An Alternative Finite Difference Stability Analysis for a Multiterm Time-Fractional Initial-Boundary Value Problem

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Abstract. A fractional initial-boundary value problem is considered, where the differential operator includes a sum of Caputo temporal derivatives, and the solution has a weak singularity at the initial time $t = 0$. The problem is solved numerically by a finite difference method based on applying the L1 method to discretise each temporal derivative on a graded mesh. Stability of this method is proved by generalising the analysis of Stynes et al., SIAM J. Numer. Anal. 55 (2017), pp. 1057-1079, where the case of a single temporal derivative was investigated. This stability result is used to prove a sharp error estimate for the finite difference method.

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1. Introduction

The numerical solution of time-fractional initial-boundary value problems has been considered in a large number of recent papers (see [2] for a detailed survey). Typical solutions of such problems exhibit a weak singularity at the initial time $t = 0$, as discussed in [7–9]. Until now, few papers have considered problems with solution singularities where the differential operator contains a sum of time-fractional derivatives, although such formulations offer greater flexibility in modelling. It is this class of problems that is the subject of our paper.

The problem that we study is the following. Set $\Omega = (0, l) \subset \mathbb{R}$, with $\bar{\Omega} = [0, l]$ and boundary $\partial \Omega = \{0, l\}$. Let $T > 0$ be fixed. Set $Q = (0, l) \times (0, T]$ and $\bar{Q} = [0, l] \times [0, T]$.

For constant $\alpha \in (0, 1)$ and any suitable function $w(x, t)$ defined on $\bar{Q}$, define the temporal Caputo fractional derivative of order $\alpha$ of $w$ by
Let $\ell$ be a positive integer. Assume that we are given constants $\alpha_i$ (for $i = 1, 2, \ldots, \ell$) which satisfy $0 < \alpha_1 < \ldots < \alpha_2 < \alpha_1 < 1$. These $\alpha_i$ are the orders of our fractional derivatives.

We shall consider the multiterm time-fractional initial-boundary value problem

$$
\sum_{i=1}^{\ell} [q_i \partial_t^{\alpha_i} u(x, t)] + Lu(x, t) = F(x, t) \quad \text{for} \quad (x, t) \in Q
$$

with initial and boundary conditions

$$u(x, 0) = u_0(x) \quad \text{for} \quad x \in \Omega,
$$
$$u(0, t) = u(l, t) = 0 \quad \text{for} \quad 0 < t \leq T,$$

where the given constants $q_i$ are positive, $F \in C(\bar{Q})$, and $u_0 \in C(\bar{Q})$ with $u_0(0) = u_0(l) = 0$.

Without loss of generality we assume that $q_1 = 1$. In (1.1a) the operator $L$ is $Lu(x, t) = -u_x(x, t) + c(x) u(x)$ for $(x, t) \in \Omega$, where $c \in C(\bar{\Omega})$ with $c \geq 0$.

The single-term case $\ell = 1$ of (1.1) was investigated in [9], and the main part of our paper is a generalisation of results from there.

After further regularity and compatibility conditions are imposed on the data of the problem, existence of a solution to (1.1) for the case $F \equiv 0$ follows from [4, Theorem 2.1], and for the case $u_0 \equiv 0$ from [4, Theorem 2.2]. Combining these results yields existence of a solution to (1.1). Uniqueness of this solution follows from [6, Theorem 4].

In [1] it is shown that, provided that the data of the problem satisfy certain regularity and compatibility conditions, the solution $u$ of (1.1) satisfies

$$\left| \frac{\partial^k u}{\partial x^k}(x, t) \right| \leq C \quad \text{for} \quad k = 0, 1, 2, 3, 4,$$
$$\left| \frac{\partial^m u}{\partial t^m}(x, t) \right| \leq C \left( 1 + t^{\alpha_1 - m} \right) \quad \text{for} \quad m = 0, 1, 2,$$

for all $(x, t) \in (0, l) \times (0, T)$, where $C$ is some fixed constant. The bound (1.2b) indicates the nature of the singularity in the solution $u$ at $t = 0$.

