The asymmetric $ABAB$ matrix model

P. Zinn-Justin

Laboratoire de Physique Théorique et Modèles Statistiques
Université Paris-Sud, Bâtiment 100
F-91405 Orsay Cedex, France

In this letter, it is pointed out that the two matrix model defined by the action $S = \frac{1}{2}(\text{tr} A^2 + \text{tr} B^2) - \frac{\alpha_A}{4} \text{tr} A^4 - \frac{\alpha_B}{4} \text{tr} B^4 - \frac{\beta}{2} \text{tr}(AB)^2$ can be solved in the large $N$ limit using a generalization of the solution of Kazakov and Zinn-Justin (who considered the symmetric case $\alpha_A = \alpha_B$). This model could have useful applications to 3D Lorentzian gravity.
1. Introduction

There are a variety of motivations to study the large \( N \) limit of matrix integrals, from statistical mechanics on random lattices and graph enumeration \([1]\) to quantum chaos and mesoscopic systems \([2]\). In \([3]\), the so-called \( ABAB \) model was introduced, as an example of model exhibiting the critical behavior of \( c = 1 \) conformal field theory coupled to gravity. It has the partition function

\[
Z(\alpha, \beta) = \int \int dA dB \exp N \left[ -\frac{1}{2} (\text{tr} A^2 + \text{tr} B^2) + \frac{\alpha}{4} (\text{tr} A^4 + \text{tr} B^4) + \frac{\beta}{2} \text{tr}(AB)^2 \right] \tag{1.1}
\]

The special form of the interaction between the two matrices, \( \text{tr} ABAB \), which is different from the usual two-matrix model \([4]\), makes it impossible to compute this partition function using the standard method – direct diagonalization of \( A \) and \( B \). Instead a new method was proposed, combining several techniques which had appeared in previous articles, including character expansion \([5]\), saddle point over Young diagrams \([6]\) and functional inversion relations \([7]\). It led to an exact expression for the “planar” (large \( N \)) free energy in terms of elliptic functions. Subsequently, this model was used in several contexts: 2D Eulerian gravity \([8]\), knot theory \([9]\) and even 3D Lorentzian gravity \([10]\). This prompted the author to consider the following natural generalization:

\[
Z(\alpha_A, \alpha_B, \beta) = \int \int dA dB \exp N \left[ -\frac{1}{2} (\text{tr} A^2 + \text{tr} B^2) + \frac{\alpha_A}{4} \text{tr} A^4 + \frac{\alpha_B}{4} \text{tr} B^4 + \frac{\beta}{2} \text{tr}(AB)^2 \right] \tag{1.2}
\]

in which the coefficients of the quartic terms are completely arbitrary. The Feynman rules of this model are as follows: we form fat graphs with edges in two colors (red and green) and quartic vertices, in such a way that colors can only cross (each other, or themselves) at each vertex. Red (resp. green, mixed) vertices are assigned a weight of \( \alpha_A \) (resp. \( \alpha_B, \beta \)). Alternatively, let us note that if one sets \( X = \frac{A + iB}{\sqrt{2}} \), then one obtains

\[
Z(b, c, d, e) = \int dX dX^\dagger \exp N \left[ -\text{tr}(XX^\dagger) + b \text{tr}(X^2 X^\dagger^2) + \frac{c}{2} \text{tr}(XX^\dagger)^2 \right. \\
+ \left. \frac{d}{4} \text{tr}(X^4 + X^\dagger^4) + e \text{tr}(XX^\dagger^3 + X^\dagger X^3) \right] \tag{1.3}
\]

with \( b = (\alpha_A + \alpha_B + 2\beta)/4 \), \( c = (\alpha_A + \alpha_B - 2\beta)/4 \) and \( e = (\alpha_A - \alpha_B)/4 \). It corresponds to a perturbation of the symmetric case \( \alpha_A = \alpha_B \) (which describes an 8-vertex model) with the extra operators \( X^3 X^\dagger + \text{h.c.} \), turning it into a 16-vertex model on random dynamical lattices. We shall come back in the concluding section to possible applications of this model. For now, let us show how, despite the more complicated analytic structure than in the symmetric case, one can solve it.
2. Character expansion and saddle point equations

