EXISTENCE OF SOLUTIONS FOR IMPULSIVE NEUTRAL FUNCTIONAL INTEGRO DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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ABSTRACT

In this paper, by using fractional power of operators and Sadovskii’s fixed point theorem, we study the existence of mild solution for a certain class of impulsive neutral functional integrodifferential equations with nonlocal conditions. The results we obtained are a generalization and continuation of the recent results on this issue.

Keywords: Sadovskii’s fixed point theorem, integrodifferential equations.

1. INTRODUCTION

Impulsive differential equations, that are differential equations involving impulsive effect, appear as a natural description of several real world problems. Many evolution process that have a sudden change in their states such as mechanical systems with impact, biological systems such as heart beats, blood flows, population dynamics, theoretical physics, radiophysics, pharmacokinetics, mathematical economy, chemical technology, electric technology, metallurgy, ecology, industrial robotics, biotechnology process, chemistry, engineering, control theory, medicine and so on. Adequate mathematical models of such processes are systems of differential equations with impulses, see the monographs of Bainov and Simeonov (13), Bainov, Lakshmikantham and Simeonov (14), the papers (10,15) and the references therein.

The theory of integrodifferential equations can be used to describe a lot of natural phenomena arising from many fields such as electronics, fluid dynamics, biological models, and chemical kinetics. Most of these phenomena cannot be described through classical differential equations. That is Why in recent years they have attracted more and more attention of several mathematicians, physicists and engineers. Impulsive integrodifferential equations has undergone rapid development over the years and played very important role in modern applied mathematical models of real process. Recently, several authors (3,9,11) have investigated the impulsive integrodifferential equations in abstract spaces. We refer to the papers Wang and Wei (20) and Guo (21) and the references cited therein. Particularly, neutral (integro) differential equations arise in many areas of applied mathematics. For instance, the system of heat conduction with finite

wave speeds, studied in (19) can be modeled in the form integrodifferential equation of neutral type. For more details on this theory and on its applications we refer to the monographs of Lakshmikantham et al. (14), and Samoilenko and Perestyuk (4) for the case of ordinary impulsive system and for partial differential and for partial functional differential equations with impulses.

The starting point of this paper is the work in papers (11, 12). Especially, authors in (12) investigated the existence of solutions for the system.

\[
\begin{align*}
\frac{d}{dt} x(t) + F(t, x(t), x(b_1t), \ldots, x(b_n t)) + A(t) x(t) \\
= G(t, x(t), x(a_1 t), \ldots, x(a_n t), 0 \leq t \leq a) \\
x(0) + g x = x_0
\end{align*}
\]

by using fractional powers of operators and Sadovskii’s fixed point theorem. And in (11), authors studied the following impulsive functional integrodifferential equation with nonlocal conditions of the form

\[
\begin{align*}
\dot{x}(t) &= A(t) x(t) \\
+ F(t, x(s_1 t, \ldots, x(s_n t), t, s, x(s_{n+1}) s ds) \\
&= G(t, x(t), x(a_1 t, \ldots, x(a_n t), 0 \leq t \leq a) \\
&= x(0) + g x = x_0
\end{align*}
\]

by using Schaefer’s fixed point theorem.
Motivated by above mentioned works (11,12), the main purpose of this paper is to prove
the existence of mild solutions for the following impulsive neutral functional integrodifferential
equations in a Banach space $X$.
\[
\frac{d}{dt} F(t, x(t), x(t-\tau), x(t-\sigma)) + \sum_{i=1}^{n} A_{i} x(t-\tau_{i}) = 0, \quad \text{for } t \neq \tau_{i},
\]
\[
\frac{d}{dt} F(t, x(t), x(t-\tau), x(t-\sigma)) + \sum_{i=1}^{n} A_{i} x(t-\tau_{i}) = x_{0}, \quad \text{for } t = \tau_{i}, i = 1, 2, \ldots, n.
\]

For our convenience let us take $F(t, x(t), x(t-\tau), x(t-\sigma))$ and $A_{i}$ to be continuous functions.

Consider the following conditions:

(i) $F(t, x(t), x(t-\tau), x(t-\sigma))$ is continuous and bounded on $[0, b] \times \mathbb{R}^{n+1}$.

