Globally F-regular type of moduli spaces

Xiaotao Sun\(^1\) · Mingshuo Zhou\(^1\)

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Abstract
We prove that moduli spaces of semistable parabolic bundles and generalized parabolic sheaves with fixed determinant on a smooth projective curve are of globally F-regular type.

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1 Introduction

Let \( X \) be a variety over a perfect field of characteristic \( p > 0 \) and \( F: X \to X \) be the Frobenius morphism. Then \( X \) is called F-split (Frobenius split) if the natural homomorphism \( \mathcal{O}_X \hookrightarrow F_*\mathcal{O}_X \) is split. It is known that some important varieties are F-split. For example, flag varieties and their Schubert subvarieties (cf. [7,13]), product of two flag varieties for the same group \( G \) (cf. [8]) and cotangent bundles of flag varieties (cf. [4]). An example, which is more closer to this article, should be mentioned. Mehta–Ramadas proved in [6] that for a generic smooth projective curve \( C \) of genus \( g \) over an algebraically closed field of characteristic \( p \geq 5 \), moduli spaces of semistable parabolic bundles of rank 2 on \( C \) are F-split. They also made a conjecture that moduli spaces of semistable parabolic bundles of rank 2 on any smooth curve \( C \) with a fixed determinant are F-split.
The notion of globally $F$-regular varieties was introduced by Smith in [16]. A variety $X$ is called globally $F$-regular if, for any effective divisor $D$, the natural homomorphism $O_X \hookrightarrow F_+^e O_X(D)$ is split for some integer $e > 0$. It is clear that a globally $F$-regular variety must be $F$-split. On the other hand, some well-known $F$-split varieties include toric varieties and Schubert varieties are proved (see \([5,16]\)) to be globally $F$-regular. Thus it is natural to extend Mehta–Ramadas conjecture: the moduli spaces $U_{\mathbb{C}, \omega}$ of semistable parabolic bundles of rank $r$ with a fixed determinant $L$ on any smooth curves $C$ (parabolic structures determined by a given data) are globally $F$-regular varieties. However, this is still a difficult open problem, we will study its characteristic zero analogy in this article.

A variety $X$ over a field of characteristic zero is called of globally $F$-regular type (resp. $F$-split type) if its modulo $p$ reduction $X_p$ is globally $F$-regular (resp. $F$-split) for a dense set of $p$. Projective varieties of globally $F$-regular type have remarkable geometric and cohomological properties (see Theorem 2.5).

Let $U_{\mathbb{C}, \omega}$ be the moduli space of semistable parabolic bundles of rank $r$ and degree $d$ on a smooth curve $C$ of genus $g \geq 0$ with parabolic structures determined by $\omega = (k, \{\vec{n}(x), \vec{a}(x)\}_{x \in I})$ and

$$\text{det} : U_{\mathbb{C}, \omega} \rightarrow J^d_C$$

be the determinant morphism. For any $L \in J^d_C$, the fiber

$$U^L_{\mathbb{C}, \omega} := \text{det}^{-1}(L)$$

is called the moduli space of semistable parabolic bundles with a fixed determinant $L$. Then one of the main results in this article is

**Theorem 1.1** (See Theorem 3.7) The moduli space $U^L_{\mathbb{C}, \omega}$ is of globally $F$-regular type.

When the projective curve $C$ has exactly one node (irreducible, or reducible), the moduli space $U_{\mathbb{C}, \omega}$ is not normal and its normalization is a moduli space $\mathcal{P}_{\omega}$ of semistable generalized parabolic sheaves (GPS) on $\widetilde{C}$ (where $\widetilde{C}$ is the normalization of $C$). There exists a similar determinant morphism $\text{det} : \mathcal{P}_{\omega} \rightarrow J^d_{\widetilde{C}}$. For any $L \in J^d_{\widetilde{C}}$, the fiber

$$\mathcal{P}^L_{\omega} := \text{det}^{-1}(L)$$

is called the moduli space of semistable generalized parabolic sheaves (GPS) with a fixed determinant $L$ on $\widetilde{C}$. Then another main result in this article is

**Theorem 1.2** (See Theorems 4.7, 4.15) The moduli space $\mathcal{P}_{\omega}^L$ is of globally $F$-regular type.
that $U_{C,\omega}^L = (R_{\omega}^{ss})^L/\text{SL}(V)$ is a Fano variety with an open subvariety $X \subset U_{C,\omega}^L$ and a morphism $X \to U_{C,\omega}^L$ satisfying $f_*\mathcal{O}_X = \mathcal{O}_{U_{C,\omega}^L}$. Since Fano varieties are of globally $F$-regular type by Proposition 6.3 of [16], so are $X$ and $U_{C,\omega}$ if the equality $f_*\mathcal{O}_X = \mathcal{O}_{U_{C,\omega}^L}$ commutes with modulo $p$ reductions for a dense set of $p$. To prove that $f_*\mathcal{O}_X = \mathcal{O}_{U_{C,\omega}^L}$ commutes with modulo $p$ reductions for a dense set of $p$, one has to show in particular that a GIT quotient over $Z$ must commute with modulo $p$ reductions for a dense set of $p$, which is Lemma 2.9 (we thought at first that Lemma 2.9 must be well-known to experts, but we are not able to find any reference). We formulate our idea in Proposition 2.10 in a general setting, then the proof of Theorems 1.1 and 1.2 is nothing but to check conditions in Proposition 2.10.

We describe briefly content of this article. In Sect. 2, some notions and properties of globally $F$-regular type varieties are collected, in particular, we formulate and prove Proposition 2.10, which is our technical tool to show that an GIT quotient is of globally $F$-regular type. In Sect. 3, we recall some facts about moduli spaces of parabolic bundles and prove Theorem 1.1. Finally, we prove Theorem 1.2 in Sect. 4.

2 Globally F-regular varieties

We recall firstly some notions and facts about globally $F$-regular varieties over a perfect field of positive characteristic, then the definition of globally $F$-regular type of varieties over a field of characteristic zero is given. Our main references here are [1,14,16].

Let $X$ be a variety over a perfect field $k$ of $\text{char}(k) = p > 0$, let

$$F : X \to X$$

be the Frobenius map and $F^e : X \to X$ be the $e$-th iterate of Frobenius map. When $X$ is normal, for any (weil) divisor $D \in \text{Div}(X)$, let

$$\mathcal{O}_X(D)(V) = \{ f \in K(X) | div_V(f) + D|_V \geq 0 \}, \forall V \subset X.$$

Then $\mathcal{O}_X(D)$ is a reflexive subsheaf of the constant sheaf $K = K(X)$. In fact, $\mathcal{O}_X(D) = j_*\mathcal{O}_{X,s}(D)$, which is an invertible sheaf if and only if $D$ is a Cartier divisor.

**Definition 2.1** A normal variety $X$ over a perfect field is called stably Frobenius $D$-split if $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$ is split for some $e > 0$. $X$ is called globally $F$-regular if $X$ is stably Frobenius $D$-split for any effective divisor $D$.

The advantage of this definition is that any open set $U \subset X$ of a globally $F$-regular variety $X$ is globally $F$-regular. Its disadvantage is the requirement of normality of $X$. When $X$ is not normal, one possible remedy of Definition 2.1 is to require that $D$ is a Cartier divisor. Then it lost the advantage that any open set $U \subset X$ is globally $F$-regular since a Cartier divisor on $U$ may not be extended to a Cartier divisor on $X$. But, when $X$ is a projective variety, we have
Proposition 2.2 (Theorem 3.10 of [16]) Let $X$ be a projective variety over a perfect field. Then the following statements are equivalent.

1. $X$ is normal and stably Frobenius $D$-split for any effective $D$;
2. $X$ is stably Frobenius $D$-split for any effective Cartier $D$;
3. For any ample line bundle $\mathcal{L}$, the section ring of $X$

$$R(X, \mathcal{L}) = \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{L}^n)$$

is strongly $F$-regular.

Proof (1) $\Rightarrow$ (2) is clear. (2) $\Rightarrow$ (3) was proved in Theorem 3.10 of [16]. That (3) $\Rightarrow$ (1) is a modification of the proof in [16]. In fact, $X$ is normal and Cohen-Macaulay by Theorem 4.1 of [16]. Then $R(X, \mathcal{L}) = R(X^{sm.}, \mathcal{L})$ where $X^{sm.} \subset X$ is the open set of smooth points. Thus $X^{sm.}$ is stably Frobenius $D \cap X^{sm.}$-split for any effective $D \in Div(X)$ (See the proof of (1) $\Rightarrow$ (3) in Theorem 3.10 of [16]), which implies that $X$ is stably Frobenius $D$-split. \hfill $\Box$

A variety $X$ is called $F$-split if $\mathcal{O}_X \to F_*\mathcal{O}_X$ is split. In particular, globally $F$-regular varieties are $F$-split. Let $X \to Y$ be a morphism such that $f_*\mathcal{O}_X = \mathcal{O}_Y$, then any splitting map $F_*\mathcal{O}_X \xrightarrow{\psi} \mathcal{O}_X$ of $\mathcal{O}_X \to F_*\mathcal{O}_X$ induces a splitting map $F_*\mathcal{O}_Y = F_*f_*\mathcal{O}_X = f_*F_*\mathcal{O}_X \xrightarrow{f_*\psi} f_*\mathcal{O}_X = \mathcal{O}_Y$. There is a generalization of above useful observation.

Lemma 2.3 (Corollary 6.4 of [14]) Let $f : X \to Y$ be a morphism of varieties over a perfect field of positive characteristic. If the natural map $\mathcal{O}_Y \xrightarrow{i} f_*\mathcal{O}_X$ splits and $X$ is globally $F$-regular, then $Y$ is stably Frobenius $D$-split for any effective Cartier divisor $D$. $Y$ is globally $F$-regular when it is normal.

Proof For Cartier divisor $D \in Div(Y)$ defined by $\mathcal{O}_Y \xrightarrow{s} \mathcal{O}_Y(D)$, let $F^e_*\mathcal{O}_X(f_*^eD) \xrightarrow{h} \mathcal{O}_X$ be a splitting of $\mathcal{O}_X \to F^e_*\mathcal{O}_X \xrightarrow{F^e_*f_*^e(s)} F^e_*\mathcal{O}_Y(f_*^eD)$, and $f_*\mathcal{O}_X \xrightarrow{j} \mathcal{O}_Y$ be a splitting of $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Then

$$F^e_*\mathcal{O}_Y(D) \xrightarrow{F^e_*1 \otimes i} F^e_*f_*\mathcal{O}_X(f_*^eD) = f_*F^e_*\mathcal{O}_X(f_*^eD) \xrightarrow{f_*h} f_*\mathcal{O}_X \xrightarrow{j} \mathcal{O}_Y$$

(induced by $\mathcal{O}_Y(D) \xrightarrow{1 \otimes i} \mathcal{O}_Y(D) \otimes f_*\mathcal{O}_X = f_*\mathcal{O}_X(f_*^eD)$) is a splitting of $\mathcal{O}_Y \to F^e_*\mathcal{O}_Y \xrightarrow{i} f_*\mathcal{O}_X$. $F^e_*\mathcal{O}_Y(D)$. When $Y$ is normal, it is globally $F$-regular if and only if $Y^{sm.}$ is stably Frobenius $D$-split for any effective Cartier divisor $D \in Div(Y^{sm.})$, which can be proved by applying above arguments to $f^{-1}(Y^{sm.}) \to Y^{sm.}$. \hfill $\Box$

For any scheme $X$ of finite type over a field $K$ of characteristic zero, there is a finitely generated $\mathbb{Z}$-algebra $A \subset K$ and an $A$-flat scheme
\[ X_A \to S = \text{Spec}(A) \]
such that \( X_K = X_A \times_S \text{Spec}(K) \cong X \). \( X_A \to S = \text{Spec}(A) \) is called an integral model of \( X/K \), and a closed fiber \( X_s = X_A \times_S \text{Spec}(k(s)) \) is called modulo \( p \) reduction of \( X \) where \( p = \text{char}(k(s)) > 0 \). For \( X \) with a line bundle \( L \) (resp. a morphism \( f : X \to Y \) of finite type), by an integral model \( (X_A, L_A) \) of \( (X, L) \) (resp. an integral model \( f_A : X_A \to Y_A \) of \( f : X \to Y \)), we mean that \( L_A \) is a line bundle on \( X_A \) such that \( L_A|_{X_K} \cong L \) (resp. \( f_A \) is a morphism over \( S = \text{Spec}(A) \) which induces \( f : X \to Y \) via the base change \( \text{Spec}(K) \to S \)). The existence of such models is well-known.

