Global Entanglement for Multipartite Quantum States

Chang-shui Yu and He-shan Song
Department of Physics, Dalian University of Technology, Dalian 116024, China
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Based on the residual entanglement [9] (Phys. Rev. A 71, 044301 (2005)), we present the global entanglement for a multipartite quantum state. The measure is shown to be also obtained by the bipartite partitions of the multipartite state. The distinct characteristic of the global entanglement is that it consists of the sum of different entanglement contributions. The measure can provide sufficient and necessary condition of fully separability for pure states and be conveniently extended to mixed states by minimizing the convex hull. To test the sufficiency of the measure for mixed states, we evaluate the global entanglement of bound entangled states. The properties of the measure discussed finally show the global entanglement is an entanglement monotone.

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I. INTRODUCTION

Entanglement is an essential ingredient in the broad field of quantum information theory. Quantification of entanglement as a central problem in quantum information theory has attracted a lot of attention in recent years.

A lot of methods to quantify entanglement have been proposed. For bipartite pure states, the partial entropy of the density matrix of a system defined by

$$E(\psi) = -\text{Tr}(\rho \log_2 \rho) = -\sum_i (\lambda_i \log_2 \lambda_i), \quad (1)$$

can provide a good measure of entanglement, where $\rho$ is the reduced density matrix obtained by trace over one of the subsystems, $\lambda_i$ is the $i$th eigenvalue of $\rho$. For mixed states, the entanglement of formation is defined by

$$E_f(\rho) = \min \sum_i p_i E(\psi_i) \quad (2)$$

with $E(\psi_i)$ the entanglement measure for the pure state $\psi_i$ corresponding to all the possible decompositions $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$. Eq.(2) is a complex constrained optimization in mathematics, hence it is hard to compute for a general mixed states, even numerically. Fortunately, Wootters [1] has shown the remarkable concurrence for a bipartite system of two spin (or pseudo spin) half particles, which can be well employed to estimate bipartite entanglement and sheds new light on the quantification of entanglement. Later, Armin Uhlmann [2] generalized the concurrence, and Koenraad Audenaert et al [3] defined a concurrence vector for bipartite systems in arbitrary dimension the length of which for pure states is proved by Wootters [4] to be equal to the $I$-concurrence introduced by Rungta et al [5], whilst they presented an effective method to extend the definition to the case of mixed states by minimizing the convex hull.

There has also been ongoing efforts to investigate entanglement measure for multipartite states [6-12]. In Ref. [6-9], the authors were focused on the genuine multipartite entanglement measure which embodies a collective property of multipartite systems. In Ref. [10], the authors extracted only the correlation between two subsystem among a given multipartite system. The authors in Ref. [11,12] only considered bipartite correlations between single subsystems and the remainder, whilst they present the concept of global entanglement because examples there showed that the entanglement measure does not vanish for semiseparable states. However, the measure is only confined to the pure quantum systems of qubits.

In this paper, starting with the residual entanglement [6,9], we present a quantity which is shown to consist of the sum of different entanglement contributions, and can also be considered as the sum of different correlations, unlike the entanglement measures given in Ref. [6-12]. In this sense, we also call it the global entanglement. The global entanglement can provide a sufficient and necessary condition of full separability for pure states. Furthermore, the global entanglement happens to be conveniently obtained by the idea of bipartite partition of a multipartite quantum states, which includes the case introduced in Ref. [11,12] (For tripartite systems of qubits, our global entanglement is equivalent to the original one). Extending the measure to mixed states by minimizing the convex hull, we obtain the lower bound analogous to those in Ref. [3] and Ref. [9]. As an application to test fully separability, we discuss the global entanglement of three bound entangled states introduced in [13,14,15], respectively. The properties of the global entanglement discussed finally shows that it is an entanglement monotone. The paper is organized as follows: In section II, firstly, we present the global entanglement for multipartite pure states; secondly, we extend it to the case of mixed states; lastly, we discuss the properties of the measure. The conclusion is drawn in section III.

