Supersymmetric Response of Bose-Fermi Mixture to Photoassociation

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(Dated: January 26, 2010)

We study supersymmetric (SUSY) responses to a photoassociation (PA) process in a mixture of Bose molecules $b$ and Fermi atoms $f$ which turn to mutual superpartners for a set of proper parameters. We consider the molecule $b$ to be a bound state of the atom $f$ and another Fermi atom $F$ with different species. The $b$-$f$ mixture and a free $F$ atom gas are loaded in an optical lattice. The SUSY nature of the mixture can be signaled in the response to a photon-induced atom-molecule transition: While two new types of fermionic excitations, an individual $b$ particle-$f$ hole pair continuum and the Nambu-Goldstone-fermion-like (or goldstino-like) collective mode, are concomitant for a generic $b$-$f$ mixture, the former is completely suppressed in the SUSY $b$-$f$ mixture and the zero-momentum mode of the latter approaches to an exact eigenstate. This SUSY response can be detected by means of the spectroscopy method, e.g., the PA spectrum which displays the molecular formation rate of $Ff\to b$.

PACS numbers: 67.85.Pq, 37.10.Jk, 11.30.Pb

Introduction — Recently, studies in the supersymmetry (SUSY) for a mixture of cold Bose and Fermi atoms have made spectacular progress \cite{1,2}. In such a cold atomic system, however, a Bose atom never transits to a Fermi atom, its superpartner or vice verse. In addition to the nonrelativity, this is another essential difference of this low-energy SUSY from the SUSY in high-energy physics. For the latter, such SUSY decay processes are always anticipated, e.g., a quark (lepton) may emit or absorb a gaugino and decays to a squark (slepton), the superpartner of the quark (lepton) \cite{3}.

To expose the interesting SUSY nature of the mixture, the effective "decay" process must be introduced. For a cold atomic SUSY mixture with Bose-Einstein condensation, there is an effective decay of SUSY generators since they behave as the fermion annihilation and creation operators \cite{2}. Therefore, the SUSY excitations can be simulated by a boson-enhanced fermionic excitation. As a result, a Nambu-Goldstone-fermion-like (or "goldstino-like") mode in the condensation phase of bosons could be observed by means of the single-particle spectroscopy \cite{2,4}.

To achieve an exact SUSY mixture, the system parameters have to be fine-tuned, which requires elaborate experimental setups and then loses the generality. In this article, we explore how to observe the SUSY response by means of a spectroscopy measurement, even if the mixture deviates slightly from the SUSY and the bosons do not condense to form a whole ordered phase. This can resolve the fine-tuning restraints in measuring the SUSY response. On the other hand, the explicit breaking of the SUSY may create new excitations, the bosonic particle-fermionic hole individual continuous excitations, other than the collective goldstino-like mode. Although our theory is nonrelativistic, the creation of these new excitations due to SUSY explicit breaking should be quite general. This may be a helpful point to the study of SUSY in relativistic theory.

We consider a mixture of Bose molecules $b$ and Fermi atoms $f$ with on-site interaction in a $d$-dimensional optical lattice ($d = 2, 3$) (see Fig.1(a)). With properly tuned interactions and hopping amplitudes, this $b$-$f$ mixture may become SUSY \cite{2}. We are interested in a special kind of molecule $b$, a bound state of $f$ and another species of Fermi atom $F$ with binding energy $E_b$, and restrict our analysis to the normal phase of the $b$-$f$ mixture \cite{2}. To probe the SUSY behaviors, we load a free Fermi atom $F$ gas, which does not interact both $b$ and $f$ directly. In a

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{(Color online) (a) Up-panel: The optical lattice with cold particles. The gray (green), black (red), and white dots denote Fermi atoms $f$, $F$, and the molecule $bf$, respectively. Low panel: The PA processes of two atoms to one molecule with the binding energy $E_b$. (b) The Feynman diagram for the linear response theory: The Green function of $Q$ is calculated by RPA. Up-panel: The wavy (red) lines and dotted (red) line denote the free Green functions of photon and $F$, respectively. Low panel: The solid (green) curves and dotted (black) curves denote the free Green functions of $f$ and $b$, respectively.}
\end{figure}
photoassociation (PA) process, the transitions between two atoms and one molecule, i.e., $Ff \leftrightarrow b$, are induced by two laser beams with frequencies $\omega_1$ and $\omega_2$. For the SUSY $b$-$f$ mixture, this resembles a high energy physics process: a ‘quark’ or a ‘lepton’ ($f$) absorbs a ‘fermionic gaugino’ ‘absorbs’ a $F$ and emits a photon and decays to a ‘squark’ or a ‘lepton’ ($b$) or vice versa. (One can also consider $f$ to be a Fermi molecule formed by the bound state of a Bose atom $b$ and a Fermi atom $F$, i.e., processes $Fb \leftrightarrow f$. We will study these processes separately.)

