ERROR ANALYSIS OF A QUASI-REVERSIBILITY METHOD FOR A TERMINAL-BOUNDARY VALUE MULTI-SPECIES MODEL WITH WHITE GAUSSIAN NOISE

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Abstract. Upon the recent development of the quasi-reversibility method for terminal value parabolic problems in [26], it is imperative to investigate the convergence analysis of this regularization method in the stochastic setting. In this paper, we positively unravel this open question by focusing on a coupled system of Dirichlet reaction-diffusion equations with additive white Gaussian noise on the terminal data. In this regard, the approximate problem is designed by adding the so-called perturbing operator to the original problem and by exploiting the Fourier reconstructed terminal data. Furthermore, we no longer need the decay behaviour of the perturbation, relying on a variant of the classical quasi-reversibility regime. By this way, Sobolev- and Gevrey-type source conditions are easily included, while we successfully maintain the logarithmic stability estimate of the corresponding stabilized operator, which is necessary for the error analysis. As the main theme of this work, we prove the error bounds for the concentrations and for the concentration gradients, driven by a large amount of weighted energy-like controls involving the expectation operator. Compared to the classical error bounds in $L^2$ and $H^1$ that we obtained in the previous studies, our analysis here needs a higher smoothness of the true terminal data to ensure their reconstructions from the stochastic fashion. Especially, using finer energy estimates our method is valid for any finite time. Two numerical examples are provided to corroborate the theoretical results.

Key words. Backward reaction-diffusion systems, Quasi-reversibility method, Gaussian white noise, Weak solvability, Global estimates, Convergence rates.

AMS subject classifications. 65J05, 65J20, 35K92, 60H35

1. Introduction. The main purpose of this paper is to carry out the error analysis of the recently proposed quasi-reversibility (QR) method in a stochastic setting for a class of terminal-boundary value multi-species model. As concluded in [26], the error bounds we obtained there can be very helpful in the finite element settings due to the variational framework we choose. Nonetheless, it still questions us about if this approach can be modified to somewhat get the convergence analysis in some specific stochastic setting involved in the partial differential equations (PDEs) we are chasing. In this work, we provide a positive answer by taking into account a nonlocal coupled system of nonlinear reaction-diffusion equations with additive white Gaussian noise on the terminal data. Besides, we significantly strengthen the applicability of this general QR framework by ameliorating some vital conditions that we set up in [26].

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1.1. Statement of the problem. Let $0 < T < \infty$ be the final time of observation and $\Omega \subset \mathbb{R}^d$ for $d \in \mathbb{N}^*$ be the domain of interest, which is open, connected, and bounded with a sufficiently smooth boundary. Denoted by $Q_T = \Omega \times (0, T)$, this work is devoted to finding $u, v : \overline{Q_T} \to \mathbb{R}$ as two solutions of the following evolution system:

\begin{align}
(1.1) \quad \begin{cases}
    u_t - D_1 (\ell_0 (u) (t)) \Delta u = F (x, t; u; v), \\
    v_t - D_2 (\ell_0 (v) (t)) \Delta v = G (x, t; u; v)
\end{cases} \quad \text{for} \quad (x, t) \in Q_T.
\end{align}

To complete the terminal-boundary value problem we wish to solve, the system (1.1) is then supplemented with the terminal conditions $u(x, T) = u_f (x)$, $v(x, T) = v_f (x)$ and the Dirichlet boundary conditions $u = v = 0$ on the boundary $\partial \Omega$. Suppose further that the terminal data are observed in the presence of white noise processes $\xi_1, \xi_2$ in terms of

\begin{align}
(1.2) \quad u_f^\epsilon (x) = u_f (x) + \epsilon \xi_1 (x), \quad v_f^\epsilon (x) = v_f (x) + \epsilon \xi_2 (x),
\end{align}

where $\epsilon \in (0, 1)$ is used for measuring the amplitude of the noise.

In this scenario, our attempt is to seek the initial values $u(x, 0) = u_0 (x)$ and $v(x, 0) = v_0 (x)$ when we only know the measurements of the terminal data $u_f$ and $v_f$. We accentuate that this seeking can only be done by regularization since our problem is, in general, exponentially unstable; cf. [4, 5, 11, 1, 20, 14] for a glimpse of statistical inverse problems. Starting from the source reconstruction model of diffusive, competitive and reactive brain tumor cells (cf. [13] and references cited therein), solving (1.1) is essential to understand the localization of the tumor source and then to possibly advance the treatment of brain tumors. This biological context conjures up the following images. Let $u$ and $v$ be the normal (healthy) and abnormally growing tissue cells densities in a brain region, respectively. In the perspective of (1.1), we assume that the movements of each kind of cells are dominantly influenced by the whole population of the corresponding type. Usually, this significant impact brings us to the nonlocal form of diffusion (see [7, 3, 9] for the mathematical background of this diffusion in the forward model), which reads as

\[ D_i (\ell_0 (w) (t)) = D_i \left( \int_{\Omega} f(x) w(x, t) dx \right), \quad \ell_0 (w) (t) := \int_{\Omega} f(x) w(x, t) dx, \]

for $i = 1, 2$, $w \in \mathbb{R}$ and where $f$ is a given sufficiently smooth weight function. Largely motivated by the typical expression, we considered in [28] a slight generalization of this diffusion for the sole purpose of extending the applicability of the QR method we are studying.

The next concern that we would point out in the continuous model (1.1) lies in the nonlinearities $F$ and $G$ which can be considered as reaction, death and proliferation rates involved in the active network under scrutiny. In fact, we are able to suppose that at the low-grade gliomas regime, one knows that the brain tumor cells start growing slowly by infiltrating into the healthy brain cells and simultaneously, they misdirect the T cells of the immune system to avoid being beaten. Besides, we can assume that for some vigorous people, those gliomas are still identified at the beginning and then are very slightly removed by some reaction rates from the immune system. Accordingly, this entire incident is well-suited to the consideration of the coupled system (1.1), provided that the assumptions on these nonlinear terms will cover most of the real-world contexts. Needless to say, one may also detail several...
well-known chemical models contained in (1.1) such as the two-species Lotka–Volterra
competition-diffusion model, the Brusselator coupled system for the autocatalytic re-
action and those already mentioned in [26]. Eventually, we remark that even though
the evolution system (1.1) can be expressed in a closed-form, it is better to contemplate
the whole mathematical treatment for every single term in there.

1.2. Settings of the statistical terminal data. Cf. [4], we recall standard
assumptions on the stochastic noise we want to address in the model. In the following,
\( \mathcal{H} \) is a Hilbert space.

**Definition 1.1.** The stochastic error is a Hilbert-space process, i.e. a bounded
linear operator \( \xi : \mathcal{H} \to L^2(\tilde{\Omega}, \mathcal{A}, P) \) where \((\tilde{\Omega}, \mathcal{A}, P)\) is a complete probability space
and \( L^2(\cdot) \) is the space of all square integrable measurable functions.

This way we are able to define the random variables \( \langle \xi, g_j \rangle \) for \( j = 1, 2 \) for all
\( g_1, g_2 \in \mathcal{H} \) by definition \( \mathbb{E} \langle \xi, g_j \rangle = 0 \). Furthermore, we define its covariance \( \text{Cov}_\xi \)
as the bounded linear operator mapping from \( \mathcal{H} \) onto itself such that \( \langle \text{Cov}_\xi g_1, g_2 \rangle = \text{Cov}(\langle \xi, g_1 \rangle, \langle \xi, g_2 \rangle) \).

**Definition 1.2.** We say that \( \xi \) is a white noise process in \( \mathcal{H} \), if \( \text{Cov}_\xi = I \) and
the induced random variables are Gaussian: for all \( g_1, g_2 \in \mathcal{H} \), the random variables
\( \langle \xi, g_j \rangle \) have distributions \( \mathcal{N}(0, \|g_j\|^2) \) and \( \text{Cov}(\langle \xi, g_1 \rangle, \langle \xi, g_2 \rangle) = \langle g_1, g_2 \rangle \).

Thus, the representation (1.2) is equivalent to
\[
\langle u_f^j, \psi \rangle = \langle u_f, \psi \rangle + \varepsilon \langle \xi_1, \psi \rangle, \quad \langle v_f^j, \psi \rangle = \langle v_f, \psi \rangle + \varepsilon \langle \xi_2, \psi \rangle \quad \text{for all } \psi \in L^2(\Omega).
\]

Accordingly, assume that the observations (1.2) can only be obtained in a discretized
or binned form. This means that we only have vectors of normally distributed random
variables \( \{u_f^j\}_{j=1}^n \) given by

\[
(1.3) \quad u_f^j := \langle u_f^j, \phi_j \rangle = \langle u_f, \phi_j \rangle + \varepsilon \langle \xi_1, \phi_j \rangle,
\]

\[
(1.4) \quad v_f^j := \langle v_f^j, \phi_j \rangle = \langle v_f, \phi_j \rangle + \varepsilon \langle \xi_2, \phi_j \rangle,
\]

where \( n \in \mathbb{N} \) is the number of steps of discrete observations and \( \phi_j \) is itself taken
from the orthonormal basis \( \{\phi_j\}_{j \in \mathbb{N}} \) of \( L^2(\Omega) \). Recall that due to the smoothness of
\( \Omega \), the existence of this basis is guaranteed, where \( \phi_j \in H^1_0(\Omega) \cap C^\infty(\overline{\Omega}) \) solves the basic
eigenvalue problem \( -\Delta \phi_j(x) = \mu_j \phi_j(x) \) for \( x \in \Omega \). Additionally, the Dirichlet
eigenvalues \( \{\mu_j\}_{j \in \mathbb{N}} \) form an infinite sequence which goes to infinity, viz.
\[
0 \leq \mu_0 < \mu_1 \leq \mu_2 \leq \ldots, \quad \text{and } \lim_{j \to \infty} \mu_j = \infty.
\]

As a consequence, one can prove that \( \langle \xi_1, \phi_j \rangle \) and \( \langle \xi_2, \phi_j \rangle \) for \( j = 1, \ldots, n \) are i.i.d.
standard Gaussian random variables.

