Abstract. A dynamical analog of the prime ideals for simple non-commutative rings is introduced. We prove a factorization theorem for the dynamical ideals. The result is used to classify the surface knots and links in the smooth 4-dimensional manifolds.

1. Introduction

The concept of an ideal is fundamental in commutative algebra. Recall that the prime factorization in the ring of integers $O_K$ of a number field $K$ fails to be unique. To fix the problem, one needs to complete the set of the prime numbers of $O_K$ by the “ideal numbers” lying in an abelian extension of $K$ [Kummer 1847] [4]. A description of the “ideal number” in terms of the ring $O_K$ leads to the notion of an ideal [Dedekind 1863] [1].

Formally, the ideal of a commutative ring $R$ is defined as a subset $I \subseteq R$, such that $I$ is additively closed and $IR \subseteq I$. However, the true power of the ideals comes from their geometry. For instance, if $R$ is the coordinate ring of an affine variety $V$, then the prime ideals of $R$ make up a topological space homeomorphic to $V$. This link between algebra and geometry is critical, e.g. for the Weil Conjectures [Weil 1949] [12].

Although the notion of an ideal adapts to the non-commutative rings using the left, right and two-sided ideals, such an approach seems devoid of meaningful geometry. The drawback is that the “coordinate rings” in non-commutative geometry are usually simple, i.e. have only trivial ideals [6, Section 5.3.1]. Moreover, such rings contain the idempotent elements (projections) and therefore the Ore localization of a domain fails in general.

The aim of our note is an analog of the ideals for simple non-commutative rings $R$ arising in the geometry of elliptic curves [6, Section 6.5.1] and topology of the 4-dimensional manifolds [8]. Our construction is similar to [Kummer 1847] [4]. Roughly, the idea is this. Instead of a subset $I \subseteq R$, we take a partition $\mathcal{D}_\alpha$ of $R$ by the orbits $\{\alpha^Z(x) | x \in R\}$ of an outer automorphism $\alpha : R \to R$. Let $\alpha$ be given by the formula:

$$\alpha(x) = u xu^{-1}, \quad \forall x \in R,$$

(1.1)

where $u$ is an “ideal number” lying outside $R$. We want to extend $R$ by $u$ so that $\alpha$ becomes an inner automorphism given by the formula (1.1). Such an extension is known to coincide with the crossed product $R \rtimes_\alpha Z$, where $u$ is the generator of $Z$. The dynamical system $\mathcal{D}_\alpha := R \rtimes_\alpha Z$ will be called a dynamical ideal (or, dynamial for short) of $R$.

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The dynamical system $\mathcal{D}_\alpha$ is called minimal, if the $\mathcal{D}_\alpha$ does not split into a union of simpler dynamical sub-systems. The $\mathcal{D}_\alpha$ is minimal if and only if the crossed product $R \rtimes_\alpha \mathbb{Z}$ is a simple algebra. The following model example shows, that the minimal dynamials are a proper generalization of the prime ideals to the case of simple non-commutative rings.

**Example 1.1.** ([6, Section 6.5.1]) Let $\mathcal{E}(K)$ be a non-singular elliptic curve over the number field $K$ having the coordinate ring $\mathcal{V}(\mathcal{E})$. Let $\mathcal{E}(\mathbb{F}_p) := \mathcal{V}(\mathcal{E})/\mathcal{P}$ be the localization of $\mathcal{E}(K)$ at the prime ideal $\mathcal{P} \subset \mathcal{V}(\mathcal{E})$ over a prime number $p$. Denote by $\mathcal{A}_\theta$ the non-commutative torus, i.e. a simple $C^*$-algebra $\mathcal{A}_\theta$ generated by the unitary operators $U$ and $V$ satisfying the relation

$$VU = e^{2\pi i \theta} UV, \quad \text{where } \theta \in \mathbb{R} - \mathbb{Q}. \quad (1.2)$$

It is verified directly, that the substitution

$$\begin{cases}
U' & = e^{\pi i ac} U^a V^c \\
V' & = e^{\pi i bd} U^b V^d
\end{cases} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \quad (1.3)$$

brings (1.2) to the form:

$$V'U' = e^{2\pi i (ad-bc)} U'V'. \quad (1.4)$$

If $\theta$ is a quadratic irrationality, then the $\mathcal{A}_\theta$ is said to have real multiplication and is denoted by $\mathcal{A}_\theta^{RM}$. In particular, the $\mathcal{A}_\theta^{RM}$ is a non-commutative coordinate ring of the elliptic curve $\mathcal{E}(K)$. Let $\alpha : \mathcal{A}_\theta^{RM} \to \mathcal{A}_\theta^{RM}$ be the shift automorphism of the $\mathcal{A}_\theta^{RM}$. Consider a minimal dynamial

$$\mathcal{D}_p := \mathcal{A}_\theta^{RM} \rtimes_\alpha p\mathbb{Z} \cong \mathcal{A}_\theta^{RM} \rtimes_{L_p} \mathbb{Z}, \quad (1.5)$$

