On the Distributional Nature of the Energy Momentum Tensor of a Black Hole or What Curves the Schwarzschild Geometry?

Herbert BALASIN
Herbert NACHBAGAUER

Institut für Theoretische Physik, Technische Universität Wien
Wiedner Hauptstraße 8–10, A - 1040 Wien, AUSTRIA

Abstract

Using distributional techniques we calculate the energy–momentum tensor of the Schwarzschild geometry. It turns out to be a well–defined tensor–distribution concentrated on the $r = 0$ region which is usually excluded from space–time. This provides a physical interpretation for the curvature of this geometry.

September 10, 2018

*e-mail: hbalasin @ email.tuwien.ac.at
**e-mail: nachb @ email.tuwien.ac.at
1. Introduction

Recently the interest in distributional solutions of general relativity \[1\], especially the gravitational field of shock-waves \[2, 3, 4\] was increased by the consideration of scattering (test) particles in these geometries. It turns out that the scattering amplitudes are exactly calculable and that they display a particular similarity to the ones obtained in string theory. This analogy is made rigorous by considering the limit of general relativity \[5\] where only a two–dimensional block of the metric is quantised and the remaining part is treated classically.

The aim of the present paper is to investigate the origin of the shock–wave geometry which arises from boosting the Schwarzschild metric. As already pointed out by Aichelburg and Sexl \[2\] the metric becomes singular at boost velocity approaching the speed of light, even in the sense of distributions. However, by performing a clever “singular” coordinate-transformation it is possible to obtain a finite result with a sensible physical interpretation, namely an energy–momentum tensor that is concentrated on a light-like line. This is somewhat puzzling since the original geometry is considered to be a vacuum solution with zero energy–momentum tensor. Thus one may ask for the origin of curvature in the Schwarzschild space-time.

As we shall show the so-called vacuum solution is such that the energy–momentum tensor is concentrated on regions usually excluded from space-time (i.e. the origin in Schwarzschild coordinates), resulting in the physically unsatisfactory situation that curvature is generated by a zero energy–momentum tensor.

In our approach, we consider the appropriately chosen manifold as the underlying object to be given (physical) structures such as a metric tensor. Thus we include these regions of space-time in the manifold now admitting also for distribution valued metrics.

The present work shows that this is in fact possible in the case of the Schwarzschild geometry. To begin with we want to give a brief sketch of the general framework of distributions on arbitrary manifolds, since we believe that this material is important but not quite standard \[6\]. Afterwards we treat some simple but instructive examples. Finally we turn to the Schwarzschild geometry and calculate its energy–momentum tensor.
2. **Mathematical Framework**

Usually a distribution \( f \) is thought of as linear functional acting on an appropriate test function space \( C_0^\infty \) (e.g. \( C^\infty \)-functions with compact support) \[7\]. This definition is motivated by the need to generalise the regular functionals that arise as integrals of locally integrable functions with a test function \( \varphi \)

\[
(f, \varphi) = \int d^n x f(x) \varphi(x), \quad \varphi \in C_0^\infty.
\] (1)

This definition makes implicit use of the test functions defined on \( \mathbb{R}^n \). In order to generalise the concept to an arbitrary manifold \( \mathcal{M} \), we consider instead of test functions the space of \( C^\infty \)-differential \( n \)-forms \( \tilde{\varphi} \) with compact support (\( \tilde{\varphi} \in \Omega_0^n \)) \[8\]. Regular functionals arise now from locally integrable functions \( f \) on \( \mathcal{M} \) in the following way:\[1\]:

\[
(f, \tilde{\varphi}) = \int_{\mathcal{M}} f \tilde{\varphi}.
\] (2)

This suggests to define distributions as linear functionals on the space of \( n \)-forms \( \Omega_0^n \). If \( \mathcal{M} \) is orientable with a volume-form \( \omega \) every \( \tilde{\varphi} \) defines uniquely a function \( \varphi \in C_0^\infty \) such that \( \tilde{\varphi} = \varphi \omega \) and (2) becomes

\[
(f, \tilde{\varphi}) = \int_{\mathcal{M}} f \varphi \omega,
\] (3)

which coincides with (1) by taking \( \omega = d^n x \).

