Cubic Algebraic Equations in Gravity Theory, Parametrization with the Weierstrass Function and Non-Arithmetic Theory of Algebraic Equations

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Abstract

A cubic algebraic equation for the effective parametrizations of the standard gravitational Lagrangian has been obtained without applying any variational principle. It was suggested that such an equation may find application in gravity theory, brane, string and Rundall-Sundrum theories. The obtained algebraic equation was brought by means of a linear-fractional transformation to a parametrizable form, expressed through the elliptic Weierstrass function, which was proved to satisfy the standard parametrizable form, but with $g_2$ and $g_3$ functions of a complex variable instead of the definite complex numbers (known from the usual (arithmetic) theory of elliptic functions and curves). The generally divergent (two) infinite sums of the inverse first and second powers of the poles in the complex plane were shown to be convergent in the investigated particular case, and the case of the infinite point of the linear-fractional transformation was investigated. Some relations were found, which ensure the parametrization of the cubic equation in its general form with the Weierstrass function.
I. INTRODUCTION

The synthesis of algebraic geometry and physics has been known for a long time, beginning from chiral Potts model, algebraic Bethe ansatz (for a review of these aspects see \(^1\)) and ending up with orbifold models of string compactification.\(^2\) In the context of string theories, the application of algebraic curves, related to Fermat’s theorem, has also been pointed out.\(^3\)

Concerning gravitational physics, which is an inherent constituent of any string, brane or ADS - theories, any applications of the theory of algebraic curves are almost absent. In this aspect perhaps one of the most serious attempts was undertaken in the recent paper of Kraniotis and Whitehouse.\(^4\) Based on a suitably chosen metric of an inhomogeneous cosmological model and introducing a pair of complex variables, the authors have succeeded to obtain a nonlinear partial differential equation for a function, entering the trial solution of the field equations. The most peculiar and important feature of the obtained equation is that it can be parametrized by the well-known \(\text{Weierstrass function}\) (for a classical introduction in the theory of elliptic and Weierstrass functions see Ref.5-7). This convenient representation enabled the authors to express important physical quantities such as the Hubble constant and the scale factor through the Weierstrass and the Jacobi theta functions. In fact, an analogy has been used with examples on the motion of a body in the field of an central force, depending on the inverse powers of the radial distance \(r\). The cases of certain inverse powers of \(r\), when the solution of the trajectory equation is expressed in terms of elliptic and Weierstrass functions, have been classified in details.\(^8\)

Three important conclusions immediately follow from the paper of Kraniotis and Whitehouse,\(^4\) and they provide an impetus towards further investigations. The first two conclusions are correctly noted by the authors themselves: 1. Other cases may exist, when solutions of nonlinear equations of General Relativity might be expressed in terms of Weierstrass or theta functions\(^9\), associated with Riemann surfaces. 2. The differential equations of General Relativity in a much broader context might be related to the mathematical theory of \textit{elliptic curves and modular forms} (for an introduction, see Refs. 10-12) and even to the famous Taniyama-Shimura conjecture, stating that every elliptic curve over the field of rational numbers is a modular one. For a short review of some of the recent developments in the \textit{arithmetic theory of elliptic curves}, the interested reader may consult also the monograph\(^13\). In fact, the eventual connection of General Relativity Theory with Number and Elliptic Functions Theory was formulated even in the form of a conjecture that ”\textit{all nonlinear exact solutions of General Relativity with a non-zero cosmological constant }\Lambda\textit{ can be given in terms of the Weierstrass Jacobi Modular Form}”.\(^4\) Of course, such a conjecture is expressed for the first time, and yet there are no other solutions derived in terms of elliptic functions, not to speak about any classification of the solutions on that bases.

The present paper will not have the purpose to present any new solutions of the Einstein’s equations by applying elliptic functions, nor will give any new physical interpretation, which in principle should be grounded on previously developed mathematical
techniques. Rather than that, in this paper an essential algebraic "feature" of the gravitational Lagrangian will be proved, which is inherent in its structure, mostly in its partial derivatives. This "algebraically inherent structure" represents the third conclusion, which in a sense may be related to the problems, discussed in Ref. 4.

However, this algebraic feature will become evident under some special assumptions. While in standard gravitational theory it is usually assumed that the metric tensor has an inverse one, in the so called theories of spaces with covariant and contravariant metrics (and affine connections)\(^{14}\) instead of an inverse metric tensor one may have another contravariant tensor \(g^{jk}\), satisfying the relation \(g_{ij}g^{jk} \equiv \delta^k_i \neq \delta^i_i\). But then, since \(\delta^i_i\) cannot be determined from any physical considerations and at the same time the important mathematical structure from a physical point of view is the Gravitational Lagrangian, a natural question arises: Is it possible that in such a theory with a more general assumption in respect to the contravariant metric tensor, the Gravitational Lagrangian is the same (scalar density) as in the usual case, provided also that the usual connection and the Ricci tensor are also given? From a physical point of view, this is the central problem, treated in this paper, and the answer, which is given, is affirmative. Namely, it has been shown that if \(e_i\) are the components of the covariant basic vectors, and \(dX^j\) are the components of a contravariant vector field (which, however, are not contravariant basic vectors and therefore \(e_i dX^k \equiv \delta^k_i \neq \delta^i_i\)), then they satisfy a cubic algebraic equation. Of course, if \(dX^i\) are to be found from this equation, then it can be shown that \(\overline{g}^{jk}\) will also be known because of the relation \(\overline{g}^{jk} = dX^j dX^k\). Also, it has been assumed that the affine connection \(\Gamma^k_{ij}\) and the Ricci tensor \(R_{ij}\), determined in the standard way through the inverse metric tensor are known.

The obtained cubic algebraic equation can be expressed in a very simple form, but unfortunately it is not easy at all to solve it. That is why a mathematical approach for dealing with such an equation has been developed, on which any further physical application will be based. The equation has been derived in two cases - when \(d^2X^i \equiv 0\) and when \(d^2X^i \neq 0\). As will be shown, the first assumption means that \(dX\) has zero-vorticity components (and non-zero divergency, however), and in physical considerations this restriction can be imposed. The second assumption would mean that \(dX\) has both non-zero divergency and non-zero vorticity components. From the mathematical theory of cubic equations, the investigation of the two cases will not be different, because in the second case only the algebraic variety from the first case (with \(dX^i\)) will be supplemented with the components \(d^2X^i\). It is worth mentioning also that the derivation of the equation does not presume zero-covariant derivatives of the covariant tensor field, thus leaving an opportunity to investigate the different kinds of transports on the space-time manifold. The algebraic equation may enable one to make a kind of a classification (from an algebraic point of view) of the contravariant tensors, satisfying the same gravitational Lagrangian.

So far, the problem investigated here may seem to be of pure "mathematical" interest, but it may have also numerous physical applications. In supergravity theories, ADS/CFT, five-dimensional and brane theories\(^{15,16,17}\), one deals with an action, consisting of a gravitational part, added to a (for example) string action of the kind \(S_{str.} \equiv \).
\[-\frac{T}{2} \int d^2 \xi \sqrt{-g} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu,\] where \(X^\mu\) are the string coordinates, \(h^{\alpha\beta}\)-the world sheet metric tensor, \(T\)- the string tension and the partial derivatives \(\partial_\alpha\) are taken in respect to the world sheet coordinates \(\xi^\alpha = (\tau, \sigma)\). One can easily guess that the above described methodology can easily be applied to the string part of the action. More concretely, \(h^{\alpha\beta}\) may be expressed as \(h^{\alpha\beta} = d\xi^\alpha d\xi^\beta\), the gravitational metric tensor may be assumed to depend on the string coordinates and the derivatives of the string coordinates will be taken in respect to the world sheet coordinates \(\xi^\alpha\). Also, the partial derivatives in the gravitational part of the action may also be taken in respect to the coordinates \(\xi^\alpha\). As a result, taking the gravitational and the string part of the action together and without applying any variational principle, one would get the same kind of a cubic algebraic equation as the one, which will be proposed in this paper. In a sense, this dependence of \(g_{ij}\) on the string coordinates is a sort of a coupling between the gravitational part of the action and the string one, and the resulting cubic equation may be called "an algebraic equation for the effective parametrization of the total Lagrangian in terms of the string coordinates". The "coupling" between the two parts of the action provides another interesting possibility, if the first variation of the Lagrangian is performed, even without taking into account any equations of motion. Provided that the gravitational Lagrangian depends also on the first and second differentials of the metric tensor, the first variation of the Lagrangian can be regarded also as an cubic algebraic equation in respect to the differentials of the vector field. \(^{18}\)

There is also another interesting problem, which is related to the current trends in ADS/CFT correspondence\(^{33}\) and the WZW theory of strings on a curved background.\(^{34,35}\) For example, in WZW theory it is not clear how to relate the two-dimensional string world-sheet symmetries to the global symmetries (global coordinates) of the three-dimensional ADS spacetime, in terms of which the parametrization of the group element is presented. If one has the ADS metric (and the ADS hyperboloid equation), then usually some parametrization of the global ADS coordinates is chosen, in terms of which the hyperboloid equation is satisfied. However, this is performed from the viewpoint of maximum simplicity and convenience. The formalism, developed in this paper on the base of cubic algebraic curves gives a possibility to find some other parametrizations. For example, the (three-dimensional) ADS spacetime has a boundary which can be found when one of the (global) coordinates tends to infinity (for example \(r \to \infty\)), and thus a two-dimensional space is obtained, which can be identified with the one, on which the world-sheet coordinates are defined. By "identification" one may mean that not the two-dimensional coordinates, but only the Ricci tensors and the Christoffel connections of the two space-times can be identified. Then, and as will further be shown in the general case, the obtained in this paper cubic algebraic equation will give an opportunity to relate one of the differentials with the Weierstrass function and the other differential - with its derivative. In such a way, one obtains a system of two equations in partial derivatives, from where the parametrizations can be found. In principle, 2+1 dimensional gravity and the WZW model of strings on an ADS background are very convenient for application of the algebro-geometric approach. Another interesting moment in these theories is that very often one has to deal with the ADS metric, written in different coordinates (includ-
ing the two-dimensional coordinates of the worldsheet), and it may be supposed that the transition from one system of coordinates to another can probably be given by a linear-fractional transformation, which will be investigated in this paper. From this algebraic point of view, it is interesting to investigate the coordinate transformations in Ref. 36 and in Ref. 37.

Until now, only the physical aspects of the implementation of the algebraic approach have been discussed. The mathematical theory of cubic algebraic equations is also worth mentioning, and creating the relevant mathematical methods will be the main purpose of this paper. In principle, the theory of cubic algebraic equations and surfaces has been an widely investigated subject for a long time. The mathematical theory of cubic hypersurfaces\textsuperscript{19} puts the emphasis especially on the classification of points on the cubic hypersurface, minimal cubic surfaces, two-dimensional birational geometry and quasi-groups. But no concrete applications of cubic curves are given. In well-known monographs,\textsuperscript{20} the general theory of affine and projective varieties and algebraic and projective plane curves is exposed, and some examples are considered too, but the theory of cubic forms is restricted only with the Pascal’s theorem. A more comprehensive introduction to the algebraic theory of second and third-rank curves, their normal forms, turning points (where the second derivatives of the curves’s equation equal to zero), rational transformations and etc. is given in the book of Walker\textsuperscript{21}. Of particular relevance to the present research will be the theorem\textsuperscript{21} that if $f(x, y) = 0$ is a non-degenerate cubic curve, then by introducing an affine set of coordinates $x_1 = \frac{x}{z}$, $y_1 = \frac{y}{z}$ and choosing the turning point at $(0, 0, 1)$, the curve can be brought to the form $y^2 = g(x)$, where $g(x)$ is a third-rank polynomial with different roots. However, the situation is much more interesting in the complex plane, where one may define the lattice $\Lambda = \{m\omega_1 + n\omega_2 | m, n \in \mathbb{Z}; \omega_1, \omega_2 \in \mathbb{C}, Im\omega_1\omega_2 > 0\}$. Let a mapping $f : \mathbb{C}/\Lambda \to \mathbb{C}P^2$ is defined of the factorized along the points of the lattice part of the complex plane into the two-dimensional complex projective space $\mathbb{C}P^2$. If under this mapping the complex coordinates $z$ are mapped as $z \to (\rho(z), \rho'(z), 1)$ when $z \neq 0$ and $z \to (0, 1, 0)$ when $z = 0$ ($\rho(z)$ is the Weierstrass elliptic function), then the mapping $f$ maps the torus $\mathbb{C}/\Lambda$ into the following affine curve $y^2 = 4x^3 - g_2x - g_3$, where $g_2$ and $g_3$ are complex numbers.\textsuperscript{13} The important meaning of this statement is that excluding the points on the lattice which may be mapped into one point of the torus (where the Weierstrass function has real values), the mapping $z \to (x, y) = (\rho(z), \rho'(z))$ parametrizes the cubic curve. The consequence from that is also essential since the solution of the resulting differential equation can be obtained in terms of elliptic functions.\textsuperscript{5,13,22} In spite of the fact that the parametrization can be presented in a purely algebraic manner, it is inherently connected to basic notions from algebraic geometry such as divisors and the Riemann-Roch theorem,\textsuperscript{23} which reveals the dimension of the vector space of meromorphic functions, having a pole of order at most $n$ at the point $z = 0$. This should be kept in mind because some results may be obtained by algebraic methods only, but the explanation may probably be found in algebraic geometry. In the present paper, a more general parametrization of a cubic curve is considered, when $g_2$ and $g_3$ are not complex numbers (the so called Eisenstein series $g_2 = 60 \sum_{\omega \in \Gamma} \frac{1}{\omega^4} = \sum_{n,m} \frac{1}{(n+m\tau)^2}$;
\( g_3 = 140 \sum_{\omega \in \Gamma} \frac{1}{\omega} = \sum_{n,m} \frac{1}{(n+mt)^3} \), but complex functions. **It has been proved that if** the Weierstrass function parametrizes again the cubic curve, then the infinite sums (in pole number terms) \( \sum_{\omega \in \Gamma} \frac{1}{\omega^n} \) for \( n = 1 \) and \( n = 2 \) turn out to be finite \((\text{convergent}) \) ones, **in spite of the fact that in the general case they might be infinite ones \((\text{divergent}) \).** The explanation of this fact from the point of view of algebraic geometry remains an open problem, but it can be supposed that standard **arithmetical theory** of elliptic functions and algebraic equations is contained in some other, more general theory, which may be called **non-arithmetical theory**, and from this theory the standard parametrization should also follow. The considered case of parametrization of a cubic curve with coefficient functions of a complex variable, although performed in this paper in a trivial algebraic manner, is the first step towards constructing such a theory. At least, a certain motivation from a physical point of view is evident for constructing such a theory.

The above-presented outlook on standard parametrization implied the use of affine coordinates, which unfortunately exclude from consideration the infinity point. But the infinity point cannot be ruled out not only from mathematical grounds, but also from physical considerations. For example, in the five-dimensional Randall-Sundrum model, one has to assume a compactification into a four-dimensional universe from an infinite extra dimension, containing also the infinity point. From this point of view, the more convenient transformation, chosen in the present paper, which brings the cubic curve into a parametrizable form, is the **linear-fractional transformation**. This transformation allows one to parametrize with the Weierstrass function the ratio of the two of the parameters, entering the linear-fractional transformation and in the case the parameters in this transformation represent complex functions. Of course, the other parameters remain unfixed, leaving the opportunity to determine them in an appropriate way. In a sense, from most general grounds the appearence of the Weierstrass function in the linear-fractional transformation might be expected, since according to a theorem in the well-known monograph of Courant and Hurwitz, an algebraic curve of the kind \( w^2 = a_0 v^4 + a_1 v^3 + a_2 v^2 + a_1 v + a_4 \) can be parametrized as \( v = \frac{\rho(z)+b}{\rho(z)+d} = \varphi(z) \) and \( w = \varphi'(z) \) by means of the transformations \( v = \frac{am_1+b}{cm_1+d} \) and \( w = \frac{ad-bc}{(cm_1+d)^2} \). In the case \( a_0 = 0 \) (which is the present case of a third-rank polynomial), \( \varphi(z) \) will be a linear function of \( \rho(z) \).

In the present paper, however, the situation is quite different, since the linear-fractional transformation is applied only in respect to one of the variables \((v)\), and in order to get the standard parametrizable form \( w^2 = 4v^3 - g_2 v - g_3 \) (with \( g_2 \) and \( g_3 - \text{complex functions} \)), an additional quadratic algebraic equation has to be satisfied. What is more interesting is that after the parametrization is performed, the linear-fractional transformation turns out to be of a more general kind \( v = \frac{A(z)\rho(z)+B(z)}{C(z)\rho(z)+D(z)+\frac{\rho^2(z)}{2}} \), where \( A, B, C, D \) are functions of \( z \), and the expression for \( v \) represents a **rational transformation of the kind** \( v(z) = \frac{P(z)}{Q(z)} \). Now from another point of view it can also be understood why it is justifiable to apply the rational transformation only in respect to \( v \) and not in respect to \( w \). The reason is in a well-known theorem from algebraic geometry that "**each non-degenerate cubic curve"
does not admit a rational parametrization”. Since each non-degenerate cubic curve can
be brought to the form $w^2 = v(v-1)(v-\lambda)$, $(\lambda \neq 0, 1)$, the essence of the above theorem
is that this (algebraic) form cannot be satisfied by a rational parametrization of both
$v = \frac{P(z)}{Q(z)}$ and $w = \frac{T(z)}{R(z)}$

In respect to the problem about finiteness of the infinite sums $\sum \frac{1}{\omega^n}$ for $n = 1$ and
$n = 2$, the application of the linear-fractional transformation has also turned out to be
useful, when the case of poles at infinity is considered. In principle, in this paper two
separate cases are distinguished - the first case of an infinite point of the linear-fractional
transformation and the second case of poles at infinity, when in the infinite limit $\omega \to \infty$
the sum $\sum \frac{1}{\omega^n}$ tends to the Riemann zeta function $\xi(n)$. For this partial case and applying
a different mathematical method, based on the Tauber’s theorem, one comes again to
the fact about the finiteness of $G_1$, proved in the general case by performing a Loran
function decomposition.

The present paper is organized as follows:

Sect. II gives some basic formulae about the so called gravitational theory with con-
travariant and covariant metric tensors. In Sect. III the third-rank algebraic equation
has been derived, starting from the standard gravitational Lagrangian. Also, the effective
parametrization problem has been formulated in an algebraic language. In Sect. IV the
general mathematical set-up for parametrization of the cubic equation has been discussed,
and some physical motivation for the application of the linear-fractional transformation
from the point of view of Randall-Sundrum theory has been presented. Sect. V shows how
the cubic algebraic equation transforms under the action of the linear-fractional transfor-
mation. Sect. VI shows how from the transformed cubic equation one can get the standard
parametrizable form of the cubic equation (with $g_2$ and $g_3$ -complex numbers) and also
the quadratic algebraic equation is derived, which has to be fulfilled if the parametrizable
form holds. The approach is valid also when $g_2(z)$ and $g_3(z)$ are complex functions.
In Sect. VII it was proved that the nonlinear and nonpolynomial transformation from the
"unbar" to the "bar" variables is also invertible, thus giving the opportunity to write
down two of the additionally imposed equations in terms of the new "bar" variables. In
Sect. VIII the Loran’s decomposition has been performed of the functions on the both
sides of the algebraic equation $(\frac{d\rho}{dz})^2 = M(z)\rho^3 + N(z)\rho^2 + P(z)\rho + E(z)$, where $\rho(z)$ is the
Weierstrass function and $M, N, P, E$ are functions of the complex variable $z$. A system
of three iterative (depending on $n$) algebraic equations has been obtained, representing a
necessary (but not sufficient!) condition for parametrization of a cubic equation of a gen-
eral form with the Weierstrass function. It is not occasional that the condition is called ”a
necessary, but not sufficient one”, because in principle more algebraic equations have to
be solved in order to prove the existence of such a parametrization. In Sect. IX the possible
parametrization of the more simplified cubic equation $[\rho'(z)]^2 = 4p^3 - g_2(z)\rho - g_3(z)$
has been considered, and of course the main motivation for considering such a case is
the close analogy with the well-known case, when $g_2$ and $g_3$ are complex numbers. By
calculating the coefficients in the negative power Loran expansion and combining them,
it has been proved that the sums $\sum \frac{1}{\omega}$ and $\sum \frac{1}{\omega^2}$ represent finite (convergent) quantities.
The other equations for the other values of $m = -3, -1$ have been presented in Appendix A; for values of $m = 2k$ in Appendix B and for $m = 2k + 1$ and $m = -k$ in Appendix C. The equations in these Appendices in fact complete the proof that all the Loran coefficient functions can be uniquely expressed through a combination of the finite sums $G_n$. The calculations are purely technical but they serve as a strict mathematical motivation and a proof of the new and basic qualitative fact that the Weierstrass function can parametrize the simplified form of the cubic equation with coefficient functions $g_2(z)$ and $g_3(z)$. This fact probably might represent one of the starting points in the creation of the so called non-arithmetic theory of algebraic equations. Sect. X investigates the positive-power decomposition of the above-mentioned equation, and from the convergency radius of the infinite sum the asymptotic behaviour of some of the Loran coefficients was found to be $-\frac{n^{l+1}}{4^{l+1}}$. Sect. XI considers the case of poles at infinity in the positive-power Loran decomposition, and from the requirement to have a certain convergency radius, expressions for some of the Loran coefficient functions are obtained. In Sect. XII a split of the original cubic equation into two equations is performed, and based on the fact from Sect. IX, it has been proved that the parametrization of the first equation leads to a parametrization of the second equation. For the two "split-up" equations, Sect. XIII presents an algebraic equation, defined on a Riemann surface, which has to be satisfied if the so called $j$-invariants of the two equations are to be equal. In Sect. XIV on the base of the Loran function decomposition of $g_2(z)$ an infinite sum is obtained in which the coefficient functions contain the sums $G_n$. In Sect. XV this formulae has been combined with a proof that the Tauber’s theorem can be applied, and this combination resulted in an expression for $G_1$ in the limit of poles at infinity. In this partial case, the expression again confirms that the sum $G_1$ is convergent. In Sect. XVI the case of the infinite point of the linear-fractional transformation is considered, and the approach essentially represents a combination of the "split-up" approach from Sect. XII and the approach from Sect. VI, based on the derivation of the additional quadratic equation. In Sect. XVII the relation between the two integration constants is found which appear in the process of integration of the two splitted-up equations. A peculiarity of the developed approach is the appropriate "fixing" of some of the functions in the linear-fractional transformation so that the simplest and most trivial form of the quadratic equation from Sect. VI is obtained. Sect. XVIII starts with an algebraic equation of a fourth rank, derived from the original equation in the case of an infinite point of the linear-fractional transformation. The main result here is that the constant Weierstrass function can parametrize this equation if it should be fulfilled in the entire complex plane. For the same equation, Sect. XIX investigates the second case, when the fourth-rank algebraic equation determines a Riemann surface for the pair of variables $(\rho(z) = w_1(z) + iw_2(z); z)$, and six values of $w_1$ are found, satisfying this equation. Sect. XX finds the necessary and sufficient condition for parametrization with a constant Weierstrass function, based again on the approach of Riemann surfaces. As a result, an integrable nonlinear equation is obtained for the coefficient functions of the algebraic equation. The coefficient functions in the solution of this equation appear in powers of non-integer (fractional) numbers.
II. Covariant and Contravariant Metric Tensor

Usually in gravitational theory it is assumed that a local coordinate system can be defined so that to each metric tensor $g_{ij}$ an inverse one $g^{jk}$ can be defined

$$g_{ij}g^{jk} \equiv \delta^k_i = \{0 \text{ if } i \neq k \text{ and } 1 \text{ if } i = k\}.$$  \hspace{1cm} (1)

However, the notion of a reference frame can be defined in different ways in Ref. 25 - coordinate, tetrad and monad. In the last case the contravariant vector field $\text{d}x^i$ of an observer, moving along a space-time trajectory, represents a reference system. In such a case one may have instead of (2)

$$e_i \text{d}x^j \equiv f^j_i \neq \delta^j_i = S(e_i, \text{d}x^j).$$  \hspace{1cm} (2)

In the context of the so called dual algebraic spaces in Ref. 26, $S(e_i, \text{d}x^j)$ is called a contraction operator. Assuming that an inverse operator of contraction $f^i_j$ exists, it can easily be obtained, as in Ref. 14

$$e^j \equiv f^j_i \text{d}x^i.$$  \hspace{1cm} (3)

Therefore, the metric tensor field $g$ can be decomposed in respect to the contravariant basic eigenvectors in the following way

$$g \equiv g_{ij}(e^i \otimes e^j) \equiv g_{ij}f^i_k f^j_l \text{d}x^k \text{d}x^l (e_k \otimes e_l) \equiv (\text{d}x^k \text{d}x^l)(e_k \otimes e_l),$$  \hspace{1cm} (4)

and the contravariant components $\tilde{g}^{ij}$ of the tensor field $g$ are represented as a contraction of the two vector fields $\text{d}x^i$ and $\text{d}x^j$:

$$\tilde{g}^{ij} \equiv \text{d}x^i \text{d}x^j.$$  \hspace{1cm} (5)

It is important to realize that this definition of a contravariant tensor field is not related to any notion of infinitesimality. In order to understand this, consider a set of global coordinates $X^\mu$, defined on the given manifold and depending also on some other (local) coordinates. Then the set of global coordinates, regarded as functions of the local ones, can be considered as a system of equations, defining some algebraic surface. Provided that the partial derivatives of the global coordinates in respect to the local ones are non-zero, at each point of this surface the corresponding tangent space can be determined, and the differentials of the global coordinates are defined on this tangent space. If one assumes that the global differentials are infinitesimally small, then either the (partial) derivatives of the global coordinates or the ”local” differentials should be small. However, the partial derivatives cannot be small, because one considers arbitrary global and local coordinates on the manifold. Also, if the local differentials are assumed to be small, then they will not be allowed to take arbitrary values. But this will mean that a large variety of integral curves on the manifold should be excluded from consideration. This will be unacceptable since one would like to define integral curves through each point of the
manifold and moreover, it would contradict to our initial assumption about existence of a tangent space at each point of the surface (or manifold). Therefore, as a partial case, each of the local differentials should be allowed to take arbitrary numerical values and of course, they may be equal also to an arbitrary function of the local coordinates.

It might be concluded, therefore, that since the partial derivatives and the local differentials cannot be infinitesimally small, then the global differentials cannot be also infinitesimally small.

Apart from the definition (5) of a contravariant tensor field, we have also the definition of a length interval in Riemannian geometry

\[ ds^2 \equiv l^2(\tau) \equiv g_{ij}dx^i dx^j \]  \hspace{1cm} (6)

If we would like to "incorporate" in this definition the standard definition of an inverse metric tensor as \( g^{ij}g_{jk} \equiv \delta^i_k \), we can set up for the ordinary inverse metric tensor

\[ g^{ij} \equiv \frac{1}{l^2}dx^i dx^j. \] \hspace{1cm} (7)

Therefore, in terms of the differentials, the ordinary inverse tensor \( g^{ij} \) can be represented in the same way as in (5), but divided by the length interval. However, usually the length interval is not known, so from a physical point of view the definition (7) is undesirable and this is the motivation to deal further with the definition (5) of a contravariant tensor field. In order to distinguish the "newly" defined tensor in (5), a "tilde" sign has been placed.

