SENDOV’S CONJECTURE: A NOTE ON A PAPER OF DÉGOT

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Abstract. Sendov’s conjecture states that if all the zeroes of a complex polynomial $P(z)$ of degree at least two lie in the unit disk, then within a unit distance of each zero lies a critical point of $P(z)$. In a paper that appeared in 2014, Dégot proved that, for each $a \in (0,1)$, there exists an integer $N$ such that for any polynomial $P(z)$ with degree greater than $N$, if $P(a) = 0$ and all zeroes lie inside the unit disk, the disk $|z - a| \leq 1$ contains a critical point of $P(z)$. Based on this result, we derive an explicit formula $N(a)$ for each $a \in (0,1)$ and, consequently obtain a uniform bound $N$ for all $a \in [\alpha, \beta]$ where $0 < \alpha < \beta < 1$. This (partially) addresses the questions posed in Dégot’s paper.

1. Introduction

In this paper we are going to prove the following result:

**Theorem 1.1.** Let $a \in (0,1)$ and define $N(a)$ to be

$$N(a) = \frac{32400}{a^7(1-a)^4}.$$ 

For any polynomial $P(z) = (z - a) \prod_{j=1}^{n-1}(z - z_j)$ with $n \geq N(a)$ and $|z_j| \leq 1$ for all $j = 1, \ldots, n-1$, the disk $|z - a| \leq 1$ contains a critical point of $P(z)$.

The Gauss–Lucas theorem tells us that the critical points of a polynomial $P(z) \in \mathbb{C}[z]$ lie in the convex hull of its zeroes. The conjecture of Sendov, which seeks to obtain a more precise location of the critical points, is the following:

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Conjecture 1.2 (Sendov, [3, p. 25]). Let \( P(z) = \prod_{j=1}^{n}(z - z_j) \) be a polynomial of degree \( n \geq 2 \) such that \( |z_j| \leq 1 \) for all \( j = 1, \ldots, n \). Then each of the disks \( |z - z_j| \leq 1, j = 1, \ldots, n \), contains a critical point of \( P(z) \).

Over the years since its inception in 1958, many special cases of the conjecture have been established. For extensive surveys of these, the reader is referred to the books [5] and [7]. To give the “modern” formulation of the conjecture though, we need a corollary of the following special case by Bojanov et al.:

Lemma 1.3 [1]. Let \( P(z) = \prod_{j=1}^{n}(z - z_j) \), with \( |z_j| \leq 1 \) for \( j = 1, \ldots, n \). For each \( j = 1, \ldots, n \), the closed disk \( |z - z_j| \leq (1 + |z_1 \cdots z_n|)^{\frac{1}{n}} \) contains a critical point of \( P(z) \).

An immediate corollary of the above lemma is that Sendov’s conjecture is true for polynomials of the form \( P(z) = z^{n-1} \prod_{j=1}^{n-1}(z - z_j) \). In 1968, Rubinstein in [6] showed that Sendov’s conjecture is also true “at the zero \( z_j \)” of the polynomial \( P(z) \) if \( |z_j| = 1 \). The above two special cases, together with the observation that a rotation of the plane preserves the relative configurations of the zeroes and critical points of the polynomial \( P(z) \), mean it is enough to consider the following reformulation of the conjecture:

Conjecture 1.4 (Sendov). Let

\[
P(z) = (z - a) \prod_{j=1}^{n-1}(z - z_j), \text{ with } a \in (0, 1), \ |z_j| \leq 1, \ j = 1, \ldots, n-1.
\]

Then the disk \( |z - a| \leq 1 \) contains a critical point of \( P(z) \).

With the above reformulation in mind, we can henceforth talk of Sendov’s conjecture being true (or false) at a particular zero \( a \in (0, 1) \) of the polynomial \( P(z) \).

1.1. An overview of Dégot’s strategy. Our paper is based on a 2014 paper [2] by Jerome Dégot. In this paper, Dégot proved that, for each \( a \in (0, 1) \), there exists an integer \( N \) such that for any polynomial \( P(z) \) with degree greater than \( N \), if \( P(a) = 0 \) and all zeroes lie inside the unit disk, the disk \( |z - a| \leq 1 \) contains a critical point of \( P(z) \). For the reader’s convenience, below we give a brief and non-technical overview of the approach in [2], as well as an outline of our paper.

Dégot starts by fixing a polynomial \( P(z) \) with a zero at \( a \in (0, 1) \) and degree \( n \), assumed to contradict Sendov’s conjecture at \( a \). By studying closely the geometry of \( P(z) \), he obtains lower and upper bounds of the form \( |P(c)| \leq 1 + a \) and \( |P(c)| \geq CK^{n-1} \) respectively, where \( c \in (0, a) \), \( C \) and \( K \) are some specifically defined parameters.
He then proceeds to show that if $n$ is greater than some integer bound $N$ (we shall refer to this bound as $N(a)$ henceforth), a contradiction on the size of $|P(c)|$ ensues, and hence the disk $|z - a| \leq 1$ must have a zero of $P'(z)$. Worthy of note is that aside from an existence proof, there was no explicit formula given for calculating $N(a)$ for any given $a \in (0,1)$. In fact, upon closer inspection, one notices that the method used to obtain it depended on additional parameters associated with the polynomial $P(z)$. More precisely, a crucial technical inequality that $N(a)$ has to satisfy depended on the quantity $m$, defined as the real part of the mean of the zeroes of $P(z)$. Dégot does indicate afterwards that this dependence can be removed by using a certain estimate on the size of $m$. This leaves much work to be done to actually give an explicit bound, which we do here. Finally, Dégot ends his paper by outlining a series of steps through which one can calculate the requisite degree bounds for some values of $a \in (0,1)$. This algorithmic procedure however does not indicate any obvious way of constructing an explicit formula.

