KÜNNETH SPLITTINGS AND CLASSIFICATION OF C*-ALGEBRAS WITH FINITELY MANY IDEALS

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ABSTRACT. The class of $AD$ algebras of real rank zero is classified by an exact sequence of $K$-groups with coefficients, equipped with certain order structures. Such a sequence is always split, and one may ask why, then, the middle group is relevant for classification. The answer is that the splitting map can not always be chosen to respect the order structures involved.

This may be rephrased in terms of the ideals of the $C^*$-algebras in question. We prove that when the $C^*$-algebra has only finitely many ideals, a splitting map respecting these always exists. Hence $AD$ algebras of real rank zero with finitely many ideals are classified by (classical) ordered $K$-theory. We also indicate how the methods generalize to the full class of $ASH$ algebras with slow dimension growth and real rank zero.

1. Introduction

In the ongoing quest to classify $C^*$-algebras of real rank zero by algebraic invariants much progress has been made on classes of $C^*$-algebras defined from a narrow class of building blocks as the smallest collection, containing these, which is closed under taking matrices, finite direct sums, and countable inductive limits. Considering the single building block $C(S^1)$ one gets the $AT$ algebras, whereas the more extensive $AH$ class is obtained by allowing every commutative $C^*$-algebra $C(Y)$ for $Y$ a finite CW-complex. The $AD$ algebras are derived from the building blocks $C(S^1)$ and the dimension drop intervals $I_{n}$, and the $ASH$ algebras are defined allowing both all finite CW complexes and the dimension drop intervals. In the cases of classes where the dimension of the topological spaces is allowed to vary ($AH$, $ASH$) one must often impose the slow dimension growth condition introduced in [BDR91].

In recent years, examples have appeared to demonstrate that ordered, graded $K$-theory is not a complete invariant for any class of stable rank one, real rank zero $C^*$-algebras with slow dimension growth that extends much beyond the $AT$ class, unless one imposes restrictions on the ideal structure of the algebras in question. The first such example – a pair of nonisomorphic $AH$ algebras with slow dimension growth having isomorphic ordered, graded $K$-theory – was found by Gong in [Gon98]. Subsequently, Elliott-Gong-Su in [EGS98, 2.19] provided similar examples involving algebras which were simultaneously $AH$ and $AD$, and Dădălaru-Loring ([DL96a]) found a pair of $AD$ algebras of real rank zero of which one was $AH$ with slow dimension growth and the other was not, (cf. [DG97, 10.21]).

These examples extinguished the hope, lit by results of Zhang, that many stable rank one, real rank zero $C^*$-algebras would be classified by this invariant, regardless of ideals. Indeed, it was proven in [Zha90] that the ideal lattice of a $C^*$-algebra $A$ in this class is reflected by the ideal lattice of $K_0(A)$, so as an order isomorphism
of $K_0$-groups hence preserves ideals, it was believed that by extending this order to the graded group $K_0(A) \oplus K_1(A)$, one could achieve that an order isomorphism $(\varphi_0, \varphi_1)$ was induced by a $*$-isomorphism respecting the pairing of ideals given by $\varphi_0$. This turned out not to be the case in general.

Several augmented invariants have been introduced recently to remedy this situation ([Eil96], [DL96b], [DG97]), all based on the ingenious definition of an order structure on $K$-theory with coefficients introduced in [DL96a]. That order structure was defined by $KK$-theoretical means, but it has subsequently proven useful to work instead with the classical order structures on certain tensor products as suggested already in [Bla85, 5.6].

These invariants are complete for large classes of stably finite, real rank zero $C^*$-algebras which may have arbitrary ideal lattices. It is the purpose of this note to show how, in the case that the ideal lattices of the $C^*$-algebras to be classified are finite, one may induce from an order isomorphism of the graded $K$-groups an isomorphism of the larger invariants, hence proving completeness of the classical invariant in this case.

