Adaptive Distributed Observer for an Uncertain Leader over General Directed Graphs
Shimin Wang

Abstract—In this paper, we propose a new technique to handle the general directed graph and we also apply it to analyze the adaptive distributed observer for an uncertain leader. The general directed graph can be found in many cooperative control problems, these problems involved the general directed graph can be solved rely on some assumptions such as undirected graph, detailed balanced graph and directed acyclic graph. We first design a different parameter for each agent to tackle the general directed graph and prove the existence of the parameter for each agent. An adaptive distributed observer for an uncertain neutrally stable leader system has been established over undirected connected graphs and directed acyclic graphs. We further investigate the validity of the adaptive distributed observer for an uncertain leader subject to general directed graphs. We first establish a useful lemma to handle the general directed graph. Then, a set of solvability conditions are given that guarantee that our proposed adaptive distributed observer can estimate the state and the uncertain parameters of the leader system.

Index Terms—Distributed Observer, Parameter Estimation, Consensus, Uncertain Leader, Multi-agent systems, Persistently Exciting, General Directed Graphs.

1. INTRODUCTION

A common assumption of cooperative control problem in [15, 19, 20, 23, 25, 37, 41], is that the communication network of the multi agent system is undirected or detailed balanced graph. Either detailed balanced graph or undirected graph requires all the communication between each agent are bidirectional. These graph assumptions limit the applications of the multi agent system. The general directed graph can be found in many cooperative control problem [17, 22, 38] and it only requires the graph contains a spanning tree, which is a weakest assumption in the multi agent systems. However, these cooperative control problem over directed graph can be solved still need some assumptions. For examples, the global optimal control problem of [22] can only be achieved for a class of specified graph they defined, namely those whose Laplacian matrix is simple, i.e. has a diagonal Jordan form. [38] proposed a controller synthesis algorithm for decentralized control problems over the directed acyclic graph which required that all the communications of the directed acyclic graph between each follower are unidirectional.

The distributed observer was first developed for solving the cooperative output regulation problem for linear multi-agent systems over static networks in [36] and over jointly connected switching networks in [35], respectively. Both [36] and [35] assumed that every follower knows the system matrix of the leader system, which was partially removed in [5]. [5] proposed a so-called adaptive distributed observer for the leader system which not only estimates the state but also the system matrix of the leader system. A key technique of the adaptive distributed observer in [5] require that the children of the leader need to know system matrix of the leader. Thus, the adaptive distributed observer in [5] is still incapable of dealing with the case where the leader system depends on the unknown system matrix and output matrix. In practice, a leader system may contain some unknown parameters. Besides, due to the communication constraint the parameters of the leader may not be known by any followers. Typically, a linear uncertain leader system takes the following form:

\[ \dot{v} = S(\omega)v, \]
\[ y = Ev, \]

where, \( v \in \mathbb{R}^m \) is the state, \( y \in \mathbb{R}^n \) is the output, \( S(\omega) \in \mathbb{R}^{m \times m} \) relies on some unknown parameter vector \( \omega \in \mathbb{R}^l \), and \( E \in \mathbb{R}^{n \times m} \) is a unknown constant matrix. The parameters of the leader have been used to design the control law to solve many cooperative control problems. For example, [43] design a control protocol with known parameters of leader to extend the flocking algorithm proposed in [33]. Besides, the demands of the leader’s parameters in the cooperative problem raises the distributed parameter estimation problem for the multi agent systems. If the signal \( v \) of the leader which contains uncertain parameters can be obtain by each follower as it is shown in Figure 1(a), we certainly can use the traditional approach proposed in [2, 7, 13, 30, 51, 34] to tackle it. Usually, multi agent systems include two groups of follower as it is shown in Figure 1(b). The first group consists of those agents can access the \( v \), and the second group consists of the rest of...
the followers. Since the followers in the second group can’t access \( v \) for designing the traditional adaptive observer, the distributed parameter estimation problem can’t be handled by the classical approach investigated [2, 7, 13, 30, 31, 34].

Some efforts have been made on designing distributed observer for an uncertain leader system of the form \([1, 25, 41–44]\). Specifically, [25] proposed a distributed dynamic compensator for uncertain leader and showed that this compensator make the following two assumptions.

Assumption 1. \( G^{-} \) contains a spanning tree with the node 0 as the root.

Assumption 2. All the eigenvalues of \( S(\omega) \) are simple with zero real part.

Under Assumption 2 without loss of generality, we can assume \( S(\omega) \) takes the following form:

\[
S(\omega) = \begin{bmatrix} 0_{m_0 \times m_0} & \text{diag} (\omega) \otimes a \end{bmatrix}
\]

where \( a = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \omega = \text{col} (\omega_{01}, \ldots, \omega_{0l}) \in \mathbb{R}^l, m_0 + 2l = m, \omega_{0k} > 0, k = 1, \ldots, l. \) Since the eigenvalues at the origin are known exactly, in what follows, like in [42] we assume that \( m_0 = 0 \). Thus, \( S(\omega) = (\text{diag} (\omega) \otimes a) \) which is skew-symmetric.

Given \([1]\) with the matrix \( E \) being known, and a digraph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) with \( \mathcal{V} = \{0, \ldots, N\} \) where the node 0 is associated with the leader system \([1]\), [42] proposed a distributed dynamic compensator as follows:

\[
\dot{\eta}_i = S(\omega_i) \eta_i + \mu_2 \sum_{j \in N_i} (\eta_j - \eta_i) \tag{5a}
\]

\[
\dot{\omega}_i = \mu_3 \phi \left( \sum_{j \in N_i} (\eta_j - \eta_i) \right) \eta_i \tag{5b}
\]

where \( \phi \) is a function \( x \mapsto \min\{\|x\|, c\} \) for some constant \( c \). 

\( \forall x \in \mathbb{R}^n, \) unless described otherwise, \( x_i \) denotes the \( i \)th component of \( x \) and let diag be such that

\[
\text{diag} (x) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.
\]

For any \( x \in \mathbb{R}^m \) with \( m \) is an even integer, the matrix functions \( \phi(\cdot) : \mathbb{R}^m \mapsto \mathbb{R}^{n \times m} \) is defined as follows:

\[
\phi(x) = \begin{bmatrix} -x_2 & x_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -x_m & x_{m-1} \end{bmatrix} \tag{3}
\]

II. PROBLEM DESCRIPTION AND PRELIMINARIES

Let us first recall the result in [42]. For this purpose, let us make the following two assumptions.

Assumption 1. \( G^{-} \) contains a spanning tree with the node 0 as the root.

Assumption 2. All the eigenvalues of \( S(\omega) \) are simple with zero real part.

Under Assumption 2 without loss of generality, we can assume \( S(\omega) \) takes the following form:

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Given \([1]\) with the matrix \( E \) being known, and a digraph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) with \( \mathcal{V} = \{0, \ldots, N\} \) where the node 0 is associated with the leader system \([1]\), [42] proposed a distributed dynamic compensator as follows:

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\]

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\]

where \( \phi \) is a function \( x \mapsto \min\{\|x\|, c\} \) for some constant \( c \). 

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\[
\phi(x) = \begin{bmatrix} -x_2 & x_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -x_m & x_{m-1} \end{bmatrix} \tag{3}
\]
digraph $G$ is undirected, for any $v(0)$, any $\eta_i(0)$ and any $\omega_i(0)$, $i = 1, \ldots, N$, 
\[
\lim_{t \to \infty} \left( \tilde{\eta}_i(t) - v(t) \right) = 0.
\]
In this sense, we say that (5) is an adaptive distributed observer for (1). It was further shown in [43] that, if $\phi(v(t))$ is persistently exciting, then 
\[
\lim_{t \to \infty} (\omega_i(t) - \omega) = 0,
\]
exponentially, that is, the adaptive distributed observer (5) can also estimate the unknown parameter vector $\omega$.

However, in many applications, the communication link among different subsystems may not be undirected. It is interesting to further remove the assumption that the digraph $G$ is undirected. Also, in many applications, the matrix $E$ may also be known. It is also interesting to estimate the matrix $E$. Thus, in this paper, we further propose the following distributed compensator:

\begin{align}
\dot{\eta}_i &= S(\omega_i) \eta_i + \mu_2 d_i \sum_{j \in \tilde{N}_i} (\eta_j - \eta_i) \quad (6a) \\
\dot{\omega}_i &= \mu_3 d_i \phi \left( \sum_{j \in \mathbb{N}} (\eta_j - \eta_i) \right) \eta_i \quad (6b) \\
\dot{E}_i &= d_i \sum_{j \in \mathbb{N}} (\tilde{y}_j - \tilde{y}_i) \eta_j^T, \quad (6c)
\end{align}

where $\eta_0, \mu_2, \mu_3, \omega_i, \eta_i$, and $\tilde{N}_i$, $i = 1, \ldots, N$, are as defined in [5], $\tilde{y}_i = y$, for $i = 1, \ldots, N$, $\bar{y}_i = E_i \eta_i$ is the output of the $ith$ follower’s observer with $E_i \in \mathbb{R}^{n \times m}$ the estimation of $E$, and $d_i$ is some real number to be specified.

Our objective is to find the design parameters $\mu_2$ and $\mu_3$, $d_i$, $i = 1, \ldots, N$, and conditions that guarantee 

\begin{align}
\lim_{t \to \infty} (\eta_i(t) - v(t)) &= 0 \quad (7a) \\
\lim_{t \to \infty} (\omega_i(t) - \omega) &= 0 \quad (7b) \\
\lim_{t \to \infty} (E_i(t) - E) &= 0. \quad (7c)
\end{align}

Remark 1. It was claimed in [25] that, for $i = 1, \ldots, N$, $E_i(t)$ converges to $E$ as time $t \to \infty$ under the following conditions:

(a) the subgraph $G$ is the detailed balanced graph

(b) all eigenvalues of the leader dynamic $S(\omega)$ are on the imaginary axis and they are non-repeated.

