Consensus with Linear Objective Maps

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Abstract—We characterize in this paper the weighted averages that can be evaluated in a decentralized way by agents in a consensus dynamics defined over a directed graph. Precisely, first recall that the usual consensus dynamics converges towards the average of the initial states of the agents. We introduce here a linear function, called the objective map, that defines the desired final state as a function of the initial states of the agents. We then provide a complete answer to the question of whether there is a decentralized consensus dynamics over a given digraph which converges to the final state specified by an objective map. In particular, we will characterize not only the set of objective maps that are feasible for a given digraph, but also the consensus dynamics that implements the objective map. In addition, we present a decentralized algorithm to design the consensus dynamics.

I. INTRODUCTION

In recent years, consensus algorithms have been recognized as an important step in a variety of decentralized and distributed algorithms, such as the rendezvous problem and distributed convex optimization. Because of their relevance, a fair amount is already known about consensus algorithms. Indeed, questions concerning sufficient and/or necessary conditions for agents to reach consensus ([11]–[9]), questions concerning time delay ([3],[4]), consensus with quantized measurements ([10],[11]), consensus with time-varying network topologies ([11]–[9]), and questions about convergence rate ([11]–[13]), robustness ([14],[15]) in the presence of an adversary have all been treated to some degree.

The problem we address in this paper is one of feasibility of weighted consensus over a given directed graph. More specifically, given a set of positive weights assigned to the agents, we say that the agents reach a weighted consensus if they converge to the weighted average of their initial conditions — a formal definition to be given shortly. As is commonly done, we assume that the information flow in the system is described by a directed graph. Our goal is to determine which weighted averages can be computed for a given information flow. Further, we shall describe how the agents can communicate over the graph to set-up the dynamical system whose evolution reaches the desired weighted consensus. Computing a weighted average rather than a simple average is a natural task when the agents in the system are not all of the same importance. For example, think of a rendezvous problem where the rendezvous position depends on the initial positions of only a small group of agents.

Broadly speaking, the problem we address in this paper is thus one of feasibility of an objective given decentralization constraints. Similar questions, but involving controllability of linear systems [22], stability of linear systems [23] and formation control [24] have also been investigated. While the general problem of feasibility of an objective under decentralization constraints is far from being completely understood, we shall see that a fairly complete characterization can be obtained in the present case. However, still open questions remain, such as: How to handle negative weights? How to handle time-varying information flow graphs? How to make sure that no-agent can “game the system” and increase or decrease its assigned weight?

We next describe the model more precisely. We assume that there are $n$ agents $\vec{x}_1, \ldots, \vec{x}_n$ evolving in $\mathbb{R}^d$, and that the underlying network topology is specified by a directed graph (or simply digraph) $G = (V, E)$, with $V = \{1, \ldots, n\}$ the set of vertices and $E$ the set of edges. We let $V_i^−$ be a subset of $V$ comprised of the outgoing neighbors of vertex $i$, i.e.,

$$V_i^− := \{j \in V | i \to j \in E\}$$

and we assume in this paper that each agent $\vec{x}_i$ can only observe its outgoing neighbors. The equations of motion for the $n$ agents $\vec{x}_1, \ldots, \vec{x}_n$ are then given by

$$\frac{d}{dt} \vec{x}_i = \sum_{j \in V_i^-} a_{ij} \cdot (\vec{x}_j - \vec{x}_i), \quad \forall i = 1, \ldots, n$$

with each $a_{ij}$ a nonnegative real number, which we call the interaction weight.

The objective of the system is characterized by positive real numbers $w_i$. We define the objective function $f : \mathbb{R}^{n \times d} \to \mathbb{R}^d$ as:

$$f(\vec{x}_1, \ldots, \vec{x}_n) := \sum_{i=1}^{n} w_i \vec{x}_i.$$  

The feasibility question we ask is the following: given a digraph $G = (V, E)$, and a weight vector $\vec{w} = (w_1, \ldots, w_n)$ in $\mathbb{R}^n$, does there exist a set of nonnegative interaction weights $\{a_{ij}|i \to j \in E\}$ such that for any initial condition $\vec{x}_1(0), \ldots, \vec{x}_n(0)$ in $\mathbb{R}^d$, all agents will converge to the same point in $\mathbb{R}^d$ specified by the objective map, i.e,

$$\lim_{t \to \infty} x_i(t) = f(\vec{x}_1(0), \ldots, \vec{x}_n(0))$$

for all $i = 1, \ldots, n$. In other words, we require that all the agents not only reach consensus, but also converge to a specific point which is a weighted sum of the initial positions of the agents. In the following section, we will convert this
problem to one of asking whether there exists a sparse, infinitesimal stochastic matrix $A$ with a fixed zero pattern (specified by the digraph) such that $A$ has a simple zero eigenvalue with the vector $\vec{w}$ being the corresponding left eigenvector.

In the paper, we will provide a complete answer to the question of weighted consensus within model (2). In particular, we will characterize not only the set of objective maps which are feasible by choices of interaction weights, but also the set of interaction weights for a feasible objective map. Note that the problem of evaluating averages in a distributed manner has also been handled using discrete-time dynamics [16]–[19]. Thus, the same question we ask here could be asked also for discrete-time dynamics, though we do not pursue that here.

Following this introduction, we proceed as follows. In section II, we introduce some key definitions, reformulate the question in precise terms and then state our main theorem. In particular, the main theorem characterizes the set of objective maps which can be realized by choices of nonnegative interaction weights. Sections III - V are devoted to the proof of this theorem. In section VI, we present a decentralized algorithm for finding a set of interaction weights associated with a feasible objective map. In particular, we relate the set of interaction weights to solutions of graph balancing. We then provide conclusions in the last section.

II. DEFINITIONS, PROBLEM REFORMULATION AND THE MAIN THEOREM

In this section, we introduce the main definitions used in this work, formulate the weighted consensus problem in precise terms and state the main result of the paper.

