Conditionally positive definite unilateral weighted shifts

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ABSTRACT. In a recent paper [15], Hilbert space operators $T$ with the property that each sequence of the form $(\|T^n h\|^2)_{n=0}^\infty$ is conditionally positive definite in a semigroup sense were introduced. In the present paper, this line of research is continued in depth in the case of unilateral weighted shifts. The conditional positive definiteness of unilateral weighted shifts is characterized in terms of formal moment sequences. The description of the representing triplet, the main object canonically associated with such operators, is provided. The backward extension problem for conditionally positive definite unilateral weighted shifts is solved, revealing a new feature that does not appear in the case of other operator classes. Finally, the flatness problem in this context is discussed, with an emphasis on unexpected differences from the analogous problem for subnormal unilateral weighted shifts.

1. Introduction

The celebrated Lambert’s criterion for subnormality states that a bounded linear operator $T$ on a complex Hilbert space $H$ is subnormal if and only if for every $h \in H$, the sequence $(\|T^n h\|^2)_{n=0}^\infty$ is positive definite as a function on the additive semigroup of all nonnegative integers $\mathbb{Z}_+$ (see [17], see also [21, Theorem 7]). In the harmonic analysis on $\ast$-semigroups presented in [4], related classes of functions appear that play an important role in various branches of mathematics and probability theory. Among them, the class of conditionally positive definite functions prevails. In a recent paper [15], a new class of operators, called conditionally positive definite (CPD for brevity), was introduced and studied in depth. The operator $T$ is said to be CPD if for every $h \in H$, the sequence $(\|T^n h\|^2)_{n=0}^\infty$ is conditionally positive definite as a function on the semigroup $\mathbb{Z}_+$. The class of CPD operators contains in particular subnormal operators [11, 8], complete hypercontractions of order 2 [7], 3-isometries [1, 2, 3] and many others.

In this paper we study CPD unilateral weighted shifts with positive weights, the issue not included in [15]. The study is preceded by necessary preparations related to selected properties of conditionally positive definite sequences (see Section 2).

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The main characterization of CPD unilateral weighted shifts given in Theorem 3.1 is written in the spirit of the classical Berger theorem on subnormal unilateral weighted shifts [10, 12]. Namely, a unilateral weighted shift $W_\lambda$ with weights $\lambda = \{\lambda_n\}_{n=0}^\infty$ is subnormal if and only if the formal moment sequence $\hat{\lambda}$ associated with $W_\lambda$ (see (3.1)) is a Stieltjes moment sequence. In other words, there is a one-to-one correspondence between subnormal unilateral weighted shifts and (non-degenerate) Stieltjes moment sequences. In turn, CPD unilateral weighted shifts are in a one-to-one correspondence with normalized CPD sequences of exponential growth whose terms are all positive and whose representing measures are supported in the closed half-line $[0, \infty)$ (see Theorems 3.1 and 4.1).

The essential difference between the above two characterizations relies on the fact that Stieltjes moment sequences have all terms nonnegative, while CPD ones do not. For this reason, we devoted one section for finding necessary and sufficient conditions for positivity of CPD sequences (see Section 5). It is worth emphasizing that, unlike Stieltjes moment sequences which are represented by single parameter (a positive measure), CPD ones are represented by three parameters coming from a Lévy-Khinchin type formula. Note that the Berger theorem (see Corollary 3.2) can be derived from Theorem 3.1 by using some scaling results from [15].

Another major difference between subnormal and CPD unilateral weighted shifts is revealed when solving the backward extension problem. Our solution given in Theorem 7.2 (see also Theorem 7.3) is the basis for constructing a CPD unilateral weighted shift with weights $(\lambda_0, \lambda_1, \ldots)$ for which the extended unilateral weighted shift with weights $(t, \lambda_0, \lambda_1, \ldots)$ is CPD for any positive real number $t$ (see Example 7.7). This differs from solving the subnormal backward extension problem (see [9, Proposition 8]).

It was Stampfli who noticed that if two consecutive weights of a subnormal unilateral weighted shift $W_\lambda$ are equal, then all weights of $W_\lambda$ (except perhaps the first one) are equal to each other (see [19, Theorem 6]). As shown in Theorems 8.1 and 8.3, this is no longer true in the case of CPD unilateral weighted shifts. Namely, the smallest number of consecutive equal weights is four. If we allow the consecutive weights to be equal to one, then the number decreases to 2 (see Theorems 8.2 and 8.4). The corresponding counterexamples confirming the minimality of these numbers are given in Examples 8.5 and 8.6.

The main object canonically associated with CPD operators is the (operator) representing triplet. The description of the representing triplet for CPD unilateral weighted shifts is provided in Theorem 6.1. The question of when the components of this triplet are compact operators is discussed in detail in Propositions 6.2, 6.3 and 6.4, and Example 6.5.

We refer the reader to the classical treatise on weighted shifts [18].

2. Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ stand for the fields of real and complex numbers respectively and let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. Denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the sets of nonnegative and positive integers respectively. We write $\mathcal{B}(X)$ for the $\sigma$-algebra of Borel subsets of a topological space $X$. Unless otherwise stated, all scalar measures we consider in this paper are assumed to be positive. Given a finite Borel measure $\nu$ on $\mathbb{R}$, we denote by $\text{supp}(\nu)$ the closed support of $\nu$. We write $\delta_t$ for the Borel probability measure on $\mathbb{R}$ concentrated at the point $t \in \mathbb{R}$. We say that a sequence $\{\gamma_n\}_{n=0}^\infty$ of
real numbers is of exponential growth if \( \limsup_{n \to \infty} |\gamma_n|^{1/n} < \infty \), or equivalently if and only if there exist \( \alpha, \theta \in \mathbb{R}_+ \) such that

\[
|\gamma_n| \leq \alpha \theta^n, \quad n \in \mathbb{Z}_+.
\]

The discrete differentiation transformation \( \Delta : \mathbb{R}^\mathbb{Z}_+ \to \mathbb{R}^\mathbb{Z}_+ \) is given by

\[
(\Delta \gamma)_n = \gamma_{n+1} - \gamma_n, \quad n \in \mathbb{Z}_+, \quad \gamma = \{\gamma_n\}_{n=0}^{\infty} \in \mathbb{R}^\mathbb{Z}_+.
\]

We denote by \( \Delta^k \) the \( k \)th composition power of \( \Delta \). Note that if \( \gamma \) is of exponential growth, so is \( \Delta \gamma \).

Given \( n \in \mathbb{Z}_+ \), we define the polynomial \( Q_n \) in real variable \( x \) by

\[
Q_n(x) = \begin{cases} 0 & \text{if } n = 0, 1, \\ \sum_{j=0}^{n-2} (n - j - 1) x^j & \text{if } n \geq 2, \quad x \in \mathbb{R}. \end{cases} \tag{2.1}
\]

It is immediate from definition that

\[
Q_n(x) \geq 1, \quad x \in \mathbb{R}_+, \quad n \geq 2. \tag{2.2}
\]

Below, \( \Delta^j Q_n(x) \) denotes the action of the transformation \( \Delta^j \) on the sequence \( \{Q_n(x)\}_{n=0}^{\infty} \) for \( x \in \mathbb{R} \) and \( j \in \mathbb{N} \). The polynomials \( \{Q_n\}_{n=0}^{\infty} \) have the following properties that will be used in the subsequent parts of the paper (see [15, Lemma 2.2.1]):

\[
Q_n(x) = \frac{x^n - 1 - n(x - 1)}{(x - 1)^2}, \quad n \in \mathbb{Z}_+, \quad x \in \mathbb{R} \setminus \{1\}, \tag{2.3}
\]

\[
Q_{n+1}(x) = x Q_n(x) + n, \quad n \in \mathbb{Z}_+, \quad x \in \mathbb{R}, \tag{2.4}
\]

\[
\frac{Q_n(x)}{n} \leq \frac{Q_{n+1}(x)}{n+1}, \quad n \geq 1, \quad x \in [0, 1], \tag{2.5}
\]

\[
\lim_{n \to \infty} \frac{Q_n(x)}{n} = \frac{1}{1-x}, \quad x \in [0, 1), \tag{2.6}
\]

\[
(\Delta Q_n(x))_n = \begin{cases} 0 & \text{if } n = 0, \quad x \in \mathbb{R}, \\ \sum_{j=0}^{n-1} x^j & \text{if } n \in \mathbb{N}, \quad x \in \mathbb{R}, \tag{2.7}
\end{cases}
\]

\[
(\Delta^2 Q_n(x))_n = x^n, \quad n \in \mathbb{Z}_+, \quad x \in \mathbb{R}. \tag{2.8}
\]

We need also the following two additional properties of polynomials \( Q_n \).

**LEMMA 2.1.** The following two assertions are valid:

\[
\frac{Q_n(x)}{n^2} \leq 1, \quad x \in [0, 1], \quad n \geq 1, \tag{2.9}
\]

\[
\lim_{n \to \infty} \int_{[0,1]} \frac{Q_n(x)}{n^2} \, d\nu = 0 \quad \text{if } \nu \text{ is a finite Borel measure on } [0, 1]. \tag{2.10}
\]

**PROOF.** It follows from (2.1) that

\[
Q_n(x) \leq 1 + 2 + \ldots + (n - 1) = \frac{n(n - 1)}{2} \leq n^2, \quad x \in [0, 1], \quad n \geq 2,
\]

which implies (2.9). By (2.6), \( \lim_{n \to \infty} \frac{Q_n(x)}{n^2} = 0 \) for all \( x \in [0, 1] \). Combined with (2.9) and Lebesgue’s dominated convergence theorem, this yields (2.10). \( \square \)
We now recall some basic facts from harmonic analysis on semigroups (see [4]; see also [15]). Let \( \gamma = \{ \gamma_n \}_{n=0}^{\infty} \) be a sequence of real numbers. We say that \( \gamma \) is positive definite (PD for brevity) if
\[
\sum_{i,j=0}^{k} \gamma_{i+j} \lambda_i \bar{\lambda}_j \geq 0,
\]
for all finite sequences \( \lambda_0, \ldots, \lambda_k \in \mathbb{C} \). If the inequality (2.11) holds for all finite sequences \( \lambda_0, \ldots, \lambda_k \in \mathbb{C} \) such that \( \sum_{j=0}^{k} \lambda_j = 0 \), then we call \( \gamma \) conditionally positive definite (CPD for brevity). Obviously, PD sequences are CPD but not conversely. We say that \( \gamma \) is a Stieltjes moment sequence if there exists a finite Borel measure \( \mu \) on \( \mathbb{R}^+ \), called a representing measure of \( \gamma \), such that
\[
\gamma_n = \int_{\mathbb{R}^+} x^n \, d\mu(x), \quad n \in \mathbb{Z}^+.
\]
By the Stieltjes theorem (see [4, Theorem 6.2.5]), \( \gamma \) is a Stieltjes moment sequence if and only if the sequences \( \{ \gamma_n \}_{n=0}^{\infty} \) and \( \{ \gamma_{n+1} \}_{n=0}^{\infty} \) are PD. A Stieltjes moment sequence \( \gamma \) is called non-degenerate if \( \gamma_n > 0 \) for all \( n \in \mathbb{Z}^+ \). The following observation is a direct consequence of the definition.

A Stieltjes moment sequence \( \gamma \) is non-degenerate if and only if \( \gamma_n > 0 \) for some \( n \geq 1 \). (2.12)

We also need the following property of non-degenerate Stieltjes moment sequences.

**Lemma 2.2.** If \( k \in \mathbb{Z}^+ \) and \( \{ \gamma_n \}_{n=0}^{\infty} \) is a non-degenerate Stieltjes moment sequence with a representing measure \( \mu \), then the following conditions are equivalent:

(i) \( \gamma_{k+1}^2 = \gamma_k \gamma_{k+2} \),
(ii) there exists \( \zeta \in (0, \infty) \) such that \( \text{supp}(\mu) = \{ \zeta \} \) if \( k = 0 \), or \( \text{supp}(\mu) \subseteq \{0, \zeta\} \) if \( k \geq 1 \).

**Proof.** Adapt the proof of [13, Lemma 3.4]. \( \square \)

In this paper we deal mainly with CPD sequences of exponential growth. They can be described as follows.

