Krein Regularization of QED

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Abstract

In this paper the electron self-energy, photon self-energy and vertex functions are explicitly calculated in Krein space quantization including quantum metric fluctuation. The results are automatically regularized or finite. The magnetic anomaly and Lamb shift are also calculated in the one loop approximation in this method. Finally, our results are compared to conventional QED results.

Keywords: Krein regularization; Krein Quantization; Quantum metric fluctuation

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1 Introduction

Recent observational data indicates that in the first approximation, background space-time is very similar to de Sitter space-time. As a result, a quantization of the linear gravitational field without infrared divergence in such a space time model may be of great importance for further developments in quantum gravity. Antoniadis, Iliopoulos and Tomaras [1] have shown that the pathological large-distance behaviour or infrared divergence of the graviton propagator in de Sitter background does not manifest itself in the quadratic part of the effective action in the one-loop approximation. This pathological behaviour of the graviton propagator may be gauge dependent and should not appear in an effective manner as a physical quantity. Linear gravity (the traceless rank-2 “massless” tensor field) in de Sitter space is indeed built up from copies of the minimally coupled scalar field [2, 3] and the above-mentioned problem also appears in the quantization of this field. One can construct a covariant quantization of the “massless” minimally coupled scalar field in de Sitter space-time, which is causal and free of any infrared divergence [4, 5]. The essential point of this construction is the unavoidable presence of negative norm states, i.e. Krein space quantization. The indefinite metric quantization or Krein space quantization is of interest to physicists for different reasons. The idea was first proposed by

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Dirac [6] and was later exploited by Gupta and Bleuler to remove infrared divergence in QED [7, 8].

The negative-norm states necessarily appear in a covariant quantization of the free minimally coupled scalar field in de Sitter space-time [4, 5]. In this process, ultraviolet divergence in vacuum energy and infrared divergence in two-point function have been automatically eliminated [5, 9] and therefore a linear graviton propagator without infrared divergence can be obtained. In order to preserve the covariant and eliminate the infrared divergence of linear gravity in de Sitter space, it is necessary to use Krein space quantization [2, 3, 10].

In the interaction QFT, the divergences appear due to the singular behavior of the Green function at short relative distances (ultraviolet divergence) or at large relative distances (infrared divergence). The ultraviolet divergence appears in the following form in the Green function, $G(x, x')$, in the limit $x \to x'$ or $\sigma \to 0$:

$$\frac{1}{\sigma}, \ln \sigma, \text{ and } \delta(\sigma), \text{ where } \sigma = \frac{1}{2}(x - x')^2.$$

It was conjectured that quantum metric fluctuations might smear out the singularities of Green functions on the light cone i.e. $\delta(\sigma)$, but it does not remove other ultraviolet divergences of quantum field theory [11]. However, we have shown that quantization in Krein space removes all ultraviolet divergences of QFT except the light cone singularity [5, 12]. In this procedure, the auxiliary negative norm states (negative frequency solutions which do not interact with the physical states or real physical world) have been utilized. It has been shown that the combination of quantum field theory in Krein space together with the consideration of quantum metric fluctuation results in a quantum field theory without any divergences [13].

A natural regularization of the one-loop interacting quantum scalar field in Minkowski space-time ($\lambda\phi^4$) has been achieved through the consideration of Krein space quantization [12]. The Casimir effect and one-loop approximation of Moller scattering have been determined in Krein space [14, 15]. This quantization has also been developed further to remove infrared divergence in linear gravity in de Sitter space [10]. Recently, the effective action for ($\lambda\phi^4$) theory and QED has been calculated in Krein space quantization including quantum metric fluctuation [16, 17]. These have proven that the physical results do not change in this method.

Although negative norm states appear in our method, by imposing the two conditions stated below, the negative norm states completely disappear and the theory becomes unitary:

i) The first condition is the "reality condition" in which the negative norm states do not appear in the external legs of the Feynmann diagram. This condition guarantees that the negative norm states only appear in the internal legs and in the disconnected parts of the diagram.

ii) The second condition is that the $S$ matrix elements must be renormalized in the following form:

$$S_{if}' \equiv \text{probability amplitude} = \frac{\langle \text{physical states, in} | \text{physical states, out} \rangle}{\langle 0, \text{in} | 0, \text{out} \rangle}.$$

This condition eliminates the negative norm states in the disconnected parts.
We must emphasize the fact that this method can be used to calculate physical observables in scenarios where the effect of quantum gravity (in the linear approximation) can not be ignored. In this model, the problem of non-renormalizability of linear quantum gravity is solved.

In this paper, we explicitly calculate the photon self energy, electron self energy and vertex function in Krein space quantization including quantum metric fluctuation in order to investigate the validity of our model. This method is very similar to the Pauli-Villars regularization and then it can be considered as a new method of regularization. Therefore we have called it "Krein regularization". In this method the Lagrangian does not need any contour terms and the QFT is automatically regularized. We also calculate the magnetic anomaly and Lamb shift in Krein space quantization. Finally our results are compared with conventional QED results.

## 2 Propagators in Krein space

Let us first recall some elementary facts about the propagators in Krein space quantization. The time-ordered product propagator for the scalar field in Krein space quantization is [5]:

\[
iG_T(x, x') = < 0 | T \phi(x) \phi(x') | 0 >= \theta(t - t') W(x, x') + \theta(t' - t) W(x', x).
\]

Writing (2.1) in terms of Feynman propagator, we obtain:

\[
G_T(x, x') = \frac{1}{2} [ G_F(x, x') + (G_F(x, x'))^* ] = \Re G_F(x, x').
\]

The Feynman Green function is defined by:

\[
G_F(x, x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik.(x-x')} \tilde{G}_F(k) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik.(x-x')}}{k^2 - m^2 + i\epsilon} = -\frac{1}{8\pi}\delta(\sigma_0) + \frac{m^2}{8\pi} \theta(\sigma_0) J_1(\sqrt{2m^2\sigma_0}) - iN_1(\sqrt{2m^2\sigma_0}) - \frac{im^2}{4\pi^2} \theta(-\sigma_0) K_1(\sqrt{-2m^2\sigma_0}),
\]

where \(2\sigma_0 = (x - x')^2 = \eta_{\mu\nu}(x^\mu - x'^\mu)(x^\nu - x'^\nu)\). So we have:

\[
G_T(x, x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik.(x-x')} PP \frac{1}{k^2 - m^2} = -\frac{1}{8\pi} \delta(\sigma_0) + \frac{m^2}{8\pi} \theta(\sigma_0) \frac{J_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}}, \quad x \neq x',
\]

