On the consequences of twisted Poincaré symmetry upon QFT on Moyal noncommutative spaces

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Abstract

We explore some general consequences of a consistent formulation of relativistic quantum field theory (QFT) on the Grönewold-Moyal-Weyl noncommutative versions of Minkowski space with covariance under the twisted Poincaré group of Chaichian et al [12], Wess [44], Koch et al [31], Oeckl [34]. We argue that a proper enforcement of the latter requires braided commutation relations between any pair of coordinates \( \hat{x}, \hat{y} \) generating two different copies of the space, or equivalently a \( \star \)-tensor product \( f(x) \star g(y) \) (in the parlance of Aschieri et al [3]) between any two functions depending on \( x, y \). Then all differences \( (x-y)^\mu \) behave like their undeformed counterparts. Imposing (minimally adapted) Wightman axioms one finds that the \( n \)-point functions fulfill the same general properties as on commutative space. Actually, upon computation one finds (at least for scalar fields) that the \( n \)-point functions remain unchanged as functions of the coordinates' differences both if fields are free and if they interact (we treat interactions via time-ordered perturbation theory). The main, surprising outcome seems a QFT physically equivalent to the undeformed counterpart (to confirm it or not one should however first clarify the relation between \( n \)-point functions and observables, in particular \( S \)-matrix elements).

These results are mainly based on a joint work [24] with J. Wess.

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1 Introduction

The idea of spacetime noncommutativity is rather old. It goes back to Heisenberg. The simplest noncommutativity one can think of is with coordinates $\hat{x}^\mu$ fulfilling the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu},$$

where $\theta^{\mu\nu}$ are the elements of a constant real antisymmetric matrix. Relations (1) have appeared in the literature under various names. For brevity, we shall denote these noncommutative spaces as Moyal spaces. For present purposes $\mu = 0, 1, 2, 3$ and indices are raised or lowered through multiplication by the standard Minkowski metric $\eta_{\mu\nu}$, so as to obtain a deformation of Minkowski space. Clearly (1) are translation invariant, but not Lorentz-invariant (in 4 dimensions there is no isotropic antisymmetric 2-tensor $\theta^{\mu\nu}$). We shall denote by $\hat{A}$ the algebra “of functions on Moyal space”, i.e. the algebra generated by $1, \hat{x}^\mu$ fulfilling (1). For $\theta^{\mu\nu} = 0$ one obtains the algebra $\mathcal{A}$ generated by commuting $x^\mu$.

Contributions to the construction of QFT on these spaces start in 1994-95. A broad attention has been devoted to the program in the last decade, with a number of different approaches. By no means are they equivalent! Roughly speaking I would divide them into the following three groups.

Doplicher-Fredenhagen-Roberts (DFR) approach

This is field quantization in (rigorous) operator formalism on Moyal-Minkowski space, with usual Poincaré transformations. The pioneering works are [17], the main developments can be found in [5]. Relations (1) are motivated by the interplay of Quantum Mechanics and General Relativity in what Doplicher calls the Principle of gravitational stability against localization of events:

The gravitational field generated by the concentration of energy required by the Heisenberg Uncertainty Principle to localise an event in spacetime should not be so

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1Heisenberg proposed it in a letter to Peierls to solve the problem of divergent integrals in relativistic quantum field theory. The idea propagated via Pauli to Oppenheimer. In 1947 Snyder, a student of Oppenheimer, published the first proposal of a quantum theory built on a noncommutative space.

2Sometimes they are called canonical, since by applying a Darboux transformation to the coordinates $\theta$ can be brought to canonical form (this depends only on its rank). More often the names contain some combination of the names of Weyl, Wigner, Grönewold, Moyal. This is due to the relation between canonical commutation relations and the $*$-product (or twisted product) of Weyl and Von Neumann, which in turn was used by Wigner to introduce the Wigner transform; Wigner’s work led Moyal to define the so-called Moyal bracket $[f \ast g] = f \ast g - g \ast f$; the $*$-product in position space [in the form of the asymptotic expansion of (10) with $x_i = x_j \equiv x$] first appeared in a paper by Grönewold.

3The arguments elaborate the well-known heuristic ones going back (as far as I know) to Wheeler.
strong to hide\footnote{By black hole formation.} the event itself to any distant observer - distant compared to the Planck scale \[16, 17\].

In the first, simplest version $\theta^{\mu\nu}$ are not fixed constants, but central operators (obeying additional conditions) which on each irreducible representation become fixed constants $\sigma^{\mu\nu}$, the joint spectrum of $\theta^{\mu\nu}$. This allows to recover Lorentz covariance for the commutation relations. However, it seems that when developing the interacting theory the wished Lorentz covariance is sooner or later lost. In more recent versions $\theta^{\mu\nu}$ is no more central, but commutation relations remain of Lie-algebra type.

According to speculations heard in conference talks by Doplicher, $\theta^{\mu\nu}$ could be finally related to the vacuum expectation value (v.e.v.) of $R^{\mu\nu}$, which in turn should be influenced by the presence of matter quantum fields in spacetime (through quantum equations of motions).

Finally, we would like to mention the work [27], which although not strictly in the DFR framework, also is based on a continuous family of fields labelled by the whole spectrum of noncommutative parameters $\theta^{\mu\nu}$, but has some overlap also with the following two approaches. A generalization of the procedure [27] has been proposed in the very recent work [10], see also Buchholz’s contribution to the present volume.

\section*{Path-integral quantization approach}

This was initiated by Filk in [21] and has been adopted by most theoretical physicists, including many string-theorists, especially after the work [37]. Useful reviews are in [18, 41]. The string-theorists’ main motivation is that such models should describe the low-energy effective limit of string theory in a constant background $B$-field. Lorentz covariance [or $SO(4)$ covariance, after Wick-rotation] is lost, but this is expected in effective string theory because of the special direction selected by the $B$-field; only covariance under a subgroup [2] of $SL(2, \mathbb{C})$, the corresponding little group, is preserved.

The (Euclidean) classical field action used in the path-integral is deformed replacing products of fields by $\star$-products, whence modified Feynman rules for perturbative QFT are derived.

