Monotone complete $C^*$-algebras and generic dynamics

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Abstract

Let $S$ be the Stone space of a complete, non-atomic, Boolean algebra. Let $G$ be a countably infinite group of homeomorphisms of $S$. Let the action of $G$ on $S$ have a free dense orbit. Then we prove that, on a generic subset of $S$, the orbit equivalence relation coming from this action can also be obtained by an action of the Dyadic Group, $\bigoplus \mathbb{Z}_2$. As an application, we show that if $M$ is the monotone cross-product $C^*$-algebra, arising from the natural action of $G$ on $C(S)$, and if the projection lattice in $C(S)$ is countably generated, then $M$ can be approximated by an increasing sequence of finite-dimensional subalgebras. On each $S$, in a class considered earlier, we construct a natural action of $\bigoplus \mathbb{Z}_2$ with a free dense orbit. Using this we exhibit a huge family of small monotone complete $C^*$-algebras, $(B_\lambda, \lambda \in \Lambda)$ with the following properties. Each $B_\lambda$ is a Type III factor that is not a von Neumann algebra. Each $B_\lambda$ is a quotient of the Pedersen–Borel envelope of the Fermion algebra and hence is strongly hyperfinite. The cardinality of $\Lambda$ is $2^c$, where $c = 2^{\aleph_0}$. When $\lambda \neq \mu$, then $B_\lambda$ and $B_\mu$ take different values in the classification semi-group; in particular, they cannot be isomorphic.

1. Introduction: monotone complete $C^*$-algebras

Let $A$ be a $C^*$-algebra. Its self-adjoint part, $A_{sa}$, is a partially ordered, real Banach space whose positive cone is $\{zz^* : z \in A\}$. If each upward directed, norm-bounded subset of $A_{sa}$ has a least upper bound, then $A$ is said to be monotone complete. Each monotone complete $C^*$-algebra has a unit element (this follows by considering approximate units). Unless we specify otherwise, all $C^*$-algebras considered in this note will possess a unit element. Every von Neumann algebra is monotone complete, but the converse is false.

Monotone complete $C^*$-algebras arise in several different areas. For example, each injective operator system can be given the structure of a monotone complete $C^*$-algebra, in a canonical way. Injective operator spaces can be embedded as ‘corners’ of monotone complete $C^*$-algebras, see [11, Theorems 6.1.3 and 6.1.6] and [17, 18].

When a monotone complete $C^*$-algebra is commutative, its lattice of projections is a complete Boolean algebra. Up to isomorphism, every complete Boolean algebra arises in this way.

We recall that each commutative (unital) $C^*$-algebra can be identified with $C(X)$, the algebra of complex-valued continuous functions on some compact Hausdorff space $X$. Then $C(X)$ is monotone complete precisely when $X$ is extremally disconnected, that is, the closure of each open subset of $X$ is also open.

Monotone complete $C^*$-algebras are a generalization of von Neumann algebras. The theory of the latter is now very well advanced. In the 1970s, the pioneering work of Connes, Takesaki and other giants of the subject transformed our knowledge of von Neumann algebras, see [39]. In contrast, the theory of monotone complete $C^*$-algebras is very incomplete with many fundamental questions unanswered. However, considerable progress has been made in recent years.

This article follows on from [35] where we introduced a classification semi-group for small monotone complete $C^*$-algebras that divides them into $2^c$ distinct equivalence classes. But it

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is not necessary to have read that paper in order to understand this one. Our aim is to be comprehensible by anyone with a grounding in functional analysis and some exposure to the more elementary parts of C*-algebra theory, say, the first chapter of [38].

A monotone complete C*-algebra is said to be a factor if its centre is one-dimensional; we may regard factors as being as far removed as possible from being commutative. Just as for von Neumann algebras, each monotone complete factor is of Type I or Type II1 or Type II∞, or Type III. Old results of Kaplansky [24–26] imply that each Type I factor is a von Neumann algebra. This made it natural for him to ask whether this is true for every factor. The answer is ‘no’, in general. We call a factor which is not a von Neumann algebra wild.

A C*-algebra is separably representable when it has an isometric *-representation on a separable Hilbert space. As a consequence of more general results, Wright [49] showed that if a monotone complete factor is separably representable (as a C*-algebra), then it is a von Neumann algebra. So, in these circumstances, Kaplansky’s question has a positive answer.

Throughout this note, a topological space is said to be separable if it has a countable dense subset; this is a weaker property than having a countable base of open sets. (But if the topology is metrizable, they coincide.) Akemann [1] showed that a von Neumann algebra has a faithful representation on a separable Hilbert space if and only if its state space is separable.

We call a C*-algebra with a separable state space almost separably representable. Answering a question posed by Akemann, see [1], Wright [48] gave examples of monotone complete C*-algebras that have separable state spaces but are NOT separably representable.

If a monotone complete factor M possesses a strictly positive functional and is not a von Neumann algebra, then, as an application of a more general result in [46], M must be of Type III, see also [29]. Whenever an algebra is almost separably representable, it possesses a strictly positive functional. (See Corollary 3.2.) So, if a wild factor is almost separably representable, then it must be of Type III.

A (unital) C*-algebra A is said to be small if there exists a unital complete isometry of A into L(H), where H is a separable Hilbert space, see [21, 33, 35].

It turns out that A is small if and only if A \( \otimes M_n(\mathbb{C}) \) has a separable state space for \( n = 1, 2, \ldots, 33 \). So, clearly every small C*-algebra is almost separably representable. We do not know whether the converse is true, but it is true for monotone complete factors, [33].

Examples of (small) wild factors were hard to find. The first examples were due to Dyer [10] and Takenouchi [37]. As a consequence of a strong uniqueness theorem of Sullivan–Weiss–Wright [36], it turned out that the Dyer factor and the Takenouchi factor were isomorphic. See also [30] where the Dyer factor was identified as a monotone cross-product of the Dixmier algebra by an action of the dyadic rationals.

Another method of finding wild factors was given by Wright [47]. He showed that each C*-algebra A could be embedded in its ‘regular σ-completion’, \( \hat{A} \). When \( A \) is separably representable, then \( \hat{A} \) is monotone complete and almost separably representable. Furthermore, when \( A \) is infinite-dimensional, unital and simple, then \( \hat{A} \) is a wild factor. But it was very hard to distinguish between these factors. Indeed one of the main results of [36] showed that all factors in what appeared to be a large class were, in fact, isomorphic to a unique (hyperfinite) factor. Some algebras were shown to be different in [19, 31, 32]. In 2001, a major breakthrough by Hamana [21] showed that there were \( 2^c \) non-isomorphic (small) wild factors, where \( c = 2^{2\aleph_0} \). This pioneering paper has not yet received as much attention as it deserves.

In [35], we introduced a quasi-ordering between monotone complete C*-algebras. From this quasi-ordering, we defined an equivalence relation and used this to construct a classification semi-group \( W \) for a class of monotone complete C*-algebras. This semi-group is abelian, partially ordered, and has the Riesz decomposition property. For each monotone complete, small C*-algebra A we assign a ‘normality weight’, \( w(A) \in W \). If \( A \) and \( B \) are algebras, then \( w(A) = w(B) \), precisely when these algebras are equivalent. It turns out that algebras that are very different can be equivalent. In particular, the von Neumann algebras correspond to
the zero element of the semi-group. It might have turned out that \( W \) is very small and fails to distinguish between more than a few algebras. This is not so; the cardinality of \( W \) is \( 2^c \), where \( c = 2^{\aleph_0} \).

One of the useful properties of \( W \) is that it can sometimes be used to replace problems about factors by problems about commutative algebras [35]. For example, let \( G_j \) be a countable group acting freely and ergodically on a commutative monotone complete algebra \( A_j \) \((j = 1, 2)\). By a cross-product construction using these group actions, we can obtain monotone complete \( C^*\)-factors \( B_j \) \((j = 1, 2)\). Then it is easy to show that \( wA_j = wB_j \). So, if the commutative algebras \( A_1 \) and \( A_2 \) are not equivalent, then \( wB_1 \neq wB_2 \). In particular, \( B_1 \) and \( B_2 \) are not isomorphic.

Influenced by \( K\)-theory, it is natural to form the Grothendieck group of the semi-group \( W \). This turns out to be futile, since this Grothendieck group is trivial, because every element of \( W \) is idempotent. By a known general theory [15], this implies that \( W \) can be identified with a join semi-lattice. The Riesz decomposition property for the semigroup turns out to be equivalent to the semi-lattice being distributive. So, the known theory of distributive join semi-lattices can be applied to \( W \).

To each monotone complete \( C^*\)-algebra \( A \) we associated a spectroid invariant \( \partial A \) [35]. Just as a spectrum is a set that encodes information about an operator, a spectroid encodes information about a monotone complete \( C^*\)-algebra. It turns out that equivalent algebras have the same spectroid. So, the spectroid may be used as a tool for classifying elements of \( W \). For a generalization of spectroid, see [52].

One of the many triumphs of Connes in the theory of von Neumann algebras was to show that the injective von Neumann factors are precisely those which are hyperfinite [6], see also [39]. It is natural to conjecture an analogous result for wild factors (See [9, 51]). But this is not true. For, by applying deep results of Hjorth and Kechris [22], it is possible to exhibit a small, wild, hyperfinite factor that is not injective. We shall give details of this, and other more general results in a sequel to this paper.

When dealing with monotone complete \( C^*\)-algebras, saying precisely what we mean by ‘hyperfinite’, ‘strongly hyperfinite’, ‘approximately finite-dimensional’ and ‘nearly approximately finite-dimensional’ requires subtle distinctions that are not needed in von Neumann algebra theory. See Section 12 for details.

(\( \# \))Let \( \Lambda \) be a set of cardinality \( 2^c \), where \( c = 2^{\aleph_0} \). Then we showed in [35] that there exists a family of monotone complete \( C^*\)-algebras \( \{B_\lambda : \lambda \in \Lambda \} \) with the following properties. Each \( B_\lambda \) is a monotone complete factor of Type III, and also a small \( C^*\)-algebra. For \( \lambda \neq \mu \), \( B_\lambda \) and \( B_\mu \) have different spectroids and so \( wB_\lambda \neq wB_\mu \) and, in particular, \( B_\lambda \) is not isomorphic to \( B_\mu \). We show, in Section 12, that, by using the machinery constructed in this paper, we may choose each \( B_\lambda \) such that it is generated by an increasing sequence of full matrix algebras.

2. Introduction: generic dynamics

An elegant account of generic dynamics is given by Weiss [41]; the term first occurred in [36]. In these articles, the underlying framework is a countable group of homeomorphisms acting on a complete separable metric space with no isolated points (a perfect Polish space). This corresponds to dynamics on a unique compact space, Hausdorff, extremally disconnected space (the Stone space of the complete Boolean algebra of regular open subsets of \( \mathbb{R} \)).

Let \( G \) be a countable group. Unless we specify otherwise, \( G \) will always be assumed to be infinite and equipped with the discrete topology. Let \( X \) be a Hausdorff topological space with no isolated points. Furthermore, suppose that \( X \) is a Baire space, that is, such that the only meagre open set is the empty set. (This holds if \( X \) is compact or a \( G\)-delta subset of a compact Hausdorff space or is homeomorphic to a complete separable metric space.) A subset \( Y \) of \( X \) is said to be generic, if \( X \setminus Y \) is meagre.
Let $\varepsilon$ be an action of $G$ on $X$ as homeomorphisms of $X$.

In classical dynamics, we would require the existence of a Borel measure on $X$ which was $G$-invariant or quasi-invariant, and discard null sets. In topological dynamics, no measure is required and no sets are discarded. In generic dynamics, we discard meagre Borel sets.

We shall concentrate on the situation where, for some $x_0 \in X$, the orbit $\{\varepsilon_g(x_0) : g \in G\}$ is dense in $X$. Of course this cannot happen unless $X$ is separable. Let $S$ be the Stone space of the (complete) Boolean algebra of regular open sets of $X$. Then, see below, the action $\varepsilon$ of $G$ on $X$ induces an action $\hat{\varepsilon}$ of $G$ as homeomorphisms of $S$, which will also have a dense orbit.

When, as in [41], $X$ is a perfect Polish space, then, as mentioned above, $S$ is unique; it can be identified with the Stone space of the regular open sets of $\mathbb{R}$. But if we let $X$ range over all separable compact subspaces of the separable space, $2^\mathbb{R}$, then we obtain $2^c$ essentially different $S$, where $S$ is compact, separable and extremally disconnected. For each such $S$, $C(S)$ is a subalgebra of $\ell^\infty$.

Let $E$ be the relation of orbit equivalence on $S$. That is, $sEt$, if, for some group element $g$, $\varepsilon_g(s) = t$. Then we can construct a monotone complete $C^*$-algebra $M_E$ from the orbit equivalence relation. When there is a free dense orbit, the algebra will be a factor with a maximal abelian subalgebra, $A$, which is isomorphic to $C(S)$. There is always a faithful, normal, conditional expectation from $M_E$ onto $A$.

For $f \in C(S)$, let $\gamma^g(f) = f \circ \varepsilon_{g^{-1}}$. Then $g \to \gamma^g$ is an action of $G$ as automorphisms of $C(S)$. Then we can associate a monotone complete $C^*$-algebra $M(C(S), G)$, the monotone cross-product (see [37]) with this action. When the action $\hat{\varepsilon}$ is free, then $M(C(S), G)$ is naturally isomorphic to $M_E$. In other words, the monotone cross-product does not depend on the group, only on the orbit equivalence relation. This was a key point in [36] where a strong uniqueness theorem was proved.

In this article, we consider $2^c$ algebras $C(S)$, each taking different values in the weight semi-group $W$. (Here $c = 2^{\omega_0}$, the cardinality of $\mathbb{R}$.)

There is no uniqueness theorem but we do show the following. Let $G$ be a countably infinite group. Let $\alpha$ be an action of $G$ as homeomorphisms of $S$ and suppose that this action has a single orbit which is dense and free. Then, modulo meagre sets, the orbit equivalence relation obtained can also be obtained by an action of $\bigoplus \mathbb{Z}_2$ as homeomorphisms of $S$.

This should be compared with the situation in classical dynamics. For example, it is shown in [7] that any action by an amenable group is orbit equivalent to an action of $\mathbb{Z}$. But, in general, non-amenable groups give rise to orbit equivalence relations which do not come from actions of $\mathbb{Z}$.

On each of $2^c$, essentially different, compact extremally disconnected spaces, we construct a natural action of $\bigoplus \mathbb{Z}_2$ with a free dense orbit. This gives rise to a family of monotone complete $C^*$-algebras, $(B_\lambda, \lambda \in \Lambda)$ with the properties (#) described above.

Let $E$ be the orbit equivalence relation arising from a free, ergodic action of $G$. Furthermore, suppose that the complete Boolean algebra of projections in $C(S)$ is countably generated. Let $N(M_E)$ be the smallest monotone closed $*$-subalgebra of $M_E$ that contains the normalizing unitaries of $A$ (that is, the set of all unitaries $u$ such that $u^*Au = A$). Then $N(M_E)$ is an AFD factor. More precisely, there is an increasing sequence of finite-dimensional, unital, $*$-subalgebras of $N(M_E)$, whose union $\sigma$-generates $N(M_E)$. (In contrast to the situation for von Neumann factors, we do not know whether we can always take these finite-dimensional subalgebras to be full matrix algebras.) As pointed out above, we need to make a number of subtle distinctions when approximating monotone complete algebras by finite-dimensional subalgebras; see Section 12 for details. For example $M_E$ is 'nearly AFD'. But in Section 11, we construct huge numbers of examples of $\bigoplus \mathbb{Z}_2$ actions on spaces $S$, which give rise to factors which we show to be strongly hyperfinite.
3. **Monotone $\sigma$-complete $C^*$-algebras**

Although our focus is on monotone complete $C^*$-algebras, we also need to consider more general objects, the monotone $\sigma$-complete $C^*$-algebras.

A $C^*$-algebra is **monotone $\sigma$-complete** if each norm bounded, monotone increasing sequence of self-adjoint elements has a least upper bound.

**Lemma 3.1.** Let $A$ be a monotone $\sigma$-complete $C^*$-algebra. Let there exist a positive linear functional $\mu : A \to \mathbb{C}$ which is faithful. Then $A$ is monotone complete. Let $\Lambda$ be a downward directed subset of $A_{sa}$ which is bounded below. Then there exists a monotone decreasing sequence $(x_n)$, with each $x_n \in \Lambda$, such that the greatest lower bound of $(x_n)$, $\bigwedge_{n=1}^{\infty} x_n$, is the greatest lower bound of $\Lambda$.

**Proof.** See [34].

**Corollary 3.2.** When an almost separably representable algebra is unital and monotone $\sigma$-complete then it is monotone complete.

**Proof.** If $A$ is almost separably representable, then we can find states $(\phi_n) (n = 1, 2, \ldots)$ which are dense in its state space. Then $\phi = \sum_{n=1}^{\infty} (1/2^n) \phi_n$ is a faithful, positive linear functional.

Let $A$ be a $C^*$-subalgebra of $L(H)$. Let $V$ be a real subspace of the real Banach space $L(H)_{sa}$. We call $V$ a $\sigma$-closed subspace of $L(H)_{sa}$ if, whenever $(a_n)$ is an upper bounded, monotone increasing sequence in $V$ then its limit in the weak operator topology is in $V$. Consider the family of all $\sigma$-closed subspaces which contain $A_{sa}$, then the intersection of this family is the (smallest) $\sigma$-closed subspace containing $A_{sa}$. By a theorem of Pedersen this is the self-adjoint part of a monotone $\sigma$-complete $C^*$-subalgebra of $L(H)$, see [28, Theorem 4.5.4].

Let $B$ be a (unital) $C^*$-algebra. Let us recall some well-known classical results [28, 38]. Let $(\pi, H)$ be the universal representation of $B$, that is, the direct sum of all the Gelfand-Naimark-Segal representations corresponding to each state of $B$. Then the second dual of $B$, $B''$, may be identified with the von Neumann envelope of $\pi(B)$ in $L(H)$. Let $B_{sa}^\infty$ be the smallest subspace of $B_{sa}'$, (the self-adjoint part of $B''$), which is closed under taking limits (in the weak operator topology) of bounded, monotonic sequences. Let $B^\infty = B_{sa}^\infty + iB_{sa}^\infty$. Then by Pedersen’s theorem $B^\infty$ is a monotone $\sigma$-complete $C^*$-subalgebra of $B''$. We call $B^\infty$ the Pedersen–Borel envelope of $B$. (Pedersen called this simply the ‘Borel envelope’, it has also, with some justice, been called the Baire envelope.)

Let $B$ be a monotone $\sigma$-complete $C^*$-algebra. We recall that $V \subset B_{sa}$ is a $\sigma$-subspace of $B_{sa}$, if $V$ is a real vector subspace of $B_{sa}$ such that, whenever $(b_n)$ is a monotone increasing sequence in $V$, which has a supremum $b$ in $B_{sa}$, then $b \in V$. (In particular, the $\sigma$-subspaces of $L(H)$ are precisely the $\sigma$-closed subspaces of $L(H)$.)

A $\sigma$-subalgebra of $B$ is a $*$-subalgebra whose self-adjoint part is a $\sigma$-subspace of $B_{sa}$. It follows from [45, Lemma 1.2] that each $\sigma$-subalgebra is closed in norm and hence is a $C^*$-subalgebra; see also [5].

Furthermore, $J$ is a $\sigma$-ideal of $B$ if $J$ is a $C^*$-ideal of $B$ and also a $\sigma$-subalgebra of $B$.

When $B$ and $A$ are monotone $\sigma$-complete $C^*$-algebras, a positive linear map $\phi : B \to A$ is said to be $\sigma$-normal if, whenever $(b_n)$ is monotone increasing and bounded above, $\phi$ maps the supremum of $(b_n)$, to the supremum of $(\phi(b_n))$, that is, $\phi(\bigvee_{n=1}^{\infty} b_n) = \bigvee_{n=1}^{\infty} \phi(b_n)$.

**Lemma 3.3.** Let $A$ be a monotone $\sigma$-complete $C^*$-algebra and let $J$ be a $\sigma$-ideal of $A$. Let $q$ be the quotient homomorphism of $A$ onto $A/J$. Then $A/J$ is monotone $\sigma$-complete and $q$ is $\sigma$-normal. Let $(c_n)$ be a monotone increasing sequence in the self-adjoint part of $A/J$ which is
bounded above by c. Then there exists a monotone increasing sequence \((a_n)\) in \(A_{sa}\) such that \(q(a_n) = c_n\) for each \(n\) and \((a_n)\) is bounded above by \(a\) where \(q(a) = c\).

Proof. This follows from [45, Proposition 1.3 and Lemma 1.1], see also [5].

The following representation theorem was proved by Wright [44]. It may be thought of as a non-commutative generalization of a theorem of Loomis and Sikorski in Boolean algebras [16].

**Proposition 3.4.** Let \(B\) be a monotone \(\sigma\)-complete \(C^*\)-algebra. Then there exists a \(\sigma\)-normal homomorphism, \(\pi\), from \(B^\infty\) onto \(B\), such that \(\pi(b) = b\) for every \(b \in B\). Let \(J\) be the kernel of the homomorphism \(\pi\). Then \(J\) is a \(\sigma\)-ideal of \(B^\infty\) and \(B = B^\infty/J\).

**Corollary 3.5.** Let \(A\) and \(B\) be \(C^*\)-algebras and let \(B\) be monotone \(\sigma\)-complete. Let \(\phi : A \to B\) be a positive linear map. Then \(\phi\) has a unique extension to a \(\sigma\)-normal positive linear map, \(\hat{\phi}\), from \(A^\infty\) into \(B\). When \(\phi\) is a \(*\)-homomorphism the following hold. First, \(\hat{\phi}\) is also a \(*\)-homomorphism. Secondly, the range of \(\hat{\phi}\) is a \(\sigma\)-subalgebra of \(B\). Thirdly, the self-adjoint part of \(\hat{\phi}[A^\infty]\) is the smallest \(\sigma\)-subspace of \(B_{sa}\) that contains \(\phi[A_{sa}]\). Finally, the kernel of \(\hat{\phi}\) is a \(\sigma\)-ideal, \(J\), such that \(A^\infty/J \approx \hat{\phi}[A^\infty]\).

For a proof, see [47, Proposition 1.1].

