Rate-Distortion Theory for Mixed States
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Abstract
This paper is concerned with quantum data compression of asymptotically many independent and identically distributed copies of ensembles of mixed quantum states. The encoder has access to a side information system. The figure of merit is per-copy or local error criterion. Rate-distortion theory studies the trade-off between the compression rate and the per-copy error. The optimal trade-off can be characterized by the rate-distortion function, which is the best rate given a certain distortion. In this paper, we derive the rate-distortion function of mixed-state compression. The rate-distortion functions in the entanglement-assisted and unassisted scenarios are in terms of a single-letter mutual information quantity and the regularized entanglement of purification, respectively. For the general setting where the consumption of both communication and entanglement are considered, we present the full qubit-entanglement rate region. Our compression scheme covers both blind and visible compression models (and other models in between) depending on the structure of the side information system.

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I. INTRODUCTION

Data compression is concerned with the task of transmitting data with high fidelity while minimizing the communication cost. The cost reduction is made possible by the structure and redundancy in the data, which is often modeled as multiple samples of a (data) source. The compression rate is the communication cost per sample. Quantum data compression was pioneered by Schumacher in [37]. He defined two notions of a quantum source. The first notion of a quantum source is a quantum system together with correlations with a purifying reference system. The second notion of a quantum source is called an ensemble of pure states where a pure quantum state is drawn according to some known probability distribution. Both notions are “pure state sources.” Subsequently, the notion of ensemble sources was generalized to ensembles of mixed states in [22, 23, 32, 33]. An ensemble source is equivalently defined as a classical-quantum state where the classical system plays the role of an inaccessible reference system. All these seemingly distinct definitions of a quantum source were unified and generalized in [5,
where a general quantum source is defined as a quantum system together with correlations with a most general reference system (which is not necessarily a purifying system nor a classical system). We note that the reference system plays a crucial role in defining the correctness or fidelity of the transmission; it has to preserve the correlations between the transmitted data and the reference. Without considering such correlations, any known classical or quantum system can be produced locally by the receiver without any input from the sender. The source specifies what correlations must be preserved in the compression task, as well as what data is to be transmitted.

Data compression, which achieves high fidelity transmission, can be extended to a rate-distortion model which allows for a potential decrease of compression rate at the expense of having a larger error [15, 38]. In the quantum setting, Schumacher’s quantum data compression model was generalized by Barnum [8] to a rate-distortion model for pure state sources. Even though both Schumacher and Barnum considered asymptotically many i.i.d. (independent identically distributed) copies of a quantum source, their set-ups involve different error definitions. More specifically, Schumacher considered the error for the whole block of data (called block or global error) whereas Barnum considered the error for each copy of the source (called per-copy or local error). Under the global error definition in Schumacher’s quantum data compression model for pure state sources, the strong converse theorems state that any reduction of rate below the optimal value causes the fidelity to vanish for asymptotically large blocks of data [25, 42]. Recently, strong converse for mixed state sources was also studied [26]. In contrast, under the local error definition in Barnum’s quantum rate-distortion model, compression rate trades more smoothly with error. Considering both global and local error criteria, quantum data compression and rate-distortion have been extensively investigated [1, 2, 5–14, 16, 17, 19, 21–32, 34–37, 41, 43].

The ultimate goal of quantum rate distortion theory is to elucidate the optimal trade-off between the compression rate and the allowed local error. This can be characterized by the rate-distortion function, which is the lowest rate achieved by a protocol satisfying an allowed distortion of the data. For pure-state sources, Barnum derived a lower bound on the rate-distortion function [8]. Subsequent work in references [16, 41] characterized the rate-distortion function of pure-state sources in various scenarios with and without entanglement assistance and side information. They found that the rate-distortion functions of entanglement assisted compression and unassisted compression are characterized by the quantum mutual information and the entanglement of purification between the decoder’s systems and the reference systems, respectively. (See Section II for the definitions of these correlation measures.) Moreover, for pure-state sources where both the encoder and the decoder have access to a side information system, the rate-distortion function is characterized in [27, 31]. However, quantum rate-distortion theory in the case of mixed-state sources remains largely unexplored. In the special case for the blind compression of ensembles of mixed states, [32] showed that the optimal rate is the same for vanishing local and global error, but the rate can be sensitive to a small allowed finite local error with potentially large rate reduction [2, 35].

In this paper, we investigate rate distortion coding for mixed-state sources, considering both scenarios with and without assistance of entanglement. We assume a mixed-state quantum ensemble source with a side information system available to the encoder, which covers both of blind compression and visible compression as special cases. First, we consider the case in which we have free entanglement. We derive the rate-distortion function for this case, which is an optimized expression of quantum mutual information between the decoder’s system and a composite system consisting of the classical reference system and other systems that purify the source (Theorem 16). Surprisingly, the rate-distortion function can be given in a single-letter form using quantum mutual information. Then, we explore the case without any entanglement assistance. In this case, we prove that the rate-distortion function is given by the entanglement of purification between the decoder’s systems and the same composite system as in the entanglement-assisted case (Theorem 17). We note that this expression for the rate-distortion function is a multi-letter formula. So far, our results are derived for the most general side information. To simplify the expression for the unassisted case, we apply it to the two special scenarios of visible and blind compressions. For visible compression, the expression simplifies with fewer systems involved (Corollary 20). For blind compression with vanishing local error, we show that the rate-distortion function becomes a single-letter formula involving the entanglement of purification (Theorem 21). As a corollary, we obtain an information theoretic identity by equating this expression to the optimal blind compression rate obtained in [32]. As a final result on deriving rate-distortion functions for the most general side information, and generalizing both entanglement-assisted and unassisted cases, we consider general qubit-entanglement rate region. We derive the necessary and sufficient condition for the qubit and entanglement rates with which rate-distortion compression is achievable (Theorem 22).

Our work contributes to a rich body of quantum rate-distortion theory by showing the optimal trade-offs of mixed-state rate-distortion compression with and without entanglement, which have not been explored before. Our achievability protocols share similarities with those in [27] but the source, the rate expressions, and the converse proofs differ significantly. Our results also have interesting implications for visible and blind compression by considering the previously known optimal compression rates for these cases. The rest of this paper is organized as follows. We review basic concepts and notations used in this paper in Sec. II. Then, we introduce our setup of rate distortion coding for mixed state ensemble sources in Sec. III. Finally, we show our main results in Sec. IV. We first show the optimal rate of entangled assisted rate distortion coding in Sec. IV-A, and then we analyze the optimal rate of unassisted case in Sec. IV-B. Moreover, we derive the full rate region of rate distortion coding for ensemble sources in Sec. IV-C. We conclude the paper with a discussion in Sec. V, further comparing our work with previous results and considering additional applications.
II. Background and Notations

In this section, we review background materials and introduce notations for the paper. Throughout, we use capital letters $A, B, \ldots$ to represent quantum systems. For a quantum system $A$, we let $\mathcal{H}_A$ denote the corresponding complex Hilbert space. We only consider finite-dimensional quantum systems in this paper. Given a quantum system $A$, we use $\dim(A)$ to denote the dimension of the corresponding Hilbert space.

Given two quantum systems $A$ and $B$ with $\dim(A) \leq \dim(B)$, we write an isometry from $A$ to $B$ as $U_{A \rightarrow B}^A$. Whenever $A = B$ we abbreviate $A \rightarrow B$ by $A$ alone, for example, $1^A$ denotes the identity operator on $A$. For a linear operator $Y$ on system $A$, let $s_1(Y), \cdots, s_{\dim(A)}(Y)$ denote the singular values of $Y$. Then, the trace norm of $Y$ is defined as $\|Y\|_1 = \sum_{i=1}^{\dim(A)} s_i(Y)$.

For systems $A$ and $B$, we write a quantum channel from $A$ to $B$, that is, a completely positive and trace-preserving linear map from square matrices on $\mathcal{H}_A$ to those on $\mathcal{H}_B$ as $N_{A \rightarrow B}$. (A quantum channel is also called a CPTP map.) We also write the Stinespring dilation [39] of the channel $N_{A \rightarrow B}$ as $U^{A \rightarrow B}_{X_N}$ with $E$ being the corresponding environment system.

Let $\Sigma$ be an alphabet. A quantum ensemble is defined as follows.

**Definition 1.** Let $A$ be a quantum system, and let $\Sigma$ be an alphabet. A quantum ensemble is a set of pairs of a positive real number and a quantum state

$$\{p_x, \rho_x^A\}_{x \in \Sigma},$$

where $\{p_x : x \in \Sigma\}$ forms a probability distribution; that is, $0 \leq p_x \leq 1$ for all $x \in \Sigma$ and $\sum_{x \in \Sigma} p_x = 1$. In addition, given a quantum ensemble $\{p_x, \rho_x^A\}_{x \in \Sigma}$, we use a classical-quantum state

$$\rho^{AX} := \sum_{x \in \Sigma} p_x \rho_x^A \otimes |x\rangle\langle x|^X$$

to represent this ensemble, where $X$ is a classical reference system with orthonormal basis $\{|x\rangle\}_{x \in \Sigma}$.

Given a quantum ensemble, we can consider a decomposition of each state into a classical part, a quantum part, and a redundant part, which is known as Koashi-Imoto (KI) decomposition [33] named after the authors.

**Theorem 2 ([33]).** Let $\rho^{AX}$ be a classical-quantum state associated with a quantum ensemble. Then, there exist a joint quantum system $CNQ$ and a corresponding isometry $U_{K} : A \rightarrow CNQ$ satisfying the following conditions.

(i) The state $\omega^{CNQX} = (U_{K} \otimes 1^X)\rho^{AX} (U_{K}^\dagger \otimes 1^X)$ can be expressed as

$$\omega^{CNQX} = \sum_x p_x c |c\rangle \langle c|^C \otimes \omega_c^N \otimes \rho_{c,x}^Q \otimes |x\rangle\langle x|^X$$

where the set of vectors $\{|c\rangle\}_{c \in \Xi}$ form an orthonormal set for system $C$, and for each $x$, $p_{c|x}$ is a distribution over $c$ conditioned on $x$. The states $\omega_c^N$ and $\rho_{c,x}^Q$ live in systems $N$ and $Q$, respectively.

(ii) Consider an arbitrary $c \in \Xi$. Let $Q_c$ be the smallest subspace of $Q$ that contains the support of $\rho_{c,x}^Q$ for each $x \in \Sigma$. If a projector $P^{Q_c}$ on $Q_c$ satisfies

$$P^{Q_c} (p_{c|x} \rho_{c,x}^Q) = (p_{c|x} \rho_{c,x}^Q) P^{Q_c},$$

for all $x$, then, $P^{Q_c} = 1^{Q_c}$ or $P^{Q_c} = 0$.

(iii) Consider an arbitrary pair of distinct $c, c' \in \Xi$ and any collection of positive numbers $\{\alpha_x^{(c,c')}\}_{x \in \Sigma}$. There is no unitary operator $V^Q$ satisfying

$$V^Q (p_{c|x} \rho_{c,x}^Q) = \alpha_x^{(c,c')} (p_{c'|x} \rho_{c',x}^Q) V^Q$$

simultaneously for all $x \in \Sigma$.

Henceforth, we call the isometry $U_{K}$ and the state $\omega^{CNQX}$ in Theorem 2 the Koashi-Imoto (KI) isometry and KI-decomposition of the state $\rho^{AX}$, respectively. In the KI decomposition, all states in a given ensemble are written in a block-diagonal form with the same block structure. Register $C$ contains the label for the block; thus $C$ is called the classical part of the ensemble. The state in register $N$ depends only on $c$, and it can be recovered using $c$ without the state label $x$; in this sense, $N$ is called the redundant part of this ensemble. On the other hand, the state in register $Q$ depends on both $x$ and $c$, and $Q$ is called the (non-redundant) quantum part of the ensemble. Items (ii) and (iii) in Theorem 2 state that the decomposition is maximal (see [33] for definition). Intuitively, item (ii) prevents further decomposition of each block, while item (iii) prohibits the non-redundant parts of one block to be related to that of another, which may generate larger redundant parts. Together, this maximality prevents more redundant parts to be created from the non-redundant parts, a condition crucial for evaluating the optimal compression rate.

Since $U_{K}$ can be applied by the encoder and be reversed by the decoder without affecting the compression task, so, without loss of generality, we assume $\rho^{AX} = \omega^{CNQX}$ for the rest of the paper.
We can also determine the form of quantum channels that preserve a given quantum ensemble using its KI decomposition.

**Lemma 3 ([33]).** Let $\rho^{AX}$ be a classical-quantum state with KI decomposition

$$\omega^{CNQX} = (U_{KI} \otimes 1_X) \rho^{AX} (U_{KI}^\dagger \otimes 1_X) = \sum_x p_x \sum_c p_c |c_x\rangle \langle c_x | \otimes \omega^c_N \otimes \rho_{c0}^Q \otimes |x \rangle \langle x |^X.$$

Let $\Lambda : A \rightarrow A$ be a quantum channel that preserves the state $\rho^{AX}$; that is, $(\Lambda^A \otimes \mathcal{I}^X) (\rho^{AX}) = \rho^{AX}$. Let $U^{A \rightarrow AE}_\Lambda$ be any Stinespring dilation of $\Lambda^A$ with environment system $E$. Then, $U^{A \rightarrow AE}_\Lambda$ has the form

$$(U_{KI} \otimes 1^E) U^{A \rightarrow AE}_\Lambda (U_{KI}^\dagger) = \sum_c |c\rangle \langle c | \otimes U^c \rightarrow NE \otimes 1^Q,$$

where for each $c$, $U_c : N \rightarrow NE$ is an isometry satisfying $\text{Tr}_E [U_c \omega_c U_c^\dagger] = \omega_c$.

Given a quantum ensemble, we can freely remove and attach the redundant part as shown in the following lemma.

**Lemma 4 (KI operations [33]).** Let $\rho^{AX}$ be a classical-quantum state and $\omega^{CQNX}$ be its KI decomposition as given in Theorem 2. Then, there are quantum channels $(\kappa^{A \rightarrow CQ}_\text{off}, \kappa^{CQ \rightarrow A}_\text{on})$ such that

$$\kappa^{A \rightarrow CQ}_\text{off} (\rho^{AX}) = \omega^{CQX},$$

$$\kappa^{CQ \rightarrow A}_\text{on} (\omega^{CQX}) = \rho^{AX}.$$

For a quantum state $\rho^A$ on system $A$, the von Neumann entropy of $\rho^A$ is defined as

$$S(\rho^A) := S(A)_\rho := - \text{Tr}(\rho^A \log \rho^A),$$

where log is taken base 2 throughout the paper. The von Neumann entropy has a dimension bound given by

$$S(\rho^A) \leq \log \dim(A).$$

The von Neumann entropy is subadditive as shown in the following lemma.

**Lemma 5 ([3]).** Let $\rho^{AR}$ be a quantum state on a composite system $AR$. Then,

$$S(AR)_\rho \leq S(A)_\rho + S(R)_\rho.$$

The von Neumann entropy is asymptotically continuous.

**Lemma 6 (Fannes Inequality [20]).** Let $\rho$ and $\sigma$ be density matrices on a Hilbert space of dimension $d$. If $\frac{1}{2} \| \rho - \sigma \|_1 \leq \epsilon \leq 1 - \frac{1}{d}$, then, $|S(\rho) - S(\sigma)| \leq \epsilon \log(d - 1) + h_2(\epsilon)$ where $h_2$ is the binary entropy function.

For a quantum state $\rho^{AR}$ on a composite system $AR$, the quantum mutual information of $\rho^{AR}$ is defined as

$$I(A : R)_\rho := S(A)_\rho + S(R)_\rho - S(AR)_\rho.$$

The quantum mutual information also has a dimension bound:

$$I(A : R)_\rho \leq 2 \log( \min(\dim(A), \dim(R)) ).$$

We will use the following property of the quantum mutual information:

**Lemma 7 ([16]).** Let $\rho^{AR}$ be a quantum state on a composite system $AR$. Let $N_1$ and $N_2$ be quantum channels from system $A$ to system $B$, and let $\lambda \in (0, 1)$. Then,

$$I(B : R)_{\lambda(N_1 \otimes \mathcal{I})(\rho) + (1 - \lambda)(N_2 \otimes \mathcal{I})(\rho)} \leq \lambda I(B : R)_{(N_1 \otimes \mathcal{I})(\rho)} + (1 - \lambda) I(B : R)_{(N_2 \otimes \mathcal{I})(\rho)}.$$

In addition, the quantum mutual information follows the data-processing inequality:

**Lemma 8.** Let $\rho^{AR}$ be a quantum state on a composite system $AR$, and let $N^{A \rightarrow B}$ and $M^{R \rightarrow R'}$ be quantum channels. Then,

$$I(A : R)_\rho \geq I(B : R')_{(N \otimes M)(\rho)}.$$

The quantum mutual information also satisfies a property called superadditivity.

