On the fixed-parameter tractability of the maximum 2-edge-colorable subgraph problem

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Abstract
A k-edge-coloring of a graph is an assignment of colors \{1, ..., k\} to edges of the graph such that adjacent edges receive different colors. In the maximum k-edge-colorable subgraph problem we are given a graph and an integer k, the goal is to find a k-edge-colorable subgraph with maximum number of edges together with its k-edge-coloring. In this paper, we consider the maximum 2-edge-colorable subgraph problem and present some results that deal with the fixed-parameter tractability of this problem.

Keywords: Edge-coloring, maximum 2-edge-colorable subgraph, exact algorithm, fixed-parameter tractability

1. Introduction
In this paper, we consider finite, undirected graphs without loops or multiple edges. The set of vertices and edges of a graph G is denoted by V(G) and E(G), respectively. \(d_G(u)\) denotes the degree of a vertex u of G. Let \(\delta(G)\) and \(\Delta(G)\) be the minimum and maximum degree of vertices of G. Let \(rad(G)\) and \(diam(G)\) be the radius and diameter of G.

A matching in a graph G is a subset of E(G) such that no vertex of G is incident to two edges from it. A maximum matching is a matching that contains the largest possible number of edges.

For \(k \geq 0\), a graph G is k-edge colorable, if its edges can be assigned colors from a set of k colors so that adjacent edges receive different colors. The smallest k, such that G is k-edge-colorable is called chromatic index of G and is denoted by \(\chi'(G)\). The classical theorem of Shannon states that for any multi-graph G \(\Delta(G) \leq \chi'(G) \leq \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor\) [27][31]. Moreover, the classical theorem of Vizing states that for any multi-graph G \(\Delta(G) \leq \chi'(G) \leq \Delta(G) + \mu(G)\) [31][32]. Here \(\mu(G)\) denotes the maximum multiplicity of an edge of G. A multi-graph is class I if \(\chi'(G) = \Delta(G)\), otherwise it is class II.

If \(k < \chi'(G)\), we cannot color all edges of G with k colors. Therefore, it is natural to investigate the maximum number of edges that one can color with k colors. A subgraph H
of $G$ is called maximum $k$-edge-colorable, if $H$ is $k$-edge-colorable and contains maximum number of edges among all $k$-edge-colorable subgraphs of $G$. For $k \geq 0$ and a graph $G$ let

$$\nu_k(G) = \max\{|E(H)| : H \text{ is a } k\text{-edge-colorable subgraph of } G\}.$$ 

Clearly, a $k$-edge-colorable subgraph is maximum if it contains exactly $\nu_k(G)$ edges. Observe that $\nu_1(G)$ is the size of a maximum matching of $G$. We will shorten this notation to $\nu(G)$.

There are many papers where the ratio $\frac{\nu_k(G)}{|E(G)|}$ has been investigated. It has been shown that for regular graphs the bounds are improved in [11]. Albertson and Haas investigated the problem in [12], where it is shown that for every cubic graph $G \nu_2(G) \geq \frac{3}{2}|V(G)|$ and $\nu_3(G) \geq \frac{7}{6}|V(G)|$. Moreover, [8] proves that for any cubic graph $G \nu_2(G) + \nu_3(G) \geq 2|V(G)|$, and in [21] Mkrtchyan et al. showed that for any cubic graph $G \nu_2(G) \leq \frac{|V(G)| + 2\nu_3(G)}{4}$. Finally, in [17], it is shown that the sequence $\nu_k$ is convex in the class of bipartite graphs.

Bridgeless cubic graphs that are not 3-edge-colorable are called snarks [9], and the ratio for snarks is investigated by Steffen in [29, 30]. This lower bound has also been investigated in the case when the graphs need not be cubic in [13, 18, 26]. Kosowski and Rizzi have investigated the problem from the algorithmic perspective [19, 26]. Since the problem of finding a $k$-edge-colorable graph in an input graph is NP-complete for every fixed $k \geq 2$, it is natural to investigate the (polynomial) approximability of the problem. In [19], for each $k \geq 2$ an algorithm for the problem is presented. There for each fixed value of $k \geq 2$, algorithms are proved to have certain approximation ratios and these ratios are tending to 1 as $k$ goes to infinity.

Some structural properties of maximum $k$-edge-colorable subgraphs of graphs are proved in [6, 24]. There it is shown that every set of disjoint cycles of a graph with $\Delta = \Delta(G) \geq 3$ can be extended to a maximum $\Delta(G)$-edge colorable subgraph. Moreover, there it is shown that any maximum $\Delta(G)$-edge colorable subgraph of a graph is always class I. Finally, if $G$ is a graph of girth (the length of the shortest cycle) $g \in \{2k, 2k+1\}$ ($k \geq 1$) and $H$ is a maximum $\Delta(G)$-edge colorable subgraph of $G$, then $\frac{|E(H)|}{|E(G)|} \geq \frac{2k}{2k + 1}$. The bound is best possible as there is an example attaining it.

In this paper, we deal with the exact solvability of the maximum $k$-edge-colorable subgraph problem. Its precise formulation is the following:

**Problem 1.** (Maximum $k$-edge-colorable subgraph) Given a graph $G$ and an integer $k$, find a $k$-edge-colorable subgraph with maximum number of edges together with its $k$-edge-coloring.

We investigate this problem from the perspective of fixed-parameter tractability. Recall that an algorithmic problem $\Pi$ is fixed-parameter tractable with respect to a parameter $\theta$, if there is an exact algorithm solving $\Pi$, whose running-time is $f(\theta) \cdot \text{poly}(\text{size})$. Here $f$ is some (computable) function of $\theta$, size is the length of the input and poly is a polynomial. In [14] the $k$-edge-coloring problem is considered, which is formulated as follows:

**Problem 2.** ($k$-edge-coloring) Given a graph $G$ and an integer $k$, check whether $G$ is $k$-edge-colorable.
Table 1: Summary of the main results obtained the paper about maximum 2-edge-colorable problem.

| Results          | Parameter                                      | in FPT? |
|------------------|-----------------------------------------------|---------|
| Theorem 2        | Radius rad$(G)$                               | NP-hard |
| Remark 1         | Diameter diam$(G)$                            | NP-hard |
| Theorem 5        | $\delta, \Delta, |V| - \Delta$, Number of maximum-degree vertices | NP-hard |
| Proposition 1    | $|V| - \delta$                                | Yes     |
| Theorem 3        | Pathwidth                                     | Yes     |
| Corollary 1      | Maximum matching $\nu(G)$                     | Yes     |
| Theorem 6        | Carving-width                                 | Yes     |
| Theorem 4        | Dimension of the cycle space                  | Yes     |

