Random dense countable sets: characterization by independence

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Abstract

A random dense countable set is characterized (in distribution) by independence and stationarity. Two examples are Brownian local minima and unordered infinite sample. They are identically distributed; the former ad hoc proof of this fact is now superseded by a general result.

Introduction

Random dense countable sets arise naturally from various probabilistic models. Their examination is impeded by the singular nature of the set DCS(0,1) of all dense countable subsets of (say) the interval (0,1). This set is not a Polish space, not even a standard Borel space. Nevertheless the idea of random elements of DCS(0,1) and their distributions can be formalized. An appropriate framework proposed in [2, Sect. 1] is used here.

Two examples of random dense countable sets are compared in [2]. One example, ‘Brownian local minima’, is the random set

\[ M = \{ s \in (0,1) : \exists \varepsilon > 0 \ \forall t \in (s-\varepsilon, s) \cup (s, s+\varepsilon) \ B_s < B_t \} \]

of local minimizers on (0,1) of the Brownian motion \((B_t)_t\). The other example, ‘unordered infinite sample’, is the random set

\[ S = \{ U_1, U_2, \ldots \} = \{ s \in (0,1) : \exists n \ U_n = s \} \]

where \(U_1, U_2, \ldots\) are independent random variables distributed uniformly on (0,1). The main result of [2] states that \(M\) and \(S\) are identically distributed, which means existence of such a joining between the Brownian motion \((B_t)_t\) and the sequence \((U_n)_n\) that \(M = S\) a.s.

Independence of Brownian increments on disjoint time intervals \((a, b)\) and \((c, d)\) implies independence of ‘fragments’ \(M \cap (a, b)\) and \(M \cap (c, d)\) of \(M\) (see...
Independence of $S \cap (a, b)$ and $S \cap (c, d)$ is less evident but true \cite[2.2]{2}. The same holds for any number of fragments.

Stationarity of Brownian increments should imply stationarity of $M$. However, time shifts do not preserve the time interval $(0, 1)$. We have two options: either replace $(0, 1)$ with the whole $\mathbb{R}$, or replace linear shifts $t \mapsto t + s$ of $\mathbb{R}$ with cyclic shifts $t \mapsto t + s \mod 1$ of $(0, 1)$; I choose the latter option. The cyclic shift is nothing but an interval exchange transformation: $(0, 1 - s) \mapsto (s, 1)$ and $(1 - s, 1) \mapsto (0, s)$. Brownian increments on $(0, 1 - s)$ and $(s, 1)$ are distributed identically; taking independence into account we get cyclic stationarity of $M$ (see \ref{1.9} for the definition).

Cyclic stationarity of $S$ is evident.

Thus, the main result of \cite{2} is a special case of the following. (See Definitions \ref{1.2}, \ref{1.9} and Theorem \ref{1.11}.)

**Theorem.** All random dense countable subsets of $(0, 1)$, satisfying the conditions of independence and cyclic stationarity, are identically distributed.

Waiving stationarity we get some other distributions; see Counterexample \ref{1.6} and Theorem \ref{1.8}.

## 1 Definitions and claims

Throughout, either by assumption or by construction, all probability spaces are standard. Recall that a standard probability space (known also as a Lebesgue-Rokhlin space) is a probability space isomorphic (mod 0) to an interval with the Lebesgue measure, a finite or countable collection of atoms, or a combination of both.

According to \cite[Sect. 1]{2}, the set $\text{DCS}(0, 1)$ is of the form $B/E$, the quotient set of a standard Borel space $B$ by an equivalence relation $E \subset B \times B$. Namely, $B = (0, 1)^\infty$ consists of all sequences $(u_n)_n$ of pairwise different points of $(0, 1)$, and $E$ consists of all pairs $((u_n)_n, (u_{\sigma(n)})_n)$ where $(u_n)_n$ runs over $B$ and $\sigma$ runs over all permutations of $\{1, 2, \ldots \}$.

A map $X : \Omega \to B/E$ is called measurable \cite[Def. 1.3]{2} if it admits a measurable lifting $Y : \Omega \to B$:

$$
\begin{array}{ccc}
\Omega & \xrightarrow{Y} & B \\
\downarrow{X} & & \downarrow{\text{canonical projection}} \\
B/E & & \\
\end{array}
$$

By a random dense countable subset of $(0, 1)$ we mean a measurable map $\Omega \to \text{DCS}(0, 1)$ (or rather an equivalence class of such maps). Its measurable
lifting $Y = (Y_1, Y_2, \ldots)$ may be called also a measurable enumeration of $X$; $X = \{Y_1, Y_2, \ldots\}$ in the sense that $X(\omega) = \{Y_1(\omega), Y_2(\omega), \ldots\}$ for almost all $\omega$.

Of course, another interval may be used instead of $(0, 1)$.

Two measurable maps $X_1 : \Omega_1 \to B/E$, $X_2 : \Omega_2 \to B/E$ are called identically distributed [2 Def. 1.4] if they can be matched by a joining $J$ between $\Omega_1$ and $\Omega_2$ (in other words, a measure with given marginals on $\Omega_1 \times \Omega_2$):

\[ (\Omega_1 \times \Omega_2, J) \]

\[ \begin{array}{c}
\Omega_1 \\
\downarrow \\
X_1 \\
\downarrow \\
B/E \\
\downarrow \\
\Omega_2 \\
\downarrow \\
X_2
\end{array} \]

**1.1 Definition.** Two measurable maps $X_1 : \Omega \to B/E$, $X_2 : \Omega \to B/E$ are independent, if they admit independent measurable liftings $Y_1, Y_2 : \Omega \to B$.

Independence of more than two maps is defined similarly.

**1.2 Definition.** Let $X$ be a random dense countable subset of $(0, 1)$ such that

\[ P\left( t \in X \right) = 0 \quad \text{for each } t \in (0, 1) . \]

We say that $X$ satisfies the independence condition, if for all $n = 1, 2, \ldots$ and all $0 = t_0 < t_1 < \cdots < t_n = 1$ the random dense countable subsets $X_k$ of $(t_{k-1}, t_k)$ defined by $X_k = X \cap (t_{k-1}, t_k)$ are independent.

