Scalar Correlator at $O(\alpha_s^4)$, Higgs Decay into $b$-quarks and Bounds on the Light Quark Masses

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We compute, for the first time, the absorptive part of the massless correlator of two quark scalar currents in five loops. As physical applications we consider the $O(\alpha_s^4)$ corrections to the decay rate of the Standard Model Higgs boson into quarks, as well as the constraints on the strange quark mass following from QCD sum rules.

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INTRODUCTION

Within the Standard Model (SM) the scalar Higgs boson is responsible for the mechanism of the mass generation. Particularly interesting for the observation of the Higgs boson with an intermediate mass $M_H < 2M_W$ is the dominant decay channel into a pair of bottom quarks, $H \to bb$. The decay of the Higgs boson into a quark–antiquark pair $(ff)$ proceeds through its coupling to the corresponding quark scalar current and reads (for a review see e.g. [1])

$$\Gamma(H \to ff) = \frac{G_F M_H^2}{4\sqrt{2}\pi} m_f^2 \tilde{R}(s = M_H^2),$$  \hspace{1cm} (1)

with $\tilde{R}(s) = \text{Im} \Pi(-s - i\epsilon)/(2\pi s)$ standing for the absorptive part of the scalar two-point correlator:

$$\tilde{\Pi}(Q^2) = (4\pi)^2 i \int dx e^{iqx} \langle 0 | T [ J^S_f(x) J^S_f(0) ] | 0 \rangle.$$  \hspace{1cm} (2)

Here $Q^2 = -q^2$ and $J^S_f = \bar{\Psi}_f \Psi_f$ is the scalar current for quarks with flavour $f$ and mass $m_f$, coupled to the scalar Higgs boson.

The currently known fixed order perturbative predictions for $\tilde{R}$ can be shortly summarized as follows [2][3] (we have put the number of effective flavours $n_f = 5$)

$$\tilde{R} = 1 + 5.66667 a_s + 29.1467 a_s^2 + 41.7576 a_s^3,$$  \hspace{1cm} (3)

with $a_s = \alpha_s(M_H)/\pi$. Note that eq. (3) is given in the high energy limit with all power-suppressed terms of order $m_f^2/M_H^2$ and higher neglected. In fact, the full dependence on the quark mass $m_f$ is known up to and including the $O(\alpha_s^4)$ contribution [4]. We will not discuss the power suppressed terms in the present publication.

For $M_H \geq 70$ GeV already the term of order $m_f^2/M_H^2 \alpha_s^2$ is numerically by an order of magnitude less than the massless $O(\alpha_s^4)$ term displayed in eq. (3). Due to the large mass $M_H$ the couplant $a_s$ is less than 0.04 which results in a good apparent convergence of the perturbation series in eq. (3).

On the other hand, for energy scales, say, of order of a few GeV’s, relevant for QCD sum rules the higher order corrections to the correlator (2) are numerically quite important (see, e.g. [2][3][4][5]). We will discuss this issue in some detail later in the text for an example of bounds for the light quark masses.

The motivation of the present publication is fourfold. First, presenting the first complete $O(\alpha_s^4)$ result for a massless QCD correlator shows that the new theoretical methods used in [2][3][4][5] indeed do deliver genuine QCD results in five-loop approximation. Second, the results are important for the QCD sum rules based on a (pseudo)-scalar correlator [2]; they also provide an accurate prediction of the Higgs decay rate into hadrons.

Third, the case of the scalar correlator should be considered a necessary preparation step before computing the $O(\alpha_s^4)$ contribution to the vector correlator. The importance of the latter calculation for the precise determination of the value of $\alpha_s$ from the $\tau$-lepton and $Z$-boson decay rates is well-known. Last, by comparing the exact $\alpha_s^4$ result with the estimates based on various optimization procedures one obtains important insights into the quality of different approaches, confirming some and refuting others.

CALCULATION AND RESULTS

To compute the absorptive part of $\tilde{\Pi}$ we proceed in full analogy with our previous calculations described in [12]. First, using the criterion of irreducibility of Feynman integrals [13][14], the set of irreducible integrals involved
diagrams were generated with the help of QGRAF \[19\]. Third, the exact answer, i.e.
a rational function of \( D \), was reconstructed from this expansion.