**Lemma 9 ([16]).** Let $\rho^{A_1 R_1}$ and $\sigma^{A_2 R_2}$ be pure quantum states on composite systems $A_1 R_1$ and $A_2 R_2$. Let $N^{A_1 A_2 \rightarrow B_1 B_2}$ be a quantum channel, and $\omega^{B_1 B_2 R_1 R_2} := N^{A_1 A_2 \rightarrow B_1 B_2} (\rho^{A_1 R_1} \otimes \sigma^{A_2 R_2})$. Then,

$$I(B_1 B_2 : R_1 R_2)_\omega \geq I(B_1 : R_1)_\omega + I(B_2 : R_2)_\omega.$$
Lemma 10. For any integer \( n \geq 2 \), let \( \rho_1^{A_1 R_1}, \ldots, \rho_n^{A_n R_n} \) be pure quantum states on \( A_1 R_1, \ldots, A_n R_n \). Let \( N^{A_1 \ldots A_n \rightarrow B_1 \ldots B_n} \) be a quantum channel, and \( \omega^{B_1 \ldots B_n R_1 \ldots R_n} := N^{A_1 \ldots A_n \rightarrow B_1 \ldots B_n}(\rho_1^{A_1 R_1} \otimes \cdots \otimes \rho_n^{A_n R_n}) \). Then,
\[
I(B_1 \cdots B_n : R_1 \cdots R_n)_\omega \geq I(B_1 : R_1) + \cdots + I(B_n : R_n)_\omega.
\]

Proof. We sketch a proof by induction on \( n \). Lemma 9 is the base case for \( n = 2 \). Furthermore, using Lemma 9,
\[
I(B_1 \cdots B_n : R_1 \cdots R_n)_\omega \geq I(B_1 \cdots B_{n-1} : R_1 \cdots R_{n-1})_\omega + I(B_n : R_n)_\omega.
\]

Let \( \tilde{\omega} = \text{Tr}_{B_n R_n} \omega \). We can see that \( \tilde{\omega} = N^{A_1 \cdots A_{n-1} \rightarrow B_1 \cdots B_{n-1}}(\rho_1^{A_1 R_1} \otimes \cdots \otimes \rho_{n-1}^{A_{n-1} R_{n-1}}) \) for some quantum channel \( \tilde{N} \), which acts by attaching \( \rho_n^{A_n R_n} \), applying \( N \), and then tracing out \( B_n R_n \). Thus we can apply the induction hypothesis to \( I(B_1 \cdots B_{n-1} : R_1 \cdots R_{n-1})_\omega \). Finally, noting that \( I(B_i : R_i)_\omega = I(B_i : R_i)_\omega \) completes the induction step.

To characterize the rate-distortion function, we use the entanglement of purification [40] which can be expressed as follows.

Theorem 11 ([40]). The entanglement of purification of a bipartite state \( \rho^{AR} \) is given by
\[
E_p(A : R)_\rho = \min_{\rho'^{A'}} S(\rho^{A'})_{\sigma},
\]
where \( \rho^{AR} \) is first purified to \( |\psi\rangle^{AARB} \) and then a quantum channel \( N^{A' \rightarrow A''} \) is applied to the purifying system \( A' \) to minimize the entropy of \( \sigma^{A''} = (I^A \otimes N^{A' \rightarrow A''})(\rho^{A'}) \).

The entanglement of purification is upper-bounded by the von Neumann entropy.

Lemma 12 ([40]). Let \( \rho^{AR} \) be a quantum state on a composite system \( AR \). Then,
\[
E_p(A : R)_\rho \leq \min\{S(A)_\rho, S(R)_\rho]\.
\]

The entanglement of purification also satisfies the monotonicity property, which is analogous to the data processing inequality for quantum mutual information.

Lemma 13 ([40]). Let \( \rho^{AR} \) be a quantum state on a composite system \( AR \), and let \( N^{A \rightarrow B} \) be a quantum channel. Then,
\[
E_p(A : R)_\rho \geq E_p(B : R)_{(N \otimes I)(\rho)}.
\]

Our results rely on quantum state redistribution [18, 44], which can be summarized as follows.

Theorem 14 (Quantum state redistribution [18, 44]). Consider an arbitrary tripartite state on \( ACB \), with purification \( |\psi\rangle^{ACBR} \). Consider \( n \) copies of the state for large \( n \), on systems \( A_1, \ldots, A_n, C_1, \ldots, C_n, B_1, \ldots, B_n, R_1, \ldots, R_n \). Suppose initially Alice has systems \( A_1, \ldots, A_n, C_1, \ldots, C_n \), and Bob has systems \( B_1, \ldots, B_n \). Then, there is a protocol transmitting \( nQ \) qubits from Alice to Bob for \( Q = \frac{1}{2} I(C : R|B) + \eta_n \), and consuming \( nE \) ebits shared between them, where \( Q + E = S(C|B) + \eta_n \), so that the final state is \( \gamma_n \)-close to \( (|\psi\rangle^{ACBR})^\otimes n \) but now \( C_1, \ldots, C_n \) are in the possession of Bob, and such that \( \{\eta_n\} \) are vanishing non-negative sequences.

In our paper, we only consider the situation when Alice and Bob share ebits, and \( B \) is trivial, so, there is no conditioning on \( B \) in the rates.

III. SETUP FOR RATE DISTORTION CODING FOR ENSEMBLE SOURCES

In this section, we define our setup for rate distortion coding. This is summarized in Fig. 1 below, and described in detail in the rest of the section.

We consider a source given by the ensemble \( \{p_x, \rho_x^A \otimes |x\rangle \langle x| \}_{x \in \Sigma} \) with side information in the system \( J \). Note that the side information can be quantum, as the states \( |x\rangle \langle x| \) need not be orthonormal. The most general side information need not be pure, and can be entangled with the system \( A \). We consider the simpler pure state side information when deriving most of our results; in Section V, we formalize the most general side information, and outline the extensions of our results to the general setting. When \( J \) is one-dimensional, there is no side information and the task is called blind compression. In contrast, when \( |x\rangle \) contains a classical description of the state \( \rho_x \), meaning that the sender knows the state to be compressed, then, the task is called visible compression. Our general model for side information includes both blind and visible compression as special cases. The ensemble source can be equivalently defined as a classical-quantum (cq) state where the system \( X \) plays the role of a classical reference system
\[
\rho^{AXX'} := \sum_{x \in \Sigma} p_x \rho_x^A \otimes |x\rangle \langle x|^{X'}, \quad (5)
\]
and symbols in \( \Sigma \) are associated with an orthonormal basis \( \{|x\rangle \}_{x \in \Sigma} \) on \( X \). We also consider a purification of the source:
\[
|\psi\rangle^{AXX'R} := \sum_{x \in \Sigma} \sqrt{p_x} \phi_x^{AR} |x\rangle^X |x\rangle^{X'}, \quad (6)
\]
We consider the following purifications for the states $\sigma$ and $\rho$ convex, and (3) vanishing on the Lipschitz constant in the definition above. We also want to discuss the requirement of convexity and continuity separately.

Moreover, consider the Stinespring dilations $U_{\mathcal{E}_{n}}^{A_{n}J_{n}A_{0}M_{W_{A}}}$ and $U_{\mathcal{D}_{n}}^{MB_{0}B_{n}W_{B}}$ for the encoding and the decoding maps. We consider the following purifications for the states $\sigma_{n}$ and $\xi_{n}$.

$$|\sigma_{n}\rangle R_{A_{n}M_{W_{A}}W_{A_{0}}B_{0}} := \sum_{x^{n}\in \Sigma^{n}} \sqrt{p_{x^{n}}} |\sigma_{x^{n}}\rangle^{M_{W_{A}}W_{A}} |x^{n}\rangle^{X^{n}} X^{n},$$

$$|\xi_{n}\rangle B_{n}^{W_{A}W_{B}}X_{n}^{A_{n}W_{n}}R_{A_{n}M_{W_{A}}W_{A_{0}}B_{0}} := \sum_{x^{n}\in \Sigma^{n}} \sqrt{p_{x^{n}}} |\xi_{x^{n}}\rangle B_{n}^{W_{A}W_{B}} |x^{n}\rangle^{X^{n}} X^{n},$$

A common method to certify the quality of data transmission in many protocols revolves around bounding the trace distance or the fidelity between the state to be transmitted and the output state of the protocol (including the references). In this paper, we consider a general notion of distortion which includes both of these common measures, and we consider an arbitrary amount or the fidelity between when we study the entanglement-qubit rate pair. Alice applies an encoding channel $\mathcal{E}_{n}^{A_{n}J_{n}A_{0} \rightarrow M}$, and sends system $M$ to Bob. Receiving $M$, Bob applies a decoding channel $\mathcal{D}_{n}^{M_{W_{A}}B_{n}W_{B}}$. We define

$$\sigma_{n}^{MB_{0}X^{n}} := (\mathcal{E}_{n}^{A_{n}J_{n}A_{0} \rightarrow M} \otimes \mathcal{I}_{X^{n}B_{0}})(\rho^{A_{n}J_{n}X^{n}} \otimes |\Phi\rangle \langle \Phi| A_{0}B_{0}),$$

$$\xi_{n}^{B_{n}^{W_{A}W_{B}}X^{n}} := (\mathcal{D}_{n}^{M_{W_{A}}B_{n}W_{B}} \otimes \mathcal{I}_{X^{n}})(\sigma_{n}^{MB_{0}X^{n}}).$$

Furthermore, consider the Stinespring dilations $U_{\mathcal{E}_{n}}^{A_{n}J_{n}A_{0}M_{W_{A}}}$ and $U_{\mathcal{D}_{n}}^{MB_{0}B_{n}W_{B}}$ for the encoding and the decoding maps. We consider the following purifications for the states $\sigma_{n}$ and $\xi_{n}$.

$$|\sigma_{n}\rangle^{M_{W_{A}}W_{A_{0}}B_{0}} := \sum_{x^{n}\in \Sigma^{n}} \sqrt{p_{x^{n}}} |\sigma_{x^{n}}\rangle^{M_{W_{A}}W_{A}} |x^{n}\rangle^{X^{n}} X^{n},$$

$$|\xi_{n}\rangle^{B_{n}^{W_{A}W_{B}}X_{n}^{A_{n}W_{n}}R_{A_{n}M_{W_{A}}W_{A_{0}}B_{0}}} := \sum_{x^{n}\in \Sigma^{n}} \sqrt{p_{x^{n}}} |\xi_{x^{n}}\rangle^{B_{n}^{W_{A}W_{B}} |x^{n}\rangle^{X^{n}} X^{n}},$$

A common method to certify the quality of data transmission in many protocols revolves around bounding the trace distance or the fidelity between the state to be transmitted and the output state of the protocol (including the references). In this paper, we consider a general notion of distortion which includes both of these common measures, and we consider an arbitrary amount of distortion. Recall that the systems $A, X$ and the source $\rho^{AX}$ are specified by the compression problem. The ideal output state of the protocol is $\rho^{BX}$ and $B \sim A$. The distortion quantifies how different any state $\tau^{BX}$ is from the ideal state $\rho^{BX}$, formalized as follows:

**Definition 15.** Let $B, X$ be some fixed quantum systems, and $\rho^{BX}$ a fixed state on $BX$. The distortion of a quantum state $\tau^{BX}$ with respect to $\rho^{BX}$, denoted by $\Delta(\tau^{BX})$, is any non-negative real-valued function $\Delta$ on $\tau^{BX}$ which is (1) continuous, (2) convex, and (3) vanishing on $\rho^{BX}$. For concreteness, we express continuity as $|\Delta(\tau^{BX}) - \Delta(\tilde{\tau}^{BX})| \leq \|\tau^{BX} - \tilde{\tau}^{BX}\|_{1} K_{\Delta}$ where $K_{\Delta}$ is a constant for a given $\Delta$.

We note that a convex function on a finite dimensional space is always continuous, but we opt to introduce the notation for the Lipschitz constant in the definition above. We also want to discuss the requirement of convexity and continuity separately,
in the hope that convexity can be removed in future studies. The third property, \( \Delta(\rho^{B^n}) = 0 \), ensures that all relevant optimizations in this paper are feasible. Since the state space is compact, the distortion defined above is always bounded. The dependency of the distortion on \( \rho^{B^n} \) is embedded in the choice of the function \( \Delta \). Note that each choice of the distortion determines a specific rate-distortion theory. A canonical example of a distortion under definition 15 is \( 1 - F \) where \( F \) is the fidelity of \( \tau^{B^n} \) with respect to \( \rho^{B^n} \). In this case, the distortion is also faithful (that is, \( \Delta(\tau) > 0 \) for \( \tau \neq \rho \)). Another choice of the distortion is the constant function with value 0 which is independent of \( \rho^{B^n} \). (See further discussion later in this section.) We derive and present our results for our general definition of distortion.

For the compression setup described above, with protocol described by \( \mathcal{E}^{A^n} : A_0^n \rightarrow M^n \) and \( \mathcal{D}^{M^n} : B_0^n \rightarrow B^n \), we define

\[
\Delta^{(n)}(\xi_n) = \max_i \Delta(\xi_n),
\]

\[
\Delta^{(n)}_{\text{ave}}(\xi_n) = \frac{1}{n} \sum_{i=1}^{n} \Delta(\xi_n),
\]

where \( \xi_n \) is the reduced state of \( \xi_n^{B^n} \) to the \( i \)-th system. Note that with the above definitions, we quantify distortion letter-wise, adopting a worst case or an average case local error criterion in our rate distortion theory. The worse case local error criterion is suitable for any use of the resulting states that do not correlate between the copies, and the average case criterion is suitable in applications such as parameter estimation via averaging. We use \( \Delta^{(n)} \) to denote one of \( \Delta^{(n)}_{\text{max}} \) or \( \Delta^{(n)}_{\text{ave}} \).

In the unassisted scenario, for a given positive number \( D > 0 \), we say that a pair of qubit rate \( R \) and distortion \( D \) is achievable if there exists a sequence of codes \( \{ (\mathcal{E}_n, \mathcal{D}_n) \} \) such that

\[
\Delta^{(n)}(\xi_n^{B^n}) \leq D + \delta_n \quad \text{and} \quad \frac{1}{n} \log \dim(M) \leq R + \eta_n
\]

for some vanishing non-negative sequences \( \{ \delta_n \} \) and \( \{ \eta_n \} \). The rate-distortion function is defined as

\[
\mathcal{R}(D) := \inf \{ R : (R, D) \text{ is achievable} \}.
\]

The entanglement-assisted scenario is essentially the same with unlimited \( \dim(A_0), \dim(B_0) \) and arbitrary choice of entangled state \( |\Phi\rangle \) shared on \( A_0B_0 \), and with the notation \( \mathcal{R}_{\text{ea}}(D) \) for the distortion function.

We also consider the most general scenario when we count both the qubit rate and the entanglement rate. In this case, we say that a tuple of qubit rate \( R \), entanglement rate \( E \), and distortion \( D \) is achievable if there exists a sequence \( \{ (\mathcal{E}_n, \mathcal{D}_n) \} \) where \( \Delta^{(n)}(\xi_n^{B^n}) \leq D + \delta_n \),

\[
\frac{1}{n} \log \dim(M) \leq R + \eta_n,
\]

\[
\frac{1}{n} \log \dim(A_0) \leq E + \eta_n,
\]

for some vanishing non-negative sequences \( \{ \delta_n \} \) and \( \{ \eta_n \} \). Similarly, the rate-distortion function in this case is defined as

\[
\mathcal{R}(D, E) := \inf \{ R : (R, E, D) \text{ is achievable} \}.
\]

Note that we use the symbol \( E \) to denote some environment system elsewhere in the paper, and occasionally we use \( E \) to denote the entanglement rate. Similarly, we use the symbol \( R \) to denote some purifying system; we also use \( R \) to denote the qubit rate for compression. We hope that the different contexts will minimize the chance of any confusion.

Although the worst-case distortion \( \Delta^{(n)}_{\text{max}} \) imposes a more stringent error criterion than the average-case distortion \( \Delta^{(n)}_{\text{ave}} \), remarkably, these two error criteria yield the same rate-distortion function if Alice and Bob share a sublinear amount of randomness. We will discuss this observation in more detail in Section V.

We illustrate the setup with some examples. Consider the unassisted case. If the distortion is \( 1 - F \) with \( F \) being the fidelity, \( \mathcal{R}(D) \) captures the qubit rate needed to attain the per-copy fidelity \( 1 - D \). If the distortion is the constant zero function, \( \mathcal{R}(D) = 0 \) for all \( D \). While mathematically permissible, this distortion does not lead to an interesting theory.