**Remark.** Let \(S\) be a subset of a monotone \(\sigma\)-complete \(C^*\)-algebra \(B\). Let \(A\) be the smallest (unital) \(C^*\)-subalgebra of \(B\) which contains \(S\). Let \(\phi\) be the inclusion map from \(A\) into \(B\). By applying the preceding result, the smallest \(\sigma\)-subspace of \(B_{sa}\) that contains \(A_{sa}\) is the self-adjoint part of a \(\sigma\)-subalgebra \(C\) of \(B\). It is now natural to describe \(C\) as the \(\sigma\)-subalgebra of \(B\) which is \(\sigma\)-generated by \(S\).

4. Extending continuous functions

We gather together some topological results which will be useful later. The most important of these is Theorem 4.7. We hope that the presentation here is clear enough to ensure that the reader can reconstruct any missing proofs without difficulty. If we have misjudged this, we apologize and refer the reader to [14], see also [2].

Throughout this section, \(K\) is a compact Hausdorff space and \(D\) is a dense subset of \(K\), equipped with the relative topology induced by \(K\). It is easy to see that \(K\) has no isolated points if and only if \(D\) has no isolated points.

Let us recall that a topological space \(T\) is extremally disconnected if the closure of each open subset is still an open set.

When \(K\) is extremally disconnected, then whenever \(Z\) is a compact Hausdorff space and \(f : D \to Z\) is continuous, there exists a unique extension of \(f\) to a continuous function \(F : K \to Z\). In other words, \(K\) is the Stone–Cech compactification of \(D\). (This is Theorem 4.7.)

For any compact Hausdorff space \(K\), the closed subsets of \(D\), in the relative topology, are all of the form \(F \cap D\) where \(F\) is a closed subset of \(K\). For any \(S \subseteq K\), we denote the closure of \(S\) (in the topology of \(K\)) by \(\text{cl}(S)\). For \(S \subseteq D\), we note that the closure of this set in the relative topology of \(D\) is \(\text{cl}(S) \cap D\). We denote this by \(\text{cl}_D(S)\). We also use \(\text{int}_S\) for the interior of \(S\) and, when \(S \subseteq D\), the interior with respect to the relative topology is denoted by \(\text{int}_D(S)\).

The following lemmas are routine point-set topology.

**Lemma 4.1.** Let \(K\) be a compact Hausdorff space and \(D\) a dense subset of \(K\).

(i) For any open subset \(U\) of \(K\), we have \(\text{cl}(U) = \text{cl}(U \cap D)\).
(ii) Let $U, V$ be open subsets of $K$. Then $V \subset \text{cl}(U)$ if and only if $V \cap D \subset \text{cl}(U \cap D) \cap D = \text{cl}_D(U \cap D)$.

(iii) Let $U$ be an open subset of $K$. Then $D \cap \text{int}(\text{cl}U) = \text{int}_D(\text{cl}_D(U \cap D))$.

(iv) If $U$ is a regular open subset of $K$, then $U \cap D$ is a regular open subset of $D$. Conversely, if $E$ is a regular open subset of $D$ in the relative topology of $D$, then $E = V \cap D$, where $V$ is a regular open subset of $K$.

For any topological space $Y$, we let Reg$Y$ denote the Boolean algebra of regular open subsets of $Y$.

**Lemma 4.2.** The function $H$, when restricted to Reg$K$, is a Boolean isomorphism of Reg$K$ onto Reg$D$.

**Lemma 4.3.** A Hausdorff topological space $T$ is extremally disconnected if and only if each regular open set is closed, and hence clopen.

**Corollary 4.4.** Let $D$ be a dense subset of a compact Hausdorff extremally disconnected space $S$. Let $D$ be equipped with the relative topology. Then $D$ is an extremally disconnected space.

**Proof.** Let $V$ be a regular open subset of $D$. Then, by Lemma 4.1, part (iv), there exists $U$, a regular open subset of $S$, such that $V = U \cap D$. By Lemma 4.3, $U$ is a clopen subset of $S$. Hence $V$ is a clopen subset of $D$ in the relative topology. Again appealing to Lemma 4.3, we have that $D$ is an extremally disconnected space.

**Lemma 4.5.** Let $D$ be an extremally disconnected topological space. Also let $D$ be homeomorphic to a subspace of a compact Hausdorff space. Then $\beta D$, its Stone–Czech compactification, is extremally disconnected.

**Lemma 4.6.** Let $D$ be a dense subspace of a compact Hausdorff extremally disconnected space $Z$. When $A$ is a clopen subset of $D$ in the relative topology, then $\text{cl}A$ is a clopen subset of $Z$. Let $A$ and $B$ be disjoint clopen subsets of $D$, in the relative topology. Then $\text{cl}A$ and $\text{cl}B$ are disjoint clopen subsets of $Z$.

The following result was given by Gillman and Jerison, see [14, p. 96], as a byproduct of other results. The argument given here may be slightly easier and more direct.

**Theorem 4.7.** Let $D$ be a dense subspace of a compact Hausdorff extremally disconnected space $S$. Then $S$ is the Stone–Czech compactification of $D$. More precisely, there exists a unique homeomorphism from $\beta D$ onto $S$ that restricts to the identity homeomorphism on $D$.

**Proof.** Since $D$ is a subspace of the compact Hausdorff space, $S$, $D$ is completely regular and hence has a well-defined Stone–Czech compactification. By the fundamental property of $\beta D$, there exists a unique continuous surjection $\alpha$ from $\beta D$ onto $S$, which restricts to the identity on $D$.

Let $a$ and $b$ be distinct points in $\beta D$. Then there exist disjoint clopen sets $U$ and $V$ such that $a \in U$ and $b \in V$. Let $A = U \cap D$ and $B = V \cap D$, then $U = \text{cl}_B A$ and $V = \text{cl}_B B$. So, $\alpha[U] \subset \text{cl}_S A$ and $\alpha[V] \subset \text{cl}_S B$. By Lemma 4.6, $\text{cl}_S A$ and $\text{cl}_S B$ are disjoint. Hence $\alpha(a)$ and
$\alpha(b)$ are distinct points of $S$. Thus $\alpha$ is injective. It now follows from compactness that $\alpha$ is a homeomorphism.

5. Ergodic discrete group actions on topological spaces

In this section, $Y$ is a Hausdorff topological space that has no isolated points. For example, a compact Hausdorff space with no isolated points, or a dense subset of such a space.

When $G$ is a group of bijections of $Y$, and $y \in Y$, we denote the orbit $\{g(y) : g \in G\}$ by $G[y]$.

**Lemma 5.1.** Let $G$ be a countable group of homeomorphisms of $Y$.

(i) If there exists $x_0 \in Y$ such that the orbit $G[x_0]$ is dense in $Y$, then every $G$-invariant open subset of $Y$ is either empty or dense.

(ii) If every non-empty open $G$-invariant subset of $Y$ is dense, then, for each $x$ in $Y$, the orbit $G[x]$ is either dense or nowhere dense.

**Proof.** (i) Let $U$ be a $G$-invariant open set which is not empty. Since $G[x_0]$ is dense, for some $g \in G$, we have $g(x_0) \in U$. But $U$ is $G$-invariant. So $x_0 \in U$. Hence $G[x_0] \subset U$. So, $U$ is dense in $Y$.

(ii) Suppose that $y$ is an element of $Y$ such that $G[y]$ is not dense in $Y$. Then $Y \setminus \text{cl}\, G[y]$ is a non-empty $G$-invariant open set. So, it is dense in $Y$. So, $\text{cl}\, G[y]$ has empty interior.

When $G$ is a group of homeomorphisms of $Y$, its action is said to be ergodic if each $G$-invariant open subset of $Y$ is either empty or dense in $Y$.

**Lemma 5.2.** Let $Y$ be an extremally disconnected space. Let $G$ be a group of homeomorphisms of $Y$. Then the action of $G$ is ergodic, if and only if the only $G$-invariant clopen subsets are $Y$ and $\emptyset$.

**Proof.** Let $U$ be a $G$-invariant open set. Then $\text{cl}\, U$ and $Y \setminus \text{cl}\, U$ are $G$-invariant clopen sets. Then $U$ is neither empty nor dense if and only if $\text{cl}\, U$ and $Y \setminus \text{cl}\, U$ are non-trivial clopen sets.

6. Induced actions

Let $X$ be a compact Hausdorff space. Then, see [35, Lemma 13], $X$ is separable if and only if $C(X)$ is isomorphic to a closed (unital) *-subalgebra of $\ell^\infty$.

The regular $\sigma$-completion of an arbitrary $C^*$-algebra was defined in [47]. For the commutative algebra, $C(X)$, its regular $\sigma$-completion can be identified with the monotone $\sigma$-complete $C^*$-algebra $B_0^\infty(X)/M_0(X)$, where $B_0^\infty(X)$ is the algebra of bounded Baire measurable functions on $X$ and $M_0(X)$ is the ideal of all $f$ in $B_0^\infty(X)$ for which $\{x : f(x) \neq 0\}$ is meagre. Let $S$ be the structure space of $B_0^\infty(X)/M_0(X)$, that is, this algebra can be identified with $C(S)$.

Let $j : C(X) \to B_0^\infty(X)/M_0(X)$ be the natural embedding. This is an injective (isometric) *-homomorphism.

Suppose that $X$ is separable. Then there exists an injective *-homomorphism $h : C(X) \to \ell^\infty$. Since $\ell^\infty$ is monotone complete, $h$ extends to a homomorphism $H : C(S) \to \ell^\infty$. From standard properties of regular $\sigma$-completions [47], $H$ is also injective. Hence $C(S)$ supports a strictly positive linear functional. By Lemma 3.1, it follows that $C(S)$ is monotone complete and hence $S$ is extremally disconnected.

Since $H$ is an injective homomorphism, it follows that there is a surjective continuous map from $\beta\mathbb{N}$ onto $S$. So, $S$ is separable.
Remark. Because $C(S)$ is monotone complete, $S$ is the Stone structure space of the complete Boolean algebra of regular open subsets of $X$. It follows from the Birkhoff–Ulam Theorem in Boolean algebras, and the linearization arguments in [42] that $C(S)$ can be identified with $B^\infty(X)/M(X)$ where $B^\infty(X)$ is the algebra of bounded Borel measurable functions on $X$ and $M(X)$ is the ideal of all bounded Borel functions with meagre support. In other words, $B^\infty(X)/M(X)$ is isomorphic to $B^\infty_0(X)/M_0(X)$ which is isomorphic to $C(S)$.

By the usual duality between compact Hausdorff spaces and commutative (unital) $C^*$-algebras, there is a continuous surjection $\rho$ from $S$ onto $X$ such that $j(f) = f \circ \rho$ for each $f$ in $C(X)$.

By the basic properties of regular $\sigma$-completions, for each self-adjoint $b$ in $C(S)$, the set \{\(j(a) : a \in C_R(X) \text{ and } j(a) \leq b\}\} has $b$ as its least upper bound in $C(S)_{sa} = C_R(S)$.

**Lemma 6.1.** Let $Y$ be a subset of $S$ such that $\rho[Y]$ is dense in $X$. Then $Y$ is dense in $S$.

**Proof.** Let us assume that $Y$ is not dense in $S$. Then there exists a non-empty clopen set $E$ which is disjoint from $cl Y$.

Let $j(a) \leq \chi_E$. Then $j(a)(s) \leq 0$ for $s \in Y$. So, $a(\rho(s)) \leq 0$ for $\rho(s) \in \rho[Y]$. Hence $a \leq 0$. But this implies $\chi_E \leq 0$ which is a contradiction $\blacksquare$.

Let $Y$ be any compact Hausdorff space. Let Homeo($Y$) be the group of all homeomorphisms from $Y$ onto $Y$. Let Aut $C(Y)$ be the group of all $\ast$-automorphisms of $C(Y)$. For $\phi \in$ Homeo($Y$), let $h_\phi(f) = f \circ \phi$ for each $f \in C(Y)$. Then $\phi \rightarrow h_\phi$ is a bijection from the group Homeo($Y$) onto Aut $C(Y)$ which switches the order of multiplication. In other words, it is a group anti-automorphism.

Let $\theta$ be a homeomorphism of $X$ onto $X$. As above, let $h_\theta$ be the corresponding $\ast$-automorphism of $C(X)$. Also $f \rightarrow f \circ \theta$ induces an automorphism $\hat{h}_\theta$ of $B^\infty(X)/M(X)$. Since $B^\infty(X)/M(X)$ can be identified with $C(S)$, there exists $\theta$ in Homeo($S$) corresponding to $h_\theta$. Clearly, $h_\theta$ restricts to the automorphism, $h_\theta$, of $C(X)$.

**Lemma 6.2.** The automorphism $\hat{h}_\theta$ is the unique automorphism of $C(S)$ which is an extension of $h_\theta$. Hence $\hat{\theta}$ is uniquely determined by $\theta$. Furthermore, the map $\theta \rightarrow \hat{\theta}$ is an injective group homomorphism from Homeo($X$) into Homeo($S$).

**Proof.** Let $H$ be an automorphism of $B^\infty(X)/M(X) = C(S)$, which is an extension of $h_\theta$. Let $b$ be a self-adjoint element of $B^\infty(X)/M(X)$. Then, for $a \in C_R(X)$, $a \leq b$ if and only if $Ha \leq Hb$, that is, $h_\theta a \leq Hb$. So, $Hb$ is the supremum of $\{h_\theta(a) : a \in C_R(X), a \leq b\}$. Hence $H = h_\theta$. That is, $h_\theta$ is the unique extension of $h_\theta$ to an automorphism of $C(S)$.

Let $h_1$ and $h_2$ be in Aut $C(X)$. Then, for $a \in C(X)$, we have $\hat{h}_1 h_2(a) = h_1 h_2(a) = \hat{h}_1 h_2(a) = \hat{h}_1 h_2(a)$.

By uniqueness, it now follows that $\hat{h}_1 = \hat{h}_1 h_2 = \hat{h}_1 h_2$. Hence $h \rightarrow \hat{h}$ is an injective group homomorphism of Aut $C(X)$ into Aut $C(S)$. Therefore, the map $\theta \rightarrow \hat{\theta}$ is the composition of a group anti-isomorphism with an injective group homomorphism composing with a group anti-isomorphism. Hence, it is an injective group homomorphism $\blacksquare$.

**Corollary 6.3.** $\theta(\rho s) = \rho(\theta s)$ for each $s \in S$.

**Proof.** For $a \in C(X)$, $s \in S$,

$$a \circ \theta(\rho s) = h_\theta(a)(\rho s) = \hat{h}_\theta(h(a)(s)) = h(a)(\theta s) = a(\rho(\theta s)).$$

Hence $\theta(\rho s) = \rho(\theta s)$ $\blacksquare$. 


Throughout this paper, unless we specify otherwise, \(G\) is a countable infinite group. Let \(\varepsilon : G \to \text{Homeo}(X)\) be a homomorphism into the group of homeomorphisms of \(X\). That is, \(\varepsilon\) is an action of \(G\) on \(X\). For each \(g \in G\), let \(\hat{\varepsilon}_g\) be the homeomorphism of \(S\) onto \(S\) induced by \(\varepsilon_g\). Then \(\hat{\varepsilon}\) is the action of \(G\) on \(S\) induced by \(\varepsilon\).

Let us recall that an action \(\varepsilon : G \to \text{Homeo}(X)\) is non-degenerate if it is injective. We shall normally only use non-degenerate actions.

**Proposition 6.4.** Let \(x_0\) be a point in \(X\) such that the orbit \(\{\varepsilon_g(x_0) : g \in G\}\) is dense in \(X\). Let \(s_0 \in S\) such that \(\rho s_0 = x_0\). Then \(\{\hat{\varepsilon}_g(s_0) : g \in G\}\) is an orbit which is dense in \(S\).

**Proof.** By Corollary 6.3, \(\varepsilon_g(x_0) = \rho(\hat{\varepsilon}_g(s_0))\). It now follows from Lemma 6.1 that the orbit \(\{\hat{\varepsilon}_g(s_0) : g \in G\}\) is dense in \(S\).

**Definition 6.5.** An orbit \(\{\varepsilon_g(x_0) : g \in G\}\) is said to be free if, for \(g \neq 1\), \(\varepsilon_g(x_0) \neq x_0\). Equivalently, for \(g \neq 1\), \(\varepsilon_g\) leaves no point of the orbit fixed.

It is easy to see that the existence of at least one free orbit implies that the action is non-degenerate.

**Definition 6.6.** Let \(Y\) be a subset of \(X\) which is invariant under the action \(\varepsilon\). Then the action \(\varepsilon\) is free on \(Y\) if, for each \(y \in Y\), the orbit \(\{\varepsilon_g(y) : g \in G\}\) is free.

**Lemma 6.7.** Let \(G\), \(X\) and \(\varepsilon\) be as above. Let \(x_0 \in X\) be such that the orbit \(\{\varepsilon_g(x_0) : g \in G\}\) is both dense and free. Then there exists a \(G\)-invariant \(Y\), which is a dense \(G\)-invariant subset of \(X\) such that for \(g \neq 1\), \(\varepsilon_g\) has no fixed point in \(Y\). Also \(x_0 \in Y\).

**Proof.** Fix \(g \neq 1\), let \(K_g = \{x \in X : \varepsilon_g(x) = x\}\). Then \(K_g\), the fix-point set of \(\varepsilon_g\), is closed. Let \(U\) be the interior of \(K_g\). Then the orbit \(\{\varepsilon_h(x_0) : h \in G\}\) is disjoint from \(K_g\). So, its closure is disjoint from \(U\). Since the orbit is dense, this means that \(K_g\) has empty interior.

Let \(Z = \bigcup \{K_g : g \in G, g \neq 1\}\). Then \(Z\) is the union of countably many closed nowhere dense sets. A calculation shows that

\[
\varepsilon_h[K_g] = K_{gh^{-1}}
\]

and from this it follows that \(Z\) is \(G\)-invariant.

Put \(Y = X \setminus Z\). Then \(Y\) has all the required properties.

**Theorem 6.8.** Let \(G\), \(X\) and \(\varepsilon\) be as above. Let \(\hat{\varepsilon}\) be the action of \(G\) on \(S\) induced by the action \(\varepsilon\) on \(X\). Let \(x_0 \in X\) such that the orbit \(\{\varepsilon_g(x_0) : g \in G\}\) is both dense and free. Let \(s_0 \in S\) such that \(\rho s_0 = x_0\). Then \(\{\hat{\varepsilon}_g(s_0) : g \in G\}\) is a dense free orbit in \(S\).

Furthermore, there exists \(Y\), a \(G\)-invariant, dense \(G\)-invariant subset of \(S\), with \(s_0 \in Y\), such that the action \(\hat{\varepsilon}\) is free on \(Y\).

**Proof.** By Corollary 6.3, \(\varepsilon_g(\rho s_0) = \rho(\varepsilon_g s_0)\). That is, \(\varepsilon_g(x_0) = \rho(\hat{\varepsilon}_g(s_0))\).

It now follows from Lemma 6.1 that the orbit \(\{\hat{\varepsilon}_g(s_0) : g \in G\}\) is dense in \(S\).

Now suppose that \(\hat{\varepsilon}_h s_0 = s_0\). Then \(\rho(\hat{\varepsilon}_h s_0) = \rho(s_0)\). So, \(\varepsilon_h(x_0) = x_0\). Hence \(h = 1\). It now follows that \(\{\hat{\varepsilon}_g(s_0) : g \in G\}\) is a dense free orbit in \(S\).

The rest of the theorem follows by applying Lemma 6.7.
Remark. Let $D$ be a countable dense subset of a compact Hausdorff space $K$. Let $\alpha$ be a homeomorphism of $D$ onto $D$. Then, in general, $\alpha$ need not extend to a homeomorphism of $K$. But, from the fundamental properties of the Stone–Čech compactification, $\alpha$ does extend to a unique homeomorphism of $\beta D$, say $\theta_\alpha$. Let $S_1$ be the Gelfand–Naimark structure space of $B^\infty(\beta D)/M(\beta D)$. Then, from the results of this section, $\theta_\alpha$ induces a homeomorphism $\hat{\theta}_\alpha$ of $S_1$. Let $S$ be the structure space of $B^\infty(K)/M(K)$. Then, by Lemma 4.2, $S$ is homeomorphic to $S_1$. Hence each homeomorphism of $D$ induces a canonical homeomorphism of $S$. Therefore, each action of $G$, as homeomorphisms of $D$, induces, canonically, an action of $G$ as homeomorphisms of $S$.

7. Orbit equivalence

Let $S$ be a compact Hausdorff extremally disconnected space with no isolated points. Let $\varepsilon$ be an action of $G$ as homeomorphisms of $S$ which is non-degenerate.

Definition 7.1. Let $Z$ be a $G$-invariant subset of $S$. Then the action $\varepsilon$ is said to be pseudo-free on $Z$ if, for every $g \in G$, the fixed point set $\{z \in Z : \varepsilon_g(z) = z\}$ is a clopen subset of $Z$ in the relative topology.

Remark. If an action is free on $Z$ then, for $g \neq i$, its fixed point set is empty. So, each free action is also pseudo-free. In particular, each free orbit is also pseudo-free.

In the rest of this section, $s_0 \in S$ such that the orbit $D = \{\varepsilon_g(s_0) : g \in G\}$ is dense in $S$. To simplify our notation, we shall write ‘$g$’ for $\varepsilon_g$. The restriction of $g$ to $D$ is a homeomorphism of $D$ onto $D$. We shall abuse our notation by also denoting this restriction by ‘$g$’.

From the results of Section 4, $S$ is the Stone–Čech compactification of $D$. So, any homeomorphism of $D$ has a unique extension to a homeomorphism of $S$.

Lemma 7.2. Let $O$ be a non-empty open subset of $S$. Then $O \cap D$ is an infinite set.

Proof. Suppose that $O \cap D$ is a finite set, say $\{p_1, p_2, \ldots, p_n\}$.

Then $O \setminus \{p_1, p_2, \ldots, p_n\}$ is an open subset of $S$ which is disjoint from $D$. But $D$ is dense in $S$. Hence $O = \{p_1, p_2, \ldots, p_n\}$. So, $\{p_1\}$ is an open subset of $S$. But $S$ has no isolated points. So, this is a contradiction.

Let $Z$ be a $G$-invariant dense subset of $S$ and let $h$ be a bijection of $Z$ onto itself. Then $h$ is said to be strongly $G$-decomposable over $Z$ if there exist a sequence of pairwise disjoint clopen subsets of $Z$, $(A_j)$ where $Z = \bigcup A_j$, and a sequence $(g_j)$ in $G$ such that

$$h(x) = g_j(x) \quad \text{for } x \in A_j.$$ 

When this occurs, $h$ is a continuous, open map. Hence it is a homeomorphism of $Z$ onto $Z$.

