Barcodes for closed one form - an alternative to Novikov theory

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Abstract

We extend the configurations $\delta_r$ and $\gamma_r$, discussed in [1] and [4], equivalently the closed, open and closed-open bar codes from real- or angle-valued maps, to topological closed one forms on compact ANRs. As a consequence one provides an extension of the classical Novikov complex associated to a closed smooth one form and a vector field the form is Lyapunov for, to a considerably larger class of situations. We establish strong stability properties and Poincaré duality properties when the underlying space is a closed manifold. Applications to Geometry, Dynamics and Data Analysis are the targets of our research. A different approach towards such bar codes was proposed in Usher-Zhang’s work cf. [10].

This paper is essentially my lecture at the workshop “Topological data analysis meets symplectic topology”, Tel Aviv, April 29–May 3, 2018. A beamer version can be downloaded from https://people.math.osu.edu/burghelea.1/notes/index.html. A detailed version of this work will be posted shortly.

1 Classical Novikov theory

Classical Novikov theory considers
- a smooth manifold $M^n$ (for the purpose of this paper a closed manifold),
- a pair $(X, \omega)$, $X$ a smooth Morse-Smale vector field on $M$, $\omega \in \Omega^1(M)$ a smooth closed one form Lyapunov for $X$,
- a field $\kappa$

and relates
- the rest points of $X$ and
- the instantons (isolated trajectories) between rest points of $X$
to the topology of $(X; [\omega])$ and the zeros of $\omega$.

The Morse-Smale property of $X$ implies that
- each rest point of $X$ is hyperbolic and has a Morse index $r$, an integer between 0 and $n$, and
- all isolated trajectories, i.e. instantons between rest points, exist only from rest points of index $r$ to rest points of index $(r - 1)$.

The Lyapunov property of $\omega$ identifies the rest points of $X$ to the zeros of $\omega$.

The theory constructs:
- a field $\kappa(\omega)$, extension of $\kappa$, called the Novikov field associated to $\omega$,
- a chain complex of finite dimensional vector spaces over $\kappa(\omega)$ whose
  a) components $C_r$ are the vector spaces generated by the rest points of index $r$, and

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1 $[\omega]$ denotes the cohomology class represented by $\omega$
b) boundary maps \( \partial_r : C_r \rightarrow C_{r-1} \) are given by matrices whose entries are derived from the instantons between rest points appropriately counted, called "instanton-counting matrices". The result of the "appropriate counting", even when the cardinality of the instantons is infinite, is an element in \( \kappa(\omega) \). The rank \( \rho_r \) of \( \partial_r \) indicates that there are at least \( \rho_r \) different rest points of index \( r \) which are the origin of instantons and at least \( \rho_{r-1} \) different rest points of index \( r - 1 \) which are the end of instantons.

The main result of the Novikov theory is that the dimensions of the homologies of this complex are the Novikov-Betti numbers \( \beta^N_r(X; \omega) \) and that the complex, up to a non canonical isomorphism, depends only on the closed one form \( \omega \). Elementary linear algebra arguments imply that the rank \( \rho_r \) plus the rank \( \rho_{r-1} \) is equal to the number \( c_r \) of rest points of \( X \) of index \( r \) minus the Novikov-Betti number \( \beta^N_r(X; \omega) \), hence the complex, up to an isomorphism, is determined by either two of these three types of numbers: \( \beta^N_r, \rho_r, c_r \).

Recall that any chain complex of vector spaces over a field \( \kappa \) is non canonically isomorphic to a complex of the following type:

\[
\cdots \rightarrow (C_r = H_r \oplus C^+_r \oplus C^-_r) \xrightarrow{\partial_r} (C_{r-1} = H_{r-1} \oplus C^+_r \oplus C^-_{r-1}) \xrightarrow{\partial_{r-1}} \cdots
\]

with \( C^-_r = C^+_r \), and

\[
\partial_r = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & Id \\
0 & 0 & 0
\end{pmatrix},
\]

called Hodge type.

The chain complex \((C_*, \partial_*)\)
- provides the exact number of rest points of Morse index \( r \),
- give a good information about the instantons between rest points,
- establishes the existence of closed trajectories (when combined with standard Betti numbers),
results of interest even outside mathematics if the problem under consideration can be modeled by the input of Novikov theory.

The drawbacks of the theory for applications outside mathematics are:

1. reduced generality (manifold structure for the underlying space, Lyapunov closed smooth one form which is Morse used to represent the "action" controlling the dynamics defined by the vector field,
2. the infinite cardinality of the mathematical objects involved in the definition of the field \( \kappa(\omega) \) and of the matrix \( \partial_r \) which makes these objects not computer friendly.

In view of potential interest outside mathematics we like:

1. to derive the collection of numbers \( \beta_r, \rho_r, c_r \) as computer friendly invariants and without any reference to the field \( \kappa(\omega) \),
2. to extend the result to a larger class of situations, i.e. a compact ANR denoted from now on by \( X \) instead of manifold \( M^n \), topological closed one form \( \omega \) on \( X \) (to be defined in section 2) instead of a smooth closed one form \( \omega \) on \( M^n \).