IV. MAIN RESULTS

In this section, we present our main results. We start with the entanglement-assisted rate-distortion function in Theorem 16, followed by the unassisted case in Theorem 17. The full region of achievable qubit and entanglement rate pairs, as a function of distortion, is given by Theorem 22.
A. Entanglement Assisted Rate Distortion Theory

In this scenario, Alice and Bob have free access to unlimited entanglement for the compression task. We prove in the following that the entanglement-assisted rate-distortion function for an ensemble source is given by an optimized expression involving the quantum mutual information in single-letter form.

**Theorem 16** (Entanglement-assisted rate-distortion theory). We use the setting in Section III and consider the purification $|\psi\rangle^{AJXX'}R = \sum_\lambda \sqrt{p_\lambda} |\psi_\lambda\rangle^{AR} |j_\lambda\rangle^I |x\rangle^N$ of the source $\rho^{AJX}$ (see Eqs. (5) and (6)). For both error criteria $\Delta_{\text{max}}^{(n)}$ and $\Delta_{\text{ave}}^{(n)}$, for $D \geq 0$, the entanglement-assisted rate-distortion function is given by

$$\mathcal{R}_{\text{ea}}(D) = \min_{\mathcal{N}^{AJX} \rightarrow B} \frac{1}{2} I(B : XX'R),$$

where the minimum is taken over all quantum channels $\mathcal{N} : AJ \rightarrow B$ such that $\Delta\left((\mathcal{N} \otimes I^X)\left(\rho^{AJX}\right)\right) \leq D$. We first show achievability ($\mathcal{R}_{\text{ea}}(D) \leq f(D)$) and then the converse ($\mathcal{R}_{\text{ea}}(D) \geq f(D)$).

To show achievability, we construct a sequence of codes $\{\mathcal{C}_n, D_n\}_n$ based on quantum state redistribution [18, 44] (Theorem 14) such that $\Delta_{\text{max}}^{(n)}(\xi_{B^nX^n}) \leq D + \delta_n$ and $\frac{1}{n} \log \dim(M) \leq f(D) + \epsilon_n$ for some vanishing non-negative sequences $\{\delta_n\}, \{\eta_n\}$. Let $\mathcal{N}^{AJX} \rightarrow B$ be a quantum channel satisfying $\Delta\left((\mathcal{N}^{AJX} \rightarrow B)\left(\rho^{AJX}\right)\right) \leq D$, and $U^{AJ \rightarrow BE}$ be its Stinespring dilation. Suppose that the source generates $n$ copies of the ensemble, resulting in the state $|\psi^n\rangle^{A^nJ^nX^nX'nR^n}$. Alice applies $U^{AJ \rightarrow BE}$ to each of her $n$ copies of her system, then, the state becomes

$$|\tau_n\rangle^{B^nE^nX^nX'nR^n} := \left((U^{AJ \rightarrow BE} \otimes 1^{XX'R}) |\psi\rangle^{AJXX'}\right)^{\otimes n} = \sum_{x^n \in \Sigma^n} \sqrt{p_{x^n}} |\tau_{x^n}\rangle^{B^nE^nX^n} |x^n\rangle^{X'n} |x^n\rangle^{X'n}.$$ 

In the above, we have defined $|\tau_{x^n}\rangle := (1^R \otimes U) |\psi\rangle^{AR} |j_\lambda\rangle^I$ and for $x^n = x_1x_2 \cdots x_n$, $|\tau_{x^n}\rangle := \otimes_{i=1}^n |\tau_{x_i}\rangle$. Alice and Bob then perform quantum state redistribution as follows. Recall that Alice and Bob share entanglement in systems $A_0B_0$. At this point, Bob has the system $B_0$, the referee has $R^nX^nX'n$, and Alice has $A_0B_0E^n$. Alice tries to transmit $B^n$ to Bob. There exists a quantum state redistribution protocol with qubit rate at most $\frac{1}{2} I(B : RX') + \eta_n$ and block error $\epsilon_n$. Therefore, the state after quantum state redistribution, denoted $|\tilde{\tau}_n\rangle^{B^nE^nX^nX'nR^n}$, satisfies

$$\|\tilde{\tau}_n - \tau_n\|_1 \leq \epsilon_n$$

which by monotonicity also implies, for each $i$,

$$\|\tilde{\tau}_{B_iX_i} - \tau_{B_iX_i}\|_1 \leq \epsilon_n.$$ 

By continuity of the distortion, we have

$$\Delta_{\text{max}}^{(n)}(\tilde{\tau}_{B_iX_i}) \leq D + \epsilon_n \Delta_{\text{max}}.$$

The above holds for each $i$, so, both $\Delta_{\text{max}}^{(n)}$ and $\Delta_{\text{ave}}^{(n)}$ are upper-bounded by $D + \epsilon_n \Delta_{\text{max}}$. Since $\frac{1}{n} I(B : RX')$ is an achievable rate for quantum state redistribution, $\{\epsilon_n\}$ and $\{\eta_n\}$ are vanishing. Letting $\delta_n = \epsilon_n \Delta_{\text{max}}$, $\{\tilde{\delta}_n\}$ is also vanishing. Thus $f(D) \leq \frac{1}{2} I(B : RX')$ and $D$ are achievable.

Next, we show the converse. Suppose a rate $R$ is achievable because of a sequence of codes $\{\mathcal{C}_n, D_n\}_n$ such that $\Delta_{\text{max}}^{(n)}(\xi_{B^nX^n}) \leq D + \delta_n$ and $\frac{1}{n} \log \dim(M) \leq R + \eta_n$ for some vanishing non-negative sequences $\{\delta_n\}, \{\eta_n\}$. We have to show that $R \geq f(D)$. Consider the following chain of inequalities:

$$2n(R + \eta_n) = 2 \log \dim(M) \geq I(M : R^nX^nX'nB_0)_{\sigma_n} = S(M)_{\sigma_n} + S(R^nX^nX'nB_0)_{\sigma_n} - S(MR^nX^nX'nB_0)_{\sigma_n} \geq S(MB_0)_{\sigma_n} + S(R^nX^nX'n)_{\sigma_n} - S(MR^nX^nX'nB_0)_{\sigma_n} = I(MB_0 : R^nX^nX'n)_{\sigma_n}.$$
\[ \geq I(B^n : R^n X^n X'^n)_{\xi_n} \]
\[ \geq \sum_{i=1}^{\infty} I(B_i : R_i X_i X'_i)_{\xi_n} \]
\[ \geq 2n f(D + \delta_n) . \]

The second line follows from the dimension upper bound of the quantum mutual information. The fourth line holds because preshared entanglement is independent of the data, so, system \( B_0 \) is independent of \( R^n X^n X'^n \). The fifth line follows from subadditivity of the von Neumann entropy (Lemma 5). The seventh line follows from the data-processing inequality (Lemma 8). Note that from the second to the sixth line, the entanglement assistance is represented by \( B_0 \) which is originally grouped with \( R^n X^n X'^n \) and is finally grouped with \( M \), and \( MB_0 \) is then turned into \( B^n \) in the seventh line. The eighth line follows from our generalized lemma for superadditivity of quantum mutual information (Lemma 10). To obtain the last inequality, let \( N^{A_j, K_j \rightarrow B^j} \) denote the quantum channel on the \( i \)-th copy of the source obtained by attaching the other \( n - 1 \) copies of the source, followed by applying the protocol \( (E_n, D_n) \) and then tracing out all but the \( i \)-th copy. Since the protocol attains the distortion (Eq. 7)), \( \Delta(\xi^n_{B_i, X_i}) \leq D + \delta_n \). Therefore, the channel \( N^{A_j, K_j \rightarrow B^j} \) satisfies the constraint \( \Delta \left( (N_i \otimes \mathcal{I}) (\rho^{A_j X_i}) \right) \leq D + \delta_n \). Thus \( I(B_i : R_i X_i X'_i)_{\xi_n} \geq 2f(D + \delta_n) \) by the minimizing definition of the function \( f \). Finally, as \( \delta_n \rightarrow 0, f(D + \delta_n) \rightarrow f(D) \) because of the continuity proved in Appendix A. This completes the proof for the converse for the worse case local error criterion. Combining with the achieving protocol, the theorem is established.

The theorem also holds for the average local error criteria. To see this, the above achieving protocol satisfies the global error criteria. We only need to modify the last step of the above converse proof. We have instead
\[ \sum_{i=1}^{\infty} I(B_i : R_i X_i X'_i)_{\xi_n} \geq \sum_{i=1}^{\infty} 2f \left( \Delta(\xi^n_{B_i, X_i}) + \delta_i \right) \geq 2nf \left( \frac{1}{n} \sum_{i=1}^{\infty} \Delta(\xi^n_{B_i, X_i}) + \delta_i \right) \geq 2nf(D + \delta_n) \]
where the first inequality holds term-wise, similarly to the last step of the original converse proof, but with the \( i \)-th distortion \( \Delta(\xi^n_{B_i, X_i}) + \delta_i \) replacing \( D + \delta_n \). The second inequality is due to the convexity of \( f(D) \), and this is proved in Appendix A. The last step follows from the local error criterion (Eq. 8) and the fact that, by definition, \( f(D) \) is decreasing with \( D \). This completes the proof for the theorem for the local error criterion. As a final remark, with entanglement assistance, the same rate-distortion function holds for both the worse case and the average case local error criteria.

\[ \square \]

B. Unassisted Rate-Distortion Theory

In this subsection, we analyze optimal rate-distortion trade-off when compressing an ensemble source without any entanglement assistance, under the average local error criterion. In this case, the unassisted rate-distortion function is given by a regularized and optimized expression involving the entanglement of purification.

**Theorem 17** (Unassisted rate-distortion theory). We use the setting in Section III and consider the purification \( |\psi\rangle^{\text{A}j \text{X}X'\text{R}} = \sum_x \sqrt{p_x} |x\rangle \otimes |j_x\rangle |X_x\rangle |X'_x\rangle \) of the source \( \rho^{\text{A}j \text{X}} \) (see Eqs. (5) and (6)). For \( \Delta(n) = \Delta(n)_{\text{ave}} \), and for \( D > 0 \), the unassisted rate-distortion function is given by
\[ R(D) = \lim_{k \to \infty} \frac{1}{k} \min_{N_k} \min_{E_p} \frac{1}{k} \mathbb{E} p(B_k: X_k X'^k R_k) \tau_k , \]
(11)
where the minimum is taken over all quantum channels \( N_k : \text{A}j \text{X} \rightarrow B_k \) such that
\[ \Delta_k(\text{ave}) \left( (N_k \otimes \mathcal{I}) (\rho^{\text{A}j \text{X}}) \right) \leq D , \]
(12)
and the entanglement of purification is evaluated on the state
\[ \gamma_k^{X_k X'^k R_k} := \left( N_k \otimes \mathcal{I} \right) \left( |\psi\rangle \langle \psi|^{\text{A}j \text{X}X'\text{R}} \right) . \]
(13)
**Proof.** We first show achievability, and then the converse. We label various expressions on the RHS of Eq. (11) as follows. Let \( g_k(D) = \frac{1}{k} \min_{N_k} \min_{E_p} \mathbb{E} p(B_k: X_k X'^k R_k) \tau_k \) where \( N_k^{j \rightarrow B_k} \) is subject to the constraint \( \Delta_k(\text{ave}) \left( (N_k \otimes \mathcal{I}) (\rho^{\text{A}j \text{X}}) \right) \leq D \), and \( g(D) = \lim_{k \to \infty} g_k(D) \). We prove important properties of \( g(D) \) and \( g_k(D) \) in Appendix B.

To show achievability, we construct a sequence of codes \( \{ (E_n, D_n) \}_n \) such that \( \Delta(\text{ave})(\xi^n_{B_k, X_k X'^k}) \leq D + \delta_n \) and \( \frac{1}{n} \log \dim(M) \leq g(D) + \eta_n \) for some vanishing non-negative sequences \( \{ \delta_n \}, \{ \eta_n \} \). In these codes, Alice applies Schumacher compression, and she processes the state before the compression to minimize the cost. To construct one of these codes, first pick \( \gamma > 0 \). We use the fact that \( g_k(D) \) converges uniformly to \( g(D) \) (see Appendix B) to choose \( k \) so that \( g_k(D) \leq g(D) + \gamma \) for all \( D \). Consider \( k \) copies of the source given by the state \( |\psi\rangle^{j \rightarrow B_k} \). Let \( N_k^{j \rightarrow B_k} \) be a quantum channel minimizing the expression in \( g_k(D) \) and satisfying \( \Delta(\text{ave}) \left( (N_k^{j \rightarrow B_k}) \right) \leq D \). Alice applies the quantum channel \( N_k^{j \rightarrow B_k} \)
to her system $A_k^k$. Let $U_{\Lambda_{k}}^{A_k^k \rightarrow B_k^k} \rightarrow B_k^k E$ be the Stinespring dilation of the channel $N_{k}^{A_k^k \rightarrow B_k}$. Then, Alice produces a $k$-copy state with purification

$$|\tau_k\rangle^B E^X X^k R^k := (U_{\Lambda_{k}}^{A_k^k \rightarrow B_k^k} \otimes 1^X X^k R^k) |\psi\rangle^{A_k^k} X^k X^k R^k = \sum_{x \in \mathbb{X}^k} \sqrt{p_x} |\tau_x\rangle^B E^X X^k |x\rangle^{X^k}.$$

Note that Alice has system $E$ because she locally applies $U_{\Lambda_{k}}^{A_k^k \rightarrow B_k^k}$. She further applies a quantum channel $\Lambda_k^{E \rightarrow E_B}$ with isometry $U_{\Lambda_{k}}^{E \rightarrow E_B \otimes E_A}$ such that $S(B_k^k E_B^{\otimes m})$ is minimized.

Now take a sufficiently large $m$ and consider $n = mk$ copies of the source. In the protocol, Alice repeats the above processing $m$ times to obtain $m$ copies of $|\tau_k\rangle^B E^X X^k R^k$, applies $U_{\Lambda_{k}}^{E \rightarrow E_B \otimes E_A}$ to each copy, and transmits system $(B_k^k E_B^{\otimes m})$ to Bob using Schumacher compression [36, 37]. Let $\epsilon_m$ be the error in the compression. Let $\tilde{r}_{mk}$ denote the resulting state. Then, $\|\tilde{r}_{mk} - r_{mk}\|_1 \leq \epsilon_m$, so, each copy of $|\tau_k\rangle^B E^X X^k R^k$ is transmitted with error at most $\epsilon_m$. As in Theorem 16, by monotonicity of the trace distance and continuity of the distortion, for each $i$,

$$\Delta(\tilde{r}_{k_i}^B, X_i) \leq D + \epsilon_m K_\Delta,$$

where $K_\Delta$ is some constant independent of $k$. Using the known achievable rate for Schumacher compression, our rate-distortion code can have rate $\frac{1}{k} S(B_k^k E_B^{\otimes m}) + \gamma_m$, so that $\{\epsilon_m\}$ and $\{\gamma_m\}$ are both vanishing sequences. We can see that

$$S(B_k^k E_B^{\otimes m}) + \gamma_m = \min_{\Lambda_k^{'}} S(\Lambda_k^{'}) = \max_{\Sigma_k^{'}} I(\Sigma_k^{'}) = E_p(\Sigma_k: R_k^X X_k^k),$$

where the minimum is taken over all channels $\Lambda'$ on $E$. The rate is therefore $g_k(D) + \gamma_m \leq g(D) + \gamma + \gamma_m$. The quantity $\gamma + \gamma_m$ can be made arbitrarily small as $k,m \rightarrow \infty$. Recall that our rate-distortion code has block-length $n = mk$. Let $\delta_m \equiv \epsilon_m K_\Delta$. We have a subsequence of codes with distortion $D + \delta_m$ and rate $\frac{1}{k} S(B_k^k E_B^{\otimes m}) + \gamma_m$ where $\{\delta_m\}$ and $\{\gamma_m\}$ are both vanishing sequences. For block-lengths that are not a multiple of $k$, we can use the above subsequence of codes for most copies, and use a noiseless code for the remaining (fewer than $k$) copies. This preserves the distortion and the effect on the rate is achievable. Therefore, $g(D)$ is achievable.

