A level-set-based topology optimisation of carpet cloaking devices with the boundary element method

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Abstract
We investigate a topology optimisation based on the level set method for a design problem of carpet cloaking devices. In topology optimisation of carpet cloaking devices, we need to calculate the electromagnetic response in a semi-infinite domain. We use the boundary element method (BEM) for the electromagnetic analysis since the BEM is more suitable to topology optimisation in wave field than the finite element method; the infinite domain can be evaluated exactly and mesh generation is required only on a boundary with the BEM. Along with the detailed formulation of the BEM-based topology optimisation, we present some numerical examples of design for carpet cloaks. We have confirmed that the designed carpet cloaking devices effectively work for the cases where permittivity, wavelength and angle of incidence are fixed. In addition, we have confirmed that cloaking devices for perturbed permittivity and wavelength can be obtained by redefining the objective function with the KS function.

Key words: Topology optimisation, Carpet cloak, Boundary element method, Level set method

1. Introduction

After the possibility of realisation of a material with negative permeability is shown by Pendry (Pendry, et al., 1999), metamaterials have extensively been researched. Because of its distinguished characteristics, the metamaterial has been used in many applications, such as for waveguides (Hrabar, et al., 2005) and superlens (Fang and Zhang, 2003), etc. Of such potential applications, a cloaking device (Pendry, et al., 2006) is one of the most attractive ones. Cloaking is a technology to make an object invisible by putting materials forming a cloaking device such as a dielectric element around the object. The light path is bent by the cloaking device in such a way to go around the object. Pendry theoretically showed for the first time that a cloaking device can be constructed by using metamaterials. Pendry also showed another strategy of optical cloaking: a carpet cloak (Fig. 1), which makes an object on a flat conducting sheet invisible (Li and Pendry, 2008). Smith experimentally created a cloaking device (Schurig, et al., 2006) and a carpet cloaking device (Liu, et al., 2009) by allocating metamaterials periodically. Such cloaking devices with the metamaterials, however, have some drawbacks for engineering applications. Firstly, the configuration of the cloaking device is not necessarily optimal because the structure of the metamaterial is made by trial and error. Besides, the cloaking device uses the resonance of the electromagnetic wave, which causes loss of energy. To overcome these issues, construction of the carpet cloaking devices without metamaterials are also reported. Valentine, et al. proposed a method to construct a carpet cloaking device with a dielectric photonic crystal. They determined distribution of effective refractive index by quasi-conformal mapping and realised the distribution by a photonic crystal fabricated on a silicon-on-insulator (SOI) wafer. (Valentine, et al., 2010). Hsu, et al. designed a dielectric metasurface for carpet cloaking which is composed of cylinders on a substrate made of Teflon (Hsu, et al., 2015). The height of each cylinder is adjusted so that the reflection angles which is expressed by the generalised Snell’s law will be the same with that for the flat conducting sheet. These devices achieve cloaking effect to some extent without energy loss. The effectiveness, however, highly depends on the wave length and permittivity of dielectric elements.

On the other hand, topology optimisation has extensively been researched as one of the promising methods to determine the optimal configuration with computer simulation. In the topology optimisation, the configuration of an object
is usually expressed as a distribution of materials with the characteristic function that distinguishes void from material domains. This strategy makes it possible to determine not only the optimal shape but also the optimal topology. With the help of the topology optimisation, we can design cloaking devices using only scattering and refractive effect, which does not cause energy loss. Topology optimisation with a characteristic function, however, may suffer from an instability problem such as the checkerboard problem (Sigmund and Petersson, 1998). To avoid the problem, some regularisation methods are introduced such as homogenisation (Bendsoe and Kikuchi, 1988). Andkjær and Sigmund achieved to obtain some symmetric cloak designs with dielectric elements which work for some angles of incidence by using density-based topology optimisation (Andkjær and Sigmund, 2011). Also, Andkjær, et al. succeeded in designing a cloaking device that is effective for both transverse electric (TE) and transverse magnetic (TM) polarisations by using density-based topology optimisation (Andkjær, et al., 2012). Vial and Hao (2015) designed cloaking devices with dielectric elements using density-based topology optimisation. They conducted a modal analysis for optimised cloaking devices and explained cloaking effect as an interaction between different eigenmodes. It is, however, difficult to fabricate their cloaking devices since they include grayscales. Alghough it is possible to remove the grayscales by using a density filter (Bourdin, 2001, Bruns and Tortorelli, 2001) or a Heaviside projection filter (Guest, et al., 2004, Wang, et al. 2011), some parameters in such filters are empirically determined, and the obtained configurations could be too complicated to fabricate according to the parameters.

In this study, we employ the level set method (Sethian and Wiegmann, 2000), in which the boundary of the design objects is expressed by zero contour of the level set function. The level set method enables us to generate a smooth boundary mesh without any grayscale. Furthermore, it is easy to arrange the complexity of the configuration of design objects by using the strategy with the reaction-diffusion equation (Yamada, et al., 2010, Choi, et al., 2011). Fuji, et al. applied the level set method to topology optimisation of cloaking devices with a dielectric material and obtained a cloaking device design that works in a TM polarised field (Fuji, et al., 2013). Otomori, et al. proposed a cloaking device with a ferrite material by using the level set-based topology optimisation and succeeded in designing a cloaking device and a carpet cloaking device in a TM polarised field (Otomori, et al., 2013). Also, Fuji and Nakamura succeeded to design carpet cloaking devices with dielectric element on a wall of finite size in TM polarised field (Fuji and Nakamura, 2015, Fuji and Ueta, 2016). One common issue in all the researches mentioned above, the finite element method (FEM) is used to solve boundary value problems involved in the sensitivity analysis. It is, however, the use of FEM is not necessarily appropriate for topology optimisation of cloaking (resp. carpet cloaking) devices because the boundary value problems are often defined in the infinite (resp. semi-infinite) domain. To treat such problems with the FEM, the infinite domain is approximated by a large domain with an appropriate boundary condition such as a perfect matched layer or absorbing boundary condition. Even when such special boundary conditions are applied, a large analysis domain is required for enough accuracy, which is computationally burdensome. Also, in the case that the FEM is applied to the carpet cloaking, flat conducting sheet is assumed to have finite size. Hence, we cannot design a carpet cloaking device which makes a bump on semi-infinite conducting sheet invisible with the FEM.

