Twisting to Abelian $BF$/Chern-Simons Theories

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Abstract

Starting from a $D = 3$, $N = 4$ supersymmetric theory for matter fields, a twist with a Grassmann parity change is defined which maps the theory into a gauge fixed, abelian $BF$ theory on curved 3-manifolds. After adding surface terms to this theory, the twist is seen to map the resulting supersymmetric action to two uncoupled copies of the gauge fixed Chern-Simons action. In addition, we give a map which takes the $BF$ and Chern-Simons theories into Donaldson-Witten TQFT’s. A similar construction, but with $N = 2$ supersymmetry, is given in two dimensions.

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1 Introduction

This paper deals with the problem of mapping supersymmetric field theories into topological field theories (TFT’s) [1-6] and of mapping different classes of TFT’s among themselves. TFT’s fall under two classes. The first of the TFT’s are the Schwarz-type [4], commonly known as $BF$, theories. Chern-Simons theory in three dimensions is a special case of $BF$ theory. The second are Donaldson-Witten or Topological Quantum Field Theories (TQFT’s) [3]. A sub-class of the TQFT’s, the topological Yang-Mills (TYM) theories are gauge invariant. Another sub-class of the TQFT’s is given by the topological sigma models which do not possess gauge invariances.

To date, these two classes of theories have had vastly different origins. On the one hand, the $BF$ theories have non-trivial classical actions and first order equations of motion. Their classical (abelian) actions on manifolds of dimension $D$ are metric independent as they are of the form $\int_M B_{(k)} \wedge F_{(D-k)}$, where $F_{(D-k)} = dA_{(D-k-1)}$ and the subscript denotes the form’s degree. These theories are invariant under Maxwell (or Yang-Mills) gauge symmetries. They are also symmetric under the $k$-form symmetry which shifts $B_{(k)}$ into the exterior derivative of a $(k-1)$-form. On the other hand, the TQFT’s classical lagrangian are either 0 or a total derivative and are devoid of classical equations of motion. Apart from the possible surface term, the entire lagrangian of a TQFT is obtained [3, 8, 9] as a BRST gauge fixing of a symmetry (topological symmetry) which manifests itself as compactly supported shifts of some field in the theory (for example, the gauge field in TYM). A large class of the latter theories may also be obtained from $N=2$ [3, 10] or even $N \geq 2$ [11] supersymmetric theories via a procedure known as twisting.

We will work in three and two dimensions restricting ourselves to abelian BF theories. Placed in this context, we will solve a problem which has existed since the birth of these theories; namely, how to obtain the BF theories via the twisting of some supersymmetric theory. Furthermore, we will make substantial progress towards solving an equally long-standing problem; namely, what (if any) is the relation between $BF$ theories and TQFT’s.

As the twisting process will play an important role in our work, it is appropriate to give a quick review [3] using the example of $R^4$. Starting with a $N=2$ supersymmetric field theory and writing the Lorentz group as $SO_L(4) \simeq SU_l(2) \times SU_r(2)$, we then take the diagonal sum of $SU_l(2)$ with the automorphism group of the $N=2$ superalgebra, $SU_l(2)$. The
result is a $SU_d(2)$, which we use to form a new Lorentz group, $SO_L'(4) \simeq SU_d(2) \times SU_r(2)$. As a result, spin-$\frac{1}{2}$ fields, which also transformed as doublets of $SU_f(2)$, now become integer spinned, Grassmann odd fields. In three dimensions, the Lorentz group is $SO_L(3) \simeq SU_L(2)$. In order to define a twist, the supersymmetric theory will have to possess a $SU_f(2)$ automorphism group so that the new Lorentz group may be taken to be the diagonal sum of the two $SU(2)$’s. This means that the $D = 3$ theory should be $N = 4$ supersymmetric. In two dimensions, we will require a $U(1)$ automorphism group, hence an $N = 2$ supersymmetric theory.

Glancing at the $BF$ lagrangian (see above), we see that the Grassmann even fields are first order in derivatives. Whereas, upon gauge fixing, the Grassmann odd fields are second order in derivatives. This is an inversion of the usual structure in supersymmetric theories. Scaling this hurdle will be achieved by a second stage of the twisting wherein we will change the Grassmann parity of the fields; (bosons) fermions will become (anti-)commuting. As the supersymmetric theory we will apply our twisting procedure to will not be gauge invariant, the $BF$/Chern-Simons theory obtained will be gauge fixed. In this way, we will obtain the abelian $BF$ and Chern-Simons theories from $N = 4$ supersymmetric theories in three dimensions. Similarly, $N = 2$ theories will be twisted to the abelian $D = 2$ $BF$ theory. As an artifact of the process, we will actually obtain two (uncoupled) copies of Chern-Simons theory.

Previously, it had been shown that the gauge fixed Chern-Simons theories \[\text{[12]}\] (along with a related construction for the $BF$ theories \[\text{[13, 14, 15]}\]) are invariant under a set of symmetries generated by a pair of scalar and a pair of vector charges, all Grassmann odd. The algebra of these charges allows a $SL_f(2, R) \simeq SU_f(2)$ automorphism group. The number of components of these charges matches the number of components of four Majorana fermions and it was shown that this algebra is a twisted version of a $D = 3$, $N = 4$ supersymmetry algebra\[\text{[1]}\]. As part of our work, we will find the missing $N = 4$ supersymmetric theory which realizes the untwisted algebra. Since, as we will show, the supersymmetric theory may also be twisted to a TQFT, we will then formally relate a subset of TQFT’s to the abelian $BF$ theory.