Next, we show the converse, that any achievable rate is at least $g(D)$. First pick $\gamma > 0$, and using (2) and (3) proved in Appendix B, choose $k$ so that $g_k(D) \leq g(D) + \gamma$ for all $D$ and that $g_k(D)$ is continuous. For a code on $k$-copies with rate $R$ and distortion $D + \delta$, we have the following chain of inequalities:

$$k R = \log \dim (M) \geq S(M)_{\sigma_k} \geq E_p(M : R_k^X X_k^k)_{\sigma_k} \geq E_p(R_k^X X_k^k),$$

The first inequality follows from the dimension bound of the von Neumann entropy. The second inequality follows from Lemma 12. The third inequality follows from monotonicity of entanglement of purification (Lemma 13). Minimizing over the choice of $\xi_k$ under the distortion constraint for $D + \delta$, the last expression is no less than $g_k(D + \delta)$. Using the continuity of $g_k$ and the uniform convergence of $g_k$ to $g$, $g_k(D + \delta) - g(D)$ can be made to vanish, as $k$ increases and as $\delta$ decreases.

Combining the achievability and the converse, the rate distortion function for unassisted compression $R(D)$ is equal to $g(D)$. Note that the achieving protocol attains worse case local error criterion, but the last steps of the converse proof requires the results in Appendix B which only holds for the average local error criterion.

**Remark 18.** The $D = 0$ case requires some special treatments, and the theorem can still be established in some cases. First, $g(0)$ remains achievable for the most general side information system, but the proof for the converse cannot be established without the continuity and convergence results in Appendix B. Second, for blind compression with trivial side information system, and using a more standard distance measure for the distortion function $\Delta$ such as the trace distance or $1 - F$ where $F$ is the fidelity, we can adapt other results in the literature to conclude Theorem 17. In particular, references [32, 33] imply $R(0) = S(CQ)$ where the state on system $CQ$ is as defined in the decomposition in Theorem 2. Meanwhile, when $D = 0$, for each $k$, $g_k$ is minimized over $N_{k}^{A_k^k \rightarrow B_k^k}$ that are characterized by Lemma 3, and $g_k(0) = S(CQ)$ so Theorem 17 still holds.

**Remark 19.** While the rate distortion function for entanglement-assisted compression is given by a single-letter quantum mutual information expression, for unassisted compression is characterized by multi-letter entanglement of purification. There are two main hurdles to single-letterize $g(D)$: first, it is still open if the entanglement of purification is additive, second, the minimization is over CPTP maps acting on $k$ copies of the source.
We have derived the rate-distortion function in the most general unassisted setting, with side information spanning all possible scenarios between visible and blind compression. The rate-distortion function is quite complex, and for the rest of this subsection, we derive simplifications for two special cases, namely, visible compression and blind compression. In the case of visible compression, we show that the rate-distortion function can be simplified to the entanglement of purification between Bob’s systems $B^k$ and the reference systems $X^k$. In the case of blind compression, in the limit of $D \to 0$, we prove that our multi-letter rate-distortion function becomes single-letter, which can be related to the previously known optimal rate of blind data compression.

1) Unassisted Visible Compression:

**Corollary 20.** Assuming the set-up in Theorem 17, but specializing to visible compression so that $|x_j| = |x|$ in the register $J$. The rate distortion function given by Eq. (11) can be simplified to

$$\mathcal{R}(D) = \frac{1}{k} \lim_{k \to \infty} \min_{N_{k}^{a_k} : J \to B^k} E_p(B^k : X^k)_{\tau_k},$$

with Eqs. (12) and (13) continue to define $\tau_k$ and the constraints on $N_{k}^{a_k} : J \to B^k$.

**Proof.** We will prove that for each $k$,

$$\min_{N_{k}^{a_k} : J \to B^k} E_p(B^k : X^k X^k R^k)_{\tau_k} = \min_{\mathcal{M}_{k}^{a_k} : J \to B^k} E_p(B^k : X^k)_{\tau_k}.$$  

Note that by the monotonicity of entanglement of purification, the RHS is no greater than the LHS. It remains to show the opposite inequality. Such an inequality means that the correlation between $R^k$ and $B^k$ does not increase the rate. The physical intuition is that, the error constraint Eq. (12) does not involve $R^k$ for mixed-state ensembles. Using her side information $J^k$, Alice’s encoding map $N_k$ does not need to operate on the given system $(AR)^k$; instead, she can use $J^k$ to generate new systems $(AR')^k$ (with state identical to that in $(AR)^k$) before transforming $(JX')^k$ to $B^k$. The output in $(X'B)^k$ still satisfies the error constraint, but now $B^k$ is not correlated with $R^k$. We provide a rigorous proof in the following. The value of $k$ does not affect the proof idea, so, we first focus on $k = 1$, with the source

$$|\psi\rangle^{XX'RAJ} := \sum_{x \in \Sigma} \sqrt{p_x} |x\rangle^X |x\rangle^{X'} |\varphi_x\rangle^AR |x\rangle^J.$$  

Consider the RHS of Eq. (15). Let $\mathcal{M}_{A^J}^{A^J}$ be the channel minimizing $E_p(B : X)_{\tau}$ while satisfying the error criterion

$$\Delta \left( (\mathcal{M} \otimes \mathcal{I}_X) \left( \rho^{AJX} \right) \right) \leq D.$$  

Since $J$ is classical, without loss of generality, the Stinespring dilation for $\mathcal{M}_{A^J}^{A^J}$ can be chosen to be $V_{A^J \to EJ}$ such that $|x\rangle^{J}$ is preserved. In other words, $V_{A^J \to EJ} = \sum_{x} |x\rangle_{x}^{J} \otimes V_{x}^{A \to EB}$ for some isometries $V_x$. Let

$$|\tilde{\tau}\rangle^{XX'RJEB} := \left| (XX'\otimes V_{A^J \to EJ}) |\psi\rangle^{XX'RAJ} \right| \left. = \sum_{x \in \Sigma} \sqrt{p_x} |x\rangle^X |x\rangle^{X'} |\varphi_x\rangle^AR \left((I_XX' \otimes V_{A^J \to EJ}) \right) |x\rangle^J \right\rangle.$$  

Furthermore, when evaluating $E_p(B : X)_{\tau}$, let $\tilde{\Lambda}_{X'RJEB}^{X'RJEB}$ be the channel that minimizes $S(FB)_{(I_{XX'} \otimes \tilde{\Lambda}_{X'RJEB}^{X'RJEB})}(\tilde{\tau}) = \alpha$.

We now show that the LHS of Eq. (15), $\min_{\mathcal{M}_{A^J}^{A^J} : B} E_p(XX'R : B)_{\tau}$, is at most $\alpha$. The proof idea is summarized in Fig. 2.

![Fig. 2](image-url)

Fig. 2: The right column has operations defined by the RHS of Eq. (15). The left column is in terms of the operations in the right column, and presents a feasible solution for the minimization of the LHS of Eq. (15) with value same as the RHS.
To show that the LHS of Eq. (15), \( \min_{N^{A\rightarrow B}} E_p(XX' R : B), \) is at most \( \alpha \), we pick a valid (feasible) \( N^{A\rightarrow B} \) to produce a valid \( \tau^{XX'R} \) and a map \( \Lambda \) from \( \tau^{XX'R} \)'s purifying system to system \( F \) so that the resulting \( S(BF) = \alpha \). These choices will be in terms of the isometry \( V \) and the map \( \Lambda^{X'JRE \rightarrow F} \) developed above in evaluating the RHS of Eq. (15). So, let \( U^{A\rightarrow JEB} \) be the Stinespring dilation of \( N^{A\rightarrow B} \) and \( U \) acts on the source as follows:

\[
|\psi\rangle^{XX'RAJ} = \sum_{x \in \Sigma} \sqrt{p_x} |x\rangle^X |x\rangle^{X'} |\varphi_x\rangle^AR |x\rangle^J \tag{19}
\]

\[
\rightarrow \sum_{x \in \Sigma} \sqrt{p_x} |x\rangle^X |x\rangle^{X'} |\varphi_x\rangle^AR |x\rangle^J \tag{20}
\]

\[
\rightarrow \sum_{x \in \Sigma} \sqrt{p_x} |x\rangle^X |x\rangle^{X'} |\varphi_x\rangle^AR |x\rangle^J \left( (I^R \otimes V_x^{A'\rightarrow EB}) |\varphi_x\rangle^{A'R'} \right) =: |\tau\rangle^{XX'ARR'JEB}, \tag{21}
\]

where \( |x\rangle^{X'} \) generates \( |\varphi_x\rangle^{A'R'} \) from Eq. (19) to Eq. (20), and then \( V^{A'\rightarrow JEB} \) is applied to give Eq. (21) (note that here, \( V \) acts on \( A'J \)). To see that \( N^{A\rightarrow B} \) is a valid map, note that the final reduced state on \( XB \) is the same as in \( |\tau\rangle^{XX'RJEB} \) so by Eq. (17) \( N^{A\rightarrow B} \) satisfies the error criterion. We can upper-bound the entanglement of purification \( E_p(XX'R : B) \), by choosing a channel \( \Lambda^{ARR'JEB} \) to act on \( |\tau\rangle^{XX'ARR'JEB} \) and evaluating \( S(BF) \). This \( \Lambda^{ARR'JEB} \) creates \( X'' \) from \( J \) and then applies \( \Lambda^{X''JEB} = \text{Tr}^A \). The resulting \( S(BF) \) is indeed \( \alpha \). This completes the proof that the LHS is no bigger than the RHS in Eq. (15) for \( k = 1 \). The proof for other values of \( k \) can be obtained by replacing \( A, X, X', R, B, J, X'', R', A' \) by \( A^k, X^k, (X')^k, R^k, B^k, J^k, (X'')^k, (R')^k, (A')^k \) respectively in the above proof.

2) Unassisted Blind Compression: Next we consider blind compression, in which Alice does not know the state she receives. We use the results in Ref. [33], and the notations in Theorem 2 and Theorem 17 throughout this subsection. We take \( U_{KI} = I \) without loss of generality. Blind compression is modeled by a trivial side information system with \( \dim(J) = 1 \), and the source simplifies to

\[
|\psi\rangle^{AXX'R} = \sum_{x \in \Sigma} \sqrt{p_x} |\varphi_x\rangle^AR |x\rangle^X |x\rangle^{X'}. \tag{22}
\]

For simplicity of notation, we omit writing out the identity channels acting on some states; for example, we write \( N^{A\rightarrow B}(|\psi\rangle\langle\psi|) \) or \( N(|\psi\rangle\langle\psi|) \) omitting writing out the identity channel on \( XX'R \).

Reference [33] shows that, under vanishing local or global error criterion, the optimal rate of blind quantum data compression is \( S(CQ)_{|\psi\rangle\langle\psi|} \) where \( C \) and \( Q \) are the classical and quantum parts of the ensemble defined in Theorem 2. We can connect this result to Theorem 17 if we choose the distortion to be

\[
\Delta(\tau^{BX}) = 1 - F(\rho^{BX}, \tau^{BX}),
\]

where \( \rho^{BX} \) is obtained from \( \rho^{AX} \) by an identity channel from \( A \) to \( B \), and if we take \( D \rightarrow 0 \). Under this distortion, and using Theorem 17 and Remark 18,

\[
\mathcal{S}(0) = S(CQ)_{|\psi\rangle\langle\psi|} = \lim_{D \rightarrow 0} g(D) = \lim_{D \rightarrow 0} \lim_{k \rightarrow \infty} g_k(D).
\]

In this subsection, using techniques separate from Theorem 17, we show the following, which removes the regularization over \( k \) in the above. This also gives semicontinuity of \( g(D) \) at \( D = 0 \) for the unassisted blind setting (see Remark 18).

**Theorem 21.** Consider the blind compression of the source given by Eq. (22). Then,

\[
\lim_{D \rightarrow 0} \min_{N^{A\rightarrow B}} E_p(B : XX'R|N(|\psi\rangle\langle\psi|)) = S(CQ)_{|\psi\rangle\langle\psi|}, \tag{23}
\]

where the minimum is taken over all channels \( N^{A\rightarrow B} \) such that

\[
1 - F(\rho^{BX}, N^{A\rightarrow B}(\rho^{AX})) \leq D.
\]

**Proof.** Fix \( D \geq 0 \). We first make a useful observation. Let \( \hat{N}^{A\rightarrow B} \) be an optimal channel achieving the minimum in the left-hand side of Eq. (23). Consider the KI operations from Lemma 4

\[
\begin{align*}
K^{A \rightarrow CQ}_{\text{off}}(\rho^{AX}) &= \omega^{CQX}, \\
K^{CQ \rightarrow A}_{\text{on}}(\omega^{CQX}) &= \rho^{AX}.
\end{align*}
\]

and let \( K^{B \rightarrow \hat{B}}_{\text{off}} \) and \( K^{\hat{B} \rightarrow \hat{B}}_{\text{off}} \) be the above maps with system \( B \) replacing \( A \) and \( \hat{B} \) replacing \( CQ \). Let \( N^{*}_{A \rightarrow B} = K^{\hat{B} \rightarrow \hat{B}}_{\text{off}} \circ K^{B \rightarrow \hat{B}}_{\text{off}} \circ N^{A \rightarrow B} \). We now show that \( N^{*}_{A \rightarrow \hat{B}} \) is also an optimal map. First, we check that \( N^{*} \) also satisfies the distortion condition

\[
F(\rho^{BX}, N^{*}_{A \rightarrow B}(\rho^{AX})) = F(K^{CQ \rightarrow A}_{\text{on}} \circ K^{A \rightarrow CQ}_{\text{off}}(\rho^{BX}), K^{\hat{B} \rightarrow \hat{B}}_{\text{off}} \circ K^{B \rightarrow \hat{B}}_{\text{off}} \circ N^{A \rightarrow B}(\rho^{AX})) \]
\[
\geq F(\rho^{BX}, \hat{N}^{A \rightarrow B}(\rho^{AX})) \geq 1 - D,
\]
where the first line comes from Eqs. (24) and (25) and the definition of $\mathcal{N}_*$, the second line follows from monotonicity of the fidelity, and the last line holds because the optimal map $\hat{N}^{A\rightarrow B}$ must satisfy the distortion constraint.

Second, we show that $\mathcal{N}_*$ also attains the minimum in the left-hand side of Eq. (23). By monotonicity of the entanglement of purification

$$E_p(B : XX'R)_{\mathcal{N}_*(\psi)} \leq E_p(B : XX'R)_{\hat{N}(\psi)}.$$ 

Since $\hat{N}^{A\rightarrow B}$ is an optimal map achieving the minimum, $\mathcal{N}_*^{A\rightarrow B}$ is also an optimal map, as claimed.

We now show that

$$\lim_{D \to 0} \min_{\mathcal{N}_*^{A\rightarrow B}} E_p(B : XX'R)_{\mathcal{N}_*(\psi)} \leq S(CQ)|\psi\rangle\langle\psi|$$

using the following chain of inequalities

$$\min_{1-F(\rho^{BX}, \hat{N}^{A\rightarrow B}(\rho^{AX})) \leq D} E_p(B : XX'R)_{\mathcal{N}_*(\psi)} = E_p(B : XX'R)_{\mathcal{N}_*(\psi)} \leq E_p(\hat{B} : XX'R)_{\mathcal{K}_{\text{out}} \circ \hat{N}(\psi)} \leq S(CQ)|\psi\rangle\langle\psi| - h_2(D) - D \log(\dim(CQ)),$$

where the first line comes from the optimality of $\mathcal{N}_*$, the second and third lines come from the monotonicity and dimension bound of the entanglement of purification, and the last line follows from Fannes inequality (Lemma 6). Thus, in the limit of $D \to 0$, the inequality (26) holds.

Next, we show the opposite inequality

$$\lim_{D \to 0} \min_{\mathcal{N}_*^{A\rightarrow B}} E_p(B : XX'R)_{\mathcal{N}_*(\psi)} \geq S(CQ)|\psi\rangle\langle\psi|.$$ 

(27)

Note that the above follows from the results in [33], since our proof for Theorem 17 can be applied to the LHS to show that it is an achievable rate under vanishing local error. But we also present a direct proof in the following.