This was already done in \[1\] and implicitly in \[4\] in the case the topological closed one form \( \omega \) is exact and in the case \( \omega \) represents an integral cohomology class. These two situations represent the case of topological closed one form of degree of irrationality 0 and 1. In the work summarized in this paper we go

\(^2\) in this paper the adjective computer friendly refers to concepts consistent with or invariants computed by a computer implementable algorithm.
much further and we treat the case of an arbitrary topological closed one form from now on abbreviated to TC1-form of arbitrary degree of irrationality.

In the theory referred to as AMN (Alternative to Morse-Novikov theory) the underlying space is always a compact ANR (absolute neighborhood retract), cf. [1] chapter 1 for definition. All spaces homeomorphic to finite simplicial complexes and in particular the compact manifolds, or more general the compact stratified spaces, are compact ANRs.

The concept of tame TC1-form, a replacement (generalization) of a Morse closed one form, is recalled in section 2, the set $O(\omega)$, orbits of critical values associated to a TC1-form $\omega$, are introduced in section 3 and the main results are formulated in section 4 below.

Note that in [1] to a tame integral TC1-form, equivalently an angle-valued map, $\omega$ one associates a principal $\mathbb{Z}$–covering $\tilde{X} \to X$ and lifts $f : \tilde{X} \to \mathbb{R}$ of $\omega$. These are continuous maps which are proper, with the homology of the levels $f^{-1}(t)$ finite dimensional vector spaces and with the set of critical values discrete subsets of $\mathbb{R}$. All these properties were used in [1].

In this paper to an arbitrary $\omega$ one associates the principal $\mathbb{Z}^k$–covering $\tilde{X} \to X$ and lifts $f : \tilde{X} \to \mathbb{R}$ but in the case the degree of irrationality (the number $k$) is $> 1$,
- the map $f$ is never proper,
- the levels $f^{-1}(t)$ might not have the homology as finite dimensional vector spaces,
- the set of critical values if nonempty is never discrete but always dense.

The approach of level persistence considered in [1] chapter 4, via graph representations, is apparently not applicable.

However, the treatment described in [1] sections 5 and 6, can be refined and leads to barcodes which are recorded as real numbers $^3$, namely as configurations of points in $\mathbb{R}$ and $\mathbb{R}_+$ rather than $\mathbb{R}^2$ or $\mathbb{R}^2/\mathbb{Z}$.

## 2 Topological closed one forms (TC1-forms) and tameness

A topological closed one form (TC1-form) extends the concept of smooth closed one form on a smooth manifold to an arbitrary topological space. One way to obtain this is to view it as an equivalence class of multivalued maps (first definition) an other way is as an equivalence class of equivariant maps on an associated principal $\mathbb{Z}^k$–covering (second definition).

### First definition

**Definition 2.1**

1. A **multi-valued map** is a systems $\{U_\alpha, f_\alpha : U_\alpha \to \mathbb{R}, \alpha \in A\}$ s.t.
   - $U_\alpha$ are open sets with $X = \bigcup U_\alpha$,
   - $f_\alpha$ are continuous maps s.t $f_\alpha - f_\beta : U_\alpha \cap U_\beta \to \mathbb{R}$ is locally constant

2. Two multi-valued maps are equivalent if together remain a multi-valued map.

3. A **TC1-form** is an equivalence class of multi-valued maps.

**Examples:**

$^3$this because a closed one form is view as a real-valued function up to a translation, cf definition 2 in section 2, so the barcodes of $f$ are intervals (i.e. barcodes of real-valued functions) up to a translation
1. A smooth closed one form \( \omega \in \Omega^1(M) \) with \( d\omega = 0 \) defines a TC1-form \( \omega \).

Indeed in view of Poincaré Lemma for any \( x \in M \) choose an open neighborhood \( U_x \ni x \) and \( f_x : U_x \to \mathbb{R} \) a smooth map s.t. \( \omega_{|U_x} = df_x \). The system \( \{U_x, f_x : U_x \to \mathbb{R}\} \) provides a representative of the TC1-form \( \omega \).

2. A simplicial one cocycle \( \omega \) on the simplicial complex \( X \) defines a TC1-form.

Indeed, if \( X \) is a simplicial complex, \( \mathcal{X}_0 \) the collection of vertices and \( S \subset \mathcal{X}_0 \times \mathcal{X}_0 \) consists of pairs \((x, y) \in \mathcal{X}_0 \) s.t. \( x, y \) are the boundaries of a 1-simplex of \( X \) then a simplicial one cocycle is a map \( \delta : S \to \mathbb{R} \) with the properties \( \delta(x, y) = -\delta(y, x) \) and for any three vertices \( x, y, z \) with \((x, y), (y, z), (x, z) \in S \) one has \( \delta(x, y) + \delta(y, z) + \delta(z, x) = 0 \). The collection of open sets \( U_x, U_y \) the open star of the vertex \( x \), and the maps \( f_x : U_x \to \mathbb{R} \), the linear extension on each open simplex of \( U_x \) of the map defined by \( f_x(y) = \delta(x, y) \) and \( f_x(x) = 0 \), provide a representative of the TC1-form defined by the one cocycle \( \delta \).

Clearly a TC1-form \( \omega \) determines a cohomology class \( \xi(\omega) \in H^1(X; \mathbb{R}) \).

One denotes by \( Z^1(X) \) the set of all TC1-forms and by \( Z^1(X; \xi) := \{ \omega \in Z^1(X) \mid \xi(\omega) = \xi \} \).