As in [3], we immediately transform the expression (1.2) by means of a character expansion of the term \( \exp[N \frac{A}{2} \text{tr}(AB)^2] \) as a function of \( AB \). We obtain:

\[
Z(\alpha_A, \alpha_B, \beta) = \sum_h (N\beta/2)^{\#h/2} \frac{\Delta(h^{\text{even}}/2)}{\prod_i (h_i^{\text{even}}/2)!} \frac{\Delta((h_i^{\text{odd}}-1)/2)}{\prod_i ((h_i^{\text{odd}}-1)/2)!} \int \int dA dB \exp \left[ -\frac{1}{2} (\text{tr} A^2 + \text{tr} B^2) + \frac{\alpha_A}{4} \text{tr} A^4 + \frac{\alpha_B}{4} \text{tr} B^4 \right] \chi_h(AB)
\]

where the sum is over a set \( h = \{h_1, \ldots, h_N\} \) of integers that satisfy \( h_1 > h_2 > \ldots > h_N \geq 0 \), and \( \#h = \sum h_i - \frac{N(N-1)}{2} \) is the number of boxes of the Young diagram. \( \Delta(\cdot) \) is the Van der Monde determinant, \( \chi_h \) is the \( GL(N) \) character associated to the set of shifted highest weights \( h \), and the \( h_i^{\text{even/odd}} \) are the even/odd \( h_i \), which must be in equal numbers. It is now possible, using character orthogonality relations, to integrate over the relative angle between \( A \) and \( B \); this leads to a separation into one-matrix integrals:

\[
Z(\alpha_A, \alpha_B, \beta) = \sum_h (N\beta/2)^{\#h/2} c_h R_h(\alpha_A) R_h(\alpha_B)
\]

where \( c_h \) is a coefficient: \( c_h = \frac{1}{\prod_i (h_i/2)! \prod_{i,j} (h_i^{\text{even}} - h_j^{\text{odd}})} \), and \( R_h(\alpha) \) is the one-matrix integral

\[
R_h(\alpha) = \int dM \chi_h(M) \exp \left[ -\frac{1}{2} \text{tr} M^2 + \frac{\alpha}{4} \text{tr} M^4 \right]
\]

which was studied in detail in [3].

We can now consider the large \( N \) limit: the summation over the \( h_j \) is dominated by a saddle point which is characterized by a continuous density \( \rho(h) \) of the rescaled variables \( h_j/N \). There is generically a saturated region \([0, h_-]\) where \( \rho(h) = 1 \) and an unsaturated region \([h_-, h_+]\) where \( 0 < \rho(h) < 1 \). In order to write down the saddle point equation we need to differentiate with respect to the \( h_i \); we denote by \( \partial/\partial h \) the operation \((1/N)\partial/\partial h_i\) in the large \( N \) limit with \( h_i/N = h \). Then we have \( 2(\partial/\partial h) \log R_h(\alpha_A) = \mathcal{L}(h), 2(\partial/\partial h) \log R_h(\alpha_B) = \mathcal{M}(h), (\partial/\partial h) \log (c_h/N^{\#h/2}) = -\mathcal{H}(h) + \log \frac{h}{\beta}/2 \), where the slash notation means \( \mathcal{D}(h) = \frac{1}{2} (\mathcal{D}(h+i0) + \mathcal{D}(h-i0)) \) for any analytic function \( \mathcal{D}(h) \) with a cut on the real axis, \( \mathcal{L}(h) \) and \( \mathcal{M}(h) \) are unknown functions whose analytic structure will be discussed below, and \( H(h) = \lim_{N \to \infty} \sum_i \frac{1}{h-h_i} = \int \frac{dh'}{h-h'} \rho(h') \). This results in the following saddle point equation:

\[
\mathcal{L}(h) + \mathcal{M}(h) - \mathcal{H}(h) = \log \frac{h}{\beta} \quad h \in [h_1, h_2]
\]
3. Analytic structure and solution

