On the U(1)-Problem of QED$_2$

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Abstract

QED$_2$ with mass and flavor has in common many features with QCD, and thus is an interesting toy model relevant for four dimensional physics. The model is constructed using Euclidean path integrals and mass perturbation series. The vacuum functional is carefully decomposed into clustering states being the analogue of the $\theta$-vacuum of QCD. Finally the clustering theory can be mapped onto a generalized Sine-Gordon model. Having at hand this bosonized version, several lessons on the $\theta$-vacuum, the U(1)-problem and Witten-Veneziano-type formulas will be drawn. This sheds light on the corresponding structures of QCD.

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1 Introduction

QED$_2$ is a model which has many features in common with (4-dimensional) QCD. U(1) gauge fields in two dimensions allow for topologically nontrivial configurations which are the counterparts of the instantons of 4-dimensional Yang-Mills theory. Thus the formal linear combination of such sectors to the $\theta$-vacuum of QCD can be repeated for QED$_2$. Furthermore the U(1)-axial current acquires an anomaly and the corresponding pseudoscalar particle turns out to be massive as it is the case for QCD. The quoted features allow to take over the formal reasoning that leads to the formulation of the U(1)-problem.

The purpose of this work is to construct QED$_2$ with mass and flavor and study the announced topics ($\theta$-vacua, U(1)-problem) without making use of poorly defined concepts such as the formal superposition of topologically nontrivial sectors to the $\theta$-vacuum or the definition of non-gauge invariant currents (as it is done in the standard formulation of the U(1)-problem).

Clustering $\theta$-vacua will be defined using a mathematically rigorous limit procedure which replaces the formal instanton arguments. The properties of this vacuum state will be compared to the $\theta$-vacuum of QCD. Using the method of bosonization we investigate the axial U(1)-symmetry on the physical Hilbert space and discuss the status of the U(1)-problem. Finally Witten-Veneziano-type formulas that relate the masses of the pseudoscalars to the topological susceptibility will be tested.

2 The model and techniques applied

The Euclidean action of the model we consider is given by $S = S_G + S_h + S_F + S_M$. The gauge field action reads

$$S_G[A] = \int d^2 x \left[\frac{1}{4} F_{\mu\nu}(x) F_{\mu\nu}(x) + \frac{1}{2} \lambda \left( \partial_\mu A_\mu(x) \right)^2 \right].$$  (1)

A gauge fixing term is included that will be considered in the limit $\lambda \to \infty$ which ensures $\partial_\mu A_\mu = 0$ (transverse gauge). As usual, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, denotes the field strength tensor. The fermion action is a sum over N flavor degrees of freedom

$$S_F[\bar{\psi}, \psi, A, h] = \sum_{f=1}^{N} \int d^2 x \, \bar{\psi}^{(f)}(x) \gamma_\mu \left( \partial_\mu - i e A_\mu(x) - i \sqrt{g} h_\mu(x) \right) \psi^{(f)}(x).$$  (2)

Since the mass term will be treated perturbatively we denote it separately

$$S_M[\bar{\psi}, \psi] = - \sum_{f=1}^{N} m^{(f)} \int d^2 x \, \chi_\Lambda(x) \bar{\psi}^{(f)}(x) \psi^{(f)}(x).$$  (3)
$m^{(f)}$ are the fermion masses for the various flavors. In order to prove the convergence of the mass perturbation series for more than one flavor, the fields in the mass term have to be cut off outside some finite rectangle $\Lambda$ in space-time. $\chi_{\Lambda}$ denotes the characteristic function of $\Lambda$.

In addition to the gauge field an auxiliary vector field $h_\mu$ is coupled to the fermions (compare Equation (2)). Its action is given by

$$S_h[h] = \frac{1}{2} \int d^2 x h_\mu(x) \left( \delta_{\mu\nu} - \lambda' \partial_\mu \partial_\nu \right) h_\nu(x).$$

$S_h[h]$ is simply a white noise term plus a term that makes $h_\mu$ transverse in the limit $\lambda' \to \infty$. When integrating out $h_\mu$ it generates a Thirring term for the $U(N)$ flavor singlet current. The purpose of this Thirring term is to make the short distance singularity of $\bar{\psi}^{(f)}(x)\psi^{(f)}(x)\bar{\psi}^{(f)}(y)\psi^{(f)}(y)$ integrable. This expression is a typical term showing up in a power series expansion of the mass term (3). It has to be integrated over $d^2 x d^2 y$ which is possible only if an ultraviolet regulator such as the Thirring term is included.

