Rationally 4-periodic biquotients

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Abstract

An \( n \)-dimensional manifold \( M \) is said to be rationally 4-periodic if there is an element \( e \in H^4(M; \mathbb{Q}) \) with the property that cupping with \( e \cdot \cup e : H^*(M; \mathbb{Q}) \to H^{*+4}(M; \mathbb{Q}) \) is injective for \( 0 < * \leq \dim M - 4 \) and surjective when \( 0 \leq * < \dim M - 4 \). We classify all compact simply connected biquotients which are rationally 4-periodic.

1 Introduction

A biquotient is any manifold which is diffeomorphic to the quotient of a Riemannian homogeneous space by a free isometric action. Biquotients are important in the study of positively curved manifolds. In fact, with the exception of the positively curved manifold found independently by Dearricott [6] and by Grove, Verdiani, and Ziller [19], all known examples of manifolds admitting positive curvature are diffeomorphic to biquotients [4, 1, 35, 11, 12, 3, 19, 6]. Further, all known examples of manifolds admitting quasi or almost positive curvature are diffeomorphic to biquotients [9, 10, 13, 18, 25, 24, 26, 30, 33, 37, 38].

Recall that an \( n \)-dimensional manifold \( M \) is said to be \( k \)-periodic with coefficient ring \( R \) if there is an element \( e \in H^k(M; R) \) with the property that cupping with \( e \cdot \cup e : H^*(M) \to H^{*+k}(M) \) is injective when \( 0 < * \leq n - k \) and surjective when \( 0 \leq * < n - k \). Due to Wilking’s connectedness lemma [39], periodicity appears when \( M \) admits a metric of positive sectional curvature with large symmetry group. For example, Wilking [39] has shown that if \( M^n \) with \( n \geq 6000 \) is positively curved and admits an effective isometric \( T^d \) action for \( d \geq \frac{1}{6} n + 1 \), then \( M \) is 4-periodic with respect to any field coefficients. Further, if the period \( k \) is small compared to the dimension of the manifold, Kennard [22] shows that \( k \)-periodic integral cohomology implies 4-periodic rational cohomology. As an application, in [23], Kennard shows that a simply connected closed manifold with positive sectional curvature and large symmetry rank has 4-periodic rational cohomology in small degrees.

If a manifold \( M^n \) has a \( k \)-periodic rational cohomology ring with \( k = 1 \) or 2, or if \( k = 4 \) and the second Betti number vanishes,
then it is easy to see $H^*(M; \mathbb{Q})$ is isomorphic to either $H^*(S^n; \mathbb{Q})$, $H^*(\mathbb{C}P^n/2; \mathbb{Q})$, or $H^*(\mathbb{H}P^n/4; \mathbb{Q})$. In particular, the cohomology ring of such examples are generated by a single element. Simply connected biquotients with singly generated rational cohomology rings have been classified by Kapovitch and Ziller [21]. In particular, every simply connected biquotient with singly generated cohomology is 4-periodic, with the exception of $\mathbb{O}P^2$. Thus, for biquotients, 4-periodicity may be viewed as a generalization of having singly generated cohomology.

We note that if $M$ has dimension at most 4, then it is vacuously 4-periodic. Similarly if $M$ is 5 dimensional and simply connected, then it is 4-periodic. Compact simply connected biquotients of dimension at most 5 were classified in [33] [29] [7], and are given, up to diffeomorphism, by $S^2$, $S^3$, $S^4$, $\mathbb{C}P^2$, $\mathbb{C}P^2 \# \mathbb{C}P^2$, $\mathbb{C}P^2 \# - \mathbb{C}P^2$, $S^2 \times S^2$, $S^5$, $SU(3)/SO(3)$, $S^2 \times S^3$, and $S^3 \times S^2$, the unique nontrivial linear $S^3$ bundle over $S^2$. Thus, we will assume the dimension of our biquotients to be at least 6.

Our first theorem applies to a larger class of spaces, rationally elliptic spaces. Recall that a simply connected CW complex $X$ is called rationally elliptic if the rational vector spaces $H^*(X; \mathbb{Q})$ and $\bigoplus \pi_*(X) \otimes \mathbb{Q}$ are finite dimensional. Simply connected biquotients are well known to be rationally elliptic. We use the notation $X \simeq \mathbb{Q} Y$ to indicate that $X$ and $Y$ have the same rational homotopy type.

**Theorem 1.1.** Suppose $X$ is an $n$-dimensional simply connected CW-complex with $n \geq 6$ which is rationally elliptic. Further, assume $H^*(X; \mathbb{Q})$ is 4-periodic but not singly generated. Then precisely one of the following occurs.

(a) $X \simeq \mathbb{Q} \mathbb{H}P^m \times S^3$
(b) $X \simeq \mathbb{Q} \mathbb{H}P^m \times S^2$
(c) $X \simeq \mathbb{Q} S^3 \times S^3$.

Under the assumption that $X$ is diffeomorphic to a biquotient and $X \simeq \mathbb{Q} S^3 \times S^3$, the author showed in [8] Corollary 3.5] that $X$ is be diffeomorphic to $S^3 \times S^3$. Using Kapovitch and Ziller’s [21] classification of biquotients with singly generated cohomology, we obtain a similar characterization in case (a), when $X \simeq \mathbb{Q} \mathbb{H}P^m \times S^3$ is a biquotient.

**Theorem 1.2.** Assume $M$ is a simply connected manifold diffeomorphic to a biquotient. If $M \simeq \mathbb{Q} \mathbb{H}P^m \times S^3$, then $M$ is diffeomorphic to exactly one of the following:

1. the total space of one of two (respectively three) $S^3$ bundles over $\mathbb{H}P^m$ for $m \geq 2$ (respectively $m = 1$),
2. if $m = 2$, the total space of one of three $S^3$ bundles over $G_2/SO(4)$, or
(3) if \( m \geq 3 \) is odd, the total space of one of two \( S^3 \) bundles over the biquotient \( \Delta SU(2) \backslash SO(2m + 3)/SO(2m + 1) \).

In particular, in dimension 7 there are 3 examples, in dimension 11 there are 5, in higher dimensions of the form \( 8m + 3 \) there are 2 examples, and in higher dimensions of the form \( 8m - 1 \) there are 4, all distinct up to diffeomorphism. In (3), if we allow \( m = 1 \), then these biquotients are diffeomorphic to examples in (1).

Unfortunately, we can only partially classify biquotients with the rational homotopy type of \( \mathbb{H}P^m \times S^2 \).

**Theorem 1.3.** Assume \( M \) is a simply connected manifold diffeomorphic to a biquotient. If \( M \simeq \mathbb{H}P^m \times S^2 \), then \( M \) is diffeomorphic to the exactly one of the following:

1. The total space of one of the two linear bundles over \( S^2 \) with fiber \( \mathbb{H}P^m \) for \( m \geq 1 \).
2. if \( m = 2 \), \( [G_2/\text{SO}(4)] \times S^2 \)
3. if \( m \geq 3 \) is odd, the total space of some bundle over \( S^2 \) with fiber the biquotient \( \Delta SU(2) \backslash SO(2m + 3)/SO(2m + 1) \).

Further, in case (3), for each \( m \) there are only finitely many diffeomorphism types occurring.

This paper is organized as follows. In Section 2, we show that a biquotient with a rational cohomology ring which is 4-periodic but not singly generated must have the rational homotopy type of \( \mathbb{H}P^m \times S^l \) for \( l \in \{2, 3\} \) or of \( S^3 \times S^3 \), proving Theorem 1.1. In Section 3, we completely analyze the case where \( M \) is rationally \( \mathbb{H}P^m \times S^3 \), proving Theorem 1.2. In Section 4, we prove Theorem 1.3.

We would like to thank Manuel Amann and Lee Kennard for suggesting this problem, as well as for several stimulating discussions.

## 2 Background

In this section, we recall the necessary background regarding rational homotopy theory, biquotients, and the Kapovitch-Ziller [21] classification of biquotients with singly generated rational cohomology rings.

### 2.1 Rational homotopy theory

In this section, we prove Theorem 1.1 after introducing the relevant notions from rational homotopy theory [15, 14]. We use the notation \( \pi_\ast(X)_{\mathbb{Q}} \) as shorthand for the rational homotopy group \( \pi_\ast(X) \otimes \mathbb{Q} \) and, as stated in the introduction, we will also use the notation \( X \simeq_{\mathbb{Q}} Y \) to mean \( X \) and \( Y \) have the same rational homotopy type. That is,
$X \simeq_{\mathbb{Q}} Y$ means there is a zigzag of maps $X \to Z_1 \leftarrow Z_2 \to \ldots \leftarrow Y$
eq \ldots \leftarrow Y$

eq \ldots \leftarrow Y$ each of which induces isomorphisms on all rational homotopy groups.

We note that if $X \simeq_{\mathbb{Q}} Y$, then $H^*(X; \mathbb{Q}) \cong H^*(Y; \mathbb{Q})$.

