On the class of chiral symmetry representations with scalar and pseudoscalar fields

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Abstract  
In the following few pages an account is given of a theme, which I began in 1966 and continued to the present.
1 \[ \Sigma = \frac{1}{\sqrt{2}} \left( \sigma - i \pi \right) \] scalar - pseudoscalar fields and the class of their chiral symmetry representations

Let's denote by \( t, s, n \ldots \) quark flavor indices with
\[ t, s, n \ldots = 1, \ldots, N \equiv N_{fl} \] (1)
and by \( \bar{\lambda}^a \) the \( N^2 \) hermitian \( U_N \) matrices with the normalization
\[ \bar{\lambda}^a = \left( \bar{\lambda}^a \right)_{ts} ; \quad \text{tr} \bar{\lambda}^a \bar{\lambda}^b = \delta_{ab} \]
\[ a = 0, 1, \ldots, N^2 - 1 ; \quad \bar{\lambda}^0 = \sqrt{N}^{-1/2} \left( \mathbf{1} \right)_{N \times N} \] (2)

In order to maintain clear quark field association we choose the convention and restriction projecting out color and spin degrees of freedom from the complete set of \( \bar{q} q \) bilinears
\[ \Sigma_{s i} \sim \bar{q}_i \frac{1}{2} \left( 1 + \gamma_5 R \right) q^c_s \]
\[ \gamma_5 R = \frac{1}{2} \gamma_0 \gamma_1 \gamma_2 \gamma_3 ; \quad c, \bar{c} = 1, 2, 3 \text{ color} \] (3)

The logical structure of \( \Sigma \) - variables is different, when used to derive the dynamics of quarks, i.e. QCD, or before this, when used in their own right as by M. Gell-Mann and M. Lévy [1], or else associating chiral symmetry with superconductivity as by Y. Nambu and G. Jona-Lasinio [2].

Here the chiral \( U N_{fl} R \times U N_{fl} L \) transformations correspond to
\[ UN_{fl} R : \frac{1}{2} \left( 1 + \gamma_5 R \right) q^c_s \rightarrow V_{ss'} \frac{1}{2} \left( 1 + \gamma_5 R \right) q^c_{s'} \]
\[ UN_{fl} L : \frac{1}{2} \left( 1 - \gamma_5 R \right) q^c_s \rightarrow W_{ss'} \frac{1}{2} \left( 1 - \gamma_5 R \right) q^c_{s'} \]
\[ \downarrow \]
\[ \Sigma \rightarrow V \Sigma W^{-1} \] (4)

The construction in eq. (4) can be interpreted as group-complexification, discussed below. The \( \Sigma \)-variables arise as classical field configurations, Legendre transforms of the QCD generating functional driven by general \( x \)-dependent complex color neutral mass terms.
The latter represent external sources with $UN_{fl \ R} \times UN_{fl \ L}$ substitutions aligned with the $\Sigma$-variables

$$- \mathcal{L}_m = m \ i_s (x) \ \{ \ \bar{q} \ \frac{i}{2} (1 - \gamma_5 \ R) \ q \ \} + h.c.$$  

$$\propto \ tr ( m \ \Sigma^\dagger + \Sigma \ m^\dagger )$$

$$m \rightarrow V \ m \ W^{-1} \quad \Sigma \rightarrow V \ \Sigma \ W^{-1}$$

The so defined (classical) target space variables form

- upon the exclusion of values for which $\text{Det} \ \Sigma = 0$

the group

$$GL ( N , C ) = \{ \ \Sigma \ | \ \text{Det} \ \Sigma \neq 0 \}$$

the general linear group over the complex numbers in N dimensional target-space.

