Continuity of the Solution to the $L_p$ Minkowski Problem in Gaussian Probability Space

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Abstract  In this paper, it is proved that the weak convergence of the $L_p$ Gaussian surface area measures implies the convergence of the corresponding convex bodies in the Hausdorff metric for $p \geq 1$. Moreover, continuity of the solution to the $L_p$ Gaussian Minkowski problem with respect to $p$ is obtained.

Keywords  $L_p$ Gaussian Minkowski problem, $L_p$ Gaussian surface area measure, continuity, convex body

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1 Introduction

A compact convex set, with non-empty interior in $n$-dimensional Euclidean space $\mathbb{R}^n$, is called a convex body. Let $\mathcal{K}^n$ denote the set of all convex bodies in $\mathbb{R}^n$, and let $\mathcal{K}_o^n$ denote the set of all convex bodies in $\mathbb{R}^n$ containing the origin in their interiors.

The Brunn–Minkowski theory, based on the addition and scalar product on the set of convex bodies in $\mathcal{K}^n$, is the very core of convex geometric analysis. Minkowski problem is one of main parts of the Brunn–Minkowski theory and characterizes a geometric measure generated by convex bodies. For smooth case, it corresponds to Monge–Ampère type equation in partial differential equations. The study of Minkowski problem promotes greatly developments of the Brunn–Minkowski theory (see [46]) and fully non-linear partial differential equations (see [49]).

In the 1990s, Lutwak [38] introduced the $L_p$ surface area measure $S_p(K, \cdot)$ of convex body $K \in \mathcal{K}_o^n$ which is a fundamental concept in convex geometric analysis defined by the variational formula of the $n$-dimensional volume (Lebesgue measure) $V_n$ as follows:

For $p \in \mathbb{R} \setminus \{0\}$,

$$
\lim_{t \to 0^+} \frac{V_n(K +_p t \cdot L) - V_n(K)}{t} = \frac{1}{p} \int_{S^{n-1}} h_L^p(u) dS_p(K, u),
$$

where $K +_p t \cdot L$ is the $L_p$ Minkowski combination of $K, L \in \mathcal{K}_o^n$ (see the details in (2.2)), and $h_L$ is the support function of $L$ on the unit sphere $S^{n-1}$ (see the details in (2.1)). Note that the case for $p = 0$ can be defined by similar way. When $p = 1$, the $L_1$ surface area measure $S_1(K, \cdot)$ is the well-known classical surface area measure $S_K$, that is, $S_1(K, \cdot) = S_K$. 

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The following Minkowski problem for the $L_p$ surface area measure is called $L_p$ Minkowski problem:

**$L_p$ Minkowski Problem:** For a fixed $p$ and a given non-zero finite Borel measure $\mu$ on $S^{n-1}$, what are the necessary and sufficient conditions on $\mu$ such that there exists a convex body $K$ in $\mathbb{R}^n$ such that its $L_p$ surface area measure $S_p(K, \cdot)$ is equal to $\mu$, that is,

$$\mu = S_p(K, \cdot)?$$

When $p = 1$, the $L_p$ Minkowski problem is the classical Minkowski problem studied by Minkowski [43, 44], Alexandrov [1, 2], Fenchel–Jensen [22] and others. Besides, the centro-affine Minkowski problem ($p = -n$) and the logarithmic Minkowski problem ($p = 0$) are other two special cases of the $L_p$ Minkowski problem, see [9, 12, 15, 34, 47, 48, 59, 61]. For the existence, uniqueness and regularity of the (normalized) $L_p$ Minkowski problem, one can see [14, 30, 33, 36, 38, 39, 41, 60, 63]. As an important application, the solutions to the $L_p$ Minkowski problem play a vital role in discovering some new (sharp) affine curvature flows.

The $L_p$ surface area measure has an important property: If the sequence $\{K_i\} \subseteq \mathcal{K}_o^n$ converges to $K_0 \in \mathcal{K}_o^n$ in the Hausdorff metric, then $\{S_p(K_i, \cdot)\}$ converges to $S_p(K_0, \cdot)$ weakly. This is the continuity of the $L_p$ surface area measure with respect to the Hausdorff metric. The reverse question of this result is interesting:

**Problem A** Does the sequence $\{K_i\} \subseteq \mathcal{K}_o^n$ converge to $K_0 \in \mathcal{K}_o^n$ in the Hausdorff metric as $\{S_p(K_i, \cdot)\}$ converges to $S_p(K_0, \cdot)$ weakly?

This problem is related closely to the $L_p$ Minkowski problem and can be restated as follows:

**Problem B** Suppose that $p \in \mathbb{R}$, $K_i \in \mathcal{K}_o^n$ is the solution to the $L_p$ Minkowski problem associated with Borel measure $\mu_i$ on $S^{n-1}$ and $K_0 \in \mathcal{K}_o^n$ is the solution to the $L_p$ Minkowski problem associated with Borel measure $\mu_0$ on $S^{n-1}$. Does the sequence $\{K_i\}$ converge to $K_0$ in the Hausdorff metric as $\{\mu_i\}$ converges to $\mu_0$ weakly?