Let \( \xi \in H^1(X; \mathbb{R}) = Hom(H_1(X; \mathbb{Z}), \mathbb{R}) \) and \( \Gamma = \Gamma(\xi) := (img \xi) \subset \mathbb{R} \). If \( X \) is a compact ANR then \( \Gamma \simeq \mathbb{Z}^k \) and one refers to \( k \) as the degree of irrationality of \( \xi \).

The surjective homomorphism \( \xi : H_1(X; \mathbb{Z}) \to \Gamma \) defines the associated \( \Gamma \)-principal cover, \( \pi : \tilde{X} \to X \) i.e. the free action \( \mu : \Gamma \times \tilde{X} \to \tilde{X} \) with \( \pi \) the quotient map \( \tilde{X} \to \tilde{X}/\Gamma = X \).

Consider \( f : \tilde{X} \to \mathbb{R} \), \( \Gamma \)-equivariant maps, i.e. satisfying \( f(\mu(g, x)) = f(x) + g \).

**Second definition**

**Definition 2.2**

A TC1-form \( \omega \) of cohomology class \( \xi \) is an equivalence class of \( \Gamma \)-equivariant maps \( f : \tilde{X} \to \mathbb{R} \) with \( f_1 \sim f_2 \) iff \( f_1 - f_2 \) is locally constant.

Such map \( f \) is referred to as a lift or a representative of \( \omega \) and one writes \( f \in \omega, \omega \in [\omega] = \xi(\omega) \) where \( \xi(\omega) = \xi \).

One denotes by \( Z^1(X; \xi) \) the set of TC1 forms in the class \( \xi \) and by \( Z^1(X) = \bigcup_{\xi \in H^1(X; \mathbb{R})} Z(X; \xi) \).

Clearly the two definitions, first and second, are equivalent.

**Tameness for TC1-forms**

**Definition 2.3**

A continuous map \( f : \tilde{X} \to \mathbb{R} \) is called weakly tame if the following hold:

(i) For any \( I \subset \mathbb{R} \) closed interval \( f^{-1}(I) \) is an ANR.

(ii) For \( t \in \mathbb{R} \), \( R^f(t) := \dim H_r(\tilde{X}^f_t, \tilde{X}^f_{<t}) + \dim H_r(\tilde{X}^{>t}_f, \tilde{X}^{>t}_{\geq t}) < \infty \)

where \( \tilde{X}^f_t = f^{-1}((-\infty, t)), \tilde{X}^f_{<t} = f^{-1}((-\infty, t)) \) and \( \tilde{X}^{>t}_f = f^{-1}([t, \infty)), \tilde{X}^{>t}_{\geq t} = f^{-1}([t, \infty)) \).

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4 It suffices to show that a representative \( \{U_\alpha, f_\alpha : U_\alpha \to \mathbb{R}\} \) of \( \omega \) defines, for any closed continuous path \( \gamma : [a, b] \to X \), the number \( \int_\gamma \omega \to \mathbb{R} \) independent on the homotopy class rel. boundary of \( \gamma \) and additive w.r. to juxtaposition of paths. Indeed if \( \gamma[a, b] \subset U_\alpha \) for some \( \alpha \) then \( \int_\gamma \omega = f_\alpha(b) - f_\alpha(a) \). If not, one choses a subdivision of \([a, b], a = t_0 < t_1 < \ldots < t_r = b\), such that \( \gamma_{t_i} := \gamma|_{[t_i, t_{i+1}]} \) lie in some open set \( U_\alpha \) and put \( \int_\gamma \omega := \sum \int_{t_i} \omega \). This assignment defines an homomorphism \( \xi(\omega) : H_1(X; \mathbb{Z}) \to \mathbb{R} \), equivalently a cohomology class \( \xi(\omega) \).
The value \( t \in \mathbb{R} \) is called regular if \( R^f_t = 0 \) and critical otherwise.

(iii) The set of critical values \( CR(f) \) is at most countable.

When there is no specification of coefficients in the notation it is understood that the homology \( H_r(\cdot, \cdot) \) is with coefficients in the field \( \kappa \) and \( H_r(\cdot, \cdot) \) is a \( \kappa \)-vector space.

Note that if \( f : \tilde{X} \to \mathbb{R} \) is \( \Gamma \)-equivariant then \( CR(f) \) is \( \Gamma \)-invariant.

Definition 2.4

Let \( X \) be a compact ANR. The TC1-form \( \omega \in Z^1(X : \xi) \), is called tame if one lift \( f : \tilde{X} \to \mathbb{R} \) (and then any other lift) is weakly tame and the set \( CR(f) / \Gamma \) is finite.

One denotes by \( Z^1_{\text{tame}}(X ; \xi) \) the subset of \( Z^1(X ; \xi) \) consisting of tame TC1 forms by \( Z^1_{\text{tame}}(X) := \bigcup_{\xi \in H^1(X ; \mathbb{R})} Z^1_{\text{tame}}(X ; \xi) \).

Topology on the set \( Z^1(X : \xi) \)

Suppose \( X \) is a compact ANR. The set \( Z^1(X : \xi) \) can be equipped with the complete metric \( D(\omega_1, \omega_2) \) (whose induced topology is referred to as the compact open topology) defined as follows:

\[
D(\omega_1, \omega_2) = \inf_{f_1 \in \omega_1, f_2 \in \omega_2} D(f_1, f_2)
\]

with \( D(f_1, f_2) := \sup_{x \in \tilde{X}} |f_1(x) - f_2(x)| \).

