Abstract

The increasing number of protein-based metamaterials demands reliable and efficient methods to study the physicochemical properties they may display. In this regard, we develop a simulation strategy based on Molecular Dynamics (MD) that addresses the geometric degrees of freedom of an auxetic two-dimensional protein crystal. This model consists of a network of impenetrable rigid squares linked through massless rigid rods, thus featuring a large number of both holonomic and nonholonomic constraints. Our MD methodology is optimized to study highly constrained systems and allows for the simulation of long time dynamics with reasonably large timesteps. The data extracted from the simulations shows a persistent motional interdependence among the protein subunits in the crystal. We characterize the dynamical correlations featured by these subunits and identify two regimes characterized by their locality or nonlocality, depending on the geometric parameters of the crystal. From the same data, we also calculate the Poisson’s (longitudinal to axial strain) ratio of the crystal, and learn that, due to holonomic constraints (rigidity of the rod links), the crystal remains auxetic even after significant changes in the original geometry. The nonholonomic ones (collisions between subunits) increase the number of inhomogeneous deformations of the crystal, thus driving it away from an isotropic response. Our work provides the first simulation of the dynamics of protein crystals and offers insights into promising mechanical properties afforded by these materials.
1 Introduction

Protein-based materials profit from the immense tunability and combinatorial diversity of their modular building blocks to create new and versatile functionalities.\textsuperscript{1–4} For instance, Suzuki and collaborators have recently synthesized the very first Protein Crystals (PC) with auxetic behavior,\textsuperscript{5} i.e., stretching (shrinking) them along one axis results in their expansion (compression) along a perpendicular one. The unusual features displayed by auxetic materials make them appealing for applications in personal protective clothing,\textsuperscript{6} clinical prosthesis,\textsuperscript{7} filtration mechanisms,\textsuperscript{8} mechanical lungs,\textsuperscript{6} controlled release of drugs,\textsuperscript{6} and reinforcement of composite materials,\textsuperscript{9} among others. Moreover, auxetic PCs display coherent dynamics, as suggested by Transmission Electron Microscopy studies, which reveal a continuous transition from an open (porous) to a closed (tight packed) configuration.\textsuperscript{5} Unexpectedly, the geometrical rearrangements associated to this transition span across entire single crystals, without identifiable formation of local configurational domains. Although the geometrical arrangement in this system is closely related to that of the simple model of rotating squares introduced by Grima,\textsuperscript{10} the inclusion of finite length linkages between the building blocks introduces further degrees of freedom that increase the complexity of the system. Hence the persistence of auxeticity or coherence in these structures follow trivially anymore.

Among the many questions posed by such a system, two of them are particularly appealing from a theoretical standpoint. The first one deals with the extent of inter-subunit coherent dynamics throughout the crystal, and the conditions leading to it, while the second one is concerned with the dependence of the auxetic behavior on the geometric parameters of the lattice. Both questions might be approached from a computational standpoint with the aid of Molecular Dynamics (MD) simulations. An overview of the geometry of the system suggests the sufficiency of a coarse-grained framework in which the particles need not be treated with atomic accuracy; instead, the protein subunits may be suitably represented by rigid squares whose generalized coordinates allow for phase space exploration within the formalism of constrained Lagrangian mechanics. Several methods have been developed to address problems akin to the one described so far; including: adaptive timestep,\textsuperscript{11} explicit minimization of discretized action,\textsuperscript{12} penalty functions,\textsuperscript{12} event-driven dynamics,\textsuperscript{13} and impulsive constraints,\textsuperscript{14} to name a few.

On the other hand, in most MD simulations, constraints are employed to freeze out only high-frequency vibrational modes, such as hydrogen atom bond vibrations in solvents and macromolecules, and are rarely
applied to the primary degrees of freedom of the system.\textsuperscript{15} Furthermore, this typically serves to simply allow a slightly longer integration timestep (typically from 1 to 2 fs), and does not constitute a major element of the simulation.\textsuperscript{16,17} Hence, working examples of highly-constrained MD simulations are markedly underrepresented relative to their unconstrained counterparts. The recent realization of auxetic PCs together with an increased interest in soft mechanical materials demands new MD methodologies that can address the simulation of highly articulated structures involving large numbers of geometrical constraints.\textsuperscript{18} In these instances, fulfillment of the constraints is achieved by correcting an unconstrained update of the configuration at each timestep.\textsuperscript{19} Currently, the most widespread methods perform such corrections iteratively,\textsuperscript{20,21} which can be, from a computational standpoint, a shortcoming for highly constrained problems.\textsuperscript{22} There are alternative methodologies that address these concerns\textsuperscript{23,24} by eliminating the iterations, but are prone to drift; thus requiring adaptive timestep schemes\textsuperscript{11} which themselves can become quite computationally costly.

In this work, we tackle the aforementioned questions, regarding coherence and auxetic behavior, by performing a statistical analysis of trajectories simulated through a performance enhanced MD scheme. We demonstrate that corrections to the unconstrained velocities do not need to be calculated iteratively if the actions of the constraints are regarded as impulses,\textsuperscript{14} as it is commonly done for collision responses.\textsuperscript{25} Moreover, we introduce a noniterative method that corrects configurations to fulfill constraints in coordinate space, and whose implied error is small enough that the procedure can be implemented in the events within the simulations that require the evaluation of shorter timesteps. This new machinery, along with slight improvements to tools originated within the methods metioned above, produces a significantly optimized approach that is able to deal with complex tasks such as collision event handling.

