From Fermion Mass Matrices to Neutrino Oscillations

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Abstract

We present an ansatz for the quark and lepton mass matrices, derivable from SO(10) type GUTs, which accommodates a heavy (> 92GeV) top quark and permits large mixings in the $\nu_\mu \leftrightarrow \nu_\tau$ sector (as suggested by the recent Kamiokande and IMB data on the atmospheric neutrinos). The well known asymptotic relations $m_b = m_\tau$, $m_s = \frac{1}{3} m_\mu$ and $m_d m_s = m_e m_\mu$ all hold to a good approximation. Depending on $\nu_\mu \leftrightarrow \nu_\tau$ mixing which can even be maximal, the mixing angle relevant for solar neutrino oscillation lies in the range $7.8 \times 10^{-3} \lesssim \sin^2 2\theta_{e\mu} \lesssim 2.1 \times 10^{-2}$. For the $^{71}$Ga experiment the event rate, normalized against the standard solar model prediction of 132 SNU, is estimated to be between 80 and 20 SNU.

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There is currently a surge of experimental activity around the exciting possibility that one or more species of the neutrinos may possess a tiny mass. The most recent analysis\(^1\) of the Kamiokande\(^2\) and the IMB data\(^3\) on atmospheric neutrino interactions suggest a discrepancy with theoretical expectations\(^4\) which can be nicely explained in terms of neutrino oscillations\(^{1,5}\). Assuming a two flavor \((\nu_\mu \leftrightarrow \nu_\tau)\) oscillation, the relevant mixing angle satisfies the constraint \(\sin^2 2\theta_{\mu\tau} \gtrsim 0.42\). The mass difference squared lies in the range \((10^{-1} - 10^{-3})\text{eV}^2\).

Perhaps the most surprising aspect of this observation, if true, is the relatively large value suggested for the \(\nu_\mu - \nu_\tau\) mixing angle \(\theta_{\mu\tau}\). Within the framework of grand unified theories with the minimal higgs structure, for instance, one typically finds that\(^{6,7}\) \(\theta_{\mu\tau} \simeq |V_{cb}| \simeq 0.05\), where \(V_{cb}\) is the \((cb)\) element of the well known KM matrix of the quark sector. By making some minor modifications in the higgs structure, one could perhaps make \(\theta_{\mu\tau}\) three times larger than \(|V_{cb}|\), but this still is much too small compared to what the atmospheric neutrino data suggests.

The purpose of this paper is to address this and some important related issues within a more general framework, which is inspired by GUTs as well as some earlier work of Fritzsch\(^8\) and others. The idea is to write down an ansatz for the fermion mass matrices which i) is predictive in the quark sector and admits a heavy (> 92GeV) top quark, ii) preserves to a good approximation some well known asymptotic relations including \(m_b \simeq m_\tau, m_\mu \simeq \frac{1}{3}m_\mu, m_d m_\tau \simeq m_\mu m_\mu\) and \(\theta_\ell \simeq (m_d/m_s)^{1/2}\), and iii) allows for large mixings in the \(\nu_\mu \leftrightarrow \nu_\tau\) sector. It turns out that in the quark sector the new ansatz we propose has essentially the same predictive capacity as the original one of ref. \(^8\). A particularly important feature of the new scenario is the restriction on the mixing angle relevant for the solar neutrino problem. One finds a significant deficit relative to the standard solar model prediction, which will soon be tested in the ongoing SAGE\(^9\) and GALLEX experiments.

The ansatz that we will consider can be motivated within a grand unified framework (e.g., SO(10))\(^{10}\) with a non-minimal higgs structure. It works both with supersymmetric (SUSY) as well as with non-SUSY GUTs. In the latter case, in order for the proton lifetime to be compatible with the experimental lower bound, an intermediate step (e.g., \(SU(4)_c \times SU(2)_L \times SU(2)_R\))\(^{11}\) would be needed. Consider then the following ansatz for the quark and lepton mass matrices:

\[
M_d = \begin{pmatrix}
0 & A & 0 \\
A & D e^{i\alpha} & B \\
0 & B & C
\end{pmatrix} \quad M_u = \begin{pmatrix}
0 & A' & 0 \\
A' & 0 & B' \\
0 & B' & C'
\end{pmatrix}
\]
\[
M_l = \begin{pmatrix}
0 & A & 0 \\
A & -3De^{i\alpha} & -3B \\
0 & -3B & C
\end{pmatrix} \quad M^\text{Dirac}_\nu = \begin{pmatrix}
0 & A' & 0 \\
A' & 0 & -3B' \\
0 & -3B' & C'
\end{pmatrix}
\]
\[
M^\text{Majorana}_\nu = \begin{pmatrix}
M_1e^{i\gamma_1} & 0 & 0 \\
0 & M_2e^{i\gamma_2} & M_3e^{i\gamma_3} \\
0 & M_3e^{i\gamma_3} & 0
\end{pmatrix}.
\]

Several comments are in order:

1. We have written down the matrices of Eq. (1) after a suitable redefinition of the phases of the fermion fields. The non-zero elements of the matrices, before this redefinition, are allowed to have arbitrary phases. All but one phase \(\alpha\) can be removed from the Dirac mass matrices. The charged current interaction in this basis is not proportional to the identity matrix, but has the generation structure given in terms of two phase parameters \(\sigma\) and \(\tau\) by

\[
\begin{pmatrix}
1 \\
e^{i\sigma} \\
e^{i\tau}
\end{pmatrix}.
\]

The entries \(A, A', B, B',...\) can be chosen to be real and positive without loss of generality.

