HILBERT-KUNZ DENSITY FUNCTION AND ASYMPTOTIC HILBERT-KUNZ MULTIPLICITY FOR PROJECTIVE TORIC VARIETIES

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Abstract. For a toric pair \((X, D)\), where \(X\) is a projective toric variety of dimension \(d−1\geq 1\) and \(D\) is a very ample \(T\)-Cartier divisor, we show that the Hilbert-Kunz density function \(HKd(X, D)(\lambda)\) is the \(d−1\) dimensional volume of \(\overline{P}_D \cap \{z = \lambda\}\), where \(\overline{P}_D \subset \mathbb{R}^d\) is a compact \(d\)-dimensional set (which is a finite union of convex polytopes).

We also show that, for \(k \geq 1\), the function \(HKd(X, kD)\) can be replaced by another compactly supported continuous function \(\varphi_{kD}\) which is ‘linear in \(k\)’. This gives the formula for the associated coordinate ring \((R, m)\):

\[
\lim_{k \to \infty} \frac{e_{HK}(R, m^k) - e_0(R, m^k)/d!}{k^{d-1}} = e_0(R, m) \int_0^\infty \varphi_D(\lambda) d\lambda,
\]

where \(\varphi_D\) (see Proposition 1.2) is solely determined by the shape of the polytope \(P_D\), associated to the toric pair \((X, D)\). Moreover \(\varphi_D\) is a multiplicative function for Segre products.

This yields explicit computation of \(\varphi_D\) (and hence the limit), for smooth Fano toric surfaces with respect to anticanonical divisor. In general, due to this formulation in terms of the polytope \(P_D\), one can explicitly compute the limit for two dimensional toric pairs and their Segre products.

We further show that (Theorem 6.3) the renormalized limit takes the minimum value if and only if the polytope \(P_D\) tiles the space \(M = \mathbb{R}^{d−1}\) (with the lattice \(M = \mathbb{Z}^{d−1}\)). As a consequence, one gets an algebraic formulation of the tiling property of any rational convex polytope.

1. Introduction

Let \(R\) be a Noetherian ring of prime characteristic \(p > 0\) and of dimension \(d\) and let \(I \subset R\) be an ideal of finite colength. Then we recall that the Hilbert-Kunz multiplicity of \(R\) with respect to \(I\) is defined as

\[
e_{HK}(R, I) = \lim_{n \to \infty} \frac{\ell(R/I^{[q^n]})}{q^d},
\]

where \(q = p^n\), \(I^{[q]} = n\)-th Frobenius power of \(I\) = the ideal generated by \(q\)-th powers of elements of \(I\). This is an ideal of finite colength and \(\ell(R/I^{[q]}\) denotes the length of the \(R\)-module \(R/I^{[q]}\). Existence of the limit was proved by Monsky [Mo1]. This invariant has been extensively studied, over the years (see the survey article [Hu]). As various standard techniques, used for studying multiplicities, are not applicable for the invariant \(e_{HK}\), it has been difficult to compute (there is no general formula even for a hypersurface).

In order to study \(e_{HK}\), when \(R\) is a standard graded ring (\(\dim R \geq 2\)) and \(I\) is a homogeneous ideal of finite colength, the second author (in [T2]) has defined the notion of Hilbert-Kunz Density function and its relation with the HK-multiplicity (stated in this paper as Theorem 4.1): the HK density function is a compactly supported continuous function \(HKd(R, I) : [0, \infty) \to [0, \infty)\) such that

\[
e_{HK}(R, I) = \int_0^\infty HKd(R, I)(x) \, dx.
\]

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Further using this relation, the asymptotic behaviour of $e_{HK}(R, I^k)$ as $k \to \infty$, was studied in [T3].

The asymptotic behaviour of $e_{HK}$ was first studied by Watanabe-Yoshida in [WY1], for a Noetherian local ring $(R, \mathfrak{m})$ of dimension $d \geq 2$ and an $\mathfrak{m}$-primary ideal $I$. In particular, in [WY1] it is shown that
\[
\frac{e_0(R, I^k)}{d!} \leq e_{HK}(R, I^k) \leq \frac{(k+d-1)}{k^d} e_0(R, I^k),
\]
and as a corollary they get
\[
e_{HK}(R, I^k) = \frac{e_0(R, I)}{d!} k^d + o(k^d).
\]
Later Hanes in [Ha] (Theorem 3.2) improved this as follows:
\[
\ell(R/I^{[q]k}) = \left[ \frac{e_0(R, I)}{d!} k^d + O(k^{d-1}) \right] q^d.
\]

In other words
\[
e_{HK}(R, I^k) - \frac{e_0(R, I^k)}{d!} = O(k^{d-1}).
\]
In [T2] (Theorem 3.6), the second author proved the following result:

**Theorem** Let $R$ be a standard graded ring of dimension $d \geq 2$ over a perfect field $K$ of characteristic $p > 0$, and let $I \subset R$ be a homogeneous ideal of finite co-length, which has a set of generators of the same degree. Let $M$ be a finitely generated graded $R$-module. Then
\[
\lim_{k \to \infty} \frac{e_{HK}(M, I^k) - e_0(M, I^k)/d!}{k^{d-1}} = \frac{e_0(M, I)}{2(d-2)!} - \frac{E_1(M, I)}{(d-1)!},
\]
where
\[
E_1(M, I) := \lim_{q \to \infty} e_1(M, I^{[q]}/q^d) exists.
\]
In particular, it implies
\[
\ell(M/I^{[q]k}M) = \left[ \frac{e_0(M, I)}{d!} k^d + \left( \frac{e_0(M, I)}{2(d-2)!} - \frac{E_1(M, I)}{(d-1)!} \right) k^{d-1} + o(k^{d-1}) \right] q^d + O((kq)^{d-1}).
\]

The above limit can be computed in the case of a nodal plane curve (due to [Mo2]), and in the case of elliptic curves and full flag varieties (due to [NT]). Other known cases are Hirzebruch surfaces ([T1]).

In this paper we study the same question for a projective toric variety $X$ of dimension $d-1 \geq 1$ over an algebraically closed field $K$ of characteristic $p > 0$, with a very ample $T$-Cartier divisor $D$. Here, by $HKd(X, D)$ (or $e_{HK}(X, D)$) for a pair $(X, D)$ we mean the HK density function (or HK multiplicity, respectively) of the associated homogeneous coordinate ring with respect to its graded maximal ideal.

It is well known that such a pair $(X, D)$ corresponds to a lattice polytope (that is, the convex hull of a finite set of lattice points) $P_D \subseteq \mathbb{M}_d \simeq \mathbb{R}^{d-1}$ (see [2.1]) for the definition).

For a pair $(K[H], I)$, where $K[H]$ is a toric ring (= normal semigroup ring) and $I$ is a monomial ideal $I$ (such that $\ell(K[H]/I) < \infty$), K. Watanabe (Theorem 2.1 of [W]) has proved that the $e_{HK}(K[H], I)$ is a rational number.

Later K. Eto (in [E]) proved the following result:

**Theorem** (Theorem 2.2, [E]) : Let $S$ be an affine semigroup and $a_1, \ldots, a_{\ast} \in S(\subset \mathbb{Z}^{N})$ elements such that $K[S]/J$ has finite length, where $J = (x^{a_1}, \ldots, x^{a_{\ast}})$. Let $C$ denote the convex rational polyhedral cone spanned by $S$ in $\mathbb{R}^{N}$ and $\mathcal{P} = \{p \in C \mid p \notin a_j + C$ for each $j\}$. Then
\[
e_{HK}(K[S]/J) = Vol(\mathcal{P}),
\]
where $\mathcal{P}$ is the closure of $\mathcal{P}$ and $Vol$ denotes the relative volume ([St2], p. 569).
For a toric pair \((X, D)\) as above (see Section 2 for the detailed theory), if \(C_D\) is the convex rational polyhedral cone spanned by \(P_D \times 1\) in \(M_\mathbb{R} \times \mathbb{R}\) and if
\[
P_D = \{p \in C_D \mid p \notin (u_j, 1) + C_D \text{ for every } u_j \in P_D \cap \mathbb{Z}^{d-1}\}
\]
then, by the above theorem of K. Eto, we have
\[
e_{HK}(X, D) = \text{Vol}(\overline{P}_D) \quad \text{and} \quad e_{HK}(X, kD) = \text{Vol}(\overline{P}_{kD}).
\]

As in [T3], we will study the asymptotic behaviour of \(e_{HK}(X, kD)\) (as \(k\) varies), via HK-density functions. However we do not use the results of [T3]; instead we directly interpret the HK density function (as in [T2]) for a toric pair \((X, D)\), in terms of \(P_D\):

**Theorem 1.1.** Let \(P_D\) denote the convex rational polyhedral cone spanned by \(P_D \times 1\) in \(M_\mathbb{R} \times \mathbb{R}\). Let
\[
P_D = \{p \in C_D \mid p \notin (u_j, 1) + C_D \text{ for every } u_j \in P_D \cap \mathbb{Z}^{d-1}\}.
\]
Then the Hilbert-Kunz density function \(HK_d(X, D)\) is given by the sectional volume of \(\overline{P}_D\) ([\(\overline{P}_D\) is the closure of \(P_D\)], i.e. precisely,
\[
HK_d(X, D)(\lambda) = \text{Vol}_{d-1}(\overline{P}_D \cap \{z = \lambda\}),
\]
for \(\lambda \geq 0\) (note that the relative volume and the volume are same here).

We prove the following key proposition:

**Proposition 1.2.** Let \((X, D)\) be a toric pair. Then, for \(\lambda \geq 0\),
\[
HK_d(X, kD)(\lambda + 1) = \frac{e_0(X, D)k^{d-1}}{(d-1)!}\varphi_{kD}(\lambda) + O(k^{d-2}),
\]
where \(\varphi_{kD} : [0, \infty) \to [0, 1]\) is the compactly supported continuous function given by
\[
\varphi_{kD}(\lambda) = \text{Vol}_{d-1}([W_v \times \{z = \lambda\}] \setminus \bigcup_{u \in \mathbb{Z}^{d-1}} [(u, 1) + C_{kD}]),
\]
for any vertex \(v \in \mathbb{Z}^{d-1}\), where \(W_v \subset \mathbb{R}^{d-1}\) (as in Notations 5.1) is the \(d-1\) dimensional unit cell at the vertex \(v\).

Note that \((u, 1) + C_{kD}\) is the translate of the cone \(C_{kD} \subseteq \mathbb{R}^d\) from its vertex \(0\) to the vertex \((u, 1)\). In fact (see Remark 5.6)
\[
(1.1) \quad \varphi_{kD}(\lambda) = \varphi_D(k\lambda), \quad \text{for all } \lambda \geq 0.
\]
Hence
\[
(1.2) \quad \int_0^\infty \varphi_{kD}(\lambda)d\lambda = \frac{1}{k} \int_0^\infty \varphi_D(\lambda)d\lambda.
\]
In otherwords, we have ‘replaced’ the continuous function \(HK_d(X, kD)\) by another continuous function \(\varphi(kD)\) which is ‘linear in \(k\’\) in the sense of (1.1) and (1.2).

Now the equality given in the above Proposition combined with Theorem 1.1 of [T2] gives the main result of this paper:

**Theorem 1.3.** For a projective toric variety \(X\) with a very ample \(T\)-Cartier divisor \(D\), we have
\[
\lim_{k \to \infty} \frac{e_{HK}(R, m^k) - e_0(R, m^k)/d!}{k^{d-1}} = \frac{e_0(R, m)}{(d-1)!} \int_0^\infty \varphi_D(\lambda)d\lambda,
\]
where \(\varphi_D : [0, \infty) \to [0, \infty)\), defined as before (for \(k = 1\)), is a compactly supported continuous function and is solely determined by the shape of the polytope \(P_D\) (as in (2.1)) associated to the toric pair \((X, D)\).
In fact, Theorem 6.3 states that among the set of $d-1$ dimensional toric pairs $(X, D)$, (the renormalized) limit of Theorem 1.3 achieves the minimum if and only if the polytope $P_D$ tiles the space $M_k = \mathbb{R}^{d-1}$ with lattice $M$. In other words the asymptotic behaviour of $e_{\text{HK}}(R, m^k)$ (relative to its usual multiplicity $e_0(R, m)$) as $k \to \infty$, characterizes the tiling property of the associated polytope $P_D$ (with the canonical lattice $\mathbb{Z}^{d-1}$). Similarly, the tiling property of any rational convex polytope can be formulated (Remark 6.6) in terms of this algebraic invariant (the renormalized limit).

It is easy to see that the polytope associated to the $d-1$ dimensional Segre self-product of the toric pair $(P^1, \mathcal{O}(m_0))$, for any $m_0 \geq 1$ tiles the space $\mathbb{R}^{d-1}$. Hence the renormalized limit for such toric pair achieves the minimum in any dimension. In particular this result (Remark 6.5) is also in the spirit of the well known conjecture of Watanabe-Yoshida (Conjecture 4.2, [WY2]).

Moreover, similar to the HKd functions, the function $\varphi_D$ turns out to have a multiplicative property on the set of toric pairs:

**Proposition 1.4.** Let $(X, D)$ and $(Y, D')$ be two toric pairs defined over the same perfect field $k$. Then

$$(1 - \varphi_{X \times Y, D \otimes D'}) = (1 - \varphi_{X, D})(1 - \varphi_{Y, D'}),$$

where $(X \times Y, D \otimes D')$ is the toric variety given by the Segre product of the toric varieties $(X, D)$ and $(Y, D')$ and $D \otimes D'$ denotes a divisor corresponding to the line bundle $\pi_1^* \mathcal{O}_X(D) \otimes \pi_2^* \mathcal{O}_Y(D')$, where $\pi_1 : X \times Y \to X, \pi_2 : X \times Y \to Y$ are the two projection morphisms.

We also compute the function $\varphi_D$ for all five smooth Fano toric surfaces with respect to their respective anticanonical divisors. Similarly one can explicitly compute $\varphi_D$, for every two-dimensional toric pair $(X, D)$. Hence, due to the multiplicative property (Proposition 1.4), one can compute $\varphi_{X, D}$, where $(X, D)$ is a Segre product of the two dimensional toric pairs. In particular, one can compute the limit (Theorem 1.3) in these cases.

The organization of the paper is as follows.

In Section 2 we recall some notations about toric varieties (following Fulton [Fu]), in a form useful for us.

In Section 3 we give a self contained proof of the fact that the sectional volume function $\phi(\lambda) = \text{Vol}_{d-1} \hat{P} \cap \{z = \lambda\}$, where $\hat{P}$ is a $d$-dimensional convex polytope in $\mathbb{R}^d$ with no facets lying in hyperplanes parallel to $\{z = 0\}$, is a continuous piecewise polynomial function of $\lambda$.

