EXPONENTIAL BOUNDARY STABILIZATION FOR NONLINEAR WAVE EQUATIONS WITH LOCALIZED DAMPING AND NONLINEAR BOUNDARY CONDITION

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Abstract. Let $D \subset \mathbb{R}^d$ be a bounded domain in the $d$-dimensional Euclidian space $\mathbb{R}^d$ with smooth boundary $\Gamma = \partial D$. In this paper we consider exponential boundary stabilization for weak solutions to the wave equation with nonlinear boundary condition:

\[
\begin{cases}
  u_{tt}(t) - \rho(t)\Delta u(t) + b(x)u_t(t) = f(u(t)), \\
  u(t) = 0 \text{ on } \Gamma_0 \times (0, T), \\
  \partial u(t)/\partial \nu + \gamma(u_t(t)) = 0 \text{ on } \Gamma_1 \times (0, T), \\
  u(0) = u_0, u_t(0) = u_1,
\end{cases}
\]

where $\|u_0\| < \lambda_\beta$, $E(0) < d_\beta$, where $\lambda_\beta$, $d_\beta$ are defined in (21), (22) and $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$.

1. Introduction. The nonlinear wave equations with damping or and source terms are well investigated by many authors and there are many results in the literatures (e.g. see Temam [14] and Malek et al. [11], and references therein). Let $D \subset \mathbb{R}^d$ be a bounded domain in the $d$-dimensional Euclidian space $\mathbb{R}^d$ with smooth boundary $\Gamma$. The local or global existence of the weak solutions to the wave equation

\[ u_{tt}(t) - \Delta u(t) + \beta |u_t(t)|^q u_t(t) = \alpha |u(t)|^q u(t), \quad \alpha, \beta \geq 0, \]

with an initial value $(u_0, u_1)$ and the Dirichlet condition have been extensively investigated.

In [6] V. Georgiev and G. Todorova considered local existence, global existence and blown up of weak solutions for $\alpha \beta \neq 0$. In [12], Messaoudi proved that if $q > p$ and $1 < p \leq \frac{2(d-1)}{d-2}$ and $E(0) := \frac{1}{2} |u_1|^2_d + \frac{1}{2} \|u_0\|^2 - \frac{\alpha}{\gamma^2} |u_0|^{\gamma+2}_q < 0$, then the weak solution blows up in finite time. Roughly speaking, the source term $\alpha |u(t)|^q u(t)$ causes blown up of weak solutions. It is worth mentioning that in [3], M.M. Cavalcanti, V.N. Domingos Cavalcanti considered the existence and asymptotic stability for evolution equations with damping and source terms, and furthermore in [4], M.M. Cavalcanti, V.N. Domingos Cavalcanti and I. Lasiecka investigated well-posedness.
and optimal decay rates for the wave equation with nonlinear damping and source interaction.

On the other hand, the wave equations with the nonlinear boundary condition are considered by many authors ([1, 2, 5, 7, 8, 9, 10, 15, 17] and references therein). We are concerned with the stability of the weak solution to the wave equations with the source term. Thus it is seemed that it is worth investigating the stability of the weak solutions by the boundary stabilization for the wave equations with the source term. We refer [2] as the investigations of this area.

Let \( \Gamma_0 \) and \( \Gamma_1 \) be nonempty subsets of \( \Gamma = \Gamma_0 \cup \Gamma_1 \) and \( \bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset \). We consider the following wave equation with nonlinear boundary condition:

\[
\begin{cases}
  u_{tt}(t) - \rho(t)\Delta u(t) + b(x)u_t(t) = f(u(t)), \\
  u(t) = 0 \text{ on } \Gamma_0 \times (0, T), \\
  \frac{\partial u(t)}{\partial \nu} + \gamma(u_t(t)) = 0 \text{ on } \Gamma_1 \times (0, T), \\
  u(0) = u_0, u_t(0) = u_1,
\end{cases}
\]

where \( \Delta = \sum \frac{\partial^2}{\partial x_i^2} \) is a Laplace operator, the function \( f \) satisfies Condition 5 and \( \gamma \) is a monotone nondecreasing function with Conditions 2 and 3. Let \( 1 \geq \rho(t) \geq \rho_0 > \frac{2}{q+2} \) and \( \rho'(t) := \frac{d}{dt}\rho(t) \leq 0 \). We prove the following theorem:

**Theorem 1.1.** Let \( 0 < q \leq \frac{2}{d-2} \) and let \( \|u_0\| < \lambda_\beta, E(0) < d_\beta \), where \( \lambda_\beta \) and \( d_\beta \) are defined in (21) and (22). Assume that Conditions 2-8 are satisfied. If the constant \( \alpha > 0 \) such that \( f(s) \leq \alpha |s|^q s \), \( \alpha > 0, s \in \mathbb{R} \) satisfies that

\[
0 < \alpha < \frac{\rho_0}{8(d-1)c_{q+2}^2} \left( \frac{\rho_0(q + 2) - 2}{2(q + 2)d_\beta} \right)^\frac{q}{2},
\]

then the energy \( E(t) \) of the weak solution \((u, u_t)\) to (1) with the initial value \((u_0, u_1)\) converges to zero exponentially as \( t \to \infty \), where \( c_{q+2} \) is the Poincaré constant (Lemma 2.1) and \( E(t) \) is defined in (20).

The exponential stability of solutions to the damped wave equations without the source term but with nonlinear boundary condition is well considered by many authors. In [7] S. Gerbi and B. Said-Houari investigated local existence and exponential growth of solutions for a semilinear damping wave equation with nonlinear boundary conditions. In [17] Zhang and Miao considered the existence and exponential decay of the energy of weak solutions to some damping wave equation with nonlinear boundary conditions. In [16], E. Vitillaro discussed global existence for the wave equation with nonlinear boundary damping and source terms. See also [1, 5, 8, 10] and the references therein.

In [2] M.M. Cavalcanti, V.N.D. Cavalcanti and P. Martinez investigated the existence, uniqueness and the uniform decay of the energy of weak solutions to the weakly damped wave equation with the source term \( f(s) = |s|^q s \) and the nonlinear boundary condition. We will extend the some results in [2] by investigating exponential boundary stabilization of the weak solutions to (1).

The contents of this paper are as follows. In Section 2 we give Preliminaries. In Sections 3 using the the Galerkin method we prove existence and uniqueness of local weak solutions in time to (1). In Section 4 we consider the global existence of weak solutions to (1) and in Section 5 exponential decay of the energy \( E(t) \) of the weak solution \( u(t) \) to (1) is investigated.
In this paper the prime $'$ denotes the derivative with respect to time $t$. The $c$ and $c_\ast$ denote positive constants depending only on $q$ and the domain $D$ which change from line to line.

