Quasiperiodic Atom Optics, Focusing, and Wave packet Rephasing

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(September 23, 1998)

We propose a laser field configuration which acts as a quasiperiodic atom optical diffraction grating. Analytical and computational results for the atomic center-of-mass wavefunction after the grating reveal a quasiperiodic density pattern, a semiclassical focusing effect, and a quasiperiodic self-imaging of the atomic wavefunction analogous to a Talbot effect.

Increased attention has been given to periodic atom optical systems and to drawing parallels between solid state physics and the corresponding atom optics in an easily-controlled environment. In the transient interaction regime, periodic atom optical elements have been used for spatial- and time-domain interferometry, atom focusing and lithography, and the Fresnel self-imaging (Talbot effect) of an atomic wave packet. In addition, optical lattices have been used to simulate solid state effects using atomic de Broglie waves. These atoms exhibit quantized motion, extended/localized state transitions, and Bloch oscillations. To further the connections with condensed matter and classical optics, the extension of atom optical experiments to quasiperiodic systems seems natural.

Recently, Guidoni and coworkers trapped cesium atoms into a quasiperiodic optical lattice formed by a three-dimensional laser configuration with incommensurate spatial components of intensity. Qian Niu and coworkers have been analyzing such optical lattices in one- and two-dimensions to understand their eigenstructure. We propose to extend atom lithography and interferometry experiments to create atomic beams or cold trapped atoms with quasiperiodic center-of-mass wave functions. We present analytical and computational results for the atomic wave function and density after interaction with a one-dimensional, quasiperiodic atom optical diffraction grating. Momentum distributions reveal the quasiperiodic nature of the wave packets. Furthermore, we show the possibility for wave packet revivals, essentially a quasiperiodic Talbot effect, where the initial atomic wave function can be (nearly) recovered.

A schematic of an experiment, similar to typical atomic focusing and Talbot arrangements, is shown in Fig. 1. Two pairs of off-resonant laser beams of width w intersect at a point along a monoenergetic, transversely-cooled atomic beam with velocity \( v_z \) propagating in the \( z \)-direction (or alternatively in the \( y \)-direction). Assuming the laser beam pairs are detuned from one another, we can ignore the cross-terms in the intensity pattern. The atoms are modeled as two-level center-of-mass plane waves with upper-state lifetimes of \( \Gamma^{-1} \). Hence, this optical configuration forms a light-shift potential for the atomic ground state,

\[
V(x, z) = V_1(z) \cos 2kx + V_2(z) \cos \sqrt{2}kx, \tag{1}
\]

where \( V_n(z) \approx h|\Omega_n(z)|^2 / 8\Delta_n \) for Rabi frequencies \( \Omega_n(z) \) and atom-field detunings \( \Delta_n \) of the laser beam pairs \( n = 1, 2 \). Spontaneous emission during the atom-field interaction is ignored by assuming that \( |\Omega_n(z)|^2 \Gamma w / 4v_z \Delta_n^2 \ll 1 \). The quasiperiodicity arises from the incommensurate wave vectors of the optical potential, and phase stability between the laser fields is essential for the potential’s integrity.

The atomic motion is described by an effective Schrödinger equation for the transverse wave function \( \phi(x, t) \) in the atomic rest frame, \( z = v_z t \). Assuming \( 1/2Mv_z^2 \gg (1/2Mv_x^2 + 1/2Mv_y^2 + V(x, z)) \), this equation reads

\[
\frac{i\hbar}{2M} \frac{\partial \phi}{\partial t} = \left[ \frac{p^2}{2M} + V(x, t) \right] \phi. \tag{2}
\]

The potential in the interaction region appears as a pulse of duration \( \tau = w/v_z \) in the atomic rest frame. The pulse shape is determined by the transverse laser profiles.

