A GENERALIZATION OF TWO CLASSICAL CONVERGENCE TESTS FOR FOURIER SERIES, AND SOME NEW BANACH SPACES OF FUNCTIONS

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ABSTRACT. The norms of these spaces fill the gap between the uniform and the variation norms. Their duals are described in terms of generalized variation. One application of these spaces is a new convergence test for Fourier series which includes both the Dirichlet-Jordan and the Dini-Lipschitz tests [1].

1. The \( \kappa \)-entropy. \( \kappa(s) \) will always denote a nondecreasing concave function on \([0,1]\) such that \( \kappa(0) = 0, \kappa(1) = 1 \); this implies that \( \kappa(s) \) is continuous except, perhaps, at \( s = 0 \).

DEFINITION. Let \( E = \{x_1 < x_2 < \cdots < x_n\} \subset [a,b] \) be a finite nonempty set. The following quantity will be called the \( \kappa \)-entropy of \( E \) (relative to \([a,b]\)):

\[
\kappa(E) = \kappa(E;[a,b]) = \sum_{j=1}^{n+1} \kappa((x_j - x_{j-1})/(b-a)),
\]
where \( x_0 = a, x_{n+1} = b \). For an arbitrary closed set \( F \subset [a,b] \) we set

\[
\kappa(F) = \kappa(F;[a,b]) = \sup\{\kappa(E): E \subset F \text{ finite}\}.
\]

Finally, we set \( \kappa(\emptyset) = 0 \).

The following properties of the \( \kappa \)-entropy are easily derived.

(i) \( F_1 \subset F_2 \) implies \( \kappa(F_1) \leq \kappa(F_2) \).

(ii) \( \kappa(F_1 \cup F_2) \leq \kappa(F_1) + \kappa(F_2) \).

(iii) If \( \text{card } E = n \), then \( \kappa(E) \leq (n + 1)\kappa(1/(n + 1)) \); the estimate is sharp and attained for \( x_1 - x_0 = x_2 - x_1 = \cdots = x_{n+1} - x_n \).

2. Examples of \( \kappa \)-entropy.

(a) \( \kappa(s) = s \). We have in this case \( \kappa(F) = 1 (F \neq \emptyset), \kappa(\emptyset) = 0 \).

(b) \( \kappa(s) = 1 (0 < s \leq 1) \). Here we have

\[
\kappa(F) = \text{card}(F \cup \{a,b\}) - 1 \quad (F \neq \emptyset).
\]

(c) \( \kappa(s) = s(1 - \log s) \). The corresponding entropy will be denoted by \( \kappa_s(F) \) and called the Shannon entropy of \( F \) (relative to \([a,b]\)).

(d) \( \kappa(s) = s^\alpha \). Here \( \kappa(F) = \kappa_{l,\alpha}(F) \) is the Lipschitz entropy \( (0 < \alpha < 1) \).

(e) \( \kappa(s) = (1 - \frac{1}{\log s})^{-1}; \kappa(F) = \kappa_d(F) \) is the Dini entropy.

Received by the editors March 2, 1983.

1980 Mathematics Subject Classification. Primary 42A20, 46E15.

\(^1\)Supported by NSF grant MCS82-01460

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0273-0979/83 $1.00 + $.25 per page
3. The $\kappa$-entropy norm.

**Definition.** The $\kappa$-*entropy* norm (or simply the $\kappa$-*norm*) of a real continuous function $x(t)$ on $[a, b]$ is

\[
\|x\|_\kappa = \|x\|_C + \int_{-\infty}^{\infty} \kappa(E_y, [a, b]) \, dy,
\]

where $\|x\|_C = \max\{|x(t)| : a \leq t \leq b\}$ and $E_y = E_y[x] = \{t \in [a, b] : x(t) = y\}$ is the level set of $x(t)$.

**Examples.** (a) $\kappa(s) = s$. Here we have $\|x\|_\kappa = \|x\|_C + \max t(t) - \min x(t)$; thus $\|x\|_C \leq \|x\|_\kappa \leq 3\|x\|_C$, so that the $\kappa$-norm in this case is equivalent to the uniform norm.

(b) $\kappa(s) = 1 \ (0 < s \leq 1)$. We have

\[
\|x\| = \|x\|_C + \int_{m}^{M} (\text{card } E_y + 1) \, dy = \|x\|_C + M - m + \text{Var } x,
\]

where $M = \max x(t), \ m = \min x(t)$; thus

\[
\|x\|_C + \text{Var } x \leq \|x\|_\kappa \leq 3\|x\|_C + \text{Var } x.
\]

(c) The $\kappa$-norm corresponding to the Shannon, Lipschitz and Dini entropies is denoted respectively by $\| \cdot \|_s, \| \cdot \|_{l, \alpha}$ and $\| \cdot \|_d$ and called the Shannon-, Lipschitz- and Dini-entropy norm.

