B-type Landau-Ginzburg models with one-dimensional target

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Abstract. We summarize the main results of our investigation of B-type topological Landau-Ginzburg models whose target is an arbitrary open Riemann surface. Such a Riemann surface need not be affine algebraic and in particular it may have infinite genus or an infinite number of Freudenthal ends. Under mild conditions on the Landau-Ginzburg superpotential, we give a complete description of the triangulated structure of the category of topological D-branes in such models as well as counting formulas for the number of topological D-branes considered up to relevant equivalence relations.

1. Introduction

It is well-known [1, 2] that classical B-type topological Landau-Ginzburg (LG) models with D-branes can be associated to any pair \((X, W)\), where \(X\) is a non-compact Kähler manifold and \(W : X \to \mathbb{C}\) is a non-constant holomorphic function defined on \(X\). When the canonical line bundle of \(X\) is holomorphically trivial and the critical set of \(W\) is compact, it is expected that the quantization of such models produces a non-anomalous two-dimensional topological field theory (2d TFT) which obeys the general axioms introduced in [3] and hence is characterized entirely by an open-closed TFT datum, an algebraic structure subject to certain axioms which encode the sewing constraints of the TFT. As shown in [2], path integral arguments lead to a proposal for the open-closed TFT datum of the quantum B-type LG model defined by such a pair \((X, W)\), a proposal which was clarified and analyzed mathematically in [4, 5] (see [6] for a brief summary). When \(X\) is Stein and Calabi-Yau, this proposal simplifies as explained in [7] and summarized in [8].

As already pointed out in [4], an interesting particular case arises when \(X\) is complex one-dimensional, i.e. an open Riemann surface (a non-compact Riemann surface without boundary, which we assume to be connected). Since any open Riemann surface is Stein and holomorphically parallelizable (hence Calabi-Yau), this fits into the class of models considered in [7], whose TFT datum admits the simplified description discussed in loc. cit. and summarized in [8].

In this contribution, we outline our investigation of B-type LG models with open Riemann surface target [9], which in turn relies on algebraic results derived for a large class of rings in references [10, 11]. We make free use of certain concepts and terminology which were introduced in [3] and [1, 2, 4, 7] and which are summarized in [6] and [8].
The open Riemann surfaces considered in this contribution are fully general. In particular, they can have infinite genus and an infinite number of Freudenthal ends\textsuperscript{1}. As such, the topological LG models discussed herein are the most general non-anomalous B-type LG models with non-singular complex one-dimensional target.

2. Open Riemann surfaces

In this section, we recall some well-known facts regarding open Riemann surfaces.

**Definition 2.1.** A Riemann surface (in the sense of Weyl-Radó) is a Hausdorff, borderless and connected complex manifold $\Sigma$ such that $\text{dim}_\mathbb{C} \Sigma = 1$.

**Theorem 2.2** (Radó). The following mutually-equivalent statements hold for any Riemann surface $\Sigma$:

(a) $\Sigma$ is paracompact.
(b) $\Sigma$ has a countable basis.
(c) $\Sigma$ is countable at infinity.

**Definition 2.3.** A Riemann surface is called open if it is non-compact.

2.1. The topological type of open Riemann surfaces

Unlike the compact case, the topological classification of open Riemann surfaces is quite involved, since such a surface can have infinite genus as well as an infinity of Freudenthal ends (a.k.a. “ideal points”). The ends of Riemann surfaces were studied in the classical work of Kerékjártó [12] and Stoilow [13] (see [14] for a summary in modern mathematical language).

**Definition 2.4.**

- The transfinite genus of $\Sigma$ is the cardinal number:

$$g(\Sigma) = \frac{1}{2} \text{rk}_\mathbb{Z} H^1(\Sigma, \mathbb{Z})$$

- The ideal boundary $\partial_\infty \Sigma$ of $\Sigma$ is the set of Freudenthal ends of $\Sigma$, endowed with its natural topology.

**Proposition 2.5** (Kerékjártó). The following statements hold for any open Riemann surface $\Sigma$:

1. $g(\Sigma)$ is finite or countable.
2. $\partial_\infty \Sigma$ is a finite or countable compact Hausdorff topological space which is totally disconnected.

