Some Remarks on $\phi$-Dedekind rings and $\phi$-Prüfer rings

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Abstract

In this paper, the notions of nonnil-injective modules and nonnil-FP-injective modules are introduced and studied. Especially, we show that a $\phi$-ring $R$ is an integral domain if and only if any nonnil-injective (resp., nonnil-FP-injective) module $R$-module is injective (resp., FP-injective). Some new characterizations of $\phi$-von Neumann regular rings, nonnil-Notherian rings and nonnil-coherent rings are given. We finally characterize $\phi$-Dedekind rings and $\phi$-Prüfer rings in terms of $\phi$-flat modules, nonnil-injective modules and nonnil-FP-injective modules.

Key Words: nonnil-injective modules; nonnil-FP-injective modules; $\phi$-Dedekind rings; $\phi$-Prüfer rings.

2010 Mathematics Subject Classification: Primary: 13A15; Secondary: 13F05.

Recall from [5] that a commutative ring $R$ is an NP-ring if the nilpotent radical $\text{Nil}(R)$ is a prime ideal, and a ZN-ring if $Z(R) = \text{Nil}(R)$ where $Z(R)$ is the set of all zero-divisors of $R$. A prime ideal $P$ of $R$ is called divided prime if $P \subsetneq (x)$, for every $x \in R - P$. Set $\mathcal{H} = \{R|R$ is a commutative ring and $\text{Nil}(R)$ is a divided prime ideal of $R\}$. A ring $R$ is a $\phi$-ring if $R \in \mathcal{H}$. Moreover, a ZN $\phi$-ring is said to be a strong $\phi$-ring. Denote by $T(R)$ the localization of $R$ at the set of all regular elements. For a $\phi$-ring $R$, there is a ring homomorphism $\phi : T(R) \to R_{\text{Nil}(R)}$ such that $\phi(a/b) = a/b$. Denote by the ring $\phi(R)$ the image of $\phi$ restricted to $R$. In 2001, Badawi [6] investigated $\phi$-chained rings ($\phi$-CRs for short) which are $\phi$-rings $R$ such that for every $x, y \in R - \text{Nil}(R)$ either $x|y$ or $y|x$. In 2004, Anderson and Badawi [1] extended the notion of Prüfer domains to that of $\phi$-Prüfer rings which are $\phi$-rings $R$ satisfies that each finitely generated nonnil ideal is $\phi$-invertible. The authors in [1] characterized $\phi$-Prüfer rings from the perspective of ring structures, which says that a $\phi$-ring $R$ is $\phi$-Prüfer if and only if $R_{m}$ is a $\phi$-chained ring for any maximal ideal $m$ of $R$ if and only if $R/\text{Nil}(R)$ is a Prüfer domain if and only if $\phi(R)$ is Prüfer. Later in 2005, the authors in [2] generalized the concepts of Dedekind domains to the context of rings that are in the class $\mathcal{H}$. A $\phi$-ring is called a $\phi$-Dedekind ring
provided that any nonnil ideal is $\phi$-invertible. They also showed that a $\phi$-ring $R$ is $\phi$-Dedekind if and only if $R$ is nonnil-Noetherian and $R_m$ is a discrete $\phi$-chained ring for any maximal ideal $m$ of $R$, if and only if $R$ is nonnil-Noetherian, $\phi$-integral closed and of Krull dimension $\leq 1$, if and only if $R/\text{Nil}(R)$ is a Dedekind domain. Some generalizations of Noetherian domains, coherent domains, Bezout domains and Krull domains to the context of rings that are in the class $\mathcal{H}$ are also introduced and studied (see [1, 2, 4, 7, 8]).

The module-theoretic studies of rings in $\mathcal{H}$ started more than a decade ago. In 2006, Yang [19] introduced nonnil-injective modules by replacing the ideals in Baer’s criterion for injective modules with nonnil ideals, and obtained that a $\phi$-ring $R$ is nonnil-Noetherian if and only if any direct sum of nonnil-injective modules is nonnil-injective. In 2013, Zhao et al. [23] introduced and studied the conceptions of $\phi$-von Neumann rings which can be defined as the following characterizations: a $\phi$-ring $R$ is $\phi$-von Neumann if and only if its Krull dimension is 0, if and only if any $R$-module is $\phi$-flat, if and only if $R/\text{Nil}(R)$ is a von Neumann regular ring. In 2018, Zhao [22] gave a homological characterization of $\phi$-Prüfer rings: a strong $\phi$-ring $R$ is $\phi$-Prüfer if and only if each submodule of a $\phi$-flat module is $\phi$-flat, if and only if each nonnil ideal of $R$ is $\phi$-flat.

The main motivation of this paper is to give some characterizations of $\phi$-Dedekind rings and $\phi$-Prüfer rings in terms of some new versions of injective modules and FP-injective modules. We first introduce and study the notions of nonnil-injective modules and nonnil-FP-injective modules, and show that a $\phi$-ring $R$ is an integral domain if and only if any nonnil-injective module $R$-module is injective, if and only if any nonnil-FP-injective module $R$-module is FP-injective (see Theorem 1.6). Some new characterizations of $\phi$-von Neumann regular rings, nonnil-Noetherian rings and nonnil-coherent rings in terms of $\phi$-flat modules, nonnil-injective modules and nonnil-FP-injective modules are also given (see Theorem 1.7, Proposition 1.8 and Proposition 1.9 respectively). We obtain that a strong $\phi$-ring $R$ is a $\phi$-Dedekind ring if and only if any divisible module is nonnil-injective, if and only if any $h$-divisible module is nonnil-injective, if and only if any nonnil ideal of $R$ is projective (see Theorem 2.8). We also obtain that a strong $\phi$-ring $R$ is $\phi$-Prüfer, if and only if any divisible module is nonnil-FP-injective, if and only if any finitely generated nonnil ideal of $R$ is projective, if and only if any ideal of $R$ is $\phi$-flat, if and only if any $R$-module has an epimorphism $\phi$-flat envelope (see Theorem 2.13).
1. NONNIL-INJECTIVE MODULES AND NONNIL-FP-INJECTIVE MODULES

