Optimal Transportation in the presence of a prescribed pressure field

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Abstract

The optimal (Monge-Kantorovich) transportation problem is discussed from several points of view. The Lagrangian formulation extends the action of the Lagrangian \( L(v, x, t) \) from the set of orbits in \( \mathbb{R}^n \) to a set of measure-valued orbits. The Eulerian, dual formulation leads an optimization problem on the set of sub-solutions of the corresponding Hamilton-Jacobi equation. Finally, the Monge problem and its Kantorovich relaxation are obtained by reducing the optimization problem to the set of measure preserving mappings and two point distribution measures subjected to an appropriately defined cost function.

In this paper we concentrate on mechanical Lagrangians \( L = \frac{|v|^2}{2} + P(x, t) \) leading, in general, to a non-homogeneous cost function. The main results yield existence of a unique flow of homomorphisms which transport the optimal measure valued orbit of the extended Lagrangian, as well as the existence of an optimal solution to the dual Euler problem and its relation to the Monge- and Kantorovich formulations.

1 Introduction

1.1 Motivation

Consider the Schrodinger equation on a domain \( \Omega \subseteq \mathbb{R}^n \)

\[
\frac{i\hbar}{\partial t} \psi = \hbar^2 \Delta_x \psi + P \psi ; \quad (x, t) \in \Omega \times [0, T]
\]

(1.1)

where \( P = P(x, t) \) is a given potential (pressure field). If \( \partial \Omega \neq \emptyset \) we specify a boundary condition under which equation (1.1) is well posed and generate a unitary semigroup. We are interested in the following question: Given a pair of probability measures \( \mu_0, \mu_1 \) on \( \Omega \) (say, absolutly continuous subjected to the densities \( \rho_0, \rho_1 \)), can one solve the two point boundary problem, given only the data \( \rho_0(x) = |\psi|^2(x, 0) \) and \( \rho_1(x) = |\psi|^2(x, T) \)?

This problem is a prototype for a variety of wave equations admitting a Lagrangian formulation. However, in spite of the underlying Lagrangian structure associated with (1.1), we do not yet know weather such a two point boundary value problem is well posed for this equation.

We turn, therefore, to the semi-classical approximation \( \hbar \to 0 \), where \( \psi \) is written as \( \sqrt{\rho} e^{i\phi/\hbar} \). Then the Schrodinger equation is reduced to the Hamilton-Jacobi (HJ) equation

\[
\phi_t + \frac{1}{2} |\nabla_x \phi|^2 = P
\]

(1.2)
coupled with the continuity equation
\[ \rho_t + \nabla_x \cdot [\rho \nabla_x \phi] = 0 \quad (1.3) \]

The two-point boundary problem for the Schrödinger equation in the semi-classical approximation is, therefore, reduced to solving (1.2) and (1.3) under the sole data of the initial and end probability densities \( \rho_0 = \rho(0) \) and \( \rho_1 = \rho(T) \), without any data on the velocity field \( \nabla_x \phi \) whatsoever.

We attempt to associate a Lagrangian on the state space of orbits of probability densities \( \rho(t) \) defined for each \( t \in [0,T] \). Since we wish to generalize the standard Lagrangian formalism defined on classical orbits \( x : [0,T] \to \Omega \), we shall extend this to a space of "relaxed orbits" of the form \( \mu_t = \mu_t(dx)dt \), where \( \mu_t \) is a Borel probability measure on \( \Omega \) for any \( t \in [0,T] \). The special class of deterministic orbits is embedded in this set as \( \mu_t = \delta_{x(t)} \), where \( x \) is a classical orbit.

The next step is to define a metric on this set of relaxed orbits which generalizes the action \( \int_0^T |\dot{x}(t)|^2 dt \) for deterministic orbits. It turns out that the above problem is closely related to the classical Monge-Kantorovich problem of optimal transportation subjected to quadratic cost. An historical background of this is given below.

1.2 Historical Background

The classical problem of optimal mass transportation was suggested by Monge in the 18'th century \([M]\): given a cost function \( c(x,y) \) (originally, \( c = |x-y| \)) and a pair of Borel probability measures \( \mu_0, \mu_1 \) on (say) a common probability space \( \Omega \), minimize

\[ \int c(x,T(x))\mu_0(dx) \quad (M) \]

along all Borel mappings \( T : \Omega \to \Omega \) which transport \( \mu_0 \) into \( \mu_1 \) \( (T#\mu_0 = \mu_1) \), namely

\[ \mu_0(T^{-1}A) = \mu_1(A) \quad \forall \text{ Borel sets } A \subset \Omega \quad (1.4) \]

The Monge problem was revived in the last century. In particular, Kantorovich \([K]\) introduced in 1942 a relaxation, reducing the Monge problem to a linear programming in a cone of two-point distributions over \( \Omega \) whose marginals are \( \mu_0, \mu_1 \) respectively:

\[ \min \lambda \int \int c(x,y)\lambda(dx,dy) \quad \pi^{(0)}_\# \lambda = \mu_0 , \quad \pi^{(1)}_\# \lambda = \mu_1 \quad (K) \]

Here \( \pi^{(i)}, i = 0,1 \) are the natural projections of \( \Omega \times \Omega \) on its factors. A particular attention is given to the Wasserstein metrics

\[ W_p(\mu_0, \mu_1) = \left[ \min \lambda \int \int |x-y|^p\lambda(dx,dy) \quad \pi^{(0)}_\# \lambda = \mu_0 , \quad \pi^{(1)}_\# \lambda = \mu_1 \right]^{1/p} \quad (1.5) \]

where \( p \geq 1 \).

The most striking advantage of the relaxed problem is that a minimizer always exists by the compactness of the set of probability measures (assuming \( c \) is continuous and \( \Omega \) is
compact). It can be shown [Am] that, if \( c \) is continuous and \( \mu_0 \) contains no atoms, then
the minimum of the Kantorovich problem coincides with the infimum of the Monge problem. The existence of an optimal Monge mapping is reduced to existence of such a minimizer of
the Kantorovich problem which is supported on a graph of a Borel map.

Further progress was achieved in the last few decades. In the late 80’s and early 90’s
Brenier [B] studied the Monge problem with a quadratic cost \( c = |x - y|^2 \) on \( \mathbb{R}^n \) and showed
the existence of a unique, optimal Monge map provided \( \mu_0, \mu_1 \) possess finite second moments
and \( \mu_0 \) is absolutely continuous with respect to Lebesgue. Moreover, he showed that this
map is the gradient of a convex function \( \Phi \) which solves the dual problem

\[
\inf \left\{ \int_{\mathbb{R}^n} \Phi \mu_0(dx) + \int_{\mathbb{R}^n} \Psi d\mu_1(dx) \right\} ; \ \Phi, \Psi \in C(\mathbb{R}^n) \ ; \ \Phi(x) + \Psi(y) \geq x \cdot y \ \forall \ x, y \in \mathbb{R}^n ,
\]

(1.6)

It turned out that any map \( T \) which transports \( \mu_0 \) to \( \mu_1 \) must be of the form \( \nabla \Phi \circ S \)
where \( S \) perseveres \( \mu_0 \). This results is interpreted as a Polar factorization for mappings,
generalizing the matrix polar factorization. Another interpretation of \( \nabla \Phi \) is as a monotone
vectorized rearrangement in the class of maps transporting \( \mu_0 \) to \( \mu_1 \). A generalization of this
result also holds for more general, strictly convex (and concave) homogeneous cost functions
\( c(x, y) = h(x - y) \), using special definitions of convexity ([GM], [C] and references therein).

1.3 Objectives and main results

In general, if \( \mu_0 \) contains an atom, then there is, in general, no deterministic mapping \( T \) of
any type which maps \( \mu_0 \) into \( \mu_1 \), so there is no sense to compare the deterministic Monge
problem (M) with the probabilistic Kantorovich problem (K). However, we may still consider
the following alternative formulation in terms of an optimal flow with respect to some family
of cost functions \( c_{t_1, t_2} = J(x, y, t_1, t_2) \):

\[
\text{(F) : Find a relaxed orbit } \mu = \mu(t) dt \text{ and a flow of diffeomorphisms } T_{t_1}^{t_2} : \Omega \rightarrow \Omega \text{ for }
\quad t_1, t_2 \in (0, T) \text{ such that}
\]

(i) \( T_{t_1}^{t_2} \) is the optimal Monge mapping with respect to \( c_{t_1, t_2} \) transporting \( \mu(t_1) \) to \( \mu(t_2) \) for
any \( t_1, t_2 \in (0, T) \).

(ii) \( \lim_{t \to 0} \mu(t) = \mu_0 \) and \( \lim_{t \to T} \mu(t) = \mu_1 \) in the weak sense of measures.

(iii) The limits \( \lim_{t \to T} T_{t}^{T} =: T_{t}^{T} \) exists uniformly and \( T_{t}^{T} \) is a continuous mappings for any
\( \tau \in (0, T) \).

It is feasible that, once a solution to the flow problem \( F \) is provided, a \( c_{0, T} \) optimal
solution to the Monge problem \( M \) with respect to \( \mu_0, \mu_1 \) exists by \( T = \lim_{\tau \to 0} T_{\tau}^{T} \) provided
the later limit exists as a Borel map.

Our starting point is the definition of a norm \( ||\mu||_p \) of a measure-valued orbit as the minimal
\( L^p_\mu \)-norm of the velocity fields \( v \) which satisfy the weak form of the continuity equation

\[
\left\{ \begin{array}{l}
v = v(x, t) ; \ \int_{0}^{T} \int_{\Omega} [\phi_t + v \cdot \nabla_x \phi] \mu(t)(dx) dt = 0 \ ; \ \forall \phi \in C_0^1(\Omega \times [0, T]) \end{array} \right. \quad . \quad (1.7)
\]
and
\[
\|\mu\|_p := \left[ \inf_v \int_{\Omega \times [0,T]} |v|^p \mu(t)(dx) dt \right]^{1/p}
\] (1.8)

where the infimum is taken over all \(\mu\)-measurable vectorfield \(v\) satisfying (1.7). Denote the set for which \(\|\mu\|_p < \infty\) as \(H_p\). This is a normed cone. In section 2 we shall indicate some of its properties and prove a compactness embedding of \(H_p\) (for \(p > 1\)) in a set of orbits which satisfies Holder continuity in an appropriate topology. In particular, the end conditions \(\mu_0 := \mu(0), \mu_1 := \mu(T)\) are uniquely defined for \(\mu \in H_p\) where \(p > 1\).

In the rest of the paper we concentrate on the case \(p = 2\). The connection between the cost function \(c_{t_1,t_2} = J\) posted in formulation (F) above and the pressure \(P\) is as follows:

The function \(J = J_P\) is the action associated with the Lagrangian
\[
J_P(x,y,t_1,t_2) = \inf_{\overline{x}} \left\{ \int_{t_1}^{t_2} \left[ \frac{|\overline{\phi}(t)|^2}{2} + P(\overline{\phi}(t), t) \right] dt ; \overline{\phi} : [t_1, t_2] \to \Omega, \overline{\phi}(t_1) = x, \overline{\phi}(t_2) = y \right\}
\]

The main result of this paper, formulated in section 3 reveals a connection between the following approaches:

**L**: The Lagrangian approach: Minimize a Lagrangian \(L_P\) on the space of orbits \(\mu \in H_2\):
\[
\mathcal{L} := \inf_{\mu} L_P(\mu) \ ; \ L_P(\mu) := \frac{1}{2} \|\mu\|^2_2 + \int_0^T \int_{\Omega} P(\mu(t), dx) dt \ , \ \mu \in H_2 \ , \ \mu(0) = \mu_0, \mu(T) = \mu_1.
\]

**E**: The Eulerian approach: Maximize on the set of velocity potentials \(\phi\)
\[
\mathcal{E} := \sup_{\phi} \left[ \int_{\Omega} \phi(x,T)\mu_1(dx) - \int_{\Omega} \phi(x,0)\mu_0(dx) \right]
\] (1.9)

where the supremum is taken in the set of all functions \(\phi = \phi(x,t)\) which are subsolutions of the Hamilton-Jacobi (HJ) equation (1.2) in a sense to be defined.

**M**: The Monge approach: Minimize on the set of mappings verifying (1.4)
\[
\mathcal{M} := \inf_{T} \left\{ \int_{\Omega} J_P(x,T(x),0,T)\mu_0(dx) ; \ T\#\mu_0 = \mu_1 \right\}
\]

**K**: The Kantorovich approach: Minimize on the set of 2-point probability measures with prescribed marginal
\[
\mathcal{K} := \min_{\lambda} \left\{ \int \int J_P(x,y,0,T)\lambda(dx,dy) ; \ \pi_0(x)\lambda = \mu_0, \ \pi_1(x)\lambda = \mu_1 \right\}
\]

Our first result reveals the relation between the above formulation: If \(P \in C^1(\Omega \times [0,T])\) then
\[
\mathcal{L} = \mathcal{E} = \mathcal{K}
\]
holds for arbitrary (probability, Borel) end measures $\mu_0, \mu_1$. As discussed above, the Monge problem may not have a solution at all (e.g., if $\mu_0$ contains an atomic measure and the set of transporting mappings $T_{\#}\mu_0 = \mu_1$ is empty).

The second part of our main result shows the relation between the flow problem (F) and the Lagrangian formulation L. This is the relation between the optimal velocity field $v$ realizing (1.8) and the induced flow
\[
\frac{d}{dt} T_{t_1}^t (x) = v \left( T_{t_1}^t, t \right). \tag{1.10}
\]
To elaborate, we shall prove

1) There exists a minimizer $\mu \in H_2$ of L which satisfies the end conditions. This minimizer may be non-unique.

2) There exists a maximizer $\psi$ of E which is a Lipschitz function on $\Omega \times [0, T]$ and satisfies the equation
\[
\psi_t + \frac{1}{2} |\nabla_x \psi|^2 = P \tag{1.11}
\]
almost everywhere. Again, such a maximizer may be non-unique.

3) The vector field $v = \nabla_x \psi$ is defined everywhere on some relatively closed set $K_0 \subset \Omega \times (0, T)$ which contains the support of any minimal path $\mu$ of L given by (1).

Under some additional assumption on $P$ (see Main Theorem in section 3) we also get

4) The vector field $v = \nabla_x \psi$ is locally Lipschitz continuous on $K_0$.

5) The restriction of $v$ to the support of any minimal orbit of L is uniquely determined.

6) The flow $T$ induced by $v \ (1.10)$ leaves $K_0$ invariant.

7) The flow $T_{t_1}^t$ transports $\mu_{(t_1)}$ to $\mu_{(t_2)}$ for any minimizer $\mu$ of L and any $t_1, t_2 \in (0, T)$. Moreover, it is an optimal Monge transport with respect to the action $J_P(\cdot, \cdot, t_1, t_2)$.

8) The maps $\lim_{r \to T} T_{t}^r := T_{t}^T : \Omega \to \Omega$ and $\lim_{r \to 0} T_{t}^r := T_{0}^t : \Omega \to \Omega$ exist and are continuous for any $t \in (0, T)$. Moreover, $[T_{t}^T]_{\#}$ (res. $[T_{0}^t]_{\#}$) is an optimal Monge map with respect to the action $J_P(\cdot, t, T)$ (res. $J_P(\cdot, 0, t)$) transporting $\mu_{(t)}$ to $\mu_1$ (res. $\mu_0$ to $\mu_{(t)}$).

9) If $\lim_{t \to T} T_{0}^t := \overline{T}$ exists as a Borel map, then $\overline{T}$ transports $\mu_0$ to $\mu_1$ and is an optimal solution of the Monge problem M. In this case
\[
\mathcal{M} = \mathcal{L} = \mathcal{E} = \mathcal{K}
\]

A particular case is the pressureless flow $P \equiv 0$. Here the optimal potential satisfies
\[
\psi_t + \frac{1}{2} |\nabla_x \psi|^2 = 0 \tag{1.12}
\]
and the associated action is

\[ J_0(x, y, t_1, t_2) = \frac{|x - y|^2}{2(t_2 - t_1)} , \]

reducing the Monge-Kantorovich problem to the Wasserstein metric \( W^2 \) for quadratic costs \(^{1.5}\). The associated flow, claimed in (6), is given in this case by

\[ T_{t_1}^{t_2}(x) = x + (t_2 - t_1) \nabla_x \psi(x, t_1) \]

where \( \nabla_x \psi \) is defined and Lipschitz everywhere. In particular it follows that, for a quadratic cost, an optimal Monge map \( T_{#} \mu_0 = \mu_1 \) exists and is unique provided \( \nabla_x \psi(x, 0) \) is \( \mu_0 \) measurable.\(^2\) In this case, Brenier representation \( T = T_{#}^0 = \nabla_x \Phi \) of the optimal map [see \(^{1.6}\) and the proceeding discussion] is recovered via

\[ \Phi(x) = x^2/2 + T\psi(x, 0) \]

The connection between the Monge-Kantorovich problem in the quadratic case and the flow problem \( L(P \equiv 0) \), as well as the dual relation \( E \) together with the Hamilton-Jacobi equation \(^{1.12}\) was indicated by several authors (see [BB], [BBG])\(^3\) as well as in the excellent monograph of Villani [V]. However, to the best of my knowledge, the existence and uniqueness result for the flow \( T_{t_1}^{t_2} \) without any regularity assumptions on the end measures \( \mu_0, \mu_1 \) is new even in the case \( P \equiv 0 \). In fact, the existence and uniqueness of the flow holds even if there is no optimal Monge map.