In Section 4 we give a proof of Theorem 1.1 relating the HKd function for $(X, D)$ with the sectional volume of $P_D$. We prove that $P_D$ can be written as a finite union of convex polytopes with disjoint interiors, and none of the facets of the involved polytopes lie in the $\{z = \lambda\}$ hyperplane for any $\lambda$. Now, owing to the fact that the HKd function and the sectional volume function (as given in Section 3) are both continuous, we only need to check the equality for a suitable dense set, namely the set of rationals $\{m/p^n \mid m, n \in \mathbb{Z}_{\geq 0} \subseteq \mathbb{R}_{\geq 0}.$

Section 5 involves purely convex geometry. In this section we prove that, for any integer $k \geq 1$, $P_{kD} \cap \{z = 1 + \lambda\} = \emptyset$, for $\lambda \geq l/k$ (where $l$ is the number of the vertices of the polytope $P_D$). This implies $P_{kD} \subseteq P_{(k+1)D} \times \mathbb{R}_{\geq 0}$, i.e., $P_{kD}$ lies ‘approximately’ in a cylinder over the polytope $P_{kD}$. We also prove various properties of the function $\varphi_{kD}$ here. Lemma 5.5 implies that, in the definition of $\varphi_{kD}$ (defined with respect to a fixed unit cell $W_v$ given by a vertex $v \in \mathbb{Z}^{d-1}$, as in Proposition 1.2), we can replace the infinite set $\{u \in \mathbb{Z}^{d-1} \mid B(v, r) \} \subseteq \mathbb{R}_{\geq 0}$ by a finite set of a fixed size, i.e., by

$$\{u \in \mathbb{Z}^{d-1} \cap B(v, r) \mid W_v \subseteq B(v, r),$$

where $r$ is independent of $k$. This implies that

$$\bigcup_{v \in S} ((W_v \times \mathbb{R}_{\geq 1}) \cap P_{kD}) \subseteq P_{kD} \cap \{z \in \mathbb{R}_{\geq 1}\} \subseteq \bigcup_{v \in S'} ((W_v \times \mathbb{R}_{\geq 1}) \cap P_{kD}),$$

where the set $S$ and $S' \subseteq \mathbb{Z}^{d-1}$ have sufficiently large overlap (note that $W_v \times \mathbb{R}_{\geq 1}$ is the cylinder over the unit cell $W_v$), and for ‘general’ $v$ from either set,

$$\text{Vol}_{d-1} ((W_v \times \{z = 1 + \lambda\}) \cap P_{kD}) = \varphi_{kD}(\lambda), \text{ for all } \lambda \geq 0.$$
where we already know that $\varphi_{kD}$ is ‘linear’ with respect to $k$ (see (1.1) and (1.2)).

In Section 6, we use the above results to prove the key Proposition 1.2, which replaces $HKd(X, kD)$ by $\varphi_{kD}$ upto $O(k^{d-2})$. Next in this section we prove the main Theorem 1.3 and Proposition 1.4 gives the multiplicative property of the function $\varphi_D$. Theorem 6.3 and Remark 6.6 relate the tiling of $P_D$ with lattice $M$ and the asymptotic growth of the HK multiplicity for $(X, D)$.

Section 7 consists of examples. We prove Theorem 1.3 for a toric pair $(\mathbb{P}^1, D)$, which takes care of one dimensional toric pairs. We also compute $\varphi_D$ (and hence the limit (Theorem 1.3), for the smooth Fano toric surfaces with respect to their anticanonical divisors.

2. Preliminaries

Henceforth we assume that $K$ is an algebraically closed field of char. $p > 0$. We follow the notations from [Fu]. Let $N$ be a lattice (which is isomorphic to $\mathbb{Z}^n$) and let $M = Hom(N, \mathbb{Z})$ denote the dual lattice with a dual pairing $\langle \cdot, \cdot \rangle$. Let $T = Spec(K[M])$ be the torus with character lattice $M$. Let $(X, D)$ denote a complete toric variety over $K$ with fan $\Delta \subset N_\mathbb{R}$ and very ample $T$-divisor $D$ on $X$.

We recall that the $T$-divisors on $X$ (the irreducible subvarieties of codimension 1 which are $T$-stable) correspond to one dimensional cones (which are edges/rays of $\Delta$) of $X$. If $\tau_1, \ldots, \tau_n$ denote the edges of the fan $\Delta$, then these divisors are the orbit closures $D_i = V(\tau_i)$. A $T$-divisor $D = \sum_{i} a_i D_i$ (note that $a_i$ are integers) determines a lattice polytopes in $M_\mathbb{R}$ defined by

$$P_D = \{ u \in M_\mathbb{R} \mid \langle u, \tau_i \rangle \geq -a_i \text{ for all } i \}$$

and the induced embedding of $X$ in $\mathbb{P}^{r-1} \mathbb{K}$ is given by

$$\phi = \phi_D : X \to \mathbb{P}^{r-1} \mathbb{K}, \quad x \mapsto (\chi^{u_1}(x) : \cdots : \chi^{u_r}(x)),$$

where $P_D \cap M = \{ u_1, u_2, \ldots, u_r \}$.

Moreover the global sections of the line bundle $\mathcal{O}(D)$ are

$$\Gamma(X, \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} K \chi^u.$$

For any integer $m \geq 1$, we have $P_{mD} = mP_D$ (see Page 67 in [Fu]).

For $(X, D)$ and $P_D$ as above. Consider $\sigma$ the cone in $N \times \mathbb{Z}$ whose dual $\sigma^\vee$ is the cone over $P_D \times 1$ in $M \times \mathbb{Z}$. Then the affine variety $U_{\sigma}$ corresponding to the cone $\sigma$ is the affine cone of $X$ in $k^*_{\mathbb{K}}$.

If $S$ is the semigroup generated by $\{(u_1, 1), \ldots, (u_r, 1)\}$ then the homogeneous coordinate ring of $X$ (with respect to this embedding) is $K[S] = K[\chi^{(u_1, 1)}, \ldots, \chi^{(u_r, 1)}]$. Note that there is an isomorphism of graded rings (see Proposition 1.1.9, [CLS])

$$\frac{K[Y_1, \ldots, Y_r]}{I} \simeq K[\chi^{(u_1, 1)}, \ldots, \chi^{(u_r, 1)}] = K[S],$$

where, the kernel $I$ is generated by the binomials of the form

$$Y_1^{a_1}Y_2^{a_2} \cdots Y_r^{a_r} - Y_1^{b_1}Y_2^{b_2} \cdots Y_r^{b_r},$$

where $a_1, \ldots, a_r, b_1, \ldots, b_r$ are nonnegative integers satisfying the equations

$$a_1u_1 + \cdots + a_ru_r = b_1u_1 + \cdots + b_ru_r \quad \text{and} \quad a_1 + \cdots + a_r = b_1 + \cdots + b_r.$$

**Definition 2.1.** By a toric pair $(X, D)$, we mean $X$ is a projective variety of dimension $d-1 \geq 1$ over a field $K$ with a very ample $T$-divisor $D$. Moreover $P_D$ denotes the associated lattice convex polytope as defined by (2.1). The homogeneous coordinate ring of $X$ with respect to this embedding is

$$K[S] = K[\chi^{(u_1, 1)}, \ldots, \chi^{(u_r, 1)}],$$

where $P_D \cap M = \{ u_1, \ldots, u_r \}$ and $S$ is the semigroup generated by $\{(u_1, 1), \ldots, (u_r, 1)\}$. 
Note that due to this isomorphism, we can consider $K[S]$ as a standard graded ring, where \( \deg \chi^{(n_\nu)} = 1. \) While dealing with the cone in \( M_\mathbb{R} \times \mathbb{R} \simeq \mathbb{R}^d \), we denote the last co-ordinate as \( x_d \) or \( z \), interchangeably.

**Remark 2.2.** We recall the following well known fact (see [St2], Exercise 33 and [St1], Proposition 4.6.30).

If \( P \) is a \( d \)-dimensional rational convex polytope in \( \mathbb{R}^m \) and \( i(P, n) = \#(nP \cap \mathbb{Z}^m) \) then

\[
i(P, n) = c_d(n)n^d + c_{d-1}n^{d-1} + \cdots + c_0(n),
\]

where \( c_0, \ldots, c_d \) are periodic functions of \( n \) and \( c_d(n) = \text{Vol}_d(P) \).

### 3. Volume of “slices” of convex polytope

Let \( P \) be a \( d \)-dimensional convex polytope in \( \mathbb{R}^d \). For \( Q \subseteq \mathbb{R}^d \) we denote \( Q \cap \{ z = \lambda \} = Q \cap \{ (x, \lambda) \mid x \in \mathbb{R}^{d-1} \} \subseteq \mathbb{R}^d \).

Our goal in this section, is to describe the behaviour of the function \( \phi : (\infty, \infty) \rightarrow [0, \infty) \) given by \( \phi(t) = \text{Vol}_{d-1}(P \cap \{ z = t \}) \).

**Definition 3.1.** Let \( \pi : \mathbb{R}^d \rightarrow \mathbb{R} \) be the projection map given by projecting to the last co-ordinate \( z \). Then we denote the set \( \pi(\text{vertex set of } P) = \{ \tau_1, \ldots, \tau_m \} \), where \( \tau_1 < \tau_2 < \cdots < \tau_m \).

**Lemma 3.2.**

1. The support of \( \phi(t) \) is a compact connected interval.

2. Suppose \( P \) has no supporting hyperplane parallel to the hyperplane \( \{ z = 0 \} \). Then \( \phi(\tau_0) = \phi(\tau_m) = 0 \).

**Proof.** First we prove that, if \( \{ z = \alpha \} \) is a hyperplane in \( \mathbb{R}^d \) such that dimension \( \{ z = \alpha \} \cap P \leq d - 2 \), then it cannot pass through the interior of the polytope and therefore \( P \) lies entirely in one of the closed half spaces defined by \( \{ z = \alpha \} \).

Suppose by contradiction, \( x \in \{ z = \alpha \} \cap \text{int}(P) \). Let \( B^d(x, \epsilon) \) be a small ball around \( x \) of radius \( \epsilon \) inside \( P \). Then \( B^d(x, \epsilon) \cap \{ z = \alpha \} \cap P \) is a nonempty \( d - 1 \) dimensional ball. Hence dimension \( \{ z = \alpha \} \cap P \) is \( d - 1 \), which is a contradiction.

Suppose the support of \( \phi \) is not connected then we have \( a < x < b \) in \( \mathbb{R} \) such that \( \phi(a) \neq 0, \phi(b) \neq 0 \) and \( \phi(x) = 0 \). But then dimension \( \{ z = x \} \cap P \leq d - 2 \), therefore \( P \) lies in one side of the hyperplane \( \{ z = x \} \), which is a contradiction since both \( \phi(a) \) and \( \phi(b) \) are nonzero. Further, since \( P \) is a bounded polytope, support of \( \phi \) is a compact interval \( \subseteq [\tau_0, \tau_m] \).

Suppose \( \phi(\tau_m) \neq 0 \) then \( \dim P \cap \{ z = \tau_m \} = d - 1 \). Since for any \( \epsilon > 0 \), \( P \cap \{ z = \tau_{m+\epsilon} \} = 0 \), the hyperplane \( \{ z = \tau_m \} \) does not pass through the interior of \( P \). Hence \( \{ z = \tau_m \} \) is a supporting hyperplane of \( P \) parallel to \( \{ z = 0 \} \), which is a contradiction. Similar proof shows \( \phi(\tau_0) \) is 0.

A volume formula \( \phi \) for “slices” of a simplex has been derived by C.A. Micchelli ([Mi], Chapter 4) in more general context, using the univariate B-splines. For details about B-splines and volume of slices, see [CS]. Here we give a simpler self contained proof, which is suited to our case.

**Lemma 3.3.** Let \( S_{ik} \subseteq \mathbb{R}^d \) be a \( d \)-simplex such that the set of vertices of \( S_{ik} \) are contained in \( \{ z = \tau_i \} \cup \{ z = \tau_{i+1} \} \). Then the function \( \phi_{ik} : [\tau_i, \tau_{i+1}] \rightarrow \mathbb{R} \), given by \( \lambda \mapsto \text{Vol}(S_{ik} \cap \{ z = \lambda \}) \) is a polynomial function of degree \( \leq d - 1 \) in \( \lambda \).

**Proof.** By the hypothesis \( S_{ik} \cap \{ z = \tau_i \} \simeq \Delta_r \), is \( r \)-simplex given by the vertices \( v_0, \ldots, v_r \) and \( S_{ik} \cap \{ z = \tau_{i+1} \} \simeq \Delta_s \), is \( s \)-simplex given by the vertices \( w_0, \ldots, w_s \), where \( \{ v_0, \ldots, v_r, w_0, \ldots, w_s \} \) the vertex set of \( S_{ik} \). Note that since \( r + 1 + s + 1 = d + 1 \), we have \( r + s = d - 1 \).

Let \( \lambda \in [\tau_i, \tau_{i+1}] \). Let \( \lambda_1 = \frac{\tau_{i+1} - \lambda}{\tau_{i+1} - \tau_i} \), and \( \lambda_2 = \frac{\lambda - \tau_i}{\tau_{i+1} - \tau_i} \). Then

**Claim 1** \( S_{ik} \cap \{ z = \lambda \} = \{ \lambda_1(p_0 + v_0) + \lambda_2(p_1 + w_0) \mid p_0 \in \Delta_r - v_0, \ p_1 \in \Delta_s - w_0 \} \).
Proof of the claim: Any element \( p \) of \( S_{ik} \cap \{ z = \lambda \} \) can be written as \( p = \sum_{i=0}^{r} a_i v_i + \sum_{j=0}^{s} b_j w_j \), where \( a_i, b_j \geq 0 \) and \( \sum_{i=0}^{r} a_i + \sum_{j=0}^{s} b_j = 1 \). Therefore

\[
p = \lambda_1 \left( \frac{\sum_{i=0}^{r} a_i (v_i - v_0)}{\lambda_1} \right) + \lambda_1 v_0 + \lambda_2 \left( \frac{\sum_{j=0}^{s} b_j (w_j - w_0)}{\lambda_2} \right) + \lambda_2 w_0.
\]

This proves the claim.

**Claim (2)** \( \{ v_1 - v_0, \ldots, v_r - v_0, w_1 - w_0, \ldots, w_s - w_0 \} \) is a basis of \( \mathbb{R}^{d-1} \).

Proof of the claim: Note that for a choice of \( \lambda \in (\tau_i, \tau_{i+1}) \), the convex polytope \( S_{ik} \cap \{ z = \lambda \} \) is \( d - 1 \) dimensional (as the hyperplane \( \{ z = \lambda \} \) contains no vertices of \( S_{ik} \), but the hyperplanes \( \{ z = \tau_i \} \) and \( \{ z = \tau_{i+1} \} \) both contain some vertices of \( S_{ik} \), we deduce that the hyperplane \( \{ z = \lambda \} \) intersects the interior of \( S_{ik} \)). By Claim (1), the set of \( d - 1 \) vectors \( \{ v_1 - v_0, \ldots, v_r - v_0, w_1 - w_0, \ldots, w_s - w_0 \} \) generate the \( d - 1 \) dimensional convex set \( S_{ik} \cap \{ z = \lambda \} \). This proves the claim.

Let \( \Delta_{rs} \) denote the image of the map \( \psi_{r,w} : \Delta_r \times \Delta_s \rightarrow \mathbb{R}^{d-1} \) given by \( (p_0, p_1) \mapsto p_0 + p_1 \).

