REASONING ABOUT POLYMORPHIC MANIFEST CONTRACTS

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Abstract. Manifest contract calculi, which integrate cast-based dynamic contract checking and refinement type systems, have been studied as foundations for hybrid contract checking. In this article, we study techniques to reasoning about a polymorphic manifest contract calculus, including a few program transformations related to static contract verification. We first define a polymorphic manifest contract calculus $F^H$, which is much simpler than a previously studied one with delayed substitution, and a logical relation for it and prove that the logical relation is sound with respect to contextual equivalence. Next, we show that the upcast elimination property, which has been studied as correctness of subtyping-based static cast verification, holds for $F^H$. More specifically, we give a subtyping relation (which is not part of the calculus) for $F^H$ types and prove that a term obtained by eliminating upcasts—casts from one type to a supertype of it—is logically related and so contextually equivalent to the original one. We also justify two other program transformations for casts: selfification and static cast decomposition, which help upcast elimination. A challenge is that, due to the subsumption-free approach to manifest contracts, these program transformations do not always preserve well-typedness of terms. To address it, the logical relation and contextual equivalence in this work are defined as semityped relations: only one side of the relations is required to be well typed and the other side may be ill typed.

1. Introduction

1.1. Software contracts. Software contracts [25] are a promising program verification tool to develop robust, dependable software. Contracts are agreements between a supplier and a client of software components. On one hand, contracts are what the supplier guarantees. On the other hand, they are what the client requires. Following Eiffel [25], a pioneer of software contracts, contracts in this work are described as executable Boolean expressions written in the same language as the program. For example, the specification that both numbers $x$ and $y$ are either positive or negative is described as Boolean expression “$x \times y > 0$”.

Contracts can be verified by two complementary approaches: static and dynamic verification. Dynamic verification is possible due to executability of contracts—the runtime system can confirm that a contract holds by evaluating it. Since Eiffel advocated
“Design by Contracts” [25], there has been extensive work on dynamic contract verification [33, 20, 11, 12, 43, 38, 8, 6, 7, 39]. Dynamic verification is easy to use, while it brings possibly significant run-time overhead [12] and, perhaps worse, it cannot check all possible execution paths, which may lead to missing critical errors. Static verification [32, 45, 5, 40, 27, 41] is another, complementary approach to program verification with contracts. It causes no run-time overhead and guarantees that contracts are always satisfied at run time, while it is difficult to use—it often requires heavy annotations in programs, gives complicated error messages, and restricts the expressive power of contracts.

1.2. Manifest contracts. To take the best of both, hybrid contract verification—where contracts are verified statically if possible and, otherwise, dynamically—was proposed by Flanagan [13], and calculi of manifest contracts [13, 15, 19, 4, 36, 35, 34] have been studied as its theoretical foundation. Manifest contracts refer to contract systems where contract information occurs as part of types. In particular, contracts are embedded into types by refinement types \{x:T | e\},\(^1\) which denote a set of values \(v\) of \(T\) such that \(v\) satisfies Boolean expression \(e\) (which is called a contract or a refinement), that is, \(e[v/x]\) evaluates to \true\. For example, using refinement types, a type of positive numbers is represented by \{x:int | x > 0\}.

Dynamic verification in manifest contracts is performed by dynamic type conversion, called casts. A cast \(T_1 \Rightarrow T_2\)\(^\ell\) checks that, when applied to value \(v\) of source type \(T_1\), \(v\) can behave as target type \(T_2\). In particular, if \(T_2\) is a refinement type, the cast checks that \(v\) satisfies the contract of \(T_2\). If the contract check succeeds, the cast returns \(v\); otherwise, if it fails, an uncatchable exception, called blame, will be raised. For example, let us consider cast \(\{x:int | \text{prime}\ x\} \Rightarrow \{x:int | x > 2\}\)\(^\ell\), where \text{prime}\ is a Boolean function that decides if a given integer is a prime number. If this cast is applied to a prime number other than 2, the check succeeds and the cast application returns the number itself. Otherwise, if it is applied to 2, it fails and blame is raised. The superscript \(\ell\) (called blame label) on a cast is used to indicate which cast has failed.

Static contract verification is formalized as subtyping, which statically checks that any value of a subtype behaves as a supertype. In particular, a refinement type \{x:T_1 | e_1\} is a subtype of another \{x:T_2 | e_2\} if any value of \(T_1\) satisfying \(e_1\) behaves as \(T_2\) and satisfies \(e_2\). For example, \{x:int | \text{prime}\ x\} is a subtype of \{x:int | x > 0\} because all prime numbers should be positive.

Hybrid contract verification integrates these two verification mechanisms of contracts. In the hybrid approach, for every program point where a type \(T_1\) is required to be a subtype of \(T_2\), a type checker first tries to solve the instance of the subtyping problem statically. Unfortunately, since contracts are arbitrary Boolean expressions in a Turing-complete language, the subtyping problem is undecidable in general. Thus, the type checker may not be able to solve the problem instance positively or negatively. In such a case, it inserts a cast from \(T_1\) to \(T_2\) into the program point in order to dynamically ensure that run-time values of \(T_1\) behave as \(T_2\). For example, let us consider function application \(f\ x\) where \(f\) and \(x\) are given types \{y:int | \text{prime}\ y\} \rightarrow \text{int}\) and \(T \overset{\text{def}}{=} \{y:int | 2 < y < 8 \text{ and odd}\ y\}\), respectively. Given this expression, the type checker tries to see if \(T\) is a subtype of

\(^{1}\)Although in the context of static verification the underlying type \(T\) of a refinement type \{x:T | e\} is restricted to be a base type usually, this work allows it to be arbitrary; this extension is useful to describe contracts for abstract data types [4, 35].
If the checker is strong enough, it will find out that values of \( T \) are only three, five, and seven and that the subtyping relation holds and accept \( f \); otherwise, cast \( \langle T \Rightarrow \{ y:\text{Int} \mid \text{prime? } y \} \rangle^\ell \) is inserted to check \( x \) satisfies contract \( \text{prime? } \) at run time and the resulting expression \( f(\langle T \Rightarrow \{ y:\text{Int} \mid \text{prime? } y \} \rangle^\ell x) \) will be evaluated.

1.3. Our work. In this article, we study program reasoning in manifest contracts. The first goal of the reasoning is to justify hybrid contract verification. As described in Section 1.2, a cast is inserted if an instance of the subtyping problem is not solved statically. Unfortunately, due to undecidability of the subtyping problem, it is possible that casts from a type to its supertype—which we call upcasts—are inserted, though they are actually unnecessary. How many upcasts are inserted rests on a prover used in static verification: the more powerful the prover is, the less upcasts are inserted. In other words, the behavior of programs could be dependent on the prover due to the insertion of upcasts, which is not very desirable because the dependency on provers would make it difficult to expect how programs behave when the prover is modified. We show that it is not the case, that is, the presence of upcasts has no influences on the behavior of programs; this property is called the upcast elimination.

In fact, the upcast elimination has been studied in the prior work on manifest contracts [13, 19, 4], but it is not satisfactory. Flanagan [13] and Belo et al. [4] studied the upcast elimination for a simply typed manifest contract calculus and a polymorphic one, respectively, but it turned out that their calculi are flawed [19, 35]. While Knowles and Flanagan [19] has resolved the issue of Flanagan, their upcast elimination deals with only closed upcasts; while Sekiyama et al. [35] fixed the flaw in Belo et al., they did not address the upcast elimination; we discuss in more detail in Section 7. As far as we know, this work is the first to show the upcast elimination for open upcasts.

We introduce a subsumption-free polymorphic manifest contract calculus \( F_H \) and show the upcast elimination for it. \( F_H \) is subsumption-free in the sense that it lacks a typing rule of subsumption, that is, to promote the type of an expression to a supertype (in fact, subtyping is not even part of the calculus) and casts are necessary everywhere a required type is not syntactically equivalent to the type of an expression. In this style, static verification is performed “post facto”, that is, upcasts are eliminated post facto after typechecking. A subsumption-free manifest contract calculus is first developed by Belo et al. [4] to avoid the circularity issue of manifest contract calculi with subsumption [19, 4]. However, their metatheory turned out to rest on a wrong conjecture [35]. Sekiyama et al. [35] revised Belo et al.’s work and resolved their issues by introducing a polymorphic manifest contract calculus equipped with delayed substitution, which suspends substitution for variables in casts until their refinements are checked. While delayed substitution ensures type soundness and parametricity, it makes the metatheory complicated. In this work, we adopt usual substitution to keep the metatheory simple. To ensure type soundness under usual substitution, we—inspired by Sekiyama et al. [36]—modify the semantics of casts so that all refinements in the target type of a cast are checked even though they have been ensured by the source type, whereas checks of refinements which have been ensured are skipped in the semantics by Belo et al. [4] and Sekiyama et al. [35]. For example, given \( \{x:\text{Int} \mid \text{prime? } x \} \Rightarrow \{ y;\{x:\text{Int} \mid \text{prime? } x \} \mid y > 2 \}^\ell \), our “fussy” semantics checks both \( \text{prime? } x \) and \( y > 2 \), while Belo et al.’s “sloppy” semantics checks only \( y > 2 \) because \( \text{prime? } x \) is ensured by the source type. Our fussy semantics resolves the issue of type soundness in Belo et al. and is arguably simpler than Sekiyama et al.
In addition to the upcast elimination, we study reasoning about casts to make static contract verification more effective. In particular, this work studies two additional reasoning techniques. The first is selfification [28], which embeds information of expressions into their types. For example, it gives expression \( e \) of integer type \( \text{Int} \) a more informative refinement type \( \{ x : \text{Int} \mid x = \text{Int} e \} \) (where \( = \text{Int} \) is a Boolean equality operator on integers). The selfification is easily extensible to higher-order types, and it is especially useful when given type information is not sufficient to solve subtyping instances; see Section 6.2 for an example. We formalize the selfification by casts: given \( e \) of \( T \), we show that \( e \) is equivalent to a cast application \( \langle T \Rightarrow \text{self}(T, e) \rangle \ell e \), where \( \text{self}(T, e) \) is the resulting type of embedding \( e \) into \( T \). In other words, \( e \) behaves as an expression of \( \text{self}(T, e) \). The second is static cast decomposition, which leads to elimination of more upcasts obtained by reducing nonredundant casts.

We show correctness of three reasoning techniques about casts—the upcast elimination, the selfification, and the cast decomposition—based on contextual equivalence: we prove that (1) an upcast is contextually equivalent to an identity function, (2) a cast application \( \langle T \Rightarrow \text{self}(T, e) \rangle \ell e \) is to \( e \), and (3) a cast is to its static decomposition. We have to note that contextual equivalence that relates only terms of the same type (except for the case of type variables) is useless in this work because we want to show contextual equivalence between terms of different types. For example, an upcast and an identity function may not be given the same type in our calculus for the lack of subsumption: a possible type of an upcast \( \langle T_1 \Rightarrow T_2 \rangle \ell \) is only \( T_1 \rightarrow T_2 \), whereas types of identity functions take the form \( T \rightarrow T \), which is syntactically different from \( T_1 \rightarrow T_2 \) for any \( T \) if \( T_1 \neq T_2 \). Instead of such usual contextual equivalence—which we call typed contextual equivalence—we introduce semityped contextual equivalence, where a well-typed term and a possibly ill-typed term can be related, and show correctness of cast reasoning based on it.

Since, as is well known, it is difficult to prove contextual equivalence of programs directly, we apply a proof technique based on logical relations [30, 31]. We develop a logical relation for manifest contracts and show its soundness with respect to semityped contextual equivalence. We also show completeness of our logical relation with respect to well-typed terms in semityped contextual equivalence, via semityped CIU-equivalence [23]. The completeness implies transitivity of semityped contextual equivalence, which is nontrivial in manifest contracts.

1.4. Organization and proofs. The rest of this paper is organized as follows. We define our polymorphic manifest contract calculus \( F_H \) equipped with fussy cast semantics in Section 2. Section 3 introduces semityped contextual equivalence and Section 4 develops a logical relation for \( F_H \). We show that the logical relation is sound with respect to semityped contextual equivalence and complete for well-typed terms in Section 5. Using the logical relation, we show the upcast elimination, the selfification, and the cast decomposition in Section 6. After discussing related work in Section 7, we conclude in Section 8.

Most of our proofs are written in the pencil-and-paper style, but the proof of cotermination, which is a key, but often flawed, property of manifest contracts, is given by Coq proof script coterm.v at https://skymountain.github.io/work/papers/fh/coterm.zip.

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As we will discuss later, showing transitivity of typed contextual equivalence is not trivial, either.
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Types

\[ B ::= \text{Bool} \mid \ldots \]
\[ T ::= B \mid \alpha \mid x : T_1 \to T_2 \mid \forall \alpha. T \mid \{x : T\} e \]

Typing Contexts

\[ \Gamma ::= \emptyset \mid \Gamma, x : T \mid \Gamma, \alpha \]

Values and Terms

\[ v ::= k \mid \lambda x : T. e \mid \Lambda \alpha. e \mid \langle T_1 \Rightarrow T_2 \rangle^\ell \]
\[ e ::= v \mid x \mid \text{op} (e_1, \ldots, e_n) \mid e_1 e_2 \mid e T \]
\[ \langle \langle \{x : T_1 \mid e_1\}, e_2 \rangle \rangle^{\ell} \mid \langle \langle x : T_1 \mid e_1\rangle, e_2, v \rangle^{\ell} \mid \uparrow \ell \]

Figure 1: Syntax.

2. Polymorphic Manifest Contract Calculus \( F_H \)

This section formalizes a polymorphic manifest contract calculus \( F_H \) and proves its type soundness. As described in Section 1.3, our run-time system checks even refinements which have been ensured already, which enables us to prove cotermination, a key property to show type soundness and parametricity without delayed substitution. We compare our fussy cast semantics with the sloppy cast semantics provided by Belo et al. \([4]\) in Section 2.2. Greenberg \([14]\) provides a few motivating examples of polymorphic manifest contracts such as abstract datatypes for natural numbers and string transducers; see Section 3.1 in the dissertation for details.

2.1. Syntax. Figure 1 shows the syntax of \( F_H \), which is based on Belo et al. \([4]\). Types, ranged over by \( T \), are from the standard polymorphic lambda calculus except dependent function types and refinement types. Base types, denoted by \( B \), are parameterized, but we suppose that they include Boolean type \( \text{Bool} \) for refinements. We also assume that, for each \( B \), there is a set \( \mathcal{K}_B \) of constants of \( B \); in particular, \( \mathcal{K}_{\text{Bool}} = \{ \text{true}, \text{false} \} \). Refinement types \( \{x : T\mid e\} \), where variable \( x \) of type \( T \) is bound in Boolean expression \( e \), denotes the set of values \( v \) of \( T \) such that \( e [v/x] \) evaluates to \( \text{true} \). As the prior work \([4, 36, 35]\), our refinement types are general in the sense that any type \( T \) can be refined, while some work \([28, 13]\) allows only base types to be refined. Dependent function types \( x : T_1 \to T_2 \) bind variable \( x \) of domain type \( T_1 \) in codomain type \( T_2 \), and universal types \( \forall \alpha. T \) bind type variable \( \alpha \) in \( T \). Typing contexts \( \Gamma \) are a sequence of type variables and bindings of the form \( x : T \), and we suppose that term and type variables bound in a typing context are distinct.

Values, ranged over by \( v \), consist of casts and usual constructs from the call-by-value polymorphic lambda calculus—constants (denoted by \( k \)), term abstractions, and type abstractions. Term abstractions \( \lambda x : T. e \) and type abstractions \( \Lambda \alpha. e \) bind \( x \) and \( \alpha \) in the body \( e \), respectively. Casts \( \langle T_1 \Rightarrow T_2 \rangle^\ell \) from source type \( T_1 \) to target type \( T_2 \) check that arguments of \( T_1 \) can behave as \( T_2 \) at run time. Label \( \ell \) indicates an abstract location of the cast in source code and it is used to identify failure casts; in a typical implementation, it would be a pair of the file name and the line number where the cast is given. We note that casts in \( F_H \) are not equipped with delayed substitution, unlike Sekiyama et al. \([35]\). We discuss how this change affects the design of the logical relation in Section 7.

The first line of terms, ranged over by \( e \), are standard—values, variables (denoted by \( x, y, z, \) etc.), primitive operations (denoted by \( \text{op} \)), term applications, and type applications. We assume that each base type \( B \) is equipped with an equality operator \( =_B \) to distinguish different constants.

The second line presents terms which appear at run time for contract checking. Waiting checks \( \langle \langle \{x : T_1 \mid e_1\}, e_2 \rangle \rangle^{\ell} \), introduced for fussy cast semantics by Sekiyama et al. \([36]\), check that the value of \( e_2 \) satisfies the contract \( e_1 \) by turning themselves to active checks. An
Reduction Rules

\[ e_1 \leadsto e_2 \]

\[
\begin{align*}
\text{op} (k_1, ..., k_n) & \leadsto [\text{op}] (k_1, ..., k_n) & \text{R\_Op} \\
(\lambda x: T.e) v & \leadsto e [v/x] & \text{R\_Beta} \\
(\Lambda \alpha. e) T & \leadsto e [T/\alpha] & \text{R\_TBeta} \\
\langle B \Rightarrow B \rangle & \leadsto v & \text{R\_Base} \\
\langle x: T_{11} \Rightarrow T_{12} \Rightarrow x: T_{21} \Rightarrow T_{22} \rangle v & \leadsto \\
\lambda x: T_{21}. & \langle (T_{21} \Rightarrow T_{11}) \Rightarrow \ell \rangle x \text{ in } (T_{12}[y/x] \Rightarrow T_{22}) \Rightarrow \ell (v y) \\
\text{where } y & \text{ is a fresh variable} \\
\langle \forall \alpha. T_1 \Rightarrow \forall \alpha. T_2 \rangle & \leadsto (\forall \alpha. \langle T_1 \Rightarrow T_2 \rangle \Rightarrow \ell (v \alpha)) & \text{R\_Forall} \\
\langle \{ x: T_1 \mid e_1 \} \Rightarrow T_2 \rangle & \leadsto (\{ T_1 \Rightarrow T_2 \} \Rightarrow \ell \rangle v) & \text{R\_Forget} \\
\langle T_1 \Rightarrow \{ x: T_2 \mid e_2 \} \rangle & \leadsto \langle \{ \{ x: T_2 \mid e_2 \}, \langle T_1 \Rightarrow T_2 \rangle \Rightarrow \ell \rangle \rangle & \text{R\_PreCheck} \\
\langle \{ x: T \mid e \}, \text{true}, \ell \rangle & \leadsto \langle x: T \mid e \rangle, e [v/x], \ell \rangle & \text{R\_Check} \\
\langle \{ x: T \mid e \}, \text{false}, \ell \rangle & \leadsto \uparrow \ell & \text{R\_Fail}
\end{align*}
\]

Evaluation Rules

\[ e_1 \longrightarrow e_2 \]

\[
\frac{E[e_1] \longrightarrow E[e_2]}{E[e_1] \longrightarrow E[e_2]} \quad \text{E\_Red}
\]

\[ E \not\equiv [] \]

\[
\frac{E[\uparrow \ell] \longrightarrow \uparrow \ell}{E[\uparrow \ell] \longrightarrow \uparrow \ell} \quad \text{E\_BLAME}
\]

Figure 2: Operational semantics.

active check \( \langle \{ x: T_1 \mid e_1 \}, e_2, v \rangle \) denotes an intermediate state of the check that \( v \) of \( T_1 \) satisfies contract \( e_1; e_2 \) is an intermediate term during the evaluation of \( e_1 [v/x] \). If \( e_2 \) evaluates to true, the active check returns \( v \); otherwise, if \( e_2 \) evaluates to false, the check fails and uncatchable exception \( \uparrow \ell \), called blame \([11]\), is raised.

We introduce usual notation. We write \( FV(e) \) and \( FTV(e) \) for the sets of free term variables and free type variables that occur in \( e \), respectively. Term \( e \) is closed if \( FV(e) \cup FTV(e) = \emptyset \). \( e [v/x] \) and \( e [T/\alpha] \) denote terms obtained by substituting \( v \) and \( T \) for variables \( x \) and \( \alpha \) in \( e \) in a capture-avoiding manner, respectively. These notations are also applied to types, typing contexts, and evaluation contexts (introduced in Section 2.2). We write \( \text{dom}(\Gamma) \) for the set of term and type variables bound in \( \Gamma \). We also write \( T_1 \Rightarrow T_2 \) for \( x: T_1 \Rightarrow T_2 \) if \( x \) does not occur free in \( T_2 \), \( e_1 \text{ op } e_2 \) for \( \text{op} (e_1, e_2) \), and let \( x: T = e_1 \) in \( e_2 \) for \( (\lambda x: T. e_2) e_1 \).

2.2. Operational Semantics. \( F_H \) has call-by-value operational semantics in the step-style style, which is given by reduction \( \leadsto \) and evaluation \( \longrightarrow \) over closed terms. We write \( \leadsto^* \) and \( \longrightarrow^* \) for the reflexive transitive closures of \( \leadsto \) and \( \longrightarrow \), respectively. Reduction and evaluation rules are shown in Figure 2.

\( \text{(R\_Op)} \) says that reduction of primitive operations depends on function \( \llbracket \cdot \rrbracket \), which gives a denotation to each primitive operation and maps tuples of constants to constants; for example, \( \llbracket + \rrbracket (1, 3) \) denotes 4. We will describe requirements to \( \llbracket \cdot \rrbracket \) in Section 2.3. Term and type applications evaluate by the standard \( \beta \)-reduction (\( \text{(R\_Beta)} \) and \( \text{(R\_TBeta)} \)).

Cast applications evaluate by combination of cast reduction rules, which are from Sekiyama et al. [36] except \( \text{(R\_Forall)} \). Casts between the same base type behave as
an identity function \((R_{\text{BASE}})\). Casts for function types produce a function wrapper involving casts which are contravariant on the domain types and covariant on the codomain types \((R_{\text{FUN}})\). In taking an argument, the wrapper converts the argument with the contravariant cast so that the wrapped function \(v\) can accept it; if the contravariant cast succeeds, the wrapper invokes \(v\) with the conversion result and applies the covariant cast to the value produced by \(v\). \((R_{\text{FUN}})\) renames \(x\) in the codomain type \(T_{12}\) of the source function type to \(y\) because \(T_{12}\) expects \(x\) to be replaced with arguments to \(v\) but they are actually denoted by \(y\) in the wrapper. Casts for universal types behave as in the previous work \([4, 35]\); it produces a wrapper which, applied to a type, invokes the wrapped type abstraction and converts the result \((R_{\text{FORALL}})\). Casts for refinements types first peel off all refinements in the source type \((R_{\text{FORGET}})\) and then check refinements in the target type with waiting checks \((R_{\text{PRECHECK}})\). After checks of inner refinements finish, the outermost refinement will be checked by an active check \((R_{\text{CHECK}})\). If the check succeeds, the checked value is returned \((R_{\text{OK}})\); otherwise, the cast is blamed \((R_{\text{FAIL}})\).

Evaluation uses evaluation contexts \([10]\), given as follows, to reduce subterms \((E_{\text{RED}})\) and lift up blame \((E_{\text{BLAME}})\).

\[
E ::= [] \mid \text{op}(v_1, \ldots, v_n, E, e_1, \ldots, e_m) \mid E\ e\ |\ v\ E\ |\ E\ T\ |\ \langle \{x:T|e\}, E\rangle^\ell\ |\ \langle\{x:T|e\}, E, v^\ell\rangle
\]

This definition indicates that the semantics is call-by-value and arguments evaluate from left to right.

Fussy versus sloppy. Our cast semantics is fussy in that, when \(\langle T_1 \Rightarrow T_2\rangle^\ell\) is applied, all refinements in target type \(T_2\) are checked even if they have been ensured by source type \(T_1\). For example, let us consider reflexive cast \(\langle\{x:\{y:\text{Int}\ |\ y > 2\} | \text{prime}\? x\}\Rightarrow\{x:\{y:\text{Int}\ |\ y > 2\} | \text{prime}\? x\}\rangle^\ell\). When applied to \(v\), the cast application forgets the refinements in the source type of the cast \((R_{\text{FORGET}})\):

\[
\langle\{x:\{y:\text{Int}\ |\ y > 2\} | \text{prime}\? x\}\Rightarrow\{x:\{y:\text{Int}\ |\ y > 2\} | \text{prime}\? x\}\rangle^\ell\ v\rightarrow^* \langle\text{Int} \Rightarrow \{x:\{y:\text{Int}\ |\ y > 2\} | \text{prime}\? x\}\rangle^\ell\ v
\]

and then refinements in the target type are checked from the innermost through the outermost by using waiting checks \((R_{\text{PRECHECK}})\):

\[
\ldots \rightarrow^* \langle\{x:\{y:\text{Int}\ |\ y > 2\} | \text{prime}\? x\}, \{\{y:\text{Int}\ |\ y > 2\}, \langle\text{Int} \Rightarrow \text{Int}\rangle^\ell v\rangle^\ell\rangle^\ell
\]

even though \(v\) would be typed at \(\{x:\{y:\text{Int}\ |\ y > 2\} | \text{prime}\? x\}\) and satisfy the refinements.

In contrast, Belo et al.’s semantics \([4]\) is sloppy in that checks of refinements that have been ensured are skipped, which is represented by two cast reduction rules:

\[
\langle T \Rightarrow T\rangle^\ell v \rightarrow^\delta v \\
\langle T \Rightarrow \{x:T|e\}\rangle^\ell v \rightarrow^\delta \langle\{x:T|e\}, e[v/x], v\rangle^\ell
\]

where \(\rightarrow^\delta\) is the reduction relation in the sloppy semantics. The first rule processes reflexive casts as if they are identity functions and the second checks only the outermost refinement because others have been ensured by the source type. Under the sloppy semantics, \(\langle\{x:\{y:\text{Int}\ |\ y > 2\} | \text{prime}\? x\}\Rightarrow\{x:\{y:\text{Int}\ |\ y > 2\} | \text{prime}\? x\}\rangle^\ell v\) reduces to \(v\) in one step. The sloppy semantics allows a logical relation to take arbitrary binary relations on terms for interpretation of type variables \([4]\).

It is found that, however, naive sloppy semantics does not satisfy the so-called coterminality (Lemma 2.6), a key property to show type soundness and parametricity in manifest contracts; Sekiyama et al. investigated this problem in detail \([35]\). Briefly speaking, the
coterminal requires that reduction of subterms preserves evaluation results, but the sloppy semantics does not satisfy it. For example, let \( T \overset{\text{def}}{=} \{ x : \text{Int} \mid \text{not} \} \) where \text{not} is a negation function on Booleans. Since reflexive cast \( (T \Rightarrow T)^{\ell} \) behaves as an identity function in the sloppy semantics, \( (T \Rightarrow T)^{\ell} v \) evaluates to \( v \) for any value \( v \). Since \text{not} \( \rightarrow \) \text{false}, the coterminal requires that \( \{ x : \text{Int} \mid \text{false} \Rightarrow T \}^{\ell} v \) also evaluate to \( v \) because reduction of subterm not \text{false} to \text{false} must not change the evaluation result. However, \( \{ x : \text{Int} \mid \text{false} \Rightarrow T \}^{\ell} v \) checks refinement not \text{true} in \( T \), which gives rise to blame; thus, the coterminal is invalidated.

The problem above does not happen in the fussy semantics. Under the fussy semantics, since all refinements in a cast are checked, both casts \( (T \Rightarrow T)^{\ell} \) and \( \{ x : \text{Int} \mid \text{false} \Rightarrow T \}^{\ell} \) check refinement not \text{true} and raise blame.

