Multirole Logic

(Extend Abstract)

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Abstract

We identify multirole logic as a new form of logic in which conjunction/disjunction is interpreted as an ultrafilter on the power set of some underlying set (of roles) and the notion of negation is generalized to endomorphisms on this underlying set. We formalize both multirole logic (MRL) and linear multirole logic (LMRL) as natural generalizations of classical logic (CL) and classical linear logic (CLL), respectively, and also present a filter-based interpretation for intuitionism in multirole logic. Among various meta-properties established for MRL and LMRL, we obtain one named multiparty cut-elimination stating that every cut involving one or more sequents (as a generalization of a (binary) cut involving exactly two sequents) can be eliminated, thus extending the celebrated result of cut-elimination by Gentzen.

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1 Introduction

While the first and foremost inspiration for multirole logic came to us during a study on multiparty session types in distributed programming (Xi et al., 2016; Xi & Wu, 2016), it seems natural in retrospective to introduce multirole logic by exploring (in terms of a notion referred to as role-based interpretation) the well-known duality between conjunction and disjunction in classical logic. For instance, in a two-sided presentation of the classical
sequent calculus (LK), we have the following rules for conjunction and disjunction:

\[
\begin{align*}
\frac{A \vdash B, A \land B}{A \vdash B} & \quad \text{(conj-r)} \\
\frac{A, A \land B \vdash B}{A, A \vdash B} & \quad \text{(conj-l-1)} \\
\frac{A, B \vdash B}{A, A \land B \vdash B} & \quad \text{(conj-l-2)} \\
\frac{A \vdash B, A \lor B}{A, A \vdash B} & \quad \text{(disj-l)} \\
\frac{A \lor B \vdash B}{A, A \vdash B} & \quad \text{(disj-l-1)} \\
\frac{A \vdash B, A \lor B}{A, B \vdash B} & \quad \text{(disj-l-2)}
\end{align*}
\]

where \(A\) and \(B\) range over sequents (that are essentially sequences of formulas). One possibility to explain this duality is to think of the availability of two roles 0 and 1 such that the left side of a sequent judgment (of the form \(A \vdash B\)) plays role 1 while the right side does role 0. In addition, there are two logical connectives \(\land_0\) and \(\land_1\); \(\land_r\) is given a conjunction-like interpretation by the side playing role \(r\) and disjunction-like interpretation by the other side playing role \(1 - r\), where \(r\) ranges over 0 and 1. With this explanation, it seems entirely natural for us to introduce more roles into classical logic.

Multirole logic is parameterized over a chosen underlying set of roles, which may be infinite, and we use \(\emptyset\) to refer to this set. Given a subset \(R\) of \(\emptyset\), we use \(\overline{R}\) for the complement of \(R\) in \(\emptyset\). Also, we use \(R_1 \uplus R_2\) for the disjoint union of \(R_1\) and \(R_2\) (where \(R_1\) and \(R_2\) are assumed to be disjoint).

For the moment, let us assume that \(\overline{\emptyset}\) consists all of the natural numbers less than \(N\) for some given \(N \geq 2\). Intuitively, a conjunctive multirole logic is one in which there is a logical connective \(\land_r\) for each \(r \in \overline{\emptyset}\) such that \(\land_r\) is given a conjunction-like interpretation by a side playing role \(r\) and a disjunction-like interpretation otherwise. If we think of the universal quantifier \(\forall\) as an infinite form of conjunction, then what is said about \(\land\) can be readily applied to \(\forall\) as well. In fact, additive, multiplicative, and exponential connectives in linear logic (Girard, 1987) can all be treated in a similar manner. Dually, a disjunctive multirole logic can be formulated (by giving \(\land_r\) a disjunction-like interpretation if the side plays the role \(r\) and a conjunction-like interpretation otherwise). For brevity, we primarily focus on conjunctive multirole logic in this paper.