The trajectories simulated produce a set of time series which are furtherly analyzed to determine the degree of synchronization among the components of the network. Additionally, we exploit the discretized character of MD as a means to create a sample space of structural strains that allows taking a more in-depth look at the geometrical response of the array.

This manuscript is organized as follows: In Section 2, we begin by introducing the coarse-grained model and identify the essential degrees of freedom required to emulate the dynamics of the auxetic PC by explicit simulation. In Section 3, we describe the formalism of constrained Lagrangian dynamics -which gives rise to the corresponding Equations Of Motion (EOM)-, the numerical and computational methods developed
to handle the EOM integration, and the collision detection and response protocols. In Section 4, we briefly sketch implementation details and provide a simple example. In Section 5, we apply these tools to examine the spread across the network of coherence among subunits under varying conditions. In section 6, we show the dependence of the auxetic behavior displayed by this system on the geometric parameters of the crystal. Finally, we present a summary and conclusions of the work in Section 7.

2 Definition of the coarse-grained model

The PC synthesized by Suzuki and co-workers was prepared through the self-assembly of L-rhamnulose-1-phosphate aldolase units containing surface-exposed cysteines (C98RhuA), which form intermolecular disulfide bonds to yield extended two-dimensional crystals. Upon tessellation, the protein assemblage adopts a checkerboard pattern (p4212 symmetry) where each of the C4-symmetric C98RhuA subunits is linked to its four nearest neighbors through its vertices (see Fig. 1). In our coarse-grained model, we consider the aforementioned articulated structure as a collection of rigid impenetrable squares connected by massless rigid rods. Labelling each square with indices $ij$ denoting its Cartesian position in the grid (Figure 2), its evolution is described by a 3D vector state $\vec{s}_{ij}$ and its time derivatives:

![Figure 1: Closed (a) and open (b) configurations of C98RhuA](image)

Figure 1: Closed (a) and open (b) configurations of C98RhuA
\( \vec{s}_{ij} = \vec{r}_{ij} + \theta_{ij} \hat{e}_z, \)
\( \dot{\vec{s}}_{ij} = \vec{v}_{ij} + \omega_{ij} \hat{e}_z, \)
\( \ddot{\vec{s}}_{ij} = \vec{a}_{ij} + \alpha_{ij} \hat{e}_z, \)

where \( \vec{r}_{ij}, \vec{v}_{ij} \) and \( \vec{a}_{ij} \) denote the location, velocity, and linear acceleration, respectively of the square’s centroid in two-dimensional Euclidean space; \( \theta_{ij} \) is the square’s orientation with respect to a fixed lab frame while \( \omega_{ij}, \) and \( \alpha_{ij} \) are its angular velocity and angular acceleration, respectively.

The squares’ vertices play a fundamental role in the dynamics of the system. The two-dimensional position of the \( k \)-th vertex belonging to the \( ij \)-th square (see Fig 2) is given by

\[ \vec{r}_{ijk} = \vec{r}_{ij} + \vec{\rho}_k(\theta_{ij}) \]
\[ = \vec{r}_{ij} + \frac{l}{\sqrt{2}} \left[ \cos(k\pi) \mathbf{1}_2 + \left( \cos \frac{k\pi}{2} - \sin \frac{k\pi}{2} \right) \mathbf{G} \right] \cdot \left[ \cos(\theta_{ij}) \hat{e}_x + \sin(\theta_{ij}) \hat{e}_y \right], \]

where \( l \) is the length of the square’s half-diagonal, the two-dimensional projector operator is given by,

\[ \mathbf{1}_2 = \hat{e}_x \hat{e}_x^T + \hat{e}_y \hat{e}_y^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]
and
\[
G = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
is the generator of rotations belonging to the $SO(2)$ group. The velocity of each vertex is calculated using the square’s center of mass velocity, as well as the corresponding tangential velocity within the subunit frame,

\[
\vec{v}_{ijk} = \vec{v}_{ij} + \omega_{ij} \hat{e}_z \times \vec{p}_k(\theta_{ij}).
\]

Defining the additional projector,

\[
B_{ijk} = 1_2 + G\vec{p}_k(\theta_{ij})\hat{e}_z^T,
\]

Eq. (3) can be written in terms of the velocity vector state for the center of mass of each square,

\[
\vec{v}_{ijk} = B_{ijk} \dot{s}_{ij}.
\]

On the other hand, the acceleration of a given vertex must account for the center of mass, radial (centripetal), and tangential accelerations,

\[
\vec{a}_{ijk} = \vec{a}_{ij} - \omega_{ij}^2 \vec{p}_k(\theta_{ij}) + \alpha_{ij} \hat{e}_z \times \vec{p}_k(\theta_{ij}) = B_{ijk} \ddot{s}_{ij} - \omega_{ij}^2 \vec{p}_k(\theta_{ij}).
\]

A given vertex with indices $ijk$ is linked to another one with primed indexes $i'j'k'$ (Fig. 2). The self-inverse relations between the primed and unprimed indices are summarized in Table 1.

| $k$ | even $(i+j)$ | odd $(i+j)$ |
|-----|--------------|------------|
|     | $i'$         | $j'$       | $k'$       |
| 1   | $i$          | $j-1$      | 2          |
| 2   | $i-1$        | $j$        | 4          |
| 3   | $i+1$        | $j$        | 1          |
| 4   | $i$          | $j+1$      | 3          |