3 The set \( \mathcal{O}(\omega) \) and spaces of configurations

As pointed out, if \( f : \tilde{X} \to \mathbb{R} \) is \( \Gamma \)-invariant then the set \( CR(f) \) of critical values is \( \Gamma \)-invariant. Denote \( \mathcal{O}(f) := CR(f) / \Gamma \). If \( f_1 \) and \( f_2 \) are two such lifts and \( X \) is connected (otherwise the considerations are done component-wise), hence \( f_2 = f_1 + s \), then the translation by \( s \) gives an identification \( \theta(s) : \mathcal{O}(f_1) \to \mathcal{O}(f_2) \).

The set \( \mathcal{O}(\omega) \), of orbits of critical values is defined by \( \mathcal{O}(\omega) := \sqcup_{f \in \mathcal{O}(f)} \sim \) with \( o \sim o' \), \( o \in \mathcal{O}(f_1) \) \( o' \in \mathcal{O}(f_2) \), iff \( f_2 = f_1 + s \) and \( \theta(s)(o_1) = o_2 \).

Constructions

A map \( \delta : Y \to \mathbb{Z}_{\geq 0} \), \( Y \) topological space, \( \delta \) a map with finite support is called configuration. One defines \( \text{supp} \delta := \{ x \in Y \mid \delta(x) \neq 0 \} \) and \( \sharp \text{supp} \delta := \sum_{x \in Y} \delta(x) \).

Denote by \( \text{Conf}_Y \) the set of all configurations on \( Y \) and by \( \text{Conf}_Y \) the subset \( \text{Conf}_Y := \{ \delta \in \text{Conf}_Y \mid \sharp \text{supp} \delta = N \} \). Since \( Y \) is a topological space \( \text{Conf}_Y = Y^N / \Sigma_N \), \( \Sigma_N \) the permutation group of \( N \) elements, is equipped with the obvious topology, the same as the collision topology and when \( Y = Z \setminus K, K \) closed subset of \( Z, \text{Conf}(Y) \) is equipped with the bottleneck topology w.r. to the pair \( (Z, K) \).

A fundamental system of neighborhoods for a configuration \( \delta \in \text{Conf}_N(Y) \) with support \( \{ y_1, y_2, \ldots, y_r \} \) in the collision topology is provided by the sets of configurations \( \mathcal{U}(\delta, \{ U_i \}) := \{ \delta' \in \text{Conf}_Y \mid \sum_{y \in U_i} \delta'(y) = \delta(y_i) \} \) with \( U_i \) a disjoint collection of open neighborhoods of \( y_i \).

A fundamental system of neighborhoods for a configuration \( \delta \in \text{Conf}_N(Y) \) with support \( \{ y_1, y_2, \ldots, y_r \} \) in the bottleneck topology is provided by the sets \( \mathcal{U}(\delta, \{ U_i \}, V) := \{ \delta' \in \text{Conf}_Y \mid \sum_{y \in U_i} \delta'(y) = \delta(y_i), \text{supp} \delta' \subset (V \sqcup \bigcup_{i=1,\ldots,r} U_i) \} \) with \( U_i \) and \( V \) a disjoint collection of open sets of \( Y, U_i \) neighborhood of \( y_i \) and \( V \) neighborhood of \( y_i \).

5
4 The Alternative Novikov theory for topological closed one form of arbitrary degree of irrationality

Denote by $\mathcal{Z} \tame (\cdot \cdot \cdot) := \{ \omega \in \mathcal{Z}(\cdot \cdot \cdot) \mid \omega \text{ tame} \}$. In what follows for any tame TC-1 form $\omega \in \mathcal{Z}(X; \xi)$, field $\kappa$ and nonnegative integer $r$ one defines two configurations $\delta^\omega_r : \mathbb{R} \to \mathbb{Z}_{\geq 0}$ and $\gamma^\omega_r : \mathbb{R}_+ \to \mathbb{Z}_{\geq 0}$.

The blue circles in Figure 1. represent the closed $r$–bar codes located at $\mathbb{R} \ni x \leq 0$ and the red circles represent the open $(r - 1)$–barcodes located at $\mathbb{R} \ni x < 0$. The brown circles in Figure 2. represent the closed-open $r$–barcodes located on $\mathbb{R}_+ = (0, \infty)$.

Define also the numbers $\beta^\omega_r, \rho^\omega_r$ and $c^\omega_r$,

$$
\beta^\omega_r := \sum_{t \in \mathbb{R}} \delta^\omega_r (t), \quad \rho^\omega_r := \sum_{t \in \mathbb{R}_+} \gamma^\omega_r (t)
$$

and

$$
c^\omega_r = \beta^\omega_r + \rho^\omega_r + \rho^\omega_{r-1}.
$$

These configurations satisfy the Theorems 4.1, 4.3 and 4.4 below.