We also need a slightly weaker condition. Let $h$ be a homeomorphism of $S$ onto itself. Then $h$ is $G$-decomposable (over $S$) if there exist a sequence of pairwise disjoint clopen subsets of $S$, $(K_j)$ where $\bigcup K_j$ is dense in $S$, and a sequence $(g_j)$ in $G$ such that

$$h(x) = g_j(x) \quad \text{for } x \in K_j.$$ 

Remark. The set $\bigcup K_j$ is an open dense set, hence its compliment is a closed nowhere dense set.

Lemma 7.3. Let $h$ be a homeomorphism of $D$ onto $D$. Let $h$ be strongly $G$-decomposable over $D$. Let $\hat{h}$ be the unique extension of $h$ to a homeomorphism of $S$. Then $\hat{h}$ is $G$-decomposable over $S$. 


Thirdly exists a homeomorphism \( h \) equivalent over \( Y \) from \( D \). So, \( \Gamma \) is strongly \( G \)-decomposable over \( D \). Moreover, \( h \) decomposes with respect to \( \Gamma \) over \( D \). By Lemma 7.3, for each \( \gamma \), there exists a clopen set \( A \) of \( \Gamma \) such that \( \{ \lambda[W] : \lambda \in \Lambda \} \). Then \( Y \) is the required \( G\delta \) set.

Corollary 7.5. Let \( \Gamma \) and \( G \) be strongly equivalent over \( D \). Then there exists a \( G\delta \) set \( D \), where \( D \subset Y \subset S \), and \( Y \) is both \( G \)-invariant and \( \Gamma \)-invariant, such that \( \Gamma \) and \( G \) are strongly equivalent over \( D \).

Proof. Let \( \Lambda \) be the countable group generated by \( \Gamma \) and \( G \).

By Lemma 7.3, for each \( \gamma \in \sigma \), there is a clopen set \( A \) such that \( \{ \lambda[W] : \lambda \in \Lambda \} \). Then \( Y \) is the required \( G\delta \) set.

Corollary 7.5. Let \( \Gamma \) and \( G \) be strongly equivalent over \( D \). Then there exists a \( G\delta \) set \( D \), where \( D \subset Y \subset S \), and \( Y \) is both \( G \)-invariant and \( \Gamma \)-invariant, such that \( \Gamma \) and \( G \) are strongly equivalent over \( D \).

Proof. Let \( \lambda \) be the countable group generated by \( \Gamma \) and \( G \).

By Lemma 7.3, for each \( \gamma \in \sigma \), there is an open set \( A \) such that \( \{ \lambda[W] : \lambda \in \Lambda \} \). Then \( Y \) is the required \( G\delta \) set.

Let \( \lambda \) be a countable group generated by \( \Gamma \) and \( G \).

By Lemma 7.3, for each \( \gamma \), there exists a clopen set \( A \) such that \( \{ \lambda[W] : \lambda \in \Lambda \} \). Then \( Y \) is the required \( G\delta \) set.

Proof. Let \( \gamma \) be a countable group generated by \( \Gamma \) and \( G \).

By Lemma 7.3, for each \( \gamma \), there exists an open set \( A \) such that \( \{ \lambda[W] : \lambda \in \Lambda \} \). Then \( Y \) is the required \( G\delta \) set.

Proof. Let \( \gamma \) be a countable group generated by \( \Gamma \) and \( G \).

By Lemma 7.3, for each \( \gamma \), there exists an open set \( A \) such that \( \{ \lambda[W] : \lambda \in \Lambda \} \). Then \( Y \) is the required \( G\delta \) set.

Proof. Let \( \gamma \) be a countable group generated by \( \Gamma \) and \( G \).

By Lemma 7.3, for each \( \gamma \), there exists an open set \( A \) such that \( \{ \lambda[W] : \lambda \in \Lambda \} \). Then \( Y \) is the required \( G\delta \) set.

Proof. Let \( \gamma \) be a countable group generated by \( \Gamma \) and \( G \).

By Lemma 7.3, for each \( \gamma \), there exists an open set \( A \) such that \( \{ \lambda[W] : \lambda \in \Lambda \} \). Then \( Y \) is the required \( G\delta \) set.

Proof. Let \( \gamma \) be a countable group generated by \( \Gamma \) and \( G \).

By Lemma 7.3, for each \( \gamma \), there exists an open set \( A \) such that \( \{ \lambda[W] : \lambda \in \Lambda \} \). Then \( Y \) is the required \( G\delta \) set.
Proof. Since \(a\) and \(b\) are in the same orbit of \(G\), there exists \(g_1\) in \(G\) such that \(g_1(a) = b\). Then \(A \cap g_1^{-1}[B]\) is a clopen neighbourhood of \(a\) which is mapped by \(g_1\) into \(B\). Since \(S\) is extremely disconnected and has no isolated points and by making use of Lemma 4.6, we can find a strictly smaller clopen neighbourhood of \(a\), say \(A_1\). By dropping to a clopen sub-neighbourhood if necessary, we can also demand that \(g_1[A_1]\) is a proper clopen subset of \(B\). Let \(B_1 = g_1[A_1]\).

By Lemma 7.2, \(A\) and \(B\) are infinite sets. Since they are subsets of \(D\), they are both countably infinite. Enumerate them both. Let \(a_1\) be the first term of the enumeration of \(A\) which is not in \(A_1\) and let \(b_1\) be the first term of the enumeration of \(B\) which is not in \(B_1\). Then there exists \(g_2\) in \(G\) such that \(g_2(a_1) = b_1\). Now let \(A_2\) be a clopen neighbourhood of \(a_2\), such that \(A_2\) is a proper subset of \(A\) and \(g_2[A_2]\) is a proper subset of \(B\). Proceeding inductively, we obtain a sequence, \((A_n)\) of disjoint clopen subsets of \(A\); a sequence \((B_n)\) of disjoint clopen subsets of \(B\) and a sequence \((g_n)\) from \(G\) such that \(g_n\) maps \(A_n\) onto \(B_n\). Furthermore, \(A = \bigcup A_n\) and \(B = \bigcup B_n\).

We define \(h\) as follows. For \(s \in A_n\), \(h(s) = g_n(s)\). For \(s \in B_n\), \(h(s) = g_n^{-1}(s)\). For \(s \in D \setminus (A \cup B)\), \(h(s) = s\). Then \(h\) has all the required properties. \(\Box\)

Lemma 7.8. Let \(\alpha\) and \(\beta\) be homeomorphisms of \(D\) onto itself. Suppose that each homeomorphism is strongly \(G\)-decomposable. Then \(\beta \alpha\) is strongly \(G\)-decomposable.

Proof. Let \(\{A_i : i \in \mathbb{N}\}\) be a partition of \(D\) into clopen sets and \((g_i^\alpha)\) a sequence in \(G\) which gives the \(G\)-decomposition of \(\alpha\). Similarly, let \(\{B_j : j \in \mathbb{N}\}\) be a partition of \(D\) into clopen sets and \((g_j^\beta)\) a sequence in \(G\) which gives the \(G\)-decomposition of \(\beta\). Then \(\{A_i \cap \alpha^{-1}[B_j] : i \in \mathbb{N}, j \in \mathbb{N}\}\) is a partition of \(D\) into clopen sets.

Let \(s \in A_i \cap \alpha^{-1}[B_j]\). Then \(\beta \alpha(s) = g_j^\beta(\alpha(s)) = g_j^\beta g_i^\alpha(s)\). \(\Box\)

Lemma 7.9. Let \(D\) be enumerated as \((s_0, s_1, \ldots)\). Let \((D_k)(k = 1, 2, \ldots)\) be a monotone decreasing sequence of clopen neighbourhoods of \(s_0\) such that \(s_n \notin D_n\) for any \(n\). Then the following statements hold.

(a) There is a sequence \((h_k)\) \((k = 1, 2, \ldots)\) of homeomorphisms of \(D\) onto \(D\) where \(h_k = h_{k-1}^{-1}\). For \(1 \leq k \leq n\), the \(h_k\) are mutually commutative. Each \(h_k\) is strongly \(G\)-decomposable over \(D\).

(b) For each positive integer \(n\), there exists a finite family of pairwise disjoint, clopen subsets of \(D\),

\[\{K^n(\alpha_1, \alpha_2, \ldots, \alpha_n) : (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_2^n\}\]

whose union is \(D\).

(c) Let \(K_0 = D\). For \(1 \leq p \leq n - 1\)

\[K^p(\alpha_1, \alpha_2, \ldots, \alpha_p) = K^{p+1}(\alpha_1, \alpha_2, \ldots, \alpha_p, 0) \cup K^{p+1}(\alpha_1, \alpha_2, \ldots, \alpha_p, 1)\]

(d) For \(1 \leq p \leq n\), \(K^p(0, 0, \ldots, 0) \subset D_p\) and \(s_0 \in K^p(0, 0, \ldots, 0)\).

(e) Let \((\alpha_1, \alpha_2, \ldots, \alpha_p) \in \mathbb{Z}_2^p\) where \(1 \leq p \leq n\). Then the homeomorphism \(h_1^{\alpha_1}h_2^{\alpha_2} \cdots h_p^{\alpha_p}\) interchanges \(K^p(\beta_1, \beta_2, \ldots, \beta_p)\) with \(K^p(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_p + \beta_p)\).

(f) For each \(n\), \(\{s_0, s_1, \ldots, s_n\} \subset \{h_1^{\alpha_1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}(s_0) : (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_2^n\}\).

(g) For each \(s \in D\), if \(h_1^{\alpha_1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}(s) = s\), then \(\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0\).

Proof. We give an inductive argument. First, let \(A = D_1\) and let \(B = D \setminus D_1\). By applying Lemma 7.7, there exists a homeomorphism \(h_1\) of \(D\) onto itself, where \(h_1\) interchanges \(D_1\) and \(D \setminus D_1\), and maps \(s_0\) to \(s_1\). (So (f) holds for \(n = 1\).) Also \(h_1 = h_1^{-1}\).
For any $s \in D$, $h_1(s)$ and $s$ are elements of disjoint clopen sets. Hence (g) holds for $n = 1$.

Now let $K^1(0) = D_1$ and $K^1(1) = D \setminus D_1$. Let us now suppose that we have constructed the homeomorphisms $h_1, h_2, \ldots, h_n$ and the clopen sets

$$\{K^p(\alpha_1, \alpha_2, \ldots, \alpha_p) : (\alpha_1, \alpha_2, \ldots, \alpha_p) \in \mathbb{Z}_2^p\}$$

for $p = 1, 2, \ldots, n$.

We now need to make the $(n+1)$th step of the inductive construction. For some $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \{0, 1\}^n$, $s_{n+1} \in K^n(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Let $c = h_1^{\alpha_1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}(s_{n+1})$. Then $c \in K^n(0, \ldots, 0)$.

If $c \neq s_0$, let $b = c$. If $c = s_0$, then let $b$ be any other element of $K^n(0, 0, \ldots, 0)$. Now let $A$ be a clopen subset of $K^n(0, 0, \ldots, 0) \cap D_{n+1}$ such that $s_0 \in A$ and $b \notin A$. Let $B = K^n(0, 0, \ldots, 0) \setminus A$. We apply Lemma 7.7 to find a homeomorphism $h$ of $D$ onto itself, which interchanges $A$ and $B$, leaves every point outside $A \cup B$ fixed, maps $s_0$ to $b$ and $h = h^{-1}$.

Also, $h$ is strongly $G$-decomposable.

Let $K^{n+1}(0, 0, \ldots, 0) = A$ and $K^{n+1}(0, 0, \ldots, 1) = B$. By construction, (d) holds for $p = n + 1$. Let $K^{n+1}(\alpha_1, \alpha_2, \ldots, \alpha_p, 0) = h_1^{\alpha_1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}[A]$ and $K^{n+1}(\alpha_1, \alpha_2, \ldots, \alpha_p, 1) = h_1^{\alpha_1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}[B]$. Then (b) holds for $n + 1$ and (c) holds for $p = n$.

We now define $h_{n+1}$ as follows. For $s \in K^n(\alpha_1, \alpha_2, \ldots, \alpha_n)$,

$$h_{n+1}(s) = h_1^{\alpha_1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}h_1^{\alpha_1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}(s).$$

**Claim 1.** $h_{n+1}$ commutes with $h_j$ for $1 \leq j \leq n$.

To simplify our notation we shall take $j = 1$, but the calculation works in general, since each of $\{h_r : r = 1, 2, \ldots, n\}$ commutes with the others.

Let $s \in D$. Then $s \in K^n(\alpha_1, \alpha_2, \ldots, \alpha_n)$ for some $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_2^n$.

So, $h_1s \in K^n(\alpha_1 + 1, \alpha_2, \ldots, \alpha_n)$. Then

$$h_{n+1}(h_1s) = h_1^{\alpha_1+1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}h_1^{\alpha_1+1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}(h_1s).$$

So

$$h_{n+1}h_1(s) = h_1h_2^{\alpha_2} \cdots h_n^{\alpha_n}hh_1^{\alpha_1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}h_1h_1(s)$$

$$= h_1h_{n+1}(s).$$

From this, we see that $h_{n+1}$ commutes with $h_1$. Similarly, $h_{n+1}$ commutes with $h_j$ for $2 \leq j \leq n$.

**Claim 2.** $h_{n+1}$ is $G$-decomposable.

By Lemma 7.8, $h_1^{\alpha_1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}h_1^{\alpha_1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}$ is $G$-decomposable. So, on restricting to the clopen set $K^n(\alpha_1, \alpha_2, \ldots, \alpha_n)$, this gives that $h_{n+1}$ is $G$-decomposable over each $K^n(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Hence $h_{n+1}$ is $G$-decomposable.

So, by Claims 1 and 2, (a) holds for $n + 1$. It is straightforward to show that (b), (c), (d) and (e) hold for $n + 1$.

Now consider (f). Either $c = s_0$ in which case, $s_0 = h_1^{\alpha_1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}(s_{n+1})$ which gives $s_{n+1} = h_1^{\alpha_1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}(s_0)$, or $c \neq s_0$, in which case

$$h_{n+1}(h_1^{\alpha_1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}(s_{n+1})) = h(h_1^{\alpha_1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}(s_{n+1})) = s_0.$$ 

This gives $s_{n+1} = h_{n+1}h_1^{\alpha_1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}(s_0)$. Because the homeomorphisms commute, this gives (f) for $n + 1$.

Finally consider (g). Let $s \in D$ with $h_1^{\alpha_1}h_2^{\alpha_2} \cdots h_n^{\alpha_n}(s) = s$. If $\alpha_{n+1} = 0$, then (g) implies $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. So, now suppose that $\alpha_{n+1} = 1$. Let $h_1^{\alpha_1}h_2^{\beta_2} \cdots h_n^{\beta_n}(s) \in K^n(0, 0, \ldots, 0)$. Then, since the $h_r$ all commute, we can suppose, without loss of generality, that $s \in K^n(0, 0, \ldots, 0)$. Then $h_{n+1}(s) = h_1^{\alpha_1}h_2^{\beta_2} \cdots h_n^{\beta_n}(s)$. But $h_{n+1}$ maps $K^n(0, 0, \ldots, 0)$ to itself and $h_1^{\alpha_1}h_2^{\beta_2} \cdots h_n^{\beta_n}$ maps $K^n(0, 0, \ldots, 0)$ to $K^n(\alpha_1, \alpha_2, \ldots, \alpha_n)$. So,

$$h_{n+1}(s) \in K^n(0, 0, \ldots, 0) \cap K^n(\alpha_1, \alpha_2, \ldots, \alpha_n).$$
But this intersection is only non-empty if \( \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0 \). So, \( h_{n+1}(s) = s \). But \( h_{n+1} \) acting on \( K^n(0,0,\ldots,0) \), interchanges \( K^{n+1}(0,0,\ldots,0) \) with \( K^{n+1}(0,0,\ldots,1) \).

So, \( s \in K^{n+1}(0,0,\ldots,1) \cap K^{n+1}(0,0,\ldots,0) \), which is impossible. \( \square \)

Let us recall that \( \bigoplus \mathbb{Z}_2 \) is the direct sum of an infinite sequence of copies of \( \mathbb{Z}_2 \). So, each element of the group is an infinite sequence of zeroes and ones, with 1 occurring only finitely many times. We sometimes refer to it as the dyadic group.

**Theorem 7.10.** Let \( S \) be a compact Hausdorff extremally disconnected space with no isolated points. Let \( G \) be a countably infinite group. Let \( \varepsilon : G \rightarrow \text{Homeo}(S) \) be a non-degenerate action of \( G \) as homeomorphisms of \( S \). Let \( s_0 \) be a point in \( S \) such that the orbit

\[
\{\varepsilon_g(s_0) : g \in G\} = D
\]

is dense and pseudo-free. Then there exist an action \( \gamma : \bigoplus \mathbb{Z}_2 \rightarrow \text{Homeo}(S) \) and \( Y \), a \( G \)-delta subset of \( S \) with \( D \subset Y \), such that the following properties hold.

1. The orbit \( \{\gamma_\delta(s_0) : \delta \in \bigoplus \mathbb{Z}_2\} \) is free and coincides with the set \( D \).
2. The groups of homeomorphisms \( \varepsilon[G] \) and \( \gamma[\bigoplus \mathbb{Z}_2] \) are strongly equivalent over \( Y \), which is invariant under the action of both these groups.
3. The orbit equivalence relations corresponding, respectively, to \( \varepsilon[G] \) and \( \gamma[\bigoplus \mathbb{Z}_2] \) coincide on \( Y \).
4. \( \gamma \) is an isomorphism.

**Proof.** We replace ‘\( G \)’ by ‘\( \varepsilon[G] \)’ in the statement of Lemma 7.9 to find a sequence \( \langle h_r \rangle \) of homeomorphisms of \( D \) onto itself with the properties listed in that lemma. Each \( h_r \) has a unique extension to a homeomorphism \( h_r \) of \( S \) onto itself. For each \( \alpha \in \bigoplus \mathbb{Z}_2 \), there exist a natural number \( n \) and \( (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_2^n \) such that \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots) \). We define \( \gamma_\alpha = h_{\alpha_1}^1 h_{\alpha_2}^2 \cdots h_{\alpha_n}^n \). Then \( \gamma \) is a homomorphism of \( \bigoplus \mathbb{Z}_2 \) into \( \text{Homeo}(S) \).

By Lemma 7.9(f), the orbit \( \{\gamma_\alpha(s_0) : \alpha \in \bigoplus \mathbb{Z}_2\} \) coincides with \( D \). By Lemma 7.9(g), this orbit is free for the action \( \gamma \).

By Lemmas 7.9 and 7.4, there exists a \( G \)-set \( Y \), \( D \subset Y \subset S \), which is invariant under the action of both \( \gamma[\bigoplus \mathbb{Z}_2] \) and \( \varepsilon[G] \) also \( \varepsilon[G] \) and \( \gamma[\bigoplus \mathbb{Z}_2] \) are strongly equivalent over \( Y \). The statement (3) now follows from Corollary 7.5. Finally, statement (4) follows from part (g) of Lemma 7.9. \( \square \)

We shall make no direct use of \( \mathbb{Z} \)-actions in this article, but the following corollary seems worth including. We sketch an argument that makes use of the above result and the notation used in Lemma 7.9.

**Corollary 7.11.** There exists a homeomorphism \( \phi : S \rightarrow S \) and a dense \( G_\delta \)-subset \( S_0 \subset S \) with the following properties. First, \( S_0 \) is invariant under the action of \( G \). Secondly, \( \phi[S_0] = S_0 \). Thirdly, the orbit equivalence relation coming from \( G \) and the \( \mathbb{Z} \)-orbit equivalence relation coming from \( \phi \), coincide on \( S_0 \).

**Proof.** (Sketch) It follows from the preceding theorem that we may replace \( G \) by \( \bigoplus \mathbb{Z}_2 \). More precisely, we shall let \( G \) be the group of homeomorphisms of \( D \) generated by \( \langle h_r \rangle (r = 1, 2, \ldots) \).

Let \( E_1 = K^1(0) \) and \( F_1 = K^1(1) \). Let \( E_{j+1} = K^{j+1}(1, 0) \) where \( 1 \in \mathbb{Z}_2^l \) and let \( F_{j+1} = K^{j+1}(0, 1) \) where \( 0 \in \mathbb{Z}_2^j \).
We observe that, if $K^n(\alpha_1, \alpha_2, \ldots, \alpha_n)$ has non-empty intersection with $K^{n+p}(\beta_1, \beta_2, \ldots, \beta_{n+p})$, then $\alpha_1 = \beta_1, \ldots, \alpha_n = \beta_n$. From this it follows that $(E_n)(n = 1, 2, \ldots)$ is a sequence of pairwise disjoint clopen subsets of $D$. Similarly, $(F_n)(n = 1, 2, \ldots)$ is a sequence of pairwise disjoint clopen subsets of $D$. We find that $D = \bigcup_{n=1}^\infty E_n$ and $\bigcup_{n=1}^\infty F_n = D \setminus \{s_0\}$.

For $s \in E_n$ let $\phi(s) = h_1h_2 \cdots h_n(s)$. Then $\phi$ is a continuous map of $E_n$ onto $F_n$. From this it is straightforward to see that $\phi$ is a continuous bijection of $D$ onto $D \setminus \{s_0\}$. Similarly, $\phi^{-1}$ is a continuous bijection from $D \setminus \{s_0\}$ onto $D$.

Since $S$ has no isolated points, $D \setminus \{s_0\}$ is dense in $S$. But, see Section 4, $S$ can be identified with the Stone–Čech compactification of any dense subset of itself. Applying this to $\phi^{-1}$ and $\phi$, we find continuous extensions that are homeomorphisms of $S$ onto $S$ and which are inverses of each other. We abuse notation and denote the extension of $\phi$ to the whole of $S$ by $\phi$. Then $j \to \phi^j$ is the $\mathbb{Z}$-action considered here; let $\Delta$ be the group generated by $\phi$. We shall further abuse our notation by writing $h_r$ for $h_r$, the extension to a homeomorphism of $S$.

On applying Lemma 4.6, we see that $(\text{cl} E_n)(n = 1, 2, \ldots)$ is a sequence of pairwise disjoint clopen subsets of $S$. So, its union is a dense open subset of $S$ which we shall denote by $O_1$. By continuity, for $s \in \text{cl} E_n$, we have $\phi(s) = h_1^1h_2^1 \cdots h_n^1(s)$. Similarly, $(\text{cl} F_n)(n = 1, 2, \ldots)$ is a sequence of pairwise disjoint clopen subsets of $S$ whose union, $O_2$, is also dense in $S$.