For an $n_s \times n_s$ lattice with open-boundary conditions, there will be $n_b = 2n_s(n_s - 1)$ rigid rods (disulfide bonds) connecting the square (protein) units. With the notation developed so far, the vector along the linkage
between the $ijk$-th vertex and the $i'j'k'$-th one can be written as:

$$\Delta \vec{r}_{ijk} = \vec{r}_{ijk} - \vec{r}_{i'j'k'}. \tag{7}$$

The constraints corresponding to the rigidity of the rods can be expressed as,

$$|\Delta \vec{r}_{ijk}|^2 = r_0^2, \tag{8}$$

where $r_0 \approx (3/22)l$ is the equilibrium length of the disulfide bonds. Differentiation of Eq. (8) with respect to time yields a velocity

$$\Delta \vec{r}_{ijk}^T \Delta \vec{v}_{ijk} = 0, \tag{9}$$

and acceleration constraints,

$$\Delta \vec{r}_{ijk}^T \Delta \vec{a}_{ijk} + |\Delta \vec{v}_{ijk}|^2 = 0. \tag{10}$$

The impenetrability between squares is imposed by demanding that no vertex of any square shall be found inside the area of another square. By identifying the $ij$-th square as a lurker, and the $mn$-th square as a target, the previous statement is equivalent to

$$\vec{\rho}_\ell(\theta_{mn})^T \vec{u}_k(\theta_{ij}) + |\vec{\rho}_\ell(\theta_{mn})^T G^T \vec{u}_k(\theta_{ij})| - l^2 \geq 0, \tag{11}$$

where $\vec{u}_k(\theta_{ij}) = \vec{r}_{ijk} - \vec{r}_{mn}$ is the location of the lurker’s $k$-th vertex in the target’s frame of reference, and $\ell$ labels the target’s vertex that is closest to the lurker. Having settled the notation and main features of the model we proceed to discuss its dynamical evolution.

### 3 Simulation of the dynamics

Since this work focuses mainly on the geometrical features of the system, we do not take into account additional potentials which modulate the actual dynamics of the PC nor do we consider dissipative mechanisms; nevertheless, they both can be readily included in generalizations of computational schemes compatible with the one suggested below. In this regard, the system is described by a Lagrangian function per mass unit of
the form:

$$
\mathcal{L} = \frac{1}{2} \sum_{ij} \left[ \dot{\vec{s}}_{ij}^T \left( 1_2 + I \dot{\vec{e}}_z \dot{\vec{e}}_z^T \right) \dot{\vec{s}}_{ij} + \sigma_{ij} + \sum_{mn} \varsigma_{ij(mn)} \right]
$$

(12)

where $I = l^2/3$ is the moment of inertia for every square. The functions $\sigma_{ij}$ and $\varsigma_{ij(mn)}$ are, respectively, holonomic and nonholonomic constraints indicating the rigid-rod linkages and the impenetrability of the squares.

3.1 Holonomic constraints

Constraints due to Eq. (8) are introduced into the Lagrangian in the form:

$$
\sigma_{ij} = \sum_{k=1}^{4} \lambda_{ijk} \left( \left| \Delta \vec{r}_{ijk} \right|^2 - r_0^2 \right) = 0,
$$

(13)

where $\lambda_{ijk}$ are Lagrange multipliers characterizing the rigid-rod linkages. Including only these constraints for the time being, evaluation of the Euler-Lagrange equations with the Lagrangian from Eq. (12) yield

$$
\ddot{\vec{s}}_{ij} = \sum_{k=1}^{4} \lambda_{ijk} \left[ 1_2 + \dot{\vec{e}}_z \rho_k^T (\theta_{ij}) \mathbf{G}^T \right] \Delta \vec{r}_{ijk}
$$

\begin{equation}
= \sum_{k=1}^{4} \lambda_{ijk} \mathbf{C}_{ijk} \Delta \vec{r}_{ijk},
\end{equation}

(14)

where the operator $\mathbf{C}_{ijk}$ projects the vector $\Delta \vec{r}_{ijk}$ onto the acceleration $\vec{a}_{ijk}$ in Eq. (6). Here, it becomes evident that the Lagrange multipliers $\lambda_{ijk}$ play the role of adaptive stiffness constants. To calculate the multipliers, the vertex accelerations can be written in terms of the bond vectors $\Delta \vec{r}_{ijh}$:

$$
\vec{a}_{ijk} = \sum_{h=1}^{4} \lambda_{ijh} \mathbf{B}_{ijh} \mathbf{C}_{ijh} \Delta \vec{r}_{ijh} - \omega_{ij}^2 \vec{\rho}_k (\theta_{ij})
$$

\begin{equation}
= \sum_{h=1}^{4} \lambda_{ijh} \mathbf{A}_{hh}^{(ij)} \Delta \vec{r}_{ijh} - \omega_{ij}^2 \vec{\rho}_k (\theta_{ij}),
\end{equation}

(15)

and then plugged into the acceleration constraint of Eq. (10),
\begin{align*}
\Delta r'_{ijk} = & \sum_{h=1}^{4} \lambda_{ijh} A^{(ij)}_{hk} \Delta r_{ijh} - \sum_{\ell=1}^{4} \lambda'_{ij\ell} A^{(ij')}_{hk'} \Delta r_{ij\ell} + \omega_{ij}^2 \rho_k(\theta_{ij}) - \omega_{ij'}^2 \rho'_k(\theta_{ij}) \right] = -|\Delta \vec{v}_{ijk}|^2. \quad (16)
\end{align*}