Since \( \Delta_r \) and \( \Delta_s \) are convex polytopes, the set \( \Delta_{rs} \) is a convex polytope and of dimension \( d - 1 \). Now for a given \( \lambda \in (\tau_i, \tau_{i+1}) \), we can define the linear transformation \( T_{\lambda} : \mathbb{R}^{d-1} 
\rightarrow \mathbb{R}^{d-1} \) given by \( \sum \alpha_i (v_i - v_0) + \sum \beta_j (w_j - w_0) \mapsto \lambda_1 \sum \alpha_i (v_i - v_0) + \lambda_2 \sum \beta_j (w_j - w_0) \) (this is a well defined map due to Claim (2)).

Note that, for any \( \lambda \in (\tau_i, \tau_{i+1}) \), \( S_{ik} \cap \{ z = \lambda \} \) is a polytope of degree \( d - 1 \) on \( \tau_i, \tau_{i+1} \), for \( i = 0, \ldots, m - 1 \). Moreover \( \phi \) is continuous on all of \( \mathbb{R} \).

**Theorem 3.4.** Let \( P \) be a bounded full dimensional convex polytope in \( \mathbb{R}^d \) which has no supporting hyperplane parallel to the hyperplane \( \{ z = 0 \} \). Then

1. the function \( \phi(t) = \text{Vol}_{d-1}(P \cap \{ z = t \}) \) is a polynomial of degree \( d - 1 \) on \( (\tau_i, \tau_{i+1}) \), for \( i = 0, \ldots, m - 1 \). Moreover \( \phi \) is continuous on all of \( \mathbb{R} \).

**Proof.** (1) Let \( P_{[\tau_i, \tau_{i+1}]} = \{ p \in P \mid \tau_i \leq \pi(p) \leq \tau_{i+1} \} \). Note that \( P_{[\tau_i, \tau_{i+1}]} \) is a convex polytope with vertices only at the level \( \{ z = \tau_i \} \) and \( \{ z = \tau_{i+1} \} \). For \( t \in (\tau_i, \tau_{i+1}) \), we have \( \phi(t) = \phi \mid_{P_{[\tau_i, \tau_{i+1}]}}(t) \). Therefore, it is enough to show that \( \phi_i := \phi \mid_{P_{[\tau_i, \tau_{i+1}]}} : (\tau_i, \tau_{i+1}) \rightarrow (0, \infty) \) is a polynomial function in \( \lambda \) of degree \( d - 1 \), for \( i = 0, \ldots, m - 1 \).

We take a triangulation (see [L]) of \( P_{[\tau_i, \tau_{i+1}]} \) such that vertices of each triangulating simplex are vertices of \( P_{[\tau_i, \tau_{i+1}]} \) itself.

Hence we can triangulate \( P_{[\tau_i, \tau_{i+1}]} = \bigcup_{k=1}^{L_i} S_{ik} \) in \( d \)-simplices such that the vertex set of each simplex \( S_{ik} \) is a subset of the vertex set of \( P_{[\tau_i, \tau_{i+1}]} \). Since vertices of \( S_{ik} \) lie in \( \{ z = \tau_i \} \) and \( \{ z = \tau_{i+1} \} \), if \( t \in (\tau_i, \tau_{i+1}) \), the plane \( \{ z = t \} \) does not contain any face of \( S_{ik} \). Therefore dimension of \( S_{ik} \cap S_{ik'} \cap \{ z = t \} \) is \( < d - 1 \). For \( t = \tau_i \), if dimension of \( S_{ik} \cap S_{ik'} \cap \{ z = \tau_i \} \) is \( d - 1 \), then dimension of both \( S_{ik} \cap \{ z = \tau_i \} \) and \( S_{ik'} \cap \{ z = \tau_i \} \) is \( d - 1 \), it follows that \( S_{ik} \cap \{ z = \tau_i \} = S_{ik'} \cap \{ z = \tau_i \} \). Hence for \( x \in S_{ik} \cap \{ z = \tau_i \} \), one can find \( \epsilon > 0 \), such that \( B^d(x, \epsilon) \cap \{ z \geq \tau_i \} \subset S_{ik} \cap S_{ik'} \), a contradiction. We show that, for \( t \in (\tau_i, \tau_{i+1}] \)

\[
\phi(t) = \sum_{k=1}^{L_i} \phi_{ik}(t),
\]

where \( \phi_{ik}(t) = \text{Vol}_{d-1}(S_{ik} \cap \{ z = t \}) \) is the volume function for the simplex \( S_{ik}, k = 1, \ldots, L_i \). Enough to show

\[
\text{Vol}_{d-1}(\bigcup_{k=1}^{L_i} S_{ik} \cap \{ z = t \}) = \sum_{k=1}^{L_i} \phi_{ik}(t)
\]
for $1 \leq l \leq L_i$. This easily follows by induction because dimension of $S_{i_k} \cap S_{i_k'} \cap \{z = t\}$ is $< d - 1$. This proves part one of the theorem.

For the second part, it is enough to show that $\phi$ is continuous at $\tau_0$ and $\tau_m$. Since $\phi(\tau_0) = \phi_0(\tau_0) = 0$, so is $\phi_{ok}(\tau_0)$, for $k = 1, \ldots, L_0$. By Lemma 3.3 each $\phi_{ok}$ is continuous at $\tau_0$. Hence so is $\phi$. Similarly, $\phi$ is continuous at $\tau_m$. $\square$

4. Hilbert-Kunz-Density Function

In [T2], the second author has defined the notion of Hilbert-Kunz Density function, and given its relation with the HK-multiplicity. We use the following interpretation of the HK multiplicity via the HK density function.

**Theorem 4.1.** (Theorem 1.1 in [T2]) Let $R$ be a standard graded Noetherian ring of dimension $d \geq 2$ over an algebraically closed field $K$ of char $p > 0$, and let $I \subset R$ be a homogeneous ideal such that $l(R/I) < \infty$. For $n \in \mathbb{N}$ and $q = p^n$, let $f_n : [0, \infty) \rightarrow [0, \infty)$ be defined as

$$f_n(R, I)(x) = \frac{1}{q^d - 1} l(R/I^q)|_{x}.$$  

Then $\{f_n(R, I)\}$ converges uniformly to a compactly supported continuous function $f_{R, I} : [0, \infty) \rightarrow [0, \infty)$, where $f_{R, I}(x) = \lim_{n \rightarrow \infty} f_n(R, I)(x)$ and

$$e_{HK}(R, I) = \int_0^\infty f_{R, I}(x) \, dx.$$  

**Definition 4.2.** For a given pair $(X, D)$ (Definition 2.1) we have the associated standard graded ring $K[S]$. We define the associated density function $HK_d(X, D) = HK_d(K[S], m)$, where $m$ is the graded maximal ideal of $K[S]$. Therefore, for $q = p^n$ where $n \geq 1$,

$$HK_d(X, D)(\lambda) = \lim_{n \rightarrow \infty} f_n(\lambda) = \lim_{n \rightarrow \infty} \frac{1}{q^d - 1} l\left(\frac{K[S]}{m^q}\right)|_{x=\lambda}.$$  

**Notations 4.3.** In $\mathbb{R}^d$, we denote the last ($d^{th}$) coordinate by $z$. Let $\lambda \in \mathbb{R}_{\geq 0}$. Then

1. For $P \subseteq \mathbb{R}^{d-1}$ we denote $P \times \{z = \lambda\} = (P \times \mathbb{R}) \cap \{(z, \lambda) \mid z \in \mathbb{R}^{d-1}\} \subseteq \mathbb{R}^d$.
2. For $Q \subseteq \mathbb{R}^d$ we denote $Q \cap \{z = \lambda\} = Q \cap \{(z, \lambda) \mid z \in \mathbb{R}^{d-1}\} \subseteq \mathbb{R}^d$.

**Remark 4.4.** In the proof of the earlier stated Theorem of K. Eto in [E] (see introduction), he has asserted that $P$ is a finite union of rational polytopes, which do not intersect at interior points. In the following lemma we give a detailed proof of this in Lemma 4.5 (1).

**Lemma 4.5.** Let

$$P_D = C_D \setminus \left( \bigcup_{u_i \in P_D \cap \mathbb{Z}^{d-1}} ((u_i, 1) + C_D) \right).$$

Then

1. $P_D$ is a finite union of rational polytopes $P_1, P_2, \ldots, P_s$ containing the origin such that $P_i \cap P_j$ is a rational polytope of dimension $< d$ if $i \neq j$. Moreover
2. (a) $\dim(\partial(P_j) \cap \{z = a\}) < d - 1$, where for a closed set $A \subseteq \mathbb{R}^d$, the set $\partial(A)$ denotes its boundary,
   
   (b) $\dim(P_i \cap P_j \cap \{z = a\}) < d - 1$, for any $a \in \mathbb{R}$.

**Proof.** For $d - 1 = 1$, the toric pair $(X, D) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$, for some integer $n \geq 1$. Therefore the lemma is obvious from Example 7.1. Henceforth we can assume that $d - 1 \geq 2$.

Let $P_D \subseteq \mathbb{R}^{d-1}$ be the convex polytope of dimension $d - 1$ associated to the pair $(X, D)$. Without loss of generality we assume that $P_D$ has the origin as one of the vertices. Let $C_D = \text{Cone}(P_D \times \{z = 1\})$.

Part (1): Let $S = \{F_j\}_j$ be the set of all subcones of $C_D$ obtained by dividing $C_D$ by the set of hyperplanes

$$W_D = \{H_{iu} \mid C_{0i} \in \{d - 3 \text{ faces of } P_D\}, \ u \in P_D \cap \mathbb{Z}^{d-1}\},$$
where
\[ H_iu = \text{the affine span of } \{(v_{ik}, 1), (u, 1), (\emptyset) \mid v_{ik} \in \text{the vertex set of } C_{0i}\}. \]

Thus, the \( F_j \) are the closures of the connected components of \( C_D \cup_{H \in W_D} H \).

**Claim** For each \( F_j \in S \) and for each \( u \in P_D \cap \mathbb{Z}^{d-1} \), the set \( F_j \cap [(u, 1) + C_D]^c \) is convex.

We assume the claim for the moment.

Now we have
\[
P_D = C_D \setminus \cup_{u_i \in P_D \cap \mathbb{Z}^{d-1}} ((u_i, 1) + C_D) = \cup_{j} F_j \setminus \{\cup_{u_i \in P_D \cap \mathbb{Z}^{d-1}} ((u_i, 1) + C_D)\}.
\]

Hence, Part (1) of the lemma follows by taking
\[
P_j = F_j \setminus \cup_{u_i \in P_D \cap \mathbb{Z}^{d-1}} ((u_i, 1) + C_D) = \bigcap_{u_i \in P_D \cap \mathbb{Z}^{d-1}} F_j \setminus ((u_i, 1) + C_D).
\]

**Proof of the claim:** First we prove that for given \( F_j \in S \) and \( u \in P_D \cap \mathbb{Z}^{d-1} \), there is a facet \( C_i \) of \( P_D \) such that \( F_j \subset C^d(C_i, u) \), where \( C^d(C_i, u) \) is the cone generated by \( (\emptyset), (u, 1) \) and all \( (v, 1) \), where \( v \) is a vertex of \( C_i \).

Consider the set \( \{C^d(C_i, u) \mid C_i \text{ is a facet of } P_D, \ u \notin C_i\} \), so that, by construction, \( C^d(C_i, u) \) is a \( d \)-dimensional cone. The facets of any such \( C^d(C_i, u) \), other than \( C^{d-1}(C_i) = \text{Con} \overline{C_i} \), are given by the set \( \{H(C_{ij}) \cap C^d(C_i, u) \mid C_{ij} \in \{\text{facets of } C_i\}\} \), where
\[
H(C_{ij}) = \text{the affine span of } \{(v, 1), (\emptyset), (u, 1) \mid v \in \text{vertex set of } C_{ij}\}
\]
are hyperplanes. Since any such \( H_{ij} \in W_D \), any such cone \( C^d(C_i, u) \) is a union of some subset of \( S \). On the other hand note that, for a given \( u \in P_D \) we have \( C_D = \bigcup_i (C^d(C_i, u)) \), where \( C_i \) are the facets of \( P_D \), and the interiors of the \( C^d(C_i, u) \) are disjoint.

Hence given \( F_j \in S \) and \( u \in P_D \cap \mathbb{Z}^{d-1} \) there is a facet \( C_i \) of \( P_D \) such that \( F_j \subset C^d(C_i, u) \).

Now we prove the convexity of the set \( F_j \cap [(u, 1) + C_D]^c \).

Fix a facet \( C_i \) with \( F_j \subset C^d(C_i, u) \). Let \( x, y \in F_j \cap [(u, 1) + C_D]^c \). Then \( x, y \in C^d(C_i, u) \cap [(u, 1) + C_D]^c \), therefore we must have expressions
\[
x = \alpha_1(u, 1) + c_1 \quad \text{and} \quad y = \alpha_2(u, 1) + c_2, \quad \text{where} \quad c_1, c_2 \in C^{d-1}(C_i) \quad \text{and} \quad 0 \leq \alpha_1, \alpha_2 < 1.
\]

This implies that if \( z \) is any point in the line segment joining \( x \) and \( y \) then \( z = l_0(u, 1) + c_3 \), where \( 0 \leq l_0 < 1 \) and \( c_3 \in C^{d-1}(C_i) \).

Since \( F_j \) is convex, \( z \in F_j \). So we need to prove that \( z \in [(u, 1) + C_D]^c \).

Suppose \( z \in [(u, 1) + C_D] \). Then we have \( z = (u, 1) + c \), where \( c \in C_D \). This implies
\[
(1 - l_0)(u, 1) + c = c_3 \in C^{d-1}(C_i) \cap [(1 - l_0)(u, 1) + C_D].
\]

Now \( C^d(C_i, u) \) is a \( d \)-dimensional cone, which implies \( (u, 1) \notin C^{d-1}(C_i) \). Moreover \( C^{d-1}(C_i) \) is a facet of \( C_D \). Hence we have a contradiction by the claim given below. Therefore we deduce that \( z \in [(u, 1) + C_D]^c \). This proves that \( z \in F_j \cap [(u, 1) + C_D]^c \).

Now the convexity of the set \( F_j \cap [(u, 1) + C_D]^c \) follows from Lemma 4.6 given below.

**Part (2):** If \( C_1 \) and \( C_2 \) are sets in \( \mathbb{R}^d \) then \( \partial(C_1 \cap C_2) \subseteq \partial(C_1) \cup \partial(C_2) \), (where \( \partial(C) \) denotes the boundary of \( C \)). Therefore for \( P_j \) as above, we have
\[
\partial(P_j) \subseteq \partial(F_j) \cup_{u_i \in P_D \cap \mathbb{Z}^{d-1}} \partial((u_i, 1) + C_D)^c \subseteq \partial(F_j) \cup_{u_i \in P_D \cap \mathbb{Z}^{d-1}} \partial((u_i, 1) + C_D).
\]

Therefore
\[
\partial(P_j) \subseteq \text{facet of } (F_j) \cup_{u_i \in P_D \cap \mathbb{Z}^{d-1}} \text{facet of } ((u_i, 1) + C_D).
\]

We note that any facet of \( (u_i, 1) + C_D \) is a translate of a facet of \( C_D \) by the point \( (u_i, 1) \). On the other hand any facet of \( F_j \) is a subset of an element of \( W_D \), where the set \( W_D \) is defined as in (7.2) above. In particular for any facet \( F \) from these set of facets, we have \( \dim (F \cap \{z = a\}) < d - 1 \), for any \( a \in \mathbb{R} \). This proves part (2)(a). Part (2)(b) follows from (a), as for \( i \neq j \), the convex polytopes \( P_i \) and \( P_j \) intersects only at their boundary. This completes the proof of the lemma. \( \square \)
Lemma 4.6. Let \( \hat{u} = (u, 1) \in C_D \) such that \( \hat{u} \notin F \), where \( F \) is a facet of \( C_D \). Then for any \( \epsilon > 0 \), we have \([\hat{u} + C_D] \cap F = \phi\).