A. Background

1) Graph theory: We consider in this paper only simple directed graphs, that is directed graphs with no self loops, and with at most one edge between each ordered pair of vertices. We denote by $G = (V, E)$ a directed graph where $V$ is the node or vertex set and $E \subset V \times V$ the edge set. We say that $G$ is rooted if there exists a vertex $i \in V$ such that for any vertex $j \in V$, there is a path from $j$ to $i$. The vertex $i$ is said to be a root of $G$. In case $G$ consists of only one vertex, then the vertex is a root. We denote by $V_r \subset V$ the set of roots of $G$. The digraph $G$ is strongly connected if for any ordered pair of vertices $(i, j)$, there is a path from $i$ to $j$. In this case, all vertices of $G$ are roots, i.e., $V_r = V$. It is well known that if the digraph $G$ associated with system (2) is rooted, then all agents converge to the same state for all initial conditions (see, for example, [6]). Conversely, if for any initial condition, all agents of system (2) converge to the same state, then the underlying digraph must be rooted. Hence we only consider rooted digraphs as the underlying digraphs of system (2). For a subset $V' \subset V$, we call $G'$ a subgraph of $G$ induced by $V'$ if $G' = (V', E')$ and $E'$ contains all edges of $E$ whose end-vertices are in $V'$.

Let $G = (V, E)$ be a rooted digraph, and let $V'$ be a subset of $V$. We say $V'$ is relevant to $G$ (or simply relevant) if it satisfies two conditions:

a) the set $V'$ is contained in the root set $V_r$;

b) the subgraph $G'$ induced by $V'$ is strongly connected.

We label all the relevant subsets of $V$ as $V_{r_1}, \ldots, V_{r_q}$.

2) The unit simplex: Let $\Delta^{n-1}$ be the $(n-1)$-simplex contained in $\mathbb{R}^n$, i.e., the convex hull of the standard basis vectors $\vec{e}_1, \ldots, \vec{e}_n$ of $\mathbb{R}^n$. We say that the subset $F \subset \Delta^{n-1}$ is a face of $\Delta^{n-1}$ if $F$ is a convex hull of a nonempty subset of $\{\vec{e}_1, \ldots, \vec{e}_n\}$.

Let $V = \{1, \ldots, n\}$ be the set of vertices of $\Delta^{n-1}$ (this is the same notation used as the one used for the set of vertices of a digraph), and let $V'$ be a subset of $V$. We will then denote by $F_{V'}$ the face of $\Delta^{n-1}$ spanned by vectors $\vec{e}_i$ with $i \in V'$, and denote by $\mathrm{int} F_{V'}$ the interior of $F_{V'}$, i.e.,

$$\mathrm{int} F_{V'} := \left\{ \sum_{i \in V'} \alpha_i \vec{e}_i \mid \sum_{i \in V'} \alpha_i \leq 1, \alpha_i > 0 \right\}.$$

(5)

In the case where $V'$ consists only of one vertex, say vertex $i$, we then set $\mathrm{int} F_{V'} = \{ \vec{e}_i \}$.

3) infinitesimal stochastic matrices: We say a matrix $A$ is an Infinitesimal stochastic matrix (or in short ISM) if its off-diagonal entries are nonnegative, and its rows sum to zero.

With a digraph $G$ of $n$ vertices, we associate a set $G$ of $n$-by-$n$ ISM’s as follows: $A = (a_{ij}) \in G$ if

$$a_{ij} = \begin{cases} \geq 0 & \text{if } i \to j \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

(6)

for $i \neq j$. If $G$ consists only of one vertex, then $G = \{0\}$ is a singleton. It should be clear that the set $G$ is a convex set.

Let $\mathbf{1}$ be a vector of all ones in $\mathbb{R}^n$; then for each matrix $A$ in $G$, we have $A\mathbf{1} = 0$. So each matrix $A$ has at least one zero eigenvalue. On the other hand, it is well known that if $A$ is an ISM, then as a consequence of the Gershgorin circle theorem, the real parts of eigenvalues of $A$ are less than or equal to zero. In particular, if the digraph $G$ is rooted and $a_{ij} > 0$ for each $(i, j) \in E$, then the matrix $A$ has a simple zero eigenvalue.

Let $\vec{w} \in \Delta^{n-1}$. We define the subset $G_{\vec{w}} \subset G$ as the set of ISMs $A \in G$ satisfying the next two conditions:

1) The matrix $A$ has a simple zero eigenvalue. (Hence, all the other eigenvalues of $A$ have negative real parts.)

2) The vector $\vec{w}$ is the left eigenvector of $A$ corresponding to the zero eigenvalue, i.e., $A^\top \vec{w} = 0$.

We will show below that $G_{\vec{w}}$ is furthermore a convex set.

B. Main results

We start by formulating the targeted consensus problem in view of the facts introduced above. First, note that we can rewrite (2) into a matrix form as follows: let $X$ be an $n$-by-$d$ matrix with $x_i^\top$ the $i$-th row of $X$. Then, system (2) is equivalent to

$$X = AX$$

(7)

with matrix $A$ contained in $G$. For the purpose of reaching consensus, we require that the matrix $A$ has a simple zero
eigenvalue. Let \( \bar{w} \in \Delta^{n-1} \) be the left eigenvector of \( A \) corresponding to the zero eigenvalue. Then for any initial condition \( X(0) \), we have
\[
\lim_{t \to 0} X(t) = \mathbf{1} \cdot \bar{w}^T X(0)
\] (8)
If we write \( \bar{w} = (w_1, \cdots, w_n) \), then
\[
\lim_{t \to \infty} \bar{x}_i(t) = \sum_{j=1}^n w_j \bar{x}_j(0)
\] (9)
Conversely, if the expression above holds for all initial conditions, then the matrix \( A \) must be contained in \( \mathcal{G}_{\bar{w}} \). So the question we raised in the first section can be restated as follows. For a given digraph \( G \) and a vector \( w \in \Delta^{n-1} \), is the set \( \mathcal{G}_{\bar{w}} \) empty? We answer this question in Theorem 1:

**Theorem 1.** Let \( G = (V, E) \) be a rooted digraph, and let \( V_1, \cdots, V_q \) be subsets of \( V \) which are relevant to \( G \). Let \( W \) be a subset of \( \Delta^{n-1} \) comprised of vectors \( \vec{w} \) with \( \mathcal{G}_{\bar{w}} \) nonempty. Then, \( W = \bigcup_{i=1}^q \text{int} F_{V_i} \).