In general, if $H^*(X; \mathbb{Q}) \cong H^*(Y; \mathbb{Q})$, there is no reason to expect that $X \simeq_{\mathbb{Q}} Y$. However, there is a certain class of spaces, the so-called formal spaces [14, pg. 156], where this implication does hold. For example, spheres and projective spaces are formal and the product of two spaces is formal if one of them has rational homology of finite type. In particular, $S^3 \times S^3$, $\mathbb{H}P^m \times S^2$, and $\mathbb{H}P^m \times S^3$ are formal.

Thus, to establish Theorem 1.1, it is enough to show that if $X$ has 4-periodic rational cohomology ring which is not singly generated, then the cohomology ring is isomorphic to that of $S^3 \times S^3$, $\mathbb{H}P^m \times S^2$ or $\mathbb{H}P^m \times S^3$.

Recall that a simply connected CW complex $X$ is said to be rationally elliptic if both $H^*(X; \mathbb{Q})$ and $\pi_*(X) \mathbb{Q}$ are finite dimensional rational vector spaces. Lie groups and spheres are well known to be rationally elliptic. Further, it is clear that in any fiber bundle $F \to E \to B$, if two of the three spaces are rationally elliptic, then so is the third. In particular, every biquotient is rationally elliptic.

Now, as shown in [14, Proposition 32.10], a rationally elliptic space always has non-negative Euler characteristic and has positive Euler characteristic iff all odd rational Betti numbers vanish.

We are now ready to prove Theorem 1.1 which we recall for convenience.

**Theorem 1.1.** Suppose $X$ is an $n$-dimensional simply connected CW-complex with $n \geq 6$ which is rationally elliptic. Further, assume $H^*(X; \mathbb{Q})$ is 4-periodic but not singly generated. Then precisely one of the following occurs,

(a) $X \simeq_{\mathbb{Q}} \mathbb{H}P^m \times S^3$

(b) $X \simeq_{\mathbb{Q}} \mathbb{H}P^m \times S^2$

(c) $X \simeq_{\mathbb{Q}} S^3 \times S^3$.

**Proof.** As mentioned above, it is sufficient to show $H^*(X; \mathbb{Q})$ is isomorphic to the rational cohomology ring of $S^3 \times S^3$ or $\mathbb{H}P^m \times S^l$ with $l \in \{2, 3\}$.

First, assume $n = 6$ and that the fourth rational Betti number, $b_4$, is equal to 0. Then Poincaré duality implies $b_2 = 0$ as well, so $0 \leq \chi(X) = 2 - b_3$. If $\chi(X) > 0$, then $b_3 = 0$ so $X$ is rationally $S^6$, so we assume $b_3 = 2$. Then Poincaré duality implies $X$ has the rational cohomology ring of $S^3 \times S^3$.

We next assume $b_4 > 0$, so, by periodicity, the $b_4 = 1$. Poincaré duality then implies $b_2 = 1$ and thus, $0 \leq \chi(X) = 4 - b_3$. If $b_3 \leq 3$, $\chi(X) > 0$ so $b_3 = 0$ and $X$ has the rational cohomology groups of...
$S^4 \times S^2$. If the square of a non-zero element of $H^2(X; \mathbb{Q})$ is non-zero, it follows that the rational cohomology ring of $X$ is isomorphic to that of $\mathbb{C}P^3$, so is singly generated. Thus, the square of any element in $H^2(X; \mathbb{Q})$ is 0. Then, using Poincaré duality, it is easy to see the rational cohomology ring is isomorphic to that of $S^4 \times S^2 = \mathbb{H}P^1 \times S^2$.

Thus, we are left with the case $b_3 = 4$. However, we now show this cannot occur for a rationally elliptic manifold. By the rational Hurewicz theorem [27], the map $\pi_3(X)_\mathbb{Q} \to H_3(X; \mathbb{Q})$ is surjective, so we must have $\dim \pi_3(X)_\mathbb{Q} \geq 4$. But, according to [14, pg. 434], for any rationally elliptic space, $\sum_k (2^k + 1) \dim \pi_{2k+1}(X)_\mathbb{Q} \leq 2n - 1$. Because $n = 6$, but $\dim \pi_3(X)_\mathbb{Q} \geq 4$, we have a contradiction. This concludes the case of $n = 6$.

We now assume $n > 6$. Then it is easy to see that if $b_4 = 0$ and $X$ is rationally 4-periodic, then $X$ has the cohomology ring of $S^n$. Hence, we will assume $b_4 > 0$ so, by periodicity, $b_4 = 1$.

If the dimension of $X$ is of the form $4m$ or $4m+1$, then Wilking [39, Proposition 7.13] has proven that 4-periodicity implies the cohomology ring is singly generated, so we may assume the dimension of $X$ is of the form $4m + 2$ or $4m + 3$.

Suppose first that $X$ has dimension $4m + 3$. Then $b_{4m} = 1$ by periodicity, so $b_3 = 1$ by Poincaré duality. Periodicity and Poincaré duality then imply the rational cohomology ring is isomorphic to that of $\mathbb{H}P^m \times S^3$.

Next, if $X$ has dimension $4m + 2 \geq 10$, then Amann and Kennard [2, Lemma 3.1] prove $\chi(X) > 0$ so all odd rational Betti numbers vanish. Using Poincaré duality, it follows that $b_{2k} = 1$ for all $k$. As in the $n = 6$ case above, if a non-zero element of $H^2(X; \mathbb{Q})$ has a non-zero square, then the rational cohomology ring of $X$ is isomorphic to that of $\mathbb{C}P^{2m+1}$, so is singly generated. Thus, every element of $H^2(X; \mathbb{Q})$ squares to 0. Now, it is easy to see the rational cohomology ring is isomorphic to that of $\mathbb{H}P^m \times S^2$.

\[\square\]

**2.2 Biquotients and their classification**

As mentioned in the introduction, a biquotient is any manifold which is diffeomorphic to the quotient of a homogeneous space by a free isometric action. There is an alternative characterization given in terms of Lie groups. Suppose $f = (f_1, f_2) : U \to G \times G$ is a homomorphism. This defines an action of $U$ on $G$ by $u \ast (g) = f_1(u)g f_2(u)^{-1}$. When this action is free, the orbit space, denoted $G//U$ is also called a biquotient. In the special case when $U = U_1 \times U_2$ with each factor
embedded into a single factor of $G \times G$, we write $G\!/\!/U \cong U_1\backslash G/U$.

We note that the diffeomorphism type of $G\!/\!/U$ depends only on the conjugacy class of the image of $U$ in $G \times G$. More specifically, if $C_g : G \times G \to G \times G$ denotes conjugation by $g = (g_1, g_2) \in G \times G$, then the actions defined by $f : U \to G \times G$ and $C_g \circ f$ are equivalent. In fact, the map $\phi : G \to G \times G$ given by $\phi(h) = g_1hg_2^{-1}$ is an equivariant diffeomorphism.

In [34], Totaro provides the framework for the classification of biquotients.

**Theorem 2.1.** [34, Lemma 3.3] Suppose $M \cong G\!/\!/H$ is a simply connected biquotient. Then $M$ can be written as as reduced biquotient $G'\!/\!/H'$, meaning $G'$ is simply connected, $H'$ is connected, and no simple factor of $H'$ acts transitively on any simple factor of $G'$.

By definition, a simple factor of $H$ is the image of any simple factor of the universal cover of $H$ under the canonical projection map. We will always assume our biquotients are reduced. Further, by allowing the $H$ action on $G$ to have a finite ineffective kernel, we may replace $H$ by a finite cover. In particular, we can and will assume $H = H' \times T^k$ where $H'$ is simply connected and semisimple.

To describe Totaro’s main result, first recall that every Lie group is rationally a product of odd spheres, $G \cong_{\mathbb{Q}} S^{2d_1-1} \times \ldots \times S^{2d_k-1}$ where $k$ is the rank of the group. The integers $d_i$ are referred to as degrees of $G$. Table 2.1 lists all the simple Lie groups together with their degrees.

| Group         | Restriction | Degrees          |
|---------------|-------------|------------------|
| $SU(n)$       | $n \geq 2$  | $2, 3, \ldots, n$|
| $SO(2n+1)$    | $n \geq 3$  | $2, 4, \ldots, 2n$|
| $Sp(n)$       | $n \geq 2$  | $2, 4, \ldots, 2n$|
| $SO(2n)$      | $n \geq 4$  | $2, 4, \ldots, 2n - 2, n$|
| $G_2$         |             | $2, 6$           |
| $F_4$         |             | $2, 6, 8, 12$    |
| $E_6$         |             | $2, 5, 6, 8, 9, 12$|
| $E_7$         |             | $2, 6, 8, 10, 12, 14, 18$|
| $E_8$         |             | $2, 8, 12, 14, 18, 20, 24, 30$|

Table 2.1: Degrees of Lie groups

Now, suppose $G_1$ is a simple factor of $G$. We say $G_1$ contributes degree $d$ to $G\!/\!/H$ if in the long exact sequence of rational homotopy
groups associated to the fibration $H \to G \to G\!/H$, the induced map $\pi_{2d-1}(H)_{\mathbb{Q}} \to \pi_{2d-1}(G_1)_{\mathbb{Q}}$ is not surjective.

