Centrality of nodes in multiplex networks

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We extend the concept of eigenvector centrality to multiplex networks, and introduce several alternative parameters that quantify the importance of nodes in a multi-layered networked system, including the definition of vectorial-type centralities. In addition we rigorously show that, under reasonable conditions, such centrality measures exist and are unique. Computer experiments and simulations demonstrate that the proposed measures provide substantially different results when applied to the same multiplex structure, and highlight the non-trivial relationships between the different measures of centrality introduced.

Many biological, social, and technological systems find a suitable representation as complex networks, where nodes represent the system’s constituents and edges account for the interactions between them [11–15]. In the general case, the nodes’ interactions need a more accurate mapping than simple links, as the constituents of a system are usually simultaneously connected in multiple ways. For instance, in social networks, one can consider several types of different actors’ relationships: friendship, vicinity, kinship, membership of the same cultural society, partnership or coworkership, etc. In such a case, it is useful to endow our network with a multiplex network structure. This representation reflects the interaction of nodes through multiple layers of links, interaction which cannot be captured by the classical single-layer network representation. This multiplex representation has long been considered by sociologists (multiplex tie ([6–8])), and although some results concerning multiplex networks’ modeling and structure have been recently given [9–17], the study of centrality parameters in such networks has not yet been addressed satisfactorily. The aim of this paper is to propose a definition of centrality in multiplex networks, and illustrate potential applications.

I. NOTATIONS

Along this paper, we consider a multiplex network \( \mathcal{G} \), made of \( m \in \mathbb{N} \) layers \( G_1, \ldots, G_m \), such that each layer is a (directed or undirected) un-weighted network \( G_k = (X, E_k) \), with \( X = \{e_1, \ldots, e_n\} \) (i.e. all layers have the same \( n \in \mathbb{N} \) nodes). The transpose of the adjacency matrix of each layer \( G_k \) is denoted by \( A_k = (a_{ij}^k) \in \mathbb{R}^{n \times n} \), where

\[
a_{ij}^k = \begin{cases} 1 & \text{if } (e_j, e_i) \in E_k, \\ 0 & \text{otherwise}, \end{cases}
\]

for \( 1 \leq i, j \leq n \) and \( 1 \leq k \leq m \). The projection network associated to \( \mathcal{G} \) is the graph \( \overline{G} = (X, E) \), where

\[
E = \bigcup_{k=1}^{m} E_k.
\]

The transpose of the adjacency matrix of \( \overline{G} \) will be denoted by \( \overline{A} = (\overline{a}_{ij}) \in \mathbb{R}^{n \times n} \). Note that for every \( 1 \leq i, j \leq n \)

\[
\overline{a}_{ij} = \begin{cases} 1 & \text{if } a_{ij}^k \neq 0 \text{ for some } 1 \leq k \leq m, \\ 0 & \text{otherwise}. \end{cases}
\]

The paper is structured as follows. In the next section, we will introduce different heuristic arguments suggesting proper ways of measuring centrality in multiplex networks. Section III is devoted to establish, under reasonable conditions, the existence and consistency of the proposed measures of centrality. In Section IV, we report some computer experiments and simulations showing how the measures introduced provide substantially different results when applied to the same multiplex networks. Finally, the last section is devoted to discussion.

II. MATHEMATICAL MODELS FOR EIGENVECTOR CENTRALITY IN CONNECTED MULTIPLEX NETWORKS

In the case of a multiplex network, the central question to be addressed is the following: How can one take into account all the interactions between the different sub-networks (channels, communities, layers...) bearing in mind that not all of them have the same importance? It is essential, indeed, to remark that in order to get the centrality of a node it is necessary to take into account how the centrality (importance, influence,...) of a node is propagated within the whole network through different channels (layers) that are non necessarily additives. For instance, worldwide social networks (such as Facebook or Twitter) are made up of very heterogeneous interactions, which also characterizes the units’ interactions in fields
as climate systems [11], game theory [12, 13], interacting infrastructures [14, 15] and many others ([9, 16]).

With reference to the network $G$, one can consider (for each layer) the classical eigenvector centrality $G_k$ as the principal eigenvector of $A_k$ (if it exists). Specifically, the eigenvector centrality of a node $e_i$ within the layer $G_k$ would be the $i^{th}$ entry of the positive definite and normalized vector $c_k \in \mathbb{R}^n$ corresponding to the largest eigenvalue of the matrix $A_k$. In a similar way, the eigenvector centrality of the projection network $\bar{G}$ will be the principal eigenvector of $\bar{A}$. The existence and uniqueness of these vectors are guaranteed by the Perron-Frobenius theorem for any symmetric matrix with positive entries.