**Definition 2.4** A variety \( X \) over a field of characteristic zero is said to be of **globally F-regular type** (resp. **F-split type**) if its modulo \( p \) reduction of \( X \) are globally F-regular (resp. F-split) for a dense subset of closed points \( s \in S \).

It is not difficult to see that the definition is independent of choices of \( X_A \) (see Remark 2.5 of [10]). For any dense subset \( T \subset S = \text{Spec}(A) \), the subset \( \{ p_s = \text{char}(k(s)) \mid s \in T \} \) of primes is a dense subset of \( \text{Spec}(\mathbb{Z}) \) since the morphism \( S \to \text{Spec}(\mathbb{Z}) \) (\( m_s \mapsto (p_s) := m_s \cap \mathbb{Z} \) where \( m_s \subset A \) is the maximal ideal of \( s \)) induced by \( \mathbb{Z} \hookrightarrow A \) is dominant.

**Theorem 2.5** (Corollary 5.3 and Corollary 5.5 of [16]) Let \( X \) be a projective variety over a field of characteristic zero. If \( X \) is of globally F-regular type, then we have

1. \( X \) is normal, Cohen-Macaulay with rational singularities. If \( X \) is \( \mathbb{Q} \)-Gorenstein, then it has log terminal singularities.
2. For any nef line bundle \( L \) on \( X \), we have \( H^i(X, L) = 0 \) when \( i > 0 \). In particular, \( H^i(X, \mathcal{O}_X) = 0 \) whenever \( i > 0 \).

A normal projective variety \( X \) is called a **Fano variety** if

\[ \omega_X^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{O}_X) \]
is an ample line bundle. A important example of **globally F-regular type** varieties is

**Proposition 2.6** [16, Proposition 6.3] A Fano variety (over a field of characteristic zero) with at most rational singularities is of globally F-regular type.

To show that a variety \( Y \) is of **globally F-regular type**, one possible approach is to construct an open set \( X \) of a Fano variety (thus \( X \) is of **globally F-regular type**) with a morphism \( f : X \to Y \) such that \( f_*\mathcal{O}_X = \mathcal{O}_Y \). Then \( Y \) is of **globally F-regular type** if the following characteristic zero analogy of Lemma 2.3 is true.
Question 2.7 Let $X \xrightarrow{f} Y$ be a morphism of varieties over a field $K$ of $\text{char}(K) = 0$ such that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is split and $X$ is of globally F-regular type. Is $Y$ of globally F-regular type?

Let $f_*\mathcal{O}_X \xrightarrow{\beta} \mathcal{O}_Y$ be a splitting of $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Then Question 2.7 consists:

1. Can we choose a model $f_A : X_A \to Y_A$ of $f : X \to Y$ such that the $\mathcal{O}_Y$-homomorphism $(f_A)_*\mathcal{O}_{X_A} \xrightarrow{\beta_A} \mathcal{O}_{Y_A}$ can be extended to $f_A_*\mathcal{O}_{X_A} \xrightarrow{\beta_A} \mathcal{O}_{Y_A}$?

2. Is there a dense set of closed points $\text{Spec}(k(s)) \to S = \text{Spec}(A)$ such that $i_* f_{A*}\mathcal{O}_{X_A} = f_{s*} j_{s*}\mathcal{O}_{X_A}$ where $Y_s = Y_A \times_A k(s) \xrightarrow{i_s} Y_A$, $X_s = X_A \times_A k(s) \xrightarrow{j_s} X_A$ and

$$
egin{array}{ccc}
X_s & \xrightarrow{j_s} & X_A \\
\downarrow{f_s} & & \downarrow{f_A} \\
Y_s & \xrightarrow{i_s} & Y_A.
\end{array}
$$

Definition 2.8 A morphism $X \xrightarrow{f} Y$ of varieties over a field $K$ of $\text{char}(K) = 0$ is called $p$-compatible if there is an integral model

$$
X_A \xrightarrow{f_A} Y_A
$$

such that $i_* f_{A*}\mathcal{O}_{X_A} = f_{s*} j_{s*}\mathcal{O}_{X_A}$ for $s \in \text{Spec}(A)$.

It is clear that (1) has an affirmative answer when either $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module or the splitting map $\beta : f_*\mathcal{O}_X \to \mathcal{O}_Y$ is a homomorphism of $\mathcal{O}_Y$-algebras. (2) has an affirmative answer for flat morphisms $f : X \to Y$ with coherent $R^i f_*\mathcal{O}_X$ ($i \geq 0$). It is also clear that any affine morphism must be $p$-compatible. When $X, Y$ are open set of GIT quotients and $f : X \to Y$ is induced by a $G$-invariant $p$-compatible morphism $\tilde{f} : \mathcal{R}' \to \mathcal{R}$ of parameter spaces, we will show that $f : X \to Y$ is $p$-compatible in Proposition 2.10, which will need the following lemma.

Lemma 2.9 Let $X \to S = \text{Spec}(A)$ be a flat projective morphism where $A$ is an integral $\mathbb{Z}$-algebra of finite type, let $G \to S$ be a $S$-flat reductive group scheme with an action on $X$ over $S$. If $L$ is a relative ample line bundle on $X$ linearizing the action of $G$, let

$$
X^{ss}(L) \xrightarrow{\pi} X^{ss}(L)/G \cong Y
$$

be the GIT quotient over $S$. Assume that the geometrically generic fiber of $X^{ss}(L) \to S$ is an irreducible normal variety. Then there is a dense open subset $U \subset S$ such that for any $s \in U$

$$
Y \times_S k(s) \cong X^{ss}(L_s)/G_s
$$

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where $X_s = X \times_S \overline{k(s)}$ (resp. $G_s = G \times_S \overline{k(s)}$) is the geometrically closed fiber of $X \to S$ (resp. $G \to S$) at Spec$(\overline{k(s)}) \to S$.

**Proof** Let $X^{ss}_s(L) \times_S \overline{k(s)} \xrightarrow{\pi_s} Y_s$ be the pullback of $X^{ss}(L) \to Y$ under the base change Spec$(\overline{k(s)}) \to S$. By Proposition 7 of [15], we have

$$X^{ss}(L) \times_S \overline{k(s)} = X^{ss}_s(L_s).$$

Then there is a unique $k(s)$-morphism $X^{ss}_s(L_s) \xrightarrow{\theta} Y_s$ such that $X^{ss}_s(L_s) \xrightarrow{\pi_s} Y_s$ is commutative. Let $\overline{Y}_s := X^{ss}_s(L_s) \xrightarrow{\theta} Y_s$, $k = \overline{k(s)}$, it is known that $\theta$ induces a bijective map $\overline{Y}_s(k) \xrightarrow{\theta} Y_s(k)$ on the sets of $k$-points (cf. Proposition 9 (i) of [15]). By the assumption, geometrically generic fiber of $Y \to S$ is an irreducible normal projective variety. Thus there is a dense open subset $U \subset S$ such that for any closed point Spec$(k(s)) \to U$ we have (1) $Y_s$ is normal and (2) the morphism $X^{ss}_s(L_s) \xrightarrow{\pi_s} Y_s$ is generic smooth. In fact, (1) is true by (iv) of Théoréme (12.2.4) in [2] and (2) is true since $K = Q(A) = k(S)$ is a field of characteristic zero. Then generic smoothness of $\pi_s$ implies the generic smoothness of $\overline{Y}_s \xrightarrow{\theta} Y_s$. Therefore $\overline{Y}_s \xrightarrow{\theta} Y_s$ must be an isomorphism by Zariski main theorem since $Y_s$ is normal. \qed

Let $(\hat{Y}, L), (\hat{Z}, L')$ be polarized projective varieties over an algebraically closed field $K$ of characteristic zero with actions of a reductive group scheme $G$ over $K$, and $\hat{Y}^{ss}(L) \subset \hat{Y}$ (resp. $\hat{Y}^{ss}(L) \subset \hat{Y}^{ss}(L)$) be the open set of GIT semi-stable (resp. GIT stable) points of $\hat{Y}$. Then there are projective GIT quotients

$$\hat{Y}^{ss}(L) \xrightarrow{\psi} Y := \hat{Y}^{ss}(L) \quot G, \quad \hat{Z}^{ss}(L') \xrightarrow{\varphi} Z := \hat{Z}^{ss}(L') \quot G. \quad (2.1)$$

**Proposition 2.10** Let $Z, Y$ be the GIT quotients in (2.1). Assume

1. there are $G$-invariant normal open subschemes $\mathcal{R} \subset \hat{Y}, \mathcal{R}' \subset \hat{Z}$ such that $\hat{Y}^{ss}(L) \subset \mathcal{R}, \hat{Z}^{ss}(L') \subset \mathcal{R}'$;

2. there is a $G$-invariant $p$-compatible morphism $\mathcal{R}' \xrightarrow{\hat{f}} \mathcal{R}$ such that $\hat{f}_*\mathcal{O}_{\mathcal{R}'} = \mathcal{O}_{\mathcal{R}}$;

3. there is an $G$-invariant open set $W \subset \hat{Z}^{ss}(L')$ such that

$\text{Codim}(\mathcal{R}' \setminus W) \geq 2, \quad \hat{X} = \varphi^{-1}\varphi(\hat{X})$

where $\hat{X} = W \cap \hat{f}^{-1}(\hat{Y}^{ss}(L))$. 

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Then $Y$ is of globally F-regular type if $Z$ is so.

