A MULTIVARIABLE CASSON-LIN TYPE INVARIANT

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ABSTRACT. We introduce a multivariable Casson-Lin type invariant for links in $S^3$. This invariant is defined as a signed count of irreducible SU(2) representations of the link group with fixed meridional traces. For 2-component links with linking number one, the invariant is shown to coincide with the multivariable signature. We also obtain some results concerning deformations of SU(2) representations of link groups.

1. Introduction

The Casson-Lin invariant $h(K)$ of a knot $K$ was originally defined by Lin as a signed count of conjugacy classes of trace-free irreducible SU(2) representations of $\pi_1(S^3 \setminus K)$ [37]. Lin proved furthermore that $h(K)$ equals half the (Murasugi) signature of $K$. Allowing for more general trace conditions, this result was later generalized by Herald [27] and Heusener-Kroll [29] who defined an invariant $h_K(\alpha)$ for those $\alpha \in (0, \pi)$ which satisfy $\Delta_K(e^{2i\alpha}) \neq 0$. They also related their invariant to the Levine-Tristram signature $\sigma_K$ by showing that

$$h_K(\alpha) = \frac{1}{2} \sigma_K(e^{2i\alpha}).$$

Similar invariants have been constructed for links: Harper-Saveliev [24] defined a signed count of a certain type of projective SU(2) representations for 2-component links $L = K_1 \cup K_2$ and showed that their invariant coincides with the linking number $\pm \ell k(K_1, K_2)$. The sign was later determined by Boden-Herald [6] and the construction was extended to $n$-component links by Boden-Harper [5]. We also refer to [5] for a construction involving the group SU($n$) and to [25, 12] for further gauge theoretic developments.

The first aim of this article is to produce a multivariable generalization of the Casson-Lin invariant. Namely, building on the approach of Lin [37] and Heusener-Kroll [29], we consider conjugacy classes of SU(2) representations with fixed meridional traces. More precisely, given an $n$-component ordered link $L$ and an $n$-tuple $(\alpha_1, \ldots, \alpha_n) \in (0, \pi)^n$ such that the multivariable Alexander polynomial $\Delta_L(t_1, \ldots, t_n)$ does not vanish at $(e^{2i\alpha_1}, \ldots, e^{2i\alpha_n})$, we define a multivariable Casson-Lin invariant

$$h_L(\alpha_1, \ldots, \alpha_n).$$

Generalizing the aforementioned authors’ approach, this invariant is defined using (colored) braids. The invariance of $h_L$ is then proved by showing independence under the two colored Markov moves (Propositions 3.8 and 3.9). By construction, $h_L$ recovers the invariant of Heusener-Kroll [29] if $L$ is a knot, while Proposition 6.6 shows that $h_L$ is locally constant. Note that since we are counting SU(2) representations and not projective SU(2) representations, our invariant $h_L$ is distinct from the link invariant constructed by Harper-Saveliev [24] and Boden-Harper [5]. The following paragraphs shall make this difference more concrete.
In [37, 29], the invariants under consideration were related to signature invariants by studying the effect of crossing changes and computing the invariant on a “base case”, namely the unknot. In our setting, this task is complicated by the following fact: if $L$ and $L'$ are related by a crossing change and $\Delta_L$ is not identically zero, it might well be that $\Delta_L' \equiv 0$, and in this case, $h_{L'}$ is not defined. Furthermore, since the Alexander polynomial of the $n$-component unlink is trivial (for $n \geq 2$), there is no obvious “base case”.

While we have not managed to circumvent this issue in general, we nevertheless provide a formula relating $h_L(\alpha_1, \ldots, \alpha_n)$ to $h_L'(\alpha_1, \ldots, \alpha_n)$ whenever $L$ and $L'$ differ by a crossing change within a component of $L$. In particular, for 2-component links with linking number 1, we are then able to relate $h_L$ to the multivariable signature $\sigma_L$ of Cimasoni-Florens [10]: for this class of links, the Hopf link can be used as the “base case”. In fact, since $\sigma_L$ is defined on $T^n = (S^3 \setminus \{1\})^n$, we shall think of $h_L$ as being defined on $T^n$ instead of on $(0, \pi)^n$.

**Theorem 1.1.** If $L$ is a 2-component ordered link with linking number 1, then the following equality holds on $T^n \setminus \{(\omega_1, \omega_2) \mid \Delta_L(\omega_1, \omega_2) = 0\}$:

$$h_L(\omega_1, \omega_2) = \frac{1}{2} \sigma_L(\omega_1, \omega_2).$$

In fact, throughout this article, we work with colored links: an $n$-component oriented link $L$ is $\mu$-colored if its components are partitioned into sublinks $L_1 \cup \ldots \cup L_\mu$. For instance, taking $\mu = n$, a $\mu$-colored link is an ordered link, while a 1-colored link is simply an oriented link. In particular, in this latter case, our construction defines a one variable Casson-Lin invariant which reduces to Heusener-Kroll’s invariant if $L$ is a knot.

**Remark 1.2.** Theorem 1.1 does not hold for 1-colored links with more than one component. Reformulating, if $L$ is an oriented link with at least two components, then $-\sigma_L(\omega)/2$ is not equal to $h_L(\omega)$ (for knots, the equality holds by Heusener and Kroll’s result; the sign difference is discussed in Remark 6.5). For instance, regardless of the number of colors, $h_J$ vanishes for the Hopf link $J$ (since $\pi_1(S^3 \setminus J)$ is abelian), while the 1-variable signature of $J$ (i.e. the Levine-Tristram signature) is equal to 1 or $-1$ depending on the orientation.

The next remark shows that the linking number hypothesis is also necessary in Theorem 1.1.

**Remark 1.3.** Theorem 1.1 does not hold for arbitrary 2-component ordered links for parity reasons. Indeed if it held for a 2-component ordered link $L = K_1 \cup K_2$ with $\ell k(K_1, K_2) = \ell$, then $\sigma_L(\omega_1, \omega_2)$ would be even. As $(\omega_1, \omega_2)$ is not a root of $\Delta_L(t_1, t_2)$, this contradicts the fact that $\sigma_L(\omega_1, \omega_2)$ has the same parity as $\ell + 1$ [10, Lemma 5.7].

Summarizing, the multivariable Casson-Lin invariant coincides with the multivariable signature for knots and 2-component ordered links with linking number 1, but is a priori a new invariant in general. Note that the resemblance between (abelian invariants of) 2-component links with linking number 1 and knots was already observed and exploited in [21].

**Remark 1.4.** It should now be clear that our multivariable Casson-Lin invariant $h_L$ differs from the invariant of Harper-Saveliev [24] and Boden-Harper [5]. As an additional remark in this direction, it is interesting to note that this latter count of projective representations might be a link homotopy invariant [5, discussion following Conjecture 4.7], while this seems unlikely for our $h_L$: the statement is already incorrect for 2-component links with linking number 1 since the multivariable signature is not a link homotopy invariant.
The second aim of this paper is to provide some results on deformations of SU(2) representations of link groups. In other words, we study whether an abelian SU(2) representation of a link group is a limit point of irreducible representations. Before providing some history and stating our results, we introduce some notation. Given an $n$-component ordered link $L = K_1 \cup \ldots \cup K_n$ (whose exterior in $S^3$ is denoted by $M_L$) and $\omega = (\omega_1, \ldots, \omega_n) \in T^*_n$, we consider the abelian representation

$$\rho_\omega: \pi_1(M_L) \to SU(2), \quad \rho_\omega(\gamma) = \begin{pmatrix} \prod_{i=1}^{n} \omega_i^{k(\gamma,K_i)} & 0 \\ 0 & \prod_{i=1}^{n} \omega_i^{-k(\gamma,K_i)} \end{pmatrix}.$$ 

In the knot case (i.e. $n = 1$), it is known since Burde [7] and de Rham [19] that if $\rho_\omega$ is a limit of irreducible SU(2) representations, then $\Delta_K(\omega^2) = 0$. Frohman and Klassen have shown that the converse holds if $\omega$ is a simple root of $\Delta_K(t)$ [22]. This result was generalized by Herald [27] and Heusener-Kroll [29]: these authors used Casson-Lin type invariants to show that if $\omega$ is a root of $\Delta_K(t)$ and if the Levine-Tristram signature $\sigma_K$ changes value at $\omega$, then $\rho_\omega$ is a limit of irreducible representations. We refer to [3] for other results in this direction and to [31, 30] (and references therein) for deformations of $SL_n(\mathbb{C})$ representations.

In the case of links, these questions seemed to have received less attention. Our first result in this context is a multivariable generalization of the theorem of Burde and de Rham. While our results hold for colored links and also concern $SL_2(\mathbb{C})$ representations (Theorems 2.4 and 2.5), we only state the following result on SU(2) representations, see Corollary 2.6:

**Proposition 1.5.** Let $L$ be an $n$-component link and let $\omega = (\omega_1, \ldots, \omega_n) \in T^*_n$. If the abelian representation $\rho_\omega$ is a limit of irreducible SU(2) representations, then $\Delta_L(\omega_1, \ldots, \omega_n) = 0$.

Just as in the knot case, one might now wonder about the converse of Proposition 1.5. Our final result uses Theorem 1.1 to provide a partial converse for 2-component links with linking number 1 (in the spirit of Herald’s and Heusener-Kroll’s result [27, 29] which involved the Levine-Tristram signature). To state our result, we use $V(\Delta_L) \subset T^*_n$ to denote the variety described by the intersection of $T^*_n$ with the zero-locus of the multivariable Alexander polynomial of an $n$-component link $L$.

**Theorem 1.6.** Let $L$ be a 2-component ordered link with linking number 1, and let $(\omega_1, \omega_2)$ lie in $V(\Delta_L)$. Assume that for any open subset $U \subset T^*_2$ containing $\omega := (\omega_1, \omega_2)$, the multivariable signature $\sigma_L$ takes several distinct values on the connected components of $U \setminus (V(\Delta_L) \cap U)$. Then the abelian representation $\rho_\omega$ is a limit of irreducible representations.

This paper is organized as follows. In Section 2, we review some facts about representation spaces and prove Proposition 1.5. In Section 3, we define the multivariable Casson-Lin invariant $h_L$. In Section 4, we review some facts about the colored Gassner representation and the multivariable potential function. In Section 5, we study the effect of crossing changes on $h_L$ and, in Section 6, we prove Theorems 1.1 and 1.6.

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2. SL\(_2(\mathbb{C})\) representations and the multivariable Alexander polynomial

The aim of this section is to obtain a necessary condition for the existence of reducible non-abelian SL\(_2(\mathbb{C})\) representations of link groups and prove Proposition 1.5 from the introduction. In Subsection 2.1, we review representation spaces, in Subsection 2.2, we recall some facts about the multivariable Alexander polynomial and, in Subsection 2.3, we state and prove our results on reducible representations of link groups. Note that while most of this paper deals with SU(2) representations, we hope that the more general SL\(_2(\mathbb{C})\) statements of Theorems 2.4 and 2.5 might be of independent interest.

2.1. Representation spaces. In this subsection, we review some basics facts and notations about representation spaces. References include [35, 2, 44].

Let \(\pi\) be a finitely generated group and let \(G\) be either SU(2) or SL\(_2(\mathbb{C})\). The representation space of \(\pi\) is the set \(R_G(\pi) := \text{Hom}(\pi, G)\) endowed with the compact open topology. Choosing a set \(\{x_1, \ldots, x_n\}\) of generators of \(\pi\), the map \(R_G(\pi) \to G^n, \rho \mapsto (\rho(x_1), \ldots, \rho(x_n))\) realizes \(R_G(\pi)\) as an algebraic subset of \(G^n\). A representation \(\rho \in R_G(\pi)\) is abelian if its image is an abelian subgroup of \(G\). The closed subset of abelian representations will be denoted by \(S(\pi)\). A representation is reducible if it admits a non-trivial invariant subspace and irreducible otherwise.

Remark 2.1. For SU(2) representation spaces, a representation \(\rho\) is reducible if and only if it is abelian. This is well known not to be the case for other Lie groups such as SL\(_2(\mathbb{C})\).

When \(G = SU(2)\), we write \(R(\pi)\) instead of \(R_{SU(2)}(\pi)\). The group SU(2) acts on \(R(\pi)\) by conjugation and the SU(2)-character variety of \(\pi\) consists of the quotient \(X(\pi) = R(\pi)/SU(2)\). After removing abelian representations, SO(3) = SU(2)/± Id acts freely and properly on \(R(\pi) \setminus S(\pi)\). In practice, we shall frequently consider (subspaces of) the set of conjugacy classes of non abelian representations:

\[\hat{R}(\pi) = (R(\pi) \setminus S(\pi))/SO(3).\]

When \(G = SL_2(\mathbb{C})\), the quotient \(R_{SL_2}(\pi)/SL_2\) is not Hausdorff in general. In this case, the SL\(_2\)-character variety of \(\pi\) is the algebro-geometric quotient

\[X_{SL_2}(\pi) = R_{SL_2}(\pi)/SL_2(\mathbb{C}).\]

While references for the algebro-geometric quotient include [39, 18, 44], all we need in the sequel is the following fact: two representations \(\rho\) and \(\rho'\) are identified in \(X_{SL_2}(\pi)\) if their traces are equal, i.e. if for all \(\gamma \in \pi\) one has \(\text{Tr} \rho(\gamma) = \text{Tr} \rho'(\gamma)\). As a consequence, this quotient is the usual one when restricted to the set of irreducible representations (see for instance [18, Proposition 1.5.2]).

Finally, when \(\pi\) is the fundamental group of a manifold \(M\), we write \(R(M)\) instead of \(R(\pi)\). In fact, we are particularly interested in the case where \(M\) is a link exterior.

2.2. The multivariable Alexander polynomial and reducible SL\(_2(\mathbb{C})\) representations. In this subsection, we first briefly review the multivariable Alexander polynomial before stating a criterion for the existence of reducible non-abelian SL\(_2(\mathbb{C})\)-representations. We also prove Proposition 1.5 from the introduction.
A \( \mu \)-colored link is an oriented link \( L \) in \( S^3 \) whose components are partitioned into \( \mu \) sublinks \( L_1 \cup \cdots \cup L_\mu \). Given an \( n \)-component \( \mu \)-colored link \( L \), we let \( M_L \) denote its exterior, we consider the homomorphism \( \varphi: \pi_1(M_L) \rightarrow \mathbb{Z}^\mu, \gamma \mapsto (\ell k(L_1, \gamma), \ldots, \ell k(L_\mu, \gamma)) \) and use \( m_1, \ldots, m_\mu \) to denote the meridians of \( L \).

**Remark 2.2.** Any reducible representation \( \rho: \pi_1(M_L) \rightarrow \text{SL}_2(\mathbb{C}) \) is conjugated to one which satisfies \( \rho(m_i) = \begin{pmatrix} \lambda_i & * \\ 0 & \lambda_i^{-1} \end{pmatrix} \) for some \( \lambda_i \in \mathbb{C} \) and for \( i = 1, \ldots, n \). Using \( K_1, \ldots, K_n \) to denote the connected components of \( L \), observe that for \( \gamma \in \pi_1(M_L) \), this representation satisfies

\[
\rho(\gamma) = \begin{pmatrix}
\prod_{i=1}^n \lambda_i^{\ell k(\gamma, K_i)} & * \\
0 & \prod_{i=1}^n \lambda_i^{-\ell k(\gamma, K_i)}
\end{pmatrix}.
\]

Next we bring the colors into play. Assume that \( \lambda = (\lambda_1, \ldots, \lambda_\mu) \) lies in \( (\mathbb{C}^*)^\mu \) and let \( \rho_\lambda: \pi_1(M_L) \rightarrow \text{SL}_2(\mathbb{C}) \) be the representation which maps \( m_j \) to \( \begin{pmatrix} \lambda_j & * \\ 0 & \lambda_j^{-1} \end{pmatrix} \) if \( m_j \) belongs to the sublink \( L_j \). Note that if \( \mu = n \), this recovers the representation described in (1). Consider the composition \( \varphi_\lambda: \pi_1(M_L) \xrightarrow{\varphi} \mathbb{Z}^\mu \rightarrow \mathbb{C} \), where the second map sends the canonical basis element \( e_i \) to \( \lambda_i \). If \( \gamma \in \pi_1(M_L) \), then \( \rho_\lambda(\gamma) \) can be written explicitly as

\[
\rho_\lambda(\gamma) = \begin{pmatrix}
\varphi_\lambda(\gamma) & * \\
0 & \varphi_\lambda(\gamma)^{-1}
\end{pmatrix} = \begin{pmatrix}
\prod_{i=1}^\mu \lambda_i^{\ell k(\gamma, L_i)} & * \\
0 & \prod_{i=1}^\mu \lambda_i^{-\ell k(\gamma, L_i)}
\end{pmatrix}.
\]

As observed in Remark 2.1, reducible \( \text{SL}_2(\mathbb{C}) \)-representations need not to be abelian. In order to describe this situation in more details, we recall the definition of the Alexander polynomial of a colored link. The previously described epimorphism \( \varphi: \pi_1(M_L) \rightarrow \mathbb{Z}^\mu \) induces a regular \( \mathbb{Z}^\mu \)-covering \( \tilde{M}_L \rightarrow M_L \). The homology of \( \tilde{M}_L \) is a module over \( \Lambda_\mu := \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}] \), and the \( \Lambda_\mu \)-module \( H_1(\tilde{M}_L) \) is called the Alexander module of the colored link \( L \).

**Definition 2.3.** The Alexander polynomial \( \Delta_L(t_1, \ldots, t_\mu) \) of a \( \mu \)-colored link \( L \) is the order of its Alexander module.

The Alexander polynomial is only well defined up to units of \( \Lambda_\mu \), that is, up to multiplication by powers of \( \pm t_i \). We refer to [33, 9] for further information on \( \Delta_L \). The main theorem of this section is the following.

**Theorem 2.4.** Let \( L \) be a \( \mu \)-colored link and let \( \lambda = (\lambda_1, \ldots, \lambda_\mu) \) lie in \( (\mathbb{C}^* \setminus \{1\})^\mu \). There exists a reducible, non abelian \( \text{SL}_2(\mathbb{C}) \)-representation of the form \( \rho_\lambda \) if and only if \( \Delta_L(\lambda^2) = 0 \).

We delay the proof of Theorem 2.4 to Subsection 2.3 and give some applications instead.

**Theorem 2.5.** Let \( L \) be a \( \mu \)-colored link and let \( \lambda = (\lambda_1, \ldots, \lambda_\mu) \) lie in \( (\mathbb{C}^* \setminus \{1\})^\mu \). If \( \Delta_L(\lambda^2) \neq 0 \), then a sufficiently small neighborhood of the representation \( \rho_\lambda \) in \( R_{\text{SL}_2(\mathbb{C})}(M_L) \) consists entirely of reducible representations.

**Proof.** The strategy of the proof follows [44, Lemma 3.9, (iii)]). A representation \( \rho: \pi \rightarrow \text{SL}_2(\mathbb{C}) \) is reducible if and only if for any \( \gamma, \delta \in \pi \), one has \( \text{Tr} \rho(\gamma \delta^1 \gamma^{-1} \delta^{-1}) = 2 \) [18, Lemma 1.2.1]. Consequently, reducibility is well defined at the level of character varieties, and the
set of irreducible characters is open in both the representation variety $R_{SL_2}(M_L)$ and in the character variety $X_{SL_2}(M_L)$.

Since $\Delta_L(\chi^2) \neq 0$, Theorem 2.4 implies that $\rho_\lambda$ is abelian. In fact, we claim that every representation $\rho'$ with the same character as $\rho_\lambda$ is abelian and is conjugated to $\rho_\lambda$. To see this, first note that since $\rho'$ has the same character as $\rho_\lambda$, the previous paragraph implies that $\rho'$ is reducible. Using Theorem 2.4, $\rho'$ must in fact be abelian. Since $\rho_\lambda$ and $\rho'$ are abelian and have the same character, they must be conjugated, concluding the proof of the claim.