Throughout this paper, $R$ denotes an NP-ring with identity and all modules are unitary. We say an ideal $I$ of $R$ is nonnil if there exists a non-nilpotent element in $I$. Denote by $\text{NN}(R)$ the set of all nonnil ideals of $R$. It is easy to verify that $\text{NN}(R)$ is a multiplicative system of ideals. That is, $R \in \text{NN}(R)$ and $IJ \in \text{NN}(R)$ for any $I$ and $J$ both in $\text{NN}(R)$. Let $M$ be an $R$-module. Define

$$\phi\text{-tor}(M) = \{x \in M|Ix = 0 \text{ for some } I \in \text{NN}(R)\}.$$ 

An $R$-module $M$ is said to be $\phi$-torsion (resp., $\phi$-torsion free) provided that $\phi\text{-tor}(M) = M$ (resp., $\phi\text{-tor}(M) = 0$). Clearly, the class of $\phi$-torsion modules is closed under submodules, quotients, direct sums and direct limits. Thus an NP-ring $R$ is $\phi$-torsion free if and only if every flat module is $\phi$-torsion free if and only if $R$ is a ZN-ring (see [22, Proposition 2.2]). The classes of $\phi$-torsion modules and $\phi$-torsion free modules constitute a hereditary torsion theory of finite type. Recall that an ideal $I$ of $R$ is regular if there exists a regular element (i.e., non-zero-divisor) in $I$.

**Lemma 1.1.** Let $R$ be a $\phi$-ring and $I$ an ideal of $R$. Then the following assertions are equivalent:

1. $I$ is a nonnil ideal of $R$;
2. $I/\text{Nil}(R)$ is a nonzero ideal of $R/\text{Nil}(R)$;
3. $\phi(I)$ is a regular ideal of $\phi(R)$;

**Proof.** (1) $\iff$ (2): Obvious.

(1) $\Rightarrow$ (3): Let $s$ be a non-nilpotent element in $I$. Then $\frac{s}{1} \in \phi(I)$ is regular in $\phi(R)$. Indeed, suppose $\frac{s^k}{1} = 0$ in $\phi(R)$, then there exists a non-nilpotent element $u \in R$ such that $ust = 0$. Since $R$ is a $\phi$-ring, $us$ is non-nilpotent. Thus $\frac{t}{1} = 0$ in $\phi(R)$.

(3) $\Leftarrow$ (1): Let $\frac{s}{1}$ be an regular element in $\phi(I)$ with $s \in I$. Then $s$ is non-nilpotent. Indeed, if $s^n = 0$ in $R$, then $(\frac{s}{1})^n = \frac{s^n}{1} = 0$ in $\phi(R)$ which implies $\frac{s}{1}$ is not regular in $\phi(R)$. $\square$

Recall that an $R$-module $M$ is injective (resp., FP-injective) if $\text{Ext}_R^1(N, M) = 0$ for any (resp., finitely presented) $R$-module $N$. Now we investigate the notions of nonnil-injective modules and nonnil-FP-injective modules using $\phi$-torsion modules.

**Definition 1.2.** Let $R$ be an NP-ring and $M$ an $R$-module.

1. $M$ is called nonnil-injective provided that $\text{Ext}_R^1(T, M) = 0$ for any $\phi$-torsion module $T$.

2. $M$ is called nonnil-FP-injective provided that $\text{Ext}_R^1(T, M) = 0$ for any finitely presented $\phi$-torsion module $T$. 

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Certainly, an $R$-module $M$ is nonnil-injective if and only if $\text{Ext}^1_R(R/I, M) = 0$ for any nonnil ideal $I$ of $R$ (see [24, Theorem 1.7]). The class of nonnil-injective modules is closed under direct summands, direct products and extensions, and the class of nonnil-FP-injective modules is closed under pure submodules, direct sums, direct products and extensions.

Recall from [23] that an $R$-module $M$ is $\phi$-flat if $\text{Tor}^1_R(T, M) = 0$ for any $\phi$-torsion module $T$. It is well-known that an $R$-module $M$ is $\phi$-flat if and only if $\text{Tor}^1_R(R/I, M) = 0$ for any (finitely generated) nonnil ideal $I$ of $R$ (see [23, Theorem 3.2]).

**Proposition 1.3.** Let $R$ be an NP-ring, then the following assertions are equivalent:

1. $M$ is $\phi$-flat;
2. $\text{Hom}_R(M, E)$ is nonnil-injective for any injective module $E$;
3. $\text{Hom}_R(M, E)$ is nonnil-FP-injective for any injective module $E$;
4. if $E$ is an injective cogenerator, then $\text{Hom}_R(M, E)$ is nonnil-injective.
5. if $E$ is an injective cogenerator, then $\text{Hom}_R(M, E)$ is nonnil-FP-injective.

**Proof.** (1) $\Rightarrow$ (2): Let $T$ be a $\phi$-torsion $R$-module and $E$ an injective $R$-module. Since $M$ is $\phi$-flat, $\text{Ext}^1_R(T, \text{Hom}_R(M, E)) \cong \text{Hom}_R(\text{Tor}^1_R(T, M), E) = 0$. Thus $\text{Hom}_R(M, E)$ is nonnil-injective.

(2) $\Rightarrow$ (3) $\Rightarrow$ (5): Trivial.