**Theorem 2.3 ([15, Theorem 2.2.5]).** Let \( \gamma = \{ \gamma_n \}_{n=0}^{\infty} \) be a sequence of real numbers. Then the following conditions are equivalent:

(i) \( \gamma \) is a CPD sequence of exponential growth,
(ii) there exist \( b \in \mathbb{R}, c \in \mathbb{R}^+ \) and a compactly supported finite Borel measure \( \nu \) on \( \mathbb{R} \) such that \( \nu(\{1\}) = 0 \) and
\[
\gamma_n = \gamma_0 + bn + cn^2 + \int_{\mathbb{R}} Q_n(x) \, d\nu(x), \quad n \in \mathbb{Z}^+.
\]
Moreover, if (ii) holds, then the triplet \( (b, c, \nu) \) is unique and
\[
\text{supp}(\nu) \subseteq \left[ -\limsup_{n \to \infty} |\gamma_n|^{1/n}, \limsup_{n \to \infty} |\gamma_n|^{1/n} \right].
\]
Call \( (b, c, \nu) \) appearing in Theorem 2.3(ii) the representing triplet of \( \gamma \). By (2.1) and (2.13) we see that if \( (b, c, \nu) \) is the representing triplet of \( \gamma \), then
\[
b = \gamma_1 - \gamma_0 - c.
\]
For \( k \in \mathbb{Z}_+ \), we define the shifted sequence \( \gamma^{(k)} = \{\gamma_n^{(k)}\}_{n=0}^{\infty} \) by
\[
\gamma_n^{(k)} = \gamma_{k+n}, \quad n \in \mathbb{Z}_+, \quad k \in \mathbb{Z}_+.
\] (2.15)

Observe that
\[
\Delta^p(\gamma^{(k)}) = (\Delta^p \gamma)^{(k)}, \quad p, k \in \mathbb{Z}_+.
\] (2.16)

It follows straightforwardly from definition that if \( \gamma \) is CPD and \( k \) is an even positive number, then \( \gamma^{(k)} \) is CPD. This is not true for any positive odd number \( k \) (e.g., \( \gamma_n = (-1)^n \) for \( n \in \mathbb{Z}_+ \)). The description of the representing triplet of \( \gamma^{(k)} \) is provided in Lemma 2.4 below. We give a direct proof of this lemma. Another way of proving it is to use induction and the recurrence formula (2.4); we leave the details to the interested reader.

**Lemma 2.4.** Let \( \gamma = \{\gamma_n\}_{n=0}^{\infty} \) be a CPD sequence of exponential growth and let \( (b, c, \nu) \) be the representing triplet of \( \gamma \). Suppose that \( k \in \mathbb{N} \). Then \( \gamma^{(k)} \) is of exponential growth and the following statements hold:

(i) if \( k \) is even, then \( \gamma^{(k)} \) is CPD,
(ii) if \( k \) is odd, then \( \gamma^{(k)} \) is CPD if and only if \( \text{supp}(\nu) \subseteq \mathbb{R}_+ \),
(iii) if \( \gamma^{(k)} \) is CPD, then
\[
b_k = \gamma_{k+1} - \gamma_k - c = b + 2kc + \sum_{j=0}^{k-1} \int_{\mathbb{R}} x^j d\nu(x),
\] (2.17)
\[
c_k = c, \quad (2.18)
\]
\[
\nu_k(\Delta) = \int_{\Delta} x^k d\nu(x), \quad \Delta \in \mathcal{B}(\mathbb{R}),
\] (2.19)

where \( (b_k, c_k, \nu_k) \) is the representing triplet \( \gamma^{(k)} \).

**Proof.** It follows from (2.8) and (2.13) that
\[
(\Delta^2 \gamma)_n = 2c + \int_{\mathbb{R}} x^n d\nu(x), \quad n \in \mathbb{Z}_+.
\] (2.20)

This implies that
\[
(\Delta^2 \gamma^{(k)})_n \overset{(2.16)}{=} (\Delta^2 \gamma)_{k+n} \overset{(2.20)}{=} 2c + \int_{\mathbb{R}} x^n d\tilde{\nu}_k(x)
\]
\[
= \int_{\mathbb{R}} x^n (\tilde{\nu}_k + 2c\delta_1)(dx), \quad n \in \mathbb{Z}_+,
\] (2.21)

where \( \tilde{\nu}_k \) is the signed Borel measure on \( \mathbb{R} \) given by
\[
\tilde{\nu}_k(\Delta) = \int_{\Delta} x^k d\nu(x), \quad \Delta \in \mathcal{B}(\mathbb{R}).
\] (2.22)

If \( k \) is odd and \( \text{supp}(\nu) \subseteq \mathbb{R}_+ \), then the measures \( \tilde{\nu}_k \) and \( \tilde{\nu}_k + 2c\delta_1 \) are positive. Combined with [15, Proposition 2.2.9] and the fact that \( \gamma^{(k)} \) is of exponential growth, we infer from (2.21) that the sequence \( \gamma^{(k)} \) is CPD, which proves the “if” part of (ii).

Suppose now that \( \gamma^{(k)} \) is CPD. Applying (2.20) to \( \gamma^{(k)} \), we obtain
\[
(\Delta^2 \gamma^{(k)})_n = 2c_k + \int_{\mathbb{R}} x^n d\nu_k(x) = \int_{\mathbb{R}} x^n (\nu_k + 2c_k\delta_1)(dx), \quad n \in \mathbb{Z}_+.
\] (2.23)
Since the measures \( \tilde{\nu}_k + 2c_\delta_1 \) and \( \nu_k + 2c_\delta_1 \) are compactly supported (the first of them may not be positive), we deduce from (2.21), (2.23) and [6, Lemma 4.1] that
\[
\tilde{\nu}_k + 2c_\delta_1 = \nu_k + 2c_\delta_1.
\]
Since \( \tilde{\nu}_k(\{1\}) = \nu_k(\{1\}) = 0 \), this implies that \( c_k = c \) and \( \tilde{\nu}_k = \nu_k \), which yields (2.18) and (2.19). Applying (2.14) to \( \gamma^{(k)} \) and then using (2.13) and (2.7), we obtain (2.17). This proves (iii). If moreover \( k \) is odd, then using (2.22) and the fact that \( \tilde{\nu}_k = \nu_k \) is a positive measure, we deduce that \( \text{supp}(\nu) \subseteq \mathbb{R}_+ \). This proves the “only if” part of (ii).

As mentioned just before Lemma 2.4, the statement (i) holds, and so the proof is complete. \( \square \)

Given a (complex) Hilbert space \( \mathcal{H} \), we write \( B(\mathcal{H}) \) for the \( C^* \)-algebra of bounded linear operators on \( \mathcal{H} \). The range of \( T \in B(\mathcal{H}) \) is denoted by \( \mathcal{R}(T) \). By a semispectral measure on \( \mathbb{R}_+ \) we mean a map \( F : B(\mathbb{R}_+) \rightarrow B(\mathcal{H}) \) defined on the \( \sigma \)-algebra \( B(\mathbb{R}_+) \) of Borel subsets of \( \mathbb{R}_+ \) for which \( \langle F(\cdot)h, h \rangle \) is a measure for every \( h \in \mathcal{H} \). The closed support of \( F \) is denoted by \( \text{supp}(F) \). Following [15], we say that an operator \( T \in B(\mathcal{H}) \) is conditionally positive definite (CPD for brevity) if the sequence \( \{||T^n||^2\}_{n=0}^\infty \) is CPD for every \( h \in \mathcal{H} \). By Lambert’s criterion for subnormality, subnormal operators are CPD. The CPD operators can be characterized as follows.

**Theorem 2.5 ([15, Theorem 3.1.1]).** Let \( T \in B(\mathcal{H}) \). Then the following statements are equivalent:

(i) \( T \) is CPD,

(ii) there exist operators \( B, C \in B(\mathcal{H}) \) and a compactly supported semispectral measure \( F : B(\mathbb{R}_+) \rightarrow B(\mathcal{H}) \) such that \( B = B^* \), \( C \geq 0 \), \( F(\{1\}) = 0 \) and
\[
T^{*n}T^n = I + nB + n^2C + \int_{\mathbb{R}_+} Q_n(x) F(dx), \quad n \in \mathbb{Z}_+.
\]

Moreover, if (ii) holds, then the triplet \( (B, C, F) \) is unique and \( \text{supp}(F) \subseteq [0, r(T)^2] \).

Furthermore, \( \langle \langle Bh, h \rangle, \langle Ch, h \rangle, \langle F(\cdot)h, h \rangle \rangle \) is the representing triplet of the CPD sequence \( \{||T^n||^2\}_{n=0}^\infty \) for every \( h \in \mathcal{H} \).

Following [15], we call the triplet \( (B, C, F) \) appearing in Theorem 2.5(ii) the representing triplet of \( T \).

### 3. A characterization of CPD unilateral weighted shifts

In this paper we consider unilateral weighted shifts with positive weights (see [18] for the fundamental properties of weighted shifts). As usual \( \ell^2 \) stands for the Hilbert space of all square summable complex sequences \( \{\alpha_n\}_{n=0}^\infty \). Given a bounded sequence \( \lambda = \{\lambda_n\}_{n=0}^\infty \) of positive real numbers, there is a unique \( W_\lambda \in B(\ell^2) \) such that
\[
W_\lambda e_n = \lambda_n e_{n+1}, \quad n \in \mathbb{Z}_+.
\]
where \( \{e_n\}_{n=0}^\infty \) stands for the standard orthonormal basis of \( \ell^2 \). We call \( W_\lambda \) a unilateral weighted shift with weights \( \lambda \). If \( \lambda_n = 1 \) for all \( n \in \mathbb{Z}_+ \), then we call \( W_\lambda \) the unilateral shift. In this paper we consider only bounded unilateral weighted shifts with positive real weights.
Suppose that $W_\lambda$ is a unilateral weighted shift with weights $\lambda = \{\lambda_n\}_{n=0}^\infty$. The sequence $\hat{\lambda} = \{\hat{\lambda}_n\}_{n=0}^\infty$ associated to $W_\lambda$ is defined by
\begin{equation}
\hat{\lambda}_n = \begin{cases} 1 & \text{if } n = 0, \\ \lambda_0^2 \cdots \lambda_{n-1}^2 & \text{if } n \geq 1, \ n \in \mathbb{Z}_+. \end{cases}
\end{equation}

It is easily seen that $\hat{\lambda}$ is of exponential growth and the following identity holds:
\begin{equation}
\hat{\lambda}_n = \|W_\lambda^ne_0\|^2, \ n \in \mathbb{Z}_+.
\end{equation}

We also have
\begin{equation}
(\hat{\theta}_\lambda)_n = \theta^{2n} \hat{\lambda}_n, \ n \in \mathbb{Z}_+, \ \theta \in (0, \infty),
\end{equation}
where $\theta\lambda := \{\theta\lambda_n\}_{n=0}^\infty$. The sequence $\lambda$ of weights of $W_\lambda$ can be recaptured immediately from the sequence $\hat{\lambda}$ via the following formula
\begin{equation}
\lambda_n = \sqrt{\frac{\hat{\lambda}_{n+1}}{\hat{\lambda}_n}}, \ n \in \mathbb{Z}_+.
\end{equation}

Note that the transformation $\lambda \mapsto \hat{\lambda}$ given by (3.1) is a one-to-one correspondence between the set of sequences of positive real numbers and the set of sequences of positive real numbers with the first term equal to 1. The celebrated Berger theorem states that a unilateral weighted shift $W_\lambda$ is subnormal if and only if the corresponding sequence $\hat{\lambda}$ is a Stieltjes moment sequence (see [10, 12]). In other words, the transformation $\lambda \mapsto \hat{\lambda}$ is a one-to-one correspondence between the set of weights of subnormal unilateral weighted shifts and the set of non-degenerate Stieltjes moment sequences with compactly supported representing probability measures. We will show that a similar effect occurs in the case of CPD unilateral weighted shifts. Namely, the transformation $\lambda \mapsto \hat{\lambda}$ gives a one-to-one correspondence between the set of weights of CPD unilateral weighted shifts and the set of normalized CPD sequences of exponential growth with positive terms and with representing measures supported in $\mathbb{R}_+$ (see Theorem 3.1, see also Theorem 4.1). The main difference between these two characterizations is that the terms of a non-degenerate Stieltjes moment sequence are automatically positive (cf. (2.12)), while some terms of a CPD sequence may not be. What is more, it is not easy to determine CPD sequences with positive weights. One of our goals in this paper is to describe CPD sequences of exponential growth with positive terms.

**Theorem 3.1.** Let $\lambda = \{\lambda_n\}_{n=0}^\infty$ be a bounded sequence of positive real numbers. Then the following conditions are equivalent:
(i) the operator $W_\lambda$ is CPD,
(ii) there exist $b \in \mathbb{R}$, $c \in \mathbb{R}_+$ and a compactly supported finite Borel measure $\nu$ on $\mathbb{R}_+$ such that $\nu\{1\} = 0$ and
\begin{equation}
\hat{\lambda}_n = 1 + bn + cn^2 + \int_{\mathbb{R}_+} Q_n(x) d\nu(x), \ n \in \mathbb{Z}_+.
\end{equation}

Moreover, the triplet $(b, c, \nu)$ appearing in (ii) is unique and
\[ \text{supp}(\nu) \subseteq [0, \text{lim sup } \lambda_n^{1/n}]. \]
We call the triplet \((b,c,\nu)\) appearing in Theorem 3.1(ii) the scalar representing triplet of \(W_\lambda\).

**Proof of Theorem 3.1.** We begin by observing that since \(\lambda\) is bounded, the sequence \(\hat{\lambda}\) is of exponential growth.

(i)\(\Rightarrow\)(ii) Apply (3.2) and Theorem 2.5.

