A MATHEMATICAL APPROACH TO THE
HIERARCHICAL STRUCTURE OF LANGUAGES

NILS A. BAAS

1. Introduction

We will in this paper suggest how the mathematical concept of hyperstructures may be a useful tool in the study of the higher, hierarchical structure of languages.

By a language \( \mathcal{L} \) we will mean a spoken, written, formal or geometric language. We will assume that in the context of a language there are some basic units: sounds, letters, geometric forms, signs, etc. They represent the “atoms” of the language. We suggest that Hyperstructures [1–9] represent a useful framework in which to represent a language structure, and suggest the following procedure.

2. Hyperstructures

Let us denote the collection or set of basic linguistic units by \( X_0 \). Let \( \mathcal{P}(X_0) \) stand for all (finite) subsets or a subcollection. We will then have an assignment (technically a presheaf in many cases):

\[
\Omega_0 : \mathcal{P}(X_0) \rightarrow \text{Sets},
\]

for example “giving some meaning” to a collection of sounds (or letters). This is what we call the Observation part (property, state).

Next we consider collections of basic units with properties

\[
(S_0, \omega_0)
\]

and we want to bind these together to meaningful entities

\[
\Gamma_1 = \{(S_0, \omega_0) \mid S_0 \subseteq X_0, \omega_0 \in \Omega_0(S_0)\}.
\]

\( B_0 : \Gamma_1 \rightarrow \text{Sets} \) is another assignment (presheaf) such that \( B_0(S_0, \omega_0) \) is the set of bonds, ways of gluing the element in \( S_0 \) together to meaningful units, like words.

This is the process we will iterate in order to get a hyperstructure. We think of

\[
\Omega_0 \text{ as the semantic assignment, “meaning”}
\]

and

\[
B_0 \text{ as the syntactic (“grammatical”) structure assignment. Admissible generation of words.}
\]

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In hyperstructures we can also define composition of bonds, by a "gluing" process (see [6]) which is a huge generalization of concatenation of letters for example. From bonds $b_1, b_2, \ldots, b_k$ we can generate new bonds:

$$ b = b_1 \boxdot b_2 \boxdot \cdots \boxdot b_k. $$

These assignments give a "grammar" both for syntax and semantics. For example Chomsky grammars and their generative rules will be covered by bond composition in the form of concatenation. We may also consider higher dimensional geometric alphabets and higher dimensional gluing. The semantic part may also play a role in the composition.

Now we can proceed to produce the higher tiers of the language:

Put

$$ X_1 = \{ b_0 \mid b_0 \in B_0(S_0, \omega_0) \}. $$

Then we choose or construct (a "presheaf"), specific for each language:

$$ \Omega_1 : \mathcal{P}(X_1) \to \text{Sets} $$

from

$$ \Gamma_1 = \{ (S_1, \omega_1) \} $$

and

$$ B_1 : \Gamma_1 \to \text{Sets}. $$

This means that $B_1(S_1, \omega_1)$ represent bonds of bonds, for example words of words bound together in meaningful admissible sentences given by the bond presheaves.

Then we can construct arbitrarily many tiers getting a formal hyperstructure $\mathcal{H}(\mathcal{L})$.

We also have "boundary" maps connecting the tiers:

$$ \partial_i : X_{i+1} \to \mathcal{P}(X_i) $$

which dissolve the bonds — or put differently: "a bond knows what it binds" or "a sentence knows its words" (linguistic version).

Starting with a top bond and iteratively applying the $\partial_i$'s is like given a sentence and constructing a parsing tree:
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\[
\begin{align*}
  C_n & \xrightarrow{\partial_{n-1}} \\
  \{C^j_{n-1}\} & \xrightarrow{\partial_{n-2}} \\
  \{C^k_{n-2}\} & \xrightarrow{\partial_{n-3}} \\
  \vdots
\end{align*}
\]

i.e., describing \(n\)-levels of generalized “morphemes”.

We suggest that this is the basic general structure of language, and that any reasonable language is governed by a Hyperstructure, and it may be very useful to identify it.

Hyperstructures are really organizing tools, and so are languages for communication, thought and learning.

3. CONSEQUENCES

Notice that even starting with a finite set of basic units, infinite numbers of bonds and meanings can be generated for any tier. Our approach covers semiotic systems as well.

Another advantage with hyperstructures is the possibility of passing from local to global via an extended Grothendieck type topology (see [6] where this is introduced).

This means that we consider tiers of bonds

\[
\begin{align*}
  B_n & \xrightarrow{\Lambda_n} \text{Sets} \\
  B_{n-1} & \xrightarrow{\Lambda_{n-1}} \text{Sets} \\
  \vdots \\
  B_0 & \xrightarrow{\Lambda_0} \text{Sets}
\end{align*}
\]

where the \(\Lambda_i\)'s are “meaning” functions or assignments at each level. We have developed rules for when we from these levelwise meanings can deduce a global meaning. This is done by what we call a globalizer (generalizing sections in sheaves), and shows how global meanings can be integrated and also that in many situations you cannot just pass from local to global. Tiers or levels are needed. This extends the linguistic notion of “duality of patterning”. But duality is just for two
levels, in general many levels are needed in an efficient language. Our approach is “multiplicity of patterning”!

In general: Data, Stimuli, Inputs, Outputs may be unstructured — have no immediate meaning.

Hence put a Hyperstructure on the situation (which may be inherent in the living brain or a synthetic one) and via a suitable “globalizer” structure and meaning will appear (unless random!).

This is a way to handle the “Generalized Binding Problem” (see [7]). If two languages are $\mathcal{H}$-structured communication will (should) take place in an $\mathcal{H}$-structured way and understanding represents some kind of structural compatibility or resonance.

Finally, these structures also relate to memory and learning. For example a machine organized as an $\mathcal{H}$-structure will take data (stimuli), process them and adjust according to a global goal state or meaning.

This is automatic learning. Learning in general requires or at least profits from a Hyperstructure. Hyperstructured libraries of events or data are important in learning and memory and $\mathcal{H}$-structures facilitates the process.

We hope that this note will initiate a further study of hyperstructures in languages.

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Department of Mathematical Sciences, NTNU, NO-7491 Trondheim, Norway

E-mail address: nils.baas@ntnu.no