Slope filtration on Banach-Colmez spaces

Jérôme Plût

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Abstract

We give a new proof of the “weakly admissible implies admissible” theorem of Colmez and Fontaine describing the semi-stable $p$-adic representations. We study Banach-Colmez spaces, i.e. $p$-adic Banach spaces with the extra data of a $\mathbb{C}_p$-algebra of analytic functions. The “weakly admissible” theorem is then a result of the existence of dimension and height functions on these objects. Furthermore, we show that the subcategory of Banach-Colmez spaces corresponding to crystalline representations is naturally filtered by the positive rationals.

Introduction

The “weakly admissible implies admissible” theorem of Colmez and Fontaine [CF00, Théorème A] gives an equivalence of categories between the semi-stable $p$-adic representations and an explicitly described category of filtered $(\varphi, N)$-modules. There exist at least five proofs of the “weakly admissible theorem”:

- the original proof by Colmez and Fontaine [CF00] uses the “almost surjectivity” of certain formal series with coefficients in $\mathbb{C}_p$;
- Colmez’ proof [Col02], which is a sheafification of the proof of [CF00];
- Berger [Ber08] related the filtered $\varphi$-module and the $(\varphi, \Gamma)$-module attached to a crystalline representation and used Kedlaya’s filtration [Ked04] by the slopes of the Frobenius;
- Kisin [Kis06] constructed a functor from filtered $(\varphi, \Gamma)$-modules to a category of differential modules with a connexion;
- most recently, Genestier and V. Lafforgue [GL12] announced a generalization of the proof by Kisin.

Of the above, our work is most closely related to the proof by Colmez [Col02]: although our proofs are different from his, the category of Banach-Colmez spaces we introduce here is actually equivalent to the category of “Espaces de Banach de dimension finie”.
Following previous work [Plû10], we adopt a geometric point of view and consider spectral varieties, which are topological spaces with the data of a $\mathbb{C}_p$-Banach algebra of analytic functions and an homeomorphism to the spectrum of this algebra. In [Plû10], we gave an analytic interpretation of the “fundamental lemma” [CF00, 2.1]: we defined the category of spectral Banach spaces as the “spectral varieties in $p$-adic Banach spaces”. Finite-dimensional $\mathbb{C}_p$- and $\mathbb{Q}_p$-vector spaces naturally embed into this category, and the category of effective Banach-Colmez spaces is the full subcategory of spectral Banach spaces which are analytic extensions of finite-dimensional $\mathbb{C}_p$-vector spaces by finite-dimensional $\mathbb{Q}_p$-vector spaces. The fundamental lemma [Plû10, Prop. 2.4.2] essentially states that the dimension of the $\mathbb{C}_p$- and $\mathbb{Q}_p$-parts of an effective Banach-Colmez space $E$ are well-defined. We name them respectively the dimension and height of $E$.

We define here the larger category of Banach-Colmez spaces as quotients of Banach-Colmez spaces by finite-dimensional $\mathbb{Q}_p$-vector spaces. Using the fundamental lemma, we show that this category is abelian (Theorem 1.2.6) and that the dimension and height functions naturally extend to the larger category (Corollary 1.2.4).

We then describe a functor from the category of $\varphi$-modules to Banach-Colmez spaces. Let $K$ be a finite extension of $\mathbb{Q}_p$ and $K_0$ be the unramified subfield of $K$. For any $\varphi$-module $D$ with coefficients in $K_0$, we define

$$E(D) = \text{Hom}_{K_0}(D, B^+_{\text{cris}}).$$

We show that $E(D)$ has a natural structure as a Banach-Colmez space (Prop. 2.1.5), of dimension equal to the Newton number of $D$, and height equal to the rank of $D$. Moreover, for any filtration $\text{Fil}$ on $D_K = D \otimes_{K_0} K$, we define

$$M(D, \text{Fil}) = \text{Hom}_K(D_K, B^+_{\text{dR}})/\text{Hom}_{K, \text{Fil}}(D_K, B^+_{\text{dR}}).$$

then $M(D, \text{Fil})$ is a $B^+_{\text{dR}}$-module of length equal to the Hodge number of $D$ and thus has a natural Banach-Colmez structure with dimension $t_H(D)$. Moreover, the canonical map $E(D) \to M(D, \text{Fil})$ is analytic, and the “weakly admissible” theorem is then obtained simply by counting dimensions and heights.

We finally study the essential image of the functor from $\varphi$-modules to Banach-Colmez spaces: it is the full subcategory of oblique Banach-Colmez spaces, whose objects are extension of a finite-length $B^+_{\text{dR}}$-module by a finite-dimensional $\mathbb{Q}_p$-vector space. We give a direct construction of an analogue of Kedlaya’s filtration by the slopes of the Frobenius for certain modules over the Robba ring: namely, we show that oblique Banach-Colmez spaces have a canonical Harder-Narasimhan filtration (Theorem 3.4.10) and that the stable objects of slope $\mu = d/h$ are the Banach-Colmez spaces corresponding to the isocrystal $K_0[\varphi]/(\varphi^h - p^d)$ (Prop. 3.4.8).
1 Banach-Colmez spaces

1.1 Spectral Banach spaces

We remind from [Plû10] that effective spectral Banach spaces are $p$-adic Banach spaces with an analytic structure provided by an algebra of analytic functions.

Let $\mathbf{BS}^+$ be the category of effective spectral Banach spaces, and $\mathbf{C}^{(-1,0)}\mathbf{BS}^+$ be the category of complexes $V \to E$, where $V$ is a finite-dimensional $\mathbb{Q}_p$-vector space, $E$ is an effective spectral Banach space, and the map $V \to E$ is injective. Then the family $\mathbf{Qis}$ of morphisms $f : (V \to E) \to (V' \to E')$ of $\mathbf{C}^{(-1,0)}\mathbf{BS}^+$ such that the sequence

$$0 \to V \to E \oplus V' \to E' \to 0$$

is exact is a left multiplicative system in the sense of [Ver96, II.2.1].

**Definition 1.1.1.** The category $\mathbf{BS}$ of spectral Banach spaces is the localization of the category $\mathbf{C}^{(-1,0)}\mathbf{BS}^+$ relatively to the multiplicative system $\mathbf{Qis}$.

The natural functor from $\mathbf{BS}^+$ to $\mathbf{BS}$, sending the effective spectral Banach space $E$ to the map $(0 \to E)$, is fully faithful.

For any spectral Banach space $X = (V \xrightarrow{d} E)$, let $H^0(X)$ be the $p$-adic Banach space $E/d(V)$. This construction is functorial and defines a faithful functor from the category $\mathbf{BS}$ to that of $p$-adic Banach spaces. Moreover, let $X = (V \xrightarrow{d} E), X' = (V \xrightarrow{d} E')$ be two spectral Banach spaces; then a continuous linear map $f : H^0(X) \to H^0(X')$ is in the image of $H^0$ if, and only if, the graph of $f$, i.e. the fibre product $E \times_{H^0(X')} E'$, is an analytic sub-space of the effective spectral Banach space $E \times E'$.

Thereafter, by abuse of language, we shall identify the category $\mathbf{BS}$ with its essential image by the faithful functor $H^0$, i.e. we see the spectral Banach spaces as $p$-adic Banach spaces. We call analytic the continuous linear maps between spectral Banach spaces that are morphisms of the category $\mathbf{BS}$.

**Definition 1.1.2.** A spectral Banach space $X$ is étale if there exists a representative $V \to E$ of $X$ such that $E$ is étale; it is connected if there exists a representative such that $E$ is connected.

In particular, these definitions are compatible with the similar definitions on the full subcategory of effective spectral Banach spaces. Moreover, if $X$ is étale, then for all its representatives $V \to E$, $E$ is étale; moreover, it is an effective spectral Banach space.

From the connected-étale sequence for effective spectral Banach spaces, we deduce the analogous result for spectral Banach spaces.

**Proposition 1.1.3.** Any analytic morphism from an étale spectral Banach space to a connected one is zero.
Let $X$ be a spectral Banach space. Then $X$ has a largest connected subspace $X^0$, a largest étale quotient $\pi_0(X)$, and the sequence

$$0 \rightarrow X^0 \rightarrow X \rightarrow \pi_0(X) \rightarrow 0$$

is exact. Moreover, this sequence is (non-canonically) split.

### 1.2 The abelian category of Banach-Colmez spaces

**Definition 1.2.1.** The category $\text{BC}$ of Banach-Colmez spaces is the full subcategory of $\text{BS}$ whose objects have one representative of the form $V \rightarrow E$, where $V$ is an étale spectral Banach space, and $E$ is an effective Banach-Colmez space.

If $X$ is a Banach-Colmez space, then for any representative $V \rightarrow E$ of $X$, $E$ is an effective Banach-Colmez space.