**Proof** Let $X = \varphi(\hat{X}) \subset Z$, which is an open set of $Z$ since

$$\varphi(Z_{ss}(L') \setminus \hat{X}) = Z \setminus X$$

(by the condition $\varphi^{-1}(X) = \hat{X}$) and $Z_{ss}(L') \setminus \hat{X}$ is a $G$-invariant closed subset. There is a morphism $X \xrightarrow{f} Y$ such that

$$\begin{array}{ccc}
\hat{X} & \xrightarrow{\varphi} & X \\
\downarrow f & & \downarrow f \\
\hat{Y}_{ss}(L) & \xrightarrow{\psi} & Y
\end{array}$$

is commutative. For any open set $U \subset Y$, since $f_* \mathcal{O}_{Y'} = \mathcal{O}_Y$, we have

$$\mathcal{O}_Y(U) = \mathcal{O}_Y(\psi^{-1}(U))^{inv} = \mathcal{O}_Y(\hat{X}^{-1} \psi^{-1}(U))^{inv} = \mathcal{O}_X(\hat{X}^{-1} \psi^{-1}(U))^{inv} = \mathcal{O}_X(f^{-1}(U)) = f_* \mathcal{O}_X(U)$$

where the third equality holds because $\hat{X}^{-1} \psi^{-1}(U) \setminus W \cap \hat{X}^{-1} \psi^{-1}(U) = \hat{X}^{-1} \psi^{-1}(U) \cap (\mathcal{R}' \setminus W)$ has codimension at least two. Thus we have

$$\mathcal{O}_Y = f_* \mathcal{O}_X, \text{ where } X \text{ is of globally F-regular type.} \quad (2.2)$$

To show that $Y$ is of globally F-regular type, it is enough to show that the morphism $X \xrightarrow{f} Y$ is $p$-compatible.

Let $(\hat{Y}_A, L), (\hat{Z}_A, L')$ be integral models of $(\hat{Y}, L), (\hat{Z}, L')$ with actions of a reductive group scheme $G_A$ over $S = \text{Spec}(A)$, and $\hat{Y}_{ss}^A(L) \subset \hat{Y}_A$ (resp. $\hat{Z}_{ss}^A(L') \subset \hat{Z}_A$) be the open subscheme of GIT semi-stable (resp. GIT stable) points of $\hat{Y}_A$. Then there are GIT quotients

$$\hat{Y}_{ss}^A(L) \xrightarrow{\psi_A} Y_A := \hat{Y}_{ss}^A(L)//G_A, \quad \hat{Z}_{ss}^A(L') \xrightarrow{\psi_A} Z_A := \hat{Z}_{ss}^A(L')//G_A,$$

which are projective over $S = \text{Spec}(A)$. Moreover, $\psi_A$ and $\varphi_A$ are surjective $G_A$-invariant affine morphisms (cf. Theorem 4 of [15]).

We can choose $G_A$-invariant open subschemes $\mathcal{R}_A \subset \hat{Y}_A, \mathcal{R}'_A \subset \hat{Z}_A, W_A \subset Z_{ss}^A(L'), X_A \subset Z_A$ and a $G_A$-invariant morphism $\mathcal{R}'_A \xrightarrow{\hat{f}_A} \mathcal{R}_A$ such that $\hat{Y}_{ss}^A(L) \subset \mathcal{R}_A, Z_{ss}^A(L') \subset \mathcal{R}'_A, \hat{f}_A \mathcal{O}_{\mathcal{R}'_A} = \mathcal{O}_{\mathcal{R}_A}$. Let

$$\hat{X}_A = \varphi_A^{-1}(X_A), \quad \mathcal{R}'_s = \mathcal{R}'_A \times_A \overline{k(s)}, \quad \mathcal{R}_s = \mathcal{R}_A \times_A \overline{k(s)}$$
and \( \hat{f}_s = \hat{f}_A \otimes \overline{k(s)} \) (\( \forall s \in S \)). Then we have \( \hat{f}_s^* \mathcal{O}_{\mathcal{R}'_s} = \mathcal{O}_{\mathcal{R}_s} \),

\[
\text{Codim}(\mathcal{R}'_s \setminus W_s) \geq 2, \quad \hat{X}_s = W_s \cap \hat{f}_s^{-1}(\hat{Y}^{ss}_A(\mathcal{L}) \times_A \overline{k(s)})
\]  

(2.3)

(by shrinking \( S \)) where \( W_s = W_A \times_A \overline{k(s)} \)
\( \hat{X}_s = \hat{X}_A \times_A \overline{k(s)} \) and
\[
\hat{Y}^{ss}_A(\mathcal{L}) \times_A \overline{k(s)} = \hat{Y}^{ss}_A(\mathcal{L}_s), \quad \hat{Z}^{ss}_A(\mathcal{L}') \times_A \overline{k(s)} = \hat{Z}^{ss}_A(\mathcal{L}'_s)
\]

(cf. Proposition 7 of [15]). Then, by Lemma 2.9, we have
\[
Z_s = Z^{ss}_s(\mathcal{L}'_s) / / G_s, \quad Y_s = Y^{ss}_s(\mathcal{L}_s) / / G_s.
\]

Thus, for any open sets \( U \subset Z_s, V \subset Y_s \), one has
\[
\mathcal{O}_Z(U) = \mathcal{O}_{\mathcal{R}'_s}(\varphi_s^{-1}(U))^{inv}, \quad \mathcal{O}_Y(V) = \mathcal{O}_{\mathcal{R}_s}(\psi_s^{-1}(V))^{inv}.
\]

which imply \( \mathcal{O}_Y(V) = \mathcal{O}_{\mathcal{R}_s}(\psi_s^{-1}(V))^{inv} = \mathcal{O}_{\mathcal{R}'_s}(\hat{f}_s^{-1}(\psi_s^{-1}(V)))^{inv} \) where the second equality holds since \( \hat{f}_s^* \mathcal{O}_{\mathcal{R}'_s} = \mathcal{O}_{\mathcal{R}_s} \). Because
\[
\hat{f}_s^{-1}(\psi_s^{-1}(V)) \setminus W_s \cap \hat{f}_s^{-1}(\psi_s^{-1}(V)) = \hat{f}_s^{-1}(\psi_s^{-1}(V)) \cap (\mathcal{R}'_s \setminus W_s)
\]

has codimension at least two and \( \hat{X}_s = W_s \cap \hat{f}_s^{-1}(\hat{Y}^{ss}_s(\mathcal{L}_s)) \), we have
\[
\mathcal{O}_Y(V) = \mathcal{O}_{\hat{X}_s}(\hat{f}_s^{-1}(\psi_s^{-1}(V)))^{inv} = \mathcal{O}_{\hat{X}_s}(\varphi_s^{-1}(f^{-1}_{s})(V))^{inv} = \mathcal{O}_{\hat{X}_s}(f^{-1}_{s}(V)) = (f_s)_* \mathcal{O}_{X_s}(V)
\]

where \( X_s \subset Z_s \) and \( \varphi_s^{-1}(X_s) = \hat{X}_s \subset \mathcal{R}'_s \) are open sets, \( \hat{f}_s : \hat{X}_s \to \hat{Y}^{ss}_s(\mathcal{L}_s) \) and \( \varphi_s : \hat{X}_s \to X_s \) are restrictions of \( f : \mathcal{R}_s' \to \mathcal{R}_s \) and \( \varphi_s : Z^{ss}_s(\mathcal{L}'_s) \to Z_s \). Thus \( \mathcal{O}_Y = (f_s)_* \mathcal{O}_{X_s} \), which implies that \( f : X \to Y \) is \( p \)-compatible and \( Y \) is of globally F-regular type since \( X \) is so. \( \square \)

3 Globally F-regular type of Moduli spaces of parabolic bundles

In this section, we prove that moduli spaces of parabolic bundles with a fixed determinant on a smooth curve are of globally F-regular type. Let \( C \) be an irreducible projective curve of genus \( g \geq 0 \) over an algebraically closed field of characteristic zero, which has at most one node \( x_0 \in C \). Let \( I \) be a finite set of smooth points of \( C \), and \( E \) be a torsion-free sheaf of rank \( r \) and degree \( d \) on \( C \) (the rank \( r(E) \) is defined to be dimension of \( E_x \) at generic point \( x \in C \), and \( d = \chi(E) - r(1 - g) \)).

**Definition 3.1** By a quasi-parabolic structure of \( E \) at a smooth point \( x \in C \), we mean a choice of flag of quotients
\[
E_x = Q_{l_{x+1}}(E)_x \to Q_{l_x}(E)_x \to \cdots \to Q_1(E)_x \to Q_0(E)_x = 0
\]
of fibre $E_x$, $n_i(x) = \dim(\ker{Q_i(E)_x \to Q_{i-1}(E)_x}) \ (1 \leq i \leq l_x + 1)$ are called type of the flags. If, in addition, a sequence of integers

$$0 \leq a_1(x) < a_2(x) < \cdots < a_{l_x+1}(x) < k$$

are given, we call that $E$ has a parabolic structure of type

$$\vec{a}(x) = (a_1(x), a_2(x), \ldots, a_{l_x+1}(x))$$

and weight $\vec{a}(x) = (a_1(x), a_2(x), \ldots, a_{l_x+1}(x))$ at $x \in C$.

**Definition 3.2** For any subsheaf $F \subset E$, let $Q_i(E)_x^F \subset Q_i(E)_x$ be the image of $F$ and $n_i^F(x) = \dim(\ker{Q_i(E)_x^F \to Q_{i-1}(E)_x^F})$. Let

$$\text{par}_X(E) := \chi(E) + \frac{1}{k} \sum_{x \in I} \sum_{i=1}^{l_x+1} a_i(x)n_i(x).$$

Then $E$ is called semistable (resp., stable) for $\omega = (k, \{\vec{a}(x), \vec{a}(x)\}_{x \in I})$ if for any nontrivial $E' \subset E$ such that $E/E'$ is torsion free, one has

$$\text{par}(E') \leq \frac{\text{par}(E)}{r} \cdot r(E') \ (\text{resp., } <).$$

**Theorem 3.3** (Theorem X1 of [11] or Theorem 2.13 of [19] for arbitrary rank) There exists a seminormal projective variety

$$U_{C, \omega} := U_C(r, d, \{k, \vec{a}(x), \vec{a}(x)\}_{x \in I}),$$

which is the coarse moduli space of $s$-equivalence classes of semistable parabolic sheaves $E$ of rank $r$ and $\chi(E) = \chi = d + r(1 - g)$ with parabolic structures of type $\{\vec{a}(x)\}_{x \in I}$ and weights $\{\vec{a}(x)\}_{x \in I}$ at points $\{x\}_{x \in I}$. If $C$ is smooth, $U_{C, \omega}$ is a normal variety with only rational singularities.