**Remark 1.1.** To discern the structure of $u$, one can imitate the analysis of [5]. In the notation of that paper one has $n = 1$; taking $\theta = 0$ in [5, equation (5)] leads to a representation of $u - u_0$ as the solution of a weakly singular Volterra integral equation given in [5, equations (12) and (26)], where the kernel of the integral operator is a finite sum of terms (one for each derivative $\partial_t^{\alpha_i} u$). Then [5, Theorem 3] reveals the complicated structure of $u(x, t) - u_0(x)$, which we now describe. Set $\Delta := \{0, \alpha_1 - \alpha_2, \alpha_1 - \alpha_3, \ldots, \alpha_1 - \alpha_\ell \}$. Then for some positive integer $m$ whose value depends on the smoothness and compatibility of the data of (1.1), for each $x$ one has

$$u(x, t) = u_0(x) + \sum_{(j,k) \in \Delta} \gamma_{j,k} t^{\alpha_1 + \delta_{\ell_0} + \delta_{\ell_1} + \cdots + \delta_{\ell_j} + k} + Z_m(t),$$
for some function $z$.

A finite difference method for solving (1.1) is described and analysed in [1]. We shall consider this method here also. In [1] the stability of the method is proved using techniques from [3] which are very different from the older stability proof for the single-term case $\ell = 1$ in [9, Section 4]. Our new contribution in the current paper is the following: we show how the stability analysis of [9, Section 4] can be extended to the multiterm problem (1.1). This extension is nontrivial, as is evident from the analysis in Section 3 below.

The paper is structured as follows. Section 2 describes the finite difference method that will be used to solve (1.1). Section 3 is the heart of our paper; in it we prove the stability of our method. Then in Section 4 we use our stability bounds to prove a sharp convergence result for the finite difference scheme.

**Notation:** In this paper $C$ denotes a generic constant that depends on the data of the problem (1.1) but is independent of $(x, t)$ and of any mesh used to solve (1.1) numerically; note that $C$ can take different values in different places.

### 2. The Finite Difference Method

The derivative bounds (1.2) show that, while typical solutions of (1.1) have a weak singularity in the temporal direction at $t = 0$, in the spatial direction one has good regularity. Consequently it is not necessary to employ any special numerical technique to handle the spatial derivative $u_{xx}$; only the fractional time-derivatives in (1.1a) need special treatment.

Thus, to solve (1.1) numerically, we use a mesh that is uniform in space but graded in time. Let $M$ and $N$ be positive integers. The mesh on $\Omega$ is uniform: set $h = l/M$ and $x_m = mh$ for $m = 0, 1, \ldots, M$. For the temporal mesh, set $t_n = T(n/N)^\gamma$ for $n = 0, 1, \ldots, N$, where the mesh grading $r \geq 1$ can be chosen by the user. Set $\tau_n = t_n - t_{n-1}$ for $n = 1, 2, \ldots, N$. Then the space-time mesh is $\{(x_m, t_n) : m = 0, 1, \ldots, M$ and $n = 0, 1, \ldots, N\}$.

Denote the computed solution at each mesh point $(x_m, t_n)$ by $u^n_m$. The term $u_{xx}$ in (1.1a) is approximated by the standard 3-point second-order discretisation

$$u_{xx}(x_m, t_n) \approx \delta^2 u^n_m := \frac{u^n_{m+1} - 2u^n_m + u^n_{m-1}}{h^2}.$$

For each $\alpha \in (0, 1)$, the Caputo fractional derivative $D^\alpha_t u$ can be written as

$$D^\alpha_t u(x_m, t_n) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{-\alpha} \frac{\partial u(x_m, s)}{\partial s} \, ds.$$
Approximate this by the standard L1 discretisation

\[ D^{\alpha} u^n_m = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \frac{u^{k+1}_m - u^k_m}{\tau_{k+1}} \int_{t_k}^{t_{k+1}} (t_n - s)^{-\alpha} ds \]

