Application of spectral elements method to calculation of stress-strain state of anisotropic laminated shells

K A Petrovskiy¹, A V Vershinin¹ and V A Levin²

¹ LLC "Fidesys" Moscow State University Science Park named after M. V. University, office 402, Leninskie gory d. 1, str. 77, Moscow 119234, Russia
² Moscow State University, GSP-1, Leninskie gory d. 1, Moscow 119991, Russia

E-mail: constantine.89@yandex.ru

Abstract. A model of a high-order triangular spectral shell element is presented to calculate a stress-strain state of a shell construction under small deformations considering the material inhomogeneity. Consideration of shear deformations, changes in the thickness of a shell is implemented through the use of 7-parameter model of shell elements. The results of problems solving are to illustrate the possibilities of the element.

1. Introduction

Most of the materials studied in modern technology experience deformation during operation. The development of new structural materials require an availability of adequate mechanical models and analytic complexes on their basis for the assessment of the strength of structural elements made of such materials. For a numerical simulation of a wide range of physical phenomena the finite element method is one of the most powerful tools for accurate, efficient and stable approximate solutions. At present, the finite element method is widely known as the basic calculation method for the numerical simulation of problems of solid mechanics.

Since the beginning of the development of practical procedures using the finite element analysis, one of the main objective was to develop methods for the calculation of the complex shell structures. In modern practice, the shell structures are used in many branches of industry, including automotive, aviation and civil construction domains. In finite element analysis, there are several basic approaches to shell elements modeling [1].

The behavior of shells of the first type is considered as a superposition of membrane and bending behaviors. Finite elements are built by simple combining of rigidity matrices for plates bending and plane stress state. Such elements have low accuracy, since the bending and membrane behavior are combined at the nodal points only.

The second category includes elements based on the use of a particular theory of shells and discretization of a corresponding variational formulation. One of disadvantages of this approach is that if the shell theory is applicable to a specific geometry or analysis conditions only, then the composition of finite elements is subject to the same restrictions.

The third approach to shells modeling is based on the use of 3D elastic elements. Three-dimensional elastic elements are the most common elements, but their use became no longer practicable for subtle and multi-layered membranes.
At present, the most common and researched elements are the shells continuum elements and elements of the 2D theory of shells. Shell continuum elements are built by displaying two-dimensional master-element to a surface in $\mathbb{R}^3$ constituting the mid-surface of a shell. An approximated three-dimensional geometry of the shell element can usually be recovered through the continuum approach, specifying the unit normal at each node, which is interpolated using the standard basic functions of an element. The formulation is completed through applying the relevant kinematic assumptions about the displacement field. Although no one of theories of shells is used, the resulting formulation can be defined as a shell element with properties corresponding to a shell model with shear deformations of the first-order.

Unlike the continuous shell elements, the shell elements based on 2D theory of shells, are formulated using an exact analytical description of undeformed mid-surface. In the theory of shells the basic kinematic assumptions, measures the stress and strain, constitutive equations are expressed in general curvilinear coordinates $(\xi, \eta, \zeta)$, which are used to describe the geometry of a three-dimensional shell. Formulations of elements of the theory of shells and elements based on continuous medium are based on the same mechanical assumptions and differ in the type of decomposition to the elements only.

In the theory of shells of Kirchhoff-Love type and in a 5-parameter shells of Reissner-Mindlin the three-dimensional constitutive equations are simplified through an assumption of stress equality to zero in the thickness direction. Return to the three-dimensional continuous medium is required if the stress and strain in the thickness direction are of great value. For example, in case of delamination of composite. To do this, an assumption of constant normal stress in the thickness direction is added to the theory of shells, and the formulation is extended to 6 parameters. But this leads to complex equations of deformation and inability to draw the explicit expression of defining relations. The difference between the distribution of stresses and strains over the thickness forced to add a linearly changing strain member, i.e. expand the formulation to 7-parameter theory. A linear deformation in the thickness direction is introduced as an additional variable and discretized in the usual way via a finite element procedure. The main advantage of the shell 7-parameter model is the use of defining relations for the three-dimensional body with no changes [2].