As was noted in Introduction, the independence condition is satisfied both for $M$ (Brownian local minima) and $S$ (unordered infinite sample).

**1.4 Lemma.** For every random dense countable subset of $(0, 1)$ there exists a measure $\mu$ on $(0, 1)$ such that

\[ X \cap A = \emptyset \quad \text{a.s. if and only if } \mu(A) = 0 \]

for all Borel sets $A \subset (0, 1)$.

**Proof.** A measure lifting $Y : \Omega \to (0, 1)^\infty$ enumerates the random dense countable set $X$ by random variables $Y_1, Y_2, \ldots$. The measure $\mu$ defined by

\[ \mu(A) = \sum_n \frac{1}{n^2} P\left( Y_n \in A \right) \]

fits evidently.
The measure \( \mu \) is determined by \( X \) up to equivalence (mutual absolute continuity). For \( M \) (Brownian local minima) and \( S \) (unordered infinite sample) we may take \( \mu = \text{mes} \) (the Lebesgue measure). In general \( \mu \) is nonatomic, otherwise arbitrary.

It may seem that all random dense countable subsets of \((0,1)\) satisfying the independence condition and having \( \mu = \text{mes} \) are identically distributed. However, they are not.

1.6 Counterexample. We choose a (nonrandom) dense open set \( G \subset (0,1) \) such that \( \text{mes} G < 1 \); its complement \( C = (0,1) \setminus G \) is a nowhere dense compact set of positive measure. We take the usual Poisson random subset \( P \) of \((0,1)\) (whose intensity measure is the Lebesgue measure) and combine it with \( S \) (unordered infinite sample, independent of \( P \)):

\[
X = (P \cap C) \cup (S \cap G).
\]

It is easy to see that \( X \) is a random dense countable subset of \((0,1)\) satisfying (1.3), the independence condition, and (1.5) for \( \mu = \text{mes} \). However, \( X \) and \( S \) are not identically distributed.

In order to exclude such cases we introduce a condition (stronger than (1.5) for \( \mu = \text{mes} \)):

\[
\begin{align*}
X \cap A &= \emptyset \text{ a.s.} & \text{if } \text{mes}(A) = 0, \\
X \cap A &\text{ is infinite a.s.} & \text{if } \text{mes}(A) > 0
\end{align*}
\]

for all Borel sets \( A \subset (0,1) \).

1.8 Theorem. All random dense countable subsets of \((0,1)\) satisfying (1.7) and the independence condition are identically distributed.

The proof is given in Sect. 5.

We turn to stationarity, that is, invariance under cyclic shifts \( T_s : (0,1) \to (0,1) \) defined for \( s \in (0,1) \) by

\[
T_s(t) = t + s \mod 1 = \begin{cases} 
  t + s & \text{for } t \in (0,1-s), \\
  t + s - 1 & \text{for } t \in (1-s,1);
\end{cases}
\]

\( T_s \) is undefined at \( 1-s \), which does not matter as long as our random sets satisfy (1.3).

Given a random dense countable set \( X \) satisfying (1.3) and a number \( s \in (0,1) \), we get another random dense countable set \( T_s(X) = \{T_s(t) : t \in X \} \). Moreover, we may randomize \( s \) as follows. We multiply the given probability space by the interval \((0,1)\) (equipped with Lebesgue measure) and define on the new probability space a measurable function \( Y : \Omega \times (0,1) \to \text{DCS}(0,1) \) by \( Y(\omega,s) = T_s(X(\omega)) \). Clearly, \( Y \) is also a random dense countable set.
1.9 Definition. A random dense countable subset $X$ of $(0, 1)$ is stationary if it satisfies (1.3) and $X, Y$ are identically distributed where $Y$ is constructed from $X$ by the random shift, as described above.

Two evident examples are $M$ (Brownian local minima) and $S$ (unordered infinite sample).

1.10 Proposition. If a random dense countable subset of $(0, 1)$ is stationary then it satisfies (1.7).

The proof is given in Sect. 5.

1.11 Theorem. All stationary random dense countable subsets of $(0, 1)$ satisfying the independence condition are identically distributed.

Proof. Follows immediately from 1.10 and 1.8.

\[\Box\]

2 Existence of measures with given marginals

Random sets do not appear at all in this section.

Denote by $M$ the set of all positive Borel measures $m$ on the square $(0, 1) \times (0, 1)$ such that

$$m(U \times (0, 1)) \leq \text{mes}(U), \quad m((0, 1) \times V) \leq \text{mes}(V)$$

for all Borel sets $U, V \subset (0, 1)$. In other words, both marginals of $m$ are bounded by the Lebesgue measure.

For every Borel subset $W$ of the square we define two numbers,

$$\alpha(W) = \sup \{m(W) : m \in M\}, \quad \beta(W) = \inf \{\text{mes}(U) + \text{mes}(V) : (U \times (0, 1)) \cup ((0, 1) \times V) \supset W\};$$

the infimum is taken over Borel sets $U, V \subset (0, 1)$ satisfying the indicated condition. In other words, $W$ does not intersect the product of their complements, $((0, 1) \setminus U) \times ((0, 1) \setminus V)$.

A well-known Strassen’s theorem states that $\alpha(W) = \beta(W)$ for all closed sets $W$. The same holds for all $F_\sigma$-sets, as is shown below.

2.1 Lemma. If $W_n \uparrow W$ (that is, $W_1 \subset W_2 \subset \ldots$ and $W_1 \cup W_2 \cup \cdots = W$; of course, these are Borel subsets of the square) then $\alpha(W_n) \uparrow \alpha(W)$.
Proof. Follows immediately from the fact that \( m(W_n) \uparrow m(W) \) for \( m \in M \).

\[ \square \]

2.2 Proposition. If \( W_n \uparrow W \) then \( \beta(W_n) \uparrow \beta(W) \).

The proof is given after Prop. 2.4.

It follows immediately that \( \alpha(W) = \beta(W) \) for all \( F_\sigma \)-sets \( W \).

