The dynamics of message passing on dense graphs, with applications to compressed sensing

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Abstract—‘Approximate message passing’ algorithms proved to be extremely effective in reconstructing sparse signals from a small number of incoherent linear measurements. Extensive numerical experiments further showed that their dynamics is accurately tracked by a simple one-dimensional iteration termed state evolution. In this paper we provide the first rigorous foundation to state evolution. We prove that indeed it holds asymptotically in the large system limit for sensing matrices with iid gaussian entries.

While our focus is on message passing algorithms for compressed sensing, the analysis extends beyond this setting, to a general class of algorithms on dense graphs. In this context, state evolution plays the role that density evolution has for sparse graphs.

I. INTRODUCTION AND MAIN RESULTS

Given an $n \times N$ matrix $A$, the compressed sensing reconstruction problem requires to reconstruct a sparse vector $x_0 \in \mathbb{R}^N$ from a (small) vector of linear observations $y = Ax_0 \in \mathbb{R}^n$. Recently [DMM09] suggested the following first order approximate message-passing (AMP) algorithm for reconstructing $x_0$ given $A, y$. Start with an initial guess $x^0 = 0$ and proceed by

$$x^{t+1} = \eta_t(A^* z^t + x^t),$$

$$z^t = y - A x^t + \frac{1}{\delta} |x^{t-1}|^{\eta_{t-1}} (A^* z^{t-1} + x^{t-1}),$$

for an appropriate sequence of threshold functions $\{\eta_t\}_{t \geq 0}$. The goal is to show that $x^t$ converges to $x_0$ (cf. [DMM09] for details). Here we assume that the columns of $A$ have $\ell_2$ norm (approximately) equal to 1, and, given a vector $v \in \mathbb{R}^N$ we write $f(x)$ for the vector obtained by applying $f$ componentwise. Further, $\delta = n/N$, $\langle v \rangle \equiv N^{-1} \sum_{i=1}^N v_i$ and $A^*$ is the transpose of matrix $A$.

Two type of findings were presented in [DMM09]: (1) For random or pseudo-random matrices $A$, the behavior of AMP algorithms is accurately described by the so called ‘state evolution’ (SE) formalism; (2) The sparsity-undersampling tradeoff of AMP as derived from SE coincides, for an appropriate choice of the functions $\eta_t$, with the one of (much more complex) convex optimization approaches.

These findings were based on heuristic arguments and extensive numerical simulations. In this paper we provide the first rigorous support to finding (1), by proving that SE holds in the large system limit, for a special class of sensing matrices.

Note that AMP is an approximation to the following message-passing algorithm. For all $i, j \in [N]$ and $a, b \in [n]$ (here and below $[N] \equiv \{1, 2, \ldots, N\}$) start with messages $x^0_{j \rightarrow a} = 0$ and proceed by

$$z^t_{a \rightarrow i} = y_a - \sum_{j \in [N]\setminus i} A_{aj} x^t_{j \rightarrow a},$$

$$x^{t+1}_{i \rightarrow a} = \eta_t \left( \sum_{b \in [n]\setminus a} A_{bi} z^{t}_{b \rightarrow i} \right).$$

As argued in [DMM10], AMP accurately approximates message passing in the large system limit. An important tool for the analysis of message passing algorithms is provided by density evolution [RU08]. Density evolution is known to hold asymptotically for sequences of sparse graphs that are locally tree-like. The factor graph underlying the algorithm (1.2) is dense: indeed it is the complete bipartite graph. State evolution plays the role of density evolution for dense graphs, and can be regarded (in a very precise sense) as the limit of density evolution for dense graphs.

For the sake of concreteness, we will focus on the algorithm (I.1). Nevertheless our analysis applies to a much larger family of message passing algorithms on dense graphs, for instance the multi-user detection algorithm studied in [Kab03], [NS05], [MT06]. It is important to mention that the algorithms (I.1) and (I.2) are completely different from gaussian belief propagation (BP). More generally, none of the existing rigorous results for BP can be used here.

It is truly remarkable that density evolution (in its special incarnation, SE) holds for dense graphs. This upsets a very popular piece of wisdom: ‘density evolution (and message passing) works because the graph is locally tree-like, and does not work on graphs with many short loops.’