Let $X$ be a Banach-Colmez space, and $V_0 \rightarrow E$ be a representative of $X$. Then there exists a presentation of $E$, of the form $0 \rightarrow V_+ \rightarrow E \rightarrow L \rightarrow 0$, where $V_+$ is a finite-dimensional $\mathbb{Q}_p$-vector space, and $L$ is a finite-dimensional $\mathbb{C}_p$-vector space. We then say that the diagram

$$0 \rightarrow V_0 \rightarrow (V_+ \rightarrow E \rightarrow L) \rightarrow X \rightarrow 0$$

is a presentation of $X$. Given such a presentation, one defines the dimension of the presentation as being $\dim_{\mathbb{C}_p} X$, and the height of the presentation as $\dim_{\mathbb{Q}_p} V_+ - \dim_{\mathbb{Q}_p} V_0$.

From the fundamental lemma for effective Banach-Colmez spaces [Plû10, 2.4.2], we deduce the following:

**Proposition 1.2.3.** Let $f : X \rightarrow X'$ be a surjective morphism of Banach-Colmez spaces, where $X$ has a presentation of dimension $d$ and height $h$, and $X'$ has a presentation of dimension $d'$ and height $h'$.

Then $X'' = \text{Ker} f$ is a Banach-Colmez space, and it has a presentation of dimension $d'' = d - d'$ and height $h'' = h - h'$.

**Proof.** The fundamental lemma corresponds to the case where $X$ is effective of dimension one and $X' = \mathbb{C}_p$. From this we deduce the general case as follows.

Let $V_0 \rightarrow (V_+ \rightarrow E \rightarrow L) \rightarrow X$ and $V_0' \rightarrow (V_+^\prime \rightarrow E' \rightarrow L') \rightarrow X'$ be presentations of $X$ and $X'$. By replacing $f : X \rightarrow X'$ by the map $E \times_X E' \rightarrow E'$, we may assume that $X'$ is effective and thus $V_0' = 0$.

Let $L' = \bigoplus L_i'$ be a decomposition of the $\mathbb{C}_p$-vector space $L'$ as a direct sum of $\mathbb{C}_p$-vector subspaces. Then $X_i' = X' \times_{L'} L_i'$ is an effective Banach-Colmez space and has the presentation $0 \rightarrow V' \rightarrow X_i' \rightarrow L_i' \rightarrow 0$. Let $X_i = X \times_{L'} L_i = X \times_{X'} X_i'$; then the result is true for the map $f : X \rightarrow X'$ if, and only if, it is true for all maps $f_i : X_i \rightarrow X_i'$. We may therefore assume that $L' = \mathbb{C}_p$.

Moreover, replacing $f$ by the composed map $X \xrightarrow{f} X' \rightarrow L'$ preserves the result, and we may thus further assume that $X' = L' = \mathbb{C}_p$.

Let now $L = \bigoplus L_i$ be a decomposition of $L$, and let $E_i = E \times_L L_i$ and $V_{-i} = V_\times E_i$. Then $E_i$ is an effective Banach-Colmez space, having the
presentation $0 \to V_+ \to E_i \to L_i \to 0$, and $V_{-,i} \hookrightarrow E_i$ defines a Banach-Colmez space $X_i$, as well as a morphism $X_i \to X$. The result for $X \to X'$ is then equivalent to the result for all composed maps $X_i \to X'$. We may therefore assume that $L = \mathbb{C}_p$.

Let $\pi_0(E)$ and $E^0$ be the étale and connected components of $E$; then $E$ is isomorphic to $\pi_0(E) \times E^0$, and $E^0$ is a connected, effective Banach-Colmez space of dimension one. The composed map $g : E^0 \hookrightarrow E \to X \xrightarrow{f} X' = \mathbb{C}_p$ is then analytic; since $f(X)$ is not étale by hypothesis, $g$ is non-zero. By the fundamental lemma for the connected effective Banach-Colmez space $E$, $g$ is surjective and $\ker g$ is a $\mathbb{Q}_p$-vector space of dimension $\dim \mathbb{Q}_p \pi_0(V) - \dim \mathbb{Q}_p V_+ - \dim \mathbb{Q}_p V_-$. $\triangleright$

**Corollary 1.2.4.** Let $X$ be a Banach-Colmez space. The dimension and height of any two presentations of $X$ coincide.

**Proof.** Apply 1.2.3 to the identity morphism of $X$. $\triangleright$

We call these the dimension and height of $X$, and we write $\dim X$ and $\text{ht} X$.

**Proposition 1.2.5.** Let $X$ be a Banach-Colmez space and $X'$ be a sub-Banach-Colmez space of $X$. There exists a quotient Banach-Colmez space $X'' = X/X'$; moreover, $\dim X = \dim X' + \dim X''$ and $\text{ht} X = \text{ht} X' + \text{ht} X''$.

**Proof.** Let $V_\cdot \to (V_+ \to E \to L) \to X$ and $V'_\cdot \to (V'_+ \to E' \to L') \to X'$ be presentations of $X$ and $X'$. As in the proof of Prop. 1.2.3, we may reduce to the case where $X = \mathbb{C}_p$.

If $X'$ is étale, then the quotient $\mathbb{C}_p/X'$ exists and the result is true. If on the other hand $X'$ is connected and non-zero, then let $0 \to V' \to E' \to X' \to 0$ be a presentation of $X'$, with $E$ connected; by the fundamental lemma for $E'$, the composed morphism $E' \to X' \to \mathbb{C}_p$ is surjective. In this case, the quotient $\mathbb{C}_p/X'$ is zero. $\triangleright$

**Theorem 1.2.6.** The category of Banach-Colmez spaces is abelian. Moreover, there exist unique functions dimension and height, additive on short exact sequences, and such that

$$\dim \mathbb{C}_p = 1, \quad \text{ht} \mathbb{C}_p = 0; \quad \dim \mathbb{Q}_p = 0, \quad \text{ht} \mathbb{Q}_p = 1.$$
injective $\sigma$-linear endomorphism $\varphi : D \to D$. The rank of $D$ is its dimension $h$ as a $K_0$-vector space. The $K_0$-line $\Lambda^h D$ is again an isocrystal, and the valuation $v_p(\Lambda^h \varphi)$ does not depend on the basis of $D$. This integer is called the Newton number of $D$, and is written $t_N(D)$.

Let $D$ be an isocrystal. We define a contravariant functor $E$ by

$$E(D) = \text{Hom}_{K_0[\varphi]}(D, B_{\text{cris}}).$$

(2.1.1)

The isocrystal $D$ is effective if there exists a lattice $\mathcal{D}$ of $D$ such that $\varphi(\mathcal{D}) \subset \mathcal{D}$. In this case, all elements of $E(D)$ have their image contained in $B_{\text{cris}}^+$. If the residual field $k$ of $K_0$ is algebraically closed, then the category of $K_0$-isocrystals is semi-simple, and its simple objects are the objects

$$D_{d,h} = K_0[\varphi]/(\varphi^h - p^d), \quad d \in \mathbb{Z}, \, h \geq 1.$$

(2.1.2)

In the general case, the rationals $\mu = d/h$ appearing in the decomposition of $D \otimes_{K_0} \mathbb{Q}_p^m$ are called the slopes of $D$. For any left $K_0[\varphi]$-module $(M, \varphi)$ and for $r \in \mathbb{Z}$, we write $M(r)$ for the twisted module $(M, p^r \varphi)$. Finally, we define

$$E_{d,h} = E(D_{d,h}) = \{ x \in B_{\text{cris}}^+, \varphi^h(x) = p^d x \}$$

(2.1.3)

and, for all $d$, $E_d = E_{d,1}$.

Lemma 2.1.4. Let $D$ be an isocrystal with slopes $\geq 1$ and of rank $h$. Then there exists a $K_0[\varphi]$-linear map $\eta : D \to B_{\text{cris}}^+(1)^h$ such that the composed map

$$\theta \circ E(\eta) : E_1^h \to E(D) \to \text{Hom}_{K_0}(D, \mathbb{C}_p)$$

is surjective.

Proof. Here we understand $E_1$ as the space $\text{Hom}_{K_0[\varphi]}(B_{\text{cris}}^+(1), B_{\text{cris}})$; thus $E(\eta) : E_1^h \to E(D)$ is actually the image of $\eta : D \to B_{\text{cris}}^+(1)^h$ under the contravariant functor $E$.

If the lemma is true for any unramified extension $K'_0$ of $K_0$, then it is true for $K_0$; therefore we may assume that $K_0 = \mathbb{Q}_p$. Let $d = t_N(D)$; then there exists $e \in D$ such that $(e, \varphi(e), \ldots, \varphi^{h-1}(e))$ is a basis of $D$ and $\varphi^h(e) = p^d e$. Since $D$ has slopes $\geq 1$, we have $d \geq h$, thus there exists $a \in B_{\text{cris}}^+$ such that $\varphi^h(a) = p^d - h a$ and $\theta(\det \varphi^i(a)) = 0$ implies that $\theta \circ E(\eta)$ is surjective. 

Proposition 2.1.5. For any isocrystal $D$, $E(D)$ has a canonical structure as a Banach-Colmez space, of dimension $d = t_N(D)$ and height $h = \dim_{K_0} D$. Moreover, for any $K_0[\varphi]$-linear map $f : D \to D' \otimes_{K_0} B_{\text{cris}}^+$, the deduced map $E(f) : E(D') \to E(D)$ is analytic.