Following [2, Sect. 5] we may avoid topological notions (closed sets, \( F_\sigma \)-sets). To this end we denote by \( A \) the algebra of subsets of \((0, 1) \times (0, 1)\) generated by all product sets \( U \times V \) where \( U, V \) are Borel subsets of \((0, 1)\). We consider the class \( A_\delta \) of all sets of the form \( W_1 \cap W_2 \cap \ldots \) where \( W_1, W_2, \ldots \in A \). (All closed sets belong to \( A_\delta \).) Further, we consider the class \( A_{\delta\sigma} \) of all sets of the form \( W_1 \cup W_2 \cup \ldots \) where \( W_1, W_2, \ldots \in A_\delta \). (All \( F_\sigma \)-sets belong to \( A_{\delta\sigma} \).)

The equality \( \alpha(W) = \beta(W) \) holds for all \( W \in A_\delta \) [2, 5.6]. Therefore (by 2.1, 2.2) it holds for all \( W \in A_{\delta\sigma} \).

The square \((0, 1) \times (0, 1)\) can be replaced with the product \( B_1 \times B_2 \) of two standard Borel spaces (these are Borel isomorphic to \((0, 1)\)) equipped with probability measures. Moreover, we may start with the product \((\Omega, \mathcal{F}, P)\) of two probability spaces \((\Omega_1, \mathcal{F}_1, P_1), (\Omega_2, \mathcal{F}_2, P_2)\) and consider the \( \sigma \)-field \( \mathcal{F} \) modulo (the \( \sigma \)-ideal of all) sets of the form \((A_1 \times \Omega_2) \cup (\Omega_1 \times A_2)\) where \( P_1(A_1) = 0, P_2(A_2) = 0 \) (and their subsets). All results of this section can be reformulated readily to this more general framework, together with their proofs.

2.3 Lemma. Every sequence of pairs of measurable subsets of \((0, 1)\) contains a subsequence \((A_n, B_n)_n\) such that the limits

\[
\begin{align*}
f(x) &= \lim_{n \to \infty} \frac{\# \{ k \leq n : x \in A_k \}}{n}, \\
g(y) &= \lim_{n \to \infty} \frac{\# \{ k \leq n : y \in B_k \}}{n}, \\
h(x, y) &= \lim_{n \to \infty} \frac{\# \{ k \leq n : x \in A_k, y \in B_k \}}{n}
\end{align*}
\]

exist for almost all \( x, y \in (0, 1) \) and

\[ h(x, y) = f(x)g(y) \]

for almost all \( x, y \in (0, 1) \).

Proof. A result of Aldous [1] (applied to the products) gives us a subsequence \((A_n, B_n)_n\) and a measurable map \( P = (P_{00}, P_{01}, P_{10}, P_{11}) \) from the square \((0, 1) \times (0, 1)\) to the set of probability measures on the four-point set \( \{0, 1\} \times \{0, 1\} \) such that for almost every point \((x, y) \in (0, 1) \times (0, 1)\) the sequence

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\((1_{A_n}(x), 1_{B_n}(y))_{n=1}^\infty\) of elements of \(\{0, 1\} \times \{0, 1\}\) is \(P(x, y)\)-normal in the following sense.

Given a probability measure \(p = (p_{00}, p_{01}, p_{10}, p_{11})\) on the four-point set \(\{0, 1\} \times \{0, 1\}\) and a sequence of pairs \(((a_n, b_n))_n\) where \(a_n, b_n \in \{0, 1\}\), we say that the sequence is \(p\)-normal, if it has appropriate frequencies of finite blocks, namely,

\[
\frac{1}{n} \# \{k \leq n : (a_k, b_k) = (s_0, t_0), \ldots, (a_{k+i}, b_{k+i}) = (s_i, t_i) \} \to p_{s_0, t_0} \cdots p_{s_i, t_i}
\]

as \(n \to \infty\), for all \(i = 0, 1, 2, \ldots\) and all \(s_0, \ldots, s_i, t_0, \ldots, t_i \in \{0, 1\}\).

The case \(i = 0\) gives \(\frac{1}{n} \# \{k \leq n : x \in A_k, y \in B_k \} \to p_{11}\) and \(\frac{1}{n} \# \{k \leq n : x \in A_k, y \notin B_k \} \to p_{10}\), thus,

\[
\frac{1}{n} \# \{k \leq n : x \in A_k \} \to P_{10}(x, y) + P_{11}(x, y);
\]

note that \(P_{10}(x, y) + P_{11}(x, y)\) appears to be a function of \(x\) only. We see that \(f = P_{10} + P_{11}\), \(g = P_{01} + P_{11}\), \(h = P_{11}\). It remains to prove that \(h = fg\).

We apply the bounded convergence theorem to the relation (integrated in \(x, y\))

\[
\frac{1}{n} \# \{k \leq n : (1_{A_k}(x), 1_{B_k}(y)) = (s_0, t_0), \ldots, (1_{A_{k+i}}(x), 1_{B_{k+i}}(y)) = (s_i, t_i) \} 
\to P_{s_0, t_0}(x, y) \cdots P_{s_i, t_i}(x, y),
\]

getting

\[
\frac{1}{n} \sum_{k=1}^{n} \text{mes}_2 \{(x, y) : 1_{A_k}(x) = s_0, 1_{B_k}(y) = t_0, \ldots, 1_{A_{k+i}}(x) = s_i, 1_{B_{k+i}}(y) = t_i \} 
\to \int \int P_{s_0, t_0}(x, y) \cdots P_{s_i, t_i}(x, y) \, dx \, dy.
\]

(Here \(\text{mes}_2\) stands for the two-dimensional Lebesgue measure.) Summation over \(t_0, \ldots, t_i\) gives

\[
\frac{1}{n} \sum_{k=1}^{n} \text{mes} \{x : 1_{A_k}(x) = s_0, \ldots, 1_{A_{k+i}}(x) = s_i \} \to \int P_{s_0}(x) \cdots P_{s_i}(x) \, dx
\]

where \(P_{s,*} = P_{s,0} + P_{s,1}\); similarly,

\[
\frac{1}{n} \sum_{k=1}^{n} \text{mes} \{y : 1_{B_k}(y) = t_0, \ldots, 1_{B_{k+i}}(y) = t_i \} \to \int P_{*,t_0}(y) \cdots P_{*,t_i}(y) \, dy.
\]
Applying Ramsey’s theorem (before Aldous’ theorem) we ensure existence of limits
\[
\lim_{k \to \infty} \text{mes}\{x : 1_{A_k}(x) = s_0, \ldots, 1_{A_{k+i}}(x) = s_i\},
\]
\[
\lim_{k \to \infty} \text{mes}\{y : 1_{B_k}(y) = t_0, \ldots, 1_{B_{k+i}}(y) = t_i\}.
\]

