Are rogue waves really unexpected?

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ABSTRACT

An unexpected wave is defined by Gemmrich and Garrett (2008) as a wave that is much taller than a set of neighboring waves. Their definition of “unexpected” refers to a wave that is not anticipated by a casual observer. Clearly, unexpected waves defined in this way are predictable in a statistical sense. They can occur relatively often with a small or moderate crest height, but large unexpected waves that are rogue are rare. Here, this concept is elaborated and statistically described based on a third-order nonlinear model. In particular, the conditional return period of an unexpected wave whose crest exceeds a given threshold is developed. This definition leads to greater return periods or on average less frequent occurrences of unexpected waves than those implied by the conventional return periods not conditioned on a reference threshold. Ultimately, it appears that a rogue wave that is also unexpected would have a lower occurrence frequency than that of a usual rogue wave. As specific applications, the Andrea and WACSIS rogue wave events are examined in detail. Both waves appeared without warning and their crests were nearly 2-times larger than the surrounding O(10) wave crests, and thus unexpected. The two crest heights are nearly the same as the threshold \( h_0 \sim 1 \times 10^6 \) exceeded on average once every \( 0.3 \cdot 10^6 \) waves, where \( H \) is the significant wave height. In contrast, the Andrea and WACSIS events, as both rogue and unexpected, would occur slightly less often and on average once every \( 3 \cdot 10^6 \) and \( 0.6 \cdot 10^6 \) waves respectively.

1. Introduction

A rogue wave is defined as such if the crest-to-trough height is at least 2.2 times the significant wave height \( H \) or if the crest height exceeds the threshold \( 1.25H_s \), where \( H_s = 4\sigma \) and \( \sigma \) is the standard deviation of surface elevations (Dysthe et al. 2008). Evidences given for the occurrence of such waves in nature include the Draupner and Andrea events. In particular, the Andrea wave was measured on November 9 2007 by a LASAR system mounted on the Ekofisk platform in the North Sea in a water depth of \( d = 74 \) m (Magnusson and Donelan 2013). The Draupner freak wave was measured by Statoil at a nearby platform in January 1995 (Haver 2001). In the last decade, the properties of the Draupner and Andrea waves have been extensively studied (Dysthe et al. 2008; Osborne 1995; Magnusson and Donelan 2013; Bitner-Gregersen et al. 2014; Dias et al. 2015 and references therein).

The Andrea wave occurred during a sea state with significant wave height \( H_s = 4\sigma = 9.2 \) m, mean period \( T_0 = 13.2 \) s and wavelength \( L_0 = 220 \) m. The Andrea crest height is \( h = 1.63H_s = 15 \) m and the crest-to-trough height \( H = 2.3H_s = 21.1 \) m. The sea state during which the Draupner wave occurred had a significant wave height \( H_s = 11.9 \) m, mean period \( T_0 = 13.1 \) s and wavelength \( L_0 = 250 \) m. The Draupner crest height is \( h = 18.5 \) m \((h/H_s = 1.55)\) and the associated crest-to-trough height \( H = 25.6 \) m \((H/H_s = 2.15)\) (Magnusson and Donelan 2013). Observations of such large extreme waves show that they tend to extend above the surrounding smaller waves without warning and thus unexpectedly. Further, both waves were twice as high as the immediately preceding as well as following groups of waves. In describing the unexpectedness of ocean waves, Gemmrich and Garrett (2008) define as unexpected a wave \( \alpha \)-times larger than a set of one-sided (preceding) waves or two-sided (preceding and following) waves (see Fig. 1). Note that their definition of unexpectedness refers to the time interval of apparent calm before or during which a wave is much taller than the neighboring waves. Hereafter, the term “unexpected” refers to a wave that is not anticipated by a casual observer as emphasized by Gemmrich and Garrett (2010). Clearly, unexpected waves defined in this way are predictable in a statistical sense as one can estimate the associated return period or frequency of occurrence.

Indeed, unexpected waves occur often with a small or average wave height, but they are rarely the largest waves in a record or rogue waves (Gemmrich and Garrett 2010). In this regard, Gemmrich and Garrett (2008) performed
Monte Carlo simulations of second order nonlinear seas characterized with the typical JONSWAP ocean spectrum and initial homogeneous random conditions. They estimated that a wave with height at least twice that of any of the preceding 30 waves occurs once every $10^3$ waves on average. Also unexpected crest heights are more probable than unexpected wave heights as they occur on average once every $7 \cdot 10^4$ in Gaussian seas and once every $10^4$ waves in second-order nonlinear seas (see Fig. 2 in Gemmrich and Garrett (2008)). Thus, their numerical predictions indicate that in weakly nonlinear seas unexpected waves occur frequently and more often than in Gaussian seas.

Further, Gemmrich and Garrett (2008) noted in their simulations that among the unexpected waves 2-times larger than the surrounding 30 waves, only about $q = 10 - 20\%$ were rogues. With reference to second order crest heights, this means that in a sample population of $10^6$ waves a set of 100 waves are unexpected, as they occur once every $10^4$ waves on average. However, only about 10 - 20 waves of the set have crest heights that are rogue, i.e. larger than $1.34H_e$, as the rogue threshold adopted by Gemmrich and Garrett (2008). This implies that unexpected wave crests that are rogue would occur less often, i.e. once every $10^2$ waves on average. Further, the percentage $q$ of rogue occurrences can be interpreted as the probability that the crest of an unexpected wave exceeds the threshold $1.34H_e$. Consequently, unexpected crest heights larger than $1.34H_e$ would occur rarely.

The preceding results provide the principal motivation here to consider a statistical model for describing unexpected waves and their rogueness. We will show that Gemmrich and Garrett (2008)’s definition of return period is unconditional. In particular, it is the harmonic mean of the return periods of all unexpected waves with any amplitude. Thus, unexpected waves of moderate amplitude occur relatively often. However, unexpected waves that are rogue have a lower occurrence frequency, and this is in agreement with Gemmrich and Garrett (2008)’s numerical predictions.

The remainder of the paper is structured as follows. First, we introduce a new theoretical model for the statistics of unexpected waves that accounts for both second- and third-order nonlinearities. We also study the effects of nonstationarity and stochastic dependence among successive waves. In particular, we present analytical solutions for the return period of unexpected waves and associated unconditional and conditional averages for crest and wave heights. Then, the conceptual framework is validated by way of Monte Carlo simulations and the theoretical predictions are compared to oceanic measurements. As a specific application here, we capitalize on the numerical simulations of the Andrea sea state (Bitner-Gregersen et al. 2014; Dias et al. 2015) and examine the unexpectedness of the Andrea wave in detail. Summary and conclusions follow subsequently.

