CONDITIONAL SAMPLING FOR MAX-STABLE PROCESSES WITH A MIXED MOVING MAXIMA REPRESENTATION

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Abstract

This paper deals with the question of conditional sampling and prediction for the class of stationary max-stable processes which allow for a mixed moving maxima representation. We develop an exact procedure for conditional sampling using the Poisson point process structure of such processes. For explicit calculations we restrict ourselves to the one-dimensional case and use a finite number of shape functions satisfying some regularity conditions. For more general shape functions approximation techniques are presented. Our algorithm is applied to the Gaussian extreme value process and the Brown-Resnick process. Finally, we compare our computational results to other approaches.

Keywords: conditional sampling; extremes; max-stable process; mixed moving maxima; Poisson point process

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1. Introduction

Over the last decades, several models for max-stable processes have been developed and applied, including times series models and stochastically continuous processes. In view of the wide range of potential applications of max-stable processes for modelling extreme events, the question of prediction and conditional sampling arises. Davis and Resnick ([5], [6]) proposed prediction procedures for time series which basically aim to minimize a suitable distance between observation and prediction. Further approaches for time series or random fields have been rare for a long time, apart from a few exceptions. Cooley et al. [3] introduced an approximation of the conditional density. Recently, Wang and Stoev [22] proposed an exact and efficient algorithm for conditional sampling for max-linear models

\[ Z_i = \max_{j=1,...,p} a_{ij} Y_j, \quad i = 1, \ldots, n, \]

where \( Y_j \) are independent Fréchet random variables. Dombry et al. [8] presented an algorithm for conditional simulation of Brown-Resnick processes based on more general results on conditional distributions of max-stable processes given in [7].

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Here, we will consider stationary max-stable processes with standard Fréchet margins that allow for a mixed moving maxima representation (see, for instance, [17], [20]). Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(F : (\Omega, \mathcal{F}) \to (G, \mathcal{G})\) be a random function such that \(E(\int dF(x) dx) = 1\). We assume that \(G\) is a countable set of measurable functions \(f : \mathbb{R}^d \to [0, \infty)\) and \(G = 2^G\).

Then, we consider the stationary max-stable process

\[
Z(t) = \max_{(s,u,f) \in \Pi} uf(t-s), \quad t \in \mathbb{R}^d,
\]

where \(\Pi\) is a Poisson point process on \(S = \mathbb{R}^d \times (0, \infty)\times G\) with intensity measure

\[
\Lambda(A \times B \times C) = |A| \cdot P_F(C) \cdot \int_B \frac{du}{u^2}, \quad A \in \mathcal{B}^d, \ B \in \mathcal{B} \cap (0, \infty), \ C \in \mathcal{G},
\]

and \(P_F\) is the probability measure belonging to \(F\).

We aim to sample from the conditional distribution of \(Z\) given \(Z(t_1), \ldots, Z(t_n)\) for fixed \(t_1, \ldots, t_n \in \mathbb{R}^d\). As \(Z\) is entirely determined by the Poisson point process \(\Pi\), we analyse the distribution of \(\Pi\) given some values of \(Z\). The idea to use a Poisson point process structure for calculating conditional distributions has already been implemented in the case of a bivariate min-stable random vector [23]. A very general Poisson point process approach was recently used by [7] yielding formulae for conditional distributions in terms of the exponent measure. Some of these results are independently found here, as well.

The paper is organized as follows. In Section 2, we introduce a random partition of \(\Pi\) into three measurable point processes. This partition allows to focus on the critical points of \(\Pi\) which determine \(Z(t_1), \ldots, Z(t_n)\). Similarly to [22] and [7], we will call realisations of these point configuration scenarios and figure out the conditional distribution of these scenarios coping with the problem that we work on events of probability zero (Section 3). In Section 4, the conditional distribution of \(\Pi\) is calculated explicitly for the case \(d = 1\) and some regularity assumptions on a finite number of random shape functions. In Section 5, the results are applied to the Gaussian extreme value process [19] and compared to other algorithms. Section 6 deals with an approximation procedure in the case of a countable and uncountable number of random shape functions. A prominent example, the Brown-Resnick process [1], is the matter of a comparison study for different algorithms in Section 6. In the last section, we give a brief overview of the results for a discrete mixed moving maxima process restricted to \(pZ^d\).

2. Random partition of \(\Pi\) and measurability

In this section, we will consider random sets of points within \(\Pi\) which essentially determine the process \(Z\). Separating these critical points of \(\Pi\) from the other ones, we get a random partition of \(\Pi\). We will show that this partition is measurable, which allows for further investigation of this partition.

For some fixed \((t, z) \in \mathbb{R}^d \times (0, \infty),\) define the set

\[
K_{t,z} = \left\{ (x, y, f) \in S : f(t-x) > 0, \ y = \frac{z}{f(t-x)} \right\},
\]
which we call the set of points generating \((t, z)\) due to the fact that
\[
Z(t) = z \iff |\Pi \cap K_{t, z}| \geq 1 \land \Pi \cap \overline{K_{t, z}} = \emptyset.
\]

Here, \(\overline{K} = \bigcup_{(x, y, f) \in K} \{x\} \times (y, \infty) \times \{f\}\) for a set \(K \subseteq S\).

In a next step, we consider \(n\) fixed points \((t_1, z_1), \ldots, (t_n, z_n) \in \mathbb{R}^d \times (0, \infty)\). For any vector \(t = (t_1, \ldots, t_n) \in \mathbb{R}^d\) and any mapping \(g\) with domain \(\text{dom}(g) \subseteq \mathbb{R}^d\) we will write \(g(t)\) instead of \((g(t_1), \ldots, g(t_n))\), for short. Similarly, \(t > 0\) is understood as \(t_i > 0, i = 1, \ldots, n\).

Now, we define the set of points generating \((t, z)\) as
\[
K_{t, z} = \left\{(x, y, f) \in S : \max_{i=1, \ldots, n} f(t_i - x) > 0, \ y = \min_{i=1, \ldots, n} \frac{z_i}{f(t_i - x)}\right\}
\]

This implies
\[
\overline{K_{t, z}} = \left\{(x, y, f) \in S : yf(t_j - x) \leq z, \ yf(t_j - x) = z_j \text{ for some } j \in \{1, \ldots, n\}\right\}
\]

and
\[
K_{t, z} \cap K_{t_i, z_i} = \left\{(x, y, f) \in S : yf(t_i - x) = z_i, \ yf(t_j - x) \leq z\right\}.
\]

Therefore, we have
\[
Z(t) \leq z \iff \Pi \cap \overline{K_{t, z}} = \emptyset
\]

and
\[
Z(t) = z \iff |\Pi \cap K_{t_i, z_i} \cap K_{t, z}| \geq 1, \ i = 1, \ldots, n \land \Pi \cap \overline{K_{t, z}} = \emptyset. \quad (2.1)
\]

Now we define a random partition of \(\Pi\) by
\[
\Pi_1 := \Pi \cap \overline{K_{t, Z(t)}}
\]
\[
\Pi_2 := \Pi \cap K_{t, Z(t)},
\]

and
\[
\Pi_3 := \Pi \setminus (\Pi_1 \cup \Pi_2).
\]

Relation (2.1) implies that \(\Pi_1 = \emptyset\) and \(|\Pi_2 \cap K_{t_i, Z(t_i)}| \geq 1\) a.s. for \(i \in \{1, \ldots, n\}\). Note that the partition of \(\Pi\) into \(\Pi_1, \Pi_2\) and \(\Pi_3\) corresponds to the classification of \(\Phi_R^2\) and \(\Phi_R^-\) in [7].

Before proceeding any further, we need to prove that \(\Pi_1, \Pi_2\) and \(\Pi_3\) are well-defined. We will do this by showing the measurability of a further partition of \(\Pi_2\), namely the restriction of \(\Pi\) to intersection sets. For any \(A \in 2^{\{1, \ldots, n\}} \setminus \emptyset\), these are defined as
\[
I_A(z) = K_{t, z} \cap \left(\bigcap_{i \in A} K_{t_i, z_i} \setminus \bigcup_{j \in A^c} K_{t_j, z_j}\right)
\]

\[
= \{(x, y, f) \in S : yf(t_i - x) = z_i, \ i \in A, \ yf(t_j - x) < z_j, \ j \notin A\}
\]
By construction $K_{t,z}$ is a disjoint union of $I_A(z), A \in 2^{1,\ldots,n} \setminus \emptyset$.

To prove the measurability of these restrictions of $\Pi$ to the intersection sets, let $\mathcal{C}$ be the $\sigma$-algebra on $\mathbb{R}^d$ generated by the cylinder sets

$$C_{s_1,\ldots,s_m}(B) = \{ f \in \mathbb{R}^d : (f(s_1),\ldots,f(s_m)) \in B \},$$

where $s_1,\ldots,s_m \in \mathbb{R}^d, B \in \mathcal{B}^m, m \in \mathbb{N}$.

**Proposition 2.1.** Let $t_1,\ldots,t_n \in \mathbb{R}^d$ be fixed.

1. The mapping

$$\Psi : S \to \mathbb{R}^d, (x,y,f) \mapsto yf(\cdot - x)$$

is $(\mathcal{B} \times (B \cap (0,\infty))) \times 2^G, \mathcal{C})$-measurable.

2. Let $A \in 2^{1,\ldots,n} \setminus \emptyset$ and $B \subset S$ a bounded Borel set. Then, $|\Pi \cap I_A(Z(t)) \cap B|$ is a random variable.

3. $\Pi_1, \Pi_2$ and $\Pi_3$ are point processes.

**Proof.** 1. It suffices to verify that $\Psi^{-1}(C_{s_1,\ldots,s_m}(\chi_{i=1}^{m}(a_i, b_i)))$ is measurable for any $s_j \in \mathbb{R}^d, a_j < b_j \in \mathbb{R}, j = 1,\ldots,m, m \in \mathbb{N}$. We have

$$\Psi^{-1}(C_{s_1,\ldots,s_m}(\chi_{i=1}^{m}(a_i, b_i)))$$

$$= \bigcup_{f \in G} \bigcup_{t \in \mathbb{R}^d} \left\{ t : (q_1, q_2) \in \left( \max_{i=1,\ldots,m} \frac{a_i}{f(t_i - t)}, \min_{i=1,\ldots,m} \frac{b_i}{f(t_i - t)} \right) \times (q_1, q_2) \right\}$$

As each $f \in G$ is measurable, sets of the type $\{ t \in \mathbb{R}^d : f(t_i - t) \in B \}$ are measurable for any $B \in \mathcal{B}$. Therefore, $\Psi^{-1}(C_{s_1,\ldots,s_m}(\chi_{i=1}^{m}(a_i, b_i))) \in \mathcal{B} \times (B \cap (0,\infty))) \times 2^G$.

2. We consider

$$\{ \omega : |\Pi \cap I_A(Z(t)) \cap B| = k \}$$

$$= \bigcup_{n_0 \in \mathbb{N}} \bigcap_{m=n_0} \bigcup_{y \in \mathbb{Q}^n} \left\{ \omega \in \Omega : Z(t) \in \chi_{i=1}^{n} \left( y_i - \frac{1}{m}, y_i + \frac{1}{m} \right), \right.$$  

$$\left| \Pi \cap \Psi^{-1}\left( \left\{ f \in \mathbb{R}^d : f(t_i) \in \left( y_i - \frac{1}{m}, y_i + \frac{1}{m} \right), i \in A, \right. \right.$$  

$$f(t_j) \leq y_j - \frac{1}{m}, j \notin A \right\} \cap B \right| = k \} \right\}$$

By the first part of this proposition, $\Psi$ is a measurable mapping and we get that $\{ \omega : |\Pi \cap I_A(Z(t)) \cap B| = k \}$ is measurable.

3. For any bounded Borel set $B \subset S$ the second part of this proposition yields that $|\Pi_1 \cap B| = 0, |\Pi_2 \cap B| = \sum_{A \in 2^{1,\ldots,n} \setminus \emptyset} |\Pi \cap I_A(Z(t)) \cap B|$ and $|\Pi_3 \cap B| = |\Pi \cap B| - |\Pi_2 \cap B|$ are measurable. Thus, $\Pi_1, \Pi_2$ and $\Pi_3$ are point processes ([4], Cor. 6.1.IV).
3. Blurred sets, scenarios and limit considerations

This section mainly deals with the analysis of the distribution of the set of critical points, \( \Pi_2 \). As this set has the intensity measure zero conditional on an event \( Z(t) = z \) of probability zero, this distribution cannot be calculated straightforward. We need to borrow arguments from martingale theory, taking limits of probabilities conditional on the observations being in small intervals containing \( z \). By this condition, the set of critical points gets blurred. We distinguish between different scenarios denoting which points influence the different observations. Using general bounds for the rate of convergence of the intensity of these sets, we can prove that each observation is generated by exactly one point of \( \Pi \) (Proposition 3.1), which restricts the number of scenarios that occur with positive probability. According to the blurred sets the scenarios get blurred, as well. The blurred scenarios are not exactly the same as the scenarios conditional on blurred observations, but much more tractable. However, it can be shown that both events asymptotically yield the same conditional probability (Theorem 3.1). Based on these considerations, the independence of \( \Pi_1, \Pi_2 \) and \( \Pi_3 \) conditional on \( Z(t) \) is shown (Theorem 3.2). This allows us to simulate \( \Pi_2 \) and \( \Pi_3 \) independently. Furthermore, \( \Pi_3 \) turns out to be easily simulated (Theorem 3.2).