New complications seem to appear, like non-unitarity after naive Wick-rotation when $\theta^0 \neq 0$ [25], violation of causality [30, 9], mixing of UV and IR divergences [33] and subsequent non-renormalizability, alleged change of statistics, etc. Some of these problems, like non-unitarity or the very occurrence of divergences [5], may be due simply to naive (and unjustified) applications of commutative QFT rules (path-integral methods, Feynman diagrams, analytic continuation, etc) and could disappear adopting the sounder field-operator approach. As for UV-IR mixing, while planar Feynman diagrams remain as the undeformed (apart from a phase factor), in particular have the
same UV divergences, nonplanar Feynman diagrams which were UV divergent become finite for generic non-zero external momentum, but diverge as the latter go to zero, even with massive fields: these are the IR divergences. As a dramatic effect, infinitely many counterterms are necessary, making these theories non-renormalizable.

As a cure to the UV-IR mixing problem Grosse, Wulkenhaar \[28\] and collaborators add a $x$-dependent harmonic potential terms (e.g. $\sim \Omega^2 x^2 \varphi \star \varphi$ for a scalar field) to the Lagrangian (for a review see Grosse’s contribution to the present volume, and references therein). Then the theory becomes renormalizable; actually $\Omega^2 x^2 \varphi \star \varphi$ is the only other marginal/relevant operator in the renormalization group flow. However these terms spoil the translation invariance of the theory.

Moreover, up to now no notion of Wick rotation between such QFT on Moyal-Euclidean space and QFT on Moyal-Minkowski noncommutative space has been found (there might be none).

**Twisted Poincaré covariant approaches**

These recover Poincaré covariance in a deformed version, following the observation \[12, 44, 31, 34\] that \[1\] are twisted Poincaré group covariant. Field quantization is done either in a path-integral (on the Euclidean) or in an operator approach. The latter is the framework adopted in the present contribution; this is mainly based on the joint work \[24\] with J. Wess, who unfortunately has recently passed away.

How to implement twisted Poincaré covariance in QFT has been subject of debate and different proposals \[14, 42, 6, 7, 8, 11, 47, 30, 1\], two main issues being whether one should: a) take the $\star$-product of fields at different spacetime points; b) deform the canonical commutation relations (CCR) of creation and annihilation operators $a, a^\dagger$ for free fields.

Our answers to questions a), b) are affirmative and related to each other. The first arises from a proper analysis of twisted Poincaré transformations (section 2). In section 3 we adapt Wightman axioms to the noncommutative setting replacing all products by $\star$-products and analyze the consequences for Wightman and Green’s functions. Motivated by the construction of normalizable states generated by the application to the vacuum of smeared fields (here we show why test functions in the Schwarz space are fine for smearing - a point we only briefly mentioned in \[24\]), we choose a setting where $\star$-products involve also the (Fock space) operator part of the fields; for free fields (section 4) this corresponds to choosing the second of the two options which were found admissible in \[24\] (they both lead to a $\star$-commutator of the fields equal to the undeformed counterpart). In section 4 we also briefly describe how the time-ordered perturbative computation of Green functions of a scalar $\varphi^n$-interacting theory gives the same results as the undeformed theory (the Feynman rules being unchanged). In section 5 we comment on what we can learn from these results, on which aspects still
need investigation, and draw the conclusions.

2 Twisting Poincaré group and Minkowski spacetime

As already noted (1) are translation invariant, but not Lorentz-invariant. In [12, 44, 31, 34] it has been recognized that they are however covariant under a deformed version of the Poincaré group, namely a triangular noncocommutative Hopf $*$-algebra $H$ obtained from the Universal Enveloping algebra (UEA) $U\mathcal{P}$ of the Poincaré Lie algebra $\mathcal{P}$ by twisting [19]. This means that (up to isomorphisms) $H$ and $U\mathcal{P}$ (extended over the formal power series in $\theta^{\mu\nu}$) have

1. the same $*$-algebra and counit $\varepsilon$ (i.e. trivial representation);
2. coproducts $\Delta, \hat{\Delta}$ related by

$$\Delta(g) = \sum_I g^I(1) \otimes g^I(2) \rightarrow \hat{\Delta}(g) = \mathcal{F} \Delta(g) \mathcal{F}^{-1} = \sum_I g^I(1) \otimes g^I(2)$$

for any $g \in H \equiv U\mathcal{P}$. Fixed $\hat{\Delta}$, the so-called twist $\mathcal{F} \in H \otimes H$ is not uniquely determined, but what follows does not depend on its choice. The simplest is

$$\mathcal{F} = \sum_I \mathcal{F}^I(1) \otimes \mathcal{F}^I(2) := \exp \left( \frac{i}{2} \theta^{\mu\nu} P^I_\mu \otimes P^I_\nu \right).$$

$P^I_\mu$ denote the generators of translations, and in [2], [3], we have used Sweedler notation; the $\sum_I$ may be a series, e.g. $\sum_I \mathcal{F}^I(1) \otimes \mathcal{F}^I(2)$ is the series arising from the power expansion of the exponential;

3. antipodes $S, \hat{S}$ related by a similarity transformation; this is trivial for the above $\mathcal{F}$, so $\hat{S} = S$.

For readers not familiar with Hopf algebras, we recall that the coproduct is the abstract operation by which one constructs the tensor product of any two representations. For the cocommutative Hopf algebra $Ug$ ($g$ being a generic Lie algebra)

$$\Delta(1) = 1 \otimes 1, \quad g \in g \rightarrow \Delta(g) = (g \otimes 1 + 1 \otimes g) \equiv g_1 + g_2$$

and $\Delta$ is extended as a $*$-algebra map

$$\Delta : Ug \rightarrow Ug \otimes Ug, \quad \Delta(ab) = \Delta(a)\Delta(b), \quad \Delta(a^*) = [\Delta(a)]^{*\otimes*}.$$ (4)

The extension is unambiguous, as $\Delta([g, g']) = [\Delta(g), \Delta(g')]$ if $g, g' \in g$. Also $\hat{\Delta}$ fulfills (1), $\hat{\Delta}(1) = 1 \otimes 1$, as well as compatibility with $\varepsilon$ and $*$ (as $\mathcal{F}$ is unitary). Then $\hat{\Delta}$

In section 4.4.1 of [34] this was formulated in terms of the dual Hopf algebra.
can replace $\Delta$ in constructing the tensor product of two representations of $U\mathfrak{g}$. The antipode is the abstract operation by which one constructs the contragredient of any representation; it is uniquely determined by the coproduct, if it exists. In the present case, it is determined by $S(g) = -g$ if $g \in \mathfrak{g}$, $S(1) = 1$, $S(ab) = S(b)S(a)$. Altogether, the structures $(\mathcal{UP}, \cdot, \Delta, \epsilon, S)$, $(H, \cdot, \ast, \Delta, \epsilon, S)$ are examples of Hopf $*$-algebras (here we have explicitly indicated the algebra product by $\cdot$, but for brevity everywhere we shorten $a \cdot b = ab$).