We proceed to define the hermitian chiral currents generating $UN_{fl \ R} \times UN_{fl \ L}$ (global) pertaining to $\Sigma$

$$j^a_{\mu \ R} = tr \ \Sigma^\dagger \left( \frac{1}{2} \lambda^a \ i \ \overd \mu \right) \ \Sigma \ \sim \ \bar{q} \ \gamma_\mu \ \frac{1}{2} \lambda^a P_R \ q$$

$$j^a_{\mu \ L} = tr \ \Sigma^\dagger \ i \ \overd \mu \ \Sigma \ (- \frac{1}{2} \lambda^a) \ \sim \ \bar{q} \ \gamma_\mu \ \frac{1}{2} \lambda^a P_L \ q$$

$$A \ \overd \mu \ B = A \ \partial_\mu B - ( \partial_\mu \ A ) \ B ; \ P_R (L) = \frac{1}{2} \ (1 \pm \gamma_5 \ R)$$

We avoid here to couple external sources to all other $\bar{q} q$ bilinears except the scalar - pseudoscalar ones as specified in eq. 5 for two reasons

1) – to retain a minimum set of external sources capable to reproduce spontaneous chiral symmetry breaking alone as a restricted but fully dynamical spontaneous phenomenon.

2) – in order to avoid a nonabelian anomaly structure. The latter would force either the consideration of leptons in addition to quarks, or the inclusion of nonabelian Wess-Zumino terms obtained from connections formed from the $\Sigma$ fields.$^3$
For completeness we display the equal time current algebra relations inherited from $\mathbf{\overline{q}} q$

\[
\left[ j^a_R \left( t , \vec{x} \right) , j^b_R \left( t , \vec{y} \right) \right] = i f_{a b n} j^n_R \left( t , \vec{x} \right) \delta^3 \left( \vec{x} - \vec{y} \right) \\
\left[ j^a_L \left( t , \vec{x} \right) , j^b_L \left( t , \vec{y} \right) \right] = i f_{a b n} j^n_L \left( t , \vec{x} \right) \delta^3 \left( \vec{x} - \vec{y} \right) \\
\left[ j^a_R \left( t , \vec{x} \right) , j^b_L \left( t , \vec{y} \right) \right] = 0 \\
\left[ \frac{1}{2} \lambda^a , \frac{1}{2} \lambda^b \right] = i f_{a b n} \frac{1}{2} \lambda^n
\]

The $GL \left( N , C \right)$ group structure defined in eq. $[4]$ enables bilateral multiplication of the $\Sigma$, $Det \Sigma \neq 0$ elements, of which the left- and right-chiral currents defined in eq. $[7]$ are naturally associated with the Lie-algebra of $UN_{fl R} \times UN_{fl L}$ through the exponential mapping with subgroups of $GL \left( N , C \right)_R \times GL \left( N , C \right)_L$. These (sub)groups act by multiplication of the base-group-manifold by respective multiplication from the left $\leftrightarrow G_R$ and from the right $\leftrightarrow G_L$. The reverse association – here – is accidental

\[
GL \left( N , C \right)_{R \left( L \right)} \rightarrow G_{R \left( L \right)} = G \\
\Sigma \in G ; \ g \in G_R ; \ h \in G_L : \\
G_R \bullet G \leftrightarrow \Sigma \rightarrow g \Sigma \\
G_L \bullet G \leftrightarrow \Sigma \rightarrow \Sigma h^{-1} \\
G_R \otimes G_L \bullet G \leftrightarrow \Sigma \rightarrow g \Sigma h^{-1} \\
\Sigma = \Sigma \left( x \right) ; \ g , h : x\text{-independent or 'rigid'}
\]

The exponential mapping and compactification(s) of $G \left( \Sigma \right)$

The condition $Det \Sigma \neq 0$ in the restriction to $GL \left( N , C \right)$ (eq. $[8]$) is very special and surprising in conjunction with the field variable definition.

