IRREDUCIBLE TRIANGULATIONS OF LOW GENUS SURFACES

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Abstract. The complete sets of irreducible triangulations are known for the orientable surfaces with genus of 0, 1, or 2 and for the nonorientable surfaces with genus of 1, 2, 3, or 4. By examining these sets we determine some of the properties of these irreducible triangulations.

1. Introduction

The irreducible triangulations of a surface provide a basis for obtaining all the triangulations of that surface. We can sequentially contract edges of a triangulation until an irreducible triangulation is produced. Reversing this sequence we can produce any triangulation of a surface with a sequence of vertex splittings starting with an irreducible triangulation. Thus all the triangulations of a surface can be generated from the irreducible triangulations of that surface by vertex splittings. The irreducible triangulations of a surface can be used to actually generate the triangulations of the surface.

Irreducible triangulations can also be used to check properties which are preserved by vertex splitting. For example, let \( P \) be a property possessed by some of the triangulations of the surface \( S \) which is preserved by vertex splitting such as “contains a cycle which separates the surface \( S \)”. If every irreducible triangulations of \( S \) possesses \( P \) then every triangulations of \( S \) possesses \( P \). Conversely, if there is a counterexample to “all triangulations of \( S \) possess \( P \)” then there is a counterexample among the irreducible triangulation of \( S \).

For any fixed surface the number of irreducible triangulations is finite [1]. Irreducible triangulations have been determined and displayed by a number of authors: the single irreducible triangulation of the sphere \((S_0)\) by Steinitz and Rademacher [19]; the two irreducible triangulations of the projective plane or the cross surface \((N_1)\) by Barnette [2]; the 21 irreducible triangulations of the torus \((S_1)\) by Lawrencenko [9]; and the 29 irreducible triangulations of the Klein bottle \((N_2)\) by Lawrencenko and Negami [10] and Sulanke [23]. The irreducible triangulations of the double torus \((S_2)\), the triple cross surface \((N_3)\), and the quadruple cross surface \((N_4)\) have been generated by the author using a computer program [22].

2. Definitions

A triangulation of a closed surface is a simple graph embedded in the surface such that each face is a triangle and any two faces share at most one edge.

In a triangulation \( T \) let \( abc \) and \( acd \) be two faces which have \( ac \) as a common edge. The contraction of \( ac \) is obtained by deleting \( ac \), identifying vertices \( a \) and \( c \), removing one of the multiple edges \( ab \) or \( cb \), and removing one of the multiple

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edges ad or cd. The edge ac of a triangulation T is *contractible* if the contraction of ac yields another triangulation of the surface in which T is embedded. If the edge ac is contained in a 3-cycle other than the two which bound the faces which share it then its contraction would produce multiple edges. Thus, for a triangulation T, not \( K_4 \) embedded in the sphere, an edge of T is not contractible if and only if that edge is contained in at least three 3-cycles. A triangulation is said to be *irreducible* if it has no contractible edges.

The operation of *splitting* a vertex is the reverse of contracting an edge. In a triangulation let ab and ac be two distinct edges. The *splitting* of the vertex a (along the edges ab and ac) is obtained by creating a new vertex \( a' \), three new edges \( a'a, a'b, \) and \( a'c \), and two new faces \( a'ab \) and \( a'ac \). The triangulation obtained by splitting a vertex is embedded in the same surface as the original triangulation.

We denote the orientable surface with genus \( g \), the sphere with \( g \) handles attached, as \( S_g \) and the nonorientable surface with genus \( g \), the sphere with \( g \) crosscaps attached, as \( N_g \). Define the *Euler genus* of the surface \( S \) to be \( 2 - \chi(S) \). For orientable surfaces the Euler genus is twice the genus and for nonorientable surfaces the Euler genus is the same as the genus.

### 3. Generating irreducible triangulations

The author has recently developed an algorithm [22] for generating irreducible triangulations of a surface by using the irreducible triangulations of other surfaces with smaller Euler genera. This algorithm was implemented as a computer program. The irreducible triangulations of \( S_2 \), \( N_3 \), and \( N_4 \) were generated and are available as computer files [24].

