INTERACTIONS BETWEEN AUTOEQUIVALENCE, STABILITY CONDITIONS, AND MODULI PROBLEMS.

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ABSTRACT. We begin by discussing various ways autoequivalences and stability conditions associated to triangulated categories can interact. Once an appropriate definition of compatibility is formulated, we derive a sufficiency criterion for this compatibility. We next apply this criterion to derived categories associated to Galois covers of the Weierstrass nodal cubic, known as n-gons and denoted by $E_n$. These are singular non-irreducible genus 1 curves naturally arising in variety of contexts, including as certain degenerations of elliptic curves.

In particular, fixing the stability condition to be the natural extension of classical slope to $E_n$, we explicitly compute the moduli space of stable objects and its compactification (given by S-equivalence). The compactification of stable objects with a fixed slope is isomorphic to a disjoint union of $E_m$ and $\mathbb{Z}/n\mathbb{Z}$ where $m|n$; $m$ varies as the slope varies and all such $m$ occur.

This computation is made possible by explicitly constructing the group of all autoequivalences compatible with the choice of stability condition. It is found that this group is an extension of the modular group $\Gamma_0(n)$ by a direct product of $\text{Aut}(E_n)$, $\text{Pic}^0(E_n)$, and $\mathbb{Z}$.

1. INTRODUCTION

Triangulated categories appear throughout geometry, with the different incarnations containing varying geometric data. The most studied is $D^b(X)$, the bounded derived category of coherent sheaves on a (locally noetherian) scheme $X$. Generally, it is difficult to abstract geometric information from triangulated categories. This paper is centered on using interactions between stability conditions and exact autoequivalences to derive information about each other, revealing a great deal of geometric information about the underlying triangulated category in the process. In particular, we study when an autoequivalence and a stability condition are well adapted to each other.

The prominent example of how a well adapted autoequivalence can elucidate structure in a triangulated category is the Fourier-Mukai transform on an elliptic curve, $E$. This is an autoequivalence obtained through the integral transform $\Phi_P$ of $D^b(E)$ with the kernel $P$ isomorphic to the Poincaré bundle. This autoequivalence does not preserve the geometric t-structure $\text{Coh}(E)$, yet it is known (and can be verified using Theorem 3.13) that $\Phi_P$ is compatible (to be defined below) with all stability conditions on $D^b(E)$, including classical slope stability. One can describe its action on $D^b(E)$ through the equality $\Phi_P \cdot (Z, P) = (Z, P) \cdot e^{i\pi/2}$ on the stability manifold (for any stability condition). This description allows one to reinterpret the method used in Atiyah’s classification of the moduli of semistable vector bundles on $E$ as a transitive action of $\text{SL}(2, \mathbb{Z})$ (via $\Phi_P$ and $\otimes L$ for any principle polarization $L$) on the set of phases of $(Z, P)$ [Ati57, BBDG06].

We must describe what is meant by “interactions” between stability conditions and autoequivalences. As a consequence of their definition, stability conditions are refinements of bounded t-structures: if $I = [a, a + 1]$ (or $(a, a + 1)$) and $\mathcal{P}(I)$ designates the extension
closed subcategory generated by semistable objects \( F \) with \( \phi(F) \in I \), then \( \mathcal{P}(I) \) is the heart of a bounded t-structure. Therefore, a stability condition gives a \( \mathbb{R} \)-indexed family of t-structures. This family yields an enhanced relationship between t-structures: two t-structures are “related” if they originate from one stability condition. This relationship can be rephrased as a “change of basis” using the \( GL^+(2, \mathbb{R}) \) action on the moduli of stability conditions.

Understanding when an autoequivalence takes one t-structure to a related one is the fundamental idea to our framework. Loosely, we define an autoequivalence \( \Phi \) and a stability condition \((Z, \mathcal{P})\) to be compatible if for each t-structure \( \mathcal{A} \) originating from \((Z, \mathcal{P})\), \( \Phi(\mathcal{A}) \) also originates from \((Z, \mathcal{P})\); we will discuss the precise definition shortly. We developed the following criterion giving sufficiency conditions for compatibility.

**Theorem 3.13.** Let \( \Phi \in \text{Aut}(\mathcal{T}) \), where \( \mathcal{T} \) is a triangulated category. Given \((Z, \mathcal{P}) \in \text{Stab}(\mathcal{T})\), a locally finite stability condition, one has the t-structure \( \mathcal{D}^{\leq 0} := \mathcal{P}(0, \infty) \), with heart \( \mathcal{A} \). If \( \Phi \) satisfies

1. \( \Phi(\mathcal{A}) \subset \mathcal{D}^{\leq M} \cap \mathcal{D}^{\geq M-1}, M \in \mathbb{Z} \).
2. \( \Phi \) descends to an automorphism \( [\Phi] \) of \( \text{im}(Z) \cong K(\mathcal{T})/\text{ker}(Z) \).
3. \( \Phi \) preserves the ordering on \( \text{im}(Z)_{\text{eff}, \text{comp}} \subset \text{im}(Z) \) induced by \((Z, \mathcal{P})\).

then \( \Phi \) is compatible with \((Z, \mathcal{P})\).

Here, \( \text{im}(Z)_{\text{eff}, \text{comp}} \) (defined in [\S] is a minimal subset for which we must check that \( \Phi \) strictly preserves the ordering. We note that the first condition is a necessary one. A fact that will be evident from the definitions. The beauty of this criterion is its ability to reduce compatibility to a series of linear algebra calculations.

To demonstrate the usefulness of our criterion we apply it to \( \text{D}^b(\mathbb{E}_n) \), where \( \mathbb{E}_n \) \((n \in \mathbb{N}^{>0})\) denotes a singular scheme, given by simple combinatorial data: \( \mathbb{E}_n \) is a singular reducible genus 1 curve that can be envisioned as a cycle of \( n \) projective planes with transverse intersections. They arise in many contexts, including as Galois covers of the Weierstrass nodal cubic, degenerations of elliptic curves, and in constructing Néron models. Our choice of a singular variety is appropriate: the pathologies in \( \text{D}^b(X) \) presented by singular and non-irreducible \( X \) have thwarted most attempts at understanding even the most basic examples. This makes the \( n \)-gon a good starting point to demonstrate why these techniques are an important contribution to our mathematical toolkit.

It is well known that given an elliptic curve \( \mathbb{E} \), \( \text{Aut}(\text{D}^b(\mathbb{E})) \) is an extension of \( \text{SL}(2, \mathbb{Z}) \). This fact is central to many of the nice results about the elliptic curves, their derived categories, and mirror symmetry. Using our criterion, we provide an analog for \( \text{D}^b(\mathbb{E}_n) \). Given \( n \), let \( \text{Aut}_{cd}(n) \subset \text{Aut}(\text{D}^b(\mathbb{E}_n)) \) consist of all autoequivalences compatible with \((Z_{cd}, \mathcal{P}_{cd})\). Then

\[
\begin{array}{cccccc}
1 & \longrightarrow & \mathbb{Z}/n\mathbb{Z} \times \mathbb{C}^* \times \mathbb{Z} \times \mathbb{C}^* & \longrightarrow & \text{Aut}_{cd}(n) & \longrightarrow & \Gamma_0(n) & \longrightarrow & 1
\end{array}
\]

where \( \Gamma_0(n) \subset \text{SL}(2, \mathbb{Z}) \) and the group on the left is generated by \( \text{Pic}^0(\mathbb{E}_n) \), \( \text{Aut}(\mathbb{E}_n) \), and the double shift \( [2] \). Under the action of \( \text{Aut}_{cd}(n) \) there are \( \Sigma_{d|n, d>0} \phi((d, n/d)) \) equivalence classes of phases, where \( \phi \) is Euler’s function.
This is a departure from the niceties of elliptic curves and nodal cubics: there is one equivalence class in those cases. Further, the modular groups $\Gamma_0(n)$ are not finitely generated for most $n$.

To demonstrate how compatible autoequivalences can reveal information about stability conditions, we provide an extension of Atiyah’s classification to the $n$-gon. It can be shown that the semistable objects in $(Z\ell, P\ell)$ correspond to Simpson semistable objects for a polarization on $E_n$ [RMGP09, Sim94]. It therefore makes sense to classify the coarse moduli space of semistable objects of given phase $\alpha$, $M_{\cl}(n, a)$. Using the action of $\Aut_{\cl}(n)$, the analog of Atiyah’s work is

**Theorem 6.1.** Given a nontrivial slice $P(\alpha)$ of $(Z\ell, P\ell)$ on $E_n$ ($n > 1$), let $M_{\cl}^{st}(n, a) \subset M_{\cl}(n, a)$ denote the subscheme whose closed points correspond to stable objects and $\overline{M}_{\cl}^{st}(n, a)$ its closure. Then $\overline{M}_{\cl}^{st}(n, a) \cong E_n \coprod \mathbb{Z}/n\mathbb{Z}$ where $s|n$ ($s$ depends on $a$) and each component of $\overline{M}_{\cl}^{st}(n, a)$ is a component of $M_{\cl}(n, a)$. 

Here we restrict to the case of $n > 1$ since for the nodal cubic it is known that $M_{\cl}^{st}(1, a) \cong E_1$ [BK06a]. The case of $n = 2$ was calculated in [RMGP09].

The disconnectedness of $\overline{M}_{\cl}^{st}(n, a)$ is a result of the rigidity of indecomposable torsion-free, but not locally free, sheaves on $E_n$. If $a$ is such that locally free objects $\mathcal{V} \in M_{\cl}^{st}(n, a)$ are line bundles, then these “rigid” elements correspond to line bundles restricted to $s$ consecutive components. There is an action by the Galois group of $\pi_n : E_n \to E_1$ on $M_{\cl}^{st}(n, a)$ that cyclically permutes the rigid elements and factors through $\mathrm{Gal}(E_n \to E_1)$ on the positive dimensional component. Our classification, like Atiyah’s, relies on the reduction of the problem to a finite number of specific cases. This is provided by Theorem 5.1. The shortcoming of Atiyah’s method applied to the $n$-gon is now clear: for $n > 4$ taking quotients of semistable sheaves by trivial subbundles is not enough to reduce the classification problem to a finite number of cases. One must “reduce” by an infinite number of sheaves to get a finite number of classes. This highlights the importance of having a general criterion.

Although we restricted to stable objects in Theorem 6.1, the methods used can be extended to classify the entire coarse moduli space of semistable objects. This is done by comparing $M_{\cl}(n, a)$ to $M_{\cl}(s, 1)$ for a suitable $s$. The structure of the latter space is completely known: $M_{\cl}(s, 1) \cong \coprod \Sym^s(E_n)$. Using this one should be able to write $M_{\cl}(n, a)$ in terms of $\coprod \Sym^s(E_n)$. This work should allow substantial progress in classifying moduli spaces obtained from other Simpson stability conditions and more general genus 1 curves.

We conclude this introduction with a quick discussion on the definition of a compatible autoequivalence, and how this definition influences Theorem 5.13. As a starting point, the most general compatibility between an autoequivalence and a stability condition is simply that our autoequivalence preserves semistable objects. Without more knowledge about the categorical structure of our triangulated category, finding easily verifiable conditions is not possible. Ideally, we want to avoid almost all knowledge of our triangulated category in the calculations. To obtain such low level conditions, one must consider more than just semistable sheaves. A stability condition gives a binary relation on semistable sheaves, induced by the charge on $K(T)$. To take this data into account, we define a compatible autoequivalence as a semistable preserving autoequivalence that preserves this binary relation. This new definition has the following strong consequence: if $a := \phi(\Phi(F))$ for any semistable object $F \in P(1)$, then $\Phi$ induces an equivalence
$\mathcal{P}(0, 1) \cong \mathcal{P}(a - 1, a)$. To understand the origin of the criterion, observe that the equivalence gives $\Phi(\mathcal{P}(0, 1)) \subset \mathcal{P}(a, a + 1) \subset \Phi(M, M - 2)$ for some $M \in \mathbb{Z}$. Thus $\Phi(\mathcal{P}(0, 1))$ is concentrated in a maximum of 2 cohomological degrees (in the original t-structure). This shows that Theorem 3.13(i) is a necessary condition. Moreover, with (i) and (ii) in place, it is clear that a compatible autoequivalence will satisfy (iii). From this perspective, (ii) is the only strong assumption in our criterion. It is this assumption that allows one to use the full data supplied by the stability condition. This in turn allows us to ignore most categorical structure in our criterion.

The layout of this paper is as follows. §2 is an overview of stability conditions, the inherent group action on their moduli, and integral transforms. With these preliminaries out of the way, in §3 we begin our discussion of compatibility between an autoequivalence and a stability condition. We show some easy consequences of the definitions and begin formulating properties that we can expect compatible autoequivalences to have. These basic properties give the criterion in Theorem 3.13, and the section is concluded with the proof of Theorem 3.13. §4 contains a review of basic terminology and results about $D^b(\mathbb{E}_n)$. With the terminology set, in §5 we use Theorem 3.13 to construct compatible autoequivalences of $D^b(\mathbb{E}_n)$, study basic properties of the group they generate. Lastly, in §6 we use our compatible autoequivalences to understand aspects of the stability condition $(Z_\mathcal{P}, \mathcal{P}_\mathcal{P})$: the compactification of the moduli of stable objects of a given phase, and number of phases for a given $n$.

1.1. A brief note on notation. Throughout this paper $\mathcal{T}$ will be a essentially small triangulated category. By $\text{Aut}(\mathcal{T})$ we mean the group of exact autoequivalences of $\mathcal{T}$, i.e., autoequivalences that preserve the triangulated structure.

We let $\mathbb{H}$ denote $\{z \in \mathbb{C} | \text{Im} z > 0 \}$ (the standard upper half plane) and $\mathbb{H}' := \mathbb{H} \cup \mathbb{R}_{<0}$.

2. Background.

2.1. Stability conditions. Stability conditions on triangulated categories were first defined and studied by Bridgeland in [Bri07], [Bri09]. Although stability conditions are the culmination of a long standing attempt to extend geometric invariant theory (GIT) to the setting of triangulated categories, the original motivation for Bridgeland was to provide a mathematical basis for Douglas’s $\Pi$-stability [Dou01, Dou02].

