Evolution of the resonances of two parallel dielectric cylinders with distance between them

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We study behavior of resonant modes under change of the distance between two parallel dielectric cylinders. The processes of mutual scattering of Mie resonant modes by cylinders result in an interaction between the cylinders which lifts a degeneracy of resonances of the isolated cylinders. There are two basic scenarios of evolution of resonances with the distance. For strong interaction of cylinders resonances are unbound with avoided crossings from the of the Mie resonances with increasing of the distance. That scenario is typical for low exciting resonances (monopole and dipole). For weak interaction of cylinders the resonances are bound around highly excited Mie resonances. Both scenarios demonstrate a significant enhancement of the $Q$ factor compared to the case of isolated cylinder.

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I. INTRODUCTION

It is rather challenging for optical resonators to support resonances of simultaneous subwavelength mode volumes and high $Q$ factors. The traditional way for increasing the $Q$ factor of optical cavities is a suppression of leakage of resonance mode into the radiation continua. That is achieved usually by decreasing the coupling of the resonant mode with the continua by the use of metals, photonic band gap structures, or whispering-gallery-mode resonators. All of these approaches lead to reduced device efficiencies because of complex designs, inevitable metallic losses, or large cavity sizes. On the contrary, all-dielectric subwavelength nanoparticles have recently been suggested as an important pathway to enhance capabilities of traditional nanoscale resonators by exploiting the multipolar Mie resonances being limited only by radiation losses $^{[1]} [2]$. The decisive breakthrough came with the paper by Friedrich and Wintgen $^{[3]}$ which put forward the idea of destructive interference of two neighboring resonant modes leaking into the continuum. Based on a simple generic two-level model they formulated the condition for the bound state in the continuum (BIC) as the state with zero resonant width for crossing of eigenlevels of the cavity or avoided crossing of resonances. This principle was later explored in open plane wave resonator where the BIC occurs in the vicinity of degeneracy of the closed integrable resonator $^{[4]}$.

However, these BICs exist provided that they embedded into a single continuum of propagating modes of a directional waveguide. In photonics the optical BICs embedded into the radiation continuum can be realized by two ways. The first way is realized in an optical cavity coupled with the continuum of 2d photonic crystal (PhC) waveguide $^{[5]}$ that is an optical variant of microwave system $^{[6]}$. Alternative way is the use periodic PhC systems (gratings) or arrays of dielectric particles in which resonant modes leak into a restricted number of diffraction continua $^{[7] [10]}$. Although the exact BICs can exist only in infinite periodical arrays $^{[11] [12]}$, finite arrays demonstrate resonant modes with the very high $Q$ factor which grows quadratically $^{[13]}$ or even cubically $^{[14]}$ with the number of particles (quasi-BICs).

Isolated subwavelength high-index dielectric resonators are more advantageous from an applied point of view to achieve high $Q$ resonant modes (super cavity modes) $^{[2] [15] [16]}$. Such super cavity modes originate from avoided crossing of the resonant modes, specifically the Mie-type resonant mode and the Fabry-Pérot resonant mode under variation of the aspect ratio of the dielectric disk which could result in a significant enhancement of the $Q$ factor. It is worthy also to notice the idea of formation of long-lived, scar like modes near resonances $^{[17]}$. The dramatic $Q$ factor enhancement was predicted by Boriskina $^{[18] [19]}$ for avoided crossing of very highly excited whispering gallery modes in symmetrical photonic molecules of dielectric disks on a surface.

In the present paper we consider a similar way to enhance the $Q$ factor by variation of the distance between two identical dielectric cylinders parallel each other as sketched in Fig. 1. As different from papers $^{[17] [21]}$ we consider the avoided crossing of low excited resonant modes (monopole, dipole and quadruple) with variation of the distance between two cylinders. Because of lifting of the axial symmetry many Mie resonances contribute into the resonances of two cylinders which show two basic scenarios of evolution with the distance, bound to the Mie resonances and unbound.