Recall the construction of $U_{C, \omega} = U_C(r, d, \omega)$. Fix a line bundle $\mathcal{O}(1) = \mathcal{O}_C(c \cdot y)$ on $C$ of deg($\mathcal{O}(1)$) = $c$, let $\chi = d + r(1 - g)$, $P(m) = c + \chi$, $\mathcal{O}_C(-N) = \mathcal{O}(1)^{-N}$ and $V = \mathbb{C}^{P(N)}$. Let $Q$ be the Quot scheme of quotients $V \otimes \mathcal{O}_C(-N) \to F \to 0$ on $C$ with rank $r(F) = r$ and deg($F$) = $d$. There exists a universal quotient

$$V \otimes \mathcal{O}_C(-N) \to F \to 0$$

on $C \times Q$. Let $\mathcal{F}_x = F|_{x \times Q}$ and $\text{Flag}_{\vec{a}(x)}(\mathcal{F}_x) \to Q$ be the relative flag scheme of type $\vec{a}(x)$. Then the reductive group $\text{SL}(V)$ acts on

$$\mathcal{R} = \times_{x \in I} \text{Flag}_{\vec{a}(x)}(\mathcal{F}_x) \to Q$$
and the weight \((k, \{\tilde{a}(x)\}_{x \in I})\) determines a polarisation

\[
\Theta_{R, \omega} = (\det R \pi R \mathcal{E})^{-k} \otimes \bigotimes_{x \in I} (\bigotimes_{i=1}^{l_x} \det (Q_{\{x\} \times R, i})^{d_i(x)}) \otimes \bigotimes_{q} \det (E_q)^{\ell}
\]

which linearizes the \(SL(V)\)-action on \(R\), where \(\mathcal{E}\) is the pullback of \(\mathcal{F}\) under \(C \times R \to C \times Q\), \(\det R \pi R \mathcal{E}\) is the determinant line bundle of cohomology, \(d_i(x) = a_i + 1(x) - a_i(x)\),

\[
\mathcal{E}_x = Q_{\{x\} \times R, l_x+1} \to Q_{\{x\} \times R, l_x} \to Q_{\{x\} \times R, l_x-1} \to \cdots \to Q_{\{x\} \times R, 1} \to 0
\]

are universal quotients on \(R\) of type \(\tilde{n}(x), r_i(x) = r(Q_{\{x\} \times R, i})\) and

\[
\ell := \frac{k \chi - \sum_{x \in I} \sum_{i=1}^{l_x} d_i(x) r_i(x)}{r}.
\]

It is proved that open subset \(\mathcal{R}_{ss, \omega}^s (\text{resp. } \mathcal{R}_{\omega}^s)\) of GIT semistable (resp. GIT stable) points are precisely the set of semistable (resp. stable) parabolic sheaves on \(C\) (see [19]). Then \(\mathcal{U}_{C, \omega}\) is the GIT quotient

\[
\mathcal{R}_{ss, \omega}^s \xrightarrow{\psi} \mathcal{U}_{C, \omega} := \mathcal{U}_C(r, d, \omega)
\]

and the polarisation \(\Theta_{\mathcal{R}_{ss, \omega}^s}\) descends to an ample line bundle \(\Theta_{\mathcal{U}_{C, \omega}}\) on \(\mathcal{U}_{C, \omega}\) (so called theta line bundle) when \(\ell\) is an integer.

**Definition 3.4** When \(C\) is a smooth projective curve, let

\[
\det : \mathcal{U}_{C, \omega} \to J_C^d, \quad E \mapsto \det(E) := \bigwedge^r E
\]

be the determinant map. Then, for any \(L \in J_C^d\), the fiber

\[
\det^{-1}(L) := \mathcal{U}_{C, \omega}^L
\]

is called moduli space of semistable parabolic bundles with a fixed determinant.

Let \(\mathcal{R}_{ss, \omega}^L \subset \mathcal{R}_{\omega}^s\) be the sub-scheme of locally free quotients with a fixed determinant \(L\), and \((\mathcal{R}_{ss, \omega}^L)^L \subset \mathcal{R}_{ss, \omega}^L\), \((\mathcal{R}_{\omega}^s)^L \subset \mathcal{R}_{\omega}^s\) be the closed subsets of locally free quotients with the fixed determinant \(L\). Then \(\mathcal{U}_{C, \omega}^L\) is the GIT quotient \((\mathcal{R}_{ss, \omega}^L)^L \psi \to (\mathcal{R}_{\omega}^s)^L://SL(V) := \mathcal{U}_{C, \omega}^L\). The proof of globally F-regular type of \(\mathcal{U}_{C, \omega}^L\) needs essentially the following two results.

**Proposition 3.5** Let \(|I|\) be the number of parabolic points. Then, for any data \(\omega = (k, \{\tilde{n}(x), \tilde{a}(x)\}_{x \in I})\), we have

1. \(\text{Codim}(\mathcal{R}_{ss, \omega}^L) \backslash (\mathcal{R}_{\omega}^s)^L) \geq (r - 1)(g - 1) + \frac{1}{k} |I|\),
(2) \( \text{Codim}(\mathcal{R}^L_f \setminus (\mathcal{R}^{ss}_{\omega^i})^L) > (r - 1)(g - 1) + \frac{1}{k}|I| \).

**Proof** This is in fact Proposition 5.1 of [17] where we did not fix determinant and the term \( \frac{1}{k}|I| \) was omitted. However, the proof there also works for the case of fixed determinant. \( \square \)

**Proposition 3.6** Let \( \omega_c = (2r, \{\tilde{n}(x), \tilde{a}_c(x)\}_{x \in I}) \) where

\[
\tilde{a}_c(x) = (\tilde{a}_1(x), \tilde{a}_2(x), \ldots, \tilde{a}_{l_{x}+1}(x))
\]

satisfy \( \tilde{a}_{i+1}(x) - \tilde{a}_i(x) = n_i(x) + n_{i+1}(x) (1 \leq i \leq l_{x}) \). Then, when

\[
(r - 1)(g - 1) + \frac{|I|}{2r} \geq 2,
\]

the moduli space \( \mathcal{U}^L_{C, \omega_c} = (\mathcal{R}^{ss}_{\omega_c})^L / \text{SL}(V) \) is a normal Fano variety with only rational singularities.

**Proof** It is in fact a reformulation of Proposition 2.2 of [17] where a formula of anti-canonical bundle \( \omega_{\mathcal{R}^F}^{-1} \) (thus a formula of \( \omega_{\mathcal{R}^F}^{-1} L \)) was given (see also Proposition 4.2 of [19] for a tidy formula). The anti-canonical bundle \( \omega_{\mathcal{R}^F}^{-1} \) is nothing but the polarisation \( \Theta_{\mathcal{R}^F, \omega_c} \), which descends to an ample line bundle \( \Theta_{\mathcal{U}^L_{C, \omega_c}} \). Then, by a result of F. Knop (see [3]), we have \( \omega_{\mathcal{U}^L_{C, \omega_c}}^{-1} = \Theta_{\mathcal{U}^L_{C, \omega_c}} \) when \( \text{Codim}((\mathcal{R}^{ss}_{\omega_c})^L \setminus (\mathcal{R}^{ss}_{\omega_c})^L) > 2 \). Thus we are done by the condition (3.1) and (1) of Proposition 3.5. \( \square \)

**Theorem 3.7** The moduli space \( \mathcal{U}^L_{C, \omega} \) is of globally F-regular type. If Jacobian \( J^0_C \) of \( C \) is of F-split type, so is \( \mathcal{U}^L_{C, \omega} \).

**Proof** Choose a subset \( I' \subset C \) such that \( I' \cap I = \emptyset \) and

\[
(r - 1)(g - 1) + \frac{|I| + |I'|}{2r} \geq 2.
\]

Let \( \hat{Y} \subset \mathcal{R} \) be the Zariski closure of \( \mathcal{R}^L_f \) and \( \hat{Z} = \hat{f}^{-1}(\hat{Y}) \subset \mathcal{R}' \) where

\[
\mathcal{R}' = \times_{x \in I' \cup I'} Q \text{Flag}_{\tilde{n}(x)}(\mathcal{F}_x) = \mathcal{R} \times_{x \in I'} Q \times \left( \times_{x \in I'} Q \text{Flag}_{\tilde{n}(x)}(\mathcal{F}_x) \right) \overset{\hat{f}}{\to} \mathcal{R}
\]

is the projection. Then \( \mathcal{R}'^L_F := \hat{f}^{-1}(\mathcal{R}^L_f) \subset \hat{Z} \) and \( \mathcal{R}^L_F \subset \hat{Y} \) are smooth \( \text{SL}(V) \)-invariant open sub-schemes such that

\[
\hat{Y}^{ss}_{\omega} = (\mathcal{R}^{ss}_{\omega_c})^L \subset \mathcal{R}^L_F, \quad \hat{Z}^{ss}_{\omega'} = (\mathcal{R}^{ss}_{\omega^i})^L \subset \mathcal{R}'^L_F
\]

holds for any polarizations determined by data \( \omega, \omega' \). It is clear that \( \mathcal{R}'^L_F \overset{\hat{f}}{\to} \mathcal{R}^L_F \) is a \( \text{SL}(V) \)-invariant \( p \)-compatible morphism with

\[
\hat{f}_* \mathcal{O}_{\mathcal{R}'^L_F} = \mathcal{O}_{\mathcal{R}^L_F}.
\]
Thus \( \mathcal{R}_F^L \subset \hat{Z}, \mathcal{R}_F^L \subset \hat{Y} \) and \( \mathcal{R}_F^L \overset{\hat{f}}{\to} \mathcal{R}_F^L \) satisfy the conditions (1) and (2) of Proposition 2.10. To verify condition (3) in Proposition 2.10, let

\[
W := \hat{Z}_{\omega}^s = (\mathcal{R}_\omega^s)^L \subset \mathcal{R}_F^L, \quad \hat{X} = \hat{f}^{-1}((\mathcal{R}_\omega^s)^L) \cap W,
\]

\[
\hat{Z}_{\omega}^{ss} \overset{\varphi}{\to} Z := \hat{Z}_{\omega}^{ss} // \text{SL}(V) \text{ and } X = \varphi(\hat{X}) \subset Z.
\]

It is clear that \( \hat{X} = \varphi^{-1}(X) \).

Choose \( \omega' = (2r, [\bar{n}(x), \bar{a}_c(x)]_{x \in I \cup I'}) \) as in Proposition 3.6, then \( Z \) is a normal Fano variety with only rational singularities, which is of globally F-regular type by Proposition 2.6. On the other hand, we have

\[
\text{Codim}(\mathcal{R}_F^L \setminus W) \geq (r - 1)(g - 1) + \frac{|I| + |I'|}{2r} \geq 2
\]

by Proposition 3.5. Thus \( \mathcal{U}_C, \omega^L = (\mathcal{R}_\omega^s)^L // \text{SL}(V) = \hat{Y}_{\omega}^{ss} // \text{SL}(V) \) is of globally F-regular type by Proposition 2.10.

If \( J_C^0 \) is of F-split type, so is \( J_C^0 \times \mathcal{U}_C, \omega^L \). The \( r^{2g} \)-fold covering

\[
J_C^0 \times \mathcal{U}_C, \omega^L \overset{f}{\to} \mathcal{U}_C, \omega, \quad f(L_0, E) = L_0 \otimes E
\]

implies that \( \mathcal{U}_C, \omega \) is of F-split type. \( \square \)

### 4 Globally F-regular type of moduli spaces of generalized parabolic sheaves

In this section, we prove that moduli spaces of generalized parabolic sheaves with a fixed determinant on a smooth curve are of globally F-regular type. We will continue to use notation of last section.

Let \( \{x_1, x_2\} \subset C \setminus I \) be two different points, a generalized parabolic sheaf (GPS) \((E, Q)\) of rank \( r \) and degree \( d \) on \( C \) consists of a sheaf \( E \) of degree \( d \) on \( C \), which is torsion free of rank \( r \) outside \( \{x_1, x_2\} \) with parabolic structures at the points of \( I \), and an \( r \)-dimensional quotient

\[
E_{x_1} \oplus E_{x_2} \overset{q}{\to} Q \to 0.
\]

**Definition 4.1** A GPS \((E, Q)\) on an irreducible smooth curve \( C \) is called semistable (resp. stable), if for every nontrivial subsheaf \( E' \subset E \) such that \( E/E' \) is torsion free outside \( \{x_1, x_2\} \), we have

\[
\text{par} \chi(E') - \text{dim}(Q^{E'}) \leq r(E') \cdot \frac{\text{par} \chi(E) - \text{dim}(Q)}{r(E)} \quad \text{(resp., <),}
\]

\( \square \)
where $Q^{E'} = q(E'_{x_1} \oplus E'_{x_2}) \subset Q$.

