Restrictions on Dirac mass matrix by approximate chiral ($Z_2$) symmetry associated with the lightest neutrino in the type-I seesaw mechanism

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In this letter, we investigate an approximate $U(1)_L$ lepton number symmetry associated with the lightest neutrino mass $m_{1\, or\, 3}$ in the type-I seesaw mechanism. If $U(1)_L$ is exact and the mass matrix of right-handed neutrinos $M_R$ is regular $\det M_R \neq 0$, the Dirac mass matrix $m_D$ has a chiral ($Z_2$) symmetry $S m_D = m_D$ due to a left-handed lepton number. Interestingly, this constraint is independent of the form and basis of $M_R$.

We search how this condition can be relaxed by considering a chiral perturbative analysis for $m_{1\, or\, 3}$. As a result, flavor structures of the first and second generation must be approximately orthogonal to the eigenvector of the lightest neutrino, and two complex parameters for $m_D$ are directly bounded from the eigenvector.

I. INTRODUCTION

The type-I seesaw mechanism is one of the key topics in the study of lepton mixing and neutrino mass matrix. While studies based on flavor symmetries have been actively discussed, there are also many papers on textures that explore the structure of mass matrices themselves. In particular, in the (constrained) sequential dominance, the smallness of the lightest neutrino mass $m_{1\, or\, 3}$ is due to the mass of the heaviest right-handed neutrino $M_3$.

A $U(1)_L$ lepton number symmetry is restored in the limit of zero mass for the lightest neutrino. Relations between such a massless neutrino and flavor symmetries have also been studied to some extent. However, the role of the $U(1)_L$ symmetry in the type-I seesaw mechanism has not been discussed previously.

Thus, in this letter, we investigate restrictions on flavor structures due to an approximate

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lepton number symmetry associated with the lightest neutrino in the type-I seesaw mechanism. It is found that the Dirac mass matrix \( m_D \) has an approximate chiral symmetry \( S m_D \simeq m_D \) due to a left-handed lepton. Since this chiral symmetry holds only for \( m_D \) and is not necessarily a symmetry of the entire theory, it is treated as a kind of remnant symmetry. Chiral perturbative analysis for finite \( m_{1\alpha 3} \) shows that flavor structures of the first and second generation are directly bounded from the eigenvectors of the lightest neutrino mass.

II. THE MASSLESS LIGHTEST NEUTRINO AND \( U(1)_L \) LEPTON NUMBER SYMMETRY

The Dirac mass matrix \( m_D \) and the symmetric Majorana mass matrix \( M_R \) of the right-handed neutrinos \( \nu_Ri \) are defined as

\[
m_D = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} \equiv (A, B, C), \quad M_R = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{pmatrix}.
\]

(1)

These matrix elements \( A_i, B_i, C_i \) and \( M_{ij} \) are general complex parameters. In the type-I seesaw mechanism, the mass matrix of light neutrinos \( m_\nu \) is given by

\[
m_\nu = m_D M_R^{-1} m_D.
\]

(2)

To discuss the situation where the lightest neutrino mass \( m_{1\alpha 3} \) is very light, we first consider the massless limit \( m_{1\alpha 3} = 0 \). The singular value decomposition of \( m_\nu \) is defined as

\[
m_\nu = U m_{\text{diag}} U^T \equiv (u, v, w) \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (u, v, w)^T,
\]

(3)

where \( \{u, v, w\} \) is the three-dimensional orthonormal basis that constitutes the unitary matrix \( U \).

