D–MODULES ON THE AFFINE GRASSMANNIAN AND REPRESENTATIONS OF AFFINE KAC-MOODY ALGEBRAS

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Dedicated to Victor Kac on his 60th birthday

1. Introduction

1.1. Let \( g \) be a simple Lie algebra over \( \mathbb{C} \), and \( G \) the corresponding algebraic group of adjoint type. Given an invariant inner product \( \kappa \) on \( g \), let \( \hat{g}_\kappa \) denote the corresponding central extension of the formal loop algebra \( g \otimes \mathbb{C}[[t]] \), called the affine Kac-Moody algebra \( \hat{g}_\kappa \),

\[
0 \to \mathbb{C}1 \to \hat{g}_\kappa \to g \otimes \mathbb{C}[[t]] \to 0,
\]

with the two-cocycle defined by the formula

\[
x \otimes f(t), y \otimes g(t) \mapsto -\kappa(x, y) \cdot \text{Res}_{t=0} f dg.
\]

Denote by \( \hat{g}_\kappa\text{–mod} \) the category of \( \hat{g}_\kappa \)-modules which are discrete, i.e., any vector is annihilated by the Lie subalgebra \( g \otimes \mathbb{C}[[t]] \) for sufficiently large \( N \geq 0 \), and on which \( 1 \in \mathbb{C} \subset \hat{g}_\kappa \) acts as the identity. We will refer to objects of these category as modules at level \( \kappa \).

Let \( \text{Gr}_G = G((t))/G[[t]] \) be the affine Grassmannian of \( G \). For each \( \kappa \) there is a category \( D_\kappa(\text{Gr}_G)\text{–mod} \) of \( \kappa \)-twisted right D-modules on \( \text{Gr}_G \) (see [BD]). We have the functor of global sections

\[
\Gamma : D_\kappa(\text{Gr}_G)\text{–mod} \to \hat{g}_\text{crit}\text{–mod}, \quad \mathcal{F} \mapsto \Gamma(\text{Gr}_G, \mathcal{F}).
\]

Let \( \kappa_{\text{Kil}} \) be the Killing form, \( \kappa_{\text{Kil}}(x, y) = \text{Tr}(\text{ad}_g(x) \circ \text{ad}_g(y)) \). The level \( \kappa_{\text{crit}} = -\frac{1}{2}\kappa_{\text{Kil}} \) is called critical. A level \( \kappa \) is called positive (resp., negative, irrational) if \( \kappa = c \cdot \kappa_{\text{Kil}} \) and \( c + \frac{1}{2} \in \mathbb{Q}^{\geq 0} \) (resp., \( c + \frac{1}{2} \in \mathbb{Q}^{< 0}, c \notin \mathbb{Q} \)).

It is known that the functor of global sections cannot be exact when \( \kappa \) is positive. In contrast, when \( \kappa \) is negative or irrational, the functor \( \Gamma \) is exact and faithful, as shown by A. Beilinson and V. Drinfeld in [BD], Theorem 7.15.8. This statement is a generalization for affine algebras of the famous theorem of A. Beilinson and J. Bernstein, see [BB], that the functor of global sections from the category of \( \lambda \)-twisted D-modules on the flag variety \( G/B \) is exact when \( \lambda - \rho \) is anti-dominant and it is faithful if \( \lambda - \rho \) is, moreover, regular.

The purpose of this paper is to consider the functor of global sections in the case of the critical level \( \kappa_{\text{crit}} \). (In what follows we will slightly abuse the notation and replace the subscript \( \kappa_{\text{crit}} \) simply by \( \text{crit} \).) Unfortunately, it appears that the approach of [BD] does not extend to the critical level case, so we have to use other methods to analyze it. Our main result is that the functor of global sections remains exact at the critical level:

**Theorem 1.2.** The functor \( \Gamma : D_{\text{crit}}(\text{Gr}_G)\text{–mod} \to \hat{g}_{\text{crit}}\text{–mod} \) is exact.

In other words, we obtain that for any object \( \mathcal{F} \) of \( D_{\text{crit}}(\text{Gr}_G)\text{–mod} \) we have \( H^i(\text{Gr}_G, \mathcal{F}) = 0 \) for \( i > 0 \). Moreover, we will show that if \( \mathcal{F} \neq 0 \), then \( H^0(\text{Gr}_G, \mathcal{F}) = \Gamma(\text{Gr}_G, \mathcal{F}) \neq 0 \). This property is sometimes referred to as “D-affineness” of \( \text{Gr}_G \).

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In fact, we will prove a stronger result. Namely, we note after [BD], that for a critically twisted D-module \( \mathcal{F} \) on \( \text{Gr}_G \), the action of \( \hat{g}_{\text{crit}} \) on \( \Gamma(\text{Gr}_G, \mathcal{F}) \) extends to an action of the renormalized enveloping algebra \( U^{\text{ren}}(\hat{g}_{\text{crit}}) \) of Sect. 5.6 of loc. cit. Following a conjecture and suggestion of Beilinson, we show that the resulting functor from \( D_{\text{crit}}(\text{Gr}_G) - \text{mod} \) to the category of \( U^{\text{ren}}(\hat{g}_{\text{crit}}) \)-modules is fully-faithful.

1.3. Our method of proof of Theorem 1.2 uses the chiral algebra of differential operators \( \mathcal{D}_{G, \kappa} \) introduced in [AG]. Modules over \( \mathcal{D}_{G, \kappa} \) should be viewed as (twisted) D-modules on the loop group \( G((t)) \). In particular, the category of \( \kappa \)-twisted D-modules on \( \text{Gr}_G \) is equivalent to the subcategory in \( \mathcal{D}_{G, \kappa} - \text{mod} \), consisting of modules, which are integrable with respect to the right action of \( g[[t]] \) (see Theorem 2.5). The functor of global sections on \( \text{Gr}_G \) corresponds, under this equivalence, to the functor of \( g[[t]] \)-invariants. Therefore, we need to prove that this functor of invariants is exact.

This approach may be applied both when the level \( \kappa \) is negative (or irrational) and critical. In the case of the negative or irrational level the argument is considerably simpler, and so we obtain a proof of the exactness of \( \Gamma \), which is different from that of [BD] (see Sect. 2).

In the case of the negative or irrational level the argument is considerably simpler, and so we understand a module over the sheaf of differential operators acting on the line bundle \( G \times_B \lambda \). Let \( \pi \) denote the natural projection \( G \rightarrow G/B \), and observe that the pull-back functor (in the sense of quasicoherent sheaves) lifts to a functor \( \pi^* : D^\lambda(G/B) - \text{mod} \rightarrow D(G) - \text{mod} \). Furthermore, for a D-module \( \mathcal{F}' \) on \( G \), the space of its global sections \( \Gamma(G, \mathcal{F}') \) is naturally a bimodule over \( g \) due to the action of \( G \) on itself by left and the right translations.

For \( \mathcal{F} \in D^\lambda(G/B) - \text{mod} \) we have

\[
\Gamma(G/B, \mathcal{F}) \simeq \text{Hom}_b \left( \mathbb{C}^{-\lambda}, \Gamma(G, \pi^*(\mathcal{F})) \right),
\]

where \( b \) is the Borel subalgebra of \( g \), \( \mathbb{C}^{-\lambda} \) its one-dimensional representation corresponding to weight \( -\lambda \), and \( \Gamma(G, \pi^*(\mathcal{F})) \) is a \( b \)-module via \( b \hookrightarrow g \) and the right action of \( g \). But the \( g \)-module \( \Gamma(G, \mathcal{F}') \), where \( \mathcal{F}' = \pi^*(\mathcal{F}) \) (with respect the right \( g \)-action), belongs to the category \( \mathcal{O} \). Thus, we obtain a functor

\[
\Gamma' : D^\lambda(G/B) - \text{mod} \rightarrow \mathcal{O}, \quad \mathcal{F} \mapsto \Gamma(G, \pi^*(\mathcal{F})),
\]

and

\[
\Gamma(G/B, \mathcal{F}) \simeq \text{Hom}_\mathcal{O} (M(-\lambda), \Gamma'(\mathcal{F})),
\]

where \( M(-\lambda) \) is the Verma module with highest weight \( -\lambda \).

The functor \( \Gamma' \) is exact because \( G \) is affine, and it is well-known that \( M(\mu) \) is a projective object of \( \mathcal{O} \) precisely when \( \mu + \rho \) is dominant. Hence \( \Gamma \) is the composition of two exact functors, and, therefore, is itself exact.

This reproves the Beilinson-Bernstein exactness statement. Note, however, that the methods described above do not give the non-vanishing assertion of [BB].

1.4. The proof of the exactness result in the negative (or irrational) level case is essentially a word for word repetition of the above argument, once we are able to make sense of the category of D-modules on \( G((t)) \) as the category of \( \mathcal{D}_{G, \kappa} \)-modules. The key fact that we will use will be the same: that the corresponding vacuum Weyl module \( V_{g, \kappa'} \) is projective in the appropriate category \( \mathcal{O} \) if \( \kappa' \) is positive or irrational.

This argument does not work at the critical level, because in this case the corresponding Weyl module \( V_{g, \text{crit}} \) is far from being projective in the category \( \hat{g}_{\text{crit}} - \text{mod} \). Roughly, the
picture is as follows. Modules over $\hat{g}_{\text{crit}}$ give rise to quasicoherent sheaves over the ind-scheme $\text{Spec}(\hat{g}_{\text{crit}})$, where $\hat{g}_{\text{crit}}$ is the center of the completed universal enveloping algebra of $\hat{g}_{\text{crit}}$ (this is the ind-scheme of $L\hat{g}$-opers on the punctured disc, where $L\hat{g}$ is the Langlands dual Lie algebra to $\hat{g}$). The ind-scheme $\text{Spec}(\hat{g}_{\text{crit}})$ contains a closed subscheme $\text{Spec}(\hat{g}_{\text{crit}})$ (this is the scheme of $L\hat{g}$-opers on the disc). The module $\mathbb{V}_{\hat{g}_{\text{crit}}}$ is supported on $\text{Spec}(\hat{g}_{\text{crit}})$ and is projective in the category of $\hat{g}_{\text{crit}}$-modules, which are supported on $\text{Spec}(\hat{g}_{\text{crit}})$ and are $G(\hat{0})$-integrable.

The problem is, however, that the $\hat{g}_{\text{crit}}$-modules of the form $\Gamma(G((t)), \pi^*(\mathcal{F}))$, where $\pi$ is the projection $G((t)) \rightarrow G((t))/G[[t]] \simeq \text{Gr}_G$, are never supported on $\text{Spec}(\hat{g}_{\text{crit}})$. Therefore we need to show that the functor of taking the maximal submodule of $\Gamma(G((t)), \pi^*(\mathcal{F}))$, which is supported on $\text{Spec}(\hat{g}_{\text{crit}})$, is exact. We do that by showing that the action of $\hat{g}_{\text{crit}}$ on $\Gamma(G((t)), \pi^*(\mathcal{F}))$ automatically extends to the action of the renormalized chiral algebra $A_{\hat{g}}^{\text{ren}, \tau}$, which is closely related to the renormalized enveloping algebra $U^{\text{ren}}(\hat{g}_{\text{crit}})$, mentioned above.

Consider the following analogy. Let $X$ be a smooth variety and $Y$ its smooth closed subvariety. Then we have a natural functor, denoted $i^!$, from the category of $\mathcal{O}_X$-modules, set-theoretically supported on $Y$, to the category of $\mathcal{O}_Y$-modules: this functor takes an $\mathcal{O}_X$-module $\mathcal{F}$ to its maximal submodule supported scheme-theoretically on $Y$. This is not an exact functor. But the corresponding functor from the category of right $D$-modules on $X$, also set-theoretically supported on $Y$, to the category of right $D$-modules on $Y$ is exact, according to a basic theorem due to Kashiwara.

In our situation the role of the category of $\mathcal{O}_X$-modules is played by the category $\hat{g}_{\text{crit}} - \text{mod}$, and the role of the category of $D$-modules is played by the category of modules over the chiral algebra $A_{\hat{g}}^{\text{ren}, \tau}$. We show that the above functor of taking the maximal submodule of $\Gamma(G((t)), \pi^*(\mathcal{F}))$, which is supported on $\text{Spec}(\hat{g}_{\text{crit}})$, factors through the latter category, and this allows us to prove the required exactness.

1.5. Contents. Let us briefly describe how this paper is organized. In Sect. 2 we treat the negative level case. In Sect. 3 we recall some facts about commutative $D$-algebras and the description of the center of the Kac-Moody chiral algebra at the critical level. In Sect. 4 we discuss several versions of the renormalized universal enveloping algebra at the critical level in the setting of chiral algebras. In Sect. 5 we study the chiral algebra of differential operators $D_{G,\kappa}$ when $\kappa = \kappa_{\text{crit}}$. In Sect. 6 we derive our main Theorem 1.2 from two other statements, Theorems 6.11 and 6.13. In Sect. 7 we prove Theorem 6.13 generalizing Kashiwara’s theorem about $D$-modules supported on a subvariety. In Sect. 8 we prove Theorem 6.11 and describe the category of $\hat{g}_{\text{crit}}$-modules, which are supported on $\text{Spec}(\hat{g}_{\text{crit}})$ and are $G(\hat{0})$-integrable. Finally, in Sect. 9 we prove that the functor $\Gamma$ is faithful.

1.6. Conventions. Our basic tool in this paper is the theory of chiral algebras. The foundational work [CHA] on this subject will soon be published (in our references we use the most recent version: a previous one is currently available on the Web). In addition, an abridged summary of the results of [CHA] that are used in this paper may be found in [AG]. We wish to remark that all chiral algebras considered in this paper are universal in the sense that they come from quasi-conformal vertex algebras by a construction explained in [FB], Ch. 18. Therefore all results of this paper may be easily rephrased in the language of vertex algebras. We have chosen the language of chiral algebras in order to be consistent with the language used in [AG].

We also use some the results from [BD], which is still unpublished, but available on the Web.

The notation in this paper mainly follows that of [AG]. Throughout the paper, $X$ will be a fixed smooth curve; we will denote by $\mathcal{O}_X$ (resp., $\omega_X$, $T_X$ and $D_X$) its structure sheaf (resp., the sheaf of differentials, the tangent sheaf and the sheaf of differential operators).
We will work with D-modules on $X$, and in our notation we will not distinguish between left and right D-modules, i.e., we will denote by the same symbol a left D-module $M$ and the corresponding right D-module $M \otimes \omega_X$. The operations of tensor product, taking symmetric algebra, and restriction to a subvariety must be understood accordingly.

We will denote by $\Delta$ the diagonal embedding $X \to X \times X$, and by $j$ the embedding of its complement $X \times X - \Delta(X) \to X \times X$. If $x \in X$ is a point, we will often consider D-modules supported at $x$. In this case, our notation will not distinguish between such a D-module and the underlying vector space.

We will use the notation $A \times C$ for a fiber product of $A$ and $C$ over $B$, and the notation $\mathcal{P} \times_G V$ for the twist of a $G$-module $V$ by a $G$-torsor $\mathcal{P}$.

Finally, if $\mathcal{C}$ is a category and $C$ is an object of $\mathcal{C}$, we will often write $C \in \mathcal{C}$.

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2. The case of affine algebras at the negative and irrational levels

2.1. In this section we will show that the functor of global sections

$$\Gamma : D_\kappa(\text{Gr}_G) - \text{mod} \to \hat{\mathfrak{g}}_\kappa - \text{mod}$$

is exact when $\kappa$ is negative or irrational. A similar result has been proved by Beilinson and Drinfeld in [BD, Theorem 7.15.8], by other methods. The setting of [BD] is slightly different: they consider twisted D-modules on the affine flag variety $\text{Fl}_G = G((t))/I$ instead of $\text{Gr}_G = G((t))/G[[t]]$, where $I \subset G[[t]]$ is the Iwahori subgroup, i.e., the preimage of a fixed Borel subgroup $B \subset G$ under the projection $G[[t]] \to G$. Here is the precise statement of their theorem:

Recall that for any affine weight $\lambda = (\lambda, 2\hbar \cdot c)$ (where $\lambda$ is a weight of $\mathfrak{g}$, $c \in \mathbb{C}$ and $\hat{\hbar}$ is the dual Coxeter number), we can consider the corresponding category $D_\lambda(\text{Fl}_G) - \text{mod}$, of right $\hat{\lambda}$-twisted D-modules on $\text{Fl}_G$. A weight $\hat{\lambda}$ is called anti-dominant if the corresponding Verma module $M(\hat{\lambda})$ over $\hat{\mathfrak{g}}_\kappa$ (where $\kappa = c \cdot \kappa_K(d)$) is irreducible. According to a theorem of Kac and Kazhdan (see [KK]), this condition can be expressed combinatorially as $\langle \hat{\lambda} + \rho_{aff}, \alpha_\text{aff} \rangle \notin \mathbb{Z}^> 0$, where $\alpha_\text{aff}$ runs over the set of all positive affine coroots. We have:

**Theorem 2.2.** If $\hat{\lambda}$ is anti-dominant, then the functor of sections $\Gamma : D_\lambda(\text{Fl}_G) - \text{mod} \to \hat{\mathfrak{g}}_\kappa - \text{mod}$ is exact.

Theorem 2.2 formally implies the exactness statement on $\text{Gr}_G$ (i.e., Theorem 2.1 below) only for $\kappa = c \cdot \kappa_K(d)$ with $c$ either irrational, or $c + \frac{1}{2} < -1 + \frac{1}{2\hbar}$; so our exactness result is slightly sharper than that of [BD]. The proof of Theorem 2.2 given below can be extended in a rather straightforward way to reprove Theorem 2.1. In contrast, in the case of the critical level, it is essential that we consider D-modules on $\text{Gr}_G$ and not on $\text{Fl}_G$; in the latter case the naive analogue of the exactness statement is not true.

Finally, note that Theorem 7.15.8 of [BD] contains also the assertion that for $0 \neq \mathcal{F} \in D_\lambda(\text{Fl}_G) - \text{mod}$, then the space of sections $\Gamma(\text{Gr}_G, \mathcal{F})$ is non-zero, implying a similar statement
for $\mathcal{F} \in D_{\kappa}(\Gr_G)$. In Sect. [10] we will reprove this fact as well, by a different method. This proof is the same in the negative and the critical level cases.

2.3. Thus, our goal in this section is to prove the following theorem:

**Theorem 2.4.** The functor $\Gamma : D_{\kappa}(\Gr_G) - \text{mod} \to \hat{g}_{\kappa} - \text{mod}$ is exact when $\kappa$ is negative or irrational.

The starting point of our proof is the following. Recall the chiral algebra $\mathcal{D}_{G,\kappa}$ (on our curve $X$), introduced in [AG]. Let $\mathcal{D}_{G,\kappa} - \text{mod}$ denote the category of chiral $\mathcal{D}_{G,\kappa}$-modules concentrated at a point $x \in X$. In [AG] it was shown that $\mathcal{D}_{G,\kappa} - \text{mod}$ is a substitute for the category of twisted D-modules on the loop group $G((t))$, where $t$ is a formal coordinate on $X$ near $x$.

In particular, we have the forgetful functor

$$\mathcal{D}_{G,\kappa} - \text{mod} \to (\hat{g}_{\kappa} \times \hat{g}_{2\kappa_{\text{crit}} - \kappa}) - \text{mod},$$

where $\hat{g}_{\kappa} - \text{mod}$ (resp., $\hat{g}_{2\kappa_{\text{crit}} - \kappa} - \text{mod}$) is the category of representations of the affine algebra at the level $\kappa$ (resp., $2\kappa_{\text{crit}} - \kappa$). This functor corresponds to the action of the Lie algebra $g((t))$ on $G((t))$ by left and right translations. In what follows, for a module $M \in \mathcal{D}_{G,\kappa} - \text{mod}$, we will refer to the corresponding actions of $\hat{g}_{\kappa}$ and $\hat{g}_{2\kappa_{\text{crit}} - \kappa}$ on it as “left” and “right”, respectively.

Let $\hat{G}_x \simeq \mathbb{C}[\![t]\!]$ be the completed local ring at $x$. Consider the subalgebra $g((\hat{G}_x)) \subset \hat{g}_{2\kappa_{\text{crit}} - \kappa}$. Let $\hat{g}_{2\kappa_{\text{crit}} - \kappa} - \text{mod}^{G((\hat{G}_x))}$ be the subcategory of $\hat{g}_{2\kappa_{\text{crit}} - \kappa} - \text{mod}$ whose objects are the $\hat{g}_{2\kappa_{\text{crit}} - \kappa}$-modules, on which the action of $g((\hat{G}_x))$ may be exponentiated to an action of the corresponding group $G((\hat{G}_x))$. Let $\mathcal{D}_{G,\kappa} - \text{mod}^{G((\hat{G}_x))}$ denote the full subcategory of $\mathcal{D}_{G,\kappa} - \text{mod}$ whose objects belong to $\hat{g}_{2\kappa_{\text{crit}} - \kappa} - \text{mod}^{G((\hat{G}_x))}$ under the right action of $\hat{g}_{2\kappa_{\text{crit}} - \kappa} - \text{mod}$.

The following result has been established in [AG]:

**Theorem 2.5.** There exists a canonical equivalence of categories

$$D_{\kappa}(\Gr_G) - \text{mod} \simeq \mathcal{D}_{G,\kappa} - \text{mod}^{G((\hat{G}_x))}.$$ 

If $\mathcal{F}$ is an object of $D_{\kappa}(\Gr_G) - \text{mod}$, and $M_{\mathcal{F}}$ the corresponding object of $\mathcal{D}_{G,\kappa} - \text{mod}^{G((\hat{G}_x))}$, then the $\hat{g}_{\kappa}$-module $\Gamma(\Gr_G, \mathcal{F})$ identifies with $(M_{\mathcal{F}})^{g((\hat{G}_x))}$, the space of invariants in $M_{\mathcal{F}}$ with respect to the Lie subalgebra $g((\hat{G}_x)) \subset \hat{g}_{2\kappa_{\text{crit}} - \kappa}$ under the right action.

2.6. To prove the exactness of the functor $\Gamma : D_{\kappa}(\Gr_G) - \text{mod} \to \hat{g}_{\kappa} - \text{mod}$, for negative or irrational $\kappa$, we compose it with the tautological forgetful functor $\hat{g}_{\kappa} - \text{mod} \to \text{Vect}$. By Theorem 2.4, this composition can be rewritten as

$$\mathcal{D}_{G,\kappa} - \text{mod}^{G((\hat{G}_x))} \to \hat{g}_{2\kappa_{\text{crit}} - \kappa} - \text{mod}^{G((\hat{G}_x))} \to \text{Vect},$$

where the first arrow is the forgetful functor, and the second arrow is $M \mapsto M^{g((\hat{G}_x))}$.

For an arbitrary level $\kappa'$, let $V_{\mathfrak{g},\kappa'}$ be the vacuum Weyl module, i.e.,

$$V_{\mathfrak{g},\kappa'} \simeq \text{Ind}_{\mathfrak{g}/\hat{G}_x}^{\mathfrak{g}}(1) \simeq \mathbb{C}[\mathfrak{g}(\hat{G}_x) \otimes \mathbb{C}1] \simeq \mathbb{C},$$

where $\mathfrak{g}(\hat{G}_x)$ acts on $\mathbb{C}$ by zero and $1$ acts as the identity. Tautologically, for any $M \in \hat{g}_{\kappa} - \text{mod}$, we have:

\begin{equation}
\text{Hom}_{\hat{g}_{\kappa}}(V_{\mathfrak{g},\kappa'}, M) \simeq M^{g((\hat{G}_x))}.
\end{equation}

Moreover, $V_{\mathfrak{g},\kappa'}$ is $G((\hat{G}_x))$-integrable, i.e., belongs to $\hat{g}_{\kappa'} - \text{mod}^{G((\hat{G}_x))}$. 