Our paper is organized as follows. In the next section, the two $N = 4$ supersymmetric actions (which differ only by surface terms) we will use

\[\text{1This algebra was termed } N = 2 \text{ in ref. [12].}\]
throughout our three dimensional discussion will be presented. Following this, in section 3, we will twist the first of these actions to the abelian \( BF \) theory in three dimensions. After writing down the action for a \( D = 2, N = 2 \) scalar supermultiplet, we will show how to twist this theory to the two dimensional \( BF \) theory, in sub-section 3.3. In section 4, we shall return to three dimensions and use the second action from section 2, which we will twist to the abelian Chern-Simons theory. The structure and transformations generated by the three dimensional twisted superalgebra will be given in section 5. In section 6, we will show how to connect TQFT’s obtained from our supersymmetric theory with \( BF \) theories via a change in Grassmann parity. We conclude in section 7. The conventions used in this paper may be found in the appendix.

2 The \( N = 4 \) Supersymmetric Actions

Let us begin by introducing the two \( N = 4 \) supersymmetric actions we will be using in our discussion of the three dimensional topological theories. In order to establish the main features of the twisting process it is best to work on a flat manifold. Later, we will extend the procedure to curved manifolds (see sub-section 3.2). Although the actions constructed in this section exist in either Minkowski space-time or \( \mathbb{R}^3 \), in the rest of the paper we will restrict our discussion to manifolds with Euclidean signature.

Our supersymmetric matter multiplet contains the following complex fields:

| FIELD | SPIN | GRASSMANN PARITY |
|-------|------|------------------|
| \( \phi \) | 0 | even |
| \( \lambda \) | 0 | even |
| \( \psi_\alpha \) | 1/2 | odd |
| \( \chi_\alpha \) | 1/2 | odd |

There are a number of possible actions we could write down for these fields. Even within a given action, we can add surface terms. We will see the
importance of this later. As our basic action we take

\[ S_{\text{SU SY}} = \int d^3x [\partial^a \bar{\phi} \partial_a \lambda + \partial^a \phi \partial_a \bar{\lambda} + i \frac{1}{2} \chi^a (\gamma^a)_{\alpha \beta} \partial_a \bar{\psi}_\beta - i \frac{1}{2} \bar{\chi}^a (\gamma^a)_{\alpha \beta} \partial_a \psi_\beta] , \]  

(2.1)

where the bar denotes complex conjugation. This action is invariant under

the following rigid supersymmetry transformations

\[ \begin{align*}
[Q_\alpha, \phi] &= i\psi_\alpha, & [Q_\alpha, \lambda] &= i\chi_\alpha, \\
\{Q_\alpha, \bar{\psi}_\beta\} &= -2(\gamma^a)_{\alpha \beta} \partial_a \bar{\phi}, & \{Q_\alpha, \bar{\chi}_\beta\} &= 2(\gamma^a)_{\alpha \beta} \partial_a \bar{\lambda}, \\
[\bar{Q}_\alpha, \bar{\phi}] &= -i\bar{\chi}_\alpha, & [\bar{Q}_\alpha, \lambda] &= -i\bar{\psi}_\alpha, \\
\{\bar{Q}_\alpha, \psi_\beta\} &= -2(\gamma^a)_{\alpha \beta} \partial_a \phi, & \{\bar{Q}_\alpha, \chi_\beta\} &= 2(\gamma^a)_{\alpha \beta} \partial_a \lambda. \end{align*} \]  

(2.2)

The \( Q \)-super–charges form the \( N = 2 \) supersymmetry algebra

\[ \begin{align*}
\{Q_\alpha, Q_\beta\} &= -i2(\gamma^a)_{\alpha \beta} \partial_a, & \{Q_\alpha, Q_\beta\} &= 0, & \{Q_\alpha, \bar{Q}_\beta\} &= 0. \end{align*} \]  

(2.3)

The action is invariant under the interchange \( \lambda \leftrightarrow \phi \). From this, it follows

that there is a second \( N = 2 \) supersymmetry of (2.1),

\[ \begin{align*}
[S_\alpha, \phi] &= i\chi_\alpha, & [S_\alpha, \lambda] &= i\psi_\alpha, \\
\{S_\alpha, \bar{\psi}_\beta\} &= -2(\gamma^a)_{\alpha \beta} \partial_a \bar{\phi}, & \{S_\alpha, \bar{\chi}_\beta\} &= 2(\gamma^a)_{\alpha \beta} \partial_a \bar{\lambda}, \\
[S_\alpha, \bar{\phi}] &= -i\bar{\chi}_\alpha, & [S_\alpha, \lambda] &= -i\bar{\psi}_\alpha, \\
\{S_\alpha, \psi_\beta\} &= -2(\gamma^a)_{\alpha \beta} \partial_a \phi, & \{S_\alpha, \chi_\beta\} &= 2(\gamma^a)_{\alpha \beta} \partial_a \lambda. \end{align*} \]  

(2.4)

The \( S \)-super–charges also form an \( N = 2 \) supersymmetry algebra. The automorphism group of each of the supersymmetry algebras is \( U(1) \).