2.1.1. Definition. Given a triangulated category $\mathcal{T}$, we define the K-group $K(\mathcal{T})$ as the abelian group freely generated on $\text{Iso}(\mathcal{T})$ (isomorphism classes of objects of $\mathcal{T}$), subject to the relations $[B] = [A] + [C]$ if there exists a triangle $A \to B \to C \to A[1] \to \ldots$.

Definition 2.1. [Bri07] A stability condition $(Z, \mathcal{P})$ on a triangulated category $\mathcal{T}$ consists of a group homomorphism $Z : K(\mathcal{T}) \to \mathbb{C}$ called the central charge, and full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{T}$ for each $\phi \in \mathbb{R}$, satisfying the following axioms:

(a) if $E \in \mathcal{P}(\phi)$ then $Z(E) = m(E) \exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$,
(b) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,
(c) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ then $\text{Hom}_\mathcal{T}(A_1, A_2) = 0$,
(d) for each nonzero object $E \in \mathcal{T}$ there is a finite sequence of real numbers $\phi_1 > \phi_2 > \cdots > \phi_n$. 

and a collection of triangles

\[
0 \xrightarrow{\kappa} E_0 \xrightarrow{\kappa} E_1 \xrightarrow{\kappa} E_2 \rightarrow \cdots \rightarrow E_{n-1} \xrightarrow{\kappa} E_n = E
\]

with \( A_j \in \mathcal{P}(\phi_j) \) for all \( j \).

Given a stability condition on \( T \), to designate the semistable objects in the Harder-Narasimhan filtration on an object \( F \) we write \( A_j(F) \), or \( A_j \) when no confusion can arise, and if the number of semistable factors is irrelevant we denote the lowest phase as \( \phi_- \) and the accompanying semistable factor as \( A_- \).

The triangles in Definition 2.1(d) are called the Harder-Narasimhan filtration of the object \( E \). In the case that our triangulated category is a derived category, and our object a sheaf (or module), then the classical notion coincides with this one.

We denote by \( \mathcal{P}(a, b) \) the full extension-closed subcategory of \( T \) generated by \( \mathcal{P}(c) \subset T \), \( c \in [a, b] \). It is clear from Definition 2.1(b), that all semistable objects, up to shift, are determined by any interval of length 1. In fact, the subcategory \( \mathcal{P}(a, a + 1) \) is the heart of the bounded t-structure given by the subcategory \( \mathcal{P}(a, \infty) \) (and therefore abelian). The subcategory \( \mathcal{P}(0, 1) \) will play a special role. Our charge, \( Z \), has the property that \( Z([A]) \in \mathbb{H}' \) for \( A \in \mathcal{P}(0, 1) \). Since \( \mathbb{H}' \) is contained in a proper open domain of \( \mathbb{C}^* \), we can assume the existence of a log branch on \( \mathbb{H}' \) such that \( \frac{1}{2} \Im \log \) agrees with the phases in \( (0, 1] \). We can use this log branch to assign phases to all objects in \( \mathcal{P}(0, 1) \), including those that are unstable. This fact will be used extensively in our proof of Theorem 3.13.

An important feature of a stability condition is that we can recover our semistable objects given the abelian category \( \mathcal{P}(0, 1) \) and the charge \( Z : K(A) \cong K(T) \to \mathbb{H}' \). This viewpoint motivates the equivalent definition of a stability condition: the heart of a bounded t-structure \( A \subset T \), and a homomorphism (called a stability function) \( Z : K(A) \to \mathbb{H}' \) which satisfies a certain finiteness condition similar to Definition 2.1(d), oddly enough, called the Harder-Narasimhan property. All the stability functions/charges in this thesis will have this property.

**Example 2.2.** Let \( X \) be an smooth algebraic curve. Define \( Z(\mathcal{F}) = -\deg(\mathcal{F}) + i \rank(\mathcal{F}) \). Then \( \text{Coh}(X) \) and \( Z \) define a stability condition on \( \text{D}^b(X) \). The semistable objects are the same as those calculated by slope.

### 2.1.2. Group actions on the moduli of stability conditions

Let \( \Phi \in \text{Aut}(T) \). The relations on \( K(T) \) depend only on the triangulated structure of \( T \). Thus, it is clear that \( \Phi \) descends to an automorphism \( [\Phi] \) of \( K(T) \). The following lemma explicitly details a commuting left/right group action on the moduli of stability conditions. We include the proof since this paper is partially concerned with relationships between these two actions.

**Lemma 2.3** ([Bri07] Lemma 8.2). The moduli space of (locally finite) stability conditions \( \text{Stab}(T) \) carries a right action of the group \( \widetilde{\text{GL}}^+(2, \mathbb{R}) \), the universal covering space of \( \text{GL}(2, \mathbb{R}) \), and a left action by the group \( \text{Aut}(T) \) of exact autoequivalences of \( T \). These two actions commute.

**Proof.** First note that the group \( \widetilde{\text{GL}}^+(2, \mathbb{R}) \) can be thought of as the set of pairs \((T, f)\) where \( f : \mathbb{R} \to \mathbb{R} \) is an increasing map with \( f(\phi + 1) = f(\phi) + 1 \), and \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is an orientation-preserving linear isomorphism, such that the induced maps on \( S^1 = \mathbb{R}/2\mathbb{Z} = \mathbb{R}^2 / \mathbb{R}_{>0} \) are the same.
Given a stability condition $\sigma = (Z, \mathcal{P}) \in \text{Stab}(\mathcal{T})$, and a pair $(T, f) \in \tilde{GL}^+(2, \mathbb{R})$, define a new stability condition $\sigma' = (Z', \mathcal{P}')$ by setting $Z' = T^{-1} \circ Z$ and $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$. Note that the semistable objects of the stability conditions $\sigma$ and $\sigma'$ are the same, but the phases have been relabeled.

For the second action, we know that an element $\Phi \in \text{Aut}(\mathcal{T})$ induces an automorphism $[\Phi]$ of $K(\mathcal{T})$. If $\sigma = (Z, \mathcal{P})$ is a stability condition on $\mathcal{T}$ define $\Phi(\sigma)$ to be the stability condition $(Z \circ [\Phi]^{-1}, \mathcal{P}')$, where $\mathcal{P}'(t) = \Phi(\mathcal{P}(t))$. The reader can easily check that this action is by isometries and commutes with the first. \hfill $\Box$

We can think of the action of $\tilde{GL}^+(2, \mathbb{R})$ as a “change of coordinates”: it is a change of basis on the range of our charge $Z$, followed by an appropriate reassigning of the phases so that Definition 2.1 is satisfied. An alternative way to understand the left group action is via the abelian subcategory/stability function viewpoint. In particular, our autoequivalence $\Phi \in \text{Aut}(\mathcal{T})$ takes $\mathcal{A}$ to another abelian category, $\Phi(\mathcal{A})$. One then obtains a stability function on $\Phi(\mathcal{A})$ by pullback, $\Phi \cdot Z = Z([\Phi^{-1}](\bullet))$, thus giving us a new stability condition.

2.2. Integral transforms. Let $X$ be a locally noetherian scheme. For any object $\mathcal{K} \in D^b(X \times X)$, using the natural projections

$$
\begin{array}{ccc}
X & \xrightarrow{\rho_1} & X \\
\downarrow & & \downarrow \\
X & \xrightarrow{\rho_2} & X
\end{array}
$$

one can define an exact functor $\Phi_\mathcal{K} \in \text{End}(D^-(X))$ called an integral transform with kernel $\mathcal{K}$. Explicitly,

$$
\Phi_\mathcal{K}(F) = \rho_{2*}(\rho_1^* F \otimes \mathcal{K}).
$$

In the case that $\mathcal{K}$ is either perfect or flat over $X$ (under the map $\rho_1$), we can restrict this to an endomorphism of $D^b(X)$, see [RMeS09] or [BK06]. Given two integral transforms, we have the composition formula $\Phi_\mathcal{K} \circ \Phi_{\mathcal{K}'} = \Phi_{\mathcal{K} \otimes \mathcal{K}'}$ where $\mathcal{K} \otimes \mathcal{K} := \rho_{13*}(\rho_{12}^* \mathcal{K} \otimes \rho_{23}^* \mathcal{K})$, with $\rho_{ij}$ the obvious projections of $X \times X \times X$ [Muk81].

The next proposition gives commutation relations between integral transforms and autoequivalences of $X$. This result is well known, and proven by standard techniques.

**Proposition 2.4.** Let $\Phi_\mathcal{K}$ denote the endomorphism of $D^b(X)$ with kernel $\mathcal{K} \in D^b(X \times X)$. If $\alpha$ is an automorphism of $X$, then

1. $\alpha_* \circ \Phi_\mathcal{K} \cong \Phi_{(\alpha \times \alpha)_* \mathcal{K}}$
2. $\Phi_\mathcal{K} \circ \alpha_* \cong \Phi_{(\alpha \times \alpha)^* \mathcal{K}}$
3. $\Phi_\mathcal{K} \circ \alpha^* \cong \Phi_{(\alpha \times \alpha)^* \mathcal{K}}$
4. $\alpha^* \circ \Phi_\mathcal{K} \cong \Phi_{(\alpha \times \alpha)^* \mathcal{K}}$

3. **Compatible Autoequivalences**

Throughout this section we let $\Phi$ be an exact autoequivalence of a triangulated category $\mathcal{T}$. Given a stability condition, we denote the heart of the t-structure $D^{\geq 0} = \mathcal{P}(0, \infty)$ by $\mathcal{A} = \mathcal{P}(0, 1]$. We begin by defining compatibility and discussing its consequences. With the definition in place, we derive our criterion and prove its sufficiency.

3.1. **Discussion of compatibility.** We begin with a basic definition of a “nice” interaction between an autoequivalence and a stability condition:

**Definition 3.1.** Let $\Phi \in \text{Aut}(\mathcal{T})$ and $(Z, \mathcal{P}) \in \text{Stab}(\mathcal{T})$. $\Phi$ is semistable preserving with regard to $(Z, \mathcal{P})$ if for every semistable object $F$, $\Phi(F)$ is semistable.
This definition has many drawbacks. First, the definition does not allow us to easily describe the action of $\Phi$ on $\mathcal{T}$. Knowledge that $\Phi$ preserves semistable objects is not enough to specify (or formulate) coherence data between $\Phi(A)$ and any bounded t-structure originating from $(Z, \mathcal{P})$. Without this data, our ability to describe $\Phi$ is severely limited. For instance, it is reasonable for an unstable object to be mapped to a stable object, showing that $\Phi^{-1}$ is not semistable preserving. Second, without specific information about $\Phi$, $(Z, \mathcal{P})$, and $\mathcal{T}$, finding an easily verifiable criterion that tests if $\Phi$ is semistable preserving is nearly impossible.

The problem with our definition is its insensitivity towards additional structure supplied by a stability condition. Stability conditions dictate more than the class of semistable objects: semistable objects are grouped together into ordered slices. Additionally, these slices are adapted to a function on the $K$-group. To improve our definition, our compatibility will respect this additional structure. It will become clear in the next sections why this allows us to find a criterion.

Definition 3.2. A semistable preserving autoequivalence $\Phi$ is compatible with $(Z, \mathcal{P})$ if for every pair of semistable objects $F$ and $G$,

1. $\phi(F) < \phi(G)$ implies $\phi(\Phi(F)) < \phi(\Phi(G))$,
2. $\phi(F) = \phi(G)$ implies $\phi(\Phi(F)) = \phi(\Phi(G))$.

We abbreviate these two conditions: $\phi(F) \prec \phi(G)$ implies $\phi(\Phi(F)) \prec \phi(\Phi(G))$. The ordering of slices imparts a binary relation on semistable objects. By abuse of terminology, we will refer to this binary relation as an ordering. We can rephrase our definition of a compatible autoequivalence as a semistable preserving autoequivalence that strictly preserves the ordering. The following proposition helps convey why we chose “compatible” to describe this type of autoequivalence.

Proposition 3.3. Let $\Phi$ and $(Z, \mathcal{P})$ be compatible. For all nontrivial slices $\mathcal{P}(a)$, $\Phi$ induces an equivalence $\Phi : \mathcal{P}(a) \xrightarrow{\sim} \mathcal{P}(b)$ for some $b \in \mathbb{R}$.

Proof. The conditions for compatibility ensure that $\Phi$ preserves our semistable objects, and strictly preserves the binary relation. We can recover the semistable objects of a particular slice via this relation (short of the zero object). Since slices are full subcategories, the result follows due to the fact that $\Phi$ is an equivalence. $\square$

The preceding proposition shows that the global behavior of a compatible autoequivalence can be explicitly described as an ordered phase rearrangement, enabling one to roughly describe its action in very simple terms. This allows us to understand relationships between different phases, helping to understand important symmetries in $\mathcal{T}$ and $(Z, \mathcal{P})$. Many times this phase rearrangement can be extended to an element in $\text{Diff}^+(\mathbb{R})$. An important example of this is when the element of $\text{Diff}^+(\mathbb{R})$ is in fact obtained by a linear change of coordinates on $\mathbb{C}$. More precisely, we would like to know if there exists an element $g \in \widetilde{\text{GL}}^+(2, \mathbb{R})$ such that $\Phi \cdot (Z, \mathcal{P}) = (Z, \mathcal{P}) \cdot g$. This will be our strongest form of compatibility.

Definition 3.4. $\Phi$ is strongly compatible if $\Phi \cdot (Z, \mathcal{P}) = (Z, \mathcal{P}) \cdot g$ for some $g \in \widetilde{\text{GL}}^+(2, \mathbb{R})$.

We list several properties of compatible autoequivalences that will be used in latter sections.

Proposition 3.5. The composition of compatible autoequivalences are compatible. Further, if $\Phi$ is compatible with $(Z, \mathcal{P})$, then
(1) \( \Phi(F) \) is semistable if and only if \( F \) is.

(2) \( \Phi(F) \) is stable if and only if \( F \) is.

(3) \( \Phi \) is compatible if and only if \( \Phi^{-1} \) is compatible.