2. Within an SO(10) type framework, the entries \(A, A', C, C'\) arise from the higgs \(10\) plets, while \(B, B'\) and \(D\) arise from the \(126\) plets. The reason for the form of the \(D\) terms is to retain, to a good approximation, the well-known asymptotic relation \(m_s \simeq \frac{1}{3}m_\mu\) \(^{(12)}\). The \(B, B'\) terms are motivated by the desire to accommodate both large mixings in the 2-3 lepton sector and a moderately heavy top quark \((m_t \lesssim 160\text{GeV})\). When the parameter \(B\) is set to zero, our ansatz reduces to that of Ref. 12, a detailed analysis of which is presented in Ref. 13;

3. With \(C \gg A, B, D\), one would recover \(m_b = m_\tau\) \(^{(14)}\) to a good approximation. However, a small violation induced through mixing of this asymptotic equality plays an important role in our analysis;

4. The form of the 2-3 sector of the heavy Majorana mass matrix is directly related to the corresponding sectors of the quark and the charged lepton matrices. In other words, the same \(126\) plet of higgs contributes
to these three sectors. The (22) elements $D$ and $M_2$ of Eq. (1) arise from the same Yukawa coupling to a 126 of Higgs. Similarly, $B, B'$ and $M_3$ result from a common Yukawa coupling to another Higgs 126 plet. (Note that the 126 generating (22) elements $D$ and $M_2$ should be distinct from the 126 that generates (23) element $B, B'$ and $M_3$, if it were the same, a non-zero (33) element will also be induced. Consequently, there is no relation of the type $M_2/M_3 = D/B$.) We need an additional (independent) contribution in the Majorana sector (the 1-1 entry) to give a large mass to the heavy $\nu_{eR}$;

5. Since the charged fermion sector is described by 11 fundamental parameters including $\tan \beta$, the ratio of the two Higgs vacuum expectation values, we will have 3 predictions. Additionally, since some of these parameters are phase angles whose moduli are constrained to be less than unity, we also have approximate relations among the physical observables (see below).

It should be emphasized that the above form of the mass matrices is prescribed at the GUT scale $M_U$.

The real symmetric matrix $M_u$ is diagonalized by an orthogonal transformation, $\tilde{O}_u M_u O_u = M_u^{diag}$ (tilde denotes transpose), where

$$O_u \simeq \begin{pmatrix}
1 & -\sqrt{\frac{m_u}{m_c}} & \sqrt{\frac{m_u}{m_t}} m_t \\
\sqrt{\frac{m_u}{m_c}} & 1 & \sqrt{\frac{m_u}{m_t}} \\
-\sqrt{\frac{m_u}{m_t}} & -\sqrt{\frac{m_u}{m_t}} & 1 
\end{pmatrix}. \quad (3)
$$

Here $m_u, -m_c$ and $m_t$ denote the eigenvalues of $M_u$ and we have used the relations $A' \simeq \sqrt{m_u m_c}, B' \simeq \sqrt{m_u m_t}$ and $C' \simeq m_t$.

Since $M_d$ and $M_\ell$ are complex symmetric matrices, their diagonalization is achieved via bi-unitary transformations of the type $\tilde{U}_d M_d U_d = M_d^{diag}$, and $\tilde{U}_\ell M_\ell U_\ell = M_\ell^{diag}$. Denoting the eigenvalues as $(m_d, -m_s, m_b)$ and $(m_e, -m_\mu, m_\tau)$ respectively, the appropriate unitary matrices are:

$$U_d \simeq \begin{pmatrix}
\epsilon^{-i\phi/2} & -\epsilon \sqrt{\frac{m_d}{m_s}} e^{-i\phi/2} & \epsilon \sqrt{\frac{m_d}{m_b}} \\
\epsilon \sqrt{\frac{m_d}{m_s}} e^{i\phi/2} & e^{i\phi/2} & \epsilon \\
-\epsilon \sqrt{\frac{m_d}{m_s}} & -\epsilon & 1 
\end{pmatrix}, \quad (4)$$

4
\[ U_1 \simeq \begin{pmatrix} e^{-i\tilde{\phi}/2} & -\sqrt{\frac{m_e}{m_\mu}} e^{-i\tilde{\phi}/2} & -3\epsilon \sqrt{\frac{m_e}{m_r}} \sqrt{\frac{m_\mu}{m_r}} \\ \sqrt{\frac{m_e}{m_\mu}} e^{i\tilde{\phi}/2} & e^{i\tilde{\phi}/2} & -3\epsilon e^{i\tilde{\phi}/2} \\ 3\epsilon \sqrt{\frac{m_e}{m_\mu}} & 3\epsilon e^{i\tilde{\phi}/2} & 1 \end{pmatrix} . \] (5)