It will prove useful to re-write \( S_{\text{SU SY}} \) in terms of real/imaginary fermions rather than the complex ones. To do this, we define the real and imaginary parts of the fermions via: \( \chi_\alpha \equiv \chi_{\alpha 1} + i\chi_{\alpha 2} \) and \( \psi_\alpha \equiv \psi_{\alpha 1} + i\psi_{\alpha 2} \). Consequently, the action becomes

\[ S_{\text{SU SY}} = \int d^3x [\partial^a \bar{\phi} \partial_a \lambda + \partial^a \phi \partial_a \bar{\lambda} + \chi^A (\gamma^A)_{\alpha \beta} \partial_a \bar{\psi}_\beta B_{\alpha \beta}] . \]  

(2.5)

The lagrangian in this action is equivalent to that in (2.1); \( i.e. \), no surface terms were incurred in this re-writing. In twisting to the \( BF \) theory, we will use this form of the action.

\[ \text{2} \text{The ordering of the fields in the various terms is important since our twisting procedure involves changing the Grassmann character of the fields. We will take the ordering as given in this action throughout.} \]

\[ \text{3} \text{Throughout this paper we will discard surface terms while establishing the existence of supersymmetries.} \]
From $\psi_{\alpha A}$ and $\chi_{\alpha A}$, we can construct another action whose lagrangian differs from (2.1) by a total derivative term. To do this we define $\Psi_{\alpha A} \equiv \psi_{\alpha A} + i \chi_{\alpha A}$ ($\Psi_{\alpha A} \equiv \psi_{\alpha A} - i \chi_{\alpha A}$). As $\Psi_{\alpha A}$ is a complex doublet, we take it to transform as a 2 of $SU(2)$ while $\bar{\Psi}_{\alpha A}$ is in the conjugate representation.

Using this in the action (2.1) we arrive at

$$S'_{SUSY} = \int d^3 x [\partial^a \bar{\phi} \partial_a \lambda + \partial^a \phi \partial_a \bar{\lambda} + i \frac{1}{2} \bar{\Psi}^{\alpha B} (\gamma^a)_{\alpha \beta} \partial_a \Psi_{\beta B}]$$

and discard the surface terms. The original two $N = 2$ supersymmetries now become invariances of the action under the following transformations

$$[Q_{\alpha A}, \phi] = i(\Psi_{\alpha A} + \bar{\Psi}_{\alpha A})$$

$$[Q_{\alpha A}, \lambda] = i(\Psi_{\alpha A} - \bar{\Psi}_{\alpha A})$$

$$\{Q_{\alpha A}, \Psi_{\beta B}\} = -2\epsilon_{AB}(\gamma^a)_{\alpha \beta} \partial_a (\bar{\phi} - \bar{\lambda})$$

$$\{Q_{\alpha A}, \bar{\Psi}_{\beta B}\} = -2\epsilon_{AB}(\gamma^a)_{\alpha \beta} \partial_a (\phi + \lambda)$$

$$\{\bar{Q}_{\alpha A}, \phi\} = -i(\bar{\Psi}_{\alpha A} + \Psi_{\alpha A})$$

$$\{\bar{Q}_{\alpha A}, \lambda\} = -i(\bar{\Psi}_{\alpha A} - \Psi_{\alpha A})$$

$$\{\bar{Q}_{\alpha A}, \Psi_{\beta B}\} = -2\epsilon_{AB}(\gamma^a)_{\alpha \beta} \partial_a (\phi - \lambda)$$

$$\{\bar{Q}_{\alpha A}, \bar{\Psi}_{\beta B}\} = -2\epsilon_{AB}(\gamma^a)_{\alpha \beta} \partial_a (\phi + \lambda).$$

(2.7)

This shows explicitly that the both actions, $S_{SUSY}$ and $S'_{SUSY}$ are invariant under an $N = 4$ supersymmetry. Indeed, the algebra of charges defined by (2.7) is

$$\{\bar{Q}_{\alpha A}, Q_{\beta B}\} = i4\delta_{AB}(\gamma^a)_{\alpha \beta} \partial_a.$$

(2.8)

This algebra has a $SU(2)$ automorphism invariance with the $Q_{\alpha A}$ transforming

in the doublet representation.

### 3 Mapping to $BF$ Theories

This section is divided into three parts. First, in sub-section (3.1), we present the twisting procedure while working with the action $S_{SUSY}$. As advertised, we will find the twisted action to be the gauge fixed, abelian $BF$
theory on $R^3$. Then, in sub-section (3.2), we will discuss how to obtain the $BF$ theory on curved manifolds. Finally, in sub-section (3.3), as another example of the procedure, we will write down a $D = 2$, $N = 2$ supersymmetric action from which the two-dimensional abelian $BF$ theory may be obtained via twisting.

3.1 $g$–Twisting $S^{SUSY}$

The Lorentz algebra in three dimensions is $SO_L(3) \simeq SU_L(2)$. As the first stage of our twisting we take all internal indices to be $SU_L(2)$ indices. This amounts \[12\] to re-defining the Lorentz group to be the diagonal subgroup of $SU_L(2) \times SU_I(2)$. With this, the original scalar fields remain Lorentz singlets while the real spin-$\frac{1}{2}$ fields become Lorentz bi-spinors: $\psi_{\alpha B} \rightarrow \psi_{\alpha \beta}$ and $\chi_{\alpha B} \rightarrow \chi_{\alpha \beta}$. This means that we can decompose $\psi_{\alpha \beta}$ as a real vector plus a scalar field; similarly for $\chi_{\alpha \beta}$.

