Matrix De Rham Complex and Quantum A-infinity algebras

S. BARANNIKOV
Ecole Normale Superieure, 45 rue d’Ulm, 75230 Paris Cedex 05, France.
e-mail: barannikov@math.jussieu.fr

Received: 21 January 2010 / Revised: 21 November 2013 / Accepted: 21 November 2013
Published online: 9 January 2014 – © Springer Science+Business Media Dordrecht 2014

Abstract. I establish the relation of the non-commutative BV-formalism with super-invariant matrix integration. In particular, the non-commutative BV-equation, defining the quantum $A_\infty$-algebras, introduced in Barannikov (Modular operads and non-commutative Batalin–Vilkovisky geometry. IMRN, vol. 2007, rnm075. Max Planck Institute for Mathematics 2006–48, 2007), is represented via de Rham differential acting on the super-matrix spaces related with Bernstein–Leites simple associative algebras with odd trace $q(N)$, and $gl(N|N)$. I also show that the matrix Lagrangians from Barannikov (Noncommutative Batalin–Vilkovisky geometry and matrix integrals. Isaac Newton Institute for Mathematical Sciences, Cambridge University, 2006) are represented by equivariantly closed differential forms.

Mathematics Subject Classification (2010). 05A05, 14N35, 53D45, 53D37.

Keywords. cyclic homology, non-commutative geometry, matrix integrals, super Lie algebras, homotopy associative algebras, Batalin–Vilkovisky formalism, mirror symmetry.

The relation of the cyclic differential with $gl(N)$-invariant tensors on matrix spaces $gl(N) \otimes \Pi V$ was one of the origins of the cyclic homology [9,13]. Motivated by the ideas of supersymmetry, in particular the importance of the odd symplectic and Batalin–Vilkovisky structures for construction of Lagrangians of physical theories, I propose to include for consideration other series of simple associative super algebras, such as Bernstein–Leites algebra $q(N)$ [8], which is the odd analogue of the general linear matrix algebra.

I start by establishing the relation of the non-commutative BV-formalism, introduced in [3], with invariant super-matrix integration. In particular, I express the non-commutative Batalin–Vilkovisky equation, defining the quantum $A_\infty$-algebras, via the de Rham differential acting on the super-matrix spaces

$q(N) \otimes \Pi V$

Submitted for publication on 20/01/2010, preprint HAL-00378776 (04/2009).
constructed from the Bernstein–Leites algebra in the even scalar product case, and on the super-matrix spaces

\[ gl(N|N) \otimes \Pi V \]

in the odd scalar product case. It implies, in particular, that the cohomology of the Batalin–Vilkovisky differential from [3] is zero. As another immediate consequence, I prove that the Lagrangians of the supersymmetric matrix integrals, introduced in [4], represent closed \( gl(N) \)-equivariant differential forms. In the even scalar product case, the Lagrangian is

\[ \exp \left( \frac{1}{\hbar} \left( -\frac{1}{2} \langle [\Xi, X], X \rangle + S_q(X) \right) \right) \]

where \( \Xi \in q(N)_1 \) is an odd element of the Lie algebra \( q(N) \), and

\[ S_q(X) = \sum_{\alpha, g, i} h^{2g-1+i} c_{g,\alpha_1\ldots\alpha_i} \text{otr}(X^{\alpha_1}) \ldots \text{otr}(X^{\alpha_i}) \]

is a \( q(N) \)-invariant function associated by the invariant theory with a sum of exterior products of cyclic co-chains, representing a solution to the noncommutative BV-equation, and \( \text{otr}(Y) \) is the odd trace on \( q(N) \), the odd analogue of the matrix algebra. In the odd scalar product case, the Lagrangian is

\[ \exp \left( \frac{1}{\hbar} \left( -\frac{1}{2} \langle [\Xi, X], X \rangle + S_{gl}(X) \right) \right) \]

where \( \Xi \in gl(N|N)_1 \) is an odd element, and

\[ S_{gl}(X) = \sum_{\alpha, g, i} h^{2g-1+i} c_{g,\alpha_1\ldots\alpha_i} \text{tr}(X^{\alpha_1}) \ldots \text{tr}(X^{\alpha_i}) \]

is a \( gl(N|N) \)-invariant function associated by the invariant theory with a sum of symmetric products of cyclic co-chains, representing a solution to the noncommutative BV-equation in the odd scalar product case, and the trace is the supertrace on \( gl(N|N) \).