By defining the \( n_b \)-dimensional vectors whose \( ijk \)-th component is given by

\begin{align*}
(b)_{ijk} &= \Delta r'_{ijk} \left[ \omega^2_{ij} \rho_k(\theta_{ij}) - \omega^2_{ij'} \rho'_k(\theta_{ij'}) \right], \\
(w)_{ijk} &= |\Delta \vec{v}_{ijk}|^2, \\
(\Lambda)_{ijk} &= \lambda_{ijk}, \\
(M\Lambda)_{ijk} &= \Delta r'_{ijk} \left[ \sum_{h=1}^{4} \lambda_{ijh} A^{(ij)}_{hk} \Delta r_{ijh} - \sum_{\ell=1}^{4} \lambda'_{ij\ell} A^{(ij')}_{hk'} \Delta r_{ij\ell} \right], \quad (17)
\end{align*}

Eq. (16) can be written in the form

\begin{align*}
M\Lambda = b - w, \quad (18)
\end{align*}

which allows for the solution of the Lagrange multipliers upon inversion of the matrix \( M \).

### 3.2 Nonholonomic constraints

Inequality (11) leads us into a set of constraints for the allowed (i.e., non-interpenetrated) configurations of the system,

\begin{align*}
\varsigma_{ij(mn)} = & \sum_{k,\ell} \lambda_{ij(mn)} \left[ \bar{\rho}_\ell(\theta_{mn})^T \bar{u}_k(\theta_{ij}) + |\bar{\rho}_\ell(\theta_{mn})^T G^T \bar{u}_k(\theta_{ij})| - l^2 \right] \\
= & 0, \quad (19)
\end{align*}

where the coefficients \( \lambda_{ij(mn)} \) are known as Karush-Kuhn-Tucker (KKT) multipliers in the mathematical optimization literature.\cite{28, 29} These objects are different from Lagrange multipliers in the sense that, for Eq. (19) to hold, the corresponding KKT multiplier must vanish whenever Eq. (11) is a strict inequality, thus justifying our disregard for constraints of this kind in the previous section. When Eq. (11) is strictly zero, i.e., at a collision, the corresponding KKT multiplier is allowed to adopt positive values. With the same treatment as before, the inclusion of these constraints to the EOMs modifies the effective forces of Eq. (14) to
\[
\ddot{s}_{ij} = \sum_{k=1}^{4} \lambda_{ijk} C_{ijk} \Delta \vec{r}_{ijk} - \sum_{mn} C_{ij(mn)} \vec{\rho}^{j(mn)}_{\text{col}},
\]

\[
\ddot{s}_{mn} = \sum_{k=1}^{4} \lambda_{mkn} C_{mkn} \Delta \vec{r}_{mkn} + \sum_{ij} C_{mn(ij)} \vec{\rho}^{j(mn)}_{\text{col}},
\]

where

\[
\vec{\rho}^{j(mn)}_{\text{col}} = \lambda_{ij(mn)} \vec{\rho}_l(\theta_{mn}) + \vec{\rho}_\nu(\theta_{mn}) \frac{\sqrt{2}}{2l}
\]

is an impulsive force accounting for the collisions, with \(\nu\) labelling the second vertex in the target closest to the lurker vertex, and the projection operators

\[
C_{ij(mn)} = 1_2 + \frac{\hat{e}_z \vec{\rho}_{k(mn)}(\theta_{ij})^T \mathbf{G}^T}{I},
\]

\[
C_{mn(ij)} = 1_2 + \frac{\hat{e}_z \vec{u}_k(\theta_{ij})^T \mathbf{G}^T}{I},
\]

have analogous interpretations as those defined in Eq. (14).

### 3.3 Time integration

Given \(s\), the set of all state vectors \(\vec{s}_{ij}\), its dynamics is propagated by means of a Störmer-Verlet (SV) time integration:

\[
s(t + \Delta t) = s(t) + \dot{s}(t) \Delta t + \frac{\ddot{s}(t)}{2} \Delta t^2,
\]

\[
\dot{s}(t + \Delta t) = \dot{s}(t) + \frac{\ddot{s}(t) + \ddot{s}(t + \Delta t)}{2} \Delta t.
\]

It is worth noting that the Lagrange multipliers computed with Eq. (18) are exact. Nevertheless, the error introduced by the time discretization propagates during the dynamics, resulting in drift; therefore, the constraints are not exactly fulfilled throughout the dynamics. This issue is solved by computing the Lagrange multipliers so that the constraints are satisfied at each time step. Since there are constraints in both coordinate and velocity spaces, there will be a set of Lagrange multipliers, \(\lambda^{(q)}\), obtained from subjecting the coordinates in Eq. (23) to the constraint Eq. (8), in addition to a set, \(\lambda^{(p)}\), obtained from enforcing
the fulfillment of Eq. (9) by the velocities in Eq. (24). The strategies employed for each set are described below.

3.3.1 Collision-free dynamics

Although the numerical accuracy is significantly improved by using a control function dependent on the Lagrange multipliers to set the size of the timestep in an adaptive framework, we have found that a constant timestep, with numerical value \( \Delta t = 0.01 \), provides a fair balance between stability and implementation time for the purposes of this work. By equating the number of timesteps it takes to complete a transition between the open and the closed configuration in the coarse-grained simulation to the time it takes for the same event to happen in all-atom simulations for various system sizes, we have determined that the timestep size is in the order of magnitude of nanoseconds.