Proof. Note that \( F = H \cap C_D \), for some hyperplane \( H = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_i = 0, \ a_i \in \mathbb{R}\} \). Without loss of generality, we assume that \( C_D \subseteq H_+ = \{x \in \mathbb{R}^d \mid \sum a_i x_i \geq 0\} \). Therefore for any \( m = (m_1, \ldots, m_d) \in C_D \) we have \( \sum a_i m_i \geq 0 \). Moreover, since \( \hat{u} \notin F \), we have \( \sum a_i (\epsilon(u_i) + m_i) > 0 \). Hence \( \epsilon(u) + m \in H_+ \setminus H \subseteq F^c \). In particular \( \{\epsilon(u) + C_D \} \cap F = \phi \). This proves the lemma. \( \square \)

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We note that
\[
np_D = \{p \in C_D \mid p \notin n(u_i, 1) + C_D \text{ for all } i = 1, \ldots, r\}.
\]
Let \( S' \) be the normalization of the monoid \( S \). Hence \( K[S'] \) is the integral closure of \( K[S] \) (Theorem 4.39, [BG]). Hence there exists \( N_0 \in \mathbb{Z} \) such that \( K[S]_n = K[S']_n \) for all \( n \geq N_0 \) (by Exercise 5.14, [Hart]). Hence, for every \( \lambda \in \mathbb{R} \), there exists \( n_\lambda \in \mathbb{N} \) such that for all \( n \geq n_\lambda \), we have
\[
\ell_{K[S]} \left( \frac{K[S]}{(Y^n_1, \ldots, Y^n_r)} \right)_{[n\lambda]} = \ell_{K[S']} \left( \frac{K[S']}{(Y^n_1, \ldots, Y^n_r)} \right)_{[n\lambda]}.
\]
Since \( C_D \cap Z^d = S' \) (by Proposition 2.22, [BG]),
\[
np_D \cap Z^d = \{p \in S' \mid p \notin n(u_i, 1) + C_D \text{ for every } u_i\}.
\]
Thus for \( n \geq n_\lambda \),
\[
\ell_{K[S]} \left( \frac{K[S]}{(Y^n_1, \ldots, Y^n_r)} \right)_{[n\lambda]} = \#(np_D \cap \{z = [n\lambda]\}).
\]
We denote
\[
i(P_D, n) = \#(np_D \cap \mathbb{Z}^d) \text{ and } i(P_D, n, m) = \#(np_D \cap \{z = m\} \cap \mathbb{Z}^{d-1})\).
\]

By Lemma 4.5, we have \( P_D = P_1 \cup P_2 \cup \cdots \cup P_s \), where \( P_1, P_2, \ldots, P_s \) are convex rational polytopes such that \( \dim (P_i \cap \mathbb{Z}^d \cap \{z = a\}) < d - 1 \) and \( \dim (\partial(P_i) \cap \{z = a\}) < d - 1 \), for every \( a \in \mathbb{R} \).

Claim. If \( Q \) is a \( d \)-dimensional convex polytope then for given \( \lambda = m_0/q_0 \), where \( q_0 = p_{n_0}^n \), for some \( n_0 \geq 1 \) and \( q = p^n \), we have,
\[
\lim_{q \to \infty} \frac{i(Q, q, [q\lambda])}{q^{d-1}} = \text{Vol}_{d-1}(Q \cap \{z = \lambda\}).
\]
Proof of the claim. Let \( q = p^n \), where \( n \geq n_0 \). Note that we have \([q\lambda] = q m_0/q_0 \). Therefore
\[
i(Q, q, [q\lambda]) = i(qQ \cap \{z = q m_0/q_0\}) = i(Q', q/q_0),
\]
where \( Q' = (q_0 Q \cap \{z = m_0\}) \). Now, by Remark 2.2
\[
\lim_{q \to \infty} \frac{i(Q, q, [q\lambda])}{q^{d-1}} = \lim_{q \to \infty} \frac{i(Q', q/q_0)}{q^{d-1}} = \frac{\text{Vol}_{d-1}(Q')}{q_0^{d-1}} = \text{Vol}_{d-1}(Q \cap \{z = \lambda\}).
\]
This proves the claim.

Let \( P_{\leq j} = P_1 \cup \cdots \cup P_j \) for \( 1 \leq j \leq s \). Then
\[
i(P_{\leq j}, q, [q\lambda]) = i(P_{\leq j-1}, q, [q\lambda]) + i(P_{j}, q, [q\lambda]) - i([P_{\leq j-1} \cap P_j], q, [q\lambda]).
\]
Now
\[
i([P_{\leq j-1} \cap P_j], q, [q\lambda]) = i \left( \frac{q}{q_0} [q_0 (P_{\leq j-1} \cap P_j) \cap \{z = m_0\}] \right).
\]
Therefore, by Lemma 4.5
\[
\lim_{q \to \infty} \frac{i([P_{\leq j_0-1} \cap P_{j_0}]; q, [q\lambda])}{q^{d-1}} = \text{Vol}_{d-1}([P_{\leq j_0-1} \cap P_{j_0}] \cap \{z = \frac{m_0}{q_0}\}) = 0.
\]
Therefore, by Theorem 1.1 of [T2], we have
\[
HKd(X, D)(\lambda) = \lim_{n \to \infty} f_n(\lambda) = \lim_{n \to \infty} \frac{i(P_D, q_1, [q\lambda])}{q^{d-1}} = \lim_{n \to \infty} \sum_j \frac{i(P_j, n, [q\lambda])}{q^{d-1}}
\]
where, for the \(d\) dimensional polytope \(P_j\), the function \(\phi_{P_j} : (-\infty, \infty) \rightarrow (-\infty, \infty)\) is the sectional volume function, given by \(t \mapsto \text{Vol}_{d-1}(P_j \cap \{z = t\})\).

Note that, by Theorem 1.1 of [T2], \(HKd(X, D)\) is a continuous function. And by Theorem 3.4 the function \(\sum_{j=1}^s \phi_{P_j}\) is also continuous. Since both \(HKd(X, D)\) and \(\sum_{j=1}^s \phi_{P_j}\) agree on the dense subset \(\{m/q \mid m \in \mathbb{Z}_{\geq 0}, q = p^n, n \in \mathbb{Z}_{\geq 0}\} \subset \mathbb{R}_{\geq 0}\), we conclude that, for every \(\lambda \in \mathbb{R}\),
\[
HKd(X, D)(\lambda) = \sum_j \text{Vol}_{d-1}(P_j \cap \{z = \lambda\}) = \sum_j \text{Vol}_{d-1}(P_j \cap \{z = \lambda\}),
\]
where the last equality follows from part (2) of Lemma 4.5 This completes the proof of the theorem.

**Remark 4.7.** For \(\lambda \in \mathbb{Q}_{\geq 0}\), we remark that a generalised (in the sense of Conca [Co]) HK density function exists. Define
\[
\hat{f}_n(\lambda) = \ell_K[S]\left( \frac{K[S]}{(Y_1^n, \ldots, Y_n^n)} \right)_{[n\lambda]}.
\]

**Claim:** If \(\lambda \in \mathbb{Q}_{\geq 0}\) then \(HKd(X, D)(\lambda) = \lim_{n \to \infty} \hat{f}_n(\lambda)\).

**Proof of the claim:** Enough to prove that for \(\lambda \in \mathbb{Q}_{\geq 0}\), the sequence \(\{\hat{f}_n(\lambda)\}\) (which contains \(\{f_n(\lambda)\}\) as a subsequence) converges. Suppose \(\lambda = r/s\) with \(r \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}_{\geq 0}, (r, s) = 1\). For \(n \in \mathbb{N}\), by division algorithm we write \(n = l_n s + s_n,\) for \(l_n \in \mathbb{N}, 0 \leq s_n < s\). Write \(r_n = [s_n \frac{r}{s}]\). Then \([n\frac{r}{s}] = l_n r + r_n\). Write
\[
Q_{jn} = \frac{l_n s + s_n}{l_n r + r_n} P_j \cap \{z = 1\} \quad \text{and} \quad Q_{j0} = \frac{s r}{r} P_j \cap \{z = 1\}
\]
for each \(P_j\) as in the proof above. Then
\[
nP_j \cap \{z = [n\lambda]\} = (l_n r + r_n) \left( \frac{(l_n s + s_n)}{(l_n r + r_n)} P_j \cap \{z = 1\} \right) = (l_n r + r_n) Q_{jn}.
\]
Note that \(Q_{jn} \supseteq Q_{j0}\), since \(P_j\) contains \(0 \in \mathbb{R}^d\) and \(\frac{l_n s + s_n}{l_n r + r_n} \geq \frac{s}{r}\). Hence
\[
\lim_{n \to \infty} \frac{i(P_j, n, [n\lambda])}{n^{d-1}} \leq \lim_{n \to \infty} \frac{i(Q_{jn}, (l_n r + r_n) Q_{jn}, 1)}{(l_n s + s_n)^{d-1}} = \lim_{n \to \infty} \frac{i(Q_{j0}, (l_n r + r_n) Q_{j0}, 1)}{(l_n s + s_n)^{d-1}} = \lim_{n \to \infty} \frac{i(Q_{jn}, (l_n r + r_n) Q_{jn}, 1)}{(l_n s + s_n)^{d-1}}
\]
\[
\geq \lim_{n \to \infty} \frac{i(Q_{j0}, (l_n r + r_n) Q_{j0}, 1)}{(l_n s + s_n)^{d-1}} = \frac{(r/s)^{d-1}}{(l_n s + s_n)^{d-1}} \text{Vol}_{d-1}(Q_{j0}) = \text{Vol}_{d-1}(P_j \cap \{z = \frac{r}{s}\}).
\]
Now for each \(m \in \mathbb{N}\), for \(n \gg 0\), we have \((l_n s + s_n)/(l_n r + r_n) \leq (s)(r) + (1)(m)\). As before \(Q_{jn} \leq (\frac{s}{r} + \frac{1}{m}) P_j \cap \{z = 1\}\). Hence
\[
\lim_{n \to \infty} \frac{i(P_j, n, [n\lambda])}{(l_n s + s_n)^{d-1}} = \lim_{n \to \infty} \frac{i((l_n r + r_n) Q_{jn}, 1)}{(l_n s + s_n)^{d-1}}
\]
\[
\leq \lim_{n \to \infty} \frac{i((\frac{s}{r} + \frac{1}{m}) P_j \cap \{z = 1\}, l_n r + r_n)}{(l_n s + s_n)^{d-1}} = \frac{(r/s)^{d-1}}{(s/r + 1/m)} \text{Vol}_{d-1}\left( \left( \frac{s}{r} + \frac{1}{m} \right) P_j \cap \{z = 1\} \right)
\]
\[
= \frac{(r/s)^{d-1}}{(s/r + 1/m)} \text{Vol}_{d-1}\left( P_j \cap \{z = \frac{1}{(s/r + 1/m)}\} \right).
\]
Letting \( m \to \infty \), we see that \( \lim_{n \to \infty} \frac{i((P, n_i), |n\lambda|)}{n^{d-1}} \leq \text{Vol}_{d-1}(P_j \cap \{ z = \frac{1}{x} \}) \). Thus for \( \lambda \in \mathbb{Q}_{\geq 0} \),

\[
\lim_{n \to \infty} \frac{i((P, n_i), |n\lambda|)}{n^{d-1}} = \text{Vol}_{d-1}(P_j \cap \{ z = \lambda \}).
\]

The claim follows easily from previous observations in the proof of the theorem.

5. INTEGRAL CONVEX POLYTOPE AND DENSITY FUNCTION

Let \( P_D \) be a convex integral polytope in \( \mathbb{R}^{d-1} \) of dimension \( d-1 \), without loss of generality we can assume one of the vertex is \( 0 \in \mathbb{R}^{d-1} \).

**Notations 5.1.**

1. Let \( \{v_1, \ldots, v_l\} \subseteq \mathbb{R}^{d-1} \) be the set of vertices of \( P_D \).
2. Let \( C \) be the cone given by \((\{(v_i, 1)\}) \) and the origin \( 0 \in \mathbb{R}^d \).
3. Let \( C = \mathbb{R}D \setminus \cup_{u \in \mathbb{R}D \cap \mathbb{Z}^{d-1}} ((u, 1) + C_D) \). Similarly, for an integer \( m \geq 1 \), \( C_m = C = \mathbb{R}D \setminus \cup_{u \in \mathbb{R}D \cap \mathbb{Z}^{d-1}} ((u, 1) + C_mD) \).
4. Let \( W_0 \subseteq \mathbb{R}^{d-1} \) be the unit cell \([0, 1]d^{-1} \). For a point \( v \in \mathbb{R}^{d-1} \), the set \( W_v \) denotes the cell which is the translate of \( W_0 \) by \( v \), that is \( W_v = v + W_0 \).
5. Let \( l_D := \min\{ t \in \mathbb{R}_{\geq 0} \mid B(0, t) \supseteq P_D \} \), where \( B(0, t) \) is the closed ball of radius \( t \) at the origin.

**Remark 5.2.** Note that

\[ C_mD \cap \{ z = \lambda + 1 \} = P_\{1 + \lambda\}D \times \{ z = 1 + \lambda \} = \{ \sum_i b_iv_i, 1 + \lambda \} \subseteq \mathbb{R}^d \mid \sum_i b_i = m + m\lambda \}.
\]

**Lemma 5.3.** For an integer \( m \geq 1 \), where \( l \) is the number of vertices of \( P_D \), we have

1. \( P_mD \cap \{ z = 1 + \lambda \} = \phi, \) for all \( \lambda \geq l/m \).
2. In particular
   
   (a) \( P_mD \cap \{ z = 1 + \lambda \} \subseteq P_{(m+l)}D \times \{ z = 1 + \lambda \}, \) for \( \lambda \geq 0 \) and
   
   (b) \( P_mD \cap \{ z \in [1, \infty) \} \subseteq P_{(m+l)}D \times \{ z \in [1, 1 + l/m] \} \).