(ii)\(\Rightarrow\)(i) It follows from (3.2) that

\[
\|W_\lambda^n h\|^2 = \sum_{j=0}^{\infty} |\langle h, e_j \rangle|^2 \|W_\lambda^n e_j\|^2 = \sum_{j=0}^{\infty} |\langle h, e_j \rangle|^2 \frac{\hat{\lambda}^{n+j}}{\hat{\lambda}_j}, \quad n \in \mathbb{Z}_+, \; h \in \ell^2. \tag{3.6}
\]

According to Theorem 2.3 and Lemma 2.4, the sequence \(\{\|W_\lambda^n h\|^2\}_{n=0}^{\infty}\) is CPD for all \(j \in \mathbb{Z}_+\). It is a matter of routine to deduce from (3.6) that the sequence \(\{\|W_\lambda^n h\|^2\}_{n=0}^{\infty}\) is CPD for all \(h \in \ell^2\), which means that \(W_\lambda\) is CPD. The “moreover” part is a direct consequence of Theorem 2.3. \(\square\)

It is of some interest that the famous Berger criterion \([10, 12]\) for subnormality of unilateral weighted shifts can be deduced easily from Theorem 3.1 via the theory of CPD operators.

**Corollary 3.2** (Berger theorem). Let \(\lambda = \{\lambda_n\}_{n=0}^{\infty}\) be a bounded sequence of positive real numbers. Then the following conditions are equivalent:

(i) the operator \(W_\lambda\) is subnormal,

(ii) there exists a Borel measure \(\mu\) on \(\mathbb{R}_+\) such that

\[
\hat{\lambda}_n = \int_{\mathbb{R}_+} x^n d\mu(x), \quad n \in \mathbb{Z}_+. \tag{3.7}
\]

Moreover, the measure \(\mu\) in (ii) is unique, compactly supported, and finite.

**Proof.** (i)\(\Rightarrow\)(ii) Fix \(\theta \in (0, \infty)\) such that

\[
\theta^2 \limsup_{n \to \infty} (\hat{\lambda}_n)^{1/n} < 1. \tag{3.8}
\]

Since \(W_{i\lambda} = tW_\lambda\) is subnormal and consequently CPD for all \(t \in (0, \infty)\), we infer from (3.2) applied to \(W_{i\lambda}\) that \(\hat{\theta}\lambda\) is CPD. By (3.3), (3.8) and [15, Theorem 2.2.13], \(\hat{\lambda}\) is PD. Applying the above to \(W_\alpha\) with \(\alpha = \{\lambda_{n+1}\}_{n=0}^{\infty}\) (\(W_\lambda\) is unitarily equivalent to \(W_\lambda|_{H(W_\lambda)}\)) and noting that \(\hat{\lambda}_{n+1} = \lambda_n^2 \hat{\alpha}_n\) for all \(n \in \mathbb{Z}_+\), we see that \(\{\hat{\lambda}_{n+1}\}_{n=0}^{\infty}\) is PD. Using [4, Theorem 6.2.5] yields (ii).

(ii)\(\Rightarrow\)(i) Fix \(\theta \in (0, \infty)\) such that \(\|\theta W_\lambda\| < 1\). By assumption and (3.3), the sequence \(\hat{\theta}\lambda\) is a Stieltjes moment sequence and by [15, Theorem 2.2.12] it is CPD with the representing triplet \((b_0, 0, \nu_0)\) such that \(\text{supp}(\nu_0) \subseteq \mathbb{R}_+\). Using Theorem 3.1, we deduce that \(W_\lambda\) is a CPD operator such that \(\|W_\lambda\| = \|\theta W_\lambda\| < 1\). By [20, Theorem 4.1], \(\theta W_\lambda\) and consequently \(W_\lambda\) are subnormal.

The “moreover” part is a direct consequence of the fact that \(\hat{\lambda}\) is of exponential growth (see e.g., [5, (1.4)]) \(\square\)

**Remark 3.3.** In the case when \(W_\lambda\) is subnormal, the relationship between the measure \(\mu\) appearing in Corollary 3.2(ii), which is called the Berger measure of \(W_\lambda\), and the triplet \((b,c,\nu)\) in Theorem 3.1(ii) is as follows:

\[
b = \int_{\mathbb{R}_+} (x-1) d\mu(x), \quad c = 0,
\]
\[ \nu(\Delta) = \int_\Delta (x-1)^2 d\mu(x), \quad \Delta \in \mathfrak{B}(\mathbb{R}_+), \]
\[ \mu(\Delta) = \int_\Delta \frac{1}{(x-1)^2} q(x) + \left( \gamma_0 - \int_{\mathbb{R}_+} \frac{1}{(x-1)^2} d\nu(x) \right) \delta_1(\Delta), \quad \Delta \in \mathfrak{B}(\mathbb{R}_+). \]

We refer the reader to [15, Theorem 2.2.12] for more details. ♦

4. Modelling CPD unilateral weighted shifts

It follows from Theorem 3.1 that if a unilateral weighted shift \( W_\lambda \) is CPD, then the sequence \( \gamma = \hat{\lambda} \) has positive real terms and takes the form (3.5). Now we show the converse, namely if \( \gamma \) is a sequence of positive real numbers which is of the form (3.5), then \( \gamma = \hat{\lambda} \), where \( \lambda \) is a bounded sequence of positive real numbers and so the unilateral weighted shift \( W_\lambda \) is CPD.

**Theorem 4.1.** Let \( \gamma = \{\gamma_n\}_{n=0}^{\infty} \) be a sequence of real numbers such that
\[ \gamma_n > 0, \quad n \in \mathbb{Z}_+, \quad (4.1) \]
\[ \gamma_n = 1 + bn + cn^2 + \int_{\mathbb{R}^+} Q_n(x) d\nu, \quad n \in \mathbb{Z}_+, \quad (4.2) \]
where \( b \in \mathbb{R}, \ c \in \mathbb{R}_+ \) and \( \nu \) is a compactly supported finite Borel measure on \( \mathbb{R}_+ \) such that \( \nu(\{1\}) = 0 \). Then the sequence \( \lambda = \{\lambda_n\}_{n=0}^{\infty} \) defined by
\[ \lambda_n = \sqrt{\frac{\gamma_{n+1}}{\gamma_n}}, \quad n \in \mathbb{Z}_+, \quad (4.3) \]
is bounded and the unilateral weighted shift \( W_\lambda \) is CPD. Moreover, \( \hat{\lambda} = \gamma \).

**Proof.** In view of Theorem 3.1 and the obvious identity \( \hat{\lambda} = \gamma \), it suffices to show that the sequence \( \lambda \) is bounded, or equivalently that
\[ \limsup_{n \to \infty} \frac{\gamma_{n+1}}{\gamma_n} < \infty. \quad (4.4) \]
Before we proceed with the proof, we will make the necessary preparations. Set \( \vartheta = \sup \text{supp}(\nu) \) with the standard convention that \( \sup \emptyset = -\infty \). First, we define the polynomials \( p \) and \( q \) in \( n \) by
\[ p(n) = 1 + b(n+1) + n\nu(\mathbb{R}_+) + c(n+1)^2, \]
\[ q(n) = 1 + bn + cn^2. \]
Secondly, we observe that according to (2.2), (2.4), (4.1) and (4.2), we have
\[ \frac{\gamma_{n+1}}{\gamma_n} = \frac{p(n) + \int_{\mathbb{R}_+} x Q_n(x) d\nu(x)}{q(n) + \int_{\mathbb{R}_+} Q_n(x) d\nu(x)} \leq \frac{p(n) + \max\{0, \vartheta\} \int_{\mathbb{R}_+} Q_n(x) d\nu}{q(n) + \int_{\mathbb{R}_+} Q_n(x) d\nu}, \quad n \in \mathbb{Z}_+. \quad (4.5) \]
Thirdly, using (2.2), (2.5) and (2.6) and applying Lebesgue’s monotone convergence theorem, we deduce that
\[ \lim_{n \to \infty} \int_{\mathbb{R}_+} \frac{Q_n(x)}{n} d\nu = \int_{\mathbb{R}_+} \frac{1}{1-x} d\nu(x) \quad \text{whenever} \ \vartheta \leq 1. \quad (4.6) \]
Now we go to the main part of the proof. We will split it into several cases.

**Case 1.** \( \vartheta > 1 \).
Take any \( \theta \in (1, \vartheta) \). Since \( \vartheta \in \text{supp}(\nu) \), we see that \( \nu([\theta, \vartheta]) > 0 \). Using (2.2), we get

\[
\int_{\mathbb{R}_+} Q_n d\nu \geq \int_{[\theta, \vartheta]} Q_n d\nu \\
= \int_{[\theta, \vartheta]} (x^{n-2} + 2x^{n-3} + \ldots + (n-1)x^0) d\nu(x) \\
\geq \int_{[\theta, \vartheta]} x^{n-2} d\nu(x) \\
\geq \theta^{n-2} \nu([\theta, \vartheta]), \quad n \geq 2. \tag{4.7}
\]

This in turn yields

\[
\lim_{n \to \infty} \frac{n^2}{\int_{\mathbb{R}_+} Q_n d\nu} = 0. \tag{4.8}
\]

Dividing the numerator and denominator of the last fraction in (4.5) by \( \int_{\mathbb{R}_+} Q_n d\nu \) and using (4.8), we deduce that (4.4) holds.

**Case 2.** \( c > 0 \).

One can show that there exists \( \alpha \in (1, \infty) \) such that

\[
0 < p(n) \leq \alpha q(n) \quad \text{for} \quad n \quad \text{large enough.} \tag{4.9}
\]

Hence by (4.5) we see that for \( n \) large enough,

\[
\frac{\gamma_{n+1}}{\gamma_n} \leq \frac{p(n) + \max\{0, \vartheta\} \int_{\mathbb{R}_+} Q_n d\nu}{q(n) + \int_{\mathbb{R}_+} Q_n d\nu} \leq \max\{\alpha, \vartheta\},
\]

which yields (4.4).

**Case 3.** \( \vartheta \leq 1, \ c = 0 \) and \( \int_{\mathbb{R}_+} \frac{1}{1-x} d\nu(x) = \infty \).

Since \( \int_{\mathbb{R}_+} Q_n d\nu > 0 \) for all integers \( n \geq 2 \) (see (2.2)), we get

\[
\frac{\gamma_{n+1}}{\gamma_n} \leq \frac{p(n) + \int_{\mathbb{R}_+} Q_n d\nu}{q(n) + \int_{\mathbb{R}_+} Q_n d\nu} = \frac{\frac{p(n)}{n} + 1}{\frac{q(n)}{n} + \int_{\mathbb{R}_+} Q_n d\nu}, \quad n \geq 2.
\]

This together with (4.6) implies that \( \lim\sup_{n \to \infty} \frac{\gamma_{n+1}}{\gamma_n} \leq 1 \).

**Case 4.** \( \vartheta \leq 1, \ c = 0 \), \( \int_{\mathbb{R}_+} \frac{1}{1-x} d\nu(x) < \infty \) and \( b + \int_{\mathbb{R}_+} \frac{1}{1-x} d\nu(x) \neq 0 \).

We can argue as follows:

\[
\lim_{n \to \infty} \frac{\gamma_{n+1}}{\gamma_n} = \lim_{n \to \infty} \frac{\frac{1}{n} + \frac{n+1}{n} b + \frac{n+1}{n} \int_{\mathbb{R}_+} \frac{Q_{n+1}}{n+1} d\nu}{\frac{1}{n} + b + \int_{\mathbb{R}_+} \frac{Q_n}{n} d\nu} = \frac{b + \int_{\mathbb{R}_+} \frac{1}{1-x} d\nu(x)}{b + \int_{\mathbb{R}_+} \frac{1}{1-x} d\nu(x)} = 1. \tag{4.10}
\]

**Case 5.** \( \vartheta \leq 1, \ c = 0 \), \( \int_{\mathbb{R}_+} \frac{1}{1-x} d\nu(x) < \infty \) and \( b + \int_{\mathbb{R}_+} \frac{1}{1-x} d\nu(x) = 0 \).

First, observe that

\[
\gamma_n = 1 + n \left( b + \int_{[0,1]} \frac{Q_n}{n} d\nu \right).
\]
\[
1 + n \left( b + \int_{[0,1]} \frac{1}{1-x} \, d\nu(x) \right) - \int_{[0,1]} \frac{1-x^n}{n(1-x)^2} \, d\nu(x), \quad n \in \mathbb{N}.
\]

(4.11)

This implies that the sequence \( \{\gamma_n\}_{n=1}^{\infty} \) is monotonically decreasing and so by (4.1), \( \frac{\gamma_{n+1}}{\gamma_n} \leq 1 \) for all \( n \geq 1 \), which completes the proof. \( \square \)

Note that Theorem 4.1 provides a method of obtaining CPD unilateral weighted shifts from positive CPD sequences of exponential growth. The question of positivity of CPD sequences will be discussed in the next section.

5. Positivity of CPD sequences

Compared to Berger’s characterization of subnormal unilateral weighted shifts (see Corollary 3.2), the one given in Theorem 3.1 (see also Theorem 4.1) which characterizes CPD unilateral weighted shifts requires answering the question of when all terms of the sequence appearing on the right-hand side of (3.5) are positive. In Theorem 3.1(ii) there are three parameters \( b, c \) and \( \nu \). Due to the number of cases occurring in the proof of Theorem 4.1 in which the parameter \( c \) is equal to 0, treating the parameter \( b \) as a variable seems to be the only possible approach that may guarantee finding the solution of this problem. That this is the case is shown in Theorem 5.2.