The next three sections are devoted to the proof of Theorem 1 and are organized as follows. In section III, we will focus on relevant subsets of \( V \). In particular, we show that if the set \( \mathcal{G}_{\bar{w}} \) is nonempty, then necessarily the vector \( \vec{w} \) will be in the union of \( \text{int} F_{V_1}, \cdots, \text{int} F_{V_q} \). In section IV, we will deviate a bit by relaxing certain conditions on the digraph \( G \), as well as the set \( \mathcal{G}_{\bar{w}} \). We assume in the section that \( G \) is an arbitrary digraph while \( \bar{w} \) is constrained in the interior of \( \Delta^{n-1} \), and we then consider the set
\[
\hat{G}_{\bar{w}} := \{ A \in \mathcal{G} | A^\top \bar{w} = 0 \}
\] (10)
In other words, a matrix \( A \in \mathcal{G}_{\bar{w}} \) is allowed to have multiple zero eigenvalues. By relating the set \( \hat{G}_{\bar{w}} \) to cycles of the digraph \( G \), we will be able to prove that the set \( \hat{G}_{\bar{w}} \) is a convex cone. This analysis is important because in section V, we will show that if \( G \) is strongly connected and if \( \vec{w} \) is in the interior of \( \Delta^{n-1} \), then \( G_{\bar{w}} \) is a nonempty convex set with its closure being \( \hat{G}_{\bar{w}} \). The complete proof of theorem 1 will be given at the end of section V.

### III. On Relevant Subsets of \( V \)

In this section, we will mainly focus on proving the following result.

**Proposition 2.** Let \( G = (V, E) \) be a rooted digraph. Let \( \vec{w} \) be a vector in \( \Delta^{n-1} \), and let \( V_{\bar{w}} \) be a subset of \( V \) collecting indices of nonzero entries of \( \vec{w} \). If \( \mathcal{G}_{\bar{w}} \) is nonempty, then \( V_{\bar{w}} \) is relevant.

The proof of Proposition 2 proceeds by first showing that a relevant subset \( V' \) is contained in the root set \( V_r \) of \( G \), and then showing that the subgraph \( \mathcal{G}_{\bar{w}}' \) derived by restricting \( G \) to \( V_{\bar{w}} \), is strongly connected. This is done in Lemmas 3 and 4 below.

**Lemma 3.** Let \( G \) be a rooted digraph, and let \( \vec{w} \) be a vector in \( \Delta^{n-1} \). If \( \mathcal{G}_{\bar{w}} \) is nonempty, then \( V_{\bar{w}} \) is a subset of \( V_r \).

**Proof.** Without loss of generality, we may assume that the root set \( V_r \) consists of the first \( m \) vertices. Then, each matrix \( A \) in \( G \) is a lower block-triangular matrix, i.e,
\[
A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}
\] (11)
with \( A_{11} \) a \( m \)-by-\( m \) square matrix.

In view of the above, the exponential \( \exp(At) \), as the transition matrix of system \( G \), is also a lower block-triangular matrix with blocks of the same dimensions as the blocks of \( A \). Furthermore, since the matrix \( A \) has a simple zero eigenvalue while all of its other eigenvalues have negative real parts, we have
\[
\lim_{t \to \infty} \exp(At) = \mathbf{1} \cdot \bar{w}^T
\] (12)
Using Equations (11) and (12), we know that \( w_i = 0 \) for all \( i = m + 1, \cdots, n \). In other words, the set \( V_{\bar{w}} \) is contained in \( V_r \).

We now show that the subgraph \( G_{\bar{w}} \) is strongly connected.

**Lemma 4.** Let \( G \) be a rooted digraph, and let \( \vec{w} \) be a vector in \( \Delta^{n-1} \). If \( \mathcal{G}_{\bar{w}} \) is nonempty, then the subgraph \( G_{\bar{w}} \) is strongly connected.

**Proof.** As in the proof of the previous lemma, we may assume without loss of generality that the set \( V_{\bar{w}} \) consists of the first \( m \) vertices of \( G \). Let \( A \) be a matrix in \( \mathcal{G}_{\bar{w}} \), and partition \( A \) into blocks as
\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}
\] (13)
with \( A_{11} \) being a \( m \)-by-\( m \) matrix and correspondingly, partition \( \vec{w} \) into
\[
\vec{w} = (\vec{w}_1, 0)
\] (14)
with \( \vec{w}_1 \) a vector in \( \mathbb{R}^m \). By assumption, each entry of \( \vec{w}_1 \) is nonzero.

Since \( A \) is in \( \mathcal{G}_{\bar{w}} \), we have \( \vec{w}_1^T \vec{w} = 0 \), so then \( \vec{w}_1^T A_{12} = 0 \). Since each entry of \( A_{12} \) is nonnegative while each entry of \( \vec{w}_1 \) is positive, we must have \( A_{12} = 0 \). This then implies that \( A_{11} \) is an \( m \)-by-\( m \) ISM.

Let \( G' = (V', E') \), with \( V' := \{1, \cdots, m\} \), be a subgraph of \( G \) induced by the block \( A_{11} \), i.e, an edge \( i \to j \) is in \( E' \) if and only if \( a_{ij} > 0 \). It suffices to show that \( G' \) is strongly connected.

First we note that the digraph \( G' \) must be rooted because otherwise \( A_{11} \), and hence \( A \), has at least two zero eigenvalues. Now, if \( G' \) is not strongly connected, from Lemma 3 there must exist at least a zero entry in vector \( \vec{w}_1 \), which is a contradiction.

We conclude this section by describing the relevant sets of some families of digraphs, namely cycle graphs and complete graphs.