Let 
\[
F_m = \sigma \left( \left\{ Z(t_i) \in \left( \frac{k}{2^m}, \frac{k+1}{2^m} \right], \ i = 1, \ldots, n, \ k \in \mathbb{N}_0 \right\} \right).
\]

Then, \( \{F_m\}_{m \in \mathbb{N}} \) is a filtration and 
\[
F_\infty := \bigcap_{m \in \mathbb{N}} F_m = \sigma(Z(t)).
\]

For \( z > 0 \), let \( j_m(z) \in \mathbb{N} \) be such that 
\[
z \in A_m(z) = \left( \frac{j_m(z)}{2^m}, \frac{j_m(z)+1}{2^m} \right).
\]

Then, we have \( A_m(z) \xrightarrow{m \to \infty} \{z\} \) monotonically and \( \{\omega \in \Omega : Z(t) \in A_m(z)\} \in F_m \). Furthermore, let
\[
K_{t,z}^{(m)} = \bigcup_{\tilde{z} \in A_m(z)} K_{t,\tilde{z}} = \{ (x, y, f) \in S : yf(t-x) \in A_m(z) \},
\]
\[
K_{t,z}^{(m)} = \bigcup_{\tilde{z} \in A_m(z)} K_{t,\tilde{z}} = \{ (x, y, f) \in S : yf(t-x) \leq \frac{j_m(z) + 1}{2^m} \},
\]
\[
yf(t_i-x) \in A_m(z_i) \text{ for some } i \in \{1, \ldots, n\}\},
\]
and
\[
K_{t,z}^{(m)} = \bigcap_{\tilde{z} \in A_m(z)} K_{t,\tilde{z}} = \{ (x, y, f) \in S : yf(t_i-x) > \frac{j_m(z_i) + 1}{2^m} \text{ for some } i \in \{1, \ldots, n\} \} = K_{t,j_m(z)+1}^{(m)}.
\]

These definitions imply that
\[
K_{t,z}^{(m)} \cap \overline{K_{t,z}^{(m)}} = \emptyset \quad \text{and} \quad K_{t,z}^{(m)} \cup \overline{K_{t,z}^{(m)}} = K_{t,j_m(z)}^{(m)}.
\]
Furthermore, as

\[
Z(t) \leq \frac{j_m(z) + 1}{2m} \quad \iff \quad \Pi \cap \overline{K_{t,z}^{(m)}} = \emptyset.
\]

We call these sets the blurred sets belonging to \( Z(t) \) conditional on \( Z(t) \in A_m(z) \). This notation is due to the fact that we have

\[
\Pi \cap \overline{K_{t,z}^{(m)}} = \emptyset.
\]

we get that

\[
K_{t,z}^{(m)} \cap K_{t_i,z_i}^{(m)} = \left\{ (x, y, f) \in S : yf(t_i - x) \in A_m(z_i), yf(t - x) \leq \frac{j_m(z) + 1}{2m} \right\},
\]

we get that

\[
|\Pi \cap K_{t_i,z_i}^{(m)} \cap K_{t,z}^{(m)}| \geq 1, \ i = 1, \ldots, n \implies Z(t) > \frac{j_m(z)}{2m}.
\]

Thus, we obtain

\[
Z(t) \in A_m(z) \iff |\Pi \cap K_{t_i,z_i}^{(m)} \cap K_{t,z}^{(m)}| \geq 1, \ i = 1, \ldots, n \land \Pi \cap \overline{K_{t,z}^{(m)}} = \emptyset. \tag{3.1}
\]

In particular, for fixed \( z \in (0, \infty)^n \), the point process \( \Pi \setminus (K_{t,z}^{(m)} \cup \overline{K_{t,z}^{(m)}}) \) is independent of the event \( Z(t) \in A_m(z) \).

Based on these blurred sets, we define the blurred intersection sets

\[
I_A^{(m)}(z) = K_{t,z}^{(m)} \cap \bigcap_{i \in A} K_{t_i,z_i}^{(m)} \setminus \bigcup_{j \in A^c} K_{t_j,z_j}^{(m)}, \quad A \in 2^{\{1, \ldots, n\} \setminus \emptyset}.
\]

We note that \( K_{t,z}^{(m)} \) can be written as a disjoint union of \( I_A^{(m)}(z), \ A \in 2^{\{1, \ldots, n\} \setminus \emptyset} \).

**Lemma 3.1.** For any \( A \in 2^{\{1, \ldots, n\} \setminus \emptyset} \) and \( z \geq 0 \) we have \( \Lambda \left( I_A^{(m)}(z) \right) \in \mathcal{O}(2^{-m}) \), where \( \Lambda(\cdot) \) is given by \( \text{12} \).

**Proof.** It suffices to show that

\[
\Lambda \left( K_{t_i,z_i}^{(m)} \right) \in \mathcal{O}(2^{-m}), \quad i = 1, \ldots, n. \tag{3.2}
\]

By a straightforward computation we get

\[
\Lambda \left( K_{t_i,z_i}^{(m)} \right) = \mathbb{E}_F \left( \int_{\mathbb{R}^d} \frac{\frac{1}{2m} + 1}{u^2} \, du \, dx \right)
\]

\[
= \mathbb{E}_F \left( \int_{\mathbb{R}^d} F(t_i - x) \, dx \right) \cdot \left( \frac{1}{\frac{j_m(z_i)}{2m}} - \frac{1 - \frac{j_m(z_i)}{2m}}{\frac{j_m(z_i) + 1}{2m}} \right) = \frac{1}{\frac{j_m(z_i)}{2m} \cdot \frac{j_m(z_i) + 1}{2m}}.
\]

Using the fact that \( \lim_{m \to \infty} \frac{j_m(z_i)}{2m} = z_i \), the assertion of the Lemma follows.

In Section 4 a more precise notion about the speed of convergence of \( 2^{-m}j_m(z_i) \to z_i \) will be useful.
Lemma 3.2. For any \( \varepsilon > 0 \), with probability one we have
\[
\liminf_{m \to \infty} 2^{m(1+\varepsilon)} \min_{i=1, \ldots, n} \left( \frac{j_m(Z(t_i)) + 1}{2^m} - Z(t_i) \right) = \infty
\]
and
\[
\limsup_{m \to \infty} 2^{m(1+\varepsilon)} \max_{i=1, \ldots, n} \left( \frac{j_m(Z(t_i))}{2^m} - Z(t_i) \right) = -\infty.
\]

Proof. We present the proof of the first assertion. The second one can be shown analogously. For the first assertion it suffices to show that
\[
\liminf_{m \to \infty} 2^{m(1+\varepsilon)} \left( \frac{j_m(Z(t_i)) + 1}{2^m} - Z(t_i) \right) = \infty, \quad i = 1, \ldots, n.
\]
Let \( a \in (0, \frac{1}{2}) \) and \( m \in \mathbb{N} \) large enough such that
\[
\sum_{k=1}^{\infty} 2^{-m} \frac{1}{(k2^{-m})^2} \exp \left( -\frac{1}{k2^{-m}} \right) \leq 2 \int_{0}^{\infty} \frac{1}{x^3} \exp \left( -\frac{1}{x} \right) dx = 2. \quad (3.3)
\]
Then, we have
\[
P \left( \frac{j_m(Z(t_i)) + 1}{2^m} - Z(t_i) \leq a2^{-m} \right)
\]
\[
= \sum_{k=1}^{\infty} P \left( (k-a)2^{-m} \leq Z(t_i) \leq k2^{-m} \right)
\]
\[
= \sum_{k=1}^{\infty} \exp \left( -\frac{1}{k2^{-m}} \right) \cdot \left( 1 - \exp \left( -\frac{1}{k2^{-m}} \cdot \frac{a/k}{1 - a/k} \right) \right)
\]
\[
\leq \sum_{k=1}^{\infty} \exp \left( -\frac{1}{k2^{-m}} \right) \cdot \left( \frac{1}{k2^{-m}} \cdot \frac{2a}{k} \right),
\]
where we used the fact that \( 1 - \exp(-x) \leq x \) for all \( x > 0 \). Employing (3.3), we get for \( a = C2^{-m\varepsilon} \) with \( C > 0 \) and \( m \) large enough that
\[
P \left( \frac{j_m(Z(t_i)) + 1}{2^m} - Z(t_i) \leq C \cdot 2^{-m(1+\varepsilon)} \right) \leq 4C \cdot (2^\varepsilon)^{-m}.
\]
Therefore, the probabilities above are summable with respect to \( m \) and the Borel-Cantelli lemma yields
\[
P \left( \liminf_{m \to \infty} 2^{m(1+\varepsilon)} \left( \frac{j_m(Z(t_i)) + 1}{2^m} - Z(t_i) \right) < C \right) = 0
\]
for any \( C > 0 \). This completes the proof.

Now, we relate the points in \( K_{t, z}^{(m)} \) to the corresponding “blurred” intersection sets by introducing disjoint “blurred” scenarios
\[
E_{\{n_A\}}^{(m)}(z) = E_{\{n_A, A \in 2^{\{1, \ldots, n\}} \setminus \emptyset\}}^{(m)}(z)
\]
\[
= \{ \omega \in \Omega : |\Pi \cap I_A^{(m)}(z)| = n_A, A \in 2^{\{1, \ldots, n\}} \setminus \emptyset, |\Pi \cap \overline{K}_{(t, z)}^{(m)}| = 0 \}.
\]
with \( \{n_A, A \in 2^{\{1, \ldots, n\} \setminus \emptyset} \} \in N_1 \) where
\[
N_1 = \left\{ \{n_A, A \in 2^{\{1, \ldots, n\} \setminus \emptyset}, \sum_{A: i \in A} n_A \geq 1, i = 1, \ldots, n \right\}.
\]

Thus, we have
\[
\{Z(t) \in A_m(z)\} = \bigcup_{\{n_A\} \in N_1} E_{\{n_A\}}^{(m)}(z)
\]
and the union is obviously disjoint. In the same way, dropping the \((m)\) in the definition, we specify scenarios \(E_{\{n_A\}}(z)\).

Now, we show that
\[
P(\exists i \in \{1, \ldots, n\} : |\Pi \cap K_{t_i,z(t_i)}| \geq 2) = 0.
\]

To this end, we first verify that, with probability one, \(\Pi_2\) has no points which can be removed without any effect on \(Z(t)\). We consider scenarios \(E_{\{n_A\}}(Z(t))\) with \(\{n_A\} \in N_2\). Here, \(N_2\) is the ensemble of all sets \(\{n_A, A \in 2^{\{1, \ldots, n\} \setminus \emptyset}\}\) such that there exists some \(A^* \in 2^{\{1, \ldots, n\} \setminus \emptyset}\) with
\[
n^*_A = \begin{cases} n_A^* - 1, & A = A^*, \\ n_A, & \text{else.} \end{cases}
\]
and \(\{n_A^*\} \in N_1\). Thus, \(\{n_A\} \in N_2\) if and only if \(E_{\{n_A\}}(Z(t))\) remains an allowable scenario after removing one point.

**Lemma 3.3.** With probability one, we have
\[
P\left( \bigcup_{\{n_A\} \in N_2} E_{\{n_A\}}(Z(t)) \mid Z(t) = z \right) = \lim_{m \to \infty} P\left( \bigcup_{\{n_A\} \in N_2} E_{\{n_A\}}^{(m)}(z) \mid Z(t) \in A_m(z) \right) = 0.
\]

In particular, \(\lim_{m \to \infty} P\left( |\Pi \cap K_{t,z}^{(m)} \setminus \Pi_2| \geq 2 \right. | Z(t) \in A_m(z) = 0.

**Proof.** By Lévy’s “Upward” Theorem (see \cite{[16]}, Thm. 50.3), we a.s. have
\[
\sum_{\{n_A\} \in N_2} P\left( E_{\{n_A\}}(Z(t)) \mid Z(t) = z \right) = \lim_{m \to \infty} \sum_{\{n_A\} \in N_2} P\left( E_{\{n_A\}}(Z(t)) \mid Z(t) \in A_m(z) \right).
\]

Noting that
\[
\bigcup_{\{n_A\} \in N_2} E_{\{n_A\}}(Z(t)) \subset \bigcup_{\{n_A\} \in N_2} E_{\{n_A\}}^{(m)}(z),
\]
it suffices to show the equation
\[
\lim_{m \to \infty} \sum_{\{n_A\} \in N_2} P(E_{\{n_A\}}^{(m)}(z) | Z(t) \in A_m(z)) = \lim_{m \to \infty} \sum_{\{n_A\} \in N_2} \frac{P(E_{\{n_A\}}^{(m)}(z))}{P(E_{\{n_A\}}^{(m)}(z))} = 0. \ (3.4)
\]
for every \( z > 0 \). Corresponding to each \( E_{(n_A)}^{(m)}(z) \) with \( \{n_A\} \in N_2 \) there exists some \( E_{(n_A^*)}^{(m)}(z) \) with \( \{n_A^*\} \in N_1 \) as in the definition of \( N_2 \). For \( A^* \in 2^{(1, \ldots, n)} \setminus \emptyset \) let \( N_{2, A^*} \) be the set of all \( \{n_A\} \in N_2 \) with \( n_{A^*} = n_{A^*} - 1 \). Then, for fixed \( m \in \mathbb{N} \), the left-hand side of (3.1) is less than or equal to

\[
\sum_{A^* \in 2^{(1, \ldots, n)} \setminus \emptyset} \frac{\sum_{\{n_A\} \in N_{2, A^*}} P(E_{(n_A)}^{(m)}(z))}{\sum_{\{n_A\} \in N_1} P(E_{(n_A)}^{(m)}(z))} \leq \sum_{A^* \in 2^{(1, \ldots, n)} \setminus \emptyset} \frac{\Lambda(I_{A^*}^{(m)}(z)) \sum_{\{n_A\} \in N_1} \prod_{A \in 2^{(1, \ldots, n)} \setminus \emptyset} \frac{\Lambda(I_{A}^{(m)}(z))^{n_A}}{n_A!}}{\sum_{\{n_A\} \in N_1} \prod_{A \in 2^{(1, \ldots, n)} \setminus \emptyset} \frac{\Lambda(I_{A}^{(m)}(z))^{n_A}}{n_A!}}
\]

where we imbedded \( N_{2, A^*} \) into \( N_1 \) by identifying \( \{n_A\} \) with the corresponding \( \{n_A^*\} \). By Lemma 3.1 we get equality (3.3), i.e. the first assertion of the lemma. Furthermore, \( |\Pi \cap K_{t, z}^{(m)}| \setminus \Pi_2 > 0 \) and \( Z(t) \in A_m(z) \) imply that \( \Pi \cap K_{t, z}^{(m)} \) contains points which can be removed without affecting \( Z(t) \in A_m(z) \). This is,

\[
\lim_{m \to \infty} P \left( |\Pi \cap K_{t, z}^{(m)}| \setminus \Pi_2 > 0 \mid Z(t) \in A_m(z) \right) \leq \lim_{m \to \infty} \sum_{\{n_A\} \in N_2} P \left( E_{(n_A)}^{(m)}(z) \mid Z(t) \in A_m(z) \right) = 0.
\]

Thus, the second assertion of this lemma is verified.

The following proposition is also stated in a more general setting in [7], Prop. 2.2.

**Proposition 3.1.** For any fixed \( t_1, \ldots, t_n \in \mathbb{R}^d \) we have

\[
P(\|\Pi \cap K_{t_i, Z(t_i)}\| \geq 2 \text{ for some } i \in \{1, \ldots, n\}) = 0.
\]

**Proof.** It suffices to show \( P(\|\Pi \cap K_{t_i, Z(t_i)}\| \geq 2) = 0 \) for all \( i \in \{1, \ldots, n\} \). Now, let \( i \in \{1, \ldots, n\} \) be fixed. Then, conditioning on \( Z(t_i) \) only, we get

\[
P \left( \|\Pi \cap K_{t_i, Z(t_i)}\| \geq 2 \mid Z(t_i) = z \right) = P \left( \bigcup_{\{n_A\} \in N_2} E_{(n_A)}(Z(t_i)) \mid Z(t_i) = z \right) = 0
\]

for almost every \( z > 0 \) by Lemma 3.3. This yields the desired result.
Theorem 3.1. With probability 1 we have

\[ \sum_{K} | \cap \Pi | 0.5, 1.0, 1.5, 2.0 \]

And again, Lévy’s “Upward” Theorem ([16], Thm. 50.3) yields.

