Thermal behavior, entanglement entropy and parton distributions

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The apparent thermalization of the particles produced in hadronic collisions can be obtained by quantum entanglement of the partons of the initial state once a fast hard collision is produced. The scale of the hard collision is related to the thermal temperature. As the probability distribution of these events is of the form $np(n)$, as a consequence, the von Neumann entropy is larger than in the minimum bias case. The leading contribution to this entropy comes from the logarithm of the number of partons $n$, all with equal probability, making maximal the entropy. In addition there is another contribution related to the width of the parton multiplicity. Asymptotically, the entanglement entropy becomes the logarithm of $\sqrt{n}$, indicating that the number of microstates changes with energy from $n$ to $\sqrt{n}$.

I. INTRODUCTION

Recently, it has been emphasized the importance of the entanglement of the parton wave function of the initial state [1–7], showing that the thermalization of the particles produced in collisions of small systems objects can be achieved by quantum entanglement of the partons of the initial state. The apparent thermalization in high energy collisions is achieved during the rapid quench induced by the hard collision induced by the collision due to the high degree of entanglement inside the wave functions of the initial protons. In this way, we expect that the hard scale $T_h \sim p_t$ is related to the thermal component. The thermal component of charged hadron transverse momentum distributions in $pp$ collisions at $\sqrt{s} = 13$ TeV can be parameterized as [8–10]

$$\frac{1}{N_{ev}} \frac{1}{2\pi p_t} \frac{d^2N_{ev}}{d\eta dp_t} = A_{th} \exp \left( -\frac{m_t}{T_{th}} \right),$$

(1)
and the hard scattering as

$$\frac{1}{N_{ev}} \frac{1}{2\pi p_t} \frac{d^2 N_{ev}}{d\eta dp_t} = A_h \frac{1}{\left(1 + \frac{m^2}{\pi T_{th}^2}\right)^n},$$

(2)

where $T_{th}$ is the effective temperature and $T_h$ can be considered as a hard temperature which settles the hard scale. The index $n$, $T_h$ and $T_{th}$ were determined from the fit to the experimental data. One finds

$$\frac{T_h}{T_{th}} \simeq 4.2,$$

(3)

with $T_{th} \simeq 0.17$ GeV at $\sqrt{s} = 13$ TeV. The ratio between the hard and soft scales at Eq. (3) approximately holds for any centrality and energy in $pp$ collisions as well as PbPb collisions even if $T_h$ and $T_{th}$ have different values for any centrality, energy and type of collision [5]. This relation between scales have been also studied in the Higgs boson transverse momentum distribution, in the case of Higgs boson decay to $\gamma\gamma$ and in the case of Higgs decay to four leptons [2]. In these two cases the hard scale is around twenty times larger than in the previous cases but the ratio of (3) still holds.

Concerning the index $n$, it was found that it depends on the energy, centrality and colliding objects, decreasing with multiplicity for not very high energy density and increasing with multiplicity in the case of PbPb collisions at $\sqrt{s} = 2.76$ TeV [5]. This behavior and the ratio between $T_h$ and $T_{th}$ can be naturally explained as a consequence of the clustering of color sources (strings) model [11–13]. In this approach, $n$ is the inverse of the normalized fluctuations of the temperature $T_h$. As the multiplicity increases, the number of different clusters increases and thus the $T_h$ fluctuations (to each cluster corresponds a local temperature $T_h$). In this approach the ratio between temperatures has a defined value $T_h/T_{th} = \pi/\sqrt{2}$.

Concerning the entropy, the pure initial parton state $|\psi\rangle$ with density matrix $\hat{\rho} = |\psi\rangle\langle\psi|$ have zero von Neumann entropy $S = -\text{Tr}(\hat{\rho} \ln \hat{\rho}) = 0$. If the partons were truly free and thus incoherent, as it is assumed in the infinite momentum frame, they would have a non-zero entropy. In a hard collision, characterized by the transverse momentum $p_t$, is probed only a part of the proton wave function, localized in a region $H$ within a cone of radius $\sim 1/p_t$ and length $l \sim 1/m x$, where $m$ is the proton mass and $x$ is the fraction of energy carried by the hard parton. If we sum over the complementary unobserved region $S$ we can calculate the reduced density matrix