In fact such a condition is completely untenable and shall be discussed below. This was a stumbling block for a while.
This condition is equivalent to the relation with the Lie algebra of $GL(N, C)$ through the exponential mapping and its inverse (log):

\[ \Sigma = \exp b = b^a \lambda^a = \frac{1}{2} \lambda^0 = (2N)^{-1/2} \left( \begin{array}{cc} \cdot & \cdot \\
\cdot & \cdot \end{array} \right)_{N \times N} \]

\[ \det \Sigma = \exp \left( \text{tr} \ b \right) = \exp \beta = \sqrt{2N} b^0 \]

\[ \det \Sigma = 0 \leftrightarrow \Re \beta = -\infty ; \beta \sim \beta + 2\pi i \nu ; \nu \in \mathbb{Z} \quad (10) \]

Of course eliminating – from general dynamical $\Sigma$-variables – the subset with $\det \Sigma = 0$ affects only the non-solvable (and non-semi-simple\(^2\)) part of the associated group, whence the former are interpreted as a manifold, which simply is not a group. It may thus appear that the restriction in order to enforce a group structure is characterized by the notion of ‘group-Plague’, infecting the general structure at hand.

This said we continue to treat $\Sigma$-variables as if they were identifiable with $GL(N, C)$.

The next reductive step is to consider the solvable (simple) subgroup

\[ SL(N, C) \subset GL(N, C) \subset \left\{ \Sigma \right\} \]

\[ SL(N, C) = \left\{ \hat{\Sigma} \mid \det \hat{\Sigma} = 1 \right\} \quad (11) \]

\[ \hat{\Sigma} \sim \Sigma / (\det \Sigma)^{1/N} \quad \text{allowing all} \ N \ \text{roots} \]

The advantage of the above reduction to $SL(N, C)$ is that it allows the exponential mapping to an irreducible (simple) Lie-algebra, refining eq. 10

\[ \hat{\Sigma} = \exp \hat{b} = \hat{b}^a \lambda^a ; a = 1, 2, \ldots, N^2 - 1 \]

\[ \hat{b}^0 = 0 ; \text{tr} \lambda^a = 0 \quad (12) \]

i.e. eliminating the unit matrix $\propto \lambda^0$ from the latter.

\(^2\) The words testify to the fight for definite mathematical notions.
1.1 Relaxing the condition $\text{Det} \Sigma \neq 0$ and the unique association

\[
\Sigma \xrightarrow{\text{Det} \Sigma \neq 0} GL(N,C)
\]

We transform $\Sigma_{si}$ as defined or better associated in eq. 3 by means of the $N^2$ hermitian matrices $\overline{\lambda}^a$ in eq. [2].

\[
\Sigma_{si} = \Sigma^a \left( \overline{\lambda}^a \right)_{si}
\]

\[
\Sigma^a = \text{tr} \overline{\lambda}^a \Sigma ; \ a = 0, 1, \ldots, N^2 - 1
\]

The complex (field valued) quantities $\Sigma^a$ are components of a complex $N^2$-dimensional space $C_{N^2}$ and in one to one correspondence with the matrix elements $\Sigma_{si}$

\[
C_{N^2} = \left\{ \left( \Sigma^0, \Sigma^1, \ldots, \Sigma^{N^2-1} \right) \right\}
\]

This serves to become aware of the second algebraic relation ($\oplus$), beyond ($\otimes$), i.e. to add matrices and not to just multiply them.

The $\oplus$ operation is also encountered upon 'shifting' general (pseudo)scalar fields relative to a spontaneous vacuum expected value. This is relevant here for spontaneous breaking of chiral symmetry.

It arises independently for the $SU_2$ $L$-doublet scalar (Higgs) fields.

Hence the idea that the combination of $\oplus$ and $\otimes$ -- which form the full motion group (of matrices) -- are related to 'fields' (Körper' in german). Thus we are led to consider quaternion- and octonion-algebras in the next sections.