The discussion here is restricted to the Raman-Nath regime for square pulses, \( V(x, t) = V(x) \) for \( -\tau \leq t \leq 0 \) and \( V(x, t) = 0 \) for all other times, where

\[
V(x) = V_1 \cos 2kx + V_2 \cos \sqrt{2}kx. \tag{3}
\]

In the Raman-Nath approximation one assumes that \( \omega_k |V(x)| \tau^2 / \hbar \ll 1 \), where \( \omega_k = 2\hbar k^2 / M \) is a two-photon recoil frequency; the kinetic energy is ignored during the interaction, allowing an immediate integration of Eq. (2) using \( \phi(x, -\tau) = 1 \) and Eq. (3). We can write \( \phi(x, 0) = \exp[-iV(x)\tau / \hbar] \) or...
\[ \phi(x, 0) = \exp \left[ i(A_1 \cos 2kx + A_2 \cos \sqrt{2}kx) \right], \]  

where the pulse area \( A_n = -V_n \tau/h \). The standing-wave light fields acts as a quasiperiodic atomic phase grating.

To follow the evolution of the wave packet after the interaction, we can expand the exponentials in Eq. \( \text{[4]} \) in a plane wave representation. The resulting, initial wave function for the time-dependent Schrödinger equation is a superposition of free particle eigenstates \( \exp[ip_{m,n}x/h] \) with energies \( E_{m,n} = p_{m,n}^2/2M \), giving the result

\[ \phi(x, t > 0) = \sum_{m,n=-\infty}^{\infty} i^{m+n} J_m(A_1)J_n(A_2) \exp \left[ i2kx(m + n/\sqrt{2}) \right] \exp [-i\varphi_{m,n}(\omega_k\tau)], \]

where

\[ \varphi_{m,n}(\omega_k\tau) = E_{m,n}\tau/h = (m + n/\sqrt{2})^2 \omega_k\tau \]

is the phase of the momentum component

\[ p_{m,n} = 2\hbar k(m + n/\sqrt{2}) \]

and \( J_m \) is a Bessel function of the first kind. The momentum space wave function superposes a set of regularly spaced, but not periodic, components which are integer combinations of momentum exchanges between the atom and fields. By squaring Eq. \( \text{[4]} \) and using a sum rule for Bessel function products, the transverse atomic density, \( \rho(x, t) = \phi^*(x, t)\phi(x, t) \), can be written as

\[ \rho(x, t) = \sum_{m,n} \rho_{m,n}(\omega_k\tau) \exp \left[ i2kx(m + n/\sqrt{2}) \right]. \]

The density has time-dependent Fourier amplitudes

\[ \rho_{m,n}(\omega_k\tau) = J_m \left( 2A_1 \sin[(m + n/\sqrt{2})\omega_k\tau] \right) \times J_n \left( 2A_2 \sin[(m/\sqrt{2} + n/2)\omega_k\tau] \right), \]

creating a spatial pattern which evolves in time. Thus, the atomic density is not only a quasiperiodic function of the coordinate \( x \), but of the coordinate \( z = v_z t \) as well.

The density Fourier transform (DFT),

\[ \rho(q, t) = \int \frac{dx}{2\pi} \rho(x, t)e^{-iqx} = \sum_{m,n} \rho_{m,n}(\omega_k\tau)\delta(q - p_{m,n}/\hbar), \]

has peaks at \( q = 2k(m + n/\sqrt{2}) \) by Eq. \( \text{[6]} \). When squared, \( \rho(q, t) \) gives the time-dependent structure factor of the atomic distribution. In realistic experiments the delta-function lineshape of each spectral component would be broadened according to the initial momentum distribution of the transverse atomic beam. For a thermal velocity distribution with most probable speed \( u \), the replacement \( \rho_{m,n}(\omega_k\tau) \rightarrow \rho_{m,n}(\omega_k\tau) \exp \left[-(p_{m,n}ut/2\hbar)^2\right] \) in Eq. \( \text{[6]} \) is sufficient to account for Doppler dephasing.