We begin by isolating hypothesis necessary to state the main result of this section. Let \( \nu \) be a compactly supported finite Borel measure on \( \mathbb{R}^+ \) such that \( \nu(\{1\}) = 0, c \in \mathbb{R}^+ \) and \( \{\gamma_n\}_{n=0}^{\infty} \) be the sequence of continuous real functions on \( \mathbb{R} \) defined by

\[
\gamma_n(t) = 1 + tn + cn^2 + \int_{\mathbb{R}^+} Q_n \, d\nu, \quad t \in \mathbb{R}, \; n \in \mathbb{Z}^+.
\]

(5.1)

Set

\[
\vartheta := \sup \text{supp}(\nu) \quad (\text{convention: } \sup \emptyset = -\infty),
\]

\[
\Gamma_j := \int_{\mathbb{R}^+} \frac{1}{(1-x)^j} \, d\nu(x), \quad j = 1, 2 \quad (\text{whenever the integrals make sense}),
\]

\[
\Omega := \{ t \in \mathbb{R} : \gamma_n(t) > 0 \text{ for all } n \in \mathbb{Z}^+ \},
\]

\[
b := \inf \Omega.
\]

Clearly, \( \Omega = \{ t \in \mathbb{R} : \gamma_n(t) > 0 \text{ for all } n \in \mathbb{N} \} \). Observing that the function \( \gamma_n(\cdot) \) is monotonically increasing for every \( n \in \mathbb{Z}^+ \), we get

\[
\text{if } t \in \Omega, \text{ then } [t, \infty) \subseteq \Omega.
\]

(5.2)

Since by (2.2), \([0, \infty) \subseteq \Omega \) and \( \lim_{t \to -\infty} \gamma_n(t) = -\infty \) for every \( n \in \mathbb{N} \), we deduce that \( b \in \mathbb{R} \). Combined with (5.2), this implies that the set \( \Omega \) is an open or a closed subinterval of \( \mathbb{R} \) such that

\[
(b, \infty) \subseteq \Omega \subseteq [b, \infty).
\]

Our goal is to determine whether or not \( b \) belongs to \( \Omega \). The answer is given in Theorem 5.2 below. Before doing it, we prove the following lemma.
Lemma 5.1. Under the above assumptions, if \( b_0 \in (-\infty, b) \) is such that
\[
\lim_{n \to \infty} \gamma_n(t) = \infty, \quad t \in (b_0, \infty),
\]
then \( b \in (-\infty, 0) \) and \( \Omega = (b, \infty) \); in particular, \( \gamma_n(b) \geq 0 \) for all \( n \in \mathbb{Z}_+ \) and \( \gamma_n(b) = 0 \) for some \( n_0 \in \mathbb{N} \).

Proof. Since \( 0 \in \Omega \), it remains to show that for every \( b' \in \Omega \), there exists \( \varepsilon \in (0, \infty) \) such that \( (b' - \varepsilon, \infty) \subseteq \Omega \). For, assume that \( b' \in \Omega \). Take \( b_1 \in (b_0, b') \) and set
\[
J_{b_1} = \{ n \in \mathbb{Z}_+ : \gamma_n(b_1) > 0 \}.
\]
Applying (5.3) to \( t = b_1 \), we see that the set \( \mathbb{Z}_+ \setminus J_{b_1} \) is finite. This and
\[
\lim_{t \to b'} \gamma_n(t) = \gamma_n(b') > 0, \quad n \in \mathbb{Z}_+,
\]
imply that there exists \( \eta \in (0, \infty) \) such that
\[
\gamma_n(t) > 0, \quad n \in \mathbb{Z}_+ \setminus J_{b_1}, \ t \in (b' - \eta, b').
\]
Since the affine function \( \gamma_n \) is monotonically increasing for every \( n \in \mathbb{Z}_+ \), we get
\[
0 < \gamma_n(b_1) \leq \gamma_n(t), \quad n \in J_{b_1}, \ t \in [b_1, \infty).
\]
Combining (5.2), (5.4) and (5.5), we get \( (b' - \varepsilon, \infty) \subseteq \Omega \) for some \( \varepsilon \in (0, \infty) \). This completes the proof. \( \square \)

Now we are ready to state and prove the main result of this section.

Theorem 5.2. Assume that \( \{ \gamma_n \}_{n=0}^{\infty} \) is as in (5.1). Then

(i) \( \Omega = (b, \infty) \) with \( -\infty < b < 0 \) if any of the following conditions holds:
   (i-a) \( \vartheta > 1 \),
   (i-b) \( \vartheta \leq 1 \) and \( c > 0 \),
   (i-c) \( \vartheta \leq 1 \), \( c = 0 \) and \( \Gamma_1 = \infty \),
(ii) if \( \vartheta \leq 1 \), \( c = 0 \) and \( \Gamma_1 = \infty \), then
   (ii-a) \( \Omega = (b, \infty) \) with \( -\Gamma_1 < b < 0 \) if \( 1 < \Gamma_2 \leq \infty \),
   (ii-b) \( \Omega = (-\Gamma_1, \infty) \) if \( \Gamma_2 = 1 \) and \( \nu = \delta_0 \),
   (ii-c) \( \Omega = (-\Gamma_1, \infty) \) if \( \Gamma_2 = 1 \) and \( \nu \neq \delta_0 \),
   (ii-d) \( \Omega = (-\Gamma_1, \infty) \) if \( \Gamma_2 < 1 \).

Proof. (i) We claim that if any of the cases (i-a), (i-b) and (i-c) holds, then (5.3) is valid for \( b_0 = -\infty \). Indeed, if \( \vartheta > 1 \), then using (4.7) and (4.8), we get
\[
\lim_{n \to \infty} \gamma_n(t) = \lim_{n \to \infty} \int_{\mathbb{R}^+} Q_n d\nu \left( \frac{1 + tn + cn^2}{\int_{\mathbb{R}^+} Q_n d\nu} + 1 \right) = \infty, \quad t \in \mathbb{R}.
\]
If \( \vartheta \leq 1 \) and \( c > 0 \), then by Lemma 2.1 we have
\[
\lim_{n \to \infty} \gamma_n(t) = \lim_{n \to \infty} n^2 \left( \frac{1 + tn + cn^2}{n^2} + \int_{\mathbb{R}^+} \frac{Q_n}{n^2} d\nu \right) = \infty, \quad t \in \mathbb{R}.
\]
In turn, if \( \vartheta \leq 1 \), \( c = 0 \) and \( \Gamma_1 = \infty \), then (4.6) implies that
\[
\lim_{n \to \infty} \gamma_n(t) = \lim_{n \to \infty} n \int_{\mathbb{R}^+} \frac{Q_n}{n} d\nu \left( \frac{1 + tn}{n \int_{\mathbb{R}^+} \frac{Q_n}{n} d\nu} + 1 \right) = \infty, \quad t \in \mathbb{R}.
\]
This proves our claim. Applying Lemma 5.1, we get (i).

\footnote{Obviously, \( \nu = \delta_0 \) implies that \( \Gamma_1 = \Gamma_2 = 1 \) and \( \vartheta = 0 \).}
(ii) Assume now that \( \vartheta \leq 1, c = 0 \) and \( \Gamma_1 < \infty \). Using (4.6), we obtain

\[
\lim_{n \to \infty} \gamma_n(t) = \lim_{n \to \infty} n \left( \frac{1}{n} + t + \int_{\mathbb{R}_+} \frac{Q_n}{n} \, d\nu \right) = \begin{cases} 
-\infty & \text{if } t \in (-\infty, -\Gamma_1), \\
+\infty & \text{if } t \in (-\Gamma_1, \infty). 
\end{cases} 
\]  
(5.6)

We infer from (5.6) that

\[ -\Gamma_1 \leq b. \]  
(5.7)

By the continuity of \( \gamma_n \), we have

\[ \gamma_n(b) \geq 0, \quad n \in \mathbb{Z}_+. \]  
(5.8)

It follows from (4.11) that (recall that \( \gamma_0 \equiv 1 \))

\[ \gamma_n(-\Gamma_1) = 1 - \int_{[0,1)} \frac{1-x^n}{(1-x)^2} \, d\nu(x), \quad n \in \mathbb{Z}_+. \]  
(5.9)

Applying Lebesgue’s monotone convergence theorem to (5.9), we see that

\[ \gamma_n(-\Gamma_1) \downarrow 1 - \Gamma_2 \text{ as } n \to \infty. \]  
(5.10)

(ii-a) Suppose that \( \Gamma_2 \in (1, \infty) \). Then by (5.10), \( \lim_{n \to \infty} \gamma_n(-\Gamma_1) \in [-\infty, 0) \). Combined with (5.7) and (5.8), this implies that \( -\Gamma_1 < b \). Hence using (5.6) and applying Lemma 5.1 to \( b_0 = -\Gamma_1 \), we get (ii-a).

(ii-b) If \( \nu = \delta_0 \), then the strict monotonicity of \( \gamma_n \) for \( n \in \mathbb{N} \) implies that

\[ \gamma_n(t) > \gamma_n(-1) = \gamma_n(-\Gamma_1) \overset{(5.9)}{=} 0, \quad n \in \mathbb{N}, \ t \in (-1, \infty). \]

Therefore, by (5.2), \( \Omega = (-1, \infty) \).

(ii-c) Assume now that \( \Gamma_2 = 1 \). We claim that

\[ \gamma_n(-\Gamma_1) > 0 \text{ for all } n \in \mathbb{N} \text{ if and only if } \nu \neq \delta_0. \]  
(5.11)

Indeed, if \( n \in \mathbb{N} \) is fixed, then

\[
\gamma_n(-\Gamma_1) \overset{(5.9)}{=} \int_{[0,1)} \frac{1}{(1-x)^2} \, d\nu(x) - \int_{[0,1)} \frac{1-x^n}{(1-x)^2} \, d\nu(x) \\
= \int_{[0,1)} \frac{x^n}{(1-x)^2} \, d\nu(x) \geq 0,
\]

which implies that \( \gamma_n(-\Gamma_1) = 0 \) if and only if \( \nu = \delta_0 \) (use the fact that \( \Gamma_2 = 1 \)). This proves our claim. Since \( \gamma_0(z) = 1 \), it follows from (5.11) that if \( \nu \neq \delta_0 \), then \(-\Gamma_1 \in \Omega\), which together with (5.2) and (5.7) yields (ii-c).

(ii-d) Suppose that \( \Gamma_2 < 1 \). It follows from (5.10) that \( -\Gamma_1 \in \Omega \), which together with (5.2) and (5.7) implies that \( \Omega = [-\Gamma_1, \infty) \). This completes the proof. \( \square \)

The quantity \( b \) that appears in Theorem 5.2 can be expressed in terms of a new sequence \( \{\zeta_n\} \) depending only on the parameters \( c \) and \( \nu \).

**Proposition 5.3.** Let \( \{\gamma_n\}_{n=0}^{\infty} \) be as in (5.1). Then

\[ b = -\inf_{n \in \mathbb{N}} \zeta_n, \text{ where } \zeta_n = \frac{1}{n} + cn + \int_{\mathbb{R}_+} \frac{Q_n}{n} \, d\nu, \quad n \in \mathbb{N}. \]

Moreover, under the notation of Theorem 5.2, the following statements are valid:

(i) in the cases (i-a), (i-b), (i-c) and (ii-a), \( b = -\min_{n \in \mathbb{N}} \zeta_n \),

(ii) in the case (ii-b), \( b = -1 = -\zeta_n \) for all \( n \in \mathbb{N} \),

(iii) in the cases (ii-c) and (ii-d), the sequence \( \{\zeta_n\}_{n=1}^{\infty} \) is strictly decreasing.
PROOF. (i) It follows from Theorem 5.2 that in each of the cases (i-a), (i-b), (i-c) and (ii-a), \( \Omega = (b, \infty) \) with \(-\infty < b < 0\). By the continuity of \( \gamma_n \), we have
\[
\gamma_n(b) \geq 0, \quad n \in \mathbb{N},
\]
which implies that
\[
b \geq -\inf_{n \in \mathbb{N}} \zeta_n.
\]

Since \( b \notin \Omega \), we infer from (5.12) that there exists \( n_0 \in \mathbb{N} \) such that \( \gamma_{n_0}(b) = 0 \). This yields \( b = -\zeta_{n_0} \). Combined with (5.13), this implies that \( b = -\min_{n \in \mathbb{N}} \zeta_n \).

(ii) Since \( c = 0 \), this statement is easily seen to be true (see Theorem 5.2(ii-b)).

(iii) By Theorem 5.2, in the cases (ii-c) and (ii-d), \( \gamma_n(-\Gamma_1) > 0 \) for all \( n \in \mathbb{Z}_+ \). This together with (5.10) implies that the sequence \( \left\{ \frac{\gamma_n(-\Gamma_1)}{n} \right\}_{n=1}^{\infty} \) is strictly decreasing to \( 0 \). Since
\[
\zeta_n = \frac{\gamma_n(-\Gamma_1)}{n} + \Gamma_1, \quad n \in \mathbb{N},
\]
the sequence \( \{\zeta_n\}_{n=1}^{\infty} \) is strictly decreasing to \( \Gamma_1 \), which by Theorem 5.2 is equal to \( -b \). Hence \( b = -\min_{n \in \mathbb{N}} \zeta_n \).

Summarizing the above considerations, we conclude that \( b = -\inf_{n \in \mathbb{N}} \zeta_n \) in all cases. This completes the proof. \( \square \)

We conclude this section by making the following observation related to Theorem 5.2 and Proposition 5.3.

