The numbers game, geometric representations of Coxeter groups, and Dynkin diagram classification results

Robert G. Donnelly

Department of Mathematics and Statistics, Murray State University, Murray, KY 42071

Abstract

The numbers game is a one-player game played on a finite simple graph with certain “amplitudes” assigned to its edges and with an initial assignment of real numbers to its nodes. The moves of the game successively transform the numbers at the nodes using the amplitudes in a certain way. This game has been studied previously by Proctor, Mozes, Björner, Eriksson, and Wildberger. We show that those connected such graphs for which the numbers game meets a certain finiteness requirement are precisely the Dynkin diagrams associated with the finite-dimensional complex simple Lie algebras. As a consequence of our proof we obtain the classifications of the finite-dimensional Kac-Moody algebras and of the finite Weyl groups. We use Coxeter group theory to establish a more general result that applies to Eriksson’s E-games: an E-game meets the finiteness requirement if and only if a naturally associated Coxeter group is finite. To prove this and some other finiteness results we further develop Eriksson’s theory of a geometric representation of Coxeter groups and observe some curious differences of this representation from the standard geometric representation.

Keywords: numbers game, generalized Cartan matrix, Dynkin diagram, Coxeter graph, Coxeter/Weyl group, geometric representation, semisimple Lie algebra, Kac-Moody algebra

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1. Introduction and main results

The numbers game is a one-player game played on a finite simple graph with weights (which we call “amplitudes”) on its edges and with an initial assignment of real numbers (which we call initial “populations”) to its nodes. At the outset, each of the two edge amplitudes (one for each direction) will be negative integers; later we will relax this integrality requirement. The move a player can make is to “fire” one of the nodes with a positive population. This move transforms the population at the fired node by changing its sign, and it also transforms the population at each adjacent node in a certain way using an amplitude along the incident edge. The player fires the nodes in some sequence of the player’s choosing, continuing until no node has a positive population. This numbers game formulated by Mozes [Moz] has also been studied by Proctor [Pr1], [Pr2], Björner [Bjö], BB, Eriksson [Erik1], [Erik2], [Erik3], and Wildberger [Wi1], [Wi2], [Wi3]. Wildberger studies a dual version which he calls the “mutation game.” See Alon et al [AKP] for a brief and readable treatment of the numbers game on “unweighted” cyclic graphs. Much of the numbers game discussion in Chapter 4 of BB can be found in Erik2. Proctor developed this process in Pr1.
to compute Weyl group orbits of weights with respect to the fundamental weight basis. For this reason we prefer his perspective of firing nodes with positive, as opposed to negative, populations.

The motivating question for this paper is: for which such graphs does there exist a nontrivial initial assignment of nonnegative populations such that the numbers game terminates in a finite number of steps? For graphs with integer amplitudes, our answer to this question (Theorem 1.1) is that the only such connected graphs are the Dynkin diagrams of Figure 1.1. Moreover, from Eriksson’s Strong Convergence Theorem (Theorem 3.1 of [Erik3]) we are able to conclude that for any initial assignment of populations to the nodes of a Dynkin diagram and for any legal sequence of node firings, the numbers game will terminate in the same finite number of steps and ultimately yield at each node the same nonpositive terminal population. Our proof of Theorem 1.1 in Section 2 requires some Coxeter/Weyl group theory, but only implicitly — in particular the proof of Eriksson’s Comparison Theorem (Theorem 4.5 of [Erik2]). As a consequence of our proof of Theorem 1.1 and with the help of another result of Eriksson we re-derive in Section 3 the classifications of the finite-dimensional Kac-Moody algebras (the finite-dimensional complex semisimple Lie algebras cf. [Hum1], [Kac]) and of the finite Weyl groups (the finite crystallographic Coxeter groups of [Hum2]).

Our second main result (Theorem 1.3) answers our motivating question for a class of graphs (the “E-games” of [Erik2]) whose amplitudes are allowed to be certain real numbers. The classification obtained in this theorem uses the classification of finite irreducible Coxeter groups by connected positive definite Coxeter graphs (cf. §2.3-2.7 in [Hum2]). The connection to Coxeter groups is made via a particular geometric representation studied in [Erik2], referred to here as a “quasi-standard geometric representation.” This has many similarities to the standard geometric representation, but also some surprising differences in regard to some finiteness properties. Results we develop about quasi-standard geometric representations in Section 4 lead to our proof of Theorem 1.3 and to some further E-game results in Section 5. These include a method for computing (in certain circumstances) the positive roots in the root system for a quasi-standard geometric representation of a finite Coxeter group (Theorem 5.5) and a classification (Theorem 5.8) of those E-games for which the choices of node firings are “interchangeable” in some sense. The latter applies a classification result of Stembridge [Stem] about “fully commutative” elements in finite Coxeter groups. In Section 6, we remark on other connections.

We formulate the problem statement and solution precisely as follows. Fix a positive integer \( n \) and a totally ordered set \( I_n \) with \( n \) elements (usually \( I_n := \{1 < \ldots < n\} \)). A generalized Cartan matrix (or GCM) is an \( n \times n \) matrix \( M = (M_{ij})_{i,j \in I_n} \) with integer entries satisfying the requirements that each main diagonal matrix entry is 2, that all other matrix entries are nonpositive, and that if a matrix entry \( M_{ij} \) is nonzero then its transpose entry \( M_{ji} \) is also nonzero. Generalized Cartan matrices are the starting point for the study of Kac-Moody algebras: beginning with a GCM, one can write down a list of the defining relations for a Kac-Moody algebra as well as the associated Weyl group (see Section 3). To an \( n \times n \) generalized Cartan matrix \( M = (M_{ij})_{i,j \in I_n} \) we associate a finite graph \( \Gamma \) (which has undirected edges, no loops, and no multiple edges) as follows: The nodes \((\gamma_i)_{i \in I_n}\) of \( \Gamma \) are indexed by the set \( I_n \), and an edge is placed between nodes \( \gamma_i \) and \( \gamma_j \) if and only if \( i \neq j \) and the matrix entries \( M_{ij} \) and \( M_{ji} \) are nonzero. We call the pair \((\Gamma, M)\) a GCM graph.
We consider two GCM graphs \((\Gamma, M) = (M_{ij})_{i,j \in I_n}\) and \((\Gamma', M') = (M'_{pq})_{p,q \in I'_n}\) to be the same if under some bijection \(\sigma : I_n \rightarrow I'_n\) we have nodes \(\gamma_i\) and \(\gamma_j\) in \(\Gamma\) adjacent if and only if \(\gamma'_{\sigma(i)}\) and \(\gamma'_{\sigma(j)}\) are adjacent in \(\Gamma'\) with \(M_{ij} = M'_{\sigma(i),\sigma(j)}\). With \(p = -M_{12}\) and \(q = -M_{21}\), we depict a generic connected two-node GCM graph as follows:

We use special names and notation to refer to two-node GCM graphs which have \(p = 1\) and \(q = 1, 2,\) or \(3\) respectively:

When \(p = 1\) and \(q = 1\) it is convenient to use the graph \(\gamma_1 \leftrightarrow \gamma_2\) to represent the GCM graph \(A_2\). A GCM graph \((\Gamma, M)\) is a Dynkin diagram if each connected component of \((\Gamma, M)\) is one of the graphs of Figure 1.1. We number our nodes as in §11.4 of [Hum1]. In these cases the GCMs are “Cartan” matrices.

A position \(\lambda = (\lambda_i)_{i \in I_n}\) is an assignment of real numbers to the nodes of the GCM graph \((\Gamma, M)\); the real number \(\lambda_i\) is the population at node \(\gamma_i\). The position \(\lambda\) is dominant (respectively, strongly...
dominant) if \( \lambda_i \geq 0 \) (resp. \( \lambda_i > 0 \)) for all \( i \in I_n \); \( \lambda \) is nonzero if at least one \( \lambda_i \neq 0 \). For \( i \in I_n \), the fundamental position \( \omega_i \) is the assignment of population 1 at node \( \gamma_i \) and population 0 at all other nodes. Given a position \( \lambda \) for a GCM graph \( (\Gamma, M) \), to fire a node \( \gamma_i \) is to change the population at each node \( \gamma_j \) of \( \Gamma \) by the transformation

\[
\lambda_j \mapsto \lambda_j - M_{ij} \lambda_i,
\]

provided the population at node \( \gamma_i \) is positive; otherwise node \( \gamma_i \) is not allowed to be fired. Since the generalized Cartan matrix \( M \) assigns a pair of amplitudes \( (M_{ij}, M_{ji}) \) to each edge of the graph \( \Gamma \), we sometimes refer to GCMs as amplitude matrices. The numbers game is the one-player game on a GCM graph \( (\Gamma, M) \) in which the player (1) Assigns an initial position to the nodes of \( \Gamma \); (2) Chooses a node with a positive population and fires the node to obtain a new position; and (3) Repeats step (2) for the new position if there is at least one node with a positive population.\(^*\)

Consider now the GCM graph \( B_2 \). As we can see in Figure 1.2, the numbers game terminates in a finite number of steps for any initial position and any legal sequence of node firings, if it is understood that the player will continue to fire as long as there is at least one node with a positive population. In general, given a position \( \lambda \), a game sequence for \( \lambda \) is the (possibly empty, possibly infinite) sequence \( (\gamma_{i1}, \gamma_{i2}, \ldots) \), where \( \gamma_{ij} \) is the \( j \)th node that is fired in some numbers game with initial position \( \lambda \). More generally, a firing sequence from some position \( \lambda \) is an initial portion of some game sequence played from \( \lambda \); the phrase legal firing sequence is used to emphasize that all node firings in the sequence are known or assumed to be possible. Note that a game sequence \( (\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{il}) \) is of finite length \( l \) (possibly with \( l = 0 \)) if the population is nonpositive at each node after the \( l \)th firing; in this case we say the game sequence is convergent and the resulting position is the terminal position. We say a connected GCM graph \( (\Gamma, M) \) is admissible if there exists a nonzero dominant initial position with a convergent game sequence. Our first main result (to be proved in Section 2) is:

**Theorem 1.1** A connected GCM graph \( (\Gamma, M) \) is admissible if and only if it is a connected Dynkin diagram. In these cases, for any given initial position every game sequence will converge to the same terminal position in the same finite number of steps.

In [Erik1], Eriksson proves the following related result using combinatorial reasoning and a result from the Perron-Frobenius theory for eigenvalues of nonnegative real matrices: For a connected GCM graph \( (\Gamma, M) \) whose amplitude products are unity, every initial position has a convergent game sequence if and only if \( (\Gamma, M) \) is one of \( A_n \), \( D_n \), \( E_6 \), \( E_7 \), or \( E_8 \) from Figure 1.1 (and if and only if the “dominant eigenvalue” of \( M \) is less than two). Wildberger generalizes this assertion to all connected GCM graphs (see [Wil3]); the resulting GCM graphs are the connected Dynkin diagrams. His proof also uses the Perron-Frobenius theory and, in particular, does not depend on the classification of finite Weyl groups. In the language of this paper, say a GCM graph is strongly admissible if every nonzero dominant position has a convergent game sequence. Then [Erik1] gives an “ADE” version of the following result:

\(^*\)Mozes studied numbers games on GCM graphs for which the amplitude matrix \( M \) is symmetrizable (i.e. there is a nonsingular diagonal matrix \( D \) such that \( D^{-1} M \) is symmetric); in [Moz] he obtained strong convergence results and a geometric characterization of the initial positions for which the game terminates.
Theorem 1.2 (Wildberger) A connected GCM graph is strongly admissible if and only if it is a connected Dynkin diagram.

Theorem 1.1 does not require “strongly,” and its proof does not refer to eigenvalues. We will also demonstrate a more general version of Theorem 1.1 that applies to what Eriksson calls “E-games” in [Erik2].∗ There Eriksson drops the integrality requirement for off-diagonal entries of the amplitude matrix and asks: For which such graphs will the numbers game be “strongly convergent”? (This concept is defined below at the beginning of Section 2.) His answer (see Theorem 2.2 of [Erik2] or Theorem 3.1 of [Erik3]) is that the amplitude matrix must be what we will call here an E-generalized Cartan matrix or E-GCM: This is an $n \times n$ matrix $M = (M_{ij})_{i,j \in I_n}$ with real entries satisfying the requirements that each main diagonal matrix entry is 2, that all other matrix entries are nonpositive, that if a matrix entry $M_{ij}$ is nonzero then its transpose entry $M_{ji}$ is also nonzero,

∗Eriksson uses “E” for edge; he also allows for “N-games” where, in addition, nodes can be weighted.
Figure 1.3: Families of connected E-Coxeter graphs.
(For adjacent nodes, the notation ⋆ means that the amplitude product on the edge is $4 \cos^2(\pi/m)$; for an unlabelled edge take $m = 3$. The asterisks for $\mathcal{E}_6$, $\mathcal{E}_7$, and $\mathcal{H}_3$ pertain to Theorem 5.8.)

$A_n$ $(n \geq 1)$

$B_n$ $(n \geq 3)$

$D_n$ $(n \geq 4)$

$E_6$

$E_7$

$E_8$

$F_4$

$\mathcal{H}_3$

$\mathcal{H}_4$

$I_2^{(m)}$ $(m \geq 4)$

and that if $M_{ij}M_{ji}$ is nonzero then $M_{ij}M_{ji} \geq 4$ or $M_{ij}M_{ji} = 4 \cos^2(\pi/m_{ij})$ for some integer $m_{ij} \geq 3$. An E-GCM graph is the pair $(\Gamma, M)$ for an E-generalized Cartan matrix $M$. As before, we depict a generic two-node E-GCM graph as follows:

In this graph, $p = -M_{12}$ and $q = -M_{21}$. We use $\gamma_1 \overrightarrow{\otimes} \gamma_2$ for the collection of all two-node E-GCM graphs for which $M_{12}M_{21} = pq = 4 \cos^2(\pi/m)$ for an integer $m \geq 3$; when $m = 3$ (i.e. $pq = 1$), we use an unlabelled edge $\gamma_1 \overrightarrow{\otimes} \gamma_2$ as before. An E-Coxeter graph will be any E-GCM graph whose connected components come from one of the collections of Figure 1.3. Other terminology of this section used for GCM graphs will also be used in reference to E-GCM graphs and their numbers games (firing, position, admissible, etc). Our second main result generalizes Theorem 1.1.
Theorem 1.3  A connected E-GCM graph \((\Gamma, M)\) is admissible if and only if it is a connected E-Coxeter graph. In these cases, for any given initial position every game sequence will converge to the same terminal position in the same finite number of steps.

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2. Proof of our first main result

Our proof of the “only if” direction of the first claim of Theorem 1.1 uses a series of reductions that are typical in Dynkin diagram classification arguments. These reductions are implemented using several results of Eriksson. This helps us to minimize the use of Coxeter group theory.

Proof of the “only if” direction of the first claim of Theorem 1.1:

Step 1: Strong convergence. Following [Erik3], we say the numbers game on a GCM graph \((\Gamma, M)\) is strongly convergent if given any initial position, every game sequence either diverges or converges to the same terminal position in the same number of steps. The next result follows from Theorem 3.1 of [Erik3].

Theorem 2.1 (Eriksson’s Strong Convergence Theorem) The numbers game on a connected GCM graph is strongly convergent.

For this part of our proof of Theorem 1.1, we only require the following weaker result, which also applies when the GCM graph is not connected:

Lemma 2.2  In any GCM graph, if a game sequence for an initial position \(\lambda\) diverges, then all game sequences for \(\lambda\) diverge.

Step 2: Comparison. The next result is an immediate consequence of Theorem 4.5 of [Erik2]. Eriksson’s proof of this result uses some Coxeter group theory.

Theorem 2.3 (Eriksson’s Comparison Theorem) Given a GCM graph, suppose that a game sequence for an initial position \(\lambda = (\lambda_i)_{i \in I_n}\) converges. Suppose that a position \(\lambda' := (\lambda'_i)_{i \in I_n}\) has the property that \(\lambda'_i \leq \lambda_i\) for all \(i \in I_n\). Then some game sequence for the initial position \(\lambda'\) also converges.

Lemma 2.4 Let \(r\) be a positive real number. If \((\gamma_{i_1}, \ldots, \gamma_{i_l})\) is a convergent game sequence for an initial position \(\lambda = (\lambda_i)_{i \in I_n}\), then \((\gamma_{i_1}, \ldots, \gamma_{i_l})\) is a convergent game sequence for the initial position \(r\lambda := (r\lambda_i)_{i \in I_n}\).

