On the gravitational energy of the Kaluza Klein monopole

Robert B. Mann\textsuperscript{1} and Cristian Stelea\textsuperscript{2}

\textsuperscript{1}Perimeter Institute for Theoretical Physics
31 Caroline St. N, Waterloo, Ontario N2L 2Y5, Canada
\textsuperscript{1,2}Department of Physics, University of Waterloo
200 University Avenue West, Waterloo, Ontario N2L 3G1, Canada

Abstract

We use local counterterm prescriptions for asymptotically flat space to compute the action and conserved quantities in five-dimensional Kaluza-Klein theories. As an application of these prescriptions we compute the mass of the Kaluza-Klein magnetic monopole. We find consistent results with previous approaches that employ a background subtraction.

1 Introduction

The problem of defining energy in theories involving gravity has a long-standing history. One would like for instance to be able to evaluate the total energy of an isolated object. Throughout the years many expressions have been proposed for computing the total energy. However, contrary to initial expectations, it was soon realised that finding satisfactory quantities is a very difficult task. The essential idea in computing the energy is to consider the values of the fields far away from the object and compare them with a background configuration, that is, with a ‘no-fields situation’. This is for instance the approach considered when defining the ADM mass (see for instance [1]).

A related problem is that of computing the gravitational action of a non-compact space-time. The gravitational action consists of the bulk Einstein-Hilbert term and it must be

\textsuperscript{1}E-mail: rbmann@sciborg.uwaterloo.ca
\textsuperscript{2}E-mail: cistelea@uwaterloo.ca
supplemented by the boundary Gibbons-Hawking term in order to have a well-defined vari-
ational principle. When evaluated on non-compact solutions of the field equations it turns 
out that both terms diverge. The general remedy for this situation is to consider the values 
of these quantities relative to those associated with some background reference spacetime, 
whose boundary at infinity has the same induced metric as that of the original spacetime. 
The background is chosen to have a topological structure that is compatible with that of 
the original spacetime and also one requires that the spacetimes approaches it sufficiently 
rapidly at infinity.

Unfortunately such background subtraction procedures are marred with difficulties: even 
if some choices of such reference background spaces present themselves as ‘natural’, in gen-
eral these choices are by no means unique. Moreover, it is not always possible to embed 
a boundary with a given induced metric into the reference background and for different 
boundary geometries one needs different reference backgrounds [2]. A good example of the 
difficulties one might encounter in such an endeavour is that of the celebrated Taub-NUT 
solution (see for instance [3]-[8]).

Similar difficulties and ambiguities are encountered when trying to compute the action 
and the conserved charges of the Kaluza-Klein monopole [9,10], and in particular its gravita-
tional energy. Many such expressions for the conserved charges have been analysed in detail 
[12,13,14]; the consistent answers they yield when applied to the Kaluza-Klein monopole 
solution are for a definite choice of the reference background, one that is not a solution of 
the field equations. Moreover it is not a flat background, so that the energy expression for a 
Kaluza-Klein monopole is problematic. In general potential ambiguities arise in computing 
energy and other conserved quantities in dimensionally-reduced gravitation theories. This is 
partly because there are many distinct topological sectors, each of which requires a different 
background, and partly because within a given fixed topological sector, there may not be 
suitable background.

Motivated by recent results in the AdS/CFT conjecture, Balasubramanian and Kraus [15] 
proposed adding a term (referred to as a counterterm) to the boundary at infinity, which 
is a functional only of curvature invariants of the induced metric on the boundary. Such 
terms will not interfere with the equations of motion because they are intrinsic invariants of 
the boundary metric. By choosing appropriate counterterms, which cancel the divergences, 
one can then obtain well-defined expressions for the action and the energy momentum of 
the spacetime. Unlike background subtraction, this procedure is intrinsic to the spacetime 
of interest and is unambiguous once the counterterm is specified. While there is a general 
algorithm for generating the counterterms for asymptotically (A)dS spacetimes [16,17], the 
asymptotically flat case is considerably less-explored (see however [18] for some new results 
in this direction). Early proposals [19,20,21] engendered study of proposed counterterm 
expressions for a class of (d + 1)-dimensional asymptotically flat solutions whose boundary 
topology is $S^n \times R^{d-n}$ [16]. This counterterm method has been applied to the five-dimensional 
black ring [23] and to an asymptotically Melvin spacetime [24].

