DIFFUSIVE SEARCH WITH RESETTING VIA THE BROWNIAN BRIDGE

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ABSTRACT. Fix $D > 0$. For a parameter $r > 0$, let $X^{(r)}(\cdot)$ be a Brownian motion with diffusion coefficient $D$, equipped with an exponential clock with rate $r$, so that when the clock rings the process jumps to the origin, where it resets and begins anew according to the same rule. Denote expectations with respect to this process by $E^{(r)}_0$. This process, by now rather well-studied, is called Brownian motion with resetting. For a parameter $T > 0$, consider also a process $X^{\text{bb}; T}(\cdot)$ that performs a Brownian bridge with diffusion coefficient $D$ and bridge interval $T$, and then at time $T$ resets and starts anew from the origin according to the same rule. Denote expectations with respect to this process by $E^{\text{bb}; T}_0$. The two resetting processes, one with jumps and the other continuous, search for a random target $a \in \mathbb{R}$ that has a known distribution $\mu$. Letting $\tau_a$ denote the hitting time of $a$, the expected time to locate the target is

$$\int_{\mathbb{R}} (E^{(r)}_0 \tau_a) \mu(da)$$

for the first process and

$$\int_{\mathbb{R}} (E^{\text{bb}; T}_0 \tau_a) \mu(da)$$

for the second process. It is known that $E^{(r)}_0 \tau_a = \frac{\sqrt{\pi |a|}}{r} - \frac{1}{r}$. We calculate $E^{\text{bb}; T}_0 \tau_a$. Then we calculate

$$\int_{\mathbb{R}} (E^{(r)}_0 \tau_a) \mu(da)$$

and

$$\int_{\mathbb{R}} (E^{\text{bb}; T}_0 \tau_a) \mu(da)$$

in the case that $\mu$ is a centered Gaussian distribution with variance $\sigma^2$, which is the mean-squared distance of the target from the origin. Finally, we compare $\inf_{r > 0} \int_{\mathbb{R}} (E^{(r)}_0 \tau_a) \mu(da)$ to $\inf_{T > 0} \int_{\mathbb{R}} (E^{\text{bb}; T}_0 \tau_a) \mu(da)$, obtaining

$$\inf_{r > 0} \int_{\mathbb{R}} (E^{(r)}_0 \tau_a) \mu(da) \approx 3.548 \frac{\sigma^2}{D}$$

and

$$\inf_{T > 0} \int_{\mathbb{R}} (E^{\text{bb}; T}_0 \tau_a) \mu(da) \approx 4.847 \frac{\sigma^2}{D}.$$
problems frequently involve resetting include animal foraging \[11, 17\] and internet search algorithms.

Over the past decade or so, a variety of stochastic processes with resetting have attracted much attention. See \[9\] for a rather comprehensive, recent overview. Prominent among such processes is the diffusive search process with resetting, the one dimensional version of which we now describe. Consider a random stationary target \( a \in \mathbb{R} \) with known distribution \( \mu \), and consider a search process that sets off from the origin and performs Brownian motion with diffusion coefficient \( D > 0 \), which is fixed once and for all. The search process is also equipped with an exponential clock with rate \( r \), so that if it has failed to locate the target by the time the clock rings, then its position is reset to the origin and it continues its search anew independently with the same rule. We consider \( r \) as a parameter that can be varied.

Denote the process by \( X^{(r)}(\cdot) \), let \( E^{(r)}_0 \) denote expectations for the search process starting from 0, and define

\[
\tau_a = \inf \{ t \geq 0 : X^{(r)}(t) = a \}, \quad \text{for } a \in \mathbb{R},
\]

the hitting time of \( a \). One may be interested in several statistics, the most important one probably being the expected time to locate the target, \( \int_{\mathbb{R}} (E^{(r)}_0 \tau_a) \mu(da) \). See, for example, \[5, 6, 7, 8, 12, 11, 4, 15, 14\] for a sampling of articles on this model and related ones.

The resetting in the above model is of course discontinuous—at the ring of the exponential clock, the search process instantaneously jumps back to its initial position at the origin. In certain applications, this is a reasonable assumption, but in others, it is more reasonable to consider a type of resetting for which the search process remains continuous. For example, while the instantaneous jump model might be reasonable for internet search algorithms, perhaps a continuous type of resetting would be more reasonable for animal foraging. In this paper, we introduce a continuous search process with resetting via the Brownian bridge, and compare its efficiency to the above search process. Of course, this continuous search will be less efficient since it diffuses rather than jumps back to its initial point.