A. Main result

Given a probability distribution $p_{X_0}$, let $\tau^2_0 \equiv \mathbb{E}\{X_0^2\}/\delta$, and define recursively for $t \geq 0$,

$$\tau^2_{t+1} = \frac{1}{\delta} \mathbb{E} \left\{ |\eta_t (X_0 + \tau_t Z) - X_0|^2 \right\},$$

with $X_0 \sim p_{X_0}$ and $Z \sim N(0, 1)$ independent. Also a function $\phi : \mathbb{R}^m \rightarrow \mathbb{R}$ is called pseudo-Lipschitz if there exist a constant $L$ such that for all $x, y \in \mathbb{R}$, $|\phi(x) - \phi(y)| \leq \max(\|x\|, \|y\|, L)\|x - y\|$. 


Theorem 1. Let \( \{A(N)\}_N \) be a sequence of sensing matrices \( A \in \mathbb{R}^{n \times N} \) indexed by \( N \), with iid entries \( A_{ij} \sim \mathcal{N}(0, 1/n) \), and assume \( n/N \to \delta \in (0, \infty) \). Consider further a sequence of signals \( \{x_0(N)\}_N \), whose empirical distributions converges weakly to a probability measure \( p_{X_0} \) on \( \mathbb{R} \), and have uniformly bounded fourth moment. Then, for any pseudo-Lipschitz function \( \psi : \mathbb{R}^2 \to \mathbb{R} \) and all \( t \), almost surely

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \psi(x_{i,t}^t, x_{0,t}) = \mathbb{E}[\psi(y_{t-1}(X_0 + \tau_{t-1} Z), X_0)]. \tag{1.4}
\]

Up to a trivial change of variables, this is a formalization of the finding of [DMM09] (cf. in particular Eqs. [7], [8] and Finding 2 in that paper).

Note 1. The empirical distribution of the vector \( x_0 \in \mathbb{R}^N \) is the probability distribution that puts a point mass \( 1/N \) at each of the \( N \) entries of the vector.

B. Alternative representation of AMP

Let \( h^t = x_0 - (A^t z^{t-1} + x^{t-1}) \), \( f(x) = f(x, x_0) = x_0 - \eta(x_0 - x) \), and \( g(x) = -x \). Also define \( m^t = g(z^t) \), \( q^t = f(h^t) \), and \( \lambda_t = \frac{1}{2} f'(h^t) \). Therefore, we will obtain the following equivalent version of AMP. Start with \( z^0 = A x_0 \) (or \( \lambda_0 = 0, q^0 = x_0 \)) and proceed by

\[
h^{t+1} = A^t m^t + q^t
\]

\[
z^t = A^t q^t - \lambda_t m^{t-1}
\]

(1.5)

Note 2. (a) It is simple to see that algorithms (1.5) and (1.1) are equivalent with a simple change of variable. We only use (1.5) to simplify the analysis of (1.1). However, since \( q_0 = x_0 \) is unknown in practice, one should use (1.1) to recover \( x_0 \).

(b) Due to symmetry, for each \( t \), all coordinates of the vector \( h^t \) have the same distribution \( (z^t, q^t, m^t) \) are similar.

(c) Our proof holds for all systems of algorithms of the type (1.5) and all functions \( f, g \) that have bounded derivate. For this general case the coefficient of \( q^t \) in the first equation should be changed from \( 1 - g'(z^t) \).

(d) Also, the proof applies to any sequence of scalar functions \( \{f_t, g_t\}_{t \geq 0} \) with \( h^t = f_t(q^t) \) and \( m^t = g_t(z^t) \). For simplicity, we shall drop the time dependence of \( f, g \).

II. ANALYSIS

The proof is based on a conditioning technique developed by Erwin Bolthausen for the analysis of the so-called TAP equations in spin glass theory [Bol98]. Related ideas can also be found in [Don06].

First we introduce some new notations and then state and prove a more general result than Theorem 1.

A. Definitions

When the update equation for \( h^{t+1} \) in (1.5) is used, all values of \( z^0, \ldots, z^t \) and also \( h^1, \ldots, h^t \) have been previously calculated. Additionally any deterministic function of them \( (m^0, \ldots, m^t \) and \( q^0, \ldots, q^t \) is known as well. Hence, we can consider the distribution of \( h^{t+1} \) when it is conditioned on all these known variables and \( x_0 \). In particular, define \( \mathcal{G}_{t_1, t_2} \) to be the \( \sigma \)-algebra generated by \( z^0, \ldots, z^{t-1} \) (thus including \( m^0, \ldots, m^{t-1} \) and \( x^0, h^1, \ldots, h^t \) (thus including \( q^0, \ldots, q^t \)). We are interested in finding the distributions of random variables \( z^t | \mathcal{G}_{t_1, t_2} \) and \( h^{t+1} | \mathcal{G}_{t_1, t_2} \).