As another consequence of the relation of non-commutative BV-formalism with super-invariant matrix integration, I prove an analogue of Morita equivalence for non-commutative BV-formalism and construct from solutions of the noncommutative BV-equations corresponding to vector space with scalar product \((V, l)\), the new solutions corresponding to the vector spaces \((V \otimes gl(k|k), l \otimes \text{tr})\) and \((V \otimes q(N), l \otimes \text{otr})\).

The integrals of Equation (1), and their odd dimensional counterparts of Equation (2), are the higher dimensional generalisations of the matrix Airy function. The case of the matrix Airy function corresponds to the simplest, zero-dimensional, solution, associated with the algebra generated by the identity element: \( e \cdot e = e \).
The results of the current paper suggest that, in a sense, any topological matrix Lagrangian, with symmetry algebras given by the super Lie algebras \( q(N) \) or \( gl(N|N) \), comes from the construction from [4].

The study of integrals of Equations (1), (2) can be viewed as a generalisation for families depending on non-commuting parameters (i.e., points over non-commutative super algebras) of the study of the \( A_\infty \)-periods from [2]:

\[
\int \left( \exp \frac{1}{\hbar} \gamma_{A_\infty} \right) \omega,
\]

where \( \gamma_{A_\infty} \) represents the \( A_\infty \) structure depending on commuting parameters. For such families of integrals there is an analogue, in the non-commutative setting, of the theory of Hodge structures and their variations, which involves the notion of semi-infinite Hodge structures, see [2]. The development of an analogue of the theory of variations of Hodge structures in the setting of families depending on non-commuting parameters is an interesting open problem to which I plan to return in future publications.

Here, is the short description of the sections of the paper. In the first two sections, the \( gl(N|N) \)-invariant geometry of the odd symplectic affine space \( gl(N|N) \otimes \Pi V \) is studied and the noncommutative Batalin–Vilkovisky differential on \( \text{Symm}(\bigoplus_{j=0}^{\infty} (V \otimes j)^{Z/jZ}) \) from [3] is identified with \( gl(N|N) \)-invariant BV-differential on this affine space. In the next section I prove an analogous result in the even scalar product case, corresponding to the noncommutative BV-differential on the exterior product of cyclic chains \( \text{Symm}(\bigoplus_{j=0}^{\infty} \Pi((\Pi V) \otimes j)^{Z/jZ}) \) and the \( q(N) \)-invariant BV-differential on the odd symplectic affine space \( q(N) \otimes \Pi V \). The main technical result used here is the analogue of the invariant theory for the algebra \( q(N) \) from [14]. By the standard odd Fourier transform, the BV-differentials on affine matrix spaces are identified with de Rham differentials on the similar affine matrix spaces. This implies the triviality of the cohomology of the noncommutative Batalin–Vilkovisky differentials on \( \text{Symm}(\bigoplus_{j=0}^{\infty} (V \otimes j)^{Z/jZ}) \) and \( \text{Symm}(\bigoplus_{j=0}^{\infty} \Pi((\Pi V) \otimes j)^{Z/jZ}) \). I also prove a kind of Morita equivalence, which, given solutions to the noncommutative BV equation corresponding to a space \( V \), allows one to construct solutions corresponding to the spaces \( V \otimes gl(k|l) \) and \( V \otimes q(N) \). In the Section 5 the Hamiltonians for the adjoint actions of super Lie algebras \( q(N) \) and \( gl(N|N) \) are written, and it is shown that their sum with BV-differential corresponds under the odd Fourier transform to the equivariant cohomology differential. The Section 6 is devoted to a noncommutative super-equivariant AKSZ-type symplectic \( \sigma \)-model interpretation of the Lagrangian from [4]. It is the invariance with respect to the supergroups \( GQ(N) \) and \( GL(N|N) \) that plays the essential role in this interpretation.

