ARITHMETICAL CONSERVATION RESULTS

BENNO VAN DEN BERG\textsuperscript{1} AND LOTTE VAN SLOOTEN\textsuperscript{2}

Abstract. In this paper we present a proof of Goodman’s Theorem, a classical result in the metamathematics of constructivism, which states that the addition of the axiom of choice to Heyting arithmetic in finite types does not increase the collection of provable arithmetical sentences. Our proof relies on several ideas from earlier proofs by other authors, but adds some new ones as well. In particular, we show how a recent paper by Jaap van Oosten can be used to simplify a key step in the proof. We have also included an interesting corollary for classical systems pointed out to us by Ulrich Kohlenbach.

1. Introduction

The axiom of choice has a special status in constructive mathematics. On the one hand, it is arguably justified on the constructive interpretation of the quantifiers. Indeed, one could argue that a constructive proof of $\forall x \in X \exists y \in Y \varphi(x,y)$ should contain, implicitly, an effective method for producing, given an arbitrary $x \in X$, an element $y \in Y$ such that $\varphi(x,y)$. Such an effective method can then be seen as a constructive choice function $f : X \to Y$ such that $\varphi(x,f(x))$ holds for any $x \in X$. In fact, it is precisely for this reason that the type-theoretic axiom of choice is provable in Martin-L"{o}f’s constructive type theory (see [16]).

On the other hand, many standard systems for constructive mathematics do not include the axiom of choice. One example of such a system is Aczel’s constructive set theory CZF. Indeed, an argument due to Diaconescu shows that in CZF the set-theoretic axiom of choice AC implies a restricted form of the Law of Excluded Middle (see, for example, [1]). One thing one learns from this is that the status of the axiom of choice may depend on the way it is formulated as well as on the background theory. (For an interesting perspective on these matters, see [17].)

In the present paper we concentrate on $\text{HA}^\omega$, Heyting arithmetic in all finite types. This system dates back to the work by Kreisel from the late fifties [15] and has since become important in the study of constructivism. Currently, it is also playing an essential rôle in the work on the extraction of programs from proofs and proof mining, as can be seen from the recent books [23] [14]. In addition, it is also starting to attract attention in the Reverse Mathematics community, as can be seen from some recent papers on higher-order reverse mathematics like [16] [10] [22].

The precise formulation of the axiom of choice that we will look at is the following axiom of choice for all finite types:

$$\text{AC}: \forall x^\sigma \exists y^\tau \varphi(x,y,z) \to \exists f^{\sigma \to \tau} \forall x^\sigma \varphi(x,f(x),z).$$

\textsuperscript{1} INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION (ILLC), UNIVERSITY OF AMSTERDAM, P.O. BOX 94242, 1090 GE AMSTERDAM, THE NETHERLANDS. E-MAIL: bennovdberg@gmail.com.

\textsuperscript{2} MATHEMATICAL INSTITUTE, UTRECHT UNIVERSITY, P.O. BOX 80010, 3508 TA UTRECHT, THE NETHERLANDS. E-MAIL: lottevanslooten@live.nl.

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This version of the axiom of choice is not provable in HAω; however, one can show that HAω + AC and HAω are equiconsistent (for example, by using Kreisel’s modified realizability from [15]).

This is in marked contrast to what happens in the case of PAω, Peano arithmetic in all finite types. Indeed, using classical logic one can derive comprehension axioms from the axiom of choice: if \( \varphi(x^\sigma) \) is any formula in the language of PAω, then

\[
\forall x^\sigma \exists n^0 \left( n = 0 \leftrightarrow \varphi(x^\sigma) \right)
\]

is derivable using classical logic, from which

\[
\exists f^{\sigma \rightarrow 0} \forall x^\sigma \left( f(x) = 0 \leftrightarrow \varphi(x^\sigma) \right)
\]

follows using the axiom of choice. For this reason the system PAω + AC has the strength of full higher-order arithmetic, a much stronger system that PAω. This is another manifestation of the special status of the axiom of choice in constructivism.

In the constructive case, more is true. Not only are HAω + AC and HAω equiconsistent, but they also prove the same arithmetical sentences (where a sentence is arithmetical if its quantifiers range over natural numbers and the equalities it contains are between natural numbers). This is a classical result in the metamathematics of constructivism and goes by the name Goodman’s Theorem. As the name suggests, it was first proved by Nicholas Goodman in 1976 (see [7]). The original proof was based on a rather complicated theory of constructions and after this proof was published, various people have sought simpler proofs. One such proof was given by Goodman himself using a new proof-theoretic interpretation combining ideas from forcing and realizability [8]. Beeson showed how this can be understood as the composition of forcing and realizability and extended Goodman’s theorem to the extensional setting (see [2] and also [3]). Other proofs have been given by Gordeev [9], Mints [18], Coquand [4] and Renardel de Lavalette [21]; the authors of this paper are unsure whether this list is complete.

(An interesting observation, due to Kohlenbach, is that Goodman’s Theorem can fail badly for fragments: see [12].)

What we have done in this paper is to give yet another proof of Goodman’s Theorem. Our reasons for doing so are that we feel that despite its classic status, complete and rigorous proofs of this result are surprisingly rare, while some of the proofs that are complete are not the simplest or most transparent possible. What we have sought to do here is to give a proof which dots all the is. But we should stress that many of the ideas of our proof can already be found in the sources mentioned above. The main novelty may be in some of the details of the presentation and the observation that a recent paper by Jaap van Oosten (see [20]) can be used to simplify a key step in the proof. We have also included an interesting corollary of Goodman’s Theorem for classical systems pointed out to us by Ulrich Kohlenbach. We are grateful for his permission to include it here.

Like most of the proofs mentioned above, ours relies on a string of proof-theoretic interpretations starting from HAω + AC and ending with HA, keeping the set of provable arithmetical sentences fixed. The string of interpretations is quite long: we could easily have made the proof shorter, but we felt that this would make the argument less transparent. Indeed, the proof combines many ideas and by making sure that each proof-theoretic interpretation relies on a single idea only, the whole structure of the argument becomes a lot easier to follow and far more intelligible.

In the intermediate stages we will make use of a theory of operations similar to Beeson’s EON and Troelstra’s APP (for which see [3, 28, 26]); our version of this is a system we have
called HAP. Systems of this form go back to the pioneering work of Feferman [5, 6] and we hope that with this paper we honour the memory of more than one great foundational thinker.