To detect the density as a function of \( t \), one can scatter a transient probe off of the atoms at \( t \) to record the time evolution of certain Fourier components of the density. For example, a weak probe pulse with duration \( < (ku)^{-1} \) and wave vector \( k_p = -kx \) backscatters a field \( E_{bs} \) proportional to \( \rho_{1,0}(\omega_k\tau) \) in the +\( \hat{x} \)-direction: \( E_{bs} \sim J_1 (2A_1 \sin[\omega_k\tau]) J_0 (2A_2 \sin[\omega_k\tau/\sqrt{2}]) \exp \left[-(kut)^2\right] \). This is a type of free induction decay experiment to detect ground state population gratings [13].

More importantly, either direct atomic deposition or lithography using the atomic beam to impinge on a prepared substrate would reconstruct the atomic density at a fixed time. Atomic lithography has advanced to the point where atoms can carve nanostructures in materials such as silicon, silicon dioxide, and gold [21]. Such quasiperiodic surfaces could be used for solid state surface and transport studies. The implications for quantum and optical properties, including photon localization, may be profound owing to the quasiperiodic boundary conditions for the electron or optical waves [21].
We now examine two different phenomena in the pattern formed by the atoms (8), ignoring dephasing. Semiclassical (near-field) dynamics explain a focusing effect, similar to that seen after periodic phase gratings [2]. In Fig. 2 the optical potential is shown for \( A_1 = 5 \) and \( A_2 = 10 \). Each potential well acts as a lens which can focus atoms using the impulsive (dipole) force, \( F(x) = M \Delta v(x)/\tau = -\partial V(x)/\partial x, \) where \( \Delta v(x) \) is an impulsive velocity kick. To illustrate this effect, \( V(x) \) is Taylor expanded around its minimum at \( x = 0 \) to give the focusing force near this point, \( F(x) = M \Delta v(x)/\tau \approx (4V_1 + 2V_2)kx. \) Solving for \( \Delta v(x) \) and setting \( tf = x/\Delta v(x) \), this geometrical argument yields a focus at the time \( \omega k t_f = (2A_1 + A_2)^{-1} \) that translates into a spatial distance \( z_f = v_t t_f \). The ratio of pulse areas in Fig. 2, \( A_2/A_1 = 2 \), was chosen so that each standing wave contributes an equal semiclassical force.

The density at this "quasi"-focus with its peak at \( x = 0 \) is also shown in Fig. 2. Additional density peaks result from focusing by the shallower wells which occur at the quasiperiods of the potential. For example, the peak at \( 2kx \approx 7 \times 2\pi \approx 5\sqrt{2} \times 2\pi \) occurs near a potential well where the incommensurate standing waves are nearly in phase. In general, the irrational wave vector ratio, \( \sqrt{2} \), in this case, can be approximated as the ratio \( a_s/b_s \), where \( a_s \) and \( b_s \) are positive integers without common factors. Quasiperiods will then be defined by \( 2kx \approx 2\pi j a_s \approx 2\pi j b_s \sqrt{2} \) for any integer \( j \). A converging sequence \( G_s \) which approximates \( \sqrt{2} \) is given in Table 1 [22].

| \( s \) | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| \( a_s \) | 3 | 7 | 17 | 41 | 99 |
| \( b_s \) | 2 | 5 | 12 | 29 | 70 |

Table 1. Sequence of approximations for \( \sqrt{2} \) defines the spatial quasiperiods, \( 2kx \approx 2\pi j a_s \), and the quasi-Talbot times \( t_s \) in Eq. (11).

Several peaks in Fig. 2 are labeled by their values of \( (ja_s, jb_s) \). For smaller differences between \( ja_s \) and \( jb_s \sqrt{2} \), the quasiperiodicity is more pronounced (i.e., the density peaks near \( 2kx \approx 2\pi ja_s \) approach the size of the peak at \( x = 0 \)). Thus, the density reflects the quasiperiodicity of the system.