The interesting properties of the Kaluza-Klein monopole merit further study in this 
context. In the present letter we propose using a local counterterm prescription to compute 
its action and its conserved quantities in the five-dimensional Kaluza-Klein theory. In the 
next section we introduce the counterterm action and the expression for the conserved mass
using the boundary stress-energy tensor. In the third section we apply this method to compute the action and the conserved mass of the Kaluza Klein monopole from the five-dimensional point of view, while in the fourth section we compute the monopole energy from the four-dimensional perspective of the dimensionally reduced theory, using two distinct counterterm prescriptions. The last section is dedicated to conclusions, in which we comment on the relationships between the various approaches.

2 The counterterm action

In $(d+1)$-dimensions, the gravitational action is generally taken to be:

$$I_g = -\frac{1}{16\pi G} \int_M d^{d+1}x \sqrt{-g} R - \frac{1}{8\pi G} \int_{\partial M} d^d x \sqrt{-h} K$$  \hspace{1cm} (1)

Here $M$ is a $(d+1)$-dimensional manifold with metric $g_{\mu\nu}$, $K$ is the trace of the extrinsic curvature $K_{ij} = \frac{1}{2} h_{ij} \nabla_k n^k$ of the boundary $\partial M$ with unit normal $n^i$ and induced metric $h_{ij}$.

For asymptotically flat 4-dimensional spacetimes, the counterterm $I_{ct} = \frac{1}{8\pi G} \int d^3 x \sqrt{-h} \sqrt{2R}$ was proposed \cite{19, 20} to eliminate divergences that occur in (1). An analysis of the higher dimensional case \cite{16} suggested in 5 dimensions the counterterm

$$I_{ct} = \frac{1}{8\pi G} \int d^4 x \sqrt{-h} \sqrt{\frac{R^2}{R^2 - R_{ij}\hat{R}^{ij}}}$$  \hspace{1cm} (3)

where $\hat{R}_{ij}$ is the Ricci tensor of the induced metric $h_{ij}$ and $\hat{R}$ is the corresponding Ricci scalar. This counterterm removes divergencies in the action for an asymptotically flat spacetime with boundary topology $S^3 \times R$ and also for a $S^2 \times R^2$ boundary topology.

By taking the variation of the action (3) with respect to the boundary metric $h_{ij}$ we obtain the following boundary stress-energy tensor:

$$8\pi G (T_{ct})^{ij} = \frac{\hat{R}^2}{(R^2 - R_{kl} R^{kl})^2} \left[ 3 R^{ij} \hat{R}_{kl} R^{kl} - R^{ij} R^2 + 2 \hat{R} R R^{ik} R^j_k + R^3 h^{ij} - R R_{kl} R^{kl} h^{ij} \right]$$

$$+ \Phi^{(i,j)k}_{,k} - \frac{1}{2} \Box \phi^{ij} - \frac{1}{2} h^{ij} \Phi^{kl}_{,kl},$$

where:

$$\Phi^{ij} = \frac{\hat{R}^2}{(R^2 - R_{kl} R^{kl})^2} \left[ 2 R R^{ij} + (R^2 - 3 R_{kl} R^{kl}) h^{ij} \right],$$

so that the final boundary stress energy tensor is given by:

$$T_{ij} = \frac{1}{8\pi G} (K_{ij} - K h_{ij} + (T_{ct})_{ij})$$  \hspace{1cm} (4)
For a five-dimensional asymptotically flat solution with a fibred boundary topology $R^2 \rightarrow S^2$, we find that the action (1) can also be regularised using the following equivalent counterterm

$$I_{ct} = \frac{1}{8\pi G} \int d^4x \sqrt{-h} \sqrt{2R}$$

(5)

