On the solution of the Volterra integral equation of second type for the error term in an asymptotic formula for arithmetical functions

by

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Abstract. In 2010, J. Kaczorowski and K. Wiertelak considered the Volterra integral equation of second type for the remainder term in the asymptotic formula for the Euler totient function. The author found that the consideration made by them holds for other remainder terms in the asymptotic formula having certain common properties. In this paper, we first consider the pair of complex-valued arithmetical functions \((a(n), b(n))\) satisfying \(b(n) = \sum_{d|n} a(d) n/d\). We prove that the solution of the Volterra integral equation of second type for the error term in the asymptotic formula for \(b(n)\) can be obtained when \(a(n)\) satisfies some special condition.

1. Introduction and the statement of the main result

Let \(\varphi(n)\) denote the Euler totient function and let

\[
E(x) = \sum_{n \leq x} \varphi(n) - \frac{3}{\pi^2} x^2
\]

be the associated error term. J. Kaczorowski and K. Wiertelak studied the following Volterra integral equation of second type for \(E(x)\) (see [2]):

\[
F(x) - \int_0^x K(x, t) F(t) dt = E(x) \quad (x \geq 1),
\]

where \(F(x)\) is the unknown function and the kernel \(K(x, t)\) is defined as follows:

\[
K(x, t) = \begin{cases} 
1/t & (0 < t \leq x), \\
0 & (1 \leq x < t). 
\end{cases}
\]

The equation (1.2) can be solved explicitly. Let us put

\[
f(x) = -\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left\{ x \right\} 
\]

for every \(x \geq 0\), where \(\mu(n)\) denotes the Möbius function and \(\{ x \} = x - [x]\) is the fractional part of a real number \(x\). Then the general solution of (1.2) is

\[
F(x) = (f(x) + A)x,
\]

where \(A\) is an arbitrary constant. (In the paper [2], \(F(x) = xf(x)\) is claimed to be the unique solution of the integral equation (1.2), but this uniqueness
does not hold even assuming the initial value condition at \( x = 0 \). Probably, the term \( Ax \) is missing to give the general solution.)

In the present article we generalize the observation of J.Kaczorowski and K.Wiertelak by slightly changing the settings, and propose the following conjecture.

**Conjecture.** Let \( \{a(n)\} \) be a complex-valued arithmetical function satisfying some suitable conditions, and \( \{b(n)\} \) be the arithmetical function given by

\[
b(n) = \sum_{d|n} a(d) \frac{n}{d}.
\]

Assume for \( x \) tending to infinity

\[
\sum_{n \leq x} b(n) = M(x) + \text{Er}(x),
\]

where

\[
M(x) := \alpha x^2 \quad (\alpha \text{ is a certain complex number}), \quad \text{Er}(x) := \sum_{n \leq x} b(n) - M(x).
\]

Now, we consider the following Volterra integral equation of second type

\[
F_1(x) - \int_0^x F_1(t) \frac{dt}{t} = \text{Er}(x) \quad (x \geq 0).
\]

Then, for every complex number \( A \), the function

\[
F_1(x) = (f_1(x) + A)x \quad (x \geq 0),
\]

is a solution of the integral equation and these exhaust all solutions. Here,

\[
f_1(x) = -\sum_{n=1}^{\infty} \frac{a(n)}{n} \left\{ \frac{x}{n} \right\}
\]

for every \( x \geq 0 \).

In the above previous research by J.Kaczorowski and K.Wiertelak, the arithmetical functions \( a(n) \) and \( b(n) \) in this conjecture are \( \mu(n) \) and \( \varphi(n) \) respectively, and all the hypothesis are satisfied. As for the error term \( \text{Er}(x) \), we have a bound similar to \( \text{Er}(x) = o(x^2) \) as \( x \) tends to infinity in mind. Assuming certain appropriate hypothesis, in the present paper we prove this conjecture under a certain additional condition.

**Theorem.** Let \( \{a(n)\} \) be a complex-valued arithmetical function for which the series

\[
\sum_{n=1}^{\infty} \frac{a(n)}{n^2}
\]

is convergent with the sum \( 2\alpha \), where \( \alpha \) is an arbitrary complex number. Let \( \{b(n)\} \) be the arithmetical function defined by

\[
b(n) = \sum_{d|n} a(d) \frac{n}{d}.
\]

Assume for \( x \) tending to infinity

\[
\sum_{n \leq x} b(n) = M(x) + \text{Er}(x),
\]
where
\begin{align}
M(x) &:= \alpha x^2, \\
\text{Er}(x) &:= \sum_{n \leq x} b(n) - M(x).
\end{align}

Now, we consider the following Volterra integral equation of second type
\begin{equation}
F_1(x) - \int_0^x \frac{F_1(t) \, dt}{t} = \text{Er}(x) \quad (x \geq 0).
\end{equation}
Then, for every complex number $A$, the function
\begin{equation}
F_1(x) = (f_1(x) + A)x \quad (x \geq 0),
\end{equation}
is a solution of the integral equation (1.10) and these exhaust all solutions of (1.10). Here,
\begin{equation}
f_1(x) = -\sum_{n=1}^{\infty} \frac{a(n)}{n} \left\{ \frac{x}{n} \right\}
\end{equation}
for every $x \geq 0$.

