A Diagrammatic Theory of Random Scattering Matrices  
for Normal–Superconducting Mesoscopic Junctions

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Abstract

The planar–diagrammatic technique of large–$N$ random matrices is extended to evaluate averages over the circular ensemble of unitary matrices. It is then applied to study transport through a disordered metallic “grain”, attached through ideal leads to a normal electrode and to a superconducting electrode. The latter enforces boundary conditions which coherently couple electrons and holes at the Fermi energy through Andreev scattering. Consequently, the leading order of the conductance is altered, and thus changes much larger than $e^2/h$ are observed when, e.g., a weak magnetic field is applied. This is in agreement with existing theories. The approach developed here is intermediate between the theory of dirty superconductors (the Usadel equations) and the random–matrix approach involving transmission eigenvalues (e.g. the DMPK equation) in the following sense: even though one starts from a scattering formalism, a quantity analogous to the superconducting order–parameter within the system naturally arises. The method can be applied to a variety of mesoscopic normal–superconducting structures, but for brevity we consider here only the case of a simple disordered N–S junction.

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I. INTRODUCTION

Dissipative single–electron transport in mesoscopic structures involving normal and superconducting elements has in recent years become a topic of lively experimental [1–6] and theoretical [7–12] activity. From both of these points of view, this is a natural extension of the interest in usual (i.e., normal) mesoscopic systems. Indeed, many of the technological and mathematical techniques used here are familiar from that field: lithography, lock–in amplifiers, diagrammatic perturbation theory, and random matrices, to name a few. However, this extension has often lead to surprises.

Consider for example the effect of a magnetic field on a normal mesoscopic metallic system, which can be described in the leading–order approximation by classical dynamics of non–interacting electrons at the Fermi surface. The conductance is then determined by the density of states and the geometric details of the sample, and is of the order of $N e^2/h$ where $N$ is the number of propagating modes in the system, $e$ is the electronic charge and $h$ is Planck’s constant. In the present work, $N$ is taken to be large, whereas additional factors which appear in the conductance, such as the factor $l/L$ where $l$ is the transport mean–free–path and $L$ is the length of the system, are taken to be of order unity and considered as fixed geometric characteristics of the system.

If the magnetic field in such a system is so weak that it does not affect the classical motion of the electrons, it may still shift the phases of different interfering partial waves in the Schrodinger equation, thus leading to changes in the conductance which are well–known [13] to be universal, of the order of the quantum unit of conductance, $e^2/h$. This contrasts sharply with the situation in the presence of one or more superconducting electrodes (we use “electrode” rather than the more technical nomenclature — “particle reservoir”). By the process of Andreev reflection [14,15], an electron–like excitation in the normal part of the system may reflect off the normal–superconducting (N–S) boundary and give rise to a hole–like excitation. The extra charge of two electrons is thus transformed into a Cooper pair, which is carried away by the superconductor. The Andreev–reflected hole is necessarily found
to be in the state time–reversed to the impinging electron. If perfect coherence is maintained between electrons and holes, an analogy may be drawn between the N–S boundary and a phase–conjugating mirror, and one finds that the hole retraces the possibly complicated motion of the electron. A weak magnetic field may break the symmetry between electrons and holes, leading to large effects in the conductance, of the order of $N e^2/h$.

Our purpose here is to suggest a new theoretical technique for calculating such effects in dirty mesoscopic N–S structures, which complements the existing theoretical tools in interesting ways. In this approach the system is described as a collection of ergodic scatterers, each possessing a random scattering matrix. The scattering cavities are connected to each other and to the external electrodes through leads of various widths. In this first paper, we demonstrate the new approach by studying a system described by only one scattering cavity, attached to a single normal electrode and a single superconducting electrode through two ideal leads (see Fig. 1). The discussion of more complicated structures (having more than one scattering cavity, more than two electrodes, and non–ideal leads) is deferred to a future publication [16].

At first sight, it might seem that the conductances of such systems would always be larger in the case with electron–hole symmetry than in the non–symmetric case, because whenever an electron happens to undergo Andreev reflection, the charge transport is “automatically” doubled by the corresponding motion of the hole. However, only some of the partial waves representing an electron entering the system from a normal electrode will end up impinging on a superconducting electrode, and thus the hole only imperfectly reproduces the wavefronts and the motion of the electron. Furthermore, the presence of Andreev scattering actually reduces the density of states at the Fermi surface — an aspect of the proximity effect which allows the gap of the superconductor to be manifest in the adjacent normal material — and thus the conductance may sometimes be lowered more than it is enhanced. In the approach to be described below, the magnitude of both of these effects is described by a single quantity $f$, which in principle may depend on the position within the (normal part of the) system, and on the energy of the pertinent electrons. We define $f$ as the (averaged, leading–order
of the) probability amplitude for an electron at that position and energy to propagate to a superconducting electrode, be Andreev-reflected, and then to coherently propagate back to the same position as a hole. The density of states at the Fermi level is then equal to $(1 - |f|^2)/(1 + |f|^2)$ times the normal one \([17]\) (i.e. the density of states in the absence of induced superconductivity).

The present theoretical approach creates a link between the available calculational tools for disordered mesoscopic N–S structures. On the one hand, the starting point is the expression of the conductance in terms of scattering properties (an appropriate Landauer formula), which themselves are assumed to be described well by certain ensembles of scattering matrices. In this respect, the approach is similar in spirit to a large number of recent articles (e.g. those studying the Dorokhov–Mello–Pereyra–Kumar equation \([9–11,18]\)), in which the distribution of transmission-matrix eigenvalues for various structures was studied and used to obtain the conductance \([19]\) (the transmission matrix is simply related to the scattering matrix of a structure, and carries the same information). On the other hand, the large-\(N\) diagrams used naturally lead to the definition of the probability-amplitude \(f\), which is closely related to the space- and frequency-dependent superconducting order parameter of the Usadel equations \([20]\). This set of equations, which is widely used in the theory of dirty superconductors, is ultimately based on the Keldysh technique of diagrammatic perturbation theory.

Another approach which proves useful for disordered mesoscopic systems is the semiclassical approach of the Gutzwiller formula or the van Vleck formula \([21]\). However, the semiclassical approximation for disordered systems is not strictly unitary, and this turns out to be a fatal flaw in the present context (see the next section). Additional approaches developed for mesoscopic systems in general, such as the nonlinear \(\sigma\)-model \([22]\), do not suffer from this flaw and are, in principle, applicable.

The outline of the paper is as follows. The more technical part of the introduction is given in the following subsection. In section II, we develop the formalism of planar diagrams for averaging over large-\(N\) random unitary matrices: the diagrammatic elements are discussed,
and used to form “self–energies” and “effective couplings”. This section is general, in the sense that its results are used for both the simple junction studied in section III and the more complicated structures (networks of cavities) to be discussed in Ref. [16]. In section III we proceed to apply the formalism to the simplest possible system — that having only two leads — and compare our results with the literature. Finally, conclusions and an outlook are given in section IV.

Preliminaries

As discussed above, only normal–superconducting systems with simple geometries — those describable by a single “grain” (Fig. 1) — are considered in the present work. The grain is taken to be an ideal scattering cavity with ergodic single–electron internal dynamics, i.e., its scattering matrix $S$ is assumed to be a random member of the Circular Orthogonal Ensemble (COE) of random matrix theory [23] (the ensemble of unitary symmetric matrices — the symmetry or “orthogonality” corresponds physically to time–reversal symmetry). This ensemble is known to describe a wide universality class of physical systems, including at least two different relevant categories: clean cavities with shapes which lead to chaotic classical dynamics, and disordered materials for which the chaotic scattering is provided by impurities. In both of these cases one assumes that the escape time for an electron in the cavity is much longer than the time required for ergodicity — otherwise the description using a single random scattering matrix becomes unjustifiable.

The grain is connected to external electrodes via leads which are assumed to be ideal (“waveguides” of electrons), and can have different widths, denoted $W_n$. The physical widths are assumed to be much larger than the Fermi wavelength, and the $W_n$’s are thus large integers which give the number of propagating modes in each lead. The total number of modes, $N = \sum_n W_n$, gives the size of the scattering matrix $S$. In the simplest case, to be discussed in detail in section III, there are only two leads — one “superconducting lead” of width $W_1$ connects the system to a superconducting electrode, and one “normal lead”
of width $W_2$ connects it to a normal electrode. In this case, the $S$ matrix of the cavity is written as

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix},$$

(1)

where the reflection matrices $r$ (of size $W_1 \times W_1$) and $r'$ (of size $W_2 \times W_2$) are not equal in size, and the two transmission matrices $t$ and $t'$ are rectangular (except for the case $W_1 = W_2$).

In general, $S$ may have more than four blocks, describing scattering from and into more than two leads (at least one normal lead and at least one superconducting lead are necessary for the conductances calculated here to be of relevance). However, the statistical properties of the $S$–matrix elements, according to the COE, do not depend on the number and sizes of the blocks.

Once an electron has scattered in the cavity it may enter the superconducting lead and impinge upon the superconducting electrode. As we will assume that the electron is propagating at the Fermi level, it can not enter the superconductor because of the gap, and it will be completely reflected. However, it may in principle undergo normal reflection, and return as an electron, or Andreev reflection [14] and return as a hole. In the present work we will assume for simplicity that the probability of normal reflection vanishes, so that the probability amplitude for Andreev reflection is of magnitude unity (this assumption may be relaxed by introducing tunnel barriers in the ideal leads). The phase of this probability amplitude is determined from the Bogoliubov – de Gennes equations [24], and the amplitude is found to be $ie^{-i\phi}$ where $\phi$ is the phase of the superconducting order parameter in the electrode. The scattering of holes in the cavity is described by $S^*$, the complex conjugate of the scattering matrix for electrons. The amplitude for a hole impinging on the N–S boundary to be Andreev–reflected back into an electron is $ie^{i\phi}$.