Proposition 3.1 ensures that almost surely one of the scenarios \( E_{(n,A)}(z) \) with \( \sum_{A: i \in A} n_A = 1 \) for all \( i \in \{1, \ldots, n\} \) occurs.

**Theorem 3.1.** With probability 1 we have

1. \( P(E_{(n,A)}(Z(t)) \mid Z(t) = z) = \lim_{m \to \infty} P(E_{(n,A)}^{(m)}(z) \mid Z(t) \in A_m(z)) \)
   for any \( \{n_A\} \in N_1 \),

2. \( P(\Pi_2 \cap B_j \mid = r_j, j = 1, \ldots, k, \mid \Pi_3 \cap B_j = r_j, j = k + 1, \ldots, l \mid Z(t) = z) \)
   \( = \lim_{m \to \infty} P(\Pi \cap K_{t,z}^{(m)} \cap B_j \mid = r_j, j = 1, \ldots, k, \mid \Pi \cap B_j \cap (K_{t,z}^{(m)} \cup K_{t,z}^{(m)}) \mid = r_j, j = k + 1, \ldots, l \mid Z(t) \in A_m(z)) \)
   for any \( B_j \subset S, r_j \in \mathbb{N}, j \in \{1, \ldots, l\} \).

**Proof.** 1. Again, Lévy’s “Upward” Theorem ([16], Thm. 50.3) yields

\[ P(E_{(n,A)}(Z(t)) \mid Z(t) = z) = \lim_{m \to \infty} P(E_{(n,A)}^{(m)}(Z(t)) \mid Z(t) \in A_m(z)) \]

with probability 1. It remains to verify

\[ \lim_{m \to \infty} P(E_{(n,A)}^{(m)}(Z(t)) \mid Z(t) \in A_m(z)) = \lim_{m \to \infty} P(E_{(n,A)}^{(m)}(Z(t)) \mid Z(t) \in A_m(z)) \]

(3.5)

where \( P(E_{(n,A)}^{(m)}(Z(t)) \mid Z(t) \in A_m(z)) \) equals \( P(E_{(n,A)}^{(m)}(z) \mid Z(t) \in A_m(z)) \) by definition.

To this end, we consider the symmetric difference \( \Delta_m \) of the events \( E_{(n,A)}^{(m)}(Z(t)) \) and \( E_{(n,A)}(Z(t)) \). Note that any element of \( \Delta_m \) satisfies \( \| \Pi \cap K_{t,z}^{(m)} \cap \Pi_2 \| > 0 \) (cf. Figure [II] left) or \( \Pi \cap \cap A(Z(t)) \| > \| \Pi \cap I_A^{(m)}(Z(t)) \| \) for some \( A \in 2^\{1, \ldots, m\} \) \( \\emptyset \) (cf. Figure [II] right). The second kind of event happens if there is a point of \( \Pi \)
in $I_A(Z(t)) \cap (\bigcup_{j \notin A} K_{t_j, Z(t_j)})$. As this set vanishes for any $Z(t) > 0$ as $m \to \infty$, we get that

$$\lim_{m \to \infty} \{ \omega \in \Omega : |\Pi \cap I_A(Z(t))| > |\Pi \cap I_A^{(m)}(Z(t))| \} \searrow \emptyset, \quad m \to \infty,$$

for any $A \in 2^{\{1, \ldots, n\}} \setminus \emptyset$, and therefore

$$\lim_{m \to \infty} P(|\Pi \cap I_A(Z(t))| > |\Pi \cap I_A^{(m)}(Z(t))|) = 0.$$

This yields

$$\int_0^\infty \lim_{m \to \infty} P\left(|\Pi \cap I_A(Z(t))| > |\Pi \cap I_A^{(m)}(z)| \mid Z(t) \in A_m(z)\right) P(Z(t) \in dz)$$

$$= \lim_{m \to \infty} P\left(|\Pi \cap I_A(Z(t))| > |\Pi \cap I_A^{(m)}(z)| \mid Z(t) \in A_m(z)\right) P(Z(t) \in A_m(z))$$

using dominated convergence and the fact that

$$\int_0^\infty P\left(|\Pi \cap I_A(Z(t))| > |\Pi \cap I_A^{(m)}(z)| \mid Z(t) \in A_m(z)\right) P(Z(t) \in dz)$$

$$= \sum_{z \in (2^{\{1, \ldots, n\}})^n} P\left(|\Pi \cap I_A(Z(t))| > |\Pi \cap I_A^{(m)}(z)| \mid Z(t) \in A_m(z)\right) P(Z(t) \in A_m(z))$$

$$= P\left(|\Pi \cap I_A(Z(t))| > |\Pi \cap I_A^{(m)}(z)| \mid Z(t) \in A_m(z)\right).$$

Therefore, we have

$$\lim_{m \to \infty} P\left(|\Pi \cap I_A(Z(t))| > |\Pi \cap I_A^{(m)}(z)| \mid Z(t) \in A_m(z)\right) = 0 \quad (3.6)$$

for any $A \in 2^{\{1, \ldots, n\}} \setminus \emptyset$ and almost all $z > 0$. All in all, we end up with

$$\lim_{m \to \infty} \left| P\left(R_{n_1}^{(m)}(Z(t)) \mid Z(t) \in A_m(z)\right) - P\left(R_{n_2}^{(m)}(Z(t)) \mid Z(t) \in A_m(z)\right) \right|$$

$$\leq \lim_{m \to \infty} \sum_{A \in 2^{\{1, \ldots, n\}} \setminus \emptyset} P\left(|\Pi \cap I_A(Z(t))| > |\Pi \cap I_A^{(m)}(z)| \mid Z(t) \in A_m(z)\right)$$

$$+ \lim_{m \to \infty} P\left(|\Pi \cap K_{t,z}^{(m)} \mid 2 \mid > 0 \mid Z(t) \in A_m(z)\right) = 0 \quad a.s.$$

by (3.6) and by the second part of Lemma 3.3. Thus, we get (3.5) which completes the proof.

2. Let $B_1, \ldots, B_k, B_{k+1}, \ldots, B_l \in B^d \times (B \cap (0, \infty)) \times 2^G$. Then, each of the events

$$\{|\Pi_2 \cap B_j| \neq |\Pi \cap K_{t,z}^{(m)} \cap B_j| \text{ for any } j = 1, \ldots, k\}$$

and

$$\{|\Pi_3 \cap B_j| \neq |\Pi \cap B_j \setminus (K_{t,z}^{(m)} \cup K_{t,z}^{(m)})| \text{ for any } j = k + 1, \ldots, l\}$$
Theorem 3.2. \[ \text{in [7, Thm. 3.1, obtained by a different approach.} \]

\[ \text{Z} \]

\[ \text{M. Oesting, M. Schlather} \]

2. The process \( \Pi_{j} = 1 \) by the second part of Lemma 3.3. This verifies the assertion.

These results enable us to show the independence of \( \Pi_{1}, \Pi_{2} \) and \( \Pi_{3} \) conditional on \( Z(t) \) and to calculate the conditional distribution of \( \Pi_{3} \). Similar results can be found in [11, Thm. 3.1, obtained by a different approach.

\textbf{Theorem 3.2.} 1. With probability 1 the point processes \( \Pi_{1}, \Pi_{2} \) and \( \Pi_{3} \) conditional on \( Z(t) \) are stochastically independent.

2. The process \( \Pi_{3} \mid Z(t) = z \) has the same distribution as \( \Pi \setminus (K_{t,z} \cup \overline{K}_{t,z}) \) with probability 1.

\textbf{Proof.} 1. Since \( \Pi_{1} = \emptyset \) a.s., we only have to show the independence of \( \Pi_{2} \) and \( \Pi_{3} \) conditional on \( Z(t) \). Let \( B_{1}, \ldots, B_{k}, B_{k+1}, \ldots, B_{l} \in \mathcal{B} \times ((0, \infty) \cap \mathcal{B}) \times \mathcal{G} \) and \( r_{1}, \ldots, r_{l} \in \mathbb{N}_{0} \). The second part of Theorem 3.1 yields

\[ P(\Pi_{2} \cap B_{j} = r_{j}, \ \ j = 1, \ldots, k, \ \ | \Pi_{3} \cap B_{j} = r_{j}, \ \ j = k + 1, \ldots, l | Z(t) = z) = \lim_{m \to \infty} P(\Pi \cap K_{t,z}^{(m)} \cap B_{j} = r_{j}, \ \ j = 1, \ldots, k, \ \ | \Pi \cap B_{j} \setminus (K_{t,z}^{(m)} \cup \overline{K}_{t,z}^{(m)}) = r_{j}, \ \ j = k + 1, \ldots, l | Z(t) \cap A_{m}(z)). \]

By 3.1 the process \( \Pi \setminus (K_{t,z}^{(m)} \cup \overline{K}_{t,z}^{(m)}) \) is independent of the event \( Z(t) \in A_{m}(z) \). Hence,

\[ P(\Pi_{2} \cap B_{j} = r_{j}, \ \ j = 1, \ldots, k, \ \ | \Pi_{3} \cap B_{j} = r_{j}, \ \ j = k + 1, \ldots, l | Z(t) = z) = \lim_{m \to \infty} P(\Pi \cap K_{t,z}^{(m)} \cap B_{j} = r_{j}, \ \ j = 1, \ldots, k | Z(t) \in A_{m}(z)) \]

\[ \cdot \lim_{m \to \infty} P(\Pi \cap B_{j} \setminus (K_{t,z}^{(m)} \cup \overline{K}_{t,z}^{(m)}) = r_{j}, \ \ j = k + 1, \ldots, l | Z(t) \in A_{m}(z)) \]

\[ = P(\Pi_{2} \cap B_{j} = r_{j}, \ \ j = 1, \ldots, k | Z(t) = z) \cdot P(\Pi_{3} \cap B_{j} = r_{j}, \ \ j = k + 1, \ldots, l | Z(t) = z), \]

where we use the same arguments as before.
2. For any sets $B_1, \ldots, B_l \in \mathcal{B}^d \times \mathcal{G}$ and $r_1, \ldots, r_l \in \mathbb{N}_0$, the second part of Theorem 3.3 implies

$$P([\Pi_3 \cap B_j] \leq r_j, \ j = 1, \ldots, l \mid Z(t) = z) = \lim_{m \to \infty} P([\Pi \cap B_j \setminus (K_{t,z}^{(m)} \cup K_{t,z}^{(m)}]) \leq r_j, \ j = 1, \ldots, l \mid Z(t) \in A_m(z))$$

$$= \lim_{m \to \infty} P([\Pi \cap B_j \setminus (K_{t,z}^{(m)} \cup K_{t,z}^{(m)}]) \leq r_j, \ j = 1, \ldots, l)$$

$$= P([\Pi \cap B_j \setminus (K_{t,z} \cup K_{t,z})] \leq r_j, \ j = 1, \ldots, l).$$

Here, we use that the process $\Pi \setminus (K_{t,z}^{(m)} \cup K_{t,z}^{(m)})$ is independent of the event $Z(t) \in A_m(z)$ and the fact that we have a monotone limit. This completes the proof.

By the second part of Theorem 3.2, the process $\Pi_3 \mid Z(t) = z$ can be easily simulated by unconditionally simulating $\Pi$ and restricting it to $\mathbb{R}^d \times (0, \infty) \times G \setminus (K_{t,z} \cup K_{t,z})$.

At the end of this section we note that there exists a more general version of Theorem 3.3 which we will need for simulation. Let $B_1, \ldots, B_k \in \mathcal{B}^d \times ((0, \infty) \cap \mathcal{B}) \times \mathcal{G}$ be pairwise disjoint with $\bigcup_{j=1}^k B_j = S$. We introduce generalized "blurred" scenarios

$$E^{(m)}_{\{n_A^{(j)} \}}(z) = E^{(m)}_{\{n_A^{(j)} \}}(z)$$

$$= \{[\Pi \cap I^{(m)}_A(z) \cap B_j] = n^{(j)}_A, \ A \in 2^{\{1, \ldots, n\} \setminus \emptyset}, \ j = 1, \ldots, k, \ [\Pi \cap K_{t,z}^{(m)}] = 0\}$$

with $n^{(j)}_A \in \mathbb{N}_0$ such that $\sum_{j=1}^k \sum_{i \in A} n^{(j)}_A \geq 1$ for $i \in \{1, \ldots, n\}$.

Analogously, generalized scenarios $E^{(m)}_{\{n_A^{(j)} \}}(z)$ are defined. Then, in exactly the same way as Theorem 3.3, the following way theorem can be shown.

**Theorem 3.3.** With probability 1 we have

$$P(E^{(m)}_{\{n_A^{(j)} \}}(Z(t)) \mid Z(t) = z) = \lim_{m \to \infty} P(E^{(m)}_{\{n_A^{(j)} \}}(z) \mid Z(t) \in A_m(z))$$

for any scenario $E^{(m)}_{\{n_A^{(j)} \}}(z)$ with $\sum_{j=1}^k \sum_{i \in A} n^{(j)}_A \geq 1$ for all $i \in \{1, \ldots, n\}$.

The remainder of the paper will address the problem of simulating $\Pi_2 \mid Z(t) = z$. We propose a procedure consisting of two steps. First, we draw a scenario $E^{(m)}_{\{n_A^{(j)} \}}(Z(t))$ conditional on $Z(t) = z$. Then, the points of $\Pi_2$ corresponding to this scenario are simulated.

4. Calculations in the case of a finite number of shape functions

As shown in Section 3, all we need for calculating $P(E^{(m)}_{\{n_A^{(j)} \}}(Z(t)) \mid Z(t) = z)$ is the exact asymptotic behaviour of $\Lambda(I^{(m)}_A(z))$. In particular, we have to analyze the behaviour of the intersection of two curves $K_{t,z+i} \cap K_{t,z+j}$ for $|\delta_i|, |\delta_j|$ small. Explicit calculations turn out to be quite involved. Therefore, we restrict ourselves to
the case $d = 1$. Furthermore, the intersection depends on the derivative of the shape function. We calculate the asymptotics of $\Lambda(I^{(m)}_A(z))$ for $|A| = 1$ (Proposition 4.3), $|A| = 2$ (Proposition 4.1) and $|A| \geq 3$ (Proposition 4.2), see Figure 2. In the latter case, the rate of convergence of $\Lambda(I^{(m)}_A(z))$ cannot be determined exactly. Nevertheless, the conditional probability of any scenario can be calculated (Theorem 4.1).

