SELF-IMPROVING PROPERTIES OF VERY WEAK SOLUTIONS TO DOUBLE PHASE SYSTEMS

SUMIYA BAASANDORJ, SUN-SIG BYUN, AND WONTAE KIM

Abstract. We prove the self-improving property of very weak solutions to non-uniformly elliptic problems of double phase type in divergence form under sharp assumptions on the non-linearity.

1. Introduction

In this paper, we consider self-improving properties of very weak solutions to the non-uniformly elliptic problems of double phase type in the divergence of the following form

\[-\text{div} \mathcal{A}(x, \nabla u) = -\text{div}(|F|^{p-2}F + a(x)|F|^{q-2}F) \quad \text{in} \quad \Omega \] (1.1)

for a bounded domain \( \Omega \subset \mathbb{R}^n \) with \( n \geq 2 \), where the map \( \mathcal{A}(x, \xi) : \Omega \times \mathbb{R}^{Nn} \rightarrow \mathbb{R}^{Nn} \) is a Carathéodory vector field satisfying the following structure assumptions with fixed constants \( 0 < \nu \leq L < \infty \):

\[
\begin{align*}
\nu(|\xi|^p + a(x)|\xi|^q) &\leq \langle \mathcal{A}(x, \xi), \xi \rangle, \\
|\mathcal{A}(x, \xi)| &\leq L(|\xi|^{p-1} + a(x)|\xi|^{q-1})
\end{align*}
\] (1.2)

for a.e. \( x \in \Omega \) and every \( \xi \in \mathbb{R}^{Nn} \), whereas \( F : \Omega \rightarrow \mathbb{R}^{Nn} \) is a given vector field. Throughout the paper, we shall assume that exponents \( 1 < p < q < \infty \) and the coefficient function \( a : \Omega \rightarrow \mathbb{R} \) satisfy the following main assumptions

\[
\frac{q}{p} < 1 + \frac{\alpha}{n}, \quad 0 \leq a(\cdot) \in C^{0, \alpha}(\Omega) \quad \text{for some} \quad \alpha \in (0, 1].
\] (1.3)

The primary model system (1.1) originates from the functional given by

\[
W^{1,1}(\Omega, \mathbb{R}^N) \ni v \mapsto \mathcal{P}(v, \Omega) = \int_{\Omega} \langle |F|^{p-2}F + a(x)|F|^{q-2}F, Dv \rangle \, dx,
\]

where the double phase functional is of the form

\[
W^{1,1}(\Omega, \mathbb{R}^N) \ni v \mapsto \mathcal{P}(v, \Omega) := \int_{\Omega} H(x, |\nabla v|) \, dx.
\] (1.4)

Here and in the rest of the paper we denote

\[
H(x, z) := |z|^p + a(x)|z|^q.
\] (1.5)

The function \( H(x, z) \), with some ambiguity of notation, will be considered in all cases \( z \in \mathbb{R}, \ z \in \mathbb{R}^N \) and \( z \in \mathbb{R}^{Nn} \). The double phase functional was introduced first by Zhikov \([38, 39]\) in order to produce models of strongly anisotropic materials in the settings of homogenization and nonlinear elasticity. The main feature of the functional \( \mathcal{P} \) in (1.4) is that its growth and ellipticity

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ratio depend on the modulating coefficient function \(a(\cdot)\), which exhibits the mixture of two different materials. The functional in (1.4) itself is a significant example of functionals belonging to a family of functionals having non-standard \((p, q)\)-growth conditions, as it was introduced first by Marcellini in [34, 35]. Recently, regularity properties of weak solutions to the double phase problems have been extensively investigated in a series of papers [6, 7, 16, 15, 17, 19]. Among them, the gradient H"older regularity of a minimizer of the functional \(P\) has been proved under the assumption (1.3) in [6, 7]. Essentially, the condition (1.3) is sharp in the sense of the regularity theory considered there, see for instance [22, 23, 24]. Our purpose in this paper is to consider very weak solutions to the system of double phase type of equations (1.1), whose definition is given as follows.

**Definition 1.1.** For a given vector field \(F : \Omega \to \mathbb{R}^{Nn}\) such that
\[
\int_{\Omega} \left( |F|^{p-1} + a(x)|F|^{q-1} \right) \, dx < \infty, \tag{1.6}
\]
a map \(u \in W^{1,1}(\Omega, \mathbb{R}^N)\) with
\[
\int_{\Omega} \left( |\nabla u|^{p-1} + a(x)|\nabla u|^{q-1} \right) \, dx < \infty \tag{1.7}
\]
is called a very weak solution to the system (1.1) under the assumptions (1.2) and (1.3) if
\[
\int_{\Omega} \langle A(x, \nabla u), \nabla \varphi \rangle \, dx = \int_{\Omega} \langle |F|^{p-2}F + a(z)|F|^{q-2}F, \nabla \varphi \rangle \, dx \tag{1.8}
\]
holds for every \(\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)\).

Note that if we replace the assumptions (1.6) and (1.7) with
\[
\int_{\Omega} H(x, |F|) \, dx < \infty \tag{1.9}
\]
and
\[
\int_{\Omega} H(x, |\nabla u|) \, dx < \infty \tag{1.10}
\]
in the above definition, respectively, where the function \(H\) is defined in (1.5), then this very weak solution \(u\) of (1.1) is called a classical weak solution naturally. Moreover, it has been shown that, for a weak solution \(u\) to (1.1), the equality (1.8) under the assumption (1.3) still holds for any function \(\varphi \in W_{0}^{1,1}(\Omega, \mathbb{R}^N)\) with \(H(x, |\nabla \varphi|) \in L^1(\Omega)\), see [17]. In fact, the required integrability of \(\nabla u\) in (1.10) is needed in order to apply energy estimates for deriving existence, uniqueness and further regularity results. But by weakening the integrability of \(\nabla u\) as in Definition 1.1, we are still able to consider so-called a very weak solution \(u\) to the elliptic problem (1.1). At this stage, we are not allowed to use energy estimate methods due to the lack of integrability of \(\nabla u\). In this paper, we are interested in validity of the self-improving property of very weak solutions to the system (1.1) modeled on the operator of double phase structure, that is, if \([H(x, |\nabla u|)]^\delta \in L^1(\Omega)\) holds for some constant \(\delta \in (0, 1)\) being enough close to 1 and the non-homogeneous term \(F\) has a certain integrability as in (1.9), then this very weak solution \(u\) becomes a weak solution.

In the case of \(p\)-Laplace \((a(\cdot) \equiv 0\) in (1.1)), the self-improving property of very weak solutions have been achieved in [28] for the homogeneous case \((F \equiv 0)\). In [32], this result has been extended to the \(p\)-Laplace type system under more general structure assumptions involving non-homogeneous data \(F \in L^p\) based on techniques of Lipschitz truncation, which provides a way for constructing admissible test functions by applying the Whitney covering lemma. Moreover, the techniques of Lipschitz truncation have been extended and applied to Calderon-Zygmund type estimates [4, 5], the existence of a very weak solution [12, 13, 37] and the setting of Orlicz space [14], the variable exponent \(p(\cdot)\) space [10], as well as parabolic \(p\)-Laplace type systems [3, 8, 11, 31]. See also [1, 2] for the application of Lipschitz truncation method on elliptic problems.

Going back to the double phase system with the non-homogeneous data, we aim at proving the self-improving property for the double phase problems by revisiting Lipschitz truncation techniques in [32, 31]. The main theorem in this paper reads as follows:
Theorem 1.2. Let $F : \Omega \to \mathbb{R}^N$ be a vector field satisfying
\[
\int_\Omega H(x, |F|) \, dx < \infty
\]
under the assumptions (1.2) and (1.3). There exists an exponent $\delta_0 \in (1 - 1/q, 1)$ depending only on $n, N, p, q, \alpha, \nu, L$ and $[a]_{0, \alpha}$ such that for every $\delta \in (\delta_0, 1)$ and every very weak solution $u$ of (1.1) with
\[
\int_\Omega [H(x, |u|)]^\delta \, dx < \infty,
\]
there exists a positive radius $R_0$ depending only on $n, N, p, q, \alpha, \nu, L, [a]_{0, \alpha}, \delta$ and $\|\nabla u\|_{L^p(\Omega)}$ such that
\[
\int_{B_r(x_0)} H(x, |\nabla u|) \, dx \leq c \left( \int_{B_{2r}(x_0)} [H(x, |\nabla u|)]^\delta \, dx \right)^{\frac{1}{\delta}} + c \int_{B_{2r}(x_0)} H(x, |F|) \, dx + c
\]
holds for some constant $c \equiv c(n, N, p, q, \alpha, \nu, L, [a]_{0, \alpha})$, whenever $B_{2r}(x_0) \subset \Omega$ is a ball with $2r \leq R_0$.

The result of the above theorem is new in the sense of the self-improving property of very weak solutions to double phase problems as far as we are concerned. Clearly, the result of the above theorem can be reduced to the standard $p$-Laplace case [28, 32] when $a(\cdot) \equiv 0$. Let us neatly explain the key ideas of the proof of Theorem 1.2. The first target is to obtain a Caccioppoli type inequality of Lemma 3.9 via the techniques of Lipschitz truncation. With a fixed ball $B_{2R} \subset \Omega$ of a suitable size $R$ and a positive number $\lambda$ appropriately large, we construct an admissible function $\phi_\lambda \in W^{1, \infty}_0(B_{2R})$ satisfying
\[
\frac{\|\phi_\lambda(x)\|}{R} + |\nabla \phi_\lambda(x)| \lesssim \lambda^p \text{ in } B_R
\]
(see Lemma 3.3 and Lemma 3.5 below) in order to estimate the key term
\[
I := \int_{\{x \in B_R : H(x, |\nabla u|) > \lambda\}} (|\nabla u|^{p-1} + a(x)|\nabla u|^{q-1}) |\nabla \phi_\lambda| \, dx
\]
in a proper way (see Lemma 3.8). Using only (1.12), it can be seen
\[
I \lesssim \int_{\{x \in B_R : H(x, |\nabla u|) > \lambda\}} |\nabla u|^{p-1} \lambda^p \, dx
+ \int_{\{x \in B_R : H(x, |\nabla u|) > \lambda\}} a(x)|\nabla u|^{q-1} \lambda^p \, dx.
\]
However, it is possible to deal with the first term of (1.14) as in the $p$-Laplacian case, but the second term in (1.14) causes a trouble to make further estimates. To overcome such a difficulty, we apply (1.13) to estimate (1.14) in the following way
\[
I \lesssim \int_{\{x \in B_R : H(x, |\nabla u|) > \lambda\}} |\nabla u|^{p-1} \lambda^p \, dx
+ \int_{\{x \in B_R : H(x, |\nabla u|) > \lambda\}} [a(x)]^\frac{q-1}{p-1} |\nabla u|^{q-1} \lambda^p \, dx.
\]
Then the second term of the above display can be treated as usually done for the $q$-Laplacian case.
Remark 1.3. We remark that $\delta_0 \in (1 - 1/q, 1)$ in the statement of Theorem 1.2 is close enough to 1 due to a counter example constructed for the p-Laplace equation $a(\cdot) \equiv 0$, see [18]. We also would like to point out that the main assumption (1.3) is optimal in the sense that, for a given very weak solution $u \in W^{1,1}(\Omega, \mathbb{R}^N)$ to the system (1.1) with

$$
\int_{\Omega} |H(x, |\nabla u|)|^\delta \, dx < \infty
$$

(1.15)

for any exponent $\delta \in (0, 1)$ sufficiently close to 1, we would have

$$
\int_{\Omega} |H(x, |\nabla u|)|^\delta \, dx \approx \int_{\Omega} [\nabla u]^\delta + a_\delta(x)|\nabla u|^\delta] \, dx =: \mathcal{P}_\delta(u, \Omega),
$$

(1.16)

where the coefficient function defined by $a_\delta(x) := |a(x)|^\delta$ is a member of $C^{0,\alpha\delta}(\Omega)$, and we need the following condition for the new functional $\mathcal{P}_\delta$ in (1.16)

$$
\delta q \leq \delta p + \frac{(\alpha\delta)(\delta p)}{n}
$$

in order to have the absence of Lavrentiev phenomenon, see for instance [16, Theorem 4.1] and [38, 39] for details. In turn, the last display yields that

$$
q \leq p + \frac{\alpha \delta p}{n} < p + \frac{\alpha p}{n}
$$

for every $\delta \in (0, 1)$ enough close to 1. In this regard, the delicate borderline case $q = p + \frac{\alpha p}{n}$ is not considered in this paper.

Remark 1.4. Here we point out that if the display (1.11) is valid for some $\delta \in (0, 1)$ sufficiently close to 1, then (1.7) holds true by means of Hölder inequalities that

$$
\int_{\Omega} (|\nabla u|^{p-1} + a(x)|\nabla u|^{q-1}) \, dx
\leq |\Omega|^{\frac{s-p+1}{sp}} \left( \int_{\Omega} |\nabla u|^{\delta p} \, dx \right)^{\frac{p-1}{sp}}
+ \left( \int_{\Omega} |a(x)|^{\frac{s-q+1}{q-1}} \, dx \right)^{\frac{s-p+1}{s}} \left( \int_{\Omega} [H(x, |\nabla u|)]^{\delta} \, dx + 1 \right)^{\frac{s}{s-1}},
$$

where we have also used the simple identity $a(x) = |a(x)|^{\frac{\delta}{s}}[a(x)]^{\frac{s-1}{s}}$.

Finally, we outline the organization of the paper. In the next section, we introduce the notations and basic tools to be used throughout the paper. Section 3 is devoted to obtaining the reverse Hölder inequality of a very weak solution to the system (1.1). In the last section, we provide the proof of the main Theorem 1.2.

2. Preliminaries

We denote by $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ the open ball in $\mathbb{R}^n$ centered at $x_0 \in \mathbb{R}^n$ with radius $r > 0$. If the center is clear in the context, we shall denote it by $B_r \equiv B_r(x_0)$. As usual, we denote by $c$ a generic positive constant, possibly varying from line to line; special constants will be denoted by symbols such as $c_1, c_*, c_c$, and so on. All such constants will always be larger than one; moreover, relevant dependencies on parameters will be emphasized using brackets, that is, for example, $c \equiv c(n, p, q, \nu, L)$ means that $c$ is a constant depending only on $n, p, q, \nu, L$. With $f : B \to \mathbb{R}^k$ ($k \geq 1$) being a measurable map for a measurable subset $B \subset \mathbb{R}^n$ having a finite and positive measure, we denote by

$$
(f)_B \equiv \int_B f(x) \, dx = \frac{1}{|B|} \int_B f(x) \, dx
$$
to mean its integral average over \( B \). For a given number \( \alpha \in (0, 1] \), the space \( C^{0, \alpha}(\Omega) \) consists of measurable functions \( f : \Omega \to \mathbb{R} \) such that

\[
\|f\|_{C^{0, \alpha}(\Omega)} := \|f\|_{0, \alpha; \Omega} + \|f\|_{L^\infty(\Omega)} < \infty,
\]

where for any open subset \( B \subset \Omega \), the semi-norm is defined by

\[
[f]_{0, \alpha; B} := \sup_{x, y \in B, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} \quad \text{and} \quad [f]_{0, \alpha} \equiv [f]_{0, \alpha; \Omega}.
\]

Next, we introduce some auxiliary tools to be used in the later parts of the paper. For a given function \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), the uncentered Hardy-Littlewood maximal function of \( f \) is defined as

\[
M(f)(x) := \sup_{x \in B_r(x_0)} \int_{B_r(x_0)} |f(y)| \, dy,
\]

(2.1)

where supremum is taken over all balls containing the point \( x \). The first tool we report here is the following classical strong type estimate of the maximal operator, see for instance [27, Lemma 2.1.6].