There it is shown that for each fixed $k$, the $k$-edge-coloring problem is fixed-parameter tractable with respect to the number of maximum degree vertices of the input graph. Observe that the maximum $k$-edge-colorable subgraph problem is harder than $k$-edge-coloring, as if we can construct a maximum $k$-edge-colorable subgraph $H_k$ of the input graph $G$, then in order to see that whether $G$ is $k$-edge-colorable, we just need to check whether $E(H_k) = E(G)$. If one considers the edge-coloring problem, where for an input graph $G$, we need to find a $\chi'(G)$-edge-coloring of $G$, then in [20] it is stated that a major challenge in the area is to find an exact algorithm for this problem whose running-time is $2^{O(n)} = O(c^n)$. Observe that the maximum $k$-edge-colorable subgraph problem is harder than edge-coloring. If we are able to solve the maximum $k$-edge-colorable subgraph problem in time $O(f(size))$, then we can solve the Edge-Coloring problem in time $O(f(size)) \cdot \log(|V(G)|)$. In order to see this, just observe that we can do a binary search on $k = 1, 2, ..., |V(G)|$, solve the maximum $k$-edge-colorable problem and find an edge-coloring of $G$ with the smallest number of colors. Here we used the fact that any graph $G$ is $|V(G)|$-edge-colorable.

In this paper, we focus on the maximum 2-edge-colorable subgraph problem which is the restriction of the problem to the case $k = 2$. We present some results that deal with the fixed-parameter tractability of this problem with respect to various graph-theoretic parameters. The results obtained in this paper are summarized in Table 1. For the notions, facts and concepts that are not explained in the paper the reader is referred to [10, 34].

2. Some auxiliary results

In this section, we present some results that will be used in obtaining the main results of the paper. Below we assume that $N$ is the set of natural numbers.

**Lemma 1.** ([28]) Let $\Pi$ be an algorithmic problem, and let $k_1$ and $k_2$ be some parameters. Assume that there is a (computable) function $g : N \rightarrow N$, such that for any instance $I$ of $\Pi$, we have $k_1(I) \leq g(k_2(I))$. Then if $\Pi$ is FPT with respect to $k_1$, then it is FPT with respect to $k_2$. 

3
In [16], Holyer has shown that checking whether a cubic graph is 3-edge-colorable is an NP-complete problem. For a cubic graph $G$, let $r_3(G)$ be defined as:

$$r_3(G) = |E| - \nu_3(G).$$

This parameter is introduced and is investigated in [30]. In particular, their it is observed there that $r_3(G) \neq 1$ for any cubic graph $G$. This means that $r_3(G)$ can be zero or at least two, and the 3-edge-coloring problem in cubic graphs amounts to deciding which of these two cases holds. For our purposes we will consider the following restriction of 3-edge-coloring problem in cubic graphs:

**Problem 1:** For a fixed integer $l \geq 1$, consider a decision problem, whose input is a cubic graph $G$, in which $r_3(G)$ is from the set $\{0, l, l + 1, l + 2, \ldots\}$. The goal is to check whether $G$ is 3-edge-colorable, that is, whether $r_3(G) = 0$.

**Lemma 2.** For each fixed $l \geq 1$, Problem 1 is NP-complete.

**Proof.** The case when $l \leq 2$ corresponds to the usual 3-edge-coloring problem in cubic graphs. Thus, we can assume that $l \geq 3$. We reduce the 3-edge-coloring problem of cubic graphs to this problem. Let $G$ be any cubic graph. Consider a cubic graph $H$ obtained from $l$ vertex disjoint copies of $G$. Observe that $|V(H)| = l \cdot |V(G)|$, hence $H$ can be constructed from $G$ in linear time. Now, it is easy to see that $G$ is 3-edge-colorable if and only if $H$ is 3-edge-colorable. Moreover, $r_3(H) = l \cdot r_3(G)$. Hence, $r_3(H)$ is either zero or at least $l$. The proof is complete. \hfill $\square$

We will also need the following result obtained in [21, 22]:

**Theorem 1.** For any cubic graph $G$ $\nu_2(G) \leq \frac{|V(G)| + 2\nu_3(G)}{4}$.

3. Main results

In this section, we present our main results about the maximum 2-edge-colorable subgraph problem. If $m$ is the number of edges of the input graph $G$, then clearly we can generate all subgraphs/subsets of $E(G)$, and check each of them for 2-edge-colorability. In great contrast with $k$-edge-colorability with $k \geq 3$, checking 2-edge-colorability can be done in polynomial time. A subgraph $F$ of $G$ is 2-edge-colorable if and only if it has maximum degree at most two, and it contains no component that is an odd cycle. Clearly this can be checked in polynomial time. The running time of this trivial, brute-force algorithm is $O^*(2^m)$. We will refer to this algorithm as trivial or brute-force algorithm.

The first parameter with respect to which we will investigate our problem is the radius of the graph.

**Theorem 2.** If $P \neq NP$, then the maximum 2-edge-colorable subgraph problem cannot be parameterized with respect to the $rad(G)$. 

4
Proof. Assume the opposite, that is the problem is FPT with respect to the $\text{rad}(G)$. Consider Problem 1 with $l \geq 6$. By Lemma 2 it is NP-complete. Let us take an arbitrary cubic graph $G$ with $r_3(G)$ either zero or at least $l$. Take a new vertex $z$, who is joined to every vertex of $G$. Let $G'$ be the resulting graph.

Let us show that $\nu_2(G') \geq |V(G)|$ if and only if $G$ is 3-edge-colorable. Let $G$ be a 3-edge-colorable. Then it admits a pair of edge-disjoint perfect matchings. Hence, these perfect matchings form a 2-edge-colorable subgraph in $G'$. Thus, $\nu_2(G') \geq |V(G)|$. Now, assume that $G$ is not 3-edge-colorable, hence $r_3(G) \geq l \geq 6$. By Theorem 1:

$$\nu_2(G') \leq 2 + \nu_2(G) \leq 2 + \frac{|V(G)| + 2 \cdot \nu_3(G)}{4} = 2 + |V(G)| - \frac{r_3(G)}{2} \leq |V(G)| - 1,$$

since $r_3(G) \geq l \geq 6$. Hence, if $\nu_2(G') \geq |V(G)|$, then $G$ is 3-edge-colorable.

