Maximal right smooth extension chains

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2011.02.26

Abstract. If $w = u\alpha$ for $\alpha \in \Sigma = \{1, 2\}$ and $u \in \Sigma^*$, then $w$ is said to be a simple right extension of $u$ and denoted by $u \prec w$. Let $k$ be a positive integer and $P^k(\varepsilon)$ denote the set of all $C^\infty$-words of height $k$. Set $u_1, u_2, \cdots, u_m \in P^k(\varepsilon)$, if $u_1 \prec u_2 \prec \cdots \prec u_m$ and there is no element $v$ of $P^k(\varepsilon)$ such that $v \prec u_1$ or $u_m \prec v$, then $u_1 \prec u_2 \prec \cdots \prec u_m$ is said to be a maximal right smooth extension (MRSE) chains of height $k$. In this paper, we show that MRSE chains of height $k$ constitutes a partition of smooth words of height $k$ and give the formula of the number of MRSE chains of height $k$ for each positive integer $k$. Moreover, since there exist the minimal height $h_1$ and maximal height $h_2$ of smooth words of length $n$ for each positive integer $n$, we find that MRSE chains of heights $h_1 - 1$ and $h_2 + 1$ are good candidates to be used to establish the lower and upper bounds of the number of smooth words of length $n$ respectively, the method of which is simpler and more intuitionistic than the previous ones.

Keywords: smooth word; primitive; height; MRSE chain.
1. Introduction

Let $\Sigma = \{1, 2\}$, $\Sigma^*$ denotes the free monoid over $\Sigma$ with $\varepsilon$ as the empty word. If $w = w_1 w_2 \cdots w_n$, $w_i \in \Sigma$ for $i = 1, 2, \cdots, n$, then $n$ is called the length of the word $w$ and denoted by $|w|$. For $i = 1, 2$, let $|w_i|$ be the number of $i$ which occurs in $w$, then $|w| = |w_1| + |w_2|$.

Given a word $w \in \Sigma^*$, a factor or subword $u$ of $w$ is a word $u \in \Sigma^*$ such that $w = xuy$ for $x, y \in \Sigma^*$, if $x = \varepsilon$, then $u$ is said to be a prefix of $w$. A run or block is a maximum factor of consecutive identical letters. The complement of $u = u_1 u_2 \cdots u_n \in \Sigma^*$ is the word $\overline{u} = \overline{u}_1 \overline{u}_2 \cdots \overline{u}_n$, where $\overline{1} = 2$, $\overline{2} = 1$.

The Kolakoski sequence $K$ which Kolakoski introduced in [13], is the infinite sequence over the alphabet $\Sigma$

$$K = 1 \overline{22} \overline{11} 2 \overline{11} 2 \overline{22} 1 \overline{11} 2 \overline{11} 2 \overline{22} \cdots = K$$

which starts with 1 and equals the sequence defined by its run lengths.

I would like to thank Prof. Jeffrey O. Shallit for introducing me the Kolakoski sequence $K$ and raising eight open questions on it in personal communications (Feb. 15, 1990), the fourth and eighth problems of them are respectively as follows:

(1) Prove or disprove: $|K_i|_1 \sim |K_i|_2$, which is almost equivalent to Keane’s question.

(2) Prove or disprove: $|K_i| \sim \alpha (3/2)^i$ (This would imply (1)), where $\alpha$ seems to be about 0.873. Does $\alpha = (3 + \sqrt{5})/6$?

where $K_0 = 2$ and define $K_{n+1}$ as the string of $1'$ and $2'$ obtained by using the elements of $K_n$ as replication factors for the appropriate prefix of the infinite sequence $1212 \cdots$.

The intriguing Kolakoski sequence $K$ has received a remarkable attention [1, 3, 5, 11, 15, 16]. For exploring two unsolved problems, both whether $K$ is recurrent and whether $K$ is invariant under complement, raised by Kimberling in [12], Dekking proposed the notion of $C^\infty$-word in [6]. Chvátal in [4] obtained that the letter frequencies of $C^\infty$-words are between 0.499162 and 0.500838.