In section 4 we shall start to develop the tools needed for the proof of our main results. Section 4.1 deals with a dual formulation for the norm \( \|\mu\|_2 \) for an orbit of measure \( \mu = \mu(t) dt \in H_2 \). It follows that

\[ \|\mu\|_2 = \sqrt{\sup \left[ \left( \int \int \phi_t \mu(dxdt) \right)^2 \right]} \]

where the supremum is taken on the set of test functions \( \phi(x, t) = \phi \in C_0^1(\Omega \times [0, T]) \). An equivalent definition turns out to be

\[ \frac{1}{2} \|\mu\|_2^2 = \sup_{\phi, P} \left\{ - \int_{\Omega} P(x, t) \mu(dxdt) - \int_{\Omega} \phi(x, 0) \mu_0(dx) + \int_{\Omega} \phi(x, T) \mu_1(dx) \right\} \quad (1.13) \]

where the infimum above is on the pairs of "velocity potentials" \( \phi \in C^1(\Omega \times [0, T]) \) and "pressures" \( P = P(x, t) \) which are related via the Bernulli-type (or Hamilton-Jacobi) equation \(^{1.2}\). In case of a prescribed pressure \( P \) (as in this paper), this identity reveals the relation between the Lagrangian formulation \( L \) and the Eulerian one \( E \). In section 4.2 we imply a dual formulation to a strict convex perturbation of the Lagrangian formulation \( L \), leading to an approximation of the Euler formulation \( E \), to be used in the proof of the main result.

For the proof of the main result we shall also need a series of auxiliary Lemmas and definitions related to the Hamilton-Jacobi equation. In subsection 5.1 we list these definitions.

\(^2\)Since \( \psi(0, 0) \) is a Lipschitz function, \( \nabla_x \psi(x, 0) \) is a measurable function defined a.e, so we recover the existence of an optimal map if \( \mu_0 \) is a continuous w.r to Lebesgue measure.

\(^3\)I wish to thank Prof. D. Kinderlehrer for turning my attention to these publications.
and Lemmas, concerning forward (maximal), backward (minimal) and reversible solutions of the Hamilton-Jacobi equation, which are essential to the proof of the main results. The proofs of the Lemmas are given in subsection 5.2. In 5.3 we utilize these results for the proofs of our main Theorem.

In the rest of the paper we shall restrict ourselves to the flat torus $\Omega = \mathbb{R}^n / \mathbb{Z}^n$. The reason is that we wish to avoid compactness problems originated from measures on $\mathbb{R}^n$, on the one hand, and the boundary conditions for the Hamilton-Jacobi equation required in case of a bounded domain $\Omega \subset \mathbb{R}^n$. The flat torus is the simplest example in the sense that it is compact manifold with no boundary, on the one hand, and it inherits the Euclidean geometry from $\mathbb{R}^n$ on the other. Any function (or probability measure) on $\Omega$ is understood as a periodic function (or periodic, normalized per-period measure) on $\mathbb{R}^n$, unless otherwise is explicitly specified. In particular, a mapping $T : \Omega \to \Omega$ is understood as a mapping on the covering $\mathbb{R}^n$ which satisfies $T(x + z) = T(x) + z$ for any $x \in \mathbb{R}^n$ and any $z \in \mathbb{Z}^n$. 
List of symbols and definitions

- $\Omega := \mathbb{R}^n / \mathbb{Z}^n$
- $I = [0, T] ; \quad I_0 = (0, T)$
- $\Omega_I = \Omega \times I , \quad \Omega_{I_0} = \Omega \times I_0$.
- $LIP_1$ is the set of all locally Lipschitz functions in $\Omega)I_0$.
- $\mathcal{M}_I$ is the set of all probability Borel measures supported in $\Omega_I$.
- $\mathcal{M}$ is the set of all probability Borel measures supported in $\Omega$.
- $\pi(0)$ (res. $\pi(1)$) is the natural projection of $\Omega \times \Omega$ on its first (res. second) factor $\Omega$.
- For any pair $\mu_0, \mu_1 \in \mathcal{M}$, the Wasserstein-$p$ metric is defined by
  \[ W_p(\mu_0, \mu_1) := \inf_{\lambda} \int_{\Omega} \int_{\Omega} |x - y|^p \lambda(dx \, dy) \]
  where the infimum is on all probability measures on $\Omega \times \Omega$ such that $\pi(#) \lambda = \mu_0, \pi(1) \lambda = \mu_1$.
- $\mathcal{E}_\mu(\psi) := \int_0^T \int_{\Omega} \psi(x, t) \mu(t)(dx)dt$. Likewise, $\mathcal{E}_{\mu(t)}(\psi) = \int_{\Omega} \psi(x, t) \mu(t)(dx)$.
- A lifting $\nu$ of $\mu \in \mathcal{M}_I$ is a Borel measure on $\Omega_I \times \mathbb{R}^n$ such that
  \[ \mathcal{E}_\nu(\psi) := \int_0^T \int_{\Omega} \int_{\mathbb{R}^n} \psi(x, t) \nu(dx \, dt \, dv) = \mathcal{E}_\mu(\psi) \quad \text{and} \quad \mathcal{E}_\nu(\psi_t + v \cdot \nabla_x \psi) = 0 \]
  for all $\psi \in C_0^1(\Omega_I)$.

2 A metric space for measure’s orbits

We start with the following

Definition 2.1. Let $\mu \in \mathcal{M}_I$. Then $\mu \in \mathcal{H}_p(I, \mathcal{M})$ if there exists a lifting $\nu$ of $\mu$ such that $\mathcal{E}_\nu(|v|^p) < \infty$. We shall also define the $\mathcal{H}_p$ norm of $\mu \in \mathcal{H}_p$ by:

\[ ||\mu||_p = \inf_{\nu} \left[ \mathcal{E}_\nu(|v|^p) \right]^{1/p} \]

where the infimum is taken over all liftings of $\mu$.
Lemma 2.1. $H_p$ is complete and locally compact under the weak $C^\ast$ topology if $p > 1$. That is, for any bounded sequence $\mu_n$ in $H_p$ we can extract a subsequence which converges in $C^\ast(I)$ to some $\mu \in H_p$. In addition:

$$\lim_{n \to \infty} ||\mu_n||_p \geq ||\mu||_p .$$

Proof. By definition there exists a set of liftings $\nu_n$ corresponding to $\mu_n$. Moreover, this sequence can be chosen so that $E_{\nu_n}(|v|^p) < C$, so $\nu_n$ and $\nu \nu_n$ are tight on $I \times \mathbb{R}$. Hence the weak limit $\nu$ of $\nu_n$ is a lifting of the weak limit $\mu$ of $\mu_n$, and $E_{\nu}(|v|^p) < C$, hence $\mu \in H_p$. The same argument also yields the lower-semi-continuity of $H_p$. □

Lemma 2.2. If $\mu = \mu(t) dt \in H_p$, $p > 1$ then the map $t \to \mu(t)$ is a Holder $(p-1)/p$ continuous function from $I$ into $M$ with respect to the weak ($C^\ast$) topology equipped with the Wasserstein-1 norm $W_1$:

$$W_1(\mu_0, \mu_1) = \sup_{|\phi| \leq 1} \int \phi(\mu_1(dx) - \mu_0(dx)) .$$

(2.1)

Proof. We know that an optimal lifting $\nu$ exists for $\mu \in H_p$. The measure $\nu$ can be decomposed, by the Theorem of measure’s decomposition [AFP], into $\nu = \mu(t)(dx)\nu_{x,t}(dv)dt$, for $\mu$ a.a. $(x,t)$. We may define now the velocity field

$$v(x,t) = E_{\nu_{x,t}}(v)$$

for $\mu$ a.a. $(x,t)$. It follows that $v \in L^p_\mu$ and, moreover,

$$||\mu||_p = \left[ \int |v|^p \mu(dxdt) \right]^{1/p} .$$

By assumption:

$$\int_I \int_\Omega \frac{\partial \phi}{\partial t} \mu(t)(dx)dt = - \int_I \int_\Omega v \cdot \nabla x \phi \mu(t)(dx)dt$$

(2.2)

where $\phi = \phi(x,t)$ is in $C^1_0(I)$. Let $\phi(x,t) = h(t)\Phi(x)$ with $\Phi \in C^1(\Omega)$ and $h \in C^1_0(I)$. Then $f_\Phi(t) := \int_\Omega \Phi(x)\mu(t)(dx)$ satisfies

$$\int_0^T f_\Phi(t)h'(t)dt = - \int_0^T h(t) \int_\Omega \nabla \Phi \cdot v \mu(t)(dx)dt .$$

By Holder inequality

$$\int_0^T h(t) \int_\Omega \nabla \Phi \cdot v \mu(t)(dx)dt \leq |\nabla \Phi|_\infty \left[ \int_0^T h^q(t)dt \right]^{1/q} ||\mu||_p$$

with $q = p/(p-1)$. It follows that $f_\Phi \in W^{1,p}(I)$ and, moreover, $||f_\Phi||_{1,p} \leq C||\Phi||_{1,\infty}$. This implies the result by Sobolev imbedding together with the dual formulation of the $W_1$ norm □.
Given $\mu_0$ and $\mu_1 \in \mathcal{M}$, define the set
\[ \Lambda_p(\mu_0, \mu_1) := \{ \mu = \mu(t) dt \in \mathbf{H}_p \; ; \; \mu(0) = \mu_0 \; , \; \mu(T) = \mu_1 \; \} . \]

**Corollary 2.1.** The set $\Lambda_p(\mu_0, \mu_1)$ where $p > 1$ is closed and locally compact in $C(I; C^*(\Omega))$.

Similar versions of the Lemma and Proposition below can be found in [Am]. We also note that Proposition 2.1 in the case $p = 2$ is a special case of our main Theorem (see section 3).

**Lemma 2.3. (Regularization Lemma):** If $\mu \in \mathbf{H}_p$ then there exists a sequence $\mu^\varepsilon \in \mathbf{H}_p$ of smooth density so that $\mu = \lim_{\varepsilon \rightarrow 0} \mu^\varepsilon$ holds in $C^*(\Omega_I)$ and, moreover,
\[ \lim_{\varepsilon \rightarrow 0} ||\mu^\varepsilon||_p = ||\mu||_p . \]

In addition, for any $t_0, t_1 \in I$,
\[ \lim_{\varepsilon \rightarrow 0} W_p(\mu^\varepsilon(t_0), \mu^\varepsilon(t_1)) = W_p(\mu(t_0), \mu(t_1)) . \]

We next consider the relation between $\mathbf{H}_p$ and the optimal solution of the Kantorovich problem.

**Proposition 2.1.** Assume $p \geq 1$. Let $\mu_0, \mu_1 \in \mathcal{M}$. Then $\Lambda_p(\mu_0, \mu_1) \neq \emptyset$, and
\[ \inf_{\mu \in \Lambda_p(\mu_0, \mu_1)} ||\mu||_p = W_p(\mu_0, \mu_1) . \]

The proof is similar to the proof of Theorem 4.2 of Ambrosio [Am] for the metric case ($p = 1$).

We note that Corollary 2.1 is not valid in the case $p = 1$. To see it, consider the measure:
\[ \mu = \sum_j \alpha_j(t) \delta_{x_j(t)} dt \]
where $x_j = x_j(t) \in C^1(I; \Omega)$ and $\alpha_j \in C^1(I, \mathbb{R})$ such that $\sum_j \alpha_j(t) = 1 \; \forall t \in I$. We can approximate $\mu$ by a sequence of measures $\mu_m \in \Lambda_1(\mu_0, \mu_1)$ as follows: For each $m \in \mathbb{N}$ consider the division $t_k^{(m)} = k/m$, $0 \leq k \leq m$ of $I$. Let $\lambda_{m,k}$ be the optimal solution of Kantorovich problem due to $W_1(\mu(t_k^{(m)}), \mu(t_{k+1}^{(m)}))$, and $T_{m,k} := \mathbf{Id} + (t - t_k^{(m)}) (T_{m,k} - \mathbf{Id}) / (t_{k+1}^{(m)} - t_k^{(m)})$.

Define $\mu_m$ as follows:
\[ \mu_m(t_k) = \mu(t_k) \; ; \; \mu_m(t) = T_{m,k}^{(t)} \mu_m(t_k) \; ; \; t_k^{(m)} \leq t \leq t_{k+1}^{(m)} . \]

Then, by Proposition 2.1, $\mu_m$ are bounded in $\mathbf{H}_1$ and $\mu_m \rightarrow \mu$. However, $\mu \notin \mathbf{H}_1$ unless $\alpha_j$ are constants in $t$. To see it, note that the continuity equation takes the form
\[ 0 = \sum_j \int_I (\alpha_j(t) \phi_t(x_j(t), t) + v_j(t) \cdot \nabla x \phi(x_j(t), t)) dt = \sum_j \int_I -\dot{\alpha}_j \phi(x_j(t), t) + [v_j(t) - \dot{x}_j] \cdot \nabla x \phi(x_j(t), t) dt \]
where $v_j(t)$ are the velocities attributed to $x_j$. It is evident that, unless $\dot{\alpha}_j \equiv 0$, for any possible choice of $v_j$ one can find $\phi = \phi(x, t)$ for which the integral on the right does not vanish.
3 Main results

Let the pressure \( P = P(x,t) \in C^1(\Omega_I) \) and the associated action:

\[
L_P(\mu) := \frac{1}{2} ||\mu||_2^2 + \int_{\Omega_I} P\mu(dxdt) ; \quad \mu \in \mathcal{H}_2 .
\]

Let us recall the definition of the action \( J_P \):

\[
J_P(x,y,t_1,t_2) = \inf_{\mathcal{T}} \left\{ \int_{t_1}^{t_2} \left[ \frac{||\mathcal{T}(t)||^2}{2} + P(\mathcal{T}(t),t) \right] dt ; \quad \mathcal{T} : [t_1,t_2] \to \Omega, \quad \mathcal{T}(t_1) = x, \mathcal{T}(t_2) = y \right\} .
\]

Remark: Note that \( J_P \) is not a function on \( \Omega = \mathbb{R}^n/\mathbb{Z}^n \) in each of the variables \( x,y \), separately. However, for each \( q \in \mathbb{Z}^n \) and each \( x,y \in \mathbb{R}^n, t_1,t_2 \in I \), \( J_P(x+q,y+q,t_1,t_2) = J_P(x,y,t_1,t_2) \).

Definition 3.1. \( (L) \) (the relaxed Lagrangian):

\[
\mathcal{L}(\mu_0,\mu_1) := \inf_{\mu \in \Lambda^2(\mu_0,\mu_1)} L_P(\mu) .
\]

Definition 3.2. \( (M) \). (the Monge problem):

\[
\mathcal{M}(\mu_0,\mu_1) := \inf_{T \# \mu_0 = \mu_1} \int_{\Omega} J_P(x,T(x),0,T)\mu_0(dx) .
\]

Definition 3.3. \( (K) \). (the Kantorovich problem):

\[
\mathcal{K}(\mu_0,\mu_1) := \inf_{\lambda} \int_{\Omega} J_P(x,y,0,T)\lambda(dx\lambda)
\]

among all probability measures on \( \Omega \times \Omega \) with the same \( \Omega \) marginals \( \mu_0,\mu_1 \).

We now introduce the Hamilton-Jacobi (HJ) equation

\[
\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla_x \phi|^2 = P .
\]

Let us denote the set of classical sub-solutions of the H-J equation as

\[
\Lambda_P^* := \left\{ \phi \in C^1(\Omega_I) ; \quad \phi_t + \frac{1}{2} |\nabla_x \phi|^2 \leq P \quad \forall (x,t) \quad \text{in} \quad \Omega_I \right\} .
\]

For our purpose we need a generalization of the concept of a classical sub-solution. The concept of viscosity sub-solution (see, e.g. [E]) is too restrictive for us. So, we define a generalized sub-solution of the H-J equation as follows:

The set of generalized sub solution of the H-J equation is given by

\[
\overline{\Lambda}_P^* := \left\{ \phi \in \text{LIP}(\Omega_I) ; \quad \forall \mathcal{T} \in C^1(I;\Omega), \right. \\
\left. \frac{d}{dt}\phi(\mathcal{T}(t),t) \leq \frac{1}{2} |\mathcal{T}(t)|^2 + P(\mathcal{T}(t),t) \quad \text{holds for Lebesgue a.e \( t \in I \)} \right\} .
\]
Remark (i): Note that $\phi(x(t), t)$ is a Lipschitz function on $I$ if $\phi$ is Lipschitz and $\bar{x} \in C^1(I)$. Hence it is a.e. differentiable (as a function of $t$) on $I$ by Rademacher's Theorem (see, e.g., [E]).

Remark (ii): It is not difficult to see that any classical sub-solution is also generalized sub-solution, so $\Lambda_P^\ast \subset \overline{\Lambda}_P$. The concept of generalized sub-solution is more general than that of a viscosity sub-solution. The relation between generalized sub-solutions and viscosity (and anti-viscosity) sub-solutions is discussed in section 5.1.