That is, for \( i = 1, 2, \ldots, \ell \), each fractional time-derivative term \( q_i \partial^{\alpha_i} u(x_m, t_n) \) in (1.1a) is discretised by

\[ q_i D^{\alpha_i} u^n_m := \frac{q_i}{\Gamma(2-\alpha_i)} \sum_{k=0}^{n-1} \frac{u^{k+1}_m - u^k_m}{\tau_{k+1}} \left[ (t_n - t_k)^{1-\alpha_i} - (t_n - t_{k+1})^{1-\alpha_i} \right] \]

Combining these formulas, we discretise (1.1) by

\[ L_{M,N} u^n_m := \sum_{i=1}^{\ell} q_i D^{\alpha_i} u^n_m - \delta^2 u^n_m + c(x_m) u^n_m = F(x_m, t_n) \]

for \( 1 \leq m \leq M - 1, \ 1 \leq n \leq N \), \( u^n_0 = u^{n-1}_m = 0 \) for \( 1 \leq n \leq N \), \( u^0_0 = u_0(x_m) \) for \( 0 \leq m \leq M \).

### 3. Stability of the Finite Difference Method

The stability analysis presented here imitates the stability analysis of [9, Section 4] for the single-term case \( \ell = 1 \), but the presence of multiple fractional derivatives (i.e., \( \ell \geq 1 \)) introduces several complications.

One can express the discretisation of each Caputo derivative in (2.1) as

\[ D^{\alpha_i} u^n_m = \frac{d^{i}_{n,1}}{\Gamma(2-\alpha_i)} u^n_m - \frac{d^{i}_{n,n}}{\Gamma(2-\alpha_i)} u^0_m + \frac{1}{\Gamma(2-\alpha_i)} \sum_{k=1}^{n-1} u^{n-k}_m \left[ \frac{d^{i}_{n,k+1} - d^{i}_{n,k}}{\tau_{n-k+1}} \right] \]

for \( i = 1, \ldots, \ell \), where

\[ d^{i}_{n,k} := \frac{\left( t_n - t_{n-k} \right)^{1-\alpha_i} - \left( t_n - t_{n-k+1} \right)^{1-\alpha_i}}{\tau_{n-k+1}} \]

Note that \( d^{i}_{n,1} = t_n^{-\alpha_i} \) for \( i = 1, \ldots, \ell \). An easy application of the mean value theorem shows that

\[ d^{i}_{n,k+1} \leq d^{i}_{n,k} \text{ for all } i, n, k. \]
Rewrite the scheme (2.1a) as

\[
\left[ \sum_{i=1}^\ell \frac{q_i d_{n,1}^i}{\Gamma(2-\alpha_i)} + \frac{2}{h^2} + c(x_m) \right] u_m^n = \sum_{i=1}^\ell \frac{q_i d_{n,n}^i}{\Gamma(2-\alpha_i)} u_m^0 + F(x_m, t_n) + \frac{1}{h^2} v_{m+1}^n + \frac{1}{h^2} v_{m-1}^n + \sum_{i=1}^\ell \frac{q_i}{\Gamma(2-\alpha_i)} \sum_{k=1}^{n-1} [d_{n,k+1}^i - d_{n,k}^i] u_{m-k}^n
\]

for \( m = 1, 2, \ldots, M - 1 \) and \( n = 1, 2, \ldots, N \).

We now begin the stability analysis of (2.1). For any mesh function \( \{z_m^n\} \), set

\[
\|z^n\|_\infty := \max_{0 \leq m \leq M} |z_m^n|, \quad \|z\|_\infty := \max_{0 \leq n \leq N} \max_{0 \leq m \leq M} |z_m^n|.
\]

Suppose that

\[
L_{M,N} v_m^n = g_m^n \quad \text{for} \quad 1 \leq m \leq M - 1, \quad 1 \leq n \leq N
\]

with \( v_N^n = v_N^0 = 0 \) for \( 0 < n \leq N \) and \( v_0^0 \) given for \( 0 \leq m \leq M \). For \( 1 \leq n \leq N \), we seek a bound on \( \|v^n\|_\infty \) in terms of \( \|v^0\|_\infty \) and \( \|g^j\|_\infty \) for \( j = 1, \ldots, n \).