Since \( h^t, z^t \) are column vectors, the equations for \( z^0, \ldots, z^{t-1} \) and \( h^1, \ldots, h^t \), can be written in matrix form as:

\[
\left[ h^t - q^0 \right] h^{0:t} - q^{t-1}, = A^t \left[ m^0 | \ldots | m^{t-1} \right],
\]

\[
\left[ z^t + \lambda_1 m^0 | \ldots | z^{t-1} + \lambda_{t-1} m^{t-2} \right], = A \left[ q^0 | \ldots | q^{t-1} \right].
\]

or in short \( Y_t = A Q_t \) and \( x_t = A^* M_t \). For each matrix \( M \) we define \( \tilde{M} = M^* M \).

We also introduce the notation \( m^t_i \) for the projection of \( m^t \) onto column space of \( M_t \) and define \( m^t_i = m^t - m^t_i \). Similarly, define \( q^t_i, q^t_j \) to be the parallel and orthogonal projections of \( q^t \) onto column space of \( Q_t \).

For vectors \( u, v \in \mathbb{R}^m \) define \( \langle u, v \rangle = \sum_{i=1}^m u_i / m \) and \( \langle u, v \rangle = \sum_{i=1}^m u_i / m \). For random variables \( X, Y \) the notation \( X \overset{d}{=} Y \) means that \( X \) and \( Y \) are equal almost surely, \( X \overset{d}{=} Y \) that they are equal in distribution.

B. Main technical Lemma

We prove the following more general result.

Lemma 1. Let \( \{A(N)\}_N \) be a sequence of sensing matrices as in Theorem 1, with \( n/N = \delta. \) Assume \( x_0 \) to have i.i.d. entries with distribution \( p_{X_0} \) having finite fourth moment. Then the following hold for all \( t \in \mathbb{N} \cup \{0\} \)

(a)

\[
h^{t+1} | \mathcal{G}_{t+1, t} = d \sum_{i=0}^{t+1} \alpha_i h^{t+1} + A^t m_L + Q_t \hat{g}_t(1) \tag{II.1}
\]

\[
z^t | \mathcal{G}_{t+1, t} = d \sum_{i=0}^{t+1} \beta_i z^t + A^t q_L + M_t \hat{g}_t(1) \tag{II.2}
\]

where \( \hat{A} \) is an independent copy of \( A \) and coefficients \( \alpha_i, \beta_j \) satisfy \( m_L^t = \sum_{i=0}^{t+1} \alpha_i m^t \) and \( q_L^t = \sum_{i=0}^{t+1} \beta_i q^t \). Here \( \hat{g}_t(1) \in \mathbb{R}^t \) is a finite dimensional random vector that converges to 0 almost surely as \( N \to \infty \).

(b) For any pseudo-Lipschitz function \( \phi : \mathbb{R}^{t+1} \to \mathbb{R} \)

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \phi(h^1, \ldots, h^t, x_0,i) \overset{a.s.}{=} \mathbb{E} \left[ \phi(t_0 Z_0, \ldots, t_l Z_l, X_0) \right] \tag{II.3}
\]

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \phi(z^0, \ldots, z^t) \overset{a.s.}{=} \mathbb{E} \left[ \phi(t_0 Z_0, \ldots, t_l Z_l) \right], \tag{II.4}
\]