*Electronic address: hssong@dlut.edu.cn
II. GLOBAL ENTANGLEMENT FOR MULTIPARTITE QUANTUM STATES

A. Global entanglement for multipartite pure states

At first, let us recall the definition of the concurrence vectors [3, 16] for bipartite states. Considering a bipartite pure state defined in $n_1 \times n_2$ dimension, written by

$$|\psi\rangle_{AB} = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} a_{ij} |i\rangle_A \otimes |j\rangle_B,$$

with $\sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} |a_{ij}|^2 = 1$. The concurrence vector $C$ can be defined by

$$C = (C_{00}, C_{01}, \cdots, C_{n_1(n_2-1)} n_2(2-n_2-1)),$$

where $C_{\alpha\beta} = \langle \psi_{AB}^\alpha | s_{\alpha\beta} | \psi_{AB}^\beta \rangle$ with $s_{\alpha\beta} = L_\alpha \otimes L_\beta; L_\alpha, \\alpha = 1, 2, \cdots, n_1(n_1-1)/2$ and $L_\beta, \beta = 1, 2, \cdots, n_2(n_2-1)/2$ are the generators of $SO(n_1)$ and $SO(n_2)$, respectively. The length of the vector $C$ (we call it concurrence throughout the paper) hence be given by

$$C(|\psi\rangle_{AB}) = \sqrt{\sum_{\alpha\beta} |C_{\alpha\beta}|^2}.$$

$C(|\psi\rangle_{AB})$ is an entanglement measure for bipartite pure states. In particular, $C(|\psi\rangle_{AB})$ can be reduced to Wooters’ concurrence for $n_1 = n_2 = 2$. For multipartite quantum states, considering the bipartite partitions, the multipartite states can be considered as bipartite quantum states. Therefore, by the definition above, one can also obtain the concurrence of such bipartite states.

For any bipartite mixed state $\rho_{AB} = \sum_{k=1}^{K} \omega_k |\psi_{AB}^k\rangle \langle \psi_{AB}^k|$, defined in $n_1 \times n_2$ dimension, the concurrence $C(\rho_{AB})$ [9] can be given by

$$C(\rho_{AB}) = \max_{z \in C_{\alpha\beta}} \lambda_1(z) - \sum_{i=1}^n \lambda_i(z).$$

Here $\lambda_i(z)$ are the singular values of $n_1(n_1-1)/2 n_2(n_2-1)/2$ $A_{\alpha\beta} = M^{1/2} \Phi S_{\alpha\beta} \Phi M^{1/2}$ with $\rho_{AB} = \Phi M \Phi^\dagger$ being the eigenvalue decomposition, and $z_{\alpha\beta} = y_{\alpha\beta} e^{i\phi_{\alpha\beta}}$ are optimal parameters with $y_{\alpha\beta} > 0, \sum_{\alpha\beta} y_{\alpha\beta}^2 = 1$.

For tripartite pure quantum states of qubits, based on Wooters’ concurrence, the authors in Ref. [6] have defined the residual entanglement (or 3 tangle) given by

$$\tau_{ABC} = C_{A(BC)}^2 - C_{AB}^2 - C_{AC}^2,$$

where $C_{AB}$ and $C_{AC}$ are the concurrences of the original pure state $\rho_{ABC}$ with traces taken over qubits $C$ and $B$, respectively. $C_{A(BC)}$ is the concurrence of $\rho_{A(BC)}$ with qubits $B$ and $C$ regarded as a single object. $\tau_{ABC}$ is shown to be the genuine tripartite entanglement. As an extension of $\tau_{ABC}$ (3 tangle), hyperdeterminant in Ref. [17] has been shown to be an entanglement monotone and represent the genuine multipartite entanglement. However, it is easy to find that the hyperdeterminant for higher dimensional systems and multipartite system can not be explicitly given conveniently. In particular, so far the hyperdeterminant as an entanglement measure has not been able to be extended to mixed systems. Therefore, the hyperdeterminant is difficult to find the connection with the distribution of multipartite entanglement. As an extension of eq. (3) or the distribution of entanglement, by considering the concurrence of bipartite states in arbitrary dimension, Ref. [9] has generalized eq. (3) to the higher-dimensional systems, multipartite systems and mixed systems. However, it is unfortunate that so far no one has been able to show whether the residual entanglements obtained in Ref. [9] corresponding to different foci [6,9] are equal or not. Hence, strictly speaking, the generalized residual entanglement $\tau$ can not be called the exact genuine multipartite entanglement measure before $\tau$s are proved to be equal. However, we can confirm that $\tau$s are relevant quantities to genuine multipartite entanglement no matter whether they are equal or not. For simplification, we call $\tau$ n-tangle corresponding to the n subscripts of $\tau$, e.g. $\tau_{1234}$ is called 4-tangle.

