Two-dimensional interactions between a 
BF-type theory and a collection of vector fields

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Abstract
Consistent interactions that can be added to a two-dimensional, 
free abelian gauge theory comprising a special class of BF-type models 
and a collection of vector fields are constructed from the deformation 
of the solution to the master equation based on specific cohomological 
techniques. The deformation procedure modifies the Lagrangian action, the gauge transformations, as well as the accompanying algebra 
of the interacting model.
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1 Introduction
A key point in the development of the BRST formalism was its cohomological 
understanding, which allowed, among others, a useful investigation of many 
interesting aspects related to the perturbative renormalization problem [1]–
[4], the anomaly-tracking mechanism [4]–[8], the simultaneous study of local 
and rigid invariances of a given theory [9], as well as to the reformulation 
of the construction of consistent interactions in gauge theories [10]–[13] in terms 
of the deformation theory [14]–[16], or, actually, in terms of the deformation 
of the solution to the master equation.

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The scope of this paper is to investigate the consistent interactions that can be added to a free, abelian, two-dimensional gauge theory consisting of a collection of vector fields and a BF-type model [17] involving a set of scalar fields, two collections of one-forms and a system of two-forms. This work enhances the previous Lagrangian [18] and Hamiltonian [19]–[20] results on the study of self-interactions in certain classes of BF-type models. The resulting interactions are accurately described by a gauge theory with an open algebra of gauge transformations. The interacting model reveals a geometric interpretation in terms of a Poisson structure present in various models of two-dimensional gravity [21]–[27] and also some interesting algebraic features. The analysis of Poisson Sigma Models, including their relationship to two-dimensional gravity and the study of classical solutions, can be found in [28]–[34] (see also [35]).

Our strategy goes as follows. Initially, we determine in Section 2 the antifield-BRST symmetry of the free model, that splits as the sum between the Koszul-Tate differential and the exterior derivative along the gauge orbits, \( s = \delta + \gamma \). Next, in Section 3 we determine the consistent deformations of the solution to the master equation for the free model. The first-order deformation belongs to the local cohomology \( H^0(s|d) \), where \( d \) is the exterior space-time derivative. The computation of the cohomological space \( H^0(s|d) \) proceeds by expanding the co-cycles according to the antighost number, and by further using the cohomological groups \( H(\gamma) \) and \( H(\delta|d) \). We find that the first-order deformation is parametrized by some functions of the undifferentiated scalar fields, which become restricted to fulfill certain equations in order to produce a consistent second-order deformation. With the help of these equations, we then infer that the remaining deformations, of order three and higher, can be taken to vanish. The identification of the interacting model is developed in Section 4. The cross-couplings between the collection of vector fields and the BF field spectrum are described, among others, by generalized cubic and quartic Yang-Mills vertices in some “backgrounds” of scalar fields. Meanwhile, both the gauge transformations corresponding to the coupled model and their algebra are deformed with respect to the initial abelian theory in such a way that the new gauge algebra becomes open. Section 5 comments on two classes of solutions to the equations satisfied by the various functions of the scalar fields (that parametrize the deformed solution to the master equation) and Section 6 closes the paper with the main conclusions.
2 Free model. Antibracket-antifield BRST symmetry

The starting point is represented by the free Lagrangian action

\[ S_0 \left[ A^a_\mu, H^a_\mu, \phi_a, B^a_\mu\nu, V^A_\mu \right] = \int d^2 x \left( H^a_\mu \partial^\mu \phi_a + \frac{1}{2} B^a_\mu\nu F^a_\mu\nu - \frac{1}{4} F^A_\mu\nu F^A_\mu\nu \right), \]

where we used the notations

\[ F^a_\mu\nu = \partial_\mu V^A_\nu, \quad F^a_\mu\nu = \partial_\mu A^a_\nu, \]

and the symbol \([\mu\nu \cdots]\) denotes full antisymmetrization with respect to the indices between brackets, but without normalization factors. Capital Latin indices \(A, B, \) etc., are raised with a constant, symmetric and field-independent, non-degenerate matrix \(k^{AB}\). We observe that (1) is written like a sum between the action of a two-dimensional abelian BF theory (involving two sets of one-forms \(\{A^a_\mu, H^a_\mu\}\), a collection of scalar fields \(\{\phi_a\}\) and a sequence of two forms \(\{B^a_\mu\nu\}\)) and the action corresponding to a set of abelian vector fields \(\{V^A_\mu\}\). The collection indices \(a, b, \) etc., and respectively \(A, B, \) etc., are assumed to run independently ones to the others. A generating set of gauge transformations for the action (1) can be taken under the form

\[ \delta_\epsilon^a A^a_\mu = \partial_\mu \epsilon^a, \quad \delta_\epsilon^a H^a_\mu = 0 = \delta_\epsilon^a B^a_\mu\nu, \quad \delta_\epsilon^a V^A_\mu = \partial_\mu \epsilon^A, \]

where all the gauge parameters are bosonic, with \(\epsilon^a_\mu\) antisymmetric in their Lorentz indices. In \(D = 2\) spacetime dimensions, where this model evolves, the abelian gauge transformations (3–4) are irreducible. In conclusion, (1) and (3–4) describe a linear (the field equations are linear in the fields) gauge theory of Cauchy order equal to two.

In order to construct the antifield-BRST symmetry for the free gauge theory under study, we need to identify the algebra on which the BRST differential acts. The generators of the BRST algebra are, besides the original bosonic fields

\[ \Phi^{a_0} = \left( A^a_\mu, H^a_\mu, \phi_a, B^a_\mu\nu, V^A_\mu \right), \]

the fermionic ghosts

\[ \eta^{a_1} = \left( \eta^a, \eta^a_\mu\nu, C^A \right), \]
respectively associated with the gauge parameters from (3–4), as well as their antifields

\[ \Phi^*_{\alpha_0} = (A^*_\mu^a, H^*_{a\mu}, \varphi^*_{\alpha^a}, V^*_{A\mu}, B^*_\mu^a), \quad \eta^*_{\alpha_1} = (\eta^*_{a}, \eta^*_{a\mu^a}, C^*_A). \] (7)

The Grassmann parity of an antifield is opposite to that of the corresponding field/ghost. It is understood that \( \eta^a_{\mu^a} \) and \( \eta^{a\mu^a} \) are antisymmetric, just like the gauge parameters \( \epsilon^a_{\mu^a} \). Since the gauge generators of this model are field-independent, it follows that the BRST differential \( s \) simply reduces to

\[ s = \delta + \gamma, \] (8)

where \( \delta \) represents the Koszul-Tate differential, graded by the antighost number \( \text{agh} (\delta) = -1 \), and \( \gamma \) stands for the exterior derivative along the gauge orbits, whose degree is named pure ghost number \( \text{pgh} (\gamma) = 1 \). Naturally, these two degrees do not interfere \( \text{agh} (\gamma) = 0, \text{pgh} (\delta) = 0 \). The overall degree that grades the BRST complex, known as the ghost number \( \text{gh} \), is defined like the difference between the pure ghost number and the antighost number, such that \( \text{gh} (s) = \text{gh} (\delta) = \text{gh} (\gamma) = 1 \). According to the standard rules of the BRST method, the corresponding degrees of the generators from the BRST complex are valued like

\[
\begin{align*}
\text{pgh} (\Phi^*_{\alpha_0}) &= \text{pgh} (\Phi^*_{\alpha_0}) = 0, \quad \text{pgh} (\eta^*_{\alpha_1}) = 1, \quad \text{pgh} (\eta^*_{\alpha_1}) = 0, \quad (9) \\
\text{agh} (\Phi^*_{\alpha_0}) &= 0, \quad \text{agh} (\Phi^*_{\alpha_0}) = 1, \quad \text{agh} (\eta^*_{\alpha_1}) = 0, \quad \text{agh} (\eta^*_{\alpha_1}) = 2. \quad (10)
\end{align*}
\]

Actually, (8) is a decomposition of the BRST differential according to the antighost number and it shows that \( s \) contains only components of antighost number equal to minus one and zero. Consequently, the equation expressing the nilpotency of \( s \) projected on the distinct values of the antighost number is equivalent with the nilpotency and anticommutation of its components

\[ s^2 = 0 \Leftrightarrow (\delta^2 = 0, \quad \delta \gamma + \gamma \delta = 0, \quad \gamma^2 = 0). \] (11)

The Koszul-Tate differential is imposed to realize an homological resolution of the algebra of smooth functions defined on the stationary surface of the field equations for the action (1), while the exterior longitudinal derivative is related to the gauge symmetries (3–4) through its cohomology at pure ghost number zero computed in the cohomology of \( \delta \), which is required to be the algebra of physical observables for the free model under consideration. The
actions of $\delta$ and $\gamma$ on the generators from the BRST complex, which enforce all the above mentioned properties, are given by

$$\delta \Phi_{\alpha} = 0 = \delta \eta_{\alpha 1}, \quad (12)$$

$$\delta A^a_\mu = \partial_\alpha B^a_\mu, \quad \delta H_a^\mu = -\partial^\mu \varphi_a, \quad \delta \varphi^a = \partial^\mu H^a_\mu, \quad \delta B^{*a}_{\mu \nu} = -\frac{1}{2} F^a_{\mu \nu}, \quad (13)$$

$$\delta V^*_A^\mu = -\partial_\alpha F^a_\alpha, \quad \delta \eta^*_a = -\partial_\mu A^*_a \mu, \quad \delta \eta^{*a}_{\mu \nu} = \frac{1}{2} \partial^\mu H^a_{\mu \nu}, \quad \delta C^a_\mu = -\partial_\mu V^*_A, \quad (14)$$

$$\gamma \Phi_{\alpha 0}^* = 0 = \gamma \eta_{\alpha 1}^*, \quad \gamma V^*_A = \partial_\mu C^A, \quad (15)$$

$$\gamma A^a_\mu = \partial_\mu \eta^a, \quad \gamma H_a^\mu = \partial^\nu \eta^{a}_{\mu \nu}, \quad \gamma \varphi_a = \gamma B^{*a}_{\mu \nu} = \gamma \eta^{\alpha 1} = 0, \quad (16)$$

The Lagrangian BRST differential admits a canonical action in a structure named antibracket and defined by decreeing the fields/ghosts conjugated with the corresponding antifields, $s \cdot (\cdot, S)$, where $(\cdot, \cdot)$ signifies the antibracket and $S$ denotes the canonical generator of the BRST symmetry. It is a bosonic functional of ghost number zero, involving both field/ghost and antifield spectra, that obeys the master equation

$$(S, S) = 0. \quad (17)$$

The master equation is equivalent with the second-order nilpotency of $s$, where its solution $S$ encodes the entire gauge structure of the associated theory. Taking into account the formulas (12–16), as well as the standard actions of $\delta$ and $\gamma$ in canonical form, we find that the complete solution to the master equation for the model under study reads

$$S = S_0 \left[A^a_\mu, H^a_\mu, \varphi_a, B^{*a}_{\mu \nu}, V^*_A\right] + \int d^2 x \left(A^a_\mu \partial_\mu \eta^a + H^a_\mu \partial^\nu \eta^{a}_{\mu \nu} + V^*_A \partial_\mu C^A\right), \quad (18)$$

such that it contains pieces of antighost number zero and one. The absence of components with antighost numbers higher than one is due to the abelianity and irreducibility of the chosen generating set of gauge transformations. If the gauge algebra were non-abelian, then the solution to the master equation would also include terms of antighost number two that are quadratic in the ghosts (6); ones linear in the antifields $\eta^*_{\alpha 1}$ and proportional with the structure functions appearing at the commutators between the gauge generators, and others quadratic in the antifields $\Phi^*_{\alpha 0}$. The latter kind of elements is
present only if these commutators strictly close on-shell, \textit{i.e.}, on the stationary surface of field equations, or, which is the same, only if the gauge algebra is open. In the case where the gauge algebra is open, the solution to the master equation may in principle continue to be non-vanishing at antighost numbers higher than two, the corresponding elements being related to the higher-order structure functions and to the identities satisfied by them.

