Is the Space-Time Non-commutativity Simply
Non-commutativity of Derivatives?*

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Recently, some problems have been found in the definition of the partial
derivative in the case of the presence of both explicit and implicit functional
dependencies in the classical analysis. In this talk we investigate the influ-
ence of this observation on the quantum mechanics and classical/quantum
field theory. Surprisingly, some commutators of the coordinate-dependent
operators are not equal to zero. Therefore, we try to provide mathemati-
cal foundations to the modern non-commutative theories. We also indicate
possible applications in the Dirac-like theories.

The assumption that operators of coordinates do not commute \([\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}\) (or,
alternatively, \([\hat{x}_\mu, \hat{x}_\nu] = iC_{\mu\nu}^{\beta}x_\beta\) has been first made by H. Snyder [1]. Later it was shown
that such an anzatz may lead to non-locality. Thus, the Lorentz symmetry may be broken.
Recently, some attention has again been paid to this idea [2] in the context of “brane
theories”.

On the other hand, the famous Feynman-Dyson proof of Maxwell equations [3] con-
tains intrinsically the non-commutativity of velocities. While \([x^i, x^j] = 0\) therein, but
\([\dot{x}^i(t), \dot{x}^j(t)] = \frac{e^{ijk}}{m^2}B_k \neq 0\) (at the same time with \([x^i, \dot{x}^j] = \frac{e^{ijk}}{m}\delta^{ij}\) that also may be
considered as a contradiction with the well-accepted theories. Dyson wrote in a very clever

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way: “Feynman in 1948 was not alone in trying to build theories outside the framework of conventional physics... All these radical programs, including Feynman’s, failed... I venture to disagree with Feynman now, as I often did while he was alive.”

Furthermore, it was recently shown that notation and terminology, which physicists used when speaking about partial derivative of many-variables functions, are sometimes confusing [4] (see also the discussion in [5]). They referred to books [6]: “...one identifies sometime $f_1$ and $f$, saying, that is the same function represented with the help of variables $x_1$ instead of $x$. Such a simplification is very dangerous and may result in very serious contradictions” (see the text after Eq. (1.2.5) in [6b]; $f = f(x), f_1 = f(u(x_1))$). In [4] the basic question was: how should one define correctly the time derivatives of the functions $E[x_1(t), \ldots x_{n-1}(t), t]$ and $E(x_1, \ldots x_{n-1}, t)$? Is there any sense in $\frac{\partial}{\partial t} E[r(t), t]$ and $\frac{d}{dt} E(r, t)$?  

Those authors claimed that even well-known formulas

$$\frac{df}{dt} = \{\mathcal{H}, f\} + \frac{\partial f}{\partial t}, \quad \text{and} \quad \frac{dE}{dt} = (\mathbf{v} \cdot \nabla)E + \frac{\partial E}{\partial t}$$  

(1)

can be confusing unless additional definitions present.  

Another well-known physical example of the situation, when we have both explicite and implicite dependences of the function which derivatives act upon, is the field of an

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1The quotation from [4c, p. 384]: “the [above] symbols are meaningless, because the process denoted by the operator of partial differentiation can be applied only to functions of several independent variables and $\frac{\partial}{\partial t} E[r(t), t]$ is not such a function.”

2As for these formulas the authors of [4] write: “this equation [cannot be correct] because the partial differentiation would involve increments of the functions $r(t)$ in the form $r(t) + \Delta r(t)$ and we do not know how we must interpret this increment because we have two options: either $\Delta r(t) = r(t) - r^*(t)$, or $\Delta r(t) = r(t) - r(t^*)$. Both are different processes because the first one involves changes in the functional form of the functions $r(t)$, while the second involves changes in the position along the path defined by $r = r(t)$ but preserving the same functional form.” Finally, they gave the correct form, in their opinion, of (1). See in [4d].
accelerated charge [7]. First, Landau and Lifshitz wrote that the functions depended on the retarded time \( t' \) and only through \( t' + R(t')/c = t \) they depended implicitly on \( x, y, z, t \). However, later they used the explicit dependence of \( R \) and fields on the space coordinates of the observation point too. Of course! Otherwise, the “simply” retarded fields do not satisfy the Maxwell equations [4b]. In the same work Chubykalo and Vlayev claimed that the time derivative and curl did not commute in their case. Jackson, in fact, disagreed with their claim on the basis of the definitions (“the equations representing Faraday’s law and the absence of magnetic charges ... are satisfied automatically”; see his Introduction in [5b]). But, he agrees with [7] that one should find “a contribution to the spatial partial derivative for fixed time \( t \) from explicit spatial coordinate dependence (of the observation point)”. So, actually the fields and potentials are the functions of the following forms:

\[
A^\mu(x, y, z, t'(x, y, z, t)), \quad E(x, y, z, t'(x, y, z, t)), \quad B(x, y, z, t'(x, y, z, t)).
\]

The convincing abilities of Dr. Jackson are excellent! Škovrlj and Ivezić [5c] call this partial derivative as ‘complete partial derivative’; Chubykalo and Vlayev [4b], as ‘total derivative with respect to a given variable’; the terminology suggested by Brownstein [5a] is ‘the whole-partial derivative’. We shall denote below this whole-partial derivative operator as \( \hat{\partial}/\hat{\partial}x^i \), while still keeping the definitions of [4c,d].