Observe that the condition that $\kappa$ is negative or irrational is equivalent to $\kappa' := 2\kappa_{\text{crit}} - \kappa$ being positive or irrational. Therefore, to prove the exactness of $\Gamma$, it is enough to establish the following:

**Proposition 2.7.** If $\kappa'$ is positive or irrational, the module $\mathcal{V}_{g,\kappa'}$ is projective in $\hat{g}_{\kappa'} - \text{mod} G(\hat{O}_x)$.

This proposition is well-known, and the proof is based on considering eigenvalues of the Segal-Sugawara operator $L_0$. We include the proof for completeness.

**Proof.** Recall that for every non-critical value of $\kappa'$, the vector space underlying every object $M \in \hat{g}_{\kappa'} - \text{mod}$ carries a canonical endomorphism $L_0$ obtained via the Segal-Sugawara construction, such that the action of $\hat{g}_{\kappa'}$ commutes with $L_0$ in the following way:

$$[L_0, x \otimes t^n] = -n \cdot x \otimes t^n, \quad x \in \mathfrak{g}, n \in \mathbb{Z}. \quad (2.2)$$

Explicitly, let $\{x^a, x_a\}$ be bases in $\mathfrak{g}$, dual with respect to $\kappa_{Kd}$. The operator

$$S_0 = \sum_a x^a \cdot x_a + 2 \sum_a \sum_{n>0} x^a \otimes t^{-n} \cdot x_a \otimes t^n \quad (2.3)$$

is well-defined on every object of $\hat{g}_{\kappa'} - \text{mod}$, and it has the following commutation relation with elements of $\hat{g}_{\kappa'}$:

$$[S_0, x \otimes t^n] = -(2c' + 1) \cdot n \cdot x \otimes t^n, \quad x \in \mathfrak{g}, n \in \mathbb{Z}, \quad (2.4)$$

where $c'$ is such that $\kappa' = c' \cdot \kappa_{Kd}$. Therefore, for $c' \neq -\frac{1}{2}$, the operator $L_0 := \frac{1}{2c' + 1} \cdot S_0$ has the required properties.

For an integral dominant weight $\lambda$ of $\mathfrak{g}$, let $V^\lambda$ be the finite-dimensional irreducible $\mathfrak{g}$-module with highest weight $\lambda$ and $\mathcal{V}^\lambda_{g,\kappa'}$ the corresponding Weyl module over $\hat{g}_{\kappa'}$.

$$\mathcal{V}^\lambda_{g,\kappa'} = \text{Ind}^{\hat{g}_{\kappa'}}_{\hat{g}(\hat{O}_x)} \otimes \mathcal{C}_1 (V^\lambda),$$

where $\mathfrak{g}(\hat{O}_x)$ acts on $V^\lambda$ through the homomorphism $\mathfrak{g}(\hat{O}_x) \to \mathfrak{g}$ and $1$ acts as the identity. Then we find from formula (2.3) that $L_0$ acts on the subspace $V^\lambda \subset \mathcal{V}^\lambda_{g,\kappa'}$ by the scalar $\frac{c(\lambda)}{2c' + 1}$, where $C_{\mathfrak{g}}(\lambda)$ is the scalar by which the Casimir element $\sum_a x^a \cdot x_a$ of $U(\mathfrak{g})$ acts on $V^\lambda$. Note that $C_{\mathfrak{g}}(\lambda)$ is a non-negative rational number for any dominant integral weight $\lambda$, and $C_{\mathfrak{g}}(\lambda) \neq 0$ if $\lambda \neq 0$.

Since $\mathcal{V}^\lambda_{g,\kappa'}$ is generated from $V^\lambda$ by the elements $x \otimes t^n \in \hat{g}_{\kappa'}$, $n < 0$, we obtain that the action of $L_0$ on $\mathcal{V}^\lambda_{g,\kappa'}$ is semi-simple. Moreover, since every object $M \in \hat{g}_{\kappa'} - \text{mod} G(\hat{O}_x)$ has a filtration whose subquotients are quotients of the $\mathcal{V}^\lambda_{g,\kappa'}$’s, the action of $L_0$ on any such $M$ is locally-finite.

Suppose now that we have an extension

$$0 \to M \to \tilde{M} \to \mathcal{V}^\lambda_{g,\kappa'} \to 0 \quad (2.5)$$

in $\hat{g}_{\kappa'} - \text{mod} G(\hat{O}_x)$. Let $\tilde{v}^0 \in \tilde{M}$ be a lift to $\tilde{M}$ of the generating vector $v^0 \in \mathcal{V}^\lambda_{g,\kappa'}$. Without loss of generality we may assume that $\tilde{v}^0$ has the same generalized eigenvalue as $v^0$, i.e., 0, with respect to the action of $L_0$. It is sufficient to show that then $\tilde{v}^0$ belongs to $(\tilde{M}) \hat{g} \otimes \mathcal{C}[t]$. Indeed, if this is so, then $\tilde{v}^0$ is annihilated by the entire Lie subalgebra $\mathfrak{g}(\hat{O}_x)$, due to the eigenvalue condition, which would mean that the extension (2.5) splits.
Suppose that this is not the case, i.e., that \( \bar{v}^0 \) is not annihilated by \( g \otimes tC[[t]] \). Then we can find a sequence of elements \( x_i \otimes t^{n_i} \in g \otimes tC[[t]] \), which we can assume to be homogeneous, automatically of negative degrees with respect to \( L_0 \), such that the vector

\[
w = x_1 \otimes t^{n_1} \cdot \ldots \cdot x_k \otimes t^{n_k} \cdot \bar{v}^0 \in M
\]

is non-zero and is annihilated by \( g \otimes tC[[t]] \). But then, on the one hand, the eigenvalue of \( L_0 \) on \( w \) is \( \deg(x_1 \otimes t^{n_1}) + \ldots + \deg(x_k \otimes t^{n_k}) = -(n_1 + \ldots + n_k) \in \mathbb{Z}^< 0 \), but on the other hand, it must be of the form \( \frac{C_2(\lambda)}{\epsilon - z} \), which is not in \( \mathbb{Q}^< 0 \), by our assumption.

\[\square\]

3. CENTER OF THE KAC-MOODY CHIRAL ALGEBRA AT THE CRITICAL LEVEL

3.1. Let \( \mathcal{A} \) be a unital chiral algebra on \( X \). In what follows we will work with a fixed point \( x \in X \) and denote by \( \mathcal{A} \mod \) the category of chiral \( \mathcal{A} \)-modules, supported at \( x \).

Recall that the center of \( \mathcal{A} \), denoted by \( z(\mathcal{A}) \), is by definition the maximal \( \mathcal{D} \)-submodule of \( \mathcal{A} \) for which the Lie-* bracket \( z(\mathcal{A}) \otimes \mathcal{A} \to \Delta(\mathcal{A}) \) vanishes. It is easy to see that \( z(\mathcal{A}) \) is a commutative chiral subalgebra of \( \mathcal{A} \). For example, the unit \( \omega_X \hookrightarrow \mathcal{A} \) is always contained in \( z(\mathcal{A}) \).

Let \( \mathcal{A}_{g,\kappa} \) be the chiral universal enveloping algebra of the Lie-* algebra \( L_{g,\kappa} := g \otimes D_X \otimes \omega_X \) at the level \( \kappa \) (modulo the relation equating the two embeddings of \( \omega_X \)). We have the basic equivalence of categories:

\[ A_{g,\kappa} \mod \simeq z_{g,\kappa} \mod. \]

It is well-known that when \( \kappa \neq \kappa_{\text{crit}} \), the inclusion \( \omega_X \hookrightarrow z(\mathcal{A}_{g,\kappa}) \) is an isomorphism. Let us denote by \( \mathcal{A}_{\kappa} \) the commutative chiral algebra \( z(\mathcal{A}_{g,\kappa}) \). In Theorem 3.4 below we will recall the description of \( \mathcal{A}_{\kappa} \) obtained in [PF].

Let \( \mathcal{A}_{g,\kappa} \) be the fiber of \( \mathcal{A}_{\kappa} \) at \( x \); this is a commutative algebra. We have the natural maps

\[ \mathfrak{z}_{g,\kappa} \rightarrow (\mathcal{V}_{g,\kappa})^g(\mathfrak{g}) \longrightarrow \text{End}_{\mathcal{V}_{g,\kappa}}(\mathcal{V}_{g,\kappa}), \]

where the left arrow is obtained from the definition of the center of a chiral algebra, and the right arrow assigns to \( e \in \text{End}_{\mathcal{V}_{g,\kappa}}(\mathcal{V}_{g,\kappa}) \) the vector \( e \cdot v^0 \), where \( v^0 \) is the canonical generator of \( \mathcal{V}_{g,\kappa} \).

The resulting homomorphism of algebras \( \mathcal{A}_{g,\kappa} \rightarrow \text{End}_{\mathcal{V}_{g,\kappa}}(\mathcal{V}_{g,\kappa}) \) is an isomorphism. In fact, for any chiral algebra \( \mathcal{A} \), its center \( z(\mathcal{A}) \) identifies with the \( \mathcal{D} \)-module of endomorphisms of \( \mathcal{A} \) regarded as a chiral \( \mathcal{A} \)-module. At the level of fibers, we have a map in one direction \( z(\mathcal{A})_x \rightarrow \text{End}_{\mathcal{A} \mod}^{\text{discrete}}(\mathcal{A}_x) \). This map is an isomorphism if a certain flatness condition is satisfied. This condition is always satisfied if \( \mathcal{A} \) is “universal”, i.e., comes from a quasi-conformal vertex algebra, which is the case of \( \mathcal{A}_{g,\kappa} \).

3.2. For a chiral algebra \( \mathcal{A} \), let \( \mathcal{A}_x \) be the canonical topological associative algebra attached to the point \( x \), see [CHA], Sect. 3.6.2. By definition, the category \( \mathcal{A} \mod \) endowed with the tautological forgetful functor to the category of vector spaces, is equivalent to the category of discrete continuous \( \mathcal{A}_x \)-modules, denoted \( \mathcal{A}_x \mod \).

For example, when \( \mathcal{A} = A_{g,\kappa} \), the corresponding algebra \( \mathcal{A}_{g,\kappa,x} \) identifies with the completed universal enveloping algebra of \( \hat{g}_\kappa \) modulo the relation \( 1 = 1 \). We denote this algebra by \( U'(\hat{g}_\kappa) \).

When \( \mathcal{A} = \mathcal{B} \) is commutative, the algebra \( \hat{\mathcal{B}}_x \) is commutative as well, see [CHA], Sects. 3.6.6 and 2.4.8. In fact, \( \hat{\mathcal{B}}_x \) can be naturally represented as \( \lim_{\leftarrow} \mathcal{B}_x^i \), where \( \mathcal{B}_x^i \) are subalgebras of \( \mathcal{B} \), such that \( \mathcal{B}_x^i|_{X-x} \simeq \mathcal{B}|_{X-x} \). In particular, we have a surjective homomorphism \( \hat{\mathcal{B}}_x \rightarrow \mathcal{B}_x \).
the subcategory \( \mathcal{B}_x - \text{mod} \subset \hat{\mathcal{B}}_x - \text{mod} \) is the full subcategory of \( \mathcal{B} - \text{mod} \), whose objects are central \( \mathcal{B} \)-modules, supported at \( x \in X \). (Recall that a \( \mathcal{B} \)-module \( M \) is called central if the action map \( j_!^*(\mathcal{B} \otimes M) \to \Delta_!^!(\mathcal{B}) \) comes from a map \( \mathcal{B} \otimes M \to M \), i.e., factors through \( j_!^*(\mathcal{B} \otimes M) \to \Delta_!(\mathcal{B} \otimes M) \).

We will view \( \text{Spec}(\hat{\mathcal{B}}_x) \) as an ind-scheme \( \lim_{\longrightarrow} \text{Spec}(\mathcal{B}_x^i) \); we have a closed embedding \( \text{Spec}(\mathcal{B}_x) \hookrightarrow \text{Spec}(\hat{\mathcal{B}}_x) \).

By taking \( \mathcal{B} = \mathfrak{g}_0 \), we obtain a topological commutative algebra \( \mathfrak{j}_{\mathfrak{g}_0,x} \), which we will also denote by \( \mathfrak{j}_{\mathfrak{g},x} \). The corresponding map \( \text{Spec}(\mathfrak{j}_{\mathfrak{g}_0,x}) \hookrightarrow \text{Spec}(\mathfrak{j}_{\mathfrak{g},x}) \) will be denoted by \( i \).

For any chiral algebra \( \mathcal{A} \) we have a homomorphism

\[
\mathfrak{j}(\mathcal{A})_x : \text{Spec}(\mathcal{A}) \to Z(\mathcal{A}_x),
\]

where \( Z(\mathcal{A}_x) \) is the center of \( \mathcal{A}_x \). We do not know whether this map is always an isomorphism, but can show that it is an isomorphism for \( \mathcal{A} = A_{\mathfrak{g}, \text{crit}} \), using the description of \( \mathfrak{j}_{\mathfrak{g}} \), given by Theorem 3.4.1 below (see [BD], Theorem 3.7.7). In other words, \( \mathfrak{j}_{\mathfrak{g},x} \) maps isomorphically to the center of \( U'(\mathfrak{g}_{\text{crit}}) \).

3.3. Let us recall the explicit description of \( \mathfrak{j}_{\mathfrak{g}} \) and \( \mathfrak{j}_{\mathfrak{g},x} \) due to [FF, F]. Let \( L^G \) be the algebraic group of adjoint type whose Lie algebra is the Langlands dual to \( \mathfrak{g} \). Denote by \( \text{Op}_{L^G}(\mathcal{D}_x) \) the affine scheme of \( L^G \)-opers on the disc \( \mathcal{D}_x = \text{Spec}(\mathcal{O}_x) \). These are triples \( (\mathcal{F}, \mathcal{F}_B, \nabla) \), where \( \mathcal{F} \) is a \( L^G \)-torsor over \( \mathcal{D}_x \), \( \mathcal{F}_B \) is its reduction to a fixed Borel subgroup \( L^B \subset L^G \) and \( \nabla \) is a connection on \( \mathcal{F} \) (automatically flat) such that \( \mathcal{F}_B \) and \( \nabla \) are in a special relative position (see, e.g., [EF] for details).

There exists an affine \( D_X \)-scheme \( J(\text{Op}_{L^G}(X)) \) of jets of opers on \( X \), whose fiber at \( x \in X \) is \( \text{Op}_{L^G}(\mathcal{D}_x) \) (see [BD], Sect. 3.3.3), and so the corresponding sheaf of algebras of functions \( \text{Fun}(J(\text{Op}_{L^G}(X))) \) on \( X \) is a commutative chiral algebra. (In what follows, \( \text{Fun}(\mathcal{Y}) \) stands for the ring of regular functions on a scheme \( \mathcal{Y} \).)

The canonical topological algebra associated to \( \text{Fun}(J(\text{Op}_{L^G}(X))) \) at the point \( x \) is nothing but the topological algebra of functions on the ind-affine space \( \text{Op}_{L^G}(\mathcal{D}_x^\times) \) of \( L^G \)-opers on the punctured disc \( \mathcal{D}_x^\times = \text{Spec}(\mathcal{K}_x) \), where \( \mathcal{K}_x \) is the field of fractions of \( \mathcal{O}_x \). The following was established in [FF, F].

**Theorem 3.4.**

1. There exists a canonical isomorphism of \( D_X \)-algebras

\[
\mathfrak{j}_{\mathfrak{g}} \cong \text{Fun}(J(\text{Op}_{L^G}(X))).
\]

In particular, we have an isomorphism of commutative algebras \( \mathfrak{j}_{\mathfrak{g},x} \cong \text{Fun}(\text{Op}_{L^G}(\mathcal{D}_x)) \) and of commutative topological algebras \( \mathfrak{j}_{\mathfrak{g},x} \cong \text{Fun}(\text{Op}_{L^G}(\mathcal{D}_x^\times)) \).

2. On the associated graded level, we have a commutative diagram of isomorphisms:

\[
\begin{array}{c}
\text{gr}(\mathfrak{j}_{\mathfrak{g},x}) \\
\downarrow \\
\text{Fun} \left( \left( \mathfrak{g}^* \times_{\mathbb{G}_m} \Gamma(\mathcal{D}_x, \Omega_X) \right)^{G(\mathcal{O}_x)} \right)
\end{array} \leftarrow \begin{array}{c}
\text{gr} \left( \text{Fun}(\text{Op}_{L^G}(\mathcal{D}_x)) \right) \\
\downarrow \\
\text{Fun} \left( \left( L^G/\mathcal{L}^G \right) \times_{\mathbb{G}_m} \Gamma(\mathcal{D}_x, \Omega_X) \right)
\end{array},
\]

where \( L^G/\mathcal{L}^G = \text{Spec}(\text{Fun}(L^G/\mathcal{L}^G)) \).
Note that in the lower left corner of the above commutative diagram we have used the identification $\text{gr}(\mathcal{V}_{B,crit}) \simeq \text{Sym} (\mathfrak{g} \otimes (\hat{\mathcal{K}}_x / \hat{\mathcal{O}}_x)) \simeq \text{Fun} (g^* \times_{G_m} \Gamma(D_x, \Omega_X))$, and

$$g^*/G \simeq \mathfrak{h}^*/W \simeq \mathcal{L}_{\mathfrak{h}} \mathcal{L}_{G}.$$ 

3.5. To proceed we need to recall some more material from [CHA] about commutative $D$-algebras (which, according to our conventions, we do not distinguish them from commutative chiral algebras).

If $B$ is a commutative $D_X$-algebra, consider the $B$-module $\Omega^1(B)$ of (relative with respect to $X$) differentials on $B$, i.e., $\Omega^1(B) \simeq I_B/I_B^2$, where $I_B$ is the kernel of the product $\mathcal{B} \otimes B \to B$.

From now on we will assume that $B$ is finitely generated as a $D_X$-algebra; in this case $\Omega^1(B)$ is finitely generated as a $B \otimes D_X$-module.

Recall that geometric points of the scheme $\text{Spec}(B_x)$ (resp., of the ind-scheme $\text{Spec} (\hat{B}_x)$) are the same as horizontal sections of $\text{Spec}(B)$ over the formal disc $D_x$ (resp., the formal punctured disc $D^*_x$), see [CHA], Sect. 2.4.9. Let us explain the geometric meaning of $\Omega^1(B)$ in terms of these identifications.

Let $z$ be a point of $\text{Spec}(B_x)$, corresponding to a horizontal section $\phi_z : \hat{\mathcal{O}}_x \to B_x$. Evidently, we have: $\phi^*_z(\Omega^1(B))_z \simeq T^*_z(\text{Spec}(B_x))$, where $T^*_z$ denotes the cotangent space at $z$.

From the definition of $\hat{B}_x$ we obtain a map

$$H^0_{DR}(D^*_x, \phi^*_z(\Omega^1(B))) \to T^*_z(\text{Spec}(\hat{B}_x)).$$

(Since the $D$-module $\phi^*_z(\Omega^1(B))$ on $D_x$ is finitely generated, its de Rham cohomology over the formal and formal punctures disc makes obvious sense.) One can show that the map of (3.1) is actually an isomorphism.

From the short exact sequence

$$0 \to H^0_{DR}(D_x, \phi^*_z(\Omega^1(B))) \to H^0_{DR}(D^*_x, \phi^*_z(\Omega^1(B))) \to \phi^*_z(\Omega^1(B)) \to 0,$$

we obtain also an identification

$$H^0_{DR}(D_x, \phi^*_z(\Omega^1(B))) \simeq N^*_z(B_x),$$

where $N^*_z(B_x)$ denotes the conormal to $\text{Spec}(B_x)$ inside $\text{Spec}(\hat{B}_x)$ at the point $z$.

Assume now that $B$ is smooth (see [CHA], Sect. 2.3.15 for the definition of smoothness). In this case $\Omega^1(B)$ is a finitely generated projective $B \otimes D_X$-module.

Consider the dual of $\Omega^1(B)$, i.e.,

$$\Theta(B) := \text{Hom}_{B \otimes D_X}(\Omega^1(B), B \otimes D_X).$$

This is a central $B$-module, called the tangent module to $B$. Moreover, $\Theta(B)$ carries a canonical structure of Lie-* algebroid over $B$ (see below). Evidently, $\Theta(B)$ is also projective and finitely generated as a $B \otimes D_X$-module.

By dualizing the members of the short exact sequence (3.2), we obtain the identifications (cf. [CHA], Sect. 2.5.21):

$$H^0_{DR}(D_x, \phi^*_z(\Theta(B))) \simeq T^*_z(\text{Spec}(B_x)), \quad H^0_{DR}(D^*_x, \phi^*_z(\Theta(B))) \simeq T^*_z(\text{Spec}(\hat{B}_x)),$$

and $\phi^*_z(\Theta(B))_z \simeq N^*_z(B_x)$.

The next definition will be needed in Sect. 6. Let $I$ denote the kernel $\hat{B}_x \to B_x$. The quotient $I/I^2$ is a topological module over $B_x$, and the normal bundle, $N(B_x)$, to $\text{Spec}(B_x)$ inside $\text{Spec}(\hat{B}_x)$ can always be defined as the group ind-scheme $\text{Spec}(\text{Sym}_{B_x}(I/I^2))$. Let now $\mathcal{E} \subset N(B_x)$ be a group ind-subscheme, and let $\mathcal{E}^\perp$ be its annihilator in $I/I^2$. 


We introduce the subcategory \( \hat{\mathcal{B}}_x \text{– mod}_E \) inside the category \( \hat{\mathcal{B}}_x \text{– mod} \) of all chiral \( \mathcal{B} \)-modules supported at \( x \) by imposing the following two conditions:

1. We require that a module \( M \), viewed as a quasicoherent sheaf on \( \text{Spec}(\hat{\mathcal{B}}_x) \), is supported on the formal neighborhood of \( \text{Spec}(\mathcal{B}_x) \). In particular, \( M \) acquires a canonical increasing filtration \( M = \bigcup \mathcal{M}_i \), where \( \mathcal{M}_i \subset M \) is the submodule consisting of sections annihilated by \( \mathcal{P}^i \).

2. We require that the natural map \( J/J^2 \otimes \mathcal{M}_{i+1}/\mathcal{M}_i \to \mathcal{M}_i/\mathcal{M}_{i-1} \) vanish on \( E^\perp \subset J/J^2 \).

Note that the category \( \hat{\mathcal{B}}_x \text{– mod}_E \) is in general not abelian.

3.6. Let us now recall the notion of \( \text{Lie-*} \) algebroid over a commutative \( D_X \) algebra \( \mathcal{B} \) (cf. \[CHA\], Sect. 2.5).

Let \( L \) be a central \( \mathcal{B} \)-module. A structure of a \textit{Lie-*} algebroid over \( \mathcal{B} \) on \( L \) is the data of a \textit{Lie-*} bracket \( L \otimes L \to \Delta_1(L) \) and an action map \( L \otimes \mathcal{B} \to \Delta_1(\mathcal{B}) \), which satisfy the natural compatibility conditions given in \[CHA\], Sect. 1.4.11 and 2.5.16.