The contents of this paper are based on a Master thesis written by the second author and supervised by the first author [24]. Finally, we would like to thank the referee for a useful report.

2. The systems HAP and HAPε

The aim of this section is to introduce the systems HAP and HAPε and show that they are conservative over HA. Both these systems are formulated using the logic of partial terms LPT (also called $E^+$-logic), due to Beeson (see [3, 27, 26]). For the convenience of the reader we present an axiomatisation following [26].

2.1. Logic of partial terms LPT. The idea of the logic of partial terms is that we want to have a logic in which we can reason about terms which do not necessarily denote (think Santa Claus or the present king of France). To express that a term $t$ denotes, or “$t$ exists”, we will write $t \downarrow$. In fact, here we will consider this as an abbreviation for $t = t$.

Having terms around which do not denote, forces us to change the usual rules for the quantifiers. Where normally we can deduce $A[t/x]$ from $\forall x A$ for any term $t$, in LPT this is only possible if $t$ denotes; conversely, we can only deduce $\exists x A$ from $A[t/x]$ if $t$ denotes.

More precisely, the language of LPT is that of standard intuitionistic predicate logic with equality, with $t \downarrow$ as an abbreviation for $t = t$. Its axioms and inference rules are all substitution instances of:

$$
\varphi \rightarrow \varphi \\
\varphi, \varphi \rightarrow \psi \Rightarrow \varphi \rightarrow \psi \\
\varphi \rightarrow \psi, \varphi \rightarrow \chi \Rightarrow \varphi \rightarrow \chi \\
\varphi \land \psi \rightarrow \varphi, \varphi \land \psi \rightarrow \psi \\
\varphi \rightarrow \psi, \varphi \rightarrow \chi \Rightarrow \varphi \rightarrow \psi \land \chi \\
\varphi \rightarrow \varphi \lor \psi, \psi \rightarrow \varphi \lor \psi \\
\varphi \rightarrow \chi, \psi \rightarrow \chi \Rightarrow \varphi \lor \psi \rightarrow \chi \\
(\varphi \land \psi) \rightarrow \chi \Rightarrow \varphi \lor \psi \rightarrow \chi \\
\varphi \rightarrow (\psi \rightarrow \chi) \Rightarrow (\varphi \land \psi) \rightarrow \chi \\
\bot \Rightarrow \varphi \\
\varphi \rightarrow \psi \Rightarrow \varphi \rightarrow \forall x \psi \quad (x \notin \text{FV}(\varphi)) \\
\forall x \varphi \land t \downarrow \Rightarrow \varphi[t/x] \\
\varphi[t/x] \land t \downarrow \Rightarrow \exists x \varphi \\
\varphi \rightarrow \psi \Rightarrow \exists x \varphi \rightarrow \psi \quad (x \notin \text{FV}(\psi))
$$

For equality we have the following rules:

$$
\forall x (x = x), \forall xy (x = y \rightarrow y = x), \forall xyz (x = y \land y = z \rightarrow x = z) \\
\forall \varphi \varphi[z = y \rightarrow F(x) \downarrow \rightarrow F(y) = F(x)) \land \forall \varphi \varphi[Rz \land z = y \rightarrow Ry]
$$

In addition, all the basic function and relation symbols will be assumed to be strict:

$$c \downarrow, \quad F(t_1, \ldots, t_n) \downarrow \rightarrow t_i \downarrow, \quad R(t_1, \ldots, t_n) \rightarrow t_i \downarrow.
$$

This includes equality:

$$s = t \rightarrow s \downarrow \land t \downarrow.
$$

It will be convenient to introduce the following weaker notion of equality:

$$s \simeq t := (s \downarrow \lor t \downarrow) \rightarrow s = t.
$$
So \( s \simeq t \) expresses that \( s \) and \( t \) are equally defined and equal whenever defined. For this weaker notion of equality we can prove the following Leibniz schemata:

\[
s \simeq t \rightarrow F_s \simeq F_t, \quad s \simeq t \land A[t/x] \rightarrow A[s/x].
\]

**Remark 2.1.** Below we will frequently exploit the following fact. Suppose \( T \) is some theory based on the logic of partial terms and we can prove in \( T \) that for some formula \( \varphi(x,y) \) we have

\[
\varphi(x,y) \land \varphi(x,y') \rightarrow y = y'.
\]

Then we may extend \( T \) to a theory \( T' \) by introducing a new function symbol \( f_\varphi \) and adding a new axiom

\[
\varphi(x,y) \leftrightarrow y = f_\varphi(x).
\]

The resulting theory \( T' \) will then be conservative over \( T \). For a proof, see [27, Section 2.7].

2.2. **The system HAP.** In this paper a key rôle is played by a formal system which we will call HAP. It is a minor variation on Beeson’s EON and Troelstra’s APP (for which see again [3, 28, 26]).

The language is single-sorted and the logic is based on LPT. There are the usual arithmetical operations \( 0, S, +, \times \), with the usual axioms:

\[
\begin{align*}
Sx &= x + y \downarrow, x \times y \downarrow, \\
Sx &= Sy \rightarrow x = y, 0 \neq Sx, \\
x + 0 &= x, x + Sy = S(x + y), \\
x \times 0 &= x, x \times Sy = (x \times y) + y
\end{align*}
\]

HAP-formulas in this fragment of the language will be called arithmetical. In addition, there will be an application operation written with a dot \( \cdot \) and combinators

\[
k, s, p, p_0, p_1, \text{succ}, r
\]

Instead of \( t_1 \cdot t_2 \) we will often simply write \( t_1t_2 \) and application associates to the left (so \( t_1t_2t_3 \) stands for \( (t_1t_2)t_3 \)). Note that our assumption that all function symbols are strict implies that we have

\[
s \cdot t \downarrow \rightarrow s \downarrow \land t \downarrow.
\]

The axioms for the combinators are

\[
\begin{align*}
kxy &= x, sxy \downarrow, sxxyz \simeq xz(yz), \\
p_0x \downarrow, p_1x \downarrow, p_0(pxy) &= x, p_1(pxy) = y, p(p_0x)(p_1x) = x, \\
\text{succ} \cdot x &= Sx, rxy0 = x, rxy(Sz) = yz(rxyz).
\end{align*}
\]

Finally, we have the induction scheme:

\[
\varphi[0/x] \land \forall x (\varphi \rightarrow \varphi[Sx/x]) \rightarrow \forall x \varphi
\]

for all HAP-formulas \( \varphi \).