Proof. For game play from initial position \(\lambda\), let \(c_j\) be the positive population at node \(\gamma_{i_j}\) when that node is fired in the game sequence \((\gamma_{i_1}, \ldots, \gamma_{i_l})\). Now \(rc_j\) will be the positive population at the same node when the same game sequence is applied to the initial position \(r\lambda\). □

The next result follows immediately from Theorem 2.3 together with Lemma 2.4:
Lemma 2.5  Suppose \((\Gamma, M)\) is admissible. Let \(\lambda = (\lambda_i)_{i \in I_n}\) be a dominant initial position such that \(\lambda_j > 0\) for some \(j \in I_n\). Suppose that a game sequence for \(\lambda\) converges. Then some game sequence for the fundamental position \(\omega_j\) also converges.

Step 3: A catalog of connected GCM graphs that are not admissible. The following immediate consequence of Lemmas 2.2 and 2.5 is useful in the proof of Lemma 2.7:

Lemma 2.6   A GCM graph is not admissible if and only if for each fundamental position there is a divergent game sequence.

Lemma 2.7  The connected GCM graphs of Figure 2.1* are not admissible.

Sketch of proof. By Lemma 2.6, it suffices to show that for each graph in Figure 2.1 one game sequence for each fundamental position diverges. Our approach is to start with a fundamental position assigned to one of these graphs and then proceed with a numbers game, firing in some predictable manner until a pattern emerges. This exercise can be completed by hand in a reasonable amount of time. We illustrate the nature of our arguments with two examples.

First, we show why GCM graphs in the “\(\tilde{D}\)” family are not admissible. The smallest GCM graph in this family has five nodes. Figure 2.2 shows that for each fundamental position there exists a game sequence that repeats indefinitely. In Figure 2.2.1, we fired each node once. In finding sequences of node firings in Figure 2.2.2, we analyzed the \(k = 0\) and \(k > 0\) cases separately; for the latter we fired the center node twice. In the case of more than five nodes, Figure 2.3 shows that for each fundamental position there exists a game sequence that repeats indefinitely. In Figure 2.3.1, we started by firing at the node with positive population, then fired all nodes to the right in succession (and returned), then fired all nodes to the left in succession (and returned); we only fired the starting node once. In Figure 2.3.2, we fired first at the positive population node, then proceeded to fire the nodes to its right in succession (and returned), then fired the starting node again, and finally fired the two leftmost nodes. In Figure 2.3.3, we traversed the “isthmus” of the graph twice.

Second, we show why GCM graphs of the form \(\begin{align*}
s &\begin{array}{c}
\tilde{D} \\
p_1 &\tilde{D} \\
p_2 &\tilde{D} \\
q_1 &\tilde{D} \\
q_2 &\tilde{D}
\end{array}
\end{align*}\) are not admissible. We assume that

the amplitude products \(p_1q_1\) and \(p_2q_2\) are at least two (at Step 6 below we will see that it suffices in Figure 2.1 to assume these products are at most 3). Assign populations \(a, b,\) and \(c\) as follows:

\[
\kappa := \left(2p_1 + 2p_2 - \frac{1}{q_1}\right)a + \left(p_1 + 2p_2 - \frac{1}{q_2}\right)b + c.
\]

Assume for now that \(a \geq 0\), \(b \geq 0\), \(c \leq 0\), and \(\kappa > 0\); when these inequalities hold we will say the position \((a, b, c)\) meets condition \((\ast)\). Under condition \((\ast)\) notice that \(a\) and \(b\) cannot both be zero. Begin by firing only at the two rightmost nodes. When this is no longer possible, fire at the leftmost node. The resulting corresponding populations are \(a_1 = q_1(\kappa + \frac{1}{q_2}b)\), \(b_1 = q_2(\kappa + \frac{1}{q_1}a)\), and \(c_1 = -\kappa - \frac{1}{q_1}a - \frac{1}{q_2}b\). In particular, \(a_1 > 0\).

*The infinite “\(\tilde{A}\)” family of GCM graphs of Figure 2.1 is the family of cycles with amplitude products of unity on all edges. Such cycles were in fact the graphs that motivated Mozes’ study of the numbers game in [Moz].
Figure 2.1: Some connected GCM graphs that are not admissible.

The “$\tilde{A}$” family of GCM graphs

The “$\tilde{B}$” family of GCM graphs

The “$\tilde{C}$” family of GCM graphs

The “$\tilde{D}$” family of GCM graphs
Figure 2.1 (continued): Some connected GCM graphs that are not admissible.

The \( \tilde{E} \) family of GCM graphs

\[
\begin{align*}
\text{The } \tilde{F} \text{ family of GCM graphs} & \\
\text{The } \tilde{G} \text{ family of GCM graphs}
\end{align*}
\]
Figure 2.1 (continued): Some connected GCM graphs that are not admissible.

Families of small cycles

\[ p_1q_1 \geq 1, p_2q_2 \geq 1 \]
\[ p_1q_1 \geq 2, p_2q_2 \geq 2 \]
\[ p_1q_1 \geq 3, p_2q_2 \geq 3 \]

Figure 2.2: The five-node GCM graph in the \( \tilde{D} \) family is not admissible.
(In each part of this figure, for any \( k \geq 0 \), the position on the right can be obtained from the position on the left by a sequence of legal node firings.)

2.2.1

\[ \begin{align*}
-k & \rightarrow 2k + 1 \\
-k & \rightarrow k
\end{align*} \]

2.2.2

\[ \begin{align*}
k + 1 & \rightarrow -2k \\
k & \rightarrow k
\end{align*} \]

Figure 2.3: GCM graphs in the \( \tilde{D} \) family with more than five nodes are not admissible.
(In each part of this figure, for any \( k \geq 0 \), the subsequent position can be obtained from the first position by a sequence of legal node firings.)

2.3.1

\[ \begin{align*}
0 & \rightarrow -k \\
0 & \rightarrow 2k + 1 \\
0 & \rightarrow 0
\end{align*} \]
The following is proved easily with an induction argument on the number of nodes. Observe that $\kappa > 0$, and $c_1 < 0$. Next we check that $\kappa_1 := (2p_1 + 2p_2 - \frac{1}{q_1})a_1 + (p_1 + 2p_2 - \frac{1}{q_2})b_1 + c_1$ is also positive. Now

$$\kappa_1 = \left(2p_1 + 2p_2 - \frac{1}{q_1}\right)\left[q_1 \left(2p_1 + 2p_2 - \frac{1}{q_1}\right) + q_2 \left(p_1 + 2p_2 - \frac{1}{q_2}\right) - 1\right]a + \left(p_1 + 2p_2 - \frac{1}{q_2}\right)\left[q_1 \left(2p_1 + 2p_2 - \frac{1}{q_1}\right) + q_2 \left(p_1 + 2p_2 - \frac{1}{q_2}\right) - 1\right]b$$

$$+ \left[q_1 \left(2p_1 + 2p_2 - \frac{1}{q_1}\right) + q_2 \left(p_1 + 2p_2 - \frac{1}{q_2}\right) - 1\right]c$$

Observe that $q_1 \left(2p_1 + 2p_2 - \frac{1}{q_1}\right) + q_2 \left(p_1 + 2p_2 - \frac{1}{q_2}\right) - 1$ is positive. Since $\frac{1}{q_1} + \frac{q_2}{q_1} \left(p_1 + 2p_2 - \frac{1}{q_2}\right)$, $\frac{1}{q_2} + \frac{q_1}{q_2} \left(2p_1 + 2p_2 - \frac{1}{q_1}\right)$, and $\kappa$ are positive as well, then $\kappa_1 > 0$. Thus, $(a_1, b_1, c_1)$ meets condition (*), so we can repeat the above firing sequence from position $(a_1, b_1, c_1)$ to obtain a position $(a_2, b_2, c_2)$ that meets condition (*), etc. Since the fundamental positions $(a, b, c) = (1, 0, 0)$ and $(a, b, c) = (0, 1, 0)$ meet condition (*), then we see that the indicated firing sequence can be repeated indefinitely from these positions. For the fundamental position $(a, b, c) = (0, 0, 1)$, begin by firing at the leftmost node to obtain the position $(q_1, q_2, -1)$. This latter position meets condition (*), and so the firing sequence indicated above can be repeated indefinitely from this position.

**Step 4:** Every node is fired. The following is proved easily with an induction argument on the number of nodes.
Lemma 2.8  Let \( \lambda \) be a nonzero dominant position assigned to the nodes of a connected GCM graph \((\Gamma, M)\). Then in any game sequence for \( \lambda \), every node of \( \Gamma \) is fired at least once.

Step 5: Subgraphs. If \( I_m' \) is a subset of the node set \( I_m \) of a GCM graph \((\Gamma, M)\), then let \( \Gamma' \) be the subgraph of \( \Gamma \) with node set \( I_m' \) and the induced set of edges, and let \( M' \) be the corresponding submatrix of the amplitude matrix \( M \); we call \((\Gamma', M')\) a GCM subgraph of \((\Gamma, M)\). In light of Lemma 2.8, the following result amounts to an observation.

Lemma 2.9 If a connected GCM graph is admissible, then any connected GCM subgraph is also admissible.

Step 6: Amplitude products must be 1, 2, or 3.

Lemma 2.10 If \( \gamma_i \) and \( \gamma_j \) are adjacent nodes in a connected admissible GCM graph \((\Gamma, M)\), then the product of the amplitudes \( M_{ij}M_{ji} \) is 1, 2, or 3. That is, the GCM subgraph of \((\Gamma, M)\) with nodes \( i \) and \( j \) is in this case one of \( A_2, B_2, \) or \( G_2 \).

Proof. By Lemma 2.9 we may restrict attention to the admissible GCM subgraph \((\Gamma', M')\) with node set \( \{i, j\} \). A nonzero dominant position with a convergent game sequence might not begin with positive populations at both nodes; nonetheless, by examining the proof one sees that Lemma 3.7 of Eriksson still applies to show that the product \( M_{ij}M_{ji} \) of amplitudes in the admissible GCM graph \((\Gamma', M')\) is 1, 2, or 3.

Conclusion of the “only if” part of the first claim of Theorem 1.1. Putting Steps 1 through 6 together, we see that the only possible connected admissible GCM graphs are the Dynkin diagrams.

Proof of the remaining claims of Theorem 1.1:

Let \((\Gamma, M)\) be a connected Dynkin diagram. Theorem 2.1 (Eriksson’s Strong Convergence Theorem) shows that if a game sequence for some initial position \( \lambda \) converges, then all game sequences from \( \lambda \) converge to the same terminal position in the same finite number of steps. Then in light of Theorem 2.3 (Eriksson’s Comparison Theorem), it suffices to show that for any strongly dominant initial position assigned to the nodes of \((\Gamma, M)\), there is a convergent game sequence. For the exceptional graphs \((E_6, E_7, E_8, F_4, \) and \( G_2 \)) this can be checked by hand (requiring 36, 63, 120, 24, and 6 firings respectively). For the four infinite families of Dynkin diagrams, the next result can be proved by induction.

Lemma 2.11 For any positive integer \( n \) (respectively, any integer \( n \geq 2, n \geq 3, n \geq 4 \)) and for any strongly dominant position \((a_1, \ldots, a_n)\) assigned to the nodes of \( A_n \) (respectively, \( B_n, C_n, D_n \)), one can obtain the position \( a_1 + \cdots + a_n, -a_n, \ldots, -a_3, -a_2 \) (respectively \( a_1 + 2a_2 + \cdots + 2a_n, -a_n, -a_2, -a_3, \ldots, -a_2 \)) \( \) (respectively \( a_1 + 2a_2 + \cdots + 2a_n, -a_n, -a_2, -a_3, \ldots, -a_2 \)) where \( b_{n-1} := a_{n-1} \) and \( b_n := a_n \) when \( n \) is odd and where \( b_{n-1} := a_n \) and \( b_n := a_{n-1} \) when \( n \) is even) by a sequence of \( \frac{n(n-1)}{2} \) (resp. \((n - 1)^2, (n - 1)^2, (n - 1)(n - 2)\)) node firings.

From this we can obtain the following (again by induction):

Lemma 2.12 For any positive integer \( n \) (respectively, any integer \( n \geq 2, n \geq 3, n \geq 4 \)) and for any strongly dominant position \((a_1, \ldots, a_n)\) assigned to the nodes of \( A_n \) (respectively, \( B_n, C_n, D_n \)), one can obtain the terminal position \(-a_n, -a_2, -a_1\) (respectively \(-a_1, -a_2, \ldots, -a_n\), \(-a_1, -a_2, \ldots, -a_n, -b_{n-1}, -b_n\)) where \( b_{n-1} := a_{n-1} \) and \( b_n := a_n \) when \( n \)
is even and where \(b_{n-1} := a_n\) and \(b_n := a_{n-1}\) when \(n\) is odd) by a sequence of \(n(n+1)/2\) (resp. \(n^2, n^2, n(n-1)\)) node firings.

This completes the proof of Theorem 1.1.

3. Classifications of finite-dimensional Kac-Moody algebras and finite Weyl groups

Since our first proof of Theorem 1.1 does not depend on the classifications of the finite-dimensional Kac-Moody algebras or of the finite Weyl groups, we can use Theorem 1.1 to obtain these results. In fact only the weaker result Theorem 1.2 due to Wildberger is needed for our proof. This is recorded as Corollary 3.2 below. These classifications are obtained in [Kac] and [Hum2] respectively by carefully studying properties of the generalized Cartan matrix (or a closely related matrix). The definitions we use here basically follow [Kum] (but see also [Kac]). The Lie algebra that is constructed next does not depend on the specific choices made. Given a GCM graph \((\Gamma, M)\) with \(n\) nodes, choose a complex vector space \(\mathfrak{h}\) of dimension \(n + \text{corank}(M)\). Choose \(n\) linearly independent vectors \(\{\beta_i^\vee\}_{1 \leq i \leq n}\) in \(\mathfrak{h}\), and find \(n\) linearly independent functionals \(\{\beta_i\}_{1 \leq i \leq n}\) in \(\mathfrak{h}^*\) satisfying \(\beta_j(\beta_i^\vee) = M_{ij}\). The Kac-Moody algebra \(\mathfrak{g} = \mathfrak{g}(\Gamma, M)\) is the Lie algebra over \(\mathbb{C}\) generated by the set \(\mathfrak{h} \cup \{x_i, y_i\}_{i \in I_n}\) with relations \([h, x_i] = \beta_i(h)x_i\) and \([h, y_i] = -\beta_i(h)y_i\) for all \(h \in \mathfrak{h}\) and \(i \in I_n\); \([x_i, y_j]\) = \(\delta_{i,j}\beta_i^\vee\) for all \(i, j \in I_n\); \((\text{ad} x_i)^{1-M_{ij}}(x_j) = 0\) for \(i \neq j\); and \((\text{ad} y_i)^{1-M_{ij}}(y_j) = 0\) for \(i \neq j\), where \((\text{ad} z)^k(w) = [z, [z, \ldots, [z, w] \ldots]]\). It is known (see for example Proposition 1.3.21 of [Kum]) that the associated Weyl group \(W = W(\Gamma, M)\) with identity denoted \(e\) has the following presentation by generators and relations: generators \(\{s_i\}_{i \in I_n}\), and relations \(s_i^2 = e\) for \(i \in I_n\) and \((s_is_j)^{m_{ij}} = e\) where the integers \(m_{ij}\) are determined as follows: \(m_{ij} = 2\) if \(M_{ij}M_{ji} = 0\), \(m_{ij} = 3\) if \(M_{ij}M_{ji} = 1\), \(m_{ij} = 4\) if \(M_{ij}M_{ji} = 2\), \(m_{ij} = 6\) if \(M_{ij}M_{ji} = 3\), and \(m_{ij} = \infty\) if \(M_{ij}M_{ji} \geq 4\). This group is the same as the Coxeter group Eriksson associates to the GCM graph \((\Gamma, M)\) in §3 of [Erik2]. If the graph \(\Gamma\) has connected components \(\Gamma_1, \ldots, \Gamma_k\) with corresponding amplitude matrices \(M_1, \ldots, M_k\), then \(\mathfrak{g}(\Gamma, M) \approx \mathfrak{g}(\Gamma_1, M_1) \oplus \cdots \oplus \mathfrak{g}(\Gamma_k, M_k)\) and \(W(\Gamma, M) \approx W(\Gamma_1, M_1) \times \cdots \times W(\Gamma_k, M_k)\). For a firing sequence \((\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_k})\) from some initial position \(\lambda\) in a numbers game on \((\Gamma, M)\), the corresponding element of \(W\) is \(s_{i_1} \cdots s_{i_k}s_{i_1}\). The next result follows from Proposition 4.1 of [Erik2] and is a key step in our proof of Corollary 3.2.