For $\mathcal{UP}$ a straightforward computation gives

$$
\hat{\Delta}(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu = \Delta(P_\mu),
\hat{\Delta}(M_\omega) = M_\omega \otimes 1 + 1 \otimes M_\omega + P[\omega, \theta] \otimes \mathcal{P} \neq \Delta(M_\omega),
$$

where we have set $M_\omega := \omega^{\mu\nu} M_{\mu\nu}$ and used a row-by-column matrix product on the right. The left identity shows that the Hopf $P$-subalgebra remains undeformed and equivalent to the abelian translation group $\mathbb{R}^d$. Therefore, denoting by $\triangleright, \triangleleft$ the actions of $\mathcal{UP}, H$ (on $\mathcal{A}$) amounts to the action of the corresponding algebra of differential operators, e.g. $P_\mu$ can be identified with $i \partial_\mu := i \partial / \partial x^\mu$, they coincide on first degree polynomials $a, b$ in $x^\nu, \hat{x}^\nu$,

$$
P_\mu \triangleright x^\rho = i \delta_\mu^\rho = P_\mu \triangleleft \hat{x}^\rho, \quad M_\omega \triangleright x^\rho = 2i(x_\omega)^\rho, \quad M_\omega \triangleleft \hat{x}^\rho = 2i(\hat{x}_\omega)^\rho, \quad (5)
$$

but $\triangleright, \triangleleft$ differ on higher degree polynomials in $x, \hat{x}$, as they are extended by the rules at the lhs of

$$
g \triangleright (ab) = \sum_I (g_{(1)} \triangleright a)(g_{(2)} \triangleright b) \quad (6)
$$

$$
g \triangleleft (\hat{a}\hat{b}) = \sum_I (g_{(1)}^I \triangleleft \hat{a})(g_{(2)}^I \triangleleft \hat{b}) \quad \Leftrightarrow \quad g \triangleright (a \ast b) = \sum_I (g_{(1)}^I \triangleright a) \ast (g_{(2)}^I \triangleright b) \quad (7)
$$

resp. involving the coproducts $\Delta(g), \hat{\Delta}(g)$ (these resp. reduce to the usual or a deformed Leibniz rule if $g = P_\mu, M_{\mu\nu}$). Moreover, $(g \triangleright a)^* = (Sg)^* \triangleright a^*$ as usual. Summarizing, the $H$-module unital $*$-algebra $\hat{\mathcal{A}}$ is obtained by twisting the $\mathcal{UP}$-module unital $*$-algebra $\mathcal{A}$.

**Several spacetime algebra. Formulation through $*$-products.**

For $n \geq 1$ we denote the $n$-fold tensor product algebra of $\mathcal{A}$ by $\mathcal{A}^n$ and $x^\mu \otimes 1 \otimes \ldots \otimes 1 \otimes x^h \otimes \ldots$ respectively by $x_1^\mu, x_2^\mu, \ldots$. In other words, $\mathcal{A}^n$ is the algebra of functions of $n$ sets of Minkowski coordinates $x_i^\mu, i = 1, 2, \ldots, n$. The proper noncommutative deformation of $\mathcal{A}^n$ is the noncommutative unital $*$-algebra $\hat{\mathcal{A}}^n$ generated by real variables $\hat{x}_i^\mu$ fulfilling the commutation relations at the lhs of

$$
[\hat{x}_i^\mu, \hat{x}_j^\nu] = 1 i \theta^{\mu\nu} \quad \Leftrightarrow \quad [x_i^\mu, x_j^\nu] = 1 i \theta^{\mu\nu}, \quad (8)
$$

6
For instance, for \( \theta \) ill-defined for \( \theta \) to the commutative ones for \( \theta \) to reproduce all the product operations used in ordinary QFT, with results reducing a or conversely, as well as for a definition

This can be used as a termwise well-defined but also convergent. Clearly enough for quantum-field-theoretic purposes. On the other hand, if \( A^n \) the algebra obtained by endowing the vector space underlying \( A^n \) with a new product, the \( \ast \)-product, related to the product in \( A^n \) by

\[
    a \ast b := \sum_i \langle F_i^{(1)} \triangleright a \rangle (F_i^{(2)} \triangleright b),
\]

with \( F \equiv F^{-1} \). This encodes both the usual \( \ast \)-product within each copy of \( A \), and the “\( \ast \)-tensor product” between different copies \([2, 3]\). As a result one finds the isomorphic \( \ast \)-commutation relations at the rhs of (8) [this follows from computing (8) \( x_i^\mu \ast x_j^\nu, \) which e.g. for the specific choice \([3]\) gives \( x_i^\mu x_j^\nu + i \theta^{\mu \nu}/2 \) and that \( A^n, A^n_0 \) are isomorphic \( H \)-module unital \( \ast \)-algebras, in the sense of the equivalences (7), (8).

The \( \ast \)-product (9) can be extended from polynomials \( a, b \) to power series. More explicitly, on analytic functions \( a(x_i), b(x_j) \) (10) reads

\[
    a(x_i) \ast b(x_j) = \exp[\frac{i}{2} \partial_{x_i} \partial_{x_j}] a(x_i)b(x_j)
\]

(for any 4-vectors \( p, q \) we define \( p \theta q := p_\mu \theta^{\mu \nu} q_\nu \), what must be followed by the indenification \( x_i = x_j \) after the action of the bi-pseudodifferential operator \( \exp[\frac{i}{2} \partial_{x_i} \partial_{x_j}] \) if \( i = j \). Strictly speaking, the last formula makes sense if \( a, b \) belong to some suitable subalgebra \([20]\) \( A^{nt} \) of the algebra of analytic functions such that the \( \theta \)-power series is not only termwise well-defined but also convergent. Clearly \( A^{nt} \) will not be large enough for quantum-field-theoretic purposes. On the other hand, if \( a(x_i), b(x_j) \in A^{nt} \) admit Fourier transforms \( \hat{a}(k_i), \hat{b}(k_j) \) then

\[
    a(x_i) \ast b(x_j) = \int d^4 k \int d^4 q \, \hat{a}(k) \hat{b}(q) \exp[i(k \cdot x_i + q \cdot x_j - k \theta q/2)].
\]

This can be used as a definition of a \( \ast \)-product for \( a(x), b(x) \in L^1(\mathbb{R}^4) \cap \mathcal{L}^1(\mathbb{R}^4) \), for \( a(x) \in \mathcal{S}(\mathbb{R}^4) \) (Schwarz space) and \( b(x) \in \mathcal{S}'(\mathbb{R}^4) \) (the space of tempered distributions), or conversely, as well as for \( a(x), b(x) \in \mathcal{S}'(\mathbb{R}^4) \) provided \( i \neq j \). These are in fact enough to reproduce all the product operations used in ordinary QFT, with results reducing to the commutative ones for \( \theta^{\mu \nu} = 0 \).

