The Covariant $W_3$ Action

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Abstract

Starting with $Sl(3, \mathbb{R})$ Chern–Simons theory we derive the covariant action for $W_3$ gravity.
1. Introduction

Two dimensional gravity has been extensively studied during the last few years. Three different approaches to the subject, namely (i) study of the induced action of 2D gravity in both the conformal (where it reduces to the Liouville action) as well as the light cone gauge, (ii) the discretized approach of the matrix models, and (iii) topological gravity, have all been very powerful (at least for $c < 1$), giving equivalent results.

Higher spin extensions of 2D gravity can also be studied using the above methods. These theories are commonly denoted as theories of $W$ gravity. Especially $W_3$ gravity in the light cone gauge has been the subject of many recent research [12, 13, 14, 15]. In this paper we will also concern ourselves with the study of $W_3$ gravity, but from a different angle. Believing that the ‘$W_3$ moduli space’ is somehow related to the moduli space of flat $Sl(3, \mathbb{R})$ bundles, we will study $W_3$ gravity starting from $Sl(3, \mathbb{R})$ Chern–Simons theory whose classical phase space is the space of flat $Sl(3, \mathbb{R})$ bundles.

Our analysis resembles the one in [1]. In this reference H. Verlinde showed how the physical state condition in $Sl(2, \mathbb{R})$ Chern–Simons theory can be reduced to the conformal Ward identity, giving as a by-product the fully covariant action of 2D gravity. We will start with $Sl(3, \mathbb{R})$ Chern–Simons theory, and derive the covariant action for $W_3$ gravity. It will turn out that this action describes $Sl(3, \mathbb{R})$ Toda theory coupled to a ‘$W_3$ background,’ confirming general beliefs. Although we restrict ourselves to the case of $W_3$ in this paper, we believe that many of our results can be generalized. This will be reported elsewhere [9].

2. Chern–Simons theory

Chern–Simons theory on a three manifold $M$ is described by the action

$$S = \frac{k}{4\pi i} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

(2.1)

where the connection $A$ is a one form with values in the Lie algebra $g$ of some Lie group $G$, and $d$ denotes the exterior derivative on $M$. In this paper $M$ will be of the
form \( M = \Sigma \times \mathbb{R} \), \( \Sigma \) being a Riemann surface, for which \( A \) and \( d \) can be decomposed into space and time components, i.e. \( A = A_0 dt + \tilde{A} \), with \( \tilde{A} = \tilde{A}_z dz + \tilde{A}_{\bar{z}} d\bar{z} \), and \( d = dt \partial/\partial t + \tilde{d} \). Rewriting the action as

\[
S = \frac{k}{4\pi i} \int dt \int_\Sigma \text{Tr}(\tilde{A} \wedge \partial_t \tilde{A} + 2A_0(\tilde{d}\tilde{A} + \tilde{A} \wedge \tilde{A})),
\]

we recognize that \( A_0 \) acts as a Lagrange multiplier which implements the constraint \( \tilde{F} = \tilde{d} \tilde{A} + \tilde{A} \wedge \tilde{A} = 0 \). Furthermore, we deduce from this action the following non-vanishing Poisson brackets

\[
\{\tilde{A}^a_z(z), \tilde{A}^b_{\bar{z}}(w)\} = \frac{2\pi i}{k} \eta^{ab} \delta(z - w),
\]

where \( \tilde{A}_z = \sum_a \tilde{A}^a_z T^a \), with \( \text{Tr}(T^a T^b) = \eta^{ab} \).

Upon quantizing the theory we have to replace the above Poisson bracket by a commutator, and we have to choose a ‘polarization.’ This simply means that we have to divide the set of variables \((\tilde{A}^a_z, \tilde{A}^a_{\bar{z}})\) into two subsets. One subset will contain fields \( X_i \) and the other subset will consist of derivatives \( \frac{\delta}{\delta X_i} \), in accordance with (2.3). The choice of these subsets is called a choice of polarization. Of course we also have to incorporate the Gauss law constraints \( \tilde{F}(\tilde{A}) = 0 \). Following [1, 3, 7] we will impose these constraints after quantization. So we will first consider a ‘big’ Hilbert space obtained by quantization of (2.3), and then select the physical states \( \Psi \) by requiring \( \tilde{F}(\tilde{A})\Psi = 0 \).