where $R$ is the Ricci scalar of the induced metric on the boundary, $h_{ij}$. By taking the variation of this total action with respect to the boundary metric $h_{ij}$, it is straightforward to compute the boundary stress-tensor, including (5):

$$T_{ij} = \frac{1}{8\pi G} (K_{ij} - Kh_{ij} - \Psi(R_{ij} - R h_{ij}) - h_{ij} \Box \Psi + \Psi_{,ij})$$

where we denote $\Psi = \sqrt{\frac{2}{R}}$. If the boundary geometry has an isometry generated by a Killing vector $\xi^i$, then $T_{ij} \xi^j$ is divergence free, from which it follows that the quantity

$$Q = \oint_{\Sigma} d^3S^i T_{ij} \xi^j,$$

associated with a closed surface $\Sigma$, is conserved. Physically, this means that a collection of observers on the boundary with the induced metric $h_{ij}$ measure the same value of $Q$, provided the boundary has an isometry generated by $\xi$. In particular, if $\xi^i = \partial/\partial t$ then $Q$ is the conserved mass $M$.

The counterterm (3) was proposed in [16] for spacetimes with boundary $S^2 \times R^2$, or $S^3 \times R$. On the other hand, the counterterm (3) is essentially equivalent to (5) for $S^2 \times R^2$ boundaries. We find that when the boundary is taken to infinity both expressions cancel the divergences in the action. Our choice of using (5) can be motivated by the fact that the expression for the boundary stress-tensor is considerably simpler. However, different counterterms can lead to different results when computing the energy, seriously constraining the various choices of the boundary counterterms (see for instance [25, 26] for a general study of the counterterm charges and a comparison with charges computed by other means in $AdS$ context). As we shall see in the next section, both expressions lead to a background-independent Kaluza-Klein mass that agrees with other answers previously known in the literature; however, we do find slight discrepancies in the diagonal components of the boundary stress-tensor.

### 3 The mass of the Kaluza-Klein Monopole

We begin by reviewing the original magnetic monopole solution in four dimensions that arises as a Kaluza-Klein compactification of a five dimensional vacuum metric [9, 10] (see also [11]). The essential ingredient used in the monopole construction is a four-dimensional version of the Taub-NUT solution, with Euclidean signature. To construct the monopole solution, we start with the Euclidean form of the Taub-NUT solution [27, 28, 29]:

$$ds^2 = F_E(r)(d\chi - 2n \cos \theta d\varphi)^2 + F^{-1}_E(r)dr^2 + (r^2 - n^2)d\Omega^2$$

where

$$F_E(r) = \frac{r^2 - 2mr + n^2}{r^2 - n^2}$$

(6)
In general, the $U(1)$ isometry generated by the Killing vector $\frac{\partial}{\partial \chi}$ (that corresponds to the coordinate $\chi$ that parameterizes the fibre $S^1$) can have a zero-dimensional fixed point set (referred to as a ‘nut’ solution) or a two-dimensional fixed point set (correspondingly referred to as a ‘bolt’ solution). The regularity of the Euclidean Taub-nut solution requires that the period of $\chi$ be $\beta = 8\pi n$ (to ensure removal of the Dirac-Misner string singularity), $F_E(r = n) = 0$ (to ensure that the fixed point of the Killing vector $\frac{\partial}{\partial \chi}$ is zero-dimensional) and also $\beta F_E'(r = n) = 4\pi$ in order to avoid the presence of the conical singularities at $r = n$.

With these conditions we obtain $m = n$, yielding

$$F_E(r) = \frac{r - n}{r + n} \quad (7)$$

Taking now the product of this Euclidean space-time with the real line, we obtain the Kaluza-Klein monopole, as described by the following five-dimensional Ricci flat metric:

$$ds^2 = -dt^2 + F_E(r)(d\chi - 2n \cos \theta d\varphi)^2 + F_E^{-1}(r)dr^2 + (r^2 - n^2)d\Omega^2 \quad (8)$$

