Mapping the $q$-voter model: From a single chain to complex networks

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(Dated: January 27, 2015)

We propose and compare six different ways of mapping the $q$-voter model to complex networks. Considering square lattices, Barabasi-Albert, Watts-Strogatz and real Twitter networks, we ask the question whether a particular choice of the influence group of a fixed size $q$ leads to different behavior on the macroscopic level. Using Monte Carlo simulations we show that the answer depends strongly on the relative average path length of the network and for real-life topologies the differences between the considered mappings may be negligible.

PACS numbers: 64.60.Ht, 05.70.Ln, 05.50.+q

I. INTRODUCTION

Ordering dynamics is not only a classical subject of non-equilibrium statistical physics [1, 2] but also one of the most studied issues in a field of sociophysics [3–5]. It often represents opinion dynamics under the most common type of the social influence, known as conformity. A general model of opinion dynamics was introduced by Castellano et al. [6] under the name $q$-voter model. Originally the model was defined on the one-dimensional lattice. Without going into details, its basic feature is the fact that only an unanimous group of $q$ neighbors can influence a voter. Therefore Sznajd and the linear voter model are particular cases of this model corresponding to $q = 2$ and $q = 1$, respectively.

For the one-dimensional lattice quite a few results has been already obtained both numerically, as well as analytically using several approximate methods [6–13]. Moreover, heterogeneous mean field approach (MFA) has been used to study $q$-voter on random regular networks [2]. Also in the case of a complete graph the definition of the $q$-voter model was straightforward since on this topology all spins are neighbors [14, 15].

In the context of opinion dynamics it would be however desirable to consider the models on top of more complex networks, as they are better representations of contact patterns observed in social systems [16, 17]. There are already several attempts to generalize the $q$-voter model to complex networks [18–21]. However, even in the simple case of transferring the model from 1D chain to a 2D square lattice there is no unique rule of choosing the influence group [22]. Thus, the main goal of this paper is to investigate, how different ways of picking up the group may impact the macroscopic behavior of the model.

II. THE MODEL

Within the $q$-voter model we consider a set of $N$ agents called spinsons [15]. Each spinson has an opinion on some issue that at any given time can take one of two values $S_i = \pm 1$ (“up” and “down”). The opinion of a spinson may be changed under the influence of his neighbors according to two different types of social influence [23]:

- **Independence**, i.e. unwillingness to yield to group pressure. It introduces indetermination in the system through an autonomous behavior of the spinsons [24].
- **Conformity** is the act of matching spinson’s opinion to a group norm. It is motivated by the psychological observations of the social impact [23].

Other types of social influence are possible as well [15], but the above two are of particular interest for studying opinion dynamics.

We study the model by means of Monte Carlo (MC) simulations with a random sequential updating scheme. Each MC step consists of $N$ elementary events, each of which may be divided into the following steps: (1) pick a spinson at random, (2) decide with probability $p$, if the spinson will act independent, (3) if independent, change its opinion with probability $1/2$, (4) if not independent, let the spinson take the opinion of its randomly chosen influence group, provided the group is unanimous. More details on the dynamic rules of the model may be found in Ref. [26].

Introduction of a group pressure as one of the rules governing the dynamics assumes some form of interactions between the spinsons. Those interactions are best illustrated as connections between nodes of a graph the spinsons are living on. Until now the $q$-voter model and its modifications were analyzed on the 1D lattice [6–13], random regular networks [2] and complete graph [14, 15]. Many results has been obtained for the Sznajd model on the square lattice which corresponds to $q$-voter model with $q = 4$ [3, 4, 22, 24].
However, it is not clear how to define the $q$-voter model on arbitrary graph. We use here both the Watts-Strogatz (WS) [27] and the Barabasi-Albert (BA) networks [28] as the underlying topology of spinson-spinson interactions, since they nicely recover the small world property of many real social systems [29]. We set $q = 4$ for two reasons: (1) to reflect the empirically observed fact that a group of four individuals sharing the same opinion has a high chance to ‘convince’ the fifth, even if no rational arguments are available [25, 30] and (2) to compare results with those obtained on the square lattice.