In [1] it was shown that these physical state conditions for \( Sl(2, \mathbb{R}) \) Chern–Simons theory with a certain choice of polarization are equivalent to the conformal Ward identities satisfied by conformal blocks in Conformal Field Theory (CFT). More precisely, it was shown that two of the three constraints in \( \tilde{F}(\tilde{A})\Psi = 0 \) could be explicitly solved, leaving one constraint which is equivalent to the conformal Ward identity. In this paper we will generalize these results to the case of \( Sl(3, \mathbb{R}) \) with a choice of polarization that leads to the Ward identities of the \( W_3 \) algebra [2]. (A different choice of polarization leading to the related \( W_3^2 \) algebra was made in [11].) To explain our strategy, we will in the next section first reconsider the case of \( Sl(2, \mathbb{R}) \) Chern–Simons theory.
3. $Sl(2, \mathbb{R})$

In order to understand how one obtains the Virasoro Ward identity from $Sl(2, \mathbb{R})$ Chern–Simons theory, let us first recall how one can in general obtain Ward identities from zero-curvature constraints. Given operator valued connections $A$ and $\bar{A}$, the zero curvature condition reads:

$$F \Psi = \partial \bar{A} - \bar{\partial} A + [A, \bar{A}] : \Psi = 0.$$

(3.1)

Here the dots denote normal ordering, which simply amounts to putting all $\delta/\delta X$ to the right. (Here $X$ is some arbitrary field.) If we take $\dagger$:

$$A = \begin{pmatrix} 0 & 1 \\ \delta/\delta\mu & 0 \end{pmatrix},$$

(3.2)

and put $\bar{A}_{12} = \mu$, we can solve for the remaining components of $A$, if we require that the curvature operator must have the form

$$F = \begin{pmatrix} 0 & 0 \\ \ast & 0 \end{pmatrix}.$$  

(3.3)

Equation (3.1) reduces to one equation which is precisely the Virasoro Ward identity for $c = 6k$

$$\left[ (\bar{\partial} - \mu \partial - 2(\partial\mu)) \frac{\delta}{\delta\mu} + \frac{1}{2} \partial^3\mu \right] \Psi = 0.$$  

(3.4)

If we start with $Sl(2, \mathbb{R})$ Chern–Simons theory and pick a polarization, then $A$ and $\bar{A}$ consist of three fields and their variational derivatives. On the other hand, the Virasoro Ward identity contains just one field. So in order to obtain the Virasoro Ward identity as zero curvature constraint of Chern–Simons theory, we must introduce two extra degrees of freedom without changing the contents of the zero curvature constraints. We can in this case simply introduce extra degrees of freedom by performing a gauge transformation. Under such a gauge transformation the curvature transforms homogeneously: $F \rightarrow g^{-1}Fg$, and $F = 0$ will still give the

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*Note that from now on $\bar{A}$ will be denoted as $A$.

†In the following $\frac{\delta}{\delta X}$ should in fact read $\frac{2\pi}{k} \frac{\delta}{\delta X}$, for all fields $X$. 
Virasoro Ward identity. If we require $F$ to remain of type \((3.3)\), $g$ must be of the form
\[
g = \begin{pmatrix} g_{11} & 0 \\ g_{21} & g_{11}^{-1} \end{pmatrix}.
\] (3.5)

We can parametrize $g$ via a Gauss decomposition:
\[
g = \begin{pmatrix} 1 & 0 \\ \chi & 1 \end{pmatrix} \begin{pmatrix} e^\phi & 0 \\ 0 & e^{-\phi} \end{pmatrix},
\] (3.6)

and we find that $F \to Fe^{2\phi}$. The gauge transformed $A, \bar{A}$ are
\[
A^g = \begin{pmatrix} \chi + \partial \phi & e^{-2\phi} \\ e^{2\phi}(\partial \chi - \chi^2 + \delta \mu) & -\chi - \partial \phi \end{pmatrix},
\] (3.7)
\[
\bar{A}^g = \begin{pmatrix} e^{2\phi}(\mu \delta \mu - \frac{1}{2} \partial^2 \mu - \mu \chi^2 + \chi \partial \mu + \bar{\partial} \phi) & -\frac{1}{2} \partial \mu - \chi \mu - \bar{\partial} \phi \\ \] (3.8)

If we pick the following polarization:
\[
A = \begin{pmatrix} A_{11} & A_{12} \\ \delta A_{12} & -A_{11} \end{pmatrix},
\] (3.9)
\[
\bar{A} = \begin{pmatrix} -\frac{1}{2} \delta A_{12} & \bar{A}_{12} \\ \frac{1}{2} \delta A_{12} & \frac{1}{2} \delta A_{11} \end{pmatrix},
\] (3.10)

and let $F(A, \bar{A})$ act on a wavefunction $e^S\Psi[\mu]$, where $S$ still has to be determined, we find that $F(A, \bar{A})e^S\Psi = 0 \Leftrightarrow e^S F(A', \bar{A}')\Psi = 0$, with $A' = \begin{pmatrix} \delta S \delta A_{12} \\ \delta A_{12} \end{pmatrix}$ and $\bar{A}' = \begin{pmatrix} \frac{1}{2} \delta S \delta A_{12} \\ \delta A_{12} \end{pmatrix}$. This gives
\[
A_{11} = \chi + \partial \phi,
A_{12} = e^{-2\phi},
\bar{A}_{12} = \mu e^{-2\phi}.
\] (3.13)