The main ingredients of the antifield-BRST symmetry derived in this section will be useful in the sequel at the analysis of consistent interactions that can be added to the action (11) without changing its number of independent gauge symmetries.

3 Deformation of the solution to the master equation

3.1 General idea

A consistent deformation of the free action (1) and of its gauge invariances (3–4) defines a deformation of the corresponding solution to the master equation that preserves both the master equation and the field/antifield spectra. Let us denote by $g$ the coupling constant and assume that the local functional $ar{S}_0 \left[ A_\mu^a, H_\mu^a, \varphi_a, B_\alpha^{\mu\nu}, V^A_\mu \right] = S_0 + g \int d^2 x a_0 + g^2 \int d^2 x b_0 + O \left( g^3 \right)$ represents a consistent deformation of (11), subject to the deformed gauge transformations

$\tilde{\delta}_\epsilon A^\alpha_\mu = \partial_\mu e^\alpha + g A^\alpha_\mu + g^2 \chi^\alpha_\mu + O \left( g^3 \right)$,  
$\tilde{\delta}_\epsilon H_\mu^a = \partial^\nu \epsilon^a_\mu + g \rho^a_\mu + g^2 \rho^a_\mu + O \left( g^3 \right)$,  
$\tilde{\delta}_\epsilon \varphi_a = g \sigma_a + g^2 \sigma_a + O \left( g^3 \right)$,  
$\tilde{\delta}_\epsilon B_\alpha^{\mu\nu} = g \sigma^{\mu\nu}_a + g^2 \sigma^{\mu\nu}_a + O \left( g^3 \right)$,  
$\tilde{\delta}_\epsilon V^A_\mu = \partial_\mu \epsilon^A + g v^A_\mu + g^2 v^A_\mu + O \left( g^3 \right)$ (by consistent we mean that $\bar{S}_0$ is invariant under the modified gauge transformations $\tilde{\delta}_\epsilon \Phi^{\alpha_0}$ at all orders in the coupling constant). Accordingly, we find that

$$\bar{S} = S + g \int d^2 x a + g^2 \int d^2 x b + O \left( g^3 \right),$$

is a consistent deformed solution to the master equation for the interacting theory, \textit{i.e.}, it satisfies the equation

$$(\bar{S}, \bar{S}) = 0,$$

at all orders in the coupling constant (with $S$ given by (18)). Moreover, the objects $a$ and $b$ start like

$$a = a_0 + A^\alpha_\mu \tilde{\epsilon}_\mu^a + H^a_\mu \tilde{\rho}^a_\mu + V^A_\mu \tilde{\sigma}^A_\mu + \varphi^a \tilde{\sigma}_a + B^{\mu\nu}_a \tilde{\sigma}^{\mu\nu}_a + \text{“more”},$$

(21)
\[ b = b_0 + A^a_\mu \lambda^a_\mu + H^a_\mu \rho^a_\mu + V^a_A \delta^A + \varphi^{aa} \tilde{\sigma}_a + B^a_\mu \delta^a_\mu + \text{“more”}, \quad (22) \]

where the “bar” quantities are obtained by replacing the gauge parameters \( \epsilon^a, \epsilon^a_\mu \) and \( \epsilon^A \) respectively with the fermionic ghosts \( \eta^a, \eta^a_\mu \) and \( C^A \) in the functions \( \lambda^a_\mu, \lambda^a_\mu, \rho^a_\mu, \rho^a_\mu, \sigma^a, \sigma^a, \sigma^a_\mu \), \( v^A \) and \( v^A \) contained in the deformed gauge transformations.

### 3.2 First-order deformation

#### 3.2.1 Basic cohomological results

Using the development (19) and the Eq. (17) satisfied by \( S \), we obtain that the master equation (20) of the deformed theory holds to order \( g \) if and only if
\[ sa = \partial_\mu j^\mu, \quad (23) \]

for some local \( j^\mu \). This means that the non-integrated density of the first-order deformation of the solution to the master equation, \( a \), belongs to the local cohomology of the BRST differential, \( H^0 (s|d) \), where \( d \) is the exterior spacetime derivative. In the case where \( a \) is a \( s \) coboundary modulo \( d \) \( (a = sc + \partial_\mu k^\mu) \), then the deformation is trivial (it can be eliminated by a redefinition of the fields). As a consequence, \( a \) is unique only up to replacing it with an element from the same cohomological class, \( a \rightarrow a + sc + \partial_\mu k^\mu \), and, on the other hand, if \( a \) is purely trivial, \( a = sc + \partial_\mu k^\mu \), then it can be removed from \( S \) by setting \( a = 0 \). For obvious reasons, we are interested only in smooth, local, Lorentz-covariant and Poincaré-invariant deformations. In order to investigate the solution to (23), we develop \( a \) according to the antighost number
\[ a = a_0 + a_1 + \cdots + a_J, \quad \text{agh} (a_k) = k = \text{pgh} (a_k), \quad \epsilon (a_k) = 0, \quad (24) \]

and assume that the expansion (24) stops at a finite, but otherwise arbitrary, value of the antighost number, \( J \). (The notation \( \epsilon (F) \) signifies the Grassmann parity of \( F \).) This result can be argued like in [36] (Section 3), under the sole assumption that the interacting Lagrangian at the first order in the coupling constant, \( a_0 \), is local, so it contains a finite, but otherwise arbitrary, number of derivatives. Replacing (24) into the Eq. (23) and taking into account the decomposition (8), we obtain that the Eq. (24) is equivalent to a tower of local equations, corresponding to the different decreasing values
of the antighost number

$$\gamma a_J = \partial_\mu j_\mu^J, \quad (25)$$

$$\delta a_J + \gamma a_{J-1} = \partial_\mu j_{\mu-1}^J, \quad (26)$$

$$\delta a_k + \gamma a_{k-1} = \partial_\mu j_{\mu-1}^k, \quad J - 1 \geq k \geq 1, \quad (27)$$

where \((j_\mu^k)_{k=\overline{0,J}}\) are some local currents with \(\text{agh} (j_\mu^k) = k\). As \(\text{pgh} (a_J) = J\), the Eq. (25) shows that \(a_J\) belongs to the local cohomology of the exterior derivative along the gauge orbits at pure ghost number \(J\), \(H^J (\gamma | d)\). Following a reasoning similar to that from [36]–[38], it can be shown that one can replace the Eq. (25) at strictly positive antighost numbers with

$$\gamma a_J = 0, \quad J > 0. \quad (28)$$

In other words, for \(J > 0\) the last representative from (24) can always be considered to pertain to the cohomological group of the exterior derivative along the gauge orbits at pure ghost number \(J\), \(H^J (\gamma)\). As a consequence, it is unique up to \(\gamma\)-exact contributions, \(a_J \rightarrow a_J + \gamma c_J\), while the purely \(\gamma\)-trivial solutions \(a_J = \gamma c_J\) can be safely removed by taking \(a_J = 0\). In conclusion, the Eq. (23) associated with the local form of the first-order deformation is completely equivalent to the tower of Eqs. (28) and (26–27).

Thus, we need to know the cohomology of \(\gamma, H (\gamma)\), in order to determine the terms of highest antighost number in \(a\). From the definitions (15–16) it is simple to see that this cohomology is generated by \(F_A^{\mu \nu}, F_a^{\mu \nu}, \varphi_a, B_a^{\mu \nu}, \partial^\mu H^a_\mu\), by the antifields \(\{ \Phi_{\alpha a}^* \}\), by all their derivatives, as well as by the undifferentiated ghosts \(\eta^{a_1}\). (The derivatives of the ghosts are \(\gamma\)-exact, as can be observed from the last relation in (15) and the first two formulas in (16), so we can discard them as being trivial in \(H (\gamma)\).) If we denote by \(e^M (\eta^{a_1})\) the elements of pure ghost number equal to \(M\) of a basis in the space of polynomials in \(\eta^{a_1}\), which is finite-dimensional due to the anticommutation of the ghosts, it follows that the general local solution to the Eq. (28) takes the form (up to irrelevant, \(\gamma\)-exact contributions)

$$a_J = \alpha_J \left( \left[ F_A^{\mu \nu} \right], \left[ F_a^{\mu \nu} \right], \left[ \varphi_a \right], \left[ B_a^{\mu \nu} \right], \left[ \partial^\mu H^a_\mu \right], \left[ \Phi_{\alpha a}^* \right], \left[ \eta^{a_1} \right] \right) e^J (\eta^{a_1}), \quad J > 0, \quad (29)$$

where \(\text{agh} (\alpha_J) = J\) for \(a_J\) to have the ghost number equal to zero, and \(\alpha_J\) must display the same Grassmann parity like \(e^J\) in order to ensure that \(a_J\) is bosonic. Here and in the sequel the notation \(f ([q])\) signifies that \(f\) depends
on \( q \) and its spacetime derivatives up to a finite order. The index-notation \( J \) is generic, in the sense that it may include unspecified Lorentz and/or collection indices. As they have both finite antighost number and derivative order, the elements \( \alpha_J \), which are non-trivial in \( H^0(\gamma) \), are polynomials in the antifields, their derivatives, and in the allowed derivatives of the fields, but may contain an indefinite number of undifferentiated fields \( \varphi_\alpha \) and \( B^{\mu\nu}_a \). They will be called “invariant polynomials”. At zero antighost number, the invariant polynomials are nothing but the local, gauge-invariant quantities of the free model under study. The fact that we can replace the Eq. (25) for \( J > 0 \) with (28) is a consequence of the triviality of the cohomology of the exterior spacetime differential in the space of invariant polynomials at strictly positive antighost numbers. So, if \( \alpha_J \) is an invariant polynomial with \( \text{agh} (\alpha_J) = J > 0 \) that is \( d \)-closed, \( d\alpha_J = 0 \), then \( \alpha_J = d\beta_J \), with \( \beta_J \) also an invariant polynomial.