For a negative detuning $\delta_0 = \omega_2 - \omega_1 - E_b$, we show that the molecule dissociation process $b \rightarrow Ff$ is forbidden. In the formation process $Ff \rightarrow b$, two types of new fermionic excitations, an individual (bosonic) particle-(fermionic) hole pair continuum and a collective mode, emerge when the SUSY in the $b$-$f$ mixture is slightly broken. For a SUSY $b$-$f$ mixture, the former is completely suppressed while the latter in zero-momentum becomes an exact eigenstate, the Goldstino mode \[3]\). In this sense, we regard these excitations as the SUSY responses. The PA spectrum is directly related to the molecular formation rate varying as the detuning and faithfully describes these two types of excitations. The position of peak in the PA spectrum determines the frequency of the collective zero-momentum mode. This molecular formation rate is measured by the number variation of the $F$ atoms in time. Experimentally, the number counting of atoms is much simpler than detecting the single atom spectrum.

**Model setup** — The system illustrated in Fig. 1(a) is described by a Hamiltonian $H = H_0 + H_{ex}$, where $H_0 = H_{bf} + H_F$ with $H_{bf} = H_b + H_f + V$. By means of the Feshbach resonance \[2]\), the scattering lengths between $F$ and the $b$-$f$ mixture can be adjusted to negligibly small. In the tight-binding approximation, one has

\[
H_{\alpha} = -\sum_{i<j} t_{\alpha} a_{i\alpha}^\dagger a_{j\alpha} - \mu_{\alpha} \sum_i a_{i\alpha}^\dagger a_{i\alpha},
\]

\[
V = U_{bf} \sum_i n_i^b (n_i^f - 1) + U_{hf} \sum_i n_i^b n_i^f,
\]

where $a_{i\alpha}^\dagger = b_{i\alpha}$, $f_{i\alpha}$, and $F_{i\alpha}$ ($\alpha = b, f, F$) are the annihilation operators of $b, f$ and $F$ at site $i$; $\mu_{\alpha}$ and $n_{i\alpha}^\alpha = a_{i\alpha}^\dagger a_{i\alpha}$ are chemical potentials and the number operators at site $i$. The definitions of the hopping amplitudes $t_{\alpha}$ and the interaction strengths $U_{\alpha\beta}$ by the Wannier function $w_{\alpha}(r)$ can be found in the literature \[10\]. The spatial inhomogeneity of optical lattices trapping the atoms and molecules has been omitted. For the subsystem $b$-$f$ mixture, the Hamiltonian $H_{bf}$ is SUSY invariant if $t_b = t_f$, $U_{bf} = U_{fb}$ and $\mu_f = \mu_f$ \[1, 3\]. In order to prevent the phase separation, the parameters obey, e.g., $4\pi t_f t_b U_{bf} > U_{bf}^2$ in 2-dimensions, where $\rho_f$ is the density of $f$ atoms \[1\]. We choose the parameters of the system obeying this condition.

The PA processes are realized by simultaneously shining two laser beams with frequencies $\omega_1$ and $\omega_2$ into the lattice (shown in Fig. 1(a)). The $\omega_1$-beam may turn two free atoms $f$ and $F$ into a higher energy bound state $|1\rangle$ which then may transit to the molecule $b$ by emitting a photon with frequency $\omega_2$. Meanwhile, the molecule $b$ may also be excited to $|1\rangle$ by the $\omega_2$-beam and then is bound with some probability by emitting a photon with frequency $\omega_1$. For large detuning $\Delta_0$, the state $|1\rangle$ can be eliminated adiabatically, so that the PA is modeled by the tight-binding Hamiltonian

\[
H_{ex} = \sum_i (g_i b_i^\dagger f_i F_i e^{i\delta_0 t} + H.c.),
\]

where the detuning $\delta_0 = \omega_2 - \omega_1 - E_b$ with the effective driven frequency $\omega_c = \omega_2 - \omega_1$, and $g_i = g_0 \exp(-i\mathbf{k}_0 \cdot \mathbf{r}_i)$ with $g_0 \propto \int d^2 \mathbf{r} \exp(-i\mathbf{k}_0 \cdot \mathbf{r}) \psi^\dagger \psi \psi \psi$ being the coupling intensity independent of the site.

