Multiple uncertainty relation for accelerated quantum information

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The uncertainty principle, first introduced by Heisenberg in inertial frames, clearly distinguishes quantum theories from classical mechanics. In non-inertial frames, its information-theoretic expressions, namely entropic uncertainty relations, have been extensively studied through delocalized quantum fields. However, measurements are infeasible on a delocalized quantum field, as field excitations could be created anywhere in spacetime. Building on advances in quantum field theories and theoretical developments in quantum memories, we demonstrate the uncertainty principle in the presence of localized quantum fields inside cavities. Here bounds for entropic uncertainty relations in localized quantum fields are given in terms of the Holevo quantity, which immediately implies how acceleration of agents affects the monogamy of entanglement in tripartite systems.

I. INTRODUCTION

Quantum mechanics offers us a new way to transform information, i.e., encoding information with the help of the state of a quantum system at a given time and decoding information through the implementation of measurements, which rises the curtain of quantum information theory. Besides the advantages provided by quantum mechanics, quantum mechanics also imposes strict restrictions on what we can gain from the measurements. These restrictions are known as Heisenberg uncertainty principle, which lies at the heart of quantum theory. A decade ago, entropic uncertainty relations including quantum memories, which allows entanglement with a measured system, was introduced [1] and paved the way to applications involving entanglement witnesses and quantum key distribution [2]. In particular, the entropic uncertainty relation of tripartite systems quantifies the information tradeoff for two incompatible observables stored in two separate quantum memories [3] and reveals monogamy of entanglement, which lies at the centre of the debate of fundamental puzzles such as black-hole paradox [4].

Relativistic quantum information is a fast-growing field that hopefully could explain these puzzles [5,10]. Relativistic effects on entropic uncertainty relations with quantum memories is not new for bipartite systems [11–15], but some important aspects are infeasible. On the one hand, previous work on uncertainty relations under relativistic effects are based on delocalized quantum fields, whose excitations could thus be created anywhere in the whole universe. Regarding this point, applying operations on quantum information over all spacetime for a point-like observer is unexplained. On the other hand, the acceleration horizon under the Uruh effect exists only when an observer accelerates from the asymptotic past to the asymptotic future [16]. In this setting, how to describe observer’s motion if the observer only accelerates until the measurements are performed has not been made clear. Consequently, to study clearly relativistic effects on uncertainty relations, we localize quantum-field information.

In addition to these gaps in knowledge, multipartite uncertainty relations for non-inertial frames have not been considered yet. Previous lower bounds of entropic uncertainty relations for tripartite systems are state-independent [3], which do not depend on relativistic motion of quantum memories. For puzzles like the black-hole paradox, relativistic effects on bipartite uncertainty relations must be generalized to the case of multipartite systems.

Therefore, our aim in this paper is to investigate multipartite entropic uncertainty relation for localized quantum information in non-inertial frames. Here we focus on fermionic quantum fields localized in rigid cavities, which can be accelerated. For simplicity, we consider spinless fermions, i.e., fermionic fields with one direction of spin fixed [17]. These settings can be realized if we take a cavity made of laser light and an atomic field inside it [18]. The model of an accelerating cavity has been used to study relativistic effects on quantum entanglement [19].

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quantum teleportation [22], quantum secret sharing [23] and quantum clock [24]. Here we combine the transformation of localized fermionic field in non-inertial frames with multipartite entropic uncertainty relation to formulate a relativistic protocol.

Our protocol is depicted in Fig. 2, where the cavities, serving as quantum memories go through relativistic motion [6]. We offer an irreducible bound for entropic uncertainty bound in terms of Holevo quantities for multipartite systems for quantum information stored in accelerating quantum memories. The difference between uncertainty and its lower bound is independent of the acceleration of quantum memories, whereas the lower bound itself changes with acceleration. We find that, as entanglement between Alice and Bob increases, Bob has a higher possibility of guessing Alice’s outcome correctly in case Alice measures $M_1$, whereas the possibility for Charlie to have a good guess in case Alice measures $M_2$ decreases, yielding a non-zero lower bound for the uncertainty relation.