In this study, we employ the boundary element method (BEM) for solving the electromagnetic wave problem. In the BEM, the infinity is treated as one of the boundaries, and the radiation condition at the infinity is rigorously treated with the help of the Green function. Also, the BEM can be applied to a design problem of carpet cloaking devices since semi-infinite domain can exactly be treated exactly with the help of the Green function. Furthermore, mesh generation is necessary only on the boundary, which leads to lower meshing costs than that of the FEM. Our previous research has shown the effectiveness of the topology optimisation with the BEM in the electromagnetic field (Isakari, et al., 2016), sound field problems (Isakari, et al., 2014). In this paper, we give some results of topology optimisation of carpet cloaking devices in TE polarised field.

The rest of the paper is organised as follows: In the second section, we show the formulation of the electromagnetic field analysis in a semi-infinite domain and the sensitivity analysis. In the third section, we formulate the topology optimisation problem of carpet cloaking devices. This section includes an expression and an update of the configuration of the target object with the level set method and an algorithm for the topology optimisation. In the fourth section, we provide some numerical examples of topology optimisation of carpet cloaking devices. We discuss the dependency of the obtained optimal configuration on the shape of the scatterer and the frequency of the incident wave. We also show that we can obtain robust with respect to perturbation of material constant and wideband carpet cloaking device by using topology optimisation with a Kreisselmeier-Steinhauser (KS) function (Kreisselmeier, 1979).
2. Electromagnetic field analysis in the semi-infinite domain

In the configuration optimisation problem for carpet cloaking devices (Fig. 1), we treat a situation in which the target object we are trying to hide (i.e. the bump in Fig. 1) lies on a semi-infinite conducting sheet. We, therefore, need to solve electromagnetic wave problems in a semi-infinite domain. This section describes the basic equations of the electromagnetic wave problems in 2D semi-infinite domain and the related boundary element analysis.

We consider the domain in Fig. 2, i.e., we assume that a bump domain \( \Omega_p \) exists on the boundary \( \Gamma_h \) which is the boundary of the semi-infinite domain \( \Omega_1 \). Also, we assume that a dielectric domain \( \Omega_2 \) exists around the bump. \( \Gamma_p \) and \( \Gamma_d \) denote the boundary of the bump and the dielectric domain, respectively. We are interested in evaluating the response of the magnetic field in the TE polarised wave when the domain is impinged by an incident wave \( u^{inc} \). In each domain, the tangential component of the magnetic field \( u \) in the TE polarised wave solves the following boundary value problem:

\[
\begin{align*}
\nabla^2 u(x) + k_i^2 u(x) &= 0, & x \in \Omega_i, \\
\nabla^2 u(x) + k_2^2 u(x) &= 0, & x \in \Omega_2, \\
\n\frac{1}{\varepsilon_1} \left( \frac{\partial u}{\partial n} \right)^1 &= \frac{1}{\varepsilon_2} \left( \frac{\partial u}{\partial n} \right)^2, & x \in \Gamma_d, \\
\frac{\partial u}{\partial n} &= 0, & x \in \Gamma_p \cup \Gamma_h, \\
\text{Radiation condition} & \quad ||x|| \to \infty,
\end{align*}
\]

where \( \partial/\partial n := n \cdot \nabla \) denotes the normal derivative. The normal vector \( n \) is defined positive when \( n \) is directed from \( \Omega_1 \). The superscripted indices \( i \) in the boundary conditions (3) and (4) mean the limit value as \( x \) goes to the boundary \( \Gamma_d \) from the domain \( \Omega_i \) ( \( i = 1, 2 \) ). \( k_i \) is the wave number in the domain \( \Omega_i \), which is defined as

\[
k_i = \omega \sqrt{\mu_i \varepsilon_i},
\]

where \( \mu_i \) and \( \varepsilon_i \) denote the permeability and permittivity in \( \Omega_i \), respectively. In this study, as is often the case, we assume that \( \mu_i = 1 \) for \( i = 1, 2 \).
The solution $u$ of the boundary value problem (1)–(6) has the following integral representation in $\Omega_1$:

$$u(x) = u^{inc}(x) + \int_{\Gamma_d} G_1^0(x, y) \frac{\partial u(y)}{\partial n_y} \, ds_y - \int_{\Gamma_N} \frac{\partial G_1^0(x, y)}{\partial n_y} u(y) \, ds_y \quad x \in \Omega_1, \quad (8)$$

where $G_1^0$ is the Green function for the semi-infinite problem defined as

$$G_1^0(x, y) + k_1^2 G_1^0(x, y) = -\delta(x - y) \quad x, y \in \Omega_0, \quad (9)$$
$$\frac{\partial G_1^0(x, y)}{\partial n_y} = 0 \quad y \in \Gamma_h, \quad (10)$$

Radiation condition

$$||x|| \to \infty, \quad (11)$$

where $\Omega_0$ denotes the semi-infinite domain. The Green function $G_1^0$ has the following representation:

$$G_1(x, y) = G_1(x, y) + G_1(x, y'), \quad (12)$$

where $G_1(x, y)$ is the fundamental solution of the Helmholtz equation in 2D, which is expressed as

$$G_1(x, y) = \frac{1}{4} H_0^{(1)}(k||x - y||), \quad (13)$$

where $H_0^{(1)}$ is the 0th Hankel function of the first kind. Also, $y'$ is the mirror image of $y$ against the dashed line in Fig. 3. By using Eq. (12), Eq. (8) is transformed to the following integral representation:

$$u(x) = u^{inc}(x) + u^{inc'}(x) + \int_{\Gamma_r \cup \Gamma_p} G_1(x, y) \frac{\partial u(y)}{\partial n_y} \, ds_y - \int_{\Gamma_r \cup \Gamma_p} \frac{\partial G_1(x, y)}{\partial n_y} u(y) \, ds_y \quad x \in \Omega_1, \quad (14)$$

where $\Gamma_r$, $\Gamma_p$ and $u^{inc}$ denote a mirror image of $\Gamma_d$, $\Gamma_p$ and the incident wave $u^{inc}$ against the dashed line in Fig. 3, respectively. In the case of $x \in \Omega_2$, we obtain the following integral representation by using the fundamental solution (13):

$$u(x) = -\int_{\Gamma_r \cup \Gamma_p} G_2(x, y) \frac{\partial u(y)}{\partial n_y} \, ds_y + \int_{\Gamma_r \cup \Gamma_p} \frac{\partial G_2(x, y)}{\partial n_y} u(y) \, ds_y \quad x \in \Omega_2. \quad (15)$$

