GRADIENT ESTIMATES OF MEAN CURVATURE EQUATIONS WITH NEUMANN BOUNDARY CONDITION

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Abstract. In this paper, we use the maximum principle to get the gradient estimate for the solutions of the prescribed mean curvature equation with Neumann boundary value problem, which gives a positive answer for the question raised by Lieberman [16] in page 360. As a consequence, we obtain the corresponding existence theorem for a class of mean curvature equations.

1. Introduction

Gradient estimate for the prescribed mean curvature equation has been extensively studied. The interior gradient estimate, for the minimal surface equation was obtained in the case of two variables by Finn [3]. Bombieri, De Giorgi and M.Miranda [1] obtained the estimate for high dimension case. For the general mean curvature equation, such estimate had also been obtained by Ladyzhenskaya and Ural’tseva [10], Trudinger [22] and Simon [19]. All their methods were test function argument and a resulting Sobolev inequality. In 1983, Korevaar [7] introduced the normal variation technique and got the maximum principle proof for the interior gradient estimate on the minimal surface equation. Wang [24] gave a new proof for the mean curvature equation via standard Bernstein technique.

The Dirichlet problem for the prescribed mean curvature equation had been studied by Jenkins-Serrin [6] and Serrin [18]. More detailed history could be found in Gilbarg and Trudinger [5].

For the mean curvature equation with prescribed contact angle boundary value problem, Ural’tseva [23] first got the boundary gradient estimates and the corresponding existence theorem. At the same time, Simon-Spruck [20] and Gerhardt [4] also obtained existence theorem on the positive gravity case. For more general quasilinear divergence structure equation with conormal derivative boundary value problem, Lieberman [11] gave the gradient estimate. They obtained these estimates also via test function technique.

Spruck [21] used the maximum principle to obtain boundary gradient estimate in two dimension for the positive gravity capillary problems. Korevaar [8] generalized his normal

Research of the first author was supported by NSFC No.11125105 and Wu Wen-Tsun Key Laboratory of Mathematics.

2010 Mathematics Subject Classification: Primary 35B45; Secondary 35J92, 35B50.
variation technique and got the gradient estimates for the positive gravity case in high dimension case. In \cite{12,13}, Lieberman developed the maximum principle approach on the boundary gradient estimates to the quasilinear elliptic equation with oblique derivative boundary value problem, and in \cite{14} he got the maximum principle proof for the gradient estimates on the general quasilinear elliptic equation with capillary boundary value problems.

In a recent book written by Lieberman \cite{16}, in page 360, he posed the following question, how to get the gradient estimates for the mean curvature equation with Neumann boundary value problem. In this paper we use the technique developed by Spruck \cite{21}, Lieberman \cite{14} and Wang \cite{24} to get a positive answer. As a consequence, we obtain an existence theorem for a class of mean curvature equations with Neumann boundary value problem.

We first consider the boundary gradient estimates for the mean curvature equation with Neumann boundary value problem. Now let’s state our main gradient estimates.

**Theorem 1.1.** Suppose \( u \in C^2(\Omega) \cap C^3(\Omega) \) is a bounded solution for the following boundary value problem

\[
\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = f(x, u) \quad \text{in} \quad \Omega,
\]

\[
\frac{\partial u}{\partial \gamma} = \psi(x, u) \quad \text{on} \quad \partial \Omega,
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded domain, \( n \geq 2 \), \( \partial \Omega \in C^3 \), \( \gamma \) is the inward unit normal to \( \partial \Omega \).

We assume \( f(x, z) \in C^1(\overline{\Omega} \times [-M_0, M_0]) \) and \( \psi(x, z) \in C^3(\overline{\Omega} \times [-M_0, M_0]) \), and there exist positive constants \( M_0, L_1, L_2 \) such that

\[
|u| \leq M_0 \quad \text{in} \quad \overline{\Omega},
\]

\[
f_z(x, z) \geq 0 \quad \text{in} \quad \overline{\Omega} \times [-M_0, M_0],
\]

\[
|f(x, z)| + |f_x(x, z)| \leq L_1 \quad \text{in} \quad \overline{\Omega} \times [-M_0, M_0],
\]

\[
|\psi(x, z)|_{C^3(\overline{\Omega} \times [-M_0, M_0])} \leq L_2.
\]

Then there exists a small positive constant \( \mu_0 \) such that we have the following estimate

\[
\sup_{\overline{\Omega}_{\mu_0}} |Du| \leq \max\{M_1, M_2\},
\]

where \( M_1 \) is a positive constant depending only on \( n, \mu_0, M_0, L_1 \), which is from the interior gradient estimates; \( M_2 \) is a positive constant depending only on \( n, \Omega, \mu_0, M_0, L_1, L_2 \), and \( d(x) = \text{dist}(x, \partial \Omega), \Omega_{\mu_0} = \{x \in \Omega : d(x) < \mu_0\} \).
As we stated before, there is a standard interior gradient estimates for the mean curvature equation.

**Remark 1.2** (5). If \( u \in C^3(\Omega) \) is a bounded solution for the equation (1.1) with (1.3), and if \( f \in C^1(\Omega \times [-M_0, M_0]) \) satisfies the conditions (1.4)-(1.5), then for any subdomain \( \Omega' \subset \subset \Omega \), we have

\[
\sup_{\Omega'} |Du| \leq M_1,
\]
where \( M_1 \) is a positive constant depending only on \( n, M_0, \text{dist}(\Omega', \partial \Omega), L_1 \).

From the standard bounded estimates for the prescribed mean curvature equation in Concus-Finn [2] (see also Spruck [21]), we can get the following existence theorem for the Neumann boundary value problem of mean curvature equation.

**Theorem 1.3.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain, \( n \geq 2 \), \( \partial \Omega \in C^3 \), \( \gamma \) is the inward unit normal to \( \partial \Omega \). If \( \psi \in C^3(\overline{\Omega}) \), is a given function, then the following boundary value problem

\[
\begin{align*}
\text{div}(\frac{Du}{\sqrt{1 + |Du|^2}}) = u & \quad \text{in} \quad \Omega, \\
\frac{\partial u}{\partial \gamma} = \psi(x) & \quad \text{on} \quad \partial \Omega,
\end{align*}
\]

exists a unique solution \( u \in C^2(\overline{\Omega}) \).

The rest of the paper is organized as follows. In section 2, we first give the definitions and some notations. We prove the main Theorem 1.1 in section 3 under the help of one lemma. In section 4, we give the proof of Theorem 1.3.

## 2. PRELIMINARIES

We denote by \( \Omega \) a bounded domain in \( \mathbb{R}^n \), \( n \geq 2 \), \( \partial \Omega \in C^3 \), set

\[
d(x) = \text{dist}(x, \partial \Omega),
\]
and

\[
\Omega_\mu = \{ x \in \Omega : d(x) < \mu \}.
\]

Then it is well known that there exists a positive constant \( \mu_1 > 0 \) such that \( d(x) \in C^3(\overline{\Omega}_{\mu_1}) \). As in Simon-Spruck [20] or Lieberman [16] in page 331, we can take \( \gamma = Dd \) in \( \Omega_{\mu_1} \) and note that \( \gamma \) is a \( C^2(\overline{\Omega}_{\mu_1}) \) vector field. As mentioned in [14] and the book [16], we also have the following formulas...
\[ |D\gamma| + |D^2\gamma| \leq C(n,\Omega) \quad \text{in} \quad \Omega_{\mu_1}, \]
\[ \sum_{1 \leq i \leq n} \gamma^i D_j \gamma^j = 0, \quad \sum_{1 \leq i \leq n} \gamma^i D_i \gamma^j = 0, \quad |\gamma| = 1 \quad \text{in} \quad \Omega_{\mu_1}. \]

As in [16], we define
\[ c^{ij} = \delta^{ij} - \gamma^i \gamma^j \quad \text{in} \quad \Omega_{\mu_1}, \]
and for a vector \( \zeta \in \mathbb{R}^n \), we write \( \zeta' \) for the vector with \( i \)-th component \( \sum_{1 \leq j \leq n} c^{ij} \zeta_j \). So
\[ |D'u|^2 = \sum_{1 \leq i,j \leq n} c^{ij} u_i u_j. \]

Let
\[ a^{ij}(Du) = v^2 \delta^{ij} - u_i u_j, \quad v = (1 + |Du|^2)^{\frac{1}{2}}. \]

Then the equations (1.1), (1.2) are equivalent to the following boundary value problem
\[ \sum_{i,j=1}^n a^{ij} u_{ij} = f(x,u)v^3 \quad \text{in} \quad \Omega, \]
\[ u_\gamma = \psi(x,u) \quad \text{on} \quad \partial \Omega. \]

3. Proof of Theorem 1.1

Now we begin to prove Theorem 1.1 as mentioned in introduction, using the technique developed by Spruck [21], Lieberman [14] and Wang [24]. We shall choose an auxiliary function which contains \( |Du|^2 \) and other lower order terms. Then we use the maximum principle for this auxiliary function in \( \Omega_{\mu_0}, 0 < \mu_0 < \mu_1 \). At last, we get our estimates.

**Proof of Theorem 1.1**

Setting \( w = u - \psi(x,u)d \), we choose the following auxiliary function
\[ \Phi(x) = \log |Dw|^2 e^{1+M_0+u} e^{\alpha_0 d}, \quad x \in \Omega_{\mu_0}, \]
where \( \alpha_0 = |\psi|_{C^0(\overline{\Omega} \times [-M_0, M_0])} + C_0 + 1, \) \( C_0 \) is a positive constant depending only on \( n, \Omega \).

In order to simplify the computation, we let
\[ \varphi(x) = \log \Phi(x) = \log \log |Dw|^2 + h(u) + g(d). \]

In our case, we take
\[ h(u) = 1 + M_0 + u, \quad g(d) = \alpha_0 d. \]

We assume that \( \varphi(x) \) attains its maximum at \( x_0 \in \overline{\Omega}_{\mu_0} \), where \( 0 < \mu_0 < \mu_1 \) is a sufficiently small number which we shall decide it later.