As usual, if we say a function $F_1$ is a solution of (1.10), then we implicitly assume that the integral in (1.10) exists in the sense that the limit
\begin{equation}
\lim_{\epsilon \to 0^+} \int_{\epsilon}^{x} \frac{F_1(t) \, dt}{t}
\end{equation}
exists. We use the same convention throughout this paper. The formula (1.11) is a generalization of the result of Kaczorowski and Wiertelak [2], except that the term $Ax$ is missing. Also, the function $f_1(x)$ is locally bounded. In fact, by the condition of theorem
\begin{equation}
f_1(x) = -\sum_{n=1}^{\infty} \frac{a(n)}{n} \left\{ \frac{x}{n} \right\}
\end{equation}
Here, we mention a previous research related to [2]. For $x \geq 0$ let us put
\begin{equation}
g(x) = \sum_{n=1}^{\infty} \mu(n) \left\{ \frac{x}{n} \right\}^2.
\end{equation}
Then, $E(x)$ in (1.1) can be splitted as follows:
\begin{equation}
E(x) = E_{AR}(x) + E_{AN}(x),
\end{equation}
where
\begin{equation}
E_{AR}(x) = xf(x) \quad \text{and} \quad E_{AN}(x) = \frac{1}{2}g(x) + \frac{1}{2}
\end{equation}
with $f(x)$ and $g(x)$ given by (1.3) and (1.14) respectively. ($E_{AR}(x)$ is the case $A = 0$ in (1.4).) We call $E_{AR}(x)$ and $E_{AN}(x)$ the arithmetic and the analytic
part of $E(x)$ respectively. J.Kacorzowski and K.Wiertelak showed the $\Omega$-estimates for each of $E^R(x)$ and $E^A(x)$ (see [2, 4]). The decomposition (1.15) itself can obtain from certain calculations starting with

$$\sum_{n \leq x} \varphi(n) = \frac{1}{2} \sum_{d \leq x} \mu(d) \left[ \frac{x}{d} \right] \left( \left\lfloor \frac{x}{d} \right\rfloor + 1 \right).$$

2. Related considerations for the twisted Euler $\varphi$-function

J.Kaczorowski and K.Wiertelak obtained a better decomposition for the remainder term in the asymptotic formula for a generalization of the Euler totient function (see [3]) : For a non-principal real Dirichlet character $\chi \pmod{q}$, $q > 2$, let $\varphi(n, \chi)$ denote the twisted Euler $\varphi$-function

$$\varphi(n, \chi) = n \prod_{p \mid n} \left( 1 - \frac{\chi(p)}{p} \right).$$

J.Kaczorowski and K.Wiertelak made a similar consideration to [2] for the remainder term in the asymptotic formula of the above twisted Euler $\varphi$-function.

Let

$$(2.1) \quad E(x, \chi) = \sum_{n \leq x} \varphi(n, \chi) - \frac{x^2}{2L(2, \chi)},$$

and

$$(2.2) \quad E_1(x, \chi) = \begin{cases} E(x, \chi) & (x \notin \mathbb{N}), \\ \frac{1}{2}(E(x - 0, \chi) + E(x + 0, \chi)) & \text{(otherwise)} \end{cases}$$

be the corresponding error term. Here, as usual, $L(s, \chi)$ denotes the Dirichlet $L$-function associated to $\chi$. It is easy to see that $E(x, \chi) = O(x \log x)$ for $x \geq 2$.

