Mathematical modeling of monochromatic acoustic wave diffraction on a system of bodies and on flat screens

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Abstract. Some problems of diffraction of a monochromatic acoustic wave on surfaces of complex shapes are considered. To solve such problems, an approach is applied, in which the problem is reduced to a boundary hypersingular integral equation, where the integral is understood in the sense of a finite value according to Hadamard. Such approach allows solving diffraction problems both on solid objects and on thin screens. To solve the integral equation, the method of piecewise constant approximations and collocations, developed in the previous works of the author, is used. In the present study, examples of modeling the diffraction of an acoustic wave by bodies with partial filling are given. It is shown how the filling of bodies influences the acoustic pressure field, and the field direction patterns are given. An example of applying this approach to solving the problem of sound propagation in an urban area is also given: the diffraction of an acoustic wave from a point source on a system of buildings is considered. The presented results demonstrate that this method allows constructing reflected fields and analyze their characteristics on surfaces of complex shapes.

1. Introduction
Mathematical modeling of acoustic wave diffraction is a topical problem. Such problem can arise both when simulating complex antenna systems and when simulating noise propagation from a source.

To solve such problems, Fredholm integral equations of the second kind are usually used [1]. However, a serious disadvantage of this approach is that there is no possibility to solve problems on thin screens, where the boundary conditions should be fulfilled on both sides of the screen.

In order to be able to solve the problem of acoustic wave diffraction both on solid bodies and on flat screens, integrals with singular and hypersingular parts are used. In recent times, the methods have become very popular that use hypersingular equations, in particular, pseudodifferential equations solved using projection methods [2-8], and special quadrature formulas and algorithms of singularity smoothing [9-13].

In the present study, a method is used that was proposed by Lifanov [14], where hypersingular integrals in the sense of the Hadamard finite value are considered. This approach was used to solve the Neumann boundary value problem for the Helmholtz equation using the collocation method and piecewise constant approximations. This method was developed for the problems of external acoustics in [15-17]. Also in [18-20], a method was proposed and substantiated, in which the kernel was divided into two parts: one with a strong singularity and the other with a weak singularity. This makes it possible to reduce the part of the solution with a strong singularity to an analytical solution and solve numerically only the part with a weak singularity.
2. Methods
Let us consider solving the problem of scattering of a monochromatic acoustic wave on a system of bodies and flat screens of complex shapes. Let \( \Omega \) be the domain outside bodies and screens, and let \( \Sigma \) be its total boundary.

The solution to the problem of sound wave scattering is reduced to finding the total acoustic pressure field \( u_{\text{full}}(x)e^{i\omega t} \), defined in the domain \( \Omega \), which has the form:

\[
u(x) = u_0(x)e^{i\omega t} + u(x)e^{i\omega t}
\]

where \( u_0(x)e^{i\omega t} \) is the primary field, \( u(x)e^{i\omega t} \) is the unknown scattered field. The plane wave \( u_0(x) = Ae^{ikr} \) can act as the primary field, where \( k \) is the wave vector, \( r \) is the radius-vector of the point \( x \).

It is assumed that the reflective bodies are absolutely rigid. In this case, the scattered field will be a solution to the Neumann boundary value problem for the Helmholtz equation.

\[
\Delta u + k^2 u = 0 \quad \text{in the domain} \, \Omega, \quad \frac{\partial u}{\partial n} = f \quad \text{on the surface} \, \Sigma
\]

To solve this problem, the approach is used developed in [17-20].

The solution is sought in the form of the double layer potential:

\[
u(x) = \int_\Sigma g(y) \frac{\partial F(x-y)}{\partial n_y} dy, \quad F(x-y) = \frac{e^{ikr}}{r}
\]

After substituting expression (2) for the function \( u(x) \) into the boundary condition (1), the following integral equation arises.

\[
\int_\Sigma g(y) \frac{\partial}{\partial n_x} \frac{\partial}{\partial n_y} F(x-y) d\sigma_y = f(x), x \in \Sigma
\]

where the integral is understood in the sense of the finite Hadamard value [21].

Equation (3) is transformed by explicitly singling out the major singularity in it. Let us set:

\[
\Delta F(x-y) = F(x-y) - F_0(x-y), \quad F_0(x-y) = \frac{1}{4\pi |x-y|}
\]

where \( F_0(x-y) \) is the fundamental solution to the Laplace equation.