However, [25] didn’t prove the above claim. Besides, the estimated errors of the dynamic compensator are uniformly ultimately bounded for the general directed graph in [25]. For these reasons, in what follows, we will rigorously show that, for the general directed graphs, $E_i(t)$ converges to $E$ as time $t \to \infty$, for $i = 1, \ldots, N$.

Let $e_{vi} = \sum_{j \in \mathbb{N}} (\eta_j - \eta_i)$, $e_v = \text{col} (e_{v1}, \ldots, e_{vN})$, $\eta = \text{col} (\eta_1, \ldots, \eta_N)$, $\tilde{v} = 1_N \otimes v$, $D = \text{block diag} (d_1, \ldots, d_N)$

\footnote{A weighted digraph $G$ is called detailed balanced if there exist some real numbers $k_i > 0, i = 1, \ldots, N$, such that $k_i \alpha_{ij} = k_j \alpha_{ji}$ [25] [28].}

and $\omega = \text{col} (\omega_1, \ldots, \omega_N)$. Then, the following relation can be verified:

\[
ev_v = -(H \otimes I_m)(\eta - \tilde{v}).
\]

where $H$ consists of the last $N$ rows and the last $N$ columns of the Laplacian matrix $L$ of the digraph $G$ [11]. Equations (6a) and (6b) can be put into the following compact form:

\[
\dot{\eta} = S_d(\bar{\omega}) \eta - \mu_2 (DH \otimes I_m) (\eta - \tilde{v})
\]

(9a) 

\[
\dot{\bar{\omega}} = \mu_3 \phi_d(e_v) (D \otimes I_m) \eta
\]

(9b)

where 

\[
\phi_d(e_v) = \text{block diag} (\phi(e_{v1}), \ldots, \phi(e_{vN})), \\
S_d(\bar{\omega}) = \text{block diag} (S(\omega_1), \ldots, S(\omega_N)).
\]

We now close this section with reviewing some existing results as follows.

**Definition 1.** [28] A bounded piecewise continuous function $f : [0, +\infty) \to \mathbb{R}^{m \times n}$ is said to be persistently exciting (PE) in $\mathbb{R}^m$ with a level of excitation $\alpha_0$ if there exist positive constants $\alpha_1$, $\alpha_2$, and $\alpha$ such that 

\[
\alpha_1 I_m \geq \frac{1}{T} \int_t^{t+T} f(s)f^T(s)ds \geq \alpha_0 I_m, \quad \forall t \geq T
\]

Remark 2. The properties and various other equivalent definitions of persistently exciting are given in [6] [14] [27] [28] [39] [40].

**Lemma 1.** [43] Suppose $f(t), g(t) : [0, +\infty) \to \mathbb{R}^{m \times n}$ are bounded over $t \geq 0$, and $\lim_{t \to \infty} \|g(t) - f(t)\| = 0$. Then, $f(t)$ is persistently exciting if and only if $g(t)$ is.

**Lemma 2.** [43] For any $z \in \mathbb{R}^l$ and $x, y \in \mathbb{R}^m$ with $m = 2l$,

\[
\phi(x)y = -\phi(y)x, \quad (10)
\]

\[
S(z)x = -\phi^T(x)z, \quad (11)
\]

where $S(z) = \text{diag} (z) \otimes a$.

**Lemma 3.** [47] Let $\tilde{\Omega}(t) \in \mathbb{R}^{m \times n}$ be bounded over $t \geq 0$ and 

\[
\int_t^{t+T} \tilde{\Omega}(s)\tilde{\Omega}^T(s)ds \geq \delta I_m,
\]

for some positive constants $\alpha$ and $\alpha$. If $x(t) \in \mathbb{R}^m$ is such that 

\[
\lim_{t \to \infty} \tilde{x}(t) = 0
\]

then, 

\[
\lim_{t \to \infty} x^T(t)\tilde{\Omega}(t)\tilde{\Omega}^T(t)x(t) = 0.
\]

**III. SOME TECHNICAL LEMMAS**

In this section, we will establish a few technical lemmas. The following lemma is the basis for choosing $d_i$, $i = 1, \ldots, N$.

**Lemma 4.** Under Assumption [1] there exists a positive diagonal matrix $D$ such that all the eigenvalues of $DH$ are real, positive, and distinct.
Proof. Under Assumption 1, $H$ is a nonsingular, all the eigenvalues of $H$ have positive real parts, and all the off-diagonal entries of $H$ are non-positive. Thus, $H$ is nonsingular $M$-matrix (see Definition 4). By Lemma 11 in the appendix, $H^{-1}$ exists and each entry of $H^{-1}$ is nonnegative. Then, all leading principal minors of $H$ are positive (Definition 5 in the Appendix). From Lemma 12 in the Appendix, there exists positive definite diagonal matrix $D = \text{block diag}(d_1, \ldots, d_N)$ such that all the eigenvalues of $DH$ are real, positive, and distinct. ∎

Remark 3. Since, under Assumption 1, there exists positive definite diagonal matrix $D = \text{block diag}(d_1, \ldots, d_N)$, such that all the eigenvalues of $DH$ are real, positive, and distinct. By Lemma 13 in the appendix, there exists positive definite symmetric matrix $W$ such that $WDH$ is positive definite symmetric matrix.

Let $P = WDH$ and $Q = PDH + H^TDP$. It can be verified that

$$Q = P \bar{D}H + H^TDP = 2H^TWDH > 0.$$  \hfill (12)

Since $H$ is an $M$-matrix and $D$ is a positive definite diagonal matrix, $DH$ is an $M$-matrix. By Theorem 2.5.3 of [24], there exists a positive definite diagonal matrix $B = \text{block diag}(b_1, \ldots, b_N)$ such that

$$\bar{H} = BDH + H^TDB$$

is positive definite. Let $\lambda_{\min}$ denote the minimum eigenvalue of $\bar{H}$. Let $b_M$ and $b_m$ be the maximum and minimum number of the set $\{b_i, i = 1, \ldots, N\}$.

Remark 4. If the subgraph $\bar{G}$ is undirected, which implies $H$ is a symmetric matrix. Under Assumption 1, the matrix $H$ is positive definite and symmetric matrix. We can choose $D = I$ and $W = I$, such that $WDH$ is a positive definite symmetric.

If the subgraph $\bar{G}$ is detailed balanced graph, by Remark 1 of [22], there exists a positive diagonal matrix $D = (\text{diag}(d))$ with $d = \text{col} (d_1, \ldots, d_N)$, $L^T d = 0$, such that $DH = H^T D$ and $W = I$, makes $WDH$ a positive definite symmetric matrix, where $L$ is the Laplacian matrix of the subgraph $\bar{G}$.

There are two limitations in a detailed balanced graph. Both a detailed balanced graph and an undirected graph require that the communication between each follower be bidirectional. If there exists unidirectional communication in the topology, then there exists no positive diagonal matrix $D$ such that $DH = H^T D$. Besides, when the Laplacian matrix of a graph is bidirectional connection is not diagonalizable, then we can not find $D$ such that $DH = H^T D$. We now list two examples.

Example 1

![Figure 2. Communication topology $\bar{G}$](image)

In the Figure 2, the subgraph $\bar{G}$'s Laplacian matrix

$$L = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

It can be easily verified that $d = \text{col} (1, 1, 1)$ satisfies $d^T L = 0$ with $D = I_3$, however $H^T D \neq DH$.

Example 2

![Figure 3. Communication topology $G$](image)

In the Figure 3, the graph $G$'s Laplacian matrix

$$L = \begin{bmatrix} 3 & -2 & -1 \\ -1 & 3 & -2 \\ -3 & -1 & 4 \end{bmatrix}$$

It can be easily verified that $L$ is not diagonalizable. From the Theorem 4 of [22], there exists no positive definite symmetric matrix (including the positive diagonal matrix) $D$, such that $DL = L^T D$.

If $G$ is a directed graph, For example: In the Figure 2, the graph $G$ is cyclic with

$$H = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$  

We can find $D = \text{block diag}(1, 2, 3)$ with

$$W = \begin{bmatrix} 1.1020 & -0.5510 & 0.2449 \\ -0.5510 & 0.9184 & -0.1224 \\ 0.2449 & -0.1224 & 0.3878 \end{bmatrix},$$

$$P = \begin{bmatrix} 3.3061 & -2.9388 & 0.7347 \\ -2.9388 & 4.0408 & -1.6531 \\ 0.7347 & -1.6531 & 1.1633 \end{bmatrix}.$$  

Remark 5. For any $v(0) \in \mathbb{R}^m$, we have $v(t) = e^{S(\omega)^T} v(0)$ and $\dot{v}(t) = S(\omega) e^{S(\omega)^T} v(0)$. Under Assumption 1, $S(\omega)$ is skew symmetric, $\|v(t)\| = \|v(0)\|$ and $\|\dot{v}(t)\| = \|\omega\| \|v(0)\|$. For convenience, let $\xi(t) = \text{col} (v(t), \eta(t), \omega(t)) \in \mathbb{R}^{5N}$.

Lemma 5. Consider systems (7) and (6). Under Assumptions 1 and 2, for any $\xi(0) \in \mathbb{R}^{5N}$, $\mu_2 > 1$ and $\mu_3 > 0$, $\eta(t)$, $\dot{\eta}(t)$, $S(\omega)\eta$ are uniformly bounded.

