Ultra-Relativistic Expansion of Ideal Fluid with Linear Equation of State

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Abstract

We study solutions of the relativistic hydrodynamical equations, which describe spherical or cylindrical expansion of ideal fluid. We derived approximate solutions involving two arbitrary functions, which describe asymptotic behavior of expanding fireballs in ultra-relativistic limit. In case of a linear equation of state $p(\varepsilon) = k\varepsilon - c_1$, ($0 < k < 1$) we show that the solution may be represented in form of an asymptotic series in negative powers of radial variable; recurrence relations for the coefficients are obtained. This representation is effective, if $k > 1/(2J + 1)$ ($J = 2$ for spherical expansion and $J = 1$ for the cylindrical one); in this case the approximate solutions have a wave-like behavior.

1 Introduction

The problem of the relativistic ideal fluid expansion arise, e.g., in the theory of gamma-ray bursts, that represent the most powerful explosions in the Universe [1]. The other applications are due to hydrodynamical theory of multiparticle production [2]. Here the relativistic ideal fluid model is utilized to describe the behavior of relativistic fireballs. Detailed investigation of this problem involves complicated 3-D numerical simulations. Nevertheless, considerable information may be obtained from analytical studies, which deal typically with one-dimensional flows having spherical, cylindrical and plane symmetry [3], [2]. One of such well-known results is due to Blandford & McKee [2], who studied self-similar ultra-relativistic gas expansion following a strong shock. Blandford & McKee solution is widely used for interpretation.
of the gamma-ray bursts and their afterglows \[1\]. Certain drawbacks of this solution have been pointed out in \[5\].

It should be noted, that investigation of ultra-relativistic asymptotics of ideal flows is rather specific; it requires some care to provide approximation correctness. For example, if we leave only the main terms in the hydrodynamical equations in ultra-relativistic limit, these become degenerate (see Sect. 2). To be sure about approximation involved, one must study the higher order corrections in the equations. Evidently, it is desirable to have a sufficiently general form of the solutions that contain necessary number of arbitrary functions. This is important, e.g., in order to describe models of expanding fluid with different radial density profiles.

In this connection we propose an approximation method to construct general solutions that describe ultra-relativistic radial expansion of ideal fluid with necessary degree of accuracy. Most detailed result is obtained in case of a linear equation of state (EOS); in this case we represent the solution in form of asymptotic series (Sect. 3), which is workable at least for a sufficiently stiff EOS. These results are compared in Sect. 4 with a self-similar solutions.

2 Main equations

The equations of motion of ideal relativistic fluid follow from the conservation laws (see, e.g., \[2\] \[3\])

\[ \partial_{\nu} T^{\mu\nu} = 0, \]

for the energy-momentum tensor \( T^{\mu\nu} = (p + \varepsilon)u^\mu u^\nu - pg^{\mu\nu}; \) \( u^\nu \) is the fluid four-velocity, \( g^{\mu\nu} = \text{diag}(1, -1, -1, -1), \) \( p \) is the pressure, \( \varepsilon \) is the invariant energy density, the light speed \( c = 1. \) We confine ourselves to EOS \( p = p(\varepsilon). \) Some kind of the continuity equation must be addedIn in a more general case of two-parametric EOS.

In case of cylindrical \((J = 1)\) and spherical \((J = 2)\) symmetry of the flow we deal with only two independent variables (time \( t \) and radial variable \( r \)). The equations take on the form \[3, \] \[2\]:

\[ \frac{\partial}{\partial t} \left[ (u^t)^2 \varepsilon + (u^r)^2 p \right] + \frac{\partial}{\partial r} \left[ u^tu^r(p + \varepsilon) \right] = -\frac{J(p + \varepsilon)}{r} u^tu^r, \] \[1\]

\[ \frac{\partial}{\partial t} \left[ u^tu^r(p + \varepsilon) \right] + \frac{\partial}{\partial r} \left[ (u^t)^2 p + (u^r)^2 \varepsilon \right] = -\frac{J(p + \varepsilon)}{r} (u^r)^2, \] \[2\]
where \( u^r \) is the radial component of the four-velocity, the other spatial four-velocity components are equal to zero.