**Lemma 2.1.** For any \( f \in L^s(\mathbb{R}^n) \) with \( 1 < s < \infty \), there exists a constant \( c(n, s) \equiv 2^\frac{n}{s} s^{-1} \) such that

\[
\int_{\mathbb{R}^n} |M(f)|^s \, dx \leq c(n, s) \int_{\mathbb{R}^n} |f|^s \, dx.
\]

Note that since the constant \( c \equiv c(n, s) \) defined in Lemma 2.1 is continuous in \( 1 < s < \infty \).

Thus, for fixed exponents \( 1 < s_1 < s_2 < \infty \), there exists a constant \( c \equiv c(n, s_1, s_2) \) such that

\[
\int_{\mathbb{R}^n} |M(f)|^s \, dx \leq c \int_{\mathbb{R}^n} |f|^s \, dx
\]

holds, whenever \( f \in L^s(\mathbb{R}^n) \) with \( 1 < s_1 < s < s_2 \).

Next, we need a Poincaré type inequality in the following, see for instance [20, Proposition 2.1] and [29, Theorem 5.47].

**Lemma 2.2.** Let \( f \in W^{1,s}_{0}(B_R) \) with \( 1 < s < \infty \) and a ball \( B_R \subset \mathbb{R}^n \). Then there exists a constant \( c \equiv c(n, s) \), which is continuous with respect to \( s \)-variable, such that

\[
\int_{B_{r_1}(y) \cap B_R} |f|^s \, dx \leq c \int_{B_{r_2}(y) \cap B_R} |\nabla f|^s \, dx
\]

holds, whenever \( B_{r_1}(y) \subset \mathbb{R}^n \) is a ball with \( B_{r_2}(y) \cap B_R \neq \emptyset \).

We end up the present section with the following classical iteration lemma, which can be found in [26, Lemma 6.1].

**Lemma 2.3.** Let \( 0 < R_0 < R_1 < \infty \) be given numbers and let \( h : [R_0, R_1] \to \mathbb{R} \) be a non-negative and bounded function. Furthermore, let \( \theta \in (0, 1) \) and \( A, B, \gamma_1, \gamma_2 \geq 0 \) be fixed constants and suppose that

\[
h(s) \leq \theta h(t) + \frac{A}{(s - t)^{\gamma_1}} + \frac{B}{(s - t)^{\gamma_2}}
\]

holds for all \( R_0 \leq t < s \leq R_1 \). Then there exists a positive constant \( c \equiv c(\theta, \gamma_1, \gamma_2) \) such that

\[
h(R_0) \leq c \left( \frac{A}{(R_1 - R_0)^{\gamma_1}} + \frac{B}{(R_1 - R_0)^{\gamma_2}} \right).
\]

3. **Reverse Hölder inequality**

Throughout this section, let \( u \in W^{1,1}(\Omega, \mathbb{R}^N) \) always be a very weak solution to the system (1.1) in the sense of Definition 1.1 under the assumptions (1.2) and (1.3). A main purpose of this section is to derive a Caccioppoli type inequality and reverse Hölder type inequality for very weak solution \( u \).
3.1. Basic settings and notations. We introduce the basic settings and notations to be used throughout the present section under which we will work on.

1. Firstly, we fix constants $\gamma, \tilde{\gamma}$ with
   \[
   \frac{1}{p} < \gamma := \frac{p + 2}{3p} < \frac{2p + 1}{3p} < 1,
   \]
   (3.1)

2. Next, we select a constant $\delta_0 \in (0, 1)$ to satisfy
   \[
   \frac{1}{p} < \delta_0 < 1 \quad \text{and} \quad q < p + \frac{(\delta_0 \alpha)p}{n},
   \]
   (3.2)
   where the possibility of such a choice of $\delta_0$ comes from the main assumption (1.3). The validity of (3.2) implies
   \[
   \frac{\alpha}{q} - \frac{n}{\delta_0 p} \left(1 - \frac{p}{q}\right) = \frac{n}{\delta_0 q} \left(\frac{\delta_0 \alpha p}{n} - (q - p)\right) > 0.
   \]
   (3.3)
   Note that the inequalities (3.2)-(3.3) hold true with $\delta_0$ replaced by any number $\delta \in (\delta_0, 1)$ once they are valid for $\delta_0 \in (0, 1)$ as in (3.2).

3. With those constants fixed in (3.1)-(3.2), let $B_{2R} \equiv B_{2R}(x_0) \subset \Omega$ be a fixed ball with $R \leq 1$ such that
   \[
   \int_{B_{2R}} \left[H(x, |\nabla u|) + H(x, |F|)\right]^{\frac{1}{\delta}} dx \leq \Lambda^\delta
   \]
   (3.4)
   and
   \[
   \int_{B_R} \left[H(x, |\nabla u|) + H(x, |F|)\right]^{\delta} dx = \Lambda^\delta
   \]
   (3.5)
   for some fixed constants $\delta \in (\delta_0, 1)$ and $\Lambda \geq 1$.

4. With the fixed ball $B_{2R}$ satisfying (3.4)-(3.6) and the fixed constant $\gamma$ in (3.1), we denote
   \[
   G(x) := [M (G_p + G_q)(x)]^{\frac{1}{\gamma}},
   \]
   (3.7)
   where $M$ is the maximal operator introduced in (2.4),
   \[
   G_p(y) := \left(\frac{|u(y) - (u)_{B_{2R}}|^{p\gamma}}{2R} + |\nabla u(y)|^{p\gamma} + |F(y)|^{p\gamma}\right) \chi_{B_{2R}}(y)
   \]
   (3.8)
   and
   \[
   G_q(y) := [a(y)]^{-\frac{1}{q\gamma}} \left(\frac{|u(y) - (u)_{B_{2R}}|^{q\gamma}}{2R} + |\nabla u(y)|^{q\gamma} + |F(y)|^{q\gamma}\right) \chi_{B_{2R}}(y).
   \]
   (3.9)

5. We denote the lower-level set of the function $G$ defined in (3.7) as
   \[
   E(\lambda) := \{x \in \mathbb{R}^n \mid G(x) \leq \lambda\} \quad (\lambda > 0).
   \]
   (3.10)

6. Let $0 \leq \eta \in W^{1, \infty}_0(B_{2R})$ be a cut-off function with
   \[
   \eta \equiv 1 \text{ in } B_R \quad \text{and} \quad |\eta| + R|\nabla \eta| \leq c(n) \text{ in } B_{2R},
   \]
   (3.11)
   where $B_{2R}$ is the same ball satisfying (3.4)-(3.6). Also, we define another function $v$ by
   \[
   v(x) := (u(x) - (u)_{B_{2R}}) \eta(x)
   \]
   (3.12)

7. Here we point out that the function $v$ in (3.12) can not be a test function in the equation (1.1) due to the lack of integrability assumption of $\nabla u$. In this regard, we truncate the function $v$ in (3.12) on the lower-level set of $v$ and $\nabla v$ and extend the truncated function to be Lipschitz in the upper-level set. Clearly, the set $E(\lambda)^c$ in (3.10) is an open set for every $\lambda > 0$ by the lower semi-continuity of the maximal function. In the remaining part of the section, we shall always consider the number $\lambda$ such that
   \[
   \lambda^{1/p} \geq c_\lambda \Lambda^{1/p}
   \]
   (3.13)
with an universal constant $c_\ast$ to be determined via Lemma 3.6 below and the number $\Lambda$ satisfying (3.5)-(3.6). As a consequence of Lemma 3.2 below together with (3.1)-(3.2), we observe that $B_{2R} \cap E(\lambda^{1/p}) \neq \emptyset$. To extend the function $v|_{E(\lambda^{1/p})}$ to be locally Lipschitz over $\mathbb{R}^n$, we revisit a classical Whitney covering lemma to the open set $E(\lambda^{1/p})^c$, which has been widely used for parabolic problems previously, see for instances [8, 31], and we shall use a version applied in [21, Section 2.3]. Therefore, there exists a countable family of open balls $\{B_i\}_{i \in \mathbb{N}}$, $B_i \equiv B_{r_i}(x_i) \subset \mathbb{R}^n$ satisfying the following properties:

**W1:** $\bigcup_{i \in \mathbb{N}} \frac{1}{2} B_i = E(\lambda^{1/p})^c$.

**W2:** $8B_i \subset E(\lambda^{1/p})^c$ and $16B_i \cap E(\lambda^{1/p}) \neq \emptyset$ for all $i \in \mathbb{N}$.

**W3:** If $B_i \cap B_j \neq \emptyset$ for some $i, j \in \mathbb{N}$, then $\frac{1}{2} r_j \leq r_i \leq 2 r_j$.

**W4:** $\frac{1}{4} B_i \cap \frac{1}{4} B_j = \emptyset$ for all $i, j \in \mathbb{N}$ with $i \neq j$.

**W5:** Every point $x \in E(\lambda^{1/p})^c$ belongs to at most $c(n)$ balls of the family $\{4B_i\}_{i \in \mathbb{N}}$. With the covering $\{B_i\}_{i \in \mathbb{N}}$ fixed as above, there exists a partition of unity $\{\psi_i\}_{i \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ such that

**P1:** $\chi_{4B_i} \leq \psi_i \leq \chi_{2B_i}$.

**P2:** $|\psi_i(x)| + r_i|\nabla \psi_i(x)| \leq c(n)$ for every $x \in \mathbb{R}^n$.

Next we define the set

$$A_i := \left\{ j \in \mathbb{N} \mid \frac{3}{4} B_i \cap \frac{3}{4} B_j \neq \emptyset \right\}$$

for every $i \in \mathbb{N}$. Therefore, the following properties are also satisfied:

**P3:** $\sum_{j \in A_i} \psi_j = 1$ in $B_i$.

**W6:** $|B_i \cap B_j| \geq c(n)^{-1} \max\{|B_i|, |B_j|\}$ whenever $j \in A_i$.

**W7:** $\frac{3}{4} B_i \cap \frac{3}{4} B_j \geq c(n)^{-1} \max\{|B_i|, |B_j|\}$ whenever $j \in A_i$.

**W8:** the cardinality of $A_i$, $\#A_i$, is uniformly bounded with $\#A_i \leq c(n)$.

8. Under this Whitney covering as above, we shall consider a truncated function $v_\lambda$ given by

$$v_\lambda(x) := v(x) - \sum_{i \in \mathbb{N}} \psi_i(x) (v(x) - v_i),$$

and

$$v_i := \begin{cases} \int_{\frac{1}{2} B_i} v \, dx & \text{if } \frac{3}{4} B_i \subset B_{2R}, \\ 0 & \text{otherwise}. \end{cases}$$

### 3.2. Lipschitz truncation and Caccioppoli inequality

Under the basic settings and notations in the previous subsection, our next purpose is to construct admissible test functions to the system (1.1) by using the techniques of Lipschitz truncation in order to obtain a Caccioppoli type inequality below, see Lemma 3.9. To go further, first we provide a Poincaré type inequality for a very weak solution $u$ to (1.1).

**Lemma 3.1.** Under the settings and notations of Subsection 3.1, there exists a constant $c \equiv c(n, N, p, q, |a|_{0, \alpha})$ such that

$$\int_{B_{2R}} \left[ H \left( x, \frac{|u - (u)_{B_{2R}}|}{2R} \right) \right]^\delta \, dx \leq c \int_{B_{2R}} [H(x, |\nabla u|)]^\delta \, dx.$$
Proof. Let $a(x_m) = \inf_{x \in B_{2R}} a(x)$ for some point $x_m \in \overline{B_{2R}}$. Then we have

$$\int_{B_{2R}} \left[ H \left( x, \left| \frac{u - (u)_{B_{2R}}}{2R} \right| \right) \right]^{\delta} \, dx \leq 2 \int_{B_{2R}} \left[ H \left( x_m, \left| \frac{u - (u)_{B_{2R}}}{2R} \right| \right) \right]^{\delta} \, dx$$

$$+ 2 \int_{B_{2R}} |a(x) - a(x_m)|^{\delta} \left| \frac{u - (u)_{B_{2R}}}{2R} \right|^{q \delta} \, dx \tag{3.17}$$

Then applying the classical Poincaré inequality in [33, Theorem 1.51] and recalling $1 < p \gamma < p \delta < p$ and $1 < q \gamma < q \delta < q$ by (3.1), we find

$$I_1 \leq 2 \int_{B_{2R}} \left| u - (u)_{B_{2R}} \right|^{p \delta} \, dx + 2 \int_{B_{2R}} |a(x_m)|^{\delta} \left| \frac{u - (u)_{B_{2R}}}{2R} \right|^{q \delta} \, dx$$

for some constant $c \equiv c(n, N, p, q)$. For the estimate of the term $I_2$ in (3.17), using the Hölder continuity of the coefficient function $a(\cdot)$ and classical Sobolev-Poincaré inequality in [33, Corollary 1.64] with observing that $\frac{q}{p} < 1 + \frac{\alpha}{n} < \frac{n}{n - 1}$ to have

$$I_2 \leq 4[a]^{\delta}_{0,\alpha} R^{\alpha \delta} \int_{B_{2R}} \left( \left| \frac{u - (u)_{B_{2R}}}{2R} \right|^{p \delta} \right)^{\frac{1}{p}} \, dx$$

$$\leq c[a]^{\delta}_{0,\alpha} R^{\alpha \delta} \left( \int_{B_{2R}} |\nabla u|^{p \delta} \, dx \right)^{\frac{1}{p}}$$

$$\leq c([a]^{\delta}_{0,\alpha} + 1) R^{\alpha \delta - n \left( \frac{1}{p} - 1 \right)} \left( \int_{B_{2R}} |\nabla u|^{p \delta} \, dx \right)^{\frac{1}{p} - 1} \int_{B_{2R}} |\nabla u|^{p \delta} \, dx$$

$$\leq c \int_{B_{2R}} |\nabla u|^{p \delta} \, dx$$

for some constant $c \equiv c(n, N, p, q, [a]^{\delta}_{0,\alpha})$, where we have also used the assumption (3.4). Inserting the estimates in the last two displays into (3.17), we arrive at the conclusion of Lemma 3.1. \(\square\)

Next, using the Poincaré type inequality of Lemma 3.1, we derive an estimate for the function $G$ introduced in (3.7).