In what follows, we first briefly review background of entropic uncertainty relations with quantum memories and evolution of quantum fields inside accelerated cavities in §II and then give a detailed introduction to our relativistic protocol and new lower bound in §III. We present a specific case of the protocol and show our calculation results in §IV. Finally, we discuss our results in §V and provide future lines of research in §VI. The proofs of this paper, we use units in which $c = \hbar = 1$ and base-2 logarithms.

II. PRELIMINARIES

In this section we give a review of entropic uncertainty relation, especially the results in relativistic case, after which we review the physical model used in this paper, namely fermionic field inside an accelerating cavity.

A. Entropic uncertainty relation in relativistic systems

Here we offer a brief review of entropic uncertainty relation with quantum memories and its development in relativistic systems. An entropic uncertainty relation with quantum memories was first introduced for entangled bipartite systems [1].

Consider three agents, Alice, Bob and a dealer. The dealer prepares a bipartite system $\rho_{AB}$, which could be entangled, then sends one subsystem $A$ to Alice, and another subsystem $B$ to Bob. $B$ is regarded as a quantum memory. Alice randomly chooses either observable $M_1$ or $M_2$ on Hilbert space $\mathcal{H}_A$ as her measurement observable, applies a corresponding measurement to $A$ and broadcasts her measurement choice to Bob. Alice keeps the measurement outcome confidential, and the entanglement between $A$ and $B$ reduces uncertainty for Bob to guess the outcome kept by Alice correctly. The uncertainty relation conditioned on quantum memory is

$$H(M_1|B) + H(M_2|B) \geq \log \frac{1}{c} + H(A|B)$$  \hspace{1cm} (1)

with $c$ the maximal overlap between measurements $M_1$ and $M_2$ and

$$H(A|B) := H(\rho_{AB}) - H(\rho_B)$$  \hspace{1cm} (2)

the conditional entropy. The Shannon entropy of a density matrix is

$$H(\rho) = -\sum_{i=1}^{n} p_i \log p_i$$  \hspace{1cm} (3)

with $\rho$ an $n \times n$ density matrix and $\{p_i\}$ eigenvalues of $\rho$. Here Shannon entropy is equivalent to von Neumann entropy $H(\rho) = -\text{Tr}(\rho \log \rho)$.

For an entangled tripartite system, we consider four agents, named Alice, Bob, Charlie and a dealer. The dealer prepares a state $\rho_{ABC}$ and sends $A$ to Alice, $B$ to Bob and $C$ to Charlie. Alice implements measurement $M_i$ ($i = 1, 2$) to $A$ and broadcasts her choice to Bob and Charlie. The ability of Bob and Charlie to infer Alice’s measurement outcome correctly is limited by

$$H(M_1|BC) + H(M_2|BC) \geq \log \frac{1}{c}$$  \hspace{1cm} (4)

Remarkably, the bound (4) not only expresses the uncertainty principle but also monogamy of entanglement [3].

In recent years, entropic uncertainty relations are studied in relativistic systems with delocalized fields [11–15]. In this context bipartite entropic uncertainty relations for non-inertial frames depend on the acceleration of quantum memory in non-inertial frames due to the Unruh effect [11]. When acceleration increases, uncertainty increases due to degradation of entanglement. The result is applicable to the uncertainty relation for Hawking radiation of a Schwarzschild black hole [12, 13, 15].