The other possibility to explore is using the Taub-bolt solution in four-dimensions instead of the nut solution. In this case the Killing vector $\frac{\partial}{\partial \chi}$ has a two-dimensional fixed point set in the four-dimensional Euclidean sector. The regularity of the solution is then ensured by demanding that $r \geq 2n$, while the period of the coordinate $\chi$ is $8\pi n$ and for the bolt solution we obtain (with $m = 5n/2$) [30]:

$$F_E(r) = \frac{(r - 2n)(r - \frac{1}{2}n)}{r^2 - n^2} \quad (9)$$

As in the case of the nut solution, we take the product with the real line and obtain a metric in five-dimensions that is a solution of the vacuum Einstein field equations. The physical interpretation of this last solution was recently clarified by Liang and Teo [31]. It corresponds to a pair of coincident extremal dilatonic black holes with opposite unequal magnetic charges.

Before we apply the counterterm prescription to compute the components of the boundary stress-tensor, let us notice that the boundary topology of the KK monopole for constant, finite values of the radial coordinate $r$ is that of a squashed 3-sphere times a real line. Therefore, one might expect that the proper counterterm action to use should be the one corresponding to an $S^3 \times R$ topology. However using that counterterm we find that it is impossible to cancel out the divergences as $r \to \infty$. Rather we note that, as $r \to \infty$, the boundary topology is that of a fibre bundle $R \times S^1 \hookrightarrow S^2$ as the radius of $S^2$ grows with $r$, while the radius of $S^1$ reaches a constant value. Thence, asymptotically, the choice of the counterterm [13] is natural and indeed, we find that using this counterterm we can eliminate the divergences in the action and obtain finite values for the total mass.
Using the metric with the general expression (3) for the function $F_E(r)$ we find

\begin{align*}
8\pi GT^t_t &= \frac{m}{r^2} + O(r^{-3}) \\
8\pi GT^x_x &= \frac{2m}{r^2} + O(r^{-3}) \\
8\pi GT^x_\phi &= \frac{4mn \cos \theta}{r^2} + O(r^{-3}) \\
8\pi GT^\theta_\theta &= -\frac{m^2 - 2n^2}{2r^3} + O(r^{-4}) \\
8\pi GT^\phi_\phi &= -\frac{m^2 - 2n^2}{2r^3} + O(r^{-4})
\end{align*}

the rest of the terms being of order $O(r^{-3})$ or higher. Then the conserved mass associated with the Killing vector $\xi = \partial/\partial t$ is found to be:

$$M = \frac{4\pi mn}{G}$$

However using the counterterm (3) and the boundary stress-tensor (4) we obtain

\begin{align*}
8\pi GT^t_t &= \frac{m}{r^2} + O(r^{-3}) \\
8\pi GT^x_x &= \frac{2m}{r^2} + O(r^{-3}) \\
8\pi GT^x_\phi &= \frac{4mn \cos \theta}{r^2} + O(r^{-3}) \\
8\pi GT^\theta_\theta &= -\frac{m^2 - 2n^2}{2r^3} + O(r^{-4}) \\
8\pi GT^\phi_\phi &= -\frac{m^2 - 2n^2}{2r^3} + O(r^{-4})
\end{align*}

(11)

It is easy to see that this boundary stress-energy tensor leads to the same mass as above. Notice however that some of the components of the stress-energy tensor (11) are different from the ones obtained in (10).

For Kaluza-Klein monopole we have $m = n$ and we obtain $M = \frac{4\pi n^2}{G}$, which is easily seen to be the same with the one derived in [12, 13] by using a background subtraction procedure.\footnote{The parameter $\lambda_\infty$ used in [12] corresponds in our case to $4n$, while $k = 8\pi G$.}

For the bolt monopole we have $m = 5n/2$ and using either prescription (3) or (5) we obtain $M = \frac{10\pi n^2}{G}$. In both cases the regularized action takes the form $I = \beta M$, where $\beta$ is the periodicity of the Euclidian time $\tau = it$. Upon application of the Gibbs-Duhem relation $S = \beta M - I$ we find that the entropy is zero, as expected since there are no horizons.