**Proposition 3.1 (Eriksson’s Reduced Word Result)** If \((\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_k})\) is a legal sequence of node firings in a numbers game played from some initial position on a GCM graph \((\Gamma, M)\), then \(s_{i_1} \cdots s_{i_k}s_{i_1}\) is a reduced expression for the corresponding element of the Weyl group \(W(\Gamma, M)\).

**Corollary 3.2** Given a generalized Cartan matrix, the associated Weyl group is finite if and only if the associated Kac-Moody algebra is finite-dimensional if and only if the associated GCM graph is a Dynkin diagram.

**Proof.** Suppose a connected GCM graph \((\Gamma, M)\) is not a Dynkin diagram. Then by Theorem 1.1 or 1.2, we may pick a nonzero dominant position \(\lambda\) as an initial position and play a nonterminating numbers game. By Eriksson’s Reduced Word Result, it follows that the sequence of the first \(k\) firings (for any \(k > 0\)) of the corresponding game sequence will correspond to a reduced word in the corresponding Weyl group \(W\). Thus our numbers game generates an infinite number of Weyl group elements, and hence \(|W| = \infty\). From Proposition 1.4.2 of [Kum], it follows that the set of roots of
the Kac-Moody algebra $\mathfrak{g}$ associated to $(\Gamma, M)$ is infinite. From the root space decomposition of $\mathfrak{g}$ (see §1.2 of [Kum]), it now follows that $\mathfrak{g}$ is infinite-dimensional. Conversely, that the Kac-Moody algebra $\mathfrak{g}$ associated to a connected Dynkin diagram $(\Gamma, M)$ is finite-dimensional follows from Serre’s Theorem (Theorem 18.3 of [Hum1]) in combination with the construction in §12 of that text of the root system for the connected Dynkin diagram; finiteness of this root system implies the finiteness of the associated Weyl group (again Proposition 1.4.2 of [Kum]).

It is well known that the Kac-Moody algebras associated to the Dynkin diagrams of Figure 1.1 are the complex finite-dimensional simple Lie algebras (see for example [Hum1] §18). It is also well known that Lie algebras corresponding to distinct Dynkin diagrams of Figure 1.1 are non-isomorphic; for the associated Weyl groups, the only redundancy is that the groups corresponding to the $B_n$ and $C_n$ graphs for $n \geq 3$ are the same.

4. Quasi-standard geometric representations of Coxeter groups and a generalization to E-games

In this section we study a certain geometric representation of a Coxeter group associated to a given E-GCM. This geometric representation is developed in [Erik2] §3, 4 and in [BB] §4.1, 4.2; it is similar to but not in general the same as the standard geometric representation (as developed in [Hum2] Ch. 5, for example). Results we derive about geometric aspects of this representation extend the standard theory and will lead to a proof of the classification result Theorem 1.3. The classification of finite Coxeter groups ([Hum2] §6.4, 2.7, 2.4) is used in our proof of Theorem 1.3.

Now and for the remainder of this section fix an E-GCM graph $(\Gamma, M)$. Define the associated Coxeter group $W = W(\Gamma, M)$ to be the Coxeter group with identity denoted $e$, generators $\{s_i\}_{i \in I_n}$, and relations $s_i^2 = e$ for $i \in I_n$ and $(s_i s_j)^{m_{ij}} = e$ for all $i \neq j$, where the integers $m_{ij}$ are determined as follows:

$$m_{ij} = \begin{cases} k & \text{if } M_{ij} M_{ji} = 4 \cos^2(\pi/k_{ij}) \text{ for some integer } k_{ij} \geq 2 \\ \infty & \text{if } M_{ij} M_{ji} \geq 4 \end{cases}$$

Observe that any Coxeter group on a finite set of generators is isomorphic to $W(\Gamma, M)$ for some E-GCM graph $(\Gamma, M)$; the Coxeter group is irreducible if $\Gamma$ is connected. By the classification of finite Coxeter groups, an irreducible Coxeter group is finite if and only if it is the Coxeter group associated to an E-Coxeter graph from Figure 1.3. Before proceeding toward our proof of Theorem 1.3, we note the following. In Propositions 4.1 and 4.2 of [Deo], Deodhar gives a number of statements equivalent to the assertion that a given irreducible Coxeter group is finite. As an immediate consequence of Theorem 1.3 and the classification of finite Coxeter groups, we add to that list the following equivalence.

**Corollary 4.1** An irreducible Coxeter group $W$ is finite if and only if there is an admissible E-GCM graph whose associated Coxeter group is $W$ if and only if any E-GCM graph is admissible when its associated Coxeter group is $W$. □

We say two nodes $\gamma_i$ and $\gamma_j$ in $(\Gamma, M)$ are odd-adjacent if $m_{ij}$ is odd, even-adjacent if $m_{ij} \geq 4$ is even, and $\infty$-adjacent if $m_{ij} = \infty$. When $m_{ij}$ is odd and $M_{ij} \neq M_{ji}$, we say that the adjacent nodes $\gamma_i$ and $\gamma_j$ form an odd asymmetry. Note that $m_{ij} = 2$ (respectively 3, 4, 6) when $M_{ij} M_{ji} = 0$ (resp. 1, 2, 3). We let $\ell$ denote the length function for $W$. For $J \subseteq I_n$, let $W_J$ be the subgroup
generated by \( \{ s_i \}_{i \in J} \), a parabolic subgroup, and \( W^J := \{ w \in W \mid \ell(ws_{ij}) > \ell(w) \text{ for all } j \in J \} \) is the set of minimal coset representatives. If \( J = \{ i, j \} \), then \( W_J \) is a dihedral group of order \( 2m_{ij} \).

Let \( V \) be a real \( n \)-dimensional vector space freely generated by \( \{ \alpha_i \}_{i \in I_n} \) (elements of this ordered basis are simple roots). Equip \( V \) with a possibly asymmetric bilinear form \( B : V \times V \to \mathbb{R} \) defined on the basis \( \{ \alpha_i \}_{i \in I_n} \) by \( B(\alpha_i, \alpha_j) := \frac{1}{2}M_{ij} \). For each \( i \in I_n \) define an operator \( S_i : V \to V \) by the rule \( S_i(v) := v - 2B(\alpha_i, v)\alpha_i \) for each \( v \in V \). One can check that \( S_i^2 = I \) (the identity transformation), so \( S_i \in GL(V) \). With \( V_{i,j} := \text{span} \{ \alpha_i, \alpha_j \} \), observe that \( S_k(V_{i,j}) \subseteq V_{i,j} \) for \( k = i, j \). Let \( \mathfrak{B} \) be the ordered basis \( \{ \alpha_i, \alpha_j \} \) for \( V_{i,j} \), and for any linear mapping \( T : V_{i,j} \to V_{i,j} \) let \( [T]_{\mathfrak{B}} \) be the matrix for \( T \) relative to \( \mathfrak{B} \). Then
\[
[S_i|V_{i,j}]_{\mathfrak{B}} = \begin{pmatrix} -1 & -M_{ij} \\ 0 & 1 \end{pmatrix}, [S_j|V_{i,j}]_{\mathfrak{B}} = \begin{pmatrix} 1 & 0 \\ -M_{ji} & -1 \end{pmatrix}, [S_iS_j|V_{i,j}]_{\mathfrak{B}} = \begin{pmatrix} M_{ij}M_{ji} - 1 & M_{ij} \\ -M_{ji} & -1 \end{pmatrix}
\]

Analysis of the eigenvalues for \( X_{i,j} := [S_iS_j|V_{i,j}]_{\mathfrak{B}} \) as in the proof of Proposition 1.3.21 of [Kum] shows that \( X_{i,j} \) has infinite order when \( M_{ij}M_{ji} \geq 4 \), and hence \( S_iS_j \) has infinite order as an element of \( GL(V) \). When \( 0 < M_{ij}M_{ji} < 4 \), write \( M_{ij}M_{ji} = 4\cos^2 \theta \) for \( 0 < \theta < \pi/2 \). In this case check that \( X_{i,j} \) has two distinct complex eigenvalues \( (e^{i2\theta} \text{ and } e^{-i2\theta}) \). It follows that \( X_{i,j} \) has finite order \( m_{ij} \) if and only if \( \theta = \pi/m_{ij} \), which coincides with Eriksson’s constraints on the amplitude products of the E-GCM. When \( M_{ij}M_{ji} = 0 \), then \( X_{i,j} = 0 \), which clearly has order \( m_{ij} = 2 \). With \( 0 \leq M_{ij}M_{ji} < 4 \), one can easily see that \( V = V_{i,j} \oplus V'_{i,j} \), where \( V'_{i,j} := \{ v \in V \mid B(\alpha_i, v) = 0 = B(\alpha_j, v) \} \). Since \( S_iS_j \) acts as the identity on \( V'_{i,j} \), it follows that \( S_iS_j \) has order \( m_{ij} \) as an element of \( GL(V) \). Then there is a unique homomorphism \( \sigma_M : W \to GL(V) \) for which \( \sigma_M(s_i) = S_i \); we call \( \sigma_M \) a quasi-standard geometric representation of \( W \). We now have \( W \) acting on \( V \), and for all \( w \in W \) and \( v \in V \) we write \( w.v \) for \( \sigma_M(w)(v) \). Define \( \Phi_M := \{ \alpha \in V \mid \alpha = w.\alpha_i \text{ for some } i \in I_n \text{ and } w \in W \} \). Elements of \( \Phi_M \) are roots; if \( \alpha = \sum c_i \alpha_i \) is a root with all \( c_i \) nonnegative (respectively nonpositive), then say \( \alpha \) is a positive (resp. negative) root, and write \( \alpha >_M 0 \) (resp. \( \alpha <_M 0 \)). Let \( \Phi^+_M \) and \( \Phi^-_M \) denote the collections of positive and negative roots respectively; it is a consequence of Proposition 4.3 below that \( \Phi_M \) is partitioned by \( \Phi^+_M \) and \( \Phi^-_M \). The possible asymmetry of the bilinear form is a crucial difference between this quasi-standard geometric realization and the standard geometric realization; for example, \( \sigma_M(W) \) preserves the form \( B \) if and only if \( M \) is symmetric, in which case the representation studied here is the same as the standard geometric representation. Under this quasi-standard action of \( W \) on \( V \), sometimes \( K\alpha_x \) is a root for \( K \neq \pm 1 \), as Example 4.10 shows (see also Exercise 4.9 of [BB]).* Our proofs of the main results of this section require us to understand how this \( W \)-action generates scalar multiples in \( \Phi \). First we analyze how \( s_i \) and \( s_j \) act in tandem on \( V_{i,j} \). Our next result strengthens Lemma 4.2.4 of [BB].

**Lemma 4.2** Fix \( i \neq j \) in \( I_n \), and let \( k \) be a positive integer. If \( m_{ij} = \infty \), then \( (s_is_j)^k.\alpha_i = a\alpha_i + b\alpha_j \) and \( s_j(s_is_j)^k.\alpha_i = c\alpha_i + d\alpha_j \), for positive coefficients \( a, b, c, \) and \( d \). Now suppose \( m_{ij} < \infty \). If \( 2k < m_{ij} \), then \( (s_is_j)^k.\alpha_i = a\alpha_i + b\alpha_j \) with \( a \geq 0 \) and \( b > 0 \). Moreover, this is a multiple of \( \alpha_j \) if and only if \( m_{ij} \) is odd and \( k = (m_{ij} - 1)/2 \), in which case \( (s_is_j)^k.\alpha_i = \frac{-M_{ij}}{2\cos(\pi/m_{ij})}\alpha_j \). Similarly, if

*Just prior to the statement of Proposition 4.4 in [Erik], it is mistakenly asserted that the only multiples in \( \Phi_M \) of a simple root \( \alpha_x \) are \( \pm \alpha_x \). This only affects Proposition 4.4 of that paper.
If \(2k < m_{ij} - 1\), then \(s_i(s_is_j)^k\cdot \alpha_i = a\alpha_i + b\alpha_j\) with \(a > 0\) and \(b \geq 0\). Moreover, this is a multiple of \(\alpha_i\) if and only if \(m_{ij}\) is even and \(k = \frac{m_{ij}}{2}\); in which case \(s_i(s_is_j)^k\cdot \alpha_i = \alpha_i\).

**Proof.** The details are somewhat tedious, but routine. For convenience set \(p := -M_{ij}\) and \(q := -M_{ji}\). Start with \(m_{ij} < \infty\), and set \(\theta := \pi/m_{ij}\). We will work with the ordered basis \(\mathcal{B}\) for \(V_{i,j}\) and operators \(X_i := [S_i|V_{i,j}]_{\mathcal{B}}\) and \(X_j := [S_j|V_{i,j}]_{\mathcal{B}}\) as above. Then to understand \((s_i(s_is_j)^k\cdot \alpha_i\) and \(s_j(s_is_j)^k\cdot \alpha_i\), we compute \(X_i^k\) and \(X_jX_i^k\). In the case that \(pq = 4\cos^2(\theta)\), then \(X_i\) can be written as \(X_i = PDP^{-1}\) for a nonsingular matrix \(P\) and diagonal matrix \(D\) in the following way:

\[
\frac{1}{q(e^{2i\theta} - e^{-2i\theta})} \begin{pmatrix}
 e^{2i\theta} + 1 & e^{-2i\theta} + 1 \\
 q & q
\end{pmatrix} \begin{pmatrix}
 e^{2i\theta} & 0 \\
 0 & e^{-2i\theta}
\end{pmatrix} \begin{pmatrix}
 q & -e^{-2i\theta} \\
 -q & e^{2i\theta} + 1
\end{pmatrix}
\]

Then for any positive integer \(k\) we have

\[
X_i^k = PD^kP^{-1} = \frac{1}{\sin(2\theta)} \begin{pmatrix}
 \sin(2(k+1)\theta) + \sin(2k\theta) & -p\sin(2k\theta) \\
 q\sin(2k\theta) & -\sin(2k\theta) - \sin(2(k-1)\theta)
\end{pmatrix}
\]

and

\[
X_jX_i^k = \frac{1}{\sin(2\theta)} \begin{pmatrix}
 \sin(2(k+1)\theta) + \sin(2k\theta) & -p\sin(2k\theta) \\
 q\sin(2k\theta) & (1-pq)\sin(2k\theta) + \sin(2(k-1)\theta)
\end{pmatrix}
\]

Use the first column of \(X_i^k\) and \(X_jX_i^k\) to see that \((s_i(s_is_j)^k\cdot \alpha_i = \frac{1}{\sin(2\theta)}[\sin(2(k+1)\theta) + \sin(2k\theta)]\alpha_i + \frac{q\sin(2\theta)}{\sin(2\theta)}\sin(2k\theta)\alpha_i\) and \(s_j(s_is_j)^k\cdot \alpha_i = \frac{1}{\sin(2\theta)}[\sin(2(k+1)\theta) + \sin(2k\theta)]\alpha_i + \frac{q}{\sin(2\theta)}\sin(2(k+1)\theta)\alpha_j\). As long as \(2(k+1) < m_{ij}\), then all the coefficients of these linear combinations will be positive. So now suppose \(2(k+1) \geq m_{ij}\). First we consider \((s_i(s_is_j)^k\cdot \alpha_i = a\alpha_i + b\alpha_j\) for some positive \(k\) with \(2k < m_{ij}\). There are two possibilities now: \(2(k+1) = m_{ij}\) or \(2(k+1) = m_{ij} + 1\). In the former case both \(a\) and \(b\) are positive. In the latter case we have \(m_{ij}\), then all the coefficients of these linear combinations will be positive. So now suppose \(2(k+1) \geq m_{ij}\). First we consider \((s_i(s_is_j)^k\cdot \alpha_i = a\alpha_i + b\alpha_j\) for some positive \(k\) with \(2k < m_{ij} - 1\).

Now the fact that \(2(k+1) \geq m_{ij}\) implies we have \(2(k+1) = m_{ij}\). In particular, \(m_{ij}\) is even. With \(k = (m_{ij} - 2)/2\) now, one can check that \(b = 0\) and \(a = 1\).