**Notations.** I work in the tensor category of \( \mathbb{Z}/2\mathbb{Z} \)-graded vector spaces, over an algebraically closed field \( k \), \( \text{char}(k) = 0 \). Let \( V = V_0 \oplus V_1 \) be a \( \mathbb{Z}/2\mathbb{Z} \)-graded vector space. I denote by \( \overline{\alpha} \) the parity of an element \( \alpha \) and by \( \Pi V \) the super vector space.
with inversed parity. For a finite group $G$ acting on a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $U$, I denote via $U^G$ the space of invariants with respect to the action of $G$ and by $U_G$ the space of coinvariants $U_G = U/(gv - v|v \in V, g \in G)$. If $G$ is finite, then the averaging $(v) \rightarrow 1/|G| \sum_{g \in G} gv$ gives a canonical isomorphism $U_G \simeq U^G$. Element $(a_1 \otimes a_2 \otimes \cdots \otimes a_n)$ of $A^\otimes n$ is denoted by $(a_1, a_2, \ldots, a_n)$. Cyclic words, i.e., elements of the subspace $(U^\otimes n)_{\mathbb{Z}/n\mathbb{Z}}$ are denoted via $(a_1 \ldots a_n)^c$. The symbol $\delta^\beta_\alpha$ denotes the Kronecker delta tensor: $\delta^\beta_\alpha = 1$ for $\alpha = \beta$ and zero otherwise. I denote by tr the super trace linear functional on $	ext{End}(U)$, $\text{tr}(U) = \sum_a (-1)^{\bar{a}} U_a^a$. The isomorphism of the tensor category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces, $X \otimes Y \simeq Y \otimes X$, is realized via $(x, y) \rightarrow (-1)^{\bar{x} \bar{y}}(y, x)$. Throughout the paper, unless it is stated explicitly otherwise, $(-1)^{\varepsilon k}$ in the formulas denotes the standard Koszul sign, which can be worked out by counting $(-1)^{\overline{a} \overline{b}}$ every time the objects $a$ and $b$ are interchanged to obtain the given formula.

1. The Vector Space $F$

I start with the case of the odd symmetric scalar product on $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V$. In this case I put $F = \bigoplus_{n=0}^\infty F_n$ where

$$F_n = (V^\otimes n \otimes k[S_n])_{\overline{S}_n}.$$ 

Here, $k[S_n]$ is the group algebra of the symmetric group $S_n$, and $S_n$ acts on $k[S_n]$ by conjugation.

The odd symmetric scalar product on $V$ defines a differential on $F$, [3], which equips $F$ with the Batalin–Vilkovisky algebra structure. My aim below is to use the invariant theory approach to cyclic homology ([9,13] and references therein) in order to represent this differential on $F$ via $gl(N|N)$-invariant geometry on the affine spaces $gl(N|N) \otimes \Pi V$. First, I start by interpreting $F$ as the space of $gl(N|\widetilde{N})$-invariant symmetric tensors on $gl(N|\widetilde{N}) \otimes \Pi V$.