As the second stage of our twist, we declare the fields to have opposite Grassmann parity to those of the parent supersymmetric theory. This second step does not exist in the known \[3\] twisting of supersymmetric theories to obtain Donaldson-like topological quantum field theories (TQFT’s). We call this two stage mapping a “$g$–twist” and define it by the map

\[T_g: \psi_{\alpha B} \rightarrow \psi_{\alpha \beta} \equiv \frac{1}{\sqrt{2}}[i(\gamma^a)_{\alpha \beta}A_a - C_{\alpha \beta} \Sigma],\]
\[T_g: \chi^{\alpha B} \rightarrow \chi^{\alpha \beta} \equiv \frac{1}{\sqrt{2}}[(\gamma^a)^{\alpha \beta}B_a + iC^{\alpha \beta} \Lambda],\]
\[T_g: \phi \rightarrow \frac{1}{\sqrt{2}}(c - ib'),\]
\[T_g: \bar{\phi} \rightarrow \frac{1}{\sqrt{2}}(c + ib'),\]
\[T_g: \lambda \rightarrow \frac{1}{\sqrt{2}}(c' + ib),\]
\[T_g: \bar{\lambda} \rightarrow \frac{1}{\sqrt{2}}(c' - ib),\]
\[T_g: \epsilon_{AB} \rightarrow iC_{\alpha \beta}.\]  

The fields on the right hand side of the arrows are defined by this map to have Grassmann parity opposite to those on the left. The factors of “$i$” have
been inserted so that the process of complex conjugation commutes with $T_g$. Additionally, the other numerical factors are for later convenience. We summarize the new field content in the following table:

| FIELD | SPIN | GRASSMANN PARITY |
|-------|------|------------------|
| $A_a$ | 1    | even             |
| $\Sigma$ | 0    | even             |
| $B_a$ | 1    | even             |
| $\Lambda$ | 0    | even             |
| $c$ | 0    | odd              |
| $b$ | 0    | odd              |
| $c'$ | 0    | odd              |
| $b'$ | 0    | odd              |

Performing the map, $T_g$, on the action $S^{\text{SUSY}}$ as given in eqn. (2.5) we find, up to surface terms,

$$S_{\text{BF}} = \int d^3 x [\epsilon^{abc} B a \partial_b A_c + (\partial^a A_a) \Lambda + (\partial^a B_a) \Sigma + c' c + b' b] . \quad (3.2)$$

This is the action of the fully gauge fixed abelian $BF$ theory in three dimensions$^4$. The first term is the classical $BF$ action. In this term, the Levi-Cevita tensor arises from a trace on the product of three gamma matrices. The second and third terms represent the gauge fixings of the local $U(1)$ and 1-form symmetry on $B_a$ (see section (5) for details). In these terms, the Lorentz dot product arises from the trace of products of two gamma matrices. The ghost actions for these gauge fixings are given by the last two terms in (3.2). Note that only the Landau gauge appears in this procedure. The surface terms mentioned above appear only from the gauge fixing and ghost terms. They are needed in order to write these terms in their conventional forms.

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$^4$The ordering of the fields in the gauge fixing terms is chosen so as not to introduce additional minus signs when we later map to the TQFT.
3.2 Curved 3-Manifolds

The classical $BF$ action is topological. It is only after gauge fixing that a metric appears in the action. We would like to recover this peculiar metric dependence.

We could simply $g$–twist the action $S^{SUSY}$ on $\mathbb{R}^3$ to obtain (3.2) and then covariantize it with respect to some background metric on a curved manifold, $M$. By definition, the subsequent action,

$$S^M_{BF} = \int d^3x \epsilon^{abc} B_a \partial_b A_c + \int d^3x \sqrt{g} \left[ (\nabla^a A_a) \Lambda + (\nabla^a B_a) \Sigma + c' \triangle c + b' \triangle b \right],$$

is the gauge fixed $BF$ theory on $M$. The derivative $\nabla_a$ is covariant with respect to diffeomorphisms of $M$: $\nabla_a \equiv e_a^m \partial_m + \omega_a^{\ b} J_b$. Here $e_a^m$ is the dreibein with determinant $e$. The object $\omega_a^{\ b}(e)$ is the dual of the Lorentz spin-connection for which the dual of the Lorentz generator is $J_a$.

Instead, suppose we started with the $N = 4$ gauged supergravity version of $S^{SUSY}$. Among the new fields introduced would be four gravitini and a $SU_I(2)$ gauge field, $V_a$. As an example, the gravitini appear in the spin-connection in the covariant derivative. The latter is also covariant with respect to local $SU_I(2)$ gauge transformations due to the introduction of $V_a$. The action (3.3) does not contain either of these fields as it is neither $N = 4$ locally supersymmetric or $SU_I(2)$ gauge invariant. Thus, in the $g$–twisting, we must set the gravitini to zero. In order to maintain this ansatz, however, we must restrict the local supersymmetry of the action so that the gravitini may not be transformed away from zero. Since the local supersymmetry variations of the gravitini, $\zeta_{aA}^A$, are given by the covariant derivative of the local supersymmetry parameter, we must find a covariantly constant anti-commuting parameter:

$$\delta \zeta_{aA}^A = D_a \epsilon \epsilon^A = \partial_a \epsilon^A - \omega_{a\beta}^{\ \beta} \epsilon^A + V_{aB}^A \epsilon^B = 0 . \quad (3.4)$$

To do this, we accentuate our procedure in analogy with the twisting in $D = 4, N = 2$ conformal supergravity backgrounds [16]. We introduce a scalar anti-commuting parameter, $\epsilon$ by $\epsilon^A_\alpha = \epsilon \delta^A_\alpha$ having embedded the $SU_I(2)$ gauge field in the $SU(2)$ spin connection: $\omega_{a\beta}^{\ \beta} \delta^A_\beta \equiv V_{aB}^A \delta^B_\alpha$. All

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5The construction of $D = 3, N = 4$ gauged supergravity along with its explicit couplings to matter is beyond the scope of this work.
supersymmetries are then lost with the exception of the one generated by
the scalar charges. The corresponding transformations will be given later.
We then identify this curved background with the geometry of $M$.