**Remarks.**

1. $\partial_\infty \Sigma$ can be a Cantor space.
2. Adjoining $\partial_\infty \Sigma$ to $\Sigma$ produces the so-called Kerékjártó-Stoilow compactification $\hat{\Sigma}$ of $\Sigma$ (see [12, 13, 14]).

There exists a natural disjoint union decomposition $\partial_\infty \Sigma = \partial_1^\infty \Sigma \sqcup \partial_2^\infty \Sigma$ where the ends belonging to $\partial_1^\infty \Sigma$ and $\partial_2^\infty \Sigma$ are called of first and second kind.

\textsuperscript{1} An open Riemann surface coincides with the analyticization of a complex affine curve iff it has finite genus and a finite number of ends, in which case it can be obtained by removing a finite set of points from a compact Riemann surface.
2.2. The complex geometry of open Riemann surfaces

The following statements summarize classical results due to KerékJartó and Stoilow:

**Theorem 2.6** (KerékJartó). Let $\Sigma$ and $\Sigma'$ be two open Riemann surfaces. Then the following statements are equivalent:

(a) $\Sigma$ and $\Sigma'$ are unoriented-homeomorphic.
(b) $g(\Sigma) = g(\Sigma')$ and there exists a homeomorphism $\psi: \partial_{\infty} \Sigma \to \partial_{\infty} \Sigma'$ such that $\psi(\partial_{1}^{\infty} \Sigma) = \partial_{1}^{\infty} \Sigma'$.

**Theorem 2.7** (Richards-Stoilow). The following statements hold:

(i) The unoriented homeomorphism type of an open Riemann surface $\Sigma$ is entirely determined by the triplet $(g(\Sigma), \partial_{1}^{\infty} \Sigma, \partial_{\infty} \Sigma)$.
(ii) Consider any triplet $(g, F_{1}, F)$ with $g \in \mathbb{Z}_{\geq 0} \sqcup \{\aleph_{0}\}$, $F$ a non-empty compact, Hausdorff and totally disconnected countable topological space and $F_{1} \subset F$ a (possibly empty) subset of $F$, endowed with the topology induced from $F$. Then there exists an open Riemann surface $\Sigma$ with $g(\Sigma) = g$, $\partial_{\infty} \Sigma \simeq_{\text{homeo}} F$ and $\partial_{1}^{\infty} \Sigma \simeq_{\text{homeo}} F_{1}$.

The following results are crucial in the context of B-type LG models:

**Theorem 2.8** (Behnke-Stein). Every open Riemann surface is Stein.

**Theorem 2.9** (Grauert-Röhrl). Any holomorphic vector bundle on an open Riemann surface $\Sigma$ is holomorphically trivial. In particular, the analytic Picard group $\text{Pic}(\Sigma)$ vanishes and the canonical line bundle $K_{\Sigma}$ is holomorphically trivial.

It follows that any open Riemann surface is Stein (and hence Kählerian) and holomorphically Calabi-Yau. Hence any open Riemann surface can be used as the target space of a B-type LG model whose TFT datum admits the simplified description discussed in [7] and summarized in [8]. Since the holomorphic tangent bundle $T\Sigma$ is trivial by Grauert-Röhrl theorem, any open Riemann surface is holomorphically parallelizable.

**Proposition 2.10** (various authors). Every non-compact Riemann surface $\Sigma$ admits a holomorphic embedding in $\mathbb{C}^{3}$ and a holomorphic immersion in $\mathbb{C}^{2}$.

**Remark.** It is not known if any open Riemann surface admits a proper holomorphic embedding in $\mathbb{C}^{2}$.

2.3. The ring $O(\Sigma)$

Let $\Sigma$ be an open Riemann surface. This subsection recalls certain results regarding the ring $O(\Sigma)$ of complex-valued holomorphic functions defined on $\Sigma$.

**Theorem 2.11** (Bers). Let $\Sigma_{1}$ and $\Sigma_{2}$ be two connected non-compact Riemann surfaces. Then $\Sigma_{1}$ and $\Sigma_{2}$ are biholomorphic iff their rings of holomorphic functions $O(\Sigma_{1})$ and $O(\Sigma_{2})$ are isomorphic as unital $\mathbb{C}$-algebras.