**Lemma 3.2.** Under the settings and notations of Subsection 3.1, there exists a constant $c_\gamma \equiv c_\gamma(n, N, p, q, [a]^{\delta}_{0,\alpha})$ satisfying the following inequality

$$\int_{B_{2R}} [G(x)]^{\delta p} \, dx \leq c_\gamma \Lambda^\delta.$$ 

for constants $\delta \in (\delta_0, 1)$ and $\Lambda \equiv 1$ satisfying the conditions (3.4)-(3.6).

**Proof.** Clearly, we have

$$\int_{B_{2R}} [G(x)]^{\delta p} \, dx \leq \int_{\mathbb{R}^n} [M (G_p + G_q)]^{\frac{\delta}{\gamma}} (x) \, dx,$$

where the functions $G_p$ and $G_q$ have been defined in (3.8)-(3.9). Recalling that $1 < \frac{\gamma}{\gamma} < \frac{\delta}{\gamma} < \frac{1}{\gamma}$ by (3.1)-(3.2), we are able to apply Lemma 2.1 with $s \equiv \frac{\delta}{\gamma}$ to obtain

$$\int_{B_{2R}} [G(x)]^{\delta p} \, dx \leq c \int_{B_{2R}} \left( [G_p(x)]^{\frac{\delta}{\gamma}} + [G_q(x)]^{\frac{\delta}{\gamma}} \right) \, dx$$
for a constant \( c \equiv c(n, p) \). Then, using the Poincaré type inequality of Lemma 3.1 and (3.5), we have
\[
\int_{B_{2R}} [G(x)]^{\delta p} \, dx \leq c \int_{B_{2R}} \left[H(x, |\nabla u|) + H(x, |F|)\right]^{\delta} \, dx \leq c\Lambda^{\delta}
\]
for some constant \( c \equiv c(n, N, p, q, |a|_{0, \alpha}) \).

Next we discuss some important properties of the truncated function \( v_\lambda \) defined in (3.15).

**Lemma 3.3.** Under the settings and notations of Subsection 3.1, there exist constants \( c_p \equiv c_p(n, p) \) and \( c_q \equiv c_q(n, N, p, q, \alpha, |a|_{0, \alpha}) \) such that the following inequalities hold true:
\[
|v_\lambda(y)| \leq c_p R \lambda^\frac{1}{p} \text{ for all } y \in E(\lambda^{1/p})^c
\]
and
\[
[a(y)]^\frac{1}{p} |v_\lambda(y)| \leq c_q R \lambda^\frac{1}{p} \text{ for all } y \in B_{2R} \cap E(\lambda^{1/p})^c,
\]
whenever \( \lambda \geq 1 \) is a number satisfying (3.13).

**Proof.** Let \( y \in E(\lambda^{1/p})^c \) be any fixed point and \( \lambda \geq 1 \) be any fixed number as defined in (3.13). Then, the condition W1 yields that \( y \in B_i \) for some \( i \in \mathbb{N} \). Using P3, we have
\[
|v_\lambda(y)| \leq \sum_{j \in A_i} |v_j| = \sum_{j \in A_i} \int_{B_j} |u - (u)_{B_{2R}}| \eta \, dx
\]
\[
\leq c(n) \sum_{j \in A_i} \int_{B_j} |u - (u)_{B_{2R}}| \chi_{B_{2R}} \, dx,
\]
where the summation is taken over the indices \( j \in A_i \) with \( \frac{3}{4} B_j \subset B_{2R} \) and the set \( A_i \) has been defined in (3.14). The last display together with applying Hölder’s inequality and observing that there exists a point \( y_j \in E(\lambda^{1/p}) \cap 16B_j \) for every \( j \in \mathbb{N} \) via W2 yield
\[
|v_\lambda(y)| \leq c(n) \sum_{j \in A_i} \left( \int_{16B_j} |u - (u)_{B_{2R}}|^{p\gamma} \chi_{B_{2R}} \, dx \right)^{\frac{1}{p\gamma}}
\]
\[
\leq c(n) R \sum_{j \in A_i} \left[ M \left( \frac{|u - (u)_{B_{2R}}|}{2R} \right)^{\frac{p\gamma}{2}} \chi_{B_{2R}} (y_j) \right]^\frac{1}{p\gamma}
\]
\[
\leq c(n) R \sum_{j \in A_i} G(y_j) \leq c(n) R \lambda^\frac{1}{p} \sum_{j \in A_i} 1 \leq c(n) R \lambda^\frac{1}{p},
\]
where we have also used the definition of the function \( G \) in (3.7) and W8. So the first part of (3.18) is proved. Now we turn our attention to the second inequality of (3.18). In fact, it suffices to show that there exists a constant \( c \equiv c(n, N, p, q, \alpha, |a|_{0, \alpha}) \) such that
\[
\|a\|_{L^\infty(B_{2R} \cap B_i)}^{\frac{1}{\gamma}} |v_j| \leq c R \lambda^\frac{1}{p}
\]
holds for every \( j \in A_i \), where \( A_i \) and \( v_j \) have been defined in (3.14) and (3.16), respectively. We may assume that \( \frac{3}{4} B_j \subset B_{2R} \). Otherwise, the inequality (20) is trivial by the definition of \( v_j \) in (3.16). Then, using W3 and Hölder continuity of the coefficient function \( a(\cdot) \), we find
\[
\|a\|_{L^\infty(B_{2R} \cap B_i)} \leq \inf_{x \in B_{2R} \cap B_j} a(x) + 8 |a|_{0, \alpha} r_j^\alpha
\]
for all \( j \in A_i \).

Therefore, using the last display and recalling the definition of \( v_j \) in (3.16), we have
\[
\|a\|_{L^\infty(B_{2R} \cap B_i)}^{\frac{1}{\gamma}} |v_j(z)| \leq c \int_{\frac{3}{4} B_j} \left[ \inf_{x \in B_{2R} \cap B_j} a(x) \right]^{\frac{1}{\gamma}} |u - (u)_{B_{2R}}| \, dx
\]
\[
+ c r_j^\alpha \int_{B_j} |u - (u)_{B_{2R}}| \, dx =: c (I_1 + I_2)
\]
for some constant \( c \equiv c(n, q, \alpha, |a|_{0, \alpha}) \). Now we shall estimate the resulting terms in the last display. Applying Hölder’s inequality together with the observation 1 \( \gamma p < \gamma q \), we see

\[
I_1 \leq c R \left( \int_{\frac{3}{4} B_j} \left| a(x) \frac{|u - (u)_{B_{2R}}|}{2R} \right|^q dx \right)^{\frac{1}{q}}
\]

\[
\leq c R \left( \int_{\frac{3}{4} B_j} \left( a(x) \frac{|u - (u)_{B_{2R}}|}{2R} \right)^q \chi_{B_{2R}} dx \right)^{\frac{1}{\gamma q}}
\]

\[
\leq c R \left[ M \left( \left( a(x) \frac{|u - (u)_{B_{2R}}|}{2R} \right)^q \chi_{B_{2R}} \right)(y_j) \right]^{\frac{1}{\gamma q}}
\]

for some constant \( c \equiv c(n, p, q) \), where \( y_j \in E(\lambda^{1/p}) \cap 16 B_j \) is a point determined by \( \text{W}2 \). Again applying Hölder’s inequality repeatedly together with recalling that

\[
1 < p \gamma < p \delta < p, \quad \frac{n}{q} - \frac{n}{\delta p} \left( 1 - \frac{p}{q} \right) < 1 \quad \text{and} \quad \frac{1}{\delta p} \left( 1 - \frac{p}{q} \right) < 1,
\]

we estimate the term \( I_2 \) in (3.21) as

\[
I_2 \leq 2 R r_j \left( \int_{\frac{3}{4} B_j} \left| a(x) \frac{|u - (u)_{B_{2R}}|}{2R} \right|^\delta dx \right)^{\frac{1}{\delta p} (1 - \frac{\delta}{q})} \left( \int_{\frac{3}{4} B_j} \left| a(x) \frac{|u - (u)_{B_{2R}}|}{2R} \right|^{\gamma p} dx \right)^{\frac{1}{\gamma p} \frac{\gamma}{q}}
\]

\[
\leq c R r_j^{\frac{\delta}{\delta p} (1 - \frac{\delta}{q})} \left( \int_{B_{2R}} \left| a(x) \frac{|u - (u)_{B_{2R}}|}{2R} \right|^\delta dx \right)^{\frac{1}{\delta p} (1 - \frac{\delta}{q})} \left( \int_{B_{2R}} \left| a(x) \frac{|u - (u)_{B_{2R}}|}{2R} \right|^{\gamma p} \chi_{B_{2R}} dx \right)^{\frac{1}{\gamma p} \frac{\gamma}{q}}
\]

\[
\leq c R \left( \int_{16 B_j} \left| \nabla u \right|^{\gamma p} \chi_{B_{2R}} dx \right)^{\frac{1}{\gamma p} \frac{\gamma}{q}}
\]

for some constant \( c \equiv c(n, N, p) \), where we have also used a classical Poincaré inequality and the fact that \( \frac{3}{4} B_j \subset B_{2R} \). Using (3.19) in the last display, we conclude

\[
I_2 \leq c R \lambda_\gamma^{\frac{1}{\gamma}}
\]

for a constant \( c \equiv c(n, N, p) \). Finally, we combine the estimates (3.22)-(3.23) in (3.21) to obtain the second inequality of (3.18). The proof is completed. \( \square \)

**Lemma 3.4.** Under the settings and notations of Subsection 3.1, there exists \( c \equiv c(n, N, p) \) such that

\[
\int_{B_i} |v - v_i| dx \leq c \min \{r_i, R\} \lambda^{\frac{1}{\gamma}} \quad \text{for every} \ i \in \mathbb{N}
\]

and

\[
|v_i - v_j| \leq c \min \{r_i, R\} \lambda^{\frac{1}{\gamma}} \quad \text{for every} \ j \in A_i \ \text{and} \ i \in \mathbb{N},
\]

where the set \( A_i \) has been defined in (3.14).

**Proof.** Let us fix any index \( i \in \mathbb{N} \). First, we focus on proving (3.24). For this, we consider several cases depending on a position of the balls \( \frac{3}{4} B_i \) and \( B_{2R} \).
Case 1: $\frac{3}{4}B_i \subset B_{2R}$ and $B_i \subset B_{2R}$. By the triangle inequality and the Poincaré inequality, we see
\[
\int_{B_i} |v - v_i| \, dx \leq c \int_{B_i} |v - (v)_{B_i}| \, dx \\
\leq c r_i \int_{B_i} |\nabla v| \, dx \\
\leq c r_i \left( \int_{B_i} |\nabla u| + \frac{|u - (u)_{B_{2R}}|}{2R} \right) \, dx \\
\leq c r_i \left( \int_{B_i} |\nabla u|^\gamma \right)^\frac{1}{\gamma} + \frac{|u - (u)_{B_{2R}}|}{2R} \chi_{B_{2R}} \, dx)^\frac{1}{\gamma}
\]
for some constant $c \equiv c(n, N)$, where we have applied Hölder’s inequality with the exponent $1 < \gamma p$ and the definition of the function $G$ introduced in (3.7) together with $W^2$ in such a way that there exists a point $y_i \in \mathcal{E}(\lambda^{1/p}) \cap 16B_i$.

Case 2: $\frac{3}{4}B_i \subset B_{2R}$ and $B_i \cap B_{2R}' \neq \emptyset$. In this case, we apply the Poincaré inequality of Lemma 2.2 together with recalling $1 < \gamma p$ to have
\[
\int_{B_i} |v - v_i| \, dx \leq c \int_{4B_i} |v| \, dx \leq c r_i \left( \int_{4B_i} |\nabla v|^\gamma \, dx \right)^\frac{1}{\gamma} \\
\leq c r_i \left( \int_{16B_i} |\nabla u|^\gamma \, dx \right)^\frac{1}{\gamma} \\
\leq c r_i \left( \int_{16B_i} \left( |\nabla u|^\gamma + \frac{|u - (u)_{B_{2R}}|^\gamma}{2R} \right) \chi_{B_{2R}} \, dx \right)^\frac{1}{\gamma}
\]
for some constant $c \equiv c(n, N, p)$. Arguing similarly as in (3.26), we arrive at (3.24) in this case.

Case 3: $\frac{3}{4}B_i \cap B_{2R}' \neq \emptyset$ and $r_i \leq R$. Again applying Lemma 2.2 and arguing similarly as in (3.27), we find
\[
\int_{B_i} |v - v_i| \, dx = \int_{B_i} |v| \, dx \leq c r_i \left( \int_{4B_i} \left( |\nabla u|^\gamma + \frac{|u - (u)_{B_{2R}}|^\gamma}{2R} \right) \chi_{B_{2R}} \, dx \right)^\frac{1}{\gamma}
\]
Then (3.24) follows from the last display together with the argument in (3.26).

Case 4: $\frac{3}{4}B_i \cap B_{2R}' \neq \emptyset$ and $r_i > R$. Recalling $B_{2R} \cap \mathcal{E}(\lambda^{1/p}) \neq \emptyset$ and $v \equiv 0$ on $B_{2R}'$, by the definition of the function $v$ in (3.12), we have
\[
\int_{B_i} |v - v_i| \, dx = \int_{B_i} |v| \, dx \leq c \int_{B_{2R}} |v| \, dx \\
\leq c R \int_{B_{2R}} \frac{|u - (u)_{B_{2R}}|}{2R} \chi_{B_{2R}} \, dx \leq cRG(y)
\]
for some constant $c \equiv c(n)$ and a point $y \in B_{2R} \cap \mathcal{E}(\lambda^{1/p})$. Taking into account all the cases we have discussed above, the estimate (3.24) is proved. Now we turn our attention to proving (3.25). For every $i, j \in \mathbb{N}$ with $j \in A_i$, we observe that
\[
|v_i - v_j| \leq c \int_{B_i \cap B_j} |v_i - v(x)| \, dx + c \int_{B_i \cap B_j} |v_j - v(x)| \, dx \\
\leq c \int_{B_i} |v - v_i| \, dx + c \int_{B_j} |v - v_j| \, dx
\]
for some constant \( c \equiv c(n) \), where we have also applied W6. Finally, applying (3.24) on the resulting term of the last display together with recalling \( j \in A_i \), we arrive at the desired estimate (3.25). The proof is completed.

\[ \square \]

**Lemma 3.5.** Under the settings and notations of Subsection 3.1, there exist universal constants \( c_{p} \equiv c_{p}(n, N, p) \) and \( c_{q} \equiv c_{q}(n, N, p, \alpha, [a]_{0, \alpha}) \) such that

\[
|\nabla v_{\lambda}(y)| \leq c_{p} \lambda^{rac{1}{n}} \text{ for all } y \in E(\lambda^{1/p})^{c} \\
\text{and} \\
|a(y)|^{rac{1}{q}}|\nabla v_{\lambda}(y)| \leq c_{q} \lambda^{rac{1}{n}} \text{ for all } y \in B_{2R} \cap E(\lambda^{1/p})^{c}.
\]

(3.28)

**Proof.** First, we fix any point \( y \in E(\lambda^{1/p})^{c} \). Then, there exists an index \( i \in \mathbb{N} \) with \( y \in \frac{3}{4} B_{i} \) via W1. Therefore, it follows from P3 that

\[
0 = \nabla \left( \sum_{j \in A_i} \psi_j \right) = \sum_{j \in A_i} \nabla \psi_j \text{ in } B_{i}.
\]

Using this one and recalling the definition of the truncated function \( v_{\lambda} \) in (3.15), we see

\[
\nabla v_{\lambda}(y) = \sum_{j \in A_i} v_j \nabla \psi_j(y) = \sum_{j \in A_i} (v_j - v_i) \nabla \psi_j(y),
\]

which implies

\[
|\nabla v_{\lambda}(y)| \leq \sum_{j \in A_i} |v_j - v_i| |\nabla \psi_j(y)|.
\]

Then applying Lemma 3.4 together with W3, W8 and P2, we arrive at the first part of (3.28). Now we shall deal with proving the second estimate of (3.28). For this, it is enough to show that there exists a constant \( c \equiv c(n, N, p, q, \alpha, [a]_{0, \alpha}) \) such that

\[
I_0 := \|a\|^{rac{1}{q}}_{L^{\infty}(B_{2R} \cap B_{i})} \int_{B_{i}} |v - v_{i}| \, dx \leq c \min\{r_i, R\} \lambda^{rac{1}{n}}.
\]

(3.29)

In fact, the arguments for proving the above inequality are similar to the ones used in the proof of Lemma 3.4. We shall divide into several cases depending on a position of the balls \( \frac{3}{4} B_{i} \) and \( B_{2R} \) as in the proof of Lemma 3.4.

