j \simeq \mathbb{C} \mathbb{P}^1 \) (see Figure 1). Let \( \tilde{\phi}_t \in \text{Diff}(\mathbb{C},0) \) be given by \( \tilde{\phi}_t(x) := \tilde{\phi}(t,x) \) for some \( t \in U_j \), then the classical Leau-Fatou flower theorem says that \( \tilde{\phi}_t \) has a parabolic fixed point at the origin, unless it is the identity. The result then follows by analytic continuation. 

\[ \square \]

**Figure 1.** A dicritical component of \( \tilde{\mathcal{F}} \).

### 5.2. The existence of an algebraic-topological criterion.

Here shall finish the proof of Theorem 3. For this sake, let us first recall some facts proved along this work and introduce some terminology. First notice that any \( \omega \in \text{Gen}(\mathcal{X}(\mathbb{C}^3,0)) \) admitting a holomorphic first integral must satisfy condition \((*)\) in Definition 3 (cf. [10]). Assume the curve \( S_X \) is the \( z \)-axis, let \( \Sigma_z := (z = \text{const.}) \) be a section transverse to \( S_X \), and \( \text{Hol}(\mathcal{F}(X), S_X, \Sigma_z) \) be the holonomy of \( \mathcal{F}(X) \) with respect to \( S_X \) evaluated at \( \Sigma_z \).

**End of the proof of Theorem 3.** First suppose all the leaves of \( \mathcal{F}(X) \) are closed off \( \text{Sing}(\mathcal{F}(X)) = \{0\} \subset \mathbb{C}^3 \) and there is an adapted flag \( \mathcal{F}(X) \subset \mathcal{F}_\omega \). Given a leaf \( L \) of \( \mathcal{F}(X) \) it follows that the closure \( \overline{L} \) of \( L \cup \text{Sing}(\mathcal{F}(X)) \) is an analytic subset of pure dimension one (17) in \( \mathbb{C}^3 \). Since this leaf is transversal to \( \Sigma_z \), one concludes that \( \overline{L} \cap \Sigma_z \) is a finite set. On the other hand, given a point \( x \in L \cap \Sigma_z \), its orbit in the holonomy group is also contained in \( L \cap \Sigma_z \), so that it is a finite set. Thus the orbits of \( H_z \) are finite. By hypothesis, for any \( z_0 \in S(X) \) the foliation \( \mathcal{F}_\omega|_{\Sigma_{z_0}} \) has a dicritical component. Now consider a simple loop \( \gamma \) around the origin, inside the \( z \)-axis, starting from \( z_0 \). Pick a leaf \( L \) of \( \mathcal{F}_\omega|_{\Sigma_{z_0}} \) and consider the liftings of \( \gamma \) starting at points of \( L \), along the trajectories of \( \mathcal{F}(X) \). Then these liftings form a three dimensional real variety, say \( S_L \), whose intersection with \( \Sigma_{z_0} \) is given by \( L \) and \( L' \) (see Figure 2). In particular if \( h := h_\gamma \) is the generator of \( \text{Hol}(\mathcal{F}(X), S_X, \Sigma_z) \), then \( L' = h(L) \). For the one-form \( \omega \), one has that \( S_L \) is tangent to \( \text{Ker}(\omega) \), and \( S_L \cap \Sigma_{z_0} \) is tangent to the induced foliation \( \mathcal{F}_\omega|_{\Sigma_{z_0}} \). Thus \( L' \) is a leaf of \( \mathcal{F}_\omega|_{\Sigma_{z_0}} \). Since \( \mathcal{F}_\omega|_{\Sigma_{z_1}} \) has a dicritical component and \( h \) is a diffeomorphism with resonant linear part having finite orbits, then Lemma 1 ensures that \( h \) is periodic (in particular linearizable and finite). Since \( \mathcal{F}(X) \in \text{Gen}(\mathcal{X}(\mathbb{C}^3,0)) \) has linearizable periodic holonomy, then it follows from [15] that the foliation \( \mathcal{F}(X) \) is also analytically linearizable. Therefore, one may suppose without loss of generality that \( X(x,y,z) = mx \frac{\partial}{\partial x} + ny \frac{\partial}{\partial y} - kz \frac{\partial}{\partial z} \). This vector field has...
a holomorphic first integral. From the above linearization, it is easy to see that the flag foliation $\mathcal{F}_\omega$ containing $\mathcal{F}(X)$ must have a linear dicritical Kupka transverse type along the $z$-axis. In particular, $\mathcal{F}_\omega$ is of radial type. This proves that (5) implies (1)-(4) and also (6). Since the converse is immediate, this proves that the first four conditions in Theorem 3 are equivalent to conditions (5) and (6). □

Figure 2. The lifting of $\gamma$ along the leaves of $\mathcal{F}$ starting at points of $L$.

Remark 1 (Parabolic curves and smooth sets of fixed points cf. [1]). In [10] it is stated an integrability result, mentioning only the fact that the leaves of $\mathcal{F}(X)$ are closed off $\text{Sing}(\mathcal{F}(X))$. Nevertheless, as we shall see in the next sections, this result is not correct. Indeed, there are such vector fields without holomorphic first integral (cf. Example 4). Let us identify precisely the missing point in [10] and to determine some further topological conditions in order to correct the statements of the main theorems therein (Theorems 1.2 and 1.3 in [10]). Along these lines, we shall keep all the notations introduced in [10]. First, let us deal with the missing point in [10]. In Theorem 3.6 of [10], we have stated that every non trivial complex map germ fixing the origin admits a parabolic curve. Javier Ribon draw our attention to the fact that this is not true with the following example:

Let $X^o = py\frac{\partial}{\partial y} - qx\frac{\partial}{\partial x}$ with $p, q \in \mathbb{Z}^+$ and $X = xyX^o$, then the orbits of the map $\Phi(x, y) = \exp[1]X(x, y)$ are confined in the level sets of the first integral $f(x, y) = x^p y^q$ to the vector field $X$. Therefore $\Phi$ has no orbit attracting to the origin, and thus does not admit any parabolic curve at the origin.

Some time after that, Marco Abate communicated us the same fact showing that Theorem 3.6 in [10] contradicts Proposition 2.1, p. 185, in [1]. As a matter of fact, Lemma 3.5 (and thus Theorem 3.6) is not correct. This is due to the authors misinterpretation of the proof of Corollary 3.1 in [1] wrongly stated as Theorem 3.2 in [10]. Indeed, the correct statement is the following: Let $G \in \text{Diff}_1(\mathbb{C}^2, 0)$ and suppose that $S := \text{Fix}(G)$ is a smooth curve through the origin such that $\text{ind}_0(G, S) \notin \mathbb{Q}^+$. Then $G$ admits $\nu(f) - 1$ parabolic curves.

More precisely, one can check that this would be the appropriate hypothesis looking to the proof of Theorem 3.1 in [1]. Now one can check that the diffeomorphism in the proof of Lemma 3.5 in [10] does not satisfy the conditions of the above theorem.

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