On the Convergence to a Statistical Equilibrium
in the Crystal Coupled to a Scalar Field

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Abstract

We consider the dynamics of a field coupled to a harmonic crystal with \( n \) components in dimension \( d, \, d, \, n \geq 1 \). The crystal and the dynamics are translation-invariant with respect to the subgroup \( \mathbb{Z}^d \) of \( \mathbb{R}^d \). The initial data is a random function with a finite mean density of energy which also satisfies a Rosenblatt- or Ibragimov-Linnik-type mixing condition. Moreover, initial correlation functions are translation-invariant with respect to the discrete subgroup \( \mathbb{Z}^d \). We study the distribution \( \mu_t \) of the solution at time \( t \in \mathbb{R} \). The main result is the convergence of \( \mu_t \) to a Gaussian measure as \( t \to \infty \), where \( \mu_\infty \) is translation-invariant with respect to the subgroup \( \mathbb{Z}^d \).

Key words and phrases: Field coupled to a harmonic crystal; random initial data; mixing condition; covariance matrices; characteristic functional; convergence to equilibrium.

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1Partially supported by the research grants of DFG (436 RUS 113/615/0-1) and of RFBR (03-01-00189) and by the Austrian Science Foundation (FWF) Project (P16105-N05)

2On leave from Department of Mechanics and Mathematics at the Moscow State University. Partially supported by Max-Planck-Institute for Mathematics in the Sciences (Leipzig) and by the Wolfgang Pauli Institute (Wien)
1 Introduction

The paper concerns problems of long-time convergence to an equilibrium distribution in a coupled system which is similar to the Born-Oppenheimer model of a solid state. In [5, 6, 7, 10] we have started the convergence analysis for partial differential equations of hyperbolic type in $\mathbb{R}^d$. In [8, 9] we have extended the results to harmonic crystals.

Here we treat a harmonic crystal coupled to a scalar Klein-Gordon field. In this case, the corresponding problem in the unit cell is an infinite-dimensional Schrödinger operator, whereas in [8, 9] (and in [5, 6, 7, 10]) it was a finite-dimensional matrix. This situation usually arises in the solid-state problems similar to that for the Schrödinger equation with space-periodic potential [18]. The main novelty in our methods consists in that they yield exact estimates of trace norms for the problem in the unit cell.

We assume that an initial state $Y_0$ of the coupled system is a random element of a Hilbert phase space $\mathcal{E}$, see Definition 2.4. The distribution of $Y_0$ is a probability measure $\mu_0$ of mean zero satisfying conditions S1-S3. In particular, the measure $\mu_0$ is invariant with respect to translations by vectors of $\mathbb{Z}^d$. For a given $t \in \mathbb{R}$, we denote by $\mu_t$ the probability measure defining the distribution of the solution $Y(t)$ to the dynamical equations with random initial state $Y_0$. We study the asymptotics of $\mu_t$ as $t \to \pm \infty$.

Our main result gives the (weak) convergence of the measures $\mu_t$ to a limit measure $\mu_\infty$:

$$\mu_t \rightharpoonup \mu_\infty, \quad t \to \infty. \quad (1.1)$$

The measure $\mu_\infty$ is Gaussian and translation-invariant with respect to the group $\mathbb{Z}^d$. We give the explicit formulas for the covariance of the measure $\mu_\infty$. The dynamical group is ergodic and mixing with respect to the limit measure $\mu_\infty$. Similar results hold as $t \to -\infty$ because the dynamics is time-reversible.

Similar results have been established in [1, 24] for one-dimensional chains of harmonic oscillators (with $d = 1$) and in [11, 13, 16, 21] for one-dimensional chains of anharmonic oscillators coupled to heat baths. For $d$-dimensional harmonic crystals, with $d \geq 1$, the convergence (1.1) was proved in [8, 9, 17]. The mixing condition was first introduced by R. Dobrushin and Yu. Suhov for an ideal gas in [3]. The condition can replace the (quasi-)ergodic hypothesis when proving the convergence to the equilibrium distribution, and this plays a crucial role in our approach. Developing a Bernstein-type approach, we have proved the convergence for the wave and Klein-Gordon equations and for harmonic crystals with translation-invariant initial measures, [5, 6, 8]. In [7, 9, 10] we have extended the results to two-temperature initial measures. The present paper extends our previous results to the scalar Klein-Gordon field coupled to the nearest neighbor crystal.

Let us outline our main result and the strategy of the proof. (For the formal definitions and statements, see Section 2.) Consider the Hamilton system with the following Hamiltonian functional:

$$H(\psi, u, \pi, v) = \frac{1}{2} \int \left( |\nabla \psi(x)|^2 + |\pi(x)|^2 + m_0^2 |\psi(x)|^2 \right) dx + \frac{1}{2} \sum_{k \in \mathbb{Z}^d} \left( \sum_{j=1}^d |u(k + e_j) - u(k)|^2 + |v(k)|^2 + \nu_0^2 |u(k)|^2 \right) + \sum_{k \in \mathbb{Z}^d} \int R(x - k) \cdot u(k) \psi(x) \, dx, \quad (1.2)$$
involving a real scalar field \( \psi(x) \) and its momentum \( \pi(x) \), \( x \in \mathbb{R}^d \), coupled to a "simple lattice" described by the deviations \( u(k) \in \mathbb{R}^n \) of the "atoms" and their velocities \( v(k) \in \mathbb{R}^n \), \( k \in \mathbb{Z}^d \). The symbol \( R(x) \) stands for a \( \mathbb{R}^n \)-valued function and \( e_j \in \mathbb{Z}^d \) for the vector with the coordinates \( e_j := \delta_j^i \). Taking the variational derivatives of \( H(\psi, u, \pi, v) \), we formally obtain the following system for \( x \in \mathbb{R}^d \) and \( k \in \mathbb{Z}^d \):

\[
\begin{array}{l}
\dot{\psi}(x, t) = \frac{\delta H}{\delta \pi} = \pi(x, t), \\
\dot{u}(k, t) = \frac{\partial H}{\partial v} = v(k, t), \\
\dot{\pi}(x, t) = -\frac{\delta H}{\delta \psi} = (\Delta - m_0^2)\psi(x, t) - \sum_{k' \in \mathbb{Z}^d} u(k', t) \cdot R(x - k'), \\
\dot{v}(k, t) = -\frac{\partial H}{\partial u} = (\Delta_L - \nu_0^2)u(k, t) - \int R(x' - k)\psi(x', t) \, dx'.
\end{array}
\] (1.3)

Here \( m_0, \nu_0 > 0 \), and \( \Delta_L \) denotes for the discrete Laplace operator on the lattice \( \mathbb{Z}^d \),

\[
\Delta_L u(k) := \sum_{e,|e|=1} (u(k + e) - u(k)).
\]

Note that for \( n = d \) and \( R(x) = -\nabla \rho(x) \), the interaction term in the Hamiltonian is the linearized Pauli-Fierz approximation of the translation-invariant coupling

\[
\sum_k \int \rho(x - k - u(k))\psi(x) \, dx.
\] (1.4)

A similar model was analyzed by Born and Oppenheimer [2] as a model of a solid state (coupled Maxwell-Schrödinger equations for electrons in the harmonic crystal; see, e.g., [18]). The traditional analysis of the coupled field-crystal system (1.3) is based on an iterative perturbation procedure using the adiabatic approximation. Namely, in the zero approximation, the crystal and the (electron) field are discoupled. In the first one, the electron field defines a slow displacement of nuclei. The displacements give the corresponding contribution to the field via the static Coulombic potentials, which means a non-relativistic approximation, etc. The iterations converge if the motion of the nuclei is sufficiently slow, i.e., the nuclei are rather heavy as compared with the electrons. A similar procedure applies to the corresponding stationary problem of finding the dispersion relations.

Our analysis of the dispersion relations is a bit different and holds for small displacements. Namely, we linearize the translation-invariant coupling (1.4) at the zero displacements of the nuclei and obtain the equations (1.3) corresponding to the Pauli-Fierz approximation. On the other hand, we analyze the dispersion relations of the linearized equations without any adiabatic or non-relativistic approximation. We give an exact nonperturbative spectral analysis of the coupled system (1.3).

We study the Cauchy problem for the system (1.3) with the initial data

\[
\begin{array}{l}
\psi(x, 0) = \psi_0(x), \quad \pi(x, 0) = \pi_0(x), \quad x \in \mathbb{R}^d, \\
u(k, 0) = u_0(k), \quad v(k, 0) = v_0(k), \quad k \in \mathbb{Z}^d.
\end{array}
\] (1.5)
Let us write
\[ \psi^0 := \psi, \quad \psi_1 := \pi, \quad u^0 := u, \quad u^1 := v, \]
\[ Y(t) := (Y^0(t), Y^1(t)) \quad Y^0(t) := (\psi^0(x, t), u^0(k, t)) := (\psi(x, t), u(k, t)), \]
\[ Y^1(t) := (\psi_1(x, t), u^1(k, t)) := (\pi(x, t), v(k, t)). \]

In other words, \( Y(\cdot, t) \) are functions defined on the disjoint union \( \mathbb{P} := \mathbb{R}^d \cup \mathbb{Z}^d \),
\[ Y^i(t) = Y^i(p, t) := \begin{cases} \psi^i(x, t), & p = x \in \mathbb{R}^d, \\ u^i(k, t), & p = k \in \mathbb{Z}^d, \end{cases} \quad i = 0, 1. \]

In this case, the system (1.3), (1.5) becomes a dynamical problem of the form
\[ \dot{Y}(t) = \mathcal{A}Y(t), \quad t \in \mathbb{R}; \quad Y(0) = Y_0. \]

Here \( Y_0 = (\psi_0, u_0, \pi_0, v_0) \) and
\[ \mathcal{A} = J \nabla H(Y) = \begin{pmatrix} 0 & 1 \\ -\mathcal{H} & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} -\Delta + m_0^2 & S \\ S^* & -\Delta_L + v_0^2 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]
where \( Su(x) = \sum_{k \in \mathbb{Z}^d} R(x - k)u(k) \), \( S^* \psi(k) = \int_{\mathbb{R}^d} R(x - k) \psi(x) \, dx \), and
\[ \langle \psi, Su \rangle_{L^2(\mathbb{R}^d)} = \langle S^* \psi, u \rangle_{L^2(\mathbb{R}^d)^n}, \quad \psi \in L^2(\mathbb{R}^d), \quad u \in [L^2(\mathbb{Z}^d)]^n. \]

We assume that the initial datum \( Y_0 \) is a random function, and the initial correlation matrix
\[ Q_0(p, p') := E\left( Y_0(p) \otimes Y_0(p') \right), \quad p, p' \in \mathbb{P}, \]
is translation invariant with respect to translations by \( \mathbb{Z}^d \), i.e.,
\[ Q_0(p + k, p' + k) = Q_0(p, p'), \quad p, p' \in \mathbb{P}, \]
for any \( k \in \mathbb{Z}^d \). We also assume that the initial mean energy densities are uniformly bounded,
\[ e_F(x) := E(|\nabla \psi_0(x)|^2 + |\psi_0(x)|^2 + |\pi_0(x)|^2) \leq \bar{e}_F < \infty, \quad a.a. \ x \in \mathbb{R}^d, \tag{1.10} \]
\[ e_L := E(|u_0(k)|^2 + |v_0(k)|^2) < \infty, \quad k \in \mathbb{Z}^d. \tag{1.11} \]

Finally, we assume that the measure \( \mu_0 \) satisfies a mixing condition of a Rosenblatt- or Ibragimov-Linnik type, which means that
\[ Y_0(p) \text{ and } Y_0(p') \text{ are asymptotically independent as } |p - p'| \to \infty. \tag{1.12} \]

Our main result gives the (weak) convergence (1.1) of \( \mu_t \) to a limit measure \( \mu_\infty \), which is a stationary Gaussian probability measure.

Let us comment on the methods of the proof. The key role in our proof is played by the standard reduction of system (1.7) to the Bloch problem on the torus. Namely, we split
x ∈ ℝ^d in the form x = k + y, k ∈ ℤ^d, y ∈ K_1^d := [0, 1]^d, and apply the Fourier transform
F_{k→θ} to the solution Y(k, t) := (ψ(k + y, t), u(k, t), π(k + y, t), v(k, t)),

\( \hat{Y}(θ, t) := F_{k→θ}Y(k, t) \equiv \sum_{k∈ℤ^d} e^{ikθ}Y(k, t) = (\hat{ψ}(θ, y, t), \hat{u}(θ, t), \hat{π}(θ, y, t), \hat{v}(θ, t)), \ θ ∈ ℝ^d, \)

which is a version of the Bloch-Floquet transform. The functions \( \hat{ψ} \), \( \hat{π} \) are periodic with respect to \( θ \) and quasi-periodic with respect to \( y \), i.e.,

\( \hat{ψ}(θ, y + m, t) = e^{-imθ}ψ(θ, y, t), \quad \hat{π}(θ, y + m, t) = e^{-imθ}π(θ, y, t), \quad m ∈ ℤ^d. \)

Further, introduce the Zak transform of \( Y(·, t) \) (which is also known as Lifshitz-Gelfand-Zak transform) (cf [18, p.5]) as

\[
ZY(·, t) \equiv \hat{Y}_{Π}(θ, t) := (\hat{ψ}_{Π}(θ, y, t), \hat{u}_{Π}(θ, t), \hat{π}_{Π}(θ, y, t), \hat{v}_{Π}(θ, t)),
\]

(1.13)

where \( \hat{ψ}_{Π}(θ, y, t) := e^{iyθ}\hat{ψ}(θ, y, t) \) and \( \hat{π}_{Π}(θ, y, t) := e^{iyθ}\hat{π}(θ, y, t) \) are periodic functions with respect to \( y \) (and quasi-periodic with respect to \( θ \)). Denote by \( T^d_1 := ℝ^d/ℤ^d \) the real \( d \)-torus and write \( \mathcal{R} := T^d_1 \cup \{0\} \). Set

\[
\hat{Y}_{Π}(θ, r, t) \equiv \hat{Y}_{Π}(θ, t) := \begin{cases} (\hat{ψ}_{Π}(θ, y, t), \hat{π}_{Π}(θ, y, t)), & r = y \in T^d_1, \\ (\hat{u}(θ, t), \hat{v}(θ, t)), & r = 0. \end{cases}
\]

Problem (1.7) is now equivalent to the problem on the unit torus \( y ∈ T^d_1 \) with the parameter \( θ ∈ K^d \equiv [0, 2π]^d \),

\[
\begin{cases} \dot{\hat{Y}}_{Π}(θ, t) = \hat{A}(θ)\hat{Y}_{Π}(θ, t), \ t ∈ ℝ \quad | \quad θ ∈ K^d. \\ \hat{Y}_{Π}(θ, 0) = \hat{Y}_{0Π}(θ) \end{cases}
\]

(1.14)

Here

\[
\hat{A}(θ) = \begin{pmatrix} 0 & 1 \\ -\hat{H}(θ) & 0 \end{pmatrix},
\]

(1.15)

and \( \hat{H}(θ) := ZHZ^{-1} \) is the “Schrödinger operator” on the torus \( T^d_1 \),

\[
\hat{H}(θ) = \begin{pmatrix} (i∇_y + θ)^2 + m^2_0 & \hat{S}(θ) \\ \hat{S}^*(θ) & \omega^2_s(θ) \end{pmatrix},
\]

(1.16)

where

\[
ω^2_s(θ) := 2(1 - \cos θ_1) + \ldots + 2(1 - \cos θ_d) + ν^2_0,
\]

(1.17)

\[
(\hat{S}(θ)\hat{u}(·))(θ, y) := \hat{R}_{Π}(θ, y) · \hat{u}(θ), \quad (\hat{S}^*(θ)\hat{ψ}_{Π}(θ, ·))(θ) := \int_{T^d_1} \hat{R}_{Π}(−θ, y)\hat{ψ}_{Π}(θ, y) \, dy,
\]

(1.18)

\[
\langle \hat{ψ}_{Π}(θ, ·), (\hat{S}(θ)\hat{u}(θ, ·)) \rangle_{L^2(T^d_1)} = \langle \hat{S}^*(θ)\hat{ψ}_{Π}(θ, ·) · \hat{u}(θ), \hat{ψ}_{Π}(θ, ·) \rangle_{H^1(T^d_1)}, \quad \hat{u}(θ) ∈ ℂ^n.
\]

Then, formally,

\[
\hat{Y}_{Π}(θ, t) = e^{\hat{A}(θ)t}\hat{Y}_{0Π}(θ), \quad θ ∈ K^d.
\]

(1.19)

To justify the definition of the exponential, we note that \( \hat{H}(θ) \) is a self-adjoint operator with a discrete spectrum. Indeed, if \( R = 0 \), then this follows from elliptic theory, and, if \( R \neq 0 \),
then the operators \( \tilde{S}(\theta) \) and \( \tilde{S}^*(\theta) \) are finite-dimensional for a fixed \( \theta \). We assume that \( \mathcal{H}(\theta) > 0 \) (condition \textbf{R2}) which corresponds to the hyperbolicity of problem (1.3).