(ii) $A_{i} x(t-\tau_{i})$ is continuous and bounded on $[0, b] \times \mathbb{R}^{n+1}$.

(iii) $x_{0}$ is continuous and bounded on $[0, b] \times \mathbb{R}$.

(H1) There exist positive constants $L_{1}$ and $L_{2}$ such that
\[
\|x\| \leq L_{1} \|y\| + L_{2}, \quad \text{for all } x, y \in \Omega
\]
and $g : \Omega \rightarrow X$ is completely continuous.

(H2) $F(t, x(t), x(t-\tau), x(t-\sigma))$ and $A_{i}$ are continuous functions.

(H3) There exist positive constants $L_{1}$ and $L_{2}$ such that
\[
\|g(x, t)\| \leq L_{1} \|x\| + L_{2}, \quad \text{for all } x \in \Omega
\]
and $g : \Omega \rightarrow X$ is completely continuous.

(H4) There exist positive constants $L_{1}$ and $L_{2}$ such that
\[
\|g(x, t)\| \leq L_{1} \|x\| + L_{2}, \quad \text{for all } x \in \Omega
\]
and $g : \Omega \rightarrow X$ is completely continuous.

(H5) $F(t, x(t), x(t-\tau), x(t-\sigma))$ and $A_{i}$ are continuous functions.

Definition 2.1

A family of linear operator $U(t, s) : \Omega \rightarrow X$ is a strongly continuous semigroup if
\[
\lim_{t \to s} \|U(t, s) x - U(s, s) x\| = 0, \quad \text{for all } x \in \Omega
\]
and $U(t, s)$ is continuous and bounded on $[0, b] \times \mathbb{R}$.

Definition 3.1

A continuous function $x(t) : [0, b] \rightarrow X$ is said to be a mild solution of the nonlocal Cauchy
problem 6 – (8), if the function
\[
A_{s} U(t, s) F(x(b), x(b-\tau), x(b-\sigma)) + \sum_{i=1}^{n} A_{i} x(t-\tau_{i}) = x_{0}, \quad \text{for } t \neq \tau_{i},
\]
\[
A_{s} U(t, s) F(x(b), x(b-\tau), x(b-\sigma)) + \sum_{i=1}^{n} A_{i} x(t-\tau_{i}) = x_{0}, \quad \text{for } t = \tau_{i}, i = 1, 2, \ldots, n.
\]
is integrable on $[0, b]$ and the integral equation
\[
\int_{0}^{b} g(s, t) \, ds = y < \infty
\]
Proof:

Let us write,

\[ t, x b_1 t, \ldots, x b_m t, \quad t, s, x b_{m+1} s ds \]
\[ = (t, v(t)) \quad \text{and} \]
\[ t, x a_1 t, \ldots, x a_n t, e t, s, x a_{n+1} s ds \]
\[ = (t, u(t)) \]

Define the operator \( P \) on \( \Omega \) by the formula

\[ p x t = U t, 0 x_0 - g x \]
\[ - F t, v t \]
\[ - t A s U t, s F s, v s \]
\[ = (0, 0) \]
\[ + U t, t l x t \]
\[ 0 \leq t \leq b \]

For each positive integer \( r \), let

\[ B_r = \left\{ x \in \Omega : \| x \| \leq r, 0 \leq t \leq b \right\} \]

Then for each \( r, B_r \) is clearly a bounded closed convex set in \( \Omega \). Since by (9) and (H1) the following relation holds:

\[ \| A t U t, s F s, v s \| \]
\[ = \| -A^{1-\beta} t U t, s -A^{1-\beta} t F(s, v(s)) \| \]
\[ \leq \frac{C_{1-\beta}}{(t-s)^\beta} L N + 1 r + L C_1 + C_2 \]