**Remark 2.2.** The system HAP can be obtained from Troelstra’s system APP (or Beeson’s EON) by making the following changes:

(a) APP has a unary predicate \( N \) for being a natural number, which is dropped in HAP. Indeed, in HAP every element acts as a natural number, in that the induction scheme is valid over the entire domain. For that reason HAP proves that equality is decidable and that there is an element \( e \) such that \( exy = 0 \) precisely when \( x = y \).

(b) APP has an if-then-else construct \( d \) instead of a recursor \( r \). This is a minor difference, as these are interderivable (see [28 Lemma 9.3.8]).
(c) In HAP the arithmetical operations are primitive, allowing us to define the arithmetical fragment of HAP. In addition, it allows us to change the interpretation of the application operation, while keeping the interpretation of the arithmetical fragment fixed (as in Lemma 2.6 below).

**Proposition 2.3.** For each term \( t \) in the language of HAP and variable \( x \), one can construct a term \( \lambda x.t \), whose free variables are those of \( t \) excluding \( x \), such that HAP \( \vdash \lambda x.t \downarrow \) and HAP \( \vdash (\lambda x.t)x \simeq t \).

**Proof.** In case \( t \) is a term built from variables, application and combinators, then there is a well-known abstraction algorithm (see, for example, [28, Proposition 9.3.5] and also [26, p. 423]). We define \( \lambda x.t \) by induction on the structure of \( t \). If \( t \) is \( x \) itself, then \( \lambda x.x \) is skk, while if \( t \) is a variable or constant different from \( t \), then \( \lambda x.t \) is \( kt \). Finally, if \( t = t_1 t_2 \), then \( \lambda x.t \) is \( s(\lambda x.t_1)(\lambda x.t_2) \).

Of course, general terms in HAP can also be built using the arithmetical operations \( S, +, \times \). However, since for \( \text{succ, plus, times} = \lambda xy.\text{rx}(\lambda uv.\text{succ \cdot v})y \) and \( \text{times} = \lambda xy.\text{r0}(\lambda uv.\text{plus \cdot v})y \) the system HAP proves

\[ \text{succ} \cdot x = Sx, \quad \text{plus} \cdot x \cdot y = x + y, \quad \text{times} \cdot x \cdot y = x \times y, \]

any term is provably equal in HAP to one built purely from variables, the application operation and combinators. \( \square \)

**Proposition 2.4.** The system HAP is conservative over HA.

**Proof.** Indeed, we can interpret HAP in HA by exploiting the fact that one can develop basic recursion theory inside HA. This allows one to interpret \( x \cdot y \) as Kleene application: that is, \( x \cdot y \) is the result of the partial recursive function coded by \( x \) on input \( y \) (whenever this is defined). More details can be found in, for instance, [28, Proposition 9.3.12]. \( \square \)

2.3. **The system HAP.** In the remainder of this section we will study an extension of HAP. This extension, which we will call HAP\(_e\), is obtained from HAP by adding for each arithmetical formula \( \varphi(x, y) \) a constant \( \varepsilon_\varphi \) as well as the following axioms:

\[ \exists y \varphi(x, y) \rightarrow \varepsilon_\varphi \cdot x \downarrow , \quad \varepsilon_\varphi \cdot x \downarrow \rightarrow \varphi(x, \varepsilon_\varphi \cdot x) \]

The goal of this subsection is to prove that the resulting system is still conservative over HA.

**Proposition 2.5.** Suppose \( \psi(x, y) \) is a formula in the language of HAP and suppose HAP\(_f\) is the extension of HAP with a function symbol \( f \) and the following axioms:

\[ \exists y \psi(x, y) \rightarrow f(x) \downarrow , \quad f(x) \downarrow \rightarrow \psi(x, f(x)) \]

Then HAP\(_f\) is conservative over HAP.

**Proof.** In view of Remark 2.7 it suffices to show that we can conservatively add to HAP a relation symbol \( F \) satisfying the formulas

\[ F(x, y) \land F(x, y') \rightarrow y = y', \quad \exists y \psi(x, y) \rightarrow \exists y F(x, y), \quad F(x, y) \rightarrow \psi(x, y). \]

Let us call this system HAP\(_F\). To show the conservativity of this extension we use forcing, with as forcing conditions finite approximations to the relation \( F \). To be precise, a condition \( p \) is a (coded) finite sequence of pairs

\[ \langle (x_0, y_0), \ldots , (x_{n-1}, y_{n-1}) \rangle \]
such that $\psi(x_i, y_i)$ holds for any $i < n$ and all $x_i$ are distinct. We will write $q \leq p$ if $p$ is an initial segment of $q$, and $(x, y) \in r$ if there is some $i < n$ such that $(x, y_i) = (x_i, y_i)$.

For any $\text{HAP}_F$-formula $\varphi$ we define a $\text{HAP}$-formula $p \models \varphi$ by induction on $\varphi$, as follows:

\[
\begin{align*}
p \models \varphi : & = \varphi \text{ if } \varphi \text{ is an atomic } \text{HAP}_F\text{-formula} \\
p \models F(x, y) : & = (\forall q \leq p) (3r \leq q) (x, y) \in r \\
p \models \varphi \land \psi : & = p \models \varphi \land p \models \psi \\
p \models \varphi \lor \psi : & = (\forall q \leq p) (\exists r \leq q) (r \models \varphi \lor r \models \psi) \\
p \models \varphi \rightarrow \psi : & = (\forall q \leq p) (q \models \varphi \rightarrow q \models \psi) \\
p \models \forall x \varphi(x) : & = (\forall x) (\forall q \leq p) q \models \varphi(x) \\
p \models \exists x \varphi(x) : & = (\forall q \leq p) (\exists r \leq q) (\exists x) r \models \varphi(x).
\end{align*}
\]

Note that we have

\[
\begin{align*}
\text{HAP} & \models p \leq q \land q \models \varphi \rightarrow p \models \varphi \text{ and} \\
\text{HAP} & \models (\forall q \leq p) (\exists r \leq q) r \models \varphi \rightarrow p \models \varphi
\end{align*}
\]

for all $\text{HAP}_F$-formulas $\varphi$ and

\[
(1) \quad \text{HAP} \models (p \models \varphi) \leftrightarrow \varphi
\]

if $\varphi$ is a $\text{HAP}_F$-formula. The idea now is to prove

\[
\text{HAP}_F \models \varphi \equiv \text{HAP} \models p \models \varphi
\]

by induction on the derivation of $\varphi$ in $\text{HAP}_F$. We leave the verification of the $\text{HAP}_F$-axioms to the reader and only check that the interpretations of the axioms we have added to $\text{HAP}$ are provable in $\text{HAP}$; for this we reason in $\text{HAP}$.