We briefly note that because we allow the $H$ action on $G$ to have finite ineffective kernel $K$, we really only have a fibration of the form $H/K \to G \to G\!/H$. However, since $H$ is a cover of $H/K$, the rational homotopy groups of $H$ and $H/K$ are canonically isomorphic, so the above definition makes sense with respect to this slight abuse of notation.

We may now state Totaro’s main classification theorem.

**Theorem 2.2.** \cite[Theorem 4.8]{Totaro} Suppose $M = G\!/H$ is a reduced biquotient. Let $G_1$ denote a simple factor of $G$. Then one of the following happens.

1. $G_1$ contributes its highest degree to $M$.

2. $G_1$ contributes its second highest degree to $M$, and there is a simple factors $H_1$ of $H$ for which $H_1$ acts only on one side of $G_1$, with $G_1/H_1$ isomorphic to $SU(n)/Sp(n)$ with $n \geq 2$, $Spin(7)/G_2 = S^7$, $Spin(8)/G_2 = S^7 \times S^7$, or $E_6/F_4$. The second highest degrees are given by $2n - 1, 4, 9$ respectively.

3. $G_1 \cong Spin(2n)$ and there is a simple factor $H_1$ of $H$ with $H_1 = Spin(2n - 1)$, acting only on one side of $G_1$ via the standard inclusion, so $G_1/H_1 = S^{2n-1}$. In this case, $G_1$ contributes degree $n$ to $M$.

4. $G_1 = SU(2n + 1)$ contributes degrees $2, 4, 6, \ldots, 2n$ to $M$ and there is a simple factor $H_1$ of $H$ with $H_1 \cong SU(2n + 1)$ acting on $G_1$ by $h \ast g = hgh^t$.

In particular, each simple factor contributes at least one degree. Thus, we have the following immediate corollary.

**Corollary 2.3.** Suppose $M \cong G\!/H$ is a simply connected reduced biquotient. Then the number of simple factors of $G$ is bounded by $\dim \pi_{\text{odd}}(M)_{\mathbb{Q}} = \dim \bigoplus_{k \geq 1} \pi_{2k-1}(M)_{\mathbb{Q}}$.

### 2.3 Biquotients with singly generated cohomology rings

In this section, we describe results of Kapovitch and Ziller \cite{Kapovitch-Ziller} on the classification of biquotients with singly generated cohomology, specifically in the case of $\mathbb{H}P^m$.

**Theorem 2.4.** (Kapovitch-Ziller) Suppose $M = G\!/H$ is a reduced simply connected biquotient with $H^*(M; \mathbb{Q}) \cong H^*(\mathbb{H}P^m; \mathbb{Q})$. Then $M$ is diffeomorphic to $\mathbb{H}P^m$ or $M = \Delta SU(2)\!/SO(4m + 1)/SO(4m - 1)$ or $M = G_2/ SO(4)$.
We will use the notation $N^m$ to denote the biquotient $\Delta SU(2) \backslash SO(2m+3)/SO(2m+1)$ when $m$ is odd. The notation $\Delta SU(2)$ refers to the following embedding into $SO(2m+3)$. First, the canonical map $\mathbb{C}^2 \to \mathbb{R}^4$ induces a map $SU(2) \to SO(4)$. Now, we embed $SO(4)$ into $SO(2m+3)$ via the block embedding $A \to \text{diag}(A, A, \ldots, A, 1)$. Then the embedding of $SU(2)$ into $SO(2m+3)$ is the composition of these two maps.

For biquotients diffeomorphic to $\mathbb{H}P^m$, we have the following classification. If the $H$ action on $G$ is homogeneous, that is, $H$ acts on only one side of $G$ then, up to cover, $G = Sp(m+1)$, $H = Sp(m) \times Sp(1)$ [28 pg. 264]. If the $H$ action on $G$ is two sided, then the action has been classified by Eschenburg [12 Table 101]. Writing $H = H' \times SU(2)$, we have

$$(G, H') \in \{(Spin(4m), Spin(4m-1)), (SU(2m), SU(2m-1)), (Sp(m), Sp(m-1))\}.$$ 

In all cases, $H'$ acts on $G$ on one side with $G/H' = S^{4m-1}$; the $SU(2)$ factor then acts via the Hopf action. So, regardless of whether the action is homogeneous or not, $H$ always has the form $H = H_1 \times SU(2)$, up to cover.

Recalling that $SO(4)$ is double covered by $SU(2)^2$, the case of $G_2/\SO(4)$ also has $H = H_1 \times SU(2)$.

Thus, we have the following corollary.

**Corollary 2.5.** If $M = G \parallel H$ is a reduced biquotient and $M \simeq \mathbb{H}P^m$, then, up to cover, $H = H_1 \times SU(2)$ with $(G, H_1)$ an element of

$$\{(Spin(4m), Spin(4m-1)), (SU(2m), SU(2m-1)), (Sp(m), Sp(m-1))\}$$

or $m$ is odd and $(G_1, H) = (Spin(2m+3), Spin(2m+1))$ or $m = 2$ and $(G, H_1) = (G_2, SU(2))$.

We now discuss the topology of each of these rational quaternionic projective spaces, $\mathbb{H}P^m$, $N^m$, and $G_2/\SO(4)$. Of course, the cohomology ring of $\mathbb{H}P^m$ is well known to be $\mathbb{Z}[x]/x^{m+1}$ with $|x| = 4$.

The cohomology ring of $N^m = \Delta SU(2) \backslash SO(2m+3)/SO(2m+1)$ with $m$ odd is computed in [21 Section 3]. The cohomology groups of $N^m$ are isomorphic to those of $\mathbb{H}P^m$, but, for $m > 1$, the ring structure is different. If $x \in H^4(N^m)$ is a generator, then $x^m$ is twice a generator of $H^{4m}(N^m)$.

For the last space $G_2/\SO(4)$, according to [20 pg. 242], we have $H^*(G_2/\SO(4)) \cong \mathbb{Z}[u_3, u_4]/I$ where $[u_i] = i$ and $I$ is the ideal generated by $2u_3, u_3u_4, u_3^2$, and $u_3^3$. Finally, we mention the cohomology

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ring $H^*(G_2/SO(4); \mathbb{Z}/2\mathbb{Z})$ is computed in [5, pg. 529]. We have
\[ H^*(G_2/SO(4); \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}[u_2, u_3]/I \]
where $|u_i| = i$ and where $I$ is the ideal generated by $u_2^3 = u_2^3$ and $u_3u_2^2 = 0$.

### 3 Classification when $M \simeq_{\mathbb{Q}} \mathbb{H}P^m \times S^3$

In this section, we classify biquotients $M = G/H$ for which $M \simeq_{\mathbb{Q}} \mathbb{H}P^m \times S^3$ up to diffeomorphism, proving Theorem 1.2. In Subsection 3.1, we show half of Theorem 1.2, that every simply connected biquotient which is rationally $\mathbb{H}P^m \times S^3$ has the form given. In Subsection 3.2, we show the spaces listed in Theorem 1.2 are distinct up to diffeomorphism.

#### 3.1 Classification of actions

In this section, we assume that $M \simeq_{\mathbb{Q}} \mathbb{H}P^m \times S^3$. Hence, $\dim \pi_{3/4}(M)_{\mathbb{Q}} = \dim \pi_{4/5}(M)_{\mathbb{Q}} = \dim \pi_{4m+3}(M)_{\mathbb{Q}} = 1$ and $\pi_k(M)_{\mathbb{Q}} = 0$ for all other $k$.

**Proposition 3.1.** Suppose $M \simeq_{\mathbb{Q}} \mathbb{H}P^m \times S^3$ and that $M = G/H$ is a reduced biquotient.

1. $G$ has at most two simple factors.
2. $G$ and $H$ have the same number of simple factors.
3. The rank of $H$ is one less than the rank of $G$.

**Proof.** The assertion (1) follows from Corollary 2.3 since $\dim \pi_{odd}(M)_{\mathbb{Q}} = 2$.

For (2) and (3), first recall that for every Lie group, the even degree rational homotopy groups vanish. Thus, the long exact sequence in rational homotopy groups associated to the bundle $H \to G \to G//H$ breaks into exact sequences of the form
\[ 0 \to \pi_{2i}(M)_{\mathbb{Q}} \to \pi_{2i}(H)_{\mathbb{Q}} \to \pi_{2i-1}(G)_{\mathbb{Q}} \to \pi_{2i-1}(M)_{\mathbb{Q}} \to 0. \]

Now, recall that $\dim \pi_{3}(G)_{\mathbb{Q}}$ counts the number of simple factors of $G$. Setting $i = 2$ in the above exact sequence, and using the fact that $\pi_{3}(M)_{\mathbb{Q}} \cong \pi_{4}(M)_{\mathbb{Q}} \cong \mathbb{Q}$, we deduce $\dim \pi_{3}(G)_{\mathbb{Q}} = \dim \pi_{3}(H)_{\mathbb{Q}}$, so $G$ and $H$ have the same number of simple factors. This proves (2).

