ON THE PRANDTL-KOLMOGOROV 1-EQUATION MODEL OF TURBULENCE

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ABSTRACT. We prove an estimate of total (viscous plus modelled turbulent) energy dissipation in general eddy viscosity models for shear flows. For general eddy viscosity models, we show that the ratio of the near wall average viscosity to the effective global viscosity is the key parameter. This result is then applied to the 1-equation, URANS model of turbulence for which this ratio depends on the specification of the turbulence length scale. The model, which was derived by Prandtl in 1945, is a component of a 2-equation model derived by Kolmogorov in 1942 and is the core of many unsteady, Reynolds averaged models for prediction of turbulent flows. Away from walls, interpreting an early suggestion of Prandtl, we set

\[ l = \sqrt{2k+1/2} \tau, \]

where \( \tau \) is a selected time scale. In the near wall region analysis suggests replacing the traditional \( l = 0.41d \) (\( d \) = wall normal distance) with \( l = 0.41d \sqrt{d/L} \) giving, e.g.,

\[ l = \min \left\{ \sqrt{2k+1/2} \tau, \ 0.41d \sqrt{d/L} \right\}. \]

This \( l(\cdot) \) results in a simpler model with correct near wall asymptotics. Its energy dissipation rate scales no larger than the physically correct \( O(U^3/L) \), balancing energy input with energy dissipation.

1. INTRODUCTION

Predicting turbulent flows in practical settings means solving models intended to predict averages of solutions of the Navier-Stokes (NS) equations. Among a wide variety of approaches, summarized in Wilcox [38], eddy viscosity URANS (unsteady Reynolds Averaged NS) models are used in many applications. Many are based on the 1-equation model of Prandtl [26] and Kolmogorov [17], considered herein and given by

\begin{align*}
\nu_t + \nu \cdot \nabla \nu - \nabla \cdot (2\nu + \nu_T(\cdot)) \nabla^h v &+ \nabla p = f(x, y, z), \\
\nabla \cdot v & = 0, \quad \text{and} \quad \nu_T = \mu l \sqrt{k}, \\
k_t + \nu \cdot \nabla k - \nabla \cdot ([\nu + \nu_T(\cdot)] \nabla k) + \frac{1}{l} k \sqrt{k} = \nu_T(\cdot) \nabla^h v \|^2.
\end{align*}

Following for example [22] and [38] p.37 eq. (3.9), \( v \) approximates a finite time window average of the Navier-Stokes velocity \( u \)

\[ v(x, y, z, t) \approx \overline{u}(x, y, z, t) = \frac{1}{\tau} \int_{t-\tau}^{t} u(x, y, z, t') dt'. \]

The fluctuation is \( u' = u - \overline{u} \). Its associated turbulent kinetic energy, approximated by the \( k \)–equation solution, is \( k_{true} = \frac{1}{2} [u - \overline{u}]^2(x, y, z, t) \). In [4] \( \nu \) is the kinematic
greater universality and improved model dissipation to lower Reynolds number solutions. Herein we analyze a specification of viscosity models, is that model dissipation often exceeds energy input and leads to the classical Von Karman constant, 0.41, is retained in (3). Prandtl [27] described \( l(\cdot) \) as "... the diameter of the masses of fluid moving as a whole in each individual case". This diameter is constrained by nearby walls leading to the classical \( l = 0.41d \) and here \( 0.41d \sqrt{d/L} \). Prandtl also mentioned a second possibility, "... or again, as the distance traversed by a mass of this type before it becomes blended in with neighboring masses..." This remark can be interpreted as \( l = |u'(x,t)| \), i.e., the distance a fluctuating eddy travels in one time unit. As \( |u'| \approx \sqrt{2} k^{1/2} \), away from walls we specify the kinematic relation

\[
(4) \quad l(\cdot) = \sqrt{2} k(\cdot)^{1/2} \tau.
\]

1.1. Justification of \( l = 0.41d \sqrt{d/L} \). The (dimensionally consistent) near wall \( l = 0.41d \sqrt{d/L} \) is a deviation from accepted practice, so justification is necessary. The true turbulent kinetic energy \( k_{\text{true}} = \frac{1}{2} |\vec{u} - \overline{\vec{u}}|^2 \to 0 \) like \( O(d^2) \) at walls. This rate implies that \( k_{\text{true}} \) satisfies

\[
k_{\text{true}} = 0 \quad \text{and} \quad \nabla k_{\text{true}} \cdot n = 0 \quad \text{at the wall}.
\]

The eddy viscosity should have a similar near wall behavior since, modulo pressure terms, \( \mu l \sqrt{k} \nabla^2 v \approx u' u' \to 0 \) at walls like \( O(d^2) \). If \( k_{\text{true}} \) replaces \( k \) in \( \nu_T(\cdot) \), then \( \mu l \sqrt{k_{\text{true}}} \nabla^2 v = O(d^2) \) near walls with \( l = 0.41d \). However, the solution to the \( k \)-equation satisfies only one boundary condition, \( k = 0 \) at the wall. Thus, the solution to the \( k \) equation (intended to model \( k_{\text{true}} \)) has only

\[
k = 0 \quad \text{at the wall, and} \quad k(d) = O(d) \quad \text{as the wall is approached}.
\]
This (incorrectly) implies \( \mu l \sqrt{k} \nabla^s v \to 0 \) at walls like \( O(d^{+1.5}) \) when \( l = 0.41d \). This is one reason for evaluations such as Pope [25] p. 434 Section 11.7.2 that "... the specification \( l = 0.41y \) is too large in the near wall region...” as well as ad hoc addition of van Driest damping. The modification \( l = 0.41d \sqrt{d/L} \) in (3) ensures \( \nu_T(\cdot) = O(d^2) \) correctly in the model.