where $L_p$ an endomorphism (1.3) of the $\mathcal{A}_\theta^{RM}$ corresponding to the matrix:

$$\begin{pmatrix}
\text{tr}(\epsilon^{\pi(p)}) & y \\
-1 & 0
\end{pmatrix} \in M_2(\mathbb{Z}), \quad (1.6)$$

see [6, Section 6.5.3.2] for the notation. Let $K_0(\mathcal{D}_p)$ be the $K_0$-group of the $C^*$-algebra $\mathcal{D}_p$. Then for all but a finite set of primes $p$, there exists a canonical isomorphism of the finite abelian groups:

$$K_0(\mathcal{D}_p) \cong \mathcal{V}(\mathcal{E})/\mathcal{P}. \quad (1.7)$$

**Remark 1.2.** Letting $p$ in (1.7) run through the set of all primes, one gets a bijective map between the minimal dynamials $\mathcal{D}_p$ of the non-commutative ring $\mathcal{A}_\theta^{RM}$ and the prime ideals $\mathcal{P}$ of the commutative ring $\mathcal{V}(\mathcal{E})$. In fact, such a map is a functor between the respective categories [6, Section 6.5.1]. A relation of $K_0(\mathcal{D}_p)$ to the Brauer group is discussed in remark 4.7. Formula (1.7) is a motivation of the following definition.

**Definition 1.3.** By the Dedekind-Hecke ring $R$ we understand a simple non-commutative topological ring, such that $\mathbb{Z} \subseteq \text{Out } R$, where $\text{Out } R$ are the outer automorphisms of $R$.

**Example 1.4.** The $\mathcal{A}_\theta$ is a Dedekind-Hecke ring in the norm topology. Indeed, one gets from (1.3) and (1.4), that $\text{Out } \mathcal{A}_\theta \cong \text{SL}_2(\mathbb{Z})$. The upper triangular matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is the generator of a group $\mathbb{Z} \subset \text{Out } \mathcal{A}_\theta$. 
Remark 1.5. If $R$ is a Dedekind-Hecke ring, then
\[ R \rtimes \alpha m \mathbb{Z} \cong R \rtimes \alpha Z := R \rtimes \alpha m \mathbb{Z}, \tag{1.8} \]
where $\alpha_m : R \to R_m \subseteq R$ is an endomorphism of degree $m \geq 1$. Indeed, the $R \rtimes \alpha m \mathbb{Z}$ gives rise to an automorphism of $R$ acting by the formula $x \mapsto u^m xu^{-m}$. Denote by $\hat{\alpha}_m$ an extension of $\alpha_m$ to the $R \rtimes \alpha Z$. Such that $\rho = (\alpha, xh, t\phi)$. Notice that the crossed product $R \rtimes \alpha_m Z$ is undefined, since $\alpha_m$ is not an automorphism of $R$, if $m \neq 1$. Hence the $R \rtimes \alpha_m Z$ in (1.8) is a symbolic notation for either $R \rtimes \alpha m Z$ or $R_m \rtimes \alpha Z$, see (1.5).

Remark 1.6. The endomorphism (1.6) of the ring $A_{RM}$ is known to commute with the Hecke operator $T_p$ on a lattice $\Lambda \subset \mathbb{C}$, such that $\mathcal{O}(K) \cong \mathbb{C}/\Lambda$, see [6, Section 6.5.1]. Hence our terminology in 1.3.

Our main result can be formulated as follows. Fix a generator $\alpha : R \to R$ of the cyclic group $Z \subseteq \text{Out } R$. For brevity, we write $\mathcal{D}_m := R \rtimes \alpha m Z$, where $m \geq 1$ is an integer. The product of the dynamials $\mathcal{D}_{m_1} \cong (R, G_1, \pi)$ and $\mathcal{D}_{m_2} \cong (R, G_2, \pi)$ will be defined as the direct sum of the corresponding transformation groups, i.e.
\[ \mathcal{D}_{m_1} \mathcal{D}_{m_2} := (R, G_1 \oplus G_2, \pi). \tag{1.9} \]

Theorem 1.7. The dynamials $\mathcal{D}_m$ of the Dedekind-Hecke ring $R$ satisfy the fundamental theorem of arithmetic, i.e.
\[ \mathcal{D}_m = \mathcal{D}_{p_1^{k_1}} \mathcal{D}_{p_2^{k_2}} \cdots \mathcal{D}_{p_n^{k_n}}, \tag{1.10} \]
where $\prod_{i=1}^n p_i^{k_i}$ is the prime factorization of $m$. The product (1.10) is unique up to the order of the factors. In particular, the dynamial $\mathcal{D}_m$ is minimal if and only if $m = p$ is a prime number.

The paper is organized as follows. The preliminary results can be found in Section 2. Theorem 1.7 is proved in Section 3. An application of theorem 1.7 to the classification of the surface knots and links in the 4-dimensional manifolds is considered in Section 4.