Let us substitute expressions (4) into the integral equation (3):

\[
\int_\Sigma g(y) \frac{\partial}{\partial n_x} \frac{\partial}{\partial n_y} F_0(x-y) d\sigma_y - \int_\Sigma g(y) K_1(x,y) d\sigma_y = f(x), K_1(x,y) = \frac{\partial}{\partial n_x} \frac{\partial}{\partial n_y} \Delta F(x-y), x \in \Sigma
\]

An approximate solution to the equation is sought in the form of a piecewise constant function \( \tilde{g}(x) \), assuming a constant value \( \bar{g}_i \) on each of the cells of the partition \( \sigma_i \). Collocation points \( \chi_i \) are selected on each cell of the partition as the geometric center of mass of the cell vertices. This gives rise to a system of linear algebraic equations:

\[
\sum_{j=1}^{n_i} g_j a_{ij} = f(x)_i, a_{ij} = a_{ij}^0 + \tilde{a}_{ij}, i = 1, \ldots, n
\]

The integrals in the matrix \( a_{ij}^0 \) are calculated analytically according to the Biot-Savart law [14]. The integrals in the matrix \( \tilde{a}_{ij} \) are weakly singular and are calculated numerically [18,19].

Let us calculate the effective scattering area [18] in the direction of the unit vector \( \tau \) by the formula:

\[
\nu(\tau) = 4nk^2 \left| \int_\Sigma g(y) \left[ e^{ikr(x-y)} \cdot (n_y, \tau) \right] d\sigma_y \right|^2.
\]
The effective scattering area can be expressed in dB:
\[ \vartheta = 10 \log \frac{\vartheta}{S} \text{ dB} \]
where \( S \) is the area of the object.
In the numerical method, expression (5) can be approximated by the following formula:
\[ \vartheta(x) = 4\pi k^2 \sum_{j=1}^{n} g_j \left[ e^{ik(x_j^2 + (n_j \cdot x) - 2r - r^2)} \right] S_j \]

3. Results and discussions

3.1. Diffraction of an acoustic wave on a disk and a partially filled disk
Let us consider the diffraction of an acoustic wave by three objects: a completely filled disk, which is a flat screen, a disk of rings, and a disk of plates, which are systems of flat screens (Fig. 1). The radius of all disks equals one.

![Fig. 1](image)

**Fig. 1.** On the left there is a completely filled disk; in the center there is a disk of rings; on the right there is a disk of plates.

Table 1 shows the number of subdivision cells and the total cell area for each disk.

| Type of a disk       | Number of subdivision cells | Total area of all cells | Ratio of the total area of cells to the total area of a completely filled disk |
|----------------------|-----------------------------|-------------------------|--------------------------------------------------------------------------------|
| Completely filled disk | 6720                        | 3.141                   | 1                                                                              |
| Disk of rings        | 3840                        | 1.795                   | 0.571                                                                          |
| Disk of plates       | 1920                        | 0.897                   | 0.286                                                                          |

For calculations, an example was considered when the incident field is a plane wave with a wave vector \( \mathbf{k} \) codirectional with the disk axis (Fig. 2). The reflected wave is the vector \( \mathbf{r} \) which is reflected at an angle \( \varphi \).

![Fig. 2](image)

**Fig. 2.** Incident and reflected waves in the diffraction on the disk.

Direction patterns were plotted for various types of disks for wave numbers \( k = 3; 10; 20 \). \[ k = |\mathbf{k}| \] Fig. 3 shows the dependencies of the scattering cross-section \( \vartheta \) on the angle \( \varphi \).
Fig. 3. Direction patterns for different types of disks for $k = 3$ – on the left, for $k = 10$ – in the center, for $k = 20$ – on the right (blue line is a completely filled disk, red line is a disk of rings, green line is a disk of plates).

A comparison was carried out between the disk of rings and the disk of plates in relation to the completely filled disk; the average and maximum deviations with respect to scattering cross-section were calculated ($\bar{\psi}_{\text{disk}} - \bar{\psi}_{\text{disk}_r}$ is the deviation of the disk of rings, $\bar{\psi}_{\text{disk}} - \bar{\psi}_{\text{disk}_p}$ is the deviation of the disk of plates). The calculations are presented in Table 2.

Table 2. The values of average and maximum deviations when comparing the direction patterns for the disk and the disk of rings, and for the disk and the disk of plates.

| Value                                    | $k=3$ | $k=10$ | $k=20$ |
|------------------------------------------|-------|--------|--------|
| $\bar{\psi}_{\text{disk}} - \bar{\psi}_{\text{disk}_r}$ | 19.709 dB | 10.872 dB | 5.976 dB |
| $\max(\bar{\psi}_{\text{disk}} - \bar{\psi}_{\text{disk}_r})$ | 23.745 dB | 28.698 dB | 26.094 dB |
| $\bar{\psi}_{\text{disk}} - \bar{\psi}_{\text{disk}_p}$ | 29.981 dB | 17.675 dB | 6.509 dB |
| $\max(\bar{\psi}_{\text{disk}} - \bar{\psi}_{\text{disk}_p})$ | 38.464 dB | 46.522 dB | 30.961 dB |

It is seen from the table that the larger the value of the wave number, the less the influence of the cut regions on the scattering cross-section.