From (3) and (7) it follows

\[ \left[ \frac{1}{l^2} - g_{kl} f^k_i f^l_j \right] dx^i dx^j \equiv 0. \] \hspace{1cm} (8)

Clearly, the requirement for existence of an inverse contraction operator is equivalent to putting \( l = 1 \), i.e. assuming that there is a unit length interval, which is again physically unacceptable, and it is more natural to assume that the length interval is varying. Let us assume that \( l^2 \) and \( f^i_k \) are known in advance, then it can be be investigated which is the algebraic variety of values of \( dx^i \), satisfying this quadratic form. The main difficulty in this approach is that \( f^i_k \) cannot be determined from physical considerations. That is why the aim in the next section will be to derive an algebraic equation, in which known physical quantities will enter - the metric tensor \( g_{ij} \), the Christoffel connection \( \Gamma^k_{ij} \) and the Ricci tensor \( R_{ij} \).
III. CUBIC ALGEBRAIC EQUATION
NOT FOLLOWING FROM A VARIATIONAL PRINCIPLE

Further in this paper it shall be assumed that if \( X^i \) are some generalized coordinates, defined on a \( n \)-dimensional manifold with coordinates on it \((x^1, x^2, \ldots, x^n)\), then the differential \( dX^i \) is defined in the corresponding tangent space \( T_X \) of the generalized coordinates \( X^i \equiv X^i(x^1, x^2, x^3, \ldots, x^n) \). Even if written with a small letter, it shall be understood that \( x^i \) represent generalized coordinates.

Our starting point for the derivation of the cubic equation will be the assumption that in spite of the choice for the contravariant metric tensor, the gravitational Lagrangian \( L = -\sqrt{-g}R \) should be the same, provided also that the Ricci tensor does not change under the definition of the contravariant metric tensor. The meaning of this statement is the following.

Essentially, the gravitational Lagrangian will have two representations. The first representation is based on the standardly defined Christoffell connection \( \Gamma^l_{ik} \)

\[
\Gamma^l_{ik} \equiv \frac{1}{2} g^{ls} (g_{ks,i} + g_{is,k} - g_{ik,s})
\]  
(9)

and the Ricci tensor

\[
R^l_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^j} - \frac{\partial \Gamma^l_{ij}}{\partial x^k} + \Gamma^l_{ik} \Gamma^m_{jm} - \Gamma^m_{il} \Gamma^l_{km} .
\]  
(10)

The second representation of the gravitational Lagrangian will be based on the definition (5) of the contravariant metric tensor \( \tilde{g}^{ik} = dx^i dx^k \). Therefore, the Christoffell connection and the Ricci tensor will be different from the previous ones and will be denoted respectively by \( \tilde{\Gamma}^l_{ik} \) and \( \tilde{R}^l_{ik} \)

\[
\tilde{\Gamma}^l_{ik} \equiv \frac{1}{2} \tilde{g}^{ls} (g_{ks,i} + g_{is,k} - g_{ik,s}) = \frac{1}{2} dx^l dx^s (g_{ks,i} + g_{is,k} - g_{ik,s}) + \frac{1}{2} dx^l dx^s g_{is,k} - \frac{1}{2} dx^l dg_{ik} ,
\]  
(11)

\[
\tilde{R}^l_{ik} = \frac{\partial \tilde{\Gamma}^l_{ik}}{\partial x^j} - \frac{\partial \tilde{\Gamma}^l_{ij}}{\partial x^k} + \tilde{\Gamma}^l_{ik} \tilde{\Gamma}^m_{jm} - \tilde{\Gamma}^m_{il} \tilde{\Gamma}^l_{km} .
\]  
(12)

The gravitational Lagrangian in this second representation is

\[
L_2 \equiv -\sqrt{-\tilde{g}}R = -\sqrt{-g} \tilde{g}^{ik} \tilde{R}_{ik} = -\sqrt{-g} dx^i dx^k(\frac{\partial \tilde{\Gamma}^l_{ik}}{\partial x^l} - \frac{\partial \tilde{\Gamma}^l_{il}}{\partial x^k}) - \\
-\sqrt{-g} dx^i dx^k (\tilde{\Gamma}^l_{ik} \tilde{\Gamma}^m_{lm} - \tilde{\Gamma}^m_{il} \tilde{\Gamma}^l_{km}) .
\]  
(13)

Note that physical meaning of this Lagrangian will depend not only on the properties of the (covariant) metric tensor, but also on the first and the second differentials \( dx^l \) and
$d^2x^i$. It should be mentioned also that the notion of a metric tensor, depending on generalized coordinates, understood in the sense of a hypersurface (an infinite-dimensional manifold of all space-like hypersurfaces, embedded in a given Riemannian spacetime), has been introduced long time ago by Kuchar in Ref.31. In such an approach, the description of the gravitational field essentially depends on the \textit{tangential and normal deformations} of the embedded hypersurface. In our case, we do not restrict to space-like hypersurfaces, but the notion of the differentials begins to play a self-consistent role, similarly to the dynamics and the deformations of the hypersurface in Kuchar’s approach. Yet, the standard gravitational physics with the usual inverse metric tensor is contained in the proposed in this paper approach, because one can identify the components of the usually known inverse metric tensor with the components of the contravariant metric tensor, defined in terms of the differentials. Thus one can obtain a \textit{system of first-order nonlinear differential equations in partial derivatives}. The solution of this system may enable one to choose such global (generalized) coordinates, in terms of which the usual inverse tensor will be equivalent to the contravariant one in terms of the differentials.

Let us now use expressions (5) for the contravariant metric tensor $\tilde{g}^{ij}$ and (11) for the Christoffel connection $\tilde{\Gamma}^k_{ij}$ in order to rewrite the gravitational Lagrangian in the second representation. The first two terms in (13) can be calculated to be

$$-\sqrt{-g}dx^i dx^k \left( \frac{\partial \tilde{\Gamma}^l_{ik}}{\partial x^l} - \frac{\partial \tilde{\Gamma}^l_{il}}{\partial x^k} \right) = -\sqrt{-g}dx^i dx^k \left\{ \frac{\partial(dx^s)}{\partial x^k} \frac{\partial(dx^s)}{\partial x^k} - \frac{1}{2} pg_{ik,l} + \frac{1}{2} g_{il,s} \frac{\partial(dx^s)}{\partial x^l} \right\} =$$

$$= -\sqrt{-g}dx^i dx^k \left\{ p \Gamma^r_{ik} g_{kr} - \Gamma^r_{ik} g_{ir} d^2 x^k - \Gamma^r_{il} g_{kr} d^2 x^k \right\} , \quad (14)$$

where $p$ is the scalar quantity

$$p \equiv \text{div}(dx) \equiv \frac{\partial(dx^i)}{\partial x^i} , \quad (15)$$

which “measures” the divergency of the vector field $dx$. It will be more interesting to calculate the contribution of the second term in (12)

$$-\sqrt{-g}dx^i dx^k (\tilde{\Gamma}^l_{ik} \tilde{\Gamma}^m_{lm} - \tilde{\Gamma}^m_{il} \tilde{\Gamma}^l_{km}) = -\frac{1}{2} \sqrt{-g}dx^i dx^k dx^l dx^m (-dg_{lm}dx^s g_{ks,i} - dg_{ik}dx^r g_{mr,l} +$$

$$+dg_{lr}dx^r g_{mr,k} + dg_{rm}dx^s g_{is,i}) - \sqrt{-g}dx^i dx^k dx^l dx^m dx^r [g_{ks,i} g_{mr,l} - g_{ls,i} g_{mr,k}] = 0 \quad (16)$$

and the \textit{first differential} $dg_{ij}$ is represented as $dg_{ij} \equiv \frac{\partial g_{ij}}{\partial x^s} dx^s \equiv \Gamma^r_{s(i} g_{j)r} dx^s$ and $\Gamma^r_{si}$ is the standard Christoffell connection. Therefore, the second two terms in (13) give no contribution to the gravitational Lagrangian. This is not surprising, since the ”factorization” of the contravariant metric tensor as $dx^i dx^j$ introduces an additional ”symmetry”, due to which all the terms in (16) cancel. That is why the \textbf{second representation} of the gravitational Lagrangian will be given only by the first two terms $-\sqrt{-g}dx^i dx^k \left( \frac{\partial \tilde{\Gamma}^l_{ik}}{\partial x^l} - \frac{\partial \tilde{\Gamma}^l_{il}}{\partial x^k} \right)$ in expression (13).
Concerning the first representation of the gravitational Lagrangian, it was based on the standard Christoffel connection \( \Gamma^k_{ij} \), the Ricci tensor \( R_{ik} \) and the usual inverse metric tensor \( g^{ij} \). The basic assumption at the beginning concerned the gravitational Lagrangian and the Ricci tensor, which means that together with the inverse metric tensor \( g^{ij} \), another contravariant tensor \( \tilde{g}^{ij} = dx^i dx^j \) exists, which enters the expression for the first representation of the gravitational Lagrangian

\[
L_1 = -\sqrt{-\tilde{g}} \tilde{g}^{ik} R_{ik} = -\sqrt{-g} dx^i dx^k R_{ik}. \tag{17}
\]

Comparing this representation with the second one, given by expression (13)

\[
L_2 = -\sqrt{-\tilde{g}} \tilde{R}_{il} = -\sqrt{-g} dx^i dx^l \{ p \Gamma^r_{il} g_{kr} dx^k - \Gamma^r_{ik} g_{rl} q^2 x^k - \Gamma^r_{il} (g_{k})_{r} d^2 x^k \} \tag{18}
\]

and remembering the initial assumption, according to which the Lagrangian should be one and the same in both the representations (i.e. \( L_1 = L_2 \)), one arrives at the following algebraic equation in respect to the first differential \( dx^k \) and the second differential \( d^2 x^k \)

\[
dx^i dx^j (p \Gamma^r_{il} g_{kr} dx^k - \Gamma^r_{ik} g_{rl} q^2 x^k - \Gamma^r_{il} (g_{k})_{r} d^2 x^k) - dx^i dx^j R_{il} = 0. \tag{19}
\]

In the limit \( d^2 x_k = 0 \) this equation assumes the form of a manifestly cubic in respect to \( dx^i \) algebraic equation

\[
dx^i dx^j dx^k p \Gamma^r_{jr} (g_{k} r) - R_{ij} dx^i dx^j = 0. \tag{20}
\]

Equation (20) is the basic equation, which shall be investigated further in this paper. Most importantly, it is manifestly cubic in the differentials \( dx^i \). Due to this reason, one qualitative argument can be given in favour of such a Lagrangian. In 1988, Witten derived the Lagrangian for 2+1 dimensional gravity in Ref. 32, which is also manifestly cubic in the chosen gauge variables \( A_\mu \). The Lagrangian was obtained under the assumption that there is an isomorphism between an abstractly introduced \((d-1,d)\) vector bundle with a structure group \( SO(d-1,d) \) and the tangent bundle of the given manifold, on which the metric is the induced one from the metric on the vector bundle. Besides, the verbein was assumed to be invertible, but as Witten remarks "permitting the verbein to not be invertible seems like a minor change." In the present case, we don’t have at all any symmetry on the tangent bundle, neither is anything supposed about the dimensionality of spacetime or even about the existence of the usual inverse tensor, but yet the Lagrangian exhibits the same cubic structure. Therefore, it may be concluded that the cubic structure of Chern-Simons theory\(^{32} \) is inherent in the structure of the gravitational Lagrangian itself, and not in the additional assumptions in Ref.32, which affect the choice of the gauge variables. In view of this, it might be interesting to investigate whether there is a transition from the Lagrangian in our case to the Lagrangian for 2+1 dimensional gravity, presented in Ref. 32.

Of course, one might slightly modify the basic assumption, concerning the first representation of the gravitational Lagrangian. For example, instead of assuming that the
Ricci tensor will be the same in both representations, one might instead assume that the Ricci tensor should not change. In such a case again in the limit \(d^2x^k = 0\) the cubic algebraic equation will be in a form without the quadratic in \(dx^i\) term

\[
\begin{align*}
    dx^i dx^j dx^k p \Gamma^r_{ij} (g_k)_r - R &= 0 . \\
\end{align*}
\]  

(21)

One can write down also the vacuum Einstein’s equations when the contravariant tensor is defined as \(\bar{g}^{ij} = dx^i dx^j\)

\[
    0 = \bar{R}_{ij} - \frac{1}{2} g_{ij} \bar{R} = \bar{R}_{ij} - \frac{1}{2} g_{ij} dx^m dx^n \bar{R}_{mn} =
\]

\[
    = -\frac{1}{2} pg_{ij} \Gamma^r_{mn} g_{kr} dx^k dx^m dx^n + \frac{1}{2} g_{ij}(\Gamma^r_{km} g_{nr} + \Gamma^r_{n(m)jr}) d^2x^k dx^m dx^n +
\]

\[
    + p \Gamma^r_{ij} g_{kr} dx^k - (\Gamma^r_{ik} g_{jr} + \Gamma^r_{j(i)g_k}) d^2x^k .
\]  

(22)

Note the following subtle moment: since we have an expression equal to zero, this time it is not necessary to assume that the above algebraic equation is valid under the assumption that the Ricci tensor does not change. Therefore, equation (22) provides the interesting possibility for classification of all solutions of the vacuum Einstein's equations with a given metric tensor \(g_{ij}\) and unknown contravariant tensor \(\bar{g}^{ij} = dx^i dx^j\). In spite of the presence of the second differentials \(d^2x^k\), equation (22) can be treated on an equal footing as an algebraic equation simply by “extending” the algebraic variety for the \(\{dx^k\}\) variables with the new variable \(dy^k = d^2x^k\). However, if additionally it is assumed that the Ricci tensor does not change under the definition of the contravariant tensor (i.e. \(\bar{R}_{ij} = R_{ij}\)), then one has

\[
    (\Gamma^r_{ik} g_{jr} + \Gamma^r_{j(i)g_k}) d^2x^k = p \Gamma^r_{ij} g_{kr} dx^k - R_{ij}
\]  

(23)

and consequently all the terms with \(d^2x^k\) in the Einstein’s vacuum equations (22) drop out and the algebraic equation becomes a cubic one in respect to the variables \(dx^k\) only. The above analyses have the purpose to demonstrate that depending on the initial assumptions about the Ricci tensor or scalar curvature, the structure of the algebraic equation also changes.

In an algebraic language,\(^{20,27,28}\) the investigated problem can be formulated in the following way:

**Proposition 1** Let the differentials \(dx^i (i = 1, \ldots, n; n \) is the space-time dimension) represent elements of an algebraic variety \(\mathbf{X} = (dx^1, dx^2, \ldots, dx^n)\). For different metric tensors (and therefore - different connections \(\Gamma^k_{ij}\) and Riemannian tensors \(R_{ik}\)), a set of polynomials (cubic algebraic equations) \(F(\mathbf{X}) \equiv 0\) may be obtained, which are defined on the algebraic variety \(\mathbf{X}\) and belong to the ring \(R[dx^1, dx^2, \ldots, dx^n]\) of all third-rank polynomials. Then finding all the possible parametrizations of some introduced generalized coordinates \(X^i(x^1, x^2, x^3, \ldots, x^n)\) is equivalent to: 1. Finding all the elements \(dX^i\) of the algebraic
variety $\overline{X}$, satisfying the equation $F(\overline{X}) \equiv 0$. These elements will be represented in the following way

$$
\frac{dx^i}{\delta} = \Phi^i(x^1, x^2, \ldots, x^n), g_{ij}(x^1, x^2, \ldots, x^n), \Gamma_{ij}^k(x^1, x^2, \ldots, x^n), R_{ij}(x^1, x^2, \ldots, x^n)).
$$

(24)

2. Finding all the solutions of the above system of partial differential equations.

In the present case, the algebraic equation is obtained before performing the variation of the Lagrangian, unlike the considered in Ref.18 another case, when again a cubic algebraic equation had been obtained after performing a variation.

Let us comment briefly on the important from a physical point of view assumption $d^2x^i \equiv 0$, under which the cubic equation (20) was derived. Suppose that for the set of generalized coordinates $X^i \equiv X^i(x^1, x^2, \ldots, x^n)$ one has

$$
\frac{dX}{\delta} \equiv a_i dx^i
$$

(25)

and let us assume that the Poincare’s theorem is fulfilled in respect to $dx^i$, i.e. $d^2x^i = 0$. Then

$$
d^2X = da_i dx^i + a_i d^2x^i = \frac{\partial a_i}{\partial x^j} dx^j \wedge dx^i = \left( \frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i} \right) dx^i dx^j.
$$

(26)

Clearly, $d^2X = 0$ only in the following two cases: 1. $a_i = \text{const.}$, i.e. $dX^i$ is a full differential.

2. $(\text{rota})_{ij} \equiv \frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i} \equiv 0$. The last means that if $dx^i$ are considered to be basic eigenvectors, then $dX^i$ have zero-vorticity components. Throughout the whole paper $dX^i$ shall be considered as vector field’s components in the tangent space $T_X$.

Note also that the algebraic equation (19) with the first and the second differentials $dx^i$ and $d^2x^i$ takes into account two important physical characteristics of the vector field $dx^i$ - the divergency $p$ and the vorticity (through the term $d^2x^i$). It might be required that these characteristics vanish, i.e. $p = d^2x^i = 0$. In such a case one is left only with the equation

$$
R_{ik}dx^i dx^k \equiv 0.
$$

(27)

If additionally the requirement for the existence of the (usual) inverse metric tensor is imposed then the intersection variety of the quadratic form (27) with the quadratic forms (one-when $\delta^j_i = 0$, and the other - when $\delta^j_i = 1$).

$$
g_{ik} dx^k dx^j \equiv \delta^j_i.
$$

(28)

has to be found. From the two last equations one easily obtains

$$
(R_{ik} - \frac{1}{2} g_{ik} R) dx^k dx^i \equiv -\frac{1}{2} R \delta^j_i,
$$

(29)
in which the left-hand side is identically zero for every \( dx^i \) in view of the Einstein’ equations

\[
R_{ik} - \frac{1}{2} g_{ik} R \equiv 0,
\]

but the right-hand side is zero only for \( i \neq j \), but not also when \( i = j \). Therefore, the Einstein’s equations are obtained only in one case and not in the other case. In fact, it shouldn’t be surprising that the Einstein’s equations cannot be obtained for both the cases \( i \neq j \) and \( i = j \). One should remember that the usual variational procedure in general relativity takes into account also the variation of the volume factor \( \sqrt{-g} \), while in our purely algebraic treatment and without any variation this volume factor was not subjected to any changes at all. Moreover, it is one standard procedure to perform the variational procedure with the usual gravitational Lagrangian and the inverse metric tensor (when the Einstein’s equations are obtained) and its quite a different procedure to start from the other representation of the gravitational Lagrangian (where the variables to be variated are \( g_{ij}, \Gamma^k_{ij} \) (or \( g_{ij,k} \)) and \( dx^i \) and \( d^2 x^i \)) and afterwards to impose the requirement for identification of the contravariant metric tensor with the inverse one in the form of another, additional equation. So one should not even hope to obtain anything similar to the Einstein’s equations. However, as already shown, if one has the Einstein’s equations, one may still ask the question are they satisfied under the new definition of the contravariant tensor.

**IV. A GENERAL MATHEMATICAL SETUP FOR TREATING THE CUBIC ALGEBRAIC EQUATION (20)**

The subsequent investigation of equation (20) will be restricted to the case of a 5-dimensional space-time, although the approach of course may be applicable to any dimensions. The main reason for choosing a 5-d spacetime is related to the widely discussed Randall-Sundrum (R - S) model\(^{29,30}\) in which the process of compactification of the five-dimensional universe to our present four-dimensional universe is related to the existence of a *large extra dimension*. In the original R - S scenario the metric was chosen to be

\[
ds^2 = e^{-2kr_c r_5} \eta^{\mu\nu} dx^\mu dx^\nu + r_c^2 dx^2_5, \tag{30}
\]

where \( r_c \) is a compactification radius, \( \eta^{\mu\nu} \) is the ordinary Minkowski metric, \( x_5 \subset [0, \pi] \) is a periodic coordinate, \( \mu\nu \) are four dimensional indices and \( k \) is a scale of order of the Planck scale. Instead of the coordinate \( x_5 \), one may chose for example a fifth coordinate \( X_5 = kr_c x_5 \), which in view of the largeness of the scale factor \( k \) may be assumed to range to infinity. *But the infinity point, from a purely mathematical point of view, may be treated on an equal footing with all other points in the framework of projective geometry*.\(^{7,21,38,39}\) In the present case the infinity point shall be realized in respect to \( dx^5 \) after performing the *linear-fractional transformation*

\[
dx^5 \equiv \frac{\tilde{a} dx + b}{\tilde{c} dx + d}, \tag{31}
\]
where \(a\), \(b\), \(c\) and \(d\) will be assumed to be functions, depending on the complex variable \(z\) (or on two complex variables). Also, the remaining four-dimensional space-time with coordinates \((x^1, x^2, x^3, x^4)\) may be complexified in the following way

\[
z_1 = x_1 + ix_2; \quad z_2 = x_3 + ix_4.
\]

It is easily seen that the infinity point in respect to \(dx^5\) is situated at \(d\tilde{x}^5 \equiv -d\tilde{c}\) and it is a zero point for the complex plane \(d\tilde{x}^5\). The convenience of the linear-fractional transformation from a physical point of view matches also the mathematical requirements of the problem. In order to parametrize the third-rank algebraic equation (15), written in a two-dimensional form, one has to bring it to the form

\[
(d\tilde{x}^5)^2 \equiv 4(d\tilde{x}^4)^3 - g_2(d\tilde{x}^4) - g_3.
\]

In the case when \(g_2\) and \(g_3\) are complex numbers \(g_2 = 60G_4 = 60\sum \varpi \omega_2\) and \(g_3 = 140G_6 = \sum \varpi \omega_2\), standard algebraic geometry contains a well-known prescription how to parametrize this algebraic curve\(^{13}\) by introducing the variables

\[
d\tilde{x}^4 \equiv \rho(z) \quad \quad \quad \quad d\tilde{x}^5 \equiv \rho'(z),
\]

where \(z\) is a complex variable and \(\rho(z)\) is a complex meromorphic function - the Weierstrass function

\[
\rho(z) \equiv \frac{1}{z^2} + \sum_{\varpi} \left[ \frac{1}{(z - \varpi)^2} - \frac{1}{\varpi^2} \right],
\]

and the summation is over all non-null elements

\[
\varpi \subset \Lambda = \{(m\varpi_1 + n\varpi_2) \mid m, n \subset Z \text{ (integer numbers)}, \varpi_1, \varpi_2 \subset C, Im > 0\}.
\]

Since further in the text the parametrization (34) will be repeatedly used, it is instructive to give just an idea how in classical textbooks it is proven that the parametrization (34) satisfies eq.(33). Let us take for example the proof, given in Ref.13, where the basic idea is to compare the Loran expansions for the non-positive degrees of \(z\) for the function \([\rho'(z)]^2\) and for the polynomial \(a\rho^3(z) + b\rho^2(z) + c\rho(z) + d\), where \(a, b, c\) and \(d\) are complex numbers. If the corresponding coefficients in the Loran expansion of these two expressions are equal, this would mean that the expressions themselves are equal. Also, it should be accounted that the function \([\rho'(z)]^2\) is an even one, and consequently only the even (non-positive) powers of \(z\) in the Loran decomposition of the two expressions should be taken into account. After performing the Loran decomposition, it becomes evident that equality of the two expressions is possible only if \(a = 4, b = 0, c = -60G_4, d = -140G_6\). Since these coefficients give exactly the algebraic equation (33), it follows that the Weierstrass function and its derivative (34) satisfy equation (33), thus representing ”uniformization variables” for the equation (33), i.e. variables, which are functions of the complex variable \(z\) and at the same time satisfy the given algebraic relation.
It is important to stress that the “tilda” differentials $d\tilde{x}^4$ and $d\tilde{x}^5$, which are related through the algebraic relation (33) and the parametrization (34) with the Weierstrass function, do not result in any dependence between the original differentials $dx^4$ and $dx^5$, which should remain independent since are related to the independent coordinates in the gravitational Lagrangian. The reason for this independence between the tilda and the non-tilda differentials is that the linear-fractional transformation (31), which relates $d\tilde{x}^5$ and $dx^5$, introduces an additional arbitrariness in the non-tilda differentials due to the arbitrary complex functions $a$, $b$, $c$ and $d$.

In the present case, however, there are some specific facts about the parametrization of the curve: 1. The parametrization shall be carried out not in respect to two of the variables, but in respect to $d\tilde{x}^5$ and another variable, which is $\tilde{x}^5$ - the two of the parameters, entering the linear-fractional transformation. The rest of the variables, entering the cubic algebraic equation shall be “hidden” in the free term, which is a function. So actually the final result will be for $d\tilde{x}^5$ (or $\tilde{x}^5$), expressed through the Weierstrass function, but in order the parametrization to be consistent, the remaining four differentials ($dx^1, dx^2, dx^3, dx^4$) should be related to $a, b, c$ and $d$ in a complicated way. 2. After performing the transformation (31) with the purpose of choosing $a, b, c$ and $d$ in to eliminate the highest (third) power of $d\tilde{x}^5$, the obtained equation will be like equation (33), but with $g_2$ and $g_3$ - functions and not complex numbers. On the other hand, the standard parametrization (34) with the Weierstrass function and its derivatives is valid only for $g_2$ and $g_3$ complex numbers. However, it will be proved in the next sections, that in such a case the formalism and the parametrizable equation can also be used. It will be shown that all the coefficient functions (those standing before the pole terms) in the Loran expansion can be found if the sums $G_n$ are known.

V. TRANSFORMED CUBIC EQUATION

WITH THE HELP OF THE LINEAR - FRACTIONAL TRANSFORMATION

In order to derive this equation, all the terms with $dx^5$ in equation (20) shall be singled out and it can be written in the following way

$$A(dx^5)^3 + B(dx^5)^2 + C(dx^5) + C^{(4)}(dx^4, ...dx^1, g_{ij}, \Gamma^k_{ij}, R_{ik}) \equiv 0,$$

(36)

where $A$, $B$ and $C$ are the following functions, depending on $g_{ij}$, $\Gamma^k_{ij}$, $R_{ij}$ and the differentials $dx^\alpha$, $dx^\beta$ ; the indices $\alpha, \beta = 1, 2, 3, 4$; $r = 1, 2, ..., 5$.

$$A \equiv 2p\Gamma^r_{55}g_5r$$

(37)

$$B \equiv 6p\Gamma^r_{a5}g_5, dx^\alpha$$

(38)
and

\[ C \equiv -2R_{\alpha 5}dx^\alpha + 2p(2\Gamma_{\alpha \beta r}^r g_{5r} + \Gamma_{5\alpha 5\beta r}^r)dx^\alpha dx^\beta. \quad (39) \]

The function \( G^{(4)}(\ldots) \) is of the following form

\[ G^{(4)}(dx^4, \ldots dx^1, g, \Gamma_{ij}^k, R_{ik}) \equiv -R_{\alpha \beta}dx^\alpha dx^\beta + pdx^\gamma dx^\alpha dx^\beta \Gamma_{\gamma(\alpha \beta r)}^r. \quad (40) \]

In (40) the indice \( \gamma = 1, 2, 3, 4 \) and \((\alpha, \beta)\) means symmetrization in respect to the two indices. Further, after performing the linear-fractional transformation (31), one easily obtains the new cubic algebraic equation, written in terms of the new variables \( \tilde{dx}^5 \):

\[
(G^{(4)} c^3 + aQ)(\tilde{dx}^5)^3 + (bQ + aT + 3c^2 dG^{(4)})(\tilde{dx}^5)^2 + \\
+ (aS + bT + 3cd^2 G^{(4)})(\tilde{dx}^5) + (bS + G^{(4)} d^3) \equiv 0 ,
\]

where \( Q, T, S \) denote the following expressions

\[ Q \equiv Aa^2 + Ce^2 + Bac + 2cdC, \quad (42) \]
\[ T \equiv 2Aab + Bbc + Bad + 2cdC, \quad (43) \]
\[ S \equiv Ab^2 + Bbd + Cd^2. \quad (44) \]

In fact, the linear-fractional transformation is performed with the purpose of setting up to zero the expression before \((\tilde{dx}^5)^3\), from where \( G^{(4)} \) is expressed as

\[ G^{(4)} = -\frac{aQ}{c^3}. \quad (45) \]

This equation is the first additional equation, which is imposed in order to receive the parametrizable form of the cubic equation. Let us write down in more details equation (45), in order to understand its meaning. Making use of the expressions for \( G^{(4)} \) and \( Q \), it can be written in the form again of a cubic algebraic equation in respect to the remaining four differentials

\[
p\Gamma_{\gamma(\alpha \beta r)}^r dx^\gamma dx^\alpha dx^\beta + K_{\alpha \beta}^{(1)} dx^\alpha dx^\beta + K_{\alpha}^{(2)} dx^\alpha + 2p \left( \frac{a}{c} \right)^3 \Gamma_{55 r}^r g_{5r} = 0, \quad (46) \]

where \( K_{\alpha \beta}^{(1)} \) and \( K_{\alpha}^{(2)} \) are the corresponding quantities

\[ K_{\alpha \beta}^{(1)} \equiv -R_{\alpha \beta} + 2p \frac{a}{c} (1 + 2 \frac{d}{c}) (2\Gamma_{\alpha \beta r}^r g_{5r} + \Gamma_{5\alpha 5\beta r}^r) \quad (47) \]

and

\[ K_{\alpha}^{(2)} \equiv 2 \frac{a}{c} \left[ 3p \frac{a}{c} \Gamma_{\alpha 5 r}^r g_{5r} - (1 + 2 \frac{d}{c}) R_{5\alpha} \right]. \quad (48) \]
The indices $\alpha, \beta, \gamma = 1, 2, 3, 4$ (but $r = 1, 2, \ldots, 5$) and $(\alpha, \beta)$ means symmetrization in respect to the two indices. In other words, the imposed ("by hand") equation (45) simply fixes the cubic algebraic equation in respect to the remaining four differentials, if one would like to parametrize the differential of the fifth coordinate with the Weierstrass function. No ratios $\frac{a}{c}$ and $\frac{d}{c}$ are to be determined from this equation - later on from the equation in respect to the fifth coordinate they will be determined.