Problem A (or Problem B) is called continuity of the solution to the $L_p$ Minkowski problem. Since the $L_p$ surface area measure is positively homogeneous of degree $n - p$, then Zhu [62] showed that this question is not positive for $p = n$ by the following counterexample: Let $p = n$, $K_1$ be an origin-symmetric convex body and $K_i = \frac{1}{i}K_1$. Then $S_n(K_i, \cdot) = S_n(K_1, \cdot)$ for all $i$ but $\{K_i\}$ converges to the origin as $i \to +\infty$. Moreover, Zhu [62] gave an affirmative answer to Problem A for $p > 1$ with $p \neq n$. The part results of Problem A for $p = 0$ and $0 < p < 1$ were obtained in [51, 52]

In 2016, Huang et al. [31] defined the dual curvature measure by the variational formula of the dual volume (see [37]) for $L_1$ Minkowski combination and studied the corresponding Minkowski problem called dual Minkowski problem. The existence, uniqueness and regularity of this problem and its generalization were studied in [8, 10, 13, 42, 53, 56–58]. The continuity of the solution to this problem for $q < 0$ was studied in [50].

Recently, the Brunn–Minkowski theory for the Gaussian probability measure $\gamma_n$ has hot
attention defined by
\[ \gamma_n(E) = \frac{1}{(\sqrt{2\pi})^n} \int_E e^{-\frac{|x|^2}{2}} dx, \]
where \( E \) is a subset of \( \mathbb{R}^n \) and \(|x|\) is the absolute value of \( x \in E \). \( \gamma_n(E) \) is called the Gaussian volume of \( E \). Since the Gaussian volume \( \gamma_n \) does not have translation invariance and homogeneity, then there are more difficulties to study the corresponding Brunn–Minkowski theory. This makes Brunn–Minkowski theory for \( \gamma_n \) quite mysterious and further stimulates people’s interest. The Brunn–Minkowski inequality and the Minkowski inequality for the Gaussian volume \( \gamma_n \) were studied in [4–7, 11, 16, 18, 25, 45].

By the variational formula of the Gaussian volume \( \gamma_n \) for \( L_p \) Minkowski combination, the \( L_p \) Gaussian surface area measure \( S_{p,\gamma_n}(K,\cdot) \) of convex body \( K \in \mathcal{K}_0^n \) was defined in [32, 35] as follows:

For \( K, L \in \mathcal{K}_0^n \) and \( p \neq 0 \),
\[ \lim_{t \to 0} \frac{\gamma_n(K + p t \cdot L) - \gamma_n(K)}{t} = \frac{1}{p} \int_{S^{n-1}} h_L^p(u) dS_{p,\gamma_n}(K, u). \quad (1.2) \]
When \( p = 1 \), it is the Gaussian surface area measure defined in [32], that is, \( S_{1,\gamma_n}(K,\cdot) = S_{\gamma_n,K} \).

Note that the \( L_p \) Gaussian surface area measure \( S_{p,\gamma_n}(K,\cdot) \) is not positively homogeneous.

The corresponding Minkowski problem in Gaussian probability space is called the \( L_p \) Gaussian Minkowski problem (see [23, 24, 32, 35]) as follows:

**\( L_p \) Gaussian Minkowski Problem:** Suppose \( p \in \mathbb{R} \) is fixed. For a given non-zero finite Borel measure \( \mu \) on \( S^{n-1} \), what are the necessary and sufficient conditions on \( \mu \) such that there exists a convex body \( K \in \mathcal{K}_0^n \) such that
\[ \mu = S_{p,\gamma_n}(K,\cdot)? \]

If \( f \) is the density of the given measure \( \mu \), then the corresponding Monge–Ampère type equation on \( S^{n-1} \) is as follows:

For \( u \in S^{n-1} \),
\[ \frac{1}{(\sqrt{2\pi})^n} e^{-|\nabla h(u)|^2/2} h^{1-p}(u) \det(\nabla^2 h(u) + h(u) I) = f(u), \]
where \( h : S^{n-1} \to (0, +\infty) \) is the function to be found, \( \nabla h, \nabla^2 h \) are the gradient vector and the Hessian matrix of \( h \) with respect to an orthonormal frame on \( S^{n-1} \), and \( I \) is the identity matrix.

In this paper, we mainly consider the continuity of the solution to the \( L_p \) Gaussian Minkowski problem and obtain the following result:

**Theorem 1.1** Suppose \( p \geq 1 \) and \( K_i \in \mathcal{K}_0^n \) with \( \gamma_n(K_i) \geq 1/2 \) for \( i = 0, 1, 2, \ldots \). If the sequence \( \{S_{p,\gamma_n}(K_i,\cdot)\} \) converges to \( S_{p,\gamma_n}(K_0,\cdot) \) weakly, then the sequence \( \{K_i\} \) converges to \( K_0 \) in the Hausdorff metric.

