\textbf{$\alpha$-STABLE DENSITIES ARE HYPERBOLICALLY COMPLETELY MONOTONE FOR $\alpha \in [0, 1/4] \cup [1/3, 1/2]$}

SONIA FOURATI

\textbf{Abstract.} We investigate the problem raised by L. Bondesson in \cite{1}, about the hyperbolic complete monotonicity of $\alpha$-stable densities. We prove that densities of subordinators of order $\alpha$ are HCM for $\alpha \in [0, 1/4] \cup [1/3, 1/2]$.

1. Introduction

Hyperbolically completely monotone functions (HCM in short) were introduced by L. Bondesson \cite{1} in order to analyze infinitely divisible distributions, and particularly the so-called generalized gamma convolutions introduced by O. Thorin \cite{4}. We recall their definition in section 1 below.

Bondesson showed that the densities of $\alpha$-stable positive random variables are HCM for $\alpha = n^{-1}$, for any integer $n \geq 2$. Furthermore, he conjectured that the HCM property actually holds for all $\alpha \in [0, 1/2]$. Recently, Wissem Jedidi and Thomas Simon \cite{2} investigated some aspects of the problem. I thank them for pointing out this question to me.

In this paper we prove this conjecture for values of $\alpha$ in $[0, 1/4] \cup [1/3, 1/2]$.

For this we introduce the functions $G_{\alpha}(x) = x^{-\frac{1}{\alpha}} g_{\alpha}(x^{-\frac{1}{\alpha}})$ where $g_{\alpha}$ is the density of the positive $\alpha$-stable distribution. We show that $G_{\alpha}$ extends to an analytic function on the slit plane $\mathbb{C} \setminus [-\infty, 0]$. By analyzing its behaviour at infinity and near the cut, we are able to prove that it has the following form

\begin{equation}
G_{\alpha}(z) = ce^{-\delta z} \exp \left( \int_{0}^{+\infty} \left( \frac{1}{z + t} - \frac{1}{1 + t} \right) \theta(t) dt \right)
\end{equation}

where $c, \delta$ are positive constants and $\theta$ takes values in $[0, 1]$.

In order that $G_{\alpha}$ be HCM it is then enough that the function $\theta$ be increasing, which we prove for $\alpha \in [1/3, 1/2]$. The HCM property for the remaining values of $\alpha$ is obtained by a multiplicative convolution argument.

This paper is organized as follows. In the section 2 we recall some results of Zolotarev on densities of stable distributions. These are used in the next section to obtain the asymptotic behaviour of the function $G_{\alpha}$ in the complex plane. In section four we establish the integral representation (1.1). Finally, in section five, we prove that $\theta$ is increasing for $1/3 < \alpha < 1/2$, and we finish the proof of this part of the conjecture.

\textit{Date:} 04-09-2013.

\textit{1991 Mathematics Subject Classification.} Primary 60H05, 60H10; Secondary 46L53.
2. Hyperbolically completely monotone functions

We recall here the basic definition and properties of the class of hyperbolically completely monotone functions, and refer to [1] for more details.

A real valued function $H$ defined on $]0, +\infty[$ is called hyperbolically completely monotone (HCM) if for every $u > 0$ the function $H(uv)H(1/v)$ is a completely monotone function of the variable $v + 1/v$. Bondesson [1] introduced this property in order to analyze infinitely divisible distributions, and particularly the so-called generalized gamma convolutions introduced by O. Thorin.

Proposition 2.1.

(i) $H$ is HCM if and only if $H(x^{-1})$ is HCM.

(ii) $H$ is HCM if and only if it admits the following representation

\[ H(x) = cx^{\beta-1} \exp \left( -a_1x - \int_1^\infty \log \frac{x + t}{1 + t} \mu_1(dt) - a_2x^{-1} - \int_1^\infty \log \frac{x^{-1} + t}{1 + t} \mu_2(dt) \right) \]

where $\beta$ is real, $a_1, a_2$ are positive constants and $\mu_1, \mu_2$ positive measures.