Let $\Gamma$ be the countable group generated by $\phi$ and $G$. Let $S_0$ be the intersection $\bigcap \{\gamma O_1 \cap O_2 : \gamma \in \Gamma\}$. Then $S_0$ is a dense $G_\delta$-subset of $S$ which is invariant under the action of $\Gamma$. From the definition of $\phi$, it is clear that $\phi$ is strongly $G$-decomposable over $S_0$. (Recall that we have identified $G$ with $\bigoplus \mathbb{Z}_2$.) Similarly, $\phi^{-1}$ is also strongly $G$-decomposable over $S_0$. Hence each element of $\Delta$ is strongly $G$-decomposable over $S_0$.

Let $H(\Delta)$ be the group of all homeomorphisms $h$, of $S$ onto $S$, such that $h$ is strongly $\Delta$-decomposable with respect to a finite partition of $S$ into clopen sets. We shall show that $h_1$ is in $H(\Delta)$.

For $s \in E_1 = K^1(0)$ we have $\phi(s) = h_1(s)$ and, for $s \in F_1 = K^1(1)$, $\phi^{-1}(s) = h_1(s)$. We observe that $\text{cl} E_1$ and $\text{cl} F_1$ are disjoint clopen sets whose union is $S$. Also,

$$h_1 = \phi(\text{cl} E_1) + \phi^{-1}(\text{cl} F_1).$$

So $h_1 \in H(\Delta)$.

We now suppose that $h_1, \ldots, h_n$ are in $H(\Delta)$. We wish to show $h_{n+1} \in H(\Delta)$. Let $s \in K^{n+1}(\beta, 0)$ where $\beta \in \mathbb{Z}_2^n$. By Lemma 7.9(e) $h_1^\beta_{n+1} \cdots h_n^\beta_{n+1}(s) \in K^{n+1}(1, 0)$. So, from the definition of $\phi$,

$$\phi(h_1^\beta_{n+1} \cdots h_n^\beta_{n+1}(s)) = h_1^\beta_1 \cdots h_n^\beta_n h_{n+1}^\beta(s).$$

Making use of the commutativity of the $h_r$, we obtain $h_{n+1}(s) = h_1^\beta_1 \cdots h_n^\beta_n \phi h_1^\beta_{n+1} \cdots h_n^\beta_{n+1}(s)$. Then, by using continuity, this holds for each $s \in \text{cl} K^{n+1}(\beta, 0)$. By a similar argument, for $s \in \text{cl} K^{n+1}(\beta, 1)$, we obtain $h_{n+1}(s) = h_1^\beta_1(1) \cdots h_n^\beta_n \phi^{-1} h_1^\beta_{n+1} \cdots h_n^\beta_{n+1}(s)$.

Since $\{\text{cl} K^{n+1}(\alpha) : \alpha \in \mathbb{Z}_2^{n+1}\}$ is a finite collection of disjoint clopen sets whose union is $S$, it follows that $h_{n+1} \in H(\Delta)$.

So, by induction, $G \subset H(\Delta)$.

It now follows that, on $S_0$, the orbit equivalence relation coming from the action of $G$ coincides with the orbit equivalence relation arising from the $\mathbb{Z}$-action generated by $\phi$. 

**Remarks.** In the above, $D$ is not a subset of $S_0$. Indeed, it is not obvious that $S_0$ has a dense orbit. So, is it possible to modify the construction of $\phi$ so that it becomes a bijection of $D$ onto itself?
8. Monotone complete $C^*$-algebra of an equivalence relation

The idea of constructing a $C^*$-algebra or a von Neumann algebra from a groupoid has a long history and a vast literature; there is an excellent exposition in [39]. Here, instead of general groupoids, we use an equivalence relation with countable equivalence classes. Our aim is to construct monotone complete (monotone $\sigma$-complete) algebras by a modification of the approach used in [36]. We try to balance conciseness with putting in enough detail to convince the reader that this is an easy and transparent way to construct examples of monotone complete $C^*$-algebras. It makes it possible to obtain all the algebras which arise as a monotone cross-product by a countable discrete group acting on a commutative algebra, but without needing to use monotone tensor products.

In this section, $X$ is a topological space, where $X$ is either a $G$-delta subset of a compact Hausdorff space or a Polish space (that is, homeomorphic to a complete separable metric space). Then $X$ is a Baire space, that is, the Baire category theorem holds for $X$. Let $B(X)$ be the set of all bounded complex-valued Borel functions on $X$. When equipped with the obvious algebraic operations and the supremum norm, it becomes a commutative $C^*$-algebra.

In the following it would be easy to use a more general setting, where we do not assume a topology for $X$, replace the field of Borel sets with a $\sigma$-field and use $\mathcal{T}$-measurable bijections instead of homeomorphisms. But we stick to a topological setting which is what we need later.

Let $G$ be a countable group of homeomorphisms of $X$ and let

$$E = \{(x, y) \in X \times X : \exists g \in G \text{ such that } y = g(x)\}.$$ 

Then $E$ is the graph of the orbit equivalence relation on $X$ arising from the action of $G$. We shall identify this equivalence relation with its graph. We know, from the work of Section 7, that the same orbit equivalence relation can arise from actions by different groups.

Let us recall that for $A \subset X$, the saturation of $A$ (by $E$) is

$$E[A] = \{x \in X : \exists z \in A \text{ such that } xEz\} = \{x \in X : \exists g \in G \text{ such that } g(x) \in A\} = \bigcup\{g[A] : g \in G\}.$$ 

It follows from this that the saturation of a Borel set is also a Borel set.

**Definition 8.1.** Let $\mathcal{I}$ be a $\sigma$-ideal of the Boolean algebra of Borel subsets of $X$ with $X \notin \mathcal{I}$.

**Definition 8.2.** Let $B_\mathcal{I}$ be the set of all $f$ in $B(X)$ such that $\{x \in X : f(x) \neq 0\}$ is in $\mathcal{I}$. Then $B_\mathcal{I}$ is a $\sigma$-ideal of $B(X)$. (See Section 3.) Let $q$ be the quotient homomorphism from $B(X)$ onto $B(X)/B_\mathcal{I}$.

**Lemma 8.3.** Let $A \in \mathcal{I}$. Then $E[A] \in \mathcal{I}$ if and only if $g[A] \in \mathcal{I}$ for every $g \in G$.

**Proof.** For each $g \in G$, $g[A] \subset E[A]$. Since $\mathcal{I}$ is an ideal, if $E[A] \in \mathcal{I}$ then $g[A] \in \mathcal{I}$.

Conversely, if $g[A] \in \mathcal{I}$ for each $g$, then $E[A]$ is the union of countably many elements of the $\sigma$-ideal and hence in the ideal. 

In the following we require that the action of $G$ maps the ideal $\mathcal{I}$ into itself. Equivalently, for any $A \in \mathcal{I}$, its saturation by $E$ is again in $\mathcal{I}$. This is automatically satisfied if $\mathcal{I}$ is the ideal of meagre Borel sets but we do not wish to confine ourselves to this situation.
Following the approach of [36], we indicate how orbit equivalence relations on $X$ give rise to monotone complete $C^*$-algebras. A key point, used in [36], is that these algebras are constructed from the equivalence relation without explicit mention of $G$. But in establishing the properties of these algebras, the existence of an underlying group is used. This construction (similar to a groupoid $C^*$-algebra) seems particularly natural and transparent. For the reader’s convenience we give a brief, explicit account which is reasonably self-contained. For reasons explained in Section 9, the work of this section makes it possible for the reader to safely avoid the details of the original monotone cross-product construction. We could work in greater generality (for example, we could weaken the condition that the elements of $G$ be homeomorphisms or consider more general groupoid constructions) but for ease and simplicity we have avoided this. We are mainly interested in two situations. First, where $X$ is an ‘exotic’ space as considered in [35] but $I$ is only the ideal of meagre subsets of $X$. Secondly, where $X$ is just the Cantor space but $I$ is an ‘exotic’ ideal of the Borel sets. In this paper, only the first situation will be considered but, since we will make use of the second situation in a later work and no extra effort is required, we add this small amount of generality.

Since $G$ is a countable group, each orbit is countable; in other words, each equivalence class associated with the equivalence relation $E$ is countable. (Countable Borel equivalence relations and their relationship with von Neumann algebras were penetratingly analysed in [12, 13].)

For each $x \in X$, let $[x]$ be the equivalence class generated by $x$. Let $[X]$ be the set of all equivalence classes. Let $\ell^2([x])$ be the Hilbert space of all square summable, complex-valued functions from $[x]$ to $\mathbb{C}$. For each $y \in [x]$ let $\delta_y \in \ell^2([x])$ be defined by

$$\delta_y(z) = 0 \quad \text{for} \quad z \neq y; \quad \delta_y(y) = 1.$$ 

Then $\{\delta_y : y \in [x]\}$ is an orthonormal basis for $\ell^2([x])$ which we shall call the canonical basis for $\ell^2([x])$. For each $x \in X$, $L(\ell^2([x]))$ is the von Neumann algebra of all bounded operators on $\ell^2([x])$. We now form a direct sum of these algebras by

$$S = \bigoplus_{[x] \in [X]} L(\ell^2([x])).$$

This is a Type I von Neumann algebra, being a direct sum of such algebras. It is of no independent interest but is a framework in which we embed an algebra of ‘Borel matrices’ and then take a quotient, obtaining monotone complete $C^*$-algebras. To each operator $F$ in $S$ we can associate, uniquely, a function $f : E \to \mathbb{C}$ as follows. First we decompose $F$ as

$$F = \bigoplus_{[x] \in [X]} F_{[x]}.$$ 

Here each $F_{[x]}$ is a bounded operator on $\ell^2([x])$. Now recall that $(x, y) \in E$ precisely when $y \in [x]$. We now define $f : E \to \mathbb{C}$ by

$$f(x, y) = \langle F_{[x]} \delta_x, \delta_y \rangle.$$ 

When $f$ is restricted to $[x] \times [x]$, then it becomes the matrix representation of $F_{[x]}$ with respect to the canonical orthonormal basis of $\ell^2([x])$. It follows that there is a bijection between operators in $S$ and those functions $f : E \to \mathbb{C}$ for which there is a constant $k$ such that, for each $[x] \in [X]$, the restriction of $f$ to $[x] \times [x]$ is the matrix of a bounded operator on $\ell^2([x])$ whose norm is bounded by $k$. Call such an $f$ matrix bounded. For each matrix bounded $f$, let $L(f)$ be the corresponding element of $S$.

When $f$ and $h$ are such functions from $E$ to $\mathbb{C}$, then straightforward matrix manipulations give

$$L(f)L(h) = L(f \circ h),$$

where $L(f)$ is the corresponding element of $S$. 

\[\text{[continued]}\]
where \( f \circ h(x, z) = \sum_{y \in [z]} f(x, y)h(y, z) \). Also \( L(f)^* = L(f^*) \), where \( f^*(x, y) = \overline{f(y, x)} \) for all \((x, y) \in E\).

Let \( \|f\| = \|L(f)\| \). Then the matrix-bounded functions on \( E \) form a \( C^* \)-algebra isomorphic to \( S \).

Let \( \Delta \) be the diagonal set \( \{(x, x) : x \in X\} \). It is closed, because the topology of \( X \) is Hausdorff. It is an easy calculation to show that \( L(\chi_\Delta) \) is the unit element of \( S \). For each \( g \in G \), the map \((x, y) \mapsto (x, g(y))\) is a homeomorphism.

So, \( \{(x, g(x)) : x \in X\} \) is a closed set.

Let us recall that

\[
E = \bigcup_{g \in G} \{(x, g(x)) : x \in X\}.
\]

Since \( G \) is countable, \( E \) is the union of countably many closed sets. Hence \( E \) is a Borel subset of \( X \times X \).

**Definition 8.4.** Let \( \mathcal{M}(E) \) be the set of all Borel measurable functions \( f : E \to \mathbb{C} \) which are matrix bounded.

**Lemma 8.5.** The set \( \{L(f) : f \in \mathcal{M}(E)\} \) is a \( C^* \)-subalgebra of \( S \) which is sequentially closed with respect to the weak operator topology of \( S \). We denote this algebra by \( L(\mathcal{M}(E)) \).

When equipped with the appropriate algebraic operations and norm, \( \mathcal{M}(E) \) is a \( C^* \)-algebra isomorphic to \( L[\mathcal{M}(E)] \).

**Proof.** See [36, Lemma 2.1].

**Lemma 8.6.** Let \( (f_n) \) be a sequence in the unit ball of \( \mathcal{M}(E) \) which converges pointwise to \( f \). Then \( f \in \mathcal{M}(E) \) and \( L(f_n) \) converges to \( L(f) \) in the weak operator topology of \( S \). Also \( f \) is in the unit ball of \( \mathcal{M}(E) \).

**Proof.** The weak operator topology gives a compact Hausdorff topology on the norm closed unit ball of \( S \).

Let \( K_n = \{L(f_j) : j \geq n\} \) and let \( \text{cl} K_n \) be its closure in the weak operator topology of \( S \). Then, by the finite intersection property, there exists \( T \in \bigcap_{n \in \mathbb{N}} \text{cl} K_n \). Let \( U \) be an open neighbourhood of \( T \), then \( U \cap K_n \) is non-empty for all \( n \).

Fix \((x, y) \in E\). Fix \( \varepsilon > 0 \). Let \( U = \{S \in S : |(S - T)\delta_x, \delta_y| < \varepsilon\} \). Then we can find a subsequence \((f_{n(r)})\) \((r = 1, 2, \ldots)\) for which \( L(f_{n(r)}) \in U \) for each \( r \). Thus \(|(f_{n(r)}(x, y) - (T\delta_x, \delta_y)| < \varepsilon\).

So, \(|f(x, y) - (T\delta_x, \delta_y)| \leq \varepsilon\). Since this holds for all positive \( \varepsilon \), we have \( f(x, y) = (T\delta_x, \delta_y) \).

Hence \( T = L(f) \).

Let \( T \) be the locally convex topology of \( S \) generated by all seminorms of the form \( V \to |(V\delta_x, \delta_y)| \) as \((x, y) \) ranges over \( E \). This is a Hausdorff topology that is weaker than the weak operator topology. Hence it coincides with the weak operator topology on the unit ball, because the latter topology is compact. But \( L(f_n)\delta_x, \delta_y \to f(x, y) = L(f)\delta_x, \delta_y \) for all \((x, y) \in E\).

It now follows that \( L(f_n) \to L(f) \) in the weak operator topology of \( S \). Since \( f \) is the pointwise limit of a sequence of Borel measurable functions, it too is Borel measurable. So, \( f \in \mathcal{M}(E) \) and, since \( T \) is in the unit ball of \( S \), \( f \) is in the unit ball of \( \mathcal{M}(E) \).

Let \( p \) be the homeomorphism of \( \Delta \) onto \( X \), given by \( p(x, x) = x \). So, \( B(X) \), the algebra of bounded Borel measurable functions on \( X \), is isometrically \( * \)-isomorphic to \( B(\Delta) \) under the map \( h \to h \circ p \).
For each \( f \in \mathcal{M}(E) \) let \( Df \) be the function on \( E \) that vanishes off the diagonal, \( \Delta \), and is such that, for each \( x \in X \),
\[
Df(x, x) = f(x, x).
\]

Then \( D \) is a linear idempotent map from \( \mathcal{M}(E) \) onto an abelian subalgebra which we can identify with \( B(\Delta) \), which can, in turn, be identified with \( B(X) \). Let \( \tilde{D}f \) be the function on \( X \) such that \( \tilde{D}f(x) = Df(x, x) \) for all \( x \in X \). We shall sometimes abuse our notation by using \( Df \) instead of \( \tilde{D}f \).

Let \( \pi : B(X) \to \mathcal{M}(E) \) be defined by \( \pi(h)(x, x) = h(x) \) for \( x \in X \) and \( \pi(h)(x, y) = 0 \) for \( x \neq y \). Then \( \pi \) is a \( * \)-isomorphism of \( B(X) \) onto an abelian \( * \)-subalgebra of \( \mathcal{M}(E) \), which can be identified with the range of \( D \).

We have \( \pi \tilde{D}f = Df \) for \( f \in \mathcal{M}(E) \). Also, for \( g \in B(X) \), \( \tilde{D}\pi(g) = g \).

**Lemma 8.7.** \( D \) is a positive map.

**Proof.** Each positive element of \( \mathcal{M}(E) \) is of the form \( f \circ f^* \). But
\[
(\#) \quad D(f \circ f^*)(x, x) = \sum_{y \in [x]} f(x, y)f^*(y, x) = \sum_{y \in [x]} f(x, y)f(y, x) = \sum_{y \in [x]} |f(x, y)|^2 \geq 0. \quad \square
\]

**Definition 8.8.**

Let \( I_\mathcal{I} = \{ f \in \mathcal{M}(E) : q\tilde{D}(f \circ f^*) = 0 \} \)
\[= \{ f \in \mathcal{M}(E) : \exists A \in \mathcal{I} \text{ such that } \tilde{D}(f \circ f^*)(x) = 0 \text{ for } x \notin A \}.\]

**Lemma 8.9.** \( I_\mathcal{I} \) is a two-sided ideal of \( \mathcal{M}(E) \).

**Proof.** In any \( C^* \)-algebra,
\[
(a + b)(a + b)^* \leq 2(aa^* + bb^*).
\]

So,
\[
D((f + g) \circ (f + g)^*) \leq 2D(f \circ f^*) + 2D(g \circ g^*).
\]

From this it follows that if \( f \) and \( g \) are both in \( I_\mathcal{I} \) then so also is \( f + g \).

In any \( C^* \)-algebra,
\[
fz(fz)^* = fzz^*f^* \leq \|z\|^2ff^*.
\]

From this it follows that \( f \in I_\mathcal{I} \) and \( z \in \mathcal{M}(E) \) implies that \( f \circ z \in I_\mathcal{I} \).

Now suppose that \( f \in I_\mathcal{I} \). Then for some \( A \in \mathcal{I} \), \( E(f \circ f^*)(x) = 0 \) for \( x \notin A \). Since \( E[A] \in \mathcal{I} \) we can suppose that \( A = E[A] \). Hence, if \( x \notin E[A] \), then \([x] \cap E[A] = \emptyset \). For \( x \notin E[A] \), we have
\[
0 = D(f \circ f^*)(x) = \sum_{y \in [x]} |f(x, y)|^2.
\]

Thus \( f(x, y) = 0 \) for \( x \in E[y] \) and \( x \notin A \). Then, for \( z \notin A \), we have \( D(f^* \circ f)(z) = \sum_{y \in [z]} |f^*(z, y)|^2 = \sum_{y \in [z]} |f(y, z)|^2 = 0 \). So \( f^* \in I_\mathcal{I} \). So \( I_\mathcal{I} \) is a two-sided ideal of \( \mathcal{M}(E) \). \( \square \)

**Lemma 8.10.** If \( y \in I_\mathcal{I} \) then \( q\tilde{D}(y) = 0 \). Furthermore, \( y \in I_\mathcal{I} \) if and only if \( q\tilde{D}(y \circ a) = 0 \) for all \( a \in \mathcal{M}(E) \).
Proof. Let $T$ be the (compact Hausdorff) structure space of the algebra $B(X)/B_I$. By applying the Cauchy–Schwarz inequality, we see that for $x, y$ in $\mathcal{M}(E)$ and $t \in T$,

$$|qD(x^* \circ y)(t)| \leq qD(x^* \circ y)(t)^{1/2} qD(y^* \circ y)(t)^{1/2}. $$

Let $x = 1$ and let $y \in I_T$. Then $y^* \in I_T$. So, $qD(y^* \circ y) = 0$. From the above inequality, it follows that $qD(y) = 0$. Since $I_T$ is an ideal, if $y \in I_T$ then $y \circ a$ is in the ideal for each $a \in \mathcal{M}(E)$. It now follows from the above that $qD(y \circ a) = 0$. Conversely, if $qD(y \circ a) = 0$ for all $a \in \mathcal{M}(E)$ then, on putting $a = y^*$, we see that $y \in I_T$.

**Lemma 8.11.** $L[I_T]$ is a (two-sided) ideal of $L[\mathcal{M}(E)]$ which is sequentially closed in the weak operator topology of $S$.

**Proof.** Let $(f_r)$ be a sequence in $I_T$ such that $(L(f_r))$ is a sequence that converges in the weak operator topology to an element $T$ of $S$. Then it follows from the Uniform Boundedness Theorem that the sequence is bounded in norm. By Lemma 8.5, there exists $f \in \mathcal{M}(E)$ such that $L(f) = T$ where $L(f_r) \to L(f)$ in the weak operator topology. So, $f_r \to f$ pointwise. Hence $D(f_r) \to D(f)$ pointwise. For each $r$, there exists $A_r \in I$ such that $x \notin A_r$ implies $D(f_r)(x) = 0$. Since $I$ is a Boolean $\sigma$-ideal of the Boolean algebra of Borel subsets of $X$, $\cup\{A_r : r = 1, 2 \ldots\}$ is in $I$. Hence $qD(f) = 0$.

For any $a \in \mathcal{M}(E)$, $(f_r \circ a)$ is a sequence in $I_T$ such that $(L(f_r \circ a))$ converges in the weak operator topology to $L(f)L(a)$. So, as in the preceding paragraph, $qD(f \circ a) = 0$. By appealing to Lemma 8.10, we see that $f \in I_T$. Hence $L[I_T]$ is sequentially closed in the weak operator topology of $S$.

**Corollary 8.12.** $\mathcal{M}(E)$ is monotone $\sigma$-complete and $I_T$ is a $\sigma$-ideal.

**Proof.** Each norm bounded monotone increasing sequence in $L[\mathcal{M}(E)]$ converges in the strong operator topology to an element $T$ of $S$. By Lemma 8.5, $T \in L[\mathcal{M}(E)]$. Then $T = L(f)$ for some $f \in \mathcal{M}(E)$. Hence $L[\mathcal{M}(E)]$ (and its isomorphic image, $\mathcal{M}(E)$) are monotone $\sigma$-complete. It now follows from Lemma 8.11 that $I_T$ is a $\sigma$-ideal.