Carefully following the treatment in Dégot’s paper, we extract information from and modify Theorems 5, 6 and 7 from [2]. Each of these theorems introduced conditions which, for a given $a \in (0,1)$, an integer bound $N_1$ ($N_2$ and $N_3$ respectively) has to satisfy in order to draw the requisite conclusions on the size of $|P(c)|$. By studying closely these conditions, we systematically remove the extra dependencies on other parameters, and obtain explicit and continuous analogues of the bounds $N_1$, $N_2$ and $N_3$. We shall refer to these new formulas as $N_1(a)$, $N_2(a)$ and $N_3(a)$ respectively.

This allows us to obtain the conclusions of each of Dégot’s main theorems and hence, ultimately his main result, but now with explicit constants which depend continuously on $a$. So we can then obtain, as a by-product of the continuity of our functions, a uniform bound $N$ independent of $a \in [\alpha, \beta]$ for any $0 < \alpha < \beta < 1$. At the end of his paper, Dégot asked if it is possible to obtain a uniform bound $N$ which works for all $a \in (0,1)$, or at least an explicit formula $N(a)$ which produces a large enough degree for any given $a$.

In the interest of ultimately obtaining a “simple” explicit formula $N(a)$, we will often in intermediate steps of our arguments replace complicated formulae with simpler estimates. This of course comes at the expense of sharpness.

The results in this paper came from investigations carried out in the author’s Master Thesis.

2. On the paper of Dégot

We begin this section with a result that allows us to bound the arithmetic mean of the roots of a polynomial assumed to contradict Sendov’s conjecture at $a \in (0,1)$.
Before proceeding to our result, we would like to point out to the reader that in Dégot’s paper, the (real part of the) mean of the zeroes of the polynomial $P$ was denoted by $m$ whilst the real part of the critical points of the same polynomial was denoted by $s$. As it turns out, these two quantities always coincide, hence we shall only and always use the notation $m$ regardless of whether it was originally $m$ or $s$ Dégot was referring to. For this fact the reader can consult [4, Example 4, p. 53]).

2.1. The mean of a polynomial assumed to contradict Sendov’s conjecture. By the real part of the mean of the roots of a polynomial $P(z)$ of degree $n$, we are referring to the quantity $m = \frac{1}{n} \Re(\sum_{j=1}^{n} z_j)$.

**Lemma 2.1** [2, Corollary 1]. Let $P(z)$ be a polynomial of degree $n$ assumed to contradict Sendov’s conjecture at $a \in (0, 1)$. Then

$$m \leq \inf_{\delta \in (0, a)} \left( \frac{\delta}{2} - \frac{1}{\delta n} \log(1 - \sqrt{1 + \delta^2 - \delta a}) \right).$$

From the above lemma, we come up with a formula $N_0(a)$ such that whenever $n \geq N_0(a)$, then $m \leq \frac{a}{4}$. Upon gaining this control over the size of $m$, we can then remove the dependence on $m$ from other future parameters. This shall become clear when we call upon this new quantity later. In the meantime, let us extract the formula.

**Lemma 2.2.** Let $N_0(a) = \frac{32 \log(\frac{40}{a^2})}{a^2}$. If $n \geq N_0(a)$, then $m \leq \frac{a}{4}$.

**Proof.** Let $n \geq N_0(a)$. By noting that $\frac{a^2}{10} < 4 - \sqrt{16 - 3a^2}$ for $a \in (0, 1)$, we have that $n \geq \frac{32 \log(\frac{4}{1 - \sqrt{16 - 3a^2}})}{a^2 \delta}$. That is,

$$n \geq \frac{-4 \log(1 - \sqrt{1 + \delta^2 - \delta a})}{a \delta - 2\delta^2} \bigg|_{\delta = \frac{a}{4}}.$$

For this fixed value of $\delta$, the previous inequality is equivalent to

$$\frac{\delta}{2} - \frac{1}{\delta n} \log \left(1 - \sqrt{1 + \delta^2 - \delta a}\right) \leq \frac{a}{4}.$$

By Lemma 2.1, we can conclude that for any polynomial $P$ assumed to contradict Sendov’s conjecture at $a \in (0, 1)$, if $\deg(P) = n \geq N_0(a)$, then $m \leq \frac{a}{4}$. \hfill \Box

2.2. Towards explicit analogues of Dégot’s bounds. In this section we study Theorems 5 and 6 from Dégot’s paper. From Theorem 5, we study closely the quantities that go into the definition of $N_1$. We will
see that the bound \(N_1\) as originally defined depends on the real part of the mean of the zeroes of the polynomial \(P(z)\). This section deals with how to circumvent this dependence.

The end result is that we come up with the formulas \(N_1(a)\) and \(N_2(a)\), the explicit and continuous analogues of Dégot’s \(N_1\) and \(N_2\) respectively, which depend only on \(a\). We begin with [2, Theorem 5].

Towards \(N_1(a)\).

**Lemma 2.3** [2, Theorem 5]. Suppose \(P(z)\) contradicts Sendov’s conjecture at \(a\). Let \(q = \frac{\frac{a}{2} - m}{1 + \frac{a}{2}}\) and let \(N_1\) be the smallest integer such that

\[
\left(\frac{1 + \frac{a}{2}}{1 + a}\right)^q \leq \left(\frac{1 - \sqrt{1 - a^2/4}}{an}\right)^{\frac{1}{n-1}} \quad \text{for all } n \geq N_1.
\]

If \(n \geq N_1\), then \(|P'(a)| \leq \frac{16a}{a^2}\) and \(|P(0)| \geq \frac{a^2}{16}\).