\section{The monopole mass from the four dimensional perspective}

It is instructive to compute the conserved mass after we perform the dimensional reduction along the $\chi$ direction down to four-dimensions. While both the metric and the fields in gen-
eral have singularities at the origin, this is not necessarily an obstruction since the conserved charges are in general computed as surface integrals at infinity.

Using the metric ansatz:

$$ds^2_{5d} = e^\phi ds^2_{4d} + e^{\frac{2\phi}{\sqrt{3}}} (d\chi + \mathcal{A})^2$$

we obtain the four-dimensional fields:

$$ds^2_{4d} = -F_E^\frac{1}{2} dt^2 + F_E^{-\frac{1}{2}} (r^2 - n^2) d\Omega^2$$

$$\mathcal{A} = -2n \cos \theta d\phi, \quad e^{\frac{\phi}{\sqrt{3}}} = F_E^{-\frac{1}{2}}$$

(12)

It is clear that the metric is asymptotically flat and the form of the electromagnetic potential \(\mathcal{A}\) describes the electromagnetic field generated by a magnetic monopole.

In four-dimensions we can use the counterterm (2), whose only difference from (5) is that we are integrating now over a three-dimensional boundary instead of a four-dimensional one. A similar computation with the one performed in five-dimensions yields

$$8\pi G_4 T^t_t = \frac{m}{r^2} + O(r^{-3})$$

$$8\pi G_4 T^\theta_\theta = \frac{n^2 - m^2}{2r^3} + O(r^{-4})$$

$$8\pi G_4 T^\phi_\phi = \frac{n^2 - m^2}{2r^3} + O(r^{-4})$$

(13)

for boundary stress-energy tensor, where \(G_4\) is Newton’s constant in four-dimensions. Then the conserved mass associated with the Killing vector \(\xi = \partial/\partial \tau\) is found to be:

$$\mathcal{M} = m^{2G_4}$$

Noting that we have the relation \(G = 8\pi n G_4\) we find that the mass computed using the four-dimensional geometry agrees precisely with the one computed in the five-dimensional theory.

Finally, we shall compute the mass using the methods from [18]. In that work, Mann and Marolf put forward a new counterterm that is also given by a local function of the boundary metric and its curvature tensor. The new counterterm is taken to be the trace of a symmetric tensor \(\hat{K}_{ij}\) that is defined implicitly in terms of the Ricci tensor \(\mathcal{R}_{ij}\) of the induced metric on the boundary via the relation

$$\mathcal{R}_{ik} = \hat{K}_{ik} \hat{K} - \hat{K}_i^m \hat{K}_{mk},$$

(14)

In contrast to previous counterterm proposals (such as (3)) this new counterterm assigns an identically zero action to the flat background in any coordinate systems while giving finite values for asymptotically flat backgrounds. The renormalized action leads to the usual conserved quantities that can also be expressed in terms of a boundary stress-tensor whose
expression involves the electric part of the Weyl tensor:

\[ T_{ij}^0 u^j = \frac{1}{8 \pi G_4} r E_{ij} u^j \]

Here \( E_{ij} \) is the pull-back to the boundary of the contraction of the bulk Weyl tensor with the induced metric while \( u^i \) is the normal to the spacelike surface \( \Sigma \). Computing this expression in the \( r \to \infty \) limit and contracting with the Killing vector \( \xi = \partial/\partial t \) we obtain:

\[ T_{ij}^0 \xi^i u^j = \frac{1}{8 \pi G_4} \frac{m}{r^2} + O(r^{-3}) \]

while the conserved mass is found by simple integration to be:

\[ \mathcal{M} = \frac{m}{2G_4} \]

in agreement with previous computations.

5 Discussion

In General Relativity there are many known expressions for computing the energy in asymptotically flat spacetimes. The general idea is to study the asymptotic values of the gravitational field, far away from an isolated object, and compare them with those corresponding to a gravitational field in the absence of the respective object. However, most of these proposals will provide results that are relative to the choice of a reference background (be it a spacetime metric or merely a connection). The background must be chosen such that its topological properties match the solution whose action and conserved charges we want to compute. However, this does not fix the choice of the background and moreover, there might be cases in which the topological properties of the solution rule out any natural choice of the background.

