Nonlinear transverse vibration of nano-strings based on the differential type of nonlocal theory

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Abstract. The nonlinear vibration responses of nano-strings are studied based on the theory of Eringen’s nonlocal elasticity. Firstly, the nonlocal differential constitutive model in one-dimensional form which is suitable for a string structure is used, and then the governing equation of motion for the nonlinear vibration of nano-strings is derived by considering the expression of classical Lagrangian strain. In order to solve the non-dimensional nonlinear governing equation of motion, the Galerkin method or Rayleigh-Ritz method is applied, and the nonlinear partial differential equation is approximately transformed into the a set of ordinary differential equations. The ordinary differential equations are then solved by a numerical method, and the nonlinear vibration responses under different time histories are thus obtained. Subsequently, the approximate numerical solution of the nonlinear displacement is solved by the second-order multi-scale method. The nonlinear phenomena in the transverse displacement and the influences of nonlocal scale parameter on the nonlinear vibration characteristics of nano-strings are analyzed accordingly. The results will provide a basis for understanding and controlling the nonlinear dynamics of nano-strings which may act as key components in the booming intelligent nano-systems.

1. Introduction
The nanotechnology is in the ascendant at present. In science, a variety of interdisciplinary and hot academic frontiers have been formed, and in engineering, the development of emerging industries and the progress of industrial technology have been promoted. Among them, nano-mechanics is a new and important research area, and the mechanical properties of various nano-materials and structures have become a research hotspot [1-3]. The distinctive nano-materials have the characteristics of high strength, high toughness and low density. Hence, they are widely used in micro/nano-wires, micro/nano-switches, high-sensitive sensors, detectors and resonators. Predictions of the special nanoscale properties of these structures require new theoretical methods. This is because the classical continuum theory has failed at the nanoscale.

Among the new theoretical methods, one popular research approach named nonlocal theory firstly proposed by Eringen [4,5] is extensively used. This theory belongs to the generalized continuum mechanics. The new idea that the interaction between atoms is a long-range force is directly introduced into the classical continuum mechanics, forming the
nonlocal theory. It is considered that the stress at a point in a continuum is not only related to the strain at that point, but also to the strains at all points in the continuum. Therefore, in the nonlocal constitutive relation, the stress tensor at a certain point can be expressed as an integral functional form with regard to all strains. By introducing the internal characteristic scale parameters, the nonlocal theory can reveal the mechanical behaviors of nano-materials and structures perfectly. When the internal characteristic parameter is negligible compared with the external characteristic parameter, the nonlocal effect can be ignored. Otherwise, when the internal characteristic parameter is almost the same order of magnitude as the external characteristic parameter, the nonlocal effect must be taken into consideration. The theoretical predictions are in good agreement with molecular dynamics simulation, lattice dynamics and nanoscaled experiments [4,5]. For example, the non-singularity of stress at the tip of a micro-crack can not be predicted by the classical continuum mechanics, but can be obtained by the nonlocal theory. As we know, the stress non-singularity at the crack tip is a reasonable result in physics. Therefore, the nonlocal theory has become a hot topic of current theoretical research in nano-mechanics. There are a large number of literatures using the nonlocal theory to study the mechanical properties of nano-materials and structures, such as [6-20]. Generally speaking, in these literatures the gaps between atoms of the nano-materials and structures are neglected, and the nano-materials are regarded and modeled as micro-continuums. Subsequently, the one- or two-dimensional nano-model is developed to simulate the nano-materials and structures respectively. After that, the nonlocal theory is applied to describe the nano-materials and structures, and then obtain the important mechanical properties by solving the established differential equilibrium or governing equations. These literatures [6-20] are no longer presented one by one due to limited space. However, there are two phenomena in the current literatures. One is that there are many studies on nano-beams, carbon nano-tubes and even graphene sheets [6-20], but the research on nano-strings is very rare. Although the macro-scale strings or beams have been well investigated [21-23], the dynamic characteristics of nano-strings will change significantly due to the scale effect. Consequently, it is necessary to investigate the dynamics of nano-strings. Secondly, the linear mechanical behaviors of nano-materials and structures are thoroughly studied, but the study of nonlinear nano-mechanics, especially the complicated nonlinear vibration, is still rare currently. For the above reasons, this paper studies the nonlinear vibration characteristics of nano-strings based on the Eringen’s nonlocal elastic theory. First, the nonlocal differential constitutive equation and Lagrangian strain are introduced. The governing equation of the nonlinear transverse vibration of nano-strings is derived based on the classical transverse vibration of macro-strings involving the nonlocal scale effect together. Following that, the nonlinear partial differential governing equation is discretized into ordinary differential equations based on the Galerkin method. The numerical method is utilized to solve the ordinary differential equations in order to reveal the macroscopic performances of the transverse nonlinear vibration behaviors at nanoscale. Finally, the nonlinear transverse vibration displacement is solved using the method of multiple scales, and the influences of nonlinear factor and nonlocal scale effect on the transverse displacement are presented. This work can provide important reference for the nonlinear design, optimization and vibration control of nano-strings and many nanoscaled devices based on nano-strings.