Actually, for \( i = j \) and some \( a(x), b(x) \in \mathcal{S}'(\mathbb{R}^4) \) it may even happen that (11) is ill-defined for \( \theta^{\mu \nu} = 0 \), but well-defined \([26]\) (and thus “regularized”) for \( \theta^{\mu \nu} \neq 0 \).

\[^6\]For instance, for \( a(x) = \delta^4(x) = b(x) \) and invertible \( \theta \) one easily finds \( a(x_i) \ast b(x_j) = \)
\( \mathcal{S}(\mathbb{R}^4) \) is a \( \ast \)-module of the \( \ast \)-algebra underlying both \( U \mathcal{P}, H \). As usual, the irreducible submodules are the eigenspaces of the Casimir \( p \cdot p \); one can endow those characterized by a positive eigenvalue \( m^2 \) and a positive spectrum for \( P^0 \) by the usual pre-Hilbert space structure. By completion, one obtains unitary irreducible representations (irreps) of the \( \ast \)-algebra underlying both \( U \mathcal{P}, H \), that describe scalar particles. (Generalized) eigenfunctions of \( P_\mu \) or \( M_{\mu\nu} \) exist instead within \( \mathcal{S}'(\mathbb{R}^4) \), which is a larger \( \ast \)-module of the \( \ast \)-algebra underlying both \( U \mathcal{P}, H \). Unitary irreps describing higher spin particles can be obtained in the standard way as some \( \mathbb{C}^k \otimes \mathcal{S}(\mathbb{R}^4) \) or projective modules thereof (spinor bundles, 4-vector bundles, etc). Summarizing, one obtains the same \([12]\) classification (à la Wigner) of elementary particles as unitary irreps of either \( U \mathcal{P} \) or \( H \).

The generalization of the definition \((11)\) to functions/distributions depending nontrivially on several (possibly all the) \( x_i \) is straightforward. In particular the \( \ast \)-product \( a \ast b \) is well-defined for any \( a \in \mathcal{S}(\mathbb{R}^{4n}) \) and \( b \in \mathcal{S}'(\mathbb{R}^{4n}) \) (or viceversa). Also \( \mathcal{S}(\mathbb{R}^{4n}) \), \( \mathcal{S}'(\mathbb{R}^{4n}) \) are \( \ast \)-modules of the \( \ast \)-algebra underlying both \( U \mathcal{P}, H \). In fact, we shall need to embed them in an even larger module \( \ast \)-algebra \( \Phi^e \) of operator-valued (instead of \( c \)-number valued) distributions. The action \( \triangleright \) fulfills the ordinary (resp. deformed) Leibniz rule \((6)\) [resp. \((7)\)] if \( a, b \) are multiplied (resp. \( \ast \)-multiplied). This implies that the action of \( U \mathcal{P}, H \) on tensor products of modules is constructed using the ordinary (resp. deformed) coproduct.

In the sequel we shall formulate the noncommutative spacetime only in terms of \( \ast \)-products and construct QFT on it replacing all products by \( \ast \)-products.

The differential calculus is not deformed, as \( P_\mu \triangleright \partial x_\nu = 0 \) implies \( \partial x_\nu \ast = \partial x_\nu \ast = \ast \partial x_\nu \star \ast \partial x_\nu \star \partial x_\nu \star = \partial x_\nu \star \partial x_\nu \star = 0 \)

(\( \partial x_\nu \star \partial x_\nu \star \) on \( \mathcal{A}^n \) is isomorphic). In the sequel we shall drop the symbol \( \ast \) beside a derivative, as it has no effect. Also integration over the space is not deformed:

\[
\int d^4 x \ a(x) \star b(x) = \int d^4 x \ a(x) b(x)
\]

(12)

[This holds in particular for all \( a(x) \in \mathcal{S}(\mathbb{R}^4) \) and \( b(x) \in \mathcal{S}'(\mathbb{R}^4) \).] Stokes’s theorem still applies. Using \((11)\) it is easy to check the property

\[
\int dx_\mu b \ast a(x_i) = b \star \int dx_\mu a(x_i), \quad \text{if } b \text{ is independent of } x_i,
\]

(13)

\((\pi^4 \det \theta)^{-1} \exp[2i x_\mu \theta^{-1} x_\mu]; \) in particular for \( i = j \) the exponential becomes 1 by the antisymmetry of \( \theta^{-1} \), and one finds a diverging constant as \( \det \theta \to 0 \), cf. [26] [20]. In [26] the largest algebra of distributions for which the \( \ast \)-product is well-defined and associative is determined. In [20] the subalgebra of analytic functions for which \((10)\) gives an asymptotic expansion of \((11)\) is determined.
analogous to the commutative conterpart [of course, if \( a(x_i) \) is a \( c \)-number valued function/distribution depending only on \( x_i \), the integral at the rhs is a \( c \)-number and the \( \star \)-product at the rhs can be dropped]. Therefore, for our purposes we can consider integration over any set of coordinates \( x \) as an operation commuting with the \( \star \)-product.

Let \( a_i \in \mathbb{R} \) with \( \sum_i a_i = 1 \). An alternative set of real generators of \( \mathcal{A}_b^\theta \) is:

\[
\xi_i^\mu := x_i^\mu - x_{i+1}^\mu, \quad i = 1, \ldots, n-1, \quad X^\mu := \sum_{i=1}^n a_i x_i^\mu. \tag{14}
\]