2. Preliminaries. Throughout this paper we use

$$
H^1_{\Gamma_0}(D) := \{v \in H^1(D); v = 0 \text{ on } \Gamma_0\},
$$

$$
D(-\Delta) := \{v \in H^1_{\Gamma_0}(D) \cap H^2(D); \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_1\},
$$

where $\nu = \nu(x)$ is the normal outward vector on $x$. Let $H^1_{\Gamma_0}(D)$ be Hilbert space with scalar product and norm:

$$(u,v) = \int_D uv dx, \|u\|^2 = \int_D |\nabla u|^2 dx, ((u,v)) = (\nabla u, \nabla v).$$

We use also the notations

$$
|u|^p = \left(\int_D |u|^p dx\right)^{\frac{1}{p}}, |u|_{L^p(\Gamma_1)} = \left(\int_{\Gamma_1} |u|^p dx\right)^{\frac{1}{p}}, p > 1.
$$

We use the next known lemma:

**Lemma 2.1 (Poincaré).** If $2 \leq k \leq 2d/(d-2)$, then there exists a constant $c_\ast = C(D,k)$ such that

$$
|u|_{k} \leq c_\ast \|u\|
$$

for any $u \in H^1(D)$.

We often use the $c_\ast$ instead of the constant $c_k$. The proof of the following lemma is given in Ono [13].

**Lemma 2.2 (Modified Gronwall inequality).** Let $\phi$ and $g$ be nonnegative functions on $[0, +\infty)$ satisfying

$$
0 \leq \phi(t) \leq M + \int_0^t g(s)\phi(s)^{r+1}ds
$$

with $M > 0$ and $r > 0$. Then

$$
\phi(t) \leq \left\{M^{-r} - r \int_0^t g(s)ds\right\}^{-\frac{1}{r}}
$$

as long as the right-hand side exists.

Finally we give the definition of a weak solution to (1).

**Definition 2.3.** Let $(u_0, u_1) \in H^1_{\Gamma_0}(D) \times L^2(D)$. Then $(u, u_t)$ is called a weak solution to (1) with an initial value $(u_0, u_1)$ if
(1) the following equality holds: for any \( \phi \in H^1_0(D) \)
\[
\begin{align*}
(u(t), \phi) &= (u(0), \phi) + \int_0^t (u_t(s), \phi)ds, \\
(u_t(t), \phi) &= (u_t(0), \phi) - \int_0^t \rho(s)(\nabla u(s), \nabla \phi) - \rho(s)\gamma(u_t(s), \phi)_{\Gamma_1}, ds \\
&\quad - \int_0^t (b(x)u_t(s), \phi)ds + \int_0^t (f(u(s)), \phi)ds \\
u(t) &= 0 \text{ on } \Gamma_0, \quad \frac{\partial u(t)}{\partial \nu} + \gamma(u_t(t)) = 0 \text{ on } \Gamma_1 \\
u(0) &= u_0 \in H^1_0(D), \quad u_t(0) = u_1 \in L^2(D).
\end{align*}
\]

We need the following conditions in this paper.

**Condition 1.** \( q \) satisfies \( 0 < q \leq \min \left\{ \frac{1}{2}, \frac{2}{3}\right\} \).

**Condition 2.** The function \( \gamma \) is a monotone nondecreasing function with \( \gamma(0) = 0 \) and \( \beta_0 |s| \leq \gamma(s) \leq \beta_1 |s|, \ s \in R, \ 0 < \beta_0 < \beta_1. \)

**Condition 3.** There exists a positive constant \( d_0 > 0 \) such that \( (\gamma(s) - \gamma(r))(s - r) \geq d_0(s - r)^2 \) for all \( s, r \geq 0 \).

**Condition 4.** There exist positive constants \( b_0, b_1, b_2 > 0 \) such that \( b_0 \leq b(x) \leq b_1 \) and \( |\nabla b(x)|_\infty < b_2 \) for all \( x \in D \), where \( |\cdot|_\infty \) denotes the sup-norm.

**Condition 5.** The function \( f \) is monotone nondecreasing and satisfies \( f \in C^1(R; R) \) and for any \( s, r \in R \), there exist positive constants \( \alpha, \alpha_1, \alpha_2, \alpha_3, c > 0 \) such that
\[
\begin{align*}
f(s) &\leq \alpha |s|^{q+1} + \alpha_1, \\
f'(s) &\leq \alpha_2 |s|^q + \alpha_3, \\
|f(s) - f(r)| &\leq c(1 + |s|^q + |r|^q) |s - r|.
\end{align*}
\]

**Condition 6.** The function \( \rho \) satisfies \( \rho(t) > 0 \) and \( \rho'(t) \leq 0 \) for all \( t \geq 0 \).

3. **Existence of local weak solutions.** In this section we consider the existence of the local weak solution \( (u, u_t) \) to the equation (1) for an initial value \( (u_0, u_1) \in H^1_{\Gamma_0}(D) \times L^2(D) \) by using the Galerkin method. Let \( \{e_1, e_2, e_3, \ldots, e_n, \ldots\} \) be the special orthonormal basis of \( H^1_{\Gamma_0}(D) \cap H^2(D) \) taken as in the proof of Proposition 3.1(p.127, [2]). Set
\[
u^n(t) = \sum_{k=1}^n \varphi_{nk}(t)e_k \quad \text{and} \quad u^n_t(t) = \sum_{k=1}^n \frac{d}{dt} \varphi_{nk}(t)e_k,
\]
where \( \varphi_{nk}(t) \) are the local solutions of the next ordinary differential equation
\[
\begin{align*}
\begin{cases}
(u^n_t(t), e_k) + \rho(t)(\nabla u^n(t), \nabla e_k) + \rho(t)(\gamma(u^n_t(t), e_k)_{\Gamma_1} \\
+ (b(x)u^n_t(t) - f(u^n(t)), e_k) = 0,
\end{cases}
\end{align*}
\]
k = 1, 2, 3, \ldots with an initial value \( u^n(0) = \sum_{k=1}^n (u_0, e_k) e_k \) and \( u^n_t(0) = \sum_{k=1}^n (u_1, e_k) e_k \), where \( u_i(t) := \frac{d}{dt} u(t) \) and \( u_t(t) := \frac{d}{dt} \). Then it holds that
\[
\begin{align*}
\begin{cases}
(u^n_t(t), w) + \rho(t)(\nabla u^n(t), \nabla w) + \rho(t)(\gamma(u^n_t(t)), w)_{\Gamma_1} \\
+ (b(x)u^n_t(t) - f(u^n(t)), w) = 0, \ w \in H^1_0(D).
\end{cases}
\end{align*}
\]
By the Green theorem
\[ (\rho(t)\Delta u^n(t), w) = -\rho(t)(\nabla u^n(t), \nabla w) + \rho(t) \left( \frac{\partial u^n(t)}{\partial \nu}, w \right)_\Gamma. \]

We have
\[ (u^n_{tt}(t), w) - (\rho(t)\Delta u^n(t), w) + (b(x)u^n_t(t) - f(u^n(t)), w) = 0. \tag{6} \]

**Theorem 3.1.** Let \((u_0, u_1) \in H^1_{\Gamma_0}(D) \times L^2(D)\) with the compatible condition \(\frac{\partial u_0}{\partial \nu} + \gamma(u_1) = 0\) on \(\Gamma_1\). Assume that Conditions 1-6 are satisfied. Then there exist a time \(T_0 > 0\) and a unique local weak solution \((u, u_t)\) in time to (1) with the initial value \((u_0, u_1)\) satisfying
\[
\begin{align*}
&u \in C(0, T_0; H^1_{\Gamma_0}(D) \cap H^2(D)), \\
&u_t \in C(0, T_0; L^2(D)), \\
&u_{tt} \in L^\infty(0, T_0; L^2(D)).
\end{align*}
\]