Given a formula \(A\) and a set \(R\) of roles, we write \([A]_R\) for an i-formula, which is some sort of interpretation of \(A\) based on \(R\). For instance, the interpretation of \(\land_r\) based on \(R\) is conjunction-like if \(r \in R\) holds, and it is disjunction-like otherwise. A crucial point, which we learned when studying multiparty session types (Xi & Wu, 2016), is that interpretations should be based on sets of roles rather than just individual roles. In other words, one side is allowed to play multiple roles simultaneously. A sequent \(\Gamma\) in multirole logic is a multiset of i-formulas, and such a sequent is inherently many-sided as each \(R\) appearing in \(\Gamma\) represents precisely one side. As can be readily expected, the cut-rule in (either conjunctive or disjunctive) multirole logic is of the following form:

\[
\frac{\Gamma, [A]_R^\Gamma}{\Gamma, [A]_R^\Gamma}
\]
The cut-rule can be interpreted as some sort of communication between two parties in distributed programming (Abramsky, 1994; Bellin & Scott, 1994; Caires & Pfenning, 2010; Wadler, 2012). For communication between multiple parties, it is natural to seek a generalization of the cut-rule that involves more than two sequents. In conjunctive multirole logic, the admissibility of the following rule of the name \( n \)-cut-conj can be established:

\[
\begin{align*}
R_1 \cup \cdots \cup R_n = \emptyset & \vdash \Gamma, [A]_{R_1} \quad \cdots \quad \vdash \Gamma, [A]_{R_n} \\
& \vdash \Gamma 
\end{align*}
\]

In disjunctive multirole logic, the admissibility of the following rule of the name \( n \)-cut-disj can be established:

\[
\begin{align*}
R_1 \cup \cdots \cup R_n = \emptyset & \vdash \Gamma, [A]_{R_1} \quad \cdots \quad \vdash \Gamma, [A]_{R_n} \\
& \vdash \Gamma 
\end{align*}
\]

We may use the name \( n \)-cut to refer to either \( n \)-cut-conj or \( n \)-cut-disj.

In classical logic, the negation operator is clearly one of a kind. With respect to negation, the conjunction and disjunction operators behave dually, and the universal and existential quantifiers behave dually as well. For the moment, let us write \( \neg A \) for the negation of \( A \). It seems rather natural to interpret \( \neg [A]_R \) as \( [A]_{\neg R} \). Unfortunately, such an interpretation of negation immediately breaks \( n \)-cut for any \( n \geq 3 \). What we discover regarding negation is that the notion of negation can be generalized to endmorphisms on the underlying set \( \emptyset \) of roles.

## 2 Multirole Logic

Let \( \emptyset \) be the underlying set of roles for the multirole logic MRL presented in this section. Strictly speaking, this MRL should be referred to as first-order predicate multirole logic.

We use \( t \) for first-order terms, which are standard (and thus not formulated explicitly for brevity).

### Definition 2.1

A filter \( \mathcal{F} \) on \( \emptyset \) is a subset of the power set of \( \emptyset \) such that

- \( \emptyset \in \mathcal{F} \)
- \( R_1 \in \mathcal{F} \) and \( R_1 \subseteq R_2 \) implies \( R_2 \in \mathcal{F} \)
- \( R_1 \in \mathcal{F} \) and \( R_2 \in \mathcal{F} \) implies \( R_1 \cap R_2 \in \mathcal{F} \)

A filter on \( \emptyset \) is an ultrafilter if either \( R \in \mathcal{F} \) or \( \neg R \in \mathcal{F} \) holds for every subset \( R \) of \( \emptyset \). We use \( \mathcal{U} \) to range over ultrafilters on \( \emptyset \). Note that each \( \mathcal{U} \) on \( \emptyset \) is of the form \( \{ R \subseteq \emptyset \mid r \in R \} \) for some \( r \in \emptyset \) if \( \emptyset \) is finite.

Given an endmorphism \( f \) on \( \emptyset \), we use \( \neg f \) for a unary negative connective. Given an ultrafilter \( \mathcal{U} \) on \( \emptyset \), we \( \land \mathcal{U} \) for a binary conjunctive connective and \( \forall \mathcal{U} \) for a universal quantifier. Given an endomorphism \( f \) and an ultrafilter on \( \emptyset \), we use \( \supseteq \mathcal{U} \) for a binary implicative connective. The formulas in MRL are defined as follows:

\[
\text{formulas } A ::= a \mid \neg f(A) \mid A_1 \land \mathcal{U} A_2 \mid A \supseteq \mathcal{U} B \mid \forall \mathcal{U}(\lambda x.A)
\]