If \( \mathcal{B} \) is smooth, then \( \Theta(\mathcal{B}) \) is well-defined, and it carries a canonical structure of \textit{Lie-*} algebroid over \( \mathcal{B} \). It is universal in the sense that for any \textit{Lie-*} algebroid \( L \), its action on \( \mathcal{B} \) factors through a canonical map of \textit{Lie-*} algebroids \( \varpi : L \to \Theta(\mathcal{B}) \), called the anchor map.

Recall now that a structure on \( \mathcal{B} \) of chiral-Poisson (or, coisson, in the terminology of \[CHA\]) algebra is a \textit{Lie-*} bracket (called chiral-Poisson bracket) \( \mathcal{B} \otimes \mathcal{B} \to \Delta_1(\mathcal{B}) \), satisfying the Leibniz rule with respect to the multiplication on \( \mathcal{B} \) (cf. \[CHA\], Sect. 1.4.18 and 2.6.).

If \( \mathcal{B} \) is a chiral-Poisson algebra, \( \Omega^1(\mathcal{B}) \) acquires a unique structure of \textit{Lie-*} algebroid, such that the de Rham differential \( d : \mathcal{B} \to \Omega^1(\mathcal{B}) \) is a map of \textit{Lie-*} algebras, and the composition

\[
\mathcal{B} \otimes \mathcal{B} \xrightarrow{d \text{id}} \Omega^1(\mathcal{B}) \otimes \mathcal{B} \to \Delta_1(\mathcal{B})
\]

coincides with the chiral-Poisson bracket.

Following \[CHA\], Sect. 2.6.6, we call a chiral-Poisson structure on \( \mathcal{B} \) elliptic if (a) \( \mathcal{B} \) is smooth, (b) the anchor map \( \varpi : \Omega^1(\mathcal{B}) \to \Theta(\mathcal{B}) \) is injective, and (c) \( \text{coker}(\varpi) \) is a projective \( \mathcal{B} \)-module of finite rank.

3.7. Finally, let us recall the definition of the chiral-Poisson structure on \( \delta_g \). Consider the flat \( \mathbb{C}[[\hbar]] \)-family of chiral algebras \( A_{g,h} \), corresponding to the pairing \( \kappa_h = \kappa_{\text{crit}} + \hbar \cdot \kappa_0 \), where \( \kappa_0 \) is an arbitrary fixed non-zero inner product.

For two sections \( a, b \in \delta_g \), consider two arbitrary sections \( a_h, b_h \in A_{g,h} \), whose values modulo \( \hbar \) are \( a \) and \( b \) respectively, and consider \( [a_h, b_h] \in \Delta_1(A_{g,h}) \). By assumption, the last expression vanishes modulo \( \hbar \). Therefore the section \( \frac{1}{\hbar}[a_h, b_h] \in \Delta_1(A_{g,h}) \) is well-defined. Moreover, its value mod \( \hbar \) does not depend on the choice of \( a_h \) and \( b_h \).

Therefore we obtain a map

\[
a, b \in \delta_g \mapsto \frac{1}{\hbar}[a_h, b_h] \mod \hbar \in \Delta_1(A_{g,\text{crit}}),
\]

and it is easy to see that its image belongs to \( \Delta(\delta_g) \). Furthermore, it is straightforward to verify that the resulting map \( \delta_g \otimes \delta_g \to \Delta(\delta_g) \) satisfies the axioms of the chiral-Poisson bracket, see \[CHA\], Sect. 2.7.1.

Let us now describe in terms of Theorem 3.4 above the \textit{Lie-*} algebroid \( \Omega^1(\delta_g) \), resulting from the chiral-Poisson structure on \( \delta_g \).

First, recall from \[CHA\], Sect. 2.4.11, that if \( M \) is a central module over a commutative chiral algebra \( \mathcal{B} \), then we can form a topological module, denoted \( \hat{\mathcal{H}}^2(M) \) over \( \mathcal{B} \). Applying this construction to \( \mathcal{B} = \delta_g \) and \( M = \Omega^1(\delta_g) \) we obtain a topological \textit{Lie-*} algebroid \( \mathcal{S}_{\text{crit}} := \hat{\mathcal{H}}^2(\Omega^1(\delta_g)) \) (see \[CHA\], Sect. 2.5.18 for details).
Let now $\mathfrak{F}_x$ be the universal $L_G$-torsor over $\text{Op}_{L_G}(\mathcal{D}_x)$, whose fiber over a given oper $(\mathcal{F}, \mathcal{F}_B, \nabla) \in \text{Op}_{L_G}(\mathcal{D}_x)$ is the $L_G$-torsor of horizontal sections of $\mathcal{F}$, or, equivalently, the fiber of $\mathcal{F}$ at $x \in \mathcal{D}_x$. Let us denote by $L_G \text{Op}$ the corresponding Atiyah algebroid over $\text{Op}_{L_G}(\mathcal{D}_x)$, which by definition consists of $L_G$-invariant vector fields on the total space of $\mathfrak{F}_x$. We have a short exact sequence 

$$0 \rightarrow (Lg)_{\mathfrak{F}_x} \rightarrow L_G \text{Op} \rightarrow T(\text{Op}_{L_G}(\mathcal{D}_x)) \rightarrow 0,$$

where $T(\text{Op}_{L_G}(\mathcal{D}_x))$ denotes the tangent algebroid, and $(Lg)_{\mathfrak{F}_x}$, which is the kernel of the anchor map, is the twist of the adjoint representation by the $L_G$-torsor $\mathfrak{F}_x$.

The following was established in [BD], Theorem 3.6.7 (see also [CHA], Sect. 2.6.8), using the fact that the isomorphism of $D_X$-algebras, given by Theorem 3.4(1), respects the chiral-Poisson structures on both sides, where $J(\text{Op}_{L_G}(X))$ acquires a chiral-Poisson structure by its realization via the Drinfeld-Sokolov reduction.

**Theorem 3.8.**

(1) The chiral-Poisson structure on $\mathfrak{z}_g$ is elliptic.

(2) Under the isomorphism $\text{Spec}(\mathfrak{z}_g, x) \simeq \text{Op}_{L_G}(\mathcal{D}_x)$, the algebroid $\mathfrak{S}_{\text{crit}}$ corresponds to the algebroid $L_G \text{Op}$.

4. The renormalized chiral algebra

4.1. We will now refine the structure of chiral-Poisson algebra on $\mathfrak{z}_g$ and obtain a chiral version of the renormalized universal enveloping algebra at the critical level introduced in [BD].

First, we introduce a Lie-* algebra $A^\#: \mathfrak{z}_g$, which fits in a short exact sequence

$$0 \rightarrow A_{\mathfrak{g}, \text{crit}} \rightarrow A^\#: \mathfrak{z}_g \rightarrow 0.$$

Namely, in the family of chiral algebras $A_{\mathfrak{g}, \hbar}$ consider the following subspace $A^\#: \mathfrak{g}, \hbar$ which contains $A_{\mathfrak{g}, \hbar}$ and is contained in $\frac{1}{\hbar} \cdot A_{\mathfrak{g}, \hbar}$:

$$A^\#: \mathfrak{g}, \hbar = \{ \frac{a}{\hbar} | a \in A_{\mathfrak{g}, \hbar}, a \mod \hbar \in \mathfrak{z}_g \}.$$ 

Define $\mathfrak{A}^\#: \mathfrak{g}$ as $A^\#: \mathfrak{g}, \hbar / \mathfrak{z}_g \cdot A_{\mathfrak{g}, \hbar}$. By repeating the construction of the chiral-Poisson structure on $\mathfrak{z}_g$ from Sect. 3.4 we obtain a Lie-* algebra structure on $\mathfrak{A}^\#: \mathfrak{g}$.

Note that the composition $\mathfrak{z}_g \boxtimes \mathfrak{A}^\#: \mathfrak{g} \rightarrow \mathfrak{A}^\#: \mathfrak{g} \boxtimes \mathfrak{A}^\#: \mathfrak{g} \rightarrow \Delta!(\mathfrak{A}^\#: \mathfrak{g})$ factors as $\mathfrak{z}_g \boxtimes \mathfrak{A}^\#: \mathfrak{g} \rightarrow \mathfrak{z}_g \boxtimes \mathfrak{z}_g \rightarrow \Delta!(\mathfrak{z}_g)$, where the last arrow is chiral-Poisson bracket on $\mathfrak{z}_g$.

**Proposition 4.2.** There exist a unique Lie-* algebroid $A^\#: \mathfrak{g}$ over $\mathfrak{z}_g$, which fits into the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & A_{\mathfrak{g}, \text{crit}} / \mathfrak{z}_g & \longrightarrow & A^\#: \mathfrak{g} & \longrightarrow & \Omega^1(\mathfrak{z}_g) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & A_{\mathfrak{g}, \text{crit}} & \longrightarrow & A^\#: \mathfrak{g} & \longrightarrow & \mathfrak{z}_g & \longrightarrow & 0.
\end{array}
\]

In the above diagram the rows are exact, and the rightmost vertical map is the de Rham differential $\mathfrak{z}_g \rightarrow \Omega^1(\mathfrak{z}_g)$. 

Lemma. Recall the following general construction. Let $L$ be a Lie-* algebra acting on a commutative chiral algebra $B$. Then we can form a central $B$-module $\text{Ind}_B(L) := B \otimes L$, which will carry a natural structure of Lie-* algebroid over $B$. This is analogous to the usual construction in differential geometry, when we have a Lie-* algebra $I$ acting on a manifold $Y$ and we form the algebroid $\Omega_Y \otimes I$.

By taking $B = \mathfrak{g}$ and $L = A^2_{/\mathfrak{g}}$, we thus obtain a Lie-* algebroid $\text{Ind}_{\mathfrak{g}}(A^2_{/\mathfrak{g}})$ on $\mathfrak{g}$. We have a short exact sequence
\[
0 \to \mathfrak{g} \to (A_{\mathfrak{g},\text{crit}}/\mathfrak{g}) \to \text{Ind}_{\mathfrak{g}}(A^2_{/\mathfrak{g}}) \to 0.
\]
To obtain from $\text{Ind}_{\mathfrak{g}}(A^2_{/\mathfrak{g}})$ the desired extension $A_{\mathfrak{g}}$, we need to take the quotient by two kinds of relations. First, we must pass from $\mathfrak{g} \otimes (A_{\mathfrak{g},\text{crit}}/\mathfrak{g})$ to just $A_{\mathfrak{g},\text{crit}}/\mathfrak{g}$, using the structure of $\mathfrak{g}$-module on $A_{\mathfrak{g},\text{crit}}$. Secondly, we must impose the Leibniz rule to pass from the free $\mathfrak{g}$-module $\mathfrak{g} \otimes \mathfrak{g}$ to $\Omega^1(\mathfrak{g})$. We will impose these two relations simultaneously.

Consider the following three maps $A^2_{\mathfrak{g}} \otimes A^2_{\mathfrak{g}} \to \text{Ind}_{\mathfrak{g}}(A^2_{/\mathfrak{g}})$:

1. The first map is the projection $A^2_{\mathfrak{g}} \otimes A^2_{\mathfrak{g}} \to \mathfrak{g} \otimes (A^2_{/\mathfrak{g}})$.
2. The second map is the projection $A^2_{\mathfrak{g}} \otimes A^2_{\mathfrak{g}} \to (A^2_{/\mathfrak{g}}) \otimes \mathfrak{g} \simeq \mathfrak{g} \otimes (A^2_{/\mathfrak{g}})$.
3. To define the third map, note that chiral bracket on $A_{\mathfrak{g},\mathfrak{h}}$, multiplied by $\mathfrak{h}$, induces a map $j_*j^*(A^2_{\mathfrak{g}} \otimes A^2_{\mathfrak{g}}) \to \Delta(A^2_{\mathfrak{g}})$. Composing the latter with the projection $\Delta(A^2_{\mathfrak{g}}) \to \Delta(A^2_{/\mathfrak{g}})$, we obtain a map that vanishes on $A^2_{/\mathfrak{g}} \otimes A^2_{\mathfrak{g}} \subset j_*j^*(A^2_{\mathfrak{g}} \otimes A^2_{\mathfrak{g}})$, thereby giving rise to a map $A^2_{\mathfrak{g}} \otimes A^2_{\mathfrak{g}} \to \mathfrak{g} \otimes (A^2_{/\mathfrak{g}})$.

By taking the linear combination of these three maps, namely $(1)-(2)-(3)$, we obtain a new map $A^2_{\mathfrak{g}} \otimes A^2_{\mathfrak{g}} \to \mathfrak{g} \otimes (A^2_{/\mathfrak{g}})$. We define $A_{\mathfrak{g}}$ as the quotient of $\text{Ind}_{\mathfrak{g}}(A^2_{/\mathfrak{g}})$ by the $\mathfrak{g}$-module, generated by the image of the latter map.

One checks in a straightforward way that the Lie-* bracket on $\text{Ind}_{\mathfrak{g}}(A^2_{/\mathfrak{g}})$ descends to a Lie-* bracket on $A^2_{\mathfrak{g}}$, so that it becomes a Lie-* algebroid over $\mathfrak{g}$. Moreover, by construction, we have a short exact sequence
\[
0 \to (A_{\mathfrak{g},\text{crit}}/\mathfrak{g})' \to A^b_{\mathfrak{g}} \to \Omega^1(\mathfrak{g}) \to 0,
\]
where $(A_{\mathfrak{g},\text{crit}}/\mathfrak{g})'$ is a quotient of $A_{\mathfrak{g},\text{crit}}/\mathfrak{g}$. Let us show that $A_{\mathfrak{g},\text{crit}}/\mathfrak{g} \to (A_{\mathfrak{g},\text{crit}}/\mathfrak{g})'$ is an isomorphism.

Observe that the Lie-* algebra $A^2_{\mathfrak{g}}$ acts on $A_{\mathfrak{g},\text{crit}}$. This action gives rise to an action of the Lie-* algebroid $\text{Ind}_{\mathfrak{g}}(A^2_{/\mathfrak{g}})$ on $A_{\mathfrak{g},\text{crit}}$, which is compatible with the $\mathfrak{g}$-module structure on $A_{\mathfrak{g},\text{crit}}$, and, moreover, it descends to an action of the algebroid $A^b_{\mathfrak{g}}$ on $A_{\mathfrak{g},\text{crit}}$.

The resulting Lie-* action of $A_{\mathfrak{g},\text{crit}}$ on $A_{\mathfrak{g},\text{crit}}$ obtained via
\[
A_{\mathfrak{g},\text{crit}} \to (A_{\mathfrak{g},\text{crit}}/\mathfrak{g})' \hookrightarrow A^b_{\mathfrak{g}}
\]
coincides with the initial Lie-* action of $A_{\mathfrak{g},\text{crit}}$ on itself. By the definition of the center, the kernel of the latter action is exactly $\mathfrak{g}$.

\[\square\]

4.3. Let us now introduce a category of modules over $A^2_{\mathfrak{g}}$, which will be of interest for us.

First, note that the action of $A^2_{\mathfrak{g}}$ on $A_{\mathfrak{g},\text{crit}}$, introduced in the course of the proof of Proposition 4.2, is compatible in the natural sense with the chiral bracket on $A_{\mathfrak{g},\text{crit}}$. 


We define $A^\#_g$–mod to have as objects $M \in A^\#_{g,\text{crit}}$–mod, endowed with an additional action of the Lie-* algebroid $A^\#_g$ (see [CHA], Sect. 2.5.16 and 1.4.12 for the definition of the latter), such that

(a) As a chiral module over $\mathfrak{g}$ (via $\mathfrak{g} \hookrightarrow A^\#_{g,\text{crit}}$), $M$ is central.

(b) The two induced Lie-* actions $(A^\#_{g,\text{crit}}/\mathfrak{g}) \boxtimes M \to \Delta_t(M)$ (one coming from the $A^\#_g$–action, and the other from the $A^\#_{g,\text{crit}}$–action and point (a) above) coincide.

(c) The chiral action of $A^\#_{g,\text{crit}}$ and the Lie-* action of $A^\#_g$ on $M$ are compatible with the Lie-* action of $A^\#_g$ on $A^\#_{g,\text{crit}}$.

One can show that the category $A^\#_g$–mod is tautologically equivalent to the category of (discrete) modules over the renormalized universal enveloping algebra introduced in [BD], Sect. 5.6.1.

For example, it is easy to see that if $M_\hbar$ is a flat $\mathbb{C}[[\hbar]]$-family of chiral $A_{g,\hbar}$-modules such that the chiral $\mathfrak{g}$-module $M := M/\hbar M$ is central, then this $M$ is naturally an object of $A^\#_g$–mod.

4.4. In addition to the notion of a Lie-* algebroid there is also the notion of a chiral Lie algebroid over a commutative chiral algebra $\mathcal{B}$, see [CHA], Sect. 3.9.6. A Lie-* algebra $L$ is called a chiral Lie algebroid over $\mathcal{B}$ if we are given:

(1) An action $L \boxtimes \mathcal{B} \to \Delta_t(\mathcal{B})$ of $L$ as a Lie-* algebra on the commutative chiral algebra $\mathcal{B}$,

(2) A chiral action $j_*\eta^*(\mathcal{B} \boxtimes L) \to \Delta_t(L)$, compatible with the action of $L$ on $\mathcal{B}$ and the bracket on $L$.

(3) A map $\eta : \mathcal{B} \to L$, compatible with both the $\mathcal{B}$- and $L$-actions,

such that the following conditions are satisfied:

(a) The action in (1) is $\mathcal{B}$-linear, in the sense that the two natural maps $j_*j^*(\mathcal{B} \boxtimes L) \boxtimes \mathcal{B} \to \Delta_t(\mathcal{B})$ on $X^3$ coincide,

(b) The map $L \boxtimes \mathcal{B} \xrightarrow{id \times \eta} L \boxtimes L \to \Delta_t(L)$ equals the negative of $\mathcal{B} \boxtimes L \hookrightarrow j_*j^*(\mathcal{B} \boxtimes L) \to \Delta_t(L)$.

Note that if for a chiral Lie algebroid $L$ as above, the data of $\eta$ is zero, we retrieve the notion of Lie-* algebroid. In most examples, however, the map $\eta$ is an injection. In this case, the data of (1) is completely determined by (2) and (3), and condition (a) is superfluous.

It would be interesting to find out whether there exists a chiral Lie algebroid $A^\#_{g,\text{ren}}$ over $\mathfrak{g}$, which is an extension

$$0 \to A^\#_{g,\text{crit}} \to A^\#_{g,\text{ren}} \to \Omega^1(\mathfrak{g}) \to 0,$$

such that the map $A^\#_{g,\text{crit}} \to A^\#_g$ lifts to a map $A^\#_{g,\text{crit}} \to A^\#_{g,\text{ren}}$.

However, we do not know how to construct such an object. Instead, we will construct another chiral Lie algebroid $A^\#_{g,\text{ren},d}$, which is, in some sense, a double of $A^\#_{g,\text{ren}}$. The construction of $A^\#_{g,\text{ren},d}$ below is in terms of generators and relations. In the next section we will give a natural construction of $A^\#_{g,\text{ren},d}$ via chiral differential operators on the group $G$.

Consider the Lie-* algebra $A^\#_{g,d}$ equal to

$$(A^\#_{g,x} \times A^\#_{g,y}) \times \mathfrak{g} \times \mathfrak{g},$$

where the map $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ is the anti-diagonal, i.e., $(id, -id)$. It fits into a short exact sequence

$$0 \to A^\#_{g,\text{crit}} \times A^\#_{g,\text{crit}} \to A^\#_{g,d} \to \mathfrak{g} \to 0.$$
Proposition 4.5. There exist a unique chiral algebroid $\mathcal{A}^{ren,d}_g$ over $\mathfrak{z}_g$, which fits into the following commutative diagram with exact rows:

$$
\begin{array}{c}
0 \longrightarrow A_{g,crit} \times A_{g,crit} / \mathfrak{z}_g \longrightarrow A_g^{ren,d} \longrightarrow \Omega^1(\mathfrak{z}_g) \longrightarrow 0 \\
0 \longrightarrow A_{g,crit} \times A_{g,crit} \longrightarrow A_{g,crit}^{d,d} \longrightarrow \mathfrak{z}_g \longrightarrow 0,
\end{array}
$$

where $\mathfrak{z}_g \rightarrow A_{g,crit} \times A_{g,crit}$ is the anti-diagonal embedding.

Proof. Let $L$ be a Lie-* algebra acting on a commutative chiral algebra $\mathcal{B}$, as in the proof of Proposition 1.2. Then, following [CHA], Sect. 3.9.9, one constructs a chiral Lie algebroid $\text{Ind}_\mathcal{B}(L)$, which fits into a short exact sequence

$$0 \rightarrow \mathcal{B} \rightarrow \text{Ind}_\mathcal{B}(L) \rightarrow \text{Ind}_\mathcal{B}(L) \rightarrow 0.$$

Indeed, consider the D-modules $j_*j^*(\mathcal{B} \boxtimes L)$ and $\Delta_!(\mathcal{B})$ on $X \times X$. We have the maps

$$j_*j^*(\mathcal{B} \boxtimes L) \leftarrow \mathcal{B} \boxtimes L \rightarrow \Delta_!(\mathcal{B}),$$

where the left arrow is the natural inclusion, and the right arrow is the negative of the Lie-* action. Then the quotient $(j_*j^*(\mathcal{B} \boxtimes L) \oplus \Delta_!(\mathcal{B}))/\mathcal{B} \boxtimes L$ is supported on the diagonal, and therefore corresponds to a D-module on $X$, which is by definition our $\text{Ind}_\mathcal{B}(L)$. By construction, we have the inclusions $\eta: \mathcal{B} \rightarrow \text{Ind}_\mathcal{B}(L)$ and $L \rightarrow \text{Ind}_\mathcal{B}(L)$, and a chiral action $j_*j^*(\mathcal{B} \boxtimes L) \rightarrow \Delta_!(\text{Ind}_\mathcal{B}(L))$. It is a straightforward verification to show that these data extend uniquely to a Lie-* algebra structure on $\text{Ind}_\mathcal{B}(L)$ and a chiral action of $\mathcal{B}$ on $\text{Ind}_\mathcal{B}(L)$, which satisfy the conditions of chiral Lie algebroid.

Let us view $A_{g,crit}^{d,d} \mathfrak{z}_g$ as a Lie-* algebra, which acts on $\mathfrak{z}_g$ via $A_{g,crit}^{d,d} \mathfrak{z}_g$ and the chiral-Poisson bracket on $\mathfrak{z}_g$. Consider the chiral Lie algebroid $\text{Ind}_{\mathfrak{z}_g}(A_{g,crit}^{d,d})$. As in the case of $A_{g,crit}^d$ (see the proof of Proposition 1.2), to obtain from $\text{Ind}_{\mathfrak{z}_g}(A_{g,crit}^{d,d})$ the desired chiral algebroid $A_{g,crit}^{ren,d}$, we must take the quotient by some additional relations.