Let $U$ be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space. There is the natural left group $S_n$-action on $U^\otimes n$ via

$$\sigma \in S_n, \quad \sigma : (u_1, \ldots, u_n) \rightarrow (-1)^{\varepsilon k} (u_{\sigma^{-1}(1)}, \ldots, u_{\sigma^{-1}(n)}),$$

where $(-1)^{\varepsilon k}$ is the standard Koszul sign, which is equal in this case to

$$\sum_{i < j, \sigma(i) > \sigma(j)} (-1)^{\overline{i} \overline{j}}.$$ 

This gives $k$-algebra morphism $\mu : k[S_n] \rightarrow \text{End}_k(U^\otimes n)$. The group $GL(U)$ of automorphisms of $U$ acts diagonally on $U^\otimes n$ and the image of $k[S_n]$ is in the invariant subspace of the adjoint action of $GL(U)$ on $\text{End}_k(U^\otimes n)$. If $U$ is a vector space with $\dim_k U_0 \geq n$, then the $k$-algebra morphism $\mu$ is an isomorphism:

$$\mu : k[S_n] \simeq (\text{End}_k(U^\otimes n))^{GL(U)}$$

(3)
PROPOSITION 1. The vector space $F_n$ is canonically identified via the map $\mu$ with $\text{GL}(U)$-invariant subspace of $n$-symmetric powers of the vector space $\text{End}_k(U) \otimes V$:

$$F_n \simeq \left( S^n \left( \text{End}_k(U) \otimes V \right) \right)^{\text{GL}(U)},$$

where $U$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space with $\dim_k U_0 \geq n$.

**Proof.** The proof is essentially the application of the invariant theory as in the classical definition of cyclic homology via homology of the general linear algebra (see [9,13]). I have the following sequence of isomorphisms of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces:

$$(V \otimes k[\mathcal{S}_n])^\mathcal{S}_n \simeq (V^\otimes n \otimes (\text{End}_k(U)^\otimes n))^{\text{GL}(U)} \simeq (V^\otimes n \otimes \text{End}_k(U)^\otimes n)^{\text{GL}(U)} \simeq ((V \otimes \text{End}_k(U))^\otimes n)^{\text{GL}(U)} \simeq (S^n (\text{End}_k(U) \otimes V))^{\text{GL}(U)}.$$ 

Here, I used the canonical isomorphism $\text{End}_k(U)^\otimes n \simeq \text{End}_k(U \otimes n)$, under which the permuting of $n$-tuples of endomorphisms by $\sigma$ corresponds to the conjugation by $\mu(\sigma)$. I also used the fact that $\text{GL}(U)$-action and $\mathcal{S}_n$-action mutually commute. \qed

I shall denote the isomorphism Equation (4) by $\mu_F$. Denote by $\{E^\beta_\alpha\}$, $E^\beta_\alpha = e^*_\alpha \otimes e_\beta$ the basis of elementary matrices in $\text{End}_k(U)$ corresponding to some basis $\{e_\alpha\}$ in $U$. Then the map Equation (3) is written as

$$\mu : [\sigma] \rightarrow \sum_{\alpha_1,\ldots,\alpha_n} (-1)^{\epsilon_k} E^\alpha_{\alpha_1^{-1}(1)} \otimes \cdots \otimes E^\alpha_{\alpha_n^{-1}(n)}.$$ 

For a set of elements $a_i \in V$, $i \in \{1,\ldots,n\}$ denote via

$$A^\beta_{i,\alpha} = E^\beta_\alpha \otimes a_i, \quad A^\beta_{i,\alpha} \in \text{End}_k(U) \otimes V$$

the corresponding set of generators of the symmetric algebra of $\text{End}_k(U) \otimes V$. Denote

$$\text{Tr}(A^{\rho_1}_1 \cdots A^{\rho_r}_r) = \sum_{\alpha_1,\ldots,\alpha_r} (-1)^{\epsilon_k} A^{\alpha_{\rho_1}}_{\rho_1,\alpha_1} \cdot A^{\alpha_{\rho_2}}_{\rho_2,\alpha_2} \cdots A^{\alpha_{\rho_r^{-1}}}_{\rho_r,\alpha_r}$$

the symmetric tensor of degree $r$.