Besides, we obtain that the solution to the \( L_p \) Gaussian Minkowski problem is continuous with respect to \( p \).

**Theorem 1.2** Suppose \( p_i \geq 1 \) and \( K_i \in \mathcal{K}_0^n \) with \( \gamma_n(K_i) \geq 1/2 \) for \( i = 0, 1, 2, \ldots \). If \( S_{p_i,\gamma_n}(K_i,\cdot) = S_{p_0,\gamma_n}(K_0,\cdot) \), then the sequence \( \{K_i\} \) converges to \( K_0 \) in the Hausdorff metric as \( \{p_i\} \) converges to \( p_0 \).
2 Preliminaries

In this section, we list some notations and recall some basic facts about convex bodies. According to the context of this paper, $| \cdot |$ may denote different meanings: the absolute value and the total mass of a finite measure. For vectors $x, y \in \mathbb{R}^n$, $x \cdot y$ denotes the standard inner product in $\mathbb{R}^n$. $S^{n-1}$ denotes the boundary of the Euclidean unit ball $B_n = \{ x \in \mathbb{R}^n : \sqrt{x \cdot x} \leq 1 \}$ and is called unit sphere. Let $\omega_n$ denote the $n$-dimensional volume (Lebesgue measure) of $B_n$. Let $\partial K$ and int $K$ denote the boundary and the set of all interiors of convex body $K$ in $\mathbb{R}^n$, respectively. $\partial' K$ is the subset of $\partial K$ with unique outer unit normal.

The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of $K \in \mathcal{K}_o^n$ is defined by

$$h_K(x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n.$$  \hspace{1cm} (2.1)

A convex body is uniquely determined by its support function. Support functions are positively homogeneous of degree one and subadditive. For $K \in \mathcal{K}_o^n$, its support function $h_K$ is continuous and strictly positive on the unit sphere $S^{n-1}$.

The radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ of convex body $K \in \mathcal{K}_o^n$ is another important function for $K \in \mathcal{K}_o^n$, and it is given by

$$\rho_K(x) = \max\{\lambda > 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$  

Note that the radial function $\rho_K$ of $K \in \mathcal{K}_o^n$ is positively homogeneous of degree $-1$, and it is continuous and strictly positive on the unit sphere $S^{n-1}$. For each $u \in S^{n-1}$, $\rho_K(u)u \in \partial K$.

The set $\mathcal{K}_o^n$ can be endowed with Hausdorff metric and radial metric which mean the distance between two convex bodies. The Hausdorff metric of $K, L \in \mathcal{K}_o^n$ is defined by

$$\|h_K - h_L\| = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$  

The radial metric of $K, L \in \mathcal{K}_o^n$ is defined by

$$\|\rho_K - \rho_L\| = \max_{u \in S^{n-1}} |\rho_K(u) - \rho_L(u)|.$$  

The two metrics are mutually equivalent, that is, for $K, K_i \in \mathcal{K}_o^n$,

$$h_{K_i} \to h_K \text{ uniformly } \quad \text{if and only if} \quad \rho_{K_i} \to \rho_K \text{ uniformly}.$$  

If $\|h_{K_i} - h_K\| \to 0$ or $\|\rho_{K_i} - \rho_K\| \to 0$ as $i \to +\infty$, we call the sequence $\{K_i\}$ converges to $K$, that is, $\lim_{i \to +\infty} K_i = K$.

The polar body $K^*$ of $K \in \mathcal{K}_o^n$ is given by

$$K^* = \{ x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K \}.$$  

It is clear that $K^* \in \mathcal{K}_o^n$ and $K = (K^*)^*$. There exists an important fact on $\mathbb{R}^n \setminus \{0\}$ between $K$ and its polar body $K^*$:

$$h_K = 1/\rho_{K^*} \quad \text{and} \quad \rho_K = 1/h_{K^*}.$$  

Then, for $K, K_i \in \mathcal{K}_o^n$, we can obtain the following result:

$$K_i \to K \quad \text{if and only if} \quad K_i^* \to K^*.$$  

Let $f : S^{n-1} \to (0, +\infty)$ be continuous. The Wulff shape $[f]$ of $f$ is defined by

$$[f] = \{ x \in \mathbb{R}^n : x \cdot u \leq f(u) \text{ for all } u \in S^{n-1} \}.$$
It is not hard to see that $|f|$ is a convex body in $\mathbb{R}^n$ and $h_{[f]} \leq f$. In addition, $h_K = K$ for all $K \in K_0^n$.