Here we have defined
\[ \epsilon \equiv \frac{1}{4} \sqrt{\frac{m_b^2}{1 - \frac{m_b^2}{m_r^2}}} , \] (6)

assumed \( C \gg B, D \gg A \), and used the relations
\[ A \simeq \sqrt{m_d m_s} \simeq \sqrt{m_e m_\mu} \]
\[ B \simeq \epsilon m_b \]
\[ C \simeq m_b \simeq m_\tau \] (7)
\[ D = \frac{m_b}{2} \left[ \left( 3\frac{m_s^2}{m_b^2} + \frac{1}{3} \frac{m_\mu^2}{m_b^2} \right) - 3\epsilon^4 \right]^{1/2} \]
\[ D \cos \alpha = \pm m_b \left[ e^2 + \frac{\epsilon^2}{3} \left( \frac{m_s^2}{m_b^2} - \frac{1}{3} \frac{m_\mu^2}{m_b^2} \right) \right] . \]

The phases \( \phi, \tilde{\phi} \) are determined in terms of \( B, C, D \) and \( \alpha \) through the relations
\[ (B^2 - CDe^{i\alpha}) = |B^2 - CDe^{i\alpha}| e^{-i\phi} , \] (8)
\[ (B^2 + \frac{1}{3} CDe^{i\alpha}) = |B^2 + \frac{1}{3} CDe^{i\alpha}| e^{-i\tilde{\phi}} . \]

From the relation \( D^2 \geq 0 \), one finds that
\[ \frac{3m_s^2}{m_b^2} + \frac{1}{9} \frac{m_\mu^2}{m_b^2} \geq 3\epsilon^4 . \] (9)

Employing \( |\cos \alpha| \leq 1 \), one can show that
\[ \frac{1}{2} \sqrt{\left| \frac{m_s}{m_b} \right| - \frac{1}{3} \left| \frac{m_\mu}{m_b} \right|} \leq \epsilon \leq \frac{1}{2} \sqrt{\left| \frac{m_s}{m_b} \right| + \frac{1}{3} \left| \frac{m_\mu}{m_b} \right|} . \] (10)
We therefore see that in the limit of a strict equality \( m_b = m_{\tau} \), we have \(| m_u | = 3 | m_s |\). In practice, \( \epsilon \) will be a small positive quantity which would lead to small violations of these relations.

In the quark sector the Kobayashi-Maskawa matrix is given by

\[
V_{KM} = \tilde{O}_u \begin{pmatrix}
1 & e^{i\sigma} \\
e^{i\tau} & e^{i\sigma}
\end{pmatrix} U_d^* \tag{11}
\]

which leads to the following (asymptotic!) matrix elements:

\[
\begin{align*}
| V_{us} | & = | V_{cd} | = | \sqrt{m_d/m_s} - \sqrt{m_u/m_c} e^{i(\sigma - \phi)} | \\
| V_{ub} | & = | \epsilon \sqrt{m_u/m_b} \sqrt{m_s/m_c} + \sqrt{m_u/m_c} e^{i\sigma} \left( \epsilon - \sqrt{m_u/m_t} e^{i(\tau - \sigma)} \right) | \\
| V_{cb} | & = | V_{ts} | = | \epsilon - \sqrt{m_c/m_t} e^{i(\tau - \sigma)} | \\
| V_{td} | & = | \sqrt{m_u/m_t} \sqrt{m_c/m_s} + \sqrt{m_d/m_s} e^{i(\sigma - \phi)} \left( \sqrt{m_u/m_t} - \epsilon e^{i(\tau - \sigma)} \right) | \tag{12}
\end{align*}
\]

In addition, the reparameterization invariant CP violating quantity \( J \) is given by

\[
J = \text{Im} \left( V_{us} V_{cb} V_{ub}^* V_{cs}^* \right) \\
\simeq \left[ \epsilon \sqrt{m_u/m_t} \sqrt{m_u/m_s} \sin(\phi - \tau) + \sin(\phi + \tau - 2\sigma) \right] \\
- \sqrt{m_u/m_t} \sqrt{m_c/m_s} \sin(\phi - \sigma) - \epsilon^2 \sqrt{m_u/m_t} \sqrt{m_c/m_s} \sin(\phi - \sigma) \tag{13}
\]

It is clear from Eqs. (12-13) that once the charged fermion masses are specified, all the mixing angles as well as \( J \) are determined in terms of the phases \( \sigma \) and \( \tau \). This situation is analogous to the Fritzsch ansatz, except that our modified version can accommodate a heavy top quark (due to the difference in the expression for \( | V_{cb} | \)).

As was pointed out earlier, the above relations hold at the GUT scale and so comparison with data requires that we consider their evolution with momenta. It was noted quite some time ago\(^{(16)}\) that as far as the mixing angles are concerned, significant evolution will occur for the entries involving the third family provided that the top quark is sufficiently heavy (\( \geq 100 \text{GeV} \)).