where \( Z_0, \ldots, Z_t, (\hat{Z}_0, \ldots, \hat{Z}_t) \) have \( \mathcal{N}(0, 1) \) distribution and are independent of \( X_0 \).
Lemma 2. Let \( t_1, t_2 \) be \((t, t)\) or \((t + 1, t)\). Then the distribution of the conditional random variable \( A|_{\Theta_{t_1, t_2}} \) satisfies
\[
A|_{\Theta_{t_1, t_2}} \overset{d}{=} E_{t_1, t_2} + P_{V_{t_1, t_2}}(\tilde{A}).
\]
Here \( \tilde{A} \) is a random matrix independent of \( \Theta_{t_1, t_2} \). Also, \( E_{t_1, t_2} = E(\tilde{A}|\Theta_{t_1, t_2}) \) is equal to
\[
E_{t_1, t_2} = Y_{t_1}(\tilde{Q}_{t_1})^{-1}Q_{t_1}^* + M_{t_2}(\tilde{M}_{t_2})^{-1}X_{t_2}^* - M_{t_2}(\tilde{M}_{t_2})^{-1}M_{t_2}Y_{t_1}(\tilde{Q}_{t_1})^{-1}Q_{t_1}^*.
\]
**Lemma 6** (Stein’s Lemma [Ste72]). For jointly gaussian random variables $Z_1, Z_2$ and any $C^1$ function $\varphi : \mathbb{R} \to \mathbb{R}$ the following holds $E[Z_1 \varphi(Z_2)] = \text{Cov}(Z_1, Z_2)E[\varphi′(Z_2)]$.

**E. Proof of Lemma 1**

The proof is by induction on $t$. Let $\mathcal{H}_{t+1}$ be the property that (II.1), (II.3), (II.5) and (II.7) are correct. Similarly, let $\mathcal{Z}_t$ be the property that (II.2), (II.4), (II.6) and (II.8) hold. The inductive proof consists of the following three main steps. (1) $\mathcal{Z}_0$ holds. (2) If $\mathcal{Z}_r, \mathcal{H}_s$ hold for all $r < t$ and $s \leq t$ then $\mathcal{Z}_t$ holds. (3) If $\mathcal{Z}_r, \mathcal{H}_s$ hold for all $r \leq t$ and $s \leq t$ then $\mathcal{H}_{t+1}$ holds.

**Step 1: $\mathcal{Z}_0$.** Note that $z^0 = Ax_0$.

(a) $\mathcal{S}_{0,0}$ is generated by $x_0$. Also $q^0 = q^1$ since $Q_0$ is an empty matrix. Hence $z^0|_{\mathcal{S}_{0,0}} = Ax_0 = Aq^0_0$.

(b) Let $\phi : \mathbb{R} \to \mathbb{R}$ be a pseudo-Lipschitz function with constant $L$, and assume w.l.o.g. $\phi(0) = 0$. Conditioning on $q^0 = x_0$, the random variable $z^0 = \sum_{i=1}^n \phi((Ax_0)_i)/n$ is a sum of iid random variables. By Lemma 5 ($Ax_0)_i \sim \mathcal{N}(0, 1)$, $\mathbb{E}(X_0|x) = \theta^2 < \infty$. For all $p \geq 2$, there exist a constant $C_p$ such that $\mathbb{E}(Ax_0)_i|p < C_p$. Therefore $\mathbb{E}(\phi(Ax_0)) = \sum_{i=1}^n \phi((Ax_0)_i)/n \leq C \mathbb{E}(Ax_0|p) < C$ for a constant $C$. Now, we can apply Theorem 2 to get

$$\lim_{N \to \infty} \frac{1}{n} \sum_{i=1}^n [\phi(z^0_i) - \mathbb{E}[\phi(z^0_i)]] \overset{a.s.}{=} 0,$$

whence the above calculation

$$\lim_{N \to \infty} \frac{1}{n} \sum_{i=1}^n [\phi(z^0_i) - \mathbb{E}[\phi(z_0\tau)]] \overset{a.s.}{=} 0.$$

(c) Using Lemma 5, $\lim_{n \to \infty} \langle z^0, z^0 \rangle = \lim_{n \to \infty} \|Ax_0\|^2/n \overset{a.s.}{=} \lim_{N \to \infty} \mathbb{E}(q^0|q^0)/\delta \overset{a.s.}{=} \mathbb{E}(X_0^2)/\delta$.

(d) Using part (a) for $t = 0$, and $\phi(x) = xg(x)$ we obtain $\lim_{n \to \infty} \langle z^0, g(z^0) \rangle \overset{a.s.}{=} \mathbb{E}(z_0\tau g(z_0\tau))$, which is equal to $\tau^2 \mathbb{E}[g'(\tau\tilde{Z})]$ using Lemma 6. On the other hand, in proof of (b) we showed $\lim_{n \to \infty} \langle z^0, z^0 \rangle \overset{a.s.}{=} \tau^2$. Now, applying part (b) to $\phi(x) = g'(x)$ we get $\lim_{n \to \infty} \langle g'(z^0) \rangle \overset{a.s.}{=} \mathbb{E}[g'(\tau\tilde{Z})]$.