**Definition 3.4.** (E): (The Euler formulation):

$$\mathcal{E}(\mu_0, \mu_1) := \sup_{\phi \in \overline{\Lambda}^\ast_P} \left\{ \int_\Omega \phi(x, T)\mu_1(dx) - \int_\Omega \phi(x, 0)\mu_0(dx) \right\}.$$ 

We now state our main result:

**Main Theorem:**

Assume $P \in C^1(\Omega_I)$. For any $\mu_0, \mu_1 \in \mathcal{M}$:

$$\mathcal{K}(\mu_0, \mu_1) = \mathcal{L}(\mu_0, \mu_1) = \mathcal{E}(\mu_0, \mu_1). \tag{3.5}$$

There exists minimizers $\mu \in \Lambda_2(\mu_0, \mu_1)$ of $L$ (Definition 3.1) and a maximizer $\psi \in \overline{\Lambda}^\ast_P$ of $E$ (Definition 3.4) such that

$$\psi_t + \frac{1}{2}|\nabla_x \psi|^2 = P \quad \text{a.e. on } \Omega_I. \tag{3.6}$$

Assume, in addition, there exists $C(t) > 0$ on $I_0$ so that $P(x, t) - C(t)|x|^2$ is a concave function on $\mathbb{R}^n$ for any $t \in I_0$. Then, for maximizer $\psi$ of $\mathcal{E}(\mu_0, \mu_1)$, there exists a closed set $K \subset \Omega_I$ such that

i) The restriction of $\psi$ to $K_0 := K \cap \Omega_{I_0}$ is continuously differentiable, the equality (3.6) holds for any $(x, t) \in K_0$ and $\nabla_x \psi$ is Locally Lipschitz continuous on $K_0$.

ii) Let $v$ be a Lipschitz extension of $\nabla_x \psi$ to $\Omega_{I_0}$. Let $T = T^2_{t_1}$ be the flow generated by $v$. Then $K_0$ is invariant under this flow.

iii) A minimizer $\mu \in \Lambda_2(\mu_0, \mu_1)$ of $L$ is not necessarily unique. However, any such minimizer is supported in $K$ and the vectorfield $v = \nabla_x \psi$ is uniquely defined on the support of any such minimizer.

iv) Any such minimizer is transported by the flow $T$, that is

$$[T^2_{t_1}]_{\#} \mu(t_1) = \mu(t_2)$$

holds for any $t_1, t_2 \in (0, T)$. Moreover, if $t_1 = 0$ (res. $t_2 = T$) then $T^t_0 := \lim_{\tau \to 0} T^t_\tau$ (res. $T^T_0 = \lim_{\tau \to T} T^\tau_0$) are continuous maps transporting $\mu_0$ to $\mu(t)$ (res. $\mu(t)$ to $\mu_1$).

v) The map $T^2_{t_1}$ are optimal with respect to the cost function $c(x, y) = J_P(x, y, t_1, t_2)$ and the measures $\mu(t_1), \mu(t_2)$, where either $t_1 \in I$, $t_2 \in I_0$ or $t_1 \in I_0, t_2 \in I$.

vi) If $P \equiv 0$ then the optimal solution $\psi$ of $E$ (Definition 3.4) is in $C^{1,1}_{loc}(\Omega_{I_0})$. In particular, the flow $T$ can be defined anywhere in terms of $\psi$ as

$$T^2_{t_1}(x) = x + (t_2 - t_1)\nabla_x \psi(x, t_1), \forall t_1 < t_2 \in I_0, \forall x \in \Omega.$$
4 Dual representation

The key duality argument for minimizing convex functionals under affine constraints is summarized in the following proposition whose proof is given in the appendix:

**Proposition 4.1.** Let $C$ a real Banach space and $C^*$ the its dual. Denote the duality $C \cong C^*$ relation by $<c^*,c> \in \mathbb{R}$. Let $Z$ a subspace of $C$ and $h \in C^*$. Let $Z^* \subset C^*$ given by the condition $z^* \in Z^*$ iff $<z^* - h,z> = 0$ for any $z \in Z$. Let $F : C^* \to \mathbb{R} \cup \{\infty\}$ a convex function and

$$I := \inf_{c^* \in Z^*} F(c^*) .$$

Assume further that $\overline{A}_0 := \{c^* \in C^* ; F(c^*) \leq I\}$ is compact (in the $\ast-$ topology of $C^*$).

Then

$$\sup_{z \in Z} \inf_{c^* \in C^*} [F(c^*) - <c^*,z> + <h,z>] = I .$$

In particular, both sides equal $\infty$ if $Z^* = \emptyset$.

4.1 Dual representation of $H_2$

We shall apply Proposition 4.1 were the space $C$ is all the continuous functions $q = q(x,t,v)$ on $\Omega_I \times \mathbb{R}^n$ subjected to:

$$\|q\| := \sup_{(x,t,v) \in \Omega_I \times \mathbb{R}^n} \left\{ \frac{|q(x,t,v)|}{1 + |v|^2} \right\} < \infty . \quad (4.1)$$

The dual space $C^*$ contains all finite Borel measures $\nu$ on $\Omega_I \times \mathbb{R}^n$ of finite second moments:

$$\int_{\Omega_I \times \mathbb{R}^n} |\nu|(dxdt dv) < \infty \; ; \; \int_{\Omega_I \times \mathbb{R}^n} |v|^2 |\nu|(dxdt dv) < \infty .$$

Define the subspaces $Z, Z_0$ of $C$ as

$$Z_0 := \{z = \phi_t + v \cdot \nabla_x \phi ; \; \phi \in C^1_0(\Omega_I) \} \subset Z := \{z = \phi_t + v \cdot \nabla_x \phi ; \; \phi \in C^1(\Omega_{t_0}) \cap LIP(\Omega_I)\} .$$

Given $\mu_0, \mu_1 \in \mathcal{M}$, define $h_{\mu_0,\mu_1}$ as a linear functional on $Z$ as follows:

$$<h_{\mu_0,\mu_1}, z> := \int_{\Omega} \phi(x,T)\mu_1(dx) - \int_{\Omega} \phi(x,0)\mu_0(dx) \quad \text{for} \; z \in Z , \quad (4.2)$$

(in particular, $<h_{\mu_0,\mu_1}, z> = 0$ if $z \in Z_0$).

**Lemma 4.1.** The functional $h_{\mu_0,\mu_1}$, so defined, is continuous (bounded) on $C$.

**Proof.** Let $\lambda$ be a probability distribution on $\Omega \times \Omega$ so that $\pi_{\#}^{(0)}\lambda = \mu_0$, $\pi_{\#}^{(1)}\lambda = \mu_1$. Then

$$\int_{\Omega} \phi(x,T)\mu_1(dx) - \int_{\Omega} \phi(x,0)\mu_0(dx) = \int_{\Omega} \int_{\Omega} [\phi(y,T) - \phi(x,0)] \lambda(dxdy) . \quad (4.3)$$
Lemma 4.2. The function $F_\mu$ is convex on $C^*$ for any $\mu \in H_2$. In addition, if $F_\mu(\nu) < \infty$ and $\nu \in Z_0^*$ then $\nu$ is a lifting of $\mu$. Similarly, if $F_\mu(\nu) < \infty$ and $\nu \in Z_{\mu_0,\mu_1}^*$ then $\mu \in \Lambda_2(\mu_0,\mu_1)$.

Proof. The proof of Lemma 4.2 is almost evident from the definitions. Let us just prove the last part. Since $Z_{\mu_0,\mu_1}^* \subset Z_0^*$ it follows that $\nu$ is a lifting of $\mu \in H_2$. We only have to show that $\mu \in \Lambda_2(\mu_0,\mu_1)$. Let $\phi \in C^1_0(\Omega_\mu) \cap LIP(\Omega_\mu)$, $\eta = \eta(t) \in C^1_0(I)$ satisfies $0 \leq \eta \leq 1$ on $I$ and, for some $\varepsilon > 0$, $\eta(t) = 1$ for $\varepsilon \leq t \leq T - \varepsilon$, and $\eta_t \geq 0$ on $[0,\varepsilon]$, $\eta_t \leq 0$ on $[T - \varepsilon, T]$. Set $\phi^{(\varepsilon)} = \phi$ on $\Omega \times [\varepsilon, T - \varepsilon]$ and $\phi^{(\varepsilon)}(x,t) = \phi(x,\varepsilon)$ on $t \in [0,\varepsilon]$ (res. $\phi^{(\varepsilon)}(x,t) = \phi(x,T - \varepsilon)$ on $t \in [T - \varepsilon, T]$). Then $\phi^{(\varepsilon)} \in C^1_0(\Omega_t)$, so

$$0 = \int_{\Omega_t \times \mathbb{R}^n} \left[ (\eta \phi^{(\varepsilon)})_t + \eta v \cdot \nabla_x \phi^{(\varepsilon)} \right] \nu(dxdt) = \int_{\varepsilon}^{T-\varepsilon} \int_{\Omega_\mu \times \mathbb{R}^n} \left[ \phi_t + \eta v \cdot \nabla_x \phi \right] \eta(t) \nu(dx)dt$$
\[ + \int_{0}^{\varepsilon} \int_{\Omega} \eta_{\varepsilon}(x, \varepsilon) \mu_{0}(dx)dt + \int_{T-\varepsilon}^{T} \int_{\Omega} \eta_{\varepsilon}(x, T-\varepsilon) \mu_{0}(dx)dt \]

\[ + \int_{0}^{\varepsilon} \int_{\Omega \times \mathbb{R}^{n}} \eta_{\varepsilon} \cdot \nabla \phi(x, \varepsilon) \nu_{0}(dx dv)dt + \int_{T-\varepsilon}^{T} \int_{\Omega \times \mathbb{R}^{n}} \eta_{\varepsilon} \cdot \nabla \phi(x, T-\varepsilon) \nu_{0}(dx dv)dt \ . \tag{4.6} \]

Since \( \nu \) is a lifting of some \( \mu \in \mathbb{H}_{2} \) it follows that \( \nu_{0}(dx dv) \) is a probability measure on \( \Omega \times \mathbb{R}^{n} \). By the Cauchy-Schwartz inequality we estimate the last two integrals by \( 2 \| \nabla \phi \|_{\infty} \sqrt{E_{\nu}(\| v \|^{2})} \varepsilon^{1/2} \). By Lemma 2.2 \( \mu_{0} \) is Holder continuous of exponent \( 1/2 \) in \( t \), with respect to the \( W_{1} \) topology, so

\[ \int_{0}^{\varepsilon} \int_{\Omega} \eta_{\varepsilon}(x, \varepsilon) \mu_{0}(dx)dt = \int_{0}^{\varepsilon} \int_{\Omega} \phi(x, \varepsilon) \mu_{0}(dx)dt + O(\varepsilon^{1/2}) \| \nabla_{x} \phi \|_{\infty} \int_{0}^{\varepsilon} |\eta_{\varepsilon}| dt \]

\[ = \int_{\Omega} \phi(x, \varepsilon) \mu_{0}(dx) + O(\varepsilon^{1/2}) \| \nabla_{x} \phi \|_{\infty} \int_{0}^{\varepsilon} |\eta_{\varepsilon}| dt = \int_{\Omega} \phi(x, \varepsilon) \mu_{0}(dx) + O(\varepsilon^{1/2}) \| \nabla_{x} \phi \|_{\infty} , \tag{4.7} \]

using \( \eta_{\varepsilon} \geq 0 \) on \([0, \varepsilon]\), hence \( \int_{0}^{\varepsilon} |\eta_{\varepsilon}| = \int_{0}^{\varepsilon} \eta_{\varepsilon} = 1 \). Similarly

\[ \int_{T-\varepsilon}^{T} \int_{\Omega} (\eta_{\varepsilon}) \mu_{0}(dx)dt = - \int_{\Omega} \phi(x, T-\varepsilon) \mu_{0}(dx) + O(\varepsilon^{1/2}) \| \nabla_{x} \phi \|_{\infty} . \tag{4.8} \]

Letting \( \varepsilon \to 0 \) we obtain from (4.6) (4.7) (4.8):

\[ \int_{\Omega \times \mathbb{R}^{n}} [ \phi_{\varepsilon} + v \cdot \nabla \phi ]_{\nu}(dx dv) - \int_{\Omega} [ \phi(x, T) \mu_{0}(dx) - \phi(x, 0) \mu_{0}(dx) ] = 0 . \]

The above is valid for any \( \phi \in C^{1}(\Omega_{0}) \cap LIP(\Omega_{T}) \). Since \( \nu \in Z^{*}_{\mu_{0}, \mu_{1}} \) by assumption, it follows that \( \mu_{0} = \mu_{0} \) and \( \mu_{0}(T) = \mu_{1} \), hence \( \mu \in \Lambda_{2}(\mu_{0}, \mu_{1}) \). \qed

**Corollary 4.1.** If \( \mu \in \mathbb{H}_{2} \) then

\[ \frac{1}{2} \| \mu \|_{2}^{2} = - \inf_{\phi \in C_{0}^{1}(\Omega_{0})} \left\{ \int_{\Omega_{T}} (\phi_{t} + |\nabla_{x} \phi|^{2}/2) \mu(dx dt) \right\} = \frac{1}{2} \sup_{\phi \in C_{0}^{1}(\Omega_{T})} \left( \frac{\int_{\Omega_{T}} \phi_{t} \mu(dx dt)}{\int_{\Omega} |\nabla_{x} \phi|^{2} \mu(dx dt)} \right) . \tag{4.9} \]

as well as

\[ - \inf_{\phi \in C^{1}(\Omega_{0}) \cap LIP(\Omega_{T})} \left\{ \int_{\Omega_{T}} (\phi_{t} + |\nabla_{x} \phi|^{2}/2) \mu(dx dt) + \int_{\Omega} \phi(x, 0) \mu_{0}(dx) - \int_{\Omega} \phi(x, T) \mu_{1}(dx) \right\} \]

\[ = \left\{ \begin{array}{ll}
\frac{1}{2} \| \mu \|_{2}^{2} & if \, \mu \in \Lambda_{2}(\mu_{0}, \mu_{1}) \\
\infty & if \, \mu \notin \Lambda_{2}(\mu_{0}, \mu_{1}) \end{array} \right. . \tag{4.10} \]

15
Proof. Certainly, \( F_\mu \) satisfies all the conditions of Proposition 4.1. Using Lemma 4.2 and Proposition 4.1 in the definition of \( \|\mu\|_2 \) (Definition 2.1 for \( p = 2 \)) we obtain that

\[
\frac{1}{2} \|\mu\|_2^2 = \inf_{\nu \in Z} F_\mu(\nu) = \inf_{\nu \in Z} \inf_{\phi \in C^1_\Phi} (F_\mu(\nu) - \langle \nu, z \rangle)
\]

\[
= \sup_{\phi \in C^1_\Phi} \inf_{\nu \in Z} \int_{\Omega_t \times \mathbb{R}^n} \left[ \frac{1}{2} |\nu| - \phi_t - \nu \cdot \nabla_x \phi \right] \nu(dxdtdv)
\]

\[
= \sup_{\phi \in C^1_\Phi} \inf_{\nu \in Z} \int_{\Omega_t \times \mathbb{R}^n} \left[ \frac{1}{2} |\nu - \nabla_x \phi|^2 - \phi_t - \frac{1}{2} |\nabla_x \phi|^2 \right] \nu(dxdtdv)
\]

\[
= \sup_{\phi \in C^1_\Phi} \left\{ - \int_{\Omega_t} \left[ \phi_t + \frac{1}{2} |\nabla_x \phi|^2 \right] \mu(dxdt) + \frac{1}{2} \inf_{\nu \in C^1_\Phi} \int_{\Omega_t \times \mathbb{R}^n} |\nu - \nabla_x \phi|^2 \nu(dxdtdv) \right\}.
\]

So, we set \( \nu = \mu \delta_{\nu} - \nabla_x \phi \) to annihilate the second integral and obtain the first equality in (4.9).

For the second equality in (4.9) we observe

\[
\inf_{\phi \in C^0_\Phi} \left\{ \int_{\Omega_t} (\phi_t + |\nabla_x \phi|^2/2) \mu(dxdt) \right\} = \inf_{\phi \in C^0_\Phi} \left\{ \int_{\Omega_t} (\beta \phi_t + \beta^2 |\nabla_x \phi|^2/2) \mu(dxdt) \right\}
\]

\[
= \inf_{\phi \in C^0_\Phi} \left( - \frac{1}{2} \left( \frac{\int_{\Omega_t} \phi_t \mu(dxdt)}{\int_{\Omega_t} |\nabla_x \phi|^2 \mu(dxdt)} \right)^2 \right).
\]

Finally, we obtain (4.10) using the constraint \( Z^*_\mu_0, \mu_1 \) for \( Z^* \) in Proposition 4.1.

Example: Let \( \mu = \sum_{k=1}^K \beta_k \delta_{x-x_k(t)} \) where \( x_k : I \to \Omega \) satisfies \( \int_0^T |\dot{x}_j|^2 dt = |\dot{x}_j|_2 < \infty \) and \( \beta_j \geq 0, \sum_j \beta_j = 1 \). Then

\[
\int_{\Omega_t} \phi_t \mu(dxdt) = \sum_j \beta_j \int_0^T \frac{\partial \phi}{\partial t}(x_j(t), t) dt
\]

and

\[
\int_{\Omega_t} |\nabla_x \phi|^2 \mu(dxdt) = \sum_j \beta_j \int_0^T |\nabla_x \phi|^2(x_j(t), t) dt
\]

On the other hand,

\[
\int_0^T \frac{\partial \phi}{\partial t}(x_j(t), t) dt = \int_0^T \left[ \frac{d\phi}{dt}(x_j(t), t) - \dot{x}_j(t) \cdot \nabla_x \phi(x_j(t), t) \right] dt
\]

\[
= - \int_0^T \dot{x}_j(t) \cdot \nabla_x \phi(x_j(t), t) dt
\]

so, by an application (twice) of the Cauchy-Schwartz inequality,

\[
\frac{\left( \int_{\Omega_t} \phi_t \mu(dxdt) \right)^2}{\int_{\Omega_t} |\nabla_x \phi|^2 \mu(dxdt)} = \frac{\left( \sum_j \beta_j \int_0^T \nabla_x \phi(x_j(t), t) \cdot \dot{x}_j dt \right)^2}{\sum_j \beta_j \int_0^T |\nabla_x \phi(x_j(t), t)|^2 dt} \leq \sum_j \beta_j \int_0^T |\dot{x}_j|^2 dt.
\]
In fact, it can be shown that $||\mu||_2^2$ coincides with the above sum, and that there exists a maximizing sequence $\phi_n(x,t)$ such that $\nabla_x \phi_n(x_j(t),t) \to \dot{x}_j(t)$ for all $j$ and a.e $t \in I$ (even if some of the orbits $x_j$ intersect (1)-see [W]).