All \( \xi_i^\mu \) are translation invariant, \( X^\mu \) is not. It is immediate to check that \( [X^\mu, X^\nu] = 1i\theta^\mu\nu \), so \( X^\mu \) generate a copy \( \mathcal{A}_{\theta,X} \) of \( \mathcal{A}_\theta \), whereas \( \forall b \in \mathcal{A}_b^\theta \)

\[
[\xi_i^\mu, b] = 0, \tag{15}
\]

so \( \xi_i^\mu \) generate a \( \star \)-central subalgebra \( \mathcal{A}_{\xi}^{n-1} \), and \( \mathcal{A}_b^\theta \sim \mathcal{A}_{\xi}^{n-1} \otimes \mathcal{A}_{\theta,X} \). The \( \star \)-multiplication operators \( \xi_i^\mu \) have the same spectral decomposition on all \( \mathbb{R} \) (including 0) as multiplication operators \( \xi^\mu \) by classical coordinates; the joint eigenvalues make up a space-like, or a null, or a time-like 4-vector, in the usual sense. Moreover, \( \mathcal{A}_{\xi}^{n-1}, \mathcal{A}_{\theta,X} \) are actually \( H \)-module subalgebras, with

\[
g \triangleright (a \star b) = \sum_I \left( g_I^{(1)} \triangleright a \right) \star \left( g_I^{(2)} \triangleright b \right), \quad a \in \mathcal{A}_{\xi}^{n-1}, \quad b \in \mathcal{A}_b^\theta, \quad g \in H, \tag{16}
\]

i.e. on \( \mathcal{A}_{\xi}^{n-1} \) the \( H \)-action is undeformed, including the related part of the Leibniz rule. [By (15) here \( \star \) can be also dropped].

Inverting (14), any set \( x_i \) can be expressed as a combination of the \( n-1 \) sets of \( \star \)-commutative variables \( \xi_i \) and the set \( X \) of \( \star \)-noncommutative ones, e.g. if \( X := x_n \) then

\[
x_i = \sum_{j=i}^{n-1} \xi_j + X.
\]

\( X \) therefore behaves as parametrizing a “global noncommutative translation”.

3 Revisiting Wightman axioms for QFT and their consequences

As in Ref. [40] we divide the Wightman axioms [39] into a subset (labelled by QM) encoding the quantum mechanical interpretation of the theory, its symmetry under space-time translations and stability, and a subset (labelled by R) encoding the relativistic properties. Since they provide minimal, basic requirements for the field-operator framework to quantization we try to apply them to the above noncommutative space (i.e.
replacing everywhere products by $\ast$-products) keeping the QM conditions, twisting Poincaré-covariance R1 and being ready to weaken locality R2 if necessary.

**QM1.** The states are described by vectors of a (separable) Hilbert space $H$.

**QM2.** The group of space-time translations $\mathbb{R}^4$ is represented on $H$ by strongly continuous unitary operators $U(a)$: the fields transform according to (26) with unit $A$, $U(A), \Lambda(A)$. The spectrum of the generators $P_\mu$ is contained in $\mathbb{V}_+ = \{p^2 \geq 0, p_0 \geq 0\}$. There is a unique Poincaré invariant state $\Psi_0$, the vacuum state.

**QM3.** The fields (in the Heisenberg representation) $\varphi^\alpha(x)$ [$\alpha$ enumerates field species and/or $SL(2, \mathbb{C})$-tensor components] are operator (on $H$) valued tempered distributions on Minkowski space, with $\Psi_0$ a cyclic vector for the fields, i.e. $\ast$-polynomials of the smeared fields applied to $\Psi_0$ give a set $D_0$ dense in $H$.

For a single scalar field $D_0$ is spanned by vectors of the form of a finite sum

$$\Psi_f = f_0 \Psi_0 + \varphi(f_1) \Psi_0 + \varphi \left( f_2^{(1)} \right) \ast \varphi \left( f_2^{(2)} \right) \Psi_0 + ..., \quad (17)$$

where $j^{(h)} \in S(\mathbb{R}^4), h \leq j \leq N < \infty$ and

$$\varphi(f) := \int d^4 x f(x) \ast \varphi(x) = \int d^4 x f(x) \varphi(x).$$

The (non-smeared) polynomials in the fields on commutative space make up a subalgebra $\Phi$ of what we may call the (extended) field algebra $\Phi^e = \left( \bigotimes_{i=1}^\infty S' \right) \otimes \mathcal{O}$, where the first, second,... tensor factor $S'$ is understood as the space of distributions depending on $x_1, x_2, ...$ [the dependence on $x_h$ of the polynomial appearing in (17) being trivial for $h > N$], and $\mathcal{O}$ is the $\ast$-algebra of linear operators on $H$ (e.g. for free bosonic/fermionic fields $\mathcal{O}$ is a Heisenberg/Clifford algebra with infinitely many modes). $\Phi^e$ also is a $UP\pi$-module $\ast$-algebra. We should therefore $H$-covariantly $\ast$-deform the whole $\Phi^e$ into the corresponding $\Phi^e_\theta$ (see also [23]). In analogy with the commutative case, we shall require that within $\Phi^e_\theta$ fields $\ast$-commute with $c$-number valued functions/distributions $f$

$$[\varphi^\alpha(x) \ast f(y)] \equiv \varphi^\alpha(x) \ast f(y) - f(y) \ast \varphi^\alpha(x) = 0. \quad (18)$$

For free (scalar) fields this was proposed in [24] as the second of two admissible options (we shall explicitly recall how this works in section 3); this relation, together with (13), implies

$$\Psi_f = f_0 \Psi_0 + \int d^4 x_1 f_1(x_1) \ast \varphi(x_1) \Psi_0 + \int d^4 x_1 \int d^4 x_2 f_2(x_1, x_2) \ast \varphi(x_1) \ast \varphi(x_2) \Psi_0 + ..., \quad (19)$$

$$f_j(x_1, ..., x_j) := f_1^{(1)}(x_1) \ast ... \ast f_j^{(j)}(x_j),$$

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so $\Psi_f$ is characterized by the terminating sequence $f = (f_0, f_1, \ldots f_N)$. It is immediate to check that the Fourier transform of $f_j$ differs from the commutative one only by a phase factor,

$$\tilde{f}_j(p_1, \ldots, p_j) = \tilde{f}_j^{(1)}(p_1) \ldots \tilde{f}_j^{(j)}(p_j) \exp \left[ i \frac{1}{2} \sum_{h=1}^{j} \sum_{k=h+1}^{j} p_h \theta p_k \right],$$

and therefore $f_j \in \mathcal{S}(\mathbb{R}^{4j})$. As on commutative space, $D_0$ is also dense in the set $D_1$ of all vectors of the form (19) with $f_j \in \mathcal{S}(\mathbb{R}^{4j})$.