3.3 Two Dimensions

To illustrate the generality of our $g$-twisting procedure, we offer an exam-
ple in two dimensions. As the Lorentz group in two dimensions is $U(1)$, our
supersymmetric theory must have this abelian automorphism group. This
means that the theory must be $N = 2$ supersymmetric. As our action we
take

$$S_{D=2}^{\text{SUSY}} = \int d^2x [\partial^a \phi \partial_a \lambda + \frac{i}{2} \bar{\psi}^\alpha (\gamma^a)_{\alpha\beta} \partial_a \psi_\beta] ,$$

where $\phi$ and $\lambda$ are scalar fields and $\psi$ is a complex spin-$\frac{1}{2}$ field. This action
is invariant under the supersymmetry transformations,

$$[Q_\alpha, \phi] = i\psi_\alpha , \quad [\bar{Q}_\alpha, \lambda] = i\bar{\psi}_\alpha ,$$
$$[Q_\alpha, \bar{\psi}_\beta] = -2(\gamma^a)_{\alpha\beta} \partial_a \phi , \quad [\bar{Q}_\alpha, \psi_\beta] = -2(\gamma^a)_{\alpha\beta} \partial_a \lambda .$$

These form the $D = 2$, $N = 2$ supersymmetry algebra

$$\{Q_\alpha, Q_\beta\} = -i2(\gamma^a)_{\alpha\beta} \partial_a .$$

Upon defining $\psi_\alpha = \psi_{\alpha 1} + i\psi_{\alpha 2}$ and denoting the new fermions as $\psi_{\alpha A}$,
$A = 1, 2$, we define the $g$-twist to be

$$T_g : \psi_{\alpha B} \to \psi_{\alpha\beta} \equiv [i(\gamma^a)_{\alpha\beta} A_a - (\gamma^3)_{\alpha\beta} B - \frac{1}{2} C_{\alpha\beta} \Lambda] ,$$
$$T_g : \phi \to c ,$$
$$T_g : \lambda \to c' .$$

The $D = 2$ analog of the procedure discussed in the previous sub-section but
with $SU(2)$ replaced by $U(1)$, may now be applied to the action (3.8). It
results in the $g$-twisted action

$$S_{BF}^{D=2} = 2 \int d^2x e^{ab} B_i A_b + \int d^2x \sqrt{|g|} (\nabla^a A_a) \Lambda + c' \Delta c .$$

This is the gauged fixed, abelian $BF$ action in two dimensions.
4 Mapping to Chern-Simons

As is well known, Chern-Simons theory is a special case of a $BF$ theory in which the $A_a$ and $B_a$ fields are identified\textsuperscript{6}. At the level of the fields this is a purely formal operation. However, when one considers that $A_a$ is a $U(1)$-valued gauge field and $B_a$ is a singlet under that gauge group, one realizes the absence of a representation theory prescription for the identification. At the level of symmetries, both fields transform as the exterior derivative of a scalar parameter. Thus, Chern-Simons theory is strictly a special case of $BF$ theory only at the level of the structure of the fields in the action. Although this is a phenomenon in the gauge sector of the theory, we might expect similar behaviour with the space-time symmetries, if we try to obtain Chern-Simons via the $g$-twisting of a supersymmetric theory. Indeed, we will see that if we use the naive version of $S^{\text{SUSY}}$, there is no group theoretic prescription, in terms of $SU(2)$ representations, for the $g$-twist. Our map will be purely in terms of the fields. After seeing this, we will then turn to $S^{\prime\text{SUSY}}$ (in the second sub-section), for which both the twist on the fields and the group theoretic interpretation are available.

4.1 $g$–Twisted $S^{\text{SUSY}}$ with $\chi$ and $\psi$ Identified

Since we already know that $A_a$ and $B_a$ must be identified, we start by identifying $\chi$ and $\psi$ in eqn. (2.5) so that we take the action to be

$$S^{\text{SUSY}}_0 = \int d^3x [\partial^a \phi \partial_a \lambda + \frac{1}{2} \psi^{\alpha B} (\gamma^a)_{\alpha \beta} \partial_a \psi_{\beta B}] .$$ (4.1)

Here, $\lambda$ and $\phi$ are now real bosons\textsuperscript{7} and $\psi^{\alpha A}$ represents a pair of real spin$-\frac{1}{2}$ fields, $A = 1, 2$. Naively, we might define the $g$-twist by the first line in eqn. (3.1) along with $T_g: \lambda \to c'$ and $T_g: \phi \to c$. Using this in $S^{\text{SUSY}}$ and applying the procedure outlined in sub-section (3.2), we arrive at the action

$$S_{CS} = \frac{1}{2} \int d^3 x e^{abc} A_a \partial_b A_c + \int d^3 x \sqrt{g} [(\nabla^a A_a) \Sigma + c' \Delta c] .$$ (4.2)

Once again, we have switched the Grassmann parity of the fields. Of course, this is the gauge fixed abelian Chern-Simons action.

\textsuperscript{6}In order to get the non-abelian Chern-Simons theory, a term which is cubic in the $B_a$ field must be added to the $BF$ lagrangian.

\textsuperscript{7}The real parts of the corresponding fields from the previous sub-sections.
As there are only two real fermions in this action, there is only a global $SO(2)$ invariance, not $SU(2)$. Thus we are unable to associate the Lorentz symmetry of $S_{CS}$ with the diagonal sum of two $SU(2)$'s and there is no group theoretic justification for taking the internal index on the fermions to be Lorentz spinor indices, in the definition of the twist. However, we simply point out that if this is done at the level of the fields, then the Chern-Simons action is obtained.

