MULTILEVEL ENSEMBLE KALMAN FILTERING WITH
LOCAL-LEVEL KALMAN GAINS

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ABSTRACT. We introduce a new multilevel ensemble Kalman filtering method (MLEnKF) which consists of a hierarchy of samples of the ensemble Kalman filter method (EnKF) using local-level Kalman gains. This new MLEnKF method is fundamentally different from the preexisting method introduced by Hoel, Law and Tempone in 2016, and it is also suitable for extensions towards multi-index Monte Carlo based filtering methods. Robust theoretical analysis and supporting numerical examples show that under appropriate regularity assumptions, the MLEnKF method has better complexity asymptotically, in the large-ensemble and small-numerical-resolution limit, for weak approximations of quantities of interest than EnKF. The method is developed for discrete-time filtering problems with a finite-dimensional state space and partial, linear observations polluted by additive Gaussian noise.

Key words: Monte Carlo, multilevel, convergence rates, Kalman filter, ensemble Kalman filter.

AMS subject classification: 65C30, 65Y20.

1. Introduction

We develop a new hierarchical-based multilevel ensemble Kalman filtering method (MLEnKF) for the setting of finite dimensional state space and discrete time partial observations polluted by additive Gaussian noise. Our method makes use of recent variance reduction techniques from multilevel and multi-index Monte Carlo [18, 12, 16] to improve the asymptotic efficiency of weak approximations of filtering distributions by orders of magnitude over plain ensemble Kalman filtering (EnKF). We consider settings where numerical methods must approximate the underlying time-continuous nonlinear dynamics model.

The MLEnKF method consists of a hierarchy of pairwise coupled EnKF estimators where the Kalman gain on each resolution level only depends on the local ensemble (hence the term “local-level Kalman gains”). The resolution here refers to three degrees of freedom: the numerical resolution, the ensemble size, and the sample size of iid coupled EnKF estimators on level \(\ell\) (i.e., in addition to having samples of ensembles, the MLEnKF method in this work also has samples of iid coupled EnKF estimators). Our method is fundamentally different from the “canonical” MLEnKF [20], which consists of a hierarchy of coupled ensembles on different resolution levels, all sharing one global “multilevel” Kalman gain, rather than having local-level Kalman gains.

The motivations for developing the new MLEnKF method are threefold. First, the method is closer to classic EnKF, and we therefore believe it will be easier to

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implement for practitioners. Second, imposing slightly stricter regularity assumptions, we are able to prove better asymptotic efficiency results for this method, measured in computational cost versus error, than what is known for the “canonical” one. Moreover, third, creating a rigorous theory for the localized Kalman gain MLEnKF method is a stepping stone towards a multi-index Ensemble Kalman filtering method (see [17] for highly efficient weak approximations of McKean–Vlasov dynamics using the multi-index Monte Carlo method).

The main theoretical contributions of this work are Theorems 1 and 2, which derive $L^p$ convergence rates for weak approximations in the large ensemble and fine numerical resolution limit with EnKF (Theorem 1) and MLEnKF (Theorem 2). Theorem 2 is a complete novelty, which may have fruitful extensions in multi-index filtering methods, and, although Theorem 1 is similar to [20, Theorem 3.11], this is, to the best of our knowledge, the first result with full proof of $L^p$ convergence rates for weak approximations with EnKF in this context. Corollary 2 shows how to determine the degrees of freedom in the MLEnKF estimator to achieve optimal rates in the computational work with respect of accuracy, and we derive asymptotic computational cost versus error estimates for the respective methods in Corollaries 1 and 2. We reach the conclusion that, asymptotically, MLEnKF outperforms EnKF whenever Assumptions 1 and 2 hold. Note that the exact optimisation of the degrees of freedom to improve the practical complexity is not the focus of this work.

1.1. Literature review. The EnKF method was first introduced in the seminal work [10], and, due to its ease of use and impressive performance in high dimensions, it quickly became a popular method for weather prediction, ocean-atmosphere science and oil reservoir simulations [27, 24, 1]. The standard version of EnKF with perturbed observations, which is the method we will study and extend to the multi-level setting in this work, first appeared in [23] and ensuing analysis [5] showed that adding artificial noise to the observations may be viewed as a consistency step for avoiding ensemble covariance deflation. EnKF $L^p$ convergence of first and second sample moments in the large-ensemble limit was first treated by a short argument in [35] for the linear model setting. Considering a subset of nonlinear model settings, a more technical argument [34] derived EnKF $L^p$ convergence rates for weak approximations in the large ensemble limit.

The Multilevel Monte Carlo method was first introduced for efficient weak approximations of random fields [18] and stochastic differential equations [12]. MLEnKF methods were first developed for settings with finite [20] and infinite dimensional state spaces [6]. These works were also the first to present $L^p$ convergence rates for weak approximations of MLEnKF in the large ensemble and finer numerical resolution limit. See [25, 2, 36, 15, 14, 32, 11] for recent contributions on multilevel methods in filtering.

Under sufficient regularity, the EnKF large-ensemble limit equals the mean-field Kalman filter, which typically does not equal the exact Bayes filter [33, 9, 37]. Partly due to this property, perhaps, a considerable number of works have focused on the large-time and/or continuous time limit of the fixed ensemble size EnKF, cf. [28, 42, 38, 39, 4, 8, 31].

1.2. Organization of the work. Section 2 describes the problem setting, the MLEnKF method, and states the large ensemble and high accuracy limit convergence results. Section 3 present numerical studies of EnKF and MLEnKF for three
different test problems. Appendix A contains the proofs for the convergence results in Section 2 and the algorithm used to obtain the reference solution in Section 3.2 is described in Appendix B.

2. Problem setting and main results

2.1. Problem setting. Let \((\Omega, (\mathcal{F}_t), \mathcal{F} = \mathcal{F}_\infty, \mathbb{P})\) denote a complete filtered probability space, and for any \(k \in \mathbb{N}\) and \(p \geq 1\), let \(L^p(\Omega, \mathbb{R}^k)\) denote the space of \(\mathcal{F}_t\)-measurable functions \(u : \Omega \to \mathbb{R}^k\) such that \(\mathbb{E}[|u|^p] < \infty\), where \(\mathcal{B}^k\) represents the Borel sigma-algebra on \(\mathbb{R}^k\). For a given state-space dimension \(d \in \mathbb{N}\) and initial data \(u_0 \in \cap_{p \geq 2} L^p(\Omega, \mathbb{R}^d)\), we consider the discrete stochastic dynamics for \(n = 0, 1, \ldots\)

\[
(1) \quad u_{n+1}(\omega) = \Psi_n(u_n, \omega), \quad \omega \in \Omega,
\]

for a sequence of mappings \(\Psi_n : \mathbb{R}^d \times \Omega \to \mathbb{R}^d\). The dynamics is associated with the stochastic differential equation (SDE)

\[
(2) \quad \Psi_n(u_n, \omega) = u_n + \int_0^{n+1} a(u_t)dt + \int_0^{n+1} b(u_t)dW_t(\omega),
\]

with coefficients \(a : \mathbb{R}^d \to \mathbb{R}^d\), \(b : \mathbb{R}^d \to \mathbb{R}^{d \times dW}\) and the driving noise \(W : [0, \infty) \times \Omega \to \mathbb{R}^{dW}\) denoting a \(dW\)-dimensional standard Wiener process. We further assume the coefficients \(a\) and \(b\) are sufficiently smooth so that

\[
(3) \quad u_n \in L^2(\Omega, \mathbb{R}^d) \implies u_{n+1} \in L^2(\Omega, \mathbb{R}^d),
\]

and we note, dropping the dependence on \(\omega\) without losing clarity, that \(u_{n+1}\) may be expressed as a quasi-iterated mapping of \(u_0\):

\[
(4) \quad u_{n+1} = \Psi_n \circ \Psi_{n-1} \circ \cdots \circ \Psi_0(u_0).
\]

Associated with the dynamics (1), there exists a series of noisy observations

\[
(5) \quad y_n = H u_n + \eta_n, \quad n = 0, 1, \ldots,
\]

where \(H \in \mathbb{R}^{d_0 \times d}\) and \(\eta_0, \eta_1, \ldots\) is a sequence of independent and identically distributed (iid) random variables satisfying \(\eta_0 \sim N(0, \Gamma)\) with positive definite \(\Gamma \in \mathbb{R}^{d_0 \times d_0}\) and \(\eta_j \perp u_k\) for all \(j, k \geq 0\).

Given a series of observations up to time \(k\), \(Y_k := (y_0, y_1, y_2, \ldots, y_k)\) and the initial distribution \(\mathbb{P}_{u_0|Y_k}\), the Bayes filter is a sequential procedure for determining the (conditional) prediction \(\mathbb{P}_{u_k|Y_{k-1}}\) and update \(\mathbb{P}_{u_k|Y_k}\). Assuming that the densities of said distributions exist, the stochastic dynamics (1) and the conditional independence \((u_k|u_{k-1}) \perp Y_{k-1}\) yield the proportionality

\[
\rho_{u_k|Y_{k-1}}(u) = \int_{\mathbb{R}^d} \rho_{u_k,u_{k-1}|Y_{k-1}}(u, v)dv \propto \int_{\mathbb{R}^d} \rho_{u_k|u_{k-1}}(u) \rho_{u_{k-1}|Y_{k-1}}(v) dv,
\]

and Bayesian inference implies that

\[
\rho_{u_k|Y_k}(u) \propto \mathcal{L}_{u_k|y_k}(u) \rho_{u_k|Y_{k-1}}(u),
\]

where the likelihood function is given by

\[
\mathcal{L}_{u_k|y_k}(u) = \exp\left(-\frac{|\Gamma^{-1/2}(y_k - Hu)|^2}{2}\right) / \sqrt{(2\pi)^p |\det(\Gamma)|}.
\]

In other words, if the updated distribution \(\mathbb{P}_{u_{k-1}|Y_{k-1}}\) is known and suitable assumptions are fulfilled, then the prediction and updated distribution at the next
time can be computed up to a proportionality constant (although this step may be computationally infeasible).

In settings where the initial density $u_0|Y_0$ is Gaussian, and the dynamics $\Psi$ is linear with additive Gaussian noise, Kalman filtering [26] solves the above filtering problem exactly. When $\Psi$ is nonlinear, it is often not possible to solve the filtering problem exactly. EnKF is an ensemble-based filtering method that preserves many features of Kalman filtering that may also be applied to nonlinear filtering problems. EnKF has proven to be particularly efficient when used for filtering problems where the “true” dimension is far smaller than the dimension of the state space.

A pertinent question to ask is whether the empirical measure of EnKF converges towards the true density $P_{u_t|Y_t}$ in the large ensemble limit. This holds in settings with Gaussian initial data $u_0$ and linear dynamics with Gaussian additive noise, $\Psi(u) = Au + \xi$ with $\xi \sim N(0, \Sigma)$, cf. [35]. More generally, for instance, when $\Psi$ is nonlinear, the EnKF large-ensemble limit converges to the mean-field EnKF, cf. Section 2.4, which, due to an underlying Gaussianity assumption in the Kalman update step, may not be equal to the Bayes filter [33, 9]. To the best of our knowledge, there does not exist any thorough scientific comparison of MFEnKF and the Bayes Filter. However, Figure 1 shows that for the nonlinear dynamics $\Psi$ defined by the SDE $du = -(u + \pi \cos(\pi u/5)/5)dt + \sigma dW$ and (1), the dissipative/contractive properties of the associated Fokker-Planck equation produce prediction densities for the respective filtering methods that are indistinguishable to the naked eye.

**Figure 1.** Illustration of the contracting property which produces almost identical prediction densities for the Bayes filter and MFEnKF even when the respective preceding updated densities differ notably.
Objective. For a given quantity of interest (QoI) \( \varphi : \mathbb{R}^d \to \mathbb{R} \), our objective is to construct an efficient filtering method for computing

\[
\mathbb{E}^{\mu_n}[\varphi(u)] = \int_{\mathbb{R}^d} \varphi(u)\mu_n(du),
\]

where \( \mu_n \) denotes the mean-field EnKF updated measure at time \( n \), cf. Section 2.4. In reality, one seeks a method that computes the exact expectation \( \mathbb{E}[\varphi(u_n) \mid Y_n] \), but since the Bayes filter posterior \( \mathbb{P}_{u_n \mid Y_n} \) is generally not attainable by ensemble filtering methods, and both EnKF and MLEnKF converge weakly towards mean-field EnKF (cf. Theorems 1 and 2), we will in this work focus on the simpler (but still very challenging) goal (4).

Notation 1.

- For \( f, g : (0, \infty) \to [0, \infty) \) the notation \( f \lesssim g \) implies there exists a constant \( C > 0 \) such that

\[
f(x) \leq C g(x), \quad \forall x \in (0, \infty).
\]

The notation \( f \approx g \) implies there exist a pair of constants \( C > c > 0 \) such that

\[
cg(x) \leq f(x) \leq Cg(x), \quad \forall x \in (0, \infty).
\]

- For \( r, s \in \mathbb{N}, |x| \) denotes the Euclidean norm of a vector \( x \in \mathbb{R}^s \) and for \( A \in \mathbb{R}^{r \times s}, |A| := \sup_{|x|=1} |Ax| \).

- For \( F \backslash \mathcal{B}^d \)-measurable functions \( u : \Omega \to \mathbb{R}^d \) and \( p \geq 1 \),

\[
\|u\|_p := \|u\|_{L^p(\Omega, \mathbb{R}^d)} = \left( \int_{\Omega} |u(\omega)|^p \mathbb{P}_d \, d\omega \right)^{1/p}.
\]

- For any \( \kappa \in \mathbb{N}_0^d \) and sufficiently smooth functions of the form \( f : \mathbb{R}^r \to \mathbb{R} \) and \( g : \mathbb{R}^r \times \Omega \to \mathbb{R} \) are respectively defined by

\[
\partial^\kappa f(x) = \frac{\partial^{\kappa_1}}{\partial x_1^{\kappa_1}} \ldots \frac{\partial^{\kappa_d}}{\partial x_d^{\kappa_d}} f(x) \quad \text{and} \quad \partial^\kappa g(x, \omega) = \frac{\partial^{\kappa_1}}{\partial x_1^{\kappa_1}} \ldots \frac{\partial^{\kappa_d}}{\partial x_d^{\kappa_d}} g(x, \omega)
\]

for all \( x \in \mathbb{R}^d \) and \( \mathbb{P} \)-almost all \( \omega \in \Omega \), where \( |\kappa|_1 := \sum_{i=1}^d \kappa_i \). This extends to mappings of the form \( f : \mathbb{R}^r \to \mathbb{R}^s \) by component-wise partial derivatives: \( \partial^\kappa f(x) = (\partial^\kappa f_1(x), \partial^\kappa f_2(x), \ldots, \partial^\kappa f_s(x)) \), and so on.

- For \( k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, C^k_B(\mathbb{R}^r, \mathbb{R}^s) \) denotes the space of \( k \) times continuously differentiable functions \( \eta : \mathbb{R}^r \to \mathbb{R}^s \) for which \( \eta \) and all of its partial derivatives of order up to and including \( k \) are uniformly bounded.

- \( C^k_P(\mathbb{R}^r, \mathbb{R}^s) \) denotes the space of \( k \) times continuously differentiable functions whose partial derivatives of order up to and including \( k \) have polynomial growth.

- For a mapping \( \eta : \mathbb{R}^r \to \mathbb{R}^s \), its Jacobian is denoted \( D\eta \), and its Hessian is \( D^2 \eta \).

- \( \mathbb{R}[x_1, \ldots, x_d] \) denotes the set of polynomials in \( d \) variables with coefficients in \( \mathbb{R} \) and of total degree smaller or equal to \( r \in \mathbb{N}_0 \).

2.2. Numerical approximation of the stochastic dynamics. We assume that for every time \( n \geq 0 \), there exists a collection of progressively more accurate numerical solvers \( \{\Psi_n^N : \mathbb{R}^d \times \Omega \to \mathbb{R}^d\}_{N \in \mathbb{N}} \) satisfying the following assumptions:
Assumption 1.
The initial distribution satisfies $u_0|Y_0 \in \cap_{p \geq 2} L^p(\Omega, \mathbb{R}^d)$ and for any $n \in \mathbb{N}_0$,

(i) there exists a constant $c_p > 0$ such that for any $u \in L^p_n(\Omega, \mathbb{R}^d)$,
$$
\left\| \Psi^N_n(u) \right\|_p \leq c_p (1 + \|u\|_p) \quad \forall (N \in \mathbb{N} \quad \& \quad p \geq 2);
$$

(ii) there exists an $\alpha > 0$ and a set of mappings $\mathcal{F}$ with $\mathbb{R}_2[x_1, \ldots, x_d] \subset \mathcal{F} \subset C^0_\psi(\mathbb{R}^d, \mathbb{R})$ such that if
$$
|\mathbb{E}[\varphi(u^N)] - \varphi(u)| \leq c_\varphi N^{-\alpha} \quad \forall (N \geq 1 \quad \& \quad \varphi \in \mathcal{F}),
$$
for some $u \in \cap_{p \geq 2} L^p_n(\Omega, \mathbb{R}^d)$ and $\{u^N\}_N \subset \cap_{p \geq 2} L^p_n(\Omega, \mathbb{R}^d)$ and (observable dependent constant) $c_\varphi > 0$, then there exists another (observable dependent constant) $\tilde{c}_\varphi > 0$ such that
$$
|\mathbb{E}[\varphi(\Psi^N_n(u^N))] - \varphi(\Psi_n(u))]| \leq \tilde{c}_\varphi N^{-\alpha} \quad \forall (N \geq 1 \quad \& \quad \varphi \in \mathcal{F});
$$

(iii) there exists a $c_p > 0$ such that for all $N \in \mathbb{N}$ and $u, v \in \cap_{p \geq 2} L^p_n(\Omega, \mathbb{R}^d)$
$$
\left\| \Psi^N_n(u) - \Psi^N_n(v) \right\|_p \leq c_p \|u - v\|_p \quad \forall (N \in \mathbb{N} \quad \& \quad p \geq 2);
$$

(iv) there exists a $c > 0$ such that for all $N \in \mathbb{N}$ and $u \in L^2_n(\Omega, \mathbb{R}^d)$, the computational cost of the numerical solution $\Psi^N_n$ is bounded by
$$
es_{ss} \sup_{(\omega, n) \in \Omega \times \mathbb{N}} \text{Cost}(\Psi^N_n(u, \omega)) \leq c_3 N^7.
$$

Remark 1. If the dynamics (1) is given by the SDE (2) where the coefficients $a: \mathbb{R}^d \to \mathbb{R}^d$, $b: \mathbb{R}^d \to \mathbb{R}^{d \times d\omega}$ satisfy $\partial^\gamma a_j, \partial^\gamma b_{jk} \in C_B(\mathbb{R}^d, \mathbb{R})$ for all $1 \leq j \leq d$, $1 \leq k \leq d\omega$ and $|\gamma| \geq 1$, and if $\Psi^N_n$ denotes the Euler–Maruyama numerical solution of (2) using $N$ uniform timesteps, then Assumption 1 holds with $\alpha = \gamma = 1$ and $\mathcal{F} = C^0_\psi(\mathbb{R}^d, \mathbb{R})$, cf. [13, Chapter 7] and [29, Thm 14.5.2].