Fixed $D \geq 0$ and an optimal map $\hat{N}^{A\rightarrow B}$ (under constraints as defined before), with Stinespring dilation $U^{A\rightarrow BE}_\hat{N}$. Let $|\tau_x\rangle_{BER} = (U^{A\rightarrow BE} \otimes \mathbf{1}^R)|\varphi_x\rangle^{AR}$, and

$$|\tau\rangle_{BERXX'} = (U^{A\rightarrow BE} \otimes \mathbf{1}^{RX'})|\psi\rangle^{AXX'} = \sum_{x \in \Sigma} \sqrt{p_x} |\tau_x\rangle_{BER}^{|x\rangle^X|X'}.$$ 

By the definition of the entanglement of purification,

$$E_p(B : XX'R)_{\tau} = \min_{\mathcal{M}^{A\rightarrow BE}} S(BE_B)_{\mathcal{M}(\tau)}.$$ 

Let $\Lambda^{E\rightarrow EB}$ be an optimal channel achieving the minimum above. As $D \to 0$, $S(BE_B)_{\mathcal{M}(\tau)}$ converges to $S(BE_B)_{\mathcal{M}(\rho)}$, where $\mathcal{M}^{A\rightarrow BE_B}$ is a quantum channel such that $\text{Tr}_{E_B}[\mathcal{M}^{A\rightarrow BE_B}(\rho^{AX})] = \rho^{AX}$. By Koashi-Imoto’s theorem (Theorem 2), the channel $\mathcal{M}^{A\rightarrow BE_B}$ only acts on the redundant part of the state $\rho^{AX}$; that is, we can write

$$\mathcal{M}^{A\rightarrow BE_B}(\rho^{AX}) = \sum_x p_x \sum_c p_{c|x}|c\rangle\langle c|^{C} \otimes \omega^{NE_B}_c \otimes \rho_{xx}^Q \otimes |x\rangle^{X},$$

where $\omega^{NE_B}_c$ is a state such that $\text{Tr}_{E_B}[\omega^{NE_B}^c] = \omega^{N}_c$. Therefore,

$$S(BE_B)_{\mathcal{M}(\rho)} = S(CQNE_B)_{\mathcal{M}(\rho)} = S(C)_{\mathcal{M}(\rho)} + S(QNE_B|C)_{\mathcal{M}(\rho)} = S(C)_{\mathcal{M}(\rho)} + S(Q|C)_{\mathcal{M}(\rho)} + S(NE_B|C)_{\mathcal{M}(\rho)} \geq S(C)_{\mathcal{M}(\rho)} + S(Q|C)_{\mathcal{M}(\rho)} = S(CQ)_{\rho} = S(CQ)|\psi\rangle\langle\psi|$$

where the third line is obtained because the state on $QNE_B$ is a product state when we conditioned on system $C$; in particular, for each $c$, the state on $QNE_B$ is $\omega^{NE_B}_c \otimes \sum_x p_{c|x}^Q \mathbf{1}_x$. Therefore, the inequality (27) holds. ■
C. Full Rate Region of Rate Distortion Theory

An ultimate goal in quantum information theory is to understand the fundamental performance and cost of a task. Here, we generalize our arguments from the previous sections to provide the full qubit-entanglement rate region of rate distortion coding for ensemble sources.

**Theorem 22** (Full rate region of rate-distortion compression). Rate-distortion coding of a given source $|\psi\rangle^{A_{J}X'X'R}$ with distortion $D$ is achievable if and only if its qubit rate $R$ and entanglement rate $E$ satisfy

\[
R \geq \frac{1}{k} \frac{1}{2} I(B^k E_B : R^k X^k X'^k)_{A_k(\tau_k)},
\]

(28)

\[
R + E \geq \frac{1}{k} S(B^k E_B)_{A_k(\tau_k)},
\]

(29)

for some $k$ and for the state $\tau_k$ defined in the statement of Theorem 17, involving a channel $N_k^{A_k \rightarrow B^k}$ satisfying the distortion condition in the statement of Theorem 17, and some channel $\Lambda_k^{E \rightarrow E_B}$.

**Proof.** We first show achievability, and then the converse.

The achievability of the given rate-region can be shown by considering a protocol very similar to the one used in the proof of Theorem 17, so we only describe the two differences: (1) Alice’s processing need not minimize $S(B^k E_B)_{A_k(\tau_k)}$ here. (2) Instead of Schumacher compression, $m$ copies of the $k$-copy state are transmitted using quantum state redistribution [18, 44] with error $\epsilon_m$. This quantum state redistribution can be performed with qubit rate $R + \eta_m$ and entanglement rate $E + \eta_m$ if

\[
R \geq \frac{1}{k} \frac{1}{2} I(B^k E_B : R^k X^k X'^k)_{A_k(\tau_k)},
\]

(28)

\[
R + E \geq \frac{1}{k} S(B^k E_B)_{A_k(\tau_k)},
\]

(29)

for some vanishing non-negative sequence $\{\eta_m\}$. The same continuity argument in Theorem 17 also applies to show that the protocol satisfies the distortion condition up to a vanishing amount as $\epsilon_m \rightarrow 0$. Finally, for large block-lengths that are not multiples of $k$, we can again use the above subsequence of codes on most of the copies, and transmit the remainder noiselessly, with vanishing increase to the above rates, similarly to the discussion in the proof of Theorem 17.

For the converse, consider a fixed $k$ and a compression protocol $(E_k, D_k)$ with qubit rate $R$ and entanglement rate $E$, and satisfying the distortion condition given by Definition 15 and Eq. (8), with $D$ replaced by $D + \delta_k$ for a small $\delta_k$ (and a large $k$). For this protocol, $\sigma_k$, $\xi_k$ are as defined in Section III (with $k$ instead of $n$ for the block size). We need to propose quantum channels $N_k^{A_k, J^k \rightarrow B^k}$ with Stinespring dilation $U_k^{A_k, J^k \rightarrow B^k E}$ and $\Lambda_k^{E \rightarrow E_B}$ with Stinespring dilation $U_k^{E \rightarrow E_A E_B}$ so that $N_k^{A_k, J^k \rightarrow B^k}$ satisfies the distortion condition (with $D$ replaced by $D + \delta_k$), and the inequalities (28) and (29) on the rates $R, E$ are satisfied. To this end, we take $N_k^{A_k, J^k \rightarrow B^k}$ to be the composition of the following steps:

1. appending the entangled state $\Phi_{A_k B_0}$ to $(\rho^{\otimes k})^{A_k J^k X^k}$,
2. applying the encoding operation $E_k^{A_k J^k : A_0 \rightarrow M}$ (with environment system $W_A$),
3. applying the decoding operation $D_k^{M B_0 \rightarrow B^k}$ (with environment system $W_B$),

and $N_k^{A_k, J^k \rightarrow B^k}$ has environment system $E = W_A W_B$. We take $\Lambda_k^{E \rightarrow E_B}$ as the quantum channel that first discards system $W_A$ and then renames $W_B$ as $E_B$.

We now have the following chain of inequalities for $k$ copies with qubit rate $R$:

\[
2kR = 2 \log \dim(M)
\]

\[
\geq I(M B_0 : R^k X^k X'^k)_{\sigma_k}
\]

\[
= I(B^k E_B : R^k X^k X'^k)_{\xi_k},
\]

where the first inequality is obtained with the same arguments for the first 6 lines of the chain of inequalities in the converse proof of Theorem 16. We also have the following chain of inequalities with qubit rate $R$ and entanglement rate $E$:

\[
kR + kE = \log \dim(M) + \log \dim(B_0)
\]

\[
\geq S(M)_{\sigma_k} + S(B_0)_{\sigma_k}
\]

\[
\geq S(M B_0)_{\sigma_k}
\]

\[
= S(B^k W_B)_{\xi_k}.
\]

Identifying $\xi_k$ as $\Lambda_k(\tau_k)$ and $W_B$ as $E_B$, we obtain

\[
2kR \geq I(B^k E_B : R^k X^k X'^k)_{A_k(\tau_k)},
\]

\[
kR + kE \geq S(B^k E_B)_{A_k(\tau_k)}.
\]
To see that the above bounds do not change abruptly when $D + \delta_k$ is replaced by $D$, we use an argument similar to that used in Appendix B, taking $l$ copies of a $k$-copy protocol for $D_1$ and 1 copy of a $k$-copy protocol for $D_2$, and noting that the overall achievable distortion parameter is convex, while the RHS of the above bounds are additive. This removes the abrupt changes in the bounds as $l$ becomes large.

Remark 23. Comparing the rates in Theorem 22 with those in Theorems 16 and 17. The qubit rate $rac{1}{k} I(B^k: E_B : R^k X^k X''^k)_{\Lambda_k(\tau_k)}$ in Theorem 22 is lower-bounded by the assisted rate $\frac{1}{k} I(B : X'R)_{N(|\psi\rangle\langle\psi|)}$ from Theorem 16 (by monotonicity when dropping $E_B$ and by superadditivity). Choosing a trivial $E_B$ can increase $R + E$ but that is not a concern with entanglement assistance. Operationally, if Alice and Bob share enough entanglement, Alice need not apply her local operation $\Lambda^{E\rightarrow E_B}$ to achieve the optimal qubit rate. Meanwhile, minimizing the rate sum $R + E$ in Theorem 22 matches the unassisted rate in Theorem 17; operationally, it says that transmitting the required entanglement and performing quantum state redistribution is as optimal as Schumacher compression used in Theorem 17.

V. Discussion and Conclusion

First, we summarize our main results. We investigated quantum rate-distortion compression with and without entanglement. We assumed a quantum mixed-state ensemble source with side information so that our analysis covers both visible and blind compression. First, we derived the rate-distortion function for entanglement assisted compression, and we proved that the optimal rate can be expressed using a single-letter formula of quantum mutual information. Second, we showed that the rate-distortion function for the unassisted case is expressed in terms of a regularized entanglement of purification. We showed optimal rate can be expressed using a single-letter formula of quantum mutual information. Second, we showed that the rate-distortion function can be single-letterized in the limit of $D \rightarrow 0$. Finally, we found the full qubit-entanglement rate region for rate-distortion theory of mixed states. Thus, we characterized rate-distortion compression of mixed states, which has not been covered before. We believe our work furthers the understanding of the limits of quantum data compression with finite approximations.

Second, we explore several observations and implications of our results.

- We can compare our results to those in references [16, 41] for a pure-state source

$$\rho^{AX} = \sum_{x \in \Sigma} p_x |\varphi_x\rangle\langle\varphi_x|^{A} \otimes |x\rangle\langle x|^X$$

(30)

with purification

$$|\psi^{AXX'}\rangle = \sum_{x \in \Sigma} \sqrt{p_x} |\varphi_x\rangle A |x\rangle X |x'\rangle.$$  

(31)

We omit the side-information system $J$ in this comparison since [16, 41] considered side-information differently from our treatment. Note also that the system $R$ is trivial for a pure state ensemble. First, we note that the figure of merit in this paper is different from that in Refs. [16, 41], even though both adopted the (average-case) local error criterion. The distortion function $\Delta$ in this paper evaluates the quality of the data transmission on the joint system $AX$ (Definition 15). On the other hand, the distortion function in Refs. [16, 41] is with respect to the system $AX$. (Note that we have adjusted the notation accordingly.) Our distortion condition is less stringent in that we do not require the coherence over the label $x$ to be preserved.

Now, we compare our rate-distortion functions to those in Refs. [16, 41], and observe some similarities. For example, in the entanglement-assisted scenario, our rate-distortion function for the pure-state source is

$$\mathcal{R}_{ea}(D) = \min_{N^{AX} \rightarrow B} \frac{1}{2} I(B : X X')_{(N \otimes I) (|\psi\rangle\langle\psi|)},$$

(32)

where the minimum is taken over all quantum channels $N : A \rightarrow B$ such that

$$\Delta \left( (N \otimes I^X) \left( \rho^{AX} \right) \right) \leq D.$$  

(33)

Meanwhile, the rate-distortion function in Refs. [16, 41] has the same form as Eq. (32), but the minimization is taken over quantum channels $N : A \rightarrow B$ such that

$$\Delta \left( (N \otimes I^{XX'}) \left( \langle \psi|\psi\rangle^{AXX'} \right) \right) \leq D.$$  

(34)

Even though the two rate-distortion functions have almost identical expressions, they can have potentially different values due to the different distortion conditions (Eqs. (33) and (34)). Very similar relations between the two results also hold for the unassisted case via a similar argument.
• The achievability proofs of Theorems 16, 17, and 22 indicate that the optimal rates for all our rate-distortion theories can always be achieved by faithfully transmitting an appropriately distorted source. Hence, in the rate-distortion coding, it is sufficient for the sender to prepare a good approximation of the given ensemble.

• If Alice and Bob are allowed to use shared randomness for their protocol, the worst-case distortion $\Delta_{\text{max}}^{(n)}$ and the average-case distortion $\Delta_{\text{ave}}^{(n)}$ result in the same rate distortion function. To see this, first note that since $\Delta_{\text{ave}}^{(n)} \leq \Delta_{\text{max}}^{(n)}$ by definition, the rate distortion function for $\Delta_{\text{ave}}^{(n)}$ is no greater than that for $\Delta_{\text{max}}^{(n)}$. Next, to show the opposite inequality, given any protocol $(\mathcal{E}_n, \mathcal{D}_n)$, we construct a new protocol $(\tilde{\mathcal{E}}_n, \tilde{\mathcal{D}}_n)$ in which Alice applies a random cyclic permutation $\pi$ on $A_1, J_1, \cdots, A_n, J_n$ before $\mathcal{E}_n$, and Bob reverts $\pi$ after $\mathcal{D}_n$. (More precisely, $\pi = (1, \cdots, n)^k$ for a uniformly chosen $k \in \{0, \cdots, n - 1\}$.) Note that (i) the qubit and the entanglement rates are the same as in the original protocol, and (ii) the worst-case distortion for the new protocol is no more than the average-case distortion of the original protocol. The latter holds because the new protocol results in the state $\tilde{\xi}_n^{B^n X^n}$ whose marginals $\tilde{\xi}_n^{B_i X_i}$ for all $i$, where $\tilde{\xi}_n^{B^n X^n}$ is the resulting state by the original protocol, because in the new protocol, Alice and Bob just randomly reindex the subsystems while keeping the original encoding and decoding. Thus,

$$
\Delta_{\text{max}}^{(n)} (\xi_n^{B^n X^n}) = \max_i \Delta (\tilde{\xi}_n^{B_i X_i}) = \Delta \left( \frac{1}{n} \sum_{j=1}^n \tilde{\xi}_n^{B_j X_j} \right) \leq \frac{1}{n} \sum_{j=1}^n \Delta (\xi_n^{B_j X_j}) = \Delta_{\text{ave}}^{(n)} (\xi_n^{B^n X^n}),
$$

where the inequality follows from the convexity of the distortion $\Delta$ (Definition 15). Note that in the new protocol, Alice and Bob only need $\log_2 n$ bits of shared randomness, so only negligible amount of shared randomness is required asymptotically. In particular, when Alice and Bob use a part of their shared entanglement as the shared randomness, they can still achieve the same asymptotic rate region for the average-case distortion and the worst-case distortion without sacrificing the entanglement rate since the entanglement rate consumed for the randomness is $(\log_2 n)/n$.

• Our results raise the natural question of the computability (analytically or numerically) of our rate-distortion functions for simple examples. To see this, let us formalize the rate-distortion functions as optimization problems. For example, the entanglement-assisted rate-distortion function $\mathcal{R}_{\text{ea}}(D)$ can be obtained as the optimal value of the following problem:

$$
\begin{align*}
\text{minimize} & \quad \frac{1}{2} I(B : X X' R)_{\tau} \\
\text{subject to} & \quad \tau^{BXX'XR} = (\mathcal{N} \otimes I^{XX'XR}) \left( |\psi\rangle\langle \psi|^{AJXX'XR} \right) \\
& \quad \Delta \left( (\mathcal{N} \otimes I^X) \left( \rho_{\text{AJX}} \right) \right) \leq D \\
& \quad \mathcal{N} : A \text{ quantum channel from } AJ \text{ to } B.
\end{align*}
$$

This problem is convex because
1) $I(B : RXX')_{\tau}$ is convex with respect to $\mathcal{N}$ (Lemma 7),
2) the distortion function $\Delta$ is convex with respect to $\mathcal{N}$ (Definition 15), and
3) the set of quantum channels from $AJ$ to $B$ is convex.

Hence, we can use convex-optimization algorithms to evaluate this rate-distortion function numerically. Since $A, J,$ and $B$ are finite-dimensional, the domain of optimization is compact and the optimal value can be attained (note that a feasible solution is the identity channel from $A$ to $B$ and tracing out $J$).

On the other hand, however, in the unassisted case, it is hard to compute the rate-distortion function in general because the expression involves regularization. Moreover, even if we consider the expression with some finite $k$ instead of the limit of $k \to \infty$, the corresponding optimization problem is not convex since the entanglement of purification is neither convex nor concave [40].