Suppose $q \models F(x, y)$ and $q \models F(x, y')$. Then there is some $r \leq q$ such that $(x, y) \in r$ and $(x, y') \in r$. Because $r$ is a condition, we must have $y = y'$ and therefore $q \models y = y'$. We conclude that we have $p \models F(x, y) \land F(x, y') \rightarrow y = y'$ for every condition $p$.

Suppose $p' \models \exists y \psi(x, y)$. Our aim is to show $p' \models \exists y F(x, y)$, so suppose $q' \leq p'$. Then there must be some $r \leq q'$ and some $y'$ such that $r \models \psi(x, y')$. Now there are two possibilities: either there is some $y$ such that $(x, y) \in r$, or no such $y$ exists (we are using here that equality in $\text{HAP}$ is decidable). In the former case we have $r \models F(x, y)$; in the latter, we use (1) to deduce that $\psi(x, y')$ holds. We extend $r$ by appending the pair $(x, y')$ to obtain some new condition $r' \leq r$. For this condition $r'$ we have $r' \models F(x, y')$. So in both cases there is some condition $r' \leq q'$ and some $y'$ such that $r' \models F(x, y')$. We conclude that $p' \models \exists y F(x, y)$ and hence that we have $p \models \exists y \psi(x, y) \rightarrow \exists y F(x, y)$ for every condition $p$.

Finally, if $q \models F(x, y)$, then $(x, y) \in r$ for some $r \leq q$. Since $r$ is a condition, we have $\psi(x, y)$ and hence $q \models \psi(x, y)$ by (1). So $p \models F(x, y) \rightarrow \psi(x, y)$ for every condition $p$.

\[
\text{Lemma 2.6.} \quad \text{Suppose } \psi(x, y) \text{ is an arithmetical formula in the language of } \text{HAP} \text{ and suppose } \text{HAP}_f \text{ is the extension of } \text{HAP} \text{ with a constant } f \text{ and the following axioms:}
\]

\[
\exists y \psi(x, y) \rightarrow f \cdot x \downarrow, \quad f \cdot x \downarrow \rightarrow \psi(x, f \cdot x)
\]

Then $\text{HAP}_f$ is conservative over $\text{HA}$.

\[
\text{Proof.} \quad \text{The idea is to work in the system } \text{HAP}_f \text{ from Proposition 2.5 and redefine the application in } \text{HAP}_f \text{ in such a way that we can use the partial function } f \text{ as an oracle. This has the desired effect of making the function } f \text{ representable. How this can be done is worked out in [20, Theorem 2.2]. Let us just recall from this paper how one redefines the application.}
\]
For any \(a, b\) an \(f\)-dialogue between \(a\) and \(b\) is defined to be a code of a sequence \(u = (u_0, \ldots, u_{n-1})\) such that for all \(i < n\) there is a \(v_i\) such that
\[
a \cdot (\langle b \rangle \ast u^{<i}) = p \downarrow v_i \quad \text{and} \quad f(v_i) = u_i.
\]
We say that the new application \(a \cdot f b\) is defined with value \(c\) if there is an \(f\)-dialogue \(u\) between \(a\) and \(b\) such that
\[
a \cdot (\langle b \rangle \ast u) = p \uparrow c.
\]
Here \(\langle b \rangle\) is the sequence consisting only of \(b\), \(*\) stands for concatenation and \(u^{<i}\) denotes \(\langle u_0, \ldots, u_{i-1} \rangle\) whenever \(u\) codes some finite sequence. In addition, \(\downarrow\) and \(\uparrow\) are assumed to be some choice for the booleans for which there is an if-then-else construct \(d\) such that \(d \uparrow xy = x\) and \(d \downarrow xy = y\) (for example, \(\downarrow = \lambda xy. y\) and \(\uparrow = k\) and \(d = \lambda xyz.xyz\).)

To correctly interpret the HAP-axioms the interpretation of the combinators needs to change as well (again, see [20, Theorem 2.2]), but, crucially, the interpretations of the arithmetical operations can remain the same. Therefore the interpretation of \(\psi\) is unaffected and the system remains conservative over \(HA\). □

**Theorem 2.7.** The system HAP\(_\varepsilon\) is conservative over HA.

**Proof.** It suffices to prove that for each finite set of formulas \(\varphi_1, \ldots, \varphi_n\) adding the \(\varepsilon\varphi_i\) together with their axioms is conservative over HAP, since each proof in HAP uses only finitely many symbols and finitely many axioms. In fact, by considering
\[
\varphi(x, y) := \bigwedge_{i=1}^n (p_0 x = i \to \varphi_i(p_1 x, y))
\]
one sees that it suffices to prove this for extensions with a single combinator \(\varepsilon\varphi\) only. But the statement that adding a single combinator of that form results in system conservative over HA is precisely Lemma 2.6 □

### 3. Realizability for HAP

Following Feferman [5, 6] we define abstract realizability interpretations of HAP into itself. In this section it will be convenient to regard disjunction as a defined connective, as follows:
\[
\varphi \lor \psi := \exists n \left( (n = 0 \to \varphi) \land (n \neq 0 \to \psi) \right).
\]

**Definition 3.1.** (Feferman) For each HAP-formula \(\varphi\) we define a new HAP-formula \(x r \varphi\) (“\(x\) realizes \(\varphi\)”) by induction on the structure of \(\varphi\) as follows:
\[
x r \varphi := \varphi \quad \text{if} \ \varphi \ \text{is atomic}
x r (\varphi \land \psi) := p_0 x r \varphi \land p_1 x r \psi
x r (\varphi \rightarrow \psi) := \forall y (y r \varphi \rightarrow x \cdot y \downarrow \land x \cdot y r \psi)
x r \forall y \varphi := \forall y (x \cdot y \downarrow \land x \cdot y r \varphi)
x r \exists y \varphi := p_1 x r [p_0 x/y]
\]

**Theorem 3.2.** If HAP \(\vdash \varphi(y)\), then HAP \(\vdash \exists x \forall y (x \cdot y r \varphi(y))\).

**Proof.** See, for example, [3, Theorem VII.1.5]. □
In what follows we will also need an extensional variant of this abstract form of realizability. In this form of realizability the collection of realizers of a fixed formula \( \varphi \) carries an equivalence relation, the intuition being that \( x \) and \( x' \) are equivalent if \( "x \) and \( x' \) are identical as realizers of \( \varphi^\prime \). Crucially, realizers of an implication \( \varphi \rightarrow \psi \) are required to send realizers which are equal as realizers of \( \varphi \) to realizers which are equal as realizers of \( \psi \).