Inserting the expression (29) into the Eq. (26) and recalling the definitions (15–16), we find that a necessary condition for the existence of (non-trivial) solutions \( a_J \)−1 is that the invariant polynomials \( \alpha_J \) are (non-trivial) elements from the local cohomology group of the Koszul-Tate differential at pure ghost number zero and at strictly positive antighost number \( J \), \( H_J (\delta|d) \)

\[
\delta \alpha_J = \partial_\mu k^\mu_{J-1}, \quad \text{agh} (k^\mu_{J-1}) = J - 1 \geq 0.
\]  

(30)

By “trivial elements of \( H_J (\delta|d) \)” we understand \( \delta \)-exact modulo \( d \) objects, hence of the form \( \delta d_{J+1} + \partial_\mu m^\mu_J \). As a consequence of (30), we need to investigate some of the main properties of the cohomology \( H (\delta|d) \) at strictly positive antighost numbers in order to fully determine the component \( a_J \) of highest antighost number from the first-order deformation. As we have discussed in Section 2, the free model under study is a linear gauge theory of Cauchy order equal to two. In agreement with the general results from [39] (also see [36]–[38]), one can state that \( H (\delta|d) \) (at pure ghost number zero) is trivial at antighost numbers strictly greater than its Cauchy order. The same result holds for the local cohomology of the Koszul-Tate differential in the space of invariant polynomials, \( H_{\text{inv}} (\delta|d) \), so we actually have that

\[
H_J (\delta|d) = 0, \quad H^\text{inv}_J (\delta|d) = 0, \quad J > 2.
\]  

(31)

\(^1\)We note that the local cohomology group of the Koszul-Tate differential at strictly positive pure ghost and antighost numbers is trivial, so the notations \( H_J (\delta|d) \) and \( H (\delta|d) \) automatically take into consideration only objects of pure ghost number zero (see, for instance, [39] and [40]).
An element of $H_t^\text{inv} (\delta | d)$ is defined via an equation similar to \(^{(30)}\), but with the corresponding current $k^\mu_{j-1}$ an invariant polynomial. Moreover, it can be shown \(^{(36, 38)}\) that if the invariant polynomial $\alpha_J$ with $\text{agh} (\alpha_J) = J \geq 2$ is trivial in $H_J (\delta | d)$, then it can be taken to be trivial also in $H_t^\text{inv} (\delta | d)$, i.e.,

$$ (\alpha_J = \delta d_{j+1} + \partial_\mu m^\mu_j, \text{agh} (\alpha_J) = I \geq 2) \Rightarrow \alpha_J = \delta \beta_{j+1} + \partial_\mu \gamma^\mu_j, \quad (32) $$

with both $\beta_{j+1}$ and $\gamma^\mu_j$ invariant polynomials. With the help of the definitions \(^{(12, 14)}\), we find that the most general non-trivial representative from $H_2 (\delta | d)$, which, essentially, has the same status in $H_t^\text{inv} (\delta | d)$, is

$$ \alpha_2^0 = K^\Delta \alpha^\Delta + K^{\Delta'\mu\nu} \alpha^\Delta'_{\mu\nu} + K^{\Delta''} \alpha^\Delta'', \quad (33) $$

where $M_\Delta$, $N^\Delta_{\mu\nu}$, and $P^\Delta''$ are respectively given by

$$ M_\Delta = \frac{\partial W_\Delta}{\partial \varphi_m} \eta^*_m - \frac{\partial^2 W_\Delta}{\partial \varphi_m \partial \varphi_n} \left( B^\mu_m \eta^*_m + H^*_{m\mu} A^\mu_n \right) - \frac{1}{2} \frac{\partial^3 W_\Delta}{\partial \varphi_m \partial \varphi_n \partial \varphi_p} H^*_{m\mu} H^*_{n\nu} B^\mu_p, \quad (34) $$

$$ N^\Delta_{\mu\nu} = \frac{\partial U^\Delta_{\mu\nu}}{\partial \varphi_m} \eta^*_m + \frac{1}{2} \frac{\partial^2 U^\Delta_{\mu\nu}}{\partial \varphi_m \partial \varphi_n} H^*_{m\mu} H^*_{n\nu}, \quad (35) $$

$$ P^\Delta'' = f^A_{\Delta''} C^A + \frac{\partial f^A_{\Delta''}}{\partial \varphi_m} \left( V^{*\mu}_A H^*_m + \eta^*_m F^\mu_A \right) + \frac{1}{2} \frac{\partial^2 f^A_{\Delta''}}{\partial \varphi_m \partial \varphi_n} H^*_m H^*_n F^\mu_A, \quad (36) $$

and $K^\Delta$, $K^{\Delta'\mu\nu}$ and $K^{\Delta''}$ denote some constant coefficients, with $K^{\Delta'\mu\nu}$ antisymmetric in their Lorentz indices. The generic indices $\Delta$, $\Delta'$ and $\Delta''$ are exclusively composed of collection indices (of the type $a, b$, etc., and/or $A$, $B$, etc.). All the functions $W_\Delta$, $U^\Delta_{\mu\nu}$ and $f^A_{\Delta''}$ involved in \(^{(31, 36)}\) depend in an arbitrary manner on the undifferentiated scalar fields $\varphi_a$. Moreover, the objects $M_\Delta$, $N^\Delta_{\mu\nu}$ and $P^\Delta''$ separately satisfy the equations

$$ \delta M_\Delta = \partial_\mu k^\mu_\Delta, \quad \delta N^\Delta_{\mu\nu} = \frac{1}{2} \partial_\mu k^\mu_{\Delta'\nu}, \quad \delta P^\Delta'' = \partial_\mu k^\mu_\Delta'', \quad (37) $$

where the corresponding currents are also invariant polynomials

$$ k^\mu_\Delta = - \left( \frac{\partial W_\Delta}{\partial \varphi_m} A^\mu_m + \frac{\partial^2 W_\Delta}{\partial \varphi_m \partial \varphi_n} B^\mu_m H^*_n \right), \quad (38) $$

$$ k^\mu_{\Delta'} = \frac{\partial U^\Delta'_{\mu\nu}}{\partial \varphi_m} H^*_m, \quad (39) $$

$$ k^\mu_\Delta'' = - V^{*\mu}_A f^A_{\Delta''} + \frac{\partial f^A_{\Delta''}}{\partial \varphi_m} F^\mu_A H^*_m, \quad (40) $$

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and hence we have that
\[ \delta \alpha_2^0 = \partial^\mu k_1^\mu, \]
where \( k_1^\mu \) is the invariant polynomial
\[ k_1^\mu = K^\Delta k_\Delta^\mu + K^{\Delta^{\prime}} k_{\Delta^\prime}^{\mu} + K^{\Delta^{\prime\prime}} k_{\Delta^\prime\prime}^{\mu}. \]

The previous results on \( H(\delta|d) \) and \( H^{\text{inv}}(\delta|d) \) at strictly positive antighost numbers are important because they control the obstructions to removing the antifields from the first-order deformation. Indeed, due to (31–32) and to the triviality of the cohomology of the exterior spacetime differential in the space of invariant polynomials at strictly positive antighost numbers, it follows that we can successively remove all the pieces with \( J > 2 \) from the non-integrated density of the first-order deformation by adding to it only trivial terms. In conclusion we can take, without loss of non-trivial objects, the maximum value \( J = 2 \) of the antighost number in the decomposition (24).

### 3.2.2 Determination of the first-order deformation

For \( J = 2 \), the first-order deformation (24) reduces to
\[ a = a_0 + a_1 + a_2, \]
where its last representative \( (\gamma a_2 = 0) \) is of the form (29). The elements of pure ghost number equal to two of a basis in the space of polynomials in \( \eta^{a_1} \) are spanned by
\[ e^2 : \left( \eta^a \eta^b, \eta^{a}_\mu \eta^{b}_\rho, C^A C^B, \eta^a \eta^{b}_\mu \nu, \eta^a C^A, \eta^{a}_\mu C^A \right), \]
and therefore we can write (up to \( \gamma \)-exact contributions) that
\[ a_2 = \frac{1}{2} \left( \alpha_{ab} \eta^a \eta^b + \alpha_{ab}^{\mu \nu \rho \lambda} \eta^a_\mu \eta^b_\rho \nu \lambda + \alpha_{AB} C^A C^B \right) \\
+ \alpha_{ab} \eta^a \eta^{b}_\mu \nu + \alpha_{Aa} \eta^a C^A + \alpha_{aA} \eta^{a}_\mu C^A. \]

According to the previous discussion, the objects \( \alpha_{ab}, \alpha_{ab}^{\mu \nu \rho \lambda}, \alpha_{AB}, \alpha_{ab}^{\mu \nu}, \alpha_{aA} \) and \( \alpha_{aA}^{\mu \nu} \) necessarily belong to \( H^{\text{inv}}_2(\delta|d) \), so
\[ \delta \alpha_{ab} = \partial_\mu k_\mu^{ab}, \delta \alpha_{ab}^{\mu \nu \rho \lambda} = \partial_\beta k_\beta^{\mu \nu \rho \lambda}, \delta \alpha_{AB} = \partial_\mu k_\mu^{AB}, \]
\[ \delta \alpha_{ab}^{\mu \nu} = \partial_\beta k_\beta^{\mu \nu}, \delta \alpha_{aA} = \partial_\mu k_\mu^{aA}, \delta \alpha_{aA}^{\mu \nu} = \partial_\beta k_\beta^{aA}. \]
for some currents that are invariant polynomials of antighost number one. In addition, they are subject to the “symmetry” conditions (due to the anticommutation of the ghosts)

\[ \alpha_{ab} = -\alpha_{ba}, \quad \alpha_{ab}^{\mu\rho\lambda} = -\alpha_{ba}^{\rho\mu\lambda}, \quad \alpha_{AB} = -\alpha_{BA}, \]  

\[ \alpha_{\mu\nu} = -\alpha_{\nu\mu}, \quad \alpha_{aA} = -\alpha_{Aa}, \quad \alpha_{ab}^{\mu\nu\rho\lambda} = -\alpha_{ba}^{\mu\nu\rho\lambda} = -\alpha_{ab}^{\mu\nu\lambda\rho}. \]  

(48)

If we insert the expression (45) into the Eq. (26) for \( J = 2 \)

\[ \delta a_2 + \gamma a_1 = \partial_{\mu} j_{1}^\mu, \]  

(50)

use the formulas (46–47) and recall the definitions (15–16), we obtain that the existence of \( a_1 \) demands further restrictions on the currents, namely,

\[ k_{\beta\mu\nu\rho\lambda}^{ab} \partial_{\beta} \left( \eta_{a\mu
u}^b \eta_{\rho\lambda}^b \right) = \sigma_{ab}^{\mu\rho\lambda} \left( \partial_{\mu} \eta_{a\rho}^b \right) \eta_{\nu\lambda}^b + \sigma_{ab}^{\mu\nu\lambda} \eta_{a\mu\rho}^b \left( \partial_{\nu} \eta_{\rho\lambda}^b \right), \] 

(51)

\[ k_{\beta\mu\nu}^{ab} \partial_{\beta} \eta_{a\mu\nu}^b = \mu_{ab}^{\mu\nu} \partial_{\mu} \eta_{a\nu}^b, \quad k_{\beta\mu\nu}^{Aa} \partial_{\beta} \eta_{a\mu\nu}^a = \mu_{aA}^{\mu\nu} \partial_{\mu} \eta_{a\nu}^a, \]  

(52)

for some \( \sigma \) and \( \mu \). On the other hand, in agreement with the result (37), every function from \( H_2^{inv} (\delta|d) \) entering the solution (45) can only be constructed out of the three different kinds of invariant polynomials (34–36).