Interestingly, the Perron-Frobenius theorem can be conveniently extended for multiplex networks, leading to even deeper concepts of nodes’ centrality. We remark that other extensions of Perron-Frobenius’s theorem have been given for hypergraphs and nonnegative tensors ([18, 19]).

Once all the eigenvector centralities have been computed, one can consider the independent layer eigenvector-like centrality of $G$ (abbrev. the independent layer centrality of $G$) as the matrix

$$C = \begin{pmatrix} c_1 & c_2 & \cdots & c_m \end{pmatrix} \in \mathbb{R}^{n \times m}.$$ 

Notice that $C$ is column stochastic, since $c_k \geq 0$ and $\|c_k\|_1 = 1$ for every $1 \leq k \leq m$.

If one now bears in mind that the centrality (importance) of a node must be proportional to the centrality of its neighbors (that are distributed among all the layers), and considers that all the layers have the same importance, one has that

$$\forall i, j \in X, \; c(i) \propto c(j) \; \text{if} \; (j \rightarrow i) \in G_\ell, \; \ell \in \{1, \ldots, m\},$$

and one can define the uniform eigenvector-like centrality (abbrev. the uniform centrality) as the positive and normalized eigenvector $\tilde{c} \in \mathbb{R}^n$ (if it exists) of the matrix $\tilde{A}$ given by

$$\tilde{A} = \sum_{k=1}^m A_k.$$

This situation occurs, for instance, in social networks, where different people may have different relationships with other people, while one is generically interested to measure the centrality of the network of acquaintances.

A step ahead is to consider different degree of importance (or influence) in different layers of the network, and to include such a latter information in the definition of a matrix that would establish the mutual influence between the layers. Thus, in order to calculate the importance (or influence) of a node within a specific layer, one must take into account also all the other layers, as some of them may be really relevant for that calculation. Consider, for instance, the case of a boss who goes to the same gym as one of his employees: the relationship between the two fellows within the gym layer has a totally different nature from that occurring inside the office layer, but the role of the boss (i.e. his centrality) in this case can be even bigger than if he was the only one person of the office frequenting that gym. In other words, one needs to consider the situation where the influence amongst layers is heterogeneous.

To this purpose, one can introduce an influence matrix $W = (w_{ij}) \in \mathbb{R}^{n \times m}$ as a non-negative matrix $W \geq 0$ such that $w_{ij}$ measures the influence of the layer $G_j$ on the layer $G_i$. Once $G$ and $W = (w_{ij})$ have been fixed, one then defines the local heterogeneous eigenvector-like centrality of $G$ (abbrev. the local heterogeneous centrality of $G$) on each layer $G_k$ ($1 \leq k \leq m$) as a positive and normalized eigenvector $c^*_k \in \mathbb{R}^n$ (if it exists) of the matrix

$$A^*_k = \sum_{j=1}^m w_{kj} A_j.$$

Once again, the local heterogeneous eigenvector-like centrality (abbrev. local heterogeneous centrality) matrix of the multiplex network $G$ will be defined as

$$C^* = \begin{pmatrix} c^*_1 & c^*_2 & \cdots & c^*_m \end{pmatrix} \in \mathbb{R}^{n \times m}.$$ 

Another important aspect to be elucidated is that, in general, the centrality of a node $e_i$ within a specific layer $k$ may depend not only on the neighbors that are linked to $e_i$ within the layer $k$, but also to all other neighbors of $e_i$ that belong to the other layers. That is the the case of scientific citations in different areas of knowledge; indeed, imagine two scientists (a chemist and a physicist) and one of them has been awarded the Nobel Prize: the importance of the other scientist will increase a lot even though the Nobel prize laureate had few citations within the other researcher’s area. This heuristic argument leads to the introduction of another concept of centrality: Given a multiplex network $G$ and an influence matrix $W = (w_{ij})$, the global heterogeneous eigenvector-like centrality of $G$ (abbrev. global centrality of $G$) is defined as a positive and normalized eigenvector $c^g \in \mathbb{R}^{nm}$ (if it exists) of the matrix

$$A^g = \begin{pmatrix} w_{11} A_1 & w_{12} A_2 & \cdots & w_{1m} A_m \\ w_{11} A_1 & w_{22} A_2 & \cdots & w_{2m} A_m \\ \vdots & \vdots & \ddots & \vdots \\ w_{m1} A_1 & w_{m2} A_2 & \cdots & w_{mm} A_m \end{pmatrix} \in \mathbb{R}^{(nm) \times (nm)}.$$ 

Note that $A^g$ is the Khatri-Rao product of the matrices

$$W = \begin{pmatrix} w_{11} & \cdots & w_{1m} \\ \vdots & \ddots & \vdots \\ w_{m1} & \cdots & w_{mm} \end{pmatrix} \text{ and } \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix}.$$ 