**Step 2: $\mathcal{Z}_t$.** This part is analogous to step 1 albeit more complex.

(a) Note that

$$Y_t = Z_t + [0|M_{t-1}]A_t, \quad X_t = H_t - Q_t,$$

where $Z_t = [z^0|\cdots|z^{t-1}]$, $A_t = \text{diag}(\lambda_0, \ldots, \lambda_{t-1})$ and $H_t = [h^1|\cdots|h^t]$.

**Lemma 7.** The following hold

(a) $h^t|_{\mathcal{S}_{t+1}} = H_t\tilde{M}_t^{-1}M_t^2 + P_{t+1}^t\tilde{A}P_t^t\tilde{M}_t^2 + Q_t\tilde{M}_t^2(1)$.  
(b) $z^t|_{\mathcal{S}_{t+1}} = Z_t(\tilde{Q}_t)^{-1}Q_tq^t + P_t\tilde{A}P_t^t\tilde{Q}_t^tq^t + M_t\tilde{M}_t(1)$.  

**Proof:** In light of Lemmas 2 and 4 we have $h^t|_{\mathcal{S}_{t+1}} \overset{a.s.}{=} X_t(M_t\tilde{M}_t^{-1}M_t^2 + Q_{t+1}(\tilde{Q}_t)^{-1}Y_{t+1}^2m_{t+1}^1 + P_{t+1}^t\tilde{A}P_t^t\tilde{M}_t^2 + q^t|_{\mathcal{S}_{t+1}} \overset{a.s.}{=} Y_t(\tilde{Q}_t)^{-1}Q_tq^t + M_t(\tilde{M}_t)^{-1}X_tq^t + P_t\tilde{A}P_t^t\tilde{M}_t^2\tilde{M}_t^2 + q^t|_{\mathcal{S}_{t+1}} \overset{a.s.}{=} Q_t\tilde{M}_t^2(1).$ Now using Equations (II.13), we only need to show $-Q_t\tilde{M}_t^2m_t^1 + Q_{t+1}(\tilde{Q}_t)^{-1}Y_{t+1}^2m_{t+1}^1 + q^t|_{\mathcal{S}_{t+1}} \overset{a.s.}{=} Q_t\tilde{M}_t^2(1)$. And $[0|M_{t-1}]A_t(\tilde{Q}_t)^{-1}Q_tq^t + M_t(\tilde{M}_t)^{-1}X_tq^t - \lambda t m^{-1} = M_t\tilde{M}_t(1).$ Recall that $m_t^1 = M_t\alpha$ and $q^t = Q_t\beta$. On the other hand $Y_t^1m_t^1 = Z_t^1m_t^1$ because $M_t\tilde{M}_t^2 = 0$. Similarly, $X_t^1q^t = H_t^1q^t$. Hence we need to show $-Q_t\alpha + Q_{t+1}(\tilde{Q}_t)^{-1}Z_t^1m_{t+1}^1 + q^t = Q_t\beta(1)$ (II.14) $[0|M_{t-1}]A_t(\tilde{M}_t)^{-1}H^1Q_t - \lambda t m^{-1} = M_t\tilde{M}_t(1)$. (II.15)

Here is our strategy to prove (II.15) (proof of (II.14) is similar). The left hand side is a linear combination of vectors $m^0, \ldots, m^{t-1}$. For any $t = 1, \ldots, t$ we will prove that the coefficient of each $m^0$ converges to 0. This coefficient in the left hand side is equal to $[M_t(\tilde{M}_t)^{-1}H_t^1q_t] - \lambda t(\beta t)^{2t}$ which can be written as $\sum_{r=0}^t \delta^{-1}[\tilde{M}_t/n]^{-1} - \lambda t(\beta t)^{2t}$. To simplify the notation denote the matrix $\tilde{M}_t/n$ by $G$. Therefore $\lim_{N \to \infty} \text{Coefficient of } m^t = 0$. (II.14) can be shown by

$$\lim_{N \to \infty} \{\sum_{r=0}^t(G^{-1})_{t+1}(h^r, q^t - \sum_{s=0}^t \beta_t M_t)^{(2t)} - \lambda t(\beta t)^{2t}\} = 0.$$