Now, if the problem is FPT with respect to $\text{rad}(G)$, then since in graphs $G'$ that we obtained from $G$, we have $\text{rad}(G') = 1$ ($z$ is of distance one from any other vertex), we have that in polynomial time we can find a maximum 2-edge-colorable subgraph in $G'$. Thus, by the previous remark, we can use this polynomial algorithm to decide whether $\nu_2(G') \geq |V(G)|$, or equivalently, whether $G$ is 3-edge-colorable in the class of cubic graphs, in which $r_3(G)$ is zero or at least $l$. Since by Lemma 2 the latter problem is NP-complete, we have $P = NP$. The proof is complete.

Remark 1. If $P \neq NP$, then the maximum 2-edge-colorable subgraph problem cannot be parameterized with respect to the $\text{diam}(G)$.

This follows from Theorem 2, Lemma 1 and the fact that in any graph $G$, we have

$$\text{rad}(G) \leq \text{diam}(G) \leq 2 \cdot \text{rad}(G).$$

In the following, we show that the maximum 2-edge-colorable subgraph problem is fixed-parameter tractable respect to the pathwidth of the graph. We first recall some basic defini-
tions that we use in Theorem 3. Informally, a path decomposition of a graph $G$ is a way of representing $G$ as a path-like structure.

**Definition 1 ([12]).** A path decomposition of a graph $G = (V, E)$ is a set $P = (X_1, \ldots, X_r)$ of subsets of $V$, that is $X_i \subseteq V$ for each $i \in \{1, \ldots, r\}$, called bags, such that

- (i) for every $u \in V$ there exists $i \in \{1, \ldots, r\}$ with $u \in X_i$;
- (ii) for every $(u, v) \in E$, there exists $i \in \{1, \ldots, r\}$ with $u, v \in X_i$;
- (iii) for every three bags $X_i$, $X_j$, and $X_k$, with $i \leq j \leq k$, it holds that $X_i \cap X_k \subseteq X_j$.

The width of a path decomposition equals $\max_{i \in \{1, \ldots, r\}} |X_i| - 1$, and the pathwidth of a graph $G$, is the minimum width of a path decomposition of $G$. To avoid confusion between the vertices of the graph, and the ones in the path $P$, we will call nodes the vertices $X_i$ in the path decomposition. A property of path decompositions [12], called here pathwidth separator property, is that for every three nodes $X_i$, $X_j$, and $X_k$, with $i < j < k$, each path that connects a vertex in $X_i \cup X_j$ with a vertex in $X_k \cup X_j$ contains a vertex in $X_j$. Thus, node $X_j$ separates the vertices in $X_i \cup X_j$ from the ones in $X_k \cup X_j$. Our dynamic programming algorithm works on a particular type of path decomposition called nice, which can be always constructed in linear time from any path decomposition, maintaining the same width.

**Definition 2 ([12]).** A path decomposition of a graph $G = (V, E)$ is nice if $|X_1| = |X_r| = 1$, and for every $i \in \{1, 2, \ldots, r-1\}$ there is a vertex $v \in V$, such that either $X_{i+1} = X_i \cup \{v\}$ (introduce node), or $X_{i+1} = X_i \setminus \{v\}$ (forget node).

Since property (iii) in Definition 1 says that every vertex $v \in V$ belongs to a consecutive set of bags, the number of nodes in a nice path decomposition is at most twice the number of vertices in $V$. In Figure 2 there is a path decomposition, and in Figure 3 there is a nice path decomposition, both with width 2, and both for the graph in Figure 1.

**Theorem 3.** The maximum 2-edge-colorable subgraph problem is FPT with respect to the pathwidth $h$, and can be solved in $O(2^{3h+4h^2}.|V|)$ time via a dynamic programming algorithm.

**Proof.** Let $G(V, E)$ be a graph where $V$ is the set of vertices, $E$ is the set of edges, and with a path decomposition of width $h$. Assume that $\{0, 1, 2\}$ is the set of colours where 1, and 2 are true colours, while 0 is dummy and means 'not coloured'. Let $p: \{0, 1, 2\} \rightarrow \{0, 1\}$ be a function, which is equal to 1 if and only if the input is a true colour, 0 otherwise (i.e., $p(0) = 0$, $p(1) = 1$, and $p(2) = 1$). If $c: E \rightarrow \{0, 1, 2\}$ is a function, then $c((u, v))$ is called the color assigned to the edge $(u, v)$. In order to avoid cluttered notation, in the following we will write $c(u, v)$ instead of $c((u, v))$.

The first step of the algorithm is to compute a nice path decomposition $P = (X_1, \ldots, X_r)$ with the same width $h$, which can be done in linear time [12]. Denote by $G(X_i)$ the subgraph induced by the vertices in $\bigcup_{j=i}^{r} X_j$. Let $f(X_i, A)$ be the maximum value of 2-edge-colorable subgraph problem on $G(X_i)$, where $A$ is a collection of $|X_i|$ subsets of colors $A(u)$, with $u \in X_i$, that satisfies the following constraints.
• The colors incident to the vertex $u$ are those in $A(u)$, for every $u \in X_i$.

Here, as usual, a color is incident to a vertex $u$ if it is used at least in one edge incident to $v$. Notice that only the dummy color $\emptyset$ can be incident more than once to a vertex, because it means that an edge is not colored.

At each node $X_i$ of $P$, we compute the values $f(X_i, A)$, for every possible collection $A$, via a dynamic programming algorithm that starts at $X_1$, ends at $X_r$, and exploits the pathwidth separator property. If there is no solution of the constrained problem, then we set $f(X_i, A) = -\infty$. Since in $X_1$ there is only one vertex, and in $G(X_1)$ there are no edges, one of the following conditions hold:

- $f(X_1, A) = 0$ if $A = \{A(v)\} = \{\emptyset\}$;
- $f(X_1, A) = -\infty$ otherwise.

In fact, there cannot be true colors in $G(X_1)$, so there is only one solution with $A = \{\emptyset\}$, and optimum value 0.