For \(m_{ij} = \infty\), first take \(pq = 4\). We can write \(X_{i,j} = PYP^{-1}\) for nonsingular \(P\) and upper triangular \(Y\) as follows:

\[
X_{i,j} = \frac{1}{p} \begin{pmatrix}
 p & p \\
 2 & 1
\end{pmatrix} \begin{pmatrix}
 1 & 1 \\
 0 & 1
\end{pmatrix} \begin{pmatrix}
 -1 & p \\
 2 & -p
\end{pmatrix}
\]

Then for any positive integer \(k\) we obtain \(X_{i,j}^k = \begin{pmatrix}
 2k + 1 & -kp \\
 kp & -2k + 1
\end{pmatrix}\). It follows that \((s_is_is_j)^k\cdot \alpha_i = (2k+1)\alpha_i + k\alpha_j\), with both coefficients of the linear combination positive. From the first column of the matrix \(X_jX_{i,j}^k\), we see that \(s_j(s_is_j)^k\cdot \alpha_i = (2k+1)\alpha_i + (2k+1)q\alpha_j\), with both coefficients of the linear combination positive. Next take \(pq > 4\). In this case we get distinct eigenvalues \(\lambda = \frac{1}{2}(pq - 2 + \sqrt{pq(pq - 4)}) > 1\) and \(\mu = \frac{1}{2}(pq - 2 - \sqrt{pq(pq - 4)}) < 1\) for \(X_{i,j}\) (here we have \(\lambda\mu = 1\)). As before, write \(X_{i,j} = PDP^{-1}\) for the diagonal matrix \(D = \begin{pmatrix}
 \lambda & 0 \\
 0 & \mu
\end{pmatrix}\) and a nonsingular matrix \(P\) to obtain

\[
X_{i,j}^k = \frac{1}{p(\lambda - \mu)} \begin{pmatrix}
 p & p \\
 \lambda' & \mu'
\end{pmatrix} \begin{pmatrix}
 \lambda^k & 0 \\
 0 & \mu^k
\end{pmatrix} \begin{pmatrix}
 \mu' & -p \\
 -\lambda' & p
\end{pmatrix}
\]
for any positive integer $k$, with $\lambda := \mu + 1$ and $\mu' := \lambda + 1$. This (eventually) simplifies to

$$X_{i,j}^k = \frac{1}{\lambda - \mu} \begin{pmatrix} \mu'\lambda^k - \lambda'\mu^k & -p(\lambda^k - \mu^k) \\ q(\lambda^k - \mu^k) & \mu'\mu^k - \lambda'\lambda^k \end{pmatrix}. $$

From this we also get

$$X_jX_{i,j}^k = \frac{1}{\lambda - \mu} \begin{pmatrix} \mu'\lambda^k - \lambda'\mu^k & -p(\lambda^k - \mu^k) \\ q(\lambda^k - \mu^k) & \mu'\mu^k - \lambda'\lambda^k \end{pmatrix}. $$

The factor $\frac{1}{\lambda - \mu}$ is positive, and for both matrices $X_{i,j}^k$ and $X_jX_{i,j}^k$, the first column entries are positive. So, $(s_is_j)^k$ and $s_is_j^k$ with both $a$ and $b$ positive, and $s_is_j^k$ with $c$ and $d$ both positive.

In view of this result, for odd $m_{ij}$, let $v_{ji}$ be the element $(s_is_j)^{(m_{ij}-1)/2}$, and set $K_{ji} := \frac{-M_{ji}}{2\cos(\pi/m_{ij})}$, which is positive. Then $v_{ji}\cdot\alpha_i = K_{ji}\alpha_j$. Observe that $K_{ji}K_{ji} = 1$ and moreover that $v_{ji} = v_{ji}^{-1}$.

A path with odd adjacencies (or OA-path, for short) in $(\Gamma, M)$ is a sequence $P := [\gamma_{i_0}, \gamma_{i_1}, \ldots, \gamma_{i_p}]$ of pairwise odd-adjacent nodes of $\Gamma$; this OA-path has length $p$, and we allow OA-paths to have length zero. We say $\gamma_{i_0}$ and $\gamma_{i_p}$ are the start and end nodes of the OA-path, respectively. If OA-path $Q = [\gamma_{j_0}, \gamma_{j_1}, \ldots, \gamma_{j_q}]$ has the same start node as the end node of $P$, then their concatenation $P;Q$ is the OA-path $[\gamma_{i_0}, \gamma_{i_1}, \ldots, \gamma_{i_p};\gamma_{j_0};\gamma_{j_1};\ldots;\gamma_{j_p}]$. Let $w_p \in W$ be the Coxeter group element $v_{ip_{ip-1}}\cdots v_{i_{2i1}}v_{i_{1i0}}$, and let $P_p := K_{ip_{ip-1}}\cdots K_{i_{2i1}}K_{i_{1i0}}$, where $w_p = e$ with $P_p = 1$ when $P$ has length zero. Note that $w_p\cdot\alpha_i = P_p\cdot\alpha_i$ and that $w_p;Q = w_Qw_p$.

**Proposition 4.3** Let $w \in W$ and $i \in I_n$. If $\ell(ws_i) > \ell(w)$, then $w.\alpha_i > M$, and in this case $w.\alpha_i = K\alpha_x (x \in I_n, K > 0)$ if and only if $w.\alpha_i = w_p.\alpha_i$ for some OA-path $P = [\gamma_{i_0};\gamma_{i_1};\ldots;\gamma_{i_p};\gamma_{i_p} = x]$, so $K = P_p$. Similarly, if $\ell(ws_i) < \ell(w)$, then $w.\alpha_i < M$, and in this case $w.\alpha_i = K\alpha_x (x \in I_n, K < 0)$ if and only if $w.\alpha_i = (w_p;\gamma_{i_0})\cdot\alpha_i$ for some OA-path $P = [\gamma_{i_0};\gamma_{i_1};\ldots;\gamma_{i_p};\gamma_{i_p} = x]$, so $K = -P_p$.

The assertions that $\ell(ws_i) > \ell(w) \Rightarrow w.\alpha_i > M$ and $\ell(ws_i) < \ell(w) \Rightarrow w.\alpha_i < M$ are Proposition 4.2.5 of [Hum2], whose set up we require for our analysis of roots which are scalar multiples of simple roots.

**Proof of Proposition 4.3.** The first part of our argument follows the proof of Theorem 5.4 from [Hum2]; however, the argument here is easier since we may use the fact that for $J \subseteq I_n$, the length function $\ell_J$ on $W_J$ agrees with $\ell$. Note that the second assertion of the theorem follows from the first. For the first assertion of the theorem, induct on $\ell(w)$. When $\ell(w) = 0$, there is nothing to prove. Now suppose $\ell(w) > 0$. Take any $j \in I_n$ for which $\ell(ws_j) = \ell(w) - 1$; since $\ell(ws_j) > \ell(w)$, then $i \neq j$. Let $J := \{i, j\}$, and let $v$ be the unique element in $W_J$ and $v_j$ the unique element in $W_J$ for which $w = wv_j$. Then $\ell(w) = \ell(v) + \ell(v_j)$. Observe that $\ell(v_j) > 0$ since $\ell(v) < \ell(w)$. From Humphreys’ proof, we can see that $\ell(ws_j) > \ell(v), (ws_j) > \ell(v), \ell(v_j) > \ell(v_j)$, and $\ell(v_j) > \ell(v_j)$. Apply the induction hypothesis to conclude that $v.\alpha_i > M$ and $v.\alpha_j > M$. It is possible that $\ell(w) = \ell(v_j)$, so the induction hypothesis might not apply to $v_j$. But since $\ell(v_j;\gamma_{i_0}) > \ell(v_j)$, it follows that any reduced expression for $v_j$ (necessarily an alternating product of $s_i$’s and $s_j$’s) must end in $s_j$. Then we may apply Lemma 4.2 to conclude that $v_j.\alpha_i > M$. Together these facts imply that $w.\alpha_i > M$.

Now we address the issue of scalar multiples. Suppose $w.\alpha_i = K\alpha_x$ for some $x \in I_n$ and real number $K > 0$. Write $v_j.\alpha_i = a\alpha_i + b\alpha_j$, and suppose $a > 0$ and $b > 0$. Note that $v.\alpha_i$ and $v.\alpha_j$
cannot both be multiples of the same $\alpha_x$ (otherwise $v^{-1}\alpha_x$ is a multiple of both $\alpha_i$ and $\alpha_j$). But now a simple calculation shows that in this case $w.\alpha_i$ will not be a multiple of $\alpha_x$ for any $x' \in I_n$. We conclude that $v_j.\alpha_i$ must be a scalar multiple of $\alpha_i$ or $\alpha_j$. If $v_j = s_j$, then $w.\alpha_i = v.\alpha_i$, and the induction hypothesis now applies to $v$ to obtain the desired result. So now suppose $\ell(v_j) > 1$. Then $v_j = (s_is_j)^k$ or $s_j(s_is_j)^k$ for some positive integer $k$. From Lemma 4.2, it follows that if $m_{ij} = \infty$, then $v_j.\alpha_i = a\alpha_i + b\alpha_j$ with $a$ and $b$ both positive. Therefore $m_{ij}$ is finite. In this case the longest element in $W_j$ has length $m_{ij}$ and can be written in two ways, one ending in $s_i$. Therefore $\ell(v_j) < m_{ij}$. We are again in the situation of Lemma 4.2. If $v_j.\alpha_i = \alpha_i$, then $w.\alpha_i = v.\alpha_i$, and we can apply the induction hypothesis to $v$. If $v_j.\alpha_i \neq \alpha_i$, then we see that $m_{ij}$ is odd and $v_j.\alpha_i = v_ji.\alpha_i = Kji\alpha_j$. Again apply the induction hypothesis to $v$, where now $v.\alpha_j = K_{ji}\alpha_x = w.\alpha_j$ for some OA-path $Q = [\gamma_{j0} = j, \gamma_{j1}, \ldots, \gamma_{jq} = x]$. Let $P := [\gamma_j, \gamma_j]\alpha Q$. Then $w.\alpha_i = K\alpha_x = w.\alpha_i$.

As with Corollary 5.4 of [Hum2] it follows that the representation $\sigma_M$ is faithful. It also follows that $\Phi_M$ is partitioned by the sets of positive and negative roots. For any $w \in W$, set $N_M(w) := \{\alpha \in \Phi_M^+ | w.\alpha \in \Phi_M^-\}$.

**Lemma 4.4** For any $i \in I_n$, $s_i(\Phi_M^+ \setminus \{K\alpha_i | K \in \mathbb{R}\}) = \Phi_M^- \setminus \{K\alpha_i | K \in \mathbb{R}\}$. Now let $w \in W$. If $w.\alpha_i > m_0$, then $N_M(ws_i) = s_i(N_M(w)) \cup \{\{K\alpha_i | K \in \mathbb{R}\} \cap \Phi_M^+, a \text{ disjoint union. If } w.\alpha_i < m_0$, then $N_M(ws_i) = s_i(N_M(w)) \setminus \{K\alpha_i | K \in \mathbb{R}\}$.

**Proof.** The proof of Proposition 5.6.(a) from [Hum2] is easily adjusted to prove the first claim. Proofs for the remaining claims involve routine set inclusion arguments.

For $J \subseteq I_n$, let $(\Phi_M)^J := \{\alpha \in \Phi_M^+ | \alpha \notin \text{span}_\mathbb{R}\{\alpha_j\}_{j \in J}\}$. Our next result observes that an assertion from the proof of Proposition 4.2 of [Deco] also holds here:

**Proposition 4.5** If $(\Gamma, M)$ is connected and $\Phi_M$ is infinite, then $\Phi_M^I$ is infinite when $I$ is a proper subset of $I_n$.

**Proof:** In the “$(ix) \Rightarrow (ii)$” part of the proof of Proposition 4.2 in [Deco], begin reading at the assumption “$|\Phi_I'| < \infty$,” replacing $\Phi_I'$ with $\Phi_M^I$.

Before we analyze the sets $\{K\alpha_x | K \in \mathbb{R}\} \cap \Phi_M^+$, we need some further notation. An OA-path $P = [\gamma_{i0}, \ldots, \gamma_{ip}]$ is an $OA$-cycle if $\gamma_{ip} = \gamma_{i0}$; it is a unital OA-cycle if $\Pi_P = 1$. For OA-paths $P$ and $Q$, write $P \sim Q$ and say $P$ and $Q$ are equivalent if these OA-paths have the same start and end nodes and $\Pi_P = \Pi_Q$; this is an equivalence relation on the set of all OA-paths. An OA-path $P$ is simple if it has no repeated nodes with the possible exception that the start and end nodes may coincide. We say the E-GCM graph $(\Gamma, M)$ is unital OA-cyclic if and only if $\Pi_C = 1$ for all OA-cycles $C$. Note that $(\Gamma, M)$ is unital OA-cyclic if and only if $\Pi_P \sim Q$ whenever $P$ and $Q$ are OA-paths with the same start and end nodes. If $\Gamma$ is a tree, then $(\Gamma, M)$ is unital OA-cyclic (vacuously so). From the definitions it follows that $(\Gamma, M)$ is unital OA-cyclic if it has no odd asymmetries. If $M$ is symmetricizable, then by Exercise 2.1 of [Kac], $(\Gamma, M)$ is unital OA-cyclic. However, a unital OA-cyclic E-GCM graph need not have a symmetricizable amplitude matrix $M$, as Example 4.10 shows. To check if an E-GCM graph is unital OA-cyclic, it is enough to check that each simple OA-cycle is unital. An E-GCM graph is $OA$-connected if any two nodes can be joined by an OA-path. An $OA$-connected component of an E-GCM graph $(\Gamma, M)$ is an E-GCM subgraph
(\Gamma', M') whose nodes form a maximal collection of nodes in (\Gamma, M) which can be pairwise joined by OA-paths.

**Lemma 4.6** Let \( \gamma_i \) and \( \gamma_j \) be nodes in the same OA-connected component of the E-GCM graph (\( \Gamma, M \)). Then there is a one-to-one correspondence between the sets \( \{K\alpha_i | K \in \mathbb{R}\} \cap \Phi_M^+ \) and \( \{K'\alpha_j | K' \in \mathbb{R}\} \cap \Phi_M^+ \).

Proof. Let \( \mathcal{G}_i := \{K\alpha_i | K \in \mathbb{R}\} \cap \Phi_M^+ \) and \( \mathcal{G}_j := \{K'\alpha_j | K' \in \mathbb{R}\} \cap \Phi_M^+ \). For \( \phi : \mathcal{G}_i \rightarrow \mathcal{G}_j \) define \( K\alpha_i \rightarrow K_j K\alpha_j \). This is well-defined since \( K\alpha_i \in \mathcal{G}_i \) means \( K = \Pi_p \) for some OA-path with some start node \( \gamma_x \) and end node \( \gamma_i \) (cf. Proposition 4.3). Then \( K_j K = \Pi_{p_i \gamma_i, \gamma_j} \), and hence \( K_j K\alpha_j \in \mathcal{G}_j \). A similar argument shows that \( \psi : \mathcal{G}_j \rightarrow \mathcal{G}_i \) given by \( K'\alpha_j \mapsto K_{ij} K'\alpha_i \) is well-defined. That \( \phi \) and \( \psi \) are inverses follows from the fact that \( K_{ij} K_{ji} = 1 \). \( \square \)

**Lemma 4.7** Suppose (\( \Gamma, M \)) is unital OA-cyclic. Then for any OA-path \( \mathcal{P} \) there is a simple OA-path which is equivalent to \( \mathcal{P} \).

Proof. If \( \mathcal{P} = [\gamma_{i_0}, \ldots, \gamma_{i_p}] \) is not simple, then let \( \gamma_j \) be the first repeated node, appearing again (say) as \( \gamma_{j_t} \) in position \( t > s \) of the sequence. Let \( \mathcal{P}_1 = [\gamma_{i_0}, \ldots, \gamma_{i_t}] \), \( \mathcal{Q} = [\gamma_{i_t}, \ldots, \gamma_{i_s}] \) (an OA-cycle), and \( \mathcal{P}_2 = [\gamma_{i_s}, \ldots, \gamma_{i_p}] \). Clearly \( \mathcal{P} = \mathcal{P}_1 \mathcal{Q} \mathcal{P}_2 \). Since \( \Pi_\mathcal{Q} = 1 \), then \( \Pi_\mathcal{P} = \Pi_\mathcal{P}_1 \mathcal{P}_2 \). So we have \( \mathcal{P}_1 \mathcal{P}_2 \sim \mathcal{P} \), and the former is shorter than the latter. Continuing this process we arrive at a simple OA-path equivalent to \( \mathcal{P} \). \( \square \)

**Proposition 4.8** Suppose (\( \Gamma', M' \)) is an OA-connected component of (\( \Gamma, M \)) with nodes corresponding to some subset \( J \subseteq I_n \). Then the following are equivalent:

1. (\( \Gamma', M' \)) is unital OA-cyclic;
2. \( |\{K\alpha_x | K \in \mathbb{R}\} \cap \Phi_M^+| < \infty \) for some \( x \in J \);
3. \( |\{K\alpha_x | K \in \mathbb{R}\} \cap \Phi_M^+| < \infty \) for all \( x \in J \).