We say that a finite word $w \in \Sigma^*$ in which neither 111 or 222 occurs is differentiable, and its derivative, denoted by $D(w)$, is the word whose $j$th symbol equals the length of the $j$th run of $w$, discarding the first and/or the last run if it has length one.

If a word $w$ is arbitrarily often differentiable, then $w$ is said to be a $C^\infty$-word (or smooth word) and the set of all $C^\infty$-word is denoted by $C^\infty$.

A word $v$ such that $D(v) = w$ is said to be a primitive of $w$. Thus 11, 22, 211, 112, 221, 122, 2112, 1221 are the primitives of 2. It is easy to see that for any word
$w \in \mathcal{C}^\infty$, there are at most 8 primitives and the difference of lengths of two primitives of $w$ is at most 2.

The *height* of a nonempty smooth word $w$ is the smallest integer $k$ such that $D^k(w) = \varepsilon$ and the height of the empty word $\varepsilon$ is zero. We write $ht(w)$ for the height of $w$. For example, for the smooth word $w = 12212212$, $D^4(w) = \varepsilon$, so $ht(w) = 4$.

## 2. Maximal right smooth extension chains

Let $\mathcal{N}$ be the set of all positive integers and $P^k(\varepsilon)$ denote the set of all smooth words of height $k$ for $k \in \mathcal{N}$, then

$$
P(\varepsilon) = \{1, 2, 12, 21\}, \quad \quad (1)
$$

$$
P^2(\varepsilon) = \{121, 212, 11, 22, 211, 122, 112, 221, 2112, 1221, 1211, 12112, 2122, 21221, 1121, 21121, 2212, 12212\}. \quad \quad (2)
$$

**Definition 1.** Let $w, u, v \in \Sigma^*$ if $w = uv$, then $w$ is said to be a right extension of $u$. Especially, if $v = \alpha \in \Sigma$, then $w$ is said to be a simple right extension of $u$, and is denoted by $u \prec w$.

**Definition 2.** Let $u_1, u_2, \ldots, u_m \in P^k(\varepsilon)$, where $k \in \mathcal{N}$.

$$
u_1 \prec u_2 \prec \cdots \prec u_m,
$$
and there is no element $v$ of $P^k(\varepsilon)$ such that

$$
v \prec u_1 \text{ or } u_m \prec v,
$$
then (3) is said to be a maximal right smooth extension (MRSE) chain of the height $k$. Moreover, $u_1$ and $u_m$ are respectively called the first and last members of the MRSE chain (3).

Let $H^k$ denote the set of all MRSE chains of the height $k$. For $\xi \in H^k$, $\xi = u_1 \prec u_2 \prec \cdots \prec u_m$, the complement of $\xi$ is $\bar{\xi} = \bar{u}_1 \prec \bar{u}_2 \prec \cdots \prec \bar{u}_m$, and is denoted by $\bar{\xi}$. It is clear that $\bar{\xi}$ is also a MRSE chain of the height $k$. In addition, for $A \subseteq H^k$, $\bar{A} = \{\xi : \xi \in A\}$.

**Definition 3.** For $\xi \in H^{k+1}$, $\xi = u_1 \prec u_2 \prec \cdots \prec u_m$, where $k \in \mathcal{N}$. If there is an element $\eta = v_1 \prec v_2 \prec \cdots \prec v_n \in H^k$ such that $u_1, u_2, \ldots, u_m$ are all the primitives of $v_1, v_2, \ldots, v_n$, then $\xi$ is said to be a primitive of $\eta$. 

For example, $\xi = 121 \prec 1211 \prec 12112 \in H^2$ is a primitive of $\eta = 1 \prec 12 \in H^1$, $\bar{\xi} = 212 \prec 2122 \prec 21221 \in H^2$.

