POSITIVITY PROPERTIES OF JACOBI-STIRLING NUMBERS
AND GENERALIZED RAMANUJAN POLYNOMIALS

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Abstract. Generalizing recent results of Egge and Mongelli, we show that each diagonal sequence of the Jacobi-Stirling numbers $J_c(n, k; z)$ and $JS(n, k; z)$ is a Pólya frequency sequence if and only if $z \in [-1, 1]$ and study the $z$-total positivity properties of these numbers. Moreover, the polynomial sequences

$$\left\{ \sum_{k=0}^{n} JS(n, k; z)y^k \right\}_{n \geq 0}, \quad \text{and} \quad \left\{ \sum_{k=0}^{n} Jc(n, k; z)y^k \right\}_{n \geq 0},$$

are proved to be strongly $\{z, y\}$-log-convex. In the same vein, we extend a recent result of Chen et al. about the Ramanujan polynomials to Chapoton’s generalized Ramanujan polynomials. Finally, bridging the Ramanujan polynomials and a sequence arising from the Lambert $W$ function, we obtain a neat proof of the unimodality of the latter sequence, which was proved previously by Kalugin and Jeffrey.

1. Introduction

The *Jacobi-Stirling numbers* of the first kind $Jc(n, k; z)$ and of the second kind $JS(n, k; z)$ ($n \geq k \geq 0$) are defined by the recurrence relations:

$$Jc(n, k; z) = Jc(n - 1, k - 1; z) + (n - 1)(n - 1 + z) Jc(n - 1, k; z), \quad (1.1)$$
$$JS(n, k; z) = JS(n - 1, k - 1; z) + k(k + z) JS(n - 1, k; z), \quad (1.2)$$

with the boundary conditions $JS(0, 0; z) = Jc(0, 0; z) = 1$ and $JS(j, 0; z) = JS(0, j; z) = Jc(j, 0; z) = Jc(0, j; z) = 0$ for $j \geq 1$. The first values of these two sequences are given in Tables 1 and 2. When $z = 1$, the two kinds of Jacobi-Stirling numbers are called the (unsigned) *Legendre-Stirling numbers* of the first and second kinds [2,3].

Recently, these numbers have attracted the attention of several authors [1–3,6–8,13,14]. In particular, a result of Egge [6, Theorem 5.1] implies that the diagonal sequences

$$\{JS(k + n, n; 1)\}_{n \geq 0} \quad \text{and} \quad \{Jc(k + n, n; 1)\}_{n \geq k}$$

are Pólya frequency sequences for any fixed $k \in \mathbb{N}$, while Mongelli [13] studied total positivity properties of Jacobi-Stirling numbers assuming that $z$ is a real number.
It is convenient to recall some necessary definitions. A sequence of nonnegative real numbers \( \{a_n\}_{n \geq 0} \) is \textit{unimodal} if \( a_0 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots \) for some \( m \), and is \textit{log-concave} (resp. \textit{log-convex}) if \( a_i^2 \geq a_{i-1}a_{i+1} \) (resp. \( a_i^2 \leq a_{i-1}a_{i+1} \)) for all \( i \geq 1 \). A real sequence \( \{a_n\}_{n \geq 0} \) is called a \textit{Pólya frequency sequence} (PF sequence for short) if the matrix \( M := (a_{j-i})_{i,j \geq 0} \) (where \( a_k = 0 \) if \( k < 0 \)) is \textit{totally positive} (TP for short), that is, every minor of \( M \) is nonnegative. Unimodal, log-concave and Pólya frequency sequences arise often in combinatorics \([4]\).

The following is our result about diagonal sequences of Jacobi-Stirling numbers.

\textbf{Theorem 1.1.} For any fixed integer \( k \geq 1 \), the two sequences \( \{JS(k+n,n;z)\}_{n \geq 0} \) and \( \{Jc(k+n,n;z)\}_{n \geq 0} \) are \textit{Pólya frequency sequences} if and only if \( -1 \leq z \leq 1 \).

For a sequence of polynomials in \( x = \{x_1, x_2, \ldots, x_n\} \), one can define the \( x \)-analog of log-concavity, log-convexity, total positivity and Pólya frequency sequence as follows (see \([5,12,18]\)). Let \( \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\} \). Given two polynomials \( f(x), g(x) \in \mathbb{R}_+[x] \), we define

\[
f(x) \leq_x g(x) \quad \text{if and only if} \quad g(x) - f(x) \in \mathbb{R}_+[x].
\]

A sequence of polynomials \( \{f_k(x)\}_{k \geq 0} \) in \( \mathbb{R}_+[x] \) is called \textit{\( x \)-log-concave} if

\[
f_{k-1}(x)f_{k+1}(x) \leq_x f_k(x)^2 \quad \text{for all } k \geq 1,
\]

and it is \textit{strongly \( x \)-log-concave} if

\[
f_{k-1}(x)f_{l+1}(x) \leq_x f_k(x)f_l(x) \quad \text{for all } l \geq k \geq 1.
\]

The \textit{\( x \)-log-convexity} and \textit{strong \( x \)-log-convexity} are defined similarly.

\textbf{Remark 1.} For a sequence of real numbers \( \{a_n\}_{n \geq 0} \), the log-concavity is equivalent to the strong log-concavity, that is, \( a_{k-1}a_{l+1} \leq a_ka_l \) for all \( l \geq k \geq 1 \). But, for polynomial sequences, the \( x \)-log-concavity is not equivalent to strong \( x \)-log-concavity (see \([17]\)), which is the same for \( x \)-log-convexity and strong \( x \)-log-convexity (see \([7]\)).