In analogy with what has been one before, if one intro-
roduces the notation

\[ C^\otimes = \begin{pmatrix} c_1^\otimes \\ c_2^\otimes \\ \vdots \\ c_m^\otimes \end{pmatrix}, \]

with \( c_1^\otimes, \ldots, c_m^\otimes \in \mathbb{R}^n \), then one can define the global heterogeneous eigenvector-like centrality matrix of \( G \) as the matrix given by

\[ C^\otimes = \begin{pmatrix} c_1^\otimes & c_2^\otimes & \ldots & c_m^\otimes \end{pmatrix} \in \mathbb{R}^{n \times m}. \]

Note that, in general \( C^\otimes \) is neither column stochastic nor row stochastic, but the sum of all the entries of \( C^\otimes \) is 1.

III. EXISTENCE AND CONSISTENCY

Let us now move to discuss the conditions that guarantee the existence and uniqueness of the centrality measures introduced in the previous section.

The natural question here is whether the strong connectedness of the projected graph \( \overline{G} \) or, equivalently, the irreducibility of the nonnegative matrix \( A \), is a sufficient condition for the existence and uniqueness of our centralities measures. One can make use of the basic facts on the Perron-Frobenius theorem, as well as on irreducible matrices and strongly connected graphs, for which we refer the interested reader to Ref. [20]. In fact, recalling that the graph determined by \( A = \sum_k A_k \) coincides with the projected graph of the network, in the case of the Uniform Centrality we immediately get the following

**Theorem 1** If the projected graph \( \overline{G} \) of a multiplex network \( G \) is strongly connected, then the Uniform Centrality \( C \) of \( G \) exists and is unique.

The case of the Local Heterogeneous Centrality is similar, as every row \( C^\otimes_i \) of the matrix \( C^\otimes \) is the principal normalized eigenvector of a linear combination \( A^\otimes_i = \sum_k w_{ki} A_k \). In particular, if \( W \) is positive, the graph associated to every \( A^\otimes_i \) is the projected graph of the multiplex network, hence one get also

**Theorem 2** If the projected graph \( \overline{G} \) of a multiplex network \( G \) is strongly connected, and \( W > 0 \) then the Local Heterogeneous Centrality \( C^\otimes \) of \( G \) exists and is unique.

A more delicate case is that of the Global Heterogeneous Centrality, that is constructed upon the principal normalized eigenvector of the matrix

\[ \begin{pmatrix} w_{11} A_1 & w_{12} A_2 & \cdots & w_{1m} A_m \\ w_{21} A_1 & w_{22} A_2 & \cdots & w_{2m} A_m \\ \vdots & \vdots & \ddots & \vdots \\ w_{m1} A_1 & w_{m2} A_2 & \cdots & w_{mm} A_m \end{pmatrix}. \]

Such a matrix is the transpose of the adjacency matrix of a graph with \( nm \) nodes that we denote by \( G^\otimes = (X^\otimes, E^\otimes) \), where \( X = \{e_{ik}, i = 1, \ldots, n, k = 1, \ldots, m\} \) and \( (e_{jk}, e_{ik}) \in E^\otimes \) iff \( w_{kj} a_{ij}^k \neq 0 \). Unfortunately, even if the projected graph of a multiplex network \( G \) is strongly connected and \( W \) is positive, the graph \( G^\otimes \) is not, in general, strongly connected. In fact one can easily check that this is already the case for the example in which \( G \) consists of two nodes and two layers, with matrices:

\[ A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]

Nevertheless, it is still possible to infer the existence and unicity of \( C^\otimes \) from the strong-connectedness of \( \overline{G} \) and the positivity of \( W \). Indeed, one has first to notice that, if \( G \) is strongly connected and \( W \) is positive, then \( G^\otimes \) satisfies:

\[ (e_{jk}, e_{ik}) \in E^\otimes \iff a_{ij}^k \neq 0 \iff (e_j, e_i) \in E. \]

Now, we denote a node \( e_{jk} \) of \( G^\otimes \) as a \( \otimes \)-sink when \( a_{ij}^k = 0 \) for all \( i \), so that the corresponding column of \( A^\otimes \) is identically zero. If a node \( e_{jk} \) is not a \( \otimes \)-sink, we claim that, given any other node \( e_{ik} \) there exists a path in \( G^\otimes \) going from \( e_{jk} \) to \( e_{ik} \).