Before we briefly describe the algorithm used to generate irreducible triangulations we examine how an irreducible triangulation can be reduced to an irreducible triangulation with a lower genus. For simplicity we only consider orientable surfaces here. Let \( T \) be an irreducible triangulation of \( S_g \) with \( g > 0 \). Every edge of T is on a 3-cycle which is not a face. Many of these 3-cycles do not separate \( S_g \) into two components. Pick one of these nonseparating 3-cycles. Cut T along this 3-cycle thereby cutting one of the handles of \( S_g \). Cap the resulting two holes with new triangular faces to produce a new triangulation \( T' \) of \( S_{g-1} \). Contract contractible edges until an irreducible triangulation of \( S_{g-1} \) is obtained.

To generate an irreducible triangulation of \( S_g \) we reverse these steps in effect “growing a handle”. Start with an irreducible triangulation of \( S_{g-1} \). Split vertices checking each new triangulation to see if it can be used to form an irreducible triangulation of \( S_g \). The final step is the reverse of the cut and cap described above. Remove two faces and join the resulting boundary cycles in such a way that the resulting triangulation is still orientable.

An irreducible triangulation of \( N_g \) can be generated in a similar way by “growing a handle or a crosshandle”. Start with an irreducible triangulation of \( S_{g/2-1} \) or \( N_{g-2} \) and split vertices. In the final step we remove two faces and join the resulting boundary cycles in such a way that the resulting triangulation is nonorientable.

We can also “grow a crosscap” to generate an irreducible triangulation of \( N_g \) starting with an irreducible triangulation of \( S_{(g-1)/2} \) or \( N_{g-1} \). As new triangulations are produced by edge splitting we check for vertices with degree 6. When we remove a vertex with degree 6 and its incident faces a hole with a 6-cycle as a
Vertices | $S_1$ | $S_2$ | $N_1$ | $N_2$ | $N_3$ | $N_4$
--- | --- | --- | --- | --- | --- | ---
6 | | 1 | | | | 
7 | 1 | 1 | | | | 
8 | 4 | 6 | | | | 
9 | 15 | 19 | 133 | 37 | | 
10 | 1 | 865 | 2 | 2521 | 10347 | 
11 | 26276 | 2 | 4638 | 370170 | | 
12 | 117047 | 1320 | 189157 | | | 
13 | 159205 | 946 | 2067817 | | | 
14 | 54527 | 93 | 956967 | | | 
15 | 38195 | 50 | 700733 | | | 
16 | 664 | 7 | 186999 | | | 
17 | 5 | | | | 89036 | 
18 | | | | | 19427 | 
19 | | | | | 3975 | 
20 | | | | | 832 | 
21 | | | | | 79 | 
22 | | | | | 6 | 
Total | 21 | 396784 | 2 | 29 | 9708 | 6297982 |

Table 1. Irreducible triangulation by vertices

boundary is produced. By identifying the 3 pairs of opposite vertices on this 6-cycle we check if the result is an irreducible triangulation.

4. COUNTS

Due to the large number of irreducible triangulations of $S_2$, $N_3$, and $N_4$ the irreducible triangulations are not be displayed here but some of their properties are presented. For comparison we also include similar properties for $S_0$, $S_1$, $N_1$, and $N_2$.

Table 1 shows for each surface the number of irreducible triangulations having a given number of vertices.

5. NONCONTRACTIBLE SEPARATING CYCLES

Let $v_1v_2\ldots v_n$ be an n-cycle in a graph embedded on the surface $S$ and let C be the closed curve which is the embedding of $v_1v_2\ldots v_n$ in $S$. $v_1v_2\ldots v_n$ is separating if $S - C$ is disconnected. $v_1v_2\ldots v_n$ is contractible if a component of $S - C$ is a 2-cell, otherwise, it is noncontractible. This definition of a contractible cycle should not be confused with the definition of a contractible edge given earlier. Necessary conditions for the existence of a noncontractible separating cycle or NSC have been studied [6] [13] [20]. An NSC separates a surface into two components neither of which is a 2-cell. Thus a surface having an NSC must have genus greater than 1.