where PP stands for the principal part. This equation exhibits singularity on the light cone alone. The contribution of the coincident point singularity \((x = x')\) only appears in the imaginary part of \(G_F(x, x')\) ([9] and equation (9.52) in [18])

\[
G_F(x, x') = -\frac{2i}{(4\pi)^2} \frac{m^2}{d - 4} + G_{finite}^{finite}(x, x'),
\]

where \(d\) is the space-time dimension and \(G_{finite}^{finite}(x, x')\) becomes finite as \(d \to 4\). In the momentum space for this propagator we have [19]

\[
\tilde{G}_T(k) = \frac{1}{2} [ \tilde{G}_F(k) + \tilde{G}_F(k)^* ] = \frac{1}{2} \left[ \frac{1}{k^2 - m^2 + i\epsilon} + \frac{1}{k^2 - m^2 - i\epsilon} \right] = PP \frac{1}{k^2 - m^2}.
\]
The quantum metric fluctuations \(g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\) remove the singularities of the Green functions on the light cone [11]. Therefore, the quantum field theory in Krein space, including the quantum metric fluctuation, removes all ultraviolet divergences of the theory [13]:

\[
\langle G_T(x - x') \rangle = -\frac{1}{8\pi} \sqrt{\frac{\pi}{2\langle \sigma_1^2 \rangle}} \exp \left(-\frac{\sigma^2}{2\langle \sigma_1^2 \rangle}\right) + \frac{m^2}{8\pi} \theta(\sigma_0) \frac{J_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}},
\]

where \(2\sigma = g_{\mu\nu}(x^\mu - x'^\mu)(x'^\nu - x'^\nu)\) and \(\sigma = \sigma_0 + \sigma_1 + \ldots\). In the case of \(\sigma_0 = 0\), due to the metric quantum fluctuation \(h_{\mu\nu}\), we have \(\langle \sigma_1^2 \rangle \neq 0\), so we get

\[
\langle G_T(0) \rangle = -\frac{1}{8\pi} \sqrt{\frac{\pi}{2\langle \sigma_1^2 \rangle}} + \frac{m^2}{8\pi} \frac{1}{2}. \tag{2.7}
\]

It should be noted that \(\langle \sigma_1^2 \rangle\) is related to the density of gravitons [11].

By using the Fourier transformation of the Dirac delta function

\[
-\frac{1}{8\pi} \delta(\sigma_0) = \int \frac{d^4k}{(2\pi)^4} e^{-ik.(x-x')} PP \frac{1}{k^2}, \tag{2.8}
\]

for the second part of the Green function (2.6), we obtain [16]:

\[
\frac{m^2}{8\pi} \theta(\sigma_0) \frac{J_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}} = \int \frac{d^4k}{(2\pi)^4} e^{-ik.(x-x')} PP \left(\frac{m^2}{k^2(k^2 - m^2)}\right). \tag{2.9}
\]

The first part is:

\[
-\frac{1}{8\pi} \sqrt{\frac{\pi}{2\langle \sigma_1^2 \rangle}} \exp \left(-\frac{(x - x')^4}{4\langle \sigma_1^2 \rangle}\right) = \int \frac{d^4k}{(2\pi)^4} e^{-ik.(x-x')} \tilde{G}_1(k), \tag{2.10}
\]

where \(\tilde{G}_1(k)\) is the Fourier transformation of the first part of the Green function (2.6). Therefore

\[
< \tilde{G}_T(k) > = \tilde{G}_1(k) + PP \frac{m^2}{k^2(k^2 - m^2)}. \tag{2.11}
\]

In the previous paper, we showed that in the one-loop approximation, the Green function in Krein space quantization, which appears in the s-channel contribution of transition amplitude is [12]:

\[
PP \frac{m^2}{k^2(k^2 - m^2)}. \tag{2.12}
\]

It means that in this approximation, the contribution of the first part (or quantum metric fluctuation) is negligible. It is worth mentioning that in order to improve the UV behavior in relativistic higher-derivative correction theories, the propagator (2.12) has been used by some authors [20, 21]. This propagator (2.12) also appears in the supersymmetry theory [22].

The time-order product of the spinor field is constructed by its scalar field counterpart:

\[
\langle S_T(x - x') \rangle \equiv (i \beta + m) < G_T(x, x') >, \tag{2.13}
\]

where

\[
iS_T(x, x') = < 0 | T\psi(x)\bar{\psi}(x') | 0 >.
\]
The propagator of the spinor field can be obtained as follows:

\[
\langle S_T(x - x') \rangle = \frac{1}{8\pi i} \gamma^\mu (x_\mu - x'_\mu) \left[ \frac{e^{-\frac{\sigma^2_0}{2\langle \sigma^2 \rangle}}} {2\langle \sigma^2 \rangle} \right] + m^2 \left[ \frac{J_1(\sqrt{2} \sigma_0 \langle \sigma^2 \rangle)} {\sqrt{2\sigma_0 \langle \sigma^2 \rangle}} \right] + \frac{m}{8\pi} \left[ \frac{\pi}{2\langle \sigma^2 \rangle} e^{-\frac{\sigma^2_0}{2\langle \sigma^2 \rangle}} + \theta(\sigma_0) \frac{J_1(\sqrt{2} \sigma_0 \langle \sigma^2 \rangle)} {\sqrt{2\sigma_0 \langle \sigma^2 \rangle}} \right].
\]

The Fourier transformation of spinor propagator is:

\[
(\hat{k} + m) \left[ \hat{G}_1(k) + PP \frac{m^2}{k^2(k^2 - m^2)} \right].
\]

If we ignore the first term the propagator becomes:

\[
(\hat{k} + m) \left[ PP \frac{m^2}{k^2(k^2 - m^2)} \right]. \tag{2.14}
\]

The time-ordered product propagator in the Feynman gauge for the vector field in Krein space is given by:

\[
D^T_{\mu\nu}(x, x') = -\eta_{\mu\nu} G_T(x, x'), \quad \bar{D}_{\mu\nu}(k) = PP \frac{-\eta_{\mu\nu}} {k^2}. \tag{2.15}
\]

It is trivial to show that in Krein space quantization, aside from the change of propagators, the Feynman rules for QED are similar to those of conventional QFT. In Hilbert space the propagator is \( \frac{1}{k^2 - m^2 + i\epsilon} \), whereas the propagator used in Krein space is \( PP \frac{1}{k^2} \). It is noteworthy that in tree diagrams or lines that do not appear in the loops, these two propagators are equivalent since there are no integrals on \( k^2 \) and \( k^2 \neq m^2 \).

