The Carter tensor and the physical-space analysis in perturbations of Kerr-Newman spacetime

Elena Giorgi∗

1Princeton University, Gravity Initiative, Princeton, United States

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Abstract
The Carter tensor is a Killing tensor of the Kerr-Newman spacetime, and its existence implies the separability of the wave equation. Nevertheless, the Carter operator is known to commute with the D’Alembertian only in the case of a Ricci-flat metric. We show that, even though the Kerr-Newman spacetime satisfies the non-vacuum Einstein-Maxwell equations, its curvature and electromagnetic tensors satisfy peculiar properties which imply that the Carter operator still commutes with the wave equation. This feature allows to adapt to Kerr-Newman the physical-space analysis of the wave equation in Kerr by Andersson-Blue [3], which avoids frequency decomposition of the solution by precisely making use of the commutation with the Carter operator.

We also extend the mathematical framework of physical-space analysis to the case of the Einstein-Maxwell equations on Kerr-Newman spacetime, representing coupled electromagnetic-gravitational perturbations of the rotating charged black hole. The physical-space analysis is crucial in this setting as the coupling of spin-1 and spin-2 fields in the axially symmetric background prevents the separation in modes as observed by Chandrasekhar [17], and therefore represents an important step towards an analytical proof of the stability of the Kerr-Newman black hole.

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*egiorgi@princeton.edu
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Introduction

The Kerr-Newman spacetimes \( \mathcal{M}_{g_{M,a,Q}} \) are a three-parameter family of solutions to the Einstein-Maxwell equations. They are the most general explicit black hole solutions to the Einstein equation, representing rotating and charged black holes, where \( M \) is the mass of the black hole, \( a \) its angular momentum and \( Q \) its charge, in the (sub-extremal and extremal) range \( a^2 + Q^2 \leq M^2 \). The Kerr-Newman metric in Boyer-Lindquist coordinates \( (t, r, \theta, \varphi) \) takes the form

\[
\mathcal{g}_{M,a,Q} = -\frac{\Delta}{|q|^2} (dt - a \sin^2 \theta d\varphi)^2 + |q|^2 dr^2 + |q|^2 d\theta^2 + \frac{\sin^2 \theta}{|q|^2} (adt - (r^2 + a^2) d\varphi)^2,
\]

where \( \Delta = r^2 - 2Mr + a^2 + Q^2 \) and \( |q|^2 = r^2 + a^2 \cos^2 \theta \). The Kerr-Newman spacetimes generalize the Kerr solution (for vanishing charge), the Reissner-Nordström solution (for vanishing angular momentum) and the Schwarzschild solution (for both \( a = Q = 0 \)), and are expected to be the final state of gravitational collapse.

The problem of stability of the domain of outer communication of the Kerr-Newman spacetime as solution to the Einstein-Maxwell equations is an important open problem in General Relativity and has implications for the physical relevance of the Kerr-Newman solution as a realistic black hole. Tremendous progress towards the proof of stability of the black hole solutions has been made in the past fifteen years, with works which encompass scalar, electromagnetic, gravitational perturbations of Schwarzschild, Reissner-Nordström, Kerr and Kerr-Newman spacetimes, from mode stability, to linear and fully non-linear stability, see for example \cite{26, 21, 11, 2, 38, 59, 22, 45} and references therein.

In \cite{35} we started a program aimed to prove the linear stability of the Kerr-Newman family to coupled electromagnetic-gravitational perturbations. Numerical works strongly support stability for the Kerr-Newman family \cite{33, 55, 43, 64}, but nevertheless an analytical proof of even its weakest version, the mode stability, has not been obtained. Mode solutions to the wave equation are solutions of the separated form

\[
\psi(r,t,\theta,\varphi) = e^{-i\omega t} e^{im\varphi} R(r) S(\theta), \tag{1}
\]

where \( \omega \in \mathbb{C} \) is the time frequency, and \( m \in \mathbb{Z} \) is the azimuthal mode. Mode stability consists in showing that there are no solutions of the form \( 1 \) with bounded initial energy which are exponentially growing in time.

The above separated form for \( \omega \in \mathbb{R} \) and \( m \in \mathbb{Z} \) is related to the Fourier transform of the solution with respect to the symmetries of the spacetime (the stationary vectorfield \( \partial_t \) and the axially symmetric \( \partial_\varphi \)), and therefore corresponds to its frequency decomposition. In addition to the two Killing vectorfields, the Kerr-Newman metric admits a Killing tensor, called the Carter tensor \cite{15}, which represents a hidden symmetry of the spacetime and which provides a third constant of motion which allows the full integrability of the geodesic flow. Because of such integrability, functions of the form \( 1 \) are solutions to the wave equation as long as \( R(r) \) and \( S(\theta) \) respectively satisfy a radial and an angular ODE. The Carter separation introduces, in addition to the frequency \( \omega \) and \( m \), a real frequency parameter \( \lambda_{nm}(\omega) \) parametrized by \( \ell \in \mathbb{N}_0 \), which are
the eigenvalues of an associated elliptic operator, whose eigenfunctions are the *oblate spheroidal harmonics* $S_{\omega m t}$.

As observed by Chandrasekhar [17] in the 80s, the methods developed in black hole perturbation theory for the other black hole solutions, such as Kerr or Reissner-Nordström, involving the separation in modes as in (1) and the proof of their mode stability, did not seem to be applicable for treating the coupled electromagnetic-gravitational perturbations of Kerr-Newman spacetime. In fact, in decomposing the system of equations in separated forms (1), the electromagnetic field (of spin ±1) and the gravitational field (of spin ±2) get separated in spheroidal harmonics $S_{\omega m t}^{(s)}$ of different spins $s$, which do not easily interact with the coupling operators appearing as consequence of the Einstein-Maxwell equations. On the other hand, for gravitational perturbations of Kerr there is no interaction between spheroidal harmonics of different spins, as only the spin ±2 gravitational field is considered. In the case of the spherically symmetric Reissner-Nordström, the spheroidal harmonics reduce to the standard spherical harmonics, which commute with the coupling operators thanks to the Hodge theory on the sphere. Instead, in the most general case of electromagnetic-gravitational perturbations of Kerr-Newman, the interaction between spin-1 and spin-2 fields in the axially symmetric background prevents the separability in modes, and their frequency decomposition is not possible due to the coupling of the equations of different spins, see Section 3.1.1. For more details, see also [17] [35].

As a consequence of the non separability in modes, we expect that the proof of the linear stability of the Kerr-Newman family necessitates a **physical-space analysis** of the coupled system of equations describing the interaction between gravitational and electromagnetic radiations, which in particular does not rely on decomposition in spheroidal harmonics.

The goal of this paper is to derive the mathematical framework to obtain estimates for the scalar wave equation

$$\Box_{g_{M,a,Q}} \psi = 0$$

(2)

on the exterior region of Kerr-Newman black hole exclusively involving **physical-space analysis**, i.e. no frequency decomposition of the solution, and which is robust enough to be adapted to the system of coupled Teukolsky and Regge-Wheeler equations obtained in our [35], in the framework of the proof of linear stability. A physical-space analysis has also the advantage of being potentially more adaptable to the non-linear stability problem, see [15] and [37]. We prove the following:

**Theorem.** Boundedness of the energy flux through a global foliation on the exterior of Kerr-Newman spacetime $(\mathcal{M}_{g_{M,a,Q}}, g_{M,a,Q})$ with $|a| \ll M$, and a suitable version of local integrated energy decay can be obtained exclusively through a **physical-space analysis** for solutions to the wave equation (2) arising from bounded initial energy on a suitable Cauchy surface.

More precisely, we obtain estimates through a physical-space analysis for axially symmetric solutions in the sub-extremal range $a^2 + Q^2 < M^2$, and for general solutions in the slowly rotating range $|a| \ll M$, see Theorem 2.1.

We then show that the physical-space analysis obtained for the wave equation can be extended to the coupled system of generalized Regge-Wheeler equations for $|a| \ll M$ obtained in [35] describing perturbations of Kerr-Newman black hole, see Proposition 3.4.

**Previous results on the wave equation in black hole backgrounds**

Boundedness and decay properties for scalar wave equations in Kerr and Kerr-Newman have been obtained in [25] [27] [28] [32] [3] in the slowly rotating case, and in [29] [19] in the full sub-extremal range. Nevertheless, the only proof obtained exclusively through a physical-space analysis is the proof of Andersson-Blue [3] in slowly rotating Kerr spacetimes. We now give a summary of previous results on the wave equation on black hole backgrounds.

In the spherically symmetric Schwarzschild and Reissner-Nordström spacetimes, the Killing vectorfield $\partial_t$ is timelike everywhere in the exterior region, and the orbital null geodesics, i.e. null geodesics that neither fall into the black hole nor escape to infinity and which are then an obstruction to decay, all asymptote to a physical-space timelike cylinder, called the *photon sphere*. The first property implies that superradiance is not present, as the vectorfield $\partial_t$ defines a positive definite conserved energy. The second property implies that the trapping region of the spacetime does not depend on the frequency of the solution. As the orbital null geodesics are restricted to a physical-space hypersurface, given by $\{r = r_{\text{trap}}\}$ for some $r_{\text{trap}}$ outside the...
black hole \((r_{\text{trap}} = 3M\) in Schwarzschild\), spacetime energy estimates can be obtained through a vectorfield of the form \(\mathcal{F}(r)\partial_r\), with \(\mathcal{F}\) vanishing at \(r = r_{\text{trap}}\), and the analysis of the wave equation can be performed in physical-space. Physical-space analysis of the wave equation in spherically symmetric backgrounds have been obtained in \([13, 15, 23, 24]\). See also \([61, 58, 59]\) and \([8, 9, 70]\] in the case of extremal Reissner-Nordström. Similarly, the study of the Maxwell equations has been obtained in \([12, 60, 55]\).

In the case of gravitational perturbations of Schwarzschild spacetime, the analysis of the separated mode solutions was obtained by the physics community in the 70s, see \([57, 11, 17]\). The first quantitative result for the linear gravitational perturbations of Schwarzschild through a physical-space analysis of the Teukolsky equation was obtained by Dafermos-Holzegel-Rodnianski in \([20]\), see also \([31, 32]\]. The physical-space analysis of the linearized gravity exploited, in addition to the conserved energy and the spacetime estimates which degenerate at the photon sphere, a hierarchy of wave-like equations, from the Teukolsky to the Regge-Wheeler equation \([20]\). For non-linear gravitational perturbations of Schwarzschild, see the work by Klainerman-Szeftel \([41]\] under the class of axially symmetric polarized perturbations, and the recent work by Dafermos-Holzegel-Rodnianski-Taylor \([22]\] for the full non-linear stability.

In the case of electromagnetic-gravitational perturbations of sub-extremal Reissner-Nordström spacetime, the analysis of the separated mode solutions was also obtained by the physics community \([53, 54, 17]\). The quantitative results for the linear electromagnetic-gravitational perturbations through the physical-space analysis of the Teukolsky system, and its derived Regge-Wheeler system, were obtained in our series of works \([31, 32, 33, 34]\).

In the axially symmetric Kerr and Kerr-Newman spacetimes, the analysis of the wave equation is complicated by two factors: the presence of an ergoregion, and therefore superradiance of the solution, and the dependence of the trapping region on the frequency of the solution. The Killing vectorfield \(\partial_\theta\) becomes spacelike in a region outside the event horizon known as the \(\text{ergoregion}\), hence its conserved energy is not positive definite everywhere. Moreover, the trapped null geodesics are not confined to a hypersurface in physical-space, but rather cover an open region of the spacetime which depends on the energy and angular momentum of the geodesics, and therefore the trapping degeneracy for the wave equation depends on the frequency of the solution. For this reason the high frequency obstruction to decay given by the trapping region cannot be described by the classical vectorfield method \([11]\).

These problems have been first overcome in the case of small angular momentum, \(|a| \ll M\), in which case both the superradiance and the trapping simplify. In \([27]\), Dafermos-Rodnianski showed that the superradiance can be controlled by analyzing the solution in its separated form \([41]\] and decomposing it into its superradiant and non-superradiant parts. Crucially the superradiant part is not trapped, and it satisfies a local energy decay identity obtained by perturbing the one in Schwarzschild. In \([25, 28]\), Dafermos-Rodnianski also overcame the problem of capturing the trapping region using frequency-localized generalizations of the Morawetz multipliers obtained in Schwarzschild. Even though the null geodesics are not localized on a physical-space hypersurface, they are localized in frequency-space, as the potential of the radial ODE has a unique simple maximum in the trapped frequency range, whose value is not localized on a physical-space hypersurface, they are localized in frequency-space, as the potential of the \(C\) case both the superradiance and the trapping simplify. In \([27]\), Dafermos-Rodnianski showed that the superradiance can be controlled by analyzing the solution in its separated form \([41]\] and decomposing it into its superradiant and non-superradiant parts. Crucially the superradiant part is not trapped, and it satisfies a local energy decay identity obtained by perturbing the one in Schwarzschild. In \([25, 28]\), Dafermos-Rodnianski also overcame the problem of capturing the trapping region using frequency-localized generalizations of the Morawetz multipliers obtained in Schwarzschild. Even though the null geodesics are not localized on a physical-space hypersurface, they are localized in frequency-space, as the potential of the radial ODE has a unique simple maximum in the trapped frequency range, whose value is not localized on a physical-space hypersurface, it becomes

\[
\hat{T}_a = M \left(1 - \frac{\ell}{M}\right) \hat{r},
\]

where \(\ell \ll M\). These solutions to the wave equation, i.e. \(\partial_r \psi = 0\), for \(a^2 + Q^2 < M^2\). For those solutions, superradiance is effectively absent and the trapping region collapses to a physical-space hypersurface. Although \(\partial_r\) still fails to be everywhere timelike, its associated energy through the horizon is non-negative as the Hawking vectorfield \(\hat{T} := \partial_r + 2a^2 r_{\text{trap}}^{-2} \partial_r\) is the null generator of the horizon and timelike everywhere outside it. In particular, for axially symmetric solutions, the Hawking vectorfield \(\hat{T}\) behaves like the Killing vectorfield \(\partial_r\). As the dependence on the azimuthal frequency \(m\) becomes trivial, the axially symmetric trapped null
geodesics all asymptote towards a hypersurface \( \{ r = r_{\text{trap}} \} \) in physical-space, where \( r_{\text{trap}} \) is defined as the largest root of the polynomial (see Section 2.5)

\[
T := r^3 - 3Mr^2 + (a^2 + 2Q^2)r + Ma^2 ,
\]

and therefore the construction of the current \( \mathcal{F}(r) \partial_r \), simplifies, see [25] for the frequency-space construction, and see [31] for a construction entirely in physical space. The analysis of the axially symmetric solutions to the wave equation in frequency space has also been extended to the extremal Kerr \( |a| = M \) by Aretakis in [10].

The only result at this day which extends the local energy decay estimates to the full sub-extremal range \( |a| < M \) for the wave equation in Kerr is the work by Dafermos-Rodnianski-Shlapentokh-Rothman [29] in frequency-space. See also [19] for the sub-extremal \( a^2 + Q^2 < M^2 \) Kerr-Newman. In [29], Dafermos-Rodnianski-Shlapentokh-Rothman perform a careful construction of frequency-dependent multiplier currents for the separated solutions, and crucially make use of the structure of trapping, i.e. the existence of a simple maximum \( r_{\text{trap}}(a\omega, m, \lambda_{\ell}) \) for the radial potential, and the fact that superradiant frequencies are not trapped, which they show it holds in the full sub-extremal range \( |a| < M \). They then make use of a continuity argument in \( a \) to justify the future integrability necessary to perform the Fourier transform in time.

In the case of electromagnetic or gravitational perturbations of Kerr spacetime, the analysis of the mode stability was obtained by the physics community in the 70s, see [22] [55] [17]. For a quantitative mode stability in sub-extremal Kerr see [58], and in extremal Kerr see [63].

For quantitative results for the linearized electromagnetic and gravitational perturbations of slowly rotating Kerr spacetime, see [1] [21] [17] [38], where a hierarchy of wave-like equations from Teukolsky to a generalized Regge-Wheeler equation is exploited. See also [2] [38]. For the analysis of the linearized gravity and electromagnetic perturbations in the sub-extremal range Kerr \( |a| < M \) see the work by Shlapentokh-Rothman-Teixeira da Costa [59]. For coupled linear-gravitational perturbations of Kerr-Newman spacetime, see our [35] for the derivation of the Teukolsky and Regge-Wheeler system of equations governing the perturbations.

For non-linear gravitational perturbations of Kerr, see the formalism developed in [36], and the recent work by Klainerman-Szeftel [15]. In the presence of positive cosmological constant, the non-linear stability of slowly rotating Kerr-de Sitter and Kerr-Newman-de Sitter spacetimes have been obtained in [40] [59].

Finally, electromagnetic-gravitational perturbations of Kerr-Newman spacetime have been considered in [17] and asymptotic solutions were obtained in [46]. In [59], the Teukolsky equations were derived in the phantom gauge. In [35] we derived the coupled system of equations for gauge-invariant perturbations of Kerr-Newman. See also the recent [52] for the construction of an energy functional for axisymmetric perturbations of Kerr-Newman.

The physical-space analysis of the wave equation in Kerr-Newman

As mentioned above, most of the results for scalar, electromagnetic and gravitational perturbations of Kerr or Kerr-Newman spacetimes rely on the separability in modes and the frequency-decomposition of the solution. Even though these methods are very effective, and they are at the present moment the only ones which allow for the analysis in the sub-extremal range for general solutions [29] [39], they are nevertheless not well suited for the analysis of coupled electromagnetic-gravitational perturbations of Kerr-Newman spacetime, as separability in modes cannot be obtained in that case (see the introduction of [35] for more details). The notable exception among the above-mentioned methods is the physical-space analysis for the wave equation in slowly rotating Kerr by Andersson-Blue [3], which makes crucial use of the Carter tensor in Kerr and the fact that the differential operator associated to the Carter tensor commutes with the D’Alembertian operator in Ricci-flat spacetimes [16].

The Carter tensor [15] is a symmetric 2-tensor \( K \) on Kerr and Kerr-Newman spacetimes which satisfies the Killing equation, i.e.

\[
D_{(\mu}K_{\nu\gamma)} = 0
\]

where \( D \) is the covariant derivative of the metric. In virtue of the Killing equation [11], the associated differential operator \( K(\psi) := D_\mu(K^{\nu\mu}D_\nu\psi) \) commutes with the D’Alembertian operator \( \square_k = D_\mu D^\mu \) in the case of Ricci-flat metric. We say that \( K \) is a symmetry operator for the wave equation. In [3], Andersson-Blue develop a generalized vectorfield method which allows for commutations with second order differential operators, and then apply it to the Carter differential operator \( K \) and its elliptic counterpart, together with the Killing vectorfields of the Kerr metric, to derive energy and Morawetz estimates for the solution.
The main obstruction to the application of Andersson-Blue’s method \( \mathcal{K} \) to the case of Kerr-Newman spacetime is that, even though the metric admits a Killing tensor, its associated differential operator does not in general commute with the wave equation. The first result of this paper is to show that, even though the Kerr-Newman spacetime is not Ricci-flat, the Carter differential operator \( \mathcal{K} \) associated to the Carter tensor still \textbf{commutes with the D’Alembertian operator of Kerr-Newman.} Interestingly enough, the commutation property is not a direct consequence of the Einstein-Maxwell equations, but rather of peculiar properties of the curvature and electromagnetic components in Kerr-Newman. We prove the following, see Theorem \( \text{T.1.12} \).

\textbf{Theorem.} \text{Even though the Kerr-Newman metric is not Ricci-flat, the Carter differential operator \( \mathcal{K} \) is still a symmetry operator for the wave equation.}

The above Theorem then allows to extend the physical-space analysis of Andersson-Blue \( \mathcal{K} \) to Kerr-Newman spacetime. More precisely, we show that the Carter differential operator \( \mathcal{K} \) is given by, see Proposition \( \text{T.1.12} \),

\[
\mathcal{K} = -a^2 \cos^2 \theta \square_{M,a,Q} + \mathcal{O},
\]

where \( \mathcal{O} \) is a \textit{modified Laplacian} for the Kerr-Newman metric, given in Boyer-Lindquist coordinates by

\[
\mathcal{O} = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin \theta} \partial_\varphi^2 + 2a \partial_\varphi \partial_\varphi + a^2 \sin^2 \theta \partial_t^2.
\]

In order to obtain a physical-space analysis of the wave equation in Kerr-Newman spacetime necessary to tackle the electromagnetic-gravitational perturbations of Kerr-Newman, we prove and make crucial use of the fact that the modified Laplacian \( \mathcal{O} \) obtained from the Carter differential operator is a \textit{conformal} symmetry operator for the wave equation, see Proposition \( \text{T.1.12} \).

\[
[\mathcal{O}, (r^2 + a^2 \cos^2 \theta) \square_{M,a,Q}] = 0.
\]

This allows to apply Andersson-Blue’s generalized vectorfield method \( \mathcal{K} \) to the case of Kerr-Newman. We now briefly recall how the physical-space analysis is obtained in \( \mathcal{K} \).

The vectorfield method is a robust geometrical approach to obtain energy estimates for solutions to the wave equation. The energy-momentum tensor associated to the wave equation is given by

\[
\mathcal{Q}[\psi]_{\mu\nu} = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} \partial_\lambda \psi \partial^\lambda \psi,
\]

and the wave equation is satisfied if and only if the divergence of the energy-momentum tensor \( \mathcal{Q}[\psi] \) vanishes. For a vectorfield \( X \), called multiplier, the current associated to \( X \) is defined by

\[
\mathcal{P}_\mu^{(X)}[\psi] = \mathcal{Q}[\psi]_{\mu\nu} X^\nu,
\]

and its divergence is then given by

\[
\mathbf{D}^\mu \mathcal{P}_\mu^{(X)}[\psi] = \frac{1}{2} \mathcal{Q}[\psi] \cdot (X)_{\pi},
\]

where \( (X)_{\pi\nu} = \mathbf{D}_{(\mu} X_{\nu)} \) is the deformation tensor of the vectorfield \( X \). Recall that if \( X \) is a Killing vectorfield, then \( (X)_{\pi} = 0 \).

The main idea to derive estimates through the vectorfield method is to use the divergence identity for the current \( \mathcal{P}_\mu^{(X)}[\psi] \) for appropriate vectorfields \( X \), and obtain

\[
\int_M \mathbf{D}^\mu \mathcal{P}_\mu^{(X)}[\psi] = \int_{\partial M} \mathcal{P}_\mu^{(X)}[\psi] \cdot n_{\partial M}
\]

for some causal domain \( M \).

To obtain local integrated energy decay, one wants to use a radial vectorfield \( X = F(r) \partial_r \), for a well chosen function \( F \), such that the divergence of the current \( \mathbf{D}^\mu \mathcal{P}_\mu^{(X)}[\psi] \) above is positive definite. Nevertheless, for general solutions to the wave equation, because of the complicated structure of trapping described above, this cannot be obtained. In fact, for \( X = F(r) \partial_r \), the above current gives

\[
\mathbf{D}^\mu \mathcal{P}_\mu^{(F\partial_\lambda)}[\psi] = \mathcal{A} |\partial_\lambda \psi|^2 + \mathcal{U}^{\alpha\beta}(\partial_\alpha \psi)(\partial_\beta \psi),
\]

for some positive coefficient \( \mathcal{A} \), and where \( \mathcal{U}^{\alpha\beta}(\partial_\alpha \psi)(\partial_\beta \psi) \) contains only derivatives in \( t, \theta, \varphi \) which are degenerate at trapping. More precisely, one obtains, see \( \text{T.1.2} \),

\[
\mathcal{U}^{\alpha\beta}(\partial_\alpha \psi)(\partial_\beta \psi) = \frac{u T}{(r^2 + a^2)^3} |\nabla \psi|^2 - u \frac{2ar}{(r^2 + a^2)^2} \hat{T}(\psi) \partial_\varphi \psi
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\]
where $u$ is an increasing function of $r$ which changes sign at $r_{trap}$, the root of the polynomial $\mathcal{T}$ defined in (3). Therefore the coefficient of the angular derivatives $|\nabla \psi|^2$, given by $u^2 \left(\frac{1}{r^2 + a^2}\right)^2$, is strictly positive and presents a degeneracy of multiplicity 2 at $r = r_{trap}$. On the other hand, the term $u^2 \frac{2a}{r^2 + a^2} \hat{T}(\psi) \partial_\phi \psi$ does not have a definite sign, and for this reason one fails to obtain a positive definite current by using the vectorfield $X = \mathcal{F}(r) \partial_r$. Observe that in the case of axially symmetric solutions, the term without definite sign vanishes, as $\partial_\phi \psi = 0$, and Morawetz and energy estimates can be obtained in physical space, see Section 2.7 without recurring to the commutation with the Carter operator. In this case, the difficulty is in defining the function $u$ for which all the terms of the divergence, together with a zero-th order term, are positive in the sub-extremal range $a^2 + Q^2 < M^2$, as obtained in Kerr by Stogin [61].

In the case of general solutions, Andersson-Blue [3] introduced a generalized vectorfield method which allows for higher-order symmetry operators as multipliers. In virtue of the commutation property (6), the set of the second order operators $S_\alpha$, for $\alpha = 1, 2, 3, 4$, given by

$$S_1 = \partial_r^2, \quad S_2 = \partial_\theta \partial_\phi, \quad S_3 = \partial_\phi^2, \quad S_4 = 0$$

are conformal symmetry operators for the wave equation in Kerr-Newman spacetime, and the commuted-solutions

$$\psi_\mu := S_\alpha (\psi), \quad \mu = 1, 2, 3, 4,$$

are also solutions to the wave equation. The generalized energy-momentum tensor is then defined as

$$Q[\psi]_{\mu \nu} = \partial_\mu \psi \partial_\nu \psi_\mu - \frac{1}{2} g_{\mu \nu} \partial_\lambda \psi \partial^\lambda \psi_\mu,$$

for $\mu, \nu = 1, 2, 3, 4$. Let $X$ be a double-indexed collection of vector fields $X = \{ X^\mu_\nu \}$, the generalized current associated to $X$ is defined by

$$P^\mu_\nu (X) = Q[\psi]_{\mu \nu} X^\mu_\nu,$$

and its divergence is given by

$$D^\nu P^\mu_\nu (X) = \frac{1}{4} Q[\psi]_{\mu \nu} \cdot D(\mu) X^\mu_\nu.$$

As for the standard vectorfield method, the goal is to apply the divergence identity to the above generalized currents for appropriate double-indexed collection of vector fields $X$, where one sums over the underlined indices $\underline{\mu}$. Just like in the standard vectorfield method, when applied to $X = \mathcal{F}(r) \partial_r$, the generalized current gives

$$D^\nu P^\mu_\nu (\mathcal{F} \partial_r) = A^\mu_\nu \partial_\mu \psi \partial_\nu \psi_\mu + U^{\alpha \beta \mu \nu} (\partial_\alpha \psi_\mu) (\partial_\beta \psi_\nu),$$

for some positive coefficients $A^\mu_\nu$, and where $U^{\alpha \beta \mu \nu} (\partial_\alpha \psi_\mu) (\partial_\beta \psi_\nu)$ contains only derivatives in $t, \theta, \phi$ of the $\psi_\mu$ which are degenerate at trapping.

The main advantage in going to the higher-order multipliers in the generalized vectorfield method is the fact that now one has the flexibility of interchanging the derivatives applied to $\psi$ through an integration by parts, and the trapped term $U^{\alpha \beta \mu \nu} (\partial_\alpha \psi_\mu) (\partial_\beta \psi_\nu)$ can in fact be rewritten as a positive definite term. More precisely, one can write (see Lemma 2.11)

$$U^{\alpha \beta \mu \nu} (\partial_\alpha \psi_\mu) (\partial_\beta \psi_\nu) = \frac{1}{2} h(|\partial_\lambda \psi|^2 + |\nabla \psi|^2) + \text{boundary terms},$$

for a positive function $h$ and where $\Psi$ is a trapped linear combinations of second order derivatives of $\psi$, schematically given by, see [21],

$$\Psi = - \frac{2\mathcal{T}}{(r^2 + a^2)^2} \partial_t^2 \psi - \frac{2\mathcal{T}}{(r^2 + a^2)^2} \mathcal{O}(\psi) + \frac{4a r}{(r^2 + a^2)^2} \hat{T}(\psi) \partial_\phi \psi.$$

One can then use the above positivity to express the generalized current as a positive definite current for the original $\psi$ for small angular momentum $|a| \ll M$ as in [3].

**Application to the Einstein-Maxwell equations**

In order to apply the previous techniques for the wave equation to the more interesting case of coupled electromagnetic-gravitational perturbations of Kerr-Newman, we need to extend the above physical-space analysis to the case of coupled generalized Regge-Wheeler equations describing the perturbations.
As a consequence of the Einstein-Maxwell equations, in [35] we showed that there exists a pair of gauge-invariant tensorial quantities, denoted $p$ and $q^F$, which satisfy the following schematic system of equations, see Theorem 4.4 for the precise formulation,

$$
\square_g p - i\frac{2a \cos \theta}{|q|^2} \nabla_T p - V_1 p = 4Q^2 \frac{q^3}{|q|^4} (\mathcal{D} \cdot q^F) + \text{l.o.t.}
$$

$$
\square_g q^F - i\frac{4a \cos \theta}{|q|^4} \nabla_T q^F - V_2 q^F = -\frac{1}{2} \frac{q^3}{|q|^4} (\mathcal{D} \tilde{\otimes} p) + \text{l.o.t.}
$$

(7)

where $q = r + ia \cos \theta$, $\overline{q} = r - ia \cos \theta$, $|q|^2 = r^2 + a^2 \cos^2 \theta$, and $\mathcal{D}$ and $\mathcal{D} \tilde{\otimes}$ are angular operators responsible for the coupling between the two quantities. In Section 3.1.1 we explain why those angular operators on the right hand side of the above equations prevent the separability in modes in Kerr-Newman.

Nevertheless, the precise structure of the right hand side has good properties when interpreted in physical-space, in terms of the energy-momentum tensor of the equations. Indeed, the combined energy-momentum tensor defined as

$$
\mathcal{Q}[p, q^F]_{\mu\nu} := \mathcal{Q}[p]_{\mu\nu} + 8Q^2 \mathcal{Q}[q^F]_{\mu\nu},
$$

has good divergence properties. More precisely, the highest-order coupling terms at the level of divergence of the energy-momentum tensor cancel out in physical-space, see Section 3.2.4.

There is still one problem in extending the Andersson-Blue method described above to the case of the generalized Regge-Wheeler equations as in [4]. The Carter differential operator and the modified Laplacian $O$, which is a conformal symmetry for the $D$-Alembertian $\square_g$, is not a conformal symmetry for the system of equations, again because of the presence of the angular operators on the right hand side. Those angular operators do not commute with the modified Laplacian $O$, but rather their commutator involves a modified Gauss curvature term, denoted $(K)$. The scalar $(K)$ is defined as a curvature component of the (non-integrable) horizontal structure associated to the principal null frame in Kerr or Kerr-Newman, see [36], and it reduces to the Gauss curvature of the spheres in the case of spherically symmetric background.

Even though the modified Laplacian $O$ does not commute with the right hand side of the generalized Regge-Wheeler equations, their symmetric structure allows to define modified Laplacian operators involving Gauss curvature which allows to create positive definite terms in $\mathcal{U}^{\alpha\beta\delta} \partial_\alpha \psi_\beta \partial_\gamma \psi_\delta$ can still be performed in this more general setting.

Structure of the paper

This paper is organized as follows.

In Section 1 we define Killing tensors and their associated differential operators in a general manifold, and we compute the commutator with the $D$-Alembertian operator in terms of the Ricci curvature of the metric. We then show that the Carter operator is a symmetry operator in Kerr-Newman.

In Section 2 we collect the main properties of the wave equation in Kerr-Newman and the vectorfield method. We also derive the equations of trapped null geodesics in Kerr-Newman, and obtain energy and local decay estimates in physical space for axially symmetric solutions to the wave equation in the sub-extremal range $a^2 + Q^2 < M^2$ and for general solutions for slowly rotating $|a| \ll M$ in Kerr-Newman spacetime.

In Section 3 we show how to apply the above physical-space analysis to the generalized Regge-Wheeler system describing the coupled electromagnetic-gravitational perturbations of Kerr-Newman black hole.

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1 The Carter tensor in Kerr-Newman spacetime

In this section, we recall the definitions of Killing tensors and associated Killing differential operators for a general Lorentzian manifold \((\mathcal{M}, g)\), without any assumption on the curvature of \(g\). We then collect the commutator of the Killing differential operator with the D’Alembertian \(\Box_g\) of the metric \(g\).

We then recall the main properties of the Kerr-Newman spacetime, and explicitly define its Killing tensor, known as Carter tensor. We will show that, because of the special structure of the Carter tensor and the electromagnetic and curvature components, the associated Carter operator is a symmetry of the wave equation. Using a convenient expression for the Carter operator, we then define a conformal symmetry operator which can be interpreted as a modified Laplacian for the Kerr-Newman metric.

1.1 Killing tensors and differential operators

Let \((\mathcal{M}, g)\) be a Lorentzian manifold and let \(D\) denote the covariant derivative of \(g\). Recall that a vectorfield \(X\) on \(\mathcal{M}\) is called Killing if the Lie derivative of the metric with respect to \(X\) vanishes, i.e.

\[ L_X g_{\mu\nu} = D_{\mu} X_{\nu} = 0, \]

where \(X\) is called the deformation tensor of the vectorfield \(X\).

The notion of Killing vectorfield can be extended to 2-tensors.

**Definition 1.1.** A Killing tensor for \((\mathcal{M}, g)\) is a symmetric 2-tensor \(K\) which satisfies the Killing equation:

\[ D_{(\mu} K_{\nu)} = 0. \tag{8} \]

As a consequence of \(K\) being Killing, the above operator \(K\) enjoys favorable properties of commutation with the D’Alembertian operator \(\Box_g = D^\alpha D_\alpha\), which are also related to the Ricci curvature of the metric \(g\).

**Proposition 1.3.** Let \((\mathcal{M}, g)\) be a Lorentzian manifold with a Killing tensor \(K\). Then the commutator between the differential operator \(K\) and the D’Alembertian operator \(\Box_g\) for a scalar function \(\phi\) is given by

\[ [K, \Box_g] \phi = \left( (D^\alpha R - \frac{4}{3} D^\alpha R_\mu^\alpha) K_{\mu\nu} + \frac{2}{3} (R_\mu^\alpha D^\beta K_{\alpha\beta} - R_\nu^\alpha D^\beta K_{\mu\beta} - D^\alpha R', K_{\alpha\beta}) \right) D^\nu \phi \]

where \(R\) denotes the Ricci curvature or the scalar curvature depending if it appears a 2-tensor or a scalar respectively.