A matrix \( F = (f_{ij})_{i,j \in \mathbb{N}} \), where \( f_{ij} \in \mathbb{R}_+[x] \), is called \textit{\( x \)-totally positive} if every minor of \( F \) is nonnegative with respect to \( \geq_x \). The \textit{\( x \)-Pólya frequency sequence} is defined similarly. Note that if a sequence \( \{f_k(x)\}_{k \geq 0} \) is a \( x \)-PF sequence, then it is strongly \( x \)-log-concave,

\begin{table}[h]
\centering
\begin{tabular}{|c|cccccc|}
\hline
\( k \backslash n \) & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & \( z + 1 \) & \((z + 1)^2\) & \((z + 1)^3\) & \((z + 1)^4\) & \((z + 1)^5\) & \((z + 1)^6\) \\
2 & 1 & 5 + 3z & 21 + 24z + 7z^2 & 85 + 141z + 79z^2 + 15z^3 & 341 + 738z + 604z^2 + 222z^3 + 31z^4 & 1408 + 1662z + 664z^2 + 90z^3 \\
3 & 1 & 14 + 6z & 147 + 120z + 25z^2 & 1408 + 1662z + 664z^2 + 90z^3 & 627 + 400z + 65z^2 & 55 + 15z \\
4 & 1 & 30 + 10z & 55 + 15z & 1 & 627 + 400z + 65z^2 & 55 + 15z \\
5 & 1 & 55 + 15z & 1 & 627 + 400z + 65z^2 & 55 + 15z & 1 \\
6 & 1 & 55 + 15z & 1 & 627 + 400z + 65z^2 & 55 + 15z & 1 \\
\hline
\end{tabular}
\caption{The first values of \( JS(n,k;z) \)}
\end{table}
related to Ramanujan and Lambert. It is well known that Lambert’s equation has an explicit solution \( w \) of rooted trees on \( n \) vertices. It is also known (see [9, 21]) that the \( n \)-th derivation (with respect to \( y \)) of Lambert’s function has the following formula

\[
    \frac{d^n}{dy^n} \left( \frac{1}{w} \right) = \frac{e^{nw}}{(1-w)^n} R_n \left( \frac{1}{1-w} \right),
\]

where \( R_n(y) \) are the so-called Ramanujan polynomials defined by the recurrence relation

\[
    R_1(y) = 1, \quad R_{n+1}(y) = n(1+y)R_n(y) + y^2 R'_n(y). \tag{1.3}
\]

### Table 2. The first values of \( Jc(n, k; z) \)

| \( k \backslash n \) | 1   | 2   | 3   | 4   | 5   |
|---------------------|-----|-----|-----|-----|-----|
| 1                   | 1   | \( z+1 \) | \( 2z^2 + 6z + 4 \) | \( 6z^3 + 36z^2 + 66z + 36 \) | \( 24z^4 + 240z^3 + 840z^2 + 1200z + 576 \) |
| 2                   | 1   | 3z + 5 | \( 11z^2 + 48z + 49 \) | 50z^3 + 404z^2 + 1030z + 820 | \( z^2 \) |
| 3                   | 1   | \( 6z + 14 \) | \( 35z^2 + 200z + 273 \) | 10z + 30 | \( 1 \) |
| 4                   |      |       |       |       |      |
| 5                   |      |       |       |       |      |
The first values of the polynomials $R_n$ are
\[ R_2(y) = 1 + y, \quad R_3(y) = 2 + 4y + 3y^2, \quad R_4(y) = 6 + 18y + 25y^2 + 15y^3. \]
It is clear that $R_n(y)$ is a polynomial in $y$ of degree $n-1$ with positive integral coefficients such that $R_n(0) = (n-1)!$, $R_n(1) = n^{n-1}$ and the coefficient of $y^{n-1}$ is $(2n-3)!!$. Actually all the coefficients of $R_n(y)$ have nice combinatorial interpretation on trees [21].

As we will show, the Ramanujan polynomials can be used to give a new proof of a recent unimodal result of Kalugin and Jeffrey [11]. Chapoton (see [9]) introduced the generalized Ramanujan polynomials $Q_n(x, y, z, t)$ defined by
\[ Q_1 = 1, \quad Q_{n+1} = [x + nz + (y + t)(n + y\partial_y)]Q_n. \quad (1.4) \]
For example, we have $Q_2(x, y, z, t) = x + y + z + t$, and
\[ Q_3(x, y, z, t) = x^2 + 3xy + 3xz + 3xt + 3y^2 + 4yz + 5yt + 2z^2 + 4zt + 2t^2. \]
Clearly, comparing (1.3) with (1.4) we have
\[ R_n(y) = Q_n(0, y, 1, 0). \quad (1.5) \]

Combinatorial interpretations of $Q_n$ in terms of plane trees and forests are given in [9] as well as some other remarkable properties. Motivated by the recent result of Chen et al. [5] about $\{R_n(y)\}_{n \geq 1}$, we shall prove the $x$-log-convexity of the polynomials $Q_n$.

**Theorem 1.5.** The sequence $\{Q_n(x, y, z, t)\}_{n \geq 1}$ is strongly $x$-log-convex, that is, for any $n \geq m \geq 2$,
\[ Q_{m-1}(x, y, z, t)Q_{n+1}(x, y, z, t) - Q_m(x, y, z, t)Q_n(x, y, z, t) \in \mathbb{N}[x, y, z, t]. \]

**Remark 2.** Setting $x = 0, z = 1$ and $t = 0$ we recover Chen et al.’s result about strong $y$-log-convexity of $R_n(y)$ [5].

This paper is organized as follows. In section 2 we study the PF property of diagonal Jacobi-Stirling numbers and give a proof of Theorem 1.1 with the parameter $z$ being a real number. In section 3 we investigate the $z$-total positivity of Jacobi-Stirling numbers. In section 4 we study the strong $x$-log-convexity of the generating functions of Jacobi-Stirling numbers and generalized Ramanujan polynomials. In section 5 we show that the unimodality of a sequence arising from Lambert $W$ function first proved by Kalugin and Jeffrey [11] follows easily from the log-concavity of the coefficients of Ramanujan polynomials.

**2. PF properties of diagonal Jacobi-Stirling numbers**

Our main tool is the following result, due to Brenti [4, Theorem 4.5.3], characterizing the rational formal power series whose coefficients are PF sequence.

**Lemma 2.1.** Let $\sum_{n \geq 0} a_n x^n = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are two relatively prime polynomials. Then $\{a_n\}_{n \geq 0}$ is a PF sequence if and only if
\begin{itemize}
  \item[(1)] $a_n \geq 0$ for all $n \geq 0$,
\end{itemize}
(2) \( P(x) \) has only real nonpositive zeros,
(3) \( Q(x) \) has only real positive zeros.

We start with some preliminary results about the generating function of the diagonal sequence of the Jacobi-Stirling numbers:

\[
F_k(x; z) = \sum_{n \geq 0} \text{JS}(k + n, n; z)x^n, \quad k \geq 0.
\]

**Lemma 2.2.** For any fixed \( z \in \mathbb{R} \setminus \{1\} \) and \( k \geq 0 \), there exists a polynomial \( A_k(x; z) \) in \( x \) of degree 2k such that

\[ F_k(x; z) = \frac{A_k(x; z)}{(1 - x)^{3k+1}} \]

and \( A_k(1; z) \neq 0 \).