As above, fix once and for all the diffusion coefficient \( D > 0 \). For \( T > 0 \), let \( \{B^{bb:T}_n(t), 0 \leq t \leq T\}_{n=1}^\infty \) be a sequence of independent Brownian bridges with bridge time interval \( T \) and diffusion coefficient \( D \). Recall that the Brownian bridge with bridge interval \( T \) is the Brownian motion conditioned to be at the origin at time \( T \). (As is well-known \[10, 16\], a Brownian bridge \( B^{bb:T}(\cdot) \) with bridge time interval \( T \) and diffusion coefficient \( D \) can
be represented as $B^{bb:T}(t) = \sqrt{D}(W(t) - \frac{1}{T}W(T))$, $0 \leq t \leq T$, where $W(\cdot)$ is a standard Brownian motion.) Define the search process $X^{bb:T}(\cdot)$ by

$$X^{bb:T}(t) = B^{bb:T}(t - nT), \quad t \in [nT, (n+1)T], \quad n = 0, 1, 2, \cdots.$$  

We consider $T$, the time interval between resets, to be a parameter that can be varied.

Let $E^{bb:T}$ denote expectations for the search process $X^{bb:T}(\cdot)$ starting from 0. Let

$$\tau_a = \inf\{t \geq 0 : X^{bb:T}(t) = a\}, \quad a \in \mathbb{R}.$$  

We wish to compare the expected time to locate a target for this search process to that of the standard resetting search process defined above. (We use the same notation $\tau_a$ for the hitting time of both processes; the different notation for the expectations will distinguish them.) It is known [5] that for the standard resetting search process, one has

$$E^{(r)}_0\tau_a = \frac{e^{\frac{a^2}{2D}}|a| - 1}{r}, \quad a \in \mathbb{R}. \quad (1.2)$$  

In particular, it follows that a necessary and sufficient condition for the finiteness of $\inf_{r>0} \int_{\mathbb{R}} (E^{(r)}_0\tau_a) \mu(da)$, the infimum over resetting rates of the expected time to locate the target, is that the target distribution $\mu$ possess some absolute exponential moment; that is, $\int_{\mathbb{R}} e^{\epsilon |x|} \mu(dx) < \infty$, for some $\epsilon > 0$. (We note that one can also consider spatially dependent resetting rates $r = r(x)$. It turns out that for any $l > 2$, if $\mu$ possesses its absolute $l$th moment, then one can choose a rate $r(x)$ for which the expected time to locate the target is finite [13].)

We begin with the following theorem, the counterpart of (1.2) for the Brownian bridge search process.

**Theorem 1.**

$$E^{bb:T}_0\tau_a = T(e^{\frac{a^2}{2DT}} - 1) + |a|e^{\frac{a^2}{2DT}} \int_{0}^{T} \frac{e^{-\frac{a^2}{2D(1-\frac{t}{T})}}}{\sqrt{2\pi Dt(1-\frac{t}{T})}} dt, \quad a \in \mathbb{R}. \quad (1.3)$$

It follows from Theorem 1 that a necessary and sufficient condition for the finiteness of $\inf_{T>0} \int_{\mathbb{R}} (E^{bb:T}_0\tau_a) \mu(da)$, the infimum over the time interval between resets of the expected time to locate the target, is that $\int_{\mathbb{R}} e^{\epsilon x^2} \mu(dx) < \infty$, for some $\epsilon > 0$. In particular, if the target distribution $\mu$ does not satisfy this condition, but does possess some absolute exponential moment, then the expected time to locate the target in the case of the
standard resetting process with instantaneous returns to the origin will be
finite for appropriate resetting rates \( r \), but will be infinite in the case of the
Brownian bridge resetting process for every time interval between resets \( T \).

We now come to the main point of this paper, which is to compare between
the two search processes the expected time to hit a centered Gaussian target.
Let \( \mu_{\text{Gauss}}^{\sigma^2} \) denote the centered Gaussian distribution with variance \( \sigma^2 \). Of
course, \( \sigma^2 \) is then the mean-squared displacement of the target from the
search origin. We begin with the classical search process with instantaneous
resetting.

