LOCALLY HARMONIC MAASS FORMS OF POSITIVE EVEN WEIGHT

ANDREAS MONO

Abstract. We twist Zagier’s function $f_{k,D}$ by a sign function and a genus character. Assuming weight $0 < k \equiv 2 \pmod{4}$, and letting $D$ be a positive non-square discriminant, we prove that the obstruction to modularity caused by the sign function can be corrected obtaining a locally harmonic Maass form or a local cusp form of the same weight. In addition, we provide an alternative representation of our new function in terms of a twisted trace of modular cycle integrals of a Poincaré series due to Petersson.

1. Introduction and statement of results

In 1975, Zagier [42] defined the function $f_{k,D}(\tau) := \sum_{Q \in Q(D)} \frac{1}{Q(\tau, 1)^k}, \quad \tau \in \mathbb{H} := \{z \in \mathbb{C} : \text{Im}(z) > 0\}$, to investigate the Doi–Naganuma lift. Here and throughout, $Q(D)$ is the set of all integral binary quadratic forms of discriminant $D \in \mathbb{Z}$, and $k \geq 2$. On one hand, if $D > 0$, Zagier proved that they define holomorphic cusp forms of weight $2k$ for $\Gamma := \text{SL}_2(\mathbb{Z})$, and computed their Fourier expansions. On the other hand, if $D < 0$, Bengoechea [3] proved that these are meromorphic cusp forms with respect to the same data, namely meromorphic modular forms which decay like cusp forms towards $i\infty$. The poles are precisely the CM points (sometimes called Heegner points instead) of discriminant $D$, and of order $k$.

Parson [33, Theorem 3.1] investigated Zagier’s $f_{k,D}$ function based on an individual equivalence class $[Q]_\sim \in Q(D)/\Gamma$ of indefinite integral binary quadratic forms, and twisted it by a sign function. More precisely, she defined

$$f_{k,Q}(\tau) := \sum_{Q \sim Q} \frac{\text{sgn}(\tilde{Q})}{\tilde{Q}(\tau, 1)^k}, \quad \text{sgn}(Q) = \text{sgn}([a,b,c]) := \begin{cases} \text{sgn}(a) & \text{if } a \neq 0, \\ \text{sgn}(c) & \text{if } a = 0. \end{cases}$$

Due to the presence of the sign function, we have a non-zero error to modularity, which is a finite sum, and explicitly given by

$$F_{k,Q}(\tau) := f_{k,Q}(\tau) - \tau^{-2k} f_{k,Q} \left(-\frac{1}{\tau}\right) = 2 \sum_{\substack{[a,b,c]=Q \sim Q \\ \text{sgn}(ac)<0}} \frac{\text{sgn}(\tilde{Q})}{\tilde{Q}(\tau, 1)^k}.$$

In other words, the function $f_{k,Q}$ is a modular integral of weight $2k$ for the rational period function $F_{k,Q}(\tau)$. We refer the reader to the work of Knopp [23] for more details.

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In a recent article [30], the author investigated a certain class of Eisenstein series

\[ \mathcal{E}_{k,D}(\tau,s) := \sum_{0 \neq Q \in \mathcal{Q}(D)/\Gamma} \chi_d(Q) \sum_{Q \sim \mathcal{Q}} \frac{\text{sgn}(Q)^\frac{k}{2} \text{Im}(\tau)^s}{Q(\tau,1)^\frac{k}{2} |Q(1,\tau)|^s}, \]  

(1.1)

for any \( k \in 2\mathbb{N} \), which arises by applying Hecke’s trick to Parson’s construction. The function \( \chi_d \) is a genus character (defined in Section 2). By results of Petersson [35 Satz 1, Satz 4, Satz 6], the sum converges absolutely for any \( k \) weights \( k \in \mathbb{E} \), series \( \mathcal{E} \) analytic continuation the Fourier expansion of his aforementioned function. Furthermore, we computed the of Duke, Imamoğlu, Tóth [16] after appealing to Zagier’s work [42, Appendix 2] on function \( \chi_\gamma \) on hyperbolicity, parabolicity, or ellipticity of \( \gamma \). Although we focus on the case of weights \( k \in 2\mathbb{N} \), one may also consider different weights. For instance, all three types of Eisenstein series were studied by Jorgenson, Kramer, von Pippich, Schwagenscheidt, Völz for weight \( k = 0 \), see [22 Theorem 4.2], [36 Section 4], [37 Theorem 1.2].

The paper [30] as well as the present one are devoted to the hyperbolic case. Letting \( D > 0 \) be a non-square discriminant, and \( d \) be a positive fundamental discriminant dividing \( D \), we computed the Fourier expansion of hyperbolic Eisenstein series for any integral weight \( k \in 2\mathbb{N} \) at \( s = 0 \) to prove a conjecture of Matsusaka [29 eq. (2.12)] about their analytic continuation in weight 2. This computation extends earlier work by Gross, Kohnen, Zagier [20, p. 517], who dealt with weights \( 4 \mid k > 2 \) not involving the sign function. In turn, the computation for weights \( k \in 2\mathbb{N} \) relies mainly on results of Duke, Imamoğlu, Tóth [16] after appealing to Zagier’s work [42 Appendix 2] on the Fourier expansion of his aforementioned function. Furthermore, we computed the analytic continuation \( \mathcal{E}_{2,D}(\tau,0) \) explicitly. Up to the addition of the completed Eisenstein series \( E_2^* \) and some constants, it agrees with another modular integral with rational period function, which was studied by Duke, Imamoğlu, Tóth in [15].

In addition, one can inspect the automorphic object arising from the analytic continuation to \( s = 0 \). On one hand, the parabolic and elliptic (twisted) Eisenstein series extend to an ordinary and a polar harmonic Maaß form respectively in weight 2. (We define all occurring types of Maaß forms in Section 2.) While the parabolic case is known by Roelcke [33] and Selberg [40], the elliptic case was proven by Matsusaka in [29 Theorem 2.3] by combining results of Bringmann, Kane [5] and of Bringmann, Kane, Löbrich, Ono, Rolen [10]. On the other hand, the hyperbolic Eisenstein series \( \mathcal{E}_{2,D}(\tau,0) \) (with \( D, d \) as above) coincides with a locally harmonic Maaß form for any \( \tau \) with sufficiently large imaginary part. This raises the natural question towards the obstruction of \( \mathcal{E}_{k,D}(\tau,s) \) to coincide with a local automorphic form, whenever the imaginary part of \( \tau \) is not sufficiently large. To this end, we relate \( \mathcal{E}_{k,D}(\tau,s) \) to the completed hyperbolic Eisenstein series

\[ \mathcal{E}_{k,D}(\tau,s) := \sum_{0 \neq Q \in \mathcal{Q}(D)/\Gamma} \chi_d(Q) \sum_{Q \sim \mathcal{Q}} \frac{\text{sgn}(Q)^\frac{k}{2} \text{Im}(\tau)^s}{Q(\tau,1)^\frac{k}{2} |Q(1,\tau)|^s}, \]  

(1.2)

\[ Q_\tau = [a, b, c]_\tau := \frac{1}{\text{Im}(\tau)} \left( a \right| \tau^2 + b \text{Re}(\tau) + c \),

1Explicitly given by \( Q_\gamma(x, y) := cx^2 + (d - a)xy - by^2 \) for \( \gamma = (\gamma, \frac{\gamma}{d}) \in \Gamma \).
outside the net of Heegner geodesics\
\[ \mathcal{N}(D) := \bigcup_{[a,b,c]=Q \in \mathbb{Q}(D)} \{ \tau \in \mathbb{H} : a|\tau|^2 + bu + c = 0 \} \]
by adding a correction term to \( \mathcal{E}_{k,D}(\tau) \) (see equation (3.1)). A possible connection of our correction term to quantum modular forms (introduced by Zagier [44]) is stated in Section 3.