More specifically, we shall prove:

**Theorem 1.1.** The invariant

$$\left[ K_*(\cdot); K_*(\cdot)^+; \Sigma(\cdot) \right]$$

is complete for the class of $AD$ algebras with real rank zero and finitely many ideals.

The first result on this form was given by Gong in [Gon98], and the idea of the proof given there surely may be employed in our setting also. Conversely, with a little more work our line of proof carries over to the invariants considered in [DG97], and hence covers the full case of $ASH$ algebras of real rank zero and slow dimension growth. Rather than giving the most general classification result, it is our prime objective to display a fundamental phenomenon that accounts for the fact that the generalized $K$-groups are irrelevant in the finite ideal lattice case. This comes out most clearly when one works with one of the subclasses of the $ASH$ algebras which are classified by a finite number of $K$-groups. We have chosen to work with the $AD$ algebras, which are classified by a single short exact sequence of (ordered) $K$-groups, and are rewarded for making this choice by a number of shortcuts along the way. To substantiate our claims about the general case we briefly indicate at the end of the paper what must be done here.

## 2. Invariants for $AD$ algebras

The invariant $K(\cdot; n)$ defined in [Eil96] consists of the ordered groups $K_*(A)$ and $K_0(A; \mathbb{Z} \oplus \mathbb{Z}/n)$ (see [DL96a]) along with the maps at the center of the exact sequence

$$K_0(A) \xrightarrow{\times n} K_0(A) \xrightarrow{\rho_n} K_0(A; \mathbb{Z}/n) \xrightarrow{\beta_n} K_1(A) \times n \xrightarrow{\kappa_m,n} K_1(A).$$

There are also natural maps

$$\kappa_{m,n} : K_0(A; \mathbb{Z}/n) \to K_0(A; \mathbb{Z}/m),$$
KÜNNETH SPLITTINGS AND CLASSIFICATION OF C*-ALGEBRAS WITH FINITELY MANY IDEALS

satisfying
\[ \beta_m \kappa_{m,n} = \frac{n}{(n,m)} \beta_n \]
\[ \kappa_{m,n} \rho_n = \frac{m}{(n,m)} \rho_m \]
\[ \kappa_{k,m} \kappa_{m,n} = \frac{m}{(k,m)(m,n)} \kappa_{k,n} \].

After taking inductive limits over \( \mathbb{N} \) ordered by
\[ n \leq m \iff n \uparrow m \]
we get
\[ K_0(A) \xrightarrow{id \otimes 1} K_0(A) \otimes \mathbb{Q} \xrightarrow{\rho} K_0(A; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} K_1(A) \xrightarrow{id \otimes 1} K_1(A) \otimes \mathbb{Q}. \]

where
\[ K_0(A; \mathbb{Q}/\mathbb{Z}) = \lim_{\longrightarrow} (K_0(A; \mathbb{Z}/n), \kappa_{kn,n}). \]
The orders on \( K_0(A; \mathbb{Z} \oplus \mathbb{Z}/n) \) induce one on \( K_0(A) \otimes \mathbb{Q} \oplus K_0(A; \mathbb{Q}/\mathbb{Z}) \), and we get an invariant \( \overline{K}(A) \). We then have ([Eil96, 5.3], [Eil95, 6.3.2]):

**Theorem 2.1.** \( \overline{K}(-) \) is a complete invariant for the class of AD algebras of real rank zero. \( \overline{K}(-; n) \) is a complete invariant for the subclass where \( n \) annihilates the torsion of \( K_1(-) \).

As these invariants include the maps \( \rho, \beta \), isomorphisms of the invariants are triples of positive isomorphisms which are also complex homomorphisms, i.e., commute with the maps in the complex. In the case of \( \overline{K}(-) \), we shall require that the map defined on \( K_0(-) \otimes \mathbb{Q} \) is the one induced by the map \( \varphi_0 \) on \( K_0(-) \), i.e., of the form \( \varphi_0 \otimes \text{id} \). As a consequence of torsion freeness, this is equivalent to requiring the maps to commute with \( \text{id} \otimes - \).