For definiteness, we focus on the supersymmetric case. We further simplify our analysis by assuming that the sparticles and the second higgs doublet are degenerate at 300 GeV. Employing the one loop renormalization
group equations, and fixing the top quark mass at 130 GeV, we first determine the unification scale $M_U$. As low energy inputs we use the precisely known values of \( \alpha_1 \text{ and } \alpha_2 \) at \( M_Z \):

\[
\alpha_1(M_Z) = 0.01013, \quad \alpha_2(M_Z) = 0.03322.
\]

We find the unification scale to be $M_U = 9.8 \times 10^{15}$ GeV and the common gauge coupling at $M_U$ to be $\alpha_U(M_U) = \frac{1}{3\sqrt{\pi}}$. Running backwards, we determine $\alpha_\beta(M_Z)$ to be 0.105, which is in agreement with measurements.

The evolution equations for the elements of the KM matrix can be found in ref. (16). It turns out that for $\tan \beta < \sim 7$ ($\tan \beta \equiv v_2/v_1$, the well-known ratio of the two vevs present in the minimal SUSY extension of the standard model), the coupled equations can be solved semi-analytically. Here we will content ourselves by presenting results (Table 1) showing the variation of the mixing angles with momentum, for varying values of $m_t$ and $\tan \beta$. For $m_t$ not too large ($< \sim 130$ GeV), the variation in $|V_{cb}|$, $|V_{ub}|$, $|V_{td}|$, $|V_{ts}|$ is $\sim 5\%$. However, if the top is close to its maximal allowed value of about 190 GeV, the variations in some of the quantities can exceed 20%. In Table 2 we display the dependence of the quark and charged lepton mass ratios $m(M_U)/m(\mu)$, on $m_t$ and $\tan \beta$. For the $d$ and $s$ quarks, as well as for the $(e, \mu, \tau)$ leptons these ratios are (essentially) independent of $m_t$ and $\tan \beta$ and are in the range (0.177-0.179) and (0.656-0.689) respectively.

From Tables 1-2, we can infer the predictions of our ansatz for the quark masses and mixing angles at the weak scale. From the inequality of Eq. (10) we see that the parameter $\epsilon$ is bounded by $\epsilon \leq 0.10$. There is also a lower bound on $\epsilon$, $\epsilon \geq 0.035$, if we assume that the top mass is around 130 GeV (see Eq. (12) for $|V_{cb}|$). This lower bound, however, goes away if the top quark is much heavier, near its fixed point value.\(^{13,19}\) If $\epsilon$ is set to zero, one recovers the asymptotic relation\(^{12,13,18,19}\)

\[
|V_{cb}| \approx (m_c/m_t)^{1/2}.
\]

This relation (Eq. (14)) is disfavored phenomenologically (although not excluded), both in the non-supersymmetric as well as in the supersymmetric case. It requires a top quark in the mass range 180 – 220 GeV (175 – 190 GeV) in the non-SUSY (SUSY) case. Moreover $|V_{cb}|$ must be $\geq 0.052^{13}$ to be compared with the experimentally favored range 0.043 ± 0.009.

From the constraint 0.035 $\leq \epsilon \leq 0.10$ one can deduce the allowed range for the $b$-quark mass. It follows from Table 2 that $m_b(m_b)$ lies in the range $(4.3 – 5.3) GeV$ (for $\alpha_s(M_Z) = 0.105$), the larger value corresponding to smaller $\epsilon$. This is in accord with the value quoted in Ref. 20, $m_b(m_b) = 4.25 \pm 0.1$ GeV. (We should point out that $m_b$ can be brought further down, as in Ref. 13, even if $\epsilon = 0$, provided the top quark mass is around 180 GeV. Here, however, we are interested in a moderately heavy top ($m_t \sim 130$ GeV).) Since the asymptotic relation $m_b = m_\tau$ holds to within 10%, it follows from Eq. (10) that $|m_s| = \frac{1}{3} |m_\mu|$ should hold to within 5%. This
implies \( m_s(1\text{GeV}) = 130 - 140\text{MeV} \), in relatively good agreement\(^{(20)}\) with observations. From the asymptotic relation \( m_d m_s = m_e m_\mu \), we also have a prediction for the \( d \) quark mass: \( m_d = (5.7 - 6)\text{MeV} \).

Since the variation of the Cabibbo angle with momentum is negligible, we essentially have the Fritzsch prediction for \( |V_{us}| \). On the other hand, \( |V_{ub}| \) can increase by as much as 14\% (even for \( m_t < 160\text{GeV} \)) so that at the weak scale, \( |V_{ub}| = 0.002 - 0.0037 \). The CP parameter \( J \) naturally comes out in the range \( 10^{-4} - 10^{-5} \) consistent with observations in the \( K \) meson system.