4.2 Dual representation of the Lagrangian

We shall now define a strong convex perturbation of the Lagrangian $L_P$ (Definition 3.1). Let $\mu : \mathbb{R} \to \mathbb{R}^+ \cup \{\infty\}$ such that

$$F(q) = \infty \quad \text{if} \quad q < 0 \quad \text{;} \quad F(0) = 0 \quad \text{;} \quad cq^\omega < F(q) < Cq^\omega \quad \text{if} \quad q > 0$$  \hspace{1cm} (4.11)

where $1 < \omega < 1 + 1/(n+1)$ and $c,C > 0$. The functional $I^P_\varepsilon : C^* \to \mathbb{R} \cup \{\infty\}$ is defined by:

$$I^P_\varepsilon(\nu) := \int_{\Omega \times \mathbb{R}^n} \varepsilon F(f_\nu) \, dxdt dv + \frac{1}{2} \int_{\Omega \times \mathbb{R}^n} |v|^2 \nu(dxdt dv) + \int_{\Omega \times \mathbb{R}^n} P(x,t) \nu(dxdt dv),$$  \hspace{1cm} (4.12)

if $\nu = f_\nu(x,t,v)dxdt dv$ is absolutely continuous with respect to Lebesgue measure and the density $f_\nu$ satisfies $F(f_\nu) \in L^1(\Omega \times \mathbb{R}^n)$. Otherwise $I^P_\varepsilon(\nu) = \infty$. Note that, since $F(q) = \infty$ for $q < 0$, it follows that $I^P_\varepsilon(\nu) = \infty$ if $\nu \in C^*$ is not a non-negative measure. However, $I^P_\varepsilon$ can attain a finite value also for a measure $\nu$ which is not normalized (i.e not a probability measure on $\Omega \times \mathbb{R}^n$).

Given $\mu_0, \mu_1 \in M$, define

$$I^P_\varepsilon(\mu_0, \mu_1) := \inf_{\nu \in Z^*_{\mu_0, \mu_1}} I^P_\varepsilon(\nu).$$  \hspace{1cm} (4.13)

Next, we claim

Lemma 4.3. For any $\varepsilon > 0$,

$$I^P_\varepsilon(\mu_0, \mu_1) \geq L(\mu_0, \mu_1)$$

where $L(\mu_0, \mu_1)$ as in Definition 3.1.

Proof. First, we can restrict ourselves to non-negative measures $\nu \in Z^*_{\mu_0, \mu_1}$, since otherwise $\int F(f_\nu) = \infty$ by (4.11). We only have to show that if $\nu \geq 0$ and $\nu \in Z^*_{\mu_0, \mu_1}$ then $\nu$ is a lifting of some $\mu \in \Lambda_2(\mu_0, \mu_1)$.

Using Lemma 4.2 it is, therefore, enough to prove that $\nu_t(dx dv)$ is a probability measure on $\Omega \times \mathbb{R}^n$ for a.e (Borel) $t \in I$. Setting $\phi(x,t) = \eta(t) \in C^1_0(I)$ we obtain from (4.5) that

$$\int_I \left( \int_{\Omega \times \mathbb{R}^n} \nu_t(dx dv) \right) \frac{d\eta}{dt} dt = 0$$

for any such $\eta$. This implies that $\int_{\Omega \times \mathbb{R}^n} \nu_t(dx dv)$ is constant for a.e. $t \in I$. Since $\nu \geq 0$ it implies that $\nu_t$ is a constant multiple of some probability measure on $\Omega \times \mathbb{R}^n$ for a.e. $t \in I$. This constant equals one since the $\Omega$ marginal of $\nu_t$ is $C^*$ continuous on $I$ by Lemma 2.2 and is a probability measure at $t = 0 (\mu_0)$ and $t = T (\mu_1)$. \hfill $\Box$

17
We now proceed to a dual formulation of the constraint minimization of $I_\varepsilon$. Certainly $I_\varepsilon$ satisfies the assumption on $F$ introduced in Proposition 4.1. In fact, it follows that the set $\{\nu \in C^* : I_\varepsilon^*(\nu) < C\}$ is bounded (and hence $*$--compact) for any real $C$. Then Proposition 4.1 and (4.13) yield

\[
I_\varepsilon^*(\mu_0, \mu_1) = \sup_{\phi \in C^1(\Omega_t) \cap LIP(\Omega_t)} \Psi_\varepsilon(\phi).
\]

Let $F^*$ be the Legendre transform of $F$:

\[
F^*(\lambda) = \sup_{s} [s\lambda - F(s)].
\]

By our assumption we know that $F^*$ is also convex and non-negative on $\mathbb{R}$. It satisfies $F^*(\lambda) = 0$ for $\lambda \leq 0$. Now,

\[
\inf_f H_\varepsilon(f, \phi) = -\varepsilon \int_{\Omega_t \times \mathbb{R}^n} F^* \left( \frac{\phi_t + |\nabla_x \phi|^2/2 - |v - \nabla_x \phi|^2/2 - P}{\varepsilon} \right) dxdt dv
\]

\[
= -\varepsilon^{1+n/2} \int_{\Omega_t \times \mathbb{R}^n} F^* \left( \frac{\phi_t + |\nabla_x \phi|^2/2 - P}{\varepsilon} - \frac{|v|^2}{2} \right) dxdt dv
\]

Let

\[
G(s) := \int_{\mathbb{R}^n} F^*(s - |v|^2/2) dv
\]

and

\[
\Psi_\varepsilon(\phi) := -\varepsilon^{1+n/2} \int_{\Omega_t} G \left( \frac{\phi_t + |\nabla_x \phi|^2/2 - P}{\varepsilon} \right) dxdt + \int_{\Omega} \phi(x,T) \mu_1(dx) - \phi(x,0) \mu_0(dx)
\]

We have proved:

**Lemma 4.4.** For $\varepsilon > 0$ and $\mu_0, \mu_1 \in \mathcal{M}$,

\[
I_\varepsilon^*(\mu_0, \mu_1) = \sup_{\phi \in C^1(\Omega_t) \cap LIP(\Omega_t)} \Psi_\varepsilon(\phi).
\]
Let \( \mu \) and \( \nu \) be Lipschitz continuous functions.

We shall also need the following result, whose proof is direct and omitted:

**Lemma 4.5.** If \( F \) satisfies (4.11) then, for some constant \( c > 0 \), the function \( G \) defined in (4.14) satisfies \( c q^{\omega_1} \omega - 1 < G(q) < c^{-1} q^{\omega_1} \omega - 1 \). Thus, the first integral of (4.13) is estimated by

\[
-\varepsilon^{1+n/2} \int_{\Omega_1} G \left( \frac{\phi_i + |\nabla_x \phi|^2/2 - P}{\varepsilon} \right) \, dx dt = -O(\varepsilon^{-\alpha}) \int_{\Omega_1} \left| \phi_i + \frac{|\nabla_x \phi|^2}{2} - P \right|^s \, dx dt
\]

where \( \alpha = 1/\omega - 1 - n/2 > 0 \) and \( s = \omega/\omega - 1 > 1 + n \) (c.f. 4.11).

We also need:

**Lemma 4.6.** Let \( \mu_0, \mu_1 \in \mathcal{M} \). Then there exists a connecting orbit \( \mu \in \Lambda_2(\mu_0, \mu_1) \) of finite \( H_2 \) norm and a lifting \( \nu \) such that both \( \mu \) and \( \nu \) has densities in \( L^p(\Omega_1) \) (res. \( L^p(\Omega_1 \times \mathbb{R}^n) \)), where \( 1 \leq p < 1 + 1/n \).

In particular, it follows that for such \( \nu \) as guaranteed in Lemma 4.6 each of the integrals in (4.12) is finite. Hence, there exists \( C > 0 \) (independent of \( \varepsilon \)) such that \( T_\varepsilon(\nu) < C \) for any \( \varepsilon > 0 \). In particular, \( I^p_\varepsilon(\mu(0), \mu(1)) < C \) for any such \( \varepsilon \) by (4.13). It follows from this, Lemma 4.3 and Lemma 4.4 that

**Corollary 4.2.** For any \( \mu_0, \mu_1 \in \mathcal{M} \) there exists \( C > 0 \) independent of \( \varepsilon \) where

\[
C > \sup_{\phi \in C^1(\Omega_1) \cap L^p(\Omega_1)} \Psi_\varepsilon(\phi) \geq \mathcal{L}(\mu_0, \mu_1)
\]

Lemma 4.6 is a direct result from Lemma 4.7 below. For its presentation we define the space \( H_p([t_0, t_1]) \) by restricting \( H_p = H_p(I) \) to orbits defined for a time interval \([t_0, t_1]\). The norm of \( \mu \in H_p([t_0, t_1]) \) is denoted by \( ||\mu||_{2, [t_0, t_1]} \). Lemma 4.7 is also used in the proof of Lemma 5.8.

**Lemma 4.7.** For any \( t_1 > t_2 \in I \), any path \( \overline{\tau} = \overline{\tau}(t) : [t_0, t_1] \to \Omega \) and any \( \alpha > 0 \) there exists an orbit \( \mu(t) \, dt \in H_p([t_0, t_1]) \) with \( \mu(t_0) = \delta_{x_0}, \mu(t_1) = \delta_{x_1} \) where \( x_i = \overline{\tau}(t_i), i = 0, 1 \), such that

\[
||\mu||^2_{2, [t_0, t_1]} \leq \int_{t_0}^{t_1} |\overline{\tau}|^2 \, dt + C|t_1 - t_0|\alpha^{-2}
\]

and \( \mu(t) \, dx = \rho(x, t) \, dx \) where \( \rho \in L^p(\Omega \times [t_0, t_1]) \) for any \( p \in [1, 1 + 1/n] \). Moreover

\[
|\rho|_p \leq C(p) \left( (t_1 - t_0) \left( \frac{\alpha}{t_1 - t_0} \right)^{n(p-1)} \right)^{1/p}
\]

and

\[
\text{supp}(\rho) \subset \left\{ (x, t) \in \Omega \times [t_0, t_1] : |\overline{\tau}(t) - x| \leq C \frac{t_1 - t_0}{\alpha} \quad \forall \ t \in [t_0, t_1] \right\}.
\]

In particular, the choice \( \overline{\tau}(t) = x_0 + \frac{t - t_0}{t_1 - t_0} (x_1 - x_0) \) yields

\[
||\mu||^2_{2, [t_0, t_1]} \leq \frac{|x_1 - x_0|^2}{t_1 - t_0} + C|t_1 - t_0|\alpha^{-2}.
\]

The proof of Lemma 4.7 is given in the Appendix.
5 Proof of main results

5.1 On the Hamilton-Jacobi Equation

In this section we introduce some fundamental results for the HJ equation

\[ \dot{\phi}_t + \frac{1}{2} \left| \nabla_x \phi \right|^2 = P \quad (x, t) \in \Omega_I \]  

(5.1)

where \( P \in C^1(\Omega_I) \). The book of L.Evans [E] contains a detailed exposition on the Hamilton-Jacobi equation. However, the discussion in [E] is restricted to generalized solutions of viscosity type and for time independent Hamiltonians, which excludes the application of backward solutions and time dependent pressure \( P = P(x, t) \). The results in this section are all needed for the proof of the Main Theorem in section [3].

We list below some properties of the action \( J_P \)

\[ \text{Lemma 5.1. For } P \in C^1(\Omega_I), \text{ the action } J_P \text{ satisfies the following:} \]

(a) For \( \tau_1 < \tau_2 \in [0, T] \) and \( x_1, x_2 \in \mathbb{R}^n \), the value of the action \( J_P(x_1, x_2, \tau_1, \tau_2) \) is realized along a (possibly not unique) orbit \( \omega \) which satisfies the equation

\[ \dot{\omega} = \nabla_x P(\omega(t), t) ; \ t \in [\tau_1, \tau_2] . \]  

(5.2)

(b) Assume further that there exists \( C(t) > 0 \) so that \( P(x, t) - C(t)|x|^2 \) is a concave function on \( \mathbb{R}^n \) for any \( t \in I_0 \). Let \( \omega \) be an optimizer orbit connecting \( x_1, \tau_1 \) to \( x_2, \tau_2 \). For any \( y \in \mathbb{R}^n \) and \( t \in (0, T) \)

\[ J_P(x_1, x_2, \tau_1, \tau_2) - J_P(x_1, y, \tau_1, t) \geq \]

\[ \hat{\omega}(\tau_2) \cdot (x_2 - y) + \left[ P(x_2, \tau_2) - \frac{1}{2} \hat{\omega}(\tau_2) \right] (\tau_2 - t) - O(|x_2 - y|^2) - o(t - \tau_2) , \]  

(5.3)

\[ J_P(y, x_2, \tau_2) - J_P(x_1, x_2, \tau_1, \tau_2) \leq \]

\[ \hat{\omega}(\tau_1) \cdot (x_1 - y) + \left[ P(x_1, \tau_1) - \frac{1}{2} \hat{\omega}(\tau_1) \right] (\tau_1 - t) + O(|x_1 - y|^2) + o(t - \tau_1) . \]  

(5.4)

(c) For any \( x_1, y, x_2 \in \mathbb{R}^n \), \( t_1 < \tau < t_2 \)

\[ J_P(x_1, y, t_1, \tau) + J_P(y, x_2, \tau, t_2) \geq J_P(x_1, x_2, t_1, t_2) \]  

(5.5)

holds.

(d) For any pair \( x_1, x_2 \in \mathbb{R}^n \) and a triple \( t_1 < \tau < t_2 \) there exists \( y^* \in \mathbb{R}^n \) (possibly non-unique) for which the equality holds in (5.6):

\[ J_P(x_1, y^*, t_1, \tau) + J_P(y^*, x_2, \tau, t_2) = J_P(x_1, x_2, t_1, t_2) . \]  

(5.6)

There exists a (possibly non-unique) optimal orbit \( \omega \) connecting \( (x_1, t_1) \) to \( (x_2, t_2) \) such that \( \omega(\tau) = y \). However, for any such optimal orbit, \( \omega(\tau) \) is determined uniquely.
(e) For any $t > t_1$, $x \in \mathbb{R}^n$ and a.e. $y \in \mathbb{R}^n$

$$\frac{\partial}{\partial t} J_P(x, y, t_1, t) + \frac{1}{2} |\nabla_y J_P(x, y, t_1, t)|^2 = P(y, t) . \quad (5.7)$$

**Definition 5.1.** $\phi(x, t)$ is a forward solution of (5.1) iff, for any $x \in \Omega$ and $t_1, t_0 \in I$

$$\phi(x, t_1) = \inf_{y \in \mathbb{R}^n} [J_P(y, x, t_0, t_1) + \phi(y, t_0)] . \quad (F)$$

Likewise, $\phi$ is a backward solution iff

$$\phi(x, t_0) = \sup_{y \in \mathbb{R}^n} [-J_P(x, y, t_0, t_1) + \phi(y, t_1)] . \quad (B)$$

**Remark:** It follows, by the remark proceeding (3.2), that the right sides of (F) (res. (B)) defines a function which is $\mathbb{Z}^n$ periodic on $\mathbb{R}^n$, namely defined on $\Omega$, if $\phi(\cdot, t_0)$ (res. $\phi(\cdot, t_1)$) is a function on $\Omega$.

For the special case of zero-pressure Hamilton-Jacobi equation, the action is reduced to

$$J_0(x_1, x_2, t_1, t_2) = \frac{|x_2 - x_1|^2}{2(t_2 - t_1)}$$

and definition 5.1 reduces to the (original) Hopf-Lax formula:

**Definition 5.2.** A forward solution of the pressureless Hamilton-Jacobi equation

$$\phi_t + \frac{1}{2} |\nabla_x \phi|^2 = 0$$

satisfies, for any $t_1 > t_0$ and $x \in \Omega$

$$\phi(x, t_1) = \inf_{y \in \mathbb{R}^n} \left[ \frac{1}{2} \frac{|x - y|^2}{t_1 - t_0} + \phi(y, t_0) \right] , \quad (F_0)$$

while a backward solution satisfies

$$\phi(x, t_0) = \sup_{y \in \mathbb{R}^n} \left[ -\frac{1}{2} \frac{|x - y|^2}{t_1 - t_0} + \phi(y, t_1) \right] . \quad (B_0)$$

A forward (backward) solution can be constructed from an initial (end) data at $t = 0$ ($t = T$) as follows.