3. IDEAL KÜNNETH SPLITTINGS

Unsplicing the complexes discussed above, we get Küneth sequences
\[ 0 \longrightarrow K_0(A) \otimes \mathbb{Z}/n \xrightarrow{\hat{\rho}_n} K_0(A; \mathbb{Z}/n) \xrightarrow{\hat{\beta}_n} K_1(A)[n] \longrightarrow 0 \]
\[ 0 \longrightarrow K_0(A) \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{\hat{\rho}} K_0(A; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\hat{\beta}} \text{tor}(K_1(A)) \longrightarrow 0. \]

It is clear by injectivity of \( K_0(A) \otimes \mathbb{Q}/\mathbb{Z} \) that the second sequence splits, and by results of Bödigheimer ([Böd79]), so does the first. In the second case, we shall refer to a splitting map
\[ \sigma : \text{tor}(K_1(A)) \to K_0(A; \mathbb{Q}/\mathbb{Z}) \]
as a Küneth splitting. See Remark 4.6 for what should be understood as a Küneth splitting in the first case.

Let \( A \) be an AD algebra of real rank zero and let \( I \) be an ideal of \( A \). It is well known ([DL94]) that both \( I \) and \( A/I \) are AD algebras of real rank zero and that the six term exact sequence in \( K \)-theory then becomes
\[ 0 \longrightarrow K_*(I) \longrightarrow K_*(A) \longrightarrow K_*(A/I) \longrightarrow 0 \]
Furthermore, both sequences are pure exact by [BD96, 8.11], in the case of \( K_0 \), simply because all groups are torsion free. We get that the vertical maps in
are all embeddings. This is a consequence of torsion freeness of $K_0(A/I)$ to the left, and in the middle then follows from exactness. We will identify the $K$-groups derived from $I$ with their images in the $K$-groups derived from $A$.

We say that a complex homomorphism $\Phi = (\phi_0 \otimes \text{id}, \phi, \phi_1)$ from $K(A)$ to $K(B)$ is ideal-preserving when $\phi_0 K_0(I) \subseteq K_0(J)$ implies $\Phi K_0(I) \subseteq K_0(J)$, i.e.

$$\phi K_0(I; \mathbb{Q}/\mathbb{Z}) \subseteq K_0(J; \mathbb{Q}/\mathbb{Z}), \quad \phi_1 K_1(I) \subseteq K_1(J).$$

An ideal isomorphism is an isomorphism which is ideal-preserving and has ideal-preserving inverse. This concept replaces a similar notion for $KK$-theory or $E$-theory introduced in [Gon98].

The following result, explaining the relevance of the order structures when working with ideals, is essentially contained in [DG97, 4.12, 9.2]. A detailed alternative proof will appear in [DE98].

**Proposition 3.1.** Let $A, B$ be real rank zero AD algebras and assume that $\Phi = (\phi_0 \otimes \text{id}, \phi, \phi_1)$ is an isomorphism of the complexes $K(A)$ and $K(B)$. Then the following conditions are equivalent:

(i) $\Phi$ is an order isomorphism

(ii) $\phi_0$ is an order isomorphism and $\Phi$ is an ideal isomorphism.

Because $\rho, \beta$ are natural maps, they always preserve ideals; for instance, $\rho(K_0(I)) \subseteq K_0(I; \mathbb{Z}/n)$. Künneth splittings are not natural, and so we need the following definition.

**Definition 3.2.** A $C^*$-algebra $A$ is ideally split if there is a Künneth splitting $\sigma$ preserving ideals, i.e.,

$$\sigma K_0(I) \subseteq K_0(I; \mathbb{Q}/\mathbb{Z})$$

for all ideals $I$ of $A$.