**Definition 8.13.** Let $Q$ be the quotient map from $\mathcal{M}(E)$ onto $\mathcal{M}(E)/I_T$.

**Proposition 8.14.** The algebra $\mathcal{M}(E)/I_T$ is monotone $\sigma$-complete. There exists a positive, faithful, $\sigma$-normal, conditional expectation $\hat{D}$ from $\mathcal{M}(E)/I_T$ onto a commutative $\sigma$-subalgebra, which is isomorphic to $B(X)/B_I$. Furthermore, if there exists a strictly positive linear functional on $B(X)/B_I$, then $\mathcal{M}(E)/I_T$ is monotone complete and $\hat{D}$ is normal.

**Proof.** By Corollary 8.12 and the results of Section 3, the quotient algebra $\mathcal{M}(E)/I_T$ is monotone $\sigma$-complete.

Let $g \in B(X)$. Then, as remarked before Lemma 8.7, $\hat{D}\pi(g) = g$.

Now $\pi(g) \in I_T$ if and only if $qD(\pi(g) \circ \pi(g)^*) = 0$.

But $qD(\pi(g) \circ \pi(g)^*) = qD(\pi(|g|^2)) = q(|g|^2)$.

So, $\pi(g) \in I_T$ if and only if $|g|^2 \in B_T$, that is, if and only if $g$ vanishes off some set $A \in I_T$, that is, if and only if $g \in B_I$.

So, $\pi$ induces an isomorphism from $B(X)/B_I$ onto $D[\mathcal{M}(E)]/I_T$. 


Let \( h \in I_T \). Then, by Lemma 8.10, \( q\hat{D}(h) = 0 \). That is, \( \hat{D}(h) \in B_T \). So, \( \pi\hat{D}(h) \in I_T \). But \( \pi \hat{D}(h) = Dh \).

So, \( h \in I_T \) implies \( QDh = 0 \). It now follows that we can define \( \hat{D} \) on \( \mathcal{M}(E)/I_T \) by

\[
\hat{D}(f + I_T) = QDf.
\]

It is clear that \( \hat{D} \) is a positive linear map which is faithful. Its range is an abelian subalgebra of \( \mathcal{M}(E)/I_T \). This subalgebra is \( D[\mathcal{M}(E)]/I_T \) which, as we have seen above, is isomorphic to \( B(X)/B_T \); we shall denote \( D[\mathcal{M}(E)]/I_T \) by \( \Lambda \) and call it the diagonal algebra. Furthermore, \( \hat{D} \) is idempotent, so it is a conditional expectation.

Let \( (f_n) \) be a sequence in \( \mathcal{M}(E) \) such that \( (Qf_n) \) is an upper bounded monotone increasing sequence in \( \mathcal{M}(E)/I_T \). Then, by Lemma 3.3, we may assume that \((f_n)\) is an upper bounded monotone increasing sequence in \( \mathcal{M}(E) \). Let \( Lf \) be the limit of \((Lf_n)\) in the weak operator topology. (Since the sequence is monotone, \( Lf \) is also its limit in the strong operator topology.) By Lemma 8.5, \( f \in \mathcal{M}(E) \), and \( f_n(x, x) \to f(x, x) \) for all \( x \in X \). Thus \( Df_n \to Df \) pointwise on \( X \). Also, since \( D \) is positive, \( (Df_n) \) is monotone increasing. Since \( Q \) is a \( \sigma \)-homomorphism, \( QDf \) is the least upper bound of \( (QDf_n) \). Since \( \hat{D}(f + I_T) = QDf \), it now follows that \( \hat{D} \) is \( \sigma \)-normal.

If \( \mu \) is a strictly positive functional on \( B(X)/B_T \), then \( \mu \hat{D} \) is a strictly positive linear functional on \( \mathcal{M}(E)/I_T \). It then follows from Lemma 3.1 that \( \mathcal{M}(E)/I_T \) is monotone complete. Furthermore, if \( \Lambda \) is a downward directed subset of the self-adjoint part of \( \mathcal{M}(E)/I_T \), with 0 as its greatest lower bound, then there exists a monotone decreasing sequence \( (x_n) \), with each \( x_n \) in \( \Lambda \), and \( \bigwedge x_n = 0 \). It now follows from the \( \sigma \)-normality of \( \hat{D} \) that 0 is the infimum of \( \{\hat{D}(x) : x \in \Lambda\} \). Hence \( \hat{D} \) is normal.

We now make additional assumptions about the action of \( G \) and use this to construct a natural unitary representation of \( G \). We give some technical results which give an analogue of Mercer–Bures convergence, see [4, 27]. This will be useful in Section 12 when we wish to approximate elements of \( \mathcal{M}(E)/I_T \) by finite-dimensional subalgebras.

For the rest of this section, we suppose that the action of \( G \) on \( X \) is free on each orbit, that is, for each \( x \in X, x \) is not a fixed point of \( g \), where \( g \in G \), unless \( g \) is the identity element of \( G \).

For each \( g \in G \), let \( \Delta_g = \{(x, gx) : x \in X\} \). Then the \( \Delta_g \) are pairwise disjoint and \( E = \bigcup_{g \in G} \Delta_g \).

For each \( g \in G \), let \( u_g : E \to \{0, 1\} \) be the characteristic function of \( \Delta_{g^{-1}} \). As we pointed out earlier, \( \chi_{\Delta_g} \) is the unit element of \( \mathcal{M}(E) \), so, in this notation, \( u_1 \) is the unit element of \( \mathcal{M}(E) \).

For each \( (x, y) \in E \), we have

\[
u_g \circ u_h(x, y) = \sum_{k \in G} u_g(x, kx) u_h(kx, y).\]

But \( u_g(x, kx) \neq 0 \), only if \( k = g^{-1} \) and \( u_h(g^{-1}x, y) \neq 0 \) only if \( y = h^{-1} g^{-1}x = (gh)^{-1}x \). So, \( u_g \circ u_h = u_{gh} \).

Also \( u_g^*(x, y) = \overline{u_g(y, x)} = u_{g^{-1}}(y, x) \). But \( u_g(x, y) \neq 0 \) only if \( x = g^{-1}y \), that is, only if \( y = gx \).

So, \( u_g^*(x, y) = u_{g^{-1}}(x, y) \). It follows that \( g \to u_g \) is a unitary representation of \( G \) in \( \mathcal{M}(E) \).

Let \( f \) be any element of \( \mathcal{M}(E) \). Then

\[
f \circ u_g(x, y) = \sum_{z \in [x]} f(x, z) u_g(z, y) = f(x, gy).
\]

So, for each \( x \in X \), \( D(f \circ u_g)(x, x) = f(x, gx) \). Then

\[
D(f \circ u_g) \circ u_{g^{-1}}(x, y) = \sum_{z \in [x]} D(f \circ u_g)(x, z) u_{g^{-1}}(z, y) = D(f \circ u_g)(x, x) u_{g^{-1}}(x, y) = f(x, gx) \chi_{\Delta_g}(x, y).
\]
So

(ii) \[ D(f \circ u_g) \circ u_{g^{-1}}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in \Delta_g, \\ 0 & \text{if } (x, y) \notin \Delta_g. \end{cases} \]

The identity (\#), used in Lemma 8.7, can be re-written as

(iii) \[ D(f \circ f^*)(x, x) = \sum_{g \in G} |f(x, gx)|^2 = \sum_{g \in G} |D(f \circ u_g)(x, x)|^2 = \sum_{g \in G} |\tilde{D}(f \circ u_g)(x)|^2. \]

Let \( F \) be any finite subset of \( G \). Let \( f_F = \sum_{g \in F} D(f \circ u_g) \circ u_{g^{-1}} \). Then, using (ii),

\[ (f - f_F)(x, y) = \begin{cases} 0 & \text{if } (x, y) \in \Delta_g \text{ and } g \in F, \\ f(x, y) & \text{if } (x, y) \in \Delta_g \text{ and } g \notin F. \end{cases} \]

We now replace \( f \) by \( f - f_F \) in (iii) and obtain

(iv) \[ D((f - f_F) \circ (f - f_F)^*)(x, x) = \sum_{g \notin G \setminus F} |D(f \circ u_g)(x, x)|^2 = \sum_{g \notin G \setminus F} |\tilde{D}(f \circ u_g)(x)|^2. \]

Now let \((F_n)(n = 1, 2\ldots)\) be any strictly increasing sequence of finite subsets of \( G \). Write \( f_n \) for \( f_{F_n} \). Then \( D((f - f_n) \circ (f - f_n)^*)(x, x) \) decreases monotonically to 0 as \( n \to \infty \). Since \( Q \) is a \( \sigma \)-homomorphism,

(v) \[ \bigwedge\{QD((f - f_n) \circ (f - f_n)^*) \mid n \in \mathbb{N}\} = 0. \]

For each \( g \in G \), let \( U_g = Qu_g \). Since \( Q \) is a \( \sigma \)-homomorphism onto \( \mathcal{M}(E)/I_E \), \( U_g \) is a unitary and \( g \to U_g \) is a unitary representation of \( G \) in \( \mathcal{M}(E)/I_E \). On applying the preceding paragraph, we obtain the following proposition.

**Proposition 8.15.** Let \( z \in \mathcal{M}(E)/I_E \). Let \( (F(n))(n = 1, 2\ldots) \) be a strictly increasing sequence of finite subsets of \( G \). Let \( z_n = \sum_{g \in F(n)} \tilde{D}(zU_g)U_{g^{-1}} \). Then

\[ \bigwedge\{\tilde{D}((z - z_n)(z - z_n)^*) \mid n \in \mathbb{N}\} = 0. \]

**Corollary 8.16.** Let \( z \in \mathcal{M}(E)/I_E \) such that \( \tilde{D}(zU_g) = 0 \) for each \( g \). Then \( z = 0 \).

**Proof.** This follows from Proposition 8.15 because \( z_n = 0 \) for every \( n \). \( \square \)

**Lemma 8.17.** For each \( f \in \mathcal{M}(E) \) and each \( g \in G \), \( u_g^* \circ f \circ u_g(x, y) = f(gx, gy) \). In particular, \( f \) vanishes off \( \Delta \) if and only if \( u_g^* \circ f \circ u_g \) vanishes off \( \Delta \).

**Proof.** Let \( h \in \mathcal{M}(E) \). Then, applying identity (i) we obtain \((u_g^* \circ h)(a, b) = (h^* \circ u_g)^*(a, b) = (h^* \circ u_g)(b, a) = h^*(b, ga) = h(ga, b)\).

Let \( h = f \circ u_g \). Then \( u_g^* \circ f \circ u_g(x, y) = f \circ u_g(gx, y) = f(gx, gy) \). \( \square \)

**Corollary 8.18.** For each \( a \in A \), the diagonal algebra, and for each \( g \in G \), \( U_g a U_g^* \) is in \( A \).

**Lemma 8.19.** Let \( T \) be a compact, totally disconnected space. Let \( \theta \) be a homeomorphism of \( T \) onto \( T \). Let \( \lambda_\theta \) be the automorphism of \( C(T) \) induced by \( \theta \). Let \( E \) be a non-empty clopen set such that, for each clopen \( Q \subset T \), \((\lambda_\theta(\chi_Q) - \chi_Q)\chi_E = 0 \). Then \( \theta(t) = t \) for each \( t \in E \). In other words, \( \lambda_\theta(f) = f \) for each \( f \in \chi_E C(T) \).
Proof. Let us assume that \( t_0 \in E \) such that \( \theta(t_0) \neq t_0 \). By total disconnectedness, there exists a clopen set \( Q \) with \( \theta(t_0) \in Q \) and \( t_0 \notin Q \). We have \( \lambda_\theta(\chi_Q) = \chi_{\theta^{-1}\{Q\}} \). So,
\[
\theta^{-1}\{Q\} \cap E = Q \cap E.
\]
But \( t_0 \) is an element of \( \theta^{-1}\{Q\} \cap E \) and \( t_0 \notin Q \). This is a contradiction. \( \square \)

Let \( M \) be a monotone \((\sigma-)\)complete \( C^*\)-algebra. Then we recall that an automorphism \( \alpha \) is properly outer if there does not exist a non-zero projection \( e \) such that \( \alpha \) restricts to the identity on \( eMe \).

**Proposition 8.20.** Whenever \( g \in G \) and \( g \) is not the identity, let \( a \to U_gaU_g^* \) be a properly outer automorphism of \( \hat{D}[,M(E)/I_\mathcal{I}] \). Then \( \hat{D}[,M(E)/I_\mathcal{I}] \) is a maximal abelian \( \ast \)-subalgebra of \( M(E)/I_\mathcal{I} \).

**Proof.** Let \( z \) commute with each element of \( \hat{D}[,M(E)/I_\mathcal{I}] \). Let \( a = \hat{D}(z) \). We shall show that \( z = a \). By Corollary 8.16, it will suffice to show that \( \hat{D}((z - a)U_g) = 0 \) for each \( g \in G \). We remark that \( \hat{D}(aU_g) = a\hat{D}(U_g) = 0 \) when \( g \) is not the identity element of \( G \).

So, it is enough to show that if \( g \in G \) and \( g \) is not the identity element of \( G \) then \( \hat{D}(zU_g) = 0 \). We have, for each \( b \in \hat{D}[,M(E)/I_\mathcal{I}] \), \( bz = zb \). So,
\[
b\hat{D}(zU_g) = \hat{D}(bzU_g) = \hat{D}(zbU_g) = \hat{D}(zU_g)U_g^*bU_g = \hat{D}(zU_g)U_g^*bU_g = U_g^*bU_g\hat{D}(zU_g).
\]
This implies that \( (\lambda_{g^{-1}}(b) - b)\hat{D}(zU_g) = 0 \). For shortness put \( c = \hat{D}(zU_g) \).

Assume that \( c \neq 0 \). Then, by spectral theory, there exists a non-zero projection \( e \) and a strictly positive real number \( \delta \) such that \( \delta e \leq cc^* \). Then
\[
0 \leq \delta(\lambda_{g^{-1}}(b) - b)e(\lambda_{g^{-1}}(b) - b)^* \leq (\lambda_{g^{-1}}(b) - b)cc^*(\lambda_{g^{-1}}(b) - b)^* = 0.
\]
So, \( (\lambda_{g^{-1}}(b) - b)e = 0 \) for each \( b \) in the range of \( \hat{D} \). It now follows from Lemma 8.19 that \( \lambda_{g^{-1}}(ca) = ca \) for each \( a \) in the range of \( \hat{D} \). But this contradicts the freeness of the action of \( G \). So \( \hat{D}(zU_g) = 0 \). It now follows that \( z \) is in \( \hat{D}[,M(E)/I_\mathcal{I}] \). Hence \( \hat{D}[,M(E)/I_\mathcal{I}] \) is a maximal abelian \( \ast \)-subalgebra of \( M(E)/I_\mathcal{I} \). \( \square \)

**Corollary 8.21.** When the action of \( G \) is free on \( X \) and \( \mathcal{I} \) is the Boolean ideal of meagre Borel subsets of \( X \), then \( \hat{D}[,M(E)/I_\mathcal{I}] \) is a maximal abelian \( \ast \)-subalgebra of \( M(E)/I_\mathcal{I} \).

A unitary \( w \in M(E)/I_\mathcal{I} \) is said to normalize a \( \ast \)-subalgebra \( A \) if \( wAu^* = A \). For future reference we define the normalizer subalgebra of \( M(E)/I_\mathcal{I} \) to be the smallest monotone complete \( \ast \)-subalgebra of \( M(E)/I_\mathcal{I} \) which contains every unitary which normalizes the diagonal subalgebra. It follows from Corollary 8.18 that each \( U_g \) is a normalizing unitary for the diagonal algebra, \( A = \hat{D}[,M(E)/I_\mathcal{I}] \); since each element of \( A \) is a finite linear combination of unitaries in \( A \), it follows immediately that \( A \) is contained in the normalizer subalgebra.

9. **Cross-product algebras**

First let us recall some familiar facts. Let \( A \) be a unital \( C^*\)-algebra. Let \( \alpha \) be an automorphism of \( A \). If there exists a unitary \( u \in A \) such that, for each \( z \in A \), \( \alpha(z) = zuu^* \), then \( \alpha \) is said to be an inner automorphism. When no such unitary exists in \( A \), then \( \alpha \) is an outer automorphism.

Let \( G \) be a countable group and let \( g \to \beta_g \) be an homomorphism of \( G \) into the group of all automorphisms of \( A \). Intuitively, a cross-product algebra, for this action of \( G \), is a larger
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C*-algebra, $B$, in which $A$ is embedded as a subalgebra and where each $\beta_g$ is induced by a unitary in $B$. More precisely, there is an injective *-homomorphism $j : A \to B$, and a group homomorphism $g \to U_g$ (from $G$ into the group of unitaries in $B$), such that, for each $z \in A$, $j\beta_g(z) = U_g j(z) U_g^*$. So when we identify $A$ with its image $j[A]$ in $B$, although $\beta_g$ need not be an inner automorphism of $A$ it can be extended to an inner automorphism of the larger algebra $B$. We also require that $B$ is ‘in some sense’ generated by $j[A]$ and the collection of unitaries $U_g$. When $A$ is a monotone complete C*-algebra, we can always construct a $B$ which is monotone complete.

An account of monotone cross-products when $A$ is commutative was given by Takenouchi [37], see also Saitô [30]. (This was for abelian groups, but everything extends without difficulty to non-abelian groups.) This was later generalized by Hamana [20] to the situation where $A$ is not commutative. For the purposes of this paper, we need to consider only the situation when $A$ is commutative. So, for the rest of this section, $A$ shall be a monotone complete commutative C*-algebra. Hence $A \simeq C(S)$, where $S$ is a compact, Hausdorff, extremally disconnected space. We shall outline below some of the properties of the monotone cross-product over abelian algebras. More information can be found in [30, 37]. It turns out that they can be identified with algebras already constructed in Section 8. But, historically, the construction of monotone cross-products came first.

We shall suppose for the rest of this section that $S$ has no isolated points. Then, as remarked in Section 4, any dense subset $Y$ has no isolated points.

We shall use a result from [36] to relate monotone cross-products to the monotone complete C*-algebra of an orbit equivalence relation.

Let $G$ be a countably infinite group of homeomorphisms of $S$, where the action of $G$ has a free dense orbit, then we shall show that the corresponding monotone cross-product is isomorphic to one obtained by an action of $\bigoplus \mathbb{Z}_2$.

Let $X$ be any dense subset of $S$. Then, see Section 4, $S$ can be identified with the Stone–Czech compactification of $X$. So, if $f : X \to \mathbb{C}$ is a bounded continuous function, then it has a unique extension to a continuous function $\hat{f} : S \to \mathbb{C}$. It follows that $f \to \hat{f}$ is an isometric *-isomorphism of $C_0(X)$, the algebra of bounded continuous functions on $X$, onto $C(S)$. Similarly, as remarked earlier, any homeomorphism $\theta$ from $X$ onto $X$ has a unique extension to a homeomorphism $\hat{\theta}$ from $S$ onto $S$. We may abuse our notation by using $\theta$ instead of $\hat{\theta}$, that is, using the same symbol for a homeomorphism of $X$ and for its unique extension to a homeomorphism of $S$. Slightly more generally, when $X_1$ and $X_2$ are dense subsets of $S$, if there exists an homeomorphism of $X_1$ onto $X_2$ then it has a unique extension to a homeomorphism of $S$ onto itself.

Let $g$ be any homeomorphism of $S$ onto itself. Let $\alpha_g(f) = f \circ g$ for each $f \in C(S)$. Then $\alpha_g$ is a *-automorphism of $C(S)$. All *-automorphisms of $C(S)$ arise in this way. Let $G$ be a subgroup of Homeo($S$), the group of homeomorphisms of $S$ onto itself. Then the map $g \to \alpha_g$ is an injective map from $G$ into $\mathrm{Aut}(C(S))$, the group of *-automorphisms. If $G$ is not abelian, then this is not a group homomorphism but an injective anti-homomorphism.

Let us recall that for any group $\Gamma$, the opposite group, $\Gamma^{\text{op}}$, is the same underlying set as $\Gamma$ but with a new group operation defined by $x \times y = yx$. Also $\Gamma^{\text{op}}$ and $\Gamma$ are isomorphic groups, the map $g \to g^{-1}$ gives an isomorphism. So, in the preceding paragraph, $g \to \alpha_g$ is a group isomorphism of $G^{\text{op}}$ into $\mathrm{Aut}(C(S))$. Since $G^{\text{op}}$ is isomorphic to $G$, this is not of major significance. But to avoid confusion, we shall define $\beta_g = \alpha_g^{-1}$ for each $g \in G$. Then $g \to \beta_g$ is a group isomorphism of $G$ into the automorphism group of $C(S)$.

Let $\alpha$ be a *-automorphism of $A$. Then $\alpha$ is said to be properly outer if, for each non-zero projection $p$, the restriction of $\alpha$ to $pA$ is not the identity. Let $\Gamma$ be a sub-group of $\mathrm{Aut}(A)$ such that every element, except the identity, is properly outer. Then the action of $\Gamma$ on $A$ is said to be free.

Let $G$ be a countable group of homeomorphisms of $S$. Then, see [36], if $g \to \beta_g$ is a free action of $G$ on $A$, then there exists a dense $G$-delta set $Y \subset S$, where $Y$ is $G$-invariant, such
that, whenever \( g \in G \) is not the identity, then \( g \) has no fixed points in \( Y \). In other words, for each \( y \in Y, G[y] \) is a free orbit. Conversely, we have the following lemma.

**Lemma 9.1.** Let \( X \) be a dense subset of \( S \), where \( X \) is \( G \)-invariant. Let \( G[x] \) be a free orbit for each \( x \in X \). Then \( g \rightarrow \beta_g \) is a free action of \( G \) on \( C(S) \).

**Proof.** Let \( g \in G \), such that, \( \beta_g \) is not properly outer. So, for some non-empty clopen set \( K \),

\[
\chi_K a = \beta_g(\chi_K a) = (\chi_K \circ g^{-1})(a \circ g^{-1}) = \chi_g[K](a \circ g^{-1}).
\]

In particular, \( K = g[K] \). Suppose that \( x_1 \in K \) with \( g(x_1) \neq x_1 \). Then we can find a continuous function \( \gamma \) which takes the value 1 at \( g(x_1) \) and 0 at \( x_1 \). But, from the equation above, this implies that \( 0 = 1 \). So, \( g(x) = x \) for each \( x \in K \). Because \( X \) is dense in \( S \) and \( K \) is a non-empty open set, there exists \( y \in X \cap K \). So \( y \) is a fixed point of \( g \) and \( G[y] \) is a free orbit. This is only possible if \( g \) is the identity element of \( G \). Hence the action \( g \rightarrow \beta_g \) is free. \(\square\)

In all that follows, \( G \) is a countably infinite group of homeomorphisms of \( S \) and \( Y \) is a dense \( G \)-delta subset of \( S \), where \( Y \) is \( G \)-invariant. Let \( g \rightarrow \beta_g \) be the corresponding action of \( G \) as automorphisms of \( A \). Let \( M(C(S), G) \) be the associated (Takenouchi) monotone cross-product. We shall describe the monotone cross-product below.

