Quaternionic quantum theory admits universal dynamics only for two-level systems

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We revisit the formulation of quantum mechanics over the quaternions and investigate the dynamical structure within this framework. Similar to standard complex quantum mechanics, time evolution is then mediated by a unitary which can be written as the exponential of the generator of time shifts. By imposing physical assumptions on the correspondence between the energy observable and the generator of time shifts, we prove that quaternionic quantum theory admits a time evolution for systems with a quaternionic dimension of at most two. Applying the same strategy to standard complex quantum theory, we reproduce that the correspondence dictated by the Schrödinger equation is the only possible choice, up to a shift of the global phase.

I. INTRODUCTION

Our understanding of quantum theory has significantly improved by investigating alternatives to quantum theory and analyzing how these alternatives would or would not be at variance with observations or expectations on the structure of a physical theory. Recently, these investigations are mostly based on the formalism of generalized probabilistic theories, where the fundamental objects are the convex sets of states and measurements. Different sets of assumptions have been found which are sufficient to single out quantum theory as the only possible theory1–7. A special role in these set of assumptions plays the analysis of the dynamics of such generalizations of quantum theory, see, for example, Ref. 8. Specifically, in quantum mechanics (and also in classical mechanics) there is an intimate relation between the Hamiltonian \( H \) as the energy observable and the generator of time shifts \(-\frac{i}{\hbar}H\) as it occurs in the Schrödinger equation.

Maybe the most notable early alternatives to quantum theory that have been studied in great detail are real and quaternionic quantum mechanics. Those are based on the question, why the wave function in quantum theory is complex valued and whether it would also be possible to formulate quantum theory over different fields, in those cases using real valued or quaternionic valued wave functions. The two main concerns for the real and quaternionic case are the composition of systems via a tensor product and a suitable modification of the Schrödinger equation. For real quantum theory, both topics lead to basically the same conclusion, namely that there must be a superselection rule9–12. In quaternionic quantum theory, the tensor product is complicated, at best13–16. However, the need for composing systems has also been questioned recently17. A consistent dynamics in quaternionic quantum theory has been formulated, however, also at the price of a superselection rule, where only a subspace of all self-adjoint operators can be used as the Hamiltonian of a system18. Nonetheless, Peres19 suggested a possible experiment on the basis of non-commuting phases which could reveal the characteristics of quaternionic quantum theory. Corresponding experiments were realized using neutron interferometry20 and using single photon interferometry21–23.

In this paper we use the standard quaternionic formulation24 of states and observables, that is, states are normalized vectors and observables are self-adjoint matrices over the quaternions. We ask which dynamical evolution is admissible in this case. For canonical quantum theory, the Schrödinger equation implies that the state evolves according to a unitary group parametrized by time. Each such group is determined by the Hamiltonian of the system. We seek for a similar construction with the aim to derive a Schrödinger-type equation for quaternionic quantum theory. In contrast to previous work18,24, we are interested in the case where the set of Hamiltonians is unrestricted, that is, every self-adjoint
operator must induce some dynamics. We find that this is only possible for one-level or two-level systems and that the corresponding Schrödinger-type equation is necessarily of the form

$$\hbar \frac{d}{dt} \psi(t) = [AH + HA - \text{tr}(H)A] \psi(t),$$  

where $H$ is the Hamiltonian and $A$ is a skew-adjoint operator which is independent of $H$. The term in the square brackets replaces here $-iH$ from the canonical Schrödinger equation. We arrive at this result assuming that the term in the square brackets is an $\mathbb{R}$-linear expression in $H$ and that it commutes with $H$.

The paper is organized as follows. In Section II we consider the case of canonical quantum theory. We review the connection between the Schrödinger equation, generators of time shifts, and Stone’s theorem and develop then the axioms for the correspondence between the Hamiltonian and the generator of time shifts. In Theorem 2 we establish that these axioms are sufficient to reproduce the usual Schrödinger equation. In Section III we then turn to the quaternionic case. We first summarize quantum theory over the quaternions. In Theorem 5 we characterize the possible dynamics in quaternionic quantum theory and subsequently discuss alternatives to the axioms that lead to Eq. (1) before we conclude in Section IV.

II. TIME EVOLUTION IN CANONICAL QUANTUM THEORY

In quantum mechanics, the time evolution of a system is described by the Schrödinger equation,

$$i\hbar \frac{d}{dt} \psi(t) = H_S \psi(t).$$  

For a time-independent Hamiltonian this gives rise to the unitary time evolution operator

$$U^S_t = e^{-\frac{i}{\hbar}H_S t},$$

which provides a solution of the Schrödinger equation via $\psi(t_0 + t) = U^S_t \psi(t_0)$.

A. Unitary groups and Stone’s theorem

There is a different way to obtain a time evolution operator $U_t$ of the same structure, without building on the Schrödinger equation. This is based on the assumptions that the transformation $U_t : \psi(t_0) \mapsto \psi(t_0 + t)$ is linear in $\psi(t_0)$, preserves the norm of $\psi(t_0)$, is independent of $t_0$, and is continuous in $t$. More precisely, $(U_t)_{t \in \mathbb{R}}$ must be a strongly continuous unitary group, that is,

(i) $U_t$ is unitary for all $t$, with $U_0 = 1$,

(ii) $U_{s+t} = U_s U_t$ for all $s, t$, and

(iii) $t \mapsto U_t$ is strongly continuous.