Assuming \( \overline{G} \) to be strongly connected, there exist then indices \( i_1 = j, i_2, \ldots, i_r = i \) such that, for every \( s \in \{1, \ldots, r - 1\} \), there exists an index \( l_s \in \{1, \ldots, m\} \) for which \( a_{lj_s}^{i_s} \neq 0 \). Thus, by construction, \( (e_{i_s}, e_{i_{s+1}}) \in E^\otimes \) for all \( s \), and this finishes the proof of the latter claim.

From these arguments, one may easily deduce that the normal form of the matrix \( A^\otimes \) (cf. [21, p. 46]) takes the form:

\[ N = P \cdot A^\otimes \cdot P^t = \begin{pmatrix} 0 & \cdots & 0 & \cdots & \cdots \\ \vdots & \ddots & \vdots & \ddots \cdots \cdots \\ \vdots & \cdots & 0 & \cdots & \cdots \\ \vdots & \cdots & \cdots & \ddots & \cdots \cdots \\ 0 & \cdots & \cdots & \cdots & B \end{pmatrix}, \]

where \( P \) is a permutation matrix and \( B \) is an irreducible nonnegative matrix, to which the Perron-Frobenius Theorem can be applied. It follows that the spectrum of \( A^\otimes \) is the union of the spectrum of \( B \) and \( \{0\} \), and that \( A^\otimes \) has a unique normalized eigenvector associated to \( \rho(A^\otimes) = \rho(B) \). Summing up, we get the following

**Theorem 3** If the projected graph \( \overline{G} \) of a multiplex network \( G \) is strongly connected, and \( W > 0 \) then the Global Heterogeneous Centrality \( C^\otimes \) of \( G \) exists and is unique.

The next step is discussing the consistency of our definitions in a variety of special cases.
Monoplex networks. It is straightforward demonstrating that on a monoplex network (i.e. a multiplex network consisting of only one layer) our three concepts of multiplex centrality coincide with the usual eigenvector centrality of the layer.

Identical layers. Let $\mathcal{G}$ be a multiplex network for which $A_k = A_\ell$ for every $1 \leq k, \ell \leq m$, and note that $A_k = \overline{A}$, for every $k$, so that the Uniform Centrality of $G$ coincides with the Eigenvector Centrality of every layer $G_k$. Assuming that every row of $W$ is nonnegative (in particular if $W > 0$) it is also clear that every column of the Local Heterogeneous Centrality $C^*$ coincides with the Uniform Centrality $\overline{C}$ of $\mathcal{G}$.

The case of the Global Heterogeneous Centrality is slightly different. If all the layers are identical, the matrix $A^\otimes$ coincides with the so called Kronecker product of the matrices $W$ and $\overline{A}$. It is well known (see for instance [22, Ch. 2]) that the spectral radius of $A^\otimes$ is then equal to $\rho(W)\rho(\overline{A})$ and that its normalized principal eigenvector is the Kronecker product of the normalized principal eigenvectors $C_W$ of $W$ and $\overline{C}$ of $\overline{A}$. In terms of matrices, this is equivalent to say that $C^* = C_W \cdot \overline{C}$. In particular, the normalization of all the columns of $C^*$ coincides $\overline{C}$.

Starred layers. We finally consider the case in which the multiplex network $\mathcal{G}$ contains exactly $m = n$ layers, satisfying that the layer $G_k$ consists of a set of edges coming out of the node $e_k$. In other words, $a_{ij}^k = 0$ if $j \neq k$. In this case there exists a permutation matrix $P$ such that:

$$P \cdot A^\otimes \cdot P^t = \begin{pmatrix} 0 & \cdots & 0 & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & * & \cdots & * \\ 0 & \cdots & 0 & \cdots & \cdots & \cdots \end{pmatrix},$$

where $W \circ \overline{A}$ is the Hadamard product (see, for example [24]) of $W$ and $\overline{A}$ (i.e. $(W \circ \overline{A})_{ij} = w_{ij} \overline{a}_{ij}$). In particular the Global Heterogeneous Centrality of $\mathcal{G}$ is the diagonal $n \times n$ matrix whose diagonal is the eigenvector centrality of $W \circ \overline{A}$. Note that $W \circ \overline{A}$ can be interpreted as the transpose of the matrix of the graph $\overline{G}$, in which the edge going from $e_j$ to $e_i$ has been assigned a weight equal to $w_{ij}$. In this sense the eigenvector centrality of a weighted graph can be seen as a particular case of the Global Heterogeneous Centrality.

IV. NUMERICAL TESTINGS AND DISCUSSIONS

This section is devoted to illustrating that the introduced centrality measures behave very differently, by computing them in some examples. Instead of considering particular tailor-made examples, we have chosen random networks coming from a class of scale-free assortative-inspired synthetic graphs (cf. [9]), that we will describe later on.

