On the Gravitational Energy-Momentum Vector in f(T) Theories

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This work is devoted to present and analyze an expression for the gravitational energy-momentum vector in the context of f(T) theories through field equations. Such theories are the analogous counterpart of the well known f(R) theories, except using torsion instead of curvature. We obtain a general expression for the gravitational energy-momentum vector in this framework. Using the hypothesis of the isotropy of spacetime, we find the gravitational energy for a closed Universe, since construction of real tetrads that do not constrain the functional form of the Lagrangian density was not possible for an open Universe. Thus we find a vanishing gravitational energy for the tetrad that we have used.

Keywords: Modified Gravity; Energy-momentum; Teleparallel Gravity.

1. Introduction

Teleparallel gravity and general relativity are two different theories with the same field equations. In general relativity it is not possible to construct an expression for the gravitational energy in terms of the metric tensor and its second derivatives which give rise to the approach of energy-momentum pseudo-tensors. On the other hand in TEGR there is an expression for the gravitational energy-momentum vector, tested throughout the years, in terms of the tetrad field which is the dynamical variable of the theory. This feature will be explored in this article.

The Hilbert-Einstein lagrangian density, which gives the dynamics of general relativity, can be generalized as a function of the Ricci scalar. This establishes a wide class of lagrangian densities that lead to what is known as f(R) theories\(^1\). Several cosmological observations, such as supernovae\(^2\),\(^3\), cosmic microwave background\(^4\),\(^5\), large-scale structure\(^6\),\(^7\) and baryon acoustic oscillations\(^8\) point to an exotic kind of energy which is known as dark energy. The f(R) theories have been used
successfully to explain such observations. They can explain the anomalous experimental data since it is added extra terms in the energy-momentum tensor of matter fields in the Einstein equations, working as a different source of energy density.

Such a success leads us to study the analogous generalization of the lagrangian density of teleparallel gravity, known as $f(T)$ theories. Recently, models based on $f(T)$ gravity were presented as an alternative to inflationary models. Moreover, in the literature we find some works that attempt to explain the late-time accelerated expansion of the Universe in the context of $f(T)$ theories.

Recently the energy of the Universe was obtained in the context of teleparallel gravity such a result was derived considering the Universe as described by the FRW metric as well as with the use of regularized expressions. This, in cosmology, is necessary to describe reference frames since it is possible to have remanence of torsion in the flat space-time. Thus we intend to find a general expression of the energy-momentum vector in the context of $f(T)$ theories, contained in an arbitrary volume $V$ of the three-dimensional spacelike hypersurface, $P_a$. The definition of $P_a$ proved to be invariant under coordinate transformations and transforms like a vector under global Lorentz transformations, features that are essential to a true energy-momentum vector. Then, as an application we obtain the gravitational energy of the universe taking into account a well known tetrad obtained in the literature. Since the theory is not invariant under local Lorentz transformations, choosing the best tetrad field is not a trivial task.

The paper is organized as follows: in section the $f(T)$ gravity is presented, we obtain through the field equations an expression for the gravitational energy-momentum vector. In section we apply our expression in the context of a homogeneous and isotropic Universe. We obtain a vanishing gravitational energy for all possible models, concerning the functional form of $f(T)$. We have used a “good” tetrad which is a solution of the field equations that does not constrain the functional form of the lagrangian density. Finally in the last section we present some concluding remarks.

Notation: space-time indices $\mu, \nu, \ldots$ and SO(3,1) indices $a, b, \ldots$ run from 0 to 3. Time and space indices are indicated according to $\mu = 0, i$, $a = (0), (i)$. The tetrad field is denoted by $e^a_{\mu}$ and the determinant of the tetrad field is represented by $e = \det(e^a_{\mu})$.

2. Teleparallel Gravity and $f(T)$ Theories

In this section we would like to establish the field equations of $f(T)$ theories as well as a definition of energy-momentum vector. However it is first necessary to recall some ideas of TEGR, which is an alternative theory of gravitation, that is entirely equivalent to standard general relativity in what concerns dynamical equations. The main difference between both formulations is the existence, in the
framework of Teleparallel gravity, of a true gravitational energy-momentum vector that is independent of coordinate transformations and sensible to the change of reference frame. It replaces the Riemannian curvature by the torsion in a tetrad formulation of Weitzenböck (or Cartan) space-time, an approach originally considered by Einstein himself in 1930.