**Proof.** For \( n \geq 0 \), let \( f_k(n; z) = \text{JS}(k + n, n; z) \). Then recurrence (1.2) can be written as

\[ f_k(n; z) - f_k(n - 1; z) = n(n + z)f_{k-1}(n; z) \quad (k \geq 0) \]

with \( f_0(n; z) = 1 \) and \( f_{-1}(n; z) = 0 \). We prove by induction on \( k \) that \( f_k(n; z) \) is a polynomial in \( n \) of degree 3k if \( z \neq 1 \). This is clear for \( k = 0 \). Suppose \( k \geq 1 \). By induction hypothesis the right-hand side of (2.2) is a polynomial in \( n \) of degree 3k – 1. Since the left-hand side of (2.2) is the difference of \( f_k(n; z) \), then \( f_k(n; z) \) is a polynomial in \( n \) of degree 3k. By a standard result about the generating functions of polynomials sequences (cf. [19 Corollary 4.3.1]) there exists a polynomial \( A_k(x; z) \) in \( x \) of degree \( \leq 3k \) satisfying (2.1) and \( A_k(1; z) \neq 0 \). By [19 Proposition 4.2.3], we have

\[ \sum_{n \geq 1} f_k(-n; z)x^n = -F_k(1/x; z) = -\frac{x^{3k+1}A_k(1/x; z)}{(x - 1)^{3k+1}}. \]

For \( k \geq 1 \) it is clear that the degree of \( A_k(x; z) \) must be 2k provided that

\[ f_k(0; z) = f_k(-1; z) = \ldots = f_k(-k; z) = 0 \quad \text{and} \quad f_k(-k - 1; z) \neq 0. \]  

(2.3)

We verify (2.3) by induction on \( k \geq 1 \). First, from (2.2) we derive that

\[ f_1(n; z) = \frac{n(n + 1)}{2} \left( \frac{2n + 1}{3} + z \right). \]

Hence \( f_1(0, z) = f_1(-1, z) = 0 \) and \( f_1(-2, z) = z - 1 \neq 0 \). Assume that \( k \geq 2 \) and (2.3) holds for \( k - 1 \), i.e., \( f_{k-1}(n; z) = 0 \) for \( 0 \geq n \geq -k + 1 \) and \( f_{k-1}(-k; z) \neq 0 \). By definition \( f_k(0; z) = \text{JS}(k, 0; z) = 0 \), hence we can derive (2.3) from (2.2) and the induction hypothesis. \( \square \)

By Lemma 2.2 we can write \( A_k(x; z) \) in (2.1) as

\[ A_k(x; z) = \sum_{i=1}^{2k} a_{k,i}(z)x^i. \]  

(2.4)
Proposition 2.1. The coefficients $a_{k,i}(z)$ in (2.4) satisfy the following recurrence

$$a_{k,i}(z) = i(i + z)a_{k-1,i}(z) + [2i(3k - i - 1) - (1 - z)(3k - 2i)]a_{k-1,i-1}(z) + (3k - i)(3k - i - z)a_{k-1,i-2}(z),$$

(2.5)

with $a_{0,i}(z) = \delta_{0,i}$. Thus, when $-1 < z < 1$, the coefficients $a_{k,i}(z)$ are nonnegative for $k \geq 1$ and $1 \leq i \leq 2k$.

Proof. For $k \geq 1$, by (2.2), we have

$$F_k(x; z) = \sum_{n \geq 1}(f_k(n - 1; z) + n(n + z)f_k(n; z))x^n = xF_k(x; z) + xD(x^{1-z}D(x^zF_k-1(x; z))) = \frac{x}{1-x}D(x^{1-z}D(x^zF_k-1(x; z))),$$

(2.6)

where $D = \frac{d}{dx}$ and $F_0(x; z) = (1-x)^{-1}$. Substituting (2.1) into (2.6) we obtain

$$(1-x)^{-3k-1}\sum_{i=1}^{2k} a_{k,i}(z)x^i = x(1-x)^{-1}D[x^{1-z}D[(1-x)^{-3k+2}\sum_{i=1}^{2k-2} a_{k-1,i}(z)x^{i+z}]],$$

which is simplified to

$$\sum_{i=1}^{2k} a_{k,i}(z)x^i = (3k-1)(3k-2)\sum_{i=1}^{2k-2} a_{k-1,i}(z)x^{i+2} + (1-x)^2\sum_{i=1}^{2k-2} i(i + z)a_{k-1,i}(z)x^i$$

$$+ (3k - 2)(1-x)\sum_{i=1}^{2k-2} (2i + 1 + z)a_{k-1,i}(z)x^{i+1}.$$  

Taking the coefficient of $x^i$ in both sides of the above equation, we get (2.5).

For $1 \leq i \leq 2k$ and $-1 < z < 1$ it is easy to verify that

$$i(i + z) \geq 0, \quad (3k - i)(3k - i - z) \geq 0, \quad 2i(3k - i - 1) - (1 - z)(3k - 2i) \geq 0.$$  

Hence, by (2.5), the coefficients $a_{k,i}(z)$ are nonnegative for $1 \leq i \leq 2k$. This finishes the proof of the lemma. \hfill \Box

Lemma 2.3. For $-1 \leq z \leq 1$, the zeros of the polynomial $A_k(x; z)$ in (2.1) are distinct, real and nonpositive numbers.

Proof. For $z = 1$ or $-1$, the Jacobi-Stirling numbers become the Legendre-Stirling numbers, and for these two special cases the lemma was proved in [6, Theorem 5.1]. It remains to prove the lemma for $-1 < z < 1$.

For any fixed $k \geq 1$, consider the polynomial

$$B_k(x; z) = (1-x)^{3k+2}x^{1-z}D(x^z(1-x)^{-1-3k}A_k(x; z)).$$

(2.7)

By Lemma [2.2], the polynomial $A_k(x; z)$ is of degree $2k$, it is not hard to see that $B_k(x; z)$ is a polynomial of degree $2k + 1$. Moreover, by (2.3), we have $A_k(0; z) = 0$, it follows from
that $B_k(0; z) = 0$. Next we show that the nonzero roots of $A_k(x; z)$ are distinct, real and nonpositive by showing that they are intertwined with the zeros of $B_k(x; z)$. We proceed by induction on $k \geq 1$. For $k = 1$, we have

$$A_1(x; z) = (1 + z)x + (1 - z)x^2.$$  \hfill (2.8)

Hence the two roots of $A_1(x; z)$ are $x_1 = 0$ and $x_2 = \frac{z + 1}{z - 1}$, which is negative if $z \in (-1, 1)$.