In particular, the function \( \hat{\mathcal{E}}_{k,D}(\tau,s) \) is modular of weight \( k \) outside \( \mathcal{N}(D) \). To describe the result, we let \( \mathcal{C}_k(h,Q) \) be the weight \( \kappa \) cycle integral of \( h \) associated to \( Q \) (defined in equation (2.11)). Moreover, we let \( \mathbb{P}_k(z_1,z_2) \) be a Poincaré series due to Petersson [35] (see Definition (2.21)), whose properties are collected in Lemma (2.25) below. We refer the reader to Subsection 2.7 for definitions of our local automorphic forms.

**Theorem 1.1.** Let \( 0 < k \equiv 2 \) (mod 4), and \( \tau \in \mathbb{H} \setminus \mathcal{N}(D) \). Let \( D > 0 \) be a non-square discriminant, and \( d \) be a positive fundamental discriminant dividing \( D \).

(i) The function \( \hat{\mathcal{E}}_{2,D}(\tau,0) \) is a locally harmonic Maaß form of weight 2 for \( \Gamma \) with exceptional set \( \mathcal{N}(D) \) as a function of \( \tau \).

(ii) If \( 2 < k \equiv 2 \) (mod 4) then \( \hat{\mathcal{E}}_{k,D}(\tau,0) \) is a local cusp form of weight \( k \) for \( \Gamma \) with exceptional set \( \mathcal{N}(D) \) as a function of \( \tau \).

(iii) Moreover, we have the alternative representation\
\[ \hat{\mathcal{E}}_{k,D}(\tau,0) = \sum_{Q \in \mathbb{Q}(D)/\Gamma} \chi_d(Q) \left\{ \begin{array}{ll} \frac{D}{2\pi} C_0 \left( \frac{j'(\tau)}{j(\tau)} - E_2^2(\tau), Q \right) & \text{if } k = 2, \\
C(k)C_{2-k} \left( \mathbb{P}_k(\tau,\cdot), Q \right) & \text{if } k > 2, \end{array} \right. \]
where \( C(k) \) is an explicit constant provided in equation (4.3).

Remarks.

(1) The cycle integral \( \mathcal{C}_k(\mathbb{P}_k(\cdot,\tau),Q) \) was computed by Löbrich, Schwagenscheidt [25]. Let \( Q_0 \in \mathbb{Q}(D) \), and \( \mathcal{F}_{1-k,Q_0} \) be the locally harmonic Maaß form introduced by Bringmann, Kane, Kohnen [9] (see Section 2.7) with summation restricted to quadratic forms equivalent to \( Q_0 \) under \( \Gamma \). Then [25] Theorem 4.2 states that\
\[ \mathcal{F}_{1-k,Q_0}(\tau) = \frac{D^{-\frac{k}{2}}}{2\pi} \mathcal{C}_k(\mathbb{P}_k(\cdot,\tau),Q_0). \]

In other words, a cycle integral of \( \mathbb{P}_k \) in either of its arguments yields a local automorphic form in the other argument.

(2) A natural splitting of \( z_2 \mapsto \mathbb{P}_k(z_1,z_2) \) into meromorphic and non-meromorphic parts is due to Bringmann, Kane [9] equation (3.6).

(3) Choosing \( d = 1 \), the function \( \hat{\mathcal{E}}_{2k+2,D}(\tau,0), \kappa \in 2\mathbb{N}, \) also appears in a slightly different normalization in [11] (1.7), and further properties of it are stated in [11] Section 4. In particular, \( \hat{\mathcal{E}}_{2k+2,D}(\tau,0) \) gives rise to a locally harmonic Maaß form of weight \( -2\kappa \), whose properties are discussed in [11] Theorem 1.2.

As an application of Theorem 1.1, we would like to highlight a possible connection to twisted central \( L \)-values. This goes back to Kohnen [21] Proposition 7, Corollary 3], who established an identity between the Petersson inner product of a cusp form with Zagiers \( f_{k,D} \) function, and such \( L \)-values for positive even weights. More recently, Kohnen’s work was utilized by Ehlen, Guerzhoy, Kane, Rolen [17] Theorem 1.1] to prove a criterion on the vanishing of certain twisted \( L \)-values under some technical conditions. Their argument relies on the theory of locally harmonic Maaß forms, and in particular on the connection of the \( f_{k,D} \) function to the locally harmonic Maaß form \( \mathcal{F}_{1-k,D} \), see Section...
(In addition, the theory of theta lifts comes in handy to ensure existence of an analytic continuation of $F_{1-k,D}$ to the case $k = 1$.) Being more precise, the form $F_{1-k,D}$ splits into three components (cf. [9, Theorem 7.1]). Two of them are a holomorphic and a non-holomorphic Eichler integral of the $f_{k,D}$-function, while the third component is a so called local polynomial, which captures the behaviour of $F_{1-k,D}$ between different connected components of $\mathbb{H} \setminus N(D)$. The idea of the paper [17] now is to formulate a condition on the local polynomial of $F_{1-k,D}$, evaluated at two special points on the real axis, and relate this conditions to the twisted central $L$-values via the work of Kohnen and of Bringmann, Kane, Kohnen.

Since the function $\hat{E}_{k,D}(\tau, 0)$ is a twisted version of the function $f_{k,D}$, and since we found a connection of $\hat{E}_{k,D}(\tau, 0)$ to a locally harmonic Maass form (resp. local cusp form), we expect that $\hat{E}_{k,D}(\tau, 0)$ may serve as a “building block” to detect the vanishing of suitable twisted $L$-values as well. This inspection is motivated by our remarks following Theorem 1.1.

The paper is organized as follows. We summarize the framework of this paper in Section 2. Section 3 is devoted to the properties of hyperbolic Eisenstein series. This enables us to prove Theorem 1.1 in Section 4.

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2. Preliminaries

We let $\tau = u + iv$, and $q := e^{2\pi i \tau}$ throughout.