For a set $A$, let $|A|$ denote the cardinal number of $A$. Next we establish the formula of the number of the members of $H^k$. For this reason, let

$$H_1^k = \{ \xi \in H^k : \xi = u_1 \prec u_2 \prec \cdots \prec u_m \text{ and first}(u_1) = 1 \};$$

$$H_2^k = \{ \xi \in H^k : \xi = u_1 \prec u_2 \prec \cdots \prec u_m \text{ and first}(u_1) = 2 \}.$$  \hspace{1cm} (4)

(5)

It immediately follows that

$$H_1^k = \bar{H}_2^k;$$

$$H^k = H_1^k \cup H_2^k;$$

$$|H_1^k| = |H_2^k|.$$  \hspace{1cm} (6)

(7)

(8)

From (1) and (2) we have

$$H^1 = \{ 1 \prec 12, 2 \prec 21 \};$$

$$H_1^1 = \{ 1 \prec 12 \};$$

$$H_2^1 = \{ 2 \prec 21 \};$$

$$H^2 = \{ 121 \prec 1211 \prec 12112, 212 \prec 2122 \prec 21221, 11 \prec 112 \prec 1121, 22 \prec 221 \prec 2212, 211 \prec 2112 \prec 21121, 122 \prec 1221 \prec 12212 \};$$

$$H_1^2 = \{ 121 \prec 1211 \prec 12112, 11 \prec 112 \prec 1121, 122 \prec 1221 \prec 12212 \};$$

$$H_2^2 = \{ 212 \prec 2122 \prec 21221, 22 \prec 221 \prec 2212, 211 \prec 2112 \prec 21121 \}. $$

(9)

(10)

Thus from (9) and (10), we see that every MRSE chain of height $k$ is uniquely determined by its first member $u_1$ and each member of $P^k(\varepsilon)$ exactly belongs to one MRSE chain of height $k$ for $k = 1, 2$ and

$$|H^2| = 3|H^1|.$$  \hspace{1cm} (11)

Actually, the above result holds for every $k \in \mathbb{N}$.

**Theorem 4.** $H^k$ is stated as above. Then each member of $P^k(\varepsilon)$ exactly belongs to one MRSE chain of height $k$, that is, $H^k$ gives a partition of the smooth words of height $k$ and

$$|H^k| = 2 \cdot 3^{k-1} \text{ for all } k \in \mathbb{N}.$$  \hspace{1cm} (12)

**Proof.** We proceed by induction on $k$. From (11) it follows that (12) holds for $k = 1, 2$. Assume that (12) holds for $k = n - 1 \geq 1$. 

Now we consider the case for \( k = n \). Since for each \( \eta = u_1 \prec u_2 \prec \cdots \prec u_m \in H_{1}^{n-1} \), from the definition 2 and (4), we see that \( \text{first}(u_1) = \text{first}(u_2) = \cdots = \text{first}(u_m) = 1 \), and \( u_{i+1} = u_i \alpha \) where \( i = 1, 2, \ldots, m - 1, \alpha = 1, 2 \). Thus if \( \alpha = 1 \) then the two primitives \( p(u_{i+1}) \) of \( u_{i+1} \) are

\[
p(u_{i+1}) = \overline{\beta} \Delta_{\overline{\beta}}^{-1}(u_{i+1}) \gamma
= \overline{\beta} \Delta_{\overline{\beta}}^{-1}(u_i) \gamma \gamma
= p(u_i) \gamma, \text{ where } \beta, \gamma \in \Sigma,
\]

so \( p(u_i) \prec p(u_{i+1}) \).

If \( \alpha = 2 \) then the four primitives \( p_t(u_{i+1}) \) of \( u_{i+1} \) are

\[
p_t(u_{i+1}) = \overline{\beta} \Delta_{\overline{\beta}}^{-1}(u_{i+1}) \gamma^t
= \overline{\beta} \Delta_{\overline{\beta}}^{-1}(u_i) \gamma^t \gamma^t
= p(u_i) \gamma^t, \text{ where } \beta = 1, 2, t = 0, 1,
\]

hence \( p(u_i) \prec p_t(u_{i+1}) \prec p_t(u_{i+1}) \). Therefore, \( \eta \) has exactly two primitives and the primitives of \( u_1, u_2, \cdots, u_m \) all occur in the two primitives of \( \eta \).