Note that in [8, 9], we considered the harmonic crystal without any field. In this case, the operator \( \tilde{\mathcal{A}}(\theta) \) is a finite-dimensional matrix.

Let us prove the convergence (1.1) by using the strategy of [5]-[10] in the following three steps.

I. The family of measures \( \mu_t, t \geq 0 \), is weakly compact in an appropriate Fréchet space.

II. The correlation functions converge to a limit,

\[
Q_t(p, p') := \int (Y(p) \otimes Y(p')) \mu_t(dY) \to Q_\infty(p, p'), \quad t \to \infty, \quad p, p' \in \mathbb{P}. \tag{1.20}
\]

III. The characteristic functionals converge to a Gaussian functional,

\[
\hat{\mu}_t(Z) := \int e^{i \langle Y, Z \rangle} \mu_t(dY) \to \exp \left\{ -\frac{1}{2} Q_\infty(Z, Z) \right\}, \quad t \to \infty, \tag{1.21}
\]

where \( Z \) is an arbitrary element of the dual space and \( Q_\infty \) is a quadratic form.

Property I follows from the Prokhorov Theorem. First, let us prove the uniform bound (2.15) for the mean local energy in \( \mu_t \). To this end, we shall show that the operator

\[
\left( \tilde{\Omega}^i j(\theta) \tilde{\Omega}^j \right)_{i,j=0,1}
\]

is of trace class, where \( \tilde{\Omega}^i j(\theta) \) represents the covariance of the measure \( \mu_t \) in the Zak transform (see (3.7)) and \( \Omega \equiv \Omega(\theta) := \sqrt{\mathcal{H}(\theta)} \). Moreover, we derive the uniform bound

\[
\sup_{t \geq 0} \sup_{\theta \in [0, 2\pi]} \operatorname{tr} \left( \tilde{\Omega}^i j(\theta) \tilde{\Omega}^j \right)_{i,j=0,1} < \infty. \tag{1.22}
\]

This implies the compactness of \( \mu_t \) by the Prokhorov theorem (when applying Sobolev’s embedding theorem as in [5]).

To derive property II, we study oscillatory integrals in the Zak transform by developing our cutting strategy introduced in [8]. Namely, we rewrite (1.20) in the form

\[
Q_t(Z, Z) \to Q_\infty(Z, Z), \quad t \to \infty, \tag{1.23}
\]

where \( Q_t(Z, Z) \) stands for the correlation quadratic form for the measure \( \mu_t \). Further, we prove formula (1.23) for \( Z \in \mathcal{D}^0 \) as follows: by the definition of \( \mathcal{D}^0 \), the Zak transform \( \tilde{\mathcal{Z}}_\Pi(\theta) \) vanishes in a neighborhood of a “critical set” \( \mathcal{C} \subset K^d \). In particular, the set \( \mathcal{C} \) includes all points \( \theta \in K^d \) with a degenerate Hessian of \( \omega_l(\theta) \) and the points for which the function \( \omega_l(\theta) \) is non-smooth. One can cut off the critical set \( \mathcal{C} \) by the following two crucial observations: (i) \( \operatorname{mes} \mathcal{C} = 0 \) and (ii) the initial correlation quadratic form is continuous in \( L^2 \) due to the mixing condition. The continuity follows from the spatial decay of the correlation functions in accordance with the well-known Shur lemma.

Similarly, we first prove property III for \( Z \in \mathcal{D}^0 \) and then extend it to all \( Z \in \mathcal{D} \). For \( Z \in \mathcal{D}^0 \), we use a version of the S.N. Bernstein “room-corridor” technique (cf. [5, 8]). This leads to a representation of the solution as the sum of weakly dependent random variables. Then (1.21) follows from the Central Limit Theorem under a Lindeberg-type condition.

Let us comment on the two main technical novelties of our paper. The first of them is the bound (1.22), which then ensures compactness. We derive formula (1.22) in Section 4 directly from our assumption concerning the finiteness of the mean energy density (1.10), (1.11).
The derivation uses the technique of trace class operators [23], which enables us to avoid additional continuity conditions for higher-order derivatives of the correlation functions. An essential ingredient of the proof is the “unitary trick” (4.6), which is a natural consequence of the Hamiltonian structure of system (1.3). The second main novelty is the bound (2.2) for the dynamics in weighted norms. In the Zak transform, the weighted norms become Sobolev norms with negative index. We derive (2.2) in Appendix A, by using duality arguments, from the corresponding bounds for the derivatives of the exponential (1.19). The bounds for the derivatives follow by differentiating the dynamical equations.

Let us comment on our conditions \(E_1\) and \(E_2\). The conditions are natural generalizations of similar conditions in [8, 9]. Condition \(E_1\) enables us to apply the stationary phase method to the oscillatory integral representation for the covariance. It provides that the stationary points of the phase functions are non-degenerate.

The paper is organized as follows. In Section 2 we formally state our main result. The compactness (Property I) is established in Section 4, the convergence (1.20) in Section 6, and the convergence (1.21) in Section 7. In Section 8, mixing properties for the limit measures are proved. Appendix A concerns the dynamics in the Fourier transform, in Appendix B we analyze the crossing points of the dispersion relations, and in Appendix C we discuss the covariance in the spectral representation.

## 2 Main results

### 2.1 Notation

We assume that the initial data \(Y_0\) are given by an element of the real phase space \(E\) defined below.

**Definition 2.1** Let \(H^{s,\alpha} = H^{s,\alpha}(\mathbb{R}^d)\), \(s \in \mathbb{R}\), \(\alpha \in \mathbb{R}\), be the Hilbert space of distributions \(\psi \in S' (\mathbb{R}^d)\) with finite norm

\[
\|\psi\|_{s,\alpha} \equiv \|\langle x \rangle^\alpha \Lambda^s \psi\|_{L^2(\mathbb{R}^d)} < \infty.
\]

For \(\psi \in D \equiv C_0^\infty(\mathbb{R}^d)\), write \(F\psi(\xi) = \int e^{i\xi \cdot x} \psi(x) dx\). Let \(\Lambda^s \psi := F_{\xi \to x}^{-1}(\langle \xi \rangle^s \hat{\psi}(\xi))\) and \(\langle x \rangle := \sqrt{|x|^2 + 1}\), where \(\hat{\psi} := F\psi\) stands for the Fourier transform of a tempered distribution \(\psi\).

**Remark 2.2** For \(s = 0, 1, 2, \ldots\), the space \(H^{s,\alpha}(\mathbb{R}^d)\) is the Hilbert space of real-valued functions \(\psi(x)\) with finite norm

\[
\sum_{|\gamma| \leq s} \int (1 + |x|^2)^\alpha |\mathcal{D}^\gamma \psi(x)|^2 dx < \infty,
\]

which is equivalent to \(\|\psi\|_{s,\alpha}^2\).

**Definition 2.3** Let \(L^\alpha\), \(\alpha \in \mathbb{R}\), be the Hilbert space of vector functions \(u(k) \in \mathbb{R}^n\), \(k \in \mathbb{Z}^d\), with finite norm

\[
\|u\|_{\alpha}^2 \equiv \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^\alpha |u(k)|^2 < \infty.
\]
Definition 2.4 Let \( \mathcal{E}^{s,\alpha} := H^{1+s,\alpha}(\mathbb{R}^d) \oplus L^\alpha \oplus H^{s,\alpha}(\mathbb{R}^d) \oplus L^\alpha \) be the Hilbert space of vectors \( Y \equiv (\psi, u, \pi, v) \) with finite norm

\[
\|Y\|^2_{s,\alpha} = \|\psi\|^2_{1+s,\alpha} + \|u\|^2_\alpha + \|\pi\|^2_{s,\alpha} + \|v\|^2_\alpha.
\]

Choose some \( \alpha \), \( \alpha < -d/2 \). Assume that \( Y_0 \in \mathcal{E} := \mathcal{E}^{0,\alpha} \).

Using the standard technique of pseudo-differential operators and Sobolev's Theorem (see, e.g., [14]), one can prove that \( \mathcal{E}^{0,\alpha} = \mathcal{E} \subset \mathcal{E}^{s,\beta} \) for every \( s < 0 \) and \( \beta < \alpha \), and the embedding is compact.

**Definition 2.5** The phase space of problem (1.7) is \( \mathcal{E} := \mathcal{E}^{0,\alpha}, \; \alpha < -d/2. \)

Introduce the space \( H^s_1 := H^s(T^d_1) \oplus \mathbb{C}^n \), \( s \in \mathbb{R} \), where \( H^s(T^d_1) \) stands for the Sobolev space.

We assume that the following conditions hold for the real-valued coupling vector function \( R(x) \):

**R1.** \( R \in C^\infty(\mathbb{R}^d) \) and \( |R(x)| \leq \bar{R}\exp(-\varepsilon|x|) \) with some \( \varepsilon > 0 \) and some \( \bar{R} < \infty \).

**R2.** The operator \( \mathcal{H}(\theta) \) is positive definite for \( \theta \in K^d \equiv [0, 2\pi]^d \). This is equivalent to the uniform bound

\[
(X^0, \mathcal{H}(\theta)X^0) \geq \kappa^2 \|X^0\|^2_{H^1_1} \quad \text{for } X^0 \in H^1_1, \; \theta \in K^d,
\]  

where \( \kappa > 0 \) is a constant and \((\cdot, \cdot)\) stands for the inner product in \( H^1_1 \) (see (3.13)).

**Remark 2.6**

i) Condition **R2** ensures that the operator \( i\mathcal{A}(\theta) \) is self-adjoint with respect to the energy inner product. This corresponds to the hyperbolicity of problem (1.3).

ii) Condition **R2** holds, in particular, if the following condition **R2’** holds (see Remark 9.3):

**R2’.** \( \int_{[0,1]^d} \left| \sum_{k \in \mathbb{Z}^d} R(k + y) \right|^2 dy < \nu_0^2 m_0^2/2. \)

iii) Condition **R2’** holds for functions \( R \) satisfying condition **R1** with \( \bar{R} \varepsilon^{-d} \ll 1 \).

**Proposition 2.7** Let conditions **R1** and **R2** hold. Then (i) for any \( Y_0 \in \mathcal{E} \), there exists a unique solution \( Y(t) \in C(\mathbb{R}, \mathcal{E}) \) to the Cauchy problem (1.7).

(ii) The operator \( W(t) : Y_0 \mapsto Y(t) \) is continuous in \( \mathcal{E} \) for any \( t \in \mathbb{R} \),

\[
\sup_{|t| \leq T} \|W(t)Y_0\|_{0,\alpha} \leq C(T)\|Y_0\|_{0,\alpha}
\]

if \( \alpha \) is even and \( \alpha \leq -2 \).

**Proof.** (i) **Local existence.** Introduce the matrices

\[
\mathcal{A}_0 := \begin{pmatrix} 0 & 1 \\ -\mathcal{H}_0 & 0 \end{pmatrix}, \quad \mathcal{H}_0 = \begin{pmatrix} -\Delta + m_0^2 & 0 \\ 0 & -\Delta_L + \nu_0^2 \end{pmatrix}.
\]

Then problem (1.3) can be rewritten as the Duhamel integral

\[
Y(t) = e^{\mathcal{A}_0 t} Y_0 + \int_0^t e^{\mathcal{A}_0(t-s)} BY(s) ds,
\]
where

\[ BY = \left( 0, 0, -\sum_{k' \in \mathbb{Z}^d} u(k') R(x - k'), -\int_{\mathbb{R}^d} R(x' - k) \psi(x') \, dx' \right), \quad Y = (\psi, u, \pi, v). \]

Condition R1 implies that \( \|BY(s)\|_{0, \alpha} \leq C \|Y(s)\|_{0, \alpha} \). Further, for \( 0 \leq s \leq t \leq T \), we obtain

\[ \|c^{A_0(t-s)}BY(s)\|_{0, \alpha} \leq C(T) \|Y(s)\|_{0, \alpha}, \]

(see, e.g., [8]). Hence,

\[ \max_{|t| \leq T} \|Y(t)\|_{0, \alpha} \leq C(T) \|Y_0\|_{0, \alpha} + TC(T) \max_{|s| \leq T} \|Y(s)\|_{0, \alpha} \leq (T + 1)C(T) \max_{|s| \leq T} \|Y(s)\|_{0, \alpha}. \]

We choose a \( T > 0 \) so that \( (T + 1)C(T) < 1 \). Then the contraction mapping principle implies the existence of a unique solution \( Y(t) \in C([0, T]; \mathcal{E}) \). The global existence follows from the bound (2.2).

(ii) The bounds (2.2) are proved in Corollary 9.6.

Conditions R1 and R2 imply that, for a fixed \( \theta \in K^d \), the operator \( \tilde{\mathcal{H}}(\theta) \) is positive definite and self-adjoint in \( H_0^1 \) and its spectrum is discrete. Introduce the Hermitian positive-definite operator

\[ \Omega(\theta) := \sqrt{\mathcal{H}(\theta)} > 0. \]

Denote by \( \omega_l(\theta) > 0 \) and \( F_l(\theta, \cdot) \), \( l = 1, 2, \ldots \), the eigenvalues (“Bloch bands”) and the orthonormal eigenvectors (“Bloch functions”) of the operator \( \Omega(\theta) \) in \( H_0^1 \), respectively. Note that \( F_l(\theta, \cdot) \in H_1^\infty := C^\infty(T_1^d) \oplus \mathbb{C}^n \), because these are eigenfunctions of the elliptic operator \( \mathcal{H}(\theta) \).

As is well known, the functions \( \omega_l(\cdot) \) and \( F_l(\cdot, r) \) are real-analytic outside the set of the “crossing” points \( \theta_* \), where \( \omega_l(\theta_*) = \omega_{l'}(\theta_*) \) for some \( l \neq l' \). However, the functions are not smooth at the crossing points in general if \( \omega_l(\theta) \neq \omega_{l'}(\theta) \). Therefore, we need the following lemma, which is proved in Appendix B.

**Lemma 2.8** (cf. [26]) There exists a closed subset \( \mathcal{C}_* \subset K^d \) such that (i) the Lebesgue measure of \( \mathcal{C}_* \) vanishes,

\[ \text{mes} \mathcal{C}_* = 0. \]  

(ii) For every point \( \Theta \in K^d \setminus \mathcal{C}_* \) and \( N \in \mathbb{N} \), there exists a neighborhood \( \mathcal{O}(\Theta) \subset K^d \setminus \mathcal{C}_* \) such that each of the functions \( \omega_l(\theta) \) and \( F_l(\theta, \cdot) \), \( l = 1, \ldots, N \), can be chosen to be real-analytic on \( \mathcal{O}(\Theta) \).