In particular, this defines a functor from the category of $K_0$-isocrystals to that of Banach-Colmez spaces.
Proof. For any \( r \in \mathbb{Z} \), let \( D(r) \) be the Tate twisted isocrystal \( (D, p^r \varphi) \). Then the slopes of \( D(r) \) are the slopes of \( D \), translated by \( r \). By using a Tate twist, we may assume that \( D \) is effective.

We now prove the proposition by induction on the maximal slope of \( D \). If all the slopes of \( D \) lie in the interval \([0, 1]\), then there exists a lattice \( D \) of \( D \) such that \( pD \subset \varphi(D) \subset D \), and therefore \( D \) is the Dieudonné module of a \( p \)-divisible group. In this case the result follows from [Plût10, 2.1.5].

Assume that the proposition is true for all isocrystals with slopes \( \leq m \) for some integer \( m \), and let \( D \) be an isocrystal with slopes \( \leq m + 1 \). By using the isocline decomposition for \( D \), we may assume that \( D \) has all its slopes in the interval \([m, m + 1]\). Let \( h = \dim_{\mathbb{K}_0} D \).

Define an admissible pair for \( D \) as a pair \((\Delta, \alpha)\), where \( \Delta \) is an isocrystal with slopes \( \leq m \) and \( \alpha : D \to \Delta \otimes_{\mathbb{K}_0} B^{\dagger}_{\text{cris}} \) is a \( \mathbb{K}_0[\varphi] \)-linear map such that \( E(\alpha) : E(\Delta) \to E(D) \), defined by \( E(\alpha)(f) = (\text{id}_{B^{\dagger}_{\text{cris}}} \otimes f) \circ \alpha \), is surjective and has a finite-dimensional kernel (as a \( \mathbb{Q}_p \)-vector space). Then any admissible pair for \( D \) gives a Banach-Colmez structure on \( E(D) \). Moreover, let \( f : D \to D' \) be a morphism of isocrystals, \((\Delta, \alpha)\) an admissible pair for \( D \), \((\Delta', \alpha')\) an admissible pair for \( D' \), and \( g : \Delta \to B^{\dagger}_{\text{cris}} \otimes_{\mathbb{K}_0} \Delta' \) a \( \mathbb{K}_0[\varphi] \)-linear map such that \( g \circ \alpha = \alpha' \circ f \). Then the graph of \( E(f) \) is the fibre product of the maps \( E(f) \circ E(\alpha') = E(\alpha) \circ E(g) \) and \( E(\alpha) \). By the induction hypothesis for \( \Delta \) and \( \Delta' \), it is analytically closed in \( E(\Delta) \times E(\Delta') \), and therefore \( E(f) \) is an analytic morphism. Finally, applying this to the identity morphism of \( D \) proves that the analytic structure on \( E(D) \) does not depend on the choice of the admissible pair \((\Delta, \alpha)\).

It remains to prove that the isocrystal \( D \) has an admissible pair \((\Delta, \alpha)\). Since \( D \) has slopes \( \leq m + 1 \), the isocrystal \( D(-1) \) has slopes \( \leq m \), and by the induction hypothesis, \( E(D(-1)) \) is a Banach-Colmez space. Moreover, multiplication by a generator \( t \) of \( \mathbb{Z}_p(1) \) in \( B^{\dagger}_{\text{cris}} \) gives the left exact sequence

\[
0 \to E(D(-1)) \xrightarrow{x t} E(D) \xrightarrow{\pi} \text{Hom}_{\mathbb{K}_0}(D, \mathbb{C}_p). \tag{2.1.6}
\]

Let \( \Delta = D(-1) \oplus \mathbb{K}_0(1)^h \) and \( \alpha : D \to \Delta \otimes_{\mathbb{K}_0} B^{\dagger}_{\text{cris}} \) be the map defined by \( \alpha(x) = (x \otimes t) \oplus \eta(x) \), where \( \eta : D \to B^{\dagger}_{\text{cris}}(1)^h \) is defined as in Lemma 2.1.4. Then \( \alpha \) is \( \mathbb{K}_0[\varphi] \)-linear and the image of \( E(\alpha) \) contains both \( E(D(-1)) \) and representatives of \( \text{Hom}_{\mathbb{K}_0}(D, \mathbb{C}_p) \); therefore, \( E(\alpha) \) is surjective. The kernel of \( E(\alpha) \) is isomorphic to \( \text{Hom}_{\mathbb{K}_0[\varphi]}(D, \mathbb{K}_0(1)^h) \), which is a \( \mathbb{Q}_p \)-vector space of dimension \( h \). Therefore, \((\Delta, \alpha)\) is an admissible pair for \( D \).

Since \( t_N(D(-1)) = d - h \) and \( \dim_{\mathbb{K}_0} D(-1) = h \), \( E(D(-1)) \) has dimension \( d - h \) and height \( h \) by the induction hypothesis; therefore, \( E(\Delta) \) has dimension \( d \) and height \( 2h \), and \( E(D) \) is a Banach-Colmez space of dimension \( d \) and height \( h \).

Proposition 2.1.7. Let \( D, D' \) be two isocrystals. Then

\[
\text{Hom}_{\text{BC}}(E(D'), E(D)) = \text{Hom}_{\mathbb{K}_0[\varphi]}(D, D' \otimes_{\mathbb{K}_0} B^{\dagger}_{\text{cris}}).
\]
Proof. We know that any $K_0[\varphi]$-linear map $D \to D' \otimes_{K_0} B^+_\text{cris}$ is analytic by Proposition 2.1.5. Conversely, any analytic map $f : E(D) \to E(D')$ defines a $B^+_{\text{dR}}$-linear map $f^+_{\text{dR}} : E(D)^+_{\text{dR}} \to E(D')^+_{\text{dR}}$. Since we also have $f^+_{\text{dR}}(E(D)) \subset E(D')$, we see that $f$ comes from a $B^+_{\text{cris}}[\varphi]$-linear map $D \otimes_{K_0} B^+_{\text{cris}} \to D' \otimes_{K_0} B^+_{\text{cris}}$. $\lhd$

2.2 $B^+_{\text{dR}}$-modules as Banach-Colmez spaces

Let $K$ be a closed subfield of $\mathbb{C}_p$ with discrete valuation and $K_0$ be its maximal non-ramified subfield. A filtered $K$-isocrystal is a pair $(D, \text{Fil})$, where $D$ is a $K_0$-isocrystal, and $\text{Fil}$ is an exhaustive and separated decreasing filtration on the $K$-vector space $D_K = D \otimes_{K_0} K$. By abuse, we shall sometimes write $D$ instead of $(D, \text{Fil})$ when the filtration is clear. The Hodge number of $(D, \text{Fil})$ is the integer $t_H(D, \text{Fil})$ defined by

$$t_H(D, \text{Fil}) = \sum_{i \in \mathbb{Z}} i \cdot \dim_K(\text{Fil}^i / \text{Fil}^{i+1} D_K).$$

(2.2.1)

Since $D$ is finite-dimensional, this sum is finite. We also define

$$V(D, \text{Fil}) = \text{Hom}_{\varphi, \text{Fil}}(D, B^+_{\text{cris}}),$$

(2.2.2)

$$M(D, \text{Fil}) = \text{Hom}_K(D, B^+_{\text{dR}})/\text{Hom}_K(D, B^+_{\text{dR}}).$$

Together with $E(D)$ as defined in (2.1.1), they make a left-exact sequence of $\mathbb{Q}_p$-vector spaces:

$$0 \to V(D, \text{Fil}) \to E(D) \to M(D, \text{Fil}).$$

(2.2.3)

The spaces $E(D)$ and $M(D, \text{Fil})$ are contravariant analogues of the spaces $V^0_{\text{cris}}$ and $V^1_{\text{cris}}$ of [CF00, §5].

**Proposition 2.2.4.** There exists a fully faithful functor from the category of finite-length $B^+_{\text{dR}}$-modules to that of Banach-Colmez spaces, extending the inclusion of finite-dimensional vector spaces over $\mathbb{C}_p$. Moreover, for any filtered isocrystal $D$, this analytic structure on $M(D, \text{Fil})$ makes the sequence (2.2.3) a left exact sequence of Banach-Colmez spaces.

We prove this proposition in Lemmas 2.2.5 to 2.2.12.

**Lemma 2.2.5.** Any finite-length $B^+_{\text{dR}}$-module $M$ has a structure as a Banach-Colmez space.

**Proof.** It is enough to prove this for the quotient spaces $B_m = B^+_{\text{dR}}/\text{Fil}^m B^+_{\text{dR}}$. This module inserts in the exact sequence [Fon94, 5.3.7(ii)]

$$0 \to \mathbb{Q}_p(m) \to E_m \to B_m.$$ 

(2.2.6)

where $E_m = \{x \in B^+_{\text{cris}}, \varphi(x) = p^m x\}$. Let $D_m$ be the isocrystal $K_0[\varphi]/(\varphi - p^m)$; then $E(D_m) = E_m$, thus by Proposition 2.1.5, $E_m$ is a Banach-Colmez space. Therefore, its quotient $B_m$ is a Banach-Colmez space. $\lhd$
Lemma 2.2.7. Let \((D, \Fil)\) be a filtered isocrystal. Then the map \(E(D) \to M(D)\) is analytic.