2.3. EnKF. For a fixed solver resolution $\Psi^N$ with $N \geq 1$ and a fixed ensemble size $P \geq 1$, the EnKF method consists of an ensemble of particles that are simulated independently in the prediction and update steps in an interacting fashion that is consistent with the Kalman filter in the large-ensemble limit. Assuming that the initial density can be sampled exactly, the updated ensemble at time 0, denoted by $\hat{v}^{N,P}_{0,1} := \{\hat{v}^{N,P}_{0,i}\}_{i=1}^P$, consists of $P$ iid particles with distribution $\mathbb{P}_{u_0|Y_0}$. The empirical measure induced by the ensemble $\hat{v}^{N,P}_{0,1}$ may thus be viewed as an approximation of $\mathbb{P}_{u_0|Y_0}$.

Given the updated ensemble at time $n$, denoted by $\hat{v}^{N,P}_{n,1}$, the EnKF prediction and updated ensembles for the next time satisfies the discrete dynamics
$$
\hat{v}^{N,P}_{n+1,i} = \Psi^N_n(\hat{v}^{N,P}_{n,i}) \quad \text{for} \quad i = 1, 2, \ldots, P,
$$
and
$$
\hat{v}^{N,P}_{n+1,i} = (I - K^{N,P}_{n+1} H)\hat{v}^{N,P}_{n+1,i} + K^{N,P}_{n+1} Y_{n+1, i} \quad \text{for} \quad i = 1, 2, \ldots, P.
$$
Here,
$$
K^{N,P}_{n+1} = C^{N,P}_{n+1} H^T (HC^{N,P}_{n+1} H^T + \Gamma)^{-1}
$$
denotes the Kalman gain, where

\begin{equation}
C^{N,P}_{n+1} = \text{Cov}([\hat{v}^{N,P}_{n+1}]) := \sum_{i=1}^P \frac{\hat{v}^{N,P}_{n+1,i}}{P} \left( \frac{\hat{v}^{N,P}_{n+1,i}}{P} \right)^T - \sum_{i=1}^P \frac{\hat{v}^{N,P}_{n+1,i}}{P} + \left( \sum_{i=1}^P \frac{\hat{v}^{N,P}_{n+1,i}}{P} \right)^T
\end{equation}
denotes the biased sample covariance of the ensemble \( \hat{v}_{n+1,i} \), and
\[
\hat{y}_{n+1,i} = y_{n+1} + \eta_{n+1,i}, \quad \text{for} \quad i = 1, 2, \ldots, P
\]
are iid perturbed observations with \( \eta_{n+1,1} \sim N(0, \Gamma) \) and \( \eta_{j,i} \perp u_k \) for all \( j, k \geq 0 \).

Perturbed observations were originally introduced in [5] to correct the statistical error induced in its absence in implementations following the original formulation of the ensemble Kalman filter in [10].

In other words, the mean-field Kalman gain, in its absence in implementations following the original formulation of the ensemble Kalman filter in [10].

We note that for all filtering methods considered in this work, the sequentially updated measurement sequence \( Y_n = (y_0, y_1, \ldots, y_n) \in \mathbb{R}^{d_0 \times (n+1)} \) is assumed to be given, i.e., non-random. Randomness in the ensemble will, therefore, only come from the driving noise in the dynamics and the perturbed observations. For EnKF with discrete dynamics resolution \( N \), for instance, randomness enters through \( \{\Psi_{n,P}^N\}_n \) and \( \{\eta_n^N\}_n \). Observe further that since the particles \( \hat{v}_{n,1}, \hat{v}_{n,2}, \ldots, \hat{v}_{n,P} \) are identically distributed, the distribution of \( \hat{v}_{n,i} \) will depend on the ensemble size \( P \) and the model resolution \( N \).

Provided that Assumption 1 holds, Theorem 1 implies that for any \( \varphi : \mathbb{R}^d \to \mathbb{R} \),
\[
\mu_n^{N,P}[\varphi] := \frac{1}{P} \sum_{i=1}^P \varphi(\hat{v}_{n,i}^{N,P}),
\]
with \( \delta \) denoting the Dirac-delta "function". For evaluating the empirical mean of a QoI \( \varphi : \mathbb{R}^d \to \mathbb{R} \), we introduce the following notation
\[
\mu_n^{N,P}[\varphi] := \int_{\mathbb{R}^d} \varphi(v) \mu_n^{N,P}(v) dv.
\]
In other words,
\[
\mu_n^{N,P}[\varphi] = \frac{1}{P} \sum_{i=1}^P \varphi(\hat{v}_{n,i}^{N,P}).
\]

### 2.4. Mean-field EnKF

In order to study convergence properties of EnKF when the ensemble limit is large, we now introduce that limit, i.e., the mean-field EnKF (MFEEnKF). In the mean-field limit, the EnKF Kalman gain becomes a deterministic matrix, and consequently, there is no further mixing between ensemble particles, and the particles become iid. It thus suffices to describe the distribution of a single MFEEnKF particle. For a fixed dynamics resolution \( \Psi_N \), let \( \hat{v}_n^N \) denote the updated state of a mean-field particle at time \( n \). The MFEEnKF prediction and update dynamics is given by
\[
\hat{v}_{n+1}^N = \Psi_n^N(\hat{v}_n^N),
\]
and
\[
\hat{y}_{n+1}^N = (I - \hat{K}_{n+1}^N H) \hat{v}_{n+1}^N + \hat{K}_{n+1}^N \hat{y}_{n+1},
\]
where
\[
\hat{K}_{n+1}^N = \Gamma \left( H \hat{C}_{n+1}^N + \Gamma \right)^{-1}
\]
denotes the deterministic mean-field Kalman gain,
\[
\hat{C}_{n+1}^N = \mathbb{E} \left[ (\hat{v}_{n+1}^N - \mathbb{E} [\hat{v}_{n+1}^N]) (\hat{v}_{n+1}^N - \mathbb{E} [\hat{v}_{n+1}^N])^T \right]
\]
is the mean-field prediction covariance, and
\[ \tilde{y}_{n+1} = y_{n+1} + \tilde{\eta}_{n+1} \]
denotes the perturbed observation with \( \{\tilde{\eta}_n\}_{n=1}^N \) being iid \( N(0, \Gamma) \)–distributed random variables satisfying \( \tilde{\eta}_j \perp u_k \) for all \( j, k \geq 0 \).

Provided that Assumption 1 (i) with \( |\kappa|_1 = 0 \) holds, and using that \( L^p_p(\Omega, \mathbb{R}^d) \subset L^p(\Omega, \mathbb{R}^d) \), it follows straightforwardly by induction that \( \tilde{v}^N_n, \tilde{\xi}^N_n \in \cap_{p \geq 2} L^p(\Omega, \mathbb{R}^d) \) uniformly in \( N \in \mathbb{N} \cup \{\infty\} \):

\[ \begin{align*}
\hat{\eta}_0 & \sim \mathbb{P}_{u_0|\mathcal{Y}_0} \implies \hat{\xi}_1 = \Psi_0^N(\hat{\eta}_0^N) \in \cap_{p \geq 2} L^p(\Omega, \mathbb{R}^d) \\
\implies |\hat{\xi}_1|^2 < \infty & \implies |\hat{\xi}_1|^2 < \infty \\
\implies \|\hat{\xi}_1\|^p \leq c_p(1 + |H|_2) (\|\hat{\xi}_1\|^p + \|\hat{\xi}_1\|^p) < \infty & \forall p \geq 2 \\
\implies \ldots & \implies \tilde{v}^N_n \in \cap_{p \geq 2} L^p(\Omega, \mathbb{R}^d) \quad (7)
\end{align*} \]

This ensures the existence of the mean-field measure \( \tilde{\mu}_n \), i.e., the distribution \( \tilde{\eta}_n^N \sim \tilde{\mu}_n \) satisfies \( \tilde{v}^N_n \in \cap_{p \geq 2} L^p(\Omega, \mathbb{R}^d) \) for all \( N \in \mathbb{N} \cup \{\infty\} \) and \( n \in \mathbb{N}_0 \). For evaluating the expectation of a QoI \( \varphi : \mathbb{R}^d \to \mathbb{R} \) wrt to the mean-field measure \( \tilde{\mu}_n^N \), we introduce the notation

\[ \tilde{\mu}_n^N[\varphi] := \int_{\mathbb{R}^d} \varphi(v) \tilde{\mu}_n^N(\text{d}v). \]

For a subset of nonlinear dynamics \( \Psi \) with additive Gaussian noise, it has been shown that in the large-ensemble limit, EnKF with perturbed observations converges to the Kalman filtering distribution with the standard rate \( O(P^{-1/2}) \), cf. [34]. That is, in \( L^p \) for all QoI \( \varphi : \mathbb{R}^d \to \mathbb{R} \) with sufficient regularity \( n \in \mathbb{N}_0 \), and \( p \geq 2 \)

\[ \|\tilde{\mu}_n^{\infty,P}[\varphi] - \tilde{\mu}_n[\varphi]\|_p \leq c_{\varphi,p} P^{-1/2} \quad \text{for some} \quad C_{\varphi,\psi} > 0, \]

where \( \tilde{\mu}_n := \tilde{\mu}_n^{\infty} \). The result was extended to a subset of fully non-Gaussian models [33] with numerical approximations of the dynamics in [20], i.e., \( \mu^N_n, P[\varphi] \to \mu_n[\varphi] \) as \( P, N \to \infty \) (see also Theorem 1).

2.5. MLEnKF. Before stating the main convergence results of this work, we introduce the new MLEnKF algorithm based on the estimator with localized Kalman gains. Given two exponentially increasing sequences \( \{N_\ell\}, \{P_\ell\} \subset \mathbb{N} \), the latter with the constraint \( P_\ell / P_{\ell-1} = 2 \), and an upper level \( L \in \mathbb{N} \), we denote by \( \hat{v}^{\ell,e}_{n,1:P_\ell} := \{\hat{v}^{\ell,e}_{n,i}\}_{i=1}^{P_\ell} \) and \( \hat{v}^{\ell,f}_{n,1:P_\ell} := \{\hat{v}^{\ell,f}_{n,i}\}_{i=1}^{P_\ell} \), respectively, the fine and coarse resolution updated ensembles on level \( \ell \) at simulation time \( n \). We consider the coarse ensemble as a union of two sets with \( P_{\ell-1} \) particles such that \( \hat{v}^{\ell,e}_{n,1:P_\ell} = \hat{v}^{\ell,e}_{n,1:P_{\ell-1}} \cup \hat{v}^{\ell,e}_{n,1:P_{\ell-1}} \), where \( \hat{v}^{\ell,e}_{n,1:P_{\ell-1}} := \{\hat{v}^{\ell,e}_{n,i}\}_{i=1}^{P_{\ell-1}} \) and \( \hat{v}^{\ell,e}_{n,1:P_{\ell-1}} := \{\hat{v}^{\ell,e}_{n,i}\}_{i=P_{\ell-1}+1}^{P_{\ell-1}+1} \), with the convention \( \hat{v}_{n,1} := 0 \) for all \( n \) and \( i = 1, ..., P_\ell \). Assuming that the initial distribution can be sampled exactly, \( \hat{v}^{\ell,f}_{0,i} \) are independent and identically \( \mathbb{P}_{u_0|\mathcal{Y}_0} \) distributed and coupling \( \hat{v}^{\ell,f}_{0,i} = \hat{v}^{\ell,f}_{0,i} \) for \( i = 1, ..., P_\ell \) and \( \ell \geq 1 \).

At time \( n \), given the coupled updated states \( \hat{v}^{\ell,f}_{n,i} \) and \( \hat{v}^{\ell,e}_{n,i} \) for \( i = 1, ..., P_\ell \), we denote by \( \hat{v}^{\ell,f}_{n+1,i} \) and \( \hat{v}^{\ell,e}_{n+1,i} \), respectively, one step forward fine and coarse resolution predict realizations computed via the following pairwise coupled dynamics with the
same driving noise
\begin{align}
    \hat{v}_{n+1,i}^\ell & = \Psi_n\left(\hat{v}_{n,i}^\ell\right), \quad i = 1, \ldots, P\ell, \\
    \hat{v}_{n+1,i}^{\ell,c} & = \Psi_n\left(\hat{v}_{n,i}^{\ell,c}\right), \quad i = 1, \ldots, P\ell.
\end{align}

After that, the sample covariance matrices and Kalman gains are computed correspondingly by
\begin{align*}
    C_{n+1}^{\ell,f} & = \overline{\text{Cov}}[\hat{v}_{n+1}^\ell], \quad K_{n+1}^{\ell,f} = C_{n+1}^{\ell,f} H^T (HC_{n+1}^{\ell,f} H^T + \Gamma)^{-1}, \\
    C_{n+1}^{\ell,c1} & = \overline{\text{Cov}}[\hat{v}_{n+1}^{\ell,c1}], \quad K_{n+1}^{\ell,c1} = C_{n+1}^{\ell,c1} H^T (HC_{n+1}^{\ell,c1} H^T + \Gamma)^{-1}, \\
    C_{n+1}^{\ell,c2} & = \overline{\text{Cov}}[\hat{v}_{n+1}^{\ell,c2}], \quad K_{n+1}^{\ell,c2} = C_{n+1}^{\ell,c2} H^T (HC_{n+1}^{\ell,c2} H^T + \Gamma)^{-1},
\end{align*}
where the sample covariances are defined in a similar fashion as in for EnKF, i.e.,
\begin{align}
    \overline{\text{Cov}}[\hat{v}_{n+1}^\ell] & : = \frac{1}{P\ell} \sum_{i=1}^{P\ell} \frac{\hat{v}_{n+1,i}^\ell}{P\ell} \left( \sum_{i=1}^{P\ell} \frac{\hat{v}_{n+1,i}^\ell}{P\ell} \right)^T, \\
    \overline{\text{Cov}}[\hat{v}_{n+1}^{\ell,c1}] & : = \frac{1}{P\ell-1} \sum_{i=1}^{P\ell-1} \frac{\hat{v}_{n+1,i}^{\ell,c1}}{P\ell-1} \left( \sum_{i=1}^{P\ell-1} \frac{\hat{v}_{n+1,i}^{\ell,c1}}{P\ell-1} \right)^T, \quad j = 1, 2.
\end{align}

Next, we simulate the measurements \(y_{n+1}\) and obtain the following updated particle states:
\begin{align}
    y_{n+1,i}^\ell & = y_{n+1} + \eta_{n+1,i}^\ell, \quad i = 1, \ldots, P\ell, \\
    \hat{v}_{n+1,i}^\ell & = (I - K_{n+1}^{\ell,f} H) \hat{v}_{n+1,i}^\ell + K_{n+1}^{\ell,f} \tilde{y}_{n+1,i}^\ell, \quad i = 1, \ldots, P\ell, \\
    \hat{v}_{n+1,i}^{\ell,c1} & = (I - K_{n+1}^{\ell,c1} H) \hat{v}_{n+1,i}^{\ell,c1} + K_{n+1}^{\ell,c1} \tilde{y}_{n+1,i}^{\ell,c1}, \quad i = 1, \ldots, P\ell-1, \\
    \hat{v}_{n+1,i}^{\ell,c2} & = (I - K_{n+1}^{\ell,c2} H) \hat{v}_{n+1,i}^{\ell,c2} + K_{n+1}^{\ell,c2} \tilde{y}_{n+1,i}^{\ell,c2}, \quad i = 1, \ldots, P\ell-1,
\end{align}
where \(\{\eta_{n+1,i}^\ell\}_{i=1}^{P\ell}\) are iid with \(\eta_{n+1,1}^\ell \sim N(0, \Gamma)\). Note that the pairwise correlation between corresponding fine and coarse particles is obtained in three ways. In particular, the pairwise ensembles share the same initial condition, the same driving noise in the dynamics and the same perturbed observations.