• So far, we have focused on the simple pure state side information in our framework and derivations. The most general side information for an ensemble source can be written as $\{p_x, \rho_{xJ}^{AJ} \}_{x \in \Sigma}$, where $\rho_{xJ}^{AJ}$ is an extension of $\rho_x^{AJ}$ ($\operatorname{Tr}_J \rho_x^{AJ} = \rho_x^{AJ}$). In other words, the only restriction for side information is that, removing it returns the original state. Note that $\rho_{xJ}^{AJ}$ can be entangled across $A$ and $J$, and the above definition includes the pure state side information as a special case. Theorems 16, 17, and 22, and results in Appendices A and B can be extended to the most general side information, if $R$ is revised to purify $AJ$ for each $x$. Our proofs for pure state side information then apply to the most general side information without change, because all the achieving protocols and distortion maps in the rate distortion functions operate on copies of $AJ$ without assumptions on the state on $AJ$. Detailed verifications are left to the interested readers.

• Our extension of quantum rate-distortion theory to mixed states ensembles also provides some conclusions that cannot be obtained from Refs. [16, 41]. One such result is proving the fact that the rate-distortion function is unaffected by the addition or removal of redundant parts to an ensemble. Here, we omit writing out the identity channel $I$ when considering a quantum channel acting on a subsystem of a given state. More specifically, let the distortion $\Delta$ be $1 - F$, and suppose we have a source

$$
\omega^{CNQJX} = \sum_x p_x \sum_{c} p_{c|x} |c\rangle\langle c|^{C} \otimes \omega_{c}^{N} \otimes \rho_{x}^{Q} \otimes |j_x\rangle\langle j_x|^{J} \otimes |x\rangle\langle x|^{X}.
$$

(36)
with purification
\[ |\omega\rangle_{CNQJXX'R_C R_N R_Q} := \sum_x \sqrt{p_x} \sum_{j,c} \sqrt{p_{c|x}} |c⟩^C |c⟩^C |\omega_c⟩^{N_R N} |\rho_{c|x}⟩^Q |J_{x'}⟩ |x⟩^X |x⟩^{X'}, \]
where \( R_C \) purifies \( C \), \( R_N \) purifies \( N \), \( R_Q \) purifies \( Q \), and \( X' \) purifies the distribution over \( X \). We prove that the entanglement-assisted rate-distortion function of \( \omega_{CNQX} \) is unchanged when we remove the redundant part \( N \).

**Theorem 24.** Consider the distortion \( \Delta = 1 - D \) and the source \( \omega_{CNQJX} \) with purification \( |\omega\rangle_{CNQJXX'R_C R_N R_Q} \) as defined above. Recall that the rate-distortion function for the entanglement-assisted rate-distortion function \( R_{ea}(D) \) is given by
\[ R_{ea}(D) = \min_{N(\omega|\omega)} \frac{1}{2} I(CQ : R_C R_N R_Q XX'), \]
where the minimum is taken over all quantum channels \( N : CNQJ \rightarrow CNQ \) with \( 1 - D (\omega_{CNQX}, N(\omega_{CNQJX})) \). The rate distortion function for this source can be simplified to
\[ R_{ea}(D) = \min_{\mathcal{M}(CQJ \rightarrow CQ)} \frac{1}{2} I(CQ : R_C R_N XX'), \]
where
\[ |\tilde{\omega}\rangle_{CQJXX'R_C R_N Q} := \sum_x \sqrt{p_x} \sum_{j,c} \sqrt{p_{c|x}} |c⟩^C |c⟩^C |\tilde{\omega}_c⟩^{N_R N} |\tilde{\rho}_{c|x}⟩^Q |J_{x'}⟩ |x⟩^X |x⟩^{X'}, \]
and the minimum is taken over all quantum channels \( \mathcal{M} : CQJ \rightarrow CQ \) with \( 1 - D (\omega_{CQX}, \mathcal{M}(\omega_{CQJX})) \). We provide a detailed proof in Appendix C.

A similar conclusion can be made in the unassisted case, with a detailed proof given in Appendix D.

**Theorem 25.** Consider the source \( \omega_{CNQJX} \) with purification \( |\omega\rangle_{CNQJXX'R_C R_N R_Q} \) as defined above. Recall that the unassisted rate-distortion function \( R(D) \) is given by
\[ R(D) = \lim_{k \rightarrow \infty} \frac{1}{k} \min_{N_{k}} E_p(C^k N^k Q^k : R_C^{k} R_N^{k} R_Q^{k} X^k Y^k)_{N_k}(|\omega|\omega^{\otimes k}), \]
where the minimum is taken over all quantum channels \( N_k : C^k N^k Q^k J^k \rightarrow C^k N^k Q^k \) with
\[ 1 - \frac{1}{k} \sum_{j=1}^{k} F \left( \omega_{C_j N_j Q_j X_j}, (N_{k}(|\omega|\omega^{\otimes k}))^{C_j N_j Q_j X_j} \right) \leq D. \]
Then, the rate distortion function for this source can be simplified to
\[ R(D) = \lim_{k \rightarrow \infty} \frac{1}{k} \min_{M_{k}} E_p(C^k Q^k : R_C^{k} R_Q^{k} X^k Y^k)_{M_k}(|\tilde{\omega}|\tilde{\omega}^{\otimes k}), \]
where the minimum is taken over all quantum channels \( M_k : C^k Q^k J^k \rightarrow C^k Q^k \) with
\[ 1 - \frac{1}{k} \sum_{j=1}^{k} F \left( \omega_{C_j Q_j X_j}, (M_{k}(|\tilde{\omega}|\tilde{\omega}^{\otimes k}))^{C_j Q_j X_j} \right) \leq D. \]

The above discussion relies on the structure of the ensemble, so it is more difficult to consider the most general side information. It turns out that the above results concerning the addition and removal of redundant parts in the KI decomposition indeed still apply to general side information that entangles only with the quantum part \( Q \). More specifically, suppose that we have a source
\[ \omega_{CNQJX} := \sum_x p_x \sum_c p_{c|x} |c⟩^C |c⟩^C |\omega_c⟩^{N_R N} |\rho_{c|x}⟩^Q |J_{x'}⟩ |x⟩^X |x⟩^{X'}, \]
with purification
\[ |\omega\rangle_{CNQJXX'R_C R_Q} := \sum_x \sqrt{p_x} \sum_{j,c} \sqrt{p_{c|x}} |c⟩^C |c⟩^C |\omega_c⟩^{N_R N} |\rho_{c|x}⟩^Q |J_{x'}⟩ |x⟩^X |x⟩^{X'}, \]
Then, the statements in Theorems 24 and 25 hold for this source as well. For example, the entanglement-assisted rate distortion function \( R_{ea}(D) \) with the fidelity distortion \( \Delta = 1 - F \) is given as
\[ R_{ea}(D) = \min_{\mathcal{M}(CQJ \rightarrow CQ)} \frac{1}{2} I(CQ : R_C R_Q XX'), \]
with
\[ |\tilde{\omega}\rangle_{CQJXX'R_C R_Q} := \sum_x \sqrt{p_x} \sum_{j,c} \sqrt{p_{c|x}} |c⟩^C |c⟩^C |\tilde{\omega}_c⟩^{N_R N} |\tilde{\rho}_{c|x}⟩^Q |J_{x'}⟩ |x⟩^X |x⟩^{X'}, \]
where the minimum is taken over all quantum channels $\mathcal{M} : CQJ \to CQ$ with $1 - F (\omega^{CQX}, \mathcal{M} (\omega^{CQJX})) \leq D$. The proof is almost the same as that of Theorem 24, for the channel $\mathcal{M}^{CQJ\to CQ}$ in Theorem 24 can take any state on $CQJ$, in particular, even if the state on $QJ$ is entangled. The unassisted rate distortion function can also be characterized similarly.

We end the paper with several interesting future directions. In Theorem 21, we proved a relation between Koashi and Imoto’s optimal blind compression rate and our rate-distortion function. On the other hand, since the optimal visible compression rate was analyzed under global error criterion, there is no direct connection between the optimal rate and our rate-distortion function in the visible case. In addition, it would be interesting to see if our rate-distortion functions are additive or not. Some of these rate-distortion functions are expressed using multi-letter formulae, and it would be highly useful if there are simplified characterizations. In this paper, we consider the average (and worst-case) local error, for which we obtained continuous rate distortion functions in all the settings considered, so, strong converse does not hold. For mixed state compression under global error, significant progress in obtaining a strong converse has been reported in the unassisted case [26], and results in [13] provide a promising direction for the assisted case. Further research in these, and also in the general $(Q, E)$ rate region are important open problems to be tackled. More recently, an alternative approach to rate-distortion theory has been proposed in [4]. Their framework differs significantly from most of the existing work: a distortion map is applied to the reference system, and it will be interesting to find potential connections between their framework and ours. Finally, we believe that our approach taken in this paper can be applied to other setups that involves data processing of mixed states. It would be highly important to see what type of trade-off can be seen in other quantum information processing setups.

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**REFERENCES**

[1] A. Abeyesinghe, P. Hayden, G. Smith, and A.J. Winter. “Optimal Superdense Coding of Entangled States”. *IEEE Transactions on Information Theory* 52.8 (2006), pp. 3635–3641.

[2] Anurag Anshu, Debbie Leung, and Dave Touchette. “Incompressibility of Classical Distributions”. *IEEE Transactions on Information Theory* 68.3 (2022), pp. 1758–1771.

[3] H. Araki and E.H. Lieb. “Entropy inequalities”. *Commun. Math. Phys.* 18 (1970), 160–170.

[4] Touheed Anwar Atif, Mohammad Aamir Sohail, and S. Sandeep Pradhan. *Lossy Quantum Source Coding with a Global Error Criterion based on a Posterior Reference Map*. 2023. arXiv: 2302.00625 [quant-ph].

[5] Z. B. Khanian. “From Quantum Source Compression to Quantum Thermodynamics”. arXiv:2012.14143 [quant-ph]. PhD Thesis. Spain: Universitat Autònoma de Barcelona, Department of Physics, 2020.

[6] Zahra Baghali Khanian and Andreas Winter. “Distributed Compression of Correlated Classical-Quantum Sources or: The Price of Ignorance”. *IEEE Transactions on Information Theory* 66.9 (2020), pp. 5620–5633.

[7] Zahra Baghali Khanian and Andreas Winter. “General Mixed-State Quantum Data Compression With and Without Entanglement Assistance”. *IEEE Transactions on Information Theory* 68.5 (2022), pp. 3130–3138.

[8] Howard Barnum. “Quantum rate-distortion coding”. *Phys. Rev. A* 62 (4 2000), p. 042309.

[9] Howard Barnum, Carlton M Caves, Christopher A Fuchs, Richard Jozsa, and Benjamin Schumacher. “On quantum coding for ensembles of mixed states”. *J. Phys. A* 34.35 (2001), pp. 6767–6785.

[10] Howard Barnum, Christopher A. Fuchs, Richard Jozsa, and Benjamin Schumacher. “General fidelity limit for quantum channels”. *Phys. Rev. A* 54 (6 1996), pp. 4707–4711.

[11] C.H. Bennett, P. Hayden, D.W. Leung, P.W. Shor, and A. Winter. “Remote preparation of quantum states”. *IEEE Transactions on Information Theory* 51.1 (2005), pp. 56–74.

[12] C.H. Bennett, P.W. Shor, J.A. Smolin, and A.V. Thapliyal. “Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem”. *IEEE Transactions on Information Theory* 48.10 (2002), pp. 2637–2655.

[13] Charles H. Bennett, Igor Devetak, Aram W. Harrow, Peter W. Shor, and Andreas Winter. “The Quantum Reverse Shannon Theorem and Resource Tradeoffs for Simulating Quantum Channels”. *IEEE Transactions on Information Theory* 60.5 (2014), pp. 2926–2959.

[14] Charles H. Bennett, David P. DiVincenzo, Peter W. Shor, John A. Smolin, Barbara M. Terhal, and William K. Wootters. “Remote State Preparation”. *Phys. Rev. Lett.* 87 (7 2001), p. 077902.

[15] Toby Berger. “Rate Distortion Theory and Data Compression”. *Advances in Source Coding*. Vienna: Springer Vienna, 1975, pp. 1–39.
A. Convexity and continuity of the entanglement-assisted rate-distortion function.
Lemma 26. Let $f(D)$ be defined as in the beginning of the proof of Theorem 16. For $D_1, D_2 > 0$ and $\lambda \in [0,1]$, 
\[ f(\lambda D_1 + (1-\lambda)D_2) \leq \lambda f(D_1) + (1-\lambda)f(D_2). \]

Proof. For any state $\psi^{A_{1J}X'R}$, let $\mathcal{N}_1^{A_1J \rightarrow B}$ and $\mathcal{N}_2^{A_1J \rightarrow B}$ be quantum channels achieving 
\[ f(D_1) = \frac{1}{2}I(B : XX'R)_{(\mathcal{N}_1 \otimes I)(\psi)}, \]
\[ f(D_2) = \frac{1}{2}I(B : XX'R)_{(\mathcal{N}_2 \otimes I)(\psi)}, \]
with $\Delta(\text{Tr}_{X'R}(\mathcal{N}_i \otimes I)(\psi)) \leq D_i$ for $i = 1, 2$. Then, consider the channel $\lambda \mathcal{N}_1 + (1-\lambda)\mathcal{N}_2$. By convexity of $\Delta$, 
\[ \Delta(\text{Tr}_{X'R}(\lambda \mathcal{N}_1 + (1-\lambda)\mathcal{N}_2 \otimes I)(\psi)) \leq \lambda D_1 + (1-\lambda)D_2. \] (45)

We have the following chain of inequalities 
\[ f(\lambda D_1 + (1-\lambda)D_2) \leq \frac{1}{2}I(B : XX'R)_{(\lambda \mathcal{N}_1 + (1-\lambda)\mathcal{N}_2 \otimes I)(\psi)} \]
\[ \leq \lambda \frac{1}{2}I(B : XX'R)_{(\mathcal{N}_1 \otimes I)(\psi)} + (1-\lambda)\frac{1}{2}I(B : XX'R)_{(\mathcal{N}_2 \otimes I)(\psi)} \]
\[ = \lambda f(D_1) + (1-\lambda)f(D_2), \]
where the first inequality comes from Eq. (45) and the minimizing definition of $f(D)$, and the second inequality follows from Lemma 7.

This lemma implies the continuity of $f(D)$. Using this lemma and Theorem 16, $\mathcal{A}_{\text{Sa}}(D)$ is convex and continuous.

B. Properties of the unassisted rate-distortion function

Lemma 27. Consider the functions $g(D), g_k(D)$ defined at the beginning of the proof of Theorem 17 and $D > 0$. Then, (1) $g(D)$ is continuous, (2) $g_k(D)$ converges to $g(D)$ uniformly, and (3) for sufficiently large $k$, $g_k(D)$ is continuous.

Proof. We first observe and sketch the derivations of some useful properties of $g_k(D)$. Next, we show that for each $D$, $g_k(D)$ converges, so $g(D)$ is well-defined. Then we establish some additional properties of $g(D)$.

Since the distortion $\Delta$ is convex and bounded (see the discussion after Definition 15), the domain for each $g_k(D)$ is an interval $[0, D_{\max}]$. Furthermore, each $g_k(D)$ is bounded, and decreasing with $D$, with $g_k(D) = 0$ in some interval $[D_k, D_{\max}]$ for some $D_k \geq 0$. Consider a fixed $D$ and the sequence $\{g_k(D)\}_k$. Note that the sequence may not be decreasing (with $k$). Instead, we have the property that for any natural number $l$, $g_{k_l}(D) \leq g_k(D)$; to see this, let $\mathcal{N}_k^{A_k \rightarrow J \rightarrow B_k} \otimes I$ be a minimizing map for $g_k$, then $(\mathcal{N}_k^{A_k \rightarrow J \rightarrow B_k} \otimes I)^{\otimes l}$ is a feasible solution for the minimization for $g_{k_{l}}$ because both types of local error criteria are preserved, and because the entanglement of purification is subadditive. Similarly, $g_{k+r}(D) \leq \frac{k}{l} g_k(D) + \frac{r}{l} g_{k+r}(D)$. In particular, if we compare $g_{k+r}(D)$ with $g_{k}(D)$ for some $r \in \{1, 2, \ldots, k-1\}$, $g_{k+r}(D)$ cannot be larger than $g_k(D)$ by a constant divided by $l$. The above properties imply that $\{g_k(D)\}_k$ converges. To see this, first there are converging subsequences of $\{g_k(D)\}_k$ because the range is compact. Second if the limsup is strictly larger than the liminf, say, by an amount $\alpha$, we can take $k$ with $g_k(D)$ no more than $\alpha/100$ from the liminf, and take some very large $l$ so that $g_{k+l}(D)$ is no less than $\alpha/100$ from the limsup, which gives a contradiction for very large $l$. So, $g(D)$ is well-defined. For each $k$, from the definition of $g_k(D)$, $g_k(D)$ is decreasing with $D$ so $g(D)$ is also a decreasing function of $D$.