With the help of the quantity \(N_0(a)\) obtained in the previous steps, we do this in the following steps:

- First, we note that, by construction, \(N_0(a)\) gives us a high enough degree bound such that any polynomial with degree \(n \geq N_0(a)\) has \(m \leq 0.25a\).
- We have \(\frac{1 + \frac{a}{2}}{1 + a} \in (0, 1)\), hence for any \(0 < q_1 < q_2\)

\[
\left(\frac{1 + \frac{a}{2}}{1 + a}\right)^{q_1} \leq \left(\frac{1 + \frac{a}{2}}{1 + a}\right)^{q_2}.
\]

- Therefore this implies \(\left(\frac{1 + \frac{a}{2}}{1 + a}\right)^q \leq \left(\frac{1 + \frac{a}{2}}{1 + a}\right)^{\frac{\frac{a}{4}}{1 + \frac{a}{2}}}\), whenever \(m \leq \frac{a}{4}\). Thus, by inequality (1), if we have

\[
\left(\frac{1 + \frac{a}{2}}{1 + a}\right)^{\frac{\frac{a}{4}}{1 + \frac{a}{2}}} \leq \left(\frac{1 - \sqrt{1 - a^2/4}}{an}\right)^{\frac{1}{n-1}},
\]

then it follows that

\[
\left(\frac{1 + \frac{a}{2}}{1 + a}\right)^q \leq \left(\frac{1 + \frac{a}{2}}{1 + a}\right)^{\frac{\frac{a}{4}}{1 + \frac{a}{2}}} \leq \left(\frac{1 - \sqrt{1 - a^2/4}}{an}\right)^{\frac{1}{n-1}},
\]

whenever \(m \leq \frac{a}{4}\).

Hence, the version of \(N_1(a)\) obtained by replacing \(m\) with \(\frac{a}{4}\) (and hence \(q = \frac{a}{4 + 2a}\)) works for all \(m \leq \frac{a}{4}\). With this in mind, we replace the quantity \(q(a, m)\) with the new quantity \(q^*(a) := \frac{a}{4 + 2a}\) which only depends on \(a \in (0, 1)\).
Proposition 2.4. Let \( N_1(a) = \max \{9\left(\frac{40+20a}{a^3}\right)^2, N_0(a)\} \), then for all \( n \geq N_1(a) \) we have

\[
\left(1 + \frac{a}{1+a}\right) q^* \leq \left(1 - \frac{\sqrt{1-a^2/4}}{an}\right)^{\frac{1}{n-1}}.
\]

Proof. By assumption, if \( n \geq N_1(a) \), then, in particular, \( n \geq 9\left(\frac{40+20a}{a^3}\right)^2 = [3n_1]^2 \), with \( n_1 := \frac{40+20a}{a^3} \). Since \([3n_1]^2 \geq [n_1 + (1 + n_1 + n_1^2)^{1/2}]^2\), we have that \( n \geq [n_1 + (1 + n_1 + n_1^2)^{1/2}]^2 \). By “reversing the completion of the square”, we have that \( n \geq 1 + n_1 + 2\sqrt{n_1} \). Let

\[
m_1(a) = \frac{\log(1 - \sqrt{1 - a^2/4}) - \log a}{q^* \log(\frac{1+a}{1+a})} \quad \text{and} \quad m_2(a) = \frac{-1}{q^* \log(\frac{1+a}{1+a})}.
\]

We claim that both \( m_1(a) \) and \( m_2(a) \) are less than \( n_1(a) \), noting that \( n_1(a) \) can be expressed in terms of \( q^* \) as \( \frac{10}{a^2 q^*} \). We verify this claim for \( m_1(a) \), the corresponding verification for \( m_2(a) \) is “easier” and follows the same strategy. Note that the claim is equivalent to showing that the function \( s(a) := \frac{m_1(a)}{n_1(a)} \) is less than one for \( a \in (0, 1) \).

We note that \( s'(a) > 0 \) if

\[
\frac{a}{(a+1)(a+2)} \log \left(\frac{a}{1 + \sqrt{1-a^2/4}}\right) + 2 \log \left(\frac{a}{1 + \sqrt{1-a^2/4}}\right) \log \left(\frac{1+a}{1+a/2}\right) - \frac{\log(\frac{1+a}{1+a/2})}{\sqrt{1-a^2/4}} > 0.
\]

Indeed, the above inequality is positive since (in particular), for \( a \in (0, 1) \),

\[
2 \log \left(\frac{a}{1 + \sqrt{1-a^2/4}}\right) > \frac{1}{\sqrt{1-a^2/4}},
\]

and the rest of the other terms are positive. This implies that \( s(a) \) is an increasing function on \((0, 1)\). It can also be verified that \( s(1) \approx 0.699 < 1 \). Hence, \( m_1(a) < n_1(a) \) as required.

We thus have that \( n \geq 1 + m_1(a) + m_2(a) \log n \). That is,

\[
n \geq 1 + \frac{\log(1 - \sqrt{1-a^2/4}) - \log a}{q^* \log(\frac{1+a}{1+a})} - \frac{\log n}{q^* \log(\frac{1+a}{1+a})},
\]

equivalently,

\[
q^* \log \left(\frac{1+a}{1+a}\right) \leq \frac{1}{n-1} \log \left(\frac{1 - \sqrt{1-a^2/4}}{an}\right).
\]
Taking the exponential on both sides of the above inequality completes the proof. □

Towards \( \mathcal{N}_2(a, c) \). Having obtained the explicit formula \( \mathcal{N}_1(a) \), we now turn our attention to [2, Theorem 6], wherein conditions to be satisfied by the second bound \( \mathcal{N}_2 \) were stipulated. The statement given below stipulates such a condition. We state our version, the only difference from his being that we replaced the appearance of \( q \) with \( q^* \).