If we take a network of $n$ nodes $\{e_1, \cdots, e_n\}$ and consider two centrality measures $c, c' \in \mathbb{R}^n$ such that the $i$-th coordinate of $c$ and $c'$ measure the centrality of node $v_i$ for every $1 \leq i \leq n$, one way of measuring the correlation between $c$ and $c'$ is by computing $\|c - c'\|$ for some norm $\| \cdot \|$. Despite the fact that $\|c - c'\|$ measures the discrepancy between $c$ and $c'$, its value is not representative of the real information about the correlation between $c$ and $c'$. Note, indeed, that one of the main features of the centrality measures is the fact that they produce rankings, i.e. in many cases the crucial information obtained from a centrality measure is the fact that a node $v_i$ is more relevant than another node $v_j$, and this ordering is more important than the actual difference between the corresponding centrality of nodes $v_i$ and $v_j$. Hence, if we want to analyze the correlations among a set of centrality measures, we should study in detail the correlations between the associated rankings, which requires a different kind of methods.

There are several options in the classic literature that can be useful for studying the correlations between two rankings $r$ and $r'$, but some of the most standard are the Spearman’s rank correlation coefficient $\rho(r, r')$ and the Kendall’s rank correlation coefficient $\tau(r, r')$. If we consider two centrality measures $c, c' \in \mathbb{R}^n$ of a network with nodes $\{e_1, \cdots, e_n\}$, then each centrality measure $c$ and $c'$ produces a ranking of the nodes that will be denoted by $r$ and $r'$ respectively. The Spearman’s rank correlation coefficient [24] between two centrality measures $c$ and $c'$ is defined as

$$\rho(c, c') = \rho(r, r') = \frac{\sum_{i=1}^n (r(v_i) - \overline{r})(r'(v_i) - \overline{r}')}{\sqrt{\sum_{i=1}^n (r(v_i) - \overline{r})^2 (r'(v_i) - \overline{r}')^2}},$$

where $r(v_i)$ and $r'(v_i)$ are the ranking of node $v_i$ with respect to the centrality measures $c$ and $c'$ respectively, $\overline{r} = \frac{1}{n} \sum_{i=1}^n r(v_i)$ and $\overline{r}' = \frac{1}{n} \sum_{i=1}^n r'(v_i)$. Similarly, the Kendall’s rank correlation coefficient [25] between two centrality measures $c$ and $c'$ is defined as

$$\tau(c, c') = \tau(r, r') = \frac{\hat{K}(r, r') - K(r, r')}{\binom{n}{2}},$$

where $\hat{K}(r, r')$ is the number of pairs of nodes $\{v_i, v_j\}$ such that they appear in the same ordering in $r$ and $r'$ and $K(r, r')$ is the number of pairs of nodes $\{v_i, v_j\}$ such that they appear in different order in rankings $r$ and $r'$. Note that either $\rho(c, c')$ and $\tau(c, c')$ give values in $[-1, 1]$. The closer $\rho(c, c')$ is to 1 the more correlated $c$ and $c'$ are, while the closer $\tau(c, c')$ is to 0 the more independent $c$ and $c'$ are (and similarly for $\tau(c, c')$). In addition to this, if $\rho(c, c')$ (or $\tau(c, c')$) is close to $-1$ then $c$ and $c'$ are anti-correlated.

A further remark comes from the fact that the centrality measures introduced so far are very different from one another, and therefore one has to carefully describe
how to compare them. Indeed, from one side some scalar measures introduced in section II (the centrality of the node in the network) associate a single number to each node of the network, while, from the other side, other vectorial measures assign a vector to each node \( v_i \) (with each coordinate of the vector measuring the centrality of the node \( v_i \) as an actor of a different layer of the multiplex network). Actually, for a multiplex network \( G \) of \( n \) nodes, two scalar centralities have been introduced (the eigenvector centrality \( c \in \mathbb{R}^n \) of the projection graph, and the uniform eigenvector-like centrality \( \tilde{c} \in \mathbb{R}^n \)) and three vectorial centralities have been proposed (the independent layer centrality \( C \in \mathbb{R}^{n \times m} \), the local heterogeneous centrality \( C^* \in \mathbb{R}^{n \times m} \), and the global heterogeneous centrality \( C^\otimes \in \mathbb{R}^{n \times m} \)). In order to compare such different measures, the information contained in each vectorial-type centrality must be aggregated to associate a number to each node.