**Remark 3.3.**

1. A simple AD algebra is ideally split.
2. An AD algebra with torsion free $K_1$ is ideally split.
3. An AD algebra with divisible $K_0$ is ideally split. For then $\tilde{\beta}$ is invertible, and $\tilde{\beta}^{-1}$ provides a natural, hence ideal-preserving, splitting map.
4. An AD algebra for which every proper ideal has torsion free $K_1$ is ideally split. For as the map from $K_1(I)$ to $K_1(A)$ is an embedding, the image of $K_1(I)$ misses $tor K_1(A)$.
5. Not all AD algebras of real rank zero are ideally split. Consider for instance the algebra $D_p$ in [DL96a, 3.3]. We have $K_1(D_p) = \mathbb{Z}/p$, and

$$K_0(D_p; \mathbb{Z}/p) = \left\{ (\overline{a}, \overline{b}, \overline{\tau}_i) \in \mathbb{Z}/p \oplus \mathbb{Z}/p \oplus \prod_{\mathbb{Z}} \mathbb{Z}/p \bigg| \overline{\tau}_i = \overline{a} \text{ as } i \to \infty, \overline{\tau}_i = \overline{b} \text{ as } i \to -\infty \right\}$$
There are ideals $I_n$ with $K_1(I_n) = \mathbb{Z}/p$ and $K_0(I_n; \mathbb{Z}/p) = \{(a,b,c_i) \mid c_i = 0, |i| \leq n\}$, so an ideal splitting must map into $\mathbb{Z}/p \oplus \mathbb{Z}/p \oplus 0$. As this set intersects trivially with $K_0(D_p; \mathbb{Z}/p)$, there is no ideal splitting. (In fact, the algebra $C_p$ considered in [DL96a, 3.3] is ideally split).

The reader has probably already guessed why we are interested in ideally split algebras. Here is the accurate statement.

**Proposition 3.4.** If $A$ and $B$ are ideally split $\AD$ algebras of real rank zero, and $[K_\star(A), K_\star(A)^+, \Sigma(A)] \simeq [K_\star(B), K_\star(B)^+, \Sigma(B)]$ then $A \simeq B$.

**Proof:** Let $A, B$ be ideally split $\AD$ algebras of real rank zero, and let $(\varphi_0, \varphi_1)$ be an order isomorphism of $K_\star(A)$ with $K_\star(B)$. There is a diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & K_0(A) \otimes \mathbb{Q}/\mathbb{Z} & \xrightarrow{\varphi_0} & K_0(A, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\varphi} & K_0(A) \xrightarrow{\varphi_1} & 0 \\
& & \downarrow{\varphi_0} & & \downarrow{\varphi} & & \downarrow{\varphi_1} & \\
0 & \rightarrow & K_0(B) \otimes \mathbb{Q}/\mathbb{Z} & \xrightarrow{\varphi_0} & K_0(B, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\varphi_1} & K_0(B) \xrightarrow{\varphi_1} & 0 \\
& & \downarrow{\varphi_0} & & \downarrow{\varphi_1} & & \downarrow{\varphi_1} & \\
& & \varphi_0 & & \varphi & & \varphi_1 & \\
\end{array}
\]

in which $\sigma$ and $\tau$ are ideal splittings and $\varphi$ is induced by

$$\varphi(\tilde{\rho}(x) + \sigma(y)) = \tilde{\rho}(\varphi_0(x)) + \tau(\varphi_1(y)).$$

As when proving Proposition 3.1, one gets that since $(\varphi_0, \varphi_1)$ is an order isomorphism, we have

$$\varphi_0 K_0(I) \subseteq K_0(J) \Rightarrow \varphi_1 K_1(I) \subseteq K_1(J)$$

(and similarly for $(\varphi_0^{-1}, \varphi_1^{-1})$). We hence only need to prove that $\varphi$ (and $\varphi^{-1}$) preserves ideals, and by its definition, this follows by the ideal-preserving properties of $\varphi_1, \sigma,$ and $\tau$. The triple $(\varphi_0, \varphi, \varphi_1)$ will be an order isomorphism by Proposition 3.1, and we may apply Theorem 2.1.