For diagrams with loops there are two possibilities: i) the diagram is convergent or ii) the diagram is divergent due to the singularity of the delta function in the propagator. In the first case we select the propagator to be \( PP \frac{1}{k^2} \). In the second case, in order to eliminate the delta function singularity, the quantum metric fluctuation is included and the propagator (2.11) or alternatively an approximation of it in the form of (2.12) is used.

### 3 Electron Self Energy in Krein Space

In this part, the electron self energy in the Krein space quantization is calculated. It is:

\[
i\Sigma_{kr}(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} i\bar{D}^T_{\mu\nu}(p - k) \gamma^\mu i\bar{S}_T(k) \gamma_\nu, \tag{3.1}
\]

which is divergent at the ultraviolet limit. This divergence is due to the delta function singularity in the propagators. By including the quantum metric fluctuation we have

\[
i\Sigma_{kr}(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} i < \bar{D}^T_{\mu\nu}(p - k) > \gamma^\mu i < \bar{S}_T(k) > \gamma_\nu, \tag{3.2}
\]
which is convergent. Since we do not have the explicit form of the Fourier transformation of $\tilde{G}_1$, the integral (3.2) is calculated by the following approximation: for the electron propagator the Green function (2.14) and for the photon propagator Green function (2.15) are used. Then we obtain:

$$i\Sigma_{kr}(p) = \frac{e^2}{4} \int \frac{d^4k}{(2\pi)^4} \gamma^\mu(k + m)\gamma_\mu PP\left(\frac{1}{k^2 - m^2} - \frac{1}{k^2}ight) PP\left(\frac{1}{(p - k)^2}\right).$$  \hspace{1cm} (3.3)

In this approximation, the integral (3.3) can be calculated. In order to solve this integral, Feynman parameters are used:

$$i\Sigma_{kr}(p) = e^2 \int_0^1 dx \int \frac{d^4l}{(2\pi)^4} (-2x \not{p} + 4m) \left[\frac{1}{(l^2 - \Delta)^2} - \frac{1}{(l^2 - \Delta')^2}\right].$$  \hspace{1cm} (3.4)

where $k = l + xp$, $\Delta = x^2p^2 - xp^2 + (1 - x)m^2$ and $\Delta' = x(x - 1)p^2$ \hspace{1cm} [Appendix A] [23].

The integral over $l$ is no longer divergent. By calculating the integral over $l$, we obtain \hspace{1cm} [Appendix A]:

$$\Sigma_{kr}(p) = \frac{e^2}{8\pi^2} \int_0^1 dx (x \not{p} - 2m) \ln \frac{x(x - 1)p^2 + (1 - x)m^2}{x(x - 1)m^2}. $$  \hspace{1cm} (3.5)

Solving the integral over $x$, we have the following result:

$$\Sigma_{kr}(p) = \frac{e^2}{8\pi^2} \left\{ \ln \left(-\frac{p^2}{m^2}\right) \left(2m - \frac{\not{p}}{2}\right) \right. \\
- \frac{\not{p}}{2} \left[ \frac{m^2}{p^2} + \frac{m^4 - (p^2)^2}{(p^2)^2} \ln \left(1 - \frac{p^2}{m^2}\right) \right] + 2m \left[ \frac{m^2 - p^2}{p^2} \ln \left(1 - \frac{p^2}{m^2}\right) \right] \left\}\right. \hspace{1cm} (3.6)

The result in the Hilbert space is \hspace{1cm} [19]:

$$\Sigma_{Hi}(p, \Lambda, M = 0) = \frac{e^2}{8\pi^2} \left\{ \ln \left(\frac{\Lambda^2}{m^2}\right) \left(2m - \frac{\not{p}}{2}\right) + \left(2m - \frac{\not{p}}{4}\right) \right. \\
- \frac{\not{p}}{2} \left[ \frac{m^2}{p^2} + \frac{m^4 - (p^2)^2}{(p^2)^2} \ln \left(1 - \frac{p^2}{m^2}\right) \right] + 2m \left[ \frac{m^2 - p^2}{p^2} \ln \left(1 - \frac{p^2}{m^2}\right) \right] \left\}\right. \hspace{1cm} (3.7)

It is clear that the result is finite and also the last two terms of (3.6) and (3.7) are equal.

### 4 Photon Self Energy in Krein space

The photon self energy in the Krein space quantization is:

$$i\Pi_{\mu\nu}^{kr}(k) = (-1)(-ie)^2 \int \frac{d^4p}{(2\pi)^4} Tr \left[ \gamma_\mu \tilde{S}_T(p) \gamma_\nu \tilde{S}_T(p - k) \right].$$  \hspace{1cm} (4.1)
which is divergent. By including the quantum metric fluctuation, we have

\[ i\Pi_{\mu\nu}^r(k) = (-ie)^2 \int \frac{d^4p}{(2\pi)^4} (-1) Tr \left[ \gamma_{\mu} < S_T(p) > \gamma_{\nu} < S_T(p-k) > \right] , \]  

(4.2)

which is convergent. By using the Green function (2.14), we obtain:

\[ i\Pi_{\mu\nu}^{kr}(k^2) = -\frac{e^2}{4} \int \frac{d^4p}{(2\pi)^4} Tr \left[ \gamma^{\mu}(\not{p} + m) PP \left( \frac{1}{p^2 - m^2} - \frac{1}{p^2} \right) \right. \]

\[ \left. \gamma^{\nu}(\not{k} + \not{p} + m) PP \left( \frac{1}{(p+k)^2 - m^2} - \frac{1}{(p+k)^2} \right) \right] . \]  

(4.3)

After using the Feynman parameter, we change the variable \( p \) to \( l - xk \), perform the Wick rotation in a manner that if \( i\epsilon \) exists in the denominator, the substitution \( l^0_E = -i\epsilon \) is used, and if \(-i\epsilon\) is present in the denominator, the change of variable \( l^0_E = i\epsilon \) is applied. We have [Appendix B] [23]:

\[ i\Pi_{\mu\nu}^{kr}(k^2) = -4ie^2 \int_0^1 dx \int \frac{d^4l_E}{(2\pi)^4} \left[ \frac{1}{(l^2_E + \Delta_1)^2} - \frac{1}{(l^2_E + \Delta_2)^2} - \frac{1}{(l^2_E + \Delta_3)^2} + \frac{1}{(l^2_E + \Delta_4)^2} \right] \]

\[ \left( \frac{1}{2}g^{\mu\nu}l^2_E - 2x(1-x)k^\mu k^\nu + g^{\mu\nu}(m^2 + x(1-x)k^2) \right) . \]  

(4.4)

