Continuous approximations of a class of piece-wise continuous systems

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Abstract

In this paper we provide a rigorous mathematical foundation for continuous approximations of a class of piece-wise continuous functions. By using techniques from the theory of differential inclusions, the underlying piece-wise functions can be locally or globally approximated. The approximation results can be used to model piece-wise continuous dynamical systems of integer or fractional-order. In this way, by overcoming the lack of numerical methods for differential equations of fractional-order with discontinuous right-hand side, unattainable procedures for systems modeled by this kind of equations, such as chaos control, synchronization, anticontrol and many other, can be easily implemented. Several examples are presented and three comparative applications are studied.

Keywords: piece-wise continuous function, fractional-order system, differential inclusion, approximate selection, sigmoid function

1 Introduction

Despite the 17th century doubt regarding the practical utility of fractional derivative and despite fractional derivatives have no a clear geometrical interpretation[1], there are nowadays a lot of works on systems of fractional-order and their related applications in many domains, such as physics, engineering, mathematics, finance, chemistry, and so on (see for example, the books [2, 3] or the papers of Caputo [4]).

On the other side, discontinuous systems can be found in two-dimensional mechanical systems such as systems with dry friction, oscillating systems with combined dry and viscous damping, forced vibrations, brake processes with locking phase, control synthesis of uncertain systems, control theory, calculus of variations, systems with stick and slip modes, braking processes with locking phases, PDEs, elastoplasticity, but also in games theory, optimization, biological and physiological systems, electrical (chaotic) circuits, networks, power electronics etc (see e.g. [5, 6] or [7] and the references therein).

Therefore, dynamical systems of fractional-order, modeled with piece-wise continuous functions, represent a logical consequence, since among the scientists, the idea that the real systems follow behaviors modeled better with fractional-order equations than of integer order, gaining more and more interest.

Even there are numerical methods for fractional DE (see e.g. [8, 9, 10]) and also for DE with discontinuous right-hand side (see e.g. [11, 12, 13, 14]), unfortunately, on our knowledge, there are not numerical methods for DE of fractional-order with discontinuous right-hand side. Consequently, modeling continuously or smoothly the underlying systems, could be of a real interest for example in chaos control, synchronization, anticontrol and so on, but also for quantitative analysis.
The class of the piece-wise continuous functions \( f : \mathbb{R}^n \to \mathbb{R}^n \) defining these systems, and which will be continuously approximated, has the following form

\[
f(x(t)) = g(x(t)) + A(x(t))s(x(t)),
\]
with \( g : \mathbb{R}^n \to \mathbb{R}^n \) a vector single-valued, at least continuous function, and \( s : \mathbb{R}^n \to \mathbb{R}^n \), \( s(x) = (s_1(x_1), s_2(x_2), \ldots, s_n(x_n))^T \) a vector valued piece-wise function, with \( s_i : \mathbb{R} \to \mathbb{R} \), \( i = 1, 2, \ldots, n \) real piece-wise constant functions. \( A_{n \times n} \) is a square matrix of real functions.

The following assumption will be considered

\([H1]\) As is discontinuous in at least one of its components

The smoothness of the functions \( x^m \text{sgn}(x) \), for different values of \( m \) has been discussed in [15].

This form of \( f \), appears in the great majority of nonlinear piece-wise continuous systems of fractional or integer order, being modeled by the following Initial Value Problem (IVP)

\[
D_q^t x(t) = f(x(t)), \quad x(0) = x_0, \quad t \in I = [0, \infty).
\]

Here, with \( D_q^t \), \( 0 < q \leq 1 \) \((q = 1\) for the integer order), is denoted the operator commonly used in fractional calculus: Caputo’s differential operator of order \( q \) (called also smooth fractional derivative) with starting point 0

\[
D_q^t x(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-\tau)^{-q} \frac{d}{dt} x(\tau) d\tau,
\]
\( \Gamma \) being the Euler’s Gamma function

\[
\Gamma(z) = \int_0^1 t^{z-1} e^{-t} dt, \quad z \in \mathbb{C}, \quad \text{Re}(z) > 0.
\]

To overcome the problem of numerical integration of systems modeled by [2], the discontinuous problem will be transformed into a continuous one. For this purpose Filippov’s approach [17] will be used as well some basic results from the theory of fractional differential inclusions [18, 19].

The obtained approximation results, target the piece-wise constant functions \( s \), results which are valid for a large class of functions, such as Heaviside function \( H \), rectangular function (as difference of two Heaviside functions), or signum, one of the most encountered in practical applications.