Now we divide three cases to complete the proof of Theorem 1.1.
Case I: If \( \varphi(x) \) attains its maximum at \( x_0 \in \partial \Omega \), then we shall use the Hopf Lemma to get the bound of \( |Du|(x_0) \).

Case II: If \( \varphi(x) \) attains its maximum at \( x_0 \in \partial \Omega \cap \Omega \), then we shall get the estimates via the standard interior gradient bound [5].

Case III: If \( \varphi(x) \) attains its maximum at \( x_0 \in \Omega \), in this case for the sufficiently small constant \( \mu_0 > 0 \), then we can use the maximum principle to get the bound of \( |Du|(x_0) \).

Now all computations work at the point \( x_0 \).

**Case 1.** If \( \varphi(x) \) attains its maximum at \( x_0 \in \partial \Omega \), we shall get the bound of \( |Du|(x_0) \).

We differentiate \( \varphi \) along the normal direction.

\[
\frac{\partial \varphi}{\partial \gamma} = \sum_{1 \leq i \leq n} (|Dw|^2)_i \gamma^i + h'u_\gamma + g'.
\]

(3.3)

Since

\[
w_i = u_i - \psi u_i d - \psi_{x_i} d - \psi^i,
\]

(3.4)

\[|Dw|^2 = |D'w|^2 + w_\gamma^2,\]

(3.5)

we have

\[
w_\gamma = u_\gamma - \psi u_\gamma d - \sum_{1 \leq i \leq n} \psi_{x_i} \gamma^i d - \psi = 0 \quad \text{on} \quad \partial \Omega,
\]

(3.6)

\[(|Dw|^2)_i = (|D'w|^2)_i \quad \text{on} \quad \partial \Omega.
\]

(3.7)

Applying (2.1), (2.3) and (3.7), it follows that

\[
\sum_{1 \leq i \leq n} (|Dw|^2)_i \gamma^i = \sum_{1 \leq i \leq n} (|D'w|^2)_i \gamma^i
\]

(3.8)

\[= 2 \sum_{1 \leq i,k,l \leq n} c^{lk} w_{ki} w_{l}\gamma^i
\]

\[= 2 \sum_{1 \leq i,k,l \leq n} c^{lk} u_{ki} u_{l}\gamma^i - 2 \sum_{1 \leq k,l \leq n} c^{kl} u_i D_k \psi,
\]

where

\[D_k \psi = \psi_{x_k} + \psi u_k.
\]

Differentiating (2.6) with respect to tangential direction, we have

\[
\sum_{1 \leq k \leq n} c^{kl} (u_\gamma)_k = \sum_{1 \leq k \leq n} c^{kl} D_k \psi.
\]

(3.9)

It follows that

\[
\sum_{1 \leq i,k \leq n} c^{kl} u_{ki} \gamma^i = - \sum_{1 \leq i,k \leq n} c^{kl} u_i (\gamma^i)_k + \sum_{1 \leq k \leq n} c^{kl} D_k \psi.
\]

(3.10)
Inserting (3.10) into (3.8) and combining (2.6), (3.3), we have
\[
|Dw|^2 \log |Dw|^2 \frac{\partial \varphi}{\partial \gamma}(x_0) = (g'(0) + h' \psi)|Dw|^2 - 2 \sum_{1 \leq i,k \leq n} c^{kl} u_i u_l(\gamma^i)_k.
\]
From (3.3), we obtain
\[
|Dw|^2 = |Du|^2 - \psi^2 \quad \text{on} \quad \partial \Omega.
\]
Assume \(|Du|(x_0) \geq \sqrt{100 + 2|\psi|^2_{C^0(\Omega \times [-M_0,M_0])}}\), otherwise we get the estimates. At \(x_0\), we have
\[
\frac{1}{2} |Du|^2 \leq |Dw|^2 \leq |Du|^2,
\]
\[
|Dw|^2 \geq 50.
\]
Inserting (3.13) and (3.14) into (3.11), we have
\[
|Dw|^2 \log |Dw|^2 \frac{\partial \varphi}{\partial \gamma}(x_0) \geq (\alpha_0 - |\psi|_{C^0(\Omega \times [-M_0,M_0])} - C_0)|Dw|^2 \log |Dw|^2
\]
\[
= |Dw|^2 \log |Dw|^2 > 0.
\]
On the other hand, from Hopf Lemma, we have
\[
\frac{\partial \varphi}{\partial \gamma}(x_0) \leq 0,
\]
it is a contradiction to (3.15).

Then we have
\[
|Du|(x_0) \leq \sqrt{100 + 2|\psi|^2_{C^0(\Omega \times [-M_0,M_0])}}.
\]

Case 2. \(x_0 \in \partial \Omega_{\mu_0} \cap \Omega\). This is due to interior gradient estimates. From Remark 1.2 we have
\[
\sup_{\partial \Omega_{\mu_0} \cap \Omega} |Du| \leq \tilde{M}_1.
\]
where \(\tilde{M}_1\) is a positive constant depending only on \(n, M_0, \mu_0, L_1\).

Case 3. \(x_0 \in \Omega_{\mu_0}\).
In this case, \(x_0\) is a critical point of \(\varphi\). We choose the normal coordinate at \(x_0\), by rotating the coordinate system suitably, we may assume that \(u_i(x_0) = 0, 2 \leq i \leq n\) and \(u_1(x_0) = |Du| > 0\). And we can further assume that the matrix \((u_{ij}(x_0))(2 \leq i,j \leq n)\) is diagonal. Let
\[
\mu_2 \leq \frac{1}{100 L_2}
\]
such that
\[ |\psi_u| \mu_2 \leq \frac{1}{100}, \quad \text{then} \quad \frac{99}{100} \leq 1 - \psi_u \mu_2 \leq \frac{101}{100}. \]

We can choose
\[ \mu_0 = \frac{1}{2} \min\{\mu_1, \mu_2, 1\}. \]

In order to simplify the calculations, we let
\[ w = u - G, \quad G = \psi(x, u) d. \]

Then we have
\[ w_k = (1 - G_u) u_k - G_{x_k}. \]

Since at \( x_0 \),
\[ |Dw|^2 = w_1^2 + \sum_{2 \leq i \leq n} w_i^2, \]
\[ w_i = -G_{x_i} = -\psi_{x_i} d - \psi \gamma^i, \quad i = 2, \ldots, n, \]
\[ w_1 = (1 - G_u) u_1 - G_{x_1} = (1 - G_u) u_1 - \psi_{x_1} d - \psi \gamma^1. \]

So from the above relation, at \( x_0 \), we can assume
\[ u_1 = |Du|(x_0) \geq \sqrt{3000(1 + |\psi|^2_{C^1(\Omega \times [-M_0, M_0])})}, \]
then
\[ \frac{9}{10} u_1^2 \leq |Dw|^2 \leq \frac{11}{10} u_1^2, \quad \frac{9}{10} u_1^2 \leq w_1^2 \leq \frac{11}{10} u_1^2 \]
and by the choice of \( \mu_0 \) and (3.18), we have
\[ \frac{99}{100} \leq 1 - G_u \leq \frac{101}{100}. \]

From the above choice, we shall prove Theorem 1.1 with three steps, as we mentioned before, all the calculations will be done at the fixed point \( x_0 \).

**Step 1:** We first get the formula (3.53).

Taking the first derivatives of \( \varphi \),
\[ \varphi_i = \frac{(|Dw|_i^2)}{|Dw|^2 \log |Dw|^2} + h' u_i + g' \gamma^i. \]

From \( \varphi_i(x_0) = 0 \), we have
\[ (|Dw|_i^2) = -|Dw|^2 \log |Dw|^2 (h' u_i + g' \gamma^i). \]

Take the derivatives again for \( \varphi_i \),
\[ \varphi_{ij} = \frac{(|Dw|_{ij}^2)}{|Dw|^2 \log |Dw|^2} - (1 + \log |Dw|^2) \frac{(|Dw|_i^2)(|Dw|_j^2)}{|(Dw|_i^2)(Dw|_j^2)} \right) 
+ h' u_{ij} + h'' u_i u_j + g'' \gamma^i \gamma^j + g' (\gamma^i)_j. \]
Using (3.27), it follows that

\begin{equation}
\varphi_{ij} = \frac{(|Dw|^2)_{ij}}{|Dw|^2 \log |Dw|^2} + h'u_{ij} + \left[ h'' - (1 + \log |Dw|^2)h'^2 \right] u_i u_j \\
+ \left[ g'' - (1 + \log |Dw|^2)g'^2 \right] \gamma^i \gamma^j - (1 + \log |Dw|^2)h' g'(\gamma^i u_j + \gamma^j u_i) + g'(\gamma^i)\gamma^j.
\end{equation}

Then we get

\begin{equation}
0 \geq \sum_{1 \leq i, j \leq n} a^{ij} \varphi_{ij} =: I_1 + I_2,
\end{equation}

where

\begin{equation}
I_1 = \frac{1}{|Dw|^2 \log |Dw|^2} \sum_{1 \leq i, j \leq n} a^{ij} (|Dw|^2)_{ij},
\end{equation}

and

\begin{equation}
I_2 = \sum_{1 \leq i, j \leq n} a^{ij} \left\{ h'u_{ij} + \left[ h'' - (1 + \log |Dw|^2)h'^2 \right] u_i u_j + \left[ g'' - (1 + \log |Dw|^2)g'^2 \right] \gamma^i \gamma^j \\
- 2(1 + \log |Dw|^2)h' g'(\gamma^i u_j + \gamma^j u_i) \right\}.
\end{equation}

From the choice of the coordinate, we have

\begin{equation}
a^{11} = 1, \ a^{ii} = v^2 = 1 + u^2_1 \ (2 \leq i \leq n), \ a^{ij} = 0 \ (i \neq j, 1 \leq i, j \leq n).
\end{equation}

Now we first treat $I_2$.