**Coordinate constraints.** Eq. (8) together with the integration step of Eq. (23) give rise to the SHAKE equations for this system:

\[
\vec{s}_{ij}(t + \Delta t) = \vec{s}_{ij}(t) + \dot{\vec{s}}_{ij}(t) \Delta t + \sum_{k=1}^{4} \lambda_{ijk}(t) C_{ijk}(t) \Delta r_{ijk}(t) \frac{\Delta t^2}{2},
\]

\[
0 = |\Delta r_{ijk}(t + \Delta t)|^2 - r_0^2,
\]

where the unknowns are the updated coordinates and the Lagrange multipliers. The quadratic character of the constraint, as well as the trigonometric functions involved in the calculation of the inter-vertex distances, make this scheme highly nonlinear. Fortunately, the values of the Lagrange multipliers obtained with Eq. (18), although inaccurate, usually require but small corrections. In this work, such values are used to initialize a Levenberg-Marquardt algorithm carried out by the built-in MATLAB gradient-based nonlinear solver `fsolve`.

**Velocity constraints.** The velocity step of the time integration of Eq. (24) together with the velocity constraint of Eq. (9) form the RATTLE equations of the system. The former is a set of linear equations
that can be written in matrix form as,

\[
\begin{pmatrix}
1_{3n_s^2} & \mathbf{C} \\
\mathbf{B} & 0_{n_b}
\end{pmatrix}
\begin{pmatrix}
\dot{s}(t + \Delta t) \\
\Lambda^{(p)} \frac{\Delta t}{2}
\end{pmatrix}
= \begin{pmatrix}
\dot{s}(t) + \dot{s}(t) \frac{\Delta t}{2} \\
0_{n_b \times 1}
\end{pmatrix},
\]

(26)

where \( \mathbf{C} \) is a \( 3n_s^2 \times n_b \) matrix with blocks \( \text{sgn}(i'j' - ij) \mathbf{C}_{ijk}(t + \Delta t)\Delta \vec{r}_{ijk}(t + \Delta t)\Delta t/2 \), and \( \mathbf{B} \) is a \( n_b \times 3n_s^2 \) matrix with blocks given by \( \text{sgn}(ij - i'j')\Delta \vec{r}^T_{ijk}(t + \Delta t)\mathbf{B}_{ijk}(t + \Delta t) \). Since the effects of the constraints in the EOM can be integrated as a set of effective impulsive forces,\(^{14}\) the leftmost matrix in Eq. 26 will be referred to as the Momentum Transfer Matrix (MTM). From Eq. 26, it follows that the updated velocities as well as the Lagrange multipliers can be computed from the inversion of the MTM in a single step.

### 3.3.2 Dynamics including collisions

To detect whether Eq. (11) is fulfilled, the first step is a pairwise pruning procedure, which checks the distances between centers of mass for each pair of squares. If this distance is larger than the squares’ diagonal \( 2l \), the collision is rejected; otherwise, all the vertices of the pair of squares being considered are tested in detail with Eq. (11), trying and swapping the roles of lurker and target for both squares. If, during the dynamics, the configuration \( s(t + \Delta t) \) contains violations of Eq. (11), the collisions are accepted, and the indices of both, the target and the lurker’s vertex, are stored to be used later in the collision handling protocol.

Having identified the overlapping squares and vertices, the next step is to find the exact time at which the contact takes place. Such task requires solving the system of nonlinear Eqs. (25) multiple times, which, given the iterative nature of the method hitherto employed, might render it computationally expensive.

On the other hand, for very small timesteps \( \delta t < \Delta t \), we have \( \left[ \cos \left( \omega \delta t + \alpha \frac{\delta t^2}{2} \right), \sin \left( \omega \delta t + \alpha \frac{\delta t^2}{2} \right) \right] \approx \left[ 1, \omega \delta t + \alpha \frac{\delta t^2}{2} \right] \). This fact allows us to formulate a method to calculate Lagrange multipliers in a non-iterative fashion, but that brings the coordinate constraints closer to fulfillment than the multipliers obtained from Eq. (18). Since the bond lengths need to remain fixed, a reasonable solution to the EOM is given by the following ansatz:

\[
\Delta \vec{r}_{ijk}(t + \delta t) = \Delta \vec{r}_{ijk}(t) \cos[\phi_{ijk}(t, \delta t)] + \frac{\Delta \vec{v}_{ijk}(t)}{\omega_{ijk}(t)} \sin[\phi_{ijk}(t, \delta t)]
\]

(27)
where
\[ \omega_{ijk}(t) = \frac{\Delta \vec{v}_{ijk}(t)}{\Delta \vec{r}_{ijk}(t)} = \frac{\Delta \vec{v}_{ijk}(t)}{r_0}. \quad (28) \]