The massless lightest neutrino \( m_{1\alpha 3} = 0 \) leads to the normal hierarchy (NH) or the inverted hierarchy (IH), respectively. For example, in the case of \( m_1 = 0 \), the diagonalized mass matrix
$m^\text{diag}_\nu$ has the following $U(1)_L$ lepton number symmetry for the massless direction $[20]$

$$(m^\text{diag}_\nu)' = R_1 m^\text{diag}_\nu R_1 \equiv \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & m_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ e^{i\theta} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = m^\text{diag}_\nu, \quad (4)$$

where $R_1$ is a phase matrix. Returning to the basis of original $m_\nu$ by $U$,

$$UR_1 U^\dagger m_\nu U^* R_1 U^T = m_\nu. \quad (5)$$

This $UR_1 U^\dagger$ is a unitary matrix such that

$$UR_1 U^\dagger = e^{i\theta} u \otimes u^\dagger + v \otimes v^\dagger + w \otimes w^\dagger = 1 - (1 - e^{i\theta}) u \otimes u^\dagger. \quad (6)$$

Similarly, for $m_3 = 0$, there is a $U(1)_L$ symmetry by $UR_3 U^\dagger = 1 - (1 - e^{i\theta}) w \otimes w^\dagger$.

The condition that $m_\nu$ is invariant under the symmetry can be rewritten as

$$(1 - P + e^{i\theta} P) m_\nu (1 - P + e^{i\theta} P)^T = m_\nu, \quad (7)$$

where $P = u \otimes u^\dagger$ is a projection to the eigenvector of the massless mode $u = u^\text{NH}$ or $u^\text{IH}$. By separating the symmetry condition with and without the dependency of $\theta$,

$$(1 - P) m_\nu (1 - P)^T = m_\nu, \quad (8)$$

$$e^{i\theta} (1 - P) m_\nu P^T + e^{i\theta} P m_\nu (1 - P)^T + e^{2i\theta} P m_\nu P^T = 0. \quad (9)$$

Adding and subtracting these two conditions lead to

$$(1 - P) m_\nu = m_\nu (1 - P)^T = m_\nu, \quad P m_\nu = m_\nu P^T = 0. \quad (10)$$

In other words, a massless mode associated with $u$ appears if $m_\nu$ does not have a projective component of this direction. In particular, from $u \neq 0$,

$$m_\nu P^T = m_\nu u^* \otimes u^T = 0 \iff m_\nu u^* = 0. \quad (11)$$

In the type-I seesaw mechanism, this condition becomes

$$P m_D M_R^{-1} m_D^T = m_D M_R^{-1} m_D^T P^T = 0. \quad (12)$$
The existence of $M_R^{-1}$ with $\det M_R \neq 0$ implies $P m_D = 0$ and $m_D^T u^* = 0$. That is, $m_D$ also respects the chiral $U(1)$ symmetry due to the left-handed lepton number\(^1\),

$$(1 - P + e^{i\theta} P) m_D = m_D,$$  

and $m_D$ has no projection in the massless direction. Since the chiral symmetry holds only for $m_D$ and is not necessarily a symmetry of the entire theory, it is treated as a kind of remnant symmetry.

The symmetry is also confirmed by the natural representation by Ref. [25]. In the diagonal basis of $M_R$, the neutrino mass matrix $m_\nu$ is\(^2\)

$$m_\nu = m_D M_R^{-1} m_D^T = \frac{1}{M_1} A \otimes A^T + \frac{1}{M_2} B \otimes B^T + \frac{1}{M_3} C \otimes C^T.$$  

The vectors $A, B, C$ are linearly dependent from $m_\nu u^* = 0$. Since the remaining two neutrino masses $m_i$ must be finite, the rank of $m_D$ must be two. If $A$ and $B$ are linearly independent, the cross product $A \times B$ is proportional to the eigenvector $u^*$ and $m_\nu (A \times B) = 0$ is satisfied.

The relation between eigenvectors of $m_\nu$ and the residual $Z_2$ symmetry has been well discussed\(^{28-31}\). For $m_D$ to satisfy the chiral symmetry (13), it is sufficient to have this subgroup of a chiral $Z_2$ symmetry with $\theta = \pi$;

$$S m_D = +m_D, \quad S = 1 - 2 u \otimes u^\dagger, \quad u \equiv \frac{(A \times B)^*}{|A \times B|}.$$  

This fact is also discussed in Refs. [29] and the anti-symmetric condition $S m_D = -m_D$ is unsuitable because the rank of $m_D$ becomes unity. Since it is obvious that this chiral $Z_2$ symmetry reduces the rank of $m_D$ by one, this symmetry is a necessary and sufficient condition for the massless lightest neutrino.