For many problems of mechanics of a deformable solid the finite-element procedures of high-order offer a variety of theoretical and practical advantages as compared with finite-element methods of lower order, which for the past few decades began to dominate in researches and commercial software [3]. In particular, it is possible to avoid various forms of locking, which without adequate stabilization often spoil the finite element model of lower order in the weak formulation of the Galerkin method for elastic and inelastic solids.

Lots of disadvantages that may be encountered in the finite element model based on the principle of unconditional minimization can be avoided mainly or completely through the use of adequate polynomial of $p$-order when creating a finite element approximation $u_{hp}$ inside each element. In particular, one can obtain effective finite element procedures which do not require the use of complicated special techniques, which are often required in the finite element formulation of the lower order to improve the accuracy of numerical solutions. As a result, one can use the formula of full integration and functional space of finite-element of a high order avoids any inconsistencies peculiar to a low-order approximation, which often lead to locking.

2. Spectral triangular element
The basis of spectral elements method is an integration scheme of Gauss-Lobatto [4]. It is not only a method of numerical integration. Integration points also determine a node basis, which gives the approximately diagonal matrix of the masses. This makes it possible to use the unsteady simulations which do not require the computation of inverse matrix of masses, but to obtain the optimal convergence rate. These properties allow the method of spectral elements to achieve a high order of accuracy. However, this rule of integration does not apply to triangles.
One of the alternatives to the Gauss-Lobatto-Legendre (GLL) points for triangles are the Fekete points [5]. These points are the solution of extreme problems and should be calculated numerically, but in most cases the minimization will be a simpler task than the calculation of optimal quadrature points. Fekete points on the border match with the GLL points and have natural extension to three dimensions. They were calculated for the tetrahedron, and for a wedge one can use the tensor product of Fekete points on a triangle with GLL points in the third direction. Figure 1 shows the location of the nodes in the triangular element of the 5th and 8th orders.

**Figure 1. Location of nodal points in spectral triangle of the 5th and 8th orders**

Fekete points are determined in accordance with the finite-dimensional space in which the interpolating function is built. For triangle a polynomial space of multiple two-dimensional polynomials is commonly used with a dimension $N_t = (N + 1)(N + 2)/2$, and in order to build interpolation polynomials exactly $N_t$ points are required. For the selected space the Fekete points are determined by the Vandermonde matrix $V$. Let $\{D_p\}$ - is the basis for a set of polynomials of degree $N$, where $p=1,...,N_t$, and $\{(\xi_p,\eta_p)\}$ - is a set of points in a right triangle with base and height $[-1, 1]$. Then $D_q(\xi_p,\eta_p)$ - component of $(q,p)$ Vandermonde matrix. The Fekete points are the points maximizing the Vandermonde determinant of the matrix, where the maximum is taken over all possible sets of points in the triangle. For the selection of the orthogonal basis in the space of polynomials on the triangle the Dubiner polynomials are used.

To use the Fekete points for the spectral element method it is necessary to calculate the Lagrange interpolation functions and their derivatives [4]. Let’s express these interpolation functions through Dubiner polynomials:

$$
\psi_p(\xi,\eta) = \sum_{i=1}^{N_t} a_{p,i} D_i(\xi,\eta).
$$

(1)

Coefficients $a_{p,i}$ represent the $(p,i)$ elements of the inverse Vandermonde matrix. Derivatives of Lagrange interpolation functions are as follows:

$$
\frac{\partial \psi_p(\xi,\eta)}{\partial \xi} = \sum_{i=1}^{N_t} a_{p,i} \frac{\partial D_i(\xi,\eta)}{\partial \xi}, \quad \frac{\partial \psi_p(\xi,\eta)}{\partial \eta} = \sum_{i=1}^{N_t} a_{p,i} \frac{\partial D_i(\xi,\eta)}{\partial \eta}.
$$