Now suppose we wish to approximate the expected value of a QoI \(\varphi : \mathbb{R}^d \to \mathbb{R}\) of the updated ensemble at the most refined level \(\hat{v}_{n}^{L,f}\). Due to the linearity of the expectation, this may be written as a telescoping sum
\begin{align}
    \mathbb{E}\left[\varphi(\hat{v}_{n+1}^{L,f})\right] & = \mathbb{E}\left[\varphi(\hat{v}_{n}^{L,f})\right] + \sum_{\ell=1}^{L} \mathbb{E}\left[\varphi(\hat{v}_{n}^{\ell,f}) - \varphi(\hat{v}_{n}^{\ell,c})\right].
\end{align}

since \(\mathbb{E}\left[\varphi(\hat{v}_{n}^{\ell,f})\right] = \mathbb{E}\left[\varphi(\hat{v}_{n}^{\ell+1,c})\right]\) for any \(\ell\). Note first that the \(\ell\)-th summand on the right-hand side in \(11\) may be approximated by a sample average of \(P\ell\) coupled realizations as follows
\begin{align*}
    \frac{1}{P\ell} \sum_{i=1}^{P\ell} \varphi(\hat{v}_{n,i}^{\ell,f}) - \frac{1}{2} \left( \frac{1}{P\ell-1} \sum_{i=1}^{P\ell-1} \varphi(\hat{v}_{n,i}^{\ell,c1}) + \frac{1}{P\ell-1} \sum_{i=1}^{P\ell-1} \varphi(\hat{v}_{n,i}^{\ell,c2}) \right).
\end{align*}

Since the ensemble \(\hat{v}_{n,1:P\ell}^{\ell,f}\) induces an empirical measure \(\mu_n^{\ell,f}\) and \(\hat{v}_{n,1:P\ell-1}^{\ell,c} = \hat{v}_{n,1:P\ell-1}^{\ell,c1} \cup \hat{v}_{n,1:P\ell-1}^{\ell,c2}\) induces \(\mu_n^{\ell,c} := (\mu_n^{\ell,c1} + \mu_n^{\ell,c2})/2\) with the convention \(\mu_n^{\ell,c} = \mu_n^{\ell,c1} = \mu_n^{\ell,c2}\)
\( \mu_{n,\ell} = 0 \) for \( \ell = 0 \), we can observe that

\[
\frac{1}{P_{\ell}} \sum_{i=1}^{P_{\ell}} \varphi(\hat{v}_{n,i}^{\ell,f}) - \frac{1}{P_{\ell-1}} \sum_{i=1}^{P_{\ell-1}} \varphi(\hat{v}_{n,i}^{\ell,c_1}) + \varphi(\hat{v}_{n,i}^{\ell,c_2}) \quad = \quad \left( \mu_{n,\ell,1} + \mu_{n,\ell,2} \right) \frac{1}{2}. \]

Further, in order to balance the variance contribution from the different levels and optimize the variance reduction, we introduce the sequence of EnKF sample sizes \( \{M_\ell\} \subset \mathbb{N} \) as an additional degree of freedom. Let \( \{(\hat{v}_{n,i,1}^{\ell,f}, \hat{v}_{n,i,1}^{\ell,c})\}_{m=1}^{M_\ell} \) denote \( M_\ell \) iid random set of coupled ensemble \((\hat{v}_{n,i,1}^{\ell,f}, \hat{v}_{n,i,1}^{\ell,c})\), which correspondingly induce a set of empirical measures \( \{(\mu_{n,\ell,m}^{f}, \mu_{n,\ell,m}^{c})\}_{m=1}^{M_\ell} \). This yields the MLEnKF estimator of a QoI \( \varphi : \mathbb{R}^d \to \mathbb{R} \) given by

\[
\mu_{n,\text{MLE}}[\varphi] := \sum_{m=1}^{M_0} \frac{\mu_{n,0,m}^{f}}{M_0} + \sum_{\ell=1}^{L} \sum_{m=1}^{M_\ell} \left( \mu_{n,\ell,m}^{f} - \mu_{n,\ell,m}^{c} \right) \frac{1}{2} \varphi(\hat{v}_{n,i}^{\ell,f,m}) \varphi(\hat{v}_{n,i}^{\ell,c,m}) \]  

which depends on the following parameters:

- the upper level \( L \geq 0 \) of the estimator,
- an exponentially increasing sequence of resolutions \( \{N_\ell\} \subset \mathbb{N} \) representing a hierarchy of stochastic solvers \( \{\Psi_{n,\ell}^N\} \) for every \( n \in \mathbb{N}_0 \),
- \( \{P_\ell\} \subset \mathbb{N} \) satisfying \( P_{\ell+1} = 2P_\ell \) for all \( \ell \geq 0 \), where \( P_\ell \) represents the ensemble size at resolution \( N_\ell \),
- \( \{M_\ell\} \subset \mathbb{N} \), where \( M_\ell \) represents the number of iid realizations of EnKF estimators on level \( \ell \).

Alternatively, the evaluation of the MLEnKF empirical measure in (12) on the QoI \( \varphi \) can be represented in the form

\[
\mu_{n,\text{MLE}}[\varphi] = \sum_{\ell=0}^{L} \sum_{m=1}^{M_\ell} \frac{1}{P_{\ell}} \varphi(\hat{v}_{n,i}^{\ell,f,m}) - \varphi(\hat{v}_{n,i}^{\ell,c,m}). \]

Recall that \( \varphi(\hat{v}_{n,i}^{\ell,f,m}) \) and \( \varphi(\hat{v}_{n,i}^{\ell,c,m}) \) in (13) come from the same driving noise, but sampled with fine and coarse time steps, respectively. In this way, the variance of the difference \( \varphi(\hat{v}_{n,i}^{\ell,f,m}) - \varphi(\hat{v}_{n,i}^{\ell,c,m}) \) gets smaller as the resolution level \( \ell \) increases, where we can achieve the overall variance reduction and potential efficiency gains. Note also that here, the updated distribution is our primary interest, although this can be done for predicting distributions in a similar manner.

In order to achieve performance gains with MLEnKF, we need to impose the following assumptions:

**Assumption 2.** For any exponentially increasing sequence \( \{N_\ell\} \) of natural numbers, there exists a constant \( c_\varphi > 0 \) and \( \beta > 0 \) such that for all \( \ell \in \mathbb{N}_0 \cup \{\infty\} \), \( n \in \mathbb{N}_0 \), and \( p \geq 2 \),

(i) for all \( |\kappa| = 1 \),

\[
\left\| \partial^{\kappa} \Psi_{n}^{N_\ell}(u) \right\|_{p} \leq c_{\varphi}(1 + \|u\|_{p}) \quad \forall u \in L^{p}_{\varphi}(\Omega, \mathbb{R}^{d}), \]

(ii) for all \( |\kappa| = 2 \),

\[
\left\| \partial^{\kappa} \Psi_{n}^{N_\ell}(u) \right\|_{p} \leq c_{\varphi}(1 + \|u\|_{2p}) \quad \forall u \in L^{2p}_{\varphi}(\Omega, \mathbb{R}^{d}), \]
(iii) for all \(|\kappa| \leq 1\),
\[
\|\partial^\alpha \Psi_n^{N+1}(u) - \partial^\alpha \Psi_n^N(u)\|_p \leq c_\Psi (1 + \|u\|_p) N^{-\beta/2} \quad \forall u \in L^p_n(\Omega, \mathbb{R}^d).
\]

**Remark 2.** If the dynamics (1) is given by the SDE in Remark 1 and \(\Psi_n^N\) denotes the corresponding Euler-Maruyama numerical solution, then Assumption 2 holds with \(\beta = 1\), cf. [41, 21, 22, 19].

2.6. Main results. We now present the main large-ensemble limit convergence theorems and computational cost versus accuracy corollaries for EnKF and MLEnKF. The proofs for these results are provided in Appendix A.

**Theorem 1** (Convergence of EnKF). If Assumption 1 holds, then for any \(\varphi \in \mathbb{F}\), \(p \geq 2\) and \(n \in \mathbb{N}_0\),
\[
\|\mu_n^{N,P}[\varphi] - \mu_n[\varphi]\|_p \lesssim N^{-1/2} + n^{\alpha},
\]
where the hidden constant in \(\lesssim\) depends on \(p\), \(n\) and \(d\), but not on \(N\) and \(P\).

**Corollary 1.** If Assumption 1 holds, then \(N \approx \epsilon^{-1/\alpha}\) and \(P \approx \epsilon^{-2}\) ensure that
\[
\|\mu_n^{N,P}[\varphi] - \mu_n[\varphi]\|_p \lesssim \epsilon
\]
for any \(\varphi \in \mathbb{F}\), \(p \geq 2\) and \(n \in \mathbb{N}_0\). The resulting computational cost is
\[
\text{Cost(EnKF)} \approx \epsilon^{-(2 + \frac{1}{\alpha})}.
\]

**Remark 3.** The proof of Theorem 1 shows that said theorem and the above corollary apply even if the constraint \(\mathbb{F} \subset C^2_P(\mathbb{R}^d, \mathbb{R})\) in Assumption 1(ii) is weakened to \(\mathbb{F} \subset C^1_P(\mathbb{R}^d, \mathbb{R})\).

**Theorem 2** (Convergence of MLEnKF). If Assumptions 1 and 2 hold, then for any triplet of sequences \(\{M_\ell\}, \{N_\ell\}, \{P_\ell\}\) \(\in \mathbb{N}\) as described in Section 2.5 and \(L \geq 1\), it holds for any \(\varphi \in \mathbb{F}\), \(n \geq 0\) and \(p \geq 2\) that
\[
\|\mu_n^{ML}[\varphi] - \bar{\mu}_n[\varphi]\|_p \lesssim N_L^{-\beta/2} P_L^{-1/2} + P_L^{-1} + N_L^{-\alpha} + \sum_{\ell=0}^{L} M_\ell^{-1/2} (N_\ell^{-\beta/2} P_\ell^{-1/2} + P_\ell^{-1}),
\]
where the hidden constant in \(\lesssim\) depends on \(p\), \(n\) and \(d\), but not on \(\{M_\ell\}, \{N_\ell\}, \{P_\ell\}\) and \(L\).

**Remark 4.** In Appendix A the Theorems 1 and 2 are for simplicity proved for EnKF and MLEnKF methods using biased sample covariances, precisely as introduced in (5) and (9), respectively. However, the proofs can be easily extended to both methods rather using unbiased sample covariances without altering the convergence rates, cf. Remark 5.

**Corollary 2.** If Assumptions 1 and 2 hold, then for any \(\epsilon, s > 0\), the configuration \(L = \left[\frac{\log_2(\epsilon^{-1})}{\min(1, (1+\beta s)/(2s))}\right]\), \(P_\ell \approx 2^\ell\), \(N_\ell \approx 2^{s \ell}\) and
\[
M_\ell \approx \begin{cases} 
\epsilon^{-2} 2^{-\frac{3+2s+\min(\beta s, 1)}{3}} \ell + 1 & \text{if } \min(\beta s, 1) > s, \\
\epsilon^{-2} 2^{-\frac{1+(s+1)\ell}{2}} + 1 & \text{if } \min(\beta s, 1) = s, \\
\epsilon^{-2} 2^{-\frac{2(s-\min(\beta s, 1))}{3}} 2^{-\frac{3+2s+\min(\beta s, 1)}{3}} \ell + 1, & \text{if } \min(\beta s, 1) < s,
\end{cases}
\]
ensures that for any \(\varphi \in \mathbb{F}\), \(n \geq 0\) and \(p \geq 2\),
\[
\|\mu_n^{ML}[\varphi] - \bar{\mu}_n[\varphi]\|_p \lesssim \epsilon.
\]
Moreover, if one sets the parameter $s$ such that

\[
s \in \begin{cases} 
\{\alpha^{-1}, 1\} & \text{if } \beta > 1 \text{ and } \alpha > 1, \\
\{\alpha^{-1}, 1\} & \text{if } (\beta > 1 \text{ and } \alpha = 1) \text{ or } (\beta = 1 \text{ and } \alpha \geq 1), \\
\{\alpha^{-1}\} & \text{if } (\beta \geq 1 \text{ and } \alpha < 1) \text{ or } (\beta < 1 \text{ and } \alpha \leq \beta), \\
(2\alpha - \beta)^{-1} & \text{if } \beta < 1 \text{ and } \alpha > \beta,
\end{cases}
\]

then the cost of the MLEnKF estimator is bounded by

\[
\text{Cost}(\text{MLEnKF}) \approx \begin{cases} 
\epsilon^{-2} & \text{if } \beta > 1 \text{ and } \alpha > 1, \\
\epsilon^{-2} \log(\epsilon)^3 & \text{if } (\beta > 1 \text{ and } \alpha = 1) \text{ or } (\beta = 1 \text{ and } \alpha \geq 1), \\
\epsilon^{-(1+1/\alpha)} & \text{if } (\beta \geq 1 \text{ and } \alpha < 1) \text{ or } (\beta < 1 \text{ and } \alpha \leq \beta), \\
\epsilon^{-(2+(1-\beta)/\alpha)} & \text{if } \beta < 1 \text{ and } \alpha > \beta.
\end{cases}
\]

Figure 2. Top row: comparison of the runtime versus RMSE for the QoIs mean (left) and variance (right) over $N = 10$ observation times for the problem in Section 3.1. The solid line represents MLEnKF, the dashed line is a fitted $O(\log(1+\text{Runtime})^{1/2}/\text{Runtime}^{-1/2})$ reference line, the solid-bulleted line represents EnKF and the dashed-bulleted line is a fitted $O(\text{Runtime}^{-1/3})$ reference line. Bottom row: similar plots over $N = 20$ observation times.

The optimization of the parameter $s$ is stated in terms of inclusion sets in (16) as that may be useful for implementations. For further details on this optimization and the computational cost of the MLEnKF estimator, see Appendix A.3.

The a priori cost versus error results of Corollaries 1 and 2 show that in settings where both rates are sharp, MLEnKF will, asymptotically, have better complexity than EnKF in computing weak approximations of observables (when viewing the mean-field limit $\bar{\mu}[^{\phi}]$ as the reference solution).
3. Numerical examples

In this section, we numerically compare the performance of MLEnKF and EnKF, verifying the optimal work rates predicted by our theory. To this end, we consider a couple of test problems with stochastic dynamics

\[ u_{n+1} = \Psi(u_n) := \int_n^{n+1} -V'(u_t) dt + \int_n^{n+1} \sigma dW_t, \]

where the potential function \( V \in C^\infty_{\text{loc}}(\mathbb{R}) \) satisfies \( V'' \in C^\infty_{\text{loc}}(\mathbb{R}) \), the diffusion constant \( \sigma = 0 \), and observations are in the form (3) with \( H = 1 \) and \( \Gamma = 0 \).

For any \( N \geq 1 \), \( \Psi_N \) will in this section denote the Milstein numerical scheme with uniform timestep \( \Delta t = 1/N \), yielding the convergence rate exponents \( \alpha = 1 \), \( \beta = 2 \) and the work rate exponent \( \gamma = 1 \).

For any \( x \in \mathbb{R} \), let \( \text{Round}(x) \) denote the nearest integer to \( x \). Given an input \( \epsilon > 0 \), we set, in order to equilibrate the asymptotic rates of error terms,

\[ P = \text{Round}(8\epsilon^{-2}) \quad \text{and} \quad N = \text{Round}(\epsilon^{-1}) \]

for EnKF, cf. (14), and for MLEnKF, \( L = \text{Round}(\log_2(\epsilon^{-1})) - 1 \) and

\[ N_\ell = 2^{\ell+1}, \quad P_\ell = 2^{\ell+2}, \quad M_\ell = \text{Round}(\epsilon^{-2}L^22^{-2\ell-3}), \quad \ell = 0, \ldots, L, \]

by Corollary 2 (with \( s = 1 \)).

Over a sequence of iterations \( n = 1, \ldots, N \), we compare the performance of the methods in terms of error versus computer runtime (wall-clock time), where the error is measured in terms of time averaged root mean squared error (RMSE):

\[ \text{RMSE(EnKF)} := \left( \frac{1}{100(N+1)} \sum_{i=1}^{100} \sum_{n=0}^{N} \left| \mu_{n,i}^{N,P} - \bar{\mu}_n \right|^2 \right)^{1/2} \lesssim \text{Runtime}^{-1/3}, \]

where for a given \( \epsilon > 0 \) with \( \alpha = 1 \), \( \beta = 2 \) and \( \gamma = 1 \), the last approximative asymptotic inequality follows from Corollary 1

\[ \text{RMSE(EnKF)} \approx \left( \frac{1}{N+1} \sum_{n=0}^{N} \left( \frac{\|\mu_{n,i}^{N,P} - \bar{\mu}_n\|_2^2}{N+1} \right) \right)^{1/2} \lesssim \epsilon \approx \text{Cost(EnKF)}^{-1/3}. \]

Similarly, with the same setting for MLEnKF

\[ \text{RMSE(MLEnKF)} := \left( \frac{1}{100(N+1)} \sum_{i=1}^{100} \sum_{n=0}^{N} \left| \mu_{n,i}^{\text{ML}} - \bar{\mu}_n \right|^2 \right)^{1/2} \lesssim \log(10 + \text{Runtime})^{3/2} \text{Runtime}^{-1/2}, \]

where the last inequality follows from Corollary 2

\[ \text{RMSE(MLEnKF)} \lesssim \epsilon \approx \log(\text{Cost(MLEnKF)})^{3/2} \text{Cost(MLEnKF)}^{-1/2}. \]

Note, \( \{\mu_{n,i}^{N,P} \}_{i=1}^{100} \) and \( \{\mu_{n,i}^{\text{ML}} \}_{i=1}^{100} \) are computed using iid sequences of EnKF and MLEnKF empirical measures, respectively.

For the first test problem, when the dynamics \( \Psi \) is linear, the reference solution \( \bar{\mu}_n \) is computed straightforwardly using the Kalman filter formulas. For the second test problem, which has nonlinear dynamics, a pseudo-reference solution is
obtained by the deterministic mean-field EnKF (DMFEnKF) algorithm described in [33], cf. Appendix B. Notice also that the optimisation of constant factors in the parameters (19) and (20) to improve the practical complexity is beyond the scope of this work.

The numerical simulations were computed in parallel on five cores on an iMac Pro 2018 with Intel Xeon W-2140B 8-core processor with 32 GB of RAM. The computer code is written in the Julia programming language [3], and it can be downloaded from https://github.com/GaukharSH/mlenkf.

3.1. Ornstein-Uhlenbeck process. We consider the SDE (18) with the potential $V(u) = u^2/2$, and the initial condition is $u_0|Y_0 \sim N(0, \Gamma)$, which is an Ornstein-Uhlenbeck (OU) process with the exact solution

$$\Psi(u_n) = u_ne^{-t} + \int_0^t \sigma e^{s-t}dW_s.$$ 

For a sequence of inputs, $\epsilon = [2^{-4}, 2^{-5}, \ldots, 2^{-10}]$ for EnKF and $\epsilon = [2^{-4}, 2^{-5}, \ldots, 2^{-11}]$ for MLEnKF, we study the performance of the two methods in terms of runtime versus RMSE over timeframes of $N = 10$ and $N = 20$ observation times in Figure 2. For both QoI considered (the mean and the variance), the observations are in agreement with Corollaries 1 and 2, as we observe that MLEnKF outperforms EnKF for small values of RMSE.