**Lemma 5.2.** For any continuous initial data $\phi(\cdot, 0)$ on $\Omega$ and $P \in LIP(\Omega_I)$,

$$\phi(x, t) = \inf_{y \in \mathbb{R}^n} [J_P(y, x, 0, t) + \phi(y, 0)] \quad (5.8)$$

is a forward solution and satisfies (5.1) a.e. Moreover, $\phi \in LIP(\Omega \times (0, T])$ and

$$\frac{|\phi(x, t) - \phi(y, t)|}{|x - y|} \leq \frac{C}{t} \quad (5.9)$$
where $C$ is a constant independent on $\phi(0)$. Likewise, for any continuous end data $\phi(T)$

$$\phi(x, t) = \sup_{y \in \mathbb{R}^n} [-J_P(x, y, 0, t) + \phi(y, 1)]$$

is a backward solution and satisfies (5.11) a.e., $\phi \in LIP(\Omega \times [0, T])$ and

$$\frac{|\phi(x, t) - \phi(y, t)|}{|x - y|} \leq \frac{C}{T - t}$$

(5.10)

If, in either cases, the end data $\phi(0)$ (res. $\phi(T)$) is Lipschitz on $\Omega$, then the corresponding forward (backward) solution is in $LIP(\Omega_I)$.

Next, we establish the connection between generalized sub-solutions, as defined in (3.4), and forward/backward solutions:

**Lemma 5.3.** Both forward and backward solutions are generalized sub-solutions in the sense of (3.4). A forward (backward) solution is a maximal (minimal) generalized sub-solution in the following sense: If $\psi$ is a generalized sub-solution and $\phi$ is a forward (backward) solution so that $\psi(x, \tau) = \phi(x, \tau)$ for all $x \in \Omega$ and some $\tau \in I$, then $\phi(x, t) \geq \psi(x, t)$ ($\phi(x, t) \leq \psi(x, t)$) for all $x \in \Omega$ and $t \geq \tau$ ($t \leq \tau$) in $I$.

An immediate corollary from Lemma 5.3 is:

**Corollary 5.1.** Let $\phi$ be a forward solution and $\psi$ is a backward solution on $\Omega_I$.

i) If $\phi(x, T) = \psi(x, T)$ holds $\forall x \in \Omega$ then $\psi(x, t) \leq \phi(x, t)$ $\forall (x, t) \in \Omega_I$.

ii) Similarly, if $\phi(x, 0) = \psi(x, 0)$ holds $\forall x \in \Omega$ then $\psi(x, t) \leq \phi(x, t)$ $\forall (x, t) \in \Omega_I$.

Next, we wish to address the notion of a reversible solution:

**Definition 5.3.** A reversible pair $\{\phi, \phi\}$ where $\phi$ is a forward (backward) solution on $\Omega_I$ such that $\phi(x, 0) = \phi(x, 0)$ and $\phi(x, T) = \phi(x, T)$ for any $x \in \Omega$. By Corollary 5.1, $\phi \geq \phi$ on $\Omega_I$. For any such reversible pair we denote the reversibility set of the pair as the relatively closed set $K_0(\phi, \phi) \subset \Omega_I$ given by

$$K_0(\phi, \phi) := \{(x, t) \in \Omega_I \ ; \ \phi(x, t) = \phi(x, t)\} \subset \Omega_I$$

Likewise,

$$K_0^{(t)}(\phi, \phi) = K_0(\phi, \phi) \cap [\Omega \times \{t\}] \quad \text{for any } t \in (0, T).$$

If $\phi \equiv \phi$ then $\phi := \phi = \phi$ is called a reversible solution.

From Corollary 5.1 we obtain a way to create reversible pairs. It turns out that, in the case $P \equiv 0$, this way yields reversible solutions:

**Lemma 5.4.** Given $\phi_0 \in LIP(\Omega)$, let $\phi$ be the forward solution subjected to $\phi(x, 0) = \phi_0(x)$. Let $\psi$ be the backward solution subjected to $\psi(x, T) = \phi(x, T)$, and $\bar{\psi}$ the forward solution subjected to $\bar{\psi}(x, 0) = \psi(x, 0)$.

Then $\{\bar{\psi}, \psi\}$ is a reversible pair. Moreover, if $P \equiv 0$ then $\bar{\psi} = \psi$ is a reversible solution.
The next Lemmas indicate that reversible pairs (in particular, reversible solutions) are closely related to classical solutions of the Hamilton-Jacobi equation.

**Lemma 5.5.** If $\phi \in C^1(\Omega_1)$ is a classical solution of \((5.1)\) then $\phi$ is a reversible solution.

Using Lemma 5.5 we show that the converse of Lemma 5.5 also holds, in some sense:

**Lemma 5.6.** If \(\{\phi, \phi\} \) is a reversible pair then both $\phi$ and $\phi$ are differentiable on $K_0 := K_0(\phi, \phi)$ (cf., Definition 5.3). Moreover, $\nabla \tilde{\phi} := \{\nabla_x \phi, \phi_t\}$ and the H.J equation is satisfied on this set. If, in addition, $P$ satisfies the condition of Lemma 5.1-(b) then $\nabla \phi$ is locally Lipschitz continuous on $K_0$ and $\phi$ satisfies (5.1) pointwise on this set.

**Lemma 5.7.** Assume $P$ satisfies the condition of Lemma 5.1-(b). Let $v(x,t)$ be a Lipschitz extension of $\nabla_x \phi$ from $K_0$ to $\Omega_{t_0}$. Then the set $K_0$ is invariant with respect to the (unique) flow generated by the vectorfield $v$.

Finally, we introduce the two following results, to be needed in Section 5.3.

**Lemma 5.8.** If $\phi \in C^1(\Omega \times [t_0,t_1])$ satisfying

$$
\phi_t + \frac{1}{2} |\nabla_x \phi|^2 = P + \xi \quad ; \quad (x,t) \in \Omega \times [t_0,t_1]
$$

where $P, \xi \in \text{Lip}(\Omega \times [t_0,t_1])$, $s > n + 1$ and $||\xi||_s$ stands for the $L^s(\Omega \times [t_0,t_1])$ norm of $\xi$ then, for any $x_0, x_1$ in $\Omega$, any $t_1 > t_0$ and any orbit $\tilde{x} = \tilde{x}(t) : [t_0,t_1] \to \Omega$ satisfying $\tilde{x}(t_0) = x_0, \tilde{x}(t_1) = x_1$:

$$
\phi(x_1,t_1) - \phi(x_0,t_0) \leq \frac{1}{2} \int_{t_0}^{t_1} |\tilde{x}|^2 + \int_{t_0}^{t_1} P(\tilde{x}(t),t) dt + C_1 ||\xi||_s^{2\beta}(t_1-t_0)^\lambda + C_2 ||P||_{\text{lip}}(t_1-t_0)^\eta ||\xi||_s^\beta,
$$

where $\beta = \frac{p}{2p+n(p-1)}$, $s = \frac{s-1}{s}$, $\lambda = \frac{2+n-np}{2p+n(p-1)}$, $\eta = \frac{4p+(n-1)(p-1)}{2p+n(p-1)}$.

From Lemma 5.8 and Definition 5.1 we also obtain

**Corollary 5.2.** Let $\phi \in C^1(\Omega_1)$ be a solution and $\psi$ a forward solution of the respective equations on $\Omega_1$:

$$
\phi_t + \frac{1}{2} |\nabla_x \phi|^2 = P + \xi \quad ; \quad \psi_t + \frac{1}{2} |\nabla_x \psi|^2 = P
$$

such that $\psi(x,0) = \phi(x,0)$ on $\Omega$. Then

$$
\psi(x,T) \geq \phi(x,T) - \left[ C_1 ||\xi||_s^{2\beta} + C_2 ||P||_{\text{lip}} ||\xi||_s^\beta \right]
$$

where $s, \beta$ as defined in Lemma 5.8.
5.2 Proof of auxiliary results

Proof. (of Lemma 5.1) We shall only establish part (b), since all the rest follows by standard arguments.

To establish this part, let us redefine the action $J_P$ in terms of parameterized orbits $x(s) \rightarrow (y(s), \tau(s)), s \in [0, 1]$. The optimal orbit is denoted $\bar{x} := \{\bar{y}, \bar{\tau}\}$. The action takes the form

$$J_P(x_1, x_2, \tau_1, \tau_2) = \inf_{y, \tau} \int_0^1 \left[ \frac{1}{2\tau'(s)} \left| y'(s) \right|^2 + P(y(s), \tau(s))\tau'(s) \right] ds \quad (5.11)$$

where $'$ stands for $s$ derivative while $\{y, \tau\}$ are the set of orbits satisfying $y(0) = x_1, y(1) = x_2,$ $\tau(0) = \tau_1, \tau(1) = \tau_2$ and $\tau' > 0$ on $[0, 1]$. An optimal orbit $\{\bar{y}, \bar{\tau}\}$ satisfies the Euler-Lagrange equations:

$$\frac{d}{ds} \left( \frac{1}{\tau'} \frac{dy}{ds} \right) = \tau' P_x(\bar{y}, \bar{\tau}) \quad ; \quad \frac{d}{ds} \left[ \frac{1}{2} \frac{y'}{\tau'} - P(\bar{y}, \bar{\tau}) \right] = -P(\bar{y}, \bar{\tau})\tau' \quad (5.12)$$

Now let us replace $x_2$ by $y$ and $\tau_2$ by $\tau$. Deform the optimal orbit $\{\bar{y}, \bar{\tau}\}$ into

$$\tilde{y}(s) = \bar{y}(s) + (y - x_2)s \quad ; \quad \tilde{\tau}(s) = \bar{\tau}(s) + (t - \tau_2)s$$

Now, $\{\tilde{y}, \tilde{\tau}\}$ is an admissible orbit for the action with end-points $(x_1, \tau_1)$ and $(y, t)$. In addition, our assumption on $P$ yields:

$$P(\tilde{y}(s), \tilde{\tau}(s))\tilde{\tau}'(s) \leq P(\bar{y}(s), \bar{\tau}(s))\bar{\tau}'(s) + \bar{\tau}'(s)P_x(\bar{y}(s), \bar{\tau}(s))(\tilde{y} - \bar{y})$$

$$+ \bar{\tau}'(s)P(\bar{y}(s), \bar{\tau}(s))(\tilde{\tau} - \bar{\tau}) + (\tilde{\tau}' - \bar{\tau}')P(\bar{y}(s), \bar{\tau}(s)) + O^2(|\tilde{y} - \bar{y}|) + o(\tilde{\tau} - \bar{\tau}) \quad (5.13)$$

as well as

$$\frac{|\tilde{y}'|^2}{2\tilde{\tau}'} \leq \frac{|\bar{y}'|^2}{2\bar{\tau}'} + \frac{\tilde{y}' - \bar{y}'}{\tilde{\tau}' - \bar{\tau}'} - \frac{|\bar{y}'|}{2\bar{\tau}'} + O^2(|y - x_2|) + O^2(|t - \tau_2|)$$

so, by substitution of $\{\tilde{y}, \tilde{\tau}\}$, integration by parts, (5.11) and (5.12)

$$J_P(x_1, y, \tau_1, t) \leq \int_0^1 \left[ \frac{1}{2\bar{\tau}'}(\bar{y}'(s))^2 + P(\bar{y}(s), \bar{\tau}(s))\bar{\tau}'(s) \right] ds$$

$$= J_P(x_1, x_2, \tau_1, \tau_2) + \frac{1}{\tau'(1)}(y - x_2) + \left[ P(\bar{y}(1), \bar{\tau}(1)) - \frac{|\bar{y}'|^2}{2(\bar{\tau}')^2} \right] (t - \tau_2) + O^2(|y - x_2|) + o(|t - \tau_2|).$$

Using $\bar{y}(1) = x_2$, $\bar{\tau}(1) = \tau_2$ and $\bar{y}'/\bar{\tau}' = \bar{\tau}$ we obtain (5.14). The second inequality (5.4) is obtained similarly.

Proof. (of Lemma 5.2)

Assuming $\phi$ given by (5.8) and $t_1 > t_0 \geq 0$ we have to prove (F) in Definition 5.1 namely

$$\phi(y, t_1) = \inf_{w \in \mathbb{R}^n} [J_P(w, y, t_0, t_1) + \phi(w, t_0)] \quad (5.14)$$
However, from (5.8)
\[
\phi(y, t_1) = \inf_{z \in \mathbb{R}^n} [J_P(z, y, 0, t_1) + \phi(z, 0)] \quad \text{and} \quad \phi(y, t_0) = \inf_{z \in \mathbb{R}^n} [J_P(z, y, 0, t_0) + \phi(z, 0)] .
\]
from (5.5), (5.6):
\[
J_P(z, y, 0, t_1) = \inf_{w \in \mathbb{R}^n} \{J_P(z, w, 0, t_0) + J_P(w, y, t_0, t_1)\}
\]
so
\[
\phi(y, t_1) = \inf_{z, w \in \mathbb{R}^n} \{J_P(z, w, 0, t_0) + J_P(w, y, t_0, t_1) + \phi(z, 0)\}
\]
and (5.14) follows. The proof for the backward equation is completely analogous. To prove the second part we proceed, as in the proof of Lemma 5.1, to consider the orbit 
\[
\tilde{x}(\tau) = \overline{x}(\tau) + (y - x) \tau / t
\]
where \( \overline{x} \) is an optimal orbit connecting \((x, t)\) with \((\overline{x}(0), 0)\), namely
\[
\phi(x, t) = \int_0^t \left[ |\dot{x}|^2 / 2 + P(\overline{x}(\tau), \tau) \right] d\tau + \phi(\overline{x}(0), 0) = J_P(\overline{x}(0), x, 0, t) + \phi(\overline{x}(0), 0) .
\]
Since \( \tilde{x}(t) = y \) we obtain, using (5.2) and integration by parts
\[
\phi(y, t) \leq \int_0^t \left[ |\dot{x}|^2 / 2 + P(\tilde{x}(\tau), \tau) \right] d\tau + \phi(\overline{x}(0), 0) = \phi(x, t) + \tilde{x}(t) \cdot (y - x) + \frac{|x - y|^2}{2t} (1 + O(t^2)) .
\]
In particular we obtain that
\[
\frac{\phi(y, t) - \phi(x, t)}{|x - y|} \leq \frac{\tilde{x}(t)}{2t} (1 + O(t^2)) \leq \frac{C}{t}
\]
where \( C = T \max |\overline{x}(t)| + \text{Diam}(\Omega) \left[ \frac{T}{2} + O(T^2) \right] \). Here \( \max |\overline{x}(t)| \) is the maximal possible value over all possible optimal orbits \( \overline{x} \). We now observe that there is a bound \(|\overline{x}| < D\) for any optimal orbit \( \overline{x} \) where \( D \) depends only on the bounds of \( P \) and \( P_x \) on \( \Omega_I \). This follows since there is a bound on the variation of \( \overline{x} \) in terms of \( \max(|P_x|) \) via (5.2), and a bound on the action \( J_P \) itself in terms of \( \max(|P|) \) and \( \max \left( |\overline{x}| \right) \) by its definition.

The reverse inequality of (5.15) and the result for the backward solutions follows analogously. 

Proof. (of Lemma 5.3) Let \( \phi \) be a forward solution. We know that \( \phi \) is Lipschitz by Lemma 5.2. If \( \phi \) is not a generalized sub-solution, then there exists \( \varepsilon > 0 \) and a \( C^1 \) orbit \( \overline{x} \) so that the set of points
\[
\left\{ t; \frac{d}{dt} \phi(\overline{x}(t), t) > \frac{|\overline{x}|^2}{2} + P(\overline{x}(t), t) + \varepsilon \right\} \subset I
\]
is of positive Lebesgue measure in $I$. Let $\tau_0$ be a density point of this set. Then, for $t_1 < \tau_0 < t_2$ sufficiently close, and $x_1 = \overline{x}(t_1), x_2 = \overline{x}(t_2)$, we obtain
\[
\phi(x_2, t_2) - \phi(x_1, t_1) = \int_{t_1}^{t_2} \frac{d}{dt} \phi(\overline{x}(t), t) dt > \int_{t_1}^{t_2} \left[ \frac{|\overline{x}(t)|^2}{2} + P(\overline{x}(t), t) \right] dt \geq J_P(x_1, x_2, t_1, t_2)
\]
which contradicts Definition 5.1. Hence, a forward solution is necessarily a generalized sub-solution.

We now show that a forward solution $\phi$ is a maximal generalized sub-solution. Let $\psi$ be a generalized sub-solution, $(x, t) \in \Omega_I$. Let $\tau < t$ and $\overline{x} : [\tau, t] \to \Omega$ be a $C^1$ orbit satisfying $\overline{x}(t) = x$. Assume $\phi$ be a forward solution in $\Omega \times [\tau, t]$ such that $\phi(x, \tau) = \psi(x, \tau)$ on $\Omega$. Then, by definition
\[
\psi(x, t) = \psi(\overline{x}(\tau), \tau) + \int_{\tau}^{t} \frac{d}{ds} \psi(\overline{x}(s), s) ds \leq \psi(\overline{x}(\tau), \tau) + \int_{\tau}^{t} \left[ \frac{1}{2} |\overline{x}(s)|^2 + P(\overline{x}(s), s) ds \right] \\
\leq \phi(\overline{x}(\tau), \tau) + J_P(\overline{x}(\tau), x, \tau, t).
\]
(5.16)

However, the same inequality survive if we take the infimum of the r.h.s of (5.16) over all such orbits $\overline{x}$. By definition, this infimum is nothing but the value of the forward solution $\phi(x, t)$. This implies the maximality of $\phi$. The statement for a backward solution is proved analogously.

Proof. (of Lemma 5.4):
From Corollary 5.1 and by construction it follows that $\underline{\psi} \leq \phi$ and $\overline{\psi} \geq \psi$ on $\Omega_I$, while $\overline{\psi}(x, 0) = \psi(x, 0)$ for $x \in \Omega$. On the other hand, since both $\overline{\psi}$ and $\phi$ are forward solutions and $\overline{\psi}(x, 0) = \psi(x, 0) \leq \phi_0(x)$ on $\Omega$, then necessarily $\overline{\psi} \leq \phi$ on $\Omega_I$.

and the equality $\overline{\psi}(x, T) = \psi(x, T)$ follows.