The major part of the calculations was performed on the Silicon
Graphics Altix 3700 computer (32 Itanium-2 1.3 GHz processors) using the parallel
version of FORM \[16, 17, 18\] and took about 18 months in total. The
diagrams were generated with the help of QGRAF \[19\].

It is convenient to introduce the Adler function as
\[
\tilde{D}(Q^2) = \frac{Q^2}{6} \frac{d}{dQ^2} \tilde{\Pi}(Q^2) = \int_0^\infty \frac{Q^2 \tilde{R}(s) ds}{(s + Q^2)^2}, \tag{4}
\]

The \( n_f^3 \) and \( n_f^2 \) terms are in agreement with \[20\] and \[9\] respectively. Once the constants \( \tilde{d}_1 \) to \( \tilde{d}_4 \) are known it is
a matter of straightforward analytic continuation to find \( \tilde{R} \), given below for brevity in the numerical form only:
\[
\tilde{R} = 1 + 5.6667 a_s + [35.94 - 1.359 n_f] a_s^2 + a_s^3 [164.14 - 25.77 n_f + 0.259 n_f^2] + a_s^4 [39.34 - 220.9 n_f + 9.685 n_f^2 - 0.0205 n_f^3]. \tag{6}
\]

In order to better understand the structure of the \( \alpha_s^4 \)
term in \( \tilde{D} \) it is instructive to separate the genuine five-loop contributions from the Adler function \( \tilde{D} \) from essentially “kinematical”, so-called \( \pi^2 \)-terms originating from the analytic continuation. We have in mind that for a
given order in \( \alpha_s \) these extra contributions are completely predictable from the standard evolution equations applied to the “more leading” terms in \( D \) proportional to some smaller powers of \( \alpha_s \). The corresponding expression for \( \tilde{R} \) reads

\[
\tilde{R} = 1 + 5.6667 a_s + a_s^2 [51.57 - 15.63 - n_f(1.907 - 0.548)] + a_s^3 [648.7 - 484.6 - n_f(63.74 - 37.97) + n_f^2(0.929 - 0.67)] + a_s^4 [9470.8 - 9431.4 - n_f(1454.3 - 1233.4) + n_f^2(54.78 - 45.10) - n_f^3(0.454 - 0.433)], \tag{7}
\]

where we have underlined the contributions coming from analytic continuation.

The inclusion of the \( \pi^2 \)-terms from higher orders thus
does not necessarily improve the quality of the perturba-
tive prediction for the scalar correlator. It remains to be
seen whether a similar pattern arises in the case of the
contour improved perturbation theory \[21\] \[22\] applied to the \( \tau \)-lepton decay rate.

At last, specifying \( n_f = 5 \) we get the corresponding
generalization of eq. (3)

\[ \tilde{R} = 1 + 5.6668 a_s + 29.147 a_s^2 + 41.758 a_s^3 - 825.7 a_s^4 \]

\[ = 1 + 0.2075 + 0.0391 + 0.0020 - 0.00148. \] (8)

In the last equation we have substituted \( a_s = 0.0366 \) which corresponds the Higgs mass value \( M_H = 120 \) GeV. The comparable sizes of the \( O(a_s^3) \) and the \( O(a_s^4) \) terms (the fourth and the fifth terms in eq. (8)) may be interpreted as a consequence of the accidentally small coefficient for the \( a_s^4 \) term.

**COMPARISONS TO PREVIOUS ESTIMATIONS**

Following [23], predictions for higher order coefficients of QCD correlators are usually made first for an Euclidian quantity and the result for the corresponding Minkowskian one is then derived as a direct consequence. On the other hand, a quite remarkable feature of our result for \( \tilde{R} \) are almost complete cancellations which take place for every power of \( n_f \) between the genuine five-loop contributions and the “trivial” ones from analytic continuation as illustrated by the corresponding decomposition displayed in eq. (7). This fact alone might indicate problems for the traditional way of obtaining predictions for \( \tilde{R} \). With the exact result in hand one can easily check this suspicion.