The photon self energy in Krein space is gauge invariant:

\[ k^\mu \Pi_{\mu\nu}^{kr}(k^2) = 0, \]  

(4.5)

and integral (4.4) can be written in the following form [Appendix B]:

\[ \Pi_{\mu\nu}^{kr}(k^2) = (k^2 g^{\mu\nu} - k^\mu k^\nu) \Pi_{kr}(k^2) , \]  

(4.6)

where

\[ \Pi_{kr}(k^2) = \frac{e^2}{2\pi^2} \int_0^1 dx (1-x) x \ln \left( -\frac{k^2}{m^2} \right) + \frac{e^2}{2\pi^2} \int_0^1 dx (1-x) x \ln \left( 1 - x(1-x) \frac{k^2}{m^2} \right) \]

\[ -\frac{e^2}{2\pi^2} \int_0^1 dx (1-x) x \ln \left( 1 - \frac{k^2}{m^2} \right) - \frac{e^2}{2\pi^2} \int_0^1 dx (1-x) x \ln \left( 1 - (1-x) \frac{k^2}{m^2} \right) . \]  

(4.7)

The first integral yields \( \frac{e^2}{12\pi^2} \ln \left( -\frac{k^2}{m^2} \right) \), and the third and fourth integrals are equal to

\[ \frac{4e^2}{\pi^2} \left\{ -\frac{5}{36} - \frac{m^2}{3k^2} + \frac{m^4}{3k^4} + \left( \frac{1}{6} - \frac{m^4}{2k^4} + \frac{m^6}{3k^6} \right) \ln \left( 1 - \frac{k^2}{m^2} \right) \right\} . \]

Therefore the photon self energy in Krein space can be written as follows:

\[ \Pi_{kr}(k^2) = \frac{e^2}{12\pi^2} \ln \left( -\frac{k^2}{m^2} \right) + \frac{e^2}{2\pi^2} \int_0^1 dx (1-x) x \ln \left( 1 - x(1-x) \frac{k^2}{m^2} \right) - \]
\[
\frac{4e^2}{\pi^2} \left\{ -\frac{5}{36} - \frac{m^2}{3k^2} + \frac{m^4}{3k^4} + \left(\frac{1}{6} - \frac{m^4}{2k^4} + \frac{m^6}{3k^6}\right) \ln \left(1 - \frac{k^2}{m^2}\right) \right\}.
\] (4.8)

In Hilbert space, the photon self energy is [23]:
\[
\Pi_{H}(k^2) = -\frac{e^2}{12\pi^2} \ln \left(\frac{\Lambda^2}{m^2}\right) + \frac{e^2}{2\pi^2} \int_0^1 dx (1-x) x \ln \left(1 - x(1-x) \frac{k^2}{m^2}\right).
\] (4.9)

The small part of the Lamb shift is calculated by using the following limit:
\[
\lim_{k^2 \to 0} k^2 \Pi_{kr}(k^2) = -\frac{e^2}{60\pi^2} \frac{k^4}{m^2},
\] (4.10)
which is equal to the result in Hilbert space [23].

5 Vertex Function in Krein space

The vertex function in the Krein space quantization is:
\[
\Lambda^\mu_{kr}(p', p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} D^T_{\nu\rho}(p-k) \gamma^\nu (-i) S_T(k') \gamma^\rho i S_T(k) \gamma^\mu,
\] (5.1)
and by including the quantum metric fluctuation we have
\[
\Lambda^\mu_{kr}(p', p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} < D^T_{\nu\rho}(p-k) > \gamma^\nu (-i) < S_T(k') > \gamma^\rho i < S_T(k) > \gamma^\mu,
\] (5.2)
where \(k' - k = q\) and \(p' - p = q\). After entering the results of the Fourier transformation of propagators into Krein space and using similar techniques, we have [Appendix C] [23]:
\[
\Lambda^\mu_{kr}(p', p) = F^{kr}_1(q^2) \gamma^\mu + \frac{i\sigma^{\mu\nu} q^\nu}{2m} F^{kr}_2(q^2),
\] (5.3)
where \((M\) is the mass of the photon):
\[
F^{kr}_1(q^2) = \frac{e^2}{8} \int_0^1 dz_3 dz_2 dz_1 \delta(1-z_1 - z_2 - z_3) \int \frac{d^4k}{(2\pi)^4} PP \frac{1}{(k-p)^2 - M^2} PP \frac{1}{k^2 - m^2} PP \frac{1}{k^2 - m^2} \left[ -\frac{1}{2}(k + z_2 q - z_1 p)^2 + q^2 (1-z_3)(1-z_2) + m^2(1-4z_1 + z_1^2) \right],
\] (5.4)
\[
F^{kr}_2(q^2) = \frac{ie^2}{8} \int_0^1 dz_3 dz_2 dz_1 \delta(1-z_1 - z_2 - z_3) \int \frac{d^4k}{(2\pi)^4} \left( 2m^2 z_1(1-z_1) \right) PP \frac{1}{(k-p)^2 - M^2} PP \frac{1}{k^2 - m^2} PP \frac{1}{k^2 - m^2}.
\] (5.5)
It can be seen that the numerators in (5.4) and (5.5) are the same as they are in Hilbert space. \(F^{kr}_2(q^2)\) is convergent while \(F^{kr}_1(q^2)\) is divergent. This divergence is due to the Dirac delta function; it is therefore possible to remove the divergence via quantum metric fluctuation. In order to achieve this, \(PP \frac{m^2}{k^2 - m^2}\) is used as a propagator for \(F^{kr}_1(q^2)\).
5.1 $F^{kr}_1$ Term

Due to the existence of divergence terms in this part, the Krein regularization is used, which means the propagator (2.14) is used for the spinor field and (2.15) is used for the photon field:

$$F^{kr}_1(q^2) = \frac{e^2}{8\pi^2} \int dz_1dz_2dz_3\delta(1-z_1-z_2-z_3) \int \frac{d^4k}{(2\pi)^4} PP \frac{1}{(k-p)^2-M^2} PP \left( \frac{1}{k^2-m^2} - \frac{1}{k^2} \right)$$

$$\left[ -\frac{1}{2} (k+z_2q-z_1p)^2 + q^2(1-z_3)(1-z_2) + m^2(1-4z_1+z_1^2) \right] PP \left( \frac{1}{k^2-m^2} - \frac{1}{k^2} \right). \quad (5.6)$$