The results

**Theorem 4.1** (Topological properties)

Let $\omega \in \mathcal{Z}_\tame^1 (X)$. One has:

1. $\delta^\omega_r (t) \neq 0$ or $\gamma^\omega_r (t) \neq 0$ implies that for any $f \in \omega$, $t = e' - e''$ with $e', e'' \in \text{CR}(f)$,

2. $\beta^\omega_r = \sum_{t \in \mathbb{R}} \delta^\omega_r (t) = \beta_r^{\mathcal{N}} (X; \xi)$.
3. for any \( f \in \omega \) and \( a_0 \in o \in \mathcal{O}(f) \),

\[
C_r^\omega = \sum_{t \in \mathbb{R}} \delta_r^\omega(t) + \sum_{t \in \mathbb{R}_+} \gamma_r^\omega(t) + \sum_{t \in \mathbb{R}_+} \gamma_{r-1}^\omega(t) = \sum_{o \in \mathcal{O}(f)} \dim H_r(\tilde{X}_{a_0}^f, \tilde{X}_{<a_0}^f)
\]

where \( \tilde{X}_{a_0}^f = f^{-1}((-\infty, a_0]) \), \( \tilde{X}_{<a_0}^f = f^{-1}( (-\infty, a_0) ) \).

Note that if \( X \) is a smooth closed manifold and \( \omega \) is a smooth Morse form Lyapunov for a Morse-Smale vector field then the number of zeros of \( \omega \) (= the rest points of the vector field) of Morse index \( r \) is equal to \( \sum_{o \in \mathcal{O}(f)} \dim H_r(\tilde{X}_{a_0}^f, \tilde{X}_{<a_0}^f) \).

**Corollary 4.2** The Novikov complex of the Morse form \( \omega \) in Hodge form is given by \( C_r = C_r^- \oplus C_r^+ \oplus H_r \)
with

\[
H_r = \kappa^\beta \omega_r, \\
C_r^- = C_{r-1}^+ = \kappa^\beta \omega_r, \\
\partial_r = \begin{pmatrix} 0 & 0 & 0 \\ \text{Id} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

As in [1] in the case of an angle-valued map, this complex \((C_*, \partial_*)\) in Hodge form can be referred to as the AN complex associated to \( \omega \).

**Relation to other work** [10] and [4]

The points in \( \text{supp} \delta_r^\omega \cup \text{supp} \gamma_r^\omega \cup \text{supp} \gamma_{r-1}^\omega \) with their multiplicity correspond bijectively to UZ (=Usher-Zhang) verbose \( r \)-bar codes, and in the case \( \xi(\omega) \) integral to the BD (=Burghelea-Dey) closed \( r \)-barcodes plus BD-open \((r-1)\)-barcodes plus closed-open \( r \)-barcodes plus closed-open \((r-1)\)-barcodes cf [2] or [4]. In the case \( \xi(\omega) \) is trivial the points in \( \text{supp} \gamma^\omega \) with their multiplicity correspond bijectively to the finite barcodes in classical ELZ-persistence (cf. [6]) and in the case \( \xi(\omega) \) is integral to the closed-open \( r \)-barcodes in angle valued persistence.

**Theorem 4.3** (Poincaré duality properties)

Suppose \( X \) is a closed topological \( n \)-dimensional manifold and \( \omega \in Z^1_{\text{tame}}(M) \). Then

1. \( \delta_r^\omega(t) = \delta_{n-r}^\omega(-t) \),
2. \( \gamma_r^\omega(t) = \gamma_{n-r-1}^\omega(t) \).

Recall that:

- The set of TC-1 forms \( Z^1(X; \xi) \) is an \( \mathbb{R} \)-vector space equipped with the norm given by the distance \( D(\omega_1, \omega_2) \) defined above.
- The space of configurations \( Conf_N(Y) := \{ \delta : Y \to \mathbb{Z}_{\geq 0} \mid \sum \delta(y) = N \} \) is equipped with the obvious collision topology.
- For \( K \) closed subset of \( Y \) the space of configurations \( Conf(Y \setminus K) \) is equipped with the bottleneck topology provided by the pair \( K \subset Y \). With respect to these topology one has

**Theorem 4.4** (stability properties)

1. The assignment \( Z^1_{\text{tame}}(X; \xi) \ni \omega \mapsto \delta_r^\omega \in Conf_{\beta_N^r(X; \xi)}(\mathbb{R}) \) is continuous.
2. The assignment $\mathcal{Z}_{\text{tame}}^1(X; \xi) \ni \omega \mapsto \hat{\omega} \in \text{Conf}([0, \infty) \setminus 0)$ is continuous.

Description of the configurations $\hat{\omega}$ and $\hat{\omega}_\infty$.

Fix a field $\kappa$. Let $\pi : \hat{X} \rightarrow X$ be a $\Gamma$–principal covering with the free action $\mu : \Gamma \times \hat{X} \rightarrow \hat{X}$ where $\Gamma \subset \mathbb{R}$ is a f.g. subgroup of $\mathbb{R}$. Let $f : \hat{X} \rightarrow \mathbb{R}$ be a $\Gamma$–equivariant map, i.e. $f(\mu(g, x)) = f(x) + g$ which is weakly tame and $\mathcal{O}(f)$ finite.