REMARK 5.4. It follows from Theorem 5.2 that \(-\infty < b < 0\) in all cases except \( c = 0 \) and \( \nu = 0 \). \( \Diamond \)

6. The representing triplet of \( W_\lambda \)

In this section we provide an explicit description of the representing triplet \((B, C, F)\) of a CPD unilateral weighted shift \( W_\lambda \). Recall that an operator \( T \in B(\mathcal{H}) \) is said to be diagonal with diagonal terms \( \{\xi_n\}_{n=0}^{\infty} \subseteq \mathbb{C} \) with respect to an orthonormal basis \( \{f_n\}_{n=0}^{\infty} \) of \( \mathcal{H} \) if \( Tf_n = \xi_n f_n \) for every \( n \in \mathbb{Z}_+ \). It is well known that a diagonal operator \( T \) with diagonal terms \( \{\xi_n\}_{n=0}^{\infty} \) is compact if and only if \( \lim_{n \to \infty} \xi_n = 0 \).

THEOREM 6.1. Let \( W_\lambda \) be a CPD unilateral weighted shift with weights \( \lambda = \{\lambda_k\}_{k=0}^{\infty} \) and let \((B, C, F)\) be the representing triplet of \( W_\lambda \). Then \( B, C \) and \( F(\Delta) \), where \( \Delta \in \mathfrak{B} (\mathbb{R}_+) \), are diagonal operators with respect to \( \{e_k\}_{k=0}^{\infty} \) with diagonal terms \( \{b_k\}_{k=0}^{\infty}, \{c_k\}_{k=0}^{\infty} \) and \( \{\nu_k(\Delta)\}_{k=0}^{\infty} \), respectively, given by
\[
b_k = \frac{\hat{\lambda}_{k+1} - \hat{\lambda}_k - c}{\lambda_k}, \quad c_k = \frac{c}{\lambda_k}, \quad \nu_k(\Delta) = \frac{1}{\lambda_k} \int_{\Delta} x^k dv(x),
\]
where \( \hat{\lambda}_k \) are as in (3.1) and \((b, c, \nu)\) is as in Theorem 3.1(ii). Moreover, the operator \( C \) is compact.

PROOF. All diagonal operators appearing in this proof are regarded with respect to the standard orthonormal basis \( \{e_k\}_{k=0}^{\infty} \) of \( \ell^2 \). First observe that for
n ∈ ℤ+, \( W_\lambda^n W_\lambda^n \) is the diagonal operator with diagonal terms \( \{ \frac{\hat{\lambda}_{k+n}}{\lambda_k} \}_{k=0}^\infty \). Applying Theorem 3.1 and Lemma 2.4 to \( \gamma = \hat{\lambda} \), we deduce that

\[
\frac{\hat{\lambda}_{k+n}}{\lambda_k} = 1 + b_k n + c_k n^2 + \int_{\mathbb{R}_+} Q_n(x) dv_k(x), \quad k, n ∈ ℤ_+.
\]  

(6.2)

It follows from the proof of Theorem 5.2(i) that if any of the cases (i-a) and (i-b) holds, then \( \lim_{k \to \infty} \hat{\lambda}_k = \infty \). This implies that whatever \( c \) is, \( \lim_{k \to \infty} c_k = 0 \). Hence, the diagonal operator with diagonal terms \( \{ c_k \}_{k=0}^\infty \), say \( \tilde{C} \), is a compact positive operator. Since the sequence \( \{ \lambda_k \}_{k=0}^\infty \) is bounded and

\[
\frac{\hat{\lambda}_{k+n}}{\lambda_k} = \lambda_k^2 \cdots \lambda_{k+n-1}^2, \quad k ∈ ℤ_+, n ∈ ℤ,
\]  

(6.3)

we deduce that for every \( n ∈ ℤ_+ \), the sequence \( \{ \frac{\hat{\lambda}_{k+n}}{\lambda_k} \}_{k=0}^\infty \) is bounded. This together with (6.1) implies that the sequence \( \{ b_k \}_{k=0}^\infty \) is bounded. Therefore, the diagonal operator with diagonal terms \( \{ b_k \}_{k=0}^\infty \), say \( \tilde{B} \), is bounded and selfadjoint. As a consequence of (6.2) and (6.3), we conclude that for every \( n ∈ ℤ_+ \), the sequence \( \{ \int_{\mathbb{R}_+} Q_n(x) dv_k(x) \}_{k=0}^\infty \) is bounded. Since \( Q_2 \equiv 1 \), we see that for every \( \Delta \in \mathscr{B}(\mathbb{R}_+) \), the sequence \( \{ \nu_k(\Delta) \}_{k=0}^\infty \) is bounded. Thus for every \( \Delta \in \mathscr{B}(\mathbb{R}_+) \), the diagonal operator with diagonal terms \( \{ \nu_k(\Delta) \}_{k=0}^\infty \), say \( \tilde{F}(\Delta) \), is bounded and positive. Note that

\[
\langle \tilde{F}(\Delta) f, f \rangle = \sum_{k=0}^\infty |\langle f, e_k \rangle|^2 \nu_k(\Delta), \quad \Delta ∈ \mathscr{B}(\mathbb{R}_+), f ∈ ℓ^2.
\]  

(6.4)

This together with (6.1) implies that \( \tilde{F} \) is a compactly supported semispectral measure such that \( \tilde{F}([1]) = 0 \). Hence, using standard measure and integration techniques, we obtain

\[
\left\langle \left( \int_{\mathbb{R}_+} Q_n d\tilde{F} \right) f, f \right\rangle = \int_{\mathbb{R}_+} Q_n(x) \langle \tilde{F}(dx) f, f \rangle
\]

\[
= \sum_{k=0}^\infty |\langle f, e_k \rangle|^2 \int_{\mathbb{R}_+} Q_n(x) dv_k(x), \quad f ∈ ℓ^2, n ∈ ℤ_+.
\]  

(6.5)

This in turn yields

\[
\langle W_\lambda^n W_\lambda^n f, f \rangle = \sum_{k=0}^\infty |\langle f, e_k \rangle|^2 \frac{\hat{\lambda}_{k+n}}{\lambda_k}
\]

\[
= \left( I + \tilde{B} + \tilde{C} n^2 + \int_{\mathbb{R}_+} Q_n d\tilde{F} \right) f, f \rangle, \quad f ∈ ℓ^2, n ∈ ℤ_+.
\]  

(6.2)\&(6.5)

Applying the uniqueness part of Theorem 2.5, we see that \( (B, C, F) = (\tilde{B}, \tilde{C}, \tilde{F}) \). This completes the proof. \[ \square \]

Recalling that the operator \( C \) appearing in Theorem 6.1 is compact, the natural question arises as to whether \( B \) and \( F(\mathbb{R}_+) \) are compact operators. In general, the answer is in the negative. Let us discuss this in more detail. We begin with the affirmative case.
Proposition 6.2. Under the assumptions and notation of Theorem 6.1, suppose that \( W_\lambda \) satisfies Case 4 of the proof of Theorem 4.1. Then \( B \) is compact and \( W_\lambda \) is not subnormal.

Proof. That \( B \) is compact follows immediately from (6.1) and (4.10) applied to \( \gamma = \lambda \). To see that \( W_\lambda \) is not subnormal, observe that by \([15, \text{Theorem 3.4.1}], \) there is no loss of generality in assuming that \( \frac{1}{x-1} \in L^1(F) \). Then

\[
\langle Be_0, e_0 \rangle = b \neq \int_{\mathbb{R}_+} \frac{1}{x-1} d\nu(x) = \left( \int_{\mathbb{R}_+} \frac{1}{x-1} F(dx) \right) e_0, e_0
\]

which contradicts \([15, \text{Theorem 3.4.1(ii-b)}] \). \( \square \)

The case when \( B \) is not compact is considered below.

Proposition 6.3. Under the assumptions and notation of Theorem 6.1, suppose that \( \vartheta := \text{sup} \text{supp}(\nu) > 1 \), \( \nu((1, \vartheta)) = 0 \) for some \( \theta \in (1, \vartheta] \) and

\[
\int_{[0,1)} \frac{1}{(x-1)^2} d\nu(x) < \infty.
\]

Then \( B \) is not compact.

Proof. By the compactness of \( C \) and (6.1), it suffices to show that

\[
\liminf_{k \to \infty} \frac{\hat{\lambda}_{k+1} - \hat{\lambda}_k}{\hat{\lambda}_k} > 0.
\]

Using the Cauchy-Schwarz inequality, we infer from (6.6) that \( \int_{[0,1]} \frac{1}{x} d\nu(x) < \infty \). Hence, we can define the functions \( f, g: \mathbb{N} \to \mathbb{R} \) by

\[
f(k) = b + c(2k + 1) - \int_{\mathbb{R}_+} \frac{1}{x-1} d\nu(x) + \int_{[0,1)} \frac{x_k}{x-1} d\nu(x), \quad k \in \mathbb{N},
\]

\[
g(k) = 1 + bk + ck^2 - \int_{\mathbb{R}_+} \frac{1 + k(x-1)}{(x-1)^2} d\nu(x) + \int_{[0,1)} \frac{x_k}{(x-1)^2} d\nu(x), \quad k \in \mathbb{N}.
\]

It is easily seen that there is a positive real number \( \alpha \) such that

\[
|f(k)| \leq \alpha k \quad \text{and} \quad |g(k)| \leq \alpha k^2 \quad \text{for all} \quad k \in \mathbb{N}.
\]

Using the assumption that \( \nu((1, \theta)) = 0 \), we deduce that

\[
\int_{(1,\infty)} \frac{x_k}{(x-1)^2} d\nu(x) \geq \theta^k \int_{[\theta,\vartheta]} \frac{1}{(x-1)^2} d\nu(x), \quad k \in \mathbb{N}.
\]

Since \( \nu((\theta, \vartheta)) \) is positive, so is \( \int_{[\theta,\vartheta]} \frac{1}{(x-1)} d\nu(x) \). Combined with (6.8), this yields

\[
\lim_{k \to \infty} \frac{f(k)}{\int_{(1,\infty)} \frac{x_k}{(x-1)^2} d\nu(x)} = 0 \quad \text{and} \quad \lim_{k \to \infty} \frac{g(k)}{\int_{(1,\infty)} \frac{x_k}{(x-1)^2} d\nu(x)} = 0.
\]

Observe also that

\[
\frac{\int_{(1,\infty)} \frac{x_k}{x-1} d\nu(x)}{\int_{(1,\infty)} \frac{x_k}{(x-1)^2} d\nu(x)} \geq \theta - 1, \quad k \in \mathbb{N}.
\]

(6.10)
It follows from (2.3), (2.7) and (3.5) that
\[
\frac{\hat{\lambda}_{k+1} - \hat{\lambda}_k}{\lambda_k} = \frac{b + c(2k + 1) + \int_{\mathbb{R}^+} \frac{x^k - 1}{x^k} \, d\nu(x)}{1 + bk + ck^2 + \int_{\mathbb{R}^+} \frac{x^k - 1 - k(x - 1)}{(x - 1)^2} \, d\nu(x)} = \frac{f(k) + \int_{(1, \infty)} \frac{x^k}{x^k} \, d\nu(x)}{g(k) + \int_{(1, \infty)} \frac{x^k}{(x - 1)^2} \, d\nu(x)} = \frac{\int_{(1, \infty)} \frac{x^k}{(x - 1)^2} \, d\nu(x)}{\int_{(1, \infty)} \frac{1}{(x - 1)^2} \, d\nu(x)} + 1, \quad k \in \mathbb{N}. \tag{6.11}
\]

Finally, we deduce (6.7) from (6.9), (6.10) and (6.11). This completes the proof. \qed

Now we discuss the question of compactness of \( F(\mathbb{R}^+) \).

**Proposition 6.4.** Under the assumptions and notation of Theorem 6.1, suppose that \( \theta := \sup \text{supp}(\nu) > 1 \) and \( 1 \notin \text{supp}(\nu) \). Then \( F(\mathbb{R}^+) \) is not compact.