In any introduce node $X_{i+1} = X_i \cup \{v\}$, the value $f(X_{i+1}, A)$, for a specific collection $A$ of color sets, is computed by solving the following maximisation problem.

$$\begin{align*}
\max \quad & f(X_i, B) + \sum_{u \in (X_i \cap V(v))} p(c(u, v)) \\
\text{s.t.} \quad & B(u) = A(u) \quad \forall u \in X_i \setminus V(v) \\
& B(u) = A(u) \quad \forall u \in (X_i \cap V(v)) \mid c(u, v) = 0 \\
& B(u) = A(u) \setminus \{c(u, v)\} \quad \forall u \in (X_i \cap V(v)) \mid c(u, v) \neq 0 \\
& A(v) = \bigcup_{u \in X_i} c(u, v)
\end{align*}$$

where $V(v)$ is the set of the vertices incident to $v$. The fourth constraint makes explicit how the colours incident to $v$ are used on the edges $(u, v)$, with $u \in X_i$. The first three constraints, instead, are used to properly define $B$, that is a collection of color subsets for $X_i$ derived from $A$. In particular, the first one states that $B(u)$ is equal to $A(u)$ for every vertex $u \in X_i$ non adjacent to $v$; the second states that $B(u)$ is equal to $A(u)$ if the color used for the edge $(u, v)$ is $\emptyset$; and the third states that $B(u)$ is equal to $A(u)$ minus the true colour used for the edge $(u, v)$, because it cannot be used again for the edges in $G(X_i)$ incident to $u$. The objective function sums the already computed optimum $f(X_i, B)$ with the number of edges in $G(X_i)$ incident to $v$ that receive a true color.

For any forget node $X_{i+1} = X_i \setminus \{v\}$, the value $f(X_{i+1}, A)$ for a specific collection of incident color sets $A(u)$, with $u \in X_{i-1}$, is computed by solving the following maximisation problem.

$$\begin{align*}
\max \quad & f(X_i, A \cup B(v)) \\
\text{s.t.} \quad & A(u) \cap B(v) \subseteq \{\emptyset\} \quad \forall u \in X_{i+1} \cap V(v) \\
& B(v) \subseteq \{\emptyset, 1, 2\}
\end{align*}$$
In fact, the value $f(X_{i+1}, A)$ is essentially the maximum value of $f(X_i, A \cup B(v))$ for every possible subset of colors $B(v)$ compatible with every $A(u)$, that means $A(u) \cap B(v) \subseteq \{0\}$ for every $u$ adjacent to $v$.

Once this dynamic programming algorithm stops, the optimum of 2-edge-colorable subgraph problem is the maximum value $f(X_r, A)$, for every possible collection of color subsets $A(u) \subseteq \{0, 1, 2\}$ of the incident colors in each vertex $u \in X_r$. Now, we compute the complexity of the algorithm. At each introduce node, we solve at most $2^{3(h+1)}$ problems once for every possible subset of incident colors $A(u)$, and for every $u \in X_{i+1}$ that are at most $h+1$. Moreover, for each specific $A$ in $[\square]$, the maximum number of edges $(u, v)$ with $u \in (X_i \cap V(v))$ is $h$, because $|X_i \cap V(v)| \leq |X_i| \leq h$. If $h \geq 3$, we can color these edges only if $\theta \in A(v)$, and we can do it in at most $h(h-1)$ different ways. In fact, if $A(v)$ contains one true colour, there are $h$ possibilities to select the edge associated with it; while, if $A(v)$ contains both the true colours, the possibilities are $h(h-1)$. Clearly, if $h \leq 2$, there can be less ways of colouring these edges. This means that, for a specific $A$, Problem $[\square]$ has at most $h(h-1)$ solutions, and the complexity for an introduce node is $O(2^{3(h+1)}h(h-1)) = O(2^{3(h+1)}h^2)$.

At any forget node, the complexity is at most $2^{3(h+1)}$, given by all the subsets of colors $A(u)$, with $u \in X_{i+1}$, and all the possible subsets of colors $B(v)$ for the forget vertex $v$. In conclusion, since there are at most $|V|$ introduce nodes and $|V|$ forget nodes, the time complexity of the dynamic programming algorithm is $O(2^{3h+4}h^2 \cdot |V|)$.

**Corollary 1.** The maximum 2-edge-colorable subgraph problem is FPT with respect to $\nu(G)$.

This just follows from Theorem $[\square]$ and Lemma $[\square]$ as it is demonstrated in $[\square]$.

Let $\alpha'(G)$ be the smallest number of edges of $G$ such that any vertex of $G$ is incident to at least one of these edges. By the classical Gallai theorem $[\square]$, we have that if the graph $G$ has no isolated vertices, then

$$\nu(G) + \alpha'(G) = |V|.$$ 

Since $\nu(G) \leq \frac{|V|}{2}$, we have

$$\nu(G) \leq \frac{|V|}{2} \leq \alpha'(G).$$

Thus, Corollary $[\square]$ and Lemma $[\square]$ imply that the maximum 2-edge-colorable subgraph problem is FPT with respect to $\alpha'(G)$. Observe that isolated vertices play no role in the maximum $k$-edge-colorable subgraph problem, thus we can assume that the input graph contains none of them.

Also, observe that the parameterization with respect to $\alpha'(G)$ can be interpreted as parameterization with respect to $|V| - \nu(G)$. One may wonder, whether we can strengthen this result, by showing that the maximum 2-edge-colorable subgraph problem is FPT with respect to $|V| - 2 \cdot \nu(G)$? The answer to this question is negative unless $P = NP$. If a cubic graph $G$ is 3-edge-colorable, then it must be bridgeless. Thus, by Holyer’s result the maximum 2-edge-colorable subgraph problem is $NP$-hard for bridgeless cubic graphs. By the classical Petersen theorem $[\square]$, bridgeless cubic graphs have a perfect matching. Thus,
in this class we have $|V| - 2\nu(G) = 0$. Hence, if $P \neq NP$, the maximum 2-edge-colorable subgraph problem cannot be FPT with respect to $|V| - 2\nu(G)$.

One can consider the decision version of the maximum 2-edge-colorable subgraph problem, where for a given graph $G$ and an integer $t$, one needs to check whether $\nu_2(G) \geq t$. It turns out that this problem is FPT with respect to $t$. In order to see this, just observe that if in the input graph $G$ $\nu(G) \geq t$, then clearly $\nu_2(G) \geq \nu(G) \geq t$, hence the instance is a “yes” instance. On the other hand, if $\nu(G) \leq t$, then the FPT algorithm with respect to $\nu(G)$ (Corollary [3]) will in fact will be an FPT algorithm with respect to $t$ (Lemma [4]).