4.2 $g$–Twisting $S^{SUSY}$

There is, however, a way to obtain the Chern-Simons action – actually two copies – while having a group theoretic justification. We start with the action $S^{SUSY}$ (2.6) which differs from $S^{SU2}$ by surface terms. Now we take the internal $SU_I(2)$ indices on $\Psi_{\alpha A}$ to be Lorentz spin-$\frac{1}{2}$ indices. Again this amounts to re-defining the Lorentz group to be the diagonal sum of the two $SU(2)$'s. Then the $g$–twist is defined by

$$T_g: \Psi_{\alpha B} \rightarrow \Psi_{\alpha \beta} \equiv \frac{1}{\sqrt{2}} [(\gamma^a)_{\alpha \beta}(A_a + iB_a) + iC_{\alpha \beta}(\Sigma + i\Lambda)], \quad (4.3)$$

along with a change of Grassmann parity. $T_g$ acts on the scalar fields as before (3.1). Performing these replacements in $S^{SUSY}$ and applying the procedure outlined in sub-section (3.2), we obtain

$$S^2_{CS} = -\frac{1}{2} \int d^3x \left[ \epsilon^{abc} A_a \partial_b A_c + \epsilon^{abc} B_a \partial_b B_c \right]$$

$$- \int d^3x \sqrt{g} \left[ (\nabla^a A_a) \Sigma + (\nabla^a B_a) \Lambda - c' \triangle c - b' \triangle b \right], \quad (4.4)$$

This is the action for two uncoupled copies of the gauge fixed Chern-Simons theory. Curiously, the appearance of more than one gauge field is a phenomena in extended supersymmetric Chern-Simons theories (14). Identifying the set of fields $(B_a, \Lambda, b', b)$ with the set $(A_a, \Sigma, c', c)$ reduces this to (twice) the action for one Chern-Simons gauge field (12).
5 The $g$–Twisted Super-Algebra

In the context of gauge fixed theories, “supersymmetry” is to be understood as a set of transformations generated by Grassmann odd charges which take fields of ghost number $n$ into fields of ghost number $n \pm 1$. Vector supercharges of ghost number 1 were discovered for the three-dimensional Chern-Simons theory in the Landau gauge in ref. [18]. It was soon thereafter realized that the same theory is further invariant under the anti-BRST transformations and another vector generator both of ghost number $-1$ [12]. The BRST generator and the ghost number $-1$ vector generator were found to close on translations, thereby forming an $N = 2$ supersymmetry algebra. In addition, the anti-BRST generator and the ghost number 1 generator form another $N = 2$ superalgebra. The $N = 2$ algebra, including the BRST generator, was then found to hold for the two- and four-dimensional non-abelian BF theories [13, 14], and was generalized to arbitrary dimensions in ref. [15]. It was used to prove the perturbative finiteness of the $D = 3$ Chern-Simons theory [19] and of the BF theory (see [15] and references therein). We will now extract these charges and algebras from our $N = 4$ supersymmetry algebra (2.8) via twisting.

The $g$–twist acts on the supercharges as

\begin{equation}
T_g : Q_{\alpha B} \rightarrow Q_{\alpha \beta} \equiv (\gamma^a)_{\alpha \beta} Q_a + iC_{\alpha \beta} Q , \\
\bar{T}_g : \bar{Q}_{\alpha B} \rightarrow \bar{Q}_{\alpha \beta} \equiv (\gamma^a)_{\alpha \beta} \bar{Q}_a + iC_{\alpha \beta} \bar{Q} .
\end{equation}

In the absence of covariantly constant vectors, only the scalar supercharges are conserved on curved manifolds. On $\mathbb{R}^3$, the full set of supercharges is conserved. Note that since the supercurrents were originally a product of a Grassmann odd and the derivative of a Grassmann even field, the Grassmann parity of the supercharges remains the same, namely odd.

Performing the map on the $N = 4$ supersymmetry algebra (2.8) we find the $g$–twisted algebra whose only non-trivial anti-commutators are

\begin{equation}
\{ \bar{Q}_a, Q_b \}, \quad \{ \bar{Q}_a, Q \}, \quad \{ Q, Q_a \} = i2\partial_a .
\end{equation}

$Q$ and its complex conjugate are nilpotent.

The supersymmetry transformations (2.7) now take the forms:
\[
\begin{align*}
[Q, A_a] &= \partial_a (c + ib') , &
[Q, B_a] &= i\partial_a (c' - ib) , \\
[Q, \Lambda] &= 0 , &
[Q, \Sigma] &= 0 , \\
\{Q, c\} &= i\Sigma , &
\{Q, b\} &= i\Lambda , \\
\{Q, c'\} &= -\Lambda , &
\{Q, b'\} &= -\Sigma , \\

[\bar{Q}, A_a] &= \partial_a (c - ib') , &
[\bar{Q}, B_a] &= -i\partial_a (c' + ib) , \\
[\bar{Q}, \Lambda] &= 0 , &
[\bar{Q}, \Sigma] &= 0 , \\
\{\bar{Q}, c\} &= -i\Sigma , &
\{\bar{Q}, b\} &= -i\Lambda , \\
\{\bar{Q}, c'\} &= -\Lambda , &
\{\bar{Q}, b'\} &= -\Sigma , \\