**Proof.** First let \((u_0, u_1) \in (H^1_{\Gamma_0}(D) \cap H^2(D)) \times H^1_{\Gamma_0}(D)\) with the compatible condition \(\frac{\partial u_0}{\partial \nu} + \gamma(u_1) = 0\) on \(\Gamma_1\). By taking \(w = -\Delta u^n_t(t)\) in (6),
\[
(u^n_{tt}(t), -\Delta u^n_t(t)) = (\rho(t)\Delta u^n(t), -\Delta u^n_t(t)) - (b(x)u^n_t(t) - f(u^n(t)), -\Delta u^n_t(t)).
\]
Thus
\[
(\nabla u^n_{tt}(t), \nabla u^n_t(t)) + (\gamma(u^n_t(t)), u^n_t(t))_\Gamma = -(\rho(t)\Delta u^n(t), \Delta u^n_t(t)) + (b(x)u^n_t(t) - f(u^n(t)), \Delta u^n_t(t))
\]
and hence by Condition 3
\[
\frac{1}{2} \frac{d}{dt} \|u^n_t(t)\|^2 + \frac{1}{2} \frac{d}{dt} \|\Delta u^n(t)\|^2_2 \leq (b(x)u^n_t(t) - f(u^n(t)), \Delta u^n_t(t)). \tag{7}
\]

By the Green theorem
\[
\begin{align*}
&b(x)u^n_t(t) - f(u^n(t)), \Delta u^n_t(t) \\
&= -(\nabla(b(x)u^n_t(t), \nabla u^n_t(t)) + (\nabla f(u^n(t), \nabla u^n_t(t)) \\
&+ \left( b(x)u^n_t(t) - f(u^n(t)), \frac{\partial}{\partial \nu} u^n_t(t) \right)_\Gamma \\
&= I_1 + I_2 + I_3.
\end{align*}
\]
Thus using the Sobolev lemma, the Hölder inequality, the Young inequality and Conditions 2 and 4-6, we have a constant \(c > 0\) such that
\[
I_1 \leq \int_D |\nabla (b(x)u^n_t(t))| |\nabla u^n_t(t)| \, dx \\
= \int_D |u^n_t(t)\nabla b(x) + b(x)\nabla u^n_t(t)| |\nabla u^n_t(t)| \, dx \\
\leq \int_D |u^n_t(t)\nabla b(x)| |\nabla u^n_t(t)| \, dx + \int_D |b(x)\nabla u^n_t(t)| |\nabla u^n_t(t)| \, dx \\
= |\nabla b(x)|_\infty \int_D |u^n_t(t)| |\nabla u^n_t(t)| \, dx + |b(x)|_\infty \int_D |\nabla u^n_t(t)|^2 \, dx \\
\leq |\nabla b(x)|_\infty \left( |u^n_t(t)|^2 + |\nabla u^n_t(t)|^2 \right) + |b(x)|_\infty |\nabla u^n_t(t)|^2,
\]
\[ I_2 \leq \int_D |\nabla f(u^n(t))\nabla u^n(t)| \, dx \]
\[ = \int_D |f'(u^n(t))\nabla u^n(t)| \, dx \]
\[ \leq \int_D (\alpha_2 |u^n(t)|^q + \alpha_3) |\nabla u^n(t)| \, dx \]
\[ \leq \alpha_2 |u^n(t)|^q |\nabla u^n(t)| + \alpha_3 \|u^n(t)\|_2 \|u^n(t)\| \]
\[ \leq c \|u^n(t)\|^2 + c |\Delta u^n(t)|^{2(q+1)} + c \|u^n(t)\|^2, \]

\[ I_3 \leq \int_\Gamma \left( |b(x)|_{\infty} |u^n(t)| + \alpha \|u^n(t)\|^{q+1} + 1 \right) |\gamma(u^n(t))| \, dx \]
\[ \leq b_1^2 |u^n(t)|_2^2 + c \alpha^2 |\Delta u^n(t)|_2^{2(q+1)} + 3\beta_1^2 |u^n(t)|_2^2 + \alpha_1^2 \mu(D). \]

Next we need the estimate of \(|u^n(t)|_2\). By taking \(w = u^n(t)\) in (6)
\[(u^n(t), u^n(t)) = (\rho(t)u^n(t), u^n(t)) - (b(x)u^n(t) - f(u^n(t)), u^n(t)).\]

Then
\[ |u^n(t)|_2^2 \leq 2\rho^2(t) |\Delta u^n(t)|_2^2 + \frac{1}{8} |u^n(t)|_2^2 + 2 \|b(x)\|_{\infty}^2 |u^n(t)|_2^2 + \frac{1}{8} |u^n(t)|_2^2 \]
\[ + 2\alpha^2 \gamma_2^{2(q+1)} |\Delta u^n(t)|_{2(q+1)} + \frac{1}{8} |u^n(t)|_2^2 + 2\alpha_2^2 \mu(D) + \frac{1}{8} |u^n(t)|_2^2. \]

This means
\[ |u^n(t)|_2^2 \leq 4 |\Delta u^n(t)|_2^2 + 4 \|b(x)\|_{\infty}^2 |u^n(t)|_2^2 \]
\[ + 4\alpha^2 \gamma_2^{2(q+1)} |\Delta u^n(t)|_{2(q+1)} + 4\alpha_2^2 \mu(D). \]

\[ \int_0^t \frac{1}{2} \rho(s) \frac{d}{ds} |\Delta u^n(s)|_2^2 \, ds \leq \frac{1}{2} \rho(t) |\Delta u^n(t)|_2^2 - \int_0^t \frac{1}{2} \rho'(s) |\Delta u^n(s)|_2^2 \, ds \]
\[ \geq \frac{1}{2} \rho(1) |\Delta u^n(t)|_2^2, \quad 0 < t \leq 1. \]

Thus from (7) we obtain positive constants \(c_7 > 0\) and \(c_8 > 0\) such that
\[ \begin{cases} \frac{1}{2} \|u^n(t)\|^2 + \frac{1}{2} \rho(1) |\Delta u^n(t)|_2^2 \leq \frac{1}{2} \|u^n(t)\|^2 + \frac{1}{2} |\Delta u^n(t)|_2^2 \\ + c_7 \int_0^t \rho(1) |\Delta u^n(s)|_2^{2(q+1)} \, ds + \frac{1}{4} \int_0^t |\nabla u^n(s)|_2^2 \, ds + c_8. \end{cases} \]

Since we assume that \((u_0, u_1) \in (H^1_0(D) \cap H^2(D)) \times H^1_0(D)\), we obtain a time \(T \in (0, 1)\), positive constants \(c > 0\) and \(B_7 > 0\) such that
\[ \|u^n(t)\|^2 + |\Delta u^n(t)|_2^2 \leq B_7 + c \int_0^T \left( \|u^n(s)\|^2 + |\Delta u^n(s)|_2^2 \right)^{q+1} \, ds \quad \text{for} \quad 0 < t \leq T. \]