There are several alternative methods for aggregating information, but we use the convex combination technique as main criterion. For a multiplex network \( G \) of \( n \) nodes and \( m \) layers, we can fix some \( \lambda_1, \ldots, \lambda_m \in [0,1] \) such that \( \lambda_1 + \cdots + \lambda_m = 1 \) and compute the aggregated scalar centralities

\[
c = c(\lambda_1, \ldots, \lambda_m) = \sum_{j=1}^m \lambda_j c_j,
\]

\[
c^* = c^*(\lambda_1, \ldots, \lambda_m) = \sum_{j=1}^m \lambda_j c^*_j,
\]

where \( c_j \) is the \( j \)-th column of the independent layer centrality \( C \) and \( c^*_j \) is the \( j \)-th column of the local heterogeneous centrality \( C^* \). Note that the value of each \( \lambda_j \) can be understood as the relative influence of the layer \( G_j \) in the aggregated scalar centrality of the multiplex network. In our numerics, the specific value \( \lambda_1 = \cdots = \lambda_m = \frac{1}{m} \) has been chosen, as we suppose that no extra information about the relative relevance of each layer is available, and therefore the influence of each of them is considered equivalent. Note that \( c \) and \( c^* \) are normalized, since \( C \) and \( C^* \) are column-stochastic.

The case of the global heterogeneous centrality is different, since \( C^\otimes \) is not column-stochastic. In this case, since the sum of all entries of \( C^* \) is 1, then it is enough to take

\[
c^\otimes = \sum_{j=1}^m c^\otimes_j,
\]

where \( c^\otimes_j \) is the \( j \)-th column of the global heterogeneous centrality \( C^\otimes \). Consequently, the relative influence of each layer \( G_j \) can be defined as \( ||c^\otimes_j||_1 \) (i.e. the sum of all the coordinates of \( c^\otimes_j \)).

Once all the vectorial measures we have been aggregated (and the setting unified), we move to discuss state the ranking comparisons. In addition to the actual correlation among the centrality measures, we analyze the influence of the matrix \( W \) (called influence matrix in section II) used in the definition of the local heterogeneous centrality \( C^* \) and in the global heterogeneous centrality \( C^\otimes \). Since this matrix \( W \in \mathbb{R}^{m \times n} \) is non-negative we will consider two families of matrices \( \{W_1(q)\} \) and \( \{W_2(q)\} \) given for every \( 0 \leq q \leq 1 \) by

\[
W_1(q) = \begin{pmatrix} 1 & q & \cdots & q \\ q & 1 & \cdots & q \\ \vdots & \vdots & \ddots & \vdots \\ q & q & \cdots & 1 \end{pmatrix}, \quad W_2(q) = \begin{pmatrix} 1 & q & \cdots & q \\ q^2 & 1 & \cdots & q \\ \vdots & \vdots & \ddots & \vdots \\ q^2 & q^2 & \cdots & 1 \end{pmatrix}.
\]

Note that while each \( W_1(q) \) corresponds to a symmetric influence among the layers, each \( W_2(q) \) models an asymmetric influence among the layers of the multiplex network.

Finally, we briefly describe the method used to construct the synthetic multiplex networks used in the numerical testing, which corresponds to the model II of Ref. [9]. The model is inspired by the Barabási-Albert preferential attachment model [24] as well as by several bipartite networks models such as the collaboration network model proposed by J.J. Ramasco et al. [28], or the sub-linear preferential attachment bipartite model introduced by M. Peltonäki and M. Alava [27]. It consists of a growing random model determined by the following rules:

(i) Model parameters. The model has three main parameters: \( n, m \) and \( p_{new} \). We set \( n \in \mathbb{N} \) as the minimal number of nodes in the multiplex network and \( 2 \leq m \leq n \) as the number of active nodes in each layer (i.e. nodes that will produce links in each layers). Note that if we take \( m = 2 \), we recover the Barabási-Albert model [26]. In this model \( m \) will be fixed, but it can also be replaced by any other non-negative integer random variable in order to produce more general models, but the results obtained are structurally similar. Finally, we set \( p_{new} \in (0,1] \) as the probability of joining a new node to the growing multiplex network during its construction.

(ii) Initial conditions. We start with a seed multiplex network made of one single layer \( G_0 \) of \( m \) nodes that are linked all to all, (i.e. \( G_0 \) is the complete graph \( K_m \)). We can replace the all-to-all structure by any other structure (such as a scale free or an Erdős-Rényi network), but the results obtained are similar. This initial layer \( G_0 \) will be removed from the final multiplex network \( G \), since the all-to-all structure would make that the eigenvector centrality of the projection graph to be the bisection.