2. Preliminaries

We briefly review the topological dynamical systems, cyclic division algebras, arithmetic topology of 3-manifolds and knotted surfaces in 4-manifolds. For a detailed exposition we refer the reader to [Gottschalk & Hedlund 1955] [2, Chapter 2], [Pierce 1982] [10, Chapter 15] , [Morishita 2012] [5] and [Piergallini 1995] [11], respectively.

2.1. Topological dynamics. Let $X$ be a topological space and let $T$ be a topological group. Consider a continuous map $\pi : X \times T \to X$, such that:

(i) $\pi(x, e) = x$ for the identity $e \in T$ and all $x \in X$;
(ii) $\pi(\pi(x, t), s) = \pi(x, ts)$ for all $s, t \in T$ and all $x \in X$.

The triple $(X, T, \pi)$ is called a transformation group (topological dynamical system). A topological isomorphism of $(X, T, \pi)$ onto $(Y, S, \rho)$ is a couple $(h, \varphi)$ consisting of a homeomorphism $h : X \to Y$ and a homomorphic group-isomorphism $\varphi : T \to S$, such that $(xh, t\varphi)\rho = (x, t)\pi h$ for all $x \in X$ and $t \in T$. The transformation
groups \((X, T, \pi)\) and \((Y, S, \rho)\) are said to be equivalent, if \((X, T, \pi) \cong (Y, S, \rho)\) are topologically isomorphic.

Let \(S \subseteq T\) be a subgroup and \(x \in X\). The \(S\)-orbit of \(x\) is a subset \(O_S(x) = \{xS \mid S \subseteq T\}\) of \(X\). If \(S = T\) we omit the subscript and write an orbit as \(O(x)\). The orbit \(O_S(x)\) is the least \(S\)-invariant subset of \(X\). The closure of an \(S\)-orbit in \(X\) is denoted by \(\overline{O_S(x)}\). The set \(A \subseteq X\) is called \(S\)-minimal provided \(O_S(x) = A\) for all \(x \in A\) and \(A\) does not contain a smaller \(S\)-orbit closure. We call \(A\) a minimal set when \(S = T\).

A subset \(S \subseteq T\) is said to be left (right, resp.) syndetic in \(T\), if \(T = SK\) \((T = KS, \text{resp.})\) for some compact subset \(K \subseteq T\). In particular, if \(T\) is discrete and if \(S\) is a subgroup of \(T\), then \(S\) is syndetic in \(T\) if and only if \(S\) is a finite index subgroup of \(T\) [Gottschalk & Hedlund 1955] [2, Remark 2.03 (7)]. We shall use the following fact.

**Theorem 2.1.** [2, Theorem 2.32] Let \(X\) be compact and minimal under \(T\). Let \(S\) be a syndetic invariant subgroup of \(T\). Then the \(S\)-orbit closures define a (star-closed) decomposition of \(X\).

### 2.2. Cyclic division algebras.

The real quaternions have been the single example of a division ring (hyper-algebraic number field) until the discovery in 1906 of the cyclic division algebras by Leonard E. Dickson. Roughly speaking, such algebras are an infinite family of the division rings generalizing the quaternions and represented by matrices over an algebraic number field. Let us review the main ideas.

Let \(K\) be a number field and let \(E\) be a finite Galois extension of \(K\). Denote by \(G = \text{Gal} (E|K)\) the Galois group of \(E\) over \(K\). Let \(n = \dim_K (E)\) be the dimension of \(E\) as a vector space over \(K\). Consider the ring \(\text{End}_K (E)\) of all \(K\)-linear transformations of \(E\). Fixing a basis of \(E\) over \(K\), one gets an isomorphism:

\[
\text{End}_K (E) \cong M_n(K).
\]

Denote by \(\mathcal{C} \subset \text{End}_K (E)\) a subring generated by multiplications by the elements \(\alpha \in E\) and the automorphisms \(\theta \in G\). It can be verified directly, that the commutation relation between the two is given by the formula:

\[
\theta \alpha = \theta (\alpha) \theta.
\]

Further we restrict to the case when \(G \cong (\mathbb{Z}/n\mathbb{Z})^\times\) is a cyclic group of order \(n\) generated by \(\theta\). Thus the relation (2.2) in \(\mathcal{C}\) is complemented by the relation \((\gamma \theta)^n = 1\). On the other hand, it is easy to see that \(\theta\) is an invertible element of \(\mathcal{C}\) along with any element of the form \(\gamma \theta\), where \(\gamma \in E\). Notice that

\[
(\gamma \theta)^n = N(\gamma) \theta^n = N(\gamma),
\]

where \(N(\gamma) \in K^\times\) is the \(K\)-norm of the algebraic number \(\gamma\).