Most of these difficulties are simply avoided once we resort to the counterterm-method \cite{15, 16, 17, 19, 20}. The main motivation for the present work was to investigate the local counterterm prescription for computing the action and the conserved charges in the five-dimensional Kaluza-Klein theory and, more specifically, for the Kaluza-Klein monopole solution. The main advantage of this approach is that it gives results that are intrinsic to the solutions considered, that is, the results are not ‘relative’ to some reference background. Using two distinct proposals for the boundary counterterm we computed the mass of the Kaluza-Klein magnetic monopole and found agreement in both cases with previous results derived by other means \cite{12, 13}. We also extended our results to the case of the Kaluza-Klein bolt-monopole solution. In the general context of Kaluza-Klein theory it is also tempting to examine the energy from the point of view of the dimensionally-reduced theory. While the metric and also the fields do have in general singularities at origin, this is not necessarily

\(^2\)Even if the four-dimensional solution is not a vacuum metric, the net effect of the matter fields is to give only sub-leading order corrections and to leading order we can still replace the bulk Riemann tensor with the Weyl tensor.
an obstruction since the conserved charges are in general computed as surface integrals at infinity. In the four-dimensional theory, using the counterterm \( \text{(2)} \) proposed by Lau \[19\] and Mann \[20\] as well as the new counterterm proposed in \[18\] we computed the mass of the monopole and found it to be equal to the five-dimensional mass. A similar result was proved in \[12\] using background subtraction methods.

Let us remark that the counterterm method for computing conserved charges might shed some light on the old problem of which compactifications are preferred in Kaluza-Klein theories. This problem involves a comparison of the gravitational energies corresponding to different vacua. The advantage of the counterterm method is that by providing results that are intrinsic to spacetime geometries it obviates the need to consider only the solutions corresponding to the same asymptotic reference background.

Finally, we believe that our results warrant further study of the counterterm method in asymptotically flat spacetimes. That the conserved charges computed from these various counterterm-supplemented actions agree is not surprising in view of the results of refs. \[15, 25, 26\]. The actions associated with the three counterterm prescriptions \( \text{(2, 3, 14)} \) all lead to well-defined variational principles (as shown on general grounds in \[18\]) and the actions are all finite on all solutions with the asymptotics of the Kaluza-Klein monopole. Consequently the energies computed from the various approaches can only differ by c-number terms \[26\]. However, while it is clear that the distinct choices \( \text{(3) and (5)} \) yield the same mass for the monopole, the diagonal components of the boundary stress-energy tensor have slightly different coefficients. The implications of this remain an interesting subject for future investigation.

Acknowledgements

This work was supported by the Natural Sciences and Engineering Council of Canada.

References

[1] R. Wald, “General Relativity,” University of Chicago Press, 1984.

[2] K.C.K. Chan, J.D.E. Creighton and R.B. Mann, Phys. Rev. D54 3892 (1996).

[3] S. W. Hawking, C. J. Hunter and D. N. Page, “Nut Charge, Anti-de Sitter Space and Entropy” Phys. Rev. D 59 (1999) 044033 [hep-th/9809035].

[4] S. W. Hawking, C. J. Hunter, “Gravitational Entropy and Global Structure” Phys. Rev. D 59 (1999) 044025 [hep-th/9808085].

[5] M. M. Taylor-Robinson, “Higher dimensional Taub-Bolt solutions and the entropy of non compact manifolds,” [hep-th/9809041].

[6] M. M. Akbar and G. W. Gibbons, “Ricci-flat metrics with U(1) action and the Dirichlet boundary-value problem in Riemannian quantum gravity and isoperimetric inequalities,” Class. Quant. Grav. 20 , 1787 (2003) arXiv:hep-th/0301026.
[7] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, “Large N phases, gravitational instantons and the NUTs and bolts of AdS holography,” Phys. Rev. D 59, 064010 (1999) [arXiv:hep-th/9808177].