**Definition 3.3.** (See [26, Definition 6.1] and [19].) For each HAP-formula \( \varphi \) we define new HAP-formulas \( x \in \varphi \) ("\( x \) extensionally realizes \( \varphi^\prime \)) and \( x = x' \in \varphi \) ("\( x \) and \( x' \) are identical as extensional realizers of \( \varphi^\prime \)) by simultaneous induction on the structure of \( \varphi \) as follows:

\[
\begin{align*}
x \in \varphi & \ := \ \varphi \quad \text{if } \varphi \text{ is atomic} \\
x = x' \in \varphi & \ := \ \varphi \land x = x' \quad \text{if } \varphi \text{ is atomic} \\
x \in (\varphi \land \psi) & \ := \ p_0x \in \varphi \land p_1x \in \psi \\
x \in (\varphi \rightarrow \psi) & \ := \ \forall y, y' (y = y' \in \varphi \rightarrow x \cdot y = x \cdot y) \\
x = x' \in (\varphi \rightarrow \psi) & \ := \ \forall y \in \varphi \land x' \in (\varphi \rightarrow \psi) \land \forall y (y \in \varphi \rightarrow x \cdot y = x' \cdot y) \\
x \in \forall y \varphi & \ := \ \forall y \in \varphi \\
x \in \exists y \varphi & \ := \ p_1x \in \varphi[p_0x/y] \\
x = x' \in \exists y \varphi & \ := \ p_1x = p_1x' \in \varphi[p_0x/y] \land p_0x = p_0x'
\end{align*}
\]

One shows by induction on the structure of \( \varphi \) that provably in HAP the relation \( x \equiv y \in \varphi \) is symmetric and transitive and \( x \in \varphi \) is equivalent to \( x = x' \in \varphi \). In addition, we have:

**Theorem 3.4.** If HAP \( \vdash \varphi(y) \), then HAP \( \vdash \exists x \forall y (x \cdot y \in \varphi(y)) \).

**Proof.** Routine. \( \square \)

For the main theorem of this section we return to the system HAP\(_e\).

**Theorem 3.5.** The system HAP\(_e\) proves \( \varphi \leftrightarrow \exists x \in \varphi \leftrightarrow \exists x \in \varphi \) for every arithmetical formula \( \varphi \).

**Proof.** In HAP\(_e\) we can define for every arithmetical formula \( \varphi \) with free variables \( \underline{\chi} \) a "canonical realizer" \( j_\varphi \), as follows:

\[
\begin{align*}
j_\varphi & \ := \ \lambda \underline{\chi}.0 \text{ if } \varphi \text{ is atomic and arithmetical} \\
j_\varphi \land \psi & \ := \ \lambda \underline{\chi}.p(j_\varphi \cdot \underline{x})(j_\psi \cdot \underline{x}) \\
j_\varphi \rightarrow \psi & \ := \ \lambda \underline{\chi}.\lambda y.(j_\varphi \cdot \underline{x}) \\
j_\forall y \varphi & \ := \ \lambda \underline{\chi}.\lambda y.j_\varphi \cdot \underline{x} \cdot y \\
j_\exists y \varphi & \ := \ \lambda \underline{\chi}.p(\varepsilon \varphi \cdot \underline{x}) (j_\varphi \cdot \underline{x} \cdot (\varepsilon \varphi \cdot \underline{x}))
\end{align*}
\]

One can now prove

\[
\text{HAP}_{\varepsilon} \vdash \varphi \leftrightarrow \exists x \in \varphi \leftrightarrow j_\varphi \cdot \underline{x} \in \varphi,
\]

as well as

\[
\text{HAP}_{\varepsilon} \vdash \varphi \leftrightarrow \exists x \in \varphi \leftrightarrow j_\varphi \cdot \underline{x} \in \varphi,
\]

by induction on \( \varphi \), assuming that \( \underline{\chi} \) lists all free variables in \( \varphi \). \( \square \)

**Corollary 3.6.** Let \( H \) be either HAP plus the schema \( \varphi \leftrightarrow \exists x \in \varphi \) for all arithmetical \( \varphi \), or HAP plus the schema \( \varphi \leftrightarrow \exists x \in \varphi \) for all arithmetical \( \varphi \). Then \( H \) is conservative over HA.
Proof. This follows from Theorem 2.7 and Theorem 3.5.

4. Applications to systems in higher types

In this section we will discuss applications to systems in higher types, in particular, Goodman’s Theorem. Various versions of finite-type arithmetic exist and the differences tend to be subtle, so first we will explain the precise version we will be working with.

Our starting point is the system \( \text{HA}_\omega \) from [28, pages 444-449]. This is a system formulated in many-sorted intuitionistic logic, where the sorts are the finite types.

**Definition 4.1.** The finite types are defined by induction as follows: 0 is a finite type, and if \( \sigma \) and \( \tau \) are finite types, then so are \( \sigma \rightarrow \tau \) and \( \sigma \times \tau \). The type 0 is the ground or base type, while the other types will be called higher types.

There will be infinitely many variables of each sort. In addition, there will be constants:

1. for each pair of types \( \sigma, \tau \) a combinator \( k^{\sigma,\tau} \) of sort \( \sigma \rightarrow (\tau \rightarrow \sigma) \).
2. for each triple of types \( \rho, \sigma, \tau \) a combinator \( s^{\rho,\sigma,\tau} \) of type \( (\rho \rightarrow (\sigma \rightarrow \tau)) \rightarrow ((\rho \rightarrow \sigma) \rightarrow (\rho \rightarrow \tau)) \).
3. for each pair of types \( \rho, \sigma \) combinators \( p^{\rho,\sigma}, p_0^{\rho,\sigma}, p_1^{\rho,\sigma} \) of types \( \rho \rightarrow (\sigma \rightarrow \rho \times \sigma), \rho \times \sigma \rightarrow \rho \) and \( \rho \times \sigma \rightarrow \sigma \), respectively.
4. a constant 0 of type 0 and a constant \( S \) of type \( 0 \rightarrow 0 \).
5. for each type \( \sigma \) a combinator \( R^{\sigma} \) (“the recursor”) of type \( \sigma \rightarrow ((0 \rightarrow (\sigma \rightarrow \sigma)) \rightarrow (0 \rightarrow \sigma)) \).