A new lower bound was derived for a bipartite entropic uncertainty relation in the presence of quantum memory via the Holevo quantity [15]. This new lower bound is advantageous because, as the memory accelerates, the difference between total uncertainty and its lower bound remains independent of acceleration. Their uncertainty relation is

$$H(M_1|B) + H(M_2|B) \geq \log \frac{1}{c} + H(A) - \mathcal{J}(B|M_1) - \mathcal{J}(B|M_2), \hspace{1cm} (5)$$

where

$$\mathcal{J}(B|M_i) = H(B) - \sum_{j}^{n} p_j H(\rho_{B|u_j^i}),$$  \hspace{1cm} (6)

$$\rho_{B|u_j^i} = \langle u_j^i | \rho_{AB} | u_j^i \rangle / \text{Tr}(\langle u_j^i | \rho_{AB} | u_j^i \rangle), \hspace{1cm} i \in \{1, 2\}. \hspace{1cm} (7)$$
and \( a^j_1 \) and \( p^j_1 \) are the \( j \)th eigenvector and eigenvalue of \( M_j \). \( \mathcal{J}(B|M_j) \) is the Holevo quantity for the quantum memory \( B \) about the measurement outcomes of \( M_j \), which reveals how much information can be encoded in a quantum system.

B. Relativistic quantum information in cavities

After introducing entropic uncertainty relations and their applications to relativistic systems, we briefly review the effect of acceleration on a fermionic field inside a cavity, which has been well-studied during this past decade [19–21].

The cavity trajectory we focus on is called the basic building block (BBB), which enables studying any arbitrary non-uniform trajectory [21]. As shown in Fig. 1, the whole trajectory comprises three steps. Initially, the cavity is at rest in region I. After \( t \geq 0 \), the cavity accelerates with constant acceleration for a certain time in region II. In region III, the cavity stops accelerating and moves uniformly. The BBB transforms from an inertial cavity, back to an inertial cavity, with a single intermediate period of uniform acceleration. The transition of the quantum field inside the cavity from inertial region to uniformly accelerating region can be represented as a linear transformation of the modes, called the Bogoliubov transformation [21].

Here we illustrate the Bogoliubov transformation in detail. Specifically, the quantum field inside the cavity of our protocol is a spinless fermionic field, so we focus on the transformation of fermionic fields. Modes of the field \( \{\psi_k^1\} \) are classified by positive and negative frequency with respect to (the future-directed Minkowski Killing vector) \( \partial_t \) [6].

In region I, the field can be expanded as

\[
\psi^1 = \sum_{k \geq 0} a_k \psi^1_k + \sum_{k < 0} b^*_k \psi^1_k, \quad (8)
\]

where \( a_k \) is the annihilation operator of a positive mode and \( b^*_k \) is the creation operator of a negative mode according to the Jordan-Wigner transformation mapping spin to fermionic fields. Different modes of a fermionic field satisfy the anti-commutation relation

\[
\{\psi_{k'}, \psi^*_{k} \} = \delta_{k'k} 1, \quad (9)
\]

and the creation and annihilation operators also have anti-commutation relations

\[
\{\hat{a}_{k'}, \hat{a}^*_{k} \} = \{\hat{b}_{k'}, \hat{b}^*_{k} \} = \delta_{k'k} 1. \quad (10)
\]

Here \( \{\psi^1_k\} \) forms a complete set of orthogonal mode functions of the fermionic field in region I.

When the field transforms from region I to region II at \( t = 0 \), the set of mode functions \( \{\psi^1_k\} \) transforms to \( \{\psi^2_k\} \). Then we have the Bogoliubov transformation from region I to region II, namely

\[
\psi^2_k = \sum_{k'} F_{k'k} \psi^1_{k'}. \quad (11)
\]

where

\[
F_{k'k} := \langle \psi^2_{k'}, \psi^*_{k} \rangle \quad (12)
\]

is called the Bogoliubov coefficient, and

\[
\langle \psi^1, \psi^2 \rangle \quad (13)
\]

denotes the Dirac inner product [6].