For example, \( \eta = 121 \prec 1211 \prec 12112 \in H_2^2 \) has exactly two primitives:

\[ \mu = 121121 \prec 1211212 \prec 12112122 \prec 121121221 \text{ and } \bar{\mu}. \]

Analogously, we can see that each member \( \eta \) of \( H_2^{n-1} \) has exactly four primitives and the primitives of \( u_1, u_2, \cdots, u_m \) all occur in the four primitives of \( \eta \).

For example, \( \eta = 212 \prec 2122 \prec 21221 \in H_2^2 \) has exactly four primitives:

\[ \xi_1 = 21222 \prec 212221 \prec 2122221 \prec 21222212 \prec 212222121; \]
\[ \xi_2 = 212221 \prec 2122211 \prec 21222211 \prec 212222112 \prec 2122221121 \text{ and } \bar{\xi}_1, \bar{\xi}_2. \]

Thus, by the induction hypothesis, it follows from (7) and (8) that

\[
|H^n| = |H_1^n| + |H_2^n|
= 2 \cdot |H_1^{n-1}| + 4 \cdot |H_2^{n-1}|
= 3 \cdot (|H_1^{n-1}| + |H_2^{n-1}|)
= 3 \cdot |H^{n-1}|
= 2 \cdot 3^{n-1}. \quad \Box
\]

3. The number of smooth words of length \( n \)

Let \( \gamma(n) \) denote the number of smooth words of length \( n \) and \( p_K(n) \) the number of subwords of length \( n \) which occur in \( K \).
Dekking in [6] proved that there is a suitable positive constant \( c \) such that
\[
c \cdot n^{2.15} \leq \gamma(n) \leq n^{7.2}
\]
and brought forward the conjecture that there is a suitable positive constant \( c \) satisfying
\[
p_K(n) \sim c \cdot n^q (n \to \infty), \quad \text{where} \quad q = (\log 3) / \log(3/2).
\]
Recall from [18] that a \( C^\infty \)-word \( w \) is left doubly extendable (LDE) if both \( 1w \) and \( 2w \) are \( C^\infty \), and a \( C^\infty \)-word \( w \) is fully extendable (FE) if \( 1w_1, 1w_2, 2w_1, \) and \( 2w_2 \) all are \( C^\infty \)-words. For each nonnegative integer \( k \), let \( A(k) \) be the minimum length and \( B(k) \) the maximum length of an FE word of height \( k \).

Weakley in [18] proved that there are positive constants \( c_1 \) and \( c_2 \) such that for each \( n \) satisfying
\[
B(k-1) + 1 \leq n \leq A(k) + 1 \quad \text{for some} \quad k, \quad c_1 \cdot n^q \leq \gamma(n) \leq c_2 \cdot n^q.
\]

It is a pity that we don’t know how many positive integers \( n \) fulfil the conditions required. Set \( \gamma'(n) = \gamma(n+1) - \gamma(n) \), Weakley in [18] gave
\[
\gamma(n) = \gamma(0) + \sum_{i=0}^{n-1} \gamma'(i) \quad \text{for} \quad n \geq 2. \tag{13}
\]

Let \( F(n) \) denote the number of LDE-words of height \( n \), Shen and Huang in [14, Proposition 3.2] established
\[
F(n) = 4 \cdot 3^{n-1} \quad \text{for each positive integer} \quad n. \tag{14}
\]

Huang and Weakley in [9] combined (13) with (14) to show that

**Theorem 5** ([9] Theorem 4). Let \( \xi \) be a positive real number and \( N \) a positive integer such that for all LDE words \( u \) with \( |u| > N \) we have \( |u|_2 / |u| > (1/2) - \xi \). Then there are positive constants \( c_1, c_2 \) such that for all positive integers \( n \),
\[
c_1 \cdot n^{\log(3/2) + \xi + (2/N)} < \gamma(n) < c_2 \cdot n^{\log(3/2) - \xi}.
\]

