Coleman Maps for Modular Forms at Supersingular Primes over Lubin-Tate Extensions

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Abstract

Given an elliptic curve with supersingular reduction at an odd prime \( p \), Iovita and Pollack have generalised results of Kobayashi to define even and odd Coleman maps at \( p \) over Lubin-Tate extensions given by a formal group of height 1. We generalise this construction to modular forms of higher weights.

0 Introduction

Let \( f \) be a normalised eigen-newform of integral weight at least 2 and \( p \) an odd supersingular prime for \( f \) (i.e. \( p \) divides \( a_p \) but not the level of \( f \)). On the one hand, the \( p \)-adic \( L \)-functions of \( f \) defined in [11] have unbounded coefficients. On the other hand, the \( p \)-Selmer group over the \( \mathbb{Q}_\infty \), the extension of \( \mathbb{Q} \) by adjoining all \( p \) power roots of unity, is not \( \Lambda \)-cotorsion where \( \Lambda \) is the Iwasawa algebra of \( \mathbb{Z}_p[[\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]] \), which can be identified with the set of power series over \( \mathbb{Z}_p[\text{Gal}(k_1/\mathbb{Q}_p)] \). It makes the Iwasawa theory for \( f \) at \( p \) difficult.

Much progress has been made in this direction. In [13], Pollack has defined the plus and minus analytic \( p \)-adic \( L \)-functions \( L_p^\pm \), which have bounded coefficients in the case \( a_p = 0 \). When \( f \) corresponds to an elliptic curve \( E \) defined over \( \mathbb{Q} \) and \( p \) is as above, Kobayashi [8] defined the even and odd Selmer groups \( \text{Sel}^\pm_p(E/\mathbb{Q}_\infty) \) by modifying the local condition of the usual Selmer group at \( p \).

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These conditions are obtained by applying Honda theory to the formal group associated to $E$ at $p$. Kobayashi then used these conditions to construct

$$\text{Col}^\pm : \lim_{\leftarrow} H^1(k_n, T_E) \to \Lambda$$

where $T_E$ is the Tate module of $E$ at $p$ and $k_n = \mathbb{Q}_p(\mu_{p^n})$. It turns out that on applying $\text{Col}^\pm$ to the Kato zeta element defined in [6], one obtains $L^\pm_p$, which can be used to show that $\text{Sel}_p^\pm(E/\mathbb{Q}_\infty)$ are $\Lambda$-cotorsion. It is then possible to formulate the “main conjecture” in the following form:

**Conjecture 0.1.** With the notation above, the characteristic ideal of the Pontryagin dual of $\text{Sel}_p^\pm(E/\mathbb{Q}_\infty)$ is generated by $L^\pm_p$.

On the one hand, the construction of $\text{Col}^\pm$ was generalised by Iovita and Pollack [5] to Lubin-Tate extensions given by formal groups of height 1. That is, we can replace $k_n$ by extensions of $\mathbb{Q}_p$ obtained by adjoining torsion points of a Lubin-Tate group of height 1 defined over $\mathbb{Z}_p$. On the other hand, Kobayashi’s construction can be generalised to modular forms of higher weights by using Perrin-Riou’s exponential map (see [10]). We will show that one can generalise the construction of the former to higher weight modular forms as well by using the Perrin-Riou’s exponential map constructed by Zhang [15].

As in [10], instead of defining the Coleman maps using local conditions obtained from the formal group, we define the Coleman maps directly using the Perrin-Riou’s exponential. We then obtain our new local conditions from $\ker(\text{Col}^\pm)$, which turn out to agree with the ones given by Kobayashi and Iovita-Pollack. We then use these conditions to define the corresponding Selmer groups.

We now outline the construction of $\text{Col}^\pm$ here. Let $V_f$ be the Deligne $p$-adic representation of $G_{\mathbb{Q}}$ associated to $f$. Write $V = V_f(1)$, the Tate twist of $V_f$ and fix $T$ a lattice in $V$ which is stable under $G_{\mathbb{Q}}$. Then, the Perrin-Riou’s exponential map enables us to define two elements

$$E_{h,V}(\mu_{\xi^\pm}) \in \mathcal{H}_{(k-1)/2} \otimes \lim_{\leftarrow} H^1(k_n, T)$$

where $\mathcal{H}_{(k-1)/2}$ denotes the set of power series over $\mathbb{Q}_p[\text{Gal}(k_1/\mathbb{Q}_p)]$ which are of order $\log_p(k-1)/2$. We then define

$$\mathcal{L}_{\xi^\pm} : \lim_{\leftarrow} H^1(k_n, T^*(1)) \to \mathcal{H}_{(k-1)/2}$$

$$z \mapsto \langle E_{h,V}(\mu_{\xi^\pm}), z \rangle$$

where $\langle, \rangle$ is a pairing on

$$\left( \mathcal{H}_{(k-1)/2} \otimes \lim_{\leftarrow} H^1(k_n, T) \right) \times \lim_{\leftarrow} H^1(k_n, T^*(1)) \to \mathcal{H}_{(k-1)/2}.$$

On computing some of its special values, we show that $\mathcal{L}_{\xi^\pm}(z)$ is divisible by $\log_p^\pm(k_{p,k})$, which is defined in [13] and has exact order $\log_p(k-1)/2$. This enables us
to define 
\[
\text{Col}^\pm : \lim_{\leftarrow} H^1(k_n, T^*(1)) \to \mathbb{Q} \otimes \Lambda \\
\mathbf{z} \mapsto \mathcal{L}_{\zeta^\pm}(\mathbf{z})/\log_{p,k}^+.
\]

The structure of this paper is as follows. We will review results of [15] in Section 1. In particular, we will state the properties of the Perrin-Riou’s exponential map which we will need for our construction of the Coleman maps. In Section 2, we will construct the Coleman maps using ideas from [10]. The kernels and images of these maps will be described in Section 3 under certain technical assumptions. In particular, we will define the even and odd Selmer groups for some $\mathbb{Z}_p$-extensions of a number field using our description of the kernels. Finally, we explain how the construction in Section 2 can be generalised to relative Lubin-Tate groups in Section 4 using ideas of Kim (see [7]).

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1 Perrin-Riou’s exponential map over height 1 Lubin-Tate extensions

In [15], Zhang has generalised the construction of Perrin-Riou’s exponential map defined in [12] to Lubin-Tate extensions. We review his results here.

We fix an odd prime $p$ and $\pi$ a uniformiser of $\mathbb{Z}_p$. Let $\alpha$ be the $p$-adic unit in $\mathbb{Z}_p^\times$ such that $\pi = \alpha p$. Let $g$ be a lift of Frobenius with respect to $\pi$, i.e. a power series over $\mathbb{Z}_p$ such that $g(X) = \pi X + (\text{higher terms})$ and $g(X) \equiv X^p \mod p$. Then, $g$ gives rise to an one-dimensional height-one formal group over $\mathbb{Z}_p$, which is independent of the choice of $g$ up to isomorphism over $\mathbb{Z}_p$. We denote this formal group by $F$.