Below we will parameterize the maximum 2-edge-colorable subgraph problem with respect to the dimension of the cycle space of a graph. Recall that if $G$ is any graph with $k$ components then the dimension of its cycle space $C(G)$ is given by the following formula:

$$\dim(C(G)) = |E| - |V| + k.$$ For our parameterization, we will require the following lemma:

**Lemma 3.** Let $G$ be a forest, and let $S$ be a set of vertices of $G$. Assume that $S = S_1 \cup S_2$ is a partition of $S$. Then there is a linear time algorithm that finds a largest 2-edge-colorable subgraph such that the vertices of $S_j$ are not incident to edges with color $j$ ($j = 1, 2$).

**Proof.** Clearly, we can assume that $G$ is a tree, otherwise we can find a largest 2-edge-colorable subgraph respecting constraints in each of the components, and by taking their union we will get an exact solution for the whole forest. Moreover, the constraints on the vertices in $S$ can be seen as: only color $2$ can be used on the edges incident to a vertex in $S_1$; and only color $1$ can be used on the edges incident to a vertex in $S_2$. In the following, we say that a color is incident to a vertex, if it is used at least on one of the edges incident to the vertex. In order to describe the algorithm, we add the extra dummy color 0 that means 'not colored', so the set of available colors becomes $\{0, 1, 2\}$, where 1, and 2 are called true colors.

Let $G(V, E)$ be a tree with $n = |V|$ vertices and $n - 1 = |E|$ edges. Assume that $W : V \rightarrow 2^{\{0, 1, 2\}}$ is an allocation of the available colors for each vertex in $V$. Let $p : \{0, 1, 2\} \rightarrow \{0, 1\}$ be a profit function that is equal to 0, if the input is the dummy color 0, and is 1, otherwise. Let $r$ be the root of $G$, and let $G(u)$ be the subgraph induced by $u$ and all the descendants of $u$ in $G$.

Now, we describe a dynamic programming algorithm that finds a largest 2-edge-colorable subgraph on $G = (V, E)$, with these additional constraints:

- the colors that can be incident to each vertex $u \in V$ are the ones in $W(u)$.

We call this problem $P$. Clearly, we can write the original problem of the lemma as $P$ with a specific color allocation, where $W(u) = \{0, 2\}$ for each vertex $u \in S_1$; $W(u) = \{0, 1\}$ for each vertex $u \in S_2$; and $W(u) = \{0, 1, 2\}$ for each vertex $u \in V \setminus S$.

In the algorithm, we compute $f(u, A)$ for each vertex $u \in V$, and for every $A \subseteq W(u)$, which is equal to the optimum value of $P$ restricted to the subgraph $G(u)$, and with the additional following constraint:
• the colors incident to \( u \) are those in \( A \).

If there is no solution, we set \( f(u, A) = -\infty \). In particular, the algorithm starts from
the leaves and goes up to the root \( r \), and the optimal value of \( P \) is the maximum \( f(r, A) \),
for every \( A \subseteq W(r) \).

If \( u \) is a leaf, and \( A \) a specific color subset of \( W(u) \), one of the following conditions holds:

• \( f(u, A) = 0 \) if \( A = \emptyset \);

• \( f(u, A) = -\infty \) otherwise.

In fact, since there are no edges in \( G(u) \) when \( u \) is a leaf, there cannot be any color
incident to \( u \) in \( G(u) \).

For each internal vertex \( u \), we suppose \( A \neq \emptyset \), as it is always possible to use the dummy
color \( 0 \). If \( u \) is an internal vertex with \( t \) sons \( \{v_1, \ldots, v_t\} \), we compute \( f(u, A) \) for
any subset \( A \subseteq W(u) \) by using the values \( f(v_i, A) \) for every \( u \)'s son \( v_i \). For every \( i \in \{1, \ldots, t\} \),
denote by \( V_i \) the set containing the vertices in the subgraph \( G(u) \) minus the vertices in each
subgraph \( G(v_j) \), with \( j \in \{i + 1, \ldots, t\} \). Let \( G(V_i) \) be the subgraph induced by the vertices
in \( V_i \). For every \( i \in \{1, \ldots, t\} \) we compute \( h(u, V_i, A) \), which equals the maximum value of
\( P \) restricted to the subgraph \( G(V_i) \), and with the following additional constraint:

• the colors incident to \( u \) are those in \( A \).

If there is no solution, we set \( h(u, V_i, A) = -\infty \). Notice that \( h(u, V_i, A) \) is equivalent to
\( f(u, A) \). Now, we see how to compute \( h(u, V_i, A) \) for every \( i \in \{1, \ldots, t\} \), recursively.

If \( i = 1 \), there is only one edge incident to \( u \) in \( G(v_1) \), i.e., \( (u, v_1) \). So, we can set
\( A = \{c(u, v_1)\} \), where \( A \) contains only the color assigned to \( (u, v_1) \). We compute \( h(u, V_1, A) \)
solving the following problem.

\[
\begin{align*}
\max \ f(v_1, C) + p(c(u, v_1)) \\
\text{s.t.} \quad C &\cap \{c(u, v_1)\} \subseteq \{0\} \\
C &\subseteq W(v_1)
\end{align*}
\]

If \( i \geq 2 \), we calculate \( h(u, V_i, A) \) by using the values \( h(u, V_{i-1}, A) \). We solve the following
maximisation problem for every non empty subset \( B \subseteq W(u) \) and any colour \( c(u, v_i) \in W(u) \)
for the edge \( (u, v_i) \), such that \( A = B \cup c(u, v_i) \), and \( B \cap \{c(u, v_i)\} \subseteq \{0\} \).

In fact, for a specific color \( c(u, v_i) \), we get the best value \( f(v_1, C) \) for every \( C \in W(v_1) \)
that is compatible with \( c(u, v_1) \). The compatibility is guaranteed by the constraint that does
not allow to choose a subset \( C \) (the problem’s variable) that contains \( c(u, v_1) \) if it is a true
color.