(iii) The eigenvalues \( \omega_l(\theta) \) have constant multiplicity in \( \mathcal{O}(\Theta) \), i.e., one can enumerate them in such a way that

\[ \omega_1(\theta) \equiv \ldots \equiv \omega_{r_1}(\theta), \quad \omega_{r_1+1}(\theta) \equiv \ldots \equiv \omega_{r_2}(\theta), \ldots, \]

\[ \omega_{r_\sigma}(\theta) \neq \omega_{r_\nu}(\theta) \quad \text{if} \quad \sigma \neq \nu, \quad r_\sigma, r_\nu \geq 1, \]

for any \( \theta \in \mathcal{O}(\Theta) \).
Corollary 2.9 The spectral decomposition holds,

\[ \Omega(\theta) = \sum_{l=1}^{\infty} \omega_l(\theta) P_l(\theta), \quad \theta \in \mathcal{O}(\Theta), \]  

(2.7)

where \( P_l(\theta) \) are the orthogonal projectors in \( H_1^0 \) onto the linear span of \( F_l(\theta, \cdot) \), and \( P_l(\theta) \) and \( \omega_l(\theta) \) depend on \( \theta \in \mathcal{O}(\Theta) \) analytically.

Assume that system (1.7) satisfies the following conditions \( E_1 \) and \( E_2 \). For every \( \Theta \in K^d \setminus \mathcal{C}_* \):

\( E_1 \) \( D_l(\theta) \neq 0, \ l = 1, 2, \ldots \), where \( D_l(\theta) := \det \left( \frac{\partial \omega_l(\theta)}{\partial \theta_i \partial \theta_j} \right)_{i,j=1}^{d}, \ \theta \in \mathcal{O}(\Theta) \), and \( \mathcal{O}(\Theta) \) is defined in Lemma 2.8.

Write \( \mathcal{C}_l := \bigcup_{\theta \in K^d \setminus \mathcal{C}_*} \{ \theta \in \mathcal{O}(\Theta) : D_l(\theta) = 0 \}, \ l = 1, 2, \ldots \)

The following lemma is also proved in Appendix B.

Lemma 2.10 Let conditions \( R_1 \) and \( R_2 \) hold. Then \( \text{mes } \mathcal{C}_l = 0, \ l = 1, 2, \ldots \)

\( E_2 \) For each \( l \neq l' \), the identity \( \omega_l(\theta) - \omega_{l'}(\theta) \equiv \text{const}_- \), \( \theta \in \mathcal{O}(\Theta) \), cannot hold for any constant \( \text{const}_- \neq 0 \), and the identity \( \omega_l(\theta) + \omega_{l'}(\theta) \equiv \text{const}_+ \) cannot hold for any constant \( \text{const}_+ \neq 0 \).

Condition \( E_2 \) could be considerably weakened (cf. [8, Remark 2.10, iii, condition \( E_5' \)]). Note that conditions \( E_1 \) and \( E_2 \) hold if \( R = 0 \).

Let us show that conditions \( E_1 \) and \( E_2 \) hold for “almost all” functions \( R \) satisfying conditions \( R_1, R_2 \). More precisely, consider finitely many coupling functions \( R_1, \ldots, R_N \) satisfying conditions \( R_1 \) and \( R_2' \) and take their linear combinations

\[ R_C(x) = \sum_{s=1}^{N} C_s R_s(x), \ C = (C_1, \ldots, C_N) \in \mathbb{R}^N. \]

For \( R_C(x) \), conditions \( R_1 \) and \( R_2' \) hold if \( \|C\| < \varepsilon \) with a sufficiently small \( \varepsilon > 0 \). Let \( M_1 := \{ C \in B_\varepsilon : \text{condition } E_1 \text{ holds for } R_C(x) \} \) and \( M_2 := \{ C \in B_\varepsilon : \text{condition } E_2 \text{ holds for } R_C(x) \} \), where \( B_\varepsilon := \{ C \in \mathbb{R}^N : \|C\| < \varepsilon \} \). In Appendix B, we prove the following lemma.

Lemma 2.11 The sets \( M_1 \) and \( M_2 \) are dense in some ball \( B_\varepsilon \) for a sufficiently small \( \varepsilon > 0 \).

2.2 Random solution. Convergence to equilibrium

Let \( (\Omega, \Sigma, P) \) be a probability space with expectation \( E \) and let \( \mathcal{B}(\mathcal{E}) \) denote the Borel \( \sigma \)-algebra in \( \mathcal{E} \). Assume that \( Y_0 = Y_0(\omega, p) \) (see (1.7)) is a measurable random function with values in \( (\mathcal{E}, \mathcal{B}(\mathcal{E})) \). In other words, the map \( (\omega, p) \mapsto Y_0(\omega, p) \) is a measurable map \( \Omega \times \mathcal{P} \rightarrow \mathbb{R}^{2+2n} \) with respect to the (completed) \( \sigma \)-algebra \( \Sigma \times \mathcal{B}(\mathcal{P}) \) and \( \mathcal{B}(\mathbb{R}^{2+2n}) \).
Then \( Y(t) = W(t)Y_0 \) is also a measurable random function with values in \((\mathcal{E}, \mathcal{B}(\mathcal{E}))\) owing to Proposition 2.7. Denote by \( \mu_0(dy_0) \) the Borel probability measure in \( \mathcal{E} \) giving the distribution of \( Y_0 \). Without loss of generality, we can assume \((\Omega, \Sigma, P) = (\mathcal{E}, \mathcal{B}(\mathcal{E}), \mu_0)\) and \( Y_0(\omega, p) = \omega(p) \) for \( \mu_0(d\omega) \times dp \)-almost all point \((\omega, p) \in \mathcal{E} \times \mathbb{P}\).

**Definition 2.12** The measure \( \mu_t \) is a Borel probability measure in \( \mathcal{E} \) giving the distribution of \( Y(t) \),

\[
\mu_t(B) = \mu_0(W(-t)B), \quad \forall B \in \mathcal{B}(\mathcal{E}), \quad t \in \mathbb{R}.
\]

Our main objective is to prove the weak convergence of the measures \( \mu_t \) in the Fréchet spaces \( \mathcal{E}^{s,\beta} \) for each \( s < 0, \beta < \alpha < -d/2 \),

\[
\mu_t \xrightarrow{\mathcal{E}^{s,\beta}} \mu_\infty \quad \text{as} \quad t \to \infty, \tag{2.8}
\]

where \( \mu_\infty \) is a limit measure on \( \mathcal{E} \equiv \mathcal{E}^{0,\alpha} \). This is equivalent to the convergence

\[
\int f(Y) \mu_t(dy) \to \int f(Y) \mu_\infty(dy) \quad \text{as} \quad t \to \infty
\]

for any bounded continuous functional \( f(Y) \) on \( \mathcal{E}^{s,\beta} \).

Let \( \mathcal{D} = [D_F \oplus D_L]^2 \) with \( D_F \equiv C_0^\infty(\mathbb{R}^d) \), and let \( D_L \) be the set of vector sequences \( u(k) \in \mathbb{R}^n, \quad k \in \mathbb{Z}^d \), such that \( u(k) = 0 \) for \( k \in \mathbb{Z}^d \) outside a finite set. For a probability measure \( \mu \) on \( \mathcal{E} \), denote by \( \hat{\mu} \) the characteristic functional (Fourier transform)

\[
\hat{\mu}(Z) \equiv \int e^{i(Y,Z)} \mu(dy), \quad Z \in \mathcal{D}.
\]

Here \( \langle \cdot, \cdot \rangle \) stands for the inner product in \( L^2(\mathbb{P}) \otimes \mathbb{R}^N \) with different \( N = 1, 2, \ldots \),

\[
\langle Y, Z \rangle := \sum_{i=0}^{1} \langle Y^i, Z^i \rangle, \quad Y = (Y^0, Y^1), \quad Z = (Z^0, Z^1),
\]

\[
\langle Y^i, Z^i \rangle := \int_{\mathbb{P}} Y^i(p)Z^i(p) dp \equiv \int_{\mathbb{R}^d} \psi^i(x)\xi^i(x) dx + \sum_{k \in \mathbb{Z}^d} u^i(k)\chi^i(k),
\]

where \( Y^i = (\psi^i, u^i), \quad Z^i = (\xi^i, \chi^i) \). A measure \( \mu \) is said to be Gaussian (with zero expectation) if its characteristic functional has the form

\[
\hat{\mu}(Z) = \exp \left\{ -\frac{1}{2} Q(Z, Z) \right\}, \quad Z \in \mathcal{D},
\]

where \( Q \) is a real nonnegative quadratic form in \( \mathcal{D} \).

**Definition 2.13** The correlation functions of the measure \( \mu_t, \quad t \in \mathbb{R} \), are defined by

\[
Q^{ij}_t(p, p') \equiv E(Y^i(p, t) \otimes Y^j(p', t)), \quad i, j = 0, 1, \quad p, p' \in \mathbb{P}, \tag{2.9}
\]

where \( E \) stands for the integral with respect to the measure \( \mu_0(dy) \) and the convergence of the integral in (2.9) is understood in the sense of distributions, namely,

\[
\langle Q^{ij}_t(p, p'), Z_1(p) \otimes Z_2(p') \rangle := E(Y^i(p, t), Z_1(p))\langle Y^j(p', t), Z_2(p') \rangle, \quad Z_1, Z_2 \in D_F \oplus D_L. \tag{2.10}
\]
2.3 Mixing condition

Let $O(r)$ be the set of all pairs of open subsets $\mathcal{A}, \mathcal{B} \subset \mathbb{P}$ such that the distance $\rho(\mathcal{A}, \mathcal{B})$ is not less than $r$, and let $\sigma(\mathcal{A})$ be the $\sigma$-algebra in $\mathcal{E}$ generated by the linear functionals $Y \mapsto \langle Y, Z \rangle$ for which $Z \in \mathcal{D}$ and $\text{supp} Z \subset \mathcal{A}$. Define the Ibragimov-Linnik mixing coefficient of a probability measure $\mu_0$ on $\mathcal{E}$ by the formula (cf. [15, Definition 17.2.2])

$$
\varphi(r) \equiv \sup_{(A,B) \in O(r)} \sup_{A \in \sigma(\mathcal{A}), B \in \sigma(\mathcal{B})} \frac{|\mu_0(A \cap B) - \mu_0(A)\mu_0(B)|}{\mu_0(B)} > 0.
$$

**Definition 2.14** A measure $\mu_0$ satisfies the strong uniform Ibragimov-Linnik mixing condition if $\varphi(r) \to 0$ as $r \to \infty$.

Below we specify the rate of the decay of $\varphi$ (see Condition S3).

2.4 Main theorem

Assume that the initial measure $\mu_0$ satisfies the following properties S0–S3:

**S0.** The measure $\mu_0$ has zero expectation value, $EY_0(p) = 0$, $p \in \mathbb{P}$.

**S1.** The correlation matrices of $\mu_0$ are invariant with respect to translations in $\mathbb{Z}^d$, i.e., Eqn (1.9) holds for a.a. $p, p' \in \mathbb{P}$.

**S2.** The measure $\mu_0$ has a finite mean “energy” density, i.e., Eqs (1.10), (1.11) hold.

**S3.** The measure $\mu_0$ satisfies the strong uniform Ibragimov-Linnik mixing condition with

$$
\int_0^{+\infty} r^{d-1} \varphi^{1/2}(r) dr < \infty. \quad (2.11)
$$

Introduce the correlation matrix $Q_\infty(p, p')$ of the limit measure $\mu_\infty$. It is translation-invariant with respect to translations in $\mathbb{Z}^d$, i.e.,

$$
Q_\infty(p, p') = Q_\infty(p + k, p' + k), \quad k \in \mathbb{Z}^d. \quad (2.12)
$$

For $Z \in \mathcal{D}$, write

$$
Q_\infty(Z, Z) := \langle Q_\infty(p, p'), Z(p) \otimes Z(p') \rangle = (2\pi)^{-d} \int_{K^d} \langle \tilde{q}_\infty(\theta), \tilde{Z}_\Pi(\theta, \cdot) \otimes \overline{Z}_\Pi(\theta, \cdot) \rangle d\theta, \quad (2.13)
$$

where $\tilde{q}_\infty(\theta)$ is the operator-valued function given by the rule

$$
\tilde{q}_\infty(\theta) := \sum_{l=1}^{+\infty} P_l(\theta) \frac{1}{2} \left( \tilde{q}_{00}^l(\theta) + \tilde{H}^{-1}(\theta) \tilde{q}_{01}^l(\theta) - \tilde{q}_{01}^l(\theta) - \tilde{q}_{00}^l(\theta) \right) P_l(\theta), \quad (2.14)
$$

for $\theta \in K^d \setminus \mathcal{L}_*$. Here the symbol $\tilde{q}_{ij}^l(\theta)$ stands for the integral operator with the integral kernel $\tilde{q}_{ij}^l(\theta, r, r')$ (see formula (3.7) with $t = 0$), $r, r' \in \mathcal{R}$, and $P_l(\theta)$ is the spectral projection operator introduced in Corollary 2.9.

**Theorem A** Let the conditions S0–S3, R1, R2, E1 and E2 hold. Then the following
assertions are valid.

(i) The convergence (2.8) holds for any \( s < 0 \) and \( \beta < -d/2 \).

(ii) The limit measure \( \mu_\infty \) is Gaussian on \( \mathcal{E} \).

(iii) The characteristic functional of \( \mu_\infty \) is Gaussian,

\[
\hat{\mu}_\infty(Z) = \exp\{-\frac{1}{2}Q_\infty(Z, Z)\}, \ Z \in \mathcal{D}.
\]

(iv) The measure \( \mu_\infty \) is invariant, i.e., \( [W(t)]^*\mu_\infty = \mu_\infty \), \( t \in \mathbb{R} \).

Assertions (i)-(iii) of Theorem A follow from Propositions 2.15 and 2.16 below.

**Proposition 2.15** The family of measures \( \{\mu_t, t \in \mathbb{R}\} \) is weakly compact in \( \mathcal{E}^{s, \beta} \) with any \( s < 0 \) and \( \beta < \alpha < -d/2 \), and the following bounds hold:

\[
\sup_{t \in \mathbb{R}} E\|W(t)Y_0\|_{0, \alpha}^2 \leq C(\alpha) < \infty.
\] (2.15)

**Proposition 2.16** The convergence (1.21) holds for every \( Z \in \mathcal{D} \).

Proposition 2.15 (Proposition 2.16) provides the existence (the uniqueness) of the limit measure \( \mu_\infty \). They are proved in Sections 4 and 7, respectively.

Theorem A (iv) follows from (2.8) because the group \( W(t) \) is continuous with respect to \( \mathcal{E} \) by Proposition 2.7 (ii).