The key fact is that, provided the \( G \)-action is free, the monotone cross-product algebra can be identified with the monotone complete \( C^* \)-algebra arising from the \( G \)-orbit equivalence relation. The end part of Section 8 already makes this plausible. The reader who is willing to assume it can skip to Theorem 9.5.

Before saying more about the monotone cross-product, we recall some properties of the Hamana tensor product, as outlined in [36]. More detailed information can be found in [19, 20, 34].

(Comment: An alternative, equivalent, approach avoiding the tensor product, is to use the theory of Kaplansky–Hilbert modules [24–26, 43], see below.)

For the rest of this section, \( H \) is a separable Hilbert space and \( H_1 \) is an arbitrary Hilbert space. Let us fix an orthonormal basis for \( H \). Then, with respect to this basis, each \( V \in L(H_1) \otimes L(H) \) has a unique representation as a matrix \( [V_{ij}] \), where each \( V_{ij} \) is in \( L(H_1) \). Let \( M \) be a von Neumann subalgebra of \( L(H_1) \). Then the elements of \( M \otimes L(H) \) are those \( V \) for which each \( V_{ij} \) is in \( M \). Let \( T \) be any set and \( \text{Bnd}(T) \) the commutative von Neumann algebra of all bounded functions on \( T \). Then, as explained in [36], \( \text{Bnd}(T) \otimes L(H) \) can be identified with the algebra of all matrices \( [m_{ij}] \) over \( \text{Bnd}(T) \) for which \( t \rightarrow [m_{ij}(t)] \) is a norm bounded function over \( T \).

We denote the commutative \( C^* \)-algebra of bounded, complex-valued Borel measurable functions on \( Y \) by \( B(Y) \). Following [34], the product \( B(Y) \otimes L(H) \) may be defined as the Pedersen–Borel envelope of \( B(Y) \otimes L(H) \) inside \( \text{Bnd}(Y) \otimes L(H) \). The elements of \( B(Y) \otimes L(H) \) correspond to the matrices \( [b_{ij}] \) where each \( b_{ij} \in B(Y) \) and \( y \rightarrow [b_{ij}(y)] \) is a norm bounded function from \( Y \) into \( L(H) \).

Let \( M^\sigma(B(Y), G) \) be the subalgebra of \( B(Y) \otimes L(H) \) consisting of those elements of the tensor product which have a matrix representation over \( B(Y) \) of the form \( [a_{\gamma,\sigma}] (\gamma \in G, \sigma \in G) \) where \( a_{\gamma,\sigma,\tau}(y) = a_{\gamma,\sigma}(\tau y) \) for all \( y \in Y \) and all \( \gamma, \sigma, \tau \) in \( G \). Let \( E \) be the orbit equivalence relation on \( Y \) arising from \( G \), that is

\[
E = \{(y, gy) : y \in Y, g \in G\}.
\]

By Sullivan, Weiss and Wright [36, Lemma 3.1], we have the following lemma.

**Lemma 9.2.** Assume that each \( g \in G \) has no fixed points in \( Y \) unless \( g \) is the identity element. Then \( M^\sigma(B(Y), G) \) is naturally isomorphic to \( \mathcal{M}(E) \).
For the reader’s convenience we sketch the argument. The correspondence between these two algebras is given as follows. Let \( f \in \mathcal{M}(E) \). For each \( \sigma, \gamma \in G \) and, for all \( y \in Y \), let
\[
a_{\gamma, \sigma}(y) = f(\gamma y, \sigma y).
\]

Then \( a_{\gamma, \sigma} \) is in \( B(Y) \). Also the norm of \( |a_{\gamma, \sigma}(y)| \) is uniformly bounded for \( y \in Y \). So, \( [a_{\gamma, \sigma}] \) is in \( B(Y) \odot L(H) \). Also
\[
a_{\gamma, \sigma}(y) = f(\gamma y, \sigma y) = a_{\gamma, \sigma}(\gamma y).
\]

It now follows that \([a_{\gamma, \sigma}]\) is in \( M^\sigma(B(Y), G) \).

Conversely, let \([a_{\gamma, \sigma}]\) be in \( M^\sigma(B(Y), G) \). We now use the freeness hypothesis on the action of \( G \) on \( Y \) to deduce that \( \langle \{(y, \tau y) : y \in Y\} \rangle(\tau \in G) \) is a countable family of pairwise disjoint closed subsets of \( E \). So, we can now define, unambiguously, a function \( f : E \to \mathbb{C} \) by \( f(y, \tau y) = a_{\gamma, \sigma}(y) \). This is a bounded Borel function on \( E \). It follows from the definition of \( M^\sigma(B(Y), G) \) that \( a_{\gamma, \sigma}(y) = a_{\gamma, \sigma,-1}(\gamma y) = f(\gamma y, \sigma y) \) for all \( \sigma, \gamma \) in \( G \) and all \( y \in Y \). From this it follows that \( f \) is in \( \mathcal{M}(E) \).

Let \( \pi \) be the quotient homomorphism from \( B(Y) \to C(S) \). (Each \( F \in B(S) \) differs only on a meagre set from a unique function in \( C(S) \). Since \( S(Y) \) is a meagre set, each \( f \in B(Y) \) corresponds to a unique element of \( C(S) \) which we denote \( \pi(f) \).

Each element of the Hamana tensor product \( C(S) \odot L(H) \) has a representation as a matrix over \( C(S) \). [Remark: the product is not straightforward.] By Saitô and Wright \([34, Theorem 2.5]\), there exists a \( \sigma \)-homomorphism \( \Pi \) from \( B(Y) \odot L(H) \) onto \( C(S) \odot L(H) \) where \( \Pi(b_{\gamma, \sigma}) = [\pi(b_{\gamma, \sigma})] \).

(Comment: As indicated above, we may use Kaplansky–Hilbert modules as an alternative approach. We replace the Hilbert space \( H \) by the separable Hilbert space \( \ell^2(G) \), consisting of all square summable complex functions on \( G \) with the standard basis \( \{\xi_\gamma \}_{\gamma \in G} \). Let \( e_{\gamma, \sigma} \) be the standard system of matrix units for \( \ell(H) \).

Let \( \mathfrak{M} = \ell^2(G, C(S)) \) be the Kaplansky–Hilbert module, over a monotone complete \( C^* \)-algebra \( C(S) \), of all \( \ell^2 \)-summable family of elements in \( C(S) \) on \( G \) and let \( \delta_\sigma \) be the standard basis for \( \mathfrak{M} \) \([26]\) and, as Kaplansky defined, let \( \{E_{\gamma, \sigma}\} \) be the standard system of matrix units for the monotone complete \( C^* \)-algebra \( \text{End}_{C(S)}(\mathfrak{M}) \) of all bounded module endomorphisms on \( \mathfrak{M} \). Then we know that \( (C(S) \odot L(H), \{1 \otimes e_{\gamma, \sigma}\}) \) is isomorphic to \( (\text{End}_{C(S)}(\mathfrak{M}), \{E_{\gamma, \sigma}\}) \) by using \([25]\). Then \( \Pi(b_{\gamma, \sigma}) \) can be identified with the \( \ell^2 \)-limit \( \sum_{\gamma \in G} \sum_{\sigma \in \sigma} \pi(b_{\gamma, \sigma}) E_{\gamma, \sigma} \) using Kadison–Pedersen order-convergence \([23]\).)

But the Takenouchi monotone cross-product is the subalgebra of \( C(S) \odot L(H) \) corresponding to matrices \([a_{\gamma, \sigma}]\) over \( C(S) \) for which \( \beta_{\gamma,-1}(a_{\gamma, \sigma}) = a_{\gamma, \sigma, \tau} \) for all \( \gamma, \sigma, \tau \) in \( G \). Equivalently, \( a_{\gamma, \sigma}(\tau s) = a_{\gamma, \sigma}(\tau) s \) for all \( \gamma, \sigma, \tau \) in \( G \) and \( s \in S \). From this it follows that the homomorphism \( \Pi \) maps \( M^\sigma(B(Y), G) \) onto \( M(C(S), G) \), see \([36, Lemma 3.2]\).

The diagonal subalgebra of \( M(C(S), G) \) consists of those matrices \([a_{\gamma, \sigma}]\) which vanish off the diagonal, that is, \( a_{\gamma, \sigma} = 0 \) for \( \gamma \neq \sigma \). Also, \( \beta_{\tau,-1}(a_{\gamma, \sigma}) = a_{\gamma, \tau} \) for each \( \tau \in G \). It follows that we can define an isomorphism from \( A \) onto the diagonal of \( M(C(S), G) \) by \( j(a) = \text{Diag}(\ldots, \beta_{\tau,-1}(a), \ldots) \). We recall, see \([30, 37]\), that the freeness of the action \( G \) implies that the diagonal algebra of \( M(C(S), G) \) is a maximal abelian \( * \)-subalgebra of \( M(C(S), G) \), alternatively, apply the results of Section 8.

Then, by Sullivan et al. \([36, Lemma 3.3]\), we have the following lemma.

**Lemma 9.3.** Let \( E \) be the graph of the relation of orbit equivalence given by \( G \) acting on \( Y \). Then there exists a \( \sigma \)-normal homomorphism \( \delta \) from \( \mathcal{M}(E) \) onto \( M(C(S), \beta, G) \).

The kernel of \( \delta \) is \( J = \{z \in \mathcal{M}(E) : D(z z^*) \text{ vanishes off a meagre subset of } Y\} \).

Furthermore, \( \delta \) maps the diagonal subalgebra of \( \mathcal{M}(E) \) onto the diagonal subalgebra of \( M(C(S), \beta, G) \). In particular, \( \delta \) induces an isomorphism of \( \mathcal{M}(E)/J \) onto \( M(C(S), G) \).
Let $C(S) \times_{\beta} G$ be the smallest monotone closed $*$-subalgebra of $M(C(S), G)$ which contains the diagonal and each unitary which implements the $\beta$-action of $G$. It will sometimes be convenient to call $C(S) \times_{\beta} G$ the ‘small’ monotone cross-product. It turns out that $C(S) \times_{\beta} G$ does not depend on $G$ only on the orbit equivalence relation. This is not at all obvious but is an immediate consequence of Theorem 10.1. This theorem shows that when $w$ is a unitary in $M(C(S), G)$ such that $w$ normalizes the diagonal then $w$ is in $C(S) \times_{\beta} G$. So, the isomorphism of $M(C(S), G)$ onto $M(E)/J$ maps $C(S) \times_{\beta} G$ onto the normalizer subalgebra of $M(E)/J$. Does the small monotone cross-product equal the ‘big’ monotone cross-product? Equivalently, is $M(E)/J$ equal to its normalizer subalgebra? This is unknown, but we can approximate each element of $M(C(S), G)$ by a sequence in $C(S) \times_{\beta} G$, in the following precise sense.

**Lemma 9.4.** Let $z \in M(C(S), G)$. Then there exists a sequence $(z_n)$ in $C(S) \times_{\beta} G$ such that the sequence $D((z-z_n)^* (z-z_n))$ is monotone decreasing and

$$\bigwedge_{n=1}^{\infty} D((z-z_n)^* (z-z_n)) = 0.$$

**Proof.** This follows from Proposition 8.15. \hfill \square

**Theorem 9.5.** Let $G_j (j = 1, 2)$ be countable, infinite groups of homeomorphisms of $S$. Let $g_j \to \beta^S_g$ be the corresponding action of $G_j$ as automorphisms of $C(S)$. Let $Y$ be a $G$-delta, dense subset of $S$ such that $G_j[Y] = Y$ and $G_j$ acts freely on $Y$. Let $E_j$ be the orbit equivalence relation on $Y$ arising from the action of $G_j$. Suppose that $E_1 = E_2$. Then there exists an isomorphism of $M(C(S), G_1)$ onto $M(C(S), G_2)$ which maps the diagonal algebra of $M(C(S), G_1)$ onto the diagonal algebra of $M(C(S), G_2)$. Furthermore, this isomorphism maps $C(S) \times_{\beta} G_1$ onto $C(S) \times_{\beta} G_2$.

**Proof.** The first part is a straightforward application of Lemma 9.3. Let $E = E_1 = E_2$. Then both algebras are isomorphic to $M(E)/J$. The final part is a consequence of Theorem 10.1. \hfill \square

In the next result, we require $S$ to be separable.

**Corollary 9.6.** Let $G$ be a countable, infinite group of homeomorphisms of $S$. Suppose that, for some $s_0 \in S, G[s_0]$ is a free, dense orbit. Then there exists an isomorphism $\phi$ of $\bigoplus Z_2$ into $\text{Homeo}(S)$, such that there exists an isomorphism of $M(C(S), G)$ onto $M(C(S), \bigoplus Z_2)$ which maps the diagonal algebra of $M(C(S), G)$ onto the diagonal algebra of $M(C(S), \bigoplus Z_2)$.

**Proof.** By Theorem 7.10, there exists an isomorphism $\phi$ from $\bigoplus Z_2$ into $\text{Homeo}(S)$ such that there exists a dense $G$-delta set $Y$ in $S$ with the following properties. First, $Y$ is invariant under the action of both $G$ and $\bigoplus Z_2$. Secondly, the induced orbit equivalence relations coincide on $Y$. Theorem 9.5 then gives the result. \hfill \square

**Remark.** By Lemma 5.1, when $G$ has a dense orbit in $S$ then the action on $S$ is such that the only invariant clopen set is empty or the whole space. This implies that the action $g \to \beta^S_g$ is ergodic, as defined in [30, 37], which implies that the algebra $M(C(S), G)$ is a monotone complete factor.
Lemma 9.7. Let \( \mathcal{B} \) be a Boolean \( \sigma \)-algebra. Let \((p_n)\) be a sequence in \( \mathcal{B} \) which \( \sigma \)-generates \( \mathcal{B} \), that is, \( \mathcal{B} \) is the smallest \( \sigma \)-subalgebra of \( \mathcal{B} \) which contains each \( p_n \). Let \( \Gamma \) be a group of automorphisms of \( \mathcal{B} \). Let \( \Gamma \) be the union of an increasing sequence of finite subgroups \((\Gamma_n)\). Then we can find an increasing sequence of finite Boolean algebras \((\mathcal{B}_n)\) where each \( \mathcal{B}_n \) is invariant under the action of \( \Gamma_n \) and \( \bigcup \mathcal{B}_n \) is a Boolean algebra which \( \sigma \)-generates \( \mathcal{B} \).

Proof. For any natural number \( k \) the free Boolean algebra on \( k \) generators has \( 2^k \) elements. So, a Boolean algebra with \( k \) generators, being a quotient of the corresponding free algebra, has a finite number of elements.

We proceed inductively. Let \( \mathcal{B}_1 \) be the subalgebra generated by \( \{g(p_1) : g \in \Gamma_1\} \). Then \( \mathcal{B}_1 \) is finite and \( \Gamma_1 \)-invariant. Suppose that we have constructed \( \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n \). Then \( \mathcal{B}_n \cup \{p_{n+1}\} \) is a finite set. So, its saturation by the finite group \( \Gamma_{n+1} \) is again a finite set. So, the Boolean algebra this generates, call it \( \mathcal{B}_{n+1} \), is finite. Clearly, \( \mathcal{B}_n \subset \mathcal{B}_{n+1} \) and \( \mathcal{B}_{n+1} \) is invariant under the action of \( \Gamma_{n+1} \).

A commutative monotone complete \( C^* \)-algebra is countably \( \sigma \)-generated if its Boolean algebra of projections is \( \sigma \)-generated by a countable subset.

Proposition 9.8. Let the Boolean algebra of projections in \( C(S) \) be countably \( \sigma \)-generated by \((p_n)\). Let \( \Gamma \) be a group of automorphisms of \( C(S) \). Let \( \Gamma \) be the union of an increasing sequence of finite subgroups \((\Gamma_n)\). Let \( g \to u_g \) be the unitary representation of \( \Gamma \) in \( M(C(S), \Gamma) \) which implements the action of \( \Gamma \) on the diagonal algebra \( A \). Let \( \pi \) be the canonical isomorphism of \( C(S) \) onto \( A \). Then the \( C^* \)-algebra generated by \( \{u_g : g \in \Gamma\} \cup \{\pi(p_n) : n = 1, 2 \ldots\} \) is the closure of an increasing sequence of finite-dimensional subalgebras.

Proof. Let \( \mathcal{B} \) be the complete Boolean algebra of all projections in \( A \). By Lemma 9.7, we can find an increasing sequence of finite Boolean algebras of projections \((\mathcal{B}_n)\), where \( \mathcal{B} \) is \( \sigma \)-generated by \( \bigcup \mathcal{B}_n \) and each \( \mathcal{B}_n \) is invariant under the action of \( \Gamma_n \).

Let \( A_n \) be the (complex) linear span of \( \mathcal{B}_n \). Then \( A_n \) is a finite-dimensional *-subalgebra of \( A \). Also, for \( g \in \Gamma_n \), \( u_g A_n u_g^* = A_n \).

Now let \( B_n \) be the linear span of \( \{bu_g : g \in \Gamma_n \text{ and } b \in A_n\} \). Then \( B_n \) is a finite-dimensional *-subalgebra. Clearly, \( (B_n) \) is an increasing sequence and \( \bigcup_{n=1}^{\infty} B_n \) is a *-subalgebra generated by \( \{u_g : g \in \Gamma\} \cup \{\pi(p_n) : n = 1, 2 \ldots\} \).

10. The normalizer algebra

After the proof of Theorem 9.5, we made some claims concerning the normalizer algebra of a monotone cross-product. They follow from Theorem 10.1.

In this section, \( M \) is a monotone complete \( C^* \)-algebra with a maximal abelian *-subalgebra \( A \) and \( D : M \to A \) a positive, linear, idempotent map of \( M \) onto \( A \). It follows from a theorem of Tomiyama [40] that \( D \) is a conditional expectation, that is, \( D(ab) = a(Dz)b \) for each \( z \in M \) and every \( a, b \in A \). Clearly, the monotone cross-product algebras considered in Section 9 satisfy these conditions.

We recall that a unitary \( w \) in \( M \) is a normalizer of \( A \) if \( wAw^* = A \). It is clear that the normalizers of \( A \) form a subgroup of the unitaries in \( M \). We use \( N(A, M) \) to denote this normalizer subgroup. Let \( MN \) be the smallest monotone closed *-subalgebra of \( M \) which contains \( N(A, M) \). Then \( MN \) is said to be the normalizer subalgebra of \( M \).

Let \( G \) be a countable group. Let \( g \to u_g \) be a unitary representation of \( G \) in \( N(A, M) \). Let \( \lambda_g(a) = u_g au_g^* \).
Let $\alpha$ be an automorphism of $A$. We recall that $\alpha$ is properly outer if, for each non-zero projection $e \in A$, the restriction of $\alpha$ to $eA$ is not the identity map. We further recall that the action $g \to \lambda_g$ is free provided, for each $g$ other than the identity, $\lambda_g$ is properly outer.

**Theorem 10.1.** Let $M_0$ be the smallest monotone closed subalgebra of $M$ which contains $A \cup \{u_g : g \in G\}$. We suppose that:

(i) the action $g \to \lambda_g$ is free.
(ii) for each $z \in M$, if $D(zu_g) = 0$ for every $g \in G$, then $z = 0$.

Then $M_0$ contains every unitary in $M$ which normalizes $A$, that is, $M_0 = M_N$.

**Proof.** Let $w$ be a unitary in $M$ which normalizes $A$. Let $\sigma$ be the automorphism of $A$ induced by $w$. Then, for each $a \in A$, we have $waw^* = \sigma(a)$. So, $wa = \sigma(a)w$. Hence, for each $g$, we have

$$D(wau_g) = D(\sigma(a)wu_g).$$

But $D$ is a conditional expectation. So,

$$D(wu_g)u_g^*au_g = D(wau_g) = D(\sigma(a)wu_g) = \sigma(a)D(wu_g).$$

Because $A$ is abelian, it follows that $(\sigma(a) - \lambda_{g^{-1}}(a))D(wu_g) = 0$.

Let $p_g$ be the range projection of $D(wu_g)D(wu_g)^*$ in $A$. So, for each $a \in A$,

$$(\lambda_h^{-1}(a) - \lambda_g^{-1}(a))e = (\sigma(a) - \lambda_{g^{-1}}(a))p_gph - (\sigma(a) - \lambda_{h^{-1}}(a))p_hp_g = 0.$$  

Fix $g$ and $h$ with $g \neq h$, and let $e$ be the projection $p_gph$. Then we have, for each $a \in A$,

$$(\lambda_h^{-1}(a) - \lambda_g^{-1}(a))e = (\sigma(a) - \lambda_{g^{-1}}(a))p_gph - (\sigma(a) - \lambda_{h^{-1}}(a))p_hp_g = 0.$$  

Let $b$ be any element of $A$ and let $a = \lambda_g(b)$. Then $(\lambda_{h^{-1}}(b) - b)e = 0$. If $e \neq 0$, then by (i) it follows that $h^{-1}g$ is the identity element of $G$. But this implies $g = h$, which is a contradiction. So, $0 = e = p_gph$. So, $\{p_g : g \in G\}$ is a (countable) family of orthogonal projections.

Let $q$ be a projection in $A$ which is orthogonal to each $p_g$. Then $qD(wu_g) = qp_gD(wu_g) = 0$. So, $D(qwu_g) = 0$ for each $g \in G$. Hence, by applying hypothesis (ii), $qw = 0$. But $ww^* = 1$. So, $q = 0$. Thus $\sum p_g = 1$.

From (\#) we see that

$$(\lambda_g\sigma(a) - a)\lambda_g(p_g) = 0.$$  

We define $q_g$ to be the projection $\lambda_g(p_g)$. Then

$$(\# \#) (a - \lambda_g\sigma(a))q_g = 0.$$  

By arguing in a similar fashion to the above, we find that $\{q_g : g \in G\}$ is a family of orthogonal projections in $A$ with $\sum q_g = 1$.

For each $g \in G$, let $v_g = u_g p_g$. Then $v_g$ is in $M_0$ and is a partial isometry with $v_g v_g^* = q_g$ and $v_g^* v_g = p_g$. By the general additivity of equivalence for AW*-algebras, see [3, p. 129], there exists a unitary $v$ in $M_0$ such that $q_g v = v_g$ and $v p_g = v_g p_g = u_g p_g$.