First we assume that $G$ is a finite space of functions $f: \mathbb{R} \to (0, \infty)$ such that the intersections

$$M_{c,t_0} = \{ t \in \mathbb{R} : f(t) = cf(t_0 + t) \} \quad (4.1)$$

are finite for all $c > 0$, $t_0 \in \mathbb{R}$, $f \in G$. This implies that each set $I_A(z)$, $A \in 2^{\{1, \ldots, n\}} \setminus \emptyset$, $|A| \geq 2$, $z > 0$, is finite. W.l.o.g. we assume that $P_F(\{f\}) > 0$ for all $f \in G$.

**Proposition 4.1.** Let $t_1, t_2 \in \mathbb{R}$, $z_1, z_2 > 0$ such that

$$I_{\{1,2\}}(z) = \{(t_0, y_0, f)\}.$$

Furthermore, let $f$ be continuously differentiable in a neighbourhood of $t_1 - t_0$ and $t_2 - t_0$ with

$$z_1 f'(t_2 - t_0) \neq z_2 f'(t_1 - t_0). \quad (4.2)$$

Then, we have

$$\Lambda(I^{(m)}_{\{1,2\}}(z)) = \frac{2^{-2m}}{y_0^{-1}f'(t_2 - t_0) - z_2 f'(t_1 - t_0)} - \frac{2^{-2m}}{y_0^{-1}f'(t_1 - t_0) - z_1 f'(t_2 - t_0)} \cdot P_F(\{f\}) + o(2^{-2m}).$$

**Proof.** We note that $(t_0, y_0, f)$ satisfies the equation

$$\frac{f(t_1 - t_0)}{z_1} = \frac{f(t_2 - t_0)}{z_2} = y_0^{-1}.$$

Let

$$H : (-\infty, \infty) \times \mathbb{R} \to \mathbb{R}, \quad (\delta, t) \mapsto \frac{f(t_1 - t)}{z_1 + \delta_1} - \frac{f(t_2 - t)}{z_2 + \delta_2}.$$
Then, $H(0,t_0) = 0$ and
\[
\frac{\partial H}{\partial t}(0,t_0) = -\frac{f'(t_1 - t_0)}{z_1} + \frac{f'(t_2 - t_0)}{z_2} \neq 0
\]
due to \ref{4.2}. The implicit function theorem yields the existence of a neighbourhood $V$ of $0$ and a continuously differentiable function $h : V \to \mathbb{R}$ such that $H(h(\delta), \delta) = 0$. Using the notation $(t_\delta, y_\delta, f) = I_{(1,2)}(z + \delta)$ we get $h(\delta) = t_\delta$ and the equality
\[
\frac{f(t_1 - t_\delta)}{z_1 + \delta_1} = \frac{f(t_2 - t_\delta)}{z_2 + \delta_2} = y_\delta^{-1}.
\]
As $h$ is continuously differentiable, we obtain $t_0 - t_\delta \in O(||\delta||)$, and a Taylor expansion of $f$ yields
\[
f(t_i - t_\delta) = f(t_i - t_0) - f'(t_i - t_0) \cdot (t_\delta - t_0) + o(||\delta||). \tag{4.3}
\]
Let $g(t) = f(t - t_0)$. Thus, using \ref{4.3}, $t_\delta$ is given implicitly by
\[
\frac{g(t_1) - g'(t_1) \cdot (t_5 - t_0)}{z_1 + \delta_1} = \frac{g(t_2) - g'(t_2) \cdot (t_5 - t_0)}{z_2 + \delta_2} + o(||\delta||),
\]
which implies the explicit representation
\[
t_\delta = t_0 + \frac{\delta_1 g(t_2) - \delta_2 g(t_1)}{z_1 g'(t_2) - z_2 g'(t_1)} + o(||\delta||). \tag{4.4}
\]
Plugging in \ref{4.4} into \ref{4.3} yields
\[
y_\delta^{-1} = \frac{f(t_1 - t_\delta)}{z_1 + \delta_1} = \frac{g(t_1)}{z_1} - \frac{g'(t_1)}{z_1} \cdot \frac{\delta_1 g(t_2) - \delta_2 g(t_1)}{z_1 g'(t_2) - z_2 g'(t_1)} + o(||\delta||).
\]
As $f$ and $\delta \mapsto t_\delta = h(\delta)$ are $C^1$-functions, all the terms $o(||\delta||)$ are continuously differentiable for small $||\delta||$. Therefore, the mapping
\[
\Phi : V \to \mathbb{R} \times (0, \infty), \quad \delta \mapsto (t_\delta, y_\delta^{-1})
\]
is continuously differentiable near the origin. Calculating the partial derivatives explicitly we obtain
\[
\det(D\Phi(\delta)) = -\frac{g^2(t_1)}{z_1^2 \cdot (z_1 g'(t_2) - z_2 g'(t_1))} + o(1). \tag{4.5}
\]
As $\det(D\Phi(0)) \neq 0$, the inverse function theorem allows to regard $\Phi$ as a diffeomorphism restricted to an appropriate neighbourhood of $0$. Thus, considering the transformed Poisson point process $\tilde{\Pi} = \sum_{(s,u) \in \Pi} \delta_{(s,u)}$ on $\mathbb{R} \times (0, \infty)$ whose intensity measure is the Lebesgue measure, we get
\[
\Lambda(I_{(1,2)}(z + \delta), z_i + \delta_i \in A_m(z_i), i = 1,2) \] 
\[
= \int_{\Phi((A_m(z_1) - z_1) \times (A_m(z_2) - z_2))} P_F(\{f\}) \, d(t, y)
\]
\[ \left( \frac{1}{|z_{1}g'(t_{2}) - z_{2}g'(t_{1})|} - \varepsilon_{m} \right) \mathbb{P}_{F}(\{f\}) \]

We note that the term \( o(1) \) is continuous w.r.t. \( \delta \) and therefore the integrands can be locally bounded by

\[ \left( \frac{1}{|z_{1}g'(t_{2}) - z_{2}g'(t_{1})|} - \varepsilon_{m} \right) \mathbb{P}_{F}(\{f\}) \]

from below, and by

\[ \left( \frac{1}{|z_{1}g'(t_{2}) - z_{2}g'(t_{1})|} + \varepsilon_{m} \right) \mathbb{P}_{F}(\{f\}) \]

from above for all \( (\delta_{1}, \delta_{2}) \in (A_{m}(z_{1}) - z_{1}) \times (A_{m}(z_{2}) - z_{2}) \) with \( m \) large enough and an appropriate sequence \( \{ \varepsilon_{m}\}_{m \in \mathbb{N}} \) with \( \varepsilon_{m} \searrow 0 \). This implies that the integral has the form

\[ 2^{-2m} \left( \frac{1}{|z_{1}g'(t_{2}) - z_{2}g'(t_{1})|} \right) \mathbb{P}_{F}(\{f\}) + o(2^{-2m}) \]

which is the desired result.

**Remark 4.1.** 1. Using \( \frac{z_{1}}{f(t_{1} - t_{0})} = \frac{z_{2}}{f(t_{2} - t_{0})} = y_{0} \) we get that the equality

\[ z_{1}f'(t_{2} - t_{0}) = z_{2}f'(t_{1} - t_{0}) \]

holds if and only if

\[ \frac{\partial}{\partial t} \left. \frac{z_{1}}{f(t_{1} - t)} \right|_{t = t_{0}} = \frac{\partial}{\partial t} \left. \frac{z_{2}}{f(t_{2} - t)} \right|_{t = t_{0}}, \]

i.e. if and only if the two sets of admissible points, \( K_{t_{1}, z_{1}} \) and \( K_{t_{2}, z_{2}} \), are tangents to each other in \((t_{0}, y_{0}, f)\) which is an event of probability zero by Assumption (I.1). Therefore, (I.2) is satisfied a.s.

2. If \( I_{\{1,2\}}(z) \) consists of a finite number of points,

\[ I_{\{1,2\}}(z_{1}, z_{2}) = \{(t_{0}^{(1)}, y_{0}^{(1)}, f_{1}), \ldots, (t_{0}^{(k)}, y_{0}^{(k)}, f_{k})\}, \]

we get

\[ \Lambda(I_{\{1,2\}}(z)) = 2^{-2m} \sum_{j=1}^{k} y_{0}^{(j)} \cdot \left( \frac{P(\{f_{j}\})}{|z_{1}f_{j}'(t_{2} - t_{0}^{(j)}) - z_{2}f_{j}'(t_{1} - t_{0}^{(j)})|} \right) + o(2^{-2m}). \]

**Proposition 4.2.** Let \( t_{1}, \ldots, t_{l} \in \mathbb{R}, z_{1}, \ldots, z_{l} > 0, l \geq 3 \) such that

\[ I_{\{1,\ldots,l\}}(z) = \{(t_{0}, y_{0}, f)\}. \]
Let \( f : \mathbb{R} \to (0, \infty) \) be continuously differentiable in a neighbourhood of \( t_1 - t_0, \ldots, t_l - t_0 \) with \((12)\). Then, we have

\[
\Lambda(I_{\{1, \ldots, l\}}^{(m)}(z)) \leq \frac{2^{-2m}}{y_0 |z_1 f'(t_2 - t_0) - z_2 f'(t_1 - t_0)|} P_{\text{F}}(\{|f| \}) + o(2^{-2m}).
\]

For any \( C > 0, \epsilon > 0 \), there exists \( m_{C, \epsilon} \in \mathbb{N} \) such that

\[
\Lambda(I_{\{1, \ldots, l\}}^{(m)}(z)) \geq C 2^{-2m(1+\epsilon)}
\]

for all \( m \geq m_{C, \epsilon} \).

**Proof.** The first assertion follows immediately from Proposition 4.1 by the fact that

\[
\bigcap_{i=1}^{l} K_{t_i, z_i}^{(m)} \subset K_{t_i, z_i}^{(m)} \cap K_{t_2, z_2}^{(m)}.
\]

In order to verify the second assertion, we recall results from the proof of Proposition 4.1: we showed the existence of a \( C^1 \)-function \( \delta_1, \delta_2 \) such that

\[
H_{1, \delta_1, \delta_2}(0) = 0 \quad \text{and} \quad H_{1, \delta_1, \delta_2}(0) = \frac{f(t_1 - t_0)}{z_1 + \delta_1} - \frac{f(t_2 - t_0)}{z_2 + \delta_2}, \quad \text{for } i \in \{3, \ldots, l\}.
\]

As \( H_{1, \delta_1, \delta_2}(0, 0) = 0 \) and \( \frac{\partial H_{1, \delta_1, \delta_2}(0, 0)}{\partial \delta_1} \neq 0 \), we get the existence of a continuously differentiable function \( \delta_1, \delta_2 \) defined on a neighbourhood of \((0, 0)\) such that

\[
\delta_1, \delta_2 \mapsto \frac{f(t_1 - t_0)}{z_1 + \delta_1} - \frac{f(t_2 - t_0)}{z_2 + \delta_2}.
\]

Using Taylor expansions of \( g(\cdot) = f(\cdot - t_0) \) of first order, employing Equation \((4.6)\), and solving Equation \((4.4)\) yields

\[
h_{i}(\delta_1, \delta_2) = \frac{g(t_i)}{g(t_1)} \delta_1 + \frac{z_i g'(t_1) - z_i g'(t_i)}{g(t_1)} \left( \frac{f(t_2) - f(t_1)}{z_1} - \frac{f(t_2) - f(t_1)}{z_2} \right) + o(|\delta_1|) + o(|\delta_2|).
\]

So, there are constants \( c_{1, i}, c_{2, i} \) such that \( h_{i}(\delta_1, \delta_2) = c_{1, i} \delta_1 + c_{2, i} \delta_2 + o(|\delta_1|) + o(|\delta_2|) \).

Let \( A_{m}^{(i)} = A_{m}(z_i) - z_i \) for \( i \in \{1, \ldots, n\} \). We are interested in those pairs \( (\delta_1, \delta_2) \in A_{m}^{(1)} \times A_{m}^{(2)} \) with \( h_{i}(\delta_1, \delta_2) \in A_{m}^{(i)} \). By Lemma \((3.2)\) for any \( C' > 0, \epsilon > 0 \), we have that \(-C' 2^{-m(1+\epsilon)}, C' 2^{-m(1+\epsilon)} \in A_{m}^{(i)} \) \( i = 1, \ldots, n \), for \( m \) large enough. Therefore, \( h_{i}(\delta_1, \delta_2) \in A_{m}^{(i)} \) is guaranteed for \( |\delta_1| < \frac{C' 2^{-m(1+\epsilon)}}{3 |c_{1, i}|} \) and \( |\delta_2| < \frac{C' 2^{-m(1+\epsilon)}}{3 |c_{2, i}|} \) if \( m \) is sufficiently large.

By the same argument for all \( i \in \{3, \ldots, l\} \) we get that the existence of all \( h_{i}(\delta_1, \delta_2) \) is ensured for

\[
|\delta_1| < \frac{C' 2^{-m(1+\epsilon)}}{3 \max_{i=3, \ldots, l} |c_{1, i}|}, \quad |\delta_2| < \frac{C' 2^{-m(1+\epsilon)}}{3 \max_{i=3, \ldots, l} |c_{2, i}|}.
\]
for $m$ large enough. Furthermore, to ensure $\delta_1 \in A^{(1)}_m$, $\delta_2 \in A^{(2)}_m$, we have to add the conditions $|\delta_1|, |\delta_2| < C' \rho - m(1+\varepsilon)$. With $C_j = \max\{1, 3 \max_{i=3,\ldots,l} |c_j, i|\}$ for $j = 1, 2$, this yields

$$
\Lambda\left(\{I_{(1,\ldots,l)}(z + \delta), \delta_1 \in A^{(i)}_m, i = 1, \ldots, l\}\right) \\
\geq \Lambda \left( \left\{ \left( t_{\delta_1, \delta_2}, \frac{z_1 + \delta_1}{f(t_1 - t_{\delta_1, \delta_2})} f \right), |\delta_j| < \frac{C_j}{C_j} 2^{-m(1+\varepsilon)}, j = 1, 2 \right\} \right) \\
= \left(\frac{C'}{C_j} \right) 2^{2-2m(1+\varepsilon)} \cdot P_F(\{\mathcal{f}\}) + o\left(2^{-2m(1+\varepsilon)}\right)
$$

where we use the same argumentation as in the proof of Proposition 4.1. This completes the proof.

**Remark 4.2.** We note that Proposition 4.2 does not provide exact asymptotics which we need to determine $P(\{(t_0, y_0, \mathcal{f}) \in \Pi \mid \Pi \cap I_{(1,\ldots,l)}(z)\} = 1, Z(t) = z)$ for $(t_0, y_0, \mathcal{f}) \in I_{(1,\ldots,l)}(z)$ and $|I_{(1,\ldots,l)}(z)| > 1$. However, we can get exact results by conditioning on $Z(t_i)$ being in intervals of different size for each $i \in \{1, \ldots, l\}$ instead of $Z(t_i) \in A_m(z_i)$ for all $i = 1, \ldots, l$. We will choose these intervals such that some restrictions on the intersection sets vanish asymptotically and we can resort to the results on the intersection of two curves.