**Theorem 4.2** (Theorem X2 of [11] or Theorem 2.24 of [19] for arbitrary rank) For any data $\omega = (k, \{\tilde{n}(x), \tilde{a}(x)\}_{x \in I})$, there exists a normal projective variety $P_\omega$ with at most rational singularities, which is the coarse moduli space of $s$-equivalence classes of semi-stable GPS on $C$ with parabolic structures at the points of $I$ given by the data $\omega$.

Recall the construction of $P_\omega$. Let $\text{Grass}_r(F_{x_1} \oplus F_{x_2}) \to Q$ and

$$\tilde{R} = \text{Grass}_r(F_{x_1} \oplus F_{x_2}) \times \mathcal{Q} \mathcal{R}^0 \to \mathcal{R}.$$ 

$\omega = (k, \{\tilde{n}(x), \tilde{a}(x)\}_{x \in I})$ determines a polarization, which linearizes the $\text{SL}(V)$-action on $\tilde{R}$, such that the open set $\tilde{R}^{ss}_\omega$ (resp. $\tilde{R}^s_\omega$) of GIT semistable (resp. GIT stable) points are precisely the set of semistable (resp. stable) GPS on $C$ (see [19]). Then $P_\omega$ is the GIT quotient

$$\tilde{R}^{ss}_\omega \xrightarrow{\psi} \tilde{R}^{ss}_\omega \sslash \text{SL}(V) := P_\omega. \quad (4.1)$$

**Notation 4.3** Let $\mathcal{H} \subset \tilde{R}$ be the open subscheme parametrising the generalised parabolic sheaves $E = (E, E_{x_1} \oplus E_{x_2} \xrightarrow{q} Q)$ which satisfy

1. the torsion $\text{Tor} E$ of $E$ is supported on $\{x_1, x_2\}$ and $q : (\text{Tor} E)_{x_1} \oplus (\text{Tor} E)_{x_2} \hookrightarrow Q$ is injective;
2. if $N$ is large enough, then $H^1(E(N)(-x - x_1 - x_2)) = 0$ for all $E$ and $x \in C$.

It is known that $\mathcal{H}$ is a reduced, normal, Gorenstein variety with at most rational singularities (see Proposition 3.2 and Remark 3.1 of [17]). Moreover, $\tilde{R}^{ss}_\omega \subset \mathcal{H}$ and there is a morphism $\text{Det}_\mathcal{H} : \mathcal{H} \to J^d_C$ that extends determinant map on open set $\tilde{R}_F \subset \mathcal{H}$ of locally free sheaves (see Lemma 5.7 of [17]). It induces a flat morphism

$$\text{Det} : P_\omega \to J^d_C. \quad (4.2)$$

**Notation 4.4** For $L \in J^d_C$, let $\mathcal{H}_L = \text{Det}_\mathcal{H}^{-1}(L) \subset \mathcal{H}$,

$$\tilde{R}_F^L = \mathcal{H}_L \cap \tilde{R}_F, \quad (\tilde{R}^{ss}_\omega)^L = \mathcal{H}_L \cap \tilde{R}^{ss}_\omega.$$

Then $P^L_\omega = \text{Det}^{-1}(L) \subset P_\omega$ is the GIT quotient

$$(\tilde{R}^{ss}_\omega)^L \xrightarrow{\psi} P^L_\omega = (\tilde{R}^{ss}_\omega)^L \sslash \text{SL}(V).$$
Proposition 4.5 (Proposition 5.2 of [17]) Let $D^f_1 = \tilde{D}_1 \cup \hat{D}_1$ and $D^f_2 = \tilde{D}_2 \cup \hat{D}_2$, where $\hat{D}_i \subset \tilde{R}$ is the Zariski closure of $\hat{D}_F, i \subset \tilde{R}_F$ consisting of $(E, Q) \in \tilde{R}_F$ that $E_{x_i} \to Q$ is not an isomorphism, and $\hat{D}_1 \subset \tilde{R}$ (resp. $\hat{D}_2 \subset \tilde{R}$) consists of $(E, Q) \in \tilde{R}$ such that $E$ is not locally free at $x_2$ (resp. at $x_1$). Then

(1) $\text{Codim}(H^L(\tilde{R}_o^{ss})^L) > (r - 1)g + \frac{|I|}{k}$;
(2) the complement in $(\tilde{R}_o^{ss})^L \setminus \{D_1 \cup D_2\}$ of the set $\tilde{R}_o^s$ of stable points has codimension $\geq (r - 1)g + \frac{|I|}{k}$.
(3) $\text{Codim}((\tilde{R}_o^{ss})^L \setminus W_o) \geq (r - 1)g + \frac{|I|}{k}$, where

$$W_o := \left\{ (E, Q) \in (\tilde{R}_o^{ss})^L \biggm| \forall E' \subset E \text{ with } 0 < r(E') < r, \text{ we have} \right.$$  

$$\frac{\text{par} \chi(E') - \dim(Q_{E'})}{r(E')} < \frac{\text{par} \chi(E) - \dim(Q)}{r(E)} \right\}.$$  

Proof The statements (1) and (2) are contained in Proposition 5.2 of [17] (where the term $\frac{|I|}{k}$ was omitted). The proof of Proposition 5.2 (2) in [17] implies statement (3) here. $\square$

Proposition 4.6 Let $\omega_c = (2r, \{\bar{n}(x), \bar{a}_c(x)\}_{x \in I})$ be the data in Proposition 3.6 and $\Theta_J^d$ be the theta line bundle on $J_C^d$. Assume

$$(r - 1)(g - 1) + \frac{|I|}{2r} \geq 2. \quad (4.3)$$

Then there is an ample line bundle $\Theta_{\mathcal{P}_{oc}}$ on $\mathcal{P}_{oc}$ such that

$$\omega_{\mathcal{P}_{oc}}^{-1} = \Theta_{\mathcal{P}_{oc}} \otimes \text{Det}^*(\Theta_{J_C^d}^{-1}).$$

In particular, for any $L \in J_C^d$, $\mathcal{P}_{oc}^L$ is a normal Fano variety with only rational singularities.

Proof Let $V \otimes \mathcal{O}_C \otimes \mathcal{H}(-N) \to \mathcal{E} \to 0$, $\mathcal{E}_x \oplus \mathcal{E}_x \to \mathcal{Q} \to 0$ and

$$\{ \mathcal{E}_{[x]} \otimes \mathcal{H} = \mathcal{Q}_{[x]} \otimes \mathcal{H}, t_{x+1} \to \mathcal{Q}_{[x]} \otimes \mathcal{H}, t_{x} \to \cdots \to \mathcal{Q}_{[x]} \otimes \mathcal{H}, 1 \to 0 \}_{x \in I}$$

be the universal quotients and universal flags. Let $\omega_C = \mathcal{O}(\sum q)$ and

$$\Theta_{J_C^d} = (\text{det} R \pi_{J_C^d} \mathcal{L}^{-2}) \otimes \mathcal{L}_{x_1} \otimes \mathcal{L}_{x_2} \otimes \mathcal{L}_{y}^{-2r} \otimes \bigotimes_{q} \mathcal{L}_{q}^{r-1}$$

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Lemma 5.6 of [17] is replaced by condition (4.3). Thus

\[ \omega^{-1}_{\mathcal{H}} = (\det R \pi_{\mathcal{H}} \mathcal{E})^{-2r} \otimes \bigotimes_{x \in I} \left( (det \mathcal{E}_x)^{n_{x+1}} \otimes \bigotimes_{i=1}^{t_x} (det \mathcal{Q}_x, i)^{n_i(x) + n_{i+1}(x)} \right) \otimes (det Q)^{2r} \]

(see Proposition 3.4 of [17]). The line bundle \( \hat{\Theta}_{\omega c} \) descends to an ample line bundle \( \Theta_{\mathcal{P}_{\omega c}} \) on \( \mathcal{P}_{\omega c} \) (see Lemma 2.3 of [17]). Thus

\[ (\psi \ast \omega^{-1}_{\mathcal{R}_{\omega c}^{ss}})^{inv.} = \Theta_{\mathcal{P}_{\omega c}} \otimes \det^*(\Theta^{-1}_{J^c}). \]

When condition (4.3) holds, the lower bounds in Proposition 3.5 and Proposition 4.5 are at least two. Then Lemma 5.6 of [17] is applicable (where assumption \( g \geq 2 \) in Lemma 5.6 of [17] is replaced by condition (4.3)). Thus \( (\psi \ast \omega^{-1}_{\mathcal{R}_{\omega c}^{ss}})^{inv.} = \omega^{-1}_{\mathcal{P}_{\omega c}} \). \( \square \)

**Theorem 4.7** For any data \( \omega = (k, \{n(x), \bar{a}(x)\}_{x \in I}) \), the moduli space \( \mathcal{P}_{\omega}^{L} \) is of globally \( F \)-regular type.

**Proof** Choose a finite subset \( I' \subset C \setminus I \) satisfying (4.3). Recall that

\[ \mathcal{R} = \times_{x \in I} \text{Flag}_{\mathfrak{n}(x)}(\mathcal{F}_x), \quad \mathcal{R}' = \times_{x \in I \cup I'} \text{Flag}_{\mathfrak{n}(x)}(\mathcal{F}_x) \to \mathcal{R} \]

where \( \hat{f} \) is the projection and \( \tilde{\mathcal{R}} = \text{Grass}_r(\mathcal{F}_x \oplus \mathcal{F}_{x'}) \times \mathcal{Q} \mathcal{R} \to \mathcal{R} \). Let

\[ \tilde{\mathcal{R}}' := \text{Grass}_r(\mathcal{F}_x \oplus \mathcal{F}_{x'}) \times \mathcal{Q} \mathcal{R}' \to \tilde{\mathcal{R}} \quad (4.4) \]

be the morphism (use the same symbol \( \hat{f} \)) induced by \( \hat{f} : \mathcal{R}' \to \mathcal{R} \) through base change \( \rho : \tilde{\mathcal{R}} \to \mathcal{R} \). Then, on \( \mathcal{H}^L \subset \tilde{\mathcal{R}} \), it is clear that

\[ (\mathcal{H}')^L := \hat{f}^{-1}(\mathcal{H}^L) \to \mathcal{H}^L \]

is \( \text{SL}(V) \)-invariant and \( p \)-compatible. Moreover, \( \hat{f} \ast \mathcal{O}_{(\mathcal{H}')^L} = \mathcal{O}_{\mathcal{H}^L} \).