$$\rho_H = \text{Tr}_S \rho,$$

(4)
where

$$\rho = |\psi_{HS}\rangle\langle\psi_{HS}|,$$  \hspace{1cm} (5)

and the wave function $|\psi_{HS}\rangle$ is the superposition of a suitable chosen orthonormal set of states $|\psi^H_n\rangle$ and $|\psi^S_n\rangle$ localized in the domains $H$ and $S$,

$$|\psi_{HS}\rangle = \sum_n \alpha_n |\psi^H_n\rangle |\psi^S_n\rangle.$$  \hspace{1cm} (6)

As

$$\rho_H \equiv \text{Tr}_S \rho = \sum_n \langle\psi^S_n|\psi_{HS}\rangle \langle\psi_{HS}|\psi^S_n = \sum_n |\alpha_n|^2 |\psi^H_n\rangle \langle\psi^H_n|,$$  \hspace{1cm} (7)

then the von Neumann entropy of this state is given by

$$S = -\text{Tr}(\rho_H \log \rho_H) = -\sum_n p_n \log p_n,$$  \hspace{1cm} (8)

with $p_n \equiv |\alpha_n|^2$.

The onset $\tau$ of the hard interaction is given by the hard scale $\tau \sim 1/p_t$, since $\tau$ is small the quench creates a highly excited multi-particle state. The produced particles have a thermalized spectrum with a temperature $T_{\text{th}} \simeq 1/2\pi\tau \simeq p_t/2\pi$. In QCD at high energies, using the Balitsky-Kovchegov (B-K) equation, it has been obtained $[2]$

$$S = \Delta \log s \simeq \Delta \log \left(\frac{l}{\epsilon}\right) = \Delta \log \left(\frac{1}{x}\right),$$  \hspace{1cm} (9)

with $\Delta = \bar{\alpha}_s \log(r^2Q^2_S)$, where $r$ is the size of the dipole, $Q_S$ the saturation momentum and $\epsilon$ the Compton wavelength of the proton. Expression $[9]$ is very similar to the result for the entanglement entropy in $(1+1)$ conformal field theory (CFT) $[14] [15]$

$$S = \frac{c}{3} \log \left(\frac{l}{\epsilon}\right),$$  \hspace{1cm} (10)

being $c$ the central charge of the CFT, which counts the number of degrees of freedom.

In this paper we study the behavior of the entanglement entropy with the hardness of the process, and the role played by the temperature fluctuations. First, we recover the result for the relation between the conditioned probability for having at last one hard collision and the full probability. We will show, using renormalization group arguments, that the conditioned probability must be a gamma distribution. Finally, we compute the entanglement entropy.
The leading term is the logarithm of the multiplicity similar to the result expressed by Eq. (1). In addition to that there is a second term which depends on the inverse of the normalized fluctuations. As the scale $p_t$ of the collision increases, this second term decreases because the size of the hard region is smaller and thus the possibility of fluctuations of the hard partons of the wave function. It is worth to compare the entanglement entropy, $S^c$, corresponding to the probability of having at least one hard parton, with the entropy $S$ corresponding to have no constraint. The difference $S^c - S$ decreases with the scale of hardness, vanishing asymptotically. $S^c - \log \langle n \rangle$ as a function of the energy density presents a maximum, which corresponds to a turnover of the behavior of the index $n$ of Eq. (2) with multiplicity. This behavior is naturally explained in the clustering of color sources model.

The plan of the paper is as follows. In the next section we obtain the conditioned probability of having a hard collision. Using the renormalization group and the scale invariance as an argument we obtain the gamma distribution as the required probability distribution. In section III we study the entanglement con Neumann entropy for hard events comparing with the non-constraint entropy, discussing the differences in connection with the fluctuations on the number of hard partons and with the clustering of color sources. Finally, in section IV the conclusions are presented.