The extra phase factors of $i$ are crucial to the description of the induced proximity effects — they lead to a relative minus sign, i.e. destructive interference, between the amplitude for an electron to reach any final state through a certain history, and the amplitude to reach the same final state through a history with two additional Andreev reflections (into a hole
and back into an electron). In the absence of any time–reversal symmetry–breaking effects, the dynamics is just such that two consecutive Andreev reflections tend to reproduce the same state, and so this destructive interference leads to a suppression of (again) the leading order expression for the density of states, by the factor of \((1 - |f|^2)/(1 + |f|^2)\) mentioned above.

According to the appropriate Landauer formula \cite{25}, the average conductance of such a structure (at zero temperature and zero bias voltage) is given by

\[
\langle \sigma_{NS} \rangle = \frac{4e^2}{h} \left\langle \text{tr} \ T_{eh} T_{eh}^\dagger \right\rangle ,
\]  

(2)

where \(T_{eh}\) is an electron–hole “transmission” matrix, giving the probability amplitudes for an electron injected into the cavity through any one of the modes of the normal lead to come back as a hole in any other of the modes of that lead (the notation \(\langle \ldots \rangle\) denotes averaging of the \(S\) matrix over the ensemble). This is similar to the standard Landauer formula, \(\sigma_N = (2e^2/h)\text{tr} \ t t^\dagger\) where the factor of \(e^2/h\) is the quantum unit of conductance, i.e., the conductance per mode of an ideal lead with no scattering, the factor of 2 comes from spin (which will play no role in this paper), and the trace sums the probabilities for electrons to be transmitted through the system. The extra factor of 2 in Eq. (2) is due to the fact that for every electron from the normal lead that comes back as a hole, by charge conservation two electrons have gone into the superconductor.

The amplitudes in \(T_{eh}\) are given by a sum over all possible histories. In the simplest kind of history, an electron injected into the cavity traverses the cavity with the transmission amplitude \(t'\), Andreev reflects off the superconductor with the amplitude \(i\) and becomes a hole, and then the hole traverses the cavity with transmission amplitude \(t^*\). This gives a contribution of \(t^* i t'\). As the system under discussion has only a single superconducting lead, we have taken here for simplicity \(\phi = 0\), and avoided the extra phase factors (because of gauge invariance, they are only important in a system with at least two superconducting leads).

In the next simplest history, after Andreev reflection, the hole is scattered back towards the superconductor (rather than traversing the cavity), with the reflection amplitude \(r^*\). It then
Andreev reflects and becomes an electron. The electron bounces off the cavity back towards the superconductor again with the reflection amplitude $r$, only to come back as a hole after Andreev reflection. The hole finally traverses the cavity with transmission amplitude $t'$. This contributes an amplitude of $t'iriri^*t'$ to $T_{eh}$. The matrix $T_{eh}$ may thus be written as

$$T_{eh} = t^*(i + iriri^*i + iriri^*i + \ldots)t'$$

This series for $T_{eh}$ could be written directly in terms of the $S$ matrix — for example, the term $t'iriri^*it'$ can be written as $i^3 S^*SS^*S$ — provided that we adopt the peculiar convention that in this product of four unitary matrices the internal indices are to be summed over only the $W_1$ indices pertaining to the lead connecting the cavity to the superconductor and the external indices are restricted to the $W_2$ indices pertaining to the external lead.

Several simplifying assumptions are made here, such as the complete absence of decoherence and symmetry breaking effects, i.e. essentially zero applied voltage, zero magnetic field, and zero temperature. The results are contrasted with those pertaining to the same system with the symmetries completely broken, but the specific form of the temperature or magnetic field dependence cannot be evaluated so simply (as in Ref. [11]), except for the case of a magnetic field in a multiply–connected geometry, with only simple Aharonov–Bohm phases. Also, only the leading–order, $O(N)$, contribution to the conductance is evaluated here, in contrast with the transmission–matrix approach which often focuses attention on the $O(N^0)$ corrections to the conductance and its fluctuations [11,18]. Another drastic simplification is the description of possibly complicated experimental geometries by a single scattering cavity, with the external leads considered as ideal. As already mentioned, work on extending the present approach to deal with more realistic situations is in progress [16].

II. PLANAR DIAGRAMS FOR LARGE $N$ UNITARY RANDOM MATRICES

In this section we develop a diagrammatic technique which may be used to evaluate the average conductance of Eq. (2), and is accurate to leading order in $1/N$. The diagrammatic
description of the unaveraged conductance is given in Fig. 2. Propagation of an electron through an ideal lead is denoted by a thin line with an arrow to the right; propagation of a hole is denoted by a line with a left–pointing arrow. An Andreev scattering event is denoted by a full semicircle. Each element of $S$ carries two indices representing incoming and outgoing transverse modes in the leads, and corresponds to a dangling double line. These double lines will eventually be connected to each other in various ways, due to averaging over the ensemble of $S$. The fact that one of the indices carried by two consecutive $S$–matrix elements is identical — the mode number is not changed by Andreev reflection — corresponds to the continuous single line which connects these two elements. The conductance itself, according to Eq. (2) and Eq. (3), corresponds to a sum over all possible “bubbles” with two electron lines leaving the left–most point and two hole lines arriving at the right–most point (see the figure), with an odd number of Andreev reflections occurring along the lower line and along the upper line. We use the convention that the upper line corresponds to the second $T_{eh}$ factor in Eq. (2), and all expressions pertaining to it are thus complex–conjugated (or Hermitean–conjugated).

In the first subsection below, we give a general discussion of averages of products of $S$–matrix elements, including our use of time–reversal symmetry to simplify the expressions, and the failure of a naive semiclassical approach to the evaluation of these averages. In the second subsection, we demonstrate that the corresponding averages may be described diagrammatically by introducing elements which couple $k$ copies of $S$ with $k$ copies of $S^\dagger$, with weights as given in Fig. 3. Although the correlations represented by the higher–order couplings of Fig. 3 are obviously weaker than those represented by the $k = 1$, “simple” coupling, they affect more matrix elements, and thus contribute to the conductance of Eq. (2) in the leading order. It is remarkable that even though large random unitary matrices possess such more subtle correlations (when compared to the Gaussian ensembles of Hermitean matrices), an analogue of Wick’s theorem still holds: a diagram composed of several of the elements of Fig. 3 carries a weight which, to leading order, is the product of the weights of the elements.
There is a close analogy to be drawn between the present diagrams and ’t Hooft’s analysis of quantum chromodynamics in the large $N$ limit [26]: the double lines are analogous to gluon propagators and the single lines to those of quarks. This analogy is useful because the index structure is similar — each continuous single line carries an index which is to be summed over $O(N)$ different modes (and each double line carries the corresponding pair of indices). Indeed, the basic rule that only “planar” diagrams contribute to the leading–order results carries over to the present situation: a coupling with a weight of order $1/N^{2k-1}$ contributes to the leading order only if it divides the plane of the diagram into $2k$ disconnected sub–regions, thus increasing the number of factors of $N$ due to independent summations by $2k – 1$. Note that the analogy with quantum chromodynamics is certainly incomplete — for example, the directions of the arrows in the diagrams is different, and there is no analogue here of gluon self–interactions (the couplings here connect $S$–matrix elements, and have no dynamics of their own). Another important difference is the fact that the mode index here can belong to different leads, with Andreev reflections occurring only in the superconducting leads, and the external points (the left–most and right–most points) imposing restrictions to modes in normal leads — each index is thus summed only over a subset of the $N$ possible values.

As often done in diagrammatic quantum field theory, we will do the averaging in two steps. First, we restrict ourselves to connecting the double lines ($S$–matrix elements) within $T_{eh}$ to each other (and similarly the double lines within $T_{eh}^\dagger$ to each other), i.e., we will sum over diagrams in which the upper and the lower lines in the conductance bubble do not interact with each other. Later, we will connect the double lines within $T_{eh}$ to those within $T_{eh}^\dagger$, and form diagrams analogous to the diffuson contribution in the calculation of the conductance of disordered metals [13]. The generic features of these two steps, involving the “self–energy” and the “effective coupling”, will be discussed in the third subsection below. Specific details which differ from one system to the next are deferred to the following section.
A. Generalities

One of the simplest examples of random–matrix ensemble averages to be considered is that of a product of only two $S$–matrix elements:

$$\langle S_{ij}S_{ji}^{\dagger} \rangle_{COE} = \frac{1 + \delta_{ij}}{N + 1}. \quad (4)$$

This is an exact result — valid for any $N$ (the subscript COE implies that the system is time–reversal invariant). Summation over repeated indices is not implied in the present notation. In the following, it will be important to distinguish between two “leading–order” approximations of this result, the “naive” one being

$$\frac{1 + \delta_{ij}}{N + 1} \simeq \frac{1 + \delta_{ij}}{N}, \quad (5)$$

and the other being

$$\frac{1 + \delta_{ij}}{N + 1} \sim \frac{1}{N} \quad (6)$$

(note the use of different relational operators). Because each of the indices $i$ and $j$ is eventually to be summed independently over a set of a size of order $N$, the $\delta_{ij}$ term will turn out to contribute less by one factor of $N$, and is omitted in the “true” leading order. In the present work, the notion of “leading order” is always used in this second sense, unless the word “naive” is explicitly added. In this example, a term which is naively of leading order disappears in the eventual evaluation of the true leading order of the conductance. In contrast, below we will encounter terms which are omitted in the naive leading order, but eventually contribute to the leading order results because they contain less $\delta$–function factors.