(iii) Layer composition. At each time step \( t \), a new layer \( G_t \) of \( m \) nodes is added to the multiplex network. We start by choosing randomly an existing node of the multiplex network proportionally to its degree (preferential election) that we will call the coordinator node. Therefore if at step \( t - 1 \), the set of
nodes of the multiplex network is \( \{v_1, \ldots, v_n\} \), and \( k_i \) denotes the degree of node \( v_i \) at time \( t = 1 \) in the projection network, then we choose the node \( v_i \) randomly and independently with probability

\[ p_i = \frac{k_i}{\sum_{j=1}^{n} k_j}. \]

Once the coordinator node has been chosen, each of the remaining \( m - 1 \) active nodes of \( G_t \) will be a new node with probability \( p_{\text{new}} \) and an existing node with probability \((1 - p_{\text{new}})\). If we have to add an already existing node, we will uniformly and independently choose it at random. Note that we can replace the uniform random selection by other random procedure (such as preferential selection), but the random tests done suggest that the multiplex network obtained have statistically the same structural properties when \( n \) is large enough (see [9]). Anyway, we should have chosen \( m \) nodes \( \tilde{v}_1, \ldots, \tilde{v}_m \) that will be the active nodes of the new layer \( G_t \) (i.e. nodes that will produce links in this layers).

(iv) **Layer inner-structure.** Once we have fixed the active nodes \( \tilde{v}_1, \ldots, \tilde{v}_m \) of the new layer \( G_t \), we have to give its links. First, we link all the active nodes to the coordinator in order to ensure that all the eigenvector-like centrality are well defined. We set new links between each pair of active nodes, say \( v_i \) and \( v_j \) (with \( 1 < i \neq j \leq m \)) by using a random assortative linking strategy (this corresponds to the Model II in [4]). For every \( 2 \leq i \neq j \leq m \), we add randomly the link \( \{\tilde{v}_i, \tilde{v}_j\} \) proportionally to the number of common layers that hold simultaneously \( \tilde{v}_i \) and \( \tilde{v}_j \). Hence if we denote by \( Q_{ij} \) the number of layers that hold simultaneously \( \tilde{v}_i \) and \( \tilde{v}_j \) at time step \( t \) (including \( G_t \)) and by \( q_i \) the number of layers that hold \( \tilde{v}_i \) at time step \( t \) (also including \( G_t \)), thus the probability of linking node \( \tilde{v}_i \) with node \( \tilde{v}_j \) is given by

\[ p_{ij} = \frac{2Q_{ij}}{q_i + q_j}, \]

for every \( 2 \leq i \neq j \leq m \). The heuristic behind this model comes from social networks, since the relationships in a new social group are correlated with the previous relationships between the actors in other social groups [5]. Hence, if two actors that belong to the new social group coincide in many (previous) groups, then the probability of linking in this new group is large. In addition to this, the model also reflects the fact that if two new actors join their first group, the probability of establishing a relationship between them is high. At the end of this step we have defined completely the new layer \( G_t \).

(v) Finally, we repeat steps (iii) and (iv) until the number of nodes of the multiplex network is at least \( n \). After fixing all the settings of the numerical testings, we perform the comparison for three multiplex networks \( G_1, G_2 \) and \( G_3 \) (constructed as above), where:

(i) \( G_1 \) is a network of 102 nodes (computed with \( n = 100 \) as initial parameter) and 13 layers of 10 nodes each \((k = 10 \) as initial parameter). The probability of \( p_{\text{new}} = 0.8 \) of adding new active nodes to each layer. This is an example of a network with a relative small number of active nodes in each layer and such as each node is active in a few number of layers (since \( p_{\text{new}} = 0.8 \)).

(ii) \( G_2 \) is a network of 108 nodes (computed with \( n = 100 \) as initial parameter) and 4 layers of 40 nodes each \((k = 40 \) as initial parameter). The probability of \( p_{\text{new}} = 0.5 \) of adding new active nodes to each layer. In this case, this is a network with a relative big number of active nodes in each layer and a balanced number of newcomers and experienced nodes as active nodes in each layer \((p_{\text{new}} = 0.5) \).

(iii) \( G_3 \) is a network of 102 nodes (computed with \( n = 100 \) as initial parameter) and 6 layers of 60 nodes each \((k = 60 \) as initial parameter). The probability of \( p_{\text{new}} = 0.1 \) of adding new active nodes to each layer. In this case, this is a network with a big number of active nodes in each layer and a very low number of newcomers in each layer \((p_{\text{new}} = 0.1) \).

For each of these networks we compute the correlation between the eigenvector centrality of the projection graph, and the uniform centrality vs. the local and global heterogeneous centralities. Figures 1 and 2 plot the dependency of these correlations with respect to the influence strength \( q \in [0, 1] \) in a family of symmetric influence matrices \( W_1(q) \) (Figure 1) and with respect to the influence strength \( q \in [0, 1] \) in a family of non-symmetric influence matrices \( W_2(q) \) (Figure 2), exhibiting a similar pattern. Similar results for the correlations between the heterogeneous centralities and the independent layer centrality are displayed in Figure 3. Finally, we also report the local heterogeneous centrality vs. the global heterogeneous centrality, under the action of the two families of influence matrices \( W_1(q) \) and \( W_2(q) \) (see Figure 4).