2.3. Type System. The type system consists of three judgments: typing context well-formedness \( \vdash \Gamma \), type well-formedness \( \Gamma \vdash T \), and term typing \( \Gamma \vdash e : T \). They are derived by rules in Figure 3. The well-formedness rules are standard or easy to understand, and the typing rules are based on previous work [4, 35]. We suppose that types of constants and primitive operations are provided by function \text{ty}(\_). Requirements to their types will be described at the end of this section. Casts are well typed when their source and target types are compatible (T.Cast). Types are compatible if they are the same modulo refinements. This is formalized by type compatibility \( T_1 \parallel T_2 \), which is derived by the rules shown at the bottom of Figure 3. The type \( T_2 [e_1/x] \) of a term application is required to be well formed (T.App). As we will see the proof in detail, this condition is introduced for showing the parametricity (Theorem 5.28). The typing rule (T.WCheck) of waiting checks \( \{ \{ x : T_1 \mid e_1 \}, e_2 \}^{\ell} \) requires \( e_2 \) to have \( T_1 \) because it is checked at run time that the evaluation result of \( e_2 \) satisfies \( e_1 \) which refers to \( x \) of \( T_1 \). Although waiting checks are run-time terms, (T.WCheck) does not require \( \{ x : T_1 \mid e_1 \} \) and \( e_2 \) to be closed, unlike other run-time typing rules such as (T.ACheck). This relaxation allows type-preserving static decomposition of \( (T_1 \Rightarrow \{ x : T_2 \mid e \})^{\ell} \) into a smaller cast \( (T_1 \Rightarrow T_2)^{\ell} \) and a waiting check for refinement \( e \) (Lemma 6.11 in Section 6.3). Active checks \( \{ \{ x : T_1 \mid e_1 \}, e_2, v \}^{\ell} \) are well typed if \( e_2 \) is an actual intermediate state of evaluation of \( e_1 [v/x] \) (T.ACheck). (T.Forget) and (T.Exact) are run-time typing rules: the former forgets a refinement and the latter adds a refinement that holds.

(T.Conv) is a run-time typing rule to show subject reduction. To motivate it, let us consider application \( v e \) where \( v \) and \( e \) are typed at \( x : T_1 \rightarrow T_2 \) and \( T_1 \), respectively. This application would be typed at \( T_2 [e/x] \) by (T.App). If \( e \) reduces to \( e' \), \( v e' \) would be at \( T_2 [e'/x] \), which is syntactically different from \( T_2 [e/x] \) in general. Since subject reduction requires evaluation of well-typed terms to be type-preserving, we need a device that allows \( v e' \) to be typed at \( T_2 [e'/x] \). To this end, Belo et al. [4] introduced a type conversion relation which relates \( T_2 [e/x] \) and \( T_2 [e'/x] \) and added a typing rule that allows terms to be retyped at convertible types. Their type conversion turns out to be flawed, but it is fixed in the succeeding work [14, 36]. Our type conversion \( \equiv \) follows the fixed version.

**Definition 2.1** (Type Conversion). The binary relation \( \Rightarrow \) over types is defined as follows: \( T_1 \Rightarrow T_2 \) if there exist some \( T, x, e_1, \) and \( e_2 \) such that \( T_1 = T[e_1/x] \) and \( T_2 = T[e_2/x] \) and \( e_1 \rightarrow e_2 \). The type conversion \( \equiv \) is the symmetric transitive closure of \( \Rightarrow \).

Finally, we formalize requirements to constants and primitive operations. We first define auxiliary function \text{unref}, which strips off refinements that are not under other type
\( \vdash \Gamma \)  
**Context Well-Formedness**

\[
\begin{align*}
  & \vdash \emptyset \quad \text{WF}_\text{EMPTY} \quad \vdash \Gamma \vdash T \quad \text{WF}_\text{EXTENDVAR} \quad \vdash \Gamma \vdash \alpha \quad \text{WF}_\text{EXTENTTVAR} \\
\end{align*}
\]

\( \Gamma \vdash T \)  
**Type Well-Formedness**

\[
\begin{align*}
  & \vdash \Gamma \quad \text{WF}_\text{BASE} \quad \Gamma \vdash \alpha \in \Gamma \quad \text{WF}_\text{VAR} \quad \Gamma, \alpha \vdash T \quad \text{WF}_\text{FORALL} \\
  & \vdash \Gamma, x : T_1 \vdash T_2 \quad \text{WF}_\text{FUN} \quad \vdash \Gamma, T \quad \text{WF}_\text{REFINE} \\
\end{align*}
\]

\( \Gamma \vdash x : T \)  
**Typing Rules**

\[
\begin{align*}
  & \vdash \Gamma \quad x : T \in \Gamma \quad \text{T}_\text{VAR} \quad \vdash \Gamma \quad \text{T}_\text{CONST} \\
  & \vdash \Gamma \quad \text{T}_\text{OP} \\
  & \vdash \Gamma, x : T_1 \vdash e : T_2 \quad \text{T}_\text{ABS} \quad \vdash \Gamma, T_1 \vdash T_2 \quad \text{T}_\text{CAST} \\
  & \vdash \Gamma \quad \text{T}_\text{APP} \\
  & \vdash \Gamma \quad \text{T}_\text{CHECK} \\
  & \vdash \Gamma \quad \text{T}_\text{CONV} \\
  & \vdash \Gamma \quad \text{T}_\text{FORGET} \\
\end{align*}
\]

\( T_1 \parallel T_2 \)  
**Type Compatibility**

\[
\begin{align*}
  & B \parallel B \quad \text{C}_\text{BASE} \\
  & \alpha \parallel \alpha \quad \text{C}_\text{VAR} \\
  & \{x : T \mid e\} \parallel T_2 \quad \text{C}_\text{REFINEL} \\
  & \{x : T_1 \mid e\} \parallel \{x : T_2 \mid e\} \quad \text{C}_\text{REFINER} \\
  & x : T_1 \rightarrow T_2 \parallel y : T_2 \rightarrow T_3 \quad \text{C}_\text{FUN} \\
  & \forall \alpha. T_1 \parallel \forall \alpha. T_2 \quad \text{C}_\text{FORALL} \\
\end{align*}
\]

**Figure 3:** Typing rules.

**constructors:**

\[
\begin{align*}
  & \text{unref}(\{x : T \mid e\}) = \text{unref}(T) \\
  & \text{unref}(T) = T \quad \text{(if } T \text{ is not a refinement type)} \\
\end{align*}
\]

Requirements to constants and primitive operations are as follows:

- For each constant \( k \in K_B \), (1) \( \text{unref}(\text{ty}(k)) = B \), (2) \( \emptyset \vdash \text{ty}(k) \) is derivable, and (3) \( k \) satisfies all refinements in \( \text{ty}(k) \), that is, \((B \Rightarrow \text{ty}(k))^f k \rightarrow^* k\).
• For each primitive operation op, ty(op) is a monomorphic dependent function type of the form \( x_1 : T_1 \rightarrow ... \rightarrow x_n : T_n \rightarrow T_0 \) where, for any \( i \in \{ 0, ..., n \} \), there exists some \( B \) such that \( \text{unref}(T_i) = B \). Furthermore, we require that \( \text{op} \) return a value satisfying the refinements in the return type \( T_0 \) when taking constants satisfying the refinements in the argument types, that is:

\[
\forall k_1, ..., k_n. \forall i \in \{ 1, ..., n \}. \left( k_i \in K_{\text{unref}(T_i)} \text{ and } \langle \text{unref}(T_i) \Rightarrow T_i[k_1/x_1, ..., k_{i-1}/x_{i-1}] \rangle^\ell k_i \rightarrow^* k_i \right) \Rightarrow \exists k \in K_{\text{unref}(T_0)}. \left( \langle \text{op} \rangle(k_1, ..., k_n) = k \text{ and } \langle \text{unref}(T_0) \Rightarrow T_0[k_1/x_1, ..., k_n/x_n] \rangle^\ell k \rightarrow^* k \right)
\]

In contrast, we assume that \( \langle \text{op} \rangle(k_1, ..., k_n) \) is undefined if some \( k_i \) does not satisfy refinements in \( T_i \), that is, \( \langle \text{unref}(T_i) \Rightarrow T_i[k_1/x_1, ..., k_{i-1}/x_{i-1}] \rangle^\ell k_i \rightarrow^* k_i \) cannot be derived.

2.4. Properties. This section proves type soundness via progress and subject reduction [44]. Type soundness can be shown as in the previous work [36, 35] and so we omit the most parts of its proof.

We start with showing the cotermination (Lemma 2.6), a key property for proving not only type soundness but also parametricity and soundness of our logical relation with respect to determinism of the reduction. In the case that it is derived by (Sekiyama et al. [35], our proof of the cotermination is based on the observation that equivalently, which means that convertible types have the same denotation. Following (Cotermination [Lemma 2.6]), we refer to the names of the lemmas in the proof script coterm.v.

**Lemma 2.2** (Unique Decomposition [lemm_redckx_decomp in coterm.v]). If \( e = E[e_1] \) and \( e_1 \rightsquigarrow e_2 \) and \( e = E'[e'_1] \) and \( e'_1 \rightsquigarrow e'_2 \), then \( E = E' \) and \( e_1 = e'_1 \).

**Proof.** By induction on \( E \).

**Lemma 2.3** (Determinism [lemm_eval_deterministic in coterm.v]). If \( e \rightarrow e_1 \) and \( e \rightarrow e_2 \), then \( e_1 = e_2 \).

**Proof.** The case that \( e \rightarrow e_1 \) is derived by (E_RED) is shown by Lemma 2.2 and the determinism of the reduction. In the case that it is derived by (E_BLAIME), let us suppose that \( e \rightarrow e_2 \) is derived by (E_RED). It is contradictory because, if \( e = E[e'] \) and \( e' \rightsquigarrow e'' \), then \( e \neq E_2[\eta \ell] \) for any \( E_2 \) and \( \ell \).

**Lemma 2.4** (Weak bisimulation, left side [lemm_coterm_left_eval in coterm.v]). If \( e_1 \rightarrow e_2 \) and \( e[e_1/x] \rightarrow e'_1 \), then there exists some \( e' \) such that \( e[e_2/x] \rightarrow^* e'[e_2/x] \) and \( e'_1 \rightarrow^* e'[e_1/x] \). (See the commuting diagram on the left in Figure 4.)

**Lemma 2.5** (Weak bisimulation, right side [lemm_coterm_right_eval in coterm.v]). If \( e_1 \rightarrow e_2 \) and \( e[e_2/x] \rightarrow e'_2 \), then there exists some \( e' \) such that \( e[e_1/x] \rightarrow^* e'[e_1/x] \) and \( e'_2 \rightarrow^* e'[e_2/x] \). (See the commuting diagram on the right in Figure 4.)

**Lemma 2.6** (Cotermination [lemm_coterm_true in coterm.v]). Suppose that \( e_1 \rightarrow e_2 \).

1. If \( e[e_1/x] \rightarrow^* v_1 \), then \( e[e_2/x] \rightarrow^* v_2 \). In particular, if \( v_1 = \text{true} \), then \( v_2 = \text{true} \).
2. If \( e[e_2/x] \rightarrow^* v_2 \), then \( e[e_1/x] \rightarrow^* v_1 \). In particular, if \( v_2 = \text{true} \), then \( v_1 = \text{true} \).
Lemma 2.12 (Term Substitution). For any closed value \( v \), if \( T \) refines \( \Gamma \), then \( \Gamma \vdash e : T \) implies \( \Gamma \vdash e [ v / x] : T'[v / x] \).

Proof. Straightforward by induction on the derivation of \( T \). The case for \( \Gamma \vdash \lambda x:T. e \) is shown by Lemma 2.8.

Lemma 2.10 (Term Weakening). Let \( x \) be a fresh variable. Suppose that \( \Gamma \vdash T \).

1. If \( \Gamma, \Gamma' \vdash e : T' \), then \( \Gamma, x:T, \Gamma' \vdash e : T' \).
2. If \( \Gamma, \Gamma' \vdash T' \), then \( \Gamma, x:T, \Gamma' \vdash T' \).
3. If \( \Gamma \vdash \Gamma' \), then \( \Gamma, x:T, \Gamma \vdash \Gamma' \).

Lemma 2.11 (Type Weakening). Let \( \alpha \) be a fresh type variable.

1. If \( \Gamma, \Gamma' \vdash e : T \), then \( \Gamma, \alpha, \Gamma' \vdash e : T \).
2. If \( \Gamma, \Gamma' \vdash T \), then \( \Gamma, \alpha, \Gamma' \vdash T \).
3. If \( \Gamma \vdash \Gamma' \), then \( \Gamma, \alpha, \Gamma \vdash \Gamma' \).

Lemma 2.12 (Term Substitution). Suppose that \( \Gamma \vdash e : T \).

1. If \( \Gamma, x:T, \Gamma' \vdash e' : T' \), then \( \Gamma, \Gamma'[e/x] \vdash e'[e/x] : T'[e/x] \).
2. If \( \Gamma, x:T, \Gamma' \vdash T' \), then \( \Gamma, \Gamma'[e/x] \vdash T'[e/x] \).

\[ \begin{align*}
\exists e', e'[e_1] & \mathcal{R} e'[e_2] \\
\exists e', e'[e_1] & \mathcal{R} e'[e_2]
\end{align*} \]

Figure 4: Lemmas 2.4 and 2.5.
We introduce semityped contextual equivalence to formalize the upcast elimination property.

\[ C ::= [\_], k | \lambda x:T^C.C | \{T^C_1 \Rightarrow T^C_2\}^\ell | \Lambda \alpha. C | x \mid \text{op}(C_1, \ldots, C_n) \mid C_1 C_2 \mid C_1 T^C_2 | \hat{\eta} \ell | \{x:T^C_1 \mid C_1\}, C_2 \}^\ell | \{x:T^C_1 \mid C_1\}, C_2, V^C \}^\ell \]
\[ V^C ::= k | \lambda x:T^C.C | \{T^C_1 \Rightarrow T^C_2\}^\ell | \Lambda \alpha. C \]
\[ T^C ::= B \mid \alpha \mid x:T^C_1 \Rightarrow T^C_2 \mid \forall \alpha. T^C \mid \{x:T^C \mid C\} \]

Figure 5: Syntax of contexts and type contexts.

(3) If \( \Gamma \vdash x:T, \Gamma' \), then \( \Gamma \vdash \Gamma'[e/x] \).

Lemma 2.13 (Type Substitution). Suppose that \( \Gamma \vdash T \).

1. If \( \Gamma, \alpha, \Gamma' \vdash e' : T' \), then \( \Gamma, \Gamma'[T/\alpha] \vdash e'[T/\alpha] : T'[T/\alpha] \).

2. If \( \Gamma, \alpha, \Gamma' \vdash T' \), then \( \Gamma, \Gamma'[T/\alpha] \vdash T'[T/\alpha] \).

3. If \( \Gamma, \alpha, \Gamma' \), then \( \Gamma, \Gamma'[T/\alpha] \).

Lemma 2.14 (Canonical Forms). Suppose that \( \emptyset \vdash v : T \).

1. If \( \text{unref}(T) = B \), then \( v \in K_B \).

2. If \( \text{unref}(T) = x:T_1 \rightarrow T_2 \), then \( v = \lambda x:T'_1.e \) for some \( x, T'_1 \), and \( e \), or \( v = (T'_1 \Rightarrow T'_2)^\ell \) for some \( T'_1, T'_2 \), and \( \ell \).

3. If \( \text{unref}(T) = \forall \alpha. T' \), then \( v = \Lambda \alpha. e \) for some \( e \).

Lemma 2.15 (Progress). If \( \emptyset \vdash e : T \), then:

- \( e \rightarrow e' \) for some \( e' \);
- \( e \) is a value; or
- \( e = \hat{\eta} \ell \) for some \( \ell \).

Lemma 2.16 (Subject Reduction). If \( \emptyset \vdash e : T \) and \( e \rightarrow e' \), then \( \emptyset \vdash e' : T \).

Theorem 2.17 (Type Soundness). If \( \emptyset \vdash e : T \), then one of the followings holds.

- \( e \) diverges;
- \( e \rightarrow^* v \) for some \( v \) such that \( \emptyset \vdash v : T \) and \( v \in \llbracket T \rrbracket \); or
- \( e \rightarrow^* \hat{\eta} \ell \) for some \( \ell \).

Proof. By the progress, the subject reduction, and the value inversion. \( \square \)

3. SEMITYPED CONTEXTUAL EQUIVALENCE

We introduce semityped contextual equivalence to formalize the upcast elimination property. It relates terms \( e_1 \) and \( e_2 \) such that (1) they are contextually equivalent, that is, behave equivalently under any well-typed program context, and (2) \( e_1 \) is well-typed. Semityped contextual equivalence does not enforce any condition on the type of \( e_2 \),\(^3\) so it can even relate terms having different types such as an upcast and an identity function.

Figure 5 shows the syntax of multi-hole program contexts \( C \), value contexts \( V^C \), and type contexts \( T^C \). Contexts have zero or more holes \([\_]\), indexed by positive numbers \( i \), and the same hole \([\_]\) can occur in a context an arbitrary number of times. Thus, any term, value, and type are contexts without holes. Replacement of the holes in program contexts, value contexts, and type contexts with terms produces terms, values, and types, respectively. For any program context \( C \) where indices of the holes range over 1 through

\(^3\)In fact, it does not even require it to be well typed.
n and any terms $e_1, ..., e_n$, we write $C[e_1, ..., e_n]$, or $C[e_1]$ simply if $n$ is clear from the context or not important, to denote a term obtained by replacing each hole $[\_]$ with term $e_i$. In particular, $e[\_] = e$ because there are zero holes in $e$. We use similar notation for value and type contexts.

Contexts having multiple holes is crucial in semityped contextual equivalence. If we restrict contexts to have a single hole, replacements of terms with contextually-equivalent ones would be performed one by one. However, a replacement with an ill-typed term produces an ill-typed program, and then, since semityped contextual equivalence requires terms on one side to be well typed, the results of the remaining replacements could not be guaranteed to be contextually equivalent to the original program. For example, the replacement of term $e_1$ in $C[e_1, e_2]$ with ill-typed term $e_1'$ produces an ill-typed program $C[e_1', e_2]$. In this case, even if there is an ill-typed term $e_2'$ contextually equivalent to $e_2$, semityped contextual equivalence cannot contain $C[e_1', e_2]$ and $C[e_1', e_2']$ because both are ill typed. The same issue arises even if we first replace $e_2$ and then $e_1$. As a result, we could not show that $C[e_1, e_2]$ and $C[e_1', e_2']$ are contextually equivalent. This is problematic also in the upcast elimination, especially when programs have multiple upcasts. We address this issue by contexts with multiple holes, which allow simultaneous replacements. In the example above, if $e_1$ and $e_2$ are shown to be contextually equivalent to $e_1'$ and $e_2'$ respectively, we can relate $C[e_1, e_2]$ to $C[e_1', e_2']$ directly, not via $C[e_1', e_2]$ nor $C[e_1, e_2']$.

The semityped contextual equivalence considers three kinds of observable results, that is, termination, blame, and being stuck—the last has to be considered because semityped contextual equivalence contains possibly ill-typed terms. We write $e \downarrow$ if $e \rightarrow^* v$ for some $v$, $e \uparrow \ell$ if $e \rightarrow^{* \ell}$, and $e \downarrow$ if $e \rightarrow^* e'$ for some $e'$ such that $e'$ cannot evaluate and it is neither a value nor blame.

**Definition 3.1 ( Observable Equivalence).** We write $e_1 \downarrow e_2$ if (1) $e_1 \downarrow$ iff $e_2 \downarrow$, (2) $e_1 \uparrow \ell$ iff $e_2 \uparrow \ell$ for any $\ell$, and (3) $e_1 \downarrow$ iff $e_2 \downarrow$.

Now, we could define semityped contextual equivalence as follows.

Terms $e_{11}, ..., e_{1n}$ and $e_{21}, ..., e_{2n}$ are contextually equivalent at $T_1, ..., T_n$ under $\Gamma_1, ..., \Gamma_n$, respectively, when (1) for any $i$, $\Gamma_i \vdash e_{1i} : T_i$ and $\text{FV}(e_{2i}) \subseteq \text{dom}(\Gamma_i)$, and (2) for any $T$ and $\Gamma$, if $\emptyset \vdash C[e_{11}, ..., e_{1n}] : T$, then $C[e_{11}, ..., e_{1n}] \downarrow C[e_{21}, ..., e_{2n}]$.

Thanks to program contexts with multiple holes, we can replace two or more well-typed terms with possibly ill-typed, contextually equivalent terms at the same time.

The semityped contextual equivalence defined in this way is well defined as it is but we find it more convenient to consider contexts as typed objects to discuss composition of contexts and terms rigorously. To this end, we introduce judgments for program context well-formedness $\Gamma \vdash C : \Gamma_i \vdash e_i : T_i \overset{*}{\rightarrow} T$ and type context well-formedness $\Gamma \vdash T^C : \Gamma_i \vdash e_i : T_i \overset{*}{\rightarrow}$, which mean that, if $e_i$ is typed at $T_i$ under $\Gamma_i$ for any $i$, $C[e_i]$ and $T^C[e_i]$ are a well-typed term of $T$ under $\Gamma$ and a well-formed type under $\Gamma$, respectively.\(^4\) These well-formedness judgments need information on terms $\Gamma_i \vdash e_i : T_i$ with which holes are replaced as well as typing context and type information because whether composition of a context with terms produces a well-typed term rests on the composed terms. For example, let us consider $C = \{x: \text{Int} \mid x > 0\} \Rightarrow \text{Int}^\ell (f [\_])$ where $f$ is typed at $y: \text{Int} \rightarrow \{x: \text{Int} \mid x > y\}$.

\(^4\)Since value contexts are a subset of program contexts, value context well-formedness is given by program context well-formedness.
Context Typing Rules

\[ \Gamma \vdash C : \Gamma_i \vdash e_i : T_i^i \mapsto T \]  
\[ \Gamma_j \vdash [ ]_j : \Gamma_i \vdash e_i : T_i^i \mapsto T_j \]  
\[ \vdash \Gamma \quad \text{CT_HOLE} \]  
\[ \vdash \Gamma \quad \text{CT_VAR} \]  
\[ \vdash \Gamma \quad \text{CT_CONST} \]  
\[ \vdash \Gamma \quad \text{CT_OP} \]  
\[ \forall j \in \{ 1, \ldots, n \}. \Gamma \vdash C_j : \Gamma_i \vdash e_i : T_i^i \mapsto T_j \left[ C_1[\tau_i]/x_1, \ldots, C_{j-1}[\tau_i]/x_{j-1} \right] \]  
\[ \Gamma \vdash \text{op}(C_1, \ldots, C_n) : \Gamma_i \vdash e_i : T_i^i \mapsto T_2 \]  
\[ \Gamma \vdash \lambda x : T_i^i.C : \Gamma_i \vdash e_i : T_i^i \mapsto \Gamma \vdash \langle x : T_i^i \mapsto T_2 \rangle \| T_2 \]  
\[ \Gamma \vdash C \Gamma_i \vdash e_i : T_i^i \mapsto \Gamma \vdash T_2 \left[ C \mid T_i^i \mapsto T_2 \right] \]  
\[ \Gamma \vdash T_2 \left[ C \mid T_i^i \mapsto T_2 \right] \]  
\[ \Gamma \vdash \text{App} : \Gamma_i \vdash e_i : T_i^i \mapsto \Gamma \vdash T_2 \left[ C \mid T_i^i \mapsto T_2 \right] \]  
\[ \Gamma \vdash \text{Abs} : \Gamma_i \vdash e_i : T_i^i \mapsto \Gamma \vdash \forall \alpha.T \]  
\[ \Gamma \vdash C \Gamma_i \vdash e_i : T_i^i \mapsto \Gamma \vdash T_2 \left[ C \mid T_i^i \mapsto T_2 \right] \]  
\[ \Gamma \vdash \text{Conv} : \Gamma_i \vdash e_i : T_i^i \mapsto \Gamma \vdash T_2 \left[ \alpha \mid T_i^i \mapsto T_2 \right] \]  
\[ \Gamma \vdash \text{WCheck} : \Gamma_i \vdash e_i : T_i^i \mapsto \Gamma \vdash \left\{ \{ x : T_i^i \mid C_1 \} \right\}_2 \]  
\[ \Gamma \vdash \text{ACheck} : \Gamma_i \vdash e_i : T_i^i \mapsto \Gamma \vdash \{ x : T_i^i \mid C_1 \} \left[ \tau_i \right] \]  
\[ \Gamma \vdash \text{Blame} : \Gamma_i \vdash e_i : T_i^i \mapsto T \]  
\[ \Gamma \vdash \text{Forget} : \Gamma \vdash \langle x : T_i^i \mid e \rangle \mapsto \langle x : T_i^i \mid e \rangle \]  
\[ \Gamma \vdash \text{Exact} : \Gamma_i \vdash e_i : T_i^i \mapsto \{ x : T_i^i \mid e \} \]  

Figure 6: Program context well-formedness rules.

C[0] is well typed because the type of f 0 matches with the source type of the cast, while C[2] is not because the type of f 2 is \{ x : \text{Int} \mid x > 2 \}, which is different from \{ x : \text{Int} \mid x > 0 \}.
Type Context Well-Formedness Rules

\[
\begin{array}{c}
\Gamma \vdash T^C : \Gamma_i \vdash e_i : T_i^i \to \ast \quad \text{Type Context Well-Formedness Rules} \\
\hline
\Gamma \vdash \Gamma \quad \text{CW_BASE} \\
\Gamma \vdash B : \Gamma_i \vdash e_i : T_i^i \to \ast \\
\Gamma \vdash \forall \alpha : \Gamma_i \vdash e_i : T_i^i \to \ast \\
\Gamma \vdash T^C : \Gamma_i \vdash e_i : T_i^i \to \ast \\
\Gamma, x : T_i^C \vdash T_2^C : \Gamma_i \vdash e_i : T_i^i \to \ast \\
\Gamma, \alpha : \Gamma_i \vdash e_i : T_i^i \to \ast \\
\Gamma \vdash \forall \alpha. T^C : \Gamma_i \vdash e_i : T_i^i \to \ast \\
\Gamma \vdash T^C : \Gamma_i \vdash e_i : T_i^i \to \ast \\
\Gamma, x : T_i^C \vdash C : \Gamma_i \vdash e_i : T_i^i \to \text{Bool} \\
\Gamma \vdash \{ x : C \mid C \} : \Gamma_i \vdash e_i : T_i^i \to \ast \\
\end{array}
\]

Figure 7: Type context well-formedness rules.

If no type information in \( C \) depends on holes, the derivation of \( \Gamma \vdash C : \Gamma_i \vdash e_i : T_i^i \to T \) refers only to \( \Gamma_i \) and \( T_i^i \), not any of \( e_i \). Inference rules for the judgments are shown in Figures 6 and 7; they correspond to term typing and type well-formedness rules given in Section 2.3.

We show a few properties of well-typed contexts: (1) composition of a well-formed context with well-typed terms produces a well-typed term, (2) free variables and free type variables are preserved by the composition, and (3) well-typed terms, well-typed values, well-formed types are well-formed contexts, value contexts, and type contexts, respectively.

**Lemma 3.2.** Suppose \( \Gamma \vdash e_1 : T_1, \ldots, \Gamma \vdash e_n : T_n \).

1. If \( \Gamma \vdash C : \Gamma_i \vdash e_i : T_i^i \to T \), then \( \Gamma \vdash C[e_i] : T \).
2. If \( \Gamma \vdash T^C : \Gamma_i \vdash e_i : T_i^i \to *, \) then \( \Gamma \vdash T^C[e_i] \).

