Boundary value problem for a fully nonlinear elliptic equation

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Abstract. In this paper, we investigate the well-posedness of a fully nonlinear elliptic problem. By using the acute angle principle for the weakly continuous operator, we obtained the $W^{2,p}$-strong solution of the fully nonlinear elliptic problem.

1. Introduction

In this paper, we study the well-posedness of the following elliptic equation

$$((|\Delta u|^4 + 1)\Delta u = h(x)e^{-2|D^2u| - |Du|} - |u| + f(x),$$

with the boundary condition

$$u|_{\partial \Omega} = 0,$$

where $\Omega \subset \mathbb{R}^n$ is $C^2$ bounded.

The nonlinear elliptic equation has been extensively studied in the past several decades, see [1-5] and the references therein. Among the above literatures, the most mathematical analysis was the well-posedness of the equation, which has been mostly discussed by using some different techniques in the above literatures. Note that Caffarelli [1] obtained a interior $W^{2,p}$-priori estimate for uniformly elliptic equations by using the Aleksandrov-Bakelman-Pucci $L^1 - L^\infty$ a priori estimate, the Krylov-Safonov Harnack inequality and the Calderón-Zygmund decomposition lemma. Moreover, Winter [2] extended Caffarelli's results on interior $W^{2,p}$-estimates for viscosity solutions and proved $W^{2,p}$-estimates at a flat boundary. In addition, Amendola, Rossi and Vitolo [3] established the weak Harnack inequality and local maximum principle to study continuous viscosity solutions for fully nonlinear elliptic equations

$$F(x,u,Du,D^2u) = f(x).$$

Also, Nakagawa researched the maximum principle for $L^p$-viscosity solutions of nonlinear equations in [4].

It is also worth mentioning that Viaclovsky [5] proved the existence of a fully nonlinear partial differential equation on smooth compact n-dimensional Riemannian manifold with $C^2$ estimate on solutions and the $C^{2,\alpha}$ estimate, which come from the work of Evans [6]. Furthermore, Tyagi and Verma [7] researched the existence of nontrivial solutions of a fully nonlinear elliptic equations with gradient nonlinearity. On the other hand, there are also a good amount of interests on the existence of viscosity solutions for elliptic equation, see [8,9]. There are also many other interesting results as for the existence of solutions for fully nonlinear equations, see [10-12] for classical results.

But, among the previous studies, most of the researchers are focused on the existence of viscosity solution, mild solutions and the axially symmetric solutions. As far as we know, the existence of a strong solution for problem (1)-(2) has not been considered before. Motivated from the above
research result, in this paper, we devote to prove the existence of strong solution for the problem (1)-(2). The method we use is the acute angle principle. This theory was proposed by Ma in [13], in which the author presented a novel inner product operator - compulsorily weakly continuous operators. This theory can solve the weak solutions and strong solutions for many nonlinear problems, see [14-16].

In this paper, we make the assumptions of $h$ and $f$ as following:

\begin{itemize}
  \item[(A_1)] $h(x) \in L^\infty(\Omega)$ and satisfying $0 < \sup_{x \in \Omega}|h(x)| < \frac{k_1}{z}$, $k_1 < K$, where $K$ is the best constant satisfying
  \[ K^2\|u\|_{H^2}^2 \leq \int_\Omega |\Delta u|^2 \, dx. \]
  \item[(A_2)] $f(x) \in L^6(\Omega)$.
\end{itemize}

For the problem (1)-(2), our result is the following.

**Theorem 1.1** Assume that $(A_1)$-$(A_2)$ hold true, then the problem (1)-(2) exist a strong solution $u \in W^{2,6}(\Omega) \cap W^{1,6}_0(\Omega)$.

The paper is organized as follows. In Section 2, we formulate the theory of the acute angle principle, especially for weakly continuous operator, and also list some basic lemmas. In Section 3, the proof of Theorem 1.1 is given.

2. Preliminaries

In this part, we introduce the acute angle principle for the compulsorily weakly continuous operator in [13], which can solve the weak solutions and strong solutions for many nonlinear problems.

Let $X$ be a linear space, $X_1, X_2$ be the completion of $X$. The norm of $X_1, X_2$ are $\|\cdot\|_1, \|\cdot\|_2$. Let $X_1$ be a separable Banach space and $X_2$ be a reflexive Banach space. $X \subset X_2$ and $X_1^*$ is the dual space of $X_1$. The linear operator $L$ satisfy

\[ L : X \rightarrow X_1 \] is a dense and one-to-one linear operator.

**Definition 2.1** If for any \( \{u_n\} \subset X_2 \), \( u_n \rightharpoonup u_0 \) in \( X_2 \), and
\[ \lim_{n \to \infty} \langle Gu_n - Gu_0, Lu_n - Lu_0 \rangle = 0, \]
we have
\[ \lim_{n \to \infty} \langle Gu_n, v \rangle = \langle Gu_0, v \rangle, \quad \text{for any } v \in X_1. \]
Then the mapping \( G : X_2 \rightarrow X_1^* \) is called compulsorily weakly continuous.