For expressions involving arbitrarily many $S$–matrix elements, the naive leading–order results are easily found; for example they can be obtained from a semiclassical approximation \[21\]. In this approximation $S$ is written as

$$S_{ij} \simeq \sum_{\mu \in \{ij\}} A_{\mu} e^{i s_{\mu}/\hbar}, \quad (7)$$
where $\mu$ is an index enumerating classical orbits in the set \{ij\}, i.e. those that have initial conditions corresponding to mode $j$ and final conditions corresponding to mode $i$ (classical orbits which enter and leave the cavity at angles specified by $j$ and $i$ respectively, and propagate at the Fermi energy). The amplitude for propagation along the orbit $\mu$ is denoted by $A_\mu$, and its classical action by $s_\mu$.

Products of elements of $S$ correspond to multiple sums over orbits, and averaging corresponds to dropping all terms which have non–trivial phase factors [27]. The approximation here is simply that $\hbar$ is small relative to the possible fluctuations in the actions $s_\mu$, regardless of whether these fluctuations occur within a certain set \{ij\} due to the classically chaotic nature of the dynamics in the cavity, or between different realizations of a disordered potential in the cavity. Consider, for example, the case without time–reversal symmetry, denoted by the subscript CUE (Circular Unitary Ensemble). In this case we have

$$\langle S_{ij} S_{ji}^\dagger \rangle_{\text{CUE}} \simeq \left \langle \sum_{\mu,\nu \in \{ij\}} A_\mu A_\nu^* e^{i(s_\mu - s_\nu)/\hbar} \right \rangle \simeq \left \langle \sum_{\mu \in \{ij\}} |A_\mu|^2 \right \rangle = \frac{1}{N} . \tag{8}$$

The last equality follows from the correspondence between $|A_\mu|^2$ and classical probabilities, and the assumption of ergodicity, i.e. that the particle “forgets” its initial conditions once it enters the cavity, and is equally likely to leave the cavity in any outgoing mode or direction.

For products of many elements of $S$ and $S^\dagger$, the obvious generalization is to pair each element of $S$ with an element of $S^\dagger$ in all possible manners, so that cancelations of the actions can occur. For example,

$$\langle S_{ij} S_{jk}^\dagger S_{kl} S_{li}^\dagger \rangle_{\text{CUE}} \simeq \frac{1}{N^2} \left( \delta_{ik} + \delta_{jl} \right) . \tag{9}$$

Such expressions may be obtained by joining the double lines in the diagrams using only the first element of Fig. 3, as displayed in the first few diagrams in Fig. 4 [specifically, that of Fig. 4(a), the first two in Fig. 4(b), and the first five in Fig. 4(c); in the last case the sixth possible permutation is omitted because it contributes at a higher order in $1/N$]. This corresponds to ignoring the unitarity of the matrices, and taking the results for Hermitian matrices instead (the Gaussian Unitary Ensemble). As we will see in the next subsection,
an additional term of order $1/N^3$ must be added to the right hand side of this equation in order not to violate unitarity, and such “naively” higher–order terms eventually affect the leading order of the conductance.

In the case with time–reversal symmetry (COE), the action differences $s_\mu - s_\nu$ vanish identically not only for $\nu$ equal to $\mu \in \{ij\}$, but also for $\nu$ equal to $\mu^T \in \{ji\}$, which is the orbit related to $\mu$ by time reversal. This leads to an extra factor of $1 + \delta_{ij}$ in the second–to–last and last expressions in Eq. (8), and similarly to additional terms in Eq. (9). However, the “true” leading order result is independent of the symmetry, e.g.,

$$\langle S_{ij} S_{ji}^\dagger \rangle_{COE} \approx \langle S_{ij} S_{ji}^\dagger \rangle_{CUE}. \quad (10)$$

The same holds also for averages involving arbitrarily many matrix elements alternating from $S$ and $S^\dagger$. For the naive leading order contribution to the “true” leading order, this follows directly from the semiclassical approximation discussed above, as can be seen by imagining all possible ways of connecting all the matrix elements with simple double lines — whenever we use the symmetry of $S$ to generate additional terms, we break the planarity of the diagram (the corresponding double line is crossed) and thus generate a contribution of a “true” higher order (containing additional $\delta$–function factors). In the next subsection we will see that the terms which are of both naive and “true” leading order, together with the property of unitarity, suffice to determine the terms which are naively of higher order but contribute to the leading order of the conductance. Thus, as both the COE and the CUE are ensembles of unitary matrices, Eq. (10) and its many–$S$–and–$S^\dagger$ generalization must hold also beyond the naive leading order. Note that it is of course crucial to use $S^\dagger$ and not $S^*$ in Eq. (10). For example $\langle S_{ij} S_{ji}^* \rangle_{CUE} = \delta_{ij}/N$ and would contribute in a higher order than $\langle S_{ij} S_{ji}^\dagger \rangle_{COE}$ given in Eq. (4).

Below we will use the time–reversal symmetry property, $S^T = S$ and $S^* = S^\dagger$, to rewrite all our expressions in terms of $S$ and $S^\dagger$ only (they naturally appear in alternating order), and then rely on Eq. (10) to avoid the necessity of actually computing in the COE. For example, take the term $\langle S_{kj}^* S_{ji} \left( S_{kn}^* S_{nm} S_{ml}^* S_{li} \right) \rangle_{COE}$ displayed in Fig. 2, which describes interference
between the amplitude for a hole produced by a single Andreev reflection and that produced by three consecutive Andreev reflections (with the multiplicative factors of $i$ omitted). This term is first transformed to read $\langle S^\dagger_{kj} S^\dagger_{ji} S^\dagger_{lm} S^\dagger_{mn} S^*_{nk} \rangle_{COE}$, and then the leading order in $N$ result for its average is obtained within the CUE. Such averages over the unitary ensemble, i.e., over the unitary group $U(N)$, have been performed, e.g., in Ref. [28], for general $N$.

We proceed to follow a somewhat heuristic diagrammatic approach for evaluating them to leading order in $1/N$, referring to this work for a mathematically rigorous derivation.

If time–reversal symmetry is broken, the replacement of $S^*$ by $S^\dagger$ is not permissible, and this affects the leading order results for the conductance, as already emphasized in the introduction. The symmetry may be broken by giving the electrons and holes a finite excitation energy, due to either a finite bias voltage or a finite temperature. In this case, the electrons travel with an energy $E_F + \epsilon$ and the corresponding holes with an energy of $E_F - \epsilon$ (where $E_F$ denotes the Fermi energy and $\epsilon$ the positive excitation energy). Thus $S^*$ is in fact the complex conjugate of $S$ evaluated at a different energy, and the conductance may be obtained by assuming that $S$ and $S^*$ are two independent members of the CUE, if $\epsilon$ is greater than the range in energy over which correlations in $S$ die out. This range or correlation energy, also called the Thouless energy, is easily evaluated from Eq. (7): one notes that the derivative of the classical action $s_\mu$ with respect to the energy is equal to $t_\mu$, the duration of the orbit $\mu$. All of these orbits are of lengths of the order of $t_{esc}$, the escape time from the cavity, and thus the Thouless energy is equal to $\hbar/t_{esc}$.

Another effect which breaks the symmetry between electrons and holes involves applying a magnetic field, stronger than the corresponding correlation field [29]. Here $S^*$ is the complex conjugate of $S$ (evaluated at the same energy), but is distinct from $S^\dagger$. However, the leading–order results for the conductance may still be obtained by considering them as independent members of the CUE (see, e.g., Ref. [11]). This can easily be demonstrated using the diagrammatic language — all the planar diagrams have in this case the property that all the double lines connect $S$’s to $S^\dagger$’s and $S^*$’s to $S^T$’s, with no intermixing [the argument is essentially the same as that used in the discussion of Eq. (10)]. In the physical discussion of
the following section, the results for the symmetric case are compared with those pertaining to such time–reversal non–symmetric situations, which turn out to be much easier to evaluate (cf. the last paragraph of this section).

B. Generalized couplings for unitary matrices

We thus would like to calculate expressions of the type \( \langle S_{ia} S_{aj}^\dagger S_{jb} S_{bk}^\dagger \ldots S_{ic} S_{ci}^\dagger \rangle \), where \( \langle \ldots \rangle \) denotes the average over the unitary group \( U(N) \). We have used letters from different sections of the alphabet to emphasize the fact that, for elements of this group, the first (or upper) index and the second (or lower) index of \( S \) transform differently (covariantly and contravariantly) under \( U(N) \) (the reverse holds for \( S^\dagger \)). It follows directly from the invariance of the probability distribution of \( S \) under such \( U(N) \) transformations that the averaged product of any \( n \) elements of \( S \) and \( m \) elements of \( S^\dagger \) may be written as

\[
\langle S_{i_1 a_1} S_{i_2 a_2} \ldots S_{i_n a_n} S_{b_1 j_1}^\dagger S_{b_2 j_2}^\dagger \ldots S_{b_m j_m}^\dagger \rangle = \delta_{nm} \sum_{u,v} C_{u,v} \delta_{i_1 j_{u(1)}} \delta_{i_2 j_{u(2)}} \ldots \delta_{i_n j_{u(n)}} \delta_{a_1 b_{v(1)}} \delta_{a_2 b_{v(2)}} \ldots \delta_{a_n b_{v(n)}} ,
\]

where \( u \) and \( v \) are permutations of \( n \) elements and \( C_{u,v} \) are coefficients to be determined (the average vanishes of course for \( m \) different from \( n \)). Thus, \( \delta \)–functions such as \( \delta_{ia} \) or products of them will never appear.