2.1. Integral binary quadratic forms. Let $Q$ be an integral binary quadratic form, and we abbreviate such forms by the terminology “quadratic form” throughout. We call a quadratic form primitive if its coefficients are coprime. The full modular group $\Gamma$ acts on the set of quadratic forms by letting

$$(Q \circ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right))(x, y) := Q(ax + by, cx + dy),$$

and this action induces an equivalence relation, which we denote by $\sim$. Moreover, the action of $\Gamma$ on $\mathbb{H}$ by fractional linear transformations is compatible with the action of $\Gamma$ on the set of quadratic forms, in the sense that

$$(Q \circ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right))(\tau, 1) = (c\tau + d)^2 Q(\gamma \tau, 1).$$

A quadratic form $Q$ may be written as $[a, b, c]$, and we denote its discriminant by

$$D([a, b, c]) := b^2 - 4ac \in \mathbb{Z}.$$ 

One can check that equivalent quadratic forms have the same discriminant. The set $Q(D)/\Gamma$ is finite, whenever $D \neq 0$, and its cardinality is called the class number $h(D)$. If $D \equiv 0 \pmod{4}$ or $D \equiv 1 \pmod{4}$, then $Q(D)/\Gamma$ is non-empty. To simplify notation, we identify an equivalence class in $Q(D)/\Gamma$ with any representative of it throughout. A good reference on this is Zagier’s book [43].
2.2. Genus characters. We follow the exposition given by Gross, Kohnen, Zagier in [20, p. 508]. Let \( Q = [a, b, c] \) be a quadratic form, and observe that \( \gcd(a, b, c) \) is invariant under \( \sim \) as well. For any \( D \neq 0 \), let \( d \) be a fundamental discriminant dividing \( D \), and stipulate \( d = 0 \) if \( D = 0 \). We say that an integer \( n \) is represented by \( Q \) if there exist \( x, y \in \mathbb{Z} \), such that \( Q(x, y) = n \), and recall the the Kronecker symbol \( (\frac{D}{n}) \). This established, an extended genus character associated to \( D \) is given by
\[
\chi_d ([a, b, c]) := \begin{cases} (\frac{d}{n}) & \text{if } \gcd(a, b, c, d) = 1, [a, b, c] \text{ represents } n, \gcd(d, n) = 1, \\ 0 & \text{if } \gcd(a, b, c, d) > 1. \end{cases}
\]
One can check that such an integer \( n \) always exists, and that the definition is independent from its choice. Since equivalent quadratic forms represent the same integers, a genus character descends to \( Q \) from its choice. Since equivalent quadratic forms represent the same integers, a genus character descends to \( Q \) from its choice. Since equivalent quadratic forms represent the same integers, a genus character descends to \( Q \) from its choice. Since equivalent quadratic forms represent the same integers, a genus character descends to \( Q \) from its choice. Since equivalent quadratic forms represent the same integers, a genus character descends to \( Q \) from its choice. Since equivalent quadratic forms represent the same integers, a genus character descends to \( Q \) from its choice. Since equivalent quadratic forms represent the same integers, a genus character descends to \( Q \) from its choice. Since equivalent quadratic forms represent the same integers, a genus
denote \( \chi_d(Q) := \chi_d(-Q) = \text{sgn}(d)\chi_d(Q) \) for every \( d \neq 0 \), linking the two choices \( \pm d \). We refer the reader to [20, Proposition 1 and 2] regarding additional properties of \( \chi_d \).

2.3. Heegner geodesics. Once more, let \( Q = [a, b, c] \), and suppose that \( D(Q) > 0 \). Hence, \( Q \) is indefinite, and \( Q(\tau, 1) \) has the two distinct zeros
\[
\frac{-b - D(Q)^{\frac{1}{2}}}{2a}, \quad \frac{-b + D(Q)^{\frac{1}{2}}}{2a} \in \mathbb{R} \cup \{\infty\}.
\]
If \( a = 0 \), then the second zero is given by \(-\frac{1}{2} \). We associate to \( Q \) the Heegner geodesic
\[ S_Q := \{ \tau \in \mathbb{H} : a|\tau|^2 + bu + c = 0 \}, \]
which connects the two zeros of \( Q(\tau, 1) \). On one hand, if \( D(Q) \) is a square and \( a \neq 0 \), then both zeros are rational. In other words, one zero of \( Q(\tau, 1) \) is \( \Gamma \)-equivalent to \( \infty \), and \( S_Q \) is a straight line in \( \mathbb{H} \), perpendicular to \( \mathbb{R} \), based on the second zero. Moreover, the stabilizer
\[ \Gamma_Q := \{ \gamma \in \Gamma : Q \circ \gamma = Q \} \]
is trivial in this case. On the other hand, if \( D(Q) > 0 \) is not a square and \( a \neq 0 \), then both zeros of \( Q(\tau, 1) \) are real quadratic irrationals, which are Galois conjugate to each other. The geodesic \( S_Q \) is an arc in \( \mathbb{H} \), which is perpendicular to \( \mathbb{R} \), and \( S_Q \) is preserved by \( \Gamma_Q \).

We stipulate that \( D \) is a positive non-square discriminant. We obtain infinitely many connected components on \( \mathbb{H} \), and finitely many such components in a fundamental domain for \( \Gamma \), because the class number of \( D \) is finite. Since \( D \) is not a square, each geodesic \( S_Q \) divides \( \mathbb{H} \) into a bounded and an unbounded component, and we denote the bounded component ("interior") of \( \mathbb{H} \setminus S_Q \) by \( A_Q \).

Moreover, there is precisely one unbounded connected component in a fundamental domain for \( \Gamma \), to which we refer as the region "above" the net of geodesics.

Furthermore, we introduce the characteristic funtion
\[
\mathbb{Z}_Q(\tau) := \begin{cases} 1 & \text{if } \tau \in A_Q, \\ 0 & \text{if } \tau \notin A_Q, \end{cases}
\]
whenever \( \tau \in \mathbb{H} \setminus N(D) \). Variants of \( \mathbb{Z}_Q \) appear in [39, Corollary 5.3.5], and in [28, p. 8].
We collect the properties of our sign functions.

**Lemma 2.1.**

(i) For every \( \gamma \in \Gamma \), we have
\[
Q_{\gamma \tau} = (Q \circ \gamma)_\tau.
\]

(ii) We have that \( \tau \in A_Q \) if and only if
\[
\text{sgn}(Q) \, \text{sgn}(Q_\tau) < 0.
\]

(iii) If \( \tau \in \mathbb{H} \setminus \mathcal{N}(D) \), then the sign functions \( \text{sgn}(Q) \), \( \text{sgn}(Q_\tau) \), and \( 1_Q(\tau) \) are related by
\[
\text{sgn}(Q_\tau) = \text{sgn}(Q) (1 - 21_Q(\tau)).
\]

**Proof.** It suffices to check the first item for the two generators
\[
S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]
of \( \Gamma \). Indeed, we calculate that
\[
Q_{S\tau} = \frac{a |S\tau|^2 + b \text{Re}(S\tau) + c}{\text{Im}(S\tau)} = a \left| -\frac{1}{\tau} \right|^2 - b \frac{u}{|\tau|^2} + c = \frac{c |\tau|^2 - b u + a}{v} = [c, -b, a]_\tau,
\]
and
\[
Q_{T\tau} = \frac{a |\tau + 1|^2 + b \text{Re}(\tau + 1) + c}{\text{Im}(\tau + 1)} = \frac{a ((u + 1)^2 + v^2) + b(u + 1) + c}{v} = [a, 2a + b, a + b + c]_\tau = (Q \circ T)_\tau.
\]
The second item is stated as a sentence directly in front of [25, Lemma 4.4], and follows by [9, (5.1), (7.12)]. The third item follows by a case by case analysis using the second item. Indeed, suppose that \( \text{sgn}(Q) = 1 \). Then the second item implies that
\[
\text{sgn}(Q_\tau) = \begin{cases} -1 & \text{if } \tau \in A_Q, \\ +1 & \text{if } \tau \not\in A_Q, \end{cases}
\]
and this coincides with \( \text{sgn}(Q) (1 - 21_Q(\tau)) \). The case \( \text{sgn}(Q) = -1 \) follows in the same manner. \( \square \)

### 2.4. Cycle integrals. Let \( Q \) be such that \( \mathcal{D}(Q) \) is positive and not a square. If \( Q = [a, b, c] \) is primitive, and \( t, r \in \mathbb{N} \) are the smallest solutions to Pell’s equation \( t^2 - \mathcal{D}(Q)r^2 = 4 \), the stabilizer \( \Gamma_Q \) is generated by
\[
\pm \left( \begin{array}{c} \frac{t a r}{\mathcal{D}(Q)} \\ -a r \\ \frac{c r}{2} \end{array} \right).
\]
If \( Q \) is not primitive, one may divide its coefficients by \( \gcd(a, b, c) \) to obtain a generator.