In these cases we have \( |\{K\alpha_x | K \in \mathbb{R}\} \cap \Phi_M^+| = |\{K\alpha_y | K \in \mathbb{R}\} \cap \Phi_M^+| \) for all \( x, y \in J \).

Proof. We show (2) \( \Rightarrow \) (1) \( \Rightarrow \) (3), the implication (3) \( \Rightarrow \) (2) being obvious. For (1) \( \Rightarrow \) (3), let \( x \in J \). Observe that if \( K\alpha_x \in \Phi_M^+ \), then by Proposition 4.3 we must have \( K = \Pi_p \) for some OA-path \( \mathcal{P} \) with end node \( \gamma_x \). Therefore \( \mathcal{P} \) is in (\( \Gamma', M' \)). By Lemma 4.7, we may take a simple OA-path \( \mathcal{Q} \) equivalent to \( \mathcal{P} \) (all OA-paths equivalent to \( \mathcal{P} \)). Let \( \mathcal{C} = [\gamma_x, \ldots, \gamma_x] \) be a non-unital OA-cycle with start/end node \( \gamma_x \) for an \( x \in J \). So necessarily \( \mathcal{C} \) has nonzero length. Note that \( w_x \alpha_x = \Pi_c \alpha_x \). Next, for \( y \in J \) (and possibly \( y = x \)) take any OA-path \( \mathcal{P} \) with start node \( \gamma_x \) and end node \( \gamma_y \). Since \( w_p \alpha_x = \Pi_p \alpha_y \), it follows that \( w_p w_x \alpha_x = \Pi_p \alpha_y \) for any integer \( k \). In particular, for all \( y \in I_n \), we have \( |\{K\alpha_y | K \in \mathbb{R}\} \cap \Phi_M^+| = \infty \). The final claim of the proposition statement follows from Lemma 4.6. \( \square \)

When the E-GCM (\( \Gamma, M \)) is OA-connected and unital OA-cyclic, let \( f_{\Gamma, M} := |\{K\alpha_x | K \in \mathbb{R}\} \cap \Phi_M^+| \) for any fixed \( x \in I_n \); then in this case Proposition 4.3, Lemma 4.4, and Proposition 4.8 allow us to modify the proof of Proposition 5.6 of [Hum2] to obtain the result that for all \( w \in W \), \( |N_M(w)| = f_{\Gamma, M}(\ell(w)) \). Proposition 4.9 below generalizes this statement. When \( \ell = \infty \), the length function must take arbitrarily large values; from Proposition 4.9 it will follow that \( \Phi_M \) is infinite as well.
**Proposition 4.9** If the E-GCM graph \((\Gamma, M)\) is unital OA-cyclic then for all \(w \in W\) we have
\[
f_1 \ell(w) \leq |N_M(w)| \leq f_2 \ell(w),
\]
where \(f_1\) is the min and \(f_2\) is the max of all integers in the set
\[
\{ f_{\Gamma',M'} | (\Gamma', M') \text{ is an OA-connected component of } (\Gamma, M) \}.
\]

**Proof.** Induct on \(\ell(w)\). For \(\ell(w) = 0\), the result is obvious. Now take \(w = w's_i\), \(\ell(w) = \ell(w') + 1\), and \(\gamma_i\) in an OA-connected component \((\Gamma', M')\) of \((\Gamma, M)\). Then by Lemma 4.4, \(|N_M(w)| = |N_M(w')| + f_{\Gamma',M'}\). Since \(f_1 \ell(w') \leq |N_M(w')| \leq f_2 \ell(w')\), the result follows. \(\square\)

**Example 4.10** In Figure 4.1 is depicted a connected, unital OA-cyclic E-GCM graph \((\Gamma, M)\) with two OA-connected components: \((\Gamma_1, M_1)\) is the E-GCM subgraph with nodes \(\gamma_i\) and \(\gamma_j\), and \((\Gamma_2, M_2)\) has nodes \(\gamma_x\), \(\gamma_y\), and \(\gamma_z\). The amplitude matrix \(M\) is not symmetrizable by Exercise 2.1 of [Kac]. Pertaining to the pair \((\gamma_y, \gamma_z)\), we have \(4 \cos^2(\pi/5) = \frac{3+\sqrt{5}}{2}\) and \(2 \cos(\pi/5) = \frac{1+\sqrt{5}}{2}\).

Since \(M_{yz} = -\frac{1+\sqrt{5}}{4}\) and \(M_{zy} = -(1+\sqrt{5})\), then \(K_{yz} = \frac{-M_{yz}}{2 \cos(\pi/5)} = \frac{1}{2}\) and \(K_{zy} = \frac{-M_{zy}}{2 \cos(\pi/5)} = 2\). For all other odd adjacencies \((\gamma_p, \gamma_q)\) in this graph, \(m_{pq} = 3\), so \(K_{pq} = -M_{pq}\) and \(K_{qp} = -M_{qp}\).

Use Proposition 4.3 to see that \(f_{\Gamma_1,M_1} = 2\) and \(f_{\Gamma_2,M_2} = 3\). For example, \(\{K\alpha_i | K \in \mathbb{R}\} \cap \Phi^+_M = \{\alpha_i, \frac{1}{5}\alpha_i\} = N_M(s_i)\) and \(\{K\alpha_x | K \in \mathbb{R}\} \cap \Phi^+_M = \{\alpha_x, \frac{1}{7}\alpha_x, \frac{2}{7}\alpha_x\} = N_M(s_x)\). By Proposition 4.9, we can see that
\[
f_{\Gamma_1,M_1} \ell(s_x s_i) = 4 \leq |N_M(s_x s_i)| \leq 6 = f_{\Gamma_2,M_2} \ell(s_x s_i).
\]

More precisely, from Lemma 4.4, we get
\[
N_M(s_x s_i) = s_i(N_M(s_x)) \cup \{K\alpha_i | K \in \mathbb{R}\} \cap \Phi^+_M,
\]
from which we see that \(|N_M(s_x s_i)| = 5\). \(\square\)

Figure 4.1: A unital OA-cyclic E-GCM graph for Example 4.10.

(In this figure, when the amplitude product on an edge is unity, we place \(\oplus\) beside the edge for emphasis.)

We have the natural pairing \(\langle \lambda, v \rangle := \lambda(v)\) for elements \(\lambda\) in the dual space \(V^*\) and vectors \(v\) in \(V\). We think of \(V^*\) as the space of positions for numbers games played on \((\Gamma, M)\): For \(\lambda \in V^*\), the populations for the corresponding position are \((\lambda_i)_{i \in I_n}\) where for each \(i \in I_n\) we have \(\lambda_i := \langle \lambda, \alpha_i \rangle\). Regard the fundamental positions \((\omega_i)_{i \in I_n}\) to be the basis for \(V^*\) dual to the basis \((\alpha_j)_{j \in I_n}\) for \(V\).
relative to the natural pairing \( \langle \cdot, \cdot \rangle \), so \( \langle \omega_i, \alpha_j \rangle = \delta_{ij} \). Given \( \sigma_M : W \to GL(V) \) the contragredient representation \( \sigma^*_M : W \to GL(V^*) \) is determined by \( \langle \sigma^*_M(w)(\lambda), v \rangle = \langle \lambda, \sigma_M(w^{-1})(v) \rangle \). From here on, when \( w \in W \) and \( \lambda \in V^* \), write \( w.\lambda \) for \( \sigma^*_M(w)(\lambda) \). Then \( s_i.\lambda \) is the result of firing node \( \gamma_i \) when the E-GCM graph is assigned position \( \lambda \), whether the firing is legal or not. Set \( P_i := \{ \lambda \in V^* \mid \langle \lambda, \alpha_i \rangle > 0 \} \), the set of positions with positive population at node \( \gamma_i \). Similarly define \( P'_i := \{ \lambda \in V^* \mid \langle \lambda, \alpha_i \rangle < 0 \} \) and \( Z_i := \{ \lambda \in V^* \mid \langle \lambda, \alpha_i \rangle = 0 \} \). Let \( C := \cap_{i \in I_n} P_i \), the set of strongly dominant positions, and let \( D := \overrightarrow{C} = \cap_{i \in I_n} (P_i \cup Z_i) \), the set of dominant positions. The Tits cone is \( U_M := \cup_{w \in W} wD \). In view of Proposition 4.3, the results of \cite{Hum2} §5.13 hold here. And in view of Propositions 4.5 and 4.9, we can use the proof of Proposition 3.2 of \cite{HRT} verbatim to get the following generalization of their result; their proof requires that all \( N_M(w) \) be finite, hence our hypothesis that \( (\Gamma, M) \) is unital OA-cyclic.

**Proposition 4.11** Suppose \((\Gamma, M)\) is connected and unital OA-cyclic. If the Coxeter group \( W = W(\Gamma, M) \) is infinite, then \( U_M \cap (-U_M) = \{0\} \).

Lemma 5.13 of \cite{Hum2} is the basis for the argument in §4 of \cite{Erik2} characterizing the set of initial positions for which the game converges. In contrast to \cite{Erik2}, here we fire at nodes with positive rather than negative populations, so we have \(-U_M\) instead of \(U_M\) in the following proposition.

**Proposition 4.12** (Eriksson) The set of initial positions for which the numbers game on the E-GCM graph \((\Gamma, M)\) converges is precisely \(-U_M\).

Although the next proposition is not used in any subsequent proofs, it is closely related to the results of this section (cf. Exercise 5.13 of \cite{Hum2}).

**Proposition 4.13** Suppose \( W = W(\Gamma, M) \) is finite. Then \( U_M = -U_M = V^* \).

Proof. Since \( W \) is finite, then by Eriksson’s Reduced Word Result for E-GCM graphs (Proposition 4.1 of \cite{Erik2}) it follows that the set of initial positions for which the numbers game on \((\Gamma, M)\) converges is all of \(V^*\). Proposition 4.12 now implies that \(-U_M = V^*\), which is therefore \(U_M\).

To understand admissibility for E-GCM graphs that are not unital OA-cyclic, we will revisit parts of the proof of Theorem 1.1 from Section 2. We note that Eriksson’s Strong Convergence Theorem (Theorem 3.1 of \cite{Erik3}, stated here for GCM graphs as Theorem 2.1), Eriksson’s Comparison Theorem (Theorem 4.5 of \cite{Erik2}, stated here for GCM graphs as Theorem 2.3), and Eriksson’s Reduced Word Result (Proposition 4.1 of \cite{Erik2}, stated above for GCM graphs as Proposition 3.1) all hold for E-GCM graphs. Moreover, Lemmas 2.2, 2.4, 2.5, 2.6, 2.8, and 2.9 all hold for E-GCM graphs since their proofs nowhere depend on the assumption of integral amplitude products. We say the \( n \)-node graph \( \Gamma \) is a loop if the nodes can be numbered \( \gamma_1, \ldots, \gamma_n \) in such a way that for all \( 1 \leq i \leq n \), \( \gamma_i \) is adjacent precisely to \( \gamma_{i+1} \) and \( \gamma_{i-1} \), understanding that \( \gamma_0 = \gamma_n \) and \( \gamma_1 = \gamma_{n+1} \).

**Lemma 4.14** Suppose that the underlying graph \( \Gamma \) of the E-GCM graph \((\Gamma, M)\) is a loop and that for any pair of adjacent nodes the amplitude product is one. Then \((\Gamma, M)\) is not admissible.

Proof. We find a divergent game sequence starting from the fundamental position \( \omega_1 \). Then by renumbering the nodes, we see that every fundamental position will have a divergent game sequence, and by Lemma 2.6 it then follows that \((\Gamma, M)\) is not admissible. Let the OA-cycle \( \mathcal{C} \) be \([\gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_1] \). From initial position \( \omega_1 \) we propose starting with the firing sequence
(γ₁, ..., γₙ₋₁, γₙ, γₙ₋₁, ..., γ₂). One can check that all of these node firings are legal and that the resulting populations are zero at all nodes other than γ₁, γ₂, and γₙ. The populations at the latter nodes are, respectively, 1 + Π_c + Π_c⁻¹, M₁₂(Π_c⁻¹), and M₁ₙ(Π_c). By repeating the proposed firing sequence (γ₁, ..., γₙ₋₁, γₙ, γₙ₋₁, ..., γ₂) from this position we obtain zero populations everywhere except at γ₁, γ₂, and γₙ, which are now 1 + Π_c + Π_c⁻¹ + Π_c² + Π_c⁻², M₁₂(Π_c⁻¹ + Π_c⁻²), and M₁ₙ(Π_c + Π_c²) respectively. After k applications of the proposed firing sequence we have populations 1 + Σ_j=1 Π_c² + Π_c⁻¹, M₁₂(Σ_j=1 Π_c⁻¹), and M₁ₙ(Σ_j=1 Π_c) at nodes γ₁, γ₂, and γₙ, and zero populations elsewhere. Thus we have exhibited a divergent game sequence.

The proof of the next lemma can be adjusted to account for loops on three nodes whose E-GCM graphs have even adjacencies.

**Lemma 4.15** Suppose (Γ, M) is the following three-node E-GCM graph:

```
   p1
   /    \
 q1   q2
   |    |
   p   q
   |    |
  p2 q2
```

Assume that all adjacencies are odd. Then (Γ, M) is not admissible.

**Proof.** Below Lemma 2.7 in Section 2 we showed that a certain three-node GCM graph is not admissible. The proof here is tedious but follows the pattern of that argument. With amplitudes as depicted in the lemma statement, assign populations a, b, and c as follows:

```
   Call this position λ = (a, b, c), so population a is at node γ₁, population b is at node γ₂, and population c is at node γ₃. Without loss of generality, assume that pq ≤ p₁q₁ and that pq ≥ p₂q₂. Set
```

\[
κ_1 := \frac{pq_2 + p_1\sqrt{pq}}{\sqrt{pq}(2 - \sqrt{pq})} \quad \text{and} \quad κ_2 := \frac{qp_1 + p_2\sqrt{pq}}{\sqrt{pq}(2 - \sqrt{pq})}.
\]

Assume that a ≥ 0, b ≥ 0, c ≤ 0, and that \((κ₁ - \frac{p}{q\sqrt{pq}})a + (κ₂ - \frac{q}{q_1\sqrt{pq}})b + c > 0\); these hypotheses will be referred to as condition (∗). Notice that a and b cannot both be zero under condition (∗). A justification of the following claim will be given at the end of the proof:

**Claim:** Under condition (∗) there is a sequence of legal node firings from initial position λ = (a, b, c) which results in the position λ' = (a', b', c') = (\(\frac{-a}{\sqrt{pq}}, \frac{-b}{\sqrt{pq}}, a, κ₁a + κ₂b + c\)).

In this case, observe that a' ≤ 0, b' ≤ 0, and c' > 0. Now fire at node γ₃ to obtain the position λ⁽¹⁾ = (a₁, b₁, c₁) with a₁ = q₁[κ₁a + (κ₂ - \(\frac{q}{q₁\sqrt{pq}}\))b + c], b₁ = q₂[(κ₁ - \(\frac{p}{q₂\sqrt{pq}}\))a + κ₂b + c], and c₁ = -(κ₁a + κ₂b + c). Now condition (∗) implies that a₁ > 0, b₁ > 0, and c₁ < 0. At this point to see that λ⁽¹⁾ = (a₁, b₁, c₁) itself meets condition (∗), we only need to show that \((κ₁ - \frac{p}{q₂\sqrt{pq}})a₁ + (κ₂ - \frac{q}{q₁\sqrt{pq}})b₁ + c₁ > 0\). As a first step, we argue that (i) \(q₁(κ₁ - \frac{p}{q₂\sqrt{pq}}) ≥ 1\) and that (ii) \(q₂(κ₂ - \frac{q}{q₁\sqrt{pq}}) ≥ 1\). We only show (i) since (ii) follows by similar reasoning. (From the inequalities (i) and (ii), a third inequality (iii) follows immediately: \(q₁(κ₁ - \frac{p}{q₂\sqrt{pq}}) + q₂(κ₂ - \frac{q}{q₁\sqrt{pq}}) - 1 > 0\).)