To summarize, in the charged fermion sector, our ansatz is quite predictive. It reproduces the well-known asymptotic relations \( m_b = m_\tau, m_s = \frac{1}{3}m_\mu \) and \( m_d m_s = m_e m_\mu \) to a good approximation. The KM angles in the quark sector are determined in terms of the fermion masses and two arbitrary phase angles. Specifically, there are three predictions in the charged fermion sector. One of them is the mass relation \( m_d m_s = m_e m_\mu \), the other two are mixing angle relations of Eq. (12) obtained by eliminating the phases \( \sigma \) and \( \tau \). The predictive capacity of our ansatz is as good as Fritzsch’s for quark mixing angles, but the top quark can be heavy in our case.

Turning now to the neutral lepton sector, the \( 3 \times 3 \) light neutrino mass matrix is obtained from the see-saw formula\(^{(21)}\)

\[
M_\nu = M_\nu^{\text{Dirac}} (M_\nu^{\text{Majorana}})^{-1} \tilde{M}_\nu^{\text{Dirac}}
\]  

(15)

where

\[
(M_\nu^{\text{Majorana}})^{-1} \equiv M^{-1} \begin{pmatrix} r_1 e^{i\phi_1} & 0 & 0 \\ 0 & 0 & r_2 e^{i\phi_2} \\ 0 & r_2 e^{i\phi_2} & 1 \end{pmatrix}.
\]

(16)

[Note that we are making the assumption that the direct contributions to the left handed Majorana neutrino masses are negligible. This is justified in a SUSY SO(10) framework, since the tree-level vacuum expectation value of the field which supplies such Majorana masses can consistently be set to zero.] Here \( M \) is an overall superheavy scale, and \( r_1, r_2 \) denote ratios of superheavy masses. The parameter \( r_2 \) is given by \( r_2 = M_3/M_2 \). It is reasonable to assume \( r_2 \) to be of order 1, since \( M_3/M_2 \sim B/D \) which is of order 1, (see eq. (7)) times ratio of vev’s. We will also discuss the case when \( r_2 \) is not of order 1. The matrix \( M_\nu \) takes the form
\[ M_\nu = \frac{m_1^2}{M} P \begin{bmatrix} 0 & -3\sqrt{\frac{m_3}{m_1}} \frac{m_2}{m_3} r_2 & \sqrt{\frac{m_3}{m_1}} \sqrt{\frac{m_2}{m_3}} r_2 \\ -3\sqrt{\frac{m_1}{m_3}} \frac{m_2}{m_1} r_2 & \frac{9}{3} \sqrt{\frac{m_3}{m_1}} + \frac{m_3}{m_1} r_1 e^{i\phi_1} & -3\sqrt{\frac{m_1}{m_3}} + \frac{9}{3} \frac{m_3}{m_1} r_2 e^{i\phi_2} \\ \sqrt{\frac{m_1}{m_3}} \sqrt{\frac{m_2}{m_1}} r_2 & -3\sqrt{\frac{m_1}{m_3}} + \frac{9}{3} \frac{m_3}{m_1} r_2 e^{i\phi_2} & 1 - 6\sqrt{\frac{m_1}{m_3}} r_2 e^{i\phi_2} \end{bmatrix} P \]  

(17)

where \( P \) is the diagonal phase matrix \( P = diag[e^{i\phi_2}, 1, 1] \). Observe that \( M_\nu \) has five parameters, viz., \( r_1, r_2, \phi_1, \phi_2 \) and \( M \). Note however, that the phase \( \phi_1 \) appears with a very small coefficient making it an irrelevant parameter. The four effective parameters describe six observables: three neutrino masses and the three neutrino mixing angles. This leads to two predictions. We shall see below that these two are predictions for mixing angles in terms of the neutrino mass ratios.

It is convenient to diagonalize \( M_\nu \) in two stages. Let \( M_\nu' \) denote the part of \( M_\nu \) which excludes the two diagonal phase matrices. We can reduce \( M_\nu' \) to an effective \( 2 \times 2 \) matrix (with zeroes along the first row and column) through the transformation \( \tilde{U}_1 M_\nu' U_1 \), where \( U_1 \) is a unitary matrix given by

\[ U_1 = \begin{pmatrix} e^{i\phi_2} & -\frac{1}{9} \sqrt{\frac{m_3}{m_1}} & -\frac{1}{3} \sqrt{\frac{m_3}{m_1}} \\ \frac{1}{9} \sqrt{\frac{m_1}{m_3}} & e^{-i\phi_2} & 0 \\ \frac{1}{3} \sqrt{\frac{m_1}{m_3}} & 0 & e^{-i\phi_2} \end{pmatrix} \].

(18)

Note that we are justified in neglecting the term \( \frac{m_3}{m_1} r_1 e^{i\phi_1} \) as long as \( r_1 \) is of order unity.