For (3), we note that from the exact sequence above, it follows that $1 = \dim \pi_{odd}(M)_{\mathbb{Q}} - \dim \pi_{even}(M)_{\mathbb{Q}} = \dim \pi_{odd}(G)_{\mathbb{Q}} - \dim \pi_{odd}(H)_{\mathbb{Q}}$. But $\dim \pi_{odd}(G)$ counts the rank of $G$, so the rank of $H$ is one smaller than the rank of $G$. 

\[ \square \]
Proposition 3.1 (1) says $G$ is simple or $G = G_1 \times G_2$ is a product of two simple factors. We now show that $G$ can not be simple.

**Proposition 3.2.** Suppose $M = G//H$ is a reduced biquotient and $M \simeq \mathbb{Q} \# P^m \times S^3$. Then $G$ cannot be simple.

**Proof.** Assume $G$ is simple. By Proposition 3.1 (2), $H$ is simple as well. Thus, $\pi_3(G) \mathbb{Q} \cong \pi_3(H) \mathbb{Q} \cong \mathbb{Q}$. Now, consider the long exact sequence in rational homotopy groups associated to the fibration $H \to G \to G//H = M$:

$$0 \to \pi_{2i}(M) \mathbb{Q} \to \pi_{2i-1}(H) \mathbb{Q} \to \pi_{2i-1}(G) \mathbb{Q} \to \pi_{2i-1}(M) \mathbb{Q} \to 0.$$

For $i \geq 3$, except $i = 2m + 2$, both $\pi_{2i}(M) \mathbb{Q}$ and $\pi_{2i-1}(M) \mathbb{Q}$ vanish so $\pi_{2i-1}(H) \mathbb{Q} \cong \pi_{2i-1}(G) \mathbb{Q}$ with $i \neq 2m + 2$.

On the other hand, when $i = 2m + 2$, $\pi_{4m+3}(M) \mathbb{Q} \cong \mathbb{Q}$. It follows that $\pi_{4m+3}(G) \mathbb{Q} \cong \mathbb{Q}$ but $\pi_{4m+3}(H) \mathbb{Q} = 0$.

Since $G$ is rationally a product of spheres whose dimension is determined by the degrees, it follows that $H^*(G; \mathbb{Q}) \cong H^*(H \times S^{4m+3}; \mathbb{Q})$. But the pairs $(G, H)$ for which this happens are classified in [21, Section 2] and, in particular, for any such pair $(G, H)$, any biquotient $G//H$ has singly generated rational cohomology.

We may now assume $G = G_1 \times G_2$ and $H = H_1 \times H_2$. Because each factor of $G$ contributes at least one degree to $M$, and $M$ only has two degrees, each $G_i$ must contribute exactly one degree. We will assume $G_1$ contributes degree $2m + 2$ and $G_2$ contributes degree 2. If 2 is the highest degree of $G_2$, then $G_2 = SU(2)$. If 2 is not the highest degree of $G_2$, then, from Theorem 2.2, we see that $G_2$ must come from case (4), so $G_2 = H_2 = SU(3)$ with $H_2$ acting on $G_2$ by $A \ast B = ABA^t$.

We now show the case where $G_2 = H_2 = SU(3)$ cannot occur.

**Proposition 3.3.** If $G = G_1 \times SU(3)$, $H = H_1 \times SU(3)$ with the $SU(3)$ factor of $H$ acting on the $SU(3)$ factor of $G$ via $A \ast B = ABA^t$, then $G//H$ does not have the rational homotopy type of $\mathbb{Q} \# P^m \times S^3$.

**Proof.** We proceed by contradiction. The homomorphism $SU(3) \to SU(3)^2$ given by $A \mapsto (A, \overline{A})$ defines the action $A \ast B = ABA^t$. This homomorphism has image a maximal connected subgroup and thus, $H_1$ must act effectively freely on the first factor $G_1$. Since $H$ has rank 1 less than that of $G$, it follows that $H_1$ has rank one fewer than that of $G_1$. In particular, $G_1//H_1$ is odd dimensional. Further, just as in the proof of Proposition 3.2, the degrees of $H_1$ and $G_1$ must agree, except for a single degree of $G_1$. As mentioned in the proof of Proposition 3.2
this situation is studied in [21, Section 2], and, in particular, $G_1/\!/H_1$ is a simply connected rational sphere.

Now, consider the restriction of the $H$ action to $H_1 \times SO(3) \subseteq H_1 \times SU(3)$. The projection of this action to the $SU(3)$ factor of $G$ fixes the identity, and so $H_1 \times SO(3)$ must act effectively freely on $G_1$. In particular, $SO(3)$ acts effectively freely on $G_1/\!/H_1$. In fact, since $SO(3)$ has no non-trivial normal subgroup, it must act freely on $G_1/\!/H_1$, so we have a principal $SO(3)$ bundle

$$SO(3) \to G_1/\!/H_1 \to G_1/\!/(H_1 \times SO(3)).$$

(∗)

Since $SO(3) \cong_q S^3$, we may apply the rational Gysin sequence this bundle, concluding that $G_1/\!/(H_1 \times SO(3)) \cong \mathbb{H}P^m$.

However, the long exact sequence in homotopy groups associated to (∗) implies that $G_1/\!/(H_1 \times SO(3))$ has nontrivial $\pi_2$. The only example in Theorem 2.4 with nontrivial $\pi_2$ is $G_2/\!/SO(4)$. It follows that $G = G_2 \times SU(3)$ and $H = SU(2) \times SU(3)$, up to cover.

But Eschenburg [12, pg. 122] classified biquotient actions of maximal rank on simple Lie groups and, in particular, showed that $G_2$ admits no free two-sided action. Hence, $H_1 = SU(2)$ and $SO(3) \subseteq SU(3)$ must both act on the same side of $G_2$. Further, the only non-trivial homomorphism $SU(3) \to G_2$ has maximal image, so $H_1$ cannot act on $G_2$. This contradiction shows the case $G = G_1 \times SU(3)$ cannot occur.

We henceforth assume $G = G_1 \times SU(2)$ and $H = H_1 \times H_2$. Because $H$ does not contain a torus factor, [8, Proposition 4.1] gives the following.

**Proposition 3.4.** Suppose $M = G/\!/H$ is a reduced biquotient and $M \cong \mathbb{H}P^n \times S^3$ with $G = G_1 \times SU(2)$ and $H = H_1 \times H_2$. Then the projection of the $H$ action to $G_1$ is effectively free. Further, if the projection of the $H$ action to the $SU(2)$ factor of $G$ is non-trivial, then, there is exactly one factor of $H$, isomorphic to $SU(2)$, which acts by conjugation.

The biquotient $G_1/\!/H$ is simply connected because $G_1$ is simply connected and $H$ is connected. Further, it is rationally 2-connected since $\pi_2(G_1) = 0$ and $\pi_1(H)$ is finite. In fact, it is rationally 3-connected. To see this, recall that, according to Singhof [31, Theorem 5.1], the Euler characteristic of equal rank biquotients is given by the quotient of the orders of their Weyl groups and is, in particular, positive. But, as mentioned in Section 2.1 for rationally elliptic topological spaces, the Euler characteristic is positive iff all odd Betti
numbers vanish. To apply this, simply note that $H$ and $G_1$ have the same rank because $\text{rank}(H) + 1 = \text{rank}(G) = \text{rank}(G_1) + 1$.

Because the $H$ action on $G_1$ is effectively free, we have a bundle $H/K \to G_1 \to G_1//H$, where $K$ is the ineffective kernel of the action. The $H$ action on the $S^3 = SU(2)$ factor of $G$ gives an associated bundle $S^3 \to G_1 \times_H S^3 = G//H \to G_1//H$. We use the associated bundle to study the topology of $G_1//H$.

**Proposition 3.5.** $G_1//H$ has the rational cohomology ring of $\mathbb{H}P^m$.

**Proof.** Consider the rational Gysin sequence associated to the bundle $S^3 \to G//H \to G_1//H$ and recall that $G//H$ is rationally $\mathbb{H}P^m \times S^3$. Since $H^3(G_1//H; \mathbb{Q}) = 0$, a portion of the rational Gysin sequence is given by

$$
\ldots \to 0 \to H^3(G//H; \mathbb{Q}) \to H^0(G_1//H; \mathbb{Q}) \xrightarrow{c_1} H^4(G_1//H; \mathbb{Q}) \to \ldots
$$

Thus, $H^3(G//H) \cong \mathbb{Q}$ must inject into $H^0(G_1//H; \mathbb{Q}) \cong \mathbb{Q}$, so this map is an isomorphism. This implies the Euler class of the bundle is 0, which, in turn, implies that projection map $G//H \to G_1//H$. Then this implies that, as groups, $H^*(G//H; \mathbb{Q}) \cong H^*(G_1//H; \mathbb{Q}) \otimes H^*(S^3; \mathbb{Q})$. In particular, the cohomology groups of $H^*(G_1//H; \mathbb{Q})$ are isomorphic to those of $\mathbb{H}P^m$.