Proof. The first statement is obvious from the discussion above. For the second, stable objects of phase \( \alpha \) are the simple objects in the abelian category \( \mathcal{P}(\alpha) \). By definition, \( \Phi \) induces an equivalence of categories between the various slices in \( \mathcal{P} \). Since simple objects are categorically defined, the equivalence must preserve them. The last statement is clear if one views a compatible autoequivalence as a rearrangement of \( \mathcal{P} \). \( \square \)

3.2. A criterion for compatibility. Now that we have made precise the definition of compatibility between autoequivalences and stability conditions, we want to derive some easily testable conditions, that when satisfied give an affirmative answer for compatibility. To reiterate our setup, \( \Phi \) will be a compatible autoequivalence and \((Z, \mathcal{P}) \in \text{Stab}(\mathcal{T})\). The criterion should be verifiable on the level of a t-structure obtained from \((Z, \mathcal{P})\), and \( K \)-group. The first condition is a necessary condition; the other two conditions reduce the verification to linear algebra.

Proposition 3.5 implies the existence of \( b \in \mathbb{R} \) such that \( \Phi(\mathcal{P}(0)) \subset \mathcal{P}(b) \). Chasing through definitions reveals \( \Phi(\mathcal{P}(1)) \subset \mathcal{P}(b+1) \). The order preserving manner in which \( \Phi \) acts implies \( \Phi \) restricts to a fully faithful map of subcategories: \( \Phi(\mathcal{P}(0,1)) \subset \mathcal{P}(b, b+1) \), and that this is an equivalence. Thus, a compatible equivalence applied to \( \mathcal{A} \) can be concentrated in a maximum of 2 consecutive cohomological degrees. By definition, this necessary condition is something that can be calculated on the level of the t-structure, and will be our starting point.

**Condition 1.** \( H^i(\Phi(A)) = 0 \) if \( i \neq \{M, M-1\} \) where \( M \in \mathbb{Z} \).

With Condition 1 in place, \((Z, \mathcal{P})\) allows the choice of a log branch on \( \mathbb{C}^* \) with the property that the phases assigned by this branch agree with those in \( \mathcal{P}(-M, -M+2) \). This branch can be chosen so it is discontinuous at the positive real axis if \( M \) is even, or the negative real axis if \( M \) is odd. Our goal is to assign a phase \( \phi_f(\Phi(F)) \) for all \( F \in \mathcal{A} \) (we use the subscript \( f \) to emphasize that \( \Phi(F) \) may or may not be semistable, thus it is a “fake” phase). Condition 1 alone does not enable this: it is possible that \( Z(\Phi(F)) = 0 \), \( F \in \mathcal{T} \) with \( F \neq 0 \). However, if we assume \( [\Phi] \) descends to an automorphism \( [[\Phi]] \) of \( \text{im}(Z) \), this concern is alleviated.

**Notation.** Given \([F] \in K(T)\), let \([[F]] \) denote its image in \( \text{im}(Z) \). Likewise, if \( A \in \text{End}(K(T)) \), let \([[A]] \) denote the resulting endomorphism on \( \text{im}(Z) \), if well defined.

This motivates our second condition:

**Condition 2.** \( [\Phi] \) descends to an automorphism \( [[\Phi]] \) of \( \text{im}(Z) \).

**Remark 3.1.** It seems plausible that this condition may be stricter than needed to prove compatibility. However, with a less stringent Condition (2) it is not likely that Condition (3) can be phrased in terms of \( K \)-group calculations.

We can now assign our “fake” phases:

**Notation.** Assume a choice of log branch on \( \mathbb{C} \). Given a nonzero \( v \in \text{im}(Z) \), \( \phi_f(v) := \frac{1}{2} \text{Im} \log(Z(v)) \). If \([[F]] = v \) for \( F \in \mathcal{T} \), we define \( \phi_f(F) \) to be \( \phi_f(v) \).

To formulate our last condition, we must fix our log branch. To do this we will assume that \( M \) is even. In the case that \( M \) is odd, compose with the shift \([1] \). Our log branch
will be compatible with the phases in $\mathcal{A}$ and discontinuous at the positive real axis. With this choice, our log branch gives two orderings on the semistable objects of $\mathcal{A}$: prior and subsequent to the application of $\Phi$. If $\Phi$ is compatible, these will be identical. Since these orderings solely rely on $\text{im}(Z) := K(\mathcal{T})/\ker(Z)$ and the log branch, we will rephrase this in terms of a subset of $\text{im}(Z)$. First, in order to limit these calculations, we restrict to the “effective” subset:

$$\text{im}(Z)_{eff} = \{ \pm v \in \text{im}(Z) \mid \exists \text{s.s. } G \in \mathcal{A} \text{ with } [G] = v \}.$$  

The discontinuity of our log branch has the unfortunate consequence of disallowing meaningfully comparison of the before and after orderings between all elements of $\text{im}(Z)_{eff}$. We want to restrict to the largest subset of $\text{im}(Z)_{eff}$ for which this comparison is valid. To do so, we need to ensure that $[[\Phi]]$ doesn’t move our objects past the discontinuity. Define

$$\text{im}(Z)_{comp} = \{ v \mid Z(v) \in \mathbb{H}' \} \cup \{ \pm v \mid Z(v) \in \mathbb{H}' \text{ and } Z([[\Phi]]v) \in \mathbb{H}' \}.$$  

**Example 3.6.** Suppose that $[[\Phi]]$ is the restriction of an element $\Upsilon \in \text{GL}^+(2, \mathbb{R})$ (via $Z$). Let $\mathcal{G} = \mathbb{H}' \cap \Upsilon^{-1}(\mathbb{H}')$. Then $\text{im}(Z)_{comp}$ is the subset of $\text{im}(Z)$ contained in $\mathbb{H}' \cup -\mathcal{G}$.

**Figure 1.** $\mathbb{H}'$ and $\Upsilon^{-1}(\mathbb{H}')$

**Figure 2.** $\mathbb{H}' \cup -\mathcal{G}$
On \( \text{im}(Z)_{\text{comp}} \) we can meaningfully compare the two aforementioned orderings. Our final condition is that these orderings are the same (i.e. the binary relation is strictly preserved under \([\Phi]\)).

**Condition 3.** Given \( v, w \in \text{im}(Z)_{\text{eff} \cap \text{comp}} := \text{im}(Z)_{\text{eff}} \cap \text{im}(Z)_{\text{comp}} \) if \( \phi_f(v) \succ \phi_f(w) \) then \( \phi_f([\Phi](v)) \succ \phi_f([\Phi](w)) \) (i.e. strict preservation).

The advantage of working with \( \text{im}(Z) \) is that we can ignore \( T \) and our stability condition. This is purely linear algebra. If \( \text{im}(Z) \) is finite rank, then this a straightforward condition to verify. Although there is no explicit mention of the log branch in Condition 3, our subset \( \text{im}(Z)_{\text{comp}} \) is only meaningful for comparison when the log branch is at the positive real axis.

Condition 1 may seem redundant after Condition 3 and the discussion in §3.1, however, we needed the former to reliably define the latter.

### 3.3. Proof that the criterion implies compatibility.

We will now assume the autoequivalence \( \Phi \) satisfies

1. \( H^i(\Phi(A)) = 0 \) if \( i \neq \{M, M - 1\} \) where \( M \in \mathbb{Z} \).
2. \([\Phi]\) descends to an automorphism \([\Phi]\) of \( \text{im}(Z) \).
3. Given \( v, w \in \text{im}(Z)_{\text{eff} \cap \text{comp}} \) the relation \( \phi_f(v) \succ \phi_f(w) \) implies \( \phi_f([\Phi](v)) \succ \phi_f([\Phi](w)) \) (i.e. strict preservation).

From the definition of a stability condition, it is clear that the shift functor is a strongly compatible autoequivalence. This, combined with Proposition 3.5 shows we can assume \( M = 0 \) (in C1). Like above, our log branch will be discontinuous on \( \mathbb{R}^{>0} \) and \( \phi_f(A) \in (0,1] \) for \( A \in \mathcal{A} \).

We begin by understanding the information contained in C3. We first lift the order preserving requirement to the level of semistable objects.

**Lemma 3.7.** Let \( F, G \in \text{Obj}(A \cup A[1]) \) be semistable with \( H^{-2}(\Phi(G)) = 0 \). Then \( \phi(F) \succ \phi(G) \) implies \( \phi_f(\Phi(F)) \succ \phi_f(\Phi(G)) \).

**Proof.** By definition, \([F],[G] \in \text{im}(Z)_{\text{eff}}\). The choice of log branch ensures the fake and real phases of \( F \) and \( G \) are equal. Assume that \( F, G \in \mathcal{A} \). Then \([F],[G] \in \text{im}(Z)_{\text{comp}}\) and applying C3 gives the result.

In the case that \( F \in \mathcal{A} \) and \( G \in \mathcal{A}[1] \), we first note that C1 and the assumption \( H^{-2}(\Phi(G)) = 0 \) imply that \( Z([G[-1]]) \in \mathbb{H}' \) and \( Z([\Phi([G[-1]])]) \in \mathbb{H}' \). This shows that \([G] \in \text{im}(Z)_{\text{comp}}\). Applying C3 gives the result.

Lastly, let \( F, G \in \mathcal{A}[1] \). We must show that \( \phi(F) \leq \phi(G) \) ensures \([F] \in \text{im}(Z)_{\text{comp}}\). The case above applies to \( F[-1] \) and \( G[-1] \), showing \( \phi_f(\Phi(F[-1])) \leq \phi_f(\Phi(G[-1])) \). Since \([G] \in \text{im}(Z)_{\text{comp}}, \phi_f(\Phi(G)) \leq 1 \). This is enough to show that \( Z([F[-1]]) \in \mathbb{H}' \) and \( Z([\Phi([-1]]) \in \mathbb{H}' \). Therefore, \([F] \in \text{im}(Z)_{\text{comp}}\) and applying C3 gives the result. \( \square \)

We would like to understand when \( \Phi \) preserves cohomological purity (i.e. conditions on an object \( F \in \mathcal{A} \) that guarantee \( H^i(\Phi(F)) = 0 \) for all but one \( i \in \mathbb{Z} \). We claim that the necessary condition is membership in one of two full subcategories that split \( \mathcal{A} \).

**Notation.** Denote by \( m_\mathcal{A} \in \mathbb{R} \) the minimal phase such that if \( \nu \in \text{im}(Z)_{\text{eff} \cap \text{comp}} \) then \( \phi_f(v) \leq m_\mathcal{A} + 1 \). When the heart \( \mathcal{A} \) is clear, we will drop the subscript.

Clearly \( m_\mathcal{A} \) depends on both \( \Phi \) and \( \mathcal{A} \). C3 ensures that \( m_\mathcal{A} \) has the following important property:
Lemma 3.8. If $F \in \mathcal{A}$ is semistable and $\phi(F) \leq m_{\mathcal{A}}$, then $Z([\Phi(F)]) \in \mathcal{H}$. Further, $m_{\mathcal{A}}$ is maximal with respect to this property.

Proof. This is a consequence of the definitions and Lemma 3.7 by definition there exists a semistable $G$ with $[G] \in \mathfrak{im}(A)_{eff \cap comp}$, $\phi(F) \leq \phi(G) \leq m_{\mathcal{A}}$, and $Z([\Phi(G)]) \in \mathcal{H}$. Lemma 3.7 then gives the lemma.

Our choice of $M$ guarantees that $m > 0$. It would be convenient to attempt to use C3 to show that if $\phi_f(G) < m$ then $\Phi(G) \in \mathcal{A}$. This in general will be a false statement. Instead we split our abelian category $\mathcal{A}$ into two pieces $\mathcal{A}_0 = \mathcal{P}(0, m]$ and $\mathcal{A}_1 = \mathcal{P}(m, 1]$. Clearly $\mathcal{A}_i$ generate $\mathcal{A}$ through extensions. Our goal is to show that

Proposition 3.9.

(i) $\Phi(\mathcal{A}^0) \subset \mathcal{A}$
(ii) $\Phi(\mathcal{A}^1) \subset \mathcal{A}[1]$

We first prove several key lemmas. The proposition is then an easy consequence of these lemmas. The first is adapted from [RMGP09].

Lemma 3.10. Let $F \in \mathcal{A}$,

(1) if $\mathcal{H}_i(\Phi(F)) = 0$ for $i \neq -1$ then $m < \phi_f(F) \leq 1$.
(2) if $\mathcal{H}_i(\Phi(F)) = 0$ for $i \neq 0$ then $0 < \phi_f(F) \leq m$.

Proof. We will prove the first statement; the second is shown by similar methods. Suppose the contrary: $\mathcal{H}_i(\Phi(F)) = 0$ for $i \neq -1$ yet $0 < \phi_f(F) \leq m$. The Harder-Narasimhan filtration of $F$ gives a short exact sequence

$$0 \to F_- \to F \to A_-(F) \to 0$$

where $0 < \phi(A_-(F)) \leq m$. The first inequality follows from the fact that $F \in \mathcal{A}$. Applying $\Phi$ gives a long exact sequence

$$\cdots \to \mathcal{H}^{-1}(\Phi(F)) \to \mathcal{H}^{-1}(\Phi(A_-(F))) \to \mathcal{H}^0(\Phi(F_-)) \to$$

$$\mathcal{H}^0(\Phi(F)) \to \mathcal{H}^0(\Phi(A_-(F))) \to 0$$

where the last 0 is due to C1.

By Lemma 3.8, we have $Z([A_-(F)]) \in \mathcal{H}$. Further, have a semistable factor $A' = A_-(\Phi(A_-(F)))$ with $\phi(A') \leq 1$ guaranteeing that $\mathcal{H}_0(\Phi(A_-(F))) \neq 0$. The exactness of the above sequence then shows $\mathcal{H}_0(\Phi(F)) \neq 0$, giving us our desired contradiction.

For an object $F \in \mathcal{A}$, the condition $0 < \phi_f(F) \leq m$ is coarser that $F \in \mathcal{P}(0, m]$. This allows us to work with objects without assuming membership a particular slice $\mathcal{A}_i$.