Condition (i) expresses linearity and isometry and $U_0 = 1$ is used to implement the identity $\psi(t_0) = U_0 \psi(t_0 + 0)$. (We do not consider antiunitary transformations as allowed by Wigner’s theorem.) Condition (ii) follows from the independence of $t_0$ and condition (iii) is a specification of the assumption of continuity. Strong continuity refers to the strong operator topology and reduces because of (i) and (ii) to $\lim_{t \to 0} ||U_t \psi - \psi|| = 0$ for all $\psi$.

The fundamental representation theorem of strongly continuous one-parameter unitary groups is due to Stone (see, for example, Ref. 25). For every such group $(U_t)_{t \in \mathbb{R}}$ there exists a unique self-adjoint operator $G$, such that

$$U_t = e^{-iGt}$$
holds. This result is valid for general Hilbert spaces, with subtleties occurring if \( t \mapsto U_t \psi \) is not differentiable at \( t = 0 \) for some \( \psi \). Clearly, in the finite-dimensional case this function is always differentiable.

B. Hamiltonians as generators of time shifts

From a physical perspective, the skew-adjoint operator \(-iG\) in Eq. (4) is responsible for time shifts and in this sense it is the generator of time shifts. The Schrödinger equation implies that the correspondence between the Hamiltonian and the generator of time shifts is obtained as \(-iG = -i\frac{\hbar}{\tau} H\). But is this the only way to establish a correspondence between the generator of time shifts and the Hamiltonian and if not, how can we classify the different possibilities?

To answer this question, we write the Schrödinger equation in the form

\[
\dot{\psi}(t) = \Phi_S(H)\psi(t)
\]

yielding the time evolution operator \( U^S_t = e^{\Phi_S(H)t} \). Here \( \Phi_S(H) \) is the generator of time shifts associated to the Hamiltonian \( H \), that is \( \Phi_S \) is a linear map from the self-adjoint matrices to the skew-adjoint matrices,

\[
\Phi_S : H \mapsto -i\frac{\hbar}{\tau} H.
\]

We denote by \( H(\mathbb{C}^n) [A(n, \mathbb{C})] \) the \( \mathbb{R} \)-vector space of self-adjoint (skew-adjoint) complex \( n \times n \) matrices. The skew-adjoint matrices obey \( A = -A^\dagger \) and together with the self-adjoint matrices, which satisfy \( H = H^\dagger \), they span the set of all matrices, that is,

\[
\text{Mat}(n, \mathbb{C}) = H(n, \mathbb{C}) \oplus_\mathbb{R} A(n, \mathbb{C}).
\]

The dimension of \( H(n, \mathbb{C}) \) equals the dimension of \( A(n, \mathbb{C}) \) and \( \Phi_S \) is a one-to-one mapping between these two vector spaces. This coincidence of dimensions is a peculiarity of the complex matrices. Over the reals as well as over the quaternions \( H \), the dimensions differ, specifically, \( \dim H(n, \mathbb{R}) = \dim A(n, \mathbb{R}) + n \) and \( \dim H(n, \mathbb{H}) = \dim A(n, \mathbb{H}) - 2n \).

In order to generalize \( \Phi_S \), we consider an arbitrary relation \( \varphi \subset H(n, \mathbb{C}) \times A(n, \mathbb{C}) \) between the Hamiltonians and the generators of time shifts. In literature, \( \varphi \) is sometimes called a dynamical correspondence\(^{6,26,27}\). We add several restrictions to this general case by requiring that \( \varphi \) is

\[
\begin{align*}
\text{(DC1)} & \quad \mathbb{R}\text{-homogeneous, } (H, A) \in \varphi \text{ implies } (\lambda H, \lambda A) \in \varphi \text{ for all } \lambda \in \mathbb{R}, \\
\text{(DC2)} & \quad \text{additive, } (H, A) \in \varphi \text{ and } (H', A') \in \varphi \text{ implies } (H + H', A + A') \in \varphi, \text{ and} \\
\text{(DC3)} & \quad \text{commuting, } (H, A) \in \varphi \text{ implies } HA = AH.
\end{align*}
\]

These assumptions have physical motivations and appeared earlier in literature, see, for example, Refs. 8 and 27. The relation \( \varphi \) should be \( \mathbb{R}\)-homogeneous (DC1) to match the intuition that higher energies correspond in direct proportion to faster time evolutions and vice versa. The assumption of additivity (DC2) can be easily justified for the case where \( H \) and \( H' \) commute since in this case we have the addition of two Hamiltonians that differ only in spectrum. Then for any eigenstate of \( H \) and \( H' \) an argument similar to the motivation for assumption (DC1) can be applied. Noncommuting Hamiltonians most prominently appear in the form of interaction Hamiltonians, where \( H = H_A + H_B + \mu H_I \). In many situations the interaction strength \( \mu \) is an external experimental parameter, for example, the strength of a magnetic field, and hence additivity is a reasonable assumption. This reasoning might be more difficult to apply, if the Hamiltonian does not emerge as an effective description, but is an unalterable property of the system. Finally, the relation \( \varphi \) should also be commuting (DC3), so that the Hamiltonian is invariant under time shifts. That is, the observable \( \tilde{H} \) should be a conserved quantity under its own time evolution,

\[
U^H_t U^H_t = e^{At}He^{-At} = H.
\]
for all \((H, A) \in \varphi\). The properties (DC1) and (DC2) make \(\varphi\) an \(\mathbb{R}\)-linear space and any relation \(\varphi\) obeys (DC1)–(DC3) if and only if the set \([(H, A) \in \varphi]\) is an \(\mathbb{R}\)-linear subspace of the normal matrices. Note that we do not include another familiar condition on \(\varphi\), namely, that it should be