From the choice of the coordinate and the equations (2.5), (3.33), we have

\begin{equation}
I_2 = h' f v^3 - h'^2 u_1^2 \log |Dw|^2 + (h'' - h'^2) u_1^2 + \left[ g'' - (1 + \log |Dw|^2)g'^2 \right] \sum_{1 \leq i \leq n} a^{ii} (\gamma^i)^2 \\
- 2(1 + \log |Dw|^2)h' g'(\gamma^i) u_1 + g' \sum_{1 \leq i \leq n} a^{ii}(\gamma^i) i \\
= h' f v^3 - (h'^2 + c^{11} g'^2) u_1^2 \log |Dw|^2 + (h'' - h'^2 + c^{11}(g'' - g'^2) + g' \sum_{2 \leq i \leq n} (\gamma^i) i) u_1^2 \\
- 2h' g' \gamma^1 u_1 \log |Dw|^2 - 2h' g' \gamma^1 u_1 + g'' - g'^2 + g' \sum_{1 \leq i \leq n} (\gamma^i) i.
\end{equation}

Hence from (3.2), we have

\begin{equation}
I_2 = f v^3 - (1 + c^{11} \alpha_0^2) u_1^2 \log |Dw|^2 + \left[ -1 - c^{11} \alpha_0^2 + \alpha_0 \sum_{2 \leq i \leq n} (\gamma^i) i \right] u_1^2 \\
- 2\alpha_0 \gamma^1 u_1 \log |Dw|^2 - 2\alpha_0 \gamma^1 u_1 - \alpha_0^2 + \alpha_0 \sum_{1 \leq i \leq n} (\gamma^i) i \\
\geq f v^3 - (1 + c^{11} \alpha_0^2) u_1^2 \log |Dw|^2 - C_1 u_1^2.
\end{equation}
here $C_1$ is a positive constant depending only on $n, \Omega, M_0, \mu_0, L_2$.

Next, we calculate $I_1$ and get the formula (3.52).

Taking the first derivatives of $|Dw|^2$, we have

$$\begin{aligned}
( |Dw|^2 )_i &= 2 \sum_{1 \leq k \leq n} w_k w_{ki}. \\
\end{aligned}$$

(3.36)

Taking the derivatives of $|Dw|^2$ once more, we have

$$\begin{aligned}
( |Dw|^2 )_{ij} &= 2 \sum_{1 \leq k \leq n} w_k w_{kij} + 2 \sum_{1 \leq k \leq n} w_{ki} w_{kj}. \\
\end{aligned}$$

(3.37)

By (3.31) and (3.37), we can rewrite $I_1$ as

$$\begin{aligned}
I_1 &= \frac{1}{|Dw|^2 \log |Dw|^2} \left[ 2 \sum_{1 \leq i,j,k \leq n} w_k a^{ij} w_{ijk} + 2 \sum_{1 \leq i,j,k \leq n} a^{ij} w_{ki} w_{kj} \right] \\
&=: \frac{1}{|Dw|^2 \log |Dw|^2} [I_{11} + I_{12}] . \\
\end{aligned}$$

(3.38)

In the following, we shall deal with $I_{11}$ and $I_{12}$ respectively.

For the term $I_{11}$. As we have let

$$w = u - G, \quad G = \psi(x, u)d.$$

(3.39)

Then we have

$$\begin{aligned}
w_k &= (1 - G_u) u_k - G_{x_k}, \\
w_{ki} &= (1 - G_u) u_{ki} - G_{uu} u_k u_i - G_{ux_k} u_k - G_{ux_k u_i} - G_{x_k u_i}, \\
w_{kij} &= (1 - G_u) u_{kij} - G_{uu} (u_k u_j + u_{kj} u_i + u_{ij} u_k) \\
&\quad - G_{ux_i} u_{kj} - G_{ux_j} u_{ki} - G_{ux_k u_ij} \\
&\quad - G_{uu} u_k u_i u_j - G_{ux_i} u_j u_k - G_{ux_j} u_i u_k - G_{ux_k u_i u_j} \\
&\quad - G_{ux_i x_j} u_k - G_{ux_k x_j} u_i - G_{ux_i x_k} u_j - G_{x_i x_j x_k} . \\
\end{aligned}$$

(3.40)
So from the choice of the coordinate and the equations (2.5), (3.33), we have
\[\sum_{1 \leq i, j, k \leq n} w_k a^{ij} w_{ijk} = \sum_{1 \leq i, j, k \leq n} [(1 - G_a)w_k a^{ij} w_{ijk} - G_{uuw} w_k a^{ij} u_k u_j - 2G_{uw} w_k a^{ij} u_k u_j - w_k G_{ux} a^{ij} u_i u_j - 2w_k u_k a^{ij} G_{uxi} u_j - w_k u_k a^{ij} G_{uxj} u_i - 2w_k a^{ij} G_{ux,xk} u_j - w_k a^{ij} G_{x,x,xk}]\]
\[= (1 - G_a) \sum_{1 \leq i, j, k \leq n} w_k a^{ij} w_{ijk} - 2(G_{uuw} u_1 + G_{ux1}) \sum_{1 \leq k \leq n} w_k u_k 1 \]
\[+ 2v^2 \sum_{2 \leq i, l \leq n} G_{ux} \sum_{1 \leq k \leq n} w_k u_k 1 - G_{uuw} f v^3 u_1 w_1 - G_{uuw} u_1^3 w_1\]
\[- v^2 u_1 w_1 \sum_{2 \leq i \leq n} G_{ux,x_i} - f v^3 \sum_{1 \leq k \leq n} w_k G_{ux_k} - 2G_{ux1} u_1^2 w_1\]
\[- u_1^2 \sum_{1 \leq k \leq n} w_k G_{uxk} - v^2 \sum_{1 \leq k \leq n} w_k \sum_{2 \leq i \leq n} G_{x_i,x_k} - G_{ux1} u_1 w_1\]
\[- 2u_1 \sum_{1 \leq k \leq n} G_{ux1x_k} w_k - \sum_{1 \leq k \leq n} G_{x_1x_k} w_k.\]

(3.42)

By the equation (2.5), we have
\[u_{11} = f v^3 - v^2 \sum_{2 \leq i \leq n} u_{ii},\]
and
\[\Delta u = f v + \frac{u_1^2}{v^2} u_{11}.\]

(3.43)

(3.44)

Differentiating (2.5), we have
\[\sum_{1 \leq i, j \leq n} a^{ij} u_{ijk} = - \sum_{1 \leq i, j, l \leq n} a^{ij} u_{ik} u_{ij} + v^3 D_k f + 3f v^2 v_k.\]

(3.45)

From (2.4), we have
\[a^{ij}_{p_l} = 2u_l \delta_{ij} - \delta_{i} u_j - \delta_{j} u_i,\]

(3.46)

By the definition of \(v\), we have
\[v v_{k} = u_{11} u_{1k}.\]

(3.47)

Since
\[D_k f = f u_k + f_{x_k},\]

(3.48)
from (3.46), (3.43) and (3.47), we have
\[ \sum_{1 \leq i, j \leq n} a^{ij} u_{ijk} = -2u_1 u_{1k} u + 2u_1 \sum_{1 \leq i \leq n} u_{1i} u_{ik} + v^3 D_k f + 3f v u_1 u_{1k}, \]
\[ = \frac{2u_1}{v^2} u_{11} u_{1k} + 2u_1 \sum_{1 \leq i \leq n} u_{1i} u_{ik} + v^3 D_k f + f v u_1 u_{1k} \]
\[ = \frac{2u_1}{v^2} u_{11} u_{1k} + 2u_1 \sum_{2 \leq i \leq n} u_{1i} u_{ik} + f u v^3 u_k + f x_k v^3 + f v u_1 u_{1k}. \]

Inserting (3.49) into (3.42), we rewrite \( I_{11} \) as
\[ I_{11} = 4(1 - G_u) u_1 u_{11} \sum_{1 \leq k \leq n} w_k u_{k1} + 4(1 - G_u) u_1 \sum_{2 \leq i \leq n} u_{1i} \sum_{1 \leq k \leq n} w_k u_{ki} \]
\[ + [2(1 - G_u)f u_1 v - 4G u u u_1 - 4G u x_1] \sum_{1 \leq k \leq n} w_k u_{k1} - 4v^2 \sum_{2 \leq i \leq n} G u x_i \sum_{1 \leq k \leq n} w_k u_{ki} \]
\[ + 2(1 - G_u) f u v^3 u_1 w_1 - 2G u u f v^3 u_1 w_1 + 2(1 - G_u) v^3 \sum_{1 \leq k \leq n} f x_k w_k \]
\[ - 2f v^3 \sum_{1 \leq k \leq n} G u x_k w_k - 2G u x_1 u_1^2 w_1 - 2v^2 u_1 w_1 \sum_{2 \leq i \leq n} G u x_i x_i - 4G u x_1 x_1 u_1^2 w_1 \]
\[ - 2u_1^2 \sum_{1 \leq k \leq n} w_k G u x_k - 2v^2 \sum_{1 \leq k \leq n} w_k \sum_{2 \leq i \leq n} G x_i x_i - 2G u x_1 x_1 u_1 w_1 \]
\[ - 4u_1 \sum_{1 \leq k \leq n} G u x_1 x_k w_k. \]

For the term \( I_{12} \): applying (3.33) and (3.40), we have
\[ I_{12} = 2 \sum_{1 \leq i,k \leq n} a^{ii} w_{ki}^2 \]
\[ = 2 \sum_{1 \leq k \leq n} w_{ki}^2 + 2v^2 \sum_{1 \leq k \leq n} \sum_{2 \leq i \leq n} w_{ki}^2 \]
\[ = 2w_{11}^2 + 2v^2 + 1 \sum_{2 \leq i \leq n} w_{1i}^2 + 2v^2 \sum_{2 \leq i,k \leq n} w_{ki}^2 \]
\[ = 2[(1 - G_u) u_{11} - (G u u u_1^2 + 2G u x_1 u_1 + G x_1 x_1)]^2 \]
\[ + 2(v^2 + 1) \sum_{2 \leq i \leq n} [(1 - G_u) u_{1i} - (G u x_i u_1 + G x_1 x_i)]^2 \]
\[ + 2v^2 \sum_{2 \leq i,k \leq n} [(1 - G_u) u_{ki} - G x_k x_i]^2 \]
\[ = 2(1 - G_u)^2 u_{11}^2 + 2(1 - G_u)^2 (v^2 + 1) \sum_{2 \leq i \leq n} u_{1i}^2 + 2(1 - G_u)^2 v^2 \sum_{2 \leq i \leq n} u_{ii}^2. \]
\[-4(1 - G_u)(G_{uu} u_1^2 + 2G_{ux} u_1 + G_{x1} u_1)u_{11} \]
\[-4(1 - G_u)(u^2 + 1) \sum_{2 \leq i \leq n} (G_{ux} u_1 + G_{x1} u_1)u_{1i} \]
\[-4(1 - G_u)u^2 \sum_{2 \leq i \leq n} G_{x_i x_i} u_{ii} + 2(G_{uu} u_1^2 + 2G_{ux} u_1 + G_{x1} u_1)^2 \]
\[(3.51) \quad + 2(u^2 + 1) \sum_{2 \leq i \leq n} (G_{ux} u_1 + G_{x1} u_1)^2 + 2u^2 \sum_{2 \leq i, k \leq n} G_{x_k x_i} \]