The calculations in the proof of Proposition 4.2 yield

$$
|h_1(\delta_1, \delta_2)| \leq (|c_{1, i}| + o(1)) \cdot |\delta_1| + (|c_{2, i}| + o(1)) \cdot |\delta_2| \leq 2^{-m}(|c_{1, i}| + |c_{2, i}| + o(1))
$$

for $(\delta_1, \delta_2) \in A^{(1)}_m \times A^{(2)}_m$. Thus, for any $\varepsilon > 0$, using the same arguments as in the proof of Lemma 3.2, we can replace $m$ by $\lfloor m(1 - \varepsilon) \rfloor$ in Equation (4.8) and get that $h_i(\delta_1, \delta_2) \in A^{(i)}_{m(1-\varepsilon)}$ holds for $|\delta_1| < C' \frac{2^{-2m(1-\varepsilon)}(1+\varepsilon)}{3|c_{1, i}|} \sim 2^{-m} 2^{-m}$ and $|\delta_2| < C' \frac{2^{-2m(1-\varepsilon)}(1+\varepsilon)}{3|c_{2, i}|} \sim 2^{-m} 2^{-m}$ for $m$ large enough. Therefore,

$$
h_i(\delta_1, \delta_2) \subset A^{(i)}_{m(1-\varepsilon)}, \quad i = 3, \ldots, l
$$

for all $\delta_1 \in A^{(1)}_m \subset (-2^{-m}, 2^{-m})$, $\delta_2 \in A^{(2)}_m \subset (-2^{-m}, 2^{-m})$ if $m$ is sufficiently large. This implies

$$
\left\{ I_{(1,\ldots,l)}(z + \delta), \delta_1 \in A^{(1)}_m, \delta_2 \in A^{(2)}_m, \delta_i \in A_{m(1-\varepsilon)}^{(i)}, i = 3, \ldots, l \right\} \\
= \left\{ I_{(1,\ldots,l)}(z + \delta), \delta_1, \delta_2 \in A^{(1)}_m \times A^{(2)}_m, h_i(\delta_1, \delta_2) \in A_{m(1-\varepsilon)}^{(i)}, i = 3, \ldots, l \right\} \\
= \left\{ \left( t_{\delta_1, \delta_2}, \frac{z_1 + \delta_1}{f(t_1 - t_{\delta_1, \delta_2})} f \right), \delta_1 \in A^{(1)}_m, \delta_2 \in A^{(2)}_m \right\}
$$

and, therefore

$$
\Lambda \left( \left\{ I_{(1,\ldots,l)}(z + \delta), \delta_1 \in A^{(1)}_m, \delta_2 \in A^{(2)}_m, \delta_i \in A_{m(1-\varepsilon)}^{(i)}, i = 3, \ldots, l \right\} \right) \\
= \frac{2^{-2m} P_F(\{\mathcal{f}\})}{y_0 |z_1 f(t_2 - t_0) - z_2 f(t_1 - t_0)|} + o(2^{-2m}).
$$
By conditioning on $Z(t) \in A_m(z_1) \times A_m(z_2) \times \ldots \times A_{m(1-\varepsilon)}(z_i)$, for $I_{(1,\ldots,l)}(z) = \{(t_1^{(1)}, y_0^{(1)}, f_1), \ldots, (t_l^{(k)}, y_0^{(k)}, f_k)\}$, $l \geq 3$, we can apply Lévy’s “Upward” Theorem ([10], Thm. 50.3) and end up with

$$P((t_0, y_0, f) \in \Pi \mid \Pi \cap I_{(1,\ldots,l)}(Z(t)) = 1, Z(t) = z) = \lim_{m \to \infty} \frac{1}{P(F(\{f\}) \leq m)} \left( \sum_{i=1}^{k} \frac{P(F(\{f\}) \leq m)}{P(F(\{f\}) \leq m)} \right).$$

(4.9)

Note that Lévy’s “Upward” Theorem implies that, with probability one, the right-hand side of (4.9) does not depend on the choice of labelling.

Thus, despite the lack of exact convergence rates of $\Lambda(I_{(1,\ldots,l)}(z))$, the distribution of $\Pi \cap I_{(1,\ldots,l)}(Z(t)) \mid \Pi \cap I_{(1,\ldots,l)}(Z(t)) = 1$ can be determined exactly.

**Proposition 4.3.** Let $f \in G, t \in \mathbb{R}^n, z > 0$ such that $f$ is continuously differentiable in a neighbourhood of $t_i - t_0$ for all $i \in \{1, \ldots, n\}$ and all $(t_0, y_0, f) \in K_{t_i, z} \cap \bigcup_{i=1}^{n} K_{t_i, z_i}$ involved in an intersection.

We denote the projection of the set $I_{(1)}(z) \cap (\mathbb{R} \times (0, \infty) \times \{f\})$ onto its first component in $\mathbb{R}$ by

$$D_{1}^{(f)} = \{t \in \mathbb{R} : (t, y, f) \in I_{(1)}(z) \text{ for some } y > 0\}.$$

Then, we have

$$\Lambda \left( I_{(1)}^{(m)}(z) \cap (\mathbb{R} \times (0, \infty) \times \{f\}) \right) = 2^{-m} \cdot \int_{D_{1}^{(f)}} \frac{f(t - t_0)}{z_1^2} \, dt + o(2^{-m}).$$

(4.10)

**Proof.** First, we note that by renumbering it suffices to show the result for $i = 1$.

The idea of this proof is to assess the set $D_{1}^{(f)}$ by the sets $D_{1, \min}^{(m)}$ from below and $D_{1, \max}^{(m)}$ from above. Here, $D_{1, \min}^{(m)}$ consists of all first components of $I_{(1)}^{(m)}(z)$ which are not part of any intersections $I_{A}^{(m)}(z)$, $A \supseteq \{1\}$ and $D_{1, \max}^{(m)}$ is the set of the first components of $\bigcup_{A \supseteq \{1\}} I_{A}^{(m)}(z)$. Analogously, $\Lambda \left( I_{(1)}^{(m)}(z) \cap (\mathbb{R} \times (0, \infty) \times \{f\}) \right)$ can be bounded from below and above by replacing $D_{1}^{(f)}$ in (4.10) by the sets $D_{1, \min}^{(m)}$ and $D_{1, \max}^{(m)}$, respectively. We will show that the difference, which consists of blurred intersections $I_{A}^{(m)}(z)$, $A \supseteq \{1\}$, vanishes asymptotically.

Let $A_0^{(m)} = A_m(z_1) - z_i$. Then, for any $\delta \in \mathbb{R}^{n-1}$ we define

$$D_{1, \delta}^{(f)} = \left\{ t \in \mathbb{R} : \left( t, \frac{z_1 + \delta_i}{f(t_1 - t)} \right) \in I_{(1)}(z + \delta) \right\} = \left\{ t \in \mathbb{R} : \frac{z_1 + \delta_i}{f(t_1 - t)} < \min_{i=2,\ldots,n} \frac{z_i + \delta_i}{f(t_i - t)} \right\}.$$

(4.11)
Thus, defining \( D_{1,\min}^{(m)} = \bigcap_{\delta \in \mathbb{R}} D_{1,\delta}^{(f)} \) and \( D_{1,\max}^{(m)} = \bigcup_{\delta \in \mathbb{R}} D_{1,\delta}^{(f)} \), we get
\[
D_{1,\min}^{(m)} \subset D_1^{(f)} \subset D_{1,\max}^{(m)}.
\]

On the other hand, we have
\[
\left\{ (t, y, f) \in \mathbb{R} \times (0, \infty) \times \{ f \} : t \in D_{1,\min}^{(f)}, yf(t_1 - t) \in A_m(z_1) \right\}
\subset I_{(1)}^{(m)}(z) \subset \left\{ (t, y, f) \in \mathbb{R} \times (0, \infty) \times \{ f \} : t \in D_{1,\max}^{(f)}, yf(t_1 - t) \in A_m(z_1) \right\}.
\]

(4.12)

Now, let \( t \in D_{1,\max}^{(m)} \setminus D_{1,\min}^{(m)} \). Then, by definition of \( D_{1,\min}^{(f)} \) and \( D_{1,\max}^{(f)} \), there exist \( \delta^{(1)} , \delta^{(2)} \in \times_{i=1}^{n} A_m^{(i)} \) such that \( t \in D_{1,\delta^{(1)}}^{(f)} \), but \( t \notin D_{1,\delta^{(2)}}^{(f)} \). That is, by Equation (4.11),
\[
\frac{z_1 + \delta^{(1)}}{f(t_1 - t)} < \min_{i=2,\ldots,n} \frac{z_i + \delta^{(1)}}{f(t_i - t)} \quad \text{and} \quad \frac{z_1 + \delta^{(2)}}{f(t_1 - t)} \geq \min_{i=2,\ldots,n} \frac{z_i + \delta^{(2)}}{f(t_i - t)}.
\]

By continuity arguments, a \( \delta \in \times_{i=1}^{n} A_m^{(i)} \) exists such that
\[
\frac{z_1 + \delta}{f(t_1 - t)} = \min_{i=2,\ldots,n} \frac{z_i + \delta}{f(t_i - t)},
\]
i.e. \( t \in T_1^{(m)} = \{ t \in \mathbb{R} : (t, y, f) \in \bigcup_{A_1 \subseteq A} I_A^{(m)}(z) \text{ for some } y > 0 \} \). Thus,
\[
D_{1,\max}^{(m)} \setminus D_{1,\min}^{(m)} \subset T_1^{(m)}.
\]

(4.13)

By definition, \( T_1^{(m)} \) denotes the set of first components involved in any blurred intersection and we have \( T_1^{(m)} \supseteq T_1^{(m)} \), \( T_1^{(m)} = \{ t \in \mathbb{R} : (t, y, f) \in \bigcup_{A_1 \subseteq A} I_A^{(m)}(z) \text{ for some } y > 0 \} \) as \( m \to \infty \) and \( T_1 \) is finite by Assumption (4.1). Therefore, dominated convergence yields
\[
\int_{T_1^{(m)}} f(t_1 - t) \, dt \searrow 0, \quad m \to \infty.
\]

(4.14)

Thus, by Equations (4.12) and (4.13) we get
\[
\Lambda \left( I_{(1)}^{(m)}(z) \right) \Delta \left( \{ (t, y, f) \in \mathbb{R} \times (0, \infty) \times \{ f \} : t \in D_{1,\min}^{(f)}, yf(t_1 - t) \in A_m(z_1) \} \right)
\leq \Lambda \left( \{ (t, y, f) \in \mathbb{R} \times (0, \infty) \times \{ f \} : t \in D_{1,\max}^{(f)}, yf(t_1 - t) \in A_m(z_1) \} \right)
\setminus \left( \{ (t, y, f) \in \mathbb{R} \times (0, \infty) \times \{ f \} : t \in D_{1,\min}^{(f)}, yf(t_1 - t) \in A_m(z_1) \} \right)
\leq \Lambda \left( \{ (t, y, f) \in \mathbb{R} \times (0, \infty) \times \{ f \} : t \in T_1^{(m)}, yf(t_1 - t) \in A_m(z_1) \} \right)
= \operatorname{Pr}(\{ f \}) \int_{T_1^{(m)}}\int_{A_m^{(i)}} \frac{f(t_1 - t)}{(z_1 + \delta_1)^2} \, dt \, d\delta_1 = \operatorname{Pr}(\{ f \}) \int_{A_m^{(i)}} \frac{o(1)}{(z_1 + \delta_1)^2} \, d\delta_1 \in o(2^{-m}).
\]

The last equality follows from Equation (4.14).
By the formulae from Propositions 4.1 and 4.3, we get
\[
\Lambda(I_{1}^{(m)}(z)) = P_F(\{f\}) \int_{D_1^{(m)}} \int_{A_1^{(m)}} \frac{f(t_1-t)}{z_1^2} \, dt \, \delta_1 \, dz + o(2^{-m})
\]
\[= 2^{-m} \cdot P_F(\{f\}) \left( \int_{D_1^{(m)}} \frac{f(t_1-t)}{z_1^2} \, dt \right) + o(1)
\]
which completes the proof.

Now, we can use Theorem 3.1 in order to compute the conditional probabilities
\[
P(E_{(n,A)}(Z(t)) \mid Z(t) = z) = \lim_{m \to \infty} P(E_{(n,A)}^{(m)}(z) \mid Z(t) \in A_m(z))
\]
\[= \lim_{m \to \infty} \frac{P(E_{(n,A)}^{(m)}(z))}{\sum_{\tilde{n},A} P(E_{(\tilde{n},A)}^{(m)}(z))}
\]
where \(N_0 = \{A : A \in 2^{\{1, \ldots, n\}} \setminus \emptyset : \sum_{A: i \in A} n_A = 1, i = 1, \ldots, n\}\).

As the sets \(I_{A}^{(m)}(z)\), \(A \in 2^{\{1, \ldots, n\}} \setminus \emptyset\), are pairwise disjoint and the sequences \(\Lambda(I_{A}^{(m)}(z))\) tend to zero for \(m \to \infty\) by Lemma 3.1, we get
\[
P(E_{(n,A)}^{(m)}(z)) \sim \exp(-\Lambda(K(z, z))) \prod_{A: n_A = 1} \Lambda(I_A(z)).
\]
(4.16)

Considering (4.15), we can restrict ourselves to those scenarios with the slowest rate of convergence to zero. Propositions 4.1, 4.2 and 4.3 yield that scenarios involving intersections of at least three sets are always of a dominating order. Therefore, the unknown terms from Proposition 4.2 are cancelled out.