From the study of $R_h(\alpha)$ performed in [3] and that we shall not repeat here, it follows that the function $L(h)$ defined by $2(\partial/\partial h) \log R_h(\alpha_A) = \mathcal{L}(h)$ on $[h_1, h_2]$ and by the fact that it has the same cut as $H(h)$ on $[h_1, h_2]$, has the following analytic structure: besides a logarithmic cut starting at $h = 0$ and the cut of $H(h)$ on $[h_1, h_2]$ (i.e. discontinuity of $2\pi i \rho(h)$), it has a semi-infinite square root cut starting at $h = h_A$, which we choose to be $[h_A, +\infty]$. Similarly, $M(h)$ defined by replacing $\alpha_A$ with $\alpha_B$ has a cut starting at $h = h_B$. Furthermore, they satisfy the following equations:

$$2\mathcal{L}(h) = H(h) + \log \frac{h}{\alpha_A} \quad h \in [h_A, +\infty]$$  \hspace{1cm} (3.1a)

$$2\mathcal{M}(h) = H(h) + \log \frac{h}{\alpha_B} \quad h \in [h_B, +\infty]$$  \hspace{1cm} (3.1b)

Now define the function $F(h) = L(h) + M(h) - H(h) - \log h$. This is a rather complicated function since it has 3 square root cuts leading to other sheets. The equations (2.4) and (3.1) define the monodromy of $F(h)$ around the branch points $h_\pm$, $h_A$ and $h_B$. If one differentiates once $F(h)$, the inhomogeneities in Eqs. (2.4) and (3.1) disappear (as well as the logarithmic cut) and we find that fortunately, $F'(h)$ has only 4 sheets: the original sheet $F'(h)$, its opposite $-F'(h)$ by going through $[h_1, h_2]$, and $\pm G'(h)$ by going around $h_A$ or $h_B$, with $G(h) = L(h) - M(h)$. Therefore it is a meromorphic function of $u$, where

$$h - \frac{h_A + h_B}{2} + u^2 + \frac{(h_A - h_B)^2}{u^2} = 0$$ \hspace{1cm} (3.2)

This shows that the problem is solvable. In practice, to stay closer to the solution of [3], we introduce $z = -u^2 = \frac{1}{2}(h - \frac{h_A + h_B}{2} \pm \sqrt{(h - h_A)(h - h_B)})$, with a cut on $[h_A, h_B]$, the $+$ sign being for the sheet of $\pm F$, the $-$ sign being for $\pm G$; also introduce the images $z_\pm$, $z_0$, $z_A$, $z_B$ on the $F$ sheet of $h_\pm$, $0$, $h_A$, $h_B$. Then the function $F(z)$ (which combines $F(h)$ and $G(h)$) has square root cuts on $[z_-, z_+]$ and $[0, +\infty]$ and satisfies

$$F(z) = -\log \beta \quad z \in [z_-, z_+]$$ \hspace{1cm} (3.3a)

$$F(z) = -\frac{1}{2} \log(\alpha_A \alpha_B) \quad z \in [0, +\infty]$$ \hspace{1cm} (3.3b)

Taking into account the extra logarithmic cut on $[z_0, z_-]$, we find that $F(z)$ is of the form:

$$F(z) = -\Phi_{z_-}(z) + 2\Phi_{z_0}(z) - \frac{\log(\beta^2/\alpha_A \alpha_B)}{2\pi i} \Phi_0(z)$$ \hspace{1cm} (3.4)
where \( \Phi_\tilde{z}(z) \) is the incomplete elliptic integral of the third kind, which can be expressed explicitly as follows: define \( k^2 = z_+ / z_- \), the complete elliptic integrals \( K, K', E, E' \), and the parameterization \( \text{sn}^2(x, k) = z_- / z \). Then \( \Phi_\tilde{z}(z) = \tilde{x}(2Z_1(x) - \frac{i\pi}{K}) + \log \frac{\Theta_1(x-\tilde{x})}{\Theta_1(x+\tilde{x})} \), where \( \Theta_a (a = 1, 2, 3, 4) \) is a rescaled theta function and \( Z_a \) its logarithmic derivative.

The parameters \( h_\pm, h_A, h_B \) are fixed by computing the asymptotic behavior of \( F(z) \) as \( z \to 0, z \to \infty \) and the normalization of the density \( \int dh \rho(h) \). This calculation is similar (though more complicated) than what is performed in [3] and we skip the details.