It can be shown that if the angle variable takes the form
\[ \phi_{ijk}(t, \delta t) = \omega_{ijk}(t) \delta t \frac{\alpha_{ijk}(t)}{2} \delta t^2, \]
the corresponding angular acceleration \( \alpha_{ijk}(t) \) becomes,
\[ \alpha_{ijk}(t) = \frac{\Delta \vec{r}_{ijk}(t)^T \Delta \vec{a}_{ijk}(t)}{|\Delta \vec{v}_{ijk}(t)|^2 r_0}, \quad (29) \]
where the acceleration differences, \( \Delta \vec{a}_{ijk}(t) \), can be found from Eq. (10) using the Lagrange multipliers calculated with Eq. (18). Since the vectors, \( \Delta \vec{r}_{ijk} \), in Eq. (27) already fulfill the coordinate constraints in Eq. (8), the latter can be replaced by
\[ \sigma_{ijk} = \vec{\chi}_{ijk}^T (\Delta \vec{r}_{ijk} - \Delta \bar{r}_{ijk}), \quad (30) \]
where \( \vec{\chi}_{ijk} \) is a vector of Lagrange multipliers. By considering such constraints as the source of acceleration in the integration step of Eq. (23), the Lagrange multipliers and the updated coordinates can be found by solving the linear system of equations:
\[ \vec{s}_{ij}(t + \delta t) = \vec{s}_{ij}(t) + \dot{\vec{s}}_{ij}(t) \delta t + \sum_{k=1}^{3} C_{ijk}(t) \vec{\chi}_{ijk}(t) \delta t^2, \quad (31) \]
\[ \sum_{h=1}^{4} A_{hh}^{(ij)} (t) \vec{\chi}_{ihk}(t) - \sum_{\ell=1}^{4} A_{\ell k}^{(ij)} (t) \vec{\chi}_{ij \ell}(t) = (b)_{ijk}(t) + \frac{2}{\delta t} \left( \Delta \bar{r}_{ijk}(t) + \frac{\Delta \vec{a}_{ijk}(t)}{\delta t} - \Delta \bar{v}_{ijk}(t) \right). \quad (32) \]
The advantage of this method over the previous one is that it requires the solution of two systems of linear equations instead of one system of mixed quadratic and trigonometric equations. Nevertheless, the equivalence between the tangential behavior of the vertices and the angular behavior of the squares assumed by this method is valid only in the infinitesimal limit \( \delta t \to 0 \). Therefore, it will be used exclusively in the search for collision times, where the timesteps are presumed to be small enough. Also, to reduce the truncation error even more, we exploit the time-reversible character of the SV integration by looking for collision times starting from both \( s(t) \) and \( s(t + \Delta t) \), pushing the dynamics forward or backward respectively, and keeping only the smallest ones. This search for collision times is performed by testing the inequality (11) with the configurations found
in Eq. (31) via the built-in MATLAB function \texttt{fzero}.

The collision handling protocol thus becomes:

1. Starting from \( s(t) \), determine the forward timestep size \( 0 < \delta^{(+)} t_i < \Delta t \) for each one of the detected collisions.

2. Starting from \( s(t + \Delta t) \), determine the backward timestep size \( 0 > \delta^{(-)} t_i > -\Delta t \) for each one of the detected collisions.

3. For each collision, \( i \), find \( h_i = \min\left(\delta^{(+)} t_i, |\delta^{(-)} t_i|\right) \).

4. Calculate \( \delta t_i = \begin{cases} h_i & h_i = \delta^{(+)} t_i, \\ \Delta t + h_i & h_i = \delta^{(-)} t_i. \end{cases} \)

5. Evolve the system from \([s(t), \dot{s}(t)]\) to \([s(t + (\delta t)), \dot{s}(t + (\delta t))]\) and perform collision response to find \([s^*(t + (\delta t)), \dot{s}^*(t + (\delta t))]\), where the star superscripts indicate ‘after collision’.

6. Evolve the system for a timestep of size \( \Delta t - \langle \delta t \rangle \) to get to a corrected \([s(t + \Delta t), \dot{s}(t + \Delta t)]\).

7. Return to collision-less dynamics.

**Collision response.** When two squares come into contact, the KKT multiplier associated with the lurker-target pair can have non-zero values. These multipliers correspond to magnitudes of impulses which provide an instantaneous modification to the velocities of the system. The effect of a bounce is the reflection of the motion with respect to the plane normal to the collision, which is consistent with the direction derived in Eq. (21). This effect can be summarized mathematically in the following system of equations:

\[
\begin{align*}
\ddot{s}_{ij} &= \dot{s}_{ij} - \sum_{mn} C_{ij(mn)} \tilde{J}_{col}^{ij(mn)}, \\
\ddot{s}_{mn} &= \dot{s}_{mn} + \sum_{mn} C_{mn(ij)} \tilde{J}_{col}^{ij(mn)}, \\
\hat{n}_{ij(mn)}^T \left( B_{ij(mn)} \dot{s}_{ij} - B_{mn(ij)} \dot{s}_{mn} \right) &= \hat{n}_{ij(mn)}^T \left( B_{mn(ij)} \dot{s}_{mn} - B_{ij(mn)} \dot{s}_{ij} \right),
\end{align*}
\]  

(33)

where the projection operators

\[
B_{ij(mn)} = 1_2 + G \tilde{p}_{k(mn)}(\theta_{ij}) e_z^T, \\
B_{mn(ij)} = 1_2 + G \tilde{u}_{k(\theta_{ij})} e_z^T,
\]  

(34)