Proof. It is enough to prove this in the case where \(D = D_{d,h} = K_0[\varphi]/(\varphi^h - p^d)\) and \(\Fil\) is the filtration \(\Fil_m\), defined by \(\Fil_m D_K = D_K\) and \(\Fil_m D_K = 0\). In this case, \(M(D, \Fil_m) = \Hom_K(D, B_m)\).

Let \(d = qr + h\) be the Euclidean division of \(d\) by \(h\), and let \(\eta = (\eta_j)_{j=1,...,h} : D_{r,h} \to B_{\cris}^+(1)\) as in Lemma 2.1.4; also let \(u \in E_1\) be such that \(\theta(u) = 1\). According to Proposition 2.1.5, the analytic structure on \(E(D) = E_{d,h}\) is given by

\[
E_{i,h}^0 \oplus E_{r,h} \longrightarrow E_{d,h}
\]

\[
(x_{i,j})_{j=0,...,q-1; \ y} \longmapsto \sum u^i \eta_j x_{i,j} + t^q y \tag{2.2.8}
\]

and by Lemma 2.2.5, that on \(B_m\) is given by

\[
E_{i,m}^0 \longrightarrow B_m
\]

\[
(x_{i})_{i=0,...,m-1} \longmapsto \sum u^i x_{i} \pmod{\Fil^m} \tag{2.2.9}
\]

The graph of the reduction map \(f : E(D) \to M(D, \Fil)\) is the fibre product of these two maps. It is analytically closed in \(E_{1,h}^{m+q} \times E_{r,h}\); therefore, \(f\) is analytic. \(<\)

Lemma 2.2.10. Let \(f : M \to M'\) be a \(B_{\cris}^+\)-linear map between finite-length \(B_{\cris}^+\)-modules. Then \(f\) is analytic.

Proof. It is enough to prove it for the multiplication map \(\mu : B_m \to B_m, x \mapsto ax\), where \(a \in B_m\). Let \(\tilde{a} \in E_m\) be a lift of \(a\). Then the graph of \(\mu\) in \(E_m \times E_m\) is the set

\[
G = \{(x, y) \in E_m \times E_m, \ \tilde{a}x - y \in \Fil^m B_{\cris}^+\}. \tag{2.2.11}
\]

Let \(v \in E_m\) be such that \(v \equiv 1 \pmod{\Fil^m}\). Then, for all \((x, y) \in E_m \times E_m\), we have \(z = \tilde{a}x - vy \in E_{2m}\), and \((x, y) \in G\) if, and only if, \(z \in \Fil^m E_{2m}\). By Lemma 2.2.7, \(G\) is analytically closed, and therefore \(\mu\) is analytic. \(<\)

In particular, when applying this lemma to the identity map of any \(B_{\cris}^+\)-module \(M\), we see that the analytic structure on \(M\) does not depend on the choice of \(B_{\cris}^+\)-generators.

Lemma 2.2.12. Let \(M, M'\) be two finite-length \(B_{\cris}^+\)-modules. Then any analytic map \(f : M \to M'\) is \(B_{\cris}^+\)-linear.

Proof. It is enough to prove that, for any \(B_{\cris}^+\)-module \(M\) of finite length, \(\Hom_{\BC}(M, C_p) = C_p\), i.e. that any analytic morphism is zero on \(\Fil^1 B_m = B_{m-1}(1)\). Moreover, given the exact sequence of Banach-Colmez spaces

\[
0 \longrightarrow B_{m-1}(1) \longrightarrow B_m \longrightarrow C_p \longrightarrow 0, \tag{2.2.13}
\]
it is again enough to prove this in the case where \( m = 2 \). According to Lemma 2.2.5, an analytic structure on \( B_2 \) is given by the exact sequence

\[
0 \to V \to E_{2.2} \to B_2 \to 0,
\]

where \( E_{2.2} = \{ x \in B_{\text{cris}}, \varphi^2 x = p^2 x \} \), and \( V = E_{2.2} \cap \text{Fil}^2 B_{\text{cris}} \) is a \( \mathbb{Q}_p \)-vector space of dimension two. Then \( \text{Hom}_{BC}(B_2, \mathbb{C}_p) \) is given by the left-exact sequence

\[
0 \to \text{Hom}_{BC}(B_2, \mathbb{C}_p) \to \text{Hom}_{BC}(E_{2.2}, \mathbb{C}_p) \xrightarrow{F} \text{Hom}_{BC}(V, \mathbb{C}_p).
\]

The space \( E_{2.2} \) is isomorphic to \( E_1 \otimes \mathbb{Q}_p \mathbb{Q}_p^2 \). Let \( \{ e, \varphi(e) \} \) be a \( \mathbb{Q}_p \)-basis of \( \mathbb{Q}_p^2 \), and define \( f_0, f_1 \in \text{Hom}_{BC}(E_{2.2}, \mathbb{C}_p) \) by

\[
f_i(x_0 e + x_1 \varphi(e)) = \theta(x_i), \quad \text{for } i \in \{0, 1\};
\]

Assume that the restriction map \( F : \text{Hom}_{BC}(E_{2.2}, \mathbb{C}_p) \to \text{Hom}_{BC}(V, \mathbb{C}_p) \) is identically zero. Then \( F(f_0) = F(f_1) = 0 \) imply that any element \( x \in V \) may be written as \( x = x_0 e + x_1 \varphi(e) \) with \( \theta(x_0) = \theta(x_1) = 0 \). Since \( x_i \in E_1 \), this implies that \( x_i \in \text{Fil}^1 E_1 = \mathbb{Q}_p(1) \) and therefore \( V \subset \mathbb{Q}_p^2(1) \), which is absurd. Therefore, \( F \) is not zero, and its kernel \( \text{Hom}_{BC}(B_2, \mathbb{C}_p) \) thus has dimension (at most) one. \( \triangleleft \)

**Lemma 2.2.17.** Let \( E \) be an effective, connected one-dimensional Banach-Colmez space. Then:

(i) any effective extension of \( \mathbb{C}_p \) by \( E \) is trivial;

(ii) the map \( \theta \circ - : \text{Hom}(E, E_1) \to \text{Hom}(E, \mathbb{C}_p) \) is surjective.

**Proof.** (i) Let \( E' \) be an effective extension of \( \mathbb{C}_p \) by \( E \). For any morphism \( u : E' \to \mathbb{C}_p, u(E) \) is either étale or \( \mathbb{C}_p \). Since \( E \) is connected, there exists \( u : E' \to \mathbb{C}_p \) such that \( u(E') = \mathbb{C}_p \). Define \( V' = \text{Ker} u \). The map \( E' \to E \) defines a morphism between the exact sequences \( 0 \to V' \to E' \xrightarrow{u} \mathbb{C}_p \to 0 \) and \( 0 \to V' \to E \to \mathbb{C}_p \otimes \mathbb{C}_p \to 0 \). Let \( \iota \) be the injection of the first summand \( \mathbb{C}_p \to \mathbb{C}_p \otimes \mathbb{C}_p \). By [Ph10, Prop. 2.2.3], there exists a \( \mathbb{C}_p \)-linear map \( \lambda : \mathbb{C}_p^2 \to V \otimes \mathbb{Q}_p \mathbb{C}_p(-1) \) such that \( E' = E(\lambda) = (V \otimes E_1(-1)) \times_{\mathbb{Q}_p} \mathbb{C}_p^2 \) and \( E = E(\lambda \circ \iota) \). Since \( E \) is connected, the transpose map \( t(\lambda \circ \iota) : \text{Hom}(V, \mathbb{Q}_p) \to \text{Hom}(\mathbb{C}_p, \mathbb{C}_p) \) is injective; therefore, for any \( \mathbb{C}_p \)-linear retraction \( \rho : \mathbb{C}_p^2 \to \mathbb{C}_p \) of \( \iota \), there exists a \( \mathbb{Q}_p \)-linear map \( \mu : V \to V \) such that the diagram

\[
\begin{array}{ccc}
\mathbb{C}_p^2 & \xrightarrow{\lambda} & V \otimes \mathbb{Q}_p \mathbb{C}_p(-1) \\
\downarrow \rho & & \downarrow \mu \otimes \mathbb{Q}_p \mathbb{C}_p(-1) \\
\mathbb{C}_p & \xrightarrow{\lambda \circ \iota} & V \otimes \mathbb{Q}_p \mathbb{C}_p(-1)
\end{array}
\]

commutes. The pair \( (\mu, \rho) \) defines a retraction of \( E' \to E \).
Let $M, M'$ be two finite-length $B_{dR}^+$-modules and $X$ be an extension of $M$ by $M'$ as Banach-Colmez spaces. Then $X$ is a finite-length $B_{dR}^+$-module.