**Case 1:** \( \frac{3}{4} B_{i} \subset B_{2R} \) and \( B_{i} \subset B_{2R} \). Using triangle inequality and Hölder continuity of the coefficient function \( a(\cdot) \), we estimate the term \( I_0 \) in (3.29) as

\[
I_0 \leq c(n) \|a\|^{rac{1}{q}}_{L^{\infty}(B_{i})} \int_{B_{i}} |v - (v)_{B_{i}}| \, dx
\]

and

\[
\|a\|_{L^{\infty}(B_{i})} \leq \inf_{x \in B_{i}} a(x) + [a]_{0, \alpha} r_{i}^{\alpha}.
\]

Then the last two displays together with the Poincaré inequality imply

\[
I_0 \leq c r_i \int_{B_{i}} \left[ \inf_{x \in B_{i}} a(x) \right]^{\frac{1}{q}} \left( |\nabla u| + \left| \frac{u - (u)_{B_{2R}}}{2R} \right| \right) \, dx \\
+ c r_i^{1 + \frac{1}{q}} \int_{B_{i}} |\nabla u| + \left| \frac{u - (u)_{B_{2R}}}{2R} \right| \, dx =: c(I_1 + I_2)
\]

(3.30)
for some constant $c \equiv c(n, N, q, [a]_{0, \alpha})$. Now we estimate the resulting terms $I_1$ and $I_2$ in the last display. For this, using Hölder’s inequality and recalling $1 < \gamma p < \gamma q$, we have

$$I_1 \leq r_i \left( \int_{B_i} \left[ \inf_{x \in B_i} a(x) \right] \left( |\nabla u| + \left| \frac{u - (u)_{B_1 2R}}{2R} \right| \right)^{\gamma} \right)^{\frac{1}{\gamma}}$$

$$\leq cr_i \left( \int_{B_i} \left( |\nabla u| + \left| \frac{u - (u)_{B_2 2R}}{2R} \right| \right)^{\gamma} \chi_{B_2 2R} \right)^{\frac{1}{\gamma}}$$

$$\leq cr_i (G(y_i))^{\frac{\alpha}{q}} \leq cr_i \lambda^{\frac{\alpha}{q}}$$

with some constant $c \equiv c(n, N, p, q, [a]_{0, \alpha})$, where we have also used the fact that there exists a point $y_i \in E(\lambda^{1/p}) \cap 16B_i$ via W2. For the term $I_2$ in (3.30), we apply Hölder’s inequality again by observing $1 < \gamma p < \gamma q$ in order to obtain

$$I_2 \leq r_i^{1 + \frac{q - \gamma}{\gamma}} \left( \int_{B_i} \left( |\nabla u| + \left| \frac{u - (u)_{B_1 2R}}{2R} \right| \right)^{\delta p} \right)^{\frac{1}{\delta p} \left( 1 - \frac{q}{\gamma} \right)}$$

$$\times \left( \int_{B_i} \left( |\nabla u| + \left| \frac{u - (u)_{B_2 2R}}{2R} \right| \right)^{\gamma p} \right)^{\frac{1}{\gamma p} \left( 1 - \frac{q}{\gamma} \right)},$$

where $\delta \in (\delta_0, 1)$ is a fixed constant satisfying (3.4)-(3.6). Recalling $\frac{3}{4} r_i \leq R$ together with applying the Poincaré inequality and (3.4), we observe that

$$r_i^{\frac{q}{\gamma} \frac{1}{\gamma}} \left( \int_{B_i} \left( |\nabla u| + \left| \frac{u - (u)_{B_1 2R}}{2R} \right| \right)^{\delta p} \right)^{\frac{1}{\delta p} \left( 1 - \frac{q}{\gamma} \right)}$$

$$\leq c R^{\frac{q}{\gamma} \frac{1}{\gamma}} \left( \int_{B_i} \left( |\nabla u| + \left| \frac{u - (u)_{B_2 2R}}{2R} \right| \right)^{\delta p} \right)^{\frac{1}{\delta p} \left( 1 - \frac{q}{\gamma} \right)}$$

$$\leq c R^{\frac{q}{\gamma} \frac{1}{\gamma}} \left( \int_{B_i} |\nabla u|^{\delta p} \right)^{\frac{1}{\delta p} \left( 1 - \frac{q}{\gamma} \right)} \leq c$$

for some constant $c \equiv c(n, N, p)$. Inserting the content of the last display into (3.31) and using the same argument as in the last part (3.26), we obtain

$$I_2 \leq cr_i \lambda^{\frac{q}{\gamma}}.$$  

(3.32)

Putting the estimates of (3.31)-(3.32) in (3.30), we arrive at the validity of (3.29) in the case.

**Case 2:** $\frac{3}{4} B_i \subset B_{2R}$ and $B_i \cap B_{2R}^c \neq \emptyset$. Using the definition of $v_i$ in (3.16) and the boundary Poincaré type inequality of Lemma 2.2, $I_0$ in (3.29) can be estimated as follows:

$$I_0 \leq c \|a\|_{L^\infty(B_{2R} \cap B_i)} \left( \int_{B_i} |v| \right)^{\frac{1}{\gamma}} \leq c \|a\|_{L^\infty(B_{2R} \cap B_i)} \left( \int_{B_i} |\nabla v|^{\gamma p} \right)^{\frac{1}{\gamma p}}$$

(3.33)

with a constant $c \equiv c(n, N, p)$. Recalling that

$$\|a\|_{L^\infty(B_{2R} \cap B_i)} \leq \inf_{x \in B_{2R} \cap B_i} a(x) + 4 [a]_{0, \alpha} r_i^\alpha,$$

we continue to estimate the resulting term in (3.33) with again applying Hölder’s inequality as follows

$$I_0 \leq cr_i \left( \int_{B_i} \left[ \inf_{x \in B_{2R} \cap B_i} a(x) \right] \left( |\nabla u| + \left| \frac{u - (u)_{B_2 2R}}{2R} \right| \right)^{\gamma} \chi_{B_{2R} 2R} \right)^{\frac{1}{\gamma}}$$

$$+ cr_i^{1 + \frac{q}{\gamma}} \left( \int_{B_i} \left( |\nabla u| + \left| \frac{u - (u)_{B_2 2R}}{2R} \right| \right)^{\gamma} \chi_{B_{2R} 2R} \right)^{\frac{1}{\gamma}}$$
for some constant \( c \equiv c(n, N, q, \alpha, [a]_{0, \alpha}) \). Once we arrive at this stage, the rest of the estimate (3.29) can be argued similarly as shown in Case 1.

**Case 3:** \( \frac{3}{4} B_i \cap B_{2R}^c \neq \emptyset \) and \( r_i \leq R \). In this case, we apply the boundary type Poincaré inequality of Lemma 2.2 to obtain

\[
I_0 = \|a\|_{L^\infty(B_{2R}^c \cap B_1)} \int_{B_1} |v| \, dx \leq \|a\|_{L^\infty(B_{2R}^c \cap B_{1/4})} r_i \left( \int_{1/4B_i} |\nabla v|^p \, dx \right)^{\frac{1}{p}}
\]

for a constant \( c \equiv c(n, N, p) \), where we have used the definition of \( v_i \) in (3.16). Then, the remaining can be done in exactly the same way as above.

**Case 4:** \( \frac{3}{4} B_i \cap B_{2R}^c \neq \emptyset \) and \( r_i > R \). By elementary calculations, we have

\[
I_0 = \|a\|_{L^\infty(B_{2R})} \int_{B_{2R}} |v| \, dx \leq \|a\|_{L^\infty(B_{2R})} \int_{B_{2R}} |v| \, dx
\]

for every \( y \in B_{2R} \cap B_{1/4} \), where we have used the fact that there exists \( y \in E(\lambda^{1/p}) \cap B_{2R} \). Finally, taking into account all the cases we considered above, the estimate (3.29) holds true. \( \square \)

**Lemma 3.6.** Under the settings and notations of Subsection 3.1, the function \( v_\lambda \) defined in (3.15) belongs to \( W^{1,\infty}(B_{2R}) \).

**Proof.** First, we show that there exists a constant \( c \equiv c(n, N, p) \) such that the following estimate

\[
\int_{B_{r}(z)} \left| \frac{v_\lambda - (v_\lambda)_{B_r(z)}}{r} \right| \, dx \leq c \lambda^{\frac{1}{p'}}
\]

holds, whenever \( B_r(z) \subset \mathbb{R}^n \) is a ball. In order to prove the inequality of (3.34), we divide several cases depending on the position of the ball \( B_r(z) \) and the upper level set of the function \( G \) in (3.7).

**Case 1:** \( B_r(z) \subset E(\lambda^{1/p}) \). Applying the mean value theorem and Lemma 3.5, we have

\[
|v_\lambda(y) - (v_\lambda)_{B_r(z)}| \leq r \sup_{x \in E(\lambda^{1/p})} |\nabla v_\lambda(x)| \leq cr \lambda^{\frac{1}{p'}}
\]

for every \( y \in B_r(z) \) with some constant \( c \equiv c(n, N, p) \).

**Case 2:** \( B_r(z) \cap E(\lambda^{1/p}) \neq \emptyset \). By the elementary inequalities, we find

\[
\int_{B_r(z)} \left| \frac{v_\lambda - (v_\lambda)_{B_r(z)}}{r} \right| \, dx \leq 2 \int_{B_r(z)} \left| \frac{v_\lambda - v}{r} \right| \, dx
\]

\[
+ 2 \int_{B_r(z)} \left| \frac{v - (v)_{B_r(z)}}{r} \right| \, dx =: 2(J_1 + J_2).
\]

Next, we shall estimate the resulting terms in the last display. By the definition of the function \( v_\lambda \) in (3.15), we have

\[
J_1 \leq \int_{B_r(z)} \sum_{i \in D} \left| \frac{v - v_i}{r} \right| \psi_i \, dx \leq \sum_{i \in D} \frac{1}{|B_r|} \int_{B_r(z) \cap \frac{3}{4}B_i} \left| \frac{v - v_i}{r} \right| \, dx,
\]

where

\[
D = \left\{ k \in \mathbb{N} : B_r(z) \cap \frac{3}{4}B_k \neq \emptyset \right\}.
\]
For any fixed ball $B_i \equiv B_r(x_i)$ of our covering introduced in Subsection 3.1 and any points $y_1 \in B_r(z) \cap \frac{3}{4}B_i$, $y_2 \in B_r(z) \cap E(\lambda^{1/p})$, we observe

$$8r_i \leq \text{dist}(x_i, E(\lambda^{1/p})) \leq |x_i - y_1| + |y_1 - y_2| \leq r_i + 2r.$$ 

Using the estimate of the last display in (3.36), we have

$$J_1 \leq \frac{c(n)}{|B_r|} \sum_{i \in D} \int_{B_r(z) \cap \frac{3}{4}B_i} \frac{|v - v_i|}{r_i} \, dx,$$  

(3.38)

where the index set $D$ has been defined in (3.37). In order to estimate the resulting term in the last display further, we consider subcases depending on a position of the balls of $\frac{3}{4}B_i$ and $B_2R$ for indices $i \in D$. For indices $i \in D$ such that $\frac{3}{4}B_i \subset B_2R$ in (3.38), recalling that there exists a point $y_i \in E(\lambda^{1/p}) \cap 16B_i$ by W2 and applying the Poincaré inequality, we estimate as

$$\int_{B_r(z) \cap \frac{3}{4}B_i} \frac{|v - v_i|}{r_i} \, dx \leq c \int_{\frac{3}{4}B_i} |\nabla v| \, dx,$$

$$\leq c \int_{\frac{3}{4}B_i} \left( |\nabla u| + \left| \frac{u - (u)_{B_{2R}}}{2R} \right| \right) \chi_{B_{2R}} \, dx,$$

$$\leq c |16B_i| \left( \int_{\frac{3}{4}B_i} \left( |\nabla u| + \left| \frac{u - (u)_{B_{2R}}}{2R} \right| \right)^{\frac{p}{p'}} \chi_{B_{2R}} \, dx \right)^{\frac{1}{p'}}$$

(3.39)

for some constant $c \equiv c(n, N)$, where we have also used the definition of the function $G$ in (3.7).