1.3. QR-based methods and our novelty. The QR method has a long re-
markable history since the pioneering monograph [21]. During the development of the
QR method for inverse PDEs with deterministic noise, people usually focus on

- the spectral methods that allow us to consider the solution in a mild presenta-
tion (cf., e.g., [28, 8, 15]), but this typical method seems hard to handle the
practical error control since one struggles with the Fourier accumulation for
the nonlinear contexts;

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• the Carleman-type estimate (cf., e.g., [19, 18]) that includes the uniqueness result and convergence analysis, but currently, it only works for the short time observation with the linear equation.

Cf. [26], our approach removes this fence by relying on the so-called perturbing and stabilized operators, benefited from the essential bound of the nonlinear diffusion coefficient.

**Definition 1.3** (perturbing operator). The linear mapping $Q^\beta : [L_2(\Omega)]^N \rightarrow [L_2(\Omega)]^N$ is said to be a perturbing operator if there exist a function space $\mathbb{W} \subset [L_2(\Omega)]^N$ and an $\varepsilon$-independent constant $C_0 > 0$ such that

$$\|Q^\beta u\|_{[L_2(\Omega)]^N} \leq \frac{C_0}{\gamma(T, \beta)} \|u\|_{\mathbb{W}} \quad \text{for any } u \in \mathbb{W}.$$  

**Definition 1.4** (stabilized operator). The linear mapping $P^\beta : [L_2(\Omega)]^N \rightarrow [L_2(\Omega)]^N$ is said to be a stabilized operator if there exists an $\varepsilon$-independent constant $C_1 > 0$ such that

$$\|P^\beta u\|_{[L_2(\Omega)]^N} \leq C_1 \log (\gamma(T, \beta)) \|u\|_{[L_2(\Omega)]^N} \quad \text{for any } u \in [L^2(\Omega)]^N.$$  

In Definitions 1.3 and 1.4, $N$ denotes the number of species involved in the model and in this work, $N = 2$. Meanwhile, we denote by $\beta = \beta(\varepsilon) \in (0, 1)$ the regularization parameter satisfying $\lim_{\varepsilon \to 0^+} \beta(\varepsilon) = 0$. The function $\gamma : [0, T] \times (0, 1) \rightarrow \mathbb{R}$ indicates the decay behaviour of the perturbing operator, which stems from the source condition measuring the high smoothness of the true solution. In our analysis, we require: for any $\beta > 0$, there holds

$$\gamma(T, \beta) \geq 1, \quad \lim_{\beta \to 0^+} \gamma(t, \beta) = \infty \quad \text{for all } t \in (0, T).$$  

In this work, we provide finer energy estimates to improve this general framework. This technique enables us to mitigate the decay behaviour of the perturbation and thus, many choices of them can be deduced, where the Sobolev source conditions in $\mathbb{W}$ can be involved. On top of that, we prove that this QR setting can work for any finite time $T$, instead of the condition $C_1 T \leq 1$ as required in [26]. Now we define a new type of perturbing operators, compared to Definition 1.3, as follows:

**Definition 1.5** (new perturbation). The linear mapping $\bar{Q}^\beta : [L_2(\Omega)]^N \rightarrow [L_2(\Omega)]^N$ is said to be a perturbing operator if there exist a function space $\bar{\mathbb{W}} \subset [L_2(\Omega)]^N$ and an $\varepsilon$-independent constant $\bar{C}_0 > 0$ such that

$$\|\bar{Q}^\beta u\|_{[L_2(\Omega)]^N} \leq \bar{C}_0 \|u\|_{\bar{\mathbb{W}}} \quad \text{for any } u \in \bar{\mathbb{W}}.$$  

Although it is clear that in the framework of deterministic noise we can prove the error estimates for the QR method, it does not ensure that one can adapt those to the statistical inverse model in this work, including the references cited above. Besides, the existing literature on regularization of the system (1.1) is very limited due to the inception stage. Starting off with the linear version of (1.1) (where $D_\ell$, $F$ and $G$ are independent of the solutions, we are aware of the presence of [23] where the authors exploited the trigonometric method in nonparametric regression to make use of the cut-off regularization in the statistical setting. This method is, however, very challenging in the nonlinear perspective as postulated in, e.g., [17]. It is worth noting in [17] that the authors essentially applied the QR-based method, where the
Gevrey source condition is taken into account, to regularize a semilinear parabolic problem. Henceforth, our contribution herein can also be seen as an improvement of [17], since the nonlocal spatial operator becomes rather challenging from the numerical standpoint in [17] and since we can slightly facilitate such very fine source conditions.

From now on, some impediments that we will meet in proofs of our analysis should be revealed, except what we were very much concerned in [26]. Due to the white noise processes we are taking into account, our observations, i.e. the measured terminal data, do not belong to $L^2(\Omega)$, but acts on $L^2(\Omega)$ by virtue of (1.3)–(1.4). In principle, the random noise is large compared to the deterministic one. Thus, the adaptation of [26] to this statistical scenario is not straightforward. Aside from the high smoothness of the true solution, we need the true terminal function to be very smooth to obtain rates of convergence in expectation.

1.4. Outline of the paper. The rest of the paper is organized as follows. In section 2, we introduce the notation and working assumptions for our analysis below. We also provide several crucial examples to validate the presence of the new perturbation we have defined in Definition 1.5. In addition, we delineate a way to reconstruct the terminal data from the stochastic setting under consideration. Our main results are reported in section 3. Based on the proposed QR framework, we establish a regularization for the time-reversed system (1.1); cf. subsection 3.1. Convergence rates of the scheme are thoroughly explored in subsection 3.2; cf. Theorems 3.3 and 3.4. Proofs of these error estimates are detailed in subsections 3.2.1 and 3.2.2, respectively. We close this main section by several discussions in subsection 3.2.3. Finally, two numerical tests are provided in section 4 to verify our convergence analysis, and the conclusions follow in section 5.

2. Preliminaries. In the sequel, wherever $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are present, we mean the $L^2$ inner product and its corresponding norm. Meanwhile, the other standard Sobolev norms will be specified, if used. Also, we use $D_i(u)(t)$ as $D_i(\ell_0(u)(t))$ for ease of presentation. In this statistical inverse problem, we assume that the true solution exists uniquely and its regularity is assumed in Theorems 3.3 and 3.4. The backward uniqueness result for this problem will be studied in the near future. To this end, we use the following assumptions:

(A1) For $i = 1, 2$, the measurable functions $D_i > 0$ is such that the mapping $\xi \mapsto D_i(\xi)$ is continuous for $\xi \in \mathbb{R}$. Moreover, there exist $\varepsilon$-independent $\underline{M}, \overline{M} > 0$ such that

$$\underline{M} \leq D_i(\xi) \leq \overline{M} \quad \text{for any } \xi \in \mathbb{R}. $$

(A2) The source functions $F, G$ are measurable and locally Lipschitz-continuous in the sense that

$$|F(x, t; u_1; v_1) - F(x, t; u_2; v_2)| + |G(x, t; u_1; v_1) - G(x, t; u_2; v_2)| \leq L(\ell)(|u_1 - u_2| + |v_1 - v_2|),$$

for $\max\{|u_1|, |u_2|, |v_1|, |v_2|\} \leq \ell$ for some $\ell > 0$.

(A3) The true final conditions $u_f, v_f$ belong to $H^{2p}(\Omega)$ for $p > 0$.

(A'_3) For $i = 1, 2$, the diffusion $D_i$ is globally Lipschitz-continuous in the sense that there exists $\varepsilon$-independent $\tilde{L}$ such that

$$|D_i(\ell_0(u_1)(t)) - D_i(\ell_0(u_2)(t))| \leq \tilde{L} \|u_1(\cdot, t) - u_2(\cdot, t)|| \quad \text{for } u_1, u_2 \in L^2(\Omega).$$

Since our observations $u_f^\varepsilon$ and $v_f^\varepsilon$ are not in general elements of $L^2(\Omega)$, we need the presence of (A3) to handle the following lemma (cf. [25]):
Lemma 2.1 (Truncated Fourier reconstruction). Given \( n \in \mathbb{N}^* \) the number of steps of discrete observations, we define

\[
\begin{align*}
L^2(\Omega) & \ni U^{\varepsilon,n}_f(x) := \sum_{j=1}^{n} \langle u^\varepsilon_f, \phi_j \rangle \phi_j(x), \\
L^2(\Omega) & \ni V^{\varepsilon,n}_f(x) := \sum_{j=1}^{n} \langle v^\varepsilon_f, \phi_j \rangle \phi_j(x).
\end{align*}
\]

Then with the aid of (A3), one has

\[
\mathbb{E} \left\| U^{\varepsilon,n}_f - u_f \right\|^2 \leq \varepsilon^2 n + \frac{\|u_f\|_{H^{2p}(\Omega)}}{\mu_n^{2p}}, \quad \mathbb{E} \left\| V^{\varepsilon,n}_f - v_f \right\|^2 \leq \varepsilon^2 n + \frac{\|v_f\|_{H^{2p}(\Omega)}}{\mu_n^{2p}},
\]

where \( \mu_n > 0 \) is the \( n \)th Dirichlet eigenvalue of the Laplacian operator.

The existence of the perturbing and stabilized operators (for Definitions 1.3 and 1.4) were commenced in [26], mimicking the stochastic gradient descent algorithm to obtain the function space \( \mathbb{W} \) as a Gevrey\(^1\) class of real-analytic functions. That perturbation satisfies our new setting in Definition 1.5, but below we exemplify many other kinds that the converse is not somewhat obtainable and that only the Sobolev source conditions are needed. Typically, we employ a variant of the classical QR method and the truncated Fourier method in the spectral setting.