[8] R. Clarkson, L. Fatibene and R.B. Mann, “Thermodynamics of (D+1)-Dimensional NUT-charged AdS Spacetimes” Nucl. Phys. B652 (2003) 348

[9] Rafael D. Sorkin “Kaluza-Klein Monopole” Phys. Rev. Lett. B 51 (1983) 87

[10] D. J. Gross and M. J. Perry “Magnetic Monopoles in Kaluza-Klein theory” Nucl. Phys. B226 (1983) 29

[11] R. Mann and C. Stelea, “Higher dimensional Kaluza-Klein monopoles,” Nucl. Phys. B729, 95 (2005), [arXiv:hep-th/0505114].

[12] L. Bombelli, R. K. Koul, G. Kunstatter, J. H. Lee and R. D. Sorkin, “On Energy In Five-Dimensional Gravity And The Mass Of The Kaluza-Klein Monopole,” Nucl. Phys. B 289, 735 (1987).

[13] S. Deser and M. Soldatc, “Gravitational Energy In Spaces With Compactified Dimensions,” Nucl. Phys. B 311, 739 (1989).

[14] V. K. Onemli and B. Tekin, “Kaluza-Klein monopole in AdS spacetime,” Phys. Rev. D 68, 064017 (2003) [arXiv:hep-th/0301027].

[15] V. Balasubramanian and P. Kraus, Commun. Math. Phys. 208, 413 (1999).

[16] P. Kraus, F. Larsen and R. Siebelink, “The gravitational action in asymptotically AdS and flat spacetimes,” Nucl. Phys. B 563, 259 (1999) [arXiv:hep-th/9906127].

[17] A.M. Ghezelbash and R.B. Mann, JHEP 0201, 005 (2002).

[18] R. B. Mann and D. Marolf, “Holographic Renormalization of Asymptotically Flat Space- times,” [arXiv:hep-th/0511096].

[19] S. R. Lau, “Lightcone reference for total gravitational energy,” Phys. Rev. D 60, 104034 (1999) [arXiv:gr-qc/9903038].

[20] R. B. Mann, “Misner string entropy,” Phys. Rev. D 60, 104047 (1999) [arXiv:hep-th/9903229].

[21] J. Ho, “Holographic counterterm actions and anomalies for asymptotic AdS and flat spaces,’ [arXiv:hep-th/9910124].

[22] M.H. Dehghani and R.B. Mann, “Quasilocal Thermodynamics of Kerr and Kerr-anti-de Sitter Spacetimes and the AdS/CFT Correspondence” Phys. Rev. D64 (2001) 044003.

[23] D. Astefanesei and E. Radu, “Quasilocal formalism and black ring thermodynamics,” [arXiv:hep-th/0509144].
[24] E. Radu, “A note on Schwarzschild black hole thermodynamics in a magnetic universe,” Mod. Phys. Lett. A 17, 2277 (2002) [arXiv:gr-qc/0211035].
[25] S. Hollands, A. Ishibashi and D. Marolf, “Comparison between various notions of conserved charges in asymptotically AdS-spacetimes,” Class. Quant. Grav. 22, 2881 (2005) [arXiv:hep-th/0503045].
[26] S. Hollands, A. Ishibashi and D. Marolf, “Counter-term charges generate bulk symmetries,” [arXiv:hep-th/0503105].
[27] A. H. Taub “Empty Space-Times Admitting a Three Parameter Group of Motions” Annal. Math. 53 (1951) 472.
[28] E. Newman, L. Tamburino, and T. Unti “Empty-space generalization of the Schwarzschild metric” J. Math. Phys. 4 (1963) 915.
[29] C. W. Misner, J. Math. Phys. 4 (1963) 924; C. W. Misner, in Relativity Theory and Astrophysics I: Relativity and Cosmology, edited by J. Ehlers, Lectures in Applied Mathematics, vol. 8 (American Mathematical Society, Providence, RI, 1967), p. 160.
[30] D. N. Page “Taub-NUT instanton with a Horizon” Phys. Lett. B 78 (1978) 249-251
[31] Y. C. Liang and E. Teo, “Black diholes with unbalanced magnetic charges,” Phys. Rev. D 64, 024019 (2001) [arXiv:hep-th/0101221].