(iii) If $H$ is HCM then $H(x^\gamma)$ is HCM for all $\gamma \leq 1$

(iv) $H$ is HCM if and only if the functions $x^\gamma H(x)$ are HCM for all values of $\gamma \in \mathbb{R}$.

(v) If $X$ and $Y$ are independent positive random variables both with an HCM density then the random variable $XY$ also has an HCM density.

In particular, from (iii) and (iv) we deduce that if $X$ is a positive random variable with HCM density, then $X^\gamma$ has HCM density for all $\gamma \geq 1$.

3. Stable random variables

Let $\alpha \in ]0, 1[,]$ and $\rho \in ]0, 1[,]$, we denote $g_{\alpha, \rho}$ the density of the strictly $\alpha$-stable distribution with asymmetry parameter $\rho$ (cf [5]). For $\rho = 1$ (and only for this value) this distribution is supported on the half axis $]0, +\infty[$, and we simply put $g_\alpha = g_{\alpha, 1}$.

The following result is an integral representation for the functions $g_{\alpha, \rho}$ on the positive axis, due to Zolotarev.

Theorem 3.1. (Zolotarev, [5], Theorem 2.4.2)

For all $x > 0, \alpha, \rho \in ]0, 1[,]$

\[ g_{\alpha, \rho}(x) = (2i\pi)^{-1} \int_0^\infty e^{-e^{-i\pi\rho}y^\alpha x^{-\alpha}} - e^{-e^{i\pi\rho}y^\alpha x^{-\alpha}} e^{-y} dy \]

The following result which is easily obtained by a subordination argument, plays an important role in the following.

Lemma 3.2. Let $X$ and $Y$ be independent positive stable random variables, with respective parameters $(\alpha, \rho)$ and $(\beta, 1)$, then $XY^{1/\alpha}$ is a stable random variable with parameter $(\alpha\beta, \rho)$.

We deduce from the preceding lemma and Proposition 2.1 that

Proposition 3.3. The set of $\alpha \in ]0, 1[,]$ such that $g_\alpha$ is HCM is a semigroup under multiplication.
Denote \( G_\alpha \) the function
\[
G_\alpha(z) = (2i\pi)^{-1} \int_0^\infty e^{-e^{-i\alpha y} y^{1-\alpha} z^{1-\alpha}} e^{-y} dy
\]
where we take (as in the rest of the paper) for \( z^h \), the determination of the power function which is positive on \([0, +\infty[\) and analytic on \( \mathbb{C} \setminus [0, +\infty[\).

This function \( G_\alpha \) is analytic in \( \mathbb{C} \setminus [-\infty, 0] \). In fact \( z^\alpha G_\alpha(z) = F_\alpha(z^{1-\alpha}) \) where \( F_\alpha \) is an entire function. One has, for all \( x > 0 \),
\[
g_\alpha(x) = x^{-\frac{1}{1-\alpha}} G_\alpha(x^{-\frac{1}{1-\alpha}})
\]
and for all \( z \in \mathbb{C} \setminus [-\infty, 0] \)
\[
G_\alpha(z) = G_\alpha(z)
\]
For \( r > 0 \) we denote
\[
G_\alpha(-r^+) = \lim_{z \to -r, \Im(z) > 0} G_\alpha(z) \quad G_\alpha(-r^-) = \lim_{z \to -r, \Im(z) < 0} G_\alpha(z) = G_\alpha(-r^+)
\]
the boundary values of \( G_\alpha \).