3.1. Diffraction of an acoustic wave on a paraboloid and a partially filled paraboloid

Similarly, let us consider the diffraction of an acoustic wave on a completely filled paraboloid, which is a solid body, a paraboloid of rings, and a paraboloid of plates, which are systems of solid bodies (Fig. 4). The paraboloid is given by the formula $x = 0.5y^2$. The wave vector $\mathbf{k}$ is directed from right to left along the rotation axis.

Fig. 4. On the left there is a paraboloid of rings; in the center there is a paraboloid of plates; on the right there is a completely filled paraboloid with an image of incident and reflected waves during diffraction.
Table 3 shows the number of subdivision cells and the total area of the cells for each paraboloid.

### Table 3. Characteristics of subdivision of various types of paraboloids

| Type of a paraboloid       | Number of subdivision cells | Total area of all cells | Ratio of the total area of cells to the total area of the completely filled paraboloid |
|---------------------------|-----------------------------|-------------------------|--------------------------------------------------------------------------------------|
| Completely filled paraboloid | 6720                        | 3.828                   | 1                                                                                   |
| Paraboloid of rings       | 3840                        | 2.231                   | 0.583                                                                                |
| Paraboloid of plates      | 1920                        | 1.115                   | 0.291                                                                                |

Direction patterns were constructed for different types of paraboloids for the values of the wave number $k = 3; 10; 20$, $k = |k|$. Fig. 5 shows the dependencies of scattering cross-section $\tilde{\gamma}$ on the angle $\varphi$.

![Direction patterns for different types of paraboloids](image)

**Fig. 5.** Direction patterns for different types of paraboloids for $k = 3$ – on the left, for $k = 10$ – in the center, for $k = 20$ – on the right (blue line – completely filled paraboloid, red line – paraboloid of rings, green line – paraboloid of plates)

A comparison was carried out between the paraboloid of rings and the paraboloid of plates in relation to the completely filled paraboloid; the average and maximum deviations with respect to scattering cross-section were calculated ($|\tilde{\gamma}_{par} - \tilde{\gamma}_{par,r}|$ is the deviation of the disk of rings, $|\tilde{\gamma}_{par} - \tilde{\gamma}_{par,p}|$ is the deviation of the disk of plates). The calculations are presented in Table 4.

### Table 4. The values of average and maximum deviations when comparing the direction patterns for the paraboloid and the paraboloid of rings, and for the paraboloid and the paraboloid of plates.

| Value       | $k=3$ | $k=10$ | $k=20$ |
|-------------|-------|--------|--------|
| $\text{avg} | |       | |       |
| $|\tilde{\gamma}_{par} - \tilde{\gamma}_{par,r}|$ | 14.943 dB | 4.247 dB | 6.864 dB |
| $|\tilde{\gamma}_{par} - \tilde{\gamma}_{par,p}|$ | 17.575 dB | 33.050 dB | 29.693 dB |
| $\text{avg} | |       | |       |
| $|\tilde{\gamma}_{par} - \tilde{\gamma}_{par,p}|$ | 24.024 dB | 12.307 dB | 6.399 dB |
| $\text{max} | |       | |       |
| $|\tilde{\gamma}_{par} - \tilde{\gamma}_{par,p}|$ | 26.538 dB | 34.001 dB | 20.495 dB |

It is seen from the table that the larger the value of the wave number, the less the influence of the cut regions on the scattering cross-section.

### 3.2. Diffraction of an acoustic wave on a system of buildings

As another example of solving diffraction problems, let us consider the problem of reflection of sound, coming from a point source, from a group of buildings.
Fig. 6 shows a group of three-dimensional buildings (total subdivision into 2,748 cells) and a point source of sound (shown by the red dot). All buildings and the sound source are located on the surface of the Earth; the frequency of the sound source is 30Hz.

![Fig. 6. A group of three-dimensional buildings and a sound source](image)

The noise level from the source was calculated as the sound pressure field determined by formula (2). Fig. 7 shows the distribution of the sound pressure modulus over the earth's surface, i.e. the areas with a low noise level (red) and with an increased noise level (blue) are shown.

![Fig. 7. Distribution of the sound pressure modulus on a group of buildings](image)

4. Conclusion
The results obtained allow observing that the developed method can be applied to calculating and analyzing reflected fields from complex systems of objects, including systems of flat screens. Also, this method can be used to calculate the noise level from a point source, to construct the distribution of sound pressure during reflection from a system of complex objects.

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