Now suppose that $k \geq 2$ and the zeros of $A_{k-1}(x; z)$ are distinct nonpositive real numbers. By Rolle’s Theorem and relation (2.7), the polynomial $B_{k-1}(x; z)$ has a root strictly between each pair of consecutive roots of $A_{k-1}(x; z)$; including 0, this accounts for $2k - 2$ of the $2k - 1$ roots of $B_{k-1}(x; z)$. To find the missing root, let $\alpha$ denote the leftmost root of $A_{k-1}(x; z)$; by (2.7) we have $B_{k-1}(\alpha; z) = \alpha(1 - \alpha) \frac{d}{dx} A_{k-1}(\alpha; z)$. Since the degree of $A_{k-1}(x; z)$ is even and its leading coefficient is positive we have $\lim_{x \to -\infty} A_{k-1}(x; z) = +\infty$. Now since the roots of $A_{k-1}(x; z)$ are distinct we find $\frac{d}{dx} A_{k-1}(\alpha; z) < 0$; hence $B_{k-1}(\alpha; z) > 0$. But the degree of $B_{k-1}(x; z)$ is odd and its leading coefficient is positive by (2.7), so $\lim_{x \to -\infty} B_{k-1}(x; z) = -\infty$, and therefore $B_{k-1}(x; z)$ has a root at the left of $\alpha$. It follows that $B_{k-1}(x; z)$ has $2k - 1$ distinct, real, nonpositive roots.

For example, if $k = 2$ then $k - 1 = 1$ and we find

$$B_1(x; z) = ((1 + z)^2x + (1 - z)(2 + z)x^2)(1 - x) - 4x^2(1 + z + (1 - z)x)$$

with $(z - 1)(6 + z)$ as the leading coefficient. So $\lim_{x \to -\infty} B_1(x; z) = -\infty$. As $B_1(x_2; z) = \frac{2(1+z)^2}{(z-1)^2} > 0$, there must be a root of $B_1(x; z)$ at the left of $x_2$.

From (2.1) and (2.7) we deduce that (2.6) is equivalent to

$$A_k(x; z) = x(1 - x)^{3k} D((1 - x)^{1 - 3k} B_{k-1}(x; z)).$$  \hfill (2.9)

Using (2.9) and the properties of zeros of $B_{k-1}(x; z)$ we can prove similarly that $A_k(x; z)$ has $2k$ distinct, real, nonpositive roots. The proof is thus complete. \hfill □

**Remark 3.** The constant term of $JS(n, k; z)$ (resp. $Jc(n, k; z)$) are the *central factorial numbers* of the second kind $T(2n, 2k)$ (resp. the first kind $t(2n, 2k)$) (see [16] pp. 213–217 and [7]), that is,

$$T(2n, 2k) = JS(n, k; 0), \quad t(2n, 2k) = Jc(n, k; 0).$$

Since $A_k(x; 0)$ can be seen as the descent polynomial of some generalized Stirling permutations (see the end of [8]), it follows from a result of Brenti [4] Theorem 6.6.3] that $A_k(x; 0)$ has only real nonnegative roots.

**Lemma 2.4.** Let $G_k(x; z) = \sum_{n \geq k} Jc(n, n - k; z)x^n$. Then

$$G_k(x; z) = (-1)^{k+1} F_k(1/x, -z).$$

**Proof.** Let $g_k(n) = Jc(n, n - k; z)$. Then by recursive formulas (1.1), for $k \geq 0$, we have

$$g_k(n) = g_k(n - 1) + (n - 1)(n - 1 + z)g_{k-1}(n - 1; z).$$
Comparing this with (2.2) we get
\[ g_k(n; z) = (-1)^k f_k(-n; -z). \]

The result follows from this relation and the standard results of generating functions (cf. [19, Proposition 4.2.3]). □

**Proof of Theorem 1.1.** By Lemma 2.4, we only need to prove the theorem for the sequence \( \{JS(n + k, n; z)\}_{n \geq 0} \). When \(-1 \leq z \leq 1\), it follows from Lemmas 2.3 and 2.1 that the sequence \( \{JS(n + k, n; z)\}_{n \geq 0} \) is a PF sequence. This proves the “if” side of the theorem.

It remains to show the “only if” side. When \( z > 1 \), by (2.8), the polynomial \( A_1(x; z) \) has a positive root \( \frac{z+1}{z-1} \). Thus, by Rolle’s Theorem and relationship (2.7), the polynomial \( B_1(x; z) \) has a positive root, and so does \( A_2(x; z) \) by relationship (2.9). It follows by induction on \( k \) and the two relationships (2.7) and (2.9) that \( A_k(x; z) \) has a positive root for any integer \( k \geq 1 \). The “only if” side of the theorem then follows from Lemmas 2.1 and 2.2. □

**Corollary 2.1.** The two sequences \( \{T(2(n + k), 2n)\}_{n \geq 0} \) and \( \{t(2(n + k), 2n)\}_{n \geq 0} \) are PF sequences.

### 3. \( z \)-total positivity of Jacobi-Stirling numbers

In this section, we show that some \( z \)-total positivity properties of Jacobi-Stirling numbers follow directly from the \( x \)-total positivity properties of the elementary and complete homogeneous symmetric functions. We begin with the observation that, similar to the classical Stirling numbers, the Jacobi-Stirling numbers are also specializations of the two symmetric functions.

The *elementary* and *complete homogeneous symmetric functions* of degree \( k \) in variables \( x_1, x_2, \ldots, x_n \) are defined by

\[
\begin{align*}
e_k(n) &:= e_k(x_1, x_2, \ldots, x_n) = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1} x_{i_2} \ldots x_{i_k}, \\
h_k(n) &:= h_k(x_1, x_2, \ldots, x_n) = \sum_{i_1 \leq i_2 \leq \ldots \leq i_k} x_{i_1} x_{i_2} \ldots x_{i_k},
\end{align*}
\]

where \( e_0(n) = k_0(n) = 1 \) and \( e_k(n) = 0 \) for \( k > n \). It is easy to deduce from the definition of \( e_k(n) \) and \( h_k(n) \) that

\[
\begin{align*}
e_k(n) &= e_k(n-1) + x_n e_{k-1}(n-1), \\
h_k(n) &= h_k(n-1) + x_n h_{k-1}(n).
\end{align*}
\]
As noticed by Mongelli [14], comparing with (1.1) and (1.2) one gets immediately the following identities: for $n \geq k \geq 0$,
\begin{align*}
J_c(n, k; z) &= e_{n-k}(1 + z), 2(2 + z), \ldots, (n - 1)(n - 1 + z), \quad (3.1) \\
J_S(n, k; z) &= h_{n-k}(1 + z), 2(2 + z), \ldots, k(k + z)). \quad (3.2)
\end{align*}

The following result is due to Sagan [18, Theorem 4.4].