Where,

\[ C = b, \quad t, s, 0, C = \]
\[ \| -A^{1-\beta} t t F(t, 0,0, \ldots, 0,0) \| \]

then from Bocher's theorem (22) it follows that \( A t U t, s F(s, v(s)) \) is integrable on \([0, b]\), so \( P \) is well defined on \( B_r \). We claim that there exist a positive integer \( r \) such that \( P B_r \subseteq B_r \), if it is not true, then for each positive integer \( r \), there is a fuction \( x(\cdot) \in B_r \), but \( P x, t \notin B_r \), that is \( \| P x, t \| > r \) for some \( t r \in [0, b] \), where \( t(r) \) denotes \( t \) is dependent of \( r \). However, on the other hand, we have

\[ r < \| P x, t \| = \| U t, 0 x_0 - g x r - F t, v r \]
\[ - t A s U t, s F s, v s ds + \]
\[ t U t, s G s, u s ds \]
\[ + t U t, t l x t \]
\[ 0 < t \leq b \]

\[ \leq M \| x_0 \| + L_1 r + L_2 + M_0 L N + \]
\[ 1 r + L C_1 + C_2 \]
\[ + t C_{1-\beta} \|
\[ 0 \leq \beta \leq 1 \]
\[ C_2 ds + M 0 t g(s) ds \]
\[ + t C_{1-\beta} \|
\[ 0 \leq \beta 

Dividing on both sides by \( r \) and taking the lower limit as \( r \to \infty \), we get

\[ M L + M L(N + 1) + \frac{C_{1-\beta}}{\beta} L(N + 1) b^\beta + M y \]
\[ + M \lambda k \geq 1 \]

For each positive integer \( r \), let

\[ B_r = \left\{ x \in \Omega : \| x \| \leq r, 0 \leq t \leq b \right\} \]

Then for each \( r, B_r \) is clearly a bounded closed convex set in \( \Omega \). Since by (9) and (H1) the following relation holds:
Then for each

This contradicts (12). Hence for some positive integer \( r \), \( PB_r \subseteq B_r \).

Next we will show that the operator \( P \) has a fixed point on \( B_r \), which implies eq 6 – (8) has a mild solution. To this end, we decompose \( P \) as

where the operator \( P_1, P_2 \) are defined on \( B_r \), respectively, by

\[ P_1 x t = -F t, v t \]

\[ P_2 x (t) = U t, 0 x_0 - g x + U(t, s) G s, u(s) ds \]

for \( 0 \leq t \leq b \), and we will verify that \( P_1 \) is a contraction while \( P_2 \) is a compact operator.

To prove that \( P_1 \) is a contraction, we take \( x_1, x_2 \in B_r \). Then for each \( t \in [0, b] \) and by condition (H1) and (11), we have

\[ \left\| P_1 (x_1) - (P_2 x_2) t \right\| \leq F t, v_1 t - F t, v_2 t \]

\[ + \left\| t A t U(t, s) F s, v s - F s, v s ds \right\| \]

\[ \leq \left\| A^{-\beta} t - A^{-\beta} t F t, v_1 t - A\beta(t) F t, v_2 t \right\| \]

\[ + \left\| t A^{-\beta} s U t, s - A^{-\beta} s F s, v s - A\beta(s) F s, v s ds \right\| \]

\[ \leq M_0 \max_{i=1,2,...,m} \| x_1 s - x_2 s \| + N \| x_1 s - x_2 s \| \]

\[ + \frac{1}{t} \max_{i=1,2,...,m} \| x_1 t s - x_2 t s \| \]

\[ \leq L(N + 1) M_0 \frac{1}{t} \beta C b^{\beta} sup_{0 \leq s \leq b} \| x_1 s - x_2 s \| \]

\[ = L_0 sup_{0 \leq s \leq b} \| x_1 s - x_2 s \| \]

Thus,

\[ \| P_1 x_1 - P_2 x_2 \|_{\Omega} \leq L_0 \| x_1 - x_2 \|_{\Omega} \]

So by assumption \( 0 < L_0 < 1 \), we see that \( P_1 \) is contraction.