**Definition 4.2.** The terms of \( \text{HA}_\omega \) are defined inductively as follows:

- each variable or constant of type \( \sigma \) will be a term of type \( \sigma \).
- if \( f \) is a term of type \( \sigma \rightarrow \tau \) and \( x \) is a term of type \( \sigma \), then \( fx \) is a term of type \( \tau \).

The convention is that application associates to the left, which means that an expression like \( fxyz \) has to be read as \( (((fx)y)z) \).

**Definition 4.3.** The formulas of \( \text{HA}_\omega \) are defined inductively as follows:

- \( \bot \) is a formula and if \( s \) and \( t \) are terms of the same type \( \sigma \), then \( s =_\sigma t \) is a formula.
- if \( \varphi \) and \( \psi \) are formulas, then so are \( \varphi \land \psi, \varphi \lor \psi, \varphi \rightarrow \psi \).
- if \( x \) is a variable of type \( \sigma \) and \( \varphi \) is a formula, then \( \exists x^\sigma \varphi \) and \( \forall x^\sigma \varphi \) are formulas.

Finally, the axioms and rules of \( \text{HA}_\omega \) are:

(i) All the axioms and rules of many-sorted intuitionistic logic (say in Hilbert-style).
(ii) Equality is an equivalence relation at all types:

- \( x = x \), \( x = y \rightarrow y = x \), \( x = y \land y = z \rightarrow x = z \)

(iii) The congruence laws for equality at all types:

- \( f = g \rightarrow fx = gx \), \( x = y \rightarrow fx = fy \)

(iv) The successor axioms:

- \( \neg S(x) = 0 \), \( S(x) = S(y) \rightarrow x = y \)
(v) For any formula \( \varphi \) in the language of \( \text{HA}^\omega \), the induction axiom:
\[
\varphi(0, y) \rightarrow \left( \forall x^0 \left( \varphi(x, y) \rightarrow \varphi(Sx, y) \right) \right) \rightarrow \forall x^0 \varphi(x, y).
\]
(vi) The axioms for the combinators:
\[
\begin{align*}
k_{xy} &= x \\
s_{xyz} &= xz(yz) \\
p_0(pxy) &= x \\
p_1(pxy) &= y \\
p(p_0x)(p_1x) &= x
\end{align*}
\]
as well as for the recursor:
\[
\begin{align*}
R_{xy0} &= x \\
R_{xy}(Sn) &= yn(R_{xy})
\end{align*}
\]
This completes the description of the system \( \text{HA}^\omega \).

**Proposition 4.4.** For each term \( t \) in the language of \( \text{HA}^\omega \) and variable \( x \), one can construct a term \( \lambda x.t \), whose free variables are those of \( t \) excluding \( x \), such that \( \text{HA}^\omega \vdash (\lambda x.t)x = t \).

**Proof.** As in Proposition 4.4. \( \square \)

Two extensions of \( \text{HA}^\omega \) will be important in what follows. One is the “intensional” variant where for each type \( \sigma \) we have a combinator \( e^\sigma \) of type \( \sigma \rightarrow (\sigma \rightarrow 0) \) satisfying
\[
e^\sigma xy \leq 1, \quad \forall x^\sigma, y^\sigma \left( e^\sigma xy = 0 \iff x =_{\sigma} y \right).
\]
This extension of \( \text{HA}^\omega \) is denoted by \( \text{I-HA}^\omega \).

In addition, we have the extensional variant, where we have for all finite types \( \sigma \) and \( \tau \) the axiom
\[
\forall f, g^\sigma \rightarrow \tau \left( \forall x^\sigma fx =_{\sigma} \tau gx \rightarrow f =_{\sigma \rightarrow \tau} g \right).
\]
This extension of \( \text{HA}^\omega \) is denoted by \( \text{E-HA}^\omega \).

This section mainly concerns the axiom of choice for all finite types, denoted by \( \text{AC} \), which is the following scheme:
\[
\forall x^\sigma \exists y^\tau \varphi(x, y, z) \rightarrow \exists f^\sigma \rightarrow \tau \forall x^\sigma \varphi(x, fx, z).
\]
Indeed, the purpose of this section is to prove that both \( \text{I-HA}^\omega + \text{AC} \) and \( \text{E-HA}^\omega + \text{AC} \) are conservative over \( \text{HA} \) (Goodman’s Theorem). Our strategy will be to show that these systems prove the same arithmetical sentences as \( \text{HAP}_\varepsilon \), by giving suitable interpretations of these systems into \( \text{HAP}_\varepsilon \).

**4.1. Goodman’s Theorem.** There is a relatively straightforward interpretation of \( \text{HA}^\omega \) inside \( \text{HAP}_\varepsilon \). The idea is to define, by induction on the finite type \( \sigma \), a predicate \( \text{HRO}_\sigma \) which picks out those elements which are suitable for representing objects of type \( \sigma \), as follows:
\[
\begin{align*}
\text{HRO}_0(x) &:= x = x \\
\text{HRO}_{\sigma \times \tau}(x) &:= \text{HRO}_\sigma(p_0x) \land \text{HRO}_\tau(p_1x) \\
\text{HRO}_{\sigma \rightarrow \tau}(x) &:= \forall y(\text{HRO}_\sigma(y) \rightarrow x \cdot y \downarrow \land \text{HRO}_\tau(x \cdot y))
\end{align*}
\]
Moreover, equality at all types is interpreted as equality, the combinators \( k, s, p, p_0, p_1 \) are interpreted as themselves, while \( r \) interprets \( R \) (indeed, in this way any term and any (atomic)
formula in the language of HA\(^\omega\) can be seen as a term or formula in the language of HAP, by forgetting the types and replacing the combinators in HA\(^\omega\) by their analogues in HAP). In addition, we can define a closed term \(e\) in HAP such that

\[ e_{xy} \leq 1, \quad e_{xy} = 0 \leftrightarrow x = y. \]

From this it follows that we can interpret all of I-HA\(^\omega\) inside HAP.