*Proof.* By induction on the derivations of \( \Gamma \vdash C : \Gamma_i \vdash e_i : T_i^i \to T \) and \( \Gamma \vdash T^C : \Gamma_i \vdash e_i : T_i^i \to * \).

**Lemma 3.3.** For any \( \Gamma_1, \ldots, \Gamma_n, e_1, \ldots, e_n, \) and \( T_1, \ldots, T_n \) such that \( \text{FTV}(e_i) \subseteq \text{dom}(\Gamma_i) \) for any \( i \),

1. if \( \Gamma \vdash C : \Gamma_i \vdash e_i : T_i^i \to T \), then \( \text{FTV}(C[e_i]) \subseteq \text{dom}(\Gamma), \) and
2. if \( \Gamma \vdash T^C : \Gamma_i \vdash e_i : T_i^i \to *, \) then \( \text{FTV}(T^C[e_i]) \subseteq \text{dom}(\Gamma). \)

*Proof.* By induction on the derivations of \( \Gamma \vdash C : \Gamma_i \vdash e_i : T_i^i \to T \) and \( \Gamma \vdash T^C : \Gamma_i \vdash e_i : T_i^i \to * \).

**Lemma 3.4.** For any \( \Gamma_1, \ldots, \Gamma_n, e_1, \ldots, e_n, \) and \( T_1, \ldots, T_n \),

1. if \( \Gamma \vdash e : T \), then \( \Gamma \vdash e : \Gamma_i \vdash e_i : T_i^i \to T \),
2. if \( \Gamma \vdash v : T \), then \( \Gamma \vdash v : \Gamma_i \vdash e_i : T_i^i \to T, \) and
3. if \( \Gamma \vdash T \), then \( \Gamma \vdash T : \Gamma_i \vdash e_i : T_i^i \to * \).

*Proof.* By induction on the derivations of \( \Gamma \vdash e : T \), \( \Gamma \vdash v : T \), and \( \Gamma \vdash T \).
Finally, we define semityped contextual equivalence by using well-formed contexts.

**Definition 3.5** (Semityped Contextual Equivalence). Terms $e_{11}, \ldots, e_{1n}$ and $e_{21}, \ldots, e_{2n}$ are *contextually equivalent* at $T_1, \ldots, T_n$ under $\Gamma_1, \ldots, \Gamma_n$, respectively, written as $\Gamma_i \vdash e_{1i} =_{\text{ctx}} e_{2i} : T_i \in \{1, \ldots, n\}$, if and only if (1) for any $i$, $\Gamma_i \vdash e_{1i} : T_i$ and $\text{FTV}(e_{2i}) \subseteq \text{dom}(\Gamma_i)$, and (2) for any $T$ and $C$, if $\emptyset \vdash C : \Gamma_i \vdash e_{1i} : T_i \rightarrow T$, then $C[e_{11}, \ldots, e_{1n}] \Downarrow C[e_{21}, \ldots, e_{2n}]$. For simplification, we write $\Gamma_i \vdash e_{1i} =_{\text{ctx}} e_{2i} : T_i$ if $n$ is not important and $\Gamma_1 \vdash e_{11} =_{\text{ctx}} e_{21} : T_1$ if $n = 1$.

We note that we state semityped contextual equivalence for *pairs* of terms and that equivalence is preserved by dropping some pairs: that is, if $\Gamma_i \vdash e_{1i} =_{\text{ctx}} e_{2i} : T_i \in \{1, \ldots, n\}$, then $\Gamma_i \vdash e_{1i} =_{\text{ctx}} e_{2i} : T_i \in \{1, \ldots, m\}$ for $m \leq n$.

Finally, we make a few remarks on semityped contextual equivalence. Although we call it semityped contextual “equivalence,” this relation is not quite an equivalence relation because symmetry does not hold (ill-typed terms cannot be on the left-hand side). More interestingly, even showing its transitivity is not trivial. For proving the transitivity, we have to show that, if $\Gamma \vdash e_1 =_{\text{ctx}} e_2 : T$ and $\Gamma \vdash e_2 =_{\text{ctx}} e_3 : T$, then $e_1$ and $e_3$ behave equivalently under any program context $C$ which is well formed for $e_1$. We might expect that $e_2$ and $e_3$ behave in the same way under $C$, but it is not clear because $C$ may not be well formed for $e_2$. Fortunately, our logical relation enables us to show (restricted) transitivity of semityped contextual equivalence via completeness with respect to semityped contextual equivalence (Corollary 5.35).

In some work [21, 29], contextual equivalence is defined for A-normal forms, where arguments to functions are restricted to values and terms are composed by let-expressions (so, they are not shorthand of term applications there) to reduce clutter. In fact, we have adopted that style at an early stage of the study but it turned out that it did not work quite well, because a term in A-normal form is not closed under term substitution. To see the problem, let us consider a typing rule for let-expression $\text{let } x : T_1 = e_1 \text{ in } e_2$, which could be given as follows:

$$\Gamma \vdash e_1 : T_1 \quad \Gamma, x : T_1 \vdash e_2 : T_2 \quad \Gamma \vdash \text{let } x : T_1 = e_1 \text{ in } e_2 : T_2[e_1/x]$$

The problem is that the index type $T_2[e_1/x]$ possibly includes refinements which are not A-normal forms if $e_1$ is neither a variable nor a value. For example, let $x : \text{Int} = 2 + 3$ in $(\text{Int} \Rightarrow \{y : \text{Int} | x > 0\})^\ell 0$ is typed at $\{x : \text{Int} | 2 + 3 > 0\}$, but the refinement $2 + 3 > 0$ is not in A-normal form. We might be able to define substitution so that $\{y : \text{Int} | \text{let } x : \text{Int} = 2 + 3 \text{ in } x > 0\}$ would be obtained, but we avoid such “peculiar” substitution.

While semitypedness of our contextual equivalence is motivated by the upcast elimination, perhaps surprisingly, it appears unclear to us how to define *typed* contextual equivalence. One naive definition of it is to demand that, for each $e_{2i}$ in Definition 3.5, $e_{2i}$ is well typed at $T_i$ under $\Gamma_i$. However, this gives rise to ill-typed terms. For example, suppose that we want to equate 0 and $(\lambda y : \text{Int}.y)$ 0. To show their contextual equivalence, we have to evaluate them in any program context. Here, a context $\langle \{x : \text{Int} | 0 < x\} \Rightarrow \text{Int} \rangle^\ell (f\ [\_])$ given above is well-formed for 0 but not for $(\lambda y : \text{Int}.y)$ 0; note that we cannot apply (CT_CONV) to $f ((\lambda y : \text{Int}.y)\ 0)$ due to the reference to free variable $f$. A better definition may be to require contexts to be well-formed for both terms that we want to equate. This definition could exclude contexts like the above whereas it seems to cause another issue: are program
contexts in such a restricted form enough to test terms? We conclude that defining typed contextual equivalence for a dependently typed calculus is still an open problem.

4. Logical Relation

We develop a logical relation for two reasons. The first is parametricity, which ensures abstraction and enables reasoning for programs in polymorphic calculi [42]. Parametricity is usually stated as “any well typed term is logically related to itself.” The second is to show contextual equivalence easily. It is often difficult to prove that given two terms are contextually equivalent since it involves quantification over all program contexts. Much work has developed techniques to reason about contextual equivalence more easily, and many of such reasoning techniques are based on logical relations. We will also use the logical relation to reason about casts in Section 6.

In this section, we first give an informal overview of main ideas in our logical relation in Section 4.1. Then, after preliminary definitions in Section 4.2, we formally define the logical relation in Section 4.3 and state its soundness and completeness with respect to semityped contextual equivalence in Section 4.4. The completeness is given in a restricted form—two contextually equivalent, well-typed terms are logically related; completeness without restrictions is left open.

4.1. Informal Overview. The definition of our logical relation follows Belo et al. [4] and Sekiyama et al. [35]. We start with two type-indexed families of relations $v_1 \simeq_v v_2 : T; \theta; \delta$ for closed values and $e_1 \simeq_e e_2 : T; \theta; \delta$ for closed terms and a relation $T_1 \simeq T_2 : *; \theta; \delta$ for (open) types. The type interpretation $\theta$ assigns value relations to type variables—which is common to relational semantics for a polymorphic language—and $\delta$, called value assignment, gives pairs of values to free term variables in $T$, $T_1$, and $T_2$. Value assignments are introduced by Belo et al. [4] to handle dependency of types on terms. Main differences from the previous work [4, 35] are that (1) our logical relation is semityped just like our con-
\((T_{3-i} \Rightarrow T_{3-i})^\ell v_{3-i}\) would be enough if we are interested only in soundness of the logical relation. Our closure condition—without \((T_{3-i} \Rightarrow T_{3-i})^\ell\)—subsumes this alternative and, in fact, is a key to proving correctness of the upcast elimination and the selfification.

The second closure condition is that \(r\) is closed under (semityped) CIU-equivalence so that the logical relation is complete with respect to contextual equivalence, following the prior work [1]. CIU-equivalence [23] relates two closed terms if they behave equivalently under any evaluation context (use of the terms), and it is extended to open terms with closing substitutions (closed instantiations). Actually, this condition subsumes the first but it will turn out so, only after we finish proving the upcast elimination property in Section 6.1. So, we have to introduce the two conditions separately. Interestingly, the closure under CIU-equivalence also enables us to show transitivity of the logical relation. We will show that CIU-equivalence, the logical relation, and contextual equivalence coincide on well-typed terms via a property similar to Pitts’ “equivalence-respecting property” [29].

4.2. Preliminaries. Here, we give a few preliminary definitions, including CIU-equivalence and the closure conditions on \(r\), to define the logical relation.

**Definition 4.1.**

- \(\text{Typ}\) is the set \(\{T \mid \emptyset \vdash T\}\) of all closed, well-formed types;
- \(\text{UTyp}\) is the set \(\{T \mid \text{FV}(T) \cup \text{FTV}(T) = \emptyset\}\) of all closed types;
- For each \(T \in \text{Typ}\), \(\text{Val}(T)\) is the set \(\{v \mid \emptyset \vdash v : T\}\) of all closed values of \(T\); and
- \(\text{UVal}\) is the set \(\{v \mid \text{FV}(v) \cup \text{FTV}(v) = \emptyset\}\) of all closed values.

In what follows, (capture-avoiding) substitutions, denoted by \(\sigma\), are maps from term and type variables to closed terms and types, respectively, and they can be extended to maps over terms, types, etc. straightforwardly. We write \(\sigma[v/x]\) and \(\sigma[T/\alpha]\) for substitutions that map \(x\) and \(\alpha\) to \(v\) and \(T\), respectively, and other term/type variables according to \(\sigma\). Then, we define the notion of closing substitutions.

**Definition 4.2 (Closing Substitutions).** Substitution \(\sigma\) is a closing substitution that respects \(\Gamma\), written \(\Gamma \vdash \sigma\), if and only if \(\sigma(x) \in \text{Val}(\sigma(\Gamma(x)))\) for any \(x \in \text{dom}(\Gamma)\) and \(\sigma(\alpha) \in \text{Typ}\) for any \(\alpha \in \text{dom}(\Gamma)\).

We define CIU-equivalence below. Our CIU-equivalence rests on static evaluation contexts \(E^S\), where holes do not occur under run-time term constructors such as active checks.

\[
E^S := [\text{op}(v_1, \ldots, v_n, E^S, e_1, \ldots, e_m), E^S e, v E^S | E^S T]
\]

Since a static evaluation context is also a (single-hole) context, we use the context well-formedness judgments also for static evaluation contexts and write \(\Gamma \vdash E^S : (\Gamma_1 \vdash e_1 : T_1) \leadsto T'\). Use of static evaluation contexts, instead of evaluation contexts, is important to show the equivalence-respecting property, especially, Lemma 5.10.

**Definition 4.3 (Semityped CIU-Equivalence).** Terms \(e_1\) and \(e_2\) are CIU-equivalent at \(T\) under \(\Gamma\), written \(\Gamma \vdash e_1 =_{\text{ciu}} e_2 : T\), if and only if (1) \(\Gamma \vdash e_1 : T\), (2) \(\text{FV}(e_2) \cup \text{FTV}(e_2) \subseteq \text{dom}(\Gamma)\), and (3) \(E^S[\sigma(e_1)] \Downarrow E^S[\sigma(e_2)]\), for any \(\sigma\), \(E^S\), and \(T'\) such that \(\Gamma \vdash \sigma\) and \(\emptyset \vdash E^S : (\emptyset \vdash \sigma(e_1) : \sigma(T)) \leadsto T'\).

Using the semityped CIU-equivalence, we define the universe \(\text{VRel}(T_1, T_2)\) of interpretations used for \(r\).
Definition 4.4 (Universe of Interpretations). For $T_1 \in \text{Typ}$ and $T_2 \in \text{UTyp}$,

$$\text{VRel}(T_1, T_2) \overset{\text{def}}{=} \{ R \subseteq \text{Val}(T_1) \times \text{UVal} \mid \forall (v_1, v_2) \in R.\}
$$

$$\exists v_1', v_2', (T_1 \Rightarrow T_1)^T v_1 \rightarrow v_1' \text{ and } (T_2 \Rightarrow T_2)^T v_2 \rightarrow v_2' \text{ and } (v_1', v_2'), (v_1, v_2) \in R, \text{ and } \forall v. \emptyset \vdash v =_{\text{ciu}} v_1 : T_1 \text{ implies } (v, v_2) \in R \}.$$

We write $(r, T_1, T_2)$ if $T_1 \in \text{Typ}$, $T_2 \in \text{UTyp}$, and $r \in \text{VRel}(T_1, T_2)$.

The conditions above on $R$ represent the closure conditions discussed in Section 4.1.

4.3. Formal Definition of Logical Relation. We formally define our logical relation, after defining type interpretations and value assignments below.

Definition 4.5 (Type Interpretations). A type interpretation $\theta$ is a finite map from type variables to tuples $(r, T_1, T_2)$ such that $(r, T_1, T_2)$. We write $\theta : \alpha \mapsto (r, T_1, T_2)$ for the same map as $\theta$ except that $\alpha$ is mapped to $(r, T_1, T_2)$. We also write $\theta_i (i \in \{1, 2\})$ for a substitution that maps type variables $\alpha$ to types $T_i$ such that $\theta(\alpha) = (r, T_1, T_2)$. dom($\theta$) denotes the set of type variables mapped by $\theta$.

Definition 4.6 (Value Assignments). A value assignment $\delta$ is a finite map from term variables to pairs $(v_1, v_2)$ such that $v_1 \in \text{Val}(T)$ for some type $T$ and $v_2 \in \text{UVal}$. We write $\delta[(v_1, v_2)/x]$ for the same mapping as $\delta$ except that $x$ is mapped to $(v_1, v_2)$. We also write $\delta_i (i \in \{1, 2\})$ for a substitution that maps term variables $x$ to values $v_i$ such that $\delta(x) = (v_1, v_2)$. dom($\delta$) denotes the set of term variables mapped by $\delta$.

Definition 4.7 (Value, Term, and Type Relations). We define the value relation $v_1 \simeq_\theta v_2 : T ; \theta; \delta$, the term relation $e_1 \simeq_\theta e_2 : T ; \theta; \delta$, and the type relation $T_1 \simeq_\theta T_2 : \star ; \theta; \delta$ by using the rules in Figure 8. In these relations, values and terms (resp. types) on the left hand side are closed and well typed (resp. well formed) and those on the right hand side are closed:

- if $v_1 \simeq_\theta v_2 : T ; \theta; \delta$, then $\emptyset \vdash v_1 : \theta_1(\delta_1(T))$ and $\text{FV}(v_2) \cup \text{FTV}(v_2) = \emptyset$;
- if $e_1 \simeq_\theta e_2 : T ; \theta; \delta$, then $\emptyset \vdash e_1 : \theta_1(\delta_1(T))$ and $\text{FV}(e_2) \cup \text{FTV}(e_2) = \emptyset$ and $\emptyset$;
- if $T_1 \simeq_\theta T_2 : \star ; \theta; \delta$, then $\emptyset \vdash \theta_1(\delta_1(T_1))$ and $\text{FV}(T_2) \cup \text{FTV}(T_2) \subseteq \text{dom}(\theta) \cup \text{dom}(\delta)$.

The definitions of value, term, and type relations are quite similar to the prior work [4, 35], but we explain them here briefly. Value relations on $B$ and $\alpha$ are standard. Related values $v_1$ and $v_2$ at function type $x : T_1 \rightarrow T_2$ have to produce related values when applied to related arguments $v_1'$ and $v_2'$ at $T_1$. Since $T_2$ may depend on arguments, $v_1'$ and $v_2'$ are recorded in value assignment $\delta$ so that the arguments can be referred to by refinements in $T_2$. Values related at all $\alpha, T$ produces related values, regardless of the interpretation $(r, T_1, T_2)$ for $\alpha$. Values related at $\{x : T | e \}$ have to be related at the underlying type $T$ and satisfy refinement $e$. What values and types should be substituted for free variables in $e$ are found in $\delta$ and $\theta$; we evaluate the refinement obtained by applying $\theta$ and $\delta$. Term relations contain terms that raise blame with the same label or evaluate to related values. Type relations, intuitively, relate types with the same “denotation.” Function types are related if both domain and codomain types are related. The codomain types may depend on values of the domain types, so we require them to be related under an extension of $\delta$ with any pair of values related at the well-formed domain type—we choose the well-formed type, not the possibly ill-formed one, since the index type in a value relation has to be
\[
\begin{align*}
\text{Value Relation} & \\
v_1 \simeq_v v_2 : T; \theta; \delta & \iff \exists r, T_1, T_2. \theta(\alpha) = (r, T_1, T_2) \text{ and } (v_1, v_2) \in r \\
v_1 \simeq_v v_2 : B; \theta; \delta & \iff v_1 = v_2 \text{ and } v_1 \in \mathcal{K}_B \\
v_1 \simeq_v v_2 : x : T_1 \rightarrow T_2; \theta; \delta & \iff \forall v'_1, v'_2. v'_1 \simeq_v v'_2 : T_1; \theta; \delta \text{ implies } v'_1 \simeq_v v'_2 : T_2; \theta; \delta[\theta(v'_1/v'_2)/x] \\
v_1 \simeq_v v_2 : \forall \alpha. T; \theta; \delta & \iff \forall T_1, T_2, r. (r, T_1, T_2) \text{ implies } v_1 \simeq_v v_2 : T; \theta \{\alpha \mapsto r, T_1, T_2\}; \delta \\
v_1 \simeq_v v_2 : \{x : T \mid e\}; \theta; \delta & \iff v_1 \simeq_v v_2 : T; \theta; \delta \text{ and } \\
& \quad \theta_1(\delta_1(e[v_1/x])) \rightarrow^* \text{true and } \theta_2(\delta_2(e[v_2/x])) \rightarrow^* \text{true}
\end{align*}
\]

\[
\begin{align*}
\text{Term Relation} & \\
e_1 \simeq_e e_2 : T; \theta; \delta & \iff e_1 \rightarrow^* \uparrow \ell \text{ and } e_2 \rightarrow^* \uparrow \ell, \text{ or } \\
& \quad e_1 \rightarrow^* v_1 \text{ and } e_2 \rightarrow^* v_2 \text{ and } v_1 \simeq_e v_2 : T; \theta; \delta
\end{align*}
\]

\[
\begin{align*}
\text{Type Relation} & \\
B \simeq B : *; \theta; \delta & \\
\alpha \simeq \alpha : *; \theta; \delta & \\
x : T_1 \rightarrow T_2 \simeq x : T_1 \rightarrow T_2 : *; \theta; \delta & \iff T_1 \simeq T_2 : *; \theta; \delta \text{ and } \\
& \quad \forall v_1, v_2. v_1 \simeq_v v_2 : T_1; \theta; \delta \text{ implies } \\
& \quad T_1 \simeq T_2 : *; \theta; \delta[\theta(v_1/v_2)/x] \\
\forall \alpha. T_1 \simeq \forall \alpha. T_2 : *; \theta; \delta & \iff \forall T'_1, T'_2, r. (r, T'_1, T'_2) \text{ implies } \\
& \quad T_1 \simeq T_2 : *; \theta \{\alpha \mapsto r, T'_1, T'_2\}; \delta \\
\{x : T_1 \mid e_1\} \simeq \{x : T_2 \mid e_2\} : *; \theta; \delta & \iff T_1 \simeq T_2 : *; \theta; \delta \text{ and } \\
& \quad \forall v_1, v_2. v_1 \simeq_v v_2 : T_1; \theta; \delta \text{ implies } \\
& \quad \theta_1(\delta_1(e_1[v_1/x])) \simeq_e \theta_2(\delta_2(e_2[v_2/x])) : \text{Bool}; \theta; \delta
\end{align*}
\]

Figure 8: Value, term, and type relations

well formed (Definition 4.7). Universal types \(\forall \alpha. T_1\) and \(\forall \alpha. T_2\) are related if \(T_1\) and \(T_2\) are related under an extension of \(\theta\) with any interpretation. Refinement types are related if both the underlying types and the refinements are related; we choose values for the bound variable from the value relation indexed by the underlying type \(T_1\) on the left hand side because it is well formed.

Now, we extend term relations for closed terms to open terms.

**Definition 4.8.** The relation \(\Gamma \vdash \theta; \delta\) is defined by: \(\Gamma \vdash \theta; \delta\) if and only if

1. \((\text{for any } \alpha \in \Gamma, \alpha \in \text{dom}(\theta))\) and
2. \((\text{for any } x : T \in \Gamma, \delta_1(x) \simeq_v \delta_2(x) : T; \theta; \delta)\).

**Definition 4.9 (Logical Relation).** Terms \(e_1\) and \(e_2\) are logically related at \(T\) under \(\Gamma\), written \(\Gamma \vdash e_1 \simeq e_2 : T\), if and only if

1. \((\text{for any } \alpha \in \Gamma, \alpha \in \text{dom}(\theta))\) and
2. \((\text{for any } x : T \in \Gamma, \delta_1(x) \simeq_v \delta_2(x) : T; \theta; \delta)\).

Similarly, types \(T_1\) and \(T_2\) are logically related under \(\Gamma\), written \(\Gamma \vdash T_1 \simeq T_2 : *\), if and only if

1. \((\text{for any } \alpha \in \Gamma, \alpha \in \text{dom}(\theta))\) and
2. \((\text{for any } x : T \in \Gamma, \delta_1(x) \simeq_v \delta_2(x) : T; \theta; \delta)\).
4.4. Soundness and Completeness. We state the soundness and the completeness of the logical relation with respect to the semityped contextual equivalence; we prove them in Section 5.

Theorem 4.10 (Soundness). For any \( \Gamma_1, \ldots, \Gamma_n, e_{11}, \ldots, e_{1n}, e_{21}, \ldots, e_{2n}, \) and \( T_1, \ldots, T_n, \) if \( \Gamma_i \vdash e_{1i} \simeq e_{2i} : T_i \) for \( i \in \{1, \ldots, n\}, \) then \( \Gamma_i \vdash e_{1i} \simeq_{\text{ctx}} e_{2i} : T_i^{\prime}. \)

Theorem 4.11 (Completeness with respect to Typed Terms). If \( \Gamma_i \vdash e_{1i} \simeq_{\text{ctx}} e_{2i} : T_i \) and \( \Gamma_j \vdash e_{2j} : T_j \) for any \( j, \) then \( \Gamma_j \vdash e_{1j} \simeq e_{2j} : T_j \) for any \( j. \)

5. Proving soundness and completeness

This section gives proofs of the soundness and the completeness of the logical relation. The readers who read this paper for the first time can skip this section.

5.1. Soundness. We start with describing an overview of the proof and then detail it.

5.1.1. Overview. Following the prior work on program reasoning with logical relations [29, 1, 9, 3], our proof of the soundness rests on so-called the fundamental property, which states that a logical relation is closed under term constructors.\(^6\) If we have the fundamental property, it is easy to show the soundness.

In manifest contracts, dependency of types on terms makes proving the fundamental property difficult. To see it, let us try to prove that the logical relation is closed under the term application constructor:

\[
\text{if } \Gamma \vdash e_{11} \simeq e_{21} : (x:T_1 \to T_2) \text{ and } \Gamma \vdash e_{12} \simeq e_{22} : T_1, \text{ then } \Gamma \vdash e_{11} e_{12} \simeq e_{21} e_{22} : T_2[e_{12}/x].
\]

A problem occurs in the case that \( T_2 \) is a refinement type \( \{y:T_2' | e_2'\}. \) In that case, we have to prove that

\[
e_{11} e_{12} \simeq_{\theta} e_{21} e_{22} : \{y:T_2' | e_2'\} | e_{12}/x \theta \delta
\]

(we omit \( \theta_i \) and \( \delta_i \) in application terms for simplicity). Specifically, we have to show that the evaluation results of both \( e_{11} e_{12} \) and \( e_{21} e_{22} \) satisfy refinement \( e_{2}'[e_{12}/x] \). On the one hand, it is trivial that \( e_{11} e_{12} \) satisfies \( e_{2}'[e_{12}/x] \) because the type of \( e_{11} e_{12} \) is \( \{y:T_2' | e_2'\} | e_{12}/x \) and well-typed terms satisfy all refinements in their types (Lemma 2.9). On the other hand, while it is easy to show \( e_{21} e_{22} \) satisfies \( e_{2}'[e_{22}/x] \), proving that \( e_{21} e_{22} \) satisfies \( e_{2}'[e_{12}/x] \) is nontrivial.

Our key idea to addressing the nontrivial case is to assume that refinement \( e_2' \) is logically related to itself. This assumption allows us to show that \( e_2'[e_{12}/x] \) and \( e_2'[e_{22}/x] \) are logically related since so are \( e_{12} \) and \( e_{22}. \) Since logically related Boolean expressions evaluate to the same value (if any), we obtain that \( e_{21} e_{22} \) satisfies \( e_2'[e_{12}/x] \) if and only if it does \( e_2'[e_{22}/x]. \) Since the latter can be shown easily, we achieve the goal. For a rigorous proof following this idea, we assume that \( \Gamma, e_{12}, T_2', \) and \( T_1 \) are also logically related to themselves.

Definition 5.1 (Self-Related Typing Contexts). \( \Gamma \) is self-related if and only if \( \Gamma_1 \vdash T \simeq T : \varepsilon \) for any \( \Gamma_1 \) and \( T \) such that \( \Gamma = \Gamma_1, x:T, \Gamma_1'. \)

\(^6\)In some work [1, 9, 3] the fundamental property means reflexivity of logical relations, but in this work it does compatibility of the logical relation as in Pitts [29].
In summary, we show that:

*Suppose that \( \Gamma \) is self-related, \( \Gamma \vdash e_{12} \simeq e_{12} : T_1 \) and \( \Gamma \vdash x : T_1 \to T_2 \simeq x : T_1 \to T_2 : * \). If \( \Gamma \vdash e_{11} \simeq e_{21} : (x : T_1 \to T_2) \) and \( \Gamma \vdash e_{12} \simeq e_{22} : T_1 \), then \( \Gamma \vdash e_{11} e_{12} \simeq e_{21} e_{22} : T_2[e_{12}/x] \).

These additional assumptions, which we call self-relatedness, are needed also in other term constructors such as type application. Self-relatedness assumptions are discharged once the parametricity, which amounts to reflexivity of the logical relation, is shown.