**Proposition 1.**

\[
\int_{\mathbb{R}} (E_0^{(r)} \tau_a) \mu_{\sigma^2}^{\text{Gauss}}(da) = \frac{1}{r} \left( 2e^{\frac{-r^2}{2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx - 1 \right).
\]

Equivalently, writing \( r = \frac{D}{\sigma^2} R \), with \( R > 0 \),

\[
\int_{\mathbb{R}} (E_0^{(\frac{D}{\sigma^2} R)} \tau_a) \mu_{\sigma^2}^{\text{Gauss}}(da) = \frac{\sigma^2}{D} \left( 2e^{\frac{R^2}{2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{R^2}{2}}}{\sqrt{2\pi R}} dx - 1 \right).
\]

One has

\[
\inf_{r>0} \int_{\mathbb{R}} (E_0^{(r)} \tau_a) \mu_{\sigma^2}^{\text{Gauss}}(da) \approx 3.548 \frac{\sigma^2}{D} \text{ with the infimum attained at } r \approx 0.491 \frac{D}{\sigma^2}.
\]

We now turn to the corresponding result for the Brownian bridge search
process.

**Theorem 2.**

\[
\int_{\mathbb{R}} (E_0^{bb; T} \tau_a) \mu_{\sigma^2}^{\text{Gauss}}(da) = \begin{cases} 
T \left( \frac{DT}{D + 4\sigma^2} \right)^{\frac{1}{2}} - T + \frac{T \sigma}{\sqrt{DT + 2\sigma}}, & T > 4\frac{\sigma^2}{D}; \\
\infty, & T \leq 4\frac{\sigma^2}{D}.
\end{cases}
\]

Equivalently, writing \( T = \frac{\sigma^2}{D} \mathcal{T} \),

\[
\int_{\mathbb{R}} (E_0^{bb; \frac{\sigma^2}{D} \mathcal{T}} \tau_a) \mu_{\sigma^2}^{\text{Gauss}}(da) = \begin{cases} 
\frac{\sigma^2}{D} \left( \mathcal{T} \left( \frac{\mathcal{T}}{\sqrt{\mathcal{T} - 4}} \right)^{\frac{1}{2}} - \mathcal{T} + \frac{\mathcal{T}}{2 + \sqrt{\mathcal{T}}}, & \mathcal{T} > 4; \\
\infty, & \mathcal{T} \leq 4.
\end{cases}
\]

One has

\[
\inf_{T>0} \int_{\mathbb{R}} (E_0^{bb; \mathcal{T}} \tau_a) \mu_{\sigma^2}^{\text{Gauss}}(da) \approx 4.847 \frac{\sigma^2}{D} \text{ with the infimum attained at } \mathcal{T} \approx 10.136 \frac{\sigma^2}{D}.
\]
From Proposition 1 and Theorem 2 it follows that the expected time to locate a centered Gaussian distributed target is about 36.6 percent longer under the Brownian bridge with optimal bridge time interval than it is under the Brownian motion with jump resetting with the optimal jump rate.

We note that a recent paper [3] introduced a hybrid version of the two search processes considered in this paper. The process in that paper is the classical search process with instantaneous resetting, conditioned to return to the origin at time $T$. The authors study various properties of this process on the time interval $[0, T]$. One can construct a search process on the entire time line $[0, \infty)$ by repeating independent copies of this process on $[0, T]$, just as was done to construct the process $X_{bb; T}(\cdot)$ from independent copies of the Brownian bridge. It would be interesting to investigate how this search process performs with respect to a centered Gaussian target. Such an analysis seems difficult because the expression for the expectation of $\tau_a$, analogous to (1.3), is quite unwieldy. One could also consider the following alternative hybrid process. Start with the Brownian bridge with bridge interval $T$ and impose on this an exponential clock with rate $r$, which upon ringing sends the process back to the origin instantaneously. Use independent copies of this process to construct a search process on $[0, \infty)$. The calculations for this process are a bit less complicated than for the hybrid process in [3].

We prove Theorem 1, Proposition 1 and Theorem 2 in sections 2, 3 and 4 respectively.