2. Theoretical model and governing equation
For a uniform, elastic nano-string of density \( \rho \), cross-sectional area \( A \) and length \( L \), which is subjected to initial tension \( P \) at both two ends. The initial tension may be caused by the surface tension or temperature effect at nanoscale. Only the transverse vibration in lateral direction (perpendicular to the axial direction) is taken into consideration. The equation that governs the transverse vibration can be derived according to Hamilton’s principle or stress analysis in lateral direction as

\[
\rho A \frac{\partial^2 u}{\partial t^2} = P \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x} \left( A \sigma \frac{\partial u}{\partial x} \right)
\]
where $u$ is transverse displacement, $x$ the axial coordinate, $t$ the time coordinate and $\sigma$ the axial normal stress.

The relation between the Lagrangian strain and the transverse displacement is given by

$$\varepsilon_L = \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2$$  \hspace{1cm} (2)

As aforementioned, the nonlocal theory is essential to the nanoscale mechanics. According to the theoretical model proposed by Eringen [4,5], the differential relationship between the strain and the stress in the nonlocal elasticity is expressed as

$$\sigma - (e_0a)^2 \frac{d^2 \sigma}{dx^2} = E\varepsilon_L$$  \hspace{1cm} (3)

From the above Equations (1) to (3) can be deduced the following result

$$\rho A \frac{\partial^2 u}{\partial t^2} = P \frac{\partial^2 u}{\partial x^2} + EA \left\{ (e_0a)^2 \left[ 4 \frac{\partial u}{\partial x} \frac{\partial^3 u}{\partial x^3} + \left( \frac{\partial^2 u}{\partial x^2} \right)^3 + \left( \frac{\partial u}{\partial x} \right)^2 \frac{\partial^4 u}{\partial x^4} \right] + 3 \left( \frac{\partial u}{\partial x} \right)^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} \right\}$$  \hspace{1cm} (4)

Equation (4) is the governing equation of the transverse vibration of the nano-string based on nonlocal theory and Lagrangian strain. The governing equation can be transformed into the non-dimensional form as

$$\bar{P} \frac{\partial^2 \bar{u}}{\partial \tau^2} + \tau \left[ 4 \frac{\partial \bar{u}}{\partial \bar{x}} \frac{\partial^3 \bar{u}}{\partial \bar{x}^3} + \left( \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} \right)^3 + \left( \frac{\partial \bar{u}}{\partial \bar{x}} \right)^2 \frac{\partial^4 \bar{u}}{\partial \bar{x}^4} \right] + 3 \left( \frac{\partial \bar{u}}{\partial \bar{x}} \right)^2 \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} - \frac{\partial^4 \bar{u}}{\partial \bar{x}^4} = 0$$  \hspace{1cm} (5)

where the non-dimensional variables are defined as

$$\bar{u} = \frac{u}{L}, \quad \bar{x} = \frac{x}{L}, \quad \bar{\tau} = \frac{t}{\left( \frac{E}{\rho L^2} \right)^{\frac{1}{2}}}, \quad \bar{P} = \frac{P}{EA}, \quad \tau = \frac{e_0a}{L}.$$