(2) $\Rightarrow$ (4) $\Rightarrow$ (5): Trivial.

(5) $\Rightarrow$ (1): Let $I$ be a finitely generated nonnil ideal of $R$ and $E$ an injective cogenerator. Since $\text{Hom}_R(M, E)$ is nonnil-FP-injective, $\text{Hom}_R(\text{Tor}^1_R(R/I, M), E) \cong \text{Ext}^1_R(R/I, \text{Hom}_R(M, E)) = 0$. Since $E$ is an injective cogenerator, $\text{Tor}^1_R(R/I, M) = 0$. Thus $M$ is $\phi$-flat. □

**Proposition 1.4.** Let $R$ be a $\phi$-ring and $E$ an $R/\text{Nil}(R)$-module. Then $E$ is injective over $R/\text{Nil}(R)$ if and only if $E$ is nonnil-injective over $R$.

**Proof.** Let $I$ be a nonnil ideal of $R$. Set $\overline{R} = R/\text{Nil}(R)$ and $\overline{I} = I/\text{Nil}(R)$. Let $E$ be an $\overline{R}$-module. The short exact sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$ induces the long exact sequence of $R$-modules:

$$0 \rightarrow \text{Hom}_R(R/I, E) \rightarrow \text{Hom}_R(R, E) \rightarrow \text{Hom}_R(I, E) \rightarrow \text{Ext}^1_R(R/I, E) \rightarrow 0. \quad (a)$$

The short exact sequence $0 \rightarrow \overline{I} \rightarrow \overline{R} \rightarrow R/I \rightarrow 0$ induces the long exact sequence of $\overline{R}$-modules:

$$0 \rightarrow \text{Hom}_{\overline{R}}(R/I, E) \rightarrow \text{Hom}_{\overline{R}}(\overline{R}, E) \rightarrow \text{Hom}_{\overline{R}}(\overline{I}, E) \rightarrow \text{Ext}^1_{\overline{R}}(R/I, E) \rightarrow 0. \quad (b)$$
By [21] Lemma 1.6, $\text{INil}(R) = \text{Nil}(R)$. Thus $I \otimes_R \overline{R} \cong I/\text{INil}(R) \cong \overline{I}$. Consequently, we have $\text{Hom}_R(I, E) \cong \text{Hom}_{\overline{R}}(I \otimes_R \overline{R}, E) \cong \text{Hom}_R(I, \text{Hom}_{\overline{R}}(R, E)) \cong \text{Hom}_R(I, E)$ by the Adjoint Isomorphism Theorem (see [18] Theorem 2.2.16). Combining (a) and (b), we have $E$ is injective over $R/\text{Nil}(R)$ if and only if $E$ is nonnil-injective over $R$ (see Lemma [11] and [1] Lemma 2.4).

Proposition 1.5. Let $R$ be a $\phi$-ring and $M$ an FP-injective $R/\text{Nil}(R)$-module. Then $M$ is nonnil-FP-injective over $R$.

Proof. Let $T$ be a finitely presented $\phi$-torsion module over $R$. Then there is a short exact sequence $0 \to K \to F \to T \to 0$ where $F$ is a finitely generated free $R$-module and $K$ is finitely generated $R$-module. Set $\overline{R} = R/\text{Nil}(R)$. By tensoring $\overline{R}$ over $R$, we obtain a long exact sequence $\text{Tor}_1^R(T, \overline{R}) \to K \otimes_R \overline{R} \to F \otimes_R \overline{R} \to T \otimes_R \overline{R} \to 0$ over $\overline{R}$. By [21] Proposition 1.7, $\overline{R}$ is $\phi$-flat over $R$ thus $\text{Tor}_1^R(T, \overline{R}) = 0$. It follows that $T \otimes_R \overline{R}$ is a finitely presented $R$-module. There exists a commutative diagram of exact rows as follows:

$$
\begin{array}{cccccc}
\longrightarrow & \text{Hom}_R(F, M) & \longrightarrow & \text{Hom}_R(K, M) & \longrightarrow & \text{Ext}^1_R(T, M) \longrightarrow 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow f \\
\longrightarrow & \text{Hom}_{\overline{R}}(F \otimes_R \overline{R}, M) & \longrightarrow & \text{Hom}_{\overline{R}}(K \otimes_R \overline{R}, M) & \longrightarrow & \text{Ext}^1_{\overline{R}}(T \otimes_R \overline{R}, M) \longrightarrow 0.
\end{array}
$$

By the Adjoint isomorphism, the left two homomorphisms are isomorphisms. It follows from the Five Lemma that $f$ is also an isomorphism. Since $M$ is FP-injective over $\overline{R}$, $\text{Ext}^1_{\overline{R}}(T \otimes_R \overline{R}, M) = 0$. Then $\text{Ext}^1_R(T, M) = 0$. Thus $M$ is nonnil-FP-injective over $R$.

Obviously, any FP-injective module is nonnil-FP-injective, and any injective module is nonnil-injective. However, the converses characterize integral domains.

Theorem 1.6. Let $R$ be a $\phi$-ring. Then the following assertions are equivalent:

1. $R$ is an integral domain;
2. any nonnil-injective module is injective;
3. any nonnil-FP-injective module is FP-injective.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) : Trivial.