These products of \( \delta \)–functions may be represented diagrammatically using the elements of Fig. 3, with a simple \( k = 1 \) coupling for every 1–cycle in the combined permutation \( uv^{-1} \), a four–sided \( (k = 2) \) coupling for every 2–cycle, etc. The condition of planarity of the diagrams does not enter at this stage because we have not yet distinguished between contributions of different orders. The purpose of this subsection is to demonstrate that the leading–order expressions for the \( C_{u,v} \) coefficients can be expressed as products of weights corresponding to the individual cycles in \( uv^{-1} \), and that these weights are indeed those given in Fig. 3. Interestingly, we show that all the necessary properties of the \( C_{u,v} \)’s follow from the unitarity of \( S \), i.e. from the equalities \( \sum_a S_{ia} S_{aj}^\dagger = \delta_{ij} \) and \( \sum_j S_{ai}^\dagger S_{ib} = \delta_{ab} \). The unitarity
of $S$ corresponds physically to the conservation of the number of particles in the cavity, i.e. to the fact that the incoming flux of electrons (or of holes) is equal to the outgoing flux.

We will now calculate the required averages iteratively, starting from averages of only two elements and gradually building up. Start with

$$\langle S_{ia} S_{ai}^\dagger \rangle = \frac{1}{N} \equiv \frac{\gamma_2}{N}. \quad (12)$$

The equality here follows from the unitarity of $S$ as we can see by summing both sides of the equality over $a$ (or $i$). This is represented in figure 4(a) (note that in this figure the arrows have been omitted, and the matrix elements appear in counter-clockwise, rather than clockwise, order). We have introduced a quantity $\gamma_2$ defined to be 1 for later convenience.

We next write

$$\langle S_{ia} S_{ij}^\dagger S_{jb} S_{bi}^\dagger \rangle \sim \delta_{ij} \langle S_{ia} S_{ai}^\dagger \rangle = \frac{\delta_{ij}}{N^2} + \frac{\gamma_4}{N^3} + \frac{\gamma_2 \gamma_4}{N^4} \quad (13)$$

where the first term corresponds to the naive leading order result, and the second term is a possible correction of naively higher order. It involves $\gamma_4$, which is an unknown "coupling constant" of a four–sided interaction [see Fig. 4(b)]. To determine $\gamma_4$ we sum this equation over $b$. The left hand side gives $\delta_{ij} \langle S_{ia} S_{ai}^\dagger \rangle = \frac{\delta_{ij}}{N}$, the right hand side $(1+\frac{\delta_{ij} N + \gamma_4}{N^3})$, and thus we obtain $\gamma_4 = -1 = -\gamma_2 \gamma_2$.

To see the recursion pattern, we go one step further and consider

$$\langle S_{ia} S_{aj}^\dagger S_{jb} S_{bk}^\dagger S_{kc} S_{ck}^\dagger \rangle \sim \langle \ldots \rangle + \frac{\gamma_2 \gamma_4}{N^4} \left( \delta_{ac} + \delta_{ik} + \delta_{bc} + \delta_{jk} + \delta_{ab} + \delta_{ij} \right) + \frac{\gamma_6}{N^5} \quad (14)$$

see Fig. 4(c). Here $\langle \ldots \rangle$ represents the naive leading order term — a sum of terms each involving two Kronecker deltas, for example, $\delta_{ab} \delta_{bc} / N^3$ or $\delta_{ik} \delta_{ab} / N^3$ — which are generated by using only the "simple" $\gamma_2$ couplings. The second term, involving a single $\delta$ function, is due to diagrams with one "simple" coupling, $\gamma_2$, and one four–sided vertex, $\gamma_4$. The third term corresponds to the six–sided vertex which must be introduced as well.

One can easily see that the terms required by unitarity at (say) the $c$ vertex are correctly reproduced by all the diagrams with a simple double line connecting $S_{kc}$ with $S_{ci}^\dagger$, and
all the previously derived elements (in this case \( \gamma_2 \) and \( \gamma_4 \)) connecting the other four \( S \)-matrix elements. Of the twelve terms in Fig. 4(c), only three are of this type, and the other nine terms must thus cancel each other when summed over \( c \). There are two types of cancelations between these “unwanted” terms. The first type occurs, for example, between the term \( \delta_{ab}\delta_{bc}/N^3 \) and the term \( -\delta_{ab}/N^4 \). More generally, such cancelations occur due to the weights of the lower–order couplings already determined (in this case the weight \( \gamma_4 \)), with a “distant” part of the diagram (in this case the simple coupling leading to the \( \delta_{ab} \) factor) playing only a passive role. There are three pairs of diagrams in Fig. 4(c) which cancel in this way. The second type of cancelation accounts for the remaining three diagrams, including the one with the new six–sided vertex, whose weight \( \gamma_6 \) we need to determine. The diagram with the six-sided vertex involves no \( \delta \)-functions, and the only terms which upon summing over \( c \) would produce no Kronecker delta are evidently the two terms \( \delta_{bc} + \delta_{ca} \). Thus, we have the recursion relation \( \gamma_6 = -2\gamma_2\gamma_4 \).

Speaking picturesquely, we see that we have a combinatorial problem in which in general \( 2k \) diners are seated around a round table and engaging in conversation. Note that the indices \( i, a, j, \ldots \) are to be associated with the spaces between the diners. Each diner is identified by two indices: thus, diner \([ia]\), whom we may think of as a gentleman, is associated with \( S_{ia} \), diner \([aj]\), whom we may think of as a lady, is associated with \( S_{\dagger aj} \), and so forth. The number of diners \( 2k \) is required to be always even. In the case \( 2k = 6 \) we just discussed, \( \gamma_6 \) corresponds to all six diners engaging in one conversational group, \( \gamma_4\gamma_2 \) corresponds to four of the diners engaging in one conversational group with the other two speaking only to each other. The rules of large \( N \) tells us that etiquette dictates that the “lines of conversation” cannot cross.

Proceeding in this way, we see that for the general case the second type of cancelation will occur if

\[
\gamma_{2k} + \sum_{l=1}^{k-1} \gamma_{2l}\gamma_{2(k-l)} = 0 .
\]  

(15)

We can readily interpret the terms as follows: in the first term all of the \( 2k \) diners are
engaged in one conversation, while in the second term the group has broken up in two, with the two diners who share the index on which we are summing (which was index $c$ in the above example) belonging to different groups. The sum is over all terms with a single $\delta$ function connecting that index ($c$) with all others of its type (covariant or contravariant). There are always an even number of diners in any conversational group, the number of “males” being equal to the number of “females” — the subscript of $\gamma$ is always even.

This last fact clearly suggests defining $c_j = (-1)^{j+1}g_{2j}$, in terms of which we have the recursion relation

$$c_k = \sum_{j=1}^{k-1} c_j c_{k-j},$$

(16)

with the explicit solution

$$c_k = \frac{(2k)!}{2(2k-1)(k!)^2}$$

(17)

[another way of stating this result is: $c_k = (4 - 6/k)c_{k-1}; c_1 = 1$.] Actually, for our later applications we don’t need the explicit solution for $c_k$ as much as the generating function defined by $w(x) = \sum_{k=1}^\infty c_k x^k$. The recursion relation above immediately implies

$$w = x + w^2;$$

(18)

the extra term $x$ on the right hand side can be seen by noting that the series for $w(x)$ starts with $x$ for small $x$ and hence $w^2$ starts with $x^2$. We thus directly obtain

$$w = \left(1 - \sqrt{1 - 4x}\right)/2,$$

(19)

which implies (17). In the appendix, we give an alternative derivation, which involves generalizing the energy–dependent Green’s functions often used for random Hermitean matrices to the case of unitary matrices.

In the above, we have tacitly assumed that the weights of diagrams with several coupling elements are given by the product of the weights of the elements (those of Fig. 3). Our argument so far does not demonstrate that this is the only way to maintain unitarity.
However, the unitarity relations between diagrams with $2k$ matrix elements and with $2k - 2$ matrix elements do in fact suffice to determine the weights uniquely \[28\], and therefore the weights we have found are the correct weights. To see this, imagine that all weights of all diagrams with $2k - 2$ or less matrix elements have been determined, and that you need to determine the weight of the diagrams with $2k$ elements. Start from all diagrams having at least one simple double line connecting two adjacent “diners” or matrix elements. By summing over the index shared by these two diners, one immediately gets the weight of that diagram as $1/N$ times the weight of the corresponding diagram with that double line omitted (as well as the two diners it connects). For diagrams which do not have such pairs, one can start from a foursome of adjacent diners, and sum over any one of the indices internal to that foursome. Obviously this can be continued until all diagrams of $2k$ elements have been accounted for, with the last diagram serving to determine $\gamma_{2k}$ according to Eq. (15). Note that the actual $C_{u,v}$ for general $N$ have more complicated expressions, with special cases for $n > N$. However, in the large $N$ limit, corrections to the above analysis (e.g. due to non-planar diagrams) are smaller than the leading term by a factor of $1/N^2$. This high degree of accuracy of the leading order is specific to the case without time-reversal symmetry — corrections of relative order $1/N$ would be present in the COE.

C. The self–energy $\Sigma$ and the effective coupling $\Gamma$

According to the diagrammatic rules established above, only planar diagrams contribute to the leading order results. In the present subsection we use these rules to define a “self–energy” and an “effective coupling” in a manner analogous to similar quantities which appear in diagrammatic analyses of perturbative field theory. We thus group together diagrams which differ from each other in their internal structure, but not in the way they connect to the rest of the diagram (or the rest of the “plane”). There are two relevant families — the “one–particle irreducible” diagrams are summed in Fig. 5(a), and the “two–particle irreducible” diagrams are summed in Fig. 5(d). These elements may then be used to calculate
the “averaged Green’s function” $G$, the “diffuson” $D$, and eventually the conductance $\sigma$ (see Fig. 5). Note that although the analogy to the usual perturbation theory of disordered metals is both striking and useful, there are also important differences: the scattering is included in a very different way, and unlike a typical Green’s function in quantum field theory, there is no energy variable involved in $G$ or $\Sigma$ (at least at the present level).