2. Statistics of unexpected waves

Consider the exceedence probability distribution of wave crests characterized by third-order nonlinearities and described by (Tayfun and Fedele 2007)

$$P(x) = \Pr[h > xH_e] = \exp \left(-8x_0^2 \left[1 + \lambda x_0^2 \left(4x_0^2 - 1\right)\right]\right),$$

(1)

where $x = h/H_e$ is the crest amplitude $h$ scaled by the significant wave height $H_s = 4\sigma$ and $x_0$ follows from the quadratic equation (Tayfun 1980)

$$x = x_0 + 2\mu x_0^2.$$

Here, the wave steepness $\mu = \lambda_3/3$ relates to the skewness of surface elevations (Fedele and Tayfun 2009) and the parameter

$$\lambda = \lambda_{40} + 2\lambda_{22} + \lambda_{04}$$

(3)

is a measure of third-order nonlinearities as a function of the fourth order cumulants $\lambda_{nm}$ of the wave surface $\eta$ and its Hilbert transform $\hat{\eta}$ (Tayfun and Fedele 2007). Mori and Janssen (2006) assume the following relations between cumulants

$$\lambda_{22} = \lambda_{40}/3, \quad \lambda_{04} = \lambda_{40},$$

(4)

which, to date, have been proven to hold for second-order narrowband waves only (Tayfun and Lo 1990). Then, $\Lambda$ in Eq. (3) is approximated in terms of the excess kurtosis $\lambda_{40}$ by

$$\Lambda_{app} = 8\lambda_{40}/3,$$

(5)

which will be used in this work. Then, Eq. (1) reduces to a modified Edgeworth-Rayleigh (MER) distribution (Mori and Janssen 2006). For realistic oceanic seas the kurtosis $\lambda_{40}$ is mainly affected by bound nonlinearities (Annenkov and Shrira 2014; Fedele 2015b,a).

Consider now a time interval $T$ during which a stationary sequence of $N_o = T/T_m$ consecutive waves occur on average. We assume that neighboring waves are stochastically independent. This assumption is convenient for the theoretic development of a probabilistic model. Furthermore, Borgman (1970) argues that "... It seems reasonable to assume that a wave height is at most interdependent with the first several wave heights occurring before and after it and essentially independent with waves further back into the past or forward into the future". We will show later that this is justified as long as the sea state is broadband so that the covariance function decays sufficiently rapid to zero after few wave periods and successive wave peaks decorrelate faster. Thus, in a sample of $N_o + 1$ successive waves it is irrelevant what wave is the unexpected wave larger than the surrounding waves. Indeed, any wave in the sample could be "p-sided" unexpected, i.e. $\alpha$-times larger than the previous $m$ waves and following $N_o - m$ waves, with $m = 1, \ldots N_o/2$ and $p = N_o/m$. For instance,
Fig. 1. WACSIS measurements: the observed largest crest height is $\alpha = 2$-times larger than the crests of the one-sided (two-sided) $N_a \sim 50$ (60) waves. Wave parameters $H_s = 4.16 \text{ m}, T_m = 6.6 \text{ s}, d = 18 \text{ m}$ (Forristall et al. 2004).

Fig. 2. Unexpected crest heights in broadbanded Gaussian seas. Left panel: empirical one-sided (thin dashed line with □) and two-sided (thin solid line with +), $N_a$ even) unexpected wave statistics versus (solid line) predicted theoretical unconditional return period $N_R$ in number of waves of a wave whose crest height is $\alpha$-times larger than the surrounding $N_a$ waves for increasing values of $\alpha = 1.5, 2$ and 2.5. Confidence bands are also shown. Right panel: empirical one-sided (thin dashed line with □) and two-sided (thin solid line with +, $N_a$ even) unexpected wave statistics versus theoretical predictions (solid line) of the mean crest height $h_{\alpha, N_a}$ of a wave whose crest height is $\alpha$-times larger than surrounding $N_a$ waves for $\alpha = 1.5, 1.75, 2$ and 2.5. Sea state parameters: fully developed JONSWAP spectrum (peak enhancement factor $\gamma = 1$), mean period $T_m = 8.3 \text{ s}$, spectral bandwidth $\nu = 0.35$, Boccotti parameters $\psi^* = 0.65$, $\psi^2 = 0.20$ and simulated $\sim 10^6$ waves (see left panel inset). The theoretical predictions accounting for the stochastic independence and dependence of successive crest heights are practically the same as the sea state is broadbanded.

The last wave in the sample could be larger than the preceding (one-sided) $N_a$ waves ($m = N_a$ and $p = 1$), or the central wave could extend above the preceding and following (two-sided) $m = N_a/2$ waves ($p = 2$ and $N_a$ even) (see Fig. 1). Note that our definition of two-sided unexpectedness is different than that in Gemmrich and Garrett (2008) as they consider $N_a$ waves on each side.

Clearly, the statistics of one- and two-sided unexpected waves, or more generally the $p$-sided statistics are the same if stochastic independence of successive waves holds. On this basis, the fraction of waves $n(x; \alpha, N_a)$ that
have a dimensionless crest height \( h/H \) within the interval \((x,x+dx)\) and that is \( \alpha \)-times larger than any of the surrounding \( N_a \) waves is given by

\[
n(x;\alpha,N_a)dx = \left[1 - P\left(\frac{x}{\alpha}\right)\right]^{N_a} p(x)dx,
\]

where \( P(x) \) is the exceedance probability given in Eq. (1) and

\[
p(x) = -\frac{dP}{dx}
\]

is the pdf of \( x \). Then the probability that the crest height \( \xi \) is in \((x,x+dx)\) follows as

\[
p_h(x;\alpha,N_a)dx = \frac{n(x;\alpha,N_a)dx}{n(\alpha,N_a)}, \tag{8}
\]

where \( n(\alpha,N_a) \) is the fraction of waves whose crest height is \( \alpha \)-times larger than the surrounding \( N_a \) waves, namely

\[
n(\alpha,N_a) = \int_0^\infty n(x;\alpha,N_a)dx = \int_0^\infty \left[1 - P\left(\frac{x}{\alpha}\right)\right]^{N_a} p(x)dx.
\]

By definition, the unconditional return period \( R \) or the average time interval between two consecutive occurrences of the unexpected wave event \( \mathcal{E} \) is

\[
R(\alpha,N_a) = \frac{\tau}{N_a n(\alpha,N_a)} = \frac{N_w T_m}{N_a n(\alpha,N_a)} = \frac{T_m}{n(\alpha,N_a)}. \tag{10}
\]

Since \( T_m \) is the mean wave period, \( \mathcal{E} \) occurs on average once every \( N_R \) waves where

\[
N_R(\alpha,N_a) = \frac{1}{n(\alpha,N_a)}. \tag{11}
\]

Another statistical interpretation of the unconditional return period \( N_R \) is as follows. Consider the average number of unexpected waves \( n_j(\alpha,N_a)\Delta x \) with a crest height between \( x_{j-1} - \Delta x/2 \) and \( x_{j} + \Delta x/2 \), where \( \Delta x \ll 1 \) is small and \( x_j \) are increasing amplitudes starting from \( x_1 = 0 \), i.e. \( x_{j+1} > x_j \), for \( j = 1, \ldots \). Then,

\[
N_{R,j}(\alpha,N_a) = \frac{1}{n_j(\alpha,N_a)\Delta x}
\]

is the return period of an unexpected wave whose crest height is nearly \( x_j \). Then, Eq. (11) is approximated as

\[
N_R(\alpha,N_a) \approx \frac{1}{\sum_{j=1}^{\infty} n_j(\alpha,N_a)\Delta x} = \frac{1}{\sum_{j=1}^{\infty} N_{R,j}(\alpha,N_a)}, \tag{12}
\]

which reveals that \( N_R \) is the harmonic mean of the return periods \( N_{R,j} \) of all unexpected waves with any crest height.