The null set of the discontinuity points of \( f \), \( \mathcal{M} \) (with zero Lebesgue measure [1]), is generated by the discontinuity points of the components \( s_i \).

Because the systems modeled by the IVP [2] are autonomous, hereafter, until mentioned, we drop the time variable.

Let next consider few examples.

The piece-wise linear one-dimensional function \( f : \mathbb{R} \to \mathbb{R} \)

\[
f(x) = 2 - 3 \text{sgn}(x),
\]
has \( \mathcal{M} = \{0\} \) and the graph in Fig. [1]

Generally, the discontinuous dynamical systems (of integer or fractional-order) can be found in the space \( \mathbb{R}^n \) for \( n \geq 2 \), such as the following fractional variant of the simple two-dimensional system which models a unit mass which is subject of a discontinuous spring force [6]

\[
D_q^t x_1 = x_2, \quad D_q^t x_2 = -\text{sgn}(x_1),
\]

where

\[
g(x) = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad s(x) = \begin{pmatrix} \text{sgn}(x_1) \\ \text{sgn}(x_2) \end{pmatrix}.
\]

In this case \( \mathcal{M} = \{(0, x_2), x_2 \in \mathbb{R} \} \).

\[\text{Legesgue measure of a point on the real line, as well of a line in } \mathbb{R}^2, \text{ or of a plane in } \mathbb{R}^3 \text{ is zero: } \mu(\mathcal{M}) = 0.\]
Another typical example of a mechanical system, which models a friction oscillator [5] and which can model wings of insects [20], is governed by the following equation

\[ \dot{x} + \lambda \dot{x} + x^3 + \varphi(x, \dot{x}) \text{sgn}(\dot{x}) = 0, \]

where \( \varphi \) is some function, \( \lambda \) is the bifurcation parameter and

\[ g(x) = \begin{pmatrix} x_2 \\ -\lambda x_1 - x_1^3 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ 0 & -\varphi(x_1, x_2) \end{pmatrix}. \]

The following three-dimensional system is a fractional variant of the discontinuous Chua system [21]

\[
\begin{align*}
D^q_{x_1} &= -2.571x_1 + 9x_2 + 3.857\text{sgn}(x_1), \\
D^q_{x_2} &= x_1 - x_2 + x_3, \\
D^q_{x_3} &= -px_3,
\end{align*}
\]

(4)

(5)

where \( p \in \mathbb{R} \) being the bifurcation parameter (see the graph of the piece-wise continuous component in Fig. 2(a)).

The paper is organize as follows: In Section 2 the approximation of \( f \) defined by (4) is presented, while in the Section 3 three applications are analyzed. The Conclusion ends the paper.

2 Approximation of \( f \)

In this section we shall see how piece-wise continuous functions \( f \) modeled by (4) can be continuously approximated. Precisely, since \( g \) is continuous, we are interested in approximating the piece-wise-constant functions \( s_i \).

For this purpose, let us consider the IVP (5) whose right-hand side will be first transformed into a set-valued function, via the Filippov regularization [17]. In this way, the single-valued initial problem is restarted as a set-valued one, namely a differential inclusion of fractional-order

\[
D^q_{t} x \in F(x), \quad x(0) = x_0, \quad \text{for a.a. } t \in I,
\]

where \( F : \mathbb{R}^n \Rightarrow \mathbb{R}^n \) is a vector set-valued function into the set of subsets of \( \mathbb{R}^n \), which can be defined in several ways (refer [22], one of few related works on fractional differential inclusions).

A simple (convex) expression of a set-valued function \( F(x) \), is obtained by the so called Filippov regularization [17, 18, 19]

\[
F(x) = \bigcap_{\varepsilon > 0} \bigcap_{\mu(M) = 0} \text{conv}(\{f(z) : |z - x| \leq \varepsilon \mu(M)\}),
\]

(6)

As can be seen, \( F(x) \) is the convex hull of \( f(x) \) (see the sketch in Fig. 3 part (a) (b)), \( \mu \) being the Lebesgue measure and \( \varepsilon \) the radius of the ball centered in \( x \). In the points where \( f \) is continuous, \( F(x) \) consists of one single point, which coincides with the value of \( f \) at this point (i.e. we get back \( f(x) \) as the right-hand side: \( F(x) = \{f(x)\} \)). In the points belonging to \( M \), \( F(x) \) is given by (6) (Fig. 3(c)).