The first set of relations is that we must identify the three copies of $\mathfrak{z}_g$ inside $\text{Ind}_{\mathfrak{z}_g}(A_{g,crit}^{d,d})$. One copy is the image of the canonical embedding $\mathfrak{z}_g \hookrightarrow \text{Ind}_{\mathfrak{z}_g}(A_{g,crit}^{d,d})$ coming from the definition of the induced algebroid. The other two copies come from $\mathfrak{z}_g \times \mathfrak{z}_g \hookrightarrow A_{g,crit}^{d,d} \hookrightarrow \text{Ind}_{\mathfrak{z}_g}(A_{g,crit}^{d,d})$. When we identify them, we obtain a new chiral algebroid over $\mathfrak{z}_g$ which we denote by $\text{Ind}_{\mathfrak{z}_g}(A_{g,crit}^{d,d})$.

The second set of relations is similar to what we had in the case of $A_{g,crit}^d$: they amount to killing the chiral $\mathfrak{z}_g$-submodule generated by the image of a certain map $A_{g,crit}^{d,d} \boxtimes A_{g,crit}^{d,d} \rightarrow \text{Ind}_{\mathfrak{z}_g}(A_{g,crit}^{d,d})$. To construct this map, we consider three morphisms from the D-module $j_*j^*(A_{g,crit}^{d,d} \boxtimes A_{g,crit}^{d,d})$ to $\Delta_!(\text{Ind}_{\mathfrak{z}_g}(A_{g,crit}^{d,d}))$:

1. The first map is $j_*j^*(A_{g,crit}^{d,d} \boxtimes A_{g,crit}^{d,d}) \rightarrow j_*j^*(A_{g,crit}^{d,d} \boxtimes A_{g,crit}^{d,d}) \rightarrow \Delta_!(\text{Ind}_{\mathfrak{z}_g}(A_{g,crit}^{d,d}))$, where the first arrow comes from the natural projection $A_{g,crit}^{d,d} \rightarrow \mathfrak{z}_g$.
2. The second map is obtained from the first one by interchanging the roles of the factors in $j_*j^*(A_{g,crit}^{d,d} \boxtimes A_{g,crit}^{d,d})$.
3. To construct the third map, note that the chiral bracket on $A_{g,crit}$ gives rise to a map

$$\hbar \cdot (\cdot, \cdot) - (\cdot, \cdot) : j_*j^*(A_{g,crit}^{d,d} \boxtimes A_{g,crit}^{d,d}) \rightarrow \Delta_!(A_{g,crit}^{d,d}),$$

and we compose it with the canonical map $\Delta_!(A_{g,crit}^{d,d}) \rightarrow \Delta_!(\text{Ind}_{\mathfrak{z}_g}(A_{g,crit}^{d,d}))$. 


Consider the linear combination (1)-(2)-(3) of the above three maps as a new map from $j_\ast j^\ast (\mathcal{A}_g^{x,d} \boxtimes \mathcal{A}_g^{x,d})$ to $\Delta_1(\text{Ind}_{\mathcal{A}_g^{x,d}}(\mathcal{A}_g^{x,d})))$. It is easy to see that the composition $\mathcal{A}_g^{x,d} \boxtimes \mathcal{A}_g^{x,d} \hookrightarrow j_\ast j^\ast (\mathcal{A}_g^{x,d} \boxtimes \mathcal{A}_g^{x,d}) \to \Delta_1(\text{Ind}_{\mathcal{A}_g^{x,d}}(\mathcal{A}_g^{x,d}))) \to \text{Ind}_{\mathcal{A}_g^{x,d}}(\mathcal{A}_g^{x,d})$ vanishes. Thus, we obtain the desired map $\mathcal{A}_g^{x,d} \otimes \mathcal{A}_g^{x,d} \to \text{Ind}_{\mathcal{A}_g^{x,d}}(\mathcal{A}_g^{x,d})$. We define $\mathcal{A}_g^{r.e.n.}$ as the quotient of $\text{Ind}_{\mathcal{A}_g^{x,d}}(\mathcal{A}_g^{x,d})$ by the chiral $\mathcal{A}_g^{x}$-module, generated by the image of this map.

By construction, $\mathcal{A}_g^{r.e.n.}$ is a chiral $\mathcal{A}_g$-module. One readily checks that the Lie-* bracket on $\text{Ind}_{\mathcal{A}_g^{x,d}}(\mathcal{A}_g^{x,d})$ descends to a Lie-* bracket on $\mathcal{A}_g^{r.e.n.}$, such that, together with the map $\mathcal{A}_g \to \text{Ind}_{\mathcal{A}_g^{x,d}}(\mathcal{A}_g^{x,d}) \to \mathcal{A}_g^{r.e.n.}$, these data define on $\mathcal{A}_g^{r.e.n.}$ a structure of chiral Lie algebroid over $\mathcal{A}_g$.

As in the case of $\mathcal{A}_g^b$, we have a short exact sequence
\[0 \to ((\mathcal{A}_g^{x,d} \otimes \mathcal{A}_g^{x,d})/\mathcal{A}_g^x) \to \mathcal{A}_g^{r.e.n.} \to \Omega^1(\mathcal{A}_g^{x,d}) \to 0,
\]where $((\mathcal{A}_g^{x,d} \otimes \mathcal{A}_g^{x,d})/\mathcal{A}_g^x)$ is a certain quotient of $(\mathcal{A}_g^{x,d} \otimes \mathcal{A}_g^{x,d})/\mathcal{A}_g^x$. Let us show that
\[\mathcal{A}_g^{x,d} \otimes \mathcal{A}_g^{x,d}/\mathcal{A}_g^x \rightarrow ((\mathcal{A}_g^{x,d} \otimes \mathcal{A}_g^{x,d})/\mathcal{A}_g^x)
\]is in fact an isomorphism.

Let $\mathcal{A}_g^{x,d}$ be the Lie-* algebroid over $\mathcal{A}_g$ equal to $\mathcal{A}_g^x/\mathcal{A}_g^x$. We have a surjection
\[\mathcal{A}_g^{x,d} \otimes \mathcal{A}_g^{x,d}/\mathcal{A}_g^x \rightarrow \ker \left(\mathcal{A}_g^{x,d} \otimes \mathcal{A}_g^{x,d}/\mathcal{A}_g^x \rightarrow \Omega^1(\mathcal{A}_g^{x,d})\right).
\]As in the case of $\mathcal{A}_g^b$, we show that $\mathcal{A}_g^{x,d}$ acts naturally on the chiral algebra $\mathcal{A}_g^{x,d} \otimes \mathcal{A}_g^{x,d}/\mathcal{A}_g^x$. This implies that the map of \ref{eq:12} is an isomorphism.

Thus, it remains to show that the canonical map $\mathcal{A}_g^x \to \mathcal{A}_g^{r.e.n.}$ is injective. If it were not so, the ideal $\ker(\mathcal{A}_g^x \to \mathcal{A}_g^{r.e.n.})$ would be stable under the chiral-Poisson bracket on $\mathcal{A}_g^x$. However, this is impossible, since the above chiral-Poisson structure is elliptic by Theorem \ref{thm:5.4}(1).

We remark that the isomorphism of \ref{eq:12} can be alternatively deduced from Theorem \ref{thm:5.4} below.

\section{4.6.}

Let $\mathcal{A}_g^{b,d}$ be the Lie-* algebroid introduced in the proof of Proposition \ref{prop:4.5}.

We introduce the category $\mathcal{A}_g^{b,d} \text{-mod}$ in a way analogous to $\mathcal{A}_g^{x,d} \text{-mod}$. Namely, the objects of $\mathcal{A}_g^{b,d} \text{-mod}$ are modules over the chiral algebra $\mathcal{A}_g^{x,d} \otimes \mathcal{A}_g^{x,d}/\mathcal{A}_g^x$ equipped with an extra Lie-* action of the Lie-* algebroid $\mathcal{A}_g^{b,d}$ such that the conditions, analogous to (a), (b) and (c) in the definition of $\mathcal{A}_g^{x,d} \text{-mod}$, hold.

Next, we will introduce an appropriate category of chiral modules over $\mathcal{A}_g^{r.e.n.}$. First, recall from \cite{CHA}, Sect. 3.9.24, the notion of chiral module over a chiral Lie algebroid.

If $L$ is a chiral algebroid over a commutative $D_X$-algebra $\mathcal{B}$, there exists a canonical chiral algebra $U(\mathcal{B}, L)$, such the category of chiral modules over $L$ (regarded as a chiral algebroid) is equivalent to the category of chiral modules over $U(\mathcal{B}, L)$ as a chiral algebra.

Now let us introduce the category $\mathcal{A}_g^{r.e.n.} \text{-mod}$. By definition, its objects are, as before, D-modules $\mathcal{M}$ on $X$ supported at $x$ equipped with
\begin{enumerate}[(1)]
  \item An action of the chiral algebra $\mathcal{A}_g^{x,d} \otimes \mathcal{A}_g^{x,d}/\mathcal{A}_g^x$,
  \item An action of the chiral algebroid $\mathcal{A}_g^{r.e.n.}$,
\end{enumerate}
such that the two induced chiral brackets $j_\ast j^\ast ((\mathcal{A}_g^{x,d} \otimes \mathcal{A}_g^{x,d}/\mathcal{A}_g^x) \boxtimes \mathcal{M}) \to \Delta_1(\mathcal{M})$ coincide.
Observe that one can reformulate the definition of $A_{\mathfrak{g}}^{\text{ren, } d}_{\mathfrak{g}}$ as modules (supported at $x \in X$) over a certain chiral algebra. Namely, let $U_{\text{ren, } d}(L_{\mathfrak{g}, \text{crit}})$ be the quotient of the chiral algebra $U(\mathfrak{z}_\mathfrak{g}, A_{\mathfrak{g}}^{\text{ren, } d})$ by the following relation:

We have a map $U((A_{\mathfrak{g}, \text{crit}} \times A_{\mathfrak{g}, \text{crit}})/\mathfrak{z}_\mathfrak{g}) \to U(\mathfrak{z}_\mathfrak{g}, A_{\mathfrak{g}}^{\text{ren, } d})$ coming from the embedding of Lie-* algebras $(A_{\mathfrak{g}, \text{crit}} \times A_{\mathfrak{g}, \text{crit}})/\mathfrak{z}_\mathfrak{g} \to A_{\mathfrak{g}, \text{crit}}^{\text{ren, } d}$. In addition, we have a map $U((A_{\mathfrak{g}, \text{crit}} \times A_{\mathfrak{g}, \text{crit}})/\mathfrak{z}_\mathfrak{g}) \to A_{\mathfrak{g}, \text{crit}} \otimes A_{\mathfrak{g}, \text{crit}}$. We need to kill the ideal in $U(\mathfrak{z}_\mathfrak{g}, A_{\mathfrak{g}}^{\text{ren, } d})$ generated by the image of the kernel of the latter map.

We define a PBW-type filtration on $U_{\text{ren, } d}(L_{\mathfrak{g}, \text{crit}})$, by setting $F^0 \left(U_{\text{ren, } d}(L_{\mathfrak{g}, \text{crit}})\right)$ to be the image of $A_{\mathfrak{g}, \text{crit}} \otimes A_{\mathfrak{g}, \text{crit}}$, and by requiring inductively that $F^{i+1} \left(U_{\text{ren, } d}(L_{\mathfrak{g}, \text{crit}})\right)$ is the smallest $D_X$-submodule such that

$$j_* j^* \left((A_{\mathfrak{g}}^{\text{ren, } d}_{\mathfrak{g}} \otimes F^i \left(U_{\text{ren, } d}(L_{\mathfrak{g}, \text{crit}})\right)\right) \to \Delta_i \left(F^{i+1} \left(U_{\text{ren, } d}(L_{\mathfrak{g}, \text{crit}})\right)\right) \text{ and}$$

$$j_* j^* \left((A_{\mathfrak{g}, \text{crit}} \otimes A_{\mathfrak{g}, \text{crit}}) \otimes F^{i+1} \left(U_{\text{ren, } d}(L_{\mathfrak{g}, \text{crit}})\right)\right) \to \Delta_i \left(F^{i+1} \left(U_{\text{ren, } d}(L_{\mathfrak{g}, \text{crit}})\right)\right).$$

In this case we automatically have also:

$$A_{\mathfrak{g}}^{\text{ren, } d}_{\mathfrak{g}} \otimes F^i \left(U_{\text{ren, } d}(L_{\mathfrak{g}, \text{crit}})\right) \to \Delta_i \left(F^{i+1} \left(U_{\text{ren, } d}(L_{\mathfrak{g}, \text{crit}})\right)\right).$$

We have a natural surjection on the associated graded level:

\begin{equation}
(A_{\mathfrak{g}, \text{crit}} \otimes A_{\mathfrak{g}, \text{crit}}) \otimes \text{Sym}_{\mathfrak{z}_\mathfrak{g}}(\Omega^1(\mathfrak{z}_\mathfrak{g})) \to \text{gr} \left(U_{\text{ren, } d}(L_{\mathfrak{g}, \text{crit}})\right).
\end{equation}

From [CHA], Theorem 3.9.12 it follows that this map is an isomorphism.

4.7. Let now $\tau$ be an automorphism of $\mathfrak{z}_\mathfrak{g}$ as a chiral-Poisson algebra. We can form the Lie-* algebra

$$A_{\mathfrak{g}}^{\tau, \text{crit}} := (A_{\mathfrak{g}}^{\tau, \text{crit}}_\mathfrak{g} \times A_{\mathfrak{g}}^{\tau, \text{crit}}_\mathfrak{g} \times \mathfrak{z}_\mathfrak{g},$$

where the map $\mathfrak{z}_\mathfrak{g} \to \mathfrak{z}_\mathfrak{g} \times \mathfrak{z}_\mathfrak{g}$ is now $(\text{id}, -\tau)$.

Repeating the construction of Proposition 4.3 we obtain a chiral algebroid $A_{\mathfrak{g}}^{\text{ren, } \tau}_{\mathfrak{g}}$, which fits in a short exact sequence

$$0 \to (A_{\mathfrak{g}, \text{crit}} \times A_{\mathfrak{g}, \text{crit}})/\mathfrak{z}_\mathfrak{g} \to A_{\mathfrak{g}}^{\text{ren, } \tau}_{\mathfrak{g}} \to \Omega^1(\mathfrak{z}_\mathfrak{g}) \to 0,$$

where $\mathfrak{z}_\mathfrak{g}$ is embedded into $A_{\mathfrak{g}, \text{crit}} \times A_{\mathfrak{g}, \text{crit}}$ also via $(\text{id}, -\tau)$.

We will denote by $A_{\mathfrak{g}}^{\tau, \text{crit}}$ the Lie-* algebroid on $\mathfrak{z}_\mathfrak{g}$ equal to the quotient $A_{\mathfrak{g}}^{\text{ren, } \tau}_{\mathfrak{g}}/\mathfrak{z}_\mathfrak{g}$. Finally, in a way similar to the above, we introduce the corresponding categories of modules, $A_{\mathfrak{g}}^{\tau, \text{crit}}_{- \text{mod}}$ and $A_{\mathfrak{g}}^{\text{ren, } \tau}_{- \text{mod}}$, and the chiral algebra $U^{\tau}_{\text{ren, } \text{crit}}(L_{\mathfrak{g}, \text{crit}})$.

Note, however, that according to [BD] (and which something that we will have to use later), every automorphism of $\mathfrak{z}_\mathfrak{g}$, respecting the chiral-Poisson structure, comes from an outer automorphism of $\mathfrak{g}$. This implies that, as abstract algebroids, $A_{\mathfrak{g}}^{\text{ren, } \tau}$ and $A_{\mathfrak{g}}^{\text{ren, } d}$ are, in fact, isomorphic.
5. Chiral differential operators at the critical level

5.1. Recall that $\mathcal{D}_{G,\kappa}$ denotes the chiral algebra of differential operators on the group $G$ at level $\kappa$ (see [AG]), and $\mathfrak{l}, \mathfrak{r}$ are the two embeddings

$$A_{\mathfrak{g},\kappa} \xrightarrow{\mathfrak{l}} \mathcal{D}_{G,\kappa} \xleftarrow{\mathfrak{r}} A_{\mathfrak{g},2\kappa_{\text{crit}}-\kappa}.$$ 

Recall that if $\mathcal{M}$ is a Lie-* module over a Lie-* algebra $L$, the centralizer of $L$ is the maximal $\mathcal{D}$-submodule $\mathcal{M}' \subset \mathcal{M}$ such that the Lie-* bracket $L \otimes \mathcal{M}' \to \Delta(\mathcal{M})$ vanishes.

**Lemma 5.2.** The centralizer of $\mathfrak{l}(A_{\mathfrak{g},\kappa})$ in $\mathcal{D}_{G,\kappa}$ equals $\mathfrak{r}(A_{\mathfrak{g},2\kappa_{\text{crit}}-\kappa})$. Conversely, the centralizer of $\mathfrak{r}(A_{\mathfrak{g},2\kappa_{\text{crit}}-\kappa})$ equals $\mathfrak{l}(A_{\mathfrak{g},\kappa})$.

**Proof.** The inclusion of $\mathfrak{l}(A_{\mathfrak{g},\kappa})$ into the centralizer of $\mathfrak{r}(A_{\mathfrak{g},2\kappa_{\text{crit}}-\kappa})$ is just the fact that the images of $\mathfrak{l}$ and $\mathfrak{r}$ Lie-* commute with each other. The fact that this inclusion is an equality is established as follows. Let $\tilde{g}_\kappa$ be the affine Kac-Moody algebra corresponding to a point $x \in X$. The fiber $A_{\mathfrak{g},\kappa,x}$ of $A_{\mathfrak{g},\kappa}$ at $x$ is a $\hat{g}_\kappa$-module, equal to the vacuum module $\mathcal{V}_{\mathfrak{g},\kappa}$.

Denote by $\mathcal{D}_{G,\kappa,x}$ the fiber of $\mathcal{D}_{G,\kappa}$ at $x$. This is a module over $\tilde{g}_\kappa \times \hat{g}_{2\kappa_{\text{crit}}-\kappa}$. Recall that as $\hat{g}_\kappa$-module, $\mathcal{D}_{G,\kappa,x}$ is the induced module

$$\text{Ind}_{\tilde{g}(\hat{\mathfrak{o}}_x) \oplus \mathbb{C}1}^{\hat{g}_\kappa(\hat{\mathfrak{o}}_x)} \left( \text{Fun} \left( G(\hat{\mathfrak{o}}_x) \right) \right).$$

Moreover, the commuting right action of $\hat{g}(\hat{\mathfrak{o}}_x) \subset \hat{g}_{2\kappa_{\text{crit}}-\kappa}$ comes by transport of structure from the right action of $\hat{g}(\hat{\mathfrak{o}}_x)$ on $\text{Fun} \left( G(\hat{\mathfrak{o}}_x) \right)$. In other words, as a right $\hat{g}(\hat{\mathfrak{o}}_x)$-module,

$$\text{Ind}_{\tilde{g}(\hat{\mathfrak{o}}_x) \oplus \mathbb{C}1}^{\hat{g}_\kappa(\hat{\mathfrak{o}}_x)} \left( \text{Fun} \left( G(\hat{\mathfrak{o}}_x) \right) \right) \simeq U(\hat{g}_\kappa) \otimes t^{-1} \mathbb{C}[t^{-1}] \otimes \text{Fun} \left( G(\hat{\mathfrak{o}}_x) \right),$$

where $\hat{g}(\hat{\mathfrak{o}}_x)$ acts through the second factor and $t$ is a uniformizer in $\hat{\mathfrak{o}}_x$. At the level of fibers, the embedding $\mathfrak{l}$ is just the natural embedding

$$\mathcal{V}_{\mathfrak{g},\kappa} \simeq \text{Ind}_{\tilde{g}(\hat{\mathfrak{o}}_x) \oplus \mathbb{C}1}^{\hat{g}_\kappa(\hat{\mathfrak{o}}_x)}(\mathbb{C}) \to \text{Ind}_{\tilde{g}(\hat{\mathfrak{o}}_x) \oplus \mathbb{C}1}^{\hat{g}_\kappa(\hat{\mathfrak{o}}_x)} \left( \text{Fun} \left( G(\hat{\mathfrak{o}}_x) \right) \right)$$

Corresponding to the unit $\mathbb{C} \to \text{Fun} \left( G(\hat{\mathfrak{o}}_x) \right)$. We have to show that $\mathcal{V}_{\mathfrak{g},\kappa} \subset \mathcal{D}_{G,\kappa,x}$ equals $(\mathcal{D}_{G,\kappa,x})^{\tilde{g}(\hat{\mathfrak{o}}_x)}$, for $\tilde{g}(\hat{\mathfrak{o}}_x) \subset \hat{g}_{2\kappa_{\text{crit}}-\kappa}$. But this immediately follows from the above description of $\mathcal{D}_{G,\kappa,x}$ as a $\tilde{g}(\hat{\mathfrak{o}}_x)$-module.

To finish the proof, observe that the roles of $\mathfrak{l}$ and $\mathfrak{r}$ in the definition of $\mathcal{D}_{G,\kappa}$ are symmetric, and in particular, $\mathcal{D}_{G,\kappa,x}$ is isomorphic to $\text{Ind}_{\tilde{g}(\hat{\mathfrak{o}}_x) \oplus \mathbb{C}1}^{\hat{g}_{2\kappa_{\text{crit}}-\kappa}} \left( \text{Fun} \left( G(\hat{\mathfrak{o}}_x) \right) \right)$ as a $\hat{g}(\hat{\mathfrak{o}}_x) \times \hat{g}_{2\kappa_{\text{crit}}-\kappa}$-module. Indeed, we have a map from the latter to the former, by the definition of the induction, and this is map is clearly an isomorphism at the level of associate graded spaces, by the PBW theorem.

Now we specialize to $\kappa = \kappa_{\text{crit}}$. Then $\mathfrak{l}$ and $\mathfrak{r}$ are two different embeddings of $A_{\mathfrak{g},\text{crit}}$ into $\mathcal{D}_{G,\text{crit}}$. Lemma 5.2 implies the following:

**Corollary 5.3.** $\mathfrak{l}(\mathfrak{z}_\mathfrak{g}) = \mathfrak{l}(A_{\mathfrak{g},\text{crit}}) \cap \mathfrak{r}(A_{\mathfrak{g},\text{crit}}) = \mathfrak{r}(\mathfrak{z}_\mathfrak{g})$.

Let $\tau$ be the involution of the Dynkin diagram of $\mathfrak{g}$, which sends a weight $\lambda$ to $-w_0(\lambda)$. We lift $\tau$ to an outer automorphism of $\mathfrak{g}$, and it gives rise to a canonically defined involution of $\mathfrak{z}_\mathfrak{g}$, which we will also denote by $\tau$. 
**Theorem 5.4.** The two compositions $\mathcal{G} \hookrightarrow A_{g,crit} \xrightarrow{\iota} D_{G,crit}$ and $\mathcal{G} \hookrightarrow A_{g,crit} \xrightarrow{\tau} D_{G,crit}$ are intertwined by the automorphism $\tau : \mathcal{G} \rightarrow \mathcal{G}$.

We have an embedding of the chiral algebroid $\mathcal{A}^{ren,\tau}_{g}$ into $D_{G,crit}$ such that the maps $\iota$ and $\tau$ are the compositions of this embedding and the canonical maps

$$A_{g,crit} \xrightarrow{} (A_{g,crit} \times A_{g,crit})/\mathcal{G} \rightarrow \mathcal{A}^{ren,\tau}_{g}.$$  

This embedding extends to a homomorphism of chiral algebras $U^{ren,\tau}(L_{g,crit}) \rightarrow D_{G,crit}$.