Expressing now each function of the type \( \alpha \) from \( a_2 \) in terms of the allowed elements (34–36) and imposing the supplementary Eqs. (51–51) at the level of the accompanying currents, after some computation we infer the solutions

\[ \alpha_{ab} = M_{ab} + P_{ab} = \frac{\partial W_{ab}}{\partial \varphi_m} \eta_{m}^* - \frac{\partial^2 W_{ab}}{\partial \varphi_m \partial \varphi_n} \left( B_{m\mu\nu} \eta_{n}^* + H_{m}^{* \mu} A_{n\mu}^* \right) \] 

\[ - \frac{1}{2} \frac{\partial^3 W_{ab}}{\partial \varphi_m \partial \varphi_n \partial \varphi_p} H_{m}^{* \mu} H_{n}^{* \nu} B_{p\mu\nu} + f_{ab} C_{A}^* \] 

\[ + \frac{\partial f_{ab}}{\partial \varphi_m} \left( V_{A}^{* \mu} H_{m\mu}^* + \eta_{m}^{* \mu} F_{A\mu} \right) + \frac{1}{2} \frac{\partial^2 f_{ab}}{\partial \varphi_m \partial \varphi_n} H_{m}^{* \mu} H_{n}^{* \nu} F_{A\mu\nu}, \]  

(53)

\[ \alpha_{ab}^{\mu\nu\rho\lambda} = 0 = \alpha_{ab}^{\mu\nu}, \]  

(54)

\[ \alpha_{AB} = P_{AB} = f_{AB} C_{C}^* + \frac{\partial f_{AB}}{\partial \varphi_m} \left( V_{C}^{* \mu} H_{m\mu}^* + \eta_{m}^{* \mu} F_{C\mu} \right) \] 

\[ + \frac{1}{2} \frac{\partial^2 f_{AB}}{\partial \varphi_m \partial \varphi_n} H_{m}^{* \mu} H_{n}^{* \nu} F_{C\mu\nu}, \]  

(55)
\[ \alpha_{ab} = N_{ab}^{\mu \nu} = \frac{\partial U_{ab}}{\partial \phi_m} \eta_{m}^{\mu \nu} + \frac{1}{2} \frac{\partial^2 U_{ab}}{\partial \phi_m \partial \phi_n} H_{m}^{* \nu} H_{n}^{* \mu}, \]  

\[ \alpha_{aA} = P_{aA} = g_{aA}^{C} C_{C}^{*} + \frac{\partial g_{aA}^{C}}{\partial \phi_m} \left( V_{C}^{* \mu} H_{m}^{* \nu} + \eta_{m}^{* \mu \nu} F_{C \mu \nu} \right) \]
\[ + \frac{1}{2} \frac{\partial^2 g_{aA}^{C}}{\partial \phi_m \partial \phi_n} H_{m}^{* \mu} H_{n}^{* \nu} F_{C \mu \nu}, \]  

where \( W_{ab}, f_{ab}^{A}, f_{AB}^{C}, U_{ab} \) and \( g_{aA}^{C} \) depend only on the scalar fields \( \phi_a \), with \( W_{ab}, f_{ab}^{A} \) and \( f_{AB}^{C} \) antisymmetric in their lower indices in order to enforce the “symmetry” properties (48–49).

(58)

In conclusion, the full expression of the last component from the first-order deformation (43), such that it leads to a consistent solution \( a_1 \) to the Eq. (50), has the form (45), with the invariant polynomials \( \alpha_{ab}, \alpha_{ab}^{\mu \nu}, \alpha_{aA}, \alpha_{ab}^{\mu \nu}, \alpha_{aA} \) from \( H_{2}^{\mu \nu} (\delta | d) \) precisely given by (53–57).

With \( a_2 \) at hand, direct calculations provide the piece of antighost number one from (44) like

(59)

The last step in completing the first-order deformation is the resolution of the equation (27) for \( k = 1 \)

\[ \delta a_1 + \gamma a_0 = \partial_{\mu} j_{0}^{\mu}. \]  

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Evaluating $\delta a_1$, we find that the Eq. (60) possesses solutions with respect to $a_0$ if the functions $f_{AB}^C$ and $g_{aA}^C$ of the undifferentiated scalar fields obey the conditions
\[ f_{ABC} = -f_{BAC} = -f_{ACB}, \quad g_{aAB} = -g_{aBA}, \]
where $f_{ABC}$ and $g_{aAB}$ are defined by
\[ f_{ABC} = k^{AE} f_{EBC}^A, \quad g_{aAB} = k^{AE} g_{EaB}^A, \]
and $k_{AE}$ denote the elements of the matrix inverse to $k^{AE}$ (used to raise the collection indices of the vector fields). Then, we get the interacting Lagrangian at the first order in the coupling constant like
\[ a_0 = \frac{1}{2} \left( \frac{\partial W_{ab}}{\partial \varphi_c} B_{c}^{\mu \nu} - f_{ab}^A F_{A}^{\mu \nu} \right) A_{\mu}^{a} A_{\nu}^{b} - \frac{1}{2} f_{BC}^A F_{A}^{\mu \nu} V_{\mu}^{B} V_{\nu}^{C} \]
\[ - g_{aB}^A f_{A}^{\mu \nu} A_{\mu}^{a} V_{\nu}^{B} - U_{ab} A^{a \mu} H_{b}^{\mu}. \]

So far we have completely determined the first-order deformation of the solution to the master equation for the model under study
\[ S_1 = \int d^2x \left( a_0 + a_1 + a_2 \right), \]
where its components are expressed by (65) (with the corresponding invariant polynomials of the form (53–57), (59) and (63)). Moreover, the various functions of the undifferentiated scalar fields are taken to obey the properties (58) and (61).

### 3.3 Higher-order deformations

#### 3.3.1 Second-order deformation

Using the notations from (19), the master equation (20) holds to order $g^2$ if and only if
\[ \Delta = -2sb + \partial_\mu u^\mu, \]
where $(S_1, S_1) = \int d^2x \Delta$. In other words, the consistency of the deformed solution to the master equation at the second order in the coupling constant requires that the integrand of $(S_1, S_1)$ should (locally) be written like a $s$-co-boundary modulo $d$. Relying on the expression of $S_1$ deduced in the above,
we can emphasize a $s$-exact part in $\Delta$ if and only if the functions $U_{ab}$ and $W_{ab}$ coincide

$$U_{ab} = W_{ab},$$

in which case we find that

$$\Delta = -2sb + \left(t_{bcd}u_{e[f]}^{bcd} + \frac{\partial t_{bcd}}{\partial \varphi_e} u_e^{bcd} + \frac{\partial^2 t_{bcd}}{\partial \varphi_e \partial \varphi_f} u_e^{bcd} + \frac{\partial^3 t_{bcd}}{\partial \varphi_e \partial \varphi_f \partial \varphi_g} u_e^{bcd}\right)
\begin{align*}
&+ \left(a_{ab}^{A}v_{|A}^{abc} + \frac{\partial \alpha_{ab}^{A}}{\partial \varphi_e} v_{e|A}^{abc} + \frac{\partial^2 \alpha_{ab}^{A}}{\partial \varphi_e \partial \varphi_f} v_{e|A}^{abc}\right) \\
&+ \left(a_{BCD}^{A}w_{|A}^{BCD} + \frac{\partial \alpha_{BCD}^{A}}{\partial \varphi_e} w_{e|A}^{BCD} + \frac{\partial^2 \alpha_{BCD}^{A}}{\partial \varphi_e \partial \varphi_f} w_{e|A}^{BCD}\right) \\
&+ \left(a_{ab}^{A}Z_{|A}^{abB} + \frac{\partial \alpha_{ab}^{A}}{\partial \varphi_e} Z_{e|A}^{abB} + \frac{\partial^2 \alpha_{ab}^{A}}{\partial \varphi_e \partial \varphi_f} Z_{e|A}^{abB}\right) \\
&+ \left(a_{aBC}^{A}q_{|A}^{aBC} + \frac{\partial \alpha_{aBC}^{A}}{\partial \varphi_e} q_{e|A}^{aBC} + \frac{\partial^2 \alpha_{aBC}^{A}}{\partial \varphi_e \partial \varphi_f} q_{e|A}^{aBC}\right). \tag{67}\end{align*}$$

The expression of $b$ in (67) is

$$b = -\frac{1}{4}Q_{\mu}^{A}k_{AB}Q_{B\nu},$$

where we performed the notation

$$Q_{\mu}^{A} = \left(\frac{\partial f_{ab}^{A}}{\partial \varphi_{e}} \eta_{e|\mu}^{*} + \frac{1}{2} \frac{\partial^2 f_{ab}^{A}}{\partial \varphi_{c} \partial \varphi_{d}} H_{c\mu}^{*} H_{d\nu}^{*}\right) \eta^{a} \eta^{b} - \left(f_{ab}^{A}A_{\mu}^{a}A_{\nu}^{b} + f_{BC}^{A}V_{\mu}^{B}V_{\nu}^{C}\right)$$

$$- \left(\frac{\partial f_{ab}^{A}}{\partial \varphi_{c}} \eta_{e|\mu}^{*} + \frac{1}{2} \frac{\partial^2 f_{ab}^{A}}{\partial \varphi_{c} \partial \varphi_{d}} H_{c\mu}^{*} H_{d\nu}^{*}\right) \eta^{a} - \left(\frac{\partial f_{BC}^{A}}{\partial \varphi_{c}} \eta_{e|\mu}^{*} + \frac{1}{2} \frac{\partial^2 f_{BC}^{A}}{\partial \varphi_{c} \partial \varphi_{d}} H_{c\mu}^{*} H_{d\nu}^{*}\right) \eta^{a} + \frac{\partial g_{ab}^{A}}{\partial \varphi_{c}} \eta_{e|\mu}^{*} + \frac{1}{2} \frac{\partial^2 g_{ab}^{A}}{\partial \varphi_{c} \partial \varphi_{d}} H_{c\mu}^{*} H_{d\nu}^{*} \eta^{a} + 2 \frac{\partial g_{ab}^{A}}{\partial \varphi_{c}} \eta_{e|\mu}^{*} + \frac{1}{2} \frac{\partial^2 g_{ab}^{A}}{\partial \varphi_{c} \partial \varphi_{d}} H_{c\mu}^{*} H_{d\nu}^{*} \eta^{a} \right). \tag{69}\end{align*}$$