2. Proof of Theorem 1

By symmetry, it suffices to consider $a > 0$. Let $W(\cdot)$ be a Brownian motion with diffusion coefficient $D$ starting from the origin (the generator of the process is $\frac{D}{2} \frac{d^2}{dx^2}$), and denote probabilities and expectations for this process by $P_0$ and $E_0$. It is well-known [10, 2] that $\tau_a$, the hitting time of $a \in \mathbb{R}$, satisfies

$$P_0(\tau_a < T | W(T) = 0) = e^{-\frac{2a^2}{DT}}.$$

Letting $P_{0;bb; T}^{bb; T}$ denote probabilities for the process $X_{bb; T}^{bb; T}(\cdot)$ defined in (1.1), it follows from the definition of the Brownian bridge and the definition of $X_{bb; T}^{bb; T}(\cdot)$ that the above equation is equivalent to

$$P_{0;bb; T}^{bb; T}(\tau_a < T) = e^{-\frac{2a^2}{DT}}.$$
It is very well-known from the reflection principle [10, 16] that the hitting time \( \tau_a \) for \( W(\cdot) \) has a density \( f(t) \) given by

\[
f(t) = \frac{1}{\sqrt{2\pi D}} \frac{ae^{-\frac{a^2}{2D}}} {t^2}, \quad t > 0.
\]

Thus, the hitting time \( \tau_a \) for \( W(t), 0 \leq t \leq T \), conditioned on \( W(T) = 0 \), has sub-density

\[
f(t) = \frac{1}{\sqrt{2\pi DT}} \frac{ae^{-\frac{a^2}{2D(1 - \frac{t}{T})}}} {t^2}, \quad 0 < t < T.
\]

Consequently,

\[
E_{0}^{bb;T}(\tau_a 1_{\tau_a < T}) = E_{0}(\tau_a 1_{\tau_a < T}|W(T) = 0) = a \int_{0}^{T} \frac{e^{-\frac{a^2}{2D(1 - \frac{t}{T})}}}{\sqrt{2\pi Dt(1 - \frac{t}{T})}} dt.
\]

From (2.1) and the definition of \( X^{bb;T}(\cdot) \), it follows that

\[
P_{0}^{bb;T}(\tau_a \in (nT, (n+1)T)) = (1 - e^{-\frac{2a^2}{DT}})^n e^{-\frac{2a^2}{DT}}.
\]

Also, from (2.1), (2.2) and the definition of \( X^{bb;T}(\cdot) \) it follows that

\[
E_{0}^{bb;T}(\tau_a |\tau_a \in (nT, (n+1)T)) = nT + E_{0}^{bb;T}(\tau_a |\tau_a < T) =
\]

\[
nT + \frac{P_{0}^{bb;T}(\tau_a 1_{\tau_a < T})}{P_{0}^{bb;T}(\tau_a < T)} = nT + ae^{2\frac{a^2}{DT}} \int_{0}^{T} \frac{e^{-\frac{2a^2}{2D(1 - \frac{t}{T})}}}{\sqrt{2\pi Dt(1 - \frac{t}{T})}} dt.
\]

Now (2.3) and (2.4) yield

\[
E_{0}^{bb;T}(\tau_a) = \sum_{n=0}^{\infty} E_{0}^{bb;T}(\tau_a |\tau_a \in (nT, (n+1)T)) P_{0}^{bb;T}(\tau_a \in (nT, (n+1)T)) =
\]

\[
\sum_{n=0}^{\infty} \left(nT + ae^{2\frac{a^2}{DT}} \int_{0}^{T} \frac{e^{-\frac{2a^2}{2D(1 - \frac{t}{T})}}}{\sqrt{2\pi Dt(1 - \frac{t}{T})}} dt \right) ((1 - e^{-\frac{2a^2}{DT}})^n e^{-\frac{2a^2}{DT}}) =
\]

\[
T(e^{\frac{2a^2}{DT}} - 1) + ae^{2\frac{a^2}{DT}} \int_{0}^{T} \frac{e^{-\frac{2a^2}{2D(1 - \frac{t}{T})}}}{\sqrt{2\pi Dt(1 - \frac{t}{T})}} dt,
\]

which completes the proof of the theorem. \( \square \)
3. Proof of Proposition 1