5. Behaviour near 0.

It follows from (4.1) that, as \( z \to 0 \),
\[
G_\alpha(z) = \Gamma(\alpha + 1) \frac{\sin(2\pi\alpha)}{\pi} z^{-\alpha}(1 + O(|z|^{1-\alpha}))
\]

6. Bounds at infinity

\textbf{Theorem 6.1.} Let \( \theta \in ]-1, 1[ \) be fixed, and
\[
\delta = (1 - \alpha)\alpha^{-\frac{1}{\alpha}} \quad c = (1 - \alpha)^{-\frac{1}{2\alpha}} \alpha^{-\frac{1}{2\alpha(\alpha-1)}}
\]
then, as \( r \to +\infty \), for \( z = re^{i\pi\theta} \), one has
\[
G_\alpha(z) \sim cz^{-\frac{1}{2} e^{-\delta z}}
\]
As \( r \to +\infty \)
\[
G_\alpha(-r^+) \sim -icr^{-\frac{1}{2}} e^{\delta r} \quad G_\alpha(-r^-) \sim icr^{-\frac{1}{2}} e^{\delta r}
\]

Furthermore, for some \( R > 0 \), the function \( G_\alpha(z)z^{1/2} e^{\delta z} \) is uniformly bounded on \( \mathbb{C} \setminus [-\infty, 0] \) \( \cap \{ |z| > R \} \).

In order to obtain this asymptotic result, observe that one can rewrite the integral defining \( G_\alpha \) as a contour integral:
\[
G_\alpha(z) = (2i\pi)^{-1} \int_\Gamma e^{y-y^n z^{1-\alpha}} \frac{dz}{z}
\]
where \( \Gamma \) is a contour which starts from \(-\infty\), following the negative axis, taking the lower branch of \( y^{\alpha} \), encircles 0 then goes back to \(-\infty\) along the negative axis, this time picking up the upper branch of \( y^{\alpha} \).

In order to obtain the asymptotics we take \( z = re^{i\theta} \) and rewrite the integral as
\[
G_\alpha(z) = (2i\pi)^{-1} \int_\Gamma e^{y-y^n e^{i(1-\alpha)\theta}} e^{-ix\theta} dy
\]
This integral is subject to the steepest descent method (see [3] for example) using the unique saddle point at $y = \alpha i e^{\pi (1-\alpha)\theta}$. This gives the point wise convergence for a fixed $\theta$. In order to obtain the uniform convergence, first notice that uniform property is clear for $\theta$ in any compact subset of $[-1, +1]$, say for $\theta \in [-7/8, 7/8]$, then, for $\theta \in [7/8, 1]$, the saddle point is over the half line $]-\infty, 0]$ and close to it, then one can use another determination of $y^\alpha$ with a cut say on the half-line $\arg(y) = -3\pi/4$, and a contour encircling the cut and going back to a neighborhood of $-\infty$ by an arc with a ray going to infinity. Then again one can deform this contour to go through the saddle point and then conclude of the uniform convergence for $\theta \in [7/8, 1]$. A symmetrical argument gives the uniformity for $\theta \in [-1, -7/8]$.

7. Behaviour of $G_\alpha$ on the cut

Lemma 7.1. For any $r > 0$

$$G_\alpha(-r^+) = (2i\pi)^{-1} \int_0^{\infty} e^{r^{1-\alpha}y^\alpha - e^{-2i\pi r^{1-\alpha}y^\alpha}} e^{-y} dy$$

(7.2) $$G_\alpha(-r^+) = (2i\pi)^{-1} \sum_{n=1}^{\infty} \frac{\Gamma(n\alpha + 1)}{\Gamma(n+1)} (1 - e^{-2i\pi n\alpha}) r^{n(1-\alpha)-1}$$

Proof. The first formula follows at once from (4.1) by letting $z \rightarrow -r$, the second one comes from expanding the exponentials in the numerator of (4.1) and integrating term by term. □

Lemma 7.2. For any $r > 0$ one has $\Im(G_\alpha(-r^+)) < 0$. Furthermore, $-r^\alpha \Im(G_\alpha(-r^+))$ is an increasing function of $r$.