Proof. Let

$$X_0(t) = \eta^T(t) \left( B \otimes I_m \right) \eta(t).$$

Then, the time derivative of $X_0(t)$ along the trajectory of (9) is

$$\dot{X}_0 = 2\eta^T \left( B \otimes I_m \right) \dot{\eta}$$
\[
= 2\eta^T (B \otimes I_m) S_d(\tilde{\omega}) \eta - 2\mu_2 \eta^T (BDH \otimes I_m) \eta
+ 2\mu_2 \eta^T (BDH \otimes I_m) \hat{v}
\]
\[
= 2\mu_2 \eta^T (BDH \otimes I_m) \hat{v} - \mu_2 \eta^T (\bar{H} \otimes I_m) \eta
+ \sum_{i=1}^{N} 2b_i \eta^T S(\omega_i) \eta
\]
Since, for \( i = 1, \ldots, N, S(\omega) \) is skew symmetric,
\[
\dot{X}_0 = -\mu_2 \eta^T (\bar{H} \otimes I_m) \eta + 2\mu_2 \eta^T (BDH \otimes I_m) \hat{v}
\leq -\mu_2 \lambda_h \eta^2 + 2\mu_2 \eta^2 \|BDH\| \eta
\leq -\mu_2 \lambda_h \eta^2 + 2\mu_2 \eta^2 \left( \frac{4}{\lambda_h} \right) \|BDH\|^2
\leq -\mu_2 \lambda_h \eta^2 - \mu_2 \eta^2 \left( \frac{4}{\lambda_h} \right) \|BDH\|^2
\]
where \( \gamma = \frac{3}{\lambda_h} \) and \( \eta_0^* = \frac{4}{\lambda_h} \eta^2 \). By Lemma 3.4 (Comparison Lemma) in [16], \( X_0(t) \) satisfies the inequality
\[
X_0(t) \leq e^{-\mu_2 \gamma t} X_0(0) + \mu_2 \eta_0^* \int_0^t \|v(t)\|^2 e^{-(\gamma t)} \mu_2 \gamma d\tau
\]
By Remark 5, we have
\[
X_0(t) \leq e^{-\mu_2 \gamma t} X_0(0) + \frac{\mu_2 \|v(0)\|^2 \eta_0^*}{\gamma} \left[ 1 - e^{-\mu_2 \gamma t} \right]
\leq e^{-\mu_2 \gamma t} X_0(0) + \frac{\eta_0^* \|v(0)\|^2}{\gamma}
\]
Thus,
\[
\|\eta(t)\|^2 \leq \frac{X_0(t)}{b_m} \leq \frac{X_0(0)}{b_m} e^{-\mu_2 \gamma t} + \frac{\eta_0^* \|v(0)\|^2}{\gamma b_m}
\]
Hence, \( \eta \) is uniformly bounded.
Next, differentiating both sides of (9a) gives
\[
\dot{\eta}(t) = S_d(\tilde{\omega}(t)) \eta(t) - \mu_2 (DH \otimes I_m) \eta(t)
+ S_d(\tilde{\omega}(t)) \eta(t) + \mu_2 (DH \otimes I_m) \dot{\omega}(t)
\]
Let
\[
X_1(t) = \eta^T (B \otimes I_m) \eta(t)
\]
The time derivative of \( X_1(t) \) along the trajectory of (15) is
\[
\dot{X}_1 = 2\eta^T (B \otimes I_m) \dot{\eta}
- \mu_2 \eta^T (H \otimes I_m) \eta + 2\mu_2 \eta^T (B \otimes I_m) S_d(\tilde{\omega}) \eta
+ 2\mu_2 \eta^T (BDH \otimes I_m) \dot{v}
\leq -\mu_2 \lambda_h \eta^2 + 2\mu_3 \eta^2 \|H\| \eta \|\eta\| + 2\mu_2 \eta^2 \|D\| \|H\| \eta \|\dot{v}\| \|\dot{\eta}\|
\]
Using equations (9b) and (8) gives
\[
\dot{X}_1 \leq -\mu_2 \lambda_h \eta^2 + 2\mu_3 \eta^2 \|H\| \eta \|\eta\| + 2\mu_2 \eta^2 \|D\| \|H\| \eta \|\dot{v}\| \|\dot{\eta}\|
\leq -\mu_2 \lambda_h \eta^2 + 2\mu_3 \eta^2 \|H\| \eta \|\eta\| + 2\mu_2 \eta^2 \|D\| \|H\| \eta \|\dot{v}\| \|\dot{\eta}\|
\]
Let \( \gamma = \frac{3}{\lambda_h} \), \( q^* = \|B\| \|H\| \|D\| \), and
\[
\rho(v, \eta, \omega) = q^* \left[ \mu_3 (N \|v\| \|\eta\|^2 + \|\eta\|^3) + \mu_2 \|\dot{v}\| \|\dot{\eta}\| \right]
\]
\[\delta = \frac{b_M}{\beta_M}\text{ and } \rho_1(\xi(t), \omega) = 2b_M \|\tilde{\eta}\| \|S_d(\tilde{\omega})\| \eta.\]

Since \(S_d(\tilde{\omega})\eta = (S_d(\tilde{\omega})\eta - (I_N \otimes S(\omega))\eta),\)

by Lemma 5 under Assumptions 1 and 2, \(v(t), S_d(\tilde{\omega})\eta\) and \(\eta\) are uniformly bounded, \(S_d(\tilde{\omega})\eta\) is also uniformly bounded. Thus, for any initial condition \(\xi(0) \in \mathbb{R}^{3N_1}\), we have

\[\rho_1(\xi(t), \omega) < \rho_1^*(\xi(0), \omega),\]

for some positive number \(\rho_1^*(\xi(0), \omega)\). From equation \((19)\), we have

\[\dot{X}_2(t) \leq -\mu_2 \delta X_2(t) + \rho_1^*(\xi(0), \omega).\]  

\[(20)\]

By Lemma 3.4 (Comparison Lemma) in [6], \(X_2(t)\) satisfies the inequality

\[X_2(t) \leq e^{-\mu_2 \delta t} X_2(0) + \frac{\rho_1^*(\xi(0), \omega)}{\mu_2 \delta} \int_0^t e^{-(t-\tau)\mu_2 \delta} d\tau\]

\[= e^{-\mu_2 \delta t} X_2(0) + \frac{\rho_1^*(\xi(0), \omega)}{\mu_2 \delta} \left[1 - e^{-\mu_2 \delta t}\right]\]

\[\leq e^{-\mu_2 \delta t} X_2(0) + \frac{\rho_1^*(\xi(0), \omega)}{\mu_2 \delta}.\]  

\[(21)\]

Since \(b_m \|\tilde{\eta}(t)\|^2 \leq X_2(t),\) we have

\[b_m \|\tilde{\eta}(t)\|^2 \leq X_2(t) \leq e^{-\mu_2 \delta t} X_2(0) + \frac{\rho_1^*(\xi(0), \omega)}{\mu_2 \delta}.\]  

\[(22)\]

Thus,

\[\lim_{t \to \infty} \|\tilde{\eta}(t)\|^2 \leq \lim_{t \to \infty} \frac{X_2(t)}{b_m} \leq \frac{\rho_1^*(\xi(0), \omega)}{\mu_2 \delta b_m}.\]

Hence, for \(i = 1, \ldots, N\), we have

\[\lim_{t \to \infty} \|\tilde{\eta}_i(t)\| \leq \alpha(\mu_2),\]

where \(\alpha(\mu_2) = \frac{\sqrt{\rho_1^*(\xi(0), \omega)}}{\sqrt{\mu_2 \delta b_m}}\). Since \(\lim_{\mu_2 \to \infty} \alpha(\mu_2) = 0\), there exist \(T_0 > 0\) such that, for \(t \geq T_0\),

\[\|\tilde{\eta}_i\| \leq 2\alpha(\mu_2), \quad i = 1, \ldots, N.\]

Then, for \(i = 1, \ldots, N\), we have, \(\forall t \geq T_0,\)

\[\left(\phi(\tilde{\eta}_i)\phi^T(\tilde{\eta}_i)\right)^{1/2} \leq \begin{bmatrix} A_1(\tilde{\eta}_i) & \cdots & A_l(\tilde{\eta}_i) \end{bmatrix} \leq 2\alpha(\mu_2) I_l\]

\[\text{where } A_k(\tilde{\eta}_i) = A_k(\eta_i - v) \geq A_k(v) - A_k(\eta_i).\]

Thus, for \(i = 1, \ldots, N\), we have, \(\forall t \geq T_0,\)

\[2\alpha(\mu_2) I_l \geq \left(\phi(\tilde{\eta}_i)\phi^T(\tilde{\eta}_i)\right)^{1/2} \geq \left(\phi(v)\phi^T(v)\right)^{1/2} - \left(\phi(\eta_i)\phi^T(\eta_i)\right)^{1/2}.\]

Simple calculation shows that, for \(k = 1, \ldots, l,\)

\[
\begin{bmatrix}
v_{2k-1}(t) \\
v_{2k}(t)
\end{bmatrix} =
\begin{bmatrix}
C_k \sin(\omega_k t + v_k) \\
C_k \cos(\omega_k t + v_k)
\end{bmatrix},
\]

where \(C_k = \sqrt{v^2_{2k-1}(0) + v^2_{2k}(0)}\) and \(v_k = \frac{v_{2k-1}(0)}{v^2_{2k}(0)}\)

Let \(\beta(\eta_i) = \min \{C_1, \ldots, C_l\}\). Since, for \(i = 1, \ldots, l,\)

\(\cos(v_{2i-1}(0), v_{2i}(0)) \neq 0, \beta(\eta_i) > 0\). In fact,

\[\left(\phi(\eta_i)\phi^T(v)\right)^{1/2} = \begin{bmatrix} C_1 & \cdots & C_l \end{bmatrix} \geq \beta(\eta_i) I_l.\]  

\[(23)\]