(2) $\Rightarrow$ (1): By [9] Theorem 3.1.6, $\text{Hom}_\mathbb{Z}(R/\text{Nil}(R), \mathbb{Q}/\mathbb{Z})$ is an injective $R/\text{Nil}(R)$-module. Thus by Proposition [12] $\text{Hom}_\mathbb{Z}(R/\text{Nil}(R), \mathbb{Q}/\mathbb{Z})$ is a nonnil-injective $R$-module, and thus an injective $R$-module. By [9] Theorem 3.2.10], $R/\text{Nil}(R)$ is a flat $R$-module. Let $K$ be a finitely generated nilpotent ideal, then $K \subseteq \text{Nil}(R) \subseteq \text{Rad}(R)$. Thus $K/K\text{Nil}(R) = \frac{K \cap \text{Nil}(R)}{K \text{Nil}(R)} = \text{Tor}_1^R(R/K, R/\text{Nil}(R)) = 0$. It follows from
the Nakayama Lemma that $K = 0$. Thus $\text{Nil}(R) = 0$, and then $R$ is an integral domain.

$(3) \Rightarrow (1)$: Similar with $(2) \Rightarrow (1)$. □

Recall from [23] that a $\phi$-ring $R$ is said to be $\phi$-von Neumann if the Krull dimension of $R$ is 0. It is well known that a $\phi$-ring $R$ is $\phi$-von Neumann if and only if $R/\text{Nil}(R)$ is a von Neumann ring, if and only if any $R$-module is $\phi$-flat (see [23, Theorem 4.1]).

**Theorem 1.7.** Let $R$ be a $\phi$-ring. Then the following assertions are equivalent:

1. $R$ is a $\phi$-von Neumann regular ring;
2. $R/\text{Nil}(R)$ is a field;
3. any non-nilpotent element in $R$ is invertible.
4. any $R$-module is $\phi$-flat;
5. any $R$-module is nonnil-FP-injective.
6. any $R$-module is nonnil-injective.

**Proof.** $(1) \iff (4)$: See [23, Theorem 4.1].

$(1) \Rightarrow (2)$: Since $\text{Nil}(R)$ is a prime ideal of $R$, $R/\text{Nil}(R)$ is a 0-dimensional domain, thus a field by [12, Theorem 3.1].

$(2) \Rightarrow (3)$: Let $a$ be a non-nilpotent element in $R$. Since $R/\text{Nil}(R)$ is a field, there exists $b \in R$ such that $1 - ab \in \text{Nil}(R)$. That is, $(1 - ab)^n = 0$ for some $n$. It is easy to verify that $a$ is invertible.

$(3) \Rightarrow (2) \Rightarrow (1)$: Trivial.

$(3) \Rightarrow (5)$: It follows from (3) that the only nonnil ideal of $R$ is $R$ itself. Let $T$ be a finitely presented $\phi$-torsion module. Then $T = \phi\text{-tor}(T) = \{x \in T | Ix = 0 \text{ for some nonnil ideal } I \text{ of } R\} = 0$. It follows that $\text{Ext}_R^1(T, M) = 0$. Consequently, $M$ is nonnil-FP-injective.

$(5) \Rightarrow (1)$: Let $I$ be a finitely generated nonnil ideal of $R$. Since for any $R$-module $M$, $\text{Ext}_R^1(R/I, M) = 0$ by (5), then $R/I$ is projective. Thus $I$ is an idempotent ideal of $R$. By [11, Proposition 1.10], $I$ is generated by an idempotent $e \in R$. Thus $R$ is a $\phi$-von Neumann regular ring by [23, Theorem 4.1].

$(3) \Rightarrow (6)$ and $(6) \Rightarrow (5)$: Obvious. □

Recall from [7] that a $\phi$-ring $R$ is called nonnil-Noetherian if any nonnil ideal of $R$ is finitely generated.

**Proposition 1.8.** Let $R$ be a $\phi$-ring. Then $R$ is nonnil-Noetherian if and only if any nonnil-FP-injective module is nonnil-injective.

**Proof.** Suppose $R$ is a nonnil-Noetherian ring. Let $I$ be a nonnil ideal of $R$ and $M$ a nonnil-FP-injective module. Then $I$ is finitely generated, and thus $R/I$ is
finitely presented \( \phi \)-torsion. It follows that \( \text{Ext}_R^1(R/I, M) \). Consequently, \( M \) is nonnil-injective by [24, Theorem 1.7]. On the other hand, since the class of nonnil-FP-injective modules is closed under direct sums, \( R \) is a nonnil-Noetherian ring by [19, Theorem 1.9].

Recall from [4] that a \( \phi \)-ring \( R \) is called nonnil-coherent if any finitely generated nonnil ideal of \( R \) is finitely presented. A \( \phi \)-ring \( R \) is nonnil-coherent if and only if any direct product of \( \phi \)-flat modules is \( \phi \)-flat, if and only if \( R^I \) is \( \phi \)-flat for any indexing set \( I \) (see [4, Theorem 2.4]). Now we give a new characterization of nonnil-coherent rings utilizing the preenveloping properties of \( \phi \)-flat modules.

**Proposition 1.9.** Let \( R \) be a \( \phi \)-ring. Then \( R \) is nonnil-coherent if and only if the class of \( \phi \)-flat modules is preenveloping.

**Proof.** Suppose \( R \) is a nonnil-coherent ring. By [4, Theorem 2.4], the class of \( \phi \)-flat modules is closed under direct products. Note that any pure submodule of a \( \phi \)-flat module is \( \phi \)-flat. Thus the class of \( \phi \)-flat modules is preenveloping by [9, Lemma 5.3.12, Corollary 6.2.2]. On the other hand, let \( \{ F_i \}_{i \in I} \) be a family of \( \phi \)-flat modules. Let \( \prod_{i \in I} F_i \to F \) is a \( \phi \)-flat preenvelope. Then there is a factorization \( \prod_{i \in I} F_i \to F \to F_i \) for each \( i \in I \). Consequently, the natural composition \( \prod_{i \in I} F_i \to F \to \prod_{i \in I} F_i \) is an identity. Thus \( \prod_{i \in I} F_i \) is a direct summand of \( F \) and then \( \prod_{i \in I} F_i \) is \( \phi \)-flat. It follows from [4, Theorem 2.4] that \( R \) is nonnil-coherent.