**Proof.** See Appendix \(A\). \(\square\)

1.2 The case of vacuum and electrovacuum spacetimes

We specialize the above commutator to the case of metrics satisfying the Einstein vacuum equation or the Einstein-Maxwell equation.

**Definition 1.4.** A Lorentzian manifold \((\mathcal{M}, g)\) is a solution to the Einstein vacuum equation if its Ricci curvature vanishes identically, i.e.

\[ R_{\mu\nu} = 0. \tag{10} \]

A Lorentzian manifold \((\mathcal{M}, g)\) is a solution to the Einstein-Maxwell equation if

\[ R_{\mu\nu} = 2 F_{\mu\lambda} F^\lambda_{\nu} - \frac{1}{2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}, \tag{11} \]

where \(F\) is a 2-form on \(\mathcal{M}\), denoted electromagnetic tensor, satisfying the Maxwell equations:

\[ D_{[\mu} F_{\nu\lambda]} = 0, \quad D^\mu F_{\mu\nu} = 0. \tag{12} \]

We will refer to the above as vacuum and electrovacuum spacetime respectively.

\[ \footnote{The computations in this section are valid in any pseudo-Riemannian manifold \((\mathcal{M}, g)\).} \]
As a consequence of Proposition 1.3, if a vacuum spacetime possesses a Killing tensor $K$, then the differential operator $\mathcal{K}$ commutes with the D’Alembertian of $g$:

$$[\mathcal{K}, \Box_g] \phi = 0.$$ 

For this reason, $\mathcal{K}$ is referred to as a symmetry operator, as it sends solution to the wave equation to solutions:

if $\Box_g \phi = 0$ then $\Box_g(\mathcal{K}(\phi)) = 0$.

As a consequence of the Einstein-Maxwell equations (11) and (12), the curvature of an electrovacuum spacetime satisfies

$$D^\mu R_{\mu\nu} = 0, \quad \mathcal{R} = 0.$$ 

Then if an electrovacuum spacetime possesses a Killing tensor $K$, the commutator between $\mathcal{K}$ and $\Box_g$ according to Proposition 1.3 becomes

$$[\mathcal{K}, \Box_g] \phi = \frac{2}{3} (R_\mu^\alpha D_\mu K_{\nu\alpha} - R_\nu^\alpha D_\mu K_{\mu\alpha} - D_\mu R^\alpha_\nu K_{\mu\nu}) D^\nu \phi.$$ (13)

Moreover, the Maxwell equations (12) do not imply the vanishing of the right hand side of (13). Therefore in this case $\mathcal{K}$ cannot be interpreted as a symmetry operator, as the commutator depends on the form of the Killing tensor $K$.

Famous examples of vacuum and an electrovacuum spacetimes which possess a Killing tensor are the Kerr and Kerr-Newman solutions respectively. The Killing tensor in Kerr was discovered by Carter [15], and it is then referred to as Carter tensor.

1.3 The Kerr-Newman spacetime

We review here the Kerr-Newman metric and associated properties, see also [19]. The Kerr-Newman metric depends on three physical parameters: the mass $M$, the angular momentum $a$, and the charge $Q$. We consider here the subextremal family of Kerr-Newman spacetimes which represent a charged rotating black hole.

1.3.1 The manifold and the metric

For $a^2 + Q^2 < M^2$, the Kerr-Newman metric in Boyer-Lindquist coordinates $(t, r, \theta, \varphi)$ takes the form

$$g_{M,a,Q} = \Delta \left( dt - a \sin^2 \theta d\varphi \right)^2 + \frac{|q|^2}{\Delta} dr^2 + |q|^2 d\theta^2 + \frac{\sin^2 \theta}{|q|^2} \left( adt - (r^2 + a^2) d\varphi \right)^2,$$ (14)

where

$$q = r + ia \cos \theta, \quad |q|^2 = r^2 + a^2 \cos^2 \theta,$$ (15)

and

$$\Delta = r^2 - 2Mr + a^2 + Q^2 = (r - r_+)(r - r_-), \quad r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}.$$ 

We recall the ambient manifold with boundary $\mathcal{M}$, diffeomorphic to $\mathbb{R}^+ \times \mathbb{R} \times S^2$, and the Kerr star coordinates $(t^*, r, \theta, \varphi^*)$ with relations $t(t^*, r) = t^* - \mathcal{T}(r)$, $\varphi(\varphi^*, r) = \varphi^* - \mathcal{P}(r)$ modulo $2\pi$. For the explicit form see [19]. When expressed in Kerr star coordinates, the metric (14) extends smoothly to the event horizon $H^+$ defined as the boundary $\partial \mathcal{M} = \{ r = r_+ \}$.

The metric $g_{M,a,Q}$, together with an appropriate 2-form $F$, satisfies the Einstein-Maxwell equations (11) and (12). Observe that the Kerr-Newman family reduces to the Kerr metric when $Q = 0$, to the Reissner-Nordström metric when $a = 0$, and to the Schwarzschild metric for $a = Q = 0$.

1.3.2 The Killing vectorfields

The coordinate vectorfields $T = \partial_t$, and $Z = \partial_\varphi$, coincide with the coordinate vectorfields $\partial_t$ and $\partial_\varphi$ in Boyer-Lindquist coordinates, which are Killing for the metric (14). The stationary Killing vectorfield $T = \partial_t$ is asymptotically timelike as $r \to \infty$, and spacelike close to the horizon, in the ergoregion $\{ \Delta - a^2 \sin^2 \theta < 0 \}$.

Since $T = \partial_t$ is not timelike in the ergoregion, we recall instead the definition of Hawking vectorfield, which is timelike in the whole exterior and null on the horizon.

\footnote{Observe that what we denote by $|q|^2 = r^2 + a^2 \cos^2 \theta$ is normally denoted in the literature as $\rho^2$. We avoid using the letter $\rho$ for this metric component as it is also used as a curvature component later.}
Proposition 1.5. The Hawking vectorfield \( \hat{T} \)

\[
\hat{T} : = \partial_t + \frac{a}{r^2 + a^2} \partial_{\varphi} 
\]

is timelike for \( \{ r > r_+ \} \) and null on the horizon \( \{ r = r_+ \} \). More precisely

\[
g_{M,a,Q}(\hat{T}, \hat{T}) = -\Delta \frac{|q|^2}{(r^2 + a^2)^2}. 
\]

Proof. Denoting \( g = g_{M,a,Q} \) and using that

\[
g_{tt} = -\Delta - a^2\sin^2\theta \frac{\Delta}{|q|^2}, \quad g_{t\varphi} = \Delta - (r^2 + a^2) a\sin^2\theta \theta, \quad g_{\varphi\varphi} = \frac{(r^2 + a^2)^2 - a^2\sin^2\theta \Delta}{|q|^2} \sin^2\theta 
\]

we obtain

\[
g(\hat{T}, \hat{T}) = g_{tt} + \frac{2a}{r^2 + a^2} g_{t\varphi} + \frac{a^2}{(r^2 + a^2)^2} g_{\varphi\varphi} = \frac{\Delta - a^2\sin^2\theta}{|q|^2} + \frac{2a\Delta - (r^2 + a^2) a\sin^2\theta \theta}{|q|^2} + \frac{a^2}{(r^2 + a^2)^2} \frac{(r^2 + a^2)^2 - a^2\sin^2\theta \Delta}{|q|^2} \sin^2\theta 
\]

\[
= \frac{\Delta}{|q|^2} + \frac{a^2}{r^2 + a^2|q|^2} \sin^2\theta - \frac{a^2}{(r^2 + a^2)^2} \frac{a^2\sin^2\theta \Delta}{|q|^2} \sin^2\theta 
\]

\[
= -\frac{\Delta}{|q|^2(r^2 + a^2)^2}(r^2 + a^2)^2 - 2a^2(r^2 + a^2) a\sin^2\theta + a^4\sin^4\theta = -\frac{\Delta |q|^4}{|q|^2(r^2 + a^2)^2} 
\]

as stated.

As a consequence of the above, we also deduce that the Killing vectorfield

\[
\hat{T}_H := T + \omega_H Z, \quad \text{with} \quad \omega_H = \frac{a}{r^2 + a^2} = \frac{a}{2Mr_+ - Q^2}, 
\]

where \( \omega_H \) is the angular velocity of the horizon, is null on the horizon and timelike in a small neighborhood of it in the exterior. Note that along \( H^+ \), we have

\[
D_{\hat{T}_H} \hat{T}_H = \kappa \hat{T}_H, \quad \kappa = \frac{r_+ - r_+}{2(r_+^2 + a^2)}, 
\]

where \( \kappa \) is the surface gravity, which is positive in the sub-extremal range and vanishes in the extremal case.

1.3.3 The principal null frames and horizontal structures

The Kerr-Newman metric is a spacetime of Petrov Type D, i.e. its Weyl curvature can be diagonalized with two linearly independent eigenvectors, the so-called principal null directions.

The vectorfields

\[
L = e_4 = \frac{r^2 + a^2}{\Delta} \partial_t + \partial_r + \frac{a}{\Delta} \partial_{\varphi}, \quad L = e_3 = \frac{r^2 + a^2}{|q|^2} \partial_t = \frac{\Delta}{|q|^2} \partial_r + \frac{a}{|q|^2} \partial_{\varphi} 
\]

define principal null directions, with the normalization \( g(e_4, e_4) = -2 \). The vectorfield \( \frac{\Delta}{r^2 + a^2} e_4 \) extends smoothly to \( H^+ \) to be parallel to the null generator, while \( \frac{\Delta}{r^2 + a^2} e_3 \) extends smoothly to \( H^+ \) to be transversal to it.

We complete the above null frame with the following vectorfields

\[
e_1 = \frac{1}{|q|} \partial_\theta, \quad e_2 = \frac{a \sin \theta}{|q|} \partial_t + \frac{1}{|q| \sin \theta} \partial_{\varphi}, 
\]

which represent an orthonormal frame on the orthogonal space spanned by \( e_3 \) and \( e_4 \).

The frame \( \{ e_a, e_3, e_4 \} \) for \( a = 1, 2 \) as defined in \ref{eq:1} and \ref{eq:2} describes a null frame which satisfies

\[
g(e_3, e_3) = g(e_4, e_4) = 0, \quad g(e_3, e_4) = -2, 
\]

\[
g(e_a, e_3) = g(e_a, e_4) = 0, \quad g(e_a, e_b) = \delta_{ab}, \quad a, b = 1, 2. 
\]
We say that a vectorfield $X$ is horizontal if $g(X, e_i) = g(X, e_3) = 0$. Observe that the commutator of two horizontal vectorfields may fail to be horizontal. We say that $\langle L, L \rangle$ is integrable if the commutator of two horizontal vectorfield is horizontal.

In the case of Kerr-Newman spacetime, the principal null frame $\{19\}$ is not integrable, and its horizontal vector space does not span a sphere, but rather a 2-plane distribution. We refer to it as a horizontal structure. On it, we can define the horizontal covariant derivative by the projection of the covariant derivative to the horizontal structure: for $X$ in the tangent space of $M$ and $Y$ horizontal

$$D_X Y := (h)(D_X Y) = D_X Y + \frac{1}{2} g(D_X Y, L)L + \frac{1}{2} g(D_X Y, L)L,$$

where $(h)(D_X Y)$ is the projection to the horizontal structure. We denote $\nabla X Y = (h)(D_L Y)$, $\nabla_3 Y = (h)(D_3 Y)$, $\nabla_a Y = (h)(D_a Y)$.

Similarly, we can extend the above definition to horizontal covariant $k$-tensors. We denote the set of horizontal tensors on $\mathcal{M}$ by $\mathfrak{s}_k$. We define the duals of $f \in \mathfrak{s}_1$ and $u \in \mathfrak{s}_2$ by

$$\div f = \varepsilon_{ab} \nabla_b f, \quad \curl f = \varepsilon^{ab} \nabla_a f,$$

$$(\nabla \hat{\otimes} f)_{ba} = \frac{1}{2} (\nabla_b f_a + \nabla_a f_b - \delta_{ab} (\div f))$$

$$\left( \div u \right)_a = \delta^{bc} \nabla_b u_{ca}.$$ 

**Definition 1.6.** Given an orthonormal basis of horizontal vectors $e_1, e_2$ we define the Hodge type operators, see $[18]$:

- $\mathcal{P}_1$ takes $\mathfrak{s}_1$ into $\mathfrak{s}_0$:

  $$\mathcal{P}_1 \xi = (\div \xi, \curl \xi),$$

- $\mathcal{P}_2$ takes $\mathfrak{s}_2$ into $\mathfrak{s}_1$:

  $$(\mathcal{P}_2 \xi)_a = \nabla^b \xi_{ab},$$

- $\mathcal{P}_1^*$ takes $\mathfrak{s}_0$ into $\mathfrak{s}_1$:

  $$\mathcal{P}_1^*(f, f) = -\nabla_a f + \varepsilon_{ab} \nabla_b f_*,$$

- $\mathcal{P}_2^*$ takes $\mathfrak{s}_1$ into $\mathfrak{s}_2$:

  $$\mathcal{P}_2^* \xi = -\nabla \hat{\otimes} \xi.$$ 

See $[36]$ or $[37]$ for more details.

### 1.3.4 The Ricci, electromagnetic and curvature components

As in $[18]$, we use standard notations to define the Ricci coefficients of the null pair frame as

$$\Sigma_{ab} = g(D_a L e_b), \quad \chi_{ab} = g(D_a L, e_b),$$

$$\xi_a = \frac{1}{2} g(D_L L e_a), \quad \xi_a = \frac{1}{2} g(D_L L, e_a),$$

$$\omega = \frac{1}{4} g(D_L L L), \quad \omega = \frac{1}{4} g(D_L L, L),$$

$$\eta_a = \frac{1}{2} g(D_L L e_a), \quad \eta_a = \frac{1}{2} g(D_L L, e_a),$$

$$\zeta_a = \frac{1}{2} g(D_a L, L).$$

using the short hand notation $D_a = D_{e_a}, a = 1, 2$.

---

3Recall that $\mathfrak{s}_0$ refers to pairs of scalar functions $(a, b)$. 

12
Observe that in the case of Kerr and Kerr-Newman spacetime, the 2-tensors $\chi_{ab}$ and $\chi_{\alpha\beta}$ associated to the principal null frame are not symmetric, as a consequence of the fact that the space which is orthogonal to the principal null frame is not integrable, see [19]. Following [20], we introduce the notations
\[
\text{tr} \chi := \delta^{ab} \chi_{ab}, \quad (a) \text{tr} \chi := \epsilon^{ab} \chi_{ab}, \\
\text{(a)} \text{tr} \chi := \epsilon^{ab} \chi_{ab}, \quad (a) \text{tr} \chi := \epsilon^{ab} \chi_{ab}.
\]
In particular we can write
\[
\chi_{ab} = \hat{\chi}_{ab} + \frac{1}{2} \delta_{ab} \text{tr} \chi + \frac{1}{2} \epsilon_{ab} (a) \text{tr} \chi,
\]
\[
\chi_{\alpha\beta} = \hat{\chi}_{\alpha\beta} + \frac{1}{2} \delta_{\alpha\beta} \text{tr} \chi + \frac{1}{2} \epsilon_{\alpha\beta} (a) \text{tr} \chi,
\]
where $\hat{\chi}$ and $\chi$ is the symmetric traceless part of $\chi$ and $\chi$, respectively.

We define the electromagnetic components relative to the null frame as
\[
(F)_{\alpha a} = F(e_{a}, e_{b}), \quad (F)_{\beta a} = F(e_{a}, e_{3})
\]
\[
(F)_{\rho} = \frac{1}{2} F(e_{3}, e_{4}), \quad *F_{\rho} = \frac{1}{2} *F(e_{3}, e_{4})
\]
where $F$ is the electromagnetic tensor in the Einstein-Maxwell equations [11] and [12] and $*F$ denotes the Hodge dual on $(\mathcal{M}, g)$ of $F$, defined by $*F_{\alpha\beta} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu}$.

We define the curvature components relative to the null frame as
\[
\alpha_{ab} = W(e_{4}, e_{a}, e_{b}), \quad \omega_{ab} = W(e_{3}, e_{a}, e_{3}, e_{b}),
\]
\[
\beta_{a} = \frac{1}{2} W(e_{a}, e_{4}, e_{4}, e_{4}), \quad \beta_{a} = W(e_{a}, e_{3}, e_{3}, e_{4})
\]
\[
\rho = \frac{1}{4} W(e_{3}, e_{3}, e_{4}, e_{4}), \quad *\rho = \frac{1}{4} *W(e_{3}, e_{3}, e_{4}, e_{4})
\]
where $W$ is the Weyl curvature, and $*W$ denotes the Hodge dual on $(\mathcal{M}, g)$ of $F$, defined by $*W_{\alpha\beta\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} W_{\alpha\beta,\rho\sigma}$.

**Lemma 1.7.** The Kerr-Newman metric has the following values of the Ricci, electromagnetic and curvature components.

- The following quantities vanish\(^4\)
  \[
  \hat{\chi} = \hat{\chi} = \xi = \xi = 0, \quad \alpha = \beta = \beta = \alpha = 0, \quad (F)_{\beta} = (F)_{\beta} = 0.
  \]
  The non-vanishing electromagnetic and curvature components take the following values:
  \[
  (F)_{\rho} = \frac{Q^{2}}{|q|^{6}} \left( -2M r^{3} + 2Q^{2} r^{2} + 6Ma^{2} \cos^{2} \theta r - 2Q^{2} a^{2} \cos^{2} \theta \right),
  \]
  \[
  *\rho = \frac{a \cos \theta}{|q|^{6}} \left( Q^{2} r^{2} - 2Ma^{2} \cos^{2} \theta \right).
  \]
- The Ricci coefficients defined with respect to the principal null frame [19] take the following values:
  \[
  \text{tr} \chi = \frac{2r}{|q|^{2}}, \quad (a) \text{tr} \chi = \frac{2a \cos \theta}{|q|^{2}}, \quad \text{(a)} \text{tr} \chi = \frac{2r \Delta}{|q|^{2}}, \quad (a) \text{tr} \chi = \frac{2a \Delta \cos \theta}{|q|^{2}},
  \]
  \[
  \omega = a \cos^{2} \theta (r - M) + M r^{2} - a^{2} r - Q^{2} r, \quad \omega = 0, \quad \eta = -\zeta.
  \]
- In the orthonormal frame $e_{a}$ for $a = 1, 2$ defined in [20], the Ricci coefficients have the following components:
  \[
  \eta_{1} = a \sin \theta \cos \theta, \quad \eta_{2} = a \sin \theta r, \quad *\eta_{1} = a \sin \theta r, \quad *\eta_{2} = a \cos \theta \cos \theta.
  \]

\(^4\)The vanishing of the quantities corresponds to the fact that the Kerr-Newman spacetime is of Petrov Type D.
\[
\eta_1 = -\frac{a^2 \sin \theta \cos \theta}{|q|^3}, \quad \eta_2 = -\frac{a \sin \theta r}{|q|^3}, \quad \eta_3 = \frac{a \sin \theta r}{|q|^3}, \quad \eta_4 = \frac{a^2 \sin \theta \cos \theta}{|q|^3}.
\]

- The derivatives of the coordinates \( r \) and \( \theta \) with respect to the frame defined in (19) and (20) satisfy the following relations:

\[
e_3(r) = \frac{|q|^2}{2r} \chi, \quad e_4(r) = \frac{|q|^2}{2r} \chi, \quad e_a(r) = 0, \quad e_a(a^2 \cos^2 \theta) = |q|^2 \eta, \quad e_3(a^2 \cos^2 \theta) = e_4(a^2 \cos^2 \theta) = 0.
\]

- Finally, the orthonormal frame \( e_a \) for \( a = 1, 2 \) defined in (20) satisfy the following:

\[
\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_1 = \Lambda e_2, \quad \nabla_{e_1} e_2 = -\Lambda e_1,
\]

where

\[
\Lambda := \frac{\kappa^2 + a^2}{|q|^3} \cot \theta.
\]

**Proof.** See [35] and [36]. \( \square \)

### 1.3.5 Complex notations

We denote by \( s_0(\mathbb{C}) \) the set of complex anti-self dual \( k \)-tensors on \( M \). More precisely, \( a + ib \in s_0(\mathbb{C}) \) is a complex scalar function on \( M \) with \( (a, b) \in s_0 \), \( F = f + i \ast f \in s_1(\mathbb{C}) \) is a complex anti-self dual 1-tensor on \( M \) with \( f \in s_1 \), and \( U = u + i \ast u \in s_2(\mathbb{C}) \) is a complex anti-self dual symmetric traceless 2-tensor on \( M \) with \( u \in s_2 \).

We extend the definitions for the Ricci, electromagnetic and curvature components to the complex case by using the anti-self dual tensors, by defining

\[
X = \chi + i \ast \chi, \quad X = \chi + i \ast \chi, \quad H = \eta + i \ast \eta, \quad H = \eta + i \ast \eta, \quad \Xi = \xi + i \ast \xi, \quad \Xi = \xi + i \ast \xi
\]

\[
(F)B = (F)B + i \ast (F)\beta, \quad (F)\beta = (F)B + i \ast (F)\beta, \quad (F)P = (F)P + i \ast (F)\rho
\]

\[
A = \alpha + i \ast \alpha, \quad A = \alpha + i \ast \alpha, \quad B = \beta + i \ast \beta, \quad B = \beta + i \ast \beta, \quad P = \rho + i \ast \rho.
\]

We also define the complexified version of the \( \nabla \) horizontal derivative as

\[
\hat{D} = \nabla + i \ast \nabla, \quad \hat{D} = \nabla - i \ast \nabla
\]

More precisely,

- For \( a + ib \in s_0(\mathbb{C}) \)

\[
\hat{D}(a + ib) := (\nabla + i \ast \nabla)(a + ib), \quad \hat{D}(a + ib) := (\nabla - i \ast \nabla)(a + ib)
\]

- For \( F = f + i \ast f \in s_1(\mathbb{C}) \),

\[
\hat{D}(f + i \ast f) := (\nabla - i \ast \nabla)(f + i \ast f) = 2(\text{div } f + i \text{curl } f), \quad \hat{D}(f + i \ast f) := (\nabla + i \ast \nabla)(f + i \ast f) = 2(\nabla \hat{\circ} f + i \ast (\nabla \hat{\circ} f)).
\]

- For \( U = u + i \ast u \in s_2(\mathbb{C}) \),

\[
\hat{D}(u + i \ast u) := (\nabla - i \ast \nabla)(u + i \ast u) = 2(\text{div } u + i \ast (\text{div } u))
\]

For \( F \in s_1(\mathbb{C}) \), the operator \(-\hat{D} \hat{\circ}\) is formally adjoint to the operator \( \hat{D} \cdot U \) applied to \( U \in s_2(\mathbb{C}) \), as shown in [35].

**Lemma 1.8** (Lemma 2.11 in [35]). For \( F = f + i \ast f \in s_1(\mathbb{C}) \) and \( U = u + i \ast u \in s_2(\mathbb{C}) \), we have

\[
(D \hat{\circ} F) \cdot U = -F \cdot (D \cdot U) - ((H + H) \hat{\circ} F) \cdot U + D_\alpha (F \cdot U)^\alpha.
\]
1.4 The Carter tensor and differential operator in Kerr-Newman

We define the Carter tensor of Kerr-Newman spacetime and show that it is a Killing tensor for its metric. We then also define the Carter differential operator associated to it and show that, even though Kerr-Newman is not Ricci-flat, the Carter operator is a symmetry operator in Kerr-Newman.

**Definition 1.9.** The Carter tensor associated to \((\mathcal{M}, g_{M,a,Q})\) is the following symmetric 2-tensor \(K\) defined by

\[
K = -(a^2 \cos^2 \theta)g_{M,a,Q} + |q|^2 (e_1 \otimes e_1 + e_2 \otimes e_2)
\]

where \(e_1\) and \(e_2\) are defined in (33).

By defining the symmetric tensor

\[
O^{\alpha \beta} := |q|^2 (e_1^\alpha e_1^\beta + e_2^\alpha e_2^\beta)
\]

we can write from (33),

\[
K = -(a^2 \cos^2 \theta)g_{M,a,Q} + O.
\]

Since \(g_{ab} = \delta_{ab}, g_{a3} = g_{a4} = 0\) for \(a = 1, 2,\) and \(g_{34} = -2,\) we obtain from (33) that the tensor \(K\) has the following components:

\[
K_{ab} = r^2 \delta_{ab}, \quad K_{34} = 2(a^2 \cos^2 \theta), \quad K_{a3} = K_{a4} = K_{33} = K_{44} = 0.
\]

In particular

\[
\text{tr}K = 2(r^2 - a^2 \cos^2 \theta).
\]

**Proposition 1.10.** The Carter tensor \(K\) defined by (33) satisfies

\[
D_{(\mu}K_{\nu)} = 0
\]

and therefore it is a Killing tensor for \((\mathcal{M}, g_{M,a,Q})\).

**Proof.** We decompose the symmetric 3-tensor \(\Pi_{\mu\nu\rho} = D_{(\mu}K_{\nu)} = \frac{1}{3}(D_{\mu}K_{\nu\rho} + D_{\nu}K_{\rho\mu} + D_{\rho}K_{\mu\nu})\) in the null frame by making use of the Ricci formulae.

\[
\begin{align*}
D_a e_b &= \nabla_a e_b + \frac{1}{2} \chi_a e_b + \frac{1}{2} \eta_a e_4, \\
D_a e_4 &= \chi_a e_b - \zeta_a e_4, \\
D_a e_3 &= \chi_a e_3 + \zeta_a e_3, \\
D_3 e_a &= \nabla_3 e_a + \eta_3 e_4 + \xi_3 e_4, \\
D_3 e_3 &= -2 \omega_3 e_4 + 2 \xi_3 e_3, \\
D_3 e_4 &= 2 \omega_3 e_4 + 2 \eta_3 e_4, \\
D_4 e_a &= \nabla_4 e_a + \eta_4 e_4 + \xi_4 e_4, \\
D_4 e_4 &= -2 \omega_4 e_4 + 2 \xi_4 e_4, \\
D_4 e_3 &= 2 \omega_4 e_3 + 2 \eta_4 e_3.
\end{align*}
\]

We compute

\[
3\Pi_{abc} = D_a K_{bc} + D_b K_{ca} + D_c K_{ab}
\]

\[
= \nabla_a K_{bc} - K_{D_a b c} - K_{a D_b c} + \nabla_b K_{ca} - K_{D_a c a} - K_{c D_a a} + \nabla_c K_{ab} - K_{D_a c b} - K_{a D_c b}
\]

\[
= \nabla_a K_{bc} + \nabla_b K_{ca} + \nabla_c K_{ab} = \nabla_a (r^2 \delta_{bc}) + \nabla_b (r^2 \delta_{ca}) + \nabla_c (r^2 \delta_{ab})
\]

\[
= 2r (\nabla_a r \delta_{bc} + \nabla_b r \delta_{ca} + \nabla_c r \delta_{ab}) = 0,
\]

since \(e_1(r) = e_2(r) = 0.\) We compute

\[
3\Pi_{abc} = D_a K_{bc} + D_b K_{ca} + D_c K_{ab}
\]

\[
= \nabla_a K_{bc} - K_{D_a b c} - K_{a D_b c} + \nabla_b K_{ca} - K_{D_a c a} - K_{c D_a a} + \nabla_c K_{ab} - K_{D_a c b} - K_{a D_c b}
\]

\[
= \frac{1}{2} \chi_{ab} K_{34} - \chi_{bc} K_{a3} - \chi_{ca} K_{b3} - \frac{1}{2} \eta_{ab} K_{34} + \nabla_a (r^2 \delta_{bc}) - \nabla_b (r^2 \delta_{ca}) + \nabla_c (r^2 \delta_{ab})
\]

\[
= -\chi_{ab} K_{34} + \chi_{bc} K_{a3} + \chi_{ca} K_{b3} - \frac{1}{2} \eta_{ab} K_{34} + \nabla_a (r^2 \delta_{bc}) - \nabla_b (r^2 \delta_{ca}) + \nabla_c (r^2 \delta_{ab})
\]

\[
= -(\chi_{ab} + \chi_{bc} + \chi_{ca}) (r^2 + a^2 \cos^2 \theta) + \nabla_a (r^2 \delta_{bc}) - \nabla_b (r^2 \delta_{ca}) + \nabla_c (r^2 \delta_{ab})
\]

\[
= -(\chi_{ab} + \chi_{bc} + \chi_{ca}) (r^2 + a^2 \cos^2 \theta) + \nabla_a (r^2 \delta_{bc}) - \nabla_b (r^2 \delta_{ca}) + \nabla_c (r^2 \delta_{ab})
\]
Theorem 1.11. The commutator between a scalar function \( \psi \) and the Kerr-Newman metric is an electrovacuum spacetime, the commutator between \( K \) and \( \Box_{\text{KM}} \) vanishes.

Proof. Since the Kerr-Newman metric is an electrovacuum spacetime, the commutator between \( K \) and \( \Box_{\text{KM}} \) satisfies [14], i.e.,

\[
[K, \Box_{\text{KM}}] \psi = 0.
\]

We now prove that because of special considerations of the metric and curvature of Kerr-Newman, the commutator between \( K \) and \( \Box_{\text{KM}} \) vanishes.

**Theorem 1.11.** In Kerr-Newman spacetime, the Carter differential operator commutes with \( \Box_{\text{KM}} \), i.e., for a scalar function \( \psi \) we have

\[
[K, \Box_{\text{KM}}] \psi = 0.
\]

In particular, the Carter differential operator \( K \) is a symmetry operator for Kerr-Newman spacetime.

Proof. Since the Kerr-Newman metric is an electrovacuum spacetime, the commutator between \( K \) and \( \Box_{\text{KM}} \) satisfies [13], i.e.,

\[
[K, \Box_{\text{KM}}] \psi = \frac{2}{3} (R_{\mu}^\nu D^\mu K_{\nu a} - R_{\nu}^\mu D^\mu K_{\nu a} - D^\mu R_{\nu \mu} K_{\nu a}) D^\nu \psi = \frac{2}{3} (I_1 - I_2 - I_3).
\]

From the Einstein-Maxwell equation [11], we compute the Ricci curvature of the Kerr-Newman metric, which is given by

\[
\begin{align*}
R_{a3} &= 2 (F)_{\rho a} \star (F)_{\beta 3} - 2 (F)_{\rho} (F)_{\beta 3} = 0, & R_{a4} &= 2 (F)_{\rho a} \star (F)_{\beta a} + 2 (F)_{\rho} (F)_{\beta a} = 0, \\
R_{33} &= 2 (F)_{33} = 0, & R_{44} &= 2 (F)_{4} = 0.
\end{align*}
\]

and

\[
R_{44} = 2 (F)_{p}^2 + \star (F)_{p}^2 = \frac{2Q^2}{|q|^4}, \quad R_{ab} = (F)_{p}^2 + \star (F)_{p}^2 \delta_{ab} = \frac{2Q^2}{|q|^4} \delta_{ab},
\]

where we used the values in Lemma [17]. Using the Ricci formulae [36], we deduce

\[
\begin{align*}
D_a R_{bc} &= \nabla_a R_{bc} - \frac{1}{2} \chi_{ab} R_{3c} - \frac{1}{2} \chi_{ac} R_{3b} - \frac{1}{2} \chi_{bc} R_{3a} - \frac{1}{2} \chi_{ab} R_{3c} - \frac{1}{2} \chi_{ac} R_{3b} - \frac{1}{2} \chi_{bc} R_{3a} \\
&= \nabla_a \left( \frac{Q^2}{|q|^4} \delta_{bc} \right) = -2(|q|^2)^{-3} \nabla_a \left(|q|^2 \right) Q^2 \delta_{bc} = -2(\eta_a + \eta_b) \frac{Q^2}{|q|^4} \delta_{bc}, \\
D_4 R_{3c} &= \nabla_4 R_{3c} - 2 \eta R_{3c} - \frac{2}{3} \eta_c R_{34} = -2 \eta_c Q^2 |q|^2 \delta_{bc} = \frac{2Q^2}{|q|^4} = -\frac{4Q^2}{|q|^4} \eta_c,
\end{align*}
\]

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Similarly, as computed in Proposition 1.10, we recall

\[ D_a R_{b3} = \nabla_a R_{b3} - \frac{1}{2} \chi_{ab} R_{33} - \frac{1}{2} \chi_{ab} R_{34} - \chi_{ac} R_{bc} - \zeta_a R_{b3} = -\operatorname{tr} \frac{Q^2}{|q|^2} \delta_{ab}, \]

\[ D_3 R_{a3} = \nabla_3 R_{a3} - 2\eta_3 R_{a3} = \nabla_3 \left( 2\frac{Q^2}{|q|^4} \right) = -\operatorname{tr} \frac{4Q^2}{|q|^2}; \]

\[ D_4 R_{a3} = \nabla_4 R_{a3} - 4\omega R_{a3} - 4\eta_3 R_{a3} = 0. \]

Similarly, as computed in Proposition 1.10, we recall

\[ D_a K_{bc} = D_4 K_{33} = D_3 K_{33} = 0, \quad D_a K_{b3} = - \frac{1}{2} \operatorname{tr} \frac{\chi}{|q|^2} \delta_{ab}, \quad D_4 K_{3c} = -2\nu_3 |q|^2. \]

Also, from [167] and using (28), we have

\[ D^\mu K_{\mu c} = -\frac{1}{2} D_c (\operatorname{tr} K) = -\frac{1}{2} \chi_{\mu} (2(r^2 - a^2 \cos^2 \theta)) = (\eta_c + \eta_3) |q|^2, \]

\[ D^\mu K_{\mu c} = -\frac{1}{2} D_3 (\operatorname{tr} K) = -\frac{1}{2} \chi_{\mu} (2(r^2 - a^2 \cos^2 \theta)) = -\operatorname{tr} \chi |q|^2, \]

\[ D^\mu K_{\mu c} = -\frac{1}{2} D_4 (\operatorname{tr} K) = -\frac{1}{2} \chi_{\mu} (2(r^2 - a^2 \cos^2 \theta)) = -\operatorname{tr} \chi |q|^2. \]