II. CONDITIONED PROBABILITY

Let us consider the probability $p_n$ of having $n$ partons in a given collision. It has been shown [16–20] that the conditioned probability $p^c_n$ of having $n$ partons and at least one giving rise to a hard collision is

$$p^c_n = \frac{n}{\langle n \rangle} p_n.$$  

This equation has been obtained not only for hard events but for events of a type, denoted by $c$, in which for a result to be considered of the type $c$ is enough to have a single $c$ event in at least one of the elementary collisions. Examples of this kind are events without a rapidity gap (non-diffractive events), hard events, annihilation events in $\bar{p}p$ collisions, events with at least one jet and $W^\pm,Z^0$ events. Let $N(n)$ be the number of events with $n$ elementary collisions observed in an hadronic or nuclear collision, we have

$$N(n) \equiv \sum_{i=0}^{n} \binom{n}{i} \alpha^i_c (1 - \alpha_c)^{n-i} N(n),$$  

(12)
where $\alpha_c$ is the probability of having an event $c$ in an elementary collision ($0 < \alpha_c < 1$). If $\alpha_c$ is small equation (12) becomes

$$N(n) = \alpha_c n N(n) + (1 - \alpha_c) n N(n),$$

(13)

where from the definition of a type $c$ event the first term of (13) is the number of events $N_c(n)$ where a $c$ occurs,

$$N_c(n) = \alpha_c n N(n).$$

(14)

If $N$ is the total number of events we have

$$\sum_n N(n) = N, \quad \sum_n n^k N(n) \equiv \langle n^k \rangle,$$

(15)

and, for the total number of events with $c$ occurring

$$\sum_n \alpha_c n N(n) = \alpha_c \langle n \rangle N.$$

(16)

This implies, for the probability distribution of having a $c$ event in $n$ collisions

$$p_c(n) = \frac{\alpha_c n N(n)}{\alpha_c \langle n \rangle N} = \frac{n}{\langle n \rangle} p(n),$$

(17)

which is of the form of Eq.(11). In this equation $n$ is the number of elementary collisions (parton-parton or nucleus-nucleus, depending on the case studied) but Eq.(17) has been applied to the multiplicity particle probability distributions, being $p(n)$ the minimum bias multiplicity distribution. Indeed, the equation (11) was checked in the case of production of $W^\pm, Z^0$ with data of CDF collaboration at Fermilab [17], for the production of jet events with UA1 collaboration data at SPS [17], for the production of Drell-Yan pairs in S-U collisions with NA38 collaboration data [18] and for the annihilation in $\bar{p}p$ collisions [17]. In all cases a good agreement with the experimental data was obtained. Notice that in Eq.(11) the right hand side is independent of $c$ and only its shape is determined by the requirement of being of the type $c$. In terms of cross sections the $c$ events are self-shadowed and their cross section can be written as a function of only the elementary cross sections of a $c$-event [21] [22].

This selection procedure of the events satisfying certain $c$ criteria can be repeatedly applied for subsequent $c$ conditions. For instance from these events with at least one particle with transverse momentum larger than $p_{t,1}$ one can further select events with at least
one particle with transverse momentum larger than $p_{t,2}$, $p_{t,2} > p_{t,1}$, and so on (there are cases in which this multiple selection procedure can not be applied more than once, like non-diffractive or annihilation events). The corresponding probability distributions to the repeated selection satisfy

$$p(n) \rightarrow \frac{n}{\langle n \rangle} p(n) \rightarrow \frac{n^2}{\langle n^2 \rangle} p(n) \rightarrow \cdots \frac{n^k}{\langle n^k \rangle} p(n).$$ \hspace{1cm} (18)

Notice that

$$\langle n \rangle_c = \frac{\langle n^2 \rangle}{\langle n \rangle},$$ \hspace{1cm} (19)

and

$$\langle n \rangle_c - \langle n \rangle = \frac{\langle n^2 \rangle - \langle n \rangle^2}{\langle n \rangle} \geq 0.$$ \hspace{1cm} (20)

Transformations of the kind of Eq.(18) were studied long time ago by Jona-Lasinio in connection with the renormalization group in probability theory \cite{23}, showing that the only stable probability distribution under such transformations are the generalized gamma distributions. The simplest one is the gamma distribution. This transformation has also been studied in connection with self-similarity condition and the KNO scaling \cite{20},

$$\langle n \rangle p_n = \psi \left( \frac{n}{\langle n \rangle} \right) = \psi(z).$$ \hspace{1cm} (21)