In [5d] I studied the case when we deal with explicit and implicit dependencies \( f(p, E(p)) \). It is well known that the energy in the relativism is connected with the 3-momentum as

\[
E = \pm \sqrt{p^2 + m^2};
\]

the unit system \( c = \hbar = 1 \) is used. In other words, we must choose the 3-dimensional hyperboloid from the entire Minkowski space and the energy is not an independent quantity anymore. Let us calculate the commutator of the whole derivative \( \hat{\partial}/\hat{\partial}E \) and \( \hat{\partial}/\hat{\partial}p_i \).

\[
\frac{\hat{\partial}f(p, E(p))}{\hat{\partial}p_i} = \frac{\partial f(p, E(p))}{\partial p_i} + \frac{\partial f(p, E(p))}{\partial E} \frac{\partial E}{\partial p_i}.
\]  

(2)

In order to make distinction between differentiating the explicit function and that which contains both explicit and implicit dependencies, the ‘whole partial derivative’ may be denoted as \( \hat{\partial} \).
Applying this rule, we surprisingly find
\[
\left[ \frac{\hat{\partial}}{\partial p_i}, \frac{\hat{\partial}}{\partial E} \right] - f(p, E(p)) = \frac{\hat{\partial}}{\partial p_i} \frac{\partial f}{\partial E} - \frac{\hat{\partial}}{\partial E} \left( \frac{\partial f}{\partial p_i} + \frac{\partial f}{\partial E} \frac{\partial E}{\partial p_i} \right) = \\
= - \frac{\partial^2 f}{\partial E \partial p_i} + \frac{\partial^2 f}{\partial E^2 \partial p_i} - \frac{\partial^2 f}{\partial p_i \partial E} - \frac{\partial f}{\partial E} \frac{\partial E}{\partial p_i} \left( \frac{\partial E}{\partial p_i} \right).
\]  
(3)

So, if \( E = \pm \sqrt{m^2 + p^2} \) and one uses the generally-accepted representation form of \( \partial E/\partial p_i = p^i/E \), one has that the expression (3) appears to be equal to \( (p_i/E^2) \frac{\partial f(p, E(p))}{\partial E} \). Within the choice of the normalization the coefficient is the longitudinal electric field in the helicity basis (the electric/magnetic fields can be derived from the 4-potentials which have been presented in [8]).

On the other hand, the commutator
\[
\left[ \frac{\hat{\partial}}{\partial p_i}, \frac{\hat{\partial}}{\partial p_j} \right] - f(p, E(p)) = \frac{1}{E^3} \frac{\partial f(p, E(p))}{\partial E} [p_i, p_j] - .
\]  
(11)

They are written in the following way:

\[
\epsilon_{\mu}(p, \lambda = +1) = \frac{1}{\sqrt{2} p} \left( \frac{p_x p_y - i p y p}{\sqrt{p_x^2 + p_y^2}}, \frac{p_y p_z + i p z p}{\sqrt{p_z^2 + p_y^2}}, -\sqrt{p_x^2 + p_y^2}, \sqrt{p_x^2 + p_y^2} \right),
\]  
(4)

\[
\epsilon_{\mu}(p, \lambda = -1) = \frac{1}{\sqrt{2} p} \left( \frac{-p_x p_y - i p y p}{\sqrt{p_x^2 + p_y^2}}, \frac{-p_y p_z + i p z p}{\sqrt{p_z^2 + p_y^2}}, +\sqrt{p_x^2 + p_y^2}, -\sqrt{p_x^2 + p_y^2} \right),
\]  
(5)

\[
\epsilon_{\mu}(p, \lambda = 0) = \frac{1}{m} \left( p, -E_p p_x, -E_p p_y, -E_p p_z \right),
\]  
(6)

\[
\epsilon_{\mu}(p, \lambda = 0_t) = \frac{1}{m} \left( E, -p_x, -p_y, -p_z \right).
\]  
(7)