### 3 Correlation matrices

To prove the compactness of the family of measures \( \{\mu_t\} \), we introduce auxiliary notations and prove necessary bounds for initial correlation matrices. Since \( Y^i(p, t) = (\psi^i(x, t), u^i(k, t)) \), we can rewrite formula (2.9) as follows:

\[
Q_t^{ij}(p, p') = E[Y^i(p, t) \otimes Y^j(p', t)] = \begin{pmatrix}
E(\psi^i(x, t) \otimes \psi^j(x', t)) & E(\psi^i(x, t) \otimes u^j(k, t)) \\
E(u^i(k, t) \otimes \psi^j(x', t)) & E(u^i(k, t) \otimes u^j(k', t))
\end{pmatrix}
\equiv \begin{pmatrix}
Q_t^{\psi \psi}(x, x') & Q_t^{\psi u}(x, k') \\
Q_t^{u \psi}(k, x') & Q_t^{u u}(k, k')
\end{pmatrix}, \ i, j = 0, 1, \ t \in \mathbb{R}.
\] (3.1)

Let us rewrite the correlation matrices \( Q_t^{ij}(p, p') \) by using the condition S1. Note that the dynamical group \( W(t) \) commutes with the translations in \( \mathbb{Z}^d \). In this case, condition S1 implies that

\[
Q_t(k + p, k + p') = Q_t(p, p'), \ t \in \mathbb{R}, \ k \in \mathbb{Z}^d.
\] (3.2)

Let us introduce the splitting \( p = k + r \), where \( k \in \mathbb{Z}^d \) and \( r \in K_1^d \cup \mathbb{R} \). In other words,

\[
r = \begin{cases}
\{ x - [x] \in K_1^d, & \text{if } p = x \in \mathbb{R}^d, \\
0, & \text{if } p = k \in \mathbb{Z}^d.
\end{cases}
\]

In this notation, (3.2) implies that

\[
Q_t^{ij}(k + r, k' + r') =: q_t^{ij}(k - k', r, r') \equiv \begin{pmatrix}
q_t^{\psi \psi}(k - k' + r, r') & q_t^{\psi u}(k - k' + r) \\
q_t^{u \psi}(k' - k + r') & q_t^{u u}(k' - k')
\end{pmatrix}.
\] (3.3)
Using the Zak transform (1.13), introduce the following matrices (cf (2.10)):

\[
\tilde{Q}_{ij}^t(\theta, r, r', t') := E[\tilde{Y}_{ij}^r(\theta, t) \otimes \tilde{Y}_{ij}^{r'}(\theta', t')], \quad \theta, \theta' \in K^d, \quad r, r' \in R \equiv T^d_1 \cup 0, \quad (3.4)
\]

where the convergence of the mathematical expectation is understood in the sense of distributions. Namely, write \( \tilde{D} = [\tilde{D}_F \oplus \tilde{D}_L]^2 \) and \( \tilde{D}_F := C^\infty(K^d \times T^d_1), \quad \tilde{D}_L := [C^\infty(T^d)]^n. \) Then

\[
\langle \tilde{Q}_{ij}^t(\theta, r, r', r'), \tilde{Z}_{ij}^r(\theta, r) \otimes \tilde{Z}_{ij}^{r'}(\theta', r') \rangle = E\langle \tilde{Y}_{ij}^r(\theta, r, t), \tilde{Z}_{ij}^r(\theta, r) \rangle (\tilde{Y}_{ij}^{r'}(\theta', r', t), \tilde{Z}_{ij}^{r'}(\theta', r')) \quad (3.5)
\]

for \( \tilde{Z}_{ij} = (\tilde{Z}_{ij}^0, \tilde{Z}_{ij}^1) \in \tilde{D}. \) Now (3.2) implies that

\[
\tilde{Q}_{ij}^t(\theta, r, r', r') = (2\pi)^d \delta(\theta - \theta')\tilde{q}^t_{ij}(\theta, r), \quad \theta, \theta' \in K^d, \quad r, r' \in R, \quad t \in R, \quad (3.6)
\]

where

\[
\tilde{q}^t_{ij}(\theta, r, r') = e^{i(r-r')\theta} \sum_{k \in Z^d} e^{ik\theta} \tilde{q}_{ij}^t(k, r, r') = \begin{pmatrix}
\tilde{q}_{ij}^{\psi \psi}(\theta, y, y') \\
\tilde{q}_{ij}^{\psi u}(\theta, y) \\
\tilde{q}_{ij}^{u \psi}(\theta, y') \\
\tilde{q}_{ij}^{u u}(\theta)
\end{pmatrix}. \quad (3.7)
\]

Recall that \( I_p \) is the disjoint union \( I_R^d \cup Z^d. \) For a measurable function \( Y(p), \)

\[
\int_{I_p} Y(p) \, dp = \int_{R^d} Y(x) \, dx + \sum_{k \in Z^d} Y(k). \quad (3.8)
\]

**Proposition 3.1** Let conditions \( S_0 \sim S_3 \) hold. Then (i) the following bounds hold

\[
\int_{I_p} |Q_0(p, p')| \, dp \leq C < \infty \quad \text{for any } p' \in I_p, \quad (3.9)
\]

\[
\int_{I_p} |Q_0(p, p')| \, dp' \leq C < \infty \quad \text{for any } p \in I_p, \quad (3.10)
\]

where the constant \( C \) does not depend on \( p, p' \in I_p. \)

(ii) \( D^{\alpha, \beta}_{y,y'} q_{ij}^{\psi \psi}(\theta, y, y'), \quad D^{\alpha, \beta}_{y,y'} q_{ij}^{\psi u}(\theta, y), \quad D^{\alpha, \beta}_{y,y'} q_{ij}^{u \psi}(\theta, y'), \quad q_{ij}^{u u}(\theta) \) are uniformly bounded in \( (\theta, y, y') \in K^d \times T^d_1 \times T^d_1, \quad |\alpha| \leq 1 - i, \quad |\beta| \leq 1 - j. \)

**Proof.** (i) By [15, Lemma 17.2.3], conditions \( S_0, S_2 \) and \( S_3 \) imply

\[
|Q_0(p, p')| \leq C \max\{e_F, e_L\} \varphi^{1/2}(|p - p'|), \quad p, p' \in I_p. \quad (3.11)
\]

Hence, the bounds (3.9) and (3.10) follow from (2.11).

(ii) Similarly to (3.11), conditions \( S_0, S_2 \) and \( S_3 \) imply

\[
|D^{\alpha, \beta}_{y,y'} q_{ij}^{\psi \psi}(\theta, y, y')| \leq \sum_{k \in Z^d} |D^{\alpha, \beta}_{y,y'} q_{ij}^{\psi}(k + y, y')| \leq C \sum_{k \in Z^d} \varphi^{1/2}(|k + y - y'|)
\]

\[
\leq C \sum_{k \in Z^d} \varphi^{1/2}(|k| - 2\sqrt{d}) \leq C < \infty.
\]

Similar arguments imply the other bounds. \( \square \)
Corollary 3.2  By the Shur lemma, Proposition 3.1, (i) implies that the following bound holds for any $F, G \in L^2 := [L^2(\mathbb{P}^1, dp)]^2 = [L^2(\mathbb{R}^d) \oplus [L^2(\mathbb{Z}^d)]^n]^2$:

$$|\langle Q_0(p, p'), F(p) \otimes G(p') \rangle| \leq C\|F\|_{L^2}\|G\|_{L^2}.$$  

Corollary 3.3  The quadratic form $Q_\infty(Z, Z)$ defined in (2.13)–(2.14) is continuous in $L^2$.

Proof. Formulas (2.13) and (2.14) imply

$$\langle Q_\infty(p, p'), Z(p) \otimes Z(p') \rangle = (2\pi)^{-d} \int_{K^d \setminus C_r} \left( \tilde{q}_\infty(\theta, r, r'), \tilde{Z}_\Pi(\theta, r) \otimes \overline{Z}_\Pi(\theta, r') \right) d\theta$$

$$= (2\pi)^{-d} \frac{1}{2} \sum_{l=1}^{\infty} \int_{K^d \setminus C_r} \left( \tilde{q}_0(\theta, r, r') + \tilde{r}_0(\theta, r, r'), P_l(\theta) \tilde{Z}_\Pi(\theta, r) \otimes P_l(\theta) \overline{Z}_\Pi(\theta, r') \right) d\theta, \ (3.12)$$

where $\tilde{r}_0(\theta, r, r')$ is the integral kernel of the operator $\tilde{r}_0(\theta) := \left( \begin{array}{cc} \tilde{H}^{-1}(\theta)\tilde{q}_0^{11}(\theta) & -\tilde{q}_0^{10}(\theta) \\ -\tilde{q}_0^{01}(\theta) & \tilde{H}(\theta)\tilde{q}_0^{00}(\theta) \end{array} \right)$.

Here and below, the symbol $(\cdot, \cdot)$ stands for the inner product in $H^0_1 \equiv H^0(K^d) \oplus \mathbb{C}^n$, i.e.,

$$(F, G) = \int_{K^d} F^1(y)G^1(y) dy + F^2 \cdot G^2, \quad F = (F^1, F^2), G = (G^1, G^2) \in H^0_1, \quad (3.13)$$

or in $H^0 \equiv [H^0_1]^2$. Consider the terms in the RHS of (3.12). Since

$$\sum_{l=1}^{\infty} \|P_l\tilde{Z}_\Pi\|_{L^2(K^d \times \mathbb{R})}^2 \leq C\|\tilde{Z}_\Pi\|_{L^2(K^d \times \mathbb{R})}^2 = C'\|Z\|_{L^2}^2, \ \ (3.14)$$

we obtain

$$\sum_{l=1}^{\infty} \int_{K^d \setminus C_r} \left( \tilde{r}_0^{ij}(\theta, r, r'), P_l(\theta) \tilde{Z}_\Pi^j(\theta, r) \otimes P_l(\theta) \overline{Z}_\Pi^i(\theta, r') \right) d\theta \leq C\|Z\|_{L^2}^2, \ \ i, j, i', j' = 0, 1, \ (3.15)$$

by Corollary 3.2. Further, consider $\tilde{r}_0^{00}$. For the terms with $\tilde{r}_0^{00}$ and $\tilde{r}_0^{10}$, estimate (3.15) holds. We rewrite the term with $\tilde{r}_0^{00}$ in the form

$$\sum_{l=1}^{\infty} \int_{K^d \setminus C_r} \left( \tilde{r}_0^{00}(\theta, r, r'), P_l(\theta) \tilde{Z}_\Pi^0(\theta, r) \otimes P_l(\theta) \overline{Z}_\Pi^0(\theta, r') \right) d\theta$$

$$= \sum_{l=1}^{\infty} \int_{K^d \setminus C_r} \left( \tilde{q}_0^{11}(\theta, r, r'), \tilde{H}^{-1}(\theta)P_l(\theta) \tilde{Z}_\Pi^0(\theta, r) \otimes P_l(\theta) \overline{Z}_\Pi^0(\theta, r') \right) d\theta. \ (3.16)$$

It follows from estimates (3.14) and (9.7) and Corollary 3.2 that the RHS of (3.16) is estimated from above by $C\|Z^0\|_{L^2(\mathbb{P})}^2$. Finally, consider the term with $\tilde{r}_0^{11}$ and represent it in the form

$$\sum_{l=1}^{\infty} \int_{K^d \setminus C_r} \left( \tilde{r}_0^{11}(\theta, r, r'), P_l(\theta) \tilde{Z}_\Pi^1(\theta, r) \otimes P_l(\theta) \overline{Z}_\Pi^1(\theta, r') \right) d\theta$$

$$= \sum_{l=1}^{\infty} \int_{K^d \setminus C_r} \left( \tilde{q}_0^{00}(\theta, r, r'), \Omega(\theta)P_l(\theta) \tilde{Z}_\Pi^1(\theta, r) \otimes \Omega(\theta)P_l(\theta) \overline{Z}_\Pi^1(\theta, r') \right) d\theta. \ (3.17)$$
By the bounds (3.14) and (9.6), the RHS of (3.17) is bounded by
\[
C \int_{K^d} \left( \| \tilde{q}_0^\omega \psi \|_{H^1(T^d_\omega)}^2 + \| \tilde{q}_0^\omega \omega \|_{H^1(T^d_\omega)} + \| \tilde{q}_0^\omega \|_{H^1(T^d_\omega)} + \| \tilde{q}_0^\omega \|_{H^1(T^d_\omega)} \right) d\theta.
\]

In turn, (3.18) is estimated by \( C \| Z^1 \|_{L^2(D)} \) by Proposition 3.1, (ii).

**Remark 3.4** The operator \( \tilde{q}_0(\theta) \) in \( H^0 \) is nonnegative and self-adjoint. Indeed, for any function \( Z \in D \), we have
\[
\int_{K^d} (\tilde{q}_0(\theta) \tilde{Z}_\Pi(\theta, \cdot) \tilde{Z}_\Pi(\theta, \cdot)) d\theta = (2\pi)^d E \langle Y, Z \rangle^2 \geq 0.
\]
Hence, \( (\tilde{q}_0(\theta) \tilde{Z}_\Pi(\theta, \cdot) \tilde{Z}_\Pi(\theta, \cdot)) \geq 0, \theta \in K^d \).

## 4 Compactness of measures \( \mu_t \)

Proposition 2.15 follows from the bound (2.15) by the Prokhorov Theorem [25, Lemma II.3.1] by using the method of [25, Theorem XII.5.2] because the embedding \( E^{0,\alpha} \subset E^{s,\beta} \) is compact if \( s < 0 \) and \( \alpha > \beta \).

**Lemma 4.1** Let conditions S0–S3 hold. Then the bounds (2.15) hold for \( \alpha < -d/2 \).

**Proof.** Step (i). By condition S1,
\[
E \| Y(t) \|_{0,\alpha}^2 = E \int_{\mathbb{R}^d} \left( 1 + |x|^2 \right)^\alpha \left( |\psi(x,t)|^2 + |\nabla \psi(x,t)|^2 + |\pi(x,t)|^2 \right) dx \\
+ \sum_{k \in \mathbb{Z}^d} \left( 1 + |k|^2 \right)^\alpha \left( |u(k,t)|^2 + |v(k,t)|^2 \right) \leq C(\alpha, d) e(t),
\]
where
\[
e(t) := E \left[ \int_{K^d_1} \left( |\psi(y,t)|^2 + |\nabla \psi(y,t)|^2 + |\pi(y,t)|^2 \right) dy + |u(0,t)|^2 + |v(0,t)|^2 \right].
\]

Denote by \( C_t \) the correlation operator of the random function
\[
(\psi(y,t)|_{y \in K^d_1}, u(0,t), \pi(y,t)|_{y \in K^d_1}, v(0,t)) \in \mathcal{E}_1 := H^1(K^d_1) \oplus \mathbb{R}^d \oplus H^0(K^d_1) \oplus \mathbb{R}^d.
\]
Then \( e(t) \) is equal to the trace of the operator \( C_t \). Note that \( C_t = \text{Op}(q_t(0,r,r')) \) is an integral operator with the integral kernel \( q_t(0,r,r') \) (see (3.7)),
\[
q_t(0,r,r') = (2\pi)^{-d} \int_{K^d} e^{-i(r-r') \theta} \tilde{q}_t(\theta,r,r') d\theta, \quad r, r' \in \mathcal{R}_1 := K^d_1 \cup \{0\}.
\]

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Denote by $\tilde{q}_t(\theta) := \text{Op}\left(\tilde{q}_t(\theta, r, r')\right)$ the integral operator with the integral kernel $\tilde{q}_t(\theta, r, r')$. In this case, (4.2) implies that

$$e(t) = \text{tr}_E C_t = (2\pi)^{-d} \int_{K^d} \text{tr}_{E_1} \left[ e^{-i\varphi \tilde{q}_t(\theta)} e^{i \varphi \theta} \right] d\theta.$$ (4.3)

**Step (ii).** Introduce the operator $\Gamma$ defined by $\Gamma \tilde{q}_t := \text{Op}(\tilde{q}_t(\theta, r, r'))$. Denote by $\tilde{\psi}_\Pi(\theta, y)$ given by (19, Theorem VI.18, (c)). Finally, since $\tilde{q}_t(\theta) \geq 0$ and the operator $e^{i \varphi \theta} : (\psi(y), u, \pi(y), v) \rightarrow (e^{i \varphi \theta} \psi, u, e^{i \varphi \theta} \pi, v)$ in $E_1$ is bounded uniformly with respect to $\theta \in K^d$, we have

$$\text{tr}_{E_1} \left[ e^{-i \varphi \tilde{q}_t(\theta)} e^{i \varphi \theta} \right] \leq C \text{tr}_{E_1} \tilde{q}_t(\theta) = C \text{tr}_H [\Gamma \tilde{q}_t(\theta) \Gamma^*].$$ (4.4)