The associated mean crest height of a wave \( \alpha \)-times larger than the surrounding \( N_a \) waves follows from Eq. (8) as

\[
\overline{h}_{\alpha,N_a} = H_s \int_0^\infty x p_h(x;\alpha,N_a)dx. \tag{13}
\]

For comparison purposes, we also consider the standard statistics \( \overline{h}_{\text{max},n} \), \( h_n \) and \( h_{1/n} \) for crest heights (Tayfun and Fedele (2007)). In particular, \( \overline{h}_{\text{max},n} \) is the mean maximum crest height of a sample of \( n \) waves

\[
\overline{h}_{\text{max},n} = H_s \int_0^\infty \{1 - [1 - P(x)]^n\} dx, \tag{14}
\]

which admits Gumbel-type asymptotic approximations (Tayfun and Fedele (2007); Fedele (2015a)). Further, \( h_n \) is the threshold exceeded by the \( 1/n \) fraction of largest crest heights and it satisfies

\[
P(h_n/H_s) = \frac{1}{n}, \tag{15}
\]

where \( P(x) \) is the unconditional nonlinear probability of exceedance for crest heights given in Eq. (1). The statistics \( h_{1/n} \) is the conditional mean \( h \) for \( h > h_{1/n} \), namely the average of the \( 1/n \) fraction of largest crest heights

\[
h_{1/n} = h_n + n H_s \int_{h_n}^\infty P(x)dx. \tag{16}
\]

One can show that \( \overline{h}_{\text{max},n} \) is always smaller than \( h_{1/n} \) and they tend to be the same as \( n \) increases (Tayfun and Fedele 2007).

We also consider the standard conditional return period \( N_h(\xi) \) (in number of waves) of a wave whose crest exceeds the threshold \( h = \xi H_s \), namely

\[
N_h(\xi) = \frac{1}{Pr[h > \xi H_s]} = \frac{1}{P(\xi)}. \tag{17}
\]

where the exceedance probability \( P(\xi) \) is that in Eq. (1). From Eq. (15), the threshold \( h_n \) exceeded with probability \( 1/n \) implies that \( N_h(h_n/H_s) = n \), i.e. on average \( h_n \) is exceeded once every \( n \) waves.

Similar statistics for the crest-to-trough height \( y = H/H_s \) of unexpected waves follow by replacing the crest exceedance probability \( P \) in Eq. (1) with the generalized Boccotti distribution (Alkhalidi and Tayfun 2013)

\[
P_H(y) = Pr[H > y H_s] = c_0 \exp \left( -\frac{4y^2}{1+y^2} \right) \left[ 1 + \frac{\Lambda y^2}{1+y^2} \left( \frac{y^2}{1+y^2} - \frac{1}{2} \right) \right], \tag{18}
\]

where

\[
c_0 = \frac{1}{\sqrt{2} \psi^* (1 + \psi^*)},
\]

and \( \psi^* = \psi(\tau^*) \) is the absolute value of the first minimum of the normalized covariance function \( \psi(\tau) = \eta(\tau)\eta(\tau + \tau)/\sigma^2 \) of the zero-mean random wave process \( \eta(t) \), which is attained at \( \tau = \tau^* \) and \( \psi^* \) the corresponding second derivative (Boccotti 2000).

The corresponding linear statistics of unexpected wave crests follow by setting \( \mu = 0 \) and \( \Lambda = 0 \) in Eq. (1), or
\[ \Lambda = 0 \] in Eq. (18) for wave heights. These will hereafter be differentiated with the superscript \( L \). In the following, we will not dwell that much on unexpected wave heights, but our main focus will be the statistics of unexpected crests in typical oceanic sea states.

Finally, we point out that our present theory of unexpected waves can be generalized to space-time extremes drawing on Fedele (2012), but this is beyond the scope of this paper.

\[ N_R \]

\[ \xi = 1.2, \ N_R(\xi) \]

\[ 0, N_R \]

\[ \alpha = 1.5 \]

\[ \beta = 0 \]

\[ \gamma = 0 \]

\[ \delta = 0 \]

\[ \epsilon = 0 \]

\[ \zeta = 0 \]

\[ \eta = 0 \]

\[ \theta = 0 \]

\[ \varphi = 0 \]

\[ \chi = 0 \]

\[ \psi = 0 \]

\[ \Omega = 0 \]

\[ \Pi = 0 \]

\[ \Sigma = 0 \]

\[ \Upsilon = 0 \]

\[ \Phi = 0 \]

\[ \Psi = 0 \]

\[ \Theta = 0 \]

\[ \tau = 0 \]

\[ \zeta_R = 0 \]

\[ \zeta_R(\xi) \]

\[ 0, N_R \]

\[ \alpha = 1.5 \]

**Fig. 3.** Conditional return period of large unexpected waves in Gaussian seas: (square) empirical one-sided unexpected wave statistics versus (solid lines) predicted theoretical conditional return periods \( N_R(\xi) \) in number of waves of unexpected waves whose crest height is greater than \( \xi H_0 \) and \( \alpha = 1.5 \)-times larger than the surrounding \( N_R \) waves for \( \xi = 1, 0.1, 0 \). Note that \( N_R(\xi = 0) \) is the unconditional return period \( N_R \). Sea state parameters: fully developed JONSWAP spectrum (peak enhancement factor \( \gamma = 1 \)), mean period \( T_m = 8.3 \) s, spectral bandwidth \( v = 0.35 \), Boccotti parameters \( \psi^* = 0.65 \), \( \psi^*_0 = 0.20 \) and simulated \( \sim 10^6 \) waves. The predictions accounting for the stochastic independence and dependence of successive crest heights are practically the same as the sea state is broadbanded.