We compute \( I_1 = R_\mu^a D^\mu K_{\alpha \nu} D^\nu \psi. \)

\[ I_1 = R_\mu^a D^\mu K_{\alpha \nu} D^\nu \psi = R_\mu^a D^\mu K_{\alpha \nu} D^\nu \psi + R_\mu^3 D^\mu K_{3\nu} D^\nu \psi + R_\mu^4 D^\mu K_{4\nu} D^\nu \psi 
\]

\[ = R_\mu^a D^\mu K_{\alpha \nu} D^\nu \psi + R_\mu^3 D^\mu K_{3\nu} D^\nu \psi + \frac{1}{2} R_{34} D^\mu K_{3\nu} D^\nu \psi + R_\mu^4 D^\mu K_{4\nu} D^\nu \psi 
\]

\[ = \frac{1}{4} R_{34} D_3 K_{4\nu} D^\nu \psi + \frac{1}{4} R_{43} D_3 K_{4\nu} D^\nu \psi + \frac{1}{4} R_{43} D_{34} K_{4\nu} D^\nu \psi 
\]

\[ = \frac{1}{4} R_{34} D_3 K_{4\nu} D^\nu \psi + \frac{1}{4} R_{43} D_3 K_{4\nu} D^\nu \psi + \frac{1}{4} R_{43} D_{34} K_{4\nu} D^\nu \psi \]

which gives, using the above values,

\[ I_1 = -\frac{R_\mu^a}{2} \left[ \frac{\operatorname{tr} \chi |q|^2}{}D^3 \psi - \frac{\operatorname{tr} \chi |q|^2}{2} \delta_{\alpha \beta} D^3 |q|^2 D^\alpha D^\beta - \frac{1}{2} R_{34} |q|^2 D^\nu - \frac{1}{2} R_{43} |q|^2 D^\nu \right] 
\]

\[ = \frac{1}{2} \frac{Q^2}{|q|^2} \left[ \frac{\operatorname{tr} \chi D^3 \psi + \operatorname{tr} \chi D^3 \psi + \frac{Q^2}{|q|^2} (\eta_3 + \eta_3) D^3 \psi \right] 
\]

\[ = \frac{1}{2} \frac{Q^2}{|q|^2} \left[ \frac{\operatorname{tr} \chi D^3 \psi + \operatorname{tr} \chi D^3 \psi + \frac{Q^2}{|q|^2} (\eta_3 + \eta_3) D^3 \psi \right] 
\]

We compute \( I_2 = R_\nu^a D^\nu K_{\mu \nu} D^\mu \psi. \)

\[ I_2 = R_\nu^a D^\nu K_{\mu \nu} D^\mu \psi = R_\nu^a D^\nu K_{\mu \nu} D^\mu \psi + R_\nu^3 D^\nu K_{3\nu} D^\mu \psi + R_\nu^4 D^\nu K_{4\nu} D^\mu \psi 
\]

\[ = R_\nu^a \left[ \frac{\operatorname{tr} \chi |q|^2}{}D^3 \psi - \frac{\operatorname{tr} \chi |q|^2}{2} \delta_{\alpha \beta} D^3 |q|^2 D^\alpha D^\beta - \frac{1}{2} R_{34} |q|^2 D^\nu - \frac{1}{2} R_{43} |q|^2 D^\nu \right] 
\]

\[ = -\frac{1}{2} \frac{Q^2}{|q|^2} \left[ \frac{\operatorname{tr} \chi D^3 \psi + \operatorname{tr} \chi D^3 \psi + \frac{Q^2}{|q|^2} (\eta_3 + \eta_3) D^3 \psi \right] 
\]

Finally, we compute \( I_3. \)

\[ I_3 = D^\nu R_\nu^a K_{\nu \alpha} D^\nu \psi = D^\nu R_\nu^a K_{\nu \alpha} D^\nu \psi + D^\nu R_\nu^3 K_{3\nu} D^\nu \psi + D^\nu R_\nu^4 K_{4\nu} D^\nu \psi 
\]

\[ = D_3 R_\nu^a K_{\nu \alpha} D^\nu \psi + \frac{1}{4} D_3 R_{4\nu} K_{3\nu} D^\nu \psi + \frac{1}{4} D_4 R_{3\nu} K_{4\nu} D^\nu \psi 
\]

\[ = D_3 R_\nu^a K_{\nu \alpha} D^\nu \psi + \frac{1}{4} D_3 R_{4\nu} K_{3\nu} D^\nu \psi + \frac{1}{4} D_4 R_{3\nu} K_{4\nu} D^\nu \psi 
\]

\[ + \frac{1}{4} D_4 R_{3\nu} K_{3\nu} D^\nu \psi + \frac{1}{4} D_4 R_{3\nu} K_{3\nu} D^\nu \psi + \frac{1}{4} D_4 R_{34} K_{3\nu} D^\nu \psi 
\]

which gives, using the above values,

\[ I_3 = -2 \frac{Q^2}{|q|^2} (\eta_3 + \eta_3) D^a \psi - \frac{Q^2}{|q|^2} 2r^2 D^3 \psi - \operatorname{tr} \frac{Q^2}{|q|^2} 2r^2 D^4 \psi 
\]

\[ = -\frac{1}{4} \frac{Q^2}{|q|^2} |q|^2 2\eta_3 (a^2 \cos^2 \theta) D^a \psi - \frac{1}{4} \operatorname{tr} \frac{4Q^2}{|q|^2} 2(a^2 \cos^2 \theta) D^3 \psi 
\]

\[ = -\frac{1}{4} \frac{Q^2}{|q|^2} |q|^2 2\eta_3 (a^2 \cos^2 \theta) D^a \psi - \frac{1}{4} \operatorname{tr} \frac{4Q^2}{|q|^2} 2(a^2 \cos^2 \theta) D^4 \psi 
\]
\[
\frac{Q^2}{|g|}(\text{tr} \chi D_4 \psi + \text{tr} \chi D_3 \psi) - 2 \frac{Q^2}{|g|}(\eta^a + \eta^b)D_a \psi = 2I_1.
\]

We therefore obtain
\[
[K, \Box_g] \psi = 2 \left( I_1 - I_2 - I_3 \right) = 0,
\]
as stated.

### 1.5 The modified Laplacian in Kerr-Newman

Even though the Carter differential operator \( K \) is a symmetry operator for the Kerr-Newman metric, it is convenient to extract from it an elliptic operator which we identify as a modified Laplacian in Kerr-Newman, see also [36]. Such modified Laplacian is then proved to be a conformal symmetry operator, as it is a symmetry operator for the conformal rescaling of the Kerr-Newman metric \(|q|^2 g_{M,a,Q} \).

**Proposition 1.12.** The Carter differential operator \( K \) in Kerr-Newman is given by
\[
K = -(a^2 \cos^2 \theta) \Box_{g_{M,a,Q}} + O,
\]
where \( O \) denotes a second order differential operator given by
\[
O(\psi) = |q|^2 \left( \triangle \psi + (\eta + \eta') \cdot \nabla \psi \right),
\]
where \( \triangle \psi = \delta^{ab} \nabla_a \nabla_b \psi \), for \( a, b = 1, 2 \). We call \( O \) the modified Laplacian of the Kerr-Newman metric. Moreover, the modified Laplacian \( O \) is a conformal symmetry operator, i.e. for a scalar function \( \psi \) we have
\[
[O, |q|^2 \Box_{g_{M,a,Q}}] \psi = 0.
\]

**Proof.** We compute
\[
K(\psi) = D_\mu (K^{\mu\nu} D_\nu \psi) = K^{\mu\nu} D_\mu \psi + D_\mu K^{\mu\nu} D_\nu \psi
\]
Using [39] and the definition of \( K \) [33], we obtain
\[
K(\psi) = \left( -(a^2 \cos^2 \theta) g^{\mu\nu} + |q|^2 (e_1 \otimes e_1 + e_2 \otimes e_2)^{\mu\nu} \right) D_\mu \psi
+ (\eta^a + \eta'^a)|q|^2 \nabla_a \psi + \frac{1}{2} |q|^2 \text{tr} \chi \nabla_3 \psi + \frac{1}{2} |q|^2 \text{tr} \nabla_4 \psi
= -(a^2 \cos^2 \theta) \Box_{g_{M,a,Q}} \psi + |q|^2 \delta^{ab} D_a D_b \psi
+ (\eta^a + \eta'^a)|q|^2 \nabla_a \psi + \frac{1}{2} |q|^2 \text{tr} \chi \nabla_3 \psi + \frac{1}{2} |q|^2 \text{tr} \nabla_4 \psi
= -(a^2 \cos^2 \theta) \Box_{g_{M,a,Q}} \psi + |q|^2 \delta^{ab} (\nabla_b \nabla_a \psi - \frac{1}{2} \eta^b \nabla_3 \psi - \frac{1}{2} \chi_{ba} \nabla_4 \psi)
+ (\eta^a + \eta'^a)|q|^2 \nabla_a \psi + \frac{1}{2} |q|^2 \text{tr} \chi \nabla_3 \psi + \frac{1}{2} |q|^2 \text{tr} \nabla_4 \psi,
\]
which gives
\[
K(\psi) = -(a^2 \cos^2 \theta) \Box_{g_{M,a,Q}} \psi + |q|^2 \left( \delta^{ab} \nabla_b \nabla_a \psi + (\eta^a + \eta'^a) \nabla_a \psi \right),
\]
where the last two terms define the operator \( O(\psi) \). This proves the first part of the Proposition.

Using [39] to write \( O = K + (a^2 \cos^2 \theta) \Box_g \), we deduce using Theorem [11],
\[
[O, \Box_g] \psi = [K, \Box_g] \psi + [(a^2 \cos^2 \theta) \Box_g, \Box_g] \psi = [(a^2 \cos^2 \theta) \Box_g, \Box_g] \psi.
\]
Recall, see for example [39], for a scalar function \( f \) we have
\[
\Box_g f = -\frac{1}{2} \Box_3 \nabla_4 + \nabla_4 \nabla_3 f + \left( \omega - \frac{1}{2} \text{tr} \chi \right) \nabla_4 f + \left( \omega - \frac{1}{2} \text{tr} \chi \right) \nabla_3 f + \Delta f + (\eta + \eta') \cdot \nabla f.
\]

Consequently, we have
\[
\Box_g (f h) = \Box_g (f h) + f \Box_g (h) - \nabla_3 f \nabla_4 h - \nabla_4 f \nabla_3 h + 2 \nabla f \cdot \nabla h.
\]

We then obtain,
\[
[O, \Box_g] \psi = (a^2 \cos^2 \theta) \Box_g (\Box_g \psi) - \Box_g \left( (a^2 \cos^2 \theta) \Box_g \psi \right)
= -\Box_g (a^2 \cos^2 \theta) \Box_g \psi - 2 \nabla (a^2 \cos^2 \theta) \cdot \nabla (\Box_g \psi).
\]
where we used that $e_3(a^2 \cos^2 \theta) = e_4(a^2 \cos^2 \theta) = 0$. Also, using (42) we compute

$$\Box_g(a^2 \cos^2 \theta) = \Delta(a^2 \cos^2 \theta) + (\eta + \bar{\eta}) \cdot \nabla(a^2 \cos^2 \theta) = |q|^{-2} \mathcal{O}(a^2 \cos^2 \theta).$$

This gives

$$[\mathcal{O}, \Box_g]\psi = -|q|^{-2} \mathcal{O}(a^2 \cos^2 \theta) \Box_g \psi - 2\nabla(a^2 \cos^2 \theta) \cdot \nabla(\Box_g \psi).$$

We can finally deduce from (11):

$$[\mathcal{O}, |q|^2 \Box_g]\psi = \mathcal{O}(|q|^2 \Box_g \psi) - |q|^2 \Box_g(\mathcal{O}(\psi))$$

where we used that

$$\mathcal{O}(\psi) = \mathcal{O}(\psi)$$

and

$$\Box_g\psi = |q|^2 \Box_g \psi - 2|q|^2 \nabla(a^2 \cos^2 \theta) \cdot \nabla(\Box_g \psi)$$

where we used that $\nabla(|q|^2) = \nabla(a^2 \cos^2 \theta) = \nabla(a^2 \cos^2 \theta)$ and $\mathcal{O}(|q|^2) = \mathcal{O}(a^2 \cos^2 \theta)$. This proves the Proposition.

We finally express the modified Laplacian $\mathcal{O}$ explicitly in Boyer-Lindquist coordinates in Kerr-Newman.

**Lemma 1.13.** The modified Laplacian $\mathcal{O}$ in Boyer-Lindquist coordinates reads

$$\mathcal{O} = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \partial_\phi^2 + 2a \partial_t \partial_\phi + a^2 \sin^2 \theta \partial_t^2$$

where $\Box_g = \Delta_g + 2a \partial_t \partial_\phi + a^2 \sin^2 \theta \partial_t^2$ is the (unit) spherical Laplacian on $S^2$.

**Proof.** We recall that the Laplacian $\Delta$ of a scalar function is given by, see [36]

$$\Delta \psi = g^{ab} \nabla_a \nabla_b \psi = \psi_{,1} \psi_{,1} + \psi_{,2} \psi_{,2} = e_1 e_1(\psi) - \nabla \psi_{,1} \psi + e_2 e_2(\psi) - \nabla \psi_{,2} \psi$$

where we used (20). Using the values (20) of $e_1$ and $e_2$, we compute

$$e_1 e_1(\psi) = \frac{1}{|q|} \partial_\theta (\frac{1}{|q|} \partial_\theta \psi) = \frac{1}{|q|} \partial_\theta^2 \psi + \frac{1}{|q|^2} \partial_\theta (\frac{1}{|q|} \partial_\theta \psi)$$

$$e_2 e_2(\psi) = \frac{a \sin \theta}{|q|} \partial_\phi + \frac{1}{|q|^2} \partial_\phi \partial_\phi \psi$$

which gives

$$\Delta \psi = \frac{1}{|q|^2} \partial_\theta^2 \psi + \frac{a^2 \sin^2 \theta \sin \theta}{|q|^4} \partial_\theta \psi + \frac{a^2 \sin^2 \theta}{|q|^2} \partial_\phi^2 \psi + \frac{a^2}{|q|^2} \partial_\theta \partial_\phi \psi + \frac{1}{|q|^2} \sin^2 \theta \partial_\phi^2 \psi$$

We also compute

$$(\eta + \bar{\eta}) \cdot \nabla \psi = (\eta_1 + \bar{\eta}_1) e_1(\psi) + (\eta_2 + \bar{\eta}_2) e_2(\psi) = -2 \frac{a^2 \cos \theta \sin \theta}{|q|^4} \partial_\theta (\psi)$$

which gives

$$\mathcal{O}(\psi) = \frac{1}{|q|^2} (\Delta \psi + (\eta + \bar{\eta}) \cdot \nabla \psi)$$

$$= \partial_\theta^2 \psi + \frac{r^2 + a^2 + a^2 \sin^2 \theta}{|q|^4} \cos \theta \partial_\theta \psi + \frac{a^2 \sin^2 \theta \partial_\theta^2 \psi}{|q|^2} + \frac{2a^2}{|q|^2} \partial_\theta \partial_\phi \psi + \frac{1}{|q|^2} \sin^2 \theta \partial_\phi^2 \psi$$

as stated.
2 The physical-space analysis of the wave equation

In this section we prove energy and local decay estimates for solutions to the scalar wave equation

$$\Box_{g_{M,a,Q}} \psi = 0$$

in Kerr-Newman spacetime for $|a| \ll M$ entirely in physical space, i.e. without recurring into frequency space or decomposition in modes. In order to do that, we will make use the approach first developed in [3] in Kerr to commute the wave equation with the Carter tensor and the modified Laplacian $\mathcal{O}$. From the fundamental commutation property obtained in Theorem 1.1, we can extend the procedure of [3] to the case of Kerr-Newman, where a resolution in physical space is crucial to tackle the problem of stability of the fundamental commutation property obtained in Theorem 1.11, we can extend the procedure of [3] to the case of Kerr-Newman, where a resolution in physical space is crucial to tackle the problem of stability of the solution to electromagnetic-gravitational perturbations for the Einstein-Maxwell equation, as we will see in Section 3.

As in [3], define

$$|\psi|^2 := |\psi|^2 + |\partial_t \psi|^2 + |\partial_r \psi|^2 + \sum_{\alpha=1}^4 |S_\alpha \psi|^2,$$

where $S_\alpha$, for $\alpha = 1, 2, 3, 4$, denote the set of the second order operators given by, see Definition 2.2.

$$S_1 = \partial_t^2, \quad S_2 = \partial_t \partial_r, \quad S_3 = \partial_r^2, \quad S_4 = \mathcal{O}.$$ (49)

We obtain the following.

Theorem 2.1. Let $\psi$ be a sufficiently regular solution to the wave equation in the slowly rotating Kerr-Newman spacetime $g_{M,a,Q}$ with $|a| \ll M$, with initial data on $\Sigma_0$ which decays sufficiently fast.

Then the following energy-Morawetz estimates, for $\tau \geq 0$, can be obtained through a physical-space analysis:

$$E_{\tau,S}[\psi] + Mor_{(\tau,\Sigma)}[\psi] \lesssim E_0,S[\psi]$$ (50)

where

$$E_{\tau,S}[\psi] := \int_{\Sigma_\tau} |\partial_t \psi|^2 + |\partial_r \psi|^2 + |\nabla \psi|^2$$

$$Mor_{(\tau,\Sigma)}[\psi] := \int_{M(\tau,\Sigma)} \frac{M}{r^3} |\partial_t \psi|^2 + \frac{M}{r^4} |\psi|^2 + \tilde{I}_{(r \neq r_{\text{trap}})} (r^{-1} |\nabla \psi|^2 + \frac{M}{r^4} |\partial_t \psi|^2)$$ (52)

where $r_{\text{trap}} = \frac{3M + \sqrt{9M^2 - 8Q^2}}{2}$ is the photon sphere of Reissner-Nordström $g_{M,Q}$ and $\tilde{I}_{(r \neq r_{\text{trap}})}$ is a function that is identically 1 for $|r - r_{\text{trap}}| > \delta$ for some $\delta > 0$ and zero otherwise, and $|\nabla \psi|^2 = |\nabla_1 \psi|^2 + |\nabla_2 \psi|^2$ with respect to the orthonormal basis in (20).

Observe that the above estimates is not optimal in terms of decay in $r$, and those weights can be improved by applying the $r^p$ hierarchy of estimates introduced by Dafermos-Rodnianski in [23] in a standard fashion.

2.1 Preliminaries

From now on, we denote $g = g_{M,a,Q}$ for $a^2 + Q^2 < M^2$. From the form of the Kerr-Newman metric in Boyer-Lindquist coordinates $(t, r, \theta, \varphi)$ as given by (14), one can deduce, see [3], that its conformal inverse $|q|^2 g^{-1}$ can be written as

$$|q|^2 g^{-\alpha \beta} = \Delta \partial_\alpha \partial_\beta + \frac{1}{\Delta} R^{\alpha \beta},$$ (53)

where

$$R^{\alpha \beta} = -(r^2 + a^2) \partial_\alpha \partial_\beta - 2a(r^2 + a^2) \partial_t (\partial^\alpha \partial_\beta) - a^2 \partial_\alpha \partial_\varphi \partial_\beta + \Delta O^{\alpha \beta},$$ (54)

$$O^{\alpha \beta} = \partial_\theta \partial_\alpha \theta + \frac{1}{\sin^2 \theta} \partial_\theta \partial_\beta \theta + 2a \partial_t (\partial^\alpha \partial_\varphi) + a^2 \sin^2 \theta \partial_\varphi \partial_\theta \partial_\varphi.$$ (55)

Observe that $O^{\alpha \beta}$ is the same tensor as defined in (33), but written here in Boyer-Lindquist coordinates using the orthonormal frame in (20).

The scalar wave equation on a Lorentzian manifold can be written in coordinates as

$$\Box_g \psi = \frac{1}{\sqrt{-\det g}} \partial_\alpha ((\sqrt{-\det g}) g^{\alpha \beta} \partial_\beta \psi) = 0,$$
and one can deduce from \([\text{Eqn}]\) that the wave operator for the Kerr-Newman metric in Boyer-Lindquist coordinates is given by

\[
|q|^2 \Box_g = \partial_t (\Delta \partial_t) + \frac{1}{\Delta} \left( - (r^2 + a^2)^2 \partial_t^2 - 2a(r^2 + a^2)\partial_t \partial_r - a^2 \partial_r^2 \right) \\
+ \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \partial_\phi^2 + 2a \partial_\phi \partial_r + a^2 \sin^2 \theta \partial_t^2
\]

(\ref{waveoperator}) where \(\mathcal{O}\) is the modified Laplacian defined in \([\text{Eqn}]\).

We now briefly formulate the initial value problem for the wave equation. We prescribe initial data on the (axisymmetric) hypersurface \(\Sigma_0 = \{t^* = 0\}\) where \(t^*\) is the Kerr star coordinate. We are interested in the behavior of the solution in the future Cauchy development of \(\Sigma_0\) which is given by \(\{t^* \geq 0\}\). Denote \(\phi_r\) the 1-parameter family of diffeomorphisms generated by the vector field \(T\), and define the spacelike hypersurfaces \(\Sigma_t = \phi_r(\Sigma_0) = \{t = t^*\}\). Each leaf of this foliation terminates at the horizon and at spatial infinity \(\partial^0\) (for more details see \([\text{Eqn}]\), \([\text{Eqn}]\)). For \(T_2 > T_1\), the leaf \(\Sigma_{T_2}\) lies in the future of \(\Sigma_{T_1}\), and we denote the region bounded by \(\Sigma_{T_1}, \Sigma_{T_2} \text{ and } \mathcal{H}^+\) by \(M(T_1, T_2) = \cup_{T_1 \leq T \leq T_2} \Sigma_T\). We also denote \(\mathcal{H}^+(T_1, T_2) = \mathcal{H}^+ \cap M(T_1, T_2)\).

The wave equation is well posed in \(M(0, \tau)\) with initial data \((\psi_0, \psi_1)\) defined on \(\Sigma_0 \in H^2_{loc}(\Sigma_0) \times H^2_{loc}(\Sigma_0), j \geq 1\), see for example \([\text{Eqn}]\). Furthermore, the solutions depends smoothly on the parameters \(a\) and \(Q\), see \([\text{Eqn}]\). By symmetry we can always assume the positivity of \(a\).

### 2.2 The vectorfield method

We recall the main definitions in applying the vectorfield method to derive energy estimates for the wave equation. The vectorfield method is based on applying the divergence theorem in a causal domain, like \(M(T_1, T_2)\), to certain energy currents, which are constructed from the energy momentum tensor.

- The energy-momentum tensor associated to the wave equation \([\text{Eqn}]\) is given by

\[
Q[\psi]_{\mu \nu} = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu \nu} \partial_\lambda \psi \partial^\lambda \psi.
\]

(\ref{energymomentum}) The wave equation \([\text{Eqn}]\) is satisfied if and only if the divergence of the energy-momentum tensor \(Q[\psi]\) vanishes.

- Let \(X\) be a vectorfield and \(w\) be a function. The current associated to \((X, w)\) is defined as

\[
\mathcal{P}_\mu^{(X, w)}[\psi] = Q[\psi]_{\mu \nu} X^\nu + \frac{1}{2} w \partial_\nu \partial_\mu \psi - \frac{1}{4} (\partial_\mu w) \psi^2.
\]

(\ref{current})

- The energy associated to \((X, w)\) on the hypersurface \(\Sigma_t\) is

\[
E^{(X, w)}[\psi](\tau) = \int_{\Sigma_{\tau}} \mathcal{P}_\mu^{(X, w)}[\psi] n_\Sigma^\mu.
\]

where \(n_{\Sigma_T}\) denotes the future directed timelike unit normal to \(\Sigma_T\).

A standard computation, see for example \([\text{Eqn}]\), then implies for the divergence of \(\mathcal{P}\):

\[
\mathbf{D}^\nu \mathcal{P}_\mu^{(X, w)}[\psi] = \frac{1}{2} Q[\psi] \cdot (X) \pi - \frac{1}{4} |\Box_g \psi|^2 + \frac{1}{2} w (\partial_\lambda \psi \partial^\lambda \psi),
\]

(\ref{divergence})
where \((X)\pi_{\mu\nu} = D_{(\mu} X_{\nu)}\) is the deformation tensor of the vector field \(X\). Recall that if \(X\) is a Killing vector field, then \((X)\pi = 0\).

For convenience we introduce the notation,

\[ \mathcal{E}^{(X,w)}[\psi] := D^\mu P^{(X,w)}_{\mu}[\psi]. \]  

(60)

By applying the divergence theorem to \(P^{(X,w)}_{\mu}\) within a region such as \(M(\tau_1, \tau_2)\) for carefully chosen \((X,w)\) one obtains the associated energy identity:

\[ E^{(X,w)}[\psi](\tau_2) + \int_{P(\tau_1, \tau_2)} P^{(X,w)}_{\mu}[\psi] n^\mu + \int_{M(\tau_1, \tau_2)} \mathcal{E}^{(X,w)}[\psi] = E^{(X,w)}[\psi](\tau_1), \]  

where the induced volume forms are to be understood. By convention, along the event horizon \(\mathcal{H}^+\) we choose \(n^\mu = \mathcal{T}_H\).

For two positive quantities \(F\) and \(G\), in what follows we write \(F \lesssim G\) to signify that there exists a universal constant \(C\), depending only on \(M, a, Q\), such that \(F \leq CG\).

### 2.3 Symmetry operators and the generalized vectorfield method

Since the Kerr-Newman metric possesses only two Killing vectorfields, \(T = \partial_t\) and \(Z = \partial_\phi\), which commute with the D’Alembertian operator associated to the metric, the control on those first derivatives is not sufficient to control all the first derivatives of a solution to the wave equation.

In order to control all the derivatives, one needs to make use of the Carter tensor, through the form of the modified Laplacian \(\mathcal{O}\) as defined in Section 1.5. Since such operator is second order, we will make use of second order differential operators as obtained from the two Killing vectorfields and the modified Laplacian.

Following \cite{3} and making crucial use of our extension of the commutation in the case of the non-Ricci flat Kerr-Newman spacetime in Theorem 1.11 and (41), we define the following second order symmetry operators.

**Definition 2.2.** The set of the second order operators \(S_\alpha\) for \(\alpha = 1, 2, 3, 4\), given by

\[ S_1 = \partial_t^2, \quad S_2 = \partial_t \partial_\phi, \quad S_3 = \partial_\phi^2, \quad S_4 = \mathcal{O} \]  

are denoted conformal symmetry operators, as for a scalar function \(\psi\) we have

\[ [S_\alpha, |q|^2 \Box_{g_{M,a,Q}}] \psi = 0, \quad \alpha = 1, 2, 3, 4. \]  

(63)

Observe that the conformal symmetry operators commute with each other, i.e.

\[ [S_\alpha, S_\beta] = 0, \quad \alpha, \beta = 1, 2, 3, 4. \]  

(64)

The modified Laplacian \(\mathcal{O}\) in (60) differs from the spherical Laplacian by terms depending on \(\partial_t^2\) and \(\partial_\phi \partial_\phi\), so the second order operators \(S_\alpha\) for \(\alpha = 1, 2, 3, 4\) together provide the spherical Laplacian, which has the elliptic properties necessary to control the angular derivatives. More precisely, the following estimates allow us to obtain pointwise boundedness and decay bounds for \(\psi\):

\[ |\psi|^2 \leq C \int_{S^2} |\psi|^2 + |\Delta_{S^2} \psi|^2 \leq C \int_{S^2} |\psi|^2 + \sum_{\alpha=1}^4 |S_\alpha \psi|^2, \]  

(65)

which follows from the spherical Sobolev inequality and \cite{15}.

In addition to the conformal symmetry operators, we also define their tensorial versions.

**Definition 2.3.** We define the following symmetric tensors

\[ S^{\alpha \beta}_1 = T^\alpha T^\beta, \quad S^{\alpha \beta}_2 = T^{(\alpha} Z^{\beta)}, \quad S^{\alpha \beta}_3 = Z^\alpha Z^\beta, \quad S^{\alpha \beta}_4 = \mathcal{O}^{\alpha \beta} \]  

(66)

With the above definition, from \cite{3}, one can write as in \cite{3}

\[ R^{\alpha \beta} = -((r^2 + a^2)^2 S_1^{\alpha \beta} - 2a(r^2 + a^2) S_2^{\alpha \beta} + a^2 S_3^{\alpha \beta} + \Delta_{\mathcal{O}}^{\alpha \beta}) = R^2 S^{\alpha \beta}, \]  

(67)

with

\[ R^1 = -(r^2 + a^2)^2, \quad R^2 = -2a(r^2 + a^2), \quad R^3 = -a^2, \quad R^4 = \Delta. \]  

(68)

Observe that the symmetric tensors \(S_\alpha\) are easily related to the conformal symmetry operators \(S_\alpha\).
Lemma 2.4. The symmetric tensors defined in Definition 2.3 and the conformal symmetry operators defined in Definition 2.2 are related by the following:

\[ S_\alpha = |q|^2 D_\alpha (|q|^{-2} S^\alpha \beta D_\beta), \quad \alpha = 1, 2, 3, 4. \]  

(69)

Proof. Observe that for \( \alpha = 1, 2, 3 \), we have \( |q|^2 D_\alpha (|q|^{-2} S^\alpha \beta D_\beta) = D_\alpha (S^\alpha \beta D_\beta) \), and for example for \( \alpha = 1 \) we have

\[ D_\alpha (S^\alpha \beta D_\beta) = D_\alpha (T^\alpha \beta D_\beta) = (D_\alpha T^\alpha) T^\beta D_\beta + (T^\alpha D_\alpha) (T^\beta D_\beta) = \partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha \]

since \( D_\alpha T^\alpha = \text{tr}(T) = 0 \). Similarly for \( \alpha = 2, 3 \). Using (65) and (69), we obtain

\[
|q|^2 D_\alpha (|q|^{-2} O^{\alpha \beta} D_\beta) = \partial_\alpha D_\beta - \partial_\beta D_\alpha + \partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha - \partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha,
\]

as stated.

Define

\[
\psi_\alpha := S_\alpha (\psi), \quad \alpha = 1, 2, 3, 4.
\]

(70)

Then if \( \psi \) is a solution to the wave equation, by (63) \( \psi_\alpha \) is also a solution,

\[
\square_{g_{M, \alpha} \beta} \psi_\alpha = 0, \quad \alpha = 1, 2, 3, 4.
\]

(71)

Following [3], we recall here a generalized vectorfield method which incorporates the commutation of the wave equation by the conformal symmetry operators.

- The generalized energy-momentum tensor for \( \psi \) solutions to (47) is defined as

\[
Q[\psi]_{ab \mu
\nu} = \partial_\mu \psi_a \partial_\nu \psi_b - \frac{1}{2} g_{ab} \partial_\lambda \psi_a \partial_\lambda \psi_b
\]

for \( \alpha = 1, 2, 3, 4. \)

Remark 2.5. Since from (71) each \( \psi_\alpha \) is a solution to the wave equation, the above energy-momentum tensor can be interpreted as a symmetrization with respect to \( \psi_\alpha \) and \( \psi_\beta \) of their respective energy-momentum tensors. The underlined indices of \( Q \) denote the conformal symmetry operators applied to \( \psi \) while the Greek indices denote the spacetime indices as in the standard energy-momentum tensor.

- Let \( X \) be a symmetric double-indexed collection of vector fields \( X_{ab} \) and \( w \) be a symmetric double-indexed collection of functions \( w_{ab} \). The generalization associated to \( (X, w) \) is defined as

\[
P_{\mu} (X, w)[\psi] = Q[\psi]_{ab \mu \nu} X^{ab \mu \nu} + \frac{1}{2} w_{ab} \psi_a \partial_\mu \psi_b - \frac{1}{4} (\partial_\lambda w_{ab}) \psi_a \partial_\lambda \psi_b.
\]

(72)

for \( \alpha = 1, 2, 3, 4. \)

- The energy associated to \( (X, w) \) on the hypersurface \( \Sigma_{\tau} \) is

\[
E^{(X, w)}[\psi](\tau) = \int_{\Sigma_{\tau}} P_{\mu} (X, w)[\psi] H_{\tau}^\mu.
\]

As in (60), we obtain for the divergence of the generalized \( P \):

\[
D^\mu P_{\mu} (X, w)[\psi] = \frac{1}{2} \square_{g_{M, \alpha} \beta} X^{\alpha \beta} + \frac{1}{4} \square_{g_{M, \alpha} \beta} \psi_a \partial_\lambda \psi_b + \frac{1}{2} \partial_\lambda (\partial_\mu \psi_a \partial_\mu \psi_b),
\]

(73)

and we denote

\[
E^{(X, w)}[\psi] := D^\mu P_{\mu} (X, w)[\psi].
\]

(74)

By applying the divergence theorem to \( P_{\mu} (X, w) \) within a region such as \( M(\tau_1, \tau_2) \) for carefully chosen \( (X, w) \) one obtains the associated energy identity:

\[
E^{(X, w)}[\psi](\tau_2) + \int_{\partial \Sigma_{\tau_2}} P_{\mu} (X, w)[\psi] H_{\tau_2}^\mu + \int_{\Sigma_{\tau_1}} E^{(X, w)}[\psi] = E^{(X, w)}[\psi](\tau_1),
\]

(75)

where the induced volume forms are to be understood.
2.4 The relevant vectorfields and the spacetime current identities

In applying the vectorfield method to derive non-degenerate energy-Morawetz estimates for the wave equation, we make use of the following vectorfields:

- A radial vectorfield $X = \mathcal{F}(r)\partial_r$, for a well chosen function $\mathcal{F}$, to obtain Morawetz estimates,

- A timelike vectorfield $\hat{T}_\chi = T + \chi \omega_H Z$, where $\chi = 1$ for $\{ r < r_1 \}$ for some $r_1 > r_+$, and $\chi = 0$ for $\{ r > r_1 \}$, with a smooth decrease in $[r_1, r_2]$. In particular, $\hat{T}_\chi = \hat{T}_H$ close to the horizon, and $\hat{T}_\chi = T$ for $r > r_2$. Also, $\hat{T}_\chi$ is not Killing only for $r \in [r_1, r_2]$. This is used to obtain energy estimates.

We collect here some relevant computations involving the deformation tensors of the above vectors which will be used in the next sections.