Let us turn to the case $\overline{P} \equiv 0$. Set $\psi_0 = \psi(\cdot, 0)$ (res. $\psi_1 = \psi(\cdot, T)$). From (F$_0$), (B$_0$) in Definition 5.2
\[
\psi_1(x) = \inf_{y \in \mathbb{R}^n} \left[ \frac{|x - y|^2}{2T} + \psi_0(y) \right] ; \quad \psi_0(x) = \sup_{y \in \mathbb{R}^n} \left[ -\frac{|x - y|^2}{2T} + \psi_1(y) \right].
\]
In particular
\[
\frac{|x - y|^2}{2T} + \psi_0(x) - \psi_1(y) \geq 0
\]
(5.17)
for any $x, y \in \mathbb{R}^n$. We may now lift the functions $\psi_0, \psi_1$ from the torus $\Omega$ to periodic functions in $\mathbb{R}^n$, and set
\[
\Psi_0(x) = T\psi_0(x) + \frac{x^2}{2} ; \quad \Psi_1(y) = \frac{y^2}{2} - T\psi_1(y), \quad x, y \in \mathbb{R}^n
\]
26
as functions on $\mathbb{R}^n$. Then (5.17) is equivalent to

$$
\Psi_0(x) + \Psi_1(y) - x \cdot y \geq 0 \quad \forall x, y \in \mathbb{R}^n.
$$

(5.18)

The condition of reversible pair is manifested in $\Psi_0, \Psi_1$ being related by the Legendre transforms:

$$
\Psi_0(x) = \sup_y [x \cdot y - \Psi_1(y)] := \Psi_1^*(x) \quad ; \quad \Psi_1(x) = \inf_y [x \cdot y - \Psi_0(y)] := \Psi_0^*(x).
$$

In particular, both $\Psi_0, \Psi_1$ are convex functions on $\mathbb{R}^n$. Recall that the super-gradient $\partial_x \Psi$ of a function $\Psi$ is given by

$$
y \in \partial_x \Psi \iff \Psi(z) - \Psi(x) \geq y \cdot (z - x) \quad ; \forall z \in \mathbb{R}^n
$$

and the equality in (5.18) holds iff $y \in \partial_x \Psi_0$ (equivalently, $x \in \partial_y \Psi_1$).

Let now $x, y, z \in \mathbb{R}^n$, $\tau \in (0, T)$. The inequality

$$
\frac{|x - y|^2}{2T} \leq \frac{|x - z|^2}{2\tau} + \frac{|z - y|^2}{2(T - \tau)}
$$

(5.19)

holds, and turns into an equality iff $(T - \tau)x + \tau y = Tz$. We obtain

$$
\overline{\psi}(z, \tau) = \inf_{y \in \mathbb{R}^n} \left[ \frac{|z - y|^2}{2\tau} + \psi_0(y) \right] \geq \psi(z, \tau) = \sup_{y \in \mathbb{R}^n} \left[ -\frac{|z - y|^2}{2(T - \tau)} + \psi_1(y) \right],
$$

$$
\implies \overline{\psi}(z, \tau) - \psi(z, \tau) = \inf_{x, y \in \mathbb{R}^n} \left[ \frac{|z - x|^2}{2\tau} + \frac{|z - y|^2}{2(T - \tau)} + \psi_0(x) - \psi_1(y) \right].
$$

(5.20)

Let now $l(z, \tau) := \{x, y \in \mathbb{R}^n ; (T - \tau)x + \tau y = Tz\}$. If we restrict the infimum in (5.20) to $x, y \in l(z, \tau)$ then, by the equality in (5.19), it is estimated by

$$
\leq \inf_{x, y \in l(z, \tau)} \left[ \frac{|y - x|^2}{2T} + \psi_0(x) - \psi_1(y) \right] = T^{-1} \inf_{x, y \in l(z, \tau)} \left[ \psi_0(x) + \psi_1(y) - x \cdot y \right].
$$

Now, the last term is zero iff there exists a pair $(x, y)$ where both $(T - \tau)x + \tau y = Tz$ and $y \in \partial_x \Psi_0$ hold, namely:

$$
Tz - (T - \tau)x \in \partial_x \Psi_0 \iff \frac{T}{\tau} z := z_\tau \in \partial_x \Psi_\tau
$$

where $\Psi_\tau(x) := \Psi_0(x) + \frac{T - \tau}{T} x^2$. Finally, $z_\tau \in \partial_x \Psi_\tau$ iff $z_\tau$ is in the domain of the Legendre transform $\Psi_\tau^*$. Since $\Psi_0$ is convex, it follows that $\Psi_\tau$ is strictly convex and it’s Legendre transform is defined on $\mathbb{R}^n$.

\[\square\]

Proof. (of Lemma 5.5):
Let $\varphi : [t_0, t_1] \to \mathbb{R}^n$ be any orbit. Then

$$
\int_{t_0}^{t_1} \left[ \frac{|\dot{\varphi}|^2}{2} + P(\varphi(t), t) \right] dt = \int_{t_0}^{t_1} \left[ \frac{|\dot{\varphi}|^2}{2} + \left( \phi(t, \varphi(t)) + \frac{1}{2} |\nabla_x \phi(\varphi(t), t)| \right) \right] dt
$$
\[
\begin{align*}
\phi(x_1, t_1) &= \int_{t_0}^{t_1} \left[ \frac{1}{2} \ddot{x} - \nabla_x \phi(x, t) \right]^2 dt + \left( \phi_t(x, t) + \dot{x} \cdot \nabla_x \phi(x, t) \right) dt \\
&= \int_{t_0}^{t_1} \left[ \frac{1}{2} \ddot{x} - \nabla_x \phi(x, t) \right]^2 dt + \frac{d}{dt} \phi(x(t), t) dt \\
&= \frac{1}{2} \int_{t_0}^{t_1} \left| \ddot{x} - \nabla_x \phi(x, t) \right|^2 dt + \phi(x_1, t_1) - \phi(x_0, t_0),
\end{align*}
\]

where \( x_0 = \phi(t_0), x_1 = \phi(t_1) \). Hence

\[
\phi(x_1, t_1) = \int_{t_0}^{t_1} \left[ \frac{\ddot{x}}{2} + P(\phi(x, t)) \right] dt + \phi(x_0, t_0) - \frac{1}{2} \int_{t_0}^{t_1} \left| \ddot{x} - \nabla_x \phi(x, t) \right|^2 dt
\]

holds for any orbit \( \phi(t) \). In particular,

\[
\phi(x_1, t_1) \leq \int_{t_0}^{t_1} \left[ \frac{\ddot{x}}{2} + P(\phi(x, t)) \right] dt + \phi(\phi(t_0), t_0).
\]

Moreover, if \( \phi \) is a solution of \( \ddot{x} = \nabla_x \phi(x, t) \), \( \phi(t_1) = x_1, t_0 \leq t \leq t_1 \), then the equality holds. Hence \( \phi \) is a forward solution. By the same way it follows that \( \phi \) is a backward solution, as well. \( \square \)

**Proof.** (of Lemma 5.6)

Let \( \tau \in (0, T) \) and \( x \in \Omega \). Since \( \phi \) is a forward solution and \( \phi \) is a backward one, then

(i) \( \phi(x, \tau) - \phi(y_1, 0) \leq J_P(y_1, x, 0, \tau) \); (ii) \( \phi(y_2, T) - \phi(x, \tau) \leq J_P(x, y_2, \tau, T) \forall y_1, y_2 \in \mathbb{R}^n \),

Moreover, there exists \( y_1^*, y_2^* \in \mathbb{R}^n \) for which

(i*) \( \phi(x, \tau) - \phi(y_1^*, 0) = J_P(y_1^*, x, 0, \tau) \); (ii*) \( \phi(y_2^*, T) - \phi(x, \tau) = J_P(x, y_2^*, \tau, T) \),

while, for any \( x^* \in \Omega \) and \( t \in (0, T) \)

(i**) \( \phi(x^*, t) - \phi(y_1^*, 0) \leq J_P(y_1^*, x^*, 0, t) \); (ii**) \( \phi(y_2^*, T) - \phi(x^*, t) \leq J_P(x^*, y_2^*, t, T) \).

From (i*), (i**) and (5.3):

\[
\phi(x, \tau) - \phi(x^*, t) \geq J_P(y_1^*, x, 0, \tau) - J_P(y_1^*, x^*, 0, t) \geq
\]

\[
\dot{x}_1(\tau) \cdot (x - x^*) + \left[ P(x, \tau) - \frac{1}{2} |\dot{x}_1(\tau)|^2 \right] (\tau - t) - O^2(|x - x^*|) - o(|t - \tau|)
\]

where \( \dot{x}_1 : [0, \tau] \rightarrow \mathbb{R}^n \) is an optimal orbit realizing the action \( J_P(y_1^*, x, 0, \tau) \). Similarly, from (i**) (ii**) and (5.4)

\[
\phi(x, \tau) - \phi(x^*, t) \leq J_P(x^*, y_2^*, t, T) - J_P(x, y_2^*, \tau, T) \leq
\]

\[
\dot{x}_2(\tau) \cdot (x - x^*) + \left[ P(x, \tau) - \frac{1}{2} |\dot{x}_2(\tau)|^2 \right] (\tau - t) + O^2(|x - x^*|) + o(|t - \tau|)
\]

where \( \dot{x}_2 : [\tau, T] \rightarrow \mathbb{R}^n \) is an optimal orbit realizing the action \( J_P(x, y_2^*, \tau, T) \).
Let now \((x, \tau) \in K_0(\bar{\phi}, \phi)\). Then \(\phi(x, \tau) := \bar{\phi}(x, \tau) = \phi(x, \tau)\) and (5.21), (5.22) yield
\[
\tilde{x}_1(\tau) \cdot (x - x^*) + \left[ P(x, \tau) - \frac{1}{2} |\tilde{x}_1(\tau)|^2 \right] (\tau - t) - O^2(|x - x^*|) - o(|\tau - t|)
\leq \bar{\phi}(x, \tau) - \phi(x^*, t) \leq \\
\tilde{x}_2(\tau) \cdot (x - x^*) + \left[ P(x, \tau) - \frac{1}{2} |\tilde{x}_2(\tau)|^2 \right] (\tau - t) + O^2(|x - x^*|) + o(|\tau - t|),
\]
as well as
\[
\tilde{x}_1(\tau) \cdot (x - x^*) + \left[ P(x, \tau) - \frac{1}{2} |\tilde{x}_1(\tau)|^2 \right] (\tau - t) - O^2(|x - x^*|) - o(|t - \tau|)
\leq \phi(x, \tau) - \phi(x^*, t) \leq \\
\tilde{x}_2(\tau) \cdot (x - x^*) + \left[ P(x, \tau) - \frac{1}{2} |\tilde{x}_2(\tau)|^2 \right] (\tau - t) + O^2(|x - x^*|) + o(|t - \tau|).
\]

Now, in order to show that both \(\nabla_x \bar{\phi}(x, \tau)\) and \(\nabla_x \phi(x, \tau)\) exist for \((x, \tau) \in K_0\) we only have to show that \(\tilde{x}_1(\tau) = \tilde{x}_2(\tau)\). Adding (i) and (ii) we obtain:
\[
\phi(y_2, T) - \phi(y_1, 0) \leq J_P(y_1, x, 0, \tau) + J_P(x, y_2, \tau, T) \quad (5.23)
\]
for any \(y_1, y_2 \in \mathbb{R}^n\). From \((i^*), (ii^*)\) we obtain that equality holds in (5.23) for \(y_1^*, y_2^*:\)
\[
\phi(y_2^*, T) - \phi(y_1^*, 0) = J_P(y_1^*, x, 0, \tau) + J_P(x, y_2^*, \tau, T). \quad (5.24)
\]

On the other hand, by pair-reversibility:
\[
\phi(y_2^*, T) - \phi(y_1^*, 0) \leq J_P(y_1^*, y_2^*, 0, T). 
\]

Together with (5.24):
\[
J_P(y_1^*, x, 0, \tau) + J_P(x, y_2^*, \tau, T) \leq J_P(y_1^*, y_2^*, 0, T), \quad (5.25)
\]
Comparing the above with (5.5) we obtain an equality in (5.25). This implies that \(\tilde{x}_1(\tau) = \tilde{x}_2(\tau)\) by Lemma 5.4d. The existence of
\[
\nabla_x \phi(x, \tau) = \tilde{x}(\tau) := \tilde{x}_1(\tau) = \tilde{x}_2(\tau) \quad (5.26)
\]
and of
\[
\phi_t(x, \tau) = P(x, \tau) - \frac{1}{2} |\tilde{x}|^2(\tau)
\]
follows from (5.21), (5.22). In particular the equality (5.1) holds for any \((x, \tau) \in K_0\).

We now prove the Lipschitz continuity of \(\nabla_x \phi\) on \(K_0\). By Lemma 5.1 (b) we obtain that for any \(t \in (0, T)\) there exists \(C = C(t)\) such that both \(J_P(y, x, 0, t) - C(t)x^2\) and \(J_P(x, y, t, T) - C(t)x^2\) are concave functions of \(x\) for any \(y \in \mathbb{R}^n\). It follows by Definition 5.1 (F) that \(\overline{q}(x) := \bar{\phi}(x, t) - C(t)x^2\), being the infimum of a set of concave functions \(J_P(y, x, 0, t) - C(t)x^2\) for any \((x, \tau) \in K_0\).
$C(t)x^2 + \phi(y,0) ; y \in \mathbb{R}^n$, is concave as well. Likewise, from (B) of the same definition we obtain that $q(x) := \phi(x,t) + C(t)x^2$, being the supremum of a set of convex functions $-J_P(x,y,t,T) + C(t)x^2 + \phi(x,T) ; y \in \mathbb{R}^n$, is convex.

Let $x, x^* \in K_0$ for some $t \in I_0$. Then both $x, x^*$ are differentiability points of the convex function $q$, hence

$$\langle \nabla_x q - \nabla_{x^*} q, x - x^* \rangle \geq 0 .$$

Using $\nabla \phi = \nabla \phi$ for $x, x^*$ we obtain

$$\langle \nabla_x \phi - \nabla_{x^*} \phi, x - x^* \rangle \geq -C |x - x^*|^2 . \tag{5.27}$$

The same argument applied to the concave function $\overline{q}$ yields

$$\langle \nabla_x \phi - \nabla_{x^*} \phi, x - x^* \rangle \leq C |x - x^*|^2 . \tag{5.28}$$

We may assume, by shifting and rotating the coordinate system, that $x = (0,0 \ldots 0)$ and $x^* = (\zeta, 0 \ldots 0)$. Then (5.27, 5.28) yield

$$|\phi_{x_1}(0,0 \ldots 0,t) - \phi_{x_1}(\zeta,0 \ldots 0,t)| \leq C|\zeta| . \tag{5.29}$$

It is left to be shown that there exists a constant $C^*$ such that

$$|\phi_{x_i}(0,0 \ldots 0,t) - \phi_{x_i}(\zeta,0 \ldots 0,t)| \leq C^*|\zeta| \tag{5.30}$$

holds for any $i > 1$. By subtracting an affine function from $\overline{\phi}$ and $\phi$ we may assume that (for the fixed value of $t$)

$$\overline{\phi}(0,t) = \phi(0,t) = \overline{q}(0) = q(0) = 0 \tag{5.31}$$

and

$$\nabla_x \overline{\phi}(0,t) = \nabla_x \phi(0,t) = \nabla_x q(0) = \nabla_x \overline{q}(0) = 0 , \tag{5.32}$$

hence (5.29) is reduced to

$$|\phi_{x_1}(\zeta,0 \ldots 0,t)| < C|\zeta| , \tag{5.33}$$

and (5.30) takes the form

$$|\phi_{x_i}(\zeta,0 \ldots 0,t)| \leq C^*|\zeta| \tag{5.34}$$

for $i > 1$.

Since $\overline{q}$ is concave (convex), then (5.31, 5.32) implies that

$$\overline{q} \leq 0 \leq q \tag{5.35}$$

everywhere.