Using [13, Theorem 3.5], we have the following Lemma.

**Lemma 2.2** Let $U \subset X_2$ be a bounded open set, $0 \in U$. Let $G : X_2 \rightarrow X_1^*$ be a compulsorily weakly continuous operator, and satisfies the following condition
\[ \langle Gu, Lu \rangle \geq 0, \quad \text{for any } u \in \partial U \cap X, \]
where $L : X_2 \rightarrow X_1$ is bounded, then the equation $Gu = 0$ has a solution in $X_2$.

This Lemma is crucial to the proof of our main result.

Next, we recall the following Lemma (see [13]).

**Lemma 2.3** $\Omega \subset R^n$ is an open set, function $f : \Omega \times R^N \rightarrow R^1$ satisfy Carathéodory conditions, the following inequality
\[ |f(x, \xi)| \leq C \sum_{i=1}^N |\xi_i|^{p_i} + b(x), \]
here $C > 0$ is a constant, $p_i, p \geq 1$, $b(x) \in L^p(\Omega)$. If \( \{u_{i_k}\} \in L^p(\Omega) \) $(1 \leq i \leq N)$ is bounded and converges to \( \{u_i\} \) by measure in $\Omega_0$, $\Omega_0$ is a bounded subset of $\Omega$, then for each $v \in L^p(\Omega)$, then we have the following equality
\[ \lim_{k \to \infty} \int_\Omega f(x, u_{1_k}, \cdots, u_{N_k}) v \, dx = \int_\Omega f(x, u_1, \cdots, u_N) v \, dx. \]
where \( p' \) satisfies \( \frac{1}{p'} + \frac{1}{p} = 1 \).

3. Proof of Theorem 1.1

In this section, we prove the Theorem 1.1.

**Proof.** The prove of Theorem 1.1 will be decomposed into in the following three steps.

**Step 1.** The definition of the operator \( G \) is given.

Let

\[
X = \{ u \in C^\infty(\Omega) | u|_{\partial \Omega} = 0 \},
X_1 = L^p(\Omega),
X_2 = W^{2,6}(\Omega) \cap W^{1,6}_0(\Omega).
\]

By the definition of weak solution, we give the definition of the operator \( G: X_2 \to X_1^* \) by following:

\[
\langle Gu, v \rangle = \int_\Omega (|\Delta u|^4 + 1) \Delta u - h(x)e^{-2|\Delta u| - |D^2 u|} - f(x) \rangle v dx,
\]

where \( v \in X_1 \), \( X_1^* \) is the dual space of \( X_1 \). It is apparent that the operator \( G \) was bounded.

**Step 2.** Check the acute angle condition.

Let \( L = \Delta: X \to X_1 \).

By using the Young inequality, (A1) and (A2), we have

\[
\langle Gu, Lu \rangle = \int_\Omega (|\Delta u|^4 + 1) \Delta u - h(x)e^{-2|\Delta u| - |D^2 u|} - f(x) \rangle \Delta u dx
\]

\[
= \int_\Omega (|\Delta u|^6 + |\Delta u|^2 - h(x)e^{-2|\Delta u| - |D^2 u|} - f(x) \rangle \Delta u dx
\]

\[
\geq \int_\Omega \left[ \frac{1}{2} |\Delta u|^6 + \frac{k_1}{2} \Delta u \right] dx - C_1 \int_\Omega |f(x)|^\frac{2}{p} dx, \quad C_1 > 0 (i = 1, 2)
\]

where \( C_i > 0 \) are constants.

According to (3), it is very clear that

\[
\langle Gu, \Delta u \rangle \geq 0, \quad \forall \ u \in X_2, \ ||u||_{X_2} \text{ is great enough.}
\]

This implies that the operator \( G: X_2 \to X_1^* \) we established in **Step 1** meets the acute angle condition.

**Step 3.** Verify the operator \( G \) is a compulsorily weak continuity operator.

Let \( \{ u_n \} \subset X_2, \ u_n \to u_0 \) in \( X_2 \). According to the **Definition 2.1**, we only need to prove the following

\[
\lim_{n \to \infty} \int_\Omega (|\Delta u|^4 + 1) \Delta u_n - h(x)e^{-2|\Delta u_n| - |D^2 u_n|} - f(x) \rangle v dx = \int_\Omega (|\Delta u_0|^4 + 1) \Delta u_0 - h(x)e^{-2|\Delta u_0| - |D^2 u_0|} - f(x) \rangle v dx, \]  

for any \( v \in X_1 \).