The existence of an NSC in a triangulation and the genera of the separated surfaces are preserved by vertex splitting. Thus if every irreducible triangulation of a surface has an NSC then every triangulation of that surface has an NSC.

Barnette conjectured that every triangulation of a surface with genus greater than 1 has an NSC. Lawrencenko and Negami [10] showed that every irreducible
triangulation of $N_2$ has an NSC and thus every triangulation of $N_2$ has an NSC. Ellingham, Zha, and Jennings have shown (without any reference to irreducible triangulations) that every triangulation of $S_2$ has an NSC.

By checking that each irreducible triangulation of $S_2$, $N_2$, $N_3$, and $N_4$ has an NSC we have the following result which is new only for $N_3$, and $N_4$.

**Theorem 1.** Every triangulation of $S_2$, $N_2$, $N_3$, or $N_4$ has an NSC.

Similarly, if an NSC separates a surface with Euler genus $g$ into two surfaces with Euler genera $h$ and $g - h$ then any triangulation obtained by vertex splitting of this triangulation has an NSC which separates the surface into two surfaces with Euler genera $h$ and $g - h$.

Thomassen conjectured (page 167) that given a triangulation of an orientable surface with genus $g$ and an integer $h$ such that $1 \leq h < g$, then the triangulation must contain an NSC such that the two surfaces separated by the NSC (after capping the holes with disks) have genera $h$ and $g - h$, respectively. This conjecture is equivalent to Barnette’s conjecture for $S_2$ (and $S_3$) but we can make a similar conjecture for nonorientable surfaces.

**Conjecture 1.** Given a triangulation of a nonorientable surface with Euler genus $g$ and an integer $h$ such that $1 \leq h < g$, then the triangulation must contain an NSC such that the two surfaces separated by the NSC have Euler genera $h$ and $g - h$, respectively.

By checking the irreducible triangulations of $N_4$ we have the following.

**Theorem 2.** Every triangulation of $N_4$ has an NSC which separates the surface into two surfaces each with Euler genus 2. Every triangulation of $N_4$ has an NSC which separates the surface into two surfaces with Euler genera 1 and 3, respectively.

From Theorems 1 and 2 it follows that Conjecture 1 is true for $N_2$, $N_3$, and $N_4$. Conjecture 1 and Theorem 2 do not specify the orientability of the separated surfaces. For example, there are triangulations of $N_3$ which do not have an NSC which separates the surface into $N_1$ and $N_2$. Such an example can be constructed using any irreducible triangulation of $N_1$ and any irreducible triangulation of $S_1$. Remove a face from each of these two irreducible triangulations and identify the resulting boundaries. Let $C_1$ be the closed curve in $N_3$ where the two surfaces were joined. Assume there is an NSC which separates $N_3$ into $N_1$ and $N_2$. Let $C_2$ be the closed curve in $N_3$ corresponding to this NSC. Due to the topology of $N_3$ the curves $C_1$ and $C_2$ must cross at least four times. But this contradicts the fact that the cycle corresponding to $C_1$ has length 3. Similarly, there are also triangulations of $N_3$ which do not have an NSC which separates the surface into $N_1$ and $S_1$.

For $N_3$, there are 9184 irreducible triangulations which have an NSC which separates the surface into $N_1$ and $N_2$ and there are 8533 irreducible triangulations which have an NSC which separates the surface into $N_1$ and $S_1$.

For $N_4$, there are 6062415 irreducible triangulations which have an NSC which separates the surface into $N_2$ and $N_2$ and there are 5971981 irreducible triangulations which have an NSC which separates the surface into $N_2$ and $S_1$.