We believe that the parametricity can be shown independently of the fundamental property, but their proofs are quite similar, so we organize a proof of the soundness as follows to avoid writing similar proofs and save the amount of work.

1. Prove that the logical relation is closed under each constructor under self-relatedness assumptions.
2. Prove the parametricity with the lemmas shown in (1).
3. Prove the soundness of the logical relation by discharging the self-relatedness assumptions from the lemmas shown in (1) with the parametricity.

5.1.2. **Proof.** The proof proceeds as follows. We start with showing weakening and strengthening of the logical relation (Lemmas 5.2–5.3), which are used broadly throughout the proof. We next prove the most challenging cases in the fundamental property: term application (Lemmas 5.4–5.7) and type application (Lemmas 5.8–5.15). After showing the remaining cases of the fundamental property (Lemmas 5.16–5.26), we prove the parametricity (Theorem 5.28) and then the soundness of the logical relation (Theorem 4.10).

**Weakening and strengthening.**

**Lemma 5.2 (Value Weakening/Strengthening).** Suppose that \( x \) is a fresh variable. If \( v_1 \) is a closed well-typed value and \( v_2 \) is a closed (but not necessarily well-typed) value, then:

1. \( e_1 \simeq_e e_2 : T ; \theta ; \delta \iff e_1 \simeq_e e_2 : T ; \theta ; \delta[(v_1, v_2)/x] \);
2. \( T_1 \simeq T_2 : * ; \theta ; \delta \iff T_1 \simeq T_2 : * ; \theta ; \delta[(v_1, v_2)/x] \); and
3. \( \Gamma , \Gamma' \vdash \theta ; \delta \) and \( v_1 \simeq_v v_2 : T ; \theta ; \delta \iff \Gamma , x : T , \Gamma' \vdash \theta ; \delta[(v_1, v_2)/x] \).

Moreover, we have the following weakening lemmas:

4. If \( \Gamma , \Gamma' \vdash e_1 \simeq e_2 : T' \) and \( \Gamma \vdash T \), then \( \Gamma , x : T , \Gamma' \vdash e_1 \simeq e_2 : T' \).
5. If \( \Gamma , \Gamma' \vdash T_1 \simeq T_2 : * \) and \( \Gamma \vdash T \), then \( \Gamma , x : T , \Gamma' \vdash T_1 \simeq T_2 : * \).

**Proof.**

1. By straightforward induction on \( T \).
2. By straightforward induction on \( T_1 \).
3. By definition and (1).
4. By (3) and (1).
5. By (3) and (2).
Lemma 5.3 (Type Weakening/Strengthening). Suppose that $\alpha$ is a fresh type variable.

1. $e_1 \simeq e_2 : T; \theta; \delta$ and $(r, T_1, T_2)$ iff $e_1 \simeq e_2 : T; \theta \{ \alpha \mapsto r, T_1, T_2 \}; \delta$;
2. $T_1' \simeq T_2' : *; \theta; \delta$ and $(r, T_1, T_2)$ iff $T_1' \simeq T_2' : *; \theta \{ \alpha \mapsto r, T_1, T_2 \}; \delta$; and
3. $\Gamma, \Gamma' \vdash \theta; \delta$ and $(r, T_1, T_2)$ iff $\Gamma, \alpha, \Gamma' \vdash \theta \{ \alpha \mapsto r, T_1, T_2 \}; \delta$.
4. $\Gamma, \Gamma' \vdash e_1 \simeq e_2 : T$ iff $\Gamma, \alpha, \Gamma' \vdash e_1 \simeq e_2 : T$.
5. $\Gamma, \Gamma' \vdash T_1 \simeq T_2 : *$ iff $\Gamma, \alpha, \Gamma' \vdash T_1 \simeq T_2 : *$.

\[ \delta \]

Proof. Similar to Lemma 5.2.

Fundamental property: term application. To show that the logical relation is closed under term application, we have to prove that, if $v_{11} \simeq v_{21} : (x:T_1 \rightarrow T_2); \theta; \delta$ and $v_{12} \simeq v_{22} : T_1; \theta; \delta$, then $v_{11} v_{12} \simeq v_{21} v_{22} : T_2[|_{T_1}]; \theta; \delta$. However, the definition of the logical relation states only that $v_{11} v_{12} \simeq v_{21} v_{22} : T_2; \theta; \delta[\{v_{12}, v_{22}\}/x]$. Thus, we have to show that the term relation indexed by $T_2[|_{T_1}]$ with $\delta$ is equivalent to the one indexed by $T_2$ with $\delta[\{v_{12}, v_{22}\}/x]$. This property is generalized to the so-called term compositionality [4, 35].

Lemma 5.4. Given $\theta, \theta', \delta$, and $\delta'$, suppose that $\{\{\alpha, r, T_1\} \mid \exists T_2. \theta(\alpha) = (r, T_1, T_2)\} = \{\{\alpha, r, T_1\} \mid \exists T_2. \theta'(\alpha) = (r, T_1, T_2)\}$ and $\delta_1 = \delta_1'$. If $T \simeq T' : *; \theta; \delta$ and $T \simeq T' : *; \theta'; \delta'$, then $e_1 \simeq e_2 : T; \theta; \delta$ iff $e_1 \simeq e_2 : T; \theta'; \delta'$ for any $e_1$ and $e_2$.

Proof. By induction on $T$. The interesting case is that $T = \{x:T' \mid e\}$. If $e_1$ and $e_2$ raise blame, the conclusion follows straightforwardly. Otherwise, $e_1 \rightarrow^* v_1$ and $e_2 \rightarrow^* v_2$ for some $v_1$ and $v_2$, and by definition, we have to show that

\[
\begin{align*}
\text{if } v_1 & \simeq v_2 : T'; \theta'; \delta' \text{ and } \theta_1'(\delta_1'(e'[v_1/x])) \rightarrow^* \text{true} \text{ and } \theta_2'(\delta_2'(e'[v_2/x])) \rightarrow^* \text{true} \\
\text{then } v_1 & \simeq v_2 : T'; \theta'; \delta' \text{ and } \theta_1'(\delta_1'(e'[v_1/x])) \rightarrow^* \text{true} \text{ and } \theta_2'(\delta_2'(e'[v_2/x])) \rightarrow^* \text{true}.
\end{align*}
\]

We show only the left-to-right direction, but the other is also shown in a similar way. Since $\{x:T' \mid e\} \simeq \{x:T' \mid e\} : *; \theta; \delta$ and $\{x:T' \mid e\} \simeq \{x:T' \mid e\} : *; \theta'; \delta'$, we have $T' \simeq T' : *; \theta; \delta$ and $T' \simeq T' : *; \theta'; \delta'$. Since $v_1 \simeq v_2 : T'; \theta; \delta$ by the assumption in the left-to-right direction, we have

\[
v_1 \simeq v_2 : T'; \theta'; \delta'.
\]

by the IH. By the assumptions of this lemma, $\theta_1(\delta_1(e'[v_1/x])) = \theta_1'(\delta_1'(e'[v_1/x]))$. Since in the left-to-right direction we assume that $\theta_1(\delta_1(e'[v_1/x])) \rightarrow^* \text{true}$, we have

\[
\theta_1'(\delta_1'(e'[v_1/x])) \rightarrow^* \text{true}.
\]

Since $\{x:T' \mid e\} \simeq \{x:T' \mid e\} : *; \theta'; \delta'$ and $v_1 \simeq v_2 : T'; \theta'; \delta'$, we have

\[
\theta_1'(\delta_1'(e'[v_1/x])) \simeq_e \theta_2'(\delta_2'(e'[v_2/x])) : \text{Bool}; \theta'; \delta'
\]

by definition. Since terms related at $\text{Bool}$ evaluate to the same value, we have

\[
\theta_2'(\delta_2'(e'[v_2/x])) \rightarrow^* \text{true}.
\]
Lemma 5.5. Suppose that $\Gamma, x: T, \Gamma' \vdash \theta; \delta[(v_1, v_2)/x]$ and $v_1 \simeq v_2': T'; \theta; \delta[(v_1, v_2)/x]$, then $\Gamma, x: T, \Gamma' \vdash \theta; \delta[(v_1, v_2')/x]$. 

Proof. By induction on $\Gamma'$. The case that $\Gamma' = \Gamma'', y: T'$ is shown by Lemma 5.4.

Now, we show the term compositionality. In the statement, $v_1'$ and $v_2'$ correspond to $v_{12}$ and $v_{22}$, respectively, in the paragraph informally explaining this property and $e'$ to $e_{12}$ discussed in the second paragraph of Section 5.1.1.

Lemma 5.6 (Term Compositionality). Suppose that $\Gamma, x: T', \Gamma' \vdash T \simeq T' : *$. If $\Gamma, x: T', \Gamma' \vdash \theta; \delta[(v_1', v_2')/x]$ and $\theta_1(\delta_1(e')) \rightarrow^* v_1'$ and $\theta_2(\delta_2(e')) \rightarrow^* v_2'$ and $v_1' \simeq v_2' : T'; \theta; \delta$, then $e_1 \simeq e_2 : T; \theta; \delta[(v_1', v_2')/x]$ iff $e_1 \simeq e_2 : T[e'/x]; \theta; \delta$ for any $e_1$ and $e_2$.

Proof. By induction on $T$. If $e_1$ and $e_2$ raise blame, then the conclusion is obvious. In what follows, suppose that $e_1 \rightarrow^* v_1$ and $e_2 \rightarrow^* v_2$. All cases except that $T$ is a refinement type are straightforward by the IH(s).

Let us consider the case that $T = \{y: T'' | e''\}$. Without loss of generality, we can suppose that $y \notin \text{dom}(\delta)$. We have to show:

$v_1 \simeq v_2 : T''[e'/x]; \theta; \delta$

by the IH. Since $\theta_1(\delta_1(e')) \rightarrow^* v_1$ and $\theta_1(\delta_1(e''[v_1'/x]))[v_1/y] \rightarrow^* \text{true}$ (the assumption in the left-to-right direction), we have

$\theta_1(\delta_1(e''[e'/x]))[v_1/y] \rightarrow^* \text{true}$

by Cotermination (Lemma 2.6). The remaining obligation is

$\theta_2(\delta_2(e''[e'/x]))[v_2/y] \rightarrow^* \text{true}$.

Since $v''_2$ is the evaluation result of $\theta_2(\delta_2(e'))$, it suffices to show that, by Cotermination,

$\theta_2(\delta_2(e''[v_2''/x]))[v_2/y] \rightarrow^* \text{true}$.

Since $\Gamma, x: T', \Gamma' \vdash \theta; \delta[(v_1', v_2')/x]$ and $v_1 \simeq v_2 : T''[e'/x]; \theta; \delta[(v_1', v_2')/x]$, we have

$\Gamma, x: T', \Gamma', y: T'' \vdash \theta; \delta[(v_1', v_2')/x][((v_1, v_2)/y)]$ (5.1)

by the weakening (Lemma 5.2). Since $v_1' \simeq v_2'' : T'; \theta; \delta$, we have

$v_1' \simeq v_2'' : T'; \theta; \delta[(v_1', v_2')/x][((v_1, v_2)/y)]$ (5.2)

by the weakening. By applying Lemma 5.5 to (5.1) and (5.2), we have

$\Gamma, x: T', \Gamma', y: T'' \vdash \theta; \delta[(v_1', v_2''/x)][((v_1, v_2)/y)]$.

Since $\Gamma, x: T', \Gamma', y: T'' \vdash e'' \simeq \text{Bool}$ from $\Gamma, x: T', \Gamma' \vdash \{y: T'' | e''\} \simeq \{y: T'' | e''\} : *$, we have

$\theta_1(\delta_1(e''[v_1'/x]))[v_1/y] \simeq e_2(\theta_2(\delta_2(e''[v_2''/x]))[v_2/y] : \text{Bool}; \theta; \delta[(v_1', v_2'')/x]][(v_1, v_2)/y]$. 


Since the term on the left-hand side evaluates to true (the assumption in the left-to-right direction), the one on the right-hand side also evaluates to true by definition. Hence, we finish.

The other direction is shown in a similar way except the case of
\[
\theta_2(\delta_2(e''[v'_2/x]))[v_2/y] \rightarrow^* true.
\]
This case is shown as follows. From (5.1), which can be shown also in the right-to-left direction with the IH, and \( \Gamma, x:T', \Gamma', y:T'' \vdash e'' \simeq e'' : \text{Bool} \), it is found that
\[
\theta_1(\delta_1(e''[v'_1/x]))[v_1/y] \simeq \theta_2(\delta_2(e''[v'_2/x]))[v_2/y] : \text{Bool}; \theta; \delta((v_1, v_2)/x)[(v_1, v_2)/y].
\]
Since it is found that the term on the left-hand side evaluates to true by applying Cotermination to \( \theta_1(\delta_1(e''[e'/x]))[v_1/y] \rightarrow^* true \), so does the one on the right-hand side, which we want to show.

**Lemma 5.7** (Compatibility under Self-relatedness Assumption: Application). Suppose that \( \Gamma \) is self-related and \( \Gamma \vdash e_{12} \simeq e_{12} : T_1 \) and \( \Gamma \vdash x:T_1 \rightarrow T_2 \simeq x:T_1 \rightarrow T_2 : * \). If \( \Gamma \vdash e_{11} \simeq e_{21} : (x:T_1 \rightarrow T_2) \) and \( \Gamma \vdash e_{12} \simeq e_{22} : T_1 \), then \( \Gamma \vdash e_{11} e_{12} \simeq e_{21} e_{22} : T_2[e_{12}/x] \).

**Proof.** Suppose that \( \Gamma \vdash \theta; \delta \). Let \( e'_{11} = \theta_1(\delta_1(e_{11})) \), \( e'_{12} = \theta_1(\delta_1(e_{12})) \), \( e'_{21} = \theta_2(\delta_2(e_{21})) \), and \( e'_{22} = \theta_2(\delta_2(e_{22})) \). It suffices to show that
\[
e'_{11} e'_{12} \simeq e'_{21} e'_{22} : T_2[e_{12}/x]; \theta; \delta.
\]
If both \( e'_{11} \) and \( e'_{21} \) or both \( e'_{12} \) and \( e'_{22} \) raise blame, the conclusion is obvious. Otherwise, we can suppose that \( e'_{11} \rightarrow^* v_{11} \) and \( e'_{12} \rightarrow^* v_{12} \) and \( e'_{21} \rightarrow^* v_{21} \) and \( e'_{22} \rightarrow^* v_{22} \) for some \( v_{11}, v_{12}, v_{21}, \) and \( v_{22} \), and it suffices to show that
\[
v_{11} v_{12} \simeq v_{21} v_{22} : T_2[e_{12}/x]; \theta; \delta.
\]
Since \( e'_{11} \simeq e'_{21} : x:T_1 \rightarrow T_2; \theta; \delta \) and \( e'_{12} \simeq e'_{22} : T_1; \theta; \delta \), we have \( v_{11} \simeq_{v} v_{21} : x:T_1 \rightarrow T_2; \theta; \delta \) and \( v_{12} \simeq v_{22} : T_2; \theta; \delta \). Thus, \( v_{11} v_{12} \simeq_{v} v_{21} v_{22} : T_2; \theta; \delta[(v_{12}, v_{22})/x] \) by definition.

Since \( \Gamma \vdash x:T_1 \rightarrow T_2 \simeq x:T_1 \rightarrow T_2 : * \), we have \( \Gamma \vdash T_1 \simeq T_1 : * \) and \( \Gamma, x:T_1 \vdash T_2 \simeq T_2 : * \).
Since \( \Gamma \) is self-related, so is \( \Gamma, x:T_1 \). Since \( \Gamma \vdash \theta; \delta \), we have \( \Gamma, x:T_1 \vdash \theta; \delta[(v_{12}, v_{22})/x] \) by the weakening (Lemma 5.2). We have \( \theta_1(\delta_1(e_{12})) \rightarrow^* v_{12} \). Since \( \theta_1(\delta_1(e_{12})) = e'_{12} \rightarrow^* v_{12} \) and \( \Gamma \vdash e_{12} \simeq e_{12} : T_1 \), we have \( \theta_2(\delta_2(e_{12})) \rightarrow^* v'_{12} \) and \( v_{12} \simeq_{v} v'_{12} : T_1; \theta; \delta \) for some \( v'_{12} \).
Thus, by the term compositionality (Lemma 5.6), we finish.

**Fundamental property: type application.** We show that the logical relation is closed under type applications, that is, if \( v_1 \simeq_{v} v_2 : \forall \alpha.T; \theta; \delta \) and \( T_1 \simeq T_2 : *; \theta; \delta \), then \( v_1 T_1 \simeq_{v} v_2 T_2 : T[T_1/\alpha]; \theta; \delta \). To this end, for a reason similar to the case of term applications, we show the type compositionality, which states that the term relation indexed by \( T[T_1/\alpha] \) with \( \theta \) coincides with the one indexed by \( T \) with \( \theta \{ \alpha \mapsto (r, T_1, T_2) \} \) for some \( r \). Since \( r \) gives an interpretation of \( \alpha \) and \( \alpha \) is replaced with \( T_1 \) in the former, it is natural to choose the term relation \( e_1 \simeq e_2 : T_1; \theta; \delta \) indexed by \( T_1 \) as \( r \). We first show that the term relation satisfies requirements to interpretations (Lemmas 5.8–5.13) and then the type compositionality (Lemma 5.14).

The first requirement which we show that term relations satisfy is that, if \( \theta(\alpha) = (r, T_1, T_2) \) and \((v_1, v_2) \in r\), then there exists some \( v'_1 \) such that \( \langle T_1 \Rightarrow T_1 \rangle^* v_1 \rightarrow^* v'_1 \) and \((v'_1, v_2) \in r\). This is generalized to elimination of reflexive casts.
Lemma 5.8 (Elimination of Reflexive Casts on Left). If \( T_1 \simeq T_1 : *; \theta; \delta \) and \( T_2 \simeq T_2 : *; \theta; \delta \) and \( T_1 \simeq T_2 : *; \theta; \delta \) and \( T_2 \simeq T_1 : *; \theta; \delta \), then \( \theta_1(\delta_1(\langle T_1 \Rightarrow T_2 \rangle^\ell)) \simeq \theta_2(\delta_2(\langle x : T_1, x \rangle)) : T_1 \rightarrow T_2 ; \theta ; \delta \).

Proof. By course-of-values induction on the sum of sizes of \( T_1 \) and \( T_2 \). By definition, it suffices to show that, for any \( v_1 \) and \( v_2 \) such that \( v_1 \simeq_\theta v_2 : T_1 ; \theta ; \delta \),

\[
\theta_1(\delta_1(\langle T_1 \Rightarrow T_2 \rangle^\ell)) v_1 \simeq_e v_2 : T_2 ; \theta ; \delta.
\]

By case analysis on the derivation of \( T_1 \simeq T_2 : *; \theta; \delta \).

Case \( \alpha \simeq \alpha : *; \theta; \delta \): Since \( v_1 \simeq_\alpha v_2 : \alpha; \theta; \delta \), there exists some \( r' \), \( T'_1 \), and \( T'_2 \) such that \( \theta(\alpha) = \langle r', T'_1, T'_2 \rangle \) and \( \langle r', T'_1, T'_2 \rangle \) and \( \langle v_1, v_2 \rangle \in r' \). Since \( \theta(\delta_1(\langle T_1 \Rightarrow T_2 \rangle^\ell)) = \langle T'_1 \Rightarrow T'_2 \rangle^\ell \), and \( r \in \text{VRel}(T'_1, T'_2) \), there exists some \( v'_1 \) such that \( \theta_1(\delta_1(\langle T_1 \Rightarrow T_2 \rangle^\ell)) v_1 \rightarrow^* v'_1 \) and \( v'_1, v_2 \in r' \). We have \( v'_1 \simeq_\theta v_2 : \alpha; \theta; \delta \), and so we finish.

Case \( B \simeq B : *; \theta; \delta \): Obvious since \( T_1 = T_2 = B \).

Case \( x : T_1 \rightarrow T_2 \simeq x : T_1 \rightarrow T_2 : *; \theta; \delta \): Without loss of generality, we can suppose that \( x \notin \text{dom}(\delta) \). By \( \text{(E\_RED)}/(\text{R\_FUND}) \),

\[
\theta_1(\delta_1(\langle T_1 \Rightarrow T_2 \rangle^\ell)) v_1 \rightarrow \theta_1(\delta_1(\lambda x : T_2, \text{let } y : T_1 = \langle T_1 \Rightarrow T_1 \rangle^\ell x \in \langle T_1 \Rightarrow T_2 \rangle^\ell (v_1 y)))
\]

for some fresh variable \( y \). It thus suffices to show that

\[
\theta_1(\delta_1(\lambda x : T_2, \text{let } y : T_1 = \langle T_1 \Rightarrow T_1 \rangle^\ell x \in \langle T_1 \Rightarrow T_2 \rangle^\ell (v_1 y))) \simeq_\theta v_2 : x : T_2 \rightarrow T_2 ; \theta ; \delta.
\]

By definition, for any \( v'_1 \) and \( v'_2 \) such that \( v'_1 \simeq_\theta v'_2 : T_2 ; \theta ; \delta \), we have to show that

\[
\theta_1(\delta_1(\text{let } y : T_1 = \langle T_2 \Rightarrow T_1 \rangle^\ell v'_1 \in \langle T_2 \Rightarrow T_2 \rangle^\ell (v_1 y))) \simeq_e v_2 : T_2 ; \theta ; \delta[\langle v'_1, v'_2 \rangle/x].
\]

By the IH, \( \theta_1(\delta_1(\langle T_2 \Rightarrow T_1 \rangle^\ell)) \simeq_\theta \theta_2(\delta_2(\lambda x : T_2, x) : T_2 \rightarrow T_1) \). Since \( v'_1 \simeq_\theta v'_2 : T_2 ; \theta ; \delta \), we have \( \theta_1(\delta_1(\langle T_2 \Rightarrow T_1 \rangle^\ell)) v'_1 \simeq_\theta v'_2 : T_1 ; \theta ; \delta \). Thus, there exists some \( v''_1 \) such that \( \theta_1(\delta_1(\langle T_2 \Rightarrow T_1 \rangle^\ell)) v'_1 \rightarrow^* v''_1 \) and \( v''_1 \simeq_\theta v'_2 : T_1 ; \theta ; \delta \). Hence, it suffices to show that

\[
\theta_1(\delta_1(\langle T_1 \Rightarrow T_2 \rangle^\ell (v''_1/x))) (v''_1 \simeq_\theta v''_2 : T_2 ; \theta ; \delta[\langle v'_1, v'_2 \rangle/x].
\]

Since \( v_1 \simeq_\theta v_2 : x : T_1 \rightarrow T_2 ; \theta ; \delta \), we have \( v_1 v'' \simeq_\theta v_2 v'' : T_2 ; \delta[\langle v''_1, v''_2 \rangle/x] \). If \( v_1 v'' \) and \( v_2 v'' \) raise blame, we finish. Otherwise, \( v_1 v'' \rightarrow^* v''_1 \) and \( v_2 v'' \rightarrow^* v''_2 \) for some \( v''_1 \) and \( v''_2 \), and it suffices to show that

\[
\theta_1(\delta_1(\langle T_1 \Rightarrow T_2 \rangle^\ell (v''_1/x))) (v''_1 \simeq_\theta v''_2 : T_2 ; \theta ; \delta[\langle v'_1, v'_2 \rangle/x].
\]

We have \( v''_1 \simeq_\theta v''_2 : T_2 ; \theta ; \delta[\langle v''_1, v''_2 \rangle/x] \). From the assumptions, we have:

- \( T_1 \simeq T_1 : *; \theta; \delta[\langle v''_1, v''_2 \rangle/x] \)
- \( T_2 \simeq T_2 : *; \theta; \delta[\langle v''_1, v''_2 \rangle/x] \)
- \( T_1 \simeq T_2 : *; \theta; \delta[\langle v''_1, v''_2 \rangle/x] \)
- \( T_2 \simeq T_1 : *; \theta; \delta[\langle v''_1, v''_2 \rangle/x] \)

Let \( \delta' = \delta[\langle v'_1, v'_2 \rangle/x][\langle v''_1, v''_2 \rangle/y] \). Since type relations are closed under \( \alpha \)-renaming, we have:

- \( T_1 [y/x] \simeq T_1 [y/x] : *; \theta; \delta' \)
- \( T_2 \simeq T_2 : *; \theta; \delta' \)
- \( T_2 [y/x] \simeq T_2 [y/x] : *; \theta; \delta' \)
- \( T_2 \simeq T_2 : *; \theta; \delta' \)
by the weakening (Lemma 5.2). Furthermore, we can show
- $T_{12} [y/x] \simeq T_{22} : \tau; \delta'$ from $T_{12} [y/x] \simeq T_{22} [y/x] : \tau; \delta' \text{ and}$
- $T_{22} \simeq T_{12} [y/x] : \tau; \delta'$ from $T_{22} \simeq T_{12} : \tau; \delta'$
because $x$ and $y$ have the same denotation in $\delta'_2$, that is, $\delta'_2(x) = \delta'_2(y)$. Thus, by the IH,

$$
\theta_1(\delta_1((T_{12} [y/x] \Rightarrow T_{22}')) [v'_1 / x, v''_2 / y]) \\
\simeq \theta_2(\delta_2(\lambda y: T_{12} [y/x, x] [v'_2 / x, v''_2 / y]) : T_{12} [y/x] \Rightarrow T_{22}; \tau; \delta').
$$

Since $v''_1 \simeq \theta_2 (T_{12}; \tau; \delta [v'_1, v'_2 / x])$, we have $v''_1 \simeq_{\nu} v''_2 : T_{12} [y/x] ; \tau; \delta'$ (term relations
are closed under $\alpha$-renaming). Thus,

$$
\theta_1(\delta_1((T_{12} [v''_1 / x] \Rightarrow T_{22} [v'_1 / x])) v''_1 \simeq_{\nu} v''_2 : T_{22} ; \tau; \delta [v'_1, v'_2 / x]
$$

with the weakening. This is what we want to show.

Case $\forall_\alpha. T'_1 \simeq \forall_\alpha. T'_2 : \tau; \delta$: Straightforward by the IH.
Case $\{x: T'_1 | e'_1\} \simeq \{x: T'_2 | e'_2\} : \tau; \delta$: Without loss of generality, we can suppose that $x \notin \text{dom}(\delta)$. By (E\_RED)/(R\_FORGET),

$$
\theta_1(\delta_1((T_1 \Rightarrow T_2')) v_1 \longrightarrow \theta_1(\delta_1((T'_1 \Rightarrow \{x: T'_2 | e'_2\}) v_1).
$$

Thus, it suffices to show that

$$
\theta_1(\delta_1((T'_1 \Rightarrow \{x: T'_2 | e'_2\})) v_1 \simeq_{\nu} v_2 : \{x: T'_2 | e'_2\}; \tau; \delta.
$$

By the IH,

$$
\theta_1(\delta_1((T'_1 \Rightarrow T'_2')) \simeq_{\nu} \theta_2(\delta_2(\lambda y: T'_1 ; y)) : T'_1 \Rightarrow T'_2; \tau; \delta.
$$

Since $v_1 \simeq \nu v_2 : \{x: T'_1 | e'_1\}; \tau; \delta$, we have $v_1 \simeq \nu v_2 : T'_1 ; \tau; \delta$ by definition. Thus,

$$
\theta_1(\delta_1((T'_1 \Rightarrow T'_2')) v_1 \simeq_{\nu} v_2 : T'_2; \tau; \delta.
$$

By definition, there exists some $v'_1$ such that $\theta_1(\delta_1((T'_1 \Rightarrow T'_2')) v_1 \longrightarrow^* v'_1$ and $v'_1 \simeq \nu v_2 : T'_2; \tau; \delta$. By (R\_FORGET) and (R\_PRECHECK),

$$
\theta_1(\delta_1((T'_1 \Rightarrow \{x: T'_2 | e'_2\})) v_1 \longrightarrow^* \theta_1(\delta_1((\{x: T'_2 | e'_2\}, v'_2 [v'_1 / x], v'_2)))
$$

Thus, it suffices to show that

$$
\theta_1(\delta_1((\{x: T'_2 | e'_2\}, v'_2 [v'_1 / x], v'_2))) \simeq_{\nu} v_2 : \{x: T'_2 | e'_2\}; \tau; \delta.
$$

We show

$$
\theta_1(\delta_1(e'_2 [v'_1 / x])) \longrightarrow^* \text{true}.
$$

Since $v_1 \simeq \nu v_2 : \{x: T'_1 | e'_1\}; \tau; \delta$, we have $v_1 \simeq \nu v_2 : T'_1 ; \tau; \delta$ and $\theta_1(\delta_1(e'_1 [v'_1 / x])) \longrightarrow^* \text{true}.