Let $s(x)$ be the saw-tooth function

$$(2.3) \quad s(x) = \begin{cases} 0 & (x \in \mathbb{Z}) \\ \frac{1}{2} - \{x\} & \text{(otherwise)} \end{cases}$$

We write for $x \geq 0$

$$(2.4) \quad f(x, \chi) = \sum_{d=1}^{\infty} \frac{\mu(d)\chi(d)}{d} s\left( \frac{x}{d} \right),$$

$$(2.5) \quad g(x, \chi) = \sum_{d=1}^{\infty} \mu(d)\chi(d) \left( \frac{x}{d} \right) \left( \frac{x}{d} - 1 \right).$$

Then the solution of the following Volterra integral equation of second type

$$F(x, \chi) - \int_{0}^{x} K(x, t)F(t, \chi)dt = E_1(x, \chi) \quad (x \geq 0),$$

where

$$K(x, t) = \begin{cases} 1/t & (0 < t \leq x), \\ 0 & (0 \leq x < t) \end{cases},$$

is the function

$$(2.6) \quad F(x, \chi) = (f(x, \chi) + A)x,$$
where $A$ is an arbitrary constant. (In the paper [3], the unique solution is $F(x,\chi) = xf(x,\chi)$, but the comments just after (1.4) should also be applied here.)

Moreover, when $A = 0$ in (2.6), for $x \geq 0$

\begin{equation}
E_1(x,\chi) = E^{AR}(x,\chi) + E^{AN}(x,\chi),
\end{equation}

where

\begin{equation}
E^{AR}(x,\chi) = xf(x,\chi) \quad \text{and} \quad E^{AN}(x,\chi) = \frac{1}{2} g(x,\chi)
\end{equation}

with $f(x,\chi)$ and $g(x,\chi)$ given by (2.4) and (2.5) respectively.

**Remark** (1) The function (2.4) is a better solution of the Volterra integral equation (1.10) than (1.12), namely, the decomposition (2.7) coincides with the decomposition (1.15). For simplicity, let us forget about the error term (2.2) and consider $E(x,\chi)$. Let

\begin{equation}
s(x) := \frac{1}{2} - \{x\},
\end{equation}

which is not the same (2.3), but (2.3) is just the normalized version of (2.9). Then we have (after removing normalization)

\begin{equation}
f(x,\chi) := \sum_{d=1}^{\infty} \frac{\mu(d)x(d)}{d} s\left(\frac{x}{d}\right) = - \sum_{d=1}^{\infty} \frac{\mu(d)x(d)}{d} \left\{\frac{x}{d}\right\} + \frac{1}{2} \sum_{d=1}^{\infty} \frac{\mu(d)x(d)}{d}
\end{equation}

and

\begin{align*}
g(x,\chi) &= \sum_{d=1}^{\infty} \mu(d)x(d) \left\{\frac{x}{d}\right\}\left(\left\{\frac{x}{d}\right\} - 1\right) \\
&= \sum_{d=1}^{\infty} \mu(d)x(d) \left\{\frac{x}{d}\right\}^2 - \sum_{d=1}^{\infty} \mu(d)x(d) \left\{\frac{x}{d}\right\}.
\end{align*}

If $\chi$ is trivial, we have

\begin{equation}
\frac{1}{2} \sum_{d=1}^{\infty} \frac{\mu(d)x(d)}{d} = \frac{1}{2} \sum_{d=1}^{\infty} \frac{\mu(d)}{d} = 0
\end{equation}

and

\begin{align*}
- \sum_{d=1}^{\infty} \mu(d)x(d) \left\{\frac{x}{d}\right\} &= - \sum_{d=1}^{\infty} \mu(d) \left\{\frac{x}{d}\right\} \\
&= \sum_{d=1}^{\infty} \mu(d) \left[\frac{x}{d}\right] - \sum_{d=1}^{\infty} \mu(d) \cdot \frac{x}{d} \\
&= \sum_{d \leq x} \mu(d) \left[\frac{x}{d}\right] \\
&= 1.
\end{align*}

Thus, for the trivial character, we have

\begin{equation}
f(x,\chi) = f(x) \quad \text{and} \quad g(x,\chi) = g(x) + 1.
\end{equation}
However, if we use the main theorem to \( a(n) = \mu(n)\chi(n) \), then the arithmetic part becomes

\[
-x \sum_{d=1}^{\infty} \frac{\mu(d)\chi(d)}{d} \left\{ \frac{x}{d} \right\}
\]
as in (1.12) and so this differs from (2.10). This discrepancy happened, indeed, because of the missing term \( Ax \). The remaining term

\[
x \cdot \frac{1}{2} \sum_{d=1}^{\infty} \frac{\mu(d)\chi(d)}{d}
\]
of \( x \cdot f(x, \chi) \) is just of the form \( Ax \) and so such term is allowed for the general solution of the integral equation (1.10). As a summary, the function (2.4) is a better representative of the solutions of the Volterra integral equation (1.10). However, this representative is that the solution (2.4) is well-defined only when the series

\[
\sum_{d=1}^{\infty} \frac{\mu(d)\chi(d)}{d}
\]
converges.