If \( i \geq 2 \), we calculate \( h(u, V_i, A) \) by using the values \( h(u, V_{i-1}, A) \). We solve the following
maximisation problem for every non empty subset \( B \subseteq W(u) \) and any colour \( c(u, v_i) \in W(u) \)
for the edge \( (u, v_i) \), such that \( A = B \cup c(u, v_i) \), and \( B \cap \{c(u, v_i)\} \subseteq \{0\} \).
Then, this is the time complexity to compute $O(u)$ for each internal node variable), the time complexity for computing algorithm on each $G$ subsets $B$ the value already computed. If function, $f(v_i, C) + p(c(u, v_i))$ refers to the subgraph $G(v_i) \cup (u, v_i)$, while $h(u, V_{i-1}, B)$ is the time complexity to compute $h(u, V_i, A)$ for all the $r$ sons of an internal vertex $u$ is $O(3 \cdot 2^6 t) = O(t)$. In fact, for $t = 1$ we solve Problem 3 for every $c(u, v_i) \in W(u)$, so at most 3 times; at each of the $t$ steps, with $t \geq 2$, we solve Problem 4 at most for every subset $B \subseteq W(u)$, and for every color $c(u, v_i) \in W(u)$. Since there are less than $2^4$ non empty subsets $B$, at most 3 possible colors for $c(u, v_i)$, and at most $2^3$ subsets $C$ (the problems’ variable), the time complexity for computing $h(u, V_i, A)$ for every $u$’s sons, is $O(3 \cdot 2^6 t)$. Then, this is the time complexity to compute $f(u, A)$ for an internal vertex $u$ with $t$ sons.

In conclusion, starting from the leaves, we can compute $f(u, A)$ for every internal node $u$, from the lowest level of the tree until we reach $r$. Since we need $O(3 \cdot 2^6 t_v) = O(t_v)$ time for each internal node $u$ with $t_v$ sons, the total running-time will be $\sum_{u \in V} O(t_v)$ which is $O(|V|)$, as the number of edges in a tree is $|V| - 1$.

We are ready to obtain the next result.

**Theorem 4.** The maximum 2-edge-colorable subgraph problem is FPT with respect to the dimension of the cycle space.

**Proof.** First of all let us show that it suffices to parameterize the problem when $G$ is connected. Assume that $G_1, ..., G_k$ are the connected components of $G$. Let us run the FPT algorithm on each $G_j$. Since the union of maximum 2-edge-colorable subgraphs in $G_j$ will give rise to a maximum 2-edge-colorable subgraph of $G$, this algorithm will solve the problem exactly. Let us estimate its running-time. If $f$ is the monotone (computable) function for parameterizing in the class of connected graphs, for the running time we will have

$$O^*(f(|E_1| - |V_1| + 1)) + ... + O^*(f(|E_k| - |V_k| + 1)) \leq k \cdot \max_{1 \leq j \leq k} O^*(f(|E_j| - |V_j| + 1)).$$

Since for any $j$

$$|E_j| - |V_j| + 1 \leq (|E_1| - |V_1| + 1) + ... + (|E_k| - |V_k| + 1) = |E| - |V| + k = \dim(C(G)),$$

and $k \leq |V|$ and $f$ is monotone, we have that the total running-time is bounded by

$$k \cdot \max_{1 \leq j \leq k} O^*(f(|E_j| - |V_j| + 1)) \leq |V| \cdot O^*(f(\dim(C(G)))) = O^*(f(\dim(C(G))))).$$
Thus, it suffices to parameterize the problem when the input graph $G$ is connected. In this case, we need to parameterize the problem with respect to $|E| - |V| + 1$. Our parameterization will work for $|E| - |V|$, hence it will be FPT with respect to $|E| - |V| + 1$ (Lemma 1).

We consider two cases. If $|E| - |V| \geq \log_2 |V|$, then we have

$$|E| \leq |V|^2 \leq (2^{|E| - |V|})^2 = 4^{|E| - |V|}.$$ 

Thus, the trivial brute-force algorithm will have a running time that is bounded in terms of $|E| - |V|$. Thus, we can assume that $|E| - |V| \leq \log_2 |V|$. Since $G$ is connected, it has a spanning tree $T$. Observe that $T$ contains $|V| - 1$ edges. Thus, there are at most $l = 1 + \log_2 |V|$ edges of $G$ that are outside $T$. Any maximum 2-edge-colorable subgraph of $G$ colors some edges outside $T$. Thus, we can guess this subset. The number of choices is at most $2^l = 2|V|$. For each of these guesses (or subsets), we can guess the colors on them. Since we have two colors, and the number of edges outside $T$ is at most $l$, there are at most $2^l = 2|V|$ different 2-colorings of these edges. Thus, in total, we will generate

$$\leq 2|V| \cdot 2|V| = 4|V|^2$$

guesses (subsets together with 2-edge-colorings). Now, consider any of these guesses. If it contains at least three edges adjacent to the same vertex, or two edges incident to the same vertex, then we do not consider it. If it contains two edges $e$ and $f$ incident to the same vertex $z$ such that edges have different color, we remove $z$ and forbid the corresponding color on the other end-point of $e$ and $f$ in $T$. If an edge is not adjacent to any other edge in the guess, we simply remove it and forbid its color in its end-points on $T$. Having done this, we get polynomially many instances of the forest problem with constraints on vertices. By Lemma 3, we can find the largest 2-edge-colorable subgraph respecting the constraints in polynomial time. Thus, we can compare the sizes of all these 2-edge-colorable subgraphs and get a maximum 2-edge-colorable subgraph of $G$ in polynomial time. Thus, the total running-time of our algorithm in the second case will be polynomial. The proof is complete.

4. Future work

In this section, we discuss some open problems that will be suitable for future research. For a graph $G$, let $\tau(G)$ be the size of the smallest vertex cover of $G$. Since in any graph $\nu(G) \leq \tau(G)$, Corollary 1 and Lemma 1 imply that the maximum 2-edge-colorable subgraph problem is FPT with respect to $\tau(G)$. We would like to ask:

**Question 1.** Is the maximum 2-edge-colorable subgraph problem FPT with respect to $\tau(G) - \nu(G)$?

The classical 2-approximation algorithm for the vertex cover problem and its analysis imply that for any graph $G$, we have $\tau(G) \leq 2 \cdot \nu(G)$. This inequality means that in any
graph $G$, we have:

$$\tau(G) - \nu(G) \leq \nu(G) \leq \tau(G).$$

Thus, a positive answer to Question 1 will strengthen Corollary 1 and its consequence for $\tau(G)$.

The classical Gallai theorem [34] states that in any graph $G$ we have:

$$\tau(G) + \alpha(G) = |V|.$$

Here $\alpha(G)$ is the size of the largest independent set of $G$. Since the maximum 2-edge-colorable subgraph problem is FPT with respect to $\tau(G)$, it is FPT with respect to $|V| - \alpha(G)$.