More on the Filippov regularization and generalized solutions to discontinuous equations, can be found e.g. on the review papers [6, 23].

In order to justify the use of the Filippov regularization to some physical system, we must choose small values for \( \varepsilon \) so that the motion of the physical systems is arbitrarily close to a certain solution of the underlying differential inclusion (it tends to the solution, as \( \varepsilon \to 0 \)).

If the piece-wise-constant functions \( s_i \) are \text{sgn}, their set-valued form, obtained with Filippov regularization, and denoted usually by \( \text{Sgn} : \mathbb{R} \Rightarrow \mathbb{R} \), is defined as follows (see Fig. 4(a) before regularization and Fig. 4(b) after regularization)

\[
\text{Sgn}(x) = \begin{cases} 
-1 & x < 0, \\
[-1, 1] & x = 0, \\
+1 & x > 0.
\end{cases}
\]

(7)
By applying the Filippov regularization to $f$, one obtains the following set-valued function

$$F(x) := g(x) + A(x)S(x), \quad (8)$$

with

$$S(x) = (S_1(x_1), S_2(x_2), ..., S_n(x_n))^T, \quad (9)$$

$S_i : \mathbb{R} \rightarrow \mathbb{R}$ being the set-valued variant of $s_i$, $i = 1, 2, ..., n$ ($\text{Sgn}(x_i)$ for the usual case).

Because the set-valued character of $F$ is generated by $S_i$, which are real functions, the notions and results presented next are considered in $\mathbb{R}$, for the case $n = 1$, but they are also valid in the general case $n > 1$.

Let next consider some general set-valued function $F : \mathbb{R} \Rightarrow \mathbb{R}$.

The graph of a set-valued function $F$ is defined as follows

$$\text{Graph}(F) := \{(x, y) \in \mathbb{R} \times \mathbb{R}, y \in F(x)\}.$$  

Remark 1  
Due to the symmetric interpretation of a set-valued function as a graph (see e.g. [18]) we shall say that a set-valued function satisfies a property if and only if its graph satisfies it. For instance, a set-valued function is said to be closed if and only if its graph is closed.

Definition 1  
As set-valued function is upper semicontinuous (u.s.c.) at $x_0 \in \mathbb{R}$, if for any open $B$ containing $F(x_0)$, there exists a neighborhood $A$ of $x_0$ such that $F(A) \subseteq B$.

We say that $F$ is u.s.c. if it is so at every $x_0 \in \mathbb{R}$.  
U.s.c., which is a basic property, practically means that the graph of $F$ is closed.

Definition 2  
A single-valued function $h : \mathbb{R} \rightarrow \mathbb{R}$ is called an approximation (selection) of the set-valued function $F$ if

$$\forall x \in \mathbb{R}, \ h(x) \in F(x).$$

Generally, a set-valued function admits (infinitely) many approximations (see Fig. 4(c) for the case of $\text{Sgn}$ function). For the theory of selections for set-valued functions, compare [18, 19] or [24].

Notation  
Let $\mathcal{C}_0^\epsilon(\mathbb{R})$ the class of real continuous approximations $\tilde{s} : \mathbb{R} \rightarrow \mathbb{R}$ of the set-valued function $F$ which satisfy

(i) $\text{Graph}(\tilde{s}) \subseteq \text{Graph}(B(F, \epsilon))$.

(ii) For every $x \in \mathbb{R}$, $\tilde{s}(x)$ belongs to the convex hull of the image of $F$.

Above, $B(x, \epsilon)$ is the disk centered in $x$ and of radius $\epsilon$.

The set-valued functions $S_i$, $i = 1, 2, ..., n$, can be approximated due to the Approximate Theorem, called also Cellina’s Theorem (see [18] p. 84 and [19] p. 358), which states that a set-valued function $F$, with closed graph and convex values, admits $\mathcal{C}_0^\epsilon$ approximations.

Remark 2  
Cellina’s Theorem provides locally Lipschitzian approximations. Since locally Lipschitzian functions are also continuous, in this paper we will consider $\mathcal{C}_0^\epsilon$ approximations.

2.1 Global approximation  
The global approximation of $S_i$, $i = 1, 2, ..., n$ (defined over $\mathbb{R}$), is assured by the following lemma

Lemma 1  
For every $\epsilon > 0$, the set-valued functions $S_i$, $i = 1, 2, ..., n$ admit global $\mathcal{C}_0^\epsilon$ approximations.