where \( p \) ranges over pre-defined primitive formulas. Instead of writing something like \( \forall \mathcal{U}(\lambda x.A) \), we write \( \forall \mathcal{U}(\lambda x.A) \), where \( x \) is a bound variable. Given a formula \( A \), a term \( t \) and
Let us use $\Delta$ for derivations of sequents, which are just trees containing nodes that are applications of inference rules. Given a derivation $\Delta$, $ht(\Delta)$ stands for the tree height of $\Delta$. When writing $\Delta :: \Gamma$, we mean that $\Delta$ is a derivation of $\Gamma$, that is, $\Gamma$ is the conclusion of $\Delta$. We may also use the following format to present an inference rule:

\[(\Gamma_1; \ldots; \Gamma_n) \Rightarrow \Gamma_0\]

where $\Gamma_i$ for $1 \leq i \leq n$ are the premisses of the rule and $\Gamma_0$ the conclusion.

**Lemma 2.1**
(Weakening) The following rule is admissible:

\[(\Gamma) \Rightarrow \Gamma, [A]_R\]

**Proof**
By structural induction on the derivation of $\Gamma$.

**Lemma 2.2**
The following rule is admissible:

$() \Rightarrow \Gamma, [A]_{\emptyset}$

**Proof**
By structural induction on $A$.

**Lemma 2.3**
(1-cut) The following rule is admissible:

$\left(\Gamma, [A]_{\emptyset}\right) \Rightarrow \Gamma$

**Proof**
Assume $D :: (\Gamma, [A]_{\emptyset})$. The proof proceeds by structural induction on $D$.

Note that Lemma 2.3 can be seen as a special form of cut-elimination where only one sequent is involved.

**Lemma 2.4**
The following rule is admissible:

$() \Rightarrow \Gamma, [A]_{R_1} [A]_{R_2}$

**Proof**
A proof for the lemma can be given based on structural induction on $A$ directly. Also, the lemma immediately follows from Lemma 2.2 and Lemma 2.6.

**Lemma 2.5**
(2-cut with spill) Assume that $R_1$ and $R_2$ are disjoint. Then the following rule is admissible in MRL:

$(\Gamma_1, [A]_{R_1}; \Gamma_2, [A]_{R_2}) \Rightarrow \Gamma_1, \Gamma_2, [A]_{R_1 \cap R_2}$

**Proof**
Assume that we have $D_1 :: (\Gamma, [A]_{R_1})$ and $D_2 :: (\Gamma, [A]_{R_2})$. The proof proceeds by induction on the structure of $A$ and $ht(D_1) + ht(D_2)$, lexicographically ordered.

**Lemma 2.6**
(Splitting of Roles) The following rule is admissible in MRL:

$(\Gamma, [A]_{R_1 \cup R_2}) \Rightarrow \Gamma, [A]_{R_1}, [A]_{R_2}$

**Proof**
Assume that $D$ is a derivation of $(\Gamma, [A]_{R_1 \cup R_2})$. The proof proceeds by induction on the structure of $A$ and $ht(D)$, lexicographically ordered.

**Lemma 2.7**
(mp-cut) Assume that $R_1, \ldots, R_n$ are subsets of $\mathbb{R}$ for some $n \geq 1$. If $\overline{R_1 \cup \cdots \cup R_n} = \emptyset$ holds, then the following rule is admissible:

$(\Gamma_1, [A]_{R_1}; \cdots; \Gamma_n, [A]_{R_n}) \Rightarrow (\Gamma_1, \ldots, \Gamma_n)$
\[\emptyset = R_1 \lor \ldots \lor R_n \quad (\text{Id})\]
\[\vdash [a]_{R_1}, \ldots, [a]_{R_n} \quad (\text{Id})\]
\[\vdash \Gamma, [A]_{R_1}^{-1}(R) \quad (\sim)\]
\[\vdash \Gamma, [[\neg f(A)]]_{R} \quad (-)\]
\[R \in \mathcal{U} \quad \vdash \Gamma, [A]_{R_1}^{-1}(R), [B]_{R} \quad (\supset -\text{neg})\]
\[\vdash \Gamma, [A \lor R_{\mathcal{U}} B]_{R} \quad (\supset -\text{pos})\]
\[\vdash \Gamma, [A]_{R_1}, \ldots, [A]_{R_n} \quad (\land -\text{neg-l})\]
\[\vdash \Gamma, [A \land R_{\mathcal{U}} B]_{R} \quad (\land -\text{neg-r})\]
\[\vdash \Gamma, [A \lor R_{\mathcal{U}} B]_{R} \quad (\land -\text{neg})\]
\[\vdash \Gamma, [A \land R_{\mathcal{U}} B]_{R} \quad (\land -\text{pos})\]
\[\vdash \Gamma, [A \land R_{\mathcal{U}} B]_{R} \quad (\land -\text{neg-weaken})\]
\[\vdash \Gamma, [A \land R_{\mathcal{U}} B]_{R} \quad (\land -\text{neg-derelict})\]
\[R \in \mathcal{U} \quad \vdash \Gamma, [A]_{R}, [B]_{R} \quad (\land -\text{neg-contract})\]
\[\vdash \Gamma, [\forall U (\lambda x. A)]_{R} \quad (\forall -\text{neg})\]
\[\vdash \Gamma, [\forall U (\lambda x. A)]_{R} \quad (\forall -\text{pos})\]
\[\vdash \Gamma, [\forall U (\lambda x. A)]_{R} \quad (\forall -\text{neg})\]
\[\vdash \Gamma, [\forall U (\lambda x. A)]_{R} \quad (\forall -\text{pos})\]

Fig. 2. The inference rules for LMRL.

Proof

The proof proceeds by induction on \(n\). If \(n = 1\), then this lemma is just Lemma 2.3. Assume that \(n \geq 2\) holds. Then we have \(D_i = (\Gamma_i, [A]_{R_i})\) for \(1 \leq i \leq n\). Clearly, \(\mathcal{R}_1\) and \(\mathcal{R}_2\) are disjoint. By Lemma 2.5, we have \(D_{12} = (\Gamma_1, \Gamma_2, [A]_{R_1 \lor \mathcal{U} R_2})\). By induction hypothesis, we can derive the sequent \(\Gamma_1, \Gamma_2, \ldots, \Gamma_n\) based on \(D_{12}, \ldots, D_n\).

This given proof of Lemma 2.7 clearly indicates that multiparty cut-elimination builds on top of Lemma 2.3 and Lemma 2.5. In particular, one may see Lemma 2.3 and Lemma 2.5 as two fundamental meta-properties of a logic.
3 Linear Multirole Logic

In this section, we generalize classical linear logic (CLL) to linear multirole logic (LMRL). The formulas in LMRL is defined as follows:

\[
\text{formulas} \quad A ::= a | \neg f | A_1 \land \varphi | A_2 \triangleright \varphi B | (\forall x)A | \forall \varphi (\lambda x.A)
\]

Let us write \( A \otimes \varphi B \) as a shorthand for \( A \triangleright \text{id}, \varphi B \), where \( \text{id} \) stands for the identity function on \( \emptyset \). If one likes, one may also prefer to write \( A & \varphi B \) for \( A \land \varphi B \). The inference rules for LMRL are listed in Figure 2.

Lemma 3.1
The following rule is admissible:
\[
(\) \Rightarrow [A]_{\emptyset}
\]

Proof
By structural induction on \( A \). Note that we only need positive rules to construct a proof of \([A]_{\emptyset}\). \( \square \)

Lemma 3.2
The following rule is admissible:
\[
(\Gamma, [A]_{\emptyset}) \Rightarrow \Gamma
\]

Proof
Assume \( D :: (\Gamma, [A]_{\emptyset}) \). We prove by induction on the height of \( D \) the existence of \( D' :: (\Gamma, [A]_{\emptyset}) \) such that \( \text{ht}(D') \leq \text{ht}(D) \) holds.