For the gamma distribution

$$\psi(z) = \frac{\beta^k}{\Gamma(k)} z^{k-1} e^{-\beta z}, \hspace{0.5cm} k > 1,$$ \hspace{1cm} (22)

we have the normalization condition

$$1 \equiv \sum_n p_n = \sum_n \frac{1}{\langle n \rangle} \psi \left( \frac{n}{\langle n \rangle} \right) = \int dz \psi(z) = 1,$$ \hspace{1cm} (23)

and

$$1 \equiv \sum_n \frac{n}{\langle n \rangle} p_n = \int dz z \psi(z),$$ \hspace{1cm} (24)

which forces $\beta = k$. We will use the gamma distribution in our evaluations.

The gamma distribution appears in different and related frameworks. It is the stationary solution of the Fokker-Planck equation associated to the Langevin equation formulated for
the temperature time evolution under a multiplicative white noise produced by the fast quench of a hard collision at a given $p_t$ scale [5, 6, 24, 25].

It appears as well as the size distribution of the clusters of strings formed in a $pp$, $pA$ or $AA$ collision [11–13, 26, 27]. At not very high energy, only single strings are stretched between the partons of the colliding objects. These strings can be seen as discs of radius 0.2 fm in the transverse plane of the scattering. As the energy or centrality of the collision increases, the number of strings increases and they start to overlap forming clusters with different number of strings. As the color field inside the clusters is larger, these clusters fragment into particles according to this larger color field and thus larger tensions. Above all critical string density, a large cluster is formed crossing the collision area. Arguments based on the renormalization of the color field of the clusters of strings indicate that the cluster size distribution should be a gamma distribution. By making a convolution of the gamma distribution with the function $\exp(-x p_t^2)$, which corresponds to the fragmentation of the cluster of size $x$, Eq. (2) is obtained for the transverse momentum distribution. In the same way, making the convolution of the gamma distribution with a Poisson distribution, which corresponds to the multiplicity distribution for the fragmentation of a cluster of size $x$ (the size of the cluster controls the mean value of the Poisson distribution) a negative binomial distribution for the total multiplicity distribution is obtained. As the inverse of the parameter $k$ controls the normalized width of the gamma distribution

$$\frac{1}{k} = \frac{\langle z^2 \rangle - \langle z \rangle^2}{\langle z \rangle^2}, \quad (25)$$

$k$ decreases with the string density as the number of clusters of different strings grows and the fluctuations increase. Once the large cluster is obtained, fluctuations decrease and thus $k$ increases [11, 13].

The origin of the non-extensive thermodynamics related to the equation (2) could be the fractal structure of the thermodynamical system. In reference [28] it is shown that such systems present temperature fluctuations following a gamma distribution. The repetitive fractal structure has to do with the scale transformations represented by equation (18).

In terms of the reduced matrix density (7), the transformation induced by the repeated selection $p_{t,1} < p_{t,2} < \cdots < p_{t,j}$ translates into a sum over each time a larger region of soft partons, modifying the probability $p_n = |\alpha_n|^2$ in the way prescribed by the chain of equation (18).
III. ENTANGLEMENT ENTROPY

We will use (22) to evaluate the entanglement entropy. The von Neumann entropy for minimum bias events is

\[ S = - \sum_n p_n \log p_n = - \sum_n \frac{1}{\langle n \rangle} \psi \left( \frac{n}{\langle n \rangle} \right) \log \left( \frac{1}{\langle n \rangle} \psi \left( \frac{n}{\langle n \rangle} \right) \right) = \log \langle n \rangle - \int_0^\infty dz \psi(z) \log(\psi(z)), \]

and the von Neumann entropy for type \( c \) events, containing at least one hard collision,

\[ S^c = - \sum_n p^c_n \log p^c_n = - \sum_n \frac{n p_n}{\langle n \rangle} \log \left( \frac{n p_n}{\langle n \rangle} \right) = - \sum_n \frac{n}{\langle n \rangle^2} \psi(z) \log \left( \frac{n \psi(z)}{\langle n \rangle^2} \right) \]
\[ = - \int_0^\infty dz z \psi(z) \log \left( \frac{z \psi(z)}{\langle n \rangle} \right) = \log \langle n \rangle - \int_0^\infty dz z \psi(z) \log(z) \psi(z). \]