**Definition 2.2.** The cyclic algebra \(\mathcal{C}(a)\) is a subring of the ring \(M_n(K)\) generated by the elements \(\alpha \in E\) and the element \(u := \gamma \theta\) satisfying the relations:

\[
u \alpha = \theta (\alpha) u, \quad u^n = a \in K^\times.
\]

**Example 2.3.** Let \(K \cong \mathbb{R}\) and \(E \cong \mathbb{C}\). Then \(G \cong (\mathbb{Z}/2\mathbb{Z})^\times\) and \(\theta\) is the complex conjugation. In this case \(\mathcal{C}(1) \cong M_2(\mathbb{R})\) and \(\mathcal{C}(-1) \cong \mathbb{H}\), where \(\mathbb{H}\) is the algebra of real quaternions.
The \( \mathcal{C}(a) \) is a simple algebra of dimension \( n^2 \) over \( K \). The field \( K \) is the center of \( \mathcal{C}(a) \) and \( E \) is the maximal subfield of \( \mathcal{C}(a) \). The following theorem gives the necessary and sufficient condition for the \( \mathcal{C}(a) \) to be a division algebra.

**Theorem 2.4. (Wedderburn’s Norm Criterion)** The \( \mathcal{C}(a) \) is a division algebra if and only if \( a^n \) is the least power of \( a \) which is the norm of an element in \( E \).

**Lemma 2.5.** The \( \mathcal{C}(a) \) is a Dedekind-Hecke ring.

**Proof.** Recall that all automorphisms of the matrix algebra \( M_n(K) \) are inner. Since \( \mathcal{C}(a) \subset M_n(K) \), we conclude that \( \text{Out} \mathcal{C}(a) \subset M_n(K) \) (2.5) and the group \( \text{Out} \mathcal{C}(a) \) consists of the elements \( g \in M_n(K) \), such that \( g \mathcal{C}(a) g^{-1} = \mathcal{C}(a) \). It is clear that \( \mathbb{Z} \subset \text{Out} \mathcal{C}(a) \), if \( g \) has the infinite order, e.g. is given by an upper triangular matrix. On the other hand, the \( \mathcal{C}(a) \) is a topological ring endowed e.g. with the discrete topology. Thus \( \mathcal{C}(a) \) is a Dedekind-Hecke ring, see definition 1.3 \( \Box \)

2.3. **Arithmetic topology.** Such a theory studies an interplay between the topology of 3-dimensional manifolds and the algebraic number fields \([\text{Morishita} 2012]^{\text{[5]}}\). Let \( \mathcal{M}^3 \) be a category of closed 3-dimensional manifolds, such that the arrows of \( \mathcal{M}^3 \) are homeomorphisms between the manifolds. Likewise, let \( \mathcal{K} \) be a category of the algebraic number fields, where the arrows of \( \mathcal{K} \) are isomorphisms between such fields. Let \( \mathcal{M}^3 \in \mathcal{M}^3 \) be a 3-dimensional manifold, let \( S^3 \in \mathcal{M}^3 \) be the 3-sphere and let \( O_K \) be the ring of integers of \( K \in \mathcal{K} \). An exact relation between the 3-dimensional manifolds and the number fields can be described as follows.

**Theorem 2.6. ([5, Theorem 1.2])** The exists a covariant functor \( F: \mathcal{M}^3 \to \mathcal{K} \), such that:

1. \( F(S^3) = \mathbb{Q} \);
2. each ideal \( I \subseteq O_K = F(\mathcal{M}^3) \) corresponds to a link \( \mathcal{L} \subset \mathcal{M}^3 \);
3. each prime ideal \( \mathcal{P} \subseteq O_K = F(\mathcal{M}^3) \) corresponds to a knot \( \mathcal{K} \subset \mathcal{M}^3 \).

**Remark 2.7.** The domain of \( F \) extends to the smooth 4-dimensional manifolds \( \mathcal{M}^4 \) \([8, \text{Theorem 1.1}]\). The range of \( F \) consists of the cyclic division algebras \( \mathcal{C}(a) \). Since \( \mathcal{C}(a) \) is a simple algebra, we must use the dynamical ideals \( \mathcal{D}_n(\mathcal{C}(a)) \) instead of the ideals. The \( \mathcal{D}_n(\mathcal{C}(a)) \) correspond to the surface knots and links in \( \mathcal{M}^4 \). We refer the reader to Section 4 for the details.

2.4. **Knotted surfaces in 4-manifolds.** By \( \mathcal{M}^4 \) we understand a smooth 4-dimensional manifold. Let \( S^4 \) be the 4-dimensional sphere and \( X_g \) be a closed 2-dimensional orientable surface of genus \( g \geq 0 \).

**Definition 2.8.** By the knotted surface \( \mathcal{R} := X_{g_1} \cup \ldots X_{g_n} \) in \( \mathcal{M}^4 \) one understands a transverse immersion of a collection of \( n \geq 1 \) surfaces \( X_{g_i} \) into \( \mathcal{M}^4 \), i.e.:

\[
\iota : X_{g_1} \cup \ldots X_{g_n} \hookrightarrow \mathcal{M}^4.
\]

We refer to \( \mathcal{R} \) a surface *knot* if \( n = 1 \) and a surface *link* if \( n \geq 2 \).

The following result extends the well-known theorem on the covering of the 3-dimensional sphere \( S^3 \) branched over a link in the \( S^3 \).
Theorem 2.9. ([Piergallini 1995] [11]) Each smooth 4-dimensional manifold \( \mathcal{M}^4 \) is the 4-fold PL cover of the sphere \( S^4 \) branched at the points of a knotted surface \( \mathcal{X} \subset S^4 \).