\footnote{Here we ignored terms of the type $\frac{\delta^2 S}{\delta X \delta X'}$, that involve delta-function type singularities that have to be regularized somehow. Therefore the validity of our discussion will be limited to the semi-classical level. In the full quantum theory we expect corrections to the expressions given in this paper.}
Using these expressions, we can express the variational derivatives with respect to $A$ and $\bar{A}$ in terms of those with respect to $\chi$, $\phi$ and $\mu$. One finds:

\[
\frac{\delta}{\delta A_{11}} = \frac{\delta}{\delta \chi},
\]

\[
\frac{\delta}{\delta A_{12}} = -\frac{1}{2} e^{2\phi} \left( \partial \frac{\delta}{\delta \chi} + \frac{\delta}{\delta \phi} - 2\mu \frac{\delta}{\delta \mu} \right), \tag{3.14}
\]

\[
\frac{\delta}{\delta A_{12}} = e^{2\phi} \frac{\delta}{\delta \mu}.
\]

Quite remarkably, the terms containing $\frac{\delta}{\delta \mu}$ in (3.11) and (3.12) agree precisely with those in (3.7) and (3.8). As $\Psi$ depends only on $\mu$, we can omit the $\frac{\delta}{\delta \phi}$ and $\frac{\delta}{\delta \chi}$ terms in (3.12). In conclusion, we see that (3.11) and (3.12) are exactly identical to (3.7) and (3.8) if the following relations hold:

\[
\frac{\delta S}{\delta A_{11}} = -\partial \mu - 2\chi \mu - 2\bar{\phi},
\]

\[
\frac{\delta S}{\delta A_{12}} = e^{2\phi} \left( \frac{1}{2} \partial^2 \mu + \mu \chi^2 + \chi \partial \mu - \bar{\partial} \chi \right), \tag{3.15}
\]

\[
\frac{\delta S}{\delta A_{12}} = e^{2\phi} (\partial \chi - \chi^2).
\]

Before continuing, we will first rewrite these expressions in a form that is more suitable for generalizations to other cases. Let $G$ be the subgroup of $Sl(2, \mathbb{R})$ consisting of all $g$ of type (3.3), and let $\Lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, which is equal to (3.2) up to terms containing $\frac{\delta}{\delta \mu}$. Furthermore, let $(\bar{A}_0)_{12} = \mu$ and let $\bar{A}_0$ be such that $F(\Lambda, \bar{A}_0)$ is of type (3.3). For this $\bar{A}_0$ one finds $F(\Lambda, \bar{A}_0) = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} \partial^2 \mu & 0 \end{pmatrix}$. To rewrite the polarizations, define projections $\Pi_k$ and $\Pi_i$ on the Lie algebra $sl(2, \mathbb{R})$ via

\[
\Pi_k \begin{pmatrix} b & a \\ c & -b \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix},
\]

\[
\Pi_i \begin{pmatrix} b & a \\ c & -b \end{pmatrix} = \begin{pmatrix} b & a \\ 0 & -b \end{pmatrix}.
\]

The polarization (3.8) and (3.10) is such that the fields are in $\Pi_i A$ and $\Pi_k \bar{A}$, and that the derivatives are in $\Pi_i^t \bar{A}$ and $\Pi_k^t A$, where $\Pi_i^t = 1 - \Pi_k$ and $\Pi_k^t = 1 - \Pi_i$. Now (3.7) and (3.8) read, up to terms containing $\frac{\delta}{\delta \mu}$, $A = g^{-1} \Lambda g + g^{-1} \partial g$ and $\bar{A} = g^{-1} \bar{A}_0 g + g^{-1} \bar{\partial} g$. Therefore, (3.13) can be compactly formulated as

\[
\Pi_i A = \Pi_i (g^{-1} \Lambda g + g^{-1} \partial g), \tag{3.16}
\]

\[
\Pi_k \bar{A} = \Pi_k (g^{-1} \bar{A}_0 g + g^{-1} \bar{\partial} g). \tag{3.17}
\]
and equations (3.13) become

\[
- \frac{\delta S}{\delta \Pi_i A} = \Pi_i^\dagger (g^{-1} \bar{A}_0 g + g^{-1} \partial g),
\]

(3.18)

\[
\frac{\delta S}{\delta \Pi_k A} = \Pi_k^\dagger (g^{-1} \Lambda g + g^{-1} \partial g).
\]

(3.19)

Surprisingly, these equations can be integrated\footnote{The proof of this fact will be given elsewhere [9].} to give

\[
S = \frac{k}{2\pi} \int d^2 z \text{Tr}(\Pi_i^\dagger A \Pi_k \bar{A}) - \frac{k}{2\pi} \int d^2 z \text{Tr}(\Lambda \partial gg^{-1}) - \Gamma_{WZW}[g],
\]

(3.20)

where \(\Gamma_{WZW}\) is the Wess-Zumino-Witten action

\[
\Gamma_{WZW}[g] = \frac{k}{4\pi} \int d^2 z \text{Tr}(g^{-1} \partial gg^{-1} \partial g) - \frac{k}{12\pi} \int_B \text{Tr}(g^{-1} dg)\cdot
\]

(3.21)