Taking for \( \psi(z) \) the gamma distribution, we obtain

\[ S = \log \langle n \rangle - \log k + k + \log \Gamma(k) + \frac{1 - k}{\Gamma(k)} \frac{\partial_k \Gamma(k)}{\Gamma(k)} \approx \log \langle n \rangle + \frac{1}{2} \left[ \frac{k - 1}{k} + \log \left( \frac{2\pi}{k} \right) \right] \]
\[ \rightarrow \log \frac{\langle n \rangle}{\sqrt{k}} = \log \langle n \rangle^{1/2}, \]

and

\[ S^c = \log \langle n \rangle + k + \log \Gamma(k) - \frac{k}{\Gamma(k)} \frac{\partial_k \Gamma(k)}{\Gamma(k)} \approx \log \langle n \rangle + \frac{1}{2} \left[ 1 + \log \left( \frac{2\pi}{k} \right) \right] \]
\[ \rightarrow \log \frac{\langle n \rangle}{\sqrt{k}} = \log \langle n \rangle^{1/2}, \]

where the last equality of the above relations hold for large \( k \) and \( \Gamma(k) \) is the gamma function. We observe that the leading term \( \log \langle n \rangle \), as \( \langle n \rangle \approx s^\Delta \) it is similar to the one obtained using the B-K equation. The difference between both entropies reads

\[ S^c - S = \log k - \frac{1}{\Gamma(k)} \frac{\partial_k \Gamma(k)}{\Gamma(k)} \approx \frac{1}{2k} \]

In Fig. 1 \( S - \log \langle n \rangle \) and \( S^c - \log \langle n \rangle \) are shown as a function of \( k \). As \( k > 1 \), \( S \) and \( S^c \) decrease with \( k \) and at larger values \( S^c \) approaches \( S \). As \( k > 1 \), \( S - \log \langle n \rangle \) and \( S^c - \log \langle n \rangle \) are decreasing functions of \( k \) in all the allowed domain of \( k \). These functions, according to Eqs. (28) and (29), become negative at very high \( k \). The leading term of \( S \) and \( S^c \) is \( \log \langle n \rangle \), meaning that the \( n \) partons, i.e. the \( n \) microstates of the system, are equally probable and thus the entropy is maximal. In addition to this contribution, there is one which depends only
on $k$, i.e. the inverse of the normalized fluctuations on the number of partons, Eq. (29). This contribution is a positive decreasing function of $k$ in a very broad range, becoming negative at very high $k$. In the infinite limit, the gamma function becomes the normal/Gaussian distribution and both $S$ and $S^c$ behave like $\log(n/\sqrt{k}) = \log(n^{1/2})$. This result means that the number of microstates is not $n$ any more but $\sqrt{n}$. A saturation effect occurs and the growth of microstates is suppressed as the collision energy or the centrality increases. This saturation is explained in models like the color glass condensate or the clustering of color sources. In this last model, the number of independent color sources, strings, $n$, formed from the initial partons of the colliding objects, is reduced at high energies because the number of effective independent color sources is proportional to $\sqrt{n}$ in such a way that Eq.(29), involving logarithms, is satisfied [30]. In the limit of high energy in the glasma picture of the CGC, the number of color flux tubes is also $\sqrt{n}$.

It could be thought that as $\langle n \rangle_c \geq \langle n \rangle$ the leading term of the entanglement entropy $S^c$
is larger than the corresponding to $S$. Indeed, instead of Eq.\((27)\) we could have written

$$S^c = \log\langle n \rangle_c - \int dz \psi_c(z) \log (\psi_c(z)),$$

(31)

with

$$\psi_c(z) \equiv \frac{1}{\langle n \rangle_c} p^c_n = \frac{1}{\langle n \rangle_c} \frac{n}{\langle n \rangle_c} p_n = \frac{1}{\langle n \rangle_c} \psi(z).$$

(32)

From Eqs.\((25)\) and \((19)\) we can write

$$\log\langle n \rangle_c = \log\langle n \rangle + \log \left(1 + \frac{1}{k}\right),$$

(33)

so asymptotically as $k \to \infty \langle n_c \rangle = \langle n \rangle$.