Let \( \gamma_{a,b}(n) \) denote the number of smooth words of length \( n \) over 2-letter alphabet \( \{a, b\} \) for positive integers \( a < b \), Huang in [10] obtained

**Theorem 6.** For any positive real number \( \xi \) and positive integer \( n_0 \) satisfying \( |u|_b / |u| > \xi \) for every LFE word \( u \) with \( |u| > n_0 \), there exist two suitable constants \( c_1 \) and \( c_2 \) such that
\[
c_1 \cdot n^{\log(2b-1) / \log(1+\alpha b - 2) (1-\xi)} \leq \gamma_{a,b}(n) \leq c_2 \cdot n^{\log(2b-1) / \log(1+\alpha b - 2) (1-\xi)}
\]
for every positive integer \( n \).

Since there are the minimum height \( h_1(n) \) and maximum height \( h_2(n) \) of smooth words of length \( n \) for each positive integer \( n \), so the lengths of smooth words of the height \( h_1(n) - 1 \) must be less than \( n \) and the length of smooth words of the height \( h_2(n) + 1 \) must be larger than \( n \). Now we are in a position to use the number of MRSE
chains of the suitable height $k$ to bound the number of smooth words of length $n$, which is simpler than the ones used in [9, 10]. The estimates for the heights of smooth words of length $n$ are borrowed from [10] in the following proof.

**Theorem 7.** For any positive number $\theta$ and $n_0$ satisfying $|u|_2/|u| > \theta$ for $|u| > n_0$, there are suitable positive constant $c_1, c_2$ such that

$$c_1 \cdot n^\log_3 \frac{1}{\log(2 - \theta)} \leq \gamma(n) \leq c_2 \cdot n^\log_3 \frac{1}{\log(1 + \theta)}$$

for any positive integer $n$.

**Proof.** It is obvious that

$$|w| \leq |D(w)| + |D(w)|_2$$

for each smooth word $w$. \hspace{1cm} (15)

First, since $|u|_2/|u| > \theta$ for $|u| \geq n_0$, from (15) one has

$$|w| \geq (1 + \theta)|D(w)|$$

for $|D(w)|_2/|D(w)| > \theta$, which implies

$$|D(w)| < \alpha |w|$$

for $|w| \geq N_0$, \hspace{1cm} (16)

where $N_0$ is a suitable fixed positive integer such that $|D(w)| \geq n_0$ as soon as $|w| \geq N_0$, $\alpha = 1/(1 + \theta)$. Since there are finitely many smooth words of length less than $N_0$, from (16) we see that there exists a suitable nonnegative integer $l$ such that

$$|D(w)| < \alpha |w| + l$$

for each smooth word. \hspace{1cm} (17)

Let $k_0$ be the least integer such that the length of smooth words of height $k_0$ is larger than $\frac{l}{1-\alpha}$ and $r$ be the smallest length of smooth words of height $k_0$. Let $k$ be the height of the smooth words $w$ such that $ht(w) \geq k_0$, then $ht(D^{k-k_0}(w)) = k_0$. So, from (17), we get

$$r \leq |D^{k-k_0}(w)|$$

$$< \alpha |D^{k-k_0-1}(w)| + l$$

$$< \alpha^2 |D^{k-k_0-2}(w)| + \alpha l + l$$

$$\cdots$$

$$< \alpha^{k-k_0} |w| + \alpha^{k-k_0} l + \cdots + \alpha^2 l + \alpha l + l$$

$$< \alpha^{k-k_0} |w| + \frac{l}{1-\alpha}.$$ 

Thus

$$(1/\alpha)^{k-k_0} < \frac{|w|}{l}$$

where $\lambda = r - \frac{l}{1-\alpha}$,
which means

$$ht(w) = k < \frac{\log |w|}{\log(1/\alpha)} + k_0 - \frac{\log \lambda}{\log(1/\alpha)}.$$  

Since there are only finitely many smooth words satisfying $ht(w) < k_0$, so there is a suitable constant $t_2$ such that

$$ht(w) < \frac{\log |w|}{\log(1/\alpha)} + t_2 \text{ for each smooth word.} \quad (18)$$

Therefore, the maximal height $h_2(n)$ of all smooth words of length $n$ satisfies

$$h_2(n) \leq \frac{\log n}{\log(1+\theta)} + t_2. \quad (19)$$

Put $k = h_2(n) + 1$, then the length of every smooth word of height $k$ is greater than $n$, so each smooth word of length $n$ can be right extended to get a MRSE chain of height $k$, which suggests $\gamma(n) \leq |H^k|$. Consequently, from (12) and (19) it follows the desired upper bound of $\gamma(n)$.