Example 2.2. Upon the presence of the Dirichlet eigen-elements of the Laplacian operator, we choose

\[
\bar{Q}^\beta_{\varepsilon}u = \bar{M} \sum_{\mu_j > \log(\gamma(T,\beta))} \mu_j \langle u, \phi_j \rangle \phi_j \quad \text{for} \quad u \in L^2(\Omega).
\]

With the help of the Parseval identity, it is then easy to see that \( \bar{\mathbb{W}} = H^2(\Omega) \) and \( \bar{C}_0 = \bar{M} \) in Definition 1.5. Now taking \( P^\beta_{\varepsilon} := \bar{M} \Delta + \bar{Q}^\beta_{\varepsilon} \) to absorb high frequencies in the Laplacian, we get

\[
P^\beta_{\varepsilon}u = -\bar{M} \sum_{\mu_j \leq \log(\gamma(T,\beta))} \mu_j \langle u, \phi_j \rangle \phi_j,
\]

and therefore, it holds that \( C_1 = \bar{M} \) in Definition 1.4.

Example 2.3. The classical QR method is from a typical perturbation of the Laplacian by adding the fourth-order term to the original problem, which is ill-posed. As readily expected, our perturbing operator can be chosen as

\[
\bar{Q}^\beta_{\varepsilon}u = \frac{\bar{M}}{4} \log^{-1}(\gamma(T,\beta)) \sum_{j \in \mathbb{N}^*} \mu_j^2 \langle u, \phi_j \rangle \phi_j,
\]

and thus, the stabilized operator will read as

\[
P^\beta_{\varepsilon}u := \bar{M} \Delta u + \bar{Q}^\beta_{\varepsilon}u = -\bar{M} \sum_{j \in \mathbb{N}^*} \mu_j \left(1 - \frac{\bar{M}}{4} \log^{-1}(\gamma(T,\beta)) \mu_j \right) \langle u, \phi_j \rangle \phi_j.
\]

By Parseval's identity, one can prove that \( \bar{C}_0 = \frac{\bar{M}}{4} \) and \( \bar{\mathbb{W}} = H^4(\Omega) \) in Definition 1.5. In view of the elementary inequality \( \mu_j \left(1 - \frac{\bar{M}}{4} \log^{-1}(\gamma(T,\beta)) \mu_j \right) \leq \frac{\bar{M}}{4} \log \gamma(T,\beta) \) we get \( C_1 = 1 \) in Definition 1.4.

\(^1\)See again the Gevrey-like space defined in [26, Section 5.2].
Example 2.4. We combine the above-mentioned perturbing operators to consider

\[ \mathbf{Q}_\varepsilon^2u = \frac{M^2}{4} \log^{-1}(\gamma(T,\beta)) \sum_{\mu_j \leq \log^+(\gamma(T,\beta))} \mu_j^2 \langle u, \phi_j \rangle \phi_j \\
+ \frac{M}{\mu_j} \sum_{\mu_j > \log^+(\gamma(T,\beta))} \mu_j \langle u, \phi_j \rangle \phi_j, \]

which leads to

\[ \mathbf{P}_\varepsilon^2u = \frac{M^2}{4} \log^{-1}(\gamma(T,\beta)) \sum_{\mu_j \leq \log^+(\gamma(T,\beta))} \mu_j^2 \langle u, \phi_j \rangle \phi_j \\
- \frac{M}{\mu_j} \sum_{\mu_j > \log^+(\gamma(T,\beta))} \mu_j \langle u, \phi_j \rangle \phi_j. \]

In this case, we obtain \( \tilde{\mathbb{W}} = H^2(\Omega) \) and \( \bar{C}_0 = \frac{M^2}{4} + M \) in Definition 1.5 and then \( C_1 = 1 \) in Definition 1.4.

Remark 2.5. The conventional cut-off function for the locally Lipschitz \( F \) can be taken by

\[ F_{\ell^\varepsilon}(x,t;u,v) := \begin{cases} F(x,t;\ell^\varepsilon;\ell^\varepsilon) & \text{if } \max\{u,v\} > \ell^\varepsilon, \\
F(x,t;u,v) & \text{if } \max\{u,v\} \in [-\ell^\varepsilon,\ell^\varepsilon], \\
F(x,t;\ell^\varepsilon;\ell^\varepsilon) & \text{if } \max\{u,v\} < -\ell^\varepsilon, \end{cases} \]

where \( \ell^\varepsilon = \ell(\varepsilon) > 0 \) satisfying \( \lim_{\varepsilon \to 0^+} \ell^\varepsilon = \infty \) is called the cut-off parameter. In the same manner, we have the cut-off function \( G_{\ell^\varepsilon}(x,t;u,v) \) and thus one can prove that

\[ |F_{\ell^\varepsilon}(x,t;u_1;v_1) - F_{\ell^\varepsilon}(x,t;u_2;v_2)| + |G_{\ell^\varepsilon}(x,t;u_1;v_1) - G_{\ell^\varepsilon}(x,t;u_2;v_2)| \leq 2L(\ell^\varepsilon) (|u_1 - u_2| + |v_1 - v_2|), \]

and one can observe that \( \lim_{\varepsilon \to 0^+} L(\ell^\varepsilon) = \infty \). The proof can be detailed in [26, 28].

3. Main results.

3.1. Approximation by the QR method. This part is devoted to establishing an approximate problem of (1.1) by using the QR method. We begin by adding the perturbing operator to the original system as follows:

\[
\begin{aligned}
\left\{ 
\begin{array}{ll}
u_{t}^{\varepsilon,n} - \mathcal{D}_1(u^{\varepsilon,n}) \Delta u^{\varepsilon,n} - \mathbf{Q}_\varepsilon^2u^{\varepsilon,n} = F_{\ell^\varepsilon}(x,t;u^{\varepsilon,n};v^{\varepsilon,n}), \\
v_{t}^{\varepsilon,n} - \mathcal{D}_2(u^{\varepsilon,n}) \Delta v^{\varepsilon,n} - \mathbf{Q}_\varepsilon^2v^{\varepsilon,n} = G_{\ell^\varepsilon}(x,t;u^{\varepsilon,n};v^{\varepsilon,n}) & \text{for } (x,t) \in Q_T.
\end{array}
\right.
\end{aligned}
\]

Hereby, we define new diffusion-like coefficients \( \overline{\mathcal{D}}_i \) for \( i = 1,2 \) in such a way that \( \overline{\mathcal{D}}_i := M - \mathcal{D}_i \). This way one can prove that \( \overline{\mathcal{D}}_i \in (\overline{M} - M_1, \overline{M}) \) for \( M_1 < \overline{M} \) by (A1). Thus, we rely on the choice of the stabilized operator (i.e. \( \mathbf{P}_\varepsilon^2 := \overline{M}^2 + \mathbf{Q}_\varepsilon^2 \)) to arrive at

\[
\begin{aligned}
\left\{ 
\begin{array}{ll}
u_{t}^{\varepsilon,n} + \overline{\mathcal{D}}_1(u^{\varepsilon,n}) \Delta u^{\varepsilon,n} = F_{\ell^\varepsilon}(x,t;u^{\varepsilon,n};v^{\varepsilon,n}) + \mathbf{P}_\varepsilon^2u^{\varepsilon,n}, \\
v_{t}^{\varepsilon,n} + \overline{\mathcal{D}}_2(u^{\varepsilon,n}) \Delta v^{\varepsilon,n} = G_{\ell^\varepsilon}(x,t;u^{\varepsilon,n};v^{\varepsilon,n}) + \mathbf{P}_\varepsilon^2v^{\varepsilon,n} & \text{for } (x,t) \in Q_T,
\end{array}
\right.
\end{aligned}
\]
where \( F_\varepsilon \) and \( G_\varepsilon \) are defined as in Remark 2.5. This is the coupled regularized system of (1.1) that we wish to scrutinize in this work. Due to Lemma 2.1, we associate (3.1) with the Dirichlet boundary condition and the following terminal conditions:

\[
\begin{align*}
    u^{\varepsilon,n}(x,T) &= U_1^{\varepsilon,n}(x), \\
    v^{\varepsilon,n}(x,T) &= V_1^{\varepsilon,n}(x).
\end{align*}
\]

Hence, these equations form our regularized problem \((P_{\varepsilon,n})\).

To study the weak solvability of the regularized problem \((P_{\varepsilon,n})\), we use an exponential weight function \( e^{\rho_\varepsilon(t-T)} \), where \( \rho_\varepsilon > 0 \) is called as an \( \varepsilon \)-dependent auxiliary parameter, to consider the mappings \( U^{\varepsilon,n} = e^{\rho_\varepsilon(t-T)} u^{\varepsilon,n} \) and \( V^{\varepsilon,n} = e^{\rho_\varepsilon(t-T)} v^{\varepsilon,n} \). Thus, (3.1) becomes

\[
\begin{align*}
    U_1^{\varepsilon,n} + \mathcal{F}_1 \left( e^{\rho_\varepsilon(T-t)} U^{\varepsilon,n} \right) \Delta U^{\varepsilon,n} - \rho_\varepsilon U^{\varepsilon,n} \\
    V_1^{\varepsilon,n} + \mathcal{F}_2 \left( e^{\rho_\varepsilon(T-t)} V^{\varepsilon,n} \right) \Delta V^{\varepsilon,n} - \rho_\varepsilon V^{\varepsilon,n}
\end{align*}
\]

endowed with the Dirichlet boundary condition and with the same terminal data as (3.2). Henceforth, we define a weak formulation of this transformed system in the following type.