By way of contradiction, assume that the representation $\rho_\lambda$ has irreducible representations in any of its neighborhoods. Since we argued that the set of irreducible characters is open in the character variety, the character of the representation $\rho_\lambda$ lies in an irreducible component $X \subset X_{SL_2}(M_L)$ that contains the character of an irreducible representation.

Next, consider the quotient map $t: R_{SL_2}(M_L) \to X_{SL_2}(M_L)$. If $\chi \in X$ is the character of an irreducible representation, then the fiber $t^{-1}(\{\chi\})$ is homeomorphic to $PSL(2, \mathbb{C})$ and in particular has dimension 3. Since irreducible characters form an open dense subset of $X$ and since the dimension of the fiber $t^{-1}(\{\chi\})$ is upper semi-continuous on $X$ for any character $\chi$ in $X$, the dimension of $t^{-1}(\{\chi\})$ is at least 3 for any character $\chi \in X$.

Set $\chi_\lambda := t(\rho_\lambda)$. Since $\rho_\lambda$ is abelian, the claim implies that $t^{-1}(\{\chi_\lambda\})$ is isomorphic to $SL_2(\mathbb{C})/G_{\rho_\lambda}$, where $G_{\rho_\lambda} \leq SL_2(\mathbb{C})$ is the stabilizer of $\rho_\lambda$ and has positive dimension (since $\rho_\lambda$ is abelian). Therefore the fiber $t^{-1}(\{\chi_\lambda\})$ has dimension strictly less than 3 which contradicts the previous paragraph.

In the next sections, our interest will lie in $SU(2)$ representations. In this case, as recalled in Remark 2.1, every reducible representation is abelian and the resulting eigenvalues lie on the unit circle. Using $T^\mu_r$ to denote $(S^1 \setminus \{1\})^\mu$, we obtain the following result which generalizes a theorem of Burde [7] and de Rham [19]. This proves Proposition 1.5 from the introduction.

**Corollary 2.6.** Let $L$ be a $\mu$-colored link and let $\omega$ lie in $T^\mu_r$. If $\Delta_L(\omega^2) \neq 0$, then a sufficiently small neighborhood of $\rho_\omega$ in $R(M_L)$ consists entirely of abelian representations.

**Proof.** This follows directly from Theorem 2.5 and the observation that $R(M_L)$ embeds in $R_{SL_2}(M_L)$: any $SU(2)$ representation is also an $SL_2(\mathbb{C})$ representation. □

2.3. **Proof of Theorem 2.4.** The map $\varphi_\lambda: \pi_1(M_L) \to \mathbb{C}$ described in Remark 2.2 endows $\mathbb{C}$ with a left $\mathbb{Z}[\pi_1(M_L)]$-module structure; we write $\mathbb{C}_\lambda$ for emphasis. Consider the twisted cochain complex $C^*(\pi_1(M_L), \mathbb{C}_\lambda)$ and recall that a 1-cocycle $u \in Z^1(\pi_1(M_L), \mathbb{C}_\lambda)$ is a map $u: \pi_1(M_L) \to \mathbb{C}$ that satisfies

$$u(\gamma \delta) = u(\gamma) + \varphi_\lambda(\gamma)u(\delta)$$

for every $\gamma, \delta$ in $\pi_1(M_L)$. The following lemma provides a cohomological obstruction for a reducible representation to be abelian.

**Lemma 2.7.** Given $\lambda \in (\mathbb{C}^* \setminus \{1\})^\mu$, the following assertions hold:

1. The representation $\rho_\lambda$ gives rise to a cocycle $u \in Z^1(M_L, \mathbb{C}_\lambda)$.

2. The representation $\rho_\lambda$ is abelian if and only if $[u] = 0 \in H^1(M_L, \mathbb{C}_\lambda)$.

**Proof.** Using the definition of $\rho_\lambda$, we may write $\rho_\lambda(\gamma)$ as $\begin{pmatrix} \varphi_\lambda(\gamma) & \varphi_\lambda(\gamma)^{-1}u(\gamma) \\ 0 & \varphi_\lambda(\gamma)^{-1} \end{pmatrix}$ for each $\gamma$ in $\pi_1(M_L)$ and this gives rise to a map $u: \pi_1(M_L) \to \mathbb{C}$. Given $\gamma$ and $\delta$ in $\pi_1(M_L)$, the equality $\rho(\gamma \delta) = \rho(\gamma)\rho(\delta)$ then shows that $u$ satisfies the following relation:

$$\varphi_\lambda(\gamma \delta)^{-1}u(\gamma \delta) = \varphi_\lambda(\gamma \delta^{-1})u(\delta) + \varphi_\lambda(\gamma \delta)^{-1}u(\gamma).$$
Multiplying this equation by $\varphi_\lambda(\gamma \delta)$, we deduce that $u$ must satisfy $u(\gamma \delta) = u(\gamma) \varphi_\lambda(\gamma^2) u(\delta)$ which is the cocycle condition from (3). Thus $u$ is a cocycle and the first assertion is proved.

To prove the second assertion, we must show that the reducible representation $\rho_\lambda$ is abelian if and only if the cocycle $u$ is a coboundary, that is if there exists a $z \in \mathbb{C}$ such that for all $\gamma \in \pi_1(M_L)$, one has

$$u(\gamma) = (\varphi_\lambda(\gamma^2) - 1)z.$$  

First, observe that $\rho_\lambda$ is abelian if and only if for each $\gamma \in \pi_1(M_L)$ there exists an invertible matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $DA = A\rho_\lambda(\gamma)$, where $D$ denotes the diagonal matrix $\begin{pmatrix} \varphi_\lambda(\gamma) & 0 \\ 0 & \varphi_\lambda(\gamma)^{-1} \end{pmatrix}$. Writing out this equation coordinate by coordinate, one deduces that $\rho_\lambda$ is abelian if and only if the three following equations hold:

\[
\begin{cases}
    b\varphi_\lambda(\gamma) = a\varphi_\lambda(\gamma)^{-1}u(\gamma) + b\varphi_\lambda(\gamma)^{-1}, \\
    c\varphi_\lambda(\gamma)^{-1} = c\varphi_\lambda(\gamma), \\
    d\varphi_\lambda(\gamma)^{-1} = d\varphi_\lambda(\gamma) - u(\gamma) + d\varphi_\lambda(\gamma)^{-1}.
\end{cases}
\]

If $\varphi_\lambda(\gamma) = \pm 1$, the representation is abelian if and only if there exists $a, c$ such that $au(\gamma) = 0$ and $cu(\gamma) = 0$. Since $A$ must be invertible, either $a$ or $c$ must be non-zero, and in this case $u(\gamma)$ must vanish for each $\gamma$. In particular $[u]$ vanishes in cohomology.

If $\varphi_\lambda(\gamma) \neq \pm 1$, the representation is abelian if and only if $c = 0$ and $au(\gamma) = b(\varphi_\lambda(\gamma^2) - 1)$. Since $A$ is invertible, we deduce that $a \neq 0$ and therefore $u(\gamma) = \frac{b}{a}(\varphi_\lambda(\gamma^2) - 1)$. Consequently, looking back to (4), we have obtained the defining equation for a coboundary. This concludes the proof of the second assertion and thus the proof of the lemma. \qed

As we shall see shortly, Theorem 2.4 will follow promptly from the following lemma.

**Lemma 2.8.** Let $\lambda$ lie in $(\mathbb{C}^* \setminus \{1\})^\mu$. The complex vector space $H^1(M_L; \mathbb{C}_\lambda)$ does not vanish if and only if $\lambda$ satisfies $\Delta_L(\lambda) = 0$.

**Proof.** First, since we are dealing with $\mathbb{C}$-vector spaces and since the $\lambda_i$ are not equal to 1, the universal coefficient theorem shows that the vanishing of $H^1(M_L; \mathbb{C}_\lambda)$ is equivalent to the vanishing of $H_1(M_L; \mathbb{C}_\lambda)$. To prove the lemma, we must show that $H_1(M_L; \mathbb{C}_\lambda)$ does not vanish if and only if $\Delta_L(\lambda)$ vanishes. It is enough to show that the order of this latter vector space is zero if and only if $\Delta_L(\lambda)$ is zero. This will immediately follow if we prove that

$$H_1(M_L; \mathbb{C}_\lambda) \cong H_1(M_L; \Lambda_\mu) \otimes_{\Lambda_\mu} \mathbb{C}_\lambda.$$

To prove this assertion, we will use (a particular case of) the universal coefficient spectral sequence (UCSS) whose second page is given by $E^2_{p,q} = \text{Tor}_{\Lambda_\mu}^p(H_q(M_L; \Lambda_\mu), \mathbb{C}_\lambda)$ and which converges to $H_*(M_L; \mathbb{C}_\lambda)$, see [32, Chapter 2]. We start with the following claim.

**Claim.** Endow $\mathbb{Z} = H_0(M_L; \Lambda_\mu)$ with the $\Lambda_\mu$-module structure coming from the augmentation homomorphism $\Lambda_\mu \to \mathbb{Z}, t_i \mapsto 1$. If $\lambda$ lies in $(\mathbb{C}^* \setminus \{1\})^\mu$, then the complex vector space $\text{Tor}_{k}^{\Lambda_\mu}(\mathbb{Z}, \mathbb{C}_\lambda)$ vanishes for $k = 1, 2$.

**Proof.** Using the $\Lambda_\mu$-resolution of $\mathbb{Z}$ given by the chain complex for the universal cover of the torus $\mathbb{T}^\mu$, we have $\text{Tor}_k^{\Lambda_\mu}(\mathbb{Z}, \mathbb{C}_\lambda) = H_k(\mathbb{T}^\mu; \mathbb{C}_\lambda)$. As the $\lambda_i$ are not equal to 1, the claim now follows from considerations involving cellular homology, see [43] and [14, Lemma 2.2]. \qed
Using the claim, we know that $E_{2,0}^2 = 0$. The UCSS then gives $E_{2,0}^0 = 0$ and provides a filtration $0 \subset F_1^0 \subset F_1 = H_1(M_L; \mathbb{C}_\lambda)$. As the UCSS also implies that $F_1^0 = E_{0,1} = H_1(M_L; \Lambda_\mu) \otimes \Lambda_\mu \mathbb{C}_\lambda$ and $E_{1,0}^\infty \cong F_1^0/F_1^0$, we obtain the following short exact sequence:

$$0 \to H_1(M_L; \Lambda_\mu) \otimes \Lambda_\mu \mathbb{C}_\lambda \to H_1(M_L; \mathbb{C}_\lambda) \to \text{Tor}_1^\Lambda(\rho_0(M_L; \Lambda_\mu), \mathbb{C}_\lambda) \to 0.$$ 

Since we showed in the claim that $\text{Tor}_1^\Lambda(\rho_0(M_L; \Lambda_\mu), \mathbb{C}_\lambda)$ vanishes, the lemma follows. □

Combining these two lemmas, we are now in position to conclude the proof of Theorem 2.4.

**proof of Theorem 2.4.** Let $u_{\rho_\lambda}$ be the 1-cocycle described in Lemma 2.7. Using the second point of this same lemma, the existence of a reducible non-abelian representation $\rho_\lambda$ equivalent to the cohomology class $[u_{\rho_\lambda}]$ being non zero in $H^1(M_L, \mathbb{C}_\lambda)$. Thus, if there exists a reducible non-abelian representation of the form $\rho_\lambda$, then $H^1(M_L, \mathbb{C}_\lambda)$ is non-trivial and Lemma 2.8 implies that the multivariable Alexander polynomial $\Delta_L$ vanishes at $\lambda^2$.

Conversely, if the multivariable Alexander polynomial vanishes at $\lambda^2$, then Lemma 2.8 implies that $H^1(M_L, \mathbb{C}_\lambda)$ does not vanish. Since $H^1(M_L; \mathbb{C}_\lambda) = H^1(\pi_1(M_L); \mathbb{C}_\lambda)$, we deduce that there is a non-zero cocycle $u$ in $Z^1(\pi_1(M_L); \mathbb{C}_\lambda)$. Defining a representation $\rho$ from $u$ just as in the proof of Lemma 2.7 produces the desired non-abelian representation. □

3. The multivariable Casson-Lin invariant

The goal of this section is to define the multivariable Casson-Lin invariant. More precisely, in Subsection 3.1, we review colored braids, in Subsection 3.2, we define our invariant on braids and in Subsection 3.3 we verify its invariance under the colored Markov moves.

3.1. Colored braids. In this subsection, we briefly review colored braids and discuss the action of the colored braid groups on $SU(2)^n$. References for colored braids include [41, 13], while discussions of the action of the braid group $B_n$ on $SU(2)^n$ include [37, 29, 28, 38].

The braid group $B_n$ admits a presentation with $n - 1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ subject to the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for each $i$, and $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i - j| > 2$. Topologically, the generator $\sigma_i$ is the braid whose $i$-th component passes over the $(i+1)$-th component. The closure of a braid $\beta$ is the link $\hat{\beta}$ obtained from $\beta$ by adding parallel strands in $S^3 \setminus (D^2 \times [0, 1])$.

**Figure 1.** A $(c,c')$-braid $\beta_1$, a $(c',c'')$-braid $\beta_2$ and their composition, the $(c,c'')$-braid $\beta_1\beta_2$. Here $\beta_1$ is the generator $\sigma_1$ of $B_4$, while $\beta_2$ is $\sigma_3$.

A braid $\beta$ is $\mu$-colored if each of its components is assigned (via a surjective map) an integer in $\{1, 2, \ldots, \mu\}$ (which we call a color). A $\mu$-colored braid induces a coloring on its top and
bottom boundary components. A \( \mu \)-colored braid is then called a \((c, c')\)-braid, where \( c \) and \( c' \) are the sequences of \( 1, 2, \ldots, \mu \) induced by the coloring of the braid (these sequence will be referred to as \( \mu \)-colorings). We shall denote by \( \text{id} \), the isotopy class of the trivial \((c, c)\)-braid. The composition of a \((c, c')\)-braid \( \beta_1 \) with a \((c', c'')\)-braid \( \beta_2 \) is the \((c, c'')\)-braid \( \beta_1 \beta_2 \) depicted in Figure 1. Thus, for any \( c \), we obtain a colored braid group \( B_c \) which consists of isotopy classes of \((c, c)\)-braids. For instance, if \( \mu = 1 \) (so that \( c = (1, \ldots, 1) \)), then \( B_c \) is the braid group \( B_n \), while if \( \mu = n \) and \( c_i = i \) for each \( i \), then \( B_c \) is the pure braid group \( P_n \). We shall often use the map \( i_{c_{n+1}} : B_c \to B_{(c_1, \ldots, c_n, c_{n+1})} \) which sends \( \alpha \) to the disjoint union of \( \alpha \) with a trivial strand of color \( c_{n+1} \), see Figure 2. Here, \( c_{n+1} \) can be equal to one of the \( n \) first \( c_i \)'s.

![Figure 2. An example of the inclusion map \( i_{c_4} \).](image)

Finally, the closure of a \( \mu \)-colored braid \( \beta \in B_c \) is the \( \mu \)-colored link \( \hat{\beta} \) obtained from \( \beta \) by adding colored parallel strands in \( S^3 \setminus (D^2 \times [0, 1]) \). We refer to [41, Theorem 3.3] for the colored version of Alexander’s theorem and instead focus on the colored version of Markov’s theorem, referring to [41, Theorem 3.5] for the proof.

**Proposition 3.1.** Two \((c, c)\)-braids have isotopic closures if and only if they are related by a sequence of the following moves and their inverses:

1. replace \( \gamma \beta \) by \( \beta \gamma \), where \( \gamma \) is a \((c, c')\)-braid and \( \beta \) is a \((c', c)\)-braid,
2. replace \( \gamma \) by \( \sigma_n^\varepsilon i_{c_n}(\gamma) \), where \( \gamma \) is a \((c, c)\)-braid with \( n \) strands, \( \sigma_n \) is viewed as a \(((c_1, \ldots, c_n, c_n), (c_1, \ldots, c_n, c_n))\)-braid, and \( \varepsilon \) is equal to \( \pm 1 \).

We conclude this subsection by discussing the action of (colored) braids on \( SU(2)^n \). Topologically, this action can be understood as follows. Any braid \( \beta \) can be represented by a homeomorphism of the punctured disk \( D_n \) which fixes the boundary pointwise [4]. As a consequence, the braid group induces a right action of \( B_n \) on the free group \( F_n = \pi_1(D_n) \). More explicitly, this action can be described on the generators \( x_1, \ldots, x_n \) of \( F_n \) as follows:

\[
    x_j \sigma_i = \begin{cases} 
    x_i x_{i+1} x_i^{-1} & \text{if } j = i, \\
    x_i & \text{if } j = i + 1, \\
    x_j & \text{otherwise.}
    \end{cases}
\]  

In particular, every braid \( \beta \) induces a homeomorphism \( R(D_n) \to R(D_n) \), which we still denote by \( \beta \). More concretely, identifying \( R(D_n) \) with \( SU(2)^n \), this homeomorphism maps \((X_1, \ldots, X_n)\) to \((X_1 \beta, \ldots, X_n \beta)\). So, for instance, the generator \( \sigma_1 \in B_2 \) acts as \((X_1, X_2) \sigma_1 = (X_1 X_2 X_1^{-1}, X_1) \). Note that we chose to follow Birman’s conventions [4] and to think of (5) as a right action. In particular, we obtain a homomorphism \( B_n \to \text{Aut}(F_n) \). Working with left actions would lead to an anti-homomorphism (see e.g. [11, 15]).

**Remark 3.2.** Our conventions match those of Lin [37, page 339]. On the other hand, given a braid \( \beta \in B_n \) and \( w \in F_n \), some authors, such as Long [38, page 539], choose to define
\( \beta \cdot w \) as \( w\beta^{-1} \); however given an automorphism \( \theta \) of \( F_n \), Long then sets \( \theta(X_1, \ldots, X_n) := (\theta^{-1}X_1, \ldots, \theta^{-1}X_n) \) and therefore obtains the same action of \( B_n \) on \( SU(2)^n \) as we do \cite[page 537]{38}. On the other hand, Heusener-Kroll also use the action \( \beta \cdot w = w\beta^{-1} \) \cite[Example 3.1]{29} but define \( \beta(X_1, \ldots, X_n) \) as \((\beta \cdot X_1, \ldots, \beta \cdot X_n) \) \cite[bottom of page 484]{29}.

**Remark 3.3.** The fixed point set of the homeomorphism \( \beta : SU(2)^n \to SU(2)^n \) induced by \( \beta \) can be identified with the representation space of \( X_\beta \), see for instance \cite[Lemma 1.2]{37}. Reformulating, \( R(X_\beta) \) is equal to the intersection of the diagonal \( \Lambda_n \subset SU(2)^n \times SU(2)^n \) with the graph \( \Gamma_\beta \subset SU(2)^n \times SU(2)^n \) of the homeomorphism of \( SU(2)^n \) induced by \( \beta \).

Building on the work of Lin \cite{37} and Heusener-Kroll \cite{29}, the invariant we shall define in Subsection 3.2 makes crucial use of Remark 3.3. Indeed we wish to “count” (conjugacy classes of) irreducible representations in \( R(X_\beta) = \Lambda_n \cap \Gamma_\beta \) with certain traces fixed. For this reason, given a \( \mu \)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_\mu) \) of real numbers in \((0, \pi)^\mu\) and a coloring \( c \), we shall frequently consider the following subspace of \( SU(2)^n \):

\[
R_n^{\alpha,c} = \{(X_1, \ldots, X_n) \in SU(2)^n | \text{tr}(X_i) = 2\cos(\alpha_{c_i}) \text{ for } i = 1, \ldots, n\}.
\]

In particular, observe that if \( \beta \) is an \( n \)-stranded \((c,c)\)-braid, then the aforementioned homeomorphism \( \beta : SU(2)^n \to SU(2)^n \) descends to a well defined homeomorphism \( \beta : R_n^{\alpha,c} \to R_n^{\alpha,c} \).