To proceed, we use the concavity of $\overline{q}$ to obtain

$$\overline{q}(\zeta,y \ldots 0) - \overline{q}(\zeta,0 \ldots 0) \leq \overline{q}_{x_2}(\zeta,0 \ldots 0)y = \phi_{x_2}(\zeta,0 \ldots 0,t)y . \tag{5.36}$$

From (5.35), (5.36), the inequality $\overline{\phi} \geq \phi$ and the relation between $\overline{q}$ and $\overline{\phi}$ we obtain

$$\phi_{x_2}(\zeta,0 \ldots 0,t)y \geq \overline{q}(\zeta,y,0 \ldots 0) = \overline{\phi}(\zeta,y,0 \ldots 0) - C(\zeta^2 + y^2)$$
\[ \geq \phi(\zeta, y, 0\ldots 0) - C(\zeta^2 + y^2) = \mathcal{L}(\zeta, y, 0\ldots 0) - 2C(\zeta^2 + y^2) \geq -2C(\zeta^2 + y^2) , \]

so

\[ \phi_{x_2}(\zeta, 0\ldots 0, t)y + 2C|y|^2 \geq -2C\zeta^2 \]

holds for any \( y \in \mathbb{R} \). Take now the minimum of the lhs above with respect to \( y \) to obtain

\[ -\frac{\phi^2_{x_2}}{8C} \geq -2C\zeta^2 \implies |\phi_{x_2}(\zeta, 0\ldots 0, t)| \leq 4C|\zeta| \]

and (5.30) holds with \( C^* = 4C \) where \( i = 2 \). The general case of \( i > 1 \) is evident. \( \square \)

**Proof.** (of Lemma 5.7): Let \((\bar{\phi}, \phi)\) a reversible pair. We first note that, for any \((x, t) \in \Omega_{K_0}\),

\[ 0 \leq \bar{\phi}(x, t) - \phi(x, t) \leq \phi(y, 0) - \phi(z, T) + J_P(y, x, 0, t) + J_P(x, z, t, T). \]

However, from definition of \( J_P \)

\[ J_P(y, x, 0, t) + J_P(x, z, t, T) \leq \int_0^T \left[ \frac{1}{2} |\bar{\tau}(s)|^2 + P(\bar{\tau}(s), s) \right] \, ds , \]

holds for any \( C^1 \) orbit \( \bar{\tau} \) which satisfies \( \bar{\tau}(0) = y, \bar{\tau}(t) = x \) and \( \bar{\tau}(T) = z \). It follows that

\[ 0 \leq \bar{\phi}(x, t) - \phi(x, t) \leq \int_0^T \left[ \frac{1}{2} |\bar{\tau}(s)|^2 + P(\bar{\tau}(s), s) \right] \, ds + \phi(\bar{\tau}(0), 0) - \phi(\bar{\tau}(T), T) \]

holds for any \((x, t) \in \Omega_{K_0}\).

As a result we conclude that, if there exists an orbit \( \bar{\tau} \) crossing the point \((x, t) \in \Omega_{K_0}\) such that

\[ \int_0^T \left[ \frac{1}{2} |\bar{\tau}(s)|^2 + P(\bar{\tau}(s), s) \right] \, ds + \phi(\bar{\tau}(0), 0) - \phi(\bar{\tau}(T), T) = 0 , \quad (5.37) \]

then \( \bar{\phi}(x, t) = \phi(x, t) \), i.e. \((x, t) \in K_0\). Moreover, in that case, the entire orbit is contained in \( K_0 \), namely \( \cup_{\tau \in I_K(\bar{\tau}(\tau), \tau) \subset K_0} \). In addition, it must be a minimal orbit for the action and, by Lemma 5.1 and Lemma 5.6

\[ \nabla_x \bar{\phi}(\bar{\tau}(t), t) = \nabla_x \phi(\bar{\tau}(t), t) = \bar{\tau}(t). \]

Next, let \((x, t) \in K_0 \) and let \( \bar{\tau}_1 : [0, t] \to \Omega, \bar{\tau}_2 : [t, T] \to \Omega \) the minimal orbits for which \( \bar{\phi}(x, t) = \phi(\bar{\tau}_1(0), 0) + \int_0^t \left[ \frac{1}{2} |\bar{\tau}_1(s)|^2 + P(\bar{\tau}_1(s), s) \right] \, ds \) (res. \( \bar{\phi}(x, t) = \phi(\bar{\tau}_2(T), T) - \int_0^T \left[ \frac{1}{2} |\bar{\tau}_2(s)|^2 + P(\bar{\tau}_2(s), s) \right] \, ds \)). Then \( \bar{\tau} = \bar{\tau}_1 \cup \bar{\tau}_2 \) satisfies (5.37). In particular, it is contained in \( K_0 \) and is an integral curve of the vectorfield \( \nabla_x \phi \). This implies the invariance of \( K_0 \) under the flow of \( \nabla_x \phi \). \( \square \)

### 5.3 Proof of the main Theorem

First, the existence of a minimizer for \( \mathcal{L}(\mu_0, \mu_1) \) in \( \Lambda_2(\mu_0, \mu_1) \) follows immediately by the lower-semi-continuity of \( \|\mu\|_2 \) and the local compactness of \( \mathbf{H}_2 \). Next, we shall prove the chain of inequalities:

\[ \mathcal{E}(\mu_0, \mu_1) \geq \mathcal{L}(\mu_0, \mu_1) \geq \mathcal{K}(\mu_0, \mu_1) \geq \mathcal{E}(\mu_0, \mu_1) \]

from left to right, together with the existence of a maximizer for \( \mathcal{E}(\mu_0, \mu_1) \) in \( \mathbf{K}_P^* \).
• $E(\mu_0, \mu_1) \geq L(\mu_0, \mu_1)$

From Lemma 4.5 and Corollary 4.2 there exists a sequence $\varepsilon_k \to 0$ and $\phi_k \in C^1(\Omega_I)$ such that

$$L \leq \Psi_{\varepsilon_k}(\phi_k) < -O(\varepsilon_k^{-\alpha}) ||\xi_k||_s^s + \int_{\Omega} \phi_k(x, T)\mu_1(dx) - \int_{\Omega} \phi_k(x, 0)\mu_0(dx) < C \quad (5.38)$$

where $\alpha > 0$, $s > n + 1$ and $\xi_k = \phi_{k,t} + |\nabla_x \phi_k|^2 / 2 - P$.

Let now $\Xi : \Omega_I \to \Omega$ be a flow such that

i) $\sup_{x \in \Omega} \int_0^T |\frac{\partial \Xi(x,t)}{\partial t}|^2 dt := E < \infty$,

ii) $\Xi(x, 0) = x$, $\Xi\#(\cdot, T)\mu_0 = \mu_1$

By Lemma 5.8 and (i) we obtain

$$\phi_k(\Xi(x, T), T) - \phi_k(x, 0) \leq \frac{1}{2}E + |P|_{\infty} + C_1||\xi_k||_s^{2\beta} + C_2||P||_{lip}||\xi_k||_s^{\beta}. \quad (5.39)$$

Integrate the above against $\mu_0$ on $\Omega$ and use (ii) to obtain

$$\int_{\Omega} \phi_k(x, T)\mu_1(dx) - \int_{\Omega} \phi_k(x, 0)\mu_0(dx) \leq \frac{1}{2}E + |P|_{\infty} + C_1||\xi_k||_s^{2\beta} + C_2||P||_{lip}||\xi_k||_s^{\beta} \quad (5.40)$$

Using $s > n + 1$ and $\beta < 1/2$ (c.f. Lemma 5.8) we obtain from (5.38) and (5.39) that $||\xi_k||_s \to 0$ as $\varepsilon_k \to 0$. In addition

$$\liminf_{k \to \infty} \left[ \int_{\Omega} \phi_k(x, T)\mu_1(dx) - \int_{\Omega} \phi_k(x, 0)\mu_0(dx) \right] \geq L. \quad (5.41)$$

Now, we may replace the sequence $\phi_k$ by a sequence of forward solutions $\overline{\psi}_k$ of the equation

$$\overline{\psi}_{k,t} + \frac{1}{2} |\nabla_x \overline{\psi}_k|^2 = P \quad ; \quad \overline{\psi}_k(x, 0) = \phi_k(x, 0).$$

This is also a maximizing sequence which, by Corollary 5.2 together with $||\xi_k||_s \to 0$, yields

$$\liminf_{k \to \infty} \left[ \int_{\Omega} \overline{\psi}_k(x, T)\mu_1(dx) - \int_{\Omega} \overline{\psi}_k(x, 0)\mu_0(dx) \right] \geq L. \quad (5.42)$$

From Lemma 5.2 we also obtain a uniform estimate on $\overline{\psi}_k$ in $LIP(\Omega \times (0, T])$. In particular, the sequence $\overline{\psi}_k(T)$ is uniformly Lipschitz on $\Omega$.

Now, define $\overline{\psi}_k$ to be the backward solutions of (3.3) subjected to $\overline{\psi}_k(x, T) = \overline{\psi}_k(x, T)$. From the first part of Lemma 5.2 $\overline{\psi}_k(x, 0) \leq \overline{\psi}_k(x, 0)$ on $\Omega$ so (5.41) is satisfied for $\overline{\psi}_k$ as well. Moreover, by the last part of Lemma 5.2 $\overline{\psi}_k$ are uniformly bounded in the Lipschitz norm on $\Omega_I$. So, we can extract a subsequence of $\overline{\psi}_k$ which converges uniformly on $LIP(\Omega \times [0, T])$ to a backward solution $\overline{\psi}$. In particular, both $\overline{\psi}(T)$ and
ψ(0) are Lipschitz. Let \( \bar{\psi} \) be the forward solution satisfying \( \bar{\psi}(0) = \psi(0) \). By Corollary 5.1 and definition 5.3, the pair \( (\bar{\psi}, \psi) \) is a reversible pair and both functions are in \( \mathcal{L}^\infty \mathcal{P} \) (see the first part of Lemma 5.3). Moreover, the inequality \( 5.41 \) is preserved in the limit process, so

\[
\int_\Omega \psi(x,T)\mu_1(dx) - \int_\Omega \psi(x,0)\mu_0(dx) \geq \mathcal{L}
\]

holds for both \( \psi = \bar{\psi} \) and \( \psi = \psi \) (recall \( \bar{\psi} = \psi \) on \( \Omega \times \{0\} \) and \( \Omega \times \{T\} \)).

- \( \mathcal{L}(\mu_0, \mu_1) \geq K(\mu_0, \mu_1) \)

Recall that there exists a minimizer of \( \mathcal{L}(\mu_0, \mu_1) \) by the first part of the Theorem. Let \( \mu \) be such a minimizer. We now use the regularization Lemma 2.3 to approximate \( \mu \) by smooth densities \( \mu_n = \rho_n(x,t)dxdt \). Let \( v_n \) be the regularized velocity field. Then

\[
\lim_{n \to \infty} \int_{\Omega_I} |v_n|^2(x,t)\rho_n(x,t)dxdt = ||\mu||^2_2. \tag{5.42}
\]

as well as

\[
\lim_{n \to \infty} \int_{\Omega_I} \rho_n(x,t)P(x,t)dxdt = \int_{\Omega_I} P\mu(dxdt).
\]

Define

\[
m_n(x,t) = \rho_n(x,t)v_n(x,t). \tag{5.43}
\]

Then \( m_n \in C^1(\Omega_I) \). Define now

\[
v_n^\varepsilon(x,t) = \frac{m_n(x,t)}{\rho_n(x,t) + \varepsilon}.
\]

By assumption, \( v_n^\varepsilon \) is Lipschitz on \( \Omega_I, t \in I \). Define \( \rho_n^\varepsilon(x,t) \) as the solution of

\[
\frac{\partial \rho_n^\varepsilon}{\partial t} + \nabla_x \left[ v_n^\varepsilon \rho_n^\varepsilon \right] = 0; \quad \rho_n^\varepsilon(x,0) = \rho_n(x,0). \tag{5.44}
\]

Since \( v_n^\varepsilon \) is Lipschitz, we may define the flow associated with it as \( \Gamma_n^\varepsilon : \Omega \to \Omega \) for \( t \in I \), namely \( \Gamma_n^\varepsilon(x) = y(x)(t) \) where \( \dot{y}(x) = v_n^\varepsilon(y(x)(t),t) \) and \( y(x)(0) = x \). It follows that \( \Gamma_n^\varepsilon, \# \rho_n^\varepsilon(\cdot,0)dx = \rho_n^\varepsilon(\cdot,t)dx \) for all \( t \in I \). In particular:

\[
\mathcal{K} \left( \rho_n(x,0)dx, \rho_n^\varepsilon(x,T)dx \right) \leq \int_{\Omega} \rho_n(x,0)J_P \left( x, \Gamma_n^T(x)0,T \right) dx
\]

\[
= \int_{\Omega_I} \rho_n(x,0) \frac{d}{dt} J_P \left( x, \Gamma_n^T(y(x),0,t) \right) dt dx
\]

\[
= \int_{\Omega_I} \rho_n(x,0) \left[ \frac{\partial}{\partial t} J_P \left( x, \Gamma_n^T(y(x),t) \right) + \nabla_y=\Gamma_n^T(y(x),y,0,t) \cdot v_n^\varepsilon \left( \Gamma_n^T(y(x),t) \right) \right] dx dt \tag{5.45}
\]
From (5.7)

\[
\partial_t J_P(x, y, 0, t) + \nabla_y J_P(x, y, 0, t) \cdot v_n(x, y, t) = P(y, t) - \frac{1}{2} |v_n(x, y, t)|^2 - \frac{1}{2} |\nabla_y J_P(x, y, 0, t) - v_n(x, y, t)|^2.
\]

Substitute the above in (5.45) at \( y = \Gamma_\varepsilon^t(x) \) to obtain

\[
K \left( \rho_n(x, 0) dx, \rho_n(x, T) dx \right) \leq \int_{\Omega_I} \rho_n(x, 0) \left[ P \left( \Gamma_\varepsilon^t(x), t \right) + \frac{1}{2} |v_n(x, t)|^2 \left( \Gamma_\varepsilon^t(x), t \right) \right] dx dt
\]

\[
= \int_{\Omega_I} \rho_n(x, t) \left[ \frac{1}{2} |v_n(x, t)|^2 + P(x, t) \right] dx dt \leq \int_{\Omega_I} \rho_n(x, t) \left[ \frac{1}{2} |v_n(x, t)|^2 + P(x, t) \right] dx dt
\]

where the last inequality follows from (5.43). We next show that

\[
\lim_{\varepsilon \to 0} \int_{\Omega} \left| \rho_n(x, t) - \rho_n(x, t) \right| dx = 0 \tag{5.46}
\]

for any \( t \in I \). In fact, we note that \( \rho_n(x, 0) = \varepsilon > 0 \) and \( w_n(x, 0) = \varepsilon \) solves this equation as well. Since \( w_n(x, 0) \geq 0 \) we obtain that \( w_n(x, t) \geq 0 \) over \( \Omega_I \) and, moreover,

\[
\int_{\Omega} \left| \rho_n(x, t) - \rho_n(x, t) \right| dx - |\Omega| \varepsilon \leq \int_{\Omega} w_n(x, t) dx = \int_{\Omega} w_n(x, 0) dx = |\Omega| \varepsilon
\]

for all \( t \in I \). Now we take first the limit \( \varepsilon \to 0 \) then the limit \( n \to \infty \) in (5.46). The r.h.s of (5.46) converges to \( \mathcal{L}(\mu_0, \mu_1) \). Now, \( \rho_0(x, 0, t) dx \) and \( \rho_n(x, t) dx \) converges, as \( n \to \infty \) and \( \varepsilon \to 0 \), weak-* to \( \mu_0 \) and \( \mu_1 \), respectively. Since \( K \) is lower-semi-continuous in both \( \mu_0 \) and \( \mu_1 \) we obtain the desired result from (5.46).

- \( \mathcal{E}(\mu_0, \mu_1) \leq \mathcal{K}(\mu_0, \mu_1) \).

Let \( \lambda \in C^*(\Omega \times \Omega) \) be an optimizer of \( \mathcal{K} \). Since \( \pi^{(1)} \lambda = \mu_1 \) then

\[
\int_{\Omega} \phi(x) \mu_1(dx) = \int_{\Omega} \int_{\Omega} \phi(y) \lambda(dy dx) \quad \text{and} \quad \int_{\Omega} \phi(x) \mu_0(dx) = \int_{\Omega} \int_{\Omega} \phi(x) \lambda(dx dy)
\]

for any continuous \( \phi_1, \phi_2 \). Set \( \phi_1(x) = \psi(x, T) \) and \( \phi_0(x) = \psi(x, 0) \) with \( \psi \) an optimal backward solution of problem \( \mathcal{E} \). Then

\[
\mathcal{E} = \int_{\Omega} \psi(x, T) \mu_1(dx) - \int_{\Omega} \psi(x, 0) \mu_0(dx) = \int_{\Omega} \int_{\Omega} [\psi(y, T) - \psi(x, 0)] \lambda(dx dy)
\]

Since \( \psi \) is a backward solution then

\[
\int_{\Omega} \int_{\Omega} [\psi(y, T) - \psi(x, 0)] \lambda(dx dy) \leq \int_{\Omega} \int_{\Omega} J_P(x, y, 0, T) \lambda(dx dy) = \mathcal{K}.
\]
We have proved
\[ \int_{\Omega} \psi(x,T) \mu_1(dx) - \int_{\Omega} \psi(x,0) \mu_0(dx) = \int_{\Omega} \psi(x,T) \mu_1(dx) - \int_{\Omega} \psi(x,0) \mu_0(dx) = \mathcal{L}(\mu_0, \mu_1) . \]  
(5.48)

We now turn to the proof of parts (i)-(vi) of the Theorem.

i) Let \( \mu^{(0)} \) be a minimizer of \( \mathcal{L} \). Given \( t \in I_0 \), let \( \mu_{1/2} := \mu^{(0)}_{(t)} \in \mathcal{M} \). Let us consider \( \mu^{(1)} \) to be the restriction of \( \mu^{(0)} \) to \( \Omega \times [0,t] \) and \( \mu^{(2)} \) the restriction of \( \mu^{(0)} \) to \( \Omega \times [t,T] \). Evidently, \( \mu^{(1)} \) is a minimizer of \( L_P \) on the set of orbits \( \Lambda_2(\mu_0, \mu_{1/2}) \) confined to \( [0,t] \) while \( \mu^{(2)} \) is a minimizer on \( \Lambda_2(\mu_{1/2}, \mu_1) \) with respect to the same set, confined to \( [t,T] \). In particular,
\[ L_P(\mu^{(1)}) + L_P(\mu^{(2)}) = L_P(\mu^{(0)}) = \mathcal{L}(\mu_0, \mu_1) . \]  
(5.49)

By what we know so far,
\[ \int_{\Omega} \psi(x,t) \mu_{1/2}(dx) - \int_{\Omega} \psi(x,0) \mu_0(dx) \leq L_P(\mu^{(1)}) \]  
(5.50)
\[ \int_{\Omega} \psi(x,T) \mu_1(dx) - \int_{\Omega} \psi(x,t) \mu_{1/2}(dx) \leq L_P(\mu^{(2)}) . \]  
(5.51)

However, if we sum (5.50) and (5.51) and use (5.49) and (5.48), we conclude that there is, in fact, an equality in both (5.50) and (5.51). Same argument holds for \( \overline{\psi} \) as well. Thus
\[ \int_{\Omega} \psi(x,t) \mu_{1/2}(dx) - \int_{\Omega} \psi(x,0) \mu_0(dx) = L_P(\mu^{(1)}) = \int_{\Omega} \psi(x,t) \mu_{1/2}(dx) - \int_{\Omega} \psi(x,0) \mu_0(dx) . \]

Since \( \overline{\psi}(x,0) \equiv \psi(x,0) \),
\[ \int_{\Omega} [\overline{\psi}(x,t) - \psi(x,t)] \mu_{1/2}(dx) = 0 . \]

But, \( \overline{\psi} \geq \psi \) by Lemma 5.3. Hence \( \overline{\psi}(x,t) = \psi(x,t) \) on \( \text{supp}(\mu^{(0)}_{(t)}) = \text{supp}(\mu_{1/2}) \).