For indices $i \in D$ such that $\frac{3}{4}B_i \cap B_{2R} \neq \emptyset$ in (3.38), we apply Lemma 2.2 to have

$$\int_{B_r(z) \cap \frac{3}{4}B_i} \frac{|v - v_i|}{r_i} \, dx = c \int_{\frac{3}{4}B_i} |v| \, dx \leq c |16B_i|G(y_i) \leq c |16B_i|\lambda^\frac{1}{p'}$$

(3.40)

for some constant $c \equiv c(n, N, p)$, where $y_i \in E(\lambda^{1/p}) \cap 16B_i$ is a point determined via W2. Therefore, taking into account both subcases that we have discussed above together with inserting the estimates (3.39)-(3.40) into (3.38), we have

$$J_1 \leq \frac{c \lambda^\frac{1}{p'}}{|B_r|} \sum_{i \in D} \frac{1}{4} \left| \frac{1}{4}B_i \right|.$$

Recalling the disjointedness of the family $\left\{ \frac{1}{4}B_i \right\}_{i \in \mathbb{N}}$ and $7r_i \leq 2r$ for every $i \in D$ in (3.37), we conclude

$$J_1 \leq c \lambda^\frac{1}{p}.$$  

(3.41)

It remains to estimate the term $J_2$ in (3.35). As in the previous cases, applying the Poincaré inequality, we find

$$J_2 \leq c \int_{B_r(z)} |\nabla v| \, dx,$$

$$\leq c \int_{B_r(z)} \left( |\nabla u| + \left| \frac{u - (u)_{B_{2R}}}{2R} \right| \right) \chi_{B_{2R}} \, dx$$

$$\leq cG(y) \leq c \lambda^\frac{1}{p'},$$

(3.42)

for some constant $c \equiv c(n, N)$ and a point $y \in B_r(z) \cap E(\lambda^{1/p})$. Inserting the estimates (3.41) and (3.42) into (3.35), we arrive at the inequality (3.34). Therefore, the Hölder continuity characterization of Campanato and (3.34) yield the desired result of Lemma 3.6.
Remark 3.7. Under the settings and notations of Subsection 3.1, it is immediate to show the
following
\[
\lim_{\lambda \to \infty} \int_{B_R} [H(x, |\nabla u - \nabla v|)]^\delta \, dx = 0, \tag{3.43}
\]
where the constant \(\delta \in (\delta_0, 1)\) and the function \(v\) have been defined in (3.4) and (3.15), respectively. In fact, the last display means that any very weak solution \(u\) of \((1.1)\) under the assumption \((1.3)\) can be approximated locally by Lipschitz truncated functions. For the proof of \((3.43)\), recalling \((3.11)\) and \((3.12)\), it suffices to show the following assertion
\[
\lim_{\lambda \to \infty} \int_{B_{2R}} [H(x, |\nabla v - \nabla v_\lambda|)]^\delta \, dx = 0, \tag{3.44}
\]
where the function \(v\) is defined in \((3.12)\). For this, by the definition of \(v\) and \(v_\lambda\), we first observe
\[
\int_{B_{2R}} [H(x, |\nabla v - \nabla v_\lambda|)]^\delta \, dx = \int_{E(\lambda^{1/p})^c} [H(x, |\nabla v - \nabla v_\lambda|)]^\delta \, dx
\leq c \int_{E(\lambda^{1/p})^c} [H(x, |\nabla v|)]^\delta \, dx + c \int_{E(\lambda^{1/p})^c} [H(x, |\nabla v_\lambda|)]^\delta \, dx, \tag{3.45}
\]
for a constant \(c \equiv c(p, q)\). Applying Lemma 3.1 and \((1.15)\), we find
\[
\int_{B_{2R}} [H(x, |\nabla v|)]^\delta \, dx \leq c \int_{B_{2R}} \left( [H(x, |\nabla u|)]^\delta + \left[ H \left( x, \frac{|u - (u)_{B_{2R}}|}{R} \right) \right]^\delta \right) \, dx
\leq c \int_{B_{2R}} [H(x, |\nabla u|)]^\delta \, dx < \infty
\]
for a constant \(c \equiv c(n, N, p, q, [a]_{0, \alpha})\). In turn, the last display implies
\[
\lim_{\lambda \to \infty} \int_{E(\lambda^{1/p})^c} [H(x, |\nabla v_\lambda|)]^\delta \, dx = 0.
\]
On the other hand, applying again Lemma 3.5, we get
\[
\lim_{\lambda \to \infty} \int_{E(\lambda^{1/p})^c} [H(x, |\nabla v_\lambda|)]^\delta \, dx \leq c \lim_{\lambda \to \infty} \lambda^\delta |E(\lambda^{1/p})^c|
\leq c \lim_{\lambda \to \infty} \int_{E(\lambda^{1/p})^c} [G(x)]^{n\delta} \, dx = 0.
\]
for a constant \(c \equiv c(n, N, p, q, [a]_{0, \alpha})\), where we have used \(E(\lambda^{1/p})^c = \{ x \in \mathbb{R}^n \mid G(x) > \lambda^{1/p} \}\) and the fact that \(G \in L^{\delta p}(B_{2R})\) via Lemma 3.2. Combination of the results from the last two displays in \((3.44)\) yields \((3.44)\). As a direct consequence, the assertion in \((3.43)\) holds true.

Lemma 3.8. Under the settings and notations of Subsection 3.1, the following inequality holds true:
\[
\int_{B_R \cap E(\lambda^{1/p})} H(x, |\nabla u|) \, dx
\leq c \int_{B_{2R} \cap E(\lambda^{1/p})^c} \left( |\nabla u|^{p-1} \lambda^{\frac{p}{q}} + [a(x)]^{\frac{q-1}{q}} |\nabla u|^{q-1} \lambda^{\frac{q}{p}} \right) \, dx
+ c \int_{B_{2R} \cap E(\lambda^{1/p})} H \left( x, \frac{|u - (u)_{B_{2R}}|}{2R} \right) \, dx \tag{3.46}
+ c \int_{B_{2R} \cap E(\lambda^{1/p})^c} \left( |F|^{q-1} \lambda^{\frac{q}{p}} + [a(x)]^{\frac{q-1}{q}} |F|^{q-1} \lambda^{\frac{q}{p}} \right) \, dx
+ c \int_{B_{2R} \cap E(\lambda^{1/p})} H(x, |F|) \, dx,
\]
where \(c \equiv c(n, N, p, q, \alpha, \nu, L, [a]_{0, \alpha})\).
Proof. With the functions \( \eta \) defined in (3.11) and \( \nu_\lambda \) defined in (3.15), we take \( v_\lambda \eta^q \in W_0^{1,\infty}(B_{2R}) \) as a test function to the system (1.1) by recalling Lemma 3.6. Then we have

\[
I_1 := \int_{B_{2R}} \langle A(x, \nabla u), \nabla v_\lambda \rangle \eta^q \, dx \\
= -q \int_{B_{2R}} \langle A(x, \nabla u), v_\lambda \nabla \eta \rangle \eta^{q-1} \, dx \\
+ \int_{B_{2R}} (|F|^{p-2}F + a(x)|F|^{q-2}F) \nabla (v_\lambda \eta^q) \, dx =: I_2 + I_3.
\]

Recalling the definition of \( v_\lambda \) in (3.15), we see

\[
I_1 = \int_{B_{2R} \cap E(\lambda^{1/p})} \langle A(x, \nabla u), \nabla u \rangle \eta^{q+1} \, dx \\
+ \int_{B_{2R} \cap E(\lambda^{1/p})} \langle A(x, \nabla u), (u - (u)_{B_{2R}}) \nabla \eta \rangle \eta^q \, dx \\
+ \int_{B_{2R} \cap E(\lambda^{1/p})} \langle A(x, \nabla u), \nabla v_\lambda \rangle \eta^q \, dx =: I_{11} + I_{12} + I_{13}.
\]

Using the structure assumption (1.2), we have

\[
I_{11} \geq \nu \int_{B_{2R} \cap E(\lambda^{1/p})} H(x, |\nabla u|) \eta^{q+1} \, dx. \tag{3.49}
\]

Again using (1.2) and Young’s inequality, we find

\[
I_{12} \geq -c \int_{B_{2R} \cap E(\lambda^{1/p})} (|\nabla u|^{p-1} + a(x)|\nabla u|^{q-1}) \left| \frac{u - (u)_{B_{2R}}}{2R} \right| \eta^q \, dx \\
\geq -\varepsilon \int_{B_{2R} \cap E(\lambda^{1/p})} \left( |\nabla u|^{p} \eta^{\frac{p}{p-1}} + a(x)|\nabla u|^{q} \eta^{\frac{q}{q-1}} \right) \, dx \\
- c_\varepsilon \int_{B_{2R} \cap E(\lambda^{1/p})} H \left( x, \left| \frac{u - (u)_{B_{2R}}}{2R} \right| \right) \, dx \tag{3.50}
\]

\[
\geq -\varepsilon \int_{B_{2R} \cap E(\lambda^{1/p})} H(x, |\nabla u|) \eta^{q+1} \, dx \\
- c_\varepsilon \int_{B_{2R} \cap E(\lambda^{1/p})} H \left( x, \left| \frac{u - (u)_{B_{2R}}}{2R} \right| \right) \, dx
\]

with some constant \( c_\varepsilon \equiv c_\varepsilon(p, q, \nu, L, \varepsilon) \), where \( \varepsilon > 0 \) is arbitrarily given. Here, we have also used the fact that

\[
\frac{qp}{p-1} \geq \frac{q^2}{q-1} \geq q + 1.
\]

For the estimate of \( I_{13} \) in (3.48), we apply the structure assumption (1.2) and then Lemma 3.5 to find

\[
I_{13} \geq -L \int_{B_{2R} \cap E(\lambda^{1/p})} (|\nabla u|^{p-1} + a(x)|\nabla u|^{q-1}) |\nabla v_\lambda| \, dx \\
\geq -c \int_{B_{2R} \cap E(\lambda^{1/p})} \left( |\nabla u|^{p-1} \lambda^{\frac{1}{p'}} + [a(x)]^{\frac{q-1}{q}} |\nabla u|^{q-1} \lambda^{\frac{1}{q'}} \right) \, dx \tag{3.51}
\]
for some constant $c \equiv c(n, N, p, q, \nu, L)$. Inserting the estimates obtained in (3.49), (3.50) and (3.51) into (3.48) and also selecting $\epsilon \equiv \nu/2$, we have the estimate for $I_1$ in (3.47) as

$$I_1 \geq \nu/2 \int_{B_{2R} \cap E(\lambda^{1/p})} H(x, |\nabla u|) \eta^{q+1} \, dx$$

$$- c \int_{B_{2R} \cap E(\lambda^{1/p})} H \left( x, \frac{|u - (u)_{B_{2R}}|}{2R} \right) \, dx$$

$$- c \int_{B_{2R} \cap E(\lambda^{1/p})} \left( |\nabla u|^p - \lambda^2 + [a(x)] \frac{2 - q}{q-1} |\nabla u|^{q-1} \right) \, dx$$

for some constant $c \equiv c(n, N, p, q, \nu, L)$. Now, we turn our attention to the term $I_2$ in (3.47). Using again (1.2), we see

$$I_2 \leq c \int_{B_{2R} \cap E(\lambda^{1/p})} \left( |\nabla u|^{p-1} + a(x) |\nabla u|^{q-1} \right) \frac{|u - (u)_{B_{2R}}|}{2R} \eta^q \, dx$$

$$+ c \int_{B_{2R} \cap E(\lambda^{1/p})} \left( |\nabla u|^{p-1} \frac{|\lambda|}{R} + a(x) |\nabla u|^{q-1} \frac{|\lambda|}{R} \right) \, dx$$

with a constant $c \equiv c(n, N, p, q, \nu, L)$. Then arguing similarly as we have done in (3.50) for the first term and applying Lemma 3.3 for the second one in (3.52), we continue to estimate (3.52) as

$$I_2 \leq \epsilon \int_{B_{2R} \cap E(\lambda^{1/p})} H(x, |\nabla u|) \eta^{q+1} \, dx$$

$$+ c \epsilon \int_{B_{2R} \cap E(\lambda^{1/p})} H \left( x, \frac{|u - (u)_{B_{2R}}|}{2R} \right) \, dx$$

$$+ c \int_{B_{2R} \cap E(\lambda^{1/p})} \left( |\nabla u|^{p-1} \frac{|\lambda|}{R} + [a(x)] \frac{2 - q}{q-1} |\nabla u|^{q-1} \frac{|\lambda|}{R} \right) \, dx$$

for constants $c_\epsilon \equiv c_\epsilon(n, N, p, q, \nu, L, \epsilon)$, $c \equiv c(n, N, p, q, \nu, L)$, whenever $\epsilon > 0$. Finally, we shall move onto estimating the term $I_3$ in (3.47). To this end, we first write it in the following.

$$I_3 = \int_{B_{2R} \cap E(\lambda^{1/p})} \langle F \rangle^{p-2} F + a(x) |F|^{q-2} F, \nabla (v_\lambda \eta^q) \rangle \, dx$$

$$+ \int_{B_{2R} \cap E(\lambda^{1/p})} \langle F \rangle^{p-2} F + a(x) |F|^{q-2} F, \nabla (v_\lambda \eta^q) \rangle \, dx$$

$$=: I_{31} + I_{32}.$$ 

Applying Young’s inequality, for every $\epsilon > 0$, we have

$$I_{31} \leq \epsilon \int_{B_{2R} \cap E(\lambda^{1/p})} H(x, |\nabla (v_\lambda \eta^q)|) \, dx$$

$$+ c_\epsilon \int_{B_{2R} \cap E(\lambda^{1/p})} H \left( x, |F| \right) \, dx =: \epsilon J_1 + c_\epsilon J_2$$

with some constant $c_\epsilon \equiv c_\epsilon(n, N, p, q, \nu, \epsilon)$. We shall deal with the terms appearing in the last display further. Recalling the definition of $v_\lambda$ in (3.15) and using Young’s inequality, we find

$$J_1 \leq c \int_{B_{2R} \cap E(\lambda^{1/p})} \left( |\nabla u|^p \eta^{p(q+1)} + a(x) |\nabla u|^{q-1} \eta^{q(q+1)} \right) \, dx$$

$$+ c \int_{B_{2R} \cap E(\lambda^{1/p})} H \left( x, \frac{|u - (u)_{B_{2R}}|}{2R} \right) \, dx$$

$$\leq c \int_{B_{2R} \cap E(\lambda^{1/p})} H(x, |\nabla u|) \eta^{q+1} \, dx$$

$$+ c \int_{B_{2R} \cap E(\lambda^{1/p})} H \left( x, \frac{|u - (u)_{B_{2R}}|}{2R} \right) \, dx$$
with a constant $c \equiv c(n, N, p, q, \nu)$. Therefore, inserting the resulting estimate of the above display into (3.55) and reabsorbing the terms, we conclude

$$I_{31} \leq \varepsilon \int_{B_R \cap E(\lambda^{1/p})} H(x, |\nabla u|)^{q+1} \, dx$$

$$+ c_\varepsilon \int_{B_R \cap E(\lambda^{1/p})} H \left( x, \frac{|u - (u)_{B_R}|}{2R} \right) \, dx$$

$$+ c_\varepsilon \int_{B_R \cap E(\lambda^{1/p})} H(x, |F|) \, dx$$

for a constant $c_\varepsilon \equiv c_\varepsilon(n, N, p, q, \nu, \varepsilon)$, whenever $\varepsilon > 0$ is an arbitrary number. Now we estimate the remaining term $I_{32}$ in (3.54). Recalling the definition of $v_\lambda$ in (3.15) again and applying Lemma 3.3 and Lemma 3.5, we find

$$I_{32} \leq c \int_{B_R \cap E(\lambda^{1/p})} \left[ |F|^{p-1} \left( \frac{|v_\lambda|}{R} + |\nabla v_\lambda| \right) + |a(x)|^{\frac{p-2}{2}} |F|^{q-1} \left( |a(x)| \frac{|v_\lambda|}{R} + |a(x)|^{\frac{1}{2}} |\nabla v_\lambda| \right) \right] \, dx$$

$$\leq c \int_{B_R \cap E(\lambda^{1/p})} \left( |F|^{p-1} \lambda^{\frac{q}{p}} + |a(x)|^{\frac{p-2}{2}} |F|^{q-1} \lambda^{\frac{q}{p}} \right) \, dx$$

with a constant $c \equiv c(n, N, p, q, \nu, L)$. Putting the estimates in (3.56)-(3.57) into (3.54), for every $\varepsilon > 0$, we have

$$I_3 \leq \varepsilon \int_{B_R \cap E(\lambda^{1/p})} H(x, |\nabla u|)^{q+1} \, dx$$

$$+ c_\varepsilon \int_{B_R \cap E(\lambda^{1/p})} H \left( x, \frac{|u - (u)_{B_R}|}{2R} \right) \, dx$$

$$+ c_\varepsilon \int_{B_R \cap E(\lambda^{1/p})} H(x, |F|) \, dx$$

$$+ c \int_{B_R \cap E(\lambda^{1/p})} \left( |F|^{p-1} \lambda^{\frac{q}{p}} + |a(x)|^{\frac{p-2}{2}} |F|^{q-1} \lambda^{\frac{q}{p}} \right) \, dx$$

for some constants $c_\varepsilon \equiv c_\varepsilon(n, N, p, q, \nu, L, \varepsilon)$ and $c \equiv c(n, N, p, q, \nu, L)$. Finally, collecting all the estimates obtained in (3.52), (3.53) and (3.58), inserting them into (3.47) and then choosing $\varepsilon$ small enough together with reabsorbing resulting terms, we arrive at the desired estimate (3.46). The proof is completed. \hfill \Box

Finally, we arrive at the stage of providing a Caccioppoli type inequality under the settings and notations of Subsection 3.1.