To also interpret AC we combine this idea with realizability.

**Definition 4.5.** For each HA\(^\omega\)-formula \(\varphi\) with free variables among \(y_1^{\sigma_1}, \ldots, y_n^{\sigma_n}\) we define a HAP-formula \(x \, r \varphi\) ("\(x\) realizes \(\varphi\)) with free variables among \(x, y_1, \ldots, y_n\) by induction on the structure of \(\varphi\) as follows:

- \(x \, r \varphi := \varphi\) if \(\varphi\) is atomic
- \(x \, r (\varphi \land \psi) := p_0 \, x \, r \varphi \land p_1 \, x \, r \psi\)
- \(x \, r (\varphi \rightarrow \psi) := \forall y (y \, r \varphi \rightarrow xy \downarrow \land xy \, r \psi)\)
- \(x \, r \forall y^\sigma \varphi := \forall y (\text{HRO}_{\sigma}(y) \rightarrow x \cdot y \downarrow \land xy \, r \varphi)\)
- \(x \, r \exists y^\sigma \varphi := \text{HRO}^\sigma(p_0 x) \land (p_1 \, x \, r \varphi)[p_0 x/y]\)

**Theorem 4.6.** If I-HA\(^\omega\) + AC \(\vdash \varphi(y_1^{\sigma_1}, \ldots, y_n^{\sigma_n})\), then

\[ \text{HAP} \vdash \exists x \forall y_1, \ldots, y_n (\text{HRO}_{\sigma_1}(y_1) \land \ldots \land \text{HRO}_{\sigma_n}(y_n) \rightarrow x \cdot y_1 \cdot \ldots \cdot y_n \, r \varphi). \]

**Proof.** This is proved by induction of the derivation \(\varphi(y_1^{\sigma_1}, \ldots, y_n^{\sigma_n})\) inside I-HA\(^\omega\) + AC. Note that if

\[ t = \lambda \xi. \lambda u. p(\lambda x^\sigma. p_0(ux), \lambda x^\sigma. p_1(ux)), \]

then \text{HAP} \vdash t \, r \text{AC}. \quad \square

**Theorem 4.7.** (Goodman’s Theorem) The system I-HA\(^\omega\) + AC is conservative over HA.

**Proof.** If \(\varphi\) is an arithmetical sentence, we have the following sequence of implications:

\[
\begin{align*}
\text{I-HA}^\omega + \text{AC} \vdash \varphi & \implies \text{Theorem 4.6} \\
\text{HAP} \vdash \exists x \, x \, r \varphi & \implies \text{Theorem 3.5} \\
\text{HAP} \varepsilon \vdash \exists x \, x \, r \varphi & \implies \text{Theorem 2.7} \\
\text{HA} \vdash \varphi & \implies \text{Theorem 2.7}
\end{align*}
\]

\(\square\)

4.2. An extensional version of Goodman’s Theorem. To interpret E-HA\(^\omega\) + AC inside HAP we need to make both HRO and the realizability interpretation from the previous subsection more extensional. As a first step, let us define HEO, an extensional variant of HRO, and indicate how it can be used to interpret E-HA\(^\omega\) inside HAP.

To do this, we still interpret the combinators \(k, s, p, p_0, p_1\) as themselves, while \(r\) interprets \(R\). But we need a different criterion for when an object inside HAP is suitable for representing an object of type \(\sigma\); also, we can no longer use equality in HAP to interpret equality at all finite types. To address this, define by induction on \(\sigma\) the following provably symmetric and
transitive relations in the language of HAP:

\[
\begin{align*}
x \sim_0 y & := x = y \\
x \sim_{\sigma \times \tau} y & := p_0x \sim_{\sigma} p_0y \land p_1x \sim_{\tau} p_1y \\
x \sim_{\sigma \rightarrow \tau} y & := \forall z, z' \left( z \sim_{\sigma} z' \rightarrow x \cdot z \downarrow \land x \cdot z' \downarrow \land y \cdot z \downarrow \land y \cdot z' \downarrow \land x \cdot z \sim_{\tau} y \cdot z \right).
\end{align*}
\]

Now define HEO\(_{\sigma}(x)\) as \(x \sim_{\sigma} x\). This yields an alternative way of selecting elements \(x\) in HAP which are suitable for representing objects of type \(\sigma\). The point is that if we now interpret equality of objects of type \(\sigma\) as \(\sim_{\sigma}\), the result will be an interpretation of E-HA\(^{\omega}\) inside HAP.

To interpret AC as well, we have to combine this with a suitably extensional form of realizability.

**Definition 4.8.** For each HAP\(^{\omega}\)-formula \(\varphi\) with free variables among \(y_1^{\sigma_1}, \ldots, y_n^{\sigma_n}\), we define new HAP-formulas \(x \varepsilon \varphi\) ("\(x\) extensionally realizes \(\varphi\)") and \(x = x' \varepsilon \varphi\) ("\(x\) and \(x'\) are identical as extensional realizers of \(\varphi\)") with free variables among \(x, y_1, \ldots, y_n\) by simultaneous induction on the structure of \(\varphi\) as follows:

\[
\begin{align*}
x \varepsilon \varphi & := \varphi \text{ if } \varphi \text{ is atomic} \\
x = x' \varepsilon \varphi & := x' \varepsilon \varphi \\
x \varepsilon (\varphi \land \psi) & := p_0x \varepsilon \varphi \land p_1x \varepsilon \psi \\
x = x' \varepsilon (\varphi \land \psi) & := p_0x = p_0x' \varepsilon \varphi \land p_1x = p_1x' \varepsilon \psi \\
x \varepsilon (\varphi \rightarrow \psi) & := \forall y, y' (y = y' \varepsilon \varphi \rightarrow xy \downarrow \land xy' \downarrow \land xy = xy' \varepsilon \psi) \\
x = x' \varepsilon (\varphi \rightarrow \psi) & := x' \varepsilon (\varphi \rightarrow \psi) \land x' \varepsilon (\varphi \rightarrow \psi) \land \forall y (y \varepsilon \varphi \rightarrow xy = x' y \varepsilon \psi) \\
x \varepsilon \forall y \varphi & := \forall y, y' (y \sim_{\sigma} y' \rightarrow x \cdot y \downarrow \land x \cdot y' \downarrow \land x \cdot y = x' \cdot y' \varepsilon \varphi) \\
x = x' \varepsilon \forall y \varphi & := x' \varepsilon \forall y \varphi \land x' \varepsilon \forall y \varphi \land \forall y (\text{HEO}_{\sigma}(y) \rightarrow xy = x' y \varepsilon \varphi) \\
x \varepsilon \exists y \varphi & := \text{HEO}_{\sigma}(p_0x) \land (p_1x \varepsilon \varphi)[p_0x/y] \\
x = x' \varepsilon \exists y \varphi & := (p_1x = p_1x' \varepsilon \varphi)[p_0x/y] \land p_0x \sim_{\sigma} p_0x'.
\end{align*}
\]