In the following, we denote $\Gamma_r \cup \Gamma_d$ (resp. $\Gamma_p \cup \Gamma_d$) as $\Gamma_d$ (resp. $\Gamma_p$) for the readability. After taking the limit as $x \to \Gamma_d$ and $\Gamma_p$ in Eqs. (14) and (15), we can numerically solve the obtained boundary integral equations. The obtained boundary integral equations may, however, suffer from so called fictitious eigenfrequency problem. To remove the fictitious eigenfrequency, we employ the Poggio-Miller-Chan-Harrington-Wu-Tsai (PMCHWT) (Chew, 1995) formulation and obtain the following integral equations:

$$\begin{bmatrix}
\frac{1}{2} I + D_{\Gamma_d}^1 & D_{\Gamma_d}^1 \\
D_{\Gamma_d}^1 & \frac{1}{\varepsilon_1} S_{\Gamma_d}^1 + \varepsilon_1 \varepsilon_2 \frac{\partial}{\partial n} \frac{\partial}{\partial n} \end{bmatrix} \begin{bmatrix}
\varepsilon_1 S_{\Gamma_d}^1 \\
\varepsilon_1 \varepsilon_2 \frac{\partial}{\partial n} \frac{\partial}{\partial n}
\end{bmatrix} \begin{bmatrix}
\varepsilon_1 S_{\Gamma_d}^1 \\
\varepsilon_1 \varepsilon_2 \frac{\partial}{\partial n} \frac{\partial}{\partial n}
\end{bmatrix} \begin{bmatrix}
u_p \\
u_d \\
u_d
\end{bmatrix} = \begin{bmatrix}
u^{inc} + u^{inc'}_p \\
u^{inc} + u^{inc'}_d \\
u^{inc} + u^{inc'}_d
\end{bmatrix}, \quad (16)$$

where superscripted index ‘inc’ means the incident wave, and dashed one represents its mirror image against $\Gamma_h$, and $u_d$ and $u_d$ denote the magnetic field defined on $\Gamma_d$ and $\Gamma_p$, respectively. Also, $u_d$ is defined on $\Gamma_d$ as follows:

$$w_d = \frac{1}{\varepsilon_1} \left( \frac{\partial u}{\partial n} \right)^2 = \frac{1}{\varepsilon_2} \left( \frac{\partial u}{\partial n} \right)^2 \quad x \in \Gamma_d, \quad (17)$$

where the superscripted indices indicate the limit value from $\Omega_i$. Furthermore, $I$ is the identity operator and $D_{\Gamma_d}$, $S_{\Gamma_d}^1$, $D_{\Gamma_d}^1$ and $N_{\Gamma_d}^1$ respectively denote the following operators:

$$[D_{\Gamma_d}^1 \phi](x) = \int_{\Gamma_d} \frac{\partial G_1(x, y)}{\partial n_y} \phi(y) \, ds_y, \quad (18)$$
$$[S_{\Gamma_d}^1 \psi](x) = \int_{\Gamma_d} G_1(x, y) \psi(y) \, ds_y, \quad (19)$$

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\[ [D^1_1\psi](x) = \int_{\Omega_1} \frac{\partial G_1(x,y)}{\partial n_x} \psi(y) dy, \quad (20) \]
\[ [N^1_1\phi](x) = \int_{\partial \Omega_1} \frac{\partial^2 G_1(x,y)}{\partial n_x \partial n_y} \phi(y) dy, \quad (21) \]

where \( \phi \) and \( \psi \) are density functions. The definitions of \( \Sigma^2_1 \) etc., are obtained by replacing the kernel function \( G_1 \) by the fundamental solution \( G_2 \) in Eqs. (18)–(21). We derive algebraic equations by discretising Eq. (16) with the collocation and an appropriate basis function. After solving the algebraic equations, we can calculate \( u \) in \( \Omega_1 \cup \Omega_2 \) with the integral representations either (14) or (15).

3. Topology optimisation

Topology optimisation is a method to determine the optimal structure of a device to maximise its efficiency by iteratively updating the configuration in a design domain \( D \) based on computer simulation (Fig. 4). Hence, in order to formulate a topology optimisation problem, we need to define an objective function which expresses efficiency of a device. In this section we formulate the topology optimisation of carpet cloaking devices. Our objective is to hide the bump on the boundary by appropriately allocating dielectric elements. In other words, we try to find a distribution of dielectric elements by which the magnetic field distribution is the same as that without the bump and dielectric elements (Fig. 5). We therefore define the objective function for the optimal design of carpet cloaking devices as follows:

Find \( \Omega_2 \) such that
\[ J = \sum_{m=1}^{M} \| u(x_{\text{obs}}^m) - \{ u^{\text{inc}}(x_{\text{obs}}^m) + u^{\text{inc}}(x_{\text{obs}}^m) \} \|^2, \quad (22) \]

where \( x_{\text{obs}}^m \) denotes fixed observation points, and \( M \) denotes the number of observation points. Although the objective function for cloaking devices is often defined as the volume integrals of the intensity of the scattered field (Andkjr, et al., 2012, Fujii, et al., 2013, Otomori, et al., 2013), we use the sum of the scattered field on fixed observation points. In our numerical experiments, \( M \) is set so large that the effects of the number on optimal results are not observed. \( u \) in (22) is required to satisfy the boundary value problem (1)–(6). Note that, although the design domain \( D \) is bounded, the boundary value problem (1)–(6) is defined in unbounded domain. In the following subsections, we show a method to express the configuration of dielectric element \( \Omega_2 \) and a method to update the configuration.

3.1. The level set method

We use the level set method to express the shape of the design object in the process of exploring an optimal distribution of the design object. In this method, we define the level set function \( \phi(x) \) \((-1 \leq \phi(x) \leq 1\) for \( x \in D \). Each domain is
expressed by the value of the level set function as follows:

\[
\begin{align*}
\Omega_1 &= \{ x \mid 0 < \phi(x) \leq 1 \}, \\
\Gamma_d &= \{ x \mid \phi(x) = 0 \}, \\
\Omega_2 &= \{ x \mid -1 \leq \phi(x) < 0 \}.
\end{align*}
\]  

(23) \hspace{1cm} (24) \hspace{1cm} (25)

The value of the level set function is numerically stored on a grid point of lattice expanding the design domain \( D \). The boundary element mesh is created by using the following procedure. Firstly, we linearly interpolate the value of the level set function. Secondly, we consider points at which \( \phi(x) = 0 \) as new boundary element nodes. By connecting these nodes, the boundary element mesh is generated (Fig. 6). Length of each boundary element obtained by the above procedure is uneven. Hence, we improve these boundary elements so that the length of each element becomes almost the same (Isakari, et al., 2016).