The differences between $S$ and $S^c$ are small and asymptotically tend to zero as it is shown in Fig.\((2)\). The dependence of $S - \log\langle n \rangle$ or $S^c - \log\langle n \rangle$ on the energy or on the impact parameter (centrality) is very interesting in the clustering of color sources approach due to the previously described dependence of $k$ on the string density $\xi$. At low density, $k$ decreases up to a critical string density $\xi_c$. Above this critical density $\xi > \xi_c$, $k$ increases. In this way, both $S - \log\langle n \rangle$ and $S^c - \log\langle n \rangle$ increase with $\xi$ up to the critical density $\xi_c$, and decrease for...
larger $\xi$. This decrease of $S - \log\langle n \rangle$ or $S^c - \log\langle n \rangle$ with energy or centrality is small compared with the grow of $\log\langle n \rangle$ in such a way that $S$ and $S^c$ are always growing. The exact value of $k$, which corresponds to the $\xi_c$ marking the turnover, depends on the observed rapidity range, the $p_t$ acceptance, and the profile functions of the projectile and target. We know from the data that $k$ decreases with energy and centrality in pp collisions and it increases for AuAu and PbPb collisions. The turnover of $k$ could be at very high multiplicity in pp collisions at $\sqrt{s} = 13$ TeV. Notice that $S$ or $S^c$ does not present a maximum at $\xi = \xi_c$ but a change in the dependence of $S$ or $S^c$ on $\xi$.

IV. CONCLUSIONS

Based on the results on the conditioned probability for having at least one hard collision in terms of the minimum bias probability, we show that this probability must be the gamma function. This function coincides with the stationary solution of a Fokker-Planck equation corresponding to a Langevin equation for the time evolution of the temperature under a multiplicative white noise produced by the fast quenching of a hard parton. Once the parton distribution is obtained we compute the entanglement entropy (von Neumann). In agreement with previous results, the leading term is the logarithm of the number of partons, meaning that the $n$ microstates are equally probable and the entropy is maximal. The corrections to the leading term depend only on the inverse of the normalized fluctuations. Asymptotically, the entanglement entropy becomes $\log \sqrt{n}$, meaning that the microstates are not anymore the number $n$ of partons but $\sqrt{n}$ due to saturation effects. We show that in the clustering of color sources approach $S - \log\langle n \rangle$ or $S^c - \log\langle n \rangle$ as a function of the energy or of the centrality should present a maximum corresponding to a critical density of the strings formed in the collision, which occurs when the overlapping strings cross all the collision surface.

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[1] O.K. Baker and D.E. Kharzeev, *Thermal radiation and entanglement in proton-proton collisions at the LHC*, arXiv:1712.04558.

[2] D.E. Kharzeev and E.M. Levin, *Deep inelastic scattering as a probe of entanglement*, Phys. Rev. D95 11 (2017) 114008.

[3] J. Bergers, S. Floerchinger and R. Venugopalan, *Dynamics of entanglement in expanding quantum fields*, JHEP1804 (2018) 145.

[4] J. Bergers, S. Floerchinger and R. Venugopalan, *Thermal excitation spectrum from entanglement in an expanding quantum string*, Phys. Lett. B778 (2018) 442.

[5] X. Feal, C. Pajares and R.A. Vazquez, *Thermal behavior and entanglement in Pb-Pb and p-p collisions*, arXiv:1805.12444.

[6] B. Muller and A. Schäfer, *Why does the thermal model for hadron production in heavy ion collisions work?*, arXiv:1712.03567.

[7] A. Kovner, M. Lublinsky and M. Serino, *Entanglement entropy, entropy production and time evolution in high energy QCD*, arXiv:1806.01089.

[8] A.A. Bylinkin and A.A. Rostovtsev, *Role of quarks in hadroproduction in high energy collisions*, Nucl. Phys. B888 (2014) 65.