For convenience and without loss of the generality, we first consider a 4-partite pure state $\Psi_{ABCD}$. According to Ref. [9], one can obtain the following 10 equalities corresponding to different foci,

$$C_{A(BCD)}^2 = C_{A(BC)}^2 + C_{AD}^2 + \tau_{A(BC)D},$$

$$C_{B(ACD)}^2 = C_{B(AC)}^2 + C_{BD}^2 + \tau_{B(AC)D},$$

$$C_{C(ABD)}^2 = C_{C(AB)}^2 + C_{CD}^2 + \tau_{C(AB)D},$$

$$C_{D(BCA)}^2 = C_{D(BC)}^2 + C_{DA}^2 + \tau_{D(BC)A},$$

$$C_{(AB)C(D)}^2 = C_{(AB)C}^2 + C_{(AB)D}^2 + \tau_{(AB)C(D)},$$

$$C_{(AC)B(D)}^2 = C_{(AC)B}^2 + C_{(AC)D}^2 + \tau_{(AC)BD},$$

$$C_{(AD)B(C)}^2 = C_{(AD)B}^2 + C_{(AD)C}^2 + \tau_{(AD)BC},$$

$$C_{(BC)AD}^2 = C_{(BC)A}^2 + C_{(BC)D}^2 + \tau_{(BC)AD}.$$
\[ C_{(BD)(AC)}^2 = C_{(BD)A}^2 + C_{(BD)C}^2 + \tau_{(BD)AC}. \]  

(12)

\[ C_{(CD)(AB)}^2 = C_{(CD)A}^2 + C_{(CD)B}^2 + \tau_{(CD)AB}. \]  

(13)

where the brackets in the subscripts denote single objects and \( C_{AB} \) and \( C_{AD} \) denote concurrences of the mixed state \( \rho_{AB} \) and \( \rho_{AD} \) which are obtained by tracing over qudits the lost indices correspond to from \( \Psi_{ABCD} \); the other analogous notations in equations (4-13) are defined in the similar way. It is worth noting that the permutations of the qudits in a bracket do not change the value of the left hand side of the equations. But the forms of the right hand side of the former four equations (4-7) will be changed. Hence considering all permutations of qudits in the former four equations, there exist two other analogous equations [18] corresponding to each of them. As given in Ref. [9], the analogous equation to eq. (3) has been shown to hold for mixed states. That is to say, for any mixed state \( \rho_{ab} \),

\[ \tau_{abc} = C_{a(bc)}^2 - C_{ab}^2 - C_{ac}^2, \]

where \( C_{a(bc)} \) is the concurrence of the mixed state \( \rho_{a(bc)} \), \( C_{ab} \) is the concurrence of the mixed state \( \rho_{ab} \) which is given by tracing over qudit \( c \), \( C_{ac} \) is defined analogously to \( C_{ab} \). According to the equation for tripartite mixed quantum systems given above, we can expand above equations (4-13). For example, for eq. (4), we have

\[ C_{ABCD}^2 = C_{AB}^2 + C_{AC}^2 + \tau_{ABC} + C_{AD}^2 + \tau_{A(BC)D}. \]  

(14)

The others are analogous. Summing all the equations up, one can obtain that

\[ C_{(A)CD}^2 + C_{(B)CD}^2 + C_{(C)CD}^2 + C_{(D)CD}^2 + \tau_{(ABC)D} + \tau_{(AB)CD} + \tau_{(AC)BD} = 3 \sum_{m,n\in S} C_{mn}^2 + \frac{2}{3} \sum_{m,n,p\in S} \tau_{mn(p)}, \]

\[ \sum_{m,n,p,q\in S} \tau_{mn(p)q} + \frac{1}{4} \sum_{m,n,p,q\in S} \tau_{(mn)pq}, \]

where \( S = \{ A, B, C, D \} \), \( C_{mn} \) denote the concurrence vector of the reduced state \( \rho_{mn} \), \( \tau_{mn(p)} \) are the 3-tangle of the reduced state \( \rho_{mn(p)} \) corresponding to the focus qudit \( m \), and \( \tau_{mn(p)q} \), \( \tau_{(mn)pq} \) are the 4-tangle of \( \rho_{ABCD} \) corresponding to the foci \( m \) and \( mn \) respectively, whilst the bracket is defined the same to that above. From eq. (15), it is nicely seen that the right hand side of the equation consists of the sum of the squared concurrence (the first term), 3-tangles (the second term) and 4-tangles (the last two terms). That is to say the left hand side is the sum of different entanglement contributions. It is obvious that some terms in the left hand side are repeated.