Proof. By (7.2) we get

$$-\Im(G_\alpha(-r^+)) = (2\pi)^{-1} \sum_{n=1}^{\infty} \frac{\Gamma(n\alpha + 1)}{\Gamma(n+1)} (1 - \cos(2\pi n\alpha)) r^{n(1-\alpha)-1}$$

in which all terms in the sum are positive; the two claims are clear. □

Remark 7.3. in fact one could also obtain from the integral representation that $\theta(t) - 1/2 = o(e^{-\epsilon r})$ as $r \rightarrow +\infty$ for some $\epsilon > 0$, but we will not use this).

8. Integral representation

Proposition 8.1. For all $z \in \mathbb{C}\setminus[-\infty, 0]$

$$G_\alpha(z) = a e^{-z} \exp \int_0^{\infty} \left[ \frac{1}{z + t} - \frac{1}{1 + t} \right] \theta(t) dt$$

for some $a > 0$. 

α-stable densities are hyperbolically completely monotone for $\alpha \in [0, 1/4] \cup [1/3, 1/2]$. \]^5

**Proof.** Let

$$L_\alpha(z) = \exp \int_0^\infty \left[ \frac{1}{z+t} - \frac{1}{1+t} \right] \theta(t) dt$$

This is an analytic function on $C\setminus -\infty, 0]$, and it satifies, by well known properties of Stieltjes transforms,

$$\frac{L_\alpha(-r^+)}{L_\alpha(-r^-)} = e^{-2i\pi \theta(r)}$$

Furthermore, as $z \to \infty$, since $\theta(t) \to t \to +\infty 1/2$, one has

$$L_\alpha(z) = z^{-1/2} \exp(o(\log(|z|)))$$

Near zero, one has $\theta(t) = \alpha + o(t^{1-\alpha})$ by (5.1), which implies

$$L_\alpha(z) \sim z^\alpha \quad z \to 0$$

On the other hand, for $r > 0$,

$$\frac{G_\alpha(-r^+)}{G_\alpha(-r^-)} = e^{-2i\pi \theta(r)}$$

therefore the function

$$E_\alpha(z) = e^{2z}G_\alpha(z)/L_\alpha(z)$$

is analytic on $C\setminus -\infty, 0]$, and can be extended continuously to $C \setminus \{0\}$. Since it is bounded near 0 it can be extended to an entire function and it satifies

$$E_\alpha(z) = \exp(o(\log(|z|)))$$

at infinity thus it is constant. Since both functions $G_\alpha, L_\alpha$ take positive values on $[0, +\infty]$, this constant is positive. \hfill \Box

9. THE FUNCTION $\theta$ IS MONOTONE FOR $\alpha \in [1/3, 1/2]$

**Lemma 9.1.** For $0 \leq \rho \leq \inf(1, \frac{1}{\alpha})$ the function $g_{\alpha, \rho}(x) = x^{-1-\alpha}g_{\alpha, \rho}(x^{-1})$ is decreasing on $[0, +\infty]$. \]^3

**Proof.** Recall that if $X$ is a stable variable with parameters $(2\alpha, \rho)$, and $Y$ an independent stable variable with parameters $(1/2, 1)$, then $Z = XY^{\frac{1}{\lambda}}$ is a stable variable with parameters $(\alpha, \rho)$. Since the density of $Y$ is $e^{-\frac{y}{\sqrt{2\pi}}} \frac{\alpha + 1}{y^{\alpha+1}}$ one has

$$g_{\alpha, \rho}(x) = 2\alpha \int_0^\infty g_{2\alpha, \rho}(y) e^{-\frac{1}{2}(y/x)^{2\alpha} y^\alpha} \frac{\alpha + 1}{\sqrt{2\pi x^{\alpha+1}}} dy$$