In order to solve these integrals, we use Feynman parameters:

$$\Delta_2 = m^2(1-z_1)^2 + M^2z_1 - q^2z_2z_3,$$

$$\Delta_3 = m^2(1-z_1)^2 + M^2z_1 - q^2z_2z_3 - z_3m^2,$$

$$\Delta_4 = m^2(1-z_1)^2 + M^2z_1 - q^2z_2z_3 - (z_2+z_3)m^2,$$

$$\Delta_5 = m^2(1-z_1)^2 + M^2z_1 - q^2z_2z_3 - z_2m^2. \quad (5.7)$$

By using the newly defined variables in equation (5.6), we obtain [Appendix D] [22]:

$$F^{kr}_1(q^2) = \frac{e^2}{8\pi^2} \int dz_1dz_2dz_3\delta(1-z_1-z_2-z_3) \sum_{j=2}^{5} \left[ (-1)^j \ln \Delta_j - \frac{m^2(1-4z_1-z_1^2) + q^2(1-z_2)(1-z_3)}{(-1)^j \Delta_j} \right]. \quad (5.8)$$

In Hilbert space, $F_1(q^2)$ is as below [22]:

$$F^{Hi}_1(q^2) = \frac{e^2}{8\pi^2} \int dz_1dz_2dz_3\delta(1-z_1-z_2-z_3) \left[ (-\ln \Delta_0 \right.$$

$$\left. + \frac{m^2(1-4z_1-z_1^2) + q^2(1-z_2)(1-z_3)}{\Delta_0} \right], \quad (5.9)$$

where $\Delta_0 = m^2(1-z_1)^2 + M^2z_1 - q^2z_2z_3 - i\varepsilon - (p^2-m^2)z_2(1-z_2) - (p^2-m^2)z_3(1-z_3).$ Note that infrared divergence in (5.8) is removed due to the existence of the negative norm state of the propagators in the loop and the answer is finite. After integrating, we have [Appendix D]:

$$F^{kr}_1(q^2)_{q^2\to0} = \frac{\alpha q^2}{3\pi m^2} \left( \ln \frac{m}{M} - \frac{3}{8} - \frac{1}{4} \right). \quad (5.10)$$

In Hilbert space, we have [19, 22]:

$$F^{Hi}_1(q^2)_{q^2\to0} = -\frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} + \frac{\alpha q^2}{3\pi m^2} \left( \ln \frac{m}{M} - \frac{3}{8} \right). \quad (5.11)$$

By using the value of $F_1(q^2)$ and the photon self energy in Krein space, the value of the Lamb shift is calculated to be 1018.19 MHz, whereas in Hilbert space it was 1052.1 MHz. The experimental value of the Lamb shift has been given as 1057.8 MHz [19]. This small difference between the experimental results and those presented here may be because of disregarding the linear quantum gravitational effect and also the fact that we have worked in the one-loop approximation.
5.2 $F^2_{kr}$ Term

In the one-loop calculation, the magnetic anomaly does not have any divergence terms, so the propagator (2.5), which is gained from Krein space quantization, is used.

For calculating the magnetic anomaly, at first we assume that $q = 0$ and $M = 0$ in the equation (5.5) then we change $k$ to $l + z_1 p$. The variables $z_1, z_2$ and $z_3$ are Feynman parameters. Since the imaginary parts cancel each other out, only the real part remains

$$F^2_{kr}(0) = \frac{8e^2}{8} \int \frac{d^4l}{(2\pi)^4} \int_0^1 dz_3 dz_2 dz_1 \delta(1 - z_1 - z_2 - z_3) m^2 z_1(1 - z_1) \frac{8}{(l^2 - \Delta)^3}, \quad (5.12)$$

where $\Delta = (z_1 - 1)^2 m^2$. Consequently $F^2_{kr}(0) = \frac{e^2}{8\pi \epsilon}$, which is the same as the results in Hilbert space quantization.

In calculating $F^2_{kr}(q^2)$ with propagator (2.5), the same result as the conventional method is achieved. In the one-loop approximation, $F^2_{kr}(q^2)$ does not have any divergence terms. The propagator (2.12) is used when the singularity appears. This propagator is very similar to the propagator which is used in Pauli-Villars regularization, and it is because of this similarity that we have called our method "Krein Regularization". In Hilbert space, $F^2_{kr}(q^2)$ does not require regularization because it does not have any divergence terms. In calculating $F^2_{kr}(q^2)$ by using (2.12), the result obtained is similar to the Hilbert space result when Pauli-Villars regularization is used for calculating $F_2(q^2)$.

Up to now, in addition to $F^2_{kr}(q^2)$, the electron and photon self energy the following quantities have been calculated using propagator (2.12) in Krein space quantization: the effective action of $\lambda \phi^4$ [16], the effective action of QED [17], the transition amplitude of $\lambda \phi^4$ [26], the four-point function of $\lambda \phi^4$[27]. All of these items have divergence terms and therefore using propagator (2.5) for computing them does not yield finite results. The results obtained using (2.12) are in agreement with the conventional method.

6 Conclusion

In this paper we explicitly calculated the photon self energy, electron self energy and vertex functions in the one-loop approximation in Krein space quantization. We observed that the results are finite and the Lagrangian does not need any contour terms. The magnetic anomaly and Lamb shift are calculated in this approximation. Our results are comparable to the results in conventional QED. The magnetic anomaly is exactly the same as the previous results and for the Lamb shift we have a very small difference, which may be due to the omission of $\tilde{G}^4(k)$.

Consequently for QED, we see that this quantization eliminates the singularity in the theory without changing the physical content of the theory in the one-loop approximation, similar to our previous work on the effective action for QED [17]. This method can be easily used for linear quantum gravity in the background field method, where the theory is automatically renormalized. This method of quantization may be used as an alternative way for solving the non-renormalizability of linear quantum gravity in the background field method and is instrumental in finding a new method of quantization, which would be compatible with general relativity.
It is important to note that negative norm states can be propagated in the theory, but by imposing the additional conditions on the quantum state of the theory and the probability amplitude, one can circumvent this problem and obtain physical results for measurable quantities. The physical states or the external legs of the Feynman diagram are all positive while the negative states only appear in the internal line in the Feynman diagram, which appear due to the effect of the S-matrix. Therefore in calculating the S-matrix elements or probability amplitudes for the physical states, negative norm states only appear in the internal line and in the disconnected part of the Feynman diagram. The negative norm states, which appear in the disconnected part of the S-matrix elements, can be eliminated by renormalizing the probability amplitudes

\[ S'_{fi} = \frac{\langle \text{physical states, in} | \text{physical states, out} \rangle}{\langle 0, \text{in} | 0, \text{out} \rangle}. \]

We would like to submit that through this method, one can approximately calculate physical observables when the effect of quantum gravity cannot be neglected. In this case, Krein space quantization of linear quantum gravity is similar to the relativistic spectrum of the hydrogen atom in the background field method before the construction of QFT. This model can not provide a full answer to quantum gravity, since negative norm states appear in our model and are eliminated by imposing the additional conditions but this model can be applied to linear quantum gravity easily [2, 3, 10, 24, 25]. By using the usual quantum principle, it is impossible to quantize general relativity with the two essential principles of general covariance and causality, since these two principles are closely related to locality but the quantum states in conventional QFT are defined globally. For quantization of general relativity, the quantum states or probability amplitudes must be defined such as they are compatible with the general coordinate transformation, i.e. the quantum principles for defining the probability amplitude may be changed.