Of particular interest is the graph of this homeomorphism:

\[
\Gamma_\beta^\alpha = \{(A_1, \ldots, A_n, A_1\beta, \ldots, A_n\beta) | (A_1, \ldots, A_n) \in R_n^{\alpha,c} \subset R_n^{\alpha,c} \times R_n^{\alpha,c}\}
\]

For instance, the trivial \((c,c)\)-braid \( \beta = \text{id}_c \) induces the identity automorphism on the free group and thus on \( R(F_n) = SU(2)^n \). Thus the graph \( \Gamma_{\text{id}_c} \subset SU(2)^n \times SU(2)^n \) coincides with the diagonal \( \Lambda_n \). We use the following notation for the corresponding space of fixed traces:

\[
\Lambda_n^{\alpha,c} = \{(A_1, \ldots, A_n, A_1, \ldots, A_n) | (A_1, \ldots, A_n) \in R_n^{\alpha,c}\}.
\]

As we alluded to above, our goal is to make sense of a signed count of conjugacy classes of irreducible representations \( \rho : \pi_1(X_\beta) \to SU(2) \) such that the trace of any meridian of the sublink \( \hat{\beta}_j \) of \( \hat{\beta} \) is equal to \( 2\cos(\alpha_j) \). In other words, using Remark 3.3 and the notations of Subsection 2.1, we are trying to make sense of a signed count of the elements of \( \Lambda_n^{\alpha,c} \cap \hat{\Lambda}_\beta^\alpha \).

### 3.2. Definition of the invariant

The goal of this subsection is to define the multivariable Casson-Lin invariant of a \((c,c)\)-braid. Our approach builds on the work of Lin \cite{37} and of Heusener-Kroll \cite{29}, see also \cite{28} and \cite{24}.

Let \( \beta \) be a \( \mu \)-colored \( n \)-stranded \((c,c)\)-braid and let \( \alpha = (\alpha_1, \ldots, \alpha_\mu) \) lie in \((0, \pi)^\mu\). The invariant that we shall consider requires us to make sense of the algebraic intersection of (quotients of) \( \Lambda_n^{\alpha,c} \) with \( \Gamma_\beta^\alpha \) inside (a quotient of) the following space:

\[
R_n^{\alpha,c} = \{(A_1, \ldots, A_n, B_1, \ldots, B_n) \in R_n^{\alpha,c} \times R_n^{\alpha,c} | \prod_{i=1}^n A_i = \prod_{i=1}^n B_i\}.
\]

In order to count conjugacy classes of the aforementioned irreducible representations, we first need to avoid the abelian locus of the various representation varieties. For this reason, we consider the following set which should be understood (under the isomorphism \( R(F_n) \cong SU(2)^n \)) as the subspace of abelian representations of \( R(F_n) \):

\[
S_n^{\alpha,c} = \{(A_1, \ldots, A_n, B_1, \ldots, B_n) \in R_n^{\alpha,c} \times R_n^{\alpha,c} | A_iA_j = A_jA_i, A_iB_j = B_jA_i, B_iB_j = B_jB_i\}.
\]
Slightly abusing notation, we shall write $S_n^{α,c}$ instead of $S_n^{α,c} \cap Θ_n^{α,c}$, where $Θ_n^{α,c}$ is any of the previously defined spaces $Γ_β^n$, $Λ_n^{α,c}$ or $H_n^{α,c}$. As described in Subsection 2.1, SO(3) acts freely on the resulting sets of irreducible representations and we make the following definitions:

$$\tilde{Λ}_n^{α,c} = (Λ_n^{α,c} \setminus S_n^{α,c}) / SO(3), \quad \tilde{Γ}_β^n = (Γ_β^n \setminus S_n^{α,c}) / SO(3), \quad \tilde{H}_n^{α,c} = (H_n^{α,c} \setminus S_n^{α,c}) / SO(3).$$

Observe that both $\tilde{Λ}_n^{α,c}$ and $\tilde{Γ}_β^n$ are smooth open $(2n - 3)$-dimensional manifolds: $Λ_n^{α,c}$ and $Γ_β^n$ are $2n$ dimensional (the subspaces of matrices in SU(2) with fixed trace are 2-dimensional) and the 3-dimensional Lie group SO(3) acts freely and properly on the open manifolds $Λ_n^{α,c} \setminus S_n^{α,c}$ and $Γ_β^n \setminus S_n^{α,c}$. Recalling Remark 3.3, the representations we wish to consider lie in the intersection $\tilde{Γ}_β^n \cap \tilde{Λ}_n^{α,c}$, viewed as a subspace of $\tilde{H}_n^{α,c}$. In order for a “count” to make sense, we must now check that this intersection is compact and that $\tilde{Γ}_β^n$ and $\tilde{Λ}_n^{α,c}$ are half dimensional in $\tilde{H}_n^{α,c}$. We start by proving the latter, namely we prove that $\tilde{H}_n^{α,c}$ is $4n - 6$ dimensional.

**Lemma 3.4.** The space $H_n^{α,c} \setminus S_n^{α,c}$ is a smooth open $(4n - 3)$-dimensional manifold. In particular $\tilde{H}_n^{α,c}$ is $(4n - 6)$ dimensional.

**Proof.** Consider the map $f_n : R_n^{α,c} \times R_n^{α,c} \rightarrow SU(2)$ defined by $f_n(A_1, \ldots, A_n, B_1, \ldots, B_n) = A_1 \cdots A_n B_1^{-1} \cdots B_n^{-1}$. Observe that $H_n^{α,c} = f_n^{-1}(1\text{Id})$. The same arguments as in [37, Lemma 1.5] and [29, Lemma 3.3] show that $f_n$ restricts to a submersion $f_n|_{H_n^{α,c} \setminus S_n^{α,c}}$. As a consequence, $H_n^{α,c} \setminus S_n^{α,c} = f_n|_{1\text{Id}}^{-1}(SU(2))$ is a smooth manifold whose dimension is equal to $\text{dim}(R_n^{α,c} \times R_n^{α,c}) - \text{dim}(SU(2)) = 4n - 3$. This concludes the proof of the lemma.

Next, making use of Section 2, we show that the space $\tilde{Γ}_β^n \cap \tilde{Λ}_n^{α,c}$ is compact.

**Proposition 3.5.** Let $α = (α_1, \ldots, α_µ)$ lie in $(0, π)^µ$ and let $β$ be an $n$-stranded $µ$-colored $(c, c)$-braid. If $ω = (e^{2iα_1}, \ldots, e^{2iα_µ})$ satisfies $Δ_β(ω) \neq 0$, then $\tilde{Γ}_β^n \cap \tilde{Λ}_n^{α,c}$ is compact.

**Proof.** Since SO(3) is compact, it is sufficient to prove that $(Λ_n^{α,c} \setminus S_n^{α,c}) \cap (Γ_β^n \setminus S_n^{α,c})$ is compact. As this set lies in the compact set $SU(2)^{2n}$, we are reduced to proving that it is closed. Let $(ρ_k)_{k \in \mathbb{N}}$ be a convergent sequence of representations in $(Λ_n^{α,c} \setminus S_n^{α,c}) \cap (Γ_β^n \setminus S_n^{α,c})$, with limit $ρ_∞ \in SU(2)^{2n}$. Since $Λ_n^{α,c}$ and $Γ_β^n$ are closed in $SU(2)^{2n}$, it follows that $ρ_∞$ lies in $Λ_n^{α,c} \cap Γ_β^n$. By way of contradiction, assume that $ρ_∞$ is abelian. Since $Δ_β(ω) \neq 0$, Corollary 2.6 implies that $ρ_k$ is abelian for $k$ big enough, a contradiction. We therefore deduce that $ρ_∞$ lies in $(Λ_n^{α,c} \setminus S_n^{α,c}) \cap (Γ_β^n \setminus S_n^{α,c})$, concluding the proof of the proposition.

Perturbing $Γ_β^n$ and $Λ_n^{α,c}$ if necessary, we can assume that they intersect transversely. Consequently, thanks to Proposition 3.5, we know that $\tilde{Γ}_β^n \cap \tilde{Λ}_n^{α,c}$ is a 0-dimensional manifold. We now orient these manifolds. Use $S_θ$ to denote the set of matrices in SU(2) with trace $2 \cos(θ)$. Orient this copy of $S^2$ in a fixed (but arbitrary) way. Since $R_n^{α,c}$ consists of an $n$-fold product of $S_α$, we endow it with the product orientation. The diagonal $Λ_n^{α,c}$ and the graph $Γ_β^n$ are naturally diffeomorphic to $R_n^{α,c}$ via the projection on the first factor and they are given the induced orientations. Next, consider the map

$$R_n^{α,c} \times R_n^{α,c} \rightarrow SU(2)$$

which we encountered in the proof of Lemma 3.4. Using this map, we can pull back the orientation of SU(2) to obtain an orientation on $H_n^{α,c} \setminus S_n^{α,c}$. The adjoint action of SO(3)
Lemma 3.7. Let \( \alpha \in (0, \pi)^\mu \). If \( \Delta_\beta(e^{2i\alpha_1}, \ldots, e^{2i\alpha_\mu}) \) is non-zero, then the multivariable Casson-Lin invariant of \( \beta \) at \( \alpha \) is defined as the algebraic intersection number of \( \tilde{\Gamma}_\beta^\alpha \) and \( \tilde{\Lambda}_n^{\alpha,c} \) inside \( \tilde{H}_n^{\alpha,c} \):

\[
h_\beta^\alpha(\alpha) := \langle \tilde{\Lambda}_n^{\alpha,c}, \tilde{\Gamma}_\beta^\alpha \rangle_{\tilde{H}_n^{\alpha,c}}.
\]

3.3. Invariance under Markov moves. In this subsection, we prove that \( h_\beta^\alpha(\alpha) \) is invariant under the two colored Markov moves described in Proposition 3.1. Since the key ideas of the proofs are present in [37, Theorem 1.8] and [24, Proposition 4.2 and Proposition 4.3], we place emphasis on the role of the colors, while referring to the original references for details.

The invariance under the first Markov move will follow promptly from the following lemma.

Lemma 3.7. Let \( \alpha \) lie in \( (0, \pi)^\mu \) and let \( c \) and \( c' \) be \( \mu \)-colorings. Let \( \xi_1 \) be a \( (c, c) \)-braid, let \( \xi_2 \) be a \( (c, c') \)-braid and view \( \xi_2^{-1} \) as a \( (c', c) \)-braid. The multivariable Casson-Lin invariants of the \( (c, c) \)-braid \( \xi_1 \) and the \( (c', c') \)-braid \( \xi_2^{-1}\xi_1\xi_2 \) are related by the following equation:

\[
h_{\xi_1}^c(\alpha) = h_{\xi_2^{-1}\xi_1\xi_2}^{c'}(\alpha).
\]

Proof. Recalling Subsection 3.1, the \( (c, c') \)-braid \( \xi_2 \) gives rise to an orientation preserving homeomorphism \( \xi_2 : R_n^{\alpha,c} \to R_n^{\alpha,c'} \). One can then argue that it induces a well defined orientation preserving homeomorphism \( \xi_2 : \tilde{H}_n^{\alpha,c} \to \tilde{H}_n^{\alpha,c'} \). A short computation (using right actions) shows that \( (\xi_2 \times \xi_2)(\tilde{\Lambda}_n^{\alpha,c}) = \tilde{\Lambda}_n^{\alpha,c'} \) and \( (\xi_2 \times \xi_2)(\tilde{\Gamma}_\beta^\alpha) = \tilde{\Gamma}_\beta^{\alpha_{\xi_2^{-1}\xi_1\xi_2}} \). The result now follows promptly, see [37, first part of the proof of Theorem 1.8] and [24, proof of Proposition 4.2].

Using Lemma 3.7, we can prove the invariance under the first colored Markov move.

Proposition 3.8. The multivariable Casson-Lin invariant is preserved under the first colored Markov move.

Proof. Let \( \alpha \) lies in \( (0, \pi)^\mu \), let \( \xi \) be a \( (c, c') \)-braid and let \( \eta \) be a \( (c', c) \)-braid. Applying Lemma 3.7 to the \( (c, c) \)-braid \( \eta \) and to the \( (c', c') \)-braid \( \xi \), we obtain \( h_{\xi\eta}^c(\alpha) = h_{\xi_{\xi_2^{-1}\xi_1\xi_2}}^{c'}(\alpha) = h_{\eta}^{c'}(\alpha) \). This concludes the proof of the proposition.

Proposition 3.9. The multivariable Casson-Lin invariant is preserved under the second colored Markov move.

Proof. Fix \( \alpha \in (0, \pi)^\mu \), a \( \mu \)-coloring \( c \) and a \( (c, c) \)-braid \( \beta \). For the sake of conciseness, we write \( c' \) instead of \( (c_1, \ldots, c_\mu, c_\mu) \) and we recall from Subsection 3.1 that \( i_{c_n} : B_r \to B_r \) denotes the natural inclusion which adds a trivial strand of color \( c_n \) to a given \( (c, c) \)-braid. Viewing the generator \( \sigma_n \in B_{n+1} \) as a \( (c', c') \)-braid, our goal is to show that \( h_{\sigma_n i_{c_n}(\beta)}^{c'}(\alpha) = h_\beta^{c'}(\alpha) \). Using Lemma 3.7, this is equivalent to showing that

\[
h_{i_{c_n}(\beta)\sigma_n}^{c'}(\alpha) = h_\beta^{c'}(\alpha).
\]

(7)
Recall (arranging transversality if necessary) that the right hand side of (7) is defined as the algebraic intersection of the diagonal \( \hat{\Lambda}_{n+1}^{\alpha,c} \) with the graph \( \hat{\Gamma}_\beta \). Similarly, the left hand side of (7) is the algebraic intersection of \( \hat{\Lambda}_{n+1}^{\alpha,c'} \) with \( \hat{\Gamma}_\beta \). In order to relate these various spaces, consider the embedding \( g: R_{n+1}^{\alpha,c} \times R_{n+1}^{\alpha,c'} \to R_n^{\alpha,c} \times R_{n+1}^{\alpha,c'} \) defined by

\[
(X_1,\ldots,X_n,Y_1,\ldots,Y_n) \mapsto (X_1,\ldots,X_n,Y_1,\ldots,Y_n,Y_n).
\]

One can check that \( g(H_n^{\alpha,c}) \subset H_{n+1}^{\alpha,c'} \) and that \( g \) commutes with the conjugation, thus giving rise to an embedding \( \hat{g}: \hat{\Lambda}_n^{\alpha,c}(\beta) \to \hat{\Lambda}_{n+1}^{\alpha,c'}(\beta) \). It can also be checked that \( \hat{g}(\hat{\Lambda}_n^{\alpha,c}(\beta)) \) is contained in \( \hat{\Lambda}_{n+1}^{\alpha,c'}(\beta) \), that \( \hat{g}(\hat{\Gamma}_\beta) \) is contained in \( \hat{\Gamma}_{\beta}^{\alpha,c} \) and that \( \hat{g}(\hat{\Lambda}_n^{\alpha,c}(\beta) \cap \hat{\Gamma}_\beta) \) is equal to \( \hat{\Lambda}_{n+1}^{\alpha,c'}(\beta) \cap \hat{\Gamma}_{\beta}^{\alpha,c} \). Given \( X = (X_1,\ldots,X_n) \) in \( \Lambda_n^{\alpha,c}(\beta) \), the same arguments as in [37, page 346] show that the intersection number of \( \hat{\Lambda}_{n+1}^{\alpha,c} \) and \( \hat{\Gamma}_{\beta}^{\alpha,c} \) at \( \hat{g}(X,X) \) is equal to the intersection number of \( \hat{\Lambda}_n^{\alpha,c} \) and \( \hat{\Gamma}_\beta^{\alpha,c} \) at \( (X,X) \). This proves (7) and concludes the proof of the proposition. \( \square \)

Using the invariance under Markov moves, we now define the main invariant of this paper.

**Definition 3.10.** Let \( L \) be a \( \mu \)-colored link and assume that \( \omega = (e^{2\pi i \alpha_1},\ldots,e^{2\pi i \alpha_\mu}) \in \mathbb{T}_*^\mu \) satisfies \( \Delta_L(\omega) \neq 0 \). The **multivariable Casson-Lin invariant** of \( L \) at \( \omega \) is defined as

\[
h_L(\omega) := h_\beta^c(\alpha),
\]

where \( \beta \) is any \((c,c)\)-braid whose closure is \( L \).

Our reason for defining \( h_L \) on \( \mathbb{T}_*^\mu \) instead of on \((0,\pi)^\mu \) will become apparent in Section 6. Briefly, we will compare \( h_L \) with the multivariable signature, and the latter is defined on \( \mathbb{T}_*^\mu \). We conclude this section with a remark concerning the \( \omega \) for which \( h_L \) can be defined.

**Remark 3.11.** The multivariable Casson-Lin invariant \( h_L \) can be defined for a larger subset of \( \mathbb{T}_*^\mu \) than the complement of the zero locus of \( \Delta_L \). More precisely, one can define \( h_L \) on the subset \( D_L \) of those \( \omega \in \mathbb{T}_*^\mu \) such that the abelian representation \( \rho_\omega \) (recall Section 2.2) is not a limit of irreducible representations. Indeed, looking at the proof of Proposition 3.5, this assumption is sufficient to guarantee that \( \Lambda_n^{\alpha,c} \cap \Gamma_\beta \) is compact in \( \hat{\Lambda}_n^{\alpha,c} \). In particular, note that Corollary 2.6 implies that \( D_L \) contains the complement of the zero locus of \( \Delta_L \) in \( \mathbb{T}_*^\mu \).

4. The colored Gassner matrices and the Potential function

This section is organized as follows. In Subsection 4.1, we recall the definition of the colored Gassner matrices, in Subsection 4.2, we review a result due to Long, in Subsection 4.3, we recall the definition of the multivariable potential function. Finally, in Subsection 4.4, we prove a technical result which shall frequently be used in Section 5.

4.1. The colored Gassner matrices. In this subsection, we recall the definition of the colored Gassner matrices and of the reduced colored Gassner matrices which are multivariable generalizations of the (reduced) Burau matrices. Although references include [34, 11, 15], our conventions are closest to those of [13].

Let \( F_n \) be the free group on \( x_1,\ldots,x_n \). Recall from Subsection 3.1 that the braid group \( B_n \) acts on \( F_n \) from the right and that each \( n \)-stranded braid \( \beta \) gives rise to an automorphism of \( F_n \) which is also denoted by \( \beta \). Given a \( \mu \)-coloring \( c = (c_1,\ldots,c_\mu) \), consider the map

\[
\psi_c: F_n \to \mathbb{Z}^\mu = \{t_1,\ldots,t_\mu\}
\]
which sends each \(x_i\) to \(t_{c_i}\) and extend it to a homomorphism \(\psi_c : \mathbb{Z}[F_n] \to \Lambda_{\mu}\). For later use, observe that if \(\beta\) is a \((c,c)\)-braid, then \(\psi_c(x_i)\) is equal to \(\psi_c(x_i)\beta\) and in fact, both are equal to \(t_{c_i}\). Next, consider the element \(\frac{\partial(x,\beta)}{\partial x_{ij}}\) of the group ring \(\mathbb{Z}[F_n]\), where \(\frac{\partial}{\partial x_{ij}} : \mathbb{Z}[F_n] \to \mathbb{Z}[F_n]\) denotes the Fox derivative associated to \(x_i\) (see e.g. [36, Chapter 11]).

The main definition of this section is the following.

**Definition 4.1.** The ( unreduced) colored Gassner matrix of an \(n\)-stranded \((c,c)\)-braid \(\beta\) is defined as the \(n \times n\) matrix \(G_i(\beta)\) whose \(i,j\)-coefficient is \(\psi_c\left(\frac{\partial x_i(\beta)}{\partial x_{ij}}\right)\).