We write $K = \mathbb{Q}_p$ (reason being we want to replace $\mathbb{Q}_p$ by a finite unramified extension of $\mathbb{Q}_p$ in Section 4). $K_n$ denotes the extension of $K$ obtained by adjoining the $\pi^n$th roots of $F$ and $G_n$ denotes the Galois group of $K_n$ over $K$ for $0 \leq n \leq \infty$. In particular, $G_n \cong (\mathbb{Z}/p^n)^\times$ and $G_\infty \cong G_1 \times \text{Gal}(K_\infty/K_1) \cong \mathbb{Z}/(p-1) \times \mathbb{Z}_p$.

Let $\kappa$ be the character of $G_K$ (the absolute Galois group of $K$) given by its action on the Tate module of $F$. Then, $\sigma \omega = [\kappa(\sigma)] x(\omega)$ for all $\omega \in F[\pi^\infty]$. If $\chi$ denotes the cyclotomic character of $G_K$, then $\kappa = \chi \psi$ for an unramified character $\psi$.

Let $\Xi$ denote the completion of the maximal unramified extension of $\mathbb{Q}_p$ and $\mathfrak{D}$ its ring of integers. Let $\eta : \mathbb{G}_m \to F$ be an isomorphism between the multiplicative group and $F$. Then $\eta \in \mathfrak{D}[[X]]$. Moreover, $\eta(X) = \Omega X + (\text{higher degree terms})$, where $\Omega$ is a $p$-adic unit. The lift of Frobenius $g$ satisfies $g \circ \eta = \eta^p \circ ((1 + X)^p - 1)$ where $\varphi$ is the Frobenius of $\text{Gal}(\mathbb{Q}_p^{ur}/\mathbb{Q}_p)$ which acts on $\eta$ by acting on its coefficients. In particular, $\Omega^\varphi = \alpha \Omega$. 

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Definition 1.1. We define $\Xi[[X]]^\psi$ to be the set of power series $f$, defined over $\Xi$, such that $\sigma f(X) = f((1 + X)^{\psi(\sigma) - 1})\forall \sigma \in G_K$.

In particular, [15 (1.13)] says that $\eta \in \Xi[[X]]^\psi$. The significance of this set is given by the following:

Lemma 1.2. Let $f \in \Xi[[X]]^\psi$ and $\zeta$ a $p^n$th root of unity. Then $f(\zeta - 1) \in K_n$.

Proof. By definition, $\sigma f(X) = f((1 + X)^{\psi(\sigma) - 1})$ for any $\sigma \in G_K$. Therefore, we have

$$\sigma(f(\zeta - 1)) = (\sigma f)(\zeta^\sigma - 1) = f((\zeta^\chi(\sigma))^{\psi(\sigma)} - 1) = f((\zeta^{\kappa(\sigma)} - 1).$$

If, in addition, $\sigma \in G_{K_n}$, then $\kappa(\sigma) \in 1 + p^n\mathbb{Z}_p$. Hence, $\sigma(f(\zeta - 1)) = f(\zeta - 1)$ for any $\sigma \in G_{K_n}$, so we are done. \hfill \square

From now on, we fix a primitive $p^n$th root of unity $\zeta_{p^n}$ for each positive integer $n$ such that $\zeta_{p^n}^p = \zeta_{p^n}$. This determines an element $t \in B_{dR}^+$ (see [2, Section III.1] for details). We also fix a crystalline (hence de Rham) representation $V$ of $G_K$ and write $D(V) = D_{dR}(V) = D_{cris}(V)$ for its Dieudonné module which is equipped with a de Rham filtration and an action of $\varphi$. We denote the $i$th de Rham filtration by $D^i(V)$. We write $r(V)$ for the slope of $\varphi$ on $D(V)$. Note that the action of $\varphi$ extends to $\Xi \otimes D(V)$ naturally.

We write $V(k)$ for the $k$th Tate twist of $V$. Then, $D(V(k)) = t^{-k}D(V)$ as $G_K$ acts on $t$ via $\chi$. Similarly, $D(V(\kappa^k)) = t^{-k}D(V)$ where $t_x = \Omega t$ since $G_K$ acts on $t_x$ via $\kappa$ by [15, Section 2]. Their filtrations are given by the following:

Lemma 1.3. The de Rham filtrations satisfy

$$D^i(V(\kappa^j)) = D^i(V(j)) = t_x^{-i}D^{i+j}(V).$$

Proof. Since $\Omega \in \tilde{K}^\times$, we have

$$D^i(V(\kappa^j)) = (t_x^{-i}D(V)) \cap t^iB_{dR}^+ = t_x^{-i}(D(V) \cap t^{i+j}\Omega B_{dR}^+),$$

$$= t_x^{-i}(D(V) \cap t^{i+j}B_{dR}^+),$$

$$= t_x^{-i}D^{i+j}(V).$$

Hence we are done. \hfill \square

For $r \in \mathbb{R}_{\geq 0}$, let $B$ be a Banach $p$-adic space, then $\mathcal{D}_r(\mathbb{Q}_p, B)$ denotes the set of tempered $B$-valued distributions of order $r$ (in the sense of [2 Definition I.4.2]) on the locally analytic functions with compact support in $\mathbb{Q}_p$. It is equipped with a Galois action of $G_K$ as defined in [15 (3.1)]. Similarly, if $A$ is a compact open subset of $\mathbb{Q}_p$, $\mathcal{D}_r(A, B)$ denotes the set of tempered distributions of order $r$ on $A$ with values in $B$. 
When $A = \mathbb{Z}_p$, we write the Amice transform of $\mu \in \mathcal{D}_r(\mathbb{Z}_p, \mathcal{B})$ as $\mathcal{A}_\mu \in \mathcal{B}[\![X]\!]$, i.e.

$$\mathcal{A}_\mu(X) = \int_{\mathbb{Z}_p} (1 + X)^\mu(x).$$

We define $\mathcal{D}_r(\mathbb{Q}_p, \Xi \otimes \mathcal{D}(V))$ to be the subset of $\mathcal{D}_r(\mathbb{Q}_p, \Xi \otimes \mathcal{D}(V))$ consisting of all the distributions $\mu$ satisfying:

$$\sigma \left( \int_{\mathbb{Q}_p} f \mu \right) = \int_{\mathbb{Q}_p} f(\psi(\sigma)x) \mu \forall \sigma \in G_K.$$

**Remark 1.4.** Let $\mu \in \mathcal{D}_r(\mathbb{Q}_p, \Xi \otimes \mathcal{D}(V))$. Then, $\mu \in \mathcal{D}_r(\mathbb{Q}_p, \Xi \otimes \mathcal{D}(V))$ iff its Amice transform is in $\Xi[\![X]\!] \psi \otimes \mathcal{D}(V)$ (see [13, Proposition 2.4(i)]).