**Lemma 3.9.** Under the settings and notations of Subsection 3.1, there exist constants $\delta_2 \in (1 - 1/q, 1)$ and $c > 0$ depending on $n, N, p, q, \alpha, \nu, L, |a|_{0, \alpha}$ such that if $\delta_2 \leq \delta_0 < \delta$, then

$$\int_{B_R} [H(x, |\nabla u|)]^\delta \, dx \leq c \int_{B_R} \left[ H \left( x, \frac{|u - (u)_{B_R}|}{2R} \right) \right]^\delta \, dx + c \int_{B_R} [H(x, |F|)]^\delta \, dx.$$  \hfill (3.59)

**Proof.** First applying Lemma 3.8, we have

$$J_0 \leq c_\ast (J_1 + J_2 + J_3 + J_4),$$  \hfill (3.60)
for some constant \( c_\ast \equiv c_\ast(n, N, p, q, \alpha, \nu, L, [a]_{0, \alpha}) \), where

\[
\begin{align*}
J_0 &:= \int_{c_\ast \Lambda^{1/p}} \lambda^{-\kappa} \int_{B_R \setminus E(\lambda^{1/p})} H(x, |\nabla u|) \, dx \, d(\lambda^\ast), \\
J_1 &:= \int_{c_\ast \Lambda^{1/p}} \lambda^{-\kappa} \int_{B_{2R} \cap E(\lambda^{1/p})} \left( |\nabla u|^{p-1} \lambda^\ast + |a(x)|^{\frac{q-1}{p}} |\nabla u|^{q-1} \lambda^\ast \right) \, dx \, d(\lambda^\ast), \\
J_2 &:= \int_{c_\ast \Lambda^{1/p}} \lambda^{-\kappa} \int_{B_{2R} \cap E(\lambda^{1/p})} H \left( x, \frac{u - (u)_{B_{2R}}}{2R} \right) \, dx \, d(\lambda^\ast), \\
J_3 &:= \int_{c_\ast \Lambda^{1/p}} \lambda^{-\kappa} \int_{B_{2R} \cap E(\lambda^{1/p})} \left( |F|^{p-1} \lambda^\ast + |a(x)|^{\frac{q-1}{p}} |F|^{q-1} \lambda^\ast \right) \, dx \, d(\lambda^\ast), \\
J_4 &:= \int_{c_\ast \Lambda^{1/p}} \lambda^{-\kappa} \int_{B_{2R} \cap E(\lambda^{1/p})} H(x, |F|) \, dx \, d(\lambda^\ast).
\end{align*}
\] (3.61)

Here \( \kappa := (1 - \delta) + 1/p \) and the constant \( c_\ast \equiv c_\ast(n, N, p, q, [a]_{0, \alpha}) \) has been determined in Lemma 3.2. In the following, we shall deal with all the terms appearing in (3.60) defined in (3.61). To go further, let us introduce auxiliary notations

\[
G_\Lambda(x) := \max\{c_\ast \Lambda^{1/p}, G(x)\}
\]
and

\[
U_R := \{ x \in B_R \mid H(x, |\nabla u|) \geq (1 - \delta) |G_\Lambda(x)|^p \}. \tag{3.62}
\]

Recalling (3.10) and applying Fubini’s theorem, we find

\[
J_0 = \int_{c_\ast \Lambda^{1/p}} \lambda^{-\kappa-(1-\delta)-1/p} \int_{B_R} H(x, |\nabla u|) \chi_{\{G(x) \leq \Lambda^{1/p}\}} \, dx \, d(\lambda^\ast) = \int_{B_R} H(x, |\nabla u|) \int_{G_\Lambda(x)} \lambda^{-\kappa-(1-\delta)-1/p} \, d(\lambda^\ast) \, dx = \frac{1}{p(1-\delta)} \int_{B_R} H(x, |\nabla u|) [G_\Lambda(x)]^{-p+\delta p} \, dx, \tag{3.63}
\]

where function \( G_\Lambda \) has been defined in (3.62). On the other hand, we observe that

\[
\int_{B_R} |H(x, |\nabla u|)|^\delta \, dx = \int_{U_R} |H(x, |\nabla u|)|^\delta \, dx + \int_{B_R \setminus U_R} |H(x, |\nabla u|)|^\delta \, dx \leq (1 - \delta)^{-(1-\delta)} \int_{U_R} H(x, |\nabla u|) [G_\Lambda(x)]^{-p+\delta p} \, dx
\]
\[
\quad + (1 - \delta)^{\delta} \int_{B_R \setminus U_R} [G_\Lambda(x)]^{\delta p} \, dx \leq (1 - \delta)^{-(1-\delta)} \int_{B_R} H(x, |\nabla u|) [G_\Lambda(x)]^{-p+\delta p} \, dx
\]
\[
\quad + c(1 - \delta)^{\delta} \Lambda^\ast |B_{2R}| \tag{3.64}
\]

for some constant \( c \equiv c(n, N, p, q, [a]_{0, \alpha}) \), where we have used Lemma 3.2 and \( U_R \) is the set introduced in (3.62). Therefore, combining (3.63)-(3.64), we conclude

\[
\frac{1}{p} \left( \frac{1}{1-\delta} \right)^{\delta} \int_{B_R} |H(x, |\nabla u|)|^\delta \, dx - c\Lambda^\ast |B_{2R}| \leq J_0. \tag{3.65}
\]
We now deal with the terms $J_k$ for $k \in \{1, 2, 3, 4\}$ appearing in (3.61). In turn, we have

\[
J_1 = \int_{c, A^1/p}^{\infty} \lambda^{-(1-\delta)-1/p} \int_{B_{2R}} |\nabla u|^{p-1} \lambda^\sharp d\lambda^\sharp dx \ d(\lambda^\sharp)
\]

\[
+ \int_{c, A^1/p}^{\infty} \lambda^{-(1-\delta)-1/p} \int_{B_{2R} \cap E(\lambda^1/p)c} [a(x)]^\frac{\lambda^\sharp}{p} |\nabla u|^{q-1} \lambda^\sharp d\lambda^\sharp dx \ d(\lambda^\sharp)
\]

\[
= : J_{11} + J_{12}.
\]

Therefore, applying Fubini’s theorem and recalling the definition of the level set $E(\lambda^1/p)$ in (3.10), we see

\[
J_{11} = \int_{c, A^1/p}^{\infty} \lambda^{-(1-\delta)-1/p} \int_{B_{2R}} |\nabla u|^{p-1} \lambda^\sharp \chi\{G(x) \geq \lambda^1/p\} \ d\lambda^\sharp dx \ d(\lambda^\sharp)
\]

\[
\leq \int_{B_{2R}} |\nabla u|^{p-1} \int_{c, A^1/p}^{G(x)} \lambda^{-(1-\delta)} \ d(\lambda^\sharp) dx
\]

\[
\leq \frac{1}{1 - p(1-\delta)} \int_{B_{2R}} |\nabla u|^{p-1}[G(x)]^{1-p(1-\delta)} \ d(\delta) dx
\]

\[
\leq \frac{1}{1 - p(1-\delta)} \int_{B_{2R}} |G(x)|^\delta \ d(\delta) dx,
\]

where we have used the definition of the function $G$ in (3.7) and the fact that

\[
|\nabla u(x)| \leq G(x) \quad \text{a.e. in } B_{2R}.
\]

Again using Fubini’s theorem, we estimate $J_{12}$ in (3.66) as

\[
J_{12} = \int_{B_{2R}} [a(x)]^\frac{\lambda^\sharp}{q} |\nabla u|^{q-1} \int_{c, A^1/p}^{G(x)} \lambda^{1-q-(1-\delta)-1/p} \ d(\lambda^\sharp) dx
\]

\[
\leq \frac{1}{q - p(1-\delta)} \int_{B_{2R}} ([a(x)]^\frac{\lambda^\sharp}{q} |\nabla u|^{q-1}[G(x)]^\frac{\lambda^\sharp}{q-p(1-\delta)} \ d(\delta) dx
\]

\[
\leq \frac{1}{q - p(1-\delta)} \int_{B_{2R}} [G(x)]^\delta \ d(\delta) dx.
\]

In order to obtain the last inequality, we have used the fact that

\[
a(x)|\nabla u(x)|^q \leq [G(x)]^p \quad \text{a.e. in } B_{2R}.
\]

Then combining the estimates (3.66)-(3.67), we conclude

\[
J_1 \leq \frac{c}{1 - q(1-\delta)} \int_{B_{2R}} |G(x)|^\delta \ d(\delta) dx
\]

for some constant $c \equiv c(p, q)$. Now we continue to estimate the next terms in (3.61). In turn, again using Fubini’s theorem, we see

\[
J_2 = \int_{B_{2R}} H\left(x, \left|\frac{u - (u)_{B_{2R}}}{2R}\right|\right) \int_{G(x)}^{\infty} \lambda^{-\kappa} \ d(\lambda^\sharp) dx
\]

\[
\leq \int_{B_{2R}} H\left(x, \left|\frac{u - (u)_{B_{2R}}}{2R}\right|\right) \int_{G(x)}^{\infty} \lambda^{-\kappa} \ d(\lambda^\sharp) dx
\]

\[
\leq \frac{1}{p(1-\delta)} \int_{B_{2R}} H\left(x, \left|\frac{u - (u)_{B_{2R}}}{2R}\right|\right) [G(x)]^{-p+\delta} \ d(\delta) dx
\]

\[
\leq \frac{1}{p(1-\delta)} \int_{B_{2R}} H\left(x, \left|\frac{u - (u)_{B_{2R}}}{2R}\right|\right) \ d(\delta) dx.
\]

The last inequality follows from the fact that

\[
H\left(x, \left|\frac{u - (u)_{B_{2R}}}{2R}\right|\right) \leq [G(x)]^{p\gamma} \quad \text{a.e. in } B_{2R},
\]

(3.70)
Finally, we shall treat the remaining terms $J_3$ and $J_4$ in (3.61). Essentially, arguing similarly as we have done in (3.66)-(3.68) for $I_3$ and in (3.69)-(3.70) for $I_4$, we conclude

$$J_3 + J_4 \leq \frac{c}{1-q(1-\delta)} \int_{B_{2R}} |G(x)|^\delta p \, dx + \frac{1}{p(1-\delta)} \int_{B_{2R}} |H(x,|u|)|^\delta \, dx$$

(3.71)

for some constant $c \equiv c(p,q)$. Inserting the estimates obtained in (3.65), (3.68), (3.69) and (3.71) into (3.60) and applying Lemma 3.2, we find

$$\left( \frac{1}{1-\delta} \right)^\delta \int_{B_R} |H(x,|\nabla u|)|^\delta \, dx - c \lambda^\delta \|B_{2R}\|

\leq \frac{c}{1-q(1-\delta)} \int_{B_{2R}} |G(x)|^\delta p \, dx + \frac{c}{1-\delta} \int_{B_{2R}} \left[H \left(x, \left| \frac{u-(u)_{2R}}{2R} \right| \right) + H(x,|F|) \right]^\delta \, dx

\leq \frac{c \lambda^{\delta \delta}}{1-q(1-\delta)} \lambda^\delta \|B_R\|

+ c_1 (1-\delta)^{\delta-1} \int_{B_{2R}} \left[H \left(x, \left| \frac{u-(u)_{2R}}{2R} \right| \right) \right]^\delta \, dx

+ c_2 (1-\delta)^{\delta-1} \int_{B_{2R}} |H(x,|F|)|^\delta \, dx

$$

for some constant $c \equiv c(n,N,p,q,\alpha,\nu,L,[a]_{\alpha,\delta})$. At this moment, we apply Lemma 3.2 in order to have

$$\int_{B_R} |H(x,|\nabla u|)|^\delta \, dx \leq \frac{c \lambda^{\delta \delta}}{1-q(1-\delta)} \lambda^\delta \|B_R\|

+ c_1 (1-\delta)^{\delta-1} \int_{B_{2R}} \left[H \left(x, \left| \frac{u-(u)_{2R}}{2R} \right| \right) \right]^\delta \, dx

+ c_2 (1-\delta)^{\delta-1} \int_{B_{2R}} |H(x,|F|)|^\delta \, dx

$$

for some constant $c \equiv c(n,N,p,q,\alpha,\nu,L,[a]_{\alpha,\delta})$. Now we take $\delta_2 \equiv \delta_2(n,N,p,q,\alpha,\nu,L,[a]_{\alpha,\delta})$ close enough to 1 with $\delta_2 \leq \delta_0 < \delta < 1$ so that

$$\frac{c \lambda^{\delta \delta}}{1-q(1-\delta)} \lambda^\delta \|B_R\|

\leq \frac{c \lambda^{\delta \delta}}{1-q(1-\delta_2)} \lambda^{\delta_2} \|B_R\|

\leq \frac{\lambda^{\delta \delta}}{2}.

$$

Taking into account the content of the last display together with recalling (3.6) and the fact that there exists a constant $c$ with

$$(1-t)^{t-1} = e^{(t-1)\log(1-t)} \leq e^{\left|(t-1)\log(1-t)\right|} \leq c \text{ for all } t \in (0,1),$$

we find a Caccioppoli type inequality of (3.59). The proof is completed. □

**Lemma 3.10 (Reverse Hölder inequality).** Under the assumptions and conclusions of Lemma 3.9, there exists a constant $\delta_1 \equiv \delta_1(n,N,p,q,\alpha,\nu,L,[a]_{\alpha,\delta})$ with $\delta_2 < \delta_1 < 1$ such that if $\delta_1 \leq \delta_0 < \delta < 1$, then there holds

$$\int_{B_R} |H(x,|\nabla u|)|^\delta \, dx \leq c \left( \int_{B_{2R}} |H(x,|\nabla u|)|^{\delta_1} \, dx \right)^{\frac{1}{\delta_1}} + c \left( \int_{B_{2R}} |H(x,|F|)|^{\delta_1} \, dx \right)^{\frac{1}{\delta_1}}

(3.72)

for some constant $c \equiv c(n,N,p,q,\alpha,\nu,L,[a]_{\alpha,\delta})$.