In what follows we assume that $\kappa(0^+) = 0$ and $\kappa(s)/s \to \infty \ (s \to 0)$, since otherwise the $\kappa$-norm is equivalent either to the $C$-norm or the $V$-norm.

4. The spaces $C_\kappa[a, b]$.

**Theorem 1.** Every $\kappa$-norm is homogeneous and convex: $\|\lambda x\|_\kappa = |\lambda| \|x\|_\kappa$, $\|x_1 + x_2\| \leq \|x_1\|_\kappa + \|x_2\|_\kappa$. Equipped with a $\kappa$-norm, the linear set of all real continuous functions $x(t)$ on $[a, b]$ such that $\|x\|_\kappa < \infty$ forms a (real) Banach space $C_\kappa[a, b]$; this space is separable: polynomials are dense in $C_\kappa[a, b]$.

The homogeneity of the $\kappa$-norm follows directly from the definition; however, the proof of the triangle inequality is more difficult.

5. The $\kappa$-variation.

**Definition.** The $\kappa$-*variation* of a real function $\mu(t)$ over $[a, b]$ is

\[
\text{Var}_\kappa \mu = \text{Sup} \left\{ \left( \sum_{1}^{n+1} |\mu(x_j) - \mu(x_{j-1})| / \kappa(E; [a, b]) \right) \right\},
\]

where the supremum is taken over all finite sets

\[
E = \{x_1 < x_2 < \cdots < x_n\} \subset [a, b] \text{ and } x_0 = a, \ x_{n+1} = b.
\]

It is easily seen that $\text{Var}_\kappa \mu < \infty$ implies the existence of unilateral limit values $\mu(t^+) \ (a \leq t < b)$ and $\mu(t^-) \ (a < t \leq b)$. Every such function $\mu(t)$ generates a "measure" on the set of all intervals $I \subset [a, b]$, e.g. $\mu([\alpha, \beta]) = \mu(\beta^+) - \mu(\alpha^-), \ \mu(\alpha, \beta) = \mu(\beta^+) - \mu(\alpha^+)$, and so on. If $\text{Var}_\kappa \mu < \infty$, then this measure can be extended to all (relatively) open sets $G \subset [a, b]$ such that

\[\text{This notion was first introduced in [2] for the Shannon variation (see also [3])}.\]
\(\kappa(\partial G) < \infty\) by the formula \(\mu(G) = \sum_j \mu(I_j)\), where \(I_j\) are the components of \(G\); the series is absolutely convergent. Similarly, for closed sets \(F \subset [a,b]\) we define \(\mu(F) = \mu([a,b]) - \mu([a,b] \setminus F)\).

The linear set consisting of all \(\mu(t) (a \leq t \leq b)\) such that \(\text{Var}_\kappa \mu < \infty\), provided with the norm \(\|\mu\| = \text{Var}_\kappa \mu\), is a Banach space \(V_\kappa[a,b]\); for the special cases of the Shannon, Lipschitz or Dini variation this space is denoted respectively by \(V_s, V_{l,a}\) and \(V_d\).

6. The \(\kappa\)-integral.

Definition. Let \(x(t) \in C_\kappa[a,b]\) and \(\mu(t) \in V_\kappa[a,b]\). The \(\kappa\)-integral of \(x\) with respect to \(d\mu\) is defined as follows:

\[
\int_a^b x(t) \, d\mu(t) = m\mu([a,b]) + \int_m^M \mu(F_y[x]) \, dy,
\]

where \(m = \min x(t), M = \max x(t)\), and \(F_y[x] = \{t \in [a,b]: x(t) \geq y\}\) are the Lebesgue sets of \(x(t)\).

It is easily seen that, by (3) and (4), \(\mu(F_y)\) is summable over \((m,M)\); we also deduce

\[
\left| \int_a^b x(t) \, d\mu(t) \right| \leq \|x\|_\kappa \text{Var}_\kappa \mu.
\]

If \(\mu\) is of bounded (classical) variation, then \(\int x \, d\mu\) exists as a Riemann-Stieltjes integral and its value coincides with that of the \(\kappa\)-integral.