The DFT contains the spectral information important for lithography or scattering experiments. In Fig. 3 we plot the Fourier amplitudes \( |\rho_{m,n}(\omega k t)| \) [Eq. (1)] at the wave vectors \( q = 2k(m + n/\sqrt{2}) > 0 \) and at \( t = t_f \) for \( A_1 = 5 \) and \( A_2 = 10 \) again: this is the DFT of Fig. 2. The major peaks are labeled by \( (m, n) \) to show the variation of amplitudes. The inset of Fig. 3 has the same axes as the main graph and shows the range of \( q \) values with significant amplitudes. This Fourier spectrum has qualitative scaling properties: for any Fourier wave vector of the density \( q_0 \), a wave vector \( q' = 2k(m' + n'/\sqrt{2}) \) can be found which is arbitrarily close to \( q_0 \), even if the amplitude of that component is much less than one.

Atoms that propagate after interacting with periodic atomic elements exhibit a self-imaging of their wave function - Talbot effect - and return to uniform density at times \( \omega k t = 2\pi j \) for integer \( j > 0 \) [11]. While an exact self-imaging which reproduces \( \phi(x,0) \) [Eq. (3)] is impossible owing to the dispersion in Eq. (3), the quasiperiodic nature of the wave function can lead to a quasi-rephasing when \( \varphi_{m,n} \approx 2\pi j' \) for some integer \( j' > 0 \).

The appropriate observation times for self-imaging will produce phases for each momentum component which are nearly integer multiples of \( 2\pi \). For odd values of \( b_s \), the choice, \( \omega k t_s = 4\pi b_s \), from Eq. (1) gives the phases

\[
\varphi_{m,n}(4\pi b_s) = 2\pi(2b_m m^2 + b_s n^2 + 2b_s \sqrt{2} mn),
\]

where we again refer to Table 1, and \( b_s \sqrt{2} \approx a_s \) by construction. For even values of \( b_s \), the rephasing occurs at \( \omega k t_s = 2\pi b_s \) from Eq. (11). At these times the first two terms in Eq. (11) are integers for all \( m, n \). Furthermore, the third term is nearly an integer, as required.

The wave function phase \( \theta(x, t) \) is defined by \( \phi(x, t) = |\phi(x, t)| \exp[i\theta(x, t)] \). For exact self-imaging, the wave function should be the unitary exponential, Eq. (1), which has density equal to one and phase \( \theta(x, 0) = A_1 \cos(2kx) + A_2 \cos(2\sqrt{2}kx) \). In Fig. 4 we plot \( \theta(x, t_s) \) for \( A_1 = A_2 = 1 \) and \( s = 2, 4, \) and 6. The self-imaging becomes more pronounced at longer times \( t_s \), corresponding to a better approximation of \( \sqrt{2} \) by \( G_s \). The average values (denoted by the bar) and standard deviations (denoted by \( \sigma \)) of both \( \rho(x, t_s) \) and the phase difference, \( \delta_s = \theta(x, t_s) - \theta(x, 0) \), are shown in Table 2 for the cases of Fig. 4.

\[
\omega x t_s = \frac{2\pi}{\sqrt{2}} \approx 2\pi s j a_s \sqrt{2},
\]

Table 2.
experiments have conformed with the focusing condition \([2,5,23]\). Furthermore, ultracold atoms will exhibit a quasi-self-imaging of the wave function. The quasiperiodic density pattern can be used to create a quasiperiodic surface for condensed matter studies when used for atomic lithography. The atomic density is a function of the time of flight from the diffraction grating. Atoms come to quasiperiodic function, developing momentum components which are similarly incommensurate and therefore regular, but not periodic. The atomic density is a function of the time of flight from the diffraction grating. Atoms come to semiclassical "quasi"-focuses according to the depth and curvature of the potential wells. The quasiperiodic density pattern is nearly identical to \(\phi(x,0)\). In order to see these thin lens effects, the atom beam must be cooled or collimated near the recoil limit for focusing (\(\omega_k \ll \omega_k\)) or below the recoil limit for Talbot self-imaging (\(\omega_k \ll \omega_k\)). Recent experiments have conformed with the focusing condition \([12,20]\).