**Proof.** For brevity, we define the real functions \( r \) and \( A \) on \( \mathbb{N} \) by
\[
r(k) = 1 + bk + ck^2 - \int_{\mathbb{R}^+} \frac{1 + k(x - 1)}{(x - 1)^2} \, d\nu(x), \quad k \in \mathbb{N},
\]
\[
A(k) = \int_{\mathbb{R}^+} \frac{x^k}{(x - 1)^2} \, d\nu(x), \quad k \in \mathbb{N}.
\]
Set \( \theta = \frac{1 + \vartheta}{2} \) and observe that \( \int_{[\theta, \theta]} \frac{1}{(x - 1)^2} \, d\nu(x) > 0 \). Since
\[
A(k) \geq \theta^k \int_{[\theta, \theta]} \frac{1}{(x - 1)^2} \, d\nu(x), \quad k \in \mathbb{N},
\]
we see that
\[
\lim_{k \to \infty} \frac{r(k)}{A(k)} = 0. \tag{6.12}
\]

It follows from (2.3), (3.5) and (6.1) that
\[
\nu_k(\mathbb{R}^+) = \frac{\int_{\mathbb{R}^+} x^k \, d\nu(x)}{1 + bk + ck^2 + \int_{\mathbb{R}^+} \frac{x^k - 1 - k(x - 1)}{(x - 1)^2} \, d\nu(x)} = \frac{\int_{\mathbb{R}^+} x^k \, d\nu(x)}{r(k) + A(k)} = \frac{\int_{[1, \infty)} x^k \, d\nu(x)}{A(k)} + 1 \geq \frac{\text{dist}(1, \text{supp}(\nu))^2}{A(k)} + 1, \quad k \in \mathbb{N}.
\]

Combined with (6.12), this implies that
\[
\liminf_{k \to \infty} \nu_k(\mathbb{R}^+) \geq \text{dist}(1, \text{supp}(\nu))^2 > 0.
\]
By Theorem 6.1, \( F(\mathbb{R}^+) \) is not compact. \qed

We now give an explicit example covering all cases discussed above.
Example 6.5. Fix $\theta \in (0, \infty) \setminus \{1\}$ and define the sequence $\gamma = \{\gamma_n\}_{n=0}^\infty$ of real numbers by

$$\gamma_n = 1 + Q_n(\theta), \quad n \in \mathbb{Z}_+.$$ 

Clearly, $\gamma_n > 0$ for all $n \in \mathbb{Z}_+$ (see (2.1) and (2.2)), $\gamma$ is a CPD sequence of exponential growth and its representing triplet $(b, c, \nu)$ is given by $b = c = 0$ and $\nu = \delta_\theta$. By Theorem 4.1, the unilateral weighted shift $W_\lambda$ with weights $\lambda = \{\lambda_n\}_{n=0}^\infty$ defined by (4.3) is CPD and $\hat{\lambda} = \gamma$. Consider now two cases.

Case 1. $\theta < 1$.

Then by Proposition 6.2, $B$ is compact. Since by (2.3) and (6.1),

$$\nu_k(\mathbb{R}_+) = \theta^k \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty,$$

we infer from Theorem 6.1 that $F(\mathbb{R}_+)$ is compact.

Case 2. $\theta > 1$.

Then by Proposition 6.3, $B$ is not compact. In turn, by Proposition 6.4, $F(\mathbb{R}_+)$ is not compact. ♦

7. CPD backward extensions

In this section, we solve the 1-step backward extension problem for CPD unilateral weighted shifts, which consists of finding a necessary and sufficient condition for the existence of a positive real number $t$ for which the unilateral weighted shift $W...(t, \lambda_0, \lambda_1, \ldots)$ is CPD under the assumption that $W...(\lambda_0, \lambda_1, \ldots)$ is.

We begin with the crucial lemma related to CPD sequences.

Lemma 7.1. Suppose that $\gamma = \{\gamma_n\}_{n=0}^\infty$ is a CPD sequence of exponential growth and $\text{supp}(\nu) \subseteq \mathbb{R}_+$, where $(b, c, \nu)$ is the representing triplet of $\gamma$. Let $\theta \in \mathbb{R}$. Set $\gamma_{-1} = \theta$. Then the following conditions are equivalent

(i) $\beta := \{\gamma_{n-1}\}_{n=0}^\infty$ is a CPD sequence of exponential growth and $\text{supp}(\nu_\beta) \subseteq \mathbb{R}_+$, where $(b_\beta, c_\beta, \nu_\beta)$ is the representing triplet of $\beta$,

(ii) $\int_{\mathbb{R}_+} \frac{1}{x} d\nu(x) \leq \theta + \gamma_1 - 2(\gamma_0 + c)$.

Moreover, if (i) holds, then $\nu(\{0\}) = 0$ and

$$b_\beta = \theta_0 - \theta - c,$$

$$c_\beta = c,$$

$$\nu_\beta(\Delta) = \int_\Delta \frac{1}{x} d\nu(x) + \nu_\beta(\{0\}) \delta_0(\Delta), \quad \Delta \in \mathfrak{B}(\mathbb{R}_+),$$

with

$$\nu_\beta(\{0\}) = \theta + \gamma_1 - 2(\gamma_0 + c) - \int_{\mathbb{R}_+} \frac{1}{x} d\nu(x).$$

Proof. (i)$\Rightarrow$(ii) Since $\beta_n = \gamma_{n-1}$ for $n \in \mathbb{Z}_+$, we get $\gamma = \beta^{(1)}$ (see (2.15) for the definition). Clearly, by (2.14) and Lemma 2.4 applied to $\beta$ in place of $\gamma$ with

3In the integral formula in (ii), we adhere to the convention that $\frac{1}{0} = \infty$. It will be used in the subsequent parts of this paper.
$k = 1$, we deduce that (7.1) and (7.2) hold and

$$\nu(\Delta) = \int_{\Delta} x d\nu_\beta(x), \quad \Delta \in \mathcal{B}(\mathbb{R}_+). \tag{7.5}$$

This implies that $\nu(\{0\}) = 0$, and consequently

$$\nu_\beta(\Delta) \equiv \int_{\Delta \setminus \{0\}} \frac{1}{x} d\nu(x) + \nu_\beta(\{0\}) \delta_0(\Delta), \quad \Delta \in \mathcal{B}(\mathbb{R}_+),$$

which implies (7.3). We now compute $\nu_\beta(\{0\})$. It follows from (2.13), (7.1), (7.2) and the identity $Q_2 \equiv 1$ that

$$\gamma_1 = \beta_2 = \beta_0 + 2\beta_\gamma + 4c + \nu_\beta(\mathbb{R}_+) = \theta + 2(\gamma_0 - \theta - c) + 4c + \nu_\beta(\mathbb{R}_+).$$

This yields

$$\nu_\beta(\mathbb{R}_+) = \theta + \gamma_1 - 2(\gamma_0 + c).$$

Substituting $\Delta = \mathbb{R}_+$ into (7.3) and using $\nu(\{0\}) = 0$, we conclude that (ii) and (7.4) hold. This also proves the “moreover” part.

(ii)$\Rightarrow$(i) Set $\tilde{b} = \gamma_0 - \theta - c$, $\tilde{c} = c$ and

$$\tilde{\nu}(\Delta) = \int_{\Delta} \frac{1}{x} d\nu(x) + \left(\theta + \gamma_1 - 2(\gamma_0 + c) - \int_{\mathbb{R}_+} \frac{1}{x} d\nu(x)\right) \delta_0(\Delta), \quad \Delta \in \mathcal{B}(\mathbb{R}_+).$$

In particular, we have

$$\tilde{\nu}(\mathbb{R}_+) = \theta + \gamma_1 - 2(\gamma_0 + c), \quad \Delta \in \mathcal{B}(\mathbb{R}_+). \tag{7.6}$$

Clearly, $\tilde{b} \in \mathbb{R}$, $\tilde{c} \in \mathbb{R}_+$ and by (ii), $\tilde{\nu}$ is a compactly supported finite Borel measure on $\mathbb{R}_+$ such that $\tilde{\nu}(\{1\}) = 0$. Define the sequence $\tilde{\beta} = \{\tilde{\beta}_n\}_{n=0}^\infty$ by

$$\tilde{\beta}_n = \theta + \tilde{b}n + \tilde{c}n^2 + \int_{\mathbb{R}_+} Q_n(x) d\tilde{\nu}(x), \quad n \in \mathbb{Z}_+. \tag{7.7}$$

By Theorem 2.3, $\tilde{\beta}$ is a CPD sequence of exponential growth. To complete the proof, it is enough to show that

$$\tilde{\beta}^{(1)} = \gamma. \tag{7.8}$$

For, applying Lemma 2.4 to $\tilde{\beta}$ in place of $\gamma$ with $k = 1$, we deduce that $\tilde{\beta}^{(1)}$ is a CPD sequence of exponential growth and

$$\tilde{b}_1 = (\tilde{\beta}_2 - \tilde{\beta}_1) \equiv (\tilde{b} + 3c + \tilde{\nu}(\mathbb{R}_+)) - c \equiv (\gamma_0 - \theta - c) + 3c + (\theta + \gamma_1 - 2(\gamma_0 + c)) - c \equiv \gamma_1 - \gamma_0 - c \equiv b. \tag{7.9}$$

Again, using Lemma 2.4 in the same context leads to

$$\tilde{c}_1 = \tilde{c} = c, \tag{7.10}$$

and

$$\tilde{\nu}_1(\Delta) = \int_{\Delta} x d\tilde{\nu}(x) = \nu(\Delta), \quad \Delta \in \mathcal{B}(\mathbb{R}_+). \tag{7.11}$$

Finally, we have

$$\tilde{\beta}_0^{(1)} = \tilde{\beta}_1 \equiv \theta + \tilde{b} + \tilde{c} = \gamma_0.$$

Combined with (7.9), (7.10) and (7.11), this implies (7.8), which completes the proof. □

We now solve the 1-step backward extension problem for CPD unilateral weighted shifts.

**Theorem 7.2.** Suppose that $W_\lambda$ is a CPD unilateral weighted shift with weights $\lambda = \{\lambda_n\}_{n=0}^\infty$ and $(b,c,\nu)$ is its scalar representing triplet. Let $t \in (0, \infty)$. Then the following conditions are equivalent:

(i) $W_{(t,\lambda_0,\lambda_1,...)}$ is CPD,

(ii) $\frac{1}{t} \geq \int_{\mathbb{R}_+} \frac{1}{x} d\nu(x) + 1 + c - b$.

Moreover, if $W_{(t,\lambda_0,\lambda_1,...)}$ is CPD and $(b_t,c_t,\nu_t)$ is its scalar representing triplet, then $\nu_t(\{0\}) = 0$ and

\begin{align}
    b_t & = t^2 (1 - c) - 1, \\
    c_t & = t^2 c, \\
    \nu_t(\Delta) & = t^2 \int_{\Delta} \frac{1}{x} d\nu(x) + \nu_t(\{0\}) \delta_0(\Delta), \quad \Delta \in \mathcal{B}(\mathbb{R}_+),
\end{align}

with

\begin{align}
    \nu_t(\{0\}) = 1 - t^2 \left( \int_{\mathbb{R}_+} \frac{1}{x} d\nu(x) + 1 + c - b \right). \quad (7.15)
\end{align}

**Proof.** Assume that (i) holds. Note that $t^2 \lambda = (t^2, t^4 \lambda_0, t^2 \lambda_0^2 \lambda_1, \ldots)$ is a CPD sequence of exponential growth with the representing triplet $(t^2 b, t^2 c, t^2 \nu)$. It follows from Theorem 3.1 applied to $W_{(t,\lambda_0,\lambda_1,...)}$ that the sequence $(1, t^2, t^4 \lambda_0, t^2 \lambda_0^2 \lambda_1, \ldots)$ is CPD and $\text{supp}(\nu) \subseteq \mathbb{R}_+$. Applying Lemma 7.1 to $\gamma = (t^2, t^4 \lambda_0, t^2 \lambda_0^2 \lambda_1, \ldots)$ and $\theta = 1$, we conclude that (ii) and (7.12)-(7.15) are valid. Reversing the above reasoning shows that (ii) implies (i), which completes the proof. □

Given $n \in \mathbb{N}$ and a unilateral weighted shift $W_\lambda$ with weights $\lambda = \{\lambda_k\}_{k=0}^\infty$, we say that $W_\lambda$ has a **CPD $n$-step backward extension** if there exists a sequence $\{t_j\}_{j=1}^n$ of positive real numbers such that the unilateral weighted shift $W_{(t_n,...,t_1,\lambda_0,\lambda_1,...)}$ is CPD. We call $W_{(t_n,...,t_1,\lambda_0,\lambda_1,...)}$ a **CPD $n$-step backward extension** of $W_\lambda$. In case $W_\lambda$ has a CPD $k$-step backward extension for every $k \in \mathbb{N}$, we say that $W_\lambda$ has a **CPD $\infty$-step backward extension**. Since the restriction of a CPD operator to its closed invariant subspace is CPD and $W_{(\lambda_1,\lambda_2,...)}$ is unitarily equivalent to $W_{(t_0,\lambda_0,\lambda_1,...)}$, we deuce that $W_{(\lambda_1,\lambda_2,...)}$ is CPD whenever $W_\lambda$ is. By induction, we have

\begin{align}
    \text{if } W_{(t_n,...,t_1,\lambda_0,\lambda_1,...)} \text{ is CPD, then } W_{(t_k,...,t_1,\lambda_0,\lambda_1,...)} \text{ is a CPD $k$-step backward extension of } W_{(t_n,...,t_1,\lambda_0,\lambda_1,...)} \text{ for } 1 \leq k \leq n. \quad (7.16)
\end{align}

The solution of the $n$-step backward extension problem for CPD unilateral weighted shifts takes the following form.

