Projective lag synchronization of different fractional order chaotic systems based on parameter identification

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Abstract. Projective lag synchronization (LPS) of two different structural fractional order hyperchaotic systems with unknown parameters is investigated. The stability theorem of fractional order linear systems is the basis of designing a scheme of LPS and parameter updating laws. Finally, the method is successfully applied to realize LPS and parameter identification between fractional order hyperchaotic system and fractional order hyperchaotic system, and numerical simulations illustrate the effectiveness of the obtained results. Key words: fractional derivative; hyperchaotic system; projective lag synchronization; parameter identification; nonlinear control

1. Introduction

Fractional calculus has been in history for more than 300 years [1]. As a branch of calculus, fractional calculus is a generalization of ordinary integer differential and integral to any real number order. Because the fractional calculus cannot find the practical application background and physical significance for a long time, the development of fractional calculus is slow. Until Mandelbort put forward the fractal theory in 1983, as the foundation of fractal geometry and fractal dynamics, the application research of fractional calculus has attracted worldwide attention and has gradually become the research hotspot in the world. With the development of fractional calculus, fractional chaotic systems and their control and synchronization have become a new and important research direction in nonlinear disciplines. Chaos is not only widely applied in physical, biochemistry, geology, social science and other theoretical research, but also in practical applications such as medical network, financial management, automation control, etc. Chaos control and synchronization are mainly used in the field of information science, such as communication security biomedical electronics [2].

In recent years various kinds of chaos synchronization of chaotic systems have been proposed, such as complete synchronization, phase synchronization, lag synchronization [3] projective synchronization [4] generalized synchronization, and function projective synchronization [5] etc, using different control scheme such as linear and nonlinear feedback synchronization, adaptive control [6], active control [7], sliding mode control etc. In 1996, projective synchronization phenomenon first discussed by Gonzalez-Miranda in 1999 the concept of projective was first proposed [8]. But most of all these results about the integer-order chaotic system. In recent years, projective synchronization of fractional order chaotic systems has attracted increasing attention due to it potential applications in secure communication and control processing. [9-12] Considering the time delay is inevitable factors in the nonlinear dynamic system, which makes projective lag synchronization [13, 14] more realistic.
And at the same time the complexity of the coefficient matrix of projective synchronization and the identification of unknown parameters also increases the security of secret communication. Motivated by the above discussion, the projective lag synchronization of fractional order chaotic systems is investigated in this paper. According to the stability theorem of the linear fractional order systems, a nonlinear controller is given for the synchronization of the different structural fractional order hyperchaotic systems.

The paper is organized as follows. In Section 2, some problem descriptions are given and projective lag synchronization by nonlinear control method is developed and nonlinear controller and updates rule of unknown parameter are proposed. Numerical example is used to demonstrate the effectiveness of proposed schemes in Section 3. Section 4 gives the conclusions of this paper.

2. Problem descriptions
There are four major definitions of fractional derivative. One of the most commonly used definitions is the Caputo fractional derivative which is described by

\[ C_0D_t^q x(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{q+1}} d\tau, \]

where \( \Gamma(\bullet) \) is the gamma function

\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \]

The following drive system of fractional order system is considered:

\[ D_t^q x(t) = F(x(t)) + M(x(t))\theta, \]

where \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n \) is an \( n \)-dimensional state vector of the drive system, \( q \in (0,1) \) is the order of the fractional differential equation, \( F: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuous vector function and \( M(x(t)) \in \mathbb{R}^{nd}\) is a polynomial matrix, \( \theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{R}^d \) is an unknown vector of parameter.

Choose a fractional order response system as:

\[ D_t^q y(t) = G(y(t)) + N(y(t))\delta + U, \]

where \( y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T \in \mathbb{R}^n \) is an \( n \)-dimensional state vector of the response system, \( G: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuous vector function and \( N(y(t)) \in \mathbb{R}^{nd}\) is a polynomial matrix, \( \delta = (\delta_1, \delta_2, \ldots, \delta_n) \in \mathbb{R}^d \) is an unknown vector of parameter, and \( U \) is a controller to be designed.

Definition 1. [13] The projective lag synchronization error state vector between system (3) and response system (4) is defined as

\[ e(t) = y(t) - Cx(t - \tau), \]

where \( e(t) = (e_1(t), e_2(t), \ldots, e_n(t))^T \in \mathbb{R}^n \) and \( C = \text{diag} \{ c_1, c_2, \ldots, c_n \} \in \mathbb{R}^{m\times m} \) is a real scaling matrix. Then, \( \tau > 0 \) denotes the time—delay. \( \hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n)^T, \hat{\delta} = (\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_n)^T \) is the estimated parameter vector for unknown parameters. Then

\[ e_{\bar{\theta}} = \hat{\theta}_i - \theta_i, i = 1, 2, \ldots, m; e_{\bar{\delta}} = \hat{\delta}_j - \delta_j, j = 1, 2, \ldots, n. \]

is defined as the error state vector of unknown parameters.