3. PROOF OF THEOREM 1.7

We shall split the proof in a series of lemmas.

Lemma 3.1. \( \mathcal{D}_{m_1} \mathcal{D}_{m_2} = \mathcal{D}_{m_2} \mathcal{D}_{m_1} \) for any integers \( m_1, m_2 \geq 1 \).

Proof. Consider an exact sequence of the subgroups \( G_1 \cong m_1 \mathbb{Z} \) and \( G_2 \cong m_2 \mathbb{Z} \) of the (additive) abelian group \( (m_1 m_2) \mathbb{Z} \):
\[
0 \rightarrow m_1 \mathbb{Z} \rightarrow (m_1 m_2) \mathbb{Z} \rightarrow m_2 \mathbb{Z} \rightarrow 0.
\]

(3.1)

It can be verified directly, that the exact sequence (3.1) splits. We conclude therefore that:
\[
(m_1 m_2) \mathbb{Z} \cong G_1 \oplus G_2.
\]

(3.2)

On the other hand, using formula (1.9) one gets the following sequence of isomorphisms:
\[
\mathcal{D}_{m_1} \mathcal{D}_{m_2} \cong (R, (m_1 m_2) \mathbb{Z}, \pi) \cong
\]
\[
\cong (R, (m_2 m_1) \mathbb{Z}, \pi) \cong \mathcal{D}_{m_2} \mathcal{D}_{m_1}.
\]

(3.3)

The conclusion of lemma 3.1 follows from formula (3.3).

\[\square\]

Lemma 3.2. \( \mathcal{D}_{m_1} \mathcal{D}_{m_2} = \mathcal{D}_{m_1 m_2} \) for any integers \( m_1, m_2 \geq 1 \).

Proof. We preserve the notation used in the proof of lemma 3.1. From the definition (1.9) of a product of the dynamials and formula (3.2), one gets the following sequence of the isomorphisms:
\[
\mathcal{D}_{m_1} \mathcal{D}_{m_2} \cong (R, G_1 \oplus G_2, \pi) \cong
\]
\[
\cong (R, (m_1 m_2) \mathbb{Z}, \pi) \cong \mathcal{D}_{m_1 m_2}.
\]

(3.4)

The conclusion of lemma 3.2 follows from formula (3.4).

\[\square\]

Lemma 3.3. The index map \( i(\mathcal{D}_m) = m \) is an isomorphism between the multiplicative semigroup \( (\mathcal{D}_R, \times) \) of all dynamials \( \mathcal{D}_R \) of the ring \( R \) and the multiplicative semigroup \( (\mathbb{N}, \times) \) of all positive integers \( \mathbb{N} \).

Proof. (i) The unit of the semigroup \( (\mathcal{D}_R, \times) \) is the dynamial \( \mathcal{D}_1 \cong R \times_0 \mathbb{Z} \). The \( i(\mathcal{D}) = 1 \) is the unit of the semigroup \( (\mathbb{N}, \times) \) and \( \mathcal{D}_1 \) is the unique dynamial having index 1. Therefore, the index map is injective.

(ii) It is easy to see, that the index map is surjective. Indeed, for every \( m \geq 1 \) there exists a dynamial \( \mathcal{D}_m \), such that \( i(\mathcal{D}_m) = m \).

(iii) Let us verify, that the index map preserves the products, i.e. \( i(\mathcal{D}_{m_1} \mathcal{D}_{m_2}) = i(\mathcal{D}_{m_1}) i(\mathcal{D}_{m_2}) \). Using lemma 3.2, we calculate:
\[
i(\mathcal{D}_{m_1} \mathcal{D}_{m_2}) = i(\mathcal{D}_{m_1 m_2}) = m_1 m_2 = i(\mathcal{D}_{m_1}) i(\mathcal{D}_{m_2}).
\]

(3.5)
Since the index map is (i) injective, (ii) surjective and (iii) preserves the products, we conclude that such a map is an isomorphism of the underlying semigroups. Lemma 3.3 follows.

**Lemma 3.4.** If \( m = p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n} \) is the prime factorization of an integer \( m \geq 1 \), then \( D_m = D_{p_1^{k_1}} D_{p_2^{k_2}} \ldots D_{p_n^{k_n}} \). The latter product is unique up to the order of the factors.

**Proof.** Let \( m \geq 1 \) be an integer. According to the fundamental theorem of arithmetic, there exists a prime factorization:

\[
m = p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n},
\]

where \( p_i \) are the prime numbers and \( k_i \) are positive integers. Moreover, the product (3.6) is unique up to the order of the factors.

(i) Using lemma 3.2, we calculate:

\[
D_m = D_{p_1^{k_1}} D_{p_2^{k_2}} \ldots D_{p_n^{k_n}}.
\]

(ii) Since by lemma 3.3 the index map is an isomorphism, we conclude that the product (3.7) is unique up to the order of the factors. (Otherwise such a property would fail also for the product (3.6).)