**Lemma 3.1.** Let $\{x_i\}_{i \geq 1}$ be a sequence of polynomials in $q$ with nonnegative coefficients. Then, for $k \leq l$ and $m \leq n$,
\begin{itemize}
\item[(i)] $e_{k-1}(n)e_{l+1}(m) \leq q e_k(n)e_l(m)$;
\item[(ii)] $h_{k-1}(n)h_{l+1}(m) \leq q h_k(n)h_l(m)$.
\end{itemize}
Moreover, if the sequence $\{x_i\}_{i \geq 1}$ is strongly $q$-log-concave, then
\begin{itemize}
\item[(iii)] $e_k(n+1)e_l(m-1) \leq q e_k(n)e_l(m)$;
\item[(iv)] $h_k(n+1)h_l(m-1) \leq q h_k(n)h_l(m)$.
\end{itemize}

**Proof of Theorem 1.2 (i).** By Lemma 3.1 (ii) and (iv), if $x_i \in \mathbb{R}_+[z]$ and the sequence $\{x_i\}_{i \geq 1}$ is strongly $z$-log-concave, then
\[ h_k(n)h_l(m) \geq q h_{k-1}(n+1)h_{l+1}(m-1) \quad (3.3) \]
for $k \leq l$ and $m \leq n$. As the sequence $\{i(i-1+z)\}_{i \geq 1}$ is strongly $z$-log-concave, namely,
\[ k(k-1+z)l(l-1+z) - (k-1)(k-2+z)(l+1)(l+z) \in \mathbb{N}[z] \]
for $k \geq 1$ and $k \leq l$, it follows from the specialization (3.2) and (3.3) that
\[ J_S(n+k, n; z-1)J_S(m+l, m; z-1) \geq q J_S(n+k, n+1; z-1)J_S(m+l, m-1; z-1) \]
for $k \leq l$ and $m \leq n$, which implies the strong $z$-log-concavity of the sequence $\{J_S(n, k; z-1)\}_{k \geq 1}$.

**Remark 4.** Theorem 1.2 (i) generalizes the following log-concavity result of Andrews et al. [11] and Mongelli [13]: the sequence $\{J_S(n, k; z-1)\}_{k \geq 1}^n$ is log-concave when $z \geq 0$ is a real number. They both proved the above result by showing that the polynomial
\[ f_n(x) = \sum_{k \geq 0} J_S(n, k; z-1)x^k \]
has only real simple nonpositive zeros. Note that $\{a_i\}_{i=0}^d$ is a PF sequence if and only if the polynomial $\sum_{i=0}^d a_ix^i$ has only real zeros. As the later result implies also that $\{J_S(n, k; z-1)\}_{k \geq 1}^n$ is a PF sequence, it would be interesting to see whether $\{J_S(n, k; z-1)\}_{k \geq 1}^n$ is a $z$-PF sequence or not.

The following theorem was mentioned in [18] without proof. For convenience we include a proof.

**Lemma 3.2.** For any $n \geq 1$, the two sequences $\{h_k(n)\}_{k \geq 0}$ and $\{e_k(n)\}_{k \geq 0}$ are $x$-PF sequences.
Proof. Choose any minor $M$ of the matrix $(h_{j-i}(n))_{i,j\in\mathbb{N}}$, say with rows $i_1,\ldots,i_d$ and columns $j_1,\ldots,j_d$. Define partitions $\lambda$ and $\mu$ by
\[
\lambda_k = j_d - i_k - d + k \quad \text{and} \quad \mu_k = j_d - j_k - d + k
\]
for $1 \leq k \leq d$. Then
\[
M = (h_{\lambda_k-\mu_k-k+i}(n))_{k,i=1}^d.
\]
If $\lambda_k \geq \mu_k$ for $1 \leq k \leq d$, then by the Jacobi-Trudi identity \cite{20} §7.16, we have
\[
\det(M) = s_{\lambda/\mu}(x_1,\ldots,x_n),
\]
where $s_{\lambda/\mu}$ is a polynomial in $x_1,\ldots,x_n$ with nonnegative coefficients. Otherwise, suppose $r$ is the smallest index such that $\lambda_r < \mu_r$, then $\det(M) = 0$ follows from the observation that $\lambda_k < \mu_k$ for all $k \geq r$ and $l \leq r$. This completes the proof. The proof for $\{e_k(n)\}_{k \geq 0}$ is similar, but using the dual Jacobi-Trudi identity \cite{20} §7.16.

**Lemma 3.3.** A finite sequence $\{f_0(x), f_1(x), \ldots, f_d(x)\}$ is an $x$-PF sequence if and only if $\{f_d(x), f_{d-1}(x), \ldots, f_0(x)\}$ is an $x$-PF sequence.

**Proof.** By definition, a sequence $\{f_0(x), f_1(x), \ldots, f_d(x)\}$ is an $x$-PF sequence if all the minors of the matrix $(f_{j-i})_{1 \leq i,j \leq n}$ are $x$-nonnegative. The result follows then from the fact that a matrix is $x$-totally positive if and only if its transpose is $x$-totally positive.

**Proof of Theorem 1.2** (ii) and (iii). (ii) This follows immediately from the fact that $\{h_k(n)\}_{k \geq 0}$ is an $x$-PF sequence (Lemma 3.2) and the specialization (3.2).

(iii) From the fact that $\{e_k(n)\}_{k \geq 0}$ is an $x$-PF sequence (Lemma 3.2) and (3.1), we see that $\{Jc(n,n-k;z-1)\}_{k \geq 0}$ is a $z$-Pólya frequency sequence for any fixed $n \in \mathbb{N}$. The result then follows from Lemma 3.3.