Proof  
$S_i$, for $i = 1, 2, ..., n$, are convex u.s.c. (see e.g. the Remark in [17] p. 43 or Example in [19] p. 39 for u.s.c. property) and, via Remark 1 are non-empty closed valued functions. Therefore, they verifies Cellina’s Theorem which guaranties the existence of $\mathcal{C}_0^\epsilon$ approximations on $\mathbb{R}$.

Notation  
Let denote by $\tilde{s}_i : \mathbb{R} \rightarrow \mathbb{R}$ the global approximations of $S_i$. 

For the sake of simplicity, hereafter, $\varepsilon$ is considered, as having the same value for each component $\tilde{s}_i(x_i)$, $i = 1, 2, ..., n$. Also, the index $i$ will be drop for ease of notations, unless specified.

The constructive proof of the Cellina’s Theorem allows us to ease the approximations choice. Any single-valued function on $R$ with the graph in the $\varepsilon$-neighborhood, is an approximate selection of $S$ from the Celina Theorem. However, some of the best candidates for $\tilde{s}$ are the sigmoid functions which provide the required flexibility and to which the abruptness of the discontinuity can be easily modified. If $S(x) = Sgn(x)$, one of the most utilized sigmoid approximations is the following function $\tilde{sgn}$

$$\tilde{sgn}(x) = \frac{2}{1 + e^{-\frac{x}{\delta}}} - 1 \approx Sgn(x), \quad (10)$$

where $\delta$ is a positive parameter which controls the slope in the neighborhood of the discontinuity manifold $x = 0$. In Fig. 5(a), $\tilde{sgn}$ is plotted function of $\delta$, while in Fig. 5(b), it is plotted for two distinct values.

The smallest $\varepsilon$ values necessary to embed $\tilde{sgn}$ within an $\varepsilon$-neighborhood of $Sgn$ (as stated by Cellina’s Theorem), depends proportionally on $\delta$. However, finding an explicit relation for $\delta$ function of $\varepsilon$, is a difficult task. Moreover, for $x \neq 0$, $\tilde{sgn}$ is identical to the single-valued branches of $Sgn$ (the horizontal lines $\pm 1$) only asymptotically, for $x \to \pm \infty$. For example, for $\delta = 1/100$, at the point $x = 0.06$, the difference is of order of $10^{-3}$, even the two graphs look apparently identic in the underlying points $A$ or $B$ (Fig. 5(c)). To reduce the size of $\varepsilon$ to e.g. $10^{-4}$, $\delta$ should be of order of $10^{-5}$.

For the Heaviside function, which in his piece-wise constant variant can be expressed in terms of signum function by $H(x) = \frac{1}{2}[1 + sgn(x)]$, the approximate sigmoid function (10) becomes $\tilde{H}(x) = \frac{1}{1 + e^{-\frac{x}{\delta}}}$. Now, we can enounce the following result, which assures the possibility to approximate globally $f$

**Theorem 3** Let $f$ defined by (1). If $g$ is continuous, then for every $\varepsilon > 0$, there exist global approximations of $f$, $\tilde{f}$: $\mathbb{R}^n \to \mathbb{R}^n$

$$\tilde{f}(x) = g(x) + A(x)\tilde{s}(x) \approx f(x), \quad (11)$$

Theorem 3 states that systems modeled by the IVP (2), can be continuously modeled by the following IVP

$$D^\alpha q(x) = \tilde{f}(x),$$

with $\tilde{f}$ defined by (11).

For example, the function

$$f(x) = -x^2 + sgn(x - 0.5), \quad (12)$$

can be globally continuously approximated on $\mathbb{R}$ (dotted line in Fig. 6), having the approximated form $\tilde{f}$

$$\tilde{f}(x) = -x^2 + \tilde{sgn}(x - 0.5) = -x^2 + \frac{2}{1 + e^{-\frac{x-0.5}{\delta}}} - 1. \quad (13)$$

First, $f$ has been transformed into the set-valued function $F(x) = -x^2 + Sgn(x - 0.5)$ (red line in Fig. 6) which is approximated via $\tilde{sgn}$.