**Example 4.1.** Let \(F(x) = f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\) a.s. Furthermore, let \(Z(t_1) = Z(t_2) = a\) where \(t_1 = -b\) and \(t_2 = b\) for constants \(a, b > 0\). Then, we get
\[
I_{(1)}(a, a) = \left\{ (t, a\sqrt{2\pi} \exp\left(\frac{(-b-t)^2}{2}\right), f) : t < 0 \right\}
\]
\[
I_{(2)}(a, a) = \left\{ (t, a\sqrt{2\pi} \exp\left(\frac{(b-t)^2}{2}\right), f) : t > 0 \right\}
\]
\[
I_{(1,2)}(a, a) = \left\{ (0, a\sqrt{2\pi} \exp\left(\frac{b^2}{2}\right), f) \right\}.
\]

By the formulae from Propositions 4.1 and 4.3, we get
\[
\Lambda(I_{(1)}^{(m)}(a, a)) = \frac{2^{-m}}{a^2} \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(-b-t)^2}{2}\right) \, dt + o(2^{-m}) \sim \frac{2^{-m}}{a^2} \Phi(b)
\]
(4.17)
\[
\Lambda(I_{(2)}^{(m)}(a, a)) = \frac{2^{-m}}{a^2} \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(b-t)^2}{2}\right) \, dt + o(2^{-m}) \sim \frac{2^{-m}}{a^2} \Phi(b)
\]
(4.18)
\[
\Lambda(I_{(1,2)}^{(m)}(a, a)) = \frac{2^{-2m}}{a^4} \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(b-t)^2}{2}\right) \, dt + o(2^{-2m}) \sim \frac{2^{-2m}}{a^4 \Phi(b)}
\]
(4.19)
Then, for any fixed set \( \{A\} \), we get that for any \( b \) we only consider those realisations of \( Z \) which yield

\[
\Phi(b)^2 + a(2b\sqrt{2\pi} \exp(\frac{b^2}{2}))^{-1}.
\]

Note that the probability that the observations are generated by two different points of \( \Pi \) increases as \( a \) gets smaller. This is due to the fact that \( \Pi \) gets more intense as the second component decreases.

Using the formulae above, the limits of the conditional probabilities can always be calculated explicitly except for those cases where two scenarios exist, both involving different terms which cannot be determined exactly (cf. Proposition 4.2). This may happen only if two sets \( A_1 = \{i_1, \ldots, i_r\} \) and \( A_2 = \{j_1, \ldots, j_s\} \), \( r, s \geq 3 \), \( A_1 \cap A_2 \neq \emptyset \), exist such that

\[
J_{A_1}(Z(t)) \neq \emptyset, \ J_{A_2}(Z(t)) \neq \emptyset \text{ and } J_{A_1 \cup A_2}(Z(t)) = \emptyset,
\]

where

\[
J_A(Z(t)) = \bigcup_{B \supset A} \bigcap_{i \in A} K_{i, Z(t_i)}, \quad A \in 2^{\{1, \ldots, n\}} \setminus \emptyset.
\]

In all other cases the terms as in Proposition 4.2 are cancelled out. Note that we work with sets of the type \( J_A(Z(t)) \) in order to avoid case-by-case analysis for all the sets \( I_B(Z(t)) \) with \( B \supset A \).

**Lemma 4.1.** Let \( G \) consist of functions which are continuously differentiable a.e. Then, for any fixed set \( \{t_1, \ldots, t_n\} \subset \mathbb{R} \) we have

\[
P(Z(t) \text{ satisfies (4.20)}) = 0.
\]

**Proof.** We proof that condition (4.20) has probability 0 for all fixed index sets \( A_1, A_2 \subset \{1, \ldots, n\} \). By renumbering, we may assume \( A_1 = \{1, \ldots, r\} \) and \( A_2 = \{q, \ldots, q + s - 1\} \) with \( q \leq r \). Assume that \( P(Z(t) \text{ satisfies (4.20)}) > 0 \). In a first step we only consider those realisations of \( Z(t_1), \ldots, Z(t_r) \) with \( J_{A_1}(Z(t_1), \ldots, Z(t_r)) \neq \emptyset \). Then, by the calculations in Propositions 4.1, 4.2 and 4.3 we get that

\[
P(|\Pi \cap \bigcup_{A_1} |Z_{1}|, \ldots, Z_{r})| = 1 \notin O(2^{-m(1+\varepsilon)})
\]

for any \( \varepsilon > 0 \) and

\[
P(|\Pi \cap \bigcup_{B_1} |Z_{1}|, \ldots, Z_{r})| = 1, |\Pi \cap \bigcup_{B_2} |Z_{1}|, \ldots, Z_{r})| = 1 \notin O(2^{-3m})
\]

for any \( B_1, B_2 \subset A_1 \), \( B_1 \cap B_2 = \emptyset \). This yields \( |\Pi \cap J_{A_1}(Z(t_1), \ldots, Z(t_r))| = 1 \) almost surely.
Similarly, for almost every \( Z_q, \ldots, Z_{q+s-1} \) such that \( J_{A_2}(Z(t_q), \ldots, Z(t_{q+s-1})) \neq \emptyset \), we have \( |\Pi \cap J_{A_2}(Z(t_q), \ldots, Z(t_{q+s-1}))| = 1 \). As

\[
\{ \omega : Z(t) \text{ satisfies (4.20)} \} 
\subset \{ \omega : J_{A_1}(Z(t_1), \ldots, Z(t_r)) \neq \emptyset \} \cap \{ \omega : J_{A_2}(Z(t_q), \ldots, Z(t_{q+s-1})) \neq \emptyset \},
\]

we have \( |\Pi \cap J_{A_1}(Z(t_1), \ldots, Z(t_r))| = 1 \) and \( |\Pi \cap J_{A_2}(Z(t_q), \ldots, Z(t_{q+s-1}))| = 1 \) for \( Z(t) \) satisfying (4.20) almost surely. Therefore, we get \( P(|\Pi \cap K(t_i, Z(t))| \geq 2) > 0 \) for every \( i \in A_1 \cap A_2 \) since \( J_{A_1 \cup A_2}(Z(t)) = \emptyset \). This is a contradiction to Proposition 3.1 which completes the proof.

From the considerations above and Lemma 4.1 we immediately derive the following result.

**Theorem 4.1.** Let \( G \) consist of functions which are continuously differentiable a.e. Then, with probability one,

\[
P(E_{\{n_A\}}(Z(t)) \mid Z(t) = z) = \lim_{m \to \infty} P(E_{\{m\}}^{(m)}(Z(t)) \mid Z(t) \in A_m(z))
\]

can be calculated explicitly via the formulae given in Propositions 4.1, 4.2, 4.3 and Remark 4.2.

**Remark 4.3.** We may also consider the case that \( G \) is countable. However, to transfer the results of the finite case, we have to ensure uniform convergence of the blurred intersection sets which is needed to compute \( \sum_{n_A, i \in A} P(E_{\{n_A\}}(z)) \) in the denominator of Equation (4.15). To this end, we have to impose some additional conditions. For example, we could assume that for almost every \( z > 0 \) there is only a finite number of shape functions involved in the intersection sets \( I_A(z) \), \( |A| \geq 2 \).

We are still left with simulating \( \Pi_2 \mid Z(t) = z \) given the occurrence of a scenario \( E_{\{n_A\}}(z) \) with \( \sum_{n_A, i \in A} n_A = 1 \) for all \( i \in \{1, \ldots, n\} \), that is, we are interested in

\[
P \left( \bigcap_{\lambda : n_A = 1} \{ \Pi_2 \cap (C_A \times (0, \infty) \times \{ f \}) \mid E_{\{n_A\}}(z) \right)
\]

for \( C_A \subset \mathbb{R}, f \in G \) with \( (C_A \times (0, \infty) \times \{ f \}) \cap I_A(z) \neq \emptyset \). Using Theorem 3.3 with sets \( B_A = C_A \times (0, \infty) \times \{ f \}, A \in 2^{\{1, \ldots, n\}} \setminus \emptyset \) we get that

\[
P \left( \bigcap_{\lambda : n_A = 1} \{ \Pi_2 \cap I_A(z) \cap B_A \mid E_{\{n_A\}}(z) \right) = \lim_{m \to \infty} \prod_{A : n_A = 1} \frac{\Lambda(I_A^{(m)}(z) \cap B_A)}{\Lambda(I_A^{(m)}(z))}
\]

Thus, each random vector \( (T_A, F_A) \in \mathbb{R} \times G, A \in 2^{\{1, \ldots, n\}} \setminus \emptyset \), which is defined by \( \Pi_2 \cap (\{T_{A} \times (0, \infty) \times \{ F_{A} \}) \cap I_A(Z(t)) = 1 \) if \( |\Pi_2 \cap I_A(Z(t))| = 1 \), can be simulated independently.

The distribution of \( (T_A, F_A) \) depends on the cardinal number of \( A \). If \( A = \{i\} \) for some \( i \in \{1, \ldots, n\} \), we have

\[
P(T_A \in B, F_A = f) = \frac{P_F(\{f\}) \int_{D_{\{i\} \cap B} f(t_i - t) \, dt}{\sum_{g \in G} P_F(\{g\}) \int_{D_{\{i\} \cap B} g(t_i - t) \, dt}}, \quad B \in \mathcal{B}.
\]
For $|A| \geq 2$, let $I_A(z) = \{(t^{(1)}_0, y^{(1)}_0, f_1), \ldots, (t^{(b)}_0, y^{(b)}_0, f_k)\}$. Then, we get
\[
P(T_A = t^{(j)}_0, F_A = f_j) = \frac{P_F(x_j)}{\sum_{j'=1}^k \frac{P_F(x_{j'j})}{(y^{(b)}_0)^2 |z_1 f_j'(t_2-t^{(j')}_0) - z_2 f_j'(t_1-t^{(j')}_0)|}}, \quad j = 1, \ldots, k.
\]

Thus, we end up with the following procedure for calculating the conditional distribution of $Z(t_0)$ given $Z(t)$ with $t_0, t_1, \ldots, t_n \in \mathbb{R}$, $z > 0$.

1. Compute the conditional probabilities for all the scenarios $E_{(n_A)}(z)$ and generate a random scenario following this distribution.

2. For a given scenario $E_{(n_A)}(z)$ set $\Pi_2 = \{(T_A, \min_{l=1}^n \frac{z_l}{P_A(t_{l-A})}, F_A) : n_A = 1\}$ where the law of $(T_A, F_A)$ is given above.

3. Independently, sample from $\Pi_3 = \Pi \cap (\mathbb{R} \times (0, \infty) \times G) \setminus (K(t, z) \cup K(t, z))$.
Then, $Z(t_0) = \max_{(s, u, f) \in \Pi_2 \cup \Pi_3} uf(t_0 - s)$.

In the next section, we will point out the capability of this exact approach by comparing it to other algorithms in the simple case of a deterministic shape function which is continuously differentiable.

5. Comparison with the algorithm for the max-linear model and transformation to Gaussian marginals

Recently, Wang and Stoev \cite{22} proposed an algorithm for exact and efficient conditional sampling for max-linear models
\[Z(t_i) = \max_{j=1, \ldots, p} a_{ij} Y_j, \quad i = 0, \ldots, n,\]
where $Y_j$, $j = 1, \ldots, p$, are independent standard Fréchet random variables.

Rewriting $Z$ from \cite{1,15} as an extremal integral (see \cite{20})
\[Z(t) = \sum_{f \in G} P_F(x_j) \int_{\mathbb{R}} f(t - u) M_1(du), \quad t \in \mathbb{R},\]
where $M_1$ is a random sup-measure on $\mathbb{R}$ w.r.t. the Lebesgue measure, we can see that $Z$ can be approximated arbitrarily well by a max-linear model, e.g. by
\[Z_{M,h}(t) = h \max_{l=-M, \ldots, M-1} P_F(x_j) \cdot f_j \left(t - \left(l + \frac{1}{2}\right)h\right) \cdot Y_{j,l}, \quad M \in \mathbb{N}, h > 0,\]
where $G = \{f_1, \ldots, f_k\}$. Then, we have $Z_{M,h}(t) \xrightarrow{h \to 0} Z(t)$ for any $t \in \mathbb{R}$ as $M \to \infty$.

We also consider another approach based on the assumption of a multi-gaussian model (cf. \cite{2}, p. 381). The data are transformed such that the marginal distribution is
Gaussian. As the marginals of $Z$ are known to be standard Fréchet, the corresponding transformation is given by

$$\Psi : (0, \infty) \to \mathbb{R}, \; x \mapsto \Phi^{-1}(\Phi_1(x)),$$

where $\Phi$ is the standard normal distribution function and $\Phi_1 = \exp(-1/x)$ is the standard Fréchet distribution function. The transformed random field $Y = \Psi(Z)$ is stationary and second-order. As the covariance function $C$ of $Y$ can hardly be computed for general shape functions $f_1, \ldots, f_k$, we estimate it using maximum likelihood techniques, for instance, from a large parametric class like the Whittle-Matérn class, i.e.

$$C_{\nu,c}(h) = \frac{(c||h||)^\nu}{2^{\nu-1} \Gamma(\nu)} K_\nu(c||h||), \quad \nu, \; c > 0,$$

assuming that $Y$ is a Gaussian random field. Under this assumption, the conditional distribution can be sampled easily (see [14], for instance). Afterwards, the sample has to be retransformed via

$$\Psi^{-1} : \mathbb{R} \to (0, \infty), \; y \mapsto \Phi^{-1}(\Phi(y)).$$

To compare these different methods, we need a measure for the goodness-of-fit of a distribution. Here, we use the continuous ranked probability score (CRPS) which is defined as [10]

$$CRPS(F_1, x) = -\int_{-\infty}^{\infty} (F_1(y) - 1_{\{y \geq x\}})^2 \, dy,$$

where $F_1$ is a cumulative distribution function and $x \in \mathbb{R}$. Note that $CRPS(F_1, F_2) := \int CRPS(F_1, x)F_2(dx)$ is a strictly proper scoring rule, i.e.

$$CRPS(F_2, F_2) \geq CRPS(F_1, F_2)$$

for all cumulative distribution functions $F_1, F_2$. If $F_1$ and $F_2$ both belong to measures with finite first moment, equality holds if and only if $F_1 = F_2$. Assuming that $F_1$ has a finite first moment, the CRPS can be calculated via

$$CRPS(F_1, x) = \frac{1}{2} E_{F_1} |X - X'| - E_{F_1} |X - x|, \quad (5.1)$$

which shows $CRPS(F_1, F_1) = -\frac{1}{2} E_{F_1} |X - X'| \leq 0$. Here, $X, X' \sim F_1$ are independent random variables.

In order to compare different algorithms for getting a realisation from the conditional distribution $Z(t_0) \mid Z(t)$, we consider $K$ samples $Z_1, \ldots, Z_K$ of the random field $Z$. For each method $m$, we get an empirical distribution function $F_1^{(m)}$ as the (approximated) conditional distribution of $\log(Z_i(t_0)) \mid Z_i(t)$, $i = 1, \ldots, K$, and calculate $CRPS(F_1^{(m)} \mid \log(Z_i(t_0)))$ via (5.1). Here, we do the log-transformation to Gumbel marginals to ensure that the conditional distribution has finite expectation.