Combining (3.50), (3.51), it follows that

\[ I_1 = \frac{1}{|Dw|^2 \log |Dw|^2} \left[ 2(1 - G_u)^2 u_{11}^2 + 2(1 - G_u)^2 (v^2 + 1) \sum_{2 \leq i \leq n} u_{ii}^2 \right. \]
\[ + 2(1 - G_u)^2 v^2 \sum_{2 \leq i \leq n} u_{ii}^2 + 4(1 - G_u) \frac{u_1}{v^2} u_{11} \sum_{1 \leq k \leq n} w_k u_{k1} \]
\[ + 4(1 - G_u)u_1 \sum_{2 \leq i \leq n} u_{ii} \sum_{1 \leq k \leq n} w_k u_{ki} \]
\[ + \left| 2(1 - G_u) f u_1 v - 4G_{uu} u_1 - 4G_{ux} u_1 \right| \sum_{1 \leq k \leq n} w_k u_{k1} \]
\[ - 4u^2 \sum_{2 \leq i \leq n} G_{ux} \sum_{1 \leq k \leq n} w_k u_{ki} \]
\[ - 4(1 - G_u)(G_{uu} u_1^2 + 2G_{ux} u_1 + G_{x1} u_1) u_{11} \]
\[ - 4(1 - G_u)(v^2 + 1) \sum_{2 \leq i \leq n} (G_{ux} u_1 + G_{x1} u_1) u_{1i} \]
\[ - 4(1 - G_u)v^2 \sum_{2 \leq i \leq n} G_{x_i x_i} u_{ii} + 2(1 - G_u) f u_1 v^3 u_1 u_1 - 2G_{uu} f v^3 u_1 u_1 \]
\[ + 2(G_{uu} u_1^2 + 2G_{ux} u_1 + G_{x1} u_1)^2 + 2(v^2 + 1) \sum_{2 \leq i \leq n} (G_{ux} u_1 + G_{x1} u_1)^2 \]
\[ + 2v^2 \sum_{2 \leq i, k \leq n} G_{x_k x_i} + 2(1 - G_u) v^3 \sum_{1 \leq k \leq n} f_{x_k} w_k - 2f v^3 \sum_{1 \leq k \leq n} G_{ux} w_k \]
\[ - 2G_{uu} u_1^3 w_1 - 2v^2 u_1 w_1 \sum_{2 \leq i \leq n} G_{ux} u_{1i} - 4G_{uu} u_1^3 u_{21}^2 w_1 - 2u_1^2 \sum_{1 \leq k \leq n} w_k G_{ux} w_k \]
\[ - 2v^2 \sum_{1 \leq k \leq n} w_k \sum_{2 \leq i \leq n} G_{x_i x_k} - 2G_{ux} u_{11} u_1 w_1 \]
\[(3.52) \quad - 4u_1 \sum_{1 \leq k \leq n} G_{ux} u_{1k} w_k - 2 \sum_{1 \leq k \leq n} G_{x1} u_{1k} w_k \].
Inserting (3.52) and (3.34) into (3.30), we can obtain the following formula

\[ Q_1 = \frac{1}{|Dw|^2 \log |Dw|^2} \left[ 2(1 - G_u)^2 u_1^2 + 2(1 - G_u)^2(v^2 + 1) \sum_{2 \leq i \leq n} u_i^2 + 4(1 - G_u) \sum_{1 \leq k \leq n} w_k u_{k1} + 4(1 - G_u) u_1 \sum_{2 \leq i \leq n} u_i \sum_{1 \leq k \leq n} w_k u_{ki} \right] + 2(1 - G_u)^2 v^2 \sum_{2 \leq i \leq n} u_i^2; \]

(3.54)

The linear terms of \( u_{ij} \) are

\[ Q_2 = \frac{1}{|Dw|^2 \log |Dw|^2} \left[ 2(1 - G_u) f u_1 v - 4G_{uu} u_1 - 4G_{ux} \right] \sum_{1 \leq k \leq n} w_k u_{k1} - 4v^2 \sum_{2 \leq i \leq n} G_{ux,i} \sum_{1 \leq k \leq n} w_k u_{ki} - 4(1 - G_u)(G_{uu} u_1^2 + 2G_{ux} u_1 + G_{x1,x1}) u_1 \]

(3.55)

and the remaining terms are

\[ Q_3 = I_2 + \frac{1}{|Dw|^2 \log |Dw|^2} \left[ 2(1 - G_u) f u_1^3 w_1 - 2G_{uu} f u_1^3 u_1 w_1 + \sum_{2 \leq i \leq n} u_i^2 \sum_{2 \leq i \leq n} G_{ux,i} w_k - 2f v^3 \sum_{1 \leq k \leq n} G_{ux,k} x_{k1}^2 w_1 - 2v^2 \sum_{1 \leq k \leq n} w_k G_{ux,k} x_{k1}^2 - 2v^2 \sum_{2 \leq i \leq n} G_{ux,i} x_{k1}^2 - 2G_{ux1} u_1^2 \right] \]

(3.56)

From the estimate on \( I_2 \) in (3.35), we have

\[ Q_3 \geq f v^3 - \frac{2f G_{uu}}{|Dw|^2 \log |Dw|^2} v^3 u_1 w_1 - (1 + c^{11} \alpha^2) u_1^2 \log |Dw|^2 - C_2 u_1^2, \]

(3.57)
in the computation of \( Q_3 \), we use the relation \( f_u \geq 0 \), where \( C_2 \) is a positive constant which depends only on \( n, \Omega, M_0, \mu_0, L_1, L_2 \).

**Step 2:** In this step we shall treat the terms \( Q_1, Q_2 \), using the first order derivative condition

\[
\varphi_i(x_0) = 0,
\]

and let

\[
(3.58) \quad A = |Dw|^2 \log |Dw|^2.
\]

By (3.27) and (3.30), we have

\[
(3.59) \quad \sum_{1 \leq k \leq n} w_kw_{ki} = -\frac{h'}{2} |Dw|^2 \log |Dw|^2 u_i - \frac{g'\gamma_i}{2} |Dw|^2 \log |Dw|^2
\]

Putting (3.40) into (3.59), we get

\[
(1 - G_u) \sum_{1 \leq k \leq n} w_ku_k = -\frac{h'}{2} Au_i - \frac{g'\gamma_i}{2} A + G_{uu} \sum_{1 \leq k \leq n} w_k u_k u_i + \sum_{1 \leq k \leq n} w_k u_k G_{ux_i}
\]

\[
(3.60) \quad + \sum_{1 \leq k \leq n} w_k G_{ux_k} u_i + \sum_{1 \leq k \leq n} w_k G_{x_k x_i}
\]

\[
= -\frac{h'}{2} Au_i - \frac{g'\gamma_i}{2} A + G_{uu} w_1 u_1 u_i + w_1 u_1 G_{ux_i}
\]

\[
+ \sum_{1 \leq k \leq n} w_k G_{ux_k} u_i + \sum_{1 \leq k \leq n} w_k G_{x_k x_i}, \quad i = 1, 2, \ldots, n.
\]

By (3.60), we have

\[
(3.61) \quad (1 - G_u) \sum_{1 \leq k \leq n} w_k u_k = -\frac{h'}{2} Au_1 - \frac{g'\gamma_1}{2} A + G_{uu} w_1^2 u_1 + G_{ux_1} w_1 u_1
\]

\[
\quad + \sum_{1 \leq k \leq n} w_k G_{ux_k} u_1 + \sum_{1 \leq k \leq n} w_k G_{x_k x_1},
\]

and

\[
(3.62) \quad (1 - G_u) \sum_{1 \leq k \leq n} w_ku_k = -\frac{g'\gamma_i}{2} A + G_{ux_i} w_1 u_1 + \sum_{1 \leq k \leq n} w_k G_{x_k x_i}, \quad i = 2, \ldots, n.
\]

Through (3.62) and the choice of the coordinate at \( x_0 \), we have

\[
(3.63) \quad u_1 = -\frac{1}{w_1} w_i u_{ii} = \frac{g'\gamma_i}{2(1 - G_u)} \frac{A}{w_1} + \frac{G_{ux_i}}{1 - G_u} u_1 + \frac{1}{(1 - G_u) w_1} \sum_{1 \leq k \leq n} w_k G_{x_k x_i}, \quad i = 2, 3, \ldots, n.
\]
Using (3.61) and (3.63), it follows that

\begin{align}
\sum_{2 \leq i \leq n} w_i^2 \frac{w_i}{w_1} u_{ii} &= \frac{h'}{2(1 - G_u)} \frac{Au_1}{w_1} + \frac{G_uG_u u_1^2}{1 - G_u} - \frac{g' \gamma}{2} \frac{A}{w_1} \\
&+ \frac{G_{ux} u_1}{1 - G_u} + \frac{u_1}{(1 - G_u)w_1} \sum_{1 \leq k \leq n} w_k G_{ux_k} + \frac{g'}{2} \frac{A}{w_1^2} \sum_{2 \leq i \leq n} w_i \gamma_i \\
&- \frac{u_1}{(1 - G_u)w_1} \sum_{2 \leq i \leq n} G_{ux_i} w_i + \frac{1}{(1 - G_u)w_1} \sum_{1 \leq k \leq n} w_k G_{x_k x_1} \\
&- \frac{1}{(1 - G_u)w_1^2} \sum_{1 \leq k, i \leq n} w_i w_k G_{x_k x_i} \\
&= : \sum_{2 \leq i \leq n} w_i^2 \frac{w_i}{w_1} u_{ii} - \frac{h'}{2(1 - G_u)} \frac{Au_1}{w_1} + \frac{D}{1 - G_u},
\end{align}