### a. Stochastic dependence of successive waves

The statistics of unexpected waves presented so far does not take into account the stochastic dependence of neighboring waves or wave groupness. Clearly, for large crest heights, as argued by Borgman (1970), one expects that only few neighboring crests are more or less correlated (Watson 1954). To quantify this, we draw on Fedele (2005) and model a stationary sequence of wave crests \( \{x_j = h_j\}_{j=1,N_x} \) as a one-step memory Markov chain, where each crest height \( x_j \) is only stochastically dependent on the preceding crest height \( x_{j-1} \), that is

\[ p(x_j|x_{j-1},x_{j-2},\ldots,x_2,x_1) = p(x_j|x_{j-1}). \]

Since the sequence is stationary, the conditional pdf \( p(x_j|x_{j-1}) \) is the same for any \( j \), say \( p(x_2|x_1) = p(x_1,x_2)/p(x_1) \), where the crest \( x_1 \) precedes \( x_2 \), and \( p(x_1,x_2) \) and \( p(x_1) \) are the associated joint and marginal pdfs.

On these assumptions, following Fedele (2005) the fraction of waves \( n(x;\alpha,N_x) \) that have a dimensionless crest height within the interval \( (x, x + dx) \) and that is \( \alpha \)-times larger than any of the surrounding \( N_x \) waves is given by

\[ n(x;\alpha,N_x)dx = \frac{\Pr \left( x_2 < \frac{x}{\alpha} | x_1 < \frac{x}{\alpha} \right)^{N_x-1}}{\Pr \left( x_2 < \frac{x}{\alpha} | x_1 = x \right)} p(x)dx, \]

where

\[ \Pr \left( x_2 < \frac{x}{\alpha} | x_1 = x \right) = \frac{\int_0^{\frac{x}{\alpha}} p(x_1,x_2)dx_1 dx_2}{\int_0^{x} p(x_1)dx_1}, \]

and

\[ \Pr \left( x_2 < \frac{x}{\alpha} | x_1 < \frac{x}{\alpha} \right) = \frac{\int_0^{\frac{x}{\alpha}} p(x_1,x_2)dx_1 dx_2}{p(x)}. \]

Then, the return period \( R(\alpha,N_x) \) of unexpected wave crests follows from Eqs. (9) and (11). Clearly, if successive waves were stochastically independent, \( p(x_1,x_2) = p(x_1)p(x_2) \) and Eq. (19) reduces to Eq. (6) for the stationary case.

The theoretical probability structure of two consecutive wave crests is known for Gaussian processes and it is given by the bivariate Rayleigh distribution (Fedele 2005)

\[ p_R(x_1,x_2) = 256 \frac{x_1 x_2}{1 - k^2} \exp \left[ -8 \frac{x_1^2 + x_2^2}{1 - k^2} \right] I_0 \left( 16k \frac{x_1 x_2}{1 - k^2} \right), \]

where \( I_0(y) \) is the modified Bessel function and the parameter \( k = \psi (\tau_2^* = 2\psi_2^* \) with \( \tau_2^* \) the abscissa of the second absolute maximum of the normalized covariance function \( \psi(\tau) \) of the zero-mean random wave process (Fedele 2005). Further, the marginal pdf

\[ p_R(x_1) = \int_0^\infty p_W(x_1,x_2)dx_2 = 16x_1 \exp \left( -8x_1^2 \right) \]

is the univariate Rayleigh distribution. As \( \psi_2^* \) tends to zero, successive crests become stochastically independent and the sea state tends to be broadbanded. Thus, we expect that stochastic dependence of waves is dominant in very narrowband sea states, where \( \psi_2^* \to 1 \). In particular, our numerical simulations discussed later in section 4 suggest that the dependence of consecutive crests in Gaussian seas is dominant when \( \psi_2^* > 0.7 \). This condition corresponds to unrealistic oceanic sea states characterized by a Jonswap spectrum with a peak enhancement factor \( \gamma > 100 \) and very narrowbanded as the spectral bandwidth \( v < 0.1 \). For typical oceanic seas, \( v \sim 0.3 - 0.5 \) and \( \psi_2^* \sim 0.2 - 0.5 \).
and successive waves can be assumed as stochastically independent.

Note that, the joint pdf of consecutive Gaussian wave crests in Eq. (20) can be generalized to account for second-order bound nonlinearities following Fedele and Tayfun (2009), but this is beyond the scope of this work. Since second-order bound harmonics are phase-locked to the Fourier components of the linear free surface, we expect the classical Tayfun’s (1980) enhancement of successive linear crest amplitudes, but their dependence should be unaffected by second-order nonlinearities.

b. Nonstationarity

The statistics of unexpected waves formulated so far is valid for stationary sea states. In nonstationary seas, as those during storms, our present theory can be formalized for stationary sea states. In nonstationary seas, as those during storms, our present theory can be formalized

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Fig. 4. Unexpected wave heights in Gaussian seas: (Left panel) predicted theoretical unconditional return period \( N_R \) in number of waves (solid line) versus empirical one-sided (+) and two-sided (□) statistics as a function of the number \( N_b \) of surrounding waves for \( \alpha = 1.5 \); (center panel) predicted mean unexpected wave height \( \bar{H}_{1.5,N_b} \) versus observations as a function of the return period \( N_R \). For comparison purposes, predicted mean wave height \( \bar{H}_{\text{max},N_b} \), conditional mean \( H_{1/N_b} \) and (right panel) threshold \( H_{\text{95}} \) versus observations (circles) are also shown. Sea state parameters: fully developed JONSWAP spectrum (peak enhancement factor \( \gamma = 1 \)), mean period \( T_m = 8.5 \) s, spectral bandwidth \( \nu = 0.35 \), Boccotti parameter \( \psi^* = 0.65 \) and simulated \( \sim 10^6 \) waves.

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The statistics of unexpected waves formulated so far is valid for stationary sea states. In nonstationary seas, as those during storms, our present theory can be formalized as follows. From Eq. (5), the pdf of an unexpected wave crest height \( h \) generalizes to

\[
p_h(x; \alpha, N_b)_{NS} = \int \cdots \int p_h(x; \alpha, N_b | b_1, \ldots, b_M) \cdot p(b_1, \ldots, b_M) \, db_1 \cdots db_M,
\]

where \( \{b_j\} \) are \( M \) time-varying wave parameters, e.g. \( \sigma, \mu, \lambda_{40}, \lambda_{22} \) and \( \lambda_{04} \), the conditional pdf \( p_h(x; \alpha, N_b | b_1, \ldots, b_M) \) is the stationary pdf in Eq. (8) for given values of \( b_j \) and \( p(b_1, \ldots, b_M) \) is the joint pdf of the parameters, which encodes their time variability. Eq. (21) can be interpreted as the average value of \( p_h(x; \alpha, N_b | b_1, \ldots, b_M) \) with respect to the random variables \( b_j \), that is

\[
p_h(x; \alpha, N_b)_{NS} = p_h(x; \alpha, N_b | \bar{b})^b,
\]

where the vector \( \bar{b} = [b_1, \ldots, b_M] \) and the labeled overbar denotes statistical average with respect to \( b \) only. Taylor-expanding around the mean \( \bar{b} = [\bar{b}_1, \ldots, \bar{b}_M] \), up to second order, yields

\[
p_h(x; \alpha, N_b)_{NS} \approx p_h(x; \alpha, N_b | \bar{b})^b + \sum_j g_j \cdot (\bar{b} - \bar{b})^b - \frac{\partial^2 p_h(x; \alpha, N_b | b_1, \ldots, b_M)}{\partial b_j \partial b_k} |_{b=\bar{b}} \cdot (\bar{b} - \bar{b})^b,
\]

where the superscript \( T \) denotes matrix transposition, the vector \( \mathbf{g} \) has entries

\[
|\mathbf{g}_j| = \left. \frac{\partial p_h(x; \alpha, N_b | b_1, \ldots, b_M)}{\partial b_j} \right|_{b=\bar{b}}
\]

and the Hessian matrix

\[
[H(\bar{b})]_{rs} = \left. \frac{\partial^2 p_h(x; \alpha, N_b | b_1, \ldots, b_M)}{\partial b_r \partial b_s} \right|_{b=\bar{b}}.
\]