Definition 2. [13] For the fractional order drive system (3) and response system (4), it is said to be projective lag synchronization if there exists a controller \( U \) such that

\[ \lim_{t \to \infty} \|e(t)\| = \lim_{t \to \infty} \|y(t) - Cx(t - \tau)\| = 0 \]

Remark 1. [2] If the scaling matrix \( C = \text{I} \) or \( C = -\text{I} \), the synchronization mentioned above is respectively simplified to complete synchronization or anti—phase synchronization.
Remark 2. [2] If the time—delay \( \tau = 0 \), the synchronization for fractional order chaotic systems is reduced to the projective synchronization of fractional order chaotic systems.

With the parameters above, a nonlinear controller is assumed as

\[
U = Ke(t) + CF(x(t - \tau)) - CM (x(t - \tau)) \dot{\delta} - N(y(t)) \dot{\delta},
\]

(8)

where \( K = \text{diag} \{ k_1, k_2, L, k_n \} \), \( K \in \mathbb{R}^{n \times n} \) is a feedback gain matrix to be determined later.

Combining (3) and (4) with (8), the error system is expressed as

\[
D^i e(t) = Ke(t) + CM (x(t - \tau)) e_g - N(y(t)) e_g,
\]

(9)

And the updates rules for unknown parameters are described as follows:

\[
D^i e_g = -[CM (x(t - \tau))]^T e(t),
\]

\[
D^i e_g = N^T (y(t)) e(t).
\]

(10)

Theorem 1 [2] The response system (4) and the drive system (3) for any initial conditions \( x(0) = (x_1(0), x_2(0), L, x_n(0)) \) and \( y(0) = (y_1(0), y_2(0), L, y_n(0)) \) can realize the projective lag synchronization and parameter identification by the control law (8) and the updates law (10).

It is easy to see that error function system can be obtained as follows:

\[
[D^ie(t), D^i e_g, D^i e_g]^T = A(x(t - \tau), y(t))[e(t), e_g, e_g]^T.
\]

(11)

For the fractional order error system (9), the coefficient matrix \( A(x(t - \tau), y(t)) = \begin{bmatrix} a_{ij}(x(t - \tau), y(t)) \end{bmatrix} \) is a polynomial matrix. The eigenvalues of \( A \) are \( \lambda_i, i = 1, 2, L, n \). Then, the projective lag synchronization between systems (3) and (4) is transformed into the discussion of the asymptotical stability of the zero solution of system (9). Then, according to proposition 1, a sufficient condition for projective lag synchronization between the systems (3) and (4) is gained as follows.

Theorem 2. [2] If all eigenvalues \( \lambda \) of the coefficient matrix \( A(x(t - \tau), y(t)) \) of fractional order linear autonomous error system (9) are satisfied \( \left| \arg(\lambda) \right| > \pi q / 2 \), then the zero solution of the fractional order linear autonomous error system is asymptotically stable, namely \( \lim_{t \to \infty} \| e(t) \| = 0 \).

Proposition 1. [13] For a given scaling matrix \( K = \text{diag} \{ k_1, k_2, L, k_n \} \), \( K \in \mathbb{R}^{n \times n} \) and a time delay \( \tau > 0 \), projective lag synchronization can be achieved if there exists a matrix \( K = \text{diag} \{ k_1, k_2, L, k_n \} \), \( K \in \mathbb{R}^{n \times n} \) and \( k_i < 1 / \sin(q\pi / 2), q \in (0,1) \) in controller.

Proof. Assume \( \lambda \) is an arbitrary eigenvalue of the matrix \( A(x(t - \tau), y(t)) \) and the corresponding nonzero eigenvector is \( \eta \), then

\[
A(x(t - \tau), y(t))\eta = \lambda \eta
\]

(12)

Let’s multiply both sides of the above equation by \( \eta^T \) we obtain

\[
\eta^T A(x(t - \tau), y(t))\eta = \lambda \eta^T \eta
\]

(13)

where \( H \) stands for conjugate transpose of a matrix.