**Corollary 5.** Let \( G \) be an \( n \)-cycle. If \( \mathcal{G}_{\bar{w}} \) is nonempty, then either \( w = \bar{e}_i \) for some \( i = 1, \cdots, n \) or \( \vec{w} \) has zero entry.
Corollary 6. Let $G$ be a rooted digraph, and let $W$ be the set of vectors in $\Delta^{n-1}$ with $G_{i\omega}$ being nonempty. If $W = \Delta^{n-1}$, then the digraph $G$ must be a complete graph.

Proof. It suffices to show that each induced subgraph of two vertices is a 2-cycle. Let
\[
\vec{w} = \frac{1}{2}(\vec{e}_i + \vec{e}_j).
\]
Then, the set of nonzero entries of $\vec{w}$ is given by
\[
V_{\vec{w}} = \{i, j\}
\]
So by Proposition 2 the set $V_{\vec{w}}$ is relevant. In particular, the subgraph $G_{i\omega}$ is strongly connected, and hence a 2-cycle. □

IV. ON CYCLES OF DIGRAPHS

In this section, we assume that $G = (V, E)$ is an arbitrary digraph, not necessarily rooted. Let $\vec{w}$ be a vector in the interior of $\Delta^{n-1}$, and recall that the set $\hat{G}_{i\omega}$ is defined by
\[
\hat{G}_{i\omega} = \{A \in G | A^T \vec{w} = 0\}
\]
Our goal in this section is to characterize the set $\hat{G}_{i\omega}$. This is important because as we will see in the next section when $G$ is strongly connected, the set $\hat{G}_{i\omega}$ is the closure, under the usual Euclidean topology, of $G_{i\omega}$.

Before proceeding further, we introduce a useful notion. We say that a digraph $G'$ is a cycle of $G$ if $G'$ is a subgraph of $G$, is a cycle consisting of at least two vertices.

We label the cycles of $G$ as $G_1, \cdots, G_k$. Let $\vec{w}$ be a vector in the interior of $\Delta^{n-1}$, i.e, each entry $w_i$ of $\vec{w}$ is positive. For each cycle $G_i$ of $G$, we now define an ISM $C_i$ by specifying its off-diagonal entries. Let $C_{i,st}$ be the $s$-th entry of $C_i$, and let
\[
C_{i,st} := \begin{cases} 1/w_s & \text{if } s \to t \text{ is an edge of } G_i \\ 0 & \text{otherwise} \end{cases}
\]
We are now in a position to state the main result of this section.

Proposition 7. Let $G$ be a digraph, and $\vec{w}$ be a vector in the interior of $\Delta^{n-1}$. Let $G_1, \cdots, G_k$ be the cycles of $G$, and $C_1, \cdots, C_k$ be the associated ISMs. Then the set $\hat{G}_{i\omega}$ is a convex cone spanned by $C_1, \cdots, C_k$, i.e,
\[
\hat{G}_{i\omega} := \{\sum_{i=1}^k \alpha_i C_i | \alpha_i \geq 0\}
\]
In the case where there is no cycle of $G$, then $\hat{G}_{i\omega}$ is a singleton consisting of only a zero matrix.

We note here that a similar result which relates cycles and doubly stochastic matrices can be found in [20]. We prove Proposition 7 by first investigating a special case where $G$ is acyclic, i.e, there is no cycle contained in $G$.

Lemma 8. Let $G$ be an acyclic digraph and let $\vec{w}$ be a vector in the interior of $\Delta^{n-1}$, then $\hat{G}_{i\omega} = \{0\}$.

Proof. We prove the lemma by induction on the number $n$ of vertices of $G$.

Base case. If $G$ consists of only one vertex, then there is nothing to prove.

Inductive step. Suppose the statement of the lemma holds for the case $n = m$, we show that it holds for $n = m + 1$.

Since $G$ is acyclic, there must exist a vertex, say vertex 1, such that there is no other vertex $i$ such that $i \to 1$ is an edge of $G$. Let $A \in \hat{G}_{i\omega}$, and let $a_1$ be the first column of $A$. Then $a_1$ has at most one nonzero entry, i.e, the first entry of $a_1$. Let $a_{11}$ be the first entry of $a_1$, then
\[
\vec{w}^T a_1 = w_1 a_{11} = 0
\]
Since $w_1$ is positive by assumption, we then have $a_{11} = 0$. This then implies that the first row vector of $A$ is also a zero vector. For convenience, we write matrix $A$ as
\[
A = \begin{pmatrix} 0 & 0 \\ 0 & A' \end{pmatrix}
\]
with $A'$ an $m$-by-$m$ matrix. It now suffices to show that $A'$ is a zero matrix.

Let $\vec{w}'$ be a vector in $\mathbb{R}^m$ defined by
\[
\vec{w}' := \frac{1}{1-w_1}(w_2, \cdots, w_{m+1}).
\]
Note that $\vec{w}'$ is well-defined since $w_1 < 1$. By construction, the vector $\vec{w}'$ is a vector in the interior of $\Delta^{m-1}$ because
\[
\sum_{i=2}^{m+1} \frac{w_i}{1-w_1} = \frac{1-w_1}{1-w_1} = 1
\]
each and each, with $i = 2, \cdots, m + 1$, is positive. Moreover, the vector $\vec{w}'$ satisfies the condition $\vec{w}'^T A' = 0$.

Let $G'$ be a subgraph of $G$ obtained by restricting $G$ to vertices $\{2, \cdots, m+1\}$. Let $G'$ be the set of ISMs associated with $G'$, and let
\[
\hat{G}_{i\omega}' := \{A' \in G' | A'^T \vec{w}' = 0\}
\]
Then the matrix $A'$ is contained in $\hat{G}_{i\omega}'$. Since the subgraph $G'$ is acyclic, by induction the set $\hat{G}_{i\omega}'$ contains only the zero matrix, and hence $A' = 0$. This completes the proof. □

We are now ready for proving Proposition 7.

Proof of Proposition 7. We let $H_{i\omega}$ be the convex cone spanned by $C_1, \cdots, C_k$. We first show that the set $H_{i\omega}$ is contained in $\hat{G}_{i\omega}$. It suffices to show that each matrix $C_i$ is contained in $\hat{G}_{i\omega}$.