The notation \(G_i(\beta)\) is meant to indicate that the coefficients of the colored Gassner matrix lie in \(\Lambda_{\mu} = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]\) (i.e. \(t\) is used as a shorthand for \((t_1, \ldots, t_\mu)\)). When \(\mu = 1\), the colored Gassner matrices recover the usual matrices for the Burau representation of \(B_n\). We refer the interested reader to [34, 11] for more intrinsic approaches and to [13, Example 3.5] for \((c,c')\)-braids. Instead, we note that the unreduced colored Gassner matrix of the generator \(\sigma_i \in B_n\), viewed as a \((c,c)\)-braid, is given by

\[
G_i(x) = I_{n-1} \oplus \begin{pmatrix} 1 - t_{c_i} & t_{c_i} \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}.
\]

Next, following [4] and [13, Section 3 (c)], we deal with the reduced colored Gassner matrices. Instead of working with the free generators \(x_1, x_2, \ldots, x_n\) of \(F_n\), we consider the elements \(g_1, g_2, \ldots, g_n\), defined by \(g_i := x_1 x_2 \cdots x_i\). As \(g_n\) is always fixed by the action of the braid group, the matrix whose \(i,j\)-coefficient is \(\psi_c\left(\frac{\partial (g_i,\beta)}{\partial g_j}\right)\) is equal to \(\overline{G}_i(\beta) := \begin{pmatrix} \overline{G}_i(\beta) v \\ 0 \end{pmatrix}\) for some column vector \(v\). This motivates the following definition.

**Definition 4.2.** The reduced colored Gassner matrix of an \(n\)-stranded \((c,c)\)-braid \(\beta\) is defined as the size \(n-1\) matrix \(\overline{G}_i(\beta)\) whose \(i,j\)-coefficient is \(\psi_c\left(\frac{\partial (g_i,\beta)}{\partial g_j}\right)\).

When \(\mu = 1\), the reduced colored Gassner matrices recover matrices for the reduced Burau representation of the braid group \(B_n\). We once again avoid the more intrinsic definition of the reduced colored Gassner representation which involves homology and covering spaces, but instead refer the interested reader to [34, 11] and [15, Theorem 1.2].

We conclude this subsection with a technical lemma which will be needed in Section 5.

**Lemma 4.3.** For any \((c,c)\)-braid \(\beta\), the submodule of fixed points of the unreduced colored Gassner matrix \(G_i(\beta)\) is generated by \(\overline{g}_n = (1 \ t_{c_1} \ t_{c_2} \cdots \ t_{c_{n-1}})\).

**Proof.** We first translate the statement into the \(g_1, \ldots, g_n\) basis of \(F_n\). Namely, computing the change of basis matrix between \(G_i(\beta)\) and \(\overline{G}_i(\beta)\) (see (15) below), the statement is equivalent to the claim that the submodule of fixed points of \(\overline{G}_i(\beta)\) is freely generated by \(x = (0 \cdots 01)\). Here, our convention is that the Burau matrices act on the right on row vectors.

Since \(x\) is fixed by \(\overline{G}_i(\beta)\), we suppose that \(w = (w_1 \cdots w_{n-1} \ w_n)\) is fixed by \(\overline{G}_i(\beta)\) and wish to show that \(w\) lies in the span of \(x\). Using Definition 4.2, the assumption on \(w\) implies that the reduced colored Gassner matrix \(\overline{G}_i(\beta)\) must fix \(w' := (w_1 \cdots w_{n-1})\) (recall that we are using right actions). As a consequence, the free \(\Lambda_{\mu}\)-module generated by \(w'\) is a rank 1 invariant submodule of \(\overline{G}_i(\beta)\). Since the reduced colored Gassner matrix is irreducible [1], \(w'\) must vanish. Thus, \(w = (0 \cdots 0 \ w_n)\) lies in the span of \(x\), concluding the proof of the lemma. \(\square\)
4.2. A result due to Long. The goal of this subsection is to recall a theorem due to Long [38, Theorem 2.4]. In order to state this result, we use Long’s conventions regarding automorphisms of the free group. As we observed in Remark 3.2, these conventions match ours when dealing with the action of the braid group \( B_n \) on \( SU(2)^n \).

For an automorphism \( \theta : F_n \to F_n \) of the free group, consider the diffeomorphism \( \theta^* : R(F_n) \to R(F_n), \rho \mapsto \rho \circ \theta^{-1} \). Picking free generators \( x_1, \ldots, x_n \) of \( F_n \) and identifying \( R(F_n) \) with \( SU(2)^n \), the diffeomorphism \( \theta^* \) is described as \( \theta^*(X_1, \ldots, X_n) = (\theta^{-1}X_1, \ldots, \theta^{-1}X_n) \). The assignment \( \theta \mapsto \theta^* \) gives rise to a homomorphism \( \text{Aut}(F_n) \to \text{Diff}(SU(2)^n) \). Fixing a subgroup \( H \) of \( \text{Aut}(F_n) \), the restriction of this assignment produces a homomorphism \( H \to \text{Diff}(SU(2)^n) \). To get a linear representation of \( H \), pick a function \( f : (0, \pi)^\mu \to SU(2)^n \) such that \( h^*f(\alpha) = f(\alpha) \) for every \( \alpha = (\alpha_1, \ldots, \alpha_\mu) \) in \( (0, \pi)^\mu \) and for every \( h \) in \( H \), and set

\[
\rho_\alpha : H \to \text{Aut}(T_f(\alpha) SU(2)^n)
\]

\[
h \mapsto T_f(\alpha)(h^*).
\]

The fact that \( \rho_\alpha \) is a representation follows from the chain rule [38, Theorem 2.3]. We now restrict to the colored braid group \( B_c \) and, for \( \theta \in (0, \pi) \), we set \( e^{i\theta} := \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \).

Recalling the notations and conventions discussed in Remark 3.2, we observe that \( f(\alpha) := (e^{i\alpha_1}, \ldots, e^{i\alpha_\mu}) \) satisfies \( f(\alpha)\beta = f(\alpha) \) for every \((c, c)\)-braid \( \beta \). As a consequence, we obtain representations \( \rho_\alpha \) of \( B_c \). Long [38, Theorem 2.4] proves the following result:

**Proposition 4.4.** Let \( c = (c_1, \ldots, c_n) \) be a \( \mu \)-coloring and let \((\alpha_1, \ldots, \alpha_\mu) \) lie in \((0, \pi)^\mu \). If one sets \( a = (e^{i\alpha_1}, \ldots, e^{i\alpha_\mu}) \), then the representation \( \rho_\alpha : B_c \to \text{Aut}(T_a SU(2)^n) \) is a direct sum of a permutation representation with the colored Gassner matrix evaluated at \( \omega = (e^{2i\alpha_1}, \ldots, e^{2i\alpha_\mu}) \).

Note that Long proved this result for \( \mu = 1 \) [38, Theorem 2.4] and \( \mu = n \) [38, Theorem 2.5] but his proof goes through for arbitrary colored braid groups. In order to make some further remarks on Proposition 4.4, we recall some known facts regarding the field \( \mathbb{H} \) of quaternions.

**Remark 4.5.** We think of \( \mathbb{H} \) using the isomorphisms \( \mathbb{H} \cong \mathbb{C} \oplus j\mathbb{C} \cong (\mathbb{R} \oplus i\mathbb{R}) \oplus (j\mathbb{R} \oplus k\mathbb{R}) \) and recall that a quaternion is pure if its real part is zero. Matrices in \( SU(2) \) can be identified with unit quaternions via the map which sends \( \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \) to \( a + jb \), for any \( a, b \in \mathbb{C} \) which satisfy \( |a|^2 + |b|^2 = 1 \). On the Lie algebra level, for \( r \in \mathbb{R} \) and \( z \in \mathbb{C} \), matrices \( \begin{pmatrix} ir & z \\ -z & -ir \end{pmatrix} \) in \( su_2 \) correspond to quaternions \( ir + jz \), and in particular \( su_2 \) splits as \( i\mathbb{R} \oplus j\mathbb{C} \).

Using Remark 4.5 and working with the notations of Proposition 4.4, Long’s result shows that the restriction of the differential of \( \beta : SU(2)^n \to SU(2)^n \) at \( a \) to the complex summand of \( su_2 \) is \( B_c^\alpha(\beta) \) (i.e. the colored Gassner matrix evaluated at \( \omega \)). In Section 5 however, we shall study the restriction of \( \beta \) to \( R^a_{n,c} \). Since this latter space is homeomorphic to a product of 2-spheres \( S_{\alpha_j} \) which consist of those matrices with trace 2 cos(\( \alpha_j \)), we adapt some observations from [29, Section 2.3] to the multivariable case.

**Remark 4.6.** Matrices in \( SU(2) \setminus \pm I \) can be identified with pairs \((\theta, Q)\), where \( \theta \in (0, \pi) \) and \( Q = xi + yj +zk \) is a pure quaternion of norm 1. More explicitly, the quaternion \( \cos(\theta) + i\sin(\theta)Q \) associated to a pair \((\theta, Q)\) corresponds to the \( SU(2) \)-matrix

\[
X = \begin{pmatrix} \cos(\theta) + ix \sin(\theta) & (y + iz) \sin(\theta) \\ (y - iz) \sin(\theta) & \cos(\theta) - ix \sin(\theta) \end{pmatrix}.
\]

On the Lie algebra level, using \( j^2 = -1 \) and the identification of \( su_2 \) with \( i\mathbb{R} \oplus j\mathbb{C} \), multiplication by \( -j \) picks out the complex component \( z \) of the matrix \( \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \in su_2 \). In particular, since \( R^a_{n,c} \)
is a product of $S_6$, Proposition 4.4 implies that the following diagram commutes:

$$
\begin{array}{ccc}
T_n \mathcal{P}^{\alpha,c} & \xrightarrow{-j} & \mathbb{C}^n \\
\downarrow T_n \beta & & \downarrow \mathcal{B}_\beta \\
T_n \mathcal{P}^{\alpha,c} & \xrightarrow{-j} & \mathbb{C}^n.
\end{array}
$$

On the topological level, it is helpful to think of SU(2) as foliated by the spheres $S_6$: indeed the quaternionic expression $\cos(\theta) + \sin(\theta)Q$, specifies a 2-sphere $S_6$ and a position $Q$ on this sphere. On the Lie algebra level, the complex lines are tangent to the leaves $S_6$ and the real lines are tangent to the transverse directions.

4.3. The potential function. In this section, we review some facts about the multivariable potential function. References include [16, 26, 9, 41].

As we recalled in Section 2, the multivariable Alexander polynomial $\Delta_L$ of a $\mu$-colored link $L$ is only well defined up to multiplication by units of $\Lambda$. The multivariable potential function of a $\mu$-colored link $L$ is a rational function $\nabla_L(t_1, \ldots, t_\mu)$ which satisfies

$$
\nabla_L(t_1, \ldots, t_\mu) = \begin{cases} \frac{1}{t_1 - t_1^{-1}} \Delta_L(t_1^2) & \text{if } \mu = 1, \\ \Delta_L(t_1^2, \ldots, t_\mu^2) & \text{if } \mu > 1. \end{cases}
$$

In this paper, we use a construction of the potential function which arises from the reduced colored Gassner representation [15, Theorem 1.1]. The next remark briefly recalls this result.

**Remark 4.7.** Any $(c,c)$-braid $\beta$ can be decomposed into a product $\prod_{j=1}^m \sigma_{i_j}^{\epsilon_j}$, where $\sigma_{i_j}$ denotes the $i_j$-th generator of the braid group (viewed as an appropriately colored braid) and each $\epsilon_j$ is equal to $\pm 1$. For each $j$, use $b_j$ to denote the color of the over-crossing strand in the generator $\sigma_{i_j}^{\epsilon_j}$ and consider the Laurent monomial

$$
\langle \beta \rangle := \prod_{j=1}^m t_{b_j}^{-\epsilon_j}.
$$

Define $g: \Lambda \rightarrow \Lambda$ by extending $\mathbb{Z}$-linearly the group endomorphism of $\mathbb{Z}^\mu = \langle t_1, \ldots, t_\mu \rangle$ which sends $t_i$ to $t_i^2$. Given an $n$-stranded $\mu$-colored $(c,c)$-braid $\beta$, [15, Theorem 1.1] shows that the multivariable potential function of the closure $\beta$ can be described as:

$$
\nabla_\beta(t_1, \ldots, t_\mu) = (-1)^{n+1} \cdot \frac{1}{t_{c_1} \cdots t_{c_n} - t_{c_1}^{-1} \cdots t_{c_n}^{-1}} \cdot \langle \beta \rangle \cdot g(\det(\overline{B}_i(\beta) - I_{n-1})).
$$

Note that in [15], the matrices $\overline{B}_i$ are the transposes of the ones used here (and in particular [15] deals with anti-representations). Naturally, this does not affect (10).

In the one-variable case, some care is needed with the terminology.

**Remark 4.8.** The expression $D_L(t) := \nabla(t)(t - t^{-1})$ is usually referred to as the Alexander-Conway polynomial of $L$ and satisfies $D_L(t) = \Delta_L(t^2)$. On the other hand, some authors call $D_L(\sqrt{t})$ the Conway-normalized Alexander polynomial. For instance Heusener-Kroll use $\Delta_L(t)$ to denote the Conway-normalized Alexander polynomial [29, Section 2.1]. These distinctions do matter: for a knot $K$ and $\omega \in S^1$, it is known that $D_K(\sqrt[4]{\omega})$ is real, while this statement is incorrect for $\nabla_K$ and makes no sense for $\Delta_K$ (because of the indeterminacy).
We conclude with some remarks on evaluations of $\nabla_L$ at elements of $\mathbb{T}^\mu = (S^1)^\mu$.

**Remark 4.9.** The potential function $\nabla_L$ of an $n$-component $\mu$-colored link is known to be $(-1)^n$-symmetric [9, Proposition 1]. Thus, for $\omega \in \mathbb{T}^\mu$, the evaluation $\nabla_L(\omega)$ need not be real. In fact, for $\omega \in \mathbb{T}^\mu$, the aforementioned symmetry property yields $\nabla_L(\omega) = \nabla_L(\bar{\omega}) = (-1)^n \nabla_L(\omega)$, and therefore $\nabla_L(\omega)$ belongs to $\mathbb{R}$ (resp. $i\mathbb{R}$) if $n$ is even (resp. odd). In particular, if two $\mu$-colored links differ by a crossing change within a sublink, then the quotient of the two potential functions evaluated at $\omega \in \mathbb{T}^\mu$ is real (assuming the quotient is defined).

### 4.4. A technical proposition.

The aim of this section is to prove the following multivariable generalization of [29, Lemma 4.4]. This result will be frequently used in Section 5.

**Proposition 4.10.** Let $c$ be a $\mu$-coloring such that $c_1 = c_2$, let $\alpha$ be an element of $(0, \pi)^\mu$ and set $\omega_k = e^{2i\alpha_k}$ for $k = 1, \ldots, \mu$. Given a $(c, c)$-braid $\beta$, use $A(\omega) B(\omega) C(\omega) D(\omega)$ to denote the unreduced colored Gassner matrix of $\beta$ evaluated at $\omega \in \mathbb{T}^\mu$, where $D(\omega)$ is a size $n-2$ square matrix. If we assume that $\omega_1^2 \neq 1$ and $\nabla_{\beta}^2(\omega) \neq 0$, then $\det(D(\omega) - I_{n-2}) \neq 0$.

The proof of Proposition 4.10 follows the strategy of [29, Lemma 4.4]. However several of the preliminary results require some additional work in the multivariable case.

#### 4.4.1. Rows and columns of $B_i^c(\beta)$.

We temporarily adopt the following conventions: given a matrix $\Psi$, we write $\Psi^i$ for the $(i, j)$-coefficient of $\Psi$, instead of the more standard $\Psi_{ij}$; apart if mentioned otherwise, $I$ denotes any identity matrix, regardless of its size.

The following lemma (which generalizes well known results for the Burau representation) describes the result of summing (linear combinations of) the rows and columns of the unreduced colored Gassner matrices.

**Lemma 4.11.** Given a $(c, c)$-braid $\beta$, the rows and columns of the colored Gassner matrix satisfy the following properties:

1. For each $i$, one has $\sum_{j=1}^n (t_{c_j} - 1) B_i^c(\beta)^j = t_{c_i} - 1$.
2. For each $j$, one has $\sum_{i=1}^n t_{c_1} \cdots t_{c_{j-1}} B_i^c(\beta)^j = t_{c_1} \cdots t_{c_{j-1}}$.

**Proof.** In order to prove both of these identities, we recall the so-called “fundamental lemma of Fox calculus” [8, Proposition 9.8, part c)]. Given a word $w$ in the free group $F_n$ on $x_1, \ldots, x_n$, the following identity holds in the group ring $\mathbb{Z}[F_n]$:

\begin{equation}
\sum_{j=1}^n \frac{\partial w}{\partial x_j} (x_j - 1) = w - 1.
\end{equation}

The first identity now follows by considering the word $w = x_i \beta$, applying $\psi_c$ to both sides of (11) and recalling that for a $(c, c)$-braid, both $\psi_c(x_i)$ and $\psi_c(x_i \beta)$ are equal to $t_{c_i}$. To obtain the second formula, apply the Fox derivative $\frac{\partial}{\partial x_j}$ to both sides of the equality $(x_1 \cdots x_n) \beta = x_1 \cdots x_n$ and use the derivation property repeatedly. \hfill $\Box$

Taking advantage of our unconventional notation, observe that the $i$-th column of $B_i^c(\beta)$ can be written as $B_i^c(\beta)_i$, while the $i$-th line of $B_i^c(\beta)$ can be written as $B_i^c(\beta)^i$. In particular, Lemma 4.11 implies that

$$
\sum_{i=1}^n (t_{c_i} - 1) B_i^c(\beta)_i = (T - 1), \quad \sum_{i=1}^n (t_{c_1} \cdots t_{c_{i-1}}) B_i^c(\beta)^i = v,
$$
where $T - 1$ denotes the size $n$ column vector whose $i$-th component is $t_{ci} - 1$ and $v$ denotes the size $n$ row vector whose $j$-th component is $t_{cj-1}$.

**Example 4.12.** If $c = (1, \ldots, 1)$, the first point of Lemma 4.11 implies the following known fact: the sum of the coefficients within any line of the Burau matrix is 1 (i.e. the Burau matrix is a “right stochastic matrix”). For $\sigma_1^2 \in B_{(1,2)} = P_2$, the Gassner matrix is given by

$$B_{c_1,c_2}(\sigma_1^2) = \begin{pmatrix} 1 - t_1 + t_2 t_1(1 - t_1) \\ 1 - t_2 \\ t_1 \end{pmatrix}.$$ 

Now Lemma 4.11 states in particular that $(1 - t_1)(1 - t_1 + t_2 t_1(1 - t_1) = 1 - t_1$ and $t_1(1 - t_1) + t_2 t_1 = t_1$ which can indeed be verified.

4.4.2. **Compositions with minors.** Given a square matrix $\Psi$ of size $n$, we use $\Psi_{i,j}$ to denote the size $(n - 1)$ matrix obtained from $\Psi$ by deleting its $i$-th row and $j$-th column. We also use $B_{c}(\beta,l,m)$ to denote $\det((B_{c}(\beta) - I)_{l,m})$ (the notation $c_{l,m}$ is used in [29, Section 2.4]).

The following lemma is a multivariable generalization of [29, Lemma 2.2, part 1]).