II. AVOIDED CROSSING UNDER VARIATION OF DISTANCE BETWEEN TWO CYLINDERS

The problem of scattering of electromagnetic waves from two parallel infinitely long dielectric cylinders sketched in Figs. 1 was solved long time ago $^{[22] [24]}$. 
The solutions for electromagnetic field (the component of electric field directed along the cylinders \( \psi = E_z \)) (see Fig. 1) outside the cylinders are given by

\[
\psi_1 = \sum_n A_{1n} H_n^{(1)}(kr_1)e^{in\theta_1}, \\
\psi_2 = \sum_n A_{2n} H_n^{(1)}(kr_2)e^{in\theta_2}. 
\]

(1)

(2)

Inside the cylinders we have

\[
\psi_1 = \sum_n B_{1n} J_n(\sqrt{k}r_1)e^{in\theta_1}, \\
\psi_2 = \sum_n \sum_n B_{2n} J_n(\sqrt{k}r_2)e^{in\theta_2}. 
\]

(3)

(4)

By means of the Graf formula \[22\]

\[
H_n^{(1)}(kr_1)e^{in\theta_1} = \sum_m i^{n-m} H_m^{(1)}(kL)J_m(kr_2)e^{im\theta_2}, \\
H_n^{(1)}(kr_2)e^{in\theta_2} = \sum_m i^{n-m} H_m^{(1)}(kL)J_m(kr_1)e^{im\theta_1} 
\]

(5)

(6)

the total field \( \psi = \psi_{inc} + \psi_1 + \psi_2 \) can be written completely in either coordinate system.

Applying the boundary conditions at \( r_j = a \) leads to

\[
A_{1n} = i^n S_n(k) \sum_m i^{-m} H_{n+m}(kL)A_{2m}, \\
A_{2n} = i^n S_n(k) \sum_m i^{-m} H_{n+m}(kL)A_{1m}, 
\]

(7)

where \( S_n \) are the scattering matrix amplitudes for the isolated cylinder

\[
S_n(k) = \frac{\sqrt{\epsilon J_m^\prime(\sqrt{k}a)J_m(\sqrt{k}) - J_m^\prime(\sqrt{k})J_m(\sqrt{k})}}{H_m^{(1)\prime}(k)J_m(\sqrt{k}) - \sqrt{\epsilon}J_m^\prime(\sqrt{k})H_m^{(1)}(k)}. 
\]

(8)

The resonances are given by the complex roots of the following equation

\[
Det[\hat{M}^2 - I] = 0 
\]

(9)

where matrix elements \( \hat{M} \) is given by Eq. \[7\] and equal

\[
M_{mn} = S_m(k)i^{m-n}H_{m+n}(kL) 
\]

(10)

and \( I \) is the unit matrix.

In general Eq. \[9\] has an infinite number of complex resonant frequencies (poles) \( k = k_n - i\gamma_n \) which are shown in Fig. 2. First of all one can see that major part of resonances evolves with the distance bypassing the Mie resonances of the isolated cylinder marked by crosses some part of which are collected in Fig. 3. Second, there are a small part of resonances which are bound around the Mie resonances with small imaginary parts. As a rule these Mie resonances correspond to highly excited Mie resonances.

Let us consider the asymptotic behavior of poles for \( L \to \infty \). By use of asymptotical behavior of the Hankel functions \[26\] we have for matrix \[10\] the following

\[
M_{mn} \sim \sqrt{2 \pi kL}e^{i(kL-\pi/4)}S_m(k)(-1)^n. 
\]

(11)

Let us take the eigenvector of matrix \( \hat{M} \) \( \psi^+ = (\psi_1, \psi_2, \psi_3, \ldots) \). Then Eq. \[10\] takes the following form

\[
\sqrt{2 \pi kL}e^{ikL}S_m(k)\sum_n (-1)^n \psi_n = \pm \psi_m, 
\]

(12)

which has the solution provided that

\[
\sqrt{2 \pi kL}e^{ikL}\sum_n (-1)^n S_n(k) = \pm 1. 
\]

(13)
For the absolute value we have

\[ 2e^{\gamma_n L} \frac{|k_n|}{L} = \frac{1}{\sum_m (-1)^m S_m(k_n)^2} \]  