Taking v.e.v.’s we define the Wightman functions

$$W^{\alpha_1, \ldots, \alpha_n}(x_1, \ldots, x_n) := (\Psi_0, \varphi^{\alpha_1}(x_1) \star \cdots \star \varphi^{\alpha_n}(x_n) \Psi_0),$$

which are in fact distributions, and (their combinations) the Green’s functions

$$G^{\alpha_1, \ldots, \alpha_n}(x_1, \ldots, x_n) := (\Psi_0, T[\varphi^{\alpha_1}(x_1) \star \cdots \star \varphi^{\alpha_n}(x_n)] \Psi_0)$$

where also time-ordering $T$ is defined as on commutative space (even if $\theta^0 \neq 0$), e.g.

$$T[\varphi^{\alpha_1}(x) \star \varphi^{\alpha_2}(y)] = \varphi^{\alpha_1}(x) \star \varphi^{\alpha_2}(y) \star \vartheta(x^0 - y^0) + \varphi^{\alpha_2}(y) \star \varphi^{\alpha_1}(x) \star \vartheta(y^0 - x^0)$$

for $n = 2$ ($\vartheta$ denotes the Heavyside function). This is well-defined as $\vartheta(x^0 - y^0)$ is $\star$-central: the $\star$-products preceding all $\vartheta$ could be dropped, by (15).

Arguing as for ordinary QFT (see [39]) one finds that QM1-3 (alone) imply exactly the same properties as on commutative space:

**W1.** Wightman and Green’s functions are translation-invariant tempered distributions and therefore may depend only on the $\xi^\mu_i$:

$$W^{\alpha_1, \ldots, \alpha_n}(x_1, \ldots, x_n) = W^{\alpha_1, \ldots, \alpha_n}(\xi_1, \ldots, \xi_{n-1}),$$

$$G^{\alpha_1, \ldots, \alpha_n}(x_1, \ldots, x_n) = G^{\alpha_1, \ldots, \alpha_n}(\xi_1, \ldots, \xi_{n-1}).$$

**W2.** (Spectral condition) The support of the Fourier transform $\tilde{W}$ of $W$ is contained in the product of forward cones, i.e.

$$\tilde{W}^{(\alpha)}(q_1, \ldots, q_{n-1}) = 0, \quad \text{if} \quad \exists j : \quad q_j \notin \nabla_+.$$
with $f_k, g_j$ defined as in (19). Using (22) it is straightforward to prove that in fact the previous formula holds also without $\ast$ (as on commutative space):

$$ (\Psi_{g_j}, \Psi_{f_k}) = \int d^4j x \int d^4k y \ g_j^* (x_j, ..., x_1) f_k (x_1, ..., x_k) W(x_1, ..., x_j, y_1, ..., y_k) \tag{24} $$

Using (24) (and the analogous formulæ for non-scalar fields) we find $W_3$. $W^{\{\alpha}\}$ fulfill the same Hermiticity and Positivity properties following from those of the scalar product in $\mathcal{H}$ as in the theory on commutative space.

For instance, for the Wightman functions of a single scalar field they reads as follows:

$$ [W(x_1, ..., x_n)]^* = W(x_n, ..., x_1), \text{ and for all terminating sequences } f = (f_0, f_1, ..., f_N) \text{ with } f_j \in S(\mathbb{R}^4) \tag{25} $$

The ordinary relativistic conditions on QFT are:

**R1.** (Lorentz Covariance) $SL(2, \mathbb{C})$ is represented on $\mathcal{H}$ by strongly continuous unitary operators $U(A)$, and under the Poincaré transformations $U(a, A) = U(a)U(A)$

$$ U(a, A) \varphi^\alpha (x) U(a, A)^{-1} = S^\alpha_{\beta i} (A^{-1}) \varphi^\beta (\Lambda(A)x + a), \tag{26} $$

with $S$ a finite-dimensional representation of $SL(2, \mathbb{C})$.

**R2.** (Microcausality or locality) The fields either commute or anticommute at spacelike separated points

$$ [\varphi^\alpha (x), \varphi^\beta (y)]_\mp = 0, \quad \text{for } (x - y)^2 < 0. \tag{27} $$

In ordinary QFT as a consequence of QM2, R1 one finds

**W4.** (Lorentz Covariance of Wightman functions)

$$ W^{\alpha_1:..\alpha_n} (\Lambda(A)x_1, ..., \Lambda(A)x_n) = S^\alpha_{\beta^1} (A) .. S^\alpha_{\beta^m} (A) W^{\beta_1:..\beta_n} (x_1, ..., x_n). \tag{28} $$

In particular, Wightman (and Green) functions of scalar fields are Lorentz invariant.

R1 needs a “twisted” reformulation $\ast$, which we defer. Now, however $\ast$ will look like, it should imply that $W^{\{\alpha}\}$ are $SL_\theta(2, \mathbb{C})$ tensors (in particular invariant if

\[\ast\] The $\ast$ between $W$ and the rest is ineffective by (15), (20). Also the $\ast$ between $g_j^*$ and $f_k$ is ineffective: going to the Fourier transforms, the corresponding phase factor reduces to 1 when exploiting the presence of the Dirac’s $\delta$ in the equality $\tilde{W}(p_1, ..., p_n) = (2\pi)^4 \delta^4(\sum_i p_i) \tilde{W}(p_1, p_1 + p_2, ..., p_1 + ... + p_n).$
all involved fields are scalar). But, as the $W^{(\alpha)}$ are to be built only in terms of $\xi_\mu^i$ and other $SL(2, \mathbb{C})$ tensors (like $\partial_\mu^i$, $\eta_{\mu\nu}$, $\gamma^\mu$, etc.), which are all annihilated by $P_\mu \lhd$, $\mathcal{F}$ will act as the identity and $W^{(\alpha)}$ will transform under $SL(2, \mathbb{C})$ as for $\theta = 0$. Therefore we shall require $W^4$ also if $\theta \neq 0$ as a temporary substitute of $R_1$.

The simplest sensible way to formulate the $*$-analog of locality is $R_2$. (Microcausality or locality) The fields either $*$-commute or $*$-anticommute at spacelike separated points

$$[\varphi^\alpha(x) \ast \varphi^\beta(y)]_\mp = 0, \quad \text{for } (x - y)^2 < 0. \quad (29)$$

This makes sense, as space-like separation is sharply defined, and reduces to the usual locality when $\theta = 0$. Therefore we shall adopt it. Whether there exist reasonable weakenings of $R_2$ is an open question also on commutative space, and the same restrictions will apply.