**Proof.** We assume the following claim for the moment.

**Claim** \( C_mD \cap \{ z = 1 + l/m \} \subseteq \bigcup_{u \in P_mD \cap \mathbb{Z}^{d-1}} ((u, 1) + C_mD) \).

(1) Note for any \( \lambda \geq 1/m \) and for \((w', 1 + \lambda \) \in C_mD \cap \{ z = 1 + \lambda \}, \) there exists \( w_0 \in P_mD \) such that \( w' = ((1 + \lambda)w_0, 1 + \lambda) \). Therefore we can write

\[
(w', 1 + \lambda) = ((1 + \frac{l}{m})w_0, 1 + \frac{l}{m}) + ((\lambda - \frac{l}{m})w_0, (\lambda - \frac{l}{m})) = ((1 + \frac{l}{m})w_0, 1 + \frac{l}{m}) + C_mD.
\]

Hence by the above claim, \( C_mD \cap \{ z = 1 + \lambda \} \subseteq \bigcup_{u \in P_mD \cap \mathbb{Z}^{d-1}} ((u, 1) + C_mD) \). Therefore \( P_mD \cap \{ z = 1 + \lambda \} = \phi, \) for all \( \lambda \geq l/m \). This proves the first assertion.

(2) The above claim implies that

\[
P_mD \cap \{ z \in [1, \infty) \} = P_mD \cap \{ z \in [1, 1 + l/m] \} \subseteq C_mD \cap \{ z \in [1, 1 + l/m] \}
\]

\[
\subseteq P_{(1+l/m)D} \times \{ z \in [1, 1 + l/m] \} = P_{(m+l)}D \times \{ z \in [1, 1 + l/m] \}.
\]

Note that the last inequality holds as \( 0 \notin P_D \) implies \( P\lambda D \subseteq P\lambda D, \) if \( \lambda \geq \lambda \). This proves both the parts of the second assertion.

Now we give a

**Proof of the claim:** Let \((w, 1 + l/m) \in C_mD \cap \{ z = 1 + l/m \} \). Then \( w = \sum_i a_imv_i \), where \( a_i \geq 0 \) in \( \mathbb{R} \) and \( \sum_i a_i = 1 + l/m \). We write \( ma_i = [ma_i] + \{ma_i\} \), where \([x]\) denote the integral part of a number \( x \) and \( \{x\} \) denote the fractional part of \( x \). Since \( 0 \leq \sum_i [ma_i] < l \) and \( \sum_i ma_i = m + l \), we have \( m + l \geq \sum_i [ma_i] \geq m \). Therefore we can choose nonnegative integers \( c_1, \ldots, c_l \) such that \( c_i \leq [ma_i] \) and \( \sum_i ([ma_i] - c_i) = m \). Now we can write

\[
(w, 1 + \frac{l}{m}) = \left( \sum_i ([ma_i] - c_i)mv_i, 1 \right) + \left( \sum_i \frac{\{ma_i\} + c_i}{m}mv_i, \frac{l}{m} \right) \in (P_mD \cap \mathbb{Z}^{d-1}, 1) + C_mD.
\]

This proves the claim and hence the lemma. 

\( \square \)
Lemma 5.4. Let $W_v \subset \mathbb{R}^{d-1}$ be a unit cell as given in Notations 5.1 (4) above. Then, for any fixed integer $m \geq 1$,

$$\text{Vol}_{d-1}[(W_v \times \{z = \lambda + 1\}) \setminus \bigcup_{u \in \mathbb{Z}^{d-1}} ((u, 1) + C_{mD})] = \text{Vol}_{d-1}[(W_v \times \{z = m\lambda + 1\}) \setminus \bigcup_{u \in \mathbb{Z}^{d-1}} ((u, 1) + C_D)].$$

Proof. Define

$$\psi : (W_v \times \{z = \lambda + 1\}) \setminus \bigcup_{u \in \mathbb{Z}^{d-1}} ((u, 1) + C_{mD}) \longrightarrow (W_v \times \{z = m\lambda + 1\}) \setminus \bigcup_{u \in \mathbb{Z}^{d-1}} ((u, 1) + C_D)$$

by $(x, \lambda + 1) \mapsto (x - v, m\lambda + 1)$, for $x \in W$. Note that $(x - v, m\lambda + 1) \in (u, 1) + C_D$ if and only if $(x, \lambda + 1) \in (u + v, 1) + C_m D$. Thus $\psi$ is a well defined isometry. \hfill \Box

Definition 5.5. Let $W_v$ be a $d - 1$ dimensional unit cell, for some $v \in \mathbb{Z}^{d-1}$. We define the sets, for $0 \leq \lambda$,

$$\Phi_{mD}^v(\lambda) = (W_v \times \{z = \lambda + 1\}) \setminus \bigcup_{u \in P_{mD} \cap \mathbb{Z}^{d-1}} ((u, 1) + C_{mD}).$$

and

$$\Psi_{mD}^v(\lambda) = (W_v \times \{z = \lambda + 1\}) \setminus \bigcup_{u \in \mathbb{Z}^{d-1}} ((u, 1) + C_{mD}).$$

Let $\varphi_{mD}^v(\lambda) : [0, \infty) \longrightarrow [0, 1]$ be the function $\varphi_{mD}^v(\lambda) = \text{Vol}_{d-1}(\Phi_{mD}^v(\lambda))$ and similarly let $\varphi_{mD}(\lambda) : [0, \infty) \longrightarrow [0, 1]$ be the function $\varphi_{mD}(\lambda) = \text{Vol}_{d-1}(\Psi_{mD}^v(\lambda)).$

Remark 5.6. (1) By Lemma 5.4 the $\varphi_{mD}$ is well defined (independent of choice of $v$ in \mathbb{Z}^{d-1}). Also

(2) by Lemma 5.4 $\varphi_{mD}(\lambda) = \varphi_{D}(m \lambda)$.

(3) By definition it follows that $\varphi_{mD}(\lambda) \leq \varphi_{mD}^v(\lambda) \leq 1$.

Definition 5.7. (1) Let $r \in \mathbb{R}_{\geq 1}$ such that $P_M$ contains a unit cell $W_v$, for some $v$.

(2) For a unit cell $W_v$, let

$$l(W_v) = \{u \in \mathbb{Z}^{d-1} \mid \tilde{d}(u, w) \leq lD + \sqrt{d - 1}, \text{ for all } w \in W_v\},$$

where $\tilde{d}$ denotes the Euclidean metric on $\mathbb{R}^{d-1}$ and $l$ denotes the number of vertices of $P_M$.

Lemma 5.8. For any given integer $m \geq 1$ and for $\lambda \geq 0$, we have $\Psi_{mD}^v(\lambda) = A_{m}^v(\lambda)$, where

$$A_{m}^v(\lambda) = (W_v \times \{z = \lambda + 1\}) \setminus \bigcup_{u \in l(W_v)} ((u, 1) + C_{mD}).$$

Moreover, for $\lambda \geq lr/m$,

$$\Psi_{mD}^v(\lambda) = A_{m}^v(\lambda) = \emptyset.$$ 

In particular $\varphi_{mD}$ is a compactly supported continuous function.

Proof. Let $0 \leq \lambda \leq lr/m$. Note that $\Psi_{mD}^v(\lambda) \subseteq A_{m}^v(\lambda)$, for every $\lambda$. Therefore, it is enough to show that

$$(W_v \times \{z = \lambda + 1\}) \cap \bigcup_{u \in \mathbb{Z}^{d-1}} ((u, 1) + C_{mD}) \subseteq (W_v \times \{z = \lambda + 1\}) \cap \bigcup_{u \in l(W_v)} ((u, 1) + C_{mD}).$$

Suppose, there is $x = (u, 1) + \sum b_i (m v_i, 1) \in W_v \times \{z = \lambda + 1\}$, for some $u \in \mathbb{Z}^{d-1} \setminus l(W_v)$, and $x = (w', \lambda + 1)$, for some $w' \in W_v$, and $\sum b_i = \lambda$. Therefore, there exists $w \in W_v$, such that

$$lr \cdot lD + \sqrt{d - 1} < \tilde{d}(u, w) \leq \tilde{d}(u, w') + \tilde{d}(w', w) \leq \tilde{d}(0, \sum b_i m v_i) + \sqrt{d - 1},$$

as $\tilde{d}(w, w') \leq \sqrt{d - 1}$, for any $w, w' \in W_v$ and $\tilde{d}(u, w') = \tilde{d}(u, w + \sum b_i m v_i) = \tilde{d}(0, \sum b_i m v_i)$. This implies, $l_D < \tilde{d}(0, \sum (b_i m / lr) v_i)$. Therefore, by the definition of $l_D$ (as $0 \in P_D$),

$$\sum (b_i m / lr) v_i \notin P_D \implies \sum b_i m / lr > 1 \implies \lambda > lr/m,$$
which is a contradiction. Hence $\Psi^w_{mD}(\lambda) = A^w_m(\lambda)$, for $\lambda \leq lr/m$.

Let $\lambda \geq lr/m$.

**Claim** $\Psi^w_{mD}(\lambda) = \emptyset$.

**Proof of the claim:** Let $x = (w, \lambda + 1) \in \Psi^w_{mD}(\lambda)$. Then there is $v' \in \mathbb{Z}^{d-1}$ such that $w - v' \in P_{rD}$ (as $P_{rD}$ contains a unit cell). We can write $(w, \lambda + 1) = (w - v', \lambda) + (v', 1)$. Now $w - v' \in P_{rD} \subseteq P_{\lambda mD}$ (as $r/m \leq r/l \leq m/\lambda$). This implies $(w - v', \lambda) \in P_{\lambda mD} \times \{z = \lambda\} \subseteq C_{mD}$.

Hence $(w, \lambda + 1) \in (v', 1) + C_{mD}$, where $v' \in \mathbb{Z}^{d-1}$. This implies $\Psi^w_{mD}(\lambda) = \emptyset$, for $\lambda \geq rl/m$.

Now we have $A^w_m(lr/m) = \Psi^w_{mD}(lr/m) = \emptyset$. Let $x = (w, \lambda + 1) \in A^w_m(\lambda)$, for some $\lambda \geq lr/m$. Then $(w, 1 + lr/m) \in (u, 1) + C_{mD}$, for some $u \in l(W_v)$. Therefore

$$(w, 1 + \lambda) = (w, 1 + lr/m) + (0, \lambda - lr/m) \in (u, 1) + C_{mD}, \quad \text{where } u \in l(W_v)$$

and $\{0 \times \mathbb{R}_{>0}\} \subseteq C_{mD}$. Hence $A^w_m(\lambda) = \emptyset$, for $\lambda \geq lr/m$.

Since $l(W_v)$ is a finite set, the function $\varphi_{mD}$ is continuous. This completes the proof of the lemma.

\[\square\]

**Remark 5.9.** Moreover if we write $P_{kD} \cap \mathbb{Z}^{d-1} = P'_{kD} \cup P''_{kD}$ such that $(l(W_v))$ is given as in Definition 5.7

$$P'_{kD} = \{v \in P_{kD} \cap \mathbb{Z}^{d-1} \mid l(W_v) \subseteq P_{kD}\}$$

and

$$P''_{kD} = \{v \in P_{kD} \cap \mathbb{Z}^{d-1} \mid l(W_v) \not\subseteq P_{kD}\}$$

then

$$v \in P'_{kD} \implies l(W_v) \subseteq P_{kD} \cap \mathbb{Z}^{d-1} \subseteq \mathbb{Z}^{d-1} \implies \varphi^v_{kD} = \varphi_{kD}, \quad \text{by Lemma 5.8}$$

**Notations 5.10.** For any given two closed sets $Q'$ and $Q''$ in $\mathbb{R}^{d-1}$, let $d(Q', Q'') = \min\{d(x, y) \mid x \in Q', y \in Q''\}$ denote the distance between the sets $Q'$ and $Q''$.

**Lemma 5.11.** Let $Q$ be a convex $d - 1$-dimensional rational polytope in $\mathbb{R}^{d-1}$ such that it contains the origin in its interior. Then, there is a constant $\delta_0 > 0$, depending on $Q$, such that for every rational $m \geq 1$ and for every integer $l \geq 0$, we have

$$d(\partial(mQ), \partial((m+l)Q)) \geq l\delta_0,$$

where $\partial(Q)$ denotes the boundary of $Q$ in $\mathbb{R}^{d-1}$.

**Proof.** Let $\{F_i\}_i$ be the set of facets of $Q$; then $\{mF_i\}_i$ is the set of facets of $mQ$, for any rational number $m \geq 1$. Moreover, if $F_i = H_i \cap Q$ then $mF_i = mH_i \cap mQ$, where $H_i$ denotes the supporting hyperplane of $Q$ at $F_i$. Now, since $\partial(mQ)$ and $\partial((m+l)Q)$ are compact closed sets, there exist $x_0 \in \partial(mQ)$ and $y_0 \in \partial((m+l)Q)$ such that $d(\partial(mQ), \partial((m+l)Q)) = d(x_0, y_0)$.

**Claim** $d(x_0, y_0) \geq \min\{d(mH_i, (m+l)H_i) \mid H_i \in \{\text{Supporting hyperplanes of } Q\}\}$.

**Proof of the claim:**

**Case 1** If $x_0 \in mF_i$ and $y_0 \in (m+l)F_i$, for some facet $F_i$ of $Q$ then $d(x_0, y_0) \geq d(mH_i, (m+l)H_i)$, as $mF_i \subset mH_i$ and $(m+l)F_i \subset (m+l)H_i$, where $mH_i$ and $(m+l)H_i$ are parallel hyperplanes.

**Case 2** Suppose $x_0 \in mF_i$ and $y_0 \in (m+l)F_j$, where $i \neq j$, then

$$d(x_0, y_0) \geq d(mF_i, (m+l)F_j) \geq d(mQ, (m+l)Q) = d(x_0, y_0).$$

As $mH_j$ is a supporting hyperplane for $mQ$, the entire polytope $mQ$ lies in one side of $mH_j$, say, $mQ \subset (mH_j)^+$, which implies $mF_i \subset (mH_j)^+$. On the other hand $(m+l)F_j \subset (m+l)H_j$. Hence

$$d(x_0, y_0) = d(mF_i, (m+l)F_j) \geq d((mH_j)^+, (m+l)H_j) = d(mH_j, (m+l)H_j).$$

This proves the claim.

Let $\delta_i = d(H_i, (0))$; then $\delta_i = \|x\|$, for some $x \in H_i$. Now it is easy to check that $d(mH_i, (0)) = \|mx\| = m\|x\|$ and $d(mH_i, (m+l)H_i) = d(mx, (m+l)x) = l\delta_i$, where $\delta_i > 0$.
as $H_i$ does not pass through the origin. Since, there are only finitely many facets and hence finitely many $H_i$, $\delta_0 = \min\{\delta_i\} > 0$. This proves the lemma. \hfill \square

**Lemma 5.12.** Let $S_1 = \{ v \in \mathbb{Z}^{d-1} \setminus P_{KD} \cap \mathbb{Z}^{d-1} \mid W_v \cap P_{KD} \neq \emptyset, \ W_v \cap (P_{KD})^c \neq \emptyset \}$ and let $P''_{KD}$ be as in Remark 5.9. Then $\#|P''_{KD}| = O(k^{d-2})$ and $\#|S_1| = O(k^{d-2})$.

**Proof.** Let $l$ be the number of vertices of $P_{KD}$ and let $r \geq 1$ be an integer such that the interior of $P_{rD}$ contains a unit cell $W_v$ for some $v \in \mathbb{Z}^{d-1}$. Let $l_{rD}$ be as in Notations 5.1.