Since $\{x: T'_1 | e'_1\} \simeq \{x: T'_2 | e'_2\} : \tau; \delta$, we have

$$
\theta_1(\delta_1(e'_1 [v'_1 / x])) \simeq_{\nu} \theta_2(\delta_2(e'_2 [v'_2 / x])) : \text{bool}; \tau; \delta.
$$

Since the term on the left-hand side evaluates to true, we have $\theta_2(\delta_2(e'_2 [v'_2 / x])) \longrightarrow^* \text{true}.

Since $\{x: T'_2 | e'_2\} \simeq \{x: T'_2 | e'_2\} : \tau; \delta$, we have

$$
\theta_1(\delta_1(e'_2 [v'_1 / x])) \simeq_{\nu} \theta_2(\delta_2(e'_2 [v'_2 / x])) : \text{bool}; \tau; \delta.
$$

Since the term on the right-hand term evaluates to true, we have $\theta_1(\delta_1(e'_2 [v'_1 / x])) \longrightarrow^* \text{true}.

Thus, $\theta_1(\delta_1((\{x: T'_2 | e'_2\}, e'_2 [v'_1 / x], v'_2))) \longrightarrow^* v'_1$, and so it suffices to show that

$$
\nu v_1 \simeq_{\nu} v_2 : \{x: T'_2 | e'_2\}; \tau; \delta,
$$

which follows by the facts that $v'_1 \simeq_{\nu} v_2 : T'_2; \tau; \delta$ and $\theta_1(\delta_1(e'_2 [v'_1 / x])) \longrightarrow^* \text{true}$ and $\theta_2(\delta_2(e'_2 [v'_2 / x])) \longrightarrow^* \text{true}. \square$
The other requirement about reflexive casts is shown similarly.

**Lemma 5.9** (Elimination of Reflexive Casts on Right). If $T \simeq T : *; \theta; \delta$ and $T \simeq T_1 : *; \theta; \delta$ and then $\theta_1(\delta_1(x:T_2,x)) \simeq_\theta \theta_2(\delta_2((T_1 \Rightarrow T_2)^L)) : T \rightarrow T; \theta; \delta$.

*Proof.* By course-of-values induction on the sum of the sizes of $T_1$ and $T_2$.

The final requirement is about CIU equivalence—if $\emptyset \vdash v =_{ciu} v_1 : T_1$ and $(v_1, v_2) \in r$, then $(v, v_2) \in r$. We show that term relations satisfy it by using the (restricted) equivalence-respecting property [29].

**Lemma 5.10.** If $\emptyset \vdash e_1 : T$ and $e_1 \rightarrow^* e_2$ and $\emptyset \vdash E^S : (\emptyset \vdash e_2 : T) \rightsquigarrow T'$, then $\emptyset \vdash E^S : (\emptyset \vdash e_1 : T) \rightsquigarrow T'$.

*Proof.* Straightforward by induction on the derivation of $\emptyset \vdash E^S : (\emptyset \vdash e_2 : T) \rightsquigarrow T'$.

**Lemma 5.11.** If $\Gamma \vdash e : T$ and $\emptyset \vdash C : (\Gamma \vdash e : T) \rightsquigarrow T'$ and $\emptyset \vdash E^S : (\emptyset \vdash C[e] : T') \rightsquigarrow T''$, then $\emptyset \vdash E^S[\{C\}] : (\Gamma \vdash e : T) \rightsquigarrow T''$.

*Proof.* Straightforward by induction on the derivation of $\emptyset \vdash E^S : (\emptyset \vdash C[e] : T') \rightsquigarrow T''$.

**Lemma 5.12** (Equivalence-Respecting). If $\emptyset \vdash e_1 =_{ciu} e_2 : \theta_1(\delta_1(T))$ and $e_2 \simeq_\theta e_3 : T; \theta; \delta$, then $e_1 \simeq_\theta e_3 : T; \theta; \delta$.

*Proof.* By induction on $T$. If $e_1, e_2$, and $e_3$ raise blame, then we finish. Otherwise, $e_1 \rightarrow^* v_1, e_2 \rightarrow^* v_2$, and $e_3 \rightarrow^* v_3$ for some $v_1, v_2$, and $v_3$. We have $v_2 \simeq_\theta v_3 : T; \theta; \delta$. By definition, it suffices to show that $v_1 \simeq_\theta v_3 : T; \theta; \delta$. By case analysis on $T$.

Case $T = B$: Since $v_2 \simeq_\theta v_3 : B; \theta; \delta$, we have $v_2 = v_3 = k \in \mathbb{K}_B$ for some $k$. Let $E^S = \langle \text{Bool} \Rightarrow \{x: \text{Bool} | x\} \rangle^L ([ ] =_B k)$. Since $\emptyset \vdash E^S : (\emptyset \vdash e_1 : B) \rightsquigarrow \{x: \text{Bool} | x\}$ and $\emptyset \vdash e_1 =_{ciu} e_2 : B$, we have $E^S[e_1] \downarrow E^S[e_2]$. Since $e_2 \rightarrow^* k$, we have $E^S[e_2] \rightarrow^* \text{true}$. If $v_1 \neq k$, then $E^S[e_1]$ does not terminate at values, which is contradictory to $E^S[e_1] \downarrow E^S[e_2]$. Thus, $v_1 = k$. Since $v_3 = k$, we have $v_1 \simeq_\theta v_3 : B; \theta; \delta$.

Case $T = \alpha$: Since $v_2 \simeq_\theta v_3 : \alpha; \theta; \delta$, there exists some $r$, $T_1$, and $T_2$ such that $\theta(\alpha) = (r, T_1, T_2)$ and $(v_2, v_3) \in r$. Since $r \in \text{VRel}(T_1, T_2)$, it suffices to show that $\emptyset \vdash v_1 =_{ciu} v_2 : T_1$, that is, for any $E^S$ and $T'$ such that $\emptyset \vdash E^S : (\emptyset \vdash v_1 : T_1) \rightsquigarrow T'$, $E^S[v_1] \downarrow E^S[v_2]$. Since $\emptyset \vdash e_1 : T_1$ and $e_1 \rightarrow^* v_1$, we have $\emptyset \vdash E^S : (\emptyset \vdash e_1 : T_1) \rightsquigarrow T'$ by Lemma 5.10. Since $\emptyset \vdash e_1 =_{ciu} e_2 : T_1$, we have $E^S[e_1] \downarrow E^S[e_2]$. Since $e_1 \rightarrow^* v_1$ and $e_2 \rightarrow^* v_2$, we have $E^S[v_1] \downarrow E^S[v_2]$.

Case $T = x : T_1 \rightarrow T_2$: Without loss of generality, we can suppose that $x \notin \text{dom}(\delta)$. By definition, it suffices to show that, for any $v_1'$ and $v_3'$ such that $v_1' \simeq_\theta v_3' : T_1; \theta; \delta$, $v_1' \simeq_\theta v_3' : T_2; \theta; \delta[ (v_1', v_3' ) / x ]$.

By the IH, it suffices to show that

- $\emptyset \vdash v_1' =_{ciu} v_2 v_3' : \theta_1(\delta_1(T_2[v_1'/x]))$ and
- $v_2 v_3' \simeq_\theta v_3' : T_2; \theta; \delta[ (v_1', v_3' ) / x ]$.

The second is shown by $v_2 \simeq_\theta v_3 : x : T_1 \rightarrow T_2; \theta; \delta$ and $v_1 \simeq_\theta v_3' : T_1; \theta; \delta$.

As for the first, it suffices to show that, for any $E^S$ and $T'$ such that $\emptyset \vdash E^S : (\emptyset \vdash v_1' : \theta_1(\delta_1(T_2[v_1'/x]))) \rightsquigarrow T'$,

$$E^S[v_1'] \downarrow E^S[v_2'].$$

Since

$$\emptyset \vdash v_1 : \theta_1(x : T_1 \rightarrow T_2),$$
Case $T = \forall \alpha . T'$: Similar to the case of $T = x : T_1 \to T_2$.

Case $T = \{ x : T' | e' \}$: Without loss of generality, we can suppose that $x \notin \text{dom}(\delta)$. By definition, it suffices to show that

1. $\theta_1(\delta_1(e'[v_1/x])) \to^* \text{true}$,
2. $\theta_2(\delta_2(e'[v_3/x])) \to^* \text{true}$, and
3. $v_1 \simeq_v v_3 : T'; \theta; \delta$.

Since $\emptyset \vdash v_1 : \theta_1(\delta_1(\{ x : T' | e' \}))$, we have (1) by the value inversion (Lemma 2.9). Since $v_2 \simeq_v v_3 : \{ x : T' | e' \}; \theta; \delta$, we have (2).

As for (3), by the IH, it suffices to show that

- $\emptyset \vdash e_1 = \text{ciu} e_2 : \theta_1(\delta_1(T'))$ and
- $v_2 \simeq_v v_3 : T'; \theta; \delta$.

Since $v_2 \simeq_v v_3 : \{ x : T' | e' \}; \theta; \delta$, we have the second by definition. We can show the first in a way similar to the case of $T = x : T_1 \to T_2$.

Now, we show that term relations are interpretations and then prove the type compositionality, which states that term relations indexed by $T[T_1/\alpha]$ with $\theta$ and by $T$ with $\theta \{ \alpha \mapsto r, T_1, T_2 \}$ are the same, provided that $r$ is a term relation indexed by $T_1$.

**Lemma 5.13** (Term Relation as Interpretation). Let $r = \{ (v_1, v_2) | v_1 \simeq_v v_2 : T_1; \theta; \delta \}$. If $T_1 \simeq T_1 : *; \theta; \delta$ and $T_1 \simeq T_2 : *; \theta; \delta$, then $(r, \theta_1(\delta_1(T_1)), \theta_2(\delta_2(T_2)))$.

**Proof.** $r \in VRel(\theta_1(\delta_1(T_1)), \theta_2(\delta_2(T_2)))$ by Lemmas 5.8, 5.9 and 5.12.

**Lemma 5.14** (Type Compositionality). Suppose that $\Gamma, \alpha, \Gamma' \vdash T : *$ and $T_1 \simeq T_1 : *; \theta; \delta$. Also, assume that $\Gamma, \alpha, \Gamma'$ is self-related. Let $r = \{ (v_1, v_2) | v_1 \simeq_v v_2 : T_1; \theta; \delta \}$. If $\Gamma, \alpha, \Gamma' \vdash \theta \{ \alpha \mapsto r, \theta_1(\delta_1(T_1)), T_2 \}; \delta$, then $e_1 \simeq_e e_2 : T; \theta \{ \alpha \mapsto r, \theta_1(\delta_1(T_1)), T_2 \}; \delta$ iff $e_1 \simeq_e e_2 : T[T_1/\alpha]; \theta; \delta$.

**Proof.** By induction on $T$. If $e_1$ and $e_2$ raise blame, the conclusion follows straightforwardly. Otherwise, $e_1 \to^{\ast} v_1$ and $e_2 \to^{\ast} v_2$ for some $v_1$ and $v_2$, and it suffices to show that $v_1 \simeq_v v_2 : T; \theta'; \delta$ iff $v_1 \simeq_v v_2 : T[T_1/\alpha]; \theta; \delta$

where $\theta' = \theta \{ \alpha \mapsto r, \theta_1(\delta_1(T_1)), T_2 \}$. By case analysis on $T$.

Case $T = \beta$: Suppose that $v_1 \simeq_v v_2 : \beta; \theta; \delta$. We show that $v_1 \simeq_v v_2 : \beta[T_1/\alpha]; \theta; \delta$. If $\alpha = \beta$, then $(v_1, v_2) \in r$, that is, $v_1 \simeq_v v_2 : T_1; \theta; \delta$. Since $\beta[T_1/\alpha] = T_1$, we finish. Otherwise, if $\alpha \neq \beta$, then obvious since $\beta[T_1/\alpha] = \beta$ and $\theta'$ is an extension of $\theta$ with $\alpha$.

Conversely, we suppose that $v_1 \simeq_v v_2 : \beta[T_1/\alpha]; \theta; \delta$. We show that $v_1 \simeq_v v_2 : \beta; \theta'; \delta$. If $\beta = \alpha$, we have $v_1 \simeq_v v_2 : T_1; \theta; \delta$, so $(v_1, v_2) \in r$ and $v_1 \simeq_v v_2 : \alpha; \theta'; \delta$. Otherwise, if $\beta \neq \alpha$, then obvious.

Case $T = B$: Obvious.
Case $T = x:T'_1 \rightarrow T'_2$: By the IHs.
Case $T = \forall \beta. T'$: By the IH.
Case $\{x:T' | e'\}$: Without loss of generality, we can suppose that $x \notin \text{dom} \delta$. We show:

\[
\begin{align*}
v_1 &\simeq v_2 : T'; \theta'; \delta \\
\theta_1(\delta_1(e'[T_1/\alpha][v_1/x])) &\rightarrow^* \text{true} \iff \theta_1(\delta_1(e'[T_1/\alpha][v_1/x])) \rightarrow^* \text{true} \\
\theta_2(\delta_2(e'[T_2/\alpha][v_2/x])) &\rightarrow^* \text{true} \iff \theta_2(\delta_2(e'[T_1/\alpha][v_2/x])) \rightarrow^* \text{true}
\end{align*}
\]

We show only the left-to-right direction; the other is shown similarly. Since $\Gamma, \alpha, \Gamma' \vdash \{x:T' | e'\} \simeq \{x:T'' | e''\} : \ast$, it is easy to show that $\Gamma, \alpha, \Gamma' \vdash T' \simeq T'' : \ast$. Since $v_1 \simeq v_2 : T'; \theta'; \delta$, we have

\[
v_1 \simeq v_2 : T'[T_1/\alpha]; \theta; \delta
\]

by the IH. We have the second case by the assumption of the left-to-right direction. The remaining case to be shown is:

\[
\theta_2(\delta_2(e'[T_1/\alpha][v_2/x])) \rightarrow^* \text{true}
\]

Since $\Gamma, \alpha, \Gamma' \vdash \theta'; \delta$ by the assumption of this lemma, we have

\[
\Gamma, \alpha, \Gamma', x:T' \vdash \theta'; \delta[\langle v_1, v_2 \rangle/x]\]

by the weakening (Lemma 5.2). Since $T_1 \simeq T_1 : \ast; \theta; \delta$, we have

\[
\langle r, \theta_1(\delta_1(T_1)), \theta_2(\delta_2(T_1))\rangle
\]

by Lemma 5.13. Then, we can show that

\[
\Gamma, \alpha, \Gamma', x:T' \vdash \theta\{\alpha \mapsto r, \theta_1(\delta_1(T_1)), \theta_2(\delta_2(T_1))\}; \delta[\langle v_1, v_2 \rangle/x]\]

by the weakening (Lemma 5.3) with (5.4) and Lemma 5.4. Since $\Gamma, \alpha, \Gamma', x:T' \vdash e' \simeq e' : \text{Bool}$, we have

\[
\theta_1(\delta_1(e'[T_1/\alpha][v_1/x])) \simeq \theta_2(\delta_2(e'[T_1/\alpha][v_2/x])) : \text{Bool}; \theta; \delta.
\]

Since the term on the left-hand side evaluates to true, so does the one on the right-hand side, which we want to show.

\[\square\]

**Lemma 5.15** (Compatibility under Self-relatedness Assumption: Type Application). Suppose that $\Gamma \vdash \forall \alpha. T \simeq \forall \alpha. T : \ast$ and $\Gamma \vdash T_1 \simeq T_1 : \ast$ and that $\Gamma$ is self-related. If $\Gamma \vdash e_1 \simeq e_2 : \forall \alpha. T$ and $\Gamma \vdash T_1 \simeq T_2 : \ast$, then $\Gamma \vdash e_1 \simeq e_2 : T : T[T_1/\alpha]$.

**Proof.** Suppose that $\Gamma \vdash \theta; \delta$. Also, let $e'_1 = \theta_1(\delta_1(e_1))$, $e'_2 = \theta_2(\delta_2(e_2))$, $T'_1 = \theta_1(\delta_1(T_1))$ and $T'_2 = \theta_2(\delta_2(T_2))$. It suffices to show that

\[
e'_1 T'_1 \simeq \theta; \delta.
\]

Since $\Gamma \vdash e_1 \simeq e_2 : \forall \alpha. T$, we have $e'_1 \simeq e'_2 : \forall \alpha. T; \theta; \delta$. If $e'_1$ and $e'_2$ raise blame, we finish. Otherwise, $e'_1 \rightarrow^* v_1$ and $e'_2 \rightarrow^* v_2$ for some $v_1$ and $v_2$, and it suffices to show that

\[
v_1 T'_1 \simeq v_2 T'_2 : T[T_1/\alpha]; \theta; \delta.
\]

We also have $v_1 \simeq v_2 : \forall \alpha. T; \theta; \delta$.

Let $r = \{\langle v'_1, v'_2 \rangle | v'_1 \simeq v'_2 : T_1; \theta; \delta\}$. Since $T_1 \simeq T_1 : \ast; \theta; \delta$ and $T_1 \simeq T_2 : \ast; \theta; \delta$, we have $\langle r, T'_1, T'_2 \rangle$ by Lemma 5.13. Since $v_1 \simeq v_2 : \forall \alpha. T; \theta; \delta$, we have $v_1 T'_1 \simeq v_2 T'_2 : T; \theta\{\alpha \mapsto r, T'_1, T'_2\}; \delta$. Since $\Gamma, \alpha \vdash T \simeq T : \ast$ and $\Gamma, \alpha \vdash \theta\{\alpha \mapsto r, T'_1, T'_2\}; \delta$ by the weakening (Lemma 5.3), and $T_1 \simeq T_1 : \ast; \theta; \delta$ and $\Gamma, \alpha$ is self-related, we have $v_1 T'_1 \simeq v_2 T'_2 : T[T_1/\alpha]; \theta; \delta$ by the type compositionality (Lemma 5.14). 

\[\square\]
**Fundamental property: other constructors.** We show remaining cases of the fundamental property.

**Lemma 5.16** (Compatibility: Variable). If $\Gamma \vdash x : T$, then $\Gamma \vdash x \simeq x : T$.

*Proof.* Straightforward by definition.

**Lemma 5.17** (Compatibility: Constant). If $\Gamma \vdash k$, then $\Gamma \vdash k \simeq k : \text{ty}(k)$.

*Proof.* Let $\Gamma \vdash \theta; \delta$. It suffices to show that $k \simeq_\theta k : \text{ty}(k); \theta; \delta$. By the assumptions that $\text{unref}(\text{ty}(k)) = B$ for some $B$ and that $k \in K_B$, we have $k \simeq_\theta k : \text{unref}(\text{ty}(k)); \theta; \delta$. Since constants satisfy contracts on their types and $\text{ty}(k)$ is closed, $k \simeq_\theta k : \text{ty}(k); \theta; \delta$ by definition.

**Lemma 5.18** (Compatibility under Self-relatedness Assumption: Op). Suppose that $\Gamma$ is self-related and that $\text{ty}(\text{op}) = x_1 : T_1 \rightarrow \cdots \rightarrow x_n : T_n \rightarrow T$. Moreover, assume that, for any $i \in \{1, \ldots, n\}$, $\Gamma \vdash e_i \simeq e_i^\theta : T_i[e_1/x_1, \ldots, e_{i-1}/x_{i-1}]$ and $\Gamma \vdash T_i[e_{i+1}/x_1, \ldots, e_n/x_n] \simeq T_i[e_{i+1}/x_1, \ldots, e_{i-1}/x_{i-1}] : *$. If $\Gamma \vdash e_i \simeq e^\theta_i : T_i[e_1/x_1, \ldots, e_{i-1}/x_{i-1}]$ for any $i \in \{1, \ldots, n\}$, then $\Gamma \vdash \text{op}(e_1, \ldots, e_n) \simeq \text{op}(e_1, \ldots, e_n) : T[e_1/x_1, \ldots, e_n/x_n]$.

*Proof.* Similar to the case of term application.

**Lemma 5.19** (Compatibility: Abstraction). If $\Gamma, x : T_1, T_2 \vdash e_1 \simeq e_2 : T_2$ and $\text{FV}(T_21) \subseteq \text{dom}(\Gamma)$, then $\Gamma \vdash \lambda x : T_1.e_1 \simeq \lambda x : T_2.e_2 : (x : T_1 \rightarrow T_2)$.

*Proof.* Let $\Gamma \vdash \theta; \delta$. By definition, it suffices to show that, for any $v_1$ and $v_2$ such that $v_1 \simeq_\theta v_2 : T_1; \theta; \delta, \theta_1(\delta_1(\lambda x : T_1.e_1)) v_1 \simeq_\theta \theta_2(\delta_2(\lambda x : T_2.e_2)) v_2 : T_2; \theta; \delta[ (v_1, v_2)/x ]$. Since $\Gamma, x : T_1 \vdash \theta; \delta[ (v_1, v_2)/x ]$ by the weakening (Lemma 5.2), and $\Gamma, x : T_1 \vdash e_1 \simeq e_2 : T_2$, we finish.

**Lemma 5.20** (Compatibility under Self-relatedness Assumption: Cast). Suppose that $\Gamma \vdash T_1 \simeq T_1 : *$ and $\Gamma \vdash T_2 \simeq T_2 : *$. If $\Gamma \vdash T_1 \simeq T_21 : *$ and $\Gamma \vdash T_2 \simeq T_22 : *$ and $T_1 \parallel T_2$, then $\Gamma \vdash (T_1 \Rightarrow T_2) \lhd \simeq (T_21 \Rightarrow T_22) \lhd : T_1 \rightarrow T_2$.

*Proof.* It suffices to show:

- If $T_1 \parallel T_2$, $T_1 \simeq T_1 : *; \theta; \delta, T_2 \simeq T_2 : *; \theta; \delta, T_1 \simeq T_21 : *; \theta; \delta, T_2 \simeq T_22 : *; \theta; \delta$, then
- $\theta_1(\delta_1((T_1 \Rightarrow T_2) \lhd)) \simeq_\theta \theta_2(\delta_2((T_21 \Rightarrow T_22) \lhd)) : T_1 \rightarrow T_2; \theta; \delta$.

We prove this by strong induction on the sum of sizes of $T_1$ and $T_2$ as elimination of reflexive casts (Lemma 5.8); the details are omitted.

**Lemma 5.21** (Compatibility: Type Abstraction). If $\Gamma, \alpha \vdash e_1 \simeq e_2 : T$, then $\Gamma \vdash \lambda \alpha. e_1 \simeq \lambda \alpha. e_2 : \forall \alpha. T$.

*Proof.* Let $\Gamma \vdash \theta; \delta$. By definition, it suffices to show that, for any $r$, $T_1$, and $T_2$ such that $\langle r, T_1, T_2 \rangle, \theta_1(\delta_1(\lambda \alpha. e_1)) T_1 \simeq_\theta \theta_2(\delta_2(\lambda \alpha. e_2)) T_2 : T; \theta \{ \alpha \mapsto r, T_1, T_2 \}; \delta$. Since $\Gamma, \alpha \vdash \theta \{ \alpha \mapsto r, T_1, T_2 \}; \delta$ by the weakening (Lemma 5.3), and $\Gamma, \alpha \vdash e_1 \simeq e_2 : T$, we finish.
Lemma 5.22 (Compatibility: Type Conversion). If $\Gamma \vdash \emptyset \vdash e_1 \simeq e_2 : T_1$ and $\emptyset \vdash T_2$, then $\Gamma \vdash e_1 \simeq e_2 : T_2$.

Proof. It suffices to show that, if $T_1 \equiv T_2$, then $e_1 \simeq e_2 : T_1; \theta; \delta$ iff $e_1 \simeq e_2 : T_2; \theta; \delta$. We consider the case of $T_1 \Rightarrow T_2$ (other cases are shown straightforwardly). There exist $T$, $e_1'$, $e_2'$, and $x$ such that $T_1 = T[e_1'/x]$ and $T_2 = T[e_2'/x]$ and $e_1' \rightarrow e_2'$. If $e_1$ and $e_2$ raise blame, then obvious. Otherwise, $e_1 \rightarrow^* v_1$ and $e_2 \rightarrow^* v_2$ for some $v_1$ and $v_2$, and it suffices to show that $v_1 \simeq v_2 : T[e_1'/x]; \theta; \delta$ iff $v_1 \simeq v_2 : T[e_2'/x]; \theta; \delta$. Straightforward by induction on $T$. The case that $T$ is a refinement type is shown with Cotermination (Lemma 2.6).

Lemma 5.23 (Compatibility under Self-relatedness Assumption: Active Check). Suppose that $\vdash \Gamma \emptyset \vdash \{x:T_1 | e_1\} \simeq \{x:T_1 | e_1\} : \ast$. If $\emptyset \vdash e_1' \simeq e_2' : \text{Bool}$ and $\emptyset \vdash v_1 \simeq v_2 : T_1$ and $e_1 [v_1/x] \rightarrow^* e_1'$, then $\Gamma \vdash \{x:T_1 | e_1\}, e_1', v_1) \simeq \{x:T_2 | e_2\}, e_2', v_2\} : \{x:T_1 | e_1\}$.

Proof. It suffices to show that $\{x:T_1 | e_1\}, e_1', v_1) \simeq \{x:T_2 | e_2\}, e_2', v_2\} : \{x:T_1 | e_1\}$; $\emptyset; \emptyset$. If $e_1'$ and $e_2'$ raise blame, then obvious. Otherwise, $e_1' \rightarrow^* v_1'$ and $e_2' \rightarrow^* v_2'$ for some $v_1'$ and $v_2'$, and it suffices to show that $\langle \{x:T_1 | e_1\}, v_1', e_1\rangle \simeq \{x:T_2 | e_2\}, v_2', e_2\} : \{x:T_1 | e_1\}; \emptyset; \emptyset$. Since $v_1' \simeq v_1 : \text{Bool}; \emptyset; \emptyset$, there are two cases we have to consider. If $v_1' = v_2' = \text{false}$, then $\langle \{x:T_1 | e_1\}, v_1', v_2\rangle \rightarrow^* v_1$ and $\langle \{x:T_2 | e_2\}, v_1', v_2\rangle \rightarrow^* v_2$. Thus, it suffices to show that $v_1 \simeq v_2 : T_1; \emptyset; \emptyset$, that is, (1) $v_1 \simeq v_2 : T_1; \emptyset; \emptyset$, (2) $e_1 [v_1/x] \rightarrow^* \text{true}$, and (3) $e_1 [v_1/x] \rightarrow^* \text{true}$. We have $v_1 \simeq v_2 : T_1; \emptyset; \emptyset$ by the assumption, and $e_1 [v_1/x] \rightarrow^* e_1'$ true. Since $\{x:T_1 | e_1\} \simeq \{x:T_1 | e_1\} : \ast; \emptyset; \emptyset$, we have $e_1 [v_1/x] \simeq e_1 [v_1/x] : \text{Bool}; \emptyset; \emptyset$. Thus, $e_1 [v_1/x] \rightarrow^* \text{true}$.