**Remark 3.5.** Combining Proposition 3.4 with Remark 3.3 1°–2° we regain the well-known classification results, essentially contained in [Ell93], that $\AD$ algebras which are either simple or have torsion free $K_1$ are classified by their ordered, graded $K$-groups. See [DL96a, 4.2] and [Eil96, 5.4].

4. Building ideal splittings

In this section, we shall prove

**Proposition 4.1.** Any $\AD$ algebra of real rank zero with finitely many ideals is ideally split.

Combining this with Proposition 3.4, we get Theorem 1.1. Note how the arguments are predominantly algebraic. We only use $C^\star$-algebra results to get purity of certain exact sequences as mentioned above, and to conclude that a certain lattice of subgroups is distributive, owing to the fact that the lattice of ideals of a $C^\star$-algebra has this property. Such a result can also, as in [Gon98], be obtained by appealing to the inductive limit structure of the $C^\star$-algebras in question.
Lemma 4.2. Let $I$ be an ideal of a real rank zero $AD$ algebra $A$. Suppose a splitting map

$$
0 \to K_0(I) \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{\beta_I \otimes \tau} K_0(I; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta_I} \text{tor}(K_1(I)) \to 0
$$

is given. Then there is a splitting map

$$
0 \to K_0(A) \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{\beta_A} K_0(A; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta_A} \text{tor}(K_1(A)) \to 0
$$

extending $\tau$.

Proof: Choose, by Zorn’s lemma, a subgroup $D$ of $K_0(A; \mathbb{Q}/\mathbb{Z})$ maximal with respect to the properties

$$D \cap \text{im} \tilde{\rho}_A = 0 \quad \text{im} \tau \subseteq D.$$

By maximality, $(\text{im} \tilde{\rho}_A + D)/D$ is an essential subgroup in $K_0(A; \mathbb{Q}/\mathbb{Z})/D$, and we have

$$0 \to \text{im} \tilde{\rho}_A \to K_0(A; \mathbb{Q}/\mathbb{Z})/D \to K_0(A; \mathbb{Q}/\mathbb{Z})/(\text{im} \tilde{\rho}_A + D)/D \to 0$$

where injectivity to the left is a consequence of $D \cap \text{im} \tilde{\rho}_A = 0$. This sequence splits by divisibility of $\text{im} \tilde{\rho}_A \simeq K_0(A) \otimes \mathbb{Q}/\mathbb{Z}$, and hence the quotient must vanish. Consequently, $D + \text{im} \tilde{\rho}_A = K_0(A; \mathbb{Q}/\mathbb{Z})$.

As $\ker \tilde{\beta}_A = \text{im} \tilde{\rho}_A$, we infer that $\tilde{\beta}_A | D$ is an isomorphism, and we let $\sigma = (\tilde{\beta}_A | D)^{-1}$. As $\beta_A \sigma = \beta_I \tau = \text{id}$ and $\text{im} \tau \subseteq D$, $\sigma$ does extend $\tau$.

Combining 6.1 and 8.1 of [Ell90], we get that the lattice isomorphism between ideals of $A$ and order ideals of $K_0(A)$ extends to a lattice isomorphism

$$I \mapsto K_0(I) \oplus K_1(I)$$

into the order ideals of $K_0(A) \oplus K_1(A)$ for the $C^*$-algebras we consider. As this is also a Riesz group, sums and intersections of order ideals are again order ideals. We conclude, in particular, that

$$K_1(I \cap J) = K_1(I) \cap K_1(J) \quad K_1(I + J) = K_1(I) + K_1(J)$$

Let $G_1, \ldots, G_n$ be a set of subgroups of a group $H$. There are maps

$$\Gamma^1 : \bigoplus_{i<j} G_i \cap G_j \to \bigoplus_i G_i \quad \Gamma^0 : \bigoplus_i G_i \to H$$

given on each summand by $\Gamma^0(g_i) = g_i$ and

$$\Gamma^1(g_{ij})_k = \begin{cases} g_{ij} & k = i \\ -g_{ij} & k = j \\ 0 & \text{other } k \end{cases}$$

Note that $\Gamma^0 \Gamma^1 = 0$.