**Lemma 2.6.** Let $\psi$ be a solution to the wave equation (77), then the following relations hold true.

- For $X = \mathcal{F}(r)\partial_r$, we have
  \[
  (X)_{\pi}^{\alpha\beta} = |q|^{-2} \left( 2\Delta^{3/2} \partial_r \left( \frac{\mathcal{F}}{\Delta^{1/2}} \partial_r \partial_r^{\beta} - \mathcal{F} \partial_r \left( \frac{1}{\Delta} R^{\alpha\beta} \right) \right) + |q|^{-2} \mathcal{X}(|q|^2) g^{\alpha\beta},
  \]
  and therefore
  \[
  |q|^2 \mathcal{Q}[\psi] \cdot (X)_{\pi} = 2|q|^{-2} \Delta \omega_H (\partial_r \chi) \partial_r(\alpha) \partial_r(\beta),
  \]

- For $\hat{T}_\chi = T + \chi \omega_H Z$, we have
  \[
  (\hat{T}_\chi)_{\pi}^{\alpha\beta} = 2|q|^{-2} \Delta \omega_H (\partial_r \chi) \partial_r(\alpha) \partial_r(\beta),
  \]
  and therefore
  \[
  |q|^2 \mathcal{Q}[\psi] \cdot (\hat{T}_\chi)_{\pi} = 2\Delta \omega_H (\partial_r \chi) \partial_r(\alpha) \partial_r(\beta).
  \]

**Proof.** Using the expression for the inverse metric (65), we compute
\[
\mathcal{L}_X (|q|^2 g^{\alpha\beta}) = \mathcal{L}_X (\Delta \partial_r \partial_r^{\beta}) + \mathcal{L}_X \left( \frac{1}{\Delta} R^{\alpha\beta} \right) = X(\Delta) \partial_r^{\alpha} \partial_r^{\beta} + \Delta [X, \partial_r^{\alpha}] \partial_r^{\beta} + \Delta \partial_r^{\alpha}[X, \partial_r] \partial_r^{\beta} + \mathcal{L}_X \left( \frac{1}{\Delta} R^{\alpha\beta} \right).
\]
Consequently, for $X = \mathcal{F}(r)\partial_r$, we obtain
\[
\mathcal{L}_X (|q|^2 g^{\alpha\beta}) = \mathcal{F}(\partial_r \Delta) \partial_r^{\alpha} \partial_r^{\beta} + \Delta [\mathcal{F} \partial_r, \partial_r]^{\alpha} \partial_r^{\beta} + \Delta \partial_r^{\alpha}[\mathcal{F} \partial_r, \partial_r]^{\beta} + \mathcal{F} \mathcal{L}_\partial \left( \frac{1}{\Delta} R^{\alpha\beta} \right)
\]
\[
= \mathcal{F}(\partial_r \Delta) \partial_r^{\alpha} \partial_r^{\beta} - 2\Delta (\partial_r \mathcal{F}) \partial_r^{\alpha} \partial_r^{\beta} + \mathcal{F} \partial_r \left( \frac{1}{\Delta} R^{\alpha\beta} \right)
\]
\[
= -2\Delta^{3/2} \partial_r \left( \frac{\mathcal{F}}{\Delta^{1/2}} \partial_r \partial_r^{\beta} - \mathcal{F} \partial_r \left( \frac{1}{\Delta} R^{\alpha\beta} \right) \right).
\]

For $\hat{T}_\chi = T + \chi \omega_H Z$ we obtain
\[
\mathcal{L}_{\hat{T}_\chi} (|q|^2 g^{\alpha\beta}) = \Delta [\hat{T}_\chi, \partial_r]^{\alpha} \partial_r^{\beta} + \Delta \partial_r^{\alpha}[\hat{T}_\chi, \partial_r]^{\beta} = -2\Delta \omega_H (\partial_r \chi) \partial_r(\alpha) \partial_r(\beta).
\]
In particular observe that $(\hat{T}_\chi)_{\pi}^{\alpha\beta} g_{\mu\nu} = 0$. By writing
\[
(X)_{\pi}^{\alpha\beta} = -\mathcal{L}_X (|q|^{-2} |q|^2 g^{\alpha\beta}) = -|q|^{-2} \Delta \mathcal{L}_X (|q|^2 g^{\alpha\beta}) = |q|^2 \mathcal{L}_X (|q|^2 g^{\alpha\beta}) - |q|^2 \mathcal{L}_X (|q|^{-2}) g^{\alpha\beta}
\]
we obtain the stated expressions for the deformation tensors.

Finally, for any vectorfield $X$, we write
\[
\mathcal{Q}[\psi] \cdot (X)_{\pi} = \mathcal{Q}[\psi] \cdot (X)_{\pi}^{\alpha\beta} = \left( \partial_\alpha \psi \partial_\beta \psi - \frac{1}{2} g_{\alpha\beta} \partial_\lambda \psi \partial_\lambda \psi \right) (X)_{\pi}^{\alpha\beta}
\]
\[
= (X)_{\pi}^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi - (\text{div } X) \partial_\alpha \psi \partial_\alpha \psi
\]
since $g_{\mu\nu} (X)_{\pi}^{\alpha\beta} = g_{\mu\nu} D(\mu X(\nu)} = 2 \text{div } X$. Using the above expressions for $(X)_{\pi}$ we obtain the stated identities.

□
We make use of the above computations to derive the Morawetz current for $E^{(X,w)}[\psi]$ with $X = F\partial_r$ and its generalized version $X^{ab} = F^{ab}\partial_r$. We mostly follow notations in [3]. See also [37].

**Proposition 2.7.** Let $z$ be a given function of $r$. The following identities hold true.

1. Let $u$ be a given function of $r$. Then for

$$
X = F\partial_r, \quad F = zu, \quad w = z\partial_r u,
$$

the current $E^{(X,w)}[\psi]$ satisfies

$$
|q|^2 E^{(X,w)}[\psi] = A|\partial_r \psi|^2 + U^{\alpha\beta}(\partial_\alpha \psi)(\partial_\beta \psi) + V|\psi|^2, \tag{81}
$$

where

$$
A = z^{1/2}\Delta^{3/2}\partial_r \left( \frac{z^{1/2}u}{\Delta^{1/2}} \right), \tag{82}
$$

$$
U^{\alpha\beta} = -\frac{1}{2}w_\alpha \partial_r \left( \frac{z}{\Delta} R^{\alpha\beta} \right), \tag{83}
$$

$$
V = -\frac{1}{4}\partial_r (\Delta \partial_r w) = -\frac{1}{4}\partial_r (\Delta \partial_r (z\partial_r u)). \tag{84}
$$

2. Let $u^{ab}$ be a given double-indexed function of $r$. Then for

$$
X^{ab} = F^{ab}\partial_r, \quad F^{ab} = zu^{ab}, \quad u^{ab} = z\partial_r u^{ab},
$$

the generalized current $E^{(X^{ab},w)}[\psi]$ satisfies

$$
|q|^2 E^{(X^{ab},w)}[\psi] = A^{ab}\partial_r \psi_\alpha \partial_r \psi_\beta + U^{\alpha\beta ab}(\partial_\alpha \psi_\alpha \partial_\beta \psi_\beta) + V^{ab}\psi_\alpha \psi_\beta \tag{86}
$$

where

$$
A^{ab} = z^{1/2}\Delta^{3/2}\partial_r \left( \frac{z^{1/2}u^{ab}}{\Delta^{1/2}} \right), \tag{87}
$$

$$
U^{\alpha\beta ab} = -\frac{1}{2}u^{ab} \partial_r \left( \frac{z}{\Delta} R^{\alpha\beta} \right), \tag{88}
$$

$$
V^{ab} = -\frac{1}{4}\partial_r (\Delta \partial_r u^{ab}) = -\frac{1}{4}\partial_r (\Delta \partial_r (z\partial_r u^{ab})). \tag{89}
$$

**Proof.** We prove the first part of the Proposition, as the second part in the case of generalized current follows in the same way. Using (60), (59) and (77) we compute for $(X = F\partial_r, w)_2$

$$
|q|^2 E^{(X,w)}[\psi] = \frac{1}{2}|q|^2 Q[\psi] \cdot (X)_2 - \frac{1}{4}|q|^2 \Box_g w|\psi|^2 + |q|^2 w(\partial_\lambda \psi \partial^\lambda \psi)
$$

$$
= \Delta^{3/2}\partial_\alpha \left( \frac{F}{\Delta^{1/2}} \right) |\partial_r \psi|^2 - \frac{1}{2}F\partial_\alpha \left( \frac{1}{\Delta} R^{\alpha\beta} \right) \partial_\alpha \psi \partial_\beta \psi - \frac{1}{4}|q|^2 \Box_g w|\psi|^2
$$

$$
+ \frac{1}{2} \left( X(|q|^2) - |q|^2 (\text{div } X) + |q|^2 w \right) \partial_\lambda \psi \partial^\lambda \psi.
$$

By defining an intermediate function $w_{int}$ as

$$
\frac{1}{2} \left( X(|q|^2) - |q|^2 \text{div } X + |q|^2 w \right) = \frac{1}{2}|q|^2 \left( |q|^2 X(|q|^2) - \text{div } X + w \right) = -\frac{1}{2}|q|^2 w_{int},
$$

and using (53) to write

$$
|q|^2 \partial_\lambda \psi \partial^\lambda \psi = |q|^2 g^{\lambda\mu} \partial_\lambda \psi \partial_\mu \psi = \Delta |\partial_\lambda \psi|^2 + \frac{1}{\Delta} R^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi,
$$

we simplify the above to

$$
|q|^2 E^{(X,w)}[\psi] = \Delta^{3/2}\partial_\alpha \left( \frac{F}{\Delta^{1/2}} \right) |\partial_r \psi|^2 - \frac{1}{2}F\partial_\alpha \left( \frac{1}{\Delta} R^{\alpha\beta} \right) \partial_\alpha \psi \partial_\beta \psi - \frac{1}{4}|q|^2 \Box_g w|\psi|^2
$$

$$
- \frac{1}{2}w_{red} \left( \Delta |\partial_\lambda \psi|^2 + \frac{1}{\Delta} R^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi \right)
$$

$$
= \left( \Delta^{1/2}\partial_\alpha \left( \frac{F}{\Delta^{1/2}} \right) - \frac{1}{2}w_{int}\Delta \right) |\partial_r \psi|^2 - \frac{1}{2} \left( F\partial_\alpha \left( \frac{1}{\Delta} R^{\alpha\beta} \right) + w_{int} \frac{1}{\Delta} R^{\alpha\beta} \right) \partial_\alpha \psi \partial_\beta \psi
$$

$$
- \frac{1}{4}|q|^2 \Box_g w|\psi|^2.
$$
To summarize, we write the above as
\[ |q|^2 E(\mathcal{F}, \omega)[\psi] = \mathcal{A} |\partial_r \psi|^2 + U^{\alpha\beta} (\partial_{\alpha} \psi)(\partial_{\beta} \psi) + \mathcal{V} |\psi|^2, \]
where
\[ \mathcal{A} = \Delta^{3/2} \partial_r \left( \frac{\mathcal{F}}{\Delta^{1/2}} \right) - \frac{1}{2} w_{int} \Delta \]
\[ U^{\alpha\beta} = - \frac{1}{2} \mathcal{F} \partial_r \left( \frac{1}{\Delta} \mathcal{R}^{\alpha\beta} \right) - \frac{1}{2} w_{int} \frac{1}{\Delta} \mathcal{R}^{\alpha\beta} \]
\[ \mathcal{V} = - \frac{1}{4} |q|^2 \Box g w. \]
with \( w = |q|^2 \text{div} \left( |q|^{-2} X \right) - w_{int} \). Observe that for a function \( z \) we can write
\[ U^{\alpha\beta} = - \frac{1}{2} \mathcal{F} \partial_r \left( \frac{1}{\Delta} \mathcal{R}^{\alpha\beta} \right) - \frac{1}{2} w_{int} \frac{1}{\Delta} \mathcal{R}^{\alpha\beta} \]
\[ = - \frac{1}{2} Fz^{-1} \partial_r \left( \frac{z}{\Delta} \mathcal{R}^{\alpha\beta} \right) + \frac{1}{2} (Fz^{-1} \partial_r z - w_{int}) \frac{\mathcal{R}^{\alpha\beta}}{\Delta} \]
Setting \( \mathcal{F} = zu \) for a function \( u \), and choosing \( w_{int} = Fz^{-1} \partial_r z = u \partial_r z \), the coefficient of \( \frac{\mathcal{R}^{\alpha\beta}}{\Delta} \) cancels out, and we deduce the stated expression for \( U^{\alpha\beta} \) in (83).

With these choices for \( \mathcal{F} \) and \( w_{int} \), we compute the function \( w \):
\[ w = |q|^2 D_\alpha \left( |q|^{-2} F q_\alpha \right) - w_{int} = |q|^2 \partial_r (|q|^{-2} \mathcal{F}) + \mathcal{D}_\alpha q_\alpha = u \partial_r z. \]
Observe that
\[ D_\alpha q_\alpha = \frac{1}{\sqrt{|g|}} \partial_\alpha \left( \sqrt{|g|} q_\alpha \right) = \frac{1}{\sqrt{|g|}} \partial_\alpha \left( \sqrt{|g|} \right) = \frac{1}{|q|^2} \partial_r (|q|^2), \]
and therefore
\[ w = |q|^2 \partial_r (|q|^{-2} z u) + z u |q|^{-2} \partial_r (|q|^2) - u \partial_r z = \partial_r (z u) - u \partial_r z = z \partial_r u. \]
We also compute
\[ \mathcal{A} = \partial_r \left( \frac{\mathcal{F}}{\Delta^{1/2}} \right) \Delta^{3/2} - \frac{1}{2} \Delta w_{int} = \partial_r \left( \frac{z u}{\Delta^{1/2}} \right) \Delta^{3/2} - \frac{1}{2} \Delta (\partial_r z) u \]
\[ = \frac{1}{2} \partial_r z \frac{z u}{\Delta^{1/2}} \Delta^{3/2} + z^{1/2} \partial_r \left( z^{1/2} \frac{u}{\Delta^{1/2}} \right) \Delta^{3/2} - \frac{1}{2} \Delta (\partial_r z) u = z^{1/2} \Delta^{3/2} \partial_r \left( \frac{z^{1/2} u}{\Delta^{1/2}} \right). \]
Finally, as for a function \( H = H(r) \),
\[ \Box g w = \frac{1}{\sqrt{|g|}} \partial_\alpha \left( \sqrt{|g|} g^{\beta \alpha} \partial_\beta \right) H = \frac{1}{\sqrt{|g|}} \partial_r \left( \sqrt{|g|} g^{r \alpha} \partial_\alpha \right) H = \frac{1}{|q|^2} \partial_r (\Delta \partial_r H) \]
we compute
\[ |q|^2 \Box g w = \partial_r (\Delta \partial_r w) = \partial_r (\Delta \partial_r (z \partial_r u)), \]
as stated. \( \square \)

2.5 Trapped null geodesics

We now dedicate this section to the derivation of the equation satisfied by the trapped null geodesics in Kerr-Newman spacetime, see also [37]. Trapped null geodesics are relevant for the analysis of the wave equation, as in the high frequency limit waves behave like null geodesics and the integrated local energy estimates necessarily have to degenerate at the trapping region of the black hole.

Let \( \gamma(\lambda) \) be a null geodesic in Kerr-Newman spacetime. Using the expression for the inverse of the metric given by (83), along \( \gamma(\lambda) \), since \( g(\gamma, \gamma) = 0 \) we have, with \( \dot{\gamma}_r = \partial_r \gamma, \dot{\gamma}_t = \partial_t \gamma, \dot{\gamma}_\phi = \partial_\phi \gamma \)
\[ 0 = |q|^2 g^{\alpha \beta} \dot{\gamma}_\alpha \dot{\gamma}_\beta = (\Delta \partial_r^2 + \frac{1}{\Delta} \mathcal{R}^{\alpha \beta}) \dot{\gamma}_\alpha \dot{\gamma}_\beta = \Delta \dot{\gamma}_r \dot{\gamma}_r + \frac{1}{\Delta} \mathcal{R}^{\alpha \beta} \dot{\gamma}_\alpha \dot{\gamma}_\beta \]
with
\[ \mathcal{R}^{\alpha \beta} \dot{\gamma}_\alpha \dot{\gamma}_\beta = -(r^2 + a^2) \dot{\gamma}_t \dot{\gamma}_t - 2a(r^2 + a^2) \dot{\gamma}_t \dot{\gamma}_\phi - a^2 \dot{\gamma}_\phi \dot{\gamma}_\phi + \Delta \mathcal{O}^{\alpha \beta} \dot{\gamma}_\alpha \dot{\gamma}_\beta \]
Since \( \partial_t = T \) and \( \partial_\phi = Z \) are Killing vectorfields we deduce that \( \dot{\gamma}_t = \mathbf{g}(\dot{\gamma}, T) \) and \( \dot{\gamma}_\phi = \mathbf{g}(\dot{\gamma}, Z) \) are constants of the motion i.e. constants along \( \gamma \), and respectively called the energy and the azimuthal angular momentum. We write,

\[
e := -\mathbf{g}(\dot{\gamma}, T), \quad \ell_z = -\mathbf{g}(\dot{\gamma}, Z).
\]

We also define\(^6\)

\[
k^2 := K^{\alpha\beta}\dot{\gamma}_\alpha \dot{\gamma}_\beta
\]

for the Carter tensor \( K \) in Kerr-Newman. Since \( K \) is Killing, \( k^2 \) is also a constant of motion.

With these constants from \( (90) \) we have

\[
\mathcal{R}(r; e, \ell_z, k^2) := \mathcal{R}^{\alpha\beta}\dot{\gamma}_\alpha \dot{\gamma}_\beta = -(r^2 + a^2)e^2 - 2a(r^2 + a^2)e \cdot \ell_z - a^2 \ell_z^2 + \Delta k^2
\]

which is only a function of \( r \) along any fixed null geodesic. Going back to the equation for null geodesics we infer that

\[
\Delta \left( \frac{dr}{dt} \right)^2 = -\mathcal{R}(r; e, \ell_z, k^2),
\]

which is the equation for a null geodesic with constant of motions \( e, \ell_z, k^2 \).

There exist null geodesics along which \( \mathcal{R}(r; e, \ell_z, k^2) = 0 \) i.e. \( r \) remains constant. These are called orbital null geodesics, or trapped null geodesics. The \( r \) values for which such solutions are possible must then verify the equations

\[
\mathcal{R}(r; e, \ell_z, k^2) = \partial_r \mathcal{R}(r; e, \ell_z, k^2) = 0.
\]

**Lemma 2.8.** All orbital null geodesics in Kerr-Newman spacetime are given by the equation

\[
T_{\ell_z} := (r^3 - 3Mr^2 + (a^2 + 2Q^2)r + Ma^2)e - (r - M)\ell_z = 0. \tag{91}
\]

**Proof.** We solve for

\[
\partial_r \mathcal{R}(r; e, \ell_z, k^2) = -\left( (r^2 + a^2)e + a\ell_z \right)^2 + \Delta k^2 = 0
\]

 Writing from the second equation \( k^2 = \frac{2r}{(r-M)}e \cdot (r^2 + a^2)e + a\ell_z, \) and substituting in the first equation, we obtain

\[
0 = \left( (r^2 + a^2)e + a\ell_z \right) \cdot \left( -\left( (r^2 + a^2)e + a\ell_z \right) + \Delta \frac{2r}{(r-M)}e \right)
\]

\[
= \left( (r^2 + a^2)e + a\ell_z \right) \cdot \left( \left( (r^2 + a^2)(r - M) + 2r\Delta \right)e - (r - M)a\ell_z \right)
\]

\[
= \left( (r^2 + a^2)e + a\ell_z \right) \cdot \left( \left( (r^2 - 3Mr^2 + a^2r - Ma^2) + 2r(r^2 - 2Mr + a^2 + Q^2) \right)e - (r - M)a\ell_z \right)
\]

\[
= \left( (r^2 + a^2)e + a\ell_z \right) \cdot \left( (r^3 - 3Mr^2 + (a^2 + 2Q^2)r + Ma^2)e - (r - M)a\ell_z \right).
\]

Observe that the case of \( (r^2 + a^2)e + a\ell_z = 0 \) implies \( k^2 = 0 \), and it yields the vanishing of all the constants of motion. For non-trivial trapped null geodesics, we then obtain the vanishing of the factor on the right, denoted by \( T_{\ell_z} \).

For \( a = 0 \), the above orbital null geodesic equation gives

\[
T_{\ell_z} := r(3r^2 - 2Mr + 2Q^2)e = 0,
\]

and therefore the trapped null geodesics lie on the hypersurface, called photon sphere, defined by polynomial \( r^2 - 3Mr + 2Q^2 = 0 \), which in Reissner-Nordström spacetime is at \( r_{\text{trap}}^{\text{RN}} = \frac{3M + \sqrt{9M^2 - 4Q^2}}{2} \) and in Schwarzschild for \( Q = 0 \) is at \( r_{\text{trap}}^{\text{S}} = 3M \).

For \( a \neq 0 \), as a consequence of Lemma 2.8 the values of \( r \) for which trapped null geodesics exist depends on the ratio \( \ell_z/e \). More precisely, at the trapped null geodesics we have

\[
\frac{r^3 - 3Mr^2 + (a^2 + 2Q^2)r + Ma^2}{r - M} = \frac{a\ell_z}{e}.
\]

\(^6\)Observe that \( k^2 \) is a positive constant of motion by definition of \( K \).
In particular, unlike Schwarzschild and Reissner-Nordström, which have a single radius where all trapped geodesics occur, Kerr or Kerr-Newman spacetimes have an entire radial interval where trapped geodesics can occur. However, for a fixed geodesic angular momentum \( \ell_x \), there is again only a single trapping radius for null geodesics with that angular momentum. If \( \ell_x = 0 \) the trapped region defined by (91) reduces to a unique hypersurface defined by

\[
T := r^3 - 3Mr^2 + (a^2 + 2Q^2)r + Ma^2 = 0.
\]

Geometrically, this reflects the fact that trapped null geodesics orthogonal to the axial Killing vectorfield \( Z \) must necessarily approach the root of \( T \). Observe that the polynomial \( T \) has a unique root in the exterior of the black hole region, and we denote that root by \( r_{\text{trap}} \). Moreover \( 2M < r_{\text{trap}} < 3M \), where the lower bound is reached in the extremal Reissner-Nordström case and the upper bound in the Schwarzschild case. At the extremal Kerr-Newman, for \( a^2 + Q^2 = M^2 \), the trapping hypersurface becomes

\[
T = r^3 - 3Mr^2 + (2M^2 - a^2)r + Ma^2 = (r - M)(r^2 - 2Mr - a^2)
\]

which vanishes at \( r = M + \sqrt{M^2 + a^2} \).

**Remark 2.9.** From the computations in Lemma 2.8, one can see that the polynomial \( T \) can be obtained as

\[
\partial_r \left( \frac{\Delta}{(r^2 + a^2)^2} \right) = \frac{2(r-M)(r^2 + a^2) - 4r\Delta}{(r^2 + a^2)^3} = -\frac{2T}{Q(r^2 + a^2)^4}.
\]

In particular, the trapping radius \( r_{\text{trap}} \) in axial symmetry maximizes the geodesic potential \( \frac{\Delta}{(r^2 + a^2)^4} \), which coincides with the familiar potential \( r^{-2}(1 - \frac{2M}{r}) \) in Schwarzschild.

A crucial property of the trapped null geodesics in Kerr-Newman spacetime is that they are unstable, i.e. one can show that at \( T_0, \ell_x = 0 \), we have \( \partial_r^2 R(r_0, \ell_x, k^2) \leq 0 \) in the subextremal range \( a^2 + Q^2 \leq M^2 \).

### 2.6 The choice of the function \( z \) in the Morawetz estimates

Here we motivate the choice of the function \( z \) done in [3], which we use here. Recall the expression for the Morawetz current in (51)

\[ |q|^2 e^{(X,w)}[\psi] = A|\partial_r \psi|^2 + U^{\alpha\beta}(\partial_\alpha \psi)(\partial_\beta \psi) + V|\psi|^2, \]

where the principal term \( U^{\alpha\beta}(\partial_\alpha \psi)(\partial_\beta \psi) \) is given by

\[ U^{\alpha\beta} = -\frac{1}{2} \partial_r \left( \frac{\Delta}{\nabla^2} \right). \]

We denote

\[ \tilde{R}^{\alpha\beta} := \partial_r \left( \frac{\Delta}{\nabla^2} \right), \quad \tilde{\nabla}^2 := \partial_r \left( \frac{\nabla^2}{\nabla^2} \right), \]

and thus write from (67), i.e. \( \mathcal{R}^{\alpha\beta} = \mathcal{R}^{\alpha\beta}_w \),

\[ \tilde{R}^{\alpha\beta} = \partial_r \left( \frac{\nabla^2}{\nabla^2} \right) = \partial_r \left( \frac{\nabla^2}{\nabla^2} \right) S^{\alpha\beta}_w = \tilde{\nabla}^2 S^{\alpha\beta}_w. \]

Explicitly, from (51), we deduce

\[ \tilde{R}^{\alpha\beta} = -\partial_r \left( \frac{\nabla^2}{\nabla^2} \right) \partial_\alpha \partial_\beta - 2a \partial_r \left( \frac{\nabla^2}{\nabla^2} \right) \partial_\alpha \partial_\beta - a^2 \partial_r \left( \frac{\nabla^2}{\nabla^2} \right) \partial_\alpha \partial_\beta + (\partial_r z) O^{\alpha\beta}, \]

and

\[ \tilde{R}^{\alpha\beta} = -\partial_r \left( \frac{\nabla^2}{\nabla^2} \right), \quad \tilde{\nabla}^2 = -2a \partial_r \left( \frac{\nabla^2}{\nabla^2} \right), \quad \tilde{\nabla}^2 = -a^2 \partial_r \left( \frac{\nabla^2}{\nabla^2} \right), \quad \tilde{\nabla}^4 = \partial_r z. \]

The principal term \( U^{\alpha\beta} = -\frac{1}{2} \partial_r \tilde{R}^{\alpha\beta} \) contains the angular and time derivatives of the solution \( \psi \), and therefore we expect it to be degenerate at the trapping region. The choice of \( z \) has to reflect this property. From Remark 2.9 following [3], we notice that if

\[ z_0 = \frac{\Delta}{(r^2 + a^2)^2} \]

we simultaneously obtain a degenerate coefficient at trapping for the term \( O^{\alpha\beta} \) and the vanishing of the term in \( \partial_\alpha \partial_\beta \), i.e.

\[ \tilde{R}^{\alpha\beta}[z_0] = \partial_r \left( \frac{\nabla^2}{\nabla^2} \right) = -2a \partial_r \left( \frac{\nabla^2}{\nabla^2} \right) \partial_r \partial_\beta - \frac{2T}{(r^2 + a^2)^3} O^{\alpha\beta}. \]
With this choice we obtain

\[ \tilde{R}^{i[0]}[z_0] = 0, \quad \tilde{R}^{i[2]}[z_0] = \frac{4ar}{(r^2 + a^2)^2} \delta_t \delta_\phi, \quad \tilde{R}^{i[3]}[z_0] = -\frac{4a^2 r}{(r^2 + a^2)^3} \delta_t \delta_\phi, \quad \tilde{R}^{i[4]}[z_0] = -\frac{2T}{(r^2 + a^2)^3} \] (95)

or also

\[ \tilde{R}^{i[0]}[z_0] = 0, \quad \tilde{R}^{i[2]}[z_0] = \frac{4ar}{(r^2 + a^2)^2} \delta_t \delta_\phi, \quad \tilde{R}^{i[3]}[z_0] = -\frac{4a^2 r}{(r^2 + a^2)^3} \delta_t \delta_\phi, \quad \tilde{R}^{i[4]}[z_0] = -\frac{2T}{(r^2 + a^2)^3}. \]

Recalling the definition (16) of \( \tilde{T} = \partial_t + \frac{r}{r^2 + a^2} \partial_\phi \), for the choice of \( z_0 = \frac{\Delta}{r^2 + a^2} \) we obtain

\[ \tilde{R}^{m\alpha \beta}[z_0] = -\frac{2T}{(r^2 + a^2)^3} O^{\alpha \beta} + \frac{4ar}{(r^2 + a^2)^2} \tilde{F}^{\alpha \beta}. \] (96)

Observe that if \( z = \left( \frac{\Delta}{r^2 + a^2} \right)^2 \), then the coefficient of \( \partial_t \delta_\phi \) in \( \partial_r \left( \frac{\Delta}{r^2 + a^2} \right)^2 \) will reduce to

\[ -\partial_r \left( \frac{\Delta}{r^2 + a^2} \right)^2 \partial_t \left( \frac{\Delta}{r^2 + a^2} \right) = -\partial_t \left( \frac{\Delta}{r^2 + a^2} \right)^2 = -\partial_t z_0 = -\frac{2T}{(r^2 + a^2)^3}, \]

which is also trapped. To combine the above choice of \( z_0 \) with one which allows for a trapped coefficient in the time derivative as well, following [35] we define for a sufficiently small \( \epsilon > 0 \),

\[ z_1 = z_0 - \epsilon z_0. \] (97)

With this choice we obtain

\[ \tilde{R}^{m\alpha \beta}[z_1] = \partial_t \left( \frac{z_1}{\Delta} \right)^2 R^{m\alpha \beta} = -\frac{2T}{(r^2 + a^2)^3} \partial_t \delta_\phi^m - \frac{2T}{(r^2 + a^2)^3} (1 + O(\epsilon r^{-2})) O^{m\alpha \beta} \]

or also

\[ \tilde{R}^{i[0]}[z_1] = -\frac{2T}{(r^2 + a^2)^3}, \quad \tilde{R}^{i[2]}[z_1] = -\frac{2T}{(r^2 + a^2)^3} (1 + O(\epsilon r^{-2})), \]

\[ \tilde{R}^{i[3]}[z_1] = -\frac{4a^2 r}{(r^2 + a^2)^3} (1 + O(\epsilon r^{-2})), \quad \tilde{R}^{i[4]}[z_1] = -\frac{2T}{(r^2 + a^2)^3} (1 + O(\epsilon r^{-2})). \] (99)

2.7 The case of axial symmetry

From Proposition 2.7 for \( z = z_0 = \frac{\Delta}{r^2 + a^2} \) as in (94) and \( u \) a function of \( r \), with

\[ X = F \partial_r, \quad \mathcal{F} = z_0 u, \quad w = z_0 \partial_r u, \] (100)

the current \( \mathcal{E}^{(X,w)[\psi]} \) satisfies

\[ |q|^2 \mathcal{E}^{(X,w)[\psi]} = A |\partial_r \psi|^2 + \mathcal{U}^{\alpha \beta} (\partial_\alpha \psi)(\partial_\beta \psi) + \mathcal{V} |\psi|^2, \] (101)

where, recall (90),

\[ A = \frac{1}{2} \Delta \frac{1}{\Delta} \partial_t \left( \frac{z_0^{1/2} u}{\Delta^{1/2}} \right) = \frac{\Delta^2}{r^2 + a^2} \partial_t \left( \frac{u}{r^2 + a^2} \right), \]

\[ \mathcal{U}^{\alpha \beta} = -\frac{1}{2} u \partial_r \left( \frac{z_0}{\Delta} R^{\alpha \beta} \right) = u \left( \frac{T}{r^2 + a^2} O^{\alpha \beta} - \frac{2ar}{(r^2 + a^2)^2} \tilde{F}^{\alpha \beta} \right), \]

\[ \mathcal{V} = -\frac{1}{2} \partial_r (\Delta \partial_r w). \]

For axially symmetric solutions, the term involving \( Z \) in \( \mathcal{U}^{\alpha \beta} \) vanishes, and we can write

\[ |q|^2 \mathcal{E}^{(X,w)[\psi]} = A |\partial_r \psi|^2 + \frac{u T}{(r^2 + a^2)^3} |q|^2 |\nabla \psi|^2 + \mathcal{V} |\psi|^2, \] (103)

where we used [35] to write

\[ O^{\alpha \beta} (\partial_\alpha \psi)(\partial_\beta \psi) = |q|^2 (e_1 e_1^\beta + e_2 e_2^\beta) (\partial_\alpha \psi)(\partial_\beta \psi) = |q|^2 ((\nabla_1 \psi)^2 + (\nabla_2 \psi)^2) = |q|^2 |\nabla \psi|^2. \]

Here \( |\nabla \psi|^2 = \frac{1}{|q|^2} |\nabla \psi|^2 + a^2 \sin^2 \theta |\partial_\theta \psi|^2 \), where \( |\nabla \psi|^2 \) is the norm of the gradient of \( \psi \) on the unit round sphere.

In order to obtain a positive definite Morawetz current in (103), we make use of the following construction first due to Stogin, see Lemma 5.2.6 in [61].
Lemma 2.10. It is possible to choose functions \( u \) and \( w \) such that in Kerr-Newman for the full sub-extremal \( a^2 + Q^2 < M^2 \), we have

\[
A \geq 0, \quad uT \geq 0, \quad V \geq 0, \tag{104}
\]

and therefore the current \( \mathcal{E}^{(X,w)}[\psi] \) is positive definite and satisfies for a uniform constant \( c_0 > 0 \),

\[
\mathcal{E}^{(X,w)}[\psi] \geq c_0 \left( \frac{\Delta^2}{r^2} |\partial_r \psi|^2 + r^{-1}(1 - \frac{r_{\text{trap}}}{r})^2 |\nabla \psi|^2 + \frac{M}{r^2} \right) \tag{105}
\]

where \( r_{\text{trap}} \) denotes the root of \( T \) in the exterior region and \( r_* > r_{\text{trap}} \) is defined in the construction.

Proof. In order to obtain the positivity conditions in (104), from the expressions of \( A \) and \( V \) in (102) we need to have

\[
\partial_r \left( \frac{u}{r^2 + a^2} \right) \geq 0, \tag{106}
\]

\[
u T \geq 0, \tag{107}
\]

\[
\partial_r (\Delta \partial_r w) \leq 0, \tag{108}
\]

with the compatibility condition

\[
w = z_0 \partial_r u. \tag{109}
\]

We start by imposing condition (107), i.e. \( uT \geq 0 \). Recall that \( r_{\text{trap}} \) is the root of the polynomial \( T \) in the exterior region, with \( 2M < r_{\text{trap}} < 3M \). We define \( u \) by the following:

\[
w(r_{\text{trap}}) = 0, \quad \partial_r u = \frac{1}{z_0} \frac{(r^2 + a^2)^2}{\Delta} w, \quad w \geq 0. \tag{110}
\]

This automatically implies that \( u \) vanishes at \( r_{\text{trap}} \) and is increasing, therefore satisfying (107). Also, by definition, the compatibility condition (109) is satisfied.

We now rewrite condition (109) in terms of \( w \), i.e.

\[
\partial_r \left( \frac{u}{r^2 + a^2} \right) = \frac{1}{r^2 + a^2} \partial_r u - \frac{2r}{(r^2 + a^2)^2} u = \frac{r^2 + a^2}{\Delta} w - \frac{2r}{(r^2 + a^2)^2} u \tag{111}
\]

To eliminate the dependence on \( u \), we multiply (111) by \( \frac{(r^2 + a^2)^2}{r} \), and take another derivative in \( r \). By defining \( \mathcal{K} := \frac{r}{(r^2 + a^2)^2} \partial_r \left( \frac{u}{r^2 + a^2} \right) \), we obtain

\[
\mathcal{K} = \frac{(r^2 + a^2)^2}{r} \partial_r \left( \frac{u}{r^2 + a^2} \right) = \frac{(r^2 + a^2)^3}{r \Delta} w - 2u \\
\partial_r \mathcal{K} = \partial_r \frac{(r^2 + a^2)^3}{r \Delta} w - 2 \frac{(r^2 + a^2)^2}{\Delta} w \\
= 2r \frac{(r^2 + a^2)^2}{r \Delta} w + (r^2 + a^2) \partial_r \left( \frac{(r^2 + a^2)^3}{r \Delta} w \right) \tag{112}
\]

\[
= (r^2 + a^2) \partial_r \left( \frac{(r^2 + a^2)^2}{r \Delta} w \right) = (r^2 + a^2) \partial_r \left( \frac{1}{r z_0} w \right).
\]

Since \( \mathcal{K} \) has the same sign as \( \partial_r \left( \frac{u}{r^2 + a^2} \right) \), condition (109) is satisfied if and only if \( \mathcal{K} \geq 0 \).