For the first of the inequalities (i), note that since 1 ≤ pq₁, then \(2 - \sqrt{pq} ≤ pq₁\). Since pq ≤ p₂q₂, then \(2 - \sqrt{pq} ≤ p₂q₂\). (Similarly \(2 - \sqrt{pq} ≤ p₁q₁\).) Thus \(\frac{pq₁q₂}{2 - \sqrt{pq}} - 1 ≥ 0\), and hence \(\frac{pq₁q₂}{2 - \sqrt{pq}} - \frac{1}{q₂} ≥ 0\).
Therefore, \( \frac{q_1 p_2}{\sqrt{p q (2 - \sqrt{pq})}} - \frac{q_1 p}{\sqrt{q_1^2 q_2}} \geq 0 \). Since \( \frac{p_1 q_1 \sqrt{pq}}{\sqrt{pq (2 - \sqrt{pq})}} \geq 1 \), then \( \frac{q_1 p_2}{\sqrt{p q (2 - \sqrt{pq})}} + \frac{p_1 q_1 \sqrt{pq}}{\sqrt{pq (2 - \sqrt{pq})}} - \frac{q_1 p}{\sqrt{q_1^2 q_2}} \geq 1 \). From this we get \( q_1 (\kappa_1 - \frac{p}{q_2 \sqrt{pq}}) \geq 1 \), which is \((i)\). The following identity is easy to verify:

\[
\left( \kappa_1 - \frac{p}{q_2 \sqrt{pq}} \right) a_1 + \left( \kappa_2 - \frac{q}{q_1 \sqrt{pq}} \right) b_1 + c_1
\]

\[
= \left( \kappa_1 - \frac{p}{q_2 \sqrt{pq}} \right) q_1 \left( \kappa_1 - \frac{p}{q_2 \sqrt{pq}} \right) a_1 + \left( \kappa_2 - \frac{q}{q_1 \sqrt{pq}} \right) q_2 \left( \kappa_2 - \frac{q}{q_1 \sqrt{pq}} \right) b_1
\]

\[
+ \left[ q_1 \left( \kappa_1 - \frac{p}{q_2 \sqrt{pq}} \right) + q_2 \left( \kappa_2 - \frac{q}{q_1 \sqrt{pq}} \right) \right] a_1 + q_1 \left( \kappa_1 - \frac{p}{q_2 \sqrt{pq}} \right) b_1 + c_1 > 0,
\]

as desired. This means that position \( \lambda^{(1)} = (a_1, b_1, c_1) \) meets condition (\(*)\) and none of its populations are zero. In view of our Claim, we may apply to position \( \lambda^{(1)} \) a legal sequence of node firings followed by firing node \( \gamma_3 \) as before to obtain a position \( \lambda^{(2)} = (a_2, b_2, c_2) \) that meets condition (\*) with none of its populations zero, etc. So from any such \( \lambda = (a, b, c) \) we have a divergent game sequence. In view of inequalities \((i)\) and \((ii)\), the fundamental positions \( \omega_1 = (1, 0, 0) \) and \( \omega_2 = (0, 1, 0) \) meet condition (\*). The fundamental position \( \omega_3 = (0, 0, 1) \) does not meet condition (\*); however, by firing at node \( \gamma_3 \) we obtain the position \( (q_1, q_2, -1) \), which meets condition (\*) by inequality (\(iii))\). Thus from any fundamental position there is a divergent game sequence, and so by Lemma 2.6 the three-node E-GCM graph we started with is not admissible.

It still remains to justify our Claim. Beginning with position \( \lambda = (a, b, c) \) under condition (\*), we propose to fire at nodes \( \gamma_1 \) and \( \gamma_2 \) in alternating order until this is no longer possible. We assert that the resulting population will be \( \lambda' = (a', b', c') = \left( \frac{q}{\sqrt{pq}} b, \frac{p}{\sqrt{pq}} a, \kappa_1 a + \kappa_2 b + c \right) \). There are three cases to consider: (I), \( a \) and \( b \) are both positive, (II), \( a > 0 \) and \( b = 0 \), and (III), \( a = 0 \) and \( b > 0 \). For (I), we wish to show that \( (\gamma_1, \gamma_2, \ldots, \gamma_1) \) of length \( m_{12} \) is a sequence of legal node firings. That is, we must check that

\[
(1) \quad \langle (s_2 s_1)^k \cdot \lambda, \alpha_1 \rangle = \langle \lambda, (s_1 s_2)^k \cdot \alpha_1 \rangle > 0 \quad \text{for} \quad 0 \leq k \leq (m_{12} - 1)/2,
\]

\[
(2) \quad \langle s_1 (s_2 s_1)^k \cdot \lambda, \alpha_2 \rangle = \langle \lambda, s_1 (s_2 s_1)^k \cdot \alpha_2 \rangle > 0 \quad \text{for} \quad 0 \leq k < (m_{12} - 1)/2.
\]

For (II), we wish to show that \( (\gamma_1, \gamma_2, \ldots, \gamma_1) \) of length \( m_{12} - 1 \) is a sequence of legal node firings. That is, we must check that

\[
(3) \quad \langle (s_2 s_1)^k \cdot \lambda, \alpha_1 \rangle = \langle \lambda, (s_1 s_2)^k \cdot \alpha_1 \rangle > 0 \quad \text{for} \quad 0 \leq k < (m_{12} - 1)/2,
\]

\[
(4) \quad \langle s_1 (s_2 s_1)^k \cdot \lambda, \alpha_2 \rangle = \langle \lambda, s_1 (s_2 s_1)^k \cdot \alpha_2 \rangle > 0 \quad \text{for} \quad 0 \leq k < (m_{12} - 1)/2.
\]
For (III), we wish to show that \((\gamma_2, \gamma_1, \ldots, \gamma_2, \gamma_1)\) of length \(m_{12} - 1\) is a sequence of legal node firings. That is, we must check that

\[
\langle (s_1 s_2)^k \lambda, \alpha_2 \rangle = \langle \lambda, (s_2 s_1)^k \alpha_2 \rangle > 0 \quad \text{for} \quad 0 \leq k < (m_{12} - 1)/2,
\]

and

\[
\langle s_2 (s_1 s_2)^k \lambda, \alpha_1 \rangle = \langle \lambda, s_2 (s_1 s_2)^k \alpha_1 \rangle > 0 \quad \text{for} \quad 0 \leq k < (m_{12} - 1)/2.
\]

Our justification of (1) through (6) has similarities to the proof of Lemma 4.2. Under the representation \(\sigma_M\) we have \(S_i = \sigma_M(s_i)\) for \(i = 1, 2, 3\). With respect to the ordered basis \(\mathfrak{B} = (\alpha_1, \alpha_2, \alpha_3)\) for \(V\) we have \(X_1 := [S_1]_{\mathfrak{B}} = \begin{pmatrix} -1 & p & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\) and \(X_2 := [S_2]_{\mathfrak{B}} = \begin{pmatrix} 1 & 0 & 0 \\ q & -1 & p_2 \\ 0 & 0 & 1 \end{pmatrix}\), and so

\[
X_{1,2} := [S_1 S_2]_{\mathfrak{B}} = X_1 X_2 = \begin{pmatrix} pq - 1 & -p & p_{2p} + p_1 \\ q & -1 & p_2 \\ 0 & 0 & 1 \end{pmatrix}
\]

and

\[
X_{2,1} := [S_2 S_1]_{\mathfrak{B}} = X_2 X_1 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & p & p_1 \\ -q & pq - 1 & p_1 q + p_2 \\ 0 & 0 & 1 \end{pmatrix}.
\]

For (1) through (6) above, we need to understand \(X_{1,2}^k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, X_{2,1}^k \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, X_{2,1}^k \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\), and

\(X_1 X_{2,1}^k \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\). Set \(\theta := \pi/m_{12}\). Then we can write \(X_{1,2} = PDP^{-1}\) for nonsingular \(P\) and diagonal matrix \(D\) as in

\[
\frac{1}{q(e^{2i\theta} - e^{-2i\theta})} \begin{pmatrix} e^{2i\theta} + 1 & e^{-2i\theta} + 1 & p_2 p + 2 p_1 \\ q & q & p_1 q + 2 p_2 \\ 0 & 0 & 4 - pq \end{pmatrix} \begin{pmatrix} e^{2i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q & -e^{-2i\theta} - 1 & C_1 \\ -q & e^{2i\theta} + 1 & C_2 \\ 0 & 0 & C_3 \end{pmatrix}
\]

where \(C_1 = [-q(p_2 p + 2 p_1) + (e^{-2i\theta} + 1)(p_1 q + 2 p_2)]/(4 - pq)\), \(C_2 = [q(p_2 p + 2 p_1) - (e^{2i\theta} + 1)(p_1 q + 2 p_2)]/(4 - pq)\), and \(C_3 = q(e^{2i\theta} - e^{-2i\theta} )/(4 - pq)\). With some work we can calculate \(X_{1,2}^k\), which results in

\[
X_{1,2}^k = \begin{pmatrix} \frac{\sin(2(k+1)\theta) + \sin(2k\theta)}{\sin(2\theta)} & -p \frac{\sin(2k\theta)}{\sin(2\theta)} & C_1' \\ \frac{\sin(2k\theta)}{\sin(2\theta)} & -\frac{\sin(2(k-1)\theta)}{\sin(2\theta)} & C_2' \\ 0 & 1 & \end{pmatrix}
\]

with

\[
C_1' = -\frac{p_2 p + 2 p_1}{4 - pq} \left[ \frac{\sin(2(k+1)\theta) + \sin(2k\theta)}{\sin(2\theta)} - 1 \right] + \frac{p(p_1 q + 2 p_2) \sin(2k\theta)}{(4 - pq) \sin(2\theta)}
\]

and

\[
C_2' = -\frac{q(p_2 p + 2 p_1) \sin(2k\theta)}{(4 - pq) \sin(2\theta)} + \frac{p_1 q + 2 p_2}{4 - pq} \left[ \frac{\sin(2k\theta) + \sin(2(k-1)\theta)}{\sin(2\theta)} + 1 \right].
\]
Similar reasoning (or simply interchanging the roles of \( \alpha_1 \) and \( \alpha_2 \) in the preceding calculations, or noting that \( X_{2,1}^k = (X_{1,2}^{-1})^k = X_{1,2}^{-k} \)) shows that

\[
X_{2,1}^k = \begin{pmatrix}
-\sin(2k\theta) - \sin(2(k-1)\theta) & \frac{p\sin(2k\theta)}{\sin(2\theta)} & C_1^{''} \\
\frac{q\sin(2k\theta)}{\sin(2\theta)} & \frac{\sin(2(k+1)\theta) + \sin(2k\theta)}{\sin(2\theta)} & C_2^{''} \\
0 & 0 & 1
\end{pmatrix},
\]

with

\[
C_1^{''} = \frac{p^2p + 2p_1}{4 - pq} \left[ \frac{\sin(2k\theta) + \sin(2(k-1)\theta)}{\sin(2\theta)} + 1 \right] - \frac{p(p_1q + 2p_2)\sin(2k\theta)}{(4 - pq)\sin(2\theta)}
\]

and

\[
C_2^{''} = \frac{q(p_2p + 2p_1)\sin(2k\theta)}{(4 - pq)\sin(2\theta)} - \frac{p_1q + 2p_2}{4 - pq} \left[ \frac{\sin(2(k+1)\theta) + \sin(2k\theta)}{\sin(2\theta)} - 1 \right].
\]

Then

\[
X_2X_{1,2}^k = \begin{pmatrix}
\frac{\sin(2(k+1)\theta) + \sin(2k\theta)}{\sin(2\theta)} & -\frac{p\sin(2k\theta)}{\sin(2\theta)} & C_1' \\
\frac{q\sin(2k\theta)}{\sin(2\theta)} & \frac{\sin(2(k+1)\theta) + \sin(2k\theta)}{\sin(2\theta)} & qC_1' - C_2' + p_2 \\
0 & 0 & 1
\end{pmatrix},
\]

and

\[
X_1X_{2,1}^k = \begin{pmatrix}
\frac{\sin(2(k+1)\theta) + \sin(2k\theta)}{\sin(2\theta)} & \frac{p\sin(2(k+1)\theta)}{\sin(2\theta)} & -C_1' + pC_2' + p_1 \\
\frac{q\sin(2k\theta)}{\sin(2\theta)} & \frac{\sin(2(k+1)\theta) + \sin(2k\theta)}{\sin(2\theta)} & C_2'
\end{pmatrix}.
\]

Now we can justify (1) through (6). For example, for (4) we see that since \( X_1X_{2,1}^k \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) \) is the second column of the matrix \( X_1X_{2,1}^k \), then \( \langle \lambda, s_1(s_2s_1)^k.\alpha_2 \rangle = a\frac{p\sin(2(k+1)\theta)}{\sin(2\theta)} \), which is positive since \( a > 0 \), \( p > 0 \), and (recalling that \( m_{12} \) is odd) \( 2(k+1) < m_{12} \).

Then the proposed firing sequence for each of cases (I), (II), and (III) is legal. To see in case (I) that the resulting position is the claimed \( \lambda' = (a', b', c') = \left( -\frac{q}{\sqrt{pq}}, \frac{p}{\sqrt{pq}}, a, \kappa_1a + \kappa_2b + c \right) \), we need to calculate \( s_i(s_2s_1)^k.\lambda.\alpha_i = \langle \lambda, s_i(s_2s_1)^k.\alpha_i \rangle \) for each of \( i = 1, 2, 3 \), where \( k \) is now \( (m_{12} - 1)/2 \).

With patience one can confirm that

\[
X_1X_{2,1}^k = \begin{pmatrix}
0 & -p/\sqrt{pq} & \kappa_1 \\
-q/\sqrt{pq} & 0 & \kappa_2 \\
0 & 0 & 1
\end{pmatrix},
\]

from which the claim follows. Similar computations confirm the claim for cases (II) and (III). \( \square \)

We have one more loop to rule out; since the details are by now routine, we omit the proof.

**Lemma 4.16** An E-GCM graph in the family \( \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ
\end{array} \) is not admissible. \( \square \)

We can now prove Theorem 1.3.

**Proof of Theorem 1.3.** First we use induction on \( n \), the number of nodes, to show that any connected admissible E-GCM graph \( (\Gamma, M) \) must be from one of the families of Figure 1.3. Clearly a one-node E-GCM graph is admissible. For some \( n \geq 2 \), suppose the result is true for all connected
admissible E-GCM graphs with fewer than \( n \) nodes. Let \((\Gamma, M)\) be a connected, admissible, \( n \)-node E-GCM graph. Suppose \((\Gamma, M)\) is unital OA-cyclic. Then by Propositions 4.11 and 4.12, we must have \( W(\Gamma, M) \) finite. Then by the classification of finite irreducible Coxeter groups, \((\Gamma, M)\) must be in one of the families of graphs in Figure 1.3. Now suppose \((\Gamma, M)\) is not unital OA-cyclic. First we show that any cycle (OA or otherwise) in \((\Gamma, M)\) must use all \( n \) nodes. Indeed, the (connected) E-GCM subgraph \((\Gamma', M')\) whose nodes are the nodes of a cycle must be admissible by Lemma 2.9. If \((\Gamma', M')\) has fewer than \( n \) nodes, then the induction hypothesis applies; but E-Coxeter graphs have no cycles (OA or otherwise), so \((\Gamma', M')\) must be all of \((\Gamma, M)\). Second, \((\Gamma, M)\) has an OA-cycle \( C \) for which \( \Pi_C \neq 1 \). We can make the following choice for \( C \): Choose \( C \) to be a simple OA-cycle with \( \Pi_C \neq 1 \) whose length is as small as possible. This smallest length must therefore be \( n \). We wish to show that the underlying graph \( \Gamma \) is a loop. Let the numbering of the nodes of \( \Gamma \) follow \( C \), so \( C = [\gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_1] \). If \( \Gamma \) is not a loop, then there are adjacencies amongst the \( \gamma_i \)'s besides those of consecutive elements of \( C \). But this in turn means that \((\Gamma, M)\) has a cycle that uses fewer than \( n \) nodes. So \( \Gamma \) is a loop. Of course we must have \( n \geq 3 \). Lemma 4.15 rules out the possibility that \( n = 3 \). Any E-GCM subgraph \((\Gamma', M')\) obtained from \((\Gamma, M)\) by removing a single node must now be a “branchless” E-Coxeter graph from Figure 1.3 whose adjacencies are all odd.