Interestingly, this condition gives constraints only for $m_D$ and does not depend on $M_R$. For example, for approximate massless vectors $u^{NH} = \frac{1}{\sqrt{6}} (2, -1, -1)^T$ and $u^{HI} = \frac{1}{\sqrt{2}} (0, 1, -1)^T$, generators of the symmetry $S_i$ are

$$S_1 \equiv 1 - 2 u^{NH} \otimes (u^{NH})^\dagger = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix}, \quad S_2 \equiv 1 - 2 u^{HI} \otimes (u^{NH})^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (16)$$

\(^1\) The anti-symmetry $(1 - P + e^{i\theta} P) m_D = -m_D$ has only the trivial solution $m_D = 0$.

\(^2\) Similar arguments can be made by using the $LDL^T$ decomposition without the root sign [26, 27].
Note that $S_3$ generates a chiral $\mu - \tau$ symmetry \cite{33,34}. For each of these, matrices $m_D$ with chiral $Z_2$ symmetry are

$$m_D^{(NH)} = \begin{pmatrix} A_2 + A_3 & B_2 + B_3 & C_2 + C_3 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}, \quad m_D^{(IH)} = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_2 & B_2 & C_2 \end{pmatrix},$$

and the symmetry fix a row of $m_D$. Since $m_D^{(NH,IH)}$ has no $u^{(NH,IH)}$ component, this situation would be realized by a linear combination of the vacuum expectation values of the following flavons;

$$\langle \phi_1 \rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad \langle \phi_2 \rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \phi_3 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}. \quad (18)$$

This fact indicates that a simple unification with the up-type Yukawa matrix $Y_u$ with small mixings is likely to be difficult because $m_D$ with exact chiral $Z_2$ symmetry have similar sizes of two elements in a certain column.

**III. CHIRAL PERTURBATION THEORY FOR THE LIGHTEST MASS $m_{1,3}$**

The finite lightest mass $m_{1,3}$ makes Eq. (13) an approximate chiral lepton number symmetry. Then, let us survey to how the condition (17) for $m_D$ is relaxed by a perturbatively light $m_{1,3}$ (i.e., “chiral perturbation theory” is performed for $m_\nu$).

In the current situation, there are two possibilities for the massless mode:

1. In the limit of $M_3 \to \infty$.

2. $C$ is linearly dependent on $A$ and $B$.

If given $A$ and $B$ are linearly independent, $m_D$ and $m_\nu$ (14) can be divided as follows;

$$m_D = m_{D0} + \delta m_D \equiv (A, B, C_0) + (0, 0, \delta C),$$

$$(A \times B)^T C_0 = A^\dagger \delta C = B^\dagger \delta C = 0,$$

$$m_\nu = (m_{D0} + \delta m_D)M_R^{-1}(m_{D0} + \delta m_D)^T \equiv m_{\nu 0} + \delta m_\nu + \delta^2 m_\nu$$

$$\equiv m_{D0}M_R^{-1}m_{D0}^T + \frac{1}{M_3}(C_0 \otimes \delta C^T + \delta C \otimes C_0^T + \delta C \otimes \delta C^T). \quad (21)$$
Note that in Eq. (20) $A, B$ and $(A \times B)^*$ form a basis for the Hermitian inner product. It is reasonable to treat $\delta C$ as symmetry-breaking parameters because $\det m_D = (A \times B)^T \delta C$ holds. The unitary matrix $U$ that diagonalizes $m_\nu$ is divided as

$$U = U_0 + \delta U \equiv (u_0, v_0, w_0) + (\delta u, \delta v, \delta w),$$

where $u_0 \propto (A \times B)^*$ and $v_0, w_0$ are linear combinations of $A$ and $B$. The second order perturbation $\delta^2 U$ is not considered because it does not contribute to the lowest-order calculation.