(14)

where \( \gamma_n = -2\text{Im}(k_n) \). The poles of the isolated cylinder are given by poles of the S-matrix, i.e., by equation \( S_m(k) = 0 \). Therefore from (14) it follows that, first, the poles of double cylinders do not converge to the poles of the isolated cylinder. Second, the line widths and resonant positions given by imaginary parts and real parts of the complex resonant frequencies respectively are to limit to zero as indeed Fig. 4(a) and (b) illustrates. Therefore we can conclude that the resonances of two dielectric cylinders do NOT limit to the Mie resonances of the isolated cylinders at \( L \to \infty \). The reason is related to the exponential factor \( \exp(\gamma_n L) \) of the resonant modes (the Gamov states).

Next, we consider behavior of some typical resonances in Fig. 4 in detail from the limiting case \( L = 2a \) to \( L = 1000a \). Due to the symmetry relative to \( x \to -x \) and \( y \to -y \) the resonant modes can be classified as \( \psi_{\sigma,s'} \), where the indices \( \sigma = s, a \) respond for symmetric and antisymmetric modes respectively. We start with the first two lowest symmetric and antisymmetric monopole resonant modes \( \psi_{s,s/a} \). The \( Q \) factor of the antisymmetric monopole resonance exceeds the \( Q \) factor of the isolated cylinder by one order in magnitude. The reason of that follows from the Mie resonances shown in Fig. 3. As seen from Fig. 5(a) at the closest distance between the cylinders \( L = 2a \) the symmetric resonant mode becomes the monopole mode with corresponding \( Q \) factor close to the \( Q \) factor of isolated cylinder which is rather low as marked by cross in Fig. 5(b). At the same time the antisymmetric mode of two cylinders at \( L = 2a \) becomes the dipole resonance which as seen from Fig. 3 has the \( Q \) factor exceeding the \( Q \) factor of the monopole Mie resonance by one order in magnitude.

The next resonances illustrate that their evolution strongly depend on interaction between the cylinders via the radiating dipole Mie resonances which are degenerate in the isolated cylinder. The general expressions and physical origin of the coupling of dielectric resonators was considered in Refs. [27–29]. The coupling constant can be written as

\[ \kappa = \int dxdy [\epsilon(\vec{r}) - 1] \vec{E}^*_1 \vec{E}_2 \]  

(15)

where \( \vec{E}_{1,2} \) are normalized solutions by the factor \( \sqrt{\int \epsilon(\vec{r})|\vec{E}_{1,2}|^2 dxdy} \). Here the indices 1 and 2 imply the resonant modes of isolated cylinders. One can see that the coupling constant is determined by overlapping of resonant modes which in turn depend on the distance between the particles and prevailing direction of radiation of the modes.

That conclusion is well illustrated by the Mie dipole resonant modes which are degenerate. The first dipole Mie resonant mode, symmetric relative to \( x \to -x \), radiates prevalently towards the neighboring cylinder as shown in insets in Fig. 6 while the second antisymmetric dipole Mie resonant mode radiates away from the neighboring cylinder as shown in insets of Fig. 7. As a result the interaction in former case turns out stronger compared to the latter case as it follows from Eq. (15). That
explains why the evolution of resonances shown in Fig. 6 (a) is similar to the case of interaction via the monopole resonant modes in Fig. 5 while the evolution of resonances in Fig. 6 is bound to the Mie dipole resonance 2. Respectively the gain in the $Q$ factor in the former case is smaller than in the latter case as seen from Figs. 6(b) and 7(b). With further increase of the distance $L$ the resonance bypasses the monopole Mie resonance 1 of the isolated cylinder. As a result the resonant mode becomes close to the monopole mode $\psi_{s,s/a}$. In both cases in wide range of distances $L$ except close the resonant modes can be presented as symmetric and antisymmetric superpositions of the Mie resonant modes of the isolated cylinder

$$
\psi_{s,s/a}^{(m)} = H_{m}^{(1)}(kr_{1}) \cos(m\theta_{1}) \pm H_{m}^{(1)}(kr_{2}) \cos(m\theta_{2}),
$$

$$
\psi_{a,s/a}^{(m)} = H_{m}^{(1)}(kr_{1}) \sin(m\theta_{1}) \pm H_{m}^{(1)}(kr_{2}) \sin(m\theta_{2}) \tag{16}
$$

except the close distance between the cylinders. Fig. 8 well illustrates Eq. (16) for bypassing of resonance the dipole Mie resonances 2 and 5 with $m = 1$ and the monopole Mie resonance 4 $m = 0$.