The coefficients denoted by $t$ and $\alpha$ from (67) involve only the undifferentiated scalar fields and are given by

$$t_{abc} = W_{e[abc]} \frac{\partial W_{bc}}{\partial \varphi_e},$$

$$\tag{70}$$
\[ \alpha^A_{abc} = f^A_{e[a} \frac{\partial W_{bc]}}{\partial \varphi_e} + W_{e[a} \frac{\partial f^A_{bc]}}{\partial \varphi_e} - f^A_{[ab} g^A_{c]}, \]
\[ \alpha^A_{BCD} = f^A_{[B|C} f^E_{D]}, \]
\[ \alpha^A_{ab} = g^A_{eB} \frac{\partial W_{ab}}{\partial \varphi_e} + f^A_{EB} f^E_{ab} + W_{e[a} \frac{\partial g^A_{b} \varphi_e}{\partial} + \left(g^A_{ab} g^A_{bE} - g^A_{ab} g^A_{aE}\right), \]
\[ \alpha^A_{aBC} = g^A_{aE} f^E_{BC} - W_{ea} \frac{\partial f^A_{BC}}{\partial \varphi_e} + \left(f^A_{EB} g^A_{aC} - f^A_{EC} g^A_{aB}\right), \]

while the functions of the type \( u, v, w, z \) and \( q \) contain only undifferentiated ghosts and antifields. Their expressions are listed below

\[ u^\text{bed} = \left(A^\mu A^\nu - B^{*\mu\nu} \eta^c\right) \eta^d_{\mu\nu} - \left(A^\mu H^d_{\mu} + \varphi^{*\mu} \eta^d\right) \eta^c, \]
\[ u^\text{bed}_e = \left(B^e_{\mu\nu} A^d_{\mu} A^e_{\nu} + A^d_{e\mu} A^e_{\mu} \eta^d\right) \eta^c + \left(B^\mu_{e\nu} B^{*\mu\nu} - H^d_{e\mu} H^e_{\mu} - \frac{1}{3} \eta^c_{e\mu} \eta^d\right) \eta^c \eta^d, \]
\[ u^\text{bed}_{ef} = \left(\eta^e_{\mu\nu} B_{f\nu} \eta^b - B_{e\mu\nu} H^a_{f\nu} A^a_{\mu} - \frac{1}{2} H^*_{e\mu} H^*_{f\nu} \eta^b - A^*_{e\mu} H^*_{f\nu} \eta^b\right) \eta^c \eta^d, \]
\[ u^\text{bed}_{efg} = \frac{1}{6} B_{e\mu\nu} H^*_{f\nu} H^*_{g\mu} \eta^b \eta^c \eta^d, \]
\[ v^a_{|A} = - \left(\frac{1}{3} C^*_A A^a_{\mu} + V^*_A A^a_{\mu} + F^*_A B^*_{\mu\nu}\right) \eta^b \eta^c + F^*_{A} \eta^a \eta^b A^*_{\nu}, \]
\[ v^a_{e|A} = - \left(\frac{1}{3} F^*_{e\mu} H^*_{e} \eta^a + \frac{1}{3} F^*_{A \eta_{e\mu}\eta^a} + F^*_A H^*_{e\mu} A^a_{\nu}\right) \eta^b \eta^c, \]
\[ v^a_{ef|A} = - \frac{1}{6} F^*_A H^*_{ef} H^*_{f\nu} \eta^a \eta^b \eta^c, \]
\[ w^A_{BCD} = - \left(\frac{1}{3} C^*_A C^B + V^*_A V^B_{\mu}\right) C^C C^D + F^*_A C^B V^C V^D, \]
\[ w^A_{e|BCD} = - \left(\frac{1}{3} F^*_A H^*_{e\mu} C^B + \frac{1}{3} F^*_A \eta_{e\mu\nu} C^B + F^*_A H^*_{e\mu} V^B_{\nu}\right) C^C C^D, \]
\[ w^A_{ef|BCD} = - \frac{1}{6} F^*_A H^*_{ef} H^*_{f\nu} C^B C^C C^D, \]
\[ z_{\alpha A}^{abB} = \left( F_{A}^{\mu \nu} A_{\mu}^{a} A_{\nu}^{b} + 2 V_{A}^{* \mu} A_{\mu}^{a} \eta^{a} - 2 F_{A}^{\mu \nu} B_{\mu \nu}^{* a} \right) C^{b} \]
\[ - F_{A}^{\mu \nu} V_{B_{\mu \nu}^{* a}} \eta^{a} - (C_{A}^{\mu a} + V_{A}^{* \mu} V_{B_a}^B) \eta^{a} \eta^{b}, \quad (85) \]
\[ z_{\epsilon aA}^{\alpha B} = - \left( V_{A}^{* \mu} H_{e_{\mu}}^{a} C^{B} + F_{A}^{\mu \nu} \eta_{e_{\mu}}^{a} C^{B} + F_{A}^{\mu \nu} H_{V_{\mu}^{c}} V_{B}^{B} \right) \eta^{a} \eta^{b} \]
\[ - F_{A}^{\mu \nu} H_{a \epsilon_{\mu \nu}}^{\alpha B} C^{B} \eta^{a}, \quad (86) \]
\[ z_{\epsilon A}^{abB} = - \frac{1}{2} F_{A}^{\mu \nu} H_{e_{\mu}}^{a} H_{f_{\nu}}^{c} C^{B} \eta^{a} \eta^{b}, \quad (87) \]
\[ q_{i A}^{\alpha BC} = \left( C_{A}^{i} \eta^{a} + V_{A}^{* \mu} A_{\mu}^{a} + F_{A}^{\mu \nu} B_{\mu \nu}^{* a} \right) C^{B} C^{C} \]
\[ - \left( 2 V_{A}^{* \mu} V_{\mu}^{c} C^{B} + F_{A}^{\mu \nu} V_{\mu}^{c} V_{B}^{B} \right) \eta^{a} + F_{A}^{\mu \nu} A_{[\mu V_{\nu}]}^{a} C^{B}, \quad (88) \]
\[ q_{i A}^{\alpha BC} = \left( F_{A}^{\mu \nu} \eta_{\epsilon_{\mu \nu}}^{a} \eta^{a} + F_{A}^{\mu \nu} H_{e_{\mu}}^{a} A_{\nu}^{c} + V_{A}^{* \mu} H_{e_{\mu}}^{c} \eta^{a} \right) C^{B} C^{C} \]
\[ + F_{A}^{\mu \nu} H_{a \epsilon_{\mu \nu}}^{\alpha B} \eta^{a} C^{B}, \quad (89) \]
\[ q_{e A}^{\alpha BC} = \frac{1}{2} F_{A}^{\mu \nu} H_{e_{\mu}}^{a} H_{f_{\nu}}^{c} \eta^{a} C^{B} C^{C}. \quad (90) \]

Since none of the terms from the right-hand side of \((67)\) containing the quantities \(t_{abc}, \alpha_{A}^{A}, \alpha_{A}^{A B}, \alpha_{A}^{A C}, \alpha_{A}^{A B C}, \alpha_{A}^{A B C D}\) and their derivatives with respect to the scalar fields can be written in a \(s\)-exact modulo \(d\) form, it follows that the second-order deformation of the solution to the master equation exists if and only if all these vanish

\[ t_{abc} = 0, \quad \alpha_{A}^{A} = 0, \quad \alpha_{A}^{A B C} = 0, \quad \alpha_{A}^{A B C D} = 0. \quad (91) \]

In conclusion, the deformed solution to the master equation is consistent to order \(g^2\) if and only if the functions \(W_{ab}, f_{BC}^{A}, f_{ab}^{A}\) and \(g_{ab}^{A}\) satisfy the Eqs. \((91)\), in which case the second-order deformation is expressed like in \((98)\). We will comment more on the Eqs. \((91)\) in Section 5, where we explicitly compute some particular solutions, which allow a nice geometric and algebraic interpretation.
3.3.2 Third- and higher-order deformations

If we denote the third-order deformation by $S_3 = \int d^2x c$, the master equation (20) holds to order $g^3$ if and only if

$$\Lambda = -sc + \partial_\mu w^\mu,$$

(92)

where $(S_1, S_2) = \int d^2x \Lambda$. Taking into account the expressions of the first- and second-order deformations obtained in the above, after some computation we infer that

$$\Lambda = \left( \eta^a g^A_{\mu} + \frac{1}{2} C^B f^A_{DE} Q^E_{\mu\nu} k_{ABQ_{\mu\nu}} \right) + \left( \alpha^a A_{\mu} B_{\mu} C_{\mu} D_{\mu} \right) + \left( \alpha^a A_{\mu} B_{\mu} C_{\mu} D_{\mu} \right) + \left( \alpha^a A_{\mu} B_{\mu} C_{\mu} D_{\mu} \right) + \left( \alpha^a A_{\mu} B_{\mu} C_{\mu} D_{\mu} \right),$$

(93)

where we employed the notations

$$\bar{v}_{|A}^{abc} = \frac{1}{2} k_{AM} Q^{M\mu\nu} (B^*_{\mu\nu} \eta^a - \eta^a A_{\mu} A_{\nu}^c),$$

(94)

$$\bar{v}_{|A}^{abc} = \frac{1}{2} k_{AM} Q^{M\mu\nu} \left( \frac{1}{3} \eta_{e\mu\nu} \eta^a + \frac{1}{2} H^*_{e\mu\nu} A_{\nu}^a \right) \eta^b \eta^c,$$