Choose \(\mu_2 > \mu_2^*(\xi(0), \omega)\) with

\[\mu_2^*(\xi(0), \omega) = 4\rho_1^*(\xi(0), \omega) \frac{\beta(\eta_i)}{\beta(\eta_i)^2 I_l} \]

such that, for \(t \geq T_0, \) and \(i = 1, \ldots, N,\)

\[\frac{\beta}{2} I_l \geq \left(\phi(\tilde{\eta}_i)\phi^T(\tilde{\eta}_i)\right)^{1/2} \geq \left(\phi(v)\phi^T(v)\right)^{1/2} - \frac{\beta(\eta_i)}{2} I_l\]

\[\geq \beta(\eta_i) I_l - \frac{\beta(\eta_i)}{2} I_l = \frac{\beta(\eta_i)}{2} I_l.\]  

\[(24)\]

Then, for \(i = 1, \ldots, N, t \geq T_0\) and \(\mu_2 > \mu_2^*(\xi(0), \omega),\) we have

\[\left(\phi(\tilde{\eta}_i)\phi^T(\tilde{\eta}_i)\right)^{1/2} \geq \left(\phi(v)\phi^T(v)\right)^{1/2} - \frac{\beta(\eta_i)}{2} I_l\]

\[\geq \beta(\eta_i) I_l - \frac{\beta(\eta_i)}{2} I_l = \frac{\beta(\eta_i)}{2} I_l.\]  

\[(25)\]

By integrating both side of equation \((24)\), we have, for \(\mu_2 > \mu_2^*(\xi(0), \omega), \forall t \geq T_0, \) and \(i = 1, \ldots, N,\)

\[\frac{1}{T} \int_{t+T}^{t+T} \phi(\tilde{\eta}_i(s))\phi^T(\tilde{\eta}_i(s)) ds \geq -\frac{\beta^2(\eta_i)}{4} I_l.\]  

\[(26)\]

By Lemma 2, we can rewrite equations \((18)\) into the following form:

\[\dot{\tilde{\eta}} = (I_N \otimes S(\omega) - \mu_2 D) \tilde{\eta} - \phi_d(\eta)\tilde{\omega},\]

\[(27)\]

\[\dot{\tilde{\omega}} = \mu_3 \phi_d(\eta) (D \otimes I_m) \tilde{\eta},\]

\[(28)\]

where \(\phi_d(\eta) = \text{block diag} (\phi(\eta_1), \ldots, \phi(\eta_N)).\)

Then, we can establish the following lemma.

**Lemma 7.** Consider systems \((1)\) and \((6)\). Under Assumptions 1 and 2, for any \(\xi(0) \in \mathbb{R}^{3N_1}\) with \(\cos(v_{2i-1}(0), v_{2i}(0)) \neq 0, k = 1, \ldots, l\), any \(\mu_3 > 0\), there exists a sufficiently large \(\mu_2\) such that \(\tilde{\omega}(t)\) is uniformly bounded.

**Proof.** From equation \((29)\), we have

\[\dot{\tilde{\omega}}(t) = \mu_3 \phi_d(\eta(t)) (D \otimes I_m) \tilde{\eta}(t),\]

\[- = - \frac{\mu_3}{\mu_2} \phi_d(\eta(t)) \phi^T_d(\eta(t)) \tilde{\omega}(t) - \frac{\mu_3}{\mu_2} \psi(t),\]

\[- = - \frac{\mu_3}{\mu_2} F(t) \tilde{\omega}(t) - \frac{\mu_3}{\mu_2} \psi(t)\]  

\[(30)\]

where

\[F(t) = \phi_d(\eta(t)) \phi^T_d(\eta(t))\]

\[\psi(t) = \phi_d(\eta(t)) [(I_N \otimes S(\omega)) \tilde{\eta}(t) - \tilde{\eta}(t)].\]  

\[(31)\]
We first consider the system (30) with \( \psi(t) = 0 \) as follows
\[
\dot{\omega}(t) = -\frac{\mu_3}{\mu_2} F(t) \dot{\omega}(t).
\] (31)

By Lemma 6 for sufficiently large positive number \( \mu_2 > \mu_2^*(\xi(0), \omega) \), \( \phi(\eta_l) \) is persistently exciting, that is, inequality (28) holds. Since, by Lemma 3 \( \eta_l \) is uniformly bounded, for some positive number \( \beta_1 \)
\[
\beta_1 I_l \geq \frac{1}{T} \int_t^{t+T} \phi(\eta_l(s)) \phi^T(\eta_l(s)) ds. \tag{32}
\]
Combining (32) and (28) gives
\[
\beta_1 I_l \geq \frac{1}{T} \int_t^{t+T} \phi(\eta_l(s)) \phi^T(\eta_l(s)) ds \geq \frac{\beta^2_0(0)}{4} I_l. \tag{33}
\]

Since \( \phi_d(\eta_l) = \) block diag \( (\phi(\eta_1), \cdots, \phi(\eta_{N_l})) \),
\[
\beta_1 I_{N_l} \geq \frac{1}{T} \int_t^{t+T} F(s) ds \geq \frac{\beta^2_0(0)}{4} I_{N_l}, \quad \forall t \geq T_0. \tag{34}
\]

Since
\[
F(t) = \phi_d(\eta_l) \phi_d^T(\eta_l) = \) block diag \( (f_1(t), \cdots, f_{N_l}(t)) \),
where \( f_j(t) = A^2_k(\eta_l) \) with \( j = \) \((i-1) + \)\( l \) + \( k \), for \( i = 1, \cdots, N \) and \( k = 1, \cdots, l \). For \( j = 1, \cdots, N_l \), we have
\[
\beta_1 \geq \frac{1}{T} \int_t^{t+T} f_j(t) ds \geq \frac{\beta^2_0(0)}{4}, \quad \forall t \geq T_0. \tag{35}
\]

For \( j = 1, \cdots, N_l \), by Lemma 14 in the appendix, the following system
\[
\dot{\omega}_j(t) = -\frac{\mu_3}{\mu_2} f_j(t) \omega_j(t)
\]
is exponentially stable. Thus, for \( t \geq T_0 \),
\[
\|\omega_j(t)\| \leq e^{-\frac{\mu_3^2}{4\mu_2}(t-T_0)} \|\omega_j(T_0)\|.
\]
Hence, system (31) is exponentially stable with the rate \( \frac{\mu_3^2}{4\mu_2} \). Let \( \Phi(t, \tau) \) be the transition matrix of (31), we have
\[
\|\Phi(t, \tau)\| \leq e^{-\frac{\mu_3^2}{4\mu_2}(t-\tau)}.
\]

Then, for any \( t_0 > T_0 \), the solution of (30) satisfies
\[
\hat{\omega}(t) = \Phi(t, t_0) \hat{\omega}(t_0) - \frac{\mu_3}{\mu_2} \int_{t_0}^{t} \Phi(t, \tau) \psi(\tau) d\tau \tag{36}
\]

By Lemma 3 under Assumptions 1 and 2 for sufficiently large \( \mu_2, \hat{\eta}, \ddot{\eta} \) and \( \eta \) are uniformly bounded, which implies \( \psi(t) \) is uniformly bounded. That is, for all \( t \geq 0 \), \( \|\psi(t)\| \leq \psi^* \) for some positive number \( \psi^* \). Thus,
\[
\|\hat{\omega}(t)\| \leq e^{-\frac{\mu_3^2}{4\mu_2}(t-t_0)} \|\hat{\omega}(t_0)\| + \frac{\mu_3}{\mu_2} \psi^* \int_{t_0}^{t} e^{-\frac{\mu_3^2}{4\mu_2}(t-\tau)} d\tau
\]
\[
\leq e^{-\frac{\mu_3^2}{4\mu_2}(t-t_0)} \|\hat{\omega}(t_0)\| + \frac{4\psi^*}{\beta^2_v(0)} \left[ 1 - e^{-\frac{\mu_3^2}{4\mu_2}(t-t_0)} \right] \tag{37}
\]
Hence, \( \hat{\omega}(t) \) is uniformly bounded. □

IV. MAIN RESULTS

In this section, we will present our main results.

A. Unknown Parameter Estimation

Lemma 8. Consider systems (7) and (6). Under Assumptions 1 and 2 for any \( \xi(0) \in \mathbb{R}^{5N_1} \) with \( \text{col} (v_{2k-1}(0), v_{2k}(0)) \neq 0, k = 1, \cdots, l \), and any \( \mu_3 > 0 \), there exists a sufficiently large \( \mu_2 \) such that \( \eta(t) \) and \( \omega(t) \) are uniformly bounded and satisfy,
\[
\lim_{t \to \infty} \hat{\eta}(t) = 0, \tag{38}
\]
\[
\lim_{t \to \infty} \dot{\omega}(t) = 0, \tag{39}
\]
\[
\lim_{t \to \infty} \hat{\omega}(t) = 0. \tag{40}
\]