The **edge-width** of a triangulation is the length of the shortest NSC in the triangulation. Tables 2 through 5 show the number of irreducible triangulations for a given number of vertices and a given value of the edge-width.
Table 2. Irreducible triangulation of $S_2$ by vertices and edge-width

| Vertices | Edge-width |
|----------|------------|
| 10       | 2  51  681  130  1 |
| 11       | 2  58  2249  16138  7818  11 |
| 12       | 25  1516  20507  72001  22877  121 |
| 13       | 710  13004  50814  78059  16609  9 |
| 14       | 8130  30555  12308  3328  205  1 |
| 15       | 36794  1395  3  1  2 |
| 16       | 661  3 |
| 17       | 5 |

Table 3. Irreducible triangulation of $N_2$ by vertices and edge-width

| Vertices | Edge-width |
|----------|------------|
| 8        | 1  5 |
| 9        | 1  5  2  11 |
| 10       | 1  1 |
| 11       | 2 |

Table 4. Irreducible triangulation of $N_3$ by vertices and edge-width

| Vertices | Edge-width |
|----------|------------|
| 9        | 1  119  13 |
| 10       | 1  140  1862  518 |
| 11       | 72  1248  1558  1760 |
| 12       | 502  811  4  3 |
| 13       | 912  34 |
| 14       | 93 |
| 15       | 50 |
| 16       | 7 |

6. Nonseparating cycles

Every cycle of a triangulation of $S_0$ separates and we exclude such triangulations in this section. In [22] it is shown that for every vertex of an irreducible triangulation there are at least two nonseparating 3-cycles containing that vertex. Thus every irreducible triangulation has a nonseparating cycle and, therefore, every triangulation has a nonseparating cycle. For triangulations of orientable surfaces the only topological type of a nonseparating cycle is one which cuts a handle.

Let $S$ be a triangulated nonorientable surface with a nonseparating cycle. Let $C$ be the closed curve which the embedding of that cycle in $S$. The cycle is one-sided if the neighborhood of $C$ in $S$ is homeomorphic to a Möbius band, otherwise the cycle is two-sided. The cycle is orientable-leaving if $S - C$ is orientable, otherwise the
cycle is nonorientable-leaving. For triangulations of nonorientable surfaces there are four possible topological types of nonseparating cycles depending on whether it is one- or two-sided and whether it is orientable- or nonorientable-leaving. At most three of these types of nonseparating cycles can occur for a fixed nonorientable surface since orientable surfaces have an even Euler genus.

By checking the irreducible triangulations of $N_1$, $N_2$, $N_3$, and $N_4$ we have the following theorem.

**Theorem 3.** Every triangulation of $N_1$ has a nonseparating cycle which is one-sided and orientable-leaving.

Every triangulation of $N_2$ has a nonseparating cycle which is one-sided and nonorientable-leaving and a nonseparating cycle which is two-sided and orientable-leaving.

Every triangulation of $N_3$ has a nonseparating cycle which is one-sided and orientable-leaving; a nonseparating cycle which is one-sided and nonorientable-leaving; and a nonseparating cycle which is two-sided and nonorientable-leaving.

Every triangulation of $N_4$ has a nonseparating cycle which is one-sided and nonorientable-leaving; a nonseparating cycle which is two-sided and orientable-leaving; and a nonseparating cycle which is two-sided and nonorientable-leaving.

**Conjecture 2.** If $g >= 3$ is odd then every triangulation of $N_g$ has a nonseparating cycle which is one-sided and orientable-leaving; a nonseparating cycle which is one-sided and nonorientable-leaving; and a nonseparating cycle which is two-sided and nonorientable-leaving.

If $g >= 4$ is even then every triangulation of $N_g$ has a nonseparating cycle which is one-sided and nonorientable-leaving; a nonseparating cycle which is two-sided and orientable-leaving; and a nonseparating cycle which is two-sided and nonorientable-leaving.
7. Maximal irreducible triangulations

Define $V_{\text{max}}(S)$ to be the maximum number of vertices in an irreducible triangulation of $S$. From Tables 2 through 5 we see that if an irreducible triangulation $T$ of one of these surfaces $S$ has $|V(T)| = V_{\text{max}}(S)$ then $T$ has an NSC of length 3. For $S_2$ this confirms a conjecture of Negami [13]. This suggests that maximal irreducible triangulations are made up of other triangulations joined at a single face of each.