The weight \( k \) cycle integral of a smooth function \( h \), which transforms like a modular form of weight \( k \), is defined as\(^2\)
\[
\mathcal{C}_k(h, Q) := \mathcal{D}(Q)^{\frac{k}{2} - \frac{1}{4}} \int_{\Gamma_Q \setminus S_Q} h(z) Q(z, 1)^{\frac{k}{2} - 1} dz. \quad (2.1)
\]
The integral is oriented counterclockwise if \( \text{sgn}(Q) > 0 \), and clockwise if \( \text{sgn}(Q) < 0 \).

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\(^2\)The normalization by \( \mathcal{D}(Q)^{\frac{k}{2} - \frac{1}{4}} \) is omitted by some authors.
We collect the properties of cycle integrals in the following lemma, which can be proven by calculation, and the fact that \( \Gamma_Q \) only depends on the equivalence class of \( Q \).

**Lemma 2.2.** Let \( f : \mathbb{H} \rightarrow \mathbb{C} \) be smooth, and suppose that \( f \) is modular of weight \( k \). Let \( Q \) be a quadratic form of positive, non-square discriminant. Then the weight \( k \) cycle integral \( C_k(f,Q) \) is a class invariant, namely it depends only on the equivalence class of \( Q \) under \( \sim \). Additionally, the weight \( k \) cycle integral \( C_k(f,Q) \) is invariant under modular substitutions of the integration variable.

Hence, \( \Gamma_Q \setminus S_Q \) projects to a circle in a fundamental domain of \( \Gamma \). The beautiful article \[16\] due to Duke, Imamoğlu, Tóth provides a good reference on Heegner geodesics as well as on cycle integrals.

**2.5. Maaß forms and modular forms.** We recall the definition of various classes of Maaß forms appearing in this paper. The slash operator is given by

\[
(f|k \begin{pmatrix} a & b \\ c & d \end{pmatrix}) (\tau) := \begin{cases} (c\tau + d)^{-k} f(\gamma \tau) & \text{if } k \in \mathbb{Z}, \\ \varepsilon_d (c\tau + d)^{-k} f(\gamma \tau) & \text{if } k \in \frac{1}{2} + \mathbb{Z}, \end{cases}
\]

where \( \left( \frac{d}{\alpha} \right) \) denotes the Kronecker symbol, and

\[
\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}, \\ i & \text{if } d \equiv 3 \pmod{4}. \end{cases}
\]

**Definition 2.3.** Let \( k \in \frac{1}{2} \mathbb{Z} \), choose \( N \in \mathbb{N} \) such that \( 4 \mid N \) whenever \( k \not\in \mathbb{Z} \), and let \( f : \mathbb{H} \rightarrow \mathbb{C} \) be smooth.

(i) We say that \( f \) is a weight \( k \) harmonic Maaß form for \( \Gamma_0(N) \), if \( f \) satisfies the following three properties:

(a) For all \( \gamma \in \Gamma_0(N) \) and all \( \tau \in \mathbb{H} \) we have \( (f|k \gamma)(\tau) = f(\tau) \).

(b) The function \( f \) is harmonic with respect to the weight \( k \) hyperbolic Laplacian on \( \mathbb{H} \), that is

\[
0 = \Delta_k f := \left( -v^2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + i k v \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \right) f.
\]

(c) The function \( f \) is of at most linear exponential growth towards all cusps of \( \Gamma_0(N) \).

(ii) A polar harmonic Maaß form is a harmonic Maaß form, which is permitted to posses isolated poles on the upper half plane.

(iii) A weak Maaß form satisfies conditions (a) and (c) of a harmonic Maaß form, but is allowed to have an arbitrary eigenvalue under \( \Delta_k \).

To study his forms \[26\], Hans Maaß introduced the Maaß lowering and raising operators \[27\]

\[
L_k := -2 iv^2 \frac{\partial}{\partial \tau} = iv^2 \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right), \quad R_k := 2i \frac{\partial}{\partial \tau} + \frac{k}{v},
\]

which decreases or increases the weight of a weak Maaß form by 2, and increases the eigenvalue under the hyperbolic Laplace operator by \( 2 - k \) or \( k \) respectively. A proof can be found in \[11\] Lemma 5.2] for instance. For any \( n \in \mathbb{N}_0 \), we let

\[
L^0_k := \text{id}, \quad L^n_k := L_{k-2n+2} \circ \ldots \circ L_{k-2} \circ L_k, \\
R^0_k := \text{id}, \quad R^n_k := R_{k+2n-2} \circ \ldots \circ R_{k+2} \circ R_k.
\]

\[3\]Be aware that some authors shift their dependence on \( k \), such as Maaß himself.
be the iterated Maaß lowering and raising operators respectively. 
Bruinier, Funke [13] introduced the shadow operator 
\[ \xi_k := 2iv^k \frac{\partial}{\partial \tau} = iv^k \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \]
to study harmonic Maaß forms. They proved that the Fourier expansion of a harmonic 
Maaß form splits naturally into a holomorphic and a non-holomorphic part.

We define \( M_k \) as the space of weakly holomorphic modular forms of weight \( k \), and 
it turns out that \( M_k \) is precisely kernel of \( \xi_k \) restricted to weight \( k \) harmonic Maaß 
forms. Analogously, a meromorphic modular form of weight \( k \) can be regarded as an 
element of the kernel of \( \xi_k \) restricted to weight \( k \) polar harmonic Maaß forms. The space 
of holomorphic modular forms of weight \( k \) is denoted by \( M_k \subseteq M_k \). More details on 
various Maaß forms and their properties can be found in [4] for instance.

### 2.6. Poincaré Series

A first class of examples of (weakly) holomorphic modular forms, and of Maaß forms is given by constructing suitable Poincaré series. Such functions arise 
by averaging a specific auxiliary function (“seed”). Various seeds then lead to various 
examples of Poincaré series.