**Proof of Theorem 1.3** Fix $r, l, m \in \mathbb{N}$. It is well known \cite{18} Theorem 5.4] that the following matrices
\[
(e_{j-r}(li+m))_{i,j\in\mathbb{N}} \quad \text{and} \quad (h_{j-r}(li+m))_{i,j\in\mathbb{N}}
\]
are $x$-totally positive. The $z$-total positivity of the matrices
\[
(JS((l-r)i + j + m, li + m; z-1))_{i,j \geq 0}, \quad (Jc(li + m, (r+l)i - j + m; z-1))_{i,j \geq 0}
\]
follows immediately from (3.2) and (3.1). This implies that the matrices $(JS(n,k;z-1))_{n,k \geq 0}$, $(Jc(n,n-k;z-1))_{n,k \geq 0}$ are $z$-totally positive.

It is known \cite{7} that the Jacobi-Stirling numbers are the connection coefficients of the bases $\{x^n\}_n$ and $\{\prod_{i=0}^{n-1}(x-i(z+i))\}_n$, namely,
\[
x^n = \sum_{k=0}^n JS(n,k;z) \prod_{i=0}^{k-1}(x-i(z+i)),
\]
and
\[
\prod_{i=0}^{n-1}(x-i(z+i)) = \sum_{k=0}^n (-1)^{n+k} Jc(n,k;z)x^k. \tag{3.4}
\]
It follows that the inverse of the matrix \((JS(n, k; z-1))_{n,k\geq 0}\) is \((Jc(n, k; z-1))_{n,k\geq 0}\), up to deletion of signs. As the inverse of a totally positive matrix (with polynomial entries), up to deletion of signs in all entries, is also totally positive (cf. [15, Proposition 1.6]), the matrix \((Jc(n, k; z-1))_{n,k\geq 0}\) is then \(z\)-totally positive. □

**Remark 5.** Theorem 1.2 (ii), (iii) and Theorem 1.3 are \(z\)-analog of [13, Theorem 5, Propositions 2 and 3].

### 4. Strongly \(x\)-log-convex polynomial sequences

In this section, we investigate the log-convexity property of the polynomials

\[
\sum_{k=0}^{n} JS(n, k; z)y^k, \quad \sum_{k=0}^{n} Jc(n, k; z)y^k, \quad \sum_{k=0}^{n} Q_{n,k}(x, t)y^k.
\]

We first establish a general result.

**Lemma 4.1.** For positive integers \(n\) and \(k\) we define polynomials \(T_{n,k}\) in \(\mathbb{R}_+[x]\) by

\[
T_{n,k} = a_{n,k} T_{n-1,k} + b_{n,k} T_{n-1,k-1}, \quad \text{for } 1 \leq k \leq n, \quad (4.1)
\]

and the boundary conditions \(T_{0,0} = 1\) and \(T_{n,-1} = T_{n,n+1} = 0\) for \(n \geq 1\).

(i) If the sequence \(\{T_{n,k}\}_{k=0}^{n}\) is strongly \(x\)-log-concave for each \(n\) and

\[
a_{n,k} \geq_x a_{n,k-1} \geq_x 0, \quad b_{n,k} \geq_x b_{n,k-1} \geq_x 0 \quad \text{for } 1 \leq k \leq n, \quad (4.2)
\]

then

\[
T_{m,k} T_{n,l} \geq_x T_{m,l} T_{n,k}
\]

for \(0 \leq m \leq n\) and \(0 \leq k \leq l\).

(ii) Moreover, for fixed \(j \geq 0\) and \(n \geq m \geq 0\), if \(c_{j,i} \geq_x c_{j,i-1}\) for \(i \geq 1\), then

\[
\sum_{i=0}^{j} \left(c_{j,j-i} - c_{j,i}\right) T_{n,j-i} T_{m,i} \geq_x 0.
\]

Here \(a_{n,k}, b_{n,k}\) and \(c_{n,k}\) are polynomials in \(\mathbb{R}[x]\).

**Proof.** Note that (i) implies (ii) because

\[
\sum_{i=0}^{j} \left(c_{j,j-i} - c_{j,i}\right) T_{n,j-i} T_{m,i} = \sum_{i=0}^{\left\lceil \frac{j}{2} \right\rceil} \left(c_{j,j-i} - c_{j,i}\right) \left(T_{n,j-i} T_{m,i} - T_{n,i} T_{m,j-i}\right).
\]

So we just need to prove (i).
When \( n = m \) or \( k = l \), there is nothing to prove. So we suppose that \( n > m \) and \( l > k \) and proceed by induction on \( n \). From recurrence relation (4.1), we see that
\[
T_{m,k}T_{n+1,l} - T_{m,l}T_{n+1,k} = T_{m,k}(a_{n+1,l}T_{n,l} + b_{n+1,l}T_{n,l-1}) - T_{m,l}(a_{n+1,k}T_{n,k} + b_{n+1,k}T_{n,k-1}) \\
\geq x a_{n+1,l}(T_{m,k}T_{n,l} - T_{m,l}T_{n,k}) + b_{n+1,l}(T_{m,k}T_{n,l-1} - T_{m,l}T_{n,k-1}) \quad \text{(by (4.2))}
\]
\[
= a_{n+1,l}(T_{m,k}T_{n,l} - T_{m,l}T_{n,k}) \\
+ b_{n+1,l}[(T_{m,k}T_{n,l-1} - T_{m,l-1}T_{n,k}) + (T_{m,l-1}T_{n,k} - T_{m,l}T_{n,k-1})],
\]
which is in \( \mathbb{R}_+[x] \) by the induction hypothesis provided that, for \( 1 \leq m \leq n \) and \( 1 \leq k \leq l \),
\[
T_{n,k}T_{m,l} - T_{n,k-1}T_{m,l+1} \geq x 0. \quad (4.3)
\]
It remains to prove (4.3). We proceed by induction on \( n \). As the sequence \( \{T_{n,k}\}_{k=0}^n \) is strongly \( x \)-log-concave, by definition,
\[
T_{n,k}T_{n,l} - T_{n,k-1}T_{n,l+1} \geq x 0, \quad (4.4)
\]
so, the claim is true for \( n = m \). Assume that \( n \geq m \). By recurrence relation (4.1), we see that
\[
T_{n+1,k}T_{m,l} - T_{n+1,k-1}T_{m,l+1} = a_{n+1,k}T_{n,k}T_{m,l} + b_{n+1,k}T_{n,k-1}T_{m,l} - a_{n+1,k-1}T_{n,k-1}T_{m,l+1} - b_{n+1,k-1}T_{n,k-2}T_{m,l+1} \\
\geq x a_{n+1,k}(T_{n,k}T_{m,l} - T_{n,k-1}T_{m,l+1}) + b_{n+1,k}(T_{n,k-1}T_{m,l} - T_{n,k-2}T_{m,l+1}), \quad \text{(by (4.2))}
\]
which is in \( \mathbb{R}_+[x] \) by (4.4) and the induction hypothesis. This completes the proof of the claim (4.3). \( \square \)