Chosen from the plethora of possibilities, in Fig. 1 six different groups of influence on a complex network are schematically shown. They were built in the following way:

- **Line** - after picking up a random target spinson (marked with double red circle in the figure), we randomly choose one of his neighbors, then one of the neighbors of the neighbor and finally a neighbor of the latter one. This is the natural generalization of the 1D $q$-voter model and was used e.g. in Ref. [20].

- **Block** - the group consists of a random neighbor of the target spinson, and three neighbors of the neighbor. This method resembles to some extent the $2 \times 2$-block used on square lattices in 2D and was used for instance in Ref. [21].

- **NN** - four randomly chosen nearest neighbors of the target spinson are in the group, as in the original formulation of the $q$-voter model [3].

- **NN3** - this is a slight modification of the NN method leading to an extended range of influence: the group is composed of three randomly chosen nearest neighbors of the target spinson, and a neighbor of one of those nearest neighbors.

- **RandBlock** - a spinson and its three neighbors build the influence group as in the Block method. However, the block may be located anywhere on the network.

- **Rand** - the group consists of four randomly chosen spinsons, not necessarily connected with the target spinson.

Note that in case of RandBlock and Rand methods we actually abstract away from the underlying network topology of the model. We expect both methods to be equivalent to the complete graph case if the minimum degree of a node in the network is bigger than or equal to $q = 4$ (otherwise the RandBlock may differ slightly from the complete graph, because there will be not always enough spinsons in the neighborhood to build the influence group). Although we use Rand mostly as a benchmark for our simulations, the RandBlock is much more interesting because it corresponds to a situation often encountered in many organizations, in which an informal and unknown network of interactions is over imposed on the given formal communication structure.

### III. RESULTS

Our goal is to check if and how microscopic details of the dynamics manifest at the macroscopic scale. Among macroscopic phenomena, phase transitions are certainly the most interesting ones. For the models of opinion dynamics, the most natural order parameter is an average...
opinion $m$, defined as magnetization i.e. $m = \frac{1}{N} \sum S_i$. In the case of a $q$-voter model the phase transition may be induced by the independence factor $p$. Below the critical value of independence, $p < p_c$, the majority coexists with the minority, i.e. the public opinion $m \neq 0$. For higher independence, $p > p_c$, there is a stalemate situation in the society, i.e. $m = 0$. Such results were obtained on the complete graph which corresponds to the mean field approach, as well as on the square lattice but only for the one particular choice of the influence panel which corresponds to the Sznajd model. Here we test six different methods described in the previous section (left plot in Fig. 2). The phase transition is observed for all methods except of $NN$. As expected, $RandBlock$ and $Block$ methods agree with MFA result $p_c(q) = (q - 1)/(q - 1 + 2q^{-1})$ which for $q = 4$ gives $p_c = 3/11$. Methods for which range of interaction is shorter tend to show lower critical value of $p$. This result is very intuitive, since infinite range of interactions usually correspond to MFA and gives the largest critical value.

Results on the BA network (right plot in Fig. 2) are less intuitive. The phase transition is observed for all six models. However, the differences between $Block$, $Line$, $RandBlock$ and $Rand$ models are negligible (corresponding curves collapse into a single one). Only the $NN$ and $NN3$ models may be distinguished from the others and are characterized by smaller critical values of the parameter $p$.

To explain the differences between topologies, recall that average path length $l$ for the square lattice is given by $l \sim l_{\text{rel}} = \ln N / \ln \ln N$. Thus, for the same system size the average path of BA networks is dramatically shorter than that of the square lattice. In result the range of interactions on BA is essentially much larger and this is the reason for the differences observed in Fig. 2.

WS model allows to control the average path length of a network of a fixed size via the rewiring probability $\beta$. Therefore we have simulated all 6 methods on WS networks. Results for several values of $\beta$ are presented in Fig. 3. As $\beta$ increases ($l$ decreases), the critical point $p_c$ shifts towards higher values and simultaneously differences between $Block$, $Line$, $RandBlock$ and $Rand$ methods are getting smaller and finally vanish at a threshold value $\beta = 0.5$. Above the threshold the results does not change any more. These results are consistent with our hypothesis that the average path length (or its relationship to the interaction range) may give some insight in the differences and similarities among the methods.