And,

\[
E(p, \lambda = +1) = -\frac{i E p_x}{\sqrt{2} p} \hat{p} - \frac{E}{\sqrt{2} p} \hat{p}^*, \quad B(p, \lambda = +1) = -\frac{p_z}{\sqrt{2} p} \hat{p} + \frac{i p}{\sqrt{2} p} \hat{p}^*,
\]  
(8)

\[
E(p, \lambda = -1) = +\frac{i E p_x}{\sqrt{2} p} \hat{p} - \frac{E}{\sqrt{2} p} \hat{p}^*, \quad B(p, \lambda = -1) = -\frac{p_z}{\sqrt{2} p} \hat{p} - \frac{i p}{\sqrt{2} p} \hat{p}^*,
\]  
(9)

\[
E(p, \lambda = 0) = \frac{im}{p} \hat{p}, \quad B(p, \lambda = 0) = 0,
\]  
(10)

with \( \hat{p} = \begin{pmatrix} p_y \\ -p_x \\ -ip \end{pmatrix} \). It is easy seen that the parity properties of these vectors are different comparing with the standard basis. The parity operator for polarization vectors coincides with the metric tensor of the Minkowski 4-space.
This may be considered to be zero unless we would trust to the genious Feynman. He postulated that the velocity (or, of course, the 3-momentum) commutator is equal to 

\[ [p_i, p_j] \sim i\hbar\epsilon_{ijk}B^k \], i.e., to the magnetic field.

Furthermore, since the energy derivative corresponds to the operator of time and the \( i \)-component momentum derivative, to \( \hat{x}_i \), we put forward the following anzatz in the momentum representation:

\[ [\hat{x}^\mu, \hat{x}^\nu] = \omega(p, E(p)) F^{\mu\nu}_{||} \frac{\partial}{\partial E}, \tag{12} \]

with some weight function \( \omega \) being different for different choices of the antisymmetric tensor spin basis. In the modern literature, the idea of the broken Lorentz invariance by this method is widely discussed, see e.g. [9].

Let us turn now to the application of the presented ideas to the Dirac case. Recently, we analyzed Sakurai-van der Waerden method of derivations of the Dirac (and higher-spins too) equation [10]. We can start from

\[ (EI^{(2)} - \sigma \cdot p)(EI^{(2)} + \sigma \cdot p)\Psi^{(2)} = m^2\Psi^{(2)}, \tag{13} \]

or

\[ (EI^{(4)} + \alpha \cdot p + m\beta)(EI^{(4)} - \alpha \cdot p - m\beta)\Psi^{(4)} = 0. \tag{14} \]

Of course, as in the original Dirac work, we have

\[ \beta^2 = 1, \quad \alpha^i \beta + \beta \alpha^i = 0, \quad \alpha^i \alpha^j + \alpha^j \alpha^i = 2\delta^{ij}. \tag{15} \]

For instance, their explicit forms can be chosen

\[ \alpha^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix}, \tag{16} \]

where \( \sigma^i \) are the ordinary Pauli \( 2 \times 2 \) matrices.

We also postulate the non-commutativity

\[ [E, p^i] = \Theta^{0i} = \theta^i, \tag{17} \]
as usual. Therefore the equation (14) will not lead to the well-known equation \( E^2 - p^2 = m^2 \). Instead, we have

\[
\left\{ E^2 - E(\alpha \cdot p) + (\alpha \cdot p)E - p^2 - m^2 - i\sigma \times I_2[p \times p] \right\} \Psi_{(4)} = 0
\]  

(18)

For the sake of simplicity, we may assume the last term to be zero. Thus we come to

\[
\left\{ E^2 - p^2 - m^2 - (\alpha \cdot \theta) \right\} \Psi_{(4)} = 0 .
\]  

(19)

However, let us make the unitary transformation. It is known \([11]\) that one can\(^5\)

\[
U_1(\sigma \cdot a)U_1^{-1} = \sigma_3|a| .
\]  

(20)

For \( \alpha \) matrices we re-write (20) to

\[
U_1(\alpha \cdot \theta)U_1^{-1} = [\theta] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \alpha_3|\theta| .
\]  

(21)

applying the second unitary transformation:

\[
U_2\alpha_3U_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \alpha_3 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .
\]  

(22)

The final equation is

\[
[E^2 - p^2 - m^2 - \gamma_5^{\text{chiral}}|\theta|] \Psi'_{(4)} = 0 .
\]  

(23)

In the physical sense this implies the mass splitting for a Dirac particle over the noncommutative space. This procedure may be attractive for explanation of the mass creation and the mass splitting for fermions.

\(^5\)Of course, the certain relations for the components \( a \) should be assumed. Moreover, in our case \( \theta \) should not depend on \( E \) and \( p \). Otherwise, we must take the noncommutativity \([E, p]^-\) again.
The presented ideas permit us to provide some foundations for non-commutative field theories and induce us to look for further applications of the functions with explicit and implicit dependencies in physics and mathematics.

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