In ultra-relativistic limit \( u^r \approx u^t \gg 1 \). But if we leave in Eqs. (1) and (2) only the higher order terms in \( u^r \), then both from (1) and (2) we have the same equation

\[
\frac{\partial}{\partial t} \left[ (u^t)^2 (\varepsilon + p) \right] + \frac{\partial}{\partial r} \left[ (u^t)^2 (\varepsilon + p) \right] = - \frac{J}{r} \left[ (u^t)^2 (\varepsilon + p) \right],
\]

that is we have certain degeneracy. We can determine \((u^t)^2 (\varepsilon + p)\) from this equation, but we cannot determine the velocity and the energy density separately. This shows that we cannot neglect the terms, looking at first sight as less important, without more detailed consideration.

\[\text{3 First approximation}\]

Further we use the substitution \( u^t = u + 1/(4u) \), \( u^r = u - 1/(4u) \). Then Eqs. (1) and (2) yield:

\[
\frac{\partial}{\partial \beta} \left[ \left( u^2 + \frac{1}{16u^2} \right) (\varepsilon + p) + \frac{\varepsilon - p}{2} \right] + \frac{\partial}{\partial \alpha} \left[ \left( u^2 - \frac{1}{16u^2} \right) (\varepsilon + p) \right] = - \frac{J (p + \varepsilon)}{r} \left( u^2 - \frac{1}{16u^2} \right), \tag{3}
\]

\[
\frac{\partial}{\partial \beta} \left[ \left( u^2 - \frac{1}{16u^2} \right) (\varepsilon + p) \right] + \frac{\partial}{\partial \alpha} \left[ \left( u^2 + \frac{1}{16u^2} \right) (p + \varepsilon) - \frac{\varepsilon - p}{2} \right] = - \frac{J (p + \varepsilon)}{r} \left( u^2 + \frac{1}{16u^2} - \frac{1}{2} \right). \tag{4}
\]

Consider the sum and difference of Eqs. (3) and (4) in new variables \( \alpha = t - r \), \( \beta = t + r \) and . We have

\[
\frac{\partial}{\partial \beta} (\varepsilon - p) + \frac{\partial}{\partial \alpha} \left( \frac{\varepsilon + p}{4u^2} \right) = - \frac{J (\varepsilon + p)}{\beta - \alpha} \left( 1 - \frac{1}{4u^2} \right), \tag{5}
\]

\[
\frac{\partial}{\partial \beta} \left[ u^2 (\varepsilon + p) \right] + \frac{\partial}{\partial \alpha} \left( \frac{\varepsilon - p}{4} \right) = - \frac{J (\varepsilon + p)}{\beta - \alpha} \left( u^2 - \frac{1}{4} \right). \tag{6}
\]

These are still exact equations.
In the ultra-relativistic approximation \( u \gg 1 \) we leave the terms \( \sim u^2 \) and zero orders in this value. Neglecting the terms containing \( \sim 1/u^2 \), we have from (5)

\[
\frac{\partial}{\partial \beta} (\varepsilon - p) \approx -\frac{J(\varepsilon + p)}{\beta - \alpha}.
\]  

This equation can be solved, if we know the state equation \( p = p(\varepsilon) \). In this case Eq. (7) yields

\[
(1 - dp/d\varepsilon) \frac{\partial \varepsilon}{\partial \beta} \approx -\frac{J[\varepsilon + p(\varepsilon)]}{\beta - \alpha}.
\]  

Let by definition

\[
\Phi(\varepsilon) = \exp \left[ \int \frac{1 - dp/d\varepsilon}{\varepsilon + p(\varepsilon)} d\varepsilon \right].
\]

Then the solution of Eq. (8) may be represented as

\[
\varepsilon \approx \Phi^{-1} \left\{ \frac{f(\alpha)}{(\beta - \alpha)^J} \right\},
\]  

where \( f(\alpha) \) is an arbitrary function. The validity of this approximate solution depends on smallness of the terms in Eq. (5) that have been neglected, i.e. the approximation is valid if

\[
\left| \int d\beta (\beta - \alpha)^J \frac{\partial}{\partial \alpha} \left( \frac{\varepsilon + p}{4u^2(\beta - \alpha)^J} \right) \right| \ll |f(\alpha)|.
\]

As we shall see in the next section, this condition is satisfied, e.g., in case of a linear EOS \( p(\varepsilon) = \kappa \varepsilon - c_1 \), if \( 1 > \kappa > 1/(2J + 1) \).

