Criteria for strong and weak random attractors

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The theory of random attractors has different notions of attraction, amongst them pullback attraction and weak attraction. We investigate necessary and sufficient conditions for the existence of pullback attractors as well as of weak attractors.

2000 Mathematics Subject Classification Primary 60H25 Secondary 37B25 37H99 37L55 60D05

Keywords. Random attractor; pullback attractor; weak attractor; Omega limit set; compact random set

1 Introduction

The notion of an attractor is one of the basic concepts in the theory of dynamical systems. For deterministic systems this notion has been of importance since several decades. Since about fifteen years also attractors for stochastic systems have been taken under consideration. The crucial obstacle came from the fact that the classical approach, using the Markov property of individual solutions to define the Markov semigroup and an associated generator, could not deal with joint motions of two or more points. This began to change after the introduction of stochastic flows, going back to Kunita and to Elworthy, and the introduction of the notion of random dynamical systems (see Arnold [1]). All approaches to random attractors use the theory of random dynamical systems (RDS). The notion of pullback attractors, which are referred to as strong attractors here, go back to Crauel, Flandoli, and Debussche [12, 10], and to Schmalfuß [22]. Later the notion

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of weak attractors was introduced by Ochs [18]. A comparison of these two concepts and yet another one, called forward attractor, has been investigated by Scheutzow [19], see also Crauel [8]. Several investigations, among others of notions of local random attractors, have been carried out by Ashwin and Ochs [3].

The present paper gives necessary and sufficient criteria for the existence of strong and weak attractors for compact and for bounded sets. This is developed in Theorems 3.1 and 3.2 for strong attraction of bounded sets and compact sets, respectively, and in Theorems 4.2 and 4.4 for weak attraction of bounded sets and compact sets, respectively. A version of Theorem 4.2 for RDS on the Euclidean space $\mathbb{R}^n$ goes back to Dimitroff [13]. Furthermore, it is shown that a weak attractor is a strong attractor if and only if it contains the Omega limit set of every set from the ‘domain of attraction’.

Existence of strong attractors has been verified for many concrete systems in the literature. Also for the more general non-autonomous case existence of attractors has been obtained for a large variety of systems. There the ‘almost surely’ used in the context of random attractors is replaced by ‘for all’, while the method of the construction remains the same. See [10] for this approach. The conditions for the existence of strong random attractors obtained in Theorems 3.1 and 3.2 are purely probabilistic. They have no non-autonomous version.

One may wonder whether these conditions are of use in order to obtain existence of random attractors. In this respect we refer to recent work of Dimitroff and Scheutzow [14], where the existence of strong random attractors for certain concrete systems is verified by using the conditions Theorem 3.1 and Theorem 3.2.

It should also be noted that a strong (or ‘pullback’, see Remark 2.5) attractor allows for several explicit representations. It may be constructed by taking the union of all Omega limit sets of deterministic compact sets. Alternatively, the attractor is the Omega limit set of every deterministic compact set which contains the attractor with positive probability. Explicit constructions of weak attractors, on the other hand, have not been available yet. Theorems 4.2 and 4.4 give constructions of weak random attractors. The constructions are technically considerably more involved than those for strong attractors.

## 2 Set-up and preliminaries

In this section we first introduce some basic notions of random dynamical systems (RDS), referring to Arnold [1] for a comprehensive presentation. Then we give a brief introduction to the concepts of strong and weak random attractors.

Let $E$ be a Polish space (i.e. a separable topological space which is metrisable by a complete metric), and let $\mathcal{B}$ be its Borel $\sigma$-algebra. We will often assume $E$ to be equipped with a metric inducing the topology, which will then be denoted by $d$. Furthermore, $d$ will be assumed to be complete whenever the argument needs it. We have
not tried to single out when this is the case. For $x \in E$ and for a subset $A$ of $E$ we write $d(x, A) = \inf_{a \in A} d(x, a)$.

### 2.1 Definition

(a) $(\Omega, \mathcal{F}, P, (\vartheta_t)_{t \in \mathbb{R}})$ is called a **metric dynamical system (MDS)**, if $(\Omega, \mathcal{F}, P)$ is a probability space, and the family of mappings $\{\vartheta_t : \Omega \rightarrow \Omega : t \in \mathbb{R}\}$ satisfies

(i) the mapping $(\omega, t) \mapsto \vartheta_t(\omega)$ is $(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathcal{F})$-measurable,

(ii) $\vartheta_{s+t} = \vartheta_s \circ \vartheta_t$ for every $s, t \in \mathbb{R}$, and $\vartheta_0 = \text{Id}_\Omega$, and

(iii) for each $t \in \mathbb{R}$, $\vartheta_t$ preserves the measure $P$.

(b) A **random dynamical system (RDS)** on the measurable space $(E, \mathcal{B})$ over the MDS $(\Omega, \mathcal{F}, P, (\vartheta_t))$ with time $\mathbb{R}^+$ is a mapping $\varphi : [0, \infty) \times E \times \Omega \rightarrow E$, $(t, x, \omega) \mapsto \varphi(t, x, \omega)$ with the following properties:

(i) $\varphi$ is $(\mathcal{B}([0, \infty)) \otimes \mathcal{B} \otimes \mathcal{F}, \mathcal{B})$-measurable.