We denote by $\vec{v}_{j}$ be the $j$-th column of $C_i$, then either $\vec{v}_{j}$ is a zero vector or $\vec{v}_{j}$ contains two nonzero entries. If $\vec{v}_{j}$ is a zero vector, then $\vec{w}^T \vec{v}_{j} = 0$, so we focus on the latter case. By definition of $C_i$, the $j$-th entry of $\vec{v}_{j}$ must be $-1/w_j$, and we let the other nonzero entry of $\vec{v}_{j}$ be the $k$-th entry, with its value given by $1/w_k$. We then have
\[
\vec{w}^T \vec{v}_{j} = w_j \cdot (-1/w_j) + w_k \cdot (1/w_k) = 0
\]
This equality holds for each column vector of $C_i$, and hence $\vec{w}^T C_i = 0$. □
We now show that the set $\hat{G}_{\hat{w}}$ is contained in $H_{\hat{w}}$. Let $A$ be a matrix in $G_{\hat{w}}$. Suppose that there is a positive real number $\alpha$, and a cycle $G_i$ of $G$ such that $A - \alpha C_i$ is contained in $G$. Then, the matrix $A - \alpha C_i$ is also contained in $G_{\hat{w}}$. Let

$$\alpha_i := \max \{ \alpha | (A - \alpha C_i) \in G \}$$

and let

$$A' := A - \alpha C_i$$

We then say that the matrix $A'$ is a reduction of $A$.

By construction, the matrix $A'$ has more zero entries than the matrix $A$ does. This, in particular, implies that if

$$A \to A' \to A'' \to \cdots$$

is a chain of reductions, then the sequence must be finite. Suppose this chains stops at $\hat{A}$, and that there is no reduction of $\hat{A}$ anymore. It then suffices to prove that $\hat{A}$ is a zero matrix.

Let $\hat{G}$ be a digraph of $n$ vertices induced by the matrix $\hat{A}$, i.e., an edge $i \to j$ is in $\hat{G}$ if and only if the $ij$-th entry of $\hat{A}$ is positive. Since there is no reduction of $\hat{A}$, the induced digraph $\hat{G}$ must be acyclic. Since $A' \to \hat{w} = 0$, with $\hat{w}$ in the interior of $\Delta^{n-1}$, by Lemma 8 $\hat{A}$ is a zero matrix. This completes the proof.

\section{Proof of the Main Theorem}

We will return to the proof of Theorem 1. First we show that if $G$ is strongly connected and if $\hat{w}$ is in the interior of $\Delta^{n-1}$, then the set $G_{\hat{w}}$ is nonempty. Precisely, we shall prove the following result:

\textbf{Proposition 9.} Let $G$ be a strongly connected digraph, and let $\hat{w}$ be a vector in the interior of $\Delta^{n-1}$. Then $G_{\hat{w}}$ is a nonempty convex set, with its closure being $G_{\hat{w}}$ with respect to the normal Euclidean topology.

We first have some preliminaries. Let $G$ be a strongly connected digraph of $n$ vertices with $n > 1$, and for convenience, let $C$ be the set of cycles of $G$. Then $C$ is a non-empty finite set since for each edge $i \to j$ of $G$, there is at least one cycle containing that edge. We label $G_1, \cdots, G_k$ as the cycles of $G$.

Let $C' = \{ G'_1, \cdots, G'_m \}$ be a subset of $C$, and let $G'$ be the digraph of $n$ vertices obtained by taking the union of $G'_1, \cdots, G'_m$, i.e., an edge $i \to j$ is in $G'$ if and only if $i \to j$ is an edge of $G'_k$ for some $G'_k$ in $C'$. We say the subset $C'$ is principal if the digraph $G'$ is strongly connected. We note that the paper [20] also introduced the notion of principal subset, but the definition here is different. Let us label the principal subsets of $C$ as $C'_1, \cdots, C'_l$.

Let $\hat{w}$ be a vector in the interior of $\Delta^{n-1}$. For each principal subset $C_i$ of $C$, introduce a subset $\text{int} H_i$ of $G_{\hat{w}}$. Let $G_{i_1}, \cdots, G_{i_m}$ be cycles of $G$ contained in $C_i$, and let $C_{i_1}, \cdots, C_{i_m}$ be the associated ISMs. Let $H_i$ be the convex cone spanned by $C_{i_1}, \cdots, C_{i_m}$, and $\text{int} H_i$ be the interior of $H_i$.

Equipped with definitions and notations above, we now prove the following lemma.

\textbf{Lemma 10.} Let $G$ be a strongly connected digraph of $n$ vertices with $n > 1$, and let $C_1, \cdots, C_l$ be principal subsets of $C$. Let $\hat{w}$ be a vector in the interior of $\Delta^{n-1}$. Then, we have $G_{\hat{w}} = \bigcup_{i=1}^l \text{int} H_i$.

\textit{Proof.} We first show that each set $\text{int} H_i$ is contained in $G_{\hat{w}}$. Suppose the principal subset $C_i$ consists of cycles $G_{i_1}, \cdots, G_{i_m}$. For any matrix $A$ in $\text{int} H_i$, there exists a set of positive coefficients $\alpha_{i_1}, \cdots, \alpha_{i_m}$ such that

$$A = \sum_{j=1}^m \alpha_j C_j$$

Let $G_A$ be the digraph induced by matrix $A$; then $G_A$ is strongly connected. Consequently, the matrix $A$ has a simple zero eigenvalue. On the other hand, we have $\hat{w}^T C_j = 0$ for all $j = 1, \cdots, m$, so then $\hat{w}^T A = 0$. This then implies that the matrix $A$ is contained in $G_{\hat{w}}$.

Next we show that the set $G_{\hat{w}}$ is contained in the union of $\text{int} H_1, \cdots, \text{int} H_l$. Let $A$ be a matrix in $G_{\hat{w}}$; then $A$ is also contained in $G_{\hat{w}}$. Thus by Proposition 9 there is a set of non-negative coefficients $\alpha_{1}, \cdots, \alpha_k$ such that

$$A = \sum_{i=1}^k \alpha_i C_i$$

Suppose $\alpha_{i_1}, \cdots, \alpha_{i_m}$ are the non-zero coefficients out of $\alpha_{1}, \cdots, \alpha_k$. Let $C' := \{ G_{i_1}, \cdots, G_{i_m} \}$. It now suffices to show that $C'$ is a principal subset of $C$.