The following Corollary follows from Theorem 2.8 and [9, Corollary 6.3.5].

**Corollary 1.10.** Let \( R \) be a nonnil-coherent ring. If the class of \( \phi \)-flat modules is closed under inverse limits, then the class of \( \phi \)-flat modules is enveloping.

2. \( \phi \)-Dedekind rings and \( \phi \)-Prüfer rings

Recall that an \( R \)-module \( E \) is said to be divisible if \( sM = M \) for any regular element \( s \in R \), and an \( R \)-module \( M \) is said to be \( h \)-divisible provided that \( M \) is a quotient of an injective module. Evidently, any injective module is \( h \)-divisible and any \( h \)-divisible module is divisible. It is well known that an integral domain \( R \) is a Dedekind domain if and only if any \( h \)-divisible module is injective, if and only if any divisible module is injective (see [18, Theorem 5.2.15] for example).

**Definition 2.1.** Let \( R \) be an NP-ring. An \( R \)-module \( E \) is called nonnil-divisible provided that for any \( m \in E \) and any non-nilpotent element \( a \in R \), there exists \( x \in E \) such that \( ax = m \).
Lemma 2.2. Let $R$ be an NP-ring and $E$ an $R$-module. Consider the following statements:

1. $E$ is nonnil-divisible;
2. $E$ is divisible;
3. $\text{Ext}^1_R(R/\langle a \rangle, E) = 0$ for any $a \not\in \text{Nil}(R)$.

Then we have $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$. Moreover, if $R$ is a ZN-ring, all statements are equivalent.

Proof. $(1) \Rightarrow (2)$ and $(2) \Rightarrow (1)$ for ZN-rings: Trivial.

$(1) \Rightarrow (3)$: Let $a$ be a non-nilpotent element (then regular) in $R$ and $f : \langle a \rangle \to E$ be an $R$-homomorphism. Then there exists an element $x \in E$ such that $f(a) = ax$ since $E$ is nonnil-divisible. Set $g(r) = rx$ for any $r \in R$. Then $g$ is an extension of $f$ to $R$. Thus $\text{Ext}^1_R(R/\langle a \rangle, E) = 0$.

$(1) \Rightarrow (3)$ for ZN-rings: Let $a$ be a non-nilpotent element in $R$ and $m$ an element in $E$. Set $f(ra) = rm$. Then $f$ is a well-defined $R$-homomorphism from $\langle a \rangle$ to $E$. Since $\text{Ext}^1_R(R/\langle a \rangle, E) = 0$, there exists an $R$-homomorphism $g : R \to E$ such that $g|_{\langle a \rangle} = f$. Let $x = g(1)$, then $m = f(a) = g(a) = ag(1) = ax$. Thus $E$ is nonnil-divisible. \qed

The following result is an easy corollary of Lemma 2.2.

Corollary 2.3. Let $R$ be a ZN-ring and $E$ a nonnil-FP-injective $R$-module. Then $E$ is a nonnil-divisible $R$-module.

Lemma 2.4. Let $R$ be an NP-ring and $E$ a nonnil-divisible $R$-module. Then $E_p$ is a nonnil-divisible $R_p$-module for any prime ideal $p$ of $R$.

Proof. Suppose $E$ is a nonnil-divisible $R$-module. Let $\frac{m}{s}$ be an element in $E_p$ and $\frac{r}{t}$ a non-nilpotent element in $R_p$. Then $s$, $t$ and $r$ are non-nilpotent elements in $R$. Thus there exists $y \in E$ such that $tm = sry$ in $R$. Then $\frac{m}{s} = \frac{r}{t} \cdot \frac{y}{1}$. It follows that $E_p$ is a nonnil-divisible $R_p$-module. \qed

Recall from [1] that a $\phi$-ring $R$ is called a $\phi$-chained ring if every $x \in R_{\text{Nil}(R)} - \phi(R)$, we have $x^{-1} \in \phi(R)$, equivalently, if for any $a, b \in R - \text{Nil}(R)$, either $a|b$ or $b|a$ in $R$. Moreover, a $\phi$-ring $R$ is said to be a discrete $\phi$-chained ring if $R$ is a $\phi$-chained ring with at most one nonnil prime ideal and every nonnil ideal of $R$ is principal (see [2]).

Proposition 2.5. Let $R$ be a discrete $\phi$-chained ring and $E$ a nonnil-divisible $R$-module. Then $E$ is a nonnil-injective $R$-module.
Proof. Let \( I \) be a nonnil ideal of \( R \). Since \( R \) is a discrete \( \phi \)-chained ring, then \( I \) is generated by a non-nilpotent element \( a \in R \). Let \( f : I \to E \) be an \( R \)-homomorphism. Then there exists \( x \in E \) such that \( f(a) = ax \) as \( E \) is divisible. Define \( g : R \to E \) by \( g(r) = rx \). Then \( g \) is an extension of \( f \) to \( R \). Hence \( E \) is a nonnil-injective \( R \)-module.

\[ \square \]

Recall that a regular ideal \( I \) of \( R \) is called invertible if \( II^{-1} = R \) where \( I^{-1} = \{ x \in T(R) | Ix \subseteq R \} \). It follows from [12, Lemma 18.1] and [11, Lemma 5.3] that a regular ideal is invertible if and only if it is finitely generated and locally principal, if and only if it is projective. Recall from [11] that a nonnil ideal \( I \) of a \( \phi \)-ring \( R \) is said to be \( \phi \)-invertible provided that \( \phi(I) \) is an invertible ideal of \( \phi(R) \).