For \(Sl(2, \mathbb{R})\) we find that

\[
S = \frac{k}{2\pi} \int d^2 z \left[ \mu \partial \chi - \mu \chi^2 - 2 \mu \partial \phi - \partial \phi \partial \phi \right],
\]

(3.22)

which indeed solves (3.13). To make contact with the results of [1] we have to redefine \(\phi \rightarrow -\frac{1}{2} \varphi\), and \(\chi \rightarrow \frac{1}{2} \omega + \frac{1}{2} \partial \varphi\). Then the action becomes \(S = S_0(\omega, \varphi, \mu) + S_L(\varphi, \mu)\), where

\[
S_0 = -\frac{k}{4\pi} \int d^2 z \left[ \frac{1}{2} \mu \omega^2 - \omega (\partial \varphi - \partial \mu - \mu \partial \varphi) \right],
\]

(3.23)

and \(S_L\) is a chiral version of the Liouville action:

\[
S_L = \frac{k}{4\pi} \int d^2 z \left[ \frac{1}{2} \partial \varphi \partial \varphi + \mu (\partial^2 \varphi - \frac{1}{2} (\partial \varphi)^2) \right].
\]

(3.24)

The actions (3.23), (3.24) are precisely the same as the ones found in [1]. Altogether we have now shown how the Virasoro Ward identity follows from \(Sl(2, \mathbb{R})\) Chern–Simons theory. In general, the procedure consists of three steps: (i) pick a polarization and parametrization of the components of \(A\) and \(\bar{A}\), (ii) move \(A\) and \(\bar{A}\) through a term of the form \(e^S\) and (iii), perform a gauge transformation. In the next section we will apply these steps to the case of \(Sl(3, \mathbb{R})\) Chern–Simons theory.
4. $Sl(3, \mathbb{R})$

The Ward identities of the $W_3$-algebra can be obtained in the same way as the Virasoro Ward identity was obtained in the previous section. We start with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\delta}{\delta \nu} & \frac{\delta}{\delta \mu} & 0 \end{pmatrix},$$

(4.1)

and put $\bar{A}_{13} = \nu$ and $\bar{A}_{23} = \mu$. The remaining components of $\bar{A}$ are fixed by requiring $F$ to be of the form

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F_{31} & F_{32} & 0 \end{pmatrix}.$$

(4.2)

As was shown in e.g. [12], $F_{31}$ and $F_{32}$ are directly related to the $W_3$-Ward identities. The subgroup $G$ of $Sl(3, \mathbb{R})$ that preserves this form of $F$ consists of all $g \in Sl(3, \mathbb{R})$ satisfying $g_{13} = g_{23} = 0$. To parametrize these $g$, we will again use a Gauss decomposition:

$$g = \begin{pmatrix} 1 & 0 & 0 \\ \phi_1 & 1 & 0 \\ \phi_3 & \phi_2 & 1 \end{pmatrix} \begin{pmatrix} e^\alpha & 0 & 0 \\ 0 & e^{\beta - \alpha} & 0 \\ 0 & 0 & e^{-\beta} \end{pmatrix} \begin{pmatrix} 1 & \chi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

(4.3)

Under a gauge transformation $F \rightarrow g^{-1}Fg$ we find

$$F'_{31} = e^{\alpha + \beta} F_{31} + \phi_1 e^{\alpha + \beta} F_{32},$$

(4.4)

$$F'_{32} = \chi e^{\alpha + \beta} F_{31} + (\chi \phi_1 e^{\alpha + \beta} + e^{2\beta - \alpha}) F_{32},$$

(4.5)

which clearly shows that $F \Psi[\mu, \nu] = 0 \Leftrightarrow (g^{-1}Fg) \Psi[\mu, \nu] = 0$. The polarization we choose is such that it is invariant under the subgroup $G$ of $Sl(3, \mathbb{R})$:

$$A = \begin{pmatrix} A_+ + A_- & A_{12} & A_{13} \\ A_{21} & -2A_+ & A_{23} \\ \frac{\delta}{\delta A_{13}} & \frac{\delta}{\delta A_{23}} & A_+ - A_- \end{pmatrix},$$

(4.6)

and

$$\bar{A} = \begin{pmatrix} -\frac{1}{6} \frac{\delta}{\delta A_+} - \frac{1}{2} \frac{\delta}{\delta A_-} & -\frac{\delta}{\delta A_{13}} & \bar{A}_{13} \\ -\frac{\delta}{\delta A_{12}} & \frac{1}{3} \frac{\delta}{\delta A_+} & \bar{A}_{23} \\ -\frac{\delta}{\delta A_{13}} & -\frac{\delta}{\delta A_{23}} & -\frac{1}{6} \frac{\delta}{\delta A_+} + \frac{1}{2} \frac{\delta}{\delta A_-} \end{pmatrix},$$

(4.7)
and the projections \( \Pi_i \) and \( \Pi_k \) are given by

\[
\Pi_k \left( \begin{array}{ccc}
    a + b & c & d \\
    e & -2a & f \\
    g & h & a - b
\end{array} \right) = \left( \begin{array}{ccc}
    0 & 0 & d \\
    0 & 0 & f \\
    0 & 0 & 0
\end{array} \right),
\]