Therefore, by the modified Gronwall inequality (Lemma 4.1) there exist a constant \(M_c > 0\) uniform in \(n > 0\) and a time \(0 < T_0 < T\) such that
\[ |\Delta u^n(t)|_2^2 \leq M_c \text{ and } \|u^n(t)\|^2 \leq M_c, \quad 0 < t \leq T_0. \]

And hence by (8) and (10) there exists a constant \(\hat{M}_c > 0\) such that
\[ |u^n(t)|_2 < \hat{M}_c, \quad 0 \leq t \leq T_0. \]
Thus there exist a subsequence \( \{l_k\} \) of \( \{n\} \) and a \( \chi \in L^2(0, T_0; L^2(\Gamma_1)) \) such that

- \( u^{l_k} \rightharpoonup u \) weakly-star in \( L^\infty(0, T_0; H^1_{W_0}(D) \cap H^2(D)) \),
- \( u^{l_k}_t \rightharpoonup u_t \) weakly-star in \( L^\infty(0, T_0; H^1_{W_0}(D)) \),
- \( u^{l_k}_{tt} \rightharpoonup u_{tt} \) weakly-star in \( L^\infty(0, T_0; L^2(D)) \),
- \( f(u^{l_k}) \rightharpoonup \eta \) weakly in \( L^2(0, T_0; L^2(D)) \),
- \( \gamma(u^{l_k}) \rightharpoonup \chi \) weakly in \( L^2(0, T_0; L^2(\Gamma_1)) \).

Therefore by the Aubin-Lions compactness lemma there exists a subsequence \( \{\alpha_k\} \) of \( \{l_k\} \) such that

- \( u^{\alpha_k} \rightharpoonup u \) strongly in \( L^2(0, T_0; H^1_{W_0}(D)) \),
- \( u^{\alpha_k}_t \rightharpoonup u_t \) strongly in \( L^2(0, T_0; L^2(D)) \).

Thus

\[
\left\{ \begin{array}{l}
(u_t(t), w) + \rho(t)(\nabla u(t), \nabla w) + \rho(t)(\chi(t), w)_{\Gamma_1} \\
+ (b(x)u_t(t) - \eta(t), w) = 0, \ w \in H^1_{0}(D).
\end{array} \right.
\] (12)

We also use the same notation \( \{n\} \) instead of \( \{\alpha_k\} \). Then from (5) and (12) we obtain that

\[
\lim_{n \to \infty} \sup_{t} \int_{0}^{t} (\rho(s)\gamma(u^n_t(s)), u^n_t(s))_{\Gamma_1} ds
= \lim_{n \to \infty} \sup_{t} \left( - \int_{0}^{t} (u^n_{tt}(s), u^n_t(s)) ds - \int_{0}^{t} \rho(s)(\nabla u^n(s), \nabla u^n_t(s)) ds \\
+ \int_{0}^{t} (-b(x)u^n_t(t) + f(u^n(s)), u^n_t(s)) ds \right)
\leq - \int_{0}^{t} (u_{tt}(s), u_t(s)) ds - \int_{0}^{t} \rho(s)(\nabla u(s), \nabla u_t(s)) ds
- \int_{0}^{t} (b(x)u_t(s) - \eta(s), u_t(s)) ds
= \int_{0}^{t} (\rho(s)\chi(s), u_t(s))_{\Gamma_1} ds.
\]

Thus we obtain that

\[
\lim_{n \to \infty} \sup_{t} \int_{0}^{t} \rho(s)(\gamma(u^n_t(s)), u^n_t(s))_{\Gamma_1} ds \leq \int_{0}^{t} \rho(s)(\chi(s), u_t(s))_{\Gamma_1} ds.
\]

Since \( \gamma \) is a monotone nondecreasing function(by Condition 2), for any \( \Psi \in L^2(0, T_0; L^2(\Gamma_1)) \)

\[
\int_{0}^{T_0} \rho(s)(\gamma(u^n_t(s)) - \rho(s)\gamma(\Psi(s)), u^n_t(s) - \Psi(s))_{\Gamma_1} ds \geq 0
\]

and hence taking \( \lim \inf \) as \( n \to \infty \), it holds that

\[
\int_{0}^{T_0} (\rho(s)\chi(s) - \rho(s)\gamma(\Psi(s)), u_t(s) - \Psi(s))_{\Gamma_1} ds \geq 0.
\]
Let $\delta > 0$ be any and set $\Psi(s) = u_t(s) - \delta \xi(s)$, where $\xi(s) \in L^2(0, T_0; L^2(\Gamma_1))$. Then we have that
\begin{equation}
\int_0^{T_0} \left( \rho(s) \chi(s) - \rho(s) \gamma(u_t(s) - \delta \xi(s)) \right) \delta \xi(t) \, ds \geq 0.
\end{equation}
By dividing (13) by $\delta$ and then $\delta \to 0$, we obtain that $\rho(s) \chi(s) = \rho(s) \gamma(u_t(s))$ in $(0, T)$ since $\xi(s)$ is any. Since $f$ is differentiable, similarly we obtain that $\eta(s) = f(u(s))$. Thus we have that for any $w \in H_1^0(D)$,
\begin{equation}
egin{cases}
(u_{tt}(t), w) + \rho(t)(\nabla u(t), \nabla w) + \rho(t)(\gamma(u_t(t)), w)_{\Gamma_1} \\
(b(x)u_t(t) - f(u(t)), w) = 0.
\end{cases}
\end{equation}
Second let $(u_0, u_1) \in H_{1,0}^0(D) \times L^2(D)$ with the compatible condition $\frac{\partial u_0}{\partial \nu} + \gamma(u_1) = 0$ on $\Gamma_1$, $D(-A)$ is dense in $H_{1,0}^0(D)$ and $H_1^0(D) \cap H^2(D)$ is dense in $L^2(D)$. Thus there exist sequences $\{u_{0k}\}$ and $\{u_{1k}\}$ such that $u_{0k} \to u_0$ in $H_{1,0}^0(D)$ and $u_{1k} \to u_1$ in $L^2(D)$ as $k \to \infty$, where $u_{0k} \in D(-A)$ and $u_{1k} \in H_{1,0}^0(D) \cap H^2(D)$. Since $\frac{\partial u_{0k}}{\partial \nu} + \gamma(u_{1k}) = 0$ on $\Gamma_1$, there exist the weak solutions to (1) with the initial value $(u_{0k}, u_{1k})$ for each integer $k \geq 1$. Now let $u^k(t) := u(t; 0, u_{0k}^k)$ and $u_t^k(t) := u_t(t; 0, u_{0k}^k)$.
\begin{equation}
\begin{cases}
(u_{tt}^k(t) - u_{tt}^j(t), u_t^k(t) - u_t^j(t)) \\
= (\rho(t) \Delta(u^k(t) - u^j(t)), u_t^k(t) - u_t^j(t)) \\
+ (b(x)u_t^k(t) + f(u^k(t)) + (b(x)u_t^j(t) - f(u^j(t))), u_t^k(t) - u_t^j(t)).
\end{cases}
\end{equation}
Thus by the Green theorem
\begin{equation}
\begin{cases}
\frac{1}{2} \frac{d}{dt} \left| u_t^k(t) - u_t^j(t) \right|^2_2 \\
= - (\rho(t) \nabla(u^k(t) - u^j(t)), \nabla(u_t^k(t) - u_t^j(t))) \\
+ (\rho(t) \frac{\partial}{\partial \nu}(u^k(t) - u^j(t)), u_t^k(t) - u_t^j(t))_{\Gamma_1} \\
+ (b(x)u_t^k(t) + f(u^k(t)) + (b(x)u_t^j(t) - f(u^j(t))), u_t^k(t) - u_t^j(t)).
\end{cases}
\end{equation}
By Condition 3, we have that
\begin{equation}
\begin{aligned}
(\rho(t) \gamma(u_t^k(t)) - \rho(t) \gamma(u_t^j(t)), u_t^k(t) - u_t^j(t))_{\Gamma_1} \\
\leq 0.
\end{aligned}
\end{equation}
Setting $\theta := \frac{2d}{\sigma - 2}$, by Condition 5
\begin{equation}
(f(u^k(t)) - f(u^j(t)), u_t^k(t) - u_t^j(t)) \\
\leq c \left( 1 + \left| u^k(t) \right|_{q_d}^q + \left| u^j(t) \right|_{q_d}^q \right) \left| u_t^k(t) - u_t^j(t) \right|_{\theta} \left| u_t^k(t) - u_t^j(t) \right|_2.
\end{equation}
Thus by the generalized Hölder inequality
\begin{equation}
\begin{cases}
\frac{1}{2} \frac{d}{dt} \left| u_t^k(t) - u_t^j(t) \right|^2_2 + \frac{1}{2} \frac{d}{dt} \left| u^k(t) - u^j(t) \right|^2 \\
\leq |b(x)|_{\infty} \left| u_t^k(t) - u_t^j(t) \right|^2_2 + c \left( 1 + \left| u^k(t) \right|_{q_d}^q + \left| u^j(t) \right|_{q_d}^q \right) \\
\times \left| u^k(t) - u^j(t) \right|_{\theta} \left| u_t^k(t) - u_t^j(t) \right|_2.
\end{cases}
\end{equation}
Then by the Sobolev lemma, there exists a constant $c_*>0$ such that $|u^k(t)|_{q_d} \leq c_*|\Delta u^k(t)|_2$. Using (10) we obtain a constant $M_d>0$ such that