The 2-3 sector of the transformed matrix coincides with that of \( M_\nu' \). It can be diagonalized by a unitary matrix \( U_2 \) of the type

\[ U_2 = e^{i\gamma} \begin{pmatrix} 1 & 0 & 0 \\ 0 & ce^{i(\rho + \delta)} & se^{i(\rho - \delta)} \\ 0 & -se^{-i(\rho - \delta)} & ce^{-i(\rho + \delta)} \end{pmatrix} \]

(19)

where \( c \equiv \cos \theta \) and \( s \equiv \sin \theta \). In particular, the expressions for \( \theta \) and \( \rho \) are as follows:
\[
\tan 2\theta = \frac{6\sqrt{\frac{m_2}{m_1}} \left| \frac{m_3}{m_1} (1 - 3\sqrt{\frac{m_3}{m_1}} r_2 e^{-i\phi_2} + (1 - 3\sqrt{\frac{m_3}{m_1}} r_2 e^{i\phi_2}) (1 - 6\sqrt{\frac{m_3}{m_1}} r_2 e^{-i\phi_2}) \right|}{\left| 1 - 6\sqrt{\frac{m_3}{m_1}} r_2 e^{i\phi_2} \right|^2 - (9\frac{m_3}{m_1})^2}
\]

(20)

\[
\tan 2\rho = \frac{-3\sqrt{\frac{m_3}{m_1}} r_2 \sin \phi_2 \left( 1 + 9\sqrt{\frac{m_3}{m_1}} m_2 \right)}{1 - 9\sqrt{\frac{m_3}{m_1}} \cos \phi_2 + 18\frac{m_3}{m_1} r_2^2 - 27\frac{m_2^2}{m_1} \cos \phi_2}
\]

(21)

The light neutrino mass eigenvalues (mass eigenstates are denoted \(\nu_1,2,3\)) turn out to be:

\[
m_{\nu_1} = \frac{m_1^2}{8M} r_1
\]

\[
m_{\nu_2} = 81\frac{m_2^2}{M} r_2^2 \left| 1 - 6\sqrt{\frac{m_2}{m_1}} r_2 e^{i\phi_2} \right|^2 + 2 \left| -3\sqrt{\frac{m_2}{m_1}} r_2 e^{i\phi_2} \right|^2 + \left( 9\frac{m_2}{m_1} \right)^2 \right|^{-\frac{1}{2}}
\]

\[
m_{\nu_3} = \frac{m_2^2}{M} \left| 1 - 6\sqrt{\frac{m_2}{m_1}} r_2 e^{i\phi_2} \right|^2 + 2 \left| -3\sqrt{\frac{m_2}{m_1}} r_2 e^{i\phi_2} \right|^2 + \left( 9\frac{m_2}{m_1} \right)^2 \right|^{\frac{1}{2}}
\]

(22)

We are now in a position to write down the elements of the lepton KM matrix \(V_{KM}^{\text{lepton}}\). Let \(U_\nu \equiv U_1 U_2\). Then

\[
V_{KM}^{\text{lepton}} = U_\nu^\dagger \left( \begin{array}{ccc} e^{i\phi_2} & e^{i\sigma} & e^{i\tau} \end{array} \right) U_1
\]

(23)

and one obtains for the off-diagonal elements the expressions:
\[ |V_{\nu 1\mu}| = \left| \sqrt{\frac{m}{m_\mu}} - \frac{1}{9} \sqrt{\frac{m}{m_e}} e^{i(\sigma - \tilde{\phi})} \right| \]

\[ |V_{\nu 1\tau}| = \left| \frac{1}{3} \sqrt{\frac{m}{m_c}} e^{i\sigma} \left( \epsilon - e^{i(\tau - \sigma)} \sqrt{\frac{m}{m_\tau}} \right) + 3\epsilon \sqrt{\frac{m}{m_\tau}} \sqrt{\frac{m}{m_e}} \right| \]

\[ |V_{\nu 2e}| = \cos \theta \left| \sqrt{\frac{m}{m_\mu}} - \frac{1}{9} \sqrt{\frac{m}{m_e}} e^{-i(\sigma - \tilde{\phi})} \right| \]

\[ |V_{\nu 2\tau}| = |V_{\nu 3\mu}| = |3\epsilon \cos \theta + \sin \theta e^{i(\tau - \sigma + 2\rho)}| \]

\[ |V_{\nu 3e}| = \left| \sin \theta \left( \sqrt{\frac{m}{m_\mu}} - \frac{1}{9} \sqrt{\frac{m}{m_c}} e^{i(\tilde{\phi} - \sigma)} \right) + 3\epsilon \cos \theta \sqrt{\frac{m}{m_\mu}} e^{i(\frac{\tilde{\phi}}{2} - \sigma - 2\phi_2)} \right| 
\]

\[ -\frac{1}{3} \cos \theta \left( \sqrt{\frac{m}{m_\mu}} e^{i(2\rho + \tilde{\phi})} \right). \]

Several comments are in order:

1. Unlike the quark sector, the mixing angles in the lepton sector do not run between \( M_U \) and the weak scale since the right-handed neutrinos are superheavy.

2. The mass ratio \( m_{\nu_2}/m_{\nu_3} \) is enhanced by almost two orders of magnitude relative to what one naively expects from the see-saw mechanism. This is again related to the factor 3 in the ansatz for the fermion mass matrices which, in turn, was motivated by the desire to include large mixings. The enhancement will be important when we discuss atmospheric and solar neutrino oscillations.