Second, since the complement of any smooth word is a smooth word of the same length, the theorem’s hypothesis implies that $|D(w)|_1/|D(w)| \geq \theta$, so $|D(w)|_2/|D(w)| \leq 1 - \theta$. From (15) it follows that

$$|w| \leq \beta|D(w)| + q \text{ for each } C^\infty\text{-word } w, \quad (20)$$

where $\beta = 2 - \theta$, $q$ is a suitable positive constant. Thus

$$|w| \leq \beta^{k-1}|D^{k-1}(w)| + q\frac{\beta^{k-1} - 1}{\beta - 1} < 2\beta^{k-1} + \frac{q\beta^{k-1}}{\beta - 1} = (2 + \frac{q}{\beta - 1})\beta^{k-1} = t\beta^{k-1},$$

where $t = 2 + q/(\beta - 1)$, $k$ is the height of $|w|$. Wherefore, the length $|w|$ of a smooth word $w$ with height $k$ is less than $t\beta^{k-1}$ and $k - 1 > (\log |w| - \log t)/\log \beta$. Hence, the smallest height $h_1(n)$ of smooth words of length $n$ meets

$$h_1(n) > \frac{\log n}{\log(2 - \theta)} + t_1, \text{ where } t_1 = 1 - \frac{\log t}{\log \beta}. \quad (21)$$

Then the length of all smooth words with height $m = h_1(n) - 1$ is less than $n$, which means that the length of the last member last ($\xi$) is less than $n$ for each $\xi \in H^m$. Since each smooth words of length no more than $n - 1$ can be extended right to a smooth word of length $n$, we see $\gamma(n) \geq |H^m|$. Herewith, from (12) and (21) we get the desired lower bound of $\gamma(n)$. □
4. Concluding remarks

Let $a$ and $b$ be positive integers of different parities and $a < b$. Lately, Sing in [15] conjectured:

*There are positive constants $c_1$, $c_2$ such that*

$$c_1 \cdot n^\delta \leq \gamma_{a,b}(n) \leq c_2 \cdot n^\delta,$$

where $\delta = \log((a + b)\log((a + b)/2))$.

Theorem 6 means Sing’s conjecture should be revised to be of the following form

$$c_1 \cdot n^\theta \leq \gamma_{a,b}(n) \leq c_2 \cdot n^\theta,$$

where $\theta = \frac{\log(2b - 1)}{\log((a + b)/2)}$.

For 2-letter alphabet $\Sigma = \{a, b\}$ with $a < b$, let $P_j(\varepsilon)$ denote the set of smooth words of height $k$ for $j \in \mathcal{N}$. For $\alpha \in \Sigma$, set

$$\xi_i = \alpha^i \prec \alpha^i \bar{\alpha} \prec \alpha^i \bar{\alpha}^2 \prec \cdots \prec \alpha^i \bar{\alpha}^{b-1} \text{ for } 1 \leq i \leq b - 1.$$ 

and

$$H^1 = \{\eta|\eta = \xi_i \text{ or } \xi_i, i = 1, 2, \cdots, b - 1\}.$$ 

Let $H^2$ denote the set of primitives of the members in $H^1$, then it is easy to see $H^2$ constitutes a partition of $P^2(\varepsilon)$. So continue, we can define the set $H^k$ for each $k \in \mathcal{N}$ and $H^k$ constitutes a partition of $P^k(\varepsilon)$. Using the method similar to Theorem 7, we could establish the corresponding result to Theorem 6.

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