For any data \( \omega = (k, \{n(x), \bar{a}(x)\}_{x \in I}), \omega_c = (2r, \{\bar{n}(x), \bar{a}_c(x)\}_{x \in I \cup I'}) \), we have \( (\tilde{\mathcal{R}}^ss)^L \subset \mathcal{H}^L \), \( (\tilde{\mathcal{R}}^ss_{\omega c})^L \subset (\mathcal{H}')^L \). Recall

\[ (\tilde{\mathcal{R}}^ss_{\omega c})^L \to \mathcal{P}_{\omega}^L := Y, \quad (\tilde{\mathcal{R}}^ss_{\omega c})^L \to \mathcal{P}_{\omega c}^L := Z. \]

To apply Proposition 2.10, let \( W = W_{\omega c} \subset (\tilde{\mathcal{R}}^ss_{\omega c})^L \) and

\[ \hat{X} = W \cap \hat{f}^{-1}((\tilde{\mathcal{R}}^ss_{\omega c})^L). \]
By Proposition 4.5, Codim((\mathcal{H})^L \setminus W) \geq (r-1)g + \frac{|I|+|I'|}{2r} \geq 2. Thus it is enough to check the condition that \( \hat{X} = \varphi^{-1} \varphi(\hat{X}) \). This is equivalent (see Remark 1.2 of [17]) to proving

\[
\forall (E, Q) \in (\tilde{R}_\omega^{ss})^L, \quad (E, Q) \in \hat{X} \iff gr(E, Q) \in \hat{X}. \tag{4.5}
\]

In fact, for any \((E, Q) \in (\tilde{R}_\omega^{ss})^L\), it is clear that we have

\[(E, Q) \in W \iff gr(E, Q) = (\tilde{E}, \tilde{Q}) \oplus (x_1 \tau_1 + x_2 \tau_2, \tau_1 \oplus \tau_2)\]

where \((\tilde{E}, \tilde{Q})\) is a stable GPS (see Definition 1.5 of [17]). Thus either

\[0 \to (x_1 \tau_1 \oplus x_2 \tau_2, \tau_1 \oplus \tau_2) \to (E, Q) \to (\tilde{E}, \tilde{Q}) \to 0\]

or \[0 \to (E', Q') \to (\tilde{E}, \tilde{Q}) \to (x_i, C, C) \to 0\]. Then \((E, Q)\) is semi-stable (respect to \(\omega\)) if and only if \((\tilde{E}, \tilde{Q})\) is semi-stable (respect to \(\omega\)). Thus (4.5) is proved and we are done. \(\Box\)

When \(C = C_1 \cup C_2\) is reducible with two smooth irreducible components \(C_1\) and \(C_2\) of genus \(g_1\) and \(g_2\) meeting at only one point \(x_0\) (which is the only node of \(C\)), we fix an ample line bundle \(\mathcal{O}(1)\) of degree \(c\) on \(C\) such that \(deg(\mathcal{O}(1)|_{C_i}) = c_i > 0\) \((i = 1, 2)\).

For any coherent sheaf \(E\), \(P(E, n) := \chi(E(n))\) denotes its Hilbert polynomial which has degree 1. The rank of \(E\) is defined to be

\[r(E) := \frac{1}{deg(\mathcal{O}(1))} \cdot \lim_{n \to \infty} \frac{P(E, n)}{n}.
\]

Let \(r_i\) denote the rank of \(E|_{C_i}\) (the restriction of \(E\)) \((i = 1, 2)\), then

\[P(E, n) = (c_1 r_1 + c_2 r_2)n + \chi(E), \quad r(E) = \frac{c_1}{c_1 + c_2} r_1 + \frac{c_2}{c_1 + c_2} r_2.
\]

We also call sometimes that \(E\) has rank \((r_1, r_2)\) on \(C\).

Fix a finite set \(I = I_1 \cup I_2\) of smooth points on \(C\), where \(I_i \subset C_i\) \((i = 1, 2)\), and parabolic data \(\omega = \{k, \bar{n}(x), \bar{a}(x)\}_{x \in I}\) with

\[\ell := \frac{k \chi - \sum_{x \in I} \sum_{i=1}^{l_x} d_i(x) r_i(x)}{r}
\]

(recall \(d_i(x) = a_{i+1}(x) - a_i(x), r_i(x) = n_1(x) + \cdots + n_i(x)\)). Let

\[n_{ij}^\omega = \frac{1}{k} \left( r \frac{c_j}{c_1 + c_2} \ell + \sum_{x \in I_j} \sum_{i=1}^{l_x} d_i(x) r_i(x) \right) \quad (j = 1, 2).
\]

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Definition 4.8 For any coherent sheaf \(F\) of rank \((r_1, r_2)\), let

\[
m(F) := \frac{r(F) - r_1}{k} \sum_{x \in I_1} a_{i_1 + 1}(x) + \frac{r(F) - r_2}{k} \sum_{x \in I_2} a_{i_2 + 1}(x),
\]

the modified parabolic Euler characteristic and slop of \(F\) are

\[
\text{par} \chi_m(F) := \text{par} \chi(F) + m(F), \quad \text{par} \mu_m(F) := \frac{\text{par} \chi_m(F)}{r(F)}.
\]

A parabolic sheaf \(E\) is called semistable (resp. stable) if, for any subsheaf \(F \subset E\) such \(E/F\) is torsion free, one has

\[
\text{par} \chi_m(F) \leq \frac{\text{par} \chi_m(E)}{r(E)} r(F) \quad (\text{resp.}<).
\]

Theorem 4.9 (Theorem 1.1 of [18] or Theorem 2.14 of [19]) There exists a reduced, seminormal projective scheme

\[
\mathcal{U}_{C, \omega} := \mathcal{U}_C(r, d, \mathcal{O}(1), \{k, \tilde{n}(x), \tilde{a}(x)\}_{x \in I_1 \cup I_2})
\]

which is the coarse moduli space of \(s\)-equivalence classes of semistable parabolic sheaves \(E\) of rank \(r\) and \(\chi(E) = \chi = d + r(1 - g)\) with parabolic structures of type \([\bar{n}(x)]_{x \in I}\) and weights \([\bar{a}(x)]_{x \in I}\) at points \([x]_{x \in I}\). Moreover, \(\mathcal{U}_{C, \omega}\) has at most \(r + 1\) irreducible components.

The normalization of \(\mathcal{U}_{C, \omega}\) is a moduli space of semistable GPS on \(\tilde{C} = C_1 \sqcup C_2\) with parabolic structures at points \(x \in I\). Recall

Definition 4.10 A GPS \((E, E_{x_1} \oplus E_{x_2} \rightarrow Q)\) is called semistable (resp. stable), if for every nontrivial subsheaf \(E' \subset E\), such \(E/E'\) is torsion free outside \(\{x_1, x_2\}\), we have

\[
\text{par} \chi_m(E') - \dim(Q^{E'}) \leq r(E') \cdot \frac{\text{par} \chi_m(E) - \dim(Q)}{r(E)} \quad \text{(resp., <)},
\]

where \(Q^{E'} = q(E'_{x_1} \oplus E'_{x_2}) \subset Q\).

Theorem 4.11 (Theorem 2.1 of [18] or Theorem 2.26 of [19]) For any data \(\omega = ([k, \bar{n}(x), \bar{a}(x)]_{x \in I_1 \cup I_2}, \mathcal{O}(1))\), the coarse moduli space \(\mathcal{P}_\omega\) of \(s\)-equivalence classes of semi-stable GPS on \(\tilde{C}\), with parabolic structures at the points of \(I\) given by the data \(\omega\) is a disjoint union of at most \(r + 1\) irreducible, normal projective varieties \(\mathcal{P}_{\chi_1, \chi_2}(\chi_1 + \chi_2 = \chi + r, n_{j}^{\omega} \leq \chi_j \leq n_{j}^{\omega} + r)\) with at most rational singularities.

For fixed \(\chi_1, \chi_2\) satisfying \(\chi_1 + \chi_2 = \chi + r\) and \(n_{j}^{\omega} \leq \chi_j \leq n_{j}^{\omega} + r\) \((j = 1, 2)\), recall the construction of \(\mathcal{P}_\omega := \mathcal{P}_{\chi_1, \chi_2}\). Let

\[
P_i(m) = c_i rm + \chi_i, \quad \mathcal{W}_i = \mathcal{O}_{C_i}(-N), \quad V_i = \mathbb{C}^{P_i(N)}
\]

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where \(O_{C_i}(1) = \mathcal{O}(1)|_{C_i}\) has degree \(c_i\). Consider the Quot schemes \(Q_i = Quot(V_i \otimes \mathcal{W}_i, P_i)\), the universal quotient \(V_i \otimes \mathcal{W}_i \rightarrow \mathcal{F}^1 \rightarrow 0\) on \(C_i \times Q_i\) and the relative flag scheme

\[
\mathcal{R}_i = \times_{x \in I_i} \text{Flag}^\varnothing(x)(\mathcal{F}_x^1) \rightarrow Q_i.
\]

Let \(\mathcal{F} = \mathcal{F}^1 \oplus \mathcal{F}^2\) denote direct sum of pullbacks of \(\mathcal{F}^1, \mathcal{F}^2\) on

\[
\tilde{C} \times (Q_1 \times Q_2) = (C_1 \times Q_1 \times Q_2) \cup (C_2 \times Q_1 \times Q_2)
\]

and \(\mathcal{E}\) be the pullback of \(\mathcal{F}\) to \(\tilde{C} \times (\mathcal{R}_1 \times \mathcal{R}_2)\). Consider

\[
\rho : \tilde{\mathcal{R}} = \text{Grass}_r(\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}) \rightarrow \mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2 \rightarrow \mathcal{Q} = Q_1 \times Q_2.
\]

For the given \(\omega = ([k, \tilde{n}(x), \tilde{a}(x)]_{x \in I_1 \cup I_2}, \mathcal{O}(1))\), let \(\tilde{\mathcal{R}}^{ss}_{\omega}\) (resp. \(\tilde{\mathcal{R}}^s_{\omega}\)) denote the open subset of GIT semi-stable (resp. GIT stable) points under the action of \(G = (\text{GL}(V_1) \times \text{GL}(V_2)) \cap \text{SL}(V_1 \oplus V_2)\) on \(\tilde{\mathcal{R}}\) (respect to the polarization determined by \(\omega\)). Let \(\mathcal{H} \subset \tilde{\mathcal{R}}\) be the open subset defined in Notation 4.3, then, for any data \(\omega\), we have \(\tilde{\mathcal{R}}^s_{\omega} \subset \tilde{\mathcal{R}}^{ss}_{\omega} \subset \mathcal{H}\). The moduli space in Theorem 4.11 is nothing but the GIT quotient

\[
\psi : \tilde{\mathcal{R}}^{ss}_{\omega} \rightarrow \mathcal{P}_{\omega} := \tilde{\mathcal{R}}^{ss}_{\omega} / \!/ G.
\]

There exists a morphism \(\text{Det}_\mathcal{H} : \mathcal{H} \rightarrow J_C^d = J_{C_1} \times J_{C_2}^d\), which extends

\[
\text{Det}_\mathcal{H}_F : \mathcal{H}_F \rightarrow J_{C_1}^{d_1} \times J_{C_2}^{d_2}, \quad (E, Q) \mapsto (\det(E|_{C_1}), \det(E|_{C_2}))
\]

on the open set \(\mathcal{H}_F \subset \mathcal{H}\) of GPB (i.e. GPS \((E, Q)\) with \(E\) locally free) and induces a flat determinant morphism

\[
\text{Det}_{\mathcal{P}_\omega} : \mathcal{P}_\omega \rightarrow J_{C_1}^{d_1} \times J_{C_2}^{d_2}
\]

(see page 46 of [19] for detail). In fact, for any \(L \in J_{C_1}^{d_1} \times J_{C_2}^{d_2}\), let

\[
\mathcal{P}^L_{\omega} := \text{Det}_{\mathcal{P}_\omega}^{-1}(L) \subset \mathcal{P}_\omega,
\]