**Lemma 4.13.** Let $c$ be a $\mu$-coloring. Given an $n$-stranded $(c,c)$-braid $\beta$ and positive integers $1 \leq l, l', m, m' \leq n$, the following equality holds in $\Lambda_{\mu}$:

$$(t_{cm'} - 1)(t_{c_1} \cdots t_{cm-1})B_{c}(\beta,l,m) = (-1)^{m+m'+l'+l}(t_{cm} - 1)(t_{c_1} \cdots t_{cm-1})B_{c}(\beta,l',m').$$

**Proof.** To prove the lemma, it suffices to prove (12) when $l = l'$ and when $m = m'$. We therefore start by assuming that $l = l'$ and claim that

$$(t_{cm'} - 1)B_{c}(\beta,l,m) = (-1)^{m+m'}(t_{cm} - 1)B_{c}(\beta,l,m').$$

Recall that $T - 1$ denotes the size $n$ column vector whose $i$-th component is $t_{ci} - 1$ and assume that $i$ differs from $m$. Using the first point of Lemma 4.11, a short computation shows that

$$(t_{ci} - 1)(B_{c}(\beta) - I)_i = \left((T - 1) - \sum_{k \neq i} (t_{ck} - 1)B_{c}(\beta)_k\right) - (t_{ci} - 1)I_i = - \sum_{k \neq i} (t_{ck} - 1)(B_{c}(\beta) - I)_k.$$

We now use this identity to compute the determinant of the matrix $B_{c}(\beta) - I)$ obtained by removing the $l$-th row and the $m$-th column from $B_{c}(\beta) - I$. Multiplying the $i$-th column of $B_{c}(\beta) - I$ by $t_{ci} - 1$, using (13), removing the $m$-th column of $B_{c}(\beta) - I$, invoking the multilinearity of the determinant and switching back the $i$-th column to its original place (this produces a sign $(-1)^{i+m-1}$ since we now have one column less), we obtain

$$(t_{ci} - 1)B_{c}(\beta,l,m) = (t_{cm} - 1)(-1)^{i+m}B_{c}(\beta,l,mi).$$

The claim now follows by taking $i = m'$. To prove (12) for $m = m'$, one uses the second point of Lemma 4.11 and follows the exact same steps as above with rows instead of columns. This concludes the proof of the lemma. $\square$

Lemmas 4.11 and 4.13 involve the colored Gassner matrices in the basis arising from the choice of generators $x_1, \ldots, x_n$ of the free group $F_n$. In order to work with the reduced colored Gassner matrices, we need the corresponding statements for the basis $g_1, \ldots, g_n$ of $F_n$. 


Proposition 4.15. That the conclusion will then follow from Remark 4.14 and Lemma 4.13 which imply respectively: It suffices to show that

\[ \psi \circ \gamma \colon \tilde{B}_1^r(\beta, 1, n-1) = (t_{c_1} \cdots t_{c_n} - 1) \tilde{B}_1^r(\beta, n, n). \]

The proof of (14) is entirely analogous to the proof of Lemma 4.13: it suffices to use the equality \( \psi_c(g_i) = t_{c_1} \cdots t_{c_i} \) instead of \( \psi_c(x_i) = t_{c_i} \). Finally note that (14) can be rewritten using the reduced colored Gassner representation. Indeed, using Definition 4.2, we have \( \tilde{B}_1^r(\beta, n, n) = \det(\tilde{B}_1^r(\beta) - I_{n-1}) \), where \( I_{n-1} \) denotes the size \( n-1 \) identity matrix.

We now relate \( \det(\tilde{B}_1^r(\beta) - I_{n-1}) \) and \( \tilde{B}_1^r(\beta, 1, 1) \), generalizing [29, Lemma 2.2.2].

Proposition 4.15. Given an \( n \)-stranded \((c, c)\)-braid \( \beta \), the following equation holds:

\[ \frac{1}{t_{c_1}} \cdots \frac{1}{t_{c_n}} \tilde{B}_1^r(\beta, 1, 1) = \det(\tilde{B}_1^r(\beta) - I_{n-1}). \]

Proof. It suffices to show that \( \tilde{B}_1^r(\beta, n, n)(t_{c_1} \cdots t_{c_n} - 1) = (t_{c_1} \cdots t_{c_n} - 1) \tilde{B}_1^r(\beta, n, n) \): the conclusion will then follow from Remark 4.14 and Lemma 4.13 which imply respectively that \( \tilde{B}_1^r(\beta, n, n) = \det(\tilde{B}_1^r(\beta) - I_{n-1}) \) and \( (t_{c_1} - 1)(t_{c_2} \cdots t_{c_{n-1}}) \tilde{B}_1^r(\beta, 1, 1) = (t_{c_1} - 1) \tilde{B}_1^r(\beta, n, n) \). A computation involving Fox calculus shows that the change of basis matrix from \( \tilde{B}_1^r(\beta) \) to \( \tilde{B}_1^r(\beta) \) is given by

\[ P_n = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & t_{c_1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & 0 \\
1 & \cdots & \cdots & \cdots & t_{c_1} \cdots t_{c_{n-2}} \cdots t_{c_{n-2}} t_{c_{n-1}}
\end{pmatrix}. \]

Given a matrix \( M \), recall that we use \( M_{n,n} \) to denote the matrix obtained by deleting the \( n \)-th row and \( n \)-th column of \( M \). Until the end of this proof, we use \( I \) to denote the size \( n \) identity matrix. With this notation, observe that \( \tilde{B}_1^r(\beta, n, n) = \det((P_n \tilde{B}_1^r(\beta) P_n^{-1} - I)_{n,n}) \).

A tedious computation now shows that

\[ \det((P_n \tilde{B}_1^r(\beta) P_n^{-1} - I)_{n,n}) = \det(P_{n-1}(\tilde{B}_1^r(\beta)_{n,n} P_{n-1}^{-1} - I_{n-1}) - \tilde{B}_1^r(\beta, n, n - 1). \]

Using the definition of \( \tilde{B}_1^r(\beta) \) and the fact that the determinant is invariant under conjugation, this can be rewritten as \( \tilde{B}_1^r(\beta, n, n) = \tilde{B}_1^r(\beta, n, n) - \tilde{B}_1^r(\beta, n, n - 1) \). The conclusion then follows by using (14). This concludes the proof of the proposition.

4.4.3. Relation to the potential function. As in Remark 4.7, \( g \colon \Lambda_\mu \to \Lambda_\mu \) is defined by extending \( \mathbb{Z} \)-linearly the group endomorphism of \( \mathcal{D}^\mu = \langle t_1, \ldots, t_\mu \rangle \) which sends \( t_i \) to \( t_i^2 \). The following lemma expresses \( \nabla_{x_\beta} \) using a minor of the un reduced colored Gassner matrix.

Lemma 4.16. Given a \( \mu \)-colored \( n \)-stranded \((c, c)\)-braid \( \beta \), we have

\[ \frac{1}{t_{c_1}^2 - 1} \frac{1}{t_{c_2}^2 - 1} \nabla_{x_\beta}(t_1, \ldots, t_\mu) = \frac{(-1)^{n+1}(\beta)}{t_{c_1} \cdots t_{c_n} - t_{c_1} \cdots t_{c_n} \cdots t_{c_{n-1}} g(\tilde{B}_1^r(\beta, 1, 1)), \]

where \( \langle \beta \rangle \) is the Laurent monomial described in Remark 4.7.
Proof. Using successively Remark 4.7 and Proposition 4.15, we obtain
\begin{equation}
\nabla_\beta(t_1, \ldots, t_\mu) = \frac{(-1)^{n+1}(\beta)}{t_{c_1} \cdots t_{c_n} - t_{c_1}^{-1} \cdots t_{c_n}^{-1}} g(\det(B^c_i(\beta) - I_{n-1}))
\end{equation}
\begin{equation}
= \frac{(-1)^{n+1}(\beta)}{t_{c_1} \cdots t_{c_n} - t_{c_1}^{-1} \cdots t_{c_n}^{-1}} \frac{t_{c_1}^2 \cdots t_{c_n}^2 - 1}{t_{c_1}^2 - 1} g(B^c_i(\beta, 1, 1)).
\end{equation}

This concludes the proof of the lemma. \hfill \Box

We need one last lemma in order to prove Proposition 5.7, namely we require a multivariable generalization of [29, Lemma 2.2, part 3]. For that purpose, we write the (unreduced) colored Gassner matrix of \( \beta \) as \( (A \ B \ D) \), where \( D \) is a square matrix of size \( n - 2 \).

**Lemma 4.17.** Let \( c \) be a \( \mu \)-coloring with \( c_1 = c_2 \) and let \( \beta \) be an \( n \)-stranded \((c, c)\)-braid. If the generator \( \sigma_1 \in B_n \) is viewed as a \((c, c)\)-braid, then the following equality holds in \( \Lambda_\mu \):
\begin{equation}
B^c_i(\sigma_1^2 \beta, 1, 1) = t_{c_1}^2 B^c_i(\beta, 1, 1) + (t_{c_1} - 1) \det(D - I_{n-2}).
\end{equation}

**Proof.** First, a short computation shows that \( B^c_i(\sigma_1^2 \beta) = \left( \begin{array}{ccc}
(1 - t_{c_1}^{-1})^2 & 1 & -1 \\
1 & t_{c_1}(1 - t_{c_1}) & t_{c_1} \\
0 & 0 & 1
\end{array} \right) \oplus I_{n-2} \). Next, recalling that we decomposed the colored Gassner matrix of \( \beta \) as \( (A \ B \ D) \), we write the matrix \( A \) as \( \left( \begin{array}{ccc}
b_1 & a_1 & a_2 \\
a_1 & b_1 & a_2 \\
a_2 & b_2 & b_2
\end{array} \right) \) where each \( b_i \) is a size \( n - 2 \) row vector and the matrix \( C \) as \( (c_1, c_2) \), where each \( c_i \) is a size \( n - 2 \) column vector. As (17) does not involve the first lines and columns of the aforementioned matrices, we are reduced to proving
\begin{equation}
\det\left(\begin{array}{ccc}
(1 - t_{c_1})a_{12} + t_{c_1} a_{22} - 1 & 1 - t_{c_1} & b_1 + t_{c_1} b_2 \\
D - I
\end{array}\right) = t_{c_1}^2 \det\left(\begin{array}{ccc}
a_{22} - 1 & b_2 \\
c_2 & D - I
\end{array}\right) + (t_{c_1} - 1) \det(D - I),
\end{equation}

where we use \( I \) as a shorthand for the identity matrix \( I_{n-2} \). Expanding the left hand side of (18) along the first row, we obtain
\begin{equation}
((1 - t_{c_1})a_{12} + t_{c_1} a_{22} - 1) \det(D - I) + \sum_{j=2}^{n-1} (-1)^{j+1} ((1 - t_{c_1})b_1^j + t_{c_1} b_2^j) \det(L_j),
\end{equation}

where, for \( j \) greater than one, \( L_j \) denotes the size \( n - 2 \) square matrix obtained from \((c_2 \ D - I)\) by removing the \( j \)-th column. Keeping these notations in mind and expanding the determinant in the right hand side of (18) along its first line, we obtain
\begin{equation}
t_{c_1}^2 \left( (a_{22} - 1) \det(D - I) + \sum_{j=2}^{n-1} (-1)^{j+1} j b_2^j \det(L_j) \right) + (t_{c_1} - 1) \det(D - I).
\end{equation}

Subtracting (20) from (19) and simplifying the extraneous \( t_{c_1} - 1 \) factors, we see that (18) in fact reduces to proving the equation \(-B^c_i(\beta, 2, 1) - t_{c_1} B^c_i(\beta, 1, 1) = 0\). Since the latter equation holds thanks to Lemma 4.13, the proof is concluded. \hfill \Box

4.4.4. **Conclusion of the proof.**

**Proof of Proposition 4.10.** Let \( \omega \in \mathbb{T}^n \) be such that \( \nabla_\beta(\omega) \) is non-zero. Our goal is to show that \( \det(D(\omega) - I_{n-2}) \) is non-zero. Use \( B^c_i(\beta) \) to denote the unreduced colored Gassner matrix of \( \beta \) evaluated at \( \omega \). Assume by way of contradiction that \( \det(D(\omega) - I_{n-2}) \) vanishes. Using
Lemma 4.17, this implies that $B^c_\omega(\sigma^2_1 \beta, 1, 1) = \omega^2_{c_1} B^c_\omega(\beta, 1, 1)$. Combining this equality with Lemma 4.16 and the fact that $(\sigma^2_1 \beta) = t^{-2}_c(\beta)$ (and assuming that $\omega^2_{c_1} \cdots \omega^2_{c_n} \neq 1$) we get

$$\frac{\omega^2_{c_1} - 1}{\omega^2_{c_1} \cdots \omega^2_{c_n} - 1} \nabla_{\sigma^2_1 \beta}(\omega) = \frac{(-1)^{n+1} \langle \sigma^2_1 \beta \rangle}{\omega_{c_1} \cdots \omega_{c_n} - \omega^2_{c_1} \cdots \omega^2_{c_n} - 1}(\omega^2_{c_1} - 1)g(B^c_\omega(\sigma^2_1 \beta, 1, 1))$$

(21)

$$= \frac{\omega^2_{c_1} - 1}{\omega^2_{c_1} \cdots \omega^2_{c_n} - 1} \nabla_\beta(\omega).$$

Note that we slightly abused notation by thinking of $g$ as being defined on $\mathbb{C}$ and noting that $g(B^c_\omega(\sigma^2_1 \beta, 1, 1)) = \omega^2_{c_1} g(B^c_\omega(\beta, 1, 1))$. Regardless of this fact, simplifying the extraneous terms, we obtain the equality $\nabla_{\sigma^2_1 \beta}(\omega) = \omega^2_{c_1} \nabla_\beta(\omega)$. We let the reader verify that this conclusion also holds if $\omega^2_{c_1} \cdots \omega^2_{c_n} = 1$. Since we assumed that $\nabla_\beta(\omega) \neq 0$, we deduce that $\nabla_{\sigma^2_1 \beta}(\omega) \neq 0$.

As Remark 4.9 implies that the quotient $\nabla_{\sigma^2_1 \beta}(\omega)/\nabla_\beta(\omega)$ is real, we obtain a contradiction when $\omega^2_{c_1}$ is different from 1. This concludes the proof of Proposition 4.10. □

Note that in their equivalent of (21), Heusener and Kroll work with the Conway-normalized Alexander polynomial which they denote $\Delta(\beta)$. This explains why they obtain the equality $\Delta_K(t) = \omega \Delta_k(t)$ [29, last equation of p.494], while we have a $\omega^2_{c_1}$ factor.

5. The multivariable Casson-Lin invariant and crossing changes

The goal of this section is to understand the behavior of the multivariable Casson-Lin invariant under a crossing change within a sublink. In Subsection 5.1, we reduce this analysis to a computation in a space $\tilde{H}^{\alpha_i}_2$, in Subsection 5.2, we perform calculations in $\tilde{H}^{\alpha_i}_2$ which are then reformulated in Subsection 5.3 in terms of the multivariable potential function.

5.1. Reduction to a “pillowcase-like” space. Let $c$ be a $\mu$-coloring such that $c_1 = c_2 = j$. Let $\beta$ be an $n$-stranded $(c, c)$-braid and view the generator $\sigma_1 = B_n$ as a $(c, c)$-braid. Let $\alpha$ be an element of $(0, \pi)^\mu$ such that $\alpha_{c_1} = \alpha_{c_2}$ and $\Delta_\beta(\omega), \Delta_{\sigma^2_1 \beta}(\omega) \neq 0$, where $\omega_k = e^{2\alpha_k}$ for $k = 1, \ldots, \mu$. In order to understand the effect of a single crossing change within a sublink on the multivariable Casson-Lin invariant $h_L$, we will study

$$h_{\sigma^2_1 \beta}(\alpha) - h_\beta(\alpha).$$

(22)

Indeed the links $L := \tilde{\beta}$ and $\sigma^2_1 \beta$ differ by a single crossing change within the sublink $L_j$ and any such (negative to positive) crossing change within a colored link can be realized in this way, see the proof of Proposition 5.9 below for further details. The first step in understanding (22) is to consider the following set:

$$V^{\alpha, c}_n = \{(A_1, \ldots, A_n, B_1, \ldots, B_n) \in H^{\alpha, c}_n | A_i = B_i \text{ for } i = 3, \ldots, n\}.$$  

Use $c'$ to denote $(c_3, \ldots, c_n)$ so that $c = (c_1, c_2, c')$. Note that $V^{\alpha, c}_n$ is homeomorphic to $H^{\alpha_1}_2 \times A^{\alpha, c'}_{n-2}$ and set $\hat{V}^{\alpha, c}_n := (H^{\alpha_1}_2 \setminus S^{\alpha_1}_{n-2})/\text{SO}(3)$. Using Lemma 3.4, we deduce that this latter space is a smooth submanifold of $\hat{H}^{\alpha, c}_n$ whose dimension is $2n - 2$. We then consider the projection $\hat{p}: \hat{V}^{\alpha, c}_n \to \hat{H}^{\alpha_1}_2$ induced by the following map:

$$p(X_1, X_2, X_3, \ldots, X_n, Y_1, Y_2, Y_3, \ldots, Y_n) = (X_1, X_2, Y_1, Y_2).$$

Arguing as in [37], and perturbing if necessary, we can assume that $\hat{V}^{\alpha, c}_n$ and $\Gamma^{c}_{\beta}$ intersect transversally and (22) can be computed by considering curves inside the 2-dimensional
space \( \hat{H}^{\alpha_j}_2 \). More precisely, using \((\cdot,\cdot)\) to denote the algebraic intersection number, the same arguments as in [37, Lemma 2.3] and [29, Equation (4))] show that

\[
(23) \quad h^c_{\sigma_1^\beta}(\alpha) - h^c_\beta(\alpha) = (\hat{\Gamma}^{\alpha_j}_{\sigma_1^2} - \hat{\Lambda}^{\alpha_j}_2, \hat{p}(\hat{V}^{\alpha,c}_{\sigma} \cap \hat{\Gamma}^{\alpha}_\beta))_{\hat{p}^c}.
\]

Note that we are adopting the following convention: we are writing \(\hat{\Gamma}^{\alpha_j}_{\sigma_1^2}, \hat{\Lambda}^{\alpha_j}_2\) and \(\hat{H}^{\alpha_j}_2\) instead of \(\hat{p}(\hat{V}^{\alpha,c}_{\sigma} \cap \hat{\Gamma}^{\alpha}_\beta)\) which would be more coherent with the previous notation. Summarizing, (23) shows that the difference of the multivariable Casson-Lin invariants (which are defined via algebraic intersections in \(\hat{H}^{\alpha,c}_n\)) can be understood in the more manageable space \(\hat{H}^{\alpha_j}_2\) by studying intersections with the difference cycle \(\hat{\Gamma}^{\alpha_j}_{\sigma_1^2} - \hat{\Lambda}^{\alpha_j}_2\).

5.2. Computations in \(\hat{H}^{\alpha_j}_2\). The goal of this subsection is to understand whether the projection \(\hat{p}(\hat{V}^{\alpha,c}_{\sigma} \cap \hat{\Gamma}^{\alpha}_\beta)\) intersects the difference cycle \(\hat{\Gamma}^{\alpha_j}_{\sigma_1^2} - \hat{\Lambda}^{\alpha_j}_2\): using (23), this will provide a formula for the difference \(h^c_{\sigma_1^\beta}(\alpha) - h^c_\beta(\alpha)\).

We first recall the parametrization of \(\hat{H}^{\alpha_j}_2 = \{(X_1, X_2, Y_1, Y_2) \in \text{SU}(2)^4 \mid \text{Tr}(X_i) = \text{Tr}(Y_i) = 2\cos(\alpha_j), X_1X_2 = Y_1Y_2\}\) which was obtained by Lin for \(\alpha_j = \pi/2\) [37, Lemma 2.1] and by Heusener-Kroll for \(\alpha_j \neq \pi/2\) [29, Lemma 4.1]. Although the proofs may be found in the aforementioned references, we provide an outline of the arguments in order to introduce some notation which we shall use throughout the section.