(ii) For all $s, t \in [0, \infty)$

$$\varphi(t + s, \omega) = \varphi(t, \vartheta_s \omega) \circ \varphi(s, \omega) \quad \text{for all } \omega \in \Omega,$$

and $\varphi(0, \omega) = \text{Id}_E$ for all $\omega \in \Omega$.

The RDS $\varphi$ is called **continuous** if, in addition,

(iii) the mapping $x \mapsto \varphi(t, x, \omega)$ is continuous for all $(t, \omega) \in [0, \infty) \times \Omega$.

The present definition of an RDS is formulated for continuous time only, i.e. $t \in \mathbb{R}$ for the base flow, and $t \in [0, \infty)$ for the RDS itself. The general definition also allows for discrete time.

A map $B : \Omega \rightarrow 2^E$, where $2^E$ denotes the power set of $E$, is said to be a **random set** if its graph $\{(x, \omega) : x \in B(\omega)\} \subset E \times \Omega$ is an element of the product $\sigma$-algebra $\mathcal{B} \otimes \mathcal{F}$. We will assume without further mentioning that a random set is nonempty almost surely. A map $B : \Omega \rightarrow 2^E$ is called a **closed random set** or a **compact random set**, respectively, if $\omega \mapsto B(\omega)$ takes values in the closed subsets or in the compact subsets, respectively, of $E$ and if, in addition, $\omega \mapsto d(x, B(\omega))$ is measurable for every $x \in E$. A closed random set is always a random set, whereas a random set taking values in the closed subsets is a closed random set if $(\Omega, \mathcal{F})$ is universally measurable. See, for instance, Hu and Papageorgiou [15] Chapter 2.1–2, also for further characterizations. In particular, a non-empty compact random set can also be characterized by considering it as a random variable taking values in the metric space of all compact subsets of $E$, equipped with the Hausdorff metric, see below.

The following definition is (essentially) due to Crauel and Flandoli [12], see also Crauel, Debussche and Flandoli [10] and Schmalfuß [21, 22].
2.2 Definition  Let $\varphi$ be an RDS on $E$ over the MDS $(\Omega, \mathcal{F}, (\vartheta_t)_{t \in \mathbb{R}}, P)$. Let $B \subset 2^E$ be an arbitrary subset of the power set of $E$. A random set $\omega \mapsto A(\omega)$ is called a \textit{strong $B$-attractor} for $\varphi$ if

(a) $\omega \mapsto A(\omega)$ is a compact random set.

(b) $A$ is \textit{strictly $\varphi$-invariant}, that is, for each $t \geq 0$ there exists a set $\Omega_t$ of full measure, such that $\varphi(t, \omega)(A(\omega)) = A(\vartheta_t \omega)$ for all $\omega \in \Omega_t$.

(c) $\lim_{t \to \infty} \sup_{x \in B} d(\varphi(t, \vartheta_{-t} \omega)(x), A(\omega)) = 0$ almost surely for every $B \in B$.

In particular, a strong $B$-attractor is called

- \textit{$B$-attractor} in case that $B$ is the set of all bounded subsets of $E$
- \textit{$C$-attractor} in case that $B$ is the set of all compact subsets of $E$.

It should be mentioned that the set of all bounded subsets of $E$ depends on the choice of the metric $d$ on $E$, whereas this is not the case for the set of all compact subsets of $E$.

2.3 Remark  The notion of a $B$-attractor may be modified to allow also for random sets $\omega \mapsto B(\omega)$ by demanding instead of (c)

$$\lim_{t \to \infty} \sup_{x \in B(\vartheta_{-t} \omega)} d(\varphi(t, \vartheta_{-t} \omega)(x), A(\omega)) = 0$$

almost surely for every $B \in B$.

This is of interest, in particular, when dealing with \textit{local} random attractors. For global attractors it does not yield much more generality, since whenever a general $B$ comprises the set of all deterministic compact sets, then in case a $B$-attractor exists also a $C$-attractor exists, it is unique, and the $B$-attractor coincides with the $C$-attractor, see Crauel [6].

It has turned out that there are situations in which the notion of a strong attractor is not the best choice. The notion of a weak attractor has first been considered by Ochs [18].

2.4 Definition  Let $\varphi$ be an RDS on $E$ over the MDS $(\Omega, \mathcal{F}, (\vartheta_t)_{t \in \mathbb{R}}, P)$. Let $B \subset 2^E$ be an arbitrary subset of the power set of $E$. A random set $\omega \mapsto A(\omega)$ is called a \textit{weak $B$-attractor} for $\varphi$ if $A$ satisfies conditions (a) (compactness) and (b) (almost sure strict invariance) of Definition 2.2, and if

(c) $\lim_{t \to \infty} \sup_{x \in B} d(\varphi(t, \omega)(x), A(\vartheta_t \omega)) = 0$ in probability for every $B \in B$.

2.5 Remark   (i) The notion of a strong attractor is also often referred to as a \textit{pullback attractor}.

(ii) Asking for convergence almost surely in (c) gives yet another concept, referred to as a \textit{forward attractor}. Clearly both pullback attractors and forward attractors are weak attractors. However, a weak attractor need neither be a strong (pullback) nor a forward
attractor, see Scheutzow [19]. Also uniform exponential attraction does not suffice to imply that a forward attractor is also a pullback attractor, see Crauel [8].

For further literature dealing with the notion of weak attractors see Ashwin and Ochs [3], Arnold and Schmalfuß [2], Chueshov and Scheutzow [5], Crauel, Duc and Siegmund [11], and Scheutzow [20].