**Proposition 2.6.** Let \( R \) be a \( \phi \)-ring and \( I \) a nonnil ideal of \( R \). If \( I \) is projective over \( R \), then \( I \) is \( \phi \)-invertible.

**Proof.** Since \( I \) is a projective \( R \)-ideal, \( I \) is a direct summand of a free \( R \)-module \( R^{(x)} \). Then \( \phi(I) \) is a direct summand of a free \( \phi(R) \)-module \( \phi(R)^{(x)} \). Thus \( \phi(I) \) is a projective \( \phi(R) \)-ideal. Since \( I \) is a nonnil ideal of \( R \), \( \phi(I) \) is a regular ideal of \( \phi(R) \) by Lemma [11]. By [11, Lemma 5.3], \( \phi(I) \) is an invertible ideal of \( \phi(R) \). Thus \( I \) is \( \phi \)-invertible. \( \square \)

Recall that an integral domain \( R \) is a Dedekind domain if any nonzero ideal is invertible. Utilizing \( \phi \)-invertible, the authors in [2] introduce \( \phi \)-Dedekind rings which are generalizations of Dedekind domains to the context of rings that are in the class \( \mathcal{H} \).

**Definition 2.7.** A \( \phi \)-ring \( R \) is called \( \phi \)-Dedekind provided that any nonnil ideal of \( R \) is \( \phi \)-invertible.

**Theorem 2.8.** Let \( R \) be a \( \phi \)-ring. Then the following statements are equivalent for \( R \):

1. \( R \) is a \( \phi \)-Dedekind ring and a strong \( \phi \)-ring;
2. any divisible module is nonnil-injective;
3. any \( h \)-divisible module is nonnil-injective;
4. any nonnil ideal of \( R \) is projective.

**Proof.** (1) \( \Rightarrow \) (2): Let \( E \) be a divisible module and \( I \) a nonnil ideal of \( R \). By [2, Theorem 2.10], \( R \) is non-nil-Noetherian. Then \( I \) is finitely generated, and thus \( R/I \) is finitely presented. Let \( m \) be a maximal ideal of \( R \). Then \( E_m \) is a divisible module over \( R_m \) by Lemma [2.2] and Lemma [2.4]. By [2, Theorem 2.10] again, \( R_m \) is
a discrete \( \phi \)-chained ring, thus \( E_m \) is a nonnil-injective \( R_m \)-module by Proposition 2.5. By [13, Theorem 3.9.11], \( \text{Ext}^1_{R_m}(R/I, E)_m = \text{Ext}^1_{R_m}(R_m/I_m, E_m) = 0 \). Thus \( \text{Ext}^1_R(R/I, E) = 0 \). Therefore, \( E \) is nonnil-injective.

(2) \( \Rightarrow \) (3): Trival.

(3) \( \Rightarrow \) (4): Let \( N \) be an \( R \)-module, \( I \) a nonnil ideal of \( R \). There exists a long exact sequence as follows:

\[
0 = \text{Ext}^1_R(R, N) \rightarrow \text{Ext}^1_R(I, N) \rightarrow \text{Ext}^2_R(R/I, N) \rightarrow \text{Ext}^2_R(R, N) = 0.
\]

Let \( 0 \rightarrow N \rightarrow E \rightarrow K \rightarrow 0 \) be an exact sequence where \( E \) is the injective envelope of \( N \). There exists a long exact sequence as follows:

\[
0 = \text{Ext}^1_R(R/I, E) \rightarrow \text{Ext}^1_R(R/I, K) \rightarrow \text{Ext}^2_R(R/I, N) \rightarrow \text{Ext}^2_R(R/I, E) = 0.
\]

Thus \( \text{Ext}^1_R(I, N) \cong \text{Ext}^2_R(R/I, N) \cong \text{Ext}^1_R(R/I, K) = 0 \) as \( K \) is nonnil-injective. It follows that \( I \) is a projective ideal of \( R \).

(4) \( \Rightarrow \) (1): It follows from Proposition 2.6 that we just need to show \( R \) is a strong \( \phi \)-ring. Indeed, Let \( a \) be non-nilpotent element in \( R \). Then \( \langle a \rangle \) is a projective ideal of \( R \). It follows [13, Corollary 2.6] that \( R \) is a strong \( \phi \)-ring. \( \square \)

The next example shows that every divisible module is not necessary nonnil-injective for \( \phi \)-Dedekind rings. Thus the condition that \( R \) is a strong \( \phi \)-ring in Theorem 2.8 cannot be removed.

**Example 2.9.** Let \( D \) be non-field Dedekind domain and \( K \) its quotient field. Let \( R = D(+)K/D \) be the idealization construction. Then \( \text{Nil}(R) = 0(+)K/D \). Since \( D \cong R/\text{Nil}(R) \) is a Dedekind domain, \( R \) is a \( \phi \)-Dedekind ring by [2, Theorem 2.5]. Denote by \( U(R) \) and \( U(D) \) the sets of unit elements of \( R \) and \( D \) respectively. Since \( Z(R) = \{(r, m) | r \in Z(D) \cup Z(K/D)\} = [R - U(D)](+)K/D = R - U(R) \) by [3 Theorem 3.5, Theorem 3.7], \( R \) is a total ring of quotient. Thus any \( R \)-module is divisible. However, since \( \text{Nil}(R) \) is not a maximal ideal of \( R \), there exists an \( R \)-module \( M \) which is not nonnil-injective by Theorem 1.7.

Recall that an integral domain \( R \) is a Prüfer domain if any finitely generated nonzero ideal is invertible. The following definition is a generalization of Prüfer domains to the context of rings that are in the class \( \mathcal{H} \) (see [1]).