Taking the averages in Eq. (22) yields

\[
p_h(x; \alpha, N_b)_{NS} \approx \frac{\left. \sum_r \left[ \frac{\partial^2 p_h(x; \alpha, N_b | b_1, \ldots, b_M)}{\partial b_r \partial b_s} \right]_{b=\bar{b}} \right] B_{rs}}{B_{rs}},
\]

where \( B_{rs} = \left. \frac{\partial^2 p_h(x; \alpha, N_b | b_1, \ldots, b_M)}{\partial b_r \partial b_s} \right|_{b=\bar{b}} \) and \( B_{rs} = \left. \frac{\partial^2 p_h(x; \alpha, N_b | b_1, \ldots, b_M)}{\partial b_r \partial b_s} \right|_{b=\bar{b}} \) are the covariances of the random variables \( b_r \) and \( b_s \).
Return period $N$ is the sum of i) the pdf in Eq. (8) evaluated using the mean parameters $\bar{b}$ and ii) additional terms that account for the spreading of the parameters from their mean. A similar formula can be obtained for $n(\alpha, N_R)$ in Eq. (9). The statistical moments of $p_b$ can then be obtained by integrating Eq. (22) and the nonstationary return period $N_R^{NS}$ follows from Eq. (11).

In our applications (see section 4), time wave measurements at a point are subdivided in a sequence of optimal 30-min intervals during which the sea state can be assumed as stationary. We observed that shorter time intervals lead to unstable estimates of higher order moments, whereas longer intervals violate the stationarity assumption. The variability of the standard deviation $\sigma$ was taken into account by normalizing the surface height measurements in each 30-min interval by the respective observed $\sigma$. In our data analysis, wave parameters are estimated as the average values over the available time record. Then, the statistics of unexpected waves can be based on Eq. (23), where the $B_{rs}$ terms accounting for non-stationarity are neglected.

where

$$B_{rs} = (b_r - \bar{b}_r) (b_s - \bar{b}_s)^\beta$$

are correlation coefficients, in particular $B_{rr} = \sigma^2_r$ is the variance of $b_r$. These can be easily estimated from the nonstationary time series. Thus, $p_b$ is the sum of i) the pdf in Eq. (8) evaluated using the mean parameters $\bar{b}$ and ii) additional terms that account for the spreading of the parameters from their mean. A similar formula can be obtained for $n(\alpha, N_R)$ in Eq. (9). The statistical moments of $p_b$ can then be obtained by integrating Eq. (22) and the nonstationary return period $N_R^{NS}$ follows from Eq. (11).

In our applications (see section 4), time wave measurements at a point are subdivided in a sequence of optimal 30-min intervals during which the sea state can be assumed as stationary. We observed that shorter time intervals lead to unstable estimates of higher order moments, whereas longer intervals violate the stationarity assumption. The variability of the standard deviation $\sigma$ was taken into account by normalizing the surface height measurements in each 30-min interval by the respective observed $\sigma$. In our data analysis, wave parameters are estimated as the average values over the available time record. Then, the statistics of unexpected waves can be based on Eq. (23), where the $B_{rs}$ terms accounting for non-stationarity are neglected.

For example, observations indicate that the Andrea rogue wave appeared without warning suddenly, attained a crest height $h_{obs} = 1.62H_s$, and it was as nearly two-times larger than the surrounding O(30) waves (Mannusson and Donelan 2013). Thus, the Andrea wave is unexpected in accordance with the definition of Gemmrich and Garrett (2008). However, as it will be discussed later in section 6, an application of our present theory using Eq. (11) predicts that a wave with a crest height at least twice as that of any of the surrounding $N_a = 30$ waves occurs on average once every $N_R \sim 10^4$ waves. This is clearly observed in the left panel of Fig. 10. Further, the right panel of the same Figure shows that the actual Andrea crest height is nearly the same as the threshold $h_{0.3 \cdot 10^6} \sim 1.6H_s$ exceeded by the $1/(0.3 \cdot 10^6)$ fraction of largest crests. Eq. (17) also suggests that the Andrea wave is likely a rare event as the crest threshold $1.6H_s$ is exceeded once every $N_h = 0.3 \cdot 10^6$ waves on average. In contrast, our present theory predicts that the Andrea event would occur relatively often as an unexpected wave, i.e. on average once every $N_R \sim 10^4$ waves.

The difference in occurrence rates is explained by first noting that the return period $N_R$ is the average time interval between two consecutive waves whose crest height $h$, of any possible amplitude, is $\alpha$-times larger than the surrounding $N_a$ wave crests. In other words, Eq. (12) reveals that $N_R$ is the harmonic mean of the return periods of all unexpected waves of any crest amplitude, and it is smaller than the return period of large (rare) unexpected waves. Thus, unexpected waves as defined by Gemmrich et al. (2008).
rich and Garrett (2008) occur relatively often with small or moderate amplitude. However, unexpected waves that are rogue are rare, in agreement with their numerical predictions (see also Gemmrich and Garrett (2010)).

To quantify the difference in occurrence frequencies of small and large unexpected waves, it is natural to define the conditional return period \( N_R(\xi, \alpha, N_a) \) of an unexpected wave whose crest height \( h \) exceeds the threshold \( \xi H_s \) and it is \( \alpha \)-times larger than the surrounding \( N_a \) wave crests. This is given by

\[
N_R(\xi; \alpha, N_a) = \frac{1}{\int_{\xi}^{\infty} n(x; \alpha, N_a) dx} = \frac{N_R(\alpha, N_a)}{P_h(\xi; \alpha, N_a)}, \tag{24}
\]

where

\[
P_h(x; \alpha, N_a) = \int_{x}^{\infty} p_h(s; \alpha, N_a) ds \tag{25}
\]

is the exceedance probability of the unexpected crest height \( h \) from Eq. (8). Clearly, for given \( \alpha \) and \( N_a \), the conditional return period \( N_R(\xi) \) is always greater than the unconditional \( N_R \) for any \( \xi > 0 \), and they are the same if \( \xi = 0 \). The left panel of Fig. 10 shows that the Andrea rogue wave as an unexpected wave that exceeds \( \xi H_s = 1.6 H_s \) would occur rarely, i.e. on average once every \( N_R(\xi = 1.6) \approx 6 \times 10^8 \). Instead, unexpected waves of any amplitude occur more often, and on average once every \( N_R \approx 10^4 \).