Lemma 3.11. [RMGP09] Let $F \in \mathcal{A}$. Then there exists a short exact sequence

$$0 \to \Phi^{-1}(\mathcal{H}^{-1}(\Phi(F))[1]) \to F \to \Phi^{-1}(\mathcal{H}^0(\Phi(F))) \to 0$$

of objects in $\mathcal{A}$.

Proof. We will include our own proof. Let $G := \Phi(F)$. We have the triangle

$$\mathcal{H}^{-1}(G)[1] \to G \to \mathcal{H}^0(G)$$
coming from the t-structure. Applying $\Phi^{-1}$ results in the long exact sequence:

$$
\cdots \to \mathcal{H}^{-1}(\Phi^{-1}(\mathcal{H}^0(G))) \to \mathcal{H}^0(\Phi^{-1}(\mathcal{H}^{-1}(G)[1])) \to F \to \mathcal{H}^0(\Phi^{-1}(\mathcal{H}^0(G))) \to \mathcal{H}^1(\Phi^{-1}(\mathcal{H}^{-1}(G)[1])) \to \cdots
$$

C1 ensures $\mathcal{H}^k(\Phi^{-1}(\mathcal{H}^{-1}(G)[1])) = 0$ for $k \neq 0, -1$ and $\mathcal{H}^k(\Phi^{-1}(\mathcal{H}^0(G))) = 0$ for $k \neq 0, 1$. Thus the two end groups in this sequence vanish.

We are left showing the isomorphisms $\mathcal{H}^0(\Phi^{-1}(\mathcal{H}^{-1}(G)[1])) \cong \Phi^{-1}(\mathcal{H}^{-1}(G)[1])$ and $\mathcal{H}^0(\Phi^{-1}(\mathcal{H}^0(G))) \cong \Phi^{-1}(\mathcal{H}^0(\Phi(F)))$. The first isomorphism is a consequence of $\mathcal{H}^{-1}(\Phi^{-1}(\mathcal{H}^{-1}(G)[1])) = 0$, which can be seen from further up on the same sequence:

$$
\mathcal{H}^{-2}(\Phi^{-1}(\mathcal{H}^0(G))) \to \mathcal{H}^{-1}(\Phi^{-1}(\mathcal{H}^{-1}(G)[1])) \to \mathcal{H}^{-1}(F)
$$

Clearly the left and right groups are zero, thus giving the result. A similar argument holds for the case of $\mathcal{H}^0(\Phi^{-1}(\mathcal{H}^0(G))) \cong \Phi^{-1}(\mathcal{H}^0(\Phi(F)))$, concluding the proof. \(\square\)

We will now analyze how $\Phi$ acts on semistable objects of $\mathcal{A}$.

**Lemma 3.12.** Let $F \in \mathcal{A}$ be semistable. Then, $\Phi(F)$ is cohomologically pure, i.e., $\mathcal{H}^i(\Phi(F)) = 0$ for all but one integer.

**Proof.** We separate the proof into two cases.

0 < $\phi(F)$ ≤ $m$: Suppose that both $\mathcal{H}^0(\Phi(F)) \neq 0$ and $\mathcal{H}^{-1}(\Phi(F)) \neq 0$. The exact sequence in Lemma 3.11 shows that $\Phi^{-1}(\mathcal{H}^{-1}(\Phi(F))[1])$ is a subsheaf of $F$ (and non-zero by assumption). $F$ being semistable, combined with its assumed phase ensures $\phi_f(\Phi^{-1}(\mathcal{H}^{-1}(\Phi(F))[1])) \leq m$. On the other hand, $\Phi^{-1}(\mathcal{H}^{-1}(\Phi(F))[1])$ satisfies Lemma 3.10(1), ensuring $\phi_f(\Phi^{-1}(\mathcal{H}^{-1}(\Phi(F))[1])) > m$, a contradiction.

$m < \phi(A) \leq 1$: Again, suppose the contrary. Then $\Phi^{-1}(\mathcal{H}^0(\Phi(F)))$ is a quotient sheaf of $F$, ensuring that $\phi_f(\Phi^{-1}(\mathcal{H}^0(\Phi(F)))) > m$. However, $\Phi^{-1}(\mathcal{H}^0(\Phi(F)))$ satisfies Lemma 3.10(2), giving $\phi_f(\Phi^{-1}(\mathcal{H}^0(\Phi(F)))) \leq m$, again, a contradiction. \(\square\)

Proposition 3.9 is an easy consequence of Lemma 3.12 since it is generated from semistable elements through extension.

**Proof of Proposition 3.9** Assume $F \in \mathcal{A}_i$, $i \in \{0, 1\}$ and $A_j(F)$ are the semistable objects in the Harder-Narasimhan filtration of $F$. By definition of $\mathcal{A}_i$, $A_j(F) \in \mathcal{A}$. Lemma 3.12 and Lemma 3.10 imply that $\Phi(A_j(F))$ is cohomologically pure and are concentrated in degree $-i$ for all $j$ (since $-0$ is 0). Since $\mathcal{A}$ and $\mathcal{A}[-1]$ are extension closed, we have that $\Phi(F)$ is concentrated in degree $-i$, completing the proof. \(\square\)

From Lemma 3.12 we know that $\Phi(F)$ is cohomologically pure. However, this t-structure was arbitrary. Since semistable objects are the objects that are cohomologically pure in every t-structure generated from $(Z, \mathcal{P})$, one can show that $\Phi(F)$ is semistable if $\Phi$ satisfies our conditions for all t-structures obtained from $(Z, \mathcal{P})$. This motivates the proof of our main theorem. Before stating the next theorem, we recall the definition of two key subsets of $\text{im}(Z)$:

$$
\text{im}(Z)_{\text{eff}} = \{ \pm v \mid \exists \text{s.s. } G \in \mathcal{A} \text{ with } Z([G]) = v \}.
$$

$$
\text{im}(Z)_{\text{comp}} = \{ v \mid Z(v) \in \mathbb{H}' \} \cup \{ \pm v \mid Z(v) \in \mathbb{H}' \text{ and } Z([\Phi])v \in \mathbb{H}' \}
$$

**Theorem 3.13.** Let $\Phi \in \text{Aut}(\mathcal{T})$. Given $(Z, \mathcal{P}) \in \text{Stab}(\mathcal{T})$, a locally finite stability condition, one has the t-structure $\mathcal{D}^{\leq 0} = \mathcal{P}(0, \infty)$, with heart $\mathcal{A}$. If $\Phi$ satisfies

(i) $\Phi(\mathcal{A}) \subset \mathcal{D}^M \cap \mathcal{D}^{\geq M-1}, M \in \mathbb{Z}$.
(ii) $\Phi$ descends to an automorphism $[[\Phi]]$ of $\text{im}(Z) = K(T) / \ker(Z)$.

(iii) for $v, w \in \text{im}(Z)_{\text{eff}} \cap \text{comp}$, the relation $\phi_f(v) \preceq \phi_f(w)$ implies $\phi_f([[\Phi]](v)) \preceq \phi_f([[\Phi]](w))$: here $\phi_f$ is the "implied phase" obtained from a log branch.

then $\Phi$ is compatible with $(Z, \mathcal{P})$.

Proof. Let $(Z_\theta, \mathcal{P}_\theta)$ be the stability condition $(Z, \mathcal{P}) \cdot e^{i\pi \theta}$, with heart $A_\theta \cong \mathcal{P}(\theta, \theta + 1] = \mathcal{P}_\theta(0, 1]$. We use the notation $M_\theta, m_\theta, \phi_f^\theta$, and $\text{im}(Z)_{\text{comp}}$ for the obvious objects associated to $(Z_\theta, \mathcal{P}_\theta)$. Note that generally, $\text{im}(Z)_{\text{comp}}$ will not be equal to $\text{im}(Z)^0_{\text{comp}}$: the log branch will change, thus changing which elements of $\text{im}(Z)$ are "comparable". We claim if $\Phi$ satisfies (i), (ii), and (iii) above for $(Z, \mathcal{P})$, then it will also satisfy them for $(Z_\theta, \mathcal{P}_\theta), \theta \in \mathbb{R}$. Only (i) and (iii) are unclear. Once this is shown, it is easy to see that $\Phi$ is semistable preserving: if not, we can choose $\theta$ to ensure $\Phi(F)$ is not cohomologically pure (in the t-structure with heart $A_\theta$). The full compatibility easily follows.

To show that $\Phi$ satisfies (i), (ii), and (iii) with regard to $(Z_\theta, \mathcal{P}_\theta)$ it is enough to assume that $\theta \in (0, 1]$, for the shift $[1]$ is strongly compatible. We split the argument into the cases $\theta < m_\theta$ and $\theta \geq m_\theta$. The main difficulty with the latter is showing (i) is satisfied: one encounters problems understanding $\Phi(\mathcal{P}(m_\theta, \theta)[1])$. We first handle $\theta \leq m_\theta$.

By definition, $A_\theta$ is the extension-closed subcategory of $T$ generated by $\mathcal{P}(\theta, m_\theta], \mathcal{P}(m_\theta, 1]$, and $\mathcal{P}(0, \theta)[1]$. Let $m_\theta' \in \mathbb{R}$ be the maximal number such that for semistable $F \in \mathcal{P}(\theta, 1], \phi_f(\Phi(F)) > m_\theta'$ (it exists for the same reason $m_\theta$ exists). Our choice of $\theta$ and $M_\theta$ ensures $0 < m_\theta' \leq 1$. If $m_\theta' \geq \theta$, then by Proposition [3,9] and condition (iii)

$$\Phi(\mathcal{P}(\theta, m_\theta]) \subset \mathcal{P}(\theta, 1] \subset A_\theta,$$

$$\Phi(\mathcal{P}(0, \theta)[1]) \subset \mathcal{P}(1, 2] \subset D_{\theta}^0 \cap D_{\theta}^{-1},$$

$$\Phi(\mathcal{P}(m_\theta, 1]) \subset \mathcal{P}(1, 2] \subset D_{\theta}^0 \cap D_{\theta}^{-1}.$$

Thus showing that $\Phi$ satisfies (i) in $(Z_\theta, \mathcal{P}_\theta)$ with $M_\theta = 0$. On the other hand, if $m_\theta' < \theta$, then

$$\Phi(\mathcal{P}(\theta, m_\theta']) \subset \mathcal{P}(\theta - 1, 1] \subset D_{\theta}^0 \cap D_{\theta}^{-1},$$

$$\Phi(\mathcal{P}(0, \theta)[1]) \subset \mathcal{P}(1, m_\theta') \subset D_{\theta}^{-1} \cap D_{\theta}^{2},$$

$$\Phi(\mathcal{P}(m_\theta, 1]) \subset \mathcal{P}(1, m_\theta' + 1] \subset D_{\theta}^{-1} \cap D_{\theta}^{2}.$$

Again showing that $\Phi$ satisfies (i) ($M_\theta = 1$) with respect to $(Z_\theta, \mathcal{P}_\theta)$.

To show (iii) is satisfied; we only need to check the case of strict inequality since equality is clearly satisfied. Any choice of log branch splits $\mathbb{C}^*$ into two symmetric pieces. $\mathbb{H'}$ is adapted to the branch with discontinuity at the positive real axis. By definition, $\text{im}(Z)^0_{\text{comp}}$ is defined using $\mathbb{H}' \cdot e^{i\pi \theta}$ and a log branch with discontinuity at the ray of angle $\pi \theta$ (relative to the positive real axis). We can assume that on the positive real axis, the phases of the two branches agree. With this description, we can rephrase (iii) for $\text{im}(Z)^0_{\text{comp}}$ in terms of $\text{im}(Z)_{\text{eff}} \cap \text{im}(Z)^0_{\text{comp}}$.

Let $m_\theta' \geq \theta$. For $v, w \in \text{im}(Z)_{\text{eff}} \cap \text{im}(Z)^0_{\text{comp}} \cap \text{im}(Z)^0_{\text{comp}}$, (iii) is clear: $v, w, [[\Phi]]v$, and $[[\Phi]]w$ will have the same phase assignment in both branches. In the case $w \notin \text{im}(Z)^0_{\text{comp}}$, $Z([[\Phi]]w) \in \mathbb{H}'$. Our choice of log branch for $\phi_f^\theta$ ensures $\phi_f^\theta([[\Phi]]w) > 2$. Since $v \notin \text{im}(Z)^0_{\text{comp}}$, $\phi_f^\theta([[\Phi]]v) \leq 2$, thus handling this case. To handle $v, w \notin \text{im}(Z)^0_{\text{comp}}$ one first notes Lemma [3,7] implies $m_\theta + 1 < \phi_f(v)$. Therefore, $v, w \in \text{im}(Z)^0_{\text{comp}} \cap \text{im}(Z)^0_{\text{comp}}$. The linearity of $[[\Phi]]$, and the above case then gives the result.
To handle $m'_{\theta} < \theta$ it suffices to prove (iii) for $[1] \circ \Phi$. For this autoequivalence, identical arguments as above can be used.

We have now shown $\Phi$ satisfies (i), (ii), and (iii) with regard to $(Z_\theta, P_\theta)$ for $\theta \leq m_\theta$. Consider the sequence $\omega_1 = m_0$, $\omega_2 = m_\omega$, etc. The sequence has the property that $\Phi$ satisfies the conditions stated above for each $(Z_\omega, P_\omega)$. Note if the conditions are met for $(Z_\omega, P_\omega)$ with $\theta - \omega \leq m_\omega$, then the above arguments ensure that it is true for $\theta$. This implies if $\lim_{i \to \infty} \omega_i = \infty$, then we are done. Thus to finish our claim that $\Phi$ satisfies (i), (ii), (iii) for all $(Z_\theta, P_\theta)$ it suffices to show if $\lim_{i \to \infty} \omega_i = \kappa < \infty$ then $\Phi$ satisfies the condition for $(Z_{\kappa + \rho}, P_{\kappa + \rho})$ for $\rho \in [0,1)$.