**Definition 3.1.** For each \( \varepsilon > 0 \), a pair of functions \((U^{\varepsilon,n}, V^{\varepsilon,n})\) is said to be a weak solution of (3.3) if

\[
(U^{\varepsilon,n}, V^{\varepsilon,n}) \in \left[ L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega)) \right]^2
\]

and it holds that

\[
\begin{align*}
    \frac{d}{dt} (U^{\varepsilon,n}, \psi_1) - \mathcal{D}_1 \left( e^{\rho_\varepsilon(T-t)} U^{\varepsilon,n} \right) \int_\Omega \nabla U^{\varepsilon,n} : \nabla \psi_1 dx - \rho_\varepsilon \langle U^{\varepsilon,n}, \psi_1 \rangle \\
    = e^{\rho_\varepsilon(t-T)} \left\langle F_\varepsilon \left( e^{\rho_\varepsilon(T-t)} U^{\varepsilon,n}; e^{\rho_\varepsilon(T-t)} V^{\varepsilon,n} \right), \psi_1 \right\rangle + \langle \mathbf{P}_\varepsilon^1 U^{\varepsilon,n}, \psi_1 \rangle,
\end{align*}
\]

\[
\begin{align*}
    \frac{d}{dt} (V^{\varepsilon,n}, \psi_2) - \mathcal{D}_2 \left( e^{\rho_\varepsilon(T-t)} V^{\varepsilon,n} \right) \int_\Omega \nabla V^{\varepsilon,n} : \nabla \psi_2 dx - \rho_\varepsilon \langle V^{\varepsilon,n}, \psi_2 \rangle \\
    = e^{\rho_\varepsilon(t-T)} \left\langle G_\varepsilon \left( e^{\rho_\varepsilon(T-t)} U^{\varepsilon,n}; e^{\rho_\varepsilon(T-t)} V^{\varepsilon,n} \right), \psi_2 \right\rangle + \langle \mathbf{P}_\varepsilon^2 V^{\varepsilon,n}, \psi_2 \rangle,
\end{align*}
\]

for all \( \psi \in H_0^1(\Omega) \).

It is worth mentioning that to be successful with the Galerkin-type aid we choose in [26], the weight \( \rho_\varepsilon \) must be large, controlled by the largeness of the magnitude of the regularized problem (cf. [24]) and of the cut-off parameter \( \ell_\varepsilon \). In the following, we only state the well-posedness result of the “scaled” problem (3.3), while details of proof can be deduced as in [26].

**Theorem 3.2.** Assume \((A_1) - (A_3)\) hold. For each \( \varepsilon > 0 \), the regularized problem (3.3) admits a pair of weak solutions \((U^{\varepsilon,n}, V^{\varepsilon,n})\) in the sense of Definition 3.1. Moreover, one has \( U^{\varepsilon,n}, V^{\varepsilon,n} \in C([0,T]; L^2(\Omega)) \) and \( U_1^{\varepsilon,n}, V_1^{\varepsilon,n} \in L^2(0,T; (H^1(\Omega))') \).

In general, one can get the strong solution of (3.3) in the sense that \( U_1^{\varepsilon,n}, V_1^{\varepsilon,n} \) and \( \Delta U^{\varepsilon,n}, \Delta V^{\varepsilon,n} \) belong to \( L^2(0,T; L^2(\Omega)) \) by increasing the regularity of the corresponding terminal conditions \( U_1^{\varepsilon,n}, V_1^{\varepsilon,n} \) (cf. (3.2)) in \( H^1(\Omega) \); see [3, Theorem 3.3] for
detailed techniques that we can adapt. However, this augment is not practical as in the context we are dealing with the noisy data. In other words, their gradients cannot be measured and even if it is possible, it eventually requires very much effort and expense. Accordingly, this explains why at present, we cannot fully adapt the error estimates for the finite element solution of (3.3) obtained in, e.g., [9, 6] in section 4. This open question will be explored in the near future.

3.2. Error analysis. In the following, \( n \) will be dependent of \( \varepsilon \) (i.e. \( n := n(\varepsilon) \in \mathbb{N} \)) due to the argument obtained in Lemma 2.1, showing that \( n \) cannot be arbitrarily large. Unlike the previous section where we design the approximate problem, here we denote by \( u^\varepsilon_n, v^\varepsilon_n \) the regularized solutions due to the involved regularization parameter \( \beta \), recalled from subsection 1.3. Note here that the weight \( \rho \) we use in proofs of the main results plays the same role as the weight \( \rho_\varepsilon \) considered in subsection 3.1, i.e. it must be large as driven by the smallness of \( \varepsilon \). Additionally, we below assume that

\[
(C6) \quad \lim_{\varepsilon \to 0^+} \gamma^{C_1T}(T, \beta)\varepsilon^{1/n} = K_0 \in (0, \infty), \quad \lim_{\varepsilon \to 0^+} \gamma^{C_1T}(T, \beta)\mu_n^P = K_1 \in (0, \infty).
\]

Note that we no longer need \( C_1T \leq 1 \) as in [26] and thus, our error bounds work for any finite time \( T > 0 \).

**Theorem 3.3.** (error estimate for \( 0 < t < T \)). Assume \((A_1)-(A_3)\) and \((A_1')\) hold. Consider the coupled Dirichlet system (1.1) with terminal data (1.2) satisfying (1.3)-(1.4). Suppose that it has a unique pair of solutions satisfying the source condition

\[
\begin{align*}
&u, v \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; \mathbb{W}) \cap L^\infty(0, T; H_0^1(\Omega) \cap L^\infty(\Omega)), \\
&\text{where the function space } \mathbb{W} \text{ is obtained from the choice of the perturbing operator } Q_\varepsilon^\beta \text{ in Definition 1.5. By the resulting stabilized operator } P_\varepsilon^\beta \text{ in Definition 1.4, we consider } (u^\varepsilon_n, v^\varepsilon_n) \text{ as a pair of solutions of the approximate system } (3.1)-(3.2).
\end{align*}
\]

Then for \( 0 < \kappa \leq 2C_1T \) the following estimate holds:

\[
\begin{align*}
&\mathbb{E} \left( \| u^\varepsilon_n (\cdot, t) - u (\cdot, t) \|^2 + \| v^\varepsilon_n (\cdot, t) - v (\cdot, t) \|^2 \right) \\
&+ (\overline{M} - M_1) \mathbb{E} \int_0^T \left( \| \nabla (u^\varepsilon_n - u) (\cdot, s) \|^2 + \| \nabla (v^\varepsilon_n - v) (\cdot, s) \|^2 \right) ds \\
&\leq \left[ K_0^2 + (K_1^2 + C_0^2) \bar{C}_p \right] \gamma^{-2C_1T}(T, \beta) \log^\kappa (\gamma(T, \beta)) e^{2(T-t)C_3},
\end{align*}
\]

where \( C_3 \geq 0, \bar{C}_p > 0 \) are not dependent of \( \varepsilon \).

There is no doubt that it is hard to attain the convergence of regularization at \( t = 0 \). Hereafter, by requiring more information of the source condition we find \( t^\varepsilon > 0 \) in such a way that \( u^\varepsilon_n (\cdot, t^\varepsilon) \) will be a good approximation of \( u(\cdot, t = 0) \), involving the expectation operator.

**Theorem 3.4.** (error estimate for \( t = 0 \)). Under the assumptions of Theorem 3.3, we further assume that \( u, v \in C^1([0, T]; L^2(\Omega)) \). Then there exists \( t^\varepsilon \in (0, T) \) approaching 0 such that

\[
\begin{align*}
&\mathbb{E} \left( \| u^\varepsilon_n (\cdot, t^\varepsilon) - u (\cdot, 0) \|^2 + \| v^\varepsilon_n (\cdot, t^\varepsilon) - v (\cdot, 0) \|^2 \right) \\
&\leq \left[ K_0^2 + (K_1^2 + C_0^2) \bar{C}_p \right] C_4^{-1} \log^{\kappa-1} (\gamma(T, \beta)) e^{2Tc_3} + C_4^{-1} \log^{-1} (\gamma(T, \beta)) C_4,
\end{align*}
\]

where \( C_4 > 0 \) is independent of \( \varepsilon \) and \( 0 < \kappa^\varepsilon < \min \{2C_1t^\varepsilon, 1 \} \).
3.2.1. Proof of Theorem 3.3. Put \( \psi x, t = e^{\rho x(t-T)} [u_{\beta}^x(x,t) - u(x,t)] \), \( \psi x, t = e^{\rho x(t-T)} [v_{\beta}^x(x,t) - v(x,t)] \) for \( \rho \beta > 0 \). Then the “scaled” difference equation for \( u \) is given by

\[
\frac{\partial \psi x, t}{\partial t} + D_1 (u_{\beta}^x) \Delta \psi x, t - \rho \beta \psi x, t = F_{\beta}^x (u_{\beta}^x, v_{\beta}^x) + e^{\rho x(t-T)} Q_{\beta}^x u
\]

In parallel, we also derive that the difference equation is endowed with the zero Dirichlet boundary condition \( \psi x, T = 0 \) on the boundary \( \partial \Omega \) and with the terminal condition \( \psi x, T = U_{\beta}^x (x) - u_\ell (x) \) for \( x \in \Omega \). Hereby, we multiply this equation by \( \psi x, t \) and then integrate the resulting equation over the domain of interest \( \Omega \) to arrive at

\[
\frac{1}{2} \frac{d}{dt} \| \psi x, t \|^2 - D_1 (u_{\beta}^x) \| \nabla \psi x, t \|^2 - \rho \beta \| \psi x, t \|^2 = F_{\beta}^x (u_{\beta}^x, v_{\beta}^x) - F (u, \psi x, t) + e^{\rho x(t-T)} Q_{\beta}^x u
\]

Cf. Definition 1.4, we bound \( I_1 \) from below by

\[
I_1 \geq -C_1 \log (\gamma (T, \beta)) \| \psi x, t \|^2,
\]

aided by the Cauchy–Schwarz inequality. Meanwhile, we use the Young inequality in combination with Definition 1.5 to get

\[
I_2 \geq -e^{2\rho x(t-T)} C_2^2 \| u \|^2 - \frac{1}{4} \| \psi x, t \|^2.
\]

Now, by decreasing \( \varepsilon \) we can choose the cut-off parameter \( \ell \varepsilon \) large in such a way that \( \ell \varepsilon \geq \max \left\{ \| u \|_{L^\infty (0,T;L^\infty (\Omega))}, \| v \|_{L^\infty (0,T;L^\infty (\Omega))} \right\} \). Thus, it holds that \( F_{\beta}^x (u, v) = F (u, v) \), according to Remark 2.5. This way we bound \( I_3 \) from below by

\[
I_3 \geq -4L (\ell \varepsilon) \left( \| \psi x, t \|^2 + \| \psi x, t \|^2 \right).
\]