Then using the fact that the Euler class is 0, we see the projection map $G//H \to G_1//H$ embeds $H^*(G_1//H; \mathbb{Q})$ as a subalgebra of $H^*(G//H; \mathbb{Q})$. This shows that a generator of $H^4(G_1//H; \mathbb{Q})$ has non-trivial powers until $x^{m+1} = 0$. Thus, $H^*(G_1//H; \mathbb{Q}) \cong H^*(\mathbb{H}P^m; \mathbb{Q})$.

Propositions 3.4 and 3.5 and Corollary 2.5 now combine to give the following characterization of biquotients having the rational homotopy type of $\mathbb{H}P^m \times S^3$.

**Theorem 3.6.** Every simply connected biquotient $M = G//H$ with $M \cong \mathbb{H}P^m \times S^3$ is diffeomorphic to a biquotient having the form $G//H = (G_1 \times SU(2))/(H_1 \times SU(2))$ with $(G_1, H_1)$ one of the pairs of groups listed in Corollary 2.5. Further, the projection of the $H$ action onto $G_1$ is effectively free with quotient a rational $\mathbb{H}P^m$ and, in addition, either the projection of the $H$ action on the $SU(2)$ factor of $G$ is trivial or at most one factor of $H$, isomorphic to $SU(2)$, acts by conjugation.

If $m = 1$, $G//H$ is 7-dimensional. Such biquotients were classified in [8] (Propositions 4.27 and 4.28). In particular among those with the rational homotopy of $\mathbb{H}P^1 \times S^3$, the homotopy and diffeomorphism classification agree and there are precisely three diffeomorphism types.
We now assume $m \geq 2$. Among the choices for $H_1$ in Theorem 3.6, the only time when $H_1 = SU(2)$ when $G_1 = G_2$. Thus, in this case, we get three biquotients: the trivial product $(G_2/SO(4)) \times S^3$, as well as two biquotients $(G_2/H_1) \times_{H_2} S^3$ and $(G_2/H_2) \times_{H_1} S^3$.

In all other cases, $H_1 \neq SU(2)$. Thus, we end up with four forms of biquotients: the trivial products $\mathbb{H}P^m \times S^3$, $N^m \times S^3$, as well as $S^{4m+3} \times SU(2) S^3$ and $(SO(2m+3)/SO(2m+1)) \times SU(2) S^3$.

We have now shown half of Theorem 1.2 that each such biquotient is diffeomorphic to one of the listed spaces. In the next section, we prove these biquotients are distinct up to diffeomorphism.

### 3.2 Diffeomorphism classification

In this section, we show each of the biquotients listed in Theorem 1.2 is distinct up to diffeomorphism.

Recall that, in the proof of Proposition 3.5, we showed that if $M = G/\!/H \approx \mathbb{H}P^m \times S^3$, then we have a bundle $S^3 \to G/\!/H \to G_1/\!/H$ with $G_1/\!/H \in \{\mathbb{H}P^m, N^m, G_2/\!/SO(4)\}$ and with Euler class 0. Thus, the Gysin sequence breaks into short exact sequences of the form

$$0 \to H^*(G_1/\!/H) \to H^*(G/\!/H) \to H^{*-3}(G_1/\!/H) \to 0.$$  

If $G_1/\!/H \neq G_2/\!/SO(4)$, then for any $\ast$, either $H^*(G_1/\!/H) = 0$ or $H^{*-3}(G_1/\!/H) = 0$. It follows that the integral cohomology ring $H^*(G/\!/H)$ is isomorphic to $H^*((G_1/\!/H) \times S^3)$. If, on the other hand, $G_1/\!/H = G_2/\!/SO(4)$, then we see the projection map induces an isomorphism $\mathbb{Z}/2\mathbb{Z} \to H^3(G_1/\!/H) \to H^3(G/\!/H)$, so $H^*(G/\!/H)$ has torsion in this case. Together with the observation in Section 2.3 that the cohomology rings of $\mathbb{H}P^m$ and $N^m$ are different, we have shown the following proposition.

**Proposition 3.7.** Suppose $B_1, B_2 \in \{\mathbb{H}P^m, N^m, G_2/\!/SO(4)\}$ with $B_1 \neq B_2$. Let $M_i$ be the total space of a linear $S^3$ bundle over $B_i$ with Euler class 0. Then $M_1$ is not homotopy equivalent to $M_2$.

Thus, in terms of the diffeomorphism classification, each of cases (1), (2), and (3) of Theorem 1.2 are distinct up to homotopy.

We recall that for each $B \in \{\mathbb{H}P^m, N^m, G_2/\!/SO(4)\}$, the bundle $S^3 \to G/\!/H \to B$ is associated to a principal bundle $F \to G_1/\!/H_1 \to G_1/\!/H$. 

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When $B = \mathbb{H}P^m$, the bundle is the Hopf bundle, having fiber $SU(2)$. When $B = N^m$, we have the principal $SU(2)$-bundle

$$SO(2m + 3)/SO(2m + 1) \to N^m.$$  

When $B = G_2/SO(4)$, there are two relevant bundles. The two normal $SU(2)$s in $SO(4)$ have Dynkin indices 1 and 3 [21], so we denote them $SU(2)_1$ and $SU(2)_3$. Consider the homogeneous space $G_2/SU(2)_1$. Then $SU(2)_3$ acts effectively free on this space with ineffective kernel $\{\pm I\} \in SU(2)$. Thus, we have a principal $SO(3)$-bundle $G_2/SU(2)_1 \to G_2/SO(4)$. The same argument gives a principal $SO(3)$-bundle $G_2/SU(2)_3 \to G_2/SO(4)$.

We let $F$ denote the fiber of the principal bundle, so $F = SO(3)$ when $B = G_2/SO(4)$ and $F = SU(2)$ otherwise. Now, for any choice of $B$, the action of $F$ on $S^3$ is either trivial or by conjugation. Hence, we must distinguish $B \times S^3$, when the action is trivial, from $(G_1//H_1) \times F S^3$ where the $F$ action on $S^3$ is by conjugation. We do this by computing Pontrjagin classes.

Our main tool for computing the first Pontrjagin class of the biquotient $(G_1//H_1) \times F S^3$ is the following proposition, which can be found, for example, in [17, pg. 383].

**Proposition 3.8.** Suppose $S^n \to E \xrightarrow{\pi} B$ is an oriented linear sphere bundle and let $\xi = \mathbb{R}^{n+1} \to \tilde{E} \xrightarrow{\tilde{\pi}} B$ denote the corresponding vector bundle. Then $T\tilde{E} \cong \tilde{\pi}^*(\xi) \oplus \tilde{\pi}^*(TB)$. In addition, the inclusion $i : E \to \tilde{E}$ gives $i^*(T\tilde{E}) = TE \oplus 1$, with 1 denoting the rank 1 trivial bundle. In particular,

$$TE \oplus 1 \cong \pi^*(\xi) \oplus \pi^*(TB).$$

Using this, we now show $B \times S^3$ and $(G_1//H_1) \times F S^3$ have different diffeomorphism types.

**Proposition 3.9.** For any

$$B \in \{\mathbb{H}P^m, N^m, G_2/SO(4)\},$$

the trivial bundle $B \times S^3$ and the non-trivial bundle $(G_1//H_1) \times F S^3$ have distinct Pontrjagin classes.

**Proof.** The $F$ action on $S^3$ is linear, so extends to a representation on $\mathbb{R}^4$. Let $\xi$ denote the rank 4 vector bundle $(G_1//H_1) \times F \mathbb{R}^4 \to G_1//H$.

From Proposition 3.8, we see

$$p_1((G_1//H_1) \times SU(2) S^3) = \pi^*(p_1(\xi)) + \pi^*(p_1(B)).$$

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and so we need only show $\pi^*(p_1(\xi)) \neq 0$. Since the associated bundle $S^3 \to G/H \to B$ has trivial Euler class, $\pi^*$ is injective on $H^1$, so it is enough to show $p_1(\xi) \neq 0$.

Consider the universal principal $F$ bundle $EF \to BF$. Then we have a commutative diagram of fibrations

\[
\begin{array}{ccc}
F & \longrightarrow & G_1/H_1 \\
\downarrow & & \downarrow \phi \\
F & \longrightarrow & EF \\
\downarrow & & \downarrow \\
\mathbb{R}^4 & \longrightarrow & (G_1/H_1) \times_F \mathbb{R}^4
\end{array}
\]

where $\phi : G_1/H \to BF$ is the classifying map. This gives rise to a map of associated bundles

\[
\begin{array}{ccc}
\mathbb{R}^4 & \longrightarrow & (G_1/H_1) \times_F \mathbb{R}^4 \\
\downarrow & & \downarrow \phi \\
\mathbb{R}^4 & \longrightarrow & EF \times_F \mathbb{R}^4 \\
\downarrow & & \downarrow \\
\mathbb{R}^4 & \longrightarrow & BF.
\end{array}
\]

Thus, if $\eta$ denotes the vector bundle $\mathbb{R}^4 \to EF \times_F \mathbb{R}^4 \to BF$, then $\xi$ is the pull back of $\eta$, $\xi = \phi^* \eta$. Hence, in order to show $p_1(\xi) \neq 0$, it is enough to show $p_1(\eta) \neq 0$ and that $\phi^*$ is non-zero on $H^4$.