From (1.2), we have

\[ \int_{\mathbb{R}} (E_{0}^{\frac{D}{\tau_{2}}})^{\text{Gauss}}(da) = 2\int_{0}^{\infty} e^{\sqrt{2\pi} a} - 1 e^{-\frac{a^2}{2\pi^2}} da. \]

Also,

\[ \int_{0}^{\infty} e^{\sqrt{2\pi} a} e^{-\frac{a^2}{2\pi^2}} da = e^{\frac{a^2}{2\pi^2}} \]

We obtain (1.4) from (3.1) and (3.2). A change of variables in (1.4) yields (1.5). Finally, (1.6) was obtained from (1.5) using the Desmos graphing calculator.

\[ \square \]

4. Proof of Theorem 2

From (1.3), we have

\[ \int_{\mathbb{R}} (E_{0}^{\frac{b\tau_{1}}{T}})^{\text{Gauss}}(da) = 2T \int_{0}^{\infty} e^{\frac{a^2}{2T}} - 1 e^{-\frac{a^2}{2\pi^2}} da + \]

\[ 2 \int_{0}^{\infty} a e^{\frac{a^2}{2T}} \left( \int_{0}^{T} e^{-\frac{a^2}{2\pi^2 (1 - \frac{t}{T})}} dt \right) e^{-\frac{a^2}{2\pi^2}} da. \]

The first integral on the right hand side of (4.1) is infinite if \( T \leq \frac{4\sigma^2}{D} \). From now on, we assume that \( T > \frac{4\sigma^2}{D} \). We have

\[ 2 \int_{0}^{\infty} e^{\frac{a^2}{2T}} e^{-\frac{a^2}{2\pi^2}} da = 2 \int_{0}^{\infty} e^{-\frac{a^2}{2\pi^2}} \frac{a^2}{2\pi^2} da = \sqrt{2\pi} DT \sigma^2. \]

Thus, the first term on the right hand side of (4.1) satisfies

\[ 2T \int_{0}^{\infty} \left( e^{\frac{a^2}{2T}} - 1 \right) e^{-\frac{a^2}{2\pi^2}} da = T \left( \frac{DT}{DT - 4\sigma^2} \right)^{\frac{1}{2}} - T. \]

We now turn to the second term on the right hand side of (4.1). We have

\[ \int_{0}^{\infty} a e^{\frac{a^2}{2T}} e^{-\frac{a^2}{2\pi^2 (1 - \frac{t}{T})}} e^{-\frac{a^2}{2\pi^2}} da = \int_{0}^{\infty} a e^{-\frac{a^2}{2\pi^2}} \frac{DT}{DT - 4\sigma^2} \frac{a^2}{T^2} \frac{1}{T^2\sigma^2 + t(T - t)(DT - 4\sigma^2)}. \]
Using (4.3), we can write the second term on the right hand side of (4.1) as
\( (4.4) \)
\[
2 \int_0^\infty ae^{2t^2} \left( \int_0^T e^{-\frac{a^2}{2(Dt(1-\frac{t}{T}))}} dt \right) e^{-\frac{a^2}{2\pi}} da =
\]
\[
\frac{T^{\frac{3}{2}}}{\pi} \int_0^T \frac{\sqrt{Dt(T-t)}}{T^2\sigma^2 + t(T-t)(DT-4\sigma^2)} dt = \frac{2T^{\frac{3}{2}}}{\pi} \int_0^T \frac{\sqrt{Dt(T-t)}}{T^2\sigma^2 + t(T-t)(DT-4\sigma^2)} dt.
\]

Make the substitution \( x = \sqrt{t(T-t)} \). Then \( t = \frac{1}{2}(T - (T^2 - 4x^2)^{\frac{1}{2}}) \) and \( dt = 2x(T^2 - 4x^2)^{-\frac{1}{2}} dx \). We obtain
\( (4.5) \)
\[
\int_0^T \frac{\sqrt{Dt(T-t)}}{T^2\sigma^2 + t(T-t)(DT-4\sigma^2)} dt = 2D \int_0^T \frac{1}{(T^2 - 4x^2)^{\frac{1}{2}}} \left( \frac{x^2}{T^2\sigma^2 + x^2(DT-4\sigma^2)} \right) dx.
\]