**Remark 2.** The notations Equations (5), (6) are justified by the fact that if one identifies $\text{End}_k(U)$ with its dual space $\text{Hom}(\text{End}_k(U), k))$ using the super trace.
and, therefore, identifies \( S^n(\text{End}_k(U) \otimes V) \) with polynomial functions of degree \( n \) on the vector space \( \text{Hom}(V, \text{End}_k(U)) \), so that the isomorphism \( \mu_F \) becomes

\[
F_n \cong \left( S^n(\text{Hom}(V, \text{End}_k(U)), k) \right)^{GL(U)}
\]

then \( A^\beta_{\alpha_i} \) is the linear function on \( \text{Hom}(V, \text{End}_k(U)) \): \( \varphi \rightarrow \varphi(a_i)^{\beta}_{\alpha} \), where \( \varphi(a_i) e_{\alpha} = \varphi(a_i)^{\beta}_{\alpha} e_{\beta} \), and Equation (6) is the super trace of the action of the product of the matrices \( \varphi(a_{\rho_1}) \ldots \varphi(a_{\rho_r}) \) on \( U \).

The space \( F_n \) is generated linearly by \( S_n \)-invariant elements of the form

\[
(a_{\rho_1} \ldots a_{\rho_r})^c \cdot \ldots \cdot (a_{\tau_1} \ldots a_{\tau_t})^c \otimes \sigma,
\]

where \( \sigma = (\rho_1 \ldots \rho_r) \ldots (\tau_1 \ldots \tau_t) \) is the cycle decomposition of a permutation \( \sigma \in S_n \).

**Lemma 3.** The isomorphism Equation (4) sends the element Equation (7) to the \( GL(U) \)-invariant symmetric tensor

\[
\text{Tr}(A_{\rho_1} \ldots A_{\rho_r}) \cdot \ldots \cdot \text{Tr}(A_{\tau_1} \ldots A_{\tau_t}).
\]

**Proof.** By definition, \( \mu_F \) sends such element to

\[
\sum_{\alpha_1, \ldots, \alpha_n} (-1)^{\epsilon_k} A^{\alpha_{\sigma^{-1}(1)}}_{1, \alpha_1} \ldots A^{\alpha_{\sigma^{-1}(n)}}_{n, \alpha_n}.
\]

(8)

It is sufficient now to rearrange Equation (8), so that the pairs of terms with the same repeating upper and lower indexes are placed one after the other. 

\[\square\]

2. The BV-Differential and the Bracket

Let us now assume that \( V \) has an odd symmetric non-degenerate scalar product

\[
l: V \otimes^2 \rightarrow \Pi k, (x, y) = (-1)^{\bar{\alpha}\bar{\beta}} l(y, x).
\]

It follows in particular that the even and odd components of \( V \) are of the same dimension, \( \dim_k V = (r|r) \).

I assume from now on that \( U \) is the \( \mathbb{Z}/2\mathbb{Z} \)-graded vector space which also has even and odd components of the same dimension:

\[
\dim_k U = (N|N).
\]

The super-trace functional

\[
\text{tr}(E^\alpha_{\bar{\alpha}}) = (-1)^{\bar{\sigma}} \delta^\alpha_{\bar{\alpha}}
\]

defines the natural even scalar product on the vector space \( \text{End}_k(U) \):

\[
\text{tr}(E^\alpha_{\bar{\alpha}}, E^\beta_{\bar{\beta}}) = (-1)^{\bar{\sigma}} \delta^\alpha_{\bar{\alpha}} \delta^\beta_{\bar{\beta}}.
\]