[9] A.A. Bylinkin, M.G. Ryskin and A.A. Rostovtsev, *Charged hadron distributions in a two component model*, Nucl. and Part. Proc. 273-275 (2016) 2746.

[10] A.A. Bylinkin, D.E. Kharzeev and A.A. Rostovtsev, *The origin of thermal component in the transverse momentum spectra in high energy hadronic processes*, Int. J. Mod. Phys. E23 (2014) 1450083.

[11] M.A. Braun et al., *De-Confinement and Clustering of Color Sources in Nuclear Collisions*, Phys. Rep. 599 (2015) 1.

[12] N. Armesto, M.A. Braun, E.G. Ferreiro and C. Pajares, *Percolation Approach to Quark-Gluon Plasma and J/Ψ Suppression*, Phys. Rev. Lett. 77 (1996) 3736.

[13] J. Dias de Deus, E.G. Ferreiro, C. Pajares and R. Ugoccioni, *Universality of the transverse momentum distributions in the framework of percolation of strings*, Eur. Phys. J. C40 (2005) 229.
[14] C. Holzhey, F. Larsen and F. Wilczek, *Geometric and Renormalized Entropy in Conformal Field Theory*, Nucl.Phys. B424 (1994) 443.

[15] P. Calabrese and J. Cardy, *Entanglement entropy and quantum field theory: a non-technical introduction*, Int.J.Quant.Inf. 4 (2006) 429.

[16] J. Dias de Deus, C. Pajares and C.A. Salgado, *Moment Analysis, Multiplicity Distributions and Correlations in High Energy Processes: Nucleus-Nucleus Collisions*, Phys. Lett. B407 (1997) 335.

[17] J. Dias de Deus, C. Pajares and C.A. Salgado, *Production Associated to Rare Events in High Energy Hadron-Hadron Collisions*, Phys. Lett. B408 (1997) 417.

[18] J. Dias de Deus, C. Pajares and C.A. Salgado, *Multiplicity and Transverse Energy Distributions Associated to Rare Events in Nucleus-Nucleus Collisions*, Phys. Lett. B409 (1997) 474.

[19] J. Dias de Deus, C. Pajares and C.A. Salgado, *Rare event triggers in hadronic and nuclear collisions*, Phys. Lett. B442 (1998) 395.

[20] M.A. Braun and C. Pajares, *Self-similarity of multiplicity distributions and the KNO scaling*, Phys. Lett. B444 (1998) 435.

[21] R. Blankenbecler et al. *Unusual shadowing effects in particle production off nuclei*, Phys. Lett. B107 (1981) 106.

[22] C. Pajares and A.V. Ramallo, *Parton model description of annihilations on nuclei*, Phys. Lett. B107 (1981) 373.

[23] G. Jona-Lasinio *The renormalization group: a probabilistic view*, Il Nuovo Cimento B 26 (1975) 99.

[24] G. Wilk and Z. Wlodarczyk, *Interpretation of the nonextensivity parameter q in some applications of Tsallis statistics and Lévy distributions*, Phys. Rev. Lett. 84 (2000) 2770.

[25] T.S. Biro and A. Jakovac, *Power-law tails from multiplicative noise*, Phys. Rev. Lett. 94 (2005) 132302.

[26] M.A. Braun and C. Pajares, *Implication of percolation of color strings on multiplicities, correlations and the transverse momentum*, Eur. Phys. J. C16 (2000) 349.

[27] M.A. Braun and C. Pajares, *Transverse Momentum Distributions and Their Forward-Backward Correlations in the Percolating Color String Approach*, Phys. Rev. Lett. 85 (2000) 4864.
[28] A. Deppman, E. Megias, D.P. Menezes and T. Frederico, \textit{Fractal structure and non extensive statistics}, arXiv:1801.01160.

[29] J.F. Grosse-Oetringhaus and K. Reygers, \textit{Charged-particle multiplicity in proton proton collisions}, J. Phys. G37 (2010) 083001.

[30] J. Dias de Deus and C. Pajares, \textit{String percolation and the glasma}, Phys. Lett. B 695 (2011) 211.