For \( 1 \leq i \leq \ell, 1 \leq n \leq N \) and \( 1 \leq j \leq N \), set

\[
\eta_{n,j}^i = \frac{q_i d_{n,j}^i}{\Gamma(2-\alpha_i)}.
\]

From (3.1) it follows that

\[
\eta_{n,j+1}^i \leq \eta_{n,j}^i \quad \text{for all} \quad i, n, j.
\]

**Lemma 3.1.** The solution of the discrete problem (3.3) satisfies

\[
\|v^n\|_\infty \leq \frac{1}{\sum_{i=1}^\ell \eta_{n,1}^i} \left[ \|g^n\|_\infty + \sum_{i=1}^\ell \sum_{k=1}^{n-1} (\eta_{n,k}^i - \eta_{n,k+1}^i) \|v^{n-k}\|_\infty + \sum_{i=1}^\ell \eta_{n,n}^i \|v^0\|_\infty \right]
\]

for \( n = 1, 2, \ldots, N \).

**Proof.** Fix \( n \in \{1, 2, \ldots, N\} \). Choose \( j_n \) such that \( |v_{j_n}^n| = \|v^n\|_\infty \). From (3.2), the equation associated with the mesh point \( (x_{j_n}, t_n) \) is

\[
\left[ \sum_{i=1}^\ell \eta_{n,1}^i + \frac{2}{h^2} + c(x_{j_n}) \right] v_{j_n}^n = \sum_{i=1}^\ell \eta_{n,n}^i v_{j_n}^0 + g_{j_n}^n + \frac{1}{h^2} v_{j_n+1}^n + \frac{1}{h^2} v_{j_n-1}^n + \sum_{i=1}^\ell \sum_{k=1}^{n-1} [\eta_{n,k}^i - \eta_{n,k+1}^i] v_{j_n}^{n-k}.
\]

Hence, by \( c \geq 0 \) and the choice of \( j_n \), one obtains

\[
\left[ \sum_{i=1}^\ell \eta_{n,1}^i + \frac{2}{h^2} \right] \|v^n\|_\infty \leq \frac{2}{h^2} \|v^n\|_\infty + |g_{j_n}^n + \sum_{i=1}^\ell \eta_{n,n}^i v_{j_n}^0 + \sum_{i=1}^\ell \sum_{k=1}^{n-1} [\eta_{n,k}^i - \eta_{n,k+1}^i] v_{j_n}^{n-k}|.
\]
This is equivalent to

\[ \|v^n\|_\infty \leq \|v^0\|_\infty + \sum_{i=1}^n \eta^n_{n,1} \|g_j\|_\infty + \sum_{i=1}^n \sum_{k=1}^{n-1} \left[ \eta^n_{n,k} - \eta^n_{n,k+1} \right] v^n_{j_h} \]

Dividing by \( \sum_{i=1}^n \eta^n_{n,1} \) and recalling (3.4), the result of the lemma follows.

Lemma 3.1 will now be used in an inductive argument to give a weighted bound for \( \|v^n\|_\infty \) in terms of the given data \( \|v^0\|_\infty \) and \( \|g^j\|_\infty \) for \( j = 1, 2, \ldots, n \).

Define the real numbers \( \theta_{n,j} \), for \( n = 1, 2, \ldots, N \) and \( j = 1, 2, \ldots, n-1 \), by

\[ \theta_{n,n} = 1, \quad \theta_{n,j} = \sum_{i=1}^{n-j} \sum_{k=1}^{n-i} \frac{1}{\eta^n_{n,k}} (\eta^n_{n,k} - \eta^n_{n,k+1}) \theta_{n-k,j}. \quad (3.5) \]

This definition is a generalisation of [9, (4.6)]. Note that (3.4) implies \( \theta_{n,j} > 0 \) for all \( n, j \).

