Coulomb drag between helical edge states

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We theoretically investigate the Coulomb drag between the edge states of two quantum spin Hall systems. Using an interacting theory of the one-dimensional helical edge modes, we show that the drag vanishes at second order in the inter-edge interaction, where it is typically finite in other systems, due to the absence of backscattering within the edges. However, in the presence of a small external magnetic field the drag is finite and scales as the fourth power of the magnetic field, a behavior that sharply distinguishes it from other systems. We obtain the temperature dependence of the drag for regimes of both linear and quadratic edge dispersion in the presence of a finite field.

\begin{equation}
\tau_D = -\lim_{I_1 \to 0} \frac{V_2^2}{\hbar L} \frac{1}{dI_1},
\end{equation}

where $V_2$ is voltage induced in the “passive” system by the current $I_1$ driven in “active” system. Here $e$ is the electron charge, $\hbar$ is a Planck’s constant, and $L$ is the length of the edge along which momentum is transferred.

Coulomb drag in non-HL one-dimensional systems has been studied both theoretically\textsuperscript{22,27} and experimentally.\textsuperscript{28–31} The HL can be viewed as a spinless Luttinger liquid (because it has the same number of degrees of freedom) without backscattering; since it is known that the backscattering governs the drag between systems with linear dispersions\textsuperscript{22,35,27} one can not expect drag between helical liquids. However, in this paper we show that an applied Zeeman field $\hbar$ opens up a backscattering process in the HL, and results in $r_D \propto \hbar^4$ for a linear spectrum. We also compute the temperature dependence of the drag over a range of temperature and field values.

The model—In second order perturbation theory in the interwire Coulomb interaction, the Coulomb drag is given
where \( \Im \Pi_i(\omega,q) \) is the imaginary part of the retarded density-density correlation function of wire \( i = 1(2), n_i \) is electron density of wire \( i \), \( T \) is the temperature, and \( U_{12}(q) \) is the Fourier transform of the interwire Coulomb interaction which is cut off at short distances by the interwire separation \( d \).

We consider two identical QSH systems, each with one Kramer’s pair on its edge, as shown in Fig. 1. As we noted earlier, if the spectrum is linear there is no contribution to the drag from forward scattering, and back scattering is forbidden by time-reversal symmetry. Therefore, one must break time-reversal symmetry in order to open up a backward scattering channel (unless there are magnetic impurities present) for a generic Dirac edge mode. Our Hamiltonian for a single HL in the presence of a Zeeman field is

\[
H_0 = \int dx \hat{\psi}^\dagger(x) \left( \hat{p}_x \sigma_z + \hat{h} \sigma_x - \mu \right) \hat{\psi}(x),
\]

where \( \hat{v} \) is the edge velocity, \( \hat{p}_x = -i \partial_x \) is the Fermi energy (which can be adjusted by gating the system), and \( \sigma_{z,x} \) are Pauli spin matrices describing the spin degree of freedom. A Zeeman field \( \hat{h} \) pointing in the \( x \)-direction opens up a gap in Dirac spectrum and tilts the spins away from the \( z \)-axis. The edge dispersion is \( \epsilon_{\pm} = \pm \sqrt{\hat{v}^2 \hat{p}_x^2 + \hat{h}^2 - \mu} \). We assume that Fermi energy is in the upper band (\( \mu > 0 \)) so that the properties of system are determined by the \( \epsilon_{+} \) band over the energy scales of interest. The wavefunction of electrons in the \( \epsilon_{+} \) band is

\[
\hat{\psi}_{\pm}(x) = \frac{\epsilon^{ipx}}{\sqrt{2}} \left( \frac{\cos(\gamma_p/2) + \sin(\gamma_p/2)}{(\cos(\gamma_p/2) + \sin(\gamma_p/2))} \right) = \epsilon^{ipx} \hat{U}_p,
\]

where \( \gamma_p = \arctan(\frac{\hat{h}}{\hat{p}_x}) \). We study \( U \) in the limit of large \( \mu \) (small \( \hbar \)) where the spectrum can be approximated as linear, and also in the opposite limit where the spectrum is approximately quadratic (i.e., \( \mu \) close to the band “bottom”). See Figs. 2 and 3.