In our work, \( F_{k'k} \) is expanded in terms of the perturbative variable

\[
h := aL, \quad (14)
\]

where

\[
L = x_2 - x_1 \quad (15)
\]

is the proper length of the cavity, up to \( h^2 \) as

\[
F_{k'k} = F_{k'k}^{(0)} + F_{k'k}^{(1)} + F_{k'k}^{(2)} + \mathcal{O}(h^3). \quad (16)
\]

Coefficients in (16) have been calculated in details [20], where \( k \) and \( k' \) are mode labels.

As the motion of the cavities in regions I and III is related by a Lorentz transformation, there is a Lorentz symmetry between regions I and III, so the transformation between regions II and III is simply the inverse transformation \( F^{-1} \). We choose the same phase conventions for region III solutions at \( \eta = \eta_1 \) as region I solutions are at \( \eta = 0 \) (we set the phase at \( \eta = 0 \) to be 0). The phases acquired by the Rindler modes in region II are accounted
for by a diagonal matrix $G(\eta_1)$, where the $n^\text{th}$ diagonal entry of the matrix is

$$G_n(\eta_1) = \exp(i\Omega_n \eta_1) = \exp\left(\frac{i\pi(n + s)\eta_1}{\ln(x_2/x_1)}\right), \quad (17)$$

and $s \in [0, 1)$ is a parameter characterizing the phase shifts of reflection from the two walls of the cavity.

Overall, the whole Bogoliubov transformation from region I to region III is represented by

$$F = F^{-1}G(\eta_1)F = F^1G(\eta_1)F. \quad (18)$$

Therefore, the transformation of fermionic fields from region I to region III is

$$\psi_{k'}^3 = \sum_k F_{k'k}^3 \psi_k^3, \quad (19)$$

where $F_{k'k}$ can also be expanded perturbatively in $h$ as

$$F_{k'k} = F_{k'k}^0 + F_{k'k}^{(1)} + F_{k'k}^{(2)} + \mathcal{O}(h^3). \quad (20)$$

with

$$F_{k'k}^0 = \delta_{k'k}G_k, \quad (21a)$$

$$F_{k'k}^{(1)} = (G_{k'} - G_k)F_{k'k}^{(1)}, \quad (21b)$$

$$F_{k'k}^{(2)} = G_k F_{k'k}^{(2)} + F_{k'k}^{(2)} G_k + \sum_{k''} F_{k'k''}^* G_{k''} F_{k''k'}^*. \quad (21c)$$

by using Eq. (18) and the coefficients in Eq. (16). Since our calculation below doesn’t include $F_{k'k}^{(2)}$ ($k' \neq k$), we just show the expansion of $F_{k'k}^{(2)}$ here.

For specific calculations to come, we consider the Bogoliubov transformation of the vacuum and one-particle states inside the cavity from region I to region III, which was used to study how entanglement is affected due to relativistic motion [20, 21]. Firstly, we suppose that all the modes chosen in our calculations are positive modes. Vacuum state in region I is denoted by $|0\rangle^1$, then the one-particle state is $|1\rangle^1 = \tilde{d}_k^1 |0\rangle^1$. When the cavity stops accelerating at $\eta_1$ and moves uniformly in region III, we denote vacuum state of the field inside it by $|0\rangle^3$, and the one-particle state is $|1\rangle^3 = \tilde{d}_k^3 |0\rangle^3$, where $\tilde{d}_k^3$ is the creation operator in region III.

When $k \geq 0$ and other modes except $k$ are in vacuum states initially, by tracing out these modes except $k$ (denoted by $-k$), we get the transformation of the vacuum and one-particle states of the field from region I to region III as [20]

$$\text{Tr}_{-k} |0\rangle^1 \langle 0| = (1 - f^{-}) |0_k\rangle^3 \langle 0_k| + f^{-} |1_k\rangle^3 \langle 1_k|, \quad (22a)$$

$$\text{Tr}_{-k} |0\rangle^1 |1_k\rangle = \left(G_k + F_{k'k}^{(2)}\right) |0_k\rangle^3 \langle 1_k|, \quad (22b)$$