![Fig. 6 Generation of a boundary mesh from the level set function.](image)

### 3.2. Update of the level set function

In the process of topology optimisation, the shape of the target object will be updated repeatedly. In this study, we use the reaction diffusion equation for the update. That is, we update the level set function by the following equation with appropriate initial and boundary conditions (Yamada, et al., 2010):

\[
\frac{\partial \phi}{\partial t} = C \text{sgn}(\phi(x)) T(x) + \tau l^2 \nabla \phi(x),
\]

(26)

where the first term of RHS denotes the direction and scale of the update. \( C \) is a constant used to arrange the scale. \( T \) denotes a topological derivative which is the differential of the objective function due to allocation of an infinitesimal circle. For the detail of the topological derivative, see Section 3.3 and appendices A and B. The second term of RHS is the so called Tikhonov regularisation term and work as a perimeter constraint. Hence, the complexity of obtained configuration after the update can be arranged by adjusting \( \tau \). (Choi, et al., 2011). \( l \) denotes characteristic length.

### 3.3. Topological derivative

In the process of topology optimisation, a configuration is iteratively updated based on the topological derivative. In this section, we derive the topological derivative in the transmission problem in 2D electromagnetic wave problems (Carpio and Rapún, 2008).

![Fig. 7 Allocation of an infinitesimal circular domain in the design domain causes a change in the objective function.](image)
where \( s(\varepsilon) \) is the measure of an infinitesimal circle, and \( \delta J \) is the variation of the objective function evaluated as follows:

\[
\delta J = \Re \left[ \frac{\partial J}{\partial \delta u} \right],
\]

(28)

where \( \delta u \) denotes the variation of the magnetic field and solves the following boundary value problem:

\[
\nabla^2 \delta u(x) + k_1^2 \delta u(x) = 0 \quad x \in \Omega_1,
\]

(29)

\[
\nabla^2 \delta u(x) + k_2^2 \delta u(x) = 0 \quad x \in \Omega_2,
\]

(30)

\[
\nabla^2 \tilde{u}(x) + k_2^2 \tilde{u}(x) = 0 \quad x \in \Omega_e,
\]

(31)

\[
\frac{\partial \delta u}{\partial n} = 0 \quad x \in \Gamma_p,
\]

(32)

\[
\delta u_1 = \delta u_2 \quad x \in \Gamma_d,
\]

(33)

\[
\frac{1}{\varepsilon_1} \left( \frac{\partial \delta u}{\partial n} \right) = \frac{1}{\varepsilon_2} \left( \frac{\partial \tilde{u}}{\partial n} \right)^2 \quad x \in \Gamma_d,
\]

(34)

\[
u_1 + \delta u_1 = \tilde{u} \quad x \in \Gamma_e,
\]

(35)

\[
\frac{1}{\varepsilon_1} \left( \frac{\partial (\delta u_1 + \delta u_1)}{\partial n} \right) = \frac{1}{\varepsilon_2} \left( \frac{\partial \tilde{u}_1}{\partial n} \right)^2 \quad x \in \Gamma_e,
\]

(36)

Radiation condition \( ||x|| \rightarrow \infty \),

(37)

where the normal vector \( n \) on \( \Gamma_e \) is defined positive when \( n \) is directed from \( \Omega_1 \). To evaluate the right hand side of Eq. (28), we use the adjoint variable method. According to the objective function in Eq. (22) and the constrain conditions in Eqs. (1)–(6), it is natural to define the adjoint boundary value problem as follows:

\[
\nabla^2 \tilde{u}(x) + k_2^2 \tilde{u}(x) + \sum_{m=1}^{M} 2\left[ \mu(x_m^{\text{obs}}) - \left[ \mu_{\text{inc}}(x_m^{\text{obs}}) + \mu_{\text{inc}}(x_m^{\text{obs}}) \right] \right] \delta(x - x_m^{\text{obs}}) = 0 \quad x \in \Omega_1,
\]

(38)

\[
\nabla^2 \tilde{u}(x) + k_2^2 \tilde{u}(x) = 0 \quad x \in \Omega_2,
\]

(39)

\[
\frac{\partial \tilde{u}}{\partial n} = 0 \quad x \in \Gamma_p,
\]

(40)

\[
\tilde{u}_1 = \tilde{u}_2 \quad x \in \Gamma_d,
\]

(41)

\[
\frac{1}{\varepsilon_1} \left( \frac{\partial \tilde{u}_1}{\partial n} \right) = \frac{1}{\varepsilon_2} \left( \frac{\partial \tilde{u}_1}{\partial n} \right)^2 \quad x \in \Gamma_d,
\]

(42)

Radiation condition \( ||x|| \rightarrow \infty \),

(43)

where \( \tilde{u} \) is the adjoint variable. By using the reciprocal relations for

- \( \tilde{u} \) and \( \delta u \) in \( \Omega_1 \setminus \Omega_e \),
- \( \tilde{u} \) and \( \delta u \) in \( \Omega_2 \),
- \( \tilde{u} \) and \( \delta u \) in \( \Omega_e \),

we can rewrite \( \delta J \) in Eq. (28) as

\[
\delta J = \Re \left[ \frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2} \int_{\Omega} \frac{\partial \tilde{u}}{\partial n} d\Gamma + \int_{\Omega_e} \frac{\varepsilon_2}{\varepsilon_1} \omega^2 \tilde{u}\bar{u} d\Omega \right].
\]

(44)

The asymptotic behaviour of \( \tilde{u} \) and \( \bar{u}\partial \tilde{u}/\partial n \) on \( \Gamma_e \) is evaluated as follows:

\[
\tilde{u}(x) = \tilde{u}(x_0) + \tilde{u}_j(x_0) n_j(x) \varepsilon + o(\varepsilon) \quad x \in \Gamma_e,
\]

(45)

\[
\frac{\partial \tilde{u}(x)}{\partial n} = \frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \tilde{u}_j(x_0) n_j(x) + \left( \frac{4\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \int_{\Omega} u_{ij}(x_0) n_i n_j - \frac{1}{2} k_2^2 u(x_0) \right) \varepsilon + o(\varepsilon) \quad x \in \Gamma_e,
\]

(46)
where \( x_0 \) represents the centre of \( \Omega \). Note that, since \( d\Gamma \) is \( O(\varepsilon) \), it is sufficient to consider only the first 2 terms. Also, \( \hat{u} \) can be expanded as follows:

\[
\hat{u}(x) = u(x_0) + \frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2} u_j(x_0)(x - x_0)_j + o(\varepsilon) \quad x \in \Omega_c.
\]  

(47)