Let \( c \in (0, a) \). For \( x \in (0, 1) \) set

\[
D(x) := \max \left\{ \left( \frac{1}{1+a} \right)^x : \left( \frac{1+c}{1+a} \right)^x (\sqrt{1+c^2-ac})^{1-x} \right\}.
\]

It is easy to see that for all \( x \in (0, 1) \), \( D(x) < 1 \), \( D(x) \geq \frac{c}{1+a} \) and \( D \) is a decreasing function of \( x \).

Proceeding, define \( N_2^* \) to be the smallest integer such that

\[
D(q^*)^{n-1} \leq \frac{a}{16n} \quad \text{for all } n \geq N_2^*.
\]

**Remark 2.5.** The role of the quantity \( N_2^* \) (and \( N_1^* \)) will become apparent when bounding the quantity \(|P(c)|\) as mentioned in the introduction. We shall consider this in the next section. In the meantime, we bring the reader’s attention to the following:

**Proposition 2.6.** Let \( \mathcal{N}_2(a, c) = \max \{ 9\left( \frac{\log \frac{a}{\log (1+a)}}{\log (1+a)} \right)^2, N_0(a) \} \), then for all \( n \geq \mathcal{N}_2(a, c) \), we have that \( D(q^*)^{n-1} \leq \frac{a}{16n} \).

**Proof.** By assumption, we have that \( n \geq 9\left( \frac{\log \frac{a}{\log (1+a)}}{\log (1+a)} \right)^2 \). Since \( D(x) \geq \frac{c}{1+a} \), then \( n \geq [3n_1]^2 \) where \( n_1 = \frac{\log \sqrt{\frac{a}{\log (1+a)}}}{\log D} \). For \( n_1 \geq \frac{2}{3} \), we have that

\[
[3n_1]^2 \geq [n_1 + (2n_1 + n_1^2)^{\frac{1}{2}}]^2.
\]

For simplicity, we shall verify that indeed \( n_1 \geq \frac{2}{3} \) for \( D(x) = \left( \frac{1}{1+a} \right)^x \). The same technique can be used to verify the inequality for the second component in the definition of \( D(x) \), by using the fact that it suffices to take \( c = c(a) = a(0.1a + 0.9) \), which we show in the next section. Recalling that \( q^*(a) = \frac{a}{4+2a} \), we note that

\[
n_1(a) = \frac{\log \frac{16}{a}}{\log (1+a)} \geq \frac{\log 16}{\frac{1}{4} \log 2} = 16.
\]

Hence \( [3n_1]^2 \geq [n_1 + (2n_1 + n_1^2)^{\frac{1}{2}}]^2 \). That is, \( n \geq 2n_1 + 2\sqrt{nn_1} \). Let \( n_2 = \frac{1}{\log D} \) and note that \( n_1 > n_2 \). We thus have that

\[
n \geq 1 + n_1 + 2\sqrt{nn_2} \geq 1 + n_1 + n_2 \log n.
\]
That is, \( n \geq 1 + \frac{\log \frac{a}{10}}{\log D} - \frac{\log n}{\log D} \). Equivalently, \( \log n + (n - 1) \log D \leq \log \frac{a}{16} \).

Taking the exponential on both sides completes the proof. \( \Box \)

3. Bounds on the size of \(|P(c)|\)

3.1. The upper bound of \(|P(c)|\).

In this section, we now put to use the bounds \( N_1(a) \) and \( N_2(a,c) \) to obtain bounds on the size of \(|P(c)|\).

**Theorem 3.1.** Suppose \( P(z) \) contradicts Sendov’s conjecture at \( a \in (0,1) \) and let \( c \in (0,a) \). If \( \deg(P) = n \geq \max\{N_1(a), N_2(a,c)\} \), then \(|P(c)| \leq 1 + a\).

The proof of the above theorem is essentially a modification of Dégot’s proof of [2, Theorem 6], the only changes being the replacement of the quantities \( q \) with \( q^* \), and \( N_1, N_2 \) with \( N_1(a) \) and \( N_2(a,c) \) respectively.

3.2. Towards the lower bound of \(|P(c)|\).

In this section we look at [2, Theorem 7]. The goal is to obtain constants \( C > 0 \) and \( K > 1 \) such that, for large enough degree \( n \), the value of \( P(c) \) satisfies \(|P(c)| \geq C \cdot K^{n-1}\). To this end, Dégot defined the following new parameters:

\[
p := \frac{\frac{a}{2} - m}{1 - \frac{a}{2}}, \quad r := \frac{c(a - c)}{2(1 - c^2)}, \quad \alpha = \frac{\log \frac{a}{10}}{\log(\frac{c + r}{1 + cr})}
\]

and

\[
(2) \quad K = \min\left\{ (1 + c - ac)^p \sqrt{1 + c^2 - ac}^{1-p}; (1 + c)^q \sqrt{1 + c^2 - ac}^{1-q} \right\},
\]

where \( q := \frac{\frac{a}{2} - m}{1 + \frac{a}{2}} \). We give the statement of the theorem below, bearing in mind the definition of \( N_1 \) from Lemma 2.3.

**Lemma 3.2 [2, Theorem 7].** For the previously defined parameters, if the degree \( n \) of \( P(z) \) is such that \( n \geq N_1 \), then we have

\[
|P(c)| \geq \frac{(1 - c)(a - c)}{1 - ac}r^\alpha K^{n-1}.
\]