Proof. Let
\[
V(t) = \hat{\eta}^T(t) (P \otimes I_m) \hat{\eta}(t) + \mu_3^{-1} \dot{\omega}^T(t) (W \otimes I_l) \dot{\omega}(t) \tag{41}
\]
which is proper and positive definite. Differentiating (41) along the trajectory of (29) gives
\[
\dot{V} = 2\hat{\eta}^T (P \otimes I_m) \dot{\eta}(t) + 2\mu_3^{-1} \dot{\omega}^T (W \otimes I_l) \dot{\omega}(t)
\]
\[= 2\hat{\eta}^T (P \otimes S (\omega)) \hat{\eta} - \mu_2 \hat{\eta}^T ((PDH + HTDP) \otimes I_m) \hat{\eta}
\]
\[= 2\hat{\eta}^T (P \otimes I_m) \phi_d^T (\hat{\eta}) \dot{\omega} + 2\mu_3^{-1} \dot{\omega}^T (W \otimes I_l) \dot{\omega}
\]
\[= 2\hat{\eta}^T (P \otimes S (\omega)) \hat{\eta} - \mu_2 \hat{\eta}^T ((PDH + HTDP) \otimes I_m) \hat{\eta}
\]
\[= 2\hat{\eta}^T (P \otimes I_m) \phi_d^T (\hat{\eta}) \dot{\omega} + 2\hat{\eta}^T (P \otimes \phi^T (v)) \hat{\eta} + 2\mu_3^{-1} \dot{\omega}^T (W \otimes I_l) \dot{\omega} \tag{42}
\]
Since \( S(\omega) \) is skew symmetric and \( P \) is symmetric, \( P \otimes S (\omega) \) is skew symmetric and \( \phi_d(v) = I_N \otimes \phi(v) \). Thus, under Assumptions 12 from equation (12), we have
\[
\dot{V} = -\mu_2 \hat{\eta}^T (Q \otimes I_m) \hat{\eta} - 2\hat{\eta}^T (P \otimes \phi^T (v)) \hat{\eta} + 2\hat{\eta}^T (P \otimes \phi^T (v)) \hat{\eta} + 2\hat{\eta}^T (W \otimes I_l) \dot{\omega} \tag{43}
\]
Substituting equation (29) into equation (43) gives
\[
\dot{V} = -\mu_2 \hat{\eta}^T (Q \otimes I_m) \hat{\eta} - 2\hat{\eta}^T (P \otimes \phi^T (v)) \hat{\eta} + 2\hat{\eta}^T (W \otimes I_l) \dot{\omega}
\]
\[= -\mu_2 \hat{\eta}^T (Q \otimes I_m) \hat{\eta} - 2\hat{\eta}^T (P \otimes \phi^T (v)) \hat{\eta} + 2\hat{\eta}^T (W \otimes I_l) \dot{\omega}
\]
\[= -\mu_2 \hat{\eta}^T (Q \otimes I_m) \hat{\eta} - 2\hat{\eta}^T (P \otimes \phi^T (v)) \hat{\eta} + 2\hat{\eta}^T (W \otimes I_l) \dot{\omega}
\]
\[= -\mu_2 \hat{\eta}^T (Q \otimes I_m) \hat{\eta} - 2\hat{\eta}^T (P \otimes \phi^T (v)) \hat{\eta} + 2\hat{\eta}^T (W \otimes I_l) \dot{\omega} \tag{44}
\]
Under Assumption 1 using \( P = WDH \) gives
\[
\hat{\omega}^T (WDH \otimes \phi^T (v)) \hat{\eta} = \hat{\omega}^T (P \otimes \phi^T (v)) \hat{\eta} = \hat{\eta}^T (P \otimes \phi^T (v)) \hat{\eta} \tag{45}
\]
Hence, from equation (44) and (45), we have
\[
\dot{V} = - \mu_2 \tilde{\eta}^T (Q \otimes I_m) \tilde{\eta} - 2 \tilde{\eta}^T (P \otimes I_m) \phi_d^\top (\tilde{\eta}) \omega + 2 \gamma^2 (W \otimes I_l) \phi_d (\tilde{\eta}) (DH \otimes I_m) \tilde{\eta} \\
\leq - \mu_2 \tilde{\eta}^T (Q \otimes I_m) \tilde{\eta} + 2 \| \tilde{\omega} \| \| \phi_d (\tilde{\eta}) \| \| P \otimes I_m \| \| \tilde{\eta} \| + 2 \| \tilde{\omega} \| \| (W \otimes I_l) \| \| \phi_d (\tilde{\eta}) \| \| (DH \otimes I_m) \| \| \tilde{\eta} \| \\
\leq - \mu_2 \tilde{\eta}^T (Q \otimes I_m) \tilde{\eta} + 2 \| \tilde{\omega} \| \| P \otimes I_m \| \| \tilde{\eta} \|^2 + 2 \| \tilde{\omega} \| \| (W \otimes I_l) \| \| (DH \otimes I_m) \| \| \tilde{\eta} \|^2 . \tag{46}
\]

Under Assumption 1, $Q$ is positive definite matrix. Let $\lambda_q$ denote the minimum eigenvalue of $Q$ and
\[
m^* = 2 \| (P \otimes I_m) \| + 2 \| (W \otimes I_l) \| \| (DH \otimes I_m) \|. \tag{47}
\]

Then, from equation (46), we have
\[
\dot{V} (t) \leq - (\mu_2 \lambda_q - m^* \| \tilde{\omega} (t) \|) \| \tilde{\eta} (t) \|^2 . \tag{48}
\]

By Lemma 7, $\tilde{\omega}$ is uniformly bounded and from equation (37),
\[
\| \tilde{\omega} (t) \| < \omega^* (\xi (0), \omega),
\]
for some positive number $\omega^* (\xi (0), \omega)$. Let
\[
\mu_{\max} = \max \left\{ \frac{\omega^* (\xi (0), \omega) m^* + 1}{\lambda_q}, \mu_2 (\xi (0), \omega) \right\} .
\]

Then, from equation (47), for any $\mu_2 \geq \mu_{\max}$, we have
\[
\dot{V} (t) \leq - \| \tilde{\eta} (t) \|^2 \leq 0 . \tag{49}
\]

Since $\dot{V}$ is negative semi-definite and $V$ is positive definite and lower bounded, $V (\infty)$ exists and is finite. From (29b), $\tilde{\eta}$ is uniformly bounded, which imply $\tilde{\eta}$ is uniformly continuous. From equation (48), we have
\[
\int_0^\infty \| \tilde{\eta} (s) \|^2 ds \leq V (0) - V (\infty) . \tag{50}
\]

By Barbalat’s Lemma, we have
\[
\lim_{t \to \infty} \tilde{\eta} (t) = 0 ,
\]
which implies (38). (38) together with (29b) yields (39). To show $\lim \tilde{\omega} (t) = 0$, differentiating $\tilde{\eta}$ gives,
\[
\ddot{\eta} = (I_N \otimes \dot{S} (\omega)) \dot{\eta} + (I_N \otimes S (\omega) - \mu_2 DH \otimes I_m) \dot{\eta} + \phi_d^\top (\eta) \dot{\omega} + \phi_d^\top (\tilde{\eta}) \tilde{\omega} . \tag{51}
\]

We have shown $\tilde{\eta}$, $\tilde{\omega}$, and $\tilde{\eta}$ are all uniformly bounded. From (29), $\dot{\eta}$ and $\dot{\omega}$ are also uniformly bounded. Thus, $\tilde{\eta}$ is uniformly bounded. By Barbalat’s Lemma again, we have $\lim_{t \to \infty} \dot{\tilde{\eta}} (t) = 0$, which together with (38) implies
\[
\lim_{t \to \infty} \phi_d^\top (\eta (t)) \tilde{\omega} (t) = 0 . \tag{52}
\]

By Lemma 6 under Assumptions 1 and 2 for any $v (0)$ satisfying col $(v_{2k-1} (0), v_{2k} (0)) \neq 0$, $k = 1, \ldots, l$, and any $\mu_3$, there exists $\mu_{\max}$ such that, if $\mu_2 \geq \mu_{\max}$, then $\phi_d (\eta)$ is persistently exciting. Also note that (39) holds and
\[
\lim_{t \to \infty} \tilde{\omega}^T (t) \phi_d (\eta (t)) \phi_d^\top (\eta (t)) \tilde{\omega} (t) = 0 . \tag{53}
\]

By Lemma 5 (41) holds.

Remark 6. Lemma 8 only show that $\eta_i$ converges to zero asymptotically, we now use the following lemma to show $\eta_i$ converge to zero exponentially, for $i = 1, \ldots, N$.

Lemma 9. Consider systems (7) and (6). Under Assumptions 2 and 7 for all $k = 1, \ldots, l$, col $(v_{2k-1} (0), v_{2k} (0)) \neq 0$. Then, for any $\eta (0)$ and $\omega (0)$, sufficiently large number $\mu_2$ and any $\mu_3 > 0$, $\eta (t)$ and $\omega (t)$ exist and are bounded for all $t \geq 0$ and satisfy,
\[
\lim_{t \to \infty} \eta_i (t) = 0, \quad \lim_{t \to \infty} \tilde{\omega} (t) = 0 . \tag{54}
\]

Proof. Proof: See Appendix B. □

Remark 7. From (23), we have
\[
\mu_2^* (\xi (0), \omega) = \frac{4 \mu_1^* (v (0), \eta (0), \tilde{\omega} (0), \omega)}{\beta_2^2 (v (0)) \delta_{\mu N}} .
\]

It was shown that, for the detailed balanced graph, the dynamic compensator proposed in (23) can also estimate the state of the leader (1) asymptotically. However, (23) did not consider the convergence issue of the estimated unknown parameters of the matrix $S$ to the actual value of the unknown parameters of the matrix $S$. It is noted that the detailed balanced graph is bidirectional and is still restrictive.

Another advantages of our adaptive distributed observer are that we only need to estimate $\frac{\mu_2^*}{\mu_{N}}$ unknown parameters of $S$ by $\frac{m_N}{\mu_{N}}$ equations instead of $m \times m$ parameters of $S$ by $mnN$ equations as in [23].

Remark 8. Motivated by [12, 14, 29], we can use the following distributed observer to estimate $\omega$ instead:
\[
\dot{\eta}_i = S (\omega_i) \eta_i + \mu_2 d_i \sum_{j \in N_i} (\eta_j - \eta_i) \tag{55a}
\]
\[
\dot{\tilde{\omega}} = \mu_3 d_i \phi \left( \sum_{j \in N_i} (\eta_j - \eta_i) \right) \eta_i - \sigma_i \omega_i \tag{55b}
\]
\[
\dot{E}_i = d_i \sum_{j \in N_i} (\tilde{\eta}_j - \tilde{\eta}_i)^T \tilde{\eta}_i , \tag{55c}
\]

where $\eta_0$, $\mu_2$, $\mu_3$, $\omega_i$, $E_i$, $d_i$ and $N_i$, $i = 1, \ldots, N$, are as defined in (6) and the switching $\sigma_i$-modification is defined as follows
\[
\sigma_i = \begin{cases} 
0 & \| \omega_i (t) \| < M_0 , \\
\frac{1}{\sigma_0} \| \omega_i (t) \| & \geq M_0,
\end{cases} \tag{56}
\]

where, $\sigma_0$ is designed constant, and $M_0$ is chosen large enough such that $\| \omega \| < M_0$.