(DC4) covariant under unitary transformations,

that is, \((H, A) \in \varphi\) implies \((V^*HV, V^*AV) \in \varphi\), for any unitary \(V\).

It is conceivable that the Hamiltonian alone does not determine the time evolution completely and that the map \(\varphi_1 : H \mapsto \{ A \mid (H, A) \in \varphi\}\) is multivalued. The set \(\varphi_2(H)\) could also be empty, in which case \(H\) would not correspond to any dynamics, that is, \(H\) would be unphysical. Conversely, the same dynamics might arise from different Hamiltonians so that \(\varphi_2 : A \mapsto \{ H \mid (H, A) \in \varphi\}\) is multivalued. Also the set \(\varphi_2(A)\) could be empty and hence the corresponding time evolution \(t \mapsto U_t\) would be unphysical. In this paper, we focus on the first case and we only consider maps \(\Phi : \mathbf{H}(n, \mathbb{C}) \to \mathbf{A}(n, \mathbb{C})\) such that \(\varphi_1(H) = \{ \Phi(H) \}\). The latter condition is not a restriction, since we can always obtain \(\varphi\) from a family of maps \((\Phi_y)_y\) via \(\varphi = \{ (H, \Phi_y(H)) \}_y\).

The conditions (DC1)–(DC3) are equivalent to the condition that \(\Phi\) is a commuting \(\mathbb{R}\)-linear map, that is, \(\Phi\) is \(\mathbb{R}\)-linear with \(H\Phi(H) = \Phi(H)H\). We now demonstrate that such maps must have a very specific form. For this we use the following result by Brešar.

**Lemma 1** (Corollary 3.3 in Ref. 28). Let \(\mathcal{A}\) be a simple unital ring. Then every commuting additive map \(f\) on \(\mathcal{A}\) is of the form \(f(x) = zx + g(x)\), where \(z \in Z(\mathcal{A})\) and \(g : \mathcal{A} \to Z(\mathcal{A})\) is an additive map.

Here, \(Z(\mathcal{A})\) denotes the center of \(\mathcal{A}\), that is, the elements of \(\mathcal{A}\) that commute with all of \(\mathcal{A}\). Since \(\text{Mat}(n, \mathbb{C})\) is a simple unital ring with center \(\mathbb{C}\), we can in principle apply Lemma 1 in order to characterize all maps \(\Phi\). However, we first need to extend the domain of \(\Phi\) to be all of \(\text{Mat}(n, \mathbb{C})\). This is readily achieved by the canonical extension \(\Phi_c\) of \(\Phi\) via

\[
\Phi_c(M_1 + iM_2) = \Phi(M_1) + i\Phi(M_2),
\]

where \(X = M_1 + iM_2\) is the unique decomposition of \(X \in \text{Mat}(n, \mathbb{C})\) into its self-adjoint and skew-adjoint part and \(M_1, M_2\) are self-adjoint matrices. Clearly this extension is additive (\(\mathbb{R}\)-linear) if \(\Phi\) is additive (\(\mathbb{R}\)-linear). The extension is also commuting if \(\Phi\) is additive and commuting. Indeed we have \(\Phi([M_1], M_2) + [\Phi(M_2), M_1] = 0\) due to \([\Phi(M_1), M_2], M_1 + M_2] = 0\) and hence \(\Phi_c(X, X) = i([\Phi(M_1), M_2] + [\Phi(M_2), M_1]) = 0\) holds. This embedding together with Lemma 1 yields the following characterization of all maps obeying (DC1)–(DC3).

**Theorem 2.** Every commuting \(\mathbb{R}\)-linear map \(\Phi : \mathbf{H}(n, \mathbb{C}) \to \mathbf{A}(n, \mathbb{C})\) is of the form

\[
\Phi(H) = i\lambda H + i \text{tr}(BH) \mathbf{1},
\]

where \(\lambda \in \mathbb{R}\) and \(B \in \mathbf{H}(n, \mathbb{C})\).

**Proof.** Applying Lemma 1 to the extended map \(\Phi_c\) yields \(\Phi_c(H) = \eta H + \text{tr}(QH)\) with \(\eta \in \mathbb{C}\) and \(Q \in \text{Mat}(n, \mathbb{C})\), if \(H \in \mathbf{H}(n, \mathbb{C})\). Note that we used the \(\mathbb{R}\)-homogeneity of \(\Phi_c\) to write \(g(H) = \text{tr}(QH)\). Since \(\Phi_c(H)\) must be skew-adjoint, \(\eta + \eta^* = 0\) and \(Q + Q^\dagger = 0\) follows, that is, \(\eta \in i\mathbb{R}\) and \(Q \in i\mathbf{H}(n, \mathbb{C})\).