Recall that $\hat{X}_a^f = f^{-1}((\infty, a]), \hat{X}_c^a = f^{-1}((-\infty, a)), \hat{X}_a^f = f^{-1}((a, \infty)), \hat{X}_f^a = f^{-1}((a, \infty))$ and denote

1. $\mathbb{I}_a^f(r) := \text{img}(H_r(\hat{X}_a) \rightarrow H_r(\hat{X}))$, $\mathbb{I}_a^f(r) \subset H_r(\hat{X})$

   $\mathbb{I}_a^f(r) := \bigcup_{a' < a} \mathbb{I}_{a'}(r)$, $\mathbb{I}_a^{\infty}(r) = \bigcap_a \mathbb{I}_a^f(r)$,

2. $\mathbb{I}_a^f(r) := \text{img}(H_r(\hat{X}_a) \rightarrow H_r(\hat{X}))$, $\mathbb{I}_a^f(r) \subset H_r(\hat{X})$

   $\mathbb{I}_a^f(r) := \bigcup_{a' > a} \mathbb{I}_{a'}(r)$, $\mathbb{I}_a^{\infty}(r) = \bigcap_a \mathbb{I}_a^f(r)$,

3. For $a < b$ $\mathbb{T}_r(a, b) := \ker(H_r(\hat{X}_a) \rightarrow H_r(\hat{X}_b))$,

   $\mathbb{T}_r(a, b) := \bigcup_{b' < b} \mathbb{T}_r(a, b') \subseteq \mathbb{T}_r(a, b)$,

4. For $a' < a$ let $i : \mathbb{T}_r(a', b) \rightarrow \mathbb{T}_r(a, b)$ the induced map and

   $\mathbb{T}_r(a, b) := \bigcup_{a' < a} i^{-1}(\mathbb{T}_r(a', b)) \subseteq \mathbb{T}_r(a, b)$.

Define

for $a, b \in \mathbb{R}$

$$\hat{\delta}_a^f(a, b) := \frac{\mathbb{I}_a^f(r) \cap \mathbb{I}_b^f(r)}{\mathbb{I}_{< a}^f(r) \cap \mathbb{I}_b^f(r) + \mathbb{I}_a^f(r) \cap \mathbb{I}_{> b}^f(r)}, \hat{\delta}_a^f(a, b) := \text{dim} \hat{\delta}_a^f(a, b),$$

for $a, b \in \mathbb{R}$, $a < b$

$$\hat{\gamma}_a^f(a, b) := \frac{\mathbb{T}_r(a, b)}{\mathbb{T}_r(< a, b) + \mathbb{T}_r(a, < b)}, \hat{\gamma}_a^f(a, b) := \text{dim} \hat{\gamma}_a^f(a, b).$$

and for $t \in \mathbb{R}$

$$\mathbb{I}_r^f(t) := \sum \mathbb{I}_a^f(r) \cap \mathbb{I}_b^f(r)$$

One can show that:

**Proposition 4.5**

1. $\hat{\delta}_a^f(a, b) = \hat{\delta}_a^f(a + g, b + g)$ and $\hat{\gamma}_a^f(a, b) = \hat{\gamma}_a^f(a + g, b + g)$ for any $g \in \Gamma$,
Figure 3: the domain of $\delta_f$

1. For a regular value $\text{supp}\delta_f \cap (a \times \mathbb{R})$ and $\text{supp}\gamma_f \cap (a \times (a, \infty))$ is empty,
2. for a critical value $\text{supp}\delta_f \cap (a \times \mathbb{R})$ and $\text{supp}\gamma_f \cap (a \times (a, \infty))$ is finite,
3. for a regular value $\text{supp}\delta_f \cap (a \times \mathbb{R})$ and $\text{supp}\gamma_f \cap ((-\infty, b) \times b)$ is empty,
4. for a critical value $\text{supp}\delta_f \cap (\mathbb{R} \times b)$ and $\text{supp}\gamma_f \cap ((-\infty, b) \times b)$ is finite.

Note that:
1. For any line $\Delta'$ or $\Delta''$ parallel to either $x$–axis or $y$–axis the cardinality of $\text{supp}\delta_f \cap \Delta$ or $\text{supp}\gamma_f \cap \Delta$ is finite.
2. For any line $\Delta$ parallel to the first diagonal $\{(x, y) \in \mathbb{R}^2 \mid x - y = 0\}$ the values of each $\delta_f$ and $\gamma_f$ are the same on each point of $\text{supp}\delta_f \cap \Delta$ resp. of $\text{supp}\gamma_f \cap \Delta$.
3. Suppose the line $\Delta$ is parallel to the first diagonal $\{(x, y) \in \mathbb{R}^2 \mid x - y = 0\}$ and $\omega$ is of degree of irrationality $> 1$. If $\text{supp}\delta_f \cap \Delta$ and $\text{supp}\gamma_f \cap \Delta$ is nonempty then it is everywhere dense in $\Delta$.

**Observation 4.6** For $a_o \in o \in \mathcal{O}(f)$, $f \in \omega$, $\delta_f(a_o, a_o + t)$ and $\gamma_f(a_o, a_o + t)$ are independent on the choice of $a_o$ in $o$ and of $f$ in $\omega$.

Define $\delta_{o,r}^\omega(t) := \delta_f(a_o, a_o + t)$ and then

$$\delta^\omega(t) = \sum_{o \in \mathcal{O}(\omega)} \delta_{o,r}^\omega(t).$$

Define $\gamma_{o,r}^\omega(t) := \gamma_f(a_o, a_o + t)$ and then

$$\gamma^\omega(t) = \sum_{o \in \mathcal{O}(\omega)} \gamma_{o,r}^\omega(t).$$
5 About the proof of Theorems 4.1, 4.3 and 4.4

Proposition 5.1 Let $f : \tilde{X} \to \mathbb{R}$, with $f \in \mathbb{Z}_{tame}^1(X)$.