Proof of (1): At the beginning of the appendix of Ref. [16], the authors proved the convexity of their rate distortion function $R^\ell(D)$ for pure state compression. Their ideas for the convexity proof apply also to our expression $g(D)$. However, since we do not have a bound for the domain for the minimization for $g(D)$, the convexity cannot be used to show continuity.

Here, we provide a direct proof for the continuity of $g(D)$, partly by adapting the ideas in Ref. [16]. Fix $\epsilon > 0$. Let $D_2$ be an interior point in $[0, D_{\max}]$, and we fix some $D_1 \in [0, D_{\max}]$ so that $D_1 < D_2$. For each $i = 1, 2$, let $k_i$ satisfy $0 \leq g_{k_i}(D_i) - g(D_i) \leq \epsilon$. Let $k = k_1k_2$ so that $0 \leq g_k(D) - g_{k_i}(D_i) \leq g_k(D_i) - g(D_i) \leq \epsilon$. To show that $g$ is continuous at $D_2$, from below, let $k' = k(l + 1)$ for some natural number $l$ to be chosen later, $D' = (D_1 + 1D_2)/(l + 1)$, and $\mathcal{N}_{i,k}$ be the optimal channel for $g_{k_i}(D_i)$. Then, on $k'$ copies, the channel $\mathcal{N}_{1,k} \otimes (\mathcal{N}_{2,k} \otimes I)^{\otimes l}$ achieves an average local error criterion no more than $D'$, and an entanglement of purification at most $k' g_{k_i}(D_i) + kl g_k(D_2)$, so, $g_{k(l+1)}(D') \leq g_{k_1}(D_1) + l g_k(D_2))/(l + 1)$. Putting the above together, 
\[ 0 \leq g(D') - g(D_2) \]
\[ \leq g_{k(l+1)}(D') - g(D_2) \]
\[ \leq (g_k(D_1) + l g_k(D_2))/(l + 1) - g(D_2) \]
= (g_k(D_1) - g_k(D_2))/(l + 1) + g_k(D_2) - g(D_2) \\
\leq (g_k(D_1) - g_k(D_2))/(l + 1) + \epsilon \\
\leq 2\epsilon

if we choose \( l \geq \lceil (g_k(D_1) - g_k(D_2))/\epsilon \rceil \). Finally, by monotonicity of \( g \), \( \forall D \in (D', D_2) \), \( 0 \leq g(D) - g(D_2) \leq 2\epsilon \).

To show that \( g \) is continuous at \( D_2 \) from above, keep the above notations and further define \( D_3 \) to satisfy \( D_2 = (D' + D_3)/2 \), and define \( k_3 \) so that \( g_k(D') - g(D') \leq \epsilon \) and \( g_k(D_3) - g(D_3) \leq \epsilon \). Then, \( g(D_2) \leq g_k(D_2) \leq (g_k(D') + g_k(D_3))/2 \leq (g(D') + g(D_3))/2 + \epsilon \leq (g(D_2) + g(D_3))/2 + 2\epsilon \). Rearranging the terms, \( g(D_2) - g(D_3) \leq 4\epsilon \), and by monotonicity, \( g(D_2) - g(D) \leq 4\epsilon \) for all \( D \in [D_2, D_3] \).

We do not know if \( g(D) \) is continuous at \( D = 0 \), but \( g_k(D) \) and \( g(D) \) are continuous at \( D = D_{\text{max}} \). For blind compression with trivial side information, \( g_k(D) \) and \( g(D) \) are continuous at \( D = 0 \).

**Proof of (2):** Fix any \( \epsilon > 0 \). Since \( g(D) \) is decreasing and continuous, we can find \( 0 < D_1 \leq D_2 \leq \cdots \leq D_m \) so that \( g(D_i) - g(D_{i+1}) \leq \epsilon \) and \( g(D_m) = 0 \). Let \( k_i \) be such that \( g_k(D_i) - g(D_i) \leq \epsilon \). Let \( k = k_1 k_2 \cdots k_m \). Then, \( g_k(D_i) - g(D_i) \leq \epsilon \) for all \( i \). Then, for each \( D \in [D_i, D_{i+1}] \),

\[
0 \leq g_k(D) - g(D) \leq g_k(D_i) - g(D_{i+1}) \leq g(D_i) + \epsilon - g(D_{i+1}) \leq 2\epsilon,
\]

where the first inequality comes from the monotonicity of \( g \) and \( g_k \), and the second inequality comes from the choice of \( k \), and the last inequality comes from the choice of the \( D_i \)’s.

**Proof of (3):** Fix any \( \epsilon > 0 \). Using uniform convergence proved in part (2), there exists a sufficiently large \( k \) so that for all \( D, g_k(D) - g(D) \leq \epsilon \). Let \( D_2 \) be any interior point of \([0, D_{\text{max}}]\). By (1) let \( D_1 < D_2 < D_3 \) satisfy \( g(D_i) - g(D_{i+1}) \leq \epsilon \) for \( i = 1, 2 \). Then, \( g_k(D_i) - g_k(D_{i+1}) \leq g(D_i) + \epsilon - g(D_{i+1}) \leq 2\epsilon \). So, \( g_k \) is continuous.

Together with Theorem 17, the above establishes that \( \mathcal{R}(D) \) is convex and continuous in \( D \) for \( D > 0 \).

C. Proof of Theorem 24

In the proof, we assume all definitions and notations defined in Theorem 24. We first prove the following lemma.

**Lemma 28.** Consider the setup stated in Theorem 24. Let \( \mathcal{N}_X : \mathcal{CNQJ} \to \mathcal{CNQ} \) be a channel achieving the minimum of Eq. (38) for distortion \( D \). Let \( \mathcal{P}^C : \mathcal{C} \to \mathcal{C} \) be a channel corresponding to the projective measurement on \( C \) with respect to the basis \( \{ |c\rangle \}_{c} \); that is, \( \mathcal{P}^C(\xi) = \sum_c \langle c| \xi \rangle \langle c | \rangle^C \). Then, \( \mathcal{N} \circ \mathcal{P}^C \) also achieves the minimum of Eq. (38).

**Proof.** We first prove the feasibility and then the optimality of \( \mathcal{N} \circ \mathcal{P}^C \) in the minimization in Eq. (38). Feasibility means that \( \mathcal{N} \circ \mathcal{P}^C \) satisfies the distortion condition \( 1 - F(\omega_{\mathcal{CNQJ}X}, (\mathcal{N} \circ \mathcal{P}^C)(\omega_{\mathcal{CNQJ}X})) \leq D \). Observe from the form of \( \omega \) in Eq. (36) that

\[
(\mathcal{P}^C \otimes \mathcal{I}_{\mathcal{NQJ}X}) (\omega_{\mathcal{CNQJ}X}) = \omega_{\mathcal{CNQJ}X} \tag{46}
\]

therefore,

\[
F(\omega_{\mathcal{CNQJ}X}, (\mathcal{N} \circ \mathcal{P}^C)(\omega_{\mathcal{CNQJ}X})) = F(\omega_{\mathcal{CNQJ}X}, \mathcal{N}(\omega_{\mathcal{CNQJ}X})) \geq 1 - D, \tag{47}
\]

where the inequality holds because \( \mathcal{N} \) satisfies the distortion condition. Hence, \( \mathcal{N} \circ \mathcal{P}^C \) is feasible.

For the optimality of \( \mathcal{N} \circ \mathcal{P}^C \) in the minimization in Eq. (38), using the form of \(|\omega\rangle \) in Eq. (37), we have \( \mathcal{N} \circ \mathcal{P}^C(|\omega\rangle) = \mathcal{P}^R \circ \mathcal{N}(|\omega\rangle) \), where \( \mathcal{P}^R \) is the same channel as \( \mathcal{P}^C \) now defined on system \( R \). Therefore,

\[
\frac{1}{2} I(CNQ : R_C R_N R_Q XX')_{(N \circ \mathcal{P}^C)(|\omega\rangle)} = \frac{1}{2} I(CNQ : R_C R_N R_Q XX')_{(\mathcal{P}^R \circ \mathcal{N})(|\omega\rangle)} \tag{48}
\]

and the RHS is upper-bounded by \( \frac{1}{2} I(CNQ : R_C R_N R_Q XX')_{M(|\omega\rangle)} \) by the data processing inequality. Since \( \mathcal{N} \) achieves the minimum, \( \mathcal{N} \circ \mathcal{P}^C \) also achieves the minimum, which shows the optimality.

We now proceed to prove Theorem 24.

**Proof of Theorem 24.** Recall that we may remove and attach redundant system \( N \) from \( \omega_{\mathcal{CNQJ}X} \) by using quantum channels \( K_{\text{off}} \) and \( K_{\text{on}} \).

We first show

\[
\min_{\mathcal{N} \subset \mathcal{CNQJ} \subset \mathcal{CNQ}} \frac{1}{2} I(CNQ : R_C R_N R_Q XX')_{\mathcal{N}(|\omega\rangle)} \leq \min_{\mathcal{M} \subset \mathcal{CNQ}} \frac{1}{2} I(CQ : R_C R_Q XX')_{\mathcal{M}(|\omega\rangle)}. \tag{49}
\]
where the minimum on the LHS is subject to the constraint $1 - F(ω^{CNQX},N(ω^{CNQJX})) \leq D$, and the minimum on the RHS is subject to the constraint $1 - F(ω^{CNQX},M(ω^{CQJX})) \leq D$. Let $M$ be a quantum channel achieving the minimization on the RHS. By Lemma 28, $M \circ P^C$ also achieves the minimum. To show inequality (49), we will show that

1) $K_{\text{on}} \circ M \circ K_{\text{off}}$ is feasible for the LHS, and
2) $\frac{1}{2} I(CNQ : RCR_N RQXX')_{(K_{\text{on}} \circ M \circ K_{\text{off}})(|ω⟩|ω')} \leq \frac{1}{2} I(CQ : RCR_QXX')_{(M \circ P^C)(|ω⟩|ω')}.

The feasibility in step 1 follows from

$$F(ω^{CNQX},(K_{\text{on}} \circ M \circ K_{\text{off}})(ω^{CNQX})) = F(K_{\text{on}}(ω^{CNQX}),(K_{\text{on}} \circ M)(ω^{CQX})) \geq F(ω^{QX},M(ω^{CQX})) \geq 1 - D.$$ (50)

In the above, the first and second lines follow from $K_{\text{off}}(ω^{CNQX}) = ω^{CQX}$ and $K_{\text{on}}(ω^{CQX}) = ω^{CNQX}$ respectively. The third line is due to the monotonicity of fidelity, and the last line is due to the feasibility of $M$ for the distortion $D$. For the inequality $\frac{1}{2} I(CNQ : RCR_N RQXX')_{(K_{\text{on}} \circ M \circ K_{\text{off}})(|ω⟩|ω')} \leq \frac{1}{2} I(CQ : RCR_QXX')_{(M \circ P^C)(|ω⟩|ω')}$ in step 2, we start with the LHS and apply the data processing inequality to obtain

$$\frac{1}{2} I(CNQ : RCR_N RQXX')_{(K_{\text{on}} \circ M \circ K_{\text{off}})(|ω⟩|ω')} \leq \frac{1}{2} I(CQ : RCR_QXX')_{(M \circ P^C)(|ω⟩|ω')}.$$ (51)

Observe that

$$K_{\text{off}}(|ω⟩|ω) = \sum_{x,x'} \sum_{\mathcal{C}} \sqrt{p_x p_{x'} p_c} p_{x|x'}|c⟩⟨x|c⟩^{RC} \otimes |ρ_{cx}⟩⟨ρ_{cx}|^{QRQ} \otimes |j_{x}⟩⟨j_{x'}|^J \otimes |x⟩⟨x'|^X \otimes |x⟩⟨x'|^X'.$$ (52)

By applying $M \circ K_{\text{off}}$ to $|ω⟩|ω'$, the resulting state is

$$\sum_{x,x'} \sum_{\mathcal{C}} \sqrt{p_x p_{x'} p_c} p_{x|x'}|c⟩⟨x|c⟩^{RC} \otimes |ρ_{cx}⟩⟨ρ_{cx}|^{QRQ} \otimes |j_{x}⟩⟨j_{x'}|^J \otimes |x⟩⟨x'|^X \otimes |x⟩⟨x'|^X'.$$ (53)

In Eq. (53), the state on $CQR_N RQXX'$ is a product state between $CQR_N XX'$ and $R_N$ when conditioned on $RC$; in particular, for each $c$, the state on $CQR_N RQXX'$ is

$$\left(\sum_{x,x'} \sqrt{p_x p_{x'} p_c} p_{x|x'}|c⟩⟨x|c⟩^{RC} \otimes |ρ_{cx}⟩⟨ρ_{cx}|^{QRQ} \otimes |j_{x}⟩⟨j_{x'}|^J \otimes |x⟩⟨x'|^X \otimes |x⟩⟨x'|^X'\right) \otimes ω^{R_N}.$$ (54)

Hence,

$$I(CQ : RCR_N RQXX')_{(M \circ K_{\text{off}})(|ω⟩|ω')} = I(CQ : RCR_QXX')_{(M \circ P^C)(|ω⟩|ω')}.$$ (55)

From Eq. (53), when we trace out $R_N$ from $(M \circ K_{\text{off}})(|ω⟩|ω')$, the resulting state is

$$\sum_{x,x'} \sum_{\mathcal{C}} \sqrt{p_x p_{x'} p_c} p_{x|x'}|c⟩⟨x|c⟩^{RC} \otimes |ρ_{cx}⟩⟨ρ_{cx}|^{QRQ} \otimes |j_{x}⟩⟨j_{x'}|^J \otimes |x⟩⟨x'|^X \otimes |x⟩⟨x'|^X',$$ (56)

which is the same state as $(M \circ P^C)(|ω⟩|ω')$. Therefore, $Tr_{R_N}[(M \circ K_{\text{off}})(|ω⟩|ω')] = (M \circ P^C)(|ω⟩|ω')$, and thus

$$I(CQ : RCR_QXX')_{(M \circ K_{\text{off}})(|ω⟩|ω')} = I(CQ : RCR_QXX')_{(M \circ P^C)(|ω⟩|ω')}.$$ (57)

From Eqs. (51), (55), and (57),

$$\frac{1}{2} I(CNQ : RCR_N RQXX')_{(K_{\text{on}} \circ M \circ K_{\text{off}})(|ω⟩|ω')} \leq \frac{1}{2} I(CQ : RCR_QXX')_{(M \circ P^C)(|ω⟩|ω')}.$$ (58)

as claimed in step 2.

Next, we show the inequality opposite to (49); that is,

$$\min_{N | C \rightarrow Q \rightarrow CNQ} \frac{1}{2} I(CNQ : RCR_N RQXX')_{N(|ω⟩|ω')} \geq \min_{M | Q \rightarrow C} \frac{1}{2} I(CQ : RCR_QXX')_{M(|ω⟩|ω')}.$$ (59)

where the minimum on the LHS is subject to the constraint $1 - F(ω^{CNQX},N(ω^{CNQJX})) \leq D$, and the minimum on the RHS is subject to the constraint $1 - F(ω^{CQX},M(ω^{CQJX})) \leq D$. Let $N$ be a quantum channel achieving the minimum of LHS. By Lemma 28, $N \circ P^C$ also achieves the minimum. To show Eq. (59), we will show that

1) $K_{\text{off}} \circ N \circ K_{\text{off}}$ is feasible for RHS, and
2) $\frac{1}{2} I(CNQ : RCR_N RQXX')_{(N \circ P^C)(|ω⟩|ω')} \geq \frac{1}{2} I(CQ : RCR_QXX')_{(K_{\text{off}} \circ N \circ K_{\text{off}})(|ω⟩|ω')}$. 