The rest of this section is devoted to the proof of this theorem.

**5.5.** The first step will be to construct a map

$$\psi : \Omega^{1}(\mathcal{G}) \rightarrow D_{G,crit}/(l(A_{g,crit}) + r(A_{g,crit})).$$

Note that if $M$ is a central module over a commutative chiral algebra $B$, we have a naturally defined notion of derivation $B \rightarrow M$, which amounts to a map of $B$-modules $\Omega^{1}(B) \rightarrow M$. We take $B = \mathcal{G}$, and $M$ to be the centralizer of $\mathcal{G}$ in the chiral $\mathcal{G}$-module $D_{G,crit}/(l(A_{g,crit}) + r(A_{g,crit}))$. Thus, we need to construct a map

$$\mathcal{G} \rightarrow D_{G,crit}/(l(A_{g,crit}) + r(A_{g,crit})),$$

whose image Lie-* commutes with $\mathcal{G}$, and which satisfies the Leibniz rule.

By letting the level $\kappa$ vary in the $\mathbb{C}[[\hbar]]$-family $\kappa_{\hbar}$, we obtain a flat $\mathbb{C}[[\hbar]]$-family of chiral algebras $D_{G,\hbar}$. Note that the map $l$ extends to a map $l_{\hbar} : A_{g,\hbar} \rightarrow D_{G,\hbar}$, whereas the map $r$ gives rise to a map $r_{\hbar} : A_{g,-\hbar} \rightarrow D_{G,\hbar}$ (the negative appears due to the sign inversion in $\kappa \mapsto 2\kappa_{\hbar} - \kappa$).

Let $a$ be an element of $\mathcal{G}$, and choose elements $a'_{\hbar}, a''_{\hbar} \in A_{g,\hbar}$ and $a'_{-\hbar}, a''_{-\hbar} \in A_{g,-\hbar}$, which map to $a$ mod $\hbar$. Consider the element $l_{\hbar}(a'_{\hbar}) - r_{\hbar}(a''_{\hbar}) \in D_{G,\hbar}$. By definition, it vanishes mod $\hbar$; hence we obtain an element

$$\frac{l_{\hbar}(a'_{\hbar}) - r_{\hbar}(a''_{\hbar})}{\hbar} \mod \hbar \in D_{G,crit}$$

which is well-defined modulo $l(A_{g,crit}) + r(A_{g,crit})$. This defines the required map. The fact that it is a derivation is a straightforward verification.

Note that the chiral bracket on $D_{G,crit}$ gives rise to a well-defined Lie-* bracket

$$\mathcal{G} \otimes (D_{G,crit}/(l(A_{g,crit}) + r(A_{g,crit})) \rightarrow \Delta^{!}(D_{G,crit}),$$

where $\mathcal{G}$ is thought of as embedded into $D_{G,crit}$ via $\mathcal{G} \hookrightarrow A_{g,crit} \xrightarrow{\iota} D_{G,crit}$.

**Lemma 5.6.** The composition

$$\mathcal{G} \otimes \Omega^{1}(\mathcal{G}) \xrightarrow{\iota \otimes \psi} \mathcal{G} \otimes (D_{G,crit}/(l(A_{g,crit}) + r(A_{g,crit})) \rightarrow \Delta^{!}(D_{G,crit})$$

factors as $\mathcal{G} \otimes \Omega^{1}(\mathcal{G}) \rightarrow \Delta^{!}(\mathcal{G}) \rightarrow \Delta^{!}(D_{G,crit})$, where the first arrow is the chiral-Poisson structure on $\mathcal{G}$. A similar assertion holds for $\mathcal{G}$ mapping to $D_{G,crit}$ via $\tau$.

**Proof.** For two sections $a, b \in \mathcal{G}$, and $a'_{\hbar}, a''_{-\hbar}$ as above, we have

$$[l_{\hbar}(a'_{\hbar}) - r_{\hbar}(a''_{-\hbar}), l_{\hbar}(b_{\hbar})] = [l_{\hbar}(a'_{\hbar}), l_{\hbar}(b_{\hbar})] = l_{\hbar}([a'_{\hbar}, b_{\hbar}]),$$

because the images of $l_{\hbar}$ and $r_{\hbar}$ Lie-* commute in $D_{G,\hbar}$. Hence, the assertion follows from the definition of the chiral-Poisson structure on $\mathcal{G}$.

\[\square\]
5.7. Since the images of \( \mathfrak{j}_g \) in \( \mathcal{D}_{G,crit} \) under \( \iota \) and \( \tau \) coincide, we obtain that there exists an automorphism \( \tau' \) of \( \mathfrak{j}_g \), as a commutative chiral algebra such that \( \iota|_{\mathfrak{j}_g} = \tau|_{\mathfrak{j}_g} \circ \tau' \). Our goal now is to show that \( \tau' = \tau \).

Lemma 4.6 implies that \( \tau' \) is in fact an automorphism of \( \mathfrak{j}_g \) as a chiral-Poisson algebra. According to Proposition 3.5.13 and Theorem 3.6.7 of [BD], the chiral-Poisson structure on \( \mathfrak{j}_g \) is rigid, i.e., its group of automorphisms equals the group of automorphisms of the Dynkin diagram of \( g \).

Therefore, in order to prove that \( \tau' = \tau \), it suffices to show, that the two automorphisms coincide at the associate graded level. Recall that if \( \mathcal{C} \) is a commutative \( \mathcal{O}_X \)-algebra, \( \mathfrak{J}(\mathcal{C}) \) denotes the corresponding commutative chiral algebra, obtained by the jet construction from \( \mathcal{C} \) (see [CHA], Sect. 2.3.2). Recall that the chiral algebras \( A_{G,n} \) and \( \mathcal{D}_{G,crit} \) are naturally filtered (see [CHA], Sect. 3.7.13 and 3.9.11), and we have:

\[
\text{gr}(A_{G,n}) \simeq \mathfrak{J}(\text{Sym}(g \otimes \omega_X^{-1})) \simeq \mathfrak{J}(\text{Fun}(g^* \times G_m, \omega_X))
\]

and

\[
\text{gr}(\mathcal{D}_{G,n}) \simeq \mathfrak{J}(\mathcal{O}_G \otimes \mathcal{O}_X) \otimes \mathfrak{J}(\text{Sym}(g \otimes \omega_X^{-1})) \simeq \mathfrak{J}(\text{Fun}(T^*G \times G_m, \omega_X))
\]

so that the maps \( \text{gr}(\iota) \) and \( \text{gr}(\tau) \) come from the (moment) maps \( T^*G \to g^* \) corresponding to the action of \( g \) on \( G \) by left and right translation, respectively.

Moreover,

\[
\text{gr}(\mathfrak{j}_g) \hookrightarrow \mathfrak{J}(\text{Sym}(g \otimes \omega_X^{-1})) \simeq \mathfrak{J}(\text{Fun}(g^*/G, \omega_X))
\]

(This inclusion is, in fact, an equality, by Theorem 3.5.12.) Therefore, the required assertion follows from the fact that the two maps \( T^*G \to g^*/G \) differ by the automorphism \( \tau \).

5.8. To finish the proof of Theorem 5.3, we will identify \( A_{g,ren}^{\cdot,\cdot} \) with

\[
\Omega^1(\mathfrak{j}_g) \times \mathcal{D}_{G,crit}/(\iota(A_{G,crit}) + \tau(A_{G,crit}))
\]

Note that the construction of the map \( \psi \) gives in fact a map \( A_{g,\cdot,\cdot}^{\cdot,\cdot} / \mathfrak{j}_g \to \mathcal{D}_{G,crit} \). Indeed, a section of \( A_{g,\cdot,\cdot}^{\cdot,\cdot} \) has a form \( \frac{a'_h, a''_h}{h} \) for \( a'_h \in \mathbb{A}_{G,h}, a''_h \in \mathbb{A}_{G,-h} \), such that

\[
a' \mod h = -\tau(a'') \mod h \in \mathfrak{j}_g.
\]

We associate to it a section of \( \mathcal{D}_{G,crit} \) equal to \( \frac{b(h(a'_h) + \tau_b(a''_h))}{h} \).

In addition, \( \mathcal{D}_{G,crit} \) is obviously a chiral \( \mathfrak{j}_g \)-module, so we obtain a map \( \text{Ind}_{\mathfrak{j}_g}(A_{g,\cdot,\cdot}^{\cdot,\cdot}) \to \mathcal{D}_{G,crit} \), and it is straightforward to check that the relations, defining \( A_{g,\cdot,\cdot}^{\cdot,\cdot} \) as a quotient of \( \text{Ind}_{\mathfrak{j}_g}(A_{g,\cdot,\cdot}^{\cdot,\cdot}) \), hold.

Finally, we obtain a homomorphism of chiral algebras \( U(\mathfrak{j}_g, A_{g,\cdot,\cdot}^{\cdot,\cdot}) \to \mathcal{D}_{G,crit} \), and it is easy to see that it annihilates the ideal defining \( U_{g,\cdot,\cdot}^{\cdot,\cdot}(L_{G,crit}) \) as a quotient of \( U(\mathfrak{j}_g, A_{g,\cdot,\cdot}^{\cdot,\cdot}) \).

6. The Functor of Global Sections on the Affine Grassmannian

6.1. Let \( \mathcal{D}_{G,crit} \text{-mod}^{G(\mathfrak{g}_\mathfrak{t},\mathfrak{g}_\mathfrak{r})} \) be the category of chiral \( \mathcal{D}_{G,crit} \)-modules supported at the point \( x \in X \), which are \( G(\mathfrak{g}_\mathfrak{t}) \)-integrable with respect to the embedding \( \mathfrak{t} : A_{G,crit} \to \mathcal{D}_{G,crit} \).

Let \( \mathcal{F} \) be a critically twisted \( \mathcal{D} \)-module on \( \text{Gr}_G \), and \( \mathcal{M}_x \) the corresponding object of \( \mathcal{D}_{G,crit} \text{-mod}^{G(\mathfrak{g}_\mathfrak{t},\mathfrak{g}_\mathfrak{r})} \). According to Theorem 2.5, we have

\[
\Gamma(\text{Gr}_G, \mathcal{F}) \simeq \text{Hom}_{\mathfrak{g}(\mathfrak{g}_\mathfrak{t},\mathfrak{g}_\mathfrak{r})}(\mathcal{C}, \mathcal{M}_x) \simeq \text{Hom}_{\mathcal{D}_{G,crit}}(\mathcal{V}_{G,crit}(x), \mathcal{M}_x),
\]

where \( \mathcal{M}_x \) is regarded as a \( \mathfrak{g}_{crit} \)-module via \( x \), and \( \mathcal{V}_{G,crit} \simeq \text{Ind}_{\hat{\mathfrak{g}}_{crit}}^{\hat{\mathfrak{g}}_{crit}}(\mathfrak{g}_\mathfrak{t} \otimes \mathfrak{C}(\mathfrak{C})) \) is the vacuum module, i.e., the fiber \( A_{G,crit, x} \) of \( A_{G,crit} \) at \( x \).
Recall that \( \hat{\mathfrak{g}}_{\text{crit}} \text{-mod} \) denotes the category of all discrete \( \hat{\mathfrak{g}}_{\text{crit}} \text{-modules} \) supported at \( x \in X \), and let \( \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}^{G(\hat{\mathfrak{g}})} \) be the subcategory of \( G(\hat{\mathfrak{g}}) \)-integrable modules. Obviously, \( \mathcal{V}_{\mathfrak{g},\text{crit}} \) belongs to \( \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}^{G(\hat{\mathfrak{g}})} \), but the main difficulty in the proof of Theorem 6.3 is that, in contrast to the negative or irrational level cases, \( \mathcal{V}_{\mathfrak{g},\text{crit}} \) is not projective in this category.

Let now \( \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}_{\text{reg}} \) denote the subcategory of \( \hat{\mathfrak{g}}_{\text{crit}} \text{-mod} \) consisting of modules, which are central (cf. [50], Sect. 3.3.7) with respect to the action of \( \mathfrak{g} \). Let us denote by \( \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}_{\text{reg}}^{G(\hat{\mathfrak{g}})} \) the intersection \( \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}_{\text{reg}} \cap \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}^{G(\hat{\mathfrak{g}})} \). The module \( \mathcal{V}_{\mathfrak{g},\text{crit}} \) belongs to \( \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}_{\text{reg}}^{G(\hat{\mathfrak{g}})} \), but the modules from \( \mathcal{D}_{G,\text{crit}} \text{-mod}^{G(\hat{\mathfrak{g}})} \), regarded as objects of \( \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}^{G(\hat{\mathfrak{g}})} \), do not belong there.

The following projectivity result is essentially due to [51] (see Sect. 5 for the proof).

**Theorem 6.2.** The module \( \mathcal{V}_{\mathfrak{g},\text{crit}} \) is a projective generator of the category \( \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}_{\text{reg}}^{G(\hat{\mathfrak{g}})} \). In particular, the functor \( \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}_{\text{reg}}^{G(\hat{\mathfrak{g}})} \to \text{Vect} \) given by \( \mathcal{M} \mapsto \text{Hom}_{\hat{\mathfrak{g}}_{\text{crit}}} (\mathcal{V}_{\mathfrak{g},\text{crit}}, \mathcal{M}) \), is exact.

Consider the functors

\[
\mathcal{F} : \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}_{\text{reg}}^{G(\hat{\mathfrak{g}})} \to \mathfrak{g}_{x} \text{-mod}, \quad \mathcal{M} \mapsto \text{Hom}_{\hat{\mathfrak{g}}_{\text{crit}}} (\mathcal{V}_{\mathfrak{g},\text{crit}}, \mathcal{M}),
\]

\[
\mathcal{G} : \mathfrak{g}_{x} \text{-mod} \to \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}_{\text{reg}}^{G(\hat{\mathfrak{g}})}, \quad \mathcal{F} \mapsto \mathcal{V}_{\mathfrak{g},\text{crit}} \otimes \mathcal{F}.
\]

Now Theorem 6.2 implies the following:

**Theorem 6.3.** The functors \( \mathcal{F} \) and \( \mathcal{G} \) are mutually inverse equivalences of categories.

By combining this theorem with Theorem 3.4 we obtain that the category \( \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}_{\text{reg}}^{G(\hat{\mathfrak{g}})} \) is equivalent to the category of quasicoherent sheaves on the scheme \( \text{Op}_{\text{reg}} (\mathcal{D}_{x}) \).

### 6.4

Consider the functor \( i' : \mathfrak{g}_{x} \text{-mod} \to \mathfrak{g}_{x} \text{-mod} \), which takes a \( \mathfrak{g}_{x} \text{-module} \) to its maximal submodule, scheme-theoretically supported on \( \text{Spec}(\mathfrak{g}_{x}) \), i.e., for an object \( \mathcal{M} \in \mathfrak{g}_{x} \text{-mod} \), \( i'(\mathcal{M}) \) consists of elements annihilated by \( \ker(\mathfrak{g}_{x} \to \mathfrak{g}_{x}) \). We will denote by the same symbol \( i' \) the corresponding functors

\[
\hat{\mathfrak{g}}_{\text{crit}} \text{-mod} \to \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}_{\text{reg}}^{G(\hat{\mathfrak{g}})} \quad \text{and} \quad \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}^{G(\hat{\mathfrak{g}})} \to \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}_{\text{reg}}^{G(\hat{\mathfrak{g}})}.
\]

According to formula (6.1), the functor of global sections \( \Gamma : D_{\text{crit}}(\text{Gr}_{G} \text{-mod} \to \text{Vect} \) can be viewed as a functor \( \mathcal{D}_{G,\text{crit}} \text{-mod}^{G(\hat{\mathfrak{g}})} \to \text{Vect} \) given by \( \mathcal{M} \mapsto \text{Hom}_{\mathfrak{g},\text{crit}} (\mathcal{V}_{\mathfrak{g},\text{crit}}, \mathcal{M}) \). Since \( \mathcal{V}_{\mathfrak{g},\text{crit}} \) is supported on \( \text{Spec}(\mathfrak{g}_{x}) \), we obtain that this functor factors as

\[
\mathcal{M} \mapsto i'(\mathcal{M}) \mapsto \text{Hom}_{\hat{\mathfrak{g}}_{\text{crit}} \text{-mod}_{\text{reg}}^{G(\hat{\mathfrak{g}})} (\mathcal{V}_{\mathfrak{g},\text{crit}}, i'(\mathcal{M})).
\]

But according to Theorem 6.2, the second functor is exact. Therefore Theorem 6.2 is equivalent to the following:

**Theorem 6.5.** The composition

\[
\mathcal{D}_{G,\text{crit}} \text{-mod}^{G(\hat{\mathfrak{g}})} \to \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}^{G(\hat{\mathfrak{g}})} \xrightarrow{i'} \hat{\mathfrak{g}}_{\text{crit}} \text{-mod}_{\text{reg}}^{G(\hat{\mathfrak{g}})},
\]

where the first arrow is the forgetful functor corresponding to the embedding \( r \), is exact.
6.6. Let \( \mathcal{Z}_{g,x} \) be the full subcategory of \( \mathcal{Z}_{g} \mod \reg \), whose objects are those modules, which are set-theoretically supported on \( \Spec(\mathcal{Z}_{g,x}) \subset \Spec(\mathcal{Z}_{g}) \), i.e., modules supported on the formal neighborhood of \( \Spec(\mathcal{Z}_{g,x}) \). Let \( \hat{\mathcal{U}}_{\text{crit}} \mod \reg \), (resp., \( \hat{\mathcal{U}}_{\text{crit}} \mod \reg G(\hat{\mathcal{O}}_{x}) \)) denote the corresponding full subcategory of \( \hat{\mathcal{U}}_{\text{crit}} \mod \), (resp., \( \hat{\mathcal{U}}_{\text{crit}} \mod G(\hat{\mathcal{O}}_{x}) \)).

Let \( \hat{\mathcal{U}} : \mathcal{Z}_{g,x} \mod \to \mathcal{Z}_{g,x} \mod \reg \) be the functor that attaches to a \( \mathcal{Z}_{g,x} \)-module its maximal submodule, which is supported on the formal neighborhood of \( \Spec(\mathcal{Z}_{g,x}) \). In other words, for \( M \in \mathcal{Z}_{g,x} \mod \), \( \hat{\mathcal{U}}(M) \) consists of all sections annihilated by some power of the ideal of \( \Spec(\mathcal{Z}_{g,x}) \). We will denote in the same way the corresponding functors

\[
\hat{\mathcal{U}}_{\text{crit}} \mod \to \hat{\mathcal{U}}_{\text{crit}} \mod \reg \quad \text{and} \quad \hat{\mathcal{U}}_{\text{crit}} \mod G(\hat{\mathcal{O}}_{x}) \to \hat{\mathcal{U}}_{\text{crit}} \mod G(\hat{\mathcal{O}}_{x}).
\]

Clearly, \( \hat{\mathcal{U}}' \simeq \hat{\mathcal{U}} \circ \hat{\mathcal{U}}' \).

**Proposition 6.7.** Every object \( M \in \hat{\mathcal{U}}_{\text{crit}} \mod G(\hat{\mathcal{O}}_{x}) \) can be canonically decomposed as a direct sum \( M = M_{\reg} \oplus M_{\non-reg} \), where \( \hat{\mathcal{U}}(M_{\reg}) \simeq M_{\reg} \) and \( \hat{\mathcal{U}}(M_{\non-reg}) \) is supported away from \( \Spec(\mathcal{Z}_{g,x}) \subset \Spec(\mathcal{Z}_{g,x}) \).

**Corollary 6.8.** The functor \( \hat{\mathcal{U}} : \hat{\mathcal{U}}_{\text{crit}} \mod G(\hat{\mathcal{O}}_{x}) \to \hat{\mathcal{U}}_{\text{crit}} \mod G(\hat{\mathcal{O}}_{x}) \) is exact.

**Proof.** (of Proposition 6.7)

For an irreducible \( g \)-module \( V^{\lambda} \) with a dominant highest weight \( \lambda \in \Lambda^{+} \), let \( V^{\lambda}_{g,\text{crit}} \) be the corresponding Weyl module in \( \hat{\mathcal{U}}_{\text{crit}} \mod G(\hat{\mathcal{O}}_{x}) \), as defined in Sect. 2.6; in particular, \( V^{0}_{g,\text{crit}} = V^{0} \). Let \( \mathcal{Y}^{\lambda} \subset \Spec(\mathcal{Z}_{g,x}) \) be the closed sub ind-scheme corresponding to the annihilating ideal of \( V^{\lambda}_{g,\text{crit}} \) in \( \mathcal{Z}_{g,x} \). In particular, \( \mathcal{Y}^{0} = \Spec(\mathcal{Z}_{g,x}) \).

By definition, every object in the category \( \hat{\mathcal{U}}_{\text{crit}} \mod G(\hat{\mathcal{O}}_{x}) \) has a filtration whose successive quotients are generated by vectors, on which the subalgebra \( g \otimes tC[[t]] \subset g(\hat{\mathcal{O}}_{x}) \) acts trivially. In particular, such a subquotient is a quotient of \( V^{\lambda}_{g,\text{crit}} \) for some \( \lambda \). Therefore, the support in \( \Spec(\mathcal{Z}_{g,x}) \) of every object from \( \hat{\mathcal{U}}_{\text{crit}} \mod G(\hat{\mathcal{O}}_{x}) \) is contained in the union of the formal neighborhoods of \( \mathcal{Y}^{\lambda} \) for \( \lambda \in \Lambda^{+} \).

**Lemma 6.9.** For \( \lambda \neq 0 \), \( \mathcal{Y}^{\lambda} \cap \Spec(\mathcal{Z}_{g,x}) = \emptyset \).

This lemma implies the proposition. Indeed, for \( M \in \hat{\mathcal{U}}_{\text{crit}} \mod G(\hat{\mathcal{O}}_{x}) \) we define \( \mathcal{Z}_{g} \mod \reg \) to be direct summand of \( M \) supported on the formal neighbourhood of \( \mathcal{Y}^{0} \), and \( \mathcal{Z}_{g} \mod \non-reg \) to be the direct summand supported on the union of the formal neighborhoods of \( \mathcal{Y}^{\lambda} \) with \( \lambda \neq 0 \).

**Proof.** (of Lemma 6.9)

Recall the operator \( S_{0} \) given by formula (2.3). At the critical level this operator commutes with the action of \( \hat{\mathcal{U}}_{\text{crit}} \), i.e., it belongs to \( \mathcal{Z}_{g,x} \). But according to formula (2.3), \( S_{0} \) acts on \( V^{\lambda} \subset V^{\lambda}_{g,\text{crit}} \), and hence on the entire \( V^{\lambda}_{g,\text{crit}} \), by the scalar \( C_{g}(\lambda) \) equal to the value of the Casimir operator on \( V^{\lambda} \). This scalar is zero for \( \lambda = 0 \) and non-zero for \( \lambda \neq 0 \). This proves the lemma.

**Proof.** (of Proposition 6.7)

Recall from Sect. 5.5 that if \( E \) is a group ind-subscheme of the normal bundle \( \mathcal{N} \mathcal{E}(\mathcal{Z}_{g,x}) \), we can introduce the subcategory \( \mathcal{Z}_{g,x} \mod \reg E \), such that

\[
\mathcal{Z}_{g,x} \mod \subset \mathcal{Z}_{g,x} \mod \reg E \subset \mathcal{Z}_{g,x} \mod .
\]
We define $\mathcal{E}$ as follows. By Theorem 6.11, the $D_X$-algebra $Hg_x$ is non-canonically isomorphic to a free algebra. This implies, in particular, that the fiber $\Theta(\tilde{g}_x)$ of $\Theta(Hg_x)$ at $x$ is locally free of countable rank over $\tilde{g}_x$, and $N(\tilde{g}_x)$ can be identified with the total space of the resulting vector bundle.