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A More details about equations (3.5) and (3.7)

In this appendix, we present the details of the calculations of electron self energy in Krein space quantization. Using propagator (2.12) in equation (3.3), we obtain

\[ i\Sigma^{kr}(p) = -\frac{(-ie)^2}{4} \int \frac{d^4k}{(2\pi)^4} \gamma^\mu(k + m)\gamma_\mu \left( \frac{1}{k^2 - m^2 + i\varepsilon} - \frac{1}{k^2 + i\varepsilon} + \frac{1}{k^2 - m^2 - i\varepsilon} - \frac{1}{k^2 - i\varepsilon} \right) \]

\[ \left( \frac{1}{(p - k)^2 + i\varepsilon} + \frac{1}{(p - k)^2 - i\varepsilon} \right). \]  

(A.1)

The denominators in the above equation can be redefined as below:

\[ yk^2 - ym^2 + (x + y)i\varepsilon + xk^2 - 2xkp + xp^2 = l^2 - \Delta + i\varepsilon, \]  

(A.2)

\[ yk^2 - ym^2 + (-x + y)i\varepsilon + xk^2 - 2xkp + xp^2 = l^2 - \Delta + (1 - 2x)i\varepsilon, \]  

(A.3)

\[ yk^2 - ym^2 + (x - y)i\varepsilon + xk^2 - 2xkp + xp^2 = l^2 - \Delta - (1 - 2x)i\varepsilon, \]  

(A.4)
Now we provide further details on calculating the photon self energy in Krein space using propagator (2.1), resulting in equations (3.5) and (3.7). Conversely, if we do not have any products of the terms $i\epsilon$ in the numerators, the Wick rotation is used in a manner that if $i\epsilon$ exists in the denominator, the substitution $l_E^0 = -i l^0$ is used, and if, conversely, $-i\epsilon$ is present in the denominator, the change of variable $l_E^0 = i l^0$ is applied. The result would be equations (3.5) and (3.7).

By using the change of variable $k \rightarrow l + xp$, we obtain:

$$i\Sigma^{kr}(p) = \frac{e^2}{4} \int_0^1 dx \int \frac{d^4l}{(2\pi)^4} (-2x \cdot \not{p} + 4m) \left[ \frac{1}{(l^2 - \Delta + i\epsilon)^2} + \frac{1}{(l^2 - \Delta + (1-2x)i\epsilon)^2} + \frac{1}{(l^2 - \Delta - (1-2x)i\epsilon)^2} + \frac{1}{(l^2 - \Delta - i\epsilon)^2} \right].$$  \hspace{1cm} (A.10)

In order to solve the integral over $l$, we apply the Wick rotation because in the numerators we do not have any products of the terms $i\epsilon$ and $-i\epsilon$. Therefore the Wick rotation is used in a manner that if $i\epsilon$ exists in the denominator, the substitution $l_E^0 = -i l^0$ is used, and if, conversely, $-i\epsilon$ is present in the denominator, the change of variable $l_E^0 = i l^0$ is applied. The result would be equations (3.5) and (3.7).

### B  More details about equations (4.5) and (4.8)

Now we provide further details on calculating the photon self energy in Krein space using propagator (2.14) for the electron and propagator (2.15) for the photon, with the purpose of arriving at equations (4.3) and (4.7). The photon self energy in Krein space quantization is:

$$i\Pi^{kr}_{\mu}(k) = \frac{e^2}{4} \int \frac{d^4p}{(2\pi)^4} Tr [\gamma^\mu(p + m)\gamma^\nu(p + k + m)]$$

$$\times \left( \frac{1}{p^2 - m^2 + i\epsilon} - \frac{1}{p^2 + i\epsilon} + \frac{1}{p^2 - m^2 - i\epsilon} - \frac{1}{p^2 - i\epsilon} \right)$$

$$\times \left( \frac{1}{(p + k)^2 - m^2 + i\epsilon} - \frac{1}{(p + k)^2 + i\epsilon} + \frac{1}{(p + k)^2 - m^2 - i\epsilon} - \frac{1}{(p + k)^2 - i\epsilon} \right).$$ \hspace{1cm} (B.1)

By using the change of variable $p \rightarrow l - xp$, and the Feynman parameters $x$ and $y$, the denominators of the above integral can be defined as below:

$$yp^2 + x(p + k)^2 - (x + y)m^2 + i\epsilon(x + y) = l^2 + \Delta_1 + i\epsilon,$$ \hspace{1cm} (B.2)

$$yp^2 + x(p + k)^2 - (x + y)m^2 + i\epsilon(-x + y) = l^2 + \Delta_1 + (1-2x)i\epsilon,$$ \hspace{1cm} (B.3)

$$yp^2 + x(p + k)^2 - (x + y)m^2 + i\epsilon(x - y) = l^2 + \Delta_1 - (1-2x)i\epsilon,$$ \hspace{1cm} (B.4)

$$yp^2 + x(p + k)^2 - (x + y)m^2 - i\epsilon(x + y) = l^2 + \Delta_1 - i\epsilon,$$ \hspace{1cm} (B.5)