**Definition 2.4.**

(i) For any \( m \in \mathbb{Z} \), and any \( \kappa \in \mathbb{N}_{>2} \), let 
\[ P_{\kappa,m}(\tau) := \sum_{\gamma \in \Gamma_\infty} q^{m|_{\kappa} \gamma}, \]
be the weight \( \kappa \) Poincaré series of exponential type. 
(ii) Let \( M_{\mu,\nu} \) be the usual \( M \)-Whittaker function, \( m \in \mathbb{Z} \setminus \{0\} \), and define the seed 
\[ g_m(\tau, s) := \frac{\Gamma(s)}{\Gamma(2s)} M_{0, s-\frac{1}{2}}(4\pi |m| y) e^{2\pi i m u} \]
Then the Niebur Poincaré series [31, 32] is given by 
\[ G_m(\tau, s) := \sum_{\gamma \in \Gamma_\infty} g_m(\tau, s)|_{0} \gamma, \quad \text{Re}(s) > 1. \]
(iii) More generally, define the seed 
\[ \varphi_{\kappa,m}(\tau) := \frac{(-\text{sgn}(m))^{1-\kappa}(4\pi |m| u)^{-\frac{\kappa}{2}} \Gamma(2-\kappa)}{M_{\text{sgn}(m)\frac{\kappa}{2} - \frac{1}{2}}(4\pi |m| u)e^{2\pi \text{sgn}(\kappa) i m u}} \]
for any \( m \in \mathbb{Z} \setminus \{0\} \), and \( \kappa \in -\frac{1}{2} \mathbb{N} \). We require the Maaß-Poincaré series of 
negative integral weight \( \kappa \in -\mathbb{N} \), which are defined as 
\[ \Phi_{\kappa,m}(\tau) := \sum_{\gamma \in \Gamma_\infty} \varphi_{\kappa,m}(\tau)|_{\kappa} \gamma. \]
(iv) We encounter one of Petersson’s Poincaré series [31], namely let \( |k, z_1 \) be the weight 
k-operator acting on \( z_1 \), and let \( k \in \mathbb{N}_{>2} \). Then we define 
\[ P_k(z_1, z_2) := \text{Im}(z_2)^{k-1} \sum_{\gamma \in \Gamma} \left( \frac{1}{(z_1 - z_2)(z_1 - z_2)^{k-1}} \right) |_{k, z_1} \gamma \]
\[ = \text{Im}(z_2)^{k-1} \sum_{\gamma \in \Gamma} \left( \frac{1}{(z_1 - z_2)(z_1 - z_2)^{k-1}} \right) |_{-k, z_2} \gamma \]

We summarize their properties.
Lemma 2.5.

(i) The function $P_{k,m}$ is a holomorphic cusp form for any $m > 0$, and a weakly holomorphic modular form for any $m < 0$.
(ii) The function $G_m(\tau,s)$ is a weak Maass form of weight 0 and eigenvalue $s(1-s)$ in $\tau$.
(iii) The function $\Phi_{\kappa,m}(\tau)$ is a harmonic Maass form of weight $\kappa$. It decays like a cusp form towards all cusps inequivalent to $i\infty$, and the principal part at the cusp $i\infty$ is given by

$$\varphi_{\kappa,m}(\tau)q^m.$$ 

(iv) The function $P_k(z_1,z_2)$ is a polar harmonic Maass form of weight $2-k$ in $z_2$, and a meromorphic modular form of weight $k$ without a pole at the cusp in $z_1$. Moreover, the singularities of $P_k(z_1,z_2)$ as a function of either argument are the $\Gamma$-orbits of the other argument.

Proof. To check the claimed growth conditions, one has to compute the Fourier expansions and investigate the constant term in each expansion. We provide a reference for each item.

(i) Compare with [4, Theorems 6.8, 6.9].
(ii) This is computed in [18, Theorem 3.4] (see [19, eq. (1.13)], [16, p. 19] as well).
(iii) This can be found in [4, pp. 97]. The projection to Kohnen’s plus space was calculated in [12, Theorem 2.1].
(iv) The statement in $z_1$ is due to Petersson [34], see [6, Proposition 3.3] as well. The statement in $z_2$ is proven in [6, Proposition 3.2]. The claim dealing with the singularities of $P_k$ follows by its definition.

Modularity is obvious, and the analyticity condition is straightforward to check due to absolute convergence of each series. □

We refer the reader to the exposition in [6] for more details on $P_k$ and related functions.

2.7. Locally harmonic Maass forms and local cusp forms. In [9], Bringmann, Kane, Kohnen introduced locally harmonic Maass forms for $k > 1$, which were independently investigated for $k = 1$ (i.e. weight 0) by Hövel [21] in his PhD thesis as well. We follow [9] here.

Definition 2.6. A locally harmonic Maass form of weight $k$ for $\Gamma$ with exceptional set $X \subset \mathbb{H}$ is a function $f: \mathbb{H} \to \mathbb{C}$, which satisfies the following properties:

(i) For all $\gamma \in \Gamma$ and all $\tau \in \mathbb{H}$ we have $(f|_k \gamma)(\tau) = f(\tau)$.
(ii) For every $\tau \in \mathbb{H} \setminus X$, there exists a neighborhood of $\tau$, in which $f$ is real analytic and $\Delta_k f = 0$.
(iii) For every $\tau \in X$, we have

$$f(\tau) = \frac{1}{2} \lim_{\varepsilon \to 0} \left( f(\tau + i\varepsilon) + f(\tau - i\varepsilon) \right).$$ 

(iv) The function $f$ exhibits at most polynomial growth towards the cusp $i\infty$, namely $f \in O(v^\delta)$ for some $\delta > 0$.

The points in the exceptional set $X$ are called “jump singularities” due to a wall-crossing behaviour between any two connected components of $\mathbb{H} \setminus X$. This definition is
motivated by the peculiar first example

\[ F_{1-k,D}(\tau) := \frac{(-1)^k D^{\frac{k}{2} - k}}{(2k-2)!} \sum_{Q \in Q(D)} \text{sgn}(Q) Q(\tau, 1)^{k-1} \psi_k \left( \frac{D \tau^2}{|Q(\tau, 1)|^2} \right), \]

where \( D > 0 \) is a non-square discriminant, and

\[ \psi_k(y) := \frac{1}{2} \int_0^y t^{k-\frac{3}{2}} (1 - t)^{-\frac{1}{2}} \, dt \]

is a special value of the incomplete \( \beta \)-function. We observe that “locality” is caused precisely by the presence of the sign function in the definition of \( F_{1-k,D} \), and indeed Bringmann, Kane, Kohnen proved that \( F_{1-k,D} \) satisfies their definition with weight \(-2k \in -2\mathbb{N}\) and exceptional set \( \mathcal{N}(D) \).

**Definition 2.7.** A local cusp form of weight \( k \) for \( \Gamma \) with exceptional set \( X \subseteq \mathbb{H} \) is a function \( f : \mathbb{H} \to \mathbb{C} \), which satisfies the following properties:

(i) For all \( \gamma \in \Gamma \) and all \( \tau \in \mathbb{H} \) we have \( (f|_k \gamma)(\tau) = f(\tau) \).

(ii) For every \( \tau \in \mathbb{H} \setminus X \), there exists a neighborhood of \( \tau \), in which \( f \) is holomorphic.

(iii) For every \( \tau \in X \), we have

\[ f(\tau) = \frac{1}{2} \lim_{\varepsilon \to 0} (f(\tau + i\varepsilon) + f(\tau - i\varepsilon)). \]

(iv) The function \( f \) vanishes as \( \tau \to i\infty \).

Altogether, this motivates the definition and inspection of \( \hat{E}_{k,D}(\tau, s) \).