We would like to ask:

**Question 2.** Can we parameterize the maximum 2-edge-colorable subgraph problem with respect to $\alpha(G)$?

Note that in cubic graphs the size of the largest clique and the chromatic number are bounded with four. Thus, combined with Holyer’s result, we have that if $P \neq NP$, the maximum 2-edge-colorable subgraph problem is not FPT with respect to these two parameters.

Some lines of research that may be worth to investigate concern the generalisation to the weighted case, where each colour can have different weights even in combination with specific edges, adding different classes of constraints among colours, and analyzing them with respect to different graph topologies, like it is done in [3, 4, 5].

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Appendix

In this section we present additional results. In [14], it is shown that for each fixed $k$, the $k$-edge-coloring problem is FPT with respect to the number of maximum degree vertices of the input graph. As we have mentioned previously, the maximum $k$-edge-colorable subgraph problem is harder than $k$-edge-coloring. Thus, one can try to parameterize the latter with...
respect to the number of vertices of maximum degree. As the following theorem states, if \( P \neq NP \), this is impossible.

**Theorem 5.** If \( P \neq NP \), then the maximum 2-edge-colorable subgraph problem cannot be parameterized with respect to the number of maximum-degree vertices.

**Proof.** Consider the class of graphs \( G' \) from the proof of Theorem 2. Observe that if \( G \) is the complete graph on four vertices then \( G' \) has five vertices of degree four, which have maximum degree in \( G' \). On the other hand, if \( |V(G)| \geq 6 \), then \( z \) is the only vertex of maximum degree. Thus, if we assume that the maximum 2-edge-colorable subgraph problem is FPT with respect to the number of vertices of maximum degree, we will have a polynomial time algorithm for constructing it in the class \( G' \), hence to say whether \( \nu_2(G') \geq |V(G)| \). As in the proof of Theorem 2, this implies \( P = NP \). \( \square \)

**Remark 2.** Observe that in the above proof, there is no need for us to join \( z \) to all the vertices of \( G \). Since \( G \) is cubic we can join \( z \) to five vertices of \( G \). This will lead to the graph \( G' \), where \( z \) is the only vertex of degree five, which is maximum. All other vertices are of degree four or three. Thus, the problem remains hard even when the number of maximum degree vertices is one and the maximum degree is five.

Holyer’s result [16] implies that it is \( NP \)-hard to find a maximum 2-edge-colorable subgraph in cubic graphs. Thus, if \( P \neq NP \), we cannot parameterize the maximum 2-edge-colorable subgraph problem with respect to \( \Delta(G) \) and \( \delta(G) \). Moreover, in the proof of Theorem 2 we have \( |V(G')| = |V(G)| + 1 \) and \( \Delta(G') = d(z) = |V(G)| \), hence \( |V(G')| - \Delta(G') = 1 \) in these graphs \( G' \). Thus, one can say that if \( P \neq NP \), we cannot parameterize the maximum 2-edge-colorable subgraph problem with respect to \( |V| - \Delta \), too. On the positive side, it turns out that

**Proposition 1.** The maximum 2-edge-colorable subgraph problem is FPT with respect to \( |V| - \delta \).

**Proof.** Let \( G \) be any graph. If \( |V(G)| - \delta(G) \geq \frac{|V(G)|}{2} \), then
\[
|V(G)| \leq 2 \cdot (|V(G)| - \delta(G)).
\]

Thus,
\[
|E(G)| \leq |V(G)|^2 \leq 4 \cdot (|V(G)| - \delta(G))^2.
\]

Now, if we run the trivial algorithm, its running-time will depend solely on \( |V| - \delta \), as we have bounded the number of edges in terms of it. On the other hand, if \( |V(G)| - \delta(G) \leq \frac{|V(G)|}{2} \), then
\[
\delta(G) \geq \frac{|V(G)|}{2}.
\]

Thus, by Ore’s classical theorem [34], \( G \) has a Hamiltonian cycle \( C \). Now, if \( |V(G)| \) is even, then \( C \) is a 2-edge-colorable subgraph in \( G \). Since in any graph \( G \), \( \nu_2(G) \leq |V(G)| \), we have that \( C \) is a maximum 2-edge-colorable subgraph in \( G \). On the other hand, if \( |V(G)| \) is odd,
then any matching in $G$ has at most $\frac{|V(G)|-1}{2}$ edges, hence $\nu_2(G) \leq |V(G)| - 1$. Now, if we remove any edge from $C$, then the resulting Hamiltonian path will be a 2-edge-colorable subgraph with $|V(G)| - 1$ edges. Hence it will be a maximum 2-edge-colorable subgraph in $G$. The proof is complete.

Remark 3. Let us note that the proof of Ore’s theorem represents a polynomial time algorithm which actually finds the Hamiltonian cycle. Thus, in the second case of the previous proof, the algorithm will run in polynomial time.

Observe that in any graph $G$, we have the following relationship among vertex, edge connectivity and minimum degree:

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

Since, the maximum 2-edge-colorable subgraph problem is not FPT with respect to $\delta$ (unless $P = NP$), Lemma 1 implies that the problem is not FPT with respect to $\kappa(G)$ and $\kappa'(G)$. Moreover, since in any graph

$$|V| - \delta(G) \leq |V| - \kappa'(G) \leq |V| - \kappa(G),$$

Proposition 1 and Lemma 1 imply that the problem is FPT with respect to $|V| - \kappa(G)$ and $|V| - \kappa'(G)$.

In the following, we will see that 2-edge-colorable subgraph problem is fixed-parameter tractable respect to the carving width [7] of the graph. We first recall some basic definitions that we use in Theorem 6. Let $G = (V, E)$ be a graph, and let $S \subseteq V$. As usual, let $(S, V \setminus S)$ be the set of edges between $S$ and $V \setminus S$. Clearly, it is an edge cut of $G$. Assume that the vertices of $G$ are in 1-to-1 correspondence with the leaves of a sub-cubic tree $T$. 

Figure 4: A graph. 

Figure 5: A carving decomposition of the graph shown in Figure 4.
whose internal vertices all have degree three. This correspondence uniquely defines an edge-weight in the following way: if \( e \in E(T) \), and \( C_1 \) and \( C_2 \) are the two connected components of \( T - e \), then let \( S_i \) be the set of leaves of \( T \) that are in \( C_i \) for \( i = 1, 2 \). We have \( S_2 = V \setminus S_1 \).