In the $k$-space, the Hamiltonian $H_{ex}$ is rewritten as

\[
H_{ex} = g_0 \sqrt{\rho} \sum_k Q_k \psi_{k-\mathbf{k}_0}^\dagger F_k e^{i\delta_0 t} + H.c.,
\]

where $Q_k = \sum p \, b_{p+k/2}^\dagger F_p / \sqrt{N}$ and $a_{\alpha}^\dagger = \sum \alpha \exp(-i\mathbf{k} \cdot \mathbf{r}_j) / \sqrt{V}$ (where $V$ stands for the volume and $a_{\alpha}^\dagger$ stand for $b_{\alpha}^\dagger, f_{\alpha}^\dagger, F_{\alpha}^\dagger$). $\rho = N/V$ is the total density of $b$ and $f$ with the particle number $N = \sum_i (n_i^b + n_i^f)$.

**Molecular formation rate** — The formation rate of the molecules $b$ can be counted by the PA variation of $F$-fermion number $R = \delta_t \langle \psi(t) | N_F | \psi(t) \rangle$ for $|\psi(t)\rangle$ being the time evolution from the ground state $|\psi\rangle = |g\rangle | F \rangle$ of $H_0$. It follows from the linear response theory that

\[
R = 2g_0^2 \sum_k \text{Im} D_R(k, -\delta_0),
\]

where the retarded Green function is given by

\[
D_R(k, \omega) = \int_{-\infty}^{\infty} dx A(k - k_0, x) \frac{n_f(x) - n_f(\epsilon^F_k)}{x - \epsilon^F_k - \omega - i0^+},
\]

in terms of one loop calculations (Fig. 1(b)). The single particle dispersions $\epsilon^F_k = -2t_0 \sum_{s=1}^d \cos k_s - \mu_f$, where the lattice spacings are set to be the unit. $n_f(x)$ is the Fermi distribution at temperature $T$ and the spectral function $A(k, \omega) = -\text{Im} \Pi_R(k, \omega) / \pi$ is defined by the retarded Green function $\Pi_R(k, \omega) = -i \int_0^\infty dt \langle g \{ Q_k(t), Q_k^\dagger(0) \} | g \rangle e^{i\omega t}$.

At sufficiently low temperature, the pole and branch cut in Eq. 10 are not qualitatively affected by $T$ and nor is the molecular formation rate. For simplicity, we take a zero temperature approximation in our calculation. It follows from Eq. 11 that the rate $R = R_{b \rightarrow F} - R_{F \rightarrow b}$ contains two parts:

\[
R_{b \rightarrow F} = \sum_k 2\pi g_0^2 \rho A(k - k_0, \epsilon^F_k - \delta_0) \theta(\delta_0 - \epsilon^F_k),
\]

\[
R_{F \rightarrow b} = \sum_k 2\pi g_0^2 \rho A(k - k_0, \epsilon^F_k - \delta_0) \theta(-\epsilon^F_k).
\]
which respectively are the dissociation rate for \( b \to Ff \) and the formation rate for \( Ff \to b \).

**Collective and individual fermionic modes** — In order to obtain \( R \) for the weak interactions, we perturbatively calculate \( \Pi_R(k, \omega) = P^{-1}[\Pi_R^{-1}(k, \omega) + U_{bf}]^{-1} \), which formally results from the equation of motion of \( Q_k \). It then follows from the random phase approximation (RPA) illustrated by the “bubble” in (Fig. 1(b)) that

\[
\Pi_0(k, \omega) = \int \frac{d^d p}{(2\pi)^d} n_f(\epsilon_{k+p}^f) + n_b(\epsilon_{k+p}^b),
\]

where \( E_{kp} = \epsilon_{k+p}^f - \epsilon_{k}^b + 2p_\theta \delta U + U_{bf} \rho \) with \( \delta U = U_{bb} - U_{bf} \) and \( \rho_b = N_b / V \); \( n_f(x) \) is the Bose distribution. The isolated pole and branch cut of \( \Pi_R(k, \omega) \) describe the collective and individual SUSY excitations of \( Q_k^\dagger |g \rangle \).