Then $P_{rD}$ contains the lattice point $v$ in its interior. Let $Q = P_{rD} - v$ then $Q$ is a convex (integral) $d - 1$ dimensional polytope in $\mathbb{R}^{d-1}$ such that the origin is in the interior of $Q$. Let $\delta_0 > 0$ be a constant for $Q$, as given in Lemma 5.11.

For $\tilde{l} = (lr)/D + \sqrt{d-1}$, we can choose $l_1 \in \mathbb{Z}_{\geq 0}$ (e.g., $l_1 \geq \tilde{l}/\delta_0$) so that we have

\[
\begin{align*}
(5.1) \quad d(\partial(\frac{k}{r} + l_1)Q), \partial(\frac{k}{r}Q) \geq \tilde{l} \quad \text{and} \quad d(\partial(\frac{k}{r} - l_1)Q), \partial(\frac{k}{r} - l_1Q) \geq \tilde{l}.
\end{align*}
\]

Note that $(0) \in Q$ implies $(k/r - l_1)Q \subseteq (k/r)Q \subseteq (k/r + l_1)Q$, which is the same as

\[
P_{(k-l_1)D} - (k/r - l_1)v \subseteq P_{KD} - (k/r)v \subseteq P_{(k+l_1)D} - (k/r + l_1)v.
\]

Hence $P_{(k-l_1)D} + l_1v \subseteq P_{KD} \subseteq P_{(k+l_1)D} - l_1v$ and, by (5.1),

\[
d(\partial(P_{KD}), \partial(P_{(k-l_1)D} + l_1v)) \geq \tilde{l} \quad \text{and} \quad d(\partial(P_{(k+l_1)D} - l_1v), \partial(P_{KD})) \geq \tilde{l}.
\]

(Note that translation by $k/r$ is an isometry.)

Therefore $v_1 \in P_{(k-l_1)D} + l_1v$ implies that $l(W_{v_1}) \subseteq P_{KD}$. Hence $P''_{KD} \subseteq P_{KD} \setminus (P_{(k-l_1)D} + l_1v)$. Similarly $v_1 \in S_1$ implies that $d(v_1, \partial(P_{KD})) \leq \sqrt{d-1} \leq \tilde{l}$. Hence $S_1 \subseteq (P_{(k+l_1)D} - l_1v) \setminus P_{KD}$. Now $\#|P''_{KD}| \leq \#|P_{KD}| - \#|P_{(k-l_1)D}| = O(k^{d-2})$, and similarly for $\#|S_1|$. This proves the lemma. \hfill \square

6. **Main theorem**

First we give a proof of Proposition 1.2 which replaces $HKd(X, kD)$ by $\varphi_{kD}$ up to $O(k^{d-2})$.

**Proof of Proposition 1.2.** By Theorem 1.1, for $\lambda \geq 0$,

\[
HKd(X, kD)(\lambda + 1) = \text{Vol}(P_{KD} \cap \{ z = \lambda + 1 \}).
\]

By Lemma 5.3(2), for $\lambda \geq 0$,

\[
P_{KD} \cap \{ z = 1 + \lambda \} = [P_{(k+1)D} \times \{ z = 1 + \lambda \}] \cap [P_{KD} \cap \{ z = 1 + \lambda \}] \subseteq Q_0(\lambda) \cup Q_1(\lambda),
\]

where $Q_0(\lambda) = (P_{(k+1)D} \setminus P_{KD}) \times \{ z = 1 + \lambda \}$ and

\[
Q_1(\lambda) = (P_{KD} \times \{ z = 1 + \lambda \}) \cap (P_{KD} \cap \{ z = 1 + \lambda \}).
\]

Now one can cover $P_{KD}$ by unit cells as follows: $P_{KD} \subseteq \cup_{v \in S_1} W_v \cup \cup_{v \in P_{KD} \cap \mathbb{Z}^{d-1}} W_v$, where

\[
S_1 = \{ v \in \mathbb{Z}^{d-1} \setminus P_{KD} \cap \mathbb{Z}^{d-1} \mid W_v \cap P_{KD} \neq \emptyset, \ W_v \cap (P_{KD})^c \neq \emptyset \}.
\]

Therefore (see Definition 5.5)

\[
Q_1(\lambda) \subseteq \cup_{v \in S_1} \Phi_{kD}^v(\lambda) \cup \cup_{v \in P_{KD} \cap \mathbb{Z}^{d-1}} \Phi_{kD}^v(\lambda).
\]

Hence

\[
\text{Vol}_{d-1}P_{KD} \cap \{ z = \lambda + 1 \} \leq ((k+l)^{d-1} - k^{d-1})\text{Vol}(P_{KD}) + \sum_{v \in S_1} \varphi_{kD}^v(\lambda) + \sum_{v \in P_{KD} \cap \mathbb{Z}^{d-1}} \varphi_{kD}^v(\lambda)
\]

\[
= O(k^{d-2}) + \sum_{v \in S_1} \varphi_{kD}^v(\lambda) + \sum_{v \in P_{KD} \cap \mathbb{Z}^{d-1}} \varphi_{kD}(\lambda) + \sum_{v \in P_{KD}^c \cap \mathbb{Z}^{d-1}} [\varphi_{kD}(\lambda) - \varphi_{kD}(\lambda)],
\]

where the last equality follows as $\varphi_{kD}^v = \varphi_{kD}$, for $v \in P_{KD}^c$ (see Remark 5.9).
On the other hand, for \( v \in P'_{kD} \), we have \( W_v \subseteq P_{kD} \), therefore \( \cup_{v \in P'_{kD}} \Phi^v_{kD}(\lambda) \subseteq Q_1(\lambda) \). Hence, (note \( \dim (\Phi^v_{kD}(\lambda) \cap \Phi^w_{kD}(\lambda)) < d - 1 \), for \( v \neq w' \))

\[
\text{Vol}_{d-1}(P_{kD} \cap \{ z = \lambda + 1 \} \geq \sum_{v \in P'_{kD}} \varphi_{kD}(\lambda) = \sum_{v \in P_{kD} \cap \mathbb{Z}^{d-1}} \varphi_{kD}(\lambda) - \sum_{v \in P'_{kD}} \varphi_{kD}(\lambda).
\]

Also, by definition, \( 0 \leq \varphi^v_{kD}(\lambda), \varphi_{kD}(\lambda) \leq 1 \). Now, by Lemma 5.12, we can conclude that

\[
\text{Vol}_{d-1}(P_{kD} \cap \{ z = \lambda + 1 \} = \sum_{v \in \mathbb{Z}^{d-1}} \varphi_{kD}(\lambda) + O(k^{d-2}) = h^0(X, \mathcal{O}_X(kD)) \varphi_{kD}(\lambda) + O(k^{d-2}).
\]

This proves the proposition.

**Remark 6.1.** Let \( R = \bigoplus_{n \geq 0} R_n \) be a standard graded ring over a field \( K \). Let

\[
R^{(k)} = \bigoplus_{d \geq 0} R_{kd} = R_0 \oplus R_k \oplus R_{2k} \oplus \cdots \quad \text{and} \quad m_{R^{(k)}} = \bigoplus_{d \geq 1} R_{kd}
\]

be the k-fold Veronese ring and its homogeneous maximal ideal, respectively. Recall that we have defined \( HKD(X, D) \) (or \( e_{HK}(X, D) \)) as the HK density function (or HK multiplicity, respectively) of the associated homogeneous coordinate ring with respect to its graded maximal ideal. With this notation, if \( (R, m) \) denotes the homogeneous coordinate ring with the graded maximal ideal \( m \) for a toric pair \( (X, D) \) then we have \( e_{HK}(R, m) = e_{HK}(X, D) \) and

\[
e_{HK}(R, m^k) = e_{HK}(R^{(k)}, m_{R^{(k)}}) = e_{HK}(X, kD),
\]

\[
e_{0}(R, m^k) = e_{0}(R^{(k)}, m_{R^{(k)}}) = e_{0}(X, kD) = k^d e_{0}(X, D).
\]

Now we give a proof of the main theorem of this paper.

**Proof of Theorem 1.3** We denote the co-ordinate ring \( K[S] \) of \( (X, D) \) by \( R \). Therefore \( R = K[S] = K[\chi^{(u_1, 1)}, \ldots, \chi^{(u_r, 1)}] \).

Then, by Remark 6.1, we have

\[
e_{HK}(R, m^k) - e_{0}(R, m^k) = e_{HK}(X, kD) - k e_{0}(X, kD).
\]

But

\[
e_{0}(X, kD) = \frac{1}{d!} \text{Vol}(kD) = \int_0^1 \text{Vol}_{d-1}(P_{kD}) d\lambda = \int_0^1 \text{Vol}_{d-1}(P_{kD} \cap \{ z = \lambda \}) d\lambda.
\]

Moreover, by Theorem 1.1 of [T2] and by Proposition 1.2

\[
e_{HK}(X, kD) = \int_0^\infty HKD(X, kD)(\lambda) d\lambda = \int_0^\infty \text{Vol}_{d-1}(P_{kD} \cap \{ z = \lambda \}) d\lambda.
\]

Hence

\[
e_{HK}(X, kD) - e_{0}(X, kD) = \int_0^{l/k} \text{Vol}_{d-1}(P_{kD} \cap \{ z = 1 + \lambda \}) d\lambda,
\]

where the last equality follows by Lemma 5.3

By Proposition 1.2

\[
\int_0^{l/k} \varphi_{kD}(\lambda) d\lambda = \frac{1}{k} \int_0^l \varphi(\lambda) d\lambda \quad \text{and} \quad h^0(X, \mathcal{O}_X(kD)) = e_{0}(X, D) \frac{k^{d-1}}{(d-1)!} + O(k^{d-2}).
\]

Therefore

\[
\int_0^{l/k} \text{Vol}_{d-1}(P_{kD} \cap \{ z = 1 + \lambda \}) d\lambda = k^d e_{0}(X, D) \frac{k^{d-1}}{(d-1)!} + O(k^{d-3}).
\]
Therefore, by (6.1), we have
\[
\lim_{k \to \infty} \frac{1}{k^{d-1}} \left( e_{HK}(R, m^k) - \frac{e_0(R, m^k)}{d!} \right) = \frac{e_0(R, m)}{(d-1)!} \int_0^\infty \phi_D(\lambda) d\lambda.
\]

This proves the theorem. □

Now we give a proof of Proposition 1.4, which shows the multiplicative property of the function \( \phi \) on the set of projective toric varieties.

Proof of Proposition 1.4. Let \((X, D)\) and \((Y, D')\) be two toric pairs of dimension \(d-1 \geq 1\) and \(d' - 1 \geq 1\), respectively. Now, by Proposition 1.2
\[
HKd(X, kD)(\lambda + 1) = h^0(X, \mathcal{O}_X(kD))\varphi_{X,kD}(\lambda) + O(k^{d-2}), \quad \text{for} \quad \lambda \geq 0,
\]
\[
HKd(Y, kD')(\lambda + 1) = h^0(Y, \mathcal{O}_Y(kD'))\varphi_{Y,kD'}(\lambda) + O(k^{d'-2}), \quad \text{for} \quad \lambda \geq 0.
\]

Let
\[
e_X = \frac{e_0(X, D)}{(d-1)!}, \quad e_Y = \frac{e_0(Y, D')}{(d'-1)!} \quad \text{and} \quad e_{X \times Y} = \frac{e_0(X \times Y, D \boxtimes D')}{(d + d' - 2)!} = \frac{e_0(X, D) e_0(Y, D')}{(d-1)! (d'-1)!}.
\]

Therefore, by Proposition 2.14 (and Definition 2.13) of [T2], we have
\[
HKd(X \times Y, k(D \boxtimes D'))(\lambda + 1) = e_X [k(\lambda + 1)]^{d-1} HKd(Y, D')(\lambda + 1) + e_Y [k(\lambda + 1)]^{d'-1} HKd(X, D)(\lambda + 1)
\]
\[
- HKd(X, D)(\lambda + 1) HKd(Y, D')(\lambda + 1)
\]
\[
\quad = \left( e_X k^{d-1}(\lambda + 1)^{d-1} \right) \left( h^0(Y, \mathcal{O}_Y(kD'))\varphi_{Y,kD'}(\lambda) + O(k^{d'-2}) \right)
\]
\[
\quad + \left( e_Y k^{d'-1}(\lambda + 1)^{d'-1} \right) \left( h^0(X, \mathcal{O}_X(kD))\varphi_{X,kD}(\lambda) + O(k^{d-2}) \right)
\]
\[
\quad - \left( h^0(Y, \mathcal{O}_Y(kD))\varphi_{Y,kD'}(\lambda) + O(k^{d'-2}) \right) \left( h^0(X, \mathcal{O}_X(kD))\varphi_{X,kD}(\lambda) + O(k^{d-2}) \right).
\]

Since, \( \varphi_{X,kD}(\lambda) \) and \( \varphi_{Y,kD'}(\lambda) \in [0, 1] \) and
\[
h^0(X, \mathcal{O}_X(kD)) = e_X k^{d-1} + O(k^{d-2}) \quad \text{and} \quad h^0(Y, \mathcal{O}_Y(kD')) = e_Y k^{d'-1} + O(k^{d'-2}),
\]
we have
\[
HKd(X \times Y, k(D \boxtimes D'))(\lambda + 1)
\]
\[
\quad = \left( e_X k^{d-1}(\lambda + 1)^{d-1} \right) \left( e_Y k^{d'-1}\varphi_{Y,kD'}(\lambda) + O(k^{d'-2}) \right)
\]
\[
\quad + \left( e_Y k^{d'-1}(\lambda + 1)^{d'-1} \right) \left( e_X k^{d-1}\varphi_{X,kD}(\lambda) + O(k^{d-2}) \right)
\]
\[
\quad - \left( e_Y k^{d'-1}\varphi_{Y,kD'}(\lambda) + O(k^{d'-2}) \right) \times \left( e_X k^{d-1}\varphi_{X,kD}(\lambda) + O(k^{d-2}) \right)
\]
\[
\quad = \left( e_X e_Y k^{d+d'-2} \right) \left( (\lambda + 1)^{d-1} \varphi_{Y,kD'}(\lambda) + (\lambda + 1)^{d'-1}\varphi_{X,kD}(\lambda) - \varphi_{X,kD}(\lambda)\varphi_{Y,kD'}(\lambda) \right)
\]
\[
\quad + \left( e_X k^{d-1}(\lambda + 1)^{d-1} \times O(k^{d-2}) \right) + \left( e_Y k^{d'-1}(\lambda + 1)^{d'-1} \times O(k^{d'-2}) \right) + O(k^{d+d'-3}).
\]
By Remark 5.6, we have \( \varphi_{X,kD}(\lambda) = \varphi_{X,D}(k\lambda) \) and similarly for the pair \((Y, D')\). In particular for any \( x \in \mathbb{R}_{\geq 0} \) and any integer \( k \geq 1 \), we have (by substituting \( \lambda = x/k \)),

\[
[HKd(X \times Y, k(D \boxplus D'))(x/k + 1)]/k^{d+d'-2}
\]

\[
= (e_Xe_Y) [(x/k + 1)^{d-1}\varphi_{Y,D'}(x) + (x/k + 1)^{d'-1}\varphi_{X,D}(x) \varphi_{Y,D'}(x)]
\]

\[
+ \frac{1}{k} [e_Y(x/k + 1)^{d-1} \times O(1) + e_Y(x/k + 1)^{d'-1} \times O(1)] + O(k^{d+d'-3}).
\]