Using expression (45), the functions standing before $(\tilde{\omega}^{5_{\alpha}})^2, \tilde{\omega}^{5_{\beta}}$ in (41) and also the free term in the same equation can be written in a form of an algebraic expression in respect to $\frac{a}{c}, \frac{b}{d}$ and $\frac{b}{d}$

$$bQ + aT + 3c^2dG^{(4)} = d^3\{-3A\frac{a}{c}(\frac{a}{d})^2 + C\frac{b}{d}(\frac{c}{d})^2 + 2C\frac{b}{d} - 6C\frac{a}{d} +$$

$$+3A\frac{b}{d}(\frac{a}{d})^2 + B\frac{a}{d}\frac{b}{d} - 2B(\frac{a}{d})^2 - C\frac{a}{d}\frac{c}{d}\}, \quad (49)$$

$$aS + bT + 3cd^2G^{(4)} = d^3\{-3A\frac{a}{c}(\frac{a}{d})^2 + B\frac{b}{d}(\frac{c}{d})^2 + 2C\frac{b}{d} -$$

$$-3B\frac{a}{c} - 6C\frac{a}{d} + 2B\frac{a}{d} + 3A\frac{b}{d}(\frac{a}{d})^2 - C\frac{a}{d}\}, \quad (50)$$

and

$$bS + G^{(4)}d^3 = d^3\{-A(\frac{a}{c})^3 + A(\frac{b}{d})^3 + B(\frac{b}{d})^2 -$$

$$-B(\frac{a}{d})^2 + C\frac{b}{d} - C\frac{a}{c} - 2C\frac{a}{d}\frac{c}{d}\}. \quad (51)$$

Let us now introduce the notations

$$\frac{a}{c} \equiv m \quad \tilde{\omega}^{5_{\alpha}} \equiv n \quad \tilde{\omega}^{5_{\beta}} \equiv n . \quad (52)$$

Equations (49-51) shall be rewritten in such a way so that the terms with powers of $m$ will be singled out. The rest of the terms will be denoted by $\overline{F}, \overline{M}$ and $\overline{N}$ and they will contain only powers of $\frac{a}{c}$ and $\frac{b}{d}$ only. The transformed equations (49-51), if substituted back into equation (41), allow one to write the equation into the following form

$$-3A(\frac{c}{d})^2m^3n^2 - 3A(\frac{c}{d})m^3n + [3A(\frac{c}{d})^2\frac{b}{d} - 2B(\frac{c}{d})^2]m^2n^2 - 3B\frac{c}{d}m^2n +$$

$$+[-6C\frac{c}{d} + B\frac{b}{d}(\frac{c}{d})^2 - C(\frac{c}{d})^2]mn^2 + [-6C + 2B\frac{c}{d} + 3A\frac{b}{d}(\frac{c}{d})^2 - C\frac{c}{d}]mn +$$

$$+\overline{F}n^2 + \overline{M}n + [\overline{M} - Am^3 - Bm^2 - Cm - 2\frac{d}{c}Cm] \equiv 0 . \quad (53)$$
The terms $F$, $M$ and $N$ have the following form

$$F \equiv C \frac{b}{d} \left(\frac{c}{d}\right)^2 + 2C \frac{b}{d} \frac{c}{d}, \quad (54)$$

$$M = A \left(\frac{b}{d}\right)^3 + B \left(\frac{b}{d}\right)^2 + C \frac{b}{d}, \quad (55)$$

$$N = B \frac{c}{d} \left(\frac{b}{d}\right)^2 + 2C \frac{c}{d} \frac{b}{d}. \quad (56)$$

In other words, we have transformed the original third-rank algebraic equation of five variables $dx_1, dx_2, dx_3, dx_4, dx_5$ into an algebraic equation of two variables only ($m$ and $n$), but with a higher rank (in the case it’s five).

VI. A PROPOSAL FOR STANDARD PARAMETRIZATION OF THE CUBIC ALGEBRAIC EQUATION WITH THE WEIERSTRASS FUNCTION

By standard parametrization it shall be meant that the cubic algebraic equation should be brought to its standard parametrizable form

$$\tilde{n}^2 = 4m^3 - g_2m - g_3, \quad (57)$$

where $g_2$ and $g_3$ are the already known complex numbers. Then one has the right to set up

$$\tilde{n} = \rho'(z) = \frac{d\rho}{dz}, \quad m = \rho(z). \quad (58)$$

In order to obtain the parametrizable form (57), it is instructive to write down the obtained algebraic equation in the form of a third-rank polynomial of $m$ with coefficient functions $P_1(n)$, $P_2(n)$, $P_3(n)$ and $P_4(n)$, representing quadratic forms of $n$ and at the same time cubic algebraic expressions in respect to $\frac{c}{d}$ and $\frac{b}{d}$:

$$P_1(n)m^3 + P_2(n)m^2 + P_3(n)m + P_4(n) \equiv 0, \quad (59)$$

where

$$P_1(n) \equiv r_1n^2 + r_2n + r_3 = -3A \left(\frac{c}{d}\right)^2 n^2 - 3A \frac{c}{d}n - A, \quad (60)$$

$$P_2(n) \equiv q_1n^2 + q_2n + q_3 = \left[3A \left(\frac{c}{d}\right)^2 - 2B \left(\frac{c}{d}\right)^2\right] n^2 - 3B \frac{c}{d}n - B. \quad (61)$$
\[ P_3(n) \equiv p_1n^2 + p_2n + p_3 = \left[ -6C\frac{c}{d} + B\frac{b}{d} - C\left(\frac{c}{d}\right)^2 \right] n^2 + \]
\[ + \left[ -6C + 2B\frac{b}{d} + 3A\left(\frac{b}{d}\right)^2 - C\left(\frac{c}{d}\right) \right] n - C - 2\frac{d}{c}C , \tag{62} \]
\[ P_4(n) \equiv \overline{F}n^2 + \overline{N}n + \overline{M} . \tag{63} \]

Let us write down the last expression in the following form
\[ P_4(n) \equiv \overline{F} \left[ \left(n + \frac{\overline{N}}{2\overline{F}}\right)^2 + \frac{\overline{M}}{\overline{F}} - \left(\frac{\overline{N}}{2\overline{F}}\right)^2 \right] \equiv \tilde{n}^2 + \overline{M} - \frac{\overline{N}^2}{4\overline{F}} , \tag{64} \]
where \( \tilde{n} \) denotes
\[ \tilde{n} \equiv \sqrt{\overline{F}} \left(n + \frac{\overline{N}}{2\overline{F}}\right) . \tag{65} \]

In terms of \( \tilde{n} \), the transformed equation (59) can be written as
\[ \tilde{n}^2 = \overline{P}_1(\tilde{n}) \ m^3 + \overline{P}_2(\tilde{n}) \ m^2 + \overline{P}_3(\tilde{n}) \ m + \overline{P}_4(\tilde{n}) , \tag{66} \]
where the coefficient function \( \overline{P}_1(\tilde{n}) \) is
\[ \overline{P}_1(\tilde{n}) \equiv r_1 \tilde{n}^2 + r_2 \tilde{n} + r_3 = \]
\[ = -r_1 \tilde{n}^2 + \left[ \frac{\overline{N}}{\overline{F}^2} r_1 - \frac{r_2}{\overline{F}^2} \right] \tilde{n} + \left[ -r_1 \frac{\overline{N}^2}{4\overline{F}^2} + r_2 \frac{\overline{N}}{2\overline{F}} - r_3 \right] \tag{67} \]
and
\[ \overline{P}_4(\tilde{n}) \equiv \frac{\overline{N}^2}{4\overline{F}} - \overline{M} . \tag{68} \]

The other coefficient functions \( \overline{P}_2(\tilde{n}) \) and \( \overline{P}_3(\tilde{n}) \) can be written analogously, but with \( (q_1, q_2, q_3) \) and \( (p_1, p_2, p_3) \) in (67) instead of \( (r_1, r_2, r_3) \). Note that unlike the expressions for \( r, q \) and \( p \), representing cubic algebraic expressions in respect to \( \frac{b}{d} \) and \( \frac{c}{d} \), the corresponding "bar" quantities represent more complicated expressions, which are no longer polynomials. It is also not correct to consider the transformation from \( (p, q, r) \) to \( (\overline{p}, \overline{q}, \overline{r}) \) as a linear affine transformation. The expressions \( \overline{N} \) and \( \overline{F} \), entering the coefficient functions of the transformation depend also on \( \frac{b}{d} \) and \( \frac{c}{d} \), so presumably they could also be expressed through \( (p, q, r) \). The above transformation shall be investigated further.

**Our purpose will be to identify the investigated equation (54) \( \tilde{n}^2 = \overline{P}_1(\tilde{n}) \ m^3 + \overline{P}_2(\tilde{n}) \ m^2 + \overline{P}_3(\tilde{n}) \ m + \overline{P}_4(\tilde{n}) \) with equation (57) \( \tilde{n}^2 = 4m^3 - g_2m - g_3 \), for which we already know that the substitution (58) can be performed. In order to obtain the standard parametrizable form of the cubic equation, one has to require that**
the two equations are to be made equal, which means that the polynomials $P_1(\tilde{n})$, $P_2(\tilde{n})$, $P_3(\tilde{n})$ and $P_4(\tilde{n})$ (depending on the variable $\tilde{n}$) are to be made equal to the numerical coefficients 4, 0, $-g_2$, and $-g_3$ respectively. Therefore, the following system of equations should be fulfilled

\begin{align}
4 &= r_1 \tilde{n}^2 + r_2 \tilde{n} + r_3, \quad (69) \\
0 &= q_1 \tilde{n}^2 + q_2 \tilde{n} + q_3, \quad (70) \\
-g_2 &= p_1 \tilde{n}^2 + p_2 \tilde{n} + p_3, \quad (71) \\
-g_3 &= \frac{N^2}{4F} - M. \quad (72)
\end{align}

The last equation (72) represents the second additional equation, imposed in order to obtain the parametrizable form of the cubic equation. Note that this equation has an extremely complicated structure: since $N$, $F$ and $M$ are third-rank polynomials in respect to $b$ and $c$, the equation will be of sixth order! This causes inconvenience in investigating such equations, therefore it is appropriate to search another variables, in terms of which the algebraic treatment will be comparatively more convenient.

Let us try to find such variables. From the first and the second equations (70,71) the terms with $\tilde{n}^2$ can be excluded, and also from the second and the third equations. The obtained equations are

\begin{align}
4q_1 &= (r_2q_1 - r_1q_2)\tilde{n} + (r_3q_1 - r_1q_3), \quad (73) \\
-g_2q_1 &= (p_2q_1 - p_1q_2)\tilde{n} + (p_3q_1 - p_1q_3). \quad (74)
\end{align}

From the last two equations the terms with $\tilde{n}$ can also be excluded and a fourth-rank algebraic equation is obtained in respect to $p_i$, $q_i$ and $r_i$ ($i = 1, 2, 3$)

\begin{align}
(p_2q_1 - p_1q_2)(4q_1 - r_3q_1 + r_1q_3) + \\
+ (p_2q_1 - p_1q_2)(g_2q_1 + p_3q_1 - p_1q_3) &= 0. \quad (75)
\end{align}

The above equation represents the third additional equation, imposed in order to obtain the parametrizable form of the cubic equation. This equation is difficult to deal with, but there is a way to rewrite it in a more convenient and simple form. Let us introduce the “angular” type variables $l$ and $f$ with the corresponding components

\begin{align}
l &= (l^1, l^2, l^3) = (l_{12}, l_{23}, l_{31}) = \\
&= (p_1q_2 - p_2q_1, p_2q_3 - p_3q_2, p_3q_1 - p_1q_3), \quad (76)
\end{align}
\[ f = (f^1, f^2, f^3) = (f_{12}, f_{23}, f_{31}) = (f^1, f^2, f^3) = (f_{12}, f_{23}, f_{31}) = (r_1 q_2 - r_2 q_1, r_2 q_3 - r_3 q_2, r_3 q_1 - r_1 q_3). \]  

In terms of these variables, the fourth-rank algebraic equation (75) will be reduced to the following quadratic equation

\[ 4q_1 l_1^1 + g_2 f^1 q_1 + l_1 f^3 + f^1 l_3^1 = 0. \]  

Having found the algebraic variety for \((q_1, l_1, l_3, f^1, f^3)\), one can go back to find the algebraic variety for \((p, q, r)\). From there by means of the inverse transformation of (67)

\[ r_1 = -F r_1; \quad r_2 = -F_2 r_2 - N r_1 \]  

\[ r_3 = -N^2 / 4F r_1 - N / 2F^2 r_2 - r_3 \]  

(79)

(80)

(81)

VII. FINDING THE NONLINEAR AND NONPOLYNOMIAL INVERTIBLE TRANSFORMATION

We shall start from expressions (60-62), from where one can find

\[ r_3 = -A \quad r_2 = 3c / d r_3 \quad r_1 = r_2 / 3r_3. \]  

(82)
\[ q_3 = -B \quad q_2 = 3 \frac{c}{d} q_3 \quad q_1 = -r_2 \frac{q_2 b}{3q_3 d} + 2 \frac{q_2^2}{9q_3} \quad , \] (83)

where it has been used that \( \frac{q_2}{q_3} = \frac{r_2}{r_3} \). If expressions (83) for \( q = (q_1, q_2, q_3) \) are substituted into the defined by (67) expressions for \( q_1 \), it can be obtained

\[ \bar{q}_1 = -q_1 \frac{1}{F} = - \frac{1}{F} \left( \frac{c}{d} \right)^2 \left[ 2 - 3r_3 \frac{b}{d} \right] \quad , \] (84)

\[ \bar{q}_2 = \frac{N}{F^2} q_1 - q_2 \frac{1}{F^2} = \frac{N}{F^2} \left( \frac{c}{d} \right)^2 \left[ 2 - 3r_3 \frac{b}{d} \right] - 3q_3 \frac{c}{d} \frac{1}{F^2} \quad , \] (85)

\[ \bar{q}_3 = -q_1 \frac{N^2}{4F^2} + q_2 \frac{N}{2F} - q_3 = \]

\[ = q_3 \left[ \frac{3N c}{2F d} - 1 \right] + r_3 \left[ \frac{3N^2}{4F^2} \left( \frac{c}{d} \right)^2 \frac{b}{d} \right] - \frac{N^2}{F^2} \left( \frac{c}{d} \right)^2 \quad . \] (86)

The corresponding equations for \( r = (\bar{r}_1, \bar{r}_2, \bar{r}_3) \) are

\[ \bar{r}_1 = -3r_3 \left( \frac{c}{d} \right)^2 \quad , \] (87)

\[ \bar{r}_2 = \frac{N}{F^2} 3r_3 \left( \frac{c}{d} \right)^2 - 3r_3 \frac{\bar{r}_1}{F^2} \quad , \] (88)

\[ \bar{r}_3 = \bar{r}_1 \left[ -\frac{N}{2F^2} + \frac{N^2}{4F^2} + \frac{3}{3} \left( \frac{c}{d} \right)^2 \right] \quad . \] (89)

From the first and the second two equations it can be obtained respectively

\[ \frac{1}{\bar{r}_1} = - \frac{1}{\bar{r}_2} \frac{1_{\bar{r}_1}}{F^2} \quad \frac{1}{\bar{r}_2} \frac{1_{\bar{r}_1}}{F^2} \quad \frac{1}{\bar{r}_3} \frac{1_{\bar{r}_1}}{F^2} \quad \] (90)

and

\[ \frac{(3 + 4\bar{r}_1^2 - 6\bar{r}_1)}{12\bar{r}_1} Y^2 + \frac{(\bar{r}_1 - 1)\bar{r}_2}{2\bar{r}_1} Y + \left( \frac{\bar{r}_2^2}{4\bar{r}_1} - \bar{r}_3 \right) = 0 \quad , \] (91)

where \( Y \equiv \frac{\bar{r}_2}{\bar{r}_1} \). It is important to note that \( Y \) can be found as a solution of the above quadratic equation with coefficient functions, which consist only of \( \bar{r} \). Therefore

\[ \frac{c}{d} = \frac{1}{Y} \frac{1}{\bar{r}_1^2} = Z \bar{r}_1^2 \quad \frac{N}{O \bar{r}_1^2} = \left[ \frac{1}{\bar{r}_1} \left( -\bar{r}_2 + \frac{1}{Y} \right) \right] \bar{r}_1^2 \quad . \] (92)

Now let us write down the corresponding equations for \( p \) from (62)

\[ p_1 = -6C \frac{c}{d} + B \frac{b c}{d d} - C \left( \frac{c}{d} \right)^2 \quad , \] (93)
\[ p_2 = -6C + \left( \frac{c}{d} \right) \left[ 2B \frac{b}{d} + 3A \left( \frac{b}{d} \right)^2 - C \right], \quad (94) \]

\[ p_3 = -2C \frac{d}{c} - C \quad . \quad (95) \]

If from the last expression \( C \) is expressed and is substituted into (93), an expression for \( \frac{b}{d} \) can be obtained in the form

\[ \frac{b}{d} = - \frac{p_1}{q_3 \bar{F}_3 Z} + \frac{p_3 \overbar{F}_3 \left[ 6 + \overbar{F}_3 \right]}{q_3 \left[ 2 + Z \overbar{F}_3 \right]}, \quad (96) \]

where the derived expressions (92) have also been used. In order to obtain an expression for \( \frac{b}{d} \) in terms of the “bar” variables \( \overbar{p} = (p_1, p_2, p_3) \), the “non-bar” variables \( p_1, p_2 \) and \( p_3 \) should be expressed from the system of equations for \( \overbar{p} \):

\[ \overbar{p}_1 = - \frac{p_1}{F} \quad ; \quad \overbar{p}_2 = \frac{N}{F^2} p_1 - \frac{p_2}{F^2} \]

\[ \overbar{p}_3 = - p_1 \frac{N^2}{4F^2} + p_2 \frac{N}{2F} - p_3 \]

and substituted into (96). The result is

\[ \frac{b}{d} = \frac{1}{q_3} \left[ \frac{\overbar{F}_3}{Z} \overbar{p}_1 - \frac{(6 + Z \overbar{F}_3)}{(2 + Z \overbar{F}_3)} \left( \frac{O^2 \overbar{F}_3}{4} Z \overbar{p}_1 + \frac{O^2 \overbar{F}_3}{2} Z \overbar{p}_2 + Z \overbar{F}_3 \overbar{p}_3 \right) \right], \quad (99) \]

The only ”unbar” variable \( q_3 \) can be expressed from the first two equations (84-85) for \( \overbar{q}_1 \) and \( \overbar{q}_2 \)

\[ q_3 = - \frac{O \overbar{q}_1 + \overbar{q}_2}{3Z} \quad . \quad (100) \]

Also, from the third equation (86) for \( \overbar{q}_3 \) it can be obtained

\[ 3r_3 b \frac{b}{d} = 8 \left[ 6Z \overbar{q}_3 + 3O^2 Z^3 + (3OZ - 2)(O \overbar{q}_1 + \overbar{q}_2) \right] \quad (101) \]

and from the first equation (3) the same expression can be found to be

\[ 3r_3 b \frac{b}{d} = \frac{2Z^2 + \overbar{q}_1}{Z^2} \quad . \quad (102) \]

From the equality of the above two formulae one relation between the ’bar’ variables can be found. More concretely, since \( O \) and \( Z \) depend only on \( \overbar{r} \), the relation will concern how \( \overbar{q}_3 \) can be expressed through \( O, Z \) and \( \overbar{q}_1, \overbar{q}_2 \). This will not be used further in the text,
since our main purpose will be to find the ratio \( \frac{r_3F}{q_2} \), which is to be used in the subsequent formulae

\[ \frac{r_3F}{q_2} = - \frac{F^3(2Z^2 + \bar{q}_1)(2 + ZF^2)}{(Oq_1 + \bar{q}_2)K_1} \quad , \tag{103} \]

where \( K_1 \) is the expression

\[ K_1 \equiv \bar{p}_1(2 +ZF^2) - Z^2(6 +ZF^2)(\frac{O^2\bar{p}_1}{4} + \frac{O}{2}\bar{p}_2 + \bar{p}_3) \quad . \tag{104} \]

At this moment the only equation not yet used is the one, which can be derived from (93-95) for \( p_2 \)

\[ p_2 = \frac{2p_3 \frac{r_3}{q_3}(6 + \frac{r_3}{d})}{2 + \frac{c}{d}}(- \frac{c}{d} + \frac{r_3}{q_3}3p_1 + \frac{1}{2}) + (2p_1 - \frac{r_3}{q_3}3p_1^2) - \]

\[ - \frac{r_3}{q_3}3p_3^2(\frac{r_3}{q_3})^3(6 + \frac{r_3}{d})^2(2 + \frac{c}{d})^2) \quad . \tag{105} \]

If the \( p \)-variables are expressed from (97-98) through their "bar" counterparts and all preceding expressions are used, the following **cubic algebraic equation** in respect to \( F^2 \equiv T \) is obtained

\[ N_1T^3 + N_2T^2 + N_3T + N_4 = 0 \quad , \tag{106} \]

where \( N_1, N_2, N_3 \) and \( N_4 \) are complicated expressions of the "bar" quantities only. These expressions will be presented in Appendix D. Therefore, the roots of this cubic equation in respect to \( T \) can be found and consequently, the quantities \( \bar{N}, \bar{F}, \bar{M} \), entering the second additional equation (72) also can be expressed in terms of the "bar" variables. This in fact proves that 1. The two additional equations (72) and (78), imposed in order to obtain the parametrizable form of the cubic equation, can be expressed in terms of the "bar" variables only. 2. The nonlinear and non-polynomial transformation from the \((r,q,p)\) to the \((\bar{r},\bar{q},\bar{p})\) variables is an invertible one. This is an important fact, since one first may study the properties of the algebraic equations, given by (72) and (78), and then chose the most convenient form for the ratios \( \frac{b}{d} \) and \( \frac{c}{d} \).

**VIII. PARAMETRIZATION OF A GENERAL CUBIC CURVE WITH COEFFICIENT FUNCTIONS OF A COMPLEX VARIABLE**

In this section an attempt will be made to deal with a cubic curve of a more general kind

\[ \bar{n}^2 = M(z)m^3 + N(z)m^2 + P(z)m + E(z) \quad , \tag{107} \]
where $M, N, P$ and $E$ are functions of the complex variable $z$ and therefore not complex numbers, as usually accepted in standard complex analyses$^5$ and algebraic geometry$^{13}$. In other words, the main problem is whether it is possible to parametrize with the Weierstrass function the above equation, i.e. when does the Weierstrass function satisfy the equation

$$
\frac{d\rho}{dz}^2 = M(z)\rho^3 + N(z)\rho^2 + P(z)\rho + E(z) ?
$$

(108)

As already briefly discussed in Sect. IV for the standard and usually investigated case of $M, N, P, E$ - constants, the Weierstrass function parametrizes the cubic equation (107) only if $M = 4, N = 0, P = -60G_4$ and $Q = -140G_6$, but evidently in the present case of functions, the situation will be quite different.

Let us first decompose $\rho(z)$ into an infinite sum, assuming that $|\varpi|$ is a large number and therefore

\[
\rho(z) = \frac{1}{z^2} + \sum \left[ \frac{1}{\varpi^2(z - 1)^2} - \frac{1}{\varpi^2} \right] = \frac{1}{z^2} + \sum \frac{1}{\varpi^2} (2\frac{z}{\varpi} + 3(\frac{z}{\varpi})^2 + \ldots + (n + 1)(\frac{z}{\varpi})^n + \ldots). \tag{109}
\]

The first derivative of the Weierstrass function is

\[
\rho'(z) = \frac{d\rho}{dz} = -\frac{2}{z^3} + \sum \frac{n(n + 1)}{\varpi^{2+n}} z^{n-1} \tag{110}
\]

and its square degree is

\[
\left[ \rho'(z) \right]^2 = \frac{4}{z^6} - 4 \sum_{n=1}^{\infty} \frac{n(n + 1)}{\varpi^{2+n}} z^{n-4} + \sum_{n=1}^{\infty} \frac{n^2(n + 1)^2}{\varpi^{2(n+2)}} z^{2(n-1)}. \tag{111}
\]

Note that in the strict mathematical sense, the second sum in the last expression is in fact a double sum over $m$ and $n$ \[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{mn(m+1)(n+1)}{\varpi^{m+n}} z^{m+n-2}, \] obtained as a result of the multiplication of the two infinite sums (110) for $\rho'(z)$ with different summation indices. Of course, since the two sums are equal and infinite ones, the representation in the form of a single sum is also correct. The appearance of the double sum should be kept in mind, since the idea further will be to compare the coefficient functions in the Loran power expansion of the functions on the left- and on the right - hand sides of (108), and naturally a double sum will appear in the R.H.S. of (108).