The question arises of why not simply specify \( l(\cdot) = \sqrt{2k(\cdot) + \frac{1}{2} \tau} \) as in [20]. The positive results in [20] were for turbulence induced by a body force with \( f(x) = 0 \) on \( \partial \Omega \) which excludes shear flows. The physical difference in the settings (summarizing the introduction of Phillips [24]) is that in shear flows the near wall region produces small scales which dominate \( k_{true} \), while when shear flows are excluded in [20], small scales are produced only through the nonlinearity.

1.2. Related work. The energy dissipation rate is a fundamental statistic of turbulence, e.g., [25], [35]. Its balance with energy input rates, \( \langle \varepsilon \rangle = O(U^3/L) \), is observed in physical experiments [35]. In 1992, Doering and Constantin [8] established a direct link between phenomenology and NSE predicted energy dissipation through upper bounds consistent with the \( O(U^3/L) \) rate. This work builds on [2], [13] and has developed in many important directions, e.g., [7], [13], [35], [36], [15], [17]. Remarkably, an \( O(U^3/L) \) lower bound has recently been proven in [4] for stochastically forced shear flow.

Model over-dissipation, producing a lower \( Re \) flow, is due to the action of turbulent viscosity terms on small scales generated by breakdown of large scales through the nonlinearity or in the boundary layer. \( \langle \varepsilon \rangle \) has been analyzed for some simpler models, e.g., [18], [19] (showing a dramatic difference between shear and no shear cases), and [23]. The kinematic length scale \( l = \sqrt{2k + \frac{1}{2} \tau} \) occurred naturally in an ensemble algorithm in [14] and was highly developed by Teixeira and Cheinet [32] and [33] (see equation (7) on p. 2699), with near-wall transition to \( l = 0.41d \) by

\[
(5) \quad l = \theta(0.41d) + (1 - \theta) \left( \sqrt{2k + \frac{1}{2} \tau} \right), \text{ with } \theta = e^{-d/100}.
\]

The global specification \( l = \sqrt{2k + \frac{1}{2} \tau} \) was proven in [20] not to over-dissipate with shear excluded (and boundary layers negligible). This work leads to the problem considered herein to analyze shear/boundary layer induced model dissipation.

Since \( \tau \) in \( \tau = 0.76 / N, N = a \) selected-frequency, [6].

2. Shear Flow

We analyze energy dissipation caused by the boundary layer for shear flow with zero body force, building on analysis in the pioneering paper [8] and early work of Hopf [12]. Let the flow domain \( \Omega = (0, L)^3 \) and select \( L \)-periodic boundary conditions in \( x, y \) and no-slip at \( z = 0, z = L \). The wall is fixed at \( z = 0 \) and the
wall at \( z = L \) slides with velocity \((U, 0, 0)\):

\[
\begin{array}{ll}
\text{Boundary} & \text{Conditions:} \\
\text{moving top lid:} & v(x, y, L, t) = (U, 0, 0) \\
\text{fixed bottom wall:} & v(x, y, 0, t) = 0 \\
\text{periodic side walls:} & v(x + L, y, z, t) = v(x, y, z, t), \\
& v(x, y + L, z, t) = v(x, y, z, t),
\end{array}
\]

(6)

On this domain the wall normal distance is \( d = \min\{z, L - z\} \). Since time averages of the velocity satisfy the same shear boundary conditions as the NSE solution, the correct boundary condition for \( k(x, y, z, t) \) is

\[
k(x, y, 0, t) = k(x, y, L, t) = 0 \text{ and } L - \text{periodicity in } x, y.
\]

Since \( k \) has homogeneous boundary conditions, non-zero initial conditions must be specified; otherwise, if \( k(x, y, z, 0) = 0 \), then \( k(x, y, z, t) \equiv 0 \) thereafter.

2.1. Notation and preliminaries. The \( L^2(\Omega) \) norm and the inner product are \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \). The \( L^p(\Omega) \) norms are \( \| \cdot \|_{L^p} \). \( C \) represents a generic positive constant independent of \( \nu, Re \), and other model parameters.

**Definition 2.1.** The finite and long time averages of a function \( \phi(t) \) are

\[
\langle \phi \rangle_T = \frac{1}{T} \int_0^T \phi(t)dt \text{ and } \langle \phi \rangle_\infty = \limsup_{T \to \infty} \langle \phi \rangle_T.
\]

These satisfy \( \langle \phi \rangle_\infty/\infty = \langle \phi \rangle_\infty \) and

\[
\langle \phi \psi \rangle_T \leq \langle \phi \rangle_T^{1/2} \langle \psi \rangle_T^{1/2}, \quad \langle \phi \psi \rangle_\infty \leq \langle \phi \rangle_\infty^{1/2} \langle \psi \rangle_\infty^{1/2}.
\]

A weak solution of the model momentum equation for shear flow problem satisfies the initial condition and

\[
(v_t, w) + ([2\nu + \nu_T(\cdot)] \nabla^s v, \nabla^s w) + (v \cdot \nabla v, w) = 0
\]

for all test functions \( w \), with \( \nabla \cdot w = 0 \), \( L \)-periodic in \( x \) and \( y \) and \( w(x, y, 0, t) = 0, w(x, y, L, t) = 0 \). If \( \phi \) is a divergence free function extending the shear boundary conditions \( \ref{shear} \) into \( \Omega \), formally taking the inner product with \( w = v - \phi \) and expanding gives

\[
\frac{1}{2} \frac{d}{dt} ||v||^2 + \int_\Omega [2\nu + \nu_T(\cdot)] \nabla^s v^2 dx = \\
= (v_t, \phi) + \int_\Omega [2\nu + \nu_T(\cdot)] \nabla^s v : \nabla^s \phi dx + (v \cdot \nabla v, \phi).
\]