4.1. **Proof of Theorem 1.4** (i) Let \( J_n(z, y) = \sum_{k=0}^{n} JS(n, k; z)y^k \). In view of recurrence relation (1.2), we have
\[
J_{m-1}(z, y)J_{n+1}(z, y) - J_m(z, y)J_n(z, y) = J_{m-1}(z, y) \sum_{k=0}^{n+1} [JS(n, k-1; z) + k(k + z)JS(n, k; z)]y^k \\
- J_n(z, y) \sum_{k=0}^{m} [JS(m-1, k-1; z) + k(k + z)JS(m-1, k; z)]y^k \\
= J_{m-1} \sum_{k=0}^{n} k(k + z)JS(n, k; z)y^k - J_n(z, y) \sum_{k=0}^{m-1} k(k + z)JS(m-1, k; z)y^k.
\]
Thus the coefficient of \( y^j \) in \( J_{m-1}(z, y)J_{n+1}(z, y) - J_m(z, y)J_n(z, y) \) is
\[
\sum_{i=0}^{j} [(j - i)(j - i + z) - i(i + z)] JS(n, j - i; z) JS(m-1, i; z). \quad (4.5)
\]
By Theorem 1.2 (i), the sequence \( \{JS(n, k; z)\}_{k=0}^{n} \) is strongly \( z \)-log-concave. It follows from (1.2) and Lemma 4.1 that the expression in (4.5) is nonnegative with respect to \( \geq_{z} \) if \( n \geq m \geq 1 \), which proves (i).

(ii) By Eq. (3.4), we have \( \sum_{k=0}^{n} Jc(n, k; z) y^k = \prod_{i=0}^{n-1} (y + i(z + i)) \). The result can be verified directly from this simple expression. \( \square \)

Recall that the Stirling numbers of the second kind \( S(n, k) \) are defined by the following recurrence relation

\[
S(n, k) = S(n-1, k-1) + kS(n-1, k)
\]

with \( S(0, 0) = 1 \). Let \( B_{n}(y) = \sum_{k=0}^{n} S(n, k) y^k \) be the \( n \)-th Bell polynomial. We show that Theorem 1.4 (i) implies the following result of Chen et al. [5].

**Corollary 4.1.** The Bell polynomials \( \{B_{n}(y)\}_{n \geq 0} \) are strongly \( y \)-log-convex.

**Proof.** By Theorem 1.4 (i), the sequence \( \{J_{n}(z, y)\}_{n} \) is strongly \( \{z, y\} \)-log-convex, namely, the polynomial

\[
J_{m-1}(z, y)J_{n+1}(z, y) - J_{m}(z, y)J_{n}(z, y)
\]

has nonnegative coefficients. It is known (see [7]) that \( JS(n, k; z) \) is a polynomial in \( z \) of degree \( n - k \) with leading coefficient \( S(n, k) \). Hence the coefficient of \( z^{n-k} y^k \) in \( J_{n}(z, y) \) is \( S(n, k) \), which implies that the coefficient of \( z^{m+n-i} y^i \) in (1.6) is equal to that of \( y^i \) in \( B_{m-1}(y)B_{n+1}(y) - B_{m}(y)B_{n}(y) \) for \( 0 \leq i \leq m+n \). This completes the proof of the desired result. \( \square \)

### 4.2. An open problem.
We say that a transformation of sequences \( \{z_{n}\}_{n} \mapsto \{w_{n}\}_{n} \) preserves the log-convexity if the log-convexity of \( \{z_{n}\}_{n \geq 0} \) implies that of \( \{w_{n}\}_{n \geq 0} \). For example, Liu and Wang [12] show that the Stirling transformation \( w_{n} = \sum_{k=0}^{n} S(n, k) z_{k} \) preserves the log-convexity. In view of Theorem 1.4 we pose the following conjecture.

**Conjecture 4.1.** The Jacobi-Stirling transformation: \( \{z_{n}\}_{n} \mapsto \{w_{n}\}_{n} \), where

\[
w_{n} = \sum_{k=0}^{n} JS(n, k; z) z_{k} \quad \text{or} \quad w_{n} = \sum_{k=0}^{n} Jc(n, k; z) z_{k},
\]

preserves the log-convexity for \( z = 0, 1 \).

### 4.3. Proof of Theorem 1.5.
By (1.4) we see that \( Q_{n}(x, y, z, t) \) are homogeneous polynomials in \( x, y, z, t \) of degree \( n - 1 \). As \( z \) is just a homogeneous parameter, namely,

\[
Q_{n}(x, y, z, t) = z^{n-1}Q_{n}(x/z, y/z, 1, t/z),
\]

it suffices to study \( Q_{n}(x, y, 1, t) \). We set

\[
Q_{n}(x, y, 1, t) = \sum_{k=0}^{n-1} Q_{n,k}(x, t)y^{k}.
\]
Substituting (4.8) in (1.4) and identifying the coefficients of $y^k$ we obtain $Q_{1,0}(x, t) = 1$ and for $n \geq 2$:

$$Q_{n,k}(x, t) = [x + n - 1 + t(n + k - 1)]Q_{n-1,k}(x, t) + (n + k - 2)Q_{n-1,k-1}(x, t),$$

(4.9)

where $Q_{n,k}(x, t) = 0$ if $k \geq n$ or $k < 0$.

**Lemma 4.2.** For $n \geq 1$ and $l \geq k \geq 1$, we have

$$Q_{n,k}(x, t)Q_{n,l}(x, t) = Q_{n,0}(x, t)Q_{n,l+1}(x, t),$$

where $x = \{x, t\}$. In other words, the polynomial sequence $\{Q_{n,k}(x, t)\}_{k=0}^{n-1}$ is strongly $x$-log-concave.