Clearly, the Andrea wave is both rogue and unexpected, i.e. its crest is larger than the crests of surrounding waves and it exceeds the threshold \( 1.25 H_s \) (Dysthe et al., 2008). What is the occurrence frequency of such a bivariate event in comparison to being only rogue as a univariate event?

From Eq. (24), the following inequality holds

\[
N_R(\xi) \geq \frac{1}{\int_{\xi}^{\infty} p(x) dx} = \frac{1}{P(\xi)} = N_h(\xi), \tag{26}
\]

where we have used \( n(x; \alpha, N_a) \leq p(x) \) from Eq. (6). Here, \( N_h(\xi) \) is defined in Eq. (17) as the standard unconditional return period (in number of waves) of a wave whose crest exceeds the threshold \( h = \xi H_s \). Thus, a wave whose crest is both larger than \( \xi H_s \) and unexpected (as being larger than the surrounding waves) has a lower occurrence frequency than a wave whose crest is just larger than the same threshold.

The preceding results imply that a rogue wave that is also unexpected has a lower occurrence frequency than just being rogue. For example, for the Andrea sea state the return period of a crest larger than \( h_0 = 1.6 H_s \) is \( N_h(h_0) = 0.3 \times 10^6 \). This is smaller than the return period \( N_R \) of an unexpected wave exceeding the same threshold, i.e. \( N_R(\xi = 1.6) \sim 6 \times 10^8 \) (see left panel of Fig. 10). Similar conclusions hold for the WACSIS rogue wave (see section 3).

4. Verification and comparisons

a. Monte Carlo simulations of Gaussian seas

Drawing on Gemmrich and Garrett (2008), we performed Monte Carlo simulations of a Gaussian sea described by the average JONSWAP spectrum with peak enhancement factor \( \gamma = 1 \). The sea state is broadbanded with mean period \( T_m = 8.3 \) s, peak period \( T_p = 10 \) s, spectral bandwidth \( v \sim 0.35 \) and Boccotti parameters \( \psi^2 = 0.65 \), \( \psi^2_z = 0.3 \) (see covariance function in the panel inset of Fig. 2). A long time series of wave surface displacements was randomly generated containing a total of \( \sim 10^6 \) waves, from which unexpected waves were sampled. As the sea state is broadbanded, our theoretical predictions can be based on Eqs. (9) and (11) assuming the stochastic independence of successive crest heights.

The left panel of Fig. 2 shows the empirical return period \( N_R = R/T_m \) in number of waves of both one-sided (thin dashed line) and two-sided (thin solid line) unexpected wave crests as a function of the surrounding \( N_a \) waves for different values of \( \alpha \) (\( N_a \) even for the two-sided statistics). The two statistics are roughly the same with two-sided unexpected waves slightly less frequent than the one-sided waves. Note that for the two-sided unexpectedness Gemmrich and Garrett (2008) consider \( N_a \) waves on each side, thus their two-sided return period is larger than ours. Shown in the right panel of Fig. 2 are also the empirical statistics of mean crest heights in comparison to our theoretical predictions for stochastically independent waves. In particular, we note that the mean crest height of two-sided unexpected waves is slightly smaller than that of one-sided waves, especially as \( \alpha \) increases.

Further, in Fig. 3 there are shown the predicted conditional return periods \( N_R(\xi) \) (solid lines) of an unexpected wave whose crest height is greater than \( \xi H_s \) for \( \xi = 0, 1.0 \) and 1.2 (\( \alpha = 1.5 \)). Note that \( N_R(\xi = 0) \) is the unconditional return period \( N_R \). We find a fair agreement with the empirical one-sided unexpected wave statistics (squares). For \( \alpha = 2 \) and \( N_a = 30 \) our predicted return period is \( N_R \sim 6 \times 10^4 \) and in fair agreement with the linear predictions (\( \sim 7 \times 10^4 \)) by Gemmrich and Garrett (2008) as shown in their Fig. 2. As regard to unexpected crest-to-trough heights, our theoretical model fairly predicts the empirical wave height statistics from simulations as clearly seen in Fig. 4.

In the above comparisons, the fair agreement with our theoretical predictions indicates that the stochastic independence of waves holds approximately as the sea state is broadbanded. However, in very narrowband seas the stochastic dependence of neighboring waves cannot be neglected. Indeed, consider a linear sea state characterized by a Gaussian spectrum with spectral bandwidth \( v = 0.1 \). This is similar to an unrealistic Jonsswap spectrum with peak enhancement factor \( \gamma \sim 300 \). From the panel inset of Fig. 5 the Boccotti parameters are \( \psi^2 = 0.94 \) and
ψ^*_2 = 0.81 indicating a strong correlation between consecutive waves. Indeed, from the same figure the empirical one-sided (square) unexpected wave statistics tends to agree with our predicted theoretical return period \( N_R \) for dependent waves (thick solid line) computed using Eqs. (19) and (11). Instead, our predictions for independent waves (thin solid line) are less conservative, where we use Eqs. (9) and (11).

**b. Monte Carlo simulations of second-order random seas**

Drawing on Tayfun and Fedele (2007), we performed Monte Carlo simulations of unidirectional second-order broadband random seas in deep water described by the same average JONSWAP spectrum introduced in the previous section for simulating Gaussian seas. The associated Tayfun (1980) steepness \( \mu = \lambda_3 / 3 \approx 0.06 \), where \( \lambda_3 \) is the skewness of surface elevations (Fedele and Tayfun 2009). Our theoretical predictions are based on Eqs. (9) and (11) and assume the stochastic independence of successive crest heights as the sea state is broadband.

In Fig. 6 it is shown the comparison between the empirical return period \( N_R = R / T_m \) in number of waves of one-sided (squares) unexpected wave crests and theoretical predictions from our model as a function of the surrounding \( N_a \) waves for different values of \( \alpha \). For \( \alpha = 2 \) and \( N_a = 30 \) our predicted second-order return period \( N_R \sim 2 \cdot 10^5 \) is shorter than the linear counterpart (\( \sim 6 \cdot 10^5 \)) for Gaussian seas (see Fig. 2) as nonlinearities enhance crest heights (Tayfun and Fedele 2007; Fedele and Tayfun 2009). Further, our second-order predictions fairly agree with those by Gemmrich and Garrett (2008) in their Fig. 2.