We define $\widetilde{\mathcal{D}}_r(\mathbb{Z}_p^\times, \Xi \otimes \mathcal{D}(V))$ to be $\lim_{\leftarrow} \mathcal{D}_r(\mathbb{Z}_p^\times, \Xi \otimes \mathcal{D}(V))$ where $T_w$ is the twist map given by $\mu \mapsto (-tx)^{-1} \mu$, which is well defined by [14, Lemma 3.6]. We define $\widetilde{\mathcal{D}}_r(\mathbb{Q}_p, \Xi \otimes \mathcal{D}(V))$ similarly. In [15, Theorems 3.3 and 3.6], the generalised Perrin-Riou’s exponential is given by:

**Theorem 1.5.** Let $h$ be a positive integer such that $D^{-h}(V) = D(V)$. Then, there is a map

$$\mathbb{E}_{h,V} : \widetilde{\mathcal{D}}_r(\mathbb{Q}_p, \Xi \otimes \mathcal{D}(V))^{\psi_0 \otimes \psi = 1, \psi} \to H^1(K_{\infty}, \mathcal{D}_{r+r(V)+h}(\mathbb{Z}_p^\times, D(V)))^{G_{\infty}}$$

such that for $k \geq 1 - h$

$$\int_{\mathbb{Z}_p^\times} x^k \mathbb{E}_{h,V}(\mu) = (k + h - 1)! \exp_k \left( (1 - \varphi)^{-1} \left( 1 - \frac{\varphi^{-n}}{p} \int_{\mathbb{Z}_p} \frac{\mu}{(-tx)^n} \right) \right),$$

$$\int_{1+p^n \mathbb{Z}_p} x^k \mathbb{E}_{h,V}(\mu) = (k + h - 1)! \exp_k \left( \frac{\varphi^{-n}}{p^n} \int_{\mathbb{Z}_p} \epsilon \left( \frac{x}{p^n} \right) \frac{\mu}{(-tx)^n} \right)$$

where $\epsilon$ is as defined in [2, Section V.1] and $\exp_k$ denotes the exponential map for the $p$-adic representation $V(\kappa^k)$ as defined in [1].

**2 The construction of even and odd Coleman maps**

We construct $\text{Col}^\pm$ in three steps. First, we prove some elementary results about distributions on $\mathbb{Z}_p^\times$ in Section 2.1. In Section 2.2, we explain how to construct a measure $\mu_\xi \in \mathcal{D}_0(\mathbb{Z}_p^\times, \Xi \otimes D(V))^{\psi}$ from a given $\xi \in D(V)$ and compute some special values of $\mathbb{E}_{h,V}(\mu_\xi)$ using Theorem 1.5 and results from Section 2.1. Finally, in Section 2.3, we apply these results to a modular form $f$ by choosing two elements of $D(V_f)$, namely $\xi^\pm$. We then proceed as explained in the introduction to construct Col$^\pm$. 5
2.1 Distributions on $\mathbb{Z}_p^\times$

Let $\mu \in \mathcal{D}_r(\mathbb{Z}_p^\times, \xi \otimes \mathcal{D}(V))^\psi$, then $\mu \in \mathcal{D}_r(\mathbb{Z}_p^\times, \xi \otimes \mathcal{D}(V))^\psi$ iff

$$\sum_{\zeta = 1}^A \mathcal{A}_\mu(\zeta(1 + X) - 1) = 0.$$ 

On the space of power series satisfying this condition, $D = (1 + X)^{\frac{1}{p^k}}$ acts bijectively. Moreover, for such $\mu$, we have

$$D^k A_\mu(\zeta p^n - 1) = \int_{\mathbb{Z}_p^\times} \varepsilon \left( \frac{x}{p^n} \right) x^k \mu,$$  \hspace{1cm} (1)

see e.g. [2, Section I.5].

**Lemma 2.1.** Any $\mu \in \mathcal{D}_r(\mathbb{Z}_p^\times, \xi \otimes \mathcal{D}(V))^\psi$ can be lifted to

$$\tilde{\mu} \in \tilde{\mathcal{D}}_r(\mathbb{Q}_p, \xi \otimes \mathcal{D}(V))^{\varphi \otimes \varphi = 1, \psi}.$$ 

Moreover, the image of such a lift under $\mathcal{E}_{h,V}$ is independent of the choice of the lift.

**Proof.** [2, Lemma IX.2.8 and Remark IX.2.6(iii)] and [15, Lemma 3.5]. \hfill \square

Given any $\mu \in \mathcal{D}_r(\mathbb{Z}_p^\times, \xi \otimes \mathcal{D}(V))^\psi$, we abuse notation and write $\mathcal{E}_{h,V}(\mu) = \mathcal{E}_{h,V}(\tilde{\mu})$ where $\tilde{\mu}$ is a lift given by Lemma 2.1. The fact that $\varphi_\mathcal{D} \otimes \varphi(\tilde{\mu}) = \tilde{\mu}$ implies that

$$\int_{pA} f(x) \tilde{\mu} = \varphi \left( \int_{A} f(px) \tilde{\mu} \right)$$  \hspace{1cm} (2)

for any $f$ and $A \subset \mathbb{Q}_p$. It allows us to compute some special values of $\tilde{\mu}$.

**Lemma 2.2.** With the above notation, $\int_{\mathbb{Z}_p^\times} x^k \tilde{\mu} = (1 - p^k \varphi)^{-1} \left( D^k A_\mu(0) \right)$.

**Proof.** Since $\tilde{\mu}_\xi$ restricts to $\mu_\xi$ on $\mathbb{Z}_p^\times$, (1) implies that

$$\int_{\mathbb{Z}_p^\times} x^k \tilde{\mu}_\xi = \int_{\mathbb{Z}_p^\times} x^k \mu_\xi = D^k A_\mu(0).$$

Hence, by applying (2) to the decomposition

$$\int_{\mathbb{Z}_p} x^k \tilde{\mu} = \int_{p\mathbb{Z}_p} x^k \tilde{\mu} + \int_{\mathbb{Z}_p} x^k \tilde{\mu},$$

we have

$$\int_{\mathbb{Z}_p} x^k \tilde{\mu} = p^k \varphi \left( \int_{\mathbb{Z}_p} x^k \tilde{\mu} \right) + D^k A_\mu(0).$$ \hfill \square
Lemma 2.3. With the notation above,
\[
\int_{\mathbb{Z}_p} \epsilon \left( \frac{x}{p^n} \right) x^k \tilde{\mu} = \sum_{i=0}^{n-1} p^i \varphi^i \left( D^k A_\mu (\zeta_{p^{n-i}} - 1) \right) + p^n (1 - p^k \varphi)^{-1} (D^k A_\mu (0)).
\]

Proof. Since \(\mathbb{Z}_p = \mathbb{Z}_p^\times \cup p \mathbb{Z}_p^\times \cup \cdots \cup p^{n-1} \mathbb{Z}_p^\times \cup p^n \mathbb{Z}_p\), we have
\[
\int_{\mathbb{Z}_p} \epsilon \left( \frac{x}{p^n} \right) x^k \tilde{\mu} = \sum_{i=0}^{n-1} \int_{p^i \mathbb{Z}_p^\times} \epsilon \left( \frac{x}{p^n} \right) x^k \mu + \int_{p^n \mathbb{Z}_p} \epsilon \left( \frac{x}{p^n} \right) x^k \tilde{\mu}
\]
\[
= \sum_{i=0}^{n-1} p^i \varphi^i \left( \int_{\mathbb{Z}_p^\times} \epsilon \left( \frac{x}{p^{n-i}} \right) x^k \mu \right) + p^n \varphi^n \int_{\mathbb{Z}_p} x^k \tilde{\mu}
\]
where the last equality follows from repeated applications of (2). Hence the result by (1) and Lemma 2.2.