We now impose that \( \partial_r \mathcal{K} = 0 \) for sufficiently large \( r \), which, together with the condition that \( \mathcal{K} \geq 0 \) up to that large \( r \), implies positivity for \( \mathcal{K} \). From the above we have

\[
w = r z_0 = \frac{r \Delta}{(r^2 + a^2)^2} \tag{112}
\]

for sufficiently large \( r \). We then compute for such choice:

\[
\partial_r w = \partial_r \left( \frac{r \Delta}{(r^2 + a^2)^2} \right) = \frac{\Delta}{(r^2 + a^2)^2} - \frac{2r T}{(r^2 + a^2)^3} \\
= \frac{(r^2 - 2Mr + a^2 + Q^2)(r^2 + a^2) - 2r(r^3 - 3Mr^2 + (a^2 + 2Q^2)r + Ma^2)}{(r^2 + a^2)^3} \\
= \frac{-r^4 - 4Mr^3 + 3Q^2r^2 + 4Ma^2r - a^4 - a^2Q^2}{(r^2 + a^2)^3}.
\]
The function $w$ then has a maximum at the root of the above polynomial. Denote $r_*$ the unique root in the exterior region. Observe that in Schwarzschild $r_* = 4M$, in Reissner-Nordström $r_* = 2M + \sqrt{4M^2 - 3Q^2}$, and in Kerr $r_* = 2M + \sqrt{4M^2 - a^2}$. In the subextremal Kerr-Newman we have $3M < r_* < 4M$, and therefore in the full sub-extremal range we have $r_* > r_{\text{trap}}$.

We then impose

$$w = rz = \frac{r\Delta}{(r^2 + a^2)^2} \quad \text{for } r \geq r_*.$$  \hspace{1cm} (113)

Then for $r \geq r_*$ condition (108) reduces to

$$-\partial_r (\Delta \partial_r w) = \partial_r \left( \frac{\Delta}{(r^2 + a^2)^2} \left( r^4 - 4Mr^3 + 3Q^2r^2 + 4Ma^2r - a^4 - a^2Q^2 \right) \right) \geq 0.$$  

The above is a derivative of a product of two functions that are positive for $r > r_*$ and increasing to 1, and therefore necessarily positive.

Consider the function $\Delta \partial_r w$. This function vanishes at $r = r_+$ and at $r = r_*$ with the above choice of $w$ for $r \geq r_*$. By the mean value theorem, in order to have condition (108), satisfied for all $r \geq r_+$, we need to have the vanishing of the function in the interval $[r_+, r_*]$, and therefore $\partial_r w = 0$ there. Define

$$w(r) = w(r_*) > 0 \quad \text{for } r_+ \leq r \leq r_*$$  \hspace{1cm} (114)

At this stage, a non-negative $w$ has been chosen for all $r$, and condition (108) has been proved to be satisfied everywhere.

We are now left with proving that with the above choice $K \geq 0$ for $r \leq r_*$. In this region, we have

$$\partial_r K = (r^2 + a^2) \partial_r \left( \frac{1}{rz^0} w \right) = w(r_*)(r^2 + a^2) \partial_r \left( \frac{1}{rz^0} \right) = -w(r_*)(r^2 + a^2) \frac{1}{r^2z^0} \partial_r (rz^0).$$

Recall that the function $z^0 r_0$ had a maximum at $r_*$ and therefore $K$ decreases to a minimum at $r = r_*$. It is therefore enough to check that $K(r = r_*) > 0$, or equivalently that $\partial_r \left( \frac{w}{r^2 + a^2} \right) \big|_{r = r_*} > 0$. We have

$$\partial_r \left( \frac{w}{r^2 + a^2} \right) = \frac{1}{r^2 + a^2} \partial_r u - \frac{2r}{r^2 + a^2} u = \frac{1}{r^2 + a^2} \frac{w}{rz^0} - \frac{2r}{r^2 + a^2} \int_{r_{\text{trap}}}^r \partial_r u = \frac{1}{r^2 + a^2} \frac{w}{rz^0} - \frac{2r}{r^2 + a^2} \int_{r_{\text{trap}}}^r \frac{w}{z^0}. $$

We evaluate the above for $r$ with $r_{\text{trap}} \leq r \leq r_*$, where $w(r) = w(r_*)$ is constant, and $\partial_r (\frac{1}{z^0}) = -\frac{1}{z^0} \partial_r z^0 = \frac{2r}{z^0 (r^2 + a^2)^2} \geq 0$. Since the function $\frac{1}{z^0}$ is increasing, we can bound its integral from above by the product of the interval times the value of the function at the right end of the interval. We therefore obtain

$$\partial_r \left( \frac{w}{r^2 + a^2} \right) \big|_{r = r_*} = \frac{1}{r^2 + a^2} \frac{w(r_*)}{z^0} - \frac{2r}{r^2 + a^2} \frac{w(r_*)}{z^0} \int_{r_{\text{trap}}}^{r_*} \frac{1}{z^0} \geq \frac{1}{r^2 + a^2} \frac{w(r_*)}{z^0} - \frac{2r}{r^2 + a^2} \frac{w(r_*)}{z^0} \left( \frac{r_*}{r_{\text{trap}}} - r_{\text{trap}} \right) = \frac{1}{(r^2 + a^2)^2} \frac{w(r_*)}{z^0} \left( \frac{r_*^2 + a^2 - 2r_* (r_* - r_{\text{trap}})}{r_{\text{trap}}} \right)$$

Observe that

$$r_*^2 + a^2 - 2r_* (r_* - r_{\text{trap}}) = r_* (2r_{\text{trap}} - r_*) + a^2 > 0$$

as can be seen by comparing the range of $r_{\text{trap}}$ and $r_*$.  

To summarize, the choice for the functions $u$ and $w$ given by

$$w = \begin{cases} r_* z^0(r) & r \leq r_* \\ z^0(r) & r > r_* \end{cases}$$

and

$$u(r_{\text{trap}}) = 0, \quad \partial_r u = \frac{1}{z^0} w$$

are such that $w(r) = z^0 w(r_*) > 0$ for all $r$, and $\partial_r w = 0$ there.
satisfy in the whole exterior region of the full subextremal Kerr-Newman spacetime the conditions \([113]\).

From the analysis of the asymptotics of the functions \(u\) and \(w\) we can easily deduce the bound \([105]\).

As in \([111]\ [44]\), the above construction has to be corrected to fix some remaining issue in the bound \([105]\). We briefly describe them here, and we remind to \([111]\ [44]\) for more details, as they do not depend on the Kerr-Newman spacetime specifically.

The function \(u\) as defined above, and consequently the vectorfield \(X\), presents a logarithmic divergence near the horizon. This can be fixed by tempering it in that region through a small deviation from the original ones. More precisely one can define \((X_{\delta}, w_\delta)\) which agree with \((X, w)\) outside a neighborhood of the horizon, and that are regular up to the event horizon. This will create a negative contribution in the lower order term \(|\psi|^2\).

The coefficient of \((\partial_r \psi)^2\) in \([105]\) vanishes at the horizon. This can be fixed by the standard procedure of making use of the redshift vectorfield, which gives a positive contribution for any spacetime in the subextremal range.

The coefficient of \(|\psi|^2\) vanishes in the interval \(r_+ \leq r \leq r_*\). In addition, because of the tempering of the vectorfield close to the horizon, we have created a negative term of size \(\delta\) in the lower order term. This can be fixed by borrowing extra-positivity from the coefficient of \((\partial_r \psi)^2\) through a local Hardy inequality.

In \([105]\), there is no control on the \((\partial_r \psi)^2\) derivative. This can be fixed through a standard procedure of making use of the Lagrangian of the wave equation, i.e. applying the vectorfield method with \(X, w\). Equivalently, one can use the function \(z_1\) in \([97]\) to insert a degenerate term at trapping for \((\partial_r \psi)^2\), as done in Section \([2]\).

After applying these standard arguments, one arrives to an improved bound on the current \(\mathcal{E}^{(X, w)}[\psi]\) given by

\[
\mathcal{E}^{(X, w)}[\psi] \geq c_0 \left( \frac{M^2}{r} |\partial_r \psi|^2 + \left(1 - \frac{r_{\text{trap}}}{r}\right)^2 \left(r^{-1} |\nabla \psi|^2 + \frac{M}{r^2} (\partial_t \psi)^2 + \frac{M}{r^4} |\psi|^2\right) \right) .
\] (115)

Observe that the boundary term of the Morawetz current \(\mathcal{P}^{(X, w)}[\psi]\) will be combined in the next subsection with the boundary terms from the energy estimates multiplied by a large constant, so that the overall boundary terms are positive definite.

2.8 The Morawetz estimate

We derive here the Morawetz estimates for general solutions to the wave equation \([17]\) for \(|a| \ll M\). Observe that the main issue is that the principal term in \([90]\), i.e.

\[
U^{\alpha \beta} = -\frac{1}{2} u_T \left( \frac{Z_0}{\Delta} R^{\alpha \beta} \right) = \frac{u_T}{(r^2 + a^2)^2 \xi} O^{\alpha \beta} - u_T \frac{2ar}{(r^2 + a^2)^2} \hat{\xi} (\alpha Z^\beta),
\]

cannot be made to be positive definite, as the function \(u\) has to change sign in the trapped region. On the other hand, if one considers the generalized vectorfield method applied to \(\psi_{\underline{a}}\), as obtained in \([3]\), it is possible to extract a positive definite contribution which is degenerate at the trapped region.

In the following, we will show the main steps in the proof of the physical-space analysis in \([3]\), while referring to \([3]\) for more details.

2.8.1 Choice of double-indexed function \(u_{\underline{a}}\)

Following \([3]\), see also \([37]\), we perform an integration by parts in the principal term \(U^{\alpha \beta \underline{a} \underline{b}} \partial_{\alpha} \psi_{\underline{a}} \partial_{\beta} \psi_{\underline{b}}\), which allows to create a positive definite term for a trapped combination of \(\psi_{\underline{a}}\), which we denote \(\Psi\) here.

**Lemma 2.11.** Let \(u_{\underline{a}}\) the double-indexed function of \(r\) as described in the second part of Proposition \([2.7]\) be given by\(^7\)

\[
u_{\underline{a} \underline{b}} = -h \tilde{R}^\underline{a}_{\underline{b}}, \quad \tilde{R}^\underline{a} = \partial_r \left( \frac{Z_0}{\Delta} R^\underline{a} \right),
\] (116)

where \(h\) is a positive function, and \(L^\underline{a}\) are the coefficient of a constant symmetric tensor \(L^{\alpha \beta} := L^\underline{a} S_{\underline{a} \underline{b}}^{\alpha \beta}\). Then, defining

\[
\Psi := \tilde{R}^\underline{a} \psi_{\underline{a}},
\] (117)
the Morawetz identity is given by

\[ \left| q \right|^{2} D^{\alpha} \left( \bar{r}_{\alpha}^{X \omega} \right) = A_{\alpha} \mathcal{L}^{\alpha}_{\beta} \partial_{\omega} \partial_{\beta} \psi_{\omega} + \frac{1}{2} h L^{\alpha \beta} \partial_{\alpha} \partial_{\beta} \psi + \mathcal{V}_{\alpha} \mathcal{L}^{\alpha}_{\beta} \psi_{\omega} \psi_{\omega} \]

where

\[ A_{\alpha} := -z^{1/2} \Delta^{1/2} \partial_{r} \left( \frac{z^{1/2} h \bar{\mathcal{R}}^{a}_{\alpha}}{\Delta^{1/2}} \right) \]

and

\[ B_{\alpha} \psi := \frac{1}{2} \left| q \right|^{-2} h \mathcal{L}^{\alpha}_{\beta} \bar{\mathcal{R}}^{a}_{\beta} \psi \left( S_{\alpha}^{\beta} \partial_{\beta} \psi_{\omega} - S_{\alpha}^{\beta} \partial_{\beta} \psi_{\omega} \right) \]

denotes a boundary term.

**Proof.** From Proposition [2.7] we write the principal term as

\[ \mathcal{U}^{a \alpha \beta} \partial_{\alpha} \psi_{\omega} \partial_{\beta} \psi_{\omega} = -\frac{1}{2} u^{a b} \bar{\mathcal{R}}^{a}_{\alpha} \partial_{\alpha} \psi_{\omega} \partial_{\beta} \psi_{\omega} = -\frac{1}{2} u^{a b} \bar{\mathcal{R}}^{a}_{\alpha} S_{\alpha}^{\beta} \partial_{\alpha} \psi_{\omega} \partial_{\beta} \psi_{\omega} \]

Integrating by parts in \( \partial_{\alpha} \), we can write the above as

\[ \left| q \right|^{-2} \mathcal{U}^{a \alpha \beta} \partial_{\alpha} \psi_{\omega} \partial_{\beta} \psi_{\omega} = -\frac{1}{2} \left| q \right|^{-2} u^{a b} \bar{\mathcal{R}}^{a}_{\alpha} S_{\alpha}^{\beta} \partial_{\alpha} \psi_{\omega} \partial_{\beta} \psi_{\omega} \]

where

\[ u^{a b} = -h \mathcal{R}^{a b} \mathcal{L}^{b}_{a} \]

for some positive function \( h \) and some constant symmetric tensor \( L^{\alpha \beta} = \mathcal{L}^{\alpha}_{\beta} S_{\alpha}^{\beta} \).

Then the first term of the above relation becomes

\[ \left| q \right|^{-2} \mathcal{U}^{a \alpha \beta} \partial_{\alpha} \psi_{\omega} \partial_{\beta} \psi_{\omega} = \frac{1}{2} h \mathcal{R}^{a b} \mathcal{L}^{b}_{a} \bar{\mathcal{R}}^{a}_{\alpha} \partial_{\alpha} \psi_{\omega} \partial_{\beta} \psi_{\omega} = \frac{1}{2} h \mathcal{R}^{a b} \mathcal{L}^{b}_{a} L^{\alpha \beta} \partial_{\alpha} \psi_{\omega} \partial_{\beta} \psi_{\omega} \]

where the terms in \( \bar{\mathcal{R}}^{a}_{\alpha} \) and \( \bar{\mathcal{R}}^{b}_{\alpha} \) can be inserted inside the derivatives as they only depend on \( r \), where the same indices means summation over those. By denoting \( \Psi := \bar{\mathcal{R}}^{a}_{\alpha} \partial_{\alpha} \), we obtain

\[ \left| q \right|^{-2} \mathcal{U}^{a \alpha \beta} \partial_{\alpha} \psi_{\omega} \partial_{\beta} \psi_{\omega} = \frac{1}{2} \left| q \right|^{-2} h L^{\alpha \beta} \partial_{\alpha} \psi_{\omega} \partial_{\beta} \psi_{\omega} - \frac{1}{2} D_{\alpha} \left( \left| q \right|^{-2} h \mathcal{L}^{\alpha}_{\beta} \bar{\mathcal{R}}^{a}_{\beta} \psi \left( S_{\alpha}^{\beta} \partial_{\beta} \psi_{\omega} - S_{\alpha}^{\beta} \partial_{\beta} \psi_{\omega} \right) \right) \]

which gives

\[ \mathcal{U}^{a \alpha \beta} \partial_{\alpha} \psi_{\omega} \partial_{\beta} \psi_{\omega} = \frac{1}{2} h L^{\alpha \beta} \partial_{\alpha} \psi_{\omega} \partial_{\beta} \psi_{\omega} - \left| q \right|^{2} B_{\alpha} \psi \]
for $B_\alpha[\psi]$ defined in (120). Finally, with the choice of $w^\alpha = -h \tilde{\mathcal{R}}^\alpha_\mu \mathcal{L}_\mu$, the expression of $A$ and $\mathcal{U}$ in (57) and (59) become

$$A^\alpha = A^\alpha_\mu \mathcal{L}_\mu, \quad A_\alpha := -z^{1/2} \Delta^{3/2} \partial_\nu \left( \frac{z^{1/2} h \tilde{\mathcal{R}}^\alpha_\mu}{\Delta^{1/2}} \right)$$

$$\mathcal{V}^\alpha = \mathcal{V}^\alpha_\mu \mathcal{L}_\mu, \quad \mathcal{V}_\alpha := \frac{1}{4} \partial_\nu (\Delta \partial_\nu (z \partial_\nu (h \tilde{\mathcal{R}}^\alpha_\mu))).$$

Recalling that $|q|^2 \mathcal{E}(X, w)[\psi] = |q|^2 \mathcal{D}^a \mathcal{D}^b(X, w)[\psi]$, we obtain the relation (118), and prove the lemma.

We now put into effect the choice of function $z_1$ as given in (97), i.e.

$$z_1 = z_0 - \epsilon z_0.$$ From (98) and (99), we have

$$\tilde{\mathcal{R}}^{\alpha\beta} = -\frac{2T}{(r^2 + a^2)^2} \partial_\alpha \partial_\beta - \frac{2T}{(r^2 + a^2)^2} \left( 1 + O(\epsilon r^{-2}) \right) \tilde{O}^{\alpha\beta} + \frac{4a}{(r^2 + a^2)^2} \left( 1 + O(\epsilon r^{-2}) \right) \tilde{T}^{(\alpha \beta)},$$

and therefore for $\Psi = \tilde{\mathcal{R}} \omega_\alpha \omega_\beta$,

$$\Psi = -\frac{2T}{(r^2 + a^2)^2} \partial_\alpha \Psi - \frac{2T}{(r^2 + a^2)^2} \left( 1 + O(\epsilon r^{-2}) \right) \tilde{O}(\psi) + \frac{4a}{(r^2 + a^2)^2} \left( 1 + O(\epsilon r^{-2}) \right) \tilde{T}(\partial_\nu \psi).$$

By considering as in [3] the constant symmetric tensor given by

$$L^{\alpha\beta} = \epsilon \partial_\alpha \partial_\beta + O^{\alpha\beta}$$

then for a positive function $h$, the principal term $\frac{1}{2} h L^{\alpha\beta} \partial_\alpha \Psi \partial_\beta \Psi$ is positive and given by

$$\frac{1}{2} h L^{\alpha\beta} \partial_\alpha \Psi \partial_\beta \Psi = \frac{1}{2} h (|\partial_\nu \Psi|^2 + O^{\alpha\beta} \partial_\alpha \Psi \partial_\beta \Psi) = \frac{1}{2} h (|\partial_\nu \Psi|^2 + |q|^2 |\nabla \Psi|^2)$$

where we used (55) to write

$$O^{\alpha\beta}(\partial_\alpha \Psi)(\partial_\beta \Psi) = |q|^2 (e_1^2 e_1^2 + e_2 e_2^2)(\partial_\alpha \Psi)(\partial_\beta \Psi) = |q|^2 \left( (\nabla_1 \psi)^2 + (\nabla_2 \psi)^2 \right) = |q|^2 |\nabla \Psi|^2.$$ We can summarize the above and write $\frac{1}{2} h L^{\alpha\beta} \partial_\alpha \Psi \partial_\beta \Psi = \frac{1}{2} |D \Psi|^2$ where we denote $D = e^{1/2} \partial_\nu, |q|\nabla$. To express it in terms of the $\omega_\alpha$, we use (121) and we bound the above by (see Lemma 3.13 in [3])

$$\frac{1}{2} h L^{\alpha\beta} \partial_\alpha \Psi \partial_\beta \Psi \geq \frac{1}{2} \tilde{h} \overline{1}_{|r - r_{\text{trap}}|} |D \Psi|^2$$

$$\geq \frac{1}{2} \tilde{h} \overline{1}_{|r - r_{\text{trap}}|} \left( |\partial_\nu \Psi|^2 + \sum_{a=2}^{4} |D \psi_a|^2 - \epsilon |a| + \epsilon \sum_{a \neq b=1}^{4} D \psi_a D \psi_b \right)$$

$$-a^2 \frac{1}{2} \tilde{h} \overline{1}_{|r - r_{\text{trap}}|} \sum_{a=1}^{4} |\psi_a|^2,$$

where $\overline{1}_{|r - r_{\text{trap}}|}$ is a function that is identically 1 for $|r - r_{\text{trap}}| > \delta$ for some $\delta > 0$ and zero otherwise. Observe that once the commutation with the conformal symmetries allows to create a positive term with respect to $\Psi$, we can insert the trapped function $\tilde{h} \overline{1}_{|r - r_{\text{trap}}|}$ to express it in terms of the $\omega_\alpha$.

Choosing $a$ sufficiently small with respect to $\epsilon$ and fixing $\epsilon$ sufficiently small with respect to 1, we obtain

$$\frac{1}{2} h L^{\alpha\beta} \partial_\alpha \Psi \partial_\beta \Psi \geq \frac{1}{2} \tilde{h} \overline{1}_{|r - r_{\text{trap}}|} \left( |\partial_\nu \Psi|^2 + |q|^2 |\nabla \Psi|^2 \right) - a^2 \frac{1}{2} \tilde{h} \overline{1}_{|r - r_{\text{trap}}|} \sum_{a=1}^{4} |\psi_a|^2.$$ (123)

### 2.8.2 Choice of function $h$

We now look at the terms $A^\alpha L^{\alpha\beta} \partial_\alpha \psi_a \partial_\beta \psi_b$ and $\mathcal{V}^\alpha L^{\alpha\beta} \partial_\alpha \psi_a \partial_\beta \psi_b$. For $z_1 = z_0 - \epsilon z_0$, we compute

$$A^\alpha = -z_1^{1/2} \Delta^{3/2} \partial_\nu \left( \frac{z_1^{1/2} h \tilde{\mathcal{R}}^\alpha_\mu}{\Delta^{1/2}} \right)$$

$$= -z_0^{1/2} \Delta^{3/2} \partial_\nu \left( \frac{\left( z_0^{1/2} + O(\epsilon r^{-3}) \right) h \tilde{\mathcal{R}}^\alpha_\mu}{\Delta^{1/2}} \right) + O(\epsilon) \partial_\nu \left( \frac{z_0^{1/2} h \tilde{\mathcal{R}}^\alpha_\mu}{\Delta^{1/2}} \right)$$

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To obtain positivity of the those terms, we can for example make use of the construction in Lemma 2.10. In particular, we denote here by \( u_{a x} \) and \( w_{a x} \) the functions \( u \) and \( w = z_0 \partial_r u_{a x} \) constructed in Lemma 2.10 in the axially symmetric case. We define

\[
h := \frac{(r^2 + a^2)^3}{2T} u_{a x}
\]

(124)

Observe that this function is positive and smooth everywhere, as both \( u_{a x} \) and \( T \) vanish at \( r = r_{trap} \) of order 1. With this choice we have \( h \bar{R}^1 = -c u_{a x} \) and \( h \bar{R}^4 = -u_{a x} \), and therefore obtain

\[
A^1 = -\frac{\Delta^2}{r^2 + a^2} \partial_r \left( \frac{h \bar{R}^1}{r^2 + a^2} \right) + O(\epsilon) \partial_r \left( \frac{h \bar{R}^1}{r^2 + a^2} \right) = \epsilon \frac{\Delta^2}{r^2 + a^2} \partial_r \left( \frac{u_{a x}}{r^2 + a^2} \right) (1 + O(\epsilon r^{-2}))
\]

\[
A^4 = \frac{\Delta^2}{r^2 + a^2} \partial_r \left( \frac{u_{a x}}{r^2 + a^2} \right) (1 + O(\epsilon r^{-2}))
\]

which are positive definite for \( \epsilon \) sufficiently small by Lemma 2.10. We also have

\[
A^2 = -a \frac{\Delta^2}{r^2 + a^2} \partial_r \left( \frac{2ru_{a x}}{T} \right) + O(\epsilon r^{-2}) = O(ar^{-2}) \frac{\Delta^2}{r^2 + a^2} (1 + O(\epsilon r^{-2}))
\]

\[
A^3 = -a^2 \frac{\Delta^2}{r^2 + a^2} \partial_r \left( \frac{2ru_{a x}}{T(r^2 + a^2)} \right) + O(\epsilon r^{-3}) = O(a^2 r^{-3}) \frac{\Delta^2}{r^2 + a^2} (1 + O(\epsilon r^{-2})).
\]

We can then bound the term \( A^2 \partial_r \psi \| \psi \| \partial_r \psi \) by (see Lemma 3.9 in [3])

\[
A^2 \partial_r \psi \| \psi \| \partial_r \psi \geq \left( \frac{\epsilon^2}{4} |\partial_r \psi| + \sum_{k=1}^{4} \left| \partial_r \psi \right|^2 \right)
\]

where the term \( \partial_r \psi \| \psi \| \partial_r \psi \) can be shown to be positive, up to boundary terms, by integration by parts, and being comparable to \( |\partial_r \psi \| \psi \| \partial_r \psi \) (see Lemma 2.4 in [3]). As above, choosing \( a \) sufficiently small with respect to \( \epsilon \) and fixing \( \epsilon \) sufficiently small with respect to 1, we obtain positivity for the above term.

We similarly compute

\[
\psi = \frac{1}{4} \partial_r \left( \Delta \partial_r \left( z_0 \partial_r (h \bar{R}^1 \bar{u}) \right) \right) = \frac{1}{4} \partial_r \left( \Delta \partial_r \left( z_0 \partial_r (h \bar{R}^1 \bar{u}) \right) \right) + O(\epsilon r^{-2}).
\]

With the given choice of \( h \) we then obtain

\[
\psi^1 = -\frac{1}{4} \partial_r \left( \Delta \partial_r \left( z_0 \partial_r (h \bar{R}^1 \bar{u}) \right) \right) + O(\epsilon r^{-2}) = -\frac{1}{4} \partial_r \left( \Delta \partial_r \left( z_0 \partial_r (h \bar{R}^1 \bar{u}) \right) \right) + O(\epsilon r^{-2})
\]

\[
\psi^4 = -\frac{1}{4} \partial_r \left( \Delta \partial_r \left( z_0 \partial_r (h \bar{R}^1 \bar{u}) \right) \right) + O(\epsilon r^{-2})
\]

which are non-negative for \( \epsilon \) sufficiently small by Lemma 2.10. We also have

\[
\psi^2 = \frac{1}{4} a \partial_r \left( \Delta \partial_r \left( z_0 \partial_r \left( \frac{2ru_{a x}}{T} \right) \right) \right) + O(\epsilon r^{-2}) = O(ar^{-4}) (1 + O(\epsilon r^{-2}))
\]

\[
\psi^3 = \frac{1}{4} a^2 \partial_r \left( \Delta \partial_r \left( z_0 \partial_r \left( \frac{2ru_{a x}}{T(r^2 + a^2)} \right) \right) \right) + O(\epsilon r^{-3}) = O(a^2 r^{-5}) (1 + O(\epsilon r^{-2})).
\]

We can then bound the term \( \psi^2 \psi \| \psi \| \psi \) by (see Lemma 3.9 in [3])

\[
\psi^2 \psi \| \psi \| \psi \geq \left( \frac{\epsilon^2}{4} |\psi| + \sum_{k=1}^{4} \left| \psi \right|^2 \right)
\]

Upon applying the Hardy inequality as in the axially symmetric case, we can upgrade the bound of the first terms to be valid everywhere in the exterior region. By combining the above bound with the one obtained

\[\text{This choice differs from the one in [3] or [37], but the following procedure is identical, as it only makes use of the positivity of } A^1 \text{ and } A^4, \text{ and the fact that } A^2, A^3 = O(a).\]
in \(\mathbb{C}^2\), and choosing \(\alpha\) sufficiently small with respect to \(\epsilon\) and fixing \(\epsilon\) sufficiently small with respect to 1, we obtain positivity for the overall zero-th order term in the Morawetz bulk, which finally gives

\[
\mathcal{D}_\alpha \left( \mathcal{P}_\alpha^n \mathbf{w} \right) + \tilde{\mathcal{B}}_{\alpha} \left[ \mathbf{w} \right] \geq c_0 \left( \frac{M^2}{r^4} |\partial_r \mathbf{w}|^2 + \frac{M}{r^3} |\mathbf{w}|^2 + \tilde{\Pi}_{\left\{ r \neq r_{\text{trap}} \right\}} \left( r^{-1} |\nabla \mathbf{w}|^2 + \frac{M}{r^2} |\partial_t \mathbf{w}|^2 \right) \right) \tag{125}
\]

where \(\tilde{\mathcal{B}}_{\alpha} \left[ \mathbf{w} \right]\) incorporates the boundary term in \(\mathbb{C}^2\), together with the ones obtained in the above integration by parts.

### 2.9 The energy estimate

The energy estimates are obtained from the current associated to the vectorfield \(\tilde{T}_\chi = T + \chi \omega_H Z\) as defined in Section 2.4. Observe that for \(|\alpha|/M\) sufficiently small, \(\tilde{T}_\chi\) is timelike everywhere in the exterior region and Killing outside the region \([r_1, r_2]\). Moreover, for \(|\alpha|/M\) sufficiently small, the trapped null geodesics remain close to the hypersurface \(\{ r = \tilde{r}_{\text{trap}} \}\). In particular for \(|\alpha|/M \ll 1\), the region with \(r \in [r_1, r_2]\), for \(r_1 = \frac{\tilde{c}}{\epsilon} r_+\) and \(r_2 = \frac{\tilde{c}}{\epsilon} r_{\text{trap}}\), does not contain any trapped null geodesics. In particular, \(\tilde{T}_\chi\) is Killing in the entire region where trapped null geodesics appear. From \(\mathbb{C}^2\), we obtain

\[
E(\tilde{T}_{\chi}, 0)[\psi](\tau) + \int_{M(0, \tau)} |q|^{-2} \Delta \omega_H (\partial_r \chi) \partial_r \psi \partial_r \psi \leq E(\tilde{T}_{\chi}, 0)[\psi](0), \tag{126}
\]

where

\[
E(\tilde{T}_{\chi}, 0)[\psi](\tau) \sim \int_{\Sigma_\tau} \frac{\Delta}{r^2 + a^2} (\partial_r \psi)^2 + (\partial_r \psi)^2 + |\nabla \psi|^2.
\]

The above energy can be made to be non-degenerate at the horizon by making use of the red-shift vectorfield. By applying the energy estimates to each commuted equation \(\mathbb{C}^2\), we obtain the higher order version:

\[
E(\tilde{T}_{\chi}, 0)[\psi]|_{\psi}|(\tau) + \int_{M(0, \tau)} |q|^{-2} \Delta \omega_H (\partial_r \chi) \partial_r \psi \partial_r \psi \leq E(\tilde{T}_{\chi}, 0)[\psi]|_{\psi}|(0) \leq O(|\alpha|) \int_{M(0, \tau)} \Pi_{[r_1, r_2]} \left( |\partial_r \psi|^2 + |\partial_r \psi|^2 \right).
\]

By combining the bound obtained in \(\mathbb{C}^2\) with the energy estimates multiplied by a large constant, one can choose a constant \(\Lambda\) large enough in order to have the boundary terms of the Morawetz estimates absorbed by the positive ones from the non-degenerate energy estimates (see Lemma 3.11 in \(\mathbb{C}^2\)), and \(|\alpha|\) small enough to absorb the above on the right hand side. This concludes the proof of Theorem 2.1.

### 3 Application to the Einstein-Maxwell equations

In this section we show how the physical-space analysis relying on the commutation with the Carter differential operator can be adapted to the system of coupled Regge-Wheeler equations describing the coupled electromagnetic-gravitational perturbations of Kerr-Newman spacetime, as obtained in \(\mathbb{C}^2\). We first recall the system of generalized Regge-Wheeler equations, and then show how the commutations with adapted symmetry operators maintain the same structure of the system.

#### 3.1 The generalized Regge-Wheeler (gRW) system

We recall the main theorem in \(\mathbb{C}^2\).

**Theorem 3.1.** Consider a linear electromagnetic-gravitational perturbation of Kerr-Newman spacetime \(\mathbf{g}_{\text{M.A.Q.}}\). Then we can define a complex horizontal 1-tensor \(p \in \mathfrak{s}_1(\mathbb{C})\) and a symmetric traceless 2-tensor \(q^F \in \mathfrak{s}_2(\mathbb{C})\) that, as a consequence of the Einstein-Maxwell equations, satisfy the following coupled system of wave equations:

\[
\Box p - \frac{2a \cos \theta}{|q|^2} \nabla \mathcal{T} p - V_1 p = 4Q^2 \frac{\tilde{v}^F}{|q|^2} (\tilde{D} \cdot \tilde{q}^F) + L_2[\mathfrak{B}, \tilde{F}] \tag{127}
\]
\[ \Box_2 q^F - i \frac{4a \cos \theta}{|q|^2} \nabla_T q^F - V_2 q^F = - \frac{1}{2} \frac{q^3}{|q|^4} \left( \Box \hat{p} - \frac{3}{2} (H - H) \hat{p} \right) + L_{q^F} [\mathcal{B}, \mathcal{E}] \quad (128) \]

where

- \[ \Box_1 = g^{a\beta} \mathcal{D}_a \mathcal{D}_\beta \] and \[ \Box_2 = g^{a\beta} \hat{D}_a \hat{D}_\beta \] denote the wave operators for horizontal 1-tensors and 2-tensors, respectively.

- The potentials \( V_1 \) and \( V_2 \) are real positive scalar functions, which for \( a = 0 \) coincide with the potentials of the Regge-Wheeler system of equations in Reissner-Nordström [36], i.e.

\[ V_1 = - \frac{1}{4} \text{tr} \chi \text{tr} \chi + 5 (F)^{\rho\sigma} + O \left( \frac{|a|}{r^4} \right), \quad V_2 = - \text{tr} \chi \text{tr} \chi + 2 (F)^{\rho\sigma} + O \left( \frac{|a|}{r^4} \right), \]

- \( L_{p} [\mathcal{B}, \mathcal{E}] \) and \( L_{q^F} [\mathcal{B}, \mathcal{E}] \) are linear first order operators in \( \mathcal{B} \) and \( \mathcal{E} \), which are lower order in terms of differentiability with respect to \( \hat{p} \) and \( q^F \).

We call the system of equations (127) - (128) a system of generalized Regge-Wheeler equations.

Observe that on the right hand side of the equations, the coupling terms on the right hand side involving \( \mathcal{D} \cdot q^F \) and \( \Box \hat{p} \) are proper of the Einstein-Maxwell case, while the left hand side of equation (128) has the same structure as the generalized Regge-Wheeler equation in Kerr as obtained in [30], where the coupling term in \( \hat{p} \) does not appear.