![Diagram](https://via.placeholder.com/150)

**Fig. 7.** Left panel: WACSIS, predicted theoretical nonlinear unconditional return period \( N_R \) in number of waves (solid line) of a wave whose crest height is \( \alpha \)-times larger than the surrounding \( N_a \) waves, linear predictions (dash lines) and empirical one-sided observed statistics (□) for \( \alpha = 1.5 \) and 2. Confidence bands are also shown. Right panel: same for TERN measurements. Statistical parameters are taken from Tayfun (2006); Tayfun and Fedele (2007).

![Diagram](https://via.placeholder.com/150)

**Fig. 8.** WACSIS unexpected wave crest heights: predicted theoretical nonlinear (solid line) and linear (dash line) mean heights \( \bar{h} \) as a function of the number \( N_a \) of surrounding waves versus empirical one-sided statistics (squares) for \( \alpha = 1.5 \). Horizontal line denotes the observed maximum crest height 1.62\( H_s \). Wave parameters \( H_s = 4.16 \text{ m}, T_m = 6.6 \text{ s}, \) depth \( d = 18 \text{ m} \) (Forristall et al. 2004). Average wave parameters are taken from Tayfun (2006); Tayfun and Fedele (2007), in particular skewness \( \lambda_3 \sim 0.23 \) and excess kurtosis \( \lambda_4 \sim 0.11 \).
For example, they predict a slightly shorter nonlinear period $N_R \sim 10^4$ for $\alpha = 2$ and $N_a = 30$. This is because their second-order correction for crest heights is based on the narrowband assumption of the sea state. This yields a slightly overestimation of crest heights shortening $N_R$. In contrast, our simulated sea states are based on the exact second-order solution for unidirectional broadband waves in deep water (Tayfun 1980).

c. Oceanic observations

We will analyze two data sets. The first comprises 9 h of measurements gathered during a severe storm in January, 1993 with a Marex radar from the Tern platform located in the northern North Sea in a water depth of $d = 167$ m. We refer to Forristall (2000) for further details on the data set, hereafter referred to as TERN. The second data set is from the Wave Crest Sensor Intercomparison Study (WACSIS) (Forristall et al. 2004). It consists of 5 h of measurements gathered in January, 1998 with a Baylor wave staff from Meetpost Noordwijk in the southern North Sea (average water depth $d = 18$ m). Tayfun (2006) and Tayfun and Fedele (2007) elaborated both data sets and provided accurate estimates of statistical parameters, especially skewness and fourth-order cumulants which will be used in this work. The data analysis indicates that the statistics of unexpected waves can be based on Eq. (23), where the $B_{rs}$ terms accounting for non-stationarity are neglected. Further, successive waves can be assumed as stochastically independent as the both sea states are broadbanded as indicated by their estimated covariance functions (see panel insets in Fig. 7).

As regard to WACSIS measurements, the left panel of Fig. 7 compares the theoretical nonlinear return period $N_R$ (solid line) of unexpected wave crests $\alpha$-times larger than the surrounding $N_a$ waves, the respective linear predictions (dashed line) and the WACSIS empirical one-sided statistics for $\alpha = 1.5, 2$ (dashed line with □). The right panel of the same figure shows similar comparisons for TERN. The observed occurrence rates are close to the theoretical predictions, indicating that the assumption of stochastic independence of waves holds approximately. It is noticed that nonlinearities tend to reduce the return period of unexpected waves and increase their mean crest amplitudes. In particular, in the left panel of Fig. 8 we compare our predicted nonlinear (solid line) and linear (dash line) mean crest heights $\bar{h}(\alpha, N_a)$ and $h^{(L)}(\alpha, N_a)$ versus the WACSIS empirical one-sided statistics (□) for $\alpha = 1.5$. Clearly, our linear predictions underestimate the observed crest amplitudes, as expected. Indeed, it is well established that nonlinearities must be accounted for to obtain reliable statistics of unexpected waves (Tayfun 1980, Forristall 2000, Tayfun and Fedele 2007, Fedele and Tayfun 2009, Gemmrich and Garrett 2011). Similar trend is also observed for the WACSIS rogue wave as evident from the center panel of Fig. 9. Here, there are shown our nonlinear predicted mean crest height $\bar{h}_{\text{max},N_R}$, conditional mean $h_{1/N_R}$ and mean unexpected crest height $\bar{h}_{\alpha=2,N_a}$ versus their linear counterparts. The right panel of the same figure depicts the nonlinear threshold $h_{N_a}$ in comparison to its linear counterpart. The nonlinear and linear predictions for the Andrea rogue wave are also shown in Fig. 10.

We observe that the empirical statistics tend to deviate from the theoretical predictions for large values of $\alpha$ and $N_a$. In particular, for both TERN and WACSIS we could not produce statistically stable estimates of extreme values for $N_a > 10$ when $\alpha > 1.5$ due to the limited number of waves in the time series ($O(10^3)$ waves in comparison to the $10^6$ waves of the simulated Gaussian seas). Nevertheless, the agreement between our present theory and observations is satisfactory and it also provides evidence that successive waves in the samples are approximately stochastically independent.

5. How rogue are unexpected waves?

WACSIS observations indicate that the actual largest crest $h_{\text{obs}}$ is $1.62H_s$. Fig. 1 shows that the WACSIS rogue wave is also unexpected as it is $\alpha = 2$-times larger than the surrounding $N_a \sim 50$ waves. According to our statistical model such unexpected wave would occur often and on average once every $N_R = 4 \cdot 10^4$ waves, as seen in the left panel of Fig. 9. Here, we report the theoretical predictions of the unconditional nonlinear return period $N_R$ as a function of $N_a$ using Eqs. (9) and (11)). Further, from the center panel of Fig. 9 it is seen that the associated average nonlinear unexpected crest height $\bar{h}_{\alpha=2,N_a=50}$ is about $1.35H_s$ and smaller than the conditional mean $h_{1/N_a} \sim 1.5H_s$, which is slightly larger than the mean maximum crest height $\bar{h}_{\text{max},N_R} = 1.48H_s$ of $N_R = 4 \cdot 10^4$ waves. Note that these average values underestimate the actual maximum crest amplitude $h_{\text{obs}} \sim 1.62H_s$ observed. In contrast, the right panel of Fig. 9 shows that $h_{\text{obs}}$ is nearly the same as the threshold $h_{3,3.6} = 1.6H_s$ exceeded on average once every $N_h = 0.3 \cdot 10^6$ waves (see Eq. (17)).

We have seen that a correct statistical interpretation of the WACSIS rogue wave as an unexpected event requires considering the conditional return period $N_R(\xi)$ of an unexpected wave whose crest height is larger than $\xi H_s$ (see Eq. (24)). In particular, the left panel of Fig. 9 depicts plots of $N_R(\xi)$ as a function of $N_a$ for increasing values of $\xi = 1, 1.2, 1.4, 1.55$ and 1.6 ($\alpha = 2$). For $\xi = 1.6H_s$, we find that an unexpected wave exceeding this threshold and standing above $N_h = 50$ waves would occur rarely and once every $N_R(\xi = 1.6) \sim 0.6 \cdot 10^6$, in contrast to the smaller unconditional value $N_R \sim 4 \cdot 10^4$.