Using \( (A_1) \), the estimate of \( I_4 \) is thus given by

\[
I_4 \geq -\hat{L} \| \psi x, t \| \| \nabla u \| \| \nabla \psi x, t \| \geq -\left( \bar{M} - M_1 \right) \| \nabla \psi x, t \|^2 - \frac{\hat{L}^2 \| \nabla u \|^2}{4(M - M_1)} \| \psi x, t \|^2.
\]
Grouping the estimates of $I_i$ for $i = 1, 4$, we obtain the fact that

\[(3.7)\]

\[
\|X_{\beta}^{e,n} (\cdot, T)\|^2 - \|X_{\beta}^{e,n} (\cdot, t)\|^2 + e^{2\rho_{\beta}(T-t)}\mathcal{C}_0^2 \|u\|^2_{L^2(0,T;\hat{W})} \\
\geq (M - M_1) \int_t^T \left( \|\nabla X_{\beta}^{e,n} (\cdot, s)\|^2 \right) ds - 4L(\varepsilon) \int_t^T \left( \|X_{\beta}^{e,n} (\cdot, s)\|^2 + \|Y_{\beta}^{e,n} (\cdot, s)\|^2 \right) ds \\
+ \left( 2\rho_{\beta} - 2C_1 \log (\gamma (T, \beta)) - \frac{1}{2} \frac{\tilde{L}^2 \|\nabla u\|^2_{L^\infty(0,T;L^2(\Omega))}}{2(M - M_1)} \right) \int_t^T \|Y_{\beta}^{e,n} (\cdot, s)\|^2 ds.
\]

By doing the same token with the difference equation for $v$, we can find that

\[(3.8)\]

\[
\|Y_{\beta}^{e,n} (\cdot, T)\|^2 - \|Y_{\beta}^{e,n} (\cdot, t)\|^2 + e^{2\rho_{\beta}(T-t)}\mathcal{C}_0^2 \|v\|^2_{L^2(0,T;\hat{W})} \\
\geq (M - M_1) \int_t^T \left( \|\nabla Y_{\beta}^{e,n} (\cdot, s)\|^2 \right) ds - 4L(\varepsilon) \int_t^T \left( \|X_{\beta}^{e,n} (\cdot, s)\|^2 + \|Y_{\beta}^{e,n} (\cdot, s)\|^2 \right) ds \\
+ \left( 2\rho_{\beta} - 2C_1 \log (\gamma (T, \beta)) - \frac{1}{2} \frac{\tilde{L}^2 \|\nabla v\|^2_{L^\infty(0,T;L^2(\Omega))}}{2(M - M_1)} \right) \int_t^T \|Y_{\beta}^{e,n} (\cdot, s)\|^2 ds.
\]

Combining (3.7), (3.8), taking the expectation and recalling Lemma 2.1, we choose

\[\rho_{\beta} = C_1 \log (\gamma (T, \beta)) + \frac{1}{4} + 4L(\varepsilon) + \frac{\tilde{L}^2 (\|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|v\|^2_{L^\infty(0,T;L^2(\Omega))})}{2(M - M_1)} \]

and thus, it yields

\[
\mathbb{E} \left( \|X_{\beta}^{e,n} (\cdot, t)\|^2 + \|Y_{\beta}^{e,n} (\cdot, t)\|^2 \right) + (M - M_1) \mathbb{E} \int_t^T \left( \|\nabla X_{\beta}^{e,n}\|^2 + \|\nabla Y_{\beta}^{e,n}\|^2 \right) ds \\
\leq \varepsilon^2 n + \frac{\|u\|^2_{H^2(\Omega)} + \|v\|^2_{H^2(\Omega)}}{\mu_{\beta}^{2p}} + \mathcal{C}_0^2 e^{2\rho_{\beta}(T-t)} \left( \|u\|^2_{L^2(0,T;\hat{W})} + \|v\|^2_{L^2(0,T;\hat{W})} \right) \cdot
\]

By the back-substitutions of $X_{\beta}^{e,n}$ and $Y_{\beta}^{e,n}$, one has

\[(3.9)\]

\[
\mathbb{E} \left( \|u_{\beta}^{e,n} (\cdot, t) - u (\cdot, t)\|^2 + \|v_{\beta}^{e,n} (\cdot, t) - v (\cdot, t)\|^2 \right) \\
+ (M - M_1) \mathbb{E} \int_t^T \left( \|\nabla (u_{\beta}^{e,n} - u) (\cdot, s)\|^2 + \|\nabla (v_{\beta}^{e,n} - v) (\cdot, s)\|^2 \right) ds \\
\leq \varepsilon^2 n + \left( \frac{\mu_{\beta}^{2p} + \mathcal{C}_0^2 e^{2\rho_{\beta}(T-t)} \tilde{C}_p}{2(M - M_1)} \right) \gamma^{2c_1(T-t)} (T, \beta) e^{2(T-t)C_2(\varepsilon)},
\]

where we have denoted by $C_2(\varepsilon) = \frac{1}{4} + 4L(\varepsilon) + \frac{\tilde{L}^2 \tilde{C}_p}{2(M - M_1)}$ with $\tilde{C}_p > 0$ given by

\[
\tilde{C}_p = \|u\|^2_{H^2(\Omega)} + \|v\|^2_{H^2(\Omega)} + \|u\|^2_{L^2(0,T;\hat{W})} \\
+ \|v\|^2_{L^2(0,T;\hat{W})} + \|\nabla u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla v\|^2_{L^\infty(0,T;L^2(\Omega))}.
\]
Observe in (3.9) that if we take \( n \) and \( \gamma \) satisfying (3.6), we can go on the estimate (3.9) as follows:

\[
\begin{align*}
\mathbb{E} \left( \left\| u_{\beta}^{\varepsilon,n} (\cdot, t) - u (\cdot, t) \right\|^2 + \left\| v_{\beta}^{\varepsilon,n} (\cdot, t) - v (\cdot, t) \right\|^2 \right) \\
+ (\overline{M} - M_1) \mathbb{E} \int_t^T \left( \left\| \nabla \left( u_{\beta}^{\varepsilon,n} - u \right) (\cdot, s) \right\|^2 + \left\| \nabla \left( v_{\beta}^{\varepsilon,n} - v \right) (\cdot, s) \right\|^2 \right) ds \\
\leq \left[ K_0^2 + (K_1^2 + \overline{C}_0^2) \tilde{C}_p \right] \gamma^{-2C_1 t} (T, \beta) e^{2(T-t)C_2 (t^*)}. 
\end{align*}
\]

It now remains to deal with a fine control of \( C_2 (t^*) \) in (3.10) that involves the blow-up profile of \( L(t^*) \) (see again in Remark 2.5) as \( \varepsilon \to 0^+ \). In fact, due to the conventional logarithmic rate of convergence in regularization of parabolic problems, our attempt is to bound this \( \varepsilon \)-dependent constant in such a way that its growth does not destroy the aimed speed. To do that, we basically follow the strategy in [26, Section 5], which enables us to choose

\[
L(t^*) \leq \frac{1}{8(T-t)} \log (\log(\kappa(T, \beta))) \quad \text{for} \quad t \in (0, T),
\]

where \( \kappa > 0 \) is an \( \varepsilon \)-independent constant. Thus, it gives \( e^{8(T-t) L(t^*)} \leq \log(\kappa(T, \beta)) \) and consequently, it holds that

\[
\begin{align*}
\mathbb{E} \left( \left\| u_{\beta}^{\varepsilon,n} (\cdot, t) - u (\cdot, t) \right\|^2 + \left\| v_{\beta}^{\varepsilon,n} (\cdot, t) - v (\cdot, t) \right\|^2 \right) \\
+ (\overline{M} - M_1) \mathbb{E} \int_t^T \left( \left\| \nabla \left( u_{\beta}^{\varepsilon,n} - u \right) (\cdot, s) \right\|^2 + \left\| \nabla \left( v_{\beta}^{\varepsilon,n} - v \right) (\cdot, s) \right\|^2 \right) ds \\
\leq \left[ K_0^2 + (K_1^2 + \overline{C}_0^2) \tilde{C}_p \right] \gamma^{-2C_1 t} (T, \beta) \log(\kappa(T, \beta)) e^{2(T-t)C_2 (t^*)}.
\end{align*}
\]

where \( C_2 = \frac{1}{4} + \frac{L^2 \tilde{C}_p}{4(\overline{M} - M_1)} > 0 \) is now independent of \( \varepsilon \).

Hence, by choosing \( \kappa := \kappa(t^*) \leq 2C_1 t \) we complete the proof of the theorem.

**3.2.2. Proof of Theorem 3.4.** Using the triangle inequality and aided by Theorem 3.3, one has

\[
\begin{align*}
\mathbb{E} \left( \left\| u_{\beta}^{\varepsilon,n} (\cdot, t^*) - u (\cdot, 0) \right\|^2 + \left\| v_{\beta}^{\varepsilon,n} (\cdot, t^*) - v (\cdot, 0) \right\|^2 \right) \\
\leq 2\mathbb{E} \left\| u_{\beta}^{\varepsilon,n} (\cdot, t^*) - u (\cdot, t^*) \right\|^2 + 2\mathbb{E} \left\| v_{\beta}^{\varepsilon,n} (\cdot, t^*) - v (\cdot, t^*) \right\|^2 \\
+ 2\mathbb{E} \left\| u (\cdot, t^*) - u (\cdot, 0) \right\|^2 + 2\mathbb{E} \left\| v (\cdot, t^*) - v (\cdot, 0) \right\|^2 \\
\leq \left[ K_0^2 + (K_1^2 + \overline{C}_0^2) \tilde{C}_p \right] \gamma^{-2C_1 t^*} (T, \beta) \log(\kappa(T, \beta)) e^{2(T-t^*)C_3} \\
+ (t^*)^2 \left( \left\| u_t \right\|^2_{C(0,T;L^2(\Omega))} + \left\| v_t \right\|^2_{C(0,T;L^2(\Omega))} \right) .
\end{align*}
\]

Here, \( \kappa^* := \kappa(t^*) \in (0, 2C_1 t^*) \). Observe that since with respect to \( t^* \), the term \( \gamma^{-C_1 t^*} \) is decreasing, while \( t^* \) itself increases linearly. Thus, for each \( \varepsilon > 0 \) we can find a unique \( t^* \in (0, T) \) such that \( t^* = \gamma^{-C_1 t^*} (T, \beta) \) or equivalently, \( \log(t^*) = -C_1 \log(\gamma(T, \beta)) \). Thanks to the inequality \( \log(a) > -a^{-1} \) for all \( a > 0 \), we find that

\[
t^* < \sqrt{\frac{1}{C_1 \log(\gamma(T, \beta))}}.
\]
By the growth of $\gamma(T, \beta)$ as $\varepsilon \to 0^+$, one knows that $\lim_{\varepsilon \to 0^+} t^\varepsilon = 0$. Hence, we complete the proof of the theorem.