Taking into account that
\[
\text{mes}_2\{(x, y) : 1_{A_k}(x) = s_0, 1_{B_k}(y) = t_0, \ldots, 1_{A_{k+i}}(x) = s_i, 1_{B_{k+i}}(y) = t_i\} = \text{mes}\{x : 1_{A_k}(x) = s_0, \ldots, 1_{A_{k+i}}(x) = s_i\} \text{ mes}\{y : 1_{B_k}(y) = t_0, \ldots, 1_{B_{k+i}}(y) = t_i\}
\]
we get
\[
\int \int P_{s_0, t_0}(x, y) \ldots P_{s_i, t_i}(x, y) \, dx \, dy = \left( \int P_{s_0, *}(x) \ldots P_{s_i, *}(x) \, dx \right) \left( \int P_{*, t_0}(y) \ldots P_{*, t_i}(y) \, dy \right)
\]
for all \(i\) and \(s_0, t_0, \ldots, s_i, t_i\). It means that
\[
\int \int P_{00}^\alpha(x, y)P_{01}^\beta(x, y)P_{10}^\gamma(x, y)P_{11}^\delta(x, y) \, dx \, dy = \left( \int P_{00}^\alpha(x)P_{10}^\gamma(x) \, dx \right) \left( \int P_{00}^\beta(y)P_{11}^\delta(y) \, dy \right)
\]
for all \(\alpha, \beta, \gamma, \delta \in \{0, 1, 2, \ldots\}\). We see that all moments of (the joint distribution of) \(P_{00}, P_{01}, P_{10}, P_{11}\) are equal to moments of \(P_0, P_0, P_0, P_1, P_1, P_1, P_0, P_1, P_1, P_1\). Thus, the distribution is the same. Therefore \(P_{s, t}(x, y) = P_{s, *}(x)P_{*, t}(y)\) almost everywhere for \(s, t \in \{0, 1\}\). In particular, \(P_{11} = (P_{10} + P_{11})(P_{01} + P_{11})\), in other words, \(h = f g\).

Recall that \(\limsup_n C_n = \cap_n (C_n \cup C_{n+1} \cup \ldots)\) for arbitrary sets \(C_1, C_2, \ldots\).

2.4 Proposition. For every measurable sets \(A_1, B_1, A_2, B_2, \ldots \subset (0, 1)\) there exist measurable sets \(A, B \subset (0, 1)\) such that
\[
\text{mes} A \geq \liminf_n \text{mes} A_n, \quad \text{mes} B \geq \liminf_n \text{mes} B_n,
\]
\[
A \times B \subset \limsup_n (A_n \times B_n).
\]
Proof. Taking into account the general relation \( \limsup_k C_{n_k} \subset \limsup_n C_n \) we may replace the given sequence of pairs \((A_n, B_n)\) with any subsequence \((A_{n_k}, B_{n_k})\) such that the limits

\[
a = \lim_k \text{mes } A_{n_k}, \quad b = \lim_k \text{mes } B_{n_k}
\]
exist. We forget the original sequence and rename the subsequence into \((A_n, B_n)\). By Lemma 2.3 we may assume that the limits

\[
f(x) = \lim_{n \to \infty} \frac{\# \{ k \leq n : x \in A_k \}}{n}, \quad g(y) = \lim_{n \to \infty} \frac{\# \{ k \leq n : y \in B_k \}}{n},
\]

\[
h(x, y) = \lim_{n \to \infty} \frac{\# \{ k \leq n : x \in A_k, y \in B_k \}}{n}
\]
exist almost everywhere, and \(h(x, y) = f(x)g(y)\) almost everywhere. Note that

\[
\int f(x) \, dx = \lim_n \frac{1}{n} (\text{mes } A_1 + \cdots + \text{mes } A_n) = a,
\]

and similarly \(\int g(y) \, dy = b\).

We have

\[
\limsup_n (A_n \times B_n) \supset \{(x, y) : h(x, y) > 0\}
\]
just because a set of nonzero frequency cannot be finite. Taking

\[
A = \{x : f(x) > 0\}, \quad B = \{y : g(y) > 0\},
\]
we get

\[
\limsup_n (A_n \times B_n) \supset A \times B
\]

since \(h(x, y) = f(x)g(y) > 0\) for \((x, y) \in A \times B\). It remains to check that \(\text{mes } A \geq a\) and \(\text{mes } B \geq b\), which is easy:

\[
a = \int f(x) \, dx = \int_A f(x) \, dx \leq \text{mes } A
\]

and similarly for \(B\). \(\square\)

Proof of Prop. 2.2. We choose \(U_n, V_n\) such that \((U_n \times (0, 1)) \cup ((0, 1) \times V_n) \supset W_n\) and \(\text{mes}(U_n) + \text{mes}(V_n) \to \lim_n \beta(W_n)\). Taking a subsequence (which does not change \(\lim_n \beta(W_n)\)) we ensure existence of the limits \(\lim_n \text{mes}(U_n), \lim_n \text{mes}(V_n)\). The complements \(A_n = (0, 1) \setminus U_n, B_n = (0, 1) \setminus V_n\) satisfy

\[
(A_n \times B_n) \cap W_n = \emptyset,
\]
\[
\text{mes}(A_n) \to a, \quad \text{mes}(B_n) \to b \quad \text{as } n \to \infty,
\]
\[
(1 - a) + (1 - b) = \lim_n \beta(W_n).
\]
Prop. 2.4 gives us $A, B$ such that $\text{mes} A \geq a$, $\text{mes} B \geq b$ and $A \times B \subset \lim \sup_n (A_n \times B_n)$. Taking the complements $U = (0, 1) \setminus A$, $V = (0, 1) \setminus B$ we get $\text{mes} U + \text{mes} V \leq \lim_n \beta(W_n)$ and

$$(U \times (0, 1)) \cup ((0, 1) \times V) \supset \lim \inf_n (U_n \times (0, 1)) \cup ((0, 1) \times V_n) \supset \lim \inf_n W_n = W,$$

therefore $\beta(W) \leq \lim_n \beta(W_n)$. The converse inequality is trivial. \qed

The proof gives us the following by-product.