Figs. 9 and 10 illustrates the second scenario of evolution of resonances which are bounded by the quadruple, octuple etc Mie resonances of the isolated cylinder because of weakness of interaction of cylinders through these resonant modes. Respectively we observe only oscillating behavior of the $A$ factor with substantial enhancement compared to the isolated cylinder. Fig. 10 (c), (d) shows as the $Q$ factor can reach extremal enhancement for variation of the distance similar to the WGM resonances 17–19.

III. SUMMARY OF RESULTS AND CONCLUSIONS

For the isolated dielectric cylinder we have well known Mie resonances specified by azimuthal index $m = 0, \pm 1, \pm 2, \ldots$ (monopole, dipole, quadruple etc resonances) due to axial symmetry. Two parallel cylinder have no axial symmetry and therefore the solutions of homogeneous Maxwell equations are given by series of the Bessel (inside) or Hankel (outside cylinders) functions in $m$. By the use of Graf formula the coefficients in
FIG. 7: The behavior of resonant modes, symmetric and anti-symmetric hybridizations (16) of dipole resonant modes of isolated cylinders (a) and respective $Q$ factors (b) for variation of distance between them. The dipole Mie resonance is shown by red cross.

series satisfy linear algebraic equations and can be easily found [22,23,30]. However there were no studies of behavior of resonances of two cylinders depend on distance between the cylinders except studies of the $Q$ factor by Boriskina for extremely highly excited resonances, whispering gallery modes [18,19]. The study presented in this paper reveals surprisingly complicated behavior of the resonances with the distance which can be divided into two families. In the first family the resonances evolve beside the Mie resonances undergoing the avoided crossing. At each event of that the resonant mode inside the cylinders takes the field profile of the corresponding Mie resonant mode while the solution between the cylinders takes regular symmetric or antisymmetric packing of half wavelengths. At these moments the $Q$ factor achieves maximal magnitudes. This type of evolution of resonances flows is typical for monopole and those dipole resonance which leakages from one cylinder alongside the another.

The evolution of resonances bounded by the Mie resonances forms the second family and typical for higher resonances with $m = 2, 3, \ldots$. It is interesting that the dipole resonance which leakages apart from the other cylinder unites both families. When the leakage from the first cylinder is directed to the second the overlapping (15) that case exceeds the coupling of the Mie dipole resonant modes which leakage aside the the cylinders. As the result in the first case the resonances consequently avoid the Mie resonances being unbound while in the second case the resonances are bound to the dipole Mie resonance of isolated cylinder. There is a range of distances between the cylinders where the trajectories of resonances are bounded by the Mie dipole resonances but beyond these distances the flows are unbounded. For variation of the distance the $Q$ factor shows oscillating behavior with maxima which can exceed the $Q$ factor of the isolated cylinder three times. That enhancement is

FIG. 8: The behavior of resonant modes, symmetric and anti-symmetric hybridizations of dipole resonant modes of isolated cylinders (a) and respective $Q$ factors (b) for variation of distance between them. The respective dipole Mie resonance is shown by red cross.
FIG. 9: The behavior of resonant modes of quadruple resonant modes of isolated cylinders for variation of distance between them. Crosses mark the Mie resonances shown in Fig. 3. Red closed circle marks the case of nearest position of cylinders and open green circle marks the distance $L = 1000$.

FIG. 10: The behavior of resonant modes of octuple resonant modes of isolated cylinders and $Q$ factor for variation of distance between them. Crosses mark the Mie resonances shown in Fig. 3. Red closed circle marks the case of nearest position of cylinders and open green circle marks the distance $L = 1000$. 
typical for all types of resonances except the monopole resonance which demonstrates enhancement by one order in magnitude.

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