One can always eliminate the repeated ones by changing the factors before each term in the right hand side. The reason is as follows. Take the term \( C_{(CD)(AB)}^2 \) as an example. From eq. (8) and eq. (13), one can have

\[ C_{(CD)(AB)}^2 = \frac{C_{(CD)(AB)}^2 + C_{(AB)(CD)}^2}{2}, \]

(16)

where \( f(C_{mn}) \) (squared concurrence), \( F(\tau_{mn(p)}) \) (3-tangles) and \( G(\tau_{(mn)pq}) \) (4-tangles) are not explicitly given. \( C_{(CD)(AB)}^2 \) can be eliminated by Eq. (15) minus eq. (16). Analogously, the other repeated terms can be eliminated. Hence, we define a quantity named global entanglement for the given 4-partite pure state \( \Psi_{ABCD} \) as

\[ C(\Psi_{ABCD}) \]

(17)

\[ = (C_{(A)BCD} + C_{(B)ACD} + C_{(C)ABD} + C_{(D)ABC})) \]

\[ + C_{(AB)CD}^2 + C_{(AC)BD}^2 + C_{(AD)(BC)}^2), \]

It happens that every term in the right hand side of eq. (17) just corresponds to the squared concurrence of the bipartite states generated by bipartition of the given 4-partite pure state. In other words, so long as we consider all the concurrences of the bipartite states after bipartition of the given 4-partite state, we can obtain the global entanglement. In fact, this conclusion is not confined to the case of 4-partite systems. Following the above procedure, one can easily prove that the analogous definition of the global entanglement for a given \( n \)-partite state can be shown as the sum of all \( m \)-tangles with \( m = 2, 3, \ldots, n \). Therefore, the global entanglement for any a state can be given in the following rigorous way.

Definition:-If we consider the \( i \) to \( (N-i) \) bipartite partitions of an \( N \)-partite state \( |\psi\rangle \), there exist \( \text{Num} \) different bipartite states defined in \( n_1(i) \times n_2(N-i) \) dimension, where \( \text{Num} = \left\{ \sum_{i=1}^{(N-2)/2} C_i^2 + \frac{1}{2} C_{N/2}^2, N \text{ is even} \right\} \), \( \left\{ \sum_{i=1}^{(N-1)/2} C_i^2, N \text{ is odd} \right\} \). The global entanglement \( C(|\psi\rangle) \) can be defined by

\[ C(\psi) = \sqrt{\sum_{\alpha=1}^{p} \sum_{\beta=1}^{Q} \sum_{\alpha=1}^{\text{Num}} \left| C_{\alpha\beta}^p \right|^2}, \]

(18)

where \( C_{\alpha\beta}^p = \langle \psi | S_{\alpha\beta}^p | \psi^* \rangle \) with \( S_{\alpha\beta}^p = L_\alpha \otimes L_\beta; L_\alpha, \alpha = 1, 2, \ldots, P \) and \( L_\beta, \beta = 1, 2, \ldots, Q \) are the generators of \( SO(n_1) \) and \( SO(n_2) \), respectively, with \( P = n_1^2 (n_1^2 - 1)/2 \) and \( Q = n_2^2 (n_2^2 - 1)/2 \); \( p = 1, 2, \ldots, \text{Num} \) denotes the \( p \)-th bipartite state.

It is obvious that a multipartite pure state \( |\varphi\rangle \) is fully separable if and only if \( C(|\varphi\rangle) = 0 \). The proof is omitted here.

In particular, when the definition is reduced to the tripartite quantum pure states, \( C(\psi) \) can be expressed
by

\[ C(\psi) = \sqrt{\sum_{\alpha=1}^{1} \sum_{\beta=1}^{6} \sum_{\gamma=1}^{3} |C^p_{\alpha \beta}|^2}, \]  