3. The mixing angle relevant for the MSW explanation of the solar neutrino puzzle is given by

\[ |V_{\nu 2e}| = \cos \theta \left| \sqrt{\frac{m}{m_\mu}} - \frac{1}{9} \sqrt{\frac{m}{m_e}} e^{i(\sigma - \tilde{\phi})} \right|. \]

Note that as \( \theta \to 0 \) the expression for \( |V_{\nu 2e}| \) essentially coincides with the one derived earlier in ref. (6). If, the recent Kamiokande results on the atmospheric neutrino survive the test of time, then \( \theta \) is expected to lie in the range \( 18^\circ \lesssim \theta \lesssim 45^\circ \) (see below). Taking all this into account, we estimate that \( 7.8 \times 10^{-3} \lesssim \sin^2 2\theta_{e\mu} \lesssim 2.1 \times 10^{-2} \). The corresponding \( |\Delta m^2| \) is in the range \( (5 \times 10^{-6} - 2 \times 10^{-6})eV^2 \). For the Gallium experiments currently under way the estimated event rate\(^{22}\), normalized relative to the standard solar model value of 132 SNU, lies between 80 and 20 SNU.
4. A combined fit to the solar neutrino and the atmospheric neutrino data requires $m_{\nu_2} = (1.4 - 2.2) \times 10^{-3} eV$, $m_{\nu_3} = (0.03 - 0.6) eV$. The ratio $m_{\nu_2}/m_{\nu_3}$ should then lie in the range $(2.3 \times 10^{-3} - 7.3 \times 10^{-2})$. From Eq. (22), this constraint implies a lower limit on the parameter $r_2$, $r_2 > 0.8$. If we restrict to $r_2 = (0.8 - 3)$ so that there is no hierarchy in the superheavy masses, the 2-3 mixing angle $\theta$ in the neutrino sector is in the range $\theta = (18^\circ - 45^\circ)$. This certainly is in the right parameter range for atmospheric neutrino oscillations. For large values of $r_2$, the angle $\theta$ approaches $3/2 \sqrt{m_c/m_t} \sim 7^\circ$ (see eq. (20)). Even in this case, the atmospheric neutrino deficit can be accommodated, if $\epsilon$ is near its upper limit of 0.1 (see $V_{\nu_2\tau}$ of eq. (24)). Note that $\theta$ close to $45^0$ gives the largest count ($\sim 80$ SNU) for the SAGE/GALLEX experiment. The effective $\nu_\mu \leftrightarrow \nu_\tau$ mixing angle for this case lies in the range $0.69 \lesssim \sin^2 2\theta_{\mu\tau} \lesssim 1$.

In conclusion, the scheme presented here was guided by the desire to retain to the extent possible the simplicity of the original Fritzsch ansatz and come up with some predictions in the neutrino sector. The SAGE/GALLEX experiment should soon be able to tell if we are on the right track!

Note added:

After the submission of this paper for publication, GALLEX collaboration has announced their result: $83 \pm 19 \pm 8$ SNU. This is in remarkable agreement with our prediction (see eq. (25) and the subsequent discussions).
References

1. E.W. Beier et al., University of Pennsylvania preprint (1992).
2. K.S. Hirata et al., Kamiokande-II Collaboration, submitted to Phys. Lett. B (1992).
3. R. Becker-Szendy et al., Boston University preprint BUHEP-91-24 (1992).
4. T.K. Gaisser, Talk given at the Fermilab Workshop on the Many Aspects of Neutrino Physics, November 1991, Bartol Preprint BA-92-03 (1992) and references therein.
5. For earlier phenomenological analysis see:
   J. Learned, S. Pakvasa and T. Weiler, Phys. Lett. B207, 79 (1988);
   V. Barger and K. Whisnant, Phys. Lett. B209, 365 (1988);
   K. Hidaka, M. Honda, and S. Midorikawa, Phys. Rev. Lett. 61, 1537 (1988);
   C. Albright, Phys. Rev. D45, 725 (1992).
6. G. Lazarides and Q. Shafi, Nucl. Phys. B350, 179 (1991);
   ibid. B364, 3 (1991).
7. S. Bludman, D. Kennedy and P. Langacker, Phys. Rev. D 45, 1810 (1992).
8. H. Fritzsch, Phys. Lett. B70, 436 (1977);
   ibid. B78, 317 (1978).
9. A.I. Abazov et al. Phys. Rev. Lett. 67, 3332 (1991).
10. H. Georgi, in “Particles and Fields,” C.E. Carlson, ed., AIP, New York (1974)
    H. Fritzsch and P. Minkowski, Ann. Phys. 93, 193 (1975).
11. J.C. Pati and A. Salam, Phys. Rev. D10, 275 (1974).
12. H. Georgi and C. Jarlskog, Phys. Lett., B89, 297 (1979).
13. S. Dimopoulos, L. Hall and S. Raby, Phys. Rev. Lett. 68, 1984 (1992);
    V. Barger, M.S. Berger, T. Han and M. Zralek, University of Wisconsin preprint MAD/PH/693 (1992).
14. M. Chanowitz, J. Ellis and M.K. Gaillard, Nucl. Phys. B128, 506 (1977);
    A. Buras, J. Ellis, M.K. Gaillard and D. Nanopoulos, Nucl. Phys. B135, 66 (1978).
15. C. Jarlskog, Phys. Rev. Lett., 55, 1039 (1985).