$$yp^2 + x(p + k)^2 - ym^2 + i\epsilon(x + y) = l^2 + \Delta_2 + i\epsilon,$$ \hspace{1cm} (B.6)
\[
yp^2 + x(p + k)^2 - ym^2 + i\varepsilon(-x + y) = l^2 + \Delta_2 + (1 - 2x)i\varepsilon, \quad (B.7)
\]
\[
yp^2 + x(p + k)^2 - ym^2 + i\varepsilon(x - y) = l^2 + \Delta_2 - (1 - 2x)i\varepsilon, \quad (B.8)
\]
\[
yp^2 + x(p + k)^2 - ym^2 - i\varepsilon(x + y) = l^2 + \Delta_2 - i\varepsilon, \quad (B.9)
\]
\[
yp^2 + x(p + k)^2 - x^2 - ym^2 + i\varepsilon(x + y) = l^2 + \Delta_3 + i\varepsilon, \quad (B.10)
\]
\[
yp^2 + x(p + k)^2 - x^2 - ym^2 + i\varepsilon(-x + y) = l^2 + \Delta_3 + (1 - 2x)i\varepsilon, \quad (B.11)
\]
\[
yp^2 + x(p + k)^2 - x^2 - ym^2 + i\varepsilon(x - y) = l^2 + \Delta_3 - (1 - 2x)i\varepsilon, \quad (B.12)
\]
\[
yp^2 + x(p + k)^2 - x^2 - ym^2 - i\varepsilon(x + y) = l^2 + \Delta_3 - i\varepsilon, \quad (B.13)
\]
\[
yp^2 + x(p + k)^2 + i\varepsilon(x + y) = l^2 + \Delta_4 + i\varepsilon, \quad (B.14)
\]
\[
yp^2 + x(p + k)^2 + i\varepsilon(-x + y) = l^2 + \Delta_4 + (1 - 2x)i\varepsilon, \quad (B.15)
\]
\[
yp^2 + x(p + k)^2 + i\varepsilon(x - y) = l^2 + \Delta_4 - (1 - 2x)i\varepsilon, \quad (B.16)
\]
\[
yp^2 + x(p + k)^2 - i\varepsilon(x + y) = l^2 + \Delta_4 - i\varepsilon, \quad (B.17)
\]
\[
\Delta_1 = x^2k^2 - xk^2 + m^2, \quad \Delta_2 = x^2k^2 - xk^2 - (1 - x)m^2,
\]
\[
\Delta_3 = x^2k^2 - xk^2 + xm^2, \quad \Delta_4 = x^2k^2 - xk^2. \quad (B.18)
\]

This technique is similar to the one used in Hilbert space [23]. After performing the Wick rotation, we have:

\[
\Pi^{kr}_{\mu\nu}(k^2) = \frac{4e^2}{4} \int_0^1 dx \int \frac{d^4l_E}{(2\pi)^4} \left( \frac{g^{\mu\nu}}{2} l_E^2 - 2x(1-x)k^\mu k^\nu + g^{\mu\nu}(m^2 + x(1-x)k^2) \right)
\]

\[
\left[ \frac{1}{(l_E^2 + \Delta_1)^2} - \frac{1}{(l_E^2 + \Delta_2)^2} - \frac{1}{(l_E^2 + \Delta_3)^2} + \frac{1}{(l_E^2 + \Delta_4)^2} \right] = 4e^2 \int_0^1 dx \frac{1}{(4\pi)^{\frac{d}{2}}} \Gamma \left( 2 - \frac{d}{2} \right)
\]

\[
(k^2g^{\mu\nu} - k^\mu k^\nu) \left( \frac{1}{\Delta_1^{2-\frac{d}{2}}} - \frac{1}{\Delta_2^{2-\frac{d}{2}}} - \frac{1}{\Delta_3^{2-\frac{d}{2}}} + \frac{1}{\Delta_4^{2-\frac{d}{2}}} \right) + 4e^2 \int_0^1 dx \frac{1}{(4\pi)^{\frac{d}{2}}} \Gamma \left( 2 - \frac{d}{2} \right)
\]

\[
\left( \frac{m^2 g^{\mu\nu} x}{\Delta_1^{2-\frac{d}{2}}} + \frac{m^2 g^{\mu\nu}(1-x)}{\Delta_2^{2-\frac{d}{2}}} - \frac{m^2 g^{\mu\nu}}{\Delta_3^{2-\frac{d}{2}}} \right) = \frac{8e^2}{16\pi^2} \int_0^1 dx x(1-x) \ln \frac{\Delta_1 \Delta_4}{\Delta_2 \Delta_3} (k^2g^{\mu\nu} - k^\mu k^\nu). \quad (B.19)
\]

Which is equal to equation (4.6).
C  More details about equation (5.4) and (5.5)

In this appendix, we clarify the calculation procedure for the vertex function in Krein space quantization:

\[ \Lambda_{kr}^{\mu}(p', p) = \frac{ie^2}{8} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k - p)^2 - M^2 + i\varepsilon} + \frac{1}{(k - p)^2 - M^2 - i\varepsilon} + \frac{1}{k^2 - m^2 + i\varepsilon} + \frac{1}{k^2 - m^2 - i\varepsilon} \gamma^{\nu}(k + m)\gamma^{\mu}(k' + m)\gamma^{\rho} \]

\[ \left( \frac{1}{k^2 - m^2 + i\varepsilon} + \frac{1}{k^2 - m^2 - i\varepsilon} \right) \left( \frac{1}{k^2 - m^2 + i\varepsilon} + \frac{1}{k^2 - m^2 - i\varepsilon} \right). \] (C.1)

By using the Feynman parameters \( z_1, z_2 \) and \( z_3 \) for the numerator, changing variable \( k \) to \( l - z_2 q + z_1 p \), we obtain [23):

\[ \Lambda_{kr}^{\mu}(p, p') = \frac{32}{8} ie^2 \int \frac{d^4l}{(2\pi)^4} \int_0^1 dz_1 dz_2 dz_3 (1 - z_1 - z_2 - z_3) \left[ \gamma^{\nu} \left( -\frac{l^2}{2} + \frac{1}{k^2 - m^2 - \omega^2} \right) \right] \left( 1 - z_3)(1 - z_2)q^2 + m^2(1 - 4z_1 + z_1^2) \right) + \frac{1}{2m} \sigma^{\mu\nu} q^{\nu} F_1^{kr}(q^2) \gamma^{\rho} + i\frac{\sigma^{\mu\nu} q^{\nu}}{2m} F_2^{kr}(q^2). \] (C.2)

In the following appendices, the denominators are defined by Feynman parameters.