Further generalisation is achieved by using tensor–valued \( n \)-forms with compact support as test spaces and tensor fields to define regular functionals

\[
(t^I, \tilde{\phi}_I) = \int_{\mathcal{M}} t^I \tilde{\phi}_I, \quad I \ldots \text{collection of tensor indices}.
\] (4)

This furnishes a manifestly coordinate–independent definition of tensor–valued distributions (tensor–distributions) on a given manifold. These definitions are completely independent of any metrical structure on \( \mathcal{M} \).

\[1\]The left–hand side of (3) assumes the existence of partitions of unity in order to define the integral.
3. Simple Examples

To provide an illustration of the above concepts let us consider some instructive examples.

The first example is given by the one–form
\[
\omega = \frac{x}{x^2 + y^2} dy - \frac{y}{x^2 + y^2} dx,
\]
which is differentiable and closed on \( \mathbb{R}^2 \setminus \{0\} \). However, \([5]\) may also be understood as a tensor–distribution on \( \mathbb{R}^2 \). This is possible since both component functions are locally integrable. I.e. the integrals
\[
\int \frac{x}{x^2 + y^2} \varphi(x, y) d^2x, \quad \int \frac{y}{x^2 + y^2} \varphi(x, y) d^2x
\]
are well defined for all \( \varphi \in C_0^\infty(\mathbb{R}^2) \). Let us calculate the exterior derivative of \( \omega \) as tensor–distribution. In order to facilitate the calculation we employ the following regularisation
\[
\omega = \lim_{\lambda \to 0} \omega_\lambda = \lim_{\lambda \to 0} r^{\lambda - 2} (x dy - y dx), \quad d\omega_\lambda = \lambda r^{\lambda - 2} dx \wedge dy,
\]
\[
d\omega = \lim_{\lambda \to 0} \left( \lambda r^{\lambda - 2} \right) dx \wedge dy = 2\pi \delta^{(2)}(x, y) dx \wedge dy.
\]
The limit in the last equation was performed using the decomposition of \( r^{\lambda - 2} \)
\[
(r^{\lambda - 2}, \varphi) = 2\pi \int_0^\infty r^{\lambda - 2} S_\varphi(r) dr =
\]
\[
\frac{2\pi}{\lambda} \varphi(0) + 2\pi \int_0^\infty r^{\lambda - 2} (S_\varphi(r) - \theta(1 - r) \varphi(0)) r dr =: \frac{2\pi}{\lambda} (\delta, \varphi) + \left( \left[ r^{\lambda - 2} \right], \varphi \right)
\]
with
\[
S_\varphi(r) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \varphi(r \cos \phi, r \sin \phi)
\]
where the singular part has been isolated. This shows that the classically closed but non–exact one–form \( \omega \) turns out to be a non–closed and therefore clearly non–exact one–form distribution. The above derivation made explicit use of a regularisation the result, however, is completely regularisation independent, as will become clear in the slightly more complicated next example, the \((\mathbb{R}^3, \delta)\)–induced metric on a cone.
Using cylindrical coordinates the explicit embedding formula and metric are

\[ z = \theta(\rho) \rho \cot \vartheta, \quad dz = \theta(\rho) \cot \vartheta d\rho, \quad ds^2 = d\rho^2 (1 + \theta(\rho) \cot^2 \vartheta) + \rho^2 d\phi^2. \]  

(6)

Since the underlying manifold structure is that of \( \mathbb{R}^2 \) the above metric may be viewed as tensor–distribution on \( \mathbb{R}^2 \). Classically the curvature of this metric vanishes, since one cuts out the region that corresponds to the tip of the cone. However, from the point of view of distribution theory this is not necessary at all. In order to calculate the distributional curvature we will start with a regularisation of (6). A geometrically natural way of regularising the above metric is obtained by considering the cone to be the limit of a sequence of hyperbolic shells. These are given by the \((\mathbb{R}^3, \delta)\) embeddings \( z = \sqrt{\rho^2 + a^2 \cot^2 \vartheta} \) with the corresponding metric

\[ ds^2 = \left(1 + \frac{\rho^2}{\rho^2 + a^2} \cot^2 \vartheta\right) d\rho^2 + \rho^2 d\phi^2. \]