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are analogous to those defined in Eq. (4). Nevertheless, the impulse in Eq. (21) can only describe vertex-side collisions, the normal-to-collision plane is ill-defined for vertex-vertex collisions (side-side collisions result from a pair of reciprocating vertex-side collisions), in which case, solving for Eq. (33) gives the vector
\[
\vec{n}_{ij}^{(mn)} = \mathbf{G} \left( \frac{\vec{p}_l(\theta_{mn})\vec{p}_l(\theta_{mn})^T - \vec{p}_k(\theta_{ij})\vec{p}_k(\theta_{ij})^T}{I} \right)^{-1} \left[ (\mathbf{G} + \hat{e}_z\vec{p}_l(\theta_{mn})^T)^T \vec{s}_{mn} + (\mathbf{G} + \hat{e}_z\vec{p}_k(\theta_{ij})^T)^T \vec{s}_{ij} \right]
\]
(35)
which is parallel to the impulse. The structure of the system of equations in Eq. (33) allows for its seamless insertion into the MTM such that,
\[
\begin{pmatrix}
1_{3s^2} & \mathbf{C}_{\text{bnd}} & \mathbf{C}_{\text{col}} \\
\mathbf{B}_{\text{bnd}} & 0_{3(n_b+n_c)\times(n_b+n_c)} & \mathbf{B}_{\text{col}} \\
\mathbf{B}_{\text{col}} & \mathbf{B}_{\text{col}} & \mathbf{B}_{\text{col}}
\end{pmatrix}
\begin{pmatrix}
\dot{\mathbf{s}}^* \\
\Lambda_{\text{bnd}} \\
\Lambda_{\text{col}}
\end{pmatrix}
= \begin{pmatrix}
\dot{\mathbf{s}} \\
0_{n_b \times 1} \\
-\mathbf{B}_{\text{col}} \dot{s}
\end{pmatrix}
\]
(36)
where the subscript 'bnd' has been added to differentiate whether a quantity is related to holonomic or nonholonomic constraints.

4 Trajectory of reference

All the trajectories computed in this work were initialized in the open configuration and provided with a set of initial angular velocities for the subunits. In Figure 3 we show snapshots of the trajectory calculated for a 10 × 10 lattice with uniform sign-altered initial angular velocities such that \(|\omega_{ij}(0)| \Delta t = 0.01\). This trajectory will set the reference temperature for all the subsequent simulations, so that \(k_bT = 5 \times 10^{-3} I/\Delta t^2\).

A remarkable observation from this example is the loss of the initial uniformity of the lattice during the advancement of the dynamics, which is illustrated in Figure 3 where the angles of the subunits angles are plotted as a function of time. The former can be explained by the fact that the number of positional restraints, \(i.e.,\) the number of neighbors to which a subunit is connected, varies from two in the case of the corners, to three for squares at the edges, and four for those in the bulk. This observation illustrates how the introduction of finite-length linkages undermines the capability of the system for absolute conformational coherence when compared to the Grima’s model, thus prompting further analysis that we will show next.
Figure 3: Panels 1-5: Snapshots at various timesteps of the trajectory simulated for a $10 \times 10$ lattice with $r_0/l = 5\%$, starting from the open configuration and with uniform sign-altered initial angular velocities. Bottom-right panel: behavior of the squares’ angles as a function of time.

5 Analysis of Coherence

In this work, coherence will be understood as the synchronized evolution of the subunits’ rotational angles. A common metric for synchronization is the phase-locking condition

$$ |\theta_{ij}(t) - \theta_{mn}(t)| < c, $$

where $c$ is a constant parameter acting as an upper bound to the difference over a time interval. In Fig. 4 we showcase various values of $r_0/l$ in the $10 \times 10$ lattice and compare the phase (angle) differences for trajectories averaged over a set of random initial angular velocities. Here, the departure from the Grima’s model originates from the inclusion of finite-length linkages between squares. It is illustrated by the fact that for large values of $r_0$ the distribution of phase differences widens over time. As the bond length $r_0$ decreases, the phase differences take shorter times to reach a phase locking regime. This intuitive trend can be seen in
Fig. 4 with phase locking occurring nearly instantaneously when $r_0/l = 2\%$.

![Figure 4: Evolution of phase differences in 10 × 10 lattices. The results were computed by averaging ten trajectories per $r_0/l$ value, starting from the open configuration and random initial velocities such that $-0.01 < \omega_{ij}(0)\Delta t < 0.01$.](image)

A more accurate measure of synchronization for nonlinear systems is given by the synchronization index (SI),

$$R_{ij,mn} = \frac{1}{T} \sum_{t=1}^{T} \exp i[\theta_{ij}(t) - \theta_{mn}(t)],$$

and the partial phase synchronization index (PPSI),

$$PR_{ij,mn} = R_{ij,mn} - \sum_{Z \neq ij,mn} R_{ij,Z}(R_{Z,Z})^{-1}R_{Z,mn}.$$  

These quantities measure the effective and direct coupling, respectively, between a pair of subunits. In Fig. 5a, we show the SI between pairs of subunits for the temporal average of the trajectories mentioned above. As one would expect, the shorter the bonds, the more correlated the system is. Furthermore, when comparing with the PPSI (Fig. 5b) it becomes clear that only the structures with long bonds present direct correlations among the squares, while short bonds lead to a high frequency of collisions which redistributes correlations...
over space, making them indirect, i.e., mediated by the other squares in the network. These observations are confirmed by comparing the leading eigenvalues of the SI and PPSI matrices for each structure (Fig. 5c).

Figure 5: (a) Synchronization index and (b) Partial phase synchronization index matrices for the time and trajectory averages of the angles $\theta_{ij}$ in a $10 \times 10$ lattice for various $r_0/l$ ratios. (c) Largest eigenvalues of the SI (bullets/dotted line) and PPSI (squares/dashed line) matrices. Notice the logarithmic scale for the $r_0/l$ ratio.