First, based on the known asymptotic behavior \( L(h) = \frac{1}{2} \log \frac{\beta^2}{\alpha_A \alpha_B} - \frac{1}{2\sqrt{-\alpha_A \alpha_B}} + \frac{1}{2} + O(h^{-3/2}) \) (and similarly for \( M(h) \)) we obtain 3 conditions as \( z \to \infty, 0 \):

\[
x_0 = \frac{K}{2} + \frac{\log(\beta^2/\alpha_A \alpha_B)}{4\pi} K' \tag{3.5a}
\]

\[
\Omega_1 = \frac{1}{2\sqrt{-z_+}} (\alpha_{-1/2}^A + \alpha_{-1/2}^B) \tag{3.5b}
\]

\[
\Omega_4 = 2\sqrt{-z_+} \frac{\alpha_{-1/2}^A - \alpha_{-1/2}^B}{h_A - h_B} \tag{3.5c}
\]

where

\[
\Omega_a = \frac{\log(\beta^2/\alpha_A \alpha_B)}{2K} + 4Z_a(x_0) \tag{3.6}
\]

and \( x_0 \) is defined by \( \text{sn}^2(x_0, k) = z_- / z_0 \).

The normalization of the density, \( \int dh \rho(h) = 1 \), after integration by parts, is essentially a complete elliptic integral of the second kind:

\[
z_0 K' \Omega_4 - z_- E' \Omega_1 + \frac{z_A^2}{z_0} K' \Omega_4 - \frac{z_A^2}{z_- k^2} E' \Omega_4 = \pi \tag{3.7}
\]

Equations (3.5)–(3.7) provide an explicit parameterization of the quantities of the model in terms of \( k \) and ratios of coupling constants, such as \( \beta^2/\alpha_A \alpha_B \) and \( \alpha_A/\alpha_B \). For example, we find

\[
\alpha_A = \frac{1}{\pi} \left( \frac{E' \Omega_1 - K' \Omega_4 / \text{sn}^2 x_0}{4 \Omega_4^2} \right) \left( 1 + \sqrt{\frac{\alpha_A}{\alpha_B}} \right)^2 + \frac{E' \Omega_4 - k^2 K' \Omega_1 \text{sn}^2 x_0}{4 \Omega_4^2} \left( 1 - \sqrt{\frac{\alpha_A}{\alpha_B}} \right)^2 \tag{3.8}
\]

where we recall that \( x_0 \) and \( \Omega_a \) are known via Eqs. (3.5a) and (3.6).

From the function \( F(z) \) one can derive many correlation functions, including \( \langle \frac{\text{tr}}{N}(AB)^2 \rangle \) and \( \langle \frac{\text{tr}}{N} A^n \rangle \), but the formulae are quite cumbersome and are not reproduced here.
4. Phase diagram

Ordinary criticality, which in the context of random dynamical graphs means the emergence of very large graphs i.e. the continuum limit, appears when one of the square root singularities at $h = h_A$, $h = h_B$, $h = h_+$ degenerate into power $3/2$ singularities. Thus, there are three phases denoted by $\text{IA}$, $\text{IB}$ and $\text{II}$, which all correspond to pure gravity ($c = 0$). Criticality can easily be tested numerically and the result is Fig. 1. On this two-dimensional critical surface, there are critical lines corresponding to the transition between two of these phases and where the corresponding branch points coalesce; and these three lines meet at a critical point where all three branch points merge.

![Critical surface](image)

**Fig. 1:** Critical surface in the $(\alpha_A, \alpha_B, \beta)$ space.

The easiest transition to analyze is $\text{IA}/\text{IB}$: $h_A = h_B$ implies $\alpha_A = \alpha_B$ and we go back to the symmetric model. According to the analysis of [3], at this transition line the model is still pure gravity. More interesting is the $\text{IA}/\text{II}$ transition. It is given by the condition $h_A = h_+$: note that contrary to the symmetric case, $h_A = h_+$ does not imply that $k \to 0$ (trigonometric limit of the elliptic functions). However this condition still produces a degenerate analytic structure where two square root branch points turn into a cubic branch point: this is typical of a $c = 1/2$ (i.e. critical Ising) theory. This is confirmed by the analysis of the vicinity of the critical point that comes now (also, based on [11] it is reasonable to assume that the particular point $\alpha_B = 0$ of this line is the critical point of a modified $O(1)$ model, which is known to have $c = 1/2$).