Let us now estimate the trace $\text{tr}_H [\Gamma \tilde{q}_t(\theta) \Gamma^*]$. Introduce the matrix-valued self-adjoint operator $\Omega_{ex}$ on the space $H^0$,

$$\Omega_{ex} \equiv \Omega_{ex}(\theta) := \left( \begin{array}{cc} \Omega(\theta) & 0 \\ 0 & I \end{array} \right),$$

where $I$ stands for the identity operator on $H^0$. Note that $\Omega_{ex} \tilde{q}_t(\theta) \Gamma_{ex} \geq 0$ (recall that $\Omega_{ex}$ is a self-adjoint operator). Further, $B := (\Gamma_{ex} \Omega_{ex}^{-1})$ is a bounded operator on $H^0$ since $\Omega^{-1} : H^0 \rightarrow H^1$, and

$$\text{tr}_H [\Gamma \tilde{q}_t(\theta) \Gamma^*] = \text{tr}_H [B \Omega_{ex} \tilde{q}_t(\theta) \Omega_{ex} B^*] \leq C \text{tr}_H [\Omega_{ex} \tilde{q}_t(\theta) \Omega_{ex}].$$ (4.5)

by [23, Theorem 1.6]. Let us now estimate the trace of the operator $\Omega_{ex} \tilde{q}_t(\theta) \Gamma_{ex}$. We first use the formula $\Omega_{ex} G(\theta, t) = U(\theta, t) \Omega_{ex}$, where $G(\theta, t)$ is defined in (11.1) and

$$U(\theta, t) := \left( \begin{array}{cc} \cos \Omega t & \sin \Omega t \\ -\sin \Omega t & \cos \Omega t \end{array} \right).$$

Hence, by (11.4) we have

$$\Omega_{ex} \tilde{q}_t(\theta) \Gamma_{ex} = \Omega_{ex} G(\theta, t) \tilde{q}_0(\theta) G^*(\theta, t) \Omega_{ex} = U(\theta, t) \Omega_{ex} \tilde{q}_0(\theta) \Omega_{ex} U^*(\theta, t).$$

Since $U(t, \theta)$ is a unitary operator on $H^0$,

$$\text{tr}_H [\Omega_{ex} \tilde{q}_t(\theta) \Omega_{ex}] = \text{tr}_H [U(\theta, t) \Omega_{ex} \tilde{q}_0(\theta) \Omega_{ex} \Omega_{ex} U^*(\theta, t)] = \text{tr}_H [\Omega_{ex} \tilde{q}_0(\theta) \Omega_{ex}].$$ (4.6)

by [19, Theorem VI.18, (c)]. Finally, it is follows from (4.3)-(4.6) that

$$\sup_{t \in \mathbb{R}} e(t) \leq C_1 \int_{K^d} \text{tr}_H [\Omega_{ex} \tilde{q}_0(\theta) \Omega_{ex}] d\theta.$$ (4.7)
Step (iii). Let us now prove that the RHS of (4.7) is finite. We use the representation

\[ \Omega_{ex} \tilde{q}_0(\theta) \Omega_{ex}^* = (\Omega_{ex} \Gamma_{ex}^{-1}) \Gamma_{ex} \tilde{q}_0(\theta) \Gamma_{ex}^* (\Omega_{ex} \Gamma_{ex}^{-1})^*, \]

where \( \Gamma_{ex}^{-1} \) stands for the left inverse operator of \( \Gamma_{ex} \). On the other hand, \( \Omega_{ex} \Gamma_{ex}^{-1} \) is a bounded operator in \( H^0 \) since \( \Omega_{ex}(\theta) (\Gamma_{ex}^{-1}, \text{ resp.}) \) is (a finite-dimensional perturbation of) a pseudodifferential operator of order 1 \((-1, \text{ resp.})\) on \( K_1^d \). Moreover, \( \Omega_{ex}(\theta) \) is uniformly bounded in \( \theta \in K^d \). Hence,

\[ \text{tr}_{H^0}[\Omega_{ex} \tilde{q}_0(\theta) \Omega_{ex}^*] \leq C \text{tr}_{H^0}[\Gamma_{ex} \tilde{q}_0(\theta) \Gamma_{ex}^*] = C \text{tr}_{\mathcal{E}_1}[\tilde{q}_0(\theta)]. \]  

Finally, by inequalities (4.7) and (4.8) and by condition \( \text{S2} \), we obtain

\[
\sup_{t \in \mathbb{R}} e(t) \leq C \int_{K^d} \text{tr}_{\mathcal{E}_1}[\tilde{q}_0(\theta)] d\theta \\
\leq C_1 E \left[ \int_{K_1^d} (|\psi_0(y)|^2 + |\nabla \psi_0(y)|^2 + |\pi_0(y)|^2 + |u_0(y)|^2 + |v_0(y)|^2) dy \right] \\
\leq C_1 (\bar{e}_F + e_L) < \infty. \]  

Now the bound (2.15) follows from (4.1) and (4.9).

\[ \square \]

5 “Cutting out” the critical spectrum

**Definition 5.1** i) Introduce the critical set \( C := C_* \cup \left( \cup_k C_k \right) \) (see \( \text{E1} \)).

ii) Introduce the set \( \mathcal{D}^0 \subset \mathcal{D} \) given by

\[
\mathcal{D}^0 = \bigcup_N \mathcal{D}_N, \quad \mathcal{D}_N := \left\{ Z \in \mathcal{D} \mid P_l \tilde{Z}_\Pi(\theta, \cdot) = 0 \text{ for } \forall l \geq N, \theta \in K^d, \right. \\
\left. \tilde{Z}_\Pi(\theta, \cdot) = 0 \text{ in a neighborhood of a set } C \cup \partial K^d. \right\} \]  

**Lemma 5.2** Let \( \lim_{t \to \infty} \mathcal{Q}_t(Z, Z) = \mathcal{Q}_\infty(Z, Z) \) for any \( Z \in \mathcal{D}^0 \). Then the convergence holds for any \( Z \in \mathcal{D} \).

**Proof.** First, Definition 2.13 implies that

\[ \mathcal{Q}_t(Z, Z) := E[(Y(\cdot, t), Z)^2] = (Q_t(p, p'), Z(p) \otimes Z(p')), \quad Z \in \mathcal{D}. \]  

Therefore, by (7.2) we have \( \mathcal{Q}_t(Z, Z) = \mathcal{Q}_0(Z(\cdot, t), Z(\cdot, t)) \), and hence

\[ \sup_{t \in \mathbb{R}} |\mathcal{Q}_t(Z, Z)| \leq C \sup_{t \in \mathbb{R}} \|Z(\cdot, t)\|^2_{L^2} \]  

by Corollary 3.2. By the Parseval identity and by the bound (7.5), we obtain

\[
\|Z(\cdot, t)\|^2_{L^2} = C(d) \int_{K^d} \|e^{A^\top(\theta)t} \tilde{Z}_\Pi(\theta, \cdot)\|^2_{H^1_0 \oplus H^1_0} d\theta \leq C \int_{K^d} \|\tilde{Z}_\Pi(\theta, \cdot)\|^2_{H^1_0 \oplus H^1_0} d\theta \\
= C \|Z\|^2_Z \]  

uniformly with respect to \( t \). Here \( \mathcal{L} := L^2(\mathbb{R}^d) \oplus [L^2(\mathbb{Z}^d)]^n \oplus H^1(\mathbb{R}^d) \oplus [L^2(\mathbb{Z}^d)]^n \). Further, by Lemma 2.8, for any \( Z \in \mathcal{D} \), we can find a \( Z^N \in \mathcal{D}_N \) such that \( \|Z - Z^N\|_{\mathcal{C}} \to 0 \) as \( N \to \infty \). Namely, \( \tilde{Z}^N(\theta) = \sum_{l \leq N} P_l(\theta) \tilde{Z}_\Pi(\theta) \) if \( \tilde{Z}_\Pi(\theta) = 0 \) for \( \theta \) in a neighborhood of \( C \cup \partial K^d \). Finally, the set of such functions \( Z \) is dense in \( \mathcal{L} \). Then Lemma 5.2 follows from (5.3), (5.4), and Corollary 3.3. \[ \square \]
Lemma 5.3 The convergence (1.21) holds for any \( Z \in \mathcal{D} \) if it holds for \( Z \in \mathcal{D}^0 \).

Proof. This follows immediately from Lemma 5.2 by the Cauchy-Schwartz inequality:

\[
\begin{align*}
|\hat{\mu}_t(Z') - \hat{\mu}_t(Z'')| &= \left| \int \left( e^{i(Y,Z')} - e^{i(Y,Z'')} \right) \mu_t(dY) \right| \\
&\leq \int |\langle Y, Z' - Z'' \rangle| \mu_t(dY) \\
&\leq \sqrt{\int |\langle Y, Z' - Z'' \rangle|^2 \mu_t(dY)} \\
&= \sqrt{Q_t(Z'-Z'',Z'-Z'')} \leq C\|Z' - Z''\|_C.
\end{align*}
\]

\[\square\]

6 Convergence of the covariance

Proposition 6.1 Let conditions E1-E2, R1-R3 and S0-S3 hold. Then, for any \( Z \in \mathcal{D} \),

\[
Q_t(Z, Z) \to Q_\infty(Z, Z), \quad t \to \infty. \tag{6.1}
\]

Proof. By Lemma 5.2, it suffices to prove the convergence (6.1) for \( Z \in \mathcal{D}^0 \) only. If \( Z \in \mathcal{D}^0 \), then \( Z \in \mathcal{D}_N \) for some \( N \). Let us apply the Zak transform to the matrix \( Q_t(p, p') \),

\[
\langle Q_t(p, p'), Z(p) \otimes Z(p') \rangle = (2\pi)^{-2d} \langle \tilde{Q}_t(\theta, \theta', r, r'), \tilde{Z}_\Pi(\theta, r) \otimes \overline{\tilde{Z}_\Pi(\theta', r')} \rangle. \tag{6.2}
\]

Further, by Lemma 2.8, we can choose some smooth branches of the functions \( F_l(\theta, r) \) and \( \omega_l(\theta) \) to apply the stationary phase arguments, which requires some smoothness with respect to \( \theta \). Denote by \( \text{supp} \, \tilde{Z}_\Pi \) the closure of the set \( \{ \theta \in K^d : \tilde{Z}_\Pi(\theta, y) \neq 0, y \in T_1^d \} \). Since \( \text{supp} \, \tilde{Z}_\Pi \cap (C \cup \partial K^d) = \emptyset \), we can apply Lemma 2.8. Namely, for any point \( \Theta \in \text{supp} \, \tilde{Z}_\Pi \), there is a neighborhood \( O(\Theta) \subset K^d \setminus (C \cup \partial K^d) \) with the corresponding properties. Hence, \( \text{supp} \, \tilde{Z}_\Pi \subset \bigcup_{m=1}^M O(\Theta_m) \), where \( \Theta_m \in \text{supp} \, \tilde{Z}_\Pi \). Therefore, there is a finite partition of unity

\[
\sum_{m=1}^M g_m(\theta) = 1, \quad \theta \in \text{supp} \, \tilde{Z}_\Pi, \tag{6.3}
\]

where \( g_m \) are nonnegative functions of \( C_0^\infty(K^d) \) and \( \text{supp} \, g_m \subset O(\Theta_m) \). Further, using Definition 5.1, i) and the partition (6.3), represent the RHS of (6.2) as

\[
\langle Q_t(p, p'), Z(p) \otimes Z(p') \rangle = (2\pi)^{-d} \sum_{m=1}^M \sum_{l,l'=1}^N \langle g_m(\theta)r_{ll'}(t, \theta), A_l(\theta) \otimes \overline{A_{l'}(\theta)} \rangle, \tag{6.4}
\]

using formulas (3.6) and (11.6). Here \( A_l(\theta) = (F_1(\theta, \cdot), \tilde{Z}_\Pi(\theta, \cdot)) \), \( r_{ll'}(t, \theta) \) is the 2×2 matrix

\[
r_{ll'}(t, \theta) := \frac{1}{2} \sum_{\pm} \left\{ \cos (\omega_l(\theta) \pm \omega_{l'}(\theta)) t \left( p_{ll'}(\theta) \mp C_l(\theta)p_{ll'}(\theta)C_{l'}^T(\theta) \right) \right.
\]

\[
+ \sin (\omega_l(\theta) \pm \omega_{l'}(\theta)) t \left( C_l(\theta)p_{ll'}(\theta) \pm p_{ll'}(\theta)C_{l'}^T(\theta) \right) \right\}, \tag{6.5}
\]

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where
\[ C_l(\theta) := \begin{pmatrix} 0 & \omega_l^{-1}(\theta) \\ -\omega_l(\theta) & 0 \end{pmatrix}, \quad C_l^T(\theta) := \begin{pmatrix} 0 & -\omega_l(\theta) \\ \omega_l^{-1}(\theta) & 0 \end{pmatrix}, \]
and \((\cdot, \cdot)\) stands for the inner product in \(H_1^0 \equiv H^0(T^d) \oplus \mathbb{F}^n\) (see (3.13)) or in \(H^0 \equiv [H_1^0]^2\). By Lemma 2.8, the eigenvalues \(\omega_l(\theta)\) and the eigenfunctions \(F_l(\theta, r)\) are real-analytic functions in \(\theta \in \text{supp} \, g_m\) for every \(m\): we do not mark the functions by the index \(m\) to simplify the notation.

**Lemma 6.2** Let conditions **S0**–**S3** hold. Then \(p_{ll'}^{ij}(\theta) \in L^1(\mathcal{O}(\Theta_m)), \ i, j = 0, 1, \ l, l' = 1, 2, \ldots \) for each \(m = 1, \ldots, M\).

**Proof.** Since \(\{F_l(\theta, \cdot)\}\) is an orthonormal basis, by the Cauchy-Schwartz inequality, we have
\[
\left\| \int_{\mathcal{O}(\Theta_m)} |p_{ll'}^{ij}(\theta)| \, d\theta \right\|^2 \leq C \int_{\mathcal{O}(\Theta_m)} |p_{ll'}^{ij}(\theta)|^2 \, d\theta \leq \int_{\mathcal{O}(\Theta_m)} \left| (F_l(\theta, r), q_0^{ij} + F_l(\theta, r')) \right|^2 \, d\theta \leq \int_{\mathcal{O}(\Theta_m)} d\theta \int_{\mathcal{R}} \int_{\mathcal{R}} |q_0^{ij}(\theta, r, r')|^2 \, dr'. \]

Further, let us study the terms in (6.4), which are oscillatory integrals with respect to the variable \(\theta\). The identities \(\omega_l(\theta) + \omega_l(\theta) \equiv \text{const}_+\) or \(\omega_l(\theta) - \omega_l(\theta) \equiv \text{const}_-\) with \(const_+ \neq 0\) are impossible by condition **E2**. Moreover, the oscillatory integrals with \(\omega_l(\theta) \pm \omega_l(\theta) \neq \text{const}_\) vanish as \(t \to \infty\). Hence, only the integrals with \(\omega_l(\theta) - \omega_l(\theta) \equiv 0\) contribute to the limit because the relation \(\omega_l(\theta) + \omega_l(\theta) \equiv 0\) would imply the relation \(\omega_l(\theta) \equiv \omega_l(\theta) \equiv 0\), which is impossible by **E2**. Let us index the eigenvalues \(\omega_l(\theta)\) as in (2.5). Then \(\cos(\omega_l(\theta) - \omega_l(\theta))t = 1\) for \(l, l' \in (r_{\sigma-1}, r_{\sigma}]\), \(\sigma = 1, 2, \ldots\). Hence, for \(l, l' \in (r_{\sigma-1}, r_{\sigma}]\), we have
\[
r_{ll'}(t, \theta) = \frac{1}{2} \left( p_{ll'}(\theta) + C_l(\theta)p_{ll'}(\theta)C_l^T(\theta) \right) \\
+ \frac{1}{2} \cos 2\omega_l(\theta) t \left( p_{ll'}(\theta) - C_l(\theta)p_{ll'}(\theta)C_l^T(\theta) \right) \\
+ \frac{1}{2} \sin 2\omega_l(\theta) t \left( C_l(\theta)p_{ll'}(\theta) + p_{ll'}(\theta)C_l^T(\theta) \right). \] (6.8)

Therefore,
\[
\langle Q_t(p, p'), Z(p) \otimes Z(p') \rangle = (2\pi)^{-d} \sum_m \sum_{l, l'} \int g_m(\theta) \left( M_{ll'}(\theta) + \ldots, A_l(\theta) \otimes \overline{A_{l'}}(\theta) \right) \, d\theta, \] (6.9)
where \(M_{ll'}(\theta) = (M_{ll'}^{ij}(\theta))_{l, l' = 1, 2, \ldots, N} \) is the matrix with the continuous entries
\[
M_{ll'}^{ij}(\theta) = \chi_{ll'}^{ij} \frac{1}{2} \left( F_l(\theta, r), \left[ q_0(\theta, r, r') + C_l(\theta)q_0(\theta, r, r')C_l^T(\theta) \right] \right) F_{ll'}(\theta, r'). \] (6.10)
here the symbol $\chi_{W'}$ is given by (see (2.5))

$$\chi_{W'} := \begin{cases} 1 & \text{if } l, l' \in (r_{\sigma-1}, r_{\sigma}], \sigma = 1, 2, ..., r_0 := 0, \\ 0 & \text{otherwise.} \end{cases} \quad (6.11)$$

Further, for $\theta \in \text{supp} g_m \subset \mathcal{O}(\Theta)$ (see Lemma 2.8), we write

$$\tilde{q}_{\infty}^{ij}(\theta, r, r') = \sum_{i, j, v=1}^{+\infty} F_i(\theta, r) M_{ij}^{\delta v}(\theta) F_v(\theta, r'), \quad i, j = 0, 1. \quad (6.12)$$

The local representation (6.12) can be expressed globally in the form (2.14). Hence,

$$\langle Q_t(p, p'), Z(p) \otimes Z(p') \rangle = (2\pi)^{-d} \sum_{m} g_m(\theta) \left( \tilde{q}_{\infty}(\theta, r, r'), \tilde{Z}_m(\theta, r) \otimes \tilde{Z}_m(\theta, r') \right) d\theta + \ldots \quad (6.13)$$

where the symbol "..." stands for the oscillatory integrals which contain $\cos(\omega_l(\theta) \pm \omega_{l'}(\theta)) t$ and $\sin(\omega_l(\theta) \pm \omega_{l'}(\theta)) t$ with $\omega_l(\theta) \pm \omega_{l'}(\theta) \neq \text{const}$. The oscillatory integrals converge to zero by the Lebesgue-Riemann Theorem because the integrands in "..." are summable, and we have $\nabla (\omega_l(\theta) \pm \omega_{l'}(\theta)) = 0$ on the set of Lebesgue measure zero only. The summability follows from Lemma 6.2 because the functions $A_l(\theta)$ are smooth. The zero-measure condition follows as in (2.4) since $\omega_l(\theta) \pm \omega_{l'}(\theta) \neq \text{const}$. This completes the proof of Proposition 6.1.