Following from Fig. 5, we have a “Schwinger–Dyson” equation for the Green’s function

$$G = G_0 + G_0 \Sigma G,$$

(20)

or equivalently

$$G^{-1} = G_0^{-1} - \Sigma$$

(21)

(from here on we use $= \text{rather than } \approx$, although it should be remembered that results are being evaluated only to leading order). The notation here is such that $G_0$ represents propagation with a single Andreev reflection [30] — it is a two–by–two, off–diagonal matrix in electron–hole space, whereas it is diagonal in mode–space. Similarly, $\Sigma$ and $G$ are off–diagonal in electron–hole space. Note that the self–energy $\Sigma$ differs from the other quantities in this equation in that it is proportional to the unit matrix in mode space — the diagrams of Fig. 5(a) are independent of the (single) mode index carried by the external lines.

It is convenient to parameterize $\Sigma$ by a single complex parameter $f$,

$$\Sigma = \begin{pmatrix} 0 & if^* \\ if & 0 \end{pmatrix},$$

(22)

where the structure in mode space is not explicitly written. The fact that the electron–hole element $if$ is minus the complex conjugate of the hole–electron element $if^*$ follows directly from the fact that a single Andreev reflection, and thus $G_0$ and $G$, also share this structure. One may refer to $f$ as the amplitude for Andreev reflection in the cavity, because following from Fig. 5(a), to leading order in $N$ and for the averaged behavior, an electron entering the cavity from any one of the leads may be considered to be Andreev reflected directly into the time–reversed hole trajectory, with a probability amplitude $f$ (and the extra phase factor $i$).
The Green’s functions $G_0$ and $G$ have a more complicated internal structure. To maintain the generality of the present section, we consider here a cavity connected to several superconducting and normal leads. The bare Green’s function, $G_0$, is parameterized as in Eq. (22), with $f$ replaced by $e^{-i\phi}$, using different values of $\phi$ for modes in different superconducting leads, and $f$ replaced by 0 (and hence $G_0 = 0$) for modes in normal leads. The corresponding elements of $G$ are, according to Eq. (21), similarly described by a parameter $g = 1/(e^{i\phi} + f^*)$, and since $\phi$ varies from one superconducting lead to the next, so does $g$. This total amplitude for transforming an electron into a hole can be understood as a sum over terms with an odd number of Andreev reflections:

$$i g = ie^{-i\phi} + i^3 e^{-i\phi} f^* e^{-i\phi} + i^5 e^{-i\phi} f^* e^{-i\phi} f^* e^{-i\phi} + \ldots.$$  

The specific values of $g$, and the number of modes for which each one of them occurs, depend on the geometrical structure of the system. Such specifics will be discussed in the following section. However, one may make progress by introducing a notation for the trace of each of the off–diagonal blocks of $G$, which reads

$$i N \alpha = \sum_j N_{j,h} G_{j,h,e},$$

and leads directly to

$$\alpha = \sum_n \frac{W_n}{N} g_n = \sum_n \frac{W_n}{N} \frac{1}{e^{i\phi_n} + f^*},$$

(23)

where the sum over $n$ runs over the superconducting lead(s). The trace of the hole–electron block is similarly equal to $i N \alpha^*$. We proceed to derive a general relationship between $\alpha$ and $f$, which taken together with Eq. (23) will eventually allow us to find $f$ for particular systems.

According to Fig. 5(a), the “self energy” $\Sigma$ is given by a sum of terms involving higher and higher order couplings. Each $\gamma_{2k}$ coupling divides the plane of the diagram into an external part and $2k - 1$ internal “triangles”, which give rise to factors of $\alpha$ and $\alpha^*$ (traces over $G_{he}$ and $G_{eh}$). Mathematically, this relationship reads

$$f = \gamma_2 \alpha - \gamma_4 \alpha |\alpha|^2 + \gamma_6 \alpha |\alpha|^4 + \ldots = \sum_{k=1}^{\infty} \gamma_{2k} \alpha \left(-|\alpha|^2\right)^{k-1},$$

(24)

or, in terms of the $c_k$s,

$$f = \sum_{k=1}^{\infty} c_k \alpha |\alpha|^{2k-2}.$$  

(25)
Comparing with (19) we find

\[ f = \frac{1}{2\alpha^*} \left( 1 - \sqrt{1 - 4|\alpha|^2} \right). \]  

(26)

Actually, a more useful form comes directly from Eq. (18) with the substitution \( w \rightarrow \alpha^*f \) and \( x \rightarrow |\alpha|^2 \) (using the fact that the complex phase of \( f \) is equal to that of \( \alpha \)):

\[ \alpha = \frac{f}{1 + |f|^2}, \]  

(27)

a result which we will use repeatedly.

We now turn to a similar consideration of the “two–particle irreducible” diagrams, i.e. the “effective coupling” \( \Gamma \) depicted in Fig. 5(d), and its relationship with the “diffuson” \( D \) of Fig. 5(e). The structure in mode space is again trivial, with both \( \Gamma \) and \( D \) having two independent mode indices on the left and right (in fact one should consider \( \Gamma \) and \( D \) to have four mode indices, with the right two and the left two being equal to each other). However, the structure in electron–hole space is more complicated than before, having both diagonal and off–diagonal elements. Before any summation over the external mode indices, the diagrams in Fig. 5(d) and Fig. 5(e) are of order \( 1/N \), and we define \( \Gamma \) and \( D \) to be \( N \) times the corresponding diagrams. We parameterize \( \Gamma \) by \( \Gamma = (\Gamma_d I + \Gamma_c \tau) \), where \( I \) (often omitted) and \( \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) denote the identity matrix and the first Pauli matrix respectively. Here \( \Gamma_d \) (for “diagonal”) is the probability for an electron to remain an electron, and a hole to remain a hole, while \( \Gamma_c \) (for “change”) is the probability for an electron to become a hole and vice versa. Similarly, we write \( D = (D_d I + D_c \tau) \). According to the diagrams of Fig. 5(e), we have

\[ D = \Gamma + \Gamma t D \]  

(28)

or

\[ D^{-1} = \Gamma^{-1} - t, \]  

(29)

where
\[ t = \sum_n \frac{W_n}{N} |g_n|^2 \tau \] (30)

represents the two Green’s functions \( G \) and \( G^* \) connecting \( \Gamma \) to \( D \). The \( \sum_n W_n \) appears explicitly in our two–by–two matrix notation, whereas it would be implicit in the matrix multiplication in Eq. (28) if multiplication in the mode sub–space were implied [the factor of \( 1/N \) is introduced in \( t \) to account for our definition of \( \Gamma \) and \( D \) as \( O(N^0) \) objects]. Again, the specific values for \( t \) and \( D \) depend very much on the geometric structure, and are left for the following section. Here we proceed to derive the general relationship between \( \Gamma \) and \( \alpha \) (and therefore \( f \)), which, as in the case of the self–energy \( \Sigma \), is the only aspect of \( G \) which enters in the definition of \( \Gamma \).

Let us then look at the graphs contributing to \( \Gamma \), Fig. 5(d), in particular the terms with \( 2k \) double lines joined by a coupling constant \( \gamma_{2k} \). Those graphs in which an odd number of double lines land on the top line (and thus necessarily with an odd number of double lines landing on the bottom line) contribute to \( \Gamma_d \), and not to \( \Gamma_c \). There are \( k \) such possibilities (one double line can land on the top line, three double lines, and so on, up to \( 2k - 1 \) double lines). In contrast, those graphs in which an even number of double lines land on the top and bottom lines contribute to \( \Gamma_c \). There are \( k - 1 \) such possibilities. We thus find that

\[ \Gamma_d = \sum_{k=1}^{\infty} c_k |\alpha|^{2k - 2} \gamma_{2k}, \] (31)

where in each contribution to \( \Gamma_d \) the signs in \( (-|\alpha|^2)^{k-1} \) cancel with the sign of \( \gamma_{2k} \). Comparing with (25) we see that \( \Gamma_d = \frac{d(\alpha^* f)}{d(|\alpha|^2)} \), and thus

\[ \Gamma_d = \frac{1}{\sqrt{1 - 4|\alpha|^2}} = \frac{1 + |f|^2}{1 - |f|^2} \] (32)

from Eqs. (26) and (27). Similarly, we have

\[ -\Gamma_c = \sum_{k=1}^{\infty} c_k |\alpha|^{2k - 2} (k - 1). \] (33)

Note the minus sign in the last equation. To see its origin, consider the term involving four double lines with two landing on the top single line and two landing on the bottom line: it is given by \( \gamma_4(\iota\alpha)(-\iota\alpha^*) \) for the electron–hole term, and \( \gamma_4(\iota\alpha^*)(-\iota\alpha) \) for the
hole–electron term (they are equal; recall that expressions pertaining to the top line are complex–conjugated). Due to the negative sign of $\gamma_4$, this gives a negative contribution to $\Gamma_c$. Next, consider a typical term involving six double lines, say with two double lines going upwards and four double lines going downwards. The contribution to $\Gamma_c$ is given by, say, $\gamma_6(i^3\alpha|\alpha|^2)(-i\alpha^*)$, but $\gamma_6$ is positive. Proceeding in this way, we see that $\Gamma_c$ is given by a series with negative coefficients. In contrast, $\Gamma_d$ is given by a series with positive coefficients. Consider for example the term just mentioned with six double lines. To obtain a contribution to $\Gamma_d$ we have to move, for example, one double line from the bottom to the top part of the diagram. In the expression just given, one factor of $i\alpha$ is thus changed to $-i\alpha$. As already emphasized in the introduction, various signs of this type, and the factors of $i$ which generate them, are of crucial importance to obtaining our final results, and reflect fundamental aspects of the physics involved.