(95)

$$\bar{v}_{|A}^{abc} = \frac{1}{12} k_{AM} Q^{M\mu\nu} C_B V_{\mu} V_{\nu},$$

(96)

$$\bar{w}_{|A}^{BCD} = \frac{1}{2} k_{AM} Q^{M\mu\nu} C_B V_{\mu} V_{\nu},$$

(97)

$$\bar{w}_{|A}^{BCD} = \frac{1}{12} k_{AM} Q^{M\mu\nu} C_B V_{\mu} V_{\nu},$$

(98)

$$\bar{w}_{|A}^{BCD} = \frac{1}{12} k_{AM} Q^{M\mu\nu} C_B V_{\mu} V_{\nu},$$

(99)

$$\bar{z}_{|A}^{ab} = \frac{1}{2} k_{AM} Q^{M\mu\nu} \left( \eta^a A_{\mu} B_{\nu} + A_{\mu} A_{\nu} B + 2 B_{\mu\nu} \eta^a C \right),$$

(100)
\[ z_{e|A} = \frac{1}{2} k_{AM} Q^{M\mu\nu} \left[ \left( \eta_{e\mu} C^B \right. + \frac{1}{2} H^\mu_{e[\mu} V^B_{\nu]} \right) \eta^a \eta^b - \frac{1}{2} H^\mu_{e[\mu} A^a_{|\nu]} \eta^b \eta^a C^B \right], \quad (101) \]

\[ \tilde{z}_{e|A} = \frac{1}{4} k_{AM} Q^{M\mu\nu} k_{AM} Q^{M\mu\nu} H^\mu_{e[\mu} \eta^a \eta^b \eta^a C^B , \quad (102) \]

\[ \tilde{q}^a_{A} = -\frac{1}{2} k_{AM} Q^{M\mu\nu} \left( A^a_{[\mu} V^C_{\nu]} C^B + B^a_{A[a\mu} C^C - \eta^a V^B V^C \right), \quad (103) \]

\[ \tilde{q}_{e|A} = -\frac{1}{2} k_{AM} Q^{M\mu\nu} \left[ \left( \eta_{e\mu} \eta^a + H^a_{e[\mu} A^a_{|\nu]} \right) C^B C^C + H^a_{e[\mu} V^C_{\nu]} \eta^a C^B \right], \quad (104) \]

\[ \tilde{q}_{e|A} = -\frac{1}{2} k_{AM} Q^{M\mu\nu} k_{AM} Q^{M\mu\nu} H^a_{e[\mu} \eta^a C^B C^C . \quad (105) \]

It is now clear that none of the terms in the right-hand side of (93) can be written like in (92). However, if we take into account the Eqs. (91) and the antisymmetry properties (61) of the functions (62), we find that \( \Lambda = 0 \), so we can take \( c = 0 \) in (92), and consequently obtain that

\[ S_3 = 0. \quad (106) \]

The equation that governs the fourth-order deformation \( S_4 \) reads as

\[ s S_4 + (S_3, S_1) + \frac{1}{2} (S_2, S_2) = 0. \quad (107) \]

On the one hand, the result (106) implies that \((S_3, S_1) = 0\) and, on the other hand, if we compute \((S_2, S_2)\), where \( S_2 = \int d^2 x \, b \), with \( b \) given in (68), it follows that \((S_2, S_2) = 0\), so we can set

\[ S_4 = 0. \quad (108) \]

Meanwhile, we remark that the equations responsible for the deformations \((S_k)_{k>4}\) involve only the solutions \((S_j)_{j \geq 3}\), which further allows us to put

\[ S_k = 0, \, k > 4. \quad (109) \]

In conclusion, among the higher-order deformations of the solution to the master equation, only that of second-order is non-vanishing and non-trivial.
4 Identification of the interacting theory

Putting together the results deduced in the previous section, we can write the full deformed solution to the master equation (20), that is consistent to all orders in the coupling constant, under the form

\[
\bar{S} = S + g S_1 + g^2 S_2 = \int d^2 x \left( H^a_{\mu} D^\mu \varphi^a_a + \frac{1}{2} B^a_{\mu \nu} \bar{F}^a_{\mu \nu} + A^a_{\mu} (D^a_{\mu})_b \eta^b \right) \\
-g W_{ab} \varphi^{*a} \eta^b + g B^{*a \mu \nu} \left( W_{ab} \eta^b - \frac{\partial W_{ab}}{\partial \varphi_c} B_{c\mu \nu} \eta^b \right) \\
+ H_{i}^{*a} \left( (D^\nu)_{a}^{*b} \eta^{b} + g \left( \frac{\partial W_{bc}^{\mu}}{\partial \varphi_a} H_{c}^{*a} \eta^{b} - \frac{2}{\partial \varphi_a} B_{d\mu \nu} A^{c \nu} \right) \eta^{b} \right) \\
+ V_{A}^{*a} \left( (D^\mu)^A_B C^B + (D^\mu)^A_a \eta^a \right) \\
- \frac{1}{4} \left( F_{\mu \nu}^{A} - Q_{\mu \nu}^{A} \right) k_{AB} \left( F_{B \mu \nu}^{A} - Q_{B \mu \nu}^{A} \right) \\
+ \frac{g}{2} \left( f_{AB}^{a c} + \frac{\partial W_{ab}^{\mu}}{\partial \varphi_c} \eta^{c} + \frac{\partial f_{ab}^{A}}{\partial \varphi^c} V_{A}^{*a} H_{c}^{*a} \eta^{b} + \frac{\partial W_{ab}^{\mu}}{\partial \varphi_c} A^{a}_{d} H_{c}^{*d} \right) \\
- \frac{1}{2} \frac{\partial W_{ab}}{\partial \varphi_c} B_{c\mu \nu} \eta_{c}^{*a} \eta_{c}^{*b} - \frac{1}{2} \frac{\partial W_{ab}}{\partial \varphi_c} B_{c\mu \nu} \eta_{c}^{*a} \eta_{c}^{*b} \\
+ g \left( \frac{\partial W_{ab}}{\partial \varphi_c} \eta_{c}^{*a} \eta_{c}^{*b} + \frac{\partial W_{ab}}{\partial \varphi_c} \eta_{c}^{*a} \eta_{c}^{*b} \right) \eta_{c}^{*a} \eta_{c}^{*b} \\
+ \frac{g}{2} \left( f_{BC}^{A} C_{A}^{a} \frac{\partial f_{BC}^{A}}{\partial \varphi^c} V_{A}^{*a} H_{c}^{*a} \right) C^B C^C \\
+ g \left( g^{a b} C_{A}^{a} \frac{\partial g^{a b}}{\partial \varphi_b} V_{A}^{*a} H_{c}^{*b} \right) \eta^a \eta^b \right),
\]

(110)

where we performed the notations

\[
D^\mu \varphi_a = \partial^\mu \varphi_a + g W_{ab} A_{b}^{\mu},
\]

\[
F_{\mu \nu}^a = \partial_{\mu} A_{\nu}^{a} + g \frac{\partial W_{bc}^{\mu}}{\partial \varphi_a} A_{b}^{\nu},
\]

\[
(D^\mu)^a_{b} = \delta^a_{b} \partial^\mu - g \frac{\partial W_{bc}^{\mu}}{\partial \varphi_a} A_{b}^{\nu},
\]

\[
(D^\mu)^A_B = \delta^a_{b} \partial^\mu - g \left( f_{BC}^{A} V_{C}^{b} - g_{a b} A_{A}^{a} \right),
\]

\[
(D^\mu)^A_a = -g \left( f_{ab} A_{a}^{\mu} + g_{a b} V_{A}^{b} \right).
\]
We note that the deformed solution (110) contains components of antighost numbers ranging from zero to four, unlike the solution (18) of the master equation for the free model, which stopped at antighost number one. We stress again that the coefficients \( W_{ab}, f_{ab}^A, f_{BC}^A \) and \( g_{aB}^A \) are all functions of the undifferentiated scalar fields, that must obey the antisymmetry properties (58) and (61), as well as the Eqs. (91), where the functions \( t_{abc}, \alpha_{abc}, \alpha_{BCD}, \alpha_{aB}^A \) and \( \alpha_{aBC}^A \) are defined in the formulas (70–74).

At this stage, we have all the information necessary at the identification of the interacting gauge theory behind our deformation procedure. According to the general rules of the antifield-BRST formalism, the Lagrangian action that describes the coupled model is nothing but the antighost number zero piece from (110), so it has the expression

\[
\bar{S}_0 \left[ A^a_\mu, H^a_\mu, \varphi_a, B^{a_\mu}_a, V^A_\mu \right] = \int d^2 x \left( H^a_\mu D^\mu \varphi_a + \frac{1}{2} B^{a_\mu}_a \bar{F}_{\mu\nu}^a - \frac{1}{4} \bar{F}_{\mu\nu}^A \bar{F}^{\mu\nu}_A \right),
\]

(116)

where we used the notation

\[
F_{\mu\nu}^A = \partial_{[\mu} V_{\nu]}^A + g \left( f_{BC}^A V^B_\nu + f_{ab}^A A^b_\mu + g_{aB}^A A_{[\mu}^a V^B_{\nu]} \right).
\]

(117)

The terms of antighost number one from (110) offer us the generating set of deformed gauge transformations corresponding to the Lagrangian action (116), or, in other words, the gauge symmetries of the interacting action, namely,

\[
\bar{\delta}_\epsilon A^a_\mu = (D_\mu)^a_\mu \epsilon^b,
\]

(118)

\[
\bar{\delta}_\epsilon H^a_\mu = (D_\nu)^a_\nu \epsilon^b - g \left( \frac{\partial W_{bc}}{\partial \varphi_a} H^c_\mu - \frac{\partial^2 W_{bc}}{\partial \varphi_a \partial \varphi_d} B^{d_\mu}_a A^{\nu}_a \right) \epsilon^b + \bar{F}_{A\mu\nu} \left( \frac{\partial (D_\nu)^A_\mu}{\partial \varphi_a} \epsilon^b + \frac{\partial (D_\nu)^A_\nu}{\partial \varphi_a} \epsilon^B \right),
\]

(119)

\[
\bar{\delta}_\epsilon \varphi_a = -g W_{ab} \epsilon^b, \quad \bar{\delta}_\epsilon V^A_\mu = (D_\mu)^A_\mu \epsilon^B + (D_\mu)^A_\nu \epsilon^a,
\]

(120)

\[
\bar{\delta}_\epsilon B^{a_\mu}_a = g \left( W_{ab} \epsilon^{b_\mu} - \frac{\partial W_{ab}}{\partial \varphi_c} B^{c_\mu}_a \epsilon^b \right) + g \left( g_{aB}^A \epsilon^B + f_{ab}^A \epsilon^b \right) \bar{F}_{A\mu\nu}.
\]

(121)