**2.8. The functions** \( E^*_2, j, \) and \( j_m \). The holomorphic Eisenstein series are given by

\[ E_k(\tau) := P_{k,0}(\tau) = 1 - \frac{2}{\zeta(1-k)} \sum_{n \geq 1} \left( \sum_{\ell | n} \ell^{k-1} \right) q^n, \]

where \( \zeta \) denotes the Riemann zeta function. If \( k \geq 4 \) is even then \( E_k \in M_k(\Gamma) \), and \( E_2 \) is quasimodular. We define

\[ E^*_2(\tau) := E_2(\tau) - \frac{3}{\pi \nu}, \]

and observe that \( E^*_2 \) is a harmonic Maaß form of weight 2 for \( \Gamma \) (cf. [4, Lemma 6.2]). The modular invariant for \( \Gamma \) is the function

\[ j(\tau) := \frac{E_4(\tau)^3}{\Delta(\tau)} \in M^!(0)(\Gamma), \]

where

\[ \Delta(\tau) := q \prod_{n \geq 1} (1 - q^n)^{24} = \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728} \in S_{12}(\Gamma) \]

is the normalized modular discriminant function. We denote the normalized derivative of \( j \) by

\[ j'(\tau) := \frac{1}{2\pi i} \frac{\partial j}{\partial \tau}(\tau) = -\frac{E_4(\tau)^2 E_6(\tau)}{\Delta(\tau)} \in M^!_2(\Gamma). \]
The latter identity can be verified by Ramanujan’s differential system [14, Proposition 15]

\[
\frac{1}{2\pi i} \frac{\partial E_2}{\partial \tau} = \frac{E_2^2 - E_4}{12}, \quad \frac{1}{2\pi i} \frac{\partial E_4}{\partial \tau} = \frac{E_2 E_4 - E_6}{3}, \quad \frac{1}{2\pi i} \frac{\partial E_6}{\partial \tau} = \frac{E_2 E_6 - E_4^2}{2}.
\]

As an intermediate result, one can check that

\[
\frac{1}{2\pi i} \frac{\partial \Delta}{\partial \tau} = E_2(\tau) \Delta(\tau).
\]

For every \(m \geq 0\), let \(j_m(\tau)\) be the unique function in the space \(M_0^!(\Gamma)\) having a Fourier expansion of the form \(q^{-m} + O(q)\). For instance, we have

\[
j_0(\tau) = 1, \quad j_1(\tau) = j(\tau) - 744, \quad j_2(\tau) = j(\tau)^2 - 1488j(\tau) + 159768,
\]

and the set \(\{j_m: m \geq 0\}\) is a basis for \(M_0^!\). This was proven by Asai, Kaneko, Ninomiya [2], and they additionally established the expansion

\[
\frac{j'(\tau)}{j(w) - j(\tau)} = \sum_{m \geq 0} j_m(w) q^m, \quad \text{Im}(\tau) > \text{Im}(w).
\]

Alternatively, the functions \(j_m\) can be constructed following [10]. More precisely, the authors proved that the functions \(j_m\) form a Hecke system, that is if \(T_m\) denotes the normalized Hecke operator, then define \(j_0, j_1\) as above, and extend inductively by

\[
j_m = T_m j_1.
\]

3. Hyperbolic Eisenstein series

Let \(D > 0\) be a non-square discriminant, \(d\) be a positive fundamental discriminant dividing \(D\), and \(k \in 2\mathbb{N}\). We recall the definition of our two hyperbolic Eisenstein series from the introduction (see equations (1.1), (1.2)), and the fact that both converge absolutely for any \(s \in \mathbb{C}\) with \(\text{Re}(s) > 1 - \frac{k}{2}\).

**Remark.** Let \(d_{\text{hyp}}\) be the hyperbolic distance. Then, we have

\[
\left| Q(\tau, 1) \right|_v = D(Q)^\frac{1}{2} \cosh (d_{\text{hyp}}(\tau, S_Q)).
\]

A proof of this identity can be found in [41] Lemma 2.5.4. Note that \(z \in S_Q\) if and only if \(d_{\text{hyp}}(z, S_Q) = 0\).

As outlined in the introduction, the function \(E_{2,D}\) possesses an analytic continuation to \(s = 0\), which can be proven by computing the Fourier expansion of \(E_{2,D}\). We recall the result for convenience.

**Lemma 3.1** ([30] Theorem 1.1). Let \(D > 0\) be a non-square discriminant, \(d\) be a positive fundamental discriminant dividing \(D\). Then the function \(E_{2,D}(\tau, s)\) can be analytically continued to \(s = 0\) and the continuation is given by

\[
\lim_{s \to 0} E_{2,D}(\tau, s) = \frac{2}{D} \sum_{m \geq 0} \sum_{Q \in \mathcal{Q}(D)/\Gamma} \chi_d(Q) \zeta_0 (j_m(\cdot) - E_2^*(\tau, Q) q^m
\]
for any $\tau \in \mathbb{H}$. Furthermore, if $v$ is sufficiently large, that is $\tau$ is located above the net of geodesics $N(D)$, then we have

$$\lim_{s \to 0} \mathcal{E}_{2,D}(\tau, s) = -\frac{2}{D} \sum_{Q \in \mathcal{Q}(D) / \Gamma} \chi_d(Q) C_0 \left( \frac{j'(\tau)}{j(w) - j(\tau)} - E_2^*(\tau, Q) \right).$$

Along the lines of Lemma 2.1 (iii), we define

$$\tilde{E}_{k,D}(\tau, s) := \sum_{Q \in \mathcal{Q}(D) / \Gamma} \chi_d(Q) \sum_{Q \sim -Q} \frac{\text{sgn}(Q)^{\frac{k}{2}}}{Q(\tau, 1)^{\frac{k}{2}}} \left| \frac{Q(\tau, 1)}{Q(\tau, 1)} \right|^s.$$

**Remark.** In [44], Zagier introduces the notion of quantum modular forms, and discusses some examples. In particular, his second example involves the quantum modular form

$$\mathcal{E}_{2,D}(\tau, s) = \frac{1}{Q(\tau, 1)^{\frac{k}{2}}} \sum_{Q \in \mathcal{Q}(D) / \Gamma} \chi_d(Q) \sum_{Q \sim -Q} \frac{\text{sgn}(Q)^{\frac{k}{2}}}{Q(\tau, 1)^{\frac{k}{2}}} \left| \frac{Q(\tau, 1)}{Q(\tau, 1)} \right|^s.$$

which appears also in his paper [45]. By Lemma 2.1 (ii), we have $1_{Q(\tau)} = 1$ if and only if $\text{sgn}(Q) \text{sgn}(v Q_x) = -1$. As the zeros of $Q(\tau, 1)$ are quadratic irrationals, the limit $\lim_{\tau \to x} \left| \frac{Q(\tau, 1)}{Q(\tau, 1)} \right|$ exists for every $x \in \mathbb{Q}$. Furthermore, we note that

$$\lim_{\tau \to x} (v Q_\tau) = \lim_{\tau \to x} \left( a |\tau|^2 + bu + c \right) = Q(x, 1).$$

Altogether, this suggests that there might be a connection of the rational function (taking $d = 1$ here)

$$x \mapsto \lim_{\tau \to x} \tilde{E}_{2k-2,D}(\tau, 0) = \sum_{Q \in \mathcal{Q}(D) / \Gamma} \frac{\text{sgn}(Q)^{k-1}}{Q(x, 1)^{k-1}} = -2 \sum_{Q \in \mathcal{Q}(D) / \Gamma} \frac{1}{Q(x, 1)^{k-1}},$$

which vanishes above the net of geodesics $N(D)$, and coincides locally with $\tilde{E}_{k,D}(\tau, 0)$ up to some non-zero constant in any bounded connected component of $\mathbb{H} \setminus N(D)$. Hence, one may obtain a Fourier expansion of $\tilde{E}_{2,D}$ locally by Lemma 3.1 (The computation was presented in [30]). This establishes the existence of $\tilde{E}_{2,D}(\tau, 0)$ via the identity (3.1) from the first item, and in addition proves the third item by uniqueness of the limit.