**Lemma 5.1.** Given \(\alpha_j \in (0, \pi)\), the space \(\hat{H}^{\alpha_j}_2\) is homeomorphic to

1. a 2-sphere with four points deleted if \(\alpha_j = \pi/2\),
2. a 2-sphere with three points deleted if \(\alpha_j \neq \pi/2\).

**Proof.** For \(X, Y \in \text{SU}(2)\), consider the \(\text{SU}(2)\)-invariant distance on \(\text{SU}(2)\) given by \(d(X, Y) := \arccos\left(\frac{\text{Tr}(X^{-1}Y)}{2}\right)\). Notice that this distance realizes the distance induced by the standard spherical metric on \(\mathbb{S}^3\). Let \((X_1, X_2, Y_1, Y_2)\) lie in \(\hat{H}^{\alpha_j}_2\). Up to conjugacy, one can assume that

\[
X_1 = \begin{pmatrix}
\cos(\alpha_j) + i\sin(\alpha_j)\cos(\theta_1) & \sin(\alpha_j)\sin(\theta_1) \\
-\sin(\alpha_j)\sin(\theta_1) & \cos(\alpha_j) - i\sin(\alpha_j)\cos(\theta_1)
\end{pmatrix}, \quad X_2 = \begin{pmatrix} e^{i\alpha_j} & 0 \\ 0 & e^{-i\alpha_j}\end{pmatrix}
\]

for some \(\theta_1 \in [0, \pi]\). As the distance \(d\) is invariant, the matrices \(X_1\) and \(Y_1\) lie on a (possibly degenerate) circle given by the intersection of the sphere \(S_{\alpha_j}(1) = \{X \in \text{SU}(2) \mid d(1, X) = \alpha_j\}\) with the sphere \(S_{\alpha_j}(X_1X_2) = \{X \in \text{SU}(2) \mid d(X, X_1X_2) = \alpha_j\}\), see Figure 3. We denote by \(\theta_2 \in [0, 2\pi]\) the oriented angle between \(X_1\) and \(Y_1\) on this circle. Two cases must be treated according to whether \(\alpha_j = \pi/2\) or \(\alpha_j \neq \pi/2\). These cases are respectively discussed in [37] and [29], but here is short outline.

1. First suppose that \(\alpha_j = \pi/2\). In this case, the space \(\hat{H}^{\alpha_j}_2\) is parametrized by the two coordinates \(\theta_1 \in [0, \pi]\) and \(\theta_2 \in [0, 2\pi]\), with the identifications \((0, \theta_2) \sim (0, 2\pi - \theta_2), (\pi, \theta_2) \sim (\pi, 2\pi - \theta_2)\) and \((\theta_1, 0) \sim (\theta_1, 2\pi)\) [37, Lemma 2.1]. Let us briefly justify the appearance of these identifications.

   When \(\theta_1 = 0\), one has \(X_1 = X_2 = \begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}\) and therefore \(X_1X_2 = -1\). As a consequence, using the definition of \(d\) and the fact that \(\alpha_j = \pi/2\), the spheres \(S^2(1)\) and \(S^2_2(X_1X_2)\) coincide. Since \(X_1 = X_2\) is diagonal, after conjugating by a diagonal
matrix, one can write $Y_1(\theta_2) = \begin{pmatrix} i \cos(\theta_2) & \sin(\theta_2) \\ -\sin(\theta_2) & -i \cos(\theta_2) \end{pmatrix}$. We then notice that $Y_1(2\pi - \theta_2) = (0, 0)$, whence the announced identification.

If $\theta_1 = \pi$, then $X_1X_2 = 1$ and the same argument holds. Finally when $\theta_2 = 0$ and $\theta_2 = 2\pi$, we see that $Y_1 = X_1$ which also leads to the claimed identifications. To conclude the proof of the first assertion, note that removing the abelian representations corresponds to removing the four points $A = (0, 0), A' = (0, \pi), B = (\pi, 0), B' = (\pi, \pi)$.

(2) Next, assume that $\alpha_j \neq \pi/2$. In this case, the parametrization is given by $\theta_1 \in [0, \pi]$ and $\theta_2 \in [0, 2\pi]$ with identifications $(0, \theta_2) \sim (0, 0), (\theta_1, 0) \sim (\theta_1, 2\pi)$ and $(\pi, \theta_2) \sim (\pi, 2\pi - \theta_2)$ [29, Lemma 4.1]. We once again briefly justify the appearance of these identifications which are illustrated in Figure 4.

When $\theta_1 = 0$, we have $X_1 = X_2$ and the spheres $S_{\alpha_j}(1)$ and $S_{\alpha_j}(X_1X_2)$ are tangent, with intersection point $X_1 = X_2 = Y_1 = Y_2$ (i.e. the red circle is “degenerate”: it is a unique point). This proves the identification $(0, 0) \sim (0, \theta_2)$. The remaining identifications follow from the same argument as in the $\alpha_j = \pi/2$ case. Finally, removing the abelian representations corresponds to removing the three points $A = (0, 0), B = (\pi, 0), B' = (\pi, \pi)$.

This concludes our outline of the description of $\hat{H}_{\alpha_j}^2$ and therefore the proof of the lemma. □

**Remark 5.2.** Since we aim to consider the algebraic intersection of $\hat{\Gamma}_{\alpha_j}^\alpha$ with the difference cycle $\hat{\Gamma}_{\sigma_i^2}^\alpha - \hat{\Lambda}_{\sigma_i^2}^\alpha$, it is worth mentioning that we lose nothing by working in $\hat{V}_{\alpha,c}^\alpha$, which is a strict subset of $(V_{\alpha,c}^\alpha \setminus S_{\alpha,c}^\alpha)/SO(3)$, see [24, Lemma 5.2]. In particular this explains why the intersection at the point $A$ will not be taken into consideration in the proof of Proposition 5.7: it is contained both in $\hat{\Gamma}_{\sigma_i^2}^\alpha$ and in $\hat{\Lambda}_{\sigma_i^2}^\alpha$.

Working in the space $\hat{H}_{\alpha_j}^2$, we now observe that near the point $A = (0, 0)$ (which was described in Lemma 5.1), the projection $\hat{p}(\hat{V}_{\alpha,c}^\alpha \cap \hat{\Gamma}_{\alpha_j}^\alpha)$ is a curve.
Figure 4. The space $\hat{H}_2^\alpha_j$, for $\alpha_j \neq \frac{\pi}{2}$. On the left hand side: the left vertical edge (in red) is collapsed onto the point $A$, both the top horizontal edge and the bottom horizontal edge (marked with a dot) are identified, as are the right vertical edges above and below the point $B$, with orientations described by arrows. On the right hand side: the result of the aforementioned identifications; gluing the two boundary segments joining $B'$ to $B$ produces the desired sphere.

**Proposition 5.3.** Let $c$ be a $\mu$-coloring such that $c_1 = c_2$, let $\alpha \in (0, \pi)^\mu$ be such that $\alpha_{c_1} = \alpha_{c_2} = \alpha_j$ and set $\omega_k = e^{2i\alpha_k}$ for $k = 1, \ldots, \mu$. If $\omega_j^2 \neq 1$ and $\beta$ is a $(c, c)$-braid such that $\nabla_{\beta}(\omega) \neq 0$, then, in a neighborhood of $A$ in $\hat{H}_2^\alpha_j$, the projection $\hat{p}(\hat{\Gamma}_{\alpha} \cap \hat{V}_{\alpha,c}^\beta)$ is a curve.

**Proof.** Given $\theta$ in $(0, \pi)$, we use $e^{i\theta}$ to denote the matrix $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$. Consequently, setting $a := (e^{i\alpha_j}, e^{i\alpha_3}, \ldots, e^{i\alpha_n})$, we obtain an element in the subset $S_{\alpha,c}^a$ of abelian representations.

Next, we consider the following subspace of $R_{\alpha,c}^n \times R_{\alpha,c}^n$:

$$\Lambda_{\alpha,c}^n = \{(X_1, X_2, X_3, \ldots, X_n, Y_1, Y_2, X_3, \ldots, X_n) \in R_{\alpha,c}^n \times R_{\alpha,c}^n\}.$$

Since $\Lambda_{\alpha,c}^n$ is $(2n + 4)$-dimensional, $\Gamma_{\beta}^\alpha$ is $2n$-dimensional and $R_{\alpha,c}^n \times R_{\alpha,c}^n$ is $4n$-dimensional, we deduce that the dimension of the vector space $T_{(a,a)}\Lambda_{\alpha,c}^n \cap T_{(a,a)}\Gamma_{\beta}^\alpha$ is at least 4.

**Claim.** The dimension of $T_{(a,a)}\Lambda_{\alpha,c}^n \cap T_{(a,a)}\Gamma_{\beta}^\alpha$ is equal to 4. In particular, the manifolds $\Lambda_{\alpha,c}^n$ and $\Gamma_{\beta}^\alpha$ intersect transversally at $(a, a)$.

**Proof.** Using Remark 4.6, the tangent map of $\beta|_{\Lambda_{\alpha,c}^n}$ at $a$ can be canonically identified with the unreduced colored Gassner matrix $B_{\omega}(\beta) = \begin{pmatrix} A(\omega) & B(\omega) \\ C(\omega) & D(\omega) \end{pmatrix}$. Since the tangent space to a graph is the graph of the corresponding derivative, the space $T_{(a,a)}\Lambda_{\alpha,c}^n \cap T_{(a,a)}\Gamma_{\beta}^\alpha$ is isomorphic to the space $X$ of $n$-tuples $v = (v_1, \ldots, v_n)$ which satisfy

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} * \\ * \\ \vdots \\ v_n \end{pmatrix}.$$
The claim is therefore equivalent to the assertion that the real dimension of $X$ is 2. Since we assumed that $\nabla_\beta'(\omega) \neq 0$ and $\omega_3^2 \neq 1$, Proposition 4.10 ensures that $\det(I_{n-2} - D(\omega)) \neq 0$. Thus we deduce from (24) that the last $n-3$ components of $v$ are equal to $(I_{n-2} - D(\omega))^{-1}C(\omega)(\nu_2^{m_2})$, finishing the proof of the first assertion; the second assertion follows immediately by recalling the respective dimensions of $\Lambda_n^{\alpha,c}$ and $\Gamma_\beta^\alpha$. This concludes the proof of the claim. □

The claim implies that in a neighborhood of $(\mathbf{a},\mathbf{a})$, the space $\Lambda_n^{\alpha,c} \cap \Gamma_\beta^\alpha$ is a manifold of dimension 4. Since $\Lambda_n^{\alpha,c} \cap \Gamma_\beta^\alpha$ finishes the proof of the first assertion; the second assertion follows immediately by recalling Proposition 5.3 shows that the question of whether $\hat{p}(\hat{V}_n^{\alpha,c} \cap \hat{\Gamma}_\beta^\alpha)$ intersects the difference cycle strongly depends on the position of this curve near $A$. Thus, we let $\gamma: (\varepsilon, \varepsilon) \rightarrow \hat{H}_2^{\alpha,j}$ be (a parameterization of) a curve such that $\gamma(t)$ approaches $A$ as $t$ goes to 0. Slightly abusing notations, we sometimes write $\gamma(0) = A$. The example to keep in mind is (a perturbation of) $\hat{p}(\hat{V}_n^{\alpha,c} \cap \hat{\Gamma}_\beta^\alpha)$. Recalling the definition and parametrization of $\hat{H}_2^{\alpha,j}$, we write

$$\gamma(t) = (X_1(t), X_2(t), Y_1(t), Y_2(t)).$$

and follow [29] by introducing the velocity $\theta_1^0 = \frac{d}{dt}\theta_1|_{t=0}$ and the angle $\theta_2^0 = \frac{d}{dt}\theta_2|_{t=0}$ of such a curve $\gamma$. Still following [29], we define

$$s(\theta_2^0) := \frac{\cos(\alpha_j + \theta_2^0/2)}{\cos(\alpha_j)} e^{i\theta_2^0/2}$$

and observe that $2 \arg s(\theta_2^0) = \theta_2^0$. The following remark is used implicitly in [29].

**Remark 5.4.** If the curve $\gamma$ is non constant, then we can choose $\theta_1(t)$ such that $\theta_1^0 \neq 0$. Assume by way of contradiction that $\theta_1^0 = 0$. Since $\gamma(0) = A$, this implies that $\gamma$ is tangent to the vertical axis $\{\theta_1 = 0\}$ (recall Figure 4). As this whole axis is collapsed to the point $A$, the curve $\gamma$ must be constant, a contradiction.

From now on, we consider the path $\gamma_\beta$ given by (a perturbation of) $\hat{p}(\hat{V}_n^{\alpha,c} \cap \hat{\Gamma}_\beta^\alpha)$. Using Remark 5.4, we suppose that $\theta_1^0 = \frac{1}{\sin(\alpha_j)}$. As in the proof of Proposition 5.3, we write the unreduced colored Gassner matrix evaluated at $\omega$ as $B^\omega_\omega(\beta) = \left( \frac{A(\omega) B(\omega)}{C(\omega) D(\omega)} \right)$. The following lemma relates the angle $\theta_1^0$ to this matrix.

**Lemma 5.5.** Let $c$ be a $\mu$-coloring, let $\alpha$ be an element of $(0, \pi)^\mu$ and set $\omega_k = e^{2i\alpha_k}$ for $k = 1, \ldots, \mu$. Assume that $\omega_j^2 \neq 1$, let $\beta$ be a $(c,c)$-braid which satisfies $\nabla_\beta'(\omega) \neq 0$ and set $v = (1 - D(\omega))^{-1}C(\omega)(\frac{1}{2})$. Then $s_{\beta} := s(\theta_2^0)$ satisfies

$$B^\omega_\omega(\beta) \begin{pmatrix} 1 \\ 0 \\ v \end{pmatrix} = \begin{pmatrix} v \omega_j(1 - s_{\beta}) \\ s_{\beta} \end{pmatrix}.$$
Writing the curve $\gamma_\beta(t)$ as $(X_1(t), X_2(t), Y_1(t), Y_2(t))$, we first compute $\gamma'_\beta(0)$. Recalling the parametrization of $\hat{H}_2^{\alpha_j}$ which was described in Lemma 5.1, we see that $X'_2(0) = 0$. Additionally using that $\theta_1(0) = 0$ and that our parametrization satisfies $\theta_1^\omega = 1/\sin(\alpha_j)$, we also get $X'_1(0) = \begin{pmatrix} 0 & \cos(\theta_1(0)) \theta_1^\omega \\ -\sin(\alpha_j) \cos(\theta_1(0)) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$. Next, using [29, page 492, Equation (5)] a computation shows that $Y'_1(0) = \begin{pmatrix} 0 & \theta_1^\omega \sin(\alpha_j) s_\beta \\ \theta_1^\omega \sin(\alpha_j) & 0 \end{pmatrix} = \begin{pmatrix} 0 & s_\beta \\ -s_\beta & 0 \end{pmatrix}$. Finally, since $Y_2(t) = Y_1(t)^{-1} X_1(t) X_2(t)$, we also deduce that $Y'_2(0)$ is given by $\begin{pmatrix} 0 & e^{-2\alpha_j(1- s_\beta)} \\ -e^{-2\alpha_j(1-s_\beta)} & 0 \end{pmatrix}$.

Summarizing and recalling Remark 4.6, we obtain

$$\gamma'_\beta(0) = (1, 0, s_\beta, \overline{v}_j(1 - s_\beta)).$$

Since $\gamma_\beta$ is a parametrization of the curve $\hat{p}(\hat{\Gamma}_\beta^{a,c} \cap \hat{\Gamma}_\beta)$ and since the tangent space of a graph can be described using the graph of the derivative, we deduce that (26) holds, concluding the proof of the proposition. □

Using Lemma 5.5, we make a first observation on the angle $\theta^0_2$ near $A$.

**Proposition 5.6.** The angle $\theta^0_2$ of $\hat{p}(\hat{\Gamma}_\beta^{a,c} \cap \hat{\Gamma}_\beta)$ is not equal to 0 or $-4\alpha_j$.

**Proof.** We argue by means of contradiction. First assume that $\theta^0_2$ is equal to 0. The definition of $s_\beta$ implies that $s_\beta = 1$. Using Lemma 5.5 (and its notations), this implies that the vector $w := (1 \ 0 \ v_3 \ldots v_n)^t$ is fixed by the colored Gassner matrix. This is a contradiction since Lemma 4.3 shows that fixed vectors of the colored Gassner matrix do not contain a zero in their second coordinate.

Next, assume that $\theta^0_2 = -4\alpha_j$. In this case, $s_\beta$ is equal to $\overline{v}_j = e^{-2\alpha_j}$. Applying Lemma 5.5 to $s_\beta$, we deduce that $B^c_{\omega}(s_\beta, v) = (s_\beta \overline{v}_j(1 - s_\beta) \ v)^t$, where $v := (v_3, \ldots v_n)$ is as in Lemma 5.5. Using the multiplicativity of the colored Gassner matrix, applying Lemma 5.5 to $\beta$ and recalling the Gassner matrix for $s_\beta$ which was described in Example 4.12, we get

$$\begin{pmatrix} 1 + \omega^2 - \omega_j & -\omega_j^2 + \omega_j \\ 1 - \omega_j & \omega_j \end{pmatrix} \begin{pmatrix} s_\beta \\ \overline{v}_j(1 - s_\beta) \end{pmatrix} = \begin{pmatrix} s_\beta \omega^2 \overline{v}_j(1 - s_\beta) \\ \overline{v}_j(1 - s_\beta) \end{pmatrix}.

Since $s_\beta = \overline{v}_j$, the left-hand side of (27) is equal to $0$. As a consequence, we obtain $s_\beta = 1$, contradicting the first paragraph of the proof. This concludes the proof of the proposition. □

We now build on [29, Lemma 4.6] in order to understand how $s_\beta$ controls the behavior of the multivariable Casson-Lin invariant under a crossing change within a given sublink. Since the argument is nearly the same as in [29], we only indicate the necessary modifications.

**Proposition 5.7.** Let $c$ be a $\mu$-coloring for which $c_1 = c_2$, let $\alpha \in (0, \pi)^\mu$ be such that $\alpha_{c_1} = \alpha_{c_2} = \alpha_j$ and set $\omega_k = e^{2\alpha_k}$ for each $k$. If $\omega^2_j \neq 1$ and if $\beta$ is a $(c, c)$-braid such that $\nabla_\beta(\omega) \neq 0$ and $\nabla_{s_\beta \beta}(\omega) \neq 0$ and the induced permutation $\overline{\beta}$ is such that $\overline{\beta}(1) \neq 1$ and $\overline{\beta}(2) \neq 2$, then the following equality holds:

$$h^c_{s_\beta \beta}(\alpha) - h^c_\beta(\alpha) = \begin{cases} 0 & \text{if } \frac{\omega_j s_\beta - 1}{s_\beta} > 0, \\
1 & \text{if } \frac{\omega_j s_\beta - 1}{s_\beta} < 0. \end{cases}$$
Proof. Recall from (23) that $h^e_{\alpha,\beta}(\gamma)$ can be understood by studying the intersection of $\hat{p}(\hat{V}^{\alpha,c}_{\gamma} \cap \hat{\Gamma}^{\beta}_{\gamma})$ with the difference cycle $\hat{\Gamma}^{\alpha}_{\sigma_1} - \hat{\Lambda}^{\alpha}_{2}$ inside $\hat{H}^{\alpha}_{2}$. We also know from Proposition 5.3 that $\hat{p}(\hat{V}^{\alpha,c}_{\gamma} \cap \hat{\Gamma}^{\beta}_{\gamma})$ approaches $A$. The conclusion now depends on the behavior near $B$ and $B'$.

Claim. There is a neighborhood of $B'$ in $\hat{H}^{\alpha}_{2}$ which is disjoint from $\hat{p}(\hat{V}^{\alpha,c}_{\gamma} \cap \hat{\Gamma}^{\beta}_{\gamma})$.