The conclusion of lemma 3.4 follows from items (i) and (ii).

**Lemma 3.5.** The dynamical \( D_m \) is minimal if and only if \( m = p \) is a prime number.

**Proof.** (i) Let us assume that \( m = p \) is a prime number. In view of lemma 3.4, the dynamical system \( D_p \) cannot be a product of two or more dynamical sub-systems. In other words, the \( D_p \) is a minimal dynamical.

(ii) Conversely, let \( D_m \) be a minimal dynamical system. Let \( (R, m\mathbb{Z}, \pi) \) be the corresponding transformation group. Notice that the abelian group \( m\mathbb{Z} \) is discrete. Therefore every finite index subgroup \( S \) of \( m\mathbb{Z} \) must be syndetic, see Section 2.1. Let us now assume to the contrary, that \( m \neq p \). Then there exists a non-trivial syndetic subgroup \( S \) of the group \( m\mathbb{Z} \). By Theorem 2.1 there exists a (star-closed) decomposition of the dynamical system \( D_m \) by the \( S \)-orbit closures, i.e. by the smaller dynamical sub-systems. In other words, the dynamical system \( D_m \) is not minimal. This contradiction proves the necessary condition of lemma 3.5.

Lemma 3.5 is equivalent to items (i) and (ii).

Theorem 1.7 follows from lemmas 3.4 and 3.5.

4. **Knotted surfaces in 4-manifolds**

We apply theorem 1.7 to prove an analog of theorem 2.6 for the smooth 4-dimensional manifolds \( \mathcal{M}^4 \), see remark 2.7. Roughly speaking, we replace the ideals \( I \subseteq O_K \) by the dynamical ideals \( D_m(C(a)) \), where \( C(a) \) is a cyclic division algebra associated to \( \mathcal{M}^4 \) [8]. It is well known, that any knot or link in \( \mathcal{M}^4 \) is trivial. This fact can be derived from the simplicity of algebra \( C(a) \) and the Wedderburn Theorem, see [9]. On the other hand, there exist non-trivially knotted surfaces \( X_g \) in \( \mathcal{M}^4 \) (E. Artin). We show that the dynamials \( D_m(C(a)) \) classify the embeddings \( X_g \hookrightarrow \mathcal{M}^4 \). Let us pass to a detailed argument.
Let $\mathcal{M}^4$ be a category of smooth 4-dimensional manifolds $\mathcal{M}^4$, such that the arrows of $\mathcal{M}^4$ are homeomorphisms between the manifolds. Denote by $\mathcal{C}$ a category of the cyclic division algebras $\mathcal{C}(a)$, such that the arrows of $\mathcal{C}$ are isomorphisms between these algebras. The following result is an extension of Theorem 2.6 to $\mathcal{M}^4$.

**Theorem 4.1.** The exists a covariant functor $F : \mathcal{M}^4 \to \mathcal{C}$, such that:

(i) $F(S^4) = \mathbb{H}(\mathbb{Q})$, where $\mathbb{H}(\mathbb{Q})$ are the rational quaternions;

(ii) the dynamial $\mathcal{D}_m(F(\mathcal{M}^4))$ corresponds to a surface link $\mathcal{X} \subset \mathcal{M}^4$;

(iii) the minimal dynamial $\mathcal{D}_p(F(\mathcal{M}^4))$ corresponds to a surface knot $\mathcal{X} \subset \mathcal{M}^4$.

**Proof.** The proof of existence and a detailed construction of functor $F$ can be found in [8, Theorem 1.1]. To prove items (i)-(iii) of theorem 4.1, we shall use Theorem 2.6 and the spinning of a link construction dating back to E. Artin, see e.g. [Kamada 2002] [3, Chapter 10]. Let us prove the following lemmas.

**Lemma 4.2.** Let $E$ be a cyclic extension of the number field $K$ with the Galois group $G \cong (\mathbb{Z}/n\mathbb{Z})^\times$ of order $n$ generated by an element $\theta$, see Section 2.2. Denote by $O_E$ the ring of integers of $E$. Then

$$\mathcal{C}(a) \cong O_E \rtimes_\theta G.$$

(4.1)

**Proof.** Recall that the algebra $\mathcal{C}(a)$ is generated by multiplications by the elements $\alpha \in E$ and the automorphism $\theta \in G$, see Section 2.2. Since $E \cong O_E \otimes \mathbb{Q}$, we can always assume that $\alpha \in O_E$.

On the other hand, the commutation relation (2.2) can be written as

$$\theta(\alpha) = \theta \alpha \theta^{-1}, \quad \forall \alpha \in O_E.$$  

(4.2)

Thus the algebra $\mathcal{C}(a)$ is an extension of the ring $O_E$ by an element $\theta$, such that each automorphism of $O_E$ becomes an inner automorphism. In other words, the $\mathcal{C}(a)$ coincided with the crossed product $O_E \rtimes_\theta G$. Lemma 4.2 is proved. $\square$

**Lemma 4.3.** The dynamial $\mathcal{D}_m(\mathcal{C}(a))$ defines an ideal $I_E \subseteq O_E$, such that:

(i) $|O_E/I_E| = m$;

(ii) $\mathcal{D}_m(\mathcal{C}(a)) = \mathcal{D}(\mathcal{C}_m(a))$, where $\mathcal{C}_m(a) = I_E \rtimes_\theta G$.