**Lemma 3.2.** The solution of the discrete problem (3.3) satisfies

\[ \|v^n\|_\infty \leq \|v^0\|_\infty + \frac{1}{\sum_{i=1}^n \eta^n_{i,1}} \sum_{j=1}^n \theta_{n,j} \|g^j\|_\infty \quad (3.6) \]

for \( n = 1, 2, \ldots, N \).

**Proof.** Use induction on \( n \). The case \( n = 1 \) of (3.6) is

\[ \|v^1\|_\infty \leq \|v^0\|_\infty + \frac{1}{\sum_{i=1}^1 \eta^1_{i,1}} \|g^1\|_\infty, \]

which is the same as the case \( n = 1 \) of Lemma 3.1.

Fix \( p \in \{2, \ldots, N\} \). Assume that (3.6) is valid for \( n = 1, 2, \ldots, p-1 \). Then Lemma 3.1 and the inductive hypothesis yield

\[ \|v^p\|_\infty \leq \frac{1}{\sum_{i=1}^p \eta^p_{i,1}} \|g^p\|_\infty + \frac{1}{\sum_{i=1}^p \eta^p_{i,1}} \sum_{i=1}^p \eta^p_{i,1} \|v^0\|_\infty \]

\[ + \frac{1}{\sum_{i=1}^p \eta^p_{i,1}} \left[ \sum_{i=1}^p \sum_{k=1}^{p-1} (\eta^p_{i,k} - \eta^p_{i,k+1}) \|v^{p-k}\|_\infty \right] \]

\[ \leq \frac{1}{\sum_{i=1}^p \eta^p_{i,1}} \left[ \sum_{i=1}^p \sum_{k=1}^{p-1} (\eta^p_{i,k} - \eta^p_{i,k+1}) \left( \frac{\sum_{j=1}^{p-k} \theta_{p-k,j} \|g^j\|_\infty}{\sum_{i=1}^p \eta^p_{i,k,1}} + \|v^0\|_\infty \right) \right] \]

\[ + \frac{1}{\sum_{i=1}^p \eta^p_{i,1}} \|g^p\|_\infty + \frac{1}{\sum_{i=1}^p \eta^p_{i,1}} \sum_{i=1}^p \eta^p_{i,1} \|v^0\|_\infty \].
Let \( i \) be any index. \( \star \) is a generalisation of [9, Lemma 4.3]. Consider the first term \( \{ \ldots \} \) in (3.7): here \( 1 \leq j \leq p-k \) and \( 1 \leq k \leq p-1 \) are equivalent to \( 1 \leq k \leq p-j \) and \( 1 \leq j \leq p-1 \), and \( \sum_{i=1}^{\ell} \) is independent of \( j \) and \( k \), so this term equals

\[
\|g^p\|_\infty + \sum_{j=1}^{p-1} \left[ \sum_{i=1}^{\ell} \sum_{k=1}^{p-j} \eta_{p,k} \theta_{p-k,j} \right] \|g^j\|_\infty = \sum_{j=1}^{p} \theta_{p,j} \|g^j\|_\infty, \tag{3.8}
\]

by the definition (3.5). For the second term \( \{ \ldots \} \) in (3.7),

\[
\left\{ \frac{1}{\sum_{j=1}^{\ell} \eta_{p,1}} \sum_{i=1}^{\ell} \sum_{k=1}^{p-1} \left( \eta^i_{p,k} - \eta^i_{p,k+1} \right) \theta_{p-k,j} + \sum_{i=1}^{\ell} \eta^i_{p,p} \right\} \|v^0\|_\infty
\]

\[
= \frac{1}{\sum_{i=1}^{\ell} \eta_{p,1}} \left( \sum_{i=1}^{\ell} \sum_{k=1}^{p-1} \left( \eta^i_{p,k} - \eta^i_{p,k+1} \right) + \sum_{i=1}^{\ell} \eta^i_{p,p} \right) \|v^0\|_\infty
\]

\[
= \frac{1}{\sum_{i=1}^{\ell} \eta_{p,1}} \left( \sum_{i=1}^{\ell} \eta^i_{p,1} - \eta^i_{p,p} \right) \|v^0\|_\infty = \|v^0\|_\infty, \tag{3.9}
\]

since telescoping gives

\[
\sum_{k=1}^{p-1} \left( \eta^i_{p,k} - \eta^i_{p,k+1} \right) = \eta^i_{p,1} - \eta^i_{p,p}.
\]