Indeed, the critical point $\alpha_A = \alpha_B = \beta = \frac{1}{4\pi}$ was already studied in [3]. It describes a $c = 1$ model – a free boson coupled to gravity. However the vicinity, in the 3-dimensional
parameter space, displays a rather complex behavior since 3 critical lines merge there. Let us briefly analyze it. We define the following deviation from criticality: \( \Delta = 1 - \frac{1}{4\sqrt{n}}(\alpha_A^{-1/2} + \alpha_B^{-1/2}) \). The critical point corresponds to the trigonometric limit where the elliptic nome \( q \) goes to zero. By small \( q \) expansion of Eqs. (3.5)–(3.7), we can express \( \Delta \) as a function of \( q \) and of \( s = \frac{\log(\beta^2/\alpha_A\alpha_B)}{8\pi} \) and \( t = (\pi\alpha_A)^{-1/2} - (\pi\alpha_B)^{-1/2} \); differentiating, we obtain
\[
\frac{d}{dq}\Delta = \frac{\log q}{32q} \frac{s - 2q}{(s + 2q)^3} (t^2 q - 64(s + 2q)^4) + \cdots
\]
This shows the following scaling behaviors: \( s \propto q \) and \( t \propto q^{3/2} \), as well as defines the phases in the vicinity of the critical point: \( s = 2q \) for phase II, \( t = \pm 8q^{-1/2}(s + 2q)^2 \) for phases IA/IB. The line separating IA and II is therefore given by \( s = 2q, t = 128q^{3/2} \).

Computing \( \Delta \) on the critical surface allows to conclude that transition IA/IB is first order, whereas transitions IA/II and IB/II are third order.

Finally, a typical correlation function \( \Gamma \) follows the same pattern as \( \Delta \) itself. In particular on a generic point of the critical surface, \( \frac{d}{dq}\Gamma = 0 \) and therefore \( \Gamma = \Delta^{3/2} \) regular, which confirms that the central charge \( c \) is zero. However at the line separating IA and II one easily finds \( \frac{d^2}{dq^2}\Delta = \frac{d^2}{dq^2}\Gamma = 0 \) and therefore \( \Gamma = \Delta^{4/3} \) regular which according to the KPZ relation \[12\] means that \( c = 1/2 \).

## 5. Concluding comments

The asymmetric \( ABAB \) model which we have just solved is enlightening in several respects. First it is an interesting statistical model with a rich phase diagram which might deserve further investigation. But it is also, at a technical level, the first solution of a matrix model in which we have a non-trivial three cut structure (the multi-cut phases of standard matrix models usually have a very simple – e.g. hyperelliptic – structure). It can be hoped that for many other matrix models, the seemingly complicated monodromy equations close and allow to provide an exact solution. Finally, at the physical level we expect that the asymmetric \( ABAB \) model may be of relevance to three-dimensional Lorentzian gravity. Indeed, it is precisely the model that was introduced in \[10\] (cf Eq. (23)) to describe time slices of three-dimensional geometries. We hope that the exact solution can shed some light on some of the remaining unsolved issues.
References

[1] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, Phys. Rep. 254 (1995) 1–133.
[2] T. Guhr, A. Mueller-Groening and H.A. Weidenmueller, Phys. Rep. 299 (1998), 189–425 (cond-mat/9707301).
[3] V.A. Kazakov and P. Zinn-Justin, Nucl. Phys. B 546 (1999) 647 (hep-th/9808043).
[4] C. Itzykson and J.-B. Zuber, J. Math. Phys. 21 (1980), 411.
[5] P. Di Francesco and C. Itzykson, Ann. Inst. Henri Poincaré 59, no. 2 (1993), 117.
[6] M.R. Douglas and V.A. Kazakov, Phys. Lett. B319 (1993), 219 (hep-th/9305047).
[7] P. Zinn-Justin, Commun. Math. Phys. 194 (1998), 631 (cond-mat/9705044).
[8] P. Di Francesco, E. Guitter and C. Kristjansen, Nucl. Phys. B549 (1999), 657–667 (cond-mat/9902082).
[9] P. Zinn-Justin, proceedings of the 1999 semester of the MSRI “Random Matrices and their Applications”, MSRI Publications Vol. 40 (2001) (math-ph/9910010).
[10] J. Ambjorn, J. Jurkiewicz, R. Loll and G. Vernizzi, JHEP 0109 (2001) 022 (hep-th/0106082).
[11] L. Chekhov and C. Kristjansen, Nucl. Phys. B479 (1996), 683 (hep-th/9605013).
[12] V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, Mod. Phys. Lett. A 3, 819–826 (1988); F. David, Mod. Phys. Lett. A 3, 1651–1656 (1988); J. Distler and H. Kawai, Nucl. Phys. B 321, 509 (1989).