**Proof.** From (4.1) and definition of $\mathcal{D}_m$, one gets:

$$\mathcal{D}_m(\mathcal{C}(a)) \cong \mathcal{C}(a) \rtimes m\mathbb{Z} \cong$$

$$\cong (O_E \rtimes_\theta G) \rtimes m\mathbb{Z}.$$  

(4.3)

On the other hand, by the commutativity of crossed products by the abelian groups, we have:

$$(O_E \rtimes_\theta G) \rtimes m\mathbb{Z} \cong (O_E \rtimes m\mathbb{Z}) \rtimes_\theta G.$$  

(4.4)

Finally, repeating the argument of remark 1.5, one gets an isomorphism:

$$O_E \rtimes m\mathbb{Z} \cong I_E \rtimes \mathbb{Z}.$$  

(4.5)

where $I_E \subseteq O_E$ is an ideal, such that $|O_E/I_E| = m$.

Altogether (4.3)-(4.5) give us:

$$\mathcal{D}_m(\mathcal{C}(a)) \cong (I_E \rtimes \mathbb{Z}) \rtimes_\theta G \cong$$
\[ \cong (I_E \rtimes \theta G) \times \mathbb{Z} \cong \mathcal{D}(\mathcal{C}_m(a)), \]  
\hfill (4.6)

where \( \mathcal{C}_m(a) := I_E \rtimes \theta G \). Lemma 4.3 is proved.

\[ \square \]

**Remark 4.4.** It is not hard to see, that the crossed product
\[ \{ R \rtimes \theta G \mid R \text{ is a commutative ring} \} \]

 corresponds to a “spinning” of the underlying topological space around a 2-dimensional plane [Kamada 2002] [3, Chapter 10]. Indeed, if \( R \cong O_E \), then the underlying topological space is a 3-dimensional manifold \( \mathcal{M}^3 \), such that \( O_E = F(\mathcal{M}^3) \), see Theorem 2.6. By (4.1) the spinning \( O_E \rtimes \theta G \cong \mathcal{C}(a) \) gives us a 4-dimensional manifold \( \mathcal{M}^4 \), such that \( \mathcal{C}(a) = F(\mathcal{M}^4) \) [8, Theorem 1.1]. Likewise, if \( R \cong I_E \subseteq O_E \) is an ideal, then the underlying topological space is a link \( \mathcal{L} \subseteq \mathcal{M}^3 \), such that \( I_E = F(\mathcal{L}) \), see Theorem 2.6. In view of lemma 4.3 (ii), the spinning \( I_E \rtimes \theta G \cong \mathcal{C}_m(a) \) gives us a 2-dimensional knotted surface \( \mathcal{X} \subseteq \mathcal{M}^4 \), such that \( \mathcal{C}_m(a) = F(\mathcal{X}) \).

Returning to the proof of theorem 4.1, we shall proceed in the following steps.

(i) Consider a cyclic division algebra \( \mathcal{C}(a) \) over the number field \( K \). It is not hard to see, that the smallest cyclic division sub-algebra of \( \mathcal{C}(a) \) consists of the rational quaternions \( \mathbb{H}(\mathbb{Q}) \), see Example 2.3. In other words, each \( \mathcal{C}(a) \) can be seen as an extension of the rational quaternions:
\[ \mathbb{H}(\mathbb{Q}) \subseteq \mathcal{C}(a). \]
\hfill (4.8)

Since the \( \mathbb{H}(\mathbb{Q}) \) is an algebra over \( \mathbb{Q} \) and
\[ \mathbb{Q} \subseteq K, \]
\hfill (4.9)

we conclude from Theorem 2.6 (i) and remark 4.4, that:
\[ F(S^4) = \mathbb{H}(\mathbb{Q}), \]
\hfill (4.10)

where \( S^4 \) is the 4-dimensional sphere. Item (i) of theorem 4.1 is proved.

(ii) Let \( m = p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n} \) be the prime factorization of an integer \( m \geq 1 \). By lemma 2.5, the \( \mathcal{C}(a) \) is a Dedekind-Hecke ring. Thus we can apply the prime factorization theorem 1.7 to the dynamial \( \mathcal{D}_m(\mathcal{C}(a)) \), i.e.
\[ \mathcal{D}_m(\mathcal{C}(a)) = \mathcal{D}_{p_1^{k_1}}(\mathcal{C}(a)) \mathcal{D}_{p_2^{k_2}}(\mathcal{C}(a)) \ldots \mathcal{D}_{p_n^{k_n}}(\mathcal{C}(a)). \]
\hfill (4.11)