The approach we adopted is the quantization via Euclidean functional integrals. After having expanded the exponential of the mass term of the action $\exp(-S_M[\bar{\psi}, \psi])$, vacuum expectation values of operators $O$ are given by

$$\langle O[\bar{\psi}, \psi, A, h] \rangle = \frac{1}{Z} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle O[\bar{\psi}, \psi, A, h] \left( S_M[\bar{\psi}, \psi] \right)^n \rangle_0,$$

where the expectation values of the massless model are formally defined as

$$\langle P \rangle_0 = Z_0^{-1} \int DhDA D\bar{\psi} D\psi \mathcal{P} e^{-S_\mathcal{P} - S_h - S_F}.$$

Of course also the normalization constant $Z$ showing up in (5) has to be expanded in terms of the massless theory.

When giving a precise mathematical meaning to the so far poorly defined functional integral one usually starts with integrating out the fermions. This gives rise to the fermion determinant, which can be constructed properly when the fermions are massless (this is the reason why the mass term is treated perturbatively). The renormalized determinant can be found in e.g. [4] and reads (the potentials $A_\mu, h_\mu$ are assumed to be transverse and to satisfy some mild regularity and falloff conditions at infinity) $\det_{\text{ren}}[A_\mu, h_\mu] = \exp \left( (2\pi)^{-1} \| e A^T + \sqrt{g} h^T \|_2^2 \right)$, where the superscript $T$ denotes the transverse part of the vector fields. Adding the exponent of this expression to the action one ends up with the effective action for the gauge field and the auxiliary field. Both those terms are Gaussian and the vacuum expectation values obtain a precise mathematical meaning through Gaussian functional integrals. Since the dependence of the propagators on the external fields is exponential the functional integrals can be computed.
The vacuum expectation functional constructed for the massless model so far violates the cluster decomposition property. In particular it turns out (see e.g. [5]) that operators which are singlets under $U(1)_V \times SU(N)_L \times SU(N)_R$, but transform nontrivially under $U(1)_A$, do not cluster. This implies that the vacuum state is not unique. In order to obtain a proper vacuum, we use a limit process [5]. The original operator $\mathcal{P}$ is correlated with a test operator $\mathcal{U}_\tau(\mathcal{P})$ depending on the chiral charge $Q_5(\mathcal{P})$ of $\mathcal{P}$ and then the limit of shifting the test operator to timelike infinity is considered.

$$\langle \mathcal{P}((x)) \rangle^\theta_0 := e^{iQ_5(\mathcal{P})/2N} \lim_{\tau \to \infty} \langle \mathcal{U}_\tau(\mathcal{P}) \mathcal{P}((x)) \rangle_0.$$  (6)

The set of ‘test operators’ $\mathcal{U}_\tau(\mathcal{B})$ is defined by

$$\mathcal{U}_\tau(\mathcal{P}) := \begin{cases} \mathcal{N}^{(n)}(\{y\}) \prod_{i=1}^n O_+(\{y + \hat{\tau}\}) & \text{for } Q_5(\mathcal{P}) = \pm 2nN, \ n \geq 1, \\ 1 & \text{otherwise} \end{cases},$$  (7)

where $O_\pm(\{y\}) := \prod_{i=1}^N \psi^{(f)}(y^{(f)}) \frac{1}{2}[1 \pm \gamma_5] \psi^{(f)}(y^{(f)})$. The space-time arguments $y^{(f)}$ are arbitrary and the left hand side of (7) does not depend on them due to a normalizing factor $\mathcal{N}^{(n)}(\{y\})$. $\theta$ is a real parameter in the range $[0, 2\pi)$. For this decomposition prescription the following theorem was proven [5].

**Theorem 1.**

i) The cluster decomposition property holds for $\langle \ldots \rangle^\theta_0$.

ii) The state $\langle \ldots \rangle_0$ constructed initially is recovered by averaging over $\theta$, and thus is a mixture of the pure states $\langle \ldots \rangle^\theta_0$.

The decomposition (6) replaces the mathematically poorly defined concept of superimposing topological sectors to the $\theta$-vacuum [4].

In two dimensions one has at hand the powerful tool of bosonization. It will be used to construct the massive model. By evaluating a generating functional including vector currents and chiral densities one can establish the following bosonization prescriptions

$$J^{(I)}_{\mu}(x) \leftrightarrow \begin{cases} \frac{1}{\sqrt{\pi + gN}} \varepsilon_{\mu\nu} \partial_\mu \Phi^{(1)}(x) & I = 1 \\
\frac{1}{\sqrt{\pi}} \varepsilon_{\mu\nu} \partial_\mu \Phi^{(I)}(x) & I = 2, \ldots, N. \end{cases}$$  (8)