(2) In the paper [1], the asymptotic formulas related to the divisor function \( \sigma(n) \) and the Euler totient function \( \varphi(n) \) are mentioned (see [1], p61,62). By introducing the weight \( n \) into the formula by using partial summation and using Theorem 4 of [1], for example, we get

\[
\sum_{n \leq x} n^{1-\alpha} \sigma^\alpha(n) = C x^2 + \sum_{r=0}^{[\alpha]-1} C_r x (\log x)^{\alpha-r-1} + O(x(\log x)^{\frac{2\alpha}{\alpha}} (\log \log x)^{\frac{4\alpha}{\alpha}}).
\]
b(\(n\)) = \(n^{1-\alpha} \sigma^\alpha(n)\) fits into the setting of the main theorem. Thus, we have second main terms

\[
\sum_{r=0}^{[\alpha]-1} C_r x (\log x)^{\alpha-r-1}
\]
provided \( \alpha \) is sufficiently large. These second main terms cannot be seen in the result of [2].

3. Proof of Theorem

Now we prove the theorem. We define the auxiliary function for \( x \geq 0 \) by

\[
R(x) = E r(x) - x f_1(x).
\]

First, we prepare the following two lemmas.

LEMMA 1. For all positive \( x \),

\[
R(x) = -\int_0^x f_1(t)dt.
\]

Proof. Let us observe that \( R(x) \) is a continuous function. For \( x = 0 \) and for positive \( x \) which is not an integer, it is evident. Let \( N \) be a positive integer. By splitting the series (1.12) at \( N \), and considering the limit \( \{ (N + x)/n \} \) as \( x \) tending to 0, we see that

\[
f_1(N + 0) = -\sum_{n=1}^{\infty} a(n) \left\{ \frac{N + 0}{n} \right\},
\]
On the solution of ...

\[ f_1(N - 0) = -\sum_{n=1}^{\infty} \frac{a(n)}{n} \left\{ \frac{N - 0}{n} \right\}. \]

Since
\[
\left\{ \frac{N + 0}{n} \right\} - \left\{ \frac{N - 0}{n} \right\} = \begin{cases} 0 & (n \nmid N) \\ -1 & (n | N) \end{cases}
\]
(see [2], P2691), we have

\[ f_1(N + 0) - f_1(N - 0) = \sum_{n|N} \frac{a(n)}{n} = \frac{b(N)}{N}. \]

Therefore

\[ R(N + 0) - R(N - 0) = (\text{Er}(N + 0) - \text{Er}(N - 0)) - N(f_1(N + 0) - f_1(N - 0)) = b(N) - N \cdot \frac{b(N)}{N} = 0, \]

and hence \( R(N - 0) = R(N + 0) = R(N) \).

Let \( x \) be positive and not an integer. Take derivatives of the both sides of (3.1). Since \( x \) is not a positive integer, we have \( \text{Er}'(x) = -M'(x) = -2\alpha x. \) Therefore we have

\[ R'(x) = -2\alpha x - f_1(x) - xf_1'(x). \]

For \( x \) which is positive and not an integer, we have \( \lfloor x/n \rfloor' = 1/n \) (see [2], p2691). Considering the hypothesis on the series (1.5), differentiating term by term we obtain

\[ f_1'(x) = -\sum_{n=1}^{\infty} \frac{a(n)}{n} \cdot \frac{1}{n} = -2\alpha. \]

Consequently, we have

\[ R'(x) = -f_1(x) \]

for \( x \) which is positive and not an integer. Because of \( R(0) = 0 \) and the continuity of \( R(x) \), we have (3.2) for all positive \( x \).

\[ \boxed{\text{□}} \]

**LEMMA 2.** Let \( G \) be a complex-valued function defined on \([0, \infty)\) satisfying

\[ \int_{0}^{x} \left| G(t) \right| \frac{dt}{t} < +\infty \]

and the integral equation

\[ G(x) - \int_{0}^{x} G(t) \frac{dt}{t} = 0 \]

for all \( x \geq 0. \) Then we have

\[ G(x) = Ax \]

for some complex number \( A. \)

**Proof.** It is obvious that (3.5) satisfies (3.3) and (3.4) for all \( x \geq 0. \) Conversely, take a function \( G(x) \) arbitrarily satisfying (3.3) and (3.4) for all \( x \geq 0. \) By (3.3) and (3.4), we see that

\[ G(x) = \int_{0}^{x} G(t) \frac{dt}{t} \]
is a continuous function on $[0, +\infty)$. Thus, using integral equation again and using the fundamental theorem of calculus, we see that $G(x)$ is continuously differentiable on $(0, +\infty)$. By taking the derivative of (3.4), we have

$$G'(x) = \frac{G(x)}{x} \quad (x > 0).$$

Thus, we have $G(x) = Ax$ for $x > 0$ for some $A$ and by the continuity this holds for $x \geq 0$. □