We will need some further notions. For non-empty sets $A, B \subset E$, $E$ a metric space, with $B$ bounded we denote by

$$d(B, A) = \sup_{b \in B} d(b, A) = \sup_{b \in B} \inf_{a \in A} d(a, b)$$

the Hausdorff semi-distance. It should not cause confusion to use the same letter $d$ for the metric on $E$ and for the Hausdorff semi-distance on subsets of $E$. The Hausdorff metric between two compact sets $A, B \subset E$ is given by

$$d_H(A, B) = \max \{ d(B, A), d(A, B) \}.$$

2.6 Definition Suppose that $\varphi$ is an RDS on $E$ over $(\Omega, \mathcal{F}, (\vartheta_t)_{t \in \mathbb{R}}, P)$.

(i) A random set $\omega \mapsto K(\omega)$ is said to attract another random set $\omega \mapsto B(\omega)$ strongly, if

$$\lim_{t \to \infty} d(\varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega), K(\omega)) = 0 \quad \text{for } P\text{-almost every } \omega \in \Omega.$$

(ii) $K$ is said to attract $B$ weakly if

$$\lim_{t \to \infty} d(\varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega), K(\omega)) = 0 \quad \text{in probability.}$$

(iii) The $\Omega$-limit set of a random set $\omega \mapsto B(\omega)$ is the random set given by

$$\Omega_B(\omega) = \bigcap_{T \geq 0} \bigcup_{t \geq T} \varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega).$$

We will make use of invariance of $\Omega$-limit sets. A random set $\omega \mapsto B(\omega)$ is said to be $\varphi$-invariant if for every $t \geq 0$ there exists a set $\Omega_t$ of full measure, such that $\varphi(t, \omega)(B(\omega)) \subset B(\vartheta_t\omega)$ for all $\omega \in \Omega_t$. Compare with the notion of strict invariance introduced in Definition 2.2 (b). Note that these notions are not used consistently in the literature. Often “forward invariant/invariant” is used instead of “invariant/strictly invariant”.

2.7 Lemma Any $\Omega$-limit set $\Omega_B$ is $\varphi$-invariant.

See Crauel [6] Lemma 5.1, for the proof.

3 Criteria for strong attractors

In this section, we assume that $\varphi$ is a continuous RDS over the metric dynamical system $(\Omega, \mathcal{F}, (\vartheta_t)_{t \in \mathbb{R}}, P)$, taking values in the Polish space $E$, which is equipped with a metric $d$. 
We note that we assume continuous time here, mainly to ease notation. The results hold for discrete time as well, and the proofs remain the same. It should be emphasized that we do not assume continuity in the dependence on time.

For a subset $A$ of $E$ we denote the closed $\delta$-neighbourhood of $A$ by $A^{\delta}$.

3.1 Theorem The following are equivalent:

(i) $\varphi$ has a strong $B$-attractor.

(ii) For every $\varepsilon > 0$ there exists a compact subset $C_\varepsilon$ such that for each $\delta > 0$ and each bounded and closed subset $B$ of $E$ it holds that

$$ P\{B \subset \bigcap_{s \geq 0} \varphi(t, \vartheta_{-t}\omega)^{-1}(C_\varepsilon^{\delta})\} \geq 1 - \varepsilon. $$

(iii) There exists a compact strongly $B$-attracting set $\omega \mapsto K(\omega)$.

Proof Equivalence of (i) and (iii) is proved in Crauel [7], Theorem 3.4 and Remark 3.5.

To see that (i) implies (ii), fix $\varepsilon > 0$. Since $E$ is a Polish space and the attractor $A$ is a random variable taking values in the compact sets, there exists a compact subset $C_\varepsilon \subset E$ such that

$$ P\{A(\omega) \subset C_\varepsilon\} \geq 1 - \varepsilon $$

(see Crauel [9] Proposition 2.15). For $\delta > 0$ and a bounded and closed subset $B$ of $E$ we have

$$ P\{B \subset \bigcap_{s \geq 0} \varphi(t, \vartheta_{-t}\omega)^{-1}(A^{\delta}(\omega))\} = 1. $$

Consequently,

$$ P\{B \subset \bigcap_{s \geq 0} \varphi(t, \vartheta_{-t}\omega)^{-1}(C_\varepsilon^{\delta})\} \geq P\{B \subset \bigcap_{s \geq 0} \varphi(t, \vartheta_{-t}\omega)^{-1}(A^{\delta}(\omega))\} - P\{A(\omega) \notin C_\varepsilon\} \geq 1 - \varepsilon, $$

proving (ii).

Finally, in order to show that (ii) implies (i), let $B_0 \subset B_1 \subset \ldots$ be a sequence of bounded and closed subsets on $E$ such that for each bounded set $B$ there exists some $k \in \mathbb{N}$ with $B \subset B_k$. For instance, $B_k$ may be taken to be the ball of radius $k$ around some $x_0 \in E$. Put

$$ A(\omega) = \bigcup_{k \in \mathbb{N}} \Omega_{B_k}(\omega), $$

where $\Omega_{B_k}(\omega)$ denotes the $\Omega$-limit set of $B_k$ (see Definition 24(iii)). Let $\varepsilon > 0$. By condition (ii) there exists a compact set $C_\varepsilon \subset E$ such that for all $k$ and all $\delta > 0$ we have

$$ P\{\Omega_{B_k}(\omega) \subset C_\varepsilon^{\delta}\} \geq 1 - \varepsilon, $$
which implies $P\{A(\omega) \subset C_{\varepsilon}\} \geq 1 - \varepsilon$. In particular, $A(\omega)$ as well as $\Omega_{B_k}(\omega)$ is compact for every $k$, for $P$-almost every $\omega \in \Omega$.