Before proceeding, we would like to bring two observations to the reader’s attention. For \( K \) as defined in (2), one can always find \( c \) sufficiently close to \( a \) such that \( K > 1 \). That is, as \( c \to a \),

\[
(1 + c - ac)^p \sqrt{1 + c^2 - ac}^{1-p} \to (1 + a(1 - a))^p > 1
\]
and, similarly,

$$(1 + c)^q \sqrt{1 + c^2 - ac}^{1-q} \to (1 + a)^q > 1.$$  

This observation was enough for Dégot’s results. However, recall that we ultimately want an explicit degree bound that depends only on $a$. Thus we would like to obtain an explicit formula $c = c(a)$ which will always yield a $c$ (in terms of $a$) close enough to $a$ such that $K > 1$. We will in fact also explicitly bound $K$ from below in terms of $a$. We introduce the quantity $p^*(a) = \frac{\frac{a^4}{2}}{1 - \frac{a^2}{2}} = p^*$ to take the place of $p$ in order to avoid the dependence on $m$. First, for ease of notation, from (2) let $K_1(a, c, p) = (1 + c - ac)^p \sqrt{1 + c^2 - ac}^{1-p}$ and $K_2(a, c, q) = (1 + c)^q \sqrt{1 + c^2 - ac}^{1-q}$. Our version of $K$ then becomes

$$(3) \quad K^* := \min \{ K_1(a, c, p^*); K_2(a, c, q^*) \}.$$

One can verify that if $p_1 \geq p_2 > 0$ and $q_1 \geq q_2 > 0$, then

$$K_1(a, c, p_1) \geq K_1(a, c, p_2) \text{ and } K_2(a, c, q_1) \geq K_2(a, c, q_2).$$

Keeping in mind the above discussion, we see that the conclusion of Lemma 3.2 still holds with $K^*$ in place of $K$ whenever $n \geq N_1(a)$. This will become more clear in the discussion leading towards our Theorem 3.6, which, mutatis mutandis, is a restatement of Lemma 3.2.

We may now proceed and study how one can obtain an explicit formula $c(a)$ and the lower bound for $K^*$. In preparation for the result, we need to first recall the following useful logarithmic inequalities:

**Lemma 3.3.**

- $\log(1 + x) \geq \frac{x}{2}$ for $x \in [0, 1]$,
- $\frac{x}{x+1} \leq \log(1 + x) \leq x$ for $x > -1$.

We proceed to define the quantity $\mu_2(a)$ as follows:

$$\mu_2(a) = \left[ \left( \frac{1}{2a} \left( \frac{2}{q^*} - 2 \right) - \frac{1}{2} \right)^2 - \left( \frac{1}{a^2} - \frac{1}{a} \left( \frac{2}{q^*} - 2 \right) \right) \right]^\frac{1}{2} + \left[ \frac{1}{2} - \frac{1}{2a} \left( \frac{2}{q^*} - 2 \right) \right],$$

and note that this expresses the only positive root of the quadratic equation

$$(4) \quad x^2 + \left[ \left( \frac{1}{a} \left( \frac{2}{q^*} - 2 \right) - 1 \right) \right] x + \left( \frac{1}{a^2} - \frac{1}{a} \left( \frac{2}{q^*} - 2 \right) \right) = 0.$$  

For $a \in (0, 1)$, the formula $\mu_2(a)$ simplifies to

$$\mu_2(a) = \left( \frac{a^4 + 4a^3 + 16a^2 + 32a + 64}{4a^4} \right)^\frac{1}{2} + \frac{a^2 - 2a - 8}{2a^2}.$$  

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CLAIM 1. $0 < \mu_2(a) < 1$.

PROOF. Recalling that $q^*(a) = \frac{a}{1 + \frac{a}{2}} = \frac{a}{4 + 2a}$, equation (4) can be written as

$$f(x) = x^2 + \beta(a)x + \rho(a) = 0,$$

where

$$\beta(a) = \frac{8 + 2a - a^2}{a^2} \quad \text{and} \quad \rho(a) = \frac{-7 - 2a^2}{a^2} < 0.$$ 

One notes that $f(0) = \rho(a) < 0$ and $f(1) = \frac{1}{a^2} > 0$. Since the constant term of $f(x)$ is negative whilst the leading coefficient is positive, this implies that the other root is negative. The claim follows. \[\square\]

PROPOSITION 3.4. Let $\gamma(a) = 0.1a + 0.9$. Then $K_2(a, a\gamma(a), q^*) > 1$ for all $a \in (0, 1)$.

PROOF. For $a \in (0, 1)$ let $\gamma$ be arbitrary such that

$$1 > \gamma > \left[\left(1 \cdot 2a \cdot \left(2 \cdot q^* - 2\right) - \frac{1}{2}\right)^2 - \left(\frac{1}{a^2} - \frac{1}{a \left(2 \cdot q^* - 2\right)}\right)\right]^\frac{1}{2} + \left[\frac{1}{2} - \frac{1}{2a} \left(2 \cdot q^* - 2\right)\right] > 0.$$ 

Focusing on the middle inequality, bearing in mind that $\mu_2(a)$ is a root of equation (4), reversing the “completion of the square” with respect to $\gamma$ we get

$$\left[\gamma + \left(\frac{1}{2a} \left(2 \cdot q^* - 2\right) - \frac{1}{2}\right)\right]^2 > \left(\frac{1}{2a} \left(2 \cdot q^* - 2\right) - \frac{1}{2}\right)^2 - \left(\frac{1}{a^2} - \frac{1}{a \left(2 \cdot q^* - 2\right)}\right).$$

Continuing to simplify, we eventually arrive at

$$q^* \left(1 + \gamma^2a^2 - \gamma a^2\right) + 2(1 - q^*)(a\gamma - a) > 0,$$

and, ultimately,

$$q^* + (1 - q^*) \left(\frac{a\gamma - a}{1 + a^2\gamma^2 - a^2\gamma}\right) > \frac{q^*}{2},$$

whence

$$q^* \frac{a\gamma}{2} + \frac{(1 - q^*)}{2} \left(\frac{\gamma^2a^2 - \gamma a^2}{1 + \gamma^2a^2 - \gamma a^2}\right) > \frac{aq^*\gamma}{4}.$$ 