Then the weight \( w(e) = |(S_1, S_2)| \). The tree \( T \) is called a carving of \( G \), and \((T, w)\) is called a carving decomposition of \( G \). The width of \((T, w)\) is the maximum weight \( w(e) \) over all \( e \in E(T) \). The carving-width of \( G \), denoted by \( \text{cw}(G) \), is the minimum width over all carving decompositions of \( G \). We define \( \text{cw}(G) = 0 \) if \( |V| = 1 \).

An example of carving decomposition is depicted in Figure 5, where the red edges among the leaves correspond to the edges of the decomposed graph in Figure 4. The integers on the edges of the tree are the weights of the decomposition, that is the size of the corresponding edge-cuts.

**Theorem 6.** The maximum 2-edge-colorable subgraph problem is FPT with respect to the carving-width \( h \), and can be solved in \( O((2n - 1)3^{2h}) \) time.

**Proof.** Let \( G(V, E) \) be a graph with a carving decomposition \((T, w)\) of width \( h \), where root is the root of \( T \). Denote by \( \{0, 1, 2\} \) the set of colours, where 1, and 2 are called true colours, while 0 is dummy and means 'not coloured'. Let \( c(u, v) \) be the pair \((c, (u, v))\), where \( c \in \{0, 1, 2\} \), and \((u, v) \in E \). Also let \( p \) a function, which takes in input a pair \( c(u, v) \), and returns 1 if \( c \in \{1, 2\} \), i.e., it is a true colour, and returns 0, otherwise. We use \( c(u, v) \) instead of \((c, (u, v))\) to avoid cluttered notation. Note that only the dummy colour 0 can be used on two or more adjacent edges, because it essentially says that an edge is not coloured. We have extended the set of colours in order to make the proof more readable.

Now, we describe a dynamic programming algorithm to find a maximum 2-edge-colorable subgraph of \( G \), which exploits the structure and the properties of carving decomposition. To avoid confusion between the vertices of the graph, and the ones in the tree, we will call nodes the vertices of \( T \).

For a node \( i \) of \( T \), denote by \( T(i) \) the subtree with \( i \) and all its descendants. Let \( S_i \) be the set of vertices corresponding to the leaves in \( T(i) \), and let \( S_i^- \) be the set \( V \setminus S_i \). Define \( G(i) \) as the subgraph of \( G \) induced by \( S_i \), and let \( G(i, i^-) \) the subgraph of \( G \) induced by \( S_i \) and the edges in \((S_i, S_i^-)\). Let \( f(i, A) \) be the optimum value for maximum 2-edge-colorable subgraph problem restricted to \( G(i, i^-) \), where \( A \) is a set of \(|(S_i, S_i^-)|\) edge-colour pairs, one for each edge in \((S_i, S_i^-)\), i.e. \( A = \bigcup_{(u, v) \in (S_i, S_i^-)} \{c(u, v)\} \), that satisfies the following constraints.

- The edge \((u, v) \in (S_i, S_i^-)\) is coloured with the colour \( c \) in the edge-colour pair \((c, (u, v)) \in A \).

If \( i \) is a leaf corresponding to a vertex \( u \in V \), then one of the following conditions holds:

- \( f(i, A) = \sum_{(u, v) \in (S_i, S_i^-)} p(c(u, v)) \) if there are no two edges in \((S_i, S_i^-)\) with the same true colour assigned;
- \( f(i, A) = -\infty \) otherwise.
If \( i \) is an interior node of \( T \) with two sons \( j \) and \( k \), then we compute \( f(i, A) \) solving the following maximisation problem, where \( A = (B \cup C) \setminus (B \cap C) = B \triangle C \), i.e., the symmetric difference of \( B \) and \( C \).

\[
\max \ f(j, B) + f(k, C) - \sum_{(u, v) \in ((S_j, S_j^-) \cap (S_k, S_k^-))} p(c(u, v))
\]

\[\text{s.t. } c(u, v) \in (B \cap C) \quad \forall (u, v) \in ((S_j, S_j^-) \cap (S_k, S_k^-))\]

(5)

In fact, by definition of carving decomposition, we have that \( S_i = S_j \cup S_k \), \( S_j \cap S_k = \emptyset \), and that the edges in \((S_i, S_i^-)\) are the ones in \((S_j, S_j^-) \cup (S_k, S_k^-)\) minus the ones in \((S_j, S_j^-) \cap (S_k, S_k^-)\), i.e., \((S_j, S_j^-) \triangle (S_k, S_k^-)\). This means that, for every \( B \) and \( C \) where the edges in \((S_i, S_i^-)\) have the colors given in \( A \), and each edge in \((S_j, S_j^-) \cap (S_k, S_k^-)\) has the same colors both in \( B \) and in \( C \), \( f(i, A) \) is equal to the best value \( f(j, B) \) plus the best value \( f(k, C) \), minus the profit given by the the edges in \((S_j, S_j^-) \cap (S_k, S_k^-)\), because it is counted twice. Clearly, for a specific \( A \), the variables of the problems are the colors to be assigned to the edges in \((S_j, S_j^-) \cap (S_k, S_k^-)\), that is the pairs of edge-colours in \( B \cap C \). Notice that, there are no specific constraints to check if adjacent edges receive the same true color. This is not needed, because a solution with a true color assigned to more than one adjacent edges, has value \(-\infty\). It directly follows from the definition of \( f(i, A) \) for the leaves. Clearly, the optimum of maximum 2-edge-colorable subgraph problem is the maximum value \( f(root, \emptyset) \), because there are no edges in \( S_{root^-} \).

The time complexity at each leaf \( i \) is \( O(3^h) \), because there are at most \( h \) edges in \((S_i, S_i^-)\), and there are 3 possible colors, \( \{0, 1, 2\} \).

The time complexity at each internal node \( i \) of \( T \) is given by all the possible colors assigned to every edge in \((S_j, S_j^-) \cup (S_k, S_k^-)\). Since the maximum number of edges in each partition is \( h \), the carving width, there are at most \( 2h \) edges in \((S_j, S_j^-) \cup (S_k, S_k^-)\). Moreover, since the number of colours is 3, the time complexity at each internal node of \( T \) is \( O(3^{2h}) \).

In conclusion, since each internal node has degree three (two sons), and in a perfect binary tree there are \( 2n - 1 \) nodes, then \( T \) has at most \( 2n - 1 \) nodes. This means that the time complexity of the dynamic programming algorithm is \( O((2n - 1)^{3h}) \).