The currents are defined as $J^{(I)}_{\mu} := \sum_{f,f'} H^{(f)}_{f,f'} \gamma_\mu \psi^{(f)} \psi^{(f')}$. The mixing matrices $H^{(I)}$ in flavor space are chosen to be proportional to the unit matrix for $H^{(1)}$, and proportional to the generators of a Cartan subalgebra of $SU(N)$.
for $H^{(I)}$, $I = 2, 3, \ldots N$ (see [3] for the explicit definition). The chiral densities are bosonized by
\[
\frac{1}{2\pi} c^{(f)} : e^{\pm i2\sqrt{\pi+g N} \Phi^{(1)}(x)} :_{M(I)} \prod_{I=2}^{N} : e^{\pm i2\sqrt{\pi+g N} U_{1f} \Phi^{(I)}(x)} :_{M(I)} e^{\pm i \Phi^1}.
\]

$U_{1f}$ are some real factors that are related to the entries of the matrices $H^{(I)}$. Normal ordering of the exponentials with respect to some reference mass $M$ (see e.g. [3]) is denoted by $\ldots :_{M}$.

The Lagrangian of the bosonic theory was obtained by bosonizing the terms of the mass expansion (5) and then summing up the perturbation series. One ends up with the following generalized Sine Gordon model
\[
\mathcal{L}_{GSG} = \frac{1}{2} \sum_{I=1}^{N} \partial_{\mu} \Phi^{(I)} \partial^{\mu} \Phi^{(I)} + \frac{1}{2} (\Phi^{(1)})^2 \frac{e^2 N}{\pi + g N} - \frac{1}{\pi} \sum_{f=1}^{N} m^{(f)} c^{(f)} : \cos \left(2 \frac{\pi}{\sqrt{\pi+g N}} U_{1f} \Phi^{(1)} + 2 \sqrt{\pi} \sum_{I=2}^{N} U_{1f} \Phi^{(I)} - \frac{\theta}{N} \right) :. \quad (10)
\]

Normal ordering of the cosine is understood in the sense of the mass-expansion and thus reduces to the normal ordering of exponentials. It has to be stressed that this normal ordering is essential for the correct bosonization of the clustering states $\langle \ldots \rangle^\theta$. In particular the normal ordered exponentials of the massless fields $\Phi^{(I)}$, $I \geq 2$ have to obey the neutrality condition [4], to give nonvanishing expectation values. This ensures the correct assignment of expectation values to operators with nonvanishing chiral charge. A model similar to (10) but without Thirring term was discussed in [7] using an operator approach.

Using the bosonized version of the model we were able to prove the following theorem [3] that generalizes the N=1 proof by Fröhlich [6].

**Theorem 2.**

*The mass perturbation series converges for finite $\Lambda$ and $m^{(b)} < r(\Lambda)$.*

Due to the massless degrees of freedom the applied techniques only allowed to establish a radius of convergence $r(\Lambda)$ which shrinks to zero when sending $\Lambda$ to infinity. We believe that the proof can be made better (i.e. $r$ becomes independent from $\Lambda$) by applying some generalized cluster expansion methods.

This concludes the construction of the model and one can start to discuss its physical implications.
3 Lessons on the U(1)-problem

The following three lessons formulate the physically interesting results that were obtained for QED$_2$, and shed light on the corresponding problems in QCD.

Lesson 1 : The structure of the vacuum functional that has been suggested within the instanton picture is recovered from the mathematically rigorous construction given in formula (6).

In particular only operators with chirality $2N\nu$, $\nu \in \mathbb{Z}$ have nonvanishing vacuum expectation values. This can be seen immediately from the prescription (6) for the $\theta$-vacuum. The same has been claimed by 't Hooft for QCD [8]. Furthermore an analysis of the Lagrangian (10) shows that as long as one of the fermion masses $m^{(f)}$ vanishes physics is independent of $\theta$ (see [1] for the case of QCD).

Lesson 2 : The axial U(1)-symmetry is not a symmetry on the physical Hilbert space, and there is no U(1)-problem for QED$_2$.

Under a U(1)-axial transformation all the chiral densities $\bar{\psi}^{(f)} (1 \pm \gamma_5)/2\psi^{(f)}$ obtain phases $\exp(\pm i2\epsilon)$. This transformation property is not compatible with the structure of the coefficients $U_{ij}$ that show up in the bosonization prescription (9) (see [4] for the explicit proof). Thus even in the case of vanishing fermion masses $m^{(f)}$ the axial U(1)-symmetry is not realized on the physical Hilbert space, and thus the U(1)-problem is not there at all. The same could be true for QCD, since the current that is used to construct the axial symmetry is not gauge invariant.

Lesson 3 : The masses of the particles that correspond to the vector currents obey a Witten-Veneziano type formula.

This result was established using a semiclassical approximation of the Lagrangian (10). The approximation is necessary due to the problem with removing the cutoff $\Lambda$. Nevertheless this approximation is under good control, since for small fermion masses $m^{(f)}$ it reduces to the exact solution.

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