**Theorem 7.3.** Suppose that $W_\lambda$ is a CPD unilateral weighted shift with weights $\lambda = \{\lambda_k\}_{k=0}^\infty$ and $(b,c,\nu)$ is its scalar representing triplet. Let $n \geq 2$ be an integer and let $\{t_j\}_{j=1}^n \subseteq (0, \infty)$. Set $t_0 := \lambda_0 = (1 + b + c)^{1/2}$ and $t_k := (t_k,\ldots,t_1)$ for $k = 1, \ldots, n$. Then the following statements are equivalent:

(i) $W_{(t_n,...,t_1,\lambda_0,\lambda_1,...)}$ is a CPD $n$-step backward extension of $W_\lambda$,.
\((\text{ii})\) the sequence \(\{t_j\}_{j=1}^n\) satisfies the following conditions\(^4\):

\[
\frac{1}{t_k^2} = \left( \prod_{j=1}^{k-1} t_j^2 \right) \left( \int_{\mathbb{R}_+} \frac{1}{x^k} \, d\nu(x) + 2c \right) + 2 - t_{k-1}^2, \quad k = 1, \ldots, n - 1,
\]

\[
\frac{1}{t_n^2} = \left( \prod_{j=1}^{n-1} t_j^2 \right) \left( \int_{\mathbb{R}_+} \frac{1}{x^n} \, d\nu(x) + 2c \right) + 2 - t_{n-1}^2.
\]

Moreover, if (\text{ii}) holds, then for \(k = 1, \ldots, n\), \(W_{(t_k, \ldots, t_1, \lambda_0, \lambda_1, \ldots)}\) is a CPD \(k\)-step backward extension of \(W_\lambda\) with the scalar representing triplet \((b_{t_k}, c_{t_k}, \nu_{t_k})\) given by:

\[
b_{t_k} = t_k^2 - c \prod_{j=1}^k t_j^2 - 1,
\]

\[
c_{t_k} = c \prod_{j=1}^k t_j^2,
\]

\[
\nu_{t_k}(\Delta) = \left( \prod_{j=1}^k t_j^2 \right) \int_{\mathbb{R}_+} \frac{1}{x^k} \, d\nu(x) + \nu_{t_k}(\{0\}) \delta_0(\Delta), \quad \Delta \in \mathfrak{B}(\mathbb{R}_+),
\]

\[
\nu_{t_k}(\{0\}) = 1 - \left( \prod_{j=1}^k t_j^2 \right) \left( \int_{\mathbb{R}_+} \frac{1}{x^k} \, d\nu(x) + 2c \right) + t_k^2(t_k^2 - 2).
\]

In particular, \(\nu_{t_k}(\{0\}) = 0\) for \(k = 1, \ldots, n - 1\).

**Proof.** This can be proved by applying induction on \(n\) by using \((7.16)\) and Theorem 7.2. We leave the details to the reader. \(\square\)

**Remark 7.4.** As for the integrals \(\int_{\mathbb{R}_+} \frac{1}{x^j} \, d\nu(x)\) that appear in Theorem 7.3, it is worth noting that

if \(\int_{\mathbb{R}_+} \frac{1}{x^{k+1}} d\nu(x) < \infty\) for some \(k \in \mathbb{N}\), then \(\int_{\mathbb{R}_+} \frac{1}{x^j} d\nu(x) < \infty\) for \(j = 1, \ldots, k\).

This is because \(\nu\) is finite and compactly supported. \(\diamondsuit\)

We now give a handy criterion for the existence of a CPD \(n\)-step backward extension. It will be used in Example 8.5.

**Proposition 7.5.** Suppose that \(p \in \mathbb{N}\). Let \(W_\lambda\) be a CPD unilateral weighted shift with weights \(\lambda = \{\lambda_k\}_{k=0}^\infty\) and \((b, c, \nu)\) be its scalar representing triplet such that

\[
\int_{\mathbb{R}_+} \frac{1}{x^k} \, d\nu(x) < \infty, \quad k = 1, \ldots, p.
\]

Let \(\{\sigma_k\}_{k=1}^\infty\) be the sequence defined by the following formal\(^5\) recurrence relations:

\[
\sigma_1 = \int_{\mathbb{R}_+} \frac{1}{x} d\nu(x) + 1 + c - b, \quad (7.18)
\]

\[
\sigma_{k+1} = \frac{1}{\prod_{j=1}^k \sigma_j} \left( \int_{\mathbb{R}_+} \frac{1}{x^{k+1}} d\nu(x) + 2c \right) + 2 - \frac{1}{\sigma_k}, \quad k \in \mathbb{N}. \quad (7.19)
\]

\(^4\)with the convention that \(\prod_{j=1}^0 t_j^2 = 1\)

\(^5\)Note that \(\sigma_j\) may vanish or be infinite for some \(j\).
If $\sigma_k > 0$ for $k = 1, \ldots, p - 1$ (which can be dropped if $p = 1$) and $\sigma_p \geq 1$, then the following statements hold with\footnote{Observe that $n_\lambda$ can be equal to $+\infty$ (see Example 8.5).} $n_\lambda = \sup \left\{ k \in \mathbb{N} : \int_{\mathbb{R}^+} \frac{1}{x^k} d\nu(x) < \infty \right\}$:

(i) $\sigma_k \in (0, \infty)$ for $k = 1, \ldots, p - 1$,
(ii) $\sigma_k \in [1, \infty)$ for every integer $k$ such that $p \leq k \leq n_\lambda$,
(iii) $W_\lambda$ has a CPD $n_\lambda$-step backward extension.

**Proof.** Set $J = \{ j \in \mathbb{N} : 1 \leq j \leq n_\lambda \}$. It follows from (7.17) and Remark 7.4 that $n_\lambda \geq p$ and

$$J = \left\{ j \in \mathbb{N} : \int_{\mathbb{R}^+} \frac{1}{x^j} d\nu(x) < \infty \right\}. \tag{7.20}$$

Assume that $\sigma_j > 0$ for $j = 1, \ldots, p - 1$ and $\sigma_p \geq 1$. Using induction and (7.17)-(7.19), one can verify that

$$\sigma_j \in (0, \infty), \quad j = 1, \ldots, p, \tag{7.21}$$

which justifies (i). The proof of (ii) is by induction on $k$. Suppose that for some unspecified integer $k$ such that $p \leq k < n_\lambda$, $\sigma_j \in [1, \infty)$ for $j = p, \ldots, k$. This together with (7.21) and the induction hypothesis implies that $\sigma_j \in (0, \infty)$ for $j = 1, \ldots, k$. Since $k + 1 \in J$, we deduce from (7.19), (7.20) and $\sigma_k \geq 1$ that $\sigma_{k+1} \in \mathbb{R}$ and

$$\sigma_{k+1} - 1 = \frac{1}{\prod_{j=1}^k \sigma_j} \left( \int_{\mathbb{R}^+} \frac{1}{x^{k+1}} d\nu(x) + 2c \right) + \left( 1 - \frac{1}{\sigma_k} \right) \geq \frac{1}{\prod_{j=1}^k \sigma_j} \left( \int_{\mathbb{R}^+} \frac{1}{x^{k+1}} d\nu(x) + 2c \right) \geq 0,$$

which shows that $\sigma_{k+1} \in [1, \infty)$. This completes the induction argument. Thus (ii) holds. Finally (iii) is a direct consequence of (i),(ii) and Theorem 7.3. \hfill $\Box$

Applying Proposition 7.5 to $p = 1$, we get the following.

**Corollary 7.6.** Let $W_\lambda$ be a CPD unilateral weighted shift with weights $\lambda = \{ \lambda_k \}_{k=0}^{\infty}$ and $(b, c, \nu)$ be its scalar representing triplet such that $b \leq c$. Then $W_\lambda$ has a CPD $\infty$-step backward extension if and only if $\int_{\mathbb{R}^+} \frac{1}{x^k} d\nu(x) < \infty$ for all $k \in \mathbb{N}$.

As shown in Example 8.5, the inequality $b \leq c$ is not necessary for a unilateral weighted shift to have CPD $\infty$-step backward extension.

Recall that in the case of the 1-step backward extension problem for subnormal unilateral weighted shifts, the parameter $t \in (0, \infty)$ for which the unilateral weighted shift $W_{(t, \lambda_0, \lambda_1, \ldots)}$ is subnormal (provided it exists) has a finite upper bound depending on $(\lambda_0, \lambda_1, \ldots)$ (see [9, Proposition 8]). In the corresponding problem for completely hyperexpansive unilateral weighted shifts, the parameter $t$ has a positive lower bound (see [14, Corollary 3.3]). In contrast to these two cases, when considering CPD unilateral weighted shifts $W_{(t, \lambda_0, \lambda_1, \ldots)}$, it may happen that there is neither a finite upper nor a positive lower bound for the parameter $t$. A similar effect appears in the CPD $n$-step backward extension problem.

**Example 7.7.** Fix $k \in \mathbb{N}$ and take any $\theta \in \left( \frac{1}{k}, \frac{1}{k - 1} \right)$ (with the convention that $\frac{1}{0} = \infty$). Define the sequence $\gamma = \{ \gamma_n \}_{n=0}^{\infty}$ by

$$\gamma_n = 1 + n\theta, \quad n \in \mathbb{Z_+}.$$
Clearly, $\gamma$ is a CPD sequence of exponential growth satisfying (4.1). Its representing triplet $(b, c, \nu)$ is given by $b = \theta$, $c = 0$ and $\nu = 0$. Applying Theorem 4.1, we get a CPD unilateral weighted shift $W_\lambda$ with weights $\lambda = \{\lambda_n\}_{n=0}^\infty$ such that $\hat{\lambda} = \gamma$. The weights of $W_\lambda$ are given by

$$\lambda_n = \sqrt{\frac{1 + (n+1)\theta}{1 + n\theta}}, \quad n \in \mathbb{Z}_+.$$ 

In fact, $W_\lambda$ is a 2-isometry (see [16, Lemma 6.1(ii)]). With notation as in Proposition 7.5, we verify that $\{\sigma_n\}_{n=0}^\infty \subseteq \mathbb{R} \setminus \{0\}$ and

$$\sigma_n = \frac{1 - n\theta}{1 - (n-1)\theta}, \quad n \in \mathbb{N}.$$ 

This implies that $\sigma_n > 0$ for $n = 1, \ldots, k - 1$ and $\sigma_k < 0$. Hence, by Theorem 7.3, there exists a finite sequence $\{t_j\}_{j=1}^{k-1} \subseteq (0, \infty)$ (which can be dropped if $k = 1$) such that the unilateral weighted shift $W(T_{t_k, \ldots, t_2, \lambda_0, \lambda_1, \ldots})$ is CPD for every $t_k \in (0, \infty)$, and $W_\lambda$ has no CPD $(k + 1)$-step backward extension.

8. Flatness of CPD unilateral weighted shifts

In [19, Theorem 6], Stampfli proved that if two consecutive weights of a subnormal unilateral weighted shift $W_\lambda$ are equal, then $W_\lambda$ is flat, meaning that the sequence of its weights stabilizes starting from the second weight. In this section, we show that this is no longer true for CPD unilateral weighted shifts, namely, the number of consecutive equal weights has to be increased to four and this number is optimal (under some constrains discussed in detail below).

We begin with the case where four consecutive weights not containing the initial one are equal, and none of them is equal to one.

**Theorem 8.1.** Suppose that $W_\lambda$ is a CPD unilateral weighted shift with weights $\lambda = \{\lambda_n\}_{n=0}^\infty$ and $\kappa \in \mathbb{N}$ is such that

$$\lambda_\kappa = \lambda_{\kappa+1} = \lambda_{\kappa+2} = \lambda_{\kappa+3} \neq 1.$$ 

Then $\lambda_0 \leq \lambda_1 = \lambda_n$ for all $n \in \mathbb{N}$ and $W_\lambda$ is subnormal with the Berger measure $\mu$ given by

$$\mu = (1 - Z)\delta_0 + Z\delta_{\lambda^2_\kappa} \quad \text{with} \quad Z = \left(\frac{\lambda_0}{\lambda_1}\right)^2.$$ 

**Proof.** Set $\theta = \lambda^2_\kappa$, $\gamma_n = \hat{\lambda}_n$ for $n \in \mathbb{Z}_+$ (see (3.1)) and $\gamma = \{\gamma_n\}_{n=0}^\infty$. By (8.1), we have (see (2.15) for notation)

$$\gamma^{(n)}_\kappa = \gamma_n \theta^n, \quad n = 0, 1, 2, 3, 4.$$ 

Straightforward computations show that

$$\Delta \gamma^{(n)}_\kappa = \gamma_n (\theta - 1) \theta^n, \quad n = 0, 1, 2, 3,$$

$$\Delta^2 \gamma^{(n)}_\kappa = \gamma_n (\theta - 1)^2 \theta^n, \quad n = 0, 1, 2.$$ 