By the concept of Wulff shape, the $L_p$ Minkowski combination can be defined for all $p \in \mathbb{R}$. When $p \neq 0$, for $K, L \in K_0^n$ and $s, t \in \mathbb{R}$ satisfying that $sh_K^p + th_L^p$ is strictly positive on $S^{n-1}$, the $L_p$ Minkowski combination $s \cdot K +_p t \cdot L$ is defined by

$$s \cdot K +_p t \cdot L = [(sh_K^p + th_L^p)^{1/p}].$$

When $p = 0$, the $L_p$ Minkowski combination is defined by $s \cdot K +_p t \cdot L = [h_K^{1/p}h_L^{1/p}]$.

### 3 Main Results

In this section, we consider the continuity of the solution to the $L_p$ Gaussian Minkowski problem for $p \geq 1$. By the variational formula (1.2) and Ehrhard inequality, the following Minkowski-type inequality was obtained in [32, 35].

**Lemma 3.1** ([32, 35]) Suppose $K, L \in K_0^n$, then, for $p \geq 1$, $\frac{1}{p} \int_{S^{n-1}} h_L^p(u) - h_K^p(u) dS_p,\gamma_n(K, u) \geq \gamma_n(K) \log \frac{\gamma_n(L)}{\gamma_n(K)}$, with equality if and only if $K = L$.

The following lemmas will be needed.

**Lemma 3.2** Suppose $p \geq 1$ and $K_i \in K_0^n$ for $i = 0, 1, 2, \ldots$. If the sequence $\{S_{p,\gamma_n}(K_i, \cdot)\}$ converges to $S_{p,\gamma_n}(K_0, \cdot)$ weakly, then there exists a constant $c_1 > 0$ such that

$$\int_{S^{n-1}} (u \cdot v)^p dS_{p,\gamma_n}(K_i, v) \geq c_1,$$

for all $u \in S^{n-1}$ and $i \in \{0, 1, 2, \ldots\}$, where, $(u \cdot v)_+ = \max\{0, u \cdot v\}$.

**Proof** For $x \in \mathbb{R}^n$, let $g_i(x) = \int_{S^{n-1}} (x \cdot v)^p dS_{p,\gamma_n}(K_i, v)$.

It is not hard to see that $g_i$ is a sublinear function on $\mathbb{R}^n$. Then, there exists a convex body in $\mathbb{R}^n$ such that its support function is equal to $g_i$. By the fact that $\{S_{\gamma_n,K_i}\}$ converges to $S_{\gamma_n,K_0}$ weakly, we infer that the sequence $\{g_i\}$ converges to $g_0$ pointwise. Since pointwise and uniform convergence of support functions are equivalent on $S^{n-1}$, then $\{g_i\}$ converges to $g_0$ on $S^{n-1}$ uniformly.

Since $S_{p,\gamma_n}(K_0, \cdot)$ is not concentrated in any closed hemisphere of $S^{n-1}$, then $g_0 > 0$ on $S^{n-1}$. Together with the compactness of $S^{n-1}$ and the continuity of $g_0$ on $S^{n-1}$, we obtain there exists a constant $c_2 > 0$ such that

$$g_0(u) \geq c_2,$$

for all $u \in S^{n-1}$. Since $\{g_i\}$ converges to $g_0$ uniformly on $S^{n-1}$, then there exists a constant $c_1 > 0$ such that

$$g_i(u) \geq c_1,$$

for all $u \in S^{n-1}$ and $i = 0, 1, 2, \ldots$. 

\[\square\]
The weak convergence of the $L_p$ Guassian surface area measures implies that the sequence of the corresponding convex bodies is bounded.

**Lemma 3.3** Suppose $p \geq 1$ and $K_i \in \mathcal{K}_o^n$ with $\gamma_n(K_i) \geq 1/2$ for $i = 0, 1, 2, \ldots$. If the sequence \{\$Q_p,\gamma_n(K_i, \cdot)$\} converges to $\$Q_p,\gamma_n(K_0, \cdot)$ weakly, then the sequence \{$K_i$\} is bounded.

**Proof** For $K \in \mathcal{K}_o^n$ and $p \geq 1$, the functional $\Phi_p$ is defined by

$$
\Phi_p(K) = -\frac{1}{p\gamma_n(K)} \int_{S^{n-1}} h_{K_i}^p(u) dS_{p,\gamma_n}(K, u) + \log \gamma_n(K). \tag{3.2}
$$

From Lemma 3.1 and $\gamma_n(K_i) \geq 1/2$, we have

$$
\Phi_p(K_i) = -\frac{1}{p\gamma_n(K_i)} \int_{S^{n-1}} h_{K_i}^p(u) dS_{p,\gamma_n}(K_i, u) + \log \gamma_n(K_i)
\geq -\frac{1}{p\gamma_n(K_i)} \int_{S^{n-1}} h_{B_n}^p(u) dS_{p,\gamma_n}(K_i, u) + \log \gamma_n(B_n)
= -\frac{|S_{p,\gamma_n}(K_i, \cdot)|}{p\gamma_n(K_i)} + \log \gamma_n(B_n)
\geq -\frac{2}{p} |S_{p,\gamma_n}(K_i, \cdot)| + \log \gamma_n(B_n).
$$