The tetrad field and metric tensor are related by $g_{\mu\nu} = e^a_{\mu} e^a_{\nu}$. The familiar theory of general relativity deals with the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$, as the connection of space-time. On the other hand, TEGR is formulated in terms of Cartan connection $\Gamma^\mu_{\lambda\nu} = e^a_{\lambda} \partial_{\nu} e^a_{\mu}$. The geometric framework of both theories is related by the following identity

$$\Gamma^\mu_{\lambda\nu} = 0 \Gamma^\mu_{\lambda\nu} + K^\mu_{\lambda\nu},$$

where $K^\mu_{\lambda\nu}$ is given by

$$K^\mu_{\lambda\nu} = \frac{1}{2} (T^\lambda_{\mu\nu} + T^\nu_{\lambda\mu} + T^\mu_{\lambda\nu}).$$

$K^\mu_{\lambda\nu}$ is the contortion tensor defined in terms of the torsion tensor constructed from the Cartan connection. The torsion tensor is $T^\mu_{\lambda\nu} = e^a_{\mu} \partial_{\nu} e^a_{\lambda}$.

The curvature tensor obtained from $\Gamma^\mu_{\lambda\nu}$ is identically zero. From the identity (1) we have

$$eR(e) \equiv -e\left(\frac{1}{4} T_{\lambda\mu\nu} + \frac{1}{2} T_\lambda^{abc} T_{bac} - T^\alpha T_\alpha\right) + 2 \partial_{\mu} (eT^\mu),$$

where $R(e)$ is the scalar curvature of a Riemannian manifold in terms of the tetrad field and $T^\mu = T^b_{\ b \mu}$.

The Teleparallel Lagrangian density can be defined from (4) and it reads

$$\mathcal{L}(e_{\mu}) = -\kappa e\left(\frac{1}{4} T^{abc}_{\alpha} T_{\alpha}^{abc} + \frac{1}{2} T^{abc}_{\alpha} T_{bac} - T^a T_\alpha\right) - \mathcal{L}_M$$

$$\equiv -\kappa e \Sigma^{abc}_{\alpha} T_{\alpha} + \mathcal{L}_M,$$

where $\kappa = 1/(16\pi)$, $\mathcal{L}_M$ is the Lagrangian density of matter fields and $\Sigma^{abc}$ is given by

$$\Sigma^{abc} = \frac{1}{4} (T^{abc} + T^{bac} - T^{cub}) + \frac{1}{2} (\eta^{ac} T^{b} - \eta^{ab} T^{c}),$$

with $T^a = e^a_{\mu} T^\mu$. It is important to note that the total divergence has been dropped once it does not contribute to the field equations. Thus both theories share the same
field equations and hence are dynamically equivalent to each other, since there is an equivalence between their Lagrangian densities.

The most general Lagrangian density in the realm of teleparallelism is given by

$$\mathcal{L} = -e f(T) - \mathcal{L}_M,$$

where $T = \Sigma^{abc} T_{abc}$. Performing a variational derivative of the above Lagrangian density with respect to $e a \gamma$, the dynamical variables of the system, the field equations are

$$f'(T) \left[ \partial_{\nu} (e \Sigma^{\alpha \gamma \nu}) - e \Sigma^{b \gamma} T_{b c} \right] + e \Sigma^{\alpha \lambda} (\partial_{\lambda} T) f''(T) + \frac{1}{4} e e a \gamma f(T) = \frac{1}{4\kappa} e T^{a \gamma},$$

where $T^{a \mu} = e a \lambda T^{\mu \lambda} = \frac{\delta \mathcal{L}_M}{\delta e a \mu}$ is the energy-momentum tensor of matter fields. The prime in $f(T)$ means a derivative with respect to $T$. The field equations can be rewritten as

$$\partial_{\nu} (e \Sigma^{a \lambda \nu} f'(T)) = \frac{1}{4\kappa} e e a \mu (t^{\lambda \mu} + T^{\lambda \mu}),$$

where $t^{\lambda \mu}$ is defined by

$$t^{\lambda \mu} = \kappa \left[ 4 f'(T) \Sigma^{b \lambda \gamma} T_{b c} \mu - g^{\lambda \mu} f(T) \right].$$