2.2 Computing some special values

With the notation above, we define
\[
\bar{\eta}(X) = \eta(X) - \frac{1}{p} \sum_{\zeta = 1}^{\mathbb{Z}_p^\times} \eta(\zeta (1 + X) - 1).
\]

Then \(\sum_{\zeta = 1}^{\mathbb{Z}_p} \bar{\eta}(\zeta (1 + X) - 1) = 0\). Moreover, we have:

Lemma 2.4. We have \(\bar{\eta} \in \Xi[[X]]^\psi\).

Proof. Let \(\sigma \in G_{\mathbb{Q}_p}\) and \(\zeta\) a \(p\)th root of unity. By [15 (1.13)], \(\eta \in \Xi[[X]]^\psi\) and \(\sigma \eta(X) = \eta((1 + X)^\psi(\sigma) - 1)\). If we replace \(X\) by \(\zeta^\sigma (1 + X) - 1\), we have
\[
\sigma(\eta(\zeta^\sigma (1 + X) - 1)) = (\sigma \eta)(\zeta^\sigma (1 + X) - 1)
\]
\[
= \eta((\zeta^\sigma (1 + X))^{\psi(\sigma)} - 1) = \eta(\zeta^{\kappa(\sigma)} (1 + X)^{\psi(\sigma)} - 1)
\]
Hence, on summing over \(\zeta^p = 1\), we have
\[
\sigma \left( \sum_{\zeta = 1}^{\mathbb{Z}_p} \eta(\zeta (1 + X) - 1) \right) = \sum_{\zeta^p = 1} \sigma(\eta(\zeta (1 + X) - 1)) = \sum_{\zeta^p = 1} \eta(\zeta^{\kappa(\sigma)} (1 + X)^{\psi(\sigma)} - 1) = \sum_{\zeta^p = 1} \eta(\zeta (1 + X)^{\psi(\sigma)} - 1) \) (as \(\kappa(\sigma) \in \mathbb{Z}_p^\times\)).

Hence, the sum \(\sum_{\zeta^p = 1} \eta(\zeta (1 + X) - 1) \in \Xi[[X]]^\psi\), so we are done.
Let \( \xi \in D(V) \), then \( \bar{\eta}(X) \otimes \xi \) defines an element \( \mu_\xi \in D_0(\mathbb{Z}_p, \Xi \otimes D(V)) \) with
\[
\bar{\eta}(X) \otimes \xi = \int_{\mathbb{Z}_p} (1 + X)^\xi \mu_\xi.
\]
By Lemma 2.3 and Remark 1.4, \( \mu_\xi \in D_0(\mathbb{Z}_p, \Xi \otimes D(V)) \). On applying the Perrin-Riou's exponential, we have:

**Proposition 2.5.** With the notation above, we have for \( n \geq 1 \) and \( k \geq 1 - h \)
\[
\int_{1+p^n\mathbb{Z}_p} (-x)^k E_{h,V}(\mu_\xi) = (k + h - 1)! \exp_k (\gamma_{n,k}(\xi))
\]
where \( \gamma_{n,k}(\xi) \) is defined by
\[
\frac{1}{p^n} \left( \sum_{i=0}^{n-1} D^{-k} \bar{\eta}^i \varphi^{n-1}(\xi_p^n - 1) \otimes \varphi^{i-n}(\xi_k) + (1 - \varphi)^{-1}(D^{-k} \bar{\eta}(0) \otimes \xi_k) \right)
\]
with \( \xi_k = \xi t^{-k} \).

**Proof.** The result follows from combining Theorem 1.5 with Lemmas 2.2 and 2.3 and the fact that \( \varphi(t) = pt \).

Our assumption on the eigenvalues of \( \varphi \) implies that there is an isomorphism
\[
H^1(K_\infty, D_0(\mathbb{Z}_p^\times, V))^{G_\infty} \cong D_r(G_\infty) \otimes H^1_{Iw}(V)
\]
\[
\mu \mapsto \left( \int_{1+p^n\mathbb{Z}_p} \mu \right)_n
\]
where \( H^1_{Iw}(V) := \lim_{\leftarrow n} H^1(K_n, V) \) and \( D_r(G_\infty) = D_r(G_\infty, \mathbb{Q}_p) \) (see e.g. [2, Proposition 2]). Under this identification, we have
\[
E_{h,V}(\mu_\xi) \in D_{h+r(V)}(G_\infty) \otimes H^1_{Iw}(V).
\]

Write \( T_{w,k} : H^1_{Iw}(V) \rightarrow H^1_{Iw}(V^{k}) \) for the twist map. Recall that \( T_{w,k}(\mu) = (-tx)^{-k} \mu \), so Proposition 2.3 implies that if \( n \geq 1 \) and \( k \geq 1 - h \), the \( n \)th component of \( T_{w,k}(E_{h,V}(\mu)) \) is given by
\[
(k + h - 1)! \exp_k (\gamma_{n,k}(\xi))
\]
where \( \exp_k \) now denotes the exponential map \( K_n \otimes D(V^{k}) \rightarrow H^1(K_n, V^{k}) \).

Recall that \( G_\infty \cong G_1 \times \Gamma \) where \( \Gamma \cong \mathbb{Z}_p \). We fix a topological generator \( \gamma \) of \( \Gamma \), then \( D_r(G_\infty) \) can be identified with the set of power series in \( \gamma - 1 \) over \( \mathbb{Q}_p[G_1] \) which are \( O(\log_p^\ast) \).

We now assume that \( V \) has a \( F \)-vector space structure where \( F \) is a finite extension of \( \mathbb{Q}_p \) and the action of \( G_K \) commutes with the multiplication by \( F \).
Denote the ring of integers of $F$ by $\mathcal{O}_F$. Let $\Lambda = \mathcal{O}_F[[G_\infty]] = \lim_{\to} \mathcal{O}_F[G_n]$, then there is a pairing
\[
<,> : \mathbb{H}^1_{Iw}(V) \times \mathbb{H}^1_{Iw}(V^*(1)) \rightarrow \mathbb{Q} \otimes \Lambda
\]
\[
((x_n)_n, (y_n)_n) \mapsto \left( \sum_{\sigma \in G_n} [x_n^\sigma, y_n]^n \right)_n
\]
where $[,]_n$ is the pairing on $H^1(K_n, V) \times H^1(K_n, V^*(1)) \rightarrow F$. It extends to
\[
\left( D_m(G_\infty) \otimes \mathbb{H}^1_{Iw}(V) \right) \times \left( D_n(G_\infty) \otimes \mathbb{H}^1_{Iw}(V^*(1)) \right) \rightarrow D_{m+n}(G_\infty)
\]
for all $m, n \in \mathbb{R}_{\geq 0}$. This enables us to define the following:

**Definition 2.6.** For a fixed $\xi \in D(V)$, we define a map
\[
\mathcal{L}^h_\xi : \mathbb{H}^1_{Iw}(V^*(1)) \rightarrow D_\theta(V)_{+h}(G_\infty)
\]
\[
z \mapsto < \mathbb{E}_{h, V}(\mu_z), z > .
\]

Following the calculations of [9], we find that for $n \geq 1$, the $n$th component of $\text{Tw}_k \mathcal{L}^h_\xi(z)$ is given by:
\[
(Tw_k \mathcal{L}^h_\xi(z))_n = (h + k - 1)! \sum_{\sigma \in G_n} [\exp_k(\gamma_{n,k}(\xi)^n), z_{-k,n}]_n^\sigma
\]
\[
= (h + k - 1)! \sum_{\sigma \in G_n} \gamma_{n,k}(\xi)^n \sigma, \sum_{\sigma \in G_n} \exp_k^*(z_{-k,n}^\sigma \sigma^{-1})_n
\]
where $z_{-k,n}$ denotes the image of $z$ under
\[
\mathbb{H}^1_{Iw}(V^*(1)) \rightarrow \mathbb{H}^1_{Iw}(V^*(1)(\kappa^{-k})) \rightarrow H^1(K_n, V^*(1)(\kappa^{-k}))
\]
and $\text{Tw}_k$ acts on $D_\theta(V)_{+h}(G_\infty)$ by $\sigma \mapsto \kappa(\sigma)^k \sigma$ for $\sigma \in G_\infty$.

Let $\theta$ be a character on $G_n$ which does not factor through $G_{n-1}$. Since $D^{-k}\bar{\eta}^{-n}((\zeta_{p^n-1}) \in K_{n-i}$ by Lemma [12], we have
\[
\theta \left( \sum_{\sigma \in G_n} \gamma_{n,k}(\xi)^n \sigma \right) = \frac{1}{p^n} \sum_{\sigma \in G_n} D^{-k}\bar{\eta}^{-n}((\zeta_{p^n-1})^\sigma \theta(\sigma) \otimes \varphi^{-n}(\xi_k).
\]
Hence, as in [10] Lemma 1.4, we have
\[
\frac{1}{(h + k - 1)!} k^h \theta(\mathcal{L}^h_\xi(z))
\]
\[
= \frac{1}{p^n} \sum_{\sigma \in G_n} D^{-k}\bar{\eta}^{-n}((\zeta_{p^n-1})^\sigma \theta(\sigma) \otimes \varphi^{-n}(\xi_k)) \sum_{\sigma \in G_n} \exp_k^*(z_{-k,n}^\sigma \theta(\sigma^{-1}))_n .
\]
(4)
2.3 Modular forms

From now on, we fix a normalised newform \( f = \sum a_n q^n \) of integral weight \( k \geq 2 \) with \( p \) a supersingular prime for \( f \) and \( a_p = 0 \) (i.e. \( p \) divides \( a_p \) but not the level of \( f \)). We allow the character of \( f \) to be arbitrary, but for the sole purpose of easing notation, we assume that the character of \( f \) takes value 1 at \( p \). Let \( V_f \) be the Deligne representation of \( G_\mathbb{Q} \) defined in [3]. Let \( L = \mathbb{Q}(a_n : n \geq 1) \) be the field of coefficients of \( f \) and fix a place of \( L \) above \( p \). Then, \( V \) is a two-dimensional vector space over \( F = \mathbb{Q}_p \) and the action of \( G_\mathbb{Q} \) commutes with \( F \). If we take \( V \) to be \( V_f(1) \), the Frobenius \( \varphi \) on \( D(V) \) satisfies

\[
\varphi^2 \frac{a_p}{p} \varphi + p^{k-3} = 0.
\]

In particular, \( r(V) = (k - 1)/2 - 1 \) and the assumption that the eigenvalues of \( \varphi \) on \( D(V_f) \) are not integral powers of \( p \) is automatically satisfied. On taking \( h = 1 \) in Theorem [1.3] and writing \( \mathcal{L}_\xi \) for \( \mathcal{L}_k^h \), we have \( \text{Im}(\mathcal{L}_\xi) \subset D_{(k-1)/2}(G_\infty) \) for any \( \xi \in D(V) \).

The de Rham filtration of \( D(V_f) \) is given by

\[
D^i(V_f) = D^0(V_f(i)) = \begin{cases} 
D(V_f) & \text{if } i \leq 0 \\
0 & \text{if } i \geq k \\
F \cdot \omega & \text{if } 1 \leq i \leq k - 1.
\end{cases}
\]

where \( \omega \) is any non-zero element of \( D^1(V_f) = D^0(V) \). We fix one such \( \omega \), this corresponds to a choice of periods for \( f \) (see [3]). We have \( D^0(V(j)) = D^0(V(k\gamma)) = F \cdot \omega \) for \( 0 \leq j \leq k - 2 \).

Let \( \gamma = k(u) \), then we can define \( \log_{p,k}^+ \) as in [13]:

\[
\log_{p,k}^+ = \prod_{j=0}^{k-2} \prod_{n=1}^{\infty} \frac{\Phi_{2n}(\gamma^{-j}u)}{p},
\]

\[
\log_{p,k}^- = \prod_{j=0}^{k-2} \prod_{n=1}^{\infty} \frac{\Phi_{2n-1}(\gamma^{-j}u)}{p},
\]

where \( \Phi_n \) denotes the \( p^m \)th cyclotomic polynomial. In particular, the zeros of \( \log_{p,k}^+ \) are given by \( \kappa^j \theta \) where \( 0 \leq j \leq k - 2 \) and \( \theta \) is a character of \( G_n \) which does not factor through \( G_{n-1} \) with \( n \) odd, whereas those of \( \log_{p,k}^- \) are characters of the same form but with even \( n \). Moreover, \( \log_{p,k}^+ \) and \( \log_{p,k}^- \) have exact order \( \log_{p,k} \). We can now give a generalisation of [3] Lemma 2.2:

**Lemma 2.7.** Let \( \xi^+ = \varphi(\omega) \) and \( \xi^- = \omega \), then \( \log_{p,k}^+ | L_{\xi^+}(z) \) for all \( z \in \mathbb{H}^1 \). We have \( \varphi^{2n}(\omega) \in D^0(V(k\gamma)) \) for all integers \( n \) and \( 0 \leq r \leq k - 2 \).

**Proof.** We have \( \varphi^{2n}(\omega) \in D^0(V(k\gamma)) \) for all integers \( n \) and \( 0 \leq r \leq k - 2 \). Therefore, by [3], we have

\[
\kappa^r \theta(L_{\xi^+}(z)) = 0 \quad \text{if } n \text{ is odd,}
\]

\[
\kappa^r \theta(L_{\xi^-}(z)) = 0 \quad \text{if } n \text{ is even.}
\]
where \( \theta \) is a character of \( G_n \) which does not factor through \( G_{n-1} \). Hence, the zeros of \( \log_{p,k}^\pm \) are also zeros of \( L_{\xi}(z) \), so we are done.