2.5 Corollary. The infimum in the definition of $\beta(W)$ is always reached.

(The special case $\beta(W) = 0$ is much simpler, see [2, 5.7].)

I do not know whether the supremum in the definition of $\alpha(W)$ is reached or not in general. However it is reached in the special case stated below (only this case is used in the next section).

2.6 Proposition. Let $W \subset (0, 1) \times (0, 1)$ be a set of class $A_{\delta\sigma}$ such that $W \cap (A \times B) \neq \emptyset$ for all measurable sets $A, B \subset (0, 1)$ of positive measure. Then there exists $m \in M$ such that $m(W) = 1$.

Proof. If $W \subset (U \times (0, 1)) \cup ((0, 1) \times V)$ then $\text{mes} U = 1$ or $\text{mes} V = 1$; therefore $\beta(W) = 1$. By Prop. 2.2 $\alpha(W) = \beta(W) = 1$. It remains to prove that the supremum in the definition of $\alpha(W)$ is reached.

As was noted after Prop. 2.2, its true generality is not restricted to $(0, 1)$ with Lebesgue measure. Especially, if $\mu, \nu$ are absolutely continuous probability measures on $(0, 1)$ then clearly $W \cap (A \times B) \neq \emptyset$ whenever $\mu(A) > 0$, $\nu(B) > 0$; a generalization of Prop. 2.2 gives $\alpha_{\mu, \nu}(W) = 1$ where $\alpha_{\mu, \nu}(W) = \sup \{m(W) : m \in M_{\mu, \nu}\}$ and $M_{\mu, \nu}$ is the set of all joinings between $\mu$ and $\nu$.

The set $M_W = \{m \in M : m((0, 1) \setminus W) = 0\}$ contains the limit of each increasing sequence of elements of $M_W$. It follows easily that $M_W$ contains a maximal element $m$ (at least one). It remains to prove that $m$ is a probability measure. Assume the contrary: $m((0, 1) \times (0, 1)) < 1$. Consider the marginals $m_1, m_2$ of $m$ and the measures $\mu = \text{mes} - m_1$, $\nu = \text{mes} - m_2$ (where ‘mes’ is the Lebesgue measure on $(0, 1)$); $\mu$ and $\nu$ are positive measures on $(0, 1)$, absolutely continuous, and $\mu((0, 1)) = \nu((0, 1)) = 0$. According to the previous paragraph (applied to normalized $\mu, \nu$) there exists a positive measure $n$ on $W$ whose marginals $n_1, n_2$ satisfy $n_1 \leq \mu$, $n_2 \leq \nu$ and such that $n(W)$ is close to $\mu((0, 1))$; however, we only need to know that $n(W) > 0$. The measure $m + n$ belongs to $M$, which contradicts to the maximality of $m$. \qed
3 Selectors and independence

3.1 Definition. Let \( X : \Omega \to \text{DCS}(0, 1) \) be a random dense countable subset of \((0, 1)\). A selector (of \( X \)) is a random variable \( Z : \Omega \to (0, 1) \) such that
\[
Z(\omega) \in X(\omega) \quad \text{for almost all } \omega.
\]

In terms of a chosen measurable enumeration \( X = \{Y_1, Y_2, \ldots \} \), the general form of a selector is
\[
Z(\omega) = Y_{n(\omega)}(\omega) \quad \text{for almost all } \omega,
\]
where \( n \) runs over all measurable maps \( \Omega \to \{1, 2, \ldots \} \).

Sometimes dependence between two random variables reduces to a joint density (w.r.t. their marginal distributions). Here are two formulation in general terms.

3.2 Lemma. Let \((\Omega, \mathcal{F}, P)\) be a probability space and \( C \subset \Omega \) a measurable set. The following two conditions on a pair of sub-\( \sigma \)-fields \( \mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F} \) are equivalent:

(a) there exists a measurable function \( f : \Omega \times \Omega \to [0, \infty) \) such that
\[
P(A \cap B \cap C) = \int_{A \times B} f(\omega_1, \omega_2) P(d\omega_1) P(d\omega_2) \quad \text{for all } A \in \mathcal{F}_1, B \in \mathcal{F}_2;
\]

(b) there exists a measurable function \( g : C \times C \to [0, \infty) \) such that
\[
P(A \cap B \cap C) = \int_{(A \cap C) \times (B \cap C)} g(\omega_1, \omega_2) P(d\omega_1) P(d\omega_2) \quad \text{for all } A \in \mathcal{F}_1, B \in \mathcal{F}_2.
\]

(Note that \( f, g \) may vanish somewhere.)

Proof. \( (b) \implies (a) \): just take \( f(\omega_1, \omega_2) = g(\omega_1, \omega_2) \) for \( \omega_1, \omega_2 \in C \) and 0 otherwise.

\( (a) \implies (b) \): we consider conditional probabilities \( h_1 = \mathbb{P}(C | \mathcal{F}_1), h_2 = \mathbb{P}(C | \mathcal{F}_2) \), note that \( h_1(\omega) > 0, h_2(\omega) > 0 \) for almost all \( \omega \in C \) and define
\[
g(\omega_1, \omega_2) = \frac{f(\omega_1, \omega_2)}{h_1(\omega_1)h_2(\omega_2)} \quad \text{for } \omega_1, \omega_2 \in C.
\]

Then
\[
\int_{(A \cap C) \times (B \cap C)} g(\omega_1, \omega_2) P(d\omega_1) P(d\omega_2) =
\]
\[
= \int_{A \times B} \frac{f(\omega_1, \omega_2) 1_C(\omega_1) 1_C(\omega_2)}{h_1(\omega_1)h_2(\omega_2)} P(d\omega_1) P(d\omega_2).
\]
(The integrand is treated as 0 outside $C \times C$.) Assuming that $f$ is ($F_1 \otimes F_2$)-measurable (otherwise $f$ may be replaced with its conditional expectation) we see that the conditional expectation of the integrand, given $F_1 \otimes F_2$, is equal to $f(\omega_1, \omega_2)$. Thus, the integral is

$$\cdots = \int_{A \times B} f(\omega_1, \omega_2) P(d\omega_1)P(d\omega_2) = P(A \cap B \cap C).$$

3.3 Definition. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $C \subset \Omega$ a measurable set. Two sub-$\sigma$-fields $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F}$ are a nonsingular pair within $C$, if they satisfy the equivalent conditions of Lemma 3.2.

