ACCELERATING EXPANSION AND CHANGE OF SIGNATURE

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Abstract. We show that some types of sudden singularities admit a natural explanation in terms of regular changes of signature on brane-worlds in AdS$_5$. The present accelerated expansion of the Universe and its possible ending at a sudden singularity may therefore simply be an indication that our braneworld is about to change its Lorentzian signature to an Euclidean one, while remaining fully regular. An explicit example of this behaviour satisfying the weak and strong energy conditions is presented.

1 Introduction

As shown in (Mars et al. 2001) and (Mars et al. 2007), brane-world models are a natural scenario for the regular description of classical signature change. In brane— and thin shell—cosmology settings the spacetime corresponds to an embedded submanifold (the brane or shell) in a higher dimensional Lorentzian space (bulk). Smooth submanifolds sitting in a bulk of fixed Lorentzian signature can have varying causal character, which implies that its induced metric can undergo a change of signature while remaining perfectly regular. Even though the change of signature may appear as a dramatical event from within the brane, both the bulk and the brane remain smooth everywhere. In particular, observers living in the brane but assuming that their Universe is Lorentzian everywhere may be misled to interpret that a curvature singularity arises precisely at the signature change.

In this short contribution we show that a signature change on the brane might explain an accelerated expansion of the Universe ending in a sudden singularity of

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DOI: (will be inserted later)
big-freeze type \cite{Cattoen:2004en}, \cite{Bouhmadi-Lopez:2006}, in which the Hubble function diverges at a finite value of the scale factor, while keeping the energy density and the rest of physical variables regular and non-negative everywhere; in particular without violating the weak or strong energy conditions.

2 Generic branes in AdS\(_5\) bulks

For concreteness we restrict ourselves to anti-de Sitter (AdS\(_5\)) bulks. Choosing suitable coordinates, the metric is

\[ ds^2 = -(k + \lambda^2 r^2) dt^2 + (k + \lambda^2 r^2)^{-1} dr^2 + r^2 d\Omega_k^2, \]

where \(\lambda > 0\) is a constant and \(d\Omega_k^2\) is the 3-dimensional metric of constant sectional curvature \(k = 1, 0, -1\). When \(k = 0, 1\) the ranges of the non-angular coordinates are \(-\infty < t < \infty\) and \(r > 0\), while for \(k = -1\) we have \(r > 1/\lambda\). The cosmological constant of AdS\(_5\) is \(\Lambda_5 = -6\lambda^2\).

As proven in Corollary 2 of \cite{Mars:2007}, branes with a change of signature require an asymmetric set-up, i.e. a bulk with no \(Z_2\) symmetry. We therefore consider the gluing of a region of AdS\(_5\) with a region of another anti-de Sitter space \(\tilde{\text{AdS}}\(_5\)\). All quantities in \(\tilde{\text{AdS}}\(_5\)\) will carry an overtilde, and we assume \(\tilde{\Lambda}_5 \neq \Lambda_5\). For simplicity we consider branes \(\Sigma\) with spherical, plane or hyperboloidal symmetry so that their parametric form read (we ignore the angular variables which are merely identified) \(\Sigma : \{t = t(\xi), r = r(\xi)\}\) and \(\tilde{\Sigma} : \{\tilde{t} = \tilde{t}(\xi), \tilde{r} = \tilde{r}(\xi)\}\). The matching requires \(r(\xi) = \tilde{r}(\xi) \equiv a(\xi)\) so that the first fundamental form on the brane reads

\[ ds^2|_\Sigma = N(\xi) dt^2 + a^2(\xi) d\Omega_k^2, \tag{2.1} \]

where \(N(\xi)\) controls the embedding functions \(t(\xi), \tilde{t}(\xi)\) via

\[ \dot{	ilde{t}} = \frac{\sigma a}{k + \lambda^2 a^2} \sqrt{\dot{\xi}^2 - N \left( \frac{k}{a^2} + \lambda^2 \right)} \tag{2.2} \]

and a similar equation for \(\tilde{t}\) in terms of \(\tilde{\lambda}\). Here dot stands for \(d/d\xi\) and \(\sigma, \tilde{\sigma}\) are two free signs. \(N(\xi)\) and \(a(\xi)\) are arbitrary functions only restricted to satisfy that both square roots, in (2.2) and its tilded version, are real.

The brane metric (2.1) changes signature whenever \(N\) changes sign. The signature-changing set of \(\Sigma\), denoted by \(S\), is defined to be the collection of all “instants” \(\xi = \xi_s\), where \(N\) becomes, or stops being, zero. We also define the Lorentzian phase \(\Sigma_L\) (where \(N < 0\)), the Euclidean phase \(\Sigma_E\) (where \(N > 0\)) and the null phase \(\Sigma_0\) (where \(N = 0\)). In general, all phases are non-empty for a typical signature-changing brane.