In summary, the WACSIS wave crest as both unexpected and rogue, i.e. two-times larger than $N_a = 50$ surrounding waves and exceeding the $1.6H_s$, would occur once every $N_R = 0.6 \cdot 10^6$ waves on average. In contrast,
FIG. 9. WACSIS rogue wave: (Left panel) predicted nonlinear theoretical return periods $N_R(\xi)$, in number of waves, of unexpected crest heights greater than $\xi H$, and $\alpha = 2$-times larger than the surrounding $N_a$ waves for $\xi = 0, 1.0, 1.2, 1.4, 1.55$ and 1.6. Dashed vertical line denotes return period values at $N_a = 50$. (Center panel) predicted nonlinear mean crest height $h_{\text{max},N_a}$, conditional mean $h_{1/N_a}$ and average unexpected crest height $h_{\text{max},2N_a}$ versus their linear counterparts as a function of number of waves $N_R$. Empirical conditional mean $h_{1/N_a}$ is also shown (circles). (Right panel) predicted (solid line) and empirical (circles) nonlinear threshold $h_{\text{max}}$ versus its linear counterpart as a function of $N_R$. Dashed vertical lines denote values at $N_R = 4 \cdot 10^5, 0.3 \cdot 10^6$ and $0.6 \cdot 10^6$. The horizontal line denotes the observed maximum crest height 1.62$H_r$. Average wave parameters are taken from [Tayfun, 2006; Tayfun and Fedele, 2007], in particular skewness $\lambda_3 \sim 0.23$ and excess kurtosis $\lambda_{40} \sim 0.11$.

FIG. 10. Andrea rogue wave: (Left panel) predicted nonlinear theoretical return periods $N_R(\xi)$, in number of waves, of unexpected crest heights greater than $\xi H$, and $\alpha = 2$-times larger than the surrounding $N_a$ waves for $\xi = 0, 1.0, 1.2, 1.4, 1.55$ and 1.6. Dashed vertical line denotes return period values at $N_a = 30$. (Center panel) predicted nonlinear mean crest height $h_{\text{max},N_a}$, conditional mean $h_{1/N_a}$ and average unexpected crest height $h_{\text{max},2N_a}$ versus their linear counterparts as a function of number of waves $N_R$. Empirical conditional mean $h_{1/N_a}$ is also shown (circles). (Right panel) predicted (solid line) and empirical (circles) nonlinear threshold $h_{\text{max}}$ versus its linear counterpart as a function of $N_R$. Dashed vertical lines denote values at $N_R = 2 \cdot 10^5, 0.3 \cdot 10^6$ and $3 \cdot 10^6$. The horizontal line denotes the observed maximum crest height 1.63$H_r$. Wave parameters $H_t = 9.2$ m, $T_m = 13.2$ s, depth $d = 70$ m [Magnusson and Donelan, 2013], skewness $\lambda_3 = 0.15$ and excess kurtosis $\lambda_{40} = 0.1$ [Dias et al., 2015].

the WACSIS wave as a rogue event has a crest height that is nearly the same as the threshold $h_{0.3 \cdot 10^6} = 1.6H_r$ exceeded on average once every $N_h = 0.3 \cdot 10^6$ waves. Thus, the WACSIS rogue wave has a slightly greater occurrence
frequency than being both rogue and unexpected since $N_R < N_{tr} = 0.6 \cdot 10^6$. This implies that the threshold $h_{tr}$ exceeded by the $1/N_{tr}$ fraction of the largest crests is larger than $1.6H_s$ and nearly the same as $1.65H_s$.

6. The Andrea rogue wave and its unexpectedness

As a specific application of the present theoretical framework, the unexpected wave statistics of the 2007 Andrea rogue wave event is examined. The actual largest crest height $h_{obs}$ is $1.63H_s$ and nearly two-times larger than the surrounding $O(30)$ waves (see Fig. 12 in Magnusson and Donelan (2013)). For the hindcast Andrea sea state, the left panel of Fig. 10 shows the unconditional and conditional nonlinear return periods $N_R$ and $N_R(\xi)$ as a function of $N_o$. In particular, according to our statistical model, the theoretical predictions indicate that a wave with a crest height at least twice that of any of the surrounding $N_o = 30$ waves occurs on average once every $N_R = 2 \cdot 10^4$ waves irrespective of its crest amplitude. In contrast, an unexpected wave whose crest height exceeds the threshold $1.6H_s$ occurs less often since our predicted conditional return period $N_R(\xi = 1.6) \sim 3 \cdot 10^6$ is greater than the unconditional counterpart $N_R = 2 \cdot 10^4$, as seen in the left panel of Fig. 10. Furthermore, the crest height $1.6H_s$ is nearly the same as the threshold $h_{0.3} = 10^6$ exceeded on average once every $N_h = 0.3 \cdot 10^6$ waves, as indicated in the right panel of the same figure. Thus, the Andrea wave has a greater occurrence rate than being both rogue and unexpected since $N_h < N_{tr} = 3 \cdot 10^6$, and implying the larger threshold $h_{tr} = 1.76H_s$.

7. Concluding remarks

We have presented a third-order nonlinear model for the statistics of unexpected waves. Gemmrich and Garrett (2008) define as unexpected a wave that is taller than a set of neighboring waves. The term “unexpected” refers to a wave that is not foreseen by a casual observer (Gemmrich and Garrett 2010). Clearly, unexpected waves are predictable in a statistical sense. Indeed, they can occur relatively often with a small or moderate crest height. However, unexpected waves that are rogue are rare. This difference in occurrence frequencies is quantified by introducing the conditional return period of an unexpected wave that exceeds a given threshold crest height. The associated unconditional return period is smaller than the conditional counterpart as it refers to the harmonic mean of the return periods of unexpected waves of any crest amplitude.

Furthermore, our analysis indicate that a wave that is both rogue and unexpected has a lower occurrence frequency than just being rogue. This is proven both analytically and verified by way of an analysis of the Andrea and WACSIS rogue wave events. Both waves appeared without warning and their crests were nearly 2-times larger than the surrounding $O(10)$ wave crests, and thus unexpected. The two crest heights are nearly the same as the threshold $h_{0.3} \sim 1.6H_s$ exceeded on average once every $0.3 \cdot 10^6$ waves. In contrast, the Andrea and WACSIS events would occur less often as being both unexpected and rogue, i.e. on average once every $3 \cdot 10^6$ and $0.6 \cdot 10^6$ respectively.

Finally, we point out that our statistical model for unexpected waves supports and goes beyond the analysis by Gemmrich and Garrett (2008) based on Monte Carlo simulations. In particular, our statistical approach can be used in operational wave forecast models to predict the unexpectedness of ocean waves.

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