**Theorem 2.12** (Iss’sa). Let $\Sigma_{1}$ and $\Sigma_{2}$ be two connected non-compact Riemann surfaces. Then $\Sigma_{1}$ and $\Sigma_{2}$ are biholomorphic iff their fields of meromorphic functions $\mathcal{M}(\Sigma_{1})$ and $\mathcal{M}(\Sigma_{2})$ are isomorphic as unital $\mathbb{C}$-algebras.

**Proposition 2.13** (Henriksen-Alling). The following statements hold:

1. The cardinal Krull dimensions of all open Riemann surfaces are equal to each other (denote this cardinal number by $k$).
2. We have $k \geq 2^{|81|}$. 
Definition 2.14. An elementary divisor domain (EDD) is an integral domain $R$ such that any matrix with coefficients from $R$ admits a Smith normal form.

Theorem 2.15 (Helmer-Henriksen-Alling). For any open Riemann surface $\Sigma$, the ring $O(\Sigma)$ is an elementary divisor domain.

2.4. Special uniformizers

Let $\Sigma$ be an open Riemann surface and $\text{ord}_x : M(\Sigma) \to \mathbb{Z}$ be the map which assigns to a meromorphic function $f \in M(\Sigma)$ its order at the point $x \in \Sigma$.

Definition 2.16. A special local uniformizer for $\Sigma$ at the point $x \in \Sigma$ is a meromorphic function $t_x \in M(\Sigma)$ such that $\text{ord}_x(t_x) = 1$ and $\text{ord}_y(t_x) = 0$ for all $y \in \Sigma \setminus \{x\}$. Notice that $t_x \in O(\Sigma)$.

Proposition 2.17. The special local uniformizers of $\Sigma$ coincide with the prime elements of $O(\Sigma)$. In particular, a special local uniformizer at $x$ is determined up to multiplication by a unit of $O(\Sigma)$ (i.e. a nowhere-vanishing element of $O(\Sigma)$).

2.5. Critically-finite elements of $O(\Sigma)$

Most of our results for B-type Landau-Ginzburg models with open Riemann surface target depend on the assumption that the Landau-Ginzburg superpotential is critically finite in the sense discussed below. The notion of critically finite element was introduced in \cite{10, 11}, which studied categories of finite rank matrix factorizations for Bézout and elementary divisor domains.

Definition 2.18. A divisor $r \in O(\Sigma)$ of $f \in O(\Sigma)$ is called critical if $r^2 | f$.

Definition 2.19. A non-zero non-unit $f \in O(\Sigma)$ is called:

- non-critical, if $f$ has no critical divisors;
- critically-finite, if $f$ has a factorization of the form:

$$f = f_0 f_c$$

where $N \geq 1$, $n_i \geq 2$, $p_1, \ldots, p_N \in O(\Sigma)$ are critical prime divisors of $f$ with $(p_i) \neq (p_j)$ for $i \neq j$ and $f_0 \in O(\Sigma)$ is non-critical and coprime with $f_c$.

Lemma 2.20. Let $f \in O(\Sigma)$ be a non-zero element and $D(f)$ be the analytic divisor of $f$. Then the following statements hold:

1. $f$ is a unit of $O(\Sigma)$ iff $D(f) = 0$.
2. $f$ is a prime element of $O(\Sigma)$ (i.e. a special local uniformizer of $\Sigma$) iff $D(f) = x$ for some $x \in \Sigma$.
3. $f$ is non-critical iff $D(f)$ is multiplicity-free at any point in its support, i.e. iff it has the form:

$$D(f) = \sum_{x \in Z(f)} x$$

where $Z(f)$ denotes the zero set of $f$ (which is at most countable).
4. If $g \in O(\Sigma)$ is another holomorphic function, then $f|g$ iff $D(f) \leq D(g)$.
5. $f$ is critically-finite iff $f = gh$, where $g \in O(\Sigma)$ is non-critical and $h \in O(\Sigma)$ satisfies:

$$D(h) = \sum_{x \in Z(h)} n_x \cdot x$$

with $Z(h)$ a non-empty finite set and $n_x > 1$ for every $x \in Z(h)$. 