Next we consider the elementary excitations in the two-dimensional lattice with \( f \) atoms at half filling, i.e., \( \rho_f = 0.5 \). For the SUSY \( b-f \) mixture, i.e., \( \delta U = 0 \) and \( \delta t = t_b - t_f = 0 \), the dispersion of the collective modes, \( E_c(k) \sim \Delta - \alpha |k|^2 \) for the small \( |k| \), is read out from the poles of the retarded Green function \( \Pi_R(k, \omega) \) (see Fig. 2(a)), where \( \Delta = \mu_f - \mu_b \). For large \( |k| \), the energy \( E_c(k) = E_c(|k|, \theta) \) depends not only on \( |k| \) but also on the angle \( \theta = \arctan(k_y/k_x) \). The energy \( E_c(|k|, \theta) \) of \( Q_k^\dagger |g \rangle \) decreases as \( |k| \) increases for a fixed \( \theta \). The retarded Green’s function \( \Pi_R(0, \omega) = (\omega - \Delta + i0^+)^{-1} \) possesses a pole \( \omega = \Delta + i\epsilon \), which corresponds to the goldstino-like excitation \( \langle Q_k^\dagger |g \rangle \). This recovers the result in Ref. 2. The excitation spectrum is schematically shown in Fig. 2(b).

For the \( b-f \) mixture deviating slightly from SUSY, the retarded Green function \( \Pi_R(0, \omega) \) has an isolated pole and a branch cut, which correspond to a collective fermionic mode and individual (bosonic) particle-fermionic hole pair continuum modes, respectively. The pole in \( \omega_0 < E_0 - \delta t \) for \( \delta t > 0 \) (or \( \omega_0 < E_0 \) for \( \delta t < 0 \)) describes the shifted goldstino-like mode. The frequencies \( \omega_0 \) of collective zero-momentum mode for different \( \delta t \) are shown in Fig. 2(c). For \( k \neq 0 \), the pole of \( \Pi_R(k, \omega) \) has the form \( E_c'(k) \sim \omega_0 - \alpha |k|^2 \) for small \( |k| \). Remarkably, the branch cut \( l_0 \) of \( \Pi_R(0, \omega) \) emerges, which describes individual zero-momentum modes. Here, \( l_0 = [E_0 - 4\delta t, E_0] \) for \( \delta t > 0 \) (or \( [E_0, E_0 - 4\delta t] \) for \( \delta t < 0 \)), and \( E_0 = \Delta + 2p_\theta \delta U + U_{bf} \). Notice that for the weak interactions \( U_{bb} \) and \( U_{bf} \) the SUSY breaking from \( \delta U \) does not develop a branch cut but only shifts the positions of the pole and branch cut. The pole and the branch cut can be seen in the spectral function \( A(k, \omega) \), i.e., the peak and the hump in Fig. 2(d) for \( k = 0 \). Note that for the SUSY \( b-f \) the branch cut length \( l_0 \) of \( \Pi_R(0, \omega) \) shrinks to zero so that the individual continuum modes of zero momentum are completely suppressed. On the other hand, as the \( b-f \) mixture deviates from the SUSY, the goldstino-like mode is gradually suppressed. We examine the dependence of the spectral function on the interacting strength and find that the hump height may be depressed as the interaction becomes stronger, e.g., the height is lower than 0.5 for \( U_{bf} = U_{bb} = 0.5 \) comparing with ~ 10 in Fig. 2(d) for \( U_{bf} = U_{bb} = 0.1 \).

In order to study the PA spectrum of the molecular formation rate, we discuss the excitation spectrum shown in Fig. 2(b). For some momenta \( k \), there is a collective mode (dashed red curve) below the individual continuum. For other momenta \( k_F \), the dispersion of the collective mode merges into the continuum. However, for small momentum \( k \), there always exists a collective mode below the individual continuum. For convenience, we define a critical momentum \( k_Q(\theta) \), so that for a fixed \( \theta \), when \( |k| > k_Q(\theta) \) the negative frequencies of the mode \( Q_k \) emerge, i.e., \( A(k, \omega < 0) \neq 0 \) when \( |k| > k_Q(\theta) \), and \( A(k, \omega < 0) = 0 \) when \( |k| < k_Q(\theta) \).

**PA spectrum** — The rate \( R \) varies as detuning \( \delta_0 \) or the light frequency \( \omega_c \). Measurement of the \( b \) boson formation rate varying as \( \delta_0 \) is called the PA spectrum \( S(\delta_0) \). For a long wave photon, the coupling \( g_j \) varies slowly in space and the rates in Eq. (6) are approximately independent of \( k_0 \).