3.4 Lemma. (a) Let $C_1 \subset C_2$. If $\mathcal{F}_1, \mathcal{F}_2$ are a nonsingular pair within $C_2$ then they are a nonsingular pair within $C_1$.

(b) Let $C_1, C_2, \ldots$ be pairwise disjoint and $C = C_1 \cup C_2 \cup \ldots$. If $\mathcal{F}_1, \mathcal{F}_2$ are a nonsingular pair within $C_k$ for each $k$ then they are a nonsingular pair within $C$.

(c) Let $\mathcal{E}_1 \subset \mathcal{F}$ be another sub-$\sigma$-field such that $\mathcal{E}_1 \subset \mathcal{F}_1$ within $C$ in the sense that

$$\forall E \in \mathcal{E}_1 \exists A \in \mathcal{F}_1 (A \cap C = E \cap C).$$

If $\mathcal{F}_1, \mathcal{F}_2$ are a nonsingular pair within $C$ then $\mathcal{E}_1, \mathcal{F}_2$ are a nonsingular pair within $C$.

Proof. (a) We define two measures $\mu_1, \mu_2$ on $(\Omega, \mathcal{F}_1) \times (\Omega, \mathcal{F}_2)$ by $\mu_k(Z) = P(C_k \cap \{ \omega : (\omega, \omega) \in Z \})$ for $k = 1, 2$. Clearly, $\mu_k(A \times B) = P(A \cap B \cap C_k)$. Condition 3.2(a) for $C_k$ means absolute continuity of $\mu_k$ (w.r.t. $P \times P$). However, $\mu_1 \leq \mu_2$.

(b) Using the first definition, 3.2(a), we just take $f = f_1 + f_2 + \ldots$

(c) Immediate, provided that the second definition us used, 3.2(b). \qed

3.5 Proposition. Let $X$ be a random dense countable subset of $(0, 1)$ such that two random sets $X \cap (0, \frac{1}{2})$, $X \cap (\frac{1}{2}, 1)$ are independent. Let $Z_1, Z_2, \ldots$ be a measurable enumeration of $X \cap (\frac{1}{2}, 1)$ independent of some measurable enumeration of $X \cap (0, \frac{1}{2})$. Let $Y_1, \ldots, Y_n$ be selectors of $X$. Then the sub-$\sigma$-field $\mathcal{F}_1$ generated by $Y_1, \ldots, Y_n$ and the sub-$\sigma$-field $\mathcal{F}_2$ generated by $Z_1, Z_2, \ldots$ are a nonsingular pair within the event

$$C = \{ Y_1, \ldots, Y_n < \frac{1}{2} \}.$$  

The proof is given below after some discussion. The condition imposed on $X$ is the relevant part of the independence condition (recall Def. 1.2).
The threshold is chosen at $\frac{1}{2}$, but any other number of $(0, 1)$ could be used equally well.

It may happen that $P(C) = 1$; even in this case $F_1, F_2$ need not be independent (see Counterexample 3.6). Of course, there exist an enumeration $X \cap (0, \frac{1}{2}) = \{U_1, U_2, \ldots\}$ independent of $(Z_k)_k$ and, for instance, $Y_1(\omega) = U_{n(\omega)}(\omega)$. However, $n(\cdot)$ need not be independent of $Z_1, Z_2, \ldots$

Note the finite sequence $(Y_k)_k$ but the infinite sequence $(Z_k)_k$. The proposition may fail for an infinite sequence $(Y_k)_k$ (see Counterexample 3.6). Note also that $Y_k$ need not be pairwise different.

3.6 Counterexample. We start with independent random variables $U_k, V_k, Z_k$ ($k = 1, 2, \ldots$) such that each $U_k$ is distributed uniformly on $(0, \frac{1}{4})$, each $V_k$ on $(\frac{1}{4}, \frac{1}{2})$, and each $Z_k$ on $(\frac{1}{2}, 1)$. We construct events $A_1, A_2, \ldots$ that generate the sub-$\sigma$-field $F_2$ generated by $Z_1, Z_2, \ldots$ Now we define random variables $Y_1, Y_2, \ldots$ as follows:

$$Y_k = \begin{cases} U_k & \text{on } A_k, \\ V_k & \text{outside } A_k. \end{cases}$$

Clearly, each $Y_k$ is a selector of $X = \{U_1, U_2, \ldots\} \cup \{V_1, V_2, \ldots\} \cup \{Z_1, Z_2, \ldots\}$, and $A_k = \{Y_k < \frac{1}{4}\}$. The sub-$\sigma$-field $F_1$ generated by $Y_1, Y_2, \ldots$ contains $A_1, A_2, \ldots$, therefore $F_1 \supset F_2$.

Clearly, $F_1$ and $F_2$ are not a nonsingular pair.

Proof of Prop. 3.5. I assume that $n = 1$ (thus, $Y = Y_1$), leaving to the reader the straightforward generalization. We choose a measurable enumeration $U_1, U_2, \ldots$ of $X \cap (0, \frac{1}{2})$ independent of $Z_1, Z_2, \ldots$ and partition $C$ into $C_k = \{Y = U_k\}$. By 3.4 (b) it is sufficient to prove that $F_1, F_2$ are a nonsingular pair within each $C_k$. By 3.4 (c) we replace $F_1$ with the $\sigma$-field $\sigma(U_k)$ generated by $U_k$. By 3.4 (a) we replace $C_k$ with the whole $\Omega$. The $\sigma$-fields $\sigma(U_k), F_2$ are a nonsingular pair within $\Omega$, since they are independent.