Lemma 5.24 (Compatibility under Self-relatedness Assumption: Waiting Check). Suppose that $\vdash \{x:T_1 | e_1\} \simeq \{x:T_1 | e_1\} : \ast$. If $\vdash \Gamma \vdash \{x:T_1 | e_1\} \simeq \{x:T_2 | e_2\} : \ast$ and $\Gamma \vdash e_1' \simeq e_2' : T_1$, then $\vdash \Gamma \vdash \{x:T_1 | e_1\}, e_1') \simeq \{x:T_2 | e_2\}, e_2') : \{x:T_1 | e_1\}$.

Proof. It suffices to show that, if $\{x:T_1 | e_1\} \simeq \{x:T_1 | e_1\} : \ast; \theta; \delta$ and $\{x:T_1 | e_1\} \simeq \{x:T_2 | e_2\} : \ast; \theta; \delta$ and $e_1') \simeq e_2' : T_1; \theta; \delta$, then $\theta_1(\delta_1(\{x:T_1 | e_1\}), e_1') \simeq \theta_2(\delta_2(\{x:T_2 | e_2\}, e_2') : \{x:T_1 | e_1\}; \theta; \delta$. If $e_1'$ and $e_2'$ raise blame, then obvious. Otherwise, $e_1' \rightarrow^* v_1'$ and $e_2' \rightarrow^* v_2'$ for some $v_1'$ and $v_2'$, and it suffices to show that $\theta_1(\delta_1(\{x:T_1 | e_1\}, e_1 [v_1'/x], v_1') \simeq \theta_2(\delta_2(\{x:T_2 | e_2\}, e_2 [v_2'/x], v_2') : \{x:T_1 | e_1\}; \theta; \delta$. We have $v_1' \simeq v_2' : T_1; \theta; \delta$. Since $\{x:T_1 | e_1\} \simeq \{x:T_2 | e_2\} : \ast; \theta; \delta$, we have $\theta_1(\delta_1(e_1 [v_1'/x])) \simeq \theta_2(\delta_2(e_2 [v_2'/x])) : \text{Bool}; \theta; \delta$. The remaining proceeds as in active check (Lemma 5.23).

Lemma 5.25 (Compatibility: Exact). Suppose that $\vdash \Gamma \emptyset \vdash \{x:T | e\} \simeq \{x:T | e\} : \ast$. If $\emptyset \vdash v_1 \simeq v_2 : T$ and $e [v_1/x] \rightarrow^* \text{true}$, then $\Gamma \vdash v_1 \simeq v_2 : \{x:T | e\}$.

Proof. By the weakening (Lemmas 5.2 and 5.3), it suffices to show that $v_1 \simeq v_2 : \{x:T | e\}; \emptyset; \emptyset$. Since $v_1 \simeq v_2 : T; \emptyset; \emptyset$ and $e [v_1/x] \rightarrow^* \text{true}$, it suffices to show that $e [v_2/x] \rightarrow^* \text{true}$. Since $x:T \emptyset; \emptyset; \emptyset$ and $x:T \emptyset; \emptyset; \emptyset : \text{Bool}$, we have $e [v_1/x] \simeq e [v_2/x] : \text{Bool}; \emptyset; \emptyset$. Since $e [v_1/x] \rightarrow^* \text{true}$, we have $e [v_2/x] \rightarrow^* \text{true}$.
Lemma 5.26 (Compatibility: Forget). If \( \Gamma \vdash \emptyset \vdash v_1 \simeq v_2 : \{x:T \mid e\} \), then \( \Gamma \vdash v_1 \simeq v_2 : T \).

\( \text{Proof.} \) Straightforward by definition. \( \square \)

Parametricity. Before showing the parametricity, we prove that the logical relation for open types is closed under type substitution, which is needed to show that \( \Gamma \vdash e : T \) implies \( \Gamma \vdash T \simeq T : \ast \).

Lemma 5.27 (Type Substitutivity in Type under Self-relatedness Assumption). Suppose that \( \Gamma, \alpha, \Gamma' \) is self-related and that \( \Gamma, \alpha, \Gamma' \vdash T_{11} \simeq T_{11} : \ast \) and \( \Gamma \vdash T_{12} \simeq T_{12} : \ast \). If \( \Gamma, \alpha, \Gamma' \vdash T_{11} \simeq T_{21} : \ast \) and \( \Gamma \vdash T_{12} \simeq T_{22} : \ast \), then \( \Gamma, \Gamma'[\Gamma_{12}/\alpha] \vdash T_{11}[\Gamma_{12}/\alpha] \simeq T_{21}[\Gamma_{22}/\alpha] : \ast \).

\( \text{Proof.} \) By induction on \( T_{11} \). Let \( \Gamma, \Gamma'[\Gamma_{12}/\alpha] \vdash \theta; \delta \). We show that

\( T_{11}[\Gamma_{12}/\alpha] \simeq T_{21}[\Gamma_{22}/\alpha] : \ast; \theta; \delta. \)

Let \( r = \{ (v_1, v_2) \mid v_1 \simeq v_2 : T_{12}; \theta; \delta \} \) and \( \theta' = \theta \{ \alpha \mapsto r, \alpha_i(\delta(T_{12})), \delta(T_{22}) \} \).

Since \( \Gamma, \Gamma'[\Gamma_{12}/\alpha] \vdash \theta; \delta \), we have \( \Gamma, \alpha, \Gamma' \vdash \theta'; \delta; \theta; \delta \) it is shown by induction on \( \Gamma' \). Thus, \( T_{11} \simeq T_{21} : \ast; \theta'; \delta; \theta; \delta \); the remaining is straightforward by case analysis on the derivation of \( T_{11} \simeq T_{21} : \ast; \theta'; \delta; \theta; \delta \); we need the type compositionality (Lemma 5.14) in the case that both \( T_{11} \) and \( T_{21} \) are refinement types. \( \square \)

Theorem 5.28 (Parametricity).

1. If \( \Gamma \vdash e : T \), then \( \Gamma \vdash e \simeq e : T \) and \( \Gamma \vdash T \simeq T : \ast \) and \( \Gamma \) is self-related.
2. If \( \Gamma \vdash T \), then \( \Gamma \vdash T \simeq T : \ast \) and \( \Gamma \) is self-related.
3. If \( \Gamma \vdash \Gamma \), then \( \Gamma \) is self-related.

\( \text{Proof.} \) The three statements are simultaneously proved by induction on the derivations of the judgments with the compatibility lemmas shown above. The case of \( \Gamma \vdash T \simeq T : \ast \) by the IH since \( \Gamma \vdash T \).

Soundness. By the parametricity, we can discharge self-relatedness assumptions from the compatibility lemmas, which leads to the fundamental property, and so we are ready to show the soundness of the logical relation.

Lemma 5.29 (Adequacy). If \( e_1 \simeq e_2 : T; \theta; \delta \), then \( e_1 \Downarrow e_2 \).

\( \text{Proof.} \) Obvious. \( \square \)

Theorem 4.10 (Soundness). For any \( \Gamma_1, \ldots, \Gamma_n, e_{11}, \ldots, e_{1n}, e_{21}, \ldots, e_{2n}, \) and \( T_1, \ldots, T_n \), if \( \Gamma_i \vdash e_{ii} \simeq e_{2i} : T_i \) for \( i \in \{ 1, \ldots, n \} \), then \( \Gamma_1 \vdash [C[e_{11}, \ldots, e_{1n}] \simeq C[e_{21}, \ldots, e_{2n}] : T] \).

\( \text{Proof.} \) We can show that, for any \( C \) and \( T \) such that \( \emptyset \vdash C : \Gamma_1 \vdash e_{ii} : T_i, \) \( \emptyset \vdash C[e_{11}, \ldots, e_{1n}] \simeq C[e_{21}, \ldots, e_{2n}] : T \) using the compatibility lemmas with the parametricity (Theorem 5.28). Then, \( C[e_{11}, \ldots, e_{1n}] \Downarrow C[e_{21}, \ldots, e_{2n}] \) by the adequacy (Lemma 5.29). \( \square \)
5.2. Completeness. We also show the completeness of the logical relation with respect to typed contextual equivalence, that is, contextually equivalent terms are logically related if they are both well typed. The completeness proof is via CIU-equivalence: we show that (1) contextually equivalent terms are CIU-equivalent (Lemma 5.32) and (2) well-typed, CIU-equivalent terms are logically related (Lemma 5.33). Using these lemmas, we can show that well-typed, contextually equivalent terms are logically related (Theorem 4.11). The completeness enables us to show (restricted) transitivity of semityped contextual equivalence (Corollary 5.35).

To prove CIU-equivalence of contextually equivalent terms, we start with defining functions to close open terms according to typing contexts and closing substitutions. These functions are used to construct program contexts in semityped contextual equivalence.

**Definition 5.30.** For $e$ and $\Gamma$, $\text{Abs}(\Gamma; e)$ denotes a term that takes term and type variables bound in $\Gamma$ as arguments:

$\text{Abs}(\emptyset; e) = e$

$\text{Abs}(\Gamma, x:T; e) = \text{Abs}(\Gamma; \lambda x:T. e)$

$\text{Abs}(\Gamma, \alpha; e) = \text{Abs}(\Gamma; \Lambda \alpha. e)$

For $\sigma$ and $\Gamma$, $\text{App}(\Gamma; \sigma; e)$ denotes a term that is applied to values and types to which $\sigma$ maps:

$\text{App}(\emptyset; \sigma; e) = e$

$\text{App}(\Gamma, x:T; \sigma; e) = \text{App}(\Gamma; \sigma; e)(\sigma(x))$

$\text{App}(\Gamma, \alpha; \sigma; e) = \text{App}(\Gamma; \sigma; e)(\sigma(\alpha))$

For $T$ and $\Gamma$, $\text{AbsType}(\Gamma; T)$ denotes a type that abstracts variables bound in $\Gamma$.

$\text{AbsType}(\emptyset; T) = \emptyset$

$\text{AbsType}(\Gamma, x:T'; T) = \text{AbsType}(\Gamma; x:T' \rightarrow T)$

$\text{AbsType}(\Gamma, \alpha; T) = \text{AbsType}(\Gamma; \forall \alpha. T)$

**Lemma 5.31.**

1. If $\Gamma \vdash e : T$, then $\emptyset \vdash \text{Abs}(\Gamma; e) : \text{AbsType}(\Gamma; T)$.
2. If $\emptyset \vdash e : \text{AbsType}(\Gamma; T)$ and $\Gamma \vdash \sigma$, then $\emptyset \vdash \text{App}(\Gamma; \sigma; e) : \sigma(T)$.
3. If $\Gamma \vdash e : T$ and $\Gamma \vdash \sigma$, then $\emptyset \vdash \text{App}(\Gamma; \sigma; \text{Abs}(\Gamma; e)) : \sigma(T)$.

**Proof.** The first and second cases are shown by induction on $\Gamma$ straightforwardly. The third case is a corollary of the combination of the first and second cases. \(\blacksquare\)

**Lemma 5.32** ($=\text{ctx} \subseteq =\text{ciu}$). If $\Gamma_i \vdash e_{i_1} =\text{ctx} e_{2i} : T_i$, then $\Gamma_j \vdash e_{1j} =\text{ciu} e_{2j} : T_j$ for any $j$.

**Proof.** We show that if $\Gamma \vdash e_1 =\text{ctx} e_2 : T$, then $\Gamma \vdash e_1 =\text{ciu} e_2 : T$. By definition, it suffices to show that, for any $\sigma$ and $E^S$ such that $\Gamma \vdash \sigma$ and $\emptyset \vdash E^S : (\emptyset \vdash \sigma(e_1) : \sigma(T)) \rightarrow T'$,

$$E^S[\sigma(e_1)] \downarrow E^S[\sigma(e_2)].$$

Here, $\text{App}(\Gamma; \sigma; \text{Abs}(\Gamma; e_1)) \rightarrow^* \sigma(e_1)$ and $\text{App}(\Gamma; \sigma; \text{Abs}(\Gamma; e_2)) \rightarrow^* \sigma(e_2)$. Since $\emptyset \vdash E^S : (\emptyset \vdash \sigma(e_1) : \sigma(T)) \rightarrow T'$, we can show

$$\emptyset \vdash E^S : (\emptyset \vdash \text{App}(\Gamma; \sigma; \text{Abs}(\Gamma; e_1)) : \sigma(T)) \rightarrow T'$$

by Lemma 5.10. By context typing rules, $\emptyset \vdash \text{App}(\Gamma; \sigma; \text{Abs}(\Gamma; [i])) : (\Gamma \vdash e_1 : T) \rightarrow \sigma(T)$. Thus, by Lemma 5.11 with (5.5):

$$\emptyset \vdash E^S[\text{App}(\Gamma; \sigma; \text{Abs}(\Gamma; [i]))] : (\Gamma \vdash e_1 : T) \rightarrow T'$$
by induction on (5.5). Since $\Gamma \vdash e_1 =_{\text{ctx}} e_2 : T$, we have

$$E^S[\text{App}(\Gamma; \sigma; \text{Abs}(\Gamma; e_1))] \Downarrow E^S[\text{App}(\Gamma; \sigma; \text{Abs}(\Gamma; e_2))]$$

by definition. Since $E^S[\text{App}(\Gamma; \sigma; \text{Abs}(\Gamma; e_i))] \Downarrow E^S[\sigma(e_i)]$ for $i \in \{1, 2\}$, we finish. \(\square\)

It is shown by the equivalence-respecting property that CIU-equivalent terms are logically related. We write substitution $\theta_1 \circ \delta_1$ for the concatenation of $\theta_1$ and $\delta_1$. Note that $\Gamma \vdash \theta_1 \circ \delta_1$ if $\Gamma \vdash \theta; \delta$.

**Lemma 5.33** ($=_{\text{ciu}} \subseteq \simeq$ with respect to Typed Terms). If $\Gamma \vdash e_1 =_{\text{ciu}} e_2 : T$ and $\Gamma \vdash e_2 : T$, then $\Gamma \vdash e_1 \simeq e_2 : T$.

**Proof.** Let $\Gamma \vdash \theta; \delta$. It suffices to show that $\theta_1(\delta_1(e_1)) \simeq_e \theta_2(\delta_2(e_2)) : T; \theta; \delta$. Since $\Gamma \vdash e_2 : T$, we have $\Gamma \vdash e_2 \simeq e_2 : T$ by the parametricity (Theorem 5.28), and so $\theta_1(\delta_1(e_2)) \simeq_e \theta_2(\delta_2(e_2)) : T; \theta; \delta$. Since $\Gamma \vdash e_1 =_{\text{ciu}} e_2 : T$ and $\Gamma \vdash \theta_1 \circ \delta_1$, we have $\emptyset \vdash \theta_1(\delta_1(e_1)) =_{\text{ciu}} \theta_1(\delta_1(e_2)) : \theta_1(\delta_1(T))$. Thus, we finish by the equivalence-respecting property (Lemma 5.12). \(\square\)

**Theorem 4.11** (Completeness with respect to Typed Terms). If $\Gamma \vdash e_1 =_{\text{ctx}} e_2 : T_i$ and $\Gamma_j \vdash e_{2j} : T_j$ for any $j$, then $\Gamma_j \vdash e_{1j} \simeq e_{2j} : T_j$ for any $j$.

**Proof.** By Lemmas 5.32 and 5.33. \(\square\)

We can show transitivity of semityped contextual equivalence for well-typed terms via the completeness.

**Lemma 5.34** (Transitivity of the Logical Relation). If $\Gamma \vdash e_1 \simeq e_2 : T$ and $\Gamma \vdash e_2 \simeq e_3 : T$, then $\Gamma \vdash e_1 \simeq e_3 : T$.

**Proof.** Let $\Gamma \vdash \theta; \delta$. We show that $\theta_1(\delta_1(e_1)) \simeq_e \theta_2(\delta_2(e_3)) : T; \theta; \delta$. Since $\Gamma \vdash e_1 \simeq e_2 : T$, we have $\Gamma \vdash e_1 =_{\text{ciu}} e_2 : T$ by Theorem 4.10 and Lemma 5.32 (note that $e_2$ is well typed). Since $\Gamma \vdash \theta_1 \circ \delta_1$, we have $\emptyset \vdash \theta_1(\delta_1(e_1)) =_{\text{ciu}} \theta_1(\delta_1(e_2)) : \theta_1(\delta_1(T))$. Since $\Gamma \vdash e_2 \simeq e_3 : T$, we have $\theta_1(\delta_1(e_2)) \simeq_e \theta_2(\delta_2(e_3)) : T; \theta; \delta$. By the equivalence-respecting property (Lemma 5.12), we finish. \(\square\)

**Corollary 5.35** (Transitivity of Semityped Contextual Equivalence). If $\Gamma \vdash e_1 =_{\text{ctx}} e_2 : T$ and $\Gamma \vdash e_2 =_{\text{ctx}} e_3 : T$ and $\Gamma \vdash e_3 : T$, then $\Gamma \vdash e_1 =_{\text{ctx}} e_3 : T$.

6. **Reasoning about Casts**

This section shows correctness of three cast reasoning techniques—the upcast elimination, the selfification, and the cast decomposition—using the logical relation developed in Section 4.
Subtyping Rules

\[
\Gamma \vdash T_1 <: T_2
\]

| Rule | Description |
|------|-------------|
| \(\Gamma \vdash B <: B\) | S\_BASE |
| \(\Gamma \vdash \alpha <: \alpha\) | S\_TVAR |
| \(\Gamma, \alpha \vdash T_1 <: T_2\) | S\_FORALL |
| \(\Gamma \vdash \forall \alpha. T_1 <: \forall \alpha. T_2\) | S\_FORALL |
| \(\Gamma \vdash T_{21} <: T_{11}\) | S\_FUN |
| \(\Gamma, x:T_{21} \vdash T_{12}[(T_{21} \Rightarrow T_{11})^\ell x/x] <: T_{22}\) | S\_FUN |
| \(\Gamma \vdash x:T_{11} \rightarrow T_{12} <: x:T_{21} \rightarrow T_{22}\) | S\_FUN |
| \(\Gamma \vdash T_1 <: T_2\) | S\_REFINEL |
| \(\Gamma \vdash T_1, x:T_1 = e_2[(T_1 \Rightarrow T_2)^\ell x/x]\) | S\_REFINER |
| \(\Gamma \vdash \{x:T_1 | e_1\} <: T_2\) | S\_REFINER |

Satisfaction Rule

\[
\forall \sigma. \Gamma \vdash \sigma \text{ implies } \sigma(e) \rightarrow^* \text{true} \quad \text{S\_Satisfy}
\]

Figure 9: Subtyping rules.

6.1. **Upcast Elimination.** We first introduce subtyping for \(F_H\) and then show that an upcast and an identity function are logically related. Thanks to the soundness of the logical relation with respect to semityped contextual equivalence (Theorem 4.10), it implies that they are contextually equivalent.

Figure 9 shows subtyping rules, which are similar to Belo et al. [4] except that we decompose the subtyping rule for refinement types into two simple rules. Subtyping judgment \(\Gamma \vdash T_1 <: T_2\) takes typing context \(\Gamma\) for checking refinements in \(T_2\). Base types and type variables can be subtypes of only themselves ((S\_BASE) and (S\_TVAR)). (S\_FORALL) checks that body types of universal types are in subtyping. As for function types, subtyping is contravariant on the domain types and covariant on the codomain types (S\_FUN). The subtyping judgment on codomain types assumes that the type of argument \(x\) is \(T_{21}\), a subtype of the other domain type \(T_{11}\), but codomain type \(T_{12}\) refers to \(x\) as \(T_{11}\). Since the type system of \(F_H\) does not allow subsumption for subtyping (unlike Knowles and Flanagan [19]), we force \(x\) to be of \(T_{11}\) by inserting an upcast, which can be eliminated after showing the upcast elimination. We can refine a subtype furthermore (S\_REFINEL). By contrast, a supertype can be refined if we can prove that any value of the subtype satisfies the refinement (S\_REFINER). Term \(e\) is satisfied under \(\Gamma (\Gamma \vdash e)\) if, for any closing substitution \(\sigma\) that respects \(\Gamma\), \(\sigma(e)\) evaluates to true. (S\_REFINER) also inserts an upcast since satisfaction assumes that the type of \(x\) is subtype \(T_1\) but \(e\) refers to it as supertype \(T_2\).

We show that an upcast and an identity function are contextually equivalent via the logical relation.

**Lemma 6.1.** If \(\Gamma \vdash T_1\) and \(\Gamma \vdash T_2\) and \(\Gamma \vdash T_1 <: T_2\) and \(\Gamma \vdash \theta; \tilde{\delta}\), then \(\theta_1(\delta_1((T_1 \Rightarrow T_2)^\ell)) \simeq_{\theta_2} (\delta_2(\lambda x:T_1(x))): T_1 \rightarrow T_2; \theta; \tilde{\delta}\).

**Proof.** By induction on \(\Gamma \vdash T_1 <: T_2\). It suffices to show that, for any \(v_1\) and \(v_2\) such that \(v_1 \simeq_{\theta_1} v_2 : T_1; \theta; \tilde{\delta}\),

\[
\theta_1(\delta_1((T_1 \Rightarrow T_2)^\ell)) v_1 \simeq_e v_2 : T_2; \theta; \tilde{\delta}.
\]

We proceed by case analysis on the rule applied last to derive \(\Gamma \vdash T_1 <: T_2\).

**Case (S\_BASE):** Obvious since \((B \Rightarrow B)^\ell v_1 \rightarrow v_1\) and \(v_1 \simeq_{\theta} v_2 : B; \theta; \tilde{\delta}\).

**Case (S\_TVAR):** We are given \(\Gamma \vdash \alpha <: \alpha\). Since \(\Gamma \vdash \alpha\) and \(\Gamma \vdash \theta; \tilde{\delta}\), there exists some \(r', T_1',\) and \(T_2'\) such that \(\theta(\alpha) = (r', T_1', T_2')\). Since \(v_1 \simeq_{\theta} v_2 : \alpha; \theta; \tilde{\delta}\), we have \((v_1, v_2) \in r'\). Since
\( r' \in VRel(T_1', T_2') \), there exists some \( v_1' \) such that \( (T_1' \Rightarrow T_1')^\ell v_1 \rightarrow^* v_1' \) and \( (v_1', v_2) \in r' \). Thus, \( v_1' \simeq_{\nu} v_2 : \alpha \theta \delta \).

Case (S\_FUN): We are given \( \Gamma \vdash x : T_{11} \rightarrow T_{12} \vdash x : T_{21} \rightarrow T_{22} \). By inversion, we have \( \Gamma \vdash T_{21} \vdash x : T_{21} \vdash T_{12} \vdash (T_{21} \Rightarrow T_{11})^\ell x / x \vdash T_{22} \). Without loss of generality, we can suppose that \( x \notin \text{dom}(\delta) \). By (E\_RED)/(R\_FUN),

\[
\begin{align*}
\theta_1(\delta_1((x : T_{11} \rightarrow T_{12} \Rightarrow T_{21} \rightarrow T_{22})^\ell)) & \rightarrow \theta_1(\delta_1(\lambda x : T_{21}.\text{let} y : T_{11} = (T_{21} \Rightarrow T_{11})^\ell x \in (T_{12} [y / x] \Rightarrow T_{22})^\ell (v_1 y))) \\
& \simeq_e v_1' v_2' : T_{21} ; \theta \delta,
\end{align*}
\]

for a fresh variable \( y \). By definition, it suffices to show that, for any \( v_1' \) and \( v_2' \) such that \( v_1' \simeq_{\nu} v_2 : T_{21} ; \theta \delta \),

\[
\theta_1(\delta_1(\lambda x : T_{21}.\text{let} y : T_{11} = (T_{21} \Rightarrow T_{11})^\ell x \in (T_{12} [y / x] \Rightarrow T_{22})^\ell (v_1 y))) \simeq_e v_2' : T_{22} ; \theta \delta[(v_1', v_2') / x].
\]

Since \( \Gamma \vdash x : T_{11} \rightarrow T_{12} \) and \( \Gamma \vdash x : T_{21} \rightarrow T_{22} \), we have \( \Gamma \vdash T_{11} \) and \( \Gamma \vdash T_{21} \) by their inversion. Since \( \Gamma \vdash T_{21} \vdash T_{11} \) and \( \Gamma \vdash \theta \delta \), we have

\[
\theta_1(\delta_1((T_{21} \Rightarrow T_{11})^\ell)) \simeq_{\nu} \theta_2(\delta_2(\lambda x : T_{21}.x)) : T_{21} \rightarrow T_{11} ; \theta \delta
\]

by the IH. Since \( v_1' \simeq_{\nu} v_2' : T_{21} ; \theta \delta \), we have

\[
\theta_1(\delta_1((T_{21} \Rightarrow T_{11})^\ell)) v_1' \simeq_e v_2' : T_{11} ; \theta \delta.
\]

By definition, there exists some \( v_2'' \) such that \( \theta_1(\delta_1((T_{21} \Rightarrow T_{11})^\ell)) v_1' \rightarrow^* v_2'' \) and \( v_2'' \simeq_{\nu} v_2' : T_{11} ; \theta \delta \). Thus, it suffices to show that

\[
\theta_1(\delta_1((T_{12} [v_2'' / x] \Rightarrow T_{22} [v_2'' / x])^\ell)) (v_1 v_1'') \simeq_e v_2' : T_{22} ; \theta \delta[(v_1', v_2') / x].
\]

Since \( v_1 \simeq_{\nu} v_2 : T_{11} \rightarrow T_{12} ; \theta \delta \) and \( v_1'' \simeq_{\nu} v_2' : T_{11} ; \theta \delta \), we have

\[
v_1 v_1'' \simeq_e v_2 v_2' : T_{12} ; \theta \delta[(v_1'', v_2') / x].
\]

If \( v_1 v_1'' \) and \( v_2 v_2' \) raise blame, we finish. Otherwise, \( v_1 v_1'' \rightarrow^* v_1''' \) and \( v_2 v_2' \rightarrow^* v_2''' \) for some \( v_1''' \) and \( v_2''' \), and it suffices to show that

\[
\theta_1(\delta_1((T_{12} [v_1''' / x] \Rightarrow T_{22} [v_1''' / x])^\ell)) v_1''' \simeq_e v_2'' : T_{22} ; \theta \delta[(v_1', v_2') / x].
\]

We also have \( v_1''' \simeq_{\nu} v_2''' : T_{12} ; \theta \delta[(v_1', v_2') / x] \). By \( \alpha \)-renaming \( x \) in \( T_{12} \) to \( y \) and the weakening (Lemma 5.2 (1)),

\[
v_1''' \simeq_{\nu} v_2''' : T_{12} [y / x] ; \theta \delta[(v_1', v_2') / x] [(v_1'', v_2'') / y].
\]

Thus, it suffices to show that

\[
\begin{align*}
\theta_1(\delta_1((T_{12} [v_1''' / x] \Rightarrow T_{22} [v_1''' / x])^\ell)) & \simeq_{\nu} \theta_2(\delta_2(\lambda z : T_{12} [v_1''' / x]) \rightarrow T_{22} ; \theta \delta[(v_1', v_2') / x] [(v_1'', v_2'') / y].
\end{align*}
\tag{6.1}
\]

where \( T_{12}' = T_{12} \vdash T_{11}^\ell x / x \).