### 3.1.1 Decomposition in spheroidal harmonics and non-commutativity with the system

We now recall why the decomposition in modes fails for the gRW system in perturbations of Kerr-Newman spacetime.

Following the physics literature and standard decomposition in modes for scalar functions, we consider the scalar projection of equations (127) and (128) to the first component of the tensors \( p \) and \( q^F \), i.e. for \( \psi^{[1]} = p(e_1) \) and \( \psi^{[2]} = q^F(e_1, e_1) \), where \( \psi^{[s]} \) is a complex scalar of spin \( s \). Then the projection of the above equations gives, see Appendix E of [30],

\[ \Box \psi^{[1]} + \frac{2 \cos \theta}{|q|^2} \frac{1}{\sin^2 \theta} \partial_\theta \psi^{[1]} - i \frac{2a \cos \theta}{|q|^2} \partial_\theta \psi^{[1]} - V_1 \psi^{[1]} = - 4Q^2 \frac{q^3}{|q|^4} \Box \psi^{[2]} + \text{l.o.t.} \quad (129) \]

\[ \Box \psi^{[2]} + \frac{4 \cos \theta}{|q|^2} \frac{1}{\sin^2 \theta} \partial_\theta \psi^{[2]} - i \frac{2a \cos \theta}{|q|^2} \partial_\theta \psi^{[2]} - V_2 \psi^{[2]} = \frac{1}{2} \frac{q^3}{|q|^5} \Box \psi^{[1]} + \text{l.o.t.} \quad (130) \]

where the operators \( D_s \) and \( D'_s \) are respectively rising and lowering-spin operators, given by

\[ D_s \psi^{[s]} := \left( - \partial_\theta - i \frac{\sin \theta}{\sin \theta} \partial_\phi + s \cot \theta - ia \sin \theta \partial_\phi \right) \psi^{[s]} \]

\[ D'_s \psi^{[s]} := \left( - \partial_\theta + i \frac{\sin \theta}{\sin \theta} \partial_\phi - s \cot \theta + ia \sin \theta \partial_\phi \right) \psi^{[s]} \]

The mode decomposition of the scalar complex functions

\[ \psi^{[s]}(t, r, \theta, \varphi) = e^{-i \omega t} e^{i m \varphi} R^{[s]}(r)_{s_m} S^{[s]}_{\ell m}(a \omega, \cos \theta) \]

Involves the spin-weighted spheroidal harmonics \( S^{[s]}_{\ell m}(a \omega, \cos \theta) \) which are eigenfunctions of the spin-weighted Laplacian

\[ \Delta^{[s]} := \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \partial_\varphi^2 + 2i s \frac{\cos \theta}{\sin^2 \theta} \partial_\varphi + (s - s^2 \cot^2 \theta) + a^2 \omega^2 \cos^2 \theta - 2a \omega s \cos \theta \]

with parameter \( s \lambda_{\ell m}(a \omega) \), i.e. \( \Delta^{[s]}(S^{[s]}_{\ell m}) = s \lambda_{\ell m}(a \omega) S^{[s]}_{\ell m} \).

When \( a = 0 \), the above spin-weighted Laplacian reduce to the spherical spin-weighted Laplacian and the spheroidal harmonics reduce to the standard spherical harmonics \( S^{[s]}_{\ell m}(0, \cos \theta) = Y^{[s]}_{\ell m}(\cos \theta) \). Crucially, in spherical symmetry the operators (131) and (132) precisely relate spherical harmonics of different spin. In fact in this case one can check [13] that the spherical Laplacian can be written as

\[ D'_{\ell + 1} D_s = \Delta^{[s]}, \quad D'_{s-1} D'_s = \Delta^{[s]} - 2s \]

As a consequence, the operators \( D \) and \( D' \) simply relate spherical harmonics of different spins. More precisely,

\[ D_s Y^{[s]}_{\ell m} = ((\ell - s)(\ell + s + 1))^{1/2} Y^{[s+1]}_{\ell m}, \quad D'_s Y^{[s]}_{\ell m} = -((\ell + s)(\ell - s + 1))^{1/2} Y^{[s-1]}_{\ell m}. \]
Consequently, the operators $\mathcal{D}$ and $\mathcal{D}'$ appearing on the right hand side of (129) and (130) commute with the decomposition in spherical harmonics, and in spherical symmetry (i.e. for electromagnetic-gravitational perturbations of Reissner-Nordström) one is able to decompose the equations in modes.

On the other hand, in the general axisymmetric case, as in Kerr or Kerr-Newman, the spin-weighted spheroidal harmonics of different spins are not simply related through the angular operators $\mathcal{D}$ and $\mathcal{D}'$. In fact one can show that for separated solutions we have $\mathcal{D}_s S_{m|s} \not= \mathcal{D}'_{s'} S_{m'|s'}$.

3.2 The physical-space combined energy-momentum tensor for the system

To solve the issue of non-commutativity with the decomposition in modes, we propose instead to perform a physical-space analysis of the system by making use of a combined energy-momentum tensor for the system. A sketch of this procedure also appeared in [35].

A sketch of this procedure also appeared in [35].

As in (57), one can define the energy-momentum tensor for a complex horizontal tensor $\psi \in \mathfrak{s}_{k}(\mathbb{C})$ as

$$Q[\psi]_{\mu\nu} := \Re(\mathcal{D}_{\mu} \psi \cdot \mathcal{D}_{\nu} \overline{\psi}) - \frac{1}{2} g_{\mu\nu} \mathcal{L}[\psi],$$

where $\Re$ denotes the real part and

$$\mathcal{L}[\psi] := \mathcal{D}_{\lambda} \psi \cdot \mathcal{D}_{\nu} \overline{\psi} + V \psi \cdot \overline{\psi}.$$ 

The divergence of $Q[\psi]$ is then given by, see [36],

$$\mathcal{D}^{\nu} Q[\psi]_{\mu\nu} = \Re\left(\mathcal{D}_{\mu} \overline{\psi} \cdot \left(\mathcal{D}_{\nu} \psi - V \psi\right) + \mathcal{D}^{\nu} \psi^A R_{A\beta\mu} \overline{\psi}^{B}\right) - \frac{1}{2} \mathcal{D}_{\nu} V \psi \cdot \overline{\psi}.$$

Let $X$ be a vectorfield and $w$ a scalar. As in [58], one can define the associated current as

$$\mathcal{P}_X^{(X,w)}[\psi] := Q_{\mu\nu} X^\nu + \frac{1}{2} w \Re(\psi \cdot \mathcal{D}_{\mu} \overline{\psi}) - \frac{1}{4} \partial_{\mu} w |\psi|^2.$$

Then, its divergence is given by, see [36],

$$\mathcal{D}^{\mu} \mathcal{P}_X^{(X,w)}[\psi] = \frac{1}{2} Q[\psi] \cdot (X^\mu \mathcal{D}^\nu \psi - X^\nu \mathcal{D}^\mu \psi) + \frac{1}{2} w \Re(\mathcal{D}_{\mu} \psi \cdot (\mathcal{D}_{\nu} \overline{\psi} + \mathcal{D}^{\nu} \psi^A R_{A\beta\mu} \overline{\psi}^{B})) - \frac{1}{4} \mathcal{D}_{\nu} V \psi \cdot \overline{\psi},$$

where we used that $R_{\alpha\beta\gamma} = - \rho_{\alpha\beta\gamma}$ to write

$$\mathcal{E}^{(X,w)}[\psi] := \frac{1}{2} Q[\psi] \cdot (X^\mu \mathcal{D}^\nu \psi - X^\nu \mathcal{D}^\mu \psi) + \Re\left(\mathcal{D}^{\nu} \psi^A R_{A\beta\mu} \overline{\psi}^{B}\right),$$

$$\mathcal{R}^{(X)}[\psi] := - \epsilon_{AB} \mathcal{E}^{(X,w)}[\psi] + \Re\left(\mathcal{D}^{\nu} \psi^A - X^3 \mathcal{D}_3 \psi^A \overline{\psi}^{B}\right).$$

Applying the above to the complex tensors $\mathfrak{p} \in \mathfrak{s}_1(\mathbb{C})$ and $\mathfrak{q}^F \in \mathfrak{s}_2(\mathbb{C})$, solutions to (127) and (128), we obtain respectively

$$\mathcal{D}^\nu \mathcal{P}_X^{(X,w)}[\mathfrak{p}] = \mathcal{E}^{(X,w)}[\mathfrak{p}] + \mathcal{R}^{(X)}[\mathfrak{p}],$$

$$+ \Re\left(\mathcal{D}^{\nu} \mathcal{D}_3 \psi^A - X^3 \mathcal{D}_3 \psi^A \overline{\psi}^{B}\right).$$

9 Recall that the potentials of the gRW system are real.
\[ D^\mu P^{(X,w)}_{\mu} [q^F] = E^{(X,w)} [q^F] + R^{(X)} [q^F] \]  

(134)

\[ + \mathcal{R} \left( (X(q^F) + \frac{1}{2} q^\alpha q^\beta \nabla_\tau q^F - \frac{1}{2} \frac{q^3}{|q|^5} (D \otimes p - \frac{3}{2} (H - H) \otimes p) + L_q r [\mathbb{B}, \mathbb{G}] \right) \]

As one can observe, the divergence of each equation for one of the two tensors involves the coupling terms with the other tensor of the system. Nevertheless, there exists a combined energy-momentum tensor for the system, which can be obtained by summing the two above, modulo spacetime boundary terms. In particular, we consider

\[ Q[p, q^F]_{\mu \nu} := Q[p]_{\mu \nu} + 8Q^2 Q[q^F]_{\mu \nu} \]  

(135)

\[ P^{(X,w)}_{\mu} [p, q^F] := P^{(X,w)}_{\mu} [p] + 8Q^2 P^{(X,w)}_{\mu} [q^F] \],

\[ Q^{(X,w)}_{\mu} [p, q^F] := D^\mu P^{(X,w)}_{\mu} [p] + 8Q^2 D^\mu P^{(X,w)}_{\mu} [q^F] \]  

(136)

and we show that with the above combination the highest order terms in the coupling satisfy a cancellation.

3.2.1 The cancellation of the highest-order coupling terms in physical-space

To show how the cancellation of the highest order coupling terms takes place at the level of the energy-momentum tensor, as opposed to non-commutativity of the decomposition in spheroidal harmonics discussed in Section 3.1.1, we concentrate on the coupling terms in the divergence above. More precisely we look at

\[ E := \frac{q^3}{|q|^5} X(p) \cdot (D \cdot q^F) - X(q^F) \cdot \frac{q^3}{|q|^5} (D \otimes p - \frac{3}{2} (H - H) \otimes p) \]

Using Lemma 1.8, we write

\[ (D \otimes p) \cdot X(q^F) = -p \cdot (D \cdot X(q^F)) - ((H + H) \otimes p) \cdot X(q^F) + D_\alpha (p \cdot X(q^F))^\alpha. \]  

(137)

Using (23) to deduce that

\[ D \left( \frac{q^3}{|q|^5} \right) = \frac{1}{2} \frac{q^3}{|q|^5} (H - 5H), \]

we can write

\[ \frac{q^3}{|q|^5} (D \otimes p) \cdot X(q^F) = -\frac{q^3}{|q|^5} p \cdot (D \cdot X(q^F)) - \frac{q^3}{|q|^5} ((H + H) \otimes p) \cdot X(q^F) - \frac{1}{2} \frac{q^3}{|q|^5} (H - 5H) \otimes p \cdot X(q^F) \]

\[ + D_\alpha \left( \frac{q^3}{|q|^5} p \cdot X(q^F) \right)^\alpha \]

\[ = -\frac{q^3}{|q|^5} p \cdot (D \cdot X(q^F)) + \frac{3}{2} \frac{q^3}{|q|^5} ((H - H) \otimes p) \cdot X(q^F) + D_\alpha \left( \frac{q^3}{|q|^5} p \cdot X(q^F) \right)^\alpha. \]

We therefore obtain the cancellation of the terms in $H$ and $H$, as follows:

\[ E = \frac{q^3}{|q|^5} X(p) \cdot (D \cdot q^F) - X(q^F) \cdot \frac{q^3}{|q|^5} (D \otimes p - \frac{3}{2} (H - H) \otimes p) \]

\[ = \frac{q^3}{|q|^5} X(p) \cdot (D \cdot q^F) + \frac{q^3}{|q|^5} p \cdot (D \cdot X(q^F)) - \frac{3}{2} \frac{q^3}{|q|^5} ((H - H) \otimes p) \cdot X(q^F) - D_\alpha \left( \frac{q^3}{|q|^5} p \cdot X(q^F) \right)^\alpha \]

\[ + X(q^F) \cdot \frac{3}{2} \frac{q^3}{|q|^5} (H - H) \otimes p \]

\[ = \frac{q^3}{|q|^5} X(p) \cdot (D \cdot q^F) + \frac{q^3}{|q|^5} p \cdot (X(D \cdot q^F)) + \frac{q^3}{|q|^5} p \cdot ([[D, \nabla X]q^F] - D_\alpha \left( \frac{q^3}{|q|^5} p \cdot X(q^F) \right)^\alpha) \]

Integrating by parts in $X$ we finally obtain

\[ E = \frac{q^3}{|q|^5} X(p) \cdot (D \cdot q^F) - X \left( \frac{q^3}{|q|^5} p \cdot (D \cdot q^F) - \frac{q^3}{|q|^5} X(p) \cdot (D \cdot q^F) + \frac{q^3}{|q|^5} p \cdot (D \cdot q^F) \right) \]

\[ - D_\alpha \left( \frac{q^3}{|q|^5} p \cdot X(q^F) \right)^\alpha + \nabla X \left( \frac{q^3}{|q|^5} p \cdot (D \cdot q^F) \right) \]

\[ = -X \left( \frac{q^3}{|q|^5} p \cdot (D \cdot q^F) + \frac{q^3}{|q|^5} p \cdot (D \cdot q^F) \right) + \text{Bdr}, \]

39
where Bdr denotes boundary terms. In particular, the highest order terms (those involving up to two derivatives) got cancelled in the sum of the two terms. Similarly, we obtain

\[
F := \frac{q^3}{|q|^2} \theta \cdot (\partial\cdot \hat{q} \mathcal{F}) - \frac{1}{2} q \mathcal{F} \cdot \frac{q^3}{|q|^2} (\partial \hat{q} \mathcal{F} - \frac{3}{2}(H-H) \hat{q} \mathcal{F})
\]

\[
= \frac{q^3}{|q|^2} \theta \cdot (\partial\cdot \hat{q} \mathcal{F}) - D_{\alpha} \left( \frac{1}{2} \frac{q^3}{|q|^2} p \cdot \hat{q} \mathcal{F} \right)^{\alpha}.
\]

As \( \mathcal{R}(E) + \mathcal{R}(F) \) precisely gives the coupling term of the combined current \( \mathcal{G}^{(X,w)}[p,q^F] \), putting together (133), (134) and (136) we finally obtain

\[
\mathcal{G}^{(X,w)}[p,q^F] = \mathcal{E}^{(X,w)}[p] + 8Q^2 \mathcal{E}^{(X,w)}[q^F]
\]

\[
- \frac{2a \cos \theta}{|q|^2} \mathcal{G}_3 \left( (X[p]) + \frac{1}{2} q \mathcal{F} \right) \cdot \nabla T \mathcal{F} + 16Q^2 (X[q^F] + \frac{1}{2} q \mathcal{F} \cdot \nabla T q^F)
\]

\[
+ \text{l.o.t.} + \text{Bdr}
\]

where \( \mathcal{G}_3 \) denotes the imaginary part, using that \( \mathcal{R}(iz) = -\mathcal{G}(z) \), and we collected the lower order terms (i.e. those involving up to one derivatives of \( p \) and \( q^F \)) in the l.o.t., which is given by

\[
\text{l.o.t.} := 4Q^2 \left( - X \left( \frac{q^3}{|q|^2} \theta \cdot (\partial\cdot \hat{q} \mathcal{F}) + \frac{q^3}{|q|^2} \theta \cdot (\partial\cdot \hat{q} \mathcal{F}) \right) \right)
\]

\[
+ \mathcal{R} \left( (X[p]) + \frac{1}{2} q \mathcal{F} \cdot L_p \mathcal{B}, \mathcal{G} \right) + 8Q^2 (X[q^F] + \frac{1}{2} q \mathcal{F} \cdot L_q \mathcal{B}, \mathcal{G}) + \mathcal{R}(X)[p] + 8Q^2 \mathcal{R}(X)[q^F].
\]

For general vectorfields, we still have terms involving two derivatives of \( p \) and \( q^F \) in the second line of (138). Nevertheless, we now show that those get cancelled when \( X = T \), used for the energy estimates in the trapping region, and can instead be absorbed by Cauchy-Schwarz, for small \( |a|/M \), outside of the trapping region.

### 3.2.2 The combined-energy momentum tensor associated to \( \hat{T}_X \)

As obtained in Section 2.1, the timelike vectorfield \( \hat{T}_X = T + \chi \omega Z \) is used in the derivation of the energy estimates. Recall that \( \hat{T} \) coincides with \( T \) in the trapping region, i.e. \( r \geq \mathcal{R}_{\text{trap}} \).

We apply the combined energy-momentum tensor in (138) to the case of \( X = \hat{T}_X \), \( w = 0 \) in the context of deriving the energy estimates for the gRW system. In this case, the second line of (138) becomes

\[
\mathcal{G}^{(\hat{T}_X,0)}[p] + 8Q^2 \mathcal{E}^{(\hat{T}_X,0)}[q^F] = \Delta \omega \mathcal{R}(\hat{T}_X)[p] + \frac{1}{2} Q[p] \cdot (\hat{T}_X) \pi + 8Q^2 \frac{1}{2} Q[q^F] \cdot (\hat{T}_X) \pi
\]

\[
= \Delta \omega \mathcal{R}(\hat{T}_X)[p] + \mathcal{R}(\hat{T}_X)[p] + 8Q^2 \mathcal{R}(\hat{T}_X)[q^F],
\]

again only supported away from trapping, where \( \chi = 0 \). Also, recall (79), which gives

\[
\mathcal{E}^{(\hat{T}_X,0)}[p] + 8Q^2 \mathcal{E}^{(\hat{T}_X,0)}[q^F] = \Delta \omega \mathcal{R}(\hat{T}_X)[p] + \frac{1}{2} Q[p] \cdot (\hat{T}_X) \pi + 8Q^2 \frac{1}{2} Q[q^F] \cdot (\hat{T}_X) \pi
\]

\[
= \Delta \omega \mathcal{R}(\hat{T}_X)[p] + \mathcal{R}(\hat{T}_X)[p] + 8Q^2 \mathcal{R}(\hat{T}_X)[q^F],
\]

as the product involving \( \nabla T \) gets cancelled. In particular, the above is only supported away from trapping, where \( \chi = 0 \). Also, recall (79), which gives

\[
\mathcal{E}^{(\hat{T}_X,0)}[p] + 8Q^2 \mathcal{E}^{(\hat{T}_X,0)}[q^F] = \Delta \omega \mathcal{R}(\hat{T}_X)[p] + \frac{1}{2} Q[p] \cdot (\hat{T}_X) \pi + 8Q^2 \frac{1}{2} Q[q^F] \cdot (\hat{T}_X) \pi
\]

\[
= \Delta \omega \mathcal{R}(\hat{T}_X)[p] + \mathcal{R}(\hat{T}_X)[p] + 8Q^2 \mathcal{R}(\hat{T}_X)[q^F],
\]

again only supported away from trapping, where \( \partial_r \chi = 0 \).

From (138), we then obtain

\[
\mathcal{G}^{(\hat{T}_X,0)}[p,q^F] = \Delta \omega \mathcal{R}(\hat{T}_X)[p] + \mathcal{R}(\hat{T}_X)[p] + 8Q^2 \mathcal{R}(\hat{T}_X)[q^F]
\]

\[
- 2a \cos \theta \frac{q^3}{|q|^2} \theta \cdot (\partial\cdot \hat{q} \mathcal{F}) - \mathcal{G}_3 \left( (X[p]) + \frac{1}{2} q \mathcal{F} \cdot \nabla T \mathcal{F} + 16Q^2 (X[q^F] + \frac{1}{2} q \mathcal{F} \cdot \nabla T q^F)
\]

\[
+ 4Q^2 \frac{q^3}{|q|^2} \theta \cdot (\partial\cdot \hat{q} \mathcal{F}) + \mathcal{R}(\hat{T}_X)[p] + 8Q^2 \mathcal{R}(\hat{T}_X)[q^F]
\]
The first two lines of the above are supported away from the trapping region, and therefore can be bounded by Cauchy-Schwarz, and eventually absorbed for small \( |a| \) by a Morawetz bulk where \( \nabla_r, \nabla_Z \) and \( \nabla_T \) do not degenerate outside trapping.

Using that \( [\nabla_T, \nabla] \psi = O(\frac{a}{r^2}) \psi \) and, see \[37],
\[
R(\tilde{T}_x)[p] = R\left( - \; \ast \rho \in_{AB} \left( \tilde{T}^a_X \tilde{D}_a^p \tilde{p}^b - \tilde{T}^a_X \tilde{D}_a^q \tilde{q}^b \right) \right) = O(\frac{a}{r^2}) R(\nabla_r p \cdot \tilde{p}) ,
\]
we see that the third line of the above can also be bounded by Cauchy-Schwarz in terms of \( \nabla_r \) derivatives or zero-th order terms of \( p \) and \( q F \), which are non-degenerate in the Morawetz norm. In particular, for very small \( |a|/M \), those terms could be absorbed once combined with a Morawetz spacetime estimates.

For the analysis of the last line involving the terms \( L_p[\mathcal{B}, \tilde{g}] \) and \( L_q[\mathcal{B}, \tilde{g}] \) see \[35].

The energy estimates would then have to be combined with the Morawetz estimates to obtain boundedness of the energy. In the case of Kerr-Newman, the non-separability in modes makes this procedure even more relevant, and in order to apply the Andersson-Blue’s method \[4] described above for the scalar wave equation, we need to allow for a commutation with symmetry operators for the system.

### 3.3 Symmetry operators for the gRW system

In Section \[1.3\], we defined the modified Laplacian \( O \) for scalar functions on Kerr-Newman spacetime. Nevertheless, the definition of \( O \) as given in \[10\], i.e.
\[
O(\psi) = \left( |q|^2 (\Delta \psi + (\eta + \eta) \cdot \nabla \psi) \right),
\]
can be applied to any tensor \( \psi \in \mathfrak{g}_k \) where \( \Delta_k := \delta^{ab} \nabla_a \nabla_b \). When proving the corresponding of Proposition \[1.12\] for tensors, i.e. the commutator with the D’Alembertian operator \( \Box \), one obtains additional lower order terms involving Riemann curvature. One can show, see \[36\], that for \( \psi \in \mathfrak{g}_k \)
\[
[O, |q|^2 \Box] \psi = O(ar^{-1}) \mathcal{O}^{k \leq 1} \psi.
\]

#### 3.3.1 The modified laplacian for the gRW system

The operator \( O \), even though is a conformal symmetry (up to the lower order terms above) for \( \Box \), it is not a symmetry for gRW system of equations \[127\] and \[128\], because of the presence of the coupling terms on the right hand side of the equations. We instead have to define a corrected pair of symmetry operators from it, and show how the commutation with such modified Carter differential operators allows to maintain the same structure of the equations.

Following \[36\], we recall the following Gauss equation for horizontal structures:

**Proposition 3.2** (Proposition 2.52 in \[36\]). The following identity holds true for \( \psi \in \mathfrak{g}_k \) for \( k = 0, 1, 2 \):
\[
[\nabla_a, \nabla_b] \psi = \left( \frac{1}{2} (^{(a)} tr_X \nabla_3 + ^{(a)} tr_X \nabla_4) \psi + k ^{(h)} K \ast \psi \right) \in_{ab}
\]
where
\[
^{(h)} K := - \frac{1}{4} tr_X tr_X - \frac{1}{4} ^{(a)} tr_X ^{(a)} tr_X + \frac{1}{2} \hat{\chi} \cdot \hat{\chi} - \frac{1}{4} R_{4343}.
\]

The scalar quantity \( ^{(h)} K \) is denoted the modified Gauss curvature\[34\] of the horizontal structure. As a consequence of the above, we obtain, see \[36\], for \( \xi \in \mathfrak{g}_1, \ u \in \mathfrak{g}_2 \)
\[
\mathcal{D}_\xi \mathcal{D}_u \xi = - \frac{1}{2} \Delta_1 \xi + \frac{1}{2} [\nabla_1, \nabla_2] \ast \xi = - \frac{1}{2} \Delta_1 \xi - \frac{1}{2} ^{(h)} K \xi + \frac{1}{4} (^{(a)} tr_X \nabla_3 + ^{(a)} tr_X \nabla_4) \ast \xi
\]
and
\[
\mathcal{D}_u \mathcal{D}_u = - \frac{1}{2} \Delta_2 u - \frac{1}{2} [\nabla_1, \nabla_2] \ast u = - \frac{1}{2} \Delta_2 u + ^{(h)} K u - \frac{1}{4} (^{(a)} tr_X \nabla_3 + ^{(a)} tr_X \nabla_4) \ast u.
\]

These relations are used to compute the commutator between \( O \) and the angular operators \( \mathcal{D} \) and \( \mathcal{D} \hat{\mathcal{O}} \).

---

\[34\] In the integrable case, \( ^{(h)} K = \frac{1}{4} \) is the Gauss curvature of the sphere orthogonal to the principal null frame. As in the non-integrable picture there are no spheres in the horizontal space to the null principal direction, this is not a real Gauss curvature.
Lemma 3.3. The following commutators hold true for \( \phi \in s_1 \) and \( \psi \in s_2 \):

\[
\begin{align*}
[O, D\tilde{\otimes}]\phi &= 3|q|^2 \left\{ (h) K D\tilde{\otimes} \phi + |q|^2 D([|q|^{-2}) D\tilde{\otimes} \phi) + i|q|^2 (tr \nabla \nabla_3 + (a) tr \nabla \nabla_4) (D\tilde{\otimes} \phi) \\
+ O(ar^{-1})a\leq 1, \phi \right\}, \\
[O, D\tilde{\otimes}]\psi &= -3|q|^2 \left\{ (h) K D\tilde{\otimes} \psi + \left| D([|q|^{-2}) D\tilde{\otimes} \psi) - i|q|^2 (tr \nabla \nabla_3 + (a) tr \nabla \nabla_4) (D\tilde{\otimes} \psi) \\
+ O(ar^{-1})a\leq 1, \psi \right\}.
\end{align*}
\] (144)

Proof. Recall that \( D\tilde{\otimes} = \nabla \tilde{\otimes} + i * \nabla \tilde{\otimes} = -D_\psi - i * D_\chi \) and \( D = D_\phi + i * D_\phi \), and therefore we can compute the real part of the commutators, i.e. \([O, D_\phi] \) and \([O, D_\phi] \). We have using (142) and (143),

\[
\begin{align*}
O(D_\phi) &= |q|^2 (\Delta_1 + (\eta + \psi) \cdot \nabla (D_\phi)) \]
\end{align*}
\] (145)

By writing \( \omega \) as \( |q|^{-2} O(\omega) \), we obtain

\[
\begin{align*}
O(D_\phi) &= |q|^2 (\Delta_1 + (\eta + \psi) \cdot \nabla (D_\phi)) + 3 \left\{ (h) K D_\phi \right\} \\
&= |q|^2 \left\{ -2 \eta \cdot D_\eta \phi - \frac{1}{2}(a) \Delta_{\phi} + \eta \cdot \nabla (D_\phi) \right\} + O(ar^{-1})a\leq 1, \phi \right\}.
\end{align*}
\] (146)

By complexifying the above we obtain (144). Similarly, we compute

\[
\begin{align*}
O(D_\phi) &= |q|^2 (\Delta_2 + (\eta + \psi) \cdot \nabla (D_\phi)) \\
&= |q|^2 \left\{ -2 \eta \cdot D_\eta \phi - (a) \Delta_{\phi} + (a) \nabla \nabla_3 + (a) tr \nabla \nabla_4 \right\} + O(ar^{-1})a\leq 1, \phi \right\}.
\end{align*}
\] (147)

By writing \( \omega \) as \( |q|^{-2} O(\omega) \), we obtain

\[
\begin{align*}
O(D_\phi) &= |q|^2 (\Delta_2 + (\eta + \psi) \cdot \nabla (D_\phi)) - 3 \left\{ (h) K D_\phi \right\} \\
&= |q|^2 \left\{ -2 \eta \cdot D_\eta \phi + (a) \Delta_{\phi} + (a) \nabla \nabla_3 + (a) tr \nabla \nabla_4 \right\} + O(ar^{-1})a\leq 1, \phi \right\}.
\end{align*}
\] (148)

By complexifying the above we obtain (145).

Observe the presence of the terms \( 3|q|^2 (h) K D\tilde{\otimes} \) and \(-3|q|^2 (h) K D\tilde{\otimes} \) in the above commutators. As they involve the modified Gauss curvature, these terms are not small even in Schwarzschild. In order to obtain symmetry operators for the gRW system, we combine the modified Laplacian \( O \) with lower order terms involving the modified Gauss curvature \( K \).

Proposition 3.4. Let \( p \) and \( q^F \) be solutions to the generalized Regge-Wheeler system in Theorem 3.4. Then for any real number \( c \), the complex tensors

\[
\begin{align*}
\tilde{p} := (O + (c + 3)|q|^2 (h) K)p, \\
\tilde{q}^F := (O + c |q|^2 (h) K)q^F
\end{align*}
\] (146)
satisfy the following system of wave equations:

\begin{align*}
\Box_1 \hat{p} - i 2 a \cos \theta \left[ \frac{\nabla^2}{|q|^2} \right] \nabla_T \hat{p} - V_1 \hat{p} &= 4 Q^2 \frac{7}{|q|^3} \left( \overline{\nabla} \cdot \hat{q} F \right) + N_1[p, q F] \quad (147) \\
\Box_2 \hat{q} F - i 4 a \cos \theta \left[ \frac{\nabla^2}{|q|^2} \right] \nabla_T \hat{q} F - V_2 \hat{q} F &= - \left\{ \frac{1}{2} \frac{q^2}{|q|^3} \right\} \left( D \hat{\nabla} - \frac{3}{2} (H - H) \hat{\nabla} \right) p + N_2[p, q F] \quad (148)
\end{align*}

where the terms \( N_1[p, q F] \) and \( N_2[p, q F] \) are \( O(|a|) \) lower order in differentiability with respect to \( p \) and \( q F \), explicitly given by

\begin{align*}
N_1[p, q F] &= - 4 Q^2 \frac{7}{|q|^3} \left\{ i (^{(a)} \partial_T \nabla_3 + (^{(a)} \partial_T \nabla_4) \overline{\nabla} \cdot q F) - 2 \nabla \left( \frac{7}{|q|^3} \right) \cdot \nabla \left( \overline{\nabla} \cdot q F \right) - \frac{7}{|q|^3} D(|q|^2)^2 \cdot O(q F) \right\} + i 4 a \nabla \cos \theta \cdot \nabla (\nabla_T p) + |q|^{-2} O(|q|^2 L_p[H, \hat{g}]) + (c + 3)\left\{ |q|^{(h)} K L_p[B, \hat{g}] \right\} + O(\ar^{-3}) \left( \partial^{\leq 1} p, \partial^{\leq 1} q F \right) \\
N_2[p, q F] &= - \frac{1}{2} \frac{q^2}{|q|^3} \left\{ i (^{(a)} \partial_T \nabla_3 + (^{(a)} \partial_T \nabla_4) \overline{\nabla} \cdot q F) + 2 \nabla \left( \frac{q^2}{|q|^3} \right) \cdot \nabla \left( \overline{\nabla} \cdot q F \right) + \frac{q^2}{|q|^3} D(|q|^2)^2 \cdot O(q F) \right\} + i 4 a \nabla \cos \theta \cdot \nabla (\nabla_T q F) + |q|^{-2} O(|q|^2 L_q[H, \hat{g}]) + c |q|^{(h)} K L_q[H, \hat{g}] + O(\ar^{-3}) \left( \partial^{\leq 1} p, \partial^{\leq 1} q F \right)
\end{align*}

where \( \partial = \nabla_3, r \nabla, r \nabla \) denotes first order derivatives.

In particular, the higher order structure of equations \((147)\) and \((148)\) is the same as the gRW system of equations \((127)\) and \((128)\) for the un-commuted \( p \) and \( q F \).

Proof. We start by communting the gRW system with the operator \( |q|^2 (\triangle + (\eta + \eta^2)) \), where \( \triangle \) denotes \( \Delta_1 \) or \( \Delta_2 \) if it applies to \( p \) or \( q F \) respectively. By multiplying equations \((127)\) and \((128)\) by \(|q|^2\) we obtain

\begin{align*}
|q|^2 \Box_1 p - i 2 a \cos \theta \nabla_T p - (|q|^2 V_1) p &= 4 Q^2 \frac{7}{|q|^3} \left( \overline{\nabla} \cdot q F \right) + |q|^2 L_p[H, \hat{g}] \quad (149) \\
|q|^2 \Box_2 q F - i 4 a \cos \theta \nabla_T q F - (|q|^2 V_2) q F &= - \frac{1}{2} \frac{q^2}{|q|^3} \left( D \hat{\nabla} - \frac{3}{2} (H - H) \hat{\nabla} \right) p + |q|^2 L_q[H, \hat{g}] \quad (150)
\end{align*}

We now apply the operator \( O \) and consider the left hand side of the two equations. Using that

\[ O(\psi) = f O(\psi) + O(f) \psi + 2|q|^2 \nabla f \cdot \nabla \psi \]

we obtain for the first equation

\[ O(\text{LHS of } (149)) = \left\{ |q|^2 \Box_1 (O) p + [O, |q|^2 \Box_1] p - i O(2 a \cos \theta \nabla_T p) - O(|q|^2 V_1) p \right\} \]

\[ = \left\{ |q|^2 \Box_1 (O) p - i 2 a \cos \theta O(\nabla_T p) - i 2 a O(\cos \theta) \nabla_T p - 2 |q|^2 \nabla (2 a \cos \theta) \cdot \nabla (\nabla_T p) \right\} + O(\ar^{-1}) \partial^{\leq 1} p \]

\[ = \left\{ |q|^2 \Box_1 (O) p - i 2 a \cos \theta \nabla_T O(p) - |q|^2 V_1 O(p) - 4 a |q|^2 \nabla (\cos \theta) \cdot \nabla (\nabla_T p) \right\} + O(\ar^{-1}) \partial^{\leq 1} p, \]

where we used that \( |O, \nabla_T| \psi = O(\ar^{-3}) \partial^{\leq 1} \psi, \) see \((37)\). Similarly for the left hand side of \((150)\).