Note that \(\Phi(H)\) is covariant if and only if \(B\) is real multiple of \(\mathbf{1}\). In quantum mechanics, we have \(\lambda = \frac{-i}{\hbar}\) and \(B = 0\). The value of \(\lambda\) constitutes a constant of nature, including a sign convention. The term involving \(B\) cannot be measured on a single system since it would only cause a global phase shift on the quantum state. In an interferometer-type setup even this phase shift would be accessible but is in contradiction to observation.
III. QUATERNIONIC QUANTUM THEORY

We have outlined the formalism for obtaining the Schrödinger equation for the familiar case of complex quantum mechanics. Our choices and assumptions have been made such that we can extend our considerations to construct a dynamical quantum theory over the quaternions. We start by summarizing a quaternionic version of quantum theory (see, for example, Ref. 24) and we then proceed by characterizing possible expressions for the Schrödinger equation.

A. The quaternions

The quaternions \( \mathbb{H} \) are an extension of the real and complex numbers. They form an associative division ring where multiplication is noncommutative. Any quaternion \( q \in \mathbb{H} \) can be written in the form

\[
q = a_1 + a_2 i + a_3 j + a_4 k, \tag{11}
\]

where the coefficients \( a_\ell \) are real numbers and \( i, j, k \) are the quaternion units, which play a role similar to the complex unit \( i \). The real part of \( q \) is \( \text{Re}(q) = a_1 \) and the imaginary part is \( \text{Im}(q) = (a_2, a_3, a_4) \). The multiplication on \( \mathbb{H} \) is commutative for the real numbers and otherwise determined by

\[
i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ij = -ji, \quad jk = -kj, \quad ki = -ik. \tag{12}
\]

yielding \( ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik \). Similar to the complex numbers, conjugation is defined by

\[
q^* = a_1 - a_2 i - a_3 j - a_4 k, \tag{13}
\]

yielding the rules \((uv)^* = v^* u^*, \quad qq^* = q^* q, \quad \text{and} \quad (q^*)^* = q\). The modulus \( |q| = \sqrt{qq^*} \) induces the euclidean norm \( \| (a_1, a_2, a_3, a_4) \| = |a_1 + a_2 i + a_3 j + a_4 k| \). This way, the quaternions are a complete normed \( \mathbb{R} \)-algebra.

We identify the complex numbers as a subset of the quaternions by identifying the complex unit \( i \) with the quaternion unit \( i \). This allows us to write uniquely \( q = a + bj \) for \( a, b \in \mathbb{C} \). Similar to the representation of complex numbers as real \( 2 \times 2 \) matrices, the quaternions can be represented as complex matrices,

\[
a + bj \mapsto \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}. \tag{14}
\]

This map is a \( \dagger \)-monomorphism of the corresponding real algebras, were the involution \( \dagger \) reduces on the quaternions to the conjugation \( * \).

B. Modules and matrices

Since quantum mechanics is formulated on the basis of complex Hilbert spaces, we use a similar structure over the quaternions, but taking into account the the noncommutativity of the quaternions. We consider here the \( n \)-fold direct product of quaternions, denoted by \( \mathbb{H}^n \). It forms a free bimodule and possesses, apart from commutativity, most properties of a vector space. In particular, since it arises from a direct product, it can be equipped with the canonical basis \((e^{(1)}, e^{(2)}, \ldots, e^{(n)})\).

For \( x, y \in \mathbb{H}^n \) with \( x = \sum_i x_i e^{(i)} \) and \( y \) similar we define the inner product to be of the canonical form, \( \langle x, y \rangle = \sum_i x_i^* y_i = \langle y, x \rangle^* \), giving also rise to the norm \( \| x \| = \sqrt{\langle x, x \rangle} \) and turning \( \mathbb{H}^n \) into a Hilbert module. For scalar multiplication with \( \alpha \in \mathbb{H} \) we obtain the rules
\[ \langle xo, y \rangle = \alpha^* \langle x, y \rangle \text{ and } \langle x, yo \rangle = \langle x, y \rangle \alpha. \] This suggests that scalar multiplication in \( \mathbb{H}^n \) is preferably taken from the right, although, technically, \( \mathbb{H}^n \) is a bimodule.