(3.64)

where we have let

\begin{align}
D &= G_{uu} u_1^2 - \frac{g' \gamma}{2} \frac{A}{w_1} + G_{ux} u_1 + \frac{u_1}{w_1} \sum_{1 \leq k \leq n} w_k G_{ux_k} + \frac{g'}{2} \sum_{2 \leq i \leq n} w_i \gamma_i \frac{A}{w_1^2} \\
&- \frac{u_1}{w_1} \sum_{2 \leq i \leq n} G_{ux_i} w_i + \frac{1}{w_1} \sum_{1 \leq k \leq n} w_k G_{x_k x_1} - \frac{1}{w_1^2} \sum_{1 \leq k, i \leq n} w_i w_k G_{x_k x_i}.
\end{align}

(3.65)

It follows that

\begin{align}
|D| \leq C_3 u_1^2.
\end{align}

(3.66)

By (3.63) and (3.64), we have

\begin{align}
\sum_{2 \leq i \leq n} (v^2 + \frac{w_i^2}{w_1}) u_{ii} &= f v^3 + \frac{h'}{2(1 - G_u)} \frac{Au_1}{w_1} - \frac{D}{1 - G_u}.
\end{align}

(3.67)

Now we use the formulas (3.61), (3.64) to treat each term in $Q_1, Q_2$. At first, we treat the first four terms of $Q_1$ in (3.54), and get (3.68)-(3.71).
By (3.64), we have

\[
2(1 - G_u)^2 u_{11}^2 = 2(1 - G_u)^2 \left[ \sum_{2 \leq i \leq n} \frac{w_i^2}{w_1^2} u_{ii} - \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} + \frac{D}{1 - G_u} \right]^2
\]

\[
= 2(1 - G_u)^2 \left( \sum_{2 \leq i \leq n} \frac{w_i^2}{w_1^2} u_{ii} \right)^2 - 2(1 - G_u)^2 \left( h' \frac{A u_1}{w_1} - \frac{D}{w_1} \right) \sum_{2 \leq i \leq n} w_i^2 u_{ii}
\]

\[
+ \frac{h'^2 A^2 u_1^2}{2 w_1^2} - 2h' \frac{A u_1}{w_1} D + 2D^2
\]

\[
= 2(1 - G_u)^2 \left( \sum_{2 \leq i \leq n} \frac{w_i^2}{w_1^2} u_{ii} \right)^2 - 2(1 - G_u)^2 \left( h' \frac{A u_1}{w_1} - \frac{D}{w_1} \right) \sum_{2 \leq i \leq n} w_i^2 u_{ii}
\]

(3.68)

\[
+ \frac{h'^2 A^2 u_1^2}{2 w_1^2} + AO(u_1^2).
\]

By (3.63), we have

\[
2(1 - G_u)^2 (v^2 + 1) \sum_{2 \leq i \leq n} u_{ii}^2
\]

\[
= 2(v^2 + 1) \sum_{2 \leq i \leq n} \left[ (1 - G_u) \frac{w_i}{w_1} u_{ii} + \frac{g'^2 A}{2} - G_{ux_i} u_1 - \frac{1}{w_1} \sum_{1 \leq k \leq n} w_k G_{x_k x_i} \right]^2
\]

\[
= 2(1 - G_u)^2 (v^2 + 1) \sum_{2 \leq i \leq n} w_i^2 u_{ii}^2 + 2(1 - G_u) g' A \frac{v^2 + 1}{w_1^2} \sum_{2 \leq i \leq n} w_i^2 \gamma_i u_{ii}
\]

\[
- 4(1 - G_u) \frac{v^2 + 1}{w_1^2} \sum_{2 \leq i \leq n} \left( G_{ux_i} u_1 w_1 + \sum_{1 \leq k \leq n} w_k G_{x_k x_i} \right) w_i u_{ii}
\]

\[
+ \frac{c_{11}^2}{2} g'^2 \frac{v^2 + 1}{w_1^2} A^2 - 2g' A \frac{v^2 + 1}{w_1^2} \sum_{2 \leq i \leq n} \left( G_{ux_i} u_1 w_1 + \sum_{1 \leq k \leq n} w_k G_{x_k x_i} \right) \gamma_i
\]

\[
+ \frac{2(v^2 + 1)}{w_1^2} \sum_{2 \leq i \leq n} \left( G_{ux_i} u_1 w_1 + \sum_{1 \leq k \leq n} w_k G_{x_k x_i} \right)^2
\]

\[
= 2(1 - G_u)^2 (v^2 + 1) \sum_{2 \leq i \leq n} w_i^2 u_{ii}^2 + 2(1 - G_u) g' A \frac{v^2 + 1}{w_1^2} \sum_{2 \leq i \leq n} w_i^2 \gamma_i u_{ii}
\]

\[
- 4(1 - G_u) \frac{v^2 + 1}{w_1^2} \sum_{2 \leq i \leq n} \left( G_{ux_i} u_1 w_1 + \sum_{1 \leq k \leq n} w_k G_{x_k x_i} \right) w_i u_{ii}
\]

(3.69)

\[
+ \frac{c_{11}^2}{2} g'^2 \frac{v^2 + 1}{w_1^2} A^2 + AO(u_1^2).
\]
By (3.61) and (3.61), we have
\begin{align*}
4(1 - G_u) & \frac{u_1}{v^2} u_{11} \sum_{1 \leq k \leq n} w_k u_k \\
= & \frac{4u_1}{v^2} \left( \sum_{2 \leq i \leq n} \frac{w_i^2}{v^2} w_{ii} - \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} + \frac{D}{1 - G_u} \right) \left[ - \frac{h'}{2} A u_1 - \frac{g' \gamma^i}{2} A + G_{uu} u_1^2 w_1 \\
& + G_{ux} w_1 u_1 + \sum_{1 \leq k \leq n} w_k G_{uxk} u_1 + \sum_{1 \leq k \leq n} w_k G_{xk1} \right] \\
= & \left[ - 2h' A u_1^2 \frac{u_1^2}{v^2 w_1^2} + 4G_{uu} \frac{u_1^2}{v^2 w_1^2} - 2g' \gamma^1 A u_1 \frac{u_1^2}{v^2 w_1^2} + 4G_{ux} \frac{u_1^2}{v^2 w_1^2} + 4 \sum_{1 \leq k \leq n} w_k G_{uxk} \frac{u_1^2}{v^2 w_1^2} \\
& + 4 \sum_{1 \leq k \leq n} w_k G_{xk1} \frac{u_1}{v^2 w_1^2} \sum_{2 \leq i \leq n} \frac{w_i^2}{v^2} w_{ii} + \frac{h'^2}{1 - G_u} \sum_{1 \leq k \leq n} w_k G_{xk1} \gamma^1 A - \frac{4u_1}{v^2} \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} \right. \\
& \left. - \sum_{1 \leq k \leq n} w_k G_{uxk} u_1 - \sum_{1 \leq k \leq n} w_k G_{xk1} \right] \\
= & \left[ - 2h' A u_1^2 \frac{u_1^2}{v^2 w_1^2} + 4G_{uu} \frac{u_1^2}{v^2 w_1^2} - 2g' \gamma^1 A u_1 \frac{u_1^2}{v^2 w_1^2} + 4G_{ux} \frac{u_1^2}{v^2 w_1^2} + 4 \sum_{1 \leq k \leq n} w_k G_{uxk} \frac{u_1^2}{v^2 w_1^2} \\
& + 4 \sum_{1 \leq k \leq n} w_k G_{xk1} \frac{u_1}{v^2 w_1^2} \sum_{2 \leq i \leq n} \frac{w_i^2}{v^2} w_{ii} + \frac{h'^2}{1 - G_u} \sum_{1 \leq k \leq n} w_k G_{xk1} \gamma^1 A - \frac{4u_1}{v^2} \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} \right. \\
& \left. - \sum_{1 \leq k \leq n} w_k G_{uxk} u_1 - \sum_{1 \leq k \leq n} w_k G_{xk1} \right] \\
= & \left[ - 2h' A u_1^2 \frac{u_1^2}{v^2 w_1^2} + 4G_{uu} \frac{u_1^2}{v^2 w_1^2} - 2g' \gamma^1 A u_1 \frac{u_1^2}{v^2 w_1^2} + 4G_{ux} \frac{u_1^2}{v^2 w_1^2} + 4 \sum_{1 \leq k \leq n} w_k G_{uxk} \frac{u_1^2}{v^2 w_1^2} \\
& + 4 \sum_{1 \leq k \leq n} w_k G_{xk1} \frac{u_1}{v^2 w_1^2} \sum_{2 \leq i \leq n} \frac{w_i^2}{v^2} w_{ii} + \frac{h'^2}{1 - G_u} \sum_{1 \leq k \leq n} w_k G_{xk1} \gamma^1 A - \frac{4u_1}{v^2} \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} \right. \\
& \left. - \sum_{1 \leq k \leq n} w_k G_{uxk} u_1 - \sum_{1 \leq k \leq n} w_k G_{xk1} \right] + A O(u_1^2).
\end{align*}