(4.7)

note that abelian variety \(J_C^0 = J_{C_1}^0 \times J_{C_2}^0\) acts on \(\mathcal{P}_\omega\), the induced morphism \(\mathcal{P}^L_{\omega} \times J_C^0 \rightarrow \mathcal{P}_\omega\) is a finite cover (see the proof of Lemma 6.6 in [19]). Similarly, let \(\mathcal{H}^L = \text{Det}_\mathcal{H}^{-1}(L)\) and \((\tilde{\mathcal{R}}^{ss}_{\omega})^L = \tilde{\mathcal{R}}^{ss}_{\omega} \cap \mathcal{H}^L\), then

\[
\psi : (\tilde{\mathcal{R}}^{ss}_{\omega})^L \rightarrow \mathcal{P}^L_{\omega} = (\tilde{\mathcal{R}}^{ss}_{\omega})^L / \!/ G.
\]

We do not have good estimate of \(\text{Codim} (\mathcal{H} \setminus (\tilde{\mathcal{R}}^{ss}_{\omega})^L)\) since sub-sheaves \((E_1, \text{Tor}(E_2)), (\text{Tor}(E_1), E_2)\) of \(E = (E_1, E_2)\) with rank \((r, 0), (0, r)\) may destroy semi-stability of
(E, Q) where Tor(E_i) \subset E_i (i = 1, 2) are torsion sub-sheaves. But we have estimate of Codim(\(H_0^{s s})\), where

\[
\mathcal{H}_0 = \{ (E, Q) \in \mathcal{H} \text{ with } n_j^{s s} \leq \chi(E_j) = \chi_j \leq n_j^{s s} + r(j = 1, 2) \text{ and } \dim(\text{Tor}(E_1)) \leq n_2^{s s} + r - \chi_2, \dim(\text{Tor}(E_2)) \leq n_1^{s s} + r - \chi_1 \}.
\]

**Proposition 4.12** Let \(D_1^f = \tilde{D}_1 \cup \tilde{D}_1, D_2^f = \tilde{D}_2 \cup \tilde{D}_2\) where \(\tilde{D}_i \subset \tilde{\mathcal{R}}\) is the Zariski closure of \(\tilde{D}_F, i \subset \tilde{\mathcal{R}}_F\) consisting of \((E, Q) \in \tilde{\mathcal{R}}_F\) that \(E_{x_i} \rightarrow Q\) is not an isomorphism and \(\tilde{D}_1^f \subset \tilde{\mathcal{R}}\) (resp. \(\tilde{D}_2^f \subset \tilde{\mathcal{R}}\)) consists of \((E, Q) \in \tilde{\mathcal{R}}\) such that \(E\) is not locally free at \(x_2\) (resp. at \(x_1\)). Then

1. \(\text{Codim}(\mathcal{H}_0^{s s}(\tilde{\mathcal{R}}_0^{s s})) > \min_{1 \leq i \leq 2} \left\{ (r - 1)(g_i - r - \frac{3}{4}) + \frac{|l_i|}{k} \right\} \)
2. \(\text{Codim}(\tilde{\mathcal{R}}_0^{s s}(\tilde{D}_1^f \cup \tilde{D}_2^f) \backslash (\tilde{\mathcal{H}}_0^{s s})) > \min_{1 \leq i \leq 2} \left\{ (r - 1)(g_i - 1) + \frac{|l_i|}{k} \right\}\) when \(n_1^{s s} < \chi_1 < n_1^{s s} + r;\)
3. \(\text{Codim}(\tilde{\mathcal{R}}_0^{s s}(\tilde{D}_1^f \cup \tilde{D}_2^f) \backslash W_0) \geq \min_{1 \leq i \leq 2} \left\{ (r - 1)(g_i - 1) + \frac{|l_i|}{k} \right\}\) when \(\chi_1 = n_1^{s s}\) or \(n_1^{s s} + r,\) where

\[
W_0 := \left\{ (E, Q) \in (\tilde{\mathcal{R}}_0^{s s}) \left| \frac{\text{par} \chi(E') - \text{dim}(Q^{E'})}{r(E')} < \frac{\text{par} \chi(E) - \text{dim}(Q)}{r(E)} \right. \right\};
\]


\forall E' \subset E \text{ of rank } (r_1, r_2) \neq (0, r), (r, 0), (0, 0).

**Proof** It is in fact a reformulation of Proposition 6.3 in [19] where determinants are not fixed. \(\square\)

**Proposition 4.13** For any \(\omega,\) let \(\tilde{\mathcal{R}}_0^{s s} \xrightarrow{\psi} \mathcal{P}_0 := \tilde{\mathcal{R}}_0^{s s} // G,\) assume

\[
\min_{1 \leq i \leq 2} \left\{ (r - 1)(g_i - \frac{r + 3}{4}) + \frac{|l_i|}{k} \right\} \geq 2.
\]

Then \((\psi_0^{\omega} \tilde{\mathcal{R}}_0^{s s})^{\text{inv.}} = \omega_{\mathcal{P}_0}.\) For \(\omega_e = (2r, \tilde{n}(x), \tilde{a}_e(x))_{x \in I_1 \cup I_2}\) satisfying

\[
\min_{1 \leq i \leq 2} \left\{ (r - 1)(g_i - \frac{r + 3}{4}) + \frac{|l_i|}{2r} \right\} \geq 2,
\]

there is an ample line bundle \(\Theta_{\mathcal{P}_0}\) over \(\mathcal{P}_0\) such that

\[
\omega_{\mathcal{P}_0}^{-1} = \Theta_{\mathcal{P}_0} \otimes \text{Det}_{\mathcal{P}_0}^\ast (\Theta_{\mathcal{P}_0}^{-1}).
\]

In particular, for any \(L \in J_{C_1}^{d_1} \times J_{C_2}^{d_2};\) \(\mathcal{P}_0^{L, \text{inv.}}\) is a normal Fano variety with only rational singularities.
Proof According to a result of Knop in [3] (see Lemma 4.17 of [11] for its global formulation), to prove \((\psi_*\omega_{\tilde{R}_{ss}})^{inv} = \omega_{\tilde{R}_{ss}}\), it is enough to show that (1) the subset where the action of \(G\) is not free has codimension at least two; (2) for every prime divisor \(D\) in \(\tilde{R}_{ss}\), \(\psi(D)\) has codimension at most 1.

We verify condition (1) firstly. When \(n_1^o < \chi_1 < n_1^o + r\), one has

\[
\text{Codim}(\tilde{R}_{ss} \setminus \{D_1^f \cup D_2^f\} \setminus W_o) > \min_{1 \leq i \leq 2} \left\{ (r - 1)(g_i - 1) + \frac{|I_i|}{k} \right\} \geq 2
\]

with \(D_j^f = D_j \cup \hat{D}_j\) where \(D_j, \hat{D}_j\) are irreducible, normal subvarieties (see Proposition C.7 of [11]). Since the subsets of \(\hat{D}_j\) and \(D_j\), where the action of \(G\) is free, are open subsets, it is enough to find a \((E, Q) \in D_j^f (j = 1, 2)\) such that its automorphisms are only scales. Let \(E_i' (i = 1, 2)\) be stable parabolic bundles of rank \(r\) and \(\chi(E_i') = \chi_i'\) on \(C_i\) with parabolic structures determined by \((k, (\tilde{n}(x), \tilde{a}(x))_{x \in L_i})\). Take \(\chi_i' = \chi_i - 1, \chi_2' = \chi_2\), let \(E_1 = E_1' \oplus x_1 C \cdot \beta, E_2 = E_2'\) and \(E = (E_1, E_2)\). If \(E_1 \oplus E_2 \xrightarrow{q_1} Q\) is defined by any isomorphism \(E_{x_1} \xrightarrow{q_2} Q\) and a linear map \(E_{x_1} = (E_1'_{x_1}) \oplus C \cdot \beta \xrightarrow{q_1} Q\) such that \(q_1(\beta) \neq 0\) and \(q_1|_{(E_i')_{x_1}} \neq 0\), \((E, Q) \in D_2^f\) by definition. To see \(\text{Aut}((E, Q)) = \mathbb{C}^*\), let \(0 \to K \to E_{x_1} \oplus E_{x_2} \xrightarrow{q} Q \to 0\) and \((E, Q) \xrightarrow{\Phi} (E, Q)\) be an isomorphism, then \(E \xrightarrow{\Phi} E\) is an isomorphism of parabolic bundles such that \(\Phi_{x_1 + x_2}(K) = K\). Since \(E_1', E_2\) are stable, we must have

\[
\Phi_{E_1} = (\lambda_1' id_{E_1'}, \lambda_1 id_{C \beta}) : E_1' \oplus x_1 C \beta \to E_1' \oplus x_1 C \beta
\]

and \(\Phi_{E_2} = \lambda_2 id_{E_2} : E_2 \to E_2\) for nonzero constants \(\lambda_1', \lambda_1, \lambda_2\). The requirement \(\Phi_{x_1 + x_2}(K) = K\) implies that \(\lambda_1' = \lambda_1 = \lambda_2\). In fact,

\[
K = \{(\alpha, f(\alpha)) \in E_{x_1} \oplus E_{x_2} | \forall \alpha \in E_{x_1}\}
\]

where \(f = -q_1^{-1} q_1 : E_{x_1} \to E_{x_2}\). For any \(\alpha = \alpha' + \beta \in E_{x_1}\), we have

\[
\Phi_{x_1 + x_2}(\alpha, f(\alpha)) = (\lambda_1' \alpha' + \lambda_1 \beta, \lambda_2 f(\alpha') + \lambda_2 f(\beta)) \in K
\]

which implies that \(\lambda_2 f(\alpha') + \lambda_2 f(\beta) = \lambda_1' f(\alpha') + \lambda_1 f(\beta)\). Thus \(\lambda_1 = \lambda_2\) (by taking \(\alpha' = 0\)) and \(\lambda_1' = \lambda_2\) (by taking \(\alpha'\) such that \(q_1(\alpha') \neq 0\)). Similarly, one can find such \((E, Q) \in D_1^f\). To construct \((E, Q) \in \hat{D}_j\) with \(\text{Aut}(E, Q) = \mathbb{C}^*\), we take \(\chi_i' = \chi_i, E_i = E_i'\) and \(E = (E_1, E_2)\) with \(E_{x_1} \oplus E_{x_2} \xrightarrow{q} Q \to 0\) defined by any isomorphism \(q_2 : E_{x_2} \xrightarrow{q} Q\) and nontrivial linear map \(q_1 : E_{x_1} \xrightarrow{q} Q\) (which is not surjective). Thus \((E, Q) \in D_1^f\) and \(\text{Aut}(E, Q) = \mathbb{C}^*\). Similarly one can find such \((E, Q) \in D_2^f\). When \(\chi_1 = n_1^o\) or \(n_1^o + r\), we have

\[
\text{Codim}(\tilde{R}_{ss} \setminus \{D_1^f \cup D_2^f\} \setminus W_o) \geq 2.
\]
Lemma 4.14

Let \( V \) be a normal variety acting by a reductive group \( G \). Suppose a good quotient \( \tilde{V} \). Then \( \tilde{V} \) is a normal Fano variety with only rational singularities.

Proof

It is in fact a reformulation of Lemma 4.16 in [11], where

\[ H^0(V', \tilde{\mathcal{L}})^{inv.} \to H^0(V', \tilde{\mathcal{L}})^{inv.} \]

was shown to be an isomorphism. \( \square \)

Theorem 4.15

For any data \( \omega = ([k, \tilde{n}(x), \tilde{a}(x)]_{x \in I_1 \cup I_2}, \mathcal{O}(1)) \) and integers \( \chi_1, \chi_2 \) satisfying \( \chi_1 + \chi_2 = \chi + r, n_j^0 \leq \chi_j \leq n_j^0 + r \) \( (j = 1, 2) \), let \( \mathcal{P}^L_\omega \) be the coarse moduli space of \( s \)-equivalence classes of semi-stable GPS \( E = (E_1, E_2) \) on \( \tilde{\mathcal{C}} \) with fixed determinant \( L \), \( \chi(E_j) = \chi_j \) and parabolic structures at the points of \( I \) given by the data \( \omega \). Then \( \mathcal{P}^L_\omega \) is of globally \( F \)-regular type.