For the detailed derivation for the asymptotic expansions, the reader is referred to Appendix A. By substituting Eqs. (45), (46) and (47) into Eq. (44), we obtain the following:

\[
\delta J = \Re \left[ \frac{2(\varepsilon_2 - \varepsilon_1)}{\varepsilon_1(\varepsilon_1 + \varepsilon_2)} \nabla \hat{u} \cdot \nabla u \right] \pi \varepsilon^2 + o(\varepsilon^2) \right].
\]  

(48)

The topological derivative is derived as follows from the definition (27):

\[
\mathcal{T} = \Re \left[ \frac{2(\varepsilon_2 - \varepsilon_1)}{\varepsilon_1(\varepsilon_1 + \varepsilon_2)} \nabla \hat{u} \cdot \nabla u \right].
\]  

(49)

Note that we have chosen the measure \( s(\varepsilon) \) of \( \Omega_c \) as \( s(\varepsilon) = \pi \varepsilon^2 \) in Eq. (27). The reader is referred to Appendix B for the validation of the topological derivative.

3.4. Algorithm for the topology optimisation

Combining the formulations presented in this section, the whole algorithm of the present topology optimisation is summarised as the flowchart in Fig. 8. We use the boundary element method to solve forward and adjoint problems and to calculate the objective function, while the finite element method for updating the level set function. This is because the initial-boundary value problem (6) is defined on a bounded domain \( D \), and \( D \) is irrelevant to the fictitious time \( t \).

Solving the evolution equation (6) by the BEM is possible but not efficient since Eq. (6) involves the source term (the first term in RHS). In this case, a domain mesh is required even when the BEM used. Hence, in the present study, the FEM is employed to solve the evolution equation. Convergence criterion for topology optimisations is usually determined with the KKT condition (Yamada, et al., 2010). In our method, however, the topological derivative does not expresses a differentiation of the objective function with respect to the design variable (the level set function). Because of this, it is difficult to define the KKT condition. In this study, we judge the objective function is converged at step \( k \) if the objective function of the last 50 steps \( (J_{k-49}, \ldots, J_k) \) satisfy the following relations:

\[
J' \leq \varepsilon_1^{\text{conv}},
\]

\[
\max_{1 \leq i \leq 50} \left\{ \frac{J_{k-i+1}}{J_{k-i+1}} \right\} \leq \varepsilon_2^{\text{conv}},
\]

(50)

(51)

where \( J' \) denotes the gradient of the objective function which is evaluated with the least square for \( \log J_{k-49}, \ldots, \log J_k \). Also, \( \varepsilon_1^{\text{conv}}, \varepsilon_2^{\text{conv}} \) are parameters.

4. Numerical example

4.1. Problem statement

In this section, we show the efficiency of the present method with some numerical examples. Throughout all the examples, we consider a semi-infinite domain which is filled with perfect electrical conductor (PEC) which has a semi-circular bump (Fig. 9). The radius and centre of the bump are 5 and \((25, 0)\), respectively. The complement region of the PEC is assumed to be filled with vacuum of (relative) permittivity \( \varepsilon_1 = 1 \). The domain is impinged by a plane wave whose wavelength and the incident angle are \( \lambda = 10 \) and \( \theta = 45^\circ \), respectively. Our objective is to make the bump invisible by allocating dielectric materials in the design domain \( D := (0, 50) \times (0, 25) \). The invisibility is evaluated with Eq.(22) with 3080 observation points \( x_{\text{obs}} \) set in the orange region \( \Omega_{\text{obs}} \) which is at a distance of half the size of wavelength \( \lambda \) from the design domain \( D \) (Fig. 9). The observation points are set in equal distance, and the distance between each observation point is \( \lambda/10 \). \( u \) in (22), and \( u' \) and \( \hat{u} \) in (49) are calculated with the boundary element method, in which the boundary integral equation (16) is discretised with the quadratic elements. The discretised integral equation (algebraic equations) are solved with GMRES whose tolerance is \( 10^{-10} \). In the discretisation, the boundary of the semi-circular PEC \( \Gamma_p \) is divided into 100 boundary elements, and the boundary of dielectric materials \( \Gamma_d \) is divided into boundary elements by a procedure shown in Section 3.1.
4.2. Topology optimisation

We show the result of the topology optimisation. Firstly, we run the optimisation for carpet cloaking in the case that the cloaking is made of dielectric materials with the permittivity $\varepsilon_2$ of either 3 or 4 or 5. We define an initial configuration $\Omega_{\text{init}}^2$ by the following:

$$\Omega_{\text{init}}^2 = \left\{ x \mid \mathcal{T}^0(x) \leq 0, \| x - (25,0) \|_2 \leq 25 \right\},$$

(52)

where $\mathcal{T}^0(x)$ denotes the topological derivative for a bare bump (Fig. 10). Since the sign of the topological derivative (49) does not depend on the value of $\varepsilon_2$, the initial configuration is the same regardless of $\varepsilon_2$. In the results to follow, we denote the regularised value of the objective function $J$ in Eq.(22) as $\tilde{J}$. From the history of the objective function for $\tau = 4.0 \times 10^{-3}$ in Fig. 11, one observes that the objective functions for all the permittivity concerned are reduced successfully while vibrating. In this study, the topological derivative defined by Eq.(27) expresses the variation of the objective function when only a single infinitesimal circular dielectric element is inserted in the design domain. In the process of the topology optimisation, however, some dielectric elements or irregular shaped elements may happen to be allocated at the same time. In such a case, the objective function can be increased no matter how small the step size for (26) is. In our future publications, we may address an amelioration of our methodology to ensure the monotone decreasing of the objective function.

The optimal shapes and the distributions of the scattered magnetic field are shown in Fig. 12–Fig. 14. In these figures, the perimeters of dielectric elements $l$ are inserted. As $\tau$ becomes larger, $l$ becomes smaller while $\tilde{J}$ for the
optimal configuration becomes larger. For small $\tau$, the distributions of the magnetic fields are almost equal to those in the case without the bump. Through the comparison of Fig. 12–Fig. 14, one observes that the optimal configuration is highly dependent on $\varepsilon_2$, and the magnetic fields in the design domain are quite different from each other. Also, results when we change the radius $r$ of the bump and angle of incidence $\theta$ is shown in Fig. 15 and 16 respectively. In every cases the objective functions are successfully reduced. From these results the effectiveness of the present method for simple design problems of carpet cloaking is confirmed. In order to investigate the mechanism of the obtained cloakings, we have

Fig. 10 Configuration of the dielectric elements and corresponding scattering magnetic fields, with neither bump nor dielectric element (left), a bump (center), and a bump and dielectric elements with initial shape of $\varepsilon_2 = 2$ (right). The values of the objective function $J$ are inserted in the figures. $J$ is regularised by the one for the bare bump case.