By (3.63) and (3.62), we have
\begin{align*}
4(1 - G_u) & u_1 \sum_{2 \leq i \leq n} u_{ii} \sum_{1 \leq k \leq n} w_k u_k \\
= & 4u_1 \sum_{2 \leq i \leq n} \left[ - \frac{g' \gamma^i}{2} A + G_{ux} w_1 u_1 + \sum_{1 \leq k \leq n} w_k G_{xk1} \right] - \frac{1}{w_1} \sum_{1 \leq k \leq n} w_k u_{ki} - \frac{g' \gamma^i}{2(1 - G_u)} A \\
& + \frac{G_{ux} w_1}{1 - G_u} + \frac{1}{(1 - G_u) w_1} \sum_{1 \leq k \leq n} w_k G_{xk1} \\
= & 2g' \frac{A u_1}{w_1} \sum_{2 \leq i \leq n} w_i \gamma^i u_{ii} - 4u_1^2 \sum_{2 \leq i \leq n} w_i G_{ux} w_1 u_{ii} - 4u_1 \sum_{2 \leq i \leq n} w_i G_{xk1} u_{ii} \\
& + \frac{c_{11} s_{12} A^2 u_1}{1 - G_u} w_1 - \sum_{2 \leq i \leq n} \left[ 2g' \gamma^i A u_1 - 4G_{ux} w_1 u_1^2 - 4u_1 \sum_{1 \leq k \leq n} w_k G_{xk1} \right] \frac{G_{ux}}{1 - G_u} \frac{u_1}{w_1} \\
& + \frac{1}{(1 - G_u) w_1} \sum_{1 \leq k \leq n} w_k G_{xk1} - \frac{2g' \gamma^i u_1}{1 - G_u} w_1 \sum_{2 \leq i \leq n} \left[ G_{ux} w_1 u_1 + \sum_{1 \leq k \leq n} w_k G_{xk1} \right] \gamma^i.
\end{align*}
\[
\begin{align*}
&= 2g' \frac{A_{x_1}}{w_1} \sum_{2 \leq i \leq n} w_i \gamma' u_{ii} - 4u_1^2 \sum_{2 \leq i \leq n} w_i G_{ux}, u_{ii} - \frac{4u_1}{w_1} \sum_{1 \leq k \leq n} w_k \sum_{2 \leq i \leq n} G_{x_k x_i} u_i u_{ii} \\
&\quad + \frac{c_{11} g^2}{1 - G_u} A_{x_1}^2 u_1 + AO(u_1^2),
\end{align*}
\]

(3.71)

Now we treat the first four terms of \(Q_2\) in (3.55), and get (3.72)-(3.75).

From (3.61), we get

\[
\begin{align*}
&\left[ 2(1 - G_u)f u_1 v - 4G_{uu} u_1 - 4G_{ux} \right] \sum_{1 \leq k \leq n} w_k u_{ki} \\
&= \frac{1}{1 - G_u} \left[ 2(1 - G_u)f u_1 v - 4G_{uu} u_1 - 4G_{ux} \right] \left[ - \frac{h'}{2} A_{x_1} u_1 + G_{uu} u_1^2 w_1 \\
&\quad - \frac{h'}{2} A + G_{ux} w_1 u_1 + \sum_{1 \leq k \leq n} w_k G_{ux_k} u_1 + \sum_{1 \leq k \leq n} w_k G_{x_k x_1} \right] \\
&= - h' f v u_1^2 + 2G_{uu} f v u_1^3 w_1 - f g' \gamma^1 A u_1 \\
&\quad + 2f v u_1 [G_{ux_1} w_1 u_1 + \sum_{1 \leq k \leq n} w_k G_{ux_k} u_1 + \sum_{1 \leq k \leq n} w_k G_{x_k x_1}] \\
&\quad - \frac{4}{1 - G_u} (G_{uu} u_1 + G_{ux_1}) \left[ - \frac{h'}{2} A_{x_1} u_1 + G_{uu} u_1^2 w_1 \\
&\quad - \frac{h'}{2} A + G_{ux} w_1 u_1 + \sum_{1 \leq k \leq n} w_k G_{ux_k} u_1 + \sum_{1 \leq k \leq n} w_k G_{x_k x_1} \right] \\
&= - h' f v u_1^2 + 2G_{uu} f v u_1^3 w_1 + AO(u_1^2).
\end{align*}
\]

(3.72)

From (3.62), we have

\[
\begin{align*}
&= - 4v^2 \sum_{2 \leq i \leq n} G_{ux_1} \sum_{1 \leq k \leq n} w_k u_{ki} \\
&= - \frac{4v^2}{1 - G_u} \sum_{2 \leq i \leq n} G_{ux_1} \left[ - \frac{g' \gamma^i}{2} A + G_{ux_1} w_1 u_1 + \sum_{1 \leq k \leq n} w_k G_{x_k x_1} \right] \\
&= \frac{2g'}{1 - G_u} \sum_{2 \leq i \leq n} G_{ux_1} \gamma^i A v^2 - \frac{4}{1 - G_u} \sum_{2 \leq i \leq n} G_{ux_1}^2 v^2 u_1 w_1 \\
&\quad - \frac{4}{1 - G_u} \sum_{2 \leq i \leq n} G_{ux_1} \sum_{1 \leq k \leq n} w_k G_{x_k x_i} v^2 \\
&= AO(u_1^2).
\end{align*}
\]

(3.73)
From (3.64), we have

\[-4(1 - G_u)(G_{uu} u_1^2 + 2G_{ux} u_1 + G_{x1x1})u_{11}\]

\[= -4(G_{uu} u_1^2 + 2G_{ux} u_1 + G_{x1x1}) \left[ (1 - G_u) \sum_{2 \leq i \leq n} \frac{w_i^2}{w_1^2} u_{ii} - \frac{h' A u_1}{2 w_1} + D \right] \]

\[= -4(1 - G_u)(G_{uu} u_1^2 + 2G_{ux} u_1 + G_{x1x1}) \sum_{2 \leq i \leq n} \frac{w_i^2}{w_1^2} u_{ii} \]

\[+ 4(G_{uu} u_1^2 + 2G_{ux} u_1 + G_{x1x1})(\frac{h' A u_1}{2 w_1} - D) \]

(3.74)

\[= -4(1 - G_u)(G_{uu} u_1^2 + 2G_{ux} u_1 + G_{x1x1}) \sum_{2 \leq i \leq n} \frac{w_i^2}{w_1^2} u_{ii} + AO(u_1^2). \]

From (3.63), we have

\[-4(1 - G_u)(v^2 + 1) \sum_{2 \leq i \leq n} (G_{ux} u_1 + G_{x1x1})u_{1i}\]

\[= 4(v^2 + 1) \sum_{2 \leq i \leq n} (G_{ux} u_1 + G_{x1x1}) \left[ \frac{1 - G_u}{w_1} w_i u_{ii} + \frac{g' \gamma^i A}{2 w_1} \right] \]

\[= G_{ux} u_1 - \frac{1}{w_1} \sum_{1 \leq k \leq n} w_k G_{kx_i} \]

\[= 4(v^2 + 1) \sum_{2 \leq i \leq n} (G_{ux} u_1 + G_{x1x1}) w_i u_{ii} \]

\[+ 4(v^2 + 1) \sum_{2 \leq i \leq n} (G_{ux} u_1 + G_{x1x1}) \left[ \frac{g' \gamma^i A}{2 w_1} - G_{ux} u_1 - \frac{1}{w_1} \sum_{1 \leq k \leq n} w_k G_{kx_i} \right] \]

(3.75)

\[= 4(v^2 + 1) \sum_{2 \leq i \leq n} (G_{ux} u_1 + G_{x1x1}) w_i u_{ii} + AO(u_1^2). \]

We treat the term \(Q_1\) using the relations (3.68) - (3.71) and use the formulas (3.72) - (3.75) to treat the term \(Q_2\). By the formula on \(Q_3\) in (3.56), we can get the following new formula of (3.53).

(3.76)

\[0 \geq \sum_{1 \leq i, j \leq n} a^{ij} \varphi_{ij} =: J_1 + J_2,\]

where \(J_1\) only contains the terms with \(u_{ii}\), the other terms belong to \(J_2\). We can write

(3.77)

\[J_1 =: \frac{1}{A}[J_{11} + J_{12}],\]
here $J_{11}$ contains the quadratic terms of $u_{ii} (i \geq 2)$, and $J_{12}$ is the term including linear terms of $u_{ii} (i \geq 2)$. It follows that

$$J_{11} = 2(1 - G_u)^2 \left\{ \sum \frac{w_i^2}{w_1^2} u_{ii}^2 + (v^2 + 1) \sum \frac{w_i^2}{w_1^2} u_{ii} + v^2 \sum \frac{w_i^2}{w_1^2} u_{ii} \right\}$$

$$= 2(1 - G_u)^2 \left\{ \sum \frac{w_i^2}{w_1^2} u_{ii}^2 + 2 \sum \frac{w_i^2}{w_1^2} u_{ii} u_{jj} + \sum \left[ v^2 + (v^2 + 1) \frac{w_i^2}{w_1^2} u_{ii}^2 \right] \right\}$$

$$= 2(1 - G_u)^2 \left\{ \sum \frac{w_i^2}{w_1^2} u_{ii}^2 + 2 \sum \frac{w_i^2}{w_1^2} u_{ii} u_{jj} \right\}$$

(3.78) $$= 2(1 - G_u)^2 \left\{ \sum \frac{w_i^2}{w_1^2} u_{ii}^2 + 2 \sum \frac{w_i^2}{w_1^2} u_{ii} u_{jj} \right\},$$

where

(3.79) $$d_i = v^2 + \frac{w_i^2}{w_1^2}, \quad i = 2, 3, \ldots, n,$$

(3.80) $$e_i = 1 + \frac{w_i^2}{w_1^2}, \quad i = 2, 3, \ldots, n.$$

And

$$J_{12} = - 2(1 - G_u)(h \frac{Au_1}{w_1^2} - 2 \frac{D}{w_1^2}) \sum w_i^2 u_{ii} + 2(1 - G_u)g' A' \frac{v^2}{w_1^2} + 1 \sum w_i \gamma^i u_{ii}$$

$$- 4(1 - G_u) \frac{v^2}{w_1^2} + 1 \sum w_k \sum_{i \leq k \leq n} (G_{x_k x_i} - G_{x_k x_i} w_1) w_i u_{ii}$$

$$- (2h' \frac{Au_1}{v^2 w_1^2} - 4G_w \frac{u_1^2}{v^2 w_1^2} + 2g' \gamma \frac{Au_1}{v^2 w_1^2} - 4G_{ux} \frac{u_1^2}{v^2 w_1^2}) - 4 \sum w_k \sum_{i \leq k \leq n} w_i G_{ux_k} w_i u_{ii}$$

$$- 4 \sum w_k \sum_{i \leq k \leq n} w_i G_{x_k x_i} w_i u_{ii} - 4(1 - G_u) v^2 \sum G_{x_i x_i} u_{ii}$$

$$- 4 \sum w_k \sum_{i \leq k \leq n} G_{x_k x_i} w_i u_{ii} - 4(1 - G_u) v^2 \sum G_{x_i x_i} u_{ii}$$

$$- 4(1 - G_u)(G_{uu} u_1^2 + 2G_{ux_1} u_1 + G_{x_1 x_1}) \sum \frac{w_i^2}{w_1^2} u_{ii}$$