**Definition 2.2.** The total energy dissipation rate (per unit volume) is

\[
\varepsilon(v) = \frac{1}{|\Omega|} \int_\Omega [2\nu + \nu_T(\cdot)] ||\nabla^s v(x, t)||^2 dx.
\]

While a new \( l(\cdot) \) gives a new model, existence of weak solutions to models of this type is treated comprehensively in [3] and [1]. Herein, we assume that a weak solution of the model \( \ref{shear} \) with shear boundary conditions \( \ref{shear} \) exists, \( k \geq 0 \) and solutions satisfy the energy inequality

\[
\frac{1}{2} \frac{d}{dt} ||v||^2 + \int_\Omega [2\nu + \nu_T(\cdot)] \nabla^s v^2 dx \\
\leq (v_t, \phi) + \int_\Omega [2\nu + \nu_T(\cdot)] \nabla^s v : \nabla^s \phi dx + (v \cdot \nabla v, \phi).
\]

(9)
Using the energy inequality the appendix gives a proof of the following bounds.

**Proposition 2.1 (Uniform Bounds).** Consider the 1-equation model \(\text{(1), (3)}\) with shear boundary conditions \(\text{(6)}\). The following are uniformly bounded in \(T\):

\[
||v(T)||^2, \int_\Omega k(T)dx, \int_\Omega \nu_T(\cdot, T)dx,
\]

\[
\left\langle \frac{1}{L^3} \int_\Omega |\nabla^s v|^2 dx \right\rangle_T, \left\langle \frac{1}{L^3} \int_\Omega \frac{1}{\sqrt{k}} \sqrt{k} dx \right\rangle_T, \left\langle \frac{1}{L^3} \int_\Omega [2\nu + \nu_T(\cdot)] |\nabla^s v|^2 dx \right\rangle_T.
\]

3. **Energy dissipation in shear flows**

To formulate our first main result we first present a definition of the effective viscosity \(\nu_{eff} \geq \nu\), the average viscosity in the boundary layer \(S_\beta\), and a few related quantities. These are well defined due to the uniform bounds in Proposition 2.3.

**Definition 3.1.** The effective viscosity of solutions of \(\text{(1)}\) under \(\text{(6)}\) is

\[
\nu_{eff} := \frac{\left\langle \frac{1}{|\Omega|} \int_\Omega [2\nu + \nu_{turb}(\cdot)] |\nabla^s v|^2 dx \right\rangle}{\left\langle \frac{1}{|\Omega|} \int_\Omega |\nabla^s v|^2 dx \right\rangle}. 
\]

The large scale turnover time is \(T^* = L/U\). The Reynolds number and effective Reynolds number are

\[
Re = \frac{UL}{\nu} \quad \text{and} \quad Re_{eff} = \frac{UL}{\nu_{eff}}.
\]

Let \(\beta = \frac{1}{8} Re_{eff}^{-1}\) and denote the region \(S_\beta\) by

\[
S_\beta = \{ (x, y, z) : 0 \leq x \leq L, 0 \leq y \leq L, (1 - \beta)L < z < L \}.
\]

The average viscosity, \(\nu\), in \(S_\beta\) is denoted

\[
\nu := \left\langle \frac{1}{|S_\beta|} \int_{S_\beta} [2\nu + \nu_T(\cdot)] dx \right\rangle, \text{ where } |S_\beta| = \beta L^3.
\]

Generally, the ratio of the effective and average viscosity is an important statistic.

**Theorem 3.1.** Suppose \(\nu_T(\cdot) \geq 0\). Let \(v\) be a weak solution of

\[
v_t + v \cdot \nabla v - \nabla \cdot ([2\nu + \nu_T(\cdot)] \nabla^s v) + \nabla p = 0, \text{ and } \nabla \cdot v = 0
\]

under \(\text{(6)}\) satisfying the energy inequality \(\text{(9)}\). Then, provided \(\nu, \nu_{eff}\) are well defined,

\[
\left\langle \varepsilon \right\rangle_{\infty} \leq \left\{ \frac{5}{2} + 8 \frac{\nu}{\nu_{eff}} \right\} U^3.
\]

**Remark 3.1.** The multiplicative constants \(5/2, 8\) have not been optimized. Due to the problem symmetries and Galilean invariance, only the upper layer (near \(z = L\)) needs to be monitored. For general shear flows, the average viscosity \(\bar{\nu}\) should be defined (and thus monitored) as the average over all (here upper and lower) boundary layers present.
The proof begins with the background flow from Doering and Constantin [8], \( \phi(z) = [\phi(z), 0, 0]^T \) where
\[
\bar{\phi}(z) = \begin{cases} 
\frac{v}{\beta T}(z - (L - \beta L)), & z \in \mathbb{R}, \\
0, & z \in \mathbb{R}.
\end{cases}
\]

This function \( \phi(z) \) is piecewise linear, continuous, divergence free and satisfies the boundary conditions. We will need the following values of norms of \( \phi \).

\[ \mathrm{Lemma\ 3.1.}\quad\text{We have} \quad \nabla \cdot \phi = 0 \quad \text{and} \quad \begin{align*}
\| \phi \|_{L^\infty(\Omega)} &= U, \\
\| \nabla \phi \|_{L^\infty(\Omega)} &= \frac{v}{\beta T}, \\
\| \phi \|_2 &= \frac{1}{3} U^2 \beta L^3, \\
\| \nabla \phi \|_2 &= \frac{1}{3} U^2 \beta L.
\end{align*} \]