Indeed, in Table I our results are compared with predictions obtained in works [24, 25, 26]. Note that the Principle of Minimal Sensitivity (PMS) [27] and that of Fastest Apparent Convergence (FAC) [28] used in [24] produce identical result at order \( \alpha_s^4 \). Also note that the Asymptotic Padé-Approximant Method (APAM) was utilized in [26] to produce the prediction directly (and only) for the absorptive part \( \tilde{R} \). In contrast, the method of Naive NonAbelianization (NNA) [29] has been applied in [24] to the Adler function only. The two predictions of FAC/PMS for \( \tilde{r}_4 \) correspond to either of the consequence of the prediction for \( \tilde{d}_4 \) (the fifth line) or to the direct application of FAC/PMS to estimate \( \tilde{r}_4 \) (the sixth line). As a consequence of the large cancellations in \( \tilde{r}_4 \) the second prediction looks much better than the first, despite the fact that the estimation of the Euclidian coefficient \( \tilde{d}_4 \) is quite close (within 10%) to the exact result. In fact, at order \( \alpha_s^3 \) the cancellations in questions are much less pronounced with the result that the corresponding prediction for \( \tilde{r}_3 \), obtained from \( \tilde{d}_3 \), is significantly more accurate than the direct estimation of \( \tilde{r}_3 \) [24]. NNA predictions have correct signs and sensible magnitudes as observed earlier [24]. Finally, the APAM estimation of \( \tilde{r}_4 \) fails to reproduce even the sign of the exact result.

Predictions of the prescription proposed by Brodsky, Mackenzie and Lepage (BLM) [30] for the \( n_f \) dependent terms of order \( \alpha_s^4 \) have been communicated to the authors [30]: \( a_s^4(-260n_f + 13n_f^2 - 0.046n_f^3) \) and are also in reasonable agreement with the exact result of eq. (9).

| \( n_f \) | 3 | 4 | 5 |
|---|---|---|---|
| \( \tilde{d}_4 \) (exact) | 5588.7 | 4501.1 | 3512.2 |
| \( \tilde{d}_4 \) ([24], PMS, FAC) | 5180.3 | 4093.0 | 3100.5 |
| \( \tilde{d}_4 \) ([26], NNA) | 1592.8 | 1521.4 | 1484.1 |
| \( \tilde{r}_4 \) (exact) | -536.8 | -690.7 | -825.7 |
| \( \tilde{r}_4 \) ([24], PMS, FAC) | -945.2 | -1098.8 | -1237.4 |
| \( \tilde{r}_4 \) ([24], PMS, FAC) | -527.8 | -748.6 | -949.4 |
| \( \tilde{r}_4 \) ([25], APAM) | 252 | 147 | 64.2 |

**QUARK MASS BOUNDS**

As an application of our result for the scalar correlator we consider the well-known bounds for the light quark masses [31, 32]. The constraints follow from the known values of the \( \pi \) or \( K \) pole contributions and the positivity of the spectral function \( \tilde{R} \) and depend on the scale \( Q \) used in evaluation of the polarization operator. We are going to discuss two types of bounds suggested in [32], the “quadratic” bound

\[
[m_s(Q) + m_u(Q)]^2 \geq \frac{16\pi^2}{\kappa_2} \frac{2f_K M_K}{Q^4} \left( 1 + \frac{M_K^2}{Q^2} \right)^5 \frac{2F_0(Q^2) - \frac{4}{3} \left( 1 + \frac{M_K^2}{Q^2} \right) F_1(Q^2) + \frac{1}{3} \left( 1 + \frac{M_K^2}{Q^2} \right)^2 F_2(Q^2)}{3F_0(Q^2)F_2(Q^2) - 2 \left( F_1(Q^2) \right)^2}.
\] (9)

and the “linear” bound

\[
[m_s(Q) + m_u(Q)]^2 \geq \frac{16\pi^2}{\kappa_2} \frac{2f_K M_K}{Q^4} \left( 1 + \frac{M_K^2}{Q^2} \right)^{-3}. \] (10)

Here the functions \( F_0, F_1 \) and \( F_2 \) are defined as (normalized to one at the leading order, that is \( c_0 = 1/6, c_1 = \))
The results of both eqs. 10 for \(m_s + m_u\) depend on the choice of \(Q\) and can be transformed to bounds for \((m_s + m_u)(\mu)\) for any (not too small) \(\mu\) using the standard mass evolution equation. Following [32] we will use \(\mu = 2\) GeV as the reference point.