Lemma 4.3. Suppose $I$ and $J$ are ideals in a real rank zero $AD$ algebra $A$ such that $I + J = A$. Then

$$0 \to K_1(I \cap J) \xrightarrow{\Gamma^1} K_1(I) \oplus K_1(J) \xrightarrow{\Gamma^0} K_1(A) \to 0$$

is a pure exact extension.
Künneth Splittings and Classification of $C^*$-Algebras with Finitely Many Ideals

Proof: Assume that $n(x_1, x_2)$ lies in the image of $\Gamma^1$. There is then $x \in K_1(I \cap J)$ with $x = nx_1$. As $I \cap J$ is an ideal of $I$, the subgroup $K_1(I \cap J)$ is pure in $K_1(I)$, which implies that $x = ny$ with $y \in K_1(I \cap J)$. Clearly $n(x_1, x_2) = n\Gamma^1(y)$. ■

The following result is very similar to [Gon98, 4.11] and is proved by the exact same argument. We include it for the convenience of the reader.

Lemma 4.4. When $(I_2)$ is a finite family of comaximal ideals of a real rank zero $AD$ algebra, the complex

$$\bigoplus_{i<j} \text{tor} K_1(I_i \cap I_j) \overset{\Gamma^1}{\longrightarrow} \bigoplus_i \text{tor} K_1(I_i) \overset{\Gamma^0}{\longrightarrow} \text{tor} K_1(A) \longrightarrow 0$$

is exact.

Proof: Surjectivity of $\Gamma^0$ follows from Lemma 4.3. We prove exactness at $\bigoplus_i \text{tor} K_1(I_i)$ for $n = 2$ and $n = 3$. A straightforward induction argument will then prove the general claim. The case $n = 2$ is obvious from Lemma 4.3. For the case $n = 3$, assume that

$$y_1 + y_2 + y_3 = 0$$

with $y_i \in \text{tor} K_1(I_i)$. As the lattice of ideals in $A$ is distributive (since $I^2 = I$ for every ideal $I$), we have

$$I_1 = I_1 \cap (I_2 + I_3) = (I_1 \cap I_2) + (I_1 \cap I_3),$$

and we apply Lemma 4.3 to $I_1$ to get that $y_1 = z_2 + z_3$ for some $z_i \in \text{tor} K_1(I_1 \cap I_i)$. From the claim for $n = 2$ we get that

$$(0, y_2 + z_2, y_3 + z_3) = (0, w, -w)$$

for some $w \in \text{tor} K_1(I_2 \cap I_3)$. And then

$$(y_1, y_2, y_3) = \Gamma^1(z_2, z_3, w).$$

Proof of 4.1: Let $\Omega$ be a proper, hereditary subset of the lattice of ideals in $A$. We get from the finiteness assumption that there is an ideal $I \not\in \Omega$ with $\min(I)$, the set of proper ideals of $I$, contained in $\Omega$. This means that $\Omega \cup \{I\}$ is again hereditary, and we can prove the claim inductively by showing that when $\Omega$ is such a set, and Künneth splittings $\sigma_J$ are given for all $J \in \Omega$ with

$$J_1 \subseteq J_2, J_1 \in \Omega \Longrightarrow \sigma_{J_2} \mid_{\text{tor} K_1(J_1)} = \sigma_{J_1},$$

and $I \not\in \Omega$ is given with $\min(I) \subseteq \Omega$, then we can define $\sigma_I$ such that (4) holds true for $\Omega \cup \{I\}$. Let $N$ be the number of maximal elements of $\min(I)$. We must deal with the cases $N = 1$ and $N > 1$ separately.