Again we have \( \mathbb{R}^2 \) as underlying manifold structure. The calculation proceeds now in a straightforward fashion using the canonical dyad and spin connection

\[ e^\rho = \sqrt{1 + \frac{\rho^2}{\rho^2 + a^2} \cot^2 \vartheta} d\rho, \quad e^\phi = \rho d\phi, \quad \omega^{\rho\phi} = -\left(1 + \frac{\rho^2}{\rho^2 + a^2} \cot^2 \vartheta\right)^{-\frac{1}{2}} d\phi. \]

The corresponding curvature tensor and Ricci–scalar are

\[ R^{\rho\phi} = \frac{a^2 \cos^2 \vartheta \sin^2 \vartheta}{(a^2 \sin^2 \vartheta + \rho^2)^2} e^\rho \wedge e^\phi, \quad R = \frac{2a^2 \cos^2 \vartheta \sin^2 \vartheta}{(a^2 \sin^2 \vartheta + \rho^2)^2}, \]

where the coefficients are understood in the sense of distributions

\[ \lim_{a \to 0} (R, \varphi) = \lim_{a \to 0} 2\pi \int_0^\infty \rho d\rho S_\varphi(\rho) \frac{2a^2 \cos^2 \vartheta \sin^2 \vartheta}{(a^2 \sin^2 \vartheta + \rho^2)^2}. \]

The singular part of the last integral may be split off in a similar manner as in the previous example. This time, however, one has to subtract the order two Taylor polynomial from \( S_\varphi \). Evaluation of the limit gives

\[ R = 2\pi \cos^2 \vartheta \delta^{(2)}(x, y). \]

This result has the expected features: The curvature is concentrated on the origin and vanishes if the cone degenerates to a plane (i.e. if \( \vartheta = \frac{\pi}{2} \)).

The above calculation made explicit use of our conception of the cone as being the limit of a hyperboloid. It is nevertheless possible to regularise (6) in a more abstract
way by replacing $\theta(\rho)$ by an arbitrary regularisation function $f(\rho)$ which has to vanish at $\rho = 0$. The chosen form of (6) guarantees that the prefactor of $d\rho^2$ in the metric equals unity for any regularisation $f(\rho)$ at $\vartheta = \frac{\pi}{2}$. These requirements are met by $f(\rho) = \rho^\lambda$, Re($\lambda$) > 0 yielding the regularised metric
\[ ds^2 = (1 + \rho^\lambda \cot^2 \vartheta) d\rho^2 + \rho^2 d\varphi^2. \]

A short calculation gives
\[ R_\lambda = -\frac{1}{\rho} \frac{\lambda \rho^{\lambda-1} \cot^2 \vartheta}{(1 + \rho^\lambda \cot^2 \vartheta)^2}; \]
\[ \lim_{\lambda \to 0} (R_\lambda, \varphi) = -2\pi \int_0^\infty d\rho \frac{\lambda \rho^{\lambda-1} \cot^2 \vartheta}{(1 + \rho^\lambda \cot^2 \vartheta)^2} S_\varphi(\rho) = 2\pi \cos^2 \vartheta \varphi(0), \]
thus
\[ \lim_{\lambda \to 0} R_\lambda = 2\pi \cos^2 \vartheta \delta^{(2)}(x, y), \]
which coincides with the previous result. Actually this result is completely independent of the chosen regularising function $f(\rho)$ since repetition of the above steps with the metric
\[ ds^2 = (1 + f(\rho) \cot^2 \vartheta) + \rho^2 d\varphi^2 \]
gives the same result in the pointwise limit $f(\rho) \to \theta(\rho)$. Although the above examples made explicit use of a regularisation, i.e. to stick to classical calculus, the results are independent of the regularisation procedure employed.