*Sigmoid functions include the ordinary arctangent such as $\frac{\pi}{4} \arctan \frac{x}{\varepsilon}$, the hyperbolic tangent, the error function, the logistic function, algebraic functions like $x^\sqrt{\varepsilon+x^2}$, and so on.*
2.2 Local approximation

In order to not affect significantly the physical characteristics of the underlying system, it is desirable to approximate \( S \) only on some tight \( \varepsilon \)-neighborhoods of the discontinuity \( x = 0 \), not on the entire real axis, as global approximations given by Theorem 1 do, when the difference error between \( S \) and \( \tilde{s} \) persists along the entire real axis \( \mathbb{R} \). Graphically speaking, we want to restrict the approximation of \( S \) only within a narrow vertical band of width \( \varepsilon \) centered along the vertical axis. This is allowed by the particular (convex u.s.c.) form of the set-valued functions \( S \), and by the great flexibility afforded by continuous functions which can be glued or pasted, without altering the continuity property. In this case, the global approximations (such as the sigmoid functions) are no longer useful (see Fig. 5(b) for the case of \( \text{Sgn} \)), and other kind of selections, such as the local approximations, can be used.

The following corollary, which is a simple consequence of the results of Subsection 2.1, ensures the existence of local approximations for the set-valued functions \( S \) (see the sketch in Fig. 7).

**Corollary 1** For every \( \varepsilon > 0 \), \( S \) admits locally \( C^0 \) approximations \( \tilde{s}_\varepsilon : (-\varepsilon, \varepsilon) \to \mathbb{R} \), which verify the neighborhood continuity conditions

\[
\tilde{s}_\varepsilon(\pm\varepsilon) = S(\pm\varepsilon).
\] (14)

In this case, \( \tilde{s}_\varepsilon \) can be also continuously extended on \( \mathbb{R} \), obtaining a new global approximation \( \tilde{s} \)

\[
\tilde{s}(x) = \begin{cases} \tilde{s}_\varepsilon(x), & x \in (-\varepsilon, \varepsilon), \\ S(x), & x \notin (-\varepsilon, \varepsilon). \end{cases}
\] (15)

Among the simplest functions \( \tilde{s}_\varepsilon \) which has the advantage to be directly evaluated by computers, are the cubic polynomials \( \tilde{s}_\varepsilon : \mathbb{R} \to \mathbb{R} \) (higher order does not always improve accuracy)

\[
\tilde{s}_\varepsilon(x) = ax^3 + bx^2 + cx + d, \quad a, b, c, d \in \mathbb{R}.
\] (16)

Other possible candidate are spline functions, which are constructed via piece-wise polynomials. The fact that in (14) there are four coefficients to be determined and only two conditions, means that there are an infinity of choices for \( \tilde{s}_\varepsilon \). This implies that \( \tilde{s}_\varepsilon \), given by (15), can be even smoothly extended on \( \mathbb{R} \), by imposing near the gluing conditions (14), the supplementary differentiability conditions at the boundary of the discontinuity neighborhood

\[
\frac{d}{dx} \tilde{s}_\varepsilon(\pm\varepsilon) = \frac{d}{dx} S(\pm\varepsilon).
\] (17)

For the case of \( \text{Sgn} \) function, when the gluing and smoothing conditions are \( \tilde{s}_\varepsilon(\pm\varepsilon) = \pm 1 \) and \( \frac{d}{dx} \tilde{s}_\varepsilon(\pm\varepsilon) = 0 \) respectively, the local smooth approximate function, denoted by \( \overline{\text{sgn}}_\varepsilon \), becomes

\[
\overline{\text{sgn}}_\varepsilon(x) = \frac{-1}{2\varepsilon^3}x^3 + \frac{3}{2\varepsilon^2}x \approx \text{Sgn}(x), \quad x \in (-\varepsilon, \varepsilon),
\] (18)

and using (15), on \( \mathbb{R} \) \( \text{Sgn} \) is approximated by the following piece-wise function \( \overline{\text{sgn}} \)

\[
\overline{\text{sgn}} \approx \begin{cases} \overline{\text{sgn}}_\varepsilon(x), & x \in (-\varepsilon, \varepsilon), \\ \pm 1, (\text{or } \text{sgn}(x)), & x \notin [-\varepsilon, \varepsilon]. \end{cases}
\] (19)

**Remark 3** While \( \tilde{s} \) are not useful for locally approximations (see Fig. 5(a)), \( \tilde{s}_\varepsilon \) cannot be used for globally approximations of \( S \) since it is unbounded outside the interval \((-\varepsilon, \varepsilon)\) and tends to \( \pm \infty \) as \( x \to \pm \infty \) (see Fig. 5(a)).

The cubic functions (16) have a great flexibility, being able to connect smoothly any kind of piece-wise continuous functions on some \( \varepsilon \)-neighborhood of the discontinuity. For example, by using the smoothness conditions (17), the function (12), can be smoothly approximated in some neighborhood of the point \( x = 0.5 \), with the cubic function (16) (see Fig. 8(b)) where \( \varepsilon \) is chosen 1/5 for a clear image.