3.2. Double-well SDE. In the second example, we consider the SDE (18) with the double-well potential

$$V(u) = \frac{1}{2 + 4u^2} + \frac{u^2}{4}.$$ 

This SDE is the metastable motion of particles between the two wells, cf. [7, 40]. Figure 3 shows the dynamics of a particle over $N = 20$ observation times. We observe that the particle remains in either one of the wells for a relatively long time; in other words, transitions between wells happen relatively rarely. Figure 4 illustrates the essential well transition of an EnKF ensemble over a few predict-update cycles when initially the observations and the EnKF ensemble are located in opposite wells.
Using the same initial condition and sequences of $\epsilon$-inputs as in the preceding example, we study the performance of the two methods in terms of runtime versus RMSE over timeframes of $N = 10$ and $N = 20$ observation times in Figure 5. Once again we observe for both QoI considered, the mean and the variance, that the work rates, given by the slopes in Figure 5, are in agreement with Corollaries 1 and 2, and that MLEnKF outperforms EnKF for small values of RMSE.
Figure 5. Top row: comparison of the runtime versus RMSE for the QoIs mean (left) and variance (right) over $N = 10$ observation times for the problem in Section 3.2. The solid line represents MLEnKF, the dashed line is a fitted $O(\log(10+\text{Runtime})^{1/3}\text{Runtime}^{-1/2})$ reference line, the solid-bulleted line represents EnKF and the dashed-bulleted line is a fitted $O(\text{Runtime}^{-1/3})$ reference line. Bottom row: similar plots over $N = 20$ observation times.

Appendix A. Theoretical proofs

A.1. EnKF. In this section, we present a collection of theoretical results for EnKF that culminates with proof of Theorem 1.

Notation. The sample average of a QoI $\varphi \in F$ applied to an ensemble of $P$ identically distributed particles $\{\hat{v}_{n,i}^{N,P}\}_{i=1}^P$ is defined as

$$E_P[\varphi(\hat{v}_{n,i}^{N,P})] := \frac{1}{P} \sum_{i=1}^P \varphi(\hat{v}_{n,i}^{N,P}) \quad (= \mu_{n, P}^{N,P} [\varphi]),$$

and similarly

$$E_P[\hat{v}_{n,i}^{N,P}] := \frac{1}{P} \sum_{i=1}^P \hat{v}_{n,i}^{N,P} \quad \text{and} \quad E_P[(\hat{v}_{n,i}^{N,P}) (\hat{v}_{n,i}^{N,P})^T] := \frac{1}{P} \sum_{i=1}^P (\hat{v}_{n,i}^{N,P}) (\hat{v}_{n,i}^{N,P})^T.$$

Lemma 1 (Difference between EnKF and MFEnKF Kalman gains). If Assumption 1 holds, then

$$\|K_n^{N,P} - \tilde{K}_n^{N}\|_p \leq |\Gamma^{-1}|_2 |H|_2 (1 + 2|\tilde{K}_n^{N} H|_2) \|C_n^{N,P} - \tilde{C}_n^{N}\|_p \lesssim \|C_n^{N,P} - \tilde{C}_n^{N}\|_p$$

for any $n \in N_0$ and $p \geq 2$. 
Proof. The proof of the first inequality is analogous to [20, Lemma 3.4], and the latter inequality follows from $\tilde{v}_n^N \in C_{p,2}^p L^p(\Omega, \mathbb{R}^d)$, uniformly in $N \geq 1$, cf. (7).

We next bound the difference between $C_n^{N,P} = \text{Cov}[v_n^{N,P}]$ and $\tilde{C}_n^N = \text{Cov}[\tilde{v}_n^N]$.

**Lemma 2** (Difference between EnKF and MFEnKF covariance matrices). If Assumption 1 holds, then

$$\left\| C_n^{N,P} - \tilde{C}_n^N \right\|_p \lesssim \left\| v_n^{N,P} - \tilde{v}_n^N \right\|_{2p} + P^{-\frac{1}{2}}$$

for any $n \in \mathbb{N}_0$ and $p \geq 2$.

Proof. In order to bound $\left\| C_n^{N,P} - \tilde{C}_n^N \right\|_p$, we introduce the auxiliary mean-field ensemble $\hat{\tilde{v}}_n^{N,P} = (\hat{\tilde{v}}_{n,i}^{N,P})_{i=1}^P$ consisting of $P$ identically distributed particles whose prediction/update dynamics is given by the mean-field Kalman gain, namely

$$\begin{align*}
\hat{\tilde{v}}_n^{N,P} & = \Psi_n (\hat{\tilde{v}}_{n,i}^{N,P}), \\
\hat{\tilde{v}}_{n+1,i}^{N,P} & = (I - \tilde{K}_{n+1}^N H) \tilde{v}_{n+1,i}^{N,P} + \tilde{K}_{n+1}^N \hat{\tilde{y}}_{n+1,i},
\end{align*}$$

(22)

with the initial condition $\hat{\tilde{v}}_{0,i}^{N,P} = \hat{\tilde{v}}_{n,i}^{N,P}$ for $i = 1, ..., P$. The auxiliary ensemble satisfies the following two crucial properties: $\hat{\tilde{v}}_{n,i}^{N,P} \overset{D}{=} \hat{\tilde{v}}_n$ and $\hat{\tilde{v}}_{n,i}^{N,P}$ has the same initial data, driving noise $\mathcal{W}$ and perturbed observations as the $i$-th EnKF particle $v_{n,i}^{N,P}$, cf. Section 2.3.

Introducing the auxiliary ensemble covariance based on the mean-field prediction particles,

$$\tilde{C}_n^{N,P} := \text{Cov}[\hat{\tilde{v}}_n^{N,P}],$$

the triangle inequality implies that

$$\left\| C_n^{N,P} - \tilde{C}_n^N \right\|_p \leq \left\| C_n^{N,P} - \tilde{C}_n^{N,P} \right\|_p + \left\| \tilde{C}_n^{N,P} - \tilde{C}_n^N \right\|_p.$$

For the first term, recalling the notation (21), we have

$$\left\| C_n^{N,P} - \tilde{C}_n^{N,P} \right\|_p \leq \left\| E_P[v_n^{N,P}] - (v_n^{N,P} T - \hat{\tilde{v}}_n^{N,P} (\hat{\tilde{v}}_n^{N,P})^T) \right\|_p + \left\| E_P[(v_n^{N,P} - \tilde{v}_n^N)^2] \right\|_p + \left\| E_P[(v_n^{N,P} - \tilde{v}_n^N)^2] \right\|_p.$$

For the second term in (24), we have

$$\left\| \tilde{C}_n^{N,P} - \tilde{C}_n^N \right\|_p \leq \left\| E_P[\tilde{v}_n^{N,P} (\hat{\tilde{v}}_n^{N,P})^T] - E [\tilde{v}_n^{N,P} (\hat{\tilde{v}}_n^{N,P})^T] \right\|_p + \left\| E_P[\hat{\tilde{v}}_n^{N,P} (\hat{\tilde{v}}_n^{N,P})^T] - E [\hat{\tilde{v}}_n^{N,P} (\hat{\tilde{v}}_n^{N,P})^T] \right\|_p.$$

Jensen’s and Hölder’s inequalities and $v_n^{N,P} \in L^p(\Omega, \mathbb{R}^d)$ yield that

$$\left\| \tilde{C}_n^{N,P} - \tilde{C}_n^N \right\|_p \leq 2 \left\| v_n^{N,P} \right\|_{2p} \left\| v_n^{N,P} - \hat{\tilde{v}}_n^{N,P} \right\|_{2p} + 2 \left\| v_n^{N,P} \right\|_{2p} \left\| v_n^{N,P} - \hat{\tilde{v}}_n^{N,P} \right\|_{2p} \leq \max \left( \left\| v_n^{N,P} - \hat{\tilde{v}}_n^{N,P} \right\|_{2p} \left\| v_n^{N,P} - \hat{\tilde{v}}_n^{N,P} \right\|_{2p} \right).$$

For the second term in (24), we have

$$\left\| \tilde{C}_n^{N,P} - \tilde{C}_n^N \right\|_p \leq \left\| E_P[\tilde{v}_n^{N,P} (\hat{\tilde{v}}_n^{N,P})^T] - E [\tilde{v}_n^{N,P} (\hat{\tilde{v}}_n^{N,P})^T] \right\|_p + \left\| E_P[\hat{\tilde{v}}_n^{N,P} (\hat{\tilde{v}}_n^{N,P})^T] - E [\hat{\tilde{v}}_n^{N,P} (\hat{\tilde{v}}_n^{N,P})^T] \right\|_p.$$
where the penultimate inequality is obtained by the Marcinkiewicz-Zygmund (M-Z) inequality [30, Theorem 5.2]. The property \( \tilde{v}_n^{N,P} \in L^p(\Omega, \mathbb{R}^d) \), together with the above inequalities imply that

\[
\|C_n^{N,P} - \tilde{C}_n^{N} \|_p \lesssim \max(\|v_n^{N,P} - \tilde{v}_n^{N,P} \|_{2p}, \|v_n^{N,P} - \bar{v}_n^{N,P} \|_{2p}) + P^{-\frac{1}{2}}.
\]

Finally, by the proof of Lemma 3, it follows that \( \|v_n^{N,P} - \bar{v}_n^{N,P} \|_{2p} \lesssim 1 \) and thus

\[
\|C_n^{N,P} - \tilde{C}_n^{N} \|_p \lesssim \|v_n^{N,P} - \bar{v}_n^{N,P} \|_{2p} + P^{-\frac{1}{2}}.
\]

\[\square\]

**Remark 5.** Lemma 2 straightforwardly extends to EnKF methods using an unbiased sample prediction covariance \( C_n^{N,P} \) (from the here considered biased sample covariance approach \( C_n^{N,P} \), cf. (5)). Using the unbiased/biased sample covariance relationship

\[
C_n^{N,P} := \frac{P}{P-1} C_n^{N,P},
\]

we obtain that

\[
\|C_n^{N,P} - \tilde{C}_n^{N} \|_p \leq \|C_n^{N,P} - C_n^{N,P} \|_p + \|C_n^{N,P} - \tilde{C}_n^{N} \|_p
\]

\[
= \frac{P}{P-1} \|C_n^{N,P} - \tilde{C}_n^{N} \|_p + \frac{1}{P-1} \|\tilde{C}_n^{N,P} \|_p.
\]

The first term in the last equality is bounded by Lemma 2, and the bound for the second term follows from the property \( \tilde{v}_n^{N,P} \in L^p(\Omega, \mathbb{R}^d) \). In the proof of Theorem 1, the error contribution of the either biased or unbiased sample covariances only enter through Lemma 2, which implies that the theorem also holds (with the same convergence rate) for EnKF with unbiased sample covariances.

Moreover, in the proof of Theorem 2 for the MLEnKF setting the error contribution from sample covariances only enter through Corollary 3 and Lemma 8. By a similar argument as above, the rates of said corollary and lemma are not affected by replacing the biased sample covariances by unbiased ones, and Theorem 2 also holds (with the same convergence rate) for MLEnKF with unbiased sample covariances.

**Lemma 3 (Distance between ensembles).** If Assumption 1 holds, then

\[
\max \left( \|v_n^{N,P} - \bar{v}_n^{N} \|_p, \|v_n^{N,P} - \tilde{v}_n^{N} \|_p \right) \lesssim P^{-1/2} \quad \text{for any} \quad n \in \mathbb{N}_0 \quad \text{and} \quad p \geq 2.
\]

**Proof.** Since \( \hat{v}_0^{N,P} = \hat{v}_0^{N} \), the first half of the statement holds for \( n = 0 \). Assume that for some \( n \geq 1 \), \( \|v_{n-1}^{N,P} - \bar{v}_{n-1}^{N} \|_p \lesssim P^{-1/2} \) holds for all \( p \geq 2 \). Assumption 1(iii) then implies that

\[
\|v_n^{N,P} - \bar{v}_n^{N} \|_p \lesssim \|v_{n-1}^{N,P} - \bar{v}_{n-1}^{N} \|_p \lesssim P^{-\frac{1}{2}}.
\]

By H"older’s inequality and Lemma 1,

\[
\|v_n^{N,P} - \hat{v}_n^{N} \|_p \lesssim |I - K_n^{N} H|_2 \|v_n^{N,P} - \bar{v}_n^{N} \|_p
\]

\[
+ \|C_n^{N,P} - \tilde{C}_n^{N} \|_{2p} \left( \|v_n^{N,P} - \tilde{v}_n^{N} \|_{2p} + \|\hat{y}_n - H\tilde{v}_n^{N} \|_{2p} \right).
\]

Since \( \bar{v}_n^{N}, \tilde{v}_n^{N} \in \cap_{\gamma \geq 2} L^r(\Omega) \), inequalities (26) and (25) imply that

\[
\|v_n^{N,P} - \hat{v}_n^{N} \|_p \lesssim (|I - K_n^{N} H|_2 + \|\hat{y}_n - H\tilde{v}_n^{N} \|_{2p}) P^{-1/2} + P^{-1} \lesssim P^{-1/2}.
\]

The argument holds for any \( p \geq 2 \), and the proof follows by induction. \[\square\]
Before proving the main convergence result for EnKF, we introduce the notation \( \hat{\mu}^{N,P}_n[\varphi] \) to denote the average of the QoI \( \varphi \) over the empirical measure associated with the auxiliary mean-field ensemble \( \{\hat{\mu}^{N,i}_n\}_{i=1}^P \) cf. (22), and we recall that \( \hat{\mu}^N_0[\varphi] \) denotes the evaluation of empirical measure on QoI \( \varphi \) associated with the MFEEnKF \( \hat{n}_N \), cf. Section 2.4.

**Proof of Theorem 1.** By the triangle inequality,
\[
\left\| \mu^{N,P}_n[\varphi] - \hat{\mu}_n[\varphi] \right\|_p \leq \left\| \mu^{N,P}_n[\varphi] - \hat{\mu}^{N,P}_n[\varphi] \right\|_p + \left\| \hat{\mu}^{N,P}_n[\varphi] - \hat{\mu}^N_0[\varphi] \right\|_p + \left\| \hat{\mu}^N_0[\varphi] - \hat{\mu}_n[\varphi] \right\|_p.
\]
By Assumption 1(ii), \( \varphi \in \mathbb{F} \subseteq C^2_{\mathbb{P}}(\mathbb{R}^d, \mathbb{R}) \), and Lemma 3 implies that
\[
\left\| \mu^{N,P}_n[\varphi] - \hat{\mu}^{N,P}_n[\varphi] \right\|_p = \left\| \frac{1}{P} \sum_{i=1}^P \varphi(\hat{\mu}^{N,P}_{n,i}) - \varphi(\hat{\mu}^{N,P}_{n,0}) \right\|_p \leq c_{\varphi} \left\| \hat{\mu}^{N,P}_n - \hat{\mu}^N_0 \right\|_{2p} \lesssim P^{-1/2}.
\]
For the second term, using the Marcinkiewicz–Zygmund (M-Z) inequality, that \( \varphi(\hat{\mu}^{N}_0) \in \cap_{p \geq 2} L^p(\Omega, \mathbb{R}) \) and that \( \hat{\mu}^{N,1}_n, \ldots, \hat{\mu}^{N,P}_n \) are iid with \( \hat{\mu}^{N,P} \overset{d}{=} \hat{\mu}^N_0 \) imply that
\[
\left\| \mu^{N,P}_n[\varphi] - \hat{\mu}^N_0[\varphi] \right\|_p = \left\| \sum_{i=1}^P \frac{\varphi(\hat{\mu}^{N,P}_{n,i}) - \mathbb{E}\left[\varphi(\hat{\mu}^{N,P}_{n,0})\right]}{P} \right\|_p \lesssim P^{-1/2}.
\]
For the last term, we have that
\[
\left\| \hat{\mu}^N_0[\varphi] - \hat{\mu}_n[\varphi] \right\|_p = \left\| \mathbb{E}\left[\varphi(\hat{\mu}^{N}_0) \right] - \varphi(\hat{\mu}_n) \right\|_p,
\]
and it remains to prove by induction that the right-hand side is \( \mathcal{O}(N^{-\alpha}) \).

Since \( \hat{\mu}^{N}_0 \overset{d}{=} \hat{\mu}_0 \), it holds that \( \left\| \mathbb{E}\left[\varphi(\hat{\mu}^{N}_0) \right] - \varphi(\hat{\mu}_0) \right\| \right\|_p \right\|_p \lesssim P^{-1/2}. \)

Then, Assumption 1(ii) implies there exists a \( \tilde{c}_{\varphi} > 0 \) such that
\[
\left\| \mathbb{E}\left[\varphi(\hat{\mu}^{N}_0) \right] - \varphi(\hat{\mu}_n) \right\|_p \lesssim \tilde{c}_{\varphi} N^{-\alpha} \forall \varphi \in \mathbb{F}.
\]

In order to bound \( \left\| \mathbb{E}\left[\varphi(\hat{\mu}^{N}_0) \right] - \varphi(\hat{\mu}_n) \right\|_p \), we first recall that
\[
\hat{\mu}^{N}_0 = (I - K^N H)\bar{\mu}^{N}_0 + K^N \bar{\gamma} + \bar{\gamma} \tilde{\gamma}_k,
\]
with \( \tilde{\gamma}_k \sim N(0, \Gamma) \) and introduce the functions \( \varphi^{N}, \hat{\varphi} \in \mathbb{F} \) defined by
\[
\varphi^{N}(x) = \frac{1}{\sqrt{(2\pi)^p|\det(\Gamma)|}} \int_{\mathbb{R}^p} \varphi \left( (I - K^N H)\bar{\mu}^{N}_0 + K^N \bar{\gamma} + \bar{\gamma} \tilde{\gamma}_k \right) e^{-\frac{1}{2}z^T \Gamma^{-1} z} dz
\]
and
\[
\hat{\varphi}(x) = \frac{1}{\sqrt{(2\pi)^p|\det(\Gamma)|}} \int_{\mathbb{R}^p} \varphi \left( (I - K^N H)\bar{\mu}^{N}_0 + K^N \bar{\gamma} + \bar{\gamma} \tilde{\gamma}_k \right) e^{-\frac{1}{2}z^T \Gamma^{-1} z} dz.
\]
It then follows by the mean-value theorem that
\[
\left\| \mathbb{E}\left[\varphi(\hat{\mu}^{N}_0) \right] - \varphi(\hat{\mu}_n) \right\| \lesssim \left\| \mathbb{E}\left[\varphi^{N}(\hat{\mu}^{N}_0) - \varphi^{N}(\hat{\mu}) \right] \right\| + \left\| \mathbb{E}\left[\varphi^{N}(\hat{\mu}) - \varphi(\hat{\mu}) \right] \right\| \lesssim N^{-\alpha} + |\hat{\mu} - \hat{\mu}^{N}_0|_2.
\]
From [20, Lemma 3.4], we have that
\[
\hat{\mu} - \hat{\mu}^{N}_0 = \frac{\hat{\mu} - \hat{\mu}^{N}_0}{\hat{K} - \hat{K}^{N}} = \frac{\hat{K}H(\hat{C}^{N}_k - \hat{C}_k)H^T(H\hat{C}^{N}_kH^T + \Gamma)^{-1} + (\hat{C}_k - \hat{C}^{N}_k)H^T(H\hat{C}^{N}_kH^T + \Gamma)^{-1},}
\]
which, since $\Gamma$ is positive definite and thus $|(HC_k^N H^T + \Gamma)^{-1}|_2 \preceq |\Gamma^{-1}|_2 < \infty$, implies that
\[
|\tilde{K}_k - \tilde{K}_k^N|_2 \lesssim |\tilde{C}_k^N - \tilde{C}_k|_2 \\
\leq \|E[\tilde{v}_k^n(\tilde{v}_k^n)^T - \tilde{v}_k(\tilde{v}_k)^T]\|_2 + \|E[\tilde{v}_k^n] E[(\tilde{v}_k^n)^T - E[\tilde{v}_k] E[(\tilde{v}_k)^T]]\|_2 \\
\lesssim N^{-\alpha}.
\]
Since all monomials of degree 1 and 2 are contained in $\mathbb{F}$, the last inequality follows from (28) and the equivalence of the Euclidean and Frobenius norms in any dimension $d < \infty$. It holds by induction that for any $n \geq 0$,
\[
|E[\varphi(\tilde{v}_n^n)] - \varphi(\hat{v}_n^n)| \lesssim N^{-\alpha}.
\]

\textbf{A.2. MLEnKF.} In this section, we present a collection of theoretical results for MLEnKF, including the proof of Theorem 2.