Let $i : S^1 \to F$ denote the inclusion of a maximal torus. Then, for either choice of $F$, it is easy to see that the induced map $i^* : H^4(BF) \to H^4(BS^1)$ is an isomorphism.

It is well known [5, Theorem 10.1] that the associated vector bundle $\eta$ over $BF$ has total Pontrjagin classes given by $i^*(\prod (1 + \beta^2))$ where the product is over all weights of the representation. Here, we interpret $\beta \in H^2(BS^1)$ as follows. First, as shown in [10, Theorem 23.16], the weights of a representation are dual to $exp^{-1}(0)$ with $exp : \mathbb{R} \to S^1$ the group exponential map. Identifying $exp^{-1}(0)$ with $\pi_1(T)$, we may interpret weights as element of $\text{Hom}(\pi_1(S^1), \mathbb{Z}) = H^1(S^1)$. Finally, from the Serre spectral sequence associated to the universal fibration $S^1 \to ES^1 \to BS^1$, we use the transgression of the generator of $H^1(S^1)$ as generators of $H^2(BS^1)$. This allows us to interpret weights as elements of $H^2(BS^1)$.

For the conjugation action $F \to SO(4)$, the weight is twice a generator of $H^2(BS^1)$ for $F = SU(2)$ and is a generator of $H^2(BS^1)$ when $F = SO(3)$. In particular, if $x \in H^4(BF) \cong \mathbb{Z}$ is a generator,
then \( p_1(\eta) = 4x \) if \( F = SU(2) \) and \( p_1(\eta) = x \) if \( F = SO(3) \). Thus, in either case, \( p_1(\eta) \neq 0 \).

Thus, we need only show \( \phi^* : H^4(BF) \to H^4(G_1/H) \) is non-zero. If it is the 0 map, naturality of Serre spectral sequences implies that all differentials in the spectral sequence of the principal \( F \) bundle \( G_1/H_1 \to G_1/H \) vanish, which, in turn, implies that \( G_1/H_1 \) has the rational cohomology ring of \( \mathbb{H}P^m \times S^3 \). Since \( G_1 \) is simple, this contradicts Proposition 3.2.

Together, Propositions 3.7 and 3.9 almost completely classify biquotients which are rationally \( \mathbb{H}P^m \times S^3 \) up to diffeomorphism. The only remaining task is to distinguish the two nontrivial bundles over \( G_2/SO(4) \). Thus, once we prove the following proposition, we will have completed the proof of Theorem 1.2.

**Proposition 3.10.** The first Pontrjagin classes of the biquotients

\[
(G_2/SU(2)_1) \times_{SO(3)} S^3 \quad \text{and} \quad (G_2/SU(2)_3) \times_{SO(3)} S^3
\]

are different.

**Proof.** We have two commutative diagrams

\[
\begin{array}{ccc}
SO(3) & \longrightarrow & G_2/SU(2)_i \longrightarrow G_2/SO(4) \\
\downarrow \phi_i & & \downarrow \phi_i \\
SO(3) & \longrightarrow & ESO(3) \longrightarrow BSO(3)
\end{array}
\]

for \( i = 1, 3 \). Following the proof of Proposition 3.9, we need only show \( \phi_i^* : \mathbb{Z} = H^4(ESO(3)) \to H^4(G_2/SO(4)) = \mathbb{Z} \) for \( i = 1, 3 \) are different maps. In fact, we will show that \( \phi_i^* \) is multiplication by \( \pm i \).

We first note that in the Serre spectral sequence for the universal bundle, the differential maps \( H^3(SO(3)) \) isomorphically onto \( H^4(BSO(3)) \). Thus, if \( x \in H^3(SO(3)) \) and \( y \in H^4(BSO(3)) \) are generators, naturality implies \( dx = \phi_i^*(y) \) in the spectral sequence associated to \( SO(3) \to G_2/SU(2)_i \to G_2/SO(4) \).

From the computation of \( H^*(G_2/SO(4)) \) and \( H^*(G_2/SO(4); \mathbb{Z}/2\mathbb{Z}) \) found at the end of Section 2.3, we see the only non-zero entries of the \( E_2 \) page of the form \( E_2^{p,q-p} \) are \( E_2^{4,0} \cong \mathbb{Z} \) and \( E_2^{2,2} \cong \mathbb{Z}/2\mathbb{Z} \). Further, \( E_2^{0,3} \cong \mathbb{Z} \) is the only entry which can possibly map non-trivially into \( E_2^{4,0} \). We now claim that the differential \( d_3 : \mathbb{Z} \cong E_2^{0,3} \to E_2^{4,0} \cong \mathbb{Z} \) is multiplication by \( i \), up to sign.
For $G_2/SU(2)_1$, we note that $G_2/SU(2)_1$ is diffeomorphic to the unit tangent bundle of $S^6$, which is 5-connected. It follows that the differential $d_3$ must be an isomorphism, so it is given by multiplication by $\pm i$.

For $G_2/SU(2)_3$, we first recall that the Dynkin index is the degree of the map $\pi_3(SU(2)_3) \to \pi_3(G_2)$ induced from the inclusion $SU(2)_3 \to G_2$. Since $G_2$ and $SU(2)$ are both 2-connected, the long exact sequence in homotopy groups implies $G_2/SU(2)_3$ is 2-connected with $\pi_3(G_2/SU(2)_3) = \mathbb{Z}/3\mathbb{Z}$. Then the universal coefficients theorem implies the torsion of $H^4(G_2/SU(2)_3)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.

Since $E_2^{2,2}$ is 2-torsion, the only possible contribution of $\mathbb{Z}/3\mathbb{Z}$ to $H^4(G_2/SU(2)_3)$ in the spectral sequence is if the differential $d_3$ is multiplication by $\pm 3$.

It follows that $\phi_*^i : \mathbb{Z} \cong H^4(BO(3)) \to H^4(G_2/SO(4)) \cong \mathbb{Z}$ is multiplication by $i$. Thus, the two biquotients $(G_2/SU(2)_1) \times_{SO(3)} S^3$ and $(G_2/SU(2)_3) \times_{SO(3)} S^3$ have different first Pontrjagin classes.

\[\square\]

4 Classification when $M \simeq \mathbb{H}P^m \times S^2$

We now study biquotients $M = G//H$ with $M \simeq \mathbb{H}P^m \times S^2$. We begin by studying the possible actions of $H$ on $G$, finding a structure theorem analogous to Theorem 3.6. Then, we partially classify the resulting diffeomorphism types.

4.1 Classification of actions

In this section, we prove the following structure theorem.

**Theorem 4.1.** Every simply connected biquotient $M = G//H$ with $M \simeq \mathbb{H}P^m \times S^2$ is diffeomorphic to a biquotient having the form $G//H = (G_1 \times SU(2))/(H_1 \times SU(2) \times S^1)$ with $(G_1, H_1)$ one of the pairs in Corollary [2.3]. Further, the projection of the $H_1 \times H_2$ action onto $G_1$ is effectively free with quotient a rational $\mathbb{H}P^m$ and, in addition, the projection of the $H$ action on the $SU(2)$ factor of $G$ is trivial. Finally, the circle factor of $H$ acts, up to ineffective kernel, as the Hopf map on the $SU(2)$ factor of $G$.

We begin by reducing the classification of biquotients $M$ which are rationally $\mathbb{H}P^m \times S^2$ to those which are rationally $\mathbb{H}P^m \times S^3$.

**Proposition 4.2.** Suppose $M = G//H$ is a simply connected reduced biquotient and $M \simeq \mathbb{H}P^m \times S^2$. Then $G = G_1 \times SU(2)$ and $H = H_1 \times SU(2) \times S^1$ with $(G_1, H_1)$ one of the pairs listed in Corollary [2.3].
Further, the projection of the $H_1 \times SU(2)$ action to $G_1$ is free with quotient a rational $\mathbb{H}P^m$.

Proof. Suppose $M = G/H$ is rationally $\mathbb{H}P^m \times S^2$. Then $\dim \pi_2(M)_\mathbb{Q} = 1$. Thus, from the long exact sequence in rational homotopy groups associated to the bundle $H \to G \to G/H$, we conclude that $\pi_1(H)_\mathbb{Q}$ has dimension 1 as well. This implies $H = H' \times S^1$ with $H'$ semi-simple. Because $H'$ is semi-simple, the long exact sequence in rational homotopy groups associated to the principal bundle $H' \to G \to G/H'$ shows that $G/H'$ is rationally 2-connected.

Now, considering the Gysin sequence associated to the principal circle bundle $S^1 \to G/H' \to G/H$, we easily see the rational Euler class must be non-zero because $G/H'$ is rationally 2-connected. It follows that $H^*(G/H'; \mathbb{Q}) \cong H^*(\mathbb{H}P^m \times S^3; \mathbb{Q})$. In particular, from Theorem 3.6, we conclude $G = G_1 \times SU(2)$, $H' = H_1 \times SU(2)$, and that $H_1 \times SU(2)$ acts freely on $G$ with quotient a rational $\mathbb{H}P^m$.