$$
\begin{align*}
\left|u^k(t) - u^i(t)\right|_2^2 + \|u^k(t) - u^i(t)\|_2^2 \\
\leq |u^k_0 - u^i_0|_2^2 + \|u^k_0 - u^i_0\|_2^2 + 2b \epsilon c_*M_c \\
M_d \left( \int_0^t \left|u^k_j(s) - u^i_j(s)\right|_2^2 + \|u^k_j(s) - u^i_j(s)\|_2^2 ds \right),
\end{align*}
$$

(18)

By the Gronwall lemma, we have a weak solution $(u, u_1) \in H^1_\Gamma(D) \times L^2(D)$ to (1) with the initial value $(u_0, u_1)$. By the usual method we can prove the uniqueness of the weak solutions. This completes the proof of the theorem.

4. **Existence of global weak solutions.** For the function $f$ with Condition 5, furthermore we add the following condition:

**Condition 7.** The function $f$ satisfies that

$$|f(s)| \leq \alpha s^{q+1}, \quad \alpha > 0 \quad \text{for } s \in \mathbb{R}, \quad F(v) := \int_0^v f(s)ds \geq 0 \quad \text{for any } v \in \mathbb{R}. \quad (19)$$

**Condition 8.** Let $\rho_0$ be a positive constant. The function $\rho$ satisfies that

$$\rho'(t) \leq 0 \quad \text{and} \quad 1 \geq \rho(t) > \rho_0 > \frac{2}{q+2}. $$

In this section we consider the energy $E(t)$ of the weak solution to (1), where $E(t)$ is defined by

$$
E(t) := \frac{1}{2} \left|u(t)\right|_2^2 + \frac{1}{2} \rho(t) \left|u(t)\right|_2^2 - \int_D F(u(t))dx.
$$

(20)

If $0 < q \leq \frac{4}{q+2}$, then by the Poincaré lemma we define the constant $K_\alpha$ as

$$K_\alpha := \sup_{v \in V, v \neq 0} \left( \frac{\alpha}{q+2} \frac{|v|_{q+2}^{q+2}}{\|v\|_{q+2}^{q+2}} \right) \leq \frac{\alpha c_*^{q+2}}{q+2},$$

where $V = H^1_\Gamma(D)$. By (19) we have

$$
\sup_{v \in V, v \neq 0} \left( \int_D \int_0^v f(s)dsdx \right) \|v\|_{q+2} \leq K_\alpha.
$$

Here we define the constant $L_\beta$ by

$$L_\beta := \sup_{v \in V, v \neq 0} \left( \int_D \int_0^v f(s)dsdx \right) \|v\|_{q+2}^{-q+2}. $$

Consider the function $\Phi(\lambda) = \frac{1}{2} \lambda^2 \rho_0 - L_\beta \lambda^{q+2}$, $\lambda \geq 0$. And hence $\Phi'(\lambda) = \lambda \rho_0 - (q+2)L_\beta \lambda^{q+1}$. Then there exists the positive constant $\lambda_\beta$ such that $\Phi'(\lambda_\beta) = 0$ and it holds that

$$
\lambda_\beta = \left( \frac{\rho_0}{L_\beta(q+2)} \right)^{\frac{1}{q}}.
$$

(21)

We define the constant $d_\beta$ by

$$d_\beta := \Phi(\lambda_\beta) = \frac{1}{2} \lambda_\beta^2 \rho_0 - L_\beta \lambda_\beta^{q+2}. $$

(22)
It holds that $d_\beta > \Phi(\lambda)$ for all $\lambda \in (0, \lambda_\beta)$.

By using Conditions 3, 4, 7 and 8, we have that
\[
E'(t) = (\rho(t) \Delta u(t) + f(u(t)), u_t(t)) - (b(x) u_t(t), u_t(t)) \\
\leq (\rho(t) \Delta u(t), u_t(t)) - b_0 |u_t(t)|^2_2 \\
+ \frac{1}{2} \rho'(t) |u(t)|^2 + \rho(t)((u(t), u_t(t))) \\
\leq \frac{1}{2} \rho'(t) |u(t)|^2 - \rho(t)(\gamma(u(t)), u_t(t))_1 - b_0 |u_t(t)|^2_2 \\
\leq -d_0 |u_t(t)|^2_2 - b_0 |u_t(t)|^2_2.
\]

Thus we have the next lemma used later.

**Lemma 4.1.** Assume that Conditions 3, 4, 7 and 8. Then it holds that $E'(t) \leq -d_0 |u_t(t)|^2_2 - b(x) |u_t(t)|^2_2$ for all $t \geq 0$.