From (\#), for each $a \in A$,

$$\sigma(a)p_g = u_g^* a u_g p_g = p_g v^* a v p_g = v^* a v p_g.$$  

So, $(\sigma(a) - v^* a v)p_g = 0$. Let $y = \sigma(a) - v^* a v$. Then $y^* y p_g = 0$. So, the range projection of $y^* y$ is orthogonal to $p_g$ for each $g$, and hence is 0. So, $y = 0$. It now follows that $waw^* = waw^*$ for each $a \in A$. Then $w^* w$ commutes with each element of $A$. Since $A$ is maximal abelian in $M$, it follows that $w^* w$ is in $A$. Since $v$ is in $M_0$, it now follows that $w$ is in $M_0$.

We note that the above theorem does not require the action $g \to \lambda_g$ to be ergodic.
11. Free dense actions of the Dyadic Group

We have said a great deal about $G$-actions with a free dense orbit and the algebras associated with them. It is incumbent on us to provide examples. We do this in this section. We have seen that when constructing monotone complete algebras from the action of a countably infinite group $G$ on an extremally disconnected space $S$, what matters is the orbit equivalence relation induced on $S$. When the action of $G$ has a free, dense orbit in $S$, then we have shown that the orbit equivalence relation (and hence the associated algebras) can be obtained from an action of $\bigoplus \mathbb{Z}_2$ with a free dense orbit. So, when searching for free, dense group actions, it suffices to find them when the group is $\bigoplus \mathbb{Z}_2$.

In this section, we construct such actions of $\bigoplus \mathbb{Z}_2$. As an application, we will find $2^c$ hyperfinite factors which take $2^c$ different values in the weight semigroup $[35]$.

We begin with some purely algebraic considerations before introducing topologies and continuity. We will end up with a huge number of examples.

We use $F(S)$ to denote the collection of all finite subsets of a set $S$. We shall always regard the empty set, the set with no elements, as a finite set. We use $\mathbb{N}$ to be the set of natural numbers, excluding 0. Let $C = \{f_k : k \in F(\mathbb{N})\}$ be a countable set where $k \rightarrow f_k$ is a bijection. For each $n \in \mathbb{N}$, let $\sigma_n$ be defined on $C$ by

$$\sigma_n(f_k) = \begin{cases} f_k \setminus \{n\} & n \in k, \\ f_k \cup \{n\} & \text{if } n \notin k. \end{cases}$$

**Lemma 11.1.** (i) For each $n$, $\sigma_n$ is a bijection of $C$ onto $C$, and $\sigma_n \sigma_n = id$, where $id$ is the identity map on $C$.

(ii) When $m \neq n$, then $\sigma_m \sigma_n = \sigma_n \sigma_m$.

**Proof.** (i) It is clear that $\sigma_n \sigma_n = id$ and hence $\sigma_n$ is a bijection.

(ii) Fix $f_k$. Then we need to show $\sigma_m \sigma_n(f_k) = \sigma_n \sigma_m(f_k)$. This is a straightforward calculation, considering separately the four cases when $k$ contains neither $m$ nor $n$, contains both $m$ and $n$, contains $m$ but not $n$ and contains $n$ but not $m$.

We recall that the Dyadic Group, $\bigoplus \mathbb{Z}_2$, can be identified with the additive group of functions from $\mathbb{N}$ to $\mathbb{Z}_2$, where each function takes only finitely many non-zero values. For each $n \in \mathbb{N}$, let $g_n$, be the element defined by $g_n(m) = \delta_{m,n}$ for all $m \in \mathbb{N}$. Then $\{g_n : n \in \mathbb{N}\}$ is a set of generators of $\bigoplus \mathbb{Z}_2$.

Take any $g \in \bigoplus \mathbb{Z}_2$ then $g$ has a unique representation as $g = g_{n_1} + \cdots + g_{n_p}$ where $1 \leq n_1 < \cdots < n_p$ or $g$ is the zero. Let us define

$$\varepsilon_g = \sigma_{n_1} \sigma_{n_2} \cdots \sigma_{n_p}.$$ 

Here we adopt the notational convention that $\sigma_{n_1} \sigma_{n_2} \cdots \sigma_{n_p}$ denotes the identity map of $C$ onto itself when $\{n_1, \ldots, n_p\} = \emptyset$.

Then $g \rightarrow \varepsilon_g$ is a group homomorphism of $\bigoplus \mathbb{Z}_2$ into the group of bijections of $C$ onto $C$. It will follow from Lemma 11.2(ii) that this homomorphism is injective.

**Lemma 11.2.** (i) $C = \{\varepsilon_g(f_k) : g \in \bigoplus \mathbb{Z}_2\} = \{\sigma_{n_1} \sigma_{n_2} \cdots \sigma_{n_p}(f_k) : \{n_1, n_2, \ldots, n_p\} \in F(\mathbb{N})\}$. In other words $C$ is an orbit.

(ii) For each $k \in F(\mathbb{N})$, where $k = \{n_1, \ldots, n_p\}$,

$$\sigma_{n_1} \sigma_{n_2} \cdots \sigma_{n_p}(f_k) = f_k$$

only if $\sigma_{n_1} \sigma_{n_2} \cdots \sigma_{n_p} = id$. 


Proof. (i) Let \( k = \{n_1, \ldots, n_p\} \), where \( n_i \neq n_j \) for \( i \neq j \). Then \( \sigma_{n_1} \sigma_{n_2} \cdots \sigma_{n_p}(f_\emptyset) = f_k \).

(ii) Assume that this is false. Then, for some \( k \in F(\mathbb{N}) \), we have \( \sigma_{n_1} \sigma_{n_2} \cdots \sigma_{n_p}(f_k) = f_k \) where \( \sigma_{n_1} \sigma_{n_2} \cdots \sigma_{n_p} \) is not the identity map. So, we may assume, without loss of generality, that \( \{n_1, n_2, \ldots, n_p\} = m \) is a non-empty set of \( p \) natural numbers.

First consider the case where \( k \) is the empty set. Then \( \sigma_{n_1} \sigma_{n_2} \cdots \sigma_{n_p}(f_\emptyset) = f_\emptyset \). So, \( f_m = f_\emptyset \).

But this is not possible because the map \( k \to f_k \) is injective.

So, \( k \) cannot be the empty set; let \( k = \{m_1, m_2, \ldots, m_q\} \). Then \( \sigma_{m_1} \sigma_{m_2} \cdots \sigma_{m_q}(f_\emptyset) = f_k \). Hence

\[
\sigma_{n_1} \sigma_{n_2} \cdots \sigma_{n_p} \sigma_{m_1} \sigma_{m_2} \cdots \sigma_{m_q}(f_\emptyset) = \sigma_{m_1} \sigma_{m_2} \cdots \sigma_{m_q}(f_\emptyset).
\]

On using the fact that the \( \sigma_j \) are idempotent and mutually commutative, we find that \( \sigma_{n_1} \sigma_{n_2} \cdots \sigma_{n_p}(f_\emptyset) = f_\emptyset \). But, from the above argument, this is impossible. So, (ii) is proved. \( \square \)

In [35], we consider the ‘Big Cantor Space’ \( \{0,1\}^\mathbb{R} \), which is compact, totally disconnected and separable but not metrizable or second countable. In [35], we pointed out that each compact, separable, totally disconnected space is homeomorphic to a subspace of \( \{0,1\}^\mathbb{R} \). Let \( C \) be a countable subset of \( \{0,1\}^\mathbb{R} \) then \( \text{cl} \ C \), the closure of \( C \), is a compact separable, totally disconnected space. This implies that \( C \) is completely regular and hence has a Stone–Čech compactification \( \beta C \).

We recall from the work of Section 6 that the regular \( \sigma \)-completion of \( C(\text{cl} \ C) \) is monotone complete and can be identified with \( B^\infty(\text{cl} \ C)/M(\text{cl} \ C) \). Let \( \text{cl} \ C \) be the maximal ideal space of \( B^\infty(\text{cl} \ C)/M(\text{cl} \ C) \). Then this may be identified with the Stone space of the complete Boolean algebra of regular open subsets of \( \text{cl} \ C \). By varying \( C \) in a carefully controlled way, we exhibited 2\( ^2 \) essentially different extremally disconnected spaces in the form \( \text{cl} \ C \).

For each of these spaces \( \text{cl} \ C \) we shall construct an action of \( \oplus \mathbb{Z}_2 \) with a free dense orbit.

We need to begin by recalling some notions from [35]. A pair \((T, O)\) is said to be feasible if it satisfies the following conditions.

(i) \( T \) is a set of cardinality \( c = 2^{\aleph_0} \); \( O = (O_n)(n = 1, 2, \ldots) \) is an infinite sequence of non-empty subsets of \( T \), with \( O_m \neq O_n \) whenever \( m \neq n \).

(ii) Let \( M \) be a finite subset of \( T \) and \( t \in T \setminus M \). For each natural number \( m \), there exists \( n > m \) such that \( t \in O_n \) and \( O_n \cap M = \varnothing \).

In other words, \( \{n \in \mathbb{N} : t \in O_n \text{ and } O_n \cap M = \varnothing\} \) is an infinite set.

An example satisfying these conditions can be obtained by putting \( T = 2^\mathbb{N} \), the Cantor space and letting \( O \) be an enumeration (without repetitions) of the (countable) collection of all non-empty clopen subsets. For the rest of this section, \((T, O)\) will be a fixed but arbitrary feasible pair.

Let \((T, O)\) be a feasible pair and let \( R \) be a subset of \( T \). Then \( R \) is said to be admissible if

(i) \( R \) is a subset of \( T \), with \( |R| = |T \setminus R| = c \).

(ii) \( O_n \) is not a subset of \( R \) for any natural number \( n \).

Return to the example where \( T \) is the Cantor space and \( O \) an enumeration of the non-empty clopen subsets. Then, whenever \( R \subset 2^\mathbb{N} \) is nowhere dense and of cardinality \( c \), \( R \) is admissible. Throughout this section the feasible pair is kept fixed and the existence of at least one admissible set is assumed. For the moment, \( R \) is a fixed admissible subset of \( T \). Later on we shall vary \( R \).

Since \( F(\mathbb{N}) \times F(T) \) has cardinality \( c \), we can identify the Big Cantor space with \( 2^{F(\mathbb{N}) \times F(T)} \). For each \( k \in F(\mathbb{N}) \), let \( f_k \in 2^{F(\mathbb{N}) \times F(T)} \) be the characteristic function of the set

\[
\{(l, L) : L \in F(T \setminus R), l \in k \text{ and } O_n \cap L = \varnothing \text{ whenever } n \in k \text{ and } n \notin l\}.
\]
As in [35], let \( N(t) = \{n \in \mathbb{N} : t \in O_n\} \). By feasibility, this set is infinite for each \( t \in T \). It is immediate that

\[
f_k(1, L) = 1 \quad \text{precisely when } L \in F(T \setminus R), 1 \subset k \quad \text{and, for each } t \in L, \quad N(t) \cap (k \setminus 1) = \emptyset.
\]

Let \( X_R \) be the countable set \( \{f_k : k \in F(\mathbb{N})\} \). Let \( K_R \) be the closure of \( X_R \) in the Big Cantor space. Then \( K_R \) is a (separable) compact Hausdorff totally disconnected space with respect to the relative topology induced by the product topology of the Big Cantor space. We always suppose \( X_R \) to be equipped with the relative topology induced by \( K_R \).

Let \( C = X_R \). If the map \( k \rightarrow f_k \) is an injection, then we can define \( \sigma_n \) on \( X_R \) as before.

**Lemma 11.3.** Let \( f_k = f_m \). Then \( k = m \).

**Proof.** By definition, \( f_k(1, \emptyset) = 1 \quad \text{precisely when } 1 \subset k \). Since \( f_k(m, \emptyset) = f_m(m, \emptyset) = 1 \), it follows that \( m \subset k \). Similarly, \( k \subset m \). Hence \( m = k \). \( \square \)

For each \( (k, K) \in F(\mathbb{N}) \times F(T) \) let \( E_{(k, K)} = \{x \in K_R : x(k, K) = 1\} \). The definition of the product topology of the Big Cantor space implies that \( E_{(k, K)} \) and its compliment \( E^c_{(k, K)} \) are clopen subsets of \( K_R \). It also follows from the definition of the product topology that finite intersections of such clopen sets form a base for the topology of \( K_R \). Hence their intersections with \( X_R \) give a base for the relative topology of \( X_R \). But we saw in [35] that, in fact,

\[
\{E_{(k, K)} \cap X_R : k \in F(\mathbb{N}), K \in F(T \setminus R)\}
\]
is a base for the topology of \( X_R \). Also \( E_{(k, K)} = \emptyset \) unless \( K \subset T \setminus R \).

Since each \( E_{(k, K)} \) is clopen, it follows from Lemma 4.1 that \( E_{(k, K)} \) is the closure of \( E_{(k, K)} \cap X_R \).

To slightly simplify our notation, we shall write \( E(k, K) \) for \( E_{(k, K)} \cap X_R \) and \( E_n \) for \( E_{(\{n\}, \emptyset)} \cap X_R \). Also \( E_n^c \) is the compliment of \( E_n \) in \( X_R \), which is, \( E_{(\{n\}, \emptyset)} \cap X_R \). We shall see, below, that \( \{f_k : n \notin h\} = E_n^c \), equivalently, \( E_n = \{f_k : n \in h\} \).

When \( G \) is a subset of \( X_R \) we denote its closure in \( \beta X_R \) by \( \text{cl} \, G \). When \( G \) is a clopen subset of \( X_R \), then \( \text{cl} \, G \) is a clopen subset of \( \beta X_R \). So, the closure of \( E_n \) in \( \beta X_R \) is \( \text{cl} \, E_n \), whereas its closure in \( K_R \) is \( E_{(\{n\}, \emptyset)} \).

We need to show that each \( \sigma_n \) is continuous on \( X_R \). Since \( \sigma_n \) is equal to its inverse, this implies that \( \sigma_n \) is a homeomorphism of \( X_R \) onto itself.

Our first step to establish continuity of \( \sigma_n \) is the following.

**Lemma 11.4.** We have \( E_n = \{f_k : n \in k\} \) and \( E_n^c = \{f_m : n \notin m\} \). Also \( \sigma_n \) interchanges \( E_n \) and \( E_n^c \). Furthermore, for \( m \neq n \), \( \sigma_m \) maps \( E_n \) onto \( E_n^c \) and \( E_n^c \) onto \( E_n^c \).

**Proof.** By definition \( f_k(\{n\}, \emptyset) = 1 \) if and only if \( \{n\} \subset k \). So \( f_m \in E_n^c \) precisely when \( n \notin m \).

For \( f_k \in E_n \) we have \( \sigma_n(f_k) = f_k \setminus \{n\} \). So, \( \sigma_n \) maps \( E_n \) onto \( E_n^c \). Similarly, it maps \( E_n^c \) onto \( E_n \).

When \( m \neq n \), consider \( f_k \in E_n \). Then \( n \in k \). So, \( n \in k \cup \{m\} \) and \( n \in k \setminus \{m\} \). Thus \( \sigma_m(f_k) \) is in \( E_n \). That is, \( \sigma_m[E_n] \subset E_n \).

Since \( \sigma_m \) is idempotent, we obtain \( \sigma_m[E_n] = E_n \). Similarly, \( \sigma_m[E_n^c] = E_n^c \). \( \square \)

**Lemma 11.5.** The map \( \sigma_n : X_R \rightarrow X_R \) is continuous.

**Proof.** It suffices to show that \( \sigma_n^{-1}[E(1, L)] \) is open when \( L \subset T \setminus R \).
Let $f_h$ be in $\sigma_n^{-1}[E(1,L)]$. We shall find, $U$, an open neighbourhood of $f_h$ such that $\sigma_n[U] \subset E(1,L)$.

We need to consider three possibilities.

(1) First suppose that $n \notin h$, that is, $f_h \in E_n$. Then $f_{h \setminus \{n\}} = \sigma_n(f_h)$, which is in $E(1,L)$.

So, $I \subset h \setminus \{n\}$ which implies $n \notin I$. Also $N(t) \cap ((h \setminus \{n\}) \setminus I) = \emptyset$ for all $t \in L$. It follows that $I \cup \{n\} \subset h$ and, for all $t \in L$, $N(t) \cap (h \setminus (I \cup \{n\})) = \emptyset$. Hence $f_h \in E_n \cap E(I \cup \{n\}, L)$.

Let $f_k \in E(I \cup \{n\}, L)$. Then $I \cup \{n\} \subset k$. Also, for $t \in L$, $N(t) \cap ((k \setminus \{n\}) \setminus I) = \emptyset$.

Hence $I \subset k \setminus \{n\}$ and, for $t \in L$, $N(t) \cap ((k \setminus \{n\}) \setminus I) = \emptyset$. This implies that $\sigma_n(f_k) = f_{k \setminus \{n\}} \in E(1,L)$. Thus $E(I \cup \{n\}, L)$ is a clopen set, which is a neighbourhood of $f_h$ and a subset of $\sigma_n^{-1}[E(1,L)]$.

(2) Now suppose $n \notin h$. Then $f_{h \cup \{n\}} = \sigma_n(f_h)$ which is in $E(1,L)$. This gives

(a) $h \cup \{n\}$ contains $I$. (b) For all $t \in L$, $N(t) \cap ((h \cup \{n\}) \setminus I) = \emptyset$. (c) $f_h \in E_n^c$.

Suppose, additionally, that $n \in I$. Then $(h \cup \{n\}) \setminus I = h \setminus (I \setminus \{n\})$. So, for all $t \in L$,

$$N(t) \cap (h \setminus (I \setminus \{n\})) = \emptyset.$$ 

So, $f_h$ is in $E(1, L) \cap E_n^c$. Hence, by (c) $f_h$ is in $E_n^c \cap E(1, L) \cap E_n^c$. Now let $f_k \in E_n^c \cap E(I \setminus \{n\}, L)$. Since $f_k \in E_n^c$, it follows that $n \notin k$. So $f_{k \cup \{n\}} = \sigma_n(f_k)$.

Since $f_k \in E(I \setminus \{n\}, L)$, we have $I \setminus \{n\} \subset k$. So $I \subset k \cup \{n\}$. Also $(k \cup \{n\}) \setminus I = k \setminus \{n\}$.

So, for any $t \in L$, $N(t) \cap ((k \setminus \{n\}) \setminus I) = \emptyset$. Thus $f_{k \cup \{n\}} \in E(1,L)$. That is, $\sigma_n(f_k) = E(1,L)$.

(3) We now suppose that $n \notin h$ and $n \notin I$. As in (2), statements (a), (b) and (c) hold.

Note that $h \setminus I = (h \cup \{n\}) \setminus (I \setminus \{n\})$. It follows from (b) that $N(t) \cap (h \setminus I) = \emptyset$ for each $t \in L$.

Hence $f_h \in E(1,L) \cap E_n^c$.

We also observe that, because $n \notin h$ and $n \notin I$, (b) implies that (d) $\{n\} \cap N(t) = \emptyset$ for each $t \in L$.

Now let $f_k \in E(1,L) \cap E_n^c$. Then $n \notin k$. So, $\sigma_n(f_k) = f_{k \cup \{n\}}$.

Also $I \subset k$ and $N(t) \cap (k \setminus I) = \emptyset$ for any $t \in L$. It now follows from (d) that

$$(k \cup \{n\}) \setminus I \cap N(t) = \emptyset$$ whenever $t \in L$.

Hence $f_{k \cup \{n\}} \in E(1,L)$. Thus $E(1,L) \cap E_n^c$ is a clopen neighbourhood of $f_h$ and it is a subset of $\sigma_n[E(1,L)]$.

It follows from (1), (2) and (3) that every point of $\sigma_n[E(1,L)]$ has an open neighbourhood contained in $\sigma_n[E(1,L)]$. In other words, the set is open.

More precisely, when $n \in I$, we have $\sigma_n[E(1,L)] = E_n^c \cap E(1 \setminus \{n\}, L)$ and for $n \notin I$, we find that $\sigma_n[E(1,L)] = E(1 \setminus \{n\}, L) \cup E(1,L) \cap E_n^c$.

We recall that $X_R$ is completely regular because it is a subspace of the compact Hausdorff space $K_R$. Let $\beta X_R$ be its Stone–Čech compactification. Then each continuous function $f : X_R \to X_R$ has a unique extension to a continuous function $F$ from $\beta X_R$ to $\beta X_R$. When $f$ is a homeomorphism, then by considering the extension of $f^{-1}$ it follows that $F$ is a homeomorphism of $\beta X_R$. In particular, each $\sigma_n$ has a unique extension to a homeomorphism of $\beta X_R$. We abuse our notation by also denoting this extension by $\sigma_n$.

Let us recall from Section 6 that when $\theta$ is in $\text{Homeo}(\beta X_R)$ then it induces an automorphism $h_\theta$ of $C(\beta X_R)$ by $h_\theta(f) = f \circ \theta$. It also induces an automorphism of $B^\infty(\beta X_R)/M(\beta X_R)$ by $H_\theta([F]) = [F \circ \theta]$. Then $H_\theta$ is the unique automorphism of $B^\infty(\beta X_R)/M(\beta X_R)$ which extends $h_\theta$. Let $S_R$ be the (extremely disconnected) structure space of $B^\infty(\beta X_R)/M(\beta X_R)$; this algebra can then be identified with $C(S_R)$. Then $H_\theta$ corresponds to $\theta$, a homeomorphism of $S_R$. Then $\theta \to h_\theta$ is an automorphism of $\text{Homeo}(\beta X_R)$ onto $\text{Aut} C(\beta X_R)$; $h_\theta \to H_\theta$ is an isomorphism of $\text{Aut} C(\beta X_R)$ into $\text{Aut} C(S_R)$. Also $H_\theta \to \theta$ is a group antiautomorphism of $\text{Aut} C(S_R)$ into $\text{Homeo}(S_R)$. When $G$ is an Abelian subgroup of $\text{Homeo}(\beta X_R)$, it follows that $\theta \to H_\theta$ and $\theta \to \theta$ are group isomorphisms of $G$ into $\text{Aut} C(S_R)$ and $\text{Homeo}(S_R)$, respectively.