By Lemma 3.3, we have that

$$\log(1 + a\gamma) \geq \frac{a\gamma}{2} \quad \text{and} \quad \log(1 + \gamma^2a^2 - \gamma a^2) \geq \frac{\gamma^2a^2 - \gamma a^2}{1 + \gamma^2a^2 - \gamma a^2}.$$
Using the above inequalities and equation (5), we deduce that
\[
\log(K_2(a, a\gamma, q^*)) = q^* \log(1 + a\gamma) + \left(\frac{1 - q^*}{2}\right) \log(1 + \gamma^2 a^2 - \gamma a^2) > \frac{a q^* \gamma}{4}.
\]
Hence \( K_2(a, a\gamma, q^*) = (1 + a\gamma)^{q^*} (1 + \gamma^2 a^2 - \gamma a^2)^{1/2} > e^{\frac{a q^* \gamma}{4}} > 1. \)

Finally, one notes that the function \( \gamma \) just has to satisfy \( 1 > \gamma(a) > \mu_2(a) \) for all \( a \in (0, 1) \). It is not difficult to show that
\[
\gamma(a) > \mu_2(a) \iff a^4 + 8a^3 + 11a^2 + 60a + 20 > 0 \quad \text{for all} \quad a \in (0, 1).
\]
The second inequality in the above equivalence is evident. □

One notes from equation (6) that \( \gamma \) was constructed such that
\[
\log K_2(a, a\gamma, q^*) > \frac{a q^* \gamma}{4}.
\]

Following the same technique as used to prove Proposition 3.4 above, one can also prove the following:

**Proposition 3.5.** Let \( \gamma(a) = 0.1a + 0.9. \) Then \( K_1(a, a\gamma(a), p^*) > 1 \) for all \( a \in (0, 1) \).

**Proof.** A considerable part of the proof strategy is very much like that used in the proof of the previous proposition. We therefore omit some of the technical details and only highlight the relevant parts of the argument.

Let
\[
\mu_1(a) := \frac{(-a^3 + a^2 + 6a - 8) + (a^6 - 2a^5 + 9a^4 - 20a^3 + 48a^2 - 96a + 64)^{\frac{1}{2}}}{2a^2(1 - a)}.
\]

This expresses the (only) positive root of the quadratic equation
\[
f(r) = r^2 + \left(\frac{a^3 - a^2 - 6a + 8}{a^2 - a^3}\right)r + \frac{5a - 7}{a^2 - a^3}.
\]

For \( a \in (0, 1) \) it can be easily shown that \( 0 < \mu_1(a) < 1. \)

Let \( \rho = \rho(a) \) be such that \( 0 < \mu_1(a) < \rho < 1. \) Then we have
\[
\rho^2 + \left(\frac{a^3 - a^2 - 6a + 8}{a^2 - a^3}\right)\rho + \frac{5a - 7}{a^2 - a^3} > 0.
\]

The above inequality implies (after some algebraic manipulations) that
\[
\frac{(4 - 3a)(\rho - 1)}{1 + a^2 \rho - a^2 \rho} > \frac{a - 1}{2}.
\]
This in turn implies that

$$\frac{a(a\rho - a^2\rho)}{2} + \frac{(4 - 3a)}{2} \left(\frac{a^2\rho^2 - a^2\rho}{1 + a^2\rho^2 - a^2\rho}\right) > \frac{a^2(1 - a)\rho}{4}.$$  

Using Lemma 3.3, we can deduce from the above inequality that

$$\frac{a}{4 - 2a} \log(1 + a\rho - a^2\rho) + \frac{1}{2} \left(\frac{4 - 3a}{4 - 2a}\right) \log(1 + a^2\rho^2 - a^2\rho) > \frac{a^2(1 - a)\rho}{4(4 - 2a)}.$$  

That is,

$$\log(K_1(a, a\rho(a), p^*)) > \frac{a(1 - a)p^*\rho(a)}{4} > 0.$$  

Now, we can move on to show that $\gamma(a)$ as defined in the previous proposition suffices for the role of $\rho(a)$ as described above. That is, $\gamma(a) > \mu_1(a)$ for all $a \in (0, 1)$. It can be shown that

$$\gamma(a) > \mu_1(a) \iff a^6 + 6a^5 - 24a^4 + 86a^3 - 109a^2 + 20a + 20 > 0$$  

for all $a \in (0, 1)$. The reader can then verify that the second inequality in the above equivalence is indeed true. □

We highlight the following observation to the reader: It is desirable to take stock of the preceding discussion at this moment. The reader is reminded that the lower bound of $n$ required to obtain the conclusion of [2, Theorem 7] is the previously defined $N_1$ from [2, Theorem 5] (in our case Lemma 2.3). We have already obtained the explicit analogue of this bound in the form of $N_1(a)$. Hence, as it stands, we have all the necessary ingredients to obtain the conclusion of [2, Theorem 7].

However, since our ultimate goal is to obtain an explicit $N(a)$ independent of all the other implicit parameters, it is worthwhile to remark on the new parameters that were introduced in preparation for Lemma 3.2.

- $p$ is defined as $p = \frac{a - m}{1 - \frac{a}{2}} = p(a, m)$. The dependence on $m$ is avoided by the same argument that led to the introduction of $q^*(a)$. We simply define the alternative quantity $p^* = \frac{a}{1 - \frac{a}{2}} = p^*(a)$ and invoke the quantity $N_0(a)$ to ensure a high enough degree bound such that the results work.

- The parameter $r$ is defined in terms of $a \in (0, 1)$ and $c \in (0, a)$ as $r = \frac{c(a - c)}{2(1 - c^2)}$. The role of the function $\gamma$ as defined in Proposition 3.4 comes into play here. We will define $c$ to be $a\gamma(a)$, thus obtaining $r = r(a)$, a function depending only on $a \in (0, 1)$.