The singular value decompositions are given by

$$U_0^\dagger m_{\nu 0} U_0^* = m_\nu^{\text{diag}}, \quad U_0^\dagger m_\nu U_0^* = m_\nu^{\text{diag}} + \delta m_\nu^{\text{diag}} + \delta^2 m_\nu^{\text{diag}}. \quad (23)$$

The diagonalization of the first-order perturbation to $m_\nu m_\nu^\dagger$ is

$$(U_0 + \delta U)^\dagger (m_{\nu 0} + \delta m_{\nu})(m_{\nu 0}^\dagger + \delta m_{\nu}^\dagger)(U_0 + \delta U) = (m_{\nu 0}^{\text{diag}})^2 + 2m_{\nu 0}^{\text{diag}} \delta m_{\nu}^{\text{diag}} + (\delta m_{\nu}^{\text{diag}})^2. \quad (24)$$

By using the diagonalization (23) and the orthogonality relation $\delta U^\dagger U_0 + U_0^\dagger \delta U = 0$,

$$U_0^\dagger m_{\nu 0} m_{\nu 0}^\dagger \delta U + \delta U^\dagger m_{\nu 0} m_{\nu 0}^\dagger U_0 + U_0^\dagger m_{\nu 0} \delta m_{\nu 0}^\dagger U_0 + U_0^\dagger \delta m_{\nu 0} m_{\nu 0}^\dagger U_0 = 2m_{\nu 0}^{\text{diag}} \delta m_{\nu}^{\text{diag}}, \quad (25)$$

$$= - (m_{\nu 0}^{\text{diag}})^2 \delta U^\dagger U_0 + \delta U^\dagger U_0 (m_{\nu 0}^{\text{diag}})^2 + m_{\nu 0}^{\text{diag}} U_0^T \delta m_{\nu 0}^\dagger U_0 + U_0^\dagger \delta m_{\nu 0} U_0^* m_{\nu 0}^{\text{diag}}. \quad (26)$$

For the diagonal element of Eq. (22),

$$(U_0^T \delta m_{\nu} U_0 + U_0^\dagger \delta m_{\nu}^\dagger U_0^*)_{ii} = 2(\delta m_{\nu}^{\text{diag}})_{ii}. \quad (27)$$

In particular, $(\delta m_{\nu}^{\text{diag}})_{11} = 0$ holds and the lightest mass comes from the second order perturbation. For the off-diagonal element of Eq. (26) with $i \neq j$,

$$-m_{0i}^2 (\delta U^\dagger U_0)_{ij} + (\delta U^\dagger U_0)_{ij} m_{0j}^2 + m_{0i} (U_0^T \delta m_{\nu}^\dagger U_0)_{ij} + (U_0^\dagger \delta m_{\nu} U_0^*)_{ij} m_{0j} = 0, \quad (28)$$

where $m_{0i} = (m_{\nu 0}^{\text{diag}})_{i}$. Thus, we obtain

$$(U_0^\dagger \delta U)_{ij} = - \frac{m_{0i} (U_0^T \delta m_{\nu}^\dagger U_0)_{ij} + (U_0^\dagger \delta m_{\nu} U_0^*)_{ij} m_{0j}}{m_{0i}^2 - m_{0j}^2} \quad (29)$$

From $U_0^\dagger (U_0 + \delta U) = 1 + U_0^\dagger \delta U$, it represents a perturbative transformation in the diagonalized basis of $m_{\nu 0}^{\text{diag}}$.