Proof. It is enough to prove this when $M = M' = \mathbb{C}_p$; by commodity, we write $M' = \mathbb{C}_p(1)$. Let $X$ be an analytic extension of $\mathbb{C}_p$ by $\mathbb{C}_p(1)$. The analytic map $\mathbb{C}_p(1) \to X$ corresponds to a pair $(V' \to V, E' \to E)$, where $0 \to V' \to E' \xrightarrow{\pi'} \mathbb{C}_p(1) \to 0$ is exact and $0 \to V \to E \to X \to 0$ is a presentation of $X$. Likewise, the analytic map $X \to \mathbb{C}_p$ corresponds to a map $E_1 \to \mathbb{C}_p$, where $0 \to \mathcal{V}_1 \to E_1 \to X$ is another presentation of $X$. By taking fibre products, we may assume that $E_1 = E$; the exact sequence $0 \to \mathbb{C}_p(1) \to X \to \mathbb{C}_p \to 0$ then pulls back to the exact sequence of effective Banach-Colmez spaces

\[0 \to E' \to E \to \mathbb{C}_p \to 0. \tag{2.2.20}\]

Moreover, since $\mathbb{C}_p(1)$ is connected, we may assume that $E'$ is connected. By Lemma 2.2.17 (i), $E$ is therefore isomorphic to $E' \oplus \mathbb{C}_p$.

The analytic structure on $B_2$ is given by the exact sequence $0 \to \mathbb{Q}_p(1) \to E_1 \oplus \mathbb{C}_p(1) \to B_2 \to 0$. By 2.2.17 (ii), there exist $u : E' \to E_1$ such that $\theta \circ u = \pi' : E' \to \mathbb{C}_p(1)$. The map $u \oplus \text{id}_{\mathbb{C}_p} : E' \oplus \mathbb{C}_p \to E_1 \oplus \mathbb{C}_p(1)$ defines an analytic morphism $f : X \to B_2$ extending the identity morphism of $\mathbb{C}_p$. The extension $X$ is then given by the restriction of $f$ to $\mathbb{C}_p(1)$, which is an analytic endomorphism of $\mathbb{C}_p$ and therefore multiplication by an element of $\mathbb{C}_p$. We finally see that $\text{Ext}^1_{\mathcal{B}_C}(\mathbb{C}_p, \mathbb{C}_p(1)) = \text{Ext}^1_{B_{dR}^+}(\mathbb{C}_p, \mathbb{C}_p(1)) = \mathbb{C}_p$. \(<\)

### 2.3 Weakly admissible implies admissible

A filtered isocrystal $(D, \text{Fil})$ is called admissible if $V(D, \text{Fil})$ is a $\mathbb{Q}_p$-vector space of dimension equal to the rank of $D$. It is called weakly admissible if $t_H(D) = t_N(D)$ and, for any subobject $D'$ of $D$, $t_H(D') \leq t_N(D')$. The “weakly admissible implies admissible” theorem was first proven in [CF00]. We give an independent proof of this theorem.

**Theorem 2.3.1.** Let $K$ be a closed subfield of $\mathbb{C}_p$ with discrete valuation. Any weakly admissible filtered $K$-isocrystal is admissible.

**Proof.** For any filtered isocrystal $D$, the left-exact sequence

\[0 \to V(D, \text{Fil}) \to E(D) \to M(D, \text{Fil}) \tag{2.3.2}\]

is analytic by Lemma 2.2.7. If $(D, \text{Fil})$ is weakly admissible, then $t_H(D) = t_N(D)$ implies that $\dim M(D) = \dim E(D)$; and $t_H(D') \leq t_N(D')$ for all subobjects $D'$.
of $D$, with $K$ having discrete valuation, implies that $V(D)$ is a finite-dimensional $\mathbb{Q}_p$-vector space (an elementary proof is given in [CF00, 4.5]). Therefore, the cokernel $X$ of $E(D) \to M(D)$ has dimension zero, and is thus an étale Banach-Colmez space. Since $M(D)$ is connected, this implies that $X = 0$, i.e. that $E(D) \to M(D)$ is surjective.

Moreover, by counting heights on the exact sequence 2.3.2, we find that $\dim_{\mathbb{Q}_p} V(D, \Fil) = \text{ht} V(D, \Fil) = \text{ht} E(D) = \dim_{K_0} D$. Therefore, $(D, \Fil)$ is admissible. ⊳

3 The slope filtration

3.1 The universal extension of $\tilde{B}$ by $\mathbb{Q}_p$

The goal of this part is to define a left adjunct to the functor from finite-length $B^\text{dr}_+\text{-modules}$ to Banach-Colmez spaces.

Let $\tilde{B} = B_{\text{dR}}/B^\text{dR}_+$ and $B_c = \{ x \in B_{\text{cris}}, \varphi(x) = x \}$. Then $\tilde{B} = \lim B_m(-m)$ and $B_c = \lim E_m(-m)$. The direct limit of the exact sequence (2.2.6) reads

$$0 \to \mathbb{Q}_p \to B_c \to \tilde{B} \to 0$$

where the objects are inductive limits of Banach-Colmez spaces.

For any Banach-Colmez space $X$, define

$$X^*_{\text{dr}} = \text{Hom}_{\text{ind-BC}}(X, \tilde{B}) = \lim_{\to} \text{Hom}_{\text{BC}}(X, B_m),$$

$$X^+_{\text{dr}} = (X^*_{\text{dr}})^\vee = \text{Hom}_{B^+_{\text{dR}}}(X^*_{\text{dR}}, \tilde{B}), \quad X_{\text{dR}} = X^+_{\text{dR}} \otimes_{B^+_{\text{dR}}} B_{\text{dR}}.$$

In particular, for any finite-length $B^+_{\text{dR}}$-module $M$, we have $M^+_{\text{dR}} = M^\vee$ and $M^+_{\text{dR}} = M$; for any finite-dimensional $\mathbb{Q}_p$-vector space, $V^*_{\text{dR}} = V^\vee \otimes_{\mathbb{Q}_p} \tilde{B}$ and $V^+_{\text{dR}} = V \otimes_{\mathbb{Q}_p} B^+_{\text{dR}}$.

**Proposition 3.1.3.** Let $M$ be a finite-length $B^+_{\text{dR}}$-module and $V$ be a finite-dimensional $\mathbb{Q}_p$-vector space. Then there is a canonical functorial isomorphism

$$\text{Ext}^1_{\text{BC}}(M, V) = \text{Hom}_{B^+_{\text{dR}}}(M, V \otimes_{\mathbb{Q}_p} \tilde{B}).$$

**Proof.** Let $\lambda : M \to V \otimes_{\mathbb{Q}_p} \tilde{B}$ be a $B^+_{\text{dR}}$-linear map. Then we may form the diagram

$$\begin{array}{cccccccc}
0 & \to & V & \to & V \otimes_{\mathbb{Q}_p} B_c & \to & V \otimes_{\mathbb{Q}_p} \tilde{B} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \lambda & & \downarrow & & \\
0 & \to & V & \to & E(\lambda) & \to & M & \to & 0
\end{array}$$

(3.1.4)

Since $M$ has finite length, $\lambda$ factors through $V \otimes_{\mathbb{Q}_p} B_m(-m)$ for some integer $m$ and the diagram (3.1.4) is actually a diagram of Banach-Colmez spaces. Therefore the fibre product $E(\lambda)$ is a Banach-Colmez space.
3.2 Constructible Banach-Colmez spaces

**Proposition 3.2.1.** Let $X$ be a Banach-Colmez space. The following conditions are equivalent:

(i) $X$ is an extension of a finite-length $B_{dR}^+$-module by a finite-dimensional $\mathbb{Q}_p$-vector space;

(ii) the envelope map $X \to X_{dR}^+$ is injective;

(iii) there exists an analytic embedding of $X$ in a finite-type $B_{dR}^+$-module.

**Proof.** (i) $\Rightarrow$ (ii): by Proposition 3.1.3, we may assume that $X = E(\lambda)$ for a $B_{dR}^+$-linear map $\lambda : M \to V \otimes_{\mathbb{Q}_p} B_{dR}^+$. Let $X'$ be the $B_{dR}^+$-submodule of $V \otimes_{\mathbb{Q}_p} B_{dR}^+$ generated by $X$. Any $v \in V_{dR}^+$ extends to a $B_{dR}^+$-linear map $v : V \otimes_{\mathbb{Q}_p} B_{dR} \to B$, and the composed map $X \to X' \to V \otimes B_{dR} \to B$ is analytic. Therefore, the sequence $0 \to M_{\mathit{dR}}^+ \to X_{dR}^+ \to V_{\mathit{dR}}^+ \to 0$ is (right-)exact, and its dual sequence $0 \to V_{\mathit{dR}}^+ \to X_{dR}^+ \to M \to 0$ is exact. Since both maps $V \to V_{dR}^+$ and $M \to M$ are injective, we deduce that $X \to X_{dR}^+$ is injective.

(ii) $\Rightarrow$ (iii): since $X$ is a quotient of extensions of finite-length $B_{dR}^+$-modules and finite-dimensional $\mathbb{Q}_p$-vector spaces, we see that $X_{dR}^+$ is a finite-type $B_{dR}^+$-module.