On the other hand, as \( X \times Y \) is a toric variety, we have from Proposition 1.2

\[
HKd(X \times Y, k(D \boxplus D'))(\lambda + 1)
\]

\[
= [h^0(X \times Y, k(D \boxplus D'))] [\varphi_{X \times Y,k(D \boxplus D')}(\lambda)] + O(k^{d+d'-3})
\]

\[
= [h^0(X, O_X(kD))h^0(Y, O_Y(kD))] [\varphi_{X \times Y,k(D \boxplus D')}(\lambda)] + O(k^{d+d'-3})
\]

\[
= [e_Xk^{d-1} + O(k^{d-2})] [e_Yk^{d'-1} + O(k^{d'-2})] [\varphi_{X \times Y,k(D \boxplus D')}(\lambda)] + O(k^{d+d'-3})
\]

\[
= [\varphi_{X \times Y,k(D \boxplus D')}(\lambda)] [e_Xe_Yk^{d+d'-2}] + O(k^{d+d'-3}),
\]

Hence for any \( x \geq 0 \) and for any integer \( k \geq 1 \), we have

\[
HKd(X \times Y, k(D \boxplus D'))(x/k + 1)/k^{d+d'-2} = [\varphi_{X \times Y,D \boxplus D'}(x)] (e_Xe_Y) + O(1/k).
\]

Now we fix \( x \geq 0 \) and take lim as \( k \to \infty \), then we have

\[
e_Xe_Y [\varphi_{X,D}(x) + \varphi_{Y,D'}(x) - \varphi_{X,D}(x) \varphi_{Y,D'}(x)] = e_Xe_Y [\varphi_{X \times Y,D \boxplus D'}(x)].
\]

Therefore, for every \( x \geq 0 \), we have

\[
\varphi_{X,D}(x) + \varphi_{Y,D'}(x) - \varphi_{X,D}(x) \varphi_{Y,D'}(x) = \varphi_{X \times Y,D \boxplus D'}(x).
\]

This implies the proposition.

\[\square\]

**Definition 6.2.** A rational polytope \( P_D \) _tiles the space_ \( \mathbb{R}^{d-1} \) if for some \( \lambda > 0 \)

1. \( \bigcup_{v \in \mathbb{Z}^{d-1}} (v + P_{\lambda D}) = \mathbb{R}^{d-1} \) and
2. \( \dim [(v + P_{\lambda D}) \cap (v' + P_{\lambda D})] < d - 1 \) if \( v \neq v' \).

Equivalently

1. \( \bigcup_{v \in \mathbb{Z}^{d-1}} [(v, 1) + C_D] \cap \{z = \lambda + 1\} = \mathbb{R}^{d-1} \times \{z = \lambda + 1\} \) and
2. \( \dim [(v, 1) + C_D] \cap \{(v', 1) + C_D \} \cap \{z = \lambda + 1\} < d - 1 \) if \( v \neq v' \).

It follows from the definition that if \( P_D \) tiles the space \( \mathbb{R}^{d-1} \) at \( \lambda \) then \( \lambda = (\operatorname{Vol}_{d-1}(P_D))^{1-d} \).

In the literature this is known as a _simple tiling_ (or 1-tiling) by the polytope \( P_D \) with the lattice \( M = \mathbb{Z}^d \).

**Theorem 6.3.** Let \( (X, D) \) be a toric pair of dimension \( d - 1 \geq 1 \). Then

\[
(e_0(X, D))^{\frac{2-d}{d}} \lim_{k \to \infty} \frac{e_{HK}(X, kD) - e_0(X, kD)/d!}{k^{d-2}} \geq \left[ \frac{d-1}{d} \right] \left( [d-1] \right)^{\frac{2-d}{d}}.
\]

Moreover, the equality hold, i.e.,

\[
(6.2) \quad (e_0(X, D))^{\frac{2-d}{d}} \lim_{k \to \infty} \frac{e_{HK}(X, kD) - e_0(X, kD)/d!}{k^{d-2}} = \left[ \frac{d-1}{d} \right] \left( [d-1] \right)^{\frac{2-d}{d}}.
\]

if and only if \( P_D \) tiles the space \( \mathbb{R}^{d-1} \) for some \( \lambda > 0 \).
Proof. Let \((X, D)\) be a toric pair of dimension \(d - 1\). We choose a real number \(\alpha > 0\) such that 
\[ e_0(X, D) = \alpha^{d-1}(d-1)! . \]
For \(v \in \mathbb{Z}^{d-1}\) and \(\lambda \geq 0\), let 
\[ P_{AD}^v = (P_{AD} \times \{z = \lambda + 1\}) \cap (W_v \times \{z = \lambda + 1\}) . \]
Note that 
\[ P_{AD} \times \{z = \lambda + 1\} = ((\emptyset, 1) + C_D) \cap \{z = \lambda + 1\} = ((\emptyset, 1) + C_D) \cap (\mathbb{R}^{d-1} \times \{z = \lambda + 1\}) \]
and 
\[ P_{AD}^v = ((\emptyset, 1) + C_D) \cap (W_v \times \{z = \lambda + 1\}) . \]
Hence 
\[ (6.3) \quad P_{AD} \times \{z = \lambda + 1\} = \bigcup_{v \in \mathbb{Z}^{d-1}} P_{AD}^v . \]
Also 
\[ (6.4) \quad P_{AD}^v - (v, 0) = ((-v, 1) + C_D) \cap (W_v \times \{z = \lambda + 1\}) . \]
Therefore 
\[ (6.5) \quad \bigcup_{u \in \mathbb{Z}^{d-1}} ((u, 1) + C_D) \cap (W_v \times \{z = \lambda + 1\}) = \bigcup_{v \in \mathbb{Z}^{d-1}} P_{AD}^v - (v, 0) , \]
where by \((6.3)\),
\[ \text{Vol}_{d-1}(\bigcup_{v \in \mathbb{Z}^{d-1}} P_{AD}^v - (v, 0)) \leq \sum_{v \in \mathbb{Z}^{d-1}} \text{Vol}_{d-1}(P_{AD}^v) = \text{Vol}_{d-1}(P_{AD}) . \]
Hence 
\[ (6.6) \quad \text{Vol}_{d-1}(W_v \times \{z = \lambda + 1\}) \leq \text{Vol}_{d-1}(P_{AD}) - 1 - \lambda^{d-1} \text{Vol}_{d-1}(P_D) . \]
Therefore 
\[ \int_0^\infty \varphi_{X,D}(\lambda) \, d\lambda \geq \int_0^{1/\alpha} \varphi_{X,D}(\lambda) \, d\lambda \geq \int_0^{1/\alpha} (1 - \lambda^{d-1} \alpha^{d-1}) \, d\lambda = \frac{1}{\alpha} \int_0^1 (1 - \beta^{d-1}) \, d\beta . \]
This implies 
\[ (6.7) \quad \frac{e_0(X, D)}{(d-1)!} \int_0^\infty \varphi_{X,D}(\lambda) \, d\lambda \geq \alpha^{d-2} \int_0^1 (1 - \beta^{d-1}) \, d\beta . \]
If we denote 
\[ A(X, D) = \lim_{k \to \infty} \frac{e_{HK}(X, kD) - e_0(X, kD)}{k^{d-2}} , \]
then we have 
\[ A(X, D) \geq \left( \frac{(d-1)!}{\alpha^{d-2}} \right) \frac{\alpha^{d-2} \int_0^1 (1 - \beta^{d-1}) \, d\beta = \left[ \frac{d-1}{d} \right] \frac{(d-1)!^{\frac{d-1}{d}}}{d} . \]
This proves Assertion (1).

(2) Suppose the polytope \(P_D\) tiles the space \(\mathbb{R}^{d-1}\), for some \(\lambda_0 > 0\). Then, by \((6.5)\) and Definition \(6.2\) (1),
\[ \bigcup_{v \in \mathbb{Z}^{d-1}} P_{\lambda_0 D}^v - (v, 0) = \bigcup_{v \in \mathbb{Z}^{d-1}} ((-v, 1) + C_D) \cap (W_{\lambda_0} \times \{z = \lambda_0 + 1\}) = W_{\lambda_0} \times \{z = \lambda_0 + 1\} . \]
This implies, by \((6.4)\) and Definition \(6.2\) (2),
\[ 1 = \text{Vol}_{d-1}(\bigcup_{v \in \mathbb{Z}^{d-1}} P_{\lambda_0 D}^v - (v, 0)) = \sum_{v \in \mathbb{Z}^{d-1}} \text{Vol}_{d-1}(P_{\lambda_0 D}^v - (v, 0)) = \sum_{v \in \mathbb{Z}^{d-1}} \text{Vol}_{d-1}(P_{\lambda_0 D}) = \text{Vol}_{d-1}(P_{\lambda_0 D}) = \lambda_0^{d-1} \alpha^{d-1} . \]
This implies \( \lambda_0 = 1/\alpha \). If \( \lambda < \lambda_0 \) then \( \dim([P^v_{\lambda, D} - (v, 0)] \cap [P^v'_{\lambda, D} - (v', 0)] \cap \{z = \lambda + 1\} < d - 1 \) implies
\[
\text{Vol}_{d - 1}(\cup_{v \in \mathbb{Z}^{d - 1}} P^v_{\lambda, D} - (v, 0)) = \text{Vol}_{d - 1}(P^v_{\lambda, D}) = \lambda^{d - 1} \alpha^{d - 1},
\]
Therefore
\[
\int_0^\infty \varphi_{X,D}(\lambda)d\lambda = \int_0^{1/\alpha} \varphi_{X,D}(\lambda)d\lambda = \int_0^{1/\alpha} (1 - \lambda^{d - 1} \alpha^{d - 1})d\lambda = \frac{1}{\alpha} \int_0^1 (1 - \beta^{d - 1})d\beta.
\]
This implies
\[
(6.8) \quad \frac{e_0(X,D)}{(d - 1)!} \int_0^\infty \varphi_{X,D}(\lambda)d\lambda = \alpha^{d - 2} \int_0^1 (1 - \beta^{d - 1})d\beta.
\]
Now the equality, as given in (6.2), follows from Theorem 1.3
Conversely suppose the equality in (6.2) holds then retracing the above argument we have
\[
\int_0^\infty \varphi_{X,D}(\lambda)d\lambda = \int_0^{1/\alpha} \varphi_{X,D}(\lambda)d\lambda + \int_{1/\alpha}^\infty \varphi_{X,D}(\lambda)d\lambda = \int_0^{1/\alpha} (1 - \lambda^{d - 1} \alpha^{d - 1})d\lambda.
\]
But, by (6.6), for every \( \lambda > 0 \), we have \( \varphi_{X,D}(\lambda) \geq 1 - \lambda^{d - 1} \alpha^{d - 1} \) and \( \varphi_{X,D}(\lambda) \geq 0 \). Hence the continuity of \( \varphi_{X,D} \) (see Lemma 5.8) implies
\[
\varphi_{X,D}(\lambda) = \begin{cases} 1 - \lambda^{d - 1} \alpha^{d - 1} & \text{if } \lambda \leq 1/\alpha \\ 0 & \text{if } \lambda > 1/\alpha. \end{cases}
\]
This implies, for \( \lambda_0 = 1/\alpha \), we have
\[
1 = \text{Vol}_{d - 1}(\bigcup_{v \in \mathbb{Z}^{d - 1}} (P^v_{\lambda_0, D} - (v, 0))) \leq \sum_{v \in \mathbb{Z}^{d - 1}} \text{Vol}_{d - 1}(P^v_{\lambda_0, D}) = \text{Vol}_{d - 1}(P^v_{\lambda_0, D}) = 1.
\]
Therefore
\[
\dim \left((P^v_{\lambda_0, D} - (v, 0)) \cap (P^v'_{\lambda_0, D} - (v', 0))\right) < d - 1
\]
and
\[
\bigcup_{v \in \mathbb{Z}^{d - 1}} ((u, 1) + C_D) \cap (W_\varnothing \times \{z = \lambda_0 + 1\}) = \bigcup_{v \in \mathbb{Z}^{d - 1}} [P^v_{\lambda_0, D} - (v, 0)] = W_\varnothing \times \{z = \lambda_0 + 1\}.
\]
Now, by Lemma 5.4, we can conclude the same thing, by replacing \( W_\varnothing \) by \( W_v \), for any \( v \in \mathbb{Z}^{d - 1} \).
In particular, \( P_D \) tiles the space \( \mathbb{R}^{d - 1} \) for \( \lambda_0 = 1/\alpha \). This completes the proof of Assertion (2) and hence the theorem. \( \square \)

**Example 6.4.** Let \( (X_0, D_0) \) be the Segre self-product of \( (\mathbb{P}^1, O_{\mathbb{P}^1}(m_0)) \), taken \( d - 1 \) times, \( i.e., \)
\[
(X_0, D_0) = (\mathbb{P}^1 \times \cdots \times \mathbb{P}^1, O_{\mathbb{P}^1}(m_0) \boxtimes \cdots \boxtimes O_{\mathbb{P}^1}(m_0)),
\]
for some integer \( m_0 \geq 1 \). Then the polytope \( P_{D_0} = [0, m_0]^{d - 1} \) and \( (1/m_0)P_{D_0} = [0, 1]^{d - 1} \). This implies that \( P_{D_0} \) tiles the space \( \mathbb{R}^{d - 1} \) for \( \lambda = 1/m_0 \).

**Remark 6.5.** We recall the following conjecture of Watanabe-Yoshida (Conjecture 4.2, [WY2]):
For a Noetherian unmixed nonregular local ring \((R, \mathfrak{m}, K)\) of dimension \( d \) with \( K = \mathbb{F}_p \),
\[
e_{HK}(R, \mathfrak{m}) \geq e_{HK}(A_{p,d}, (Y_0, \ldots, Y_d)),
\]
where \( A_{p,d} \) is given by \( A_{p,d} := F_p[[Y_0, Y_1, \ldots, Y_d]]/(Y_0^2 + \cdots + Y_d^2) \).

Here Theorem 6.3 implies that for any toric pair \((X, D)\) of dimension \( d - 1 \), the asymptotic growth of the HK multiplicity (relative to its usual multiplicity, \( e_0(X, D) \)) is always \( \geq \) the asymptotic growth of the HK multiplicity (relative to its usual multiplicity \( e_0(X_0, D_0) \)) for the pair \((X_0, D_0)\), where \((X_0, D_0)\) is the Segre self-product, of any toric pair of the type \((\mathbb{P}^1, O_{\mathbb{P}^1}(m_0))\), taken \( d - 1 \) times. Note that the associated coordinate ring for any such pair \((X_0, D_0)\) is given by a set of quadratic binomials over \( K \).
Remark 6.6. Let $P$ be a rational convex polytope in $\mathbb{R}^{d-1}$; then we can formulate the property that $P$ tiles the space $\mathbb{R}^{d-1}$ in terms of HK multiplicity, as follows:

We choose $m \gg 0$ (by Corollary 2.2.18 in [CLS], any $m \geq (d-2)n_1$, where $n_1P$ is an integral polytope) such that $mP$ is a very ample integral convex polytope. In particular there is a toric pair $(X,D)$ such that the associated polytope $P_D = mP$. Then the polytope $P$ tiles the space $\mathbb{R}^{d-1}$ for some $\lambda > 0$ if and only if

$$
(e_0(X,D))^{\frac{2-d}{d-1}} \lim_{k \to \infty} \frac{e_{HK}(X,kD) - e_0(X,kD)/d!}{k^{d-2}} = \left[\left(\frac{d-1}{d}\right)(d-1)^{\frac{2-d}{d-1}}\right].
$$

Note that this criteria is independent of the choice of $m$, as left hand side of the above equation does not change if we replace $D$ by an integral multiple of $D$. Moreover, if $P$ tiles the space $\mathbb{R}^{d-1}$ then it tiles at $\lambda = (\text{Vol } P)^{1-d}$.