In order to obtain a connection between $\{\tilde{v}_{n,1}^{\ell,f_1}, \ldots, \tilde{v}_{n,1}^{\ell,f_{P_{\ell-1}}}, \tilde{v}_{n,1}^{\ell,f_{P_{\ell-1}+1}}, \ldots, \tilde{v}_{n,1}^{\ell,f_{P_{\ell-1}+1}+1}\}$ in the superindex $c_j + f_j$, we introduce
\[
\tilde{v}_{n,i}^{\ell,f_1} := \tilde{v}_{n,i}^f \quad \text{and} \quad \tilde{v}_{n,i}^{\ell,f_2} := \tilde{v}_{n,i}^{f_{P_{\ell-1}+1}+1} \quad \text{for} \quad i = 1, \ldots, P_{\ell-1},
\]
and
\[
\mu_{n,f_j}^\ell[\varphi] := E_p^{t_{\ell-1}[\varphi(\tilde{v}_{n,i}^{f_j})}] = \frac{1}{P_{\ell-1}} \sum_{i=1}^{P_{\ell-1}} \varphi(\tilde{v}_{n,i}^{f_j}) \quad \text{for} \quad j = 1, 2.
\]

The random variable $\tilde{v}_{n,i}^{f_j}$ has the same driving noise and perturbed observations as $\tilde{v}_{n,i}^{f_j}$ and that the following relationship holds:
\[
\left(\mu_n^{f_1} - \mu_n^{f_2} + \mu_n^{c_j} \frac{2}{2}\right)[\varphi] = E_p[\varphi(\tilde{v}_{n,i}^{f_1} - \varphi(\tilde{v}_{n,i}^{f_2})]
\]
\[
= \frac{1}{2} \sum_{j=1}^{2} (\mu_n^{f_j} - \mu_n^{f_j})[\varphi].
\]

We further introduce the auxiliary mean-field MLEnKF ensemble $\{\tilde{v}_{n,1}^{\ell,f_1}, \ldots, \tilde{v}_{n,1}^{\ell,f_{P_{\ell-1}+1}}\}$ with the convention that $\tilde{v}_{n,i}^{0,e} := 0$ for all $n$ and $i = 1, \ldots, P_{\ell}$, that the dynamics for $\tilde{v}_{n,i}^{f}$ and $\tilde{v}_{n,i}^{e}$ are coupled through the dynamics
\[
\tilde{v}_{n+1,i}^{\ell,f} = \Psi_{n}^{\ell}(\tilde{v}_{n,i}^{\ell,f}), \quad \tilde{v}_{n+1,i}^{\ell,e} = \Psi_{n}^{\ell}(\tilde{v}_{n,i}^{\ell,e})
\]
and
\[
\tilde{v}_{n+1,i}^{\ell,f} = (I - \tilde{K}_n^{\ell,e} H) \tilde{v}_{n,i}^{\ell,f} + \tilde{K}_n^{\ell,e} \gamma_{n,i}, \quad \tilde{v}_{n+1,i}^{\ell,e} = (I - \tilde{K}_n^{\ell,e} H) \tilde{v}_{n,i}^{\ell,e} + \tilde{K}_n^{\ell,e} \gamma_{n,i},
\]
where $\tilde{K}_n^{\ell,f} := \tilde{K}_n^{N,t}$ and $\tilde{K}_n^{\ell,e} := \tilde{K}_n^{N,t-1}$, cf. Section 2.4, and with initial conditions equal identical to MLEnKF:
\[
\tilde{v}_{0,i}^{\ell,f} := \tilde{v}_{0,i}^{\ell,e} = \tilde{v}_{0,i}^{\ell,f} = \tilde{v}_{0,i}^{\ell,e} = 0, \quad \ell \geq 1.
\]

The particles $(\tilde{v}_{n,i}^{\ell,f}, \tilde{v}_{n,i}^{\ell,e})$ are thus coupled through sharing the same initial condition, driving noise $W$, and perturbed observations. The particle pair is also coupled to the MLEnKF pair $(\tilde{v}_{n,i}^{f}, \tilde{v}_{n,i}^{e})$ in all the same ways. The ensemble
For the second term in (32), note first that
\[
\mathbb{E} \left[ \mu_n^{L, \ell} \mid \varphi \right] = \mathbb{E} \left[ \mu_n^{L, \ell, f} + \mu_n^{L, \ell, c} \right] = 0 \text{ for } \ell = 0. \text{ Similar to (12), we define the auxiliary mean-field MLEnKF estimator by}
\[
\hat{\mu}_n^{ML}[\varphi] = \sum_{\ell=0}^{L} \sum_{m=1}^{M_\ell} \frac{1}{M_\ell} \left( \hat{\mu}_n^{f, m} - \hat{\mu}_n^{c, m} \right) \mid \varphi \mid
\]
where, for \( m = 1, \ldots, M_\ell \), \((\hat{\mu}_n^{f, m} - \hat{\mu}_n^{c, m}) \mid \varphi \mid \) are iid realizations that are coupled to \((\mu_n^{f, m} - \mu_n^{c, m}) \mid \varphi \mid \) by sharing the same underlying randomness (i.e., driving noise and perturbed observations).

**Proof of Theorem 2.** By the triangle inequality,
\[
\| \mu_n^{ML}[\varphi] - \hat{\mu}_n[\varphi] \|_p \leq \| \mu_n^{ML}[\varphi] - \hat{\mu}_n^{ML}[\varphi] \|_p + \| \hat{\mu}_n^{ML}[\varphi] - \hat{\mu}_n^N[\varphi] \|_p
\]
\[
+ \| \hat{\mu}_n^N[\varphi] - \hat{\mu}_n[\varphi] \|_p.
\]
By (13) and (31), \( \Delta \hat{\mu}_n^{\ell} \mid \varphi \mid := \mathbb{E} \left[ (\hat{\mu}_n^{f, \ell} - \hat{\mu}_n^{f, \ell} + \mu_n^{c, \ell} - \mu_n^{c, \ell}) \mid \varphi \mid \right] \) and the M-Z inequality,
\[
\| \mu_n^{ML}[\varphi] - \hat{\mu}_n^{ML}[\varphi] \|_p \leq \left\| \sum_{\ell=0}^{L} \Delta \hat{\mu}_n^{\ell}[\varphi] \right\|_p
\]
\[
+ \left\| \sum_{\ell=0}^{L} \sum_{m=1}^{M_\ell} \left( \hat{\mu}_n^{f, m} - \hat{\mu}_n^{c, m} \right) \mid \varphi \mid - \Delta \hat{\mu}_n^{\ell}[\varphi] \right\|_p
\]
\[
\leq \mathbb{E} \left[ (\mu_n^{L, \ell} - \mu_n^{L, \ell}) \mid \varphi \mid \right] \mid \varphi \mid + \sum_{\ell=0}^{L} M_\ell^{-1/2} \| \left( \hat{\mu}_n^{f, \ell} - \mu_n^{f, \ell} + \mu_n^{c, \ell} - \mu_n^{c, \ell} \right) \mid \varphi \mid \|_p
\]
where we used that \( \mathbb{E} \left[ \mu_n^{c, \ell} \right] = \mathbb{E} \left[ \mu_n^{c, \ell-1} \right] \) and \( \mathbb{E} \left[ \hat{\mu}_n^{c, \ell} \right] = \mathbb{E} \left[ \hat{\mu}_n^{c, \ell-1} \right] \) for \( \ell \geq 1 \) in the last inequality. The \( L^p(\Omega) \)-convergence \( \mu_n^{L, \ell} \mid \varphi \mid \rightarrow \mu_n^{L, \ell} \mid \varphi \mid \) as \( L \rightarrow \infty \), cf. Theorem 1, further implies that
\[
(\mu_n^{L, \ell} - \mu_n^{L, \ell}) \mid \varphi \mid \mid \varphi \mid = \sum_{\ell=L+1}^{\infty} \mathbb{E} \left[ (\hat{\mu}_n^{f, \ell} - \mu_n^{f, \ell} + \mu_n^{c, \ell} - \mu_n^{c, \ell}) \mid \varphi \mid \right],
\]
and Jensen’s inequality and Lemma 11 yield
\[
\| \mu_n^{ML}[\varphi] - \hat{\mu}_n^{ML}[\varphi] \|_p \leq \sum_{\ell=0}^{L} M_\ell^{-1/2} \| (\hat{\mu}_n^{f, \ell} - \mu_n^{f, \ell} + \mu_n^{c, \ell} - \mu_n^{c, \ell}) \mid \varphi \mid \|_p
\]
\[
+ \sum_{\ell=L+1}^{\infty} \| (\hat{\mu}_n^{f, \ell} - \mu_n^{f, \ell} + \mu_n^{c, \ell} - \mu_n^{c, \ell}) \mid \varphi \mid \|_p
\]
\[
\leq N_L^{-\beta/2} P_L^{-1/2} + P_L^{-1} + \sum_{\ell=0}^{L} M_\ell^{-1/2} (N_\ell^{-\beta/2} P_\ell^{-1/2} + P_\ell^{-1}).
\]
For the second term in (32), note first that
\[
\mathbb{E} \left[ \hat{\mu}_n^N[\varphi] \right] = \sum_{\ell=0}^{L} \mathbb{E} \left[ (\hat{\mu}_n^{f, \ell} - \mu_n^{f, \ell}) \mid \varphi \mid \right] = \sum_{\ell=0}^{L} \mathbb{E} \left[ (\hat{\mu}_n^{f, \ell} - \mu_n^{f, \ell}) \mid \varphi \mid \right].
\]
By applying the M-Z inequality twice (first in $M_\ell$ and thereafter in $P_\ell$) and Lemma 6,

$$\|\bar{\mu}^{ML}_n[\varphi] - \bar{\mu}^N_n[\varphi]\|_p \leq \sum_{\ell=0}^L \frac{M_\ell \left( \sum_{m=1}^{M_\ell} (\bar{\mu}^{M,\ell}_n - \bar{\mu}^{M,\ell}_m)[\varphi] - E \left[ (\bar{\mu}^{M,\ell}_n - \bar{\mu}^N_n)[\varphi] \right] \right)}{M_\ell} \left\| E \left[ \varphi(\tilde{\xi}^{L,f}_n) - \varphi(\tilde{\xi}^{L,f}_{n+1}) - \varphi(\tilde{\xi}^{L,e}_n) \varphi(\tilde{\xi}^{L,e}_{n+1}) \right] \right\|_p$$

$$\lesssim \sum_{\ell=0}^L M_\ell^{-1/2} \left\| E \left[ \varphi(\tilde{\xi}^{L,f}_n) - \varphi(\tilde{\xi}^{L,f}_{n+1}) - \varphi(\tilde{\xi}^{L,e}_n) \varphi(\tilde{\xi}^{L,e}_{n+1}) \right] \right\|_p$$

$$\lesssim \sum_{\ell=0}^L M_\ell^{-1/2} P_\ell^{-1/2} \left\| \varphi(\tilde{\xi}^{L,f}_n) - \varphi(\tilde{\xi}^{L,f}_{n+1}) \right\|_p$$

$$\lesssim \sum_{\ell=0}^L M_\ell^{-1/2} P_\ell^{-1/2} N_\beta^{-\beta/2}.$$

For the third term in (32), it follows by (28) that

$$\|\bar{\mu}^{NL}_n[\varphi] - \bar{\mu}_n[\varphi]\|_p = E \left[ \varphi(\tilde{\xi}^{L,f}_n) - \varphi(\tilde{\xi}_n) \right] \lesssim N_\beta^{-\alpha}.$$

\[\square\]

**Lemma 4** (Continuity of mean-field Kalman gains). *It holds that*

$$|\bar{K}^{\ell,f}_n - \bar{K}^{\ell,f}_n| \leq |\Gamma^{-1}|_2 |H|_2 (1 + 2|\bar{K}^{\ell,f}_n H|_2) |\bar{C}^{\ell,e}_n - \bar{C}^{\ell,e}_n| \lesssim |\bar{C}^{\ell,e}_n - \bar{C}^{\ell,e}_n|_2.$$

**Proof.** The proof is analogous to [20, Lemma 3.4].

\[\square\]

**Lemma 5** (Continuity of mean-field covariance matrices). *If Assumptions 1 and 2 hold, then for any triplet of sequences $\{M_\ell\}, \{N_\ell\}, \{P_\ell\} \subset \mathbb{N}$ described in Section 2.5, $L \geq 0$ and $n \geq 1$ it holds that*

$$|\bar{C}^{\ell,e}_n - \bar{C}^{\ell,e}_n|_2 \lesssim \|\bar{v}^{\ell,e}_n - \bar{v}^{\ell,e}_n\|_4.$$

**Proof.** Using that $\bar{v}^{\ell,e}_n, \bar{v}^{\ell,e}_n \in L^4(\Omega, \mathbb{R}^d)$, Hölder’s and Jensen’s inequalities yield that

$$|\bar{C}^{\ell,e}_n - \bar{C}^{\ell,e}_n|_2 \leq E \left[ \bar{v}^{\ell,e}_n (\bar{v}^{\ell,e}_n)^T - \bar{v}^{\ell,e}_n (\bar{v}^{\ell,e}_n)^T \right]_2 + E \left[ \bar{v}^{\ell,e}_n \right] E \left[ (\bar{v}^{\ell,e}_n)^T \right]_2 \lesssim \|\bar{v}^{\ell,e}_n - \bar{v}^{\ell,e}_n\|_4.$$

\[\square\]

In the remaining part of this section, we will assume that Assumptions 1 and 2 hold, and that the sequences $\{N_\ell\}, \{P_\ell\} \subset \mathbb{N}$ satisfy the constraints given in Section 2.5 (namely, $P_\ell = 2P_{\ell-1}$ and $\{N_\ell\}$ exponentially increasing).

**Lemma 6** (Stability of mean-field particles). *For any $n \geq 0$ and $p \geq 2$, it holds that*

$$\max \left( \frac{\|\tilde{v}^{\ell,e}_n - \tilde{v}^{\ell,e}_{n+1}\|_p}{\|\tilde{v}^{\ell,e}_{n+1} - \tilde{v}^{\ell,e}_{n+1}\|_p} \right) \lesssim N_\beta^{-\beta/2},$$

for multilevel mean-field prediction and update particles defined as in (29) and (30).

**Proof.** Since $\tilde{v}^{\ell,e}_0 = \tilde{v}^{\ell,e}_0$, we may assume that for some $n \geq 1$,

$$\|\tilde{v}^{\ell,e}_{n+1} - \tilde{v}^{\ell,e}_{n+1}\|_p \lesssim N_\beta^{-\beta/2}.$$
Assumption 1(iii) and Assumption 2(iii) imply that 
\[
\left\| \hat{v}_{n}^{f} - \hat{v}_{n}^{c} \right\|_{p} \lesssim N_{\ell}^{-\beta/2},
\]
and Lemmas 4 and 5 and \( |\hat{v}_{n}^{f}, \hat{y}_{n}^{f} | \in \mathbb{R}^{r} \) yield that 
\[
\left\| \hat{v}_{n}^{f} - \hat{v}_{n}^{c} \right\|_{p} \leq \left( I - \tilde{K}_{n}^{f,c} H \right) \left\| \hat{v}_{n}^{f} - \hat{v}_{n}^{c} \right\|_{p} + \left\| \tilde{K}_{n}^{f,c} - \tilde{K}_{n}^{f} \right\|_{2} \left\| H \hat{v}_{n}^{f} + \hat{y}_{n}^{f} \right\| \lesssim N_{\ell}^{-\beta/2}.
\]
The statement holds by induction. □

Corollary 3 (Continuity mean-field and EnKF covariance matrices). For any \( n \geq 0 \) and \( p \geq 2 \), it holds that 
\[
\left\| C_{n}^{f} - \tilde{C}_{n}^{f} \right\|_{p} \lesssim \left\| v_{n}^{f} - \hat{v}_{n}^{f} \right\|_{2p} + P_{\ell}^{-1/2},
\]
\[
\left\| C_{n}^{c} - \tilde{C}_{n}^{c} \right\|_{p} \lesssim \left\| v_{n}^{c} - \hat{v}_{n}^{c} \right\|_{2p} + P_{\ell}^{-1/2},
\]
for multilevel prediction particles defined as in (8) and (29).

Proof. Since \( C_{n}^{f} = C_{n}^{N_{\ell},P_{\ell}} \), \( \tilde{C}_{n}^{f} = \tilde{C}_{n}^{N_{\ell}} \), \( C_{n}^{c} = C_{n}^{N_{\ell-1},P_{\ell}} \) and \( \tilde{C}_{n}^{c} = \tilde{C}_{n}^{N_{\ell-1}} \), the result follows from Lemma 2. □

Corollary 4 (Distance between ensembles II). For any \( n \geq 0 \) and \( p \geq 2 \), the following asymptotic inequality holds 
\[
\max \left( \left\| \hat{v}_{n}^{f} - \hat{v}_{n}^{c} \right\|_{p}, \left\| \hat{v}_{n}^{f} - \hat{v}_{n}^{c} \right\|_{p} \right) \lesssim P_{\ell}^{-1/2}.
\]
for multilevel update particles defined as in (10) and (30).