We take linear maps \( M: \mathbb{H}^n \to \mathbb{H}^m \) to be right-homogeneous, \( M(xo) = M(x)\alpha \) which allows for a representation of \( M \) as an \( m \times n \) matrix \( (M_{i,j})_{i,j} \) via \( M_{i,j} = \langle e^{(i)}, M e^{(j)} \rangle \). Then \( (x, M(y)) = \sum_{i,j} x_i^* M_{i,j} y_j \alpha \). We consider \( \text{Mat}(n, \mathbb{H}) \) as an \( \mathbb{R} \)-algebra, ignoring that it can be treated consistently as a left \( \mathbb{H} \)-module. Where unambiguous, we use \( \alpha \in \mathbb{H} \) also as the linear map \( \alpha \). The adjoint \( \dagger \) of a linear map is defined as usual, \( \langle x, M(y) \rangle = \langle M^\dagger(x), y \rangle \), and therefore \( (M^\dagger)_{i,j} = (M_{j,i})_{i,j} \). Since linear maps are well-represented by matrices, we mostly use the latter notion. Self-adjoint and skew-adjoint matrices are defined in the obvious way. For unitary matrices \( U \in \text{Mat}(n, \mathbb{H}) \), we note that \( U^\dagger U = 1 \) is equivalent to \( U U^\dagger = 1 \). If \( M \) is normal, \( M^\dagger M = MM^\dagger \), then there exits a diagonal matrix \( D \) with entries in \( \{ a + bi \mid a, b \in \mathbb{R}, \ b \geq 0 \} \) and a unitary \( U \) such that \( M = U D U^\dagger \). The trace \( \text{tr}(M) = \sum_{i,j} M_{i,j} \) for self-adjoint matrices is invariant under unitary transformations, \( \text{tr}(UMU^\dagger) = \text{tr}(M) \), and thus equivalent to the sum of diagonal elements of \( D \). This follows from the fact that one can readily construct matrices of the form \( (b_i b_j^\ast)_{i,j} \) the real span of which are the self-adjoint matrices.

It is sometimes convenient to use the embedding \( \Lambda: \text{Mat}(n, \mathbb{H}) \to \text{Mat}(2n, \mathbb{C}) \),

\[
\Lambda: A + Bj \mapsto \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix}.
\]

This map is a \( \dagger \)-monomorphism of the corresponding \( \mathbb{R} \)-algebras. In particular, we have \( \Lambda[(rA + BC^\dagger)] = r \Lambda(A) + \Lambda(B) \Lambda(C)^\dagger \) for \( r \in \mathbb{R} \) and \( A, B, C \in \text{Mat}(n, \mathbb{H}) \). The map \( \Lambda^{-1}: \text{Mat}(2n, \mathbb{C}) \to \text{Mat}(n, \mathbb{H}) \),

\[
\Lambda^{-1}: \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto A + Bj,
\]

is an \( \mathbb{R} \)-linear left inverse of \( \Lambda \), which, however, it is not preserving the algebraic properties of \( \text{Mat}(2n, \mathbb{C}) \).

C. Stone’s theorem

In order to study the dynamics in quaternionic quantum theory we proceed similar to the complex case by studying continuous unitary groups \( (U_t) \). Then, an analogous result to Stone’s theorem holds.

**Theorem 3.** For every continuous unitary group \( (U_t) \subset \text{Mat}(n, \mathbb{H}) \) there exist a unique skew-adjoint matrix \( A \in \text{Mat}(n, \mathbb{H}) \) such that \( U_t = e^{At} \).

This theorem has been proved in Ref. 18. For completeness, we provide here a proof for finite dimensions.

**Proof of Theorem 3.** Since the embedding \( \Lambda \) is a mapping between finite-dimensional vector spaces, the family \( \{ \Lambda(U_t) \} \subset \text{Mat}(2n, \mathbb{C}) \) is also a continuous unitary group. By virtue of Stone’s theorem there exists a skew-adjoint matrix \( B \in \text{Mat}(2n, \mathbb{C}) \) such that \( \Lambda(U_t) = e^{Bt} \). The map \( \mathcal{W}: t \mapsto \Lambda(U_t) \) is also differentiable, so that we can write for \( \mathcal{U}: t \mapsto U_t \),

\[
\Lambda^{-1}(\mathcal{W}) = \frac{d}{dt} \Lambda^{-1}(\mathcal{W}) = \dot{U}.
\]

The left hand side exists, proving that also \( \mathcal{U} \) is differentiable. By letting \( A = \dot{U}(0) \) and using \( U_{t+\delta} = U_{\delta} U_t \), we have

\[
\dot{U}(t) = \lim_{\delta \to 0} \frac{U_{t+\delta} - U_t}{\delta} = AU_t.
\]
It remains to show that $A$ is skew-adjoint and satisfies $U_t = e^{At}$. Since $\Lambda(U_t)$ is unitary, the identity $\Lambda(A)\Lambda(U_t) = \Lambda(U_t)\Lambda(A) = \frac{d}{dt}\Lambda(U_t) = WV = BW = B\Lambda(U_t)$, allows us to conclude that $\Lambda(A) = B$. This implies that $A$ is skew-adjoint, since $B$ is. Finally, applying $\Lambda^{-1}$ to

$$\Lambda(U_t) = e^{Bt} = e^{\Lambda(A)t} = \Lambda(e^{At})$$

from the left gives us $U_t = e^{At}$. Uniqueness then follows immediately from $\dot{U}(0) = A$. □

We mention that the smoothness of the map $t \mapsto U_t$, which we show in the first part of the above proof, is a simple consequence of the fact that the unitary matrices form a Lie group. Indeed, for any Lie group $G$ a continuous homomorphism $\mathbb{R} \to G$ is necessarily smooth. Also note that in contrast to the complex case, we consider here only the finite-dimensional case and therefore it is not necessary to use strong continuity.