In particular, since \( L_{\xi}(z) \in \mathcal{D}_{(k-1)/2}(G_\infty) \), we have \( L_{\xi}(z)/\log_{p,k}^\pm = O(1) \). Hence, we have:

**Definition 2.8.** The even and odd Coleman maps are defined to be

\[
\text{Col}^\pm : H^1_{Iw}(V^*(1)) \to \mathbb{Q} \otimes \Lambda, \\
z \mapsto L_{\xi}(z)/\log_{p,k}^\pm.
\]

3 Kernel

In this section, we describe the kernels of \( \text{Col}^\pm \), generalising those given in [5] and use them to define the even and odd Selmer groups. We first give some elementary linear algebra results.

3.1 Linear algebra

For any positive integer \( n \), we write \( \pi_n = \eta \phi_n - n(\zeta_{p^n} - 1) \). Then, \( g^n(\pi_n) = 0 \) where \( g(\pi_n) = \pi_n - \zeta_{p^n} \phi_n \zeta_{p^n}^{-1} \). Moreover, \( g(\pi_n) = \pi_n - 1 \) and \( K_n = K(\pi_n) \). We will from now on assume \( g \) to be a good lift of Frobenius in the sense of [5], Section 4.1. In particular, we will have to assume \( \pi \in p(1 + p\mathbb{Z}_p) \) which would exclude many Lubin-Tate extensions of \( \mathbb{Q}_p \). However, if we start with a totally ramified \( \mathbb{Z}_p \)-extension of \( \mathbb{Q}_p \), then we can always assume that it is obtained from such Lubin-Tate extensions (see [5] for details). For \( n > 1 \), let \( \pi'_n = \pi_n - \zeta_{p^n}^{-1} \phi_n \zeta_{p^n}^{-1} \pi_n \) and \( \pi'_1 = \pi_1 - \frac{1}{p-1} \phi_n \zeta_{p^n}^{-1} \pi_1 = \pi_1 + \frac{p}{p-1} \). Then, \( \text{Tr}_{n/n-1}(\pi'_n) = 0 \) for all \( n \geq 1 \).

**Lemma 3.1.** Let \( K^{(n)} \) be the kernel of the trace map from \( K_n \) to \( K_{n-1} \), then \( \{ \pi'_n : \sigma \in G_n \} \) generates \( K^{(n)} \) over \( K \).

**Proof.** Let \( x \in K^{(n)} \). By [5] Proposition 4.4, we have \( x \in K[G_n] \pi_n + K_{n-1} \). Since \( \text{Tr}_{n/n-1} \pi_n \in K_{n-1} \), we can write \( x = \sum_{\sigma \in G_n} a_\sigma \pi'_n + y \) for some \( a_\sigma \in K \) and \( y \in K_{n-1} \). Since \( \text{Tr}_{n/n-1} x = \text{Tr}_{n/n-1} \pi'_n = 0 \) for all \( \sigma \), we have \( y = 0 \). Hence we are done.

**Corollary 3.2.** Let \( n \geq 0 \) be an integer and \( \alpha = \sum_{i=0}^n x_i \pi'_i \) for some \( x_i \in K \) with \( \pi'_0 = 1 \). Then, the \( k \)-vector space generated by \( \{ \alpha^\sigma : \sigma \in G_n \} \) is given by \( \bigoplus K^{(S)} \) where \( S = \{ i : x_i \neq 0 \} \) and \( K^{(0)} = K \).

**Proof.** We proceed by induction on \( |S| \). The case \( |S| = 1 \) follows directly from Lemma 3.1.
Without loss of generality, we assume that $x_n \neq 0$. Let $\beta = \sum_{i=0}^{n-1} x_i \pi_i$. Then, by induction, $\{\beta^\tau : \tau \in G_{n-1}\}$, generates $\bigoplus_{i \in S \setminus \{n\}} K^{(i)}$ over $K$. Fix $\tau \in G_{n-1}$ and consider the following $p$ elements: $\alpha^\tau, \sigma|_{K_{n-1}} = \tau$. Then, their sum equals $p\beta^\tau + (\text{Tr}_{n-1} \pi_n^* \rho^\tau) = p\beta^\tau$. Therefore, for any $\tau \in G_{n-1}$ and $\sigma \in G_n$, $\beta^\tau$ and $\pi_n^*$ lie inside the $K$-vector space generated by $\alpha^\tau$. Hence we are done. \qed

### 3.2 Description of the kernels

We now fix a lattice $T_f$ in $V_f$ which is stable under $G_K$. Write $T = T_f(1) \subset V = V_f(1)$. To describe the kernel of $\text{Col}^\pm$, we will assume $p \geq k - 1$ as in \cite{[10]}. This implies that $(V/T(k^m))^{G_{K_n}} = 0$ for any $j$ and $n$ as in \cite{[10]} Lemma 2.5. Therefore, $H^1(K_n, T(k^m))$ injects into $H^1(K_n, V(k^m))$ under the natural map and we can treat the former as a lattice of the latter. In addition, the corestriction maps between $H^1(K_n, T(k^m))$ are surjective and the restriction maps are injective (see \cite{[8]}). We will treat $H^1(K_n, T(k^m))$ as a subset of $H^1(K_n, T(k^m))$ for $n' \geq n$.

Let $z \in H^1_{w}(T^*(1))$, then $z \in \ker(\text{Col}^\pm)$ iff $z_{-m,n}$ is in the annihilator of the $O_F$-module generated by $\{\exp_m(\gamma_{n,m}(\xi^\pm)^m) : \sigma \in G_n\}$ for all $n \geq 0$ and $0 \leq m \leq k - 2$. By \cite{[10]} Proposition 2.7, this is in fact equivalent to the same statement being true for all, $n \geq 0$ with one fixed $m \in \{0, \ldots, k - 2\}$ (we will take $m = 0$ below).

Instead of looking at the said $O_F$-module, we study the $F$-vector space generated by these elements inside $H^1_f(K_n, V(k^m))$ first. We can then intersect it with $H^1_f(K_n, T(k^m))$ to obtain the kernel.