We recall that $g \to \varepsilon_g$ is an injective group homomorphism of $\bigoplus \mathbb{Z}_2$ into the group of bijections of $C$ onto $C$. By taking the natural bijection from $C$ onto $X_R$, and by applying
Lemmas 11.2 and 11.5, we may regard \( \varepsilon_* \) as an injective group homomorphism of \( \bigoplus \mathbb{Z}_2 \) into \( \text{Homeo}(X_R) \). Since each homeomorphism of \( X_R \) onto itself has a unique extension to a homeomorphism of \( \beta X_R \) onto itself, we may identify \( \varepsilon_* \) with an injective group homomorphism of \( \bigoplus \mathbb{Z}_2 \) into the group \( \text{Homeo}(\beta X_R) \). This induces a group isomorphism, \( g \to \hat{\varepsilon}^g \) from \( \bigoplus \mathbb{Z}_2 \) into \( \text{Auto}C(S_R) \) by putting \( \hat{\varepsilon}^g = H_{\varepsilon_*} \). The corresponding isomorphism, \( g \to \hat{\varepsilon}_g \), from \( \bigoplus \mathbb{Z}_2 \) into \( \text{Homeo}(S_R) \), is defined by

\[
\hat{\varepsilon}_g = \hat{\varepsilon}_g \quad \text{for each } g \in \bigoplus \mathbb{Z}_2.
\]

As in Section 6, \( \rho \) is the continuous surjection from \( S_R \) onto \( \beta X_R \) which is dual to the natural injection from \( C(\beta X_R) \) into \( B^\infty(\beta X_R)/M(\beta X_R) \cong C(S_R) \). Let \( s_0 \in S_R \) such that \( \rho(s_0) = f_0 \).

**Theorem 11.6.** Let \( g \to \hat{\varepsilon}_g \) be the representation of \( \bigoplus \mathbb{Z}_2 \) as homeomorphisms of \( S_R \), as defined above. Then the orbit \( \{\hat{\varepsilon}_g(s_0) : g \in \bigoplus \mathbb{Z}_2\} \) is a free dense orbit in \( S_R \). There exists \( Y \), a dense \( G_\delta \) subset of \( S_R \), with \( s_0 \in Y \), such that \( Y \) is invariant under the action \( \hat{\varepsilon} \) and the action \( \hat{\varepsilon} \) is free on \( Y \).

**Proof.** By Lemma 11.2(i), \( X_R = \{\varepsilon_g(f_0) : g \in \bigoplus \mathbb{Z}_2\} \). By Proposition 6.4, this implies the orbit \( \{\hat{\varepsilon}_g(s_0) : g \in \bigoplus \mathbb{Z}_2\} \) is dense in \( S_R \).

By Lemma 11.2(ii), \( \{\varepsilon_g(f_0) : g \in \bigoplus \mathbb{Z}_2\} \) is a free orbit. This theorem now follows from Theorem 6.8.

**Corollary 11.7.** The group isomorphism, \( g \to \hat{\varepsilon}^g \), from \( \bigoplus \mathbb{Z}_2 \) into \( \text{Aut}C(S_R) \) is free and ergodic.

We shall see below that we can now obtain some additional information about this action of \( \bigoplus \mathbb{Z}_2 \) as automorphisms of \( C(S_R) \). This will enable us to construct huge numbers of hyperfinite, small wild factors.

We have seen that, for each natural number \( n \), \( \sigma_n \) is a homeomorphism of \( X_R \) onto itself with the following properties. First, \( \sigma_n = \sigma_n^{-1} \). Secondly, \( \sigma_n[E_n] = f_n \) and, for \( m \neq n \), we have \( \sigma_n[E_m] = f_m \). (This notation was introduced just before Lemma 11.4.)

We have seen that \( \sigma_n \) has a unique extension to a homeomorphism of \( \beta X_R \) which we again denote by \( \sigma_n \). Then \( \sigma_n[\text{cl}E_n] = \text{cl}E_n \) and, for \( m \neq n \), we have \( \sigma_n[\text{cl}E_m] = \text{cl}E_m \).

We define \( e_n \in C(\beta X_R) \) as the characteristic function of the clopen set \( \text{cl}E_n \).

Using the above notation, \( \hat{\varepsilon}^{\sigma_n} \) is the \( * \)-automorphism of \( B^\infty(\beta X_R)/M(\beta X_R) \cong C(S_R) \) induced by \( \sigma_n \). We have

\[
\hat{\varepsilon}^{\sigma_n}(e_n) = 1 - e_n \quad \text{and, for } m \neq n, \quad \hat{\varepsilon}^{\sigma_n}(e_m) = e_m.
\]

We recall, see the final paragraph of Section 6, that \( B^\infty(K_R)/M(K_R) \) can be identified with \( B^\infty(\beta X_R)/M(\beta X_R) \) and so with \( C(S_R) \). By Saitô and Wright [35, Proposition 13], the smallest monotone \( \sigma \)-complete \( * \)-subalgebra of \( B^\infty(\beta X_R)/M(\beta X_R) \) which contains \( \{e_n : n = 1, 2 \ldots\} \) is \( B^\infty(\beta X_R)/M(\beta X_R) \) itself. We shall see that the (norm-closed) \( * \)-algebra generated by \( \{e_n : n = 1, 2 \ldots\} \) is naturally isomorphic to \( C(2^N) \).

When \( S \subset N \) we use \( \eta_S \) to denote the element of \( 2^N \) which takes the value 1 when \( n \in S \) and 0 otherwise. Let \( G_n \) be the clopen set \( \{\eta_S \in 2^N : n \in S\} \). These clopen sets generate the (countable) Boolean algebra of clopen subsets of \( 2^N \). An application of the Stone–Weierstrass Theorem shows that the \( * \)-subalgebra of \( C(2^N) \), containing each \( \chi_{G_n} \), is dense in \( C(2^N) \).
Lemma 11.8. There exists an isometric isomorphism, \( \pi_0 \), from \( C(2^N) \) into \( C(\beta X_R) \) such that \( \pi_0(\chi_{G_n}) = e_n \).

Proof. As in Section 6 of [35] we define a map \( \Gamma \) from the Big Cantor space, \( 2^{F(\mathbb{N}) \times F(T)} \), onto the classical Cantor space, \( 2^N \), by \( \Gamma(x)(n) = x((\{n\}, \emptyset)) \). Put \( J = \{\{n\}, \emptyset : n = 1, 2, \ldots\} \). Then we may identify \( 2^N \) with \( 2^J \). So \( \Gamma \) may be regarded as a restriction map and, by definition of the topology for product spaces, it is continuous.

From the definition of \( f_k \), we see that \( f_k(\{n\}, \emptyset) = 1 \) precisely when \( n \in k \). So, \( \Gamma f_k = \eta_k \).

Hence

\[
\Gamma [E_n] = \{\eta_k : n \in k \text{ and } k \in F(N)\}.
\]

By the basic property of the Stone–Čech compactification, the natural embedding of \( X_R \) into \( K_R \) factors through \( \beta X_R \). So, there exists a continuous surjection \( \phi \) from \( \beta X_R \) onto \( K_R \) which restricts to the identity map on \( X_R \). Then \( \Gamma \phi \) maps \( \mathcal{E} \) onto \( E_n \) and \( \mathcal{E}^c \) onto \( G_n^c \). For \( f \in C(2^N) \) let \( \pi_0(f) = f \circ \phi \). Then \( \pi_0 \) is the required isometric isomorphism into \( C(\beta X_R) \subset B^\infty(\beta X_R)/M(\beta X_R) \).

Let \( \mathcal{F} \) be the action of the Dyadic Group on \( C(S_R) \) considered above. Let \( M_R \) be the corresponding monotone cross-product algebra. So, there is an isomorphism \( \pi_R \) from \( C(S_R) \) onto the diagonal subalgebra of \( M_R \) and a group representation \( g \to u_g \) of the Dyadic Group in the unitary group of \( M_R \) such that \( u_g \pi_R(a) u^*_g = \pi_R(\hat{g}^\phi(a)) \). Since each element of the Dyadic Group is its own inverse, we see that each \( u_g \) is self-adjoint. Since the Dyadic Group is Abelian, \( u_g u_h = u_h u_g \) for each \( g \) and \( h \).

As before, let \( g_n \) be the \( n \)th term in the standard sequence of generators of \( \bigoplus \mathbb{Z}_2 \), that is, \( g_n \) takes the value 1 in the \( n \)th coordinate and 0 elsewhere. We abuse our notation by writing \( 'u_n' \) for the unitary \( u_{g_n} \) and \( 'e_n' \) for the projection \( \pi_R(e_n) \) in the diagonal subalgebra of \( M_R \).

We then have

\[
u_n e_n u_n = 1 - e_n \quad \text{and, for } m \neq n, \quad u_n e_m u_n = e_m.\]

Let \( A_R = \pi_R[C(S_R)] \) be the diagonal algebra of \( M_R \). We recall that the Boolean \( \sigma \)-subalgebra of the projections of \( A_R \), generated by \( \{e_n : n = 1, 2, \ldots\} \), contains all the projections of \( A_R \).

Lemma 11.9. Let \( \mathcal{F} \) be the Fermion algebra. Then there exists an isomorphism \( \Pi \) from \( \mathcal{F} \) onto the smallest norm closed \( \ast \)-subalgebra of \( M_R \) which contains \( \{u_n : n = 1, 2, \ldots\} \) and \( \{e_n : n = 1, 2, \ldots\} \). This isomorphism takes the diagonal of \( \mathcal{F} \) onto the smallest closed abelian \( \ast \)-subalgebra containing \( \{e_n : n = 1, 2, \ldots\} \).

Proof. For any projection \( p \) we define \( p^{(0)} = p \) and \( p^{(1)} = 1 - p \).

For each choice of \( n \) and for each choice of \( \alpha_1, \ldots, \alpha_n \) from \( \mathbb{Z}_2 \), it follows from Lemma 11.8 that the product \( e_1^{\alpha_1} e_2^{\alpha_2} \cdots e_n^{\alpha_n} \) is neither 1 nor 0. In the notation of [50], \( (e_n) \) is a sequence of (mutually commutative) independent projections. The lemma now follows from (the easy part) of the proof of Proposition 2.1 [50]. In particular, for each \( n \), \( \{u_j : j = 1, 2, \ldots, n\} \cup \{e_j : j = 1, 2, \ldots, n\} \) generates a subalgebra isomorphic to the algebra of all \( 2^n \times 2^n \) complex matrices.

Definition 11.10. Let \( B_R \) be the smallest monotone \( \sigma \)-complete \( \ast \)-subalgebra of \( M_R \) which contains \( \Pi[\mathcal{F}] \).

Lemma 11.11. \( B_R \) is a monotone complete factor which contains \( A_R \) as a maximal abelian \( \ast \)-subalgebra. There exists a faithful normal conditional expectation from \( B_R \) onto \( A_R \). The
state space of $B_R$ is separable. The factor $B_R$ is wild and of Type III. It is also a small $C^*$-algebra.

Proof. Let $D_R$ be the faithful normal conditional expectation from $M_R$ onto $A_R$. The maximal ideal space of $A_R$ can be identified with the separable space $S_R$. Then, arguing as in Corollary 3.2, there exists a faithful state $\phi$ on $A_R$. Hence $\phi D_R$ is a faithful state on $M_R$ and restricts to a faithful state on $B_R$. So, by Lemma 3.1, $B_R$ is monotone complete. Let $D$ be the restriction of $D_R$ to $B_R$, then $D$ is a faithful and normal conditional expectation from $B_R$ onto $A_R$.

Since each $e_n$ is in $B_R$ it follows that $A_R$ is a $*$-subalgebra of $B_R$. Since it is maximal abelian in $M_R$, it must be a maximal abelian $*$-subalgebra of $B_R$. So, the centre of $B_R$ is a subalgebra of $A_R$. Each $u_n$ is in $B_R$ and so each central projection of $B_R$ commutes with each $u_n$. Since the action $\hat{\varepsilon}$ of the Dyadic group is ergodic, it follows that the only projections in $A_R$ which commute with every $u_n$ are 0 and 1. So, $B_R$ is a (monotone complete) factor.

The state space of every unital $C^*$-subalgebra of $M_R$ is a surjective image of the state space of $M_R$, which is separable. So, the state space of $B_R$ is separable. Equivalently, $B_R$ is almost separably representable. In fact, a slightly more elaborate argument shows that this algebra is small. See the remark preceding Theorem 6 [35].

Since $B_R$ contains a maximal abelian $*$-subalgebra which is not a von Neumann algebra, it is a wild factor. Also $M_R$ is almost separably representable; hence it possesses a strictly positive state and so is a Type III factor [46], see also [29].

It now follows immediately that the factor is a small $C^*$-algebra. For, by work of Saitô [33], a monotone, complete factor is a small $C^*$-algebra whenever it has a separable state space.

Proposition 11.12. The homomorphism $\Pi$ extends to a $\sigma$-homomorphism $\Pi^\infty$ from $F^\infty$, the Pedersen–Borel envelope of the Fermion algebra, onto $B_R$. Let $J_R$ be the kernel of $\Pi^\infty$ then $F^\infty / J_R$ is isomorphic to $B_R$.

Proof. This follows by Corollary 3.5.

12. Approximately finite dimensional algebras

We could proceed in greater generality, but for ease and simplicity, we shall only consider monotone complete $C^*$-algebras which possess a faithful state. Every almost separably representable algebra has this property and hence so does every small $C^*$-algebra.

Definition 12.1. Let $B$ be a monotone complete $C^*$-algebra with a faithful state. Then $B$ is said to be approximately finite-dimensional (AFD) if there exists an increasing sequence of finite-dimensional $*$-subalgebras $(F_n)$ such that the smallest monotone closed subalgebra of $B$ which contains $\bigcup_{n=1}^{\infty} F_n$ is $B$ itself.

Definition 12.2. If (i) $B$ is a monotone complete $C^*$-algebra which satisfies the conditions of Definition 12.1 and (ii) we can take each $F_n$ to be a full matrix algebra, then $B$ is said to be strongly hyperfinite.

Definition 12.3. Let $M$ be a monotone complete $C^*$-algebra with a faithful state. We call $M$ nearly AFD (with respect to $B$) if it satisfies the following conditions.

(i) $M$ contains a monotone closed subalgebra $B$, where $B$ is AFD.
(ii) There exists a linear map $D : M \to B$ which is positive, faithful and normal.
(iii) For each \( z \in M \), there exists a sequence \( (z_n) \) \( (n = 1, 2 \ldots) \) in \( B \), such that
\[
D((z - z_n)(z - z_n)^*) \geq D((z - z_{n+1})(z - z_{n+1})^*)\]
for each \( n \), and
\[
\bigwedge_{n=1}^{\infty} D((z - z_n)(z - z_n)^*) = 0.
\]

**Definition 12.4.** Let \( M \) be a monotone complete \( C^* \)-algebra with a faithful state. We call \( M \) **hyperfinite** if it contains a monotone closed subalgebra \( B \) such that (i) \( M \) is nearly AFD with respect to \( B \) and (ii) \( B \) is strongly hyperfinite.

**Proposition 12.5.** Let \( S \) be a compact Hausdorff extremally disconnected space. Let \( G \) be a countably infinite group and \( g \to \beta^g \) be a free action of \( G \) as automorphisms of \( C(S) \). Let \( M(C(S), G) \) be the corresponding monotone cross-product; let \( \pi[C(S)] \) be the diagonal subalgebra; let \( D : M(C(S), G) \to \pi[C(S)] \) be the diagonal map. Let \( g \to u_g \) be a unitary representation of \( G \) in \( M(C(S), G) \) such that \( \beta^g(a) = u_g au_g^* \) for each \( a \in \pi[C(S)] \). Let \( B \) be the monotone closure of the \( * \)-algebra generated by \( \pi[C(S)] \cup \{ u_g : g \in G \} \). If \( B \) is AFD then \( M(C(S), G) \) is nearly AFD.

**Proof.** By Theorem 10.1, \( B \) is the normalizer subalgebra of \( M(C(S), G) \).

By Lemma 9.4, \( M(C(S), G) \) satisfies condition (iii) of Definition 12.3, with respect to \( B \) and the diagonal map \( D \).

It follows immediately that \( M(C(S), G) \) is nearly AFD whenever \( B \) is AFD.

We recall that in [35, Section 3] we constructed a weight semigroup, \( W \), which classifies monotone complete \( C^* \)-algebras. In particular, for algebras \( B_1 \) and \( B_2 \), they are equivalent (as defined in [35]) precisely when their values in the weight semigroup, \( wB_1 \) and \( wB_2 \), are the same.

**Remark.** The theory of von Neumann algebras would lead us to expect \( M(C(S), G) = C(S) \times_{\beta} G \). But this is an open problem. However, it is easy to show that \( w(M(C(S), G)) = w(C(S) \times_{\beta} G) \), that is, the two algebras are equivalent.

Let \( (T, \mathcal{O}) \) be a feasible pair as in Section 11. Let \( \mathcal{R} \) be the collection of all admissible subsets of \( T \). For each \( R \in \mathcal{R} \), let \( A_R = C(S_R) \). Then, by Corollary 20 [35], we can find \( \mathcal{R}_0 \subset \mathcal{R} \) such that \( \# \mathcal{R}_0 = 2^c \) where \( c = 2^{86} \) with the following property. Whenever \( R_1 \) and \( R_2 \) are distinct elements of \( \mathcal{R}_0 \), then \( A_{R_1} \) is not equivalent to \( A_{R_2} \), that is, \( wA_{R_1} \neq wA_{R_2} \).

**Theorem 12.6.** There exists a family of monotone complete \( C^* \)-algebras, \( (B_\lambda, \lambda \in \Lambda) \) with the following properties: each \( B_\lambda \) is a strongly hyperfinite Type III factor, each \( B_\lambda \) is a small \( C^* \)-algebra, each \( B_\lambda \) is a quotient of the Pedersen–Borel envelope of the Fermion algebra. The cardinality of \( \Lambda \) is \( 2^c \), where \( c = 2^{86} \). When \( \lambda \neq \mu \), then \( B_\lambda \) and \( B_\mu \) take different values in the classification semi-group \( W \); in particular, they cannot be isomorphic.

**Proof.** First we put \( \Lambda = \mathcal{R}_0 \). For each \( R \in \mathcal{R}_0 \) we have a faithful normal conditional expectation from \( B_R \) onto the maximal abelian \( * \)-subalgebra \( A_R \). We use the partial ordering defined in [35]. Since \( A_R \) is a monotone closed subalgebra of \( B_R \) and \( B_R \) is a monotone closed subalgebra of \( M_R \), we obtain \( A_R \preceq B_R \preceq M_R \). Since \( D_R \) is a faithful normal map from \( M_R \) onto \( A_R \), we have \( M_R \preceq A_R \). Hence \( A_R \) is equivalent to \( B_R \) which is equivalent to \( M_R \). By using the classification weight semi-group \( W \), we obtain \( wA_R = wB_R = wM_R \). Since, for \( R_1 \neq R_2 \), we have \( wA_{R_1} \neq wA_{R_2} \), this implies \( wB_{R_1} \neq wB_{R_2} \).
The only item left to prove is that each factor $B_R$ is strongly hyperfinite. But the Fermion algebra is isomorphic to $\mathcal{II}[\mathcal{F}]$. So, $\mathcal{II}[\mathcal{F}]$ is the closure of an increasing sequence of full matrix algebras. It now follows that $B_R$ is strongly hyperfinite.

\[\square\]

**Corollary 12.7.** For each orbit equivalence relation $E(R)$, corresponding to $R$, the orbit equivalence factor $M_{E(R)}$ is hyperfinite.

**Theorem 12.8.** Let $S$ be a separable compact Hausdorff extremally disconnected space. Let $C(S)$ be countably $\sigma$-generated. Let $G$ be a countably infinite group of homeomorphisms of $S$ with a free dense orbit. Let $E$ be the orbit equivalence engendered by $G$ and $E$ be the corresponding monotone complete factor. Suppose that the Boolean algebra of projections of $C(S)$ is countably generated. Then $M_E$ is nearly AFD. Let $B$ be the smallest monotone closed $\ast$-subalgebra of $M_E$ containing the diagonal of $M_E$ and the unitaries induced by the action of $G$. Then $B$ is AFD.

**Proof.** By Corollary 9.6, we may identify $M_E$ with $M(C(S), \bigoplus \mathbb{Z}_2)$. In other words we can assume that $G = \bigoplus \mathbb{Z}_2$. We can further assume that $g \to \beta_g$, the action of $G$ as homeomorphisms of $S$, is free and ergodic. Hence the corresponding action $g \to \beta^g$, as automorphisms of $C(S)$, is free and ergodic. Let $\pi$ be the isomorphism from $C(S)$ onto the diagonal. Let $g \to u_g$ be a unitary representation of $\bigoplus \mathbb{Z}_2$ such that $u_g \pi(a) u_g^* = \pi(\beta^g(a))$ for each $a \in C(S)$.

Let $(p_n)$ be a sequence of projections in $C(S)$ which $\sigma$-generate $C(S)$. By Proposition 9.8 the $C^\ast$-algebra, $B_0$, generated by $\{p_n : n = 1, 2, \ldots\}$ is the closure of the union of an increasing sequence of finite-dimensional subalgebras. Let $B$ be the smallest monotone $\sigma$-closed subalgebra containing $B_0$. ($B$ is the normalizer subalgebra.) Then $B$ is monotone closed (because $M_E$ has a faithful state) and AFD. Hence $M_E$ is nearly AFD.

In Theorem 12.8, the hypotheses allow us to deduce that we can approximate factors by increasing sequences of finite-dimensional subalgebras (AFD) but in the $2^c$ examples constructed in Section 11, we can do better. We can approximate by sequences of full matrix algebras (strongly hyperfinite). Let $M$ be a monotone factor which is AFD. Is $M$ strongly hyperfinite? Experience with von Neumann algebras would suggest a positive answer but for wild factors this is unknown.

Is $M_E$ equal to its normalizer subalgebra? Equivalently is the ‘small’ monotone cross-product equal to the ‘big’ monotone cross-product? In general, this is unknown. This is closely related to the following question, which has been unanswered for over 30 years: Let $A$ be a closed $\ast$-subalgebra of $L(H)$. Let $A^\sigma$ be the Pedersen–Borel envelope of $A$. Let $A^2$ be the sequential closure of $A$ in the weak operator topology. By a theorem of Davies [8] $A^2$ is a $C^\ast$-algebra. Clearly $A^\sigma \subset A^2$. Are these algebras the same? For some special cases, a positive answer is known, see the discussion in [28].

Let $A$ be a monotone complete factor which is almost separably representable. When $A$ is a von Neumann algebra being AFD is equivalent to being strongly hyperfinite and is equivalent to being injective. But, as we pointed out in Section 1, when $A$ is a wild factor the relationship between injectivity and being AFD is a mystery.

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