With this choice of \( \phi \), time averaging the energy inequality [9] over \( [0, T] \) and normalizing by \( |\Omega| = L^3 \) gives
\[
\frac{1}{2T L^3} \|v(T)\|^2 + \int_\Omega [2\nu + \nu_T(\cdot)] \nabla^s v |^2 dx \leq \leq \frac{1}{2T L^3} \|v(0)\|^2 + \frac{1}{L^3} \langle v(T) - v(0), \phi \rangle + \frac{1}{L^3} \langle v \cdot \nabla v, \phi \rangle + \frac{1}{L^3} \int_\Omega [2\nu + \nu_T(\cdot)] \nabla^s v : \nabla^s \phi dx.
\]

Recall \( \beta = \frac{1}{8} \Re_{\text{eff}}^{-1} \). Due to Proposition 2.3, (10) can be written as
\[
\langle \varepsilon \rangle_T \leq \mathcal{O}(\frac{1}{T}) + \frac{1}{L^3} \langle v \cdot \nabla v, \phi \rangle + \frac{1}{L^3} \int_\Omega [2\nu + \nu_T(\cdot)] \nabla^s v : \nabla^s \phi dx.
\]

The right-hand side (RHS) has two terms shared by the NSE, \( (v \cdot \nabla v, \phi) \) and \( \int 2\nu \nabla^s v : \nabla^s \phi dx \). The main issue is thus the third term, \( \int \nu_T(\cdot) \nabla^s v : \nabla^s \phi dx \). Before treating that we recall the analysis of Doering and Constantin [8] and Wang [36] for the first two. For the nonlinear term \( \frac{1}{L^3} \langle v \cdot \nabla v, \phi \rangle \), denoted \( NLT \), we have
\[
NLT = \frac{1}{L^3} \langle v \cdot \nabla v, \phi \rangle = \frac{1}{L^3} \langle (v - \phi) \cdot \nabla v, \phi \rangle + \frac{1}{L^3} \langle \phi \cdot \nabla v, \phi \rangle \
\leq \frac{1}{L^3} \int_{S_h} |v - \phi||\nabla v||\phi| + |\phi|^2|\nabla| dx
\leq \frac{1}{L^3} \left( \frac{1}{L^3} \right)^\frac{1}{2} \left( \frac{L}{L - z} \right)^2 \|v - \phi\|^2_{L^2(S_h)} \|\nabla v\|^2_{L^2(S_h)} \|\nabla\phi\|^2_{L^2(S_h)} + \|\phi\|^2_{L^2(S_h)} \|\nabla\phi\|^2_{L^1(S_h)}.
\]

On the RHS, \( \|\phi\|^2_{L^\infty(S_h)} = U^2 \). We calculate \( \|(L - z)\phi\|_{L^\infty(S_h)} = \frac{1}{3} \beta LU \). Since \( v - \phi \) vanishes on \( \partial S_h \), Hardy’s inequality, the triangle inequality and a calculation imply
\[
\|v - \phi\|_{L^2(S_h)} \leq 2 \|\nabla (v - \phi)\|_{L^2(S_h)} \leq 2 \|\nabla v\|_{L^2(S_h)} + 2 \|\nabla\phi\|_{L^2(S_h)} \leq 2 \|\nabla v\|_{L^2(S_h)} + 2U \sqrt{\frac{L}{\beta}}.
\]
Thus we have the estimate

\[ NLT \leq \frac{\beta LU}{4} \frac{1}{L^3} \left( 2||\nabla v||_{L^2(S_R^\beta)}^2 + 2U \sqrt{\frac{L}{\beta}} ||v||_{L^2(S_R^\beta)} \right)_T + \frac{U^2}{L^3} \frac{1}{T^3} \left( ||\nabla v||_{L^1(S_R^\beta)} \right)_T. \]

For the last term on the RHS, H"older's inequality in space then in time implies

\[ \frac{U^2}{L^3} \frac{1}{T^3} \left( ||\nabla v||_{L^1(S_R^\beta)} \right)_T = \frac{U^2}{L^3} \frac{1}{T^3} \left( \int_{S_R^\beta} |\nabla v| \cdot 1 \, dx \right)_T \leq \frac{U^2}{L^3} \frac{1}{T^3} \left( \int_{S_R^\beta} |\nabla v|^2 \, dx \sqrt{\beta L^3} \right)_T \leq \frac{U^2 \sqrt{\beta}}{L^{3/2}} \left( \int_{S_R^\beta} |\nabla v|^2 \, dx \right)^{1/2}_T. \]

Increase the integral’s domain from $S_R^\beta$ to $\Omega$, use (as $\nabla \cdot v = 0$) $||\nabla v||^2 = 2||\nabla^s v||^2$ and $\beta = \frac{1}{8} Re_{eff}^{-1}$. Rearranging and using the arithmetic-geometric inequality gives

\[ \frac{U^2}{L^3} \frac{1}{T^3} \left( ||\nabla v||_{L^1(S_R^\beta)} \right)_T \leq \frac{U^2 \sqrt{\beta}}{L^{3/2}} \left( \frac{1}{T} \int_{\Omega} |\nabla^s v|^2 \, dx \right)^{1/2}_T \leq \frac{1}{2} \left( \frac{U^3}{L} \right)^{1/2} \frac{1}{2} \left( \frac{1}{L^3} \int_{\Omega} \nu_{eff} |\nabla^s v|^2 \, dx \right)^{1/2}_T \leq \frac{1}{2} \frac{U^3}{L} + \frac{1}{8} \left( \frac{1}{L^3} \int_{\Omega} \nu_{eff} |\nabla^s v|^2 \, dx \right)_T. \]

Similar manipulations yield

\[ \frac{1}{4} \beta LU \frac{1}{L^3} \left( 2U \sqrt{\frac{L}{\beta}} ||v||_{L^2(S_R^\beta)} \right)_T \leq \frac{1}{2} \beta LU \left( \frac{1}{L^3} ||v||_{L^2(S_R^\beta)} \right)_T + \frac{U^3}{8 L} \leq \frac{1}{8} \left( \frac{1}{L^3} \nu_{eff} ||\nabla^s v||^2_{L^2(S_R^\beta)} \right)_T + \frac{U^3}{8 L}. \]