**Proposition 3.3.** The vector subspace over $F$ of $H^1_f(K_n, V(k))$ generated by the set $\{\exp(\gamma_{n,0}(\xi^\pm)^m) : \sigma \in G_n\}$, is equal to

$$\{x \in H^1_f(K_n, V) : \text{cor}_{n/m+1} x \in H^1_f(K_m, V) \forall m \text{ even (odd)}\}.$$

**Proof.** Recall that by the proof of Lemma \cite{[12]} we have $\sigma f(\zeta - 1) = f(\zeta^{\kappa(\sigma)} - 1)$ for any $f \in \Xi([X])$, $\sigma \in G_K$ and $\zeta$ a $p$ power root of unity. Therefore, for $n > 1$

$$\sum_{\zeta^{p^n} = 1} f(\zeta^{p^n} - 1) = \text{Tr}_{n/n-1} f(\zeta^{p^n} - 1).$$

If $n = 1$, then

$$\sum_{\zeta^{p} = 1} f(\zeta^{p} - 1) = f(0) + \text{Tr}_{1/0} f(\zeta - 1).$$
Hence, we have
\[ p^n \gamma_{n,0}(\xi) = \sum_{i=0}^{n-1} \bar{\eta}^{i-n}(\zeta_{p^n-i} - 1) \otimes \varphi^{i-n}(\xi) + \bar{\eta}(0) \otimes (1 - \varphi)^{-1}(\xi) \]
\[ = \sum_{i=0}^{n-1} \left( \eta^{i-n}(\zeta_{p^n-i} - 1) - \frac{1}{p} \sum_{\zeta_p=1}^{\eta} \eta^{i-n}(\zeta_{p^n-i} - 1) \right) \otimes \varphi^{i-n}(\xi) \]
\[ + \left( \eta(0) - \frac{1}{p} \sum_{\zeta_p=1}^{\eta} \eta(\zeta - 1) \right) \otimes (1 - \varphi)^{-1}(\xi) \]
\[ = \sum_{i=0}^{n} (\pi_{n-i} - \frac{1}{p} \text{Tr}(\pi_{n-i})) \otimes \varphi^{i-n}(\xi) - \frac{1}{p} \text{Tr}(\pi_1) \otimes (1 - \varphi)^{-1}(\xi) \]
\[ = \sum_{i=0}^{n} \pi'_{n-i} \otimes \varphi^{i-n}(\xi) - \frac{1}{p - 1} \otimes \xi + (1 - \varphi)^{-1}(\xi). \]

Recall that \( \varphi^2 = -p^{k-3} \), so we have
\[ (1 - \varphi)^{-1} = \frac{1}{1 + p^{k-3}}(1 + \varphi). \]

In particular, \(-\frac{1}{p-1} \otimes \xi^{\pm} + (1 - \varphi)^{-1}(\xi^{\pm}) \notin D^0(V)\). Moreover, \( \varphi^r(\omega) \in D^0(V) \) iff \( r \) is even, hence \( \{ \gamma_{n,0}(\xi^{\pm})^{\sigma} \} \) generates
\[ \left( K + \sum_{i \in S^{\pm}} K^{(i)} \right) \otimes D(V)/D^0(V) \]
where \( S^{\pm} = \{ m \in [1, n] : m \text{ even (odd)} \} \) by Corollary 3.2. Hence the result by [10, Lemma 2.8].

We write \( H^f_1(K_n, V)^\pm \) for the vector space described in the proposition and define \( H^f_1(K_n, T)^\pm = H^f_1(K_n, T) \cap H^f_1(K_n, V)^\pm \). Then,
\[ H^f_1(K_n, T)^\pm = \{ x \in H^f_1(K_n, T) : \text{cor}_{n/m+1} x \in H^f_1(K_m, T) \forall m \text{ even (odd)} \} \]
and \( \ker(\text{Col}^\pm) \) is given by
\[ \mathbb{H}^{1}_{\text{tw,\pm}}(T^*(1)) := \lim_{n} H^1_{\pm}(K_n, T^*(1)) \]
where \( H^1_{\pm}(K_n, T^*(1)) \) is defined to be the annihilator of \( H^f_1(K_n, T)^\pm \) under the pairing
\[ H^1(K_n, T^*(1)) \times H^1(K_n, T) \rightarrow O_F. \]

The images of \( \text{Col}^\pm \) can be found in the same way as [10 Section 3]. Namely, \( \text{Im}(\text{Col}^+) \cong (u-1) \Lambda + \sum_{\sigma \in G_1} \Lambda \) and \( \text{Im}(\text{Col}^-) \cong \Lambda \).
3.3 The even and odd Selmer groups

Let $E$ be a number field with $[E : \mathbb{Q}] = d$. Then, the $p$-Selmer group of $f$ over $E$ is defined to be

$$\text{Sel}_p(f/E) = \ker \left( H^1(E, V/T) \to \prod_v \frac{H^1(E_v, V/T)}{H^1(E_v, V/T)} \right)$$

where $v$ runs through all places of $E$ and $V$ and $T$ are as defined above.

Assume that $p$ splits completely in $E$. Let $p_1, \ldots, p_d$ be the primes of $E$ above $p$ and $E_\infty/E$ a $\mathbb{Z}_p$-extension such that $p_i$ is totally ramified in $E_\infty$. We write $E_n$ for the $n$th layer. Note that $E_{p_i}$ is isomorphic to $\mathbb{Q}_{p_i}$ for $i = 1, \ldots , d$. By [5, Section 4.2], $E_\infty, p_i/E_{p_i}$ is contained in a Lubin-Tate extension for some uniformiser $\pi$ of $\mathbb{Q}_p$ such that $\pi \in p(1 + p\mathbb{Z}_p)$. Therefore, the $\text{Col}^\pm$ restrict to $\lim_{\leftarrow} H^1(E_{n,p_i}, T^\ast(1))$ and it easy to check that the description of the kernels generalise directly. For each $n \geq 0$, we can define

$$\text{Sel}_p^\pm(f/E_n) = \ker \left( \text{Sel}_p(f/E) \to \prod_i \frac{H^1(E_{n,p_i}, V/T)}{H^1(E_{n,p_i}, T^\ast \otimes \mathbb{Q}_p/\mathbb{Z}_p)} \right)$$

and $\text{Sel}_p^\pm(f/E_\infty) = \lim_{\leftarrow} \text{Sel}_p^\pm(f/E_n)$.

Unfortunately, unlike the cyclotomic case, $\text{Sel}_p^\pm(f/E_\infty)$ is not $\Lambda$-cotorsion in general. However, they do satisfy a control theorem (cf [8, Theorem 9.3]) and their coranks can be used to describe those of $\text{Sel}_p(f/E_n)$ (cf [5, Proposition 7.1]). Since the proofs for these results given in [8] are purely algebraic and do not involve properties of elliptic curves, they generalise to general $f$ with no difficulties.

4 Relative Lubin-Tate groups

We now assume $K$ to be a finite unramified extension of $\mathbb{Q}_p$ of degree $d$. For a fixed $\pi \in \mathbb{Z}_p$ with $p$-adic valuation $d$ and $g$ a lift of Frobenius with respect to $\pi$ in the sense of [4, Section I.1.2], then $\varphi^i(g)$ is also such a lift for any integer $i$. To ease notation, we will write $g_i$ for $\varphi^i(g)$. Each $g_i$ gives rise to an one-dimensional formal group over $O_K$ which we write as $F_{g_i}$. For any positive integer $n$, we write

$$g_i^{(n)} = \varphi^{n-1}(g_i) \circ \varphi^{n-2}(g_i) \circ \cdots \circ g_i = g_{i+n-1} \circ g_{i+n-2} \circ \cdots \circ g_i.$$ 

Let $W_{g_i}^n$ be the set of zeros of $g_i^{(n)}$ in $K$ and write $K_n = K(W_{g_i}^n)$ which is independent of the choice of $g$ and $i$. Moreover, if $\omega \in W_{g_i}^n \setminus W_{g_i}^{n-1}$, then $K_n = K(\omega)$. Let $\eta_i : \mathbb{G}_m \to F_{g_i}$ be an isomorphism, then $\eta_i \in \mathbb{G}_m[\![X]\!]$ and $\omega_i := \eta_i \circ \varphi^{n-1}(\eta_i - 1) \in W_{g_i}^n \setminus W_{g_i}^{n-1}$ (see [4, Section I.3.2]). Note that $g_{i-n}$ sends $W_{g_{i-n}}^n$ to $W_{g_{i-n+1}}^{n-1}$, we define the Tate module of $F_{g_i}$ to be

$$T_{g_i} = \lim_{\leftarrow} W_{g_i}^n_{g_{i-n}}.$$
Since \( \eta_i \) satisfies \( g_i \circ \eta_i = \eta_i^\varphi((1 + X)^p - 1) \), we have \( (\omega_n,i)_n \in T_{g_i} \).