1. For $t \in \mathbb{R}$ one has
   \[
   H_r(\tilde{X}^f_a, \tilde{X}^f_{<a}) \simeq \mathbb{I}^f_a(r)/\mathbb{I}^f_{<a}(r) \oplus \text{coker}(\mathbb{T}_r^f(< t, \infty) \to \mathbb{T}_r^f(t, \infty)) \oplus \text{ker}(H_{r-1}(\tilde{X}^f_{\leq t}) \to H_{r-1}(\tilde{X}^f_{<t})),
   \]
   and if $t$ is a regular value then $H_r(\tilde{X}^f_a, \tilde{X}^f_{<a}) = 0$.

2. $\mathbb{I}^f_t(r)/\mathbb{I}^f_{<t}(r) \simeq \oplus_{s \in \mathbb{R}} \delta^f_t(t, s)$.

3. $\text{coker}(\mathbb{T}_r^f(< t, \infty) \to \mathbb{T}_r^f(t, \infty)) \simeq \oplus_{s \in \mathbb{R}_+} \gamma^f_t(t, t + s)$.

4. $\text{ker}(H_{r-1}(\tilde{X}^f_{\leq t}) \to H_{r-1}(\tilde{X}^f_{<t})) \simeq \oplus_{s \in \mathbb{R}_+} \delta^f_{r-1}(t - s, t)$.

The proof of item 1 follows from the inspection of the homology long exact sequence of the pair $(\tilde{X}_{\leq t}, \tilde{X}_{<a})$. The proof of the other items follow from the finite dimensionality of $H_r(\tilde{X}_{\leq t}, \tilde{X}_{<a})$ and the definitions of $\delta^f_t(a, b)$ and $\gamma^f_t(a, b)$.

The results remain true for any $f : \tilde{X} \to \mathbb{R}$ weakly tame and not necessary $\Gamma$–equivariant, in particular for any restriction of $f$ to an open subset $U \subset \tilde{X}$.

In view of the action of $\Gamma$ on $\tilde{X}$ the $\kappa$–vector spaces $H_r(\tilde{X})$, $\mathbb{I}^f_{-\infty}(r), \mathbb{I}^f_{\infty}(r) \subseteq H_r(\tilde{X})$ are actually f.g. $\kappa[\Gamma]$–modules resp. submodules. Denote by $TH_r(\tilde{X})$ the submodule of torsion elements of $H_r(\tilde{X})$.

For $f \in \omega \in \mathbb{Z}_{tame}^1(X)$ consider the filtration of f.g. $\kappa[\Gamma]$–modules indexed by $t \in \mathbb{R}$
\[
\mathbb{I}_{-\infty}(r) = \mathbb{F}_r(t)(-\infty) \subseteq \cdots \mathbb{F}_r(t')(\infty) \subseteq \mathbb{F}_r(t) \cdots \subseteq \mathbb{F}_r(\infty) = H_r(\tilde{X}), t' < t
\]
and observe that in view of f.g. of $H_r(\tilde{X})$ there are finitely many $t$'s where the rank of $\mathbb{F}_r(t)$ changes.

Theorem 4.1 (Topological properties) follows from Proposition 5.2 below and the following facts:
- $\mathbb{P}_r^f(t)/TH_r(\tilde{X}) \simeq \bigoplus_{a-b \leq t} \delta^f_t(a, b)$,
- if $a$ is a regular value then $H_r(\tilde{X}^f_a, \tilde{X}^f_{<a}) = 0$ and if $a \in o \in \mathcal{O}(f)$ then
  \[
  \dim H_r(\tilde{X}^f_a, \tilde{X}^f_{<a}) = \sum_{t \in \mathbb{R}} \delta^f_{r, o}(t) + \sum_{t \in \mathbb{R}_+} \gamma^f_{r, o}(t) + \sum_{t \in \mathbb{R}_+} \gamma^f_{r, o-1}(t),
  \]
which follow from Proposition 5.1 above.

Proposition 5.2
\[
\mathbb{I}_{-\infty}^f(r) = \mathbb{I}_{\infty}^f(r) = TH_r(\tilde{X})
\]
The proof of this proposition is similar to the proof of Proposition 5.10 in [1].

To prove Theorem 4.3 one uses Borel-Moore homology. One defines the analogues $BM \delta^f_r$ and $BM \gamma^f_r$ of $\delta^f_r$ and $\gamma^f_r$ and then $BM \delta^f_r$ and $BM \gamma^f_r$, and one proves

Proposition 5.3 $BM \delta^f_t(a, b) = \delta^f_t(a, b)$ and $BM \gamma^f_r(a, b) = \gamma^f_r(a, b)$. 

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One derives these equalities using similar arguments as in the proof of Theorems 5.6 and 6.2 in [1] (which contain Theorem 4.3 as a particular case when \( \omega \) of degree of irrationality zero or one). However the arguments are longer since \( f \) is not proper in general.

To prove item 1 in Theorem 4.4 one verifies first

**Proposition 5.4** Suppose \( f \in \omega_1, g \in \omega_2, \omega_1, \omega_2 \in Z_{tame}(X; \xi) \). If \( |f - g| < \epsilon \) then

\[
F^i_f(t) \subseteq F^g(t + 2\epsilon) \subseteq F^i_f(t + 4\epsilon)
\]

Once Proposition 5.4 is established one proceeds as in [1], Proposition 5.7 and Theorem 5.2.