Compared to the case of globally approximation, the locally approximations, \( \tilde{s}_\varepsilon \), which are determined in the neighborhood of the discontinuity points, at which they are generated, are
identical with the single-valued branches of \( S \), for \(|x| \geq \varepsilon\) (see also Fig. 9 for the case of function \( f(x) = -10x^2 + \text{sgn}(x)\)).

Using \( \text{sgn} \), for example to Chua’s system [4], one obtains

\[
\begin{align*}
D_t^q x_1 &= \begin{cases} 
-2.571x_1 + 9x_2 + 3.857\text{sgn}_\varepsilon(x_1), & x \in (-\varepsilon, \varepsilon), \\
-2.571x_1 + 9x_2 + 3.857\text{sgn}(x_1), & x \notin (-\varepsilon, \varepsilon),
\end{cases} \\
D_t^q x_2 &= x_1 - x_2 + x_3, \\
D_t^q x_3 &= -px_3.
\end{align*}
\]

The first right-hand side component, which is smoothly approximated, has the image in Fig. 2(b) with, again, a large \( \varepsilon \).

The following theorem, similar to Theorem 3, states the possibility to approximate locally the right-hand side of IVP (2).

**Theorem 4** Let \( f \) defined by (7). If \( g \) is continuous, then for every \( \varepsilon > 0 \), there exist local approximations of \( f \), \( \tilde{f}_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n \),

\[
\tilde{f}_\varepsilon(x) = g(x) + A(x)\tilde{s}_\varepsilon(x) \approx f(x), \quad x \in (-\varepsilon, \varepsilon).
\]

**Remark 4** If \( g \) and \( \tilde{s}_\varepsilon \) or \( \hat{s} \) are smooth functions, then one obtains a smooth approximation of \( f \).

In this case we can consider we have approximations of class \( C^k(\mathbb{R}) \), with \( k > 1 \), and therefore, the IVP (2) can be smoothly modeled.

Summarizing, as can be seen in Fig. 10, aided by Cellina’s Theorem, and depending on \( q \) properties (continuity or smoothness), the discontinuous function \( f \), given by (1), can be continuously or smoothly approximated, by simply replacing the discontinuous function \( s \) with, either (10) or (16).

### 3 Numerical tests

In order to illustrate how this approximation apparatus is utilized, we consider two practical examples of piece-wise continuous systems and one theoretical one-dimensional piece-wise continuous system. To emphasize the rightness of the approximation results, the practical examples are of fractional-order.

The use of Caputo derivative in the IVP (2) is fully justified in practical examples since in these problems we need physically interpretable initial conditions, or Caputo derivative satisfies these demands. Even there are some applications discussed in recent years with \( q > 1 \), the great majority of the physical phenomena are modeled with \( 0 < q < 1 \). Accordingly, the initial condition can be considered in the standard form [8], such as for the IVP (2), where \( x(0) = x_0 \). Therefore, next we consider the case \( q < 1 \).

Two of the most known methods to solve fractional-order equations are the multi-step predictor-corrector Adams-Bashforth-Moulton method (see e.g. [8, 9]) and the Grünwald-Letnikov discretization method (see e.g. [25, 26]). In this paper we used the Grünwald-Letnikov discretization method with the integration step-size \( h = 0.005 \).

The Hausdorff distance \( d_H \) [27] used to underline the results rightness, is of order of \( 10^{-5} \).

1. The fractional variant of the chaotic attractor of the Chen’s piece-wise-linear system presented in [28], has the following model

\[
\begin{align*}
D_t^1 x_1 &= 1.18(x_2 - x_1), \\
D_t^1 x_2 &= \text{sgn}(x_1)(5.82 - x_3) + 0.7x_2, \\
D_t^1 x_3 &= x_1\text{sgn}(x_2) - 0.168x_3.
\end{align*}
\]

With the global approximation (10), the system becomes

\[
\begin{align*}
D_t^1 x_1 &= 1.18(x_2 - x_1), \\
D_t^1 x_2 &= 5.82\text{sgn}(x_1) - x_3\text{sgn}(x_1) + 0.7x_2, \\
D_t^1 x_3 &= \text{sgn}(x_2)x_1 - 0.1x_3,
\end{align*}
\]
and by using $\delta = 1/10000$, one obtains the phase plot if Fig. 11(a).