We first show

\[
\begin{align*}
\theta_1(\delta_1((T_{12} [v_1''' / x] \Rightarrow T_{22} [v_1''' / x])^\ell)) & \simeq_{\nu} \theta_2(\delta_2(\lambda z : T_{12} [v_1''' / x]) \rightarrow T_{22} ; \theta \delta[(v_1', v_2') / x] \tag{6.2}
\end{align*}
\]

by applying the equivalence-respecting property (Lemma 5.12). Since \( \Gamma \vdash x : T_{11} \rightarrow T_{12} \) and \( \Gamma \vdash x : T_{21} \rightarrow T_{22} \), we have \( \Gamma, x : T_{11} \vdash T_{12} \) and \( \Gamma, x : T_{21} \vdash T_{22} \). By the typing weakening (Lemma 2.10) and the term substitution (Lemma 2.12), \( \Gamma, x : T_{21} \vdash T_{12} \). Since \( \Gamma \vdash \theta \delta \) and \( v_1' \simeq_{\nu} v_2' : T_{21} ; \theta \delta \), we have \( \Gamma, x : T_{21} \vdash \theta \delta[(v_1', v_2') / x] \) by the weakening of
the logical relation (Lemma 5.2 (3)). Since \( \Gamma, x : T_2 \vdash T_2 \) and \( \Gamma, x : T_2 \vdash T_2' \ll T_2 \), we have

\[
\theta_1(\delta_1((T_2' \Rightarrow T_2)\ell [v'_1/x])) \simeq x_\theta \theta_2(\delta_2(\lambda (x : T_2', x) [v'_2/x])) : T_2' \Rightarrow T_2; \theta, \delta \theta_1((v'_1, v'_2)/x) \]  
(6.3)

by the IH. Furthermore, since \( \theta_1(\delta_1((T_2 \Rightarrow T_{11})\ell )) v_1 \rightarrow^* v_1' \), we can show

\[
\emptyset \vdash \theta_1(\delta_1((T_2 [v'_1/x] \Rightarrow T_2 [v'_2/x])\ell ))
\]

by using Cotermination. From (6.3) and (6.4), the equivalence-respecting property derives (6.2).

We show (6.1) by applying (6.2) to the term compositionality (Lemma 5.6). Since \( T_2' = T_12 [(T_21 \Rightarrow T_{11})\ell x/x] = T_12 [y/x] [(T_21 \Rightarrow T_{11})\ell x/y] \), it suffices to show that

1. \( \Gamma, x : T_21, y : T_{11} \) is self-related,
2. \( \Gamma, x : T_21, y : T_{11} \vdash T_12 [y/x] \simeq T_12 [y/x] : * \),
3. \( \Gamma, x : T_21, y : T_{11} \vdash \theta, \delta \theta_1(v'_1, v'_2/x) : (v'_1, v'_2)/y) \),
4. \( \theta_2(\delta_2((T_2 \Rightarrow T_{11})\ell )) v_2 \rightarrow^* v_2' \) and \( v'_2 \simeq v_2' : T_11; \theta, \delta (v'_1, v'_2)/x) \) for some \( v'_2 \).

Since \( \Gamma, x : T_21, y : T_{11} \), we have \( \Gamma, x : T_21, y : T_{11} \vdash T_12 [y/x] \). By the parametricity (Theorem 5.28), we have (1) and (2).

Since \( \Gamma \vdash \theta, \delta \) and \( v'_1 \simeq v_2' : T_{11}; \theta, \delta \) and \( v'_1 \simeq v'_2 : T_{21}; \theta, \delta \), we have (3) by the weakening (Lemma 5.2 (3)).

Since \( \Gamma \vdash \theta, \delta \) and \( v'_1 \simeq v'_2 : T_{21}; \theta, \delta \), we have \( \Gamma, x : T_21 \vdash \theta ; \delta (v'_1, v'_2)/x) \). Since \( \Gamma, x : T_21 \vdash (T_2 \Rightarrow T_{11})\ell x : T_{11} \), we have \( \Gamma, x : T_21 \vdash (T_2 \Rightarrow T_{11})\ell x \simeq (T_2 \Rightarrow T_{11})\ell x : T_{11} \) by the parametricity (Theorem 5.28). Thus, by definition,

\[
\theta_1(\delta_1((T_2 \Rightarrow T_{11})\ell )) v_1 \simeq x_\theta \theta_2(\delta_2((T_2 \Rightarrow T_{11})\ell )) v_2 : T_{11}; \theta, \delta (v'_1, v'_2)/x).
\]

Since \( \theta_1(\delta_1((T_2 \Rightarrow T_{11})\ell )) v_1 \rightarrow^* v_1' \), we have (4).

Case (S_FORALL): By the IH.
Case (S_REFINER): We are given \( \Gamma \vdash T_1 \llx x : T_2' \theta, \delta \). By inversion, \( \Gamma \vdash T_1 \llx T_2' \) and \( \Gamma, x : T_2 \vdash c_2' [T_2' \Rightarrow T_2] \ell x/x \). By (E_REDUCE)/(R_FORGET),

\[
\theta_1(\delta_1((T_1 \Rightarrow x : T_2' \ell x))\ell x \rightarrow^* \langle \text{unref}(\theta_1(\delta_1(T_1))) \Rightarrow \theta_1(\delta_1(x : T_2' \ell x)))\ell x
\]

By (E_REDUCE)/(R_PRECHECK),

\[
\langle \text{unref}(\theta_1(\delta_1(T_1))) \Rightarrow \theta_1(\delta_1(x : T_2' \ell x)))\ell x \rangle
\]

Thus, it suffices to show that

\[
\langle \theta_1(\delta_1((T_1 \Rightarrow T_2')\ell x)) \rangle \langle \text{unref}(\theta_1(\delta_1(T_1))) \Rightarrow \theta_1(\delta_1(T_2'))\ell x \rangle v_1 \simeq v_2 : x : T_2' \ell c_2' ; \theta, \delta.
\]

Since \( \Gamma \vdash x : T_2' \ell c_2' \), we have \( \Gamma \vdash T_2' \ell x \) by its inversion. Thus, by the IH,

\[
\theta_1(\delta_1((T_1 \Rightarrow T_2')\ell x)) \simeq x_\theta \theta_2(\delta_2(x : T_1, x) : T_1 \Rightarrow T_2' ; \theta, \delta.
\]

Since \( v_1 \simeq v_2 : T_1 ; \theta, \delta \), we have

\[
\theta_1(\delta_1((T_1 \Rightarrow T_2')\ell x)) v_1 \simeq v_2 : T_2' ; \theta, \delta.
\]

Since \( \theta_1(\delta_1((T_1 \Rightarrow T_2')\ell x)) v_1 \rightarrow^* \langle \text{unref}(\theta_1(\delta_1(T_1))) \Rightarrow \theta_1(\delta_1(T_2'))\ell x \rangle v_1 \), we have

\[
\langle \text{unref}(\theta_1(\delta_1(T_1))) \Rightarrow \theta_1(\delta_1(T_2'))\ell x \rangle v_1 \simeq v_2 : T_2' ; \theta, \delta.
\]
By definition, there exists some \( v'_1 \) such that \( \langle \text{unref}(\delta_1(T_1)) \rangle \Rightarrow \theta_1(\delta_1(T_2')) \). Thus, it suffices to show that

\[
\langle \theta_1(\delta_1(\{x:T'_2 | e'_2\})) \rangle \overset{\varepsilon}{\Rightarrow} v_2 : \{x:T'_2 | e'_2\}; \theta; \delta.
\]

Since \( \Gamma \vdash \theta; \delta \) and \( v_1 \overset{\varepsilon}{\Rightarrow} v_2 \), we have \( \Gamma, x:T_1 \vdash \theta_1 \circ \delta_1[v_1/x] \). Since \( \Gamma, x:T_1 \vdash e'_2 \{/T_1 \Rightarrow T'_2, e'/x\} \), we have \( \theta_1(\delta_1(e'_2(\{x:T'_2 \Rightarrow e'_2(x)/v_1\}))) \rightarrow_t^* v'_1 \). Since \( \theta_1(\{x:T_1 \Rightarrow T'_2 \}) \) \( v_1 \rightarrow_t^* v'_1, \) we have \( \theta_1(\delta_1(e'_2(\{x:T'_2 \Rightarrow e'_2(x)/v_1\}))) \rightarrow_t^* v'_1 \) by Cotermination (Lemma 2.6). Thus, \( \langle \theta_1(\delta_1(\{x:T'_2 | e'_2\})) \rangle, v'_1 \rangle \rightarrow_t^* v'_1 \) (by \( \text{R\_CHECK} \) and \( \text{R\_OK} \)), and so it suffices to show that

\[
v'_1 \overset{\varepsilon}{\Rightarrow} v_2 : \{x:T'_2 | e'_2\}; \theta; \delta.
\]

Since \( v'_1 \overset{\varepsilon}{\Rightarrow} v_2 : T'_2; \theta; \delta \) and \( \theta_1(\delta_1(e'_2(x)/v_1/\)) \rightarrow_t^* \text{true} \), it suffices to show that

\[
\theta_1(\delta_1(e'_2(v_2/x))) \rightarrow_t^* \text{true}.
\]

Since \( \Gamma \vdash \theta; \delta \) and \( v'_1 \overset{\varepsilon}{\Rightarrow} v_2 : T'_2; \theta; \delta \), we have \( \Gamma, x:T'_2 \vdash \theta; \delta \). Since \( \Gamma \vdash \theta; \delta \) and \( v'_1 \overset{\varepsilon}{\Rightarrow} v_2 : T'_2; \theta; \delta \), we have \( \Gamma, x:T'_2 \vdash \theta; \delta \). Thus,

\[
\theta_1(\delta_1(e'_2(v_2/x))) \overset{\varepsilon}{\Rightarrow} \theta_2(\delta_2(e'_2(v_2/x))) : \text{Bool}; \theta; \delta (v'_1, v_2).\]

Since the term on the left-hand side evaluates to \( \text{true} \), we have \( \theta_1(\delta_1(e'_2(v_2/x))) \rightarrow_t^* \text{true} \) by definition.

**Theorem 6.2 (Upcast Elimination).** If \( \Gamma \vdash T_1 \) and \( \Gamma \vdash T_2 \) and \( \Gamma \vdash T_1 <: T_2 \), then \( \Gamma \vdash \{T_1 \Rightarrow T_2\} \overset{\varepsilon}{\Rightarrow} \text{ctx} (\lambda x:T_1.x) : T_1 \rightarrow T_2 \).

**Proof.** By Lemma 6.1, \( \Gamma \vdash \{T_1 \Rightarrow T_2\} \overset{\varepsilon}{\Rightarrow} (\lambda x:T_1.x) : T_1 \rightarrow T_2 \). By the soundness of the logical relation (Theorem 4.10), we finish.

---

6.2. Selfification. Selfification embeds information of a term into its type so that we can get the singleton type that identifies the term [28]. For example, selfification of \( x \) of \( \text{Int} \) produces \( \{y:\text{Int} \mid y = \text{int} x\} \), which identifies \( x \), and that of \( e \) of \( \text{Bool} \rightarrow \text{Int} \) does \( x: \text{Bool} \rightarrow \{y:\text{Int} \mid y = \text{int} e \} \), which means functions that return the same value as the result of call to \( e \).

We expose the power of the selfification combined with the upcast elimination via an example using stacks. First of all, let us assume type \( \text{Stack} \) (which can be implemented as an abstract datatype in \( F_H \)) and the following functions:

\[
\begin{align*}
\text{is\_empty} : \text{Stack} &\rightarrow \text{Bool} \\
\text{push} : \text{Int} &\rightarrow \text{Stack} \rightarrow \{x:\text{Stack} \mid \text{not (is\_empty x)}\} \\
\text{empty} : \{x:\text{Stack} \mid \text{is\_empty x}\} &\rightarrow \text{Stack} \\
\text{pop} : \{x:\text{Stack} \mid \text{not (is\_empty x)}\} &\rightarrow \text{Stack}
\end{align*}
\]

where \( \text{is\_empty} \) returns whether a given stack is empty, \( \text{empty} \) is the empty stack, \( \text{push} \) produces a nonempty stack by adding an element at the top of a stack, and \( \text{pop} \) returns the stack without the topmost element. Since the type signature of \( \text{push} \) ensures that the result stack is never empty, expression \( \text{pop}(\text{push} 2 \text{empty}) \) would be accepted. However, the type of \( \text{pop} \) guarantees nothing about stacks that it returns. Thus, expression \( \text{pop}(\text{pop}(\text{push} 2(\text{push} 3 \text{empty}))) \) would be rejected because the outermost \( \text{pop} \) takes a possibly empty stack (\( \text{Stack} \)), not nonempty stacks (\( \{x:\text{Stack} \mid \text{is\_empty x}\} \)), even though it is actually called with a nonempty one. Insertion of cast \( \{\{x:\text{Stack} \mid \text{not (is\_empty x)}\}\} \rightarrow \text{Stack} \) makes the program acceptable, but it incurs additional, redundant overhead. (Note that the

---

7Trivial cast \( \{x:\text{Stack} \mid \text{is\_empty x}\} \Rightarrow \text{Stack} \) to empty is omitted here.
upcast elimination cannot be applied here because \( \text{Stack} \Rightarrow \{ x : \text{Stack} \mid \text{not} \ (\text{is} \_\text{empty} \ x) \}^\ell \) is not an upcast.)

Combination of the selfification and the upcast elimination solves this unfortunate situation. Selfification can give subexpression \( \text{pop} \ (\text{push} \ 2 \ (\text{push} \ 3 \ \text{empty})) \) type \( T = \{ x : \text{Stack} \mid x = \text{Stack} \ \text{pop} \ (\text{push} \ 2 \ (\text{push} \ 3 \ \text{empty})) \} \), which identifies the subexpression. Since \( T \) denotes the singleton stack with only 3, we expect that \( (T \Rightarrow \{ x : \text{Stack} \mid \text{not} \ (\text{is} \_\text{empty} \ x) \} )^\ell \) is proven to be an upcast. If so, by the upcast elimination, acceptable program

\[
\text{pop} \ ((T \Rightarrow \{ x : \text{Stack} \mid \text{not} \ (\text{is} \_\text{empty} \ x) \}^\ell) (\text{pop} \ (\text{push} \ 2 \ (\text{push} \ 3 \ \text{empty})))
\]

should be contextual equivalent to \( \text{pop} \ (\text{pop} \ (\text{push} \ 2 \ (\text{push} \ 3 \ \text{empty}))) \), and so it would be proven that it does not get stuck.

Selfification function \( \text{self}(T, e) \), which returns a type into which term \( e \) of \( T \) is embedded, is defined as follows.

**Definition 6.3 (Selfification).**

\[
\begin{align*}
\text{self}(B, e) &= \{ x : B \mid x =_B e \} & \text{(if } x \notin \text{FV}(e)) \\
\text{self}(\alpha, e) &= \alpha \\
\text{self}(x : T_1 \rightarrow T_2, e) &= x : T_1 \rightarrow \text{self}(T_2, e \, x) & \text{(if } x \notin \text{FV}(e)) \\
\text{self}(\forall \alpha. T, e) &= \forall \alpha. \text{self}(T, e \, \alpha) & \text{(if } \alpha \notin \text{FTV}(e)) \\
\text{self}(\{ x : T' \mid e' \}, e) &= \{ x : T'' \mid \text{let } x : T' = \{ T'' \Rightarrow T' \}^\ell x \in e' \} & \text{(if } x \notin \text{FV}(e)) \\
&\quad \text{where } T'' = \text{self}(T', \{ x : T' \mid e' \} \Rightarrow T')^\ell e
\end{align*}
\]

Selfification \( \text{self}(B, e) \) produces the most precise type for \( e \) in that it is the singleton type which identifies \( e \), and \( \text{self}(\alpha, e) \) returns \( \alpha \) as it is because we cannot make type variables more precise without polymorphic equality. Selfification of function types \( x : T_1 \rightarrow T_2 \) and universal types \( \forall \alpha. T \) is forwarded to \( T_2 \) and \( T \), respectively. Term \( e \) is applied to variables so that selfified types can identify what \( e \) produces. Selfifying refinement types \( \{ x : T' \mid e' \} \) appears slightly tricky: it selfifies the underlying type \( T' \) with \( \{ x : T' \mid e' \} \Rightarrow T')^\ell e \) (the cast makes \( e \) a term of \( T' \)) and refines the result with refinement \( e' \), but, since \( e' \) refers to \( x \) of \( T' \) whereas the selfified underlying type is \( T'' \), cast \( T'' \Rightarrow T' \)^\ell is inserted at the beginning of the refinement. Label \( \ell \) can be any because, as shown later, the casts never fail.

The rest of this section shows that inserting casts to selfified types causes no run-time errors, which leads to use of selfification to any expression for free. More formally, we prove that, given term \( e \) of \( T \), \( (T \Rightarrow \text{self}(T, e))^\ell \) is contextually equivalent to \( e \). We start with showing that casts to selfified types are well typed, which is implied by two facts: (1) \( T \) is compatible with \( \text{self}(T, e) \) and (2) \( \text{self}(T, e) \) is well formed if \( e \) is well typed.

**Lemma 6.4.** \( T \parallel \text{self}(T, e) \).

**Proof.** Straightforward by induction on \( T \).

**Lemma 6.5.** If \( \Gamma \vdash e : T \), then \( \Gamma \vdash \text{self}(T, e) \).

**Proof.** By induction on \( T \) with the fact that \( \Gamma \vdash T \), which is obtained from \( \Gamma \vdash e : T \).

The selfification of refinement types involves casts from selfified types to the underlying types, so we need to show that such casts also do not raise blame.

**Lemma 6.6.** If \( \Gamma \vdash e : T \), then \( \Gamma \vdash \langle \text{self}(T, e) \Rightarrow T \rangle^\ell \simeq \lambda x : T. x : \text{self}(T, e) \rightarrow T \).
Proof. By induction on $T$. Let $\Gamma \vdash \theta; \delta$. It suffices to show that, for any $v_1$ and $v_2$ such that $v_1 \sim_v v_2 : \text{self}(T, e); \theta; \delta$,

$$\theta_1(\delta_1((\text{self}(T, e) \Rightarrow T)^\ell)) \sim_e v_1 \sim_e v_2 : T; \theta; \delta.$$  

By case analysis on $T$.

Case $T = B$: Trivial.

Case $T = \alpha$: We have $\text{self}(T, e) = \alpha$. Since reflexive casts are logically related to identity functions (Lemma 5.8), we have $\theta_1(\delta_1((\alpha \Rightarrow \alpha)^\ell)) \sim_v \theta_2(\delta_2(\lambda x: \alpha. x)) : \alpha \rightarrow \alpha; \theta; \delta$. Since $v_1 \sim_v v_2 : \alpha; \theta; \delta$, we finish by definition.

Case $T = x : T_1 \rightarrow T_2$: Without loss of generality, we can suppose that $x \notin \text{dom}(\delta)$. We have $\text{self}(T, e) = x : T_1 \rightarrow \text{self}(T_2, e x)$. By (E Red)/(R Fun),

$$\theta_1(\delta_1((\text{self}(T, e) \Rightarrow T)^\ell)) \sim_e v_1 \sim_e v_2 : T_1; \theta; \delta,$$

for fresh variable $y$. Thus, it suffices to show that, for any $v_1'$ and $v_2'$ such that $v_1' \sim_v v_2' : T_1; \theta; \delta$,

$$\theta_1(\delta_1(\text{let } y : T_1 \Rightarrow T_1)^\ell x \in \text{self}(T_2, e x) [y/x] \Rightarrow T_2)^\ell (v_1 y) [v_1'/x]) \sim_e v_2' : T_2; \theta; \delta[(v_1'/x)/x].$$

Let $e' = \text{self}(T_2, e x) [y/x] \Rightarrow T_2)^\ell$ for fresh variable $y$. Since reflexive casts are logically related to identity functions (Lemma 5.8), we have $\Gamma \vdash (T_1 \Rightarrow T_1)^\ell \sim (\lambda x : T_1.x : T_1 \rightarrow T_1)$. Thus, we have

$$\Gamma, z : (x : T_1 \rightarrow \text{self}(T_2, e x)), x : T_1 \vdash \text{let } y : T_1 \Rightarrow T_1)^\ell x \in e' \sim \text{let } y : T_1 \Rightarrow T_1)^\ell x \in e' : T_2$$

by the fundamental property.

Since $v_1 \sim_v v_2 : x : T_1 \rightarrow \text{self}(T_2, e x); \theta; \delta$ and $v_1' \sim_v v_2' : T_1; \theta; \delta$, we have $\Gamma, z : (x : T_1 \rightarrow \text{self}(T_2, e x)), x : T_1 \vdash \theta; \delta[(v_1, v_2)/z][(v_1', v_2')/x]$ by the weakening (Lemma 5.2). Since logically related terms are CIU-equivalent (Theorem 4.10 and Lemma 5.32), we have

$$\emptyset \vdash \theta_1(\delta_1(\text{let } y : T_1 \Rightarrow T_1)^\ell x \in e'[v_1/z]) [v_1'/x])$$

$$= \text{ciu} \theta_1(\delta_1(\text{let } y : T_1 \Rightarrow (\lambda x : T_1.x)x \in e'[v_1/z]) [v_1'/x]) : \theta_1(\delta_1(T_2)[v_1'/x]).$$

from (6.5). Thus, by the equivalence-respecting property (Lemma 5.12), it suffices to show that

$$\theta_1(\delta_1(\text{let } y : T_1 \Rightarrow (\lambda x : T_1.x)x \in \text{self}(T_2, e x) [y/x] \Rightarrow T_2)^\ell (v_1 y) [v_1'/x])$$

$$\sim_e v_2' : T_2; \theta; \delta[(v_1', v_2')/x],$$

that is,

$$\theta_1(\delta_1((\text{self}(T_2, e x) \Rightarrow T_2)^\ell) [v_1'/x]) \sim_e v_2' : T_2; \theta; \delta[(v_1', v_2')/x].$$

Since $v_1 \sim_v v_2 : x : T_1 \rightarrow \text{self}(T_2, e x); \theta; \delta$ and $v_1' \sim_v v_2' : T_1; \theta; \delta$, we have $v_1 v_1' \sim_e v_2 v_2'$ by self(T_2, e x); \theta; \delta[(v_1', v_2')/x].$ If $v_1$ and $v_2'$ raise blame, then we finish. Otherwise, $v_1 v_1' \rightarrow v_1''$ and $v_2 v_2' \rightarrow v_2''$ for some $v_1''$ and $v_2''$, and it suffices to show that

$$\theta_1(\delta_1((\text{self}(T_2, e x) \Rightarrow T_2)^\ell) [v_1'/x]) \sim_e v_2'' : T_2; \theta; \delta[(v_1', v_2')/x].$$

We also have $v_1'' \sim_v v_2'' : \text{self}(T_2, e x); \theta; \delta[(v_1', v_2')/x]$. Since $\Gamma, x : T_1 \vdash e : T_2$, we have

$$\Gamma, x : T_1 \vdash (\text{self}(T_2, e x) \Rightarrow T_2)^\ell \sim (\lambda x : T_2.x : \text{self}(T_2, e x) \Rightarrow T_2)$$
by the IH. Since $\Gamma, x : T_1 \vdash \theta ; \delta[(v_1', v_2')/x]$, we have

$$\theta_1(\delta_1(\langle \text{self}(T_2, e \ x) \Rightarrow T_2\rangle^\ell [v_1'/x]))$$

$$\simeq_v \theta_2(\delta_2(\lambda x : T_2, x)[v_2'/x]) : \text{self}(T_2, e \ x) \Rightarrow T_2; \theta ; \delta[(v_1', v_2')/x].$$

Since $v_1' \simeq_v v_2'$, we have (6.6).

Case $T = \forall \alpha. T'$: Straightforward by the IH.

Case $T = \{ x : T' \mid e \}$: Without loss of generality, we may suppose that $x \notin \text{dom}(\delta)$. We have $\text{self}(T, e) = \{ x : T'' \mid \text{let } x : T' = T'' \Rightarrow T'\ell x \in e' \}$ where $T'' = \text{self}(T', \{ x : T' \mid e' \}) \Rightarrow T'\ell e$. By $(E_{\text{RED}})/(R_{\text{FORGET}})$,

$$\theta_1(\delta_1(\langle \text{self}(T, e) \Rightarrow T'\ell e \rangle)) v_1 \rightarrow \theta_1(\delta_1(\langle T'' \Rightarrow \{ x : T' \mid e' \}\rangle)) v_1.$$

It suffices to show that

$$\theta_1(\delta_1(\langle T'' \Rightarrow \{ x : T' \mid e' \}\rangle)) v_1 \simeq_e v_2 : \{ x : T' \mid e' \}; \theta ; \delta.$$

By the IH, $\Gamma \vdash \{ T'' \Rightarrow T'\ell \} \simeq \lambda x : T'. x : T'' \rightarrow T'$, and so $\theta_1(\delta_1(\langle T'' \Rightarrow T'\ell e \rangle)) \simeq_v \theta_2(\delta_2(\lambda x : T'. x)[v_2]) : T'' \rightarrow T'; \theta ; \delta$. Since $v_1 \simeq_v v_2 : \text{self}(T, e); \theta ; \delta$, we have $v_1 \simeq_v v_2 : T''; \theta ; \delta$. Thus, $\theta_1(\delta_1(\langle T'' \Rightarrow T'\ell e \rangle)) v_1 \simeq_e v_2 : T'; \theta ; \delta$. By definition, there exists some $v_1'$ such that $\theta_1(\delta_1(\langle T'' \Rightarrow T'\ell e \rangle)) v_1 \rightarrow* v_1'$ and $v_1' \simeq_v v_2 : T'; \theta ; \delta$. By $(R_{\text{FORGET}})$ and $(R_{\text{PRECHECK}})$, $\theta_1(\delta_1(\langle T'' \Rightarrow \{ x : T' \mid e' \}\rangle)) v_1 \rightarrow* \theta_1(\delta_1(\langle \{ x : T' \mid e' \}, e'[v_1'/x], v_1'\ell e \rangle)).$

Thus, it suffices to show that

$$\theta_1(\delta_1(\langle \{ x : T' \mid e' \}, e'[v_1'/x], v_1'\ell e \rangle)) \simeq_e v_2 : \{ x : T' \mid e' \}; \theta ; \delta.$$

Since $v_1 \simeq_v v_2 : \text{self}(T, e); \theta ; \delta$, we have $\theta_1(\delta_1(\langle \text{let } x : T' = (T'' \Rightarrow T'\ell v_1 \in e' \rangle) \rightarrow* \text{true}$. Since $\theta_1(\delta_1(\langle T'' \Rightarrow T'\ell e \rangle)) v_1 \rightarrow* v_1'$, we have $\theta_1(\delta_1(e'[v_1'/x])) \rightarrow* \text{true}$. Thus, it suffices to show that

$$v_1' \simeq_v v_2 : \{ x : T' \mid e' \}; \theta ; \delta.$$

Since $v_1' \simeq_v v_2 : T'; \theta ; \delta$ and $\theta_1(\delta_1(e'[v_1'/x])) \rightarrow* \text{true}$, it suffices to show that

$$\theta_2(\delta_2(e'[v_2'/x])) \rightarrow* \text{true}.$$

Since $\Gamma \vdash \{ x : T' \mid e' \}$, we have $\Gamma, x : T' \vdash e' \simeq v_1' : \text{Bool}$ by the parametricity (Theorem 5.28). Since $\Gamma, x : T' \vdash \delta(v_2'/x)]$, we have $\theta_1(\delta_1(e'[v_1'/x])) \simeq_e \theta_2(\delta_2(e'[v_2'/x])) : \text{Bool}; \delta ; \delta(v_2'/x)].$ Since the term on the left-hand side evaluates to true, we have $\theta_2(\delta_2(e'[v_2'/x])) \rightarrow* \text{true}. \quad \Box$

Now, we prove that casts to selfified types are redundant at run time.