We now henceforth assume $G = G_1 \times SU(2)$ and $H = H_1 \times SU(2) \times S^1$ with $(G_1, H_1)$ given Corollary 2.5. The following proposition will conclude the proof of Theorem 4.1.

Proposition 4.3. The projection of the $H_1 \times SU(2)$ action onto the $SU(2)$ factor of $G$ is trivial. The projection of the $S^1$ action to the $SU(2)$ factor of $G$ is, up to ineffective kernel, the Hopf map.

Proof. Suppose for a contradiction the projection of the $H_1 \times H_2$ action onto the $SU(2)$ factor is non-trivial. Then, by Proposition 3.4, either $H_1 \cong SU(2)$ acts by conjugation or the $SU(2)$ factor acts by conjugation. The map which defines the conjugation action is the diagonal embedding $SU(2) \to \Delta SU(2) \subseteq SU(2)^2$. The image is maximal among connected groups, so the diagonal embedding cannot be extended to $SU(2) \times S^1$ while maintaining only finite kernel. It follows that the projection of the $H$ action to the $SU(2)$ factor of $G$ fixes the identity. Thus, in order for the $H$ action on $G$ to be effectively free, the projection of the $H$ action onto $G_1$ must be effectively free.

Hence, we have an effectively free $S^1$ action on $G_1/(H_1 \times SU(2)) \cong \mathbb{H}P^m$. However, it is well known that any circle action on a space of non-zero Euler characteristic must have a fixed point. This contradicts the fact that the action is effectively free. Thus, we conclude the projection of the $H_1 \times SU(2)$ action to the $SU(2)$ factor of $G$ is trivial.

It follows from this that the biquotient $G/H$ has the form $(G_1/(H_1 \times SU(2))) \times S^1 SU(2)$, where $G_1/(H_1 \times SU(2))$ is rationally $\mathbb{H}P^m$, of Euler characteristic $m + 1 > 0$. In particular, every $S^1$ action on it must have a fixed point. Thus, in order for the $S^1$ action on
From the proposition above, we know $M$ has the form $(G_1/(H_1 \times SU(2))) \times S^1 S^3$ where $S^1$ acts on $S^3$, up to ineffective kernel, as the Hopf map. In fact, $M$ is diffeomorphic to a space of the form $(G_1/(H_1 \times SU(2))) \times S^1 S^3$ where the action on $S^3$ is effective. For, if $z \in S^1$ acts trivially on $S^3$, then it fixes a point of $(G_1/(H_1 \times SU(2))) \times S^3$ because the $S^1$ action on $G_1/(H_1 \times SU(2))$ has a fixed point. Because the action is effectively free, we conclude that $z$ fixes all points of $(G_1/(H_1 \times SU(2))) \times S^3$. Thus, dividing $S^1$ by its ineffective kernel, we obtain an effective action of $S^1$ on $(G_1/(H_1 \times SU(2))) \times S^3$ as desired.

Because $M$ is diffeomorphic to a space of the form $(G_1/(H_1 \times SU(2))) \times S^1 S^3$, see that $M$ has the structure of the total space of a bundle over $S^2$ with fiber $\mathbb{H}P^m, N^m$, or $G_2/SO(4)$. In fact, the bundle structure is associated to the Hopf bundle $S^1 \to S^3 \to S^2$ via the $S^1$ action on $G_1/(H_1 \times SU(2))$. This is in contrast to the case where $M \simeq \mathbb{H}P^m \times S^3$, where $M$ naturally has the structure of a bundle with fiber $S^3$ and base one of $\mathbb{H}P^m, N^m$, or $G_2/SO(4)$.

### 4.2 Partial diffeomorphism classification

In this section, we investigate the topology of biquotients $M = G//H$ with $M \simeq \mathbb{H}P^m \times S^2$, proving Theorem 1.3. We have already shown that such an $M$ is a bundle over $S^2$ with fiber over $B$.

$$B \in \{\mathbb{H}P^m, N^m, G_2/SO(4)\}.$$ 

When $m = 1$, such biquotients are classified in [3] Section 4.1. In particular, the homotopy and diffeomorphism classifications coincide, and there are precisely two diffeomorphism types. Thus, we now assume $m \geq 2$.

The differentials in the Serre spectral sequence associated to $B \to M \to S^2$ all vanish for trivial reasons and there are no extension problems. If $B = \mathbb{H}P^m$ or $N^m$, it follows that $H^*(M)$ is torsion free. However, the inclusion $B \to M$ induces ring isomorphisms $H^i(M) \to H^i(B)$ for $0 \leq i \leq m$. Thus, $H^*(M) \cong H^*(B \times S^2)$ when $B = \mathbb{H}P^m$ or $N^m$. In particular, if $M_1$ is a biquotient with $B_1 = \mathbb{H}P^m$ and $M_2$ is a biquotient with $B_2 = N^m$, then $\ast(M_1) \not\cong \ast(M_2)$.

In addition, when $B = G_2/SO(4)$, it follows that $H^*(M)$ has torsion. Thus, we have the following analogue of Proposition 3.7.
Proposition 4.4. Suppose

\[ B_1, B_2 \in \{ \mathbb{H}P^m, N^m, G_2/\text{SO}(4) \} \]

with \( B_1 \neq B_2 \). Let \( M_i \) denote a biquotient of the form \( B_i \times_{S^1} S^3 \) where the action of \( S^1 \) on \( S^3 \) is the Hopf map. Then \( M_1 \) is not homotopy equivalent to \( M_2 \).

Hence, in terms of proving Theorem 1.3, we may work with one choice of \( B \) at a time. We begin with the case \( B = G_2/\text{SO}(4) \).

Proposition 4.5. Suppose \( M = G/\mathbb{H} \) is a simply connected biquotient diffeomorphic to \(( G/\mathbb{H} ) \times_{S^1} S^3 \) where the \( S^1 \) action on \( S^3 \) is the Hopf action. Then \( M \) is diffeomorphic to the product \(( G_2/\text{SO}(4) ) \times S^2 \).

Proof. We have already shown \( M \) has the form \( B \times_{S^1} S^3 \) and is an associated bundle to the Hopf bundle. Equipping \( G_2 \) with a bi-invariant metric, the \( S^1 \) action on \( B \) is isometric, so we have a map \( S^1 \to \text{Iso}(B) \).

But \( G_2/\text{SO}(4) \) with bi-invariant metric is a symmetric space, so the identity component of the isometry group is given by \( G_2/(Z(G_2) \cap \text{SO}(4)) \). Since the center of \( G_2 \) is trivial, the identity component of the isometry group of \( G_2/\text{SO}(4) \) is \( G_2 \), which is simply connected.

Now, principal \( G_2 \) bundles over \( S^2 \) are in bijective correspondence with \([ S^2, BG_2 ] \). But \( G_2 \) is 2-connected, so \( BG_2 \) is 3-connected. In particular, the only \( G_2 \)-principal bundle over \( S^2 \) is the trivial bundle.

Consider the bundle over \( S^2 \) given by \( G_2 \times_{S^1} S^3 \) which is associated to the Hopf bundle. Left multiplication by \( G_2 \) is well defined so this is a principal \( G_2 \)-bundle, which is equivariantly trivial as mentioned above: \( G_2 \times_{S^1} S^3 \cong G_2 \times S^2 \) as principal \( G_2 \)-bundles.

Then we have

\[
M \cong B \times_{S^1} S^3 \\
\cong B \times G_2 ( G_2 \times_{S^1} S^3 ) \\
\cong B \times G_2 ( G_2 \times S^2 ) \\
\cong B \times S^2.
\]

\[ \square \]

Having handled the case \( B = G_2/\text{SO}(4) \), we now turn to the case where \( B = \mathbb{H}P^m \). We note that according to Theorem 2.4 and the following discussion, a bi-invariant metric on \( G_1 \) induces, up to scale, the Fubini-Study metric on \( G_1/\mathbb{H} \cap SU(2) = \mathbb{H}P^m \). Thus, the identity component of the isometry group \( \text{Iso}(B) \) is \( Sp(m+1)/Z(Sp(m+1)) = \)
\[ Sp(m+1)/(\mathbb{Z}/2\mathbb{Z}). \] Now, because \( \pi_1(\text{Iso}(B)) = \mathbb{Z}/2\mathbb{Z}, \) it follows that \([S^2, BIso(B)]\) contains two elements.

Thus, following the proof of Proposition 3.10, we conclude that for each \( m, \) there are at most two diffeomorphism types of biquotients of the form \( \mathbb{H}P^m \times S^3, \) depending on how \( S^1 \) acts on \( \mathbb{H}P^m. \) Of course, the trivial action gives the biquotient \( \mathbb{H}P^m \times S^2. \) However, for each \( m, \) we will find an action of \( S^1 \) on \( \mathbb{H}P^m \) for which the biquotient \( \mathbb{H}P^m \times S^3 \) is not even homotopy equivalent to \( \mathbb{H}P^m \times S^2: \) the two biquotients are distinguished by their Stiefel-Whitney classes, a homotopy invariant [10].