(iii) $\Rightarrow$ (i): let $X \subset N$ be an analytic embedding in a finite-type $B_{dR}^+$-module $N$. We prove that there exists a finite-length quotient $M$ of $N$ such that the composed map $X \to M$ is surjective with étale kernel. We may assume that $N = B_m$ and proceed by induction on $m$. The case $m = 0$ is trivial; assume that the case $B_m$ is known, and let $X \subset B_{m+1}$ be an analytic embedding. Let $X' = X \times_{B_{m+1}} B_m(1)$ and $X'' = X/X'$; there exists a natural injective map $\iota : X'' \to \mathbb{C}_p$. By the fundamental lemma, $X''$ is either $\mathbb{C}_p$ or étale. If $X''$ is étale, then there exists an analytic section of $X \to X''$, so that we may write $X = X' \times X''$ and the result on $X$ is immediate.

Assume that $X'' = \mathbb{C}_p$. By the induction hypothesis, there exists a finite-length quotient $B_\iota(1)$ of $N$ such that $X' \to B_{m-1}(1) \to B_\iota(1)$ is surjective with étale kernel. The same is then true of composed map $X \to B_m \to B_\iota$, which proves the result. $\triangle$

We say that a Banach-Colmez space is constructible if it satisfies the equivalent conditions of Proposition 3.2.1; and that it is oblique if it is constructible and $X \to X_{dR}^+$ is injective.

**Proposition 3.2.2.** Let $X$, $X'$ be two constructible Banach-Colmez spaces. Then $\text{Hom}_{\text{BC}}(X, X')$ identifies with the set of all $B_{dR}^+$-linear maps $f : X_{dR}^+ \to (X')_{dR}^+$ such that $f(X) \subset X'$.

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Proof. By replacing \( f \) by its graph map \( X \to X \times X' \), we may assume that \( f \) is injective. By Proposition 3.2.1, \( X' \) inserts in an exact sequence \( 0 \to V' \to X' \to M' \to 0 \), where \( M' \) is a finite-length \( B^+_{\text{dR}} \)-module and \( V' \) is étale. Let \( V = V' \times_X X \); then \( V \) is étale and therefore the map \( V \to X \) is analytic. By hypothesis, the composed map \( X \to X' \to M' \to 0 \) factorizes through \( X_{\text{dR}}^+ \) and is therefore analytic, therefore the cokernel \( X \to M \) of \( V \to X \) is a sub-module of \( M' \). Finally, we get a morphism of exact sequences between \( 0 \to V \to X \to M \to 0 \) and \( 0 \to V' \to X' \to M' \to 0 \); by Proposition 3.1.3, this comes from a commutative square

\[
\begin{array}{c}
M \longrightarrow V \otimes_{\mathbb{Q}_p} B^+_{\text{dR}} \\
\downarrow \quad \downarrow \\
M' \longrightarrow V' \otimes_{\mathbb{Q}_p} B^+_{\text{dR}}
\end{array}
\] (3.2.3)

which defines an analytic structure on the morphism \( f : X \to X' \). \( \triangleright \)

**Proposition 3.2.4.** The functor that maps a constructible Banach-Colmez space \( X \) to the \( B^+_{\text{dR}} \)-module \( X_{\text{dR}}^+ \) is exact and faithful.

**Proof.** The faithfulness is a consequence of the fact that, for any constructible \( X \), \( X \subset X_{\text{dR}}^+ \). To prove exactness, it is enough to prove that

\[
\text{Ext}^1_{\text{BC}}(X, \tilde{B}) = \lim_{\rightarrow} \text{Ext}^1_{\text{BC}}(X, B_m(-m)) = 0. \quad (3.2.5)
\]

Since \( X \) is constructible, it inserts in a presentation \( 0 \to V \to X \to M \to 0 \), where \( V \) is étale and \( M \) is a finite-length \( B^+_{\text{dR}} \)-module. Using the Ext exact sequence in the category of ind-Banach-Colmez spaces, we see that it is enough to prove the result when \( X \) is étale or a \( B^+_{\text{dR}} \)-module.

If \( X \) is étale, then any surjective map to \( X \) has a section, and therefore \( \text{Ext}^1_{\text{BC}}(X, Y) = 0 \) for all \( Y \). If \( X \) is a finite-length \( B^+_{\text{dR}} \)-module, the result follows from Proposition 2.2.19 and the fact that \( \tilde{B} \) is an injective \( B^+_{\text{dR}} \)-module. \( \triangleright \)

**Proposition 3.2.6.** Let \( X \) be a constructible Banach-Colmez space. There exists a unique filtration

\[
X = F^0 X \supset F^+ X \supset F^\infty X,
\]

such that \( F^0 X/F^+ X \) is étale, \( F^+ X/F^\infty X \) is connected and oblique, and \( F^\infty X \) is a finite-length \( B^+_{\text{dR}} \)-module.

**Proof.** We define \( F^+ X \) as the connected part of \( X \), and \( F^\infty X \) as the \( B^+_{\text{dR}} \)-torsion part of \( (F^+ X)^+_{\text{dR}} \). Let \( X' \) be the image of \( X \) in the \( B^+_{\text{dR}} \)-module \( X_{\text{dR}}^+ \). Then \( F^\infty X/(X' \cap F^\infty X) \) is both torsion and free, and therefore \( F^\infty X \subset X \). \( \triangleright \)

The object \( F^\infty X \) is essentially an extra step, corresponding to slope \( \infty \), of the filtration of Theorem 3.4.10.
3.3 Harder-Narasimhan categories

The following is a formalization of the Harder-Narasimhan filtration [HN74] in more general categories.

A Harder-Narasimhan category is an exact category $C$ (in the sense of [Lau83, 1.0.2]), together with an exact functor $\eta$ (called fibre functor) to an abelian category $C_\eta$, and additive functions $d : C \to \mathbb{N}$ and $r : C_\eta \to \mathbb{N}$, such that

(HN1) For any object $X$ of $C$, $r(\eta(X)) = 0$ if, and only if, $E = 0$;

(HN2) If $f : E \to E'$ is a morphism of $C$ such that $\eta(f)$ is an isomorphism, then $d(E') \geq d(E)$, with equality if, and only if, $f$ is an isomorphism;

(HN3) For any object $E$ of $C$, $\eta$ maps injectively the subobjects of $E$ to those of $\eta(E)$;

(HN4) For any morphism $f : F \to G$ of $C$ that is the compositum of a strict monomorphism $F \to E$ followed by a strict epimorphism $E \to G$, there exists a decomposition $f = v \circ u \circ g$ such that $u$ is strict epi, $\eta(g)$ is an isomorphism, and $v$ is strict mono.

**Proposition 3.3.1.** Assume that the category $C$ satisfies axioms (HN1) to (HN3), and moreover that it has finite fibre products of strict monomorphisms, compatible with the fibre product in the abelian category $C_\eta$. Then $C$ satisfies axiom (HN4).

**Proof.** Let $f : F \to G$ be the composite of a strict epi $F \to E$ and a strict mono $E \to G$; let $E' = \text{Ker}(E \to G)$. By the hypothesis of the proposition, the fibre product $F' = F \times_E E'$ exists and $F' \to F$ is a strict mono; therefore, $F'' = F/F'$ exists and $F'' \to F$ is a strict epi. Since $C$ is exact and $F \to E$ is a strict mono, the amalgamated sum $G' = F' +_F E$ exists, and the pushout $F' \to G'$ is a strict mono.

Finally, the arrow $F \to G$ is the composite $F \to F' \to G' \to G$, where $F \to F'$ is a strict epi, $G' \to G$ is a strict mono, and $\eta(F') \to \eta(G')$ is the isomorphism in $C_\eta$ between the coïmage and the image of $\eta(f)$. $\triangleright$

For the remainder of this part, we assume that $C$ is a Harder-Narasimhan category. As most of the proofs from [HN74] directly apply in $C$, we shall only detail here the parts that differ.

For any object $E$ of $C$, we define $r(E) = r(\eta(E))$ and $\mu(E) = d(E)/r(E)$. We call $d(E)$, $r(E)$ and $\mu(E)$ the degree, rank and slope of $E$. We write $E' \preceq E$ for a strict mono $E' \to E$, and $E' \prec E$ if moreover $E' \to E$ is not an isomorphism. We say that $E$ is semi-stable if $E \neq 0$ and $\mu(E') \leq \mu(E)$ for any $E' \preceq E$; that $E$ is stable if $E \neq 0$ and $\mu(E') < \mu(E)$ for any $E' \prec E$.

Let $F \preceq E$ be a strict subobject of $E$. We say that $F$ is costable in $E$ if, for all $F \preceq F' \preceq E$, $\mu(F') > \mu(F)$.

**Lemma 3.3.2 ([HN74, Lemma 1.3.5]).** Let $E$ be an object of $C$ and $F_1$, $F_2$ be two strict subobjects of $E$ such that $F_1$ is semi-stable and $F_2$ is costable in $E$. If $F_1$ is not a strict subobject of $F_2$, then $\mu(F_1) < \mu(F_2)$. 