- A similar reasoning as above applies to the quantity $\alpha = \log\frac{a}{\log\frac{a}{1 + cr}}$.
That being said, we arrive at our version of [2, Theorem 7], which depends only on $a \in (0, 1)$. We restate the conclusion here for the sake of continuity. We would like to point out to the reader that in Dégot’s version the condition was that $n \geq N_1$, hence the corresponding condition for us is that $n \geq N_1(a)$. Recall that $K^*$ was defined in (3).

**Theorem 3.6.** Suppose $P(z)$ contradicts Sendov’s conjecture at $a \in (0, 1)$. Let $c = a\gamma(a)$. If $\deg(P(z)) = n \geq N_1(a)$, then

$$|P(c)| \geq \frac{(1 - c)(a - c)}{1 - ac} r^\alpha K^{*n-1}.$$ 

Before proceeding, let us take yet another closer look at these parameters. This analysis will prove useful and simplify notation in the result that follows thereafter.

- The quantity $r$ is defined as $r = \frac{c(a-c)}{2(1-c^2)}$. It can be shown that $0 < r < 1$, hence $\log(r) < 0$. Also, since $1 - c^2 < 1$, in later analysis we can replace $r$ with $r^* := \frac{c(a-c)}{2}$. Theorem 3.6 will still be true for $r^*$ since $0 < r^* < r < 1$, and, as we will show below, $\alpha > 0$.
  - $\log(\frac{1+a}{a-c}) > 0$ and is always defined.
  - $\log(\frac{1-ac}{1-c}) > 0$ and is always defined.
  - Also, $0 < \frac{c+r}{1+cr} < 1$, hence $\alpha := \alpha(r) = \frac{\log \frac{c+r}{1+cr}}{\log r} > 0$. Furthermore, $\frac{c+r}{1+cr} > \frac{c+r^*}{1+cr^*}$, and therefore $\alpha^* := \alpha(r^*) > \alpha(r)$.
  - Finally, we have shown that we can express $c$ explicitly in terms of $a$. Furthermore, this $c$ is sufficiently close to $a$ such that $K^* > 1$. Hence $\log K^* > 0$.

We define the function $N_3(a, c)$ by

$$N_3(a, c) = \frac{\log(\frac{1+a}{a-c}) + \log(\frac{1-ac}{1-c}) - \alpha \log r}{\log K^*} + 1.$$ 

All the above analysis culminates in the following definition of the final degree bound, which we denote by $N_3(a)$ as follows:

$$N_3(a) = \max\{N_0(a), N_3(a, a\gamma(a))\}.$$ 

**4. Proof of the main result, Theorem 1.1**

We begin this section with our analogue of Dégot’s main result.
Theorem 4.1. Let \( P(z) = (z - a) \prod_{j=1}^{n-1} (z - z_j) \), with \( a \in (0, 1) \), \( |z_j| \leq 1 \) for all \( j = 1, \ldots, n - 1 \), where \( n \geq 2 \). If

\[
\deg P(z) = n > \max \{ N_1(a), N_2(a, a\gamma(a)), N_3(a) \},
\]

then \( P'(z) \) has a zero in the disk \( |z - a| \leq 1 \).

Proof. We follow Dégot’s approach from [2]. Let \( c = a\gamma(a) \) and suppose to the contrary, that \( P'(w) \neq 0 \) for all \( w \in |z - a| \leq 1 \). From Theorems 3.1 and 3.6, we get

\[
1 + a \geq |P(c)| \geq \frac{(1 - c)(a - c)}{1 - ac} \frac{r^\alpha}{K^{n-1}}.
\]

This implies that

\[
\frac{(1 - c)(a - c)}{(1 - ac)(1 + a)} r^\alpha K^{n-1} \leq 1.
\]

Therefore,

\[
(n - 1) \log K^* \leq \log \left( \frac{1 - ac}{1 - c} \right) + \log \left( \frac{1 + a}{a - c} \right) - \alpha \log r,
\]

hence,

\[
n \leq \log \left( \frac{1 + a}{a - c} \right) + \log \left( \frac{1 - ac}{1 - c} \right) - \alpha \log(r) \log K^* + 1 \leq N_3(a).
\]

This contradicts the assumption on the degree of \( P(z) \). Hence \( P'(w) = 0 \) for some \( w \in |z - a| \leq 1 \). \( \square \)

4.1. The explicit function \( \mathcal{N}(a) \). Thus far, we have all the ingredients that go into constructing the function \( \mathcal{N}(a) \) given in Theorem 1.1. For the convenience of the reader, we recall that

- \( \mathcal{N}_0(a) := 32\log \frac{\frac{60}{a^2}}{a^2} \) and hence \( \mathcal{N}_0(a) \leq 32\left( \frac{60}{a^2} \right) = \frac{1280}{a^4} \),
- \( \mathcal{N}_1(a) := \max \{ 9\left( \frac{40 + 20a}{a^2} \right)^2, N_0(a) \} \leq \max \{ 9, N_0(a) \} \),
- \( \mathcal{N}_2(a) := \max \{ 9\left( \frac{\log \left( \frac{1 + a}{0.1a^2 + 0.9a} \right)}{\log(0.1a^2 + 0.9a)} \right)^2, N_0(a) \} \). On the interval \((0, 1)\), we have that \( \log(\frac{1 + a}{0.1a^2 + 0.9a}) \geq \frac{2a^2}{5} \). Hence \( \mathcal{N}_2(a) \leq \max \{ 5760 \frac{a^2}{a^4}, N_0(a) \} \).
- We therefore have that \( \mathcal{N}_0(a), \mathcal{N}_1(a), \mathcal{N}_2(a) \leq 5760 \frac{a^2}{a^4} \) for all \( a \in (0, 1) \).