**4. Schwarzschild–Geometry**

The usual derivation of the Schwarzschild metric takes advantage of the fact that the metric is spherically symmetric and static. This allows to fix the coordinates in a symmetry–appropriately manner and leaves two arbitrary functions in the metric [8]. Imposition of the vacuum Einstein equations determines those functions and one obtains
\[ ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \] (7)

As already pointed out in the introduction, the reason for the curved geometry induced by the Schwarzschild solution [8] is somewhat missing. In particular, since one explicitly used the vacuum equations, the energy–momentum tensor of this geometry vanishes leaving us with the question for the physical ground of the non–Minkowskian geometry.
The situation is quite similar to our first example, where $\omega$ remained closed only when considered on the smaller manifold $\mathbb{R}^2 \setminus \{0\}$ instead of $\mathbb{R}^2$. Physically, this line of arguments may further be illustrated in the case of electrodynamics.

Let us calculate the static spherically symmetric solution of the vacuum Maxwell equations. Stationarity implies that the vector potential $A$ has to be time–independent, and rotational invariance constrains $A$ to the form $A(x) = A_0(r)dt + A_r(r)dr$ where $r$ denotes the radial distance. Using the residual gauge freedom we eliminate the spatial component. Finally, the vacuum Maxwell equations $d\ast dA = 0$ leave us with

$$\Delta A_0(r) = \frac{1}{r^2} \partial_r \left( r^2 \partial_r A_0(r) \right) = 0, \quad A_0(r) = -\frac{e}{r}, \quad (8)$$

where a possible integration–constant in the potential has been dropped by imposing natural boundary conditions $A_0(r \to \infty) = 0$. $A_0(r)$ is precisely the Coulomb potential.

The solution obtained in this way has to be restricted to the domain $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$ in order to be differentiable in the classical sense. Let us now turn the argument around and consider $A(x)$ as a distribution on $\mathbb{R}^4$. Plugging (8) into the Maxwell equations now gives a non–vanishing current density. The calculation boils down to the evaluation of $\Delta \left( 1/r \right) = -4\pi \delta^{(3)}(x)$ producing $j^0(x) = 4\pi e \delta^{(3)}(x)$. Physically this shows that the Coulomb solution may be regarded as the field of a point–charge distribution located at $r = 0$.

Let us now use this line of arguments to calculate the energy–momentum tensor of the Schwarzschild metric. Similar to the Coulomb case (8) the metric (7) is a well–defined tensor–distribution even under inclusion of the line $r = 0$. Furthermore we may restrict the support of the test functions to the ball $r < 2m$ which suggests to use interior Schwarzschild–coordinates. With regard to the experience gained in the last two sections we choose the regularisation

$$ds^2 = h(r)dt^2 - \frac{1}{h(r)} dr^2 + r^2 d\Omega^2, \quad h(r) = -1 + \frac{2m}{r} f(r),$$

which becomes flat space upon turning off the mass $m$. The calculation proceeds now in a straightforward manner, using the canonical tetrad and spin connection

$$e^t = \sqrt{h(r)} dt, \quad e^r = h(r)^{-\frac{1}{2}} dr, \quad e^\theta = r d\theta, \quad e^\phi = r \sin \theta d\phi, \quad (9)$$

$$\omega^t_r = \frac{1}{2} h'(r) dt, \quad \omega^\theta_r = \sqrt{h(r)} d\theta, \quad \omega^\phi_r = \sqrt{h(r)} \sin \theta d\phi, \quad \omega^\phi_\phi = \cos \theta d\phi. \quad (10)$$
From (10) one finds for the Ricci–tensor and the curvature scalar, with respect to the vielbein basis (9)

\[ R_t = \frac{m}{r} f''(r) e^t, \quad R_r = -\frac{m}{r^2} f''(r) e^r, \quad R_\theta = \frac{2m}{r^2} f'(r) e^\theta, \quad R_\phi = \frac{2m}{r^2} f'(r) e^\phi, \]

\[ R = 2m \left( \frac{1}{r} f''(r) + \frac{2}{r^2} f'(r) \right). \]