To check item 2. one proceeds as follows.

- Consider the configuration \( \lambda_\omega^r : \mathbb{R}_+ \to \mathbb{Z}_{\geq 0} \) defined by

\[
\lambda_\omega^r(t) = \delta_\omega^r(t) + \gamma_\omega^r(r) + \gamma_{\omega - 1}(t),
\]

and establish the continuity of the assignment

\[
Z_{tame}(X; \xi) \ni \omega \leadsto \lambda_\omega^r \in Conf([0, \infty) \setminus 0)
\]

as in the proof of the CEH-stability cf. [5].

- Use item 1 to conclude the continuity for the assignment given by \( \delta_{\omega, r}^+ \) the restriction of the configuration \( \delta_\omega^r \) to \( \mathbb{R}_+ \).

Clearly the continuity of the assignment \( \omega \leadsto \gamma_\omega^r(r) + \gamma_{\omega - 1}(t) \) and then of \( \omega \leadsto \gamma_\omega^r(r) \) is the same as the continuity of the assignment \( \omega \leadsto \lambda_\omega^r(t) - \delta_{\omega, r}^+(t) \). The continuity of \( \lambda_\omega^r(t) \) follows also from [10].

6 Comments and Applications

The results summarized above can and will be used for additional explorations / results in:

1. Dynamics,
2. Topology of smooth manifolds with symmetry,
3. Topology of the free loop space,
4. Geometrization of Data.

**Dynamics**: One can extend the results of Novikov theory from smooth flows on compact manifold with a Morse closed one form as Lyapunov to dynamics defined by a continuous flow on a compact ANR whose trajectories minimize an ”action”. Precisely one can detect rest points instantons and closed trajectories from the topology of the underlying space as in the case of classical Morse-Novikov theory. A paper on such applications is in preparation.

**Morse-Novikov theory on a G-Manifolds**: Given a smooth manifold \( M \), a closed one form and a smooth vector field with the closed form Lyapunov for the vector field which are invariant to a smooth action of a compact Lie group, by passing to the ”quotient space”, one obtains a compact ANR, a tame TC1-form and a continuous flow on the ANR. It seems interesting to compare the barcodes of the TC1-form with the complex which calculates the \( G \)-Novikov homology derived via classical G-Novikov theory. This will be done in a future work.

**Morse-Novikov theory for free loop space**: Suppose \( M \) a closed Riemannian manifold and \( M^{S^1} \) the infinite dimensional manifold of smooth free loops on \( M \). For a fixed integer \( N \) and a fixed \( \epsilon > 0 \) denote by \( M^N_\epsilon := \{(x_1, x_2, \cdots, x_n) \mid d(x_i, x_j) \leq \epsilon, d(x_n, x_1) \leq \epsilon\} \). Clearly \( M^N_\epsilon \) is a smooth compact manifold
with corners hence a compact ANR. If $\epsilon$ is small enough to insure unicity of a geodesic between two points $x, y$ with $d(x, y) \leq \epsilon$ then one has an obvious inclusion $M^{2N}_\epsilon \subset M^{2N}_{\epsilon/2} \subset M^{S^1}$. A smooth closed one form $\omega$ on $M^{S^1}$ induces the smooth closed one form $\omega_{N, \epsilon}$ on $M^n$. Clearly for any $r$ there exists an integer $K(r)$ s.t. for $k > K(r)$ the inclusion $M^{2N}_{\epsilon/2k} \subset M^{S^1}$ induces an isomorphism in $r-$ dimensional homology and cohomology. One expects that the barcodes of the pair $M^{2N}_{\epsilon/2k}, \omega|_{M^{2N}_{\epsilon/2k}}$ stabilize with $k$ large enough and permit the definition of barcodes for $(M^{S^1}, \omega)$. We plan to apply this to a system $(M, \sigma, g, J)$ with $(M, \sigma)$ a symplectic manifold equipped with a compatible almost complex structure $J$ (the Riemannian metric being determined by $(\omega, J)$) and a time dependent periodic symplectic vector field, cf [3]. These data provides a closed one form on $M^{S^1}$ determined by $\sigma$ and the closed one form induced from a time dependent periodic symplectic vector field, cf [3]. This is of interest in symplectic topology and Hamiltonian dynamics. This is work in progress.

**Geometrization of some data:** Suppose $(X, d)$ is a finite metric space (i.e. point cloud data), $S \subset X \times X$ and $f : S \rightarrow \mathbb{R}$ with the property that $(x, y) \in S$ implies $(y, x) \in S$ and $f(x, y) = -f(y, x)$. Let $\epsilon(f, d) > 0$ be the largest real number $\epsilon$ such that for any three points $x, y, z \in X$ with $(x, y), (y, z), (z, x) \in S$ and $d(x, y), d(x, z), d(y, z) < \epsilon$ one has $f(x, y) + f(y, z) + f(z, x) = 0$.

Clearly the Rips complex for any $\epsilon < \epsilon(f, d)$ is a simplicial complex equipped with a $TC1$-form $\omega(\epsilon)$ given by the simplicial cocycle defined by $f$. It might be interesting to study the bar codes based on persistent (Novikov) homology or even of the family of the $AN -$ chain complexes associated with $\omega(\epsilon)$, a considerable refinements of Novikov homology. They offer additional topological invariants to describe features of $\{(X, d), f\}$.

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