$$\text{Tr}_{-k} |1_k\rangle^1 \langle 1_k| = (1 - f^{+}) |1_k\rangle^3 \langle 1_k| + f^{+} |0_k\rangle^3 \langle 0_k|, \quad (22c)$$

where

$$f^{+} = \sum_{l \geq 0} \left|F_{kl}^{(1)}\right|^2 = \sum_{l \leq 0} \left|e^{2\pi i u(k-l)} - 1\right|^2 \left|F_{kl}^{(1)}\right|^2, \quad (23a)$$

$$f^{-} = \sum_{l < 0} \left|F_{kl}^{(1)}\right|^2, \quad (23b)$$

and

$$u = \frac{\eta_1}{2 \ln(x_2/x_1)} = \frac{\eta_1}{2 \ln[(a \cdot L + 2)/(2 - a \cdot L)]}. \quad (24)$$

Here $u$ is a factor related to acceleration of the cavity, where $a = 2/(x_1 + x_2)$ is the proper acceleration of the center of the cavity, and $L$ is the length of the cavity.

III. APPROACH

In this section, we develop a relativistic protocol for uncertainty game by employing the evolution of fermionic fields inside cavities moving with the trajectories of BBB to a protocol revealing uncertainty relation in entangled tripartite system, as depicted in Fig. 2.

Four agents, Alice, Bob, Charlie and a dealer, enact our protocol. At first, a dealer takes three cavities with one spinless fermion field inside each cavity and prepares a tripartite entangled state with one mode in each cavity. Then the dealer delivers the three cavities to Alice, Bob and Charlie respectively, and tells them their tasks, including how long Bob and Charlie should accelerate and when Alice should apply measurements. After $t = 0$, Bob and Charlie both accelerate, which is shown in region II. Then they stop accelerating at the time they are told.

When they stop accelerating, Alice stays in her lab, takes two detectors that can measure $M_1$ and $M_2$ respectively out, and chooses one of them to perform a measurement to the mode in her cavity. Then she broadcasts her choice of the measurement to Bob and Charlie while keeping outcome of the measurement secretly. After that, Bob and Charlie play a game against Alice without communication between themselves, from which they win only if both of them guess the outcome correctly. The game rule is that, Bob performs the measurement to his own cavity after Alice detects $M_1$, and Charlie performs the measurement to her own cavity after Alice detects $M_2$. Also, the protocol needs to be repeated for many rounds so that each of Bob and Charlie gets a distribution of their inferences.

The lower bound of the uncertainty relation is a limit to the ability for Bob and Charlie to both of them correctly inferring Alice’s outcome. Here we introduce a new lower bound of multiple uncertainty relation in terms of Hohlov quantities by generalizing [3]. Contrary to a previous state-independent bound like [1], our bound reveals how acceleration affects uncertainty. The difference between joint uncertainty and our new bound is independent of acceleration of memories.
IV. RESULTS

Here we suppose the dealer prepares a W state

\[
|\psi_0\rangle_{ABC} = \frac{1}{\sqrt{3}} \left( |1k_1 0k_2 0k_3\rangle + |0k_1 1k_2 0k_3\rangle + |0k_1 0k_2 1k_3\rangle \right)
\]

for Alice, Bob and Charlie initially in region I. Then Alice stays at rest while Bob and Charlie move in the trajectories of BBB. From region I to region III, the modes entangled inside Alice, Bob and Charlie’s cavities experience a Bogoliubov transformation and the initial state (27) transforms to a mixed state \(\rho_{ABC}\). We calculate \(\rho_{ABC}\) by substituting (22) into (27). Finally, when all of the three agents are in region III, Alice chooses \(\sigma_x\) or \(\sigma_y\) to measure her mode inside her cavity and broadcasts her choice to Bob and Charlie. From the Jordan-Wigner transformation, we know fermions can be mapped to a spin chain.