6 Analysis of Auxetic behavior

An auxetic material is characterized by its negative Poisson’s ratio,\textsuperscript{10}

$$\nu = -\frac{d\varepsilon_{yy}}{d\varepsilon_{xx}},$$

(40)

where $\varepsilon_{yy}$ and $\varepsilon_{xx}$ are the diagonal components of the strain tensor, each related to the deformation of the structure along the axis indicated by the subscripts. In particular, both Grima’s rotating squares model
and the coherent mode of the synthesized PC yield $\nu = -1$, which is the lowest limit for isotropic two-dimensional structures.

We assess the intrinsic auxetic behavior of the structure by determining the strain tensor over a range of configurations yielded by the simulated trajectories mentioned in previous sections. Specifically, for each timestep, the change in configuration is characterized by a deformation gradient tensor (DGT), $F$, such that

$$\vec{r}(t + \Delta t) = F(\vec{r}(t), t) \vec{r}(t),$$

where $\vec{r}(t)$ is any of all possible $\vec{r}_{ij}(t)$ and $\vec{r}_{ijk}(t)$ describing a location in the lattice. Given the small step size, it is reasonable to expect a modest change from a timestep to the next one, so that the DGT can be accurately approximated by a uniform change in the whole lattice, i.e., $F(\vec{r}(t), t) \to F(t)$. The latter can be readily obtained from the slopes of a least-square fitting of the updated coordinates as a function of the earlier ones.

The strain tensor can be computed from the DGT through

$$d\varepsilon(t) = \left[ F(t)^T F(t) \right]^{1/2} - 1,$$

and, according to (40), the Poisson’s ratio is evaluated as the slope of the least-square fitting of $-\varepsilon_{yy}$ as a function of $\varepsilon_{xx}$ (Fig. 6).

In Figure 7 we show the calculated Poisson’s ratios as a function of $r_0/l$. As it can be seen, all the structures are auxetic. However, it shows that perfect auxeticity holds for the structures with larger bonds. At first sight, such an observation contradicts the expectations drawn from Grima’s model. However, we must consider that, for the trajectories simulated, there were no collision events for the structures with larger bonds in the explored time window. The former implies that the deviations from ideal auxeticity are due to configurations with non-auxetic strains allowed by a set of collisions. If anything, the deviation from ideal auxeticity rises as a measure of the frequency of collisions in the system. This claim is supported by the appreciable decrease in the correlation coefficient, $r^2$, which accounts for the departure from a linear dependence between strains.

To confirm that the collisions produce the observed digression, we calculated ten more trajectories for
Figure 6: Behavior of the negative transverse strain, \( \varepsilon_{yy} \) as a function of the axial strain \( \varepsilon_{xx} \) for various link-to-side ratios. Bullets correspond to data extracted from the dynamics, while the line is the result from a least-square linear fitting.

The lattice with \( r_0 = l \) but with a tenfold increase in the number of the timesteps. The corresponding phase difference evolution, SI, PPSI and strain-strain relation are illustrated in Figure 8.

The phase difference evolution shows that a phase-locking regime is achieved somewhere before 1000 timesteps, where a collision dominated regime is expected to be reached. This observation is consistent with the strain-strain relation which displays a more scattered behavior and is further reinforced by the synchronization indices that become closer in appearance to their shortest bonds counterparts.

7 Summary and Conclusions

We have developed an efficient and reliable procedure to simulate the dynamical behavior of a lattice built from impenetrable squares linked by rigid rods which emulates the geometry and topology of the auxetic two-dimensional crystals assembled from \(^{C98}\)RhuA. Such a system is subject to constraints of both holonomic and nonholonomic nature. The constraints in velocity space are handled, regardless of their nature, by
means of the MTM, which we have shown to be a simple yet powerful tool. The treatment of constraints in coordinate space is improved from other approaches by applying the exact values of the Lagrange multipliers (as taken from the acceleration constraints) to initialize the iterative process. For shorter timesteps, however, a constant-length ansatz provides a boost in performance by eliminating the need for an iterative protocol. It is worth noting that this approach is limited to short timesteps only because of the nonlinearity of the angular variables considered in our model. For a system described purely by Euclidean coordinates, the reliability of the method should allow for longer step sizes.

The data extracted from the trajectories produced using our developed method were analyzed to characterize the dynamic behavior of the geometric array. In particular, the synchronization among building blocks was evaluated as a function of the bond lengths. We demonstrate that while both sets of restraints increase the overall correlations, the holonomic constraints produce highly locally correlated motion, while the non-holonomic constraints give rise to a diffuse, lattice-wide dynamical profile. Despite this, the time required to reach the collision dominated regimes increases with the length of the linkages.

Finally, we explored the auxetic capabilities of the structures. We determined that this geometric configuration always yields a negative Poisson’s ratio. However, the presence of collisions allows for deformations that decrease the average value of the strain-strain ratio, thus producing a departure from ideal auxeticity. Nevertheless, this is expected for inhomogeneous materials.

These results strongly suggest that, to recover the experimentally observed behavior, an inter-protein potential or a dissipative mechanism should be introduced. The present method allows for a seamless inclusion of the former, as long as its functional form is known, by a simple adjustment of the SV equations. However, a
proper design that takes into account the shapes of the bodies studied remains a challenging task. In the same
fashion, this method is, in principle, compatible with any dissipative scheme with which the SV integration
is already compatible.

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