Proof. Since \( v_{n}^{f} - \hat{v}_{n}^{f} = D_{n}^{N_{\ell},P_{\ell}} \) and \( v_{n}^{c} - \hat{v}_{n}^{c} = D_{n}^{N_{\ell-1},P_{\ell}} \), the result follows from Lemma 3. □

Lemma 7 (Continuity of Kalman gain double differences). For any \( n \geq 0 \) and \( p \geq 2 \), it holds that 
\[
\left\| K_{n}^{f,f} - K_{n}^{f,c} \right\|_{p} \lesssim \left\| C_{n}^{f} - C_{n}^{c} \right\|_{2p} \lesssim \frac{P_{\ell}^{-1/2} N_{\ell}^{-\beta/2} + P_{\ell}^{-1}}{2}.
\]

Proof. From the proof of [20, Lemma 3.4], one may deduce that 
\[
K_{n}^{f,f} - K_{n}^{f,c} = K_{n}^{f,f} H ( C_{n}^{f} - C_{n}^{c} ) H^{T} ( H C_{n}^{f} H^{T} + \Gamma )^{-1} \left( C_{n}^{f} - C_{n}^{c} \right) H^{T} ( H C_{n}^{f} H^{T} + \Gamma )^{-1},
\]
\[
K_{n}^{f,c} - K_{n}^{f,c} = K_{n}^{f,c} H ( \tilde{C}_{n}^{f,c} - C_{n}^{c} ) H^{T} ( H \tilde{C}_{n}^{f,c} H^{T} + \Gamma )^{-1} \left( C_{n}^{f,c} - C_{n}^{c} \right) H^{T} ( H \tilde{C}_{n}^{f,c} H^{T} + \Gamma )^{-1},
\]
and 
\[
( H \tilde{C}_{n}^{f,c} H^{T} + \Gamma )^{-1} = ( H \tilde{C}_{n}^{f,c} H^{T} + \Gamma )^{-1} + ( H \tilde{C}_{n}^{f,c} H^{T} + \Gamma )^{-1} ( \tilde{C}_{n}^{f,c} - C_{n}^{c} ) H^{T} ( H \tilde{C}_{n}^{f,c} H^{T} + \Gamma )^{-1}.
\]
The above equations imply that 
\[
K_{n}^{f,f} - K_{n}^{f,c} = K_{n}^{f,f} H ( \tilde{C}_{n}^{f,f} - C_{n}^{f} ) H^{T} ( H \tilde{C}_{n}^{f,c} H^{T} + \Gamma )^{-1}
\]
\[
\quad + K_{n}^{f,c} H ( \tilde{C}_{n}^{f,c} - C_{n}^{c} ) H^{T} ( H \tilde{C}_{n}^{f,c} H^{T} + \Gamma )^{-1} ( \tilde{C}_{n}^{f,c} - C_{n}^{c} ) H^{T} ( H \tilde{C}_{n}^{f,c} H^{T} + \Gamma )^{-1}
\]
\[
+ ( C_{n}^{f,c} - C_{n}^{c} ) H^{T} ( H \tilde{C}_{n}^{f,c} H^{T} + \Gamma )^{-1},
\]
and
\[ K_n^{\ell f} - \frac{K_n^{\ell,c_1} + K_n^{\ell,c_2}}{2} - (K_n^{\ell f} - \bar{K}_n^{\ell,c}) = K_n^{\ell f} - \bar{K}_n^{\ell,c} - \frac{1}{2} \left( K_n^{\ell,c_1} - \bar{K}_n^{\ell,c} + K_n^{\ell,c_2} - \bar{K}_n^{\ell,c} \right) \]
\[ = (I - K_n^{\ell f} H) \left( C_n^{\ell f} - \frac{C_n^{\ell,c_1} + C_n^{\ell,c_2}}{2} - \bar{C}_n^{\ell f} + \bar{C}_n^{\ell,c} \right) H^T (H \bar{C}_n^{\ell, e} H^T + \Gamma)^{-1} \]
\[ + (I - K_n^{\ell f} H) (\bar{C}_n^{\ell f} - C_n^{\ell f}) H^T (H \bar{C}_n^{\ell, e} H^T + \Gamma)^{-1} H (\bar{C}_n^{\ell f} - \bar{C}_n^{\ell, e}) H^T (H \bar{C}_n^{\ell, e} H^T + \Gamma)^{-1} \]
\[ - (K_n^{\ell,c_1} - K_n^{\ell,c_2}) H (\bar{C}_n^{\ell,e} - C_n^{\ell,e}) H^T (H \bar{C}_n^{\ell, e} H^T + \Gamma)^{-1} \]
\[ - \frac{1}{2} (K_n^{\ell,c_1} - K_n^{\ell,c_2}) H (\bar{C}_n^{\ell,c} - C_n^{\ell,c}) H^T (H \bar{C}_n^{\ell, e} H^T + \Gamma)^{-1}. \]

By the positive definiteness of $\Gamma$, Lemmas 3 and 4, and corollaries 3 and 4,
\[ \left\| K_n^{\ell f} - \bar{K}_n^{\ell,c} \right\|_p \lesssim \left\| C_n^{\ell f} - \frac{C_n^{\ell,c_1} + C_n^{\ell,c_2}}{2} - \bar{C}_n^{\ell f} + \bar{C}_n^{\ell,c} \right\|_p \]
\[ + \| \bar{C}_n^{\ell f} - C_n^{\ell f} \|_p \| \bar{C}_n^{\ell,c} - C_n^{\ell,c} \|_2 + \| K_n^{\ell,c_1} - K_n^{\ell,c_2} \|_p \| \bar{C}_n^{\ell,c} - C_n^{\ell,c} \|_2 \]
\[ + \frac{1}{2} \| K_n^{\ell,c_1} - K_n^{\ell,c_2} \|_p \| \bar{C}_n^{\ell,c} - C_n^{\ell,c} \|_2 \]
\[ \lesssim \left\| C_n^{\ell f} - \frac{C_n^{\ell,c_1} + C_n^{\ell,c_2}}{2} - \bar{C}_n^{\ell f} + \bar{C}_n^{\ell,c} \right\|_p + \| P_{\ell}^{-1/2} N_{\ell}^{-\beta/2} + P_{\ell}^{-1} \]

where the last inequality follows from
\[ \left\| K_n^{\ell,c_1} - K_n^{\ell,f} \right\|_p \lesssim \left\| C_n^{\ell,c_1} - C_n^{\ell,f} \right\|_p \]
\[ \lesssim \left\| C_n^{\ell,c_1} - \bar{C}_n^{\ell,c} \right\|_p + \left\| \bar{C}_n^{\ell,c} - C_n^{\ell,c} \right\|_2 \]
\[ \lesssim \left\| C_n^{\ell,c_1} - C_n^{\ell,c_2} \right\|_p \lesssim \left\| C_n^{\ell,c_1} - C_n^{\ell,c} \right\|_p + \left\| \bar{C}_n^{\ell,c} - C_n^{\ell,c} \right\|_p. \]

We next show how to bound the first term on the right hand side of (33).

**Lemma 8** (Continuity of covariance matrix double differences). For any $n \geq 0$ and $p \geq 2$, the following asymptotic inequality holds:
\[ \left\| C_n^{\ell f} - \frac{C_n^{\ell,c_1} + C_n^{\ell,c_2}}{2} - \bar{C}_n^{\ell f} + \bar{C}_n^{\ell,c} \right\|_p \lesssim \left\| E P_{\ell} \left[ (v_n^{\ell f_j} - v_n^{\ell c_j} - \bar{v}_n^{\ell f_j} + \bar{v}_n^{\ell c_j}) (\bar{v}_n^{\ell f_j})^T \right] \right\|_p \]
\[ + \left\| E P_{\ell} \left[ v_n^{\ell f_j} - v_n^{\ell c_j} - \bar{v}_n^{\ell f_j} + \bar{v}_n^{\ell c_j} \right] \right\|_{2p} \]
\[ + \| P_{\ell}^{-1/2} N_{\ell}^{-\beta/2} + P_{\ell}^{-1}. \]

**Proof.** Let us first recall that
\[ C_n^{\ell f} = \text{Cov}[v_n^{\ell f}], \quad \bar{C}_n^{\ell f} = \text{Cov}[\bar{v}_n^{\ell f}], \]
\[ C_n^{\ell,c} = \text{Cov}[v_n^{\ell,c}], \quad \bar{C}_n^{\ell,c} = \text{Cov}[\bar{v}_n^{\ell,c}], \]
and introduce the following covariance matrices for the auxiliary mean-field MLEnKF ensemble
\[ \tilde{C}_n^{\ell f} := \text{Cov}[^{\ell f} v_n], \quad \tilde{C}_n^{\ell,c} := \text{Cov}[^{\ell c} v_n]. \]
By the triangle inequality,

\[
\left\| C_n^{\ell,f} - \frac{C_n^{\ell,e_1} + C_n^{\ell,e_2}}{2} - \tilde{C}_n^{\ell,f} + \tilde{C}_n^{\ell,e}ight\|_p \lesssim \left\| C_n^{\ell,f} - \frac{C_n^{\ell,e_1} + C_n^{\ell,e_2}}{2} - \tilde{C}_n^{\ell,f} + \frac{\tilde{C}_n^{\ell,e_1} + \tilde{C}_n^{\ell,e_2}}{2}ight\|_p \\
+ \left\| \tilde{C}_n^{\ell,f} - \frac{\tilde{C}_n^{\ell,e_1} + \tilde{C}_n^{\ell,e_2}}{2} - \tilde{C}_n^{\ell,f} + \frac{\tilde{C}_n^{\ell,e_1} + \tilde{C}_n^{\ell,e_2}}{2}ight\|_p.
\]

Using that

\[
E_{P_{t-1}} \left[ \sum_{j=1}^2 \frac{v_n^{\ell,e_j} (v_n^{\ell,e_j})^T}{2} \right] = E_{P_{t}} [v_n^{\ell,e} (v_n^{\ell,e})^T],
\]

and

\[
E_{P_{t-1}} \left[ \sum_{j=1}^2 \frac{v_n^{\ell,e_j} (v_n^{\ell,e_j})^T}{2} \right] = E_{P_{t}} [\tilde{v}_n^{\ell,e} (\tilde{v}_n^{\ell,e})^T],
\]

we obtain

\[
\left\| C_n^{\ell,f} - \frac{C_n^{\ell,e_1} + C_n^{\ell,e_2}}{2} - \tilde{C}_n^{\ell,f} + \frac{\tilde{C}_n^{\ell,e_1} + \tilde{C}_n^{\ell,e_2}}{2} \right\|_p \lesssim \left\| E_{P_{t}} [v_n^{\ell,f} (v_n^{\ell,f})^T] - \tilde{E}_{P_{t}} [v_n^{\ell,e} (v_n^{\ell,e})^T] \right\|_p \\
+ \left\| E_{P_{t}} [v_n^{\ell,f} (v_n^{\ell,f})^T] - E_{P_{t}} [\tilde{v}_n^{\ell,f} (\tilde{v}_n^{\ell,f})^T] \right\|_p \\
- \frac{1}{2} \left( E_{P_{t-1}} [v_n^{\ell,e_1}] E_{P_{t-1}} [v_n^{\ell,e_1}] + E_{P_{t-1}} [v_n^{\ell,e_2}] E_{P_{t-1}} [v_n^{\ell,e_2}] \right) \\
- E_{P_{t-1}} [v_n^{\ell,e_1}] E_{P_{t-1}} [v_n^{\ell,e_2}] \\
=: I_{11} + I_{12}.
\]

For the first term, Lemma 6 and Corollary 4 yield

\[
I_{11} \lesssim \left\| E_{P_{t}} [(v_n^{\ell,f} - v_n^{\ell,e} - \tilde{v}_n^{\ell,f} + \tilde{v}_n^{\ell,e}) (v_n^{\ell,f})^T] \right\|_p + \left\| v_n^{\ell,f} - v_n^{\ell,e} - \tilde{v}_n^{\ell,f} + \tilde{v}_n^{\ell,e} \right\|_p \left\| v_n^{\ell,f} - \tilde{v}_n^{\ell,f} \right\|_p \\
+ \left\| v_n^{\ell,f} - v_n^{\ell,e} \right\|_p \left\| v_n^{\ell,e} - \tilde{v}_n^{\ell,e} \right\|_p + \left\| v_n^{\ell,f} - \tilde{v}_n^{\ell,f} \right\|_p \left\| v_n^{\ell,e} - \tilde{v}_n^{\ell,e} \right\|_p \\
\lesssim \left\| E_{P_{t}} [v_n^{\ell,f} - v_n^{\ell,e} - \tilde{v}_n^{\ell,f} + \tilde{v}_n^{\ell,e}] (v_n^{\ell,f})^T \right\|_p + P^{-1/2} \lambda_n^{\beta/2} + P^{-1}.
\]

For the second term, the identity \( a^T b + b^T = \frac{1}{2} [(a + b)(a + b)^T + (a - b)(a - b)^T] \) yields

\[
I_{12} \lesssim \left\| E_{P_{t}} [v_n^{\ell,f} E_{P_{t}} [v_n^{\ell,f}] - E_{P_{t}} [v_n^{\ell,e}] E_{P_{t}} [v_n^{\ell,e}]] \right\|_p \\
- E_{P_{t}} [v_n^{\ell,e}] E_{P_{t}} [v_n^{\ell,e}] E_{P_{t}} [v_n^{\ell,e}] E_{P_{t}} [v_n^{\ell,e}] \right\|_p \\
+ \left\| E_{P_{t-1}} [v_n^{\ell,e_1} - v_n^{\ell,e_2}] E_{P_{t-1}} [v_n^{\ell,e_1} - v_n^{\ell,e_2}] \right\|_p \\
+ \frac{1}{4} \left( E_{P_{t-1}} [v_n^{\ell,e_1} - v_n^{\ell,e_2}] E_{P_{t-1}} [v_n^{\ell,e_1} - v_n^{\ell,e_2}] \right) =: I_{121} + I_{122}.
\]

The term \( I_{121} \) can be bounded in a similar fashion as \( I_{11} \), and to bound the second term, we employ the identity \( a^T b - b^T = \frac{1}{2} [(a + b)(a + b)^T + (a - b)(a + b)^T] \),
Jensen’s inequality and Corollary 4:
\[ I_{122} \lesssim \left\| E_{\ell - 1}[v_{n}^{\ell, e_1} - v_{n}^{\ell, e_2} + \tilde{v}_{n}^{\ell, e_1} - \tilde{v}_{n}^{\ell, e_2}] \right\|_{2p} \left\| E_{\ell - 1}[v_{n}^{\ell, e_1} - v_{n}^{\ell, e_2} + \tilde{v}_{n}^{\ell, e_2}] \right\|_{2p} \]
\[ \lesssim \left\| v_{n}^{\ell, e} - \tilde{v}_{n}^{\ell, e} \right\|_{2p}^{2} \lesssim P_{\ell}^{-1}. \]

Consequently,
\[ I_{12} \lesssim \left\| E_{\ell} [v_{n}^{\ell, f} - v_{n}^{\ell, e} - \tilde{v}_{n}^{\ell, f} + \tilde{v}_{n}^{\ell, e}] \right\|_{2p} + P_{\ell}^{-1/2} N_{\ell}^{-\beta/2} + P_{\ell}^{-1}. \]

For the second term in (35), the equation \( E_{\ell} [\tilde{v}_{n}^{\ell, e}] = E_{\ell - 1}[(\tilde{v}_{n}^{\ell, e_1} + \tilde{v}_{n}^{\ell, e_2})/2] \) implies that

\[
\left\| \frac{\tilde{v}_{n}^{\ell, f} - \tilde{C}_{n}^{\ell, e_1} + \tilde{C}_{n}^{\ell, e_2}}{2} - C_{n}^{\ell, f} + \tilde{C}_{n}^{\ell, e} \right\|_{p} \\
\leq \left\| E_{\ell} [\tilde{v}_{n}^{\ell, f} (\tilde{v}_{n}^{\ell, e})_{T} - \tilde{v}_{n}^{\ell, e} (\tilde{v}_{n}^{\ell, f})_{T} - E_{\ell} [\tilde{v}_{n}^{\ell, f} (\tilde{v}_{n}^{\ell, e})_{T} - \tilde{v}_{n}^{\ell, e} (\tilde{v}_{n}^{\ell, f})_{T}]] \right\|_{p} \\
+ \left\| E_{\ell} [\tilde{v}_{n}^{\ell, f}] E_{\ell} [(\tilde{v}_{n}^{\ell, e})_{T} - E_{\ell} [\tilde{v}_{n}^{\ell, f}] E_{\ell} [(\tilde{v}_{n}^{\ell, e})_{T}]] \right\|_{p} \\
+ \frac{1}{4} \left\| E_{\ell - 1} [v_{n}^{\ell, e_1} - v_{n}^{\ell, e_2}] E_{\ell - 1} [(\tilde{v}_{n}^{\ell, e_1} + \tilde{v}_{n}^{\ell, e_2})] \right\|_{p} \\
= I_{21} + I_{22} + I_{23}. \]

Hölder’s inequality, Lemma 6 and the M-Z inequality imply that
\[ I_{21} \lesssim P_{\ell}^{-1/2} N_{\ell}^{-\beta/2}, \quad I_{22} \lesssim P_{\ell}^{-1/2} N_{\ell}^{-\beta/2} + P_{\ell}^{-1}, \quad \text{and} \quad I_{23} \lesssim P_{\ell}^{-1} \]
(where \( E [\tilde{v}_{n}^{\ell, e_1} - v_{n}^{\ell, e_2}] = 0 \) was used in the last inequality).

\[ \square \]