(4.8)

\[
\Pi_i \left( \begin{array}{ccc}
    a + b & c & d \\
    e & -2a & f \\
    g & h & a - b
\end{array} \right) = \left( \begin{array}{ccc}
    a + b & c & d \\
    e & -2a & f \\
    0 & 0 & a - b
\end{array} \right).
\]

(4.9)

The matrices \( \Lambda \) and \( \bar{A}_0 \) are also found completely analogously to the \( SL(2, \mathbb{R}) \)-case, one simply requires \( F(\Lambda, \bar{A}_0) \) to be of type \( [1,2] \), to find:

\[
\Lambda = \left( \begin{array}{ccc}
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    0 & 0 & 0
\end{array} \right),
\]

(4.10)

\[
\bar{A}_0 = \left( \begin{array}{ccc}
    \partial \mu + \frac{2}{3} \partial^2 \nu & \mu + \partial \nu & \nu \\
    -\partial^2 \mu - \frac{2}{3} \partial^3 \nu & \frac{1}{3} \partial^2 \nu & \mu \\
    \partial^3 \mu + \frac{2}{3} \partial^3 \nu & -\partial^2 \mu - \frac{2}{3} \partial^3 \nu & -\partial \mu - \frac{1}{3} \partial^2 \nu
\end{array} \right).
\]

(4.11)

It is straightforward to read of the explicit expressions for \( \Pi_i A \) and \( \Pi_k \bar{A} \) from (3.16) and (3.17). They are:

\[
A_+ = \frac{1}{2} \left( \phi_1 - \phi_2 + (\phi_1^2 - \phi_3) \chi \right) e^{2\alpha - \beta} - \chi \phi_1 e^{2\alpha - \beta} + \partial (\alpha - \beta),
\]

\[
A_- = \frac{1}{2} \left( \phi_1 + \phi_2 + (\phi_1^2 - \phi_3) \chi \right) e^{2\alpha - \beta} - \chi \phi_1 e^{2\alpha - \beta} + \partial (\alpha + \beta),
\]

\[
A_{12} = (2\phi_1 - \phi_2) \chi + e^{\beta - 2\alpha} + \chi^2 (\phi_1^2 - \phi_3) e^{2\alpha - \beta} - \chi^2 \phi_1 e^{2\alpha - \beta} + \chi \partial (2\alpha - \beta) + \partial \chi,
\]

\[
A_{13} = -\chi e^{2\alpha - \beta},
\]

\[
A_{21} = (\phi_3 - \phi_1^2 + \partial \phi_1) e^{2\alpha - \beta},
\]

\[
A_{23} = e^{\alpha - 2\beta},
\]

\[
\bar{A}_{13} = \chi (\nu \phi_1 - \mu) e^{\alpha - 2\beta} + \nu e^{-\alpha - \beta},
\]

\[
\bar{A}_{23} = (\mu - \nu \phi_1) e^{\alpha - 2\beta},
\]

(4.12)

Again, we want to construct an action \( S \), such that \( A' \) and \( \bar{A}' \), defined through \( F(A, \bar{A}) e^S \Psi = 0 \Leftrightarrow e^S F(A', \bar{A}') \Psi = 0 \), are equal to the connections \( A^g \) and \( \bar{A}^g \) (the generalizations of (3.7) and (3.8)) that are the gauge transforms with \( g \) as in (4.3) of the connections \( A \) and \( \bar{A} \) mentioned in and below (4.1). If \( A^g = A' \) and \( \bar{A}^g = \bar{A}' \) are satisfied, \( F(A', \bar{A}') \Psi = 0 \) is equivalent to the statement that \( \Psi \) satisfies the \( W_3 \)-Ward identities. The connections \( A' \) and \( \bar{A}' \) can be obtained from (3.9) and (3.10) by first replacing \( \frac{\delta}{\delta X_i} \) by \( \frac{\delta}{\delta X_i} + \frac{\delta S}{\delta X_i} \) everywhere, followed by putting all terms in \( \frac{\delta}{\delta X_i} \) that do
not contain $\frac{\delta}{\delta \mu}$ or $\frac{\delta}{\delta \nu}$ equal to zero, as $\Psi$ depends only on $\mu$ and $\nu$. Comparing these $A'$ and $\bar{A}'$ with $A^g$ and $\bar{A}^g$ yields a set of equations for $\frac{\delta S}{\delta X_i}$, analogous to (3.15). These equations are necessary, but not sufficient, because the $\frac{\delta}{\delta \mu}$ and $\frac{\delta}{\delta \nu}$ dependence of $A'$ and $\bar{A}'$ must also be equal to the $\frac{\delta}{\delta \mu}$ and $\frac{\delta}{\delta \nu}$ dependence of $A^g$ and $\bar{A}^g$. Whether this is the case has been verified by explicitly computing the $\frac{\delta}{\delta X_i}$ in (4.6) and (4.7) in terms of $\frac{\delta}{\delta \mu}$, $\frac{\delta}{\delta \nu}$, and the functional derivatives with respect to the fields in the Gauss decomposition (4.3). It turns out that this dependence is indeed precisely the same. The computations involved here are rather cumbersome, whether there is a more direct way to see this, is under current investigation [9]. Due to this remarkable fact, we know that if we now solve (3.18) and (3.19) for this case, $Fe^S\Psi = 0$ will be satisfied (with this choice of parametrization and polarization) if and only if $\Psi$ satisfies the $W_3$-Ward identities. Again, $S$ is given by (3.20), and reads:

$$S = \frac{k}{2\pi} \int d^2z \left[-\frac{1}{2} A^{ij} \partial \alpha_i \bar{\partial} \alpha_j - \phi_i A^{ij} \bar{\partial} \alpha_j - \bar{\partial} \chi (\partial \phi_1 + \phi_2 - \phi_3) e^{2\alpha_1 - \alpha_2} + \mu((\partial - \phi_2) \phi_1 + \phi_2 \phi_1 - \phi_3) + \nu(\partial - \phi_2)(\phi_3 - \phi_2 \phi_1)\right], \quad (4.13)$$

where $\alpha_1 = \alpha, \alpha_2 = \beta$ and $A^{ij}$ is the Cartan matrix of $Sl(3, \mathbb{R})$, $A^{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \end{pmatrix}$.

This action is important if one wants to compute inner products of wave functions $\Psi$ in $Sl(3, \mathbb{R})$ Chern–Simons theory. This will be the topic of the next section.

5. The Inner Product and $W_3$ Gravity

The wave functions $\Psi[\mu, \nu] = \exp S_W[\mu, \nu]$ that solve the $W_3$-Ward identities, can be obtained from a constrained Wess-Zumino-Witten model [3] [12]. This means that at this stage we know the complete wavefunction $e^S\Psi$. These wave functions solve the holomorphic $W_3$-Ward identities, and can therefore be seen as effective actions of chiral $W_3$-gravity. However, in ordinary gravity there is a nontrivial coupling between the holomorphic and the anti-holomorphic sectors, and this is where part of the geometry of two-dimensional quantum gravity comes in. In this section we will consider the coupling between the holomorphic and anti-holomorphic sectors of $W_3$-gravity, hoping that it will lead to an understanding of the geometry underlying the $W_3$-algebra. This nontrivial coupling appears when computing inner products.
of wave functions in $\text{SL}(3,\mathbb{R})$ Chern–Simons theory. The expression for such an inner product is

$$\langle \Psi_1 | \Psi_2 \rangle = \int D(\Pi_i A) D(\Pi_k \bar{A}) D(\Pi_k^\dagger B) D(\Pi_k^\dagger \bar{B}) e^{V+S+\bar{S}} \Psi_1[\mu, \nu] \bar{\Psi}_2[\bar{\mu}, \bar{\nu}]. \tag{5.1}$$

The nontrivial coupling is due to the Kähler potential $V$, which is associated to the symplectic form defined by (2.3). To find an expression for this Kähler potential, we first give the definitions of $B$ and $\bar{B}$, the variables on which the anti-holomorphic wave function $\bar{\Psi}_2$ depends. Let $H$ be the subgroup of $\text{SL}(3,\mathbb{R})$ consisting of all elements $h \in \text{SL}(3,\mathbb{R})$ satisfying $h_{31} = h_{32} = 0$; $H$ can be conveniently parametrized by a Gauss decomposition. Define the connection $B_0$ by requiring that $(B_0)_3 = \bar{\nu}$ and $(B_0)_{\bar{3}} = \bar{\mu}$, and that $\Pi_i^\dagger F(B_0, \Lambda^t) = 0$, where $\Lambda^t$ is the transpose of $\Lambda$. Then:

$$B = hB_0 h^{-1} - \partial hh^{-1}, \tag{5.2}$$

$$\bar{B} = h\Lambda^t h^{-1} - \partial hh^{-1}, \tag{5.3}$$

and the anti-holomorphic action $\bar{S}$ is given by

$$\bar{S} = \frac{k}{2\pi} \int d^2 z \ Tr(\Pi_k^\dagger B \Pi_k \bar{B}) + \frac{k}{2\pi} \int d^2 z \ Tr(\Lambda^t h^{-1} \partial h) - \Gamma_{WZW}[h]. \tag{5.4}$$

In terms of $A$ and $B$, the Kähler potential is given by

$$V = \frac{k}{2\pi} \int d^2 z \ Tr(\Pi_i A \Pi_i^\dagger \bar{B} - \Pi_k \bar{A} \Pi_k^\dagger B). \tag{5.5}$$