To prove that \( P_2 \) is compact, firstly we prove that \( P_2 \) is continuous on \( B_r \). Let \( x_n \rightarrow x \) in \( B_r \), then by H2(i) and H5

\[ i l_k, k = 1, 2, ..., m \] is continuous

\[ (ii) G s, u_n s \rightarrow G s, u s, n \rightarrow \infty \]

Since

\[ \left\| G s, u_n s - G s, u s \right\| \leq 2 g_r(s) \]

By the dominated convergence theorem, we have

\[ \| P_2 x_n - P_2 x \| = sup_{0 \leq s \leq b} \| U(t, 0) g x_n - g(x) + \frac{1}{t} U(t, s) G s, u(s) ds \]

\[ \| l_k x_n t_k - l_k x t_k \| \]

\[ \rightarrow 0 \text{ as } n \rightarrow \infty \]

ie, \( P_2 \) is continuous.

Next, we prove that \( P_2 x : x \in B_r \) is a family of equicontinuous functions. To see this we fix \( \epsilon > 0 \) and let \( t_2 > t_1 \), and \( \epsilon > 0 \) be enough small. Then

\[ \| (P_2 x)(t_2) - (P_2 x)(t_1) \| \leq \| U_{t_1, 0} x_0 - g(x) \|

\[ + \frac{1}{t_1} \epsilon \| U_{t_2, t_1} s - U_{t_2, 0} s \| \]

\[ + \| U_{t_1, s} G(s, u(s)) ds \|

\[ + \| U_{t_2, s} G(s, u(s)) ds \|

\[ \rightarrow 0 \text{ as } n \rightarrow \infty \]

\[ + \frac{1}{t_2} \epsilon \| U_{t_2, t_1} s - U_{t_2, 0} s \| \]

\[ + \| U_{t_2, s} G(s, u(s)) ds \| \]

\[ + \frac{1}{t_2} \epsilon \| U_{t_2, t_1} s - U_{t_2, 0} s \| \]

\[ + \| U_{t_2, s} G(s, u(s)) ds \| \]

\[ \rightarrow 0 \text{ as } n \rightarrow \infty \]

\[ + \frac{1}{t_2} \epsilon \| U_{t_2, t_1} s - U_{t_2, 0} s \| \]

\[ + \| U_{t_2, s} G(s, u(s)) ds \| \]

\[ \rightarrow 0 \text{ as } n \rightarrow \infty \]
Noting that $\|G(s, u(s))\| \leq g_r(s)$ and $g_r(s) \in L^1$, we see that $\|(P_2 x)(t_2) - (P_2 x)(t_1)\|$ tends to zero independently of $x \in B$, as $t_2 - t_1 \to 0$, since the compactness of $U(t, s)$ for $t - s > 0$, implies the continuity in the uniform operator topology. We can prove that the functions $P_2 x, x \in B$, are equicontinuous at $t = 0$. Hence $P_2$ maps $B$ into a family of equicontinuous functions.

It remains to prove that $V(t) = (P_2 x)(t) : x \in B$, is relatively compact in $X$. Let $0 < t \leq b$ be fixed and $0 < \varepsilon < t$. For $x \in B$, we define

$$
(P_2 x)(t) = \int_0^t U(t, s) G s, u(s) \, ds + \int_0^{t - \varepsilon} U(t, s) G s, u(s) \, ds
$$

Then from the compactness of $U(t, s)$ for $t - s > 0$, we obtain $V(t) = (P_2 x)(t) : x \in B$, is relatively compact in $X$ for every $\varepsilon$, $0 < \varepsilon < t$. Moreover, for every $x \in B$, we have

$$
\|(P_2 x)(t)\| \leq \int_0^t \|U(t, s) G s, u(s)\| \, ds + \int_0^{t - \varepsilon} \|U(t, s) G s, u(s)\| \, ds + M \int_0^t \varepsilon r \, ds + M t \int_0^{t - \varepsilon} g_r \, ds + M t \int_0^{t - \varepsilon} g_r \, ds +
$$

Therefore, there are relatively compact sets arbitrarily close to the set $V(t)$. Hence the set $V(t)$ is also relatively compact in $X$.

Thus, by Arzela-Ascoli theorem, $P_2$ is a compact operator. Those arguments enable us to conclude that $P = P_1 + P_2$ is a condensing map on $B$, and by the fixed point theorem of Sadovskii there exists a fixed point $x(\cdot)$ for $P$ on $B$. Therefore, the nonlocal Cauchy problem $6 - (8)$ has a mild solution and the proof is completed.

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