Then, a measure for the goodness-of-fit is given by the mean score [10]

$$CRPS_{K,m} = \frac{1}{K} \sum_{i=1}^{K} CRPS(F_1^{(m)} \mid \log(Z_i(t_0))).$$
Furthermore, we have a look at the mean absolute error of the logarithmic conditional median

\[ MAE_{K,m} = \frac{1}{K} \sum_{i=1}^{K} \left| \log \left( \left( F_i^{(m)} \right)^{-1}(0.5) \right) - \log(Z_i(t_0)) \right| . \]

Here, for computational reasons, we choose the Gaussian extreme value process [19] which has the deterministic shape function

\[ f(x) = \phi(x) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} x^2 \right) . \]

Furthermore, let \( n = 4, t = (-2, -1, 1, 2) \) and \( t_0 = 0 \). Figure 3 shows two realisations of \( Z(\cdot) \); the first one is sampled unconditionally and the second one is based on conditional sampling of the first one.

The conditional distribution is calculated based on a sample of size 100 simulated in R [11]. The performance is measured via \( CRPS_{K,1} / MAE_{K,1} \) (conditional sampling via the Poisson point process), \( CRPS_{K,2} / MAE_{K,2} \) (conditional sampling for a max-linear model with \( M = 5 \) and \( h = 0.1 \) using the R package maxLinear [21]) and \( CRPS_{K,3} / MAE_{K,3} \) (conditional sampling via transformation to Gaussian marginals) with \( K = 1000 \) samples. As already mentioned, the latter approach requires the knowledge of the covariance structure of the transformed random field. This is assessed by first simulating data from this model on a dense grid repeatedly and then estimating the parameters of a Whittle-Matérn covariance model based on maximum likelihood techniques implemented in the R package RandomFields [18]. The parameters are chosen such that the first and second method have a similar running time, the last method runs much faster.

The results of the simulation study are shown in Table II. Here, \( CRPS_{K,1} \) and \( MAE_{K,1} \) can be interpreted as reference values as the first method is exact.
Table 1: Results of the simulation study for \( f(x) = \varphi(x) \) and \( K = 100 \).

| \( m = 1 \) | \( m = 2 \) | \( m = 3 \) |
|-----------|-----------|-----------|
| CRPS \(_{K,m}\) | -0.149    | -0.337    | -0.268    |
| MAE \(_{K,m}\)   | 0.217     | 0.476     | 0.357     |

Note that conditional sampling for max-linear models performs worse than conditional sampling via transformation to Gaussian marginals.

For further analysis and comparison of these methods we do not restrict ourselves to pointwise prediction, but have a look at the sample paths. Additionally, pointwise quantile estimation of the conditional distribution can be done including the special case of the conditional median which can be seen as an analogon to Kriging. In case of conditional sampling via the Poisson point process and conditional sampling of a max-linear model the quantiles have to be estimated from the empirical conditional distribution. For sampling via Gaussian transformation the quantiles can be calculated directly by the means of Kriging result and variance.

Figure 4 shows five sample paths and the median of the Gaussian extreme value process conditional on

a. observations at four locations \(-2, -1, 1, 2\),

b. observations at eleven locations \(-2.5, -2, \ldots , 2, 2.5\).

In general, conditional simulation via the Poisson point process yields sample paths which capture the main features of the process quite well. Even in the case of four observations parts of the sample path are reconstructed exactly with a positive probability. For eleven observations most of the sample path is restored with high probability.

The results of conditional sampling of the max-linear model are similar to the first method in case of four observations. For eleven observations, however, the method fails because of model misspecification. As the data to not match the discretized model, some observations cannot be reconstructed. For some realisations of the Gaussian extreme value process this problem even occurs in case of four observations. This is the main reason for the bad results of this method in the simulation study above.

Gaussian transformation yields conditional sample paths which are structurally very different from the true ones. However, for eleven observations the deviations from the original sample path are quite small.

6. Approximation in the case of an infinite number of shape functions

Here, we drop the assumption that \( G \) is finite. Note that the measurability of \( \Pi_2 \) and \( \Pi_3 \) is still an open question if \( G \) is uncountable. We present an approximation of the distribution of \( Z(t_0) \) given \( Z(t) \) based on a finite number of shape functions.

Let \( F_1, F_2, \ldots \) be independent copies of \( F \) where \( F \) is defined as in Section 1. Then, given \( F_1, \ldots , F_N \), we define

\[
Z_N(t) = \max_{(s,u,f) \in \Pi^{(N)}} uf(t - s), \quad t \in \mathbb{R}^d,
\] (6.1)
Figure 4: Comparison of the Gaussian extreme value process with different types of conditional simulations: a. simulations conditional on four observations at $-2$, $-1$, $1$, $2$, b. simulations conditional on eleven observations at $-2.5$, $-2$, $-1$, $1$, $2$, $2.5$. In both cases the original Gaussian extreme value process (top left), conditional samples via the Poisson point process (top right) and conditional results for a max-linear approximation (bottom left) and an approximation by Gaussian transformation (bottom right) are shown. Black crosses: observations, grey lines: conditional sample paths, black line: conditional mean.
where $\Pi^{(N)}$ is a Poisson point process on $\mathbb{R}^d \times (0, \infty) \times \{F_1, \ldots, F_N\}$ with intensity measure

$$
\Lambda(A \times B \times \{F_k\}) = \frac{1}{N} \int_A \int_B \frac{1}{u^2} \text{d}u \text{d}s, \quad A \in \mathcal{B}^d, \ B \in \mathcal{B} \cap (0, \infty), \ k \in \{1, \ldots, n\}.
$$

**Theorem 6.1.** For any $z > 0$ it holds

$$
\Pi^{(N)} \mid Z(t) \leq z \xrightarrow{D} \Pi \mid Z(t) \leq z
$$
as $N \to \infty$. In particular, $Z_N(t_0) \mid Z_N(t) \leq z \xrightarrow{D} Z(t_0) \mid Z(t) \leq z$.

**Proof.** We note that it suffices to show

$$
\lim_{N \to \infty} P(|\Pi^{(N)} \cap M_j| = n_j, \ j = 1, \ldots, l \mid Z_N(t) \leq z) = P(|\Pi \cap M_j| = n_j, \ j = 1, \ldots, l \mid Z(t) \leq z)
$$

for $z > 0$ and sets $M_j = (a_j, b_j) \times (c_j, \infty) \times G_j \subset \mathbb{R}^d \times (0, \infty) \times G$ which are pairwise disjoint. First, we consider the distributions of $Z$ and $Z_N$, respectively. It holds that

$$
P(Z(t) \leq z) = \exp(-EH),
$$

where

$$
H = \int_{\mathbb{R}^d} \max_{i=1,\ldots,n} \frac{F(t_i-t)}{z_i} \text{d}t.
$$

We define $H_1, \ldots, H_N$ replacing $F$ by $F_1, \ldots, F_N$, respectively. Then, we have

$$
P(Z_N(t) \leq z) = E(P(Z_N(t) \leq z \mid F_1, \ldots, F_N)) = E \left( \exp \left( -\frac{1}{N} \sum_{k=1}^{N} H_k \right) \right).
$$

As the sets $M_1, \ldots, M_l$ are pairwise disjoint, the common distribution of $\Pi$ and $Z(t)$ can be rewritten as

$$
P(|\Pi \cap M_j| = n_j, \ j = 1, \ldots, l, \ Z(t) \leq z)
$$

$$
= P(Z(t) \leq z) \cdot \prod_{j=1}^{l} P \left( |\Pi \cap M_j \cap \{ (x, y, f) \in S : y \leq \min_{i=1,\ldots,n} \frac{z_i}{f(t_i-x)} \} | = n_j \right)
$$

$$
= P(Z(t) \leq z) \cdot \prod_{j=1}^{l} P \left( |\Pi \cap \{ (x, y, f) \in (a_j, b_j) \times (c_j, \infty) \times G_j : y \leq \min_{i=1,\ldots,n} \frac{z_i}{f(t_i-x)} \} | = n_j \right).
$$

Thus, we get

$$
P(|\Pi \cap M_j| = n_j, \ j = 1, \ldots, l, \ Z(t) \leq z)
$$

$$
= \exp(-EH) \prod_{j=1}^{l} \frac{(EH^{(j)})^{n_j}}{n_j!} \exp(-EH^{(j)}),
$$

where

$$
H^{(j)} = \int_{a_j}^{b_j} \left( \frac{1}{c_j} - \max_{i=1,\ldots,n} \frac{F(t_i-t)}{z_i} \right) \lor 0 \right) \mathbf{1}_{\{f \in G_j\}} \text{d}t, \quad j = 1, \ldots, l.
$$
Analogously, we can define $H_k^{(j)}$ replacing $F$ by $F_k$ in the definition of $H^{(j)}$ for $j = 1, \ldots, l, k = 1, \ldots, N$. This yields
\[
P(\Pi^N \cap M_j = n_j, j = 1, \ldots, l, Z(t) \leq z)
= E \left( \exp \left( -\frac{1}{N} \sum_{k=1}^{N} H_k \right) \prod_{j=1}^{l} \frac{1}{n_j} \left( \frac{1}{N} \sum_{k=1}^{N} H_k^{(j)} \right)^{n_j} \exp \left( -\frac{1}{N} \sum_{k=1}^{N} H_k^{(j)} \right) \right).
\]
As $H_1, \ldots, H_N \sim i.i.d. H$ and $H_1^{(j)}, \ldots, H_N^{(j)} \sim i.i.d. H^{(j)}$ for $j = 1, \ldots, l$ with $E|H| \leq \sum_{i=1}^{n} \frac{1}{z^i} < \infty$ and $H^{(j)} < \frac{\Pi_{(i-1,b_j-a)}^{(j)}}{\delta^j}$, the Strong Law of Large Numbers gives $\frac{1}{N} \sum_{k=1}^{N} H_k \rightarrow E H$ and $\frac{1}{N} \sum_{k=1}^{N} H_k^{(j)} \rightarrow E H^{(j)}$ a.s. as $N \rightarrow \infty$. Thus, dominated convergence yields (6.2) and $P(Z_N(t) \leq z) \rightarrow P(Z(t) \leq z)$, which verifies the first assertion.

The second assertion follows immediately by rewriting
\[
\{Z(t_0) \leq z\} = \{|\Pi \cap \{(x,y,f) \in S : yf(t_0-x) > z\}| = 0\}.
\]
If $G$ is countable, we have
\[
\lim_{N \rightarrow \infty} \Pi_2^{(N)} \mid Z(t) = z \overset{D}{=} \lim_{N \rightarrow \infty} (\Pi_N \setminus (K_{t,z} \cup K_{t,x}))
\overset{D}{=} \Pi \setminus (K_{t,z} \cup K_{t,x}) \overset{D}{=} \Pi_3 \mid (Z(t) = z),
\]
where we used the first part of Theorem 6.1 and applied the second part of Theorem 3.2 to the processes $Z$ and $Z_N$. This motivates to improve the approximation $\Pi \approx \Pi_2^{(N)}$ by $\Pi \approx \Pi_2^{(N)} \cup \Pi_3$, i.e. by the following procedure:

1. Simulate $\Pi_2^{(N)} \mid Z(t) = z$.

2. Independently of $\Pi_2^{(N)}$, sample $\Pi_3 \mid Z(t) = z$ which is defined as $\Pi \cap (\mathbb{R} \times (0, \infty) \times G) \setminus (K_{t,z} \cup K_{t,x})$ analogously to the second part of Theorem 3.2.

Then, $Z(t_0) \approx \max_{(s,u,f) \in \Pi \setminus \Pi_3} uf(t_0-s)$.

7. Application to the Brown-Resnick process

We will apply the method of conditional sampling via the Poisson point process to the Brown-Resnick process [1].

Let $\{W_x(t), t \in \mathbb{R}\}$, $x \in (0, \infty)$, be independent copies of a standard Brownian motion and — independently of the $W_x$’s — let $\Pi$ be a Poisson point process on $(0, \infty)$ with intensity measure $u^{-2}du$. Then,
\[
Z(t) = \max_{u \in \mathbb{R}} \left( u \exp \left( W_u(t) - \frac{|t|}{2} \right) \right), \quad t \in \mathbb{R},
\]
defines a stationary max-stable process with standard Fréchet margins.

Recently, this process was generalized [12] and its mixed moving maxima representation (1.1) was given explicitly [9].
This is,
\[ Z(t) \overset{d}{=} \max_{(s,u,f) \in \Pi} u \cdot \frac{1}{2} \exp(-f(t-s)), \quad t \in \mathbb{R}, \quad (7.1) \]
where \( \Pi \) is a Poisson point process on \( \mathbb{R} \times (0, \infty) \times C(\mathbb{R}) \) with intensity measure \( ds \, du \, P(R(df)) \) and \( P_R \) is the law of the process
\[ R(t) = 1_{t<0}R_1(-t) + 1_{t\geq 0}R_2(t). \]
Here, \( \{R_1(t), \ t \geq 0\} \) and \( \{R_2(t), \ t > 0\} \) are independent Bessel processes of a three-dimensional Brownian motion with drift \( \frac{1}{2} \) in its first component (cf. [15]), i.e.
\[ R_1(t) \overset{d}{=} R_2(t) \overset{d}{=} \sqrt{(W_1(t) + |t|/2)^2 + W_2(t)^2 + W_3(t)^2}, \]
where \( W_1, W_2 \) and \( W_3 \) are independent standard Brownian motions.

We will use the results obtained in the section above to sample from the conditional distribution of the Brown-Resnick process. However, the sample paths of \( \exp(-R(\cdot)) \) do not satisfy the assumptions of Propositions 4.1, 4.2 and 4.3. In particular, the sample paths are not continuously differentiable almost everywhere. To overcome this drawback, we do not use the exact sample paths \( F(\cdot) = \exp(-R(\cdot)) \), but the sample paths evaluated on a grid and interpolated linearly in between.

We show that this procedure is correct in the limit. Let \( T = \{t_z, \ z \in \mathbb{Z}\} \subset \mathbb{R} \) with \( \ldots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \ldots \) such that \( \lim_{z \to -\infty} t_z = -\infty, \lim_{z \to \infty} t_z = \infty \) and \( ||T|| = \sup_{z \in \mathbb{Z}} (t_2 - t_{-1}) \). Let \( \{(t,F_T(t)), \ t \in \mathbb{R}\} \) be the polygonal line through the points \( \{(t,F(t)), \ t \in T\} \). Furthermore, define \( Z_T(t) \) as in (7.1), replacing \( F \) by \( F_T \).

**Proposition 7.1.** For \( ||T|| \to 0 \) we have \( Z_T(t) \to Z(t) \) in probability for all \( t \in \mathbb{R} \). In particular,
\[ Z_T(t_0) \mid (Z_T(t) \in B) \overset{d}{\to} Z(t_0) \mid (Z(t) \in B) \]
for all Borel sets \( B \subset \mathbb{R}^n \) with \( P(Z(t) \in B) > 0 \) and \( P(Z(t) \in \partial B) = 0 \).