\( \lambda \) is also eigenvalue of the matrix, \( A(x(t - \tau), y(t)) \) namely

\[
\eta^T [A(x(t - \tau), y(t))]^T = \lambda \eta^T
\]

(14)

Let’s multiply both sides of the above equation by \( \eta \), we obtain

\[
\eta^T [A(x(t - \tau), y(t))]^T \eta = \lambda \eta^T \eta
\]

(15)

We know from equations (13) and (15) that the eigenvalue \( \lambda \) satisfies

\[
\lambda + \lambda = \eta^T [A(x(t - \tau), y(t)) + A^T (x(t - \tau), y(t))] \eta / \eta^T \eta
\]

\[
= \eta^T Q \eta / \eta^T \eta
\]

(16)
Where $\eta t \eta > 0$, $Q = A(x(t - \tau), y(t)) + A'(x(t - \tau), y(t))$, since $K = \text{diag} \{k_1, k_2, k_3, k_4\} \in \mathbb{R}^{n\times n}$ and the polynomial matrix $A(x(t - \tau), y(t))$ is subject to $a_{ij} = -a_{ji}, i = j$ and $a_{ij} \in \mathbb{R}^n, i \neq j$ we can derive that $Q$ is a positive--definite diagonal matrix. Thus $\eta t Q \eta > 0$ Then $\lambda + \overline{\lambda} = 2\Re(\lambda) > 0$. Hence, for the fractional order error system (9), due to the given condition $k < -(\sin(q\pi / 2)), q \in (0, 1)$ Therefore, all the eigenvalues $\lambda$ of $A$ satisfy

$$\arg(\lambda) > \pi / 2 > q\pi / 2.\quad (17)$$

According the stability theorem of the linear fractional order autonomous systems, the error system is asymptotically stable at zero point. That is \(\lim_{t \to \infty} \|\epsilon(t)\| = 0\), which implies the two different structural fractional order chaotic systems (3) and (4) realize the projective lag synchronization. This completes the proof.

3. Numerical simulations

Numerical examples of projective lag synchronization for two different structural fractional order chaotic systems are performed. And a predictor—corrector scheme is used for the approximate numerical solutions of the fractional order differential equations.

It is assumed that a fractional order Lorenz—Stenflo hyperchaotic system (18) drives a fractional order Chen hyperchaotic system (20). The drive system is described by

$$D^q x_1 = \theta_1(x_2(t) - x_1(t)) + \theta_3 x_4(t)$$
$$D^q x_2 = x_1(t)(\theta_3 - x_1(t)) - x_2(t)$$
$$D^q x_3 = x_1(t)x_2(t) - \theta_1 x_4(t)$$
$$D^q x_4 = -x_1(t) - \theta_1 x_4(t).$$

(18)

Where $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T$ is the state vector, $q \in (0, 1)$ is the order of fractional differential equation, $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ are the real positive parameters. For $q = 0.98, \theta = (1.0, 7, 26, 1.5)$ and $x(0) = (0.1, 0.2, 0.2, 0.2)^T$, the chaotic attractor of system (18) is shown in figure 1.

The drive system can be rewritten as equation (3), where

$$F(x(t)) = \begin{pmatrix} 0 & x_1 x_3 - x_2 \\ x_2 x_3 & -x_1 \end{pmatrix}, \quad M(x(t)) = \begin{pmatrix} x_2 - x_1 & 0 & 0 & x_4 \\ 0 & 0 & x_4 & 0 \\ 0 & -x_3 & 0 & 0 \\ -x_4 & 0 & 0 & 0 \end{pmatrix}.$$ \quad (19)

The response system is given as

$$D^q y_1(t) = \delta_1(y_2(t) - y_1(t)) + y_4(t) + u_1$$
$$D^q y_2(t) = \delta_2 y_1(t) - y_1(t)y_3(t) + \delta_4 y_2(t) + u_2$$
$$D^q y_3(t) = y_1(t)y_2(t) - \delta_1 y_3(t) + u_3$$
$$D^q y_4(t) = y_2(t)y_3(t) + \delta_3 y_4(t) + u_4.$$ 

(20)

Where $y(t) = (y_1(t), y_2(t), y_3(t), y_4(t))^T$ is the state vector, $\delta = (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5)$ are the real positive parameters, $U = (u_1, u_2, u_3, u_4)^T$ is the controller to be designed later. For $q = 0.98$, $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) = (35.7, 12, 3, 0.3)$ and $y(0) = (1.2, 2.1, 3.1, 0.1)^T$, the chaotic attractor of system (20) is displayed in figure 2.

The response system can be shown in the form of equation (4), where


\[ G(y(t)) = \begin{pmatrix}
 y_4 \\
 y_1 y_3 \\
 y_2 y_3
\end{pmatrix}, \quad N(y(t)) = \begin{pmatrix}
 y_2 - y_i & 0 & 0 & 0 & 0 \\
 0 & y_1 & y_2 & 0 & 0 \\
 0 & 0 & 0 & -y_3 & 0 \\
 0 & 0 & 0 & 0 & y_4
\end{pmatrix} \]