Therefore, to specify a group ind-subscheme $\mathcal{E} \subset N(\tilde{g}_x)$ it would be sufficient to specify a $\tilde{g}_x$-submodule in $\Theta(\tilde{g}_x)$, which is locally a direct summand. Such a submodule is given by the image of the anchor map $\gamma : \Omega^1(\tilde{g}_x) \to \Theta(\tilde{g}_x)$; it is locally a direct summand, as follows from Theorem 6.11.

For $\mathcal{E}$ defined in this way, let us denote by $\hat{g}_{\text{crit}} \text{-mod}_\mathcal{E}$ the subcategory of $\hat{g}_{\text{crit}} \text{-mod}$ whose objects are the $\hat{g}_{\text{crit}}$-modules, such that the underlying chiral $\hat{g}_x$-module belongs to $\tilde{g}_x \text{-mod}_\mathcal{E}$. In Sect. 8 we will prove the following

**Theorem 6.11.** The category $\hat{g}_{\text{crit}} \text{-mod}_{\text{reg}}^G(\tilde{g}_x)$ is contained in $\hat{g}_{\text{crit}} \text{-mod}_\mathcal{E}$.

In other words, this theorem says that the inclusion

$$\hat{g}_{\text{crit}} \text{-mod}^G(\tilde{g}_x) := \hat{g}_{\text{crit}} \text{-mod}^G(\tilde{g}_x) \cap \hat{g}_{\text{crit}} \text{-mod}_\mathcal{E} \subset \hat{g}_{\text{crit}} \text{-mod}^G_{\text{reg}}$$

is in fact an equivalence.

6.12. The following remark was suggested by A. Beilinson:

Let $\text{Spec}(\tilde{g}_x \text{-mod}_{\text{m.f.}})$ be the smallest formal subscheme inside $\text{Spec}(\tilde{g}_x)$, which contains $\text{Spec}(\tilde{g}_x)$, and which is preserved by the Poisson bracket. (The subscript "m.f." stands for "monodromy free"). Let $\tilde{g}_x \text{-mod}_{\text{m.f.}}$ be the subcategory of $\tilde{g}_x \text{-mod}$ consisting of modules supported on $\text{Spec}(\tilde{g}_x \text{-mod}_{\text{m.f.}})$, and let $\hat{g}_{\text{crit}} \text{-mod}_{\text{m.f.}}$ be the corresponding subcategory in $\hat{g}_{\text{crit}} \text{-mod}$. Beilinson has suggested that the following strengthening of Theorem 6.11 might be true:

**Conjecture 6.13.** The subcategory $\hat{g}_{\text{crit}} \text{-mod}_{\text{reg}}^G(\tilde{g}_x)$ is contained in $\hat{g}_{\text{crit}} \text{-mod}_{\text{m.f.}}$.

In other words, this conjecture says that any $G(\tilde{g}_x)$-integrable $\hat{g}_{\text{crit}}$-module, which is set-theoretically supported on $\text{Spec}(\tilde{g}_x)$, is supported on the formal subscheme $\text{Spec}(\tilde{g}_x \text{-mod}_{\text{m.f.}})$.

If we could prove this conjecture, the proof of Theorem 6.11 would have been more elegant, since instead of the obscure condition (2) in the definition of $\tilde{g}_x \text{-mod}_\mathcal{E}$, we would work with a clearer geometric concept of support on a subscheme.

6.14. Recall now the category $A^{\text{ren,}\tau}_x \text{-mod}$, introduced in Sect. 3.6 and 4.7. We have a natural forgetful functor $A^{\text{ren,}\tau}_x \text{-mod} \to \hat{g}_{\text{crit}} \text{-mod}$ coming from the “right” copy of $A^{\text{ren,}\tau}_x$ in $A^{\text{ren,}\tau}_x$. Let $A^{\text{ren,}\tau}_x \text{-mod}^G(\tilde{g}_x)$ (resp., $A^{\text{ren,}\tau}_x \text{-mod}_{\text{reg}}$, $A^{\text{ren,}\tau}_x \text{-mod}_{\text{crit}}$, $A^{\text{ren,}\tau}_x \text{-mod}^G_{\text{reg}}$, etc.) be the preimages of the corresponding subcategories of $\hat{g}_{\text{crit}} \text{-mod}$ under the above forgetful functor. Note, that by Theorem 6.11 the inclusion

$$A^{\text{ren,}\tau}_x \text{-mod}_{\text{reg}}^G(\tilde{g}_x) \hookrightarrow A^{\text{ren,}\tau}_x \text{-mod}_{\text{reg}}$$

is in fact an equivalence.

It is easy to see that the functor $i^! : \tilde{g}_x \text{-mod} \to \tilde{g}_x \text{-mod}_{\text{reg}}$ gives rise to a well-defined functor $i^! : A^{\text{ren,}\tau}_x \text{-mod} \to A^{\text{ren,}\tau}_x \text{-mod}_{\text{reg}}$. In particular, the corresponding functor

$$A^{\text{ren,}\tau}_x \text{-mod}^G(\tilde{g}_x) \xrightarrow{i^!} A^{\text{ren,}\tau}_x \text{-mod}_{\text{reg}}^G(\tilde{g}_x)$$

is exact, by Corollary 6.5.
Recall now the Lie-* algebroid $A^\flat,\tau_g$ and the corresponding category $A^\flat,\tau_g$–mod (see Sect. 4.6). We claim that the functor $i^! : \mathcal{F}_{g,x} – \text{mod} \to \mathcal{F}_{g,x} – \text{mod}$ gives rise to a functor from $A^\flat,\tau_g$–mod to $A^\flat,\tau_g$–mod.

Indeed, given an object $M$ of the category $A^{\text{ren},\tau}_g$–mod, we consider it as a $\mathcal{F}_{g,x}$–module and take its maximal submodule $i'(M)$ supported on $\text{Spec}(\mathcal{F}_{g,x})$. We consider $i'(M)$ as a Lie-* module over $A^{\text{ren},\tau}_g$. But now the Lie-* action of the diagonal $\mathfrak{g}_{\text{reg}} \subset (A_{g,\text{crit}} \times A_{g,\text{crit}})/\mathfrak{g}_g \subset A^{\text{ren},\tau}_g$ will be zero. Therefore, the Lie-* action of $A^{\text{ren},\tau}_g$ on $i'(M)$ will factor through the action of the Lie-* algebra $A^\flat,\tau_g = A^{\text{ren},\tau}_g/\mathfrak{g}_g$. Moreover, $i'(M)$ is clearly preserved by the chiral action of $A_{g,\text{crit}} \otimes A_{g,\text{crit}}$, and these two structures make $i'(M)$ an object of the category $A^\flat,\tau_g$–mod. By a slight abuse of notation, we denote the resulting functor $A^{\text{ren},\tau}_g \to A^\flat,\tau_g$–mod also by $i^!$.

In the next section we will prove the following theorem, which can be regarded as a version of the Kashiwara theorem in the theory of D-modules.

**Theorem 6.15.** The functor $i^! : A^{\text{ren},\tau}_g \to A^\flat,\tau_g$–mod is an equivalence of categories. In particular, it is exact.

If we denote by $A^\flat,\tau_g$–mod$^G(\mathcal{O}_x)$ the corresponding subcategory of $A^{\text{ren},\tau}_g$, we obtain that the functor

$$A^{\text{ren},\tau}_g – \text{mod}^G(\mathcal{O}_x) \to A^\flat,\tau_g – \text{mod}^G(\mathcal{O}_x)$$

is also exact (and, in fact, an equivalence).

6.16. We are now able to finish the proof of Theorem 6.15 modulo Theorems 6.11 and 6.16. Recall from Theorem 6.14 that we have a homomorphism of chiral algebras $U^{\text{ren},\tau}(L_{g,\text{crit}}) \to \mathcal{D}_{G,\text{crit}}$. Hence, the forgetful functor $\mathcal{D}_{G,\text{crit}} – \text{mod} \to \mathcal{g}_{\text{crit}} – \text{mod}$ factors as

$$\mathcal{D}_{G,\text{crit}} – \text{mod} \to A^{\text{ren},\tau}_g – \text{mod} \to \mathcal{g}_{\text{crit}} – \text{mod}.$$

We have a commutative diagram of functors

$$
\begin{array}{ccc}
A^{\text{ren},\tau}_g – \text{mod}^G(\mathcal{O}_x) & \xrightarrow{i^!} & A^\flat,\tau_g – \text{mod}^G(\mathcal{O}_x) \\
\downarrow & & \downarrow \\
\mathcal{g}_{\text{crit}} – \text{mod}^G(\mathcal{O}_x) & \xrightarrow{i^!} & \mathcal{g}_{\text{crit}} – \text{mod}^G(\mathcal{O}_x),
\end{array}
$$

(6.2)

where the vertical arrows are the forgetful functors.

Thus, to prove Theorem 6.15 it is sufficient to show that the composition

$$A^{\text{ren},\tau}_g – \text{mod}^G(\mathcal{O}_x) \xrightarrow{i^!} A^{\text{ren},\tau}_g – \text{mod}^G(\mathcal{O}_x) \simeq A^{\text{ren},\tau}_g – \text{mod}^G(\mathcal{O}_x) \xrightarrow{i^!} A^\flat,\tau_g – \text{mod}^G(\mathcal{O}_x)$$

is exact. But, as we have just seen, all the above arrows are exact functors.

Our plan now is as follows. In the next section we will prove Theorem 6.15 and hence complete the proof of Theorems 6.15 and 6.17 modulo Theorems 6.2 and 6.11. These theorems will be proved simultaneously in Sect. 8. Finally, in Sect. 9 we will prove that the functor of global sections, considered as a functor $\mathcal{D}_{\text{crit}}(\text{Gr}_G) – \text{mod} \to A^\flat,\tau_g – \text{mod}$, is fully faithful.
7. Proof of Theorem 6.15

7.1. Let us recall the setting of the original Kashiwara theorem. Let \( X \) be a smooth variety, and \( i : Y \to X \) an embedding of a smooth closed subvariety. Consider the category \( D_X \)-modules on \( X \), and its subcategory \( D_Y \)-mod of right \( D \)-modules set-theoretically supported on \( Y \). Finally, consider the category \( D_Z \)-mod of \( D \)-modules on \( Y \).

We have the functor \( i^! : D_X \to D_Y \) which sends a \( D \)-module \( M \) to its maximal \( \mathcal{O}_X \)-submodule consisting of sections annihilated by the ideal of \( Y \). Then the resulting \( \mathcal{O}_X \)-module naturally acquires a right action of the ring of differential operators on \( Y \), and so we obtain a functor \( i^! : D_X \to D_Y \)-mod. Kashiwara’s theorem asserts that this functor is an equivalence of categories.

Our Theorem 6.15 should be regarded as a generalization of the above theorem, when the ring of differential operators is replaced by a certain algebroid (cf. Sect. 7.6 below). In the proof we will use the same argument as in the proof of the original Kashiwara theorem.

7.2. We start by constructing a functor \( N : A_{g,\tau}^{\text{ren}} \to A_{g,\tau}^{\text{ren},\tau} \), which will be the left adjoint of \( i^! \).

Given an object \( N \) in \( A_{g,\tau}^{\text{ren}} \)-mod, we regard it as a Lie-* module over \( A_{g,\tau}^{\text{ren},\tau} \), considered as a Lie-* algebra. Let \( \text{Ind}(N) \) denote the induced chiral \( A_{g,\tau}^{\text{ren},\tau} \)-module (see [CHA, Sect. 3.7.15]), where \( A_{g,\tau}^{\text{ren},\tau} \) is again considered merely as a Lie-* algebra (and not as a chiral algebroid).

Thus, \( \text{Ind}(N) \) is a chiral module over the chiral universal enveloping algebra \( U(A_{g,\tau}^{\text{ren},\tau}) \). We have the surjections

\[
U(A_{g,\tau}^{\text{ren},\tau}) \twoheadrightarrow U(3g, A_{g,\tau}^{\text{ren},\tau}) \to U^{\text{ren},\tau}(L_{g,\text{crit}}),
\]

and we set \( u(N) \) to be the (maximal) quotient of \( \text{Ind}(N) \), on which the action of \( U(A_{g,\tau}^{\text{ren},\tau}) \) factors through an action of \( U^{\text{ren},\tau}(L_{g,\text{crit}}) \), and for which the two maps

\[
j_* j^*(A_{g,\text{crit}} \otimes A_{g,\text{crit}}) \boxtimes N \to \Delta_! (u(N)),
\]

one coming from the initial chiral action of \( A_{g,\text{crit}} \otimes A_{g,\text{crit}} \) on \( N \), and the other from the homomorphism \( A_{g,\text{crit}} \otimes A_{g,\text{crit}} \to U^{\text{ren},\tau}(L_{g,\text{crit}}) \), coincide.

By definition, \( u(N) \) is an object of \( A_{g,\tau}^{\text{ren},\tau} \)-mod. It is easy to see that the functor \( N \to u(N) : A_{g,\tau}^{\text{ren}} \to A_{g,\tau}^{\text{ren},\tau} \)-mod is the left adjoint to \( i^! : A_{g,\tau}^{\text{ren}} \to A_{g,\tau}^{\text{ren},\tau} \)-mod. One readily checks that for \( N = A_{g,\text{crit}} \otimes A_{g,\text{crit}} \), with the action of \( A_{g,\tau}^{\text{ren},\tau} \) introduced in the proof of Theorem 6.15, the resulting object \( u(N) \) is isomorphic to \( U^{\text{ren},\tau}(L_{g,\text{crit}}) \) itself.

7.3. Let us regard \( u(N) \) as a chiral module over \( A_{g,\text{crit}} \otimes A_{g,\text{crit}} \). We have a canonical map \( N \to u(N) \), and the PBW filtration on \( U^{\text{ren},\tau}(L_{g,\text{crit}}) \) induces an increasing filtration \( F^i(u(N)) \), \( i \geq 1 \) on \( u(N) \) with the \( F^1(u(N)) \) term being the image of \( N \). The terms of this filtration are stable under the chiral action of \( A_{g,\text{crit}} \otimes A_{g,\text{crit}} \) and the Lie-* action of the entire \( A_{g,\tau}^{\text{ren},\tau} \); the action of \( 3g \otimes 3g \) on \( \text{gr}(u(N)) \) is central.

The description of \( \text{gr}(U^{\text{ren},\tau}(L_{g,\text{crit}})) \) given by (7.1) implies that we have an isomorphism

\[
N \otimes \text{Sym}_{3g,x}(\Omega^1(3g,x)) \simeq \text{gr}(u(N)).
\]

Evidently, as a module over \( 3g \otimes 3g \), \( u(N) \) is supported on the formal neighborhood of \( \text{Spec}(3g,x) \). Recall the setting if Sect. 5.6 in particular, let \( J \) be the ideal \( J_{3g,x} \to 3g,x \).
Lemma 7.4. For $N$ as above, $F^i(t_1(N))$ equals the submodule of $t_1(N)$ annihilated by $\mathcal{J}^i$. Moreover, as a module over $\mathcal{Z}_{g,x}$, $t_1(N)$ belongs to the category $\mathcal{Z}_{g,x} - \text{mod}_\mathcal{E}$.

Proof. Since the action of $\mathcal{J}_g$ on $gr(t_1(N))$ is central, we have $\mathcal{J} : F^{i+1}(t_1(N)) \subset F^i(t_1(N))$. Therefore, by induction, every element of $F^i(t_1(N))$ is annihilated by $\mathcal{J}^i$.

To prove the inclusion in the other direction, consider the map
$$J/J^2 \otimes_{\mathcal{Z}_{g,x}} (F^{i+1}(t_1(N))/F^i(t_1(N))) \to (F^i(t_1(N))/F^{i-1}(t_1(N)))$$
and the corresponding dual map
$$(F^{i+1}(t_1(N))/F^i(t_1(N))) \to \Theta(\mathcal{J}_g) \otimes_{\mathcal{Z}_{g,x}} (F^i(t_1(N))/F^{i-1}(t_1(N))).$$

We need to show that the map of (7.2) is injective. But this follows by combining the isomorphism (7.1) and Theorem 3.8. Indeed, this map is obtained by tensoring with $F$.

Thus, we obtain that $\mathcal{N} \to t_1(N)$ is a functor $\mathcal{A}_{g,x}^{\text{ren},\tau} \to \mathcal{A}_{g,x}^{\text{ren},\tau} - \text{mod}_\mathcal{E}$, left adjoint to $i^!$, and the adjunction map $N \to i^! t_1(N)$ is an isomorphism.

To prove Theorem 5.16 it remains to show that for every $M \in \mathcal{A}_{g,x}^{\text{ren},\tau} - \text{mod}_\mathcal{E}$, the adjunction map $i^! t_1(M) \to M$ is surjective. Indeed, from the fact that $i^! t_1(N) \simeq N$, we know that the map $i^! t_1(M) \to i^! (N)$ is an isomorphism, and we conclude that $i^! (\ker(t_1 t_1(M) \to M)) = 0$.

Thus, we obtain that $N \to t_1(N)$ is a functor $\mathcal{A}_{g,x}^{\text{ren},\tau} \to \mathcal{A}_{g,x}^{\text{ren},\tau} - \text{mod}_\mathcal{E}$, left adjoint to $i^!$, and the adjunction map $N \to i^! t_1(N)$ is an isomorphism.

Consider the natural action of $H^0_{DR}(\mathcal{D}_x^\wedge, \mathcal{A}_{g,x}^{\text{ren},\tau})$ on $\mathcal{Z}_{g,x}$; we have:
$$\xi_k(f_1) = \delta_{k,l} \mod \mathcal{J}.$$
We claim that $\delta$ is in fact a scalar operator that acts as multiplication by $i - 1$. To prove this statement, we assume by induction, that $\delta$ acts as $j - 1$ on $M_j / M_{j-1}$ for all $j < i$.

Given an element $m \in M_i / M_{i-1}$, consider the finite sum $\sum_{k=1,\ldots,N} \xi_k \cdot f_k \cdot m$, which includes all the indices $k$ for which $f_k \cdot m \notin M_{i-2}$. We must show that for any index $l$,

$$f_l \cdot \left( \sum_{k=1,\ldots,N} \xi_k \cdot f_k \cdot m - (i-1) \cdot m \right) \in M_{i-2}.$$

Without loss of generality, we can assume that the initial finite set of $k$’s included $l$, as well as the corresponding set of indices for the element $f_l \cdot m \in M_{i-1} / M_{i-2}$. Then we have

$$f_l \cdot \left( \sum_{k=1,\ldots,N} \xi_k \cdot f_k \cdot m - (i-1) \cdot m \right) = \left( \sum_{k=1,\ldots,N} \xi_k \cdot f_k \cdot f_l \cdot m - (i-2) \cdot f_l \cdot m \right) + \left( \xi_l (f_l) \cdot f_k \cdot m - f_l \cdot m \right).$$

In the above expression, the first term belongs to $M_{i-2}$, by the induction hypothesis on the action of $\delta$ on $M_{i-1} / M_{i-2}$. The second term belongs to $M_{i-2}$, because for $k \neq l$ we have $\xi_k (f_l) \notin 1$, and the third term also belongs to $M_{i-2}$, because $\xi_l (f_l) = 1 \mod 3$. This completes the proof of the induction step, and hence, of Theorem 6.15.

7.6. Recall that formal scheme $\text{Spec}(\mathcal{F}_g, x, m, \tau)$ introduced in Sect. 6.12. From Theorem 6.15 we obtain the following corollary:

**Corollary 7.7.** Every object of $\mathcal{A}^{\tau}_{g, \text{ren}, \text{mod}_{\mathcal{E}}}$ is supported on $\text{Spec}(\mathcal{F}_g, x, m, \tau)$.

**Proof.** Let us write $M \in \mathcal{A}^{\tau}_{g, \text{ren}, \text{mod}_{\mathcal{E}}}$ as $\iota_! (\mathcal{N})$ for $\mathcal{N} \in \mathcal{A}_{g, \text{ren}, \text{mod}}$, and consider the filtration $F^i (\iota_! (\mathcal{N}))$.

Of course, $F^i (\iota_! (\mathcal{N}))$ is supported on $\text{Spec}(\mathcal{F}_g, x, m, \tau) \subset \text{Spec}(\mathcal{F}_g, x, m, \tau)$, and let us assume by induction that $F^{i-1} (\iota_! (\mathcal{N}))$ is supported on $\text{Spec}(\mathcal{F}_g, x, m, \tau)$. However, since $F^{i-1} (\iota_! (\mathcal{N}))$ is stable under the chiral action of $\mathcal{A}_{g, \text{crit}} \otimes \mathcal{A}_{g, \text{crit}}$, and $j_* j^* (\mathcal{A}_{g, \text{ren}, \text{mod}} \otimes F^{i-1} (\iota_! (\mathcal{N})))$ maps surjectively onto $F^i (\iota_! (\mathcal{N}))$, and taking into account that $\mathcal{A}_{g, \text{ren}, \text{mod}} / (\mathcal{A}_{g, \text{crit}} \times \mathcal{A}_{g, \text{crit}} / \mathcal{F}_g) \simeq \Omega^1 (\mathcal{F}_g)$, we obtain that $F^i (\iota_! (\mathcal{N}))$ is also supported on $\text{Spec}(\mathcal{F}_g, x, m, \tau)$. \hfill \Box

Following a suggestion of Beilinson, let us note that Theorem 6.15 can be viewed in the following general framework. (We formulate it in the finite-dimensional situation, for simplicity.)

Let $X$ and $Y$ be as in Sect. 6.14. Let $L_X$ be a Lie algebroid on $X$, and let $L_Y$ be its pull-back back to $Y$. Let $\hat{Y}$ be the formal neighborhood of $Y$ in $X$, and let $\hat{Y}' \supset \hat{Y}$ be the smallest ind-subscheme of $\hat{Y}$, stable under the action of $L_X$. Let $L_X - \text{mod}$ be the category of all right $L_X$-modules, $L_X - \text{mod}_y$ its subcategory of modules supported on $\hat{Y}$, and let $L_Y - \text{mod}$ be the category of right $L_Y$-modules. Let also $L_X - \text{mod}_y$ be the subcategory of $L_X - \text{mod}_y$, consisting of modules supported on $\hat{Y}'$.

We have the direct image functor $\iota_! : L_Y - \text{mod} \to L_X - \text{mod}_y$, but it is easy to see that its image belongs in fact to $L_X - \text{mod}_y$. And we have the right adjoint of $\iota_!$, denoted $\iota^! : L_X - \text{mod}_y \to L_Y - \text{mod}$.

Assume now that $L_X$ is $\mathcal{O}_X$-flat, and that $L_Y$ is a locally direct summand in $N(\hat{Y})$. Let $\mathcal{O}_X - \text{mod}_{L_Y}$ be the subcategory of $\mathcal{O}_X - \text{mod}$ defined as in Sect. 6.6. We have:
Theorem 7.8.
(1) An object $\mathcal{M} \in L_X - \text{mod}_g$ belongs to $L_X - \text{mod}'_g$ if and only if, as a $\mathcal{O}_X$-module, it belongs to $\mathcal{O}_X - \text{mod}_{\mathcal{L}_X}$.