Using the last two estimates in the NLT upper bound (12), we obtain

\[ NLT \leq 2 \beta LU \frac{1}{\nu_{eff}} \left( \frac{1}{L^3} \nu_{eff} ||\nabla^s v||^2_{L^2(S_R^\beta)} \right)_T + \frac{5}{8} \frac{U^3}{L} + \frac{5}{8} \frac{U^3}{L}. \]

Thus,

\[ \langle \varepsilon \rangle_T \leq C \left( \frac{1}{T} \right) + \frac{1}{4} \left( \frac{1}{L^3} \nu_{eff} ||\nabla^s v||^2_{L^2(\Omega)} \right)_T + \frac{5}{8} \frac{U^3}{L} + \frac{1}{L^3} \left( \int_{\Omega} [2\nu + \nu_T(\cdot) |\nabla^s v : \nabla^s \phi] \, dx \right)_T. \]
Consider now the last term on the RHS. Since $\phi$ is zero off $S_\beta$,

$$\left\langle \frac{1}{L^3} \int_{\Omega} [2\nu + \nu_T(\cdot)] \nabla^s v : \nabla^s \phi dx \right\rangle_T = \left\langle \frac{1}{L^3} \int_{S_\beta} [2\nu + \nu_T(\cdot)] \nabla^s v : \nabla^s \phi dx \right\rangle_T$$

$$\leq \frac{1}{2} \langle \varepsilon \rangle_T + \frac{1}{2} \left\langle \frac{1}{L^3} \int_{S_\beta} [2\nu + \nu_T(\cdot)] \left( \frac{U}{\beta L} \right)^2 dx \right\rangle_T$$

$$\leq \frac{1}{2} \langle \varepsilon \rangle_T + \frac{1}{2} \left( \frac{U}{\beta L} \right)^2 \left\langle \frac{1}{L^3} \int_{S_\beta} [2\nu + \nu_T(\cdot)] dx \right\rangle_T.$$ 

Thus

$$\frac{1}{2} \langle \varepsilon \rangle_T \leq O\left( \frac{1}{T} \right) + \frac{1}{4} \left\langle \frac{1}{L^3} \nu_{e,ff} ||\nabla^s v||^2_{L^2(\Omega)} \right\rangle_T +$$

$$+ \frac{5 U^3}{8 L} + \frac{\beta}{2} \left( \frac{U}{\beta L} \right)^2 \left\langle \frac{1}{L^3} \int_{S_\beta} 2\nu + \nu_T(\cdot) dx \right\rangle_T.$$

As $T \to \infty$

$$\left\langle \frac{1}{\beta L^3} \int_{S_\beta} 2\nu + \nu_T(\cdot) dx \right\rangle_T \to \overline{\nu} \text{ and } \left\langle \frac{1}{L^3} \nu_{e,ff} ||\nabla^s v||^2_{L^2(\Omega)} \right\rangle_T \to \langle \varepsilon \rangle_\infty.$$

Thus,

$$\left( \frac{1}{2} - 2\beta Re_{e,ff} \right) \langle \varepsilon \rangle_\infty \leq \frac{5 U^3}{8 L} + \frac{1}{2} \left( \frac{U}{\beta L} \right)^2 \beta \overline{\nu} \leq \left[ \frac{5}{8} + \frac{1}{2\beta Re_{e,ff}} \frac{\overline{\nu}}{\nu_{e,ff}} \right] \frac{U^3}{L}.$$ 

The choice $\beta = \frac{1}{8} Re_{e,ff}^{-1}$ implies $2\beta Re_{e,ff} = 1/4$, completing the proof since

$$\langle \varepsilon \rangle_\infty \leq \frac{5 U^3}{2 L} + \frac{1}{2} \left( \frac{U}{\beta L} \right)^2 \beta \overline{\nu} = \left[ \frac{5}{2} + \frac{8 \overline{\nu}}{\nu_{e,ff}} \right] \frac{U^3}{L}.$$ 

4. Application to a 1-equation URANS model

We now apply Theorem 3.2 to (1), (3). The main work will be in estimating $\nu_{e,ff}$.

Theorem 4.1. Let $\nu$ be a weak solution of (1), (3) under (6) satisfying the energy inequality (9). We have

$$\langle \varepsilon \rangle_\infty \leq \left[ 5 + 32 \frac{\nu}{\nu_{e,ff}} + \left( \frac{0.41^2 \sqrt{2}\mu^2}{4} \right) \frac{\tau}{T^*} \right] \frac{U^3}{L}.$$ 

Remark. We note that $\frac{\nu}{\nu_{e,ff}} \leq 1$ (and possibly $<< 1$) and for $\mu = 0.55$, $0.41^2 \sqrt{2}\mu^2/4 \approx 0.017978$. 