On the other hand, we consider the simply supported boundary constraints at both ends of the nano-strings. The dimensionless boundary conditions are given by

$$\bar{u}(0, \bar{\tau}) = \bar{u}(1, \bar{\tau}) = 0, \quad \frac{\partial^2 \bar{u}(0, \bar{\tau})}{\partial \bar{x}^2} = \frac{\partial^2 \bar{u}(1, \bar{\tau})}{\partial \bar{x}^2} = 0$$  \hspace{1cm} (6)

3. Dynamical behaviors via Galerkin method

The Galerkin method or Rayleigh-Ritz method can be employed to simplify the equation of motion (5) directly. Under the boundary conditions (6), the solution of governing equation (5) may be expanded into the following expression

$$\bar{u}(\bar{x}, \bar{\tau}) = \sum_{n=1}^{m} q_n(\bar{\tau}) \sin(n\pi \bar{x})$$  \hspace{1cm} (7)

where $q_n$ is the $n$-order generalized displacement and $\sin(n\pi \bar{x})$ is the $n$-order trial function. Denote the left hand of Equation (5) as $L[\bar{u}]$, a functional of $\bar{u}(\bar{x}, \bar{\tau})$. If $\sin(k\pi \bar{x})$ is chosen as the $k$-order weighting function, then application of the Galerkin method results in

$$\int_0^1 L[\bar{u}(\bar{x}, \bar{\tau})] = \sum_{n=1}^{m} q_n(\bar{\tau}) \sin(n\pi \bar{x}) \left[ \sin(k\pi \bar{x}) \right] d\bar{x} = 0 \quad (k = 1, 2, \cdots, m)$$  \hspace{1cm} (8)
For given $m$, inserting Equation (7) into the functional $L[u]$ and taking the appropriate derivatives and then substituting the resulting equation into Equation (8) yield the $m$-term Galerkin approximation of the equation of motion (5). Previous research on axially moving strings or beams shows that even order Galerkin truncation gives better results. So in present study, $m = 2$ is chosen. The expression of $L[u]$ when $m = 2$ is given by

\[
-P(q,\pi^2 \sin(\pi x) + 4q,\pi^2 \sin(2\pi x)) + \tau^2 \left[ 2q_1,\pi^6 \cos(\pi x) \sin(2\pi x) + 10q_1,\pi^2 \cos(4\pi x) \right] + 16q_2,\pi^6 \cos^2(\pi x) \sin(2\pi x) + 80q_2,\pi^6 \cos(\pi x) \sin(4\pi x) + 64q_2,\pi^6 \cos(\pi x) \cos^2(2\pi x) + 128q_3,\pi^6 \cos(2\pi x) \sin(4\pi x) - q_3,\pi^6 \sin^2(\pi x) - 64q_3,\pi^6 \sin(3(2\pi x)) - 12q_3,\pi^6 \sin(2\pi x) + 4q_3,\pi^6 \cos^2(2\pi x) \sin(\pi x) + 64q_3,\pi^6 \cos^2(\pi x) \sin(2\pi x) + q_3,\pi^6 \sin(4\pi x) \right] - \frac{3}{2} \left( q_1,\pi^2 \cos^2(\pi x) \sin(\pi x) + 4q_1,\pi^2 \cos^2(\pi x) \sin(\pi x) \right) \sin(2\pi x) + 4q_2,\pi^4 \sin(\pi x) \cos^2(2\pi x) + 16q_2,\pi^4 \cos^2(\pi x) \sin(2\pi x) + q_2,\pi^3 \sin(4\pi x) + 8q_3,\pi^3 \cos(\pi x) \sin(4\pi x) - (q_1,\sin(\pi x) + q_2,\sin(2\pi x)) = L \left\{ \sum_{n=1}^{\infty} q_n(T) \sin(n\pi x) \right\}
\]