Next if there exists a time $t_0 > 0$ such that $\|u(t_0)\| = \lambda_\beta$ and $\|u(t)\| < \lambda_\beta$ for all $t \in [0, t_0)$, then by (22) and Lemma 4.1 it follows that
\[
E(0) = E(t_0) \geq \frac{1}{2} \rho(t_0) \|u(t_0)\|^2 - \int_D F(u(t_0)) dx \\
> \|u(t_0)\|^2 \left(\frac{1}{2} \rho_0 - L_\beta \|u(t_0)\|^2\right) \\
= \left(\frac{1}{2} \lambda_\beta^q \rho_0 - L_\beta \lambda_\beta^{q+2}\right) = d_\beta,
\]
which means a contradiction. Thus we have the following lemma (See Lemma 3.2, [2]). See also [15].

**Lemma 4.2.** Let Conditions 3, 4, 7 and 8 be satisfied. Assume that $\|u_0\| < \lambda_\beta$ and $E(0) < d_\beta$. Then $\|u(t)\| < \lambda_\beta$ for all $t \geq 0$, where $\lambda_\beta$ and $d_\beta$ are defined in (21) and (22).

Furthermore, we have the following lemma (see [2], p124, (2.12)).

**Lemma 4.3.** Assume that $\|u_0\| < \lambda_\beta$ and $E(0) < d_\beta$. Assume that Conditions 3, 4, 7 and 8 are satisfied. Then it holds that
\[
\|u(t)\|^2 < \frac{2(q+2)}{\rho_0(q+2) - 2} E(t).
\]

**Proof.** By using Lemma 4.2, we have that
\[
E(t) \geq \frac{1}{2} \rho(t) |u(t)|^2 - \int_D \int_0^{u(t)} f(s) dx ds \\
> \|u(t)\|^2 \left(\frac{1}{2} \rho_0 - L_\beta \|u(t)\|^2\right) \\
\geq \|u(t)\|^2 \left(\frac{1}{2} \rho_0 - L_\beta \lambda_\beta^q\right) \\
= \|u(t)\|^2 \left(\frac{1}{2} \rho_0 - \frac{1}{(q+2)}\right).
\]

This completes the proof of the lemma.

We consider the existence of global weak solutions.
Proposition 1. Let $0 < q \leq \frac{2}{d-2}$. Let $(u_0, u_1) \in H^1_{V_0}(D) \times L^2(D)$ with the compatible condition $\frac{\partial u_0}{\partial \nu} + \gamma(u_1) = 0$ on $\Gamma_1$. Assume that $\|u_0\| < \lambda_\beta$, $E(0) < d_\beta$ and Conditions 2-8 are satisfied. Then the unique weak solution $(u, u_t) \in H^1_{V_0}(D) \times L^2(D)$ to (1) with the initial value $(u_0, u_1)$ exists globally.

Proof. Since by Theorem 3.1 there exists the local unique weak solution to (1), we set $w = u_t(t)$ in (14). Then

\[
\begin{cases}
(u_{tt}(t), u_t(t)) + \rho(t)(\nabla u(t), \nabla u_t(t)) + \rho(t)(\gamma(u_t(t)), u_t(t))_{\Gamma_1} \\
+ (b(x)u_t(t) - f(u(t)), u_t(t)) = 0.
\end{cases}
\]

Thus by Conditions 4 and 8

\[
\frac{1}{2} \frac{d}{dt} |u_t(t)|_2^2 + \frac{1}{2} \left( \rho(t) |\nabla u(t)|_2^2 \right) \\
\leq \frac{1}{2} \rho'(t) |\nabla u(t)|_2^2 + |b(x)|_{\infty} |u_t(t)|_2^2 + \int_{D} \alpha |u(t)|^q u(t)u_t(t)dx
\]

\[
\leq \frac{1}{2} \left( 1 + 2 |b(x)|_{\infty} \right) |u_t(t)|_2^2 + \frac{1}{2} \alpha^2 |u(t)|^{2(q+1)}.
\]

By Lemma 2.1 we have that

\[
|u_t(t)|_2^2 + \rho_0 \|u(t)\|^2 \leq |u_t(0)|_2^2 + \rho(0) \|u_0\|^2 + \int_0^t \left( (1 + 2 |b(x)|_{\infty}) |u_t(s)|_2^2 + \alpha^2 c^2 \|u(t)\|^{2(q+1)} \right) ds.
\]

Thus by Lemma 4.2 and the Gronwall lemma the solution $(u, u_t)$ to (1) exists globally. This completes the proof of the proposition. □

5. The Proof of Theorem 1. Assume that the boundary $\Gamma$ satisfies $\Gamma = \Gamma_0 \cup \Gamma_1$, where

\[
\Gamma_0 := \{ x \in \Gamma; m(x) \cdot \nu(x) \leq 0 \},
\]

\[
\Gamma_1 := \{ x \in \Gamma; m(x) \cdot \nu(x) > 0 \},
\]

where $m(x) := x - x^0$, $x^0, x \in \mathbb{R}^d$ and $\nu = \nu(x)$ denotes the unit outward normal at $x \in \Gamma$.

Notation. $R(x_0) := \max \{|m(x)| : x \in \bar{D}\}$.

We prove the main theorem in this paper. The real number $c_{q+2}$ in this theorem denotes the Poincaré number, that is, $|u|_{q+2} \leq c_{q+2} \|u\|$ holds by Lemma 2.1.

The proof of Theorem 1

Proof. Set

\[
\Psi(t) := 2(u_t(t), m(x) \cdot \nabla u(t)) + (d - 1)(u_t(t), u(t)).
\]

Consider the Lyapunov function

\[
E_\varepsilon(t) := E(t) + \varepsilon \Psi(t), \quad \varepsilon > 0.
\]

First

\[
\Psi'(t) = 2(u_{tt}(t), m(x) \cdot \nabla u(t)) + 2(u_t(t), m(x) \cdot \nabla u_t(t))
\]

\[
+ (d - 1)(u_{tt}(t), u(t)) + (d - 1) |u_t(t)|_2^2
\]