We first show show that $\Phi(P_{[\kappa, \kappa + 1]}) \subset P_{[\kappa + 1, \kappa + 2]}$. Let $F \in P_{[\kappa, \kappa + 1]}$ be a semistable object. Then for some $j$ with $\psi_j$ small (or possibly zero), $F \in A_{\omega_j}$ and $\phi(F) > m_{\omega_j}$. By Proposition 3.9, $\phi(F) \in \mathcal{A}_{\omega_1}$. If $\phi(F) \notin P_{[\kappa, \kappa + 1]}[1]$, then there exists at least one semistable factor of $\phi(F)$ not contained $\mathcal{B}_{\kappa}[1]$. The definition of $\kappa$ implies that we must have a $k > j$ such that $F \in A_{\omega_k}$, yet $\phi(F) \notin \mathcal{A}_{\omega_k}[1]$. Using Lemma 3.12 again, $\phi(F) \in \mathcal{A}_{\omega_k}$ resulting in $\phi(F) < m_{\omega_k}$, a contradiction.

The autoequivalence $[-1] \circ \Phi$ restricts to an element in $\text{Aut}(P_{[\kappa, \kappa + 1]})$. Clearly this is enough to show $[-1] \circ \Phi$ satisfies (i) with respect to $(Z_\kappa, P_\kappa)$. For (iii), note that given $v, w \in \text{im}(Z)_\kappa$, there exists $\omega$ such that $v, w \in \text{im}(Z)_\omega$. The condition is assumed true for $v, w \in \text{im}(Z)_\omega$. We can choose our log branches for $\omega$ and $\kappa$ in a compatible manner such that $\phi_j(\omega)$ and $\phi_j(\kappa)$ agree on $v, w, [\phi(\omega)]v$ and $[\phi(\kappa)]w$, thus showing that (iii) is satisfied.

We can therefore apply the above machinery to $\rho < \kappa$ (with respect to $[-1] \circ \Phi$). However, since $[-1] \circ \Phi(P_{[\kappa, \kappa + 1]}) \subset P_{[\kappa, \kappa + 1]}$, it is clear that if $m_{\omega_k} \neq 1$ and $\rho > \kappa$, then $(Z_{\kappa + \rho}, P_{\kappa + \rho})$ and $(Z_{\kappa + m_{\omega_k}}, P_{\kappa + m_{\omega_k}})$ differ only by a shift of “fake” phases. This proves our claim.

We will now show that $\Phi$ is compatible with $(Z, P)$. Given $F$ semistable in $(Z, P)$ with $\phi(F) = \eta$, suppose that $\Phi(F)$ is not semistable. The Harder-Narasimhan filtration of $\Phi(F)$ gives $\phi_-(\Phi(A)) < \phi_f(\Phi(F) < \phi_0(\Phi(A))$. If we set $\theta = \phi_F(\Phi(F)$, in the stability condition $(Z_\theta, P_\theta)$, $H^0(\Phi(F)) = 0$ and $H^1(\Phi(F)) = 0$. From above, we know that $\Phi$ satisfies conditions (i), (ii), and (iii) for $(Z_\theta, P_\theta)$, allowing us to apply Lemma 3.12 and get a contradiction. Therefore $\Phi(F)$ is semistable in $(Z, P)$, with phase $\theta$. We can therefore remove the “$f$” from $\phi_f$. Condition (iii) is then the condition for compatibility.

\begin{corollary}
If $\Phi$ is compatible with $(Z, P)$ then it is compatible with $(Z, P) \cdot g$ for $g \in GL^+(2, \mathbb{R})$.
\end{corollary}

\begin{corollary}
Let $\Phi$ satisfy all conditions of Theorem 3.13. Then $\Phi$ is strongly compatible if and only if the induced automorphism on $\text{im}(Z)$ extends to an orientation preserving $\mathbb{R}$-linear automorphism of $\mathbb{C}$.
\end{corollary}

In the case that $K(T)$ is finite rank, Theorem 3.13(ii) can be rephrased.

\begin{proposition}
$\Phi$ descends to an automorphism of $\text{im}(Z)$ if and only if $\Phi$ restricts to an automorphism of $\ker Z$. Further, if $\text{rank}(\ker Z)$ is finite, this condition is equivalent to $\Phi(\ker Z) \subseteq \ker Z$.
\end{proposition}

\begin{proof}
The first statement is a consequence of the triangle axioms in the derived category of $\mathbb{Z}$-modules. For the second, it is clearly a necessary condition. For sufficiency, first note that the map is injective, so we just need to show surjectivity. If not surjective, there exists an element $a \in \ker(Z) \notin \Phi(\ker Z)$, thus $\Phi^{-1}(\ker Z) \nsubseteq \ker Z$. Our assumption can be
rephrased as $\ker Z \subset \Phi^{-1}(\ker Z)$. The equivalence of rank (since both are finite rank) implies that $\Phi^{-1}(\ker Z)/\ker Z$ is a torsion group. However, $\Phi^{-1}(\ker Z)/\ker Z \subset \text{im } Z \cong \text{im } Z \subset \mathbb{C}$ which has no torsion points, a contradiction. Thus $\Phi$ is an automorphism when restricted to $\ker(Z)$. \hfill \qed

4. N-GONS

This chapter is a collection of facts and definitions about $n$-gons and their derived categories. We define the “classical” stability condition, which will be used throughout the rest of this paper.

4.1. The geometry of $n$-gons. Let $E_n$, $n \in \mathbb{N}$, denote the $n$-gon: projective singular reducible curves consisting of a cycle of $n$ components, all isomorphic to $\mathbb{P}^1$, with nodal singularities (i.e. transverse intersections). $E_m$ is a Galois cover of $E_n$ if and only if $n|m$. In particular, n-gons are Galois covers of the Weierstrass nodal cubic. We fix a consistent choice of covering maps $\{\pi_{m,n}\}_{m \times n} (\pi_{m,n} \in \text{Hom}(E_m, E_n))$ and deck transformations $\{l_{m,n}\}_{m \times n} (l_{m,n} \in \text{Gal}(E_m, E_n))$ satisfying $\pi_{m,n} \circ l_{l,m} = \pi_{l,n}$ and $\pi_{m,n} \circ l_{m,t} = \pi_{n,t}$ for $l|n|m$. Denote by $l_{m,n}$ a consistent choice of generator for the $\text{Gal}(E_m, E_n) \cong (m)/(n) \cong \mathbb{Z}_d$ (where $n = lm$). We often times omit the second number when it is 1 e.g. $\pi_{m}$ for $\pi_{m,1}$.

It is implicit in the above definitions that $\pi_1(E_n) \cong \mathbb{Z}$.

The normalization of $E_n$ is $\Pi_n \mathbb{P}^1$. Obviously the normalization map $\eta : \Pi_n \mathbb{P}^1 \to E_n$ is an isomorphism away from the singular locus. It will become important to refer to individual components. To do so, we index the components with $Z/n\mathbb{Z}$ by arbitrarily choosing the “first” component and continuing in a clockwise or counterclockwise manner, depending on our choice of $l_{m,n}$.

We will also need the projective genus 0 singular curves $\mathbb{I}_m$. These curves are a chain of $m$ reducible components, all isomorphic to $\mathbb{P}^1$. They can be obtained as partial normalizations of the $m$-gon at any one of its singular points.

![Figure 3. E_6 and I_4](image)

4.2. $K(D^b(E_n))$ and $\sigma_{cl}(n)$. To define our stability conditions, we need to calculate $K(D^b(E_n))$. This is a well known calculation: $K(D^b(E_n)) \cong \mathbb{Z}^{n+1}$. To see this, one just needs to analyze short exact sequences supplied by the adjunction map $id \to \eta_*\eta^*$, see [RGP09, Proposition 2.3]. Our preferred basis for $K(D^b(E_n))$ consists of $e_0 = [k(p)]$ for a smooth point $p \in E_n$, and $e_i = [\eta_* \mathcal{O}_{\mathbb{P}^1}(-1)]$, $0 < i \leq n$. Define $\text{rank}_{i}(\mathcal{F})$ as the dimension of the vector space obtained by restricting $\mathcal{F}$ to the generic point of the $i$th component. Clearly $[\mathcal{F}] = \chi(\mathcal{F})e_0 + \Sigma_{0 \leq i \leq n} \text{rank}_{i}(\mathcal{F})e_i$.

Let $\sigma_{cl}(n) = (Z_{cl}, \mathcal{P}_{cl})$ denote the stability condition with charge $Z_{cl}(\mathcal{F}) = -\chi(\mathcal{F}) + i \text{rank}_{tot}(\mathcal{F})$, with heart $\text{Coh}(E_n)$. Here $\text{rank}_{tot}$ designates the function $\Sigma_{0 \leq i \leq n} \text{rank}_{i}$. It is clear that $\ker(Z_{cl})$ is finite rank and $\text{im}(Z_{cl})$ is rank 2.
4. Reduction of conditions in Theorem 3.13 for applicable and natural conditions.

The purpose of showing how the conditions of Theorem 3.13 many times reduce to easily posable torsion free sheaf on $E_n$. This stability condition is an extension of classical slope to the case of $n$-gons. Given a torsion free $F \in \text{Coh}(E_n)$, one can define its slope $\psi(F) := \frac{\chi(F)}{rk(F)}$. In the case that $F$ is semi-stable (in $\sigma_{cl}(n)$), the conversion between $\psi(F)$, its slope, and $\phi(F)$, its phase, is given by

$$\psi(F) = -\cot(\pi \phi(F)).$$

4.3. Classification of torsion free sheaves.

**Theorem 4.1 ([BBDG06] Theorem 1.3).** With $E_n$ and $\mathbb{E}_k$ as above, let $E$ be an indecomposable torsion free sheaf on $E_n$. 

1. If $E$ is locally free, then there is an étale covering $\pi_{nr,n} : E_{nr} \to E_n$, a line bundle $L \in \text{Pic}(E_{nr})$, and a natural number $m \in \mathbb{N}$ such that

$$E \cong \pi_{nr,r}(L \otimes F_m),$$

where $F_m$ is an indecomposable vector bundle on $E_{nr}$, recursively defined by the sequences

$$0 \longrightarrow F_{m-1} \longrightarrow F_m \longrightarrow O_{E_{nr}} \longrightarrow 0, \quad m \geq 2, \quad F_1 = O_{E_{nr}}.$$

2. If $E$ is not locally free then there exists a finite map $p_k : I_k \to E_n$ and a line bundle $L \in \text{Pic}(I_k)$ (where $k, p_k, \Lambda$ and $L$ are determined by $E$) such that $E \cong p_{k+1}(L)$.

**Example 4.2.** If $V \cong \pi_{nr,r}(L \otimes F_m)$ with $m > 1$, then $\dim \text{End}(V) > 1$.

5. Compatible autoequivalences of $D^b(E_n)$

In this section we will apply Theorem 3.13 to the example of $n$-gons. This serves the purpose of showing how the conditions of Theorem 3.13 many times reduce to easily applicable and natural conditions.

5.1. Reduction of conditions in Theorem 3.13 for $\sigma_{cl}(n)$.

**Lemma 5.1.** Given $\Phi \in \text{Aut}(D^b(E_n))$, suppose $\Phi \circ \iota^*_n \cong \iota^*_n \circ \Phi$ for some $0 < q \leq n$, then $\Phi$ preserves the kernel of $Z_{cl}$.

**Proof.** By definition, an element $t \in K(D^b(E_n))$ is in the kernel of $Z_{cl}$ if and only $t = a_1 e_1 + \ldots + a_n e_n$ with $\Sigma a_i = 0$. We have $[\iota^*_n]e_i = [\iota^*_n][\eta e_i] = [\eta e_i] = e_i$ (if $i = n$ then $i + 1 = 1$). Thus $[\iota^*_n]$ acts as the identity on $e_n$, and cyclically permutes $\{e_i\}_{Z/nZ}$. Since $rk_{tot} = \sum_{0 \leq i \leq n} e_i$, it is clear that $[\iota^*_n] \cdot rk_{tot} = rk_{tot}$ and $[\iota^*_n] \cdot \chi = \chi$. Clearly the same holds for powers of $\iota^*_n$.

Using the commuting relation of $\Phi$ and $\iota^*_n$, we can write

$$[\Phi]t = \sum_{0 \leq i \leq n} a_i ([\iota^*_n]^{(i-1)*}([\Phi]e_1)).$$

Applying $rk_{tot}$ to the left and right yields

$$rk_{tot}([\Phi]t) = rk_{tot}([\Phi]t) = \sum_{0 \leq i \leq n} a_i ([\iota^*_n]^{(i-1)*}([\Phi]e_1))$$

$$= \sum_{0 \leq i \leq n} a_i ([\iota^*_n]^{(i-1)*}([\Phi]e_1))$$

$$= \sum_{0 \leq i \leq n} (a_i \cdot rk_{tot}([\iota^*_n]^{(i-1)*}([\Phi]e_1)))$$

$$= \sum_{0 \leq i \leq n} (a_i \cdot rk_{tot}([\Phi]e_1))$$

$$= (0 \ast rk_{tot}([\Phi]e_1))$$

$$= 0 \ast rk_{tot}([\Phi]e_1)$$

$$= 0.$$
These equations only used the additive property and invariance of \( \text{rk}_{\text{tot}} \) under \( \iota_n^* \). Thus the same will be true for \( \chi \) and \( [\Phi] \) is in the kernel of \( Z_{cl} \). □

Lemma 5.2. Assume \( \text{rank}(\text{im}(Z)) = 2 \) and \( \text{im}(Z) \otimes \mathbb{R} = \mathbb{C} \). Then \( \Phi \) satisfies Theorem \( 3.13 \) (iii) if and only if \( [\Phi] \in \text{SL}(2, \mathbb{Z}) \).

Proof. Since \( \text{im}(Z) \) is rank 2, any automorphism that descends to the image will naturally be an element of \( \text{GL}(2, \mathbb{Z}) \). Thus \( [\Phi] \) extends to a \( \mathbb{R} \)-linear automorphism of \( \mathbb{C} \). Clearly, Theorem \( 3.13 \) (iii) is true if and only if the extension of \( [\Phi] \) is orientation preserving, and as such \( [\Phi] \in \text{SL}(2, \mathbb{Z}) \). □

5.2. The autoequivalence group \( \text{Aut}_{cl}(n) \). Using these reductions we will now produce the subgroup \( \text{Aut}_{cl}(n) \subset \text{Aut}(\text{D}^b(\mathbb{E}_n)) \) of autoequivalences compatible with \( \sigma_{cl}(n) \). We obtain this subgroup by explicitly constructing autoequivalences of \( \text{D}^b(\mathbb{E}_n) \). This group will be an extension of \( \Gamma_0(n) \subset \text{SL}(2, \mathbb{Z}) \), the congruence subgroup consisting of elements that are upper triangular under reduction of coefficients \( \text{SL}(2, \mathbb{Z}) \to \text{SL}(2, \mathbb{Z}/n\mathbb{Z}) \). We begin by lifting autoequivalences of \( \text{D}^b(\mathbb{E}_1) \) to endomorphisms of \( \text{D}^b(\mathbb{E}_n) \). Once this shown, we ascertain that these endomorphisms are strongly compatible autoequivalences.