(9)

As a result, the 2-term Galerkin approximation of Equation (5) is given by

\[
\frac{1}{2} \frac{d^2 q_1}{dT^2} + \frac{1}{2} \pi^2 F q_1 + \left( \frac{3}{16} \pi^2 - \frac{\pi^6 \tau^2}{4} \right) q_1^3 + \left( \frac{3}{2} \pi^4 - 5\pi^6 \tau^2 \right) q_1 q_2^2 = 0 \quad (10a)
\]

\[
\frac{1}{2} \frac{d^2 q_2}{dT^2} + 2\pi^2 F q_2 + \left( 3\pi^4 - 16\pi^6 \tau^2 \right) q_2^3 + \left( \frac{3}{2} \pi^2 - 5\pi^6 \tau^2 \right) q_1^2 q_2 = 0 \quad (10b)
\]

Based on the results from Equation (10), the numerical results can be calculated and shown in Figures 1-4 for time is 10, 40 and 1000, respectively. It is indicated that the when the vibrating time is not long, the nano-string exhibits regular periodic vibration. When the vibrating time is long, the vibration gradually evolves into irregular non-periodic vibration. This is mainly caused by the nonlinear factor considered in this work.
Figure 1. The vibration responses of nano-strings with $T=10$.

Figure 2. The vibration responses of nano-strings with $T=40$. 
Figure 3. The vibration responses of nano-strings with $T = 100$.

Figure 4. The vibration responses of nano-strings with $T = 1000$. 
4. Transverse displacement via multiple scales analysis
The nonlinear transverse displacement can be obtained via the method of multiple scales. Suppose

\[ \overline{u}(\overline{x}, \overline{t}) = h(\overline{t}) \sin \pi \overline{x} \]  

(11)

Substituting Equation (11) into (5) yields

\[ \frac{d^2 h}{dt^2} + \pi^2 \overline{P} h - \pi^4 \varepsilon \overline{h}^3 = 0 \] 

(12)

where

\[ \varepsilon = \pi^2 \tau^2 \left( 6 \cos^2 \pi \overline{x} - 1 \right) - \frac{3}{2} \cos^2 \pi \overline{x} \]  

is a small dimensionless parameter with respect to the nonlocal nanoscale parameter \( \tau \).

The method of multiple scales will be employed to solve Equation (12) directly. A second-order uniform approximation is sought in the form

\[ h(\overline{t}, \varepsilon) = h_0(T_0, T_1, T_2) + \varepsilon h_1(T_0, T_1, T_2) + \varepsilon^2 h_2(T_0, T_1, T_2) + \cdots \]  

(13)

where \( T_0 = \overline{t} \) is a fast scale characterizing motion while \( T_1 = \varepsilon \overline{t} \) and \( T_2 = \varepsilon^2 \overline{t} \) are slow scales characterizing the modulation of the amplitudes and phases. Substituting Eq. (13) and the following relationships

\[ \frac{\partial}{\partial T_0} = \frac{\partial}{\partial \overline{t}} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + \cdots = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots \]  

(14a)

\[ \frac{\partial^2}{\partial T^2} = \frac{\partial}{\partial \overline{t}} \left( \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + \cdots \right) = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 \left( D_1^2 + 2D_0 D_2 \right) + \cdots \]  

(14b)

into Equation (12) yields

\[ \left[ D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 \left( D_1^2 + 2D_0 D_2 \right) \right] \left( \overline{h}_0 + \varepsilon \overline{h}_1 + \varepsilon^2 \overline{h}_2 \right) + \pi^2 \overline{P} \left( \overline{h}_0 + \varepsilon \overline{h}_1 + \varepsilon^2 \overline{h}_2 \right) - \pi^4 \varepsilon \overline{h}_0 \overline{h}_1 \overline{h}_2 = 0 \]  

(15)

Equating coefficients of like powers of \( \varepsilon \) gives

\[ D_0^2 \overline{h}_0 + \pi^2 \overline{P} \overline{h}_0 = 0 \]  