Eq. (6) can be written as

\[
(\beta - \alpha)^{-J} \frac{\partial}{\partial \beta} \left[ u^2(\beta - \alpha)^J (\varepsilon + p) \right] = \frac{J(\varepsilon + p)}{4(\beta - \alpha)} - \frac{\partial}{\partial \alpha} \frac{\varepsilon - p}{4}.
\]

Using \( \varepsilon \) from (9) we obtain \( u \) from

\[
u^2(p + \varepsilon)(\beta - \alpha)^J =
\]

\[
g(\alpha) + \int d\beta \frac{(\beta - \alpha)^J}{4} \left[ \frac{J(\varepsilon + p)}{(\beta - \alpha)} - \frac{\partial}{\partial \alpha}(\varepsilon - p) \right],
\]  

(10)
where \( g(\alpha) \) is an arbitrary function. This enables us to find \( \varepsilon \) and \( u \) separately.

If the integral in the right-hand side of (10) converges for \( \beta \to \infty \) and \( \alpha \) remains bounded, then for sufficiently large \( \beta \sim t \) we obtain that \( u^2(\varepsilon + p) (\beta - \alpha)' \) is a function only of \( \alpha \). We shall see below that this is just the case of the linear EOS with \( \kappa > 1/(2J + 1) \). In this case we have that the energy profile of the ultra-relativistic expanding shell \( dE/d\alpha \approx (\varepsilon + p)u^2rJ \) remains almost constant for large values of \( t \).

4 Higher orders of asymptotic series for the solutions

Consider the higher orders of approximation in case of a linear equation of state \( p(\varepsilon) = \kappa \varepsilon - c_1 \), where \( \kappa = c_0^2 < 1 \), \( c_0 \) is a speed of sound. We are looking for the asymptotic representation of the solution, which is workable for bounded \( \alpha = t - r \) and \( t \to \infty \); this means also \( \beta \sim r \to \infty \).

We put

\[
\varepsilon = \frac{\varphi(\alpha, \beta)}{(\beta - \alpha)^a} + \frac{c_1}{1 + \kappa}, \quad a = \frac{J (1 + \kappa)}{1 - \kappa},
\]

\[
u^2(\varepsilon + p) = \frac{\psi(\alpha, \beta)}{(\beta - \alpha)^J}.
\]

This substitution reduces the system (5), (6) to the form

\[
\frac{\partial \varphi}{\partial \beta} + \frac{(1 + \kappa)^2}{4 (1 - \kappa) (\beta - \alpha)^{a-J}} \frac{1}{\partial \alpha} \left( \varphi^2 \right) = - \frac{J \kappa (1 + \kappa)^2}{(1 - \kappa)^2} \frac{1}{(\beta - \alpha)^{a-J+1}} \frac{\varphi^2}{\psi}
\]

(11)

\[
\frac{\partial \psi}{\partial \beta} + \frac{1 - \kappa}{4} \frac{1}{(\beta - \alpha)^{a-J}} \frac{\partial \varphi}{\partial \alpha} = 0
\]

(12)

We shall look for solution of exact equations (11), (12) in the form of asymptotic series \((m, n = 0, 1, 2, \ldots)\)

\[
\varphi(\alpha, \beta) = \sum_{n,m} \frac{\varphi_{n,m}(\alpha)}{(\beta - \alpha)^{m+n}},
\]

(13)
\[ \psi(\alpha, \beta) = \sum_{n,m} \frac{\psi_{n,m}(\alpha)}{(\beta - \alpha)^{\gamma n + m}}, \]  

where \( \gamma = a - J - 1 \).

If we are looking for an asymptotic solution for \( r \to \infty \), this representation is effective for \( \gamma > 0 \), that is for \( \kappa > 1/(2J + 1) \).

We also introduce

\[ \chi(\alpha, \beta) = \frac{\varphi^2}{\psi} = \sum_{n,m} \frac{\chi_{n,m}(\alpha)}{(\beta - \alpha)^{\gamma n + m}}; \]

the coefficients of these series may be expressed by means of \( \varphi_{n,m}, \psi_{n,m} \). Moreover, \( \chi_{n,m} \) depends only upon \( \varphi_{n',m'}, \psi_{n',m'} \) with \( n' \leq n, m' \leq m \).

We now substitute the above representations for \( \varphi, \psi, \chi \) into Eqs. (11), (12) to find recurrence relations for \( \varphi_{n,m}, \psi_{n,m} \).