Suppose that it is not the case. Then, the digraph $G'$ obtained by taking the union of $G_{i_1}, \cdots, G_{i_m}$ is not strongly connected. On the other hand, the digraph $G'$ is the digraph induced by the matrix $A$. So if $G'$ is not strongly connected, then by Proposition 2 the set $G_{\hat{w}}$ is empty which is a contradiction. We conclude that $C'$ is a principal subset of $C$.

\textbf{Remark 1.} We note that in general, the sets $\text{int} H_1, \cdots, \text{int} H_l$ might have non-empty intersections. However, it is never the case that

$$\text{int} H_i - \bigcup_{j \neq i} \text{int} H_j = \emptyset$$

Fig. 1. The digraph $G$ in this figure is strongly connected, and it has three cycles $G_1, G_2$ and $G_3$. The principal subsets associated with $G$ are $G_{\hat{w}}$, $\{ G_1, G_2 \}$ and $\{ G_1, G_2, G_3 \}$.
i.e., there is a matrix $A$ in $\text{int } H_j$ such that $A$ is not contained in any other $\text{int } H_i$ with $j \neq i$.

With Lemma \[10\] we are now equipped to prove Proposition \[9\].

Proof of Proposition \[2\]. If $G$ consists of only one vertex, the vector $\vec{w}$ is then the scalar 1, and hence $G_{\vec{w}} = \bar{G}_{\vec{w}} = \{0\}$. Henceforth, we assume that the number of vertices of $G$ is greater than one.

We first show that $G_{\vec{w}}$ is a nonempty convex set. By Lemma \[10\] we have $G_{\vec{w}} = \bigcup_{i=1}^{k} \text{int } H_i$, and since each $\text{int } H_i$ is nonempty, then so is their union.

Let $A_i$ and $A_j$ be two matrices in $\text{int } H_i$ and $\text{int } H_j$ respectively; we show that for $a_i$ and $a_j$ positive, the matrix $a_iA_i + a_jA_j$ is contained in $G_{\vec{w}}$. Let $C_i$ and $C_j$ be the two principal subsets associated with $H_i$ and $H_j$ respectively. Then, the union

$$C_k := C_i \cup C_j$$

(32)

is also a principal subset. The matrix $a_iA_i + a_jA_j$ is then an element in $\text{int } H_k$, and hence contained in $G_{\vec{w}}$.

It now remains to show that the closure of $G_{\vec{w}}$ is $\bar{G}_{\vec{w}}$. First we notice that $G_{\vec{w}}$ is contained in $\bar{G}_{\vec{w}}$ while $\bar{G}_{\vec{w}}$ is a closed set. So the closure of $G_{\vec{w}}$ must be contained in $\bar{G}_{\vec{w}}$. We now show that the converse is also true, that is $\bar{G}_{\vec{w}}$ is contained in the closure of $G_{\vec{w}}$. Choose a matrix $A$ in $G_{\vec{w}}$, then by Proposition \[7\] we have

$$A = \sum_{i=1}^{k} a_iC_i$$

(33)

with each $a_i$ non-negative. Since for each positive real number $\epsilon > 0$, the matrix

$$A(\epsilon) := \sum_{i=1}^{k} (a_i + \epsilon)C_i$$

(34)

is contained in $G_{\vec{w}}$, $A$ is in the closure of $G_{\vec{w}}$. This completes the proof.

We are now in a position to prove Theorem \[1\] stated in Section \[11\].

Proof of Theorem \[7\]. Denote by $V_1, \cdots, V_q$ the relevant subsets of $G$, and recall that $\text{int } F_{V_1} \subset \Delta^{n-1}$ is the interior of the face spanned by the unit vectors $\{\vec{e}_i|ij \in V_1\}$. We also recall that the set $W$ is the set of vectors $\vec{w}$ in $\Delta^{n-1}$ for which $G_{\vec{w}}$ is not empty.

First note that by Proposition \[2\] the set $W$ is contained in the union $\bigcup_{i=1}^{d} \text{int } F_{V_i}$. We will now show that the converse is also true. Let $V'$ be a relevant subset of $V$, and for simplicity we may assume that $V' = \{1, \cdots, m\}$ with $m \leq n$. Let $G'$ be a subgraph of $G$ induced by $V'$; then by definition $G'$ is strongly connected. Let $\vec{w}$ be a vector contained in $\text{int } F_{V'}$, and let $\vec{w}'$ be the vector in $\mathbb{R}^m$ containing the first $m$ entries of $\vec{w}$, i.e., $\vec{w} = (\vec{w}', 0)$. Thus $\vec{w}'$ is contained in the interior of $\Delta^{m-1}$.

We will now prove that $G_{\vec{w}}$ is nonempty, by directly constructing a matrix $A$ in it. For convenience, we partition the matrix $A$ into four blocks as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

(35)

with $A_{11}$ being an $m$-by-$m$ matrix. Let $G_{\vec{w}}'$ be the set of $m$-by-$m$ ISMs associated with the digraph $G'$ and the vector $\vec{w}'$. Then by Proposition \[2\] the set $G_{\vec{w}}'$ is nonempty, and we choose $A_{11}$ such that $A_{11}$ is contained in $G_{\vec{w}}'$. Let $A_{12}$ be a zero matrix as it has to be because $\vec{w}' A_{12} = 0$. We then choose $A_{21}$ and $A_{22}$ such that if $i \rightarrow j$ is an edge of $G$ and if $i > m$, then the $ij$-th entry of $A$ is positive.