Moreover, we recall the Fourier expansion of $\tilde{E}_{k,D}(\tau, 0)$ for higher weights.
Lemma 3.3 ([30 Theorem 1.2]). Let \( D > 0 \) be a non-square discriminant, let \( d \) be a positive fundamental discriminant dividing \( D \), and suppose that \( k \geq 4 \) is even. Then, we have the Fourier expansion

\[
E_{k,D}(\tau,0) = \frac{(-1)^{\frac{k}{2}}2\pi^{\frac{k}{2}}}{D^{\frac{k+2}{4}}\Gamma\left(\frac{k}{2}\right)^2} \sum_{m \geq 1} m^{\frac{k}{2} - 1} \sum_{Q \in \mathbb{Q}(D)/\Gamma} \chi_d(Q)C_0\left(G_m\left(\cdot, \frac{k}{2}\right)\right)q^m.
\]

Since \( E_{k,D} \) converges absolutely on \( \mathbb{H} \) at \( s = 0 \) for any \( k \geq 4 \) even, we may rearrange its Fourier expansion, and study the integrand

\[
f(w,\tau) := \sum_{m \geq 1} m^{\frac{k}{2} - 1} G_m\left(w, \frac{k}{2}\right)q^m, \quad w \in \Gamma_Q\setminus\mathcal{S}_Q, \quad \tau \in \mathbb{H}
\]

inside the cycle integral. In other words, we may rewrite the Fourier expansion from the previous Lemma as

\[
E_{k,D}(\tau,0) = \frac{(-1)^{\frac{k}{2}}2\pi^{\frac{k}{2}}}{D^{\frac{k+2}{4}}\Gamma\left(\frac{k}{2}\right)^2} \sum_{Q \in \mathbb{Q}(D)/\Gamma} \chi_d(Q)C_0(f(\cdot,\tau),Q).
\]

We obtained an alternative representation of the Fourier expansion of \( E_{2,D}(\tau,0) \) already if \( \tau \) is located in the unbounded component of a fundamental domain for \( \Gamma \). The main ingredient to prove the second claim of Theorem 1.3 is to find such a representation in the case of higher weights under the same assumption on \( \tau \).

Proposition 3.4. Let \( 2 < k \equiv 2 \pmod{4} \), let \( D > 0 \) be a non-square discriminant, and \( d \) be a positive fundamental discriminant dividing \( D \). Suppose that \( v \) is sufficiently large, that is \( \tau \) is located above the net of geodesics \( \mathcal{N}(D) \). Then \( E_{k,D}(\tau,0) \) coincides with the function

\[
\sum_{Q \in \mathbb{Q}(D)/\Gamma} \chi_d(Q)C_{2-k}(\mathcal{P}_k(\tau,\cdot),Q)
\]

up to an explicit non-zero constant, which is provided in equation (4.3).

Remark (Rearranging the Fourier expansion). Let \( W_{\mu,\nu} \) be the usual \( W \)-Whittaker function. Inserting the Fourier expansion of \( G_{-m} \), next comparing with the Fourier expansion of \( P_{k,m} \) (see the proof of Lemma 2.3 for a list of references), and rearranging further, one obtains

\[
f(w,\tau) = \frac{\Gamma\left(\frac{4}{k}\right)}{\Gamma(k)} \sum_{m \geq 1} m^{\frac{k}{2} - 1} M_{0,\frac{k}{2} - \frac{1}{2}}\left(4\pi |m| \text{Im}(w)\right)e^{-2\pi i m \text{Re}(w)}q^m
\]

\[
+ \frac{2^{2-k} \pi^{\frac{k}{2}} \Gamma(k)}{(k-1)\Gamma\left(\frac{4}{k}\right)} \sin\left(\frac{\pi}{2}(1-k)\right) \text{Im}(w)^{1-\frac{k}{2}} \left(E_k(\tau) - 1\right)
\]

\[
+ i^{-k} \sum_{\mathbb{N} \neq 0} |n|^{\frac{k}{2}} W_{0,\frac{k}{2} - \frac{1}{2}}\left(4\pi |n| \text{Im}(w)\right) \left(P_{k,n}(\tau) - q^n\right)e^{-2\pi i n \text{Re}(w)}.
\]

However, we may not split the final sum involving \( P_{k,n}(\tau) - q^n \) into two separate sums over \( n \), since the resulting expressions would not converge with respect to \( \tau \). This emphasizes the error to modularity of \( E_{k,D} \) from a different viewpoint.
4. Proof of Theorem 1.1

We begin with the proof of Proposition 3.1. To this end, we write \( w = x + iy \in \Gamma_0 \setminus \mathcal{S}_Q \) for the integration variable of the cycle integral, and collect three intermediate results first. In case of ambiguity, we specify the variable a Maaß operator shall act on by an additional subscript next to the weight.

The first step is to convert \( G_m \) to a harmonic Maaß form.

**Lemma 4.1.** We have

\[
\left( L_0^\frac{k}{2} G_{-m} \right) \left( w, \frac{k}{2} \right) = C_1(k) \Gamma(k) \left( \frac{8 \pi |m|}{\sqrt{2} - 1} \right) \phi_{2-k,-m}(w), \quad C_1(k) := \prod_{j=0}^{\frac{k}{2}-1} (k + 2j).
\]

**Proof.** By absolute convergence, we may differentiate the seed directly. We calculate that

\[
L_0^{\frac{k}{2}+1} \left( M_{0,\frac{k}{2}} \left( 4\pi |m| y e^{-2\pi i mx} \right) \right) = \prod_{j=0}^{\ell} (k + 2j) \left( \frac{y}{2} \right)^{\ell+1} M_{\frac{k}{2}+1,\frac{k}{2}} \left( 4\pi |m| y e^{-2\pi i mx} \right)
\]

for every \( \ell \in 2\mathbb{N}_0 \). We compare this with the definition of the seed \( \varphi_{\kappa,m} \), and choose \( \ell = k - 4 \). This yields the claim. \( \square \)

The second step is to connect this result to the Fourier expansion of \( \mathcal{E}_{k,D}(\tau, 0) \). Thus, we need an identity involving (iterated) Maaß operators and cycle integrals. This was performed by Alfes-Neumann, Schwagenscheidt \[1\], generalizing earlier results of Bringmann, Guerzhoy, Kane \[7, 8\]. To simplify the notation, we drop the weights of the cycle integrals temporarily.