So if \( n = 4 \), \((\Gamma, M)\) must be in one of the families 

\[
\begin{align*}
\bullet & \quad \bullet & \quad \bullet \\
\bullet & \quad \bullet & \quad \bullet \\
\bullet & \quad \bullet & \quad \bullet
\end{align*}
\]

or 

\[
\begin{align*}
\bullet & \quad \bullet & \quad \bullet \\
\bullet & \quad \bullet & \quad \bullet \\
\bullet & \quad \bullet & \quad \bullet
\end{align*}
\]

, which are ruled out by Lemmas 4.14 and 4.16 respectively. If \( n \geq 5 \), the only possibility is that \((\Gamma, M)\) meets the hypotheses of Lemma 4.14 and therefore is not admissible. In all cases, we see that if \((\Gamma, M)\) is not unital OA-cyclic, then it is not admissible. This completes the induction step, so we have shown that a connected admissible E-GCM graph must be in one of the families of Figure 1.3.

On the other hand, if \((\Gamma, M)\) is from Figure 1.3, then the Coxeter group \( W \) is finite (again by the classification), so there is an upper bound on the length of any element in \( W \). So by Eriksson’s Reduced Word Result for E-GCM graphs, the numbers game converges for any initial position. The remaining claims of Theorem 1.3 now follow from Eriksson’s Strong Convergence Theorem for E-GCM graphs.

\[\square\]

5. Some further finiteness aspects of E-game play

The results of this section are independent of the classifications obtained in Theorems 1.1 and 1.3. Suppose the Coxeter group \( W \) associated to an E-GCM graph \((\Gamma, M)\) is finite. In this case we may choose the (unique) longest element \( w_0 \) in \( W \). Since we must have \( \ell(w_0 s_i) < \ell(w_0) \) for all \( i \in I_n \), it follows that \( w_0 \alpha_i <_M 0 \) for all \( i \). So if \( \alpha = \sum c_\alpha \alpha_i >_M 0 \), then \( w_0 \alpha <_M 0 \), i.e. \( N_M(w_0) = \Phi^+_M \).

**Lemma 5.1** Let \((\Gamma, M)\) be an E-GCM graph with associated Coxeter group \( W = W(\Gamma, M) \). Let \( J \subseteq I_n \), and suppose the parabolic subgroup \( W_J \) is finite. Suppose \( \alpha = \sum_{j \in J} c_j \alpha_j \) is a root in \( \Phi^+_M \). Then for some \( w_j \) in \( W_J \), we have \( w_j \alpha <_M 0 \).

**Proof.** Since \( W_J \) is finite, consider the longest element \( (w_0)_J \) in \( W_J \). Note that any element of \( W_J \) preserves the subspace \( V_J := \text{span}_\mathbb{R}\{\alpha_j\}_{j \in J} \). As seen just above, \((w_0)_J \) will send each simple root \( \alpha_j \) for \( j \in J \) to some root in \( \Phi^-_M \). So apply \((w_0)_J \) to the given \( \alpha \) to see that \((w_0)_J \alpha <_M 0 \). \[\square\]
In what follows, for any subset \( J \) of \( I_n \), let \( C_J \) be the set of all dominant positions \( \lambda \) for which \[
\lambda = \sum_{i \in I_n \setminus J} \lambda_i \omega_i \] with \( \lambda_i > 0 \) for all \( i \in I_n \setminus J \), that is, \( C_J := (\cap_{i \in I_n \setminus J} P_i) \cap (\cap_{j \in J} Z_j) \). The next result generalizes Proposition 4.2 of \[ \text{Erik}2 \].

**Proposition 5.2** For an E-GCM graph \( (\Gamma, M) \) with Coxeter group \( W = W(\Gamma, M) \), let \( J \) be any subset of \( I_n \) such that \( W_J \) is finite. Let \( s_{i_p} \cdots s_{i_q} s_{i_1} \) be any reduced expression for an element \( w' \) of the set \( W^J \) of minimal coset representatives. Let \( \lambda \) be in \( C_J \). Then for \( 1 \leq q \leq p, (\gamma_{i_1}, \ldots, \gamma_{i_q}) \) is a legal sequence of node firings for a numbers game played from initial position \( \lambda \).

**Proof.** For \( 1 \leq q \leq p \), we must show that \( \langle s_{i_q} \cdots s_{i_1} \rangle > 0 \). But \( \langle s_{i_q-1} \cdots s_{i_2} s_{i_1}, \lambda, \alpha_i \rangle = \langle \lambda, s_{i_1} s_{i_2} \cdots s_{i_{q-1}}, \alpha_i \rangle \) now \( \langle s_{i_1} s_{i_2} \cdots s_{i_{q-1}}, \alpha_i \rangle > M \) since \( \ell(s_{i_1} s_{i_2} \cdots s_{i_{q-1}}, s_{i_q}) > \ell(s_{i_1} s_{i_2} \cdots s_{i_{q-1}}) \) (this is because \( \ell(s_{i_1} s_{i_2} \cdots s_{i_{q-1}}, s_{i_q}) > \ell(s_{i_1} \cdots s_{i_2} s_{i_1}) \) for these two reduced expressions). Moreover, suppose \( w_{j-1} = s_{j_1} \cdots s_{j_r} \) is a reduced expression for some \( w_j^{-1} \in W_J \). Then since \( w_{j-1} = s_{i_p} \cdots s_{i_2} s_{i_1} s_{j_1} \cdots s_{j_r} \) is reduced (cf. Proposition 2.4.4 of \[ \text{Hj}2 \]), it follows that both expressions \( s_{j_p} \cdots s_{j_1} s_{i_1} \cdots s_{i_{q-2}} s_{i_{q-1}}, s_{j_p} \cdots s_{j_1} s_{i_1} \cdots s_{i_{q-2}} \) and \( s_{j_p} \cdots s_{j_1} s_{i_1} \cdots s_{i_{q-2}} s_{i_{q-1}} \) are reduced, with the latter longer than the former. In particular \( s_{j_p} \cdots s_{j_1} s_{i_1} \cdots s_{i_{q-2}} s_{i_{q-1}} \alpha_i = w_{j} s_{i_1} \cdots s_{i_{q-2}} s_{i_{q-1}} \alpha_i > M \) for all \( w_j \in W_J \).

We wish to show that \( s_{i_1} s_{i_2} \cdots s_{i_{q-1}} \alpha_i \) cannot be contained in span\(_{\mathbb{R}}(\alpha_j)_{j \in J} \). Suppose otherwise, so \( s_{i_1} s_{i_2} \cdots s_{i_{q-1}} \alpha_i = \sum_{j \in J} c_j \alpha_j \). We now invoke the finiteness assumption for \( W_J \) and Lemma 5.1 as follows: Let \( (w_0)_{j} \) be the longest element of \( W_J \). Then \( (w_0)_{j} s_{i_1} s_{i_2} \cdots s_{i_{q-1}} \alpha_i < M \) for some \( k \in I_n \setminus J \). So \( s_{i_1} s_{i_2} \cdots s_{i_{q-1}} \alpha_i = \sum_{i \in I_n} c_i \alpha_i = \sum_{i \in I_n} c_i \alpha_i \), which is positive since all \( c_i \)'s are nonnegative, \( \lambda_k > 0 \), and \( c_k > 0 \). Then firing at node \( \gamma_{i_{q-1}} \) from game position \( s_{i_{q-1}} \cdots s_{i_2} s_{i_1}, \lambda \) is legal.

From the previous proof we see that Proposition 5.2 could be generalized to other kinds of subsets \( J \subseteq I_n \) by extending Lemma 5.1. Doing so for all subsets \( J \subseteq I_n \) would yield a simple proof of Theorem 1.3: An E-GCM graph \( (\Gamma, M) \) would have a convergent game sequence for some \( \lambda \in C_J \) if and only if \( W^J \) is finite if and only if \( W \) if finite (by Proposition 4.2 of \[ \text{Dec} \]). For an arbitrary E-GCM graph \( (\Gamma, M) \), let \( \Psi(\lambda) \) denote the set of all positions obtainable from legal firing sequences in numbers games with initial position \( \lambda \). Clearly \( \Psi(\lambda) \subseteq W \lambda \), where the latter is the orbit of \( \lambda \) under the \( W \)-action on \( V^* \). Since the statement of Theorem 5.13 of \[ \text{Hum2} \] holds for quasi-standard geometric representations, then \( W_J \) is the full stabilizer of any \( \lambda \in C_J \), so \( W \lambda \) and \( W^J \) can be identified. So from Proposition 5.2 we see that for \( \lambda \in C_J \) with \( W_J \) finite, then \( \Psi(\lambda) = W \lambda \).

**Proposition 5.3** Suppose \( W = W(\Gamma, M) \) is finite for some E-GCM graph \( (\Gamma, M) \). Let \( J \subseteq I_n \) be any subset, and let \( (w_0)_{j} \) be the longest word in the parabolic subgroup \( W_J \). Then all game sequences for any \( \lambda \in C_J \) have length \( \ell(w_0) - \ell((w_0)_{j}) \).

**Proof.** Write \( w_0 = (w_0)^J (w_0)_{j} \), with \( (w_0)^J \) the minimal coset representative for \( w^0 W_J \). Proposition 5.2 implies that there is a game sequence for \( \lambda \) with length \( \ell((w_0)^J) = \ell(w_0) - \ell((w_0)_{j}) \). By Erikssoon’s Strong Convergence Theorem, this must be the length of any game sequence for \( \lambda \).

Our next two results expand on Remark 4.6 of \[ \text{Erik}2 \]. Let \( \lambda \) be a strongly dominant position thought of as an initial position for a numbers game on an E-GCM graph. Consider a legal sequence of node firings \( (\gamma_{i_1}, \ldots, \gamma_{i_p}) \). Let \( \alpha = \sum_{i \in I_n} c_i \alpha_i \) be the root \( s_{i_1} \cdots s_{i_p}, \alpha_k \). Then the population at node \( \gamma_k \) can be computed as \( \langle s_{i_p} \cdots s_{i_1}, \lambda, \alpha_k \rangle = \langle \lambda, s_{i_1} \cdots s_{i_p}, \alpha_k \rangle = \langle \lambda, \alpha \rangle = \sum_{i \in I_n} c_i \lambda_i \). For indeterminates \( x_1, \ldots, x_n \), let \( \phi_\alpha := \phi_\alpha(x_1, \ldots, x_n) := \sum_{i \in I_n} c_i x_i \), and call \( \phi_\alpha \) the root functional.
for \( \alpha \); a root functional is positive or negative depending on whether \( \alpha \) is positive or negative. By Proposition 5.2 (or Proposition 4.2 of [Erik2]), this game sequence is valid for all strongly dominant \( \lambda \), so we will say that \( \phi_\alpha \) is the root functional at node \( \gamma_k \) for the firing sequence \((\gamma_{i_1}, \ldots, \gamma_{i_p})\). If \( k = i_p \) and the firing sequence is understood, then we just say \( \phi_\alpha \) is the root functional at node \( \gamma_{i_p} \).

Note that \( \phi_\alpha = \phi_\beta \) for roots \( \alpha \) and \( \beta \) if and only if \( \alpha = \beta \). The same positive root functional can appear at different nodes during a given numbers game (e.g. play the numbers game on the GCM graph \( A_3 \) with a generic strongly dominant initial position \( \lambda = (a,b,c) \); fire nodes \( \gamma_3, \gamma_2, \) and \( \gamma_1 \) in that order to see that the root functional \( \phi_{\alpha_2+\alpha_3} \) appears twice). However, our next result shows that positive root functionals are not repeated among the fired nodes in any numbers game.

**Proposition 5.4** Let \( \lambda \) be a strongly dominant position and let \((\gamma_{i_1}, \gamma_{i_2}, \ldots)\) be any game sequence for a numbers game played on an E-GCM graph \((\Gamma, M)\) from initial position \( \lambda \). Then for all \( k \geq 1 \), the positive root functional at node \( \gamma_{i_k} \) is not the same as the positive root functional at any \( \gamma_{i_j} \) for \( 1 \leq j < k \).

**Proof.** Let \( \beta_j := \gamma_{i_1} \cdots \gamma_{i_{j-1}} \cdot \alpha_{i_j} \), which is necessarily positive. Then \( \phi_{\beta_j} \) is the root functional at node \( \gamma_{i_j} \) for the given game sequence. Suppose that for some \( k > j \), we have \( \phi_{\beta_k} = \phi_{\beta_j} \). Since \( \beta_j = \beta_k \), then one can see that \( s_{i_j} \cdots s_{i_k-1} \cdot \alpha_{i_k} = \alpha_{i_j} \), and so \( s_{i_{j+1}} \cdots s_{i_{k-1}} \cdot \alpha_{i_k} = \alpha_{i_j} <_M 0 \). But \( s_{i_{j+1}} \cdots s_{i_{k-1}} s_{i_k} \) is reduced and longer than \( s_{i_{j+1}} \cdots s_{i_{k-1}} \), which means we must have \( s_{i_{j+1}} \cdots s_{i_{k-1}} \cdot \alpha_{i_k} >_M 0 \). From this contradiction we conclude that we cannot have \( \phi_{\beta_k} = \phi_{\beta_j} \) for any \( k > j \).

This gives us a strategy for generating all of the positive roots when \( \Phi^+_M \) is finite. However, if \((\Gamma, M)\) has odd asymmetries, then not every positive root will be encountered as a positive root functional in a single game sequence, as the following result shows.

**Theorem 5.5** For an E-GCM graph \((\Gamma, M)\), suppose \( W = W(\Gamma, M) \) is finite. Let \((\gamma_{i_1}, \ldots, \gamma_{i_l})\) be any game sequence for a strongly dominant position \( \lambda \). Then the following are equivalent:

1. For each \( \alpha \in \Phi^+_M \), \( \phi_\alpha \) is the positive root functional at some node \( \gamma_{i_j} \) for the game sequence;
2. Each OA-connected component \((\Gamma', M')\) of \((\Gamma, M)\) is unital OA-cyclic with \( f_{\Gamma', M'} = 1 \);
3. \((\Gamma, M)\) has no odd asymmetries;
4. \( \ell(w_0) = |\Phi^+_M| = l \).

**Proof.** To show (1) \(\Rightarrow\) (2), choose an OA-connected component \((\Gamma', M')\). The proof of Proposition 4.8 shows that \((\Gamma', M')\) must be unital OA-cyclic, else \( W \) will be infinite. Let \( J \) be the subset of \( I_n \) corresponding to the nodes of the subgraph \( \Gamma' \). As in the proof of Proposition 5.3, write \( w = w_0 = (w_0)^j(w_0)_j = w^j w_j \), where \( w_j = (w_0)_j \) is the longest word in \( W_J \) and \( w^j = (w_0)^j \) is the minimal coset representative for \( w_0 W_J \). Set \( w_j = s_{i_k} \cdots s_{i_2} s_{i_1} \), a reduced expression. Using Lemma 4.4, we see that

\[
|N_M(ws_{i_1})| = |N_M(w)| - f_{\Gamma', M'},
\]
\[
|N_M(ws_{i_1} s_{i_2})| = |N_M(ws_{i_1})| - f_{\Gamma', M'} = |N_M(w)| - 2f_{\Gamma', M'},
\]

so that eventually \( |N_M(w)| = |N_M(w^j)| + \ell(w^j) f_{\Gamma', M'} \). Now by hypothesis each positive root functional appears once and therefore, by Proposition 5.4, exactly once. Thus any game sequence for \( \lambda \) has length \( |\Phi^+_M| \). So by Proposition 5.3 we see that \( \ell(w) = |\Phi^+_M| \). By Proposition 4.9,
\[ |N_M(w^j)| \geq \ell(w^j). \] Summarizing,
\[ \ell(w^j) + \ell(w_j) = \ell(w) = |N_M(w)| = |N_M(w^j)| + \ell(w_j) f_{\Gamma, M'} \geq \ell(w^j) + \ell(w_j) f_{\Gamma, M'}, \]
from which \( f_{\Gamma, M'} = 1 \). For (2) \( \Leftrightarrow \) (3), note that by Proposition 4.3 we get a nontrivial positive multiple of some simple root if and only if there are odd asymmetries. For (2) \( \Rightarrow \) (4), Proposition 4.9 and the fact that \( f_{\Gamma, M'} = 1 \) for each OA-connected component tell us that \( \ell(w) = |N_M(w)| = |\Phi^+_M| \), which by Proposition 5.3 is \( l \). For (4) \( \Rightarrow \) (1), see that when \( \ell(w) = |N_M(w)| = |\Phi^+_M| \), then Propositions 5.3 and 5.4 imply that each positive root functional must appear at least once in any game sequence for \( \lambda \).