In order to evaluate these expressions we choose an explicit regularisation function \( f(r) = r^\lambda \). Taking the distributional limit \( \lambda \to 0 \) gives

\[ R(x) = 8\pi m \delta^{(3)}(x), \]

\[ G(x) = 8\pi T(x) = -8\pi m \delta^{(3)}(x) \left( dt \otimes \partial_t + dr \otimes \partial_r - \frac{1}{2} d\theta \otimes \partial_\theta - \frac{1}{2} d\phi \otimes \partial_\phi \right). \]

The second equality employed the coordinate basis for \( T(x) \) which coincides with our choice of the tetrad (9). \( T(x) \) and \( R(x) \) have the desired features: They are concentrated at \( r = 0 \) and vanish in the limit \( m \to 0 \). Physically \( T(x) \) represents the source of the gravitational field that manifests itself in the Schwarzschild metric.

Finally, let us comment on two important issues, regularisation independence and type of the energy–momentum tensor.

Actually the above result could have been obtained by choosing a different regularisation. Examples are provided by \( f(r) = r^2/(r^2 + a^2) \), \( a \to 0 \) or by changing the dimension of the spherical part of the metric. (For a discussion of this approach the reader is referred to the Appendix.) However, all of them give the same answer. This is due to the fact that we might have done the whole calculation keeping \( f \in C^\infty \) at will and considering the limit \( f \to 1 \) in the end. Let us briefly sketch this argument with regard to the curvature scalar

\[ (R, \varphi) = 8\pi m \int_0^\infty \left( \frac{1}{r} f''(r) + \frac{2}{r^2} f'(r) \right) r^2 S_\varphi(r), \]

where the integral can be rewritten by partial integration

\[ f'(r) r S_\varphi(r) + f(r) S_\varphi(r) - f(r) r S'_\varphi(r) \bigg|_0^\infty + \int_0^\infty df(r) \left[ (r S_\varphi(r))'' - 2 f(r) S'_\varphi(r) \right]. \]

One obtains, by evaluating the boundary terms, thereby taking into account the zero of \( f(r) \) at \( r = 0 \) and the compact support of \( S_\varphi(r) \) the following result

\[ 8\pi m \int_0^\infty drr S''_\varphi(r) = 8\pi m \left( r S'_\varphi(r) \bigg|_0^\infty - \int_0^\infty dr S'_\varphi(r) \right) = 8\pi m S_\varphi(0). \]
Secondly, our calculation gave only the mixed components of the energy–momentum tensor. Due to the singular behaviour of the metric the other forms simply do not exist. However, this causes no loss of information since those forms are redundant. Moreover, the mixed form may be considered being the most “democratic” which also allows the calculation of the traces.

5. Conclusion

The present work advocates the use of distributional techniques to calculate (geo)metrical quantities of manifolds equipped with a singular metric. This approach is tested on simple (conceptive) examples, like the curvature of a cone or the integrability of a vector field on $\mathbb{R}^2$. Finally, the proper subject of our work, the energy–momentum tensor of the Schwarzschild space–time is calculated. One finds that it is a tensor–distribution supported in the singular region. From a physical point of view this allows the identification of the source of the Schwarzschild geometry putting the starting point of the shock–wave geometry of Aichelburg and Sexl on the same footing with the result of the boost. The presented techniques may also be applied to the Kerr geometry. Work in this direction is currently under progress.

Acknowledgement: The authors are greatly indebted to Prof. P. C. Aichelburg for many useful discussions.
Appendix

**Dimensional Regularisation of the Schwarzschild Metric**

This regularisation is conceptually different from the ones previously used. It takes advantage of the spherical symmetry of the Schwarzschild space–time which may be interpreted as $S^2$–bundle over the two–dimensional $(r, t)$–space containing the interesting geometry. Replacing $S^2$ by $S^{2\omega}$ the $(r, t)$–geometry remains unchanged but the $(2\omega + 2)$–dimensional quantities such as curvature are regularised.