Let us start from (9). Using as reference value \(\alpha_s(M_T) = 0.334\) and the standard evolution equations for the mass and coupling constant we get (the indices 4 and 3 stand for the order in \(\alpha_s\))

\[
[(m_s + m_u)^2]_4 \left( \frac{\mu = 2}{Q = 2\text{ GeV}} \right) > (103 \text{ MeV})^2, \tag{12}
\]

which should be compared to the three-loop bound [32]

\[
[(m_s + m_u)^2]_3 \left( \frac{\mu = 2}{Q = 2\text{ GeV}} \right) > (111 \text{ MeV})^2. \tag{13}
\]

The bound (13), if valid, is already in conflict with significantly lower value 75(8) MeV derived by one of lattice collaborations [33]. However, for a significantly larger lattice result around a hundred of MeV, see [34, 35].

The problem was “solved” in [14] by observing that the quadratic combination of the \(F\) functions appearing in the denominator of (9) displays a poor pattern of convergence:

\[
3F_0F_2 - 2(F_1)^2 = 1 + 0.83 + 0.61 + 0.51 + \ldots \tag{14}
\]

and then suggesting to increase \(Q\) in (13) till, say, 2.5 GeV, which lowers the rhs of (13) to 86 MeV and the rhs of (12) to 81 MeV.

From our point of view it is more relevant to consider the convergence pattern of the whole ratio appearing in the rhs of (12). Indeed, after expanding (9) in \(\alpha_s\) and disregarding all terms in \(\alpha_s\) higher than \(\alpha_{s4}\) we get

\[
[(m_s + m_u)^2]_4 \left( \frac{\mu = 2}{Q = 2\text{ GeV}} \right) > (179 \text{ MeV})^2 \times \\
\left\{ 1 - 6.44a_s - 12.83a_s^2 + (482.95 - 525.2)a_s^3 \\
+ (-2561.8 + 6948.34 - 4439.9)a_s^4 \right\}. \tag{15}
\]

In order to demonstrate the importance of higher order corrections we have underlined in (15) all terms originating from the contributions of order \(a_{s4}\) (eventually multiplied by a subleading \(O(a_s)\) term) to the functions \(F_0, F_1\) and \(F_2\) and, correspondingly, boxed the one originating exclusively from \(O(a_{s4})\) terms in the same functions. After substituting the value for \(\alpha_s(\text{2 GeV}) = 0.312\) into eq. (15) we arrive at

\[
[(m_s + m_u)]_4(\mu = 2\text{ GeV}) > 77 \text{ MeV} \tag{16}
\]

to be compared to

\[
[(m_s + m_u)]_3(\mu = 2\text{ GeV}) > 79 \text{ MeV}. \tag{17}
\]

If, on the other hand, we would try to use eq. (15) with only the boxed term nullified we would get an astonishingly different value

\[
[(m_s + m_u)]_3(\mu = 2\text{ GeV}) > 141 \text{ MeV}. \tag{18}
\]

In similar way one can examine the second bound (10). For the lowest choice \(Q = 1.4\) GeV used in (10) we immediately get

\[
[(m_s + m_u)](\mu = 2\text{ GeV}) > 76 \text{ MeV and 78 MeV},
\]

with the numbers corresponding to the use of five- and four-loop expressions for the function \(F_0\) correspondingly.

On the other hand, the expanded version of eq. (10) reads:

\[
[(m_s + m_u)]_4(\mu = 2\text{ GeV}) > 72 \text{ MeV}, \tag{20}
\]

\[
[(m_s + m_u)]_3(\mu = 2\text{ GeV}) > 74 \text{ MeV}. \tag{21}
\]

Again, if we were using use eq. (19) with only the boxed term nullified we would arrive at significantly different value

\[
[(m_s + m_u)]_4(\mu = 2\text{ GeV}) > 81 \text{ MeV}.
\]

**CONCLUSION**

We have computed the correction of order \(\alpha_{s4}\) to the correlator of quark scalar currents in the massless limit. Our result demonstrates a remarkable interplay between the genuine five-loop terms and the effects due to the analytical continuation. The newly computed contribution stabilizes the (quadratic) quark mass bound of [32] and pushes it significantly down, thus, avoiding any potential conflict with lattice results.

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