At the end of our proof of Theorem 1.1, we saw that for any connected Dynkin diagram, the lengths of the convergent game sequences for any two strongly dominant initial positions are the same. In view of the previous result, this common value is the number of positive roots in the standard root system associated to the Weyl group (cf. [Hum2] §2.10, 2.11). For \( H_3 \) and \( H_4 \) the lengths of the longest words are 15 and 60 respectively. To obtain these values consult [Hum2] §2.13, or in light of Theorem 5.5 just play the numbers game on the appropriate E-GCM graphs with no odd asymmetries.

In some related work with Norman Wildberger [DW], we will take an interest in what we call here “adjacency-free positions.” For a firing sequence \( (\gamma_{i_1}, \gamma_{i_2}, \ldots) \) from a position \( \lambda \), then any position \( s_{i_1} \cdots s_{i_r} \lambda \) (including \( \lambda \) itself) is an intermediate position for the sequence. A position \( \lambda \) is adjacency-free if there exists a game sequence played from \( \lambda \) such that no intermediate position has a pair of adjacent nodes with positive populations. We will see that this notion is related to the notion of “full commutativity” of Coxeter group elements studied by Stembridge in [Stem].

In the discussion that follows, we view a Coxeter group \( W \) as \( W(\Gamma, M) \) for some E-GCM graph \( (\Gamma, M = (M_{ij})_{i,j \in I_n}) \). Following §1.1 of [Stem] and §8.1 of [Hum2], we let \( W = I_n \) be the free monoid on the set \( I_n \). Elements of \( W \) are words and will be viewed as finite sequences of elements from \( I_n \); the binary operation is concatenation, and the identity \( \varepsilon \) is the empty word. Fix a word \( s := (i_1, \ldots, i_r) \). Then \( \ell_W(s) := r \) is the length of \( s \). A subword of \( s \) is any subsequence \( (i_{p}, i_{p+1}, \ldots, i_q) \) of consecutive elements of \( s \). For a nonnegative integer \( m \) and \( x, y \in I_n \), let \( \langle x, y \rangle_m \) denote the sequence \( (x, y, x, y, \ldots) \in W \) so that \( \ell_W(\langle x, y \rangle_m) = m \). We employ several types of “elementary simplifications” in \( W \). An elementary simplification of braid type replaces a subword \( \langle x, y \rangle_{m_{xy}} \) with the subword \( \langle y, x \rangle_{m_{xy}} \) if \( 2 \leq m_{xy} < \infty \). An elementary simplification of length-reducing type replaces a subword \( \langle x, x \rangle \) with the empty subword. We let \( S(s) \) be the set of all words that can be obtained from \( s \) by some sequence of elementary simplifications of braid or length-reducing type.

Since \( s_i \) in \( W \) is its own inverse for each \( i \in I_n \), there is an induced mapping \( W \to W \). We compose this with the mapping \( W \to W \) for which \( w \mapsto w^{-1} \) to get \( \psi : W \to W \) given by \( \psi(s) = s_{i_1} \cdots s_{i_i} \). Tits’ Theorem for the word problem on Coxeter groups (cf. Theorem 8.1 of [Hum2]) implies that: For words \( s \) and \( t \) in \( W \), \( \psi(s) = \psi(t) \) if and only if \( S(s) \cap S(t) \neq \emptyset \). (This theorem is the basis for Eriksson’s Reduced Word Result.) We say \( s \) is a reduced word for \( w = \psi(s) \) if \( \ell_W(s) = \ell(w) \) (assume this is the case for the remainder of the paragraph); let \( R(w) \subseteq W \) denote the set of all reduced words for \( w \). Suppose that \( t \in R(w) \). By Tits’ Theorem, \( S(s) \cap S(t) \neq \emptyset \), so that \( t \) can be obtained from \( s \) by a sequence of elementary simplifications of braid or length-reducing type. Since \( \ell_W(s) = \ell(w) = \ell_W(t) \), then no elementary simplifications of
Lemma 5.6 For an E-GCM graph \((\Gamma, M)\), an element \(w \in W(\Gamma, M)\) is fully commutative if and only if there is a commutativity class \(\mathcal{C}\) of \(\mathcal{R}(w)\) such that for all \(x, y \in I_n\) with \(3 \leq m_{xy} < \infty\), no member of \(\mathcal{C}\) contains \((x, y)_{m_{xy}}\) as a subword. The following is a variation of this result.

**Proof.** By Proposition 1.1 of Stem the “\( \Rightarrow \)” direction is clear, so we will show the “\( \Leftarrow \)” direction. Say \(\mathcal{C} = \mathcal{C}(s)\) for some \(s \in \mathcal{R}(w)\). Now take any commutativity class \(\mathcal{C}(t)\) for \(\mathcal{R}(w)\). Since \(\psi(s) = \psi(t)\), then \(S(s)\) meets \(S(t)\) by Tit’s Theorem. Since \(\ell_{\mathcal{W}}(s) = \ell(w) = \ell_{\mathcal{W}}(t)\), then \(t\) can be obtained from \(s\) using only elementary simplifications of braid type. By hypothesis, only elementary simplifications of length-reducing or commuting type can be applied to any member of the commutativity class \(\mathcal{C}\). In particular, it must be the case that \(t \in \mathcal{C}\), so \(\mathcal{R}(w)\) has only one commutativity class. Then \(w\) is fully commutative.

The significance of the next two results is discussed in the first two paragraphs of Section 6.

**Proposition 5.7** For an E-GCM graph \((\Gamma, M)\), suppose the Coxeter group \(W = W(\Gamma, M)\) is finite. Let \(J \subseteq I_n\). (1) Suppose an adjacency-free position \(\lambda\) is in \(C_J\). Then every element \(w^J\) of \(W^J\) is fully commutative, and for any reduced expression \(w^J = s_{i_k} \cdots s_{i_1}\), no intermediate position for the firing sequence \((\gamma_{i_1}, \ldots, \gamma_{i_k})\) from \(\lambda\) has positive populations on adjacent nodes. (2) Suppose each element of \(W^J\) is fully commutative. Then any position in \(C_J\) is adjacency-free.

**Proof.** Write \(w_0 = (w_0)^J (w_0)_J\), with \((w_0)^J \in W^J\) and \((w_0)_J\) longest in \(W_J\). Let \(L := \ell(w_0) - \ell((w_0)_J)\). Our proof of (1) is by induction on the lengths of elements in \(W^J\). It is clear that the identity element is fully commutative. Now suppose that for all \(v^J\) in \(W^J\) with \(\ell(v^J) < k\), it is the case that \(v^J\) is fully commutative and that for any reduced expression \(v^J = s_{i_p} \cdots s_{i_1}\), no intermediate position for the firing sequence \((\gamma_{i_1}, \ldots, \gamma_{i_p})\) starting at \(\lambda\) has positive populations on adjacent nodes. Now consider \(w^J\) in \(W^J\) such that \(\ell(w^J) = k\). Suppose that for some adjacent \(\gamma_x\) and \(\gamma_y\) in \(\Gamma\) with \(3 \leq m_{xy} < \infty\), we have \((x, y)_{m_{xy}}\) as a subword of some reduced word \(s = (i_1, \ldots, i_k) \in \mathcal{R}(w^J)\). Since \((i_1, \ldots, i_{k-1})\) is a reduced word and \(s_{i_{k-1}} \cdots s_{i_1}\) is in \(W^J\), then \((x, y)_{m_{xy}}\) cannot be a subword of \((i_1, \ldots, i_{k-1})\). Therefore it must be the case that \(s = (i_1, \ldots, i_p, (x, y)_{m_{xy}})\) for \(p = k - m_{xy}\). But by Proposition 5.2, the corresponding firing sequence is legal, so in particular after the first \(p\) firings of the sequence there must be positive populations on adjacent nodes \(\gamma_x\) and \(\gamma_y\). This contradicts the induction hypothesis for element \(v^J = s_{i_p} \cdots s_{i_1}\). So \(w^J\) is fully commutative. Now suppose that the firing sequence \((\gamma_{j_1}, \ldots, \gamma_{j_k})\) corresponding to some reduced word \(t = (j_1, \ldots, j_k)\) in \(\mathcal{R}(w^J)\) results in positive populations at two adjacent nodes, say \(\gamma_x\) and \(\gamma_y\), for which \(3 \leq m_{xy} < \infty\). This means that for some \(j_{k+m_{xy}+1}, \ldots, j_L\) the word \(s := (j_1, \ldots, j_k, (x, y)_{m_{xy}}, j_{k+m_{xy}+1}, \ldots, j_L)\) corre-
responds to a game sequence played from $\lambda$ and is a reduced word for some $u$ in $W$ of length $L$. Write $u = u^j u_j$ with $u^j \in W^J$. By Proposition 5.2, any reduced expression for $u^j$ corresponds to a legal firing sequence from $\lambda$; since $u^j \cdot \lambda = u \cdot \lambda$ is the terminal position for any game sequence played from $\lambda$, then this legal firing sequence must be a game sequence, and hence by Proposition 5.3 we have $\ell(u^j) = L$. Then $u_j = e$, so $u = u^j \in W^J$. Since $\ell(u(w_0)_j) = L + \ell((w_0)_j) = \ell(w_0)$, then $u(w_0)_j = w_0 = (w_0)^j_0(x,y)_j$, so $u = (w_0)^j$. So $s$ is a reduced word for $(w_0)^j$. By Proposition 1.1 of [Stem], we see that $(w_0)^j$ is not fully commutative. Therefore by Lemma 5.6, every commutativity class of $R((w_0)^j)$ has a member containing such a subword. But this means that every firing sequence corresponding to a reduced expression for $(w_0)^j$ must have intermediate positions with positive populations at adjacent nodes. This contradicts the hypothesis that $\lambda$ is an adjacency-free position. Therefore no reduced word $t = (j_1, \ldots, j_k)$ in $R(w^J)$ has firing sequence $(r_{j_1}, \ldots, r_{j_k})$ which results in positive populations at two adjacent nodes for an intermediate position. This completes the induction step and the proof of (1).

For part (2), assume every member of $W^J$ is fully commutative, and let $\lambda$ be any position in $C_J$. Let $(i_1, \ldots, i_L)$ be a reduced word for $(w_0)^j$. Suppose an intermediate position $s = s_i \cdots s_j \cdot \lambda$ for the game sequence $(r_{i_1}, \ldots, r_{i_L})$ has positive populations on adjacent nodes $r_i$ and $r_j$. Then by Eriksson’s Strong Convergence Theorem, there is a game sequence of length $L$ from $\lambda$ corresponding to a reduced word $s = (i_1, \ldots, i_k, (x,y)_{\beta}j_{k_1}m_{x,y}(j_{k}+m_{x,y}+1), \ldots, j_L)$ for $u = \psi(s)$. As in the proof of part (1), we can see that $u$ is just $(w_0)^j$. So $(w_0)^j$ is fully commutative (by hypothesis) and has reduced word $s$, in violation of Proposition 1.1 of [Stem]. Therefore $\lambda$ is adjacency-free.

In Theorem 5.1 of [Stem], Stembridge classifies those $W^J$ for irreducible Coxeter groups $W$ such that every member of $W^J$ is fully commutative. In view of Proposition 5.7 and the classification of finite Coxeter groups, we may apply this result here to conclude that for finite irreducible $W$, the adjacency-free dominant positions are exactly those specified in the following theorem. Observe that a dominant position $\lambda$ is adjacency-free if and only if $r\lambda := (r\lambda)_i \in I_n$ is adjacency-free for all positive real numbers $r$; call any such $r\lambda$ a positive multiple of $\lambda$.

**Theorem 5.8** Suppose the E-GCM graph $(\Gamma, M)$ is connected. If $W = W(\Gamma, M)$ is finite, then an adjacency-free dominant position is a positive multiple of a fundamental position. All fundamental positions for any E-Coxeter graph of type $A_n$ are adjacency-free. The adjacency-free fundamental positions for any graph of type $B_n$, $D_n$, or $I_2(m)$ are precisely those corresponding to end nodes. The adjacency-free fundamental positions for any graph of type $E_6$, $E_7$, or $H_3$ are precisely those corresponding to the nodes marked with asterisks in Figure 1.3. Any graph of type $E_8$, $F_4$, or $H_4$ has no adjacency-free fundamental positions.

### 6. Comments

Continue to think of a Coxeter or Weyl group as a group associated to some E-GCM graph with index set $I_n$. For $J \subseteq I_n$, let $(W^J, \leq)$ denote the Bruhat (partial) order on $W^J$ (see for example [BB] Ch. 2 for a definition). In Proposition 3.1 of [Pr1], Proctor shows that for finite irreducible Weyl groups $W$, those $(W^J, \leq)$ for which the Bruhat order is a lattice have $|J| = 1$ and correspond precisely to the adjacency-free fundamental positions for Dynkin diagrams identified in Theorem 5.8; in Proposition 3.2 of that paper, he shows that these lattices are, in fact, distributive. For
finite irreducible Coxeter groups $W$, it is a consequence of Theorems 5.1 and 6.1 of [Stem] that $(W^J, \leq)$ is a lattice if and only if $(W^J, \leq)$ is a distributive lattice if and only if each element of $W^J$ is fully commutative, in which case $|J| = 1$ and all such $J$’s correspond with the adjacency-free fundamental positions from Theorem 5.8 above. Proposition 5.7 above adds to these equivalences the property that each element of $W^J$ is fully commutative if and only if for any associated E-GCM graph, any position in $C_J$ is adjacency-free. The adjacency-free viewpoint is similar to Proctor’s original viewpoint (cf. Lemma 3.2 of [Pr1]).

In [Don] we use Theorem 1.1 to show that if a finite “edge-colored” ranked poset meets a certain condition relative to an $n \times n$ matrix $M$, then $M$ must be a Cartan matrix, i.e. $M$ must be a GCM such that $(\Gamma, M)$ is a Dynkin diagram. This so-called “structure property” is necessary for edge-colored ranked posets to carry certain information about semisimple Lie algebra representations. Indeed, identifying such combinatorial properties is part of our program for obtaining combinatorial models for Lie algebra representations. For example, in [ADLMPW], we introduced four families of finite distributive lattices whose elements are “weighted” by a simple combinatorial rule; their “weight-generating functions” are Weyl characters for the irreducible representations of the four rank two semisimple Lie algebras. In [Don] we show that the posets of join irreducibles (cf. [Sta]) for these distributive lattices are characterized by a short list of combinatorial properties. These are called “semistandard posets” in [ADLMPW]; the smallest of these posets are called “fundamental posets.” In [DW], we will say how these fundamental and semistandard posets can be constructed from information obtained by playing the numbers game on two-node Dynkin diagrams. More generally, we will show how to construct fundamental posets for the adjacency-free fundamental positions for any given Dynkin diagram. These fundamental posets can be combined to obtain semistandard posets whose corresponding distributive lattices are models (as above) for Weyl characters for certain irreducible representations of the corresponding semisimple Lie algebra. When an adjacency-free fundamental position for a Dynkin diagram corresponds to a “minuscule” fundamental weight (see [Pr1], [Pr2], [Stem]), then our fundamental poset is a vertex-colored version of the corresponding “wave” poset of [Pr2] and of the corresponding “heap” of [Stem].

Our work here leaves open some questions about the relationship between the numbers game and quasi-standard geometric representations of Coxeter groups. For example, in the notation of Section 5, if $J \subseteq I_n$ and $W_J$ is infinite, then for $\lambda \in C_J$ is it the case that $\mathfrak{P}(\lambda) = W\lambda$? In Proposition 5.7, can the assumption that $W$ is finite be relaxed? Also, when $(\Gamma, M)$ is not unital OA-cyclic, what can be said about $U_M \cap (-U_M)$? In a different direction, one could ask to what extent the proofs of Theorems 1.1 and 1.3 can be combinatorialized, avoiding Coxeter groups altogether. For Theorem 1.1, one only needs a combinatorial proof of Eriksson’s Comparison Theorem; such a proof is given for “unweighted” GCM graphs in [Erik1].

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