1.2 Octonions (or Cayleigh numbers) as pairs of quaternions

Let

\[
q = q^0 i_0 + q^a i_a ; \ a = 1, 2, 3 ; \ (q^0, \overline{q}) \in R_4
\]

\[
i_0 = 1 ; \ i_a i_b = -\delta_{ab} i_0 + \varepsilon_{a b n} i_n \quad \text{for } a, b, n = 1, 2, 3
\]

\[
\overline{q} = q^0 i_0 - q^a i_a
\]

denote a quaternion over the real numbers.

Then a single octonion is represented (modulo external automorphisms $^3$)

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$^3$ Elements of a $N \times N$-matrix can equivalently be arranged along a line.

$^4$ These automorphisms form the exceptional group $G_2$. 

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by a pair of quaternions \((p, q)\) with the nonassociative multiplication rule

\[
o = (p, q) = p^0 j_0 + p^a j_a + q^0 j_4 + q^a j_4 + a
\]

\[
o^\alpha = (p^\alpha, q^\alpha) ; \alpha = 1, 2, \ldots
\]

\[
o^1 \odot o^2 = (p^1 p^2 - q^2 q^1, q^2 p^1 + q^1 p^2)
\]

\[
\overline{o} = (\overline{p}, - q)
\]

\[
\to \text{for } o^2 = \overline{o}^1 ; \quad o^2 = (\overline{p}^1, - q^1)
\]

\[
o^1 \odot (o^2 = \overline{o}^1) = \left(p^1 \overline{p}^1 + \overline{q}^1 q^1, - q^1 p^1 + q^1 \overline{p}^1\right)
\]

\[
= \left\{ |p^1|^2 + |q^1|^2 \right\} j_0 + 0
\]

\[
j_0 = \mathbb{I}, j_1, j_7 ; \quad j_{1,2,3} \simeq i_{1,2,3}
\]

In eq. (16) we used the involutory properties

\[
\overline{q} = q ; \quad \overline{o} = o
\]

It follows that unitary quaternions \((q \overline{q} = q^2 = \mathbb{I})\) are equivalent to \(S_3 \simeq SU2 \subset \mathbb{C}^4\), whereas unitary octonions \((o \overline{o} = o^2 = \mathbb{I})\) are equivalent to \(S_7 \subset \mathbb{R}^8\).

This leads together with the complex numbers to the algebraic association of \(N = 1\) and \(N = 2 - \Sigma\) variables to the three inequivalent ‘field’-algebras

\[
\begin{align*}
1 & : N = 1 \leftrightarrow \mathbb{C} \simeq \mathbb{R}^2 \supset S_1 \\
2 & : N = 2 \leftrightarrow \mathbb{Q} \simeq \mathbb{R}^4 \supset S_3 \\
3 & : N = 2 \leftrightarrow \mathbb{O} \simeq \mathbb{R}^8 \supset S_7
\end{align*}
\]

(18)

The group structures of cases 1 - 3 in eq. (18) correspond to

\[
\begin{align*}
1 & : S_1 \simeq U1 \leftrightarrow U1_R \otimes U1_L \\
2 & : S_3 \simeq SU2 \leftrightarrow SU2_R \otimes SU2_L \\
3 & : S_7 \leftrightarrow U2_L \otimes U2_R
\end{align*}
\]

(19)

While the model introduced by M. Gell-Mann and M. Lévy \([1]\) corresponds to case 2 (eq. (18), (19)), it is case 3 (also for \(N = 2\)) which is different and the only one extendable to \(N > 2\).