16. K.S. Babu, Z. Phys. C 35, 69 (1987);
   K. Sasaki, Z. Phys. C 32, 149 (1986);
   B. Grzadkowski, M. Lindner and S. Theisen, Phys. Lett. B198, 64 (1987).

17. U. Amaldi, Wim de Boer, and H. Fürstenau, Phys. Lett. 260B, 447 (1991).

18. M. Freire, G. Lazarides and Q. Shafi, Mod. Phys. Lett. A5, 2453 (1991).

19. J. Harvey, P. Ramond and D. Reiss, Nucl. Phys. B199, 223 (1982);
   P. Ramond, University of Florida preprint 92-4 (1992);

20. J. Gasser and H. Leutwyler, Phys. Rep. 87, 77 (1982).

21. M. Gell-Mann, P. Ramond and R. Slansky, in Supergravity, ed., F. van Nieuwenhuizen and D. Freedman (North Holland, 1979), p. 315;
   T. Yanagida, in "Proceedings of the Workshop on Unified Theory and Baryon Number in the Universe", ed. A. Sawada and H. Sugawara,
   (KEK, Tsukuba, Japan, 1979).

22. See for example:
   V. Barger, R.J.N. Phillips and K. Whisnant, Phys. Rev. D 43, 1110 (1991);
   A.J. Baltz and J. Weneeser, Phys. Rev. Lett. 66, 520 (1991).
Table 1: The evolution factor $\left| \frac{V_{ij}(m_t)}{V_{ij}(m_U)} \right|$ for the quark mixing angles $|V_{ub}|$, $|V_{cb}|$, $|V_{td}|$ and $|V_{ts}|$ (as functions of $m_t$ (in GeV) and $\tan \beta$). The corresponding factor for the CP parameter $J$ is given by the square of these numbers. The running of all the other elements of $V$ are negligible. The dashes indicate that the top quark Yukawa coupling has become non-perturbative before the GUT scale.

| $m_t \rightarrow$ | 100  | 130  | 160  | 190  |
|-------------------|------|------|------|------|
| $\downarrow \tan \beta$ |      |      |      |      |
| 1                 | 1.051| 1.139|  --  |  --  |
| 2                 | 1.027| 1.055| 1.122|  --  |
| 3                 | 1.023| 1.046| 1.093| 1.359|
| 4                 | 1.022| 1.043| 1.085| 1.244|
| 5                 | 1.021| 1.041| 1.081| 1.217|
| 6                 | 1.021| 1.041| 1.080| 1.205|
| 7                 | 1.021| 1.040| 1.079| 1.198|
Table 2: The evolution factors $\frac{m_i(M_{\text{tu}})}{m_i(\mu)}$ for the quark masses as function of $m_t$ and $\tan \beta$. For $u$ quark, $\mu = 1\text{GeV}$ is chosen and for $c$ and $b$ quarks, $\mu = 1.27\text{GeV}$ and $4.25\text{GeV}$ respectively. The corresponding factors for $(d,s)$ and $(e,\mu,\tau)$ are independent of $m_t$ and $\tan \beta$ and are equal to 0.178 and 0.673 respectively.

| $m_t \rightarrow$ | 100  | 130  | 160  | 190  |
|-------------------|------|------|------|------|
| $\downarrow \tan \beta$ |      |      |      |      |
| $u$               | 0.201| 0.257|      |      |
| 1                 |      |      |      |      |
| $c$               | 0.212| 0.272|      |      |
| $b$               | 0.242| 0.263|      |      |
| 2                 |      |      |      |      |
| $u$               | 0.187| 0.205| 0.249|      |
| $c$               | 0.198| 0.216| 0.263|      |
| $b$               | 0.236| 0.244| 0.260|      |
| 3                 |      |      |      |      |
| $u$               | 0.185| 0.199| 0.230| 0.448|
| $c$               | 0.196| 0.211| 0.243| 0.474|
| $b$               | 0.236| 0.242| 0.254| 0.317|
| 4                 |      |      |      |      |
| $u$               | 0.184| 0.198| 0.225| 0.343|
| $c$               | 0.195| 0.209| 0.238| 0.363|
| $b$               | 0.235| 0.241| 0.252| 0.290|
| 5                 |      |      |      |      |
| $u$               | 0.184| 0.197| 0.223| 0.322|
| $c$               | 0.194| 0.208| 0.236| 0.340|
| $b$               | 0.235| 0.241| 0.251| 0.284|
| 6                 |      |      |      |      |
| $u$               | 0.184| 0.196| 0.222| 0.312|
| $c$               | 0.194| 0.208| 0.234| 0.330|
| $b$               | 0.235| 0.241| 0.251| 0.281|
| 7                 |      |      |      |      |
| $u$               | 0.184| 0.196| 0.221| 0.307|
| $c$               | 0.194| 0.207| 0.234| 0.325|
| $b$               | 0.235| 0.240| 0.250| 0.280|