(16a)

\[ D_0^2 \overline{h}_1 + \pi^2 \overline{P} \overline{h}_1 = \pi^4 \overline{h}_0^3 - 2D_0 \overline{h}_0 \]  

(16b)

\[ D_0^2 \overline{h}_2 + \pi^2 \overline{P} \overline{h}_2 = 3\pi^4 \overline{h}_0^2 \overline{h}_1 - 2D_0 \overline{h}_1 - D_1^2 \overline{h}_0 - 2D_0 D_2 \overline{h}_0 \]  

(16c)

From the second-order ordinary differential equation (16a) we obtain

\[ \overline{h}_0 = A(T_1, T_2) e^{i\pi \sqrt{\overline{P}}} + \overline{A}(T_1, T_2) e^{-i\pi \sqrt{\overline{P}}} \]  

(17)

where \( A \) is a complex function while \( \overline{A} \) is the conjugate function of \( A \). Substituting Equation (17) into (16b) yields

\[ D_0^2 \overline{h}_1 + \pi^2 \overline{P} \overline{h}_1 = A^3 \pi^4 e^{3i\pi \sqrt{\overline{P}}} + \left( 2i\pi \sqrt{\overline{P}} D_1 A - 3\pi^4 A^2 \overline{A} \right) e^{i\pi \sqrt{\overline{P}}} + cc \]  

(18)
where $cc$ is the conjugate complex number of its left terms, similarly hereinafter. In Equation (18), secular terms must vanish, or

$$2i\pi\sqrt{\frac{P}{8P}}D_1A - 3\pi^4A^2A = 0$$

which leads to

$$\overline{h}_1 = -\frac{A^3\pi^2}{8P} e^{i\pi}\sqrt{\frac{P}{8P}} + cc$$

Further substituting Equations (17) and (20) into (16c) yields

$$D_0\overline{h}_2 + \pi^2 \overline{P}h_2 = \frac{21\pi^6}{8P} A^4 A e^{i\pi}\sqrt{\frac{P}{8P}} - \frac{3\pi^6}{8P} A^6 e^{i\pi}\sqrt{\frac{P}{8P}} \left(2i\pi\sqrt{\frac{P}{8P}}D_1A - \frac{51\pi^6}{8P} A^3A^2\right) e^{i\pi}\sqrt{\frac{P}{8P}} + cc$$

As the same with Equation (18), in order to obtain a periodic solution for $\overline{h}_2$, one gets

$$2i\pi\sqrt{\frac{P}{8P}}D_2A - \frac{51\pi^6}{8P} A^3A^2 = 0$$

The above Equations (21) and (22) can be deduced the result

$$\overline{h}_2 = -\frac{21\pi^4}{64P^2} A^4 A e^{i\pi}\sqrt{\frac{P}{8P}} + \frac{\pi^4}{64P^2} A^6 e^{i\pi}\sqrt{\frac{P}{8P}} + cc$$

In order to determine the expression of complex function $A$, we suppose its derivative as

$$\frac{dA}{dt} = D_0A + \varepsilon D_1A + \varepsilon^2 D_2A$$

where $D_0A = 0$ and the expressions of $D_1A$ and $D_2A$ are given by Equations (19) and (22) respectively. Hence,

$$\frac{dA}{dt} = \frac{3i\varepsilon\pi^3 A^2A}{2\sqrt{\frac{P}{8P}}} - \frac{51i\varepsilon^2\pi^5 A^3A^2}{16P\sqrt{\frac{P}{8P}}}$$

We can express complex function $A$ using an exponential function as

$$A(t) = \frac{1}{2} a(t) e^{i\theta(t)}$$

where $a$ and $\theta$ are all real functions with respect to $t$. Substituting Equation (26) into (25) and separating the real-parts and imaginary- parts in the results, we can obtain the complex function $A$ after integrating those two first-order ordinary differential equations.