3. B-type Landau-Ginzburg models with one-dimensional target

**Definition 3.1.** A Landau-Ginzburg (LG) pair is a pair \((X, W)\) such that:

1. \(X\) is a non-compact complex and \(\mathbb{C}\)-Kählerian connected manifold of dimension \(d > 0\), which is holomorphically Calabi-Yau in the sense that its holomorphic canonical line bundle \(K_X\) is holomorphically trivial.
2. \(W \in \mathcal{O}(X)\) is a non-constant holomorphic complex-valued function defined on \(X\), which is called the Landau-Ginzburg superpotential.

Given an LG pair \((X, W)\), let \(Z_W \overset{\text{def}}{=} \{ x \in X | (\partial W(x) = 0) \}\) be the critical set of \(W\). Since any Stein manifold is \(\mathbb{C}\)-Kählerian, a particular case of LG pair is obtained by taking \(X\) to be Stein. This situation was studied in [7] (see [8] for a summary).

**Definition 3.2.** A one-dimensional LG pair is an LG pair \((X, W)\) such that \(\dim \mathbb{C}X = 1\).

Since any open Riemann surface is Stein and holomorphically Calabi-Yau (see Subsection 2.2), we have:

**Proposition 3.3.** An LG pair is one-dimensional iff \(X\) coincides with an open Riemann surface \(\Sigma\).

Since any compact analytic subset of a Stein manifold is finite, the critical set of \(W \in \mathcal{O}(\Sigma)\) is compact iff it is finite. Let \((\Sigma, W)\) be a one-dimensional LG pair with compact (and hence finite) critical set \(Z_W\). For any \(p \in Z_W\), let:

\[
\text{M}(\hat{W}_p) \overset{\text{def}}{=} \frac{\mathcal{O}_{\Sigma,p}}{\langle \hat{W}'_p \rangle}
\]

denote the analytic Milnor algebra of the analytic function germ \(\hat{W}_p\) of \(W\) at \(p\) and \(\nu_p \overset{\text{def}}{=} \dim \mathbb{C} \text{M}(\hat{W}_p)\) denote the analytic Milnor number of \(W\) at \(p\). Here \(\langle \hat{W}'_p \rangle\) is the ideal of \(\mathcal{O}(\Sigma)\) generated by \(\frac{dW}{dz}\), where \(z\) is any local holomorphic coordinate of \(\Sigma\) centered at \(p\) (this ideal is independent of the choice of such a local coordinate). Let \(t_p \in \mathcal{M}(\Sigma)\) be a special local uniformizer of \(\Sigma\) at \(p\).

**Proposition 3.4.** [9] We have \(\text{M}(\hat{W}_p) \cong \mathcal{O}(\Sigma) \oplus \bigoplus_{i=1}^N \mathcal{O}_{\Sigma,p}/\langle \hat{W}'_p \rangle \).

**Proposition 3.5.** [9] The bulk state space of the open-closed TFT defined by \((\Sigma, W)\) can be identified with the direct sum of analytic Milnor algebras:

\[
\mathcal{H} \cong \mathcal{O}(X) \oplus \bigoplus_{p \in Z_W} \text{M}(\hat{W}_p).
\]

In particular, we have:

\[
\dim \mathbb{C} \mathcal{H} = \sum_{p \in Z_W} \nu_p.
\]

3.1. The topological D-brane category

For any unital commutative ring \(R\), let \(\text{MF}(R, W)\) denote the category of finite rank matrix factorizations of \(W\) over \(R\) and \(\text{HMF}(R, W)\) denote its total cohomology category (which is \(\mathbb{Z}_2\)-graded and \(R\)-linear). Let \(\text{hmf}(R, W) \overset{\text{def}}{=} \text{HMF}^0(R, W)\) denote the sub-category obtained from \(\text{HMF}(R, W)\) by keeping only the even morphisms. It is well-known that \(\text{hmf}(R, W)\) has a natural structure of triangulated category with involutive shift functor.

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Proposition 3.6. [9] Let \((\Sigma, W)\) be a one-dimensional LG pair. Then the D-brane category \(T\) of the associated 2d TFT (i.e. of the quantum LG model with target \((X, W)\)) can be identified with:

\[
T \simeq_{\text{O}(\Sigma)} \text{HMF}(\text{O}(\Sigma), W)
\]

Definition 3.7. A matrix factorization \((M, D)\) of \(W\) over \(\text{O}(\Sigma)\) is called elementary if \(\text{rk}M^0 = \text{rk}M^1 = 1\).