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Proof. By hypothesis, the composite morphism \( F_1 \to E \to E/F_2 \) is not zero. By the axiom (HN4), it factorizes as \( v \circ g \circ u \), where \( v : F_1 \to F'_1 \) is strict epi, \( u : F'_2 \to E/F_2 \) is strict mono, and \( \eta(g) : \eta(F'_1) \to \eta(F'_2) \) is an isomorphism. By (HN2), this implies that \( d(F'_2) \geq d(F'_1) \); since \( r(F'_2) = r(F'_1) \), we have \( \mu(F'_2) \geq \mu(F'_1) \). Since \( F_1 \) is semi-stable, \( \mu(F_1) \leq \mu(F'_1) \); moreover, since \( F_2 \) is costable in \( E \) and \( F'_2 \lesssim E/F_2 \), \( \mu(F'_2) < \mu(F_2) \). We therefore have \( \mu(F_1) \leq \mu(F'_1) \leq \mu(F'_2) < \mu(F_2) \). \( \triangleright \)

Proposition 3.3.3 ([HN74, Prop. 1.3.4]). Let \( E \) be an object of \( C \). Then \( E \) contains a unique semi-stable and costable strict subobject \( F \).

Proposition 3.3.4 ([HN74, Lemma 1.3.8]). Let \( E \) be an object of \( C \) and \( 0 = F_0 \prec F_1 \prec \ldots \prec F_n = E \) be an increasing filtration by strict subobjects of \( E \), such that each quotient \( F_i/F_{i-1} \) is semi-stable. Then the following conditions are equivalent:

(i) For each \( i \), \( F_{i+1}/F_i \) is costable in \( E/F_i \);

(ii) The sequence \( (\mu(F_i)) \) is strictly decreasing.

A filtration satisfying these conditions is called a Harder-Narasimhan filtration on \( E \).

Proposition 3.3.5 ([HN74, Prop. 1.3.9]). Any non-zero object \( E \) of \( C \) has a unique Harder-Narasimhan filtration.

Given such a filtration \( 0 = F_0 \prec \ldots \prec F_n = E \) and \( \alpha \in \mathbb{Q} \), we define \( \text{Fil}^\alpha E = F_r \), where \( r \) is the largest index such that \( \mu(F_r) \geq \alpha \). The slope filtration of \( E \) is the decreasing filtration \( (\text{Fil}^\alpha E) \); its graded quotients are zero except for a finite number of values \( \alpha \), for which they are semi-stable.

3.4 Harder-Narasimhan structure on oblique Banach-Colmez spaces

Lemma 3.4.1. Let \( Y \) be an oblique Banach-Colmez space and \( X \subset Y \) be a sub-Banach-Colmez space. Then:

(i) \( X \) is oblique;

(ii) \( \overline{X} = X^+_{\text{dR}} \cap Y \) is a sub-oblique-space of \( Y \), and the quotient \( Y/\overline{X} \) is oblique;

(iii) if \( Y/X \) is oblique, then \( X = \overline{X} \).

Proof. (i) We have \( X \subset Y \subset Y^+_{\text{dR}} \), which is finite free; therefore, \( X \) is oblique.

(ii) Since \( X \) and \( Y \) are oblique, the natural map \( Y/\overline{X} \to Y_{\text{dR}}/X_{\text{dR}} \) is injective, and therefore \( Y/\overline{X} \) is oblique.

(iii) If \( Y/X \) is oblique, then \( Y/X \subset Y_{\text{dR}}/X_{\text{dR}} \); this arrow factors through \( Y/\overline{X} \), which means that \( Y/X \subset Y/\overline{X} \). Therefore, \( X = \overline{X} \). \( \triangleright \)
Proposition 3.4.2. Let $\text{BCO}$ be the category of oblique Banach-Colmez spaces. The category $\text{BCO}$, equipped with the functor $\eta(X) = X_{\text{dR}}$, the rank function equal to the dimension over $B_{\text{dR}}$, and the height function equal to the dimension of the Banach-Colmez space $X$, is a Harder-Narasimhan category.

Proof. The category $\text{BCO}$ is the full subcategory of $\text{BC}$ of objects $X$ such that $X_{\text{dR}}^+$ is finite free over $B_{\text{dR}}^+$. Therefore, it is an exact category, and the functor $\eta$ is exact. Moreover, the rank and degree functions are additive on exact sequences.

Axiom (HN1) is evident. Let $f : X' \to X$ be such that $\eta(f) : X'_{\text{dR}} \to X_{\text{dR}}$ is an isomorphism; then we have $X' \subset X \subset X_{\text{dR}}^+$, and therefore $X'$ is a sub-object of $X$. Since $\dim(X') = \dim(X) - \dim(X/X')$, axiom (HN2) is satisfied. Moreover, if $X', X'' \subset X$ and $(X')_{\text{dR}} = (X'')_{\text{dR}}$, then we have $X = X'$, and therefore (HN3) is satisfied.

Finally, let $X', X'' \subset Y$ be strict subobjects in $\text{BCO}$, and let $X = X' \times_Y X''$. Since $X_{\text{dR}}^+ = (X')_{\text{dR}}^+ \times_{Y_{\text{dR}}^+} (X'')_{\text{dR}}^+$, $X_{\text{dR}}^+$ is a finite free $B_{\text{dR}}^+$-module, and therefore $X$ is oblique. Let $\overline{X} = X_{\text{dR}} \cap Y$; then we have $\overline{X} = \overline{X}' \cap \overline{X}''$. Since $X'$ and $X''$ are strict, we have $\overline{X}' = X'$ and $\overline{X}'' = X''$, and therefore $\overline{X} = X$ by Lemma 3.4.1. Therefore, $X$ is a strict subobject of $Y$. By Proposition 3.3.1, (HN4) is satisfied. \(<\)

Let $M$ be a $B_{\text{dR}}^+$-module of length $d$, $V$ be a $h$-dimensional $\mathbb{Q}_p$-vector space, and $\lambda : M \to V \otimes_{\mathbb{Q}_p} \widetilde{B}$ be a $B_{\text{dR}}^+$-linear map. Then the Banach-Colmez space $X(\lambda) = (B_{\text{c}} \otimes_{\mathbb{Q}_p} V) \times_{\widetilde{B} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p} M$ is oblique if, and only if, $\lambda$ is injective. We say that $\lambda$ is stable if it is injective and $X(\lambda)$ is stable.

Lemma 3.4.3. Let $\lambda : B_d \to V \otimes_{\mathbb{Q}_p} \widetilde{B}$ be a stable $B_{\text{dR}}^+$-linear map and $h = \dim_{\mathbb{Q}_p} V$. Let $\iota : E_{d,h} \to (B_{\text{dR}}^+)^h$ be the envelope map.

Then there exists an analytic, injective map $u : \text{Hom}(V, E_{d-1,h}) \to \text{Hom}(V, E_{d,h})$ such that the square

$$
\begin{array}{ccc}
\text{Hom}(V, E_{d,h}) & \xleftarrow{\iota \otimes u} & \text{Hom}_{B_{\text{dR}}^+}(B_d, \widetilde{B}^h) \\
\downarrow u & & \downarrow \\
\text{Hom}(V, E_{d-1,h}) & \xleftarrow{\iota \otimes u} & \text{Hom}_{B_{\text{dR}}^+}(B_{d-1}, \widetilde{B}^h)
\end{array}
$$

is commutative.

Proof. Let $\iota_{\mathbb{C}_p} = \iota : E_{1,h} \to \mathbb{C}_p^h$ be the map defined by $\iota_{\mathbb{C}_p}(x) = \theta(\iota(x)) = (\theta \varphi'(x))_{r=0,...,h-1}$. Then, by the fundamental lemma [Pla10, Lemma 2.3.3], the map $\iota \otimes \iota_{\mathbb{C}_p} : \text{Hom}(V, E_{1,h}) \to \text{Hom}_{B_{\text{dR}}^+}(B_d, \mathbb{C}_p(-1))^h$ is surjective, and its kernel $V'$ is a $h$-dimensional $\mathbb{Q}_p$-vector space. Let $c$ be a non-zero element of $V'$, and define $u : \text{Hom}(V, E_{d-1,h}) \to \text{Hom}(V, E_{d,h})$ by $u(f)(v) = f(v) \cdot c(v)$. Then $u$ is analytic and injective, and the square diagram is commutative as required. \(<\)
Lemma 3.4.4. Let \( \lambda : B_d \to V \otimes_{Q_p} \tilde{B} \) be a stable \( B_{\text{dR}}^+ \)-linear map and \( h = \dim_{Q_p} V \). Let \( \iota : E_{d,h} \to (B_{\text{dR}}^+)^h \) be the envelope map. Then \( \langle \iota \lambda \rangle \) contains the image of \( \text{Hom}_{(B_{\text{dR}}^+)}(\mathbb{C}_p, \tilde{B}^h) \) in \( \text{Hom}_{B_{\text{dR}}^+}(B_d, \tilde{B}^h) \).