Since the sequence \{\$Q_{p,\gamma_n}(K_i, \cdot)$\} converges to $\$Q_{p,\gamma_n}(K_0, \cdot)$ weakly, then, \{|$S_{p,\gamma_n}(K_i, \cdot)$|\} converges to $|S_{p,\gamma_n}(K_0, \cdot)|$. Together with the fact that $|S_{p,\gamma_n}(K_0, \cdot)|$ is finite, there exists a big enough constant $M_1 > 0$ satisfying

$$
|S_{p,\gamma_n}(K_i, \cdot)| \leq \frac{p}{2} M_1,
$$

for all $i$. Hence, for $i = 1, 2, \ldots$,

$$
\Phi_p(K_i) \geq -M_1 + \log \gamma_n(B_n). \tag{3.3}
$$

Since $K_i \in \mathcal{K}_o^n$, then $\rho_{K_i}$ is continuous on $S^{n-1}$. By the fact that $S^{n-1}$ is compact, we obtain there exists $u_i \in S^{n-1}$ such that

$$
\rho_{K_i}(u_i) = \max\{\rho_{K_i}(u) : u \in S^{n-1}\}.
$$

Let $R_i = \rho_{K_i}(u_i)$. Then, $R_i u_i \in K_i$ and $K_i \subseteq R_i B_n$. By the definition of support function,

$$
h_{K_i}(v) \geq R_i(u_i \cdot v)_+,
$$

for all $v \in S^{n-1}$. Combining $p \geq 1$, Lemma 3.2 with the fact that $\gamma_n(K) \leq 1$ for any subset $K$ of $\mathbb{R}^n$, we have

$$
\Phi_p(K_i) = -\frac{1}{p\gamma_n(K_i)} \int_{S^{n-1}} h_{K_i}^p(u) dS_{p,\gamma_n}(K_i, v) + \log \gamma_n(K_i)
\leq -\frac{R_i^p}{p\gamma_n(K_i)} \int_{S^{n-1}} (u_i \cdot v)^p dS_{p,\gamma_n}(K_i, v) + \log \gamma_n(K_i)
\leq -\frac{c_1 R_i^p}{p\gamma_n(K_i)} + \log \gamma_n(K_i)
\leq -\frac{c_1 R_i^p}{p}.
$$

Then, by (3.3),

$$
-\frac{c_1 R_i^p}{p} \geq \Phi_p(K_i) \geq -M_1 + \log \gamma_n(B_n),
$$
that is,
\[ R_i \leq \left( \frac{p}{c_1} (M_1 - \log \gamma_n(B_n)) \right)^{1/p}, \]
for all \( i = 1, 2, \ldots \).

Therefore, \( \{R_i\} \) is bounded, that is, the sequence \( \{K_i\} \) is bounded. \qed

Lemma 3.4 Suppose \( K \) is a compact convex set in \( \mathbb{R}^n \). If \( \gamma_n(K) > 0 \), then \( K \) is a convex body in \( \mathbb{R}^n \), that is, \( K \in \mathcal{K}_n \).

Proof By the definition of the Gaussian volume \( \gamma_n \), we have
\[ \gamma_n(K) = \frac{1}{(\sqrt{2\pi})^n} \int_K e^{-\frac{|x|^2}{2}} \, dx \leq \frac{1}{(\sqrt{2\pi})^n} \int_K \, dx = \frac{1}{(\sqrt{2\pi})^n} V_n(K). \]
Together with \( \gamma_n(K) > 0 \),
\[ V_n(K) \geq (\sqrt{2\pi})^n \gamma_n(K) > 0. \]
Therefore, compact convex set \( K \) has nonempty interior in \( \mathbb{R}^n \), that is, \( K \) is a convex body in \( \mathbb{R}^n \). \qed

Lemma 3.5 Suppose \( K_i \in \mathcal{K}_n \) with \( \gamma_n(K_i) \geq 1/2 \) for \( i = 1, 2, \ldots \). If the sequence \( \{K_i\} \) converges to compact convex set \( L \) in the Hausdorff metric, then \( L \in \mathcal{K}_n \).

Proof By the continuity of Gaussian volume,
\[ \gamma_n(L) = \lim_{i \to +\infty} \gamma_n(K_i) \geq 1/2. \]
Together with Lemma 3.4, we have \( L \) is a convex body in \( \mathbb{R}^n \).