**Proof of Theorem.** Let a function $F_1(x)$ be the solution of the Volterra integral equation of second type (1.10) satisfying the condition (1.13). Using (3.1) and (1.10), from (3.2) we have

$$\int_0^x r^{-1}(F_1(t) - tf_1(t))dt = F_1(x) - xf_1(x) \quad (x \geq 0).$$  \hspace{1cm} (3.6)

Now we put

$$G(x) := F_1(x) - xf_1(x).$$  \hspace{1cm} (3.7)

Then, the equation (3.6) yields

$$\int_0^x r^{-1}G(t)dt = G(x) \quad (x \geq 0).$$  \hspace{1cm} (3.8)

Using Lemma 2, we must have (3.5). By substituting into (3.7), we have the solution (1.11).

Conversely, if we assume that $F_1(x)$ is a function of type (1.11). Then,

$$F_1(x) - \int_0^x F_1(t) \frac{dt}{t} = (f_1(x) + A)x - \int_0^x (f_1(t) + A)dt$$

$$= (f_1(x) + A)x - \int_0^x f_1(t)dt - Ax$$

$$= xf_1(x) - \int_0^x f_1(t)dt.$$

Using (3.1) and (3.2),

$$xf_1(x) - \int_0^x f_1(t)dt = xf_1(x) + R(x)$$

$$= xf_1(x) + Er(x) - xf_1(x)$$

$$= Er(x).$$

Therefore, the function $F_1(x)$ of type (1.11) is the solution of the integral equation (1.10) for all $x \geq 0$. Since the function $f_1(x)$ is a locally bounded as noted in section 1, and $A$ is a constant, it is clear that the function $F_1(x)$ satisfies the condition (1.13). The completes the proof. □

**Remark.** It is possible to deduce our theorem from a known result. In fact, the entry 2.1.50 of [5] with

$$A := -1, \quad \lambda := 0, \quad \mu := -1, \quad f(x) := Er(x),$$

which satisfies $\lambda + \mu + 1 = 0$, gives the solution (1.11) of the Volterra integral equation of second type (1.10). We can show the following fact:

*Let $E_1$ be a complex-valued function defined on $[0, +\infty)$ satisfying

$$\int_0^x |E_1(t)| \frac{dt}{t^2} < +\infty$$  \hspace{1cm} (3.9)*
for all $x \geq 0$. Then, the function

$$F_2(x) := E_1(x) + x \int_0^x \frac{E_1(t)}{t^2} \, dt$$

well-defined on $[0, +\infty)$ satisfies the Volterra integral equation of second type

$$F_2(x) - \int_0^x \frac{F_2(t)}{t} \, dt = E_1(x)$$

and satisfies

$$\int_0^x |F_2(t)| \frac{dt}{t} < +\infty$$

for all $x \geq 0$.

It follows immediately that the special solution of the integral equation (1.10) is expressed using (1.12) by using (3.10). The general solution of (3.11) can be expressed as follows:

Let $E_1$ be a complex-valued function defined on $[0, +\infty)$ satisfying the condition (3.9) for all $x \geq 0$. Then, the complex-valued functions $F_3$ defined on $[0, +\infty)$ satisfying the condition (3.12) and the Volterra integral equation of second type (3.11) are given by

$$F_3(x) = E_1(x) + x \int_0^x \frac{E_1(t)}{t^2} \, dt + Ax$$

with complex number $A$.

However, in this paper, we choose a self-contained method how to construct the solution concretely not using the formula in the entry 2.1.50 of [5].

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References

[1] U. Balakrishnan, Y.-F. S. Pétermann, The Dirichlet series of $\zeta(s)\zeta'(s + 1)f(s + 1)$ : On an error term associated with its coefficients, Acta Arithmetica 75 (1996), 39-69.
[2] J. Kaczorowski, K. Wiertelak, Oscillations of the remainder term related to the Euler totient function, J. Number Theory 130 (2010) 2683-2700.
[3] J. Kaczorowski, K. Wiertelak, On the sum of the twisted Euler function, Int. J. Number Theory 8 (7) (2012) 1741-1761.
[4] H. L. Montgomery, Fluctuations in the mean of Euler’s phi function, Proc. Indian Acad. Sci. Math. Sci. 97 (1-3) (1987) 239-245.
[5] A. D. Polyanin and A. V. Manzhirov, Handbook of Integral Equations 2nd edition, Chapman & Hall / CRC, (2008)

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