3.2.3. Essential remarks. Upon our main results reported in Theorems 3.3 and 3.4, we single out some important remarks where we believe that it is essential to highlight our novel QR method.

- By enjoying the exponential weight function in proofs of these theorems, it is clear that the speeds of convergence are exponentially controlled with respect to time and this control becomes larger when our reconstruction runs close to $t = 0$.

- When $F$ and $G$ are globally Lipschitz-continuous with respect to the arguments $u$ and $v$, we obtain the Hölder-type rate for $t \in (0, T)$, which means $O(\gamma^{-2C_1 t}(T, \beta))$, whilst it is of the logarithmic order $O(\log^{-1}(\gamma(T, \beta)))$ at $t = 0$. These are what we have obtained in [26, Theorem 4.5]. The presence of the term $\log^\circ(\gamma(T, \beta))$ in Theorem 3.3 shows that the locally Lipschitz-continuous nonlinearities usually encountered in real-world applications require much effort to approximate. In the near future, we will figure out a way to design a new bound for the cut-off constant $L(\ell^\circ)$ (cf. (3.11)).

- Cf. [26], if the model under consideration involves the gradient terms accounting for, e.g., haptotaxis in a population of cells (as presented in a model for haptotaxis and chemotaxis effects on cells’ motion; cf. [27]), our proposed method is still valid. A slightly lower rate of convergence is expected in controlling the gradient terms, but theoretically, it still converges logarithmically.

- The special difference between the deterministic noise in $[26, 24, 18, 28, 22]$ and the stochastic noise we are taking into account is the following. It is not fully about the involved expectation operator, but about the findings of $K_0$ and $K_1$ in (3.6). It shows that our QR method in this context really needs a very careful adaptation because not only $n$ has to be chosen properly, but also the eigenvalues driven by such $n$ need a fine control with respect to $\varepsilon$ and $\gamma$. Consider $\Omega$, for example, as an open parallelepiped $(0, a_1) \times \ldots \times (0, a_d) \subset \mathbb{R}^d$ with $a_i > 0$, $i \in \{1, \ldots, d\}$. As one of advantages of this QR method, we only need to solve the simple Dirichlet eigenvalue problem regardless of the complex structure involved in the diffusion term, whenever it is essentially bounded. Henceforth, the Dirichlet eigen-elements are given by

$$\phi_l = \prod_{j=1}^d \sqrt{\frac{2}{a_j}} \sin \left( \frac{\pi l_j}{a_j} x_j \right), \quad \mu_l = \sum_{j=1}^d \left( \frac{\pi l_j}{a_j} \right)^2 \quad \text{for } l_j \in \mathbb{N}, j \in \{1, \ldots, d\}.$$ 

In this mean, we choose $n$ in (3.6) as $|n|$ where $n := (n_1, \ldots, n_d)$. Then, we can choose $n_j \propto \varepsilon^{-2\theta}$ for $j = 1, \ldots, d$ and $\theta \in (0, 1)$. Notice that $\mu_n \propto |n|^2$, then $\gamma^{\gamma; T}(T, \beta) \propto \varepsilon^{-\max\{1-\theta, -4d\theta\}}$ is needed. Essentially, we have drawn a possibility to choose $n$ and $\gamma$ in the context of (3.6). Additionally, one can take $\beta = \varepsilon$ and thus $\gamma(t, \beta) = \varepsilon^{-\min\{1-\theta, -4d\theta\}}$. Hence, the following error bounds hold

$$\mathbb{E}\left\| \varepsilon_n u(t) - u(t) \right\|^2 + \mathbb{E}\left\| \varepsilon_n v(t) - v(t) \right\|^2 \leq C\varepsilon^{\frac{\theta}{2} - \min\{1-\theta, -4d\theta\}} \log^{\kappa(t)} \left( \varepsilon^{-\max\{1-\theta, -4d\theta\}} \right) \quad \text{for } t \in (0, T),$$

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\[
\mathbb{E}\left[\|u_{\beta}^{n}(\cdot, t^{\varepsilon}) - u(\cdot, 0)\|^2 + \mathbb{E}\left[\|v_{\beta}^{n}(\cdot, t^{\varepsilon}) - v(\cdot, 0)\|^2\right]\right] \\
\leq C \log^{\kappa-1}\left(\varepsilon^{-2\theta} \max(\theta - 1, -4\theta p)\right),
\]

where \(\kappa(t) \in (0, 2C_{1}t]\) and \(\kappa^{\varepsilon} \in (0, \min \{2C_{1}t^{\varepsilon}, 1\})\).

4. Numerical tests. Throughout the examples, we choose \(\Omega = (a, b) = (0, \pi)\) as a sample domain of tissue in one-dimensional. Accordingly, the Dirichlet eigen-values \(v_{\beta}\) of the Dirichlet conditions \(\langle u(x)\rangle_{\Omega} = \langle v(x)\rangle_{\Omega}\) are used. Here \(x\) is a sample domain of tissue in one-dimensional.

In this regard, a uniform grid of mesh-points \((x, t) = (\Delta x, \Delta t)\) is used. Here \(\Delta x = \frac{b-a}{N}\) and \(\Delta t = \frac{T}{K}\) for all \(t \in [0, T]\). Then (3.6) is satisfied with \(K_{0}, K_{1} \leq 1\). Employing Example 2.2, we define the stabilized operator \(P_{\varepsilon}^{\beta}\) as follows:

\[
P_{\varepsilon}^{\beta}u = -M \sum_{j \in \mathbb{B}_{\varepsilon}} \mu_{j} \langle u, \phi_{j} \rangle \phi_{j},
\]

where \(\mathbb{B}_{\varepsilon} := \{j \in \mathbb{N} : \mu_{j} \leq \log(\varepsilon^{-2\theta} \max(\theta - 1, -4\theta p))\}\) is the admissible set of our Fourier frequencies, and \(C_{1} = \overline{M}\).

We generate the discrete observations \(u_{f}^{\varepsilon,j}, v_{f}^{\varepsilon,j}\) for \(1 \leq j \leq n\) by the assumptions (1.3)–(1.4) where \(\langle \xi_{i}, \phi_{j} \rangle\) for \(i = 1, 2\) are random variables with normal distribution. In our illustration, we employ the manufactured solution to verify the validity of the regularization scheme. When doing so, we know the explicit expression of the true \(u_{f}\) and \(v_{f}\). Cf. [16, Section 4], the adaptive Filon-type method is rather reliable to compute the inner products \(\langle u_{f}, \phi_{j} \rangle\) and \(\langle v_{f}, \phi_{j} \rangle\) due to the low level of frequency restricted by the stabilized operator. It then enables us to obtain the reconstructions \(U_{f}^{\varepsilon,j}, V_{f}^{\varepsilon,j}\).

We now take into account the numerical regularized solution by using the backward Euler method. In this regard, a uniform grid of mesh-points \((x, t) = (x_{m}, t_{k})\) is used. Here \(x_{m} = a + m\Delta x\) and \(t_{k} = k\Delta t\) for \(0 \leq m \leq M\) and \(0 \leq k \leq K\) where \(\Delta x = \frac{b-a}{N}\), \(\Delta t = \frac{T}{K}\) and \(M, K \in \mathbb{N}\). Henceforth, we find \(u_{m,k}^{\varepsilon,n} \approx u^{\varepsilon,n}(x = x_{m}, t = t_{k})\) and \(v_{m,k}^{\varepsilon,n} \approx v^{\varepsilon,n}(x = x_{m}, t = t_{k})\) by solving the following fully discretized system:

\[
\frac{u_{m,k+1}^{\varepsilon,n} - u_{m,k}^{\varepsilon,n}}{\Delta t} + D_{1}(u_{k+1}^{\varepsilon,n}) u_{m+1,k}^{\varepsilon,n} - 2u_{m,k}^{\varepsilon,n} + u_{m-1,k}^{\varepsilon,n} = F_{\varepsilon}^{\theta} \left(u_{m,k+1}^{\varepsilon,n}, v_{m,k+1}^{\varepsilon,n}\right) + P_{\varepsilon}^{\beta} u_{m,k+1}^{\varepsilon,n},
\]

\[
\frac{v_{m,k+1}^{\varepsilon,n} - v_{m,k}^{\varepsilon,n}}{\Delta t} + D_{2}(v_{k+1}^{\varepsilon,n}) v_{m+1,k}^{\varepsilon,n} - 2v_{m,k}^{\varepsilon,n} + v_{m-1,k}^{\varepsilon,n} = G_{\varepsilon}^{\theta} \left(u_{m,k+1}^{\varepsilon,n}, v_{m,k+1}^{\varepsilon,n}\right) + P_{\varepsilon}^{\beta} v_{m,k+1}^{\varepsilon,n},
\]

for \(1 \leq m \leq M - 1\) and \(0 \leq k \leq K - 1\). Accordingly, we endow this system with the Dirichlet conditions \(u_{0,0}^{\varepsilon,n} = u_{M,0}^{\varepsilon,n} = v_{0,0}^{\varepsilon,n} = v_{M,0}^{\varepsilon,n} = 0\) and the terminal conditions \(u_{m,K}^{\varepsilon,n} = U_{f}^{\varepsilon,n}(x = x_{m}), v_{m,K}^{\varepsilon,n} = V_{f}^{\varepsilon,n}(x = x_{m})\). Note that our objective here is to solve this system backwards in time and thus, \(k\) runs from \(K\) to \(0\) in the iterations. Then,
for each $k = 0, \ldots, K - 1$ we arrive at a linear algebraic system with a Toeplitz-like matrix, as follows:

\[
\Lambda \left( u_{k+1}^{\varepsilon,n}, v_{k+1}^{\varepsilon,n} \right) \mathbb{W}_{k}^{\varepsilon,n} = \mathbb{F}_\varepsilon \left( u_{k+1}^{\varepsilon,n}, v_{k+1}^{\varepsilon,n} \right),
\]

where the solution $\mathbb{W}_{k}^{\varepsilon,n} \in \mathbb{R}^{2(M-1)}$, the block matrix $\Lambda \in \mathbb{R}^{2(M-1) \times 2(M-1)}$ and the function $\mathbb{F}_\varepsilon \in \mathbb{R}^{2(M-1)}$ are defined by

\[
\mathbb{W}_{k}^{\varepsilon,n} = \begin{bmatrix}
\begin{array}{cccccc}
  u_{1,k}^{\varepsilon,n} & u_{2,k}^{\varepsilon,n} & \cdots & u_{M-2,k}^{\varepsilon,n} & u_{1,k-1}^{\varepsilon,n} & v_{1,k}^{\varepsilon,n} & \cdots & v_{M-2,k}^{\varepsilon,n} & v_{1,k-1}^{\varepsilon,n}

e_{1,k}^{\varepsilon,n} & e_{2,k}^{\varepsilon,n} & \cdots & e_{M-2,k}^{\varepsilon,n} & e_{1,k-1}^{\varepsilon,n} & \varepsilon^{\varepsilon,n}_{1,k} & \cdots & \varepsilon^{\varepsilon,n}_{M-2,k} & \varepsilon^{\varepsilon,n}_{1,k-1}
\end{array}
\end{bmatrix}^T,
\]

\[
\mathbb{F}_\varepsilon \left( u_{k+1}^{\varepsilon,n}, v_{k+1}^{\varepsilon,n} \right) = \begin{bmatrix}
\begin{array}{cccccc}
  u_{1,k+1}^{\varepsilon,n} & u_{2,k+1}^{\varepsilon,n} & \cdots & u_{M-2,k+1}^{\varepsilon,n} & u_{1,k}^{\varepsilon,n} & v_{1,k+1}^{\varepsilon,n} & \cdots & v_{M-2,k+1}^{\varepsilon,n} & v_{1,k}^{\varepsilon,n}
\end{array}
\end{bmatrix},
\]

\[
\Lambda \left( u_{k+1}^{\varepsilon,n}, v_{k+1}^{\varepsilon,n} \right) = \begin{bmatrix}
\begin{array}{cccccc}
  D_1^{k+1} & & & & & & \\
  0 & D_2^{k+1}
\end{array}
\end{bmatrix},
\]

with $D_1^{k+1}, D_2^{k+1} \in \mathbb{R}^{K-1}$ taken by

\[
D_1^{k+1} = \begin{bmatrix}
\begin{array}{cccccc}
  1 & 2D_1^{k+1} & 0 & \cdots & \cdots & 0 \\
  -D_1^{k+1} & 1 & 2D_1^{k+1} & \cdots & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
  0 & \cdots & \cdots & \cdots & 1 & 2D_1^{k+1} \\
\end{array}
\end{bmatrix},
\]

\[
D_2^{k+1} = \begin{bmatrix}
\begin{array}{cccccc}
  1 & 2D_2^{k+1} & 0 & \cdots & \cdots & 0 \\
  -D_2^{k+1} & 1 & 2D_2^{k+1} & \cdots & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
  0 & \cdots & \cdots & \cdots & 1 & 2D_2^{k+1} \\
\end{array}
\end{bmatrix},
\]

for $D_1^{k+1} = D_1 \left( u_{k+1}^{\varepsilon,n} \right), D_2^{k+1} = D_2 \left( v_{k+1}^{\varepsilon,n} \right)$ and $\bar{\alpha} = \frac{\alpha}{M}$. We remark that for each iteration step $k$, (4.1) has a solution because the block matrix $\Lambda$ is symmetric positive definite. In this illustration, we only seek the approximation of $u^{\varepsilon}(x_m, t = t^\varepsilon)$, but usually $t^\varepsilon$ does not belong to the underlying
mesh-points. We then rely on the nearest point of this \( t^\varepsilon \), which means that for each \( \varepsilon > 0 \) the error, denoted by \( E(t_k) \), is computed for \( t_k \) such that \( t_k \geq t^\varepsilon \) and \( t_{k-1} < t^\varepsilon \). Given the number of samples \( M \in \mathbb{N}^* \) and choosing \( T = 1 \), the averaged error \( E(t_k) \) reads as

\[
E(t_k) = \frac{1}{M} \frac{1}{M} \sum_{m=1}^{M} \sum_{m_x=1}^{M} \left| u_{m,k,m_x}^{\varepsilon,n} - u_{ex}(x_m,0) \right|^2,
\]

where \( u_{ex} \) denotes the manufactured solution that we will choose in the numerical tests below. Note that for simplicity, we eventually omit the index \( m_x \) in the regularized solution in the above numerical setting. The value of \( t^\varepsilon \) can be approximately computed by the equation \( \log(t^\varepsilon) = -t^\varepsilon \log\left(\varepsilon^{\frac{\theta+1}{\theta+4}}\right) \) (cf. subsection 2.3.2).

### 4.1. Test 1.

We begin the numerical verification by the most widely used Lotka–Volterra model for tumor growth kinetics where in (1.1) one has \( D_2 \) is positive constants and the corresponding Lotka–Volterra functions are given by

\[
F = r_u u \left(1 - \frac{u}{K_u} - \frac{b_u v}{K_u}\right) + F_1(x,t), \quad G = r_v v \left(1 - \frac{v}{K_v} - \frac{b_v u}{K_v}\right) + F_2(x,t).
\]

Here, the involved parameters in \( F, G \) are positive constants and cf. [10], they possess the following biological meaning: \( b_u \) is the negative effects of tumor on normal cells, \( b_v \) is a lumped, phenomenological term presented by, e.g., the immune response; \( r_u, r_v \) are, respectively, maximum growth rates of normal and tumor cells; \( K_u, K_v \) denote their maximal densities.

Given \( D_1 = 0.5 \) and \( D_2 = 1 \) for slow diffusions, which then lead to \( M = 1 \) and \( M_1 = 0.75 \), we choose \( r_u = r_v = K_u = K_v = 1 \) and \( b_u = 2, b_v = 0.5 \), usually describing the slight domination of tumor cells. The manufactured solutions are given by \( v_{ex} = e^{-t} \sin(x) \) and \( v_{ex} = t^2 \sin(\pi - x) \), ensuring the positivity of solutions. This way we need to take

\[
F_1(x,t) = e^{-t} \sin(x) \left[e^{-t} \sin(x) + 2t^2 \sin(\pi - x) - \frac{3}{2}\right],
\]

\[
F_2(x,t) = 2t^2 + tx \left(2 - t + t^2 \sin(\pi - x) + \frac{t}{2} e^{-t} \sin(x)\right).
\]

By the choice of \( F \) and \( G \), we find that \( L(\ell^\varepsilon) = 1 + 3\ell^\varepsilon \) and hence, for every time step \( \ell^\varepsilon \) can be controlled by

\[
\ell^\varepsilon(t) = \frac{1}{3} \left(\frac{1}{8(T-t)} \log\left(\log^{\kappa(t)} \left(\varepsilon^{\frac{\theta+1}{\theta+4}}\max\{\theta+1, -4\theta\}\right)\right) - 1\right),
\]

where we choose \( \kappa(t) = C_1 t > 0 \). Thus, we can work with the cut-off functions \( F_{\ell^\varepsilon} \) and \( G_{\ell^\varepsilon} \) in this circumstance.

### 4.2. Test 2.

In the second example, we take into account the nonlocal problem as in [7], where \( F, G \) are independent of the solutions, i.e. \( F(x,t;u,v) = F(x,t) \) and \( G(x,t;u,v) = G(x,t) \). Then the diffusion terms are particularly expressed by

\[
D_i(u)(t) = d_i + \int_{\Omega, u \in [0,1]} u(1-u)dx.
\]

Basically, this context can be understood that the tumor cells are on the way to dominate the normal cells, since the evolution system is now uncoupled and

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partly driven by the total population of each species. In this test, we choose \( u^{\text{ex}} = 0.25 \log(3 + t) \sin^2(x) \) and \( v^{\text{ex}} = t \sin(x) \) with \( d_1 = 0.01 \) and \( d_2 = 0.05 \). Then, it follows that \( M = 4, M_1 = 3.5 \) and

\[
F(x, t) = \frac{1}{4(3 + t)} \sin^2(x)
- \frac{1}{8} \left( \frac{1}{25} + \frac{\pi}{2} \log (3 + t) \left( 1 - \frac{3}{16} \log (3 + t) \right) \right) \log (3 + t) \cos (2x),
\]

\[
G(x, t) = \left( 1 - t \left( \frac{t(\pi t - 4)}{2} - \frac{1}{20} \right) \right) \sin (x).
\]

Note that due to the nonlocal diffusion, we compute \( D_{k+1}^{i} (i = 1, 2) \) involved in the algebraic system (4.1) by employing the midpoint rule. It yields

\[
D_{k+1}^{i} = M - d_i \left( u^{\varepsilon,n}_{k+1} \right) \approx M - d_i - \frac{b - a}{M} \sum_{m=1}^{M-1} \tau \left( u^{\varepsilon,n}_{m,k+1} \right) \left| u^{\varepsilon,n}_{m,k+1} \right|_{\varepsilon[n,1]}
\]

where \( \tau (u) = |u (1 - u)| \). It is worth noting that the choice of the numerical integration for the diffusion term has to be careful. Although the efficient Gauss–Legendre quadrature and its variants seem applicable by choosing suitable nodes, in practice one cannot get measurements exactly at those abscissae. One can also take into account the Newton–Cotes method for the uniform grid in this implementation, but this approach is usually unstable when \( M \) is large. Therefore, one can see the impediment in handling regularization methods in numerics, especially when there are numerous nonlinear factors involved in the model.