By Lemma 3 we have $\eta_i (t)$ is bounded by some constant $M_\eta$. Then, the time derivative of $V (\tilde{\omega} (t)) = 0.5 \tilde{\omega}^T \tilde{\omega}$ along the the solution of (52) and (53) is given by
\[
\dot{V} \leq - \sigma_i \tilde{\omega}^T \omega_i + \| \tilde{\omega} \| M_\eta . \tag{57}
\]
Now,
\[ \sigma_i \omega_i^* \omega_i = \sigma_i (\omega_i - \omega) \omega_i \geq \sigma_i \|\omega_i\|^2 - \sigma_i \|\omega_i\| \|\omega\| \\geq 0 \]
where the last inequality follows from the fact that \( \sigma_i \geq 0 \), \( \sigma_i (\|\omega_i\| - M_0) \geq 0 \) and \( M_0 \geq \|\omega\| \). We can also verify
\[ -\sigma_i \omega_i^* \omega_i \leq -\sigma_0 \omega_i^* \omega_i + 2 \sigma_0 M_0^2 \]
Then, from system (54), we have
\[ \dot{V} \leq -\sigma_0 V + 3 \sigma_0 M_0^2 + \frac{2}{\sigma_0} M_n^2, \]
which implies that \( \omega_i(t) \) converges exponentially to the residual set
\[ D_\sigma = \left\{ \omega_i(t) \left| \|\omega_i\| = M_0 \right. \right\}. \]
Thus, system (52a) can always guarantee \( \omega_i(t) \) is uniformly bounded. Then, we can choose \( \mu_2 \geq \frac{6 \sigma_0 M_0^2}{\sigma_0 \delta_{\sigma}} \) and \( \mu_3 \geq \frac{2 \sigma_0 M_n^2}{\sigma_0 \delta_{\sigma}} \). The rest proof we leave for an open problem.

B. Output Matrix Estimation
We now consider the convergence of system (60). Let \( \tilde{E}_i = (E_i - \hat{E}_i) \). Then,
\[ \dot{\tilde{E}}_i = d_i \sum_{j \in N_i} a_{ij} (\tilde{E}_j - \tilde{E}_i) v_i^T + \psi_i(t), \] (56)
where, for \( i = 1, \ldots, N \),
\[ \psi_i(t) = d_i \sum_{j \in N_i} a_{ij} \left( (\tilde{E}_j^T - \tilde{E}_i^T) \eta_i^T \right) + d_i E_i e_{\eta_i} \eta_i^T \]
\[ + d_i \sum_{j \in N_i} a_{ij} (\tilde{E}_j \eta_i^T - \tilde{E}_i \eta_i^T). \] (57)
For \( i = 1, \ldots, N \), \( \zeta_i = \text{vec}(\tilde{E}_i) \), \( \zeta_0 = \text{vec}(E) \) and \( \pi_i = \text{vec}(\psi_i(t)) \), equation (56) can be written into
\[ \dot{\zeta}_i = (vv^T \otimes I_n) d_i \sum_{j \in N_i} a_{ij} (\zeta_j - \zeta_i) + \pi_i, \]
(58)
where, for \( i = 1, \ldots, N \),
\[ \pi_i = d_i \sum_{j \in N_i} a_{ij} \left( (\tilde{E}_j \eta_i^T \otimes I_n) (\zeta_j - \zeta_i) \right) \]
\[ + d_i \sum_{j \in N_i} a_{ij} (\tilde{E}_j \eta_i^T \otimes I_n) (\eta_i \tilde{E}_j^T \otimes I_n) \zeta_j \]
\[ + d_i (\eta_i e_{\eta_i}^T \otimes I_n) \zeta_0. \] (59)
Let \( \zeta = \text{col} (\zeta_1, \ldots, \zeta_N) \) and \( \pi = \text{col} (\pi_1, \ldots, \pi_N) \). Then, system (58) can be put into the following compact form:
\[ \dot{\zeta} = - (I_N \otimes (vv^T \otimes I_n)) (DH \otimes (E_i(t) - E)) \zeta + \pi, \]
\[ = - (DH \otimes (vv^T \otimes I_n)) \zeta + \pi. \] (60)

**Lemma 10.** Consider systems (1) and (6). Under Assumptions 1 and 2 for any \( \xi(0) \in U_0 \) with \( \text{col} (v_{2k-1}(0), v_{2k}(0)) \neq 0 \), \( k = 1, \ldots, l \), any \( \mu_3 > 0 \), and any \( \mu_3 > 0 \), there exists sufficiently large \( \mu_2 \) such that \( E_i(t) \) is uniformly bounded and satisfies \( \lim_{t \to \infty} (E_i(t) - E) = 0 \).

**Proof.** We first consider system (60) with \( \pi = 0 \) as follows
\[ \dot{\zeta} = B(t) \zeta, \]
(61)
where \( B(t) = -(DH \otimes (vv^T \otimes I_n)) \). As \( DH \) is diagonalizable matrix and has real and positive eigenvalues. Thus, there exist matrix \( P_H \) such that
\[ P_H D H P_H^{-1} = \text{block diag}(\lambda_1, \ldots, \lambda_N) = J_H. \]
Let \( x = (P_H \otimes I_{nm}) \zeta \), then system (61) can be transformed into the following system
\[ \dot{x} = -(J_H \otimes (vv^T \otimes I_n)) x. \] (62)
Under Assumption 2 for all \( k = 1, \ldots, l \), \( \text{col} (v_{2k-1}(0), v_{2k}(0)) \neq 0 \), from Lemma 3 of [42], \( v \) is persistently exciting which implies there exist positive constants \( \epsilon_1, \epsilon_2, t_0, T_0 \) such that, \( \forall T \geq t_0 \),
\[ \epsilon_1 I_m \geq \int_{t_1}^{t + T_0} v(s) v^T(s) ds \geq \epsilon_2 I_m. \]
Hence, for \( i = 1, \ldots, N \), there exist positive constants \( \epsilon, t_0, T_0 \) such that
\[ \epsilon_1 J_H \otimes I_{nm} \geq \int_{t_1}^{t + T_0} (J_H \otimes (vv^T \otimes I_n)) ds \geq \epsilon_2 J_H \otimes I_{nm}. \]
Thus, by Lemma 15 in the appendix, system (62) is exponentially stable. Thus, the system \( \dot{\zeta} = B(t) \zeta \) is exponentially stable. Then, there exist positive definite matrix \( M \) and \( P(t) \) satisfying, for all \( t \geq 0 \), \( \| P(t) \| \leq c_3 \) for some \( c_3 > 0 \), such that
\[ \dot{P}(t) = -P(t) B(t) - B^T(t) P(t) - M, \]
\[ c_1 \| \zeta \|^2 \leq \zeta^T P(t) \zeta \leq c_2 \| \zeta \|^2, \]
where \( c_1 \) and \( c_2 \) are positive constants.
Let \( U(t) = \tilde{z}^T(t) P(t) \tilde{z}(t) \), and \( \lambda_m \) be the minimum eigenvalue of \( M \). Then, along the trajectory (60),
\[ \dot{U} = \zeta^T \dot{P}(t) + (P(t) B(t) + B^T(t) P(t)) \zeta + 2 \zeta^T P(t) \pi \]
\[ = - \zeta^T M \zeta + 2 \zeta^T P(t) \pi \]
\[ \leq - \lambda_m \| \zeta \|^2 + 2 \| P(t) \| \| \zeta \| \| \pi \| \]
(63)
As a result of Lemma 8 for \( i = 1, \ldots, N \), \( v \) and \( \eta_i \) are bounded by \( e^* > 0 \). Thus, from (59),
\[ \| \pi_i(t) \| \leq d_i \sum_{j \in N_i} a_{ij} \| (\tilde{E}_j \eta_i^T \otimes I_n) (\zeta_j - \zeta_i) \| \]
\[ + d_i \sum_{j \in N_i} a_{ij} \left( (\tilde{E}_j \eta_i^T \otimes I_n) \zeta_j - (\eta_i \tilde{E}_j^T \otimes I_n) \zeta_j \right) \]
\[ + d_i (\eta_i e_{\eta_i}^T \otimes I_n) \zeta_0. \] (59)
Let \( \zeta = \text{col} (\zeta_1, \ldots, \zeta_N) \) and \( \pi = \text{col} (\pi_1, \ldots, \pi_N) \). Then, system (58) can be put into the following compact form:
\[ \dot{\zeta} = - (I_N \otimes (vv^T \otimes I_n)) (DH \otimes (E_i(t) - E)) \zeta + \pi, \]
\[ = - (DH \otimes (vv^T \otimes I_n)) \zeta + \pi. \] (60)
\[
\leq 2d_M \sum_{j \in N_i} a_{ij} \| \tilde{\eta}_j v \| \| \zeta \| + d_M \sum_{j \in N_i} a_{ij} \| \tilde{\eta}_j^T \| \| \zeta \| + d_M \sum_{j \in N_i} a_{ij} \| \tilde{\eta}_j v \| \| \zeta \|
\]
\[
+ d_M \sum_{j \in N_i} a_{ij} \| \tilde{\eta}_j^T \| \| \zeta \| + d_M \| \tilde{\eta} e_v \| \| \zeta_0 \|
\]
\[
\leq d_M \varepsilon^* \| \tilde{\eta} \| \sum_{j \in N_i} a_{ij} (3 \| \tilde{\eta}_j \| + \| \tilde{\eta}_j \|) + d_M \varepsilon^* \| e_v \| \| \zeta_0 \|
\]
\[
\leq 4d_M \varepsilon^* \| \tilde{\eta} \| \sum_{j \in N_i} a_{ij} + d_M \varepsilon^* \| e_v \| \| \zeta_0 \| .
\]