In the limit \(W \to 0\), the B-type LG model parameterized by \((X, W)\) reduces to a B-type non-linear sigma model with non-compact target \(X\). Intuitively, elementary factorizations correspond to ‘elementary D-branes’, i.e. those topological D-branes which are obtained by ‘topological tachyon condensation’ in a pair formed of a single brane and a single anti-brane of this non-linear sigma model, where tachyon condensation is driven by turning on the Landau-Ginzburg superpotential \(W\). See [15, 16, 17, 18] for the origin of this interpretation in open topological string field theory.

Definition 3.8. The \(\mathbb{Z}_2\)-graded cocycle category \(\text{ZMF}(\text{O}(\Sigma), W)\) has the same objects as \(\text{MF}(\Sigma, W)\) but its morphisms are the closed morphisms of \(\text{MF}(\text{O}(\Sigma), W)\).

Let \(\text{ZMF}(\text{O}(\Sigma), W)\) be the even subcategory of \(\text{ZMF}(\text{O}(\Sigma), W)\). We say that two matrix factorizations of \(W\) are strongly isomorphic if they are isomorphic in \(\text{ZMF}(\text{O}(\Sigma), W)\). Notice that an elementary factorization of \(W\) is indecomposable in the additive category \(\text{ZMF}(\text{O}(\Sigma), W)\), but it need not be indecomposable in \(T^0 = \text{hmf}(\text{O}(\Sigma), W)\).

3.2. Critically-finite superpotentials

The following result characterizes critically-finite superpotentials:

Proposition 3.9. [9] Let \((\Sigma, W)\) be a one-dimensional LG pair. Then the following statements are equivalent:

(a) \(W\) is a critically-finite element of \(\text{O}(\Sigma)\).

(b) The intersection \(Z(W) \cap Z_W\) is finite.

(c) The divisor of \(W\) has the form:

\[
D(W) = D_0 + \sum_{i=1}^{N} n ix_i
\]

where \(D_0\) is either the trivial divisor or an effective divisor whose multiplicity at every point of its support is one, the symbols \(x_1, \ldots, x_N\) (with \(N \geq 1\)) denote a finite collection of points of \(\Sigma\) which do not belong to the support of \(D_0\) and \(n_i \geq 2\) for all \(i = 1, \ldots, N\).

(d) We have \(W = W_0W_c\), where \(W_0\) has no zeros or only simple zeros, \(W_c\) has a finite number of zeros, each of which has multiplicity at least two and \(W_0, W_c\) have no common zero.

In particular, any holomorphic function \(W\) with finite critical set is a critically-finite element of \(\text{O}(\Sigma)\).

Remark. Any critically-finite superpotential \(W \in \text{O}(\Sigma)\) can be written as:

\[
W = W_0W_c\quad\text{with}\quad W_c = \prod_{i=1}^{N} t_{x_i}^{n_i},
\]

where:
(i) \( W_0 \in O(\Sigma) \) has a finite or countable number of zeros, all of which are simple.

(ii) \( x_1, \ldots, x_N \) (with \( N \geq 1 \)) are distinct points of \( \Sigma \) and \( t_{x_i} \) are special uniformizers at these points.

(iii) None of the points \( x_i \) is a zero of \( W_0 \).

In this case, we have \( D_0 = D(W_0) \) and we set \( D_v \) as follows:

\[
D_v \overset{\text{def}}{=} \sum_{i=1}^{N} n_i x_i.
\]

### 3.3. Primary factorizations and the Krull-Schmidt property of \( \text{hmf}(O(\Sigma), W) \)

Recall that an additive category \( A \) is called Krull-Schmidt if any object of \( A \) decomposes as a finite sum of indecomposable objects with quasi-local endomorphism rings. We say that a matrix factorization of \( W \) is trivial if it is a zero object in \( \text{hmf}(\Sigma, W) \).