Lemma 9. For any \( n \geq 0, p \geq 2 \), denoting Jacobian of \( \varphi \) by \( D\varphi \), it holds that
\[
\left\| E_{\ell} [D\varphi(\tilde{v}_{n}^{\ell, e})(\tilde{v}_{n}^{\ell, f} - \tilde{v}_{n}^{\ell, e} - \tilde{v}_{n}^{\ell, f} + \tilde{v}_{n}^{\ell, e})] \right\|_{p} \]
\[ \lesssim \left\| E_{\ell} [D\varphi(\tilde{v}_{n}^{\ell, e})(I - K_{n}^{\ell, f} H)(v_{n}^{\ell, f} - v_{n}^{\ell, e} - \tilde{v}_{n}^{\ell, f} + \tilde{v}_{n}^{\ell, e})] \right\|_{p} \]
\[ + \left\| E_{\ell} [(v_{n}^{\ell, f} - v_{n}^{\ell, e} - \tilde{v}_{n}^{\ell, f} + \tilde{v}_{n}^{\ell, e}) (\tilde{v}_{n}^{\ell, f})_{T}] \right\|_{2p} + \left\| E_{\ell} [v_{n}^{\ell, f} - v_{n}^{\ell, e} - \tilde{v}_{n}^{\ell, f} + \tilde{v}_{n}^{\ell, e}] \right\|_{2p} \]
\[ + P_{\ell}^{-1/2} N_{\ell}^{-\beta/2} + P_{\ell}^{-1}. \]

Proof. Let \( \tilde{y}_{n}^{\varphi,j} \) denote the perturbed observation associated with \((v_{n}^{\ell, f}, v_{n}^{\ell, e})\). By the update equations (10) and (29), and \((\tilde{v}_{n}^{\ell, f}, \tilde{v}_{n}^{\ell, e})\), we obtain the representation
\[ v_{n}^{\ell, f} - v_{n}^{\ell, e} - \tilde{v}_{n}^{\ell, f} + \tilde{v}_{n}^{\ell, e} = (I - K_{n}^{\ell, f} H) (v_{n}^{\ell, f} - v_{n}^{\ell, e} - \tilde{v}_{n}^{\ell, f} + \tilde{v}_{n}^{\ell, e}) \]
\[ + (K_{n}^{\ell, f} - K_{n}^{\ell, f} H) (\tilde{v}_{n}^{\ell, f} - \tilde{v}_{n}^{\ell, e}) - (\tilde{K}_{n}^{\ell, f} - \tilde{K}_{n}^{\ell, e}) H (v_{n}^{\ell, e} - \tilde{v}_{n}^{\ell, e}) \]
\[ - (K_{n}^{\ell, f} - K_{n}^{\ell, e}) - (\tilde{K}_{n}^{\ell, f} + \tilde{K}_{n}^{\ell, e}) H (\tilde{v}_{n}^{\ell, e} - \tilde{v}_{n}^{\ell, e}) \]
\[ - (K_{n}^{\ell, f} - K_{n}^{\ell, e} + \tilde{K}_{n}^{\ell, e} - \tilde{K}_{n}^{\ell, e}) (H \tilde{v}_{n}^{\ell, e} - \tilde{v}_{n}^{\ell, e}), \]
(36)
By (36) and Hölder’s inequality

\[ (37) \]
\[
\left\| E_{P_{\ell-1}} \left[ D\varphi(\tilde{v}_n^{\ell,i}) (\tilde{v}_n^{\ell,f} - \hat{v}_n^{\ell,c} + \tilde{v}_n^{\ell,e}) \right] \right\|_p
\]
\[ = \left\| E_{P_{\ell-1}} \left[ \sum_{j=1}^{2} D\varphi(\tilde{v}_n^{\ell,e,j}) (\tilde{v}_n^{\ell,f} - \hat{v}_n^{\ell,c} - \tilde{v}_n^{\ell,f} + \tilde{v}_n^{\ell,e,j}) \right] \right\|_p
\]
\[ \lesssim \left\| E_{P_{\ell-1}} \left[ \sum_{j=1}^{2} D\varphi(\tilde{v}_n^{\ell,e,j}) (I - K_n^{\ell,f}H) (v_n^{\ell,f,i} - v_n^{\ell,c} - \tilde{v}_n^{\ell,f} + \tilde{v}_n^{\ell,e,j}) \right] \right\|_p
\]
\[ + \left\| D\varphi(\tilde{v}_n^{\ell,e,j}) \right\|_{2p} \left\{ \left\| K_n^{\ell,f} - K_n^{\ell,f} \right\|_{4p} \left\| v_n^{\ell,f,i} - v_n^{\ell,f} \right\|_{4p} + \left\| K_n^{\ell,c} - K_n^{\ell,c} \right\|_{4p} \left\| v_n^{\ell,c} - v_n^{\ell,c} \right\|_{4p}
\right.
\]
\[ + \left( \left\| K_n^{\ell,f} - K_n^{\ell,f} \right\|_{4p} + \left\| K_n^{\ell,c} - K_n^{\ell,c} \right\|_{4p} \right) \left\| v_n^{\ell,c} - v_n^{\ell,c} \right\|_{4p} \}
\]
\[ + \left\| E_{P_{\ell-1}} \left[ \sum_{j=1}^{2} D\varphi(\tilde{v}_n^{\ell,e,j}) (K_n^{\ell,f} - K_n^{\ell,f} + \tilde{K}_n^{\ell,f}) (H\tilde{v}_n^{\ell,c} - y_n^{\ell,j}) \right] \right\|_p
\]
\[ =: \mathcal{J}_{1,\ell} + \mathcal{J}_{2,\ell} + \mathcal{J}_{3,\ell}. \]

The properties \( \varphi \in \mathcal{F} \subset C_0^2(\mathbb{R}^d, \mathbb{R}) \) and \( \hat{v}_n^{\ell,e} \in \cap_{r \geq 2} L^r(\Omega, \mathbb{R}^d) \) imply that \( \| D\varphi(\tilde{v}_n^{\ell,e,j}) \|_{2p} < \infty \), and Lemmas 4, 5, and 6, and Corollaries 3 and 4 yield that

\[ \mathcal{J}_{2,\ell} \lesssim P_{\ell}^{-1/2} \nu_{\ell}^{-\beta/2} + P_{\ell}^{-1}. \]

For the last term, we use \( K_n^{\ell,e,1} = (K_n^{\ell,e,1} + K_n^{\ell,e,2})/2 + (K_n^{\ell,e,1} - K_n^{\ell,e,2})/2 \) and the Frobenius scalar product \( \langle \cdot, \cdot \rangle_F \) on \( \mathbb{R}^{d \times d_0} \times \mathbb{R}^{d \times d_0} \) to obtain

\[ \mathcal{J}_{3,\ell} \leq \left\| D\varphi(\tilde{v}_n^{\ell,e,j}) (K_n^{\ell,f} - K_n^{\ell,f} + \tilde{K}_n^{\ell,f}) (H\tilde{v}_n^{\ell,c} - y_n^{\ell,j}) \right\|_p
\]
\[ + \left( \left\| K_n^{\ell,f} - K_n^{\ell,f} + \tilde{K}_n^{\ell,f} \right\|_{4p} + \left\| K_n^{\ell,c} - K_n^{\ell,c} \right\|_{4p} \right) \left\| v_n^{\ell,c} - v_n^{\ell,c} \right\|_{4p} \]
\[ \lesssim \left\| K_n^{\ell,f} - K_n^{\ell,f} + \tilde{K}_n^{\ell,f} \right\|_{2p} + \left\| K_n^{\ell,c} - K_n^{\ell,c} \right\|_{2p} \right\| P_{\ell}^{-1/2}
\]
\[ \lesssim \left\| E_{P_{\ell-1}} \left[ v_n^{\ell,f} - v_n^{\ell,f} - \tilde{v}_n^{\ell,e} + \tilde{v}_n^{\ell,e} \right] \right\|_{2p} + \left\| E_{P_{\ell}} \left[ v_n^{\ell,f} - v_n^{\ell,f} - \tilde{v}_n^{\ell,e} + \tilde{v}_n^{\ell,e} \right] \right\|_{2p}
\]
\[ + P_{\ell}^{-1/2} \nu_{\ell}^{-\beta/2} + P_{\ell}^{-1}. \]

Here, the second last inequality follows from the M-Z inequality applied to the right argument in the scalar product (as it is a sample average of \( P_{\ell-1} \) iid, mean-zero random matrices). The last inequality follows by Lemmas 7 and 8 and (34).

Lemma 11 shows that Theorem 2 is achieved through bounding \( \| E_{P_{\ell}} [\varphi(\tilde{v}_n^{\ell,f}) - \varphi(\tilde{v}_n^{\ell,e}) - \varphi(\tilde{v}_n^{\ell,f}) + \varphi(\tilde{v}_n^{\ell,e})] \|_{p} \) from above. To obtain this bound, we introduce the following sequence of random \( d \times d \) matrices: for \( \ell \geq 0, r, s \in \{1, \ldots, n\} \) and \( j \in \{1, 2\} \),

\[ A_{r,s}^{\ell} = \begin{cases} \prod_{i=r}^{s} D\Psi_i^{N_i}(\tilde{v}_i^{\ell,e})(I - K_i^{\ell,f}H) & \text{if } r \leq s \\ I & \text{if } r > s. \end{cases} \]
Moreover, for \( j = 1, 2 \),

\[
A_{r,s}^{\ell,j} = \left\{ \prod_{i=r}^{s} D\Psi_i^{N_i}(\hat{v}_i^{\ell,i})/(I - K_i^{\ell,f}H) \right\} 
\]

if \( r \leq s \)

\[
A_{r,s}^{\ell,j} = \left\{ \prod_{i=r}^{s} D\Psi_i^{N_i}(\hat{v}_i^{\ell,i})/(I - K_i^{\ell,f}H) \right\} 
\]

if \( r > s \).

We note that \( |A_{r,s}^{\ell,j}| \in \cap_{\ell \geq 2} L^q(\Omega) \) for all index values.

**Corollary 5.** For any \( n \geq 2 \), \( k \leq n - 1 \) and \( p \geq 2 \) it holds that

\[
\|E_{P_\ell}(D\varphi(\hat{v}_n^{\ell,e}))(I - K_n^{\ell,f}H)A_{k+1,n-1}^{\ell,j}D\Psi_{k}^{N_k}(\hat{v}_k^{\ell,e})((\hat{v}_k^{\ell,e} - \hat{v}_k^{\ell,e}) + \hat{v}_k^{\ell,e})\|_p \leq \|E_{P_\ell}(D\varphi(\hat{v}_n^{\ell,e}))(I - K_n^{\ell,f}H)A_{k+1,n-1}^{\ell,j}D\Psi_{k}^{N_k}(\hat{v}_k^{\ell,e})((\hat{v}_k^{\ell,e} - \hat{v}_k^{\ell,e}) + \hat{v}_k^{\ell,e})\|_p \\
+ \|E_{P_\ell}( [ (\hat{v}_k^{\ell,e} - \hat{v}_k^{\ell,e}) + \hat{v}_k^{\ell,e})((\hat{v}_k^{\ell,e} - \hat{v}_k^{\ell,e}) + \hat{v}_k^{\ell,e}) ]^{\ell}]_p + \|E_{P_\ell}( [ (\hat{v}_k^{\ell,e} - \hat{v}_k^{\ell,e}) + \hat{v}_k^{\ell,e})((\hat{v}_k^{\ell,e} - \hat{v}_k^{\ell,e}) + \hat{v}_k^{\ell,e}) ]^{\ell}]_p \\
+ P_{\ell}^{-1/2}N_{\ell}^{-\beta/2} + P_{\ell}^{-1}.
\]

**Sketch of proof.** By proceeding as in the proof of Lemma 9, we obtain three terms that respectively are similar to \( J_{1,\ell}, J_{2,\ell} \) and \( J_{3,\ell} \) in (37), but now with the prefactor

\[
D\varphi(\hat{v}_n^{\ell,e}))(I - K_n^{\ell,f}H)A_{k+1,n-1}^{\ell,j}D\Psi_{k}^{N_k}(\hat{v}_k^{\ell,e})
\]

The proof is obtained through the equality

\[
A_{k+1,n-1}^{\ell,j}D\Psi_{k}^{N_k}(\hat{v}_k^{\ell,e})((I - K_n^{\ell,f}H) = A_{k+1,n-1}^{\ell,j},
\]

the boundedness of \( A_{k+1,n-1}^{\ell,j} \) and \( D\varphi(\hat{v}_n^{\ell,e}))(I - K_n^{\ell,f}H) \), and by bounding the terms corresponding to \( J_{2,\ell} \) and \( J_{3,\ell} \) in this corollary similarly as in said lemma.

\[\square\]

**Corollary 6.** For any \( n \geq 1 \), \( k \leq s \leq n - 1 \) and \( p \geq 2 \) it holds that

\[
\|E_{P_\ell}(A_{k+1,s}^{\ell,j}D\Psi_{k}^{N_k}(\hat{v}_k^{\ell,e})((\hat{v}_k^{\ell,e} - \hat{v}_k^{\ell,e}) + \hat{v}_k^{\ell,e})\|_p \leq \|E_{P_\ell}(A_{k+1,s}^{\ell,j}D\Psi_{k}^{N_k}(\hat{v}_k^{\ell,e})((\hat{v}_k^{\ell,e} - \hat{v}_k^{\ell,e}) + \hat{v}_k^{\ell,e})\|_p \\
+ \|E_{P_\ell}( (\hat{v}_k^{\ell,e} - \hat{v}_k^{\ell,e}) + \hat{v}_k^{\ell,e})((\hat{v}_k^{\ell,e} - \hat{v}_k^{\ell,e}) + \hat{v}_k^{\ell,e}) ]^{\ell}]_p + \|E_{P_\ell}( (\hat{v}_k^{\ell,e} - \hat{v}_k^{\ell,e}) + \hat{v}_k^{\ell,e})((\hat{v}_k^{\ell,e} - \hat{v}_k^{\ell,e}) + \hat{v}_k^{\ell,e}) ]^{\ell}]_p \\
+ P_{\ell}^{-1/2}N_{\ell}^{-\beta/2} + P_{\ell}^{-1}.
\]

**Sketch of proof.** The statement of this corollary is similar to Corollary 5, but with the \( 1 \times d \) prefactor

\[
D\varphi(\hat{v}_n^{\ell,e}))(I - K_n^{\ell,f}H)A_{k+1,n-1}^{\ell,j}D\Psi_{k}^{N_k}(\hat{v}_k^{\ell,e})
\]

As the latter factor also is bounded (it is an element of \( \cap_{\ell \geq 2} L^r(\Omega, \mathbb{R}^{d \times d}) \)), the proof follows by a similar argument as in Corollary 5. However, since the prefactor in this case is a \( d \times d \) matrix rather than a vector, the term in (38) that corresponds to \( J_{3,\ell} \) in the proof of Lemma 9 becomes

\[
\|E_{P_{\ell-1}} \sum_{j=1}^{2} A_{k+1,s}^{\ell,j}D\Psi_{k}^{N_k}(\hat{v}_k^{\ell,e})((K_k^{\ell,e} - K_k^{\ell,e}) + K_k^{\ell,e})(H\hat{v}_k^{\ell,e} - \hat{y}_k^{\ell,j}) \|_p \\
\leq \sum_{j=1}^{d} \|E_{P_{\ell-1}} \sum_{j=1}^{2} (A_{k+1,s}^{\ell,j}D\Psi_{k}^{N_k}(\hat{v}_k^{\ell,e}))(K_k^{\ell,e} - K_k^{\ell,e}) + K_k^{\ell,e})(H\hat{v}_k^{\ell,e} - \hat{y}_k^{\ell,j}) \|_p.
\]

Here \( (A_{k+1,s}^{\ell,j}D\Psi_{k}^{N_k}(\hat{v}_k^{\ell,e}))_i \) denotes the \( i \)-th row vector of the matrix. Hence, we have \( d \) terms to bound that are similar to the \( J_{3,\ell} \)-term in Corollary 5. \[\square\]
Corollary 7. For any $n \geq 1$, $k \leq s \leq n-1$, $r \leq n$ and $p \geq 2$ it holds that

\begin{equation}
\left\|E_{\mathcal{P}_t}\left[A_{k+1,s}D\Psi_k^N(\hat{\nu}^c)(\hat{\nu}^f - \hat{\nu}^c + \hat{\nu}^e)(\hat{\nu}^f)^T\right]\right\|_p \\
\lesssim \left\|E_{\mathcal{P}_t}\left[A_{k,s}(v_k^f - v_k^c - \hat{\nu}^c + \hat{\nu}^e)(v_k^f)^T\right]\right\|_p + \left\|E_{\mathcal{P}_t}[v_k^f - v_k^c - \hat{\nu}^c + \hat{\nu}^e]\right\|_{2p} \\
+ \left\|E_{\mathcal{P}_t}[v_k^f - v_k^c - \hat{\nu}^c + \hat{\nu}^e](\hat{\nu}^f)^T\right\|_{2p} + P_t^{-1/2}N_t^{-\beta/2} + P_t^{-1}.
\end{equation}

\textbf{Sketch of proof.} The statement of this corollary is similar to Corollary 5, but here with the additional $1 \times d$ postfactor $(\nu_{\ell,i}^f)^T$. The only technicality this introduces is in bounding the term in (39) that corresponds to $J_{3,\ell}$ in the proof of Lemma 9. That is,

\begin{equation}
\left\|E_{\mathcal{P}_t-1}\left[A_{k+1,s}D\Psi_k^N(\hat{\nu}^c)(K_k - K_k^c - \tilde{K}_k^c) + \tilde{K}_k^c)(H\hat{\nu}^c - y_k^f)(\nu_{\ell,i}^f)^T\right]\right\|_p \\
\lesssim \sum_{i=1}^d \left\|E_{\mathcal{P}_t-1}\left[A_{k+1,s}D\Psi_k^N(\hat{\nu}^c)(K_k - K_k^c - \tilde{K}_k^c) + \tilde{K}_k^c)(H\hat{\nu}^c - y_k^f)(\nu_{\ell,i}^f)^T\right]\right\|_p.
\end{equation}

Here $(\nu_{\ell,i}^f)^T$ denotes the $i$-th component of the row vector. We thus have $d$ terms to bound which are similar to the $J_{3,\ell}$-term in Corollary 5. \hfill \Box