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3 Even if we consider $\delta^2 U$, terms like $U_0 m_{\nu 0} \delta^2 U$ do not contribute to $(\delta^2 m_{\nu}^{\text{diag}})_{11}$. 
What needs to be investigated is how the constraint (17) is shifted by the chiral symmetry breakings. For this purpose, the deviation $\delta u$ for the massless direction $u_0^* \propto A \times B$ is evaluated. Considering NH for simplicity and substituting $j = 1$ and $m_{01} = 0$ in Eq. (29), we obtain a perturbation to the massless direction as

$$(U_0^\dagger \delta U)_{i1} = -\frac{(U_0^T \delta m_0^\dagger U_0)_{i1}}{m_{0i}} = -\frac{1}{M_3 m_{0i}}(U_0^T C_0^*)_{i} \otimes (\delta C^\dagger U_0)_{1}. \quad (30)$$

In components,

$$v_0^\dagger \delta u = -\frac{C_v^* C_u^*}{M_3 m_{02}}, \quad w_0^\dagger \delta u = -\frac{C_w^* C_u^*}{M_3 m_{03}}, \quad (31)$$

where $C_v \equiv v_0^\dagger C = v_0^\dagger C_0$ and $C_u \equiv u_0^\dagger C = u_0^\dagger C$ by Eq. (20). A similar notation is used for $w_0$. If there is no fine-tuning between the sums in $m_{0i}$, $C_v$ and $C_w$ have upper bound about $\sqrt{m_{02} M_3}$ and $\sqrt{m_{03} M_3}$ from the zero-order diagonalization $U_0^\dagger m_D M_R^{-1} m_D^T U_0^* = m_{0i}^{\text{diag}}$. Since we will see later $C_u \sim \sqrt{m_1 M_3}$, these perturbations are suppressed by at least $\sqrt{m_1/m_2}$ and $\sqrt{m_1/m_3}$ respectively.

We examine changes of the constraint for $m_D$ (17) by the perturbed eigenvector of the lightest mode. For simplicity, let $U_0$ be the tri-bi-maximal mixing (37) and ignore the contribution of $w_0^\dagger \delta u$ in Eq. (31) by $m_3 \sim 6 m_2$. The eigenvector $u$ actually observed is

$$u = u_0 + \delta u \simeq \frac{1}{\sqrt{6}} \left( \begin{array}{c} 2 \\ -1 \\ -1 \end{array} \right) - \frac{C_v^* C_u^*}{M_3 m_{02}} \frac{1}{\sqrt{3}} \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right). \quad (32)$$

Conversely this means that $A$ and $B$ have $u$ (and $v_0$) components that are suppressed by at least $O(\sqrt{m_1/m_2})$;

$$u^\dagger A \simeq -\frac{C_v C_u}{M_3 m_{02}} v^\dagger A, \quad u^\dagger B \simeq -\frac{C_v C_u}{M_3 m_{02}} v^\dagger B. \quad (33)$$

Next, to estimate the lightest mass $m_{1\alpha 3}$, the perturbation theory proceeds to the second order. For simplicity, we directly consider a diagonalization of $m_\nu$ instead of $m_\nu m_\nu^\dagger$;

$$(U_0 + \delta U)^\dagger (m_\nu + \delta m_\nu + \delta^2 m_\nu)(U_0 + \delta U)^* = m_{0i}^{\text{diag}} + \delta m_\nu^{\text{diag}} + \delta^2 m_\nu^{\text{diag}}, \quad (34)$$

$$\delta U^\dagger m_\nu U_0 + m_\nu^\dagger \delta m_\nu U_0^* + U_0^\dagger \delta m_\nu U_0^* = \delta m_\nu^{\text{diag}}, \quad (35)$$

$$U_0^\dagger \delta^2 m_\nu U_0^* + U_0^\dagger \delta m_\nu \delta U^* + \delta U^\dagger m_\nu \delta U^* + \delta U^\dagger \delta m_\nu U_0^* = \delta^2 m_\nu^{\text{diag}}. \quad (36)$$
The sum of Eq. (35) and its complex conjugate is equivalent to Eqs. (27) and (29). By substituting Eq. (35) into Eq. (36) and eliminating $\delta m_\nu$,

\[ U_0^\dagger \delta^2 m_\nu U_0^* + (\delta m_\nu^\text{diag} - m_{\nu0}^\text{diag} U_0^T \delta U^*) U_0^T \delta U^* \\
- \delta U^\dagger U_0 m_{\nu0}^\text{diag} U_0^T \delta U^* + \delta U^\dagger U_0 (\delta m_\nu^\text{diag} - \delta U^\dagger U_0 m_{\nu0}^\text{diag}) = \delta^2 m_\nu^\text{diag}. \] (37)