This, together with Lemma 5.6, proves that \( \text{Supp}(\mu^{(0)}_{(t)}) \cap \Omega_{I_0} \subset K_0 \equiv \{(x,t) : t \in I_0 , \psi(x,t) = \psi(x,t) \} \) and, in particular, that \( \phi \) is differentiable at any point on the support of \( \mu^{(0)} \) in \( \Omega_I \).

ii) This part follows from Lemma 5.7. In addition, the limits \( \lim_{t \rightarrow T} T'_{t} \) and \( \lim_{t \rightarrow 0} T'_{t} \) exists since \( \nabla_x \psi \) is uniformly bounded on \( K_0 \). In particular, the Lipschitz extension \( v \) can be chosen to be a uniformly bounded function on \( \Omega_I \) as well.

iii) Suppose there are two optimal solutions \( \psi_1, \psi_2 \) of \( \mathcal{E}(\mu_0, \mu_1) \). To prove the uniqueness for the vector field \( v \) we claim that
\[ \int_{\Omega} [\nabla_x \psi_1 - \nabla_x \psi_2] \mu(dxdt) = 0 \]
for any minimizer $\mu \in \Lambda_2(\mu_0, \mu_1)$ of $\mathcal{L}(\mu_0, \mu_1)$. Let $\psi = \alpha \psi_1 + (1-\alpha)\psi_2$ where $\alpha \in (0,1)$. Then
\[|\nabla_x \psi|^2 = \alpha|\nabla_x \psi_1|^2 + (1-\alpha)|\nabla_x \psi_2|^2 - \alpha(1-\alpha)|\nabla_x \psi_1 - \nabla_x \psi_2|^2,\]
so $\psi_t + |\nabla_x \psi|^2/2 < P$ and
\[
\int_{\Omega_t} [\psi_t + |\nabla_x \psi|^2/2] \mu(dxdt) < \int_{\Omega_t} P\mu(dxdt)
\] (5.52)
if $\nabla_x \psi_1 \neq \nabla_x \psi_2$ at some point in the support of a minimizer $\mu$ (recall that both $\nabla \psi_i$, $i = 1, 2$ are continuous on the support of $\psi$ by Lemma 5.9).

On the other hand,
\[
\mathcal{E}(\mu_0, \mu_1) = \int_{\Omega} [\psi(x, T)\mu_1(dx) - \psi(x,0)\mu_0(dx)] = \mathcal{L}(\mu_0, \mu_1)
\]
(5.53) follows from the assumptions that both $\psi_1, \psi_2$ are maximizers of $\mathcal{E}$. From (4.10), (5.52) and (5.53) it follows that
\[
L_P(\mu) := \frac{1}{2}||\mu||^2 + \int_{\Omega_t} P\mu(dxdt) > \frac{1}{2}||\mu||^2 + \int_{\Omega_t} (\psi_t + |\nabla_x \psi|^2/2)\mu(dxdt) \geq \mathcal{E}(\mu_0, \mu_1) = \mathcal{L}(\mu_0, \mu_1),
\]
in contradiction to the assumption that $\mu$ is a minimizer of $L_P$.

iv) Let, again, $\mu \in \Lambda_2(\mu_0, \mu_1)$ a minimizer of $\mathcal{L}$ and $\psi$ a maximizer of $\mathcal{E}$. Since $\psi$ satisfies the HJ equation on a closed set $K_0$ containing the support of $\mu$ in $\Omega_{t_0}$ and is a $C^1$ function there, we can extend it as a $C^1$ function on $\Omega_{t_0}$ so $\psi \in C^1(\Omega_{t_0}) \cap LIP(\Omega_{t})$ and, by (5.53),
\[
-\int_{\Omega} \left[\psi_t + \frac{1}{2}||\nabla_x \psi||^2 - P\right] \mu(dxdt) + \int_{\Omega} [\psi(x, T)\mu_1(dx) - \psi(x,0)\mu_0(dx)] = \mathcal{L}(\mu_0, \mu_1) = L_P(\mu)
\]
(5.54)
We now use Corollary 4.11 (4.10) to observe that $\psi$ is a maximizer of the left of (5.54), so by taking the variation $\phi = \psi + \varepsilon \eta$ with $\eta \in C^1(\Omega_{t})$ we obtain
\[
\int_{\Omega} (\eta_t + \nabla_x \psi \cdot \nabla_x \eta)\mu(dxdt) + \int_{\Omega} \eta(x,0)\mu_0(dx) - \int_{\Omega} \eta(x, T)\mu_1(dx) \geq 0
\]
for any such $\eta$. Replacing $\eta$ by $-\eta$ we obtain the equality above. Moreover, by the same argument following (5.39) to (5.51) we also obtain that
\[
\int_{t_0}^t \int_{\Omega} (\eta_t + \nabla_x \psi \cdot \nabla_x \eta)\mu_t(dx) + \int_{\Omega} \eta(x, t_0)\mu_{(t_0)}(dx) - \int_{\Omega} \eta(x, t)\mu_{(t)}(dx) = 0,
\]
(5.55)
hold for any $0 < t_0 < t < T$. In particular, $\mu$ solves the weak form of the continuity equation with $v = \nabla_x \psi$.

Now, we know that, by the additional assumption on $P$, that $K_0$ is invariant with respect to the flow $T_{t_0}$ induced by the Lipschitz vectorfield $v$ extending $\nabla_x \psi$. We shall
now prove that \( \mu \) is transported by this flow. That is, for any choice of \( t_0, t \in (0, T) \), we need to show that \( \mu(t) = \gamma_t \) where

\[
\gamma_t := \left[ T^t_{t_0} \right] \# \mu_{t_0}.
\]

Since \( T \) is the flow generated by \( v \) and \( K_0 \) is invariant with respect to \( v \) it follows that \( \gamma = \gamma_t dt \) is supported on \( K \) and solves the weak form of the continuity equation as well. Setting \( \zeta_t = \mu(t) - \gamma_t, \zeta := \zeta_t dt \) we obtain from (5.55)

\[
\int_{t_0}^t \int_{\Omega} (\eta_t + v \cdot \nabla_x \eta) \zeta_t(\eta_t) dx d\eta = \int_{\Omega} \eta(x, t) \zeta_t dx
\]

for any \( \eta \in C^1([t_0, t]; \mathbb{R}) \) where we used \( \zeta_0 \equiv 0 \).

Let now \( h = h(x) \in C^1(\Omega) \). Let \( \eta = \eta(x, \tau) \) be a solution of

\[
\eta_t + v \cdot \nabla_x \eta = 0 ; \quad \eta(x, t) = h(x), \quad t_0 \leq \tau \leq t .
\]

(5.57)

Since, by Lemma 5.6 and Lemma 5.7, the vector field \( \nabla_x \psi = v \) is locally Lipschitz continuous on \( K_0 \) which is invariant with respect to the induced flow, we can find a solution of (5.57) on \( K \cap (\Omega \times [t_0, t]) \) via

\[
\eta(x, \tau) = h \left( T^\tau_t(x) \right) .
\]

(5.58)

The function \( \eta \) so defined can be extended into a \( C^1 \) function on \( \Omega \times [t_0, t] \). It satisfies (5.57) on \( K \), and recalling that \( \zeta \) is supported on \( K_0 \), we substitute now (5.57) in (5.58) to obtain \( \zeta_t \equiv 0 \) and the proof of part (iv).

v) The optimality of \( T \) is evident from the proof of (iii) and (iv).

vi) From the last part of Lemma 5.4 it follows that \( \psi \) is a reversible solution so Lemma 5.6 implies that \( \psi_t + |\nabla_x \psi|^2/2 = 0 \) is satisfied everywhere on \( \Omega_{t_0} \). The flow induced by such a solution is given by \( T^t_{t_0}(x) = x + (t - \tau) \nabla_x \psi(x, \tau) \) and, by (iv) and (v), it transports \( \mu(\tau) \) to \( \mu(t) \) optimally.

6 Appendix

Proof. of Proposition 4.1

Define

\[
\Phi(c^*, z) := \mathcal{F}(c^*) - < c^*, z > + < h, z > .
\]

First, note that

\[
I = \inf_{c^* \in C^*} \sup_{z \in Z} \Phi(c^*, z) .
\]

Indeed, if \( c^* \not\in Z^* \) then \( \sup_{z \in Z} \Phi(c^*, z) = \infty \) while, if \( c^* \in Z^* \) then \( \phi(c^*, z) = \mathcal{F}(c^*) \) by definition. We have, therefore, to show

\[
\inf_{c^* \in C^*} \sup_{z \in Z} \Phi(c^*, z) = \sup_{z \in Z} \inf_{c^* \in C^*} \Phi(c^*, z) .
\]
It is trivial that
\[ \inf_{c^* \in C^*} \sup_{z \in \mathbb{Z}} \Phi(c^*, z) \geq \sup_{z \in \mathbb{Z}} \inf_{c^* \in C^*} \Phi(c^*, z) := I, \]
so we only have to show that
\[ \inf_{c^* \in C^*} \sup_{z \in \mathbb{Z}} \Phi(c^*, z) \leq I. \]
Define, for any \( z \in \mathbb{Z} \)
\[ A_z = \{ c^* \in C^* ; \Phi(c^*, z) \leq I \}. \]
Note that (6.1) follows provided
\[ \bigcap_{z \in \mathbb{Z}} A_z \neq \emptyset. \]

The next step is to show that, for any finite set \( z_1, \ldots, z_n \in \mathbb{Z} \), the set \( \bigcap_{z_i} A_{z_i} \neq \emptyset \). The proof of this part can be taken from the proof of Theorem 2.8.1 in [Ba].

Finally, note that \( A_0 \subset A_0 \) as defined in the Proposition, since \( I \leq I \). It follows that \( A_0 \) is compact, and that the non-empty intersection of finite sets implies (6.2).

Proof. of Lemma

Let \( \rho_1(r) \) be a smooth, positive function with compact support such that
\[ |\mathbb{S}^{n-1}| \int_0^\infty r^{n-1} \rho_1(r)dr = 1 ; \int_0^\infty r^k \rho_1(r)dr := M_k ; \int_0^\infty r^{n-1} \rho_1^p dr := L(p). \]
Set also
\[ \rho_\alpha(r) = \alpha^n \rho_1(\alpha r). \]
Define
\[ v(x, t) = \begin{cases} \frac{x - \pi(t)}{t - t_0} + \pi(t) & \text{if } t_0 \leq t \leq (t_0 + t_1)/2 \\ \frac{x - \pi(t)}{t_1 - t} + \pi(t) & \text{if } (t_0 + t_1)/2 \leq t \leq t_1 \end{cases} \]
\[ \rho(x, t) = \begin{cases} \frac{1}{(t - t_0)^n} \rho_\alpha \left( \frac{|x - \pi(t)|}{t - t_0} \right) & \text{if } t_0 \leq t \leq (t_0 + t_1)/2 \\ \frac{1}{(t_1 - t)^n} \rho_\alpha \left( \frac{|x - \pi(t)|}{t_1 - t} \right) & \text{if } (t_0 + t_1)/2 \leq t \leq t_1 \end{cases}. \]
A direct calculation shows that \( \rho \) satisfies the weak form of the continuity equation:
\[ \rho_t + \nabla_x \cdot (v \rho) = 0. \]
Let us now consider the interval \([t_0, (t_0 + t_1)/2] \). The second interval \([(t_0 + t_1)/2, t_1] \) can be treated analogously. Define the lifting of \( \rho \) as
\[ f(x, t, v) = \sigma^{-n} \pi^{-2/n} \exp \left( -\frac{|v - v|}{\sigma^2} \right) \rho(x, t). \]
It follows immediately that
\[ \int_{\mathbb{R}^n} v^2 f(x, t, v)dv = \frac{\sigma n}{2} \rho(x, t) + v^2 \rho(x, t) ; \int_{\mathbb{R}^n} |f|^p dv = p^{-n/2} \pi^{-2(1-p)/n} \sigma^{n(1-p)} \rho^p(x, t). \]
Moreover:
\[
\int_{\Omega} \rho(x,t)dx = 1 ; \quad \int_{\Omega} \rho^p(x,t)dx = (t-t_0)^{n(1-p)}|S^{n-1}| \int_0^\infty r^{n-1} \rho^p_\alpha(r)dr = \alpha^{n(p-1)}(t-t_0)^{n(1-p)}|S^{n-1}| L(p)
\]
\[
\int_{\Omega} |v(x,t)|^2 \rho(x,t)dx = |\nabla \phi(t)|^2 + |S^{n-1}| \int_0^\infty r^{n+1} \rho_\alpha(r)dr = |\nabla \phi(t)|^2 + |S^{n-1}| \alpha^{-2} M_{n+1}
\]

In particular:
\[
\int_{\Omega} \int_{t_0}^{(t_0+t_1)/2} |v|^2 f = \int_{t_0}^{(t_0+t_1)/2} \left| \nabla \phi(t) \right|^2 dt + C_1 |t_1 - t_0| \alpha^{-2} + O(\sigma)
\]
and
\[
\int_{\Omega} \int_{t_0}^{(t_0+t_1)/2} \rho^p = C_3 |t_1 - t_0|^{n(1-p)+1} \alpha^{n(p-1)}
\]

\[\square\]

Proof of Lemma 5.8

We use Corollary 4.11 with \( \mu \) supported on \( \Omega \times [t_1, t_0] \) and \( \mu_{t_0} = \delta_{x_0}, \mu_{t_1} = \delta_{x_1} \), to obtain
\[
\phi(x_1, t_1) - \phi(x_0, t_0) \leq \frac{1}{2} \|\mu\|^2 + \left| \int_{\Omega} \int_{t_0}^{t_1} (\phi_t + |\nabla \phi|^2/2) \mu(t, dx)dt \right| . \tag{6.3}
\]

By Lemma 4.7, we can find such a \( \mu \) for which:
\[
\|\mu\|^2 \leq \int_{t_0}^{t_1} \left| \nabla \phi(t) \right|^2 + C_1 |t_1 - t_0| \alpha^{-2} \tag{6.4}
\]
and, for the density \( \rho = \rho_\mu \):
\[
\int_{\Omega} \int_{t_0}^{t_1} \rho^p \leq C_3 |t_1 - t_0|^{n(1-p)+1} \alpha^{n(p-1)} \tag{6.5}
\]
where \( p < 1 + 1/n \) and \( \alpha \) any positive constant. Since \( \rho \) is supported, for any \( t \), in a domain of diameter \( (t_1 - t_0)\alpha^{-1} \) it follows
\[
\int_{\Omega} \int_{t_0}^{t_1} (\phi_t + |\nabla \phi|^2/2) \rho dt dx = O \left( \frac{L(t_1 - t_0)^2}{\alpha} \right) + \int_{t_0}^{t_1} P(\phi(t), t)dt + \int_{\Omega} \int_{t_0}^{t_1} \xi |\rho| dt dx,
\]
where \( L \) is the Lipschitz norm of \( P \). By (6.5), we obtain
\[
\left| \int_{\Omega} \int_{t_0}^{t_1} \xi |\rho| dx dt \right| \leq \|\rho\| \|\xi\|_s \leq C_3^{1/p} |t_1 - t_0|^{n(1-p)+1} \alpha^{n(p-1)/p} \|\xi\|_s . \tag{6.6}
\]

Collecting (6.3) to (6.6)
\[
\phi(x_1, t_1) - \phi(x_0, t_0) \leq \frac{1}{2} \int_{t_0}^{t_1} \left| \nabla \phi(t) \right|^2 dt + C \|\xi\|_s |t_1 - t_0|^{n(1-p)+1} \alpha^{n(p-1)/p} + C_1 |t_1 - t_0| \alpha^{-2} + O \left( \frac{L(t_1 - t_0)^2}{\alpha} \right) .
\]

The choice \( \alpha = \|\xi\|_s^{-\beta} (t_1 - t_0)^{\gamma} \) where \( \gamma = \frac{(n+1)(p-1)}{2p+np(p-1)} \) and \( \beta = \frac{p}{2p+np(p-1)} \) is the optimal choice and yields the desired result. \[\square\]
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