**Proposition 3.10.** [9] Any elementary factorization of \( W \) over \( O(\Sigma) \) is strongly isomorphic to an elementary factorization of the form \( e_v \overset{\text{def}}{=} (O(\Sigma)^{[1]}, D_v) \), where \( v \in O(\Sigma) \) is a divisor of \( W \) and \( D_v \) is defined as:

\[
D_v \overset{\text{def}}{=} \begin{bmatrix} 0 & v \\ u & 0 \end{bmatrix}, \text{ with } u \overset{\text{def}}{=} W/v \in O(\Sigma).
\]

**Definition 3.11.** An element \( f \in O(\Sigma) \) is called primary if it is a power of a prime element of \( O(\Sigma) \). An elementary factorization \( e_v \) of \( W \) is called primary if \( v \) is a primary divisor of \( W \).

If \( e_v \) is a primary factorization of \( W \) over \( O(\Sigma) \), then we have \( v = at^k \) for some \( a \in O(\Sigma)^\times \), some \( k \in \mathbb{Z}_{>0} \) and some \( x \in \Sigma \). In this case, \( W \) has a zero of order at least \( k \) at the point \( x \in \Sigma \).

**Theorem 3.12.** [9] Suppose that \( W \) is critically-finite. Then the additive category \( \text{hmf}(O(\Sigma), W) \) is a Krull-Schmidt category whose non-zero indecomposables are the nontrivial primary matrix factorizations of \( W \).

Hence any finite rank matrix factorization (in particular, any non-primary elementary factorization) of \( W \) decomposes into primary factorizations in the category \( \text{hmf}(O(\Sigma), W) \). The primary factorizations of \( W \) play the role of ‘simple’ (a.k.a. ‘irreducible’) topological D-branes of the B-type LG model.

### 3.4. The triangulated structure of \( \text{hmf}(O(\Sigma), W) \)

Let \( R \) be a unital commutative ring. Let \( \text{mod}_R \) be the Abelian category of finitely-generated \( R \)-modules.

**Definition 3.13.**

(i) The projectively stable category \( \text{mod}_R \) has the same objects as \( \text{mod}_R \) and modules of morphisms given by:

\[
\text{Hom}_R(M, N) \overset{\text{def}}{=} \text{Hom}_R(M, N)/\mathcal{P}_R(M, N) \quad \forall M, N \in \text{Ob}(\text{mod}_R),
\]

where \( \mathcal{P}_R(M, N) \subset \text{Hom}_R(M, N) \) consists of those morphisms of \( \text{mod}_R \) which factor through a projective module of finite rank.

(ii) The injectively stable category \( \text{mod}_R \) has the same objects as \( \text{mod}_R \) and modules of morphisms given by:

\[
\text{Hom}_R(M, N) \overset{\text{def}}{=} \text{Hom}_R(M, N)/\mathcal{I}_R(M, N) \quad \forall M, N \in \text{Ob}(\text{mod}_R),
\]

where \( \mathcal{I}_R(M, N) \subset \text{Hom}_R(M, N) \) consists of those morphisms of \( \text{mod}_R \) which factor through an injective module of finite rank.
Remark. When \( R \) is a self-injective ring, the categories \( \text{mod}_R \) and \( \text{mod}_R \) are equivalent to each other and the category \( \text{mod}_R \) is naturally triangulated since in this case \( \text{mod}_R \) is a Frobenius category (i.e. an exact category whose projective and injective objects coincide).

The following result gives a complete description of the triangulated structure of the category \( T^0 = \text{hmf}(O(\Sigma), W) \) for the case when \( W \) is critically finite:

**Theorem 3.14.** \[9\] Let \( W \in O(\Sigma) \) be a critically-finite superpotential of the form (1). Then the following statements hold:

1. The ring \( A_i \) \( \overset{\text{def}}{=} \frac{O(\Sigma)}{(t^{n_i}_x)} \) is Artinian and Frobenius (hence also self-injective) for all \( i = 1, \ldots, N \).
2. For each \( i = 1, \ldots, N \), there exist equivalences of triangulated categories:
   \[ \text{D}_{\text{Sing}}(A_i) \simeq \text{mod}_{A_i}, \]
   where \( \text{mod}_{A_i} \) denotes the projectively stable category of finitely-generated \( A_i \)-modules.
3. The triangulated category \( \text{mod}_{A_i} \) is Krull-Schmidt with non-zero indecomposable objects given by the \( A_i \)-modules:
   \[ V_i^{(k)} \overset{\text{def}}{=} \frac{O(\Sigma)}{(t^{n_i}_x)} \simeq \frac{(t^{n_i-1}_x)}{(t^{n_i}_x)} \text{ where } k = 1, \ldots, n_i - 1. \]
   This category admits Auslander-Reiten triangles, having the Auslander-Reiten quiver shown in Figure 1. Moreover, it is classically generated by the residue field \( V_i^{(1)} = \frac{O(\Sigma)}{(t_x)} \) of \( A_i \).
4. The triangulated category \( \text{D}_{\text{Sing}}(A_i) \simeq \text{mod}_{A_i} \) is 1-Calabi-Yau for all \( i \), with involutive shift functor:
   \[ \Omega(V_i^{(k)}) = V_i^{(n_i-k)} \forall k = 1, \ldots, n_i - 1. \]
5. There exists an equivalences of triangulated categories:
   \[ \text{hmf}(O(\Sigma), W) \simeq \bigvee_{i=1}^N \text{D}_{\text{Sing}}(A_i) \simeq \bigvee_{i=1}^N \text{mod}_{A_i}. \]
6. The triangulated category \( \text{hmf}(O(\Sigma), W) \) is 1-Calabi-Yau, Krull-Schmidt and admits Auslander-Reiten triangles. Its Auslander-Reiten quiver is disconnected, with connected components given by the Auslander-Reiten quivers of the categories \( \text{mod}_{A_i} \).

**Figure 1.** Auslander-Reiten quiver for \( \text{mod}_{A_i} \) when \( n_i = 5 \). The Auslander-Reiten translation fixes all vertices and the multiplicities of all arrows are trivial.

3.5. Counting indecomposables in \( \text{hmf}(O(\Sigma), W) \)

The following result gives the count of simple topological D-branes of the LG model:

**Proposition 3.15.** \[9\] Suppose that \( W \) is critically finite of the form given in eq. (1). Then the number of isomorphism classes of indecomposable non-zero objects of the category \( \text{hmf}(O(\Sigma), W) \) equals \( \sum_{i=1}^N (n_i - 1) = -N + \sum_{i=1}^N n_i. \)
The degrees $n_i$ of the prime factors $t_x$, arising in the decomposition of $W$, define a $\mathbb{Z}_2$-grading on the set $I_N = \{1, \ldots, N\}$ whose components are given by:

$$I^0_N \overset{\text{def}}{=} \{i \in I_N \mid n_i \text{ is even}\}, \quad I^1_N \overset{\text{def}}{=} \{i \in I_N \mid n_i \text{ is odd}\}.$$

Let:

$$N^0 \overset{\text{def}}{=} |I^0_N| \quad \text{and} \quad N^1 \overset{\text{def}}{=} |I^1_N|$$

denote the cardinalities of these subsets of $I$, which satisfy $N^0 + N^1 = N$. Any non-empty subset $K \subset I_N$ is endowed with the $\mathbb{Z}_2$-grading induced from $I_N$, which has components:

$$K^0 \overset{\text{def}}{=} K \cap I^0_N, \quad K^1 \overset{\text{def}}{=} K \cap I^1_N.$$

3.6. Counting elementary factorizations in $\text{hmf}(O(\Sigma), W)$ and $\text{HMF}(O(\Sigma), W)$

The following result allows one to count elementary topological D-branes as well as elementary brane-antibrane pairs:

**Theorem 3.16.** [9] Suppose that $W$ is a critically-finite element of $O(\Sigma)$ with the decomposition given in eq. (1). Then:

1. The number of isomorphism classes of elementary factorizations in the category $\text{hmf}(O(\Sigma), W)$ is given by:

$$\hat{N}_W = \sum_{k=0}^{N^1} \sum_{K \subseteq I_N, |K^1| = k} 2^{N^0 + k} \prod_{i \in K} \left\lfloor \frac{n_i - 1}{2} \right\rfloor.$$

2. The number of isomorphism classes of elementary matrix factorizations in the category $\text{HMF}(O(\Sigma), W)$ is given by:

$$N_W = 2^\hat{N}_W + \sum_{k=0}^{N^1} 2^{N^0 + k - 1} \sum_{K \subseteq I_N, |K^1| = k} \prod_{i \in K} \left\lfloor \frac{n_i - 1}{2} \right\rfloor.$$

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