We now claim that $\Omega_{B_k}$ is strictly invariant for every $k$. Invariance of $\Omega_{B_k}$ follows from Lemma 2.7. In order to see that $\Omega_{B_k}$ is strictly invariant, fix $\varepsilon > 0$ and $t \geq 0$, and suppose that $y \in \Omega_{B_k}(\vartheta_t(\omega))$. Then $y = \lim_{n \to \infty} \varphi(t_n, \vartheta_{-t_n}(\vartheta_t(\omega)))b_n$ for some sequence $t_n \to \infty$, $b_n \in B_k$, with $n \to \infty$. Consider the sequence $\varphi(t_n - t, \vartheta_{-(t_n-t)}(\vartheta_t(\omega)))b_n$, defined for $n$ with $t_n - t \geq 0$. Then for $\omega$ with $\Omega_{B_k}(\omega) \subset C_{\varepsilon}$ we have $d(\varphi(t_n - t, \vartheta_{-(t_n-t)}(\vartheta_t(\omega)))b_n, C_{\varepsilon}) = 0$ for $n \to \infty$. This implies existence of a convergent subsequence with limit $z(\omega)$, say. Clearly $z(\omega) \in \Omega_{B_k}$. Using the same notation for the subsequence, continuity of $\varphi(t, \omega)$ implies

$$\varphi(t, \omega)z(\omega) = \lim_{n \to \infty} \varphi(t_n, \vartheta_{-(t_n-t)}(\vartheta_t(\omega)))b_n = y(\omega).$$

Consequently,

$$P\{\Omega_{B_k}(\vartheta_t(\omega)) \subset \varphi(t, \omega)\Omega_{B_k}(\omega)\} \geq 1 - \varepsilon.$$ 

This holding for every $\varepsilon > 0$, we obtain that $\Omega_{B_k}$ is strictly invariant, whence also $A$ is strictly invariant.

It remains to show that $A$ attracts every bounded set. It suffices to show that $A$ attracts each of the sets $B_k$. Now $\Omega_{B_k}$ attracts $B_k$ for every $k$, $P$-almost surely. In fact, if this would not be the case then there would be a $\delta > 0$, a sequence $t_n \to \infty$, and $b_n \in B_k$ such that $d(\varphi(t_n, \vartheta_{-t_n}(\vartheta_t(\omega)))b_n, \Omega_{B_k}(\omega)) \geq \delta$ for every $n \in \mathbb{N}$ with positive probability. In view of the fact that $(\varphi(t_n, \vartheta_{-t_n}(\vartheta_t(\omega)))b_n)_{n \in \mathbb{N}}$ has a convergent subsequence with probability larger than $1 - \varepsilon$ for $\varepsilon > 0$ arbitrary, which then must converge to a limit in $\Omega_{B_k}(\omega)$, this would yield a contradiction, completing the proof. \hfill \Box

Next we consider the case of a strong $C$-attractor. Several arguments of the previous Theorem 3.1 proceed analogously. However, the construction of the attractor has to be done in a different way, and also the argument invoked for the verification of the attraction property is different. Therefore it seems appropriate to formulate the result separately, even though the assertions appear to be very similar.

3.2 Theorem The following are equivalent:

(i) $\varphi$ has a strong $C$-attractor.

(ii) For every $\varepsilon > 0$ there exists a compact subset $C_\varepsilon$ such that for each $\delta > 0$ and each compact subset $K$ of $E$ it holds that

$$P\{K \subset \bigcup_{s \geq 0} \bigcap_{t \geq s} \varphi(t, \vartheta_{-t}(\vartheta_t(\omega)))^{-1}(C_\varepsilon)\} \geq 1 - \varepsilon.$$ 

(iii) There exists a compact strongly $C$-attracting set $\omega \mapsto K(\omega)$.

Proof Equivalence of (i) and (iii) follows again from Crauel [7], Theorem 3.4 and Remark 3.5, and (i)$\Rightarrow$(ii) is proved exactly as in Theorem 3.1.
In order to see that (ii) implies (i), consider $C_{1/k}, k \in \mathbb{N}$, where the sets $C_\varepsilon$ are from condition (ii), and put

$$A(\omega) = \bigcup_{k \in \mathbb{N}} \Omega_{C_{1/k}}(\omega).$$

As in Theorem 3.1 we obtain that $A$ is compact almost surely, and strictly invariant. It remains to show that $A$ attracts every compact set. Let $K \subset E$ be compact. Then $\Omega_K$ is strictly invariant, arguing as in the proof of Theorem 3.1. Further

$$1 - \frac{1}{n} \leq P\{\Omega_K(\omega) \subset C_{1/n}\} \leq P\{\Omega_K(\omega) \subset \Omega_{C_{1/n}}\} \quad \text{for every } n \in \mathbb{N}$$

(see Crauel [6] Proposition 5.2 for the second inequality), hence $\Omega_K \subset A$ almost surely, and therefore $A$ attracts $K$ almost surely. This completes the proof. \qed

4 Criteria for weak attractors

In this section we are interested in weak attractors. We follow the same line as in the previous section on strong attractors. We establish necessary and sufficient criteria first for the existence of weak $B$-attractors, and then for the existence of weak $C$-attractors. The structure is very similar to that of Theorems 3.1 and 3.2. The difference of the two concepts gets visible in the corresponding conditions (ii), which is eventually “uniform in time” for strong attractors, while it is eventually “pointwise in time” for weak attractors.