Substituting $(m_{\nu0}^\text{diag})_{11} = (\delta m_\nu^\text{diag})_{11} = 0$ to the 1-1 element, we obtain

\[ m_1 = (U_0^\dagger \delta^2 m_\nu U_0^*)_{11} - \delta U^\dagger U_0 m_{\nu0}^\text{diag} U_0^T \delta U^* \]

\[ = (U_0^\dagger \delta^2 m_\nu U_0^*)_{11} - \sum_i (U_0^\dagger \delta m_\nu U_0^*)_{1i} m_{0i}^\dagger (U_0^\dagger \delta m_\nu U_0^*)_{1i} m_{0i}. \] (38)

Finally, $\delta m_\nu$ and $\delta^2 m_\nu$ in Eq. (21) yields $(U_0^\dagger \delta m_\nu U_0^*)_{1i} = \frac{i}{M_3} (u_0^\dagger \delta C) \cdot (C_0^T (u_0^*, v_0^*, w_0^*))_i$ and the lightest mass is found to be

\[ m_1 = \frac{C_u^2}{M_3} [1 - \frac{C_v^2}{m_{02}} - \frac{C_w^2}{m_{03}}]. \] (39)

The singular value is strictly the absolute value of this expression. Intuitively, Eq. (39) agrees with an approximate evaluation of the perturbative contribution like the seesaw mechanism in the diagonal basis of $m_{\nu0}$. Since $C_v$ and $C_w$ has an upper bound around $\sqrt{m_{02}M_3}$ and $\sqrt{m_{03}M_3}$, $C_u \sim \sqrt{m_1M_3}$ holds. In particular, $m_1$ is proportional to $\det m_2^2/M_3$ because of $C_u = u^\dagger \delta C \propto \det m_D$.

Moreover, $|C_1| \ll |C_2| \ll |C_3|$ is expected in many unified theories. If mixings in $U_0$ are as large as the MNS matrix in this hierarchical basis, a rough expression $C \sim (0, 0, 1)^T$ leads to $|C_u| \sim |C_v| \sim |C_w| \sim \sqrt{m_1M_3}$. Therefore, $m_1$ is approximately equal to the first term $C_u^2/M_3$ in Eq. (39) and the order of $\delta u$ in Eq. (31) is changed to $m_1/m_{2,3}$ instead of $\sqrt{m_1/m_{2,3}}$. The condition that perturbations are sufficiently small in such a model is

\[ m_1^{\text{NH}} \lesssim 1 \text{ meV}, \quad m_3^{\text{IH}} \lesssim 5 \text{ meV}. \] (40)

In this case, corrections (33) to the constraints for the Dirac mass matrix (17) is about 10% or less for the lighter (the first and second) generations.

In such models with large $C_3 = (m_D)_{33}$, although the lepton number symmetry (7) and chiral symmetry (13) are largely broken, there still remains a partial chiral $(Z_2)$ symmetry [38] such that

\[ Sm_DP_3 = +m_DP_3, \quad P_3 \equiv \text{diag}(1, 1, 0). \] (41)
Equivalently, for $\mathbf{u}^{\text{NH}} = \frac{1}{\sqrt{6}} (2, -1, -1)^T$ and $\mathbf{u}^{\text{IH}} = \frac{1}{\sqrt{2}} (0, 1, -1)^T$, the constraints are

$$m_D^{(\text{NH})} \simeq \begin{pmatrix} A_2 + A_3 & \frac{B_2 + B_3}{2} & C_1 \\
 A_2 & B_2 & C_2 \\
 A_3 & B_3 & C_3 \end{pmatrix}, \quad m_D^{(\text{IH})} \simeq \begin{pmatrix} A_1 & B_1 & C_1 \\
 A_2 & B_2 & C_2 \\
 A_2 & B_2 & C_3 \end{pmatrix}. \quad (42)$$

Since $\mathbf{A}$ and $\mathbf{B}$ must be approximately orthogonal to the eigenvectors of the massless mode, the two complex parameters in the lighter generations are directly constrained.