Combining (3.7), (3.8) and (3.9) yields

\[
\|v^0\|_\infty \leq \|v^p\|_\infty + \frac{1}{\sum_{i=1}^{\ell} \eta_{p,1}} \sum_{j=1}^{p} \theta_{p,j} \|g^j\|_\infty.
\]

Thus we have proved (3.6) for \( n = p \). By the principle of induction, the lemma is proved. \( \blacksquare \)

The next result is a generalisation of \( [9, \text{Lemma 4.3}] \).

**Lemma 3.3.** Let \( i \in \{1, 2, \ldots, \ell\} \). Let the parameter \( \beta \) satisfy \( \beta \leq r \alpha_1 \). Then for \( n = 1, 2, 3, \ldots, N \), one has

\[
\frac{1}{\sum_{i=1}^{\ell} \eta_{n,1}} \sum_{j=1}^{n} j^{-\beta} \theta_{n,j} \leq \Gamma(1 - \alpha_1) T^{\alpha_1} N^{-\beta}. \tag{3.10}
\]
Proof. Use induction on $n$. When $n = 1$, then by (3.5) and $q_1 = 1$ we have
\[
\frac{1}{\sum_{j=1}^{l} \eta_{i,j}} \sum_{j=1}^{l} j^{-\beta} \theta_{i,j} = \frac{1}{\sum_{i=1}^{d} (q_i d_i^1)/(\Gamma(2 - \alpha_i))} \leq \frac{1}{d_i^1/(\Gamma(2 - \alpha_1))} \leq 1 \quad \text{since } 0 < \alpha_1 < 1, \quad \tau_1 = N^{-\beta} \quad \text{and } \beta \leq r \alpha_1.
\]

Thus (3.10) is true for $n = 1$.

Next, assume that (3.10) is true for $n = 1, 2, \ldots, l - 1$, where $l \in \{2, 3, \ldots, N\}$. We want to prove (3.10) for $n = l$. Invoking (3.5) and interchanging the order of summation, we get
\[
\frac{1}{\sum_{j=1}^{l} \eta_{i,j}} \sum_{j=1}^{l} j^{-\beta} \theta_{i,j} = \frac{1}{\sum_{i=1}^{d} \eta_{i,1}} \left\{ l^{-\beta} \theta_{i,1} + \sum_{j=1}^{l-1} j^{-\beta} \sum_{i=1}^{d} \sum_{k=1}^{l-1} \eta_{i,k} \eta_{i,k+1} \theta_{i-k,j} \right\}
\]
\[
= \frac{1}{\sum_{i=1}^{d} \eta_{i,1}} \left\{ l^{-\beta} + \sum_{i=1}^{d} \sum_{k=1}^{l-1} \eta_{i,k} \eta_{i,k+1} \sum_{j=1}^{l-k} j^{-\beta} \theta_{i-k,j} \right\}
\]
\[
\leq \frac{1}{\sum_{i=1}^{d} \eta_{i,1}} \left\{ l^{-\beta} + \sum_{i=1}^{d} \sum_{k=1}^{l-1} (\eta_{i,k} - \eta_{i,k+1}) \Gamma(1 - \alpha_1) N^{-\beta} \right\}
\]
by the inductive hypothesis. Now telescoping yields
\[
\frac{1}{\sum_{i=1}^{d} \eta_{i,1}} \sum_{j=1}^{l} j^{-\beta} \theta_{i,j} \leq \frac{1}{\sum_{i=1}^{d} \eta_{i,1}} \left\{ l^{-\beta} + \sum_{i=1}^{d} (\eta_{i,1} - \eta_{i,l}) \Gamma(1 - \alpha_1) N^{-\beta} \right\}
\]
\[
= \Gamma(2 - \alpha_1) \Gamma(1 - \alpha_1) N^{-\beta} + \frac{1}{\sum_{i=1}^{d} \eta_{i,1}} \left[ l^{-\beta} - \sum_{i=1}^{l} \eta_{i,1} \Gamma(1 - \alpha_1) N^{-\beta} \right].
\]