There also appear two types of antighost number two elements in (110). As we have stated in the end of Section 2, their presence indicates that the gauge algebra associated with the deformed gauge transformations is open, so
under the form are completely determined from the antighost number two objects in (110) deformed gauge transformations (118–121) associated with the se parameters gauge parameters. Then, the expressions of the commutators between the Indeed, let \( e^{a_1} = (e^a, e^a_{\mu}, e^A) \) and \( \xi^{a_1} = (\xi^a, e^a_{\mu}, \xi^A) \) be two different sets of gauge parameters. Then, the expressions of the commutators between the deformed gauge transformations (118–121) associated with these parameters are completely determined from the antighost number two objects in (110) under the form

\[
\left[ \tilde{\delta}_{\epsilon}, \tilde{\delta}_{\xi} \right] \varphi_a = \tilde{\delta}_A \varphi_a, \tag{122}
\]

\[
\left[ \tilde{\delta}_{\epsilon}, \tilde{\delta}_{\xi} \right] A^a_\mu = \tilde{\delta}_A A^a_\mu + g \frac{\delta S_0}{\delta H^{4a}} \frac{\partial^2 W_{bc}}{\partial \varphi_a \partial \varphi_d} e^b \xi^c, \tag{123}
\]

\[
[\tilde{\delta}_{\epsilon}, \tilde{\delta}_{\xi}] B^a_{\mu\nu} = \tilde{\delta}_A B^a_{\mu\nu} + 2 g^2 \frac{\delta S_0}{\delta B^a_{\mu\nu}} - \left( (g_{aC} g_{cD} - g_{aD} g_{cC}) \right) e^c \xi^D
\]

\[
+ f^A_{a[b} f^B_{c]} e^b \xi^c - \left( (g_{aC} f^B_{D} + f^A_{ab} g^A_{cD}) \right) \left( e^b \xi^C - \xi^b e^C \right)
\]

\[
+ g^2 \frac{\delta S_0}{\delta H^B} f^B_{a[b} \frac{\partial f^B_{D]} e^b}{\partial \varphi_d} \left( A^{aC} \frac{\partial f^B_{e]} e^b}{\partial \varphi_d} + V^{Ea} f^A_{a[b} \frac{\partial g^B_{C]} e^b}{\partial \varphi_d} \right) e^c \xi^D
\]

\[
+ \left( A^{aC} \frac{\partial f^B_{e]} e^b}{\partial \varphi_d} + V^{Ea} f^A_{a[b} \frac{\partial g^B_{C]} e^b}{\partial \varphi_d} \right) \left( e^b \xi^C - \xi^b e^C \right), \tag{124}
\]

\[
[\tilde{\delta}_{\epsilon}, \tilde{\delta}_{\xi}] H^\mu_\mu = \tilde{\delta}_A H^\mu_\mu - g \frac{\delta S_0}{\delta A^a_{\mu\nu}} \frac{\partial^2 W_{bc}}{\partial \varphi_a \partial \varphi_d} e^b \xi^c
\]

\[
+ g \frac{\delta S_0}{\delta V^{A\mu}} \left( \frac{\partial f^A_{B]} e^b}{\partial \varphi_d} \xi^C + \frac{\partial f^A_{e]} e^b}{\partial \varphi_d} \xi^c + \frac{\partial g^A_{B]} e^b}{\partial \varphi_d} \xi^b \xi^C \right)
\]

\[
+ g^2 \frac{\delta S_0}{\delta B^a_{\mu\nu}} - \left( (g_{aC} f^B_{D]} e^b + f^A_{ab} g^A_{cD]} e^b) \right) \left( e^b \xi^C - \xi^b e^C \right)
\]

\[
- V^{Ea} g^a_{aC} \frac{\partial f^B_{e]} e^b}{\partial \varphi_d} \left( A^{aC} \frac{\partial g^B_{C]} e^b}{\partial \varphi_d} + V^{Ea} f^A_{a[b} \frac{\partial g^B_{C]} e^b}{\partial \varphi_d} \right) e^c \xi^D
\]

\[
+ \left( A^{aC} \frac{\partial f^B_{e]} e^b}{\partial \varphi_d} + V^{Ea} f^A_{a[b} \frac{\partial g^B_{C]} e^b}{\partial \varphi_d} \right) \left( e^b \xi^C - \xi^b e^C \right),\]
\begin{align*}
&\times (e^b \xi^c - \xi^b e^c) + g \frac{\delta S_0}{\delta H_\mu} \left( - \frac{\partial^2 W_{bc}}{\partial \varphi_a \partial \varphi_e} (e^b \xi^c_\mu - \xi^b e^c_\mu) \\
&+ \left( \frac{\partial^3 W_{bc}}{\partial \varphi_a \partial \varphi_d \partial \varphi_e} B_{d\mu} - \frac{\partial^2 f^A_{bc}}{\partial \varphi_a \partial \varphi_e} F_{A\mu} \right) e^b \xi^c \\
&- F_{A\mu} \left( \frac{\partial^2 g^A_{bB}}{\partial \varphi_a \partial \varphi_e} (e^b \xi^B - \xi^b e^B) + \frac{\partial^2 f^A_{BC}}{\partial \varphi_a \partial \varphi_e} \epsilon^B \xi^C \right) \right) \\
&+ g^2 \delta^a_{\mu} \delta^b_{\nu} M^{ab}_{\alpha} \frac{\delta S_0}{\delta H_{b\beta}}, \tag{125}
\end{align*}

\begin{align*}
&\left[ \delta_a, \delta_b \right] V^\mu_a = \partial_a \Lambda^\mu_a - g \frac{\delta S_0}{\delta H^\mu} \left( \frac{\partial f^A_{BC}}{\partial \varphi_a} \epsilon^B \xi^C + \frac{\partial f^A_{BC}}{\partial \varphi_a} e^b \xi^c \\
&+ \frac{\partial g^A_{bB}}{\partial \varphi_a} (e^b \xi^B - \xi^b e^B) \right), \tag{126}
\end{align*}

where

\begin{align*}
&\Lambda^{a1} = \left( \Lambda^a, \Lambda^a_{\mu\nu}, \Lambda^A \right), \tag{127}
\end{align*}

with

\begin{align*}
&\Lambda^a \equiv g \frac{\partial W_{bc}}{\partial \varphi_a} e^b \xi^c, \tag{128}
\end{align*}

\begin{align*}
&\Lambda^a_{\mu\nu} \equiv g \left( \frac{\partial f^A_{bc}}{\partial \varphi_a} F_{A\mu} - \frac{\partial^2 W_{bc}}{\partial \varphi_a \partial \varphi_d} B_{d\mu} \right) e^b \xi^c + \frac{\partial f^A_{BC}}{\partial \varphi_a} F_{A\mu} \epsilon^B \xi^C \\
&+ g \frac{\partial g^A_{bB}}{\partial \varphi_a} F_{A\mu} \left( e^b \xi^B - \xi^b e^B \right) - g \frac{\partial W_{bc}}{\partial \varphi_a} \left( e^b \xi^c_\mu - \xi^b e^c_\mu \right), \tag{129}
\end{align*}

and respectively

\begin{align*}
&\Lambda^A \equiv g \left( f^A_{BC} \epsilon^B \xi^C + f^A_{ab} e^a \xi^b + g^A_{bB} \left( e^b \xi^B - \xi^b e^B \right) \right). \tag{130}
\end{align*}

At the same time, the function from (125) denoted by $M^{ab}_{\alpha}$ reads

\begin{align*}
&M^{ab}_{\alpha} = -k_{AB} \left( \left( \frac{\partial g^A_{bc}}{\partial \varphi_a} V^C_a + \frac{\partial f^A_{bc}}{\partial \varphi_a} A^e_a \right) \left( \frac{\partial g^B_{cd}}{\partial \varphi_a} V^C_a + \frac{\partial f^B_{cd}}{\partial \varphi_a} A^d_a \right) \right) \\
&- \left( \frac{\partial g^A_{bc}}{\partial \varphi_a} V^C_a + \frac{\partial f^A_{bc}}{\partial \varphi_a} A^e_a \right) \left( \frac{\partial g^B_{cd}}{\partial \varphi_a} V^C_a + \frac{\partial f^B_{cd}}{\partial \varphi_a} A^d_a \right) \epsilon^c \xi^d. \tag{23}
\end{align*}
\[\begin{align*}
&+ \left( \frac{\partial f^A}{\partial \varphi_b} V^E_{\alpha} - \frac{\partial g^{A}_{eC}}{\partial \varphi_b} A^e_{\alpha} \right) \left( \frac{\partial f^B_{DF}}{\partial \varphi_a} V^{F\nu} - \frac{\partial g^{B}_D}{\partial \varphi_a} A^{\nu}_{D} \right) \\
&- \left( \frac{\partial f^A_{DE}}{\partial \varphi_b} V^E_{\alpha} - \frac{\partial g^{A}_{eD}}{\partial \varphi_b} A^e_{\alpha} \right) \left( \frac{\partial f^B_{CF}}{\partial \varphi_a} V^{F\nu} - \frac{\partial g^{B}_D}{\partial \varphi_a} A^{\nu}_{D} \right) \right) e^C \xi^D \\
&+ \left( \frac{\partial g^{A}_{cD}}{\partial \varphi_b} V^D_{\alpha} + \frac{\partial f^A_{cE}}{\partial \varphi_b} A^e_{\alpha} \right) \left( \frac{\partial g^{B}_{dC}}{\partial \varphi_a} V^{D\nu} + \frac{\partial f^B_{dE}}{\partial \varphi_a} A^{\nu}_{D} \right) \times \left( e^C \xi^D - \xi^e C \right) \right) .
\end{align*}\] (131)

From the terms of antighost numbers three and four present in (1.10) we can recover the higher-order structure functions due to the open character of the deformed gauge algebra, as well as the accompanying identities. They have an intricate (but not illuminating) form and consequently we will omit writing their concrete expressions.