Arguing as in [39] one proves that QM1-3, W4, R2 are independent and compatible, as they are fulfilled by free fields (see below): the noncommutativity of a Moyal-Minkowski space is compatible with $R_2$! As consequences of $R_2$, one again finds

W5. (Locality) if $(x_j - x_{j+1})^2 < 0$

$$W(x_1, ...x_j, x_{j+1}, ...x_n) = \pm W(x_1, ...x_{j+1}, x_j, ...x_n). \quad (30)$$

W6. (Cluster property) For any spacelike $a$ and for $\lambda \to \infty$

$$W(x_1, ...x_j, x_{j+1} + \lambda a, ..., x_n + \lambda a) \to W(x_1, ..., x_j) W(x_{j+1}, ..., x_n), \quad (31)$$

(convergence in the distribution sense); this is true also with permuted $x_i$’s.

Summarizing: our QFT framework is based on QM1-3, W4, R2, and the technical requirement (18), or alternatively on the constraints W1-6 for $W^{(\alpha)}$, exactly as in QFT on Minkowski space. We stress that this applies for all $\theta_{\mu\nu}$, even if $\theta^{0i} \neq 0$, contrary to other approaches. Moreover, we have just seen that (contrary to [13]) we can keep the Schwarz space $\mathcal{S}^*(\mathbb{R}^4)$ as the space of test functions for smearing the fields. We shall keep it as this guarantees not only the separability of $\mathcal{H}$ but also that a finite number of subtractions is enough to define field products at the same point, i.e. essentially the possibility to renormalize the theory. However we should note that, for given $f_j^{(h)} \in \mathcal{S}^*(\mathbb{R}^4)$, the states (17) do not coincide with their undeformed counterparts. We do not know whether this might have consequences on observables (as $S$-matrix elements).
4 Free or interacting scalar field

As the differential calculus remains undeformed, so remain the equation of motions of free fields. Sticking for simplicity to the case of a scalar field of mass \( m \), the solution of the Klein-Gordon equation reads

\[
\varphi_0(x) = \int d\mu(p) \left[ e^{-ip\cdot x} \ast a^p + a^\dag_p \ast e^{ip\cdot x} \right]
\]  

(32)

where \( d\mu(p) = \delta(p^2 - m^2) \partial(p^0) d^4p = dp^0 \delta(p^0 - \omega_p) d^3p / 2 \omega_p \) is the invariant measure (\( \omega_p := \sqrt{p^2 + m^2} \)). Postulating the axioms of the preceding section, except \( \mathbf{R}2 \), one can prove that up to a positive factor (which can be always reabsorbed in a field redefinition)

\[
W(x-y) = \int d\mu(p) e^{-ip\cdot(x-y)}
\]

(33)

and therefore coincides with the undeformed counterpart. Adding also \( \mathbf{R}2 \), one can prove the free field commutation relation

\[
[\varphi_0(x) \ast \varphi_0(y)] = 2 \int d\mu(p) \sin[p\cdot(x-y)] = : i F(x-y),
\]

(34)

coinciding with the undeformed one. Applying \( \partial_{p^0} \) to (33) and setting \( y^0 = x^0 \) [this is compatible with (3)] one finds the canonical commutation relation

\[
[\varphi_0(x^0, x) \ast \varphi_0(x^0, y)] = i \delta^3(x-y).
\]

(35)

As a consequence of (34), the \( n \)-point Wightman functions not only fulfill W1-W6, but coincide with the undeformed ones, i.e. vanish if \( n \) is odd and are sum of products of 2-point functions (factorization) if \( n \) is even.

A \( \varphi_0 \) fulfilling (34) can be obtained assuming \( P_\mu \ast a^\dagger_p = p_\mu a^\dagger_p, P_\mu \ast a^p = -p_\mu a^p \), so as to extend the \( \ast \)-product law also to \( a^p, a^\dagger_p \), and plugging in (32) \( a^p, a^\dagger_p \) satisfying

\[
a^\dagger_p \ast a_q = e^{-ip\theta q} a^\dagger_q \ast a^p, \quad a^p \ast a^q = e^{-ip\theta q} a^q \ast a^p,
\]

\[
a^p \ast a^\dagger_q = e^{ip\theta q} a^\dagger_q \ast a^p + 2i \omega p \delta^3(p-q), \quad a^\dagger_p \ast e^{iq\cdot x} = e^{ip\theta q} e^{iq\cdot x} \ast a^\dagger_p.
\]

(36)

Note the nontrivial commutation relations between the \( a^p, a^\dagger_p \) and \( c \)-number valued functions, but \( [\varphi_0(x) \ast f(y)] = 0 \) as in (13). The first three relations define an example of a general deformed Heisenberg algebra [22]

\[
a^q \ast a^p = R^{qp}_{rs} a^r \ast a^s, \quad a^\dagger_p \ast a^\dagger_q = R^{qp}_{rs} a^\dagger_r \ast a^\dagger_s,
\]

\[
a^p \ast a^\dagger_q = \delta^\dagger_q p + R^{qp}_{rs} a^\dagger_r \ast a^s,
\]

(37)
covariant under a triangular Hopf algebra $H$. Here $\mathcal{R} := \mathcal{F}_1 \mathcal{F}^{-1}$ is the triangular structure of $H$, $\{|p\}\}$ is the generalized basis of the 1-particle Hilbert space consisting of (on-shell) eigenvectors of $P_\mu$, $\delta_0^p = 2\omega_p \delta^3(p-q)$ is Dirac’s delta (up to normalization), $R_p^q := \langle p | \otimes \langle q | \mathcal{R} | r \rangle \otimes | s \rangle = e^{ipq_0} \delta^3(p-q)$.

Up to normalization of $R$, and with $p, q, r, s \in \{1, ..., N\}$, relations (25) are also identical to the ones defining the older $q$-deformed Heisenberg algebras of $[35, 46]$, based on a quasitriangular $\mathcal{R}$ in (only) the $N$-dimensional representation of $H = U_q su(N)$.