Since $\Sigma^{a \lambda \nu}$ is skew-symmetric in the last two indices, it follows that

$$\partial_{\lambda} \partial_{\nu} (e \Sigma^{a \lambda \nu} f'(T)) \equiv 0.$$

Thus we get

$$\partial_{\lambda} (e t^{a \lambda} + e T^{a \lambda}) = 0$$

which yields the continuity equation

$$\frac{d}{dt} \int_V d^3x e e a \mu (t^{0 \mu} + T^{0 \mu}) = - \int_S dS_j \left[ e e a \mu (t^{j \mu} + T^{j \mu}) \right].$$

It should be noted that the above expression works as a conservation law for the sum of the energy-momentum tensor of matter fields and the quantity $t^{\lambda \mu}$. Thus $t^{\lambda \mu}$ is interpreted as the energy-momentum tensor of the gravitational field in the context of $f(T)$ theories, being more general than (and slightly different from) the usual quantity in TEGR. Therefore, one can write the total energy-momentum contained in a three-dimensional volume $V$ of space as

$$P^a = \int_V d^3x e e a \mu (t^{0 \mu} + T^{0 \mu}),$$
or using the field equations we have

\[ P^a = 4k \int_V d^3x \partial_\nu \left( e \Sigma^{a0\nu} f'(T) \right) . \]  

(14)

It is worth noting that the above expression is invariant under coordinate transformation and transforms like a vector under global Lorentz transformations. Such features are desirable and expected for a true energy-momentum vector, they are shared by other theories such as TEGR and special relativity. However at this point it is important to stress that the above expression is not invariant under local Lorentz transformations due to the lack of such invariance in the lagrangian density itself. In addition we point out that it is impossible to get a local Lorentz invariance from a quantity defined by integration such as eq. (14) since it cannot depend on coordinates, which precludes a transformation like \( P^a = \lambda^a b(x) P^b \), where \( \lambda^a b(x) \) represents local Lorentz transformations. From expression (1) we see that the scalar of curvature shares the same properties with the scalar of torsion which reflect on the field equations. On the other hand the picture is entirely different when one is dealing with \( f(T) \) theories since the total divergence is no longer in the lagrangian density. As a consequence \( f(T) \) theories are not equivalent to \( f(R) \) theories and different tetrads, even when originating the same metric tensor, can lead to different field equations. Equation (14) is unique in the sense that for each tetrad that is a solution of the field equations we have a well defined gravitational energy-momentum vector.

Therefore the question on how to choose a tetrad field arises. This problem is addressed in ref. [19] for spherical symmetry and FRW metric. There the authors define a “good” tetrad field by the following specifications: i) The solution of its field equations should reduce to a general relativity solution in the limit \( f(T) \rightarrow T \); ii) The functional form of \( f(T) \) should not be constrained by the field equations.

3. Application in Cosmological Scales: The energy of the Universe

The cosmological principle asserts that the large-scale structure of the Universe reveals homogeneity and isotropy [20]. The most general form of a line element that preserves such features may be written as [21]

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{(1 - kr'^2)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] , \]  

(15)

where \( a(t) = S(t)/|K|^{1/2} \) if \( K \neq 0 \) and \( a(t) = S(t) \) if \( K = 0 \). \( S(t) \) is the scale factor and \( K \) is the constant curvature of space. Here \( K = |K|k' \) where \( k' \) assumes the values \(+1, 0, -1\) which correspond to a space of constant positive curvature, a flat space or a space of constant negative curvature, respectively.

The simplest choice form for the tetrad field is diagonal, however it is not a good choice for it since such tetrad constrains the functional form of \( f(T) \). It forces, by
means the field equations, \( f(T) \) to reproduce TEGR. It is shown in [19] that, among all the possible tetrads, the “good” choice for \( k = 1 \) would be