Now make the substitution \( x = \frac{T}{2} \sin \theta \). Then \( dx = \frac{T}{2} \cos \theta d\theta \). We obtain
\( (4.6) \)
\[
\int_0^\infty \frac{1}{\sigma^2 + \frac{DT-4\sigma^2}{4} \sin^2 \theta} d\theta = \frac{1}{8} \int_0^\frac{T}{2} \frac{\sin^2 \theta}{\sin^2 \theta + \frac{4\sigma^2}{DT-4\sigma^2}} d\theta.
\]

We write
\[
\frac{\sin^2 \theta}{\sigma^2 + \frac{DT-4\sigma^2}{4} \sin^2 \theta} = \frac{4}{DT-4\sigma^2} \frac{\sin^2 \theta}{\sin^2 \theta + \frac{4\sigma^2}{DT-4\sigma^2}} = \frac{4}{DT-4\sigma^2} \frac{16}{(DT-4\sigma^2)^2} \frac{1}{\sin^2 \theta + \frac{4\sigma^2}{DT-4\sigma^2}}.
\]

Thus,
\( (4.7) \)
\[
\int_0^\frac{T}{2} \frac{\sin^2 \theta}{\sigma^2 + \frac{DT-4\sigma^2}{4} \sin^2 \theta} d\theta = \frac{2\pi}{DT-4\sigma^2} - \frac{16\sigma^2}{(DT-4\sigma^2)^2} \int_0^\frac{T}{2} \frac{1}{\sin^2 \theta + \frac{4\sigma^2}{DT-4\sigma^2}} d\theta.
\]

Making the substitution \( \tan \theta = s \), in which case \( \sin \theta = \frac{s}{\sqrt{1+s^2}} \) and \( d\theta = \frac{1}{1+s^2} ds \), we obtain for any \( A > 0 \),
\( (4.8) \)
\[
\int_0^\frac{T}{2} \frac{1}{\sin^2 \theta + A} d\theta = \int_0^\infty \frac{1}{A + (A+1)s^2} ds = \frac{1}{A+1} \sqrt{\frac{A+1}{A}} \arctan \sqrt{\frac{A+1}{A}} s \bigg|_0^\infty = \frac{\pi}{2\sqrt{A(A+1)}}.
\]
From (4.8), we have

\[
\int_{0}^{\frac{\pi}{2}} \frac{1}{\sin^2 \theta + \frac{4\sigma^2}{DT-4\sigma^2}} d\theta = \frac{\pi}{2} \left( \frac{4\sigma^2}{DT-4\sigma^2} \left( \frac{4\sigma^2}{DT-4\sigma^2} + 1 \right) \right)^{-\frac{1}{2}} = \frac{\pi (DT-4\sigma^2)}{4\sigma \sqrt{DT}}.
\]

From (4.4)-(4.7) and (4.9), we obtain

(4.10)

\[
2 \int_{0}^{\infty} ae^{\frac{2T^2}{\tau}} \left( \int_{0}^{T} \frac{e^{-\frac{a^2}{2D}(1-t)}}{\sqrt{2\pi Dt(1-t)}} dt \right) \frac{e^{-\frac{a^2}{2\sigma^2}}}{\sqrt{2\pi \sigma}} da =
\]

\[
\frac{2T^\frac{3}{2}\sigma}{\pi} \left( \frac{\frac{2\pi}{8}}{DT-4\sigma^2} - \frac{16\sigma^2}{(DT-4\sigma^2)^2} \right) \frac{\pi(DT-4\sigma^2)}{4\sigma \sqrt{DT}} =
\]

\[
\frac{T^\frac{3}{2}\sigma \sqrt{D}}{2\pi} \frac{2\sigma \sqrt{DT}-4\sigma \pi}{(DT-4\sigma^2)\sqrt{DT}} = \frac{T^\frac{3}{2}\sigma \sqrt{D}}{2\pi} \frac{2\pi (\sqrt{DT}-2\sigma)}{(DT-4\sigma^2)\sqrt{DT}} = \frac{T \sigma}{\sqrt{DT} + 2\sigma}.
\]

Now (1.7) follows from (4.1), (4.2) and (4.10). A change of variables in (1.7) yields (1.8). Finally, (1.9) was obtained from (1.8) using the Desmos graphing calculator.

\[\square\]

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