Fig. 11 History of $J$ for $\varepsilon_2 = 2, 3, 4$ and $\tau = 4.0 \times 10^{-3}$.

Fig. 12 Optimal configuration and scattering magnetic field for $\varepsilon_2 = 2$ when $\tau = (a) 2.0 \times 10^{-3}$, (b) $4.0 \times 10^{-3}$, (c) $8.0 \times 10^{-3}$ and (d) $1.6 \times 10^{-2}$.
Fig. 13 Optimal configuration and scattering magnetic field for $e_2 = 3$ when $\tau = (a) 2.0 \times 10^{-3}$, $l = 480$,
(b) $4.0 \times 10^{-3}$, $l = 468$,
(c) $8.0 \times 10^{-3}$, $l = 333$ and (d) $1.6 \times 10^{-2}$.

Fig. 14 Optimal configuration and scattering magnetic field for $e_2 = 4$ when $\tau = (a) 2.0 \times 10^{-3}$, $l = 394$,
(b) $4.0 \times 10^{-3}$, $l = 302$,
(c) $8.0 \times 10^{-3}$, $l = 269$ and (d) $1.6 \times 10^{-2}$.

Fig. 15 Optimal configuration and scattering magnetic field for a bump whose radius is 8 (left) and 10 (right) in the case that $e_2 = 2$ and $\tau = 1.0 \times 10^{-3}$.
plotted the Poynting vectors for each optimal configuration in the case of $\varepsilon_2 = 4 \times 10^{-3}$ (Fig. 12–Fig. 14). The Poynting vector $P$ in time domain is calculated as $P = E \times H$, where $E$ and $H$ are respectively the electric and magnetic fields in time domain. In the TE mode, $E$ and $H$ are calculated from $u$ as follows:

\[
H = \begin{pmatrix}
0, 0, u_r \cos \omega t + u_i \sin \omega t
\end{pmatrix}^T, \\
E = \frac{1}{\omega \varepsilon_0} \begin{pmatrix}
\frac{\partial u_r}{\partial y} \sin \omega t - \frac{\partial u_i}{\partial y} \cos \omega t, \\
-\frac{\partial u_r}{\partial x} \sin \omega t + \frac{\partial u_i}{\partial x} \cos \omega t, 0
\end{pmatrix}^T,
\]

where $u_r$ and $u_i$ denote the real and imaginary parts of $u$, respectively. By using the above relations, the time averaged Poynting vector $\overline{P}$ is derived as follows:

\[
\overline{P} = \frac{\omega}{2 \pi} \int_0^{2\pi/\omega} E \times H dt
\]

\[
= \frac{1}{2\omega \varepsilon_0} \begin{pmatrix}
u_r \frac{\partial u_i}{\partial x} - u_i \frac{\partial u_r}{\partial x}, u_r \frac{\partial u_i}{\partial y} - u_i \frac{\partial u_r}{\partial y}, 0
\end{pmatrix}^T,
\]

where $\frac{2\pi}{\omega}$ is the period of harmonic wave concerned. In the case of $\varepsilon_2 = 2$ and 3, the flow of the Poynting vector is controlled to bypass the PEC, which indicates that the scattering field on the surface of the PEC has almost no effect on the magnetic field around the PEC. In contrast, in the case of $\varepsilon_2 = 4$, intensive energy flow is observed near the PEC, from which the scattering on the surface of the PEC is considered to be an important factor for the effectiveness of the cloaking device. These observations are also confirmed by Fig. 18, in which the obtained cloaking devices are allocated around a starfish-shaped PEC.
Fig. 18 Attempts to hide a starfish-shaped PEC by cloaking devices obtained for a circular-shaped PEC when $\varepsilon_2 = 4$: (10)

(a) $\mathcal{J} = 1.1 \times 10^{-1}$

(b) $\mathcal{J} = 1.0 \times 10^{0}$

Fig. 18 Attempts to hide a starfish-shaped PEC by cloaking devices obtained for a circular-shaped PEC when $\varepsilon_2 = 4$: (10)

(a) $\mathcal{J} = 1.1 \times 10^{-1}$

(b) $\mathcal{J} = 1.0 \times 10^{0}$

Fig. 19 Poynting vectors and magnetic responses when permittivity of the optimal configuration for $\varepsilon_2 = 2$ (resp. $\varepsilon_2 = 4$) are changed to 4 (resp. 2). In both cases, the disturbance of the magnetic field is observed. In

the real world, however, permittivity of materials is not necessarily exactly identical to the expected value. Hence, it is desirable that a cloaking device is effective even if permittivity of the device is little bit different from the expected one. In order to design a cloaking device which is effective for a range of permittivity, we define the objective function using the KS function as follows:

$$J = \log \left( \sum_{i=1}^{n} e^{w_i J_i} \right). \tag{57}$$

where $J_i$ denotes the objective function in the case in which permittivity $\varepsilon_2 = i$. Also, $w_i$ is the weight function which is fixed to 1.0 in this paper. By the above definition, the configuration will be updated in such a way that the biggest objective function decreases in each optimisation step. The topological derivative corresponding to the KS function in Eq. (57) can be calculated as the differential of composite function as follows (Isakari, et al., 2016):

$$T = \frac{\sum_{i=1}^{n} w_i e^{w_i J_i}}{\sum_{i=1}^{n} w_i e^{w_i J_i}}. \tag{58}$$

In our implementation, the objective functions are normalised to avoid an overflow in the following manner:

$$J_i' = \frac{J_i}{c}. \tag{59}$$
\[
e = \frac{\sum_{i=1}^{n} J_i}{n},
\]

We consider to minimise the objective function (57) for \( \varepsilon_2 = 2 \) and 4. Settings of the topology optimisation other than the permittivity is the same with the previous example. Fig. 20 shows the obtained configuration as the result of topology optimisation. Also, the Poynting vector and magnetic response for the obtained configuration with permittivity \( \varepsilon_2 = 2, 4 \) are shown in Fig. 21. In both cases, the obtained structure shows a cloaking effect to a certain degree. Also, we find that scattering on the surface of the PEC is reduced compared to Fig. 17 in the case that \( \varepsilon_2 = 4 \). Through the comparison of Poynting vectors in each permittivity, we find that the distribution of the Poynting vectors in each case is similar. Hence, the obtained configuration is considered to be effective in similar mechanism for both permittivities.