However, as it follows from Fig. 4 the results are virtually independent on the network size. Since the average path increases with the network size, the results forced us to reconsider our hypothesis. After some investigations we found that in case of networks with different sizes, a relative average path length, i.e. the average path of a network divided by the average path of a corresponding random graph, is consistent with all our results. For instance, for WS networks shown in Fig. 4 the relative lengths are $l_{\text{rel}}(N = 500) = 1.654$ and $l_{\text{rel}}(N = 1000) = 1.677$, i.e. they are indeed almost size independent. For BA network of size $N = 500$ it is $l_{\text{rel}} = 0.974$. The relative path lengths of WS networks in Fig. 3 range from $l_{\text{rel}}(\beta = 0.05) = 1.654$ to $l_{\text{rel}}(\beta = 0.5) = 1.02$. This explains why results for BA (right plot in Fig. 2) and WS with $\beta = 0.5$ (Fig. 3) are almost identical.

For the sake of comparison, we run our model on some real networks as well. Results are shown in Fig. 5. In the top row we see the final magnetization $m$ as a function of independence $p$ for ego graphs of three Twitter users, in the bottom one - three artificial networks chosen to match either the average path length or the clustering coefficient of the ego graphs. All networks are of the same size, but differ in the degree distributions. Again, the results are consistent with our hypothesis - we obtain almost identical curves for networks with different topologies and/or clustering coefficients, provided they have similar path lengths. Moreover, if the path length is high compared to the interaction length, the
methods differs significantly from each other.

Some of the above results may be immediately understood within the statistical physics. Rand method for instance is equivalent to the complete graph and hence to the mean-field approach \cite{14}. A correspondence between RandBlock and mean-field approach is more subtle, but still rather clear - this method introduces an "infinite" range of interaction. And we know from statistical physics that the mean field approach should give exact results in case of such interactions.

To explain the differences between some methods, first note that in a network with an average degree \( \langle k \rangle \), every node has on average \( \langle k \rangle \) neighbors at the first level of its ego graph, \( \langle k \rangle^2 \) agents at the second one and so forth. Now, let us look at an early stage of a simulation and assume, that there are only two spinsons in the “down” state and that one of these spinsons has been chosen again in a basic MC event. However this time it is not independent. If the other “down” spinson lives at the first level of the ego graph, the probability of finding a non-unanimous group of influence containing it (i.e. of maintaining disorder in the system) is highest for the NN method, because in this case we draw 4 agents out of \( \langle k \rangle \). It is slightly smaller in NN3, and much smaller for other methods. If the other spinson lives at the second or higher level, the probabilities to find a non-unanimous group are in general much lower, because now the group is drawn from a much bigger number of agents. For this reason the methods NN and NN3 destroy the order in the systems the fastest, i.e. for relative small values of the independence parameter \( p \). A short relative path length means, that for each node most agents resides at very few levels of its ego graph. In this case: (1) it is much easier to maintain disorder locally, in the nearest neighborhood, (2) differences between methods that operate beyond the nearest neighborhood are negligible, because already the second level of an ego graph is high populated, leading to small probabilities of finding non-unanimous groups at the beginning of simulations.

IV. CONCLUSIONS

From physical point of view it is always an interesting question how microscopic details manifest at the macroscopic level. In the field of opinion dynamics such a macroscopic quantity is opinion, defined in a case of binary models as a magnetization \( m \). We examined six models that differ only in the way of selecting a group of influence but the size of this group remains fixed. We have found that: (1) methods with a shorter interaction range tend to show lower critical value of \( p \), below which \( m > 0 \); (2) different networks with (almost) the same relative path lengths reveal the same characteristics; (3) if the relative path length is around one, most of the methods do not differ from each other. Since many real-world networks are characterized by relatively short paths \cite{16}, for them the differences between the mappings are to a large extent are negligible.

Acknowledgments

This work was supported by funds from the National Science Centre (NCN) through grant no. 2013/11/B/HS4/01061.

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FIG. 5: Magnetization $m$ as a function of the independence $p$ on different networks of size $N = 233$. Top: real Twitter ego networks [31]. Bottom: BA and WS networks with different average path lengths and clustering coefficients.

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