**Lemma 4.2** (\[1\] Theorem 1.1). Let \( h : \mathbb{H} \to \mathbb{C} \) be a smooth function, which transforms like a modular form of weight \( 2 - 2\kappa \in 2\mathbb{Z} \) for \( \Gamma \). Then we have the identity

\[
\mathcal{C}(L_{2-2\kappa} h, Q) = \mathcal{C}(R_{2-2\kappa} h, Q) = \mathcal{C}(\xi_{2-2\kappa} h, Q).
\]

Moreover, if \( h \) is a weak Maaß form of weight \( 2 - 2\kappa \) with eigenvalue \( \lambda \), then we have

\[
\mathcal{C} \left( R_{2-2\kappa}^{\kappa-\ell} h, Q \right) = ((\kappa + \ell)(\kappa - \ell - 1) - \lambda) \mathcal{C} \left( R_{2-2\kappa}^{\kappa-\ell-2} h, Q \right), \quad \text{if } \ell \leq \kappa - 2, \tag{4.1}
\]

\[
\mathcal{C} \left( L_{2-2\kappa}^{\kappa-\ell+2} h, Q \right) = ((\kappa + \ell)(\kappa - \ell - 1) - \lambda) \mathcal{C} \left( L_{2-2\kappa}^{\kappa-\ell} h, Q \right), \quad \text{if } \ell \leq -\kappa. \tag{4.2}
\]

Note that the conditions on \( \ell \) in \( (4.1), (4.2) \) include the cases \( R_{2-2\kappa}^0, L_{2-2\kappa}^0 \). Thus, we may insert a suitable chain of raising or lowering operators in our cycle integrals and compensate for that by factors in \( \kappa, \ell \) from equations \( (1.1), (1.2) \).

The third step is to utilize an identity due to Bringmann, Kane \[6\].

**Lemma 4.3** (\[6\] eq. (3.10), (3.11)). We have

\[
\sum_{m \geq 1} \phi_{2-k,-m}(w) q^m = \frac{i}{2\pi} (2i)^{k-1} \varphi_k(\tau, w),
\]

whenever

\[
\text{Im}(\tau) > \max \left( \text{Im}(w), \frac{1}{\text{Im}(w)} \right).
\]

Now, we are in position to prove Proposition 3.1.
Proof of Proposition \[\text{Proposition} \] Since $\tau$ is assumed to be located above the net of geodesics, the assumption from Lemma \[\text{Lemma} \] is satisfied for every $w \in \mathcal{N}(D)$. (Im($w$) is bounded from below and above.) In addition, we have no poles of $\mathbb{P}_k$ for such $\tau$ and $w$.

We invoke Lemma \[\text{Lemma} \] and employ equation \[\text{Equation} \] backwards and iteratively to the integrand

$$f(w, \tau) = \sum_{m \geq 1} m^{\frac{k}{2} - 1} G_{-m} \left( w, \frac{k}{2} \right) q^m,$$

from the Fourier expansion of $\mathcal{E}_{k,D}$. Here, we keep $\tau$ fixed, and take $\kappa = 1, \lambda = \frac{k}{2} \left( 1 - \frac{k}{2} \right)$, and $\ell = -1, -3, \ldots, -\frac{k}{2} + 2$ using that $k \equiv 2 \pmod{4}$. This produces the constant

$$C_2(k) := \prod_{\ell = -\frac{k}{2} + 2}^{1} \frac{1}{(1 + \ell)(1 - \frac{k}{2} - \ell - \frac{k}{2})}.$$

To indicate the steps, we keep the constants until the last equation. Combining, we have

$$\mathcal{E}_{k,D}(\tau, 0) = \frac{(-1)^{\frac{k}{2} + 2} 2 \pi}{D \sqrt{\Gamma \left( \frac{k}{2} \right)}} \sum_{Q \in \mathbb{Q}(D)/\Gamma} \chi_d(Q) C_0 \left( L^k_{0}, f(\cdot, \tau), Q \right)$$

$$= \frac{(-1)^{\frac{k}{2} + 2} 2 \pi}{D \sqrt{\Gamma \left( \frac{k}{2} \right)}} \sum_{Q \in \mathbb{Q}(D)/\Gamma} \chi_d(Q) C_{2-k} \left( L^k_{0}, f(\cdot, \tau), Q \right)$$

$$= \frac{(-1)^{\frac{k}{2} + 2} 2 \pi}{D \sqrt{\Gamma \left( \frac{k}{2} \right)}} \sum_{Q \in \mathbb{Q}(D)/\Gamma} \chi_d(Q) C_{2-k} \left( \sum_{m \geq 1} \Phi_{2-k, -m}(\cdot) q^m, Q \right)$$

The constant in front of the final sum simplifies to

$$C(k) := \frac{(-1)^{k} \Gamma(k)}{2^{\frac{k}{2} - 2} D \sqrt{\Gamma \left( \frac{k}{2} \right)}} C_1(k) C_2(k). \tag{4.3}$$

This establishes the Proposition.

We conclude this section and the paper with the proof of Theorem \[\text{Theorem} \].

Proof of Theorem \[\text{Theorem} \].

(i) The case $k = 2$ was shown in \[\text{Reference} \] in the unbounded component of $\mathbb{H} \setminus \mathcal{N}(D)$ for $\mathcal{E}_{2,D}(\tau, 0)$. Since $\mathcal{\tilde{E}}_{k,D}(\tau, 0) = \mathcal{E}_{k,D}(\tau, 0)$ in the unbounded component by definition of $\mathbb{Q}_Q$, the result of \[\text{Reference} \] extends to $\mathcal{\tilde{E}}_{k,D}(\tau, 0)$ in the unbounded component of $\mathbb{H} \setminus \mathcal{N}(D)$ directly. Now, we can use modularity of $\mathcal{\tilde{E}}_{k,D}(\tau, 0)$ to obtain the claim in the other connected components of $\mathbb{H} \setminus \mathcal{N}(D)$.

(ii) Suppose that $2 < k \equiv 2 \pmod{4}$. Modularity follows by Lemma \[\text{Lemma} \] (i). By Lemma \[\text{Lemma} \] (iii), $\mathcal{\tilde{E}}_{k,D}(\tau, 0)$ is holomorphic outside $\mathcal{N}(D)$. The limit condition on $\mathcal{N}(D)$ can be verified by adapting the proof of \[\text{Reference} \] Proposition 5.2] straightforwardly. The vanishing at $i \infty$ either follows by $\text{sgn}(Q_\tau) = 1$ in the unbounded component and cuspidality of $f_{k,D}$, or by the Fourier expansions of $\mathcal{E}_{k,D}(\tau, 0)$ and $\mathcal{\tilde{E}}_{k,D}(\tau, 0)$. 

\[\square\]
We prove the explicit representation of \( \hat{E}_{k,D}(\tau,0) \) outside \( \mathcal{N}(D) \). If \( \tau \) is located above the net of geodesics \( \mathcal{N}(D) \), we have \( \hat{E}_{k,D}(\tau,0) = E_{k,D}(\tau,0) \). We apply Proposition 3.4 and obtain the claimed representation of \( \hat{E}_{k,D} \) above the net of geodesics. Finally, the representation extends to every connected component of \( \mathbb{H} \setminus \mathcal{N}(D) \) by virtue of weight \( k \) modularity of both sides of the claimed identity.

\[ \square \]

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