**Proof.** W.l.o.g. we assume \( ||T|| \leq \frac{1}{2} \). Then, we have
\[
\int_{\mathbb{R}} \frac{1}{2} F_T(t) \, dt \leq \sum_{z \in \mathbb{Z}} \sup_{t \in [z-1/2, z+1/2]} \frac{1}{2} F_T(t)
\leq \sum_{z \in \mathbb{Z}} \left( \sup_{t \in [z-1,z]} \frac{1}{2} F(t) + \sup_{t \in [z,z+1]} \frac{1}{2} F(t) \right)
= \sum_{z \in \mathbb{Z}} \sup_{t \in [0,1]} F(z+t) = \sum_{z \in \mathbb{Z}} \exp \left( - \inf_{t \in [0,1]} R(z+t) \right).
\]
We show that the rhs is $L^1$-integrable. To this end, we assess

$$
E \left( \sum_{z \in \mathbb{Z}} \exp \left( - \inf_{t \in [0,1]} R(z + t) \right) \right)
$$

$$
\leq 2 \limsup_{n \to \infty} \sum_{k=0}^{n} E \left( \exp \left( - \inf_{t \in [0,1]} R(k + t) \right) \right)
$$

$$
\leq 2 \limsup_{n \to \infty} \sum_{k=0}^{n} E \left( \exp \left( - \inf_{t \in [0,1]} W(k + t) + \frac{k + t}{2} \right) \right)
$$

$$
\leq 2 \limsup_{n \to \infty} \sum_{k=0}^{n} E \left( \exp \left( - \sup_{t \in [0,1]} |W(k) + \frac{k}{2}| + \sup_{t \in [0,1]} |W(k + t) - W(k) + \frac{t}{2}| \right) \right)
$$

$$
\leq 2 E \left( \sup_{t \in [0,1]} |W(t) + \frac{t}{2}| \right) \cdot \limsup_{n \to \infty} \left\{ 1 + \sum_{k=1}^{n} E \left( \exp \left( - \sup_{t \in [0,1]} |W(k) + \frac{k}{2}| \right) \right) \right\}.
$$

where $W$ is a standard Brownian motion.

By [13] there exists some $\varepsilon > 0$ such that

$$
E \left\{ \exp \left( \varepsilon \left( \sup_{t \in [0,1]} |W(t) + \frac{t}{2}| \right)^2 \right) \right\} < \infty.
$$

Since $|X| \leq \frac{1}{2} + \varepsilon X^2$, this implies that $E(\exp(\sup_{t \in [0,1]} |W(t) + \frac{t}{2}|))$ is finite.

Furthermore, using that $\int_{x_0}^{\infty} \exp \left( - \frac{x^2}{2} \right) \, dx < \frac{1}{x_0} \exp \left( - \frac{x_0^2}{2} \right)$ for all $x_0 > 0$, we have

$$
E \left( \exp \left( - \left| W(k) + \frac{k}{2} \right| \right) \right)
$$

$$
= \int_{-k/2}^{\infty} e^{-x-k/2} \frac{1}{\sqrt{2\pi k}} \exp \left( - \frac{x^2}{2k} \right) \, dx + \int_{-\infty}^{-k/2} e^{x+k/2} \frac{1}{\sqrt{2\pi k}} \exp \left( - \frac{x^2}{2k} \right) \, dx
$$

$$
= \frac{1}{\sqrt{2\pi k}} \left( \int_{-k/2}^{\infty} \exp \left( - \frac{(x+k)^2}{2k} \right) \, dx + \exp(k) \int_{-\infty}^{-k/2} \exp \left( - \frac{(x-k)^2}{2k} \right) \, dx \right)
$$

$$
= \frac{1}{\sqrt{2\pi k}} \left( \int_{-\frac{k}{2}}^{\infty} \exp \left( - \frac{x^2}{2} \right) \, dx + \exp(k) \int_{-\frac{3k}{2}}^{-\frac{k}{2}} \exp \left( - \frac{x^2}{2} \right) \, dx \right)
$$

$$
\leq \frac{2}{\sqrt{2\pi k}} \exp \left( - \frac{k}{8} \right) + \exp(k) \frac{2}{3\sqrt{2\pi k}} \exp \left( - \frac{9k}{8} \right) = \frac{8}{3\sqrt{2\pi k}} \exp \left( - \frac{k}{8} \right).
$$

All in all, we get

$$
E \left( \sum_{z \in \mathbb{Z}} \exp \left( - \inf_{t \in [0,1]} R(z + t) \right) \right) \leq 2C \limsup_{n \to \infty} \left( 1 + \sum_{k=1}^{n} \exp \left( - \frac{k}{8} \right) \right) < \infty.
$$
As we have an integrable majorizing random variable, dominated convergence yields

$$\lim_{||T|| \to 0} \mathbb{E} \left( \int_{\mathbb{R}} \left| \frac{1}{2} F_T(s - t) - \frac{1}{2} F(s - t) \right| \, dt \right) = \mathbb{E} \left( \lim_{||T|| \to 0} \int_{\mathbb{R}} \left| \frac{1}{2} F_T(s - t) - \frac{1}{2} F(s - t) \right| \, dt \right) = 0, \quad s \in \mathbb{R}, \quad (7.2)$$

as \( F \) is continuous and integrable.

By Theorem 2.1 in [20] Equation (7.2) is equivalent to \( Z_T(s) \xrightarrow{\mathcal{L}} Z(s) \) for all \( s \in \mathbb{R} \). As

$$P(||Z_t(s_1), \ldots, Z_T(s_m)|| - (Z(s_1), \ldots, Z(s_m))) > \varepsilon) \leq \sum_{i=1}^{m} P(|Z_T(s_i) - Z(s_i)| > \varepsilon),$$

we get that all finite-dimensional marginal distributions of \( Z_T \) converge in probability and therefore also weakly. Thus, for Borel sets \( A \subset \mathbb{R}, B \subset \mathbb{R}^n \) with \( P(Z(t) \in B) > 0 \) and \( P(Z(t_0) \in \partial B) = 0 \), we get

$$\lim_{||T|| \to 0} P(Z_T(t)_0 \in A \mid Z_T(t) \in B) = P(Z(t_0) \in A \mid Z(t) \in B)$$

which completes the proof.

Still, for any fixed \( T \), the range of \( \frac{1}{2} F_T \) is uncountable. Therefore, we have to use the approximation introduced in Section 6.

We compare this approximation to conditional sampling based on the approach of [22] and the Gaussian marginals approach, see Section 5. To compare these procedures, we simulate \( K = 100 \) independent samples of \( Z \) on the set \( \{t_0, t_1, t_2, t_3, t_4\} \) with \( t_0 = 0, t_1 = -2, t_2 = -1, t_3 = 1 \) and \( t_4 = 2 \). We calculate the CRPS by sampling 50 times from the (approximate) conditional distribution of \( \log(Z(t_0)) \) given \( Z(t) \). Let \( CRPS_{K,1a} / MAE_{K,1a} \) and \( CRPS_{K,1b} / MAE_{K,1b} \) denote the CRPS / MAE based on two variants of conditional sampling of the Poisson point process for \( K, CRPS_{K,2} / MAE_{K,2} \) the CRPS /MAE of the approach by Wang and Stoev [22] and \( CRPS_{K,3} / MAE_{K,3} \) the CRPS / MAE based on transformation to Gaussian marginals. For the Poisson point process approach, we chose \( N = 1000 \) as the number of shape functions on the grid \( T = \{-7, -6.95, -6.9, \ldots, 6.9, 6.95, 7\} \). However, if we restrict ourselves to a finite number of shape functions, the intersection set \( I_B(z) \) with \( |B| \geq 3 \) is most likely empty, even though \( Z(t_i), i \in B \), may be determined by the same \( (s, u, f) \in \Pi \). Therefore, we do not only consider “exact” intersections, but also intersections which occur if the function values differ up to a given tolerance, i.e. we assume \( (t, y, f) \in I_B(z) \) with \( y = \min_{i=1, \ldots, n} \frac{z_i}{f(t_i - t)} \) if

$$\frac{z_i}{f(t_i - t)} < \min \left\{ y + \varepsilon, y(1 + \varepsilon) \right\} \quad \iff \quad i \in B.$$

for some given tolerance \( \varepsilon > 0 \). The simulation study is performed for \( \varepsilon = 10^{-6} \) yielding \( CRPS_{K,1a} / MAE_{K,1a} \) and \( \varepsilon = 10^{-3} \) \( CRPS_{K,1b} / MAE_{K,1b} \). Thus, by construction \( CRPS_{K,1a} / MAE_{K,1a} \) does not contain any intersections of more than two curves, but \( CRPS_{K,1b} / MAE_{K,1b} \) does. For the Wang-Stoev-approach, we chose
Table 2: Results of the simulation study for the Brown-Resnick process with $N = 1000$ and $K = 100$. 

|           | $m = 1a$ | $m = 1b$ | $m = 2$ | $m = 3$ |
|-----------|----------|----------|---------|---------|
| $CRPS_{K,m}$ | -0.35    | -0.43    | -0.35   | -0.35   |
| $MAE_{K,m}$  | 0.49     | 0.57     | 0.49    | 0.48    |

$M = 5$, $h = 0.05$ and the same $N = 1000$ shape functions. The parameters are chosen such that the first and the second method have similar running times.

Then, we get the results presented in Table 2. Note that the results are quite precise as $K$ is small. However, for different runs we always got a similar behaviour. All the methods have almost the same accuracy, at least if $\varepsilon = 10^{-6}$. However, the last method runs much faster than the others. Furthermore, we notice the difference between $CRPS_{K,1a}$ and $CRPS_{K,1b}$ indicating that considering approximate intersections of at least three curves yields worse results. This is because these intersections involve incorrect shape functions. Furthermore, they lead to degenerated conditional distributions which are not supposed to occur in the case of the Brown-Resnick process. Thus, in this case the approximation of the mixed moving maxima process seems to be appropriate only if we do not consider intersections of three or more curves.

### 8. The discretized case

By now, we have considered the general model (1.1). The procedure we proposed is exact in the case of a finite number of shape functions which are sufficiently smooth. However, as the example of the Brown-Resnick process in Section 7 illustrates, we may run into problems if these assumptions are violated.

Now, we modify our general model (1.1) and use a discretized version

$$Z(t) = \max_{(s,u) \in \Pi} uf(t-s), \quad t \in p\mathbb{Z}^d, \quad (8.1)$$

where $\Pi$ is a Poisson point process on $p\mathbb{Z}^d \times (0, \infty) \times G$ where $p > 0$ and $G \subset (0, \infty)^p\mathbb{Z}^d$ is countable. The intensity measure of $\Pi$ is given by

$$\Lambda(\{s\} \times B \times \{g\}) = \sum_{z \in \mathbb{Z}^d} \delta_{pz}(ds) \times \int_B u^{-2} du \times P_F(\{g\})$$

where $P_F$ is a probability measure belonging to a $G$-valued random variable $F$ with $\mathbb{E}(\sum_{z \in \mathbb{Z}^d} F(pz)) = 1$.

Using the same notations as before, we get the same results as in Section 7. However, all the calculations can be done explicitly without any further assumptions on $f \in G$. We get the following results.

**Proposition 8.1.** Let $B = \{i\} \subset \{1, \ldots, n\}$, $z \in \mathbb{R}^n$ and

$$D_i(z) = \{(x,f) \in p\mathbb{Z}^d \times G : (x,y,f) \in I_{\{i\}}(z), \text{ for some } y \in \mathbb{R}\}.$$

Then, we have

$$\Lambda(I_{\{i\}}(z)) = \frac{1}{2^m} \sum_{(x,f) \in D_i(z)} \frac{f(t_i - x)}{z_i} P_F(\{f\}) + o(2^{-m}).$$
Proposition 8.2. Let \( B \in 2^{\{1,\ldots,n\}} \setminus \emptyset, |B| > 1 \) and \( z \in \mathbb{R}^n \) such that
\[
I_A(z) = \{(x_j, y_j, f_j), \ j = 1, \ldots, l(z)\}
\]
with \( l(z) > 0 \). Then, for \( m \) large enough, we have
\[
\Lambda(I_B^{(m)}(z)) = \sum_{j=1}^{l(z)} \frac{1}{y_j} P_F(\{f_j\}) \cdot \left( \min_{i \in A} \frac{2^m z_i}{J_m(z_i)} - \max_{i \in A} \frac{2^m z_i}{J_m(z_i)} + 1 \right)
\]
In particular, \( \Lambda(I_B^{(m)}(z)) \in O(2^{-m}) \), but \( \Lambda(I_B^{(m)}(z)) \notin O(2^{-m(1+\varepsilon)}) \) for any \( \varepsilon > 0 \).

By these formulae, all the scenario probabilities can be calculated. As the intensity of each intersection set has the same rate of convergence, only scenarios with minimal \( |\Pi \cap K_t, z| \) occur.

We note that our model is very close to the model investigated by [22]. To see this, we calculate that
\[
P(Z(t_1) \leq z_1, \ldots, Z(t_n) \leq z_n) = \exp \left( -\sum_{f \in G} \sum_{z \in \mathbb{Z}^d} \max_{i=1,\ldots,n} \frac{f(t_i - p z)}{z_i} P_F(\{f\}) \right).
\]
Therefore, we get that
\[
Z \overset{D}{=} \max_{z \in \mathbb{Z}^d} \max_{f \in F} \left( f(\cdot - p z) P_F(\{f\}) Z_f^{(z)} \right),
\]
where the random variables \( Z_f^{(z)}, z \in \mathbb{Z}^d, f \in G \), are independently standard Fréchet distributed.

This means, the model (8.1) is a max-linear model and the exact conditional distribution can be calculated via the algorithm of [22] if \( G \) is finite and the support of each \( f \in G \) is finite. However, the algorithm of conditional sampling of the Poisson point process and the algorithm of [22] do not work in exactly the same way. According to the latter algorithm one samples from each random variable \( Z_f^{(z)} \). This procedure corresponds to simulating the largest point of \( \Pi \cap (0, \infty) \times \{p z\} \times \{f\} \) for each \( z \in \mathbb{Z}, f \in G \). The first algorithm includes the simulation of points in \( \Pi \) until a terminating condition given in Theorem 4 of [17] is met. Computational experiments show that both algorithms yield identical results. Also the technical results are related.

For example, the occurrence of a scenario \( J \subset \mathbb{Z}^d \times G \) (in the notation of Wang/Stoev) corresponds to the event that \( \Pi_2 \) consists of \(|J|\) elements \((p z, \min_{i=1,\ldots,n} Z(t_i)^{Z(t_i)/(t_i - p z)}, f)\) with \((z, f) \in J\). By this correspondence, the statements

- \(|\Pi \cap K_t, z|\) is minimal a.s.
- an occurring hitting scenario \( J \) satisfies \(|J| = r(J(A, x))\) a.s. (22)

are equivalent, both claiming that the number of points generating the observation \((t, z)\) is minimal. Hence, although the approaches look quite different, there are similar observations and results in [22] and in this section.
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