\section{Bernstein’s argument}

\subsection{Oscillatory representation and stationary phase method}

To prove (1.21), we evaluate $\langle Y(\cdot, t), Z \rangle$ by duality arguments. Namely, introduce the dual space $\mathcal{E'} := H^{-1, -\alpha}(\mathbb{R}^d) \oplus L^{-\alpha} \oplus H^{0, -\alpha}(\mathbb{R}^d) \oplus L^{-\alpha}$ with finite norm

$$(\|Z\|_{0, -\alpha})^2 := \|\psi\|^2_{1, -\alpha} + \|\pi\|^2_{0, -\alpha} + \|u\|^2_{-\alpha} + \|v\|^2_{-\alpha}.$$ 

For $t \in \mathbb{R}$, introduce the “formal adjoint” operator $W'(t)$,

$$\langle W(t) Y, Z \rangle := \langle Y, W'(t) Z \rangle, \quad Y \in \mathcal{E}, \quad Z \in \mathcal{E'}, \quad (7.1)$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(\mathbb{R}^d) \oplus [l^2(\mathbb{Z}^d)]^n \oplus L^2(\mathbb{R}^d) \oplus [l^2(\mathbb{Z}^d)]^n$.

Write $Z(\cdot, t) = W'(t) Z$. Then formula (7.1) can be rewritten as

$$\langle Y(t), Z \rangle = \langle Y_0, Z(\cdot, t) \rangle, \quad t \in \mathbb{R}. \quad (7.2)$$

The adjoint group $W'(t)$ admits a convenient description.

**Lemma 7.1** The action of the group $W'(t)$ coincides with the action of $W(t)$ up to the order of components. Namely, $W'(t) = \exp(A^T t)$, where $A$ is the generator of the group $W(t)$. 

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Theorem 7.2 The following uniform bound holds:

\[ \|e^{\mathcal{A}T(\theta)} \tilde{Z}_\Pi(\theta, \cdot)\|_{H^0 \oplus H^1} \leq C \|\tilde{Z}_\Pi(\theta, \cdot)\|_{H^0 \oplus H^1}, \quad \tilde{Z}_\Pi(\theta, \cdot) \in H^0 \oplus H^1, \]

which can be proved similarly to (9.3).

Applying Lemma 7.1, we can rewrite \( Z(t) = W'(t)Z \) as the Zak transform, i.e., \( \tilde{Z}_\Pi(\theta, r, t) = \exp \left( \mathcal{A}^T(\theta) t \right) Z_\Pi(\theta, r) \). Recall that we can restrict ourselves to elements \( Z \in \mathcal{D}_N \) with a fixed index \( N \). Using the partition of unity (6.3), we obtain

\[
Z(k + r, t) = (2\pi)^{-d} \sum_{m=1}^{M} \sum_{l=1}^{N} \int g_m(\theta) e^{-i(k+r)\theta} G_l^T(\theta, t) F_l(\theta, r) A_l(\theta) \, d\theta \,
\]

Here \( A_l(\theta) = (F_l(\theta, \cdot), \tilde{Z}_\Pi(\theta, \cdot)) \),

\[
G_l(\theta, t) := \begin{pmatrix} \cos \omega_l(\theta)t & \sin \omega_l(\theta)t \\ -\omega_l(\theta) \sin \omega_l(\theta)t & \omega_l(\theta) \cos \omega_l(\theta)t \end{pmatrix}, \quad \theta \in \text{supp } g_m, \tag{7.7}
\]

and \( \omega_l(\theta) \) and \( a_l^\pm(\theta) \) are real-analytic functions in the interior of the set \( \text{supp } g_m \) for every \( m \).

Let us derive formula (1.21) by analyzing the propagation of the solution \( Z(k + r, t) \) of the form (7.6) in diverse directions \( k = vt \) with \( v \in \mathbb{R}^d \) and for \( r \in \mathcal{R} \). To this end, we apply the stationary phase method to the oscillatory integral (7.6) along the rays \( k = vt \), \( t > 0 \). Then the phase becomes \( (\theta v \pm \omega_l(\theta)) t \), and its stationary points are the solutions of the equations \( v = \mp \nabla \omega_l(\theta) \).

Note that \( \tilde{Z}_\Pi(\theta, r) = 0 \) at the points \( (\theta, r) \in K^d \oplus \mathcal{R} \) with degenerate Hessian \( D_l(\theta) \) (see E1). Therefore, the stationary phase method leads to the following two different types of asymptotic behavior of \( Z(vt, t) \) as \( t \to \infty \).

I. Let the velocity \( v \) be inside the light cone, \( v = \pm \nabla \omega_l(\theta) \), where \( \theta \in \mathcal{O}(\Theta) \setminus \mathcal{C} \). Then

\[
Z(vt, t) = \mathcal{O}(t^{-d/2}). \tag{7.8}
\]

II. Let the velocity \( v \) be outside the light cone, \( v \neq \pm \nabla \omega_l(\theta) \), where \( \theta \in \mathcal{O}(\Theta) \setminus \mathcal{C} \), \( l = 1, \ldots, N \). Then

\[
Z(vt, t) = \mathcal{O}(t^{-k}), \quad \forall k > 0. \tag{7.9}
\]
Lemma 7.3 The following bounds hold for any fixed $Z \in \mathcal{D}^0$:

i) \[ \sup_{p \in \mathcal{P}} |Z(p, t)| \leq C t^{-d/2}. \] (7.10)

ii) For any $k > 0$, there exist numbers $C_k, \gamma > 0$ such that

\[ |Z(p, t)| \leq C_k (1 + |p| + |t|)^{-k}, \quad |p| \geq \gamma t. \] (7.11)

Proof. Consider $Z(k + r, t)$ along each ray $k = vt$ with an arbitrary $v \in \mathbb{R}^d$. Substituting the related expressions into (7.6), we obtain

\[ Z(vt + r, t) = \sum_{m, \pm t=1}^N \int e^{-i(\theta v + \omega_l(\theta))t} e^{-i\theta r} a^\pm_l(\theta) F_l(\theta, r) A_l(\theta) d\theta, \quad Z \in \mathcal{D}_N. \] (7.12)

This is a sum of oscillatory integrals with phase functions of the form $\phi^\pm_l(\theta) = \theta v + \omega_l(\theta)$ and with amplitudes $a^\pm_l(\theta)$ that are real-analytic functions of $\theta$ in the interiors of the sets $\text{supp} \ g_m$. Since $\omega_l(\theta)$ is real-analytic, each function $\phi^\pm_l$ has at most finitely many stationary points $\theta \in \text{supp} \ g_m$ (solutions of the equation $v = \mp \nabla \omega_l(\theta)$). The stationary points are nondegenerate for $\theta \in \text{supp} \ g_m$ by Definition 5.1 and by E1 since

\[ \det\left( \frac{\partial^2 \phi^\pm_l}{\partial \theta_i \partial \theta_j} \right) = \pm D_l(\theta) \neq 0, \quad \theta \in \text{supp} \ g_m. \] (7.13)

At last, $\tilde{Z}_H(\theta, r)$ is smooth because $Z \in \mathcal{D}$. Therefore, we have $Z(vt + r, t) = \mathcal{O}(t^{-d/2})$ according to the standard stationary phase method of [12, 20]. This implies the bounds (7.10) in each cone $|k| \leq ct$ with any finite $c$.

Further, write $\bar{v} := \max_{m} \max_{t=1}^N \max_{\theta \in \text{supp} \ g_m} |\nabla \omega_l(\theta)|$. Then, for $|v| > \bar{v}$, there are no stationary points in $\text{supp} \ Z_H$. Hence, integration by parts (as in [20]) yields $Z(vt + r, t) = \mathcal{O}(t^{-k})$ for any $k > 0$. On the other hand, the integration by parts in (7.6) implies a similar bound, $Z(p, t) = \mathcal{O}\left(\frac{|t|}{|p|}\right)$ for any $l > 0$. Therefore, relation (7.11) follows with any $\gamma > \tau$. This shows that the bounds (7.10) hold everywhere.

\[ \square \]

7.2 “Room-corridor” partition

The remaining constructions in the proof of (1.21) are similar to [5, 8]. However, the proofs are not identical, since here we consider a non-translation-invariant case and a coupled system.

Introduce a “room-corridor” partition of the ball $\{ p \in \mathcal{P} : |p| \leq \gamma t \}$ with $\gamma$ taken from (7.11). For $t > 0$, choose $\Delta_t$ and $\rho_t \in \mathbb{N}$. Asymptotic relations between $t$, $\Delta_t$ and $\rho_t$ are specified below. Set $h_t = \Delta_t + \rho_t$ and

\[ a^j = j h_t, \quad b^j = a^j + \Delta_t, \quad j \in \mathbb{Z}, \quad N_t = \lfloor \gamma t / h_t \rfloor. \] (7.14)

The slabs $R^t_c = \{ p \in \mathcal{P} : |p| \leq N_t h_t, a^j \leq p_d < b^j \}$ are referred to as “rooms”, $C^t_c = \{ p \in \mathcal{P} : |p| \leq N_t h_t, b^j \leq p_d < a^{j+1} \}$ as “corridors”, and $L^t_t = \{ p \in \mathcal{P} : |p| > N_t h_t \}$ as “tails”. Here $p = (p_1, \ldots, p_d)$, $\Delta_t$ is the width of a room, and $\rho_t$ is that of a corridor. Denote by
the indicator of the room $R^j_t$, by $\xi^j_t$ the indicator of the corridor $C^j_t$, and by $\eta_t$ the indicator of the tail $L_t$. In this case,

$$\sum_{t} \left[ \chi^j_t(p) + \xi^j_t(p) \right] + \eta_t(p) = 1, \ p \in \mathbb{I}P,$$

(7.15)

where the sum $\sum_t$ stands for $\sum_{j=-N_t}^{N_t-1}$. Hence, we obtain the following Bernstein’s type representation:

$$\langle Y_0, Z(\cdot, t) \rangle = \sum_{t} \left[ \langle Y_0, \chi^j_t Z(\cdot, t) \rangle + \langle Y_0, \xi^j_t Z(\cdot, t) \rangle \right] + \langle Y_0, \eta_t Z(\cdot, t) \rangle.$$

(7.16)

Introduce the random variables $r^j_t$, $c^j_t$ and $l_t$ by the formulas

$$r^j_t = \langle Y_0, \chi^j_t Z(\cdot, t) \rangle, \ c^j_t = \langle Y_0, \xi^j_t Z(\cdot, t) \rangle, \ l_t = \langle Y_0, \eta_t Z(\cdot, t) \rangle.$$

(7.17)

Then relation (7.16) becomes

$$\langle Y_0, Z(\cdot, t) \rangle = \sum_{t} (r^j_t + c^j_t) + l_t.$$

(7.18)

Lemma 7.4 Let $S_0$–$S_3$ hold and $Z \in \mathcal{D}_0$. The following bounds hold for $t > 1$:

$$E|r^j_t|^2 \leq C(Z) \Delta_t/t, \ \forall j,$$

(7.19)

$$E|c^j_t|^2 \leq C(Z) \rho_t/t, \ \forall j,$$

(7.20)

$$E|l_t|^2 \leq C_k(Z) t^{-k}, \ \forall k > 0.$$

(7.21)

Proof. Relation (7.21) follows from (7.11) and Proposition 3.1, (i). We discuss (7.19) only, and relation (7.20) can be studied in a similar way. Let us express $E|r^j_t|^2$ in terms of correlation matrices. Definition (7.17) implies

$$E|r^j_t|^2 = \langle Q_0(p, p'), \chi^j_t(p)Z(p, t) \otimes \chi^j_y(p')Z(p', t) \rangle.$$

(7.22)

According to (7.10), Eqn (7.22) yields

$$E|r^j_t|^2 \leq C t^{-d} \int \chi^j_t(p) \|Q_0(p, p')\| dpdp'$$

$$= C t^{-d} \int \chi^j_t(p) dp \int \|Q_0(p, p')\| dp' \leq C \Delta_t/t,$$

(7.23)

where $\|Q_0(p, p')\|$ stands for the norm of the matrix $\left(Q_0^{ij}(p, p')\right)$. Therefore, (7.23) follows by Corollary 3.2.

7.3 Proof of Theorem A

The remaining part of the proof of the convergence (1.21) uses the Ibragimov-Linnik central limit theorem [15] and the bounds (7.19)-(7.21). For details, see [5, Sections 8,9] and [9, Sections 9,10].
8 Ergodicity and mixing for the limit measures

The limit measure $\mu_\infty$ is invariant by Theorem A, (iv). Let $E_\infty$ be the integral with respect to $\mu_\infty$.

**Theorem 8.1** Let the assumptions of Theorem A hold. Then $W(t)$ is mixing with respect to the corresponding limit measure $\mu_\infty$, i.e., for any $f, g \in L^2(E, \mu_\infty)$ we have

$$\lim_{t \to \infty} E_\infty f(W(t)Y)g(Y) = E_\infty f(Y)E_\infty g(Y). \quad (8.1)$$

In particular, the group $W(t)$ is ergodic with respect to the measure $\mu_\infty$, i.e.,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(W(t)Y)dt = E_\infty f(Y) \quad (\text{mod } \mu_\infty). \quad (8.2)$$

**Proof.** Step (i). Since $\mu_\infty$ is Gaussian, the proof of (8.1) reduces to that of the convergence

$$\lim_{t \to \infty} E_\infty \langle W(t)Y, Z \rangle \langle Y, Z^1 \rangle = 0. \quad (8.3)$$

for any $Z, Z^1 \in D$. It suffices to prove relation (8.3) for $Z, Z^1 \in D_N$. However, this follows from Corollary 3.3 and Theorem A, (iv).