We now show that the resulting matrix $A$ is contained in $G_{\vec{w}}$. By construction, we have

$$\vec{w}' A = \vec{w}' A_{11} = 0.$$ 

(36)

Let $G_A$ be the digraph induced by matrix $A$, which we next show rooted with root set $V'$. Since $A_{12} = 0$ by construction, there is no edge $i \rightarrow j'$ with $1 \leq i \leq m$ and $j' > m$ and thus for any vertex $j \notin V'$, there is no path in $G_A$ from a vertex $i \in V'$ to $j$. On the other hand, by construction of $A_{21}$ and $A_{22}$, we know that if $i \rightarrow j$, $i > m$, is an edge of the digraph $G$, then it is also an edge of the digraph $G_A$. So for any vertex $i \notin V'$, there is a path from $i$ to some vertex in $V'$. Since the subgraph $G'$ is strongly connected, the set $V'$ is then the root set of $G_A$. This completes the proof.

Remark 2. We note that it is not necessary to set $a_i > 0$ for all $i \in E$ with $i > m$. In fact, the matrix $A$ is still in $G_{\vec{w}}$ as long as the induced digraph $G_A$ is rooted with $V'$ the root set. We also note that the set $G_{\vec{w}}$ is a convex set. To see this, let $A'$ and $A''$ be in $G_{\vec{w}}$, and let $A = a'A' + a''A''$ with $a'$ and $a''$ positive. Then it follows that $A^\top \vec{w} = 0$. On the other hand, either $G_A'$ or $G_A''$ is a subgraph of $G_A$. So $G_A$ is rooted with $V'$ the root set.

VI. DECENTRALIZED IMPLEMENTATION

In this section, we will assume that $G$ is a rooted digraph, and $\vec{w}$ is a vector in $W$ for which the set $G_{\vec{w}}$ is nonempty. We present here a decentralized algorithm that allows the agents to find a matrix $A$ in $G_{\vec{w}}$. In particular, we assume here that each agent $\bar{x}_i$ only knows its own weight $w_i$, and each agent is only able to communicate/cooperate with its neighbors, which are defined as the agents connected to $i$ with either an incoming or an outgoing edge.

The implementation of the algorithm derived here relies on decentralized methods for the so-called graph balancing problem for the digraph $G$ (referred to as G-balancing). We say that the coefficients $b_{ij} \geq 0$, for $i \rightarrow j \in E$, form a solution of G-balancing if for each vertex $i$ of $G$, we have

$$\sum_{k \in V_i^+} b_{ki} = \sum_{j \in V_i^-} b_{ij}$$

(37)

with $V_i^+$ and $V_i^-$ being the incoming and the outgoing neighbors of vertex $i$, respectively. We call a solution positive (resp. non-negative) if the $b_{ij}$ are strictly positive (resp. non-negative).
Lemma 11. Let $G$ be a digraph, and let $\vec{w}$ be a vector in the interior of $\Delta^n-1$. Let $B_{\geq 0}$ be the set of nonnegative solutions of $G$-balancing. Let $\Lambda$ be a diagonal matrix with $\Lambda_{ii} = w_i$. Then $B_{\geq 0} = \Lambda^{-1} \mathcal{G}_{\vec{w}}$.

Proof. Let $\{ b_{ij} | i \rightarrow j \in E \}$ be a nonnegative solution of $G$-balancing. This solution gives rise to an ISM $B$, i.e., if we let $B_{ij}$ be the $ij$-th, $i \neq j$, entry of $B$, then

$$B_{ij} := \begin{cases} b_{ij} & \text{if } i \rightarrow j \in E \\ 0 & \text{otherwise} \end{cases} \quad (38)$$

Since $\{ b_{ij} | i \rightarrow j \in E \}$ is a nonnegative solution of the $G$-balancing problem, not only is $B$ an ISM, but also is $B^\top$. In other words, we have $B^\top 1 = 0$. Let $\Lambda$ be a diagonal matrix with $\vec{w}$ being its diagonal; then the matrix $A := \Lambda^{-1} B$ is an element in $\mathcal{G}_{\vec{w}}$ because

$$A^\top \vec{w} = B^\top \Lambda^{-1} \vec{w} = B^\top 1 = 0 \quad (39)$$

Conversely, if $A$ is a matrix in $\mathcal{G}_{\vec{w}}$, then the matrix $B := \Lambda B A$ yields a nonnegative solution of $G$-balancing. Moreover, this map between $B_{\geq 0}$ and $\mathcal{G}_{\vec{w}}$ is one-to-one and onto because the diagonal matrix $\Lambda$ is invertible.

Remark 3. It is known that $G$ is strongly connected if and only if there exists a positive solution of $G$-balancing. We also note that for $G$ strongly connected and $\{ b_{ij} | i \rightarrow j \in E \}$ a positive solution of $G$-balancing, the matrix $A = \Lambda^{-1} B$ is contained in $\mathcal{G}_{\vec{w}}$.

Now suppose that $G$ is strongly connected, and $\vec{w}$ is in the interior of $\Delta^n-1$. Suppose that there is a decentralized algorithm for agents to find a positive solution $\{ b_{ij} | i \rightarrow j \in E \}$ of $G$-balancing. Then by Lemma 11 if each agent $\bar{x}_i$ sets the interaction weights as

$$a_{ij} := b_{ij} / w_i \quad (40)$$

the resulting set $\{ a_{ij} | i \rightarrow j \in E \}$ yields a matrix $A$ in $\mathcal{G}_{\vec{w}}$. So for $G$ a strongly connected digraph $\vec{w}$ in the interior of $\Delta^n-1$, the problem of finding a matrix $A \in \mathcal{G}_{\vec{w}}$ is reduced to the problem of finding a positive solution of $G$-balancing. This is a well-studied problem, and we provide here a decentralized iterative algorithm for agents to find a positive solution of $G$-balancing.

Algorithm A1: G-balancing for G strongly connected. We let $b_{ij}[l]$ be the value of $b_{ij}$ at iteration $l \geq 0$. We assume that at every step, the agent $\bar{x}_i$ knows the values of $b_{ki}$ for all $k \in V_i^+$ and the value of $b_{ij}$ for all $j \in V_i^-$. Initialization. Each agent $\bar{x}_i$ sets $b_{ij}[0] = 1$ for all $j \in V_i^-$. Iterative step. Each agent $\bar{x}_i$ updates $b_{ij}[l]$ as

$$b_{ij}[l+1] = \frac{1}{2} \left( b_{ij}[l] + \sum_{k \in V_i^+} b_{ki}[l] / |V_i^-| \right) \quad (41)$$

We refer to [21] for a proof of convergence of the algorithm. We note that in the same paper, the authors also proposed a decentralized algorithm for finding a positive integer solution of $G$-balancing.