(19)

where \( C^p_{\alpha \beta} = \langle \psi | S^p_{\alpha \beta} | \psi^* \rangle \) with \( S^1_{\alpha \beta} = \sigma_y \otimes L_\beta \), \( L_\beta \) are the generator of \( SO(4) \); \( S^2_{\alpha \beta} = L_\beta \otimes \sigma_y \) and \( S^3_{\alpha \beta} = (I \otimes \text{swap}) \ (L_\beta \otimes \sigma_y) (I \otimes \text{swap}) \) with \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( \text{swap} \) is the swap operator defined as \( \text{swap} = \sum_{ijk'j} \delta_{ik} \delta_{jk'} |j⟩ ⟨ k'| \otimes |k⟩ ⟨ j' | \), \( j, k, j', k = 1, 2 \). Recalling the tensor cube in Ref. [19], one will find that every \( |C^p_{\alpha \beta}| \) just corresponds to a plane of the cube including the surfaces and the diagonal planes (the surfaces are corresponded to twice). However, unlike the criterion in Ref. [19], the global entanglement has good properties which will be discussed in the next section. But 3 more complex optimal parameters have to be introduced for the case of mixed states compared with Ref. [19].

In terms of the equivalent relations between the length of concurrence vectors for bipartite pure states and the I-concurrence [5], we can rewrite the global entanglement of multipartite pure states by

\[ C(|\Psi^{ABC...N}⟩) = \sqrt{\sum_{\alpha=1}^{P} \sum_{\beta=1}^{Q} \sum_{\gamma=1}^{Num} |C^p_{\alpha \beta}|^2} = \sqrt{2(Num - \sum_{\alpha=1}^{Num} Tr(\rho_p^2))}, \]  

(20)

where \( \rho_p \) denotes the reduced density matrix of bipartite pure states corresponding to the \( p \) to \( (N - p) \) partition of the given multipartite pure state \( |\Psi^{ABC...N}⟩ \). It is interesting that eq. (20) is just equivalent to that in Ref. [20] in essence. Hence, the global entanglement can also account for multi-partite correlations [20], unlike Ref. [11,12]. In fact, from the viewpoint that global entanglement consists of different entanglement contributions, the global entanglement also consists of the sum of different correlations from physics, which corresponds to full separability of a quantum state from mathematically. Furthermore, we also show the connection with the distribution of multipartite entanglement. In this sense, the global entanglement has its own merit, even though a single measure can not sufficient to capture all the properties of multipartite entanglement completely. In addition, one should note the difference between ours and that in Ref. [20] —— Ref. [20] considered all different reduced matrices, while we omit the repeated bi-partitions.

**B. Extension to multipartite mixed states**

Analogous to Ref. [3,9], our global entanglement for multipartite pure states can be extended to mixed states via minimizing their convex roofs,

\[ C(\rho) = \inf \sum_{k} \omega_k C(\psi^k), \]  

(21)

where the infimum is to be taken among all possible decompositions such that

\[ \rho = \sum_{k=1}^{P} \omega_k |\psi^k⟩ ⟨ \psi^k| \]. \]  

(22)

Considering \( C(\psi) \) for pure states, \( C(\rho) \) can be written by

\[ C(\rho) = \sum_{k} \omega_k \left( \sum_{\alpha=1}^{P} \sum_{\beta=1}^{Q} \sum_{\gamma=1}^{Num} |\langle \psi^k | S^p_{\alpha \beta} | \psi^k* \rangle|^2 \right) \]  

(23)

where \( P, Q \) and \( Num \) are defined the same to the above section.

Following the analogous procedure [9], one can get

\[ C(\rho) = \sqrt{\sum_{\alpha=1}^{P} \sum_{\beta=1}^{Q} \sum_{\gamma=1}^{Num} |\langle \psi^k | S^p_{\alpha \beta} | \psi^k* \rangle|^2} \]  

(24)

where \( A^p_{\alpha \beta} = M^{1/2} \Phi^T S^p_{\alpha \beta} \Phi M^{1/2} \) and \( z^p_{\alpha \beta} = y^p_{\alpha \beta} e^{i \phi_{\alpha \beta}} \) with \( y^p_{\alpha \beta} > 0 \), \( \sum_{\alpha \beta} (y^p_{\alpha \beta})^2 = 1 \), the superscript \( T \) denotes the transpose of a matrix; Furthermore, we consider the matrix notation of eq. (22) \( \rho = \Psi \Psi^\dagger \) and the eigenvalue decomposition \( \rho = \Phi \Phi^\dagger \) and the relation \( \Phi M^{1/2} T^\dagger = \Psi \Psi^\dagger \). Therefore, we obtain

\[ C(\rho) \geq \inf_{T} \sum_{k} \left| |T^T \left( \sum_{\alpha=1}^{P} \sum_{\beta=1}^{Q} \sum_{\gamma=1}^{Num} z^p_{\alpha \beta} A^p_{\alpha \beta} \right) |^2 \right|_{kk} \]  