D. Observables and generators of time shifts

We now head for the characterization of the dynamics in a quaternionic version of quantum theory. So far we have obtained a result about the structure of all possible dynamical evolutions. But for a dynamical evolution to be useful we need to specify a notion of states and observables. Here, we proceed in complete analogy to quantum mechanics, that is, states are represented by normalized vectors and observables by self-adjoint matrices. The expectation value of an observable $H$ for a state $\psi$ is then given by $\langle H \rangle = \langle \psi, H \psi \rangle$. Clearly, the expectation value is always real and all states $\psi_\alpha$ are equivalent for all $\alpha \in \mathbb{H}$ with $|\alpha| = 1$. The spectral theorem for self-adjoint matrices can also be written as $H = \sum_k h_k \Pi_k$, with distinct eigenvalues $h_k \in \mathbb{R}$ and self-adjoint projections $\Pi_k$ such that $\sum_k \Pi_k = 1$. The expectation values of the projections correspond then to the probability $p_k$ for observing the eigenvalue $h_k$, that is, $p_k = \langle \Pi_k \rangle$. This way we recover a large bit of the structure and physical interpretation of quantum theory.

With the same arguments as in the complex case, we assume that the time evolution of a state is generated by a continuous unitary group $(U_t)$ and by virtue of Theorem 3 we have $U_t = e^{At}$. It is worth mentioning here a significant difference to the complex case, which occurs if we add a global, time-dependent phase to a state, $\psi(t) \to \psi_\alpha(t) = \psi(t)\alpha(t)$. We obtain $\dot{\psi}_\alpha(t) = A\psi_\alpha(t) + \psi_\alpha(t)\varphi(t)$ where $\varphi(t) = \alpha(t)\dot{\alpha}(t)$. Since $\varphi(t)$ in general does not commute with $\psi_\alpha$, we cannot simply write $\dot{\psi}_\alpha(t) = (A + \varphi(t))\psi_\alpha(t)$, as it would be the case in the complex case.

In analogy to the complex case we are interested in the correspondence between observables and the generators of time shifts, in particular for the case where the observable is the Hamiltonian of the system. In the complex case, the multiplication with a purely imaginary number $i\lambda$ is the right choice to establish this correspondence. The matrix $A$ in Theorem 3 can written in the polar decomposition as $A = -XH$, where $X$ is unitary and skew-adjoint, $H$ is self-adjoint and positive semidefinite, and $[X,H] = 0$ holds. It is conceivable to identify $H$ in this decomposition with the Hamiltonian of the system while $X$ is kept constant. This limits the possible set of Hamiltonians to those with $[X,H] = 0$ which basically reduces quaternionic quantum theory to complex quantum theory.

Here, we are interested in the case where the Hamiltonian of the system can be any self-adjoint operator. The discussion in Section II B remains valid and leaves us with the task to characterize the commuting $\mathbb{R}$-linear maps $\Phi : \mathbb{H}(n,\mathbb{H}) \to \mathbb{A}(n,\mathbb{H})$. However, we cannot proceed similar to above to obtain a result akin to Theorem 2. The main difficulty here is that it is not possible to use an extension of $\Phi$ as in Section II B, since such an extended map would be no longer commuting. We hence resort to a case-by-case study for different dimensions $n$.

For this it is useful to note that determining all admissible maps $\Phi$ can be reduced to finding the kernel of an $\mathbb{R}$-linear map. Indeed, $\Phi$ is commuting if and only if the $\mathbb{R}$-bilinear map $\mathcal{Q}_\Phi : (X,Y) \mapsto [\Phi(X),Y] + [\Phi(Y),X]$ is trivial for all $X,Y \in \mathbb{H}(n,\mathbb{H})$, that is, $\mathcal{Q}_\Phi = 0$. 
Here, sufficiency follows immediately for $Y = X$ and necessity from $[\Phi(X + Y), X + Y] = 0$. Thus, the set of commuting maps $\Phi$ is determined by the kernel of the $R$-linear map $\Phi \mapsto Q_\Phi$. We perform this calculation with the help of a computer algebra system for $n = 1, 2, 3$ with the following results. For $n = 1$, all admissible maps are (obviously) given by $\Phi_1(H) = \alpha H$ for any $\alpha \in \mathbb{H}$ with $\Re(\alpha) = 0$. For $n = 2$, all admissible maps are of the form $\Phi_2(H) = AH + HA - \text{tr}(H)A$, where $A \in \mathbb{A}(2, \mathbb{H})$ is arbitrary. For $d = 3$, only the trivial map $\Phi = 0$ is commuting. This implies also that for $d > 3$ all commuting maps must be trivial, as it follows from the following contradiction.

**Lemma 4.** If there would exist a nontrivial commuting $R$-linear map $\Phi: \mathbb{H}(n, \mathbb{H}) \to \mathbb{A}(n, \mathbb{H})$ for $n \geq 3$, then there would also exist a nontrivial commuting $R$-linear map $\Phi_3: \mathbb{H}(3, \mathbb{H}) \to \mathbb{A}(3, \mathbb{H})$.