(3.81) $$=: \sum_{2 \leq i \leq n} K_i u_{ii},$$
where

\[ K_i = -2(1 - G_u)(h^' \frac{Au_i}{w_1^2} - 2 \frac{D}{w_1^2})w_i^2 + 2(1 - G_u)g^' \frac{1}{w_1^2}w_i \gamma^i \]

\[ - 4(1 - G_u) \frac{v^2 + 1}{w_1^2}(G_{ux}, u_1 w_1 + \sum_{1 \leq k \leq n} w_k G_{x_k x_i})w_i \]

\[ + \left[ - 2h^' \frac{Au_1^2}{v^2 w_1} + 4G_{uu} \frac{u_1^2}{v^2 w_1} - 2g^' \frac{Au_1}{v^2 w_1} + 4G_{ux} \frac{u_1^2}{v^2 w_1} + 4 \sum_{1 \leq k \leq n} w_k G_{uxk} \frac{u_1^2}{v^2 w_1} \right] \]

\[ + \sum_{1 \leq k \leq n} w_k G_{x_k x_i} \frac{u_1}{v^2 w_1^2}]w_i^2 + 2g^ \frac{Au_1}{w_1} w_i \gamma^i - 4u_1^2 w_i G_{ux_i} - \frac{4u_1}{w_1} \sum_{1 \leq k \leq n} w_k G_{x_k x_i}w_i \]

\[ - 4(1 - G_u)v^2 G_{x_i x_i} - 4(1 - G_u)(G_{uu} u_1^2 + 2G_{ux_1} u_1 + G_{x_1 x_1}) \frac{w_i^2}{w_1^2} \]

\[ (3.82) + 4(1 - G_u) \frac{v^2 + 1}{w_1^2}(G_{ux}, u_1 + G_{x_1 x_i})w_i. \]

It follows that

\[ |K_i| \leq C_4 A, \quad i = 2, \ldots, n. \]

We write other terms as \( J_2 \), then

\[ J_2 = Q_3 - h^' f v^2 u_1^2 + 2f G_{uu} \frac{u_1^3}{v w_1^3} w_1 + h^2 A u_1^2 \frac{w_1}{2} \]

\[ + \frac{h^2}{1 - G_u} A u_1^3 + c^{11} g^2 \frac{u_1^2}{w_1^2} A + c^{11} g^2 \frac{Au_1}{1 - G_u} + O(u_1^2). \]

So by the choice of \( \mu_0 \) and the formula on \( Q_3 \) in \((3.57), (3.58)\) and \((3.2)\), we get the following estimate on \( J_2 \),

\[ J_2 \geq - (1 + c^{11} \alpha_0^2)u_1^2 \log |Dw|^2 + \frac{h^2 A u_1^2}{2 w_1^2} + \frac{h^2}{1 - G_u} \frac{Au_1^3}{w_1} \]

\[ + \frac{c^{11}}{2} g^2 \frac{u_1^2}{w_1^2} A + c^{11} g^2 \frac{Au_1}{1 - G_u} - C_5 u_1^2 \]

\[ (3.85) \geq \frac{1}{4}(1 + c^{11} \alpha_0^2)u_1^2 \log |Dw|^2 - C_6 u_1^2, \]

where \( C_4, C_5, C_6 \) are positive constants which only depend on \( n, \Omega, \mu_0, M_0, L_1, L_2 \).

**Step 3:** In this step, we concentrate on \( J_1 \). We first treat the terms \( J_{11} \) and \( J_{12} \) and obtain the formula \((3.98)\), then we complete the proof of Theorem \((1.1)\) through Lemma \((3.1)\).

By \((3.67)\), we have

\[ (3.86) \quad u_{22} = - \frac{1}{d_2} \sum_{3 \leq i \leq n} d_i u_{ii} + \frac{1}{d_2} \left[ f v^3 + \frac{h^'}{2(1 - G_u)} \frac{Au_1}{w_1} - \frac{D}{1 - G_u} \right]. \]
We first treat the term $J_{11}$: using (3.86) to simplify (3.78), we get

\begin{align*}
(3.87) \\
J_{11} &= 2(1 - G_u)^2 \left\{ d_2 e_2 u_{22} + \sum_{3 \leq i \leq n} d_i e_i u_{ii} + 2 \frac{w_2^2}{w_1^2} u_{22} \sum_{3 \leq j \leq n} \frac{w_j^2}{w_1^2} u_{jj} \\
&\quad + 2 \sum_{3 \leq i < j \leq n} \frac{w_i^2 w_j^2}{w_1^2} u_{ii} u_{jj} \right\} \\
&= 2(1 - G_u)^2 \left\{ \frac{e_2}{d_2} \left[ \sum_{3 \leq i \leq n} d_i u_{ii} + f v^3 + \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} - \frac{D}{1 - G_u} \right]^2 + \sum_{3 \leq i \leq n} d_i e_i u_{ii}^2 \\
&\quad + 2 \frac{w_2^2}{d_2 w_1^2} \left[ - \sum_{3 \leq i \leq n} d_i u_{ii} + f v^3 + \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} - \frac{D}{1 - G_u} \right] \sum_{3 \leq j \leq n} \frac{w_j^2}{w_1^2} u_{jj} \\
&\quad + 2 \sum_{3 \leq i < j \leq n} \frac{w_i^2 w_j^2}{w_1^2} u_{ii} u_{jj} \right\} \\
&= 2(1 - G_u)^2 \left\{ \sum_{3 \leq i \leq n} \left[ e_2 d_i^2 + e_i d_i d_2 - 2 \frac{w_2^2 w_i^2}{w_1^2} d_i \right] u_{ii}^2 \\
&\quad + 2 \sum_{3 \leq i < j \leq n} \left[ e_2 d_i d_j - \frac{w_2^2 w_i^2}{w_1^2} d_j - \frac{w_2^2 w_j^2}{w_1^2} d_i + \frac{w_j^2}{w_1^2} u_{ii} u_{jj} \right] \right\} \\
&\quad - 2e_2 [f v^3 + \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} - \frac{D}{1 - G_u}] \sum_{3 \leq i \leq n} d_i u_{ii} \\
&\quad + 2 \frac{w_2^2}{w_1^2} [f v^3 + \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} - \frac{D}{1 - G_u}] \sum_{3 \leq i \leq n} w_i^2 u_{ii} \\
&\quad + e_2 [f v^3 + \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} - \frac{D}{1 - G_u}]^2 \right\}. \\
\end{align*}

We can rewrite it as the following

\begin{align*}
J_{11} &= : 2(1 - G_u)^2 \left\{ \sum_{3 \leq i \leq n} b_{ii} u_{ii}^2 + 2 \sum_{3 \leq i < j \leq n} b_{ij} u_{ii} u_{jj} \\
&\quad - 2e_2 [f v^3 + \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} - \frac{D}{1 - G_u}] \sum_{3 \leq i \leq n} d_i u_{ii} \\
&\quad + 2 \frac{w_2^2}{w_1^2} [f v^3 + \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} - \frac{D}{1 - G_u}] \sum_{3 \leq i \leq n} w_i^2 u_{ii} \\
&\quad + e_2 [f v^3 + \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} - \frac{D}{1 - G_u}]^2 \right\}, \\
\end{align*}

(3.88)
where

\[ b_{ii} = e_2d_i^2 + e_id_i d_2 - \frac{w_2^2 w_i^2}{w_1^4} d_i \]

\[ = : 2u_i^4 + A_1 u_i^2 + A_2 i, \quad i \geq 3 \]  \hspace{1cm} (3.89)

\[ b_{ij} = e_2d_i d_j - \frac{w_2^2 w_i^2}{w_1^4} d_j - \frac{w_2^2 w_j^2}{w_1^4} d_i + \frac{w_1^2 w_j^2}{w_1^4} \]

\[ = : u_i^4 + G_{ij} u_i^2 + \hat{G}_{ij}, \quad i \neq j, \quad i, j \geq 3, \]  \hspace{1cm} (3.91)

and

\[ A_{1i} = 4 + (w_2^2 + w_i^2) \frac{u_i^2}{w_1^2}, \]

\[ A_{2i} = 2 + (3w_2^2 + 5w_i^2) \frac{u_i^2}{w_1^2} + w_i^2 \left( \frac{w_2^2 + w_i^2}{w_1^2} u_i^2 + \frac{2w_2^2 + 4w_i^2}{w_1^2} + \frac{2w_2^2 w_i^2}{w_1^2} \right), \]  \hspace{1cm} (3.90)

\[ G_{ij} = 2 + w_2^2 \frac{u_i^2}{w_1^2}, \]

\[ \hat{G}_{ij} = 1 + (2w_2^2 + w_i^2 + w_j^2) \frac{u_i^2}{w_1^2} + \frac{w_2^2 + w_i^2 + w_j^2}{w_1^2} + \frac{2w_2^2 w_i^2}{w_1^2} - \frac{w_2^2 w_i^2 w_j^2}{w_1^4} \].