Again we consider the cases of $B$-attractors and of $C$-attractors separately, even if the assertions are very similar. Also certain parts of the proofs are similar, and they could be presented in one result. However, the construction of the attractor is different in those two cases. The argument is more straightforward for $B$-attractors. Therefore the presentation has been split, and the arguments are given separately.

As before, $\varphi$ is a continuous RDS over the metric dynamical system $(\Omega, \mathcal{F}, (\theta_t)_{t \in \mathbb{R}}, P)$, taking values in the Polish space $E$, equipped with a metric $d$.

We will make use of an elementary lemma. Again $B^\delta$ denotes the $\delta$-neighbourhood of a subset $B$ of a metric space.

4.1 Lemma Suppose that $\varphi : X \to Y$ is a continuous map from a metric space $X$ to a metric space $Y$. Let $C \subset X$ be compact.

(i) For every $\varepsilon > 0$ there exists a $\gamma > 0$ such that

$$\varphi(C_\gamma) \subset (\varphi(C))^\varepsilon,$$

or, equivalently, $d(\varphi(C^\delta), \varphi(C)) \to 0$ for $\delta \to 0$.

(ii) If $B \subset Y$ satisfies $B \subset \varphi(C^\delta)$ for every $\delta > 0$, then $B \subset \varphi(C)$. 

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Proof Assuming (i) not to be true, we get existence of some \( \varepsilon > 0 \) and of a sequence \((x_n)_{n \in \mathbb{N}} \) with \( x_n \in C^{1/n} \) such that
\[
d(\varphi(x_n), \varphi(C)) \geq \varepsilon \quad \text{for every } n \in \mathbb{N}.
\]
(2)

By compactness of \( C \) the sequence \((x_n)\) has a convergent subsequence, denoted by \((x_n)\) again, converging to some \( x \in C \). This implies convergence of \( d(\varphi(x_n), \varphi(C)) \) to \( d(\varphi(x), \varphi(C)) = 0 \), contradicting (2).

In order to obtain (ii) note that \( B \subset \overline{\varphi(C^\delta)} \) implies \( d(B, \varphi(C)) \leq d(\varphi(C^\delta), \varphi(C)) = d(\varphi(C^\delta), \varphi(C)) \), which converges to zero for \( \delta \to 0 \) by (i). Therefore \( d(B, \varphi(C)) = 0 \), which implies \( B \subset \varphi(C) \) since \( \varphi(C) \) is closed, in fact even compact.

4.2 Theorem The following are equivalent:

(i) \( \varphi \) has a weak \( B \)-attractor.

(ii) For every \( \varepsilon > 0 \) there exists a compact subset \( C_\varepsilon \) such that for each \( \delta > 0 \) and each bounded subset \( B \) of \( E \) there is a \( t_0 > 0 \) with the property that for all \( t \geq t_0 \),
\[
P\{\varphi(t, \omega)(B) \subset C_\varepsilon^\delta\} \geq 1 - \varepsilon.
\]

(iii) There exists a compact weakly \( B \)-attracting set \( \omega \mapsto K(\omega) \).

Proof Obviously (i) implies (iii).

To see that (iii) implies (ii), let \( \omega \mapsto K(\omega) \) be a compact weakly \( B \)-attracting set. Fix \( \varepsilon > 0 \). Since \( E \) is a Polish space and \( K \) is a random variable taking values in the compact sets, there exists a compact subset \( C_\varepsilon \) of \( E \) such that
\[
P\{K(\omega) \subset C_\varepsilon\} \geq 1 - \frac{\varepsilon}{2}
\]
(see Crauel [9] Proposition 2.15). For every \( \delta > 0 \) and every bounded subset \( B \) of \( E \) there exists \( t_0 \) such that for every \( t \geq t_0 \) one has
\[
P\{\varphi(t, \varnothing)(B) \subset K^\delta(\omega)\} \geq 1 - \frac{\varepsilon}{2}.
\]

Therefore, for every \( t \geq t_0 \)
\[
P\{\varphi(t, \omega)(B) \notin C_\varepsilon^\delta\} = P\{\varphi(t, \varnothing)(B) \notin C_\varepsilon^\delta\}
\leq P\{\varphi(t, \varnothing)(B) \notin K^\delta(\omega)\} + P\{K(\omega) \notin C_\varepsilon\}
\leq \varepsilon,
\]
which proves (ii).