(2)
In order to calculate integrals of some function \( f(\xi, \eta, \zeta) \), that is required in the method of spectral elements, over each \( \Omega_e \) element, the following formula is used:

\[
\int_{\Omega_e} f(\xi, \eta, \zeta) \, d\Omega = \int_{\Omega_e} \sum_{p=1}^{N_p} w_p J_e(\xi_p, \eta_p, \zeta_p) f(\xi_p, \eta_p, \zeta_p) \, d\zeta = \sum_{i=0}^{N_i} \sum_{p=1}^{N_p} w_p w_i J_i(\xi_p, \eta_p, \zeta_i) f(\xi_p, \eta_p, \zeta_i),
\]

where \( w_p \) - quadrature weight equal to the first Dubiner coefficient in interpolation polynomial of \( p \) order; \( J_e \) - Jacobian associated with the element mapping \( \Omega_e \) to reference triangle; \( w_i \) - quadrature weight of one-dimensional Legendre polynomial of \( r \) order with \((N_h + 1)\) GLL point on the interval \([-1, 1]\).

3. Shell element model

By definition a shell is a solid with one geometric dimension considerably smaller in relation to the other two ones. Let's consider a conventional shell element \( \Omega_e \).

3.1. Shell element geometry.

The geometry of shell spectral element in the initial moment of time is described by the formula:

\[
0_{X} = \sum_{k=1}^{n} \rho_k(\xi, \eta) 0_{X_M}^k + \frac{\zeta}{2} \sum_{k=1}^{n} \rho_k(\xi, \eta) 0_h^k 0_y^k, \tag{3}
\]

where \( 0_{X_M} \) - coordinates of a material point on the mid-surface of the shell, \( 0_h \) - thickness, \( 0_y \) - normal to the mid-surface at that point (figure 2).

Deformed geometry at time \( t \) is described as follows:

\[
'X = \sum_{k=1}^{n} \rho_k(\xi, \eta) 'X_M^k + \frac{\zeta}{2} \sum_{k=1}^{n} \rho_k(\xi, \eta) 'h^k 'y^k + \frac{\zeta}{2} \sum_{k=1}^{n} \rho_k(\xi, \eta) 0_h^k Q^k 'y^k, \tag{4}
\]

Here \( 'y \) - normal vector at the time \( t \), \( 'h \) - shell thickness at the point at the time \( t \), \( Q^k \) - degree of freedom corresponding to the quadratic distribution of displacements in the direction of \( 'y \) (figure 3).

![Figure 2](image1.png) Shell element geometry at the initial moment of time.

![Figure 3](image2.png) Quadratic function of displacement in the node \( k \).

The displacement increment from the position of the element at the time \( t \) to position at the time \( t + \Delta t \) is equal to
\[ \mathbf{u} = \epsilon_{iM} \mathbf{x} - \mathbf{x} = \sum_{k=1}^{n} \psi_k \left( \xi, \eta \right) \left( \epsilon_{iM} \mathbf{x}_M^k - \mathbf{x}_M^k \right) + \frac{\zeta}{2} \sum_{k=1}^{n} \psi_k \left( \xi, \eta \right) \left( \epsilon_{iM} \mathbf{h}^k \epsilon_{iM} \mathbf{x}_n^k - \epsilon_{iM} \mathbf{h}^k \mathbf{x}_n^k \right) + \zeta^2 \sum_{k=1}^{n} \psi_k \left( \xi, \eta \right) \left( \epsilon_{iM} \mathbf{Q}_n^k \epsilon_{iM} \mathbf{x}_n^k - \epsilon_{iM} \mathbf{Q}_n^k \mathbf{x}_n^k \right) \]

In the degrees of freedom at the time \( t \) the displacement increments can be expressed as follows:

\[ \epsilon_{iM} \mathbf{x}_n^k - \epsilon_{iM} \mathbf{x}_M^k = \mathbf{u}_M^k = u_k \mathbf{e}_k + v_k \mathbf{e}_k + w_k \mathbf{e}_k \quad (6) \]

\[ \epsilon_{iM} \mathbf{h}^k - \epsilon_{iM} \mathbf{h}^k = 0 \mathbf{h}^k \mathbf{A}_k \quad (7) \]

\[ \epsilon_{iM} \mathbf{Q}_n^k - \epsilon_{iM} \mathbf{Q}_n^k = q_n^k \quad (8) \]

Formula (6) describes the displacements in the global Cartesian coordinate system, (7) - change in thickness of the element, (8) - quadratic increment of displacements over the thickness.