We now consider the case of $G$ rooted, but not necessarily strongly connected. We assume that the vector $\vec{w}$ is chosen so that $G_{\vec{w}}$ is nonempty. For simplicity, we still assume that only the first $m$ entries of $\vec{w}$ are nonzero, and let $\vec{w}'$ be in $\mathbb{R}^m$ so that $\vec{w} = (\vec{w}', 0)$. Let $G'$ be the subgraph of $G$ by restricting $G$ to the first $m$ vertices, and let

$$G'_{\vec{w}'} := \{ A' \in G' | A'^\top \vec{w}' = 0 \} \quad (42)$$

Similarly, we partition an ISM $A$ into four blocks

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad (43)$$

with $A_{11}$ being $m$-by-$m$. We will now describe a decentralized algorithm for agents to construct four block matrices $A_{11}, A_{12}, A_{21}$ and $A_{22}$ so that the resulting matrix $A$ is in $G'_{\vec{w}'}$.

Algorithm A2: computing $A \in \mathcal{G}_{\vec{w}'}$ for $G$ rooted. We use an iterative method to derive $a_{ij}$, but since the graph $G$ is not necessarily strongly connected, we need to process the input as described below.

Initialization. Each agent $\bar{x}_i$ informs his/her neighbors (both incoming and outgoing) of his/her own weight $w_i$ and agent $\bar{x}_i$ receives the information of weights from all of his/her neighbors.

There are two different cases depending on whether the weight $w_i$ of agent $\bar{x}_i$ is zero or not.

a). If $w_i = 0$, then agent $\bar{x}_i$ sets $a_{ij}[0] = 1$ for all $j \in V_i^-$. b). If $w_i > 0$, then agent $\bar{x}_i$ needs to first create a pair of vertex sets

$$\begin{cases} V_i^{\rightarrow} := \{ k \in V_i^+ | w_k > 0 \} \\ V_i^{\leftarrow} := \{ j \in V_i^- | w_j > 0 \} \end{cases} \quad (44)$$

We note that $V_i^{\rightarrow}$ (resp. $V_i^{\leftarrow}$) is just the set of incoming (resp. outgoing) neighbors of $i$ in the digraph $G'$. Agent $\bar{x}_i$ then sets

$$a_{ij}[0] = \begin{cases} 1/w_i & \text{if } j \in V_i^{\rightarrow} \\ 0 & \text{otherwise} \end{cases} \quad (45)$$

Iterative step. We still consider two cases:

a). If $w_i = 0$, then agent $\bar{x}_i$ retains the value of $a_{ij}[l]$, i.e., $a_{ij}[l+1] = a_{ij}[l] = 1$ for all $j \in V_i^-$. b). If $w_i > 0$, then agent $\bar{x}_i$ updates $a_{ij}[l]$ as

$$a_{ij}[l+1] = \begin{cases} \frac{1}{2} \left( a_{ij}[l] + \sum_{k \in V_i^+} w_k a_{ki}[l] / w_i |V_i^-| \right) & \text{if } j \in V_i^{\leftarrow} \\ 0 & \text{otherwise} \end{cases} \quad (46)$$

In other words, agent $\bar{x}_i$ only updates $a_{ij}[l]$ with $j \in V_i^-$. In addition, if we replace $a_{ij}[l]$ with $b_{ij}[l] / w_i$, then we actually recover the algorithm A1 and obtain a positive solution of $G'$-balancing.

We now verify that by following this algorithm, we do get an ISM $A$ in $G'_{\vec{w}'}$. We check that

1. The block matrix $A_{11}$ is contained in $G'_{\vec{w}'}$.
2. The block matrix $A_{12}$ is a zero matrix.
3. The two block matrices $A_{21}$ and $A_{22}$ are constructed in a way that if $i \rightarrow j$ is an edge of $G$ and if $i > m$, then $a_{ij}$, the $ij$-th entry of $A$, is 1.
Using arguments similar to the ones in the proof of Theorem [1], we can easily see that the resulting matrix is in $G_{\Delta}$.

VII. CONCLUSIONS

In this paper, we have worked with the standard continuous-time consensus model, and addressed the question of given a rooted digraph $G$, what kind of linear objective map

\[ f(\bar{x}_1, \cdots, \bar{x}_n) = \sum_{i=1}^{n} w_i \bar{x}_i \]  

(46)

with $\bar{w} = (w_1, \cdots, w_n) \in \Delta^{n-1}$, is feasible by a choice of interaction weights $a_{ij}$? By introducing the notion of relevant subsets of vertices, we have provided a complete answer to this question, as stated in Theorem [1]. Some examples, such as circles and complete graphs, have been given for illustration. In the paper, we have also posed the question of given a feasible objective map $f$, what are the choices of interaction weights for achieving $f$? By looking at cycles of $G$, and introducing the notion of principal subsets, we have presented relevant results in Proposition[9] and Lemma[10] Besides answering the feasibility questions, we have also presented in the paper a decentralized algorithm for agents in a network to implement a selected set of interaction weights for achieving a feasible objective map. Future work may focus on the case where the interaction weights $a_{ij}$’s are allowed to be negative. Note that in the case when $a_{ij}$’s are nonnegative, the vector $\bar{w}$ associated with a feasible objective map has to be in the unit simplex. Thus, if $f$ is an objective map with $\bar{w} \notin \Delta^{n-1}$, and if there is a choice of $a_{ij}$ under which $f$ is feasible, then there must exist some $a_{ij}$ which is negative. The question about feasibility, and the question about decentralized implementation can still be raised in this context for a given digraph $G$. Other open problems, such as dealing with time-varying digraphs, dealing with nonlinear objective maps, and dealing with the presence of a malicious player who attempts to increase his/her own weight, as in [15], are all interesting topics to look at.

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