Finally, let us show that (ii) implies (i). Fix a point \( x_0 \in E \) and let \( B_k, k \in \mathbb{N} \), be closed balls in \( E \) with center \( x_0 \) and radii increasing to infinity, such that \( C^1_{2^{-k}} \subset B_k, k \in \mathbb{N} \). Define a sequence of numbers \( u_n > n, n \in \mathbb{N} \), recursively by
\[
P\{\varphi(\sum_{k=1}^n u_k, \omega)(B_n) \subset C^1_{2^{-m}}\} \geq 1 - 2^{-m}, m = 1, \ldots, n,
\]
(3)
and

\[ P\{\varphi(u - n, \omega)(B_n) \subset B_{n-1}\} \geq 1 - 2^{-n+1} \text{ for all } u \geq u_n, \ n \geq 2. \quad (4) \]

Define \( t_n = \sum_{i=1}^n u_i, \ n \in \mathbb{N} \), and put

\[ A(\omega) = \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} \varphi(t_k, \vartheta_{-t_k} \omega)(B_k). \]

Then (4) implies

\[ \sum_{k=2}^{\infty} P\{\varphi(u_k, \omega)(B_k) \not\subset B_{k-1}\} < \infty, \]

so that the first Borel-Cantelli lemma yields existence of a set \( \Omega_0 \) of full measure and a positive integer \( j_0(\omega) \) such that

\[ \varphi(u_j, \vartheta_{-t_j} \omega)(B_j) \subset B_{j-1} \quad (5) \]

for every \( j \geq j_0(\omega), \ \omega \in \Omega_0 \). Here we also have made use of the \( \vartheta_t \)-invariance of \( P \) for every fixed \( t \in \mathbb{R} \). In particular, using the cocycle property, we have

\[ A(\omega) = \bigcap_{j=j_0(\omega)}^{\infty} \varphi(t_j, \vartheta_{-t_j} \omega)(B_j) \quad \text{on } \Omega_0. \]

We claim that \( A \) is a weak \( B \)-attractor.

**Step 1** \( A \) is almost surely compact.

Fix \( m \in \mathbb{N} \). Using (3) we get, for \( n \geq m \),

\[ P\{A(\omega) \subset C_{2^{-m}}^{1/n}\} \geq P\{\varphi(t_n, \vartheta_{-t_n} \omega)(B_n) \subset C_{2^{-m}}^{1/n}\} - P\{j_0(\omega) > n\} \geq 1 - 2^{-m} - P\{j_0(\omega) > n\} \rightarrow 1 - 2^{-m} \text{ for } n \to \infty, \]

hence \( P\{A(\omega) \subset C_{2^{-m}}\} \geq 1 - 2^{-m} \) for every \( m \in \mathbb{N} \), so that \( A \) is almost surely compact.

**Step 2** \( A \) is strictly invariant.

Fix \( t > 0 \). We first establish \( \varphi(t, \omega)A(\omega) \subset A(\vartheta_t \omega) \) almost surely.

Fix \( \varepsilon > 0 \) and \( \delta > 0 \) and choose \( n \) so large that

(i) \( P\{\varphi(t_n, \vartheta_{-t_n} \omega)(B_n) \subset A^t(\omega)\} \geq 1 - \frac{\varepsilon}{3}, \)

(ii) \( P\{j_0(\omega) > n + 1\} \leq \frac{\varepsilon}{3}, \)

(iii) \( 2^{-n} \leq \frac{\varepsilon}{3}. \)
Observe that such an $n$ always exists. Then

\[ P\{\varphi(t, \omega)A(\omega) \subset A^\delta(\vartheta_{i}\omega)\} \]

\[ \geq P\{\varphi(t, \omega)\varphi(t_{n}+1, \vartheta_{-t_{n}+1}\omega)(B_{n+1}) \subset A^\delta(\vartheta_{i}\omega)\} - P\{j_{0}(\omega) > n + 1\} \]

\[ \geq P\{\varphi(t_{n}+1, \vartheta_{-t_{n}+1}\omega)(B_{n+1}) \subset \varphi(t_{n}, \vartheta_{-t_{n}}\vartheta_{t}\omega)(B_{n})\} - \frac{\varepsilon}{3} - P\{j_{0}(\omega) > n + 1\} \]

\[ \geq P\{\varphi(t_{n}+1 - t_{n}, \omega)(B_{n+1}) \subset B_{n}\} - \frac{2\varepsilon}{3} \]

\[ \geq 1 - \varepsilon, \]

where we used (i) and the cocycle property for the second inequality, (ii) and the cocycle property for the third inequality, and (iii) and (iv) for the last inequality. Since $\delta, \varepsilon > 0$ are arbitrary, this implies $\varphi(t, \omega)A(\omega) \subset A(\vartheta_{i}\omega)$ almost surely.

Next we prove $A(\vartheta_{i}\omega) \subset \varphi(t, \omega)A(\omega)$ almost surely. Fix $\varepsilon > 0$ and $\delta > 0$ and choose $i$ so large that

(i) $P\{\varphi(t_{i}, \vartheta_{-t_{i}}\omega)(B_{i}) \subset A^\delta(\omega)\} \geq 1 - \frac{\varepsilon}{3}$, and, consequently,

\[ P\{\varphi(t, \omega)\varphi(t_{i}, \vartheta_{-t_{i}}\omega)(B_{i}) \subset \varphi(t, \omega)A^\delta(\omega)\} \geq 1 - \frac{\varepsilon}{3} \quad (6) \]

(ii) $P\{j_{0}(\omega) > i + 1\} \leq \frac{\varepsilon}{3}$

(iii) $2^{-i} \leq \frac{\varepsilon}{3}$

(iv) $i \geq t - 1$.