In general, due to the Jordan-Wigner transformation, Pauli operators are not local in fermionic system. However, as only Alice’s mode undergoes Pauli measurement, we can ignore the nonlocal phase here. The Pauli operators in our paper represent measurements of superposition of particle numbers, where if we measure \(\sigma_z\) we can get 0 for "no particle" and 1 for "one particle", and if we measure \(\sigma_x\) or \(\sigma_y\) we can get phase of the superposition of "no particle" and "one particle". When Alice applies a measurement to \(\sigma_x\), \(\rho_{ABC}\) changes to a post-measurement state

\[
\rho_{\sigma_xC} = \sum_{i=\pm} (\Pi_i \otimes \mathbb{I})\rho_{ABC}(\Pi_i \otimes \mathbb{I}),
\]

where \(\{\Pi_i\}\) are projections on the eigenvectors of \(\sigma_x\) at subsystem A. Suppose Charlie guesses the outcome when Alice measures \(\sigma_x\), and it’s Bob’s turn to guess when Alice measures \(\sigma_y\). To calculate the conditional entropy \(H(\sigma_x|C)\), we express \(\rho_{\sigma_xC}\) in the basis \(\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}\) by tracing out subsystem B as

\[
\rho_{\sigma_xC} = \frac{1}{6} \begin{bmatrix}
2 - \tilde{f}_{k_3} & 0 & 0 & F_{k_3} \\
0 & 1 + \tilde{f}_{k_3} & F_{k_3}^* & 0 \\
0 & F_{k_3}^* & 2 - f_{k_3} & 0 \\
F_{k_3} & 0 & 0 & 1 + f_{k_3}
\end{bmatrix},
\]

where

\[
\tilde{f}_{k_3} = -f_{k_3}^+ + 2f_{k_3}^- F_{k_3} - G_{k_3} + F_{k_3}^{(2)},
\]

Here \(\rho_{\sigma_xC}\) has four eigenvalues

\[
\lambda_1 = \lambda_2 = \frac{3 - \sqrt{1 - 4\tilde{f}_{k_3} + 4f_{k_3}^2 + 4F_{k_3}^2}}{12},
\]

\[
\lambda_3 = \lambda_4 = \frac{3 + \sqrt{1 - 4\tilde{f}_{k_3} + 4f_{k_3}^2 + 4F_{k_3}^2}}{12}.
\]
Similarly, we calculate $H(\sigma_x \vert C) = H(\rho_{xC}) - H(\rho_C)$
\[= 2H(\lambda_1) + 2H(\lambda_3) - H\left(\frac{2 - \bar{f}_{k_3}}{3}\right) - H\left(\frac{1 + \bar{f}_{k_3}}{3}\right). \tag{32}\]

We present a numerical plot on the change of bound of the uncertainty relation with two memories in Fig. 3. As depicted in Fig. 3 the function of the bound is a two-dimensional plot, which relates to two variables $u_b$ and $u_c$ with expression of Eq. (26). $u_b$ and $u_c$ are acceleration factors for cavities of Bob and Charlie. In this plot, we have set $h = 0.1$ and chosen $k_2 = 2, k_3 = 1$, and for massless fermionic fields $s_2 = s_3 = \frac{1}{2} [20]$.

Then we calculate
\[H(\sigma_x \vert C) = H(\rho_{xC}) - H(\rho_C)\]
\[= 2H(\lambda_1) + 2H(\lambda_3) - H\left(\frac{2 - \bar{f}_{k_3}}{3}\right) - H\left(\frac{1 + \bar{f}_{k_3}}{3}\right). \tag{32}\]

Our result is shown in Fig. 3 the bound of the uncertainty relation evolves as a periodic function of the acceleration factors $u_b$ and $u_c$, for which the expression has been shown in Eq. (26). We note that when $u = 1$, the total phase rotation caused by Bogoliubov transformation returns to integral multiples of $2\pi$ and increase of the bound due to acceleration is canceled.