**Proof.** Let

$$U_n(k, l) = Q_{n,k}(x, t)Q_{n,l}(x, t).$$

We prove by induction on $n \geq 1$. For $n = 1, 2$ the inequality is trivial. For $n = 3$, we have

$$U_3(1, 1) - U_3(0, 2) = 6x^2 + 15x + 10 + 21tx + 28t + 19t^2 \geq 0.$$

Using recurrence relation (4.9) we can write

$$U_{n+1}(k, l) - U_{n+1}(k-1, l+1) = A_n + B_n + C_n + D_n,$$

where

$$A_n = (x + n + t(n + k))(x + n + t(n + l))[U_n(k, l) - U_n(k - 1, l + 1)],$$

$$B_n = (n + k - 1)(n + l - 1)[U(k - 1, l - 1) - U_n(k - 2, l)],$$

$$C_n = (x + n)(l - k + 1)[U_n(k, l - 1) - U_n(k - 1, l)]$$

$$+ (x + n + t(n + l + 1))(n + k - 2)[U_n(k, l - 1) - U_n(k - 2, l + 1)],$$

$$D_n = (l - k + 1)[t^2U_n(k - 1, l + 1) + U_n(k - 2, l) + 2U_n(k, l - 1)].$$

By induction hypothesis, the polynomials $A_n, B_n, C_n$ and $D_n$ are clearly nonnegative with respect to $\geq x$. This completes the proof. \(\Box\)

By (4.7), it suffices to prove Theorem 1.3 for the polynomial sequence $\{Q_n(x, y, 1, t)\}_{n \geq 0}$. For brevity, we write $Q_{n,k}$ for $Q_{n,k}(x, t)$, $Q_n$ for $Q_n(x, y, 1, t)$ and $Q'_n$ for $\partial_y Q_n(x, y, 1, t)$.

By recurrence relation (1.4), we have

$$Q_{m-1}Q_{n+1} - Q_mQ_n = (n - m + 1)(y + t + 1)Q_{m-1}Q_n + y(y + t)(Q'_nQ_{m-1} - Q_nQ'_{m-1}).$$

Thus, the strong $x$-log-convexity of $\{Q_n\}_{n \geq 0}$ will follow from the claim that, for all $n \geq m \geq 1$,

$$Q'_nQ_{m-1} - Q_nQ'_{m-1} \geq 0,$$

where $x = \{x, y, t\}$. The coefficient of $y^{j-1}$ in $Q'_nQ_{m-1} - Q_nQ'_{m-1}$ is

$$\sum_{i=0}^{j} \left( (j - i)Q_{n,j-i}Q_{m-1,i} - iQ_{n,j-i}Q_{m-1,i} \right) = \sum_{i=0}^{j} [(j - i) - i]Q_{n,j-i}Q_{m-1,i}.$$

(4.10)
By Lemmas 4.2, the polynomial sequence $\{Q_{n,k}(x,t)\}_{k=0}^{n-1}$ is strongly $x$-log-concave. It follows from (4.9) and Lemma 4.1 that the right-hand side of (4.10) is nonnegative with respect to $\geq x$. So the claim is true. This completes the proof of Theorem 1.5. □

5. ON A SEQUENCE ARISING FROM LAMBERT $W$ FUNCTION

In [11] Kalugin and Jeffrey consider another form of Lambert’s equation $we^w = x$. Differentiating $n$ times $w$ they obtain

$$\frac{d^n w(x)}{dx^n} = \frac{e^{-n w(x)} p_n(w(x))}{(1 + w(x))^{2n-1}},$$

where $p_n(x)$ are polynomials that satisfy $p_1(x) = 1$ and the recurrence relation

$$p_{n+1}(x) = -(nx + 3n - 1)p_n(x) + (1 + x)p'_n(x), \quad n \geq 1. \quad (5.1)$$

Based on the above recurrence, Kalugin and Jeffrey [11] prove that the coefficients of $(-1)^{n-1}p_n(x)$ are positive and form a unimodal sequence. In what follows, we show how this result follows easily from a connection with the Ramanujan polynomials $R_n$.

**Proposition 5.1.** We have

$$(-1)^{n-1}p_n(x) = (1 + x)^{n-1}R_n(1/(1 + x)).$$

**Proof.** Let

$$(-1)^{n-1}q_n(x) = (1 + x)^{n-1}R_n(1/(1 + x)). \quad (5.2)$$

and substituting (5.2) into (1.3) we get

$$q_{n+1}(x) = -n(2 + x)q_n(x) + (-1)^n(1 + x)^{n-2}R'_n(1/(1 + x)). \quad (5.3)$$

Now, differentiating the Eq. (5.2) yields

$$(-1)^n(1 + x)^{n-2}R'_n(1/(1 + x)) = (1 + x)q_n(x)' - (n - 1)q_n(x).$$

Substituting this into (5.3) we see that $q_n(x)$ satisfy recurrence (5.1). As $q_1(x) = p_1(x)$, we have $q_n(x) = p_n(x)$ for $n \geq 1$. □

A sequence $a_0, a_1, \ldots, a_n$ of real numbers is said to have no internal zeros if there do not exist integers $0 \leq i < j < k \leq n$ satisfying $a_i \neq 0$, $a_j = 0$ and $a_k \neq 0$. The following result is known, see [4, Theorem 2.5.3] or [10, Theorem 2].

**Lemma 5.1.** If the coefficients of the polynomial $A(x)$ are nonnegative without internal zeros and log-concave, then so are the coefficients of $A(x+1)$.

From the above proposition and lemma we can derive a neat proof of the following result of Kalugin and Jeffrey [11].

**Corollary 5.1.** The coefficients of the polynomial $(-1)^{n-1}p_n(x)$ are positive, log-concave, and unimodal.
Proof. First, by (1.3) it is clear that $R_n(y)$ is a polynomial in $y$ with positive coefficients. By (1.5), (4.8) and Lemma 4.2 we see that the coefficients of $R_n(y)$ are log-concave. Combining these with Proposition 5.1 and Lemma 5.1 we derive that the coefficients of polynomials $(-1)^{n-1}p_n(x)$ are positive and log-concave. Since a log-concave positive sequence is unimodal, we are done. □

Remark 6. Kalugin and Jeffrey [11] proved Corollary 5.1 through a long discussion based on recurrence (5.1).

Acknowledgments. We thank the referee for a careful reading of the manuscript. The first author was supported by the China Scholarship Council (CSC) for studying abroad. This work was also supported by CMIRA COOPERA 2012 de la Région Rhône-Alpes.

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