7. The dual of \(C_\kappa\).

Theorem 2. \(V_\kappa\) is the dual of \(C_\kappa\). This means that every linear functional \(F(x)\) in \(C_\kappa[a,b]\) has the form of a \(\kappa\)-integral (5), where \(\mu\) is uniquely (up to a constant) determined by \(F\). We also have \(\frac{1}{2} \text{Var}_\kappa \mu \leq \|F\| \leq \text{Var}_\kappa \mu\).

8. A convergence test for Fourier series. The Dirichlet-Jordan (D-J) convergence test [1] states that the (symmetrical) partial sums \(S_n(t;f)\) of the Fourier series of a \(2\pi\)-periodic function \(f(t)\) of bounded variation tend to \(\frac{1}{2}[f(t+0) + f(t-0)]\) as \(n \to \infty\); if \(f(t)\) is also continuous, then \(S_n(t) \to f(t)\) uniformly.

The Dini-Lipschitz (D-L) test [1] states that \(S_n(t;f) \to f(t)\) uniformly if the modulus of continuity \(\omega(\delta)\) of \(f(t)\) is \(o(\log \delta^{-1})(\delta \to 0)\).

The proof of the Dirichlet-Jordan test is based on the C-V duality. However, if instead of the C-V duality we use the Dini-entropy-norm—Dini-variation duality (\(C_d-V_d\)), we obtain a new test that includes both the D-J and the D-L tests.

Definition. A function \(\mu(t) \in V_\kappa[a,b]\) is said to be of vanishing \(\kappa\)-variation at \(t_0 \in [a,b]\) if \(\text{Var}_\kappa\{(\mu(t) - \mu(t_0))\chi_{\delta}(t)\} \to 0 (\delta \to 0)\), where \(\chi_{\delta}(t)\) is the characteristic function of \([t_0 - \delta, t_0 + \delta]\), and the \(\kappa\)-variation is taken over \([a,b]\). If this takes place at every point \(t_0 \in [a,b]\), then \(\mu(t)\) is said to be of vanishing \(\kappa\)-variation on \([a,b]\).

Remark. For the classical variation, if \(f(t)\) is of bounded variation on \([a,b]\) and continuous at \(t_0\), then \(f(t)\) is of vanishing variation at \(t_0\). However, for the \(\kappa\)-variation this is generally not true.
Theorem 3. Let \( f(t) \in V_d[0,2\pi] \) be \( 2\pi \)-periodic and normalized so that 
\[ f(t) = \frac{1}{2} [f(t + \tau) + f(t - \tau)]. \]
If \( \varphi(\tau; t_0) = \frac{1}{2} [f(t_0 + \tau) + f(t_0 - \tau)] \) is of vanishing Dini variation at \( \tau = 0 \), then the Fourier series of \( f(t) \) converges at \( t_0 \) to \( f(t_0) \).

If \( f(t) \) is of vanishing Dini-variation on \([0,2\pi]\), then \( S_n(t, f) \to f(t) (n \to \infty) \) uniformly. 

A short outline of the proof. We have 
\[
S_n(t_0; f) - f(t_0) = \int_0^\pi \mathcal{E}_n(t) \, d[\varphi(t; t_0) - f(t_0)],
\]
where 
\[
\mathcal{E}_n(t) = \int_t^\pi D_n(\tau) \, d\tau \quad (0 \leq t < \pi), \quad D_n(\tau) = \sin\left(n + \frac{1}{2}\right) \tau / \left(\pi \sin \frac{\tau}{2}\right).
\]
A simple computation shows that \( \mathcal{E}_n \) satisfies 
\[
|\mathcal{E}_n(t)| \leq \min\{1, 4((2n + 1)t)^{-1}\} \quad (0 \leq t \leq \pi)
\]
and is monotone in each of the intervals \((2k\pi/(2n+1), 2(k+1)\pi/(2n+1))(k = 0,1,\ldots,n-1)\) and \((2n\pi/(2n+1), \pi)\); from this we deduce that the Dini-entropy norms \( ||\mathcal{E}_n||_d \) (\( n = 1, 2, \ldots \)) are bounded if taken over \([0,\pi]\), and tend to 0 if taken over \([\delta, \pi]\) (\( \delta > 0 \)). Using this, and (7) and (6), we get the required result.

References

1. A. Zygmund, *Trigonometric series*, vol. 1, Cambridge Univ. Press, New York, 1959.
2. B. Korenblum, *An extension of the Nevanlinna theory*, Acta Math. 135 (1975), 187–219.
3. ———, *A Beurling-type theorem*, Acta Math. 138 (1977), 265–293.

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