\section*{V. DISCUSSION}

We have presented a relativistic protocol revealing entropic uncertainty relation with multiple quantum memories with quantum fields localized in cavities. On the one hand, we obtain a new mathematical expression of entropic uncertainty relation with multiple quantum memories [24], where the difference between uncertainty and its bound remains independent of acceleration. On the other hand, we formulate a relativistic protocol with localized quantum fields to show how acceleration of memories affects the bound of the uncertainty relation. However, the final result cannot be calculated analytically.
beyond the horizon, entanglement between she and observers outside the horizon will be killed [4]. To explore which postulates are right and whether there is a firewall, we can present the relativistic protocol of entropic uncertainty relation near the horizon of a black hole. This can give a prediction as to whether the monogamy is violated according to the violation of lower bound of the uncertainty relation.

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Appendix A: Derivation of the multiple uncertainty bound [12]

For $N+1$ quantum systems, we choose one system $A$ to be measured by $M_i \in \mathcal{M}_{n \times n}(\mathbb{C})$ and the rest $N$ systems are regarded as quantum memories. The conditional entropy for the $i$th memory $E_i$ is

$$H(M_i|E_i) = H(\rho_{M,E_i}) - H(\rho_{E_i}), \quad (A1)$$

where $\rho_{M,E_i} \in \mathcal{M}_{n^2 \times n^2}(\mathbb{C})$ is the joint state of system $A$ and $E_i$, it can be expanded to

$$\rho_{M,E_i} = \sum_j p_j^i |v_j^i\rangle \langle v_j^i| \otimes \rho_{E_i}^j, \quad (A2)$$

here $|v_j^i\rangle$ is one of eigenvectors of $M_i$, $p_j^i$ is the possibility to get $|v_j^i\rangle$ when we measure $M_i$ for the state of the system $\rho_A \in \mathcal{M}_{n \times n}(\mathbb{C})$, and $\rho_{E_i}^j \in \mathcal{M}_{n \times n}(\mathbb{C})$ is the final state of memory system $E_i$ when the measured state of system $A$ is $|v_j^i\rangle$. According to the joint entropy theorem [28] we have

$$H(\rho_{M,E_i}) = H(M_i) + H(\rho_{E_i|M_i}). \quad (A3)$$

Similar to the derivation for the bound in bipartite systems [15], we define a reduced density matrix $\rho_{E_i|v_j^i} \in \mathcal{M}_{n \times n}(\mathbb{C})$ which satisfies

$$H(\rho_{E_i|v_j^i}) = \sum_j \text{Tr}(\rho_{M,E_i}^j |v_j^i\rangle \langle v_j^i|) \times H(\rho_{E_i|v_j^i}). \quad (A4)$$

where

$$\rho_{E_i|v_j^i} = \langle v_j^i| \rho_{M,E_i} |v_j^i\rangle / \text{Tr}(\langle v_j^i| \rho_{M,E_i} |v_j^i\rangle). \quad (A5)$$

Thus, we define a Holevo quantity

$$\mathcal{J}(E_i|M_i) = H(\rho_{E_i}) - \sum_j \text{Tr}(\rho_{M,E_i}^j |v_j^i\rangle \langle v_j^i|) \times H(\rho_{E_i|v_j^i}) \quad (A6)$$

similar to the Holevo bound [28]. By taking $H(M_i) = \sum_j \mathcal{J}(p_j^i)$ and (A6) into (A1), we rewrite (A1) as

$$H(M_i|E_i) = \sum_j \mathcal{J}(p_j^i) - \mathcal{J}(E_i|M_i). \quad (A7)$$

According to the derivation above, the total entropy is

$$\sum_i H(M_i|E_i) = \sum_i H(M_i) - \sum_i \mathcal{J}(E_i|M_i), \quad (A8)$$

with

$$\sum_i H(M_i) \geq B, \quad (A9)$$

where the lower bound $B$ is state-independent and it has been introduced in [11]. We can derive Eq. (25) by applying (A9) to (A8).

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