**Proof.** Applying the Sobolev-Poincaré inequality and observing that $\frac{q}{p} < 1^* = \frac{n}{n-1}$, there exists a constant $\delta_1 \equiv \delta_1(n,N,p,q,\alpha,\nu,L,[a]_{\alpha,\delta}) \in (\delta_2,1)$ such that

$$\int_{B_{2R}} \left( \frac{|u-(u)_{2R}|}{2R} \right)^p + \inf_{B_{2R}} a(x) \left( \frac{|u-(u)_{2R}|}{2R} \right)^q \, dx

\leq c \left( \int_{B_{2R}} |\nabla u|^{\delta_1 p} + \left[ \inf_{B_{2R}} a(x) \right]^{\delta_1} \left| \nabla u \right|^{\delta_1 q} \, dx \right)^{\frac{1}{\delta_1}}

\leq c \left( \int_{B_{2R}} \left( \frac{H(x,|\nabla u|)}{\left| \nabla u \right|^\delta} \right) \, dx \right)^{\frac{1}{\delta_1}}

(3.73)$$
for some constant \( c \equiv c(n, N, p, q, \alpha, \nu, L, [a]_\alpha) \). In addition, if \( \delta_1 \leq \delta_0 < \delta \), then we have

\[
R^\frac{\alpha}{p} \int_{B_{2R}} \left| \frac{u - (u)_{B_{2R}}}{2R} \right|^q dx \leq c R^\alpha \left( \int_{B_{2R}} |\nabla u|^{\delta_1 p} dx \right)^{\frac{q}{p}} \\
\leq c R^\alpha \left( \int_{B_{2R}} |\nabla u|^{\delta_1 p} dx \right)^{\frac{q}{p}} \left( \int_{B_{2R}} |\nabla u|^{\delta_1 p} dx \right)^{\frac{q}{\delta_1}} \\
\leq c R^{\alpha - \frac{(q-p)\delta_1}{\delta_1}} \left( \int_{B_{2R}} |\nabla u|^{\delta_1 p} dx \right)^{\frac{q}{\delta_1}} \left( \int_{B_{2R}} |\nabla u|^{\delta_1 p} dx \right)^{\frac{q}{\delta_1}} \tag{3.74}
\]

for some constant \( c \equiv c(n, N, p, q, \alpha, \nu, L, [a]_\alpha) \), where we have also used Hölder’s inequality and (3.4). Therefore, if \( \delta_1 \leq \delta_0 < \delta \), then we are able to apply Lemma 3.9 to obtain

\[
\int_{B_R} |H(x, |\nabla u|)|^\delta dx \leq c \int_{B_R} \left( H \left( x, \left| \frac{u - (u)_{B_{2R}}}{2R} \right| \right) \right)^\delta dx + c \int_{B_R} |H(x, |F|)|^\delta dx
\]

for some constant \( c \equiv c(n, N, p, q, \alpha, \nu, L, [a]_\alpha) \). Then using the estimates (3.73), (3.74) and the Hölder continuity of the coefficient function \( a(\cdot) \), we see that if \( \delta_1 \leq \delta_0 < \delta \), then we have

\[
\int_{B_{2R}} \left( H \left( x, \left| \frac{u - (u)_{B_{2R}}}{2R} \right| \right) \right)^\delta dx \leq \left( \int_{B_{2R}} H \left( x, \left| \frac{u - (u)_{B_{2R}}}{2R} \right| \right) dx \right)^\delta \\
\leq c \left( \int_{B_{2R}} \left| \frac{u - (u)_{B_{2R}}}{2R} \right|^p dx + \inf_{B_{2R}} a(x) \int_{B_{2R}} \left| \frac{u - (u)_{B_{2R}}}{2R} \right|^q dx \right)^\delta \\
+ c \left( \int_{B_{2R}} |\nabla u|^p a(x)|\nabla u|^q dx \right)^\delta \\
\leq c \left( \int_{B_{2R}} \left| |\nabla u|^p + a(x)|\nabla u|^q \right)^\delta dx \right)^\delta \tag{3.72}
\]

We combine the last two displays to obtain (3.72). The proof is completed. \( \square \)

4. PROOF OF THEOREM 1.2

Finally, we provide the proof of our main Theorem 1.2. It is based on the Gehring lemma [25, 36] and the stopping time argument and covering lemma used in [9, 30] under the double phase settings. Let \( u \in W^{1,1}(\Omega, \mathbb{R}^N) \) be a very weak solution to the system (1.1) with

\[
\int_{\Omega} (|\nabla u|^p + a(x)|\nabla u|^q)^\delta dx < \infty
\]

for some \( \delta \in (\delta_0, 1) \), where \( \delta_0 \) is a fixed constant satisfying (3.2)-(3.3), which will be determined at the end of the proof. The remaining part of the proof consists of two main steps.

**Step 1: Stopping time argument and covering.** Let \( B_{2r} \equiv B_{2r}(x_0) \subset \Omega \) be a fixed ball with \( 2r \leq R_0 \), where the size of \( R_0 \) is selected to satisfy

\[
R_0^\frac{\alpha}{p} \left( \int_{\Omega} |\nabla u|^{\delta_1 p} dx \right)^{\frac{q}{p} \left(1 - \frac{\alpha}{q} \right)} \leq 1. \tag{4.1}
\]

We select radii \( r_1, r_2 \) such that \( r \leq r_1 < r_2 \leq 2r \) and consider the following level sets

\[
E(\Lambda, \varrho) := \{ x \in B_r(x_0) : H(x, |\nabla u|) \leq \Lambda \},
\]

\[
S(\Lambda, \varrho) := \{ x \in B_r(x_0) : H(x, |\nabla u|) > \Lambda \} \tag{4.2}
\]

and

\[
T(\Lambda, \varrho) := \{ x \in B_r(x_0) : H(x, |F|) > \Lambda \} \tag{4.3}
\]
for every \( r \leq \varrho \leq 2r \) and \( \Lambda > 0 \). Also, we define the quantity

\[
\Phi(B_\varrho(y)) := \int_{B_\varrho(y)} [H(x,|\nabla u|) + H(x,|F|)]^\delta \, dx,
\]

whenever \( B_\varrho(y) \subset B_{2r} \) is a ball. Then we observe that

\[
\lim_{s \to 0^+} \Phi(B_\varrho(y)) > \Lambda^\delta \text{ for a.e } y \in S(\Lambda^\delta, \varrho) \text{ with } r \leq \varrho \leq 2r
\]

and that if \( y \in B_{r_1} \), then we have, for every \( \varrho \in ((r_2 - r_1)/20, r_2 - r_1) \),

\[
\Phi(B_\varrho(y)) \leq \left( \frac{2r}{\varrho} \right)^n \left[ 1 + \int_{B_{2r}} [H(x,|\nabla u|) + H(x,|F|)]^\delta \, dx \right]
\]

\[
\leq \left( \frac{40r}{r_2 - r_1} \right)^n \Lambda_0^\delta,
\]

where

\[
\Lambda_0^\delta := 1 + \int_{B_{2r}} [H(x,|\nabla u|) + H(x,|F|)]^\delta \, dx.
\]

From now on, we shall always consider the values of \( \Lambda \) satisfying

\[
\Lambda^\delta > \left( \frac{40r}{r_2 - r_1} \right)^n \Lambda_0^\delta.
\]

Then for almost every \( y \in S(\Lambda^\delta, r_1) \), there exists an exit time \( \varrho_y < (r_2 - r_1)/20 \) such that

\[
\Phi(B_{\varrho_y}(y)) = \Lambda^\delta \quad \text{and} \quad \Phi(B_{\varrho(y)}(y)) < \Lambda^\delta \text{ for every } \varrho \in (\varrho_y, r_2 - r_1).
\]

Therefore, the family \( \{B_{\varrho_y}(y)\} \) covers \( S(\Lambda^\delta, r_1) \) up to a negligible set. Observing that \( \varrho_y \leq (r_2 - r_1)/20 \leq R_0 \) for almost every \( y \in S(\Lambda^\delta, r_1) \) and recalling (4.1)-(4.5), we are in a position to apply Lemma 3.10 with \( B_{2R} \) replaced by \( B_{2\varrho_y}(y) \), which implies that there exists an exponent \( \delta_1 \equiv \delta_1(n,N,p,q,\alpha,\nu,L,[a]_{[\alpha]}) \in (1 - 1/q, 1) \) such that if \( \delta_1 \leq \delta_0 < \delta < 1 \), then there holds

\[
\int_{B_{\varrho_y}(y)} [H(x,|\nabla u|) + H(x,|F|)]^\delta \, dx
\]

\[
\leq c \left( \int_{B_{2\varrho_y}(y)} [H(x,|\nabla u|)]^{\delta_1} \, dx \right)^\frac{\delta}{\delta_1} + c \int_{B_{2\varrho_y}(y)} [H(x,|F|)]^\delta \, dx
\]

for some constant \( c = c(n,N,p,q,\alpha,\nu,L,[a]_{[\alpha]}) \) and almost every \( y \in S(\Lambda^\delta, r_1) \). Then it follows from (4.5) and Hölder’s inequality that

\[
\int_{B_{\varrho_y}(y)} [H(x,|\nabla u|) + H(x,|F|)]^\delta \, dx \leq c_0 \Lambda^{\delta - \delta_1} \int_{B_{2\varrho_y}(y)} [H(x,|\nabla u|)]^{\delta_1} \, dx
\]

\[
+ c_0 \int_{B_{2\varrho_y}(y)} [H(x,|F|)]^\delta \, dx
\]

(4.6)

for a constant \( c_0 \equiv c_0(n,N,p,q,\alpha,\nu,L,[a]_{[\alpha]}) \). Now we shall deal with the terms appearing in the last display. In turn, for all \( \theta \in (0,1) \), we have

\[
I_1 \leq \Lambda^{\delta - \delta_1} \left( \theta \Lambda^\delta \right)^{\frac{\delta}{\delta_1}} + \frac{1}{|B_{2R_y}(y)|} \int_{S(\theta \Lambda^\delta, r_2) \cap B_{2\varrho_y}(y)} [H(x,|\nabla u|)]^{\delta_1} \, dx
\]

\[
\leq \theta^{\delta_1} \Lambda^\delta + \frac{\Lambda^{\delta - \delta_1}}{|B_{2R_y}(y)|} \int_{S(\theta \Lambda^\delta, r_2) \cap B_{2\varrho_y}(y)} [H(x,|\nabla u|)]^{\delta_1} \, dx
\]

and

\[
I_2 \leq \theta^\delta \Lambda^\delta + \frac{1}{|B_{2\varrho_y}(y)|} \int_{T(\theta \Lambda^\delta, r_2) \cap B_{2\varrho_y}(y)} [H(x,|F|)]^\delta \, dx,
\]
where the sets $S$ and $T$ have been defined in (4.2)-(4.3). Inserting the estimates obtained in the last two displays into (4.6), inserting (4.5) and reabsorbing terms, we find

$$
(1 - c_\delta) \int_{B_{2\rho}(y)} [H(x, |\nabla u|) + H(x, |F|)]^\delta \, dx 
\leq c_\delta \int_{S(\theta \Lambda^\delta_r, r_2) \cap B_{2\rho}(y)} [H(x, |\nabla u|)]^\delta \, dx 
+ c_\delta \int_{T(\theta \Lambda^\delta_r, r_2) \cap B_{2\rho}(y)} [H(x, |F|)]^\delta \, dx,
$$

for the same constant $c_\delta$ as appeared in (4.6). We take $\theta \equiv \theta(n, p, q, \alpha, \nu, L, [a]_{0, \alpha}) \in (0, 1)$ small enough such that

$$
1 - c_\delta - c_\delta \theta \geq 1/2.
$$

From (4.7) with the choice of $\theta$ as in the last display, we conclude that

$$
\int_{B_{2\rho}(y)} [H(x, |\nabla u|) + H(x, |F|)]^\delta \, dx 
\leq c_\delta \int_{S(\theta \Lambda^\delta_r, r_2) \cap B_{2\rho}(y)} [H(x, |\nabla u|)]^\delta \, dx 
+ c_\delta \int_{T(\theta \Lambda^\delta_r, r_2) \cap B_{2\rho}(y)} [H(x, |F|)]^\delta \, dx,
$$

for some constant $c \equiv c(n, p, q, \alpha, \nu, L, [a]_{0, \alpha})$, whenever $y \in S(\Lambda^\delta, r_1)$ in the sense of almost everywhere and $\delta_1 \equiv \delta_1(n, p, q, \alpha, \nu, L, [a]_{0, \alpha}) \in (1 - 1/q, 1)$ is a constant such that $\delta_1 \leq \delta_0$. Now we again use (4.5) together with the last display to obtain

$$
\int_{10B_{2\rho}(y)} [H(x, |\nabla u|)]^\delta \, dx \leq \Lambda^\delta \int_{B_{2\rho}(y)} [H(x, |\nabla u|) + H(x, |F|)]^\delta \, dx 
\leq c_\delta \int_{S(\theta \Lambda^\delta_r, r_2) \cap B_{2\rho}(y)} [H(x, |\nabla u|)]^\delta \, dx 
+ c_\delta \int_{T(\theta \Lambda^\delta_r, r_2) \cap B_{2\rho}(y)} [H(x, |F|)]^\delta \, dx,
$$

for a constant $c \equiv c(n, p, q, \alpha, \nu, L, [a]_{0, \alpha})$, where $\theta \in (0, 1)$ is the number given in (4.8). On the other hand, applying Vitali’s covering lemma, there exists a countable family of disjoint balls \{B_{2\rho_i}(y_i)\}_{i \in \mathbb{N}} \equiv \{B_i\}_{i \in \mathbb{N}} such that

$$
S(\Lambda^\delta, r_1) \subset \bigcup_{i \in \mathbb{N}} 10B_i \cup \text{negligible set} \subset B_{r_2}
$$