**Lemma 4.9.** For any HAP\(^{\omega}\)-formula \(\varphi\) we can prove in HAP that:

1. \(x \varepsilon \varphi\) is equivalent to \(x = x \varepsilon \varphi\);
2. the relation \(x = x' \varepsilon \varphi\) is symmetric and transitive;
3. if \(x \varepsilon \varphi\) and \(y \sim_{\sigma} y'\), then \((x \varepsilon \varphi)[y'/y]\);
4. if \(x = x' \varepsilon \varphi\) and \(y \sim_{\sigma} y'\), then \((x = x' \varepsilon \varphi)[y'/y]\).

**Proof.** The idea is to prove the conjunction of (1 – 4) by simultaneous induction on the structure of \(\varphi\). \(\square\)

**Theorem 4.10.** If E-HA\(^{\omega}\) + AC \(\vdash\) \(\varphi(y_1^{\sigma_1}, \ldots, y_n^{\sigma_n})\), then
\[
\text{HAP} \vdash \exists x \forall y_1, \ldots, y_n \left( \text{HEO}_{\sigma_1}(y_1) \land \ldots \land \text{HEO}_{\sigma_n}(y_n) \rightarrow x \cdot y_1 \cdot \ldots \cdot y_n \varepsilon \varphi \right).
\]

**Proof.** Again a straightforward induction on the length of the derivation of \(\varphi\) in E-HA\(^{\omega}\) + AC, with
\[
t = \lambda x. \lambda u. p(\lambda x^\sigma. p_0(ux), \lambda x^\sigma. p_1(ux))
\]
still realizing AC. \(\square\)

**Theorem 4.11.** (Beeson’s extensional version of Goodman’s Theorem) The system E-HA\(^{\omega}\) + AC is conservative over HA.
Proof. If \( \varphi \) is an arithmetical sentence, we have the following sequence of implications:

\[
\begin{align*}
\text{E-HA}^\omega + \text{AC} & \vdash \varphi \quad \implies \quad \text{(Theorem 4.10)} \\
\text{HAP} & \vdash \exists x \, \exists x \, \varphi \quad \implies \quad \text{(Theorem 3.9)} \\
\text{HAP}_e & \vdash \exists x \, \exists x \, \varphi \quad \implies \quad \text{(Theorem 2.7)} \\
\text{HA} & \vdash \varphi.
\end{align*}
\]

\( \square \)

4.3. Applications to classical systems. Goodman’s Theorem has interesting consequences for classical systems as well, as we will now explain. (The results in this subsection were pointed out to us by Ulrich Kohlenbach, answering a question by Fernando Ferreira.

In what follows we will call a formula in the language of \( \text{HA}^\omega \) quantifier-free if it contains no quantifiers and no equalities of higher type (hence such a formula is built from equalities of type 0 and the propositional operations \( \land, \lor, \to \)). The quantifier-free axiom of choice, denoted by \( \text{QF-AC} \), is the schema

\[
\forall x \sigma \exists y \tau \varphi(x, y, z) \rightarrow \exists f \sigma \to \tau \forall x \sigma \varphi(x, f x, z),
\]

where \( \varphi \) is assumed to be quantifier-free.

**Theorem 4.12.** (Kohlenbach) \( \text{I-PA}^\omega + \text{QF-AC} \) is conservative over \( \text{PA} \).

**Proof.** (Compare [11, Theorem 4.1].) Suppose \( \varphi \) is an arithmetical sentence provable in \( \text{I-PA}^\omega + \text{QF-AC} \). Without loss of generality, we may assume that \( \varphi \) is in prenex normal form:

\[
\varphi := \exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \varphi_{qf}(x_1, y_1, \ldots, x_n, y_n),
\]

with \( \varphi_{qf} \) quantifier-free. If

\[
\text{I-PA}^\omega + \text{QF-AC} \vdash \varphi,
\]

then also

\[
\text{I-PA}^\omega + \text{QF-AC} \vdash \varphi^H,
\]

where

\[
\varphi^H := \forall f_1, \ldots, f_n \exists x_1, \ldots, \exists x_n \varphi_{qf}(x_1, f_1(x_1), \ldots, x_n, f_n(x_1, \ldots, x_n))
\]

is the Herbrand normal form of \( \varphi \). By combining negative translation and the Dialectica interpretation (that is, the Shoenfield interpretation), it follows that

\[
\text{I-HA}^\omega \vdash \varphi^H
\]

as well. But then

\[
\text{I-HA}^\omega \vdash \neg \exists f_1, \ldots, f_n \forall x_1, \ldots, x_n \neg \varphi_{qf}(x_1, f_1(x_1), \ldots, x_n, f_n(x_1, \ldots, x_n)),
\]

and therefore

\[
\text{I-HA}^\omega + \text{AC} \vdash \neg \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \neg \varphi_{qf}(x_1, y_1, \ldots, x_n, y_n).
\]

By Goodman’s Theorem we obtain

\[
\text{HA} \vdash \neg \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \neg \varphi_{qf}(x_1, y_1, \ldots, x_n, y_n),
\]

and therefore \( \text{PA} \vdash \varphi. \)

\( \square \)

**Theorem 4.13.** (Kohlenbach) The system \( \text{E-PA}^\omega + \text{QF-AC} \) is conservative over \( \text{PA} \).
Proof. One can formalise the ECF-model of E-PAω + QF-AC inside PAω + QF-AC (see [25, Theorem 2.6.20]). This interpretation does not affect the meaning of statements mentioning only objects of type 0 and 1 = (0 → 0), and therefore E-PAω + QF-AC is conservative over PAω + QF-AC for statements of this type. So this theorem follows from the previous. □

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