D  More details about equations (5.8) and (5.10)

In this part, we explicate the method used to arrive at equation (5.8). We change \( k \) to \( l \) as \( k \rightarrow l + z_1 p - z_2 q \) and \( \omega = 1 - z_1 , z_2 = \omega \xi \), \( \theta^2 = -\frac{q^2}{m^2} \) and \( \xi = \frac{1}{2} - \frac{1}{2\tanh \varphi} \). Then the integral (5.8) can be written as [22]:

\[ F_1^{kr}(q^2) = \frac{e^2}{4\pi^2} \left[ \ln \frac{M}{m} \right] + 2 \coth \theta \int d\varphi \varphi \tanh \varphi - \frac{\theta}{2} \tanh \frac{\theta}{2} \right] + \sum_{j=3}^5 p_j, \] (D.1)

where

\[ p_3 = \frac{e^2}{8\pi^2} \int d\xi \int \omega d\omega \left( \ln \Delta_3 + \frac{m^2 + 2m^2\omega + m^2\omega^2 - 3m^2 + q^2(1 - \omega + \omega^2\xi - \omega^2\xi^2)}{m^2 - q^2(1 - \xi)\omega^2 + M^2(1 - \omega) - M^2(1 - \xi)m^2} \right), \] (D.2)

\[ p_4 = \frac{e^2}{8\pi^2} \int d\xi \int \omega d\omega \left( \ln \Delta_4 - \frac{m^2 + 2m^2\omega + m^2\omega^2 - 3m^2 + q^2(1 - \omega + \omega^2\xi - \omega^2\xi^2)}{m^2 - q^2(1 - \xi)\omega^2 + M^2(1 - \omega) - M^2(1 - \xi)m^2} \right), \] (D.3)

\[ p_5 = \frac{e^2}{8\pi^2} \int d\xi \int \omega d\omega \left( \ln \Delta_5 + \frac{m^2 + 2m^2\omega + m^2\omega^2 - 3m^2 + q^2(1 - \omega + \omega^2\xi - \omega^2\xi^2)}{m^2 - q^2(1 - \xi)\omega^2 + M^2(1 - \omega) - M^2(1 - \xi)m^2} \right). \] (D.4)

The equation (D.2), (D.3) and (D.4) can be written in the form of \( p_3 = \sum_{i=1}^4 p_3 \), \( p_4 = \sum_{i=1}^4 p_4 \), \( p_5 = \sum_{i=1}^4 p_5 \), where

\[ p_{13} = \frac{e^2}{8\pi^2} \int d\xi \int \omega d\omega \ln[(m^2 - q^2\xi(1 - \xi))\omega^2 + M^2(1 - \omega) - M^2(1 - \xi)m^2], \] (D.5)
For solving these integrals, we define the new variable \( \xi \) and
\[
p_{23} = \frac{e^2}{8\pi^2} \int d\xi \int \omega d\omega \frac{-2m^2 + q^2}{(m^2 - q^2\xi(1 - \xi))\omega^2 + M^2(1 - \omega) - \omega(1 - \xi)m^2},
\]
\[
p_{33} = \frac{e^2}{8\pi^2} \int d\xi \int \omega d\omega \frac{(2m^2 - q^2)\omega}{(m^2 - q^2\xi(1 - \xi))\omega^2 + M^2(1 - \omega) - \omega(1 - \xi)m^2},
\]
\[
p_{43} = \frac{e^2}{8\pi^2} \int d\xi \int \omega d\omega \frac{(m^2 + q^2(\xi(1 - \xi)))\omega^2}{(m^2 - q^2\xi(1 - \xi))\omega^2 + M^2(1 - \omega) - \omega(1 - \xi)m^2},
\]
\[
p_{14} = -\frac{e^2}{8\pi^2} \int d\xi \int \omega d\omega \ln[(m^2 - q^2\xi(1 - \xi))\omega^2 + M^2(1 - \omega) - \omega m^2],
\]
\[
p_{24} = -\frac{e^2}{8\pi^2} \int d\xi \int \omega d\omega \frac{-2m^2 + q^2}{(m^2 - q^2\xi(1 - \xi))\omega^2 + M^2(1 - \omega) - \omega m^2},
\]
\[
p_{34} = -\frac{e^2}{8\pi^2} \int d\xi \int \omega d\omega \frac{(2m^2 - q^2)\omega}{(m^2 - q^2\xi(1 - \xi))\omega^2 + M^2(1 - \omega) - \omega m^2},
\]
\[
p_{44} = -\frac{e^2}{8\pi^2} \int d\xi \int \omega d\omega \frac{(m^2 + q^2(\xi(1 - \xi)))\omega^2}{(m^2 - q^2\xi(1 - \xi))\omega^2 + M^2(1 - \omega) - \omega m^2},
\]
\[
p_{15} = \frac{e^2}{8\pi^2} \int d\xi \int \omega d\omega \ln[(m^2 - q^2\xi(1 - \xi))\omega^2 + M^2(1 - \omega) - \omega \xi m^2],
\]
\[
p_{25} = \frac{e^2}{8\pi^2} \int d\xi \int \omega d\omega \frac{-2m^2 + q^2}{(m^2 - q^2\xi(1 - \xi))\omega^2 + M^2(1 - \omega) - \omega \xi m^2},
\]
\[
p_{35} = \frac{e^2}{8\pi^2} \int d\xi \int \omega d\omega \frac{(2m^2 - q^2)\omega}{(m^2 - q^2\xi(1 - \xi))\omega^2 + M^2(1 - \omega) - \omega \xi m^2},
\]
\[
p_{45} = \frac{e^2}{8\pi^2} \int d\xi \int \omega d\omega \frac{(m^2 + q^2(\xi(1 - \xi)))\omega^2}{(m^2 - q^2\xi(1 - \xi))\omega^2 + M^2(1 - \omega) - \omega \xi m^2}.
\]

For solving these integrals, we define the new variable \( \xi = \frac{1}{2} - \frac{1}{2} \frac{\tanh \frac{\varphi}{2}}{\tanh \frac{\varphi}{2}} \) and
\[
d\xi = -\frac{1}{2 \tanh \frac{\varphi}{2} \cosh^2 \varphi} d\varphi.
\]

Consequently:
\[
p_{23} + p_{25} + p_{33} + p_{35} + p_{43} + p_{45} + p_{24} + p_{34} = \]
\[
\frac{\alpha}{4\pi} \theta^2 - \frac{\alpha}{3\pi} \theta^2 - \frac{2\alpha}{9\pi} \theta^2 + \frac{\alpha}{2\pi} \theta^2 + \frac{\alpha}{2\pi} \theta \coth \theta t g^2 \theta^2 \frac{1}{2} \ln \theta^2,
\]
\[
p_{14} = -\frac{\alpha}{36\pi} \theta^2, p_{15} + p_{13} = \frac{\alpha}{8\pi} \theta^2 + \text{constant},
\]
\[
p_{44} = -\frac{\alpha}{3\pi} \theta^2 - \frac{\alpha}{8\pi} \theta^2 - \frac{\alpha}{4\pi} \theta^2 - \frac{\alpha}{2\pi} \theta^2 + \frac{\alpha}{2\pi} \theta^2 + \frac{\alpha}{2\pi} \theta^2 + \frac{\alpha}{2\pi} \theta^2.
\]

When \( \theta^2 \to 0 \) and \( \frac{\theta}{\sin \theta} \to 1 \), all \( p_{ij} \) terms are added and equation (5.10) is proved.
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