Proof. Suppose this not to be the case and recall that $B' = \hat{p}(A, A)$, where $A = (e^{i\alpha_1}, e^{-i\alpha_2}, A_3, \ldots, A_n)$. Using this notation, we deduce that there is a point in $\hat{\Gamma}^{\beta}_3 \cap \hat{V}^{\alpha,c}_{\gamma}$ which is represented by the pair $(A, A)$. There are now two cases both of which lead to contradictions. If $(A, A)$ represents an irreducible point, then we obtain the same contradiction as in [29, page 491]. On the other hand, if $(A, A)$ represents a reducible point, then the same argument as [29, page 496] shows that $A$ is a fixed point of $\beta|_{R^{\alpha,c}}$. Looking back to the definition of $A$, this contradicts our assumption that $\bar{\beta}(1) \neq 1$ and $\bar{\beta}(2) \neq 2$, concluding the proof of the claim.

Using the claim, we know that $\hat{p}(\hat{V}^{\alpha,c}_{\gamma} \cap \hat{\Gamma}^{\beta}_{\gamma})$ must join $A$ and $B$. The intersection properties of this curve now depend on the angle $\theta_{2}^{0}$. If $\theta_{2}^{0}$ lies between 0 and $-4\alpha_j$, then the curve $\hat{p}(\hat{\Gamma}^{\alpha}_{\beta} \cap \hat{V}^{\alpha,c}_{\gamma})$ starts off in the connected component of $\hat{H}^{\alpha}_{2} - (\hat{\Lambda}^{\alpha}_{2} \cup \hat{\Gamma}^{-}_{\sigma_1})$ which does not contain $B$. Since this curve eventually reaches $B$, it must intersect algebraically once positively the circle $\hat{\Lambda}^{\alpha}_{2} \cup \hat{\Gamma}^{-}_{\sigma_1}$, such a situation is depicted in Figure 5.

Similarly, if $\theta_{2}^{0}$ is not between 0 and $-4\alpha_j$, then the algebraic intersection of $\hat{p}(\hat{V}^{\alpha,c}_{\gamma} \cap \hat{\Gamma}^{\beta}_{\gamma})$ with the difference cycle will be zero.

5.3. The behavior under crossing changes and the Alexander polynomial. In this subsection, we express the behavior of the multivariable Casson-Lin invariant under crossing changes in terms of the multivariable potential function.
Following closely the proof of [37, Lemma 2.7], the next result generalizes [29, Lemma 4.7]. In this latter reference, the authors use the Conway-normalized Alexander polynomial instead of the potential function: this explains the slight difference in their formula, see Remark 4.8.

**Proposition 5.8.** Let $c$ be a $\mu$-coloring for which $c_1 = c_2$, let $\alpha \in (0, \pi)^\mu$ be such that $\alpha c_1 = \alpha c_2 = \alpha_3$ and set $\omega_k = e^{2\pi i \alpha_k}$ for each $k$. Assume that $\omega_3^2 \neq 1$ and $\omega c_1 \cdots \omega c_n \neq 1$. If $\beta$ is a $(c,c)$-braid such that $\nabla_\beta(\omega) \neq 0$ and $\nabla_{\sigma_1\beta}(\omega) \neq 0$, then we have

$$\frac{\nabla_\beta(\omega)}{\nabla_{\sigma_1\beta}(\omega)} = \frac{s_\beta - 1}{\omega_3^{s_\beta} - 1}.$$ 

**Proof.** Let $\xi$ be an $n$-stranded $(c,c)$-braid such that $\nabla_\xi(\omega) \neq 0$. As in Subsection 4.4, we use $B^c_\omega(\xi)$ to denote the colored Gassner matrix of $\xi$ evaluated at $\omega$. Setting $v := (1,0,v_3,\ldots,v_n)^t$ and $x_\xi := (s_\xi,\omega_j(1-s_\xi),v_3,\ldots,v_n)^t$, Lemma 5.5 implies that $B^c_\omega(\xi)v = x_\xi$. Writing $B^c_\omega(\xi)$ as $(A(\omega) B(\omega) C(\omega) D(\omega))$, we know from Proposition 4.10 that $I_{n-2} - D(\omega)$ is invertible. Using this fact to isolate the last $n-3$ vectors in the equation $B^c_\omega(\xi)v = x_\xi$, we deduce that

$$\begin{pmatrix} s_\xi \\ \xi_j(1-s_\xi) \end{pmatrix} = \begin{pmatrix} A(\omega) + B(\omega)(I_{n-2} - D(\omega))^{-1} C(\omega) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$ 

We can therefore write $A(\omega) + B(\omega)(I_{n-2} - D(\omega))^{-1} C(\omega)$ as $(\begin{smallmatrix} s_\xi \\ \xi_j(1-s_\xi) \end{smallmatrix}) a^{t-b}$ for some $a$ and $b$. Next, we set $T = (t_00)$, where $t$ is some indeterminate. Since $I_{n-2} - D(\omega)$ is invertible, a short computation using the formula for the determinant of a $(2 \times 2)$ block matrix shows that

$$\det \left( \begin{pmatrix} T & 0 \\ 0 & I_{n-2} \end{pmatrix} - B^c_\omega(\xi) \right) = \det \left( \begin{pmatrix} 1 - s_\xi \\ \xi_j(\omega_j - 1) & t-b \end{pmatrix} \det(I_{n-2} - D(\omega)) \right).$$

We now compute the left hand side of (28). Let $M_1, \ldots, M_n$ be the columns of $M := I_{n} - B^c_\omega(\xi)$ and let $E_2$ be the column vector whose only non-zero entry is in the second position and is equal to 1. Using these notations, the second column of $(\begin{smallmatrix} T & 0 \\ 0 & I_{n-2} \end{pmatrix} - B^c_\omega(\xi))$ is equal to $(t-1)E_2 + M_2$. Using the linearity of the determinant in its second column, we get

$$\det \left( \begin{pmatrix} T & 0 \\ 0 & I_{n-2} \end{pmatrix} - B^c_\omega(\xi) \right) = \det(M) + (t-1)\det(M_1, E_2, M_3, \ldots, M_n).$$

The first summand vanishes: the matrix $M = B^c_\omega(\xi) - I_n$ has a nontrivial kernel since the colored Gassner matrix has fixed vectors. Recall from Section 4 that we use $B^c_\omega(\xi, l, m)$ to denote the determinant of the size $(n-1)$ matrix obtained by deleting the $l$-th row and $m$-th column of $M$. Expanding the second summand in (29) along the second column, we obtain

$$\det \left( \begin{pmatrix} T & 0 \\ 0 & I_{n-2} \end{pmatrix} - B^c_\omega(\xi) \right) = (t-1)B^c_\omega(\xi, 2, 2).$$

On the other hand, Lemma 4.13 gives $\omega c_1(\omega c_2 - 1)B^c_\omega(\xi, 1, 1) = (\omega c_1 - 1)B^c_\omega(\xi, 2, 2)$, while Lemma 4.15 ensures that $\frac{\omega c_1 - \omega c_2}{\omega c_1 - 1}B^c_\omega(\xi, 1, 1) = \det(B^c_\omega(\xi) - I_{n-1})$. Using (28), we obtain

$$\det \left( \begin{pmatrix} 1 - s_\xi \\ \xi_j(\omega_j - 1) & t-b \end{pmatrix} \det(I_{n-2} - D(\omega)) = \frac{\omega c_1(\omega c_2 - 1)}{\omega c_1 \cdots \omega c_n - 1}(t-1)\det(B^c_\omega(\xi) - I_{n-1}).$$

We now set $t = 1$ in (30) so that its right hand side vanishes. Since $\det(I_{n-2} - D(\omega))$ is non-zero, the leftmost determinant must vanish. But as $s_\xi$ cannot be equal to 1 (recall the proof of Proposition 5.6), we deduce that $(-a) = \omega_j(1 - b)$. A straightforward computation
Consequently, substituting $\xi$ of the reduced colored Gassner matrices. Since $\langle \xi \rangle$ of the proposition, it thus only remains to express (32) using the potential function instead of the reduced colored Gassner matrices. In order to apply (31) to $\beta$ and $\sigma_1^2 \beta$, notice that the “D blocks” of the colored Gassner matrices of $\beta$ and $\sigma_1^2 \beta$ are equal: multiplying by $\mathcal{B}_0^c(\sigma_1^2 \beta)$ only affects the $A$ and $B$ submatrices. Consequently, substituting $\xi$ with $\beta$ and $\sigma_1^2 \beta$ in (31) and taking quotients, we obtain

$$
\frac{1 - s_\beta}{1 - s_{\sigma_1^2 \beta}} = \frac{\det(\mathcal{B}_0^c(\beta) - I_{n-1})}{\det(\mathcal{B}_0^c(\sigma_1^2 \beta) - I_{n-1})}.
$$

A short computation using (27) shows that $s_{\sigma_1^2 \beta} - 1 = \omega_j(\omega_j s_\beta - 1)$. To conclude the proof of the proposition, it thus only remains to express (32) using the potential function instead of the reduced colored Gassner matrices. Since $\langle \sigma_1^2 \beta \rangle = t_{c_1}^{-2}(\beta)$, Remark 4.7 and (32) yield

$$
\frac{\nabla_\beta(\omega)}{\nabla_{\sigma_1^2 \beta}(\omega)} = \omega_j^2 \frac{\det(\mathcal{B}_0^c(\beta) - I_{n-1})}{\det(\mathcal{B}_0^c(\sigma_1^2 \beta) - I_{n-1})} = \omega_j^2 (1 - s_\beta) \omega_2 (\omega_j s_\beta - 1).
$$

Simplifying the $\omega_j^2$ terms concludes the proof of the proposition.

We can now express the effect of an intra-component crossing change on the multivariable Casson-Lin invariant $h_L$ in terms of the multivariable potential function.

**Proposition 5.9.** Let $L$ be a $\mu$-colored link and assume that $L_+$ is obtained from $L$ by changing a negative crossing within a sublink of $L$. Assume that $\omega \in T_k^\mu$ satisfies $\omega_j \neq 1$ for each $j$ and $\omega_{c_1} \cdots \omega_{c_n} \neq 1$. If $\nabla_L(\omega^{1/2}) \neq 0$ and $\nabla_{L_+}(\omega^{1/2}) \neq 0$, then the multivariable Casson-Lin invariants of $L$ and $L_+$ satisfy

$$
\nabla_L(\omega^{1/2}) \neq 0 \quad \text{and} \quad \nabla_{L_+}(\omega^{1/2}) \neq 0,
$$

then the multivariable Casson-Lin invariants $h_L(\omega)$ and $h_{L_+}(\omega)$ are defined. Assume that the crossing change occurs within the sublink $L_j$. Arguing as in [15, Remark 2.1], we can then assume that $L = \hat{\beta}$ and $L_+ = \sigma_1^2 \beta$, where $\beta$ and $\sigma_1^2 \beta$ are $\mu$-colored $(c, c)$-braids with $c_1 = c_2 = j$, see Figure 6.

\[\text{Figure 6. On the left hand side, the braid } \beta; \text{ on the right hand side, the braid } \sigma_1^2 \beta.\]

Next, write $\omega_k$ as $e^{2i\alpha_k}$ for $k = 1, \ldots, \mu$ and recall from Definition 3.10 that $h_L(\omega) = h_0^\beta(\alpha)$. Applying Proposition 5.7, we deduce that $h_{L_+}(\omega) - h_L(\omega) = \varepsilon$, where $\varepsilon = 0$ (resp. 1) according
to whether \(\frac{\omega_1 \cdot \omega_2 \cdots \omega_{s-1}}{s_{p-1}}\) is positive (resp. negative). Applying Proposition 5.8, this latter term is equal to \(\nabla_{L+}(\omega^{1/2}) / \nabla_{L}(\omega^{1/2})\), concluding the proof of the proposition.

In Proposition [29, 37], the condition in (33) is expressed as a product of polynomials instead of a quotient. Since these authors work with knots, the Conway-normalized Alexander polynomial evaluated at \(\omega \in S^1\) is real (recall Remark 4.8) and so the two formulations are in fact equivalent. The next remark describes the situation in the multivariable case.

**Remark 5.10.** If \(L\) and \(L_+\) are \(n\)-component \(\mu\)-colored links as in Proposition 5.9 and \(\omega \in \mathbb{T}^n\), then the sign of \(\nabla_{L_+}(\omega^{1/2}) / \nabla_{L}(\omega^{1/2})\) is equal to the sign of \(\nabla_{L_+}(\omega^{1/2}) \nabla_{L}(\omega^{1/2})\) up to a power \((-1)^n\).

Indeed, the quotient and the product agree up to multiplication by \(\nabla_{L}(\omega^{1/2})^2\), and recalling Remark 4.9, this latter quantity equals \((-1)^n\nabla_{L}(\omega^{1/2}) \nabla_{L}(\omega^{-1/2}) = (-1)^n |\nabla_{L}(\omega^{1/2})|^2\).

### 6. The Relation to the Multivariable Signature

In this section, we prove the main results of this paper. In more details, Subsection 6.1 gathers some facts about the multivariable signature, Subsection 6.2 proves Theorem 1.1, Subsection 6.3 shows that \(h_L\) is locally constant and Subsection 6.4 proves Theorem 1.6.

#### 6.1. The multivariable signature

In this subsection, we briefly recall the definition of the multivariable signature, the main references being [17] and [10].

A **C-complex** for a \(\mu\)-colored link \(L = L_1 \cup \cdots \cup L_\mu\) is a union \(S = S_1 \cup \cdots \cup S_\mu\) of surfaces in \(S^3\) which is connected, and such that:

1. for all \(i\), \(S_i\) is a Seifert surface for the sublink \(L_i\),
2. for all \(i \neq j\), \(S_i \cap S_j\) is either empty or a union of clasps,
3. for all \(i, j, k\) pairwise distinct, \(S_i \cap S_j \cap S_k\) is empty.

The existence of a C-complex for arbitrary colored links was established in [9, Lemma 1]. Given a sequence \(\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\mu)\) of signs \(\pm 1\), let \(i^\varepsilon: H_1(S) \to H_1(S^3 \setminus S)\) be defined as follows. Any homology class in \(H_1(S)\) can be represented by an oriented cycle \(x\) which behaves as illustrated in [10, Figure 2] whenever crossing a clasp. Define \(i^\varepsilon([x])\) as the class of the 1-cycle obtained by pushing \(x\) in the \(\varepsilon_i\)-normal direction off \(S_i\) for \(i = 1, \ldots, \mu\). Next, consider the bilinear form

\[
\alpha^\varepsilon: H_1(S) \times H_1(S) \to \mathbb{Z}, \quad (x, y) \mapsto \ell k(i^\varepsilon(x), y),
\]

where \(\ell k\) denotes the linking number. Fix a basis of \(H_1(S)\) and denote by \(A^\varepsilon\) the matrix of \(\alpha^\varepsilon\). Note that for all \(\varepsilon\), these **generalized Seifert matrices** satisfy \((A^\varepsilon)^T = A^{-\varepsilon}\). Using this fact, one easily checks that for any \(\omega = (\omega_1, \ldots, \omega_\mu)\) in the \(\mu\)-dimensional torus \(\mathbb{T}^\mu\), the matrix

\[
H(\omega) = \sum_{\varepsilon} \prod_{i=1}^\mu (1 - \omega_i^{\varepsilon_i}) A^\varepsilon
\]

is Hermitian. Since this matrix vanishes when one of the coordinates of \(\omega\) is equal to 1, we restrict ourselves to the subset \(\mathbb{T}_0^\mu = (S^1 \setminus \{1\})^\mu\) of \(\mathbb{T}^\mu\).

**Definition 6.1.** The **multivariable signature and nullity** of a \(\mu\)-colored link \(L\) are the maps \(\sigma_L, \eta_L: \mathbb{T}_0^\mu \to \mathbb{Z}\), where \(\sigma_L(\omega)\) is the signature of \(H(\omega)\) and \(\eta_L(\omega)\) its nullity.
The multivariable signature and nullity are independent of the choice of the C-complex \([10]\). Note furthermore that when \(\mu = 1\), a C-complex is nothing but a Seifert surface and \(\sigma_L\) recovers the Levine-Tristram signature of the oriented link.

### 6.2. The multivariable signature and the multivariable Casson-Lin invariant

The goal of this subsection is to relate the multivariable Casson-Lin invariant \(h_L\) to \(\sigma_L\) when \(L\) is a 2-component ordered link with linking number 1, proving Theorem 1.1 from the introduction.

The following lemma describes the parity of the multivariable signature and its behavior under crossing changes within a sublink.

**Lemma 6.2.** The multivariable signature satisfies the following properties:

1. If a \(\mu\)-colored link \(L\) has \(\nu\) components and \(\omega \in \mathbb{T}^\mu_s\) is not a root of \(\Delta_L\), then
   \[
   \sigma_L(\omega) \equiv \nu + \sum_{k<j} \ell k(L_k, L_j) - \text{sign}(\tau^\nu i^\nu \nabla_L(\omega^{1/2})) \mod 4.
   \]
   In particular, if \(L\) is a 2-component ordered link with linking number 1 and \(\omega \in \mathbb{T}^2_s\) is not a root of \(\Delta_L\), then \(\sigma_L(\omega)\) is even and
   \[
   \sigma_L(\omega) \equiv \begin{cases} 
   0 \mod 4 & \text{if } \nabla_L(\omega^{1/2}) > 0, \\
   2 \mod 4 & \text{if } \nabla_L(\omega^{1/2}) < 0.
   \end{cases}
   \]

2. If \(L_+\) is obtained from \(L\) by changing a unique negative crossing within a given sublink. If \(\omega \in \mathbb{T}^\mu_s\) is neither a root of \(\Delta_{L_+}\) nor a root of \(\Delta_L\), then
   \[
   \sigma_{L_+}(\omega) - \sigma_L(\omega) \in \{0, -2\}.
   \]

**Proof.** The first statement is contained in [10, Lemma 5.7] and directly implies the claim about 2-component links with linking number 1 (here \(\nabla_L(\omega)\) is real since \(L\) has 2 components, see Remark 4.9). We now prove the second statement. Pick C-complexes \(S_+\) and \(S\) for \(L_+\) and \(L\) which only differ at the crossing under consideration. Since the crossing change occurs within a sublink, there are bases for \(H_1(S_+)\) and \(H_1(S)\) such that the resulting generalized Seifert matrix \(A_+^\varepsilon\) only differs from \(A^\varepsilon\) at one diagonal entry which is reduced by 1. As a consequence, the Hermitian matrix \(H_+^\varepsilon(\omega)\) is the same as \(H(\omega)\) except for one diagonal entry which is reduced by the positive real number \(\sum_{i=1}^\nu (2 - \omega_i - \tau^i)\). Since only one eigenvalue can change and since we assumed both Alexander polynomials to be non-zero (i.e. there are no zero eigenvalues in \(H_+^\varepsilon(\omega)\) and \(H(\omega)\)), the result follows. \(\Box\)

Reformulating Lemma 6.2, we immediately obtain the following result.