(2) The functor $\iota$ is an equivalence of categories $L_Y - \text{mod} \to L_X - \text{mod}'_g$ with the quasi-inverse given by $i^!$.

8. Proof of Theorems 7.8 and 7.11

8.1. Let us start by observing that we have a natural map

$$\Omega^1(\mathfrak{g},x) \to \text{Ext}^1_{\mathcal{O}_{\mathfrak{g},\text{crit}}}(\mathcal{V}_{\mathfrak{g},\text{crit}}, \mathcal{V}_{\mathfrak{g},\text{crit}}).$$

This map can be constructed in the following general framework. Let $\mathcal{A}_{\mathbb{C}[\hbar]/\hbar^2}$ be a flat $\mathbb{C}[\hbar]/\hbar^2$-family of chiral algebras, and set $\mathcal{A} = (\mathcal{A}_{\mathbb{C}[\hbar]/\hbar^2})/\hbar$. We claim that there is a canonical map

$$\Omega^1(\mathfrak{g}(\mathcal{A}),x) \to \text{Ext}^1_{\mathcal{A}-\text{mod}}(\mathcal{A}_x, \mathcal{A}_x).$$

Indeed, consider $(\mathcal{A}_{\mathbb{C}[\hbar]/\hbar^2})_x$ as an extension

$$0 \to \mathcal{A}_x \to (\mathcal{A}_{\mathbb{C}[\hbar]/\hbar^2})_x \to \mathcal{A}_x \to 0$$

in the category of $\mathcal{A}_{\mathbb{C}[\hbar]/\hbar^2}$-modules. Let $\mathfrak{e}$ denote its class in $\text{Ext}^1_{(\mathcal{A}_{\mathbb{C}[\hbar]/\hbar^2})_x}(\mathcal{A}_x, \mathcal{A}_x)$.

For an element $a \in \mathfrak{g}(\mathcal{A})_x$, viewed as an endomorphism of $\mathcal{A}_x$ as a $\mathcal{A}_{\mathbb{C}[\hbar]/\hbar^2}$-module, we can produce two more elements of $\text{Ext}^1_{\mathcal{A}_{\mathbb{C}[\hbar]/\hbar^2}-\text{mod}}(\mathcal{A}_x, \mathcal{A}_x)$, namely, $a \cdot \mathfrak{e}$ and $\mathfrak{e} \cdot a$. However, it is easy to see that their difference already belongs to $\text{Ext}^1_{\mathcal{A}-\text{mod}}(\mathcal{A}_x, \mathcal{A}_x)$. Moreover, one readily checks that the resulting map $\mathfrak{g}(\mathcal{A})_x \to \text{Ext}^1_{\mathcal{A}-\text{mod}}(\mathcal{A}_x, \mathcal{A}_x)$ is a derivation, i.e., gives rise to a map in $\Omega^1$.

Explicitly, for $\mathcal{A} = \mathcal{A}_\mathfrak{g}$ the map of (8.1) looks as follows. Note that

$$\text{Ext}^1_{\mathcal{O}_{\mathfrak{g},\text{crit}}}(\mathcal{V}_{\mathfrak{g},\text{crit}}, \mathcal{V}_{\mathfrak{g},\text{crit}}) \simeq H^1(\mathfrak{g}(\widehat{\mathcal{O}}_x), \mathcal{V}_{\mathfrak{g},\text{crit}}).$$

Given an element $da \in \Omega^1(\mathfrak{g},x)$, where $a \in \mathfrak{g}(x) \subset \mathcal{V}_{\mathfrak{g},\text{crit}}$, consider its deformation $a_\hbar \in \mathcal{V}_{\mathfrak{g},\hbar}$ and define the corresponding 1-cocycle on $\mathfrak{g}(\widehat{\mathcal{O}}_x)$ by

$$g \in \mathfrak{g}(\widehat{\mathcal{O}}_x) \mapsto \frac{g \cdot a_\hbar}{\hbar} |_{\hbar = 0}.$$

This gives the desired map.

The next proposition states that the map of (8.1) is an isomorphism. This is a particular case of the following general theorem established in [FT]:

Theorem 8.2. We have a canonical isomorphism between $\Omega^i(\mathfrak{g},x)$ and the relative cohomology $H^i(\mathfrak{g}(\widehat{\mathcal{O}}_x), \mathcal{g}, \mathcal{V}_{\mathfrak{g},\text{crit}})$.

For $i = 1$ we have $H^1(\mathfrak{g}(\widehat{\mathcal{O}}_x), \mathcal{g}, \mathcal{V}_{\mathfrak{g},\text{crit}}) \simeq H^1(\mathfrak{g}(\widehat{\mathcal{O}}_x), \mathcal{V}_{\mathfrak{g},\text{crit}})$, and we obtain that (8.1) is indeed an isomorphism.

Here we will give a different proof of this fact, using some results from [BD].

Proposition 8.3. The above map $\Omega^1(\mathfrak{g},x) \to \text{Ext}^1_{\mathcal{O}_{\mathfrak{g},\text{crit}}}(\mathcal{V}_{\mathfrak{g},\text{crit}}, \mathcal{V}_{\mathfrak{g},\text{crit}})$ is an isomorphism.

Proof. Recall the setting of Sect. 5. Consider the short exact sequence

$$0 \to \mathcal{V}_{\mathfrak{g},\text{crit}} \to \mathfrak{D}_{\mathcal{G},\text{crit},x} \to \mathfrak{D}_{\mathcal{G},\text{crit},x}/\mathfrak{V}(\mathcal{V}_{\mathfrak{g},\text{crit}}) \to 0.$$
We know that $H^0(\mathfrak{g}(\tilde{O}_x), \mathcal{D}_{G, \text{crit}, x}) \simeq \mathcal{I}_g(\mathfrak{g}, \text{crit})$, and by a similar argument we obtain that $H^1(\mathfrak{g}(\tilde{O}_x), \mathcal{D}_{G, \text{crit}, x}) = 0$, since $H^1\left(\mathfrak{g}(\tilde{O}_x), \text{Fun}\left(G(\tilde{O}_x)\right)\right) = 0$. Therefore from the long exact sequence we obtain an isomorphism

$$H^1(\mathfrak{g}(\tilde{O}_x), V_{g, \text{crit}}) \simeq \left(\mathcal{D}_{G, \text{crit}, x}/\mathcal{I}(V_{g, \text{crit}})\right)/\mathcal{I}(V_{g, \text{crit}}).$$

By letting the point $x$ move, we obtain from the subspace

$$\left(\mathcal{D}_{G, \text{crit}, x}/\mathcal{I}(V_{g, \text{crit}})\right)/\mathcal{I}(V_{g, \text{crit}}) \subset \mathcal{D}_{G, \text{crit}, x}/\mathcal{I}(V_{g, \text{crit}})$$

a $D$-submodule, which we will denote by $\mathcal{O}_1(\mathfrak{g}) \subset \mathcal{D}_{G, \text{crit}}/(\mathcal{I}(A_{g, \text{crit}}) + \mathcal{I}(A_{g, \text{crit}}))$. We can form the Cartesian squares

$$\tilde{A}^g := \mathcal{O}_1(\mathfrak{g}) \times_{\mathcal{D}_{G, \text{crit}}/(\mathcal{I}(A_{g, \text{crit}}) + \mathcal{I}(A_{g, \text{crit}}))} \mathcal{D}_{G, \text{crit}}/(\mathcal{I}(A_{g, \text{crit}})),$$

$$\tilde{A}^{\text{ren}, \tau} := \mathcal{O}_1(\mathfrak{g}) \times_{\mathcal{D}_{G, \text{crit}}/(\mathcal{I}(A_{g, \text{crit}}) + \mathcal{I}(A_{g, \text{crit}}))} \mathcal{D}_{G, \text{crit}}.$$
$D_{crit}(Gr_G)$–mod; for $V \in \text{Rep}(L G)$, let $\mathcal{F}V \in D_{crit}(Gr_G)$–mod be the corresponding D-module.

Set
$$\mathcal{V}_{\mathfrak{g},x} := \text{Hom}_{\mathfrak{g},crit} (\mathcal{V}_{\mathfrak{g},crit}, \Gamma(Gr_G, \mathcal{F}V)).$$

Theorem 5.4.8 and Sect. 5.5.1 of [BD] imply that $V \mapsto \mathcal{V}_{\mathfrak{g},x}$ is a tensor functor from $\text{Rep}(L G)$ to the category of locally free finitely generated $\mathfrak{g}_{\mathfrak{g},crit}$-modules. By letting the point $x$ move along the curve, for every $V$ as above, we obtain a $\mathfrak{g}_{\mathfrak{g},crit}$-module, denoted $\mathcal{V}_x$, and the assignment $V \mapsto \mathcal{V}_x$ is the sought-for principal $L G$-bundle on $\text{Spec}(\mathfrak{g}_{\mathfrak{g},crit})$.

The assertion of Theorem 5.5.3 of [BD], combined with Theorem 3.18, implies that the Lie-* algebroid $\Omega^j_*(\mathfrak{g})$ is the universal Lie-* algebroid over $\mathfrak{g}_{\mathfrak{g},crit}$, whose action lifts to the above $L G$-bundle. However, as we have seen above, the Lie-* algebroid $\Omega^1(\mathfrak{g})$ acts on every $\mathfrak{g}_{\mathfrak{g},crit}$-module of the form $\text{Hom}_{\mathfrak{g}_{\mathfrak{g},crit}} (\mathcal{V}_{\mathfrak{g},crit}, \Gamma(Gr_G, \mathcal{F}V))$, for $V \in D_{crit}(Gr_G)$–mod. Again, globally over $X$, we obtain that $\Omega^1(\mathfrak{g})$ acts on all $\mathcal{V}_x$. This implies that we have a splitting $\Omega^1(\mathfrak{g}) \to \Omega^1(\mathfrak{g})$, which is automatically an isomorphism.

8.4. Next we will prove that $\text{Ext}^i_{\mathfrak{g}_{\mathfrak{g},crit}–mod}(\mathcal{V}_{\mathfrak{g},crit}, \mathcal{V}_{\mathfrak{g},crit})$ is flat as a $\mathfrak{g}_{\mathfrak{g},crit}$-module for any $i$. This statement can be formally deduced from Theorem 8.2, but we will give a different proof.

First, we claim that the topological Lie algebroid $\hat{h}^L_x(\Omega^1(\mathfrak{g}))$ over $\text{Spec}(\mathfrak{g}_{\mathfrak{g},crit})$ (defined as in [CHA], Sect. 2.5.18) acts on every such $\text{Ext}^i$. Indeed, consider the Lie algebra $H^0_{DR}(\mathcal{D}_x, \Omega^1(\mathfrak{g}))$. The Lie-* action of $\mathfrak{a}_g^L$ on $A_{\mathfrak{g},crit}^0$ yields an action of $H^0_{DR}(\mathcal{D}_x, \Omega^1(\mathfrak{g}))$ on the associative algebra $\hat{A}_{\mathfrak{g},crit,x}$ by outer derivations. Since the $A_{\mathfrak{g},crit}$-action on $\mathcal{V}_{\mathfrak{g},crit}$ lifts to an action of $\mathfrak{a}_g^L$, we obtain that $H^0_{DR}(\mathcal{D}_x, \Omega^1(\mathfrak{g}))$ indeed acts on every $\text{Ext}^i_{\mathfrak{g}_{\mathfrak{g},crit}–mod}(\mathcal{V}_{\mathfrak{g},crit}, \mathcal{V}_{\mathfrak{g},crit})$. Since $\mathcal{V}_{\mathfrak{g},crit}$ is a $\mathfrak{g}_{\mathfrak{g},crit}$-module, so is $\text{Ext}^i$, and the above $H^0_{DR}(\mathcal{D}_x, \Omega^1(\mathfrak{g}))$-action extends to an action of its completion $\hat{h}^L_x(\Omega^1(\mathfrak{g}))$.

By identifying $\mathfrak{X}_x$ with $\mathbb{C}(t)$, we endow $\mathfrak{g}_{\mathfrak{g},crit}$ with a $\mathbb{Z}$-grading by letting $t$ have degree $-1$. In this case $\mathcal{V}_{\mathfrak{g},crit}$ is a non-negatively graded $\mathfrak{g}_{\mathfrak{g},crit}$-module. Moreover, the terms of the standard complex computing the cohomology $\text{Ext}^i_{\mathfrak{g}_{\mathfrak{g},crit}–mod}(\mathcal{V}_{\mathfrak{g},crit}, \mathcal{V}_{\mathfrak{g},crit}) \simeq H^i(\mathfrak{g}(\mathcal{O}_x), \mathcal{V}_{\mathfrak{g},crit})$ are also non-negatively graded.

By applying Lemma 6.2.2 of [BD], we conclude that the above $\text{Ext}^i$ is free over $\mathfrak{g}_{\mathfrak{g},crit}$.

**Lemma 8.5.** The module $\mathcal{V}_{\mathfrak{g},crit}$ is flat over $\mathfrak{g}_{\mathfrak{g},crit}$.

**Proof.** The lemma is proved by passing to the associate graded. Recall from Theorem 5.5.3 that $\mathcal{V}_{\mathfrak{g},crit}$ and $\mathfrak{g}_{\mathfrak{g},crit}$ are naturally filtered, and

$$\text{gr}(\mathcal{V}_{\mathfrak{g},crit}) \simeq \text{Fun}(g^* \times G_m \Gamma(\mathcal{D}_x, \Omega X)) \simeq \mathcal{J} (\text{Fun}(g^* \times G_m \omega X))_{x} \quad \text{and} \quad \text{gr}(\mathfrak{g}_{\mathfrak{g},crit}) \simeq \text{Fun}((g^* \times G_m \Gamma(\mathcal{D}_x, \Omega X)) \simeq \mathcal{J} (\text{Fun}(g^* \times G_m \omega X))_{x}.

Now we apply Theorem A.4 of [ET], which exactly asserts that $\mathcal{J} (\text{Fun}(g^* \times G_m \omega X))$ is flat over $\mathcal{J} (\text{Fun}(g^* \times G_m \omega X))$. \(\square\)

Finally, we are ready to prove the following:

**Proposition 8.6.** For any $\mathfrak{g}_{\mathfrak{g},crit}$-module $\mathcal{L}$, the natural map

$$\text{Ext}^i_{\mathfrak{g}_{\mathfrak{g},crit}–mod}(\mathcal{V}_{\mathfrak{g},crit}, \mathcal{V}_{\mathfrak{g},crit}) \otimes \mathcal{L} \to \text{Ext}^i_{\mathfrak{g}_{\mathfrak{g},crit}–mod}(\mathcal{V}_{\mathfrak{g},crit}, \mathcal{V}_{\mathfrak{g},crit} \otimes \mathcal{L})$$

is an isomorphism.
Proof. From the identification $\text{Ext}^i_{\hat{g}, \text{crit} - \text{mod}}(V_{g, \text{crit}}, V_{g, \text{crit}} \otimes \mathcal{L}) \simeq H^i(\hat{O}_{z}), V_{g, \text{crit}} \otimes \mathcal{L})$ and the standard complex, computing cohomology of the Lie algebra $g(\hat{O}_{z})$, we obtain that the functor $\mathcal{L} \mapsto \text{Ext}^i_{\hat{g}, \text{crit} - \text{mod}}(V_{g, \text{crit}}, V_{g, \text{crit}} \otimes \mathcal{L})$ commutes with direct limits. Therefore, to prove the proposition, we can suppose that $\mathcal{L}$ is finitely presented.

Since $\mathfrak{g}_{z,x}$ is isomorphic to a polynomial algebra (by Theorem 3.3(1)), any finitely presented module $\mathcal{L}$ admits a finite resolution by projective modules:

$$0 \to P_n \to \ldots \to P_1 \to P_0 \to \mathcal{L} \to 0.$$

By Lemma 3.5, the complex

$$0 \to V_{g, \text{crit}} \otimes P_n \to \ldots \to V_{g, \text{crit}} \otimes P_1 \to V_{g, \text{crit}} \otimes P_0 \to V_{g, \text{crit}} \otimes \mathcal{L} \to 0$$

is also exact.

Thus, we have a spectral sequence, converging to $\text{Ext}^i_{\hat{g}, \text{crit} - \text{mod}}(V_{g, \text{crit}}, V_{g, \text{crit}} \otimes \mathcal{L})$, with the $E_1^{i,j}$-term isomorphic to $\text{Ext}^i_{\hat{g}, \text{crit} - \text{mod}}(V_{g, \text{crit}}, V_{g, \text{crit}} \otimes P^j)$.

Since each $P^j$ is projective, we evidently have

$$\text{Ext}^i_{\hat{g}, \text{crit} - \text{mod}}(V_{g, \text{crit}}, V_{g, \text{crit}} \otimes P^j) \simeq \text{Ext}^i_{\hat{g}, \text{crit} - \text{mod}}(V_{g, \text{crit}}, V_{g, \text{crit}}) \otimes P^j.$$

But since all $\text{Ext}^i_{\hat{g}, \text{crit} - \text{mod}}(V_{g, \text{crit}}, V_{g, \text{crit}})$ are $\mathfrak{g}_{z,x}$-flat, this spectral sequence degenerates at $E_2$, implying the assertion of the proposition.

\[\square\]

Corollary 8.7. $\text{Ext}^1_{\hat{g}, \text{crit} - \text{mod}, \text{reg}}(V_{g, \text{crit}}, V_{g, \text{crit}} \otimes \mathcal{L}) = 0$.

Proof. We have a map

$$\mathfrak{j}/\mathfrak{j}^2 \otimes \text{Ext}^1_{\hat{g}, \text{crit} - \text{mod}}(V_{g, \text{crit}}, V_{g, \text{crit}} \otimes \mathcal{L}) \to \text{Hom}_{\hat{g}, \text{crit} - \text{mod}}(V_{g, \text{crit}}, V_{g, \text{crit}} \otimes \mathcal{L}),$$

and its adjoint

$$\text{Ext}^1_{\hat{g}, \text{crit} - \text{mod}}(V_{g, \text{crit}}, V_{g, \text{crit}} \otimes \mathcal{L}) \to \Theta(\mathfrak{j}_{g,x}) \otimes \text{Hom}_{\hat{g}, \text{crit} - \text{mod}}(V_{g, \text{crit}}, V_{g, \text{crit}} \otimes \mathcal{L}).$$

It is easy to see that

$$\text{Ext}^1_{\hat{g}, \text{crit} - \text{mod}, \text{reg}}(V_{g, \text{crit}}, V_{g, \text{crit}} \otimes \mathcal{L}) \subset \text{Ext}^1_{\hat{g}, \text{crit} - \text{mod}}(V_{g, \text{crit}}, V_{g, \text{crit}} \otimes \mathcal{L})$$

is exactly the kernel of the latter map.

However, by Proposition 3.9 applied to $i = 0$ and 1, we can identify both sides in (8.4) with

$$\Omega^1(\mathfrak{j}_{g,x}) \otimes \mathcal{L} \to \Theta(\mathfrak{j}_{g,x}) \otimes \mathcal{L},$$

and the latter map is injective, since $\text{coker}(\Omega^1(\mathfrak{j}_{g,x}) \to \Theta(\mathfrak{j}_{g,x}))$ is flat as a $\mathfrak{g}$-module, by Theorem 3.8

\[\square\]
8.8. Recall the functor \( F : \hat{\mathfrak{g}}_{\text{crit}} \cdot \text{mod}^{G(\hat{\mathcal{O}})}_{\text{reg}} \to \hat{\mathfrak{g}}_{\theta,x} \cdot \text{mod} \) and its left adjoint \( G : \hat{\mathfrak{g}}_{\theta,x} \cdot \text{mod} \to \hat{\mathfrak{g}}_{\text{crit}} \cdot \text{mod}^{G(\hat{\mathcal{O}})}_{\text{reg}} \) defined in Sect. 6.3.

Note that the functor \( F \) is faithful. Indeed, a module \( M \in \hat{\mathfrak{g}}_{\text{crit}} \cdot \text{mod}^{G(\hat{\mathcal{O}})}_{\text{reg}} \) necessarily contains a non-zero vector, which is annihilated by \( \mathfrak{g} \otimes \mathbb{C}[t] \) \( \subset \hat{\mathfrak{g}}_{\text{crit}} \). Therefore we have a non-zero map \( \mathfrak{g}^\lambda_{\theta,\text{crit}} \to M \), and by Lemma 6.9 \( \lambda \) must be equal to 0.

Note that the assertion of Proposition 8.6 for \( i = 0 \) implies that the adjunction morphism

\[ \mathcal{L} \to F \circ G(\mathcal{L}) \]

is an isomorphism. We claim that this, combined with Corollary 8.7, formally implies Theorem 6.3.

We have to show that for \( M \in \hat{\mathfrak{g}}_{\text{crit}} \cdot \text{mod}^{G(\hat{\mathcal{O}})}_{\text{reg}} \) the adjunction map

\[ G \circ F(M) \to M \]

is an isomorphism.

This map is injective. Indeed, if \( M' \) is the kernel of \( G \circ F(M) \to M \), by the left exactness of \( F \), we would obtain that

\[ F(M') = \ker(F \circ G \circ F(M) \to F(M)) \sim \ker(F(M) \to F(M)) = 0. \]

But we know that the functor \( F \) is faithful, so \( M' = 0 \).

Let us prove that \( G \circ F \to \text{Id} \) is surjective. Let \( M'' \) be the cokernel of \( G \circ F(M) \to M \). We have the long exact sequence

\[ 0 \to F \circ G \circ F(M) \to F(M) \to F(M'') \to R^1F(G \circ F(M)) \to \ldots \]

However, Corollary 8.7 implies that \( R^1F(G(\mathcal{L})) = 0 \) for any \( \hat{\mathfrak{g}}_{\theta,x} \cdot \text{module} \mathcal{L} \). Therefore, the above portion of the long exact sequence amounts to a short exact sequence

\[ 0 \to F \circ G \circ F(M) \to F(M) \to F(M'') \to 0. \]

But the first arrow is an isomorphism, which implies that \( F(M'') = 0 \) and hence \( M'' = 0 \). Thus, Theorem 6.3 is proved.

As a corollary, we obtain the following result. Let \( \sigma \in \text{Spec}(\hat{\mathfrak{g}}_{\theta,x}) \) be a \( \mathbb{C} \)-point, and consider the subcategory \( \hat{\mathfrak{g}}_{\text{crit}} \cdot \text{mod}^{G(\hat{\mathcal{O}})}_{\text{reg}}(\sigma) \) of \( \hat{\mathfrak{g}}_{\text{crit}} \cdot \text{mod}^{G(\hat{\mathcal{O}})}_{\text{reg}} \) whose objects are the \( \hat{\mathfrak{g}}_{\text{crit}} \)-modules with central character equal to \( \sigma \). Theorem 6.3 implies that the category \( \hat{\mathfrak{g}}_{\text{crit}} \cdot \text{mod}^{G(\hat{\mathcal{O}})}_{\text{reg}}(\sigma) \) is equivalent to the category of vector spaces. In particular, the module

\[ V_{\theta,\sigma} := V_{\theta,\text{crit}} \otimes \mathbb{C}_\sigma \]

is irreducible.