Let $(b, c, \nu)$ be the scalar representing triplet of $W_\lambda$. It follows from (3.5) and (2.8) that

$$\Delta^2 \gamma_n = \int_{\mathbb{R}_+} x^n (\nu + 2c\delta_1)(dx), \quad n \in \mathbb{Z}_+.$$
This, in turn, implies that
\[ \alpha_n := (\Delta^2 \gamma)_n = \int_{\mathbb{R}_+} x^n d\rho_\kappa(x), \quad n \in \mathbb{Z}_+, \] (8.5)
where \( \rho_\kappa \) is a finite compactly supported Borel measure on \( \mathbb{R}_+ \) given by
\[ \rho_\kappa(\Delta) = \int_{\Delta} x^\kappa (\nu + 2c\delta_1)(dx), \quad \Delta \in \mathfrak{B}(\mathbb{R}_+). \] (8.6)
According to (2.16), (8.4) and (8.5), we have
\[ \alpha_n = \int_{\mathbb{R}_+} x^n d\rho_\kappa(x) = \gamma_\kappa (\theta - 1)^2 \theta^n, \quad n = 0, 1, 2. \] (8.7)
Since \( \theta \neq 1 \), we infer from (8.5), (8.7) and (2.12) that the Stieltjes moment sequence \( \{\alpha_n\}_{n=0}^\infty \) is non-degenerate. It follows from (8.7) that
\[ \alpha_1^2 = \alpha_0 \alpha_2. \]
Applying Lemma 2.2 with \( k = 0 \), we deduce that there exist \( \xi, \zeta \in (0, \infty) \) such that \( \rho_\kappa = \xi \delta_\zeta \). As a consequence of (8.7), we obtain
\[ \gamma_\kappa (\theta - 1)^2 \theta^n = \xi \zeta^n, \quad n = 0, 1, 2. \] (8.8)
Substituting \( n = 0 \), we get \( \xi = \gamma_\kappa (\theta - 1)^2 \), which together with (8.8) gives \( \zeta = \theta \).
Hence \( \rho_\kappa = \xi \delta_\theta \), which yields
\[ \xi \delta_\theta(\Delta) = \int_{\Delta} x^\kappa (\nu + 2c\delta_1)(dx), \quad \Delta \in \mathfrak{B}(\mathbb{R}_+). \]
Since \( \theta \neq 1 \), this implies that \( c = 0 \) and that there exist \( u \in \mathbb{R}_+ \) and \( v \in (0, \infty) \) such that
\[ \nu = u \delta_0 + v \delta_\theta. \] (8.9)
Combined with (3.5), this shows that
\[ \gamma_n = 1 + bn + uQ_n(0) + vQ_n(\theta) \]
\[ = 1 + bn + u(n - 1) + v \frac{\theta^n - 1 - n(\theta - 1)}{(\theta - 1)^2} \]
\[ = X + Y n + Z \theta^n, \quad n \in \mathbb{N}, \] (8.10)
where \( X := 1 - u - Z, \ Y := b + u - \frac{v}{\theta - 1} \) and \( Z := \frac{v}{(\theta - 1)^2} \). By (8.10), we have
\[ \gamma_n(\kappa) = (X + \kappa Y) + Y n + (Z \theta^n) \theta^n, \quad n \in \mathbb{Z}_+. \]
This together with (8.3) leads to
\[ \gamma_n(\kappa) = (X + \kappa Y) + Y n + (Z \theta^n) \theta^n, \quad n = 0, 1, 2, 3, 4. \]
Therefore, we get
\[ (X + \kappa Y) + Y n = (\gamma_n - Z \theta^n) \theta^n, \quad n = 0, 1, 2, 3, 4. \] (8.11)
Substituting \( n = 0 \) into (8.11) gives
\[ X + \kappa Y = \gamma_n - Z \theta^n. \] (8.12)
Next substituting \( n = 1, 2 \) into (8.11) and using (8.12), we obtain
\[ (X + \kappa Y) + Y = (X + \kappa Y) \theta, \]
\[ (X + \kappa Y) + 2Y = (X + \kappa Y) \theta^2. \]
Since $\theta \neq 1$, we conclude that the above homogeneous system of linear equations has only the solution $X = Y = 0$. It follows from (8.10) that

$$\gamma_n = Z\theta^n, \quad n \in \mathbb{N}. \quad (8.13)$$

(Note that $Z > 0$ because $\gamma_1 > 0$.) This implies that

$$\lambda_n \overset{(3.4)}{=} \sqrt{\frac{\gamma_{n+1}}{\gamma_n}} = \sqrt{\theta}, \quad n \in \mathbb{N}. \quad (8.14)$$

Since $1 - u - Z = X = 0$ and $u \in \mathbb{R}^+$, we infer from (8.14) that

$$\frac{\lambda_0^2}{\lambda_1^2} = \frac{\gamma_1}{\theta} \overset{(8.13)}{=} Z = 1 - u \leq 1. \quad (8.15)$$

Using (8.13), (8.14) and (8.15), we verify that the measure $\mu$ given by (8.2) satisfies (3.7), so by Corollary 3.2, $W_\lambda$ is subnormal and $\mu$ is its Berger measure. This completes the proof. $\square$

If the weights in the sequence (8.1) are equal to 1, then we can reduce their number to two ($\lambda_0$ is still excluded).

**Theorem 8.2.** Suppose that $W_\lambda$ is a CPD unilateral weighted shift with weights $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ and $\kappa \in \mathbb{N}$ is such that $\lambda_\kappa = \lambda_{\kappa+1} = 1$.

Then $\lambda_0 \leq 1 = \lambda_n$ for all $n \in \mathbb{N}$ and $W_\lambda$ is subnormal with the Berger measure $\mu$ given by

$$\mu = (1 - \lambda_0^2)\delta_0 + \lambda_0^2\delta_1. \quad \text{(3.7)}$$

**Proof.** We will modify the proof of Theorem 8.1. Set $\gamma_n = \hat{\lambda}_n$ for $n \in \mathbb{Z}^+$. Let $(b, c, \nu)$ be the scalar representing triplet of $W_\lambda$. First note that

$$\gamma_n(\kappa) = \gamma_n, \quad n = 0, 1, 2, \quad (8.16)$$

and thus

$$(\Delta^2\gamma_n(\kappa))_0 = 0. \quad (\text{3.7})$$

This implies that

$$\int_{\mathbb{R}^+} x^\kappa (\nu + 2c\delta_1)(dx) \overset{(8.6)}{=} \rho_\kappa(\mathbb{R}^+) \overset{(8.5)}{=} (\Delta^2\gamma_n(\kappa))_0 \overset{(2.16)}{=} (\Delta^2\gamma(\kappa))_0 = 0. \quad (8.17)$$

Since $\kappa \geq 1$, we deduce that $c = 0$ and $\text{supp}(\nu) \subseteq \{0\}$. Hence, we have

$$\gamma_n(\kappa) = (1 - \nu(\{0\}))+\bar{Y} n, \quad n \in \mathbb{N}, \quad (8.18)$$

where $\bar{Y} := b + \nu(\{0\})$, and consequently

$$\gamma_n(\kappa) \overset{(8.16)}{=} \gamma_n(\kappa) \overset{(8.17)}{=} \bar{X} + \bar{Y} n, \quad n = 0, 1, 2, \quad (8.19)$$

where $\bar{X} = 1 - \nu(\{0\}) + \kappa\bar{Y}$. This implies that $\bar{Y} = 0$ and thus by (8.17), we get

$$1 - \nu(\{0\}) = \gamma_n = \gamma_1 = \lambda_0^2, \quad n \in \mathbb{N}. \quad (8.20)$$

Therefore $\lambda_0 \leq 1$ and $\lambda_n = \sqrt{\frac{\gamma_{n+1}}{\gamma_n}} = 1$ for all $n \in \mathbb{N}$. The remaining part of the proof proceeds as in Theorem 8.1. $\square$
We now consider the case of the first four equal weights, none of which is equal to 1.

**Theorem 8.3.** Suppose that $W_{\lambda}$ is a CPD unilateral weighted shift with weights $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ such that

$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 \neq 1$.

Then $\frac{1}{\lambda_0} W_{\lambda}$ is the unilateral shift.

**Proof.** We again modify the proof of Theorem 8.1. For the reader’s convenience, we will point out the most important differences. First, in (8.9) we have $u = 0$. Consequently, the identity (8.10) takes the form

$\gamma_n = X + Yn + Z\theta^n, \quad n \in \mathbb{Z}_+.$

Further differences appear in the formulas (8.13) and (8.14), so now we have

$\gamma_n = Z\theta^n$ and $\lambda_n = \sqrt{\theta}$ for all $n \in \mathbb{Z}_+$.

This completes the proof. □

Finally, arguing as in the proof of Theorem 8.2, we obtain the variant of this theorem for $\kappa = 0$.

**Theorem 8.4.** Suppose that $W_{\lambda}$ is a CPD unilateral weighted shift with weights $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ such that

$\lambda_0 = \lambda_1 = 1$.

Then $W_{\lambda}$ is the unilateral shift.

We now show that the numbers of consecutive equal weights appearing in Theorems 8.1, 8.2, 8.3 and 8.4 are optimal. The first counterexample concerns the case of three consecutive equal weights. It is noteworthy that if two consecutive weights of a subnormal unilateral weighted shift are equal, all weights, except the first, are equal (see [19, Theorem 6]). Thus, for subnormal unilateral weighted shifts the optimal number of consecutive equal weights is exactly 2.

**Example 8.5.** Fix $\theta \in (1, \infty) \setminus \{3\}$ and define the triplet $(b, c, \nu)$ by $b = \theta - 1$, $c = 0$ and

$\nu = \frac{1}{2} (\theta - 1)^2 (\delta_{\frac{1}{2} \theta} + \delta_{\frac{1}{2} \theta})$.

Let $\gamma = \{\gamma_n\}_{n=0}^{\infty}$ be the CPD sequence given by (4.2). Clearly, by (2.2), $\gamma$ satisfies (4.1). Applying Theorem 4.1, we get a CPD unilateral weighted shift $W_{\lambda}$ with weights $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ such that $\hat{\lambda} = \gamma$. Straightforward computations yield

$\gamma_n = \theta^n, \quad n = 0, 1, 2, 3,$

(8.18)

and

$\gamma_4 = \frac{1}{9} \theta^2 (13 \theta^2 - 8 \theta + 4). \quad (8.19)$

(Note that the expression on the right-hand side of (8.19) is equal to $\theta^4$ if and only if $\theta = 1$, which is not the case, so $\gamma_4 \neq \theta^4$. Hence, by (4.3) and (8.18) we have

$\lambda_0 = \lambda_1 = \lambda_2 = \sqrt{\theta} \neq 1,$

(8.20)
and
\[
\lambda_3 = \frac{\sqrt{13\theta^2 - 8\theta + 4}}{3\sqrt{\theta}} \neq \sqrt{\theta}.
\] (8.21)

This shows that Theorem 8.3 is no longer true if the number of consecutive equal weights is decreased to 3.

To cover the case \(\kappa \geq 1\) discussed in Theorem 8.1, we will use Proposition 7.5. With notation as in this proposition, note that
\[
\sigma_1 = g_1(\theta),
\] (8.22)
where \(g_1 : (1, \infty) \to \mathbb{R}\) is defined by
\[
g_1(x) := \frac{4x^2 - 8x + 9}{5x}, \quad x \in (1, \infty).
\]

Since
\[
g_1(x) - 1 = \frac{(x - 1)(4x - 9)}{5x}, \quad x \in (1, \infty),
\]
we see that
\[
g_1(x) \geq 1, \quad x \in \left[\frac{9}{4}, \infty\right).
\] (8.23)

Fix any \(\theta \in \left[\frac{9}{4}, \infty\right) \setminus \{3\}\). Using (8.22) and (8.23) and applying Proposition 7.5 with \(p = 1\), we conclude that \(W_\lambda\) has a CPD \(\infty\)-step backward extension. Summarizing, we have proved that for every \(n \in \mathbb{N}\), there exists a sequence \(\{t_j\}_{n=1}^\infty \subseteq (0, \infty)\) such that the unilateral weighted shift \(W_\lambda\) with weights
\[
\lambda = (t_n, \ldots, t_1, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots),
\]
is CPD. By (8.20), (8.21) and Theorems 8.1 and 8.3, \(\lambda_0 = \lambda_1 = \lambda_2 \neq 1\) and \(t_1 \neq \lambda_0\). This means that regardless of the value of \(\kappa \geq 1\), Theorem 8.1 ceases to be true if the number of consecutive equal weights is decreased to 3.

Applying Proposition 7.5, now to any \(p \geq 1\), and using computer simulations, one can confirm that the above conclusion is true for \(\theta \in (1, \infty) \setminus \{3\}\). ♦

The second counterexample shows that the number of consecutive equal weights appearing in Theorems 8.2 and 8.4 is optimal. It is only interesting for non-subnormal CPD unilateral weighted shifts because the class of subnormal operators, unlike CPD ones, is scalable (see [15, Corollary 3.4.7]).

**Example 8.6.** Let \(\nu\) be a compactly supported finite Borel measure on \(\mathbb{R}_+\) such that \(\nu(\{1\}) = 0\) and let \(c \in (0, \infty)\). Set \(b = -c\). Let \(\gamma = \{\gamma_n\}_{n=0}^\infty\) be the CPD sequence defined by (4.2). Because of (2.2), \(\gamma\) satisfies (4.1). Applying Theorem 4.1, we obtain a CPD unilateral weighted shift \(W_\lambda\) with weights \(\lambda = \{\lambda_n\}_{n=0}^\infty\) such that \(\lambda = \gamma\). Since \(c > 0\), one can infer from (4.3) that \(\lambda_0 = 1\) and \(\lambda_1 \neq 1\), which means that \(W_\lambda\) is not the unilateral shift. In fact, \(W_\lambda\) is not subnormal because \(c > 0\) (use Theorem 6.1 and [15, Theorem 3.4.1]). This shows that Theorem 8.4 is no longer true if the first weight is equal to 1.

Finally, assuming additionally that \(\int_{\mathbb{R}_+} \frac{1}{x}d\nu(x) < \infty\) for all \(k \in \mathbb{N}\) and applying Corollary 7.6 to the above unilateral weighted shift \(W_\lambda\), we conclude that regardless of the value of \(\kappa \geq 1\), Theorem 8.2 ceases to be true if \(\lambda_\kappa = 1\) (cf. Example 8.5). ♦
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