Assume that \( o \in \partial L \). Then, there exists a \( u_0 \in S^{n-1} \) such that \( h_L(u_0) = 0 \). Since \( \{K_i\} \) converges to \( L \), we have
\[ \lim_{i \to +\infty} h_{K_i}(u_0) = h_L(u_0) = 0. \]
For arbitrary \( \varepsilon > 0 \), there exists a big enough integer \( N_\varepsilon > 1 \) so that \( h_{K_i}(u_0) < \varepsilon \) for all \( i > N_\varepsilon \). Thus,
\[ K_i \subseteq \{ x \in \mathbb{R}^n : x \cdot u_0 \leq \varepsilon \} \]
for all \( i > N_\varepsilon \). By the fact that \( \{K_i\} \) converges to \( L \) again, there exists a constant \( R > 0 \) such that \( K_i \subseteq B_n(R) \) where \( B_n(R) \) is a ball with radius \( R \) in \( \mathbb{R}^n \). Hence, for all \( i > N_\varepsilon \),
\[ K_i \subseteq B_n(R) \cap \{ x \in \mathbb{R}^n : x \cdot u_0 \leq \varepsilon \}. \]

By the following result:
\[ \gamma_n(\mathbb{R}^n) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} \, dx = 1, \]
we obtain, for halfspace \( H^-(u_0) = \{ x \in \mathbb{R}^n : x \cdot u_0 \leq 0 \} \),
\[ \gamma_n(H^-(u_0)) = \frac{1}{(\sqrt{2\pi})^n} \int_{H^-(u_0)} e^{-\frac{|x|^2}{2}} \, dx = \frac{1}{2} \gamma_n(\mathbb{R}^n) = \frac{1}{2}. \]
Together with \( \gamma_n(H^-(u_0)) = \gamma_n(H^-(u_0) \cap B_n(R)) + \gamma_n(H^-(u_0) \backslash B_n(R)) \), we have
\[ \gamma_n(H^-(u_0) \cap B_n(R)) < \frac{1}{2}. \]
Hence, for $\varepsilon$ small enough,

$$\gamma_n(K_i) \leq \gamma_n(B_n(R) \cap \{ x \in \mathbb{R}^n : x \cdot u_0 \leq \varepsilon \}) < \frac{1}{2},$$

for all $i > N_\varepsilon$. This is a contradiction to the condition $\gamma_n(K_i) \geq 1/2$ for $i = 1, 2, \ldots$. Therefore, $o$ is an interior point of $L$, that is, $L \in \mathcal{K}_o^n$.

\[\square\]

**Remark 3.6** From the proof of Lemma 3.5, we obtain that $\gamma_n(K) \geq \frac{1}{2}$ for $K \in \mathcal{K}_o^n$ means that the origin $o$ can not be close sufficiently to the boundary $\partial K$.

The weak convergence of Gaussian surface area measure was obtained in [32]:

**Lemma 3.7** ([32]) Let $K_i \in \mathcal{K}_o^n$ for $i = 1, 2, \ldots$ such that $\{K_i\}$ converges to $K_0 \in \mathcal{K}_o^n$ in the Hausdorff metric, then $\{S_{\gamma_n,K_i}\}$ converges to $S_{\gamma_n,K_0}$ weakly.

By the variational formula (1.2) of the Gaussian volume $\gamma_n$ for $L_p$ Minkowski combination, the integral expression of $L_p$ Gaussian surface area measure was obtained in [32, 35]:

**Lemma 3.8** ([32, 35]) Suppose $p \in \mathbb{R}$ and $K \in \mathcal{K}_o^n$. For each Borel set $\eta \subseteq S^{n-1}$, $L_p$ Gaussian surface area $S_{\gamma_n}(K, \cdot)$ of $K$ is defined by

$$S_{\gamma_n}(K, \eta) = \frac{1}{(\sqrt{2\pi})^n} \int_{\nu_K^{-1}(\eta)} (x \cdot \nu_K(x))^{1-p} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x)$$

$$= \int_{\eta} h_{K}^{1-p}(u)dS_{\gamma_n,K}(u).$$

Here, $\nu_K : \partial K \to S^{n-1}$ is the Gauss map of $K$ and $\mathcal{H}^{n-1}$ is an $(n - 1)$-dimensional Hausdorff measure.

Since $\{K_i\}$ converges to $K_0$ in the Hausdorff metric for $K_i \in \mathcal{K}_o^n (i = 0, 1, \ldots)$, then $\{h_{K_i}\}$ converges to $h_{K_0}$ uniformly on $S^{n-1}$. Together with the Lemmas 3.7 and 3.8, we obtain the weak convergence of $L_p$ Gaussian surface area measures as follows:

**Proposition 3.9** Suppose $p \in \mathbb{R}$ and $K_i \in \mathcal{K}_o^n$ for $i = 0, 1, \ldots$. If $\{K_i\}$ converges to $K_0$ in the Hausdorff metric, then $\{S_{\gamma_n}(K_i, \cdot)\}$ converges to $S_{\gamma_n}(K_0, \cdot)$ weakly.