Proof. By applying \( d-1 \) times Lemma 3.4.3 in succession, we get a commutative square

\[
\begin{array}{ccc}
\text{Hom}(V, E_{d,h}) & \xrightarrow{\iota \otimes t} & \text{Hom}_{B_{\text{dR}}^+}(B_d, \tilde{B}^h) \\
\uparrow & & \uparrow \\
\text{Hom}(V, E_{1,h}) & \xrightarrow{\iota \otimes t} & \text{Hom}_{B_{\text{dR}}^+}(\mathbb{C}_p, \tilde{B}^h)
\end{array}
\]

(3.4.5)

By the fundamental lemma [Plu10, Lemma 2.3.3], the bottom map is surjective. Therefore, the image of \( \iota \lambda \otimes t \) contains \( \text{Hom}(\mathbb{C}_p, \tilde{B}^h) \).

Proposition 3.4.6. Let \( \lambda : M \to V \otimes_{Q_p} \tilde{B} \) be a stable \( B_{\text{dR}}^+ \)-linear map, \( h = \dim_{Q_p} V \) and \( d \) be the length of \( M \). Let \( \iota : E_{d,h} \to (B_{\text{dR}}^+)^h \) be the envelope map.

Then \( \iota \lambda \otimes t : \text{Hom}(V, E_{d,h}) \to \text{Hom}(M, \tilde{B}^h) \) is surjective.

Proof. First assume that there exist non-zero \( M', M'' \) such that \( M = M' \oplus M'' \). Since \( \lambda \) is injective, there exist \( B_{\text{dR}}^+ \)-linear maps \( \lambda' : M' \to V' \otimes \tilde{B} \) and \( \lambda'' : M'' \to V'' \otimes \tilde{B} \) such that \( \lambda : M \to V \otimes_{Q_p} \tilde{B} \) is the direct sum of \( \lambda' \) and \( \lambda'' \). Then at least one of \( \mu(X(\lambda')) \) and \( \mu(X(\lambda'')) \) is greater than \( \mu(X(\lambda)) \), which contradicts the stability of \( \lambda \). Therefore, we may assume that \( M \) is simple, which implies that \( M = B_d \). We now prove the proposition by induction on \( d \).

The case \( d = 1 \) corresponds to the fundamental lemma [Plu10, Lemma 2.3.3].

Let \( \lambda' \) be the composed map \( B_{d-1} \xrightarrow{\iota} B_d \to V \otimes \tilde{B} \). Since \( \lambda \) is stable, \( \lambda' \) is also stable. By the induction hypothesis, \( t \lambda' \otimes t : \text{Hom}(V, E_{d-1,h}) \to \text{Hom}(B_{d-1}, \tilde{B}^h) \) is surjective. Since this map is analytic, its kernel \( V' \) is a \( h \)-dimensional \( Q_p \)-vector space. Multiplication in \( B_{\text{crys}}^+ \) induces an analytic map \( V' \otimes_{Q_p} E_{1,h} \to \text{Hom}(V, E_{d,h}) \); moreover, the image of this map is exactly the kernel of \( \text{Hom}(V, E_{d,h}) \to \text{Hom}(B_{d-1}(1), \tilde{B}^h) \). We get the following diagram:

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{} & V' \otimes_{Q_p} E_{1,h} & \xrightarrow{} & \text{Hom}(V, E_{d,h}) & \xrightarrow{} & \text{Hom}(B_{d-1}(1), \tilde{B}^h) & \xrightarrow{} & 0 \\
\downarrow & & \downarrow & & \downarrow \iota \otimes t & & \downarrow & & \downarrow \\
0 & \xrightarrow{} & \text{Hom}(\mathbb{C}_p, \tilde{B}^h) & \xrightarrow{} & \text{Hom}(B_d, \tilde{B}^h) & \xrightarrow{} & \text{Hom}(B_{d-1}(1), \tilde{B}^h) & \xrightarrow{} & 0
\end{array}
\]

(3.4.7)

By counting dimensions and heights, we see that the first line is exact. The second line is exact because \( \tilde{B} \) is an injective \( B_{\text{dR}}^+ \)-module.

By Lemma 3.4.4, the image of \( \iota \lambda \otimes t \) contains \( \text{Hom}(\mathbb{C}_p, \tilde{B}^h) \); therefore, the left vertical map is surjective. This proves that the map \( \iota \lambda \otimes t \) is surjective.

Proposition 3.4.8. Let \( X \) be a stable oblique Banach-Colmez space. Then \( X \) is isomorphic to \( E_{d,h} \), where \( d = \dim X \) and \( h = \text{ht} X \).
Proof. The proof mirrors that of [Plu10, Prop. 2.3.2]. By Prop. 3.1.3, we know that there exists a $B_{dR}^+$-linear map $\lambda : M \to V \otimes_{\mathbb{Q}_p} \mathcal{O}_L$ such that $X$ is isomorphic to $E(\lambda)$. By Prop. 3.4.6, $\lambda \otimes t : \text{Hom}(V, E_{d,h}) \to \text{Hom}(M, \mathcal{O}_L)$ is surjective. Let $L_{d,h}$ be the division algebra over $\mathbb{Q}_p$ with Brauer invariant $d/h$; then $\lambda \otimes t$ is left $L_{d,h}$-linear. By counting dimensions and heights, its kernel has dimension zero and height $h^2$ and is therefore a line over $L_{d,h}$. Let $a$ be a generator of $\text{Ker}(\lambda \otimes t)$ over $L_{d,h}$.

Define $f : V \otimes \mathbb{B}_c \to \mathcal{B}_{cris}$ by $f(v \otimes b) = b a(v)$. We readily check that $(\varphi - p^d) \circ f = 0$. To show that $f : E(\lambda) \to E_{d,h}$ is an isomorphism, it is enough to prove that $f(E(\lambda)) \subset \mathcal{B}_{cris}^+$ and $f : E(\lambda) \to E_{d,h}$ is surjective.

Let $x \in E(\lambda)$ and $y = f(x) \in \text{Fil}^{-d} \mathcal{B}_{cris}$. Since $(\lambda \otimes t)(a) = 0$, we have for all $r \in \mathbb{N}$: $\varphi^r(t^d y) = 0 \pmod{\text{Fil}^d \mathcal{B}_{dR}}$. Therefore, for all $r$, $\varphi^r(y) = t^{-d} p^{-d} \varphi^r(t^d y) \in \text{Fil}^0 \mathcal{B}_{cris}$. By [Fon94], 5.3.7 (i), this implies that $\varphi^2(y) \in \mathcal{B}_{cris}^+$ and therefore $y \in \mathcal{B}_{cris}^+$. By taking coordinates in $V$, we may write $a = (a_1, \ldots, a_h)$ with $a_i \in E_{d,h}$. Since $a$ generates $\text{Ker}(\lambda \otimes t)$ as a left $L_{d,h}$-module, the determinant $\det(\varphi^r a_i)$ is an unit in $\mathbb{Z}_p(d)$. Therefore, the map $f : E(\lambda) \to \mathcal{B}_{cris}^+$ is injective. By counting dimensions and heights, it follows that $f : E(\lambda) \to E_{d,h}$ is surjective and therefore an analytic isomorphism. $\triangle$

Proposition 3.4.9. Let $0 \leq d \leq d'$ and $h \geq 1$ be integers. Then $\text{Ext}^1_{\mathcal{B}_c}(E_{d,h}, E_{d,h}) = 0$.

Proof. ([Plu09, Proposition 9.4.4]) This result follows from Prop. 2.1.7 and counting dimensions and heights on the long Ext$^1$ sequence in the abelian category $\mathcal{B}_c$. $\triangle$

Theorem 3.4.10. Let $X$ be an oblique Banach-Colmez space. Let $Q^+ = [0, \infty]\cap \mathbb{Q}$.

(i) $X$ has a unique decreasing filtration $(\text{Fil}^\alpha X)_{\alpha \in Q^+}$, where $\text{Gr}^\alpha X = \text{Fil}^\alpha X / \varinjlim_{\beta > \alpha} \text{Fil}^\beta X$ is semi-stable of slope $\alpha$.

(ii) For all $\alpha = d/h \in Q^+$, there exists a unique integer $n_\alpha(X)$ such that $\text{Gr}^\alpha X$ is (non-canonically) isomorphic to $E_{d,h}^{n_\alpha(X)}$.

(iii) For any analytic map $f : X \to X'$ and for all $\alpha \in Q^+$, we have $f(\text{Fil}^\alpha X) \subset \text{Fil}^\alpha X'$.

(iv) The filtration Fil is (non-canonically) split.

Proof. (i) follows from Props. 3.4.2 and 3.3.5.

(ii) The stable case follows from Prop. 3.4.8, and the semi-stable case from $\text{Ext}^1_{\mathcal{B}_c}(E_{d,h}, E_{d,h}) = 0$ (Prop. 3.4.9).

(iii) follows from Prop. 2.1.7.

(iv) follows from Prop. 3.4.9. $\triangle$
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Written at Université Versailles-Saint-Quentin.
Permanent e-mail address: jerome.plut@normalesup.org