\[
= I_1 + I_2 + I_3 + I_4.
\]
Since
\[ m(x) \cdot \nabla u(t) = (m(x) \cdot \nu) \frac{\partial u(t)}{\partial \nu} \text{ and } |\nabla u(t)|^2 = \left( \frac{\partial u(t)}{\partial \nu} \right)^2 \text{ on } \Gamma_0, \quad (24) \]
it holds that
\[
2 \int_{\Gamma} \frac{\partial u(t)}{\partial \nu} [m(x) \cdot \nabla u(t)] d\Gamma - \int_{\Gamma} (m(x) \cdot \nu) |\nabla u(t)|^2 d\Gamma \\
= 2 \int_{\Gamma_0} \frac{\partial u(t)}{\partial \nu} (m(x) \cdot \nu) \frac{\partial u(t)}{\partial \nu} d\Gamma + 2 \int_{\Gamma_1} \frac{\partial u(t)}{\partial \nu} [m(x) \cdot \nabla u(t)] d\Gamma \\
- \int_{\Gamma} (m(x) \cdot \nu) |\nabla u(t)|^2 d\Gamma \\
\leq 2 \int_{\Gamma_1} \frac{\partial u(t)}{\partial \nu} [m(x) \cdot \nabla u(t)] d\Gamma \\
\leq 2R(x_0) |\gamma(u(t))|_{L^2(\Gamma_1)} |\nabla u(t)|_{L^2(\Gamma_1)} \\
\leq 2C^2 \pi R^2(x_0) |\gamma(u(t))|_{L^2(\Gamma_1)}^2 + \frac{1}{2} \|u(t)\|^2, 
\]
where there exists a constant \(C_\pi > 0\) such that \(|\text{tr}(a)|_{L^2(\Gamma)} \leq C_\pi |a|_{L^2(D)}\).
Thus by the Rellich identity and Condition 8
\[
I_{11} = 2(\rho(t) \Delta u(t), m(x) \cdot \nabla u(t)) \\
= \rho(t)(d - 2) \|u(t)\|^2 + 2\rho(t) \int_{\Gamma} \frac{\partial u(t)}{\partial \nu} [m(x) \cdot \nabla u(t)] d\Gamma \\
- \rho(t) \int_{\Gamma} (m(x) \cdot \nu) |\nabla u(t)|^2 d\Gamma \\
\leq \rho(t)(d - \frac{3}{2}) \|u(t)\|^2 + 2C^2 \pi R^2(x_0) |\gamma(u(t))|_{L^2(\Gamma_1)}^2. 
\]
Since there exists a constant \(c_\ast > 0\) such that \(|u(t)|_{2(\ast + 1)} \leq c_\ast \|u(t)\|\) by Lemma 2.1, by the Schwartz inequality and (23) we obtain an \(a(q) > 0\) such that
\[
I_{12} = 2(f(u(t)), m(x) \cdot \nabla u(t)) \\
\leq 2\alpha \int_D |u(t)|^{q+1} |m(x) \cdot \nabla u(t)| dx \\
\leq 2\alpha R(x_0)c^q_{\ast + 1} \|u(t)\|^{q+2} \\
\leq \alpha \left( 2 \left( \frac{2(q + 2)}{\rho_0(q + 2) - 2} \right)^{\frac{q+2}{2}} E(0)^{\frac{1}{2}} R(x_0)c^q_{\ast + 1} \right) E(t) \\
= \alpha a(q) E(t). 
\]
where \( a(q) := \left(2 \left( \frac{2(q+2)}{p \omega(q+2)} \right)^\frac{q+2}{2} E(0)^\frac{q}{2} R(x_0)c_{q+1} \right) \).

\[
I_2 = 2(u_t(t), m(x) \cdot \nabla u_t(t)) = \int_D m(x) \cdot \nabla (|u_t(t)|^2)
= \int_{\Gamma} \nu \cdot (|u_t(t)|^2 m(x)) d\Gamma - \int_D (\nabla \cdot m(x)) |u_t(t)|^2 dx
\leq R(x_0) |u_t(t)|_{L^2(\Gamma_1)}^2 - d |u_t(t)|_{H^1}^2.
\]

\[
I_3 + I_4 = (d-1)(u_t(t), u(t)) + (d-1) |u_t(t)|_{L^2}^2
= (d-1)(\rho \Delta u(t) - b(x) u_t(t) + f(u(t)), u(t)) + (d-1) |u_t(t)|_{L^2}^2
= -(d-1) \rho(t) \| u(t) \|^2 + (d-1) \rho(t) \left( \frac{\partial u(t)}{\partial \nu}, u(t) \right)_{\Gamma} - (d-1) (b(x) u_t(t), u(t))
+ \alpha(d-1) |u_t(t)|_{L^{q+2}}^2 + (d-1) |u_t(t)|_{L^2}^2.
\]

There exists a \( \hat{\lambda} > 0 \) such that \( |u(t)|_{L^2(\Gamma_1)}^2 \leq C_\pi |u(t)|_{L^2}^2 \leq C_\pi \hat{\lambda} \| u(t) \|^2 \) and set \( \delta := C_\pi \hat{\lambda} \). Then for any \( \eta > 0 \)
\[
(d-1) \rho(t) \left( \frac{\partial u(t)}{\partial \nu}, u(t) \right)_{\Gamma} = -(d-1) \rho(t) \int_{\Gamma_1} (\gamma(u_t(t)) u(t)) d\Gamma
\leq \frac{(d-1)^2}{4\eta} |\gamma(u_t(t))|_{L^2(\Gamma_1)}^2 + \eta \| u(t) \|^2_{L^2(\Gamma_1)}
\leq \frac{(d-1)^2}{4\eta} |\gamma(u_t(t))|_{L^2(\Gamma_1)}^2 + \eta \delta \| u(t) \|^2,
\]
where we may take an \( \eta > 0 \) such that \( \eta \delta < \frac{\delta}{2} \). Therefore it holds by Condition 8 that

\[
\Psi'(t) \leq -(d-\frac{3}{2}) \rho(t) \| u(t) \|^2 - (d-1) \rho(t) \| u(t) \|^2 + 2C^2_\pi R^2(x_0) \beta^2_1 |u_t(t)|_{L^2(\Gamma_1)}^2
+ \alpha a(q) E(t) + R(x_0) |u_t(t)|_{L^2(\Gamma_1)}^2
- d |u_t(t)|_{L^2}^2 + \frac{(d-1)^2}{4\eta} |\gamma(u_t(t))|_{L^2(\Gamma_1)}^2 + \eta |u(t)|_{L^2}^2
+ \alpha(d-1) |u(t)|_{L^{q+2}}^2 + (d-1) |u_t(t)|_{L^2}^2 - 2(b(x) u_t(t), m(x) \cdot \nabla u(t))
- (d-1) (b(x) u_t(t), u(t))
\leq - \left( \frac{1}{2} \rho_0 - \eta \delta - 2\kappa \right) \| u(t) \|^2 + \alpha(d-1) |u(t)|_{L^{q+2}}^2 - |u_t(t)|_{L^2}^2 + \alpha a(q) E(t)
+ \left( R(x_0) + 2C^2_\pi R^2(x_0) \beta^2_1 + \frac{(d-1)^2}{4\eta} \beta^2_1 \right) |u_t(t)|_{L^2(\Gamma_1)}^2
+ \left( \frac{R(x_0)^2}{4\kappa} b^2_1 + \frac{(d-1)^2}{4\kappa \lambda_1} b^2_1 \right) |u_t(t)|_{L^2}^2,
\]
where we take a sufficiently small \( \kappa > 0 \) such that