Due to the propose controller (7), let the control law as follows

\[ u_1 = k_1 e_1 - y_4 + c_1 (x_2 - x_4) \dot{\theta}_1 + c_1 x_4 \dot{\theta}_4 - (y_2 - y_i) \dot{\delta}_i \]
\[ u_2 = k_2 e_2 - c_2 x_1 x_3 - c_2 x_2 + y_1 y_3 + c_2 x_4 \dot{\theta}_3 - y_1 \dot{\delta}_2 - y_2 \dot{\delta}_3 \]
\[ u_3 = k_3 e_3 + c_3 x_1 x_2 - y_1 y_2 - c_3 x_4 \dot{\theta}_2 + y_2 \dot{\delta}_4 \]
\[ u_4 = k_4 e_4 - c_4 x_1 - y_3 y_4 - c_4 x_4 \dot{\theta}_4 - y_4 \dot{\delta}_3 \]

The updated rules for unknown parameters are described as follows:

\[ D^q e_{\theta_1} = c_1 (x_1 - x_2) e_1 + c_4 x_4 e_4 \]
\[ D^q e_{\theta_2} = -c_2 x_3 e_3 \]
\[ D^q e_{\theta_3} = -c_3 x_4 e_4 \]
\[ D^q e_{\theta_4} = (y_2 - y_1) e_1 \]
\[ D^q e_{\theta_5} = y_1 e_2 \]
\[ D^q e_{\theta_6} = y_2 e_2 \]
\[ D^q e_{\theta_7} = -y_3 e_3 \]
\[ D^q e_{\theta_8} = y_3 e_4 \]

The error system is shown as

\[ D^q e_i(t) = k_i e_i + c_i (x_2 - x_i) e_{\theta_i} + c_i x_4 e_{\dot{\theta}_4} - (y_2 - y_i) e_{\delta_i} \]
\[ D^q e_2(t) = k_2 e_2 + c_2 x_1 e_{\theta_1} - y_1 e_{\delta_1} - y_2 e_{\delta_2} \]
\[ D^q e_3(t) = k_3 e_3 - c_3 x_3 e_{\theta_3} + y_3 e_{\delta_3} \]
\[ D^q e_4(t) = k_4 e_4 + c_4 x_4 e_{\theta_4} - y_4 e_{\delta_4} \]

The initial conditions for the projective lag synchronization are set to \( x(0) = (1.2, 2.1, 3.1, 0.1)^T \), \( y(0) = (1.2, 2.1, 3.1, 0.1)^T \), \( \tau = 0.01 \) and \( \dot{\theta} = (1, 0, 7, 26, 1, 5)^T \), \( \dot{\delta} = (35, 7, 12, 3, 0.5)^T \) are real value of unknown parameters. The fractional derivatives are taken as \( q = 0.98 \), the drive system (18) and response systems (20) are chaotic. According to Proposition 1, projective lag synchronization between systems (18) and (20) can be realized with \( K = \text{diag} [-6, -5, -5.9, -6] \) and \( C = \text{diag} [2, 2, 2, 2] \). The corresponding error state curves are shown in figure 3, which also indicate that projective lag synchronization between system (18) and (20) is successfully achieved. The unknown parameters identification of drive system (18) and response system (20) are shown in figure 4 and figure 5, with the estimate of parameters \( \dot{\theta} = (3, 5, 3, 2)^T \) and \( \dot{\delta} = (5, 12, 7, 9, 10)^T \), which indicated that unknown parameters are identified successfully.
Figure 1. The chaotic attractor of fractional order hyperchaotic systems (18) with $q=0.98, t=0.01$, and $x(0) = (0.1, 0.2, 0.2, -0.2)^T$.

Figure 2. The chaotic attractor of fractional order hyperchaotic systems (20) with $q=0.98, t=0.01$, and $y(0) = (1.2, 2.1, 3.1, 0.1)^T$.

Figure 3. State trajectories of the projective lag synchronization error system $e_i (i=1,2,3,4)$ for the drive system (18) & response system (20) for case at $q_i = 0.98, i=1,2,3,4$. $e(0) = (3, 9, 7, 2)^T$.

Figure 4. State trajectories of the identification of unknown parameter vectors $e_{q_i} (t), (i=1,2,3,4)$ of drive system (18) for case at $q_i = 0.98, i=1,2,3,4$. $\hat{\Theta} = (3, 5, 3, 2)^T$. 
Figure 5. State trajectories of the identification of unknown parameter vectors $\hat{\epsilon}_i(t), (i=1, 2, 3, 4, 5)$ of repose system (20) for case at $q_i = 0.98, i = 1, 2, 3, 4, 5$.

$\hat{\delta} = (5, 12, 7, 9, 10)^r$.

4. Conclusions

The projective lag synchronization of fractional order chaotic systems is investigated based on the stability theorem of linear fractional order systems. A nonlinear controller and parameter updates ruler are provided for the projective lag synchronization of different structural fractional order chaotic systems and identification of unknown parameters of drive and response system. The proposed scheme is simple and theoretically rigorous. Finally the synchronization speed can be improved by selecting an appropriate matrix K.

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