(25)

The infimum is given by \( \max_{z \in C^p_{\alpha \beta}} \lambda_i(z) - \sum_{i>1} \lambda_i(z) \) with \( \lambda_j(z) \) are the singular values of \( \sum_{\alpha=1}^{P} \sum_{\beta=1}^{Q} \sum_{\gamma=1}^{Num} z^p_{\alpha \beta} A^p_{\alpha \beta} \). Hence we get the lower bound

\[ C(\rho) \geq \max_{z \in C^p_{\alpha \beta}} \lambda_1(z) - \sum_{i>1} \lambda_i(z), \]  

(26)

which is analogous to the result in Ref. [3,9]. It is obvious that eq. (26) is the necessary condition for full separability.

As the applications to test the sufficiency, we evaluate \( C(\rho) \) for tripartite bound entangled states of qubits similar to Ref. [19]. For the bound entangled state [13]

\[ \tilde{\rho} = \frac{1}{4} \left( 1 - \sum_{i=1}^{4} |\psi_i⟩ ⟨ \psi_i| \right), \]  

(27)
where \( \{\psi_i : i = 1, \cdots, 4\} \) corresponds to \( \{(0,1,0), (1,1,0), (0,0,1), (0,0,0)\} \) with \( \pm = (0) \pm (1)/\sqrt{2} \), one can get that \( C(\rho) = 0.1434 \).

For Dür-Cirac-Tarrach states \([14]\]

\[
\rho_{DCT} = \sum_{\sigma=\pm} \lambda_0^\sigma \langle \Psi_0^\sigma | \Psi_0^\sigma \rangle + \sum_{k=0,1,10,11} \lambda_k \left( \langle \Psi_k^+ | \Psi_k^+ \rangle + \langle \Psi_k^- | \Psi_k^- \rangle \right),
\]

where \( \langle \Psi_k^\sigma | \Psi_k^\sigma \rangle = \frac{1}{|k|} (|k_1 k_2 0\rangle \pm |\bar{k}_1 \bar{k}_2 1\rangle) \) with \( k_1 \) and \( k_2 \) the binary digits of \( k \), and \( \bar{k}_i \) denoting the flipped \( k_i \). Ref. [21] has shown that the state is bound entangled for \( \lambda_0^\pm = \frac{1}{2} ; \lambda_0^0 = \lambda_{10} = 0; \lambda_{11} = \frac{1}{b} \). In this case, one can get \( C(\rho_{DCT}) = 0.2158 \).

For the bound state \([15]\]

\[
\rho_{bound} = \frac{1}{N} \langle 2|GHZ \rangle \langle GHZ| + |000\rangle \langle 000| \nonumber
\]

\[
+ b|010\rangle \langle 010| + c|111\rangle \langle 111| + \frac{1}{c}|100\rangle \langle 100| + \frac{1}{b}|101\rangle \langle 101| + \frac{1}{a}|110\rangle \langle 110|,
\]

where \( |GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2} \) and \( N = 2a + b + c + 1/a + 1/b + 1/c \), one can easily obtain \( \text{nonzero} \ C(\rho_{bound}) \) for all \( a = b = c = 1/c \). The algorithm is the same to that in Ref. [19]. However we conjecture that the sufficiency would be weaker for higher dimensional systems.

C. The properties of the entanglement measure

Now we will show that the global entanglement is an entanglement monotone by the similar method to that in Ref. [22]. From eq. (20), it is obvious that \( C(|\Psi^{ABC\cdots N}\rangle) = C(U_i |\Psi^{ABC\cdots N}\rangle) \) where \( U_i \) is a local unitary operation on the \( i \)th subsystem. Furthermore, one can also find from eq. (20) that \( C(|\Psi^{ABC\cdots N}\rangle) \) is a concave function of all the reduced density matrix \( \rho_p \). At first, it should be noted that the local operations are only performed on a subsystem and the operations on one subsystem are independent of how to divide a multipartite state into a bipartite one. E.g. we assume the subsystem \( A \) is performed a quantum operation \( \{\varepsilon_{A,k}\} \) with \( k \) denoting different outcomes, then the final state after the operation is \( \rho_f = \sum_k p_k \varepsilon_{A,k} \rho_k \) where \( p_k = Tr[\varepsilon_{A,k}|\Psi^{ABC\cdots N}\rangle \langle \Psi^{ABC\cdots N}|] \) and \( \rho_k = (1/p_k)|\Psi^{ABC\cdots N}\rangle \langle \Psi^{ABC\cdots N}| \), which directly implies that \( \varepsilon_{A,k} \) is for the bipartition \( A - others \). For the bipartition \( AB - others \), \( \varepsilon_{A,k} \) should be considered as \( \varepsilon_{A,k} \otimes I_B \) with \( I_B \) the identity of subsystem \( B \). Therefore, for different bi-partitions, \( \varepsilon_{A,k} \) can always be considered as the kronecker product of \( \varepsilon_{A,k} \) and the identities of other subsystems which are considered as a big subsystem of the corresponding bipartite state.