(c) For $r, s < t$ we can use induction hypothesis. For $s = t, r < t$, using (II.2) for $t$ that was just proved,

$$\langle z^t, z^r \rangle|_{\mathcal{S}_{t+1}} \overset{a.s.}{=} \sum_{i=0}^{t-1} \beta_i(z^s, z^r) + \langle \tilde{A}q_1^t, z^r \rangle + \sum_{i=0}^{t-1} o(1)(m^t, z^r).$$

Now, by induction hypothesis $Z_{t-1}(d)$ each term $\langle m^t, z^r \rangle$ has a finite limit. Thus, $\lim_{n \to \infty} \sum_{s=0}^t o(1)(m^t, z^t) = 0$. Now we can use induction hypothesis $Z_{t-1}(c)$ or $Z_t(c)$ for each term of the form $\langle z^t, z^r \rangle$ and use Lemma 5 for $\langle \tilde{A}q_1^t, z^r \rangle$ to obtain

$$\langle z^t, z^r \rangle|_{\mathcal{S}_{t+1}} \overset{a.s.}{=} \lim_{N \to \infty} \sum_{i=0}^{t-1} o(1)(m^t, z^r) = 0.$$
For the case of $r = s = t$, we have
\[ \langle z^t, z^t \rangle |_{\mathcal{E}_t,r} \overset{\text{a.s.}}{=} \lim_{n \to \infty} \sum_{i,j=0}^{t-1} \beta_i \beta_j \langle z^t, z^t \rangle + \langle \hat{A} q^t_1, \hat{A} q^t_1 \rangle + o(1). \]

Note that we used similar argument (Lemma 5 and $\mathcal{Z}_{t-1}(e)$) to show the contribution of all products of the form $\langle M_t \tilde{g}(1), \cdot \rangle$ and $\langle \hat{A} q^t_1, z^t \rangle$ a.s. tend to 0. Now, using induction hypothesis and Lemma 5
\[ \lim_{n \to \infty} \langle z^t, z^t \rangle |_{\mathcal{E}_t,r} \overset{\text{a.s.}}{=} \lim_{n \to \infty} \sum_{i,j=0}^{t-1} \beta_i \beta_j \langle z^t, z^t \rangle + \lim_{n \to \infty} \|q^t_1\|^2 \]
\[ \overset{\text{a.s.}}{=} \lim_{n \to \infty} \langle z^t, z^t \rangle \]
\[ \overset{\text{a.s.}}{=} \lim_{n \to \infty} \langle q^t_1, q^t_1 \rangle \]
\[ \overset{\text{a.s.}}{=} \lim_{n \to \infty} \langle \hat{q}^t_1, q^t_1 \rangle. \]

(b) Using part (a) we can write $\phi(z_0, \ldots, z_{t-1}) |_{\mathcal{E}_t,r}$, as $\phi(z_0, \ldots, z_{t-1}) = [\sum_{r=0}^{t-1} \beta_r z^t + \hat{A} q^t_1]$. First we would like to drop the error term $\mathcal{E}_t(r)$. For simplicity let $a_{i,n} = (z_0, \ldots, z_{t-1}) |_{\mathcal{E}_t,r}$ and $b_{i,n} = (z_0, \ldots, z_{t-1}) |_{\mathcal{E}_t(r)}$. Since $\phi$ is pseudo-Lipschitz with constant $L$ we have $|\phi(a_{i,n}) - \phi(b_{i,n})| \leq \max |L(\|a_{i,n}\|, \|b_{i,n}\|)| \sum_{r=0}^{t-1} m_r^2 |o(1)|$. Therefore, the difference $n=1, \ldots, n-1 \phi(a_{i,n}) - \sum_{i=1}^{t-1} \phi(b_{i,n}) |$ is less than
\[ \max \{L(\|a_{i,n}\|, \|b_{i,n}\|)| \sum_{i=1}^{t-1} m_r^2 | |o(1)|. \]
(II.16)

using Cauchy-Schwarz inequality twice. Also note that $n=1 \sum_{i=1}^{t-1} \|a_{i,n}\|^2 \leq \sum_{r=0}^{t-1} (z_r z^t + \langle z^t, z^t \rangle) \|s_t, r\|$, which is finite almost surely, using induction hypothesis and part (c) that was just proven. Similarly $n=1 \sum_{i=1}^{t-1} \|a_{i,n}\|^2$ and $\sum_{r=0}^{t-1} (m_r, m_r)$ are finite. Hence for any finite $t$, (II.16) goes to 0 almost surely when $n$ goes to $\infty$.