and that

$$
\Phi(B_{2\rho_i}(y_i)) = \Lambda^\delta \quad \text{and} \quad \Phi(B_{\rho_i}(y_i)) < \Lambda^\delta \quad \text{for every} \quad y_i \in (\rho_i, r_2 - r_1)
$$

for every $i \in \mathbb{N}$. Therefore, it follows from the resulting estimate of (4.9) that

$$
\int_{S(\Lambda^\delta, r_1)} [H(x, |\nabla u|)]^\delta \, dx = \sum_{i \in \mathbb{N}} \int_{10B_i} [H(x, |\nabla u|)]^\delta \, dx 
\leq c \sum_{i \in \mathbb{N}} \Lambda^\delta_1 \int_{S(\theta \Lambda^\delta_r, r_2) \cap 2B_i} [H(x, |\nabla u|)]^\delta \, dx 
+ c \sum_{i \in \mathbb{N}} \int_{T(\theta \Lambda^\delta_r, r_2) \cap 2B_i} [H(x, |F|)]^\delta \, dx 
\leq c \Lambda^\delta_1 \int_{S(\theta \Lambda^\delta_r, r_2)} [H(x, |\nabla u|)]^\delta \, dx 
+ c \int_{T(\theta \Lambda^\delta_r, r_2)} [H(x, |F|)]^\delta \, dx.
for some constant $c \equiv c(n, N, p, q, \alpha, \nu, L, [a]_{0, \alpha})$. Taking into account the last display together with the observation that
\[
\int_{S(\theta^A, r_1)} [H(x, |\nabla u|)]^\delta dx \leq \Lambda_0 \int_{S(\theta^A, r_1)} [H(x, |\nabla u|)]^\delta dx
\]
for any $\Lambda_0 \equiv \left( \frac{40r}{r_2 - r_1} \right)^n \Lambda_0^\delta$, we find
\[
\int_{S(\theta^A, r_1)} [H(x, |\nabla u|)]^\delta dx = \int_{S(\theta^A, r_1)} [H(x, |\nabla u|)]^\delta dx
\]
\[
+ \int_{S(\theta^A, r_1) \setminus S(\theta^A, r_2)} [H(x, |\nabla u|)]^\delta dx
\]
\[
\leq c\Lambda_0^{\delta - 1} \int_{S(\theta^A, r_2)} [H(x, |\nabla u|)]^{\delta_1} dx
\]
\[
+ c \int_{T(\delta^A, r_2)} [H(x, |F|)]^\delta dx
\]
for some constant $c \equiv c(n, N, p, q, \alpha, \nu, L, [a]_{0, \alpha})$. In particular, we have
\[
\int_{S(\Lambda^A, r_1)} [H(x, |\nabla u|)]^\delta dx \leq c\Lambda_0^{\delta - 1} \int_{S(\Lambda^A, r_2)} [H(x, |\nabla u|)]^{\delta_1} dx
\]
\[
+ c \int_{T(\Lambda^A, r_2)} [H(x, |F|)]^\delta dx
\]
(4.10)
for some constant $c \equiv c(n, N, p, q, \alpha, \nu, L, [a]_{0, \alpha})$, whenever $\Lambda_0^\delta > \left( \frac{40r}{r_2 - r_1} \right)^n \Lambda_0^\delta$.

**Step 2: Integration and iteration.** We shall integrate on level sets and use a standard truncation argument to guarantee that the quantities involved are finite. For any fixed constant $k > 0$, we denote by
\[
H_k(x, |z|) := \min\{H(x, |z|), k\} \quad (x \in \Omega, z \in \mathbb{R}^N \text{ or } \mathbb{R}^{nN})
\]
and
\[
S_k(\Lambda, \rho) := \{x \in B_\rho(x_0) : H_k(x, |\nabla u|) > \Lambda\}, \quad r \leq \rho \leq 2r.
\]
Clearly, by the definition of the set $S_k$ in (4.11), we observe that
\[
S_k(\Lambda, \rho) = \begin{cases} 
\emptyset & \text{if } \Lambda > k, \\
S(\Lambda, \rho) & \text{if } \Lambda \leq k.
\end{cases}
\]
Recall that the estimate (4.10) can be written as
\[
\int_{S(\Lambda^A, r_1)} [H(x, |\nabla u|)]^\delta dx \leq c\Lambda^{\delta - 1} \int_{S(\Lambda^A, r_2)} [H(x, |\nabla u|)]^{\delta_1} dx + c \int_{T(\Lambda^A, r_2)} [H(x, |F|)]^\delta dx.
\]
Then using (4.12) in the last display, we deduce that
\[
\int_{S_k(\Lambda^A, r_1)} [H_k(x, |\nabla u|)]^{\delta - 1} [H(x, |\nabla u|)]^{\delta_1} dx
\]
\[
\leq c\Lambda^{\delta - 1} \int_{S_k(\Lambda^A, r_2)} [H(x, |\nabla u|)]^{\delta_1} dx
\]
\[
+ c \int_{T(\Lambda^A, r_2)} [H(x, |F|)]^\delta dx
\]
(4.13)
In what follows, we denote
\[
\Lambda_1^\delta := \left( \frac{40r}{r_2 - r_1} \right)^n \Lambda_0^\delta,
\]
(4.14)
where \( \Lambda_0 \) has been defined in (4.4). Then we multiply (4.13) by \( \Lambda^{-\delta} \) and integrate the resulting inequality with respect to \( \Lambda \) for \( \Lambda \geq \Lambda_1 \) to discover

\[
J_0 := \int_{\Lambda_1}^{\infty} \Lambda^{-\delta} \int_{S_k(A^\delta, r_1)} [H_k(x, |\nabla u|)]^{1-\delta_1} [H(x, |\nabla u|)]^{\delta_1} \ d\Lambda dx \\
\leq c_* \int_{\Lambda_1}^{\infty} \Lambda^{-\delta_1} \int_{S_k(A^\delta, r_2)} [H(x, |\nabla u|)]^{\delta_1} \ d\Lambda dx + c_* \int_{\Lambda_1}^{\infty} \Lambda^{-\delta} \int_{T(A^\delta, r_2)} [H(x, |F|)]^\delta \ d\Lambda d\Lambda =: c_*(J_1 + J_2)
\]

for some constant \( c_* \equiv c_*(n, N, \rho, q, \alpha, \nu, L, [\varrho]_{0, \alpha}) \). Now we shall estimate the terms appearing in the last display. In turn, applying Fubini’s theorem, we find

\[
J_0 = \int_{\Lambda_1}^{\infty} \Lambda^{-\delta} \int_{B_{2r}(x_0)} [H_k(x, |\nabla u|)]^{1-\delta_1} [H(x, |\nabla u|)]^{\delta_1} \chi_{\{H_k(x, |\nabla u|) \geq \Lambda\}} \ d\Lambda dx \\
= \int_{S_k(A^\delta, r_1)} [H_k(x, |\nabla u|)]^{\delta_1} [H(x, |\nabla u|)]^{\delta_1} \int_{\Lambda_1}^{\Lambda} \Lambda^{-\delta} \ d\Lambda dx \\
= \frac{1}{1-\delta} \int_{S_k(A^\delta, r_1)} [H_k(x, |\nabla u|)]^{1-\delta_1} [H(x, |\nabla u|)]^{\delta_1} \ d\Lambda dx \\
- \frac{\Lambda_1^{1-\delta}}{1-\delta} \int_{S_k(A^\delta, r_1)} [H_k(x, |\nabla u|)]^{1-\delta_1} [H(x, |\nabla u|)]^{\delta_1} \ d\Lambda dx.
\]

Recalling that \( \frac{1-\delta}{\delta} < \frac{1}{\delta_1} \) and using the definition of \( \Lambda_1 \) in (4.14), we see

\[
\Lambda_1^{1-\delta} \int_{S_k(A^\delta, r_1)} [H_k(x, |\nabla u|)]^{\delta_1} [H(x, |\nabla u|)]^{\delta_1} \ d\Lambda dx \\
\leq \left(\frac{40r}{r_2 - r_1}\right) \Lambda_0 \Lambda_1^{1-\delta} \int_{B_{2r}(x_0)} [H(x, |\nabla u|)]^\delta \ d\Lambda dx \\
\leq \left(\frac{40r}{r_2 - r_1}\right) \Lambda_0 |B_{2r}(x_0)|.
\]

From the estimate of the last display in (4.16), we conclude

\[
J_0 \geq \frac{1}{1-\delta} \int_{S_k(A^\delta, r_1)} [H_k(x, |\nabla u|)]^{1-\delta_1} [H(x, |\nabla u|)]^{\delta_1} \ d\Lambda dx \\
- \frac{1}{1-\delta} \left(\frac{40r}{r_2 - r_1}\right) \Lambda_0 |B_{2r}|.
\]

Again applying Fubini’s theorem, we see

\[
J_1 = \int_{S_k(A^\delta, r_2)} [H(x, |\nabla u|)]^{\delta_1} \int_{\Lambda_1}^{\Lambda} \Lambda^{-\delta_1} \ d\Lambda dx \\
\leq \frac{1}{1-\delta_1} \int_{S_k(A^\delta, r_2)} [H_k(x, |\nabla u|)]^{1-\delta_1} [H(x, |\nabla u|)]^{\delta_1} \ d\Lambda dx \\
\leq \frac{1}{1-\delta_1} \int_{B_{r_2}(x_0)} [H_k(x, |\nabla u|)]^{1-\delta_1} [H(x, |\nabla u|)]^{\delta_1} \ d\Lambda dx
\]

and

\[
J_2 \leq \frac{1}{1-\delta} \int_{T(A^\delta, r_2)} [H(x, |F|)]^\delta \ d\Lambda dx \leq \frac{1}{1-\delta} \int_{B_{r_2}(x_0)} [H(x, |F|)]^\delta \ d\Lambda dx.
\]

Combining the resulting estimates of (4.17)-(4.18) in (4.15), we have
\[
\int_{S_k(\Lambda_1^r,r_1)} [H_k(x,|\nabla u|)]^{1-\delta_1} [H(x,|\nabla u|)]^{\delta_1} \, dx \\
\leq \frac{c(1-\delta)}{1-\delta_1} \int_{B_{r_2}(x_0)} [H_k(x,|\nabla u|)]^{1-\delta_1} [H(x,|\nabla u|)]^{\delta_1} \, dx \\
+ c \int_{B_{r_2}(x_0)} [H(x,|F|)] \, dx + \left(\frac{40r}{r_2-r_1}\right)^{\frac{n}{\delta_1}} \Lambda_0 |B_{2r}| 
\]
for some constant \(c \equiv c(n, N, p, q, \alpha, \nu, L, [a]_{0,\alpha})\). At this moment, recalling the definition of \(\Lambda_1\) in (4.14) again, we observe

\[
\int_{B_{r_1}(x_0) \setminus S_k(\Lambda_1^r,r_1)} [H_k(x,|\nabla u|)]^{1-\delta_1} [H(x,|\nabla u|)]^{\delta_1} \, dx \\
\leq \left[ \left(\frac{40r}{r_2-r_1}\right)^{\frac{n}{\delta_1}} \Lambda_0 \right]^{1-\delta} \int_{S_k(\Lambda_1^r,r_1)} [H_k(x,|\nabla u|)]^{\delta-\delta_1} [H(x,|\nabla u|)]^{\delta_1} \, dx \\
\leq \left(\frac{40r}{r_2-r_1}\right)^{\frac{n}{\delta_1}} \Lambda_0^{1-\delta} \int_{B_{2r}(x_0)} [H_k(x,|\nabla u|)]^{\delta-\delta_1} [H(x,|\nabla u|)]^{\delta_1} \, dx \\
\leq \left(\frac{40r}{r_2-r_1}\right)^{\frac{n}{\delta_1}} \Lambda_0 |B_{2r}|. 
\]

Using the resulting inequality of the last display in (4.19) and recalling that \(\delta_0 < \delta\), we have

\[
\int_{B_{r_1}(x_0)} [H_k(x,|\nabla u|)]^{1-\delta_1} [H(x,|\nabla u|)]^{\delta_1} \, dx \\
\leq \frac{c_0(1-\delta_0)}{1-\delta_1} \int_{B_{r_2}(x_0)} [H_k(x,|\nabla u|)]^{1-\delta_1} [H(x,|\nabla u|)]^{\delta_1} \, dx \\
+ c_0 \int_{B_{r_2}(x_0)} H(x,|F|) \, dx + c_0 \left(\frac{40r}{r_2-r_1}\right)^{\frac{n}{\delta_1}} \Lambda_0 |B_{2r}| 
\]
for some constant \(c_0 \equiv c_0(n, N, p, q, \alpha, \nu, L, [a]_{0,\alpha})\). Finally, we select \(\delta_0 \in (1-1/q, 1)\) so that

\[
1-1/q < \delta_1 \leq \delta_0 < 1 \quad \text{and} \quad 0 < \frac{c_0(1-\delta_0)}{1-\delta_1} \leq \frac{1}{2}. 
\]

In turn, we have

\[
\int_{B_{r_1}(x_0)} [H_k(x,|\nabla u|)]^{1-\delta_1} [H(x,|\nabla u|)]^{\delta_1} \, dx \\
\leq \frac{1}{2} \int_{B_{r_2}(x_0)} [H_k(x,|\nabla u|)]^{1-\delta_1} [H(x,|\nabla u|)]^{\delta_1} \, dx \\
+ c_0 \int_{B_{2r}(x_0)} H(x,|F|) \, dx + c_0 \left(\frac{40r}{r_2-r_1}\right)^{\frac{n}{\delta_1}} \Lambda_0 |B_{2r}|. 
\]

Once we arrive at this stage, we apply Lemma 2.3 to a bounded function \(h : [r, 2r] \to [0, \infty)\) given by

\[
h(t) := \int_{B_t(x_0)} [H_k(x,|\nabla u|)]^{1-\delta_1} [H(x,|\nabla u|)]^{\delta_1} \, dx
\]
with the exponents \(\gamma_1 \equiv 0\) and \(\gamma_2 \equiv \frac{n}{\delta_1}\), to discover

\[
\int_{B_{r}(x_0)} [H_k(x,|\nabla u|)]^{1-\delta_1} [H(x,|\nabla u|)]^{\delta_1} \, dx \leq c \Lambda_0 |B_{2r}| + \int_{B_{2r}(x_0)} H(x,|F|) \, dx
\]
for some constant \( c \equiv c(n, N, p, q, \alpha, \nu, L, [a]_{0, \alpha}) \). Finally, letting \( k \to \infty \) in the last display and recalling the definition of \( \Lambda_0 \) in (4.4), we conclude that

\[
\int_{B_r(x_0)} H(x, |\nabla u|) \, dx \leq c \left( \int_{B_{2r}(x_0)} [H(x, |\nabla u|)]^\delta \, dx \right)^{\frac{1}{\delta}} + c \left( \int_{B_{2r}(x_0)} H(x, |F|) \, dx + 1 \right)
\]

for some constant \( c \equiv c(n, N, p, q, \alpha, \nu, L, [a]_{0, \alpha}) \). This completes the proof of Theorem 1.2.

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