By applying the local approximation (19) one obtains

\[
D^q_1 x_1 = 1.18 (x_2 - x_1),
\]

\[
D^q_2 x_2 = \begin{cases} 
5.82 \text{sgn}(x_1) - x_3 \text{sgn}(x_1) + 0.7x_2, & x_1 \in (-\varepsilon, \varepsilon), \\
5.82 \text{sgn}(x_1) - x_3 \text{sgn}(x_1) + 0.7x_2, & x_1 \notin (-\varepsilon, \varepsilon), 
\end{cases}
\]

\[
D^q_3 x_3 = \text{sgn}(x_2)x_1 - 0.168x_3, & x_2 \in (-\varepsilon, \varepsilon),
\]

Because in (21) $x_3 \text{sgn}(x_1)$ and $x_1 \text{sgn}(x_2)$ are continuous, only $5.82 \text{sgn}(x_1)$ has to be approximated.

With $\varepsilon = 1/10000$, one obtains the phase plot in Fig. 11(b) (both attractors have been generated starting from the same initial conditions). As can be seen, both attractors match very well.

In the next example, we will study a regular motion.

2. Let us consider a planar mechanical system, an "inverted" Duffing like system (due to the negativeness of the $\dot{x}$ coefficient) of fractional-order modeled by the following equation (see [29] for a general form of integer order)

\[
\ddot{x} + 0.18 \dot{x} - 1.5x + 0.8x^3 + 6.5 \text{sgn}(x) = 35 \cos(0.88t). \tag{22}
\]

The system evolves along a stable limit cycle. In Fig. 12 a, both attractors, determined with local and global approximation, are plotted superimposed, after transients neglected. Fig. 12 b and c reveal the fact that both approximations are in this case of same order of approximation.

3. Finally, we consider the following one-dimensional piece-wise system of integer-order, which can be found e.g. in [30] or [11] and which allows us to calculate, empirically, the approximations errors

\[
\dot{x}(t) = 2(h(t) - x(t)) + h'(t) + 2 - 2 \text{sgn}(x(t)), \quad t \in [0, 2], \tag{23}
\]

where

\[
h(t) = -\frac{4}{\pi} \arctan(t - 1), \quad x(0) > 0. \tag{24}
\]

As shown in [30], the problem (23) admits a unique solution, his exact expression being

\[
x(t) = \begin{cases} 
h(t), & t \in [0, 1], \\
0, & t \in [1, 2].
\end{cases} \tag{25}
\]

The numerical solutions of this system present the sliding phenomena (oscillations), which appear near the discontinuity (manifold $x = 0$), and which are in full accord with the convergence order of numerical methods for discontinuous problems (see e.g. [30] and [11]). In Fig. 13 (a) the exact solution, and the numerical solutions corresponding to the global and local approximations respectively. Fig. 13 b and c shows the difference between the two approximate trajectories and the exact solution, which is of order of $10^{-5}$ while in Fig. 13 c this difference is plotted for $t \in [1, 2]$. The qualitative difference between these two plots is due to the mentioned sliding phenomenon. However the error is the same in both intervals, namely of order $10^{-5}$, which are even better than the errors obtained when the IVP is integrated with methods for DE with discontinuous right-hand side [30].

The results for this example have been obtained with the Standard Runge-Kutta method, with $h = 10^{-5}$ (better errors can be obtained for smaller $h$, but in expense of the computer time).
4 Conclusion and discussions

In this paper we proven that piece-wise continuous functions defined by (1) can be continuously or smoothly approximated. Accordingly, the underlying systems (2), of fractional or integer order, can be modeled by continuous or smooth dynamical systems.

The approximations of the discontinuous components can be made locally or globally. This is possible due to the upper semicontinuity of the convex set-valued functions, obtained with Filippov’s regularization applied to the piece-wise constant functions $s$, property which allow the use of the Approximate Cellina’s Theorem.

For global approximations, one of the most accessible functions is the sigmoid function (10), while for local approximations, polynomials seem to be the appropriate choice. However, the steps to prove the existence of these approximations, apply for other continuous approximations.

Even the global (polynomial) approximations give, from theoretical point of view, better performances than the global approximations (due to the more realistic approximation of the underlying physical phenomenon), the aspects related to the numerical implementations, require some further studies. Thus, as shown in the last example in the last section, the errors outline that both methods give the same accuracy. Therefore the choice of one of these approximations should take into account the physical properties of the considered systems. Further studies are to be made.