Observe that such an $i$ always exists. Then

\[ P\{A(\vartheta_{i}\omega) \subset \varphi(t, \omega)A^\delta(\omega)\} \]

\[ \geq P\{A(\vartheta_{i}\omega) \subset \varphi(t, \omega)\varphi(t_{i}, \vartheta_{-t_{i}}\omega)(B_{i})\} - \frac{\varepsilon}{3} \]

\[ \geq P\{\varphi(t_{i+1}, \vartheta_{-t_{i+1}}\omega)(B_{i+1}) \subset \varphi(t, \omega)\varphi(t_{i}, \vartheta_{-t_{i}}\omega)(B_{i})\} - P\{j_{0}(\omega) > i + 1\} - \frac{\varepsilon}{3} \]

\[ \geq P\{\varphi(t_{i+1}, \vartheta_{-t_{i+1}}\omega)(B_{i+1}) \subset \varphi(t, \omega)\varphi(t_{i}, \vartheta_{-t_{i}}\omega)(B_{i})\} - P\{j_{0}(\omega) > i + 1\} - \frac{\varepsilon}{3} \]

\[ \geq P\{\varphi(t_{i+1} - t_{i}, \vartheta_{-t_{i+1}}\omega)(B_{i+1}) \subset B_{i}\} - P\{j_{0}(\omega) > i + 1\} - \frac{\varepsilon}{3} \]

\[ \geq 1 - \varepsilon, \]

where we used (ii) for the first inequality, the cocycle property for the fourth inequality, and (ii), (iii), (iv) as well as (iii) for the final inequality. Since $\varepsilon > 0$ is arbitrary, we get $A(\vartheta_{i}\omega) \subset \varphi(t, \omega)A^\delta(\omega)$ almost surely for every $\delta > 0$, which, by virtue of Lemma 4.1 (ii), implies $A(\vartheta_{i}\omega) \subset \varphi(t, \omega)A(\omega)$ almost surely.

**Step 3**  $A$ attracts every bounded $B \subset E$ in probability.

Let $B$ be a bounded subset of $E$, and fix $\delta, \varepsilon > 0$. Then there exists $j$ such that
\[ 2^{-j} \leq \varepsilon/2, \quad B \subset B_{j+1}, \quad \text{and} \quad P\{\varphi(t_j, \vartheta_{-t_j})B_j \subset A^\delta(\omega)\} \geq 1 - \varepsilon/2. \]

Then, for every \( t \geq t_{j+1} \),

\begin{align*}
P\{\varphi(t, \vartheta_{-t})B \subset A^\delta(\omega)\} &\geq P\{\varphi(t_j, \vartheta_{-t_j})B_j \subset A^\delta(\omega)\} - P\{\varphi(t, \vartheta_{-t})(B_{j+1}) \not\subset \varphi(t_j, \vartheta_{-t_j})B_j\} \\
&\geq 1 - \varepsilon,
\end{align*}

where we used (1) for the final inequality. \( \square \)

In order to obtain a corresponding result for weak \( C \)-attractors we will make use of a technical lemma.

**4.3 Lemma** Let \( \varphi : [0, \infty) \times \Omega \times E \to E \) be as before. Let \( C \subset E \) be compact and \( B \subset E \) closed. Let \( \alpha > 0 \). Then the map

\[ t \mapsto \gamma(t) := \sup \{ \eta > 0 : P\{\varphi(t, \omega)C^{\eta} \subset B\} \geq \alpha \} \]

is measurable.

**Proof** First note that for every \( r \in [0, \infty) \) the set \( \{(t, \omega) \in [0, \infty) \times \Omega : \varphi(t, \omega)C^{r} \subset B\} \) is measurable due to measurability of \( (t, \omega) \mapsto \varphi(t, \omega)x \) for every \( x \in E \) together with separability of the metrizable \( E \), which implies separability of every subset of \( E \). Put

\[ V = \bigcup_{q \in \mathbb{Q}^+} \left( \{(t, \omega) : \varphi(t, \omega)C^{\eta} \subset B\} \times [0, q] \right), \]

then \( V \) is a measurable subset of \( [0, \infty) \times \Omega \times [0, \infty) \). Therefore \( (t, \eta) \mapsto f(t, \eta) = \int 1_V(t, \omega, \eta) \, dP(\omega) \) is measurable, and so is \( W = \{(t, \eta) : f(t, \eta) \geq \alpha\} \subset [0, \infty) \times [0, \infty) \) and, consequently, also \( t \mapsto \gamma(t) = \int 1_W(t, \eta) \, d\eta. \) \( \square \)

**4.4 Theorem** The following are equivalent:

(i) \( \varphi \) has a weak \( C \)-attractor.

(ii) For every \( \varepsilon > 0 \) there exists a compact subset \( C_{\varepsilon} \) such that for each \( \delta > 0 \) and each compact subset \( C \) of \( E \) there is a \( t_0 > 0 \) with the property that for all \( t \geq t_0 \)

\[ P\{\varphi(t, \omega)(C) \subset C_{\varepsilon}^{\delta}\} \geq 1 - \varepsilon. \]

(iii) There exists a compact weakly \( C \)-attracting set \( \omega \mapsto K(\omega) \).

**Proof** Obviously (i) implies (iii), and (iii) implies (ii) by exactly the same argument as in the proof of Theorem 4.2.