Increment of directing vectors is expressed in the terms of rotary degree of freedom \( \alpha, \beta, \gamma \) according to the formula

\[ \epsilon_{iM} \mathbf{y}_i^k = Q_i^k \mathbf{y}_i^k, \quad i = 1, 2, n, \quad (9) \]

where \( Q_i^k \) - rotation tensor.

\[ Q_i^k = I + \sin \frac{\theta}{\theta} S + \frac{1}{2} \left( \begin{array}{ccc} 2 \sin \frac{\theta}{2} & & \\ & \frac{\theta}{2} & \\ & & \frac{\theta}{2} \end{array} \right) \]

\[ S = \left[ \begin{array}{ccc} 0 & 0 & \beta_k \\ 0 & 0 & -\alpha_k \\ -\beta_k & \alpha_k & 0 \end{array} \right], \quad \theta = \sqrt{\alpha_k^2 + \beta_k^2}, \]

Here \( \alpha_k, \beta_k \) - Increment of turns around the vectors \( ^i \mathbf{y}_i^k, \) respectively.

By substituting (6) - (9) to the formula (5) we obtain the increment of displacements as follows

\[ \mathbf{u}(\xi, \eta, \zeta) = \sum_{k=1}^{n} \psi_k \left( \xi, \eta \right) u_i^k \quad (10) \]

where:

\[ u_i^k = u_i^M + \zeta \left[ \epsilon_{iM} \mathbf{y}_i^k \epsilon_{iM} \mathbf{A}_k - \epsilon_{iM} \mathbf{y}_i^k \epsilon_{iM} \mathbf{A}_k + \epsilon_{iM} \mathbf{y}_i^k \epsilon_{iM} \mathbf{A}_k + \epsilon_{iM} \mathbf{y}_i^k \epsilon_{iM} \mathbf{A}_k \right] + \zeta^2 \epsilon_{iM} \mathbf{Q}_n^k \epsilon_{iM} \mathbf{x}_n^k - \epsilon_{iM} \mathbf{Q}_n^k \mathbf{x}_n^k + \epsilon_{iM} \mathbf{Q}_n^k \mathbf{x}_n^k \]

The model uses the Lagrangian formulation for small deformations. The covariant components of Green tensor of strains are defined as follows:

\[ ^i \epsilon_{iM} = \frac{1}{2} \left( ^i \mathbf{g}_i - \frac{1}{2} \epsilon_{iM} \mathbf{g}_i - \frac{1}{2} \epsilon_{iM} \mathbf{g}_i \right). \quad (11) \]

In this statement of the covariant components formulation, all members higher than linear by \( \zeta \) are discarded.

3.2. Anisotropic materials

The shells used in modern technology as structural elements are mostly natural or structurally anisotropic. Moreover, most of anisotropic shells are layered. The widespread distribution of such shells is of great interest with respect to the theory of anisotropic shells. There are many forms of relationship between strain and stress for anisotropic materials. In general, the generalized Hooke's law for homogeneous anisotropic body is as follows:
There are 21 independent elastic constants. If the body features an anisotropic elastic symmetry, the equations of generalized Hooke's law are simplified. An important class of anisotropic bodies are orthotropic bodies with three planes of elastic symmetry. In this case, relations between strain and stress may be expressed as follows:

\[
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{xz}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{E_x} & -\frac{v_{xy}}{E_x} & -\frac{v_{xz}}{E_x} & 0 & 0 & 0 \\
-\frac{v_{xy}}{E_y} & \frac{1}{E_y} & -\frac{v_{yz}}{E_y} & 0 & 0 & 0 \\
-\frac{v_{xz}}{E_z} & -\frac{v_{yz}}{E_z} & \frac{1}{E_z} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{xy}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xz}}
\end{bmatrix} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{xz}
\end{bmatrix},
\]

Here we are getting 9 independent parameters: \( E_x, E_y, E_z \) - elastic moduli; \( v_{xy}, v_{yz}, v_{xz} \) - Poisson moduli; \( G_{xy}, G_{yz}, G_{xz} \) - shear moduli.