4 Existence of selectors

In order to allow for some additional randomization, in this section we often construct a selector not on the original probability space $\Omega$ but on some extended space $\tilde{\Omega}$. In fact, the product space $\Omega \times \Omega'$ (with the product measure), where $\Omega'$ is a nonatomic probability space, may serve as $\tilde{\Omega}$. Naturally, the given random dense countable set $X$ is transferred to $\tilde{\Omega}$ by $X(\omega, \omega') = X(\omega)$.
4.1 Lemma. The following two conditions on a random dense countable set
$X : \Omega \rightarrow \text{DCS}(0, 1)$ are equivalent:

(a) there exists a selector $Z : \tilde{\Omega} \rightarrow (0, 1)$ of $X$ (on some extension $\tilde{\Omega}$ of
$\Omega$), distributed uniformly on $(0, 1)$;

(b) there exists a probability measure on the set

(4.2) $W = \{ (\omega, t) : t \in X(\omega) \} \subset \Omega \times (0, 1)$

whose two marginals are $P$ and the uniform distribution on $(0, 1)$.

Proof. (a) $\implies$ (b): The joint distribution of $\omega$ (treated as a function of
$\tilde{\omega} \in \tilde{\Omega}$) and $Z$ is the needed measure on $W$.

(b) $\implies$ (a): We take $\tilde{\Omega} = \Omega \times \Omega'$, disintegrate the given measure $m$
on $W$ into conditional measures $m_\omega$ on $(0, 1)$, represent $m_\omega$ as the distribution
of some $Z_\omega : \Omega' \rightarrow (0, 1)$ and combine these $Z_\omega$ by $Z(\omega, \omega') = Z_\omega(\omega')$.
Measurability of $Z$ can be achieved by choosing $\Omega'$ to be $(0, 1)$ (with Lebesgue
measure) and each $Z_\omega$ to be an increasing function $(0, 1) \rightarrow (0, 1)$. \(\square\)

The set $W$ defined by (4.2) should be treated modulo sets of the form
$A \times (0, 1)$, $P(A) = 0$ (and their subsets). Then $W$ appears to belong to the
class $A_{\delta\sigma}$ (introduced before Lemma 2.3), as stated below.

4.3 Lemma. Let $X : \Omega \rightarrow \text{DCS}(0, 1)$ be a random dense countable set.
Then there exists a subset $\Omega_1 \subset \Omega$ of probability 1 such that the set

$W_1 = \{ (\omega, t) : \omega \in \Omega_1, t \in X(\omega) \}$

belongs to the class $A_{\delta\sigma}$.

Proof. We take a measurable enumeration $X = \{ Y_1, Y_2, \ldots \}$ and note that
$W = W_1 \cup W_2 \cup \ldots$ where $W_k = \{ (\omega, Y_k(\omega)) : \omega \in \Omega \}$ is the graph of
$Y_k$. We choose a topology on $\Omega$ turning $\Omega$ into a compact metrizable space
(such that the given probability measure on $\Omega$ is a Borel measure, up to
negligible sets). By Lusin’s theorem there exist compact sets $C_n \subset \Omega$ such
that the restrictions $Y_k|_{C_n}$ are continuous and the set $\Omega_1 = C_1 \cup C_2 \cup \ldots$ is
of probability 1. The compact sets $W_{k,n} = \{ (\omega, Y_k(\omega)) : \omega \in C_n \}$ belong to
the class $A_\delta$, therefore their union $W_1$ belongs to $A_{\delta\sigma}$. \(\square\)

4.4 Proposition. Let $X : \Omega \rightarrow \text{DCS}(0, 1)$ satisfy (1.7). Then $X$ has a
selector $Z : \tilde{\Omega} \rightarrow (0, 1)$ (on some extension $\tilde{\Omega}$ of $\Omega$), distributed uniformly on
$(0, 1)$. 
Proof. Lemma 4.3 gives us a set $W_1 \subset W$ of class $\mathcal{A}_{\delta \sigma}$; (1.7) shows that $W_1$ intersects every $A \times B$ where $A \subset \Omega$, $P(A) > 0$ and $B \subset (0, 1)$, mes $B > 0$. Proposition 2.6 (or rather, its evident generalization) gives us a measure on $W_1$ whose marginals are $P$ and mes. Lemma 4.1 ((b) $\implies$ (a)) completes the proof.

Only the second part of (1.7) was used.

4.5 Lemma. Let $X$ be a random dense countable subset of $(0, 1)$. If for every $\varepsilon > 0$ the random dense countable subset $X \cap (\varepsilon, 1)$ of $(\varepsilon, 1)$ satisfies (1.7), then $X$ satisfies (1.7).

The (straightforward) proof is left to the reader. More generally, one may check $X \cap (0, a \varepsilon)$ and $X \cap (\varepsilon, 1)$, etc.

4.6 Proposition. Let $X$ be a random dense countable subset of $(0, 1)$, satisfying (1.7) and the independence condition. Let $Y_1, \ldots, Y_n$ be selectors of $X$. Then there exists a selector $Z$ (on some extension of the given probability space) independent of $Y_1, \ldots, Y_n$ and distributed uniformly on $(0, 1)$.

Proof. I assume that $n = 1$ (thus, $Y = Y_1$), leaving to the reader the straightforward generalization. We disintegrate the given measure $P$ on the given probability space $(\Omega, \mathcal{F}, P)$ into conditional (given $Y = y$) measures $P_y$ on $\Omega$ for $y \in (0, 1)$; $P = \int P_y \mu(dy)$, where $\mu$ is the distribution of $Y$.

Being considered w.r.t. $P_y$, the $\mathcal{F}$-measurable map $X : \Omega \to DCS(0, 1)$ is another random dense countable set; denote it $X_y$. We claim that

\begin{equation}
(4.7) \quad X_y \text{ satisfies (1.7) for } \mu\text{-almost every } y \in (0, 1).
\end{equation}

By (generalized) Lemma 4.3 it is enough to check (1.7) for $X_y \cap (0, a)$ for $\mu$-almost all $y \in (a, 1)$ as well as $X_y \cap (a, 1)$, $y \in (0, a)$; here $a$ runs over $(0, 1)$ (or only its rational points). Proposition 3.5 (generalized a bit) shows that, roughly speaking, the distribution of $X_y \cap (a, 1)$ is absolutely continuous w.r.t. the distribution of $X \cap (a, 1)$, as far as $y \in (0, a)$. Thus, (1.7) for $X \cap (a, 1)$ implies (1.7) for $X_y \cap (a, 1)$, and (4.7) is verified.