![Fig. 20 Optimal configuration obtained by the topology optimisation with the KS function.](image)

![Fig. 21 Poynting vectors and magnetic responses of the optimal configuration in the case that the KS function is used as the objective function (left: \( \varepsilon_2 = 2 \), right: \( \varepsilon_2 = 4 \)).](image)

Next, we investigate the wavelength dependency of the effectiveness of the carpet cloaking devices. We calculated \( \mathcal{J} \) for the optimal configuration in the case that \( \varepsilon_2 = 2, 3, 4 \) and \( \tau = 4.0 \times 10^{-3} \) while changing the wavelength of the incident wave from 8 to 12 by 0.01. The result is shown in Fig. 22. In every cases, the objective function changes drastically near the wavelength \( \lambda = 10 \). We show the range of wavelength in which \( \mathcal{J} \) is less than 1.0 in Table 1. To design a cloaking device that is effective for a range of frequencies, we employ again the KS function (57). We define \( J_i \) in Eq. (57) as the objective function (22) in the case that the wave length of the incident wave is \( \lambda_i \). We conducted topology optimisation

![Fig. 22 Objective function dependence on the wavelength of the incident wave.](image)
with $\lambda = \{9.4, 10.0, 10.6\}$ and $\varepsilon_2 = 2$. Fig. 23 shows a comparison between the optimal configuration in the case that the KS function is used as the objective function and that in the case that the KS function is not used. The distribution of each dielectric element is almost the same but each configuration is a little bit different. Fig. 24 shows the dependency of $J$ on the wavelength. Case 1 is the case in which the KS function is not used as the objective function, while Case 2 is the case in which the KS function is used as the objective function. The figure shows that the change of the objective function around $\lambda = 10.0$ gets smoother by using the KS function. The range in which the normalised objective function is less than 1 in each case is shown in Table 2. The distributions of the scattering magnetic field for $\lambda = \{9.4, 10.0, 10.6\}$ are shown in Fig. 25. In every case, the optimal configuration shows a cloaking effect to some extent.

### Table 1

| $\varepsilon_2$ | $\{ \lambda \mid J(\lambda) \leq 1.0 \}$ |
|-----------------|------------------------------------------|
| 2               | [9.23, 10.69]                            |
| 3               | [9.49, 10.54]                            |
| 4               | [9.53, 10.7]                            |

### Table 2

| $\{ \lambda \mid J(\lambda) \leq 1.0 \}$ | case 1 | case 2 |
|------------------------------------------|--------|--------|
| [9.23, 10.69]                            |        |        |
| [9.10, 10.95]                            |        |        |

5. **Conclusion**

In this study, we investigated a topology optimisation with the BEM to the design of carpet cloak devices. We formulated the BEM in a semi-infinite domain using Green’s function and the PMCHWT formulation. Numerical examples indicate that our method is effective for determining the shape of carpet cloaking devices for TE polarised incidence with a specific frequency. Furthermore, we showed that the dependency of the cloaking effect on permittivity of the device and frequency of the incident wave is weakened by using an objective function defined by the KS function. Extending the proposed method to three dimensional problems is an interesting future topic and will require an acceleration of the BEM. We will need to employ the LU decomposition and the $H$-matrix method which is more suitable for an optimisation problem than the current solver GMRES.
Fig. 25 Magnetic fields in the optimal configuration in the case that the KS function is used as the objective function (left: $\lambda = 9.4$, centre: $\lambda = 10.0$, and right: $\lambda = 10.6$)

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1. Appendix A: Evaluation of the asymptotic behaviour of $\hat{u}$ and $\partial \hat{u} / \partial n$ on $\Gamma_e$

We evaluate the asymptotic behaviour of $\partial \hat{u} / \partial n$ on $\Gamma_e$. Since $\hat{u}$ is the solution of Eq. (31), $\hat{u}$ in $\Omega_e$ is expressed as follows:

$$\hat{u}(x) = \sum_{n=0}^{\infty} a_n J_n(k_2 r)e^{in\theta} \quad x \in \Omega_e,$$

(A.1)

where $a_n$ denotes a complex coefficient and $J_n$ is the Bessel function of order $n$. Hence, $\hat{u}$ and its normal derivative on $\Gamma_e$ can be expressed as:

$$\hat{u}(x) = \sum_{n=0}^{\infty} a_n J_n(k_2 r)e^{in\theta} \quad x \in \Gamma_e,$$

(A.2)

$$\partial \hat{u}(x) \bigg/ \partial n = \sum_{n=0}^{\infty} a_n \left( J_n(k_2 r) \frac{d}{dr} r \bigg|_{r=e} e^{in\theta} \quad x \in \Gamma_e.$$

(A.3)

Similarly, $\hat{u}$ and its normal derivative on $\Gamma_e$ can be written as:

$$\delta u(x) = \sum_{n=0}^{\infty} b_n H_n^{(1)}(k_1 r)e^{in\theta} \quad x \in \Gamma_e,$$

(A.4)

$$\frac{\partial \delta u(x)}{\partial n} = \sum_{n=0}^{\infty} b_n \frac{d}{dr} H_n^{(1)}(k_1 r) \bigg|_{r=e} e^{in\theta} \quad x \in \Gamma_e,$$

(A.5)

where $b_n$ is a complex coefficient and $H_n^{(1)}$ denotes the Hankel function of first kind of order $n$. Also, $u$ and $\partial u / \partial n$ on $\Gamma_e$ can be expressed as:

$$u(x) = \sum_{n=0}^{\infty} c_n e^{in\theta} \quad x \in \Gamma_e,$$

(A.6)

$$\frac{\partial u(x)}{\partial n} = \sum_{n=0}^{\infty} nc_n e^{in\theta} \quad x \in \Gamma_e,$$

(A.7)

in which $c_n = O(1)$ is a complex coefficient. By substituting Eqs. (A.2)–(A.7) into Eqs. (35) and (36), we have the following estimation:

$$a_0 = \frac{\varepsilon \varepsilon_2 H_0^{(1)}(k_1 \varepsilon)}{\varepsilon \varepsilon_2 H_0^{(1)}(k_1 \varepsilon) J_0(k_2 \varepsilon) - \varepsilon_1 H_0^{(1)}(k_1 \varepsilon) J_0'(k_2 \varepsilon)} c_0.$$

(A.8)
Hence, we obtain the following estimates:

\[
a_n = \frac{\varepsilon_2 e H_0^{(1)}(k_1 e) - |n| H_0^{(1)}(k_1 e)) e^{[n]} - 1}{\varepsilon_2 H_0^{(1)}(k_1 e) J_n(k_2 e) - e_1 H_0^{(1)}(k_1 e) J_n(k_2 e)} c_n \quad (|n| \geq 1),
\]