If \( D \) consists of an application of the axiom, then the case is trivial. If \([A]_{\emptyset}\) is introduced by the last applied rule in \( D \), then the rule must be negative and the case follows from the induction hypothesis on the immediate subderivation of \( D \). If \([A]_{\emptyset}\) is not introduced by the last applied rule in \( D \), then the case is straightforward. \( \square \)

Lemma 3.3
(2-cut with spill) Assume that \( \overline{R}_1 \) and \( \overline{R}_2 \) are disjoint. Then the following rule is admissible in LMRL:
\[
(\Gamma_1, [A]_{\overline{R}_1} ; \cdots ; \Gamma_n, [A]_{\overline{R}_2}) \Rightarrow \Gamma_1, \cdots, \Gamma_n, [A]_{\overline{R}_1 \cap \overline{R}_2}
\]

Lemma 3.4
(Splitting of Roles) The following rule is admissible in LMRL:
\[
(\Gamma, [A]_{\overline{R}_1 \cup \overline{R}_2}) \Rightarrow \Gamma, [A]_{\overline{R}_1}, [A]_{\overline{R}_2}
\]

Lemma 3.5
(mp-cut) Assume that \( R_1, \ldots, R_n \) are subsets of \( R \) for some \( n \geq 1 \). If \( \overline{R}_1 \uplus \cdots \uplus \overline{R}_n = \emptyset \) holds, then the following rule is admissible:
\[
(\Gamma_1, [A]_{\overline{R}_1} ; \cdots ; \Gamma_n, [A]_{\overline{R}_n}) \Rightarrow \Gamma_1, \ldots, \Gamma_n
\]

Proof
The proof follows induction on \( n \). It is essentially parallel to the proof of Lemma 2.7. \( \square \)
4 Filter-Based Interpretation for Intuitionism

We can introduce another parameter in MRL to account for intuitionism, supporting a genuine unification of classical logic and intuitionistic logic.

Definition 4.1
Given a filter $F$ on $\emptyset$, a sequent $\Gamma$ is $F$-intuitionistic if there exists at most one $i$-formula $[A]_R$ in $\Gamma$ such that $R \in F$ holds. An inference rule is $F$-intuitionistic if its conclusion is a $F$-intuitionistic sequent.

Definition 4.2
(Intuitionistic MRL) Given an ideal $F$ on $\emptyset$, the inference rules in $\text{MRL}_F$ are those in MRL that are $F$-intuitionistic. We may refer to $\text{MRL}_F$ as the $F$-intuitionistic multirole logic.

It can be readily noted that MRL is essentially equivalent to the $\text{MRL}_F$ for $F = \{\emptyset\}$.

Lemma 4.1
Let $F$ be a filter $F$ on $\emptyset$.
- If $\vdash [A_1 \land_U A_2]_R$ is derivable in $\text{MRL}_F$ for some $R \notin U$, then $\vdash [A_i]_R$ is derivable in $\text{MRL}_F$ for either $i = 0$ or $i = 1$.
- If $\vdash [\exists_U (\lambda x. A)]_R$ is derivable in $\text{MRL}_F$ for some $R \notin U$, then there exists a term $t$ such that $\vdash [A[t/x]]_R$ is derivable in $\text{MRL}_F$.

Lemma 4.2
Given any filter $F$ on $\emptyset$, $\text{MRL}_F$ enjoys multiparty cut-elimination.

Similarly, $F$-intuitionistic LMRL ($\text{LMRL}_F$) can be defined, and both Lemma 4.1 and Lemma 4.2 have obvious corresponding versions that hold for $\text{LMRL}_F$.

5 Related Work and Conclusion

The first and foremost inspiration for multirole logic came from a study on multiparty session types in distributed programming (Xi et al., 2016; Xi & Wu, 2016), which was in turn closely related to series of earlier work (Abramsky, 1994; Bellin & Scott, 1994; Caires & Pfenning, 2010; Wadler, 2012). Also, MCP, a variant of CLL that admits a generalized cut-rule for composing multiple proofs, is first introduced in a paper by Carbone et al. (Carbone et al., 2015). In the following work (Carbone et al., 2016), a variant of MCP is introduced, and a translation from MCP to CP (Wadler, 2012) via GCP (an intermediate calculus) is given that interprets a coherence proof in MCP as an arbiter process for mediating communications in a multiparty session.

For long, studies on logics have been greatly influencing research on programming languages. In the case of multirole logic, we see a genuine example that demonstrates the influence of the latter on the former. What an influence it is! If just for one thing only, we should immediately revisit some classical results in logic and recast/reinterpret them in the framework of multirole logic.
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