To complete the inductive step, we show that $[\ldots] \leq 0$. For
\[
\sum_{i=1}^{d} \eta_{i,l} \Gamma(1 - \alpha_1) N^{-\beta} \geq \frac{d_i^1}{\Gamma(2 - \alpha_1)} \Gamma(1 - \alpha_1) N^{-\beta}
\]
\[
= d_i^1 T^{\alpha_1} N^{-\beta}
\]
\[
= \frac{1}{\Gamma(2 - \alpha_1)} \Gamma(1 - \alpha_1) T^{\alpha_1} N^{-\beta}
\]
\[
= \frac{d_i^1}{\Gamma(2 - \alpha_1)} \Gamma(1 - \alpha_1) T^{\alpha_1} N^{-\beta}
\]
\[
= \frac{d_i^1}{\Gamma(2 - \alpha_1)} \Gamma(1 - \alpha_1) T^{\alpha_1} N^{-\beta}
\]
\[
= l^{-\beta} (N^T) T^{\alpha_1} N^{-\beta}
\]
\[
\geq l^{-\beta},
\]
where we used the mean value theorem, $N \geq l$, and $r \alpha_1 \geq \beta$. Thus (3.10) holds true for $n = l$. By the principle of induction, (3.10) is true for all $n$. \qed
4. Convergence of the Method

We can now prove convergence of the finite difference method by combining our stability analysis of Section 3 with a consistency error bound from [1].

**Theorem 4.1.** The solution \( \{u^n_m\} \) of the scheme (2.1) satisfies

\[
\max_{(x_m, t_n) \in Q} |u(x_m, t_n) - u^n_m| \leq C \left( h^2 + N^{-\min\{2-\alpha_1, r\alpha_1\}} \right)
\]

for some constant \( C \) that is independent of the mesh.

**Proof.** Fix \((x_m, t_n) \in Q\). From [1, Section 3.2], the truncation error \( \chi^n_m \) of (2.1) at each mesh point \((x_m, t_n)\) satisfies

\[
|\chi^n_m| := \sum_{i=1}^{t} D^n_M \left[ u(x_m, t_n) - u^n_m \right] - \sum_{i=1}^{t} \delta^n_i \left[ u(x_m, t_n) - u^n_m \right] \leq C \sum_{i=1}^{t} (h^2 + n^{-\min\{2-\alpha_1, r\alpha_1\}}).
\]

But \( L_{M,N} [u(x_m, t_n) - u^n_m] = \chi^n_m \), so invoking Lemma 3.2 with \( v^n_m = u(x_m, t_n) - u^n_m \), we obtain

\[
\max_{(x_m, t_n) \in Q} |u(x_m, t_n) - u^n_m| \leq \frac{1}{\sum_{i=1}^{t} \eta^n_{i,1}} \sum_{j=1}^{n} \theta_{n,j} \| \chi^n_{i,j} \|_{\infty} \leq \frac{C}{\sum_{i=1}^{t} \eta^n_{i,1}} \sum_{j=1}^{n} \theta_{n,j} \left( h^2 + j^{-\min\{2-\alpha_1, r\alpha_1\}} \right) \leq C T(2 - \alpha_1) T^{\alpha_1} \left( h^2 + N^{-\min\{2-\alpha_1, r\alpha_1\}} \right) = C \left( h^2 + N^{-\min\{2-\alpha_1, r\alpha_1\}} \right),
\]

where Lemma 3.3 was used twice: once with \( \beta = 0 \) for the \( h^2 \) term and once with \( \beta = \min\{2 - \alpha_1, r\alpha_1\} \) for the \( N^{-\min\{2-\alpha_1, r\alpha_1\}} \) term. \( \square \)

Numerical results demonstrating the sharpness of Theorem 4.1 can be found in [1].

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