In summary, this letter has introduced the possibility of quasiperiodic atom optical elements made from laser intensity gratings with incommensurate wave vectors. The analytical results show that the atomic wave packet becomes a quasiperiodic function at \(t \propto \frac{\lambda M}{\Delta p_x} \sim (\omega_k)^{-1}\), the Doppler dephasing time. This condition does not present a problem for thick lens focusing and lithography schemes \([3,24]\). In order to see these thin lens effects, the atomic wave packet must be cooled or collimated near the recoil limit for Talbot self-imaging (\(\omega_k \ll \omega_k\)). The improvements in the density and phase are evident as \(\rho(x, t_s)\) and \(\delta_s = \theta(x, t_s) - \theta(0)\) for \(|2kx| \leq 16\pi\).

The authors would like to thank F. Nori, Q. Niu, G. Georgakis, and R. Merlin for discussions regarding this work. This work is supported by the National Science Foundation under Grant No. PHY-9414020, by the U.S. Army Research Office under Grant No. DAAG55-97-0113, and by the University of Michigan Rackham predoctoral fellowship.

ACKNOWLEDGMENTS

The authors would like to thank F. Nori, Q. Niu, G. Georgakis, and R. Merlin for discussions regarding this work. This work is supported by the National Science Foundation under Grant No. PHY-9414020, by the U.S. Army Research Office under Grant No. DAAG55-97-0113, and by the University of Michigan Rackham predoctoral fellowship.

| \(s\) | 2 | 4 | 6 |
|-----|---|---|---|
| \(\rho(x, t_s)\) | 1.019 | 1.00085 | 1.000024 |
| \(\sigma(\rho(x, t_s))\) | 0.505 | 0.0170 | 0.00508 |
| \(\delta_s\) (rads) | 0.0324 | 0.00645 | 0.00106 |
| \(\sigma(\delta_s)\) (rads) | 0.417 | 0.0899 | 0.0156 |

Table 2. Average and standard deviation of \(\rho(x, t_s)\) and \(\delta_s = \theta(x, t_s) - \theta(0)\) for \(|2kx| \leq 16\pi\).
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FIGURE CAPTIONS

Fig. 1. The atomic beam traverses the quasiperiodic potential formed by the laser beams and is detected by light scattering or lithography after free propagation.

Fig. 2. The optical potential in the interaction region for \( A_1 = 5 \) and \( A_2 = 10 \) and the corresponding atomic density at the quasi-focus, \( z_f = v_z [\omega_k(2A_1 + A_2)]^{-1} \). Quasiperiods are pronounced near \( 2kx \approx 2\pi ja_s \approx 2\pi jb_s\sqrt{2} \), where peaks are labeled by \((ja_s, jb_s)\).

Fig. 3. Fourier amplitudes \(|\rho_{m,n}(\omega_k t_f)|\) at \( q = 2k(m + n/\sqrt{2}) \) for the density in Fig. 2, as given by Eq. [2]. Larger amplitudes are labeled by \((m, n)\). Inset: Full range of Fourier components with significant amplitudes.

Fig. 4. Quasiperiodic Talbot Effect for \( A_1 = 1 \) and \( A_2 = 1 \). We plot the phase of the atoms at the quasi-Talbot times \( t_s \) for \( s = 2 (\ldots), 4 (\ldots), 6 (\ldots) \), versus \( \theta(x, 0) \), the initial phase (---), where \( \theta(x, 0) = A_1 \cos(2kx) + A_2 \cos(\sqrt{2}kx) \).
Interaction Region

\[-k(\hat{x}-\hat{z})/\sqrt{2}\]

\[-k\hat{x}\]

Detection/Lithography Region

\[k(\hat{x}+\hat{z})/\sqrt{2}\]

\[k\hat{x}\]

Atomic Beam

laser fields
\[ 2\pi V(x) \tau / \hbar \]

\[ \text{Density} \]

\[ \text{Atomic Density at } z = Z \]
$|\rho_{m,n}(\omega_{k_f})|$