The uniqueness of the solution to the $L_p$ Gaussian Minkowski problem was obtained in [32, 35]:

**Lemma 3.10** ([32, 35]) Let $p \geq 1$ and $K, L \in \mathcal{K}_o^n$ with $\gamma_n(K), \gamma_n(L) \geq 1/2$. If

$$S_{\gamma_n}(K, \cdot) = S_{\gamma_n}(L, \cdot),$$

then, $K = L$.

The continuity of the solution to the $L_p$ Gaussian Minkowski problem is obtained for $\gamma_n(\cdot) \geq 1/2$. The proof of Theorem 1.1 is given as follows:

**Proof of Theorem 1.1** Assume that the sequence $\{K_i\}$ does not converge to $K_0$. Then, without loss of generality, we may assume that there exists a constant $\varepsilon_0 > 0$ such that

$$\|h_{K_i} - h_{K_0}\| \geq \varepsilon_0,$$

for all $i = 1, 2, \ldots$.

By Lemma 3.3, $\{K_i\}$ is bounded. Thus, from Blaschke selection theorem, the sequence $\{K_i\}$ has a convergent subsequence $\{K_{i_j}\}$ which converges to a compact convex set $L_0$. Clearly,
$L_0 \neq K_0$. Together with the continuity of $\gamma_n$ and $\gamma_n(K_i) \geq 1/2$ for $i = 1, 2, \ldots$, we have $\gamma_n(L_0) = \lim_{j \to +\infty} \gamma_n(K_{i_j}) \geq 1/2$.

By $\lim_{j \to +\infty} K_{i_j} = L_0$ and $\gamma_n(K_{i_j}) \geq 1/2$ for $i = 1, 2, \ldots$ again, $L_0 \in \mathcal{K}_0^n$ with $L_0 \neq K_0$ from Lemma 3.5.

Since $\{K_{i_j}\}$ converges to $L_0$ in Hausdorff metric, then $\{S_{p,\gamma_n}(K_{i_j},\cdot)\}$ converges to $S_{p,\gamma_n}(L_0,\cdot)$ weakly by Lemma 3.9. Together with $\{S_{p,\gamma_n}(K_{i_j},\cdot)\}$ converges to $S_{p,\gamma_n}(K_0,\cdot)$ weakly, we have $S_{p,\gamma_n}(L_0,\cdot) = S_{p,\gamma_n}(K_0,\cdot)$.

By $\gamma_n(K_0), \gamma_n(L_0) \geq 1/2$ and Lemma 3.10, we obtain $K_0 = L_0$. This is a contradiction to $K_0 \neq L_0$. Therefore, the sequence $\{K_i\}$ converges to $K_0$ in the Hausdorff metric. \hfill $\square$

Next, we mainly prove Theorem 1.2. The following lemma will be needed.

**Lemma 3.11** Suppose $p_i \geq 1$ and $K_i \in \mathcal{K}_0^n$ with $\gamma_n(K_i) \geq 1/2$ for $i = 0, 1, 2, \ldots$. If $S_{p_i,\gamma_n}(K_i,\cdot) = S_{p_0,\gamma_n}(K_0,\cdot)$ with $\lim_{i \to +\infty} p_i = p_0$, then the sequence $\{K_i\}$ is bounded.

**Proof** By $p_i \geq 1$ and $\lim_{i \to +\infty} p_i = p_0$, without loss of generality, we may assume that

$$1 \leq p_i < 2p_0, \quad (3.4)$$

for all $i$. From (3.2), $S_{p_i,\gamma_n}(K_i,\cdot) = S_{p_0,\gamma_n}(K_0,\cdot)$, Lemma 3.1 and $\gamma_n(K_i) \geq 1/2$, we have

$$\Phi_{p_i}(K_i) = -\frac{1}{p_i \gamma_n(K_i)} \int_{S^{n-1}} h_{K_i}^{p_i}(u) dS_{p_i,\gamma_n}(K_i, u) + \log \gamma_n(K_i)$$

$$\geq -\frac{1}{p_i \gamma_n(K_i)} \int_{S^{n-1}} h_{B_i}^{p_i}(u) dS_{p_i,\gamma_n}(K_i, u) + \log \gamma_n(B_i)$$

$$= -\frac{|S_{p_i,\gamma_n}(K_i,\cdot)|}{p_i \gamma_n(K_i)} + \log \gamma_n(B_i)$$

$$\geq -\frac{2}{p_i} |S_{p_0,\gamma_n}(K_0,\cdot)| + \log \gamma_n(B_i)$$

$$\geq -2 |S_{p_0,\gamma_n}(K_0,\cdot)| + \log \gamma_n(B_i) \quad (3.5)$$

Let

$$R_i = \rho_{K_i}(u_i) = \max\{\rho_{K_i}(u) : u \in S^{n-1}\},$$

where $u_i \in S^{n-1}$. Then, $R_i u_i \in K_i$ and $K_i \subseteq R_i B_n$. Thus,

$$h_{K_i}(v) \geq R_i (u_i \cdot v),$$

for all $v \in S^{n-1}$ and $\gamma_n(K_i) \leq \gamma_n(R_i B_n)$.