7. Examples

Example 7.1. We compute the HK density function for the toric pair $(X,D) = (\mathbb{P}^1, O(n))$ for $n \in \mathbb{N}$. The polytope $P_D$ can be taken to be the line segment $[0,n]$ (up to translation by integer points). Then $P_D = \bigcup_{i=0}^{n-1} P_i$, where $P_i = \text{Conv } \{(0,0), (i,1), (i+1,1), (i+1, \frac{n+1}{n})\}$, $i = 0, \ldots, n-1$. One has

$$
HKd(X,D)(\lambda) = \begin{cases} 
  n\lambda & \text{if } 0 \leq \lambda < 1 \\
  n(1 - n(\lambda - 1)) & \text{if } 1 \leq \lambda < 1 + \frac{1}{n} \\
  0 & \text{if } \lambda \geq 1 + \frac{1}{n}.
\end{cases}
$$

Moreover $\varphi_{kD}(\lambda) = 1 - nk\lambda$ if $0 \leq \lambda < 1/nk$ and $\varphi_{kD}(\lambda) = 0$ otherwise.

Example 7.2. We compute the HK density function for the Hirzebruch surface $X = F_a$ (See [T1] for a different geometric approach for this) with parameter $a \in \mathbb{N}$, which is a ruled surface over $\mathbb{P}^1_k$, where $k$ is a field of characteristic $p > 0$. See [Fu] for a detailed description of the surface as a toric variety. The $T$-Cartier divisors are given by $D_i = V(v_i)$, $i = 1, 2, 3, 4$, where $v_1 = e_1, v_2 = e_2, v_3 = -e_1 + ae_2, v_4 = -e_2$ and $V(v_i)$ denotes the $T$-orbit closure corresponding to the cone generated by $v_i$. We know the Picard group is generated by $\{D_i : i = 1, 2, 3, 4\}$ over $\mathbb{Z}$. One can check the only relations in Pic(X) can be described by $D_3 \sim D_1$ and $D_2 \sim D_4 - aD_1$. Therefore Pic(X) = $\mathbb{Z}D_1 \oplus \mathbb{Z}D_4$. One can use standard methods in toric geometry to see that $D = cD_1 + dD_4$ is ample if and only if $a, c > 0$. Then $P_D = \{(x,y) \in M_\mathbb{R} \mid x \geq -c, y \leq d, x \leq ay\}$ and $\alpha^2 = \text{Vol}(P_D) = cd + \frac{ad^2}{2}$. To consider $HKd(X,D)$ for $D = cD_1 + dD_4$, we split it into two different cases.

1. Case 1: $c \geq d$

$$
HKd(X,D)(\lambda) = \begin{cases} 
  (cd + \frac{ad^2}{2})\lambda^2 & \text{if } 0 \leq \lambda < 1 \\
  (cd + \frac{ad^2}{2})\lambda^2 - (c + \frac{ad}{2}) (d+1)(cd + \frac{ad^2}{2})(\lambda - 1)^2 & \text{if } 1 \leq \lambda < 1 + \frac{1}{c+ad} \\
  (c + \frac{ad}{2})(d+1)\frac{1}{2\lambda}(c + 1 - c\lambda)^2 + (cd + \frac{ad^2}{2})\lambda(d+1-d\lambda) & \text{if } 1 + \frac{1}{c+ad} \leq \lambda < 1 + \frac{1}{c} \\
  (cd + \frac{ad^2}{2})\lambda(d+1-d\lambda) & \text{if } 1 + \frac{1}{c} \leq \lambda < 1 + \frac{1}{a} \\
  0 & \text{if } \lambda \geq 1 + \frac{1}{a}.
\end{cases}
$$
(2) Case 2: $c \leq d$

$$HKd(X, D)(\lambda) = \begin{cases} (cd + \frac{ad^2}{2})\lambda^2 & \text{if } 0 \leq \lambda < 1 \\ -(c + \frac{ad}{2} + 1)(d + 1)(cd + \frac{ad^2}{2})(\lambda - 1)^2 & \text{if } 1 \leq \lambda < 1 + \frac{1}{c+ad} \\ (c + \frac{ad}{2})(d + 1)\frac{1}{2a}(c + 1 - c\lambda)^2 & \text{if } 1 + \frac{1}{c+ad} \leq \lambda < 1 + \frac{1}{d} \\ +(cd + \frac{ad^2}{2})(d + 1 - d\lambda) & \text{if } 1 + \frac{1}{d} \leq \lambda < 1 + \frac{a+1}{ad+c} \\ (cd + \frac{ad^2}{2} + \frac{ad}{2})\frac{1}{2a}(a + 1 - (c + ad)(\lambda - 1))^2 & \text{if } 1 + \frac{a+1}{ad+c} \leq \lambda < \frac{1}{c}, \\ +\frac{c}{2a}(c + 1 - c\lambda)^2 & \text{if } \lambda \geq 1 + \frac{1}{c}, \\ 0 & \text{if } \lambda = 1 \end{cases}$$

Example 7.3. In this example we consider how $\varphi_{X, D}$ changes as $D$ varies over the ample cone of divisors on $X$. We consider this question for the Hirzebruch surface $X = F_a$ with parameter $a \in \mathbb{N}$, as in Example 7.2. Let $P_D = \{(x, y) \in M_R \mid x \geq -c, y \leq d, x \leq ay\}$. To consider $\varphi_{X, D}$ for $D = cD_1 + dD_4$, we split it into two different cases.

(1) When $c \geq d$:

$$\varphi_{X, D}(\lambda) = \begin{cases} 1 - \lambda^2(cd + \frac{ad^2}{2}) & \text{if } 0 \leq \lambda \leq \frac{1}{ad+c}, \\ (1 - \lambda d) + \frac{(1-\lambda)(1-\lambda)}{2a} & \text{if } \frac{1}{ad+c} \leq \lambda \leq \frac{1}{c}, \\ 1 - \lambda d & \text{if } \frac{1}{c} \leq \lambda \leq \frac{1}{d}, \\ 0 & \text{if } \lambda \geq \frac{1}{d}. \end{cases}$$

(2) When $c \leq d$:

$$\varphi_{X, D}(\lambda) = \begin{cases} 1 - \lambda^2(cd + \frac{ad^2}{2}) & \text{if } 0 \leq \lambda \leq \frac{1}{ad+c}, \\ (1 - \lambda d) + \frac{(1-\lambda)(1-\lambda)}{2a} & \text{if } \frac{1}{ad+c} \leq \lambda \leq \frac{1}{d}, \\ (1+a-\lambda)(d+\lambda))^2 & \text{if } \frac{1}{d} \leq \lambda \leq \frac{1+a}{ad+c}, \\ 0 & \text{if } \lambda \geq \frac{1+a}{ad+c}. \end{cases}$$

Example 7.4. Here we compute the $\varphi_{X, -K}$ of the smooth Fano toric varieties $X$ of dimension $d - 1 = 2$ with respect to the anticanonical divisor $-K$, namely $\mathbb{P}^2$, and blow ups of $\mathbb{P}^2$ at one, two and three points with respect to the anticanonical divisor $-K = \sum D_i$, where $D_i$ are the T-Cartier divisors on the respective varieties. We find $P_{-K}$, and eventually $\varphi_{X, -K}$. $\varphi_{X, -K}$ equals the volume of the darker shaded region at $Z = \lambda$. For each surface we denote the co-ordinate ring by $R$ and the homogeneous maximal ideal by $m$ with respect to the respective embedding. Let

$$A(X, D) = \lim_{k \to \infty} \frac{e_{HK}(R, m^k) - e_0(R, m^k)}{kd^{d-1}}.$$

(1) $\mathbb{P}^1 \times \mathbb{P}^1$

$$\varphi_{\mathbb{P}^1 \times \mathbb{P}^1, -K}(\lambda) = \begin{cases} 1 - 4\lambda^2 & \text{if } 0 \leq \lambda \leq \frac{1}{2}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\int \varphi_{\mathbb{P}^1 \times \mathbb{P}^1, -K}(\lambda)d\lambda = \frac{1}{3}$$ and $A(\mathbb{P}^1 \times \mathbb{P}^1, -K) = 2\left(\frac{1}{3}\right) = \frac{2}{3}.$
Figure 1.

Figure 2.

(2) $\mathbb{P}^2$, the projective space

\[
\varphi_{\mathbb{P}^2, -K}(\lambda) = \begin{cases} 
1 - \frac{3}{2}\lambda^2 & \text{if } 0 \leq \lambda \leq \frac{1}{3}, \\
\frac{1}{2}(2 - 3\lambda)^2 & \text{if } \frac{1}{3} \leq \lambda \leq \frac{2}{3}, \\
0 & \text{otherwise},
\end{cases}
\]

\[
\int \varphi_{\mathbb{P}^2, -K}(\lambda) d\lambda = \frac{1}{3} \quad \text{and} \quad A(\mathbb{P}^2, -K) = \left(\frac{9}{4}\right) \left(\frac{1}{3}\right) = \frac{3}{4}.
\]

(3) $X_3 =$ blow-up of $\mathbb{P}^2$ at one point

\[
\varphi_{X_3, -K}(\lambda) = \begin{cases} 
1 - 4\lambda^2 & \text{if } 0 \leq \lambda \leq \frac{1}{5}, \\
\frac{1}{2}(\lambda^2 - 6\lambda + 3) & \text{if } \frac{1}{5} \leq \lambda \leq \frac{2}{5}, \\
\frac{1}{2}(2 - 3\lambda)^2 & \text{if } \frac{2}{5} \leq \lambda \leq \frac{3}{5}, \\
0 & \text{otherwise},
\end{cases}
\]

\[
\int \varphi_{X_3, -K}(\lambda) d\lambda = \frac{25}{72} \quad \text{and} \quad A(X_3, -K) = 2 \left(\frac{25}{72}\right) = \frac{25}{36}.
\]

(4) $X_4 =$ blow-up of $\mathbb{P}^2$ at two points
\[ X_3 = \text{blow-up of } \mathbb{P}^2 \text{ at one point} \]

\[ P_{-K} \]

\[ \varphi_{X_3,-K}(\lambda) \text{ at } \lambda = \frac{1}{2} \]

\[ X_4 = \text{blow-up of } \mathbb{P}^2 \text{ at two points} \]

\[ P_{-K} \]

\[ \varphi_{X_4,-K}(\lambda) \text{ at } \lambda = \frac{1}{2} \]

\[ \varphi_{X_4,-K}(\lambda) = \begin{cases} 
1 - \frac{7}{4} \lambda^2 & \text{if } 0 \leq \lambda \leq \frac{1}{2}, \\
\frac{1}{2} (2 - 3\lambda)^2 & \text{if } \frac{1}{2} \leq \lambda \leq \frac{2}{3}, \\
0 & \text{otherwise}, 
\end{cases} \]

\[ \int \varphi_{X_4,-K}(\lambda) d\lambda = \frac{13}{36} \quad \text{and} \quad A(X_4, -K) = \left( \frac{7}{4} \right) \left( \frac{13}{36} \right) = \frac{91}{144}. \]

(5) \( X_5 = \text{blow-up of } \mathbb{P}^2 \text{ at three points} \)

\[ \varphi_{X_5,-K}(\lambda) = \begin{cases} 
1 - 3\lambda^2 & \text{if } 0 \leq \lambda \leq \frac{1}{2}, \\
(2 - 3\lambda)^2 & \text{if } \frac{1}{2} \leq \lambda \leq \frac{2}{3}, \\
0 & \text{otherwise}, 
\end{cases} \]

\[ \int \varphi_{X_5,-K}(\lambda) d\lambda = \frac{7}{18} \quad \text{and} \quad A(X_5, -K) = \left( \frac{3}{2} \right) \left( \frac{7}{18} \right) = \frac{7}{12}. \]
X_5 = blow-up of \( \mathbb{P}^2 \) at three points

\[ X_5 = \text{blow-up of } \mathbb{P}^2 \text{ at three points} \]

**Figure 5.**

**Remark 7.5.** Given \( A(X, D) \) as in the above example, if we define (see Theorem 6.3 and (6.2))

\[
B(X, D) = (e_0(X, D))^{\frac{2-d}{d}} A(X, D) - \left[ \frac{d-1}{d} \right] [(d-1)!]^{\frac{2-d}{d}},
\]

then we have

\[
B(\mathbb{P}^1 \times \mathbb{P}^1, -K) = 0 \quad \text{and} \quad B(\mathbb{P}^2, -K) > B(X_3, -K) > B(X_4, -K) > B(X_5, -K).
\]

**Remark 7.6.** The equality given by (6.2) can be achieved by a toric pair \((X, D)\) other than a self product of \((\mathbb{P}^1, O(m_0))\). However, for \(d-1 = 2\) and such a pair \((X, D)\), \(P_D\) must be a centrally symmetric hexagon (a convex body \( C \subset \mathbb{R}^{d-1} \) is said to be centrally symmetric with respect to origin, if \( x \in C \) if and only if \( -x \in C \)), see [Sc]. For \( d = 3 \), consider the fan \( \Delta \) in

**Figure 6.**

where \( v_1 = 2e_1 - e_2, v_2 = e_1 + e_2, v_3 = -e_1 + 2e_2, v_4 = -v_1, v_5 = -v_2, v_6 = -v_3 \). The fan \( \Delta \) has the maximal cones \( \sigma_i = (v_i, v_{i+1}), i = 1, \ldots, 6 \), with the convention \( v_7 = v_1 \). This gives a singular toric surface, since the cones \( \sigma_i \) are not smooth (Theorem 3.1.18, [CLS]). Consider the divisor \( D = 3 \sum D_i \) to get \( P_D \) as in Figure 6. Since dimension of \( P_D \) is 2, \( P_D \) is normal (Theorem 2.2.12, [CLS]) and hence is very ample (Proposition 2.2.18, [CLS]). By Proposition 6.1.10, [CLS] it follows that \( D \) is very ample. We see that such a \( P_D \) is indeed
possible, where $D$ is a very ample $T$-Cartier divisor on $X(\Delta)$ with $\text{Vol}(P_D) = \alpha^2$ such that $\varphi_{X,D}(\lambda/\alpha) = 1 - \lambda^2$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure7.png}
\caption{\label{fig:figure7} $\varphi_D(\lambda) = 0$ at $\lambda = 1/3$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure7.png}
\caption{\label{fig:figure7} $\varphi_D(\lambda) = 1 - \lambda^2$}
\end{figure}

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