**Lemma 6.7.** If $\Gamma \vdash e_1 \simeq e_2 : T$, then $\Gamma \vdash \langle T \Rightarrow \text{self}(T, e_1)\rangle\ell e_1 \simeq e_2 : \text{self}(T, e_1)$.

**Proof.** By induction on $T$. Let $\Gamma \vdash \theta ; \delta$. We show that

$$\theta_1(\delta_1(\langle T \Rightarrow \text{self}(T, e_1)\rangle\ell e_1)) \simeq_e \theta_2(\delta_2(e_2)) : \text{self}(T, e_1); \theta ; \delta.$$

Since $\Gamma \vdash e_1 \simeq e_2 : T$, we have $\theta_1(\delta_1(e_1)) \simeq_e \theta_2(\delta_2(e_2)) : T; \theta ; \delta$. If $\theta_1(\delta_1(e_1))$ and $\theta_2(\delta_2(e_2))$ raise blame, we finish. Otherwise, $\theta_1(\delta_1(e_1)) \rightarrow* v_1$ and $\theta_2(\delta_2(e_2)) \rightarrow* v_2$ for some $v_1$ and $v_2$, and it suffices to show that

$$\theta_1(\delta_1(\langle T \Rightarrow \text{self}(T, e_1)\rangle\ell e_1)) v_1 \simeq_e v_2 : \text{self}(T, e_1); \theta ; \delta.$$

We also have $v_1 \simeq_v v_2 : T; \theta ; \delta$. By case analysis on $T$. 

Case $T = B$: We have $\text{self}(T, e_1) = \{x : B \mid x =_B e_1\}$. Since $\theta_1(\delta_1(e_1)) \rightarrow^* v_1$, we have $\theta_1(\delta_1((T \Rightarrow \text{self}(T, e_1))^{\ell})) v_1 \rightarrow^* v_1$. Thus, it suffices to show that

$$v_1 \simeq_v v_2 : \{x : B \mid x =_B e_1\}; \theta; \delta.$$  

Since $v_1 \simeq_v v_2 : B; \theta; \delta$ and $\theta_1(\delta_1(x =_B e_1)[v_1/x]) \rightarrow^* \text{true}$, it suffices to show that

$$\theta_2(\delta_2(x =_B e_1)[v_2/x]) \rightarrow^* \text{true}.$$  

Since $\Gamma \vdash e_1 : B$, we have $\Gamma \vdash e_1 \simeq e_1 : B$ by the parametricity. Thus, by definition, $\theta_2(\delta_2(e_1)) \rightarrow^* v_1$. Since $v_1 = v_2$ from $v_1 \simeq_v v_2 : B; \theta; \delta$, we finish.

Case $T = \alpha$: Obvious since $\text{self}(T, e_1) = \alpha$ and a reflexive cast is logically related to an identity function (Lemma 5.8).

Case $T = x : T_1 \rightarrow T_2$: Similar to the case of function types in Lemma 6.6.

Case $T = \forall \alpha. T'$: Straightforward by the IH.

Case $T = \{x : T'| e'\}$: Without loss of generality, we can suppose that $x \notin \text{dom}(\delta)$. We have $\text{self}(T, e_1) = \{x : T'' \mid \text{let } x : T' = \{T'' \Rightarrow T''\} \text{ in } e'\}$ where $T'' = \text{self}(T', \{x : T' | e'\} \Rightarrow T')^{\ell} e_1$. By $(\text{E}_\text{RED}) / (\text{R}_\text{FORGET})$,  

$$\theta_1(\delta_1((T \Rightarrow \text{self}(T, e_1))^{\ell})) v_1 \rightarrow \theta_1(\delta_1((T' \Rightarrow \text{self}(T, e_1))^{\ell})) v_1.$$  

Thus, it suffices to show that

$$\theta_1(\delta_1((T' \Rightarrow \text{self}(T, e_1))^{\ell})) v_1 \simeq_{\text{e}} v_2 : \text{self}(T, e_1); \theta; \delta.$$  

We first show

$$\theta_1(\delta_1((T' \Rightarrow \text{self}(T, e_1))^{\ell})) v_1 \simeq_{\text{e}} v_2 : T''^{\ell}; \theta; \delta.$$  

by using the equivalence-respecting property (Lemma 5.12). We can show $\Gamma \vdash \{x : T' | e'\} \Rightarrow T''^{\ell} \simeq \lambda x : \{x : T' | e'\}.x : \{x : T' | e'\} \rightarrow T'$ easily from the fact that a reflexive cast and an identity function are logically related (Lemma 5.8). Since the logical relation is compatible, we have $\Gamma \vdash \{x : T' | e'\} \Rightarrow T''^{\ell} e_1 \simeq (\lambda x : \{x : T' | e'\}.x) e_2 : T'$. Thus, by the IH,

$$\Gamma \vdash (T' \Rightarrow T'')^{\ell} ((\{x : T' | e'\} \Rightarrow T'')^{\ell} e_1) \simeq (\lambda x : \{x : T' | e'\}.x) e_2 : T''.$$  

Since $\theta_1(\delta_1(e_1)) \rightarrow^* v_1$ and $\theta_2(\delta_2(e_2)) \rightarrow^* v_2$ and $\theta_1(\delta_1((\{x : T' | e'\} \Rightarrow T'')^{\ell})) v_1 \rightarrow \theta_1(\delta_1((T' \Rightarrow T'')^{\ell})) v_1$ by $(\text{E}_\text{RED}) / (\text{R}_\text{FORGET})$, we have

$$\theta_1(\delta_1((T' \Rightarrow T'')^{\ell} ((\{x : T' | e'\} \Rightarrow T'')^{\ell} v_1))) \simeq_{\text{e}} v_2 : T''^{\ell}; \theta; \delta.$$  

Since an identity function is logically related to a reflexive cast (Lemma 5.9), we have

$$\Gamma \vdash (\lambda x : T'.x) \simeq (T' \Rightarrow T'')^{\ell} : T' \rightarrow T'.$$

Thus, by the fundamental property,

$$\Gamma, x : T' \vdash (T' \Rightarrow T'')^{\ell} ((\lambda x : T'.x) x) \simeq (T' \Rightarrow T'')^{\ell} ((T' \Rightarrow T'')^{\ell} x) : T''.$$  

Since $v_1 \simeq_v v_2 : \{x : T' | e'\}; \theta; \delta$, we have $v_1 \simeq_v v_2 : T'; \theta; \delta$, and so $\Gamma, x : T' \vdash \theta; \delta[(v_1, v_2)/x]$. Thus, from the fact that logically related terms are CIU-equivalent (Theorem 4.10 and Lemma 5.32) and the definition of CIU-equivalence,

$$\emptyset \vdash \theta_1(\delta_1((T' \Rightarrow T'')^{\ell} ((\lambda x : T'.x) v_1))) =_{\text{CIU}} \theta_1(\delta_1((T' \Rightarrow T''^{\ell} ((T' \Rightarrow T'')^{\ell} v_1)) : \theta_1(\delta_1(T''))).$$  

From (6.8) and (6.9), the equivalence-respecting property derives (6.7).
From (6.7), there exists some $v'_1$ such that $\theta_1(\delta_1((T' \Rightarrow \text{T}''\ell)\ell v_1)) \to^* v'_1$ and $v'_1 \simeq_v v_2 : \ell \theta ; \ell \delta$. Thus,
\[
\begin{align*}
\theta_1(\delta_1((T' \Rightarrow \text{s}elf(T, e_1))\ell)) v_1 & \to^* v'_1 \\
\theta_1(\delta_1((\text{s}elf(T, e_1), \text{l}et x:T''\ell = (T''\ell v'_1\text{ in } e', v'_1))
\end{align*}
\]
by (R_FORGET) and (R_PRECHECK), and so it suffices to show that
\[
\theta_1(\delta_1((\text{s}elf(T, e_1), \text{l}et x:T''\ell = (T''\ell v'_1\text{ in } e', v'_1))) \simeq_v v_2 : \text{s}elf(T, e_1) ; \ell \theta ; \ell \delta.
\]
Since $\Gamma \vdash \{x:T'| e'\} \Rightarrow T''\ell e_1 : T'$, we have $\Gamma \vdash (T'' \Rightarrow T'') \ell\ell x : T'' \Rightarrow T'$ by Lemma 6.6. Since $v''_1 \simeq_v v_2 : \ell \theta \ell \delta$, we have $\theta_1(\delta_1((T'' \Rightarrow T'')\ell)) v'_1 \simeq_v v_2 : \ell \theta \ell \delta$. By definition, $\theta_1(\delta_1((T'' \Rightarrow T'')\ell)) v'_1 \to^* v''_1$ for some $v''_1$, and it suffices to show that
\[
\theta_1(\delta_1((\text{s}elf(T, e_1), e'[v''_1/x], v''_1))) \simeq_v v_2 : \text{s}elf(T, e_1) ; \ell \theta \ell \delta.
\]
We also have $v''_1 \simeq_v v_2 : T'; \ell \theta \ell \delta$.

Since $\Gamma \vdash \{x:T'| e'\}$, we have $\Gamma, x : T' \vdash e' \simeq e' : \text{Bool}$ by the parametricity (Theorem 5.28). Since $\Gamma, x : T' \vdash \theta [v''_1, v_2] / x$, we have $\theta_1(\delta_1(e'[v''_1/x])) \simeq_v \theta_2(\delta_2(e'[v_2/x])) : \text{Bool}; \ell \delta; (\ell \theta)[v''_1, v_2] / x$. Since $v_\ell \simeq_v v_2 : \ell \ell x : T'| e' \ell \ell \theta \ell \delta$, we have $\theta_2(\delta_2(e'[v_2/x])) \to^* \text{true}$, so $\theta_1(\delta_1(e'[v''_1/x])) \to^* \text{true}$. Thus, it suffices to show that $v'_1 \simeq_v v_2 : \text{s}elf(T, e_1) ; \ell \theta \ell \delta$.

We have it by the discussion above.

**Corollary 6.8 (Selfification Cast Elimination).** If $\Gamma \vdash e : T$, then $\Gamma \vdash (T \Rightarrow \text{s}elf(T, e))\ell e =_{\text{ctx}} e : \text{s}elf(T, e)$.

**Proof.** By the parametricity (Theorem 5.28) and Lemma 6.7.

\[\square\]

6.3. **Cast Decomposition.** The upcast elimination enables us to eliminate redundant casts, but there are cases that nonredundant casts produce redundant ones. For example, let us consider $\{x:\text{Int} | x \neq 0\} \to \{x:\text{Int} | \text{prime? } x\} \Rightarrow \{x:\text{Int} | x \geq 0\} \to \{x:\text{Int} | x > 0\}$, which is not an upcast because the argument check may fail. This will be decomposed into two casts at run time: one for the domain type—$\{x:\text{Int} | x \geq 0\} \Rightarrow \{x:\text{Int} | x \neq 0\}$—and one for the codomain type—$\{x:\text{Int} | \text{prime? } x\} \Rightarrow \{x:\text{Int} | x > 0\}$—and one for the codomain type. As mentioned above, the cast for the domain type cannot be eliminated because it would fail if applied to zero.

By contrast, the cast for the codomain type is an upcast and so can be eliminated without changing the behavior of a program.

Static decomposition of casts makes it possible to eliminate as many redundant casts as possible. For example, it allows us to statically decompose casts for function types into ones for domain types and codomain types and eliminate them if they are upcasts. In what follows, we show how casts can be decomposed.

**Lemma 6.9.** If $\Gamma \vdash \{x:T_{11} \to T_{12} \Rightarrow x:T_{21} \to T_{22}\} : (x:T_{11} \to T_{12}) \to (x:T_{21} \to T_{22})$, then $\Gamma \vdash \{x:T_{11} \to T_{12} \Rightarrow x:T_{21} \to T_{22}\} \ell \ell x:T_{11} \Rightarrow T_{12} \to T_{22}\ell x$ in $(T_{12} y/x) \Rightarrow T_{22}\ell (z y) : (x:T_{11} \to T_{12}) \to (x:T_{21} \to T_{22})$.

**Proof.** By following (E_RED)/(R_FUN) and the parametricity. \[\square\]

**Lemma 6.10.** If $\Gamma \vdash \{\forall \alpha . T_1 \Rightarrow \forall \alpha . T_2\} : (\forall \alpha . T_1) \to (\forall \alpha . T_2)$, then $\Gamma \vdash \forall \alpha . T_1 \Rightarrow \forall \alpha . T_2 \ell \ell \forall \alpha . T_1 \Rightarrow \forall \alpha . T_2 \ell \ell (x \alpha) : (\forall \alpha . T_1) \to (\forall \alpha . T_2)$.

**Proof.** By following (E_RED)/(R_FORALL) and the parametricity. \[\square\]
Lemma 6.11. If $\Gamma \vdash \langle T_1 \Rightarrow \{x:T_2 \mid e_2\} \rangle^\ell : T_1 \rightarrow \{x:T_2 \mid e_2\}$, then $\Gamma \vdash \langle T_1 \Rightarrow \{x:T_2 \mid e_2\} \rangle^\ell \simeq \lambda y: T_1. \langle \{x:T_2 \mid e_2\}, \langle T_1 \Rightarrow T_2 \rangle^\ell \rangle^\ell : T_1 \rightarrow \{x:T_2 \mid e_2\}$.

Proof. By following (E\_RED)/(R\_PRECHECK), the parametricity, and the fact that, if $\langle T_1 \Rightarrow T_2 \rangle^\ell v \rightarrow^* v'$, then $\langle T_1 \Rightarrow \{x:T_2 \mid e\} \rangle^\ell v \rightarrow^* \langle \{x:T_2 \mid e\}, e[v''/x] \rangle^\ell$.

Since $F_H$ allows waiting checks to be open, this decomposition is type-preserving.

We can show that $\langle \{x:T_1 \mid e_1\} \Rightarrow T_2 \rangle^\ell$ is logically related to $\langle T_1 \Rightarrow T_2 \rangle^\ell$, but it does not preserves the type, which makes further optimization based on contextual equivalence impossible; note that the transitivity of the logical relation requires the index types to be the same (see Lemma 5.34). Instead, we show that, if $\langle T_1 \Rightarrow T_2 \rangle^\ell$ is logically related to term $e$, then $\langle \{x:T_1 \mid e_1\} \Rightarrow T_2 \rangle^\ell$ is also logically related to $e$. In this formulation, we can relate $\langle \{x:T_1 \mid e_1\} \Rightarrow T_2 \rangle^\ell$ to fully optimized term $e$.

Lemma 6.12. If $\Gamma \vdash \langle \{x:T_1 \mid e_1\} \Rightarrow T_2 \rangle^\ell : \{x:T_1 \mid e_1\} \rightarrow T_2$ and $\Gamma \vdash \langle T_1 \Rightarrow T_2 \rangle^\ell \simeq e : T_1 \rightarrow T_2$, then $\Gamma \vdash \langle \{x:T_1 \mid e_1\} \Rightarrow T_2 \rangle^\ell \simeq e : \{x:T_1 \mid e_1\} \rightarrow T_2$.

Proof. Let $\Gamma \vdash \theta ; \delta$. It suffices to show that

$$\theta_1(\delta_1(\langle \{x:T_1 \mid e_1\} \Rightarrow T_2 \rangle^\ell)) \simeq_\theta \theta_2(\delta_2(e)) : \{x:T_1 \mid e_1\} \rightarrow T_2; \theta; \delta.$$ 

Since $\Gamma \vdash \langle T_1 \Rightarrow T_2 \rangle^\ell \simeq e : T_1 \rightarrow T_2$, there exists some $v$ such that $\theta_2(\delta_2(e)) \rightarrow^* v$ and $\theta_1(\delta_1(\langle T_1 \Rightarrow T_2 \rangle^\ell)) \simeq_\theta v : T_1 \rightarrow T_2; \theta; \delta$. Thus, it suffices to show that, for any $v_1$ and $v_2$ such that $v_1 \simeq_\theta v_2 : \{x:T_1 \mid e_1\}; \theta; \delta$,

$$\theta_1(\delta_1(\langle \{x:T_1 \mid e_1\} \Rightarrow T_2 \rangle^\ell)) v_1 \simeq_\theta v_2 : T_2; \theta; \delta.$$

Since $\theta_1(\delta_1(\langle \{x:T_1 \mid e_1\} \Rightarrow T_2 \rangle^\ell)) v_1 \rightarrow \theta_1(\delta_1(\langle T_1 \Rightarrow T_2 \rangle^\ell)) v_1$ by (E\_RED)/(R\_FORGET), it suffices to show that

$$\theta_1(\delta_1(\langle T_1 \Rightarrow T_2 \rangle^\ell)) v_1 \simeq_\theta v_2 : T_2; \theta; \delta.$$

Since $v_1 \simeq_\theta v_2 : \{x:T_1 \mid e_1\}; \theta; \delta$, we have $v_1 \simeq_\theta v_2 : T_1; \theta; \delta$. Since $\theta_1(\delta_1(\langle T_1 \Rightarrow T_2 \rangle^\ell)) \simeq_\theta v : T_1 \rightarrow T_2; \theta; \delta$, we finish by definition.

Finally, we show that reflexive casts are redundant.\(^8\)

Lemma 6.13. If $\Gamma \vdash T$, then $\Gamma \vdash \langle T \Rightarrow T \rangle^\ell \simeq \lambda x:T . x : T \rightarrow T$.

Proof. By the parametricity (Theorem 5.28) and Lemma 5.8.

As a byproduct of the cast decomposition, it turns out that our fussy semantics can simulate Belo et al.’s sloppy semantics. The sloppy semantics, as shown at the end of Section 2.2, eliminates reflexive casts immediately and checks only the outermost refinement if others have been ensured already. It is found that the former is simulated from Lemma 6.13 and the second from combination of Lemmas 6.11 and 6.13. As a result, the type system of $F_H$ turns out to be sound also for the sloppy semantics despite that the cotermnation (Lemma 2.6), a key property for the type soundness, does not hold under the sloppy semantics [35].

\(^8\)We believe that this is derived from the upcast elimination, but showing that subtyping is reflexive is not trivial due to substitution on the subtype side in (S\_FUN).
7. Related Work

7.1. Simply-typed Manifest Contracts. Flanagan [13] introduced a simply typed manifest contract calculus $\lambda^H$ equipped with a subsumption rule for subtyping. While the subsumption rule allows us to eliminate upcasts, its naive introduction results in an occurrence of well typedness at a negative position in the definition of the type system, especially, in the implication judgment for refinements; it is unclear whether the type system with such a negative occurrence is well defined.

To avoid the negative occurrence problem due to the subsumption rule while keeping that rule, Knowles and Flanagan [19] designed another simply typed manifest contract calculus where the implication judgment refers to denotations of types instead of well-typed values. They gave a denotation of each type as a set of terms in the simply typed lambda calculus and defined a manifest contract calculus equipped with a well-defined type system using the denotations. Flanagan and Knowles [13, 19] also developed a compilation algorithm that transforms possibly ill-typed programs to well-typed ones by inserting casts everywhere a required type is not a supertype of an actual type. The compilation result depends on an external prover that judges implication between refinements: the more powerful the prover is, the less upcasts are inserted. Although how many upcasts are inserted depends on the prover, what prover is used does not have an influence on the final results of programs because upcasts should behave as identity functions. To substantiate this idea, Knowles and Flanagan [19] proved that an upcast is contextually equivalent to an identity function via a logical relation.

Apart from parametric polymorphism, a major difference between Knowles and Flanagan [19] and our work is the treatment of the subsumption for subtyping, which has a great influence on the metatheory of manifest contract calculi. Knowles and Flanagan allow for the subsumption in the definition of their calculus. While their type system with the subsumption rule makes it possible that an upcast and an identity function have the same type, they need some device to ensure that the type system is well defined; in fact, their type system is defined based on semantic typing and semantic subtyping. By contrast, following Belo et al. [4], we consider subtyping after defining $F_H$. Since a type system defined in this “post facto” approach does not refer to the implication judgment, it is well defined naturally. As a result, we can discuss the metatheory, such as the subject reduction, of our calculus without semantic typing and semantic subtyping. However, in such a type system, an upcast and an identity function may not have the same type. To relate two terms of different types, we introduce semityped contextual equivalence. Another difference is that, while Knowles and Flanagan [19] show the upcast elimination only for cases that upcasts are closed,9 we deal with open upcasts as well.

Ou et al. [28] studied interoperability of certified, dependently-typed parts and un-certified, simply-typed ones. As in manifest contracts, coercion of simply-typed values to dependently-typed ones is achieved by run-time checking. Their dependent type system supports refinement types where refinements have to be pure (i.e., they consist of only variables, constants, and primitive operations with pure arguments), a subsumption rule for subtyping, and a typing rule for selfification, which inspires the contract reasoning in

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9Corollary 13 in Knowles and Flanagan [19] states that logically related, open terms $e_1$ and $e_2$ are contextually equivalent, but their proof shows that result terms of capture-avoiding substitution of $e_1$ and $e_2$ for a variable in any context are observationally equal; this proof is valid only if $e_1$ and $e_2$ are closed.
Section 6.2. Unlike the other work on manifest contracts [13, 19, 4, 14, 35, 34], they did not address elimination of run-time coercion.

7.2. Polymorphic Manifest Contracts. Belo et al. [4] studied parametric polymorphism in manifest contracts. In particular, they introduced a polymorphic manifest contract calculus, developed a logical relation, and showed the parametricity and the upcast elimination; details are described in Greenberg’s dissertation [14]. The semantics of their calculus is sloppy in that refinements that have been ensured already are not checked at run time. For example, a reflexive cast returns a given argument immediately because the argument should been typed at the source type of the cast and satisfy the refinements in the target type (note that the source and target types of a reflexive cast are the same). This sloppiness is important in their proof of the parametricity, especially, to show that polymorphic cast \((\alpha \Rightarrow \alpha)^\ell\) is logically related to itself. However, it turns out that their sloppy semantics does not satisfy the cotermination, a key property for both the type soundness and the parametricity [35].

Sekiyama et al. [35] resolved the problem in the sloppy semantics by equipping casts with delayed substitution, which makes it possible to show the cotermination even under sloppy semantics. Furthermore, they also proved the type soundness and the parametricity in the cast semantics with delayed substitution, while leaving proving the upcast elimination open. Although their delayed substitution works well in the sloppy semantics, it makes the metatheory of a manifest contract calculus, especially, the definition of substitution, complicated.

We define a polymorphic manifest contract calculus with fussy cast semantics, where all refinements to be satisfied are checked even if they have been ensured already. The fussy cast semantics, which is adopted also by the simply-typed manifest contract calculus [13, 19] and a manifest contract calculus for algebraic data types [36] and mutable states [34], uses usual substitution and simplifies the metatheory of manifest contract calculi. Our logical relation for the fussy cast semantics requires interpretations of type variables to be closed under reduction of applications of reflexive casts because in the fussy semantics reflexive casts may produce wrappers of given arguments. Fortunately, we can construct such an interpretation from any binary relation on closed values easily, because (well-typed) reflexive casts always succeed. We furthermore introduce semityped contextual equivalence, show the soundness and the completeness of the logical relation with respect to it, and prove correctness of reasoning techniques including the upcast elimination.

The SAGE language [16] supports key features in polymorphic manifest contracts—general refinements (i.e., refining refinement types), casts, subtyping, and parametric polymorphism—as well as recursive functions, recursive types, the dynamic type, and the Type:Type discipline, but the parametricity and the upcast elimination for SAGE have not been investigated. In particular, parametricity for languages equipped with both refinement types and the dynamic type is left open.

7.3. Gradual Typing. Gradual typing [37] is a methodology to achieve a full spectrum from dynamically typed programs to statically typed ones. A gradually typed language is considered to be an extension of a static type system with the dynamic type (or called the unknown type) and it deals with values of the dynamic type as ones of any other type and vice versa. Ahmed et al. [2] and, more recently, Igarashi et al. [17] study gradual typing with parametric polymorphism. To ensure parametricity, polymorphic gradual typing has
to prevent that ones investigate what type a type variable is instantiated with at run time. Ahmed et al. and Igarashi et al. achieved it with help of type bindings, which are similar to delayed substitution in Sekiyama et al. [35], 10 inspired by a parametric multi-language system by Matthews and Ahmed [24]. Ahmed et al. [3] actually proved the parametricity of the polymorphic gradual typing with type bindings. Adding the dynamic type to polymorphic manifest contracts is an interesting future direction.

Gradual typing allows checks of a part of types to be deferred to run-time. Lehmann and Taurer [22] apply this idea to refinement checking. They extended refinements with the unknown refinement “?”, which means that values satisfying this refinement may have some additional information but it is unknown statically. In the spirit of gradual typing, their system defers refinement checking with the unknown refinement to run-time, while checking without the unknown refinement is performed completely statically. In other words, the unknown refinement works as a marker that indicates refinements to be possibly checked at run time. Their gradual refinement type system is similar to (the simply typed) manifest contracts, but in their system the dynamic semantics depends on the subtyping whereas, conversely, in manifest contracts the subtyping refers to the dynamic semantics. In their work, casts just check that one type is a (gradual) subtype of the other using the subtyping. Hence, upcasts behave as identity functions naturally and upcast elimination is less meaningful than in manifest contracts. Instead, they showed that their calculus satisfies key properties in gradual typing.

7.4. Parametricity with Run-Time Analysis. Neis et al. [26] proved that a language with run-time type analysis can be parametric by generating fresh type names dynamically. Their language allows for run-time investigation of what types are substituted for type variables. By contrast, in $F_H$ type variables are compatible with (possibly refined) themselves and the run-time analysis on type variables is not allowed.

7.5. Program Equivalence in Dependent Type Systems. While type conversion in manifest contracts is performed explicitly by casts, there are many dependent type systems where type conversion is implicit. In such a system, term equivalence plays an important role to judge whether a required type matches with an actual type. To investigate an influence of term equivalence on dependent type checking, Jia et al. [18] equipped a dependent type system with various instances of equivalence. In particular, they introduced untyped contextual equivalence as an instance. Since the dependent type system rests on an instance of term equivalence, if their contextual equivalence has been typed, the same issue as in Flanagan [13] would happen, as discussed in Section 7.1. Although we also use contextual equivalence for type conversion, our contextual equivalence can refer to the type system without such an issue since it is given after defining the calculus.

8. Conclusion

This paper has introduced semityped contextual equivalence, which relates a well-typed term to a contextually equivalent, possibly ill-typed term, and formulated the upcast elimination in a manifest contract calculus without subtyping. We have also developed a logical relation for a polymorphic manifest contract calculus with fussy cast semantics and show

10Precisely, delayed substitution comes from type binding.
that it is sound with respect to semityped contextual equivalence and complete for well-typed terms. We have applied the logical relation to show the upcast elimination and correctness of the selfification and the cast decomposition. We are interested in extending the logical relation to step-indexed logical relations [1], which are used broadly for languages with recursive types and mutable references, and studying bisimulation-based reasoning for manifest contracts.

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