[Q_a, A_b] &= -\epsilon_{abc}\partial^c (c + ib') , &
[Q_a, B_b] &= -i\epsilon_{abc}\partial^c (c' - ib) , \\
[Q_a, \Lambda] &= -i\partial_a (c' - ib) , &
[Q_a, \Sigma] &= -\partial_a (c + ib') , \\
\{Q_a, c\} &= iA_a , &
\{Q_a, b\} &= iB_a , \\
\{Q_a, c'\} &= -B_a , &
\{Q_a, b'\} &= -A_a , \\
\end{align*}
\]

These are symmetry transformations for the three-dimensional, gauge fixed BF action. Upon defining \(Q \equiv s + is'\) we find the BRST (s) and anti-BRST (s') transformations to be

\[
\begin{align*}
[s, A_a] &= \partial_a c , &
[s, B_a] &= \partial_a b , \\
[s, \Lambda] &= 0 , &
[s, \Sigma] &= 0 , \\
\{s, c\} &= 0 , &
\{s, b\} &= 0 , \\
\{s, c'\} &= -\Lambda , &
\{s, b'\} &= -\Sigma , \\

[s', A_a] &= \partial_a b' , &
[s', B_a] &= \partial_a c' , \\
[s', \Lambda] &= 0 , &
[s', \Sigma] &= 0 , \\
\{s', c\} &= \Sigma , &
\{s', b\} &= \Lambda , \\
\{s', c'\} &= 0 , &
\{s', b'\} &= 0 .
\end{align*}
\]

Similarly, the transformations generated by the real, vector super-charges, \(s_a\) and \(s'_a\) defined by \(Q_a \equiv s_a + is'_a\) are found from (5.7) to be
\[
\begin{align*}
[s_a, A_b] &= -\epsilon_{abc} \partial^c c, & [s_a, B_b] &= -\epsilon_{abc} \partial^b b, \\
[s_a, A] &= -\partial_a b, & [s_a, \Sigma] &= -\partial_a c, \\
\{s_a, c\} &= 0, & \{s_a, b\} &= 0, \\
\{s_a, c'\} &= -B_a, & \{s_a, b'\} &= -A_a, \\
[s'_a, A_b] &= -\epsilon_{abc} \partial^c' b', & [s'_a, B_b] &= -\epsilon_{abc} \partial^c' c', \\
[s'_a, \Lambda] &= -\partial_a c', & [s'_a, \Sigma] &= -\partial_a b', \\
\{s'_a, c\} &= A_a, & \{s'_a, b\} &= B_a, \\
\{s'_a, c'\} &= 0, & \{s'_a, b'\} &= 0.
\end{align*}
\]

The vector super-charges along with the scalar BRST and anti-BRST super-charges satisfy the superalgebra

\[
\{s'_a, s_b\} = \epsilon_{abc} \partial^c, \quad \{s_a, s'_b\} = \partial_a, \quad \{s'_a, s\} = -\partial_a,
\]

with all other combinations vanishing. The BRST symmetry and the symmetry generated by the vector super-charge, \(s'_a\), are in agreement with the results of [14]. The transformations of the anti-BRST and \(s_a\) charges were not previously given for the case of \(BF\) theories. Our results verify the general statement that \(s'\) and \(s'_a\) may be obtained from \(s\) and \(s_a\), respectively, via interchanges of ghosts and anti-ghosts. Due to the first order nature of the classical \(BF\) action this takes the form \(c \to b', b' \to -c, b \to c'\) and \(c' \to -b\).

Our superalgebras close on-shell only. Superfield formulations of the supersymmetric theories in section 2 are expected to yield, upon \(g\)-twisting, off-shell closure of the algebras (5.6) and (5.10).

6 Relating \(BF\) to TQFT’s

As mentioned before, twisting a supersymmetric action to a TQFT requires only the first step in our \(g\)-twisting process in that the Grassmann parity of the fields is not changed. Performing the Grassmann parity change twice is equivalent to the identity. Thus if we perform a Grassmann parity change on the \(BF\) action, we expect to find a TQFT. Let us see this explicitly.
Upon making the replacements,

\begin{align*}
A_a & \to \rho_{a1} , & B_a & \to \rho_{a2} , \\
\Lambda & \to \xi_1 , & \Sigma & \to \xi_2 , \\
c & \to \varpi_1 , & b & \to \varpi_2 , \\
c' & \to \varphi_1 , & b' & \to \varphi_2 ,
\end{align*}

(6.1)

with the Grassmann parity assignments,

| FIELD | SPIN | GRASSMANN PARITY |
|-------|------|------------------|
| $\rho_{ai}$ | 1 | odd |
| $\xi_i$ | 0 | odd |
| $\varphi_i$ | 0 | even |
| $\varpi_i$ | 0 | even |

in the three dimensional $BF$ action (3.3), we obtain

\[ S'_{TQFT} = S_{TQFT} - i \frac{1}{2} \int d^3 x \epsilon^{abc} \rho_{ai} \partial_b \rho_{cj} \epsilon^{ij} , \]

(6.2)

where

\[ S_{TQFT} = - \int d^3 x \sqrt{g} \sum_{i=1}^{2} [\nabla^a \varpi_i \nabla_a \varphi_i + \rho_i^a \nabla_a \xi_i] . \]

(6.3)

Making the same replacements in (5.8) yields the BRST transformations under which $S'_{TQFT}$ is invariant. We record them for completeness:

\begin{align*}
\{ s, \rho_{ai} \} & = \partial_a \varpi_i , \\
[s, \varpi_i] & = 0 , \\
[s, \varphi_i] & = -\xi_i , \\
\{ s, \xi_i \} & = 0 .
\end{align*}

(6.4)