Since $S_{p_0,\gamma_n}(K_0,\cdot)$ is not concentrated in any closed hemisphere of $S^{n-1}$ and $S^{n-1}$ is a compact set, then there exists a constant $c_3 > 0$ such that

$$\int_{S^{n-1}} (u \cdot v)^{2p_0} dS_{p_0,\gamma_n}(K_0, v) \geq c_3,$$

for $u \in S^{n-1}$. Thus, together with the fact that $\gamma_n(K) \leq 1$ for any subset $K$ of $\mathbb{R}^n$, we have

$$\Phi_{p_i}(K_i) = -\frac{1}{p_i \gamma_n(K_i)} \int_{S^{n-1}} h_{K_i}^{p_i}(v) dS_{p_i,\gamma_n}(K_i, v) + \log \gamma_n(K_i)$$

$$= -\frac{1}{p_i \gamma_n(K_i)} \int_{S^{n-1}} h_{K_i}^{p_i}(v) dS_{p_0,\gamma_n}(K_0, v) + \log \gamma_n(K_i)$$

$$\geq -2 |S_{p_0,\gamma_n}(K_0,\cdot)| + \log \gamma_n(K_i) = -2 |S_{p_0,\gamma_n}(K_0,\cdot)| + \log \gamma_n(K_0).$$
By (3.5),
\[ -\frac{c_3}{2p_0} R_{i}^p \geq \Phi_{P_i}(K_i) \geq -2|S_{p_0,\gamma_n}(K_0,\cdot)| + \log \gamma_n(B_n), \]
that is,
\[ R_i \leq \left( \frac{2p_0}{c_3} (2|S_{p_0,\gamma_n}(K_0,\cdot)| - \log \gamma_n(B_n)) \right)^{\frac{1}{p}} \leq \max \left\{ 1, \frac{2p_0}{c_3} (2|S_{p_0,\gamma_n}(K_0,\cdot)| - \log \gamma_n(B_n)) \right\}, \]
for all \( i = 1, 2, \ldots \).

Hence, \( \{R_i\} \) is bounded, that is, the sequence \( \{K_i\} \) is bounded. \( \square \)

Now, the proof of Theorem 1.2 is given as follows:

**Proof of Theorem 1.2** Suppose that \( \{K_i\} \) does not converge to \( K_0 \) in the Hausdorff metric. Then, there exist a constant \( \varepsilon_1 > 0 \) and a subsequence of \( \{K_i\} \), denoted by \( \{K_i\} \) again, such that \( \|h_{K_i} - h_{K_0}\| \geq \varepsilon_1 \), for all \( i = 1, 2, \ldots \).

By Lemma 3.11, \( \{K_i\} \) is bounded. By the Blaschke selection theorem, we have \( \{K_i\} \) has a convergent subsequence \( \{K_{i_j}\} \) which converges to a compact convex set \( L_0 \). Clearly, \( L_0 \neq K_0 \).

Together with the continuity of \( \gamma_n \) and \( \gamma_n(K_{i_j}) \geq 1/2 \) for \( i = 1, 2, \ldots \), we have
\[ \gamma_n(L_0) = \lim_{j \to +\infty} \gamma_n(K_{i_j}) \geq 1/2. \]

By \( \lim_{j \to +\infty} K_{i_j} = L_0 \) and \( \gamma_n(K_{i_j}) \geq 1/2 \) for \( i = 1, 2, \ldots \) again, \( L_0 \in K_0^n \) with \( L_0 \neq K_0 \) from Lemma 3.5.

Since \( \{K_{i_j}\} \) converges to \( L_0 \) in Hausdorff metric, then \( h_{K_{i_j}} \to h_{L_0} \) uniformly and \( S_{\gamma_n,K_{i_j}} \to S_{\gamma_n,K_0} \) weakly as \( j \to +\infty \). Together with \( \lim_{i \to +\infty} p_i = p_0 \), we have \( \{S_{p_{i_j},\gamma_n}(K_{i_j},\cdot)\} \) converges to \( S_{p_0,\gamma_n}(L_0,\cdot) \) weakly. By \( S_{p_i,\gamma_n}(K_0,\cdot) = S_{p_0,\gamma_n}(K_0,\cdot) \),
\[ S_{p_i,\gamma_n}(K_0,\cdot) = S_{p_0,\gamma_n}(L_0,\cdot). \]

By \( \gamma_n(K_0), \gamma_n(L_0) \geq 1/2 \) and Lemma 3.10, we obtain \( K_0 = L_0 \). This is a contradiction to \( K_0 \neq L_0 \). Therefore, the sequence \( \{K_i\} \) converges to \( K_0 \) in the Hausdorff metric. \( \square \)

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