Finally, if the magnitude of $(m_D)_{33}$ is close to the mass of the top quark$^4$ $m_t \sim 100$ GeV, a relation for $M_3$ and $m_1$ is obtained as

$$M_3 = \frac{C_u^2}{m_1} = \frac{|u_3 100 \text{ GeV}|^2}{m_1} = |u_3|^2 \left( \frac{1 \text{ meV}}{m_1} \right) 10^{16} \text{ GeV}. \quad (43)$$

where $u_3 = (\mathbf{u}^{\text{NH}})_3$ or $(\mathbf{u}^{\text{IH}})_3$ is the third element of the lightest neutrino direction. Although a similar fact has been mentioned in Ref. [39], it is interesting that $M_3$ is quite close to the scale of grand unified theories (GUTs).

### IV. SUMMARY

In this letter, we search restrictions on flavor structures due to an approximate $U(1)_L$ lepton number symmetry associated with the lightest neutrino mass $m_{1,3}$ in the type-I seesaw mechanism. If $U(1)_L$ is exact and the mass matrix of right-handed neutrinos $M_R$ is regular $\det M_R \neq 0$, the Dirac mass matrix $m_D$ has a chiral ($Z_2$) symmetry $S m_D = m_D$ due to a left-handed lepton number. Interestingly, this constraint is independent of the form and basis of $M_R$. Since this chiral symmetry is only for $m_D$ and not necessarily a symmetry of the entire theory, it is treated as a kind of remnant symmetry. Although this chiral symmetry restricts three complex parameters of $m_D$, it is incompatible with a simple unified theory because the diagonalization of $m_D$ becomes large mixing.

Next, we investigated how this condition can be relaxed by considering a chiral perturbative analysis of $m_{1,3}$. For $m_D = (\mathbf{A}, \mathbf{B}, \mathbf{C})$, a linearly independent component $\delta \mathbf{C}$ with $\mathbf{A}$ and $\mathbf{B}$ is treated as the symmetry-breaking parameters. We found that a deviation in eigenvectors of the massless mode $\delta \mathbf{u}$ is about $\sqrt{m_1/m_{2,3}}$ for $m_{1,3} \propto \det m_D^2/M_3$.

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$^4$ This value seems reasonable in a unified theory with $Y_u \sim Y_\nu$ because $Y_\nu$ does not receive any running effect of the strong coupling.
In particular, if $m_D$ satisfies the hierarchy $|(m_D)_{13}| \ll |(m_D)_{23}| \ll |(m_D)_{33}|$ and the diagonalization of the mass matrix of light neutrinos $m_\nu$ has large mixing in this hierarchical basis, the order of $\delta u$ is reduced to $m_{1,3}/m_2$. Thus the perturbation theory works well with $m^{\text{NH}}_1 \lesssim 1\,\text{meV}$, $m^{\text{IH}}_3 \lesssim 5\,\text{meV}$. In this case, the above chiral symmetry remains partially as $S m_D P_3 = m_D P_3$ with $P_3 = \text{diag}(1, 1, 0)$.

In conclusion, in the type-I seesaw mechanism with a hierarchical Dirac mass matrix, flavor structures of the lighter generation must be approximately orthogonal to the eigenvector of the lightest neutrino, and two complex parameters for $m_D$ are directly bounded from the eigenvector. These facts are very useful in a construction of models and unified theories, and similar chiral symmetries are expected to exist in other analogs of the seesaw mechanism.

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