The feasibility can be shown as follows.

\[
F(\omega^{\text{CQX}}, (\mathcal{K}_{\text{off}} \circ \mathcal{N} \circ \mathcal{K}_{\text{con}})(\omega^{\text{CQX}})) = F(\omega^{\text{CQX}}, (\mathcal{K}_{\text{off}} \circ \mathcal{N})(\omega^{\text{CNQX}})) \\
= F(\omega^{\text{CNQX}}, (\mathcal{K}_{\text{off}} \circ \mathcal{N})(\omega^{\text{CNQX}})) \\
\geq F(\omega^{\text{CNQX}}, \mathcal{N}(\omega^{\text{CNQX}})) \\
\geq 1 - D.
\]

The first line follows because \(\mathcal{K}_{\text{con}}(\omega^{\text{CQX}}) = \omega^{\text{CNQX}}\). The second line follows because \(\mathcal{K}_{\text{off}}(\omega^{\text{CNQX}}) = \omega^{\text{CQX}}\). The third line is monotonicity of fidelity. The fourth line follows because \(\mathcal{N}\) is feasible for distortion \(D\). We then show the inequality in the second step: \(\frac{1}{2} I(CQ : RC_RN RQ XX')(\mathcal{N} \circ \mathcal{PC})(|\omega| |\omega|) \geq \frac{1}{2} I(CQ : RC_RQ XX')(\mathcal{K}_{\text{off}} \circ \mathcal{N} \circ \mathcal{K}_{\text{con}})(|\omega| |\omega|).\) By data processing inequality,

\[
\frac{1}{2} I(CQ : RC_RQ XX')(\mathcal{K}_{\text{off}} \circ \mathcal{N} \circ \mathcal{K}_{\text{con}})(|\omega| |\omega|) \leq \frac{1}{2} I(CQ : RC_RQ XX')(\mathcal{N} \circ \mathcal{PC})(|\omega| |\omega|). \tag{61}
\]

Observe that

\[
\mathcal{K}_{\text{con}}(|\omega| |\omega|) = \sum_{x, x^{'}} \sum_{c} \sqrt{p_x p_{x'}} p_{c|x} p_{c|x'} |c⟩⟨c| (|c⟩⟨c| \otimes |\omega_c⟩⟨\omega_c|) (\rho_{cx} \otimes \rho_{cx'})^{QRQ} \otimes |j_x⟩⟨j_x| ^J \otimes |x⟩⟨x| ^X \otimes |x⟩⟨x'| ^{X'} \tag{62}
\]

By applying \(\mathcal{N} \circ \mathcal{K}_{\text{con}}\) to \(|\omega⟩\langle\omega|\), the resulting state is

\[
\frac{1}{2} I(CQ : RC_RQ XX')(\mathcal{K}_{\text{off}} \circ \mathcal{N} \circ \mathcal{K}_{\text{con}})(|\omega| |\omega|) \leq \frac{1}{2} I(CQ : RC_RN RQ XX')(\mathcal{N} \circ \mathcal{PC})(|\omega| |\omega|). \tag{65}
\]

From Eqs. (61) and (65),

\[
\frac{1}{2} I(CQ : RC_RQ XX')(\mathcal{K}_{\text{off}} \circ \mathcal{N} \circ \mathcal{K}_{\text{con}})(|\omega| |\omega|) \leq \frac{1}{2} I(CQ : RC_RN RQ XX')(\mathcal{N} \circ \mathcal{PC})(|\omega| |\omega|), \tag{66}
\]

which proves the inequality in step 2.

\section{D. The proof for Theorem 25}

We retain all the notations and definitions leading up to Theorem 25. We will use the following, which follows similarly as in Lemma 28.

\textbf{Lemma 29.} Consider the setup described in Theorem 25. Let \(k\) be a fixed positive integer, and let \(N_k : C^k N^k Q^k J^k \rightarrow C^k N^k Q^k\) be a channel achieving the minimum of

\[
\min_{N_k} E_p(C^k N^k Q^k : R^k R^k R^k Q^k X^k X^k (N_k(\omega)|\omega|^{\otimes k})) \tag{67}
\]

where the minimum is taken over all quantum channels \(N_k : C^k N^k Q^k J^k \rightarrow C^k N^k Q^k\) with

\[
1 - \frac{1}{k} \sum_{j=1}^{k} F(\omega^{C^k N^k Q^k J^k}, (N_k(\omega)|\omega|^{\otimes k})) \leq D. \tag{68}
\]

Let \(\mathcal{PC} : C \rightarrow C\) be the projective measurement on \(C\) as defined in Lemma 28. Then, \(N_k \circ (\mathcal{PC})^{\otimes k}\) also achieves the minimum of Eq. (67).

\textbf{Proof Sketch.} We first show the feasibility and then show the optimality. Similar to the proof of Lemma 28, the feasibility of \(N_k \circ (\mathcal{PC})^{\otimes k}\) follows from

\[
(\mathcal{PC} \otimes \mathcal{I}^{NQJX})(\omega^{CNQJX}) = \omega^{CNQJX} \tag{69}
\]

and the feasibility of \(N_k\). The optimality also follows similarly because

\[
(N_k \circ (\mathcal{PC})^{\otimes k})(|\omega| |\omega|^{\otimes k}) = (\mathcal{PC})^{\otimes k} \circ N_k(\omega |\omega|^{\otimes k}) \tag{70}
\]
with the measurement $P^{RC}$ acting on $R_C$, as defined in Lemma 28. Indeed, we have
\[
E_p(C^k N^k Q^k : R_C^k R_N^k R_Q^k X^k X'^k)_{(\mathcal{M}_k \circ (\mathcal{P}^C)^{\otimes k})(|\omega\rangle\langle\omega|^{\otimes k})} = E_p(C^k N^k Q^k : R_C^k R_N^k R_Q^k X^k X'^k)_{(\mathcal{P}^R)^{\otimes k} \circ \mathcal{N}_k}(|\omega\rangle\langle\omega|^{\otimes k})
\]
\[
\leq E_p(C^k N^k Q^k : R_C^k R_N^k R_Q^k X^k X'^k)_{\mathcal{M}_k}(|\omega\rangle\langle\omega|^{\otimes k}),
\]
where the inequality follows from monotonicity of entanglement of purification.

**Proof Sketch of Theorem 25.** It suffices to show that for each $k$,
\[
\min_{\mathcal{N}_k} E_p(C^k N^k Q^k : R_C^k R_N^k R_Q^k X^k X'^k)_{\mathcal{N}_k}(|\omega\rangle\langle\omega|^{\otimes k}) = \min_{\mathcal{M}_k} E_p(C^k N^k Q^k : R_C^k R_N^k R_Q^k X^k X'^k)_{\mathcal{M}_k}(|\omega\rangle\langle\omega|^{\otimes k}),
\]
where the minimum on the LHS is taken over all quantum channels $\mathcal{N}_k : C^k N^k Q^k, J^k \rightarrow C^k N^k Q^k$ with
\[
1 - \frac{1}{k} \sum_{j=1}^k F\left(\omega^{\mathcal{C}_{jN_jQ_jX_j}}, (\mathcal{N}_k (|\omega\rangle\langle\omega|^{\otimes k}))^{\mathcal{C}_{jN_jQ_jX_j}\mathcal{M}_k}\right) \leq D,
\]
and the minimum on the RHS is taken over all quantum channels $\mathcal{M}_k : C^k Q^k, J^k \rightarrow C^k Q^k$ with
\[
1 - \frac{1}{k} \sum_{j=1}^k F\left(\omega^{\mathcal{C}_{jQ_jX_j}}, (\mathcal{M}_k (|\omega\rangle\langle\omega|^{\otimes k}))^{\mathcal{C}_{jQ_jX_j}\mathcal{M}_k}\right) \leq D.
\]

We first show “≤” in the above, starting from the RHS, with $\mathcal{M}_k$ achieving the minimum. By Lemma 29, $\mathcal{M}_k \circ (\mathcal{P}^C)^{\otimes k}$ also achieves the minimum. We will show that
1) $\mathcal{K}_{\text{con}}^{\otimes k} \circ \mathcal{M}_k \circ \mathcal{K}_{\text{off}}^{\otimes k}$ is feasible for the LHS, and
2) $E_p(C^k N^k Q^k : R_C^k R_N^k R_Q^k X^k X'^k)_{(\mathcal{K}_{\text{con}}^{\otimes k} \circ \mathcal{M}_k \circ \mathcal{K}_{\text{off}}^{\otimes k})(|\omega\rangle\langle\omega|^{\otimes k})} \leq E_p(C^k Q^k : R_C^k R_Q^k X^k X'^k)_{(\mathcal{M}_k \circ (\mathcal{P}^C)^{\otimes k})(|\omega\rangle\langle\omega|^{\otimes k})}.$

The feasibility in the first step can be shown by using the relation
\[
\mathcal{K}_{\text{off}}(\omega^{C^k Q^k X^k}) = \omega^{C^k Q^k X^k}, \quad \mathcal{K}_{\text{con}}(\omega^{C^k Q^k X^k}) = \omega^{C^k Q^k X^k},
\]
monotonicity of fidelity, and feasibility of $\mathcal{M}_k$. (Note that the distortion condition is now given with respect to the average-case local error criterion. In the proof for feasibility, we use the fact that for a quantum state $\rho^{A_1A_2}$ on joint system $A_1, A_2$ and a quantum channel $\mathcal{N} : A_2 \rightarrow A_3$, $Tr_{A_2}[\rho^{A_1A_2}] = Tr_{A_3}[\mathcal{N}(\rho^{A_1A_2})]$.) For the second step, for simplicity, we consider $k = 1$; that is, we will show $E_p(CNQ : R_C R_N R_Q X X')_{(\mathcal{K}_{\text{con}} \circ \mathcal{M}_1 \circ \mathcal{K}_{\text{off}})(|\omega\rangle\langle\omega|)} \leq E_p(CQ : R_C R_Q X X')_{(\mathcal{M}_1 \circ \mathcal{P}^C)(|\omega\rangle\langle\omega|)}.$ The proof for general $k$ is similar. By monotonicity of entanglement of purification,
\[
E_p(CNQ : R_C R_N R_Q X X')_{(\mathcal{K}_{\text{con}} \circ \mathcal{M}_1 \circ \math{K}_{\text{off}})(|\omega\rangle\langle\omega|)} \leq E_p(CQ : R_C R_Q X X')_{(\mathcal{M}_1 \circ \mathcal{K}_{\text{off}})(|\omega\rangle\langle\omega|)}.
\]

We would like to show
\[
E_p(CQ : R_C R_N R_Q X X')_{(\mathcal{M}_1 \circ \mathcal{K}_{\text{off}})(|\omega\rangle\langle\omega|)} \leq E_p(CQ : R_C R_Q X X')_{(\mathcal{M}_1 \circ \mathcal{P}^C)(|\omega\rangle\langle\omega|)}.
\]
The state $(\mathcal{M}_1 \circ \mathcal{P}^C)(|\omega\rangle\langle\omega|)$ can be purified as
\[
|\tilde{\psi} : \sum_x \sqrt{p_x} \sum_c \sqrt{p_{c|x}} |\mu_{cx}]^{\mathcal{C}QWR^C} |c\rangle^{R_C} |c\rangle^{R_C} |x]\langle x| X',
\]
where
\[
|\mu_{cx}]^{\mathcal{C}QWR^C} = U_{\mathcal{M}_1}(|c]\langle c|^{R_Q} |j_x\rangle^J)
\]
with Stinespring dilation $U_{\mathcal{M}_1} : CJQ \rightarrow CWQ$ of $\mathcal{M}_1$. From Theorem 11, recall that
\[
E_p(CQ : R_C R_Q X X')_{(\mathcal{M}_1 \circ \mathcal{P}^C)(|\omega\rangle\langle\omega|)} = \min_{\Lambda : WR^C \rightarrow F} S(CQF)_{\Lambda(|\tilde{\psi}\rangle\langle\tilde{\psi}|)}.
\]
Let $\tilde{\Lambda} : WR' \rightarrow F$ be a quantum channel achieving the minimum in the RHS above; that is,
\[
E_p(CQ : R_C R_Q X X')_{(\mathcal{M}_1 \circ \mathcal{P}^C)(|\omega\rangle\langle\omega|)} = S(CQF)_{\tilde{\Lambda}(|\tilde{\psi}\rangle\langle\tilde{\psi}|)}.
\]
Here, from Eq. (78), the marginal state of $\tilde{\Lambda}(|\tilde{\psi}\rangle\langle\tilde{\psi}|)$ on system $CWQ$ is
\[
\tilde{\Lambda}(|\tilde{\psi}\rangle\langle\tilde{\psi}|)_{\mathcal{C}QF} = \sum_x \sum_c \sqrt{p_x} \sum_c \sqrt{p_{c|x}} \tilde{\Lambda}(\mu_{cx}]^{\mathcal{C}QWR^C} \otimes |c\rangle|c\rangle^{R_C} X'.
\]
On the other hand, the state $(\mathcal{M}_1 \circ \mathcal{K}_{\text{off}})(|\omega\rangle\langle\omega|)$ can be purified as
\[
|\psi : \sum_x \sqrt{p_x} \sum_c \sqrt{p_{c|x}} |\mu_{cx}]^{\mathcal{C}QWR^C} |c\rangle^{R_C} |c\rangle^{R_C} |x]\langle x| X'.
\]
Recall also that
\[ E_p(CQ : R_C R_N R_Q X X') (\text{Tr}_{M_1} \circ \text{K}_{	ext{off}})(|\omega\rangle|\omega\rangle) = \min_{\Lambda : N W R_C \rightarrow F} S(CQF)_{\Lambda(|\nu\rangle|\nu\rangle)}. \] (84)

By choosing \((\tilde{\Lambda} \circ \text{Tr}_N)\) as a feasible channel for the minimization in the RHS,
\[ E_p(CQ : R_C R_N R_Q X X') (\text{Tr}_{M_1} \circ \text{K}_{\text{off}})(|\omega\rangle|\omega\rangle) \leq S(CQF)_{(\tilde{\Lambda} \circ \text{Tr}_N)(|\nu\rangle|\nu\rangle)}. \] (85)

Observing that the marginal state of \((\tilde{\Lambda} \circ \text{Tr}_N)(|\nu\rangle|\nu\rangle)\) on system \(CQ\) coincides with \(\tilde{\Lambda}(|\varphi\rangle|\varphi\rangle)\) in Eq. (82),
\[ S(CQF)_{(\tilde{\Lambda} \circ \text{Tr}_N)(|\nu\rangle|\nu\rangle)} = S(CQF)_{\tilde{\Lambda}(|\varphi\rangle|\varphi\rangle)}. \] (86)

By Eqs. (81), (85), and (86), we have Eq. (77).

Now, we show the opposite inequality “\(\geq\)" in (72) with all the associated constraints. Let \(C_k\) be a quantum channel achieving the minimum of LHS. By Lemma 29, \(C_k \circ (P C) \otimes k\) also achieves the minimum. We will show that
1) \((K_{\text{off}}) \otimes k \circ C_k \circ (K_{\text{on}}) \otimes k\) is feasible for RHS, and
2) \(E_p(C^k Q^k : R^k C^k R^k N^k Q^k X^k X^k) (K_{\text{off}} \circ C_k \circ K_{\text{on}})(|\tilde{\omega}\rangle|\tilde{\omega}\rangle) \leq E_p(C^k N^k Q^k : R^k C^k R^k N^k Q^k X^k X^k) (C_k \circ (P C) \otimes k)(|\omega\rangle|\omega\rangle).\)

The feasibility in step 1 can be shown by using the relation
\[ K_{\text{off}}(\omega^{C N Q X}) = \omega^{C Q X}, \quad K_{\text{on}}(\omega^{C Q X}) = \omega^{C N Q X}, \] (87)
the monotonicity of fidelity, and the feasibility of \(C_k\). For the inequality in the second step, we apply the monotonicity of the entanglement of purification to obtain
\[ E_p(C^k Q^k : R^k C^k R^k N^k Q^k X^k X^k) (K_{\text{off}} \circ C_k \circ K_{\text{on}})(|\tilde{\omega}\rangle|\tilde{\omega}\rangle) \leq E_p(C^k N^k Q^k : R^k C^k R^k N^k Q^k X^k X^k) (C_k \circ (P C) \otimes k)(|\omega\rangle|\omega\rangle). \] (88)

Observe that
\[ \text{Tr}_{R^k N} \left[ (C_k \circ (P C) \otimes k)(|\omega\rangle|\omega\rangle) \right] = (C_k \circ (P C) \otimes k)(|\varphi\rangle|\varphi\rangle). \] (89)

Hence, by the monotonicity of the entanglement of purification,
\[ E_p(C^k N^k Q^k : R^k C^k R^k N^k Q^k X^k X^k) (C_k \circ (P C) \otimes k)(|\tilde{\omega}\rangle|\tilde{\omega}\rangle) \leq E_p(C^k N^k Q^k : R^k C^k R^k N^k Q^k X^k X^k) (C_k \circ (P C) \otimes k)(|\omega\rangle|\omega\rangle). \] (90)

From (88) and (90), we obtain
\[ E_p(C^k Q^k : R^k C^k R^k N^k Q^k X^k X^k) (K_{\text{off}} \circ C_k \circ K_{\text{on}})(|\tilde{\omega}\rangle|\tilde{\omega}\rangle) \leq E_p(C^k N^k Q^k : R^k C^k R^k N^k Q^k X^k X^k) (C_k \circ (P C) \otimes k)(|\omega\rangle|\omega\rangle), \]
which completes the proof.