To continue with our evaluation of $\Gamma_c$ we obtain by its defining series (33) that

$$-\frac{\Gamma_c}{\alpha} = \Gamma_d - \frac{f^2}{\alpha} = \frac{|f|^2(1 + |f|^2)}{1 - |f|^2}. \quad (34)$$

Putting $\Gamma_d$ and $\Gamma_c$ together and evaluating $\Gamma^{-1}$, we find the remarkably simple formula

$$\Gamma^{-1} = \frac{\Gamma_d - \Gamma_c\tau}{\Gamma_d^2 - \Gamma_c^2} = \frac{1}{(1 + |f|^2)^2(1 + |f|^2\tau)}. \quad (35)$$

As we saw from (29), we need the inverse of $\Gamma$, rather than $\Gamma$ itself.

In the following section, we simply apply these formulas — Eqs. (23), (27), (29), (30) and (33) — to a specific geometry, and proceed to evaluate diagrammatically the average conductance of Eq. (2).

Before continuing, we mention how the procedure is changed when one wishes to evaluate the conductance in the absence of time–reversal symmetry. According to the discussion at the end of subsection A above, in this case the leading order contributions cannot contain couplings connecting electron scattering elements and hole scattering elements along the bottom line (or the top line) of the diagrams. Thus there are no contributions to $\Sigma$ (to leading order), and we find that $f = 0$ in this case (and thus also $\alpha = 0$). Similarly, only the
first “simple” diagram in Fig. 5(d) is a legitimate contribution, and $\Gamma = \Gamma^{-1} = I$. Lastly, we have $G = G_0$ in this case, and $|g_n| = |e^{-i\phi_n}| = 1$ should be used in Eq. (30). In order to obtain the conductance, one still needs to invert Eq. (29) as in the symmetric case.

III. A SIMPLE APPLICATION

The simplest possible dirty mesoscopic N–S system arguably consists of a disordered cavity or junction with just one normal and one superconducting lead. In such a system the phase of the superconducting order parameter can be gauged away, and so we assume that $\phi = 0$ and $f$ is real. We denote the relative width of the superconducting lead by $n_s = W_1/N$, so that the conductance we seek is a simple function of $0 < n_s < 1$, multiplied by the appropriate units and the total number of modes $N$.

From the Schwinger-Dyson equation, Eq. (21), we see that

$$g = \frac{1}{1 + f}, \tag{36}$$

or equivalently

$$\alpha = \frac{n_s}{1 + f}, \tag{37}$$

from Eq. (23). Putting Eqs. (27) and (36) together, we obtain

$$\frac{f}{1 + f^2} = \frac{n_s}{1 + f}, \tag{38}$$

and thus we determine $f$ completely in terms of $n_s$:

$$f = \frac{\sqrt{1 + 4n_s - 4n_s^2} - 1}{2(1 - n_s)}. \tag{39}$$

The plus root is chosen so that as the number of modes $n_s$ in the lead connecting the cavity and the superconductor goes to zero, $f$ should vanish as is physically reasonable. The amplitude $f$ grows with $n_s$, and approaches 1 when $n_s$ approaches 1. In particular, for the symmetric case, $n_s = \frac{1}{2}$ we have $f = \sqrt{2} - 1$. 

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We proceed to evaluate the diffusion for this system. Using Eq. \((29)\) with
\[
t = n_s g^2 \tau = \frac{f}{(1 + f^2)(1 + f)} \tau ,
\] we find through simple arithmetic that
\[
D = \frac{(1 + f^2)(1 + f)^2}{(1 + 2f - f^2)} \left( I + \frac{f(1 - f)}{(1 + f)} \tau \right) .
\]

Now we are ready to compute the conductance, as represented diagrammatically in Fig. 5(f). As shown in the figure, there are two distinct terms which contribute to this quantity — one in which the electrons are Andreev reflected “individually” (which involves only \(\Sigma\)), and one in which the top line and the bottom line in the diagram are connected, which involves the diffusion \(D\).

In the case of the non–diffusion contribution, one must note that the appropriate terms of \(G\) must contain only a single copy of \(\Sigma\), as opposed to the infinite series reflected in Eq. \((36)\). This is due to the fact that the external indices represent modes in the normal lead, and cannot be directly Andreev reflected (\(G_0\) for the normal lead vanishes — excitations leaving the cavity through it never return). Thus, the non–diffusion contribution to the conductance is given by
\[
\sigma_\Sigma = 4 \frac{e^2}{h} N(1 - n_s) f^2 = 4 \frac{e^2}{h} N(n_s - f) .
\]
This is plotted in Fig. 6 as a function of \(n_s\).

The conductance due to the diffusion is given by
\[
\sigma_D = 4 \frac{e^2}{h} N(1 - n_s)^2 D_c = 4 \frac{e^2}{h} N(1 - n_s)^2 \frac{f(1 - f^4)}{1 + 2f - f^2} ,
\]
which is also plotted in Fig. 6. Note that to obtain the non–diffusion contribution to the conductance we multiply by the number of modes on the external lead \(W_2 = (1 - n_s)N\), while to obtain the diffusion contribution we multiply by the square of this number. In the non–diffusion contribution to the conductance, an electron injected into a given mode comes back as a hole in the same mode: this is known as the “giant backscattering peak” \([10]\). In contrast, in the diffusion contribution, the hole comes back in general in some other mode.
The total conductance is of course given by the sum of Eqs. (42) and (43). We find that when we add these two contributions to the conductance, we obtain the surprisingly simple expression

\[ \sigma_{NS} = \sigma_{\Sigma} + \sigma_{D} = 2 \frac{e^2}{h} N \left( 1 - \frac{1}{\sqrt{1 + 4n_s - 4n_s^2}} \right). \]  

(44)

Indeed, if we introduce the asymmetry parameter \( a \) by \( n_s = \frac{1 + a}{2} \), so that \( a = 0 \) for a symmetric cavity, we obtain simply

\[ \sigma_{NS} = 2 \frac{e^2}{h} N \left( 1 - \frac{1}{\sqrt{2 - a^2}} \right). \]  

(45)

This is again plotted in Fig. 6. For the case of equal widths of the leads, \( a = 0 \), this has been derived using the transmission–eigenvalue approach in Ref. [31]. Incidentally, note that \( \sigma_{NS} \) is symmetric with respect to interchanging the widths of the leads, \( a \leftrightarrow -a \). In the transmission–eigenvalue approach, this follows directly from the fact that \( \sigma_{NS} \) depends only on the transmission eigenvalues [32] (even when these are appropriately redefined for rectangular transmission matrices).

For comparison, we consider the case in which the charge conjugation symmetry between electrons and holes is broken, by a finite bias voltage \( V \), a finite temperature, or a finite magnetic field. As explained above (see the end of Sec. II A and Sec. II C) we may then denote the scattering matrix of holes by a unitary matrix \( S' \), which to leading order can be taken as unrelated to \( S \), and average over \( S \) and \( S' \) separately. The situation becomes enormously simpler: all the non–diffuson graphs cease to exist, \( \Sigma = 0 \), and we need only to evaluate the diffuson contribution with \( \Gamma = I \) and \( G = G_0 \). We are left with the ladder graphs shown in Fig. 7, with the number of rungs on the ladder restricted to be even. The conductance is thus given by \( (1 - n_s)^2(n_s + n_s^3 + n_s^5 + \ldots) = n_s(1 - n_s) = \frac{(1-a)(1+a)}{2(3+a)} \), which we plot in Fig. 8. Notice that it is not symmetric in the asymmetry parameter \( a \). We also plot the difference between the total conductance with and without charge conjugation (or time reversal) symmetry. We see that for \( n_s < \frac{1}{2}(1 + a_c) \sim 0.65 \) charge conjugation symmetry actually lowers the conductance (\( a_c = 0.2955 \ldots \) is the real solution of \( a_c^3 + a_c^2 + 3a_c - 1 = 0 \)).

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Incidentally, the total conductance without charge conjugation symmetry can be calculated easily using elementary physics. Since the scattering of the hole in the cavity is no longer strongly correlated with the scattering of the electron, the system is analogous to two independent cavities connected in series, the motion of electrons being represented as excitations in the first cavity, and holes being represented as excitations in the second. The conductance of such a two–cavity system is given, to leading order, simply by the series addition of the resistances corresponding to the leads that connect them to each other and to the “electron electrode” and its copy, the “hole electrode”: \[ \sigma_{2C} = \frac{2e^2}{\hbar}(W^{-1}_2 + W^{-1}_1 + W^{-1}_2)^{-1} \] (the subscript 2C implies 2 cavities). According to Eq. (2), the conductance of the N–S structure is given by twice this value, which is, as noted above,

\[ \sigma_{\text{NS, non-symmetric}}^{\text{NS}} = 4Ne^2/\hbar n_s \left(1 - n_s\right). \] (46)

Another comparison may be made with the conductance the same structure would have if both electrodes were normal, \( \sigma_N = (2e^2/\hbar)N n_s(1-n_s) \). However, experimentally it is usually much easier to observe the crossover from \( \sigma_{\text{NS}} \) to \( \sigma_{\text{NS, non-symmetric}}^{\text{NS}} \) than that from \( \sigma_{\text{NS}} \) to \( \sigma_N \).