At this point, we have all the information on the gauge structure of the deformed model, whose free limit is given by the Lagrangian action (1), together with the abelian and irreducible gauge symmetries (4). We observe that there are two main types of vertices in the deformed action (1.16). The first kind
\[g \left( H^a_{\mu} W^{ab}_{\nu} A^{\nu}_{b} + \frac{1}{2} B^a_{\mu \nu} \frac{\partial W^{be}_{c}}{\partial \varphi_a} A^e_{\mu} \right) ,\] (132)
corresponds to the self-interactions among the purely BF fields in the absence of the vector fields \( \{ V^A_{\mu} \} \), being given only by terms of order one in the coupling constant. Such terms have been previously obtained in the literature and we will not insist on their structure (for a detailed analysis, see for instance [18]). The second kind of vertices can be written in the form
\[-g k_{AB} \left( \partial^{\mu} V^{\nu A} \right) \left( f^{B}_{CD} (\varphi) V^{C}_{\mu} V^{D}_{\nu} + f^{B}_{ab} (\varphi) A^{a}_{\mu} A^{b}_{\nu} + g^{B}_{AC} (\varphi) A^{a}_{[\mu} V^{C}_{\nu]} \right) \]
\[-g^2 4 k_{AD} \left( f^{A}_{BC} (\varphi) V^{B}_{\mu} V^{C}_{\nu} + f^{A}_{ab} (\varphi) A^{a}_{\mu} A^{b}_{\nu} + g^{A}_{ab} (\varphi) A^{a}_{[\mu} V^{B}_{\nu]} \right) \times \left( f^{D}_{EF} (\varphi) V^{E\mu} V^{F\nu} + f^{D}_{cd} (\varphi) A^{c\mu} A^{d\nu} + g^{D}_{cE} (\varphi) A^{c[\mu} V^{d\nu]} \right) .\] (133)

We note that (133) contains a vertex involving only the BF fields, namely
\[-\frac{1}{4} k_{AD} f^{A}_{ab} (\varphi) f^{D}_{cd} (\varphi) A^{a}_{\mu} A^{b}_{\nu} A^{c\mu} A^{d\nu} \] whose existence is induced by the presence of the vector fields \( \{ V^A_{\mu} \} \). Indeed, if the vector fields \( \{ V^A_{\mu} \} \) were
absent \((k_{AB} = 0)\), then this term would vanish. The remaining terms reveal the cross-couplings between the BF fields and the vector fields \(\{V^A_\mu\}\). Among the cross-coupling pieces in (133), we find generalized cubic and quartic Yang-Mills-like vertices in ‘backgrounds’ of the scalar fields. We remark that neither the one-forms \(\{H^a_\mu\}\) nor the two-forms \(\{B^{\mu\nu}_a\}\) can be coupled in a consistent, non-trivial manner to the vector fields. Related to the deformed gauge transformations (118–121), there appears a complementary situation, in the sense that among the BF fields, only the one-forms \(\{H^a_\mu\}\) and the two-forms \(\{B^{\mu\nu}_a\}\) gain gauge symmetries involving the parameters \(\epsilon^B\). The deformed gauge transformations of the vector fields \(\{V^A_\mu\}\) have a rich structure, including, besides other terms, the generalized covariant derivative \(\left(\delta^A_B \partial_\mu - g f^{A}_{BC} (\varphi) V^C_\mu \right)\) with respect to the parameters \(\epsilon^B\).

5 Some solutions to the Eqs. (91)

We have seen that the deformation procedure developed so far essentially relies on the existence of four types of functions depending on the undifferentiated scalar fields, namely, \(W_{ab}, f^A_{BC}, f^A_{ab}\) and \(g^A_{ab}\), which are subject on the one hand to the conditions (58) plus (61) and on the other hand to the Eqs. (91). In the sequel we analyze two classes of solutions to the above conditions and equations and emphasize that they admit an interesting geometric and algebraic interpretation. The first class of solutions corresponds to a non-vanishing \(W_{ab}(\varphi)\), while the second kind is associated with \(W_{ab} = 0\).

Related to the former type of solutions, it is clear that the first equation from (91) together with the first antisymmetry property in (58)

\[ W_{e[a} \frac{\partial W_{bc]}}{\partial \varphi_e} = 0, \quad W_{ab} = -W_{ba}, \tag{134} \]

shows that the antisymmetric functions \(W_{ab}\) of the undifferentiated scalar fields satisfy the Jacobi’s identity for a nonlinear algebra. Let us see the geometric meaning of \(W_{ab}\). To this end, we briefly review the basic notions on Poisson manifolds. If \(N\) denotes an arbitrary Poisson manifold, then this is equipped with a Poisson bracket \(\{,\}\) that is bilinear, antisymmetric, subject to a Leibnitz-like rule and satisfies a Jacobi-type identity. If \(\{X^i\}\) are some local coordinates on \(N\), then there exists a two-tensor \(\mathcal{P}^{ij} \equiv \{X^i, X^j\}\) (the Poisson tensor) that uniquely determines the Poisson
structure together with the Leibnitz rule. This two-tensor is antisymmetric and transforms covariantly under coordinate transformations. Jacobi’s identity for the Poisson bracket \( \{ \cdot, \cdot \} \) expressed in terms of the Poisson tensor reads as \( \mathcal{P}^{ij} \mathcal{P}^{kl} + \text{cyclic} (i,j,l) = 0 \), where \( \mathcal{P}^{ij} \equiv \partial \mathcal{P}^{ij} / \partial X^k \). Now, the geometric origin of \( W_{ab} \) is obvious. If, for instance, we choose a concrete form for the antisymmetric functions \( W_{ab}(\varphi) \) that satisfy (134), then we can interpret the dynamical scalar fields \( \{ \varphi_a \} \) precisely like some local coordinates on a target manifold endowed with a prescribed Poisson structure (up to the plain convention that the lower index \( a \) is a ‘covariant’ index of the type \( i \)). Conversely, any given Poisson manifold parametrized in terms of some local coordinates \( \{ \varphi_a \} \) (within the same index convention) prescribes a Poisson tensor \( W_{ab}(\varphi) \) which is antisymmetric and satisfies (134). Once we have fixed the functions \( W_{ab} \), it is easy to see that a solution for the remaining coefficients \( f^A_{BC}, f^A_{ab} \) and \( g^A_{ab} \) is represented by

\[
\begin{align*}
  f^A_{BC} &= \tilde{f}^A_{BC}, \quad g^A_{ab} = \tilde{f}^A_{BE} M^E X_a(\varphi_c); \\
  f^A_{ab} &= M^A \left( X_c \frac{\partial W_{ab}}{\partial \varphi_c} + W_{ca} \frac{\partial X_b}{\partial \varphi_c} + W_{bc} \frac{\partial X_a}{\partial \varphi_c} \right),
\end{align*}
\]

(135)

(136)

where \( M^E \) are some real constants, \( \{ X_a \} \) stands for a set of arbitrary functions depending only on the undifferentiated scalar fields \( \varphi_a \), and \( \tilde{f}^A_{BC} \) are some real, antisymmetric constants, that obey the identity

\[
\tilde{f}^A_{E[B} \tilde{f}^E_{CD]} = 0.
\]

(137)

Accordingly, \( \tilde{f}^A_{BC} \) can be viewed like the structure constants of a semi-simple Lie algebra, endowed with the Killing-Cartan metric \( k_{AB} \), while \( M^A \) can be seen like the components of an arbitrary element from this Lie algebra. In this situation, the deformed Lagrangian action (116) also includes self-interactions among the vector fields \( \{ V^A_{\mu} \} \) precisely described by cubic and quartic Yang-Mills vertices. Accordingly, the gauge transformations of \( V^A_{\mu} \) contain the well-known covariant derivative of the gauge parameters \( \epsilon^A \)

\[
\bar{\delta}_{\epsilon} V^A_{\mu} = \left( \delta^A_B \partial_{\mu} - g\tilde{f}^A_{BC} V^C_{\mu} \right) \epsilon^B + \text{“more”).
\]

(138)

Next, we examine the latter kind of solutions (corresponding to \( W_{ab} = 0 \)), in which case the Eqs. (131) become

\[
\begin{align*}
  f^A_{E[B} \tilde{f}^E_{CD]} &= 0,
\end{align*}
\]

(139)
\[ f_{[ab]}g_{c]}^E = 0, \quad f_{AB}^E f_{ab}^E + g_{AB}g_{bc}^E - g_{aB}g_{bA}^E = 0, \quad (140) \]
\[ g_{AE}^B f_{BC}^E + f_{EB}^A g_{aC}^E - f_{EC}^A g_{ab}^E = 0. \quad (141) \]

The solution to (140–141) takes the form
\[ f_{ab}^A = f_{BC}^A M_b^B M_a^C, \quad g_{AB}^A = f_{CB}^A M_a^C, \quad (142) \]
where \( M_b^B \) are some arbitrary functions of the undifferentiated scalar fields and \( f_{BC}^A \) verify the Eq. (139). In order to solve the remaining equation, namely, (139), let \( M_a^a \) be some functions of the scalar fields such that \( M_a^a M_b^B = \delta_b^a \). Then, the solution of (139) reads as
\[ f_{BC}^A = f_{bc}^a M_a^A M_b^B M_c^C, \quad (143) \]
where \( f_{bc}^a \) are the structure constants of a semi-simple Lie algebra with the Killing-Cartan metric \( k_{ab} \). It is easy to see that the Jacobi identity (139) is a consequence of the Jacobi identity for the structure constants \( f_{bc}^a \). For the functions \( g_{AB}^A \) and \( f_{BC}^A \) given in (142–143) to satisfy the antisymmetry properties (58) and (61), it is necessary that \( k_{AB} \) and \( k_{ab} \) are correlated through some relations of the type
\[ k_{AB} M_a^A M_b^B = k_{ab} \Phi (\varphi), \quad (144) \]
with \( \Phi (\varphi) \) a non-vanishing, but otherwise arbitrary function of the scalar fields. We remark that, although the functions \( f_{BC}^A \) from (143) depend in general on the scalar fields, they however verify the Jacobi identity (139). Accordingly, these functions can be regarded like some ‘structure constants’ of a Lie algebra whose generators depend on the scalar fields (generalized Lie algebra). It is interesting to note that the gauge algebra is open also for the latter kind of solutions. In both cases, the entire gauge structure of the interacting model can be obtained by substituting the solutions (135–136) and respectively (142-143) in the formulas (116) and (118–126).

6 Conclusion

To conclude with, in this paper we have investigated the consistent two-dimensional interactions that can be introduced among a set of scalar fields, two types of one-forms, a system of two-forms and a collection of vector fields, described in the free limit by an abelian BF theory and a sum of
Maxwell actions. Starting with the BRST differential for the free theory, \( s = \delta + \gamma \), we compute the consistent first-order deformation of the solution to the master equation with the help of some cohomological techniques, and obtain that it is parametrized by five kinds of functions depending on the undifferentiated scalar fields. Next, we investigate the second-order deformation, whose existence reduces the number of independent types of functions on the scalar fields to four and, meanwhile, requires that these are subject to certain equations. Based on these restrictions, we determine the expression of the second-order deformation and, moreover, show that we can take all the remaining higher-order deformations to vanish. As a consequence of our procedure, we are led to an interacting gauge theory with deformed gauge transformations and a non-abelian gauge algebra that only closes on-shell. The presence of the collection of vector fields brings in a rich structure of non-trivial terms if compared with the self-interactions that can be added to a two-dimensional abelian BF theory. Finally, we give two classes of solutions to the equations satisfied by the various functions of the scalar fields that parametrize the deformed solution to the master equation, which can be interpreted in terms of Poisson manifolds and respectively of generalized Lie algebras.

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