**Remark.** In $[24]$ we actually found also a different (and maybe more intuitive) way to construct a free field fulfilling (33). It amounts to: 1. introducing $a^p, a^\dagger_p$ satisfying

$$a^\dagger_p a^q = e^{ip\theta_q} a^\dagger_q a^p, \quad a^p a^q = e^{ip\theta_q} a^q a^p, \quad a^p a^\dagger_q = e^{-ip\theta_q} a^\dagger_q a^p + 2\omega_p \delta^3(p-q),$$

(with $\theta' = \theta$), and $[a^p, f(x)] = [a^\dagger_p, f(x)] = 0,$ (38)

(so $c$-number valued functions/distributions keep commuting with $a^p, a^\dagger_p$), as adopted e.g. in $[7, 30, 1]$; 2. restricting $\ast$-multiplication only to the functions/distributions part (i.e. elements of the extended $A^\theta_q$ of the fields). Consequently, instead of the field decomposition reads $\varphi_0(x) = \int \mu(p) [e^{-ip\cdot x} a^p + a^\dagger_p e^{ip\cdot x}]$ with such $a^p, a^\dagger_p$. This leads to the same properties W1-W6. However, as $\varphi(f)$ does no more depend on spacetime coordinates $x$, the $\ast$ in (17) and (19) becomes redundant, and we obtain

$$\Psi_f = f_0 \Psi_0 + \int d^4x_1 f_1(x_1) \varphi(x_1) \Psi_0 + \int d^4x_1 d^4x_2 f_2(x_1, x_2) \varphi_0(x_1) \varphi_0(x_2) \Psi_0 + ..., \quad \text{with} \quad f_j(x_1, ..., x_j) = f^{(1)}_j(x_1)... f^{(j)}_j(x_j).$$

As a result, scalar products $(\Psi_{g_0}, \Psi_{f_0})$ cannot be expressed in terms of Wightman functions as in (21), but in the form

$$\langle \Psi_{g_0}, \Psi_{f_0} \rangle = \int d^4x \int d^4y g^*_{x,y} f_{x,y} \mathcal{W}(x_1, ..., x_j, y_1, ..., y_k) \mathcal{W}'(x_1, ..., x_j, y_1, ..., y_k)$$

(with no $\ast$-products in the definition of $\mathcal{W}'$, as in [5]). The distributions $\mathcal{W}'$ do not fulfill all the properties W1-W6 (except of course in the undeformed case $\theta' = 0$). We also briefly consider some consequences of choosing $\theta' \neq \theta$ in (38) ($\theta' = 0$ gives CCR among the $a^p, a^\dagger_p$, assumed in most of the literature, explicitly [17] or implicitly, in operator [14] [15] or in path-integral approach to quantization) together with $\varphi_0(x) = \int \mu(p) [e^{-ip\cdot x} a^p + a^\dagger_p e^{ip\cdot x}]$ and definition (20) for the Wightman functions. One finds the non-local $\ast$-commutation relation

$$\varphi_0(x) \ast \varphi_0(y) = e^{i\theta_0(\theta-\theta')} \varphi_0(x) \ast \varphi_0(y) + i F(x - y),$$

and the corresponding (free field) Wightman functions violate W4, W6, unless $\theta' = \theta$. 

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Going back to our framework, we now define **normal ordering** as a \(A_\theta^0\)-bilinear map of field algebra into itself such that \((\Psi_0, : M : \Psi_0) = 0\) for any field polynomial \(M\), in particular \(: 1 : = 0\). Applying it to (38) we find that it is consistent to define

\[
:a^p \star a^q: = a^p \star a^q, \quad :a^\dagger_p \star a^q: = a^\dagger_p \star a^q, \quad :a^p \star a^\dagger_q: = a^\dagger_q \star a^p e^{-i\theta q}
\]

(Note the phase). More generally, by definition in any monomial this map reorders all \(a^p\) to the right of all \(a^\dagger_q\) introducing an \(e^{-i\theta p}\) for each flip \(a^p \leftrightarrow a^\dagger_q\). For \(\theta = 0\) the map reduces to the undeformed normal ordering.

As a result, one finds that the v.e.v. of any normal-ordered \(\star\)-polynomial of fields is zero, that normal-ordered \(\star\)-products of fields can be obtained from \(\star\)-products by the undeformed pattern of subtractions, and that the **same Wick theorem** as in the undeformed case holds. Applying **time-ordered perturbation theory** to an interacting field again one can heuristically derive [24], through the same arguments used on commutative space, the Gell-Mann–Low formula

\[
G(x_1, ..., x_n) = \frac{\langle \Psi_0, T \{ \varphi_0(x_1) \star ... \star \varphi_0(x_n) \star \exp \left[ -i \lambda \int dy^0 H_I(y^0) \right] \} \Psi_0 \rangle}{\langle \Psi_0, T \exp \left[ -i \int dy^0 H_I(y^0) \right] \Psi_0 \rangle} \tag{39}
\]

(which is rigorously valid under the assumption of asymptotic completeness, \(\mathcal{H} = \mathcal{H}^{in} = \mathcal{H}^{out}\)). Here \(\varphi_0, H_I(x^0)\) denote the free “in” field (i.e. the incoming field) and the interaction Hamiltonian in the interaction representation, e.g.

\[
H_I(x^0) = \lambda \int d^3x : \varphi_0^{*m}(x) : \star, \quad \varphi_0^{*m}(x) \equiv \varphi_0(x) \star ... \star \varphi_0(x), \quad m \text{ times}
\]

Thus [24] one finds that the **Green functions** (39) coincide with the undeformed ones (at least perturbatively). They can be computed by Feynman diagrams with the undeformed Feynman rules, and the theory can be regularized and renormalized in the standard ways.

## 5 Conclusions. What do we learn?

Although various approaches to relativistic QFT on Moyal-Minkowski space have been proposed, there is still no generally accepted one. Operator-based approaches look safer starting points, but twisting or not the Poincaré group, and doing it properly, makes the results radically different.

We have claimed here that a sensible theory with twisted Poincaré seems possible and avoids all complications (IV-UR, causality/unitarity violation, statistics violation, cluster property violation, loss of spacetime symmetry,...). It naturally involves a
compensation of operator \((a, a^\dagger)\) and spacetime noncommutativities, so that the free field \(*\)-commutators coincide with the undeformed ones.

The surprising and probably disappointing fact is that also the corresponding \(n\)-point functions, expressed as functions of the coordinates’ differences, coincide with the undeformed ones. The natural consequence seems that no new physics, nor a more satisfactory formulation of the old one (e.g. by an intrinsic UV regularization) is obtained (at least for scalar fields), although this can be confirmed only upon clarifying the relation between \(n\)-point functions and observables, in particular \(S\)-matrix elements.

Nevertheless we think that we can learn quite much from trying to understand the reasons of these surprising results, which are in striking contrast with the ones found in most of the literature, as well as from using our approach as a laboratory for:

1. searching and testing equivalent formulations of QFT on NC spaces: Wick rotation into EQFT, path integral quantization, etc.;
2. clarifying notions such as asymptotic states, spin-statistics, CPT, etc., on noncommutative spaces;
3. properly formulating covariance properties of fields under twisted symmetries \((R_1^\star)\), and clarify their connection to the ordinary ones;
4. properly formulating gauge field theory on noncommutative spaces.

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