For each \( k \), let \( \{r_{kl}, \psi_{kl}\} \) be a pure-state ensemble realizing \( \rho_k \) optimally such that

\[
C(\rho_k) = \sum_l r_{kl} C(\psi_{kl}), \tag{30}
\]

where \( \sum_l r_{kl} = 1, r_{kl} > 0 \) and \( \rho_k = \sum_l r_{kl} |\psi_{kl}\rangle \langle \psi_{kl}| \). Define \( \sigma_{kl} = Tr_{(A)}(|\psi_{kl}\rangle \langle \psi_{kl}|) \) with the subscript \( (A) \) denoting the big subsystem of bipartite states which include subsystem \( A \). Hence, due to the concave \( C(\rho_p) \), we can get

\[
C(\rho_f) = \sum_k p_k C(\rho_k) = \sum_k p_k r_{kl} C(\psi_{kl}) = \sum_k p_k r_{kl} \leq \sqrt{2(N \text{num} - \sum_{p=1}^{N \text{num}} \text{Tr}[\sigma_{kl}]^2)}, \tag{31}
\]

where \( (\rho_p)_{\sigma_{kl}} \) denotes the reduced density matrix of \( \rho \) corresponding to the \( p - to - (N - p) \) partition of \( \rho \). Note that for any \( p - to - (N - p) \) partition of \( \rho \)

\[
\rho_p = Tr_{(A)}(|\Psi^{ABC\cdots N}\rangle \langle \Psi^{ABC\cdots N}|) = \sum_k p_k Tr_{(A)}(\rho_k) = \sum_k p_k \sum_l r_{kl} Tr_{(A)}(|\psi_{kl}\rangle \langle \psi_{kl}|) = \sum_k p_k r_{kl} \sigma_{kl},
\]

one can obtain that

\[
C(\rho_f) \leq \sqrt{2(N \text{num} - \sum_{p=1}^{N \text{num}} \text{Tr}[\rho_p]^2)} = C(|\Psi^{ABC\cdots N}\rangle). \tag{32}
\]

For the mixed state \( \rho \), let \( \{p_k, \psi_k\} \) be a pure-state ensemble realizing \( \rho_k \) optimally such that \( C(\rho) = \sum_k p_k C(\psi_k) \). Analogous to the pure-state case, considering a quantum operation \( \{\varepsilon_{A,k}\} \), there exists the final state \( \rho_f = \sum_k p_k \rho_{fk} \) with \( \rho_{fk} \) corresponding to every \( \psi_k \). According to eq. (24), \( C(\psi_k) \geq C(\rho_{fk}) \) holds for each \( \psi_k \). It implies \( \sum_k p_k C(\psi_k) \geq \sum_k p_k C(\rho_{fk}) \geq C(\rho_f) \), where the last inequality is due to \( C(\rho_f) = \inf \sum_k p_k C(\rho_{fk}) \). All above show that the global entanglement is decreasing under local quantum operations, hence is an entanglement monotone.

III. CONCLUSION AND DISCUSSION

We have presented the global entanglement for multipartite quantum systems based on residual entanglement.
Unlike the previous measure for multipartite quantum states, the distinct characteristic of the global entanglement is that the measure consists of the sum of different entanglement contributions. Furthermore, we find that the global entanglement can be conveniently obtained by the idea of bipartite partitions of a quantum state. The measure has been shown to be an entanglement monotone. It is interesting that the global entanglement for tripartite quantum pure states of qubits has been effectively related to the tensor cube, and to that in Ref. [20]. Hopefully the global entanglement can further our understanding of multipartite entanglement.

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