**Definition 2.10.** A \( \phi \)-ring \( R \) is called \( \phi \)-Prüfer provided that any finitely generated nonnil ideal of \( R \) is \( \phi \)-invertible.

**Lemma 2.11.** Let \( R \) be an NP-ring, \( p \) a prime ideal of \( R \) and \( I \) an ideal of \( R \). Then \( I \) is nonnil over \( R \) if and only if \( I_p \) is nonnil over \( R_p \).
Proof. Let $I$ be nonnil over $R$ and $x$ a non-nilpotent element in $I$. We will show the element $x/1$ in $I_p$ is non-nilpotent in $R_p$. If $(x/1)^n = x^n/1 = 0$ in $R_p$ for some positive integer $n$, there is an $s \in R - \mathfrak{p}$ such that $sx^n = 0$ in $R$. Since $R$ is an NP-ring, $\text{Nil}(R)$ is the minimal prime ideal of $R$. In the integral domain $R/\text{Nil}(R)$, we have $\overline{sx^n} = 0$, thus $\overline{x^n} = 0$ since $s \not\in \text{Nil}(R)$. So $x \in \text{Nil}(R)$, a contradiction.

Let $x/s$ be a non-nilpotent element in $I_p$ where $x \in I$ and $s \in R - \mathfrak{p}$. Clearly, $x$ is non-nilpotent in $R$ and thus $I$ is nonnil over $R$. $\square$

Proposition 2.12. Let $R$ be an NP-ring, $\mathfrak{p}$ a prime ideal of $R$ and $M$ an $R$-module. Then $M$ is $\phi$-torsion over $R$ if and only $M_\mathfrak{p}$ is $\phi$-torsion over $R_\mathfrak{p}$.

Proof. Let $M$ be an $R$-module and $x \in M$. If $M_\mathfrak{p}$ is $\phi$-torsion over $R_\mathfrak{p}$, there is a nonnil ideal $I_p$ over $R_\mathfrak{p}$ such that $I_p x/1 = 0$ in $R_\mathfrak{p}$. Let $I$ be the preimage of $I_p$ in $R$. Then $I$ is nonnil by Lemma 2.11. Thus there is a non-nilpotent element $t \in I$ such that $tkx = 0$ for some $k \not\in m$. Let $s = tk$. Then we have $\langle s \rangle$ is nonnil and $\langle s \rangle x = 0$. Thus $M$ is $\phi$-torsion. Suppose $M$ is $\phi$-torsion over $R$. Let $x/s$ be an element in $M_\mathfrak{p}$. Then there is a nonnil ideal $I$ such that $Ix = 0$, and thus $I_p x/s = 0$ with $I_p \in \text{Nil}(R_\mathfrak{p})$ by Lemma 2.11. It follows that $M_\mathfrak{p}$ is $\phi$-torsion over $R_\mathfrak{p}$. $\square$

Theorem 2.13. Let $R$ be a $\phi$-ring. Then the following statements are equivalent for $R$: 

1. $R$ is a $\phi$-Prüfer ring and a strong $\phi$-ring.; 
2. any divisible module is nonnil-FP-injective; 
3. any $h$-divisible module is nonnil-FP-injective; 
4. any finitely generated nonnil ideal of $R$ is projective; 
5. any (finitely generated) nonnil ideal of $R$ is flat; 
6. any (finitely generated) ideal of $R$ is $\phi$-flat; 
7. any submodule of $\phi$-flat module is $\phi$-flat; 
8. any $R$-module has an epimorphism $\phi$-flat preenvelope; 
9. any $R$-module has an epimorphism $\phi$-flat envelope.

Proof. $(1) \Rightarrow (2)$: Let $T$ be a finitely presented $\phi$-torsion module and $\mathfrak{m}$ a maximal ideal of $R$. Then by Proposition 2.12, $T_\mathfrak{m}$ is a finitely presented $\phi$-torsion $R_\mathfrak{m}$-module. By [11 Corollary 2.10], $R_\mathfrak{m}$ is a $\phi$-chained ring. Since $R$ is a strong $\phi$-ring, $R_\mathfrak{m}$ is a strong $\phi$-ring. Thus $T_\mathfrak{m} \cong \oplus_{i=1}^n R_\mathfrak{m}/R_\mathfrak{m} x_i$ for some regular element $x_i \in R_\mathfrak{m}$ by [22 Theorem 4.1]. Let $E$ be a divisible module. Then $E_\mathfrak{m}$ is a divisible module over $R_\mathfrak{m}$ by Lemma 2.2 and Lemma 2.4. Thus $\text{Ext}_R^1(T, E)_\mathfrak{m} = \text{Ext}_R^1(T, E) = \oplus_{i=1}^n \text{Ext}_R^1(R_\mathfrak{m}/R_\mathfrak{m} x_i, E_\mathfrak{m}) = 0$ by Lemma 2.2 and [18 Theorem 3.9.11]. It follows that $\text{Ext}_R^1(T, E) = 0$. Therefore, $E$ is nonnil-FP-injective.
Thus there is a commutative diagram of \( R \)-modules:

\[
\begin{array}{ccccccc}
0 & \rightarrow & \text{Tor}_1^R(R/I, J) & \rightarrow & I \otimes_R J & \rightarrow & R \otimes_R J & \rightarrow & R/I \otimes_R J & \rightarrow & 0 \\
\downarrow f & & \downarrow g & & \downarrow h & & \downarrow \cong & & & \\
0 & \rightarrow & \text{Tor}_1^R(R/K, J \otimes_R \overline{R}) & \rightarrow & K \otimes_R J & \rightarrow & \overline{R} \otimes_R J & \rightarrow & \overline{R}/K \otimes_R J & \rightarrow & 0.
\end{array}
\]