Recall that \( \mathcal{N}_3(a) := \max \{ N_0(a), \mathcal{N}_3(a, a\gamma(a)) \} \) where

\[
\mathcal{N}_3(a, a\gamma(a)) = \mathcal{N}_3(a) = \frac{\log \left( \frac{1 + a}{a - a\gamma} \right) + \log \left( \frac{1 - a^2\gamma}{1 - a\gamma} \right) - \alpha \log r}{\log K^*} + 1.
\]
We would like to replace \( N_3(a) \) with a larger estimate. First we note that
\[
N_3(a) \leq \frac{\log\left(\frac{1+a}{a-a\gamma}\right) + \log\left(\frac{1-a^2\gamma}{1-a\gamma}\right) - \alpha^* \log r^*}{\log K^*} + 1,
\]
where
\[
\alpha^* = \frac{\log \frac{a}{16}}{\log \left(\frac{a\gamma + r^*}{1+a\gamma}\right)} \quad \text{and} \quad r^* = \frac{a^2(1-\gamma)\gamma}{2}.
\]
Since \( \log(x) \leq x - 1 \) for \( x > 0 \), we have
\[
\log\left(\frac{1+a}{a-a\gamma}\right) \leq \frac{1+a}{a-a\gamma} \leq \frac{2}{a-a\gamma}, \quad \log\left(\frac{1-a^2\gamma}{1-a\gamma}\right) \leq \frac{1-a^2\gamma}{1-a\gamma} \leq \frac{1}{a-a\gamma},
\]
\[
\log \frac{1}{r^*} \leq \frac{1}{r^*} = \frac{2}{a^2(1-\gamma)\gamma} \leq \frac{2}{a^3(1-\gamma)} \quad \text{since} \quad a < \gamma.
\]
One can show that \( \frac{1+ar^*\gamma}{a\gamma + r^*} > \frac{1}{\sqrt{a}} \) for \( a \in (0, 1) \). Hence
\[
\alpha^* \leq \frac{16}{a} / \frac{1}{2} \log \frac{1}{a} = \frac{32}{a \log \frac{1}{a}}.
\]
Recall that \( \gamma = 0.1a + 0.9 \) was constructed such that for \( a \in (0, 1) \)
\[
\log K^* > \min\left\{ -\frac{a^2\gamma}{4(4+2a)}, \frac{a^2(1-a)\gamma}{4(4-2a)} \right\}
\]
\[
= \frac{a^2(1-a)\gamma}{4(4-2a)} \geq \frac{a^2(1-a)\gamma}{16} \geq \frac{a^3(1-a)}{16}.
\]
Thus, we have
\[
N_3(a) \leq \left( \frac{3}{a(1-\gamma)} + \frac{2}{a^3(1-\gamma) \log \frac{1}{a}} \frac{32}{a^3(1-a)} \right) \frac{16}{a^3(1-a)} + 1
\]
\[
= \left( \frac{3a^3 \log \frac{1}{a} + 64}{a^3(1-\gamma) \log \frac{1}{a}} \frac{16}{a^3(1-a)} \right) + 1.
\]
We note that \( \lim_{a \to 0} a^3 \log \frac{1}{a} = 0 \). Also, the function \( a^3 \log \frac{1}{a} \) attains its global maximum of \( \frac{1}{3e} \) at \( a = \frac{1}{\sqrt{e}} \). Furthermore, \( 1-\gamma = 0.1(1-a) \). We conclude that
\[
N_3(a) \leq \left( \frac{650}{a^4(1-a) \log \frac{1}{a}} \right) \frac{32}{a^3(1-a)} = \frac{20800}{a^7(1-a)^2 \log \frac{1}{a}}.
\]
Table 1: Dégot’s $N$ vs $N(a)$ for certain $a \in (0, 1)$

| $a$ | Dégot’s $N$ | $N(a)$ |
|-----|-------------|--------|
| 0.1 | 15064       | $4.94 \times 10^1$ |
| 0.2 | 3587        | $6.18 \times 10^9$ |
| 0.3 | 1654        | $6.18 \times 10^8$ |
| 0.4 | 1004        | $1.53 \times 10^8$ |
| 0.5 | 718         | $6.64 \times 10^7$ |
| 0.6 | 563         | $4.53 \times 10^7$ |
| 0.7 | 560         | $4.86 \times 10^7$ |
| 0.8 | 616         | $9.66 \times 10^7$ |
| 0.9 | 1006        | $6.78 \times 10^8$ |

For $a \in (0, 1)$, $\log \frac{1}{a} \geq (1 - a)^2$. Hence $N_3(a) \leq \frac{20800}{a^2(1-a)^2}$. By letting $N(a)$ be given by $N(a) = \frac{32400}{a^2(1-a)^2}$, we obtain Theorem 1.1.

**Towards uniformity.** In [2], Dégot concludes by asking for a degree bound $N \in \mathbb{N}$ which is independent of $a \in (0, 1)$, or at least an explicit formula $N(a)$. We note that the formula $N(a)$ defined above suffices for the latter request. Furthermore, for any $0 < \alpha < \beta < 1$, the extreme value theorem tells us that $N(a)$ has a maximum on $[\alpha, \beta]$.

**Concluding remarks.** In conclusion, we would like to bring the following points to the reader’s attention: We would like a definitive result that would bridge the gaps $[0, \alpha)$ and $(\beta, 1]$. We are of the opinion that these gaps rather illustrate the limitation of this current approach, as opposed to the veracity of the conjecture.

Our main goal was to find a simple explicit formula $N(a)$, we thus overestimated many functions by replacing them with simpler formulae. In view of Dégot’s computations, the results we obtained here can still be considerably improved.

In Table 1, we see a comparison between the values of $N$ Dégot computed for certain values of $a$ and the corresponding rounded up approximate values of $N(a)$. As expected, the values produced by $N(a)$ are several orders of magnitude worse than those obtained by Dégot’s procedure. We however have the advantage of an explicit formula that works for all $a \in (0, 1)$.

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