The results reported here are in complete agreement with previous theoretical work. For example, the fact that breaking electron–hole symmetry may either decrease or increase the conductance, \( \sigma_{\text{NS}} \), is known, at least in principle [33]. This situation suggests the following experimental challenge: to fabricate a device with an ergodic scatterer connected to electrodes through leads with tunable widths, for example by changing a gate voltage in a two–dimensional electron gas device, such that \( \sigma_{\text{NS}} \) would exhibit positive magnetoconductance for some gate voltages and negative magnetoconductance for other gate voltages. The challenge here is considerable because fabricating good contacts between a two–dimensional electron gas and a superconducting electrode is not easy — there are invariably large mismatches in the Fermi velocity, not to mention Schottky barriers, etc. In fact, the goal of fabricating such contacts has been one of the driving forces behind the development of this field in the past few years. Note that related normal–normal conductances have already been
shown to exhibit changes in the sign of the magnetoconductance as a function of geometric
parameters of the system, in Ref. [2].

It is interesting to compare our results with the experimental data of Kastalsky et al. [1], who have studied an N–S system in which a relatively large piece of “normal metal” (in this case, a degenerate semiconductor) was in contact with a superconductor and a normal reservoir (the semiconducting substrate). The conductance of this junction $\sigma_{NS}$ was observed to grow by a factor of about 1.8 upon decreasing an applied magnetic field at low temperatures (the temperature dependence is complicated by not having a very large separation between the Thouless energy and the superconducting gap). This compares favorably with the results shown in Fig. 8 for large $n_s$ — in this case $f$ approaches 1 while the resistance of the wide lead, $W_1$, becomes negligible; the conductance due to “direct” Andreev reflections ($\sigma_\Sigma$) is then equal to twice that which is obtained when the holes must “find their way to the external electrodes” independently of the motion of the electrons that produced them [expanding Eqs. (44) and (46) near $n_s = 1$ gives $\sigma_{NS} \to \frac{4e^2}{h} W_2$ and $\sigma_{NS}^{\text{non-symmetric}} \to \frac{2e^2}{h} W_2$ respectively]. This experiment was in fact one of the first available in the subfield of mesoscopic N–S structures discussed here, and was initially very hard to interpret. The interpretation which was eventually accepted [34] put forward precisely the ideas of multiple Andreev reflection which are formalized in the present work and its predecessors.

IV. CONCLUSIONS AND OUTLOOK

We have developed a new technique for evaluating the dissipative conductances of disordered mesoscopic systems attached to normal and to superconducting leads. It is based on a generalization of the planar–diagrammatic technique of large–$N$ random matrix models, which allows one to consider unitary (as opposed to Hermitean) matrices, specifically the scattering matrix of a disordered mesoscopic grain. The leading order results for the conductance are affected by mesoscopic coherence, because of the possibility and importance of
an exact symmetry between electrons and holes at the Fermi level.

In the present work, we have demonstrated the method only by application to the simplest possible disordered mesoscopic N–S system: a scattering cavity attached through ideal leads to one superconducting electrode and one normal electrode. This calculation can only serve as a “toy–model” description of actual experimental geometries; it is necessary to consider many possible complications, such as the effects of potential barriers, in order to check, e.g., whether there is any physics behind the agreement with the experiment of Ref. 1 which was mentioned at the end of the previous section. This underlines the importance of generalizing the present method to deal with more complicated geometries (in analogy with the “circuit theory of Andreev conductance” put forward by Nazarov 7). Such a generalization turns out to be possible, and the results will be reported in future work 16.

An additional possible direction for future pursuit is to evaluate the fluctuations and the next order corrections to the conductances. Specifically, the much–discussed $O(N^0)$ term can in principle be found from similar large–$N$ diagrams which break the condition of planarity exactly once. A third direction is to consider excitations with finite energies, or systems in a finite magnetic field. Such calculations could turn out to be much more difficult, because the scattering matrices depend on energy and magnetic field in a continuous manner, and it should be necessary to invoke ensembles of such two–parameter families of scattering matrices. However, the experience in mesoscopic phenomena may lead to a hope that the generic behavior in such ensembles would be simple and universal 35, perhaps describable by simple modifications of the $c_k$ couplings used in the present method.

**APPENDIX**

Here we give an alternative derivation of (17). As customary in the literature on random matrix theory, we define the function of a complex variable $z$ called the “one–point” Green’s function by

$$G(z) = \left\langle \frac{1}{N} \operatorname{tr} \frac{1}{z - (S + S^\dagger)} \right\rangle \quad (47)$$
(which should not be confused with the quantity $G$ in the main text). As before, $\langle \ldots \rangle$ denotes averaging the $N$ by $N$ unitary matrix $S$ over the CUE. Diagonalizing the unitary matrix $S$ and denoting its eigenvalues by $e^{i\theta_j}$, $j = 1, 2, \ldots, N$, we may use the obvious fact that the eigenvalues are distributed uniformly over the unit circle to find that

$$G(z) = \int \frac{d\theta}{2\pi} \frac{1}{z - 2\cos\theta} = \sum_{k=0}^{\infty} \frac{1}{z^{2k+1}} \frac{(2k)!}{(k!)^2} = \frac{1}{\sqrt{z^2 - 4}},$$

(48)

where the sum converges for $|z| > 2$.

On the other hand, if we evaluate (47) by a diagrammatic expansion as shown in Fig. 9, we have the two equations

$$G(z) = \frac{1}{z - \Sigma(z)}$$

(49)

and

$$\Sigma(z) = 2 \sum_{k=1}^{\infty} \gamma_{2k} G^{2k-1}(z)$$

(50)

(the latter holds at least to leading order in $1/N$). We recognize (49) as just the Schwinger–Dyson equation again: here the bare Green’s function $G_0(z)$ is simply equal to $1/z$. In (50) we have used the definition of $\gamma_{2k}$ as the coupling constant involving $2k$ matrix elements. Note also the factor of 2: the first matrix element in the diagram for $\Sigma$, reading from left to right say, may represent either $S$ or $S^\dagger$. This feature does not appear in discussions of Hermitean random matrices.

Our goal here is to determine the coupling constants $\gamma_{2k}$, assuming that averages over elements of $S$ and $S^\dagger$ may indeed be written in terms of such diagrams, with a generalization of Wick’s theorem implying a multiplicative property of the weights of diagrams involving more than one coupling (those of Fig. 3). We do this by eliminating $z$ between (48) and (49) and thus solving for $\Sigma(z)$ in terms of $G(z)$:

$$\Sigma(z) = \frac{\sqrt{1 + 4G^2(z)} - 1}{G(z)},$$

(51)

which is identical to Eq. (19) if we identify $x = -G^2(z)$ and $w = -\Sigma(z)G(z)/2$ [cf. also Eq. (27)]. Expanding the right-hand side as a series in $G$ and comparing with (50) we obtain immediately $\gamma_{2k}$ and hence $c_k$ in agreement with (17).
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REFERENCES

[1] A. Kastalsky, A.W. Kleinsasser, L.H. Greene, R. Bhat, F.P. Milliken, and J.P. Harbison, Phys. Rev. Lett. 67, 3026 (1991).

[2] V.T. Petrashov, V.N. Antonov, S.V. Maksimov, and R.S. Shaikhaidarov, Pis’ma Zh. E. T. P. 58, 48 (1993) [JETP Lett. 58, 49 (1993)].

[3] V.T. Petrashov, V.N. Antonov, P. Delsing, and R. Claeson, Phys. Rev. Lett. 70, 347 (1993).

[4] P.G.N. de Vegvar, T.A. Fulton, W.H. Mallison, and R.E. Miller, Phys. Rev. Lett. 73, 1416 (1994).

[5] H. Pothier, S. Gueron, D. Esteve, and M.H. Devoret, Phys. Rev. Lett. 73, 2488 (1994).

[6] V.T. Petrashov, V.N. Antonov, P. Delsing, and T. Claeson, Phys. Rev. Lett. 74, 5268 (1995).

[7] Yu.V. Nazarov, Phys. Rev. Lett. 73, 1420 (1994).

[8] A.V. Zaitsev, Phys. Lett. A, 194, 315 (1994).

[9] C.W.J. Beenakker, B. Rejaei, J.A. Melsen, Phys. Rev. Lett. 72, 2470 (1994).

[10] C.W.J. Beenakker, J.A. Melsen, P.W. Brouwer, Phys. Rev. B 51, 13883 (1995).

[11] P.W. Brouwer and C.W.J. Beenakker, Phys. Rev. B 52, R3868 (1995).

[12] The above list of references is only a representative sample of the many available publications. A review is given by C.W.J. Beenakker in Mesoscopic Quantum Physics, E. Akkermans, G. Montambaux, and J.-L. Pichard, eds. (North-Holland, Amsterdam, to be published).

[13] For reviews see C.W.J. Beenakker and H. van Houten, in Solid State Physics, H. Ehrenreich and D. Turnbull, eds. (Academic Press, New York, 1991), Vol. 44, pp. 1–228; Mesoc-
scopic Phenomena in Solids, B.L. Altshuler, P.A. Lee, and R.A. Webb, eds. (North–Holland, New York, 1991).

[14] A.F. Andreev, Zh. Eksp. Teor. Fiz. 46, 1823 (1964) [Sov. Phys. JETP 19, 1228 (1964)]; 51, 1510 (1966) [24, 1019 (1967)].

[15] For an early application of Andreev scattering to mesoscopic N–S structures see G.E. Blonder, M. Tinkham, and T.M. Klapwijk, Phys. Rev. B 25, 4515 (1982).

[16] N. Argaman, in preparation. See also Ref. [7], which describes simple rules for applying the theory of dirty superconductors to network structures.