The long time averaging of $\langle 1 \rangle$ is used in the boundary layer region to estimate $\tau$ as follows

$$
\tau = \left\langle \frac{1}{\beta L^3} \int_{S_0} 2 \nu + \nu_T(\cdot) dx \right\rangle \leq 2 \nu + \left\langle \frac{1}{\beta L^3} \int_{S_0} \mu \left( 0.41d \sqrt{\frac{d}{\tau}} \right) k^4 dx \right\rangle \sim 
$$

$$
\leq 2 \nu + 0.41 \mu L^{1/2} \beta \left\langle \int_{S_0} (L-z)^{3/2} k^{1/2} dx \right\rangle \sim 
$$

$$
\leq 2 \nu + 0.41 \mu L^{1/2} \beta \left\langle \int_{S_0} (L-z)^{3} dx \right\rangle \sim 
$$

$$
\leq 2 \nu + 0.41 \mu L^{1/2} \beta \left\langle \int_{S_0} k \right\rangle \sim 
$$

$$
(13) \quad \tau \leq 2 \nu + 0.41 \mu L^{1/2} \beta \left\langle \frac{1}{L^3} \int_{\Omega} k dx \right\rangle \sim 
$$

Next use the $k$-equation to estimate $\int k dx$. We have

$$
(14) \quad \int_{\Omega} k dx + \int_{\Omega} \frac{1}{\nu} k \sqrt{k} dx = \int_{\Omega} \nu_T(\cdot) \nabla \cdot v \ dx. 
$$

By the choice of $l$, $\frac{1}{l} k \sqrt{k}$ is bounded below by $\frac{1}{\sqrt{2} \tau} k$ because

$$
\frac{1}{l} k \sqrt{k} = \max \left\{ \frac{1}{\sqrt{2} \tau}, \frac{\sqrt{k}}{0.41d \sqrt{\frac{d}{\tau}}} \right\} k \geq \frac{1}{\sqrt{2} \tau} k. 
$$

The long time averaging of $\int k dx$ is zero. Since $\frac{1}{\sqrt{2} \tau} k \leq \frac{1}{l} k \sqrt{k}$, we have

$$
\frac{1}{\sqrt{2} \tau} \left\langle \frac{1}{|\Omega|} \int_{\Omega} k dx \right\rangle \sim \left\langle \frac{1}{|\Omega|} \int_{\Omega} \frac{1}{l} k \sqrt{k} dx \right\rangle = \left\langle \frac{1}{|\Omega|} \int_{\Omega} \nu_T(\cdot) \nabla \cdot v \right\rangle \sim = \langle \epsilon \rangle \sim. 
$$

Thus, $\left\langle \frac{1}{|\Omega|} \int_{\Omega} k dx \right\rangle \sim \leq \sqrt{2} \tau \langle \epsilon \rangle \sim$. Using this upper estimate in (13) we obtain

$$
\tau \leq 2 \nu + 0.41 \mu L^{1/2} \beta \left\langle \frac{1}{L^3} \int_{\Omega} k dx \right\rangle \sim \leq 2 \nu + 0.41 \sqrt{2} \mu L^{1/2} \sqrt{T^*} \sqrt{\langle \epsilon \rangle \sim}. 
$$

Divide by $\nu_{eff}$, use $T^* = L/U, \beta = \frac{1}{8} Re_{eff}^{-1}$ and rearrange. This gives

$$
\frac{\tau}{\nu_{eff}} \leq 2 \frac{\nu}{\nu_{eff}} + 0.41 \sqrt{2} \mu L^{1/2} \sqrt{T^*} \sqrt{\langle \epsilon \rangle \sim}. 
$$

Using this estimate in Theorem 3.2 gives

$$
\langle \epsilon \rangle \sim \leq \left[ \frac{5}{2} + 16 \frac{\nu}{\nu_{eff}} \right] \frac{U^3}{L} + \left[ \frac{0.41 \sqrt{2} \mu L^{1/2}}{2} \sqrt{T^*} \sqrt{\frac{U^3}{L}} \right] \sqrt{\langle \epsilon \rangle \sim}. 
$$

The arithmetic-geometric mean inequality then completes the proof:

$$
\langle \epsilon \rangle \sim \leq \left[ 5 + 32 \frac{\nu}{\nu_{eff}} + 4 \frac{0.41 \sqrt{2} \mu^2 \tau}{T^*} \right] \frac{U^3}{L}. 
$$
This section provides a computational illustration of the theoretical results for the model. The results of the computations are consistent with the theoretical predictions. The results were obtained on a workstation with a program developed with the FEniCS software suite [21]. The code can be found on GitHub at https://github.com/kierakean/1eqnRANS-FEM.

5.1. Problem Setting. We examined the classical Taylor-Couette flow between counter-rotating cylinders for rotations well above, e.g. [10], those yielding stable patterns, [31]. The domain is given by

$$\Omega = \{(x, y, z) : r_{\text{inner}}^2 \leq x^2 + y^2 \leq r_{\text{outer}}^2, 0 \leq z \leq z_{\text{max}}\},$$

with $r_{\text{inner}} = .5$, $r_{\text{outer}} = 1$, $z_{\text{max}} = 2.2$. Figure 1 (a) depicts the domain $\Omega$.

![The Domain Ω](image)

(a) The Domain Ω. (b) The mesh viewed from the top.

**FIGURE 1.** The unstructured mesh used in the numerical experiments.

We imposed periodic boundary conditions in the $z$ direction. The outer cylinder was held fixed and the flow was driven by the rotation of the inner cylinder. The angular velocity of the inner cylinder, $\omega_{\text{inner}}$ was smoothly increased from zero at $T = 0$ to $\omega_{\text{inner}} = 4$ at $T = 5$. Plots of flow statistics indicated that statistical equilibrium was reached around $T = 20$ so we give snapshots below at $T = 30$. We chose final time $T = 40$ and time averaged over $20 \leq t \leq 40$. The time scale was chosen to be $\tau = 0.1$.

**Initialization.** The model is turned on with a non-zero $k(x,5)$ at $T = 5$ when the inner cylinder has been spun up to its full angular velocity. We use a $k$ initialization standard for turbulent flow in a square duct, Wilcox [38], given by

$$k(x,5) = 1.5|v(x,5)|^2I^2,$$

where $I$ = turbulence intensity $\simeq 0.16Re^{-1/8}$.