$$A(t) = \frac{1}{2} a_0 e^{i\theta_0} \left(\frac{3i\varepsilon\pi^3 A^2}{2\sqrt{\frac{P}{8P}}} - \frac{51i\varepsilon^2\pi^5 A^3}{256P\sqrt{\frac{P}{8P}}}\right)$$

where $a_0$ and $\theta_0$ are integral constants and they can be determined by initial conditions. From Equations (11), (13), (17), (20), (23) and (27), we gain the approximate solution as
\[ \bar{u}(\bar{x},\bar{t}) = \left\{ a_0 \cos \alpha - \frac{\pi^2}{32 \bar{P}} \left[ \pi^2 \tau^2 \left( 6 \cos^2 \pi \bar{x} - 1 \right) - \frac{3}{2} \cos^2 \pi \bar{x} \right] a_0^3 \cos 3\alpha \right. \\
+ \left. \frac{\pi^4}{1024 \bar{P}^2} \left[ \pi^2 \tau^2 \left( 6 \cos^2 \pi \bar{x} - 1 \right) - \frac{3}{2} \cos^2 \pi \bar{x} \right]^2 a_0^5 \left( -21 \cos 3\alpha + \cos 5\alpha \right) \right\} \sin (\pi \bar{x}) \]

where

\[ \alpha = \left( \pi \sqrt{\bar{P}} + \frac{\beta \pi^2 a_0^2}{8 \sqrt{\bar{P}}} - \frac{51 \beta \pi^4 a_0^4}{256 \bar{P}^{1/2}} \right) \bar{T} + \theta_0 \]

\[ \beta = \pi^2 \tau^2 \left( 6 \cos^2 \pi \bar{x} - 1 \right) - \frac{3}{2} \cos^2 \pi \bar{x} \]

Assuming the parameters \( a_0 = 1, \ \theta_0 = 1, \ \bar{P} = 1, \ \tau = 0.1 \), and we can calculate and plot the following Figure 5 which shows the evolvement of transverse displacement and non-dimensional axial coordinate and time coordinate. It is shown that the displacement varies non-linearly with respect to axial coordinate and time.

**Figure 5.** The evolvement of transverse displacement.

Next we investigate the relationship between the midpoint displacement and the nonlocal nanoscale parameter \( \tau \) at time \( \bar{T} = 1 \) as illustrated in Figure 6. It is shown that the obvious nonlinearity occurs with increasing the nonlocal nanoscale parameter. Such observation proves that the nonlocal effect plays a significant role in the nonlinear behaviors of the transverse free vibration of nano-strings. Additionally, it is shown that the transverse displacement is very sensitive to the initial axial tension. The variation of initial axial tension may lead to great change in displacement, and it can further predict the generation of period
doubling bifurcation and chaos in dynamics of nano-strings. Note that the axial tension arises from the surface or the temperature effect, hence it proves that the surface stress or the thermal effect at nanoscale plays a remarkable role in nonlinear dynamics of nano-strings.

Figure 6. Effects of nonlocal nanoscale parameter and initial tension on transverse displacement.

5. Conclusions
The present article is concerned with the nonlinear transverse vibration of a nano-string. In order to characterize the distinctive properties of such a structure at nanoscale, the differential constitution of nonlocal theory is employed. The nano-string is treated as a continuum and further modeled as a one-dimensional nonlocal nano-structure. The Lagrangian strain is introduced and the nonlinear governing equation for the nano-string is obtained in which the nonlocal nanoscale parameter is involved to represent the nonlocal scale effect. After non-dimensionalization and discretization, the nonlinear governing equation is solved by two different numerical approaches, namely the 2-term Galerkin approximation and the second-order multi-scale approximation. According to these numerical methods, the ordinary differential equations are gained from the derived partial differential equation, and they are easily to be solved. The nonlinear responses and displacements of transverse vibration of nano-strings are determined, respectively. The effects of nonlinear appearance and nonlocal nanoscale parameter on the transverse vibration are demonstrated in the numerical examples. The results may be useful for the dynamic characteristics of nano-string-based nano-electro-mechanical systems.

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