In order to prove that (ii) implies (i) we will follow the proof of Theorem 4.2 as closely as possible. Lemma 4.11 and Lemma 4.13 imply that for every strictly positive sequence \( \delta_n, \quad n \in \mathbb{N} \), there exist strictly positive sequences \( \gamma_n \), and \( u_n > n \) together with measurable
subsets $U_n$ of $[u_n/2, 2u_n]$, $n \in \mathbb{N}$, such that $u_n \in U_n$ and $U_n$ has Lebesgue measure at least $\frac{3}{2}u_n - \delta_n$ for all $n \in \mathbb{N}$, and such that the sets $B_n := C_{2^{-n}}^n$, $n \in \mathbb{N}$, satisfy

$$P\{\varphi(\sum_{i=1}^n u_i, \omega)(B_n) \subset C_{2^{-m}}^1\} \geq 1 - 2^{-m+1}, \ m = 1, \ldots, n, \quad (7)$$

and

$$P\{\varphi(u, \omega)(B_n) \subset B_{n-1}\} \geq 1 - 2^{-n+2} \text{ for all } u \in U_n, \ n \geq 2. \quad (8)$$

Now choose a summable sequence $(\delta_n)$, define $t_n = \sum_{i=1}^n u_i$, $n \in \mathbb{N}$, and put

$$A(\omega) = \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} \varphi(t_k, \vartheta_{-t_k}\omega)(B_k).$$

Defining $\Omega_0$ and $\omega \mapsto j_0(\omega)$ as in the proof of Theorem 4.2 it follows that

$$A(\omega) = \bigcap_{j=j_0(\omega)} \bigcup_{n \geq 1} \varphi(t_j, \vartheta_{-t_j}\omega)(B_j) \quad \text{on } \Omega_0. \quad (9)$$

We claim that $A$ is a weak $C$-attractor.

Almost sure compactness of $A$ follows like in the proof of Theorem 4.2.

To show strict invariance we first replace both conditions (iii) in the proof of Step 2 of Theorem 4.2 by multiplying the left hand side by 2 (this is due to the factor 2 in (8) compared to (4)). Then all estimates follow as before except that we require that $t + u_{n+1} \in U_{n+1}$ respectively $u_{i+1} - t \in U_{i+1}$. Due to the definition of the sets $U_n$ and the summability of the sequence $(\delta_n)$ it follows that for Lebesgue almost all $t > 0$ we have $t + u_n \in U_n$ and $u_n - t \in U_n$ for all but finitely many $n$. Therefore we see that the set

$$T := \{t \geq 0 : \varphi(t, \omega)A(\omega) = A(\vartheta_t\omega) \text{ almost surely}\}$$

has full Lebesgue measure (and contains 0). Further the cocycle property shows that $T$ is closed under addition, from which we conclude that $T = [0, \infty)$.

It remains to prove that $A$ attracts compact sets.

For $\delta, \epsilon > 0$ choose $k_0 \in \mathbb{N}$ with $2^{k_0} > 2/\epsilon$ such that for all $k \geq k_0$

$$P\{d(\varphi(t_k, \vartheta_{-t_k}\omega)(B_k), A(\omega)) > \delta\} < \frac{\epsilon}{2}, \quad (10)$$

this is possible due to (9). Let $C \subset E$ be compact. Since $B_k = C_{2^{-k}}^n$, condition (ii) yields existence of a time $t_C > 0$ such that for all $t \geq t_C$

$$P\{\varphi(t, \omega)(C) \subset B_k\} \geq 1 - \frac{\epsilon}{2}.$$

Using the cocycle property we get for all $t \geq t_C$

$$1 - \frac{\epsilon}{2} \leq P\{\varphi(t, \vartheta_{-t_k}\omega)(C) \subset B_k\} \leq P\{\varphi(t_k, \vartheta_{-t_k}\omega)\varphi(t, \vartheta_{-t_k}-t\omega)(C) \subset \varphi(t_k, \vartheta_{-t_k}\omega)(B_k)\} = P\{\varphi(t + t_k, \vartheta_{-t_k}-t\omega)(C) \subset \varphi(t_k, \vartheta_{-t_k}\omega)(B_k)\}.$$
Together with (10) this implies, for all $t \geq t_C + t_k$, 

$$P\left(d\left(\varphi(t, \vartheta - t\omega)(C), A(\omega)\right) > \delta\right) < \varepsilon.$$ 

This holding true for every $\delta > 0$ and $\varepsilon > 0$ we conclude that $d\left(\varphi(t, \vartheta - t\omega)(C), A(\omega)\right)$ converges to zero in probability.

Finally, we give a necessary and sufficient condition for a weak attractor to be also a strong attractor. Let $\mathcal{B}$ be an arbitrary family of deterministic subsets of $E$.

4.5 Proposition Suppose that $A$ is a weak $\mathcal{B}$-attractor. Then the following are equivalent:

(i) $A$ is a strong $\mathcal{B}$-attractor.

(ii) For every $B \in \mathcal{B}$ it holds that $\Omega_B \subset A$ almost surely.

Proof (i) $\Rightarrow$ (ii): obvious.

(ii) $\Rightarrow$ (i): let $B \in \mathcal{B}$. Due to (ii), $A$ attracts $B$ strongly. Since $A$ is a weak attractor by assumption, (i) follows.

Acknowledgement

Part of this work was done during a stay of HC and MS at the Institut Mittag-Leffler, which is gratefully acknowledged.

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