**The mesh.** We used an unstructured mesh that was refined around the inner and outer boundaries, as can be seen from the top of the mesh in Figure 1 (b). We did preliminary tests at Reynolds number $Re = 1000$ by refining the mesh until $\langle \varepsilon \rangle$
was unchanged on three successive refinements. These parameters yielded a Taylor number of
\[ Ta := \frac{\omega^2 r_{\text{inner}} (r_{\text{outer}} - r_{\text{inner}})^3}{\nu^2} = 10^6. \]
We then did all reported tests on the coarsest mesh that produced the same value of \( \langle \varepsilon \rangle \).

Tests were run with varying Reynolds numbers by varying the viscosity \( \nu \) from \( 3 \times 10^{-3} \) to \( 5 \times 10^{-4} \) (\( Ta \simeq 10^{10} \) to \( 2.5 \times 10^{17} \)). Persistent vortices, marked by the Q-criterion, are plotted for two Reynolds numbers in Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{vortices.png}
\caption{Q-criterion at \( T = 30 \).}
\end{figure}

We used the \( P^2 - P^1 \) Taylor-Hood element pair. The velocity space, \( X_h \) and pressure space, \( Q_h \) had 947,802 and 44,585 degrees of freedom, respectively. We used the timestepping scheme backward Euler plus time filter from [11] for the momentum and continuity equation. The added time filter increased accuracy and reduced numerical dissipation making the calculated \( \langle \varepsilon \rangle \) more accurate. We used Backward Euler for the \( k \) equation. This choice smoothed the \( k(x,t) \) evolution and reduced solver issues. We took \( \Delta t = 1e^{-2} \) and ran the simulation from \( T = 0 \) to \( T = 40 \).

5.2. Energy Dissipation Rate. In Figure 3 \( \varepsilon(t) \) is plotted as a function of time. The jump at \( T = 5 \) corresponds to when the \( k \) equation (and thus the turbulent viscosity) is turned on.

To find the dependence on the Reynolds number, we plotted \( \frac{\langle \varepsilon \rangle}{U^3} \) as a function of Reynolds number, and fit to \( y = a + bRe^{c} \) using Matlab’s nonlinear least squares tool. The initial guess chosen for the (iterative) solver was \( y = .05 + 5Re^{-1} \).

Figure 4 shows that the long time average of the energy dissipation rate for the model scales like a constant plus the inverse of the Reynolds number, \( \langle \varepsilon \rangle \simeq (0.5 + 4Re^{-1}) \frac{U^3}{T} \), consistent with our analysis.
6. Conclusions and open problems

The work herein was motivated by the idea that models more closely reflecting the kinetic energy balance in turbulence can be simpler and require fewer calibration parameters for accuracy. One important aspect of kinetic energy balance is the averaged energy dissipation rate, $\langle \varepsilon \rangle$, in turbulence models matching averaged energy input rates, $U^3/L$, as they do for the NSE. For (1.1) this matching, related to models not over-dissipating solutions, depends on the choice of the turbulence length scale $l(\cdot)$, the decision to include or exclude the term $-\nu \triangle k$ in the $k$–equation and (in numerics) numerical dissipation in the methods used. For the turbulence length scale, away from walls we used the simple and universal kinematic specification $l = \sqrt{2k^{1/2}/\tau}$. Near walls it is necessary to match the near wall behavior of $\nu_T(\cdot)$ to that of the Reynolds stress $-u'u'$. Including the term $-\nu \triangle k$, matching requires near wall behavior $l = O(d^{3/2})$. With this matching, model energy dissipation rates do match input rates, as desired for accuracy. For implementation, $l = \min \left\{ \sqrt{2k^{1/2}/\tau}, 0.41d\sqrt{d/L} \right\}$ retains the issue of specifying the wall distance but it does not require pre-determining fluid sub-regions.

The 1–equation model studied has been used in many numerical codes, yet open problems abound. The important analytic problems of existence and positivity of $k$, while open for the new length scale, seem within reach given the advances in theory presented in Chacon-Rebollo and Lewandowski [3]. The question of inclusion or
exclusion of $-\nu \triangle k$ is little explored. We conjecture that it is linked to the correct near wall asymptotics of $l(\cdot)$, global dissipation rates, and possible ill-posedness of the continuum model and its numerical discretization. The model parameters used in our tests were $\mu = 0.55$ and von Karman constant 0.41. These values are classical for $l = 0.41d$. The numerical illustration found that with these parameter values $(\varepsilon) \approx (0.5 + 4Re^{-1}) \frac{U^3}{L}$. The $Re \to \infty$ limiting value 0.5 includes numerical dissipation and grid effects. It is larger than the best estimate for the NSE of 0.088 of Doering and Constantin [8]. If this persists in more detailed tests, the chosen model parameters 0.55 and 0.41 should be adjusted for the new turbulence length scale herein.

When achievable, the analysis of energy dissipation rates provides a powerful tool to investigate conditions under which turbulence models do not severely over dissipate solutions. Naturally, the region between models amenable to such analysis and models used in practice remains filled with important, interesting, and challenging open problems.