\[
-2(b(x) u_t(t), m(x) \cdot \nabla u(t)) \leq \frac{R(x_0)^2}{4\kappa} b^2_1 |u_t(t)|_{L^2}^2 + \kappa \| u(t) \|^2
\]
Thus by Lemma 4.1
\[-(d-1)(b(x)u_t(t), u(t)) \leq \frac{(d-1)^2}{4\kappa\lambda_1} b_1^2 |u_t(t)|^2 + \kappa \|u(t)\|^2.
Next since $\kappa > 0$ and $\eta > 0$ are sufficiently small, we may choose $\frac{1}{8}\rho_0 > \eta\delta + 2\kappa$ and by Lemmas 4.1-4.3 and (2) it follows that
\[
\left(-\frac{1}{2}\rho_0 + \eta\delta + 2\kappa \right) \|u(t)\|^2 + \alpha(d-1)|u(t)|^{q+2}_{q+2} \\
\leq \left(-\frac{1}{2}\rho_0 + \eta\delta + 2\kappa + \alpha(d-1)c_{q+2}^q\lambda_{E(0)}^q \right) \|u(t)\|^2 \leq -\frac{1}{4}\rho_0 \|u(t)\|^2,
\]
where
\[
\lambda_{E(0)} := \left(\frac{2(q+2)E(0)}{\rho_0(q+2) - 2}\right)^\frac{1}{2}.
\]
Therefore we have that
\[
\Psi'(t) \leq -\frac{1}{4}\rho_0 \|u(t)\|^2 - \frac{1}{4}\rho_0 |u_t(t)|^2 + \alpha a(q) E(t) \\
+ \left(R(x_0) + 2C_2 R^2(x_0)\beta_1^2 + \frac{(d-1)^2}{4\eta}\beta_1^2 \right) |u_t(t)|^2_{L^2(\Gamma_1)} \\
+ \left(\frac{R(x_0)^2}{4\kappa} b_1^2 + \frac{(d-1)^2}{4\kappa \lambda_1} b_1^2 \right) |u_t(t)|^2_{L^2(\Gamma_1)}.
\]
By (19), it follows that
\[
\varepsilon\Psi'(t) \leq \frac{\varepsilon}{2} \left(-\frac{1}{2}\rho_0 \|u(t)\|^2 - \frac{1}{2}\rho_0 |u_t(t)|^2 + \rho_0 \int_D F(u(t)) dx\right) \\
+ \varepsilon\alpha a(q) E(t) + d_0 |u_t(t)|^2_{L^2(\Gamma_1)} + b_0 |u_t(t)|^2_{L^2(\Gamma_1)},
\]
where we take an $\varepsilon > 0$ such that
\[
b_0 > \varepsilon \left(\frac{R(x_0)^2 b_1^2 + (d-1)^2}{4\kappa \lambda_1} b_1^2 \right), \\
d_0 > \varepsilon \left(\frac{R(x_0) + 2C_2 R^2(x_0)\beta_1^2 + (d-1)^2}{4\eta}\beta_1^2 \right).
\]
Thus by Lemma 4.1
\[
E_\varepsilon'(t) = E'(t) + \varepsilon\Psi'(t) \\
\leq -d_0 |u_t(t)|^2_{L^2(\Gamma_1)} - b_0 |u_t(t)|^2_{L^2(\Gamma_1)} \\
+ \frac{\varepsilon}{2} \left(-\frac{1}{2}\rho_0 \|u(t)\|^2 - \frac{1}{2}\rho_0 |u_t(t)|^2 + \rho_0 \int_D F(u(t)) dx\right) \\
+ \varepsilon\alpha a(q) E(t) + d_0 |u_t(t)|^2_{L^2(\Gamma_1)} + b_0 |u_t(t)|^2_{L^2(\Gamma_1)}.
\]
Since there exist positive constants $a_2, a_1 > 0$ such that $a_2 > a_1$ and
\[
a_1 E(t) \leq E_\varepsilon(t) \leq a_2 E(t),
\]
we obtain that
\[
E_\varepsilon'(t) \leq \frac{\rho_0}{2} \frac{\rho_0 - 2\alpha a(q)E_\varepsilon(t)}{2a_2} = -\sigma E_\varepsilon(t)
\]
where $\sigma = \frac{-\varepsilon (\rho_0 - 2\alpha a(q))}{2a^2} > 0$ because $\rho_0 > 2\alpha a(q)$ holds by the assumption. Thus for all $t \geq 0$

$$E(t) \leq \frac{a_2 e^{-\sigma t} E(0)}{a_1}.$$  

This completes the proof of the theorem. \hfill $\square$

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**REFERENCES**

[1] F. D. Araruna and A. B. Maciel, Existence and boundary stabilization of the semilinear wave equation, *Nonlinear Analysis*, 67 (2007), 1288–1305.

[2] M. M. Cavalcanti, V. N. D. Cavalcanti and P. Martinez, Existence and decay rate estimates for the wave equation with nonlinear boundary damping and source term, *J. Differential Equations*, 203 (2004), 119–158.

[3] M. M. Cavalcanti, V. N. Domingos Cavalcanti, Existence and asymptotic stability for evolution problems on manifolds with damping and source terms, *J. Math. Anal. Appl.*, 291 (2004), 109–127.

[4] M. M. Cavalcanti, V. N. Domingos Cavalcanti and I. Lasiecka, Well posedness and optimal decay rates for the wave equation with nonlinear damping-source interaction, *J. Differential Equations*, 236 (2007), 407–459.

[5] M. M. Cavalcanti, V. N. D. Cavalcanti and J. A. Soriano, On existence and asymptotic stability of solutions of the degenerate wave equation with nonlinear boundary conditions, *J. Math. Anal. Appl.*, 281 (2003), 108–124.

[6] V. Georgiev and G. Todorova, Existence of a solution of the wave equation with damping and source terms, *J. Differential Equations*, 109 (1994), 295–308.

[7] S. Gerbi and B. Said-Houari, Local existence and exponential growth for a semilinear damping wave equation with dynamic boundary conditions, *Adv. Equ.*, 13 (2008), 1051–1074.

[8] B. Guo and Z-C. Shao, On exponential stability of a semilinear wave equation with variable coefficients under the nonlinear boundary feedback, *Nonlinear Analysis*, 71 (2009), 5961–5978.

[9] V. Komornik and E. Zuazua, A direct method for boundary stabilization of the wave equation, *J. Math. Pures et appl.*, 69 (1990), 33–54.

[10] A. T. Louredo, M. A. Ferreira and M. M. Miranda, On a nonlinear wave equation with boundary damping, *Math. Meth. in Applied Sciences*, 37 (2014), 1278–1302.

[11] J. Malek, J. Necas, M. Okykta and M. Ruzicka, *Weak and Measure-valued Solutions to Evolutionary PDEs*, Chapman and Hall, 1996.

[12] S. A. Messoudi, Blow up in a nonlinearly damped wave equation, *Math Nachr.*, 231 (2001), 105–111.

[13] K. Ono, On global existence, Asymptotic stability and blowing up of solutions for some degenerate non-linear wave equations of Kirchhoff type with a strong dissipation, *Math. Meth. in Applied Sciences*, 20 (1997), 151–177.

[14] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Volarg, Berlin, 1989.

[15] E. Vitillaro, A potential well method for the wave equation with nonlinear source and boundary damping terms, *Glasgow Math. J.*, 44 (2002), 375–395.

[16] E. Vitillaro, Global existence for the wave equation with nonlinear boundary damping and source terms, *J. Differential Equations*, 186 (2002), 259–298.

[17] Zai-yun Zhang and Xiu-jin Miao, Global existence and uniform decay for wave equation with dissipative term and boundary damping, *Computers and Math. Appl.*, 59 (2010), 1003–1018.

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