It is then easy to see that

\[ S_{TQFT} = \{ s, - \int d^3 x \sqrt{g} \sum_{i=1}^{2} \rho_{ai}^a \nabla_a \varphi_i \} . \]

(6.5)

Since the last term in $S_{TQFT}$ is metric independent, the energy-momentum tensor from the latter action is $s$–exact. Of course, starting with this TQFT action and inverting the replacements (6.1) leads us back to the $BF$ theory.
Alternatively, we could start with our action (2.6) and perform the usual TQFT twist defined to be the map

\[ \mathcal{T}_{TQFT} : \Psi_{\alpha B} \rightarrow \Psi_{\alpha \beta} \equiv \frac{1}{\sqrt{2}}[(\gamma^a)_{\alpha \beta}(\rho_{a1} + i\rho_{a2}) + iC_{\alpha \beta}(\xi_1 + i\xi_2)] , \quad (6.6) \]

which leaves the spin-0 fields, \( \phi \equiv \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) \) and \( \lambda \equiv \frac{1}{\sqrt{2}}(\bar{\varphi}_1 + i\bar{\varphi}_2) \) along with the Grassmann parity of the fields unchanged. With this prescription, we find that the action, \( S^{ISUSY} \) becomes \( S'_{TQFT} \) up to surface terms. If we denote the operation of changing the Grassmann parity of the fields by \( g \), then this information may be encoded in the following diagram:

\[ \begin{array}{ccc}
S^{ISUSY} & & S'_{TQFT} \\
\mathcal{T}_g & & \mathcal{T}_{TQFT} \\
S_{BF} & & S'_{TQFT} \\
\end{array} \]

Figure 1: The TFT Triangle.

The last term in \( S'_{TQFT} \) does not normally appear in topological sigma models (even flat ones). Its presence is idiosyncratic to three dimensions. It is invariant under the BRST transformations of eqn. (6.4). Although this part of the action has ghost number \(-2\), the full action remains invariant under the \( U(1) \) transformation with weights \((-)^i\) for \( \rho_{ai} \) and \((-)^{i+1}\) for \( \xi_i \).

A similar procedure may be performed using the Chern-Simons action, \( S_{CS} \). We find only \( S_{TQFT} \) instead of \( S'_{TQFT} \); that is \( g : S_{CS} \rightarrow S_{TQFT} \). The map is not invertible as we cannot obtain the Chern-Simons action from \( g : S_{TQFT} \). In other words, only the gauge fixing and ghost actions of the Chern-Simons theory may be obtained from \( S_{TQFT} \) (or \( S'_{TQFT} \)).
7 Conclusions

We have defined supersymmetric actions for matter fields which when $g$-twisted (a twist plus Grassmann parity change) yield gauge fixed, abelian $BF$ theories in three and two dimensions. In three dimensions, our theory is $N = 4$ supersymmetric while in two dimensions it is $N = 2$ supersymmetric. It has also been shown how to obtain the gauge fixed Chern-Simons theory via a $g$-twist. Furthermore, a Donaldson-Witten TQFT is obtained via the usual twisting applied to our supersymmetric action. This yields a scheme for mapping the $BF$ theories into TQFT’s. For the examples studied we can associate a topological field theory triangle explicitly illustrating the maps which relate the supersymmetric, $BF$ and TQFT actions.

The non-abelian case has not been addressed in this work. It would also be interesting to check for possible connections between the observables of the $BF$ theories (linking numbers) and those of the TQFT’s. Indeed, we expect that our procedure may be generalized to arbitrary dimensional manifolds (without torsion).

Appendix: Conventions

Our conventions are as follows. A Majorana spinor, $\psi^\alpha$, in three dimensions is real and has two components. Our gamma matrix conventions in Minkowski space are $\gamma^a \equiv (\sigma^2, -i\sigma^1, i\sigma^3)$. We have the useful identity $(\gamma^a \gamma^b)_{\alpha\beta} = \eta^{ab} C_{\alpha\beta} - i \epsilon^{abc} (\gamma_c)_{\alpha\beta}$. The charge conjugation matrix, $C_{\alpha\beta} = \gamma^0 = \sigma^2$, acts as $\psi^\alpha = C^{\alpha\beta} \psi_\beta$ with $C_{\alpha\beta} C^{\gamma\delta} = \delta^\beta_\alpha [\gamma^\delta]$. Note that since $C$ is imaginary, $\psi_\alpha$ is imaginary. The metric in Minkowski space is $\eta = \text{diag}(1, -1, -1)$. For manifolds with Euclidean signature, the gamma matrices are $\gamma^a = (\sigma^2, \sigma^1, \sigma^3)$. With these conventions, $\bar{\psi}^\alpha (\psi_\alpha)$ is still real (imaginary). The space-time Levi-Cevita tensor is defined by $\epsilon^{012} \equiv 1$ such that $\epsilon_{abc} \epsilon^{def} = \delta_a^{[d} \delta_b^{e} \delta_c^{f]}$. Internal or $SU(2)$ doublet indices are lowered with the real sympletic metric $\epsilon_{AB} \epsilon_{AB} = \psi_B \psi_B$ and raised as $\epsilon^{AB} \psi_B = \psi^A$. A bar is used to indicate complex conjugation.

In two dimensions, our gamma matrices are $\gamma^a = (\sigma^2, -i\sigma^1)$ and $\gamma^3 = \sigma^3$. These satisfy $\gamma^a \gamma^b = \eta^{ab} - \epsilon^{ab} \gamma^3$ and $\gamma^3 \gamma^a = -\epsilon^{ab} \gamma_b$. Otherwise, our conventions are in analogy with three dimensions.
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