Step (ii). Let $Z, Z^1 \in D_N$. Applying the Zak transform and the Parseval identity, we obtain

$$I(t) \equiv E_\infty \langle W(t)Y, Z \rangle \langle Y, Z^1 \rangle = E_\infty \langle Y, W'(t)Z \rangle \langle Y, Z^1 \rangle = (2\pi)^{-d} \langle \tilde{q}_\infty, G^*(\theta, t) \tilde{Z}_\Pi(\theta, r) \otimes \tilde{Z}_\Pi^1(\theta, r') \rangle. \quad (8.4)$$

Using a finite partition of unity (6.3), and relations (8.4) and (6.12), we see that

$$I(t) = (2\pi)^{-d} \sum_{m=1}^M \sum_{l, l'=1}^N \int g_m(\theta)G_l^*(\theta, t)A_l(\theta)M_{ll'}(t, \theta)A_{l'}(\theta) d\theta. \quad (8.5)$$

Here $A_l(\theta) = (F_l(\theta, \cdot), \tilde{Z}_\Pi(\theta, \cdot))$ and $A_{l'}(\theta) = (F_{l'}(\theta, \cdot), \tilde{Z}_{\Pi}^1(\theta, \cdot))$, and $G_l(t, \theta)$ is defined in (7.7). Similarly to (7.6), we have

$$I(t) = \sum_{m} \sum_{l, l'=1}^N \int g_m(\theta)e^{\pm i\omega_l(\theta)}a_l^\pm(\theta)A_l(\theta)M_{ll'}(\theta)A_{l'}(\theta) d\theta. \quad (8.6)$$

Here all the phase functions $\omega_l(\theta)$ and the amplitudes $a_l^\pm(\theta)$ are smooth functions on $\text{supp } g_m$. Further, the relation $\nabla \omega_l(\theta) = 0$ holds on a set of Lebesgue measure zero only. This follows similarly to (2.4) since $\nabla \omega_l(\theta) \not\equiv \text{const}$ by condition $E2$. Hence, $I(t) \to 0$ as $t \to \infty$ by the Lebesgue-Riemann theorem since the functions $M_{ll'}(\theta)$ are continuous. \[\blacksquare\]

**Remark.** A similar result for wave equations and for harmonic crystals was proved in [4, 8].
9 Appendix A: Dynamics in the Bloch-Fourier representation

In this appendix we prove the bound (2.2). We first construct the exponential \( \exp(\tilde{A}(\theta)t) \) for any chosen \( \theta \in K^d \equiv [0, 2\pi]^d \) and study its properties. Let us choose \( \theta \in K^d \) and \( X_0 \in H^1 := H^1_1 \oplus H^0 \), where \( H^s_1 \equiv H^s(T^d_1) \oplus \mathbb{R}^n \). Introduce the functions \( \exp(\tilde{A}(\theta)t)X_0 \) for \( X_0 \in H^1 \) as the solutions \( X(\theta,t) \) to the problem

\[
\begin{aligned}
\dot{X}(\theta, t) &= \tilde{A}(\theta)X(\theta, t), \quad t \in \mathbb{R}, \\
X(\theta, 0) &= X_0.
\end{aligned}
\]  

(9.1)

**Proposition 9.1** For any chosen \( \theta \in K^d \), the Cauchy problem (9.1) admits a unique solution \( X(\theta, t) \in C(\mathbb{R}; H^1) \). Moreover,

\[
X(\theta, t) = e^{\tilde{A}(\theta)t}X_0,
\]

(9.2)

and

\[
\|X(\theta, t)\|_{H^1} \leq C\|X_0\|_{H^1},
\]

(9.3)

where the constant \( C \) does not depend on \( \theta \in K^d \) and \( t \in \mathbb{R} \).

We prove this proposition in Subsection 9.2.

9.1 Schrödinger operator

Let us first construct solutions \( X(\theta, t) \) to problem (9.1) with a chosen parameter \( \theta \in K^d \). Write \( X(\theta, t) = (X^0(\theta, t), X^1(\theta, t)) \), where \( X^0(\theta, t) = (\varphi(\theta, t), u(\theta, t)) \) and \( X^1(\theta, t) = (\phi(\theta, t), v(\theta, t)) \). By (9.1) and (1.15) we have \( X^1(\theta, t) = \dot{X}^0(\theta, t) \), and \( X^0(\theta, t) \) is a solution to the following Cauchy problem with a chosen parameter \( \theta \in K^d \):

\[
\begin{aligned}
\ddot{X}^0(\theta, t) &= -\tilde{\mathcal{H}}(\theta)X^0(\theta, t), \quad t \in \mathbb{R}, \\
\left. \begin{array}{l}
X^0(\theta, t), \dot{X}^0(\theta, t)
\end{array} \right|_{t=0} &= \begin{array}{l}
X^0_0, X^1_0
\end{array} = X_0,
\end{aligned}
\]

(9.4)

where \( \tilde{\mathcal{H}}(\theta) \) is the “Schrödinger operator” (1.16). Hence, formally,

\[
X^0(\theta, t) = \cos \Omega(\theta)t \ X^0_0 + \sin \Omega(\theta)t \ \Omega^{-1}(\theta)X^1_0,
\]

(9.5)

where \( \Omega(\theta) = \sqrt{\tilde{\mathcal{H}}(\theta)} > 0 \).

**Lemma 9.2** For \( X^0 \in H^1_0 \), the following bounds hold:

\[
\|\Omega(\theta)X^0\|_{H^{-1}_0} \leq C\|X^0\|_{H^1_0},
\]

(9.6)

\[
\|\tilde{\mathcal{H}}^{-1}(\theta)X^0\|_{H^0_1} \leq C\|X^0\|_{H^1_0},
\]

(9.7)

where the constant \( C \) does not depend on \( \theta \in K^d \).
Proof (i) Formula (1.16) for $\tilde{H}(\theta)$ implies that
\[
\|\tilde{H}(\theta)X^0\|_{H^{-1}_1} \leq C\|X^0\|_{H^1_1}, \quad X^0 \in H^1_1.
\]
where the constant $C$ does not depend on $\theta \in K^d$. Hence,
\[
\|\Omega(\theta)X^0\|_{H^2_1}^2 = (X^0, \tilde{H}(\theta)X^0) \leq \|X^0\|_{H^1_1} \|\tilde{H}(\theta)X^0\|_{H^{-1}_1} \leq C\|X^0\|_{H^1_1}^2.
\]
(9.9)
Since, $\Omega(\theta) = \Omega^*(\theta)$, the bound (9.9) implies (9.6).

(ii) Condition \textbf{R2} implies that
\[
\|X^0\|_{H^1_1} \|\tilde{H}(\theta)X^0\|_{H^{-1}_1} \geq (X^0, \tilde{H}(\theta)X^0) \geq \kappa^2\|X^0\|_{H^1_1}^2.
\]
Hence, $\|\tilde{H}(\theta)X^0\|_{H^{-1}_1} \geq \kappa^2\|X^0\|_{H^1_1}$. Therefore, $\|\tilde{H}^{-1}(\theta)X^0\|_{H^1_1} \leq \kappa^{-2}\|X^0\|_{H^{-1}_1}$. In particular, (9.7) follows.

Remark 9.3 Condition \textbf{R2'} implies condition \textbf{R2}.

Proof. Indeed, for $X^0 = (\varphi(y), u) \in H^1_1$ and $\theta \in K^d$ we have
\[
(X^0, \tilde{H}(\theta)X^0) = \int_{T^d_1} \left[ \varphi(y) (i\nabla_y + \theta)^2 + m_0^2 \varphi(y) + \overline{\bar{R}_{\Pi}(\theta, y)} \cdot u(y) \varphi(y) 
\right. 
+ \bar{R}_{\Pi}(\theta, y) \cdot u(y) \varphi(y) \bigg] \, dy + \omega^2(\theta)|u(y)|^2 
\]
\[
= \int_{T^d_1} \left[ (i\nabla_y + \theta) \varphi(y) \bigg]^2 + m_0^2 \varphi(y) + \frac{2}{m_0^2} \bar{R}_{\Pi}(\theta, y)|u(y)|^2 \bigg] \, dy 
+ \omega^2(\theta)|u(y)|^2 - \frac{2}{m_0^2} \int_{T^d_1} |\bar{R}_{\Pi}(\theta, y)|^2 \, dy + \frac{m_0^2}{2} \int |\varphi(y)|^2 \, dy 
\]
\[
\geq \alpha \int_{T^d_1} |\nabla_y \varphi(y)|^2 \, dy + \left( \frac{m_0^2}{2} - \beta d2\pi \right) \int |\varphi(y)|^2 \, dy 
+ |u(y)|^2 \left( \nu_0^2 - \frac{2}{m_0^2} \int_{T^d_1} |\bar{R}_{\Pi}(\theta, y)|^2 \, dy \right) 
\]
for some $\beta > 0$ and $\alpha \in (0, \beta/(\beta + 1))$. Take $\beta < m_0^2/(4\pi d)$. It remains to prove that
\[
\nu_0^2 - \frac{2}{m_0^2} \int_{T^d_1} |\bar{R}_{\Pi}(\theta, y)|^2 \, dy > 0.
\]

With regard to condition \textbf{R2'}, The Parseval equality implies
\[
\int_{T^d_1} |\bar{R}_{\Pi}(\theta, y)|^2 \, dy = \int_{T^d_1} \left| \sum_{k \in \mathbb{Z}^d} e^{ik\theta} R(k + y) \right|^2 \, dy \leq \int_{T^d_1} \left| \sum_{k \in \mathbb{Z}^d} |R(k + y)| \right|^2 \, dy < \nu_0^2 m_0^2./2.
\]
9.2 Existence of the Schrödinger group

Recall that \( \omega_l(\theta) > 0 \) (\( F_l(\theta, \cdot) \)), \( l = 1, 2, \ldots \), are the eigenvalues (orthonormal eigenvectors) of the operator \( \Omega(\theta) \) in \( H_0^1 \). Let us prove the existence of solutions to the Cauchy problem (9.1). We represent \( X^0(\theta, t) \) in the form

\[
X^0(\theta, t) = \sum_{l=1}^{\infty} A_l(t) F_l(\theta, r), \quad t \in \mathbb{R}, \tag{9.10}
\]

where \( A_l(t) \equiv A_l(\theta, t) \) is the unique solution of the Cauchy problem

\[
\ddot{A}_l(t) = -\omega_l^2(\theta) A_l(t), \quad (A_l(t), \dot{A}_l(t))|_{t=0} = (A^0_{0l}, A^1_{0l}),
\]

and \( A^0_{0l} \equiv A^0_{0l}(\theta) = (F_l(\theta, \cdot), X^0_0(\cdot)) \), \( i = 0, 1 \). Hence,

\[
A_l(t) = \cos \omega_l(\theta)t A^0_{0l} + \frac{\sin \omega_l(\theta)t}{\omega_l(\theta)} A^1_{0l}. \tag{9.11}
\]

By the energy conservation, this yields

\[
\frac{|\dot{A}_l(t)|^2}{2} + \omega_l^2(\theta) \frac{|A_l(t)|^2}{2} = \frac{|A^0_{0l}|^2}{2} + \omega_l^2(\theta) \frac{|A^1_{0l}|^2}{2}.
\]

Summing up, for \( t \in \mathbb{R} \), we obtain

\[
\frac{1}{2} \|X^0(\theta, t)\|_{H_0^1}^2 + \frac{1}{2} (X^0(\theta, t), \tilde{\mathcal{H}}(\theta)X^0(\theta, t)) = \frac{1}{2} \|X^1_0\|_{H_1^1}^2 + \frac{1}{2} (X^0_0, \tilde{\mathcal{H}}(\theta)X^0_0) \leq C \|X_0\|_{\mathcal{H}^1}^2. \tag{9.12}
\]

by (9.8). Hence, the solution (9.10) exists and is unique.

Further, relation (9.11) implies (9.5). Finally, the solution to problem (9.1) exists; it is unique and can be represented by (9.2). The bound (9.3) follows from (9.12) and (2.1).

Now the exponential \( \exp\left(\tilde{A}(\theta)t\right) \) is defined for any chosen value \( \theta \in K^d \), and this exponential is a continuous operator in \( \mathcal{H}^1 \).

9.3 Smoothness of the Schrödinger group

To complete the proof of Proposition 9.1, we must prove the smoothness of the exponential with respect to \( \theta \). This is needed to define the product (1.19) of the exponential and the distribution \( \tilde{Y}_{0\mathcal{H}}(\cdot) \).

Consider the operators \( \exp\left(\tilde{A}'(\theta)t\right), \quad t \in \mathbb{R}, \) on \( \mathcal{H}^{-1} := (\mathcal{H}^1)^* = H_1^{-1} \oplus H_0^1 \), where \( \tilde{A}'(\theta) \) is the formal adjoint operator to \( \tilde{A}(\theta) \):

\[
(X, \tilde{A}'(\theta)Z)_{H_0^1} = (\tilde{A}(\theta)X, Z)_{H_0^1}, \quad X, Z \in C_0^\infty(T_1^d) \times \Phi^n.
\]

Note that

\[
\tilde{A}'(\theta) = \tilde{A}^T(\theta) = \begin{pmatrix} 0 & -\tilde{\mathcal{H}}(\theta) \\ 1 & 0 \end{pmatrix}. \tag{9.13}
\]
Lemma 9.4 For any \( \alpha \geq 0 \), the following bound holds:

\[
\sup_{|t| \leq T} \sup_{\theta \in \mathbb{R}^d} \sum_{|\gamma| \leq \alpha} \| D_0^\gamma e^{\tilde{A}(\theta)t} X_0 \|_{L^1} \leq C(T) \| X_0 \|_{L^1}. \tag{9.14}
\]

**Proof.** For \( \alpha = 0 \), the bound

\[
\| e^{\tilde{A}(\theta)t} X_0 \|_{L^1} \leq C \| X_0 \|_{L^1}, \tag{9.15}
\]

follows from the bound (9.3) by duality arguments. Consider the case \( \alpha = 1 \). Introduce the function \( X_\gamma(t) := D_0^\gamma X(\theta, t) \), where \( X(\theta, t) = e^{\tilde{A}(\theta)t} X_0 \). Then

\[
\dot{X}_\gamma(t) = \tilde{A}(\theta) X_\gamma(t) + [D_0^\gamma \tilde{A}(\theta)] X(\theta, t), \quad X_\gamma(0) = 0.
\]

Hence,

\[
X_\gamma(t) = \int_0^t e^{\tilde{A}(\theta)(t-s)} [D_0^\gamma \tilde{A}(\theta)] X(\theta, s) ds.
\]

Therefore, by the bound (9.15),

\[
\| X_\gamma(t) \|_{H^1} \leq \int_0^t \| e^{\tilde{A}(\theta)(t-s)} [D_0^\gamma \tilde{A}(\theta)] X(\theta, s) \|_{H^1} ds \\
\leq C \int_0^t \| [D_0^\gamma \tilde{A}(\theta)] X(\theta, s) \|_{H^1} ds. \tag{9.16}
\]

It follows from (9.13) that

\[
[D_0^\gamma \tilde{A}(\theta)] = \begin{pmatrix} 0 & -[D_0^\gamma \tilde{H}(\theta)] \\ 0 & 0 \end{pmatrix}, \quad [D_0^\gamma \tilde{H}(\theta)] := \begin{pmatrix} 2(D_0^\gamma \theta)(i \nabla_y + \theta) & [D_0^\gamma \tilde{S}(\theta)] \\ [D_0^\gamma \tilde{S}^*(\theta)] & 2\omega_*(\theta)D_0^\gamma \omega_*(\theta) \end{pmatrix}.
\]

Here \( [D_0^\gamma \tilde{S}(\theta)] u := [D_0^\gamma \tilde{R}_{\Pi}(\theta, \cdot)] u, \ u \in \Phi^a \), and \( [D_0^\gamma \tilde{S}^*(\theta)] \) \( \varphi(y) := \int [D_0^\gamma \tilde{R}_{\Pi}(\theta, y)] \varphi(y) dy \). Hence, if \( \theta \in K^d \), then

\[
\|[D_0^\gamma \tilde{A}(\theta)] X(\theta, s) \|_{H^1} = \|[D_0^\gamma \tilde{H}(\theta)] X(\theta, s) \|_{H^1} \leq C \| X(\theta, s) \|_{H^1} \leq C \| X_0 \|_{H^1}, \tag{9.17}
\]

by the bound (9.15). Inequalities (9.16) and (9.17) imply the bound (9.14) with \( \alpha = 1 \). For \( \alpha > 1 \), the estimate follows by induction.