8.9. Finally, let us prove Theorem 6.11. Let \( \mathcal{M} \) be an object of \( \hat{\mathfrak{g}}_{\text{crit}} \cdot \text{mod}^{G(\hat{\mathcal{O}})}_{\text{reg}} \), and let \( \mathcal{M}_i \) be the filtration as in Sect. 3.5. We need to show that the action of \( \mathcal{J}/\mathcal{J}^2 \) on \( \mathcal{M}_{i+1}/\mathcal{M}_i \), that maps \( \mathcal{M}_{i+1}/\mathcal{M}_i \to \mathcal{M}_{i}/\mathcal{M}_{i-1} \), factors through \( (\mathcal{J}/\mathcal{J}^2)/\mathcal{E}_\perp \).

We claim that for any extension in \( \hat{\mathfrak{g}}_{\text{crit}} \cdot \text{mod}^{G(\hat{\mathcal{O}})}_{\text{reg}} \)

\[ 0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0 \]

with \( \mathcal{M}_1, \mathcal{M}_3 \in \hat{\mathfrak{g}}_{\text{crit}} \cdot \text{mod}^{G(\hat{\mathcal{O}})}_{\text{reg}} \), the map \( \mathcal{J}/\mathcal{J}^2 \otimes \mathcal{M}_3 \to \mathcal{M}_1 \) factors through \( (\mathcal{J}/\mathcal{J}^2)/\mathcal{E}_\perp \).
Indeed, by Theorem 9.2 we can map surjectively onto $M^3$, a module of the form $V_{g,\text{crit}} \otimes L$,
where $L$ is a free $\mathfrak{g}[\tau]$-module. Hence, we can replace $M^3$ by $V_{g,\text{crit}}$. Again, by Theorem 9.2 $M^3$ has the form $V_{g,\text{crit}} \otimes L^3$ for some $\mathfrak{g}[\tau]$-module $L^3$.

Now, our assertion follows from Proposition 8.9 for $i = 1$.

9. FAITHFULNESS

9.1. Recall the category $\mathcal{A}_g^{\beta,\tau}$ mod $G(\hat{\mathfrak{g}})$ introduced in Sect. 6.14. Observe that the functor $F : \hat{\mathfrak{g}}_{\text{crit}} \text{mod}_{\hat{G}(\hat{\mathfrak{g}})} \to \mathfrak{g}_{\text{crit}}$ mod extends to a functor

$$A_g^{\beta,\tau} \text{mod} G(\hat{\mathfrak{g}}) \to A_g^{\beta} \text{mod}. \quad (9.1)$$

Moreover, Theorem 6.3 implies that the latter is also an equivalence of categories, with the quasi-inverse being $M \to M \otimes V_{g,\text{crit}}$.

We obtain that the functor $\Gamma : D_{\text{crit}}(\text{Gr}_G) \to \hat{\mathfrak{g}}_{\text{crit}}$ mod naturally factors as

$D_{\text{crit}}(\text{Gr}_G) \text{mod} \to A_g^{\beta} \text{mod} \to \hat{\mathfrak{g}}_{\text{crit}} \text{mod}_{\hat{G}(\hat{\mathfrak{g}})} \to \hat{\mathfrak{g}}_{\text{crit}} \text{mod}$,

where the second arrow is the tautological forgetful functor. We will denote the resulting functor $D_{\text{crit}}(\text{Gr}_G) \to A_g^{\beta}$ mod by $\Gamma^\beta$.

Remark. Suppose that $F_h$ is a $\mathbb{C}[[\hbar]]$-flat family of $\mathfrak{g}_{\hbar}$-twisted D-modules on $\text{Gr}_G$. By taking global sections, we obtain a $\mathbb{C}[[\hbar]]$-family of $\hat{\mathfrak{g}}_{\hbar}$-modules. Theorem 9.3 implies that this family is flat as well.

Set $F_0 = F/\hbar$. We obtain that the $\hat{\mathfrak{g}}_{\text{crit}}$-module $\Gamma(\text{Gr}_G, F_0)$ has two (a priori different) structures of object of $A_g^{\beta}$ mod: one such structure has been described above, and another is as in Sect. 6.13. However, it is easy to see that these structures in fact coincide.

The main result of this section is the following

**Theorem 9.2.** The above functor $\Gamma^\beta : D_{\text{crit}}(\text{Gr}_G) \to A_g^{\beta}$ mod is fully faithful.

9.3. Recall the category $\mathcal{A}_g^{\beta,\tau} \text{mod} G(\hat{\mathfrak{g}})$ by combining Theorems 6.15 and 6.11 we obtain:

**Corollary 9.4.** We have the following sequence of equivalences:

$$A_g^{\beta,\tau} \text{mod}_{\hat{G}(\hat{\mathfrak{g}})} \equiv A_g^{\beta} \text{mod},$$

where the last functor is as in (9.1).

Recall now the functor $i' : \hat{\mathfrak{g}}_{\text{crit}} \text{mod} G(\hat{\mathfrak{g}}) \to \hat{\mathfrak{g}}_{\text{crit}} \text{mod}_{\hat{G}(\hat{\mathfrak{g}})}$, and the corresponding functor

$$i' : A_g^{\beta,\tau} \text{mod}_{\hat{G}(\hat{\mathfrak{g}})} \to A_g^{\beta,\tau} \text{mod}_{\hat{G}(\hat{\mathfrak{g}})}.$$

Let us denote by $\tilde{\Gamma}$ the functor $D_{\text{crit}}(\text{Gr}_G) \to A_g^{\beta,\tau} \text{mod}_{\hat{G}(\hat{\mathfrak{g}})}$ equal to the composition

$$D_{\text{crit}}(\text{Gr}_G) \to \hat{\mathfrak{g}}_{\text{crit}} \text{mod}_{\hat{G}(\hat{\mathfrak{g}})} \equiv A_g^{\beta} \text{mod}_{\hat{G}(\hat{\mathfrak{g}})}.$$ 

The functor $\Gamma^\beta$ is the composition of $\tilde{\Gamma}$, followed by the $\mathcal{A}_g^{\beta,\tau} \text{mod}_{\hat{G}(\hat{\mathfrak{g}})} \to A_g^{\beta} \text{mod}$ of Corollary 9.4. So we are reduced to proving

**Theorem 9.5.** The functor $\tilde{\Gamma} : D_{\text{crit}} \to A_g^{\beta,\tau} \text{mod}_{\hat{G}(\hat{\mathfrak{g}})}$ is fully faithful.
9.6. Recall from Proposition 6.7 that for $\mathcal{M} \in \mathfrak{g}_{\text{crit}} \text{-}\text{mod}^{G(\mathfrak{g})}$, the object $\hat{\mathcal{M}}$ is in fact a direct summand of $\mathcal{M}$, denoted $\hat{\mathcal{M}}^{\text{reg}}$.

Let us first analyze the decomposition $\mathcal{M} \simeq \mathcal{M}^{\text{reg}} \oplus \mathcal{M}^{\text{non-reg}}$ when the module $\mathcal{M}$ equals $\mathfrak{D}_{G,\text{crit},x}$ itself. Letting $x$ move, we obtain a direct sum decomposition of D-modules $\mathfrak{D}_{G,\text{crit}} \simeq \mathfrak{D}^{\text{reg}}_{G,\text{crit}} \oplus \mathfrak{D}^{\text{non-reg}}_{G,\text{crit}}$.

Lemma 9.7. The homomorphism $U^{\text{ren},\tau}(L_{G,\text{crit}}) \to \mathfrak{D}_{G,\text{crit}}$ is an isomorphism onto $\mathfrak{D}^{\text{reg}}_{G,\text{crit}}$.

In particular, $\mathfrak{D}^{\text{reg}}_{G,\text{crit}}$ is a chiral subalgebra of $\mathfrak{D}_{G,\text{crit}}$.

Proof. It is enough to show that the homomorphism $U^{\text{ren},\tau}(L_{G,\text{crit}}) \to \mathfrak{D}_{G,\text{crit}}$ induces an isomorphism at the level of fibers. The fiber of $U^{\text{ren},\tau}(L_{G,\text{crit}})$, viewed as an object of $\mathfrak{A}_{G}^{\text{ren},\tau}-\text{mod}$, corresponds, under the equivalence of categories given by Corollary 9.4, to $V_{G,\text{crit}} \in \mathfrak{A}_{G}^{\text{ren},\tau}-\text{mod}$. Hence it remains to show that $(\mathfrak{D}_{G,\text{crit},x})^{\mathfrak{g}(\mathfrak{g})} \simeq V_{G,\text{crit}}$, but this is the content of Lemma 5.2.

This lemma implies, in particular, that for any $\mathcal{M} \in \mathfrak{D}_{G,\text{crit}} \text{-}\text{mod}^{G(\mathfrak{g})}$, the chiral action of $\mathfrak{D}^{\text{reg}}_{G,\text{crit}}$ maps $\mathcal{M}^{\text{reg}}$ to $\mathcal{M}^{\text{reg}}$ and $\mathcal{M}^{\text{non-reg}}$ to $\mathcal{M}^{\text{non-reg}}$.

Lemma 9.8. For $\mathcal{M} \in \mathfrak{D}_{G,\text{crit}} \text{-}\text{mod}^{G(\mathfrak{g})}$, the chiral action of $\mathfrak{D}^{\text{non-reg}}_{G,\text{crit}}$ maps $\mathcal{M}^{\text{reg}}$ to $\mathcal{M}^{\text{non-reg}}$.

Proof. According to Lemma 9.3, we can find an element of $\mathfrak{g}_{G,x}$ such that its action is nilpotent on $\mathfrak{D}^{0}_{G,\text{crit},x}$ and invertible on $\mathfrak{D}^{\text{non-reg}}_{G,\text{crit},x}$. We can assume that this element comes from a local section $a \in \mathfrak{g}_{x}$. For example, $a$ can be taken to be the section corresponding to the Segal-Sugawara $S_{0}$ operator.

Moreover, we can find a section $a'$ of $\mathfrak{g}_{x} \boxtimes \mathfrak{o}_{X}$ such that the $\mathfrak{o}$-module endomorphism of $\mathfrak{D}_{G,\text{crit}}$ given by

$$b \mapsto (h \boxtimes \text{id})(a' \otimes b)$$

is nilpotent on $\mathfrak{D}^{0}_{G,\text{crit}}$, and invertible on $\mathfrak{D}^{\text{non-reg}}_{G,\text{crit}}$. In the above formula $a' \otimes b$ is viewed as an element of the D-module $\mathfrak{D}_{G,\text{crit}} \boxtimes \mathfrak{D}_{G,\text{crit}}$ on $X \times X$, $[\cdot, \cdot]$ denotes the chiral bracket, and $(h \boxtimes \text{id})$ denotes the De Rham projection $\Delta^{!}(\mathfrak{D}_{G,\text{crit}}) \to \mathfrak{D}_{G,\text{crit}}$.

The chiral action gives rise to a map of D-modules on $X$:

$$\varphi : j_{x}^{*}j_{x}^{*}(\mathfrak{D}^{\text{non-reg}}_{G,\text{crit}}) \otimes \mathcal{M}^{\text{reg}} \to i_{x!}(\mathcal{M}),$$

where $i_{x}$ (resp., $j_{x}$) is the embedding of the point $x$ (resp., of its complement). For a section $b \in j_{x}^{*}j_{x}^{*}(\mathfrak{D}^{\text{non-reg}}_{G,\text{crit}})$ and an element $m \in \mathcal{M}^{\text{reg}}$, consider the section

$$a' \otimes b \otimes m \in j_{2,x}^{*}j_{2,x}^{*}(\mathfrak{g}_{x} \boxtimes \mathfrak{D}^{\text{non-reg}}_{G,\text{crit}}) \otimes \mathcal{M}^{\text{reg}},$$

where $j_{2,x}$ is the embedding of the complement to $\Delta_{X} \cup X \times x$ into $X \times X$.

By applying the Jacobi identity to the above section, we obtain that the action of $a'$ on the image of $\varphi$, given by the same formula as (1.2), is invertible. But this means that the subspace of $\mathcal{M}$ corresponding to the D-submodule $\text{Im}(\varphi) \subset i_{x!}(\mathcal{M})$ is supported off $\text{Spec}(\mathfrak{g}_{x})$. Therefore, this subspace belongs to $\mathcal{M}^{\text{non-reg}}$.

9.9. Let us assume for a moment that for any non-zero $\mathcal{M} \in \mathfrak{D}_{G,\text{crit}} \text{-}\text{mod}^{G(\mathfrak{g})}$, the component $\mathcal{M}^{\text{reg}} \simeq \Gamma(\mathcal{M})$ is necessarily non-zero. Let us show that the functor $\Gamma$ is then full.

Let $\mathfrak{D}^{\text{reg}}_{G,\text{crit},x}$ be the canonical associative algebra corresponding to the chiral algebra $\mathfrak{D}_{G,\text{crit}}$ and the point $x \in X$, see Sect. 9.4. We have a decomposition $\mathfrak{D}^{\text{reg}}_{G,\text{crit},x} = \mathfrak{D}^{\text{reg}}_{G,\text{crit},x} \oplus \mathfrak{D}^{\text{non-reg}}_{G,\text{crit},x}$. 


where the first summand is a subalgebra. For \( \mathcal{M} \in \mathcal{D}_{G,\text{crit}}{\mod}^{G(\hat{\mathcal{O}}_x)} \), the action of \( \hat{\mathcal{D}}_{G,\text{crit},x}^{\text{reg}} \) preserves the decomposition \( \mathcal{M} = \mathcal{M}^{\text{reg}} \oplus \mathcal{M}^{\text{non-reg}} \); moreover, by Lemma 9.8, the action of \( \hat{\mathcal{D}}_{G,\text{crit},x}^{\text{non-reg}} \) sends \( \mathcal{M}^{\text{reg}} \) to \( \mathcal{M}^{\text{non-reg}} \).

Observe that for \( \mathcal{M} \in \mathcal{D}_{G,\text{crit}}{\mod}^{G(\hat{\mathcal{O}}_x)} \), the map
\[
(9.3) \quad \hat{\mathcal{D}}_{G,\text{crit},x} \otimes \mathcal{M}^{\text{reg}} \rightarrow \mathcal{M}
\]
is automatically surjective. Indeed, its image is a \( \mathcal{D}_{G,\text{crit}} \)-submodule \( \mathcal{M}_1 \subset \mathcal{M} \), which satisfies \( \mathcal{M}_1^{\text{reg}} = \mathcal{M}^{\text{reg}} \). But then, for the quotient module \( \mathcal{M}_2 := \mathcal{M}/\mathcal{M}_1 \), we have: \( \mathcal{M}_2^{\text{reg}} = 0 \), which implies \( \mathcal{M}_2 = 0 \).

Similarly, for any element \( m \in \mathcal{M} \) we can always find a section \( b \) of \( \hat{\mathcal{D}}_{G,\text{crit},x} \), such that \( b \cdot m \) is a non-zero element of \( \mathcal{M}^{\text{reg}} \).

Let now \( \mathcal{M} \) and \( \mathcal{N} \) be two objects of \( \mathcal{D}_{G,\text{crit}}{\mod}^{G(\hat{\mathcal{O}}_x)} \), and let \( \phi: \mathcal{M}^{\text{reg}} \rightarrow \mathcal{N}^{\text{reg}} \) be a map in \( \mathcal{A}_{\mathcal{G}_{\text{crit}},x}^{\text{ren,}\tau} \)-mod. By Lemma 9.8, \( \phi \) is a homomorphism of \( \hat{\mathcal{D}}_{G,\text{crit},x}^{\text{reg}} \)-modules. We have to show that this map extends uniquely to a map of \( \hat{\mathcal{D}}_{G,\text{crit},x} \)-modules \( \mathcal{M} \rightarrow \mathcal{N} \).

The uniqueness statement is clear from the surjectivity of (9.3). To prove the existence, let us suppose by contradiction that the required extension does not exist. This means that there exist elements \( a_i \in \mathcal{D}_{G,\text{crit},x} \), and \( m_i \in \mathcal{M}^{\text{reg}} \), such that \( \sum a_i \cdot m_i = 0 \in \mathcal{M} \), but \( \sum a_i \cdot \phi(m_i) = n \neq 0 \) in \( \mathcal{N} \). Let \( b \in \mathcal{D}_{G,\text{crit},x} \) be an element such that \( 0 \neq b \cdot n \in \mathcal{N}^{\text{reg}} \). Let us write \( b \cdot a_i := c_i = c_i' + c_i'' \), where \( c_i' \in \hat{\mathcal{D}}_{G,\text{crit},x}^{\text{reg}} \), and \( c_i'' \in \hat{\mathcal{D}}_{G,\text{crit},x}^{\text{non-reg}} \).

By Lemma 9.8 we have \( \sum c_i' \cdot \phi(m_i) \neq 0 \). However, for the same reason, \( \sum c_i'' \cdot m_i = 0 \), which contradicts the fact that \( \phi \) was a morphism of \( \hat{\mathcal{D}}_{G,\text{crit},x}^{\text{reg}} \)-modules.

9.10. Finally, let us show that \( 0 \neq \mathcal{F} \in \mathcal{D}_{\text{crit}}(\text{Gr}_G)\mod \) implies that \( \mathcal{M}_\mathcal{F}^{\text{reg}} \neq 0 \), where \( \mathcal{M}_\mathcal{F} \) is the corresponding object of \( \mathcal{D}_{G,\text{crit}}{\mod}^{G(\hat{\mathcal{O}}_x)} \). By Theorem 1.2 and Theorem 9.4, this is equivalent to the fact that
\[
0 \neq \mathcal{F} \in \mathcal{D}_{\text{crit}}(\text{Gr}_G)\mod \Rightarrow \Gamma(\text{Gr}_G, \mathcal{F}) \neq 0.
\]

Note that the same argument works also in the negative and irrational level cases:

For a congruence subgroup \( K \subset G(\hat{\mathcal{O}}_x) \), let \( \mathcal{D}_K(\text{Gr}_G) \mod K \) be the subcategory of (strongly) \( K \)-equivariant \( \mathcal{D} \)-modules, and let \( \hat{\mathcal{G}}_\mathcal{K} \mod K \) be the subcategory of \( K \)-integrable modules. The functor \( \Gamma \) of global sections evidently maps \( \mathcal{D}_K(\text{Gr}_G) \mod K \) to \( \hat{\mathcal{G}}_\mathcal{K} \mod K \).

Recall now the setting for the Harish-Chandra action of [BD], Sect. 7.14. Namely, let \( G((t)) \) be the loop group corresponding to the point \( x \in X \), and \( G((t))/K \)-the corresponding ind-scheme. We have the convolution functor
\[
*: \mathcal{D}_K(G((t))/K)\mod \times \mathcal{D}_K(\text{Gr}_G)\mod K \rightarrow D^b(\mathcal{D}_K(\text{Gr}_G)\mod)
\]
where \( D^b(\cdot) \) stands for the bounded derived category. In addition, we have the functor
\[
*: \mathcal{D}_K(G((t))/K)\mod \times \hat{\mathcal{G}}_\mathcal{K} \mod K \rightarrow D^b(\hat{\mathcal{G}}_\mathcal{K} \mod).
\]

Moreover, the (derived) functor of global sections
\[
R\Gamma: D^b(\mathcal{D}_K(\text{Gr}_G)\mod K) \rightarrow D^b(\hat{\mathcal{G}}_\mathcal{K} \mod K)
\]
intertwines the two actions.

**Lemma 9.11.** For any non-zero object \( \mathcal{F} \in \mathcal{D}_K(\text{Gr}_G)\mod K \), there exists a \( G(\hat{\mathcal{O}}_x) \)-equivariant object \( \mathcal{F}' \in \mathcal{D}_K(G((t))/K)\mod \), such that \( \mathcal{F}' \ast \mathcal{F} \in D^b(\mathcal{D}_K(\text{Gr}_G)\mod G(\hat{\mathcal{O}}_x)) \) is non-zero.
Proof. We have an equivalence of categories
\[ \mathcal{F} \mapsto \mathcal{F}^* : D_{\kappa}(G_G) - \text{mod} K \to D_{\kappa}(G((t))/K) - \text{mod} G(\hat{\mathcal{O}}_x), \]
corresponding to the involution \( g \mapsto g^{-1} \) on \( G((t)) \). For \( \mathcal{F} \in D_{\kappa}(G_G) - \text{mod} K \) and \( \mathcal{F}' \in D_{\kappa}(G((t))/K) - \text{mod} G(\hat{\mathcal{O}}_x) \), the fiber at \( 1 \in G_G \) of the convolution \( \mathcal{F}' \star \mathcal{F} \) is canonically isomorphic to \( H_{DR}(G((t))/K, \mathcal{F}' \otimes \mathcal{F}^*) \). (Note that \( \mathcal{F}' \otimes \mathcal{F}^* \) is an object of the derived category of usual (i.e. non-twisted) right \( D \)-modules on \( G((t))/K \), therefore, global cohomology makes sense.)

In particular, this global cohomology is non-zero for \( \mathcal{F}' \) being (the direct image of) the constant \( D \)-module on a \( G(\hat{\mathcal{O}}_x) \)-orbit \( G(\hat{\mathcal{O}}_x) \cdot g \subset G((t))/K \), for some \( g \in G((t))/K \), such that the fiber \((\mathcal{F}')_g \) is non-zero.

Using this lemma, our non-vanishing assertion reduces to the fact that for a non-zero \( \mathcal{F} \in D^b(D_{\kappa}(G_G) - \text{mod} G(\hat{\mathcal{O}}_x)) \), the object \( R\Gamma(G_G, \mathcal{F}) \) is non-zero either.

To prove it, note that since the functor \( \Gamma \) is exact, we can assume that \( \mathcal{F} \) belongs to the abelian category of \( D \)-modules. By the semi-smallness result [BD], Sect. 5.3.6, the convolution \( * \) is exact on \( D_{\kappa}(G_G) - \text{mod} G(\hat{\mathcal{O}}_x) \), i.e., \( D_{\kappa}(G_G) - \text{mod} G(\hat{\mathcal{O}}_x) \) acquires a structure of monoidal category. (Note that for \( \kappa \) integral, the Satake equivalence identifies \( D_{\kappa}(G_G) - \text{mod} G(\hat{\mathcal{O}}_x) \) with the category of representation of the Langlands dual group \( ^L G \).

For an object \( \mathcal{F} \in D_{\kappa}(G_G) - \text{mod} G(\hat{\mathcal{O}}_x) \) (which we can assume to be finitely generated), let \( \mathcal{F}^* \in D_{-\kappa}(G_G) - \text{mod} G(\hat{\mathcal{O}}_x) \) be the object as in the proof of Lemma 9.11 and take \( \mathcal{F}' \in D_{\kappa}(G_G) - \text{mod} G(\hat{\mathcal{O}}_x) \) be the Verdier dual of \( \mathcal{F}^* \).

Let \( \delta_1 \) be the delta-function twisted \( D \)-module, corresponding to the unit point \( 1 \in G_G \). By adjunction, we obtain a non-zero map \( \delta_1 \to \mathcal{F}' \star \mathcal{F} \). This map is necessarily an injection, because \( \delta_1 \) is irreducible in \( D_{\kappa}(G_G) - \text{mod} \). Hence, by the exactness of \( \Gamma \), we obtain
\[ V_{g, \kappa} \simeq \Gamma(G_G, \delta_1) \neq 0 \Rightarrow \Gamma(G_G, \mathcal{F}' \star \mathcal{F}) \neq 0, \]
which, in turn, implies that \( \Gamma(G_G, \mathcal{F}) \neq 0 \).

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