Therefore

$$x^{-1-\alpha}g_{\alpha, \rho}(x^{-1}) = 2\alpha \int_0^\infty g_{2\alpha, \rho}(y) e^{-\frac{1}{2}(y/x)^{2\alpha} y^\alpha} \frac{\alpha + 1}{\sqrt{2\pi}} dy$$

which is clearly decreasing in $x$. \hfill \Box

**Lemma 9.2.** For $\alpha \in [1/3, 1/2]$ the function $r^\alpha \Re G_\alpha(-r^+) \exp(o(2\alpha))$ is decreasing.

**Proof.** Note that, by formulas (3.1) and (7.1) one has

$$\Re G_\alpha(-r^+) = r^{-1/\alpha}g_{\alpha, \frac{\alpha}{\alpha+2}}(r^{-\frac{\alpha}{\alpha+2}})$$

for $\alpha \in [1/3, 1/2]$. If it follows that

$$r^\alpha \Re G_\alpha(-r^+) = r^{\alpha-1/\alpha}g_{\alpha, \frac{\alpha}{\alpha+2}}(r^{\frac{\alpha}{\alpha+2}}) = x^{-1-\alpha}g_{\alpha, \frac{\alpha}{\alpha+2}}(x^{-1})$$
with \( x = r^{1 - \frac{\alpha}{2}} \). The result follows from the preceding lemma.

\[ x = r^{1 - \frac{\alpha}{2}} \]

\[ \Box \]

**Theorem 9.3.** For \( \alpha \in [1/3, 1/2] \), the function \( \theta \) increases from the value \( \alpha \) to the value \( 1/2 \), and

\[
G_\alpha(z) = \Gamma(\alpha + 1)e^{-\delta z}z^{-1/2}\exp - \int_0^\infty \log(1 + t/z)\theta'(t)dt
\]

Proof. One has

\[
\tan(\pi \theta(r)) = \frac{-r^\alpha \Im(G_\alpha(-r^+))}{r^\alpha \Re(G_\alpha(-r^+))}
\]

and the numerator and denominator of this formula are positive and respectively increasing and decreasing. This implies that \( \theta \) is increasing. The other claim follows by integrating by parts. \( \Box \)

10. **The HCM Property of Stable Distribution**

For \( \alpha \in [1/3, 1/2] \) one has \( \frac{1 - \alpha}{\alpha} \geq 1 \) and \( G_\alpha \) is HCM. This implies that \( g_\alpha \) is HCM. By Proposition 6.3, the set of \( \alpha \) such that \( g_\alpha \) is HCM thus contains the multiplicative semigroup generated by \([1/3, 1/2] \), which is \([0, 1/4] \cup [1/3, 1/2] \). \( \Box \)

**Remark 10.1.** Following the same arguments than above, one can prove that for \( \alpha \geq 1/2 \), the function \( \theta \) decreases from \( \alpha \) to \( 1/2 \) and consequently \( G_\alpha \) enjoys the next decomposition:

\[
G_\alpha(z) = \Gamma(\alpha + 1)e^{-\delta z}z^{-1/2}\exp \int_0^\infty \log(1 + t/z)\theta'(t)dt
\]

In other words, \( e^{-\delta z}\frac{1}{G_\alpha(z)} \) is an HCM function.

**References**

[1] Bondesson, Lennart *Generalized gamma convolutions and related classes of distributions and densities.* Lecture Notes in Statistics, 76. Springer-Verlag, New York, 1992.

[2] Jedidi, Wissem and Simon Thomas *Further examples of GGC and HCM functions.* To appear in Bernoulli Journal.

[3] Miller Peter David *Applied Asymptotic Analysis,* AMS, 2006, Vol.77

[4] Thorin, Olof *On the infinite divisibility of the Pareto distribution.* Scand. Actuar. J. 1977, no. 1, 3140.

[5] Zolotarev, V. M. *One-dimensional stable distributions.* Translated from the Russian by H. H. McFaden. Translations of Mathematical Monographs, 65. American Mathematical Society, Providence, RI, 1986.