[17] We postpone a detailed discussion of the density of states and its relationship with reduced conductances to Ref. [16] (this reduction of the density of states is also obtained in the context of the Usadel equations, as mentioned, e.g., in Ref. [7]). Note that it is much stronger (corresponding to a “leading order” correction) than the reduction due to the “edge” of the spectrum, discussed e.g. in A. Altland and M.R. Zirnbauer, preprint cond-mat/9508026 (1995). There the depression in the density of states was short–ranged, and disappeared at energies a few level–spacings above the Fermi level, because the system considered was not perfectly time–reversal symmetric. In contrast, here the density of states remains depressed through a range in energies of the order of the Thouless energy, which is much larger than the level–spacing, as explained in Sec. II.

[18] For a brief review see C.W.J. Beenakker, Mod. Phys. Lett. B 8, 469 (1994).

[19] Although the majority of recent work uses the transmission matrix, there are also exceptions in which the scattering matrix was used directly, e.g., K. Frahm and J.–L. Pichard, J. de Physique I 5, 847 (1995).

[20] K. Usadel, Phys. Rev. Lett. 25, 507 (1970); A.I. Larkin and Yu.N. Ovchinnikov, Zh. Eksp. Teor. Fiz. 68, 1915 (1975) [Sov. Phys. JETP 41, 960 (1975)]. For a review of the
early work on superconducting weak links see K.K. Likharev, Rev. Mod. Phys. 51, 101 (1979).

[21] See, e.g., R. Blumel and U. Smilansky, Phys. Rev. Lett. 60, 477 (1988). In this context, the semiclassical approximation should be understood as implying propagation along distinct classical orbits, with definite expressions for the contribution of each orbit (and an underlying chaotic classical dynamics for each specific realization of the impurity potential or the shape of the cavity). Unfortunately, confusion may arise because the word “semiclassical” is often used to imply other $\hbar \to 0$ approximations, e.g. those in which the impurity averaging has been performed before the $\hbar \to 0$ limit is taken.

[22] See K.B. Efetov, Adv. Phys. 32, 53 (1983).

[23] M.L. Mehta, Random matrices (2nd ed.), Academic Press (Boston, 1991).

[24] For a detailed exposition see K. Slevin, J.–L. Pichard, and P.A. Mello, preprint cond-mat/9507028 (1995). Note that, depending on the convention, an extra minus sign may appear in all of the Andreev reflection amplitudes; this is of no importance as interference can only occur between amplitudes differing by an even number of Andreev reflections.

[25] C.J. Lambert, J. Phys. Condens. Matter 3, 6579 (1991); Y. Takane and H. Ebisawa, J. Phys. Soc. Jpn. 60, 3130 (1991); 61, 1685 (1992); 61, 2858 (1991); see also Ref. [15].

[26] G. ’t Hooft, Nucl. Phys. B72, 461 (1974); E. Brezin and S.R. Wadia, Eds., The Large $N$ expansion in quantum field theory and statistical physics: from spin systems to 2-dimensional gravity, World Scientific (Singapore, 1993).

[27] See, e.g., M.V. Berry, Proc. R. Soc. London, A 400, 229 (1985). Note that here the times considered, or the lengths of the corresponding orbits, are of the order of the escape time $t_{esc}$ and much shorter than the Heisenberg time $\hbar/\Delta$. The fact that $\Delta$, the single–particle level spacing in the cavity, is much smaller than $\hbar/t_{esc}$, i.e. that the
cavity has a continuous rather than a discrete spectrum, follows from the fact that the total number of modes $N$ is large.

[28] S. Samuel, J. Math. Phys. 21, 2695 (1980); See also Ref. [26], and especially chapter 8 in M.J. Creutz, *Quarks, gluons and lattices* (Cambridge U. Press, Cambridge, 1983). Although our results are certainly contained in those of the large–$N$ QCD literature, we are not aware of an analogue in the literature to our simple expression for the couplings defined below, Eq. (17).

[29] Again, one may estimate the correlation field from the semiclassical analysis: the field required to shift the phases of the orbits threads approximately one flux quantum through the directed area encircled by a typical orbit. This area is of order $\sqrt{t_{\text{esc}}/t_{\text{erg}}}$ times the area of the cavity, where $t_{\text{erg}}$ is the time required for a particle to reach from one side of the cavity to the other (the assumption of ergodic scattering implies $t_{\text{esc}} \gg t_{\text{erg}}$).

[30] Note that this notation is convenient only for the single–cavity case; in Ref. [16] we use a “bare Green’s function” which represents only the motion through the leads and is diagonal in the electron–hole subspace (the Andreev reflections are then included in a suitable redefinition of $\Sigma$).

[31] R.A. Jalabert, J.–L. Pichard, and C.W.J. Beenakker, Europhys. Lett. 27, 255 (1994).

[32] C.W.J. Beenakker, Phys. Rev. B 46, 12 841 (1992).

[33] The order–$N$ effects considered here do not occur in diffusive wires with large enough $L/l$. Indeed, the emphasis on diffusive wires and tunnel contacts, e.g., in Refs. [4] and [12], may lead to the impression that electron–hole symmetry always enhances the conductance of the structure, except for the weak localization effect, which decreases the conductance but is of higher order in $1/N$. In fact, Ref. [11] went so far as to suggest the possibility of restricting the diffusive metallic regime to a range of parameters such that the weak localization correction is greater than the order–$N$ effect (which is referred
to there as a magnetic–field– and voltage–dependent “contact resistance”). From the present point of view, based on notions of ideal leads and the large–$N$ limit, such a strong emphasis on weak localization is not warranted.

[34] B.J. van Wees, P. de Vries, P. Magnee, and T.M. Klapwijk, Phys. Rev. Lett. 69, 510 (1992).

[35] See, e.g., N. Taniguchi, A. Hashimoto, B.D. Simons, and B.L. Altshuler, Europhys. Lett. 27, 335 (1994).

[36] Remarkably, corrections to $\Sigma(z)$ from higher order diagrams and from corrections to the products of coefficients $\gamma_{2k}$ (i.e., corrections to the $C_{u,v}$ coefficients), seem to cancel each other. This follows from the fact that Eqs. (48) and (49) hold for any $N$. 

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Fig. 1: Sketch of a simple normal–superconducting mesoscopic system, consisting of a single “grain” which is treated as an ideally ergodic scattering cavity, connected to a normal electrode (particle reservoir) and to a superconducting electrode through ideal leads of different widths.

Fig. 2: An example of a diagram contributing to the unaveraged conductance of Eq. (3).

Fig. 3: The different couplings, and their weights [see Eq. (17) below]. These couplings are used to connect the double lines of Fig. 2.

Fig. 4: Diagrams corresponding to (a) $\langle S_{ia}S_{aj}^{\dagger} \rangle$, (b) $\langle S_{ia}S_{aj}^{\dagger}S_{jb}S_{bi}^{\dagger} \rangle$, and (c) $\langle S_{ia}S_{aj}^{\dagger}S_{jb}S_{bk}^{\dagger}S_{kc}S_{ci}^{\dagger} \rangle$.

Fig. 5: Diagrammatic representation of: (a) the “self–energy” $\Sigma$, (b) the “bare Green’s function” $G_0$, (c) the “averaged Green’s function” $G$, (d) the “effective coupling constant” $\Gamma$, (e) the “diffuson” $D$, and (f) the conductance $\sigma$ of Eq. (2).

Fig. 6: The conductance of the simplest normal–superconducting system considered, in units of $4Ne^2/h$, as a function of the fraction of modes belonging to the superconducting lead (full line). The non–diffuson (dashed) and the diffuson (dotted) contributions are also shown separately, the former corresponding to “direct” Andreev reflections from the cavity, or the so-called giant backscattering peak.

Fig. 7: The simplified diagrams contributing to the conductance when the symmetry between electrons and holes is broken. Only vertical simple double lines can be used, because the diagram must be planar and the matrices on the even rungs of the ladder cannot be connected to the matrices on the odd rungs.
Fig. 8: A comparison of the conductance of Fig. 6 (full line) to that obtained in the absence of electron–hole symmetry (dashed line). Also shown is the difference between the two, on an expanded scale (top panel).

Fig. 9: Diagrams for the one–point Green’s function, $G(z)$ (thick lines), and its associated self–energy $\Sigma(z)$ (shaded semicircles) of the appendix. The thin lines represent $G_0(z) = 1/z$, and carry no arrows. The double lines represent an element of $S$ if the arrow is pointing away from the single line, and an element of $S^\dagger$ in the opposite case. Although the quantities $G(z)$ and $\Sigma(z)$ are not to be confused with $G$ and $\Sigma$ of the main text, the couplings are the same (see Fig. 3).
figure 1, Argaman and Zee
figure 2, Argaman and Zee
figure 3, Argaman and Zee
(a) \[ i \quad \quad \quad a = \frac{1}{N} \]

(b) \[ \frac{\delta_{ij}}{N^2} \quad + \quad \frac{\delta_{ab}}{N^2} \quad + \quad \frac{-1}{N^3} \]

(c) \[ \text{figure 4, Argaman and Zee} \]
\[ \Sigma = \alpha + \alpha^* \alpha + \ldots = \Sigma \]

\[ G_0 = \]

\[ G = \]

\[ \frac{\Gamma}{N} = \]

\[ \frac{D}{N} = \]

\[ \frac{\sigma}{4e^2 \hbar} = \]
figure 6, Argaman & Zee
\[ \sigma_{\text{non-symmetric}} = \frac{4e^2}{h} \]

*figure 7, Argaman and Zee*
figure 8, Argaman and Zee
$G(z) = \sum (z) + \sum (z) + \sum (z) + \sum (z) + \cdots = \sum (z)$

$\Sigma(z) = \sum (z) + \sum (z) + \sum (z) + \sum (z) + \cdots = \Sigma(z)$

figure 9, Argaman & Zee