As \( \| \pi(t) \| \leq \sum_{i=1}^N \| \pi_i(t) \| \), we have
\[
\| \pi(t) \| \leq \| \zeta(t) \| \omega_0(t) + \omega_1(t),
\]
where,
\[
\omega_0(t) = 4d_M \varepsilon^* \| \tilde{\eta}(t) \| \sum_{i=0}^N \sum_{j=0}^N a_{ij},
\]
\[
\omega_1(t) = N d_M \varepsilon^* \| \text{vec}(E) \| \| e_v(t) \| .
\]

Then, from equation (63), we have
\[
\dot{U} \leq -\lambda_m \| \zeta \|^2 + 2c_3 \| \zeta \|^2 \omega_0(t) + 2c_4 \| \zeta \| \omega_1(t)
\]
\[
\leq -\lambda_m \| \zeta \|^2 + 2c_3 \| \zeta \|^2 \omega_0(t) + \left[ \frac{\lambda_m}{4} \| \zeta \|^2 + \frac{4c_3^2}{\lambda_m} \omega_1^2(t) \right]
\]
\[
\leq -\left[ \frac{3\lambda_m}{4c_2} - \frac{2c_3 \omega_0(t)}{c_1} \right] U + \frac{\omega_1^2(t)}{\varepsilon}
\]
with \( \varepsilon = \frac{3\lambda_m}{4c_2} \). As a result of Lemma 8, \( \omega_0(t) \) and \( \omega_1(t) \) converge to zero, asymptotically. Then, there exists some time instant \( t_1 \), such that
\[
\frac{3\lambda_m}{4c_2} - \frac{2c_3 \omega_0(t)}{c_1} > c_4 = \frac{3\lambda_m}{8c_2} > 0, \quad t \geq t_1
\]

Hence, we have
\[
\dot{U}(t) \leq -c_4 U(t) + \frac{\omega_1^2(t)}{\varepsilon}, \quad t \geq t_1 .
\]

By Lemma 8, \( \omega_1(t) \) converges to zero, asymptotically. The system (65) could be viewed as a stable system with \( \omega_1(t) \) as the input. This input are bounded over \( t \geq 0 \) and tend to zero as \( t \to \infty \). We conclude \( U(t) \) converge to zero asymptotically, which implies \( \lim \zeta(t) = 0 \), asymptotically. (By the comparison method in [16], form Lemma 9.5 and Lemma 9.6 of [16], we can also conclude the same results)

\[
V . \text{NUMERICAL EXAMPLE}
\]

\textbf{Example}

![Figure 4. Communication topology \( \tilde{G} \)](image)

Figure 5. Trajectory of \( \| \tilde{\eta}_i(t) \|, \ i = 1, 2, 3, 4. \)

![Figure 6. Trajectory of \( \| \tilde{\omega}_i(t) \|, \ i = 1, 2, 3, 4. \)](image)

We consider a multi-agent systems with four followers and one leader. The communication topology is shown in Figure 4 which satisfies Assumption \( \Pi \) with
\[
H = \begin{bmatrix}
2 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 1 & 0 \\
0 & -1 & -1 & 2
\end{bmatrix}
\]

As there exists unidirectional edge in Figure 4, the digraph in Figure 4 is not detailed balanced. We can find \( D = \text{block diag}(1, 2, 3, 4) \) such that \( DH \) is diagonalizable and all of its eigenvalues are real, positive, and distinct. Also, we can find \( W \) and \( P \) as follows:
\[
W = \begin{bmatrix}
0.7993 & -0.4421 & -0.0671 & 0.4092 \\
-0.4421 & 1.1599 & 0.4099 & -0.8280 \\
-0.0671 & 0.4099 & 0.6599 & -0.6405 \\
0.4092 & -0.8280 & -0.6405 & 1.1979
\end{bmatrix}
\]
trajectories of distributed observer being chosen randomly. Since computer simulation with the initial condition of the adaptive performance of the distributed observer (6) is evaluated by \( \| \dot{\tilde{E}}_i(t) \| \). Figure 5 shows the consensus error \( \tilde{\eta}_i(t) \) satisfies the condition of Lemma 4 in [43], both \( \phi(v(t)) \) and \( v(t) \) are persistently exciting. Figure 5 shows the consensus error of \( \| \tilde{E}_i \| \), for \( i = 1, 2, 3, 4 \). Figure 5 and Figure 7 show the trajectories of \( \| \tilde{E}_i \| \) and \( \| \tilde{\omega}_i \| \), respectively, for \( i = 1, 2, 3, 4 \). As expected, they all converge to their respective origins asymptotically.

VI. CONCLUSION

In this paper, we have proposed a new technique to handle the general directed graph and studied adaptive distributed observer for an uncertain leader system over directed graphs. We have presented solvability conditions for guaranteeing the convergence of the state of the adaptive distributed observer to the state of the leader system and its unknown parameters. In the future, we will further consider establishing an adaptive distributed observer for an uncertain nonlinear leader system. Since nonlinear system are a class of dynamic system which include the linear system as a special case and can describe much more natural physical phenomenon. The parameter estimation problem of the leader system for each agent by using local information will become much more challenging if the communication network are time-varying and disconnected from time to time or at every time instant. Therefore, it will be much more interesting and practical to investigate such these constrained cases in our future work.

APPENDIX A

Definition 2. [2] Let \( \mathbb{Z}_n \subset \mathbb{R}^{n \times n} \) denote the set of all square matrices of dimension \( n \) with non-positive off diagonal entries. A matrix \( A \in \mathbb{R}^{n \times n} \) is said to be a nonsingular \( M \)-matrix if \( A \in \mathbb{Z}_n \) and all eigenvalues of \( A \) have real positive parts.

Lemma 11. [4] A matrix \( A \in \mathbb{Z}_n \) is a nonsingular \( M \)-matrix if and only if \( A^{-1} \) exists and each entry of \( A^{-1} \) is nonnegative.

The following lemma is rephrased from corollary of Theorem 1 in [27] which can be found in [42].

Lemma 12. Let \( A \) be a real square matrix with positive leading principle minors. Then there exists a positive diagonal matrix \( D \) such that \( DA \) has simple positive eigenvalues.

The following Definition is rephrased from Theorem 6.1. [32] which can be found in [26].

Definition 3. Let \( A \) be a square matrix whose off-diagonal entries are non-positive. Then \( A \) will be called an \( M \)-matrix if it satisfies any of the following (equivalent) conditions:

(i) All leading principal minors of \( A \) are positive.

(ii) \( A \) is nonsingular, and all entries of \( A^{-1} \) are nonnegative.

The following lemma is rephrased from Lemma 1 of [3].

Lemma 13. For the matrix \( W \in \mathbb{R}^{m \times m} \), \( C \in \mathbb{R}^{n \times m} \) and \( B \in \mathbb{R}^{n \times m} \), the following equation

\[ W(CB) = (CB)^TW \]

has a positive definite symmetric(PDS) solution for \( W \) that makes \( W(CB) \) PDS if the matrical product \( CB \) is diagonalizable and has real and positive eigenvalues.

Lemma 14. [9] Consider the scalar dynamics

\[ \dot{x}(t) = a(t)x(t). \]

If there exists \( T, \alpha_1, \alpha_2 > 0 \) such that \( \int_{t}^{t+T} a(\tau)d\tau \leq -\alpha_1 \) and \( -\infty < a(t) < \alpha_2 \) for all \( t \geq t_0 \), then we have \( \| x(t_0) \| e^{-\alpha_1(t-t_0)} \leq \| x(t) \| \).

The following lemma is rephrased from Theorem 1 of [1].

Lemma 15. Let \( F(\cdot) : [0, \infty) \rightarrow \mathbb{R}^{n \times r} \) be regulated matrix function(one-sided limits exist for all \( t \in \mathbb{R}^+ \)). Then

\[ \dot{x} = -F(t)x \]

is exponentially asymptotically stable if and only if for some positive \( \delta, \alpha_1 \) and \( \alpha_2 \), and for all \( t \geq 0 \),

\[ \alpha_1 I_n \leq \int_{t}^{t+\delta} F(s)F^T(s)ds = \alpha_2 I_n. \]

A brief introduction of graph theory is shown in the following which can be found in [10]:

A digraph \( G = (V, E) \) consists of a finite set of nodes \( V = \{1, 2, \ldots, N\} \) and an edge set \( E = \{(i, j), i, j \in V, i \neq j\} \). Weighted directed graph \( G \) is used to model communication among the \( N \) systems. Graph \( G \) consists of a node set \( V = \{1, \cdots, N\} \), and edge set \( E \subseteq (V \times V) \) and a weighted adjacency matrix \( C = [c_{ij}] \in \mathbb{R}^{N \times N} \) with \( c_{ij} \geq 0 \). If \( c_{ij} > 0 \), then \( (j, i) \in E \). The in degree of node \( i \)
is defined as \( d_k = \sum_{j \in N_i} c_{ij} \). Let \( D = diag\{d_1, \ldots, d_N\} \) be the degree matrix of \( G \). The laplacian matrix of \( G \) is defined as \( L = D - C \). A node \( i \) is called a neighbor of a node \( j \) if the edge \((i, j) \in E\). \( N_i \) denotes the subset of \( V \) that consists of all the neighbors of the node \( i \). If the graph \( G \) contains a sequence of edges of the form \((i_1, i_2), (i_2, i_3), \ldots, (i_k, i_{k+1})\), then the set \( \{i_1, i_2, i_3, \ldots, i_k, i_{k+1}\} \) is called a path of \( G \) from \( i_1 \) to \( i_{k+1} \), and node \( i_{k+1} \) is said to be reachable from node \( i_1 \).