(A.9)

where \( f'(k,e) \) indicates \( \partial f(k,e)/\partial r \). From the Eqs. (A.8) and (A.9), we can estimate the asymptotic behaviour of \( a_n \) as \( a_n = O(1) \) for all \( n \). Furthermore, \( dJ_n(k_2 r)/dr \) shows the following asymptotic behaviour:

\[
\left. \frac{dJ_0(k_2 r)}{dr} \right|_{r=a} = O(\varepsilon),
\]

(A.10)

\[
\left. \frac{dJ_n(k_2 r)}{dr} \right|_{r=a} = O(\varepsilon^{[n]}). \quad (n \geq 1).
\]

(A.11)

Hence, we obtain the following estimates:

\[
a_0 \left. \frac{dJ_0(k_2 r)}{dr} \right|_{r=a} = O(\varepsilon),
\]

(A.12)

\[
a_n \left. \frac{dJ_n(k_2 r)}{dr} \right|_{r=a} = O(\varepsilon^{[n]-1}) \quad \text{for} \quad n \geq 1.
\]

(A.13)

From Eqs. (A.3), (A.12) and (A.13) we have

\[
\frac{\partial \hat{u}(x)}{\partial n} = \sum_{n=2} \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \frac{dJ_n(k_2 r)}{dr} \quad e^{[n]} + o(\varepsilon) \quad x \in \Gamma_e.
\]

(A.14)

Hence, we just need to evaluate \( a_n \) for \( |n| \leq 2 \). On the other hand, \( u(x) \) on \( \Gamma_e \) have the following representation:

\[
u(x) = u(x_0) + \varepsilon u_{ij}(x_0)n_j + \frac{\varepsilon^2}{2} u_{ijj}(x_0)n_i n_j(x) + o(\varepsilon^2) \quad x \in \Gamma_e.
\]

(A.15)

Hence, from the boundary conditions (35) and (36), \( c_n \) \((|n| \leq 2)\) is derived as follows:

\[
c_0 = u(x_0) + o(1),
\]

(A.16)

\[
c_{n+1} = \frac{u_{nn}(x_0) + iu_{n2}(x_0)}{2} + o(1),
\]

(A.17)

\[
c_{n+2} = \frac{u_{n1}(x_0) + 2iu_{n2}(x_0) + u_{22}(x_0)}{4} + o(1).
\]

(A.18)

By substituting Eqs. (A.16)–(A.18) into Eqs. (A.8) and (A.9), we have

\[
a_0 = \frac{\varepsilon_2 H_0^{(1)}(k_1 e) - \varepsilon_2 H_0^{(1)}(k_1 e) J_0(k_2 e) - e_1 H_0^{(1)}(k_1 e) J_1(k_2 e)}{\varepsilon_2 H_0^{(1)}(k_1 e) J_1(k_2 e) - e_1 H_0^{(1)}(k_1 e) J_1(k_2 e)} u(x_0) + o(1),
\]

(A.19)

\[
a_{n+1} = \frac{\varepsilon_2 (e H_1^{(1)}(k_1 e) - H_1^{(1)}(k_1 e))}{\varepsilon_2 H_1^{(1)}(k_1 e) J_1(k_2 e) - e_1 H_0^{(1)}(k_1 e) J_1(k_2 e)} \frac{u_{nn}(x_0) + iu_{n2}(x_0)}{2} + o(1),
\]

(A.20)

\[
a_{n+2} = \frac{\varepsilon_2 (e H_2^{(1)}(k_1 e) - 2H_1^{(1)}(k_1 e))}{\varepsilon_2 H_2^{(1)}(k_1 e) J_2(k_2 e) - e_1 H_0^{(1)}(k_1 e) J_2(k_2 e)} \frac{u_{n1}(x_0) + 2iu_{n2}(x_0) + u_{22}(x_0)}{4} + o(1).
\]

(A.21)

We obtain the following representation of \( \partial \hat{u}/\partial n \) on \( \Gamma_e \) by substituting Eqs. (A.19)–(A.21) into Eqs. (A.3), and expanding the spherical functions:

\[
\frac{\partial \hat{u}(x)}{\partial n} = \frac{2\varepsilon_2 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \quad u_{ij}(x_0)n_j(x) + \frac{4\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \left( u_{ij}(x_0)n_j \right)^2 - \frac{1}{2} \varepsilon_2^2 u(x_0) + o(\varepsilon) \quad x \in \Gamma_e.
\]

(A.22)

Also, \( \hat{u}(x) \) in \( \Omega_e \) is expressed as follows by substituting Eqs. (A.19)–(A.21) into Eq. (A.1):

\[
\hat{u}(x) = u(x_0) + \frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \quad u_{ij}(x_0)(x - x_0)j + o(\varepsilon) \quad x \in \Omega_e.
\]

(A.24)
2. Appendix B: validation of the topological derivatives

We confirm the validity of the topological derivative which is derived in the section 3. We consider the same problem statement with examples in section 4 (Fig. 9). The permittivity in $\Omega_2$ is fixed to $\varepsilon_2 = 2$. The wave length of the incident wave and incident angle are $\lambda = 10$ and $\theta = 45^\circ$, respectively. We consider to compare the value of the topological derivative $T$ in (49) calculated by the boundary element method on the line $y = 10$ and 20 with the topological difference, which is defined as follows:

$$\Delta T = \frac{(J + \delta J) - J}{s(\varepsilon)},$$  \hfill (B.25)

where $\delta J$ is the variation of the objective function (22) when a small circular dielectric material of $\varepsilon_2 = 2$ whose radius $\varepsilon = 0.005$ is allocated in the design domain $D$. The results are shown in Fig. 26, from which the correctness of the calculated topological derivative by the boundary element method is confirmed.

![Fig. 26 Comparison between the topological derivative and the topological difference on y = 10 (left) and y = 20 (right).](image)

Next, we show the correctness of the topological derivative for the objective function which is defined with the KS function (27). We consider a problem of designing wide-band carpet cloaking device. Namely, we define $J_i$ in Eq. (57) as the objective function (22) for the wave length $\lambda = 9.4, 10.0, 10.6$. Figure 27 shows the result of comparison between the topological derivative and the topological difference, which indicates the correctness of the topological derivative.

![Fig. 27 Comparison between the topological derivative and the topological difference on y = 10 (left) and y = 20 (right) in the case in which the KS function is used as an objective function.](image)