We can use the following construction to obtain large irreducible triangulations which give a lower bound for $V_{\text{max}}(S)$. This construction is similar to the one given by Nakamoto and Ota [12] and the lower bound which it provides is a slight improvement.

For $N_1$ there is only one maximal irreducible triangulation $M$ which has 7 vertices. If we take two copies of $M$ and remove one face from each we can join them at the boundaries of these faces to obtain a triangulation of $N_2$. This triangulation has 11 vertices and is irreducible.

For each $g > 2$ we will construct a base triangulation $B_g$ of $S_0$ which when joined with $g$ copies of $M$ will produce an irreducible triangulation of $N_g$.

The left side of Figure 1 shows $B_3$ which is a triangulation of $S_0$ from which three faces (the shaded faces and the outside face) have been removed. Every edge of $B_3$ is on a removed face. If we join three punctured copies of $M$ at these faces we get a triangulation of $N_3$. This triangulation is irreducible since each edge of $B_3$ is now on at least three 3-cycles.

The right side of Figure 1 shows $B_4$ which is a triangulation of $S_0$ from which four faces have been removed. Again every edge of $B_4$ is on a removed face. When we join four punctured copies of $M$ at the removed faces we obtain an irreducible triangulation of $N_4$.

If we take two copies of $B_4$ and join them at removed faces then we obtain $B_6$ which is a triangulation of $S_0$ with 6 faces removed. Every edge of $B_6$ is either on a removed face or on at least three 3-cycles. Joining six punctured copies of $M$ we obtain an irreducible triangulation of $N_6$. We can repeat this construction to obtain $B_g$ for even $g > 2$. $|V(B_g)| = 6$ and each additional copy of $B_4$ adds 3 vertices such that $|V(B_g)| = 3g/2$. Each copy of $M$ adds 4 vertices. Thus for even $g > 2$ the number of vertices in the constructed irreducible triangulation of $N_g$ is $11g/2$. 

![Figure 1. Base triangulations for constructing large irreducible triangulations](image)
To obtain $B_g$ for odd $g$ we join $B_3$ to $B_{g-1}$. Then $|V(B_g)| = 3(g - 1)/2 + 1 = 3g/2 - 1/2$ and the number of vertices in the constructed irreducible triangulation of $N_g$ is $11g/2 - 1/2$.

Thus for any $g$ we have

$$V_{\text{max}}(N_g) \geq \lfloor 11g/2 \rfloor$$

For $S_1$ the only maximal irreducible triangulation has 10 vertices. Repeating the above construction with this triangulation as $M$ we obtain

$$V_{\text{max}}(S_g) \geq \lfloor 17g/2 \rfloor$$

In the above construction the triangulation $M$ does not need to be irreducible. Any edge of the removed face may be contractible and the resulting triangulation would still be irreducible.

A triangulation is almost irreducible if it is not irreducible and it has a face which is incident on all the contractible edges. If $M$ is almost irreducible then the construction still produces an irreducible triangulation. However, there are no almost irreducible triangulations of $N_1$ and there are no almost irreducible triangulations $T$ of $S_1$ for which $|V(T)| > V_{\text{max}}(S_1)$. There are 8 almost irreducible triangulations of $S_1$ but none have more than 9 vertices.

8. Pseudo-minimal triangulations

Two triangulations $T$ and $T'$ of a surface are equivalent if there is an isomorphism $h$ with $h(T) = T'$. That is, if $a$, $b$, and $c$ are vertices of $T$ then $ab$ is an edge of $T$ if and only if $h(a)h(b)$ is an edge of $T'$ and a face of $T$ is bounded by the cycle $abc$ if and only if a face of $T'$ is bounded by the cycle $h(a)h(b)h(c)$.