---

**9.4 Dual group**

Here we complete the proof of the bound (2.2) by duality arguments. Introduce the dual space \( \mathcal{E}' := H^{-1, \alpha}([R^d]) \oplus L^{-\alpha} \oplus H^{0, \alpha}([R^d]) \oplus L^{-\alpha} \) of functions \( Z \) with finite norm

\[
\| Z \|_{0, -\alpha}^2 := \| \psi \|_{1, -\alpha}^2 + \| \pi \|_{0, -\alpha}^2 + \| u \|_{-\alpha}^2 + \| v \|_{-\alpha}^2.
\]

For \( Z \in \mathcal{E}' \), we have \( \tilde{Z}_{\Pi}(\theta, \cdot) \in H^{\alpha}(K^d; H^1) \).
Lemma 9.5 Let \( \alpha \) be even and let \( \alpha \leq -2 \). Then
\[
\sup_{|t|\leq T} \|W'(t)Z\|_{0,-\alpha} \leq C(T) \|Z\|_{0,-\alpha}.
\] (9.18)

Proof. Note first that
\[
(\|Z\|_{0,-\alpha})^2 \sim \sum_{|\gamma| \leq -\alpha K_d} \int \|D_\theta^\gamma \tilde{Z}_\Pi(\theta, \cdot)\|^2_{H^{-1}} d\theta
\] (9.19)
for \( Z \in \mathcal{E}' \). Indeed,
\[
\|u\|^2_{-\alpha} = \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-2\alpha} |u(k)|^2 = C \int_{T_d} |(1 - \Delta_\theta)^{-\alpha/2} \tilde{u}(\theta)|^2 d\theta,
\]
\[
\|\psi\|^2_{-1,-\alpha} = \|\langle x \rangle^{-\alpha} \Lambda^{-1} \psi(x)\|^2_{L^2} \sim \|\Lambda^{-1} \langle x \rangle^{-\alpha} \psi(x)\|^2_{L^2} = C \|(1 + |\xi|^2)^{-1/2} (1 - \Delta_\xi)^{-\alpha/2} \hat{\psi}(\xi)\|^2_{L^2}
\]
\[
\sim \sum_{m \in \mathbb{Z}^{K_d}} \int (1 + |2\pi m + \theta|^2)^{-1/2} |(1 - \Delta_\theta)^{-\alpha/2} \hat{\psi}(2\pi m + \theta)|^2 d\theta
\]
\[
\sim \sum_{m \in \mathbb{Z}^{K_d}} \int (1 + |m|^2)^{-1/2} |(1 - \Delta_\theta)^{-\alpha/2} \hat{\psi}(2\pi m + \theta)|^2 d\theta
\]
\[
\sim \int_{K_d} \|\langle 1 - \Delta_\theta \rangle^{-\alpha/2} \tilde{\psi}_\Pi(\theta, \cdot)\|^2_{H^{-1}(T_d)} d\theta.
\] (9.20)

Hence, by Lemma 9.4 and by (9.19),
\[
(\|W'(t)Z\|'_{0,-\alpha})^2 \sim \sum_{|\gamma| \leq -\alpha K_d} \int \|D_\theta^\gamma (e^{\tilde{A}(\theta)t} \tilde{Z}_\Pi(\theta, \cdot))\|^2_{H^{-1}} d\theta
\]
\[
\leq C(t) \sum_{|\gamma| \leq -\alpha K_d} \int \|D_\theta^\gamma \tilde{Z}_\Pi(\theta, \cdot)\|^2_{H^{-1}} d\theta \sim C(t)(\|Z\|'_{0,-\alpha})^2.
\]

Corollary 9.6 The bound (2.2) follows from (9.18) by the duality considerations.

10 Appendix B: Crossing points

10.1 Proof of Lemmas 2.8 and 2.10

Let us prove Lemma 2.8. For any chosen \( \theta \in \mathbb{R}^d \), the Schrödinger operator \( \tilde{H}(\theta) \) admits the spectral resolution
\[
\tilde{H}(\theta) = \sum_{l=1}^\infty \lambda_l(\theta) P_l(\theta),
\]
where \( 0 < \lambda_1(\theta) \leq \lambda_2(\theta) \leq \ldots \), and \( P_l(\theta) \) are one-dimensional orthogonal projectors in \( H_1^0 \). Further, let us take an arbitrary point \( \Theta \in \mathbb{R}^d \) and a number \( \Lambda \in (\lambda_M(\Theta), \lambda_{M+1}(\Theta)) \), where \( M \geq N \) and \( \lambda_M(\Theta) < \lambda_{M+1}(\Theta) \). Then \( \Lambda \neq \lambda_1(\theta) \) for \( \theta \in \mathcal{O}(\Theta) \) if \( \mathcal{O}(\Theta) \) is a sufficiently small neighborhood of \( \Theta \). Write
\[
\tilde{H}^\Lambda(\theta) = \sum_{\lambda_l(\theta) < \Lambda} \lambda_l(\theta) P_l(\theta)
\]
\[
P^\Lambda(\theta) = \sum_{\lambda_l(\theta) < \Lambda} P_l(\theta), \quad \theta \in \mathcal{O}(\Theta).
\]
Further, let us choose a contour $\Gamma_\Lambda$ (in the complex plane $\mathcal{C}$) surrounding the interval $(0, \Lambda)$ such that $\Lambda \in \Gamma_\Lambda$. In this case, by the Cauchy theorem,

$$\mathcal{H}^{\Lambda}(\theta) = \int_{\Gamma_\Lambda} \frac{\lambda d\lambda}{\mathcal{H}(\theta) - \lambda}, \quad \theta \in \mathcal{O}(\Theta).$$

Finally, by (1.16)-(1.18) and by condition $\mathbf{R1}$, the mapping $\mathcal{H}(\theta)$ is an analytic operator-valued function in $\theta \in \mathcal{O}_c(\Theta)$, where $\mathcal{O}_c(\Theta)$ is a complex neighborhood of $\mathcal{O}(\Theta)$. Therefore, the same integrals converge for $\theta \in \mathcal{O}_c(\Theta)$, and the functions $P^{\Lambda}(\theta)$ and $\mathcal{H}^{\Lambda}(\theta)$ are analytic in a smaller neighborhood $\mathcal{O}'_c(\Theta)$. Reducing $\mathcal{O}'_c(\Theta)$ again, we can choose a basis $e_1(\theta), \ldots, e_M(\theta)$ in the space $R^{\Lambda}(\theta) := P^{\Lambda}(\theta)H_0^0$, where the functions $e_i(\theta)$ depend analytically on $\theta \in \mathcal{O}_c(\Theta)$. For example, it suffices to choose an arbitrary basis $e_1(\theta), \ldots, e_M(\theta)$ and set $e_i(\theta) = P^{\Lambda}(\theta)e_i(\Theta)$. The operator $\mathcal{H}(\theta)$ on the invariant space $R^{\Lambda}(\theta)$ can be identified with the corresponding matrix

$$\mathcal{H}^{\Lambda}(\theta) = \left(\mathcal{H}_{kl}(\theta)\right)_{k,l=0,\ldots,M}, \quad (10.1)$$

which depends analytically on $\theta \in \mathcal{O}_c(\Theta)$. Therefore, the eigenvalues $\lambda_1(\theta), \ldots, \lambda_M(\theta)$ and eigenvectors $F_1(\theta), \ldots, F_M(\theta)$ of this matrix can be chosen as real-analytic functions of $\theta \in \mathcal{O}_c(\Theta) \setminus C^A$, where $\mathcal{O}_c(\Theta) := \mathcal{O}_c(\Theta) \cap \mathbb{R}^d$ and $C^A$ is a subset of $\mathbb{R}^d$ of Lebesgue measure zero. This can be proved by using the methods of [8, Appendix]. It remains to pass to the limit as $\Lambda \to \infty$ and define $C_* := \cup_1^\infty C^A$. Finally, $\omega_i(\theta) := \sqrt{\lambda_i(\theta)}$. After this, relations (2.5) and (2.6) follow as in [8, Appendix].

Lemma 2.10 can be proved in a similar way.

### 10.2 Proof of Lemma 2.11

First let us show that conditions $\mathbf{E1}$, $\mathbf{E2}$ hold for $R_C(x) \equiv 0$ corresponding to $C_1 = \ldots = C_N = 0$. Indeed, in this case, relation (1.16) becomes

$$\mathcal{H}(\theta) := \begin{pmatrix} (i\nabla_y + \theta)^2 + m_0^2 & 0 \\ 0 & \omega^2(\theta) \end{pmatrix}.$$ 

Therefore, $\omega_i(\theta)$ are equal to either $\omega^*(\theta)$ or $\sqrt{(2\pi k + \theta)^2 + m_0^2}$, $k \in \mathbb{Z}^d$. Namely, $\omega^*(\theta)$ corresponds to the eigenvectors $(0,u)$ with an arbitrary $u \in \mathbb{R}^n$. The square root corresponds to the eigenvectors $F_k(\theta,y) = (e^{-2\pi ik - \theta}y,0)$ with $k \in \mathbb{Z}^d$ . It can be readily be seen that conditions $\mathbf{E1}$ and $\mathbf{E2}$ hold in this case.

Further, choose an arbitrary $l = 1, 2, \ldots$, a point $\Theta \in \mathbb{R}^d \setminus C_*$ and a bound $\Lambda \in (\lambda_M(\theta), \lambda_{M+1}(\theta))$ as above (with $M > l$). The function $R_C(x)$ and the corresponding operator $\mathcal{H}_C^{\Lambda}(\theta)$ depend analytically on $(\theta,C) \in \mathbb{C}^d \times \mathbb{C}^N$. Moreover, $R_C(x)$ satisfies conditions $\mathbf{R1}$ and $\mathbf{R2}^*$ for $C \in B_\varepsilon$ with a sufficiently small $\varepsilon > 0$. Therefore, as in the proof of Lemma 2.8, the corresponding eigenvalues $\omega_l(\theta,C)$, $l = 1, \ldots, M$, are also analytic functions of $(\theta,C)$ in the domain $\mathcal{M}_l(\Theta) = \mathcal{O}_c \setminus C$, where $\mathcal{O}_c$ is a complex neighborhood of
The set is a proper analytic subset. Therefore, $D_I(\theta, C) \neq 0$ on $M_I(\Theta)$ since $D_I(\theta, 0) \neq 0$, $\theta \in \mathbb{R}^d$. Further, introduce the set
\[
M_{II} = \{ C \in B_\varepsilon : D_I(\theta, C) \neq 0 \}.
\]
The set $B_\varepsilon \setminus M_{II}$ cannot contain any open ball, since otherwise $D_I(\theta, C) \equiv 0$. Hence, $M_{II}$ is an open dense set in $B_\varepsilon$. It remains to note that $M_1 = \cap_i M_{II}$ is thus a dense subset of $B_\varepsilon$. For $M_2$, the proof is similar.

11 Appendix C: Covariance in the spectral representation

Introduce the matrix-valued operator
\[
G(\theta, t) := e^{\hat{A}(\theta)t} = \begin{pmatrix}
\cos \Omega(\theta)t & \sin \Omega(\theta)t \\
-\Omega(\theta) \sin \Omega(\theta)t & \cos \Omega(\theta)t
\end{pmatrix}.
\]
(11.1)

Note that we can represent the matrix $G(\theta, t)$ in the form
\[
G(\theta, t) = \cos \Omega(\theta)t I + \sin \Omega(\theta)t C(\theta),
\]
where $I$ stands for the unit matrix, and
\[
C(\theta) := \begin{pmatrix}
0 & \Omega^{-1}(\theta) \\
-\Omega(\theta) & 0
\end{pmatrix}.
\]
In this case, the solution of (1.14) has the form $\tilde{Y}_I(\theta, r, t) = G(\theta, t)\tilde{Y}_0(\theta, r), \ r \in \mathcal{R}$. Using (11.2) and (3.4), we obtain
\[
\tilde{Q}_I(\theta, r, \theta', r') = E[\tilde{Y}_I(\theta, r, t) \otimes \tilde{Y}_I(\theta', r', t)] = \cos \Omega(\theta)t \tilde{Q}_0(\theta, r, \theta', r') \cos \Omega(\theta)t \\
+ \sin \Omega(\theta)t C(\theta) \tilde{Q}_0(\theta, r, \theta', r') C^T(\theta') \sin \Omega(\theta)t \\
+ \cos \Omega(\theta)t \tilde{Q}_0(\theta, r, \theta', r') C^T(\theta') \cos \Omega(\theta)t \\
+ \sin \Omega(\theta)t C(\theta) \tilde{Q}_0(\theta, r, \theta', r') \cos \Omega(\theta)t.
\]
(11.3)

By (3.6), we see that
\[
\tilde{q}_I(\theta) = G(\theta, t)\tilde{q}_0(\theta)G^*(\theta, t) = \cos \Omega(\theta)t \tilde{q}_0(\theta) \cos \Omega(\theta)t \\
+ \cos \Omega(\theta)t \tilde{q}_0(\theta)C^T(\theta) \sin \Omega(\theta)t + \sin \Omega(\theta)t C(\theta)\tilde{q}_0(\theta) \cos \Omega(\theta)t \\
+ \sin \Omega(\theta)t C(\theta)\tilde{q}_0(\theta)C^T(\theta) \sin \Omega(\theta)t,
\]
(11.4)
where $\tilde{q}_I(\theta)$ is the integral operator with the kernel $\tilde{q}_I(\theta, r, r')$ defined by (3.7).

For the simplicity of our manipulations, we assume now that the set of “crossing” points $\theta_\star$ is empty, i.e., $\omega_l(\theta) \neq \omega_{l'}(\theta)$ for any $l, l' \in \mathbb{N}$, and the functions $\omega_l(\theta)$ and $F_l(\theta, r)$ are
real-analytic. (Otherwise we need a partition of unity (6.3)). Consider the first term in the RHS of (11.4) and represent it in the form

\[
\cos \Omega(\theta) t \tilde{q}_0(\theta) \cos \Omega(\theta) t = \left( \sum_{l,l'} F_l(\theta, r) \right) \left( \cos \omega_l(\theta) t \ p_{ll'}(\theta) \cos \omega_{l'}(\theta) t \right) \overline{F}_{l'}(\theta, r')
\]

\[
= \sum_{l,l'} F_l(\theta, r) \frac{1}{2} \left[ \cos(\omega_l(\theta) - \omega_{l'}(\theta)) t + \cos(\omega_l(\theta) + \omega_{l'}(\theta)) t \right] p_{ll'}(\theta) \overline{F}_{l'}(\theta, r'),
\]

(11.5)

where \( p_{ll'}(\theta) = \left( p^{ij}_{ll'}(\theta) \right)_{i,j=0}^1 = \left( F_l(\theta, \cdot), (\tilde{q}^{ij}_{0}(\theta) F_{l'})(\theta, \cdot) \right)_{i,j=0}^1 ( p^{ij}_{ll'}(\theta) \) are introduced in (6.7)). Similarly, we can rewrite the remaining three terms in the RHS of (11.4). Finally,

\[
\tilde{q}^{ij}_{l'}(\theta, r, r') := \sum_{l,l'=1}^{\infty} F_l(\theta, r) r^{ij}_{ll'}(t, \theta) \otimes \overline{F}_{l'}(\theta, r'),
\]

(11.6)

where \( r^{ij}_{ll'}(t, \theta) = (r^{ij}_{ll'}(t, \theta))_{i,j=0}^1 \) are defined in (6.5).

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