The total exponent $K = V + S + \bar{S}$ occurring in the inner product (5.1) is now a function of $g$, $h$, and $\{\mu, \nu, \bar{\mu}, \bar{\nu}\}$. This "action" $K$ is part of the covariant action of $W_3$-gravity. The complete covariant action is given by $K + S_W[\mu, \nu] + \bar{S}_W[\bar{\mu}, \bar{\nu}]$, where $S_W[\mu, \nu]$ and $\bar{S}_W[\bar{\mu}, \bar{\nu}]$ are the chiral actions for $W_3$-gravity that were constructed in [12]. In the case of $\text{SL}(2,\mathbb{R})$, $K$ is equal to the Liouville action in a certain background metric, plus an extra term depending on $\mu, \bar{\mu}$ only. $K$ represents the $W_3$-analogon of the Quillen-Belavin-Knizhnik anomaly. Clearly, it will be interesting to have an explicit expression for it. One can work out such an explicit expression, by simply substituting all the expressions given above. The result of all this is a large, intransparant expression. However, upon further inspection, it turns out that $K$ is invariant under local $\text{SL}(2,\mathbb{R}) \times \mathbb{R}$ symmetry transformations, which can be used to gauge away four degrees of freedom. Actually, one can proof (see [9]) that $K$
only depends on the product $gh$. Using this the action can be greatly simplified by introducing a new Gauss decomposition for $gh$, which is now an arbitrary element of $Sl(3, \mathbb{R})$. More specific, we take

$$gh = \begin{pmatrix} 1 & 0 & 0 \\ \rho_1 & 1 & 0 \\ \rho_3 & \rho_2 & 1 \end{pmatrix} \begin{pmatrix} e^{\varphi_1} & 0 & 0 \\ 0 & e^{\varphi_2-\varphi_1} & 0 \\ 0 & 0 & e^{-\varphi_2} \end{pmatrix} \begin{pmatrix} 1 & -\rho_1 & -\rho_3 \\ 0 & 1 & -\rho_2 \\ 0 & 0 & 1 \end{pmatrix}. \quad (5.6)$$

Substituting this, one sees that the action depends simply quadratically on $\rho_3, \bar{\rho}_3$. Therefore, ignoring subtleties arising from the measure when changing variables from $A$ and $B$ to $gh$ and $\mu, \nu, \bar{\mu}, \bar{\nu}$, we can perform the $\int D\rho_3 D\bar{\rho}_3$ integration. The resulting action can be written in the following form:

$$K = \frac{k}{2\pi} \int d^2z \left\{ \frac{1}{2} A^{ij} \partial \varphi_i \bar{\partial} \varphi_j + \sum_i e^{-A^{ij} \varphi_j} - A^{ij} (\rho_i + \partial \varphi_i) (\bar{\rho}_j + \bar{\partial} \varphi_j) \right.$$

$$\left. - e^{\varphi_1-\varphi_2} (\mu - \frac{1}{2} \partial \nu - \nu \rho_1) (\bar{\mu} + \frac{1}{2} \bar{\partial} \bar{\nu} + \bar{\nu} \bar{\rho}_1) - e^{\varphi_1-\varphi_2} \nu \bar{\nu} \right.$$

$$\left. - e^{\varphi_2-\varphi_1} (\mu + \frac{1}{2} \partial \nu + \nu \rho_2) (\bar{\mu} - \frac{1}{2} \bar{\partial} \bar{\nu} - \bar{\nu} \bar{\rho}_2) + \mu T + \nu W + \bar{\mu} \bar{T} + \bar{\nu} \bar{W} \right\},$$

where we defined $T, W, \bar{T}, \bar{W}$ through the following Fateev-Lyukanov \cite{16} construction:

$$(\partial - \rho_2)(\partial - \rho_1 + \rho_2)(\partial + \rho_1) = \partial^3 + T \partial - W + \frac{1}{2} \partial T,$$

$$(\partial - \bar{\rho}_2)(\partial - \bar{\rho}_1 + \bar{\rho}_2)(\bar{\partial} + \bar{\rho}_1) = \bar{\partial}^3 + \bar{T} \bar{\partial} + \bar{W} + \frac{1}{2} \bar{\partial} \bar{T}, \quad (5.8)$$

and we shifted $\mu \to \mu - \frac{1}{2} \partial \nu$ and $\bar{\mu} \to \bar{\mu} + \frac{1}{2} \bar{\partial} \bar{\nu}$. The first part of $K$ is precisely a chiral $Sl(3)$ Toda action, confirming the suspected relation between $W_3$-gravity and Toda theory. Actually, one would expect that in a "conformal gauge", the covariant $W_3$-action will reduce to a Toda action. Indeed, if we put $\nu = \bar{\nu} = 0$ in $K$, then we find that $K$ is also purely quadratic in $\rho_1, \rho_2, \bar{\rho}_1, \bar{\rho}_2$. Performing the integrations over these variables as well, we find that

$$K[\varphi_1, \varphi_2, \mu, \bar{\mu}] = \frac{k}{4\pi} \int d^2z \sqrt{-\tilde{g}} \left( \frac{1}{2} \tilde{g}^{ab} \partial_a \varphi_i \partial_b \varphi_j A^{ij} + 4 \sum_i e^{-A^{ij} \varphi_j} + R \tilde{\xi} \cdot \bar{\varphi} \right) + K[\mu, \bar{\mu}], \quad (5.9)$$