**Regime of linear spectrum—** We first consider the case \( \mu > \hbar \), and linearize the spectrum near the Fermi energy, \( \epsilon_{+} = \hat{v} \hat{p}_x - \mu \) (see Fig. 2), in order to use standard bosonization procedures. We express the electron operator as a sum of left- and right- moving states: \( \hat{\psi}_+ = \hat{\psi}_R(x) + \hat{\psi}_L(x) \), where \( \hat{R}(\hat{L}) \) stands for right (left) movers. The non-interacting Hamiltonian can then be written:

\[
H_0 = \int dx \left[ \hat{\psi}^\dagger_R(x) \hat{p}_x \hat{\psi}_R(x) + \hat{\psi}^\dagger_L(x) \hat{p}_x \hat{\psi}_L(x) \right].
\]

where \( \hat{p}_x = \pm \hat{v} \hat{p}_x - \mu \). We assume intrawire interactions have the form

\[
H_{\text{int}} = \frac{U(0)}{2} \int dx dx' U(x-x') \rho(x) \rho(x'),
\]

where \( U(x-x') \) is the intrawire Coulomb interaction, and \( \rho(x) \) is the electron density:

\[
\rho(x) = \hat{\psi}^\dagger_R(x) \hat{\psi}_R + \hat{\psi}^\dagger_L(x) \hat{\psi}_L + \cos(\gamma_p) \left( \hat{\psi}^\dagger_R \hat{\psi}_L + \hat{\psi}^\dagger_L \hat{\psi}_R \right),
\]

which contains cross terms due to the presence of the magnetic field. In terms of bosonic fields \( \phi(x) \) and \( \theta(x) \), \( \hat{\psi}_R \) and \( \hat{\psi}_L \) are expressed as

\[
\hat{\psi}_R(x) = e^{ipx} \frac{\eta_R}{\sqrt{2\pi a}} e^{-i(\phi(x) - \theta(x))},
\]

\[
\hat{\psi}_L(x) = e^{-ipx} \frac{\eta_L}{\sqrt{2\pi a}} e^{-i(\phi(x) - \theta(x))},
\]

where \( \eta_R(\eta_L) \) are Klein factors, and \( a \) is a short-distance cut-off. The electron density in terms of bosonic fields takes the form

\[
\rho(x) = \frac{1}{\pi} \partial_x \phi(x) - \frac{\cos(\gamma_p)}{\pi a} \sin(2pFx - 2\phi(x)).
\]

Substituting this expression into (6), we find

\[
H_{\text{int}} = \frac{U(0) - \cos^2(\gamma_p)}{2\pi^2} \int dx \left( \partial_x \phi(x) \right)^2,
\]

where \( U(0) \) and \( U(2p_F) \) are the zero and \( 2p_F \) momentum parts of the interaction, respectively. Note that the \( 2p_F \) part has a \( \cos^2(\gamma_p) \) factor, which is proportional to \( \hbar^2 \) for small \( \hbar \). The full Hamiltonian then becomes

\[
H = \frac{1}{2\pi^2} \int dx \left[ v(\partial_x \theta(x))^2 + (v + g)(\partial_x \phi(x))^2 \right],
\]

where \( g = (U(0) - \cos^2(\gamma_p)U(2p_F))/\pi \). We observe that the Hamiltonian of an interacting HL in a Zeeman field is equivalent to a spinless Luttinger liquid where the strength of backscattering depends on the Zeeman field. Similar results were obtained in studies of Luttinger liquids with Rashba spin-orbit coupling and a Zeeman magnetic field.

We now give an expression for the retarded density-density correlation function. Because the backscattering
governs the Coulomb drag when the dispersion is linear (as it is in a Luttinger liquid), we only need the $2p_F$ part of the retarded density-density correlation function. Since our model has reduced to a spinless Luttinger liquid, the calculation is standard:  

$$
\Pi_{R}^{2p_F}(q, \omega) = \left\{ \pi \right\} \left[ 1 \right] \left[ \Pi(q + 2p_F, \omega) + \Pi(q - 2p_F, \omega) \right],
$$

with $\Pi(q, \omega)$ given by

$$
\Pi(q, \omega) = \frac{2K D}{u} \left( \beta u \right)^2 \left( \frac{2}{\pi} \right) F(q, \omega),
$$