According to Eq. (5), the dissociation rate \( R_{b \to Ff} \) does not vanish only if \( \delta_0 > \epsilon_k^f \) and \( A(k, \epsilon_k^f - \delta_0) \neq 0 \). Because the spectral function \( A(k, \omega < 0) \neq 0 \) is defined by the re-
tard Green function, it does not vanish only when the energies for collective modes or individual modes of $Q_k$ are negative for the large $|k| > k_Q(\theta)$. We consider a dilute Fermi gas $F$ with the chemical potential $\mu_F \sim -\delta F$, the dispersion relation turns out $\varepsilon^F_k = |k|^2 - \mu_F$, where $\mu_F = \mu + 4\delta F$. In this case, the fermion $F$ possesses a small Fermi momentum $k_F = \sqrt{\mu_(\text{eff})/1F}$ which is much smaller than $k_Q(\theta)$ for small deviating $\delta F$. Therefore, $\varepsilon^F_k$ is always positive when $|k| > k_Q$ (see Fig. 3(b)).

For the negative detuning $\delta_0$, the condition $\delta_0 > \varepsilon^F_k$ is not satisfied in the regime $|k| > k_Q(\theta)$. That is, $A(k, \varepsilon^F_k - \delta_0)$ and $\theta(\delta_0 - \varepsilon^F_k)$ can not be non-zero simultaneously for the negative detuning and small Fermi momentum $k_F$. This finishes our proof of $R_{b \rightarrow F} = 0$.

The vanishing of $R_{b \rightarrow F}$ for the negative $\delta_0$ and small Fermi momentum $k_F$ can be understood in a more straightforward way. The transition $b \rightarrow F$ is described by the Hermite conjugate term (H.c.) in the Hamiltonian $H_{ex}$, which is a high-frequency oscillation term when $\delta_0 < 0$. Hence, the Fermi golden rule results in that $R_{b \rightarrow F}$ vanishing under the first-order perturbation (linear response).

For the negative $\delta_0$ and small Fermi momentum $k_F$ ($\mu_{\text{eff}} \ll t_F$), the molecule formation rate now is reduced to $R = -R_{F \rightarrow b}$ and

$$R_{F \rightarrow b} \simeq \begin{cases} Z_0 g^2_0 N/[2(t_F + \alpha')], & \text{for } \delta_0 = -\omega_0, \\ 2\pi g^2_0 \rho N A(0, -\delta_0), & \text{for } |\delta_0| \in I_0, \\ 0, & \text{otherwise} \end{cases}$$

which leads to our main result: The PA spectrum $S(\delta_0) = -R_{F \rightarrow b}$ (see Fig. 3(a)) displays the spectral function of excitations $Q_0^f |g\rangle$.

For the SUSY $b$-$f$ mixture, the length of branch cut $I_0$ tends to zero and the individual modes are suppressed. Meanwhile, the residue $Z_0 = 1$ and the formation rate $R_{F \rightarrow b} = g^2_0 N/[2(t_F + \alpha')] \equiv R_0 \propto N$ at $\delta_0 = -\Delta\mu$, and vanishes for the other detunings. There is a sharp peak at $\delta_0 = -\Delta\mu$ in the PA spectrum. For a generic $b$-$f$ mixture deviating from SUSY, the residue $Z_0 < 1$ decreases as $|\delta|_0$ increases. As a result, the peak height is lowered while its position is shifted to $\delta_0 = -\omega_0$. The ratio $R_{F \rightarrow b}(\omega_0)/R_0$ is shown in Fig. 3(b) for different $\delta_0$ and a small $\mu_{\text{eff}}$, where $R_{F \rightarrow b}(\omega_0)$ is the value of $R_{F \rightarrow b}$ at $\delta_0 = -\omega_0$. Remarkably, a minor hump develops in the region $\delta_0 \in I_0$ due to the emergence of individual modes (See Fig. 3(a)). As the system deviates further from SUSY, the individual modes are enhanced due to the sum rules $\int d\omega A(|0, \omega| = 1$. The temperature may suppress and broaden the peak and hump. These characters of the PA spectrum in the $b$-$f$ mixture are experimentally measurable SUSY responses to the light field.

Conclusions — We studied how to observe the SUSY nature of the $b$-$f$ mixture in optical lattices through PA spectra. For the Bose molecules formed with two species of Fermi atoms, we showed that the photon induced atom-molecule transition displays the signal of SUSY. As the response to the PA processes, a fermionic individual continuum and the goldstino-like mode were found. The PA spectrum can explicitly witness the molecular formation rate of $F \rightarrow b$. Because the goldstino-like mode in zero momentum turns to be the exact eigen state for the SUSY mixture, the major peak in the PA spectrum reflects the SUSY response to the light field, even if the mixture is not fine-tuned to a SUSY one.

The authors thank Jinbin Li and Peng Zhang for the useful discussions. YY is grateful to Kun Yang for sharing his idea in the earlier stage of this work. This work is supported in part by National Natural Science Foundation of China, the national program for basic research of MOST of China and a fund from CAS.

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