As usual we begin with the metric and its canonical $(2\omega + 2)$–frame

$$
    ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}}dr^2 + r^2d\Omega^2_{(2\omega)},
$$

$$
    e^t = \left(\frac{2m}{r} - 1\right)^{-\frac{1}{2}}dt, \quad e^r = \left(\frac{2m}{r} - 1\right)^{-\frac{1}{2}}, \quad e^i = r\tilde{e}^i, \quad (A.1)
$$

where $d\Omega^2_{(2\omega)}$ and $\tilde{e}^i$ denote the line–element of the $2\omega$–sphere and its canonical $2\omega$–frame respectively. (In the following we will use a tilde to denote $S^{2\omega}$–quantities like frame, spin–connection and curvature). The spin–connection and curvature of (A.1) are given by

$$
    \omega^t_r = -\frac{m}{r^2}dt, \quad \omega^r_r = \left(\frac{2m}{r} - 1\right)^{\frac{1}{2}}\tilde{e}^i, \quad \omega^i_j = \tilde{\omega}^i_j,
$$

$$
    R^t_r = \frac{2m}{r^3}e^r \wedge e^t, \quad R^t_i = -\frac{m}{r^2}e^t \wedge \tilde{e}^i,
$$

$$
    R^r_i = -\frac{m}{r^2}e^r \wedge \tilde{e}^i, \quad R^i_j = \tilde{R}^i_j + \left(\frac{2m}{r} - 1\right)\tilde{e}^i \wedge \tilde{e}^j.
$$

Taking into account that $\tilde{R}^i_j = \tilde{e}^i \wedge \tilde{e}^j$ which reflects the fact that $S^{2\omega}$ is a space of constant curvature, the Ricci–tensor and the curvature scalar become

$$
    R^t = -\frac{2m}{r^3}(\omega - 1)e^t, \quad R^r = -\frac{2m}{r^3}(\omega - 1)e^r, \quad R^i = \frac{4m}{r^3}(\omega - 1)e^i,
$$

$$
    R_\omega = \frac{4m}{r^3}(\omega - 1)(2\omega - 1).
$$
The distributional limits of these quantities are calculated in the usual way. In order to see this explicitly let us calculate the curvature scalar.

\[(R, \varphi) = \lim_{\omega \to 1} (R_\omega, \varphi),\]

\[
(R_\omega, \varphi) = \int d^{2\omega+1}x R(x)\varphi(x) = 4m(\omega - 1)(2\omega - 1) \frac{2\pi^{\omega+\frac{1}{2}}}{\Gamma(\omega + \frac{1}{2})} \int_0^\infty \frac{r^{2\omega}}{r^3} S_\varphi(r) dr =
\]

\[
= \frac{8m(2\omega - 1)\pi^{\omega+\frac{1}{2}}}{2\Gamma(\omega + \frac{1}{2})} \left( r^{2\omega-2} S_\varphi(r) \bigg|_0^\infty - \int_0^\infty r^{2\omega-2} S_\varphi'(r) dr \right),
\]

where the averaged test function is defined as follows

\[S_\varphi(r) = \frac{1}{|\Omega_{2\omega}|} \int_{S^{2\omega}} d\Omega \varphi(r\Omega), \quad |\Omega_{2\omega}| = \frac{2\pi^{\omega+\frac{1}{2}}}{\Gamma(\omega + \frac{1}{2})}.\]

Therefore the limit \(\omega \to 1\) becomes

\[(R, \varphi) = 8\pi m S_\varphi(0) = 8\pi m \varphi(0)\]

which coincides with our previous result.
References

[1] R. Geroch and J. Traschen, Phys. Rev. D36 (1987) 1017.

[2] P. Aichelburg and R. Sexl, J. Gen. Rel. Grav. 2 (1971) 303.

[3] G. ’t Hooft and T. Dray, Nucl. Phys. B253 (1985) 173.

[4] C. O. Lousto and N. Sánchez, Nucl. Phys. B383 (1992) 377; Int. J. Mod. Phys. AS (1990) 915.

[5] E. Verlinde and H. Verlinde, Nucl. Phys. B371 (1992) 246.

[6] A. Lichnerowicz, Propagateurs, Commutateurs et Anticommutateurs en Relativité Générale, IHES No. 10 (1961).

[7] I. M. Gel’fand and G. E. Shilov, Generalized Functions, Vol. 1, Academic Press, (1964).

[8] R. Wald, General Relativity, University of Chicago Press, 1984.