The inherited total energy-momentum tensor on the brane, \(\tau_{\mu\nu}\), can be shown to have a vanishing, simple eigenvalue, another simple and generically non-zero eigenvalue \(\hat{\rho}\) and a triple eigenvalue \(\tilde{p}\) associated with \(d\Omega_k^2\) \cite{Mars:2007}. The energy-momentum \(\tau_{\mu\nu}\) and therefore also \(\hat{\rho}\) and \(\tilde{p}\), are affected by a normalisation freedom depending on the choice of volume element on the brane and are smooth everywhere on \(\Sigma\) (for regular \(a(\xi)\) and \(N(\xi)\)).
2.1 FLRW cosmology in the Lorentzian phase

On the Lorentzian phase $\Sigma_L$ we can change to the usual cosmic time $T$ with the change of variables $\dot{T} = \sqrt{-N}$ on $\Sigma_L$, so that the metric becomes

$$ds^2\big|_{\Sigma_L} = -dT^2 + a^2d\Omega_k^2.$$  \hfill (2.3)

At the signature changing subset $S \cap \Sigma_L$, this Lorentzian metric must show some pathology related to the fact that the signature in the brane is about to change. Indeed, since we are assuming a smooth brane, we must have $a|_S \neq 0$. Moreover, $\dot{a}|_S = 0$ because otherwise, from (2.2) it would follow $\dot{t}|_S = \dot{\tilde{t}}|_S = 0$ which is impossible. Consequently $a' = \dot{a}/\sqrt{-N}$ diverges necessarily when approaching the signature-changing set $S \cap \Sigma_L$. Equivalently, the Hubble function $H = a'/a$ diverges necessarily at $S \cap \Sigma_L$, where $a$ is finite. This behaviour cannot be found in pure Lorentzian brane cosmologies, and is a kinematic characteristic of the so-called big-freeze singularities.

This ‘singularity’ concerns exclusively the “Lorentzianity” of the brane and only appears when the geometry is analyzed from the inner point of view of the Lorentzian part of the brane. The “singularity” arises because the cosmic time is pathological near the signature change, when it stops existing. No true singularity is there in the brane. As already stressed both the bulk and the brane $\Sigma$ are totally regular everywhere for regular functions $N(\xi)$ and $a(\xi)$.

Let us describe the type of singularity that any observers living on $\Sigma_L$ would believe to see there. If the scientists on $\Sigma_L$ know the spacetime is a 4-d brane immersed in a 5-d bulk, but exclude changes of signature a priori, they would take the line element (2.3) and the associated cosmic time $T$ to describe all the Universe and its history. Moreover, the metric volume form would be chosen to normalise the energy-momentum tensor $\tau_{\mu\nu}$ as well as $\dot{\varrho}$ and $\dot{p}$. Denoting them simply by $\varrho$ and $p$, their explicit expressions read (Mars et al. 2007)

$$q' + 3\frac{a'}{a}(q + p) = 0,$$

where $' \equiv d/dT$, $\kappa_5^2$ is the 5-dimensional gravitational coupling constant and $\epsilon_1$ is a sign selecting which region bounded by $\Sigma$ in AdS$_5$ is to be matched to which region bounded by $\tilde{\Sigma}$ in $\tilde{\text{AdS}}_5$. The metric volume form on the Lorentzian phase obviously becomes singular on the signature changing set $S \cap \Sigma_L$. This may seem to imply that $q$ and $p$ also have to diverge necessarily there. However, this is not the case and, in fact, $q$ must vanish at the signature change, and $p$ can also be regular there in many situations (Mars et al. 2007). In fact, there exists a universal bound for the energy-density $\rho$ in the Lorentzian phase given by

$$\frac{\kappa_5^2}{2}|\varrho| \leq \sqrt{\bar{\lambda}^2 - \lambda^2}.$$

Consequently, appropriate choices of hypersurfaces in AdS$_5$ and $\tilde{\text{AdS}}_5$ allow one to construct signature-changing branes with both $q$ and $p$ finite and well-behaved everywhere on $\Sigma_L$. As a matter of fact, signature-changing branes satisfying some desirable energy conditions can be built, as we show next.
3 Example

This model is based on the equation of state (on \( \Sigma_L \)) \( p = C^2 \varrho^{\frac{m-2}{2}} \) with \( m \) odd. The conservation equation (2.1) implies \( \varrho = C^m \left[ \left( \frac{a_S}{m} \right)^{6/m} - 1 \right]^{m/2} \), where \( a_S > 0 \) is an integration constant. In order to construct a regular brane \( \Sigma \) we need \( N(\xi) = (\xi - \xi_e)(\xi - \xi_b)^m \), so that the Lorentzian phase \( \Sigma_L \) corresponds to \( \xi_b < \xi < \xi_e \). The function \( a(\xi) \) satisfies an ODE which admits the boundary conditions \( a(\xi_b) = a(\xi_e) = a_S \). The corresponding solution can be proven (Mars et al. 2008) to be regular all over \( \Sigma \) and satisfy \( a_{\min} \leq a \leq a_S \) on \( \Sigma_L \), which implies in particular that the weak and strong energy conditions are satisfied. Moreover, \( a'(T) \to \pm\infty \) and \( a''(T) \to \infty \) as \( a \to a_S \) and the model describes a sudden type singularity with a diverging Hubble factor for a finite non-zero scale factor \( a = a_S \). The brane can be extended past \( \xi_e \) and \( \xi_b \) smoothly so that this apparent singularity is just an artifact of the bad behaviour of \( T \), as discussed above.

Fig. 1. \( \Sigma_L \) starts at a change of signature at \( \xi_b \) where/when the cosmic time \( T \) is born. There, the scale factor \( a(T) \) has a maximum and \( \varrho \) and \( p \) vanish. From there, \( a(T) \) contracts to a minimum, \( a_0 \), where \( \varrho \) attains its maximum (regular little bang). From this bounce, the model expands with an accelerated rate and \( T \) ends in a seemingly sudden singularity, at the signature change at \( \xi_e \), where again \( \varrho = p = 0 \). The picture on the right describes the whole brane embedded in the AdS\( _5 \) (\( k = 1 \)) bulks, with \( \sigma = -\epsilon_1 = 1 \), and \( \lambda > \tilde{\lambda} \), so that \( \varrho > 0 \) on \( \Sigma_L \).

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