For \( n \geq 1, m \geq 1 \) we have

\[ (\gamma n + m) \varphi_{n,m} = \]

\[ = \frac{(1 + \kappa)^2}{4 (1 - \kappa)} \left\{ \frac{d\chi_{n-1,m}}{d\alpha} \right\} + \left[ \gamma(n - 1) + m - 1 + \frac{4J\kappa}{(1 - \kappa)} \right] \chi_{n-1,m-1}, \]

\[ (\gamma n + m) \psi_{n,m} = \]

\[ = \frac{1 - \kappa}{4} \frac{d\varphi_{n-1,m}}{d\alpha} + \frac{1 - \kappa}{4} [\gamma (n - 1) + m - 1] \varphi_{n-1,m-1}. \]

For \( n = 0, m = 1, 2, ... \)

\[ \varphi_{0,m} = \psi_{0,m} = 0, \]  

whence also \( \chi_{0,m} = 0 \).

For \( m = 0, n = 1, 2, ... \)

\[ n \gamma \varphi_{n,m} = \]

\[ = \frac{(1 + \kappa)^2}{4 (1 - \kappa)} \frac{d\chi_{n-1,0}}{d\alpha}, \]

and

\[ \gamma n \psi_{n,0} = \]

\[ = \frac{1 - \kappa}{4} \frac{d\varphi_{n-1,0}}{d\alpha}. \]
Using (15), (16), (17), (18) we can express all the coefficients of the series \((13), (14)\) by means of \(\psi_{00}(\alpha)\) and \(\varphi_{00}(\alpha)\) \((\psi_{00} \neq 0)\).

Here we write expressions for some lowest order coefficients

\[
\varphi_{1,0} = \frac{(1 + \kappa)^2}{4\gamma(1 - \kappa)} \frac{d\chi_{0,0}}{d\alpha}, \quad \psi_{1,0} = \frac{1 - \kappa}{4\gamma} \frac{d\varphi_{0,0}}{d\alpha},
\]

\[
\varphi_{2,0} = \frac{(1 + \kappa)^2}{8\gamma(1 - \kappa)} \frac{d\chi_{1,0}}{d\alpha}, \quad \chi_{1,0} = 2\varphi_{1,0} \varphi_{2,0} \frac{\varphi_{0,0}}{\psi_{0,0}} - \psi_{1,0} \varphi_{0,0}^2,
\]

\[
\psi_{2,0} = \frac{(1 + \kappa)^2}{2\gamma^2} \frac{d^2\chi_{0,0}}{d\alpha^2}, \quad \varphi_{1,1} = \frac{J\kappa (1 + \kappa)^2}{(1 - \kappa)^2(\gamma + 1)} \chi_{0,0},
\]

\[
\psi_{1,m} = 0, m = 1, 2, \ldots,
\]

5 Self-similar solutions

The representation of solution in the previous section allows us to obtain the coefficients of the series \((13), (14)\) up to any given order. Therefore, if \(\kappa > 1/(2J + 1)\), we may calculate asymptotic representation of the solution with any desired accuracy for \(r \to \infty\). Nevertheless, this does not mean that series \((13)\) and \((14)\) are convergent. Therefore, it would be useful to check the results of the previous section using the self-similar solutions, when we deal with ordinary differential equations.

We choose the similarity variable as \(\xi = r/t\). In case of the EOS \(p = \kappa \varepsilon - c_1/(1 + \kappa)\) the system \((11)-(2)\) leads to the ordinary differential equations

\[
\xi \frac{dv}{d\xi} = J\kappa (1 - v^2) \left[\frac{\mu \xi (1 - v^2) + v(1 - v\xi)}{(v - \xi)^2 - \kappa (1 - v\xi)^2}\right],
\]

\[
\xi \frac{d\sigma}{d\xi} = J(1 + \kappa) \left[\frac{\mu + [1 + (1 - \kappa)\mu]v\xi - (1 + \mu)\xi^2}{(v - \xi)^2 - \kappa (1 - v\xi)^2}\right],
\]

where \(v = u^r/u^t = (4u^2 - 1)/(4u^2 + 1)\), \(\sigma = ln\rho\), \(\varepsilon = \rho \nu + c_1/(1 + \kappa)\), \(\mu = \nu/[J(1 + \kappa)]\).