\[
e^a_\mu = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & a \cot \phi \sin \theta / \sqrt{1 - r^2} & 0 & 0 \\
0 & r a \left( \sqrt{1 - r^2} \cos \theta \cos \phi - r \sin \phi \right) & 0 & 0 \\
0 & r a \sin \theta \left( -r \cos \theta \cos \phi - \sqrt{1 - r^2} \sin \phi \right) & -a \sin \theta \sin \phi / \sqrt{1 - r^2} & a \cos \phi / \sqrt{1 - r^2} \\
r a \left( r \cos \phi + \sqrt{1 - r^2} \cos \theta \sin \phi \right) & -r a \left( r \cos \phi + \sqrt{1 - r^2} \cos \theta \sin \phi \right) & a \cos \phi \sin \theta / \sqrt{1 - r^2} & a \cos \phi / \sqrt{1 - r^2} \\
r a \sin \theta \left( 1 - r^2 \cos \phi - r \cos \theta \sin \phi \right) & r a \sin \theta \left( 1 - r^2 \cos \phi - r \cos \theta \sin \phi \right) & -r \sqrt{1 - r^2} a \sin \theta & r \sqrt{1 - r^2} a \sin \theta \\
\end{pmatrix},
\] (16)

where the above tetrad is presented with inverted lines and columns for the sake of adjustment.

The non-vanishing components of the torsion tensor are

\[
T_{101} = \frac{a}{1 - r^2} \frac{\partial a}{\partial t},
\]
\[
T_{123} = \frac{2 a^2 r^2}{\sqrt{1 - r^2}} \sin \theta,
\]
\[
T_{202} = a r^2 \left( \frac{\partial a}{\partial t} \right),
\]
\[
T_{213} = \frac{2 a^2 r^2}{\sqrt{1 - r^2}} \sin \theta,
\]
\[
T_{303} = a r^2 \left( \frac{\partial a}{\partial t} \right) \sin^2 \theta,
\]
\[
T_{312} = \frac{2 a^2 r^2}{\sqrt{1 - r^2}} \sin \theta.
\] (17)

Then after some algebraic manipulations it is possible to obtain the scalar of torsion which reads

\[
T = 6 \left[ -1 + \left( \frac{a(t)}{a(t)} \right)^2 \right].
\]

The field equations, which establish the temporal evolution of \( a(t) \), for \( k = 1 \) are given by

\[
12 (H)^2 f'(T) + f(T) = 16 \pi \rho,
\] (18)
\[
\dot{H} (12 H^2 f''(T) - f'(T)) + \left( \frac{12 H^2 f''(T) + f'(T)}{a^2} \right) = 4 \pi (\rho + p),
\] (19)
where $\rho$ and $p$ are respectively the energy density and the pressure of the cosmological perfect fluid. $H = \dot{a}/a$ is the Hubble constant.

Although the tetrad $\ell_{(16)}$ represent a good choice, it yields

$$\Sigma^{(0)i} = 0,$$

which gives a vanishing expression for the gravitational energy for all possible functional form of $f(T)$, since

$$P^{(0)} = 4k \int_V d^3x \sqrt{g} \rho \left( \Sigma^{(0)i} f'(T) \right).$$

In addition, the procedure proposed in [21] does not allow a good choice for the tetrad field when $k = -1$. It is important to point out that we do expect a vanishing gravitational energy when $k = 0$, since we would be dealing with a flat Universe.

Other attempts have been made to construct good tetrads in the cosmological context, for instance one may see the tetrad field presented in ref. [22]. It represents the same tetrad above in different coordinates, as a consequence it will lead to the same vanishing gravitational energy. This means that in the context of $f(T)$ theories the gravitational energy is not due to the curvature of the spacetime, since it yields a vanishing one even in the presence of a spacetime with constant (and positive) curvature such as a closed Universe.

4. Conclusion

In this paper we have obtained a general expression for the gravitational energy-momentum in the realm of teleparallel gravity when dynamics are governed by a general lagrangian as a function of the scalar $T = \Sigma_{abc} T_{abc}$, and thus called $f(T)$ theories (in analogy to the $f(R)$ theories). Such an expression has never appeared in the literature. We then applied our expression to the FRW metric which was established taking into account the principle of isotropy and homogeneity of the large scale of the Universe. We show that the tetrad field that does not constrain the functional form of the lagrangian density yields a vanishing energy. We apply our expression for a tetrad obtained in the literature then we show that the gravitational energy vanishes independent of the functional form of $f(T)$, which may vary from the Born-Infeld model to simpler ones that may represent small deviations of the TEGR lagrangian density such as $f(T) = T + \frac{1}{2} \lambda T^2$. Thus we conclude that in the context of $f(T)$ theories, for FRW spacetime, the dynamics of a particle is due to the matter fields, which means that the gravitational energy plays no role in such a system.

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