Proposition 5.3. Given \( K \in \text{D}^b(\mathbb{E}_1 \times \mathbb{E}_1) \) such that \( \Phi_K \in \text{Aut}(\text{D}^b(\mathbb{E}_1)) \) and \( [\Phi_K] \in \Gamma_0(n) \subset \text{SL}(2, \mathbb{Z}) \), there exists \( K_n \in \text{D}^b(\mathbb{E}_n \times \mathbb{E}_n) \) with \( (\pi_n \times id)_* K_n \cong (id \times \pi_n)^* K \). This sheaf is unique up to the action of the covering transformations of \( (\pi_n \times id) \).

Proof. First, we need to understand how \( \Phi_K \) acts on \( \text{D}^b(\mathbb{E}_1) \). As shown in [BK05], \( \text{Stab}(\mathbb{E}_1) \cong \text{GL}^+(2, \mathbb{R}) \), the isomorphism is provided by \( \sigma_{cl}(1) \) and the natural right action on the stability manifold. The left action by \( \text{Aut}(\text{D}^b(\mathbb{E}_1)) \) implies all autoequivalences are strongly compatible with all stability conditions. Since \( k(p), p \in \mathbb{E}_1 \), are stable in all stability conditions, this shows that \( \mathcal{V}_p[M] := \Phi_K(k(p)) \) is a shifted stable indecomposable vector bundle (torsion free sheaf) for \( p \) smooth (singular), with \( M, \text{rank}(\mathcal{V}_p) = n \) and \( \chi(\mathcal{V}_p) = d \) uniform for all \( p \). The assumption \( [\Phi_K] \in \Gamma_0(n) \) ensures \( d \neq 0 \). Without loss of generality, we can assume that \( M = 0 \) and by [Huy06], Lemma 3.31] that \( K \) is a sheaf. With these assumptions, it is clear that \( K|_{p \times \mathbb{E}_1} = \mathcal{V}_p \). For clarity, we will first handle the case that \( \text{rank}(\mathcal{V}_p) = n \). Once this is shown, we will comment on how the general case proceeds.

By Proposition 4.1, \( \mathcal{V}_p \cong \pi_n\mathcal{L}_p \) for some torsion free rank 1 bundle on \( \mathbb{E}_n \). Using base change and the Cartesian diagram (since \( \mathbb{E}_n \) is a Galois cover)

\[
\begin{array}{ccc}
\mathbb{Z}_n \times \mathbb{E}_n & \longrightarrow & \mathbb{E}_n \\
\downarrow & & \downarrow \\
\mathbb{E}_n & \longrightarrow & \mathbb{E}_1
\end{array}
\]

there exists an isomorphism \( \pi_n^* \pi_n\mathcal{L}_p \cong \oplus_{0 \leq k < n} t^k_n \mathcal{L}_p \). Thus, \( ((id \times \pi_n)^* K)|_{p \times \mathbb{E}_n} \cong \oplus_{0 \leq k < n} t^k_n \mathcal{L}_p \). Clearly, this isomorphism is trivially affected by the choice of \( \mathcal{L}_p \); the choice was ambiguous up to the action of \( \iota^*_n \).

From this description, the stability properties of \( \mathcal{V}_p \) gives

\[
R \Gamma^i((id \times \pi_n)^* K)|_{p \times \mathbb{E}_n}) = \begin{cases} 
0 & \text{if } i \neq 0 \text{ and } d > 0 \\
0 & \text{if } i \neq 1 \text{ and } d < 0 \end{cases}
\]

Temporarily, we assume \( d > 0 \) and surjectivity of the natural map \( \Gamma(\mathcal{V}_p) \otimes \mathcal{O}_{\mathbb{E}_1} \to \mathcal{V}_p \) for all \( p \); this will be justified at the end of the proof. Letting \( \rho_1 \) designate the projection
E_1 \times E_n \to E_1$. [Mum70 Corollary 2] implies \( \mathcal{F} := \rho_1^*((id \times \pi_n^*)z(\mathcal{K})) \) is a rank \( d \) vector bundle on \( \mathbb{E}_1 \), with fiber \( \mathcal{F}_p := \oplus_{0 \leq j < n} \Gamma((\mathcal{L}_p)^j) \). Our choice of \( d \) now implies that the natural adjunction map \( \rho^* \mathcal{F} \to (id \times \pi_n)^* \mathcal{K} \) is a surjection.

We want to find a sheaf, \( \mathcal{F}_n \) on \( \mathbb{E}_n \) such \( \pi_n^*(\mathcal{F}_n) \cong \mathcal{F} \). We will give \( \mathcal{F} \) a natural \( \pi_n^*(\mathcal{O}_{\mathbb{E}_n}) \) module structure, giving the existence of a lift. The algebraic nature of \( \pi_n \) then ensures the existence of \( \mathcal{F}_n \). One has the natural splitting \( \pi_n^*(\mathcal{O}_{\mathbb{E}_n}) \cong \oplus \mathcal{O}(\xi_i) \) where \( \mu_n \) are the \( n \)th roots of unity. Likewise, the \( \mathbb{Z}/n\mathbb{Z} \) action on \( \mathcal{F} \) induces a decomposition by eigenbundles: \( \mathcal{F} \cong \oplus \mu_n \mathcal{F}(\xi_i) \). The natural module action is then described summand wise as the natural maps \( \mathcal{O}(\xi^i) \otimes \mathcal{F}(\xi^i) \to \mathcal{F}(\xi^{i+k}) \). Clearly this makes \( \mathcal{F} \) a module over \( \mathcal{O}_{\mathbb{E}_n} \).

The cartesian diagram

\[
\begin{array}{c}
\mathbb{E}_n \\
\downarrow \rho_n \\
\mathbb{E}_n \\
\downarrow \pi_n \\
\mathbb{E}_1
\end{array}
\]

\[
\begin{array}{c}
\mathbb{E}_n \\
\downarrow \rho_n \\
\mathbb{E}_n \\
\downarrow \pi_n \\
\mathbb{E}_1
\end{array}
\]

gives a canonical surjective morphism \( \tau : (\pi_n \times id)_* \circ \rho_n^* (\mathcal{F}_n) \to (id \times \pi_n)^* \mathcal{K} \) (using flat base change and adjunction). Adjunction again gives \( v : (\pi_n \times id)^* \ker(\tau) \to \rho_n^* (\mathcal{F}_n) \). Define \( \mathcal{K}_n := \text{coker}(v) \). The flatness of \( \pi_n \times id \) gives an exact sequence

\[
(\pi \times id)_* \circ (\pi_n \times id)^* \ker(\tau) \to (\pi_n \times id)_* (\rho_n^* (\mathcal{F}_n)) \to (\pi_n \times id)_* \mathcal{K}_n \to 0.
\]

By adjunction we have a factorization \( \ker(\tau) \to (\pi \times id)_* \circ (\pi_n \times id)^* (\ker(\tau)) \to (\pi_n \times id)_* \rho_n^* \mathcal{F}_n \), giving a surjective morphism \( (id \times \pi_n)^* \mathcal{K} \to (id \times \pi_n)_* \mathcal{K}_n \). By comparing dimensions of fibers, this is an isomorphism. Thus \( (\pi_n \times id)_* (\mathcal{K}_n) \cong (id \times \pi_n)^* (\mathcal{K}) \).

We now justify our assumption on \( d \). Letting \( \rho_2 \) designate the projection of \( \mathbb{E}_1 \times \mathbb{E}_i \) onto the second factor, the ample nature of \( \mathcal{L}(1; \omega) \) implies the construction works for \( \mathcal{K} := \mathcal{K} \otimes \rho_2^* (\mathcal{L}(m; \omega)) \) for \( m \ll d \). Thus we have an isomorphism \( (\pi_n \times id)_* \mathcal{K}_n \cong (id \times \pi_n)^* (\mathcal{K}) \cong (\rho_2^* \circ \pi_n)^* (\mathcal{L}(m; \omega)) \). From this, it is clear how to find \( \mathcal{K}_n \), thus justifying our assumption. We are done for the case that \( \text{rank}(\mathcal{K}) = n \).

Finally, the case that \( n < \text{rank}(\mathcal{V}_p) = m \). The adjustments are more of a technical inconvenience: we have \( \mathcal{V}_p \cong \pi_{m*} \mathcal{L}_p \cong \pi_{m*} \circ \pi_m^* (\mathcal{L}_p) \). Therefore, rather then having a fiberwise decomposition of \( (id \times \pi_n)^* \mathcal{V}_p \) into rank 1 sheaves, we decompose into \( n \) rank \( \frac{m}{n} \) vector bundles (torsion free sheaves). Once this change is made, the construction proceeds in a straightforward manner.

\[ \square \]

**Remark 5.1.** The more natural way of viewing this construction (and the original motivation) is that \( (id \times \pi_n)^* \mathcal{K} \) has the natural splitting fiberwise given above. This gives an explicit description for the monodromy associated to the generator of \( \pi_1(\mathbb{E}_1) \). This monodromy has order \( n \) and makes the choice of a summand in the fiberwise splitting a “cyclic vector”. This implies that there is a natural lift on \( \mathbb{E}_n \times \mathbb{E}_n \) that pushes forward to \( (id \times \pi_n)^* \mathcal{K} \). The argument given in the above proof has the benefit of not losing any data contained in \( \mathcal{K} \).

Now that \( \mathcal{K}_n \in \text{D}^b(\mathbb{E}_n \times \mathbb{E}_n) \) is constructed, we aim to understand \( \Phi_{\mathcal{K}_n} \). With this in mind, we prove some basic properties of \( \mathcal{K}_n \).

**Lemma 5.4.** Let \( \mathcal{K} \) be as in Proposition 5.2. Then \( \mathcal{K}_n \) has the following properties:

1. \( \pi_n^*(\mathcal{K}_n|_{\mathbb{E}_n}) \cong \mathcal{K}|_{\mathbb{E}_n \times \mathbb{E}_n} \) and therefore is a torsion free sheaf.
2. For any two distinct \( p_1, p_2 \), \( \text{RHom}^i(\mathcal{K}_n|_{\mathbb{E}_n}, \mathcal{K}_n|_{\mathbb{E}_n}) = 0 \) for all \( i \in \mathbb{Z} \).
Proof. The first property is clear from the construction. The second follows from the inclusion

\[ \text{RHom}^j(K_n|_{p_1 \times E_n}, K_n|_{p_2 \times E_n}) \leftrightarrow \text{RHom}^j(K_n|_{p_1 \times E_n}, \oplus_{0 \leq \ell < n} i_{\ell, n}^* K_n|_{p_2 \times E_n}) \]

\[ \cong \text{RHom}^j(\pi_{n*} K_n|_{p_1 \times E_n}, \pi_{n*} L|_{p_2 \times E_n}). \]

We have two cases: \( p_2 \neq i_{n}^k p_1 \) and \( p_2 = i_{n}^k p_1 \) for some \( k \). For the former, it follows from the original family \( K \); for the latter, it follows from the stability properties of the fibers of \( K \).

\[ \square \]

Lemma 5.5. \( K_n \) is a sheaf and flat over \( E_n \) with regards to projection onto the first factor.

Proof. From Lemma 5.4 \( \Phi_{K_n}(k(p)) \) is a torsion free sheaf for all \( p \in E_n \). However, letting \( \rho_i \) denote the projections of \( E_n \times E_n \), we have

\[ \Phi_{K_n}(k(p)) = R\rho_2^* (\rho_1^* k(p) \otimes^L K_n) \]

\[ \cong R\rho_2^* (i_* O_{E_n} \otimes^L K_n) \]

\[ \cong R\rho_2^* i_* (Ri^* K_n) \]

But, \( i \circ \eta_2 = \text{id} \). Thus \( \Phi_{K_n}(k(p)) \cong Ri^* K_n \). Since the left side is a torsion free sheaf uniformly concentrated in a single cohomological degree for all \( p \), the result follows from [Huy06 Lemma 3.31].

\[ \square \]

It is now clear that \( K_n \) inherits many nice properties from \( K \). It is not surprising then that there is a nice relationship between \( \Phi_K \) and \( \Phi_{K_n} \).

Lemma 5.6. \( \Phi_{K_n} \circ \pi_{n*} \cong \pi_{n*} \circ \Phi_K \)

Proof. We have the following diagram
where every square is Cartesian. The result is a calculation using the above diagram, base change, and the projection formula.

\[
\pi_n^* \circ \Phi_{K_n} (F) = \pi_n^* \eta_{2*} (\eta_1^* F \otimes K) \\
\cong \sigma_{2*} \gamma_1^* (\eta_1^* F \otimes K) \\
\cong \sigma_{2*} (\gamma_1^* \eta_1^* F \otimes \sigma_1^* K) \\
\cong \sigma_{2*} \phi_{2*} (\phi_1^* \gamma_2^* \eta_1^* F \otimes K_n) \\
\cong \sigma_{2*} \phi_{2*} (\phi_1^* \gamma_2^* \pi_n^* F \otimes K_n) \\
\cong \Phi_{K_n} (\pi_n^* F).
\]

\[\square\]

With this initial analysis done, we now show \(\Phi_{K_n}\) is strongly compatible with \(\sigma_{cl}(n)\). In order to apply Theorem 3.13, we must first know \(\Phi_{K_n}\) is an equivalence. Once this is shown, it is simply a matter of verifying the reduced conditions derived in the previous section.