Even Cellina’s Theorem assures as small as desired approximations errors, in the numerical examples we are limited by several kind of errors, such as the convergence errors of the utilized numerical methods errors, errors arising because of the finite precision representation of real numbers on any computer etc.

The approximation errors in the case of the discontinuous equation (23), where one knows the exact solution, are consistent with the errors of the numerical schemes for discontinuous systems (see e.g. [30]).

Since there are not numerical methods for fractional piece-wise continuous systems, the approximation apparatus we provided in this paper, could be of a real interest. For example, procedures unattainable before to piece-wise continuous systems of fractional-order, like chaos control, synchronization, anticontrol, etc., can be attained in this way. Also, these approximation procedures can be utilized to piece-wise continuous systems of integer order modeled by (2), in order to obtain continuous systems.

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Figure 2: Graph of the piece-wise continuous component of Chua’s system \([4]\). (a) Before approximation. (b) After approximation.

Figure 3: (a) Graph of a set-valued function \(F\). (b) The closure of the convex hull of \(F\). (c) For \(x = x_1\) and \(x = x_3\) \(F(x)\) are segments, while for \(x = x_2\), \(F(x_2)\), is a point \(f(x_2)\).
Figure 4: (a) Graph of $sgn$. (b) Graph of $Sgn$. (c) Graph of a continuous approximation.

Figure 5: (a) Sigmoid function, $\tilde{sgn}$, for $\delta \in [1/10000, 1/10]$. (b) $\tilde{sgn}$ for $\delta = 1/50$ and $\delta = 1/1000$. (c) Graph of $Sgn$ (red) and of $\tilde{sgn}$ (blue) for $\delta = 1/100$. The detail shows the difference between the two graphs.
Figure 6: Global approximation \( \tilde{f} \) (dotted blue color) of the function \( f(x) = -x^2 + \operatorname{sgn}(x - 0.5) \). In red is plotted the set-valued form, \( F(x) = -x^2 + S\operatorname{gn}(x - 0.5) \) and by yellow is represented the \( \varepsilon \)-neighborhood (slowly enlarged) where the approximation, \( \tilde{f} \), is embedded.

Figure 7: Graph of \( S\operatorname{gn} \) (red color) and graph of a continuous local approximation \( \tilde{S}\operatorname{gn}_\varepsilon \) (blue color), defined inside of some \( \varepsilon \)-neighborhood of \( x = 0 \). Outside \( \varepsilon \)-neighborhood, \( \tilde{S}\operatorname{gn}_\varepsilon(x) = S\operatorname{gn}(x) \).

Figure 8: (a) Graph of the polynomial local approximation \( \tilde{S\operatorname{gn}}_\varepsilon \) of the set-valued \( S\operatorname{gn} \) function, defined only within the interval \([-\varepsilon, \varepsilon]\). Outside, \( \tilde{S\operatorname{gn}}_\varepsilon \to \infty \). (b) Local approximation of the function \( f(x) = -x^2 + \operatorname{sgn}(x - 0.5) \). The approximation is made smooth: at the edges of the interval \((-\varepsilon, \varepsilon)\) the supplementary conditions \([17]\) have been imposed (same tangent slope at these points).
Figure 9: Comparison between global and local approximation in the case of piece-wise continuous function $f(x) = -10x^2 + \text{sgn}(x)$ for a big $\varepsilon = 0.5$ value (in order to obtain a clear image). Local approximation $\tilde{\text{sgn}}_\varepsilon$ (green color) is smoothly connected with the graph of $f$ (red color) at the edges points of the $\varepsilon$-interval (point $B$ in the detail). Global approximation $\tilde{\text{sgn}}$ is only close to the graph of $f$ (point $A$ in the detail).

Figure 10: Sketch of the proposed approximation procedure. $\tilde{s}$ stands either for the local or global approximation.

Figure 11: Overplotted chaotic attractors of the piece-wise linear Chen system [21], obtained with both approximations.
Figure 12: (a) Superimposed regular motions of Duffing system corresponding to local (blue color) and global (red color) approximations. (b) Difference between the local and global approximation for the first component $x_1$. (c) Difference between the local and global approximation for the second component $x_2$. 
Figure 13: (a) Superimposed solutions of the uni-dimensional piece-wise continuous system corresponding to the exact solution $x$, global approximation $\bar{x}$ (blue) and local approximation $\tilde{x}_\epsilon$ (red). (b) Difference between the exact solution and the approximate solutions for $t \in [0, 1]$. (c) Difference between the exact solution and the approximate solutions for $t \in [1, 2]$. 
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