We now consider the right hand side of the equations. We make use of Lemma \((93)\) to obtain for the first equation:

\[ O(\text{RHS of } (149)) = 4 Q^2 \frac{7}{|q|^3} \left( \overline{\nabla} \cdot O(q F) - 3 |q|^{(h)} K \overline{\nabla} (q F) \right) - 4 Q^2 \frac{7}{|q|^3} \left\{ i (^{(a)} \partial_T \nabla_3 + (^{(a)} \partial_T \nabla_4) \overline{\nabla} \cdot q F) \right\} - 2 |q|^2 \nabla \left( \frac{7}{|q|^3} \right) \cdot \nabla \left( \overline{\nabla} \cdot q F \right) - \frac{7}{|q|^3} D(|q|^2)^2 \cdot O(q F) \right\} + O(\ar^{-1}) \partial^{\leq 1} q F + O(|q|^2 L_p[H, \hat{g}]), \]

and for the second equation

\[ O(\text{RHS of } (150)) = - \frac{1}{2} \frac{q^2}{|q|^3} \left( D \hat{\nabla} O(p) - \frac{3}{2} (H - H) \hat{\nabla} O(p) + 3 |q|^2 \overline{\nabla} (q F) \right) + O(\ar^{-1}) \partial^{\leq 1} p + O(|q|^2 L_q[H, \hat{g}]). \]
By combining the above computations, we obtain for $\mathcal{O} p$ and $\mathcal{O} q F$ respectively:

$$\Box_1(\mathcal{O} p) - i \frac{2 a \cos \theta}{|q|^2} \nabla_T (\mathcal{O} p) - V_1(\mathcal{O} p) = 4Q^2 \frac{q^3}{|q|^5} (\mathcal{O} q F) - 3|q|^2 (h) K (\mathcal{O} q F) + \mathcal{N}_1[p, q F]$$

$$+ O(ar^{-3}) \left( d^{\leq 1} p, d^{\leq 1} q F \right) + |q|^{-2} \mathcal{O} (|q|^2 L_p [\mathcal{B}, \mathcal{F}])$$

$$\Box_1(\mathcal{O} q F) - i \frac{4a \cos \theta}{|q|^2} \nabla_T \mathcal{O} (q F) - V_2 \mathcal{O} (q F) = \frac{1}{2} \frac{q^3}{|q|^3} \left( D \hat{\otimes} (\mathcal{O} p) - \frac{3}{2} (H - \mathcal{H}) \hat{\otimes} (\mathcal{O} p) + 3|q|^2 (h) K (D \hat{\otimes} p) \right)$$

$$+ \mathcal{N}_2[p, q F] + O(ar^{-3}) \left( d^{\leq 1} p, d^{\leq 1} q F \right) + |q|^{-2} \mathcal{O} (|q|^2 L_q F [\mathcal{B}, \mathcal{F}])$$

where

$$\mathcal{N}_1[p, q F] = -4Q^2 \left[ \frac{q^3}{|q|^3} (a) \text{tr}_\lambda \nabla_3 + (a) \text{tr}_\lambda \nabla_4 (D \cdot q F) - 2\nabla \left( \frac{q^3}{|q|^3} \right) \cdot \nabla (D \cdot q F) 

- \frac{q^3}{|q|^3} D(|q|^{-2}) \cdot \mathcal{O}(q F) \right] + 4ai \nabla \left( \cos \theta \right) \cdot \nabla (\nabla_T p),$$

$$\mathcal{N}_2[p, q F] = -\frac{1}{2} \frac{q^3}{|q|^3} (a) \text{tr}_\lambda \nabla_3 + (a) \text{tr}_\lambda \nabla_4 (D \hat{\otimes} p) + 2\nabla \left( \frac{q^3}{|q|^3} \right) \cdot \nabla (D \hat{\otimes} p)$$

$$+ \frac{q^3}{|q|^3} \left[ D(|q|^{-2}) \hat{\otimes} (\mathcal{O} p) \right] + 8ai \nabla \left( \cos \theta \right) \cdot \nabla (\nabla_T q F).$$

Observe the terms involving $(h) K$ on the right hand side of both equations. In order to absorb them, we combine the above with the gRW equations commuted with $|q|^2 (h) K = 1 + O(a^2 r^{-2})$, i.e.

$$\Box_1(|q|^2 (h) K p) - i \frac{2 a \cos \theta}{|q|^2} \nabla_T (|q|^2 (h) K p) - V_1(|q|^2 (h) K p) = 4Q^2 \frac{q^3}{|q|^5} \left( |q|^2 (h) K \mathcal{O} q F \right) + |q|^2 (h) K L_p [\mathcal{B}, \mathcal{F}]$$

$$+ O(a^2 r^{-5}) \left( d^{\leq 1} p, d^{\leq 1} q F \right)$$

$$\Box_2(|q|^2 (h) K q F) - i \frac{4a \cos \theta}{|q|^2} \nabla_T (|q|^2 (h) K q F) - V_2(|q|^2 (h) K q F) = -\frac{1}{2} \frac{q^3}{|q|^3} \left( |q|^2 (h) K \mathcal{O} q F \right) - \frac{3}{2} (H - \mathcal{H}) \hat{\otimes} (|q|^2 (h) K p)$$

$$+ |q|^2 (h) K L_q F [\mathcal{B}, \mathcal{F}] + O(a^2 r^{-5}) \left( d^{\leq 1} p, d^{\leq 1} q F \right).$$

We therefore obtain, for $\tilde{p} := (\mathcal{O} + (c + 3)|q|^2 (h) K) p$ and $\tilde{q} F := (\mathcal{O} + (c)|q|^2 (h) K) q F$, combining the above:

$$\Box_1 \tilde{p} - i \frac{2 a \cos \theta}{|q|^2} \nabla_T \tilde{p} - V_1 \tilde{p} = 4Q^2 \frac{q^3}{|q|^5} \left( \mathcal{O} q F + c|q|^2 (h) K q F \right) + \mathcal{N}_1[p, q F] + O(ar^{-3}) \left( d^{\leq 1} p, d^{\leq 1} q F \right)$$

$$+ |q|^{-2} \mathcal{O} (|q|^2 L_p [\mathcal{B}, \mathcal{F}]) + (c + 3)|q|^2 (h) K L_p [\mathcal{B}, \mathcal{F}]$$

$$\Box_2 \tilde{q} F - i \frac{4a \cos \theta}{|q|^2} \nabla_T \tilde{q} F - V_2 \tilde{q} F = -\frac{1}{2} \frac{q^3}{|q|^3} \left( \mathcal{O} q F + (c + 3)|q|^2 (h) K p \right) - \frac{3}{2} (H - \mathcal{H}) \hat{\otimes} (\mathcal{O} p + (c + 3)|q|^2 (h) K p)$$

$$+ \mathcal{N}_2[p, q F] + O(ar^{-3}) \left( d^{\leq 1} p, d^{\leq 1} q F \right)$$

$$+ |q|^{-2} \mathcal{O} (|q|^2 L_q F [\mathcal{B}, \mathcal{F}]) + c|q|^2 (h) K L_q F [\mathcal{B}, \mathcal{F}]$$

which proves the proposition. \qed

### 3.3.2 The other symmetry operators for the gRW system

The pairs of tensors obtained in Proposition 3.3.1 i.e.

$$\left( \tilde{p}, \tilde{q} F \right) = (\mathcal{O} p + (c + 3)|q|^2 (h) K) p, \quad \mathcal{O} q F + (c)|q|^2 (h) K q F$$

represents the symmetry operators associated to the Carter operator $\mathcal{O}$ for the system. In addition to them, one can define the second order operator associated to $T$ and $Z$.

To maintain clear the difference between the operators applied to the 1-tensor $p$ and to the 2-tensor $q F$, we define the following couples of operators $(\mathcal{S}, W_\alpha) = (\mathcal{S}_\alpha, W_\alpha)$ for $\alpha = 1, 2, 3, 4$:

$$\mathcal{S}_1 = \nabla_T \nabla_T \quad \mathcal{W}_1 = \nabla_T \nabla_T$$

$$\mathcal{S}_2 = a \nabla_T T Z \quad \mathcal{W}_2 = a \nabla_T T Z$$

$$\mathcal{S}_3 = a^2 \nabla_Z \nabla_Z \quad \mathcal{W}_3 = a^2 \nabla_Z \nabla_Z$$

$$\mathcal{S}_4 = \mathcal{O} + (c + 3)|q|^2 (h) K \quad \mathcal{W}_4 = \mathcal{O} + c|q|^2 (h) K$$

(151)
where \( c \) is any number, the \( S \) operators are applied to the \( p \) and the \( W \) operators are applied to \( q^F \).

We also define the corresponding symmetric tensor\(^{[1]}\)
\[
S_1^\alpha \beta = T^\alpha T^\beta, \quad S_2^\alpha \beta = a T^{(\alpha Z^\beta)}, \quad S_3^\alpha \beta = a^2 Z^\alpha Z^\beta, \quad S_4^\alpha \beta = O^\alpha \beta. \tag{152}
\]

With the above definition, from (53), one can write
\[
R^\alpha \beta = -(r^2 + a^2)^2 S_1^\alpha \beta - 2(r^2 + a^2) S_2^\alpha \beta - S_3^\alpha \beta + \Delta O^\alpha \beta =: R^\alpha \beta S_4^\alpha \beta.
\tag{153}
\]

with
\[
R^1 = -(r^2 + a^2)^2, \quad R^2 = -2(r^2 + a^2), \quad R^3 = -1, \quad R^4 = \Delta.
\tag{154}
\]

We can then relate the symmetry operators (\( S_\underline{a}, W_\underline{a} \)) to the symmetric tensors \( S^\alpha \beta \).

**Lemma 3.5.** We have for \( \underline{a} = 1, 2, 3, 4 \):
\[
S_\underline{a} = |q|^2 D_\alpha (|q|^{-2} S_\underline{a}^\alpha \beta D_\beta) + \delta_a(c + 3)|q|^2 (h) K, \tag{155}
\]
\[
W_\underline{a} = |q|^2 D_\alpha (|q|^{-2} S_\underline{a}^\alpha \beta D_\beta) + \delta_a c|q|^2 (h) K \tag{156}
\]
where \( \delta_a = 0 \) for \( \underline{a} = 1, 2, 3, \) and \( \delta_4 = 1 \).

**Proof.** For \( S_\underline{a}, Z_\underline{a} \) with \( \underline{a} = 1, 2, 3 \) it is proved as in Lemma 2.3. Using that \( |q|^2 D_\alpha (|q|^{-2} O^\alpha \beta D_\beta) = O \), we obtain the expression for \( S_4 \) and \( W_4 \) from their definition. \[ \square \]

As in the case of \( O \) in (139), in the case of tensors, the commutation between the operators \( S_\underline{a} \) and \( \hat{D}_k \) gives rise to terms involving the curvature, which are all \( O(a) \) lower order terms, more precisely, see (37).

\[
[S_1, \hat{D}_2] \psi = O(\alpha r^{-4}) (\nabla_T \nabla_1 \psi) + O(\alpha r^{-4}) (\nabla_r \nabla_T \psi) + O(\alpha r^{-5}) \delta \leq 1 \psi
\]
\[
[S_2, \hat{D}_2] \psi = O(\alpha r^{-4}) (\nabla_T \nabla_1 \psi) + O(\alpha r^{-4}) (\nabla_r \nabla_T \psi)
+ O(\alpha r^{-4}) (\nabla_Z \nabla_1 \psi) + O(\alpha r^{-4}) (\nabla_Z \nabla_r \psi) + O(\alpha r^{-3}) \delta \leq 1 \psi.
\]
\[
[S_3, \hat{D}_2] \psi = O(\alpha r^{-4}) (\nabla_Z \nabla_1 \psi) + O(\alpha r^{-4}) (\nabla_Z \nabla_r \psi) + O(\alpha r^{-4}) (\nabla_Z \nabla_r \psi) + O(\alpha r^{-3}) \delta \leq 1 \psi.
\]

As in (70), we define
\[
p_\underline{a} := S_\underline{a}(p), \quad q_\underline{a} := W_\underline{a}(q^F), \quad \underline{a} = 1, 2, 3, 4. \tag{157}
\]

Then, from Proposition 3.3 and the above commutators, the couple of tensors \((p_\underline{a}, q_\underline{a})\) for \( \underline{a} = 1, 2, 3, 4 \) satisfies the gRW system, with possibly corrections in \( O(a) \) lower order terms from the above commutators.

### 3.4 The positivity of the principal terms in the Morawetz estimates for the commuting system

We now describe how to apply the Andersson-Blue method to obtain the Morawetz estimates for the gRW system of equations for tensors. For its application to the gRW equation in Kerr see [37].

From the combined energy-momentum tensor for the system given in Section 5.2 i.e.
\[
Q[p, q^F]_{\mu \nu} := Q[p]_{\mu \nu} + 8 Q^2 Q[q^F]_{\mu \nu},
\]
we define, as for the scalar wave equation in Section 2.3 the generalized energy-momentum tensor for the gRW system:
\[
Q_{\mu \nu}[p, q^F] := \frac{1}{4}(Q_{\mu \nu}[p_\underline{a} + p_\underline{a}, q_\underline{a} + q_\underline{a}] - Q_{\mu \nu}[p_\underline{a} - p_\underline{a}, q_\underline{a} - q_\underline{a}]) \tag{158}
\]
and the generalized current:
\[
P_{\mu}^{(X, w)}[p, q^F] := Q_{\mu \nu}[p, q^F] X_{\mu \nu} + \frac{1}{2} \nu_{\mu \nu}(p_\underline{a}, D_\mu p_\underline{a} + 8 q_\underline{a}, D_\mu q_\underline{a}) - \frac{1}{4}(D_{\mu} \nu_{\mu \nu})(p_\underline{a}, p_\underline{a} + 8 q_\underline{a}, q_\underline{a}) \]
\[
\quad = P_{\mu}^{(X, w)}[p] + 8 Q^2 P_{\mu}^{(X, w)}[q^F].
\]
Its divergence \([138]\) is given by, see \([138]\),
\[
\mathcal{G}^{(X, w)}[p, q^F] = D_{\mu} P_{\mu}^{(X, w)}[p] + 8 Q^2 D_{\mu} P_{\mu}^{(X, w)}[q^F]
\]
\[
= \mathcal{D}^{(X, w)}[p] + 8 Q^2 \mathcal{D}^{(X, w)}[q^F]
\]

\(^{[1]}\)Observe the difference in the presence of \( a \) and \( a^2 \) in the definition of \( S_2 \) and \( S_3 \) respectively. This is important in the case of tensors in order to obtain \( O(a) \) lower order terms in the curvature.
\[-\frac{2a\cos\theta}{|q|^2} \left( (X^{ab}(p_a) + \frac{1}{2} u^{ab}_\alpha) \cdot \nabla_T p_a + 16Q^2 (X^{ab}(q_a) + \frac{1}{2} u^{ab}_\alpha) \cdot \nabla_T q_a \right) + \text{i.o.t.} + \text{Bdr} \]

where

\[ E^{(X,w)}[p] \] = \[ \frac{1}{2} Q[p]_{ab} \cdot D_{(\mu} X^{ab}_{\nu)} - \frac{1}{2} X^{ab}(V_i) p_a \cdot p_b - \frac{1}{4} \Box^2 u^{ab} q_a \cdot q_b + \frac{1}{2} u^{ab} L_1[p_a, p_b] \]

\[ E^{(X,w)}[q^F] \] = \[ \frac{1}{2} Q[q^F]_{ab} \cdot D_{(\mu} X^{ab}_{\nu)} - \frac{1}{2} X^{ab}(V_i) q_a \cdot q_b - \frac{1}{4} \Box^2 u^{ab} q_a \cdot q_b + \frac{1}{2} u^{ab} L_1[q_a, q_b] \]

with

\[ L_1[p_a, p_b] = g^{\alpha\beta} \tilde{\nabla}_\alpha p_a \tilde{\nabla}_\beta p_b + V_i p_a \cdot p_b \]

\[ L_2[q_a, q_b] = g^{\alpha\beta} \tilde{\nabla}_\alpha q_a \tilde{\nabla}_\beta q_b + V_2 q_a \cdot q_b \]

Through a similar derivation, we obtain the corresponding of Proposition 2.7.

**Proposition 3.6.** Let \( z \) be a given function of \( r \). Let \( \tilde{\phi}^{ab} \) be a given double-indexed function of \( r \). Then for

\[ X^{ab} = \mathcal{F}^{ab} \partial_r, \quad \mathcal{F}^{ab} = z \tilde{\phi}^{ab}, \quad \tilde{\phi}^{ab} = z \partial_r \tilde{\phi}^{ab}, \]

we obtain

\[ E^{(X,w)}[p] = \mathcal{A}^{ab} \nabla_r p_a \nabla_r p_b + U^{ab}_{\alpha\beta} \tilde{\nabla}_\alpha p_a \tilde{\nabla}_\beta p_b + V^{ab}_1 p_a \cdot p_b, \]

\[ E^{(X,w)}[q^F] = \mathcal{A}^{ab} \nabla_r q_a \nabla_r q_b + U^{ab}_{\alpha\beta} \tilde{\nabla}_\alpha q_a \tilde{\nabla}_\beta q_b + V^{ab}_2 q_a \cdot q_b \]

where

\[ \mathcal{A}^{ab} = z^{1/2} \Delta^{3/2} \partial_r \left( \frac{z^{1/2} \tilde{\phi}^{ab}}{\Delta^{1/2}} \right) \]

\[ U^{ab}_{\alpha\beta} = -\frac{1}{2} u^{ab}_\alpha \partial_r \left( \frac{z \tilde{R}_{\alpha\beta}}{\Delta^{1/2}} \right) \]

\[ V^{ab}_1 = -\frac{1}{4} \partial_r \left( \Delta \partial_r z \tilde{\phi}^{ab} \right) - \frac{1}{2} \left( X^{ab}(\partial_r V_1) + |q|^2 u^{ab} \mathcal{S}^{ab}_{\text{red}} \right) \]

\[ V^{ab}_2 = -\frac{1}{4} \partial_r \left( \Delta \partial_r \left( z \partial_r \tilde{\phi}^{ab} \right) \right) - \frac{1}{2} \left( X^{ab}(\partial_r V_1) + |q|^2 u^{ab} \mathcal{S}^{ab}_{\text{red}} \right) \]

where \( u^{ab}_{\text{red}} = u^{ab}_\alpha \partial_r z \).

Recall from Section 2.8 that the crucial step in deriving Morawetz estimates for the commuted system was to perform an integration by parts in the principal term \( U^{ab}_{\alpha\beta} \partial_r \psi_{\alpha} \partial_r \psi_{\beta} \), which allows to create a positive definite term for a trapped combination of \( \psi_{\alpha} \), denoted \( \Psi \). We now show how this property can be extended to the gRW system and its combined energy-momentum tensor.

We write the principal terms above as

\[ U^{ab}_{\alpha\beta} \tilde{\nabla}_\alpha p_a \tilde{\nabla}_\beta p_b = -\frac{1}{2} u^{ab}_\alpha \tilde{R}_{\alpha}^{\alpha\beta} \tilde{\nabla}_\alpha p_a \tilde{\nabla}_\beta p_b = -\frac{1}{2} u^{ab}_\alpha \tilde{R}_{\alpha}^{\alpha\beta} S_{\alpha\beta}^{ab} \tilde{\nabla}_\alpha p_a \tilde{\nabla}_\beta p_b \]

\[ U^{ab}_{\alpha\beta} \tilde{\nabla}_\alpha q_a \tilde{\nabla}_\beta q_b = -\frac{1}{2} u^{ab}_\alpha \tilde{R}_{\alpha}^{\alpha\beta} \tilde{\nabla}_\alpha q_a \tilde{\nabla}_\beta q_b = -\frac{1}{2} u^{ab}_\alpha \tilde{R}_{\alpha}^{\alpha\beta} S_{\alpha\beta}^{ab} \tilde{\nabla}_\alpha q_a \tilde{\nabla}_\beta q_b \]

By performing the integration by parts in \( \tilde{\nabla}_\alpha \), we obtain

\[ |q|^{-2} U^{ab}_{\alpha\beta} \tilde{\nabla}_\alpha p_a \tilde{\nabla}_\beta p_b = -\frac{1}{2} \tilde{\nabla}_\alpha (|q|^{-2} u^{ab}_\alpha \tilde{R}_{\alpha}^{\alpha\beta} S_{\alpha\beta}^{ab} \tilde{\nabla}_\beta p_b) + \frac{1}{2} u^{ab}_\alpha \tilde{R}_{\alpha}^{\alpha\beta} \tilde{\nabla}_\alpha (|q|^{-2} S_{\alpha\beta}^{ab} \tilde{\nabla}_\beta p_b) \]

Now consider \( \tilde{\nabla}_\alpha (|q|^{-2} S_{\alpha\beta}^{ab} \tilde{\nabla}_\beta p_b) \). Recall Lemma 3.5 i.e.,

\[ S_{\alpha\beta} = |q|^2 \tilde{\nabla}_\alpha (|q|^{-2} S_{\alpha\beta}^{ab} \tilde{\nabla}_\beta p_b) + \delta_{\alpha\beta}(c + 3)|q|^2 \]

Therefore we can write

\[ \tilde{\nabla}_\alpha (|q|^{-2} S_{\alpha\beta}^{ab} \tilde{\nabla}_\beta p_b) = \tilde{\nabla}_\alpha (|q|^{-2} S_{\alpha\beta}^{ab} \tilde{\nabla}_\beta p_b) \]

\[ = |q|^{-2} S_{\alpha\beta}^{ab} \tilde{\nabla}_\beta p_b - |q|^{-2} (\delta_{\alpha\beta}(c + 3)|q|^2 K \tilde{S}_{\alpha\beta}) \]

\[ = |q|^{-2} S_{\alpha\beta}^{ab} \tilde{\nabla}_\beta p_b + |q|^{-2} (|q|^2 \tilde{S}_{\alpha\beta}^{ab} \tilde{\nabla}_\beta p_b - |q|^{-2} (\delta_{\alpha\beta}(c + 3)|q|^2 K \tilde{S}_{\alpha\beta}) \]

\[ = |q|^{-2} S_{\alpha\beta}^{ab} \tilde{\nabla}_\beta p_b + |q|^{-2} (|q|^2 \tilde{S}_{\alpha\beta}^{ab} \tilde{\nabla}_\beta p_b - |q|^{-2} (\delta_{\alpha\beta}(c + 3)|q|^2 K \tilde{S}_{\alpha\beta}) \]

\[ + |q|^{-2} (\delta_{\alpha\beta}(c + 3)|q|^2 K \tilde{S}_{\alpha\beta}) - |q|^{-2} (\delta_{\alpha\beta}(c + 3)|q|^2 K \tilde{S}_{\alpha\beta}) \]
Thus, repeating the integration by parts procedure, and for \( u^b = -\hbar \tilde{R}^{\alpha \beta} L_{\alpha}^\beta \) and recalling that \( L^\beta_{\Sigma} S^\alpha_{\Sigma} = L^\alpha_{\Sigma} \), we obtain

\[
U^{\alpha \beta \delta \kappa} D_{\alpha} p_{\beta} D_{\beta} p_{\delta} = \frac{1}{2} h L^{\alpha \beta} D_{\alpha} (\tilde{R}^{\alpha \beta} p_{\delta}) D_{\beta} (\tilde{R}^{\alpha \beta} p_{\delta}) - \frac{1}{2} h \tilde{R}^{\alpha \beta} L^\beta_{\Sigma} (\tilde{R}^{\alpha \beta} p_{\delta}) \cdot [S^\alpha_{\Sigma}, S^\beta_{\Sigma}] p
\]

\[
+ |q|^2 D_{\alpha} \left( |q|^2 \frac{1}{2} h \tilde{R}^{\alpha \beta} L^\beta_{\Sigma} (\tilde{R}^{\alpha \beta} p_{\delta}) \left( S^\alpha_{\Sigma} \beta D_{\beta} p_{\delta} - S^\beta_{\Sigma} \alpha D_{\alpha} p_{\delta} \right) \right)
\]

\[
- \frac{c + 3}{2} h |q|^2 (h K (\tilde{R}^{\alpha \beta} p_{\delta}) L^\delta_{\Sigma} (\delta_{\alpha \Sigma} p_{\delta} - \delta_{\beta \Sigma} p_{\delta}),
\]

and similarly for \( q^F \). By denoting \( \Phi := \tilde{R}^{\alpha \beta} p_{\delta} \) and \( \Psi := \tilde{R}^{\alpha \beta} q_{\delta} \), we respectively obtain

\[
U^{\alpha \beta \delta \kappa} D_{\alpha} p_{\beta} D_{\beta} q_{\delta} = \frac{1}{2} h L^{\alpha \beta} D_{\alpha} \Phi D_{\beta} \Phi - \frac{1}{2} h \Phi \cdot \tilde{R}^{\alpha \beta} L^\beta_{\Sigma} (S^\alpha_{\Sigma}, S^\beta_{\Sigma}) p - |q|^2 D_{\alpha} B^\alpha_{\Sigma} p
\]

\[
- \frac{c + 3}{2} h |q|^2 (h K (\tilde{R}^{\alpha \beta} p_{\delta}) L^\delta_{\Sigma} (\delta_{\alpha \Sigma} p_{\delta} - \delta_{\beta \Sigma} p_{\delta}),
\]

\[
U^{\alpha \beta \delta \kappa} D_{\alpha} q_{\beta} D_{\beta} q_{\delta} = \frac{1}{2} h L^{\alpha \beta} D_{\alpha} \Psi D_{\beta} \Psi - \frac{1}{2} h \Psi \cdot \tilde{R}^{\alpha \beta} L^\beta_{\Sigma} (W^\alpha_{\Sigma}, W^\beta_{\Sigma}) q^F - |q|^2 D_{\alpha} B^\alpha_{\Sigma} q^F
\]

\[
- \frac{c + 3}{2} h |q|^2 (h K (\tilde{R}^{\alpha \beta} p_{\delta}) L^\delta_{\Sigma} (\delta_{\alpha \Sigma} q_{\delta} - \delta_{\beta \Sigma} q_{\delta}),
\]

For the second line in each of the above expressions, we write

\[
L^\beta_{\Sigma} (\delta_{\alpha \Sigma} p_{\delta} - \delta_{\beta \Sigma} p_{\delta}) = L^\beta_{\Sigma} (\tilde{R}^{\alpha \beta} p_{\delta}) - \tilde{R}^{\alpha \beta} L^\beta_{\Sigma} (\tilde{R}^{\alpha \beta} p_{\delta}) = L^4 \Phi - \tilde{R}^{\alpha \beta} L^\beta_{\Sigma} (\tilde{R}^{\alpha \beta} p_{\delta})
\]

\[
L^\beta_{\Sigma} (\delta_{\alpha \Sigma} q_{\delta} - \delta_{\beta \Sigma} q_{\delta}) = L^\beta_{\Sigma} (\tilde{R}^{\alpha \beta} q_{\delta}) - \tilde{R}^{\alpha \beta} L^\beta_{\Sigma} (\tilde{R}^{\alpha \beta} q_{\delta}) = L^4 \Psi - \tilde{R}^{\alpha \beta} L^\beta_{\Sigma} (\tilde{R}^{\alpha \beta} q_{\delta}).
\]

Now observe that with the choice \( z = z_1 \) as in (127), we have, see (129), \( \tilde{R}^{\alpha \beta}[z_1] = -\frac{2T}{(r^2+a^2)}(1 + O(\epsilon r^{-2})) \), and with the choice of \( \tilde{L}_{\Sigma} \) as in (122), we write, using (121):

\[
L^\beta_{\Sigma} (\delta_{\alpha \Sigma} p_{\delta} - \delta_{\beta \Sigma} p_{\delta}) = \left( \Phi - \frac{2T}{(r^2+a^2)} \right) (1 + O(\epsilon r^{-2}))(\epsilon S^\alpha_{\Sigma} \psi + O(\epsilon))
\]

\[
= \Phi - \left( \Phi - \frac{4ar}{(r^2+a^2)} \right) (1 + O(\epsilon r^{-2})) \nabla \tilde{T} \nabla Z p
\]

\[
L^\beta_{\Sigma} (\delta_{\alpha \Sigma} q_{\delta} - \delta_{\beta \Sigma} q_{\delta}) = \frac{4ar}{(r^2+a^2)} (1 + O(\epsilon r^{-2})) \nabla \tilde{T} \nabla Z q^F,
\]

Also, using that, see (87),

\[
[S^\alpha_{\Sigma}, S^\beta_{\Sigma}] \psi = [S^\alpha_{\Sigma}, S^\beta_{\Sigma}] \psi = [S^\alpha_{\Sigma}, S^\beta_{\Sigma}] \psi = 0
\]

and

\[
[S^\alpha_{\Sigma}, S^\beta_{\Sigma}] \psi = O(ma^2) \nabla Z \nabla T \psi + O(\epsilon a^2) \psi \]

\[
[S^\alpha_{\Sigma}, S^\beta_{\Sigma}] \psi = O(ma) \nabla Z \nabla T \psi + O(\epsilon a^2) \psi \]

\[
[S^\alpha_{\Sigma}, S^\beta_{\Sigma}] \psi = O(ma^2) \nabla Z \nabla T \psi + O(\epsilon a^2) \psi \]

and similarly for \( W_{\Sigma} \), we can write for the commutator terms:

\[
-\frac{1}{2} h \Phi \cdot \tilde{R}^{\alpha \beta} L^\beta_{\Sigma} [S^\alpha_{\Sigma}, S^\beta_{\Sigma}] p = \frac{1}{2} h \Phi \left( \tilde{R}^{\alpha \beta} L^\beta_{\Sigma} [S^\alpha_{\Sigma}, S^\beta_{\Sigma}] p + \tilde{R}^{\alpha \beta} L^\beta_{\Sigma} [S^\alpha_{\Sigma}, S^\beta_{\Sigma}] p + (\tilde{R}^{\alpha \beta} L^\beta_{\Sigma} - \tilde{R}^{\alpha \beta} L^\beta_{\Sigma}) [S^\alpha_{\Sigma}, S^\beta_{\Sigma}] p \right)
\]

\[
= \frac{1}{2} h \Phi \cdot \left( O(\epsilon r^{-3}) \nabla Z \nabla T p + O(\epsilon r^{-3}) \nabla Z \nabla T p + O(\epsilon r^{-3}) \nabla Z \nabla T p \right),
\]

\[
-\frac{1}{2} h \Psi \cdot \tilde{R}^{\alpha \beta} L^\beta_{\Sigma} [S^\alpha_{\Sigma}, S^\beta_{\Sigma}] q^F = \frac{1}{2} h \Psi \cdot \left( O(\epsilon r^{-3}) \nabla Z \nabla T q^F + O(\epsilon r^{-3}) \nabla Z \nabla T q^F + O(\epsilon r^{-3}) \nabla Z \nabla T q^F + O(\epsilon r^{-3}) \nabla Z \nabla T q^F \right).
\]

We conclude from (163) and (165), that the principal terms can be written as

\[
U^{\alpha \beta \delta \kappa} D_{\alpha} p_{\beta} D_{\beta} p_{\delta} = \frac{1}{2} h L^{\alpha \beta} D_{\alpha} \Phi D_{\beta} \Phi - |q|^2 D_{\alpha} B^\alpha_{\Sigma} p
\]

\[
- \frac{1}{2} h \Phi \cdot \left( O(\epsilon r^{-3}) \nabla Z \nabla T p + O(\epsilon r^{-3}) \nabla Z \nabla T p + O(\epsilon r^{-3}) \nabla Z \nabla T p + O(\epsilon r^{-3}) \nabla Z \nabla T p \right),
\]

\[
U^{\alpha \beta \delta \kappa} D_{\alpha} q_{\beta} D_{\beta} q_{\delta} = \frac{1}{2} h L^{\alpha \beta} D_{\alpha} \Psi D_{\beta} \Psi - |q|^2 D_{\alpha} B^\alpha_{\Sigma} q^F
\]
Lemma A.1. of the coupled system in Reissner-Nordström \([34]\), where the quadratic form is shown to be positive for momentum perturbations of Kerr-Newman.

**A Proof of Proposition 1.3**

The full derivation of the Morawetz-Energy estimates for the gR W system in Kerr-Newman will follow the Morawetz-Energy estimates for the gR W equation in Kerr in [37] and will appear in a future work.

**Proof.** First of all, observe that as a consequence of the definition (33), contracting \(D\mu K_{\nu\rho} + D\nu K_{\mu\rho} + D\rho K_{\mu\nu} = 0\) with \(g^{\mu\nu}\), one immediately derives
\[
D\mu K^{\nu\rho} = -\frac{1}{2}D^\nu (\text{tr} K),
\]
where \(\text{tr} K = g^{\mu\nu} K_{\mu\nu}\) is the trace of \(K\). By applying another derivative to the above and antisymmetrizing, one also obtains
\[
D^\alpha D_\alpha K^{\nu\rho} - D^\nu D_\mu K^{\mu\rho} = \frac{1}{2}D^\nu D^\rho (\text{tr} K) + \frac{1}{2}D^\rho D^\nu (\text{tr} K) = 0.
\]

To obtain the commutator \([K, \Box_g]\), we first collect the following preliminary computations.

**Lemma A.1.** The following commutation formulas hold for a scalar \(\phi\), a 1-tensor \(X\) and a 2-tensor \(\Psi\) in \(M\):
\[
[D_\mu, \Box_g] \phi = -R_{\rho\alpha} D^\rho \phi,
\]
\[
[D_\mu, \Box_g] X^\nu = R_{\rho\mu\alpha} D^\rho X^\alpha + (D_\rho R - D^\rho R_{\rho\mu\alpha}) X^\alpha,
\]
\[
[D_\mu, \Box_g] \Psi_{\nu\delta} = 2R_{\rho\mu\alpha\beta} D^\rho \Psi_{\nu\beta} + R_{\rho\alpha\beta} D^\rho \Psi_{\nu\delta} + D^\rho R_{\alpha\beta\nu} \Psi_{\delta\gamma} + (D_\rho R_{\mu\beta} - D_\nu R_{\mu\beta}) \Psi_{\nu\delta},
\]
where \(R\) denotes the Riemann curvature, the Ricci curvature or the scalar curvature depending if it appears as a 4-tensor, a 2-tensor or a scalar respectively.