Layered composite materials consist of two or more materials featuring in sum the desirable properties, which do not occur when using these materials individually. Let's consider the layered shell, wherein each layer is an orthotropic material with the principal directions of elasticity \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \). When integrating by the thickness it is necessary to calculate the elasticity tensor \( \underline{C} \) for each layer in the global coordinate system. Then, in order to write down the elasticity tensor \( \underline{C} \) components for one layer in the global coordinates system \( \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \) it is necessary to perform the conversion

\[
\underline{C}^{ijkl}_{mathrm{global}} = T_{im} T_{jn} T_{kp} T_{lq} C_{mnqp},
\]

where the components \( T_{ij} \) are calculated as follows:

\[
T_{ij} = g^i \cdot \hat{e}_j.
\]

**4. Numerical examples**

In all the examples it is believed that each layer of the shell is made of orthotropic material, and the relationship between the components of the elastic tensor is defined by Table 1.

| Table 1. Orthotropic properties adopted in the examples. |
|--------------------------------------------------------|
| \( C^{2222}/C^{1111} = 0.543103 \)                  |
| \( C^{3333}/C^{1111} = 0.530172 \)                  |
4.1. Shell under the uniform load, normal to the upper surface.
A simply supported shell under the uniformly distributed normal static load $q_0$ is considered. Numerical results presented in Table 2 are obtained for the homogeneous shell with the aspect ratio $a/b = 1.0$ and a thickness $h/a = 0.1$.

Table 2. Deflections and stress of homogeneous shell

|          | $C^{1111}_{w}/hq_0$ | $\sigma_z/q_0$ | $\sigma_y/q_0$ |
|----------|----------------------|----------------|----------------|
| Exact solution | -688.57             | 36.021         | 22.21          |
| Triangular spectral shell | -703.87             | 36.75          | 22.6           |

4.2. Shell under the uniform load, normal to the upper surface.
A simply supported shell under the uniformly distributed normal static load $q_0$ is considered. The shell consists of three layers with identical top and bottom layers. The results are presented for ratio of layers $\beta = C^{1111}_T/C^{1111}_M$, where $C^{1111}_T$ - tensor of elasticity coefficient for the top layer, and $C^{1111}_M$ - the coefficient for the middle layer. Numerical results presented in Table 3 are obtained for the homogeneous shell with the aspect ratio $a/b = 1.0$ and a thickness $h/a = 0.1$.

Table 3. Deflections and stress of three-layer shell

| $\beta = C^{1111}_T/C^{1111}_M$ | 5          | 10         |
|---------------------------------|------------|------------|
| Exact solution                  | Triangular spectral shell | Exact solution | Triangular spectral shell |
| $C^{1111}_{w}/hq_0$             | -258.97    | -159.38    | -174.55        |
| $\sigma_z/q_0$                  | -260.06    |            |               |
| The upper surface of the top layer | 60.353   | 60.958     | 65.332         | 71.351         |
| The upper surface of the bottom layer | -46.426 | -47.259    | -48.609        | -53.97         |
| $\sigma_y/q_0$                  |            |            |               |
| The upper surface of the top layer | 38.491   | 40.124     | 43.566         | 47.252         |
| The upper surface of the bottom layer | -30.322 | -32.2      | -33.756        | -37.28         |

5. Conclusion
The 7-parameter model of triangular spectral shell element is developed. The model is intended for stress analysis of shells under small strains using the method of spectral elements. The thickness
change of shell thickness is allowed, and it is assumed that strains varies linearly with the thickness of a shell. These features of the model permit one to use this model for the analysis of complex composite structures. The results of computations for some problems that can be solved analytically show a good accuracy of the proposed method.

Acknowledgements
This work was financially supported by the Ministry of Education and Science of the Russian Federation in the framework of the agreement No. 14.579.21.0112 (unique identifier of the project RFMEFI57915X0112). Investigations were carried out within the Fidesys company — a grantee of the Ministry of Education and Science of the Russian Federation.

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