**Lemma 6.3.** Let \(L\) be a 2-component ordered link with linking number 1 and assume that \(L_+\) is obtained from \(L\) by a unique crossing change within a component of \(L\). If \(\omega \in \mathbb{T}^\mu_s\) is such that \(\nabla_L(\omega^{1/2}) \neq 0\) and \(\nabla_{L_+}(\omega^{1/2}) \neq 0\), then
   \[
   \sigma_{L_+}(\omega) - \sigma_L(\omega) = \begin{cases} 
   0 & \text{if } \nabla_{L_+}(\omega^{1/2}) \nabla_L(\omega^{1/2}) > 0, \\
   -2 & \text{if } \nabla_{L_+}(\omega^{1/2}) \nabla_L(\omega^{1/2}) < 0.
   \end{cases}
   \]

For 2-component links with linking number 1, we can now relate the multivariable Casson-Lin invariant to the multivariable signature, proving Theorem 1.1 from the introduction.
Theorem 6.4. If $L$ is a 2-component ordered link with linking number 1, then the following equality holds on $T_2^\omega \setminus \{ (\omega_1, \omega_2) \mid \Delta_L(\omega_1, \omega_2) = 0 \}$:

$$h_L(\omega_1, \omega_2) = -\frac{1}{2} \sigma_L(\omega_1, \omega_2).$$

Proof. We first prove the theorem when $\omega$ is such that $\arg(\omega)$ is transcendental, $\omega_1 \omega_2 \neq 1$ and $\omega_j \neq 1$ for $j = 1, 2$. Since $L$ has 2 components and linking number 1, the Torres formula (which reads $\Delta_L(1, 1) = (t_1^{\ell_{12}} - 1)/(t_1 - 1)\Delta_K(t_1)$, where $\ell_{12} = \ell(K_1, K_2)$) shows that $|\Delta_L(1, 1)| = 1$. Thus $\Delta_L$ is not identically zero and therefore the multivariable Casson-Lin invariant $h_L(\omega)$ is well defined whenever $\Delta_L(\omega) \neq 0$. Since the fundamental group of the complement of the Hopf link $J$ is abelian, $h_J$ vanishes identically. The same conclusion holds for the multivariable signature $\sigma_J$, as $J$ admits a contractible C-complex.

Since $\arg(\omega)$ is transcendental for $i = 1, 2$, it follows that $\nabla_L(\omega^{1/2}) \neq 0$ for all $L$ as in the statement of the theorem. The equality $h_L(\omega) = -\sigma_L(\omega)/2$ is obtained by induction: both invariants vanish on the (positive) Hopf link, while Proposition 5.9, Lemma 6.3 and Remark 5.10, ensure an identical behavior under crossing changes within components. Since the links have linking number one, the Torres formula guarantees that such crossing changes do not make the Alexander polynomial vanish (consequently if $h_L$ is defined for $L$, then it is also defined for $L'$). This concludes the proof of the theorem for the $\omega \in T_2^\omega$ which were described above since the linking number is a complete link homotopy invariant for 2-component links [40].

Finally, note that both invariants are locally constant on $T_2^\omega \setminus \{ (\omega_1, \omega_2) \mid \Delta_L(\omega_1, \omega_2) = 0 \}$: for the multivariable signature, this is proved in [10, Corollary 4.2], while for $h_L$, the result is proved in Proposition 6.6 below. This concludes the proof of the theorem. \qed

The sign appearing in Theorem 6.4 depends on some conventions which we briefly discuss.

Remark 6.5. Given a knot $K$ obtained as the closure of a braid $\beta$, Lin writes $K_+ = \sigma_1^2 \beta$, while Heusener and Kroll write $K_- = \sigma_1^2 \beta$. As a consequence, while these authors agree on the sign of $h_L(\omega) = h_\beta(\omega)$, comparing [37, Theorem 2.9] with [29, Proposition 4.8] shows that the meaning of this sign differs: it depends on the conventions adopted for the generators of the braid group. We follow Lin’s conventions (recall Figures 1 and 6). On the other hand, assuming that $K_+$ is obtained from $K_-$ by changing a single negative crossing, Lin states that $0 \leq \sigma_{K_+}(\omega) - \sigma_{K_-}(\omega) \leq 2$ [37, page 356], while Heusener-Kroll state that $0 \leq \sigma_{K_+}(\omega) - \sigma_{K_-}(\omega) \leq 2$ [29, page 497]. With our notations, the proof of Lemma 6.2 (as well as [42, proof of Lemma 2.1] and [23, proof of Lemma 2.2]) yields the latter result. Summarizing, the sign differences in [37] and [29] cancel out (explaining why these authors obtain “$h_K = \sigma_K/2\mathbb{Z}$”) while our conventions account for the minus sign in Theorem 6.4.

6.3. The multivariable Casson-Lin invariant is locally constant. Recall from Remark 3.11 that $h_L$ is defined on the set $D_L$ which consists of those $\omega$ in $T_2^\omega$ such that the abelian representation $\rho_\omega$ is not a limit of irreducible representations. Since $T_2^\omega \setminus V(\Delta_L)$ is contained in $D_L$, the following proposition concludes the proof of Theorem 6.4.

Proposition 6.6. Given a $\mu$-colored link $L$, the multivariable Casson-Lin invariant is locally constant on $D_L$. Namely, if $\omega^0$ and $\omega^1$ lie in the same connected component of $D_L$, then the following equality holds:

$$h_L(\omega^0) = h_L(\omega^1).$$
We first describe the strategy of the proof which is inspired by [29, Proposition 3.8]. Write \( \omega = e^{2i\alpha} \), where \( \alpha \in (0, \pi)^\mu \). Given \( \varepsilon > 0 \), we denote by \( B(\alpha, \varepsilon) \) the ball of radius \( \varepsilon \) centered at \( \alpha \). We will show that if \( \varepsilon \) is small enough, then \( h_L(e^{2i\alpha'}) \) coincides with \( h_L(e^{2i\alpha}) \) for any \( \alpha' \) in \( B(\alpha, \varepsilon) \). Writing \( L \) as the closure of an \( n \)-stranded \( (c, c) \)-braid \( \beta \), this will be carried out by constructing a cobordism which joins \( \hat{\Lambda}_{\alpha,c}^+ \cap \hat{\Gamma}_{\beta}^+ \) to \( \hat{\Lambda}_{\alpha',c}^+ \cap \hat{\Gamma}_{\beta'}^+ \). This cobordism will take place in an ambient space whose description requires us to introduce the following spaces:

\[
R_{n,2n}^c = \{(A_1, \ldots, A_n, B_1, \ldots, B_n) \in SU(2)^{2n} | \text{ tr}(A_i) = \text{ tr}(B_i) = \text{ tr}(A_j) = \text{ tr}(B_j) \text{ if } c_i = c_j \},
\]

\[
H_{n,2n}^c = \{(A_1, \ldots, A_n, B_1, \ldots, B_n) \in R_{n,2n}^c | \prod_{i=1}^n A_i = \prod_{i=1}^n B_i \}.
\]

Recalling the notations from Section 3, observe that we have the inclusions \( R_{n,2n}^{\alpha,c} \times R_{n,2n}^{\alpha,c} \subset R_{n,2n}^c \) and \( H_{n,2n}^{\alpha,c} \subset H_{n,2n}^c \). Just as in Section 3, we then define \( S_n^c \) as the space of abelian representations in \( R_{n,2n}^c \) (i.e. we impose the same relations as in (6)) and define \( \hat{H}_{n}^c \) by removing \( S_n^c \cap H_{n}^c \) from \( H_{n}^c \) and modding out by the action of \( SO(3) \). The next lemma is an analogue of Lemma 3.4; we also refer to [28, Corollary 3.2] where a similar statement is made.

**Lemma 6.7.** The space \( \hat{H}_{n}^c \) is a smooth open manifold which contains \( \hat{H}_{n}^{\alpha,c} \) as a codimension \( \mu \) submanifold. Furthermore, the normal bundle of \( H_{n}^{\alpha,c} \) inside \( H_{n}^c \) is trivial.

**Proof.** The proof of the first statement is the same as in Lemma 3.4. Namely, the map \( f_n: R_{n,2n}^c \to SU(2) \) defined by \( f_n(A_1, \ldots, A_n, B_1, \ldots, B_n) = A_1 \cdots A_n B_n^{-1} \cdots B_1^{-1} \) restricts to a submersion \( f_n| \) on \( H_{n}^c \setminus S_n^c \) and therefore \( H_{n}^c \setminus S_n^c = f_n^{-1}(\text{Id}) \) is a smooth manifold whose dimension is equal to \( \dim(R_{n,2n}^c) - \dim(SU(2)) = 4n + \mu - 3 \). Since \( SO(3) \) acts freely on \( H_{n}^c \setminus S_n^c \), the quotient \( \hat{H}_{n}^c \) is a smooth open manifold of dimension \( 4n - 6 + \mu \). It is clear that \( \hat{H}_{n}^{\alpha,c} \) has codimension \( \mu \) in \( \hat{H}_{n}^c \) because that many traces are fixed.

We now show that \( \hat{H}_{n}^{\alpha,c} \) has trivial normal bundle in \( \hat{H}_{n}^c \). Recall that for any \( \theta \in (0, \pi) \), the 2-sphere \( S_\theta = \{ A \in SU(2) | \text{ tr}(A) = 2 \cos(\theta) \} \) has trivial normal bundle in \( SU(2) \): the Lie algebra \( su(2) \) splits as \( \mathbb{C} \oplus \mathbb{R} \), the complex line being mapped onto the tangent space of \( S_\theta \) at \( A \) by the tangent map of multiplication by \( A \) and the real direction is spanned by the tangent map of the trace function \( \text{Tr}: SU(2) \setminus \{ \pm \text{Id} \} \to (-2, 2) \) at \( A \). Denoting by \( (R_{n,2n}^c)^* \) the subspace of \( R_{n,2n}^c \) with none of its coordinates equal to \( \pm \text{Id} \), and by \( i_1, \ldots, i_\mu \) some preimages of \( 1, \ldots, \mu \) by the coloring \( c \), the following map is thus a submersion:

\[
\text{Tr}_\mu: (R_{n,2n}^c)^* \to (-2, 2)^\mu
\]

\[
(A_1, \ldots, A_n, B_1, \ldots, B_n) \mapsto (\text{ tr}(A_{i_1}), \ldots, \text{ tr}(A_{i_\mu})).
\]

Fiberwise, the normal bundle of \( R_{n,2n}^{\alpha,c} \times R_{n,2n}^{\alpha,c} \) in \( R_{n,2n}^c \) is given by \( T_x R_{n,2n}^c / T_x(R_{n,2n}^{\alpha,c} \times R_{n,2n}^{\alpha,c}) \), for any \( x \) in \( R_{n,2n}^{\alpha,c} \times R_{n,2n}^{\alpha,c} \). As a consequence, using \( \mathcal{N}( (R_{n,2n}^{\alpha,c} \times R_{n,2n}^{\alpha,c}) / R_{n,2n}^c ) \) to denote the normal bundle of \( R_{n,2n}^{\alpha,c} \times R_{n,2n}^{\alpha,c} \) inside \( R_{n,2n}^c \), the map \( \text{Tr}_\mu \) induces a fiberwise isomorphism \( \mathcal{N}( (R_{n,2n}^{\alpha,c} \times R_{n,2n}^{\alpha,c}) / R_{n,2n}^c ) \to T(-2, 2)^\mu \). Since this latter bundle is trivial, so is the former. The statement now descends to the normal bundle of \( H_{n}^{\alpha,c} \) inside \( H_{n}^c \); indeed \( H_{n}^{\alpha,c} \setminus S_n^c \) (resp. \( R_{n,2n}^{\alpha,c} \times R_{n,2n}^{\alpha,c} \)) is a submanifold of codimension \( \mu \) in \( H_{n}^c \setminus S_n^c \) (resp. \( R_{n,2n}^c \)). This concludes the proof of the lemma. \( \square \)

Using Lemma 6.7, we can now prove Proposition 6.6 which asserts that \( h_L \) is locally constant on \( D_L \). The main idea is inspired by the proof of Ehresmann’s fibration theorem [20].
Proof of Proposition 6.6. Write $\omega$ as $e^{2\alpha t}$ for $\alpha \in (0, \pi)^{\mu}$, fix $\varepsilon > 0$ and use $B(\alpha, \varepsilon)$ to denote the ball of radius $\varepsilon$ centered in $\alpha$. We want to show that if $\varepsilon$ is small enough, then $h_L(e^{2\alpha t})$ coincides with $h_L(e^{2\alpha t'})$ for any $\alpha' \in B(\alpha, \varepsilon)$. Pick an isotopy $F: \hat{H}_n^{a,c} \times [0, 1] \to \hat{H}_n^{a,c}$ which makes the intersection $\hat{\Lambda}_n^{\alpha,c} \cap \hat{\Gamma}_\beta$ transverse in $\hat{H}_n^{a,c}$. Choose a path $\alpha: [0, 1] \to (0, \pi)^{\mu}$ joining $\alpha$ to $\alpha'$ and such that $\omega(t) = e^{2\alpha t}$ lies in $D_L$ for every $t \in [0, 1]$. In order to build a cobordism joining $\hat{\Lambda}_n^{\alpha,c} \cap \hat{\Gamma}_\beta$ to $\hat{\Lambda}_n^{\alpha,c} \cap \hat{\Gamma}_\beta'$, we will prove that $F$ can be “transported” along $\alpha(t)$ so that for each $t$, the intersection $\hat{\Lambda}_n^{\alpha(t),c} \cap \hat{\Gamma}_\beta$ becomes transverse in $\hat{H}_n^{a,c}$.

Let $\mathcal{N}(\hat{H}_n^{a,c}/\hat{H}_n^c)$ denote the normal bundle of $\hat{H}_n^{a,c}$ inside of $\hat{H}_n^c$. Since Lemma 6.7 ensures that this bundle is trivial, we can pick a nowhere vanishing normal vector field $X: \hat{H}_n^{a,c} \to \mathcal{N}(\hat{H}_n^{a,c}/\hat{H}_n^c)$ whose flow we denote by $\phi_X: \hat{H}_n^c \to \hat{H}_n^{a,c}$. Since the intersection $\hat{\Lambda}_n^{\alpha(t),c} \cap \hat{\Gamma}_\beta(t)$ is compactly supported for each $t$, there is a compact set $K_0 \subset \hat{H}_n^{a,c}$ containing $\hat{\Lambda}_n^{\alpha,c} \cap \hat{\Gamma}_\beta$ and such that for each $t$, the compact set $K_t = \phi_X(K_0)$ is a subset of $\hat{H}_n^{a(t),c}$ containing $\hat{\Lambda}_n^{\alpha(t),c} \cap \hat{\Gamma}_\beta(t)$. It can in fact safely be assumed that $K_0$ is a manifold. Let $\{U_i \mid i \in I\}$ be an open cover of $\hat{H}_n^{a,c}$, with finite subcover $\{U_i \mid i = 1, \ldots, k\}$ of $K_0$. Refining this sub-cover if necessary, one can assume that each open set $U_i \subset \hat{H}_n^{a,c}$ verifies the following property: for some $t \in [0, 1]$, the set $\phi_X(U_i)$ contains only one component of the non-transverse intersection $\hat{\Lambda}_n^{\alpha(t),c} \cap \hat{\Gamma}_\beta(t)$ in $\hat{H}_n^{a(t),c}$ (there are finitely number such components because we are dealing with (semi-)algebraic sets).

Since there are only finitely many non-transverse intersections, it is enough to show that for one such $U \subset \hat{H}_n^{a,c}$, one can transport the isotopy $F$ so that, for the corresponding $t$, the non-transverse intersection point of $\hat{\Lambda}_n^{\alpha(t),c} \cap \hat{\Gamma}_\beta(t)$ in $\hat{H}_n^{a(t),c}$ is perturbed to a transverse one. To make this possible, consider the isotopy

$$(\phi_X)^*F: \phi_X(U) \times [0, 1] \to \phi_X(U)$$

$$(p, s) \mapsto \phi_X \circ F(\phi_X^{-1}(p), s).$$

As vector fields $X$ such that the isotopy $(\phi_X)^*F$ makes this intersection transverse are generic in the set of normal vector fields, this procedure can always be carried out.

We now conclude the proof. Pick $\varepsilon$ small enough so that each $\phi_X: U_i \to \hat{H}_n^c$ is an embedding. The set $K = \bigcup_{t \in [0, 1]} \phi_X(K_0)$ is therefore a compact submanifold of $\hat{H}_n^c$. The previous construction now ensures that $\bigcup_{t \in [0, 1]} \phi_X(\hat{\Lambda}_n^{\alpha,c})$ and $\bigcup_{t \in [0, 1]} \phi_X(\hat{\Gamma}_\beta)$ can be assumed to intersect transversally in a one dimensional submanifold of $K$. This latter submanifold realizes the desired cobordism between $\hat{\Lambda}_n^{\alpha,c} \cap \hat{\Gamma}_\beta$ and $\hat{\Lambda}_n^{\alpha,c} \cap \hat{\Gamma}_\beta'$. As a consequence, the corresponding intersections numbers are equal and therefore the proposition is proved.

6.4. Deformations of SU(2) representations of link groups. The goal of this subsection is to prove Theorem 1.6 from the introduction.

Recall that for a $\mu$-colored link $L$, the multivariable Alexander polynomial $\Delta_L(t_1^{\pm 1}, \ldots, t_\mu^{\pm 1})$ restricts to a polynomial on the $\mu$-dimensional torus $T^\mu$. In particular, its zero locus $V(\Delta_L) = \{ (\omega_1, \ldots, \omega_\mu) \in T^\mu_\bullet | \Delta_L(\omega_1, \ldots, \omega_\mu) = 0 \}$ is a (possibly non smooth) real algebraic subvariety of $T^\mu$, which might have several irreducible components.
Definition 6.8. A subset $X$ of $V(\Delta_L)$ is smoothly path connected if any two smooth points $\omega$ and $\omega'$ in $V(\Delta_L)$ can be joined by a path of smooth points in $V(\Delta_L)$. A smoothly path connected component of $V(\Delta_L)$ is a maximal smoothly path connected set.

Note that the smooth locus of $V(\Delta_L)$ is equal to the reunion of its smoothly path connected components. In particular, a smooth irreducible variety is smoothly path connected. Here is a slightly more involved example.

Example 6.9. The subvariety of a slightly more involved example. Components. In particular, a smooth irreducible variety is smoothly path connected. Here is a slightly more involved example.

\[ \text{Let } \sigma \text{ be a signature function.} \]

\[ \text{Since Theorem 6.4 implies that } U \subset V(\Delta_L) \text{ is contained in } \mathbb{C} \times \mathbb{C}. \]

\[ \text{Proposition 6.6 shows that } h \text{ is constant on } U. \]

\[ \text{In particular, } h \text{ is constant on } U \setminus (V(\Delta_L) \cap U). \]

\[ \text{Since the multivariable Casson-Lin invariant } h_L \text{ is defined at } \omega, \text{ and since Proposition 6.6 shows that } h_L \text{ is locally constant on } D_L, \text{ there is a small open neighborhood } U \subset D_L \text{ such that } h_L \text{ is constant on } U. \]

\[ \text{In particular, } h_L \text{ is constant on } U \setminus (V(\Delta_L) \cap U). \]

\[ \text{Since Proposition 6.6 shows that } h_L \text{ is locally constant on } D_L, \text{ there is a small open neighborhood } U \subset D_L \text{ such that } h_L \text{ is constant on } U. \]

\[ \text{In particular, } h_L \text{ is constant on } U \setminus (V(\Delta_L) \cap U). \]

\[ \text{This contradicts the hypothesis of the theorem, concluding the proof of the first assertion.} \]

Assume $\omega$ and $\omega'$ lie in the same smoothly connected component of $V(\Delta_L)$. Pick small enough open neighborhoods $U$ and $U'$ of $\omega$ and $\omega'$ so that $U \setminus (V(\Delta_L) \cap U)$ and $U' \setminus (V(\Delta_L) \cap U')$ each consist of two connected components. Write these respectively as $U_1 \sqcup U_2$ and $U'_1 \sqcup U'_2$. Since $\omega$ and $\omega'$ lie in the same smoothly connected component (renumbering if necessary) $U_1$ and $U'_1$ lie in the same connected component of $\mathbb{T}_r \setminus V(\Delta_L)$ and similarly for $U_2$ and $U'_2$. The second statement follows: since $h|_{U_i} = h|_{U'_i}$ for $i = 1, 2$, we deduce that the assumption of the first statement also holds for $\omega'$, so $\rho_{\omega'}$ is also a limit of irreducible representations. \qed

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