**Proof.** For a scalar \(\phi\), we have by definition of Riemann curvature
\[
[D_\mu, D_\beta] \phi = 0, \quad [D_\alpha, D_\beta] D_\gamma \phi = R_{\alpha\beta\gamma} \delta D_\delta \phi.
\]
We therefore compute
\[
[D_\nu, \Box_g] \phi = [D_\nu, D^\alpha D_\alpha] \phi = [D_\nu, D^\alpha] D_\alpha \phi + D^\alpha [D_\nu, D_\alpha] \phi = g^{\alpha\mu} R_{\beta\mu\alpha} \delta D_\delta \phi = -R_{\nu\delta} \delta D_\delta \phi.
\]
For a 1-tensor \(X\) we have
\[
[D_\alpha, D_\beta] X_\gamma = R_{\alpha\beta\gamma} X_\gamma, \quad [D_\alpha, D_\beta] D_\gamma X_\alpha = R_{\alpha\beta\gamma} D_\delta X_\gamma + R_{\alpha\beta\gamma} D_\delta X_\gamma.
\]
We then compute
\[
[D_\mu, \Box_g] X_\beta = [D_\mu, D^\alpha D_\alpha] X_\beta = [D_\mu, D^\alpha] D_\alpha X_\beta + D^\alpha [D_\mu, D_\alpha] X_\beta = g^{\alpha\mu} (R_{\rho\mu\beta\alpha} D_\rho X_\beta + R_{\rho\mu\alpha\beta} D_\rho X_\alpha) + R_{\rho\alpha\beta} [D_\rho, X_\alpha] = -R_{\alpha\beta} D_{\delta} X_\delta + R_{\alpha\beta} [D_\delta, X_\alpha] + D^\alpha R_{\rho\alpha\beta\delta} X_\delta + R_{\rho\alpha\beta} D^\rho X_\delta.
\]
Using the second Bianchi identity \(D^\alpha R_{\mu\alpha\nu\rho} = D_\alpha R_{\mu\gamma\nu} - D_\gamma R_{\nu\mu\alpha}, \) we obtain
\[
[D_\mu, \Box_g] X_\beta = 2R_{\rho\mu\alpha\beta} D^\rho X_\gamma - R_{\rho\alpha\beta} [D_\rho, X_\gamma] + (D^\rho R_{\mu\alpha\beta} - D_\gamma R_{\nu\mu\alpha}) X_\gamma.
\]
By contracting with \(g^{\mu\beta}\) we finally obtain
\[
[D_\mu, \Box_g] X_\alpha = 2R_{\rho\alpha\beta} D^\rho X_\gamma - R_{\rho\alpha\beta} [D_\rho, X_\gamma] + (D^\rho R_{\mu\alpha\beta} - D_\gamma R_{\nu\mu\alpha}) X_\gamma.
\]
For a 2-tensor $\Psi$ we have
\[
[D_\nu, D_\delta]\Psi_{\gamma\delta} = R_{\nu \alpha \gamma \delta} \Psi_{\alpha \delta} + R_{\nu \alpha \delta \gamma} \Psi_{\alpha \delta}.
\]
\[
[D_\nu, D_\xi]D_\gamma \Psi_{\gamma\delta} = R_{\nu \alpha \delta \xi} D_\alpha \Psi_{\gamma\delta} + R_{\xi \gamma \delta} + R_{\nu \gamma \delta} D_\alpha \Psi_{\gamma\xi}.
\]

We then compute
\[
[D_\nu, \Box_g] \Psi_{\gamma\delta} = [D_\nu, D^\alpha D_\alpha] \Psi_{\gamma\delta} = [D_\nu, D^\alpha] D_\alpha \Psi_{\gamma\delta} + D^\alpha [D_\nu, D_\alpha] \Psi_{\gamma\delta} = g^{\gamma\xi} (R_{\nu \alpha \delta \xi} D_\alpha \Psi_{\gamma\delta} + R_{\nu \gamma \delta} D_\alpha \Psi_{\gamma\delta} + D^\alpha (R_{\nu \alpha \gamma \delta} \Psi_{\alpha \delta} + R_{\nu \alpha \delta \gamma} \Psi_{\alpha \delta} + D^\alpha (R_{\nu \alpha \gamma \delta} \Psi_{\alpha \delta} + R_{\nu \alpha \delta \gamma} \Psi_{\alpha \delta} + D^\alpha (R_{\nu \alpha \gamma \delta} \Psi_{\alpha \delta} + R_{\nu \alpha \delta \gamma} D_\alpha \Psi_{\gamma\xi}).
\]
\[
= -R_{\nu \gamma} D_\delta \Psi_{\gamma\delta} + R_{\nu \gamma \delta} D_\delta \Psi_{\gamma\delta} + R_{\nu \gamma \delta} D_\alpha \Psi_{\gamma\delta} + D^\alpha (R_{\nu \gamma \delta} D_\delta \Psi_{\gamma\delta} + R_{\nu \gamma \delta} D\alpha \Psi_{\gamma\delta} + R_{\nu \gamma \delta} D^\alpha \Psi_{\gamma\xi} + D^\alpha (R_{\nu \gamma \delta} D\alpha \Psi_{\gamma\delta} + R_{\nu \gamma \delta} D^\alpha \Psi_{\gamma\xi} = 2 R_{\nu \gamma \delta} D_\delta \Psi_{\gamma\delta} + 2 R_{\nu \gamma \delta} D_\alpha \Psi_{\gamma\delta} + 2 R_{\nu \gamma \delta} D_\alpha \Psi_{\gamma\delta} + 2 R_{\nu \gamma \delta} D_\alpha \Psi_{\gamma\delta} + D^\alpha (R_{\nu \gamma \delta} D_\delta \Psi_{\gamma\delta} + R_{\nu \gamma \delta} D\alpha \Psi_{\gamma\delta} + R_{\nu \gamma \delta} D^\alpha \Psi_{\gamma\xi} + D^\alpha (R_{\nu \gamma \delta} D\alpha \Psi_{\gamma\delta} + R_{\nu \gamma \delta} D^\alpha \Psi_{\gamma\xi}.
\]

By contracting with $g^{\gamma\delta}$ we finally obtain
\[
[D_\nu, \Box_g] \Psi_{\gamma\delta} = 2 R_{\nu \gamma \delta} D_\delta \Psi_{\gamma\delta} + 2 R_{\nu \gamma \delta} D_\alpha \Psi_{\gamma\delta} - R_{\nu \gamma} D_\delta \Psi_{\gamma\delta} + D^\alpha (R_{\nu \gamma \delta} D_\delta \Psi_{\gamma\delta} + R_{\nu \gamma \delta} D\alpha \Psi_{\gamma\delta} + R_{\nu \gamma \delta} D^\alpha \Psi_{\gamma\xi} + D^\alpha (R_{\nu \gamma \delta} D\alpha \Psi_{\gamma\delta} + R_{\nu \gamma \delta} D^\alpha \Psi_{\gamma\xi}.
\]

Again using the Bianchi identity to write $D^\alpha (R_{\nu \gamma \delta} D_\delta \Psi_{\gamma\delta} + R_{\nu \gamma \delta} D\alpha \Psi_{\gamma\delta} + R_{\nu \gamma \delta} D^\alpha \Psi_{\gamma\xi} + D^\alpha (R_{\nu \gamma \delta} D\alpha \Psi_{\gamma\delta} + R_{\nu \gamma \delta} D^\alpha \Psi_{\gamma\xi}$, we prove the Lemma. \qed

We use the above Lemma to compute the commutator between the Killing differential operator $K$ as defined by (8) and the D'Alambertian operator $\Box_g$.

We compute for a scalar function $\phi$:
\[
[K, \Box_g] \phi = [D_\mu K^{\mu\nu} D_\nu, \Box_g] \phi = D_\mu [K^{\mu\nu} D_\nu, \Box_g] \phi = D_\mu (K^{\mu\nu} D_\nu, \Box_g) = [D_\mu, \Box_g] K^{\mu\nu} D_\nu \phi.
\]

Writing $\Box_g (K^{\mu\nu} D_\nu \phi) = (\Box_g K^{\nu\mu}) D_\nu \phi + 2 D^\nu K^{\nu\mu} D_\delta D_\nu \phi + K^{\mu\nu} \Box_g D_\nu \phi$, we have
\[
[K, \Box_g] \phi = D_\mu (K^{\mu\nu} [D_\nu, \Box_g] \phi - \Box_g K^{\nu\mu} D_\nu \phi - 2 D^\nu K^{\nu\mu} D_\delta D_\nu \phi) + [D_\mu, \Box_g] K^{\mu\nu} D_\nu \phi.
\]

Applying Lemma (A.1) to $K$ and to $X^\mu = K^{\mu\nu} D_\nu \phi$, we have
\[
K^{\mu\nu} [D_\nu, \Box_g] \phi = -K^{\mu\nu} R_{\nu \rho} D_\rho \phi,\]
\[
[D_\mu, \Box_g] K^{\mu\nu} D_\nu \phi = R_{\nu \rho} \Box_g (K^{\nu\rho} D_\rho \phi) + (D\rho - D^\rho R_{\nu \rho}^\gamma) K^{\nu\gamma} D_\rho \phi.
\]

We therefore obtain
\[
[K, \Box_g] \phi = D_\mu (\Box_g K^{\mu\nu} D_\nu \phi - D^\nu K^{\nu\mu} D_\rho D_\nu \phi) - D_\mu (K^{\mu\nu} R_{\nu \rho} D_\rho \phi) + R_{\nu \rho} \Box_g (K^{\nu\rho} D_\rho \phi) + (D\rho - D^\rho R_{\nu \rho}^\gamma) K^{\nu\gamma} D_\rho \phi.
\]

From (S), i.e. $D_\nu K^{\nu\mu} + D^\mu K^{\nu\alpha} + D^\nu K^{\alpha\mu} = 0$, we have
\[
(D_\alpha K^{\nu\mu}) D^\nu D_\alpha \phi = -D^\alpha K^{\nu\alpha} D^\nu D_\alpha \phi - D^\alpha K^{\nu\alpha} D^\nu D_\alpha \phi = -D^\nu K^{\nu\alpha} D^\alpha D_\alpha \phi - D_\alpha K^{\nu\alpha} D^\nu D_\alpha \phi,
\]
which gives
\[
(D_\alpha K^{\nu\mu}) D^\nu D_\alpha \phi = -\frac{1}{2} D^\nu K^{\nu\alpha} D^\alpha D_\alpha \phi.
\]

This implies
\[
[K, \Box_g] \phi = D_\mu (\Box_g K^{\mu\nu} D_\nu \phi + D^\nu K^{\nu\mu} D_\alpha D_\nu \phi) - D_\mu (K^{\mu\nu} R_{\nu \rho} D_\rho \phi) + R_{\nu \rho} \Box_g (K^{\nu\rho} D_\rho \phi) + (D\rho - D^\rho R_{\nu \rho}^\gamma) K^{\nu\gamma} D_\rho \phi.
\]

Observe that in expanding the $D_\mu$ derivative there is a cancellation of the term $(\Box_g K^{\mu\nu}) D_\alpha D_\mu \phi$:
\[
[K, \Box_g] \phi = -D_\mu (\Box_g K^{\mu\nu} D_\nu \phi - D^\nu K^{\nu\mu} D_\alpha D_\nu \phi) + D_\mu D^\nu K^{\nu\alpha} D_\alpha D_\nu \phi + D^\nu K^{\nu\alpha} D_\alpha D^\nu D_\alpha D_\nu \phi - D_\mu (K^{\mu\nu} R_{\nu \rho} D_\rho \phi) + R_{\nu \rho} \Box_g (K^{\nu\rho} D_\rho \phi) + (D\rho - D^\rho R_{\nu \rho}^\gamma) K^{\nu\gamma} D_\rho \phi \\
= -D_\mu (\Box_g K^{\mu\nu} D_\nu \phi + D^\nu K^{\nu\alpha} D_\alpha D_\nu \phi - D_\mu (K^{\mu\nu} R_{\nu \rho} D_\rho \phi) + R_{\nu \rho} \Box_g (K^{\nu\rho} D_\rho \phi)) + (D\rho - D^\rho R_{\nu \rho}^\gamma) K^{\nu\gamma} D_\rho \phi
\]

(169)

We now derive an expression for the two terms appearing in the first line above. For the second term we have, using again the Killing equation (S),
\[
(D_\alpha K^{\nu\mu}) D^\nu D_\mu D_\alpha \phi = (D^\nu K^{\nu\alpha} - D^\nu K^{\nu\alpha}) D^\nu D_\mu D_\alpha \phi = -D^\nu K^{\nu\alpha} D^\nu D_\mu D_\alpha \phi - D^\nu K^{\nu\alpha} D^\nu D_\mu D_\alpha \phi
\]
\]

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\[= -2D^\alpha K^\alpha_{\mu\nu} D^\mu D^\nu \phi = -2D^\alpha K^\alpha_{\mu\nu} (D_{\mu} D_{\nu} \phi + [D^\alpha, D_{\mu}] D_{\nu} \phi) = -2D^\alpha K^\alpha_{\mu\nu} (D_{\mu} D^\nu \phi + R^{\mu\nu}_{\rho\sigma} D_{\rho} \phi) \]

Observe that the first term on the right hand side is the same as the term on the left hand side. We therefore obtain:

\[(D_{\mu} D_{\sigma}) D_{\nu} D^\alpha \phi = -\frac{2}{3} D^\alpha K^\alpha_{\mu\nu} R^{\mu\nu}_{\rho\sigma} D_{\rho} \phi.\]

For the first term, starting with \(D_{\mu} K_{\nu\rho} + D_{\nu} K_{\rho\mu} + D_{\rho} K_{\mu\nu} = 0\) and applying \(D^\mu\) we have

\[0 = D^\mu D_{\mu} K_{\nu\rho} + D^\nu D_{\nu} K_{\rho\mu} + D^\rho D_{\rho} K_{\mu\nu} = D^\mu D_{\mu} K_{\nu\rho} + D^\nu D_{\nu} K_{\rho\mu} + R_{\nu\rho} K_{\mu\nu} + D_{\nu} D^\mu K_{\rho\mu} + R_{\mu\nu} K_{\rho\mu} + R_{\nu\rho} K_{\mu\nu}.\]

This gives

\[D^\mu D_{\alpha} K_{\mu\nu} = -D_{\nu\rho} K_{\mu\alpha} - D_{\alpha} D^\mu K_{\mu\nu} - R_{\mu\nu}^\rho K_{\alpha\rho} - R_{\nu\mu}^\rho K_{\alpha\rho} - R_{\mu\nu}^\rho K_{\alpha\rho} - R_{\nu\mu}^\rho K_{\alpha\rho}.\]

Using \eqref{168}, we can write \(D_{\mu} D^\alpha K_{\alpha\nu} + D_{\nu} D^\alpha K_{\alpha\mu} = 2D_{\mu} D^\alpha K_{\alpha\nu},\) which gives

\[D^\mu D_{\alpha} K_{\mu\nu} = -2D_{\mu} D^\alpha K_{\alpha\nu} - (R_{\mu\nu}^\rho + R_{\nu\mu}^\rho) K_{\alpha\rho} - R_{\mu\nu}^\rho K_{\alpha\rho} - R_{\nu\mu}^\rho K_{\alpha\rho}.\]

Applying \(D^\nu\) to the above we have

\[D^\nu D^\mu D_{\alpha} K_{\mu\nu} = -2D^\nu D_{\alpha} D^\nu K_{\alpha\nu} - (R_{\mu\nu}^\rho + R_{\nu\mu}^\rho) D^\nu K_{\alpha\nu} - (D^\nu R_{\mu\nu}^\rho + D^\nu R_{\nu\mu}^\rho) K_{\alpha\rho} - D^\nu (R_{\mu\nu}^\rho K_{\alpha\rho} + R_{\nu\mu}^\rho K_{\alpha\rho}).\]

which can be written as

\[D^\nu D^\mu D_{\alpha} K_{\mu\nu} + 2D^\nu D_{\mu} D^\alpha K_{\alpha\nu} = -(R_{\mu\nu}^\rho + R_{\nu\mu}^\rho) D^\nu K_{\alpha\nu} - (D^\nu R_{\mu\nu}^\rho + D^\nu R_{\nu\mu}^\rho) K_{\alpha\rho} - D^\nu (R_{\mu\nu}^\rho K_{\alpha\rho} + R_{\nu\mu}^\rho K_{\alpha\rho}).\]

The left hand side of \eqref{170} is given by

\[D^\nu D^\mu D_{\alpha} K_{\mu\nu} + 2D^\nu D_{\mu} D^\alpha K_{\alpha\nu} = 3D^\nu D_{\alpha} D^\mu K_{\mu\nu} + [D^\mu, D^\nu D_{\alpha}] K_{\mu\nu}.\]

Using Lemma \ref{1.1} to write

\[[D^\alpha, \Box_{\mu}] K_{\mu\nu} = 2R_{\mu\nu}^\alpha_{\mu\nu} D_{\alpha} K^{\alpha\nu} + R^{\alpha\nu}_{\mu\nu} D_{\alpha} K_{\mu\nu} + D^\alpha R_{\nu\alpha} K_{\mu\nu} + D^\nu R_{\alpha\mu} K^{\mu\nu}\]

we have

\[D^\nu D^\mu D_{\alpha} K_{\mu\nu} + 2D^\nu D_{\mu} D^\alpha K_{\alpha\nu} = 3D^\nu D_{\alpha} D^\mu K_{\mu\nu} + 2R_{\mu\nu}^\alpha_{\mu\nu} D_{\alpha} K^{\alpha\nu} + R^{\alpha\nu}_{\mu\nu} D_{\alpha} K_{\mu\nu} + D^\alpha R_{\nu\alpha} K_{\mu\nu} + D^\nu R_{\alpha\mu} K^{\mu\nu}.\]

Plugging in the above as the left hand side in \eqref{170}, we arrive to

\[D^\nu D_{\alpha} D^\mu K_{\mu\nu} = -\left(\frac{1}{3} R_{\mu\nu}^\rho + R_{\nu\mu}^\rho\right) D^\mu K_{\alpha\nu} + \frac{1}{3} \left[-(D^\nu R_{\mu\nu}^\rho + D^\nu R_{\nu\mu}^\rho) K_{\alpha\nu} - D^\mu (R_{\nu\mu}^\rho K_{\alpha\nu} + R_{\mu\nu}^\rho K_{\alpha\nu})\right].\]

Writing again from \eqref{171},

\[D^\nu D^\mu D_{\alpha} K_{\mu\nu} = D^\nu D_{\alpha} D^\mu K_{\mu\nu} + 2R_{\mu\nu}^\alpha_{\mu\nu} D_{\alpha} K^{\alpha\nu} + R^{\alpha\nu}_{\mu\nu} D_{\alpha} K_{\mu\nu} + D^\alpha R_{\nu\alpha} K_{\mu\nu} + D^\nu R_{\alpha\mu} K^{\mu\nu},\]

we obtain

\[D^\nu \Box_{\mu} K_{\mu\nu} = -\left(\frac{1}{3} R_{\mu\nu}^\rho - R_{\nu\mu}^\rho\right) D^\mu K_{\alpha\nu} + \frac{1}{3} \left[-(D^\nu R_{\mu\nu}^\rho + D^\nu R_{\nu\mu}^\rho) K_{\alpha\nu} - D^\mu (R_{\nu\mu}^\rho K_{\alpha\nu} + R_{\mu\nu}^\rho K_{\alpha\nu})\right].\]

Plugging the above in \eqref{169}, we obtain

\[\left[K, \Box_{\mu}\right] D_{\phi} = D^\mu K_{\alpha\nu} \left(\frac{1}{3} R_{\mu\rho}^\delta - R_{\mu\rho}^{\delta\nu}\right) D_{\delta} \phi - \frac{2}{3} D^\mu K_{\alpha\nu} R_{\mu\rho}^{\delta\nu} D_{\delta} \phi - \frac{1}{3} \left[-(D^\nu R_{\mu\nu}^\rho + D^\nu R_{\nu\mu}^\rho) K_{\alpha\nu} - D^\mu (R_{\nu\mu}^\rho K_{\alpha\nu} + R_{\mu\nu}^\rho K_{\alpha\nu}) + 2R_{\nu\alpha} D_{\mu} K_{\nu\alpha} + 2D^\nu R_{\alpha\mu} K^{\mu\nu} + 2D^\nu R_{\alpha\mu} K^{\mu\nu}\right] D_{\phi} - D_{\mu} (R_{\mu\nu}^\rho K_{\nu\alpha} D_{\phi} + R_{\mu\nu}^\rho D^\nu (K_{\nu\alpha} D^\nu \phi) + (D^\nu - D^\nu R_{\mu\nu}^\rho) K_{\nu\alpha} D^\nu \phi).\]
We now simplify the remaining three lines on the RHS of (172). We have

\[ R^{\mu}K_{\alpha} \left( \frac{1}{3} R^{\alpha \beta}_{\mu} - R^{\alpha}_{\mu} \delta_{\beta} \right) D_{\beta} \phi = \frac{1}{3} R^{\mu}K_{\alpha} \left( R^{\alpha \beta}_{\mu} - R^{\alpha}_{\mu} \delta_{\beta} \right) D_{\beta} \phi = \frac{1}{3} (D_\alpha K_{\beta} - D_{\beta} K_{\alpha} + 2R_{\alpha}D_{\beta}K_{\chi}) + 2D^{\alpha}R_{\chi}D_{\beta}K_{\chi} \]

where we wrote once again \( D_\alpha K_{\beta} - D_{\beta} K_{\alpha} = -D_{\alpha} K_{\beta} - D_{\beta} K_{\alpha} \), and observe that the first term is symmetric in \( \alpha \mu \) while the second Riemann tensor term in antisymmetric in \( \alpha \beta \), and the second term in symmetric in \( \mu \beta \) while the first Riemann tensor is antisymmetric in \( \mu \beta \). Writing \( D^{\nu}K_{\alpha}R_{\alpha \mu \nu} = D^{\nu}K_{\alpha}R_{\mu \alpha \nu} = D^{\nu}K_{\alpha}R_{\mu \alpha \nu} = -D^{\alpha}K_{\nu}R_{\mu \alpha \nu} = 0 \) we obtain the cancellation of the first line.

We now simplify the remaining three lines on the RHS of (172). We have

\[ R^{\mu}K_{\alpha} \left( \frac{1}{3} R^{\alpha \beta}_{\mu} - R^{\alpha}_{\mu} \delta_{\beta} \right) D_{\beta} \phi = \frac{1}{3} R^{\mu}K_{\alpha} \left( R^{\alpha \beta}_{\mu} - R^{\alpha}_{\mu} \delta_{\beta} \right) D_{\beta} \phi = \frac{1}{3} (D_\alpha K_{\beta} - D_{\beta} K_{\alpha} + 2R_{\alpha}D_{\beta}K_{\chi}) + 2D^{\alpha}R_{\chi}D_{\beta}K_{\chi} \]

This gives

\[ R^{\mu}K_{\alpha} \left( \frac{1}{3} R^{\alpha \beta}_{\mu} - D^{\mu}R_{\alpha \nu} - 2D^{\nu}R_{\alpha} \right) K_{\alpha} + 2D^{\mu}R^{\nu}R_{\alpha} - 2R_{\alpha}D^{\nu}D_{\beta}K_{\chi} \]

By writing \( D^{\nu}R^{\mu}_{\alpha \nu} = -D_{\alpha}R^{\mu} + 2D^{\nu}R^{\alpha}_{\nu} \) and \( D^{\nu}R^{\alpha}_{\nu \mu} = D^{\nu}R^{\mu} - D_{\nu}R^{\alpha} \) we have

\[ R^{\mu}K_{\alpha} \left( \frac{1}{3} R^{\alpha \beta}_{\mu} - D^{\mu}R_{\alpha \nu} - 2D^{\nu}R_{\alpha} \right) K_{\alpha} + 2D^{\mu}R^{\nu}R_{\alpha} - 2R_{\alpha}D^{\nu}D_{\beta}K_{\chi} \]

as stated.

References

[1] Alinhac S., Energy multipliers for perturbations of the Schwarzschild metric. Commun. Math. Phys. 288, 199-224 (2009)
[2] Andersson L., Bäckdahl T., Blue P., Ma S. Stability for linearized gravity on the Kerr spacetime. arXiv preprint arXiv:1903.03859 (2019)
[3] Andersson L., Blue P. Hidden symmetries and decay for the wave equation on the Kerr spacetime. Ann. of Math. (2) 182, no.3, 787-853 (2015)
[4] Andersson L., Blue P. Uniform energy bound and asymptotics for the Maxwell field on a slowly rotating Kerr black hole exterior. J. Hyperbolic Differ. Equ. 12(4), 689–743 (2015)
[5] Angelopoulos Y., Aretakis S., Gajic D. A vector field approach to almost-sharp decay for the wave equation on spherically symmetric, stationary spacetimes. Annals of PDE, 4:15 (2018)
[6] Angelopoulos Y., Aretakis S., Gajic D. Late-time asymptotics wave equation on spherically symmetric, stationary spacetimes. Adv. in Math. 323, 529–621 (2018)
[7] Angelopoulos Y., Aretakis S., Gajic D. Late-time asymptotics for the wave equation on extremal Reissner–Nordström backgrounds. Adv. in Math. 375, 107363 (2020).
[8] Aretakis S. Stability and instability of extreme Reissner-Nordström black hole spacetimes for linear scalar perturbations I. Communications in mathematical physics 307 (1), 17-63 (2011)
[9] Aretakis S. Stability and instability of extreme Reissner-Nordström black hole spacetimes for linear scalar perturbations II. Annales Henri Poincaré 12 (8), 1491-1538 (2011)
[10] Aretakis S. Decay of axi-symmetric solutions of the wave equation on extreme Kerr backgrounds. J. Funct. Anal. 263, 2770 (2012)
[11] Bardeen J.M., Press W.H. Radiation fields in the Schwarzschild background. J. Mathematical Phys., 14:7-19 (1973)
[12] Blue P., Decay of the Maxwell field on the Schwarzschild manifold. J. Hyperbolic Differ. Equ., 5(4):807-856 (2008)

[13] Blue P., Soffer A., Semilinear wave equations on the Schwarzschild manifold. I. Local decay estimates. Adv. Differ. Equ. 8(5), 595-614 (2003)

[14] Breuer R. A., Ryan M. P., Waller S. Some properties of spin-weighted spheroidal harmonics. Proc. R. Soc. Lond. A. 358, 71-86 (1977)

[15] Carter B., Global structure of the Kerr family of gravitational fields, Physical Review. 174 (5): 1559–1571 (1968)

[16] Carter B., Hamilton-Jacobi and Schrödinger separable solutions of Einstein’s equations, Comm. Math. Phys. 10, 268-310 (1968)

[17] Chandrasekhar S., The mathematical theory of black holes. Oxford University Press (1983)

[18] Christodoulou D., Klainerman S., The global nonlinear stability of the Minkowski space, Princeton University Press (1993).

[19] Civin D., Stability at charged rotating black holes for linear scalar perturbations, Ph.D. thesis, University of Cambridge. Available at http://www.repository.cam.ac.uk/handle/1810/247 (2014)

[20] Dafermos M., Holzegel G., Rodnianski I. The linear stability of the Schwarzschild solution to gravitational perturbations. Acta Mathematica, 222: 1-214 (2019)

[21] Dafermos M., Holzegel G., Rodnianski I. Boundedness and decay for the Teukolsky equation on Kerr spacetimes I: the case $|a| \ll M$. Ann. PDE, 5, 2 (2019)

[22] Dafermos M., Holzegel G., Rodnianski I., Taylor M. The non-linear stability of the Schwarzschild family of black holes. arXiv preprint arXiv:2104.08222 (2021)

[23] Dafermos M., Rodnianski I. A new physical-space approach to decay for the wave equation with applications to black hole spacetimes. XVIth International Congress on Mathematical Physics, 421-433, (2009)

[24] Dafermos M., Rodnianski I. The red-shift effect and radiation decay on black hole spacetimes. Comm. Pure Appl. Math, 62:859-919 (2009)

[25] Dafermos M., Rodnianski I. Decay for solutions of the wave equation on Kerr exterior spacetimes I-II: The cases $|a| \ll M$ or axisymmetry. arXiv preprint arXiv:1010.5132 (2010)

[26] Dafermos M., Rodnianski I. The black hole stability problem for linear scalar perturbations, Proceedings of the Twelfth Marcel Grossmann Meeting on General Relativity, T. Damour et al (ed.) (2011).

[27] Dafermos M., Rodnianski I., A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds, Invent. Math. 185(3), 467–559 (2011).

[28] Dafermos M., Rodnianski I. Lectures on black holes and linear waves. Evolution equations, Clay Mathematics Proceedings, Vol. 17, pages 97-205 Amer. Math. Soc. (2013)

[29] Dafermos M., Rodnianski I., Shlapentokh-Rothman Y. Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full subextremal case $|Q| \ll M$. Annals of Math. 183, no. 3, 787-913 (2016)

[30] Dias O.J.C., Godazgar M., Santos J.R. Linear Mode Stability of the Kerr-Newman Black Hole and Its Quasi-normal Modes. Phys. Rev. Lett. 114, 151101 (2015)

[31] Giorgi E. Boundedness and decay for the Teukolsky system of spin $\pm 2$ on Reissner-Nordström spacetime: the case $|Q| \ll M$. Ann. Henri Poincaré, 21, 2485 - 2580 (2020)

[32] Giorgi E. Boundedness and decay for the Teukolsky system of spin $\pm 1$ on Reissner–Nordström spacetime: the $\ell = 1$ spherical mode. Class. Quantum Grav. 36, 205001 (2019)

[33] Giorgi E. The linear stability of Reissner-Nordström spacetime for small charge. Annals of PDE, 6, 8 (2020)

[34] Giorgi E. The linear stability of Reissner-Nordström spacetime: the full sub-extremal range $|Q| < M$. Commun. Math. Phys. 380, 1313–1360 (2020)

[35] Giorgi E., Electromagnetic-gravitational perturbations of Kerr-Newman spacetime: the Teukolsky and Regge-Wheeler equations, arXiv preprint arXiv:2002.07228 (2020)

[36] Giorgi E., Klainerman S., Szeftel J. A general formalism for the stability of Kerr. arXiv preprint arXiv:2002.02740 (2020)

[37] Giorgi E., Klainerman S., Szeftel J. Morawetz estimates in perturbations of Kerr. In preparation

[38] Häfner D., Hintz P., Vasy A. Linear stability of slowly rotating Kerr black holes. Inventiones mathematae (2020)

[39] Hintz P. Non-linear stability of the Kerr-Newman-de Sitter family of charged black holes. Annals of PDE, 4(1):11 (2018)
[40] Hintz P., Vasy A. The global non-linear stability of the Kerr-de Sitter family of black holes. Acta Mathematica, 220:1-206 (2018)

[41] Hung P.-K., Keller J., Wang M.-T. Linear Stability of Schwarzschild Spacetime: The Cauchy Problem of Metric Coefficients. J. Differential Geom. 116 (3) 481 - 541 (2020)

[42] Johnson T. The linear stability of the Schwarzschild solution to gravitational perturbations in the generalised wave gauge. Ann. PDE 5, 13 (2019)

[43] Kay, B.S., Wald, R.M., Linear stability of Schwarzschild under perturbations which are nonvanishing on the bifurcation two sphere. Class. Quantum Grav. 4, 893–898 (1987)

[44] Klainerman S., Szeftel J. Global Non-Linear Stability of Schwarzschild Spacetime under Polarized Perturbations. Annals of Mathematics Studies, Princeton University Press (2020)

[45] Klainerman S., Szeftel J. Kerr stability for small angular momentum. arXiv preprint arXiv:2104.11857 (2021)

[46] Lee C. H. Coupled gravitational and electromagnetic perturbations around a charged black hole. Journal of Mathematical Physics 17, 1226 (1976)

[47] Ma S. Uniform energy bound and Morawetz estimate for extreme component of spin fields in the exterior of a slowly rotating Kerr black hole I: Maxwell field. Ann. Henri Poincaré 21, 815–863 (2020)

[48] Ma S. Uniform energy bound and Morawetz estimate for extreme component of spin fields in the exterior of a slowly rotating Kerr black hole II: linearized gravity. Commun. Math. Phys. 377, 2489–2551 (2020)

[49] Mark Z., Yang H., Zimmerman A., Chen Y., Quasinormal modes of weakly charged Kerr-Newman spacetimes, Phys. Rev. D, 91, 4, 044025 (2015)

[50] Moncrief V. Odd-parity stability of a Reissner-Nordström black hole. Phys. Rev. D 9, 2707 (1974)

[51] Moncrief V. Stability of Reissner-Nordström black holes. Phys. Rev. D 10, 1057 (1974)

[52] Moncrief V., Gudapati N. A positive-definite energy functional for the axisymmetric perturbations of Kerr-Newman black holes. arXiv preprint arXiv:2105.12632 (2021)

[53] Nathanail A., Most E., Rezzolla L. Gravitational collapse to a Kerr-Newman black hole. Monthly Notices of the Royal Astronomical Society Letters (2017)

[54] Newman E.T., Couch E., Chinnapared K., Exton E., Prakash A., Torrence R. Metric of a rotating, charged mass J. Math. Phys., 6, 918–919 (1965)

[55] Pani P., Berti E., Gualtieri L. Gravitoelectromagnetic Perturbations of Kerr-Newman Black Holes: Stability and Isospectrality in the Slow-Rotation Limit. Phys. Rev. Lett. 110, 241103 (2013)

[56] Pasqualotto F. The spin ±1 Teukolsky equations and the Maxwell system on Schwarzschild. Annales Henri Poincaré 20, 1263-323 (2019)

[57] Regge T., Wheeler J.A. Stability of a Schwarzschild singularity. Phys. Rev. 2, 108:1063-1069 (1957)

[58] Shlapentokh-Rothman Y., Quantitative Mode Stability for the Wave Equation on the Kerr Spacetime, Ann. Henri Poincaré. 16 (2015), 289–345.

[59] Shlapentokh-Rothman Y., Teixeira da Costa R. Boundedness and decay for the Teukolsky equation on Kerr in the full subextremal range |a| < M: frequency space analysis, arXiv preprint arXiv:2007.07211.

[60] Sterbenz, J., Tataru, D., Local energy decay for Maxwell fields. Part I: spherically symmetric black hole backgrounds, Int. Math. Res. Not. 11(11), 3298–3342 (2015)

[61] Stojin J., Nonlinear wave dynamics in black hole spacetimes, Ph.D. thesis, Princeton University. Available at http://www.johnstojin.com/static/thesis.pdf (2017)

[62] Tataru D., Tohaneanu M. A local energy estimate on Kerr black hole background. IMRN no.2, 248-292 (2011)

[63] Teixeira da Costa R. Mode stability for the Teukolsky equation on extremal and subextremal Kerr spacetimes, Commun. Math. Phys., 378(1), 705-781 (2020)

[64] Teukolsky S.A. Perturbations of a rotating black hole. I. Fundamental equations for gravitational, electromagnetic and neutrino-field perturbations. The Astrophysical Journal, 185: 635-647 (1973)

[65] Wang H.-T., Tang S.-P., Li P.-C., Fan Y.-Z., Quasinormal-modes of the Kerr-Newman black hole: GW150914 and fundamental physics implications. arXiv preprint arXiv:2104.07594 (2021)

[66] Whiting B. F. Mode stability of the Kerr black hole. J. Math. Phys., 30 (6):1301-1305 (1989)