$\Gamma^0$: $N = 1$: Denote the single maximal ideal by $J$ ($J$ could be the zero ideal). By Lemma 4.2, $\sigma_J$ extends to a splitting map $\sigma_J$, and (4) follows from

$$\sigma_J \mid_{\text{tor} K_1(J_1)} = \sigma_J \mid_{\text{tor} K_1(J_1)} = \sigma_{J_1}.$$

$\Gamma^0$: $N > 1$: Let $I_1, \ldots, I_N$ be the set of different maximal ideals. Clearly $I_i + I_j = I$ for all $i \neq j$, and we get a diagram

...
in which the vertical maps form a complex and the solid square is commutative by (4). By Lemma 4.4, the horizontal maps induce a map \(\sigma_I : \text{tor } K_1(I) \to K_0(I; \mathbb{Q}/\mathbb{Z})\), such that the other square commutes. This implies that (4) holds for \(\Omega \cup \{I\}\) also.

\[ \square \]

**Remark 4.5.** The method of proof given here differs from the one given in [Gon98] by us being able to argue by sheer algebraic means, and without going back into the inductive limit presentation of the \(C^*\)-algebra in question. That is of course only possible because we have a complete algebraic invariant at our service. On the other hand, the reader will note that the overall structure of the two proofs are the same, in particular in the way we deal with the lattice of ideals. We have certainly been inspired by [citegg:ccrrzuet].

**Remark 4.6.** When working with general \(ASH\) algebras of real rank zero and with slow dimension growth, the invariant consists of the full collection of groups \(K_i(A; \mathbb{Z}/n)\) along with order structures and coherence maps as described in [DG97, 4]. As mentioned above, every unsplitted sequence

\[ 0 \to K_i(A) \otimes \mathbb{Z}/n \xrightarrow{\tilde{\rho}_i} K_i(A; \mathbb{Z}/n) \xrightarrow{\tilde{\beta}_i} K_{i+1}(A)[n] \to 0 \]

is split. In fact, as in [Böd79],[Böd80] one can choose an entire family of splitting maps which is coherent in the coefficient. This means, when we denote by \(\lambda^i_{m,n}\) the natural map

\[ \lambda^i_{m,n} : K_i(A)[n] \to K_i(A)[m], \]

that \(\sigma^i_m \lambda^i_{m,n} = \kappa^i_{m,n} \sigma_n^i\). Such a coherent family is what we understand by a Künneth splitting in this setting.

We define ideal Künneth splittings as above, and get the natural analogue of Proposition 3.4 from [DG97, 9.1-2]. We can prove that \(ASH\) algebras of real rank zero with finitely many ideals are ideally split by proceeding as in the proof of Proposition 4.1. Here, our approach in case \(2^o\) easily generalizes because

\[ \bigoplus_{j<k} K_i(I_j \cap I_k)[n] \xrightarrow{\Gamma^1} \bigoplus_j K_i(I_j)[n] \xrightarrow{\Gamma^0} K_i(A)[n] \to 0 \]

is exact as a consequence of purity, but we only know of a fairly laborious way to generalize our approach in case \(1^o\). A possible method is to achieve an analogue of Lemma 4.2, by hands-on extending a given coherent family defined on the \(K\)-theory of an ideal. This is seen by inspection of the proofs of [Böd79, 2.8], [Böd80, 2],
KÜNNETH SPLITTINGS AND CLASSIFICATION OF $C^*$-ALGEBRAS WITH FINITELY MANY IDEALS

using at a crucial point that if a basis (cf. [Fuc70, 17.2]) for $K_1(I)[p^r]$ is given, it can be augmented to a basis for $K_1(A)[p^r]$ by purity arguments.

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