Since \( g \) and \( h \) are epimorphisms, \( f \) is also an epimorphism by the Five Lemma (see \cite{[18]} Theorem 1.9.9). By (5) \( J \) is flat, then \( \text{Tor}_1^R(R/I, J) = 0 \). Thus \( \text{Tor}_1^R(R/K, L) \cong \).
\[
\text{Tor}_1^R(\overline{R}/K, J \otimes_R \overline{R}) = 0. \text{ Consequently, } \overline{R} = R/\text{Nil}(R) \text{ is a Prüfer domain. By } [11 \text{ Corollary 2.10}], R \text{ is a } \phi\text{-Prüfer ring.}
\]

(5) ⇒ (7): Let \( M \) be a \( \phi \)-flat module and \( N \) a submodule of \( M \). Let \( I \) be a nonnil ideal of \( R \), then \( I \) is flat by (6). Thus \( \text{fd}_R(R/I) \leq 1 \). By considering the long exact sequence \( \text{Tor}_2^R(R/I, M/N) \to \text{Tor}_1^R(R/I, N) \to \text{Tor}_1^R(R/I, M) \), we have \( \text{Tor}_1^R(R/I, N) = 0 \) as \( \text{Tor}_2^R(R/I, M/N) = \text{Tor}_1^R(R/I, M) = 0 \). Thus \( N \) is \( \phi \)-flat.

(7) ⇒ (6) and (9) ⇒ (8): Trivial.

(8) ⇒ (7): Let \( F \) be a \( \phi \)-flat module, \( i : K \to F \) a monomorphism and \( f : K \to F' \) an epimorphism \( \phi \)-flat preenvelope. Then there exists an homomorphism \( g : F' \to F \) such that \( i = gf \). Thus \( f \) is a monomorphism. Consequently, \( K \cong F' \) is \( \phi \)-flat.

(1) + (4) + (7) ⇒ (9): Let \( R \) be a \( \phi \)-Prüfer ring and \( I \) a finitely generated nonnil ideal of \( R \). By (4), \( I \) is projective and thus finitely presented. It follows that \( R \) is nonnil-coherent. Thus the class of \( \phi \)-flat modules is preenveloping by Proposition 1.9. Let \( \{ F_i | i \in I \} \) be a family of \( \phi \)-flat modules. Then \( \prod_{i \in I} F_i \) is \( \phi \)-flat by [4, Theorem 2.4]. By (7), the class of \( \phi \)-flat modules is closed under submodules. Thus the class of \( \phi \)-flat modules is closed under inverse limits. By corollary 1.10 the class of \( \phi \)-flat modules is enveloping.

We claim that the \( \phi \)-flat envelope of any \( R \)-module \( M \) is an epimorphism. Indeed, suppose \( f : M \to F \) be \( \phi \)-flat envelope of \( M \). Let \( f = h \circ g \) with \( g : M \to \text{Im} f \) an epimorphism and \( f : \text{Im} f \to F \) the embedding map. We will show \( g \) is the the \( \phi \)-flat envelope of \( M \). For any \( f' : M \to F' \) with \( F' \phi \)-flat, there exists \( l : F \to F' \) such that \( l \circ f = f' \). Then \( g \circ h \circ l = f' \), and thus \( g \) is a \( \phi \)-flat preenvelope of \( M \) as \( \text{Im} f \) is \( \phi \)-flat by (7). Suppose \( a : \text{Im} f \to \text{Im} f \) such that \( g = a \circ g \). Then \( a \) is an epimorphism. Consider the following commutative diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{g} & \text{Im} f \xrightarrow{h} F \\
\downarrow a & & \downarrow b \\
M & \xrightarrow{g} & \text{Im} f \xrightarrow{h} F
\end{array}
\]

Since \( f = h \circ g \) is an \( \phi \)-flat envelope, there exists \( b : F \to F \) such that \( b \circ f = b \circ h \circ g = h \circ a \circ g = h \circ g = f \). Since \( g \) is an epimorphism, \( h \circ a = b \circ h \). Then \( a \) is a monomorphism, and thus \( a \) is an isomorphism. It follows that \( g \) is the \( \phi \)-flat envelope of \( M \).

\begin{remark}
Actually, Zhao [22, Theorem 4.3] showed that if \( R \) is a strong \( \phi \)-ring, then \( R \) is a \( \phi \)-Prüfer ring if and only if each submodule of a \( \phi \)-flat \( R \)-module is \( \phi \)-flat, if and only if each nonnil ideal of \( R \) is \( \phi \)-flat, if and only if finitely generated nonnil ideal of \( R \) is \( \phi \)-flat. In Theorem 2.13 we give some simple versions of [22, Theorem
\end{remark}
and several new characterizations $\phi$-Prüfer ring using divisible modules, nonnil-FP-injective modules and the epimorphic enveloping properties of $\phi$-flat modules.

The final example shows that every divisible $R$-module is not necessary nonnil-FP-injective for $\phi$-Prüfer rings. Thus the condition that $R$ is a strong $\phi$-ring in Theorem 2.13 also cannot be removed.

**Example 2.15.** Let $D$ be non-field Prüfer domain and $K$ its quotient field. Let $R = D(+)K/D$ be the idealization construction. As in Example 2.9, we can show $R$ is a $\phi$-Prüfer ring and total ring of quotient. Thus any $R$-module is divisible. However, since $\text{Nil}(R)$ is not a maximal ideal of $R$, the Krull dimension of $R > 1$. Thus there exists an $R$-module $M$ which is not nonnil-FP-injective by Theorem 1.7.

**Acknowledgement.**
The first author was supported by the Natural Science Foundation of Chengdu Aeronautic Polytechnic (No. 062026) and the National Natural Science Foundation of China (No. 12061001).

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