V. DISCUSSION AND CONCLUSIONS

The analysis of the obtained results allows to draw the following conclusions:

- The results obtained with Spearman’s and Kendall’s coefficients are qualitatively equivalent, although Spearman’s rank is always slightly higher.

- The differences between the heterogeneous (global, local) and the flat centralities (centrality of the projected network, uniform centrality) are significantly
FIG. 1. Ranking comparison for the eigenvector centrality measures for two multiplex networks with the family of symmetric influence matrices of type $W_1(q)$. Top panels (from (a) to (d)) correspond to the network $G_1$, middle panels (from (e) to (h)) report the results for the network $G_2$, and bottom panels (from (i) to (l)) are for $G_3$ (see text for the details on the network construction). Panels (a) and (b) ((c) and (d)) show the ($q$-dependent) correlations between the eigenvector centrality of the projection graph and the uniform centrality vs. the local (global) heterogeneous centrality of $G_1$, respectively. Similarly, panels (e) to (h) give the same information for $G_2$ and panels (i) to (l) correspond to $G_3$ respectively. In all panels, the Spearman coefficient is reported in red, while the Kendall coefficient is reported in black.

broader for lower values of $q$. On the other hand, the behavior of a multiplex network with high values of $q$ is similar to the corresponding, monoplex, projected network. In other words, $q$, thought of as a measure of the multiplexity of the network, is detected by heterogeneous centrality measures.

- The total variation with respect to $q$ of the correlation between a heterogeneous and a flat measure grows with the number of layers.

- The centralities introduced have a quite different behavior among them. It is significatively remarkable the statistical independence between the independent layer and the heterogeneous centralities.

- There is a non-linear relationship between the centrality measures and the strength $q$ of the influence between layers.

- The symmetry of the influence between layers does not play a critical role in the correlations among centrality measures.

- The correlations between these new eigenvector-like centralities strongly depend on the structure of the multiplex networks.

In summary, we have introduced several definitions of centrality measures for multiplex networks, and we have showed that, under reasonable conditions, these centrality measures exist and are unique. Computer experiments and simulations performed by using the model introduced in [9] show that our measures provide substantially different results when applied to the same multiplex networks. This agrees with the fact that each of these measures arises from a different heuristic. In this sense, the concept of multiplex networks may be used to model complex networks of different kinds, so that the most appropriate kind of centrality measure shall be carefully determined in each case.

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FIG. 2. Ranking comparison for the eigenvector centrality measures for two multiplex networks with the family of non-symmetric influence matrices of type $W_2(q)$. Top panels (from (a) to (d)) correspond to $G_1$, middle panels (from (e) to (h)) report the results for the network $G_2$ and bottom panels (from (i) to (l)) are for $G_3$. Panels (a) and (b) ((c) and (d)) show the $(q$-dependent) correlations between the eigenvector centrality of the projection graph and the uniform centrality vs. the local (global) heterogeneous centrality of $G_1$, respectively. Similarly, panels (e) to (h) give the same information for $G_2$ and panels (i) to (l) correspond to $G_3$ respectively. Same stipulations as in the caption of Figure 1.

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FIG. 3. Ranking comparison for the independent layer centrality vs. local and global heterogeneous centralities. Top panels (from (a) to (d)) correspond to \( G_1 \), middle panels (from (e) to (h)) report the results for the network \( G_2 \) and bottom panels (from (i) to (l)) are for \( G_3 \). The first two columns of panels on the left correspond to the symmetric family of influence matrices \( W_1(q) \) while the two on the right are for the asymmetric family of influence matrices \( W_2(q) \) (\( 0 \leq q \leq 1 \)). The first and the third columns of panels on the left show the correlations between the independent layer centrality and the local heterogeneous centrality, while the second and the forth columns of panels on the left show the correlations between the independent layer centrality and the global heterogeneous centrality. The Spearman coefficient is in red, and the Kendall coefficient is in black.

FIG. 4. Ranking comparison between the local heterogeneous centrality and the the global heterogeneous centrality for \( G_1 \) (panels (a) and (d)), for \( G_2 \) (panels (b) and (e)) and for \( G_3 \) (panels (c) and (f)). The computation has been done with the family of symmetric influence matrices of type \( W_1(q) \) (top panels) and with the family of non-symmetric influence matrices of type \( W_2(q) \) (bottom panels). Once again, the Spearman coefficient is in red, and the Kendall coefficient is in black.