**Proof.** The real span of the matrices of the form $(u_i, u_j^\ast)_{i,j}$ with $u \in \mathbb{H}^n$ is $\mathbb{H}(n, \mathbb{H})$. Hence we can choose linearly independent vectors $x, y, z \in \mathbb{H}^n$ such that $\langle y, \Phi(x)z \rangle \neq 0$ with $X_{i,j} = x_i x_j^\ast$. Then there is an isometry $\tau: \mathbb{H}^3 \to \mathbb{H}^n$ such that $x, y, z \in \tau \mathbb{H}^3$ and $\tau = \tau \tau^\ast$ acts as identity on $x, y,$ and $z$. Such an isometry can be constructed by means of the Gram–Schmidt procedure, yielding orthonormal vectors which are then used as columns of $\tau$. We define the map $\Phi_3$ as $\Phi_3: H \mapsto \tau \Phi(\tau H \tau^\ast)\tau$. By construction, this map is $R$-linear and maps self-adjoint matrices to skew-adjoint matrices. Since $\Phi$ is commuting, we have $[\Phi(\tau H \tau^\ast), \tau H \tau^\ast] = 0$ for any $H \in \mathbb{H}(3, \mathbb{H})$ and by multiplication with $\tau^\ast$ from the left and $\tau^\ast$ from the right, it follows that $\Phi_3$ is also commuting. Finally, $\Phi$ is nontrivial for $X' = \tau X \tau^\ast$, $y' = \tau y$, and $z' = \tau z$ in that we have $\langle y', \Phi_3(X')z' \rangle = \langle \pi y, \Phi(\pi X)\pi z \rangle = \langle y, \Phi(X)z \rangle \neq 0$, where $\pi X \pi = X$ follows from $\pi X \pi v = \pi x \langle x, \pi v \rangle = x \langle x, v \rangle = X v$ for all $v \in \mathbb{H}^n$.

In summary we have the following characterization.

**Theorem 5.** For $n = 1, 2$, every commuting $R$-linear map $\Phi: \mathbb{H}(n, \mathbb{H}) \to \mathbb{A}(n, \mathbb{H})$ is of the form

$$\Phi(H) = AH + HA - \text{tr}(H)A,$$

(20)

where $A \in \mathbb{A}(n, \mathbb{H})$. Conversely, for $n = 1, 2$ every map of this form is commuting. For $n > 2$ every commuting $R$-linear map is trivial, $\Phi = 0$.

For a two-level system, $n = 2$, the rank of $\Phi$ can be at most $\dim[\mathbb{H}(n, \mathbb{H})] - 1 = 5$ due to $\Phi(1) = 0$, for any choice of $A$. The maximal rank is achieved, for example, for $A$ being the diagonal matrix with diagonal $(i, 0)$. A more intuitive choice for $A$ might be $-\frac{1}{2\pi} I$, which yields the quaternionic Schrödinger equation

$$i\hbar \dot{\psi}(t) = [H_c - \frac{1}{2} \text{tr}(H_c) 1] \psi(t),$$

(21)

where $H_c = \frac{1}{2}(H + iH^\ast)$. This is the complex part of the matrix $H$. Hence the corresponding map $\Phi$ has an $R$-rank of only 3.

**Theorem 5** has been obtained using the conditions (DC1)–(DC3). In contrast to the complex case, the resulting map $\Phi$ only obeys the covariance condition (DC4) if it is trivial. This can be seen by defining $\Phi_V : H \mapsto V \Phi(V^\ast HV)^\ast = AV + HA^\ast - \text{tr}(H)A V$ with $A = V A^\ast$. Covariance requires then $\Phi_V = \Phi$ for any unitary $V$. Without loss of generality, we can choose $A = iD$ with $D$ a real diagonal matrix and then the case $V = j$ leads to $\Phi_V = -\Phi$.

However, the no-go statement in **Theorem 5** can be avoided by loosening our assumptions. For example, one can drop the assumption of additivity (DC2). Then a rather natural such candidate can be achieved as follows. We fix a spectral decomposition $H = U_H^\ast U_H$ for every $H$ such that $U_{Hi}^\ast U_H$ is independent of $r \in \mathbb{R}$. This allows us to define the $R$-homogeneous map $\Psi: \mathbb{H}(n, \mathbb{H}) \to \mathbb{A}(n, \mathbb{H})$ as

$$\Psi: H \mapsto U_H^\ast iD_H U_H,$$

(22)
which can be easily seen to be commuting, but, in general, fails to be additive, due to Theorem 5. We can even satisfy the covariance condition (DC4) by requiring that $U_{VHV} = VUH$ for all unitaries $V$. As mentioned before, another way to evade Theorem 5 is to limit the set of Hamiltonians to be purely complex\footnote{D. Finkelstein, J. Jauch, S. Schiminovich, and D. Speiser, "Foundations of quaternion quantum mechanics," J. Math. Phys. \textbf{3}, 207 (1962).}, $[H, i] = 0$. Then $\Phi_S: H \mapsto -\frac{i}{\hbar}H$ is admissible under (DC1)–(DC3) and we obtain the usual Schrödinger equation also in the quaternionic case.

IV. CONCLUSIONS

We studied the structure of universal dynamics in quantum theory using three main axioms (DC1)–(DC3). These axioms proved to be sufficient in order to recover the Schrödinger equation for the case of canonical quantum theory but when applied to quaternionic quantum theory they yield nontrivial dynamics only for dimension two (and one). For two-level systems, the resulting Schrödinger equation is not unique but can be modified by the choice of a skew-adjoint operator, see Eq. (1).