**Proposition 5.7.** \(\Phi_{K_n}\) is an equivalence.

**Proof.** Let \(K \in D^b(E_1 \times E_1)\) satisfy Proposition 5.3. Recall that \(\Phi_K\) is an autoequivalence if and only if the sheaf \(K'[1]\) satisfies \(K * K'[1] \cong O_{\Delta}\). Setting \(K' = K[1]\). Proposition 5.3 constructs two (shifted) sheaves \(K_n\) and \(K'_n\) corresponding to these two kernels. As noted above, there is some ambiguity in the definition of \(K_n\): there are in fact \(n\) choices that will satisfy the properties listed in Proposition 5.3. We will correct for this momentarily.

Lemma 5.6 gives an isomorphism of functors

\[
\pi_n^* \circ \Phi_{K'} \circ \Phi_K \cong \Phi_{K'_n} \circ \pi_n^* \circ \Phi_K \cong \Phi_{K'_n} \circ \Phi_{K_n} \circ \pi_n^*.
\]

Applying these isomorphisms to \(\oplus_{0 \leq k < n} k(t_n(p))\) for any \(p \in E_n\), yields the isomorphism \(\Phi_{K'_n} \circ \Phi_{K_n} (\oplus_{0 \leq k < n} k(t_n(p))) \cong \oplus_{0 \leq k < n} k(t_n(p))\). Correcting by some power of \(t_n\) we can ensure that \(t_n^* \circ \Phi_{K_n} \circ \Phi_{K_n}(k(p)) = k(p)\). The geometric origin of our autoequivalence, the connectedness of \(E_n\), and the fact that \(\{t_n^*(p)\}\) is a discrete subscheme combine to imply this identity holds for all \(p\) (here we are using the continuity properties that our kernel affords).

Thus we know \(t_n^* \circ \Phi_{K'_n} \circ \Phi_{K_n}\) acts as the identity on \(k(p)\) for all \(p \in E_n\). By [BK06b, Lemma 2.11] we know \(t_n^* \circ \Phi_{K'_n} \circ \Phi_{K_n}\) is isomorphic to an autoequivalence obtained by tensoring by a line bundle, thus the result.

**Proposition 5.8.** \(\Phi_{K_n}\) is strongly compatible with \(\sigma_{cl}(n)\) on \(E_n\).

**Proof.** According to Theorem 3.13 and the reductions carried out in [5.1] we need to verify that

1. \(\mathcal{H}^i(\Phi_{K_n}(F)) = 0\) for \(i \neq M, M + 1, M \in \mathbb{Z}\) and \(F \in \mathrm{Coh}(E_n)\)
2. \(\Phi_{K_n} \circ t_n^* \cong t_n^{q*} \circ \Phi_{K_n}\), for some \(0 \leq q < n\).
3. \([\Phi_{K_n}]\) descends to an element in \(\mathrm{SL}(2, \mathbb{Z})\).

(1) Without loss of generality, we can assume that \(M = 0\). Let \(\rho_1\) designate the \(i^{th}\) projection \(E_n \times E_n \xrightarrow{\rho_1} E_n\). By Lemma 5.5 \(K_n\) is flat over \(\rho_1\), and therefore \(\rho_1^* F \otimes K_n \cong \rho_1^* F \otimes K_n\). Thus, the assertion reduces to showing \(R\rho_2^*(G) = 0\) for \(G \in \mathrm{Coh}(E_n \times E_n)\), \(j > 1\). This is true since \(\rho_2\) is a fibration with fiber dimension 1.
(2) Proposition 2.4 shows the left side is isomorphic to $\Phi_{(\iota_n \times \text{id}), \mathcal{K}_n}$ and the right is isomorphic to $\Phi_{(\text{id} \times \iota_q^n) \circ \mathcal{K}_n} \cong \Phi_{(\iota_q^n) \circ \mathcal{K}_n}$. By construction $(\iota_n \times \text{id})^* \mathcal{K}_n \cong (\text{id} \times \iota_q^n)^* \mathcal{K}_n$, where $q$ is coprime to $n$ and $a = \text{rank}(\Phi(k(p)))$. Using $(\iota_n \times \text{id})_* \cong (\iota_q^n \times \text{id})_{n-1}$ then gives the result.

(3) From (2) we know that $[\Phi_{\mathcal{K}_n}]$ descends to an automorphism $[[\Phi_{\mathcal{K}_n}]]$ of the coimage; we just need to calculate the matrix. The matrix defining $[\Phi_{\mathcal{K}}]$ (here $[[\Phi_{\mathcal{K}}]]$ and $[\Phi_{\mathcal{K}}]$ coincide since there is no kernel) is of the form (in the basis $e_0 = [k(p)]$ and $e_1 = [\mathcal{O}_{E_1}(-1)]$),

$$
\begin{bmatrix}
  d & \chi(K[p \times E_1]) & a \\
  r & \text{rank}(\Phi_{\mathcal{K}}) & b
\end{bmatrix}
$$

where $db - ra = 1$. In order to calculate $[[\Phi_{\mathcal{K}_n}]]$, we choose a basis of $\text{im}(Z)$ compatible with $e_0$ and $e_1$: $e'_0 = [[k(p)]]$ for $p$ such that $\pi_{\mathcal{K}_n}(k(p)) = e_0$ and $e'_1 = [[\mathcal{O}_{P_1}(-1)]]$ for any $i$ (since under $Z_{cl}$, the image is the same for all $i$). We need to calculate $\text{rank}(\Phi_{\mathcal{K}_n} \circ [k(p)])$, $\text{rank}(\Phi_{\mathcal{K}_n} \cdot \mathcal{O}_{P_1}(-1))$, $\chi(\Phi_{\mathcal{K}_n} \cdot \mathcal{O}_{P_1}(-1))$, and $\chi(\Phi_{\mathcal{K}_n} \cdot [k(p)])$.

For $[k(p)]$ this is easy: $\Phi_{\mathcal{K}_n}(k(p))$ is a torsion free sheaf with $\text{rank}(\Phi_{\mathcal{K}_n}(k(p))) = \frac{r}{n}$, implying that $\text{rank}(\Phi_{\mathcal{K}_n}(k(p))) = \frac{r}{n} \ast n = r$, while $\chi(\Phi_{\mathcal{K}_n}(k(p))) = d$ by Lemma 5.4.

We can replace $[\mathcal{O}_{P_1}(-1)]$ by $[\mathcal{O}_{E_n}] / n$ since $[\mathcal{O}_{E_n}] = \Sigma_{1 \leq i < n} [\mathcal{O}_{P_1}(-1)]$. Since $\mathcal{O}_{E_n}$ is the pullback of $\mathcal{O}_{E_1}$, Lemma 5.5 shows that $\Phi_{\mathcal{K}_n}(\mathcal{O}_{E_n}) \cong \pi_{\mathcal{K}_n} \circ \mathcal{K}(\mathcal{O}_{E_1})$. This reduces our calculation understanding how $\pi_{\mathcal{K}_n}$ affects $\text{rank}$ and $\chi$. Clearly the rank (restricted to each component) doesn’t change, so $\text{rank}(\pi_{\mathcal{K}_n} \circ F) = n \text{rank}(F)$. Further, by pulling back to the normalization, we see that $\chi(\pi_{\mathcal{K}_n} \circ F) = n \chi(F)$.

Thus,

$$[[\Phi_{\mathcal{K}_n}]] = \begin{bmatrix} d & a \\ r & b \end{bmatrix}$$

So $[[\Phi_{\mathcal{K}_n}]] = [[\Phi_{\mathcal{K}_n}]]$, showing that $[\Phi_{\mathcal{K}_n}]$ descends to an orientation preserving automorphism.

We now have a large class of compatible autoequivalences. We are interested in the subgroup of $\text{Aut}(\mathcal{D}^b(E_n))$ containing all autoequivalences compatible with $\sigma_{cl}(n)$. We will first define $\text{Aut}_{cl}(n)$ and give some of its important properties. In the next section we show that it is maximal.

**Definition 5.9.** Let $\text{Aut}^{triv}(n)$ be the subgroup of $\text{Aut}(\mathcal{D}^b(E_n))$ generated by

1. $\otimes \pi_n^* \mathcal{L}, \mathcal{L} \in Pic^0(E_1)$,
2. $2[i] := [1] \circ [1]$, where $[1]$ is the shift functor,
3. $\text{Aut}(\mathcal{E}_n)$.

**Definition 5.10.** Let $\text{Aut}_{cl}(n)$ be the subgroup of $\text{Aut}(\mathcal{D}^b(E_n))$ generated by

$$\{ \mathcal{K}_n | \Phi_{\mathcal{K}} \in \text{Aut}(\mathcal{D}^b(E_1)), [\Phi_{\mathcal{K}}] \in \Gamma_0(n) \}$$

and $\text{Aut}^{triv}(n)$.

**Theorem 5.11.** All autoequivalences of $\text{Aut}_{cl}(n)$ are strongly compatible with $(Z_{cl}, \mathcal{P}_{cl})$. Further, $\text{Aut}_{cl}(n)$ is an extension

$$1 \longrightarrow \mathbb{Z} / n \mathbb{Z} \times \mathbb{C}^* \times \mathbb{Z} \times \mathbb{C}^* \longrightarrow \text{Aut}_{cl}(n) \longrightarrow \Gamma_0(n) \longrightarrow 1.$$

Under the action of $\text{Aut}_{cl}(n)$, there are $\Sigma_{d|n, d > 0} \phi((d, n/d))$ equivalence classes of phases, where $\phi$ is Euler’s function.
Proof. It is not difficult to see that all elements of $\text{Aut}^{\text{triv}}(n)$ act trivially on $K(\mathbb{E}_n)$. Further, it is easily shown that $\text{Aut}^{\text{triv}}(n) \cong \text{Aut}(\mathbb{E}_n) \times \mathbb{Z} \times \text{Pic}^0(\mathbb{E}_n) \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{C}^* \times \mathbb{C}^*$. We first show that the generators of $\text{Aut}_{cl}(n)$ are strongly compatible. From Definition 5.10 and Proposition 5.8, we are left to show this for elements in $\text{Aut}^{\text{triv}}(n)$. However, this is an easy application of Theorem 3.13: any autoequivalence acting trivially on $K(D^b(E_n))$ will clearly satisfy conditions (ii) and (iii). Our specific choice of autoequivalences shows they satisfy condition (i).

It is now clear that $\text{Aut}_{cl}(n)$ acts on $\text{im}(Z) \cong \mathbb{Z}^2$. Recall for a smooth projective variety $X$ any autoequivalence $\Phi$ can be represented by $\Phi_K$ for some object $K \in D^b(X \times X)$. In the case of $\mathbb{E}_1$, although not smooth, the statement remains true [BK05]. Thus, by Proposition 5.3 and Proposition 5.8, if $\Phi \in \text{Aut}(D^b(\mathbb{E}_1))$ and $[\Phi] \in \Gamma_0(n)$, then there exists an autoequivalence $\Phi_n$ strongly compatible with $\sigma_{cl}(n)$ lifting $\Phi$. These statements are enough to show that $\text{Aut}_{cl}(n)$ acts on $\mathbb{Z}^2$ via $\Gamma_0(n)$.

Let $H$ be the kernel of $\text{Aut}_{cl}(n)$ acting on $\mathbb{Z}^2$. To obtain the exact sequence above, one just needs to show $\text{Aut}^{\text{triv}}(n) \cong H$. One inclusion is clear. To show the other direction we will write any autoequivalence $\Phi \in H$ as a composition of generators of $\text{Aut}^{\text{triv}}(n)$. First, if $\alpha := \phi(\Phi(k(p)))$, then $\alpha$ is an odd integer. For if $\alpha$ was even, $\Phi$ would result in a nontrivial action on $\mathbb{Z}^2$. Let $\Psi_0 := [-\alpha + 1] \circ \Phi$. As noted in Section 3.1 we have $\Psi_0(\text{Coh}(\mathbb{E}_n)) \cong \text{Coh}(\mathbb{E}_n)$ as subcategories of $D^b(\mathbb{E}_n)$. Gabriel’s theorem [Gab62] guarantees that $\Psi_0$ is generated by automorphisms of $X$ and tensoring $\text{Coh}(\mathbb{E}_n)$ by line bundles. Since the latter elements are a normal subgroup of $\text{Aut}(\text{Coh}(\mathbb{E}_n))$, we can write this as a composition $(\otimes \mathcal{L}) \circ \alpha_i$. It is clear from this description that if $\mathcal{L} \notin \text{Aut}_{cl}^{\text{triv}}(n)$, then it will not preserve the kernel of $Z_{cl}$. Thus, we have our description of $\Phi$ as the composition of elements in $\text{Aut}^{\text{triv}}(n)$.

The last statement of the theorem follows from the well known computations for the number of cusps under the action of $\Gamma_0(n)$ on the upper half plane [Shi94]. More precisely, we use the shift [2] to reduce the computation to that of equivalence classes of slopes under the action of $\Gamma_0(n)$. In fact, a more precise statement can be made: as (redundant) representatives for the equivalence classes of phases under the action of $\Gamma_0(n)$, we can choose phases $\phi_\frac{c}{d}$ such that $\frac{c}{d} = -\cotan(\pi \phi_\frac{c}{d})$ where $d|n$, $c < \frac{n}{2}$, $c,d \in \mathbb{Z}$, and $c$ coprime to $d$. Further, the action of $\Gamma_0(n)$ on this set doesn’t change the denominator of the representative.

\[\square\]

6. THE MODULI OF STABLE BUNDLES ON $\mathbb{E}_n$

A sheaf $\mathcal{F}$ is semistable (stable) in $\sigma_{cl}(n)$ if and only if $\mathcal{F}$ is Simpson semistable (stable) with polarization $\mathcal{L}(1, \ldots, 1; 1)$. The results of [Sim94], shows the existence of a coarse moduli space $\mathcal{M}_{cl}(n,a)$ of $\text{Obj}(\mathcal{P}_{cl}(a))$. The closed points correspond to equivalence classes of objects with equivalent Jordan factors (S-equivalence). This moduli is a projective scheme over $\mathbb{C}$. As noted in the introduction, we restrict to the case $n > 1$.