APPENDIX B

Proof of Lemma 9

Proof: We first show for any initial condition, \( \lim_{t \to \infty} \tilde{\eta}(t) = 0 \). By the Lemma 8 with the following Lyapunov function candidate for (29):

\[
V = \tilde{\eta}^T (P \otimes I_m) \tilde{\eta} + \mu_3^{-1}\tilde{\omega}^T (W \otimes I) \tilde{\omega}.
\]

From (48) and sufficiently large number \( \mu_3 \), we have

\[
\dot{V}(t) \leq -|| \tilde{\eta}(t) ||^2 \leq 0,
\]

According to Lyapunov Theorem B.1.5 in [21], \((\tilde{\eta}, \tilde{\omega})\) is a uniformly bounded non-increasing function for any \( t \geq t_0 \),

\[
\lim_{t \to \infty} \int_{t_0}^{t} \| \tilde{\omega}(s) \| ds \leq \lim_{t \to \infty} \int_{t_0}^{t} -V(t)ds
\]

\[
= V(t_0) - V(\infty) \leq \infty.
\]

Applying the Barbalat’s Lemma to \( \tilde{\eta}(t) \), we conclude that

\[
\lim_{t \to \infty} \tilde{\eta}(t) = 0.
\]

Now, to show that for any initial condition

\[
\lim_{t \to \infty} \tilde{\omega}(t) = 0.
\]

i.e. for any \( \epsilon > 0 \) and \( T > 0 \), for any initial condition \( \tilde{\eta}(t_0) \) and \( \tilde{\omega}(t_0) \) there exists \( t > T \) such that \( \| \tilde{\omega} \| \leq \epsilon \).

The following proof is similar to the proof of Lemma B.2.3 in [21]. For convenience, let \( \mu_3 = 1 \).

Claim 1. Given any \( \epsilon > 0 \) and \( T > 0 \) and any initial condition \( \tilde{\eta}(t_0) \) and \( \tilde{\omega}(t_0) \), there exist \( t > T \) such that \( \| \tilde{\omega}(t) \| < \epsilon \).

Proof of the claim. We equivalently show by contradiction that for any \( \epsilon > 0 \) a time instant \( t_1 \) such that

\[
\| \tilde{\omega}(t) \| \geq \epsilon, \quad \forall t \geq t_1
\]

does not exist.

Let \( A = (I_N \otimes S(\omega) - \mu_2DH \otimes I_m) \), \( \tilde{W} = (W \otimes I_1) \) and \( \tilde{H} = (DH \otimes I_m) \). Consider the function \( (T > 0) \)

\[
\varphi(t) = \frac{1}{2} \left[ \tilde{\omega}^T(t + T)\tilde{W}\tilde{\omega}(t + T) - \tilde{\omega}^T(t)\tilde{W}\tilde{\omega}(t) \right]
\]

which is bounded since we have shown that \( \| \tilde{\omega} \| \) is bounded for any \( t \geq t_0 \) in Lemma 7. The time derivative of (69) is such that

\[
\varphi(t) = \tilde{\omega}^T(t + T)\tilde{W}\tilde{\omega}(t + T) - \tilde{\omega}^T(t)\tilde{W}\tilde{\omega}(t)
\]

\[
= \int_{t}^{t+T} \frac{d}{d\tau} \left( \tilde{\omega}^T(t)\tilde{W}\tilde{\omega}(t) \right) d\tau
\]

\[
= \int_{t}^{t+T} \frac{d}{d\tau} \left( \tilde{\omega}^T(t)\tilde{W}\tilde{\omega}(t) \right) d\tau
\]

\[
= \int_{t}^{t+T} \left( \tilde{\omega}^T(t)\tilde{W}\tilde{\omega}(t) \right) d\tau
\]

\[
= \int_{t}^{t+T} \left( \tilde{\omega}^T(t)\tilde{W}\tilde{\omega}(t) \right) d\tau
\]

\[
= \int_{t}^{t+T} \left( \tilde{\omega}^T(t)\tilde{W}\tilde{\omega}(t) \right) d\tau
\]

Under Assumptions 2 and 11 for all \( k = 1, \ldots, l \), col \((v_{2k-1}(0), v_{2k}(0)) \neq 0\), by Lemma 5 and Lemma 7, we have \( \tilde{\eta}, \tilde{\eta} v \) and \( \tilde{\omega} \) are uniformly bounded. Let \( M \) be such that \( || \tilde{\eta} || \leq M, || \tilde{\eta} || \leq M \), \( || \tilde{\omega} || \leq M, || \tilde{\eta} || \leq M \) and \( || \tilde{\omega} || \leq M \). Let \( \lambda_w \) and \( \lambda_w \) be the smallest and largest eigenvalues of \( W \). Let \( \lambda_p \) and \( \lambda_p \) be the smallest and largest eigenvalues of \( W \). Then, from equation (70), we have

\[
\int_{t}^{t+T} \left( \tilde{\omega}^T(t + T)\tilde{W}\tilde{\omega}(t + T) - \tilde{\omega}^T(t)\tilde{W}\tilde{\omega}(t) \right) d\tau
\]

\[
\leq \left( 3M^3 \| \tilde{H} \|^2 \lambda_W + 2M^2\lambda_W \| \tilde{H} \| \right) t \int_{t}^{t+T} \| \tilde{\eta} \| d\tau.
\]

On the other hand, from equation (38), there exists a time instant \( t_2 \) such that

\[
\int_{t}^{t+T} \left( \tilde{\omega}^T(t + T)\tilde{W}\tilde{\omega}(t + T) - \tilde{\omega}^T(t)\tilde{W}\tilde{\omega}(t) \right) d\tau
\]

\[
\leq \left( 3M^3 \| \tilde{H} \|^2 \lambda_W + 2M^2\lambda_W \| \tilde{H} \| \right) t \int_{t}^{t+T} \| \tilde{\eta} \| d\tau
\]

Assume now by contradiction that there exists a time instant \( t_1 \) such that (68) holds. Under Assumptions 2 and 11 for all \( k = 1, \ldots, l \), col \((v_{2k-1}(0), v_{2k}(0)) \neq 0\), \( \phi(v)\phi^T(v) \) is a diagonal constant matrix such that there exist a constant \( \beta > 0 \),

\[
\phi(v)\phi^T(v) \geq \beta I_l.
\]
Thus, we have, $\forall t \geq t_0$, $\forall c : \| c \| = 1,$
\[
\int_{t}^{t+T} c^T \left( P \otimes (\phi(v) \phi^T(v)) \right) c d\tau \geq \beta \lambda_p,
\]
which in turn, along with equation (68), implies the following equation
\[
\frac{1}{c^2} \int_{t}^{t+T} \tilde{\omega}^T \left( P \otimes (\phi(v) \phi^T(v)) \right) \tilde{\omega} d\tau \\
\geq \int_{t}^{t+T} \left( \frac{\tilde{\omega}^T}{\| \tilde{\omega} \|} \left( P \otimes (\phi(v) \phi^T(v)) \right) \frac{\tilde{\omega}}{\| \tilde{\omega} \|} d\tau \\
\geq \beta \lambda_p, \quad \forall t \geq t_1.
\]
(73)

Thus, we have, $\forall t \geq t_1$,
\[
\int_{t}^{t+T} \tilde{\omega}^T \left( P \otimes (\phi(v) \phi^T(v)) \right) \tilde{\omega} d\tau \geq \beta \lambda_p c^2,
\]
(74)

From (70), (72) and (74), we obtain
\[
\phi(t) \leq -\frac{\beta}{2} \lambda_p c^2, \quad \forall t \geq t_3
\]
(75)

with $t_3 = \max\{t_1, t_2\}$, which contradicts the boundedness of $\varphi(f)$ for any $t \geq t_0$. The Claim is proved.

By virtue of (38), for any $\epsilon > 0$, there exists a time instant $t_\epsilon$ such that
\[
\| \tilde{\eta}(t) \| \leq \frac{\epsilon}{\sqrt{2} \lambda_p}, \quad \forall t \geq t_\epsilon.
\]
(76)

By virtue of the claim, there exists a time instant $T_\epsilon \geq t_\epsilon$ such that
\[
\| \tilde{\omega}(T_\epsilon) \| \leq \frac{\epsilon}{\sqrt{2} \lambda_W}.
\]
(77)

Since by (41) and (48)
\[
\| \tilde{\omega}(t) \| \leq \sqrt{\| \tilde{\eta}(t_0) \|^2 \lambda_p + \| \tilde{\omega}(t_0) \|^2 \lambda_W},
\]
(78)

with $\tilde{\eta}(t_0)$ and $\tilde{\omega}(t_0)$ initial conditions.

From the initial conditions $\tilde{\eta}(T_\epsilon)$ and $\tilde{\omega}(T_\epsilon)$, according to (76), (77) and (78), we have
\[
\| \tilde{\omega}(t) \| \leq \sqrt{\| \tilde{\eta}(T_\epsilon) \|^2 \lambda_p + \| \tilde{\omega}(T_\epsilon) \|^2 \lambda_W} \leq \epsilon, \quad \forall t \geq T_\epsilon
\]
which implies (40). Therefore the equilibrium point $(\tilde{\eta}, \tilde{\omega}) = 0$ is attractive. Equation (67) holds uniformly with respect to $t_0$, so also do \( \lim_{t \to \infty} \tilde{\eta}(t) = 0 \) and \( \lim_{t \to \infty} \tilde{\omega}(t) = 0 \). It follows that $(\tilde{\eta}, \tilde{\omega}) = 0$ is a globally uniformly asymptotically stable equilibrium point and since system (29) is linear, by the Theorem 4.11 in [16], the equilibrium is also globally exponentially stable. \( \square \)

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