4.3. Numerical results. We take \( p = 1 \) and \( \theta = 0.3 \) in these two tests. The choice of \( \theta \) indicates that we aim to see the convergence for a not very large number of steps of discrete observations \( n \). Meanwhile, the value of \( \varepsilon \) varies from \( 10^{-3} \) to \( 10^{-5} \).

![Fig. 1. Numerical errors between the regularized solution and the true solution in Test 1 with 100 samples and \( \varepsilon \in \{10^{-3}, 10^{-4}, 10^{-5}\} \).](image)

Our convergence results are concluded in Figures 1 and 2, where we have computed the numerical errors of two tests with \( M = 100 \) samples (the number of samples
Fig. 2. Numerical errors between the regularized solution and the true solution in Test 2 with 100 samples and $\varepsilon \in \{ 10^{-3}, 10^{-4}, 10^{-5} \}$.

is implemented in the horizontal line of each figure) and $M = 15$, $K = 100$. Having these errors allows us to obtain the averaged errors corresponding to each solution, which afterward confirms our theoretical expectation. Particularly, in Test 1 the averaged error for $u$ decreases from 0.01279 (for $\varepsilon = 10^{-3}$) to 0.00319 (for $\varepsilon = 10^{-5}$), while for $v$ it slowly goes down to 0.06514 from 0.09528. Similarly, in Test 2 the averaged error for $u$ is from 0.00191 (for $\varepsilon = 10^{-3}$) to 0.00177 (for $\varepsilon = 10^{-5}$), while for $v$ it decreases from 0.04640 to 0.02689. Henceforth, one can see that our approximation is acceptable from $\varepsilon = 10^{-3}$, i.e. this time the regularized solution is close to the true solution.

5. Conclusions. We have improved and adapted the recently developed QR method in [26] to regularization of a terminal-boundary value parabolic problem with white Gaussian noise. Although this noise process is very standard in the stochastic setting, this attempt concretely consolidates the mathematical quality of the reconstruction method we have established so far, where both the well-posedness of the regularized problem and the convergence analysis are successfully obtained. Besides, it again confirms the conventional logarithmic rate that we usually meet in regularization of parabolic problems. This time, based on finer energy estimates, we prove that this approach works for any finite time interval and derive a new type of perturbation. Concretely verified by a variant of the classical QR method, this special novelty shows that the Sobolev source conditions can be involved, whilst the speed of convergence remains unchanged. Although several questions remain open in [26], we would like to add one more interesting problem. In view of the localization of the brain tumor source in this paper, one can address the model in a domain with moving boundaries as investigated in, e.g., [2]. This way may be helpful in brain pathologies, especially in improving the Magnetic Resonance Imaging method as one wants to anatomically see the motion of the brain (see, e.g., [12]).

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REFERENCES

[1] S. Agapiou, A. M. Stuart, and Y.-X. Zhang, Bayesian posterior contraction rates for linear severely ill-posed inverse problems, Journal of Inverse and Ill-posed Problems, 22 (2014), https://doi.org/10.1515/jip-2012-0071.

[2] R. M. Almeida, J. C. Duque, J. Ferreira, and R. J. Robalo, Finite element schemes for a class of nonlocal parabolic systems with moving boundaries, Applied Numerical Mathematics, 127 (2018), pp. 226–248, https://doi.org/10.1016/j.apnum.2018.01.007.

[3] R. M. Almeida, S. N. Antontsev, J. C. M. Duque, and J. Ferreira, A reaction–diffusion model for the non-local coupled system: existence, uniqueness, long-time behaviour and localization properties of solutions, IMA Journal of Applied Mathematics, 81 (2016), pp. 344–364, https://doi.org/10.1093/imamat/hxv041.

[4] L. Cavalier, Inverse Problems and High-Dimensional Estimation, Springer Berlin Heidelberg, 2011, https://doi.org/10.1007/978-3-642-19989-9.

[5] L. Cavalier, Y. Golubev, O. Lepski, and A. Tsybakov, Block thresholding and sharp adaptive estimation in severely ill-posed inverse problems, Theory of Probability & Its Applications, 62 (2017), pp. 441–462, https://doi.org/10.1137/16M1079055.

[6] S. Chaudhary, A statistical approach to some inverse problems for partial differential equations, Problems of Information Transmission, 35 (1999), pp. 136–149.

[7] S. Chaudhary, Some remarks on non local elliptic and parabolic problems, Linear Algebra and its Applications, 30 (1997), pp. 4619–4627, https://doi.org/10.1016/s0362-546x(97)00169-7.

[8] V. N. Doan, H. T. Nguyen, V. A. Khoa, and V. A. Vo, A note on the derivation of filter regularization operators for nonlinear evolution equations, Applicable Analysis, 87 (2017), pp. 3–12, https://doi.org/10.1080/00036811.2016.1276176.

[9] J. C. Duque, R. M. Almeida, S. N. Antontsev, and J. Ferreira, The Euler–Galerkin finite element method for a nonlocal coupled system of reaction–diffusion type, Journal of Computational and Applied Mathematics, 296 (2016), pp. 116–126, https://doi.org/10.1016/j.cam.2015.09.019.

[10] R. Gatenby, P. Maini, and E. Gawlinski, Analysis of tumor as an inverse problem provides a novel theoretical framework for understanding tumor biology and therapy, Applied Mathematics Letters, 15 (2002), pp. 339–345, https://doi.org/10.1016/s0893-9659(01)00141-0.

[11] G. K. Golubev and R. Z. Khass’minskii, A statistical approach to some inverse problems for partial differential equations, Problemy Peredachi Informatsii, 35 (1999), pp. 136–149.

[12] S. J. Holdsworth, M. S. Rahimi, W. W. Ni, G. Zaharchuk, and M. E. Mosley, Amplified magnetic resonance imaging (aMRI), Magnetic Resonance in Medicine, 75 (2016), pp. 2245–2254, https://doi.org/10.1002/mrm.26142.

[13] R. Jaroudi, G. Baravdish, B. T. Johansson, and F. Åström, Numerical reconstruction of brain tumours, Inverse Problems in Science and Engineering, (2018), pp. 1–21, https://doi.org/10.1080/14715977.2018.1456537.

[14] J. P. Kaipio and C. Fox, The Bayesian framework for inverse problems in heat transfer, Heat Transfer Engineering, 32 (2011), pp. 718–753, https://doi.org/10.1080/17415977.2011.525137.

[15] B. Kaltenbacher and W. Rundell, Regularization of a backwards parabolic equation by fractional operators, Inverse Problems & Imaging, 13 (2019), pp. 401–430, https://doi.org/10.3934/ipi.2019020.

[16] V. A. Khoa, M. T. N. Truong, N. H. M. Duy, and N. H. Tuan, The Cauchy problem of coupled elliptic sine–Gordon equations with noise: Analysis of a general kernel-based regularization and reliable tools of computing, Computers & Mathematics with Applications, 73 (2017), pp. 426–446, https://doi.org/10.1016/j.camwa.2016.11.001.

[17] M. Kirane, E. Nane, and N. H. Tuan, On a backward problem for multidimensional Ginzburg–Landau equation with random data, Inverse Problems, 34 (2017), p. 015008, https://doi.org/10.1088/1361-6420/aa9c2a.

[18] M. V. Klibanov, Carleman estimates for the regularization of ill-posed Cauchy problems, Applied Numerical Mathematics, 94 (2015), pp. 46–74, https://doi.org/10.1016/j.apnum.2015.02.003.

[19] M. V. Klibanov, Carleman weight functions for solving ill-posed Cauchy problems for quasilinear PDEs, Inverse Problems, 31 (2015), p. 125007, https://doi.org/10.1088/0266-5611/31/12/125007.

[20] B. T. Knapp, A. W. van der Vaart, and J. H. van Zanten, Bayesian recovery of the initial
[20] N. H. Tuan, V. A. Khoa, P. T. K. Van, and V. V. Au, *Identification of the population density of a species model with nonlocal diffusion and nonlinear reaction*, Inverse Problems, 33 (2017), p. 055019, https://doi.org/10.1088/1361-6420/aa635f.

[21] R. Lattès and J.-L. Lions, *Méthode de quasi-réversibilité et applications*, Travaux et Recherches Mathématiques, No. 15, Dunod, Paris, 1967.

[22] N. T. Long and A. P. N. Dinh, *Approximation of a parabolic non-linear evolution equation backwards in time*, Inverse Problems, 10 (1994), pp. 905–914, https://doi.org/10.1088/0266-5611/10/4/010.

[23] N. D. Minh, K. T. Duc, N. H. Tuan, and D. D. Trong, *A two-dimensional backward heat problem with statistical discrete data*, Journal of Inverse and Ill-posed Problems, 26 (2018), pp. 13–31, https://doi.org/10.1515/jiip-2016-0038.

[24] P. T. Nam, *An approximate solution for nonlinear backward parabolic equations*, Journal of Mathematical Analysis and Applications, 367 (2010), pp. 337–349, https://doi.org/10.1016/j.jmaa.2010.01.020.

[25] E. Nane and N. H. Tuan, *Convergence rates in expectation for a nonlinear backward parabolic equation with Gaussian white noise*, https://arxiv.org/abs/http://arxiv.org/abs/1711.10081v2.

[26] H. T. Nguyen, V. A. Khoa, and V. A. Vo, *Analysis of a quasi-reversibility method for a terminal value quasi-linear parabolic problem with measurements*, SIAM Journal on Mathematical Analysis, 51 (2019), pp. 60–85, https://doi.org/10.1137/18m1174064.

[27] J. A. Sherratt, E. Sage, and J. Murray, *Chemical control of Eukaryotic cell movement: A new model*, Journal of Theoretical Biology, 162 (1993), pp. 23–40, https://doi.org/10.1006/jtbi.1993.1074.

[28] N. H. Tuan, V. V. Au, V. A. Khoa, and D. Lesnic, *Identification of the population density of a species model with nonlocal diffusion and nonlinear reaction*, Inverse Problems, 33 (2017), p. 055019, https://doi.org/10.1088/1361-6420/aa635f.