Let $ac$ be an edge in a triangulation $T$ and $abc$ and $acd$ be the two faces which have $ac$ as a common edge. The diagonal flip of $ac$ is obtained by deleting $ac$, adding edge $bd$, deleting the faces $abc$ and $acd$, and adding the faces $abd$ and $bcd$. An edge $ac$ of a triangulation $T$ is flippable if the diagonal flip of $ac$ yields another triangulation of the surface in which $T$ is embedded. Thus the edge $ac$ is flippable if there is not already an edge $bd$. Two triangulations are equivalent under diagonal flips if one is equivalent to a triangulation obtained from the other by a sequence of diagonal flips.

The number of vertices of an irreducible triangulation cannot be reduced by edge contraction. Negami defines a type of triangulation for which the number of vertices cannot be reduced by a combination of diagonal flips and edge contractions. An irreducible triangulation is said to be pseudo-minimal if it is equivalent under diagonal flips only to irreducible triangulations.

A triangulation is said to be minimal if there are no triangulations of the same surface with fewer vertices. It is clear that such a triangulation is also pseudo-minimal. The number of vertices in a minimal triangulation for nonorientable surfaces was determined by Ringel and for orientable surfaces by Jungerman and Ringel. It is given for all surfaces except $N_2$, $N_3$, and $S_2$ by the formula:

$$V_{\text{min}}(S) = \left\lceil \frac{7 + \sqrt{49 - 24\chi(S)}}{2} \right\rceil$$

For the three exceptions the value is one more than the value given by the formula: $V_{\text{min}}(N_2) = 8$, $V_{\text{min}}(N_3) = 9$, and $V_{\text{min}}(S_2) = 10$. 
Let $N(S)$ be the minimum value such that two triangulations $T$ and $T'$ of $S$ are equivalent under diagonal flips if $|V(T)| = |V(T')| \geq N(S)$. Negami has shown that such a finite value exists for any $S$.

$N(S_0) = V_{\text{min}}(S_0) = 4$, $N(S_1) = V_{\text{min}}(S_1) = 7$, $N(N_1) = V_{\text{min}}(N_1) = 6$, and $N(N_2) = V_{\text{min}}(N_2) = 8$ are known.

Checking the irreducible triangulations generated for $S_2$ we have determined that the 865 minimal triangulations are the only pseudo-minimal triangulations and that these pseudo-minimal triangulations form one equivalence class under diagonal flips. Thus $N(S_2) = V_{\text{min}}(S_2) = 10$ (14, 21). Similarly, the 133 minimal triangulations $N_3$ are the only pseudo-minimal triangulations and they form one equivalence class under diagonal flips. Thus $N(N_3) = V_{\text{min}}(N_3) = 9$.

The situation for $N_4$ is more interesting. The 37 minimal triangulations are the only pseudo-minimal triangulations. However, these pseudo-minimal triangulations are partitioned into three equivalence classes under diagonal flips with cardinality 32, 3, and 2. Using this complete list of pseudo-minimal triangulations of $N_4$ it is possible to show that $N(N_4) = V_{\text{min}}(N_4) + 1 = 10$.

Suppose for a surface $S$ there exist at least two inequivalent minimal triangulations which have no flippable edges. Then $N(S) > V_{\text{min}}(S)$. There are an infinite number of surfaces which have many inequivalent triangular embeddings of complete graphs. A triangular embeddings of complete graph is minimal and a complete graph has no flippable edges. Therefore there are an infinite number of surfaces $S$ for which $N(S) > V_{\text{min}}(S)$. For each of the surfaces $S_g, 3 \leq g \leq 15$ and $N_g, 5 \leq g \leq 30$, the author has found, using random computer searching, a pair of minimal triangulations which are inequivalent under diagonal flips. The existence of these pairs shows that if $3 \leq g \leq 15$ then $N(S_g) > V_{\text{min}}(S_g)$ and that if $4 \leq g \leq 30$ then $N(N_g) > V_{\text{min}}(N_g)$.

Conjecture 3. The only surfaces $S$ for which $N(S) = V_{\text{min}}(S)$ are $S_0$, $S_1$, $S_2$, $N_1$, $N_2$, and $N_3$.

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