This makes quaternionic quantum theory for two-level systems exceptionally interesting. For higher dimensions, a possible conclusion from our analysis is to discard quaternionic quantum theory. However, it should be noted that the main reason for our no-go result is axiom (DC2), which requires that the sum of two Hamiltonian should correspond to the sum of two generators of time shifts. While this axiom is natural at least in canonical quantum theory it is an open question whether it is expendable nonetheless, and what this would imply for canonical quantum theory as well as quaternionic quantum theory.

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1L. Hardy, “Quantum theory from five reasonable axioms,” (2001), arXiv:quant-ph/0101012.
2G. Chiribella, G. M. D’Ariano, and P. Perinotti, “Informational derivation of quantum theory,” Phys. Rev. A \textbf{84}, 012311 (2011).
3B. Dakić and Č. Brukner, “Quantum theory and beyond: Is entanglement special?” in \textit{Deep Beauty. Understanding the Quantum World through Mathematical Innovation}, edited by H. Halvorson (Cambridge University Press, 2011) p. 365.
4L. Masanes and M. P. Müller, “A derivation of quantum theory from physical requirements,” New J. Phys. \textbf{13}, 063001 (2011).
5L. Hardy, “Reformulating and reconstructing quantum theory,” (2011), arXiv:1104.2066.
6A. Wilce, “A royal road to quantum theory (or thereabouts),” (2016), arXiv:1606.09306.
7P. A. Höhn and C. S. F. Wever, “Quantum theory from questions,” Phys. Rev. A \textbf{95}, 012102 (2017).
8H. Barnum, M. Müller, and C. Ududec, “Higher-order interference and single-system postulates characterizing quantum theory,” New J. Phys. \textbf{16}, 123029 (2014).
9E. Stueckelberg, “Quantum theory in real Hilbert-space,” Helv. Phys. Acta \textbf{33}, 727 (1960).
10M. McGauley, M. Mosca, and N. Gisin, “Simulating quantum systems using real hilbert spaces,” Phys. Rev. Lett. \textbf{102}, 020505 (2009).
11A. Aleksandrova, V. Borish, and W. Wootters, “Real-vector-space quantum theory with a universal quantum bit,” Phys. Rev. A \textbf{87}, 052106 (2013).
12Y. Moretti and M. Oppio, “Quantum theory in real Hilbert space: How the complex Hilbert space structure emerges from Poincaré symmetry,” Rev. Math. Phys. \textbf{29}, 1750021 (2017).
13L. Horwitz and L. Biedenharn, “Quaternion quantum mechanics: Second quantization and gauge fields,” Ann. Phys. \textbf{157}, 432 (1984).
14J. Baez, “Division algebras and quantum theory,” Found. Phys. \textbf{42}, 819 (2012).
15D. Joyce, “A theory of quaternionic algebra, with applications to hypercomplex geometry,” in \textit{Quaternionic Structures in Mathematics and Physics} (World Scientific, 2001) p. 145.
16C.-K. Ng, “On quaternionic functional analysis,” Math. Proc. Camb. Philos. Soc. \textbf{143}, 391 (2007).
17G. Chiribella, “Agents, subsystems, and the conservation of information,” Entropy \textbf{358} (2018).
18D. Finlestein, J. Jauch, S. Schiminovich, and D. Speiser, “Foundations of quaternion quantum mechanics,” J. Math. Phys. \textbf{3}, 207 (1962).
19A. Peres, “Proposed test for complex versus quaternion quantum theory,” Phys. Rev. Lett. \textbf{42}, 683 (1979).
10

20 H. Kaiser, E. George, and S. Werner, “Neutron interferometric search for quaternions in quantum mechanics,” Phys. Rev. A 29, 2276(R) (1984).

21 L. Procopio, L. Rozema, Z. J. Wong, D. Hamel, K. O’Brien, X. Zhang, B. Dakić, and P. Walther, “Single-photon test of hyper-complex quantum theories using a metamaterial,” Nat. Commun. 8, 15044 (2017).

22 S. Adler, “Peres experiment using photons: No test for hypercomplex (quaternionic) quantum theories,” Phys. Rev. A 95, 060101(R) (2017).

23 L. Procopio, L. Rozema, B. Dakić, and P. P. Walther, “Comment on ‘Peres experiment using photons: No test for hypercomplex (quaternionic) quantum theories’,” Phys. Rev. A 96, 036101 (2017).

24 S. Adler, Quaternionic Quantum Mechanics and Quantum Fields, 1st ed. (Oxford University Press, New York, 1995).

25 G. Teschl, “Quantum dynamics,” in Mathematical Methods in Quantum Mechanics (American Mathematical Society, Providence, Rhode Island, 2009) Chap. 5, p. 123.

26 E. Alfsen and F. Shultz, “Orientation in operator algebras,” Proc. Natl. Acad. Sci. USA 95, 6596 (1998).

27 D. Branford, O. Dahlsten, and A. Garner, “On defining the Hamiltonian beyond quantum theory,” Found. Phys. 48, 982 (2018).

28 M. Brešar, “Commuting maps: A survey,” Taiwan. J. Math. 8, 361 (2004).

29 F. Zhang, “Quaternions and matrices of quaternions,” Lin. Alg. Appl. 251, 21 (1997).

30 J. Lee, “The exponential map,” in Introduction to Smooth Manifolds (Springer, New York, 2012) Chap. 20, p. 515, 2nd ed.