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7. Appendix: Existence of long time limits

In this appendix, \(C\) will denoted any quantity uniformly bounded in time. We now prove the bounds given in Section 2 on

\[
\|v(T)\|^2, \int_\Omega k(T)dx, \int_\Omega \nu_T(T)dx, \left\langle \frac{1}{L^3} \int_\Omega |\nabla v|^2dx \right\rangle_T,
\]

\[
\left\langle \frac{1}{L^3} \int_\Omega 1/kdx \right\rangle_T \quad \text{and} \quad \left\langle \frac{1}{L^3} \int_\Omega [2\nu + \nu_T(\cdot)]|\nabla v|^2dx \right\rangle_T.
\]

proof: We begin with the energy equalities and inequalities for the two equations:

\[
\frac{1}{2} \frac{d}{dt} \|v\|^2 + \int_\Omega [2\nu + \nu_T(\cdot)]|\nabla v|^2dx \leq (v, \phi) + \int_\Omega [2\nu + \nu_T(\cdot)]\nabla v : \nabla \phi dx + (v \cdot \nabla v, \phi),
\]

and

\[
\int_\Omega k_v dx + \int_\Omega \frac{1}{L^3} k/kdx = \int_\Omega \nu_T(\cdot)|\nabla v|^2dx.
\]

Pick \(\theta, 0 < \theta < 1\), and add the first equation + \(\theta\) × second equation. Using \(\frac{d}{dt}\|\phi\|^2 = 0\) gives

\[
\frac{d}{dt} \left( \frac{1}{2} \|v\|^2 - (v, \phi) + \frac{1}{2} \|\phi\|^2 + \theta \int_\Omega kdx \right) + \int_\Omega [2\nu + (1 - \theta)\nu_T(\cdot)]|\nabla v|^2dx + \theta \frac{1}{L^3} k/kdx \leq \int_\Omega [2\nu + \nu_T(\cdot)]|\nabla v|^2dx + (v \cdot \nabla v, \phi).
\]

We make the same choice of \(\phi\) only with \(\beta = \frac{1}{2}Re^{-1}\) rather than \(\beta = \frac{1}{8}Re^{-1}_{eff}\). Consider now the three terms on the RHS. For the last, nonlinear term, we have proven the estimate

\[
(v \cdot \nabla v, \phi) \leq C + \beta Re \int_{\beta} 2\nu|\nabla v|^2dx \leq C + \frac{1}{8} \int_\Omega 2\nu|\nabla v|^2dx,
\]

and the second term is subsumed in the LHS of (15). The first term on the RHS is bounded by the Cauchy-Schwarz-Young inequality in a standard way as

\[
\int_\Omega 2\nu|\nabla v|^2dx \leq C + \frac{1}{8} \int_\Omega 2\nu|\nabla v|^2dx
\]

with the second term on the RHS again subsumed as above. The remaining term on the RHS involves \(\nu_T\). As a first step we again apply the Cauchy-Schwarz-Young inequality in a standard way and then use the direct calculation of \(|\nabla v|^2\) to give

\[
\int_\Omega \nu_T(\cdot)|\nabla v|^2dx \leq \frac{1 - \theta}{2} \int_\Omega \nu_T(\cdot)|\nabla v|^2dx + \frac{1}{2(1 - \theta)} \int_\Omega \nu_T(\cdot)|\nabla v|^2dx
\]

\[
\leq \frac{1 - \theta}{2} \int_\Omega \nu_T(\cdot)|\nabla v|^2dx + \frac{\mu}{2(1 - \theta)} \left( \frac{U \beta L}{\beta} \right)^2 \int_{\beta} l/kdx.
\]
Collecting these terms in (15) gives
\[
\frac{d}{dt} \left( \frac{1}{2} |v - \phi|^2 + \theta \int_\Omega kdx \right) + \int_\Omega \left[ \frac{3}{2} \nu + \frac{1-\theta}{2} \nu_T(\cdot) \right] |\nabla^s v|^2 + \theta \frac{1}{l} k\sqrt{k}dx \leq C + \frac{\mu}{2(1-\theta)} \left( \frac{U}{\beta L} \right)^2 \int_\Omega l^\frac{3}{2} dx,
\]

For the last term we apply Hölder’s inequality with exponents 3 and 3/2 as follows
\[
\int_\Omega l\sqrt{k}dx = \int_{S_\beta} l^{4/3} \cdot l^{-1/3} \sqrt{k}dx \leq \left( \int_{S_\beta} l^{-1} k^{3/2} dx \right) \frac{1}{3} \left( \int_{S_\beta} \left( l^{4/3} \right)^{3/2} dx \right)^{3/2}
\]
\[
= \frac{1}{2(1-\theta)} \left( \frac{U}{\beta L} \right)^2 \int_{S_\beta} \mu l \sqrt{k}dx \leq \frac{1}{3} \int_\Omega l^\frac{1}{2} k\sqrt{k}dx + \frac{2}{3} \left[ \frac{1}{2(1-\theta)} \left( \frac{U}{\beta L} \right)^2 \right]^{3/2} \int_{S_\beta} l^2 dx.
\]
We thus have
\[
\frac{d}{dt} \left( \frac{1}{2} |v - \phi|^2 + \theta \int_\Omega kdx \right) + \int_\Omega \left[ \frac{3}{2} \nu + \frac{1-\theta}{2} \nu_T(\cdot) \right] |\nabla^s v|^2 + \theta \frac{1}{l} k\sqrt{k}dx \leq C + C^* \int_{S_\beta} l^2 dx,
\]
where
\[
C^* = \frac{2}{3} \left[ \frac{1}{2(1-\theta)} \right]^{3/2} \left( \frac{U}{\beta L} \right)^3.
\]
The result now follows by standard differential inequalities provided there is an \( \alpha > 0 \) with
\[
\int_\Omega l^\frac{1}{2} k\sqrt{k}dx \geq \alpha \int_\Omega kdx \quad \text{and} \quad \int_{S_\beta} l^2 dx \leq C < \infty.
\]
These two depend on the choice of \( l = \min \left\{ \sqrt{2} k^{1/2}, 0.41 d \sqrt{\frac{2}{L}} \right\} \). By selecting the last argument in the minimum, the condition \( \int l^2 dx \leq C < \infty \) holds. By selecting the first term in the minimum (and noting that then \( \frac{1}{l} k\sqrt{k} = \frac{1}{\sqrt{2}} k \)) the condition \( \int \frac{1}{l} k\sqrt{k}dx \geq \alpha \int kdx \) holds. Thus the uniform bounds follows.