More precisely, for a particular action of \( S^1 \) on \( \mathbb{H}P^m, \) we will show that either \( w_2(\mathbb{H}P^m \times S^3) \neq 0 \) or \( w_6(\mathbb{H}P^m \times S^3) \neq 0. \) On the other hand, since \( S^2 \) is stably parallelizable, \( \mathbb{H}P^m \times S^2 \) has non-trivial Stiefel-Whitney classes only in dimensions divisible by 4.

We let \( S^1 = \{ e^{i\theta} : \theta \in [0, 2\pi) \} \) act on \( \mathbb{H}P^m = \{ [q_0: \ldots : q_m] | q_i \in \mathbb{H} \} \) (where \([q_0: \ldots : q_m] \simeq [p_0: \ldots : p_m] \) if \( \text{there is a } q \in \mathbb{H} \text{ with } q_j q = p_j \) for all \( j \)) as follows:
\[
e^{i\theta} [q_0 : \ldots : q_m] = [e^{i\theta/2} q_0 : \ldots : e^{i\theta/2} q_m]. \quad (*)
\]

We let \( C_m = \mathbb{H}P^m \times S^3 \) using this action of \( S^1 \) on \( \mathbb{H}P^m. \) We set \( H^*(C_m; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[u_2, u_4]/(u_2^3, u_4^{m+1}). \)

**Proposition 4.6.** For each \( m, \) either \( w_2(TC_m) \neq 0 \) or \( w_6(TC_m) \neq 0. \)

**Proof.** We first note that there is a commutative diagram of fibrations:

\[
\begin{array}{ccc}
\mathbb{H}P^m & \xrightarrow{j} & C_m \\
\downarrow & & \downarrow i \\
\mathbb{H}P^{m+1} & \xrightarrow{j} & C_{m+1}
\end{array}
\]

To see this, simply note that \( \mathbb{H}P^m \) embeds into \( \mathbb{H}P^{m+1} \) as the subset
\[
\{ [q_0: \ldots : q_m : 0] \in \mathbb{H}P^{m+1} | [q_0: \ldots : q_m] \in \mathbb{H}P^m \}.
\]

Then the \( S^1 \) action clearly preserves this set and acts as in (*)\). We point out that on cohomology, \( i^* \) is an isomorphism except that it is trivial on \( H^{4m+6}. \)

We recall that \( j^* : H^*(C_m) \to H^*(\mathbb{H}P^m) \) is an isomorphism when \( * = 4k. \) Further, the normal bundle of \( \mathbb{H}P^m \subseteq C_m \) is trivial: a neighborhood of \( \mathbb{H}P^m \) takes the form \( \mathbb{H}P^m \times \mathbb{R}^2, \) coming from a trivializing
neighborhood in \( S^2 \). It follows that \( w_2(C_m) = 0 \) iff \( w_4(\mathbb{H}P^m) = 0 \). But the total Stiefel-Whitney class of \( \mathbb{H}P^m \) is given by \( (1 + j^*u_4)^{m+1} \) \cite[Section 15.7]{5}. Thus, \( w_4(\mathbb{H}P^m) = 0 \) iff \( m \) is odd, so \( w_4(C_m) = 0 \) iff \( m \) is odd.

We next claim that \( w_2(TC_1) = u_2 \) is non-trivial. This follows because \( \mathbb{H}P^1 = S^4 \), and the \( S^1 \) action on \( S^4 \) comes from an embedding of \( S^1 \) into \( SO(5) = Sp(2)/Z(Sp(2)) \). This embedding is homotopically non-trivial because the lift to \( Sp(2) \) is not a loop. Then according to \cite[Lemma 8.2.5]{17}, \( w_2 \neq 0 \) in this case. Further, \( w_0(C_1) = 0 \) since is the reduction mod 2 of the Euler characteristic, which is 4.

Now, inductively, we have \( C_1 \subseteq C_2 \subseteq \ldots \subseteq C_m \). If \( i : C_1 \to C_m \) denotes the inclusion, then \( i^*TC_m \cong TC_1 \oplus \nu_{1,m} \) where \( \nu_{1,j} \) is the normal bundle of \( C_j \) in \( C_j \). Because of the chain of inclusions, the normal bundle \( \nu_{1,m} \) decomposes into a sum of rank 4 bundles and clearly these rank 4 bundles, when pulled back to \( C_1 \), are all isomorphic. Thus,

\[
i^*TC_m \cong TC_1 \oplus (m-1)\nu_{1,2} = TC_1 \oplus \nu_{1,2} \oplus \ldots \oplus \nu_{1,2}^m.
\]

We now compute the characteristic classes of \( \nu_{1,2} \). We first claim that \( w_4(\nu_{1,2}) = u_4 \). This follows because \( w_4(TC_2) = u_4 \) since 4 is even, so

\[0 \neq w_4(i^*(TC_2)) = w_4(TC_1 \oplus \nu_{1,2}) = w_4(TC_1) + w_4(\nu_{1,2}) + w_2(TC_1)w_2(\nu_{1,2}).\]

But \( w_4(TC_1) = 0 \) since 1 is odd. Further, either \( w_2(\nu_{1,2}) = u_2 \) or it is 0. Since \( u_2^2 \), we see that \( w_2(TC_1)w_2(\nu_{1,2}) = u_2w_2(\nu_{1,2}) = 0 \) regardless of the value of \( w_2(\nu_{1,2}) \). Thus, \( w_4(\nu_{1,2}) \neq 0 \), so \( w_4(\nu_{1,2}) = u_4 \).

We now break into cases depending on whether \( w_2(\nu_{1,2}) = 0 \) or \( w_2(\nu_{1,2}) = u_2 \). First, if \( w_2(\nu_{1,2}) = 0 \), then \( i^*w_2(TC_m) = w_2(TC_1 \oplus (\nu_{1,2})^{m-1}) = w_2(TC_1) = u_2 \) since the smallest non-trivial Stiefel Whitney class of \( \nu_{1,2} \) occurs in dimension 4. Thus, if \( w_2(\nu_{1,2}) = 0 \), each biquotient \( C_m \) has \( w_2(TC_m) \) non-trivial, so is not homotopy equivalent to \( \mathbb{H}P^m \times S^2 \).

On the other hand, if \( w_2(\nu_{1,2}) \neq 0 \), then

\[i^*w(TC_m) = w(TC_1)w((m-1)\nu_{1,2}) = (1 + u_2)(1 + u_2 + u_4)^{m-1}.
\]

The degree two part of \((1 + u_2)(1 + u_2 + u_4)^{m-1}\) is \( mu_2 \) so if \( m \) is odd, \( i^*w_2(TC_m) \neq 0 \), so \( w_2(TC_m) \neq 0 \). But, when \( m \) is even, the degree
six part of $(1 + u_2)(1 + u_2 + u_4)^{m-1}$ is $(m - 1 + (m - 1)(m - 2))u_2u_4$
which has coefficient $(m - 1)(1 + m - 2) = (m - 1)^2$. If $m$ is even, this is odd, so non-zero mod 2. In particular, $i * w_6(TC_m) \neq 0$ in this case.

Thus, no matter the value of $w_2(\nu_1, 2)$, either $w_2(TC_m) \neq 0$ or $w_6(TC_m) \neq 0$.

We now turn attention to the final case $B = N^m$. We are, unfortunately, unable to obtain a full classification. We begin with a theorem of Sullivan [32, Theorem 13.1] (see also [36, Proposition 2.3(d)]).

**Theorem 4.7.** (Sullivan) Any class of simply connected manifolds of dimension at least five with isomorphic cohomology rings, the same Pontrjagin classes, and whose minimal models are formal contains at most finitely many diffeomorphism types.

From the proof of Theorem 1.1 and the discussion leading up to it, we have already shown that rationally 4-periodic biquotients are formal. Further, at the start of Section 4.2, we showed that any biquotient of the form $M = N^m \times S^1 S^3$ has cohomology ring isomorphic to that of $N^m \times S^2$. Thus, in order to apply this theorem, we need only show all such biquotients have the same Pontrjagin classes.

**Proposition 4.8.** For each odd $m$, the Pontrjagin classes of $M = N^m \times S^1 S^3$ are independent of the choice of $S^1$ action on $N^m$.

**Proof.** Recall that we have a bundle $N^m \to M \xrightarrow{\pi} S^2$. Over a chart $\mathbb{R}^2 \cong U \subseteq S^2$, the bundle trivializes, so $\pi^{-1}(U) \cong N^m \times \mathbb{R}^2$. In particular, the normal bundle of $N^m$ in $M$ is trivial.

Now, let $i : N^m \to M$ be the inclusion of the fiber. From the discussion at the beginning of this section, we know that $i^*$ is an isomorphism on cohomology groups in degree a multiple of 4. Since the normal bundle is trivial, it follows that $i^*$ identifies the Pontryagin classes of $N^m$ with those of $M$.

This completes the proof of Theorem 1.3.

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