(9)
It allows to extend the odd symmetric scalar product \( l \) on \( V \) to the odd symmetric non-degenerate scalar product \( \hat{\mathcal{I}} \) on \( gl(N|N) \otimes V \). The latter space is, therefore, an affine space with constant odd symplectic structure. Its algebra of symmetric tensors \( \bigoplus_{n=0}^{\infty} S^n(gl(N|N) \otimes V) \) is naturally a Batalin–Vilkovisky algebra. If I choose a basis \{\( a_v \)\} in \( V \), then the Batalin–Vilkovisky operator acting on the symmetric algebra \( \bigoplus_{n=0}^{\infty} S^n(gl(N|N) \otimes V) \) is written, using the generators \( A_{v,\alpha} \), as

\[
\Delta = \sum_{v,\alpha,\beta} (-1)^{\varepsilon_K} l_{vk} \frac{\partial^2}{\partial A_{v,\alpha} \partial A_{k,\beta}},
\]

where \( l_{vk} = l(a_v, a_k) \), and the Koszul sign in this case \( \varepsilon_K = \overline{\beta} + \overline{a} \). Similarly, I have the standard odd Poisson bracket corresponding to the affine space with constant odd symplectic structure:

\[
\{ A_{v,\alpha} A_{k,\beta} \} = (-1)^{\varepsilon_K} \delta^\alpha_{\overline{\alpha}} \delta^\beta_{\overline{\beta}} l_{vk}.
\]

Since the scalar product \( \hat{\mathcal{I}} \) is \( GL(N|N) \)-invariant, therefore, both the second-order odd operator \( \Delta \) and the bracket \( \{\cdot, \cdot\} \) are \( GL(N|N) \)-invariant. Therefore, \( \Delta \) defines a differential on the \( GL(N|N) \)-invariant subspace

\[
\bigoplus_{n \leq s} S^n(gl(N|N) \otimes V)^{GL(N|N)},
\]

which coincides with \( \bigoplus_{n \leq s} F^n \) by Proposition 1 if \( \dim_k U \) is sufficiently big \( (N \geq s) \). This also gives a bracket \( \{\gamma_1, \gamma_2\} \in \bigoplus_{n \leq s} S^n(gl(N|N) \otimes V)^{GL(N|N)} \) for elements \( \gamma_i \in S^{n_i}(gl(N|N) \otimes V)^{GL(N|N)} \), \( n_1 + n_2 \leq s + 2 \).

The Batalin–Vilkovisky operator acting on \( F \) was introduced in [3]. It is the combination of “dissection-gluing” operator acting on cycles with contraction by the tensor of the scalar product. The space \( F \) is naturally identified, by considering the cycle decomposition of permutations, with the symmetric algebra of the space of cyclic words:

\[
F = \text{Symm}(\bigoplus_{j=0}^{\infty} (V \otimes j)^{\mathbb{Z}/j\mathbb{Z}}).
\]

The second order Batalin–Vilkovisky operator from ([3,4]) is completely determined by its action on the second symmetric power, and it sends a product of two cyclic words \((a_1 \ldots a_r)^c(a_{\tau_1} \ldots a_{\tau_r})^c\) to

\[
\sum_{p, q} (-1)^{\varepsilon_{1,K}} l_{p,\tau q} a_{1 \ldots a_{p-1} a_{\tau q+1} \ldots a_{\tau q-1} a_{p+1} \ldots a_r}^c
+ \sum_{p+1 < q} (-1)^{\varepsilon_{2,K}} l_{p,\rho q} a_{1 \ldots a_{p-1} a_{\rho q+1} \ldots a_r}^c(a_{p+1} \ldots a_{q-1})^c(a_{\tau_1} \ldots a_{\tau_r})^c
+ \sum_{p+1 < q} (-1)^{\varepsilon_{3,K}} l_{\tau p q} a_{1 \ldots a_r}^c(a_{\tau_1} \ldots a_{\rho q+1} \ldots a_{\tau q-1} a_{p+1} \ldots a_{\tau q-1})^c(a_{p+1} \ldots a_{q-1})^c,
\]

where \( \varepsilon_{i,K} \) are the Koszul signs, and \( q' < q \) denotes the cyclic order. It follows from [3] that \( \Delta^2 = 0 \).
THEOREM 4. The operator $\Delta$ defined on the $GL(N|N)$-invariant subspace Equation (11), with sufficiently big $N \geq s$, coincides with the BV-differential defined on $\