Proposition 4.4 gives us selectors $Z_y : \hat{\Omega} \to (0, 1)$ of $X$ such that $Z_y$ is uniformly distributed on $(\Omega, P_y)$. We want to glue them together,

\[ Z(\omega) = Z_{Y(\omega)}(\omega); \]

to this end, however, we need $\mu$-measurability of $Z_y$ in $y$, which is the goal below.

Similarly to the proof of Lemma 4.3 we take a measurable enumeration $X = \{U_1, U_2, \ldots, \}$, turn $\Omega$ into a compact metrizable space and construct
compact sets $C_n \subset \Omega$ such that the restrictions $U_k|_{C_n}$ are continuous and the set $\Omega_1 = C_1 \cup C_2 \cup \ldots$ satisfies $P(\Omega_1) = 1$. Then $P_y(\Omega_1) = 1$ for $\mu$-almost all $y$ (since $P = \int P_y \mu(dy)$). Once again, the $A_{\delta\sigma}$-set $W_1 = \{ (\omega, t) : \omega \in \Omega_1, t \in X(\omega) \}$ is the union of compact sets $W_{k,n} = \{ (\omega, U_k(\omega)) : \omega \in C_n \}$. The set $M$ of all probability measures on $W_1$ is a standard Borel space. The same holds for measures on $\Omega$ and on $(0,1)$. Denote the two marginals of a measure $m \in M$ by $\varphi(m)$ and $\psi(m)$; $\varphi, \psi$ are Borel measurable maps. We know that the set $M_y = \{ m \in M : \varphi(m) = P_y, \psi(m) = \text{mes} \}$ is nonempty for $\mu$-almost all $y$. (Indeed, the selectors $Z_y$ are constructed in the proof of Prop. 4.4 via measures that belong to $M_y$.) Taking into account that $P_y$ is (or rather, may be chosen to be) a Borel measurable function of $y$, we apply well-known uniformization theorems and get a $\mu$-measurable map $y \mapsto m_y$ such that $m_y \in M_y$ for $\mu$-almost all $y$. Now $\mu$-measurability of the map $y \mapsto Z_y$ can be achieved (recall the end of the proof of Lemma 4.1).

5 Proving the main results

Proof of Theorem 1.8. Let $X$ be a random dense countable subset of $(0,1)$ satisfying (1.7) and the independence condition. In order to prove that $X$ is distributed like the unordered infinite sample it is sufficient to find a measurable enumeration $X = \{ Y_1, Y_2, \ldots \}$ that satisfies the conditions of the main lemma (Lemma 2.1) of [2]. Namely, the conditional distribution of $Y_n$ given $Y_1, \ldots, Y_{n-1}$ must have a density $(t,\omega) \mapsto f_n(t,\omega)$ and the series $\sum_{n=1}^{\infty} f_n(t,\omega)$ must diverge for almost all pairs $(t,\omega)$.

Existence of conditional densities is ensured by (4.7) (generalized for $n \geq 1$) for every measurable enumeration.

Starting with an arbitrary measurable enumeration $(Z_k)_k$ we construct the needed enumeration $(Y_k)_k$ as follows. First, $Y_1 = Z_1$. Second, Prop. 4.6 gives us $Y_2$ independent of $Y_1$ and distributed uniformly on $(0,1)$. (The probability space is extended as needed.) Third, $Y_3 = Z_2$ unless $Y_2 = Z_2$, in which case $Y_3 = Z_3$. Prop. 4.6 gives us $Y_4$ independent of $Y_1, Y_2, Y_3$ and uniform. And so on; $Y_{2n}$ is independent of $Y_1, \ldots, Y_{2n-1}$ and uniform, while $Y_{2n+1}$ is the first $Z_k$ different from $Y_1, \ldots, Y_{2n}$. Clearly, $(Y_k)_k$ is a measurable enumeration of $X$, and $\sum f_n = \infty$ since $f_{2n}(t,\omega) = 1$.

It remains to prove Proposition 1.10. Recall the cyclic shift $T_s$ (introduced after 1.8).
5.1 Lemma. Let $A \subset (0, 1)$ be a measurable set, $\text{mes} \ A > 0$, and $L \subset (0, 1)$ a dense countable set (not random). Then the set $A \cap T_s(L)$ is infinite for almost all $s \in (0, 1)$.

Proof. First, the set $B = \{s : A \cap T_s(L) \neq \emptyset\} = \bigcup_{l \in L} T_l^{-1}(A)$ is of full measure, since $\frac{1}{\varepsilon} \text{mes}(A \cap (a, a + \varepsilon)) \leq \frac{1}{\varepsilon} \text{mes}(B \cap (b, b + \varepsilon))$ for all $\varepsilon > 0$ and $a, b \in (0, 1 - \varepsilon)$.

Second, $L \supset L_1 \cup L_2 \cup \ldots$ for some pairwise disjoint dense countable sets $L_1, L_2, \ldots$. Almost every $s$ satisfies $A \cap T_s(L_n) \neq \emptyset$ for all $n$. \hfill \Box

Proof of Proposition 1.10. Condition (1.7) is satisfied by the given $X : \Omega \rightarrow \text{DCS}(0, 1)$ if (and only if) it is satisfied by $Y : \Omega \times (0, 1) \rightarrow \text{DCS}(0, 1)$ used in 1.9 $Y(\omega, s) = T_s(X(\omega))$ (just because $X$ and $Y$ are identically distributed). If $\text{mes} A = 0$ then $A \cap Y(\omega, s) = \emptyset$ for all $s$ except for the negligible set $\bigcup_{l \in X(\omega)} T_l^{-1}(A)$. If $\text{mes} A > 0$ then $A \cap Y(\omega, s)$ is infinite for almost all $s$ by Lemma 5.1 and it holds for almost all $\omega$. \hfill \Box

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