On the other hand, each \( \mathcal{D}_i(\mathcal{C}(a)) \) defines an ideal \( I_i \subseteq O_E \), where \( E \) is the maximal number field of the cyclic division algebra \( \mathcal{C}(a) \), see lemma 4.3. Moreover, item (i) of the same lemma implies that the prime factorization (4.11) defines the prime factorization of the ideal \( I_m \subseteq O_E \) corresponding to the \( \mathcal{D}_m(\mathcal{C}(a)) \), i.e.
\[ I_m = \mathcal{D}_1^{k_1} \mathcal{D}_2^{k_2} \ldots \mathcal{D}_n^{k_n}, \]
\hfill (4.12)

where \( \mathcal{D}_i \) are the prime ideals of \( E \). We can now use Theorem 2.6 and Remark 4.4 to conclude that the dynamial \( \mathcal{D}_m(\mathcal{C}(a)) \) corresponds to a surface link \( \mathcal{X} \subseteq \mathcal{M}^4 \) obtained by the spinning of a link \( \mathcal{L} \) defined by the prime factorization (4.12). Item (ii) of theorem 4.1 is proved.

(iii) Let \( m = p \) be a prime number. We repeat the argument of item (ii) and we apply item (iii) of Theorem 2.6. In this case one gets a surface knot \( \mathcal{X} \subseteq \mathcal{M}^4 \),
since there is only one component in the prime factorization formula \((4.11)\). Item (iii) of theorem 4.1 is proved.

Theorem 4.1 is proven. □

Example 4.5. (Sphere knots in simply connected 4-manifolds)

(i) Let \(\mathcal{W}^4\) be a simply connected 4-dimensional manifold. Denote by \(K_{ab}\) an abelian extension of the field \(\mathbb{Q}\) and by \(\mathcal{C}(a, K_{ab})\) a cyclic division algebra over the \(K_{ab}\). It follows from \([8, \text{Corollary 1.2}]\), that

\[
F(\mathcal{W}^4) = \mathcal{C}(a, K_{ab}). \tag{4.13}
\]

Indeed, since the Galois group of \(\mathcal{C}(a) = F(\mathcal{W}^4)\) is abelian, so does any subfield of the \(\mathcal{C}(a)\). In particular, the center of \(\mathcal{C}(a)\) must be an abelian extension of \(\mathbb{Q}\).

(ii) Consider the sphere knot: \(S^2 \rightarrow \mathcal{W}^4\). \(\tag{4.14}\)

Theorem 4.1 (iii) says that \((4.14)\) are classified by the minimal dynamials:

\[
\mathcal{D}_p(\mathcal{C}(a, K_{ab})) = \mathcal{C}(a, K_{ab}^p) \times p\mathbb{Z} \cong \mathcal{C}(a, K_{ab}^p), \tag{4.15}
\]

where \(K_{ab}^p\) is an abelian extension of the field \(\mathbb{Q}_p\) of the \(p\)-adic numbers. The local class field theory says that the fields \(K_{ab}^p\) are classified by the open subgroups of the group \(\mathbb{Q}_p^\times\) of index \(n = \deg (K_{ab}^p|\mathbb{Q})\). Thus the classification of the sphere knots \((4.14)\) depends on the finite index open subgroups of the \(\mathbb{Q}_p^\times\).

(iii) Let us restrict to the case \(\mathcal{W}^4 \cong S^4\) is a 4-dimensional sphere. By theorem 4.1 (i), we have \(F(S^4) = \mathbb{H}(\mathbb{Q})\) and from \((4.15)\):

\[
\mathcal{D}_p(\mathbb{H}(\mathbb{Q})) \cong \mathbb{H}(\mathbb{Q}_p). \tag{4.16}
\]

Since the index \(n = 1\), we conclude that the sphere knots:

\[
S^2 \rightarrow S^4 \tag{4.17}
\]

are classified by the primes \(p\). In view of theorem 2.6 (iii) and that \(\mathcal{D} \cong p\mathbb{Z}\), we recover the Artin classification of sphere knots \(\pi_1(S^4 - S^2) \cong \pi_1(S^3 - \mathcal{W})\) by knots in \(S^3\) \([Kamada 2002]\ [3, \text{Chapter 10}]\).

Remark 4.6. It is known, that all division algebras over a local field are cyclic. The Brauer group of such algebras has the form:

\[
Br(\mathcal{C}(a, K_{ab}^p)) \cong \mathbb{Q}/\mathbb{Z}, \tag{4.18}
\]

see e.g. \([10, \text{Section 17.10}]\). The Brauer group classifies the Morita equivalence classes of the algebras \(\mathcal{C}(a, K_{ab}^p)\) and, therefore, the corresponding classes of the sphere knots \((4.14)\). However, the nature of such classes is unclear to the author.

Remark 4.7. Using \((4.15)\), one rewrite \((4.18)\) in the form:

\[
Br(\mathcal{D}_p) \cong \mathbb{Q}/\mathbb{Z}. \tag{4.19}
\]

Notice a formal resemblance of \((4.19)\) to \((1.7)\). This correspondence is part of the Merkurjev-Suslin theory linking the Brauer groups with the algebraic K-theory.
DYNAMICAL IDEALS OF NON-COMMUTATIVE RINGS

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