Proof

Let \( I'_i \subset X_i \setminus (I_i \cup \{ x_i \}) \) be a subset and \( I' = I'_1 \cup I'_2 \). Recall

\[ \mathcal{R}_i = \times_{x \in I_i} Flag_{\tilde{n}(x)}(\mathcal{F}_x^i) \to \mathcal{Q}_i \]
and \( \rho : \tilde{\mathcal{R}} = \text{Grass}_r(\mathcal{F}^1_{x_1} \oplus \mathcal{F}^2_{x_2}) \to \mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2 \), let

\[
\mathcal{R}' = \bigtimes_{x \in I_1 \cup I'_1} \text{Flag}_{\tilde{\mu}(x)}(\mathcal{F}^i_x) \to \mathcal{R}_i, \quad \mathcal{R}' = \mathcal{R}'_1 \times \mathcal{R}'_2 \xrightarrow{\tilde{f}} \mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2
\]

where \( \tilde{f} \) is projection. Let \( \tilde{\mathcal{R}}' := \tilde{\mathcal{R}} \times_{\mathcal{R}} \mathcal{R}' \xrightarrow{\tilde{f}} \tilde{\mathcal{R}} \) be induced via

\[
\tilde{\mathcal{R}}' \xrightarrow{\rho} \mathcal{R}' \\
\tilde{\mathcal{R}} \xrightarrow{\rho} \mathcal{R}.
\]

Then, on \( \mathcal{H}^L \subset \tilde{\mathcal{R}} \), it is clear that \( (\mathcal{H}')^L := \tilde{f}^{-1}(\mathcal{H}^L) \xrightarrow{\tilde{f}} \mathcal{H}^L \) is a \( G \)-invariant and \( p \)-compatible morphism such that \( \tilde{f}_* \mathcal{O}_{(\mathcal{H}')^L} = \mathcal{O}_{\mathcal{H}^L} \).

For \( \omega = (k, \{\tilde{\mu}(x), \tilde{a}(x)\})_{x \in I_1 \cup \hat{I}_1}, \mathcal{O}(1), \omega_c = (2r, \{\tilde{\mu}(x), \tilde{a}_c(x)\})_{x \in I_1 \cup I'_1}, \mathcal{O}(1) \), we have \((\tilde{\mathcal{R}}_{\omega_c}^{ss})^L \subset \mathcal{H}^L \subset (\tilde{\mathcal{R}}_{\omega_c}^{ss})^L \subset (\mathcal{H}')^L \). Moreover, for \( \omega_c \), let \( \ell^c_j = 2 \chi_j - r - \sum x \in I_1 \cup I'_1 r_i(x) \) and \( \ell^c = \ell^c_1 + \ell^c_2 = 2 \chi - \sum x \in I_1 \cup I'_1 r_i(x) \). Then

\[
\sum_{x \in I_1 \cup I'_1} \sum_{i=1}^{l_i} (\tilde{a}_{i+1}(x) - \tilde{a}_i(x))r_i(x) + r \ell^c = 2r \chi.
\]

The choices of \( \{\tilde{\mu}(x)\}_{x \in I_1} \) satisfying \( \ell^c_j = \frac{c_j}{c_1+c_2} \ell^c \) for arbitrary large \( |I'_1| \) and \( |I'_2| \) are possible and it is easy to compute that \( n'^{of}_{j} = \chi_j - \frac{r}{2} \), thus

\[
n'^{of}_{j} < \chi_j < n'^{of}_{j} + r \quad (j = 1, 2).
\]

Recall \((\tilde{\mathcal{R}}_{\omega_c}^{ss})^L \xrightarrow{\psi} \mathcal{P}_{\omega}^L := Y, \quad (\tilde{\mathcal{R}}_{\omega_c}^{ss})^L \xrightarrow{\varphi} \mathcal{P}_{\omega_c}^L := Z \), choose \( I'_i \) satisfying

\[
\min_{1 \leq i \leq 2} \left\{(r - 1)(g_i - \frac{r + 3}{4}) + \frac{|I_i| + |I'_i|}{2r}\right\} \geq 2.
\]

Then \( Z \) is a normal Fano variety with only rational singularities by Proposition 4.13, which is in particular of globally \( F \)-regular type. To apply Proposition 2.10, let

\[
W = W_{\omega_c} \subset (\tilde{\mathcal{R}}_{\omega_c}^{ss})^L, \quad \hat{X} = W \cap \tilde{f}^{-1}(\tilde{\mathcal{R}}_{\omega_c}^{ss})^L.
\]

For any \((E, Q) \in (\tilde{\mathcal{R}}_{\omega_c}^{ss})^L \setminus (\tilde{\mathcal{R}}_{\omega_c}^{ss})^L \), there is an exact sequence

\[
0 \to (E', Q') \to (E, Q) \to (\tilde{E}, \tilde{Q}) \to 0 \quad (4.8)
\]
in the category \( C_\mu \) (see Proposition 2.4 of [18]) such that \( (\widetilde{E}, \widetilde{Q}) \) is stable (respect to \( \omega_c \)). Then either \( \widetilde{E} \) is torsion free when \( r(\widetilde{E}) > 0 \) or \( (\widetilde{E}, \widetilde{Q}) = (x_i, \mathbb{C}, \mathbb{C}) \). If \( (E, Q) \in W, \widetilde{E} \) has rank \( r \) or rank \( (r, 0) \), \((0, r)\) when \( r(\widetilde{E}) > 0 \). Thus it is easy to show that \( (E, Q) \in W \) if and only if \( gr(E, Q) \) is one of the following

1. \( gr(E, Q) = (\tilde{E}, \tilde{Q}) \oplus (x_1 \tau_1 \oplus x_2 \tau_2, \tau_1 \oplus \tau_2) \) where \((\tilde{E}, \tilde{Q}) \in C_\mu \) is stable of rank \((r, r)\);
2. \( gr(E, Q) = (\tilde{E}_1, \tilde{Q}_1) \oplus (\tilde{E}_2, \tilde{Q}_2) \oplus (x_1 \tau_1 \oplus x_2 \tau_2, \tau_1 \oplus \tau_2) \) where \((\tilde{E}_1, \tilde{Q}_1) \) and \((\tilde{E}_2, \tilde{Q}_2) \in C_\mu \) are stable of rank \((r, 0)\) and \((0, r)\),

which implies that \( \varphi^{-1} \varphi(W) = W \). Hence, to check \( \hat{X} = \varphi^{-1} \varphi(\hat{X}) \), it is enough to show that \((E, Q)\) is semi-stable (respect to \( \omega \)) if and only if the above GPS \((\tilde{E}, \tilde{Q}), (\tilde{E}_1, \tilde{Q}_1) \) and \((\tilde{E}_2, \tilde{Q}_2) \) in (1) and (2) are semi-stable (respect to \( \omega \)) with the same slope \( \mu_{\omega}(E, Q) \), which is easy to check by using (4.8) when either \( \tilde{E} \) has rank \( r \) or \( r(\tilde{E}) = 0 \).

If \( \tilde{E} \) has rank \((0, r)\), \( E' \) must have rank \((r, 0)\). Then \((E, Q)\) is \( \omega \)-semistable if and only if \((E', Q')\) and \((\tilde{E}, \tilde{Q})\) are \( \omega \)-semistable with \( \mu_{\omega}(E', Q') = \mu_{\omega}(E, Q) = \mu_{\omega}(E, Q) \) since the exact sequence (4.8) is split in this case. Thus we have \( \hat{X} = \varphi^{-1} \varphi(\hat{X}) \).

Now \( \hat{X} \xrightarrow{\varphi} X := \varphi(\hat{X}) \subset Z \) is a category quotient and the \( G \)-invariant \((\mathcal{H}')^L \xrightarrow{j} \mathcal{H}^L \) induces a morphism \( f : X \rightarrow Y \) such that

\[
\begin{array}{ccc}
(\mathcal{H}')^L & \xrightarrow{\psi} & X \\
\downarrow \hat{f} & & \downarrow f \\
(\mathcal{H}^L \supset \mathcal{R}_{ss}^\omega)^L & \xrightarrow{\psi} & Y \\
\end{array}
\]

is a commutative diagram. However we do not have

\[
\text{Codim}((\mathcal{H}')^L \setminus W) \geq 2
\]

as required in Proposition 2.10, which was used to prove \( f_* \mathcal{O}_X = \mathcal{O}_Y \).

On the other hand, let \( (\mathcal{R}_{ss}^\omega)^L \subset (\mathcal{R}_{ss}^\omega)^L \), \((\mathcal{H}'_F)^L \subset (\mathcal{H}')^L \) be the \( G \)-invariant open sets of GPS \((E, Q)\) where \( E \) is locally free. Then

\[
\text{Codim}((\mathcal{H}'_F)^L \setminus W) \geq \min_{1 \leq i \leq 2} \left\{ (r - 1)(g_i - \frac{r + 3}{4}) + \frac{|I_i|}{k} \right\} \geq 2
\]

by Proposition 4.12 (1) and (2), which and Lemma 4.14 imply

\[ f_* \mathcal{O}_X = \mathcal{O}_Y. \]

In fact, let \( \hat{X}_F = \hat{X} \cap (\mathcal{H}'_F)^L \), apply Lemma 4.14 to the surjections

\[
\hat{X}_F \xrightarrow{\psi} X, \quad (\mathcal{R}_{ss}^\omega)^L \xrightarrow{\psi} Y.
\]
we have $\varphi^*_{\mathcal{O}_X} = \mathcal{O}_X$, $\psi^*_{\mathcal{O}_X}$ $(\tilde{\mathcal{R}_{G,F}})^{inv} = \mathcal{O}_Y$. It means, for any open set $U \subset Y$, one has $\mathcal{O}_Y(U) = H^0(\psi^{-1}(U), \mathcal{O}_{(\tilde{\mathcal{R}_{G,F}})^{inv}})$ and

$$\mathcal{O}_X(f^{-1}(U)) = H^0(\varphi^{-1}(f^{-1}(U)), \mathcal{O}_{\tilde{X}})^{inv}.$$

Then, by $f^*_{\mathcal{O}_X} = \mathcal{O}_X$ and the fact that

$$f^{-1}_* (\mathcal{O}_X(U)) \cap \tilde{X} = f^{-1}_* (\mathcal{O}_X(U)) \cap ((\mathcal{H}_F)^L \setminus W)$$

has at least codimension two, we have

$$\mathcal{O}_Y(U) = H^0(\psi^{-1}(U), \mathcal{O}_{(\tilde{\mathcal{R}_{G,F}})^{inv}}) = H^0(\hat{f}^{-1}_* (\mathcal{O}_X(U)), \mathcal{O}_{(\mathcal{H}_F)^L}^{inv}) = H^0(\varphi^{-1}(f^{-1}(U)), \mathcal{O}_{\tilde{X}})^{inv} = \mathcal{O}_X(f^{-1}(U)).$$

Thus $Y = \mathcal{P}_L$ is of globally $F$-regular type since $X$ is so.

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