A qualitative investigation of self-similar relativistic fluid flows with spherical and cylindrical symmetry can be found in [3]. Here we confine ourselves to asymptotical solutions of the Eqs. \((20), (21)\) near the boundary of expanding fluid, where \(u >> 1, v \to 1\). If the fluid expands to vacuum, its
boundary moves with the light speed (correspondingly, $\xi = 1$). Using the results of qualitative invesigation of [3] in the phase plane $\{v, \xi\}$, it is easy to classify possible asymptotic solutions of Eqs. (20), (21) near this boundary for $\xi \to 1$ and to obtain their asymptotics.

In case of $\kappa > 1/(2J + 1)$ we have

$$u^2 \approx A_1(1 - \xi)^{p_1}, \quad p_1 = -\frac{2\kappa J}{1 - \kappa},$$

$$\varepsilon \approx A_2 r^\nu(1 - \xi)^{p_2}, \quad p_2 = \nu + \frac{J(1 + \kappa)}{1 - \kappa},$$

(22) (23)

$A_1, A_2$ being arbitrary constants.

This corresponds to the results of the previous section, where one should assign

$$\varphi_{00}(\alpha) \sim \alpha^{p_2}, \quad \psi_{00}(\alpha) \sim \alpha^{\nu + J}.$$  

If $\kappa < 1/(2J + 1)$, this type of solution is absent.

On the other hand, there is also a solution with asymptotics

$$u^2 = A_\pm(1 - \xi)^{-1}, \quad A_\pm = (1 - \kappa)^{-1}(B \pm \sqrt{D}),$$

(24)

where

$$B = 1 + \kappa + J\kappa(1 + 2\mu), \quad D = B^2 + (1 - \kappa)[(1 + 2J)\kappa - 1].$$

For $\kappa > 1/(2J + 1)$ it is easy to see that $A_- < 0$, so only $A_+ > 0$ survives. For $\kappa < 1/(2J + 1)$ both $A_-$ and $A_+$ are positive. Depending upon $\nu$ and $\kappa$, Eq. (24) may represent either a special solution or an infinite family of solutions. Some values of the parameters may lead to $\varepsilon \to \infty$ for $\xi \to 1$ and they must be matched to another solutions through the shock wave; the other parameters correspond to solutions that can be matched to vacuum ($\varepsilon \to 0$). This asymptotics is not well described by a finite number of terms in the representation (13), (14) of the previous section.

6 Discussion

We studied non-stationary ultra-relativistic hydrodynamical flows having spherical or cylindrical symmetry in case of the equation of state $p = p(\varepsilon)$. 8
The results may be applicable also in case of a more general EOS if the fluid motion is isentropic. The solutions derived in Sect. 4 (as well as the first order approximations of Sect. 3) contain two arbitrary functions, therefore they represent some kind of a general solution of the equations of the fluid motion. However, consideration of the section 5 shows that they do not describe solutions of all possible types equally well in the ultra-relativistic limit.

Formally we may use some of the above results in case of arbitrary $\kappa \in (0, 1)$, $p(\varepsilon) = \kappa \varepsilon - c_1$, in any ultra-relativistic case, but in certain limited domain of spatial variables, where the approximations work.

The approximations work in the best way in case of the "stiff" equation of state: $\kappa > 1/5$ in case of a spherical expansion (this involves the important case $\kappa = 1/3$), and $\kappa > 1/3$ in case of a cylindrical one. In these cases the representations of Sect. 3 are most effective for $t - r \ll t \to \infty$. Corresponding solutions exhibit the wave-like behavior irrespectively of energy profile of the expanding shell. The energy of the spherical/cylindrical shell, which expands almost with the light speed, is approximately conserved. This does not mean, however, that this shell will contain all the energy of the expanding fluid. It would be interesting to study, what is the part of the total energy carried by this expanding wave-like shell, and to relate the shell profile with the initial conditions; this probably cannot be done without numerical investigation. On the other hand, in case of small values of $\kappa$ we may expect that the energy of the fluid is spread over all the values of $r \sim t$.

The values of the speed of sound $c_0^2 = 1/(2J + 1)$ may be considered as some bifurcation points, which separate different regimes of ultra-relativistic flow.

The method of Section 3 allows to derive formally the solutions up to any order of approximation. However, we do not make here any statement about the convergence of the series, taking in mind that calculation of high orders involves higher order derivatives. Nevertheless, our results show that, in spite of some degeneracy in the main order of hydrodynamical equations in ultra-relativistic limit, we do not meet any problems like small denominators in the higher orders and in this sense we may be sure about the asymptotic properties of the solutions.

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