Now we simplify the terms in \( J_{12} \): by (3.86), we can rewrite (3.81) as

\[ J_{12} = K_2 u_{22} + \sum_{3 \leq i \leq n} K_i u_{ii} \]

\[ = \frac{K_2}{d_2} \left[ - \sum_{3 \leq i \leq n} d_i u_{ii} + f v^3 + \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} - \frac{D}{1 - G_u} \right] + \sum_{3 \leq i \leq n} K_i u_{ii} \]

\[ = \sum_{3 \leq i \leq n} \left[ K_i - \frac{K_2 d_i}{d_2} \right] u_{ii} + \frac{K_2}{d_2} \left[ f v^3 + \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} - \frac{D}{1 - G_u} \right]. \]  \hspace{1cm} (3.91)

Using (3.87) and (3.91) to treat (3.77), we have

\[ J_1 = \frac{2(1 - G_u)^2}{A d_2} \left\{ \sum_{3 \leq i \leq n} b_{ii} u_{ii}^2 + 2 \sum_{3 \leq i < j \leq n} \tilde{K}_{ij} u_{ii} u_{jj} \right\} + \sum_{3 \leq i \leq n} \tilde{K}_i u_{ii} ] + R, \]  \hspace{1cm} (3.92)

where

\[ \tilde{K}_i = -2e_2 \left[ f v^3 + \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} - \frac{D}{1 - G_u} \right] d_i \]

\[ + 2 \frac{w_2^2}{w_1^2} \left[ f v^3 + \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} - \frac{D}{1 - G_u} \right] w_i^2 + K_i d_2 - K_2 d_i. \]  \hspace{1cm} (3.93)
We also have let
\[
R = \frac{2e_2(1 - G_u)^2}{Ad_2} \left[ f v^3 + \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} - \frac{D}{1 - G_u} \right]^2 \\
+ \frac{K_2}{Ad_2} \left[ f v^3 + \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} - \frac{D}{1 - G_u} \right].
\]

For \(\tilde{K}_i\) and \(R\), using the formulas on \(D, K_i\) in (3.66), (3.83); the formula of \(A\) in (3.58); \(e_i; d_i\) in (3.80)-(3.79), and \(h(u), g(d)\) in (3.2), we have the following estimates
\[
|\tilde{K}_i| \leq C_7 u_1^5, \quad i = 3, \ldots, n; \tag{3.94}
\]
\[
|R| \leq C_8 \frac{u_1^2}{\log u_1}. \tag{3.95}
\]

Now we use Lemma 3.1 if there is a sufficiently large positive constant \(C_9\) such that
\[
|Du|(x_0) \geq C_9, \tag{3.96}
\]
then we have
\[
J_1 \geq \frac{2}{Ad_2} (-C_{10} u_1^6),
\]
\[
\geq -C_{11} u_1^2, \tag{3.97}
\]
where we use the formulas \(d_2\) in (3.79) and \(A\) in (3.58).

Using the estimates on \(J_1\) in (3.97) and \(J_2\) in (3.85), from (3.76) we obtain
\[
0 \geq \sum_{1 \leq i,j \leq n} a^{ij} \varphi_{ij} \geq \frac{1}{4} (1 + c^{11} u_0^2) u_1^2 \log |Dw| - C_{12} u_1^2 \tag{3.98}
\]
\[
\geq \frac{1}{4} u_1^2 \log |Dw| - C_{12} u_1^2.
\]

There exists a positive constant \(C_{13}\) such that
\[
|Du|(x_0) \leq C_{13}. \tag{3.99}
\]

So from Case 1, Case 2, and (3.99), we have
\[
|Du|(x_0) \leq C_{14}, \quad x_0 \in \Omega_{\mu_0} \cup \partial \Omega.
\]

Since \(\varphi(x) \leq \varphi(x_0)\), for \(x \in \Omega_{\mu_0}\), there exists \(M_2\) such that
\[
|Du|(x) \leq M_2, \quad \text{in} \quad \Omega_{\mu_0} \cup \partial \Omega, \tag{3.100}
\]
where \(M_2\) depends only on \(n, \Omega, \mu_0, M_0, L_1, L_2\).

So at last we get the following estimate
\[
\sup_{\Omega_{\mu_0}} |Du| \leq \max\{M_1, M_2\},
\]
where the positive constant $M_1$ depends only on $n, \mu_0, M_0, L_1$; and $M_2$ depends only on $n, \Omega, \mu_0, M_0, L_1, L_2$.

So we complete the proof of Theorem 1.1. \hfill \Box

Now we prove the main Lemma 3.1 which was used to estimate $J_1$ defined in (3.97).

**Lemma 3.1.** We define $(b_{ij})$ as in (3.89); $d_i, e_i$ defined as in (3.79) - (3.80); $A_1, A_2, G_{ij}, \hat{G}_{ij}$ defined as in (3.90). We study the following quadratic form

\[
Q(x_3, x_4, \ldots , x_n) = \sum_{3 \leq i \leq n} b_{ii} x_i^2 + 2 \sum_{3 \leq i < j \leq n} b_{ij} x_i x_j + \sum_{3 \leq i \leq n} \tilde{K}_i x_i,
\]

where $\tilde{K}_i$ defined in (3.93) and we have the estimate (3.94) for $\tilde{K}_i$. Then there exists a sufficiently large positive constant $C_{15}$ which depends only on $n, \Omega, \mu_0, M_0, L_1, L_2$, such that if

\[
|Du|(x_0) = u_1(x_0) \geq C_{15},
\]

then the followings hold.

(I): The matrix $(b_{ij})$ is positive definite since the matrix $(b_{ij}^1) = (1 + \delta_{ij})$ is positive definite.

(II): We have

\[
Q(x_3, x_4, \ldots , x_n) \geq -C_{16} u_1^6,
\]

where positive constant $C_{16}$ also depends only on $n, \Omega, \mu_0, M_0, L_1, L_2$.

**Proof:** Let

\[
B = (b_{ij}) = B_1 + B_2, B_1 = u_1^4(b_{ij}^1), B_2 = (O(u_1^2)\delta_{ij}).
\]

We first prove (I):

\[
\sigma_k(B) = \sigma_k(B_1 + B_2) = \sigma_k(B_1) + \sigma_k(B_1, B_1, \ldots , B_1, B_2) + \cdots + \sigma_k(B_1, B_2, \ldots , B_2, B_2) + \sigma_k(B_2) = u_1^{4k} \sigma_k(b_{ij}^1) + O(u_1^{4k-2}),
\]

so if $u_1$ is sufficiently large, then $\sigma_k(B) > 0 \iff \sigma_k(b_{ij}^1) > 0$.

Now we prove (II): Since $B_1 = u_1^4(b_{ij}^1)_{3 \leq i, j \leq n}$ is positive definite, from the argument in (I), we get

\[
B^{-1} = (B_1 + B_2)^{-1} = B_1^{-1}(I + B_1^{-1}B_2)^{-1} = u_1^{-4}(b_{ij}^1)^{-1}(1 + o(1)).
\]
Then we have

\[
(b_{i}^{1})^{-1} = \begin{pmatrix}
2 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 2
\end{pmatrix}^{-1} = \frac{1}{n-1} \begin{pmatrix}
n-2 & -1 & \cdots & -1 \\
-1 & n-2 & \cdots & -1 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & n-2
\end{pmatrix}.
\]

Now we solve the following linear algebra equation

\[
\frac{\partial Q}{\partial x_k} = 0, \quad k = 3, 4, \ldots, n.
\]

We assume \((\bar{x}_3, \bar{x}_4, \ldots, \bar{x}_n)\) is the extreme point of the quadratic form \(Q(x_3, x_4, \ldots, x_n)\). From the definition of \(b_{ij}, \tilde{K}_{i}\) in (3.89), (3.93) and the estimate for \(\tilde{K}_{i}\) in (3.94), using the formulas (3.105) and (3.106), it follows that

\[
\begin{pmatrix}
\bar{x}_3 \\
\bar{x}_4 \\
\vdots \\
\bar{x}_n
\end{pmatrix} = O(u_1^5)B^{-1}\begin{pmatrix}
1 \\
1 \\
\vdots \\
1
\end{pmatrix} = O(u_1) \begin{pmatrix}
1 \\
1 \\
\vdots \\
1
\end{pmatrix}.
\]

It follows that we have the following minimum of the quadratic \(Q\),

\[
Q(\bar{x}_3, \bar{x}_4, \ldots, \bar{x}_n) \geq -C_{17}u_1^6.
\]

In this computation, the bounds in the coefficient on \(O(u_1^5), O(u_1)\) depend only on \(n, \Omega, M_0, \mu_0, L_1, L_2\). Thus we complete this proof. \(\square\)

4. **The proof of Theorem 1.3**

In this section we first prove the Theorem 1.3.

In the proof of the existence theorem for the Neumann boundary value problem, we need the a priori estimates. For the \(C^0\) estimates we use the methods introduced by Concus-Finn[2] and Spruck[21]. As in Simon-Spruck[20], we use the continuity method to complete the proof of theorem 1.3.

**Proof of Theorem 1.3.**

We consider the following family on the mean curvature equation with Neumann boundary value problem:

\[
\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = u \quad \text{in} \quad \Omega,
\]

\[
\frac{\partial u}{\partial \gamma} = \tau \psi(x) \quad \text{on} \quad \partial \Omega,
\]

where \(\tau \in [0, 1]\).
For $\tau = 0$, then $u = 0$ is the unique solution. And we need to find the solution for $\tau = 1$. By the standard existence theorem \[23, 9\], as in Simon-Spruck \[20\], if we can get the a priori estimates for the $C^2(\Omega)$ solution of the equation (4.1) and (4.2)

\begin{align*}
\sup_\Omega |u| &\leq \hat{K}_1, \\
\sup_\Omega |Du| &\leq \hat{K}_2,
\end{align*}

where $\hat{K}_1, \hat{K}_2$ are independent of $\tau$. Then we can get the existence theorem. From the interior gradient estimates and our boundary gradient estimates, we only need get the $C^0$ estimates for the solution $u$ in (4.1) and (4.2).

In the paper by Spruck \[21\], he used the comparison theorem developed by Concus-Finn \[2\] to get the $C^0$ estimates for the mean curvature equation with prescribed contact angle boundary value problem. In our case, his proof is still true, so we complete the proof of Theorem 1.3.

\begin{remark}
In X.N. Ma and J.J. Xu \[17\], we have generalized the boundary gradient estimates to Hessian equation and the higher order curvature equation with Neumann boundary value and the capillary boundary value problem. And we can reduce the condition on $\psi \in C^3$ to $\psi \in C^2$, then we could get the similar results on Theorem 1.1 and Theorem 1.3.
\end{remark}

Acknowledgments: The authors would like to thank Qiu Guohuan for pointing a flaw in our first version of this paper.

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