Lemma 10. For any $n \geq 1$, $k \leq s \leq n-1$, $r \leq n$ and $p \geq 2$ it holds that

\begin{equation}
\left\|E_{\mathcal{P}_t}[D\varphi(\hat{\nu}^c)(I - K_n^f H)A_{k,n-1}(v_{k+1}^f - v_{k+1}^c - \hat{\nu}^c + \hat{\nu}^e)]\right\|_p \\
\lesssim \left\|E_{\mathcal{P}_t}[D\varphi(\hat{\nu}^c)(I - K_n^f H)A_{k,n-1}(v_{k}^f - v_{k}^c - \hat{\nu}^c + \hat{\nu}^e)]\right\|_{2p} \\
+ \left\|E_{\mathcal{P}_t}[(v_{k}^f - v_{k}^c - \hat{\nu}^c + \hat{\nu}^e)(\nu_{\ell,i}^f)^T]\right\|_{2p} + \left\|E_{\mathcal{P}_t}[v_{k}^f - v_{k}^c - \hat{\nu}^c + \hat{\nu}^e]\right\|_{2p} \\
+ N_t^{-\beta/2}P_t^{-1/2} + P_t^{-1},
\end{equation}

and

\begin{equation}
\left\|E_{\mathcal{P}_t}[A_{k+1,s}(v_{k+1}^f - v_{k+1}^c - \hat{\nu}^c + \hat{\nu}^e)]\right\|_p \\
\lesssim \left\|E_{\mathcal{P}_t}[A_{k,s}(v_{k}^f - v_{k}^c - \hat{\nu}^c + \hat{\nu}^e)]\right\|_p + \left\|E_{\mathcal{P}_t}[v_{k}^f - v_{k}^c - \hat{\nu}^c + \hat{\nu}^e]\right\|_{2p} \\
+ \left\|E_{\mathcal{P}_t}[(v_{k}^f - v_{k}^c - \hat{\nu}^c + \hat{\nu}^e)(\nu_{\ell,i}^f)^T]\right\|_{2p} + P_t^{-1/2}N_t^{-\beta/2} + P_t^{-1},
\end{equation}

and

\begin{equation}
\left\|E_{\mathcal{P}_t}[A_{k+1,s}(v_{k+1}^f - v_{k+1}^c - \hat{\nu}^c + \hat{\nu}^e)(\nu_{\ell,i}^f)^T]\right\|_p \\
\lesssim \left\|E_{\mathcal{P}_t}[A_{k,s}(v_{k}^f - v_{k}^c - \hat{\nu}^c + \hat{\nu}^e)(\nu_{\ell,i}^f)^T]\right\|_p + \left\|E_{\mathcal{P}_t}[v_{k}^f - v_{k}^c - \hat{\nu}^c + \hat{\nu}^e]\right\|_{2p} \\
+ \left\|E_{\mathcal{P}_t}[(v_{k}^f - v_{k}^c - \hat{\nu}^c + \hat{\nu}^e)(\nu_{\ell,i}^f)^T]\right\|_{2p} + P_t^{-1/2}N_t^{-\beta/2} + P_t^{-1}.
\end{equation}
Lemma 11. For any $n \geq 0$ and $p \geq 2$, it holds that

$$\| E_P \left[ \varphi(\hat{v}_{n+1}^f) - \varphi(\hat{v}_n^f) - \varphi(\hat{v}_n^e) + \varphi(\hat{v}_n^c) \right] \|_p \lesssim N_\ell^{-\beta/2} P_\ell^{-1/2} + P_\ell^{-1}.$$  

Proof. By assumption 1(ii) and a similar application of the mean-value theorem as in (43),

$$\| E_P \left[ \varphi(\hat{v}_{n+1}^f) - \varphi(\hat{v}_n^f) - \varphi(\hat{v}_n^e) + \varphi(\hat{v}_n^c) \right] \|_p \lesssim \| E_P \left[ D\varphi(\hat{v}_n^f) (\hat{v}_{n+1}^f - \hat{v}_n^f + \hat{v}_n^e) \right] \|_p + N_\ell^{-\beta/2} P_\ell^{-1/2} + P_\ell^{-1}.$$  

By Lemma 9, and thereafter using that $A_{n-1,n-2}^f = I$ and applying Lemma 10, we obtain

$$K_\ell := \| E_P \left[ D\varphi(\hat{v}_n^f) (\hat{v}_{n+1}^f - \hat{v}_n^f + \hat{v}_n^e) \right] \|_p$$

$$\lesssim \| E_P \left[ D\varphi(\hat{v}_n^f) (I - K_{n+1}^f H) A_{n-1,n-1}^f (v_{n+1}^f - v_n^f - v_n^e + v_n^c) \right] \|_p \leq 2p + \| E_P [A_{n-1,n-1}^f (v_{n+1}^f - v_n^f - v_n^e + v_n^c) \|_p + \| E_P [A_{n-1,n-1}^f (v_{n+1}^f - v_n^f - v_n^e + v_n^c) + P_\ell^{-1/2} N_\ell^{-\beta/2} + P_\ell^{-1}.$$  

where $\theta_k \in \text{Conv}(\hat{v}_k^f, \hat{v}_k^e, \hat{v}_k^c)$, $\theta_k \in \text{Conv}(\hat{v}_k^f, \hat{v}_k^e, \hat{v}_k^c)$, and Conv($x, \bar{x}$) := \{xt + (1 - t)\bar{x} \mid t \in [0, 1]\} for $x, \bar{x} \in \mathbb{R}^d$. The expansion, Assumption 2 and Corollary 5 yield (40). And, similarly, Corollaries 6 and 7 respectively yield (41) and (42).
Recalling that $\hat{\psi}_0^f - \hat{\psi}_0^c - \hat{\psi}_0^f + \hat{\psi}_0^c = 0$ and applying Lemma 10 iteratively $n - 1$ times, we obtain that

$$K_\ell \lesssim \left\| E_P \left[ D\varphi(\hat{\psi}_0^f) (I - K_\ell^f H) A_{1,n-1}^f D\Psi_0^N (\hat{\psi}_0^f - \hat{\psi}_0^f + \hat{\psi}_0^f + \hat{\psi}_0^f) \right] \right\|_p$$

$$+ \sum_{k=1}^{n-1} \left\{ \left\| E_P \left[ A_{1,n-k}^f D\Psi_0^N (\hat{\psi}_0^f - \hat{\psi}_0^f + \hat{\psi}_0^f + \hat{\psi}_0^f) \right] \right\|_p \right\} + N_L^{-\beta/2} P_{\ell}^{-1/2} + P_{\ell}^{-1} \lesssim N_L^{-\beta/2} P_{\ell}^{-1/2} + P_{\ell}^{-1}.$$ 

\[ \square \]

A.3. Proof of Corollary 2. By Theorem 2, we recall that

$$\|\mu_n^M[\varphi] - \overline{\mu}_n[\varphi]\|_p \lesssim N_L^{-\beta/2} P_L^{-1/2} + P_L^{-1} + N_L^{-\alpha} + \sum_{\ell=0}^{L} M_{\ell}^{-1/2}(N_L^{-\beta/2} P_{\ell}^{-1/2} + P_{\ell}^{-1}),$$

with $P_\ell \approx 2^\ell$, $N_\ell \approx 2^s$ for some $s > 0$. Our objective is to prove that the parameter choices for $s, L, \{M_\ell\}$ stated in Corollary 2 ensure that the goal

$$\|\mu_n^M[\varphi] - \overline{\mu}_n[\varphi]\|_p \lesssim \epsilon$$

is reached at the asymptotic computational cost (17), where

$$\text{Cost(MLEnKF)} := \sum_{\ell=0}^{L} M_\ell N_\ell P_\ell.$$ 

Fix the value of $s > 0$. It then follows straightforwardly that in order to control the “bias”

$$N_L^{-\beta/2} P_L^{-1/2} + P_L^{-1} + N_L^{-\alpha} \lesssim \epsilon,$$

one must have $L \approx \log_2(\epsilon^{-1})/\min(1, (1 + \beta s)/2, \alpha s) + 1$. It remains to minimize

$$\sum_{\ell=0}^{L} M_\ell N_\ell P_\ell$$

subject to the constraint

$$\sum_{\ell=0}^{L} M_\ell^{-1/2}(N_L^{-\beta/2} P_{\ell}^{-1/2} + P_{\ell}^{-1}) \lesssim \epsilon,$$

and also having in mind that $M_\ell$ must be a natural number for all $\ell \leq L$.

The method of Langrange multipliers applied to

$$\mathcal{L}(\{M_\ell\}, \lambda) := \sum_{\ell=0}^{L} M_\ell N_\ell P_\ell + \lambda \left( \sum_{\ell=0}^{L} M_\ell^{-1/2} D_\ell - \epsilon \right)$$

with $D_\ell := (N_L^{-\beta/2} P_{\ell}^{-1/2} + P_{\ell}^{-1})$ yields

$$M_\ell \approx \lambda^{2/3}(N_\ell P_\ell)^{-2/3} D_{\ell}^{2/3} + 1 \quad \text{and} \quad \lambda = \epsilon^{-3} \left( \sum_{\ell=0}^{L} (N_\ell P_\ell)^{1/3} D_{\ell}^{2/3} \right)^3.$$ 

Since

$$P_\ell N_\ell \approx 2^{(1+s)\ell} \quad \text{and} \quad D_\ell \approx 2^{-(\min(\beta s, 1))\ell/2},$$
we have
\[
\sum_{\ell=0}^{L} (N_{\ell} P_{\ell})^{1/3} D_{\ell}^{2/3} \approx \sum_{\ell=0}^{L} 2^{(s-\min(\beta s,1))\ell/3} \approx \begin{cases} 
1 & \text{if } \min(\beta s,1) > s, \\
L & \text{if } \min(\beta s,1) = s, \\
2^{(s-\min(\beta s,1))L/3} & \text{if } \min(\beta s,1) < s,
\end{cases}
\]
and the optimal formula for \{M_{\ell}\} for a fixed \(s > 0\), cf. (15), follows from the last equality and (45).

By (44), the choice for \{M_{\ell}\} leads to
\[
\text{Cost}(\text{MLEnKF}) \approx \begin{cases} 
\epsilon - 2 + f(s) & \text{if } \min(\beta s,1) > s, \\
\epsilon - 2 L^{3} + \epsilon - f(s) & \text{if } \min(\beta s,1) = s, \\
\epsilon - 2 - \frac{s - \min(\beta s,1)}{\min(1, (\beta s + 1)/2, \alpha s)} & \text{if } \min(\beta s,1) < s,
\end{cases}
\]
where
\[
f(s) := \frac{1 + s}{\min(1, (\beta s + 1)/2, \alpha s)} = \begin{cases} 
\frac{1}{\alpha s} + s & \text{if } s \leq \frac{\min((1 + \beta s)/2, 1)}{\alpha}, \\
\frac{2(1 + s)}{\beta s + 1} & \text{if } \frac{1 + \beta s}{2\alpha} < s < \alpha^{-1}, \\
1 + s & \text{if } \min((1 + \beta s)/2, \alpha s) \geq 1.
\end{cases}
\]

We next consider the problem of determining the value/inclusion set of \(s\) which minimizes the asymptotic growth rate of Cost(\text{MLEnKF}). We consider three cases separately:

1. If \(\beta < 1\), then \(\min(\beta s,1) < s\) for all \(s > 0\), cf. Figure 6(a), and

\[
\min(1, (\beta s + 1)/2, \alpha s) \leq \frac{1 + \min(1, \beta s)}{2},
\]

implies that

\[
2 + \frac{s - \min(\beta s,1)}{\min(1, (\beta s + 1)/2, \alpha s)} \leq f(s).
\]

Consequently,

**Figure 6.** (a) The inequality \(\min(\beta s,1) < s\) (green line). (b) The equality \(\min(\beta s,1) = s\) (blue line). (c) The inequality \(\min(\beta s,1) > s\) (red line). The dash lines correspond to the function \(y(s) = \min(\beta s,1)\) and the dotted lines refer to the function \(y(s) = \beta s\) varying by different cases of \(\beta\) value.

\[
\text{Cost}(s) \approx \epsilon - \frac{1 + s}{\min(1, (\beta s + 1)/2, \alpha s)}.
\]
Observing that \( f \) is strictly decreasing on the set \( (s < \frac{\min((1+\beta s)/2,1)}{\alpha}) \) and strictly increasing on \( (s > \frac{\min((1+\beta s)/2,1)}{\alpha}) \), we obtain the unique minimizer

\[
\text{(46)} \quad \arg\min_{s > 0} f(s) = \begin{cases} 
\alpha^{-1} & \text{if } \alpha \leq \beta, \\
(2\alpha - \beta)^{-1} & \text{otherwise}.
\end{cases}
\]

2. If \( \beta = 1 \), then \( \min(\beta s, 1) = s \) for all \( s \in (0,1] \) and \( \min(\beta s, 1) < s \) for \( s > 1 \), cf. Figure 6(a-b). Consequently,

\[
\text{Cost}(s) \approx \begin{cases} 
\epsilon^{-2}L^3 + \epsilon^{-f(s)} & \text{if } s \in (0,1], \\
\epsilon^{-f(s)} & \text{if } s > 1.
\end{cases}
\]

By (46), now simplifying to \( f(s) = \max(1 + s, 2, (1 + s)/(\alpha s)) \), we obtain that

\[
\text{Cost}(s) \approx \begin{cases} 
\epsilon^{-2}L^3 & \text{if } \alpha \geq 1 \text{ and } s \in [\alpha^{-1}, 1] \\
\epsilon^{-f(s)} & \text{if } \alpha < 1 \text{ and } s = \alpha^{-1}.
\end{cases}
\]

3. If \( \beta > 1 \), then \( \min(\beta s, 1) > s \) for all \( s \in (0,1) \) and \( \min(\beta s, 1) = s \) for \( s = 1 \), and \( \min(\beta s, 1) < s \) for \( s > 1 \). Thus

\[
\text{Cost}(s) \approx \begin{cases} 
\epsilon^{-2} + \epsilon^{-f(s)} & \text{if } s \in (0,1), \\
\epsilon^{-2}L^3 + \epsilon^{-f(s)} & \text{if } s = 1, \\
\epsilon^{-f(s)} & \text{if } s > 1.
\end{cases}
\]

If \( \alpha > 1 \), then \( f(s) \leq 2 \iff s \in [\alpha^{-1}, 1] \). If \( \alpha = 1 \), then \( f(s) \leq 2 \iff s = 1 \).

And if \( \alpha < 1 \), then

\[
f(s) = \min_{s > 0} f(s) = (1 + \alpha^{-1}) \iff s = \alpha^{-1}.
\]

This yields

\[
\text{Cost}(s) \approx \begin{cases} 
\epsilon^{-2} & \text{if } \alpha > 1 \text{ and } s \in [\alpha^{-1}, 1] \\
\epsilon^{-2}L^3 & \text{if } \alpha = 1 \text{ and } s = 1 \\
\epsilon^{-f(s)} & \text{if } \alpha < 1 \text{ and } s = \alpha^{-1}.
\end{cases}
\]

**APPENDIX B. DMFEnKF Algorithm**

This section describes the algorithm for the density-based deterministic approximation of the MFEnKF, which iteratively computes the prediction density \( \rho_{\hat{v}_n} \) and updated density \( \rho_{\hat{v}_n} \) for \( n = 1, 2, \ldots \). Each iteration cycle consists of two steps: one transition from \( \rho_{\hat{v}_n} \) to \( \rho_{v_{n+1}} \) governed by the Fokker-Planck equation (FPE) and another from \( \rho_{v_{n+1}} \) to \( \rho_{\hat{v}_{n+1}} \) via an affine transformation and a convolution with a Gaussian density. For simplicity, we show the algorithm for the one-dimensional state-space case, i.e., \( d = 1 \).

Let \( S^{t} \rho \) denote a solution at time \( t \) of the FPE

\[
\partial_t \rho(x, t) = \partial_x (V'(x) \rho(x, t)) + \frac{\sigma^2}{2} \partial_x^2 \rho(x, t), \quad (x, t) \in \mathbb{R} \times (0, \infty),
\]

with the initial condition \( \rho(\cdot, 0) = \rho \). Note that for the mean-field dynamics (6) with \( N = \infty \) that satisfies the SDE (18), it holds that \( \Psi(\hat{v}_n) \sim S^{t} \rho_{\hat{v}_n} \) for any \( n \geq 0 \).
Since the updated mean-field ensembles can be viewed as the sum of the following independent random variables
\[
\hat{v}_n = (I - \bar{K}_n H)\bar{v}_n + \bar{K}_n y_n + \bar{K}_n \tilde{\eta}_n,
\]
the updated density can be written
\[
\rho_{\hat{v}_n}(x) = \rho_{X+Y}(x) = \int_{\mathbb{R}} \rho_X(z) \rho_Y(x-z) dz = \rho_X * \rho_Y(x),
\]
where * stands for a convolution operator.

**Algorithm 1: DMFEnKF**

**Input:** The initial updated density \(\rho_{\hat{v}_0} = \rho_{u_0|y_0}\), the number of time steps \(N_t\), the number of spatial steps \(N_x\), the discretization interval \([x_0, x_1]\), the simulation length \(N\).

**Output:** The prediction and updated density, \(\rho_{\bar{v}_n}\) and \(\rho_{\hat{v}_n}\), respectively.

1. \(\Delta t = \frac{1}{N_t}, \Delta x = \frac{x_1 - x_0}{N_x}\).
2. for \(n = 1 : N\) do
   3. Compute the prediction density \(\rho_{\bar{v}_n}(x) = S^1 \rho_{\hat{v}_{n-1}}\) by a numerical method (e.g., Crank-Nicolson) with the discretization steps \((\Delta t, \Delta x)\).
   4. Compute the prediction covariance \(\bar{C}_n = \int (x^2 \rho_{\bar{v}_n}(x) dx - (\int x \rho_{\bar{v}_n}(x) dx)^2)\) using a quadrature rule.
   5. Compute the Kalman gain \(\bar{K}_n = \bar{C}_n H^T (H \bar{C}_n H^T + \Gamma)^{-1}\).
   6. Compute the updated density \(\rho_{\hat{v}_n} = \rho_X * \rho_Y\) by discrete convolution of the two functions represented on the spatial mesh.

**Remark:** The discretization interval \([x_0, x_1]\) must be chosen such that the truncation error pertaining to the integral on the complement of said interval is close to zero, i.e.,
\[
\int_{[x_0,x_1]^{c}} \rho_{\hat{v}_n}(x) dx \approx 0.
\]

For the problem in Section 3.2, we found by numerical experiments using the Crank-Nicolson method that \([x_0, x_1] = [-5, 5], \Delta t = 10^{-3}\) and \(\Delta x = 10^{-5}\) were suitable resolution parameters to obtain solutions with negligible approximation errors relative to the statistical and bias error introduced by EnKF and MLEnKF.

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