where $F(q, \omega) = B(-i \frac{q}{2\pi} (\omega - q^2) + \frac{1}{2}) - B(-i \frac{q}{2\pi} (\omega + q u) + \frac{1}{2})$, $\beta = 1/T$, $u = v(v + g)$, $K = v/u$, and $B(x, y)$ is the Beta function. The parameter $D$ is

$$
D = \sinh(\pi K) (\frac{\pi a}{u})^{2K-2}.
$$

With \ref{14}, \ref{15} in hand, the drag resistivity is readily computed from \ref{2}, \ref{10} $F(q \pm 2p_F, \omega)$ is sharply peaked about $q = \mp 2p_F$ with peak widths proportional to temperature. Since the momentum integration in \ref{2} runs from 0 to $\infty$, we neglect the $\Pi(q + 2p_F, \omega)$ contribution. Taking the imaginary part of the retarded density-density correlation function and assuming identical helical liquids we obtain

$$
r_D \simeq \frac{24K u^2 D^2}{16\pi} (2p_F)^2 U^2_1 (2p_F) \frac{I}{n^2 T^3},
$$

where $I \equiv \int_0^\infty d\Omega \frac{(3f(\Omega))}{\sinh^2(\Omega/2)}$, with $\Omega = \omega / T$. The density of states, $n = 1/\pi\nu v$. Extracting the temperature and magnetic field dependence using \ref{15}, we find

$$
r_D \propto h^4 T^{4K-3}.
$$

\ref{17} is one of the central results of the paper. This result is valid at temperatures larger than $T^*$, below which the drag begins to exhibit an exponential dependence on temperature.\textsuperscript{22,23} Since $T^* \sim \mu e^{\frac{p_F}{\pi K}}$, the dependence on the Zeeman field via $K$ will also depend on the Zeeman field via the dependence of $K$ on $h$.

By contrast, in a spinful Luttinger liquid the magnetic field only enters the interaction constant in the spin channel and therefore the drag is only (weakly) dependent on magnetic field through the interaction parameter appearing in an exponent to the temperature. Therefore, Coulomb drag can be used as a method for experimental verification of the HL, complementing the earlier studies.\textsuperscript{13,15} We note that a spin-Coulomb drag effect in which two density mismatched Luttinger liquids can be brought into more favorable kinematic conditions for enhanced drag effects has also been studied.\textsuperscript{25} To complete our analysis of drag between two HL, we turn to the case when the spectrum is approximately quadratic.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig3}
\caption{(color online) Temperature dependence of the drag in the regime of small $\mu$ where the spectrum is approximately quadratic, as shown in the inset, and $0 < \mu - h < \frac{\nu^2}{2 \pi n^2}$. Note the non-monotonic temperature dependence above $T^*$. For the dependence of $r_D$ on the Zeeman field $h$ in each region of temperature, see the text. The second crossover from $T^*$ to $T^{-1}$ occurs for $T \sim \frac{h}{\nu} \sqrt{\frac{d}{2\pi}}$ where $d$ is the distance between wires.}
\end{figure}

**Regime of quadratic spectrum** - When $0 < \mu - h < h$, the spectrum of upper band is approximately quadratic, as shown in the inset, and $0 < \mu - h < \frac{\nu^2}{2 \pi n^2}$. Note the non-monotonic temperature dependence above $T^*$. For the dependence of $r_D$ on the Zeeman field $h$ in each region of temperature, see the text. The second crossover from $T^*$ to $T^{-1}$ occurs for $T \sim \frac{h}{\nu} \sqrt{\frac{d}{2\pi}}$ where $d$ is the distance between wires.

\begin{equation}
3\Pi_R(q, \omega) = - \frac{h}{4vq} f^2_+ (p, q) \frac{\sinh(\frac{\omega}{2h})}{\cosh(\frac{\omega q}{2h}) \cosh(\frac{\omega (p + q)}{2h})},
\end{equation}

where $p = -\frac{\nu}{2} + \frac{h}{2\pi} e^{\gamma}$, and $f_+ (p, q) = \hat{U}_p \hat{U}_{p+q}$, which we assume to be approximately equal to one. One also needs to take into account restrictions on $\omega$ defined by $\epsilon_+(p) < 0$ and $\epsilon_+(p + q) > 0$ which will give

$$
\frac{1}{2} v + \sqrt{2h(\mu - h)} > \frac{h \omega}{v q} > - \frac{1}{2} v + \sqrt{2h(\mu - h)}.
$$