where $K[\mu, \bar{\mu}]$ is the same expression as was derived in \cite{14}, namely

$$K[\mu, \bar{\mu}] = \frac{k}{\pi} \int d^2z \ (1 - \mu \bar{\mu})^{-1} (\partial \mu \bar{\partial} \bar{\mu} - \frac{1}{2} \mu (\bar{\partial} \bar{\mu})^2 - \frac{1}{2} \bar{\mu} (\partial \mu)^2), \quad (5.10)$$
and $\hat{g}$ is the metric given by $ds^2 = |dz + \mu d\bar{z}|^2$. In the case of $Sl(3, \mathbb{R})$, $\hat{\xi} \cdot \hat{\varphi}$, with $\hat{\xi}$ being one half times the sum of the positive roots, is just given by $\varphi_1 + \varphi_2$. The action (5.9) is the same Toda action that was originally present in $K$ in a chiral form, and the integration over $\rho_1, \bar{\rho}_1, \rho_2, \bar{\rho}_2$ has the effect of coupling it to a background metric $\hat{g}$.

Of course, the most interesting part of the action is the part containing $\nu, \bar{\nu}$. Unfortunately, if we do not put $\nu = \bar{\nu} = 0$, we can integrate over either $\rho_1, \bar{\rho}_1$ or over $\rho_2, \bar{\rho}_2$, but not over both at the same time, due to the presence of third order terms in $K$. Another clue regarding the contents of the action (5.8) can be obtained by treating the second and third line in (5.8) as perturbations of the first line of (5.8). This means that we try to make an expansion in terms of $\mu, \bar{\mu}, \nu, \bar{\nu}$. The saddlepoint of the $\rho$-terms is at $\rho_i = -\partial \varphi_i$ and $\bar{\rho}_i = -\bar{\partial} \varphi_i$. From (5.8) we can now see that $T, W, \bar{T}, \bar{W}$ are, when evaluated in this saddle point, the (anti)holomorphic energy momentum tensor and $W_3$-field that are present in a chiral Toda theory

$$T = -\frac{1}{2} A^{ij} \partial \varphi_i \partial \varphi_j - \hat{\xi} \cdot \partial^2 \varphi,$$
$$W = -\partial \varphi_1((\partial \varphi_2)^2 + \frac{1}{2} \partial^2 \varphi_2 - \partial^2 \varphi_1) + \frac{1}{2} \partial^3 \varphi_1 - (1 \leftrightarrow 2), \quad (5.11)$$

and similar expressions for $\bar{T}, \bar{W}$.

This suggests that the full action $K$ contains the generating functional for the correlators of the energy-momentum tensor and the $W_3$-field of a Toda theory, ”covariantly” coupled to $W_3$-gravity. The presence of the third order terms in $W, \bar{W}$ in (5.8) prevents us from computing the action of this covariantly coupled Toda theory.

Detailed proofs, that were omitted here, as well as generalizations to other $W$-algebras, will be the subjects of a future publication [9].

This work was financially supported by the Stichting voor Fundamenteel Onderzoek der Materie (FOM).
References

[1] H. Verlinde, Nucl. Phys. B337 (1990) 652.

[2] A. B. Zamolodchikov, Theor. Math. Phys, 65 (1985) 1205.

[3] M. Bershadsky and H. Ooguri, Comm. Math. Phys. 126 (1989) 49.

[4] M. Bershadsky, Comm. Math. Phys. 139 (1991) 71.

[5] S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, Nucl. Phys. B326 (1989) 108.

[6] A. M. Polyakov, Int. Journal of Mod. Phys. A5 (1990) 833.

[7] E. Witten, Comm. Math. Phys. 137 (1991) 29.

[8] E. Witten, ‘On Holomorphic Factorization of WZW and Coset Models,’ IASSNS-preprint, IASSNS-HEP-91/25 (June 1991).

[9] J. de Boer and J. Goeree, in preparation.

[10] A. Bilal, V. V. Fock and I. I. Kogan, ‘On the Origin of W-Algebras,’ CERN-preprint, CERN-TH 5965/90 (December 1990).

[11] A. Bilal, ‘W-Algebras from Chern-Simons Theory,’ CERN-preprint, CERN-TH 6145/91, LPTENS 91/17.

[12] H. Ooguri, K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, ‘The Induced Action of W3 Gravity,’ preprint, ITP-SB-91/16, RIMS-764 (June 1991).

[13] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Nucl. Phys. B349 (1991) 791, Phys. Lett. B243 (1991) 248.

[14] E. Bergshoeff, C. N. Pope, L. J. Romans, E. Sezgin, X. Shen and K. S. Stelle, Phys. Lett. B243 (1991) 330.

[15] C. M. Hull, Nucl. Phys. B353 (1991) 707.

[16] V. Fateev and S. Lukyanov, Int. Journal of Mod. Phys. A3 (1988) 507.