Now given, $z_0, \ldots, z_{t-1}$, consider the random variables $X_{i,n} = \phi(z_0, \ldots, z_{t-1}) = [\beta_r z^t + \hat{A} q^t_1]$. $X_{i,n} \overset{a.s.}{=} X_{i,n} - \mathbb{E}X_{i,n}$. Proceeding as in Step 1, and using the pseudo-Lipschitz property of $\phi$, it is easy to check the conditions of Theorem 2. We therefore get
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \phi(z_0, \ldots, z_{t-1}) \overset{\text{a.s.}}{=} \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}X_{i,n} \phi(z_0, \ldots, z_{t-1}) \overset{\text{a.s.}}{=} \mathbb{E}Z \phi(z_0, \ldots, z_{t-1} + \mathcal{E}_t(r)$. \]

Note that $[\hat{A} q^t_1]$ is a gaussian random variable with variance $\|q^t_1\|^2/\delta$. Hence we can use induction hypothesis $\mathcal{Z}_{t-1}(b)$ for $\phi(z_0, \ldots, z_{t-1}) = \mathbb{E}Z \phi(z_0, \ldots, z_{t-1}) + \sum_{r=0}^{t-1} \beta_r z^t + \mathcal{E}_t(r)$. $Z$ is an independent N(0, 1) random variable, to show
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}X_{i,n} \phi(z_0, \ldots, z_{t-1}) \overset{\text{a.s.}}{=} \mathbb{E}Z \phi(z_0, \ldots, z_{t-1} + \mathcal{E}_t(r)$. \]

Note that $\sum_{r=0}^{t-1} \beta_r \tau_r Z_r + \delta^{-5} \|q^t_1\| \mathcal{Z}$ is gaussian. All that is needed is to show that the variance of this gaussian is $\tau^2$. But using what we just proved for the pseudo-Lipschitz function $\phi(y_0, \ldots, y_t) = y^2_t$, we have
\[ \mathbb{E}\{(\sum_{r=0}^{t-1} \beta_r \tau_r Z_r + \delta^{-5} \|q^t_1\| \mathcal{Z})^2\} \overset{\text{a.s.}}{=} \lim_{n \to \infty} \langle z^t, z^t \rangle \]
(II.17)

On the other hand in part (c) we proved $\lim_{n \to \infty} \langle z^t, z^t \rangle \overset{\text{a.s.}}{=} \lim_{n \to \infty} \delta^{-1} \mathcal{F}(f(h^t), f(h^t))$. By the induction hypothesis $\mathcal{H}_t(b)$ for the pseudo-Lipschitz function $\phi(y_0, \ldots, y_t) = f(y^2_t)$ we get $\lim_{n \to \infty} \delta^{-1} \mathcal{F}(f(h^t), f(h^t)) \overset{\text{a.s.}}{=} \mathcal{E}(f(\tau_{r-1}) \mathcal{Z})^2$. So by the definition (I.3), both sides of (II.17) are equal to $\tau^2$.

(d) Very similar to the proof of $\mathcal{Z}_{t-1}(d)$, using part (b) for the pseudo-Lipschitz function $\phi : \mathbb{R}^{t-1} \to \mathbb{R}$ that is given by $\phi(y_0, \ldots, y_t) = y_t g(y_t)$ we can obtain $\lim_{n \to \infty} \langle z^t, g(z^t) \rangle \overset{\text{a.s.}}{=} \mathcal{E}(g(\tau_{r-1} \mathcal{Z}))$. And another application of part (b) for $\langle y_0, \ldots, y_t \rangle = y_t y_t$ transforms $\mathcal{E}(g(\tau_{r-1} \mathcal{Z}))$ to $\lim_{n \to \infty} \langle z^t, z^t \rangle$. Similarly, $\mathcal{E}(g(\tau_{r-1} \mathcal{Z}))$ can be transformed to $\lim_{n \to \infty} \langle g^2(z^t) \rangle$ almost surely. This finishes the proof of (d).

Step 3: $\mathcal{H}_{t+1}$. Due to symmetry, proof of this step is exactly similar to the proof of step 2.

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