This shall be illustrated for \(N = 3\) and from there back to case 3 with \(N = 2\) in the next section.
1.3 \[ \Sigma = \frac{1}{\sqrt{2}} (\sigma - i\pi) \] for \( N = N_{fl} = 3 \) \((m_u \sim m_d \sim m_s)\)

For \( N = 3 \) the \( \Sigma \) - variables describe a \( U3_{fl} \) - nonet of *scalars and pseudoscalars* (one each). I shall use the notation \( \Sigma \rightarrow \pi, K, \eta, \eta' \) labelled by the names of pseudoscalars, yet denoting associated pairs

\[
\Sigma = \left( \begin{array}{ccc}
\Sigma_{11} & \Sigma_{\pi} - \Sigma_{K^{-}} \\
\Sigma_{\pi} + \Sigma_{22} & \Sigma_{K_{0}} \\
\Sigma_{K} + \Sigma_{\bar{K}_{0}} & \Sigma_{33}
\end{array} \right)
\]

(20)

\[
\Sigma_{11} = \frac{1}{\sqrt{3}} \Sigma_{\eta}, + \frac{1}{\sqrt{2}} \Sigma_{\pi_{0}} + \frac{1}{\sqrt{6}} \Sigma_{\eta}
\]

\[
\Sigma_{22} = \frac{1}{\sqrt{3}} \Sigma_{\eta}, - \frac{1}{\sqrt{2}} \Sigma_{\pi_{0}} + \frac{1}{\sqrt{6}} \Sigma_{\eta}
\]

\[
\Sigma_{33} = \frac{1}{\sqrt{3}} \Sigma_{\eta}, - \frac{2}{\sqrt{6}} \Sigma_{\eta}
\]

In the chiral limit \( m_{u,d,s} \rightarrow 0 \) – 8 pseudoscalar Goldstone modes become massless : \( \pi, (3) ; K, \bar{K}, (4) ; \eta, (1) \), whereas \( \eta' \) and all 9 scalars remain massive.

\( \pi_{0} \leftrightarrow \eta \leftrightarrow \eta' \) – mixing – eventually different for scalars relative to pseudoscalars – is not discussed here [4].

Projecting back on case 3 and \( N = 2 \) in the limit \( m_s \rightarrow \infty \) an \( SU2_{fl} \) – singlet pair – denoted \( \Sigma_{\eta_{(2)}} \) – forms as (singlet) combinations of \( \Sigma_{\eta}, \Sigma_{\eta'} \), and a corresponding isotriplet pair \( \Sigma_{\pi} \rightarrow \bar{\Sigma}_{\pi} \).

Instead of the \( 2 \times 2 \) matrix form pertinent to case 3 and \( N = 2 \) we can equivalently display the double quaternion basis from the octonion structure (eq. 16)

\[
p \leftrightarrow \left( \sigma_{\eta_{(2)}}, \pi \right) \rightarrow [1]
\]

\[
q \leftrightarrow \left( \eta_{(2)}, \tilde{\sigma}_{\pi} \right)
\]

(21)
2 From $\langle \Sigma \rangle$ as spontaneous real parameter to $f_\pi$

As shown in section 1, the $\Sigma$ - variables are chosen such , that the spontaneous breaking of just chiral symmetry can be explicitly realized . For $N$ equal ( positive ) quark masses it folows

$$\langle \Sigma \rangle = S \gamma_{N \times N}$$

$$S = \frac{1}{\sqrt{2N}} \langle \sigma^0 \rangle ; \quad \Sigma = \frac{1}{\sqrt{2}} (\sigma - i \pi)_{N \times N}$$

$$j^a_{\mu R} = i S \text{tr} \frac{1}{2} \lambda^a \partial_\mu (\Sigma - \Sigma^\dagger) + \cdots$$

$$= S \partial_\mu \pi^a + \cdots$$

$$\Sigma - \Sigma^\dagger = -i \pi^b \lambda^b$$

$$\langle \Omega | j^a_{\mu R} | \pi^b , p \rangle = i \frac{1}{2} f_\pi p_\mu \delta^{ab} \text{ for } a,b > 0$$

$$-S = \frac{1}{2} f_\pi \leftrightarrow -\langle \sigma^0 \rangle = \left( \frac{N}{2} \right)^{1/2} f_\pi ; \quad f_\pi \sim 92.4 \text{ MeV for } \vec{\pi}$$

(22)

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