The character \( \kappa \) of \( G_K \) on \( T_{g_i} \) is independent of \( i \) by [4, Proposition I.1.8]. As in the case of absolute Lubin-Tate groups, \( \kappa \) can be decomposed as \( \kappa = \chi \psi \) where \( \chi \) is the cyclotomic character and \( \psi \) is an unramified character.

Results of [15] hold in this context with the obvious modifications, especially Theorem 1.5. In particular, for any \( \xi \in D(V) \) and \( i \) an integer, we can define a measure \( \mu_{\xi}(i) \) on \( \mathbb{Z} \times p \) whose Amice transform is given by \( \bar{\eta}_i(X) \otimes \xi \) where \( \bar{\eta}_i \) is defined in the same way as \( \bar{\eta} \) in Section 2. We can then define \( L_i^\varphi(\xi) \).

For \( V = V_f(1) \) and \( F = \mathbb{Q}_p \) (so \( \mathcal{O}_F = \mathbb{Z}_p \)), we define

\[
\text{Col}^\pm : H^1_{Iw}(V^*(1)) \to \mathbb{Q} \otimes \Lambda^d \quad z \mapsto \left( L_i^\varphi(z)/\log_{p,k} \right)_{i=0,\ldots,d-1}.
\]

We now follow [7, Section 3] to find the image of \( \text{Col}^- \). In particular, we assume that \( g \) is a polynomial of degree \( p \) and the coefficient of \( X^p - 1 \) is \( \zeta_0 \) where \( \zeta_0 \) is a root of unity in \( K \) such that \( \mathcal{O}_K = \mathbb{Z}_p[\zeta_0] \).

**Lemma 4.1.** With the above notation, \( \left( E_{h,V}(\mu_{\xi}^{(i)}) \right)_0, i = 0, \ldots, d - 1, \) is linearly independent over \( \mathbb{Q}_p \).

**Proof.** By Theorem 2.5 we have

\[
\left( E_{h,V}(\mu_{\xi}^{(i)}) \right)_0 = \exp \left( (1 - \varphi)^{-1} \left( 1 - \frac{\varphi^{-1}}{p} \right) \bar{\eta}_i(0) \otimes \xi \right).
\]

We first simplify the expression \( (1 - \varphi)^{-1}(1 - \frac{\varphi^{-1}}{p}) \). Recall that \( \varphi \) satisfies

\[
\varphi^2 + p^{k-3} = 0 \quad \text{and} \quad (1 - \varphi)^{-1} = \frac{1}{1 + p^{k-3}(1 + \varphi)}.
\]

Therefore,

\[
(1 - \varphi)^{-1} \left( 1 - \frac{\varphi^{-1}}{p} \right) = \frac{1}{1 + p^{k-3}} \left( 1 - \frac{\varphi^{-1}}{p} \right) = \frac{1}{1 + p^{k-3}} \left( \varphi - \frac{\varphi^{-1}}{p} + 1 - \frac{1}{p} \right) = \frac{1}{1 + p^{k-3}} \left( \frac{1 + \frac{1}{p^{k-2}}}{p} \varphi + 1 - \frac{1}{p} \right).
\]

We write \( \lambda = (p^{2-k} + 1)/(p^{k-3} + 1) \). Since \( \xi^~ = \omega \in D^0(V) \), we have

\[
(1 - \varphi)^{-1} \left( 1 - \frac{\varphi^{-1}}{p} \right) \bar{\eta}_i(0) \otimes \xi^~ \equiv \lambda \bar{\eta}_i^\varphi(0) \otimes \varphi(\omega) \mod D^0(V).
\]
But \( \tilde{\eta}_i^\varphi(0) \) equals to
\[
\eta_i^\varphi(0) - \frac{1}{p} \sum_{\zeta = 1}^{p} \eta_i^\varphi(\zeta - 1) = \varphi^{i+1}(\zeta_0)
\]
since the summands are the roots \( g_i^\varphi \). By definition, \( \zeta_0, \varphi(\zeta_0) \cdots, \varphi^{d-1}(\zeta_0) \) is a \( \mathbb{Z}_p \)-basis of \( \mathcal{O}_K \), so we are done. \( \square \)

**Corollary 4.2.** The image of \( \mathbb{H}_1^{\text{Iw}}(T^*(1)) \) under \( \text{Col}^\varphi \) is isomorphic to \( \Lambda^d \).

**Proof.** By [10, proof of Lemma 3.11], there exists an integer \( r \) such that
\[
p^{-r} \left( \mathbb{R}_b, \nu(\mu_{\zeta}) \right)_0 \in H^1(K, T) \setminus pH^1(K, T)
\]
for all \( i \). Hence, as in [11, proof of Proposition 3.9], their linear independence over \( \mathbb{Z}_p \) implies that
\[
\left\{ \left( p^{-r} \mathcal{L}_{\zeta}^{(i)}(z) \right)_{i=0, \ldots, d-1} : z \in H^1(K, T) \right\} = \mathbb{Z}_p^d.
\]
But the image of \( \log_{p,k}^- \) in \( \mathbb{Z}_p \) is a \( p \)-adic unit (see [10, Section 3.2]), so we have
\[
p^{-r} \text{Col}^-_{(H^1(K, T^*(1)))} = \mathbb{Z}_p^d.
\]
But the following diagram commutes (see [10, proof of Theorem 3.10]):
\[
\begin{array}{ccc}
H^1(K_m, T^*(1)) & \xrightarrow{p^{-r} \mathcal{L}_{\zeta}^{(i)}} & \mathbb{Q}_p[G_m] \xrightarrow{(\log_{p,k}^-)^{-1}} \mathbb{Z}_p[G_m] \\
\text{cor} & & \text{pr} \\
H^1(K_n, T^*(1)) & \xrightarrow{p^{-r} \mathcal{L}_{\zeta}^{(i)}} & \mathbb{Q}_p[G_n] \xrightarrow{(\log_{p,k}^-)^{-1}} \mathbb{Z}_p[G_n] \\
\end{array}
\]
where \( m > n \), hence the result by Nakayama’s lemma. \( \square \)

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