**Theorem 6.1.** Given a nontrivial slice $\mathcal{P}(a)$ of $\sigma_{cl}(n)$ with $n > 1$, let $\mathcal{M}_{cl}^\sharp(n,a) \subset \mathcal{M}_{cl}(n,a)$ denote the subscheme whose closed points correspond to stable objects and $\overline{\mathcal{M}_{cl}^\sharp(a)}$ its closure. Then $\overline{\mathcal{M}_{cl}^\sharp(a)} \cong \mathbb{E}_n \bigcup \mathbb{Z}/n\mathbb{Z}$ where $s|n$ and each component of $\overline{\mathcal{M}_{cl}^\sharp(a)}$ is a component of $\mathcal{M}_{cl}(n,a)$.

**Proof.** Through the action of $\text{Aut}_{cl}(n)$, we can assume that $a$ is one of the specific (redundant) representatives given in the proof of Theorem 5.11: $a = \phi_\frac{r}{s}$ where $s|n$, $r < \frac{n}{s}$ and $r,s$ are coprime. All other representatives of this form have the same denominator in
the index. Since we are primarily concerned with the denominator our choice of representative will not affect the calculation. The coprimality of $r, s$ ensures the existence of $A \in \text{SL}(2, \mathbb{Z})$ such that in our preferred basis

$$A = \begin{bmatrix} r & * \\ s & * \end{bmatrix}.$$  

Clearly $A \in \Gamma_0(s)$.

Proposition 5.3 and Proposition 5.8 construct $\Phi_{K_s} \in \text{Aut}(D^b(\mathbb{E}_s))$ such that for smooth $p, \Phi_{K_s}(k(p)) \in \text{Pic}(\mathbb{E}_s)$. We are interested in the functor $\pi_{n,s} \circ \Phi_{K_s} : D^b(\mathbb{E}_s) \to D^b(\mathbb{E}_s)$. By a calculation similar to Proposition 2.4 this is equivalent to $\Phi_{U}$ where $U = (\text{id} \times \pi_{n,s})^*K_s \in D^b(\mathbb{E}_s \times \mathbb{E}_s)$. It is clear that for smooth $p \in \mathbb{E}_s, \Phi_{U}(k(p)) \in \text{Pic}(\mathbb{E}_s)$.

A calculation similar to the one done in Proposition 5.8 shows for all $p \in \mathbb{E}_s, \phi_{f}(\Phi_{U}(k(p))) = a$. We claim that $\Phi_{U}(k(p))$ is stable: if it was not stable its rank only allows for Jordan factors consisting of torsion free indecomposable subbundles supported on strict subschemes of $\mathbb{E}_s$. This gives the existence of a torsion free, but not locally free, sheaf in the Jordan decomposition of $\pi_{n,s} \circ \Phi_{U}(k(p))$. This is not possible since $\pi_{n,s} \circ \Phi_{U}(k(p))$ is a direct sum of stable line bundles on $\mathbb{E}_s$.

The result of this is an inclusion $\mathbb{E}_s,\text{smooth} \xrightarrow{\Phi_{U}} M_{cl}^t(n, a)$. We want to show that its image is the locus of all stable vector bundles. We denote this latter space as $M_{cl}^{t,\text{vb}}(n, a)$.

We do this by showing that if $V$ is a stable vector bundle on $\mathbb{E}_n$ with $\phi(V) = a$ then $\iota_{n,s}^*(V) \cong V$ (clearly $\Phi_U(k(p))$ satisfies this). Assume this to be true. One calculates directly that $\phi(\pi_{n,s}^*(V)) = a$ (in $\sigma_{cl}(s)$) and that $\pi_{n,s}^*(V)$ carries a fiberwise action of $\mathbb{Z}/\mathbb{Z}$. The action gives a splitting by eigenbundles: $\pi_{n,s}^*(V) \cong \oplus \mathbb{W}_\lambda$, where $\lambda$ are the $n$th roots of unity and $\phi(\mathbb{W}_\lambda) = a$. One recovers $V$ by pulling back $\mathbb{V}_1$. To summarize, if $\iota_{n,s}^*(V) \cong V$ then there exists a vector bundle $V'$ on $\mathbb{E}_n$ such that $V \cong \pi_{n,s}^*V'$. The stability of $V$ shows $\text{End}(V) \cong \mathbb{C}$. Since $\text{Hom}_{\mathbb{E}_n}(V', V') \subset \text{Hom}_{\mathbb{E}_n}(V, V), \text{End}_{\mathbb{E}_n}(V') = \mathbb{C}$ as well. The isomorphism $\mathcal{P}(\phi_{a}) \cong \mathcal{P}(1)$ (on $\mathbb{E}_s$) shows $V'$ is a stable line bundle. Thus, $M_{cl}^{t,\text{vb}}(n, a) \cong \mathbb{E}_s,\text{smooth}$.

We now show that if $V$ is a stable vector bundle on $\mathbb{E}_n$ with $\phi(V) = a$ then $\iota_{n,s}^*(V) \cong V$. Assume $\iota_{n,s}^*V \not\cong V$ for all $0 < k < \frac{n}{2}$. Since $V$ is assumed to be stable, $\text{Hom}_{\mathbb{E}_n}(\iota_{n,s}^*V, V) = 0$. Therefore,

$$\text{Hom}_{\mathbb{E}_n}(\pi_{n,s}^*V, \pi_{n,s}^*V) \cong \text{Hom}_{\mathbb{E}_n}(\pi_{n,s}^*\pi_{n,s}^*V, V) \cong \text{Hom}_{\mathbb{E}_n}(\oplus_{0 \leq k \leq \frac{n}{2}} \mathbb{W}_\lambda^k, V) \cong \mathbb{C}$$

The previous paragraph shows $\pi_{n,s}^*V$ is stable on $\mathbb{E}_s$, with phase $a$. As such, it must be a line bundle. This clearly cannot happen, showing $\iota_{n,s}^*V \cong V$ for some $K \subset \mathbb{Z}/\mathbb{Z}$. If this is not strict, we are done. Otherwise, using the previous paragraph again, there exists $t$ such that $s|t|n$ and $V \cong \pi_{n,t}^*V'$ with $V'$ a vector bundle on $\mathbb{E}_t$. Further, $\text{End}(V') = \mathbb{C}$, $\iota_{n,t}^*V' \not\cong V'$ and $\phi(V') = a$. Recursion gives the contradiction, showing $\iota_{n,s}^*(V) \cong V$.

The scheme $M_{cl}^{t}(n, a) \setminus M_{cl}^{t,\text{vb}}(n, a)$, by definition, consists of stable torsion free, but not locally free sheaves. Let $F$ be such an object and suppose that $G := \pi_{n,s}^*F$ is not stable (it is automatically semistable). In the Jordan decomposition $G$ has a stable subbundle $G'$. If $G'$ is not a vector bundle, there exists a $F'$ on $\mathbb{E}_n$ with $\phi(F') = a, \pi_{n,s}^*F' \cong G'$, and $F' \subset F$. This contradicts that $F$ is stable. Alternatively, if $G'$ is a vector bundle, as noted above, it must be a line bundle. Thus $F' := \pi_{n,s}^*G'$ is a stable line bundle on $\mathbb{E}_n$.

By adjunction, there exists a non-trivial $F' \to F$ between stable objects which is not an isomorphism, contradicting the stability of $F$ and $F'$.

The result of this is that if $F$ is stable and not locally free, then $\pi_{n,s}^*F$ is stable. On $\mathbb{E}_s$ there are exactly $s$ of these objects (corresponding to the image of singular skyscrapers...
under $\Phi_{E_d}$. This yields $s + \frac{\pi}{\pi}$ stable non-locally free bundles in $P(a)$ on $E_n$. The continuous map $M_{cl}(n, a) \xrightarrow{\pi_{n,a}} M_{cl}(s, a) \cong M_{cl}(s, 1)$ shows that each stable non-locally free object corresponds to an open and closed point in $M_{cl}^*(n, a)$. Thus $M_{cl}^*(n, a) \cong M_{cl}^0 \coprod \bigsqcup \mathbb{Z}/n\mathbb{Z}$.

Although we have only discussed $\Phi_{E_d}$ on smooth skyscrapers, it is defined on all of $M_{cl}(s, 1)$, giving a map $\bigsqcup \text{Sym} E_s \rightarrow M_{cl}(n, a)$. Since $M_{cl}(n, a)$ is a separated scheme $M_{cl}^0(n, a) \cong E_s$. These moduli spaces are originating from a GIT quotients. Since stability is an open condition, one obtains that the components of $M_{cl}^*(n, a)$ are components of $M_{cl}(n, a)$.

**Corollary 6.2.** The group $\text{Aut}_{cl}(n)$ contains all autoequivalences compatible with $\sigma_{cl}(n)$.

**Proof.** The goal will be to show that if $\Phi$ is an autoequivalence compatible with $\sigma_{cl}(n)$, then we can write $\Phi$ as the composition of elements contained within $\text{Aut}_{cl}(n)$.

In order to carry out such a task, we must understand which phases can be the image of $\Psi(P(1))$. The proof Theorem 5.11 gives a set of (redundant) explicit representatives for the equivalence classes of phases under the action of $\text{Aut}_{cl}(n)$ on phase space. For convenience we will speak of these representatives through their negative slope, not their phase (e.g. $\psi_{\frac{1}{2}} = \frac{1}{2}$).

By Theorem 6.1, for a given $a$, the scheme $M_{cl}^0(n, a) \cong E_d$ where a representative for $a$ in the above set is $\phi_{\frac{a}{2}}$. If $\Phi(P(1)) \subset P(a)$, the representative for the equivalence class $a$ must be of the form $\phi_{\frac{a}{2}}$ with $c$ coprime to $n$. It is not difficult to see that $\Gamma_0(n)$ act transitively on the set of elements of this form. Thus, we can choose the representative to be $\frac{1}{n}$, and their exists an element $\psi_0 \in \text{Aut}_{cl}(n)$ such that $\psi_0 \circ \Phi(P(1)) \cong P(1)$. The compatibility of this morphism implies that $\psi_0 \circ \Phi(\text{Coh}(E_n)) \cong \text{Coh}(E_n)$ as subcategories of $D^b(E_n)$. Thus, $\psi_0 \circ \Phi$ is an autoequivalence of $\text{Coh}(E_n)$ that is extended to $D^b(E_n)$. From here, we use similar methods as in the proof of Theorem 5.11 to give the desired result.

We conclude this section by noting that much of the above analysis can be carried out in any Simpson stability condition on $E_n$. With the above methods it is possible to classify not only the stable objects, but also the full structure of the coarse moduli space $M_{cl}(n, a)$. This analysis will be done in future publications.

**References**

[Ati57] M. F. Atiyah, *Vector bundles over an elliptic curve*, Proc. London Math. Soc. (3) 7 (1957), 414–452. MR MR0131423 (24 #A1274)

[BBDG06] Lesya Bodnarchuk, Igor Burban, Yuriy Drozd, and Gert-Martin Greuel, *Vector bundles and torsion free sheaves on degenerations of elliptic curves*, Global aspects of complex geometry, Springer, Berlin, 2006, pp. 83–128. MR MR2264108 (2008a:14048)

[BK05] Igor Burban and Bernd Kreussler, *Fourier-Mukai transforms and semi-stable sheaves on nodal Weierstraß cubics*, J. Reine Angew. Math. 584 (2005), 45–82. MR MR2155085 (2006d:14016)

[BK06a] Igor Burban and Bernd Kreußer, *Derived categories of irreducible projective curves of arithmetic genus one*, Compos. Math. 142 (2006), no. 5, 1231–1262. MR MR2264663 (2007b:18016)

[BK06b] _______.; *On a relative Fourier-Mukai transform on genus one fibrations*, Manuscripta Math. 120 (2006), no. 3, 283–306. MR MR2243564 (2007i:14019)

[Bri07] Tom Bridgeland, *Stability conditions on triangulated categories*, Ann. of Math. (2) 166 (2007), no. 2, 317–345. MR MR2373143

[Bri09] _______.; *Spaces of stability conditions*, Algebraic geometry—Seattle 2005. Part 1, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 1–21. MR MR2483930
INTERACTIONS BETWEEN AUTOEQUIVALENCES, STABILITY CONDITIONS, AND MODULI PROBLEMS.

[1] Michael R. Douglas, *D-branes on Calabi-Yau manifolds*, European Congress of Mathematics, Vol. II (Barcelona, 2000), Progr. Math., vol. 202, Birkhäuser, Basel, 2001, pp. 449–466. MR MR1909947 (2004d:81090)

[2] _____, *D-branes and N = 1 supersymmetry*, Strings, 2001 (Mumbai), Clay Math. Proc., vol. 1, Amer. Math. Soc., Providence, RI, 2002, pp. 139–152. MR MR1940182 (2004a:81187)

[3] Pierre Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France 90 (1962), 323–448. MR MR0232821 (38 #1144)

[4] Daniel Hernández Ruipérez, Ana Cristina López Martín, and Fernando Sancho de Salas, *Relative integral functors for singular fibrations and singular partners*, J. Eur. Math. Soc. (JEMS) 11 (2009), no. 3, 597–625. MR MR2505443 (2010a:14029)

[5] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, Oxford, 2006. MR MR2244106 (2007f:14013)

[6] Shigeru Mukai, *Duality between D(X) and D(\hat{X}) with its application to Picard sheaves*, Nagoya Math. J. 81 (1981), 153–175. MR MR607081 (82f:14036)

[7] David Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Published for the Tata Institute of Fundamental Research, Bombay, 1970. MR MR0282985 (44 #219)

[8] Daniel Hernández Ruipérez, Ana Cristina López Martín, Darío Sánchez Gómez, and Carlos Tejero Prieto, *Moduli spaces of semistable sheaves on singular genus 1 curves*, Int. Math. Res. Not. IMRN (2009), no. 23, 4428–4462. MR MR2558337

[9] Goro Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, vol. 11, Princeton University Press, Princeton, NJ, 1994, Reprint of the 1971 original, Kanô Memorial Lectures, 1. MR MR1291394 (95e:11048)

[10] Carlos T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. I*, Inst. Hautes Études Sci. Publ. Math. (1994), no. 79, 47–129. MR MR1307297 (96e:14012)

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