Note that

\[
\lim_{n \to \infty} \left\{ (|\Delta u|^4 + 1) \Delta u_n - (|\Delta u_0|^4 + 1) \Delta u_0 \right\} \Delta u_n - \Delta u_0
\]

\[
- h(x) \left[ e^{-2|\Delta u_n| - |D^2 u_n|} - e^{-2|\Delta u_0| - |D^2 u_0|} \right] \langle \Delta u_n - \Delta u_0 \rangle dx = 0
\]

Obviously, (5) is equivalent to
\[
\lim_{n \to \infty} \int_{\Omega} \left\{ \left[ \left| \nabla u_n \right|^4 + 1 \right] \nabla u_n - \left( \left| \nabla u_0 \right|^4 + 1 \right) \nabla u_0 \right\} (\Delta u_n - \Delta u_0) \\
- \nabla \left( e^{-2 |u_n| - |D u_n| - |u_n|} - e^{-2 |u_0| - |D u_0| - |u_0|} \right) (\Delta u_n - \Delta u_0) \\
- \nabla \left( e^{-2 |u_0| - |D u_0| - |u_0|} \right) (\Delta u_n - \Delta u_0) \right\} dx = 0
\]

By using the compact embedding theorem, we have the following relations

\[
(u_n, D u_n) \to (u_0, D u_0) \quad \text{in} \quad \begin{cases}
C(\Omega) \times C(\Omega, R^n), & \text{as } n < p, \\
C(\Omega) \times L^p(\Omega, R^n) \quad \text{as } p < n < 2p, \\
L^2(\Omega) \times L^2(\Omega, R^n), & \text{as } n \geq 2p.
\end{cases}
\]

Under the assumption \((A_2), (7)\) and Lemma 2.3, it is easy to have the following equality

\[
\lim_{n \to \infty} \int_{\Omega} h(x) \left( e^{-2 |D u_n| - |D u_0| - |u_n|} - e^{-2 |D u_0| - |D u_0| - |u_0|} \right) (\Delta u_n - \Delta u_0) dx = 0
\]

By (6) and (8), it is easy to derive that

\[
\lim_{n \to \infty} \int_{\Omega} \left\{ \left[ \left| \nabla u_n \right|^4 + 1 \right] \nabla u_n - \left( \left| \nabla u_0 \right|^4 + 1 \right) \nabla u_0 \right\} (\Delta u_n - \Delta u_0) \\
- \nabla \left( e^{-2 |u_n| - |D u_n| - |u_n|} - e^{-2 |u_0| - |D u_0| - |u_0|} \right) (\Delta u_n - \Delta u_0) \right\} dx = 0.
\]

Combining the differential mean value theorem, Young inequality and \((A_4)\), we obtain the following relation,

\[
\int_{\Omega} \left\{ \left[ \left| \nabla u_n \right|^4 + 1 \right] \nabla u_n - \left( \left| \nabla u_0 \right|^4 + 1 \right) \nabla u_0 \right\} (\Delta u_n - \Delta u_0) \\
- \nabla \left( e^{-2 |D^2 u_n| - |D u_n| - |u_n|} - e^{-2 |D^2 u_0| - |D u_0| - |u_0|} \right) (\Delta u_n - \Delta u_0) \right\} dx \\
= \int_{\Omega} \left\{ (1 + 5 |\Delta u_n|^4) (\Delta u_n - \Delta u_0)^2 + 2 h(x) e^{-2 |D^2 u_n| - |D u_n| - |u_n|} (D^2 u_n - D^2 u_0) (\Delta u_n - \Delta u_0) \right\} dx \\
\geq \int_{\Omega} \left\{ \frac{9}{2} (\Delta u_n - \Delta u_0)^2 - k_1 |D^2 u_n - D^2 u_0|^2 \right\} dx \\
\geq \frac{k_2 - k_1^2}{2} \int_{\Omega} |D^2 u_n - D^2 u_0|^2 dx,
\]

where \(\Delta u_n\) and \(D^2 u_n\) lie between \(\Delta u_n\) and \(\Delta u_0\), between \(D^2 u_n\) and \(D^2 u_0\), respectively.

Obviously, combined with (9) and (10), we can derive that

\[
\lim_{n \to \infty} \int_{\Omega} |D^2 u_n - D^2 u_0|^2 dx = 0.
\]

which implies that \(D^2 u_n \to D^2 u_0\) in \(L^q(\Omega)\) for any \(0 < q \leq 2\).

By the \((7), (11)\) and Lemma 2.3, we prove that the \((4)\) holds true. Then we obtained that the operator \(G\) is a compulsorily weak continuity operator.

Consequently, by Lemma 2.2, it is easy to know that the problem \((1)-(2)\) has a strong solution \(u \in W^{2,6}(\Omega) \cap W^{1,6}_0(\Omega)\).

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