Plugging \ref{18} into \ref{2} and evaluating the integrals we obtain the following results. When $\mu - h < \frac{\nu^2}{2 \pi n^2}$,\textsuperscript{22} \textsuperscript{23} for small temperatures $T < \frac{\nu}{4d} \sqrt{\frac{\mu - h}{2h}}$, \textsuperscript{22} \textsuperscript{23} \textsuperscript{24}

$$
r_D \simeq \frac{1}{2^{5/2} \pi^2 h^2 v^4} \sqrt{\frac{h^5}{(\mu - h)^3}} T^2,
$$

while at large temperatures $T > \frac{\nu}{4d} \sqrt{\frac{\mu - h}{2h}}$, \textsuperscript{22} \textsuperscript{23} \textsuperscript{24}

$$
r_D \simeq \frac{h \nu g^2}{2^{8/3} \pi^3 n^2 v d^3 T}.
$$

\begin{equation}
3\Pi_R(q, \omega) = - \frac{h}{4vq} f^2_+ (p, q) \frac{\sinh(\frac{\omega}{2h})}{\cosh(\frac{\omega q}{2h}) \cosh(\frac{\omega (p + q)}{2h})},
\end{equation}

where $p = -\frac{\nu}{2} + \frac{h}{2\pi} e^{\gamma}$, and $f_+ (p, q) = \hat{U}_p \hat{U}_{p+q}$, which we assume to be approximately equal to one. One also needs to take into account restrictions on $\omega$ defined by $\epsilon_+(p) < 0$ and $\epsilon_+(p + q) > 0$ which will give

$$
\frac{1}{2} v + \sqrt{2h(\mu - h)} > \frac{h \omega}{v q} > - \frac{1}{2} v + \sqrt{2h(\mu - h)}.
$$

Plugging \ref{18} into \ref{2} and evaluating the integrals we obtain the following results. When $\mu - h < \frac{\nu^2}{2 \pi n^2}$,\textsuperscript{22} \textsuperscript{23} for small temperatures $T < \frac{\nu}{4d} \sqrt{\frac{\mu - h}{2h}}$, \textsuperscript{22} \textsuperscript{23} \textsuperscript{24}
When $\mu - h < \frac{v^2}{2\pi^2 \hbar}$ and for small temperatures $T < \frac{\hbar}{\sqrt{2\mu-h}}$, 

$$r_D \simeq \frac{\sqrt{T/\pi}}{\hbar^2 \pi^2 n^2} \gamma^2 \frac{h^{5/2}}{v^4} \sqrt{T},$$  \hspace{0.5cm} (22)

while at large temperatures $T > \frac{\hbar^4}{v^2 \mu}$, 

$$r_D \simeq \frac{d^{15/2}}{3 \pi^4 n^2} \frac{g^2}{v} \frac{h^{3/2}}{\gamma^2} \frac{1}{T^{5/2}}. \hspace{0.5cm} (23)$$

Here $g_s = -\gamma + \ln(2)$ ($\gamma \approx 0.5772$ is Euler’s constant) is an estimate of interwire Coulomb interaction at small momentum. The density of states, $n = \frac{1}{4\sqrt{\mu-h}}$. The results are summarized in Fig. 3. We emphasize that in obtaining these results we have not considered effects of interband (intra-edge) particle-hole excitations. These excitations will result in Fermi edge singularity physics.\textsuperscript{35,37}

Summary and Discussion—We studied the Coulomb drag between identical one dimensional helical liquids. We showed that the helical liquid can be mapped to a spinless Luttinger liquid where backscattering is prohibited. Since backscattering governs the drag between one-dimensional liquids with linear dispersion, there is no Coulomb drag unless there is a nonlinearity in the spectrum. Nonlinearity in the spectrum gives rise to a small Coulomb drag unless there is a nonlinearity in the spin channel.

For completeness we studied the case when the spectrum of a helical liquid in a magnetic field can not be approximated as linear, but is rather approximately quadratic (valid for a strong magnetic field). Expressions for the Coulomb drag in this case are given by (20), (22). Finally, we note that the presence of a few magnetic impurities on the edge of a QSH system would allow a finite drag contribution even in the absence of applied magnetic fields since they would allow backscattering. Inclusion of Rashba coupling would not affect our results, provided the zero-field case is still adiabatically connected to the topologically non-trivial state.

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