In reference to this, an important remark follows. Suppose one works in the framework of standard arithmetical theory of elliptic functions, when $M, N, P$ and $E$ are assumed to be just number coefficients. Since the Weierstrass functions $\rho(z)$ is an even function of the complex variable $z$ (see Ref.5 and Ref. 7), the whole expression on the right-hand side (R.H.S.) of (108) will be an even one too. On the other hand, the function $\rho'(z)$ in the left-hand side (L.H.S.) of (108) is an odd one, but its square again gives an even function. Therefore, comparing the coefficients in front of the powers in $z$ means that only the even

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powers should be included in the infinite sum decomposition

\[
\left[ \rho'(z) \right]^2 = \frac{4}{z^6} - 76G_6 - 24G_4 \frac{1}{z^2} + \ldots ,
\]

(112)

where \( G_n \) will denote the following infinite sum of the complex pole numbers

\[
G_n = \sum \frac{1}{\omega^n} .
\]

(113)

Note also another very important fact the proof of which is given in Ref.5: the infinite (in numbers of \( \omega \)) sum (113) is always convergent (i.e. finite) when \( n > 2 \), but for \( n \leq 2 \) the finiteness is not guaranteed! In the presently investigated case of \( M, N, P, E \)-complex functions, no information is available whether the R.H.S. of (108) is an even or an odd function in \( z \). Consequently, one should not use formulae (112), but just start with the more general expression (111) for \( \left[ \rho'(z) \right]^2 \).

In order to find the Loran decomposition of the functions on the right-hand side of (108), one should first find the second and the third powers of \( \rho(z) \), which may be written as

\[
\rho^2(z) = \frac{1}{z^4} + 2 \sum_{n=1}^{\infty} (n + 1) \frac{z^{n-2}}{\omega^n} + \sum_{n=1}^{\infty} (n + 1)^2 \frac{z^{2n}}{\omega^{2n}} ,
\]

(114)

\[
\rho^3(z) = \frac{1}{z^4} + 2 \sum_{n=1}^{\infty} (n + 1) \frac{z^{n-2}}{\omega^n} + \sum_{n=1}^{\infty} (n + 1)^2 \frac{z^{2n-2}}{\omega^{2n}} +
\]

\[
+ \sum_{n=1}^{\infty} (n + 1) \frac{z^{n-2}}{\omega^n} + 2 \sum_{n=1}^{\infty} (n + 1) \frac{z^{2n-2}}{\omega^{2n}} + \sum_{n=1}^{\infty} (n + 1)^2 \frac{z^{3n}}{\omega^{3n}} .
\]

(115)

Since these two expressions are to be multiplied by another infinite sums, here in (114-115) we have retained the single-sum representation.

The function \( E(z) \) has the following Loran expansion around the zero point

\[
E(z) = \sum_{m=-\infty}^{\infty} c_m^{(0)} z^m = \sum_{m=0}^{\infty} a_m^{(0)} z^m + \sum_{m=1}^{\infty} b_m^{(0)} z^m ,
\]

(116)

where \( a_m^{(0)} \) and \( b_m^{(0)} \) can be represented as integrals along some contour in the complex plane \( w \) is a complex integration variable

\[
a_m^{(0)} = \frac{1}{2\pi i} \int \frac{E(w)}{w^{m-1}} dw , \quad b_m^{(0)} = \frac{1}{2\pi i} \int E(w)w^{m-1} dw .
\]

(117)

The coefficient functions in the Loran expansion of the functions \( N(z), P(z) \) and \( Q(z) \) will be denoted respectively by \( c_m^{(1)}, c_m^{(2)} \) and \( c_m^{(3)} \). Each term of the expression for the right-hand side of (108) is a product of two infinite sums, and the final result is
\[ M(z)\rho^3 + N(z)\rho^2 + P(z)\rho + E(z) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \{c_{m+4}^{(3)} + 2(n+1)G_n c_{m+4-n}^{(3)} + (n+1)^2 G_{2n} c_{m+2-2n}^{(3)} + (n+1)^2 G_{m+2-2n}^{(3)} + 2(n+1)^2 G_{n}^{(3)} c_{m+2-2n}^{(3)} + (n+1)^3 G_{3n} c_{m-3n}^{(3)} + c_{m+4}^{(2)} + 2(n+1)G_n c_{m-n+2}^{(2)} + (n+1)^2 G_{2n} c_{m-2n}^{(2)} + c_{m+2}^{(1)} + c_{m-n}^{(1)} G_n + c_m^{(0)} \} z^m. \] (118)

In principle, the general case for an arbitrary \( m \) may also be considered. Then the above expression should be put equal to formulae (111) for \( [\rho'(z)]^2 \), where in the first sum one should set up \( 2(n-1) = m \) and in the second sum \( n - 4 = m \). In the first sum in (111) the summation will be over values of \( m = 0, 2, 4, 6, \ldots \) for \( m \leq -3 \), and in the second sum over \( m = -2, -3, -2, -1, 0, 1, 2, \ldots \). In such a case and for a given \( n \), one would have to consider a recurrent (in \( n \)) set of \textit{seven algebraic equations} in respect to the \textit{four Loran expansion coefficients} \( n, c_{m+2}^{(1)}, c_{m+2}^{(2)}, c_{m+2}^{(3)} \) and for the seven values of \( m = -6, -3, -2, -1, 0, 2k, 2k+1 \) \((k = 1, 2, \ldots)\). Therefore, the system of equations is predetermined, which enables one to find not only the unknown variables (the coefficient functions), but also certain relations about the "coefficient" expressions, represented in the case by the sums \( G_n \). This is an important moment, which shall be worked out further in this paper, and indeed certain interesting relations will be found. Moreover, since the summation over \( m \) in the R.H.S. of (118) ranges from \(-\infty\) to \(+\infty\), terms with values of \( m \), different from the above written shall be present also in the R.H.S. of (81), but not in the L.H.S. of (111). Therefore, \textit{two additional algebraic equations} may be obtained by putting \( m = -2k \) \((k \neq 0, 1, 3)\) and then \( m = -(2k+1) \) \((k \neq 0, 1)\) in the R.H.S. of (118) and then setting up the whole expression equal to zero. In fact, effectively instead of two additional equations one may have just one additional equation by putting \( m = -k \) \((k > 3 \text{ and } k \neq 6)\), so the total number of equations will be \textit{eight}. This complicated calculation for the general case has not been performed in the present paper, because due to considerable technical difficulties it would be impossible to reconstruct analytically the whole set of \textit{Loran coefficients} \( n, c_{m+2}^{(1)}, c_{m+2}^{(2)}, c_{m+2}^{(3)} \) as solutions of the above system of \textit{eight algebraic equations}. However, the calculation will be performed in Appendix A for the simplified case, which will be described also below.

In this Section we shall restrict ourselves to the case of negative-power expansion terms in the decomposition of \([\rho'(z)]^2\), obtained for values of \( m = -6, -2, 0 \), and the main motivation for this is the analogy with the standard parametrization of the cubic curve. In the next sections the case of positive-power expansion will be considered too. Unfortunately, even under this additional assumption it is impossible to resolve analytically the corresponding system of algebraic equations, if some other simplifying assumption is not added. This assumption will be given in the next Section.

The first recurrent relation for \( m = -6 \) is
\[ 4 = c_{-2}^{(3)} + 2(n + 1)G_n c_{-n-2}^{(3)} + (n + 1)^2 G_{2n} c_{-4-2n}^{(3)} + (n + 1) G_n c_{-4-n}^{(3)} + \\
+2(n + 1)^2 G_{2n} c_{-4-2n}^{(3)} + (n + 1)^3 G_{3n} c_{-6-3n}^{(3)} + c_{-2}^{(2)} + 2(n + 1) G_n c_{-n-4}^{(2)} + \\
+(n + 1)^2 G_{2n} c_{-6-2n}^{(2)} + c_{-4}^{(1)} + c_{-6-n} G_n + c_{-6}^{(0)}. \quad (119) \]

For \( m = -2 \) the relation is

\[ -76G_6 = c_{4}^{(3)} + 2(n + 1)G_n c_{-n}^{(3)} + (n + 1)^2 G_{2n} c_{-2-2n}^{(3)} + (n + 1) G_n c_{-2-n}^{(3)} + \\
+2(n + 1)^2 G_{2n} c_{-2-2n}^{(3)} + (n + 1)^3 G_{3n} c_{-3-3n}^{(3)} + c_{-2}^{(2)} + 2(n + 1) G_n c_{-n+2}^{(2)} + \\
+(n + 1)^2 G_{2n} c_{-2-2n}^{(2)} + c_{2}^{(1)} + c_{-n} G_n + c_{0}^{(0)}. \quad (120) \]

The last relation for \( m = 0 \) is

\[ -24G_4 = c_{2}^{(3)} + 2(n + 1)G_n c_{-2}^{(3)} + (n + 1)^2 G_{2n} c_{-2-2n}^{(3)} + (n + 1) G_n c_{-n}^{(3)} + \\
+2(n + 1)^2 G_{2n} c_{-2-2n}^{(3)} + (n + 1)^3 G_{3n} c_{-3-3n}^{(3)} + c_{2}^{(2)} + 2(n + 1) G_n c_{-n}^{(2)} + \\
+(n + 1)^2 G_{2n} c_{-2-2n}^{(2)} + c_{0}^{(1)} + c_{-2}^{(1)} G_n + c_{-2}^{(0)}. \quad (121) \]

To avoid the possible confusion why \( n \) appears in the R.H.S of (119-121) but not in the L. H.S., let us remind that the left-hand sides for \( [p'(z)]^2 \) in these three equations have been obtained by fixing both summation indices \( n (n = m) \) and also \( m (m = -6, -2, 0) \), while in the right-hand sides only the indice \( m \) is fixed and the indice \( n \) is left unfixed! What will be performed in the next Section will be for each value of \( m = -6, -2, 0 \) to fix in an appropriate way the possible values of \( n \). Therefore more than three algebraic equations will be obtained, in which there will be no summation left.

From the above system of three recurrent algebraic equations (119-121), the infinite sequence of coefficient functions \( c_{n}^{(0)}, c_{n}^{(1)}, c_{n}^{(2)} \) and \( c_{n}^{(3)} \) should be found and moreover, it should be proved that this sequence is convergent in the limit \( n \to \pm \infty \). Still, because of the restriction to three values of \( m \) only, even if is possible to find \( c_{n}^{(0)}, c_{n}^{(1)}, c_{n}^{(2)} \), \( c_{n}^{(3)} \), it would not be correct to assert that the Weierstrass function parametrizes an arbitrary cubic curve with coefficient functions of a complex variable. This problem probably may be resolved by means of computer simulations only.

In the next section the system of equations (119-121) shall be used for parametrizing a more simplified cubic curve (without the quadratic in \( \rho(z) \) term).
IX. PARAMETRIZATION WITH THE WEIERSTRASS FUNCTION OF THE CUBIC CURVE

\[ \left[ \rho'(z) \right]^2 = 4\rho^3 - g_2(z)\rho - g_3(z) \]

The form of the cubic curve is the same as the parametrizable cubic curve in standard algebraic geometry, but here it will be with \( g_2 \) and \( g_3 \) - functions of a complex variable. The key problem, which can be raised is: does there exist an algorithm for finding out the sequence of coefficient functions in the Lorange decomposition of \( g_2(z) \) and \( g_3(z) \), satisfying the above algebraic equation, provided that its more simple form will result in the following restrictions on the coefficient functions of the already considered general cubic equation (108)

\[
M(z) = 4 = \sum_{m=-\infty}^{m=+\infty} c_m^{(3)} z^m ; \quad N(z) = \sum_{m=-\infty}^{m=+\infty} c_m^{(2)} z^m = 0 , \tag{122}
\]

\[
N(z) = -g_2(z) = - \sum_{m=-\infty}^{m=+\infty} c_m^{(1)} z^m ; \quad E(z) = -g_3(z) = - \sum_{m=-\infty}^{m=+\infty} c_m^{(0)} z^m . \tag{123}
\]

From the first sequence of equations one obtains for the coefficients \( c_m^{(3)} \) and \( c_m^{(2)} \)

\[
c_0^{(3)} = 4 \quad c_0^{(2)} = 0 \quad \text{for all} \quad m , \tag{124}
\]

\[
c_m^{(3)} = 0 \quad \text{for all} \quad m \neq 0 . \tag{125}
\]

Taking the above relations into consideration, equation (120) for \( m = -2 \) can be written as

\[
-24G_4 = 2(n + 1)G_n c_{2-n}^{(3)} - c_0^{(1)} - G_n c_{-2-n}^{(1)} - c_{-2}^{(0)} . \tag{126}
\]

For values of \( n = 2 \) and \( n = 1 \) from the above equation the following equations are obtained

\[
-24G_4 = 24G_2 - c_0^{(1)} - c_0^{(1)} - c_{-4}^{(1)} G_2 - c_{-2}^{(0)} , \tag{127}
\]

\[
24G_4 = c_0^{(1)} + G_1 c_{-3}^{(1)} + c_{-2}^{(0)} . \tag{128}
\]

From the above two equations \( c_{-4}^{(1)} \) and \( c_{-3}^{(1)} \) can be found

\[
c_{-4}^{(1)} = \frac{1}{G_2} \left[ 24G_4 + 24G_2 - c_0^{(1)} - c_{-2}^{(0)} \right] , \tag{129}
\]
\[
c_{-3}^{(1)} = \frac{1}{G_1} \left[ c_{-3}^{(1)} + c_{-2}^{(0)} - 24G_4 \right]. \tag{130}
\]

From (89), the general recurrent relation for \( n = p > 2 \) can be obtained
\[
c_{-p}^{(1)} = \frac{1}{G_p} \left[ 24G_4 - c_{-2}^{(1)} - c_{-2}^{(0)} \right]. \tag{131}
\]

It is clear that for the determination of \( c_{-4}^{(1)}, c_{-3}^{(1)} \) and \( c_{-2}^{(1)} \) one has to know \( c_{-2}^{(0)} \) and \( c_0^{(1)} \). There is, however, one exception - in (129) \( G_2 = \sum \frac{1}{\omega^2} \) may be a divergent sum, so then one has \( c_{-4}^{(1)} = 24 \) (since \( G_2 \) is in the denominator, when \( G_2 \to \infty \), the corresponding part of the expression will tend to zero).

Further, for \( m = 0 \) and keeping in mind (124-125), equation (121) will give
\[
-76G_6 = 2(n+1)G_n c_{-4}^{(3)} \bigg( 4G_4 - c_{-4}^{(1)}G_4 - c_0^{(0)} \bigg), \tag{133}
\]
\[
-76G_6 = 12G_2 - c_{-2}^{(1)} - c_{-2}^{(1)}G_4 - c_0^{(0)} \bigg), \tag{134}
\]
\[
-76G_6 = 48G_2 - c_{-2}^{(1)} - c_{-1}^{(1)}G_1 - c_0^{(0)} \bigg), \tag{135}
\]

The above linear algebraic equations can be solved trivially linear to find the coefficients \( c_{-4}^{(1)}, c_{-2}^{(1)} \) and \( c_{-1}^{(1)} \), which depend on \( c_{-1}^{(1)} \) and \( c_0^{(0)} \)
\[
c_{-4}^{(1)} = \frac{1}{G_4} \left[ c_{-4}^{(1)}G_1 - 48G_2 + 40G_4 \right], \tag{136}
\]
\[
c_{-2}^{(1)} = -36 + \frac{G_1}{G_2} c_{-1}^{(1)} \bigg), \tag{137}
\]
\[
c_0^{(1)} = \frac{1}{G_6} \left[ c_{-1}^{(1)}(-48G_2 + 40G_4) \right]. \tag{138}
\]

Note that these coefficients can be divergent if \( G_2 \) and \( G_1 \) are divergent. Taking into account equation (138) and also (125) for the case \( m = 0 \), but for a general value of \( n = p \neq 1, 2, 4 \), an expression for \( c_{-k}^{(1)} \) can easily be found
\[
c_{-k}^{(1)} = \frac{1}{G_k} \left[ (-48G_2 + c_{-1}^{(1)}G_1) \right]. \tag{139}
\]
This formulae should be compared to the previously derived formulae (131), setting up
\[ -2 - p = -k. \]
From the two expressions \( c^{(0)}_{-2} \) can be expressed
\[
c^{(0)}_{-2} = 24G_4 - c^{(1)}_0 - \frac{G_{k-2}}{G_k} (-48G_2 + c^{(1)}_{-1}G_1). \] (140)

However, \( c^{(0)}_{-2} \) can be expressed also from the two formulaes (129) and (136) for \( c^{(1)}_{-4} \):
\[
c^{(0)}_{-2} = 24G_4 + 24G_2 - c^{(1)}_0 + \frac{G_2}{G_4} (48G_2 - c^{(1)}_{-1}G_1 - 40G_4). \] (141)

Comparing (140-141), an expression for \( c^{(1)}_{-1} \) can be found, which does not depend on any
Loran coefficient functions
\[
c^{(1)}_{-1} = \frac{16G_2 [3G_2G_k - G_4G_k - 3G_4G_{k-2}]}{G_2G_k - G_4G_{k-2}}. \] (142)

Substituting this expression into the formulae (139) for \( c^{(1)}_{-k} \), one obtains the convergent
expression \( k > 2, k \neq 4 \)
\[
c^{(1)}_{-k} = -\frac{16G_4}{G_k - \frac{G_4}{G_2}G_{k-2}}. \] (143)

The obtained expression (142) for \( c^{(1)}_{-1} \) can be substituted into (140) to find a formulae for
\( c^{(0)}_{-2} \), which will depend on \( G_k \) and only on the Loran coefficient function \( c^{(1)}_0 \)
\[
c^{(0)}_{-2} = -c^{(1)}_0 + 24G_4 + \frac{16G_{k-2}}{G_k - \frac{G_4}{G_2}G_{k-2}}. \] (144)

The above expression is well defined also when \( G_2 \to \infty \). It shall be proved subsequently
that such a case will turn out to be impossible.

Further, from (144) and expression (130) for \( c^{(1)}_{-3} \) it follows
\[
c^{(1)}_{-3} = \frac{G_4}{G_2} \frac{16G_{k-2}}{G_k - \frac{G_4}{G_2}G_{k-2}}. \] (145)

But since \( k \) in expression (143) can take a value \( k = 3 \), it follows also
\[
c^{(1)}_{-3} = -\frac{16G_4}{G_3 - \frac{G_4}{G_2}G_1}. \] (146)

The comparison of the two expressions gives the following formulae for the infinite sum
\( G_k \):
\[
G_k = \gamma G_{k-2} = \gamma^3 G_{k-2s} = \ldots = \gamma^{2p-1} G_1 \text{ for } k = 2p \]
\[
= \gamma^p G_1 \text{ for } k = 2p + 1, \] (147)
where
\[ \gamma = 2 \frac{G_4}{G_2} - \frac{G_3}{G_1}. \] (148)

Another recurrent relation for \( G_k \) can be found also from (145)
\[ G_k = \frac{(G_{k-2})^2}{G_{k-4}} = \frac{(G_{k-4})^3}{(G_{k-6})^2} = \ldots = \frac{(G_{k-2s})^{s+1}}{(G_{k-2s-2})^2}. \] (149)

This formulae for values of \( k = 2p \) and \( k = 2p + 1 \), combined with the previous formulae (147), allows one to find an expression for \( G_1 \):
\[ G_1 = \frac{G_{2p-3}}{G_{2p-5}} \frac{p}{G_{2p+1}}. \] (150)

The last formulae is interesting, because it shows that the divergent in the general case quantity \( G_1 \) in the present case is expressed through convergent quantities only - \( G_3 \) and \( G_{2p+1} \) (\( p \) is of course a finite number!). Substituting (150) into (149), one can get expressions for \( G_{2p} \) and \( G_{2p+1} \):
\[ G_{2p} = G_3 \left( \frac{G_4}{G_2} \right)^{\frac{2p-3}{2}}; \quad G_{2p+1} = G_3 \left( \frac{G_4}{G_2} \right)^{p-1}. \] (151)

It is seen that \( G_2 \) is also expressed through convergent quantities.

From equation (130) for \( m = -6 \) one obtains
\[ 4 = c_{-4}^{(1)} + c_{-6-n}^{(1)} G_n + c_{-6}^{(0)} \] (152)
\((n = 1, \ldots, \infty)\). Since \( c_{-4}^{(1)} \) and \( c_{-6-n}^{(1)} \) can be found, \( c_{-6}^{(0)} \) can also be determined. It is clear that among the coefficients \( c_{m}^{(0)} \) two of them - \( c_{-6}^{(0)} \) and \( c_{-2}^{(0)} \) can be determined from (143). The other coefficients will be determined in the Appendixes.

Let us summarize the obtained results in this Section and in Appendixes A, B and C by formulating the following

**Proposition 2** Let \( g_2(z) \) and \( g_3(z) \) are functions of a complex variable, which have a Loran function decomposition \( g_2(z) = \sum_{m=-\infty}^{\infty} c_{m}^{(1)} z^m \) and \( g_3(z) = \sum_{m=-\infty}^{\infty} c_{m}^{(0)} z^m \) and satisfy the algebraic equation \([\rho'(z)]^2 = 4\rho^3 - g_2(z)\rho - g_3(z)\), where \( \rho(z) \) is the Weierstrass (elliptic) function. Then the following statements represent (only) necessary conditions for the fulfillment of the above equation:

1. The poles of the Weierstrass function (even if they are infinite in number), must be situated in such a way so that the sums \( G_1 = \sum \frac{1}{\omega} \) and \( G_2 = \sum \frac{1}{\omega^2} \) are convergent (i.e. finite). The sum \( G_1 \) can be expressed through formulae (150).
2. All the coefficients $c^{(1)}_m$ and $c^{(0)}_m$ in the Loran positive- and negative-power expansion can be expressed uniquely from the finite sums $G_n$.

3. The sum $G_1$ is proportional to the sum $G_3$ with a coefficient of proportionality, equal to the ratio of the sums $G_2$ and $G_4$, i.e. $G_1 = \frac{G_2}{G_4} G_2$ (from A22). This formula follows also from the more general one $G_{2p+1} = G_3 (\frac{G_4}{G_2})^{p-1}$ (151) for $p = 0$.

4. As a consequence from the above relation and formulae (147-148), the sum $G_2$ can be uniquely expressed as $G_2 = \sqrt{\frac{G_1G_3}{G_2}}$.

5. All the even-number sums $G_6, G_8, G_{10}, \ldots$ equal to zero.

6. The following relation is fulfilled $G_{2k+1} = 2G_k^2 - G_k^2 G_{2k+2}$, which can be obtained from (B15) and (B16). In order this relation to comply with statement 5, additionally one should have that $G_5, G_7, G_{13}, G_{15}, G_{21}, G_{23} \ldots$ should be zero. However, $G_9, G_{11}, G_{17}, G_{19}$ are different from zero. Finally, with the help of (152), a check can also be made for the consistency of the obtained results. Substracting the two equations (152) for values of $n$ and $n + l$, one obtains

$$\frac{G_n}{G_{n+6}} - \frac{G_{n+l}}{G_{n+6+l}} = 0$$

(it’s more appropriate to divide everywhere by $G_n$). From (151) for values of $n = 2p$ and $l = 2q$, for example, it can be found

$$\frac{G_{n+l}}{G_n} = \left(\frac{G_4}{G_2}\right)^{2q}.$$ (154)

For other combinations (even and odd) of $n$ and $l$ the calculation is similar. Using the above formulae, it can easily be verified that equation (153) is identically satisfied. This confirms that the investigated in this paragraph system of equations gives consistent and noncontradictory results.

X. POSITIVE-POWER TERMS IN THE INFINITE SUM DECOMPOSITION OF EQUATION $\left[\rho'(z)\right]^2 = 4\rho^3 - g_2(z)\rho - g_3(z)$ - THE CASE OF POLES NOT AT INFINITY

For the purpose, the expansion (111) for $\left[\rho'(z)\right]^2$ has to be used, and a change in the summation indices is performed so that positive-power terms starting from $n = 1$ (without the free term) are taken into account

$$\left[\rho'(z)\right]^2 = \sum_{n=1}^{\infty} (n+1)^2(n+2)^2G_{2(n+3)}z^{2n} - 4\sum_{n=1}^{\infty} (n+4)(n+5)G_{n+6}z^n.$$ (155)
Using the Loran decomposition \( g_2(z) = \sum_{m=-\infty}^{\infty} c_m^{(1)} z^m \), it can be obtained for \(-g_2(z)\rho(z)\) for the positive terms only

\[
-g_2(z)\rho(z) = -\sum_{n=1}^{\infty} c_{n+2}^{(1)} z^n - \sum_{n=1}^{\infty} \sum_{k=-\infty}^{0} (n - k + 1)G_{n-k+2} c_k^{(1)} z^n.
\]

By means of the relevant expressions for \( \rho^3(z) \) from (115) and for \( g_3(z) = \sum_{m=-\infty}^{\infty} c_m^{(0)} z^m \), the following expression can be obtained for the positive - power terms of

\[
0 = \left[ \rho'(z) \right]^2 - 4\rho^3 + g_2(z)\rho + g_3(z) =
\]

\[
= \sum_{n=1}^{\infty} \left[ A_0(n,G) z^n + A_1(n,G) z^{2n} + A_2(n,G) z^{3n} \right],
\]

where \( A_0(n,G) \), \( A_1(n,G) \) and \( A_2(n,G) \) are the following expressions

\[
A_0(n,G) = 8(n+5)G_{n+4} + 4(n+3)G_{n+2} + 4(n+4)(n+5)G_{n+6} -
\]

\[
- c_{n+2}^{(1)} - c_n^{(0)} - \sum_{k=0}^{\infty} (n + k + 1)G_{n+k+2} c_k^{(1)},
\]

\[
A_1(n,G) = 4(n+2)^2G_{2(n+1)} + 8(n+2)G_{2(n+1)} - (n+1)^2(n+2)^2G_{2(n+3)},
\]

\[
A_2(n,G) = 4(n+1)^3G_{3n}.
\]

Two important observations can be made from these expressions.

First, the unknown coefficient functions \( c_{n+2}^{(1)} \), \( c_n^{(0)} \) and \( c_{-k}^{(1)} \) are singled out only in \( A_0(n,G) \). Therefore, it is more appropriate to write down (157) in the form

\[
\sum_{n=1}^{\infty} A_0(n,G) z^n = - \sum_{n=1}^{\infty} A_1(n,G) z^{2n} - \sum_{n=1}^{\infty} A_2(n,G) z^{3n}.
\]

The above mentioned coefficient functions can be determined in such a way so that the characteristics of the infinite sum on the left-hand side (L.H.S) would correspond to the characteristics of the infinite sum on R.H.S. Such a characteristic would be for example the "convergency radius", defined in standard complex analyses (for the infinite sum on the L.H.S.) as

\[
R_0 = \lim_{n \to \infty} \left| A_0(n,G) \right|^{-\frac{1}{n}}.
\]
Second, since the infinite sums in (158-160) are finite, it can be seen that

\[ \lim_{n \to \infty} | A_1(n, G) |^{-\frac{1}{n}} = \lim_{n \to \infty} | A_2(n, G) |^{-\frac{1}{n}} = \infty . \]  

(163)

Let us find the convergency radius of the first infinite sum on the R.H.S. of (158)

\[ R_1 = \lim_{n \to \infty} | A_1(n, G) |^{-\frac{1}{n}} = \lim_{n \to \infty} \exp\left\{ -\frac{\ln | A_1(n, G) |}{n} \right\} = \]

(164)

\[ = \exp\left\{ -\lim_{n \to \infty} \frac{d|A_1|}{dn} \right\} \]

(the Lopital’s rule has been used, since \( n \to \infty \) and \( | A_1(n, G) | \to \infty \). It can be found that

\[ \frac{d}{dn} | A_1(n, G) | = 8(n + 2)G_2(n+1) + 8G_2(n+1) + \]

\[ + 8G_2(n+1) - 2(n + 1)(n + 2)(2n + 3)G_2(n+3) . \]  

(165)

This expression is a third-rank polynomial in \( n \), while \( | A_1 | \) is a second-rank polynomial. Therefore the ratio \( \frac{d|A_1|}{|A_1|} \) will evidently tend to infinity when \( n \to \infty \) and consequently

\[ R_1 = \exp[-\infty] = 0 . \]  

(166)

The same can also be proved in an analogous way for \( R_2 \).

Now, since on the R.H.S. of (161) one has an infinite sum with a zero convergency radius, it’s natural to suppose that the same holds also for the L.H.S. A simple calculation shows that this may happen only if

\[ \lim_{n \to \infty} \frac{d|A_0(n, G)|}{dn} = \]

\[ = \lim_{n \to \infty} \left[ \frac{d_{c_n}^{(2)}}{dn} + \frac{d_{c_n}^{(3)}}{dn} \frac{1}{G_{n+6}} + \sum C_{n+k+2}^{(2)} c_{-k}^{(2)} \right] = \infty . \]  

(167)

This means that either \( c_{n+2}^{(1)} \), or \( c_n^{(0)} \), or \( c_{-k}^{(1)} \) have to be proportional to \( \frac{n^l}{(l+1)!} \), where \( l > 2 \).

In such a way, we acquired information how the coefficient functions \( c_n^{(1)} \) behave both in the positive-power \( (n > 0) \) and the negative power decomposition \( (n < 0) \).
XI. POLES AT INFINITY \((\wp \rightarrow \infty)\) IN THE
POSITIVE - POWER DECOMPOSITION
OF \[\left(\rho'(z)\right)^2 = 4\rho^3 - g_2(z)\rho - g_3(z)\]

If the period of the Weierstrass function can be represented as \(\wp = q\wp_1 + p\), then in the
limit \(\wp \rightarrow \infty\) one has

\[G_n(\wp) = \sum \frac{1}{\wp^n} = \sum \frac{1}{(q\wp_1 + p)^n} \rightarrow_{\wp \rightarrow \infty} \sum \frac{1}{p^n} = 2\zeta(n) , \quad (168)\]

where \(\zeta(n)\) denotes the Riemann zeta-function. Therefore, in the asymptotic limit \(n \rightarrow \infty\) (when this limit is used, as in the preceeding section), one should also take into account
the asymptotic limit of the zeta-function as compared to the other power-like terms of \(n\).

For the present particular case, let us calculate for example \(R_2\). It can be found that

\[\lim_{n \rightarrow \infty} \frac{d|A_2|}{dn} = \lim_{n \rightarrow \infty} \frac{3}{n + 1} \left| \frac{(3n) - n(n + 1)((3n + 1))}{\zeta(3n)} \right| = \infty , \quad (169)\]

and consequently,

\[R_2 = \lim_{n \rightarrow \infty} \exp\left\{ -\frac{\ln |A_2|}{n} \right\} = \lim_{n \rightarrow \infty} \exp\{ -\infty \} = 0 . \quad (170)\]

After performing similar calculations, the same result can also be obtained for \(R_1\).

Let us find now \(R_0\). Some more lengthy calculations will give

\[\lim_{n \rightarrow \infty} \frac{d|A_0|}{dn} = \lim_{n \rightarrow \infty} \frac{C^{(1)}(n, k, p)}{C^{(2)}(n, k, p)} , \quad (171)\]

where \(C^{(1)}(n, k, p)\) and \(C^{(2)}(n, k, p)\) will be the following expressions

\[C^{(1)}(n, k, p) = -\frac{1}{4(n + 4)(n + 5)(n + 6)} \left\{ \frac{d}{dn} \left[ c_{n+2}^{(1)} + c_n^{(0)} \right] + \sum_{k=0}^{\infty} \zeta(n + k + 2)c_{-k}^{(1)} + \right. \]

\[+ \zeta(n + 7) + \sum_{k=0}^{\infty} (n + k + 1) \left[ \zeta(n + k + 2) \frac{dc_{-k}^{(1)}}{dn} - (n + k + 2)c_{-k}^{(1)} \zeta(n + k + 3) \right] , \quad (172)\]

\[C^{(2)}(n, k, p) = -\frac{1}{4(n + 4)(n + 5)(n + 6)} \left[ c_{n+2}^{(1)} + c_n^{(0)} + \sum_{k=0}^{\infty} (n + k + 1)\zeta(n + k + 2)c_{-k}^{(1)} \right] . \quad (173)\]
It is easily seen that $R_0$ will again tend to zero if the nominator $C^{(1)}(n,k,p)$ in (172) tends to $\pm \infty$. This may happen if at least one of the equations below is fulfilled

$$\frac{dC^{(1)}_{n+2}}{dn} - \text{const.}n^l; \quad \frac{dC^{(0)}_n}{dn} - \text{const.}n^l \quad (l > 3),$$

$$\frac{d}{dn} \left[ \sum_{k=0}^{\infty} \zeta(n + k + 2)c^{(1)}_{-k} \right] - \text{const.}n^r \quad (l > 3)$$

$$\Rightarrow c^{(1)}_{-k} - \text{const.} \int n^l p^{n+k+2} dn \quad \text{for some } k \text{ and some } p,$$  \hspace{1cm} (175)

$$\sum_{k=0}^{\infty} \zeta(n + k + 3)c^{(1)}_{-k} = \sum_{k=0}^{\infty} \sum_{p \neq 0} c^{(1)}_{-k} \frac{1}{p^{n+k+3}} - \text{const.}n^r \quad (r > 1)$$

$$\Rightarrow c^{(1)}_{-k} - \text{const.} n^r p^{n+k+3} dn \quad \text{for some } k \text{ and some } p,$$  \hspace{1cm} (176)

$$\sum_{k=0}^{\infty} \sum_{p \neq 0} \frac{1}{p^{n+k+3}} \frac{dc^{(1)}_{-k}}{dn} \sim \text{const.}n^r \quad (r > 2)$$

$$\Rightarrow c^{(1)}_{-k} \sim \text{const.} \int n^r p^{n+k+2} dn.$$  \hspace{1.5cm} (177)

It is interesting to note that after performing a change of variables

$$tn = x \quad \exp(-t) = p,$$  \hspace{1cm} (178)

the integral (175) for $c^{(1)}_{-k}$ can be transformed as follows

$$\int_{1}^{\infty} n^l p^{n+k+2} dn = \frac{\exp[-t(k+2)]}{t^{l+1}} \int_{t}^{\infty} \exp(-x)x^l dx.$$  \hspace{1cm} (179)

Keeping in mind the generally known formulae for the Gamma function

$$\Gamma(l) \equiv \int_{0}^{\infty} \exp(-x)x^{l-1} dx = \int_{1}^{\infty} \exp(-x)x^{l-1} dx + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(l+n)},$$  \hspace{1cm} (180)

it can be derived for $c^{(1)}_{-k}$

$$c^{(1)}_{-k} - \text{const.} \frac{\exp[-t(k+2)]}{t^{l+1}} \left\{ - \int_{1}^{t} \exp(-x)x^l dx + \Gamma(l+1) - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(l+n+1)} \right\}.$$  \hspace{1cm} (181)

Let us denote the integral in the above expression by

$$I(l) \equiv \int_{1}^{t} \exp(-x)x^l dx.$$  \hspace{1cm} (182)
Then the following relation can be found after performing multiple integration by parts
\[ I(l) = - \left\lfloor x^l \exp(-x) \bigg|_{x=1}^{x=t} + I(l-1) = \ldots = - \left\lfloor x^l \exp(-x) \bigg|_{x=1}^{x=t} - \right. \]
\[ - l \left. \left\lfloor x^{l-1} \exp(-x) \bigg|_{x=1}^{x=t} \right. \ldots - l(l-1) \ldots (l-k+2) \bigg| x^{l-k+1} \exp(-x) \bigg|_{x=1}^{x=t} + \right. \]
\[ + l(l-1)(l-2) \ldots (l-k+1) I(l-k) . \] (183)

Continuing in the same way, one can derive
\[ I(l) = - \exp(-t)[l^t + lt^{l-1} + \ldots + lt] + \]
\[ + \exp(-1)[1 + l + l(l-1) + \ldots + l!] + l! \int_1^t \exp(-x) dx = \]
\[ = -\exp(-t) \sum_{k=0}^{l-1} \frac{d}{dt^k} t^k + \exp(-1) \sum_{k=0}^{l-1} l(l-1) \ldots (l-k) - \exp(-1)! [\exp(1-t) + 1] . \] (184)

Substituting this formulae into (181) and returning to the original variables \((n, p)\), finally an expression is derived for \(c^{(1)}_{-k} : \)
\[ c^{(1)}_{-k} = \text{const.}(-1)^{l+1} \frac{p^{l+2}}{(lnp)^{l+1}} \{l!(p + \exp(-1)) + p \sum_{k=0}^{l-1} \frac{d}{dt^k} t^k - \}
\[ - \exp(-1) \sum_{k=0}^{l-1} l(l-1) \ldots (l-k) + \Gamma(l+1) - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(l+n+1)} \} . \] (185)

The above expression is of course not an exact one, since other representations of \(c^{(1)}_{-k} \) may exist, when the convergency radius of the infinite sum tends to \(\infty\). Nevertheless, it can be used as a possible model representation.

It may happen also that the denominator \(C^{(2)}(n, k, p)\) tends to zero, and then again the convergency radius will tend to infinity. For the purpose, the following two equations have to be fulfilled together
\[ c^{(1)}_{n+2} + c^{(0)}_n = \text{const.}n^r \quad (r = 1, 2, 3) \] (186)
and
\[ \zeta^{(1)}_{n+k+2} c^{(1)}_{-k} = n \implies c^{(1)}_{-k} = \text{const.} n^p n^{n+k+2} . \] (187)

The last possible choice is when the denominator is finite, but the nominator is infinite (equations 174-177). The denominator is finite when
\[ c^{(1)}_{n+2} \sim \text{const.} n^3 \quad c^{(1)}_{-k} \sim \text{const.} n^3 \] (188)
\[ c^{(1)}_{-k} \zeta(n+k+2) \sim \text{const.} n^2 . \] (189)

But since these equations contradict equations (175-176), such a case of finiteness of the denominator and tending to zero nominator has to be excluded from consideration.
XII. APPLICATION OF THE EQUATION

\[
\left[ \rho'(z) \right]^2 = 4 \rho^3 - g_2(z) \rho - g_3(z) - \]

- ANOTHER WAY FOR PARAMETRIZATION OF THE CUBIC EQUATION (53)

In Sect. VI a parametrization of the cubic equation (53) was proposed, based on presenting the equation in the form of a cubic equation in respect to one of the variables, and more importantly, applying the cubic equation for the Weierstrass function \( \rho(z) \) and setting up \( g_2, g_3 \) – complex numbers. In this Section it shall be demonstrated how one can obtain such an equation with \( g_2, g_3 \) – functions of a complex variable.

It is instructive first to note that the last terms in the square parenthesis in (53) depend only on the variable \( n \). Therefore, let us denote the term in the parenthesis by \( \Pi_2 \) and thus let us represent (53) in the form of two cubic equations

\[
\Pi^2 = M - A\tilde{m}^3 - B\tilde{m}^2 - C(1 + 2\frac{d}{c}\tilde{m}) \tag{190}
\]

and

\[
\Pi^2 = \tilde{P}_1(n)\tilde{m}^3 + \tilde{P}_2(n)\tilde{m}^2 + \tilde{P}_3(n)\tilde{m} + \tilde{P}_4(n) . \tag{191}
\]

The quadratic forms \( \tilde{P}_i(n)(i = 1..4) \) are the same as in (60-63), but with a reversed sign of \( n \). Also, in the first three forms the last terms \((-A), (-B)\) and \((-C - 2\frac{d}{c}C)\) are absent.

In order to bring the two equations to a parametrizable form, let us perform the linear transformation

\[
\tilde{m} = r\tilde{m} + s . \tag{192}
\]

The first equation (190) transforms to the following equation

\[
\Pi^2 = (M - As^3 - Bs^2 - Cs - 2\frac{b}{d}Cs) + [-3rs^2A - Cr - 2Br s - 2\frac{b}{d}Cr]\tilde{m} + \]

\[
+ [-3r^2sA - Br^2]\tilde{m}^2 - Ar^3\tilde{m}^3 . \tag{193}
\]

In order to obtain the parametrizable form

\[
\left[ \rho'(z) \right]^2 = 4 \rho^3 - g_2(z) \rho - g_3(z) \tag{194}
\]

of the cubic equation, one should require

\[
s = -\frac{B}{3A} \quad \quad r = (-\frac{4}{A})^{\frac{1}{3}} = i^{\frac{2}{3}}(\frac{4}{A})^{\frac{1}{3}} , \tag{195}
\]

\[
g_3(z) = As^3 + Bs^2 + C(1 + 2\frac{b}{d}s - \overline{M} = -\overline{M} - \frac{B}{3A}C(1 + 2\frac{b}{d}) \tag{196}
\]

42
\[ g_2(z) = 3rs^2A + Cr + 2Br + \frac{b}{d}Cr = \frac{i\xi 2\xi}{A^{\xi}}[C(1 + \frac{b}{d}) - \frac{B^2}{3A}] \, . \tag{197} \]

Since in the preceding sections it has been proved that the Weierstrass function \( \rho(z) \) parametrizes equation (194), one has the right to set up

\[ \overline{m} = \rho(z) \quad \Pi = \rho'(z) \tag{198} \]

and consequently, the transformation (192) acquires the form

\[ \tilde{m} = r \rho(z) + s \, . \tag{199} \]

After performing this transformation, the second algebraic equation (191) assumes the following form

\[ \left[ \rho'(z) \right]^2 = \overline{Q}_1(n)\rho^3(z) + \overline{Q}_2(n)\rho^2(z) + \overline{Q}_3(n)\rho(z) + \overline{Q}_4(n) \, , \tag{200} \]

where

\[ \overline{Q}_1(n) \equiv \tilde{P}_1(n)r^3 \, , \tag{201} \]

\[ \overline{Q}_2(n) \equiv 3r^2s\tilde{P}_1(n) + r^2\tilde{P}_2(n) \, , \tag{202} \]

\[ \overline{Q}_3(n) \equiv 3rs^2\tilde{P}_1(n) + r\tilde{P}_3(n) + 2rs\tilde{P}_2(n) \equiv -\overline{f}_2 \, , \tag{203} \]

\[ \overline{Q}_4(n) \equiv \tilde{P}_4(n) + \tilde{P}_1(n)s^3 + \tilde{P}_2(n)s^2 + \tilde{P}_3(n)s \equiv -\overline{f}_3 \, . \tag{204} \]

The first equation (194) and the second equation (200) will be identical in respect to the cubic and the quadratic in \( \rho \) terms if

\[ s = -\frac{\tilde{P}_2(n)}{3\tilde{P}_1(n)} = -\frac{B}{3A} \quad r^3 = \frac{4}{\tilde{P}_1(n)} = -\frac{4}{A} \, . \tag{205} \]

In fact, this means that the first two terms (cubic and quadratic) in the transformed equations (201) and (202) are identical with the corresponding ones in the original equations (190-191):

\[ \tilde{P}_1(n) = -A \quad \tilde{P}_2(n) = -B \, , \tag{206} \]

\[ \overline{Q}_1(n) = 4 \quad \overline{Q}_2(n) = 0 \, . \tag{207} \]

In other words, the linear transformation is such that it transforms the straight line

\[ [\tilde{P}_3(n) + C(1 + \frac{d}{c})]\tilde{m} + \tilde{P}_4(n) - \overline{M} = 0 \, , \tag{208} \]
defined on the points of the original cubic equation (53) (therefore - intersecting it), into
the straight line
\[ (\overline{Q}_3(n) + g_2(z))\rho(z) + (\overline{Q}_4(n) + g_3(z)) = 0, \] (209)
defined on the algebraic variety ”points” of the transformed cubic curve (obtained from
(201) and (206))
\[ (\overline{Q}_1(n) - 4)\rho^3(z) + \overline{Q}_2(n)\rho^2(z) + \]
\[ + (\overline{Q}_3(n) + g_2(z))\rho(z) + (\overline{Q}_4(n) + g_3(z)) = 0. \] (210)
Note also that it cannot be assumed that
\[ Q_2(n) = -g_2(z) \]
\[ Q_4(n) = -g_3(z), \] (211)
because then it will turn out that the linear transformation is a degenerate one and
maps the straight line into the zero point. Of course, the choice of variables (206) does
not mean that an additional and restrictive assumption has been imposed. The variable
identification (206) simply helps to ”fix” the transformation so that an one-dimensional
submanifold (the straight line) is mapped again into an one-dimensional submanifold. In
such a way, the original cubic equations (191-192) are replaced with the transformed ones
(194) and (200), which differ yet in their last two terms.
Now, using expressions (203-204) for \( g_2(z) \) and \( g_3(z) \) and also equations (205) for \( r \)
and \( s \), the following expressions are obtained, which will be used in the next section:
\[ \overline{g}_2(z) = 4C(1 + 2 \frac{b}{d}) \frac{1}{P_1} + \frac{4\tilde{P}_2}{3P_1^2}, \] (212)
\[ \overline{g}_3(z) = -C(1 + 2 \frac{b}{d}) \frac{\tilde{P}_2}{3P_1} - \frac{2\tilde{P}_2^3}{27P_1^2}. \] (213)

**XIII. J - INVARIANT IN THE CASE OF ARBITRARY RATIO \( \frac{a}{c} \) AND IN THE GENERAL CASE**

The purpose in this section will be to see how the so called \( j \) or modular invariant of an elliptic curve, defined as:
\[ j(E) = 1728 \frac{g_3^3}{g_2^3 - 27g_3}, \] (214)
will change under some assumptions, for example when the ratio \( \tilde{m} = \frac{a}{c} \) of the parameter functions is an arbitrary one. In a broader sense, the idea is to see if there is any relation
between the possible motions and group transformations on the complex plane and the $j$--invariant. This will not be considered in this section.

If the cubic equation (53) is satisfied under arbitrary $\tilde{m}$, then all the coefficient functions in front of $\tilde{m}$ and its powers should equal to zero. Therefore, the following identification holds:

$$\overline{M} = \tilde{P}_4(n)$$
$$- C(1 + \frac{b}{d}) = \tilde{P}_3(n) \ ,$$
$$\tilde{P}_1(n) = -A$$
$$\tilde{P}_2(n) = -B \ .$$

(215)

Taking this into account, equations (196-197) for $g_2$ and $g_3$ can be written as

$$g_2(z) = \frac{2^4}{\tilde{P}_1^3} \left[ \frac{\tilde{P}_2^2}{3\tilde{P}_1} - \tilde{P}_3 \right] ,$$

(217)

$$g_3(z) = \frac{\tilde{P}_2}{\tilde{P}_1} - \tilde{P}_4 \ .$$

(218)

Eliminating $\tilde{P}_3(n)$ from the two equations and expressing $\tilde{P}_4(n)$ gives

$$\tilde{P}_1 = \frac{g_2 \tilde{P}_2}{2^7 \tilde{P}_1^3} + \frac{\tilde{P}_3^3}{3\tilde{P}_1^2} - g_3 \ .$$

(219)

Substituting $\tilde{P}_1(n)$ from (219) and $\tilde{P}_3(n)$ from (218) into equation (204) for $\overline{Q}_4(n)$, and taking into account also (195), one obtains

$$-\overline{g}_3 \equiv \overline{Q}_4(n) = -g_3 + \frac{8\tilde{P}_3^3}{27\tilde{P}_1^2} + \frac{2^7}{3} \frac{\tilde{P}_2}{\tilde{P}_1^3} g_2 \ .$$

(220)

Performing the same substitutions in respect to equation (203) for $\overline{Q}_3(n)$, a rather unexpected result is obtained

$$-\overline{g}_2 \equiv \overline{Q}_3(n) = \frac{2^7}{3} \frac{\tilde{P}_2^2}{\tilde{P}_1^3} - g_2 - \frac{2^7}{3} \frac{\tilde{P}_2^2}{\tilde{P}_1^3} = -g_2 \ .$$

(221)

Therefore, $g_2$ for the cubic equation (190) does not change if assuming that $\tilde{m}$ is arbitrary. It follows also from (220) that $\overline{g}_3 = g_3$ if and only if

$$g_2^3 = -\frac{2^5}{9^3} \frac{\tilde{P}_6^2}{\tilde{P}_1^4} \ .$$

(222)

Since

$$\tilde{P}_1 = -A = -2p\Gamma_{55}^r g_{5r} \quad \text{and} \quad \tilde{P}_2 = -B = -6p\Gamma_{a5}^r g_{5r}dx^a \ ,$$

(223)
clearly (222) will be a rather complicated algebraic equation of sixth rank in respect to the sub-algebraic variety of the variables \( dx^\alpha (\alpha = 1, 2, 3, 4) \).

However, it is more interesting to see when the \( j \)-invariants for the cubic equations in the case of arbitrary \( \frac{a}{c} = \tilde{m} \) are equal, i.e.

\[
\overline{j}(E) = j(E).
\] (224)

Making use of the defining equation (214) for the \( j \)-invariant and of equations (220-221), one easily finds

\[
\left[ g_3 - \frac{8\tilde{P}_3}{27\tilde{P}_1^2} - \frac{2\tilde{P}_2}{3\tilde{P}_1^3}g_2 \right]^2 = g_3^2.
\] (225)

This is an algebraic equation on a Riemann surface. If one denotes

\[
g_3(z) = w,
\] (226)

then in respect to \( w \) the surface has two sheaves - for \(+w\) and for \(-w\). If the positive sign is taken, then the algebraic equation is satisfied for \( \overline{\eta}_3 = g_3 \) - a case already considered (equations (223-224)).

For the case of the minus sign \(-w\), equation (225) will give another possible relation between the functions \( g_2(z) \) and \( g_3(z) \):

\[
g_3(z) = \frac{4\tilde{P}_3}{27\tilde{P}_1^2} + \frac{2\tilde{P}_2}{3\tilde{P}_1^3}g_2,
\] (227)

following from the equality of the \( j \)-invariants of the cubic equations for the case of arbitrary \( m \). If the Loran decomposition of \( g_2(z) \) and \( g_3(z) \)

\[
g_2(z) = \sum_{m=-\infty}^{\infty} c_m^{(1)} z^m \quad g_3(z) = \sum_{m=-\infty}^{\infty} c_m^{(0)} z^m
\] (228)

is substituted into (227), then some additional relations may be obtained between the Loran function coefficients.

One should keep in mind, however that the original cubic equation has been "splitted up" into two parts, so it should be seen how the \( j \)-invariants of the two parts are related to the \( j \)-invariant of the original equation.

**XIV. INTEGRAL REPRESENTATION IN THE LORAN FUNCTION DECOMPOSITION OF \( g_2(z) \) AND A SUMMATION FORMULAE WITH THE FINITE SUMS \( G_n \)**

In order to derive the summation formulae, let us remember that the coefficient functions \( c_0^{(1)} \) and \( c_m^{(1)} \) in the Loran infinite sum decomposition possess also an integral representation.
\[ c_m^{(1)} = \frac{1}{2\pi i} \int_C \frac{g_2(w)}{w^{n+1}} dw \quad \text{for} \quad m = 0, 1, 2, \ldots \] \quad \text{(229)}

For the purpose of obtaining a formula for \( g_2(z) \), one should substitute the already found coefficient functions \( c_{-1}^{(1)} \) (eq.142), \( c_{-2}^{(1)} \) (eq.137), \( c_{-4}^{(1)} \) (eq.136) and \( c_{-m}^{(1)} \) (eq.143) into the Loran’s decomposition formulae (228). The obtained expression is

\[ g_2(z) = \sum_{m=1}^{\infty} c_m^{(1)} z^m + c_0^{(1)} + \frac{1}{z} \left( -36 + \frac{G_1}{G_2} c_{-1}^{(1)} \right) \frac{1}{z^2} - \frac{16G_4}{G_3} \frac{1}{z^3} - \frac{1}{G_4} \left( c_{-1} G_1 - 48G_2 + 40G_4 \right) \frac{1}{z^4} - \sum_{m=5}^{\infty} \frac{16G_4}{G_{m-3}} \frac{1}{z^m} \] \quad \text{(230)}

Let us see whether this will remain an expression for \( g_2(z) \), or (which will turn out to be the case) the term \( g_2(z) \) on both sides of the equality will cancel out. For the purpose, let us rewrite the first two terms on the R.H.S. as

\[ \sum_{m=1}^{\infty} c_m^{(1)} z^m + c_0^{(1)} = \sum_{m=1}^{\infty} \left( \frac{1}{2\pi i} \int g_2(w) \frac{1}{w^{n+1}} dw \right) z^m + \frac{1}{2\pi i} \int \frac{g_2(w)}{w} dw \] \quad \text{(231)}

The integral and the sum in the first term can be interplaced, and also the formulae for the infinite geometric progression will be taken into account

\[ \sum_{m=1}^{\infty} \frac{1}{\xi^m} = \frac{\frac{1}{\xi}}{1 - \frac{1}{\xi}} = \frac{1}{\xi - 1} \] \quad \text{(232)}

So, expression (230) acquires the form

\[ \frac{1}{2\pi i} \int \frac{g_2(w)}{w} \left[ \sum_{m=1}^{\infty} \left( \frac{1}{w} \right)^m + 1 \right] dw = \frac{1}{2\pi i} \int \frac{g_2(w)}{w} \left[ \frac{z}{w} + 1 \right] dw = \] \quad \text{(233)}

\[ = \frac{1}{2\pi i} \int \frac{g_2(w)}{w-z} dw = g_2(z) \]

according to Coushie’s formulae. Therefore, \( g_2(z) \) on the two sides of (230) cancels out. Changing the summation index in the last term in (230) from \( m \) to \( m' = m - 3 \), one obtains the following expression for the last infinite sum in (230)

\[ \sum_{m=3}^{\infty} \frac{d_m}{z^{m+2}} = \frac{c_{-1}^{(1)}}{16G_4} z^{-1} + \left( \frac{c_{-1}^{(1)}}{16G_4 G_2} - \frac{9}{4G_4} \right) z^{-2} + \frac{1}{G_1 G_2 - G_3} z^{-3} + \left[ \frac{3G_2}{G_3} - \frac{5}{2G_4} - \frac{c_{-1}^{(1)}}{16G_4 G_2} \right] z^{-4}, \] \quad \text{(234)}
where
\[ d_m \equiv \frac{1}{G_{m+2} - \frac{G_k}{G_2} G_m}. \] (235)

The essence of the above presented proof is the following: The Lor an’s decomposition (ranging from \(-\infty\) to \(+\infty\)) is known to be convergent only in a segment. On the other hand, for the positive - power terms in \(m\) in Lor an’s sum one has the Coushie’s formulæ, where the integration is performed around a closed contour (a circle) in the complex plane. As for the other part with the negative-power in \(m\) terms, if \(z\) is not restricted in a segment area, in the general case there is no guarantee that the sum will be convergent. However, in the present case the negative-power part is represented by the infinite sum in (234), and also by terms with inverse powers of \(z\), which are convergent in the limit \(z \to \infty\). The infinite sum \(\sum d_m z^{-m}\) is also convergent, since \(G_m\) and \(G_{m+2}\), entering \(d_m\), are finite when \(m \to \infty\) (according to a well-known theorem from complex analyses), and therefore the whole sum converges to zero in this limit (and also in the limit \(z \to \infty\)). This is entirely consistent with the R.H.S. of (234), which is convergent to zero in the limit \(z \to \infty\) and naturally does not contain a free term.

**XV. AN EXPRESSION FOR \(G_1\) FROM TAUBER’S THEOREM IN THE CASE OF INFINITE POLES**

Let us consider again the case of infinite poles, when \(G_k \to \zeta(k)\). Note that for \(k = 1\) the zeta-function \(\zeta(k)\) is not defined, so further the notation \(G_1\) shall be preserved.

**Proposition 3** The coefficient function \(d_m\) in the infinite sum (234) in the limit \(m \to \infty\) and under the assumption of infinite poles has the following behavior:
\[ d_m = O\left(\frac{1}{m}\right) \] (236)

**Proof.**

The proof is based on the following representation of the Riemann zeta-function, which can be found in Ref.40:
\[
\zeta(m) = \frac{1}{m-1} + \frac{1}{2} + \sum_{k=1}^{n} B_k m(m+1)\ldots(m+k-2)\frac{1}{k!} - \frac{1}{n!} m(m+1)\ldots(m+n-1) \int_{1}^{\infty} B_n(x)x^{-m-n}dx. \] (237)
Then $d_m$ can be represented as follows

$$d_m = \left[ \zeta(m + 2) - \frac{\zeta(4)}{\zeta(2)} \zeta(m) \right]^{-1} = N^{-1}[1 + \left( \frac{1}{N(m + 1)} - \frac{\zeta(4)}{N\zeta(2)(m - 1)} \right) +$$

$$+ \frac{1}{N} \sum_{k=2}^{n} B_k \frac{F_k}{k!} (m + 2)(m + 3)...(m + k - 2) + \frac{1}{N} \frac{P_m}{n!} (m + 1)...(m + n - 1)]^{-1} ,$$

(238)

where $F_k, P_m$ and $N$ are the following expressions

$$F_k = (m + k - 1)(m + k) - m(m + 1) \frac{\zeta(4)}{\zeta(2)} ,$$

(239)

$$P_m = \int_{1}^{\infty} B_n(x)x^{-m-n}[-x^{-2}(m + 2)(m + n)(m + n + 1) + \frac{\zeta(4)}{\zeta(2)}m]dx ,$$

(240)

$$N = \frac{1}{2} - \frac{\zeta(4)}{2\zeta(2)} .$$

(241)

In order to estimate $d_m$, one needs the following inequalities for the last two terms in (238):

$$\sum_{k=2}^{n} \frac{B_k F_k}{N k!} (m + 2)(m + 3)...(m + k - 2) > \sum_{k=2}^{n} \frac{B_k F_k}{N k!} m^{k-3}$$

(242)

$$\frac{1}{N} \frac{P_m}{n!} (m + 1)...(m + n - 1) > \frac{1}{N} \frac{P_m}{n!} m^{n-1} .$$

(243)

From these inequalities, an inequality for $d_m$ also follows

$$d_m < N^{-1} \left[ 1 + (...) + \sum \frac{B_k F_k}{N k!} m^{k-3} + \frac{1}{N} \frac{P_m}{n!} m^{n-1} \right]^{-1}$$

(244)

and (...) denotes the second term in the brackets in (238). It should be kept in mind also that in the limit $m \to \infty$ due to $x^{-m-n-2}$ the ratio $\frac{F_k}{P_m}$ is a very small term.

Further, inequality (244) may be rewritten as

$$d_m < \left( \frac{P_m}{n!} m^{n-1} \right)^{-1} \left[ 1 + \text{O}\left( \frac{1}{m^2} \right) + \text{O}\left( \frac{1}{m^{n-1}} \right) + \sum \frac{B_k}{k} \frac{n!}{N} \frac{F_k}{P_m} m^{k-n-2} \right]^{-1} .$$

(245)

Since the last three terms in the square brackets are small, the expression in the brackets can be decomposed (when $X \ll 1$) according to the formulae

$$[1 + X]^{-1} = 1 - X + X^2 .... .$$

(246)
Recall also from (240) that

\[ P_m^{-1} \equiv O\left(\frac{1}{m^3}\right), \quad F_k \sim m^2. \]  

Therefore, inequality (244) for \( d_m \) can be rewritten as

\[ d_m < n! \left( O\left(\frac{1}{m^{n-4}}\right) - O\left(\frac{1}{m^4}\right) - O\left(\frac{1}{m^3}\right) - \sum_{k=2}^{n} B_k \frac{n!}{k!} m^{k-2n-1}\right). \]

Neglecting all the small terms \( O(\ldots) \) and keeping in mind that in the last term in (248) the powers of \( m \) range from \(-2n (k = 2)\) to \(-n - 2 (k = n)\), \( |d_m| \) can be estimated

\[ |d_m| < (n!)^2 \sum_{k=2}^{n} \left| \frac{B_k}{k!} \right| m^{k-2n-2} < (n!)^2 (n-1) \left| \frac{B_k}{k!} \right| m^{-n-2} = O\left(\frac{1}{m^{n+2}}\right) < \left(\frac{1}{m}\right). \]

This precludes the proof that \( d_m = O\left(\frac{1}{m}\right) \).

The purpose of the above proposition is to demonstrate the opportunity to apply the Tauber’s theorem in respect to the infinite sum (234), which for the presently investigated case and in terms of \( \bar{\tau} = \frac{1}{z} \) can be rewritten as

\[ \sum_{m=3}^{\infty} \tilde{d}_m \bar{\tau}^m = F(\bar{\tau}) = \left\{ \left( \frac{\bar{c}^{(1)}_{-1}}{16\zeta(4)\xi(2)} \right) - \frac{9}{4\zeta(4)} \right\} + \left( \frac{\bar{c}^{(1)}_{-1}}{16\zeta(4)G_1} + \bar{\tau} \frac{1}{G_1\xi(2) - \xi(3)} \right) \bar{c}^{-1} \right\}, \]  

where \( \tilde{d}_m \) and \( \bar{c}^{(1)}_{-1} \) denote

\[ \tilde{d}_m \equiv \frac{1}{\zeta(m+2) - \frac{\zeta(4)}{4\zeta(2)}\zeta(m)}, \]  

\[ \bar{c}^{(1)}_{-1} \equiv \zeta(2) \frac{48\zeta(2)\xi(k) - 16\zeta(4)\xi(k) - 48\zeta(4)\xi(k-2)}{\zeta(2)\xi(k) - \zeta(4)\xi(k-2)}. \]

Let us remind the formulation of Tauber’s theorem in Ref. 41 in terms of our notations:

**Theorem 4** If the infinite sum (250) \( \sum_{m=3}^{\infty} \tilde{d}_m \bar{\tau}^m = F(\bar{\tau}) \) converges to \( S \) when \( \bar{\tau} \to 1 \) and also \( d_m = O\left(\frac{1}{m}\right) \), then the infinite sum \( \sum_{m=3}^{\infty} \tilde{d}_m \) also converges to \( S \).
Applying this theorem to the infinite sum (250) and expressing \( G_1 \) from there, one can easily obtain the following quadratic equation for \( G_1 \)

\[
G_1^2 - \frac{\zeta(2)}{16\zeta(4)} \tilde{F} \left[ 16\zeta(3) \tilde{F} + \frac{\tilde{c}_1^{(1)}}{\zeta(2)} + 16 \right] G_1 + \frac{\tilde{c}_1^{(1)} \zeta(2) \zeta(3)}{16 \tilde{F} \zeta^2(4)} = 0 , \tag{253}
\]

where

\[
\tilde{F} \equiv \sum_{m=3}^{\infty} \tilde{d}_m + \frac{19}{4\zeta(4)} - \frac{3\zeta(2)}{\zeta^2(4)} . \tag{254}
\]

Therefore, in the asymptotic limit \( z \to 1 \) and in the case of poles at infinity, \( G_1 \) is uniquely expressed through the Riemann zeta-function. As a solution of the quadratic equation, it has two values, which of course are finite, since the values of the zeta-function are finite. This again confirms the previously established result (although for a certain partial case) that \( G_1 \) is a finite (convergent) quantity for the case of the investigated cubic equations.

**XVI. INFINITE POINT OF THE LINEAR - FRACTIONAL TRANSFORMATION (31) AND THE TWO COUPLED ALGEBRAIC EQUATIONS FOR THE WEIERSTRASS FUNCTION**

In this Section again the obtained algebraic equation (53) will be studied, but for the case of the infinite point \( n = dx^5 = -\frac{d}{c} \) of the linear-fractional transformation (31), which for convenience shall be rewritten again

\[
dx^5 \equiv \frac{adx}{c} + b
\]

Substituting (255) into equation (53), a cubic algebraic equation is obtained for the variable \( \tilde{m} = \frac{\tilde{m}}{c} \):

\[
-A\tilde{m}^3 + 3A\frac{b}{d}\tilde{m}^2 + (B\frac{b}{d} - A\frac{b^2}{d^2} - 2\frac{d}{c}C)\tilde{m} + (-G^{(4)} + 2\frac{b}{c}C + \frac{b}{d}C) \equiv 0 , \tag{256}
\]

where \( A, B, C \) and \( G^{(4)} \) are given, as previously, by formulae (37 - 40) in Sect.V.

For the purpose of investigating the above equation, it is appropriate to remind briefly the two approaches, proposed in this paper. The first one was developed in Sect.VI and it contained in itself the following steps:
1. Transforming the original cubic equation into a cubic two-variable equation of the type (59), containing cubic terms of a new variable \( \tilde{m} = \frac{a}{c} \), related to the linear-fractional transformation, and also quadratic terms of the original variable \( n = dx^5 \). In a sense, the appearance of a new variable makes the equation more complicated, but at the same time this variable turns out to be the parametrizable one (with the Weierstrass function), and this shall be successfully exploited also in this section.

2. "Fixing up" the coefficient functions in the quadratic forms \( P_1(n), P_2(n), P_3(n) \) (60-62), so that the standard parametrizable form

\[
\tilde{n}^2 = 4m^3 - g_2m - g_3
\]  

(257)

is obtained with \( g_2, g_3 \) - definite complex numbers. The consistency of the parametrization, in the sense of an additionally imposed condition, was ensured by the obtained quadratic algebraic equation (78) in terms of the "angular" type variables \( l = (l_1, l_2, l_3) = p \times q \) and \( f = (f_1, f_2, f_3) = r \times q \)

\[
4\bar{q}_1 l^1 + g_2 f^1 \bar{q}_1 + l^1 f^3 + f^1 l^3 = 0 .
\]  

(258)

Evidently, there is a certain indeterminacy and freedom in choosing the elements of the algebraic variety \((l, f)\). However, by means of an appropriate choice of the ratio of the parameters in the linear-fractional transformation, equation (258) will acquire a very simple form, and in such a way this second difficulty can also be overcome.

The second approach, outlined in Sect. XII on the base of the analytical calculations in the preceding sections, was based on parametrization with the Weierstrass function of the cubic form (257), but with \( g_2 \) and \( g_3 \) - complex functions. In order to bring the original cubic equation into the form (257), a linear transformation with an appropriately chosen parameters was needed.

In the present section, a combination of the two approaches shall be implemented. First, the original cubic equation shall be "split up" into the following two algebraic equations

\[
t^2 = -A\tilde{m}^3 - \frac{d}{c}C\tilde{m} + (-G^{(4)} + 2\frac{b}{c}) ,
\]  

(259)

\[
t^2 = -3Ab\tilde{m}^2 - A\frac{b^2}{d^2}\tilde{m} + B\frac{b}{d}\tilde{m} .
\]  

(260)

The first equation is a cubic one and the second - a quadratic one, and this fact is important. Moreover, there is no need to perform a linear transformation, and since it has been proven that an equation of the type (259) is parametrizable with the Weierstrass function, one can set up:

\[
t = \rho'(z) \quad \tilde{m} = \rho(z) .
\]  

(261)

Therefore, the second equation (260) can be written in the form

\[
\left[\rho'(z)\right]^2 = K\rho(z)[\rho(z) - \alpha] ,
\]  

(262)
where
\[ K = -3A \frac{b}{d} \quad \alpha = \frac{1}{3} \frac{b}{d} - \frac{B}{3A}. \] (263)

Introducing the notation
\[ g(z) \equiv \rho(z) - \alpha \] (264)
and performing the integration in (262), one obtains
\[ 2 \int \frac{dg^{\frac{1}{2}}}{g^{\frac{1}{2}} \sqrt{1 + \frac{\alpha}{g}}} = \int \sqrt{K} dz . \] (265)

After two subsequent changes of variables
\[ g^{\frac{1}{2}} \equiv y \quad \text{and} \quad \frac{y}{g^{\frac{1}{2}}} \equiv tg\phi , \] (266)
one derives the following equation
\[ \int \sqrt{K} dz = 2 \int \frac{1}{\cos \phi} d\phi = 2 \int \frac{dr}{1 - r^2} = \ln | \frac{1 + r}{1 - r} | , \] (267)
where
\[ r = \sin \phi . \] (268)

In terms of the original variables, the solution is easily found to satisfy the equation
\[ | \frac{2\rho - \alpha + 2\epsilon \rho^{\frac{1}{2}}(\rho - \alpha)^{\frac{1}{2}}}{\alpha} | = \lambda_1 \exp(\int \sqrt{K} dz) . \] (269)

In (269) \( \epsilon = \pm 1 \) and \( \lambda_1 \) is the integration constant. Remembering the initial equation (262) and introducing for convenience the notations
\[ P(z) \equiv \frac{1}{2} \lambda_1 \sqrt{K} \exp(\int \sqrt{K} dz) + \frac{\alpha \sqrt{K}}{2} \quad Q(z) \equiv -\frac{\epsilon}{2} \sqrt{K} , \] (270)
the first order differential equation (268) may be rewritten in the simple form
\[ \rho'(z) = P(z) + Q(z) \rho(z) . \] (271)

Performing again the integration, one obtains
\[ \rho(z) = -\frac{P}{Q} + H , \] (272)
where \( H \) denotes
\[ H \equiv \frac{\lambda_2}{Q} \exp(\int Q dz) \] (273)
and $\lambda_2$ is the second integration constant. From (271) and (272) one can find

$$\lambda_2 = \frac{\rho'(z)}{\exp(\int Q\,dz)} \quad \text{and} \quad \rho'(z) = QH , \quad (274)$$

from where the integration constant $\lambda_2$ can be expressed, if one substitutes $\rho'(z)$ with its equivalent and well-known expression

$$\rho'(z) = -2 \sum_{\omega} \frac{1}{(z-\omega)^3} . \quad (275)$$

In order to find the first integration constant, or at least a relation between the two integration constants, let us substitute $\rho'(z)$ from (272) and $\rho(z)$ from (271) into the first cubic equation (259), which shall be written in a more general form

$$\left[ \rho'(z) \right]^2 = B_1 \rho^3(z) + B_2 \rho^2(z) + B_3 \rho(z) + B_4 . \quad (276)$$

In the present case

$$B_1 \equiv -A \quad B_2 \equiv 0 \quad B_3 \equiv -2\frac{d}{c} C \quad B_4 = -G^{(4)} + 2\frac{b}{c} C . \quad (277)$$

After the substitution, equation (276) acquires the following form in respect to the variable $X = \frac{P}{Q}$, containing the first integration constant $\lambda_1$ (which is to be found)

$$\overline{B}_1 X^3 + \overline{B}_2 X^2 + \overline{B}_3 X + \overline{B}_4 \equiv 0 , \quad (278)$$

where

$$\overline{B}_1 \equiv -B_1 \quad \overline{B}_2 \equiv 3HB_1 + B_2 \quad (279)$$

$$\overline{B}_3 \equiv -3H^2B_1 - 2HB_2 - B_3 \quad \overline{B}_4 \equiv B_1 + B_3 H + B_2 H^2 + B_1 H^3 - Q^2 H^2 . \quad (280)$$

Note that the other integration constant $\lambda_2$ through $H$ enters the coefficient functions $\overline{B}_1, \overline{B}_2, \overline{B}_3$ and $\overline{B}_4$.

**XVII. FINDING THE RELATION BETWEEN THE TWO INTEGRATION CONSTANTS**

The analyses in the preceding section concerned the parametrization of the two coupled equations, and performing an integration only of the second one. Now we shall find
the relation between the two integration constants, inserting the found as a result of the integration solution into the first equation (276-277).

Let us first observe that if one makes the formal identification

\[ A \leftrightarrow B_1, \quad B \leftrightarrow B_2, \quad C \leftrightarrow B_3, \quad D \leftrightarrow G^{(4)}, \] (281)

then equation (281) is analogous to equation (36) from Sect. V. Therefore, one may apply the developed there approach in the same manner.

After performing the linear-fractional transformation

\[ X = \frac{\pi \tilde{X} + \tilde{b}}{\tilde{c} \tilde{X} + \tilde{d}} \] (282)

(where naturally \( \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \) are different from \( a, b, c, d \) in the transformation (53 ,255)), and introducing the notation \( \chi = \frac{\pi}{\tilde{c}} \), the transformed equation (278) is obtained in the form

\[ C_1(\tilde{X}) \chi^3 + C_2(\tilde{X}) \chi^2 + C_3(\tilde{X}) \chi + C_4(\tilde{X}) \equiv 0. \] (283)

Since the subsequent step concerns the coefficient functions \( C_1(\tilde{X}), C_2(\tilde{X}), C_3(\tilde{X}), C_4(\tilde{X}) \), which are completely similar to the expressions (60-63) and (54-56), they shall be represented below in terms of the new notations

\[ C_1(\tilde{X}) \equiv r_1 \tilde{X}^2 + r_2 \tilde{X} + r_3 = -3 \tilde{B}_1(\frac{\tilde{c}}{\tilde{d}})^2 \tilde{X}^2 - 3 \tilde{B}_1 \frac{\tilde{c}}{\tilde{d}} \tilde{X} - \tilde{B}_1, \] (284)

\[ C_2(\tilde{X}) \equiv q_1 \tilde{X}^2 + q_2 \tilde{X} + q_3 = \left[ 3 \tilde{B}_1(\frac{\tilde{c}}{\tilde{d}})^2 \tilde{b} - 2 \tilde{B}_2(\frac{\tilde{c}}{\tilde{d}})^2 \right] \tilde{X}^2 - 3 \tilde{B}_2 \frac{\tilde{c}}{\tilde{d}} \tilde{X} - \tilde{B}_2, \] (285)

\[ C_3(\tilde{X}) \equiv p_1 \tilde{X}^2 + p_2 \tilde{X} + p_3 = \left[ -6 \tilde{B}_3 \frac{\tilde{c}}{\tilde{d}} + \tilde{B}_2 \frac{\tilde{b} \tilde{c}}{\tilde{d}} - \tilde{B}_3(\frac{\tilde{c}}{\tilde{d}})^2 \right] \tilde{X}^2 + \]

\[ + \left[ -6 \tilde{B}_3 + 2 \tilde{B}_2 \frac{\tilde{b} \tilde{c}}{\tilde{d}} + 3 \tilde{B}_1 \frac{\tilde{c} \tilde{b} \tilde{c}}{\tilde{d}} - \tilde{B}_3(\frac{\tilde{c}}{\tilde{d}})^2 \right] \tilde{X} - \tilde{B}_3 - 2 \frac{\tilde{d}}{\tilde{c}} \tilde{B}_3, \] (286)

\[ C_4(\tilde{X}) \equiv \mathcal{F} \tilde{X}^2 + \mathcal{N} \tilde{X} + \mathcal{M}, \] (287)

where in the last expression

\[ \mathcal{F} \equiv \tilde{B}_3 \frac{\tilde{c}}{\tilde{d}}(\frac{\tilde{c}}{\tilde{d}})^2 + 2 \tilde{B}_3 \frac{\tilde{c} \tilde{b}}{\tilde{d}} \frac{\tilde{c}}{\tilde{d}}, \] (288)

\[ \mathcal{N} = \tilde{B}_2 \frac{\tilde{c}}{\tilde{d}}(\frac{\tilde{c}}{\tilde{d}})^2 + 2 \tilde{B}_3 \frac{\tilde{c} \tilde{b}}{\tilde{d}} \frac{\tilde{c}}{\tilde{d}}, \] (289)
and

\[ M = B_1 \left( \frac{b}{d} \right)^3 + B_2 \left( \frac{b}{d} \right)^2 + B_3 \frac{b}{d} . \] (290)

Now in turn is the following important assumption: In order to simplify the expressions (284-286) and to make the most appropriate choice so that a relation is imposed only on the coefficients \( a, b, c, d \) of the linear-fractional transformation (and not on \( B_1, B_2, B_3, B_4 \), for example), let us suppose that \( C_2(\tilde{X}), C_3(\tilde{X}) \) are not quadratic, but \( \text{linear} \) in \( \tilde{X} \), i.e. \( q_1 = p_1 = 0 \). From (285) for \( C_2(\tilde{X}) \) and (286) for \( C_3(\tilde{X}) \) it follows

\[ \frac{\tilde{b}}{d} = \frac{2B_2}{3B_1} , \] (291)

\[ \frac{\tilde{c}}{d} = -6 + \frac{2B_2^2}{3B_1B_3} . \] (292)

After performing the same change of variables as in (65), but in terms of our notations

\[ X = \sqrt{F} \left( \tilde{X} + \frac{N}{2F} \right) \Rightarrow \tilde{X} = \frac{X}{\sqrt{F}} - \frac{N}{2F} , \] (293)

the transformed equation (283) will be

\[ X^2 = D_1(X)\chi^3 + D_2(X)\chi^2 + D_3(X)\chi + D_4(X) . \] (294)

Let us find also the quadratic algebraic equation (258) for \( (l_1, l_2, l_3) \) and \( (f_1, f_2, f_3) \) for the case \( p_1 = q_1 = 0 \). We have

\[ l = (l_1, l_2, l_3) = (0, p_2, 0) \quad f = (f_1, f_2, f_3) = (\overline{r}_1 \overline{q}_2, \overline{r}_2 \overline{q}_3, -\overline{r}_1 \overline{q}_3) . \] (295)

It can easily be checked that eq.(258) is \( \text{identially satisfied!} \) In other words, we have succeeded to find such a (trivial!) algebraic variety \( (p, q, r) \), so that the quadratic algebraic equation (221) holds and therefore, the parametrization with the Weierstrass function can be applied:

\[ \overline{X} \equiv \rho'(v) \quad \chi = \frac{\bar{\pi}}{\bar{\epsilon}} = \rho(v) . \] (296)

In principle, \( v \) should be another complex variable, but here for simplicity it shall be assumed that \( v = z \). Now it is interesting to note that relations (291) for \( \frac{\tilde{b}}{d} \), (292) for \( \frac{\tilde{c}}{d} \) and (296) for \( \frac{\bar{\pi}}{\bar{\epsilon}} \) allow us to determine all the ratios between \( \overline{a}, \overline{b}, \overline{c}, \overline{d} \). Consequently, \( \overline{F} \) and \( \overline{N} \) in (288 - 289) are also determined, and taking into account transformation (293) and expression (274) for \( \lambda_2 \) (expressed through \( \rho'(z) \)), the linear-fractional transformation (282) can be represented as

\[ X = \frac{\frac{\bar{\pi}}{\bar{\epsilon}} \overline{X} + \frac{\bar{\pi}}{\bar{\epsilon}}}{\overline{X} + \frac{\bar{\pi}}{\bar{\epsilon}}} = \frac{\rho(z)[\overline{F}^{\frac{1}{2}}\lambda_2 \exp(\int Qdz) - \frac{\overline{N}}{2\overline{F}}] + \frac{\bar{\pi}}{\bar{\epsilon}}}{\overline{F}^{\frac{1}{2}}\lambda_2 \exp(\int Qdz) + \frac{\bar{\pi}}{\bar{\epsilon}}} . \] (297)
As already mentioned, the ratios \( b : c, d : c \) can easily be found from (291-292)
\[
\frac{b}{c} = \frac{B_2 B_3}{B_2^2 - 9B_1 B_2}, \quad \frac{d}{c} = \frac{3B_1 B_2}{2(B_2^2 - 9B_1 B_2)}.
\] (298)

On the other hand, \( X = \frac{P}{Q} \) can be determined also from (270)
\[
X = \frac{P}{Q} = -\epsilon \lambda_1 \exp(\int \sqrt{K} dz) - \epsilon \alpha.
\] (299)

From the two equations (297) and (299), the first integration constant \( \lambda_1 \) can be expressed through the other constant \( \lambda_2 \). Moreover, (297) can be substituted into (292) and after some transformations, the following quadratic equation can be found for \( \lambda_2 \)
\[
\lambda_2^2 + K_1 \lambda_2 + K_2 (\rho(z) - \frac{b}{d}) = 0,
\] (300)
where
\[
K_1 \equiv \frac{1}{2\sqrt{F} \sqrt{c}} \frac{d}{c} \exp(\frac{\epsilon}{2} \int \sqrt{K} dz) \left[ 2F - N \frac{c}{d} \right],
\] (301)
\[
K_2 \equiv \frac{\epsilon}{2} \sqrt{F} \frac{d}{c} \sqrt{K} \exp(\epsilon \int \sqrt{K} dz),
\] (302)
The two roots of the quadratic equation (300) can be expressed as
\[
\lambda_2 = \frac{-K_1 + \epsilon \sqrt{K_1^2 - 4(\rho(z) - \frac{b}{d})K_2}}{2}.
\] (303)
But \( \lambda_2 \) can also be expressed from (274) and (262) in the following way
\[
\lambda_2^2 = \frac{[\rho'(z)]^2}{\exp[-\epsilon \int \sqrt{K} dz]} = \frac{K \rho^2(z) - K \alpha \rho(z)}{\exp(-\epsilon \int \sqrt{K} dz)}.
\] (304)
Now taking the square of \( \lambda_2 \) from (303) and expressing the equality of (303) and (304), one can obtain
\[
2K \rho^2(z) + 2(K_2 - \alpha K) \rho(z) - \overline{L} =
\]
\[
= -\epsilon K_1 \exp(-\epsilon \int \sqrt{K} dz) \sqrt{K_1^2 - 4(\rho(z) - \frac{b}{d})K_2},
\] (305)
where \( \overline{L} \) denotes
\[
\overline{L} \equiv 2K_2 \frac{b}{d} + \exp(-\epsilon \int \sqrt{K} dz) K_1^2.
\] (306)
In order to get rid of the square root in (305) and thus to obtain an algebraic equation, let us take the square of both sides of (305). The result is a fourth-order algebraic equation in respect to the Weierstrass function \( \rho(z) \)

\[
E_1 \rho^4(z) + E_2 \rho^3(z) + E_3 \rho^2(z) + E_4 \rho(z) + E_5 = 0 ,
\]

where

\[
E_1 \equiv 4K^2 , \quad E_2 \equiv 8K(K_2 - \alpha K) , \quad E_3 \equiv 4[(K_2 - \alpha K)^2 - \mathcal{L}K] , \quad (308)
\]

\[
E_4 \equiv 4[K_1^2K_2\exp(-2\varepsilon \int \sqrt{K}dz) - \mathcal{L}(K_2 - \alpha K)] , \quad (309)
\]

\[
E_5 \equiv - \left[ \mathcal{L}^2 + K_1^4\exp(-2\varepsilon \int \sqrt{K}dz) + \frac{b}{d}K_2K_1^2\exp(-2\varepsilon \int \sqrt{K}dz) \right] . \quad (310)
\]

In a sense, it is surprising that (307) is a fourth-rank algebraic equation, while after parametrizing the "original" system of two coupled equations (259 - 260), one is left with a third-rank algebraic equation. It can be rewritten as

\[
\rho^3(z) = 3 \frac{b}{d} \rho^2(z) + \left( \frac{Bb}{A} - \frac{b^2}{A^2} - 2 \frac{C}{A} \right) \rho(z) + \left( - \frac{G^{(4)}}{A} + 2 \frac{b}{c} \frac{C}{A} + \frac{b}{d} \frac{C}{A} \right) . \quad (311)
\]

It should be remembered, however that the fourth-rank algebraic equation (307) has been obtained after the differential equation (262) has been solved, and through the integration constant this amounts to a redefinition of the complex variable and the functions. Also, a more "specific" form of the linear-fractional transformation (282) has been performed, which in a way 'adjusts' the parametrizable variable in (294) \( \mathcal{X} = \rho'(v) \), and it was assumed that \( v = z \). What is perhaps interesting, from a purely mathematical point of view, is that this "redefinition" and transformation resulted in a higher-rank algebraic equation.

It is important to note that in writing down the system of the two coupled equations (259-260), only the proven in the previous sections fact about the parametrization of a cubic equation of the kind \( t^2 = 4m^3 - g_2(z)m - g_3(z) \) has been used. Since \((t, m)\) turn out to be parametrized with \( (\rho'(z), \rho(z)) \), and \((t, m)\) enter also the second equation (260), this explains why the Weierstrass function "parametrizes" the second equation. But this does not mean yet that the more general cubic equation (256) can be parametrized with the Weierstrass function. As it has been already mentioned in Section VI, this fact needs to be proved, if this is possible at all. In the next Section, some necessary conditions will be found for parametrizing the cubic equation (256) of a more general kind (with all non-zero coefficient functions) . Although for a particular equation, the investigation in the next section might provide a partial answer to the raised in Sect. VIII problem whether it is possible to parametrize a cubic equation in its general form.
XVIII. FINDING THE NECESSARY CONDITIONS FOR PARAMETRIZATION OF THE CUBIC EQUATION (256)

The analyses in this section concern equations (307) and (311). If $\rho^3(z)$ is substituted into equation (307), then the following equation is obtained in respect to $\rho(z)$

$$S_1 \rho^4(z) + S_2 \rho^2(z) + S_3 \rho(z) + S_4 = 0 ,$$

where

$$S_1 \equiv 4K^2, \quad S_2 \equiv 4 \left[ (K_2 - \alpha K)^2 + 6 \frac{b}{d} K (K_2 - \alpha K) \right] ,$$

$$S_3 \equiv 4[K_1^2 K_2 \exp(-2\varepsilon \int K dz) - \overline{L}(K_2 - \alpha K) + 2K(K_2 - \alpha K) \left( \frac{b}{c} - \frac{b^2}{d^2} - 2 \frac{C}{A} \right) ,$$

$$S_4 \equiv 8K(K_2 - \alpha K) \left( \frac{bc}{A} + \frac{bC}{dA} - \frac{G^{(4)}}{A} \right) - \overline{L}^2 - K_1^4 \exp(-2\varepsilon \int \sqrt{K} dz) - 4 \frac{\overline{B}_3^2}{d} K_2 \rho^2(z) \exp(-2\varepsilon \int \sqrt{K} dz) .$$

Two possibilities result from the above equation (312), and they shall be investigated here.

The first possibility is when (312) holds for arbitrary values of $z$ and $\rho(z)$, and therefore the coefficient functions $S_1, S_2, S_3, S_4$ must be identically zero. This means that

$$S_1 = K \equiv 0 .$$

But if $K = 0$, then from the defining equality (302) for $K_2$ it follows

$$K_2 = 0 .$$

If (316-317) are satisfied, then it will follow also

$$S_2 = S_3 = 0 ,$$

and this is exactly what is needed. Now it remains only to impose the condition

$$0 \equiv S_4(K=0) = -\overline{L}^2 - K_1^4 = -2K_1^4 .$$

Using expressions (301) for $K_1$ and (288-289) for $F$ and $N$, it can be shown that the above equation is equivalent to the following equation

$$\overline{b} \frac{d}{\overline{B}_3} \left[ 4\overline{B}_3 - \overline{B}_2 \frac{\overline{b}}{d} \right] = 0 .$$
The first opportunity for this equation to be satisfied is when
\[ \frac{\partial}{\partial d} \frac{2B_2}{3B_1} = \frac{2HB_1}{3B_1} \equiv 0 . \]  
(321)

Since we do not want to impose the rather restrictive condition \( B_1 = -A = 0 \), it remains (with account also of (273)) that
\[ 0 \equiv H \equiv -\frac{2\lambda_2}{\varepsilon \sqrt{K}} \exp\left(\frac{-\varepsilon}{2} \int \sqrt{K} dz\right) \Rightarrow \lambda_2 = 0 . \]  
(322)

If \( \lambda_2 = 0 \), then from (297) and (299) it can be found that
\[ \lambda_1 = -\alpha + \varepsilon \rho(z) . \]  
(323)

But perhaps it is more interesting to see from (270)) and (271) that for the particular case it will follow
\[ P = Q = 0 \quad \rho'(z) = 0 \quad \Rightarrow \rho(z) = \text{const.} . \]  
(324)

This represents the first found very nice and simple condition under which a parametrization of (256) with the Weierstrass function is possible. The second possibility for equation (320) to hold is when
\[ 4B_3 - B_2 \frac{\tau}{dz} \equiv 0 . \]  
(325)

Additionally, if equations (277), (279-280) and (273) for \( H \) are taken into account, the above equation can be represented in the following form in respect to the integration constant \( \lambda_2 \)
\[ -16.3^3 A^2 \lambda_2^2 = 0 . \]  
(326)

Again, since \( A \neq 0 \), we come to the case \( \lambda_2 = 0 \).

**XIX. ANOTHER NECESSARY CONDITION FOR PARAMETRIZATION OF (256) AND APPLICATION OF THE THEORY OF RIEMANN SURFACES**

Now the second possibility when equation (312) holds shall be investigated. If one denotes \( \rho(z) \equiv w \), then the equation
\[ S_1 w^4 + S_2 w^2 + S_3 w + S_4 = 0 \]  
(327)
and the couple of complex variables \((w, z)\) determine a Riemann surface for the equation
\[
w = F(z, S_1(z), S_2(z), S_3(z), S_4(z)) ,
\] (328)
which represents a solution of the complex algebraic equation (327). Since the equation always has complex solutions and the Weierstrass function in principle is also a complex function, let us write \(w\) as
\[
\rho(z) \equiv w \equiv w_1(z) + iw_2(z) .
\] (329)
Remember also that no matter that from the original definition of the Weierstrass function in (35) it is not evident that the "imaginary" part \(w_2(z)\) is present, this can be guessed from the presence of complex poles, which after decomposing into real and imaginary numbers will give the "\(w_1(z)\)" and "\(w_2(z)\)" terms. If one substitutes (329) into (327), then equation (327) for the Riemann surface "splits up" into two equations for a couple of Riemann surfaces, determined by the complex pairs \((w_1, z)\) and \((w_2, z)\). The obtained equations for the real and imaginary parts of the complex algebraic equation (327) are respectively
\[
S_1(w_1^2 - w_2^2)^2 - 4S_1w_1^2w_2^2 + S_2(w_1^2 - w_2^2) + S_3w_1 + S_4 = 0 ,
\] (330)
\[
w_2 \left[ 4S_1w_1(w_1^2 - w_2^2) + 2S_2w_1 + S_3 \right] = 0 .
\] (331)
The last equation for the imaginary part is identically satisfied when \(w_2 = 0\). Then the corresponding \(w_1\) has to be determined from (330), which becomes a quartic equation
\[
S_1w_1^4 + S_2w_1^2 + S_3w_1 + S_4 = 0
\] (332)
with roots \(x_1, x_2, x_3, x_4\), satisfying the relation \(-x_3 - x_4 = x_1 + x_2 = u\), and according to the general theory\(^{13}\) the roots can be found after solving the equation
\[
u^6 + 2\frac{S_2}{S_1}u^4 - \left[\left(\frac{S_2}{S_1}\right)^2 - 4\frac{S_4}{S_1}\right]u^2 - \left(\frac{S_3}{S_1}\right)^2 = 0 .
\] (333)
The other case when (331) is identically zero is when the expression in the square brackets is zero, and from there
\[
w_2^2 = \frac{4S_1w_1^3 + 2S_2w_1 + S_3}{4S_1w_1} .
\] (334)
Again, substituting (334) into (330) for the real part and setting up \(w_1^2 \equiv \bar{w}_1\), one can obtain the following third-rank equation for \(\bar{w}_1\)
\[
64S_1^3\bar{w}_1^3 + 32S_1^2S_2\bar{w}_1 + (4S_1S_2^2 - 16S_1^2S_4)\bar{w}_1 - S_1S_3^2 = 0 .
\] (335)
This equation has three complex roots for \(\bar{w}_1\) and therefore - six roots for \(w_1\). Note also that each time when \(\varepsilon\) appears in the coefficient functions, then for each function \(S_1, S_2, S_3, S_4\) and for every root, the corresponding expressions for the roots have to be taken once with \(\varepsilon = 1\) and once with \(\varepsilon = -1\). So there will be a finite number of combinations, but more than six.
**XX. NECESSARY AND SUFFICIENT CONDITIONS FOR PARAMETRIZATION WITH A CONSTANT WEIERSTRASS FUNCTION AND THE RESULTING NONLINEAR EQUATIONS**

In this section it shall be assumed that a constant Weierstrass function \( \rho(z) \equiv const \equiv w \equiv w_1(z) + iw_2(z) \) parametrizes the algebraic equation (327), defined over the Riemann surface \((w, z)\). If the Weierstrass function is constant then it follows that

\[
\frac{\partial w}{\partial z} = \frac{\partial \bar{w}}{\partial z}, \quad (336)
\]

and therefore

\[
\frac{\partial w_1}{\partial z} + i \frac{\partial w_2}{\partial z} = \frac{\partial w_1}{\partial z} - i \frac{\partial w_2}{\partial z} = 0 . \quad (337)
\]

Consequently,

\[
\frac{\partial w_1}{\partial z} = \frac{\partial w_2}{\partial z} = 0 . \quad (338)
\]

Taking this into account and differentiating by \( z \) equation (334) for \( w_2 \), one can obtain

\[
w_1 = -\frac{X'}{Y'}, \quad (339)
\]

where \( X \) and \( Y \) denote

\[
X = \frac{S_3}{4S_1}, \quad Y = \frac{S_2}{2S_1}, \quad (340)
\]

and ‘ means a derivative in respect to the complex variable \( z \). Denoting \( \beta = w_2^2 = const \) and inserting back \( w_1 \) from (339) into (334), one can obtain the following *nonlinear equation*

\[
\left(\frac{X'}{Y'}\right)^2 - X \frac{Y'}{X'} + Y - \beta = 0 . \quad (341)
\]

Introducing the notation \( Z = \frac{X'}{Y'} \) allows us to rewrite the equation as a *cubic algebraic equation* in respect to \( Z \)

\[
Z^3 + (Y - \beta)Z - X = 0 . \quad (342)
\]

Finding the roots of this equation, one can easily express \( X' \) as

\[
X' = \bar{F}(X, Y)Y', \quad (343)
\]
where \( \tilde{F}_1(X, Y) \) is the expression for the first root of the cubic equation

\[
\tilde{F}_1(X, Y) = \sqrt[3]{\frac{X}{2}} + \sqrt{\frac{X^2}{4} + \frac{(Y - \beta)^3}{27}} + \sqrt[3]{\frac{X}{2} - \sqrt{\frac{X^2}{4} + \frac{(Y - \beta)^3}{27}}} .
\] (344)

The other roots are correspondingly

\[
\tilde{F}_2(X, Y) = -\sqrt[3]{\frac{X}{2}} + \sqrt{\frac{X^2}{4} - \frac{(Y - \beta)^3}{27}} - \sqrt[3]{\frac{X}{2} - \sqrt{\frac{X^2}{4} - \frac{(Y - \beta)^3}{27}}} ,
\] (345)

\[
\tilde{F}_3(X, Y) = \sqrt[3]{\frac{X}{2}} + \sqrt{\frac{X^2}{4} + \frac{(Y - \beta)^3}{27}} - \sqrt[3]{\frac{X}{2} - \sqrt{\frac{X^2}{4} + \frac{(Y - \beta)^3}{27}}} .
\] (346)

Further the calculations will be done only for the first root \( \tilde{F}_1(X, Y) \).

Now let us perform a differentiation by \( z \) of equation (330) for the real part, taking into account also the defining expressions (340), (339) and introducing also the notation \( T \equiv \frac{S_4}{S_1} \). The result be another nonlinear differential equation

\[
2YY'' - 4\tilde{F}^2Y' + T' = 0 .
\] (347)

Equations (341) and (347) represent a system of coupled nonlinear differential equations for \( X \) and \( Y \). Now we are going to establish an interesting mathematical property of this system: due to the peculiar structure of equation (341) as an algebraic equation with respect to the ratio of the derivatives of \( X \) and \( Y \) and subsequently expressing \( X' \) as linearly proportional to \( Y' \) (with a coefficient function \( \tilde{F} \) depending only on \( X \) and \( Y \)), this system will turn out to be an integrable one! So, we shall treat equations (343) and (347) instead of (341) and (347). Substituting (343) into (347), we receive

\[
2YY'' - 4\tilde{F}^2Y' + T' = 0 .
\] (348)

Fortunately, this equation can be integrated to give

\[
T + f(X,...) = -Y^2 + 4 \frac{4}{27} \beta^2 (Y - \beta)^2 + 4I_1 + 4I_2
\] (349)

and the integrals can be written as

\[
I_1, I_2 \equiv \int \left( \frac{X}{2} \mp \sqrt{\frac{X^2}{4} + \frac{(Y - \beta)^3}{27}} \right)^{\frac{2}{3}} dY .
\] (350)

The integration function \( f(X,...) \) depends on \( X \) and other constants or functions, different from \( Y \). Now it remains to calculate (analytically) the remaining two integrals. For the purpose, let us perform the following three consequent variable changes

\[
\frac{Y - \beta}{3} \equiv t ,
\] (351)
\[ \tilde{t} \equiv \left( \frac{4}{X^2} \right)^{\frac{3}{2}} t \]  

and

\[ \tilde{t} \equiv \sqrt{1 + \tilde{t}^3} = \sqrt{1 + \frac{4}{X^2} \frac{(Y - \beta)^3}{27}}. \]  

It is straightforward to check that the integrals can be brought to the following form

\[ I_1 = \int (1 + \tilde{t})^{\frac{3}{2}} \tilde{t} d\tilde{t} = \frac{3}{5} \tilde{t}(1 + \tilde{t})^{\frac{5}{2}} \mid_{\tilde{t}=z_1}^{\tilde{t}=z_2} - \frac{3}{5} \int (1 + \tilde{t})^{\frac{3}{2}} d\tilde{t}, \]  

\[ I_2 = \int (1 - \tilde{t})^{\frac{3}{2}} \tilde{t} d\tilde{t} = \frac{3}{5} \tilde{t}(1 - \tilde{t})^{\frac{5}{2}} \mid_{\tilde{t}=z_1}^{\tilde{t}=z_2} - \frac{3}{5} \int (1 - \tilde{t})^{\frac{3}{2}} d\tilde{t}. \]  

Note that if a closed contour is chosen, then the first terms in the above two integrals will be zero. If the integrals are calculated and expression (353) is used, then the following expression is obtained

\[ I_1 + I_2 = -\frac{9}{40} \left\{ \left( 1 + \sqrt{1 + \frac{4}{X^2} \frac{(Y - \beta)^3}{27}} \right)^{\frac{5}{2}} + \left( 1 - \sqrt{1 + \frac{4}{X^2} \frac{(Y - \beta)^3}{27}} \right)^{\frac{5}{2}} \right\}. \]  

Substituting back into expression (349), one obtains a solution of the nonlinear differential equation (348), which depends on terms in powers of fractional numbers.

**XXI. CONCLUSION**

Let us summarize the obtained results.

In this paper a cubic algebraic equation has been obtained in respect to the differentials \( dX^i \) of some generalized coordinates \( X^i \). The derivation of the equation was possible due to the representation of the contravariant metric tensor in terms of differential quantities. Also, in Sect. III the equation was derived upon assuming that \( dX^i \) is either an exact differential, or that \( dX^i \) are zero-helicity vector field components.

The derived equation (20) clearly reflects the structure of the gravitational Lagrangian, and can be regarded as an equation for its all possible coordinate transformations (admissible parametrizations), provided the Christoffel’s connection \( \Gamma^k_{ij} \) and the Ricci tensor \( R_{ij} \) are given.

The main problems, which one encounters when investigating such algebraic equations are several, and in this paper only one of them is resolved in more details.

The first and most serious problem is that the equation is defined on an algebraic variety of several variables, since in gravity theory one usually deals with at least four-dimensional (and higher-dimensional also) manifolds. At the same time, the standard and known methods from algebraic geometry for parametrizing algebraic curves by means of
the Weierstrass function concern only algebraic curves of two variables. That is why in the paper one of the variables - $dx_5$ has been singled out on the base of the physical consideration of the Randall-Sundrum models, and the other variable for convenience is chosen to be the ratio $\frac{a}{c}$ of the functions $a(z)$ and $c(z)$, which enter in the linear-fractional transformation of $dx_5$. The rest of the variables $dx_1, dx_2, dx_3, dx_4$ enter the cubic equation in scalar quantities (functions). Of course, if a two-dimensional manifold is considered, then the variables might be related to the two variables of the manifold. Such an analyses of a two dimensional algebraic equation and its parametrization may find application in string and brane theory (also in gravity theory).

For the purpose of higher-dimensional algebraic varieties and equations, probably methods from the theory of abelian varieties and hyper-elliptical (Weierstrass) functions have to be applied. However, these methods are developed at an abstract mathematical level, far from being adjusted to any concrete application.

The second problem concerns the methods for bringing the algebraic equation to a parametrizable form. The standard approach of applying a linear-fractional transformation has been chosen for the purpose, but in Sect. XII it was demonstrated also how a linear transformation can also be applied. In the last case, a couple of cubic equations is investigated, and what is interesting is that by means of parametrization of the first equation with the Weierstrass function $\rho(z)$ the second equation is obtained in a parametrizable form, but in the general form of a cubic equation in respect to $\rho(z)$. This justifies the performed investigation in Sect. VIII, concerning the (eventual) possibility for the Weierstrass function to satisfy a cubic equation of the general kind (108). Sects. XVI - XIX demonstrate how a differential equation with the Weierstrass function can be integrated, and after performing an additional linear-fractional transformation and finding the relation between the two integration constants in Sect. XVII, one comes to a purely algebraic (and not differential) equation. Interestingly, in some special cases, for example the case of parametrization with a constant in Weierstrass function in Sect. XX, one can effectively work with the approach of Riemann surfaces and even to obtain an integrable equation at the end. It should be noted here that this is possible in some cases only. For example, if one changes the formulation of the problem and would like to find the conditions for parametrization with a real-valued Weierstrass function, then it may be shown that the equation will not be an integrable one. This is not performed in this paper.

As for the linear-fractional transformation, its advantages are the following: 1. It contains more parameters (in the case - functions $a, b, c, d$ of a complex variable) and it makes possible to take account of the point at infinity. In principle, in complex analyses and projective geometry this is a well-developed procedure, but as far as physical applications in gravity theory and in relativistic hydrodynamics are concerned - this is still un unexplored area. 2. In Sect. VI it was proved that by means of a suitable change of variables it is possible to derive a second-order (quadratic) algebraic equation in terms of "angular"-type variables from the initial cubic algebraic equation. Since a quadratic equation is easier to deal with, this simplifies the analyses and moreover, a transition to the original variables can also be performed.
The third problem, investigated in more details in the present paper, is the form of the parametrizable cubic curve. This was discussed in the Introduction, and evidently a concrete physical problem from gravity theory has shown the necessity to investigate the case when $g_2$ and $g_3$ are complex functions and not complex numbers, as it is in the standard theory of elliptic functions. In regard to this, in Sect. VIII and Sect. IX two mutually related with each other problems for resolving are being stated: 1. Can the Weierstrass function $\rho(z)$ parametrize an arbitrary cubic curve with coefficient functions of a complex variable? 2. Can the Weierstrass function parametrize the well-known parametrizable form of the cubic equation, but again with coefficient functions depending on a complex variable? Although the explicit form of the equations for the Loran coefficient functions are presented in Sect. VIII, the first question still remains unanswered, and perhaps computer simulations only can help for its resolution. As for the second question, the answer is affirmative, and after solving a system of algebraic equations for various values of $m$ and $n$, the explicit form of all the Loran coefficient functions $c_{m}^{(1)}$, $c_{m}^{(0)}$ was found, both from the negative-power and the positive-power expansion. A confirmation of the consistency of the derived equations is equation (153) for a value of $m = -6$, which is being satisfied by the previously derived equations. However, the values of the coefficient functions are perhaps not so important as the result, which follows from this calculation, namely: the infinite sums $G_1$ and $G_2$, which in the general case might be divergent, in the particular case of the "parametrizable" form of the cubic algebraic equation with $g_2 = g_2(z)$ and $g_3 = g_3(z)$, should be convergent! This fact, although of pure mathematical nature, probably deserves more attention and further elaboration from another point of view and by applying different mathematical approaches. The finiteness of $G_1$ and $G_2$ is not imposed "by hand", but is obtained as a consequence of the fulfillment of the above mentioned equation with the Weierstrass function.

It is to be noted also that if some other assumptions are made - for example in Sect. XI about poles at infinity in the positive-power decomposition of $g_2(z)$ and $g_3(z)$, then within the large $n$ asymptotic approximation, a method for comparing the convergency radius of two infinite sums may be used for the determination of the coefficient functions $c_{-k}^{(1)}$. Comparing the expressions for $c_{-k}^{(1)}$ in the negative-power Loran decomposition and in the other case of positive-power Loran decomposition with poles at infinity, it is seen that in the two cases $c_{-k}^{(1)}$ are expressed in different ways. For example, in the first case from (143) $c_{-k}^{(1)}$ is expressed through the sums $G_k$, while in the second (asymptotic) case $G_n$ do not appear (185), and instead of them the Gamma function $\Gamma(l)$ appears and some finite and infinite summation formulae of the kind $\sum_{k=0}^{l-1} l(l-1)...(l-k)$ and $\sum_{n=0}^{\infty} \frac{(-1)^n}{n! (l+1+n)}$. Therefore, the assumption about poles at infinity and the consequent appearance of the zeta-function substantially changes the calculations. However, while the infinite point of the linear-fractional transformation may have some physical justification, the "poles at infinity" case for the moment does not have a definite physical meaning.
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APPENDIX A: ADDITIONAL SYSTEM OF EQUATIONS FOR \( m = -3, -1 \)

In this Appendix the system of equations for \( m = -3, -1 \) will be presented, which were not investigated in Sect. IX. However, the method for their derivation is completely the same, but some new interesting consequences will appear.

For the purpose, let us first rewrite the two sums in expression (111) for \( [\rho'(z)]^2 \), putting in the first sum \( 2(n-1) = m \) and in the second term \( n - 4 = m \). Then expression (111) acquires the form

\[
[\rho'(z)]^2 = \frac{4}{z^6} + \frac{1}{16} \sum_{m=0,2,4,...}^\infty G_{m+6}(m+2)^2(m+4)^2z^m -
-4 \sum_{m=-3,-2,-1,0,1,...} G_{m+6}(m+4)(m+5)z^m. \quad (A1)
\]

For \( m = -3 \) only the term from the second sum in (A1) will contribute. Putting also \( m = -3 \) in the R.H.S. of expression (118) for \( M(z)\rho^3 + N(z)\rho^2 + P(z)\rho + E(z) \) for the case of \( M = 4, N = 0, P(z) \equiv -g_2(z) \) and \( E(z) \equiv -g_3(z) \) (i.e. \( c_0^{(3)} = 0 \) for all \( m \), \( c_0^{(3)} \equiv 4 \) and \( c_m^{(3)} \equiv 0 \) for \( m \neq 0 \)), one has to take into account that terms with a ”negative-valued” indice like \( c_{-1-2n}^{(3)}, c_{-1-n}^{(3)}, c_{-3-3n}^{(3)} \) \( (n = 1, 2,...) \) are zero.

The obtained equation for \( m = -3 \) is

\[
-8G_3 = 2(n+1)G_n c_{-1-n}^{(3)} + c_{-1}^{(1)} + c_{-3-n}^{(1)} G_n + c_{-3}^{(0)}. \quad (A2)
\]

For \( n = 1 \), when the first term on the R.H.S. is non-zero, the equation is

\[
-8G_3 = 16G_1 + c_{-1}^{(1)} + c_{-4}^{(1)} G_1 + c_{-3}^{(0)}. \quad (A3)
\]
Since from expressions (142) and (136) from Section 7 \( c_{-3}^{(1)} \) and \( c_{-4}^{(1)} \) can be found, from the above equation (A3) \( c_{-3}^{(0)} \) can be expressed.

For \( n = p > 1 \) equation (A2) is

\[
-8G_3 = c_{-1}^{(1)} + c_{-3-p}^{(1)}G_p + c_{-3}^{(0)}.
\]

Subtracting the two equations (A3-A4), one can find the following expression for \( c_{-3-p}^{(1)} \):

\[
c_{-3-p}^{(1)} = \frac{G_1}{G_p}(16 + c_{-4}^{(1)}) = 16 \frac{G_1}{G_p} \left[ \frac{G_4}{G_3}G_p + 3\left(\frac{G_4}{G_3}\right)^2G_p - (3\frac{G_4}{G_3} + 21)G_p^{-2} \right] \frac{G_3}{G_4}G_p - G_p^{-2} .
\]

An expression for \( c_{-3-p}^{(1)} \) can also be found from formulae (143) for \( k = p + 3 \)

\[
c_{-3-p}^{(1)} = -\frac{16G_4}{G_{p+3} - \frac{G_4}{G_3}G_{p+1}}.
\]

From the equality of the two expressions, \( G_1 \) can again be expressed as a convergent expression. Note also that the formulae \( G_2_p = G_3\left(\frac{G_4}{G_3}\right)^{2p-1} \) and \( G_{2p+1} = G_3\left(\frac{G_4}{G_3}\right)^{p-1} \) from (151) satisfy the equality expression since then the denominators in (A5) and (A6) will be zero. This precludes the investigation of the system of equations for \( m = -3 \).

For \( m = -1 \), the general equation can be written as

\[
-48G_5 = 2(n + 1)G_n\epsilon_{-3-n}^{(3)} + (n + 1)G_n\epsilon_{-1-n}^{(3)} + G_n\epsilon_{-1-n}^{(1)} + c_1^{(1)} + c_{-1}^{(0)},
\]

and for \( n = 1 \) and \( n = 2 \) the corresponding equations are

\[
-48G_5 = 8G_1 + c_{-2}^{(1)}G_1 + c_1^{(1)} + c_{-1}^{(0)},
\]

\[
-48G_5 = G_2c_{-3}^{(1)} + c_1^{(1)} + c_{-1}^{(0)}.
\]

The coefficient \( c_{-3}^{(1)} \) can also be found from (143) for \( k = 3 \)

\[
c_{-3}^{(1)} = -\frac{16G_4}{G_3 - \frac{G_4}{G_2}G_1},
\]

. Substituting (A10) into (A9) gives an opportunity to express \( c_1^{(1)} + c_{-1}^{(0)} \) as

\[
c_1^{(1)} + c_{-1}^{(0)} = -48G_5 + \frac{16G_2G_4}{G_3 - \frac{G_4}{G_2}G_1}.
\]

This expression, together with formulae (137) for \( c_{-2}^{(1)} \) and (143) for \( c_{-1}^{(1)} \), represented as \( c_{-1}^{(1)} = \frac{F}{G_1} \), can be substituted into the first equation (A8) to obtain the following quadratic equation for \( G_1 \)

\[
(G_3 - \frac{G_4}{G_2}G_1)(\frac{G_1}{G_2}F - 24G_1) + 16G_2G_4 = 0 .
\]
In a similar way, one can write down the equation for \( n = 3 \)
\[
-48G_5 = 32G_3 + c_{-4}^{(1)}G_3 + c_3^{(1)} + c_{-1}^{(0)}.
\]
(A13)

Substituting \( c_{-4}^{(1)} \) and \( c_3^{(1)} + c_{-1}^{(0)} \) from (136) and (A11), one can derive
\[
2L = \frac{G_2}{G_3} \frac{G_k}{G_{k-2}} - \frac{1}{G_3}
\]
where
\[
L = \frac{G_3 - \frac{G_1}{G_2}G_1}{7G_3(G_3 - \frac{G_4}{G_2} + 2G_2G_4)}.
\]
(A15)

From the above two equations a relation, similar to (147) can be obtained
\[
G_k = \beta G_{k-2} = \beta^{\frac{2p-1}{2}}G_1 \text{ for } k = 2p
\]
(A16)

where
\[
\beta = \frac{G_3G_4}{G_2}(2L + \frac{1}{G_3}).
\]
(A17)

Of course, in order to have an unique determination of \( G_k \), one has to require \( \beta = \gamma \), where from (148) \( \beta = 2\frac{G_4}{G_2} - \frac{G_3}{G_1} \). This will result again in a quadratic equation for \( G_1 \).

Much more important and informative in the investigated case \( m = -1 \) turns out to be the equation for a general \( n > 3 \)
\[
-48G_5 = c_{-1-n}^{(1)}G_n + c_1^{(1)} + c_{-1}^{(0)}.
\]
(A18)

Let us remind that an expression for \( c_{-1-n}^{(1)} \) can be written from (143)
\[
c_{-n-1}^{(1)} = -\frac{16}{G_{n+1} - \frac{G_1}{G_2}G_{n-1}}.
\]
(A19)

Also, from (A11) one has an expression for \( c_1^{(1)} + c_{-1}^{(0)} \). These two expressions can be substituted into equation (A18), which acquires the form
\[
G_n(G_3 - \frac{G_4}{G_2}G_1) = G_2G_4(G_{n+1} - \frac{G_2}{G_4}G_{n-1}).
\]
(A20)

Now it is interesting to note that using the formulae
\[
G_{n+1} = G_{2p+1} = G_3(\frac{G_4}{G_2})^{p-1}; \quad G_{n-1} = G_{2p-1} = G_3(\frac{G_4}{G_2})^{p-2}
\]
(A21)

from (151), it can easily be checked that the R.H. S. of (A20) is equal to zero for the case \( n = 2p \). The other case \( n = 2p + 1 \) gives the same result. Therefore, from (A20) the following concise relation is obtained, expressing the proportionality of \( G_1 \) and \( G_2 \) with a coefficient of proportionality the ratio \( \frac{G_3}{G_4} \)
\[
G_1 = \frac{G_3}{G_4}G_2.
\]
(A22)
APPENDIX B: ADDITIONAL SYSTEM OF EQUATIONS FOR \( m=2k \)

This Appendix will preclude the proof, started in Sect. IX that all the coefficient functions in the Loran function decomposition of the equation \( [\rho(z)]^2 = 4\rho^3 - g_2(z)\rho - g_3(z) \) can be uniquely expressed, and especially those from the positive-power decomposition.

For \( m = 2k > 0 \), the corresponding equation is

\[
(k+2)[(k+1)^2(k+2)-8(2k+5)]G_{2k+6} = 2(n+1)G_n^2 + \frac{1}{2k-1 - n} (n+1)G_n^3 + 2(n+1)^2G_{2n}^3 + (n+1)^3G_{3n}^3 + c^{(1)}_{2k-n}G_n + c^{(0)}_{2k}. \tag{B1}
\]

Additionally fixing the value of \( n = k + 1 \), one can obtain from (B1)

\[
(k+2)[(k+1)^2(k+2)-8(2k+5)]G_{2k+6} = 12(k+2)^2G_{2(k+1)} + c^{(1)}_{2(k+1)} + G_{k+1}c^{(1)}_{k-1} + c^{(0)}_{2k}. \tag{B2}
\]

For \( n = 2(k+2) \) the equation is

\[
(k+2)[(k+1)^2(k+2)-8(2k+5)]G_{2k+6} = 8(2k+5)G_{2(k+2)} + c^{(1)}_{2(k+1)} + c^{(1)}_{2k} + c^{(0)}_{2k}. \tag{B3}
\]

Subtracting the two equations, one can express \( c^{(1)}_{k-1} \) as

\[
c^{(1)}_{k-1} = \frac{1}{G_{k-1}}[-12(k+2)^2G_{2(k+1)} + 8(2k+5)G_{2(k+2)} + c^{(1)}_{2(k+2)}]. \tag{B4}
\]

Since \( k \geq 1 \), from this formulae it is clear that all the Loran coefficients \( c^{(1)}_m \) in the positive-power decomposition can be expressed, including the coefficient \( c^{(1)}_0 \), through which the coefficients from the negative-power decomposition in Sect. IX were expressed. Also, from (B4) \( c^{(1)}_{2(k+1)} \) can be expressed (by performing the indice change \( k-1 \rightarrow 2(k+1) \)). If \( c^{(1)}_{2(k+1)} \) is substituted back into equation (B3), one can express also the even positive-power coefficients \( c^{(0)}_{2k} \) as

\[
c^{(0)}_{2k} = (k+2)[(k+1)^2(k+2) - 16(k+3)]G_{2k+6} - [c^{(1)}_{-4} + 8(2k+5)]G_{2(k+2)} + \frac{1}{G_{2(k+2)}}[12(2k+5)^2G_{4(k+2)} - (32k + 88 + c^{(1)}_{-4})G_{2(2k+5)}]. \tag{B5}
\]

Now let us write down equation (B1) for another possible value of \( n = 2(k+1) \)

\[
(k+2)[(k+1)^2(k+2)-8(2k+5)]G_{2k+6} = 4(2k+3)G_{2(k+1)} + c^{(1)}_{2(k+1)} + c^{(0)}_{2k} + c^{(1)}_{-2}G_{2(k+1)}. \tag{B6}
\]
Combining (B3) and (B6), $c^{(1)}_{-2}$ can be expressed as

$$c^{(1)}_{-2} = \frac{1}{G_{2(k+1)}}[-4(2k+3)G_{2(k+1)} + 8(2k+5)G_{2(k+2)} + c^{(1)}_{-4}G_{2(k+2)}]. \quad (B7)$$

The coefficient $c^{(1)}_{-4}$ can easily be calculated from (136) and (142) to be

$$c^{(1)}_{-4} = \frac{16}{G_{2(k)} - G_{4}G_{k-2}} \frac{3G_{4}(G_{2}G_{k} - G_{4}G_{k-2}) - 20G_{4}G_{k-2}}{G_{2}G_{k} - G_{4}G_{k-2}}. \quad (B8)$$

On the other hand, it is important to observe that $c^{(1)}_{-2}$ can be calculated independently also from equations (100) and (142)

$$c^{(1)}_{-2} = \frac{12(G_{2}G_{k} - G_{4}G_{k-2}) - 16G_{4}G_{k}}{G_{2}G_{k} - G_{4}G_{k-2}}. \quad (B9)$$

Note also that from relations (151) for $G_{2p}$ and $G_{2p+1}$ it follows

$$\frac{G_{k-2}}{G_{k}} = \frac{G_{2}}{G_{4}}, \quad \text{for } k = 2p \text{ and } k = 2p + 1 \quad (B10)$$

or written in another way - $G_{2}G_{k} - G_{4}G_{k-2} = 0$.

Setting up equal the two expressions (B7) and (B9) for $c^{(1)}_{-2}$, cancelling the equal denominators and subsequently taking into account (B10), one can obtain the following concise recurrent relation

$$G_{k} = \alpha(k)G_{k-2}, \quad (B11)$$

where $\alpha(k)$ denotes

$$\alpha(k) = 20\frac{G_{2(k+2)}}{G_{2(k+1)}}. \quad (B12)$$

Continuing further the recurrent relation (B11), one can derive

$$G_{k} = \alpha(k)\alpha(k-2)\ldots\alpha(k-(k-3))G_{k-(k-1)} = \ldots$$

$$= \text{coeff} \frac{G_{2(k+2)}}{G_{2(k+1)}} \frac{G_{2k}}{G_{2(k+1)}} \frac{G_{2(k-2)}}{G_{2(k+1)}} \ldots \frac{G_{2,5}}{G_{2,4}} G_{1}. \quad (B13)$$

If $k = 2p$, the numerical coefficient in (B13) will be $20^p$.

A similar relation can be obtained by fixing $n = 2(k + 1)$. Then the corresponding equation is

$$(k + 2)[(k+1)^2(k+2) - 8(2k+5)]G_{2k+6} =$$

$$= 4(2k+3)G_{2(k+1)} + c^{(1)}_{-1}G_{2(k+1)} + c^{(1)}_{2(k+1)} + c^{(0)}_{2k}. \quad (B14)$$
Substracting this equation from (B16) for \( n = 2(k+1) \) and taking into account expression (B4) for \( c_{k-1}^{(1)} \), one can obtain

\[
\frac{G_{k+1}}{G_{k-1}}[c_{-4}^{(1)} + 8(2k + 5)]G_{2(k+2)} =
\]

\[
= G_{2(k+1)}[c_{-2}^{(1)} + 4(2k + 3) + 12(k + 2)^2\frac{(G_{k+1} - G_{k-1})}{G_{k-1}}],
\]

(B15)

where \( c_{-4}^{(1)} \) is given by (B8) and \( c_{-1}^{(1)} \) by (B9). Substituting the above expressions into (B15) and again taking into account that \( G_2 G_k - G_4 G_{k-2} = 0 \), one derives the following recurrent relation

\[
G_{2(k+1)} = 20\frac{G_{k-1}}{G_{k+1}}.
\]

(B16)

If this relation is substituted into (B11), then it can be derived that

\[
G_{2}G_{k+1} = 20G_{k-1}G_{k-2}G_{k+2}.
\]

(B17)

This equality is valid for \( k \geq 3 \). For \( k = 3, 5, 7, 9 \) the above relation may be written as

\[
G_3^2 G_8 = 20G_1 G_5 \]

\[
G_5^2 G_{12} = G_3 G_7 \]

(B18)

\[
G_7^2 G_{16} = G_5 G_9 \]

\[
G_9^2 G_{20} = G_7 G_{11} \]

(B19)

Since on the L.H.S. of (B18) and (B19) \( G_8, G_{12}, G_{16} \) and \( G_20 \) are zero, the R. H. S. should also be zero. If \( G_1 \neq 0 \), \( G_3 \neq 0 \), the R. H. S. of the first pair of equations (B18) equals to zero if \( G_5 = G_7 = 0 \). But since \( G_5 \) and \( G_7 \) appear also in the R.H.S. of the second pair of equations (B19), the R.H.S. will be zero and therefore \( G_9 \) and \( G_{11} \) may be different from zero. The treatment of the subsequent equations is analogous. That is why one may conclude that a pair of even sums \( G_{2l+1}, G_{2l+3} \) (\( l \geq 2 \)) is zero, but the next pair \( G_{2l+5}, G_{2l+7} \) may be different from zero.

The last fixing of the value of \( n = \frac{2k}{3} \) for the case \( m = 2k \) gives the equation

\[
(k+2)[(k+1)^2(k+2) - 8(2k+5)]G_{2k+6} = 4(1 + \frac{2k}{3})^3G_{2k} + c_{2(k+1)}^{(1)} + c_{2k}^{(1)} G_{2k} + c_{2k}^{(0)} .
\]

(B20)

Subtracting from (B20) equation (B6) for \( n = 2(k+1) \) and setting up \( \frac{2k}{3} = p \), one can derive

\[
[12(p+1) + c_{-2}^{(1)}] G_{3p+2} - 4(1 + p)^3G_{3p} - c_{2p}^{(1)} G_{p} = 0 .
\]

(B21)

Similarly, subtracting from (B20) equation (B3) for \( n = 2(k+2) \), one obtains

\[
4(1 + p)^3G_{3p} + c_{2p}^{(1)} G_{p} - 8(3p + 5)G_{3p+4} - c_{-4}^{(1)} G_{3p+4} = 0 .
\]

(B22)
From the two equations it follows

\[ 12(p + 1) + c_{-2}^{(1)} G_{3p+2} = 8(3p + 5) + c_{-4}^{(1)} G_{3p+4} . \]  

(B23)

Again taking into account (B8) for \( c_{-2}^{(1)} \) and (B8) for \( c_{-4}^{(1)} \) for value of \( k = 3p \), one can obtain

\[ G_{3p+4} = 20 \frac{G_{3p+2}}{G_{3p-2}} G_{3p} . \]  

(B24)

However, in view of the relations (B18-19) and the consequences from them, the last relation will make sense only when each of the indices \( 3p + 4, 3p + 2, 3p - 2, 3p \) equals one of the indices \( 2l + 5, 2l + 7 \) (\( l \geq 2 \)) and then the relation (B24) will be nonzero.

**APPENDIX C: ADDITIONAL SYSTEM OF EQUATIONS FOR \( m=2k+1 \) AND \( m=-k \)**

For \( m = 2k + 1 \) the corresponding equation is

\[-8(2k + 5)(k + 3)G_{2k+7} = c_{2k+5}^{(3)} + 2(n + 1)G_n c_{2k+5-n}^{(3)} + (n + 1)^2 G_{2n} c_{2k+3-2n}^{(3)} +
\]

\[ + (n + 1) G_n c_{2k+3-n}^{(3)} + 2(n + 1)^2 G_{2n} c_{2k+3-2n}^{(3)} + (n + 1)^3 G_{3n} c_{2k+1-3n}^{(3)} +
\]

\[ + c_{2k+3}^{(1)} + c_{2k+1-n}^{(1)} G_n + c_{2k+1}^{(0)} . \]  

(C1)

The important conclusion, which can be made from this equation is the following: if \( c_{2k+3}^{(1)} \) is calculated from (B4) for value of \( k' - 1 = 2k + 3 \), then the odd number coefficients \( c_{2k+1}^{(0)} \) can also be found! Remember also that in Appendix B only the even number coefficients \( c_{2k}^{(0)} \) were found (form. B5). In order to express \( c_{2k+1}^{(0)} \), it is enough to set up \( n = 2k + 3 \) in (C1), when from all the coefficients \( c_{m}^{(3)} \) only the second term on the R.H.S. will be non-zero. Then

\[ c_{2k+1}^{(0)} = -c_{2k+3}^{(1)} - c_{-4}^{(1)} G_{2k+5} - 8(k + 3)(2k + 5)G_{2k+7} - 16(k + 3)G_{2k+5} . \]  

(C2)

For another value of \( n = \frac{2k+3}{2} \), equation (C1) acquires the following form

\[-8((k + 3)(2k + 5)G_{2k+7} = 3(2k + 5)^2 G_{2k+3} + c_{2k+3}^{(1)} + c_{2k-1}^{(1)} G_{2k+3} + c_{2k+1}^{(0)} . \]  

(C3)

But since \( c_{2k+1}^{(1)} \) and \( G_{2k+3} \) have to be integer numbers, this will be possible if for example \( 2k - 1 = 2p \). For this value of \( k \), one can express \( c_{2p+2}^{(0)} \) from (C3)

\[ c_{2p+2}^{(0)} = -c_{2p+4}^{(1)} - c_{p}^{(1)} G_{p+2} - 12(p + 3)^2 G_{2p+4} - 8(p + 3)(2p + 7)G_{2p+1} . \]  

(C4)
However, \( c_{2p+4}^{(0)} \) can be expressed also from equation (B2) for values of \( m = 2k, n = k + 1 \) and \( k = p + 2 \)

\[
c_{2p+2}^{(0)} = -c_{2p+4}^{(1)} - c_p^{(1)} G_{p+2} - 12(p + 3)^2 G_{2p+4} + (p + 3) [ (p + 3)(p + 2)^2 - 8(2p + 7)] G_{2p+8} .
\] (C5)

From the two equations (C4-C5), one easily obtains

\[
16(p + 3)(2p + 7) G_{2p+8} = 0 .
\] (C6)

Since the coefficient in front of \( G_{2p+8} \) is a positive one, (C6) will be fulfilled if

\[
G_{2p+8} = G_{2k+6} = 0 .
\] (C7)

The last means that the even-number sums \( G_6, G_8, G_{10}, \ldots \) are zero!

Again fixing the value of \( n = 2k + 3 \), one derives from (C1) the equation

\[
-8(k + 3)(2k + 5) G_{2k+7} = [8(k + 2) + c_{-2}^{(1)}] G_{2k+3} + c_{2k+3}^{(1)} + c_{2k+1}^{(0)} .
\] (C8)

Combining this equation with (C3), setting up \( 2k - 1 = 2p \), one can express \( c_p^{(1)} \)

\[
c_p^{(1)} = -\frac{(A(p) - c_{-2}^{(1)}) G_{2p+4}}{G_{p+2}} ,
\] (C9)

where

\[
A(p) = 3(2p + 1)^2 + 26(2p + 1) + 59 .
\] (C10)

Comparing this expression with formulae (B4) for \( c_p^{(1)} \) and taking into account (B8) and (B9), one derives the following relation

\[
G_{2(p+3)} = \frac{1}{20} \frac{G_{2(p+2)}}{G_{2(p-2)}} .
\] (C11)

Expressing by means of (B15) \( G_{2(p+2)} \) and \( G_{2(p-2)} \) and substituting into (C11), one can obtain the relation also in another form

\[
G_{2(p+3)} = \frac{1}{20} \frac{G_{p+2}}{G_p} \frac{G_p}{G_{p-2}} .
\] (C12)

and from the two expressions one can obtain also the ratio \( \frac{G_{2(p+2)}}{c_{2(p-2)}^{(1)}} \) without any numerical coefficients. Note also that (C11-C12) refer to non-zero even numbers of \( G_m \) since we have \( 2p = 2k - 1 \), and the relations should be written in respect to \( k \) and not \( p \).

The corresponding equation is

\[
-8(k + 3)(2k + 5) G_{2k+7} = \frac{4}{27} (2k + 4)^3 G_{2k+1} + c_{2k+3}^{(1)} +\frac{G_{2k+1}}{3} c_{2k+2}^{(1)} + c_{2k+1}^{(0)} .
\] (C13)
Setting up \(2k + 1 = 3p\) and keeping in mind that \(c_{2p}^{(1)}\) and \(c_{3p+2}^{(1)}\) can be found from (B4), one can express \(c_{3p}^{(0)}\) as

\[
\frac{4}{3}(3p + 4)(3p + 5)G_{3p+6} - 4(p + 1)^3G_{3p} - c_{3p+2}^{(1)} - Gpc_{2p}^{(1)}.
\] (C14)

The equation for the last case of \(m = -k\) \((k > 3, k \neq 6)\) is

\[
c_{4-k}^{(3)} + 2(n + 1)Gnc_{-k+4-n}^{(3)} + (n + 1)^2G_{2n}c_{-k+2-2n}^{(3)} + (n + 1)Gnc_{-k+2-n}^{(3)} +
\]

\[
+2(n + 1)^2G_{2n}c_{-k+2-2n}^{(3)} + (n + 1)^3G_{3n}c_{-k-3n}^{(3)} + c_{-k+2}^{(1)} + c_{-k-n}^{(1)}G_{n} + c_{-k}^{(0)} = 0.
\] (C15)

Since it has been shown already how all the Loran coefficient functions can be expressed and the treatment of this equation is completely analogous to the preceding ones, equation (C15) shall not be considered.

1 **APPENDIX D: COEFFICIENT FUNCTIONS**

\(N_1, N_2, N_3\) and \(N_4\) **DEPENDING ON THE "BAR" VARIABLES**

The coefficient functions \(N_1, N_2, N_3\) and \(N_4\) in the cubic algebraic equation (106) for \(T\) in Section VII are the following

\[
N_1 \equiv 2\bar{p}_1Z^2 - 2Z^4\bar{p}_1\left(\frac{O^2\bar{p}_1}{4} + \frac{O}{2}\bar{p}_2 + \bar{p}_3\right) - Z^4\bar{p}_1\left(O^2\bar{p}_1 + O\bar{p}_2 + 2\bar{p}_3\right) +
\]

\[
+Z^6\left(O^2\bar{p}_1 + O\bar{p}_2 + 2\bar{p}_3\right)\left(\frac{O^2\bar{p}_1}{4} + \frac{O\bar{p}_2}{2} + \bar{p}_3\right)
\] (D1)

\[
N_2 \equiv 8Z\bar{p}_1^2 - Z^2\bar{p}_1(\bar{p}_2 + O\bar{p}_1) + [Z^4(\bar{p}_2 + O\bar{p}_1) - 16Z^3\bar{p}_1]\left(\frac{O^2\bar{p}_1}{4} + \frac{O}{2}\bar{p}_2 + \bar{p}_3\right) +
\]

\[
+(O^2\bar{p}_1 + O\bar{p}_2 + 2\bar{p}_3)\left[-\frac{15}{2}Z^3\bar{p}_1 + \frac{23}{2}Z^5\left(\frac{O^2\bar{p}_1}{4} + \frac{O}{2}\bar{p}_2 + \bar{p}_3\right)\right]
\] (D2)

\[
N_3 \equiv \left(\frac{O^2}{4}\bar{p}_1 + \frac{O}{2}\bar{p}_2 + \bar{p}_3\right)\left[30Z^4(O^2\bar{p}_1 + O\bar{p}_2 + 2\bar{p}_3) + 8Z^3(\bar{p}_2 + O\bar{p}_1) - 24\bar{p}_1Z^2\right] +
\]

\[
+8\bar{p}_1^2 - 4Z\bar{p}_1(\bar{p}_2 + O\bar{p}_1) - 8Z^2\bar{p}_1(O^2\bar{p}_1 + O\bar{p}_2 + 2\bar{p}_3)
\] (D3)

\[
N_4 \equiv -4\bar{p}_1(\bar{p}_2 + O\bar{p}_1) + 6Z\bar{p}_1(O^2\bar{p}_1 + O\bar{p}_2 + 2\bar{p}_3) +
\]
\[ + [-18Z^3(O^2p_1 + O^2p_2 + 2p_3) + 12Z^2(p_2 + O^2p_1)] \left( \frac{O^2}{4}p_1 + \frac{O}{2}p_2 + p_3 \right) . \] (D4)

1S. Roan, Proceedings of the Third Asian Conference 2000, edited by T. Sunada, P. W.Sy, and Y. Lo (World Scientific, Singapore, 2002) [arXiv:math-ph/0011038].

2M. Kaku, Introduction to Superstrings (Springer-Verlag, Berlin-Heidelberg, 1988).

3M. Green, J. Schwartz, E. Witten, Superstring Theory, vol. 1 and 2 (Cambridge University Press, Cambridge, 1987).

4G. V. Kranriotis, S. B. Whitehouse, Class. Quant. Grav. 19, 5073 (2002) [arXiv:gr-qc/0105022].

5A. I. Markushevich, Theory of Analytical Functions (State Publ.House, Moscow, 1950).

6N. I. Ahiezer, Elements of the Elliptic Functions Theory (Nauka Publish. House, Moscow, 1979).

7I. I. Privalov, Introduction to the Theory of Complex Variable Functions (Higher School. Publish. House, Moscow, 1999).

8E. I. Whittaker, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies (Cambridge University Press, Cambridge, 1927).

9V. V. Golubev, Lectures on the Integration of the Equations of Motions of a Heavy Rigid Body Around a Fixed Point (Moscow, 1953).

10S. Lang, Elliptic Functions (Addison-Wesley Publishing Company, Inc., London-Amsterdam, 1973).

11N. Koblitz, Introduction to Elliptic Curves and Modular Forms (Springer-Verlag, New-York - Berlin, 1984).

12S. Lang, Introduction to Modular Forms (Springer-Verlag, Berlin-Heidelberg, 1976).

13V. V. Prasolov, Y. P. Soloviev, Elliptic Functions and Algebraic Equations (Factorial Publishing House, Moscow, 1997).

14S. Manoff, Part. Nucl. 30, 517 (1999) [Rus. Edit. Fiz. Elem. Chast.Atomn.Yadra. 30 (5), 1211 (1999) arXiv:gr-qc/0006024].

15C. Johnson, D-Branes Primer. Lectures, given at ICTP, TASI and SUSSTEPP arXiv:hep-th/0007170.

16P. Di Vecchia, A. Liccardo, D-Branes in String Theory I. Lectures, presented at the 1999 NATO-ASI on Quantum Geometry in Akureyri arXiv:hep-th/9912161.

17P. Di Vecchia, A. Liccardo, D-Branes in String Theories II. Lectures, presented at the YITP Workshop on Developments in Superstring and M-Theory, Kyoto, Japan, October 1999 /arXiv:hep-th/9912275.

18B. G. Dimitrov, in Perspective of Complex Analyses, Differential Geometry and Mathematical Physics. Proceedings of the 5th International Workshop on Complex Structures and Vector Fields”, edited by S. Dimiev, and K. Sekigawa (World Scientific, Singapore, 2001) [arXiv:gr-qc/0107089].
19 Y. I. Manin, *Cubic Forms: Algebra, Geometry, Arithmetic* (North Holland, Amsterdam, 1974).
20 W. Fulton, *Algebraic Curves. An Introduction to Algebraic Geometry* (W.A. Benjamin, Inc., New York, Amsterdam, 1969).
21 R. J. Walker, *Algebraic Curves* (Princeton, New Jersey, 1950).
22 A. I. Markushevich, *Introduction to the Classical Theory of Abelian Functions* (Nauka Publishing House, Moscow, 1979).
23 M. Schlichenmaier, *An Introduction to Riemann Surfaces, Algebraic Curves and Moduli Spaces*, Lecture Notes in Physics 322 (Springer-Verlag, Berlin-Heidelberg, 1989).
24 A. Hurwitz, R. Courant, *Allgemeine Funktionentheorie und Elliptische Funktionen* (Springer-Verlag, Berlin-Heidelberg, 1964).
25 S. Manoff, Class. Quant. Grav. **18**, 1111 (2001) [arXiv:gr-qc/9908061].
26 N. Efimov, E.P. Rosendorn, *Linear Algebra and Multidimensional Geometry* (Nauka Publishing House, Moscow, 1974).
27 M. Reid, *Undergraduate Algebraic Geometry*, London Math. Soc. Student Texts 12 (Cambridge University Press, Cambridge).
28 D. Mumford, *Algebraic Geometry. Complex Projective Varieties* (Springer-Verlag, New York, Berlin, Heidelberg, 1976).
29 L. Randall, R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999) [arXiv:hep-th/9905221].
30 L. Randall, R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999) [arXiv:hep-th/9906064].
31 K. Kuchar, Journ.Math.Phys. **17**, 777 (1976); **17**, 792 (1976); **17**, 801 (1976); **18** 1589 (1977).
32 E. Witten, Nucl. Phys. B **311**, 46 (1988).
33 O. Aharony, S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, Phys. Reports **323**, 183 (2000) [arXiv:hep-th/9905111].
34 J. Maldacena and H. Ooguri, J. Math. Phys. **42**, 2929 (2001) [arXiv:hep-th/0001053].
35 J. Maldacena, H. Ooguri and J. Son, Journ. Math. Phys. **42**, 2961 (2001) [arXiv:hep-th/0005183].
36 A. Giveon, D. Kutasov, N. Seiberg, Adv. Theor. Math. Phys. **2**, 733 (1998) [arXiv:hep-th/9806194].
37 M. Spradlin, A. Strominger, A. Volovich, Lectures at the LXXVI Houches School "Unity from Duality: Gravity, Gauge Theory and Strings", August 2001 [arXiv:hep-th/0110007].
38 N.V. Efimov, *A Higher Geometry* (State Publish. House, Moscow, 1961).
39 D. Cox, J. Little, D. O’Shea, *Ideals, Varieties, and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra* (Springer-Verlag, New York, 1998).
40 From *Number Theory to Physics*, edited by M. Waldschmidt, P. Moussa, J.-M. Luck, and C. Itzykson (Springer-Verlag, Berlin, Heidelberg, 1992).
41 E. C. Titchmarsh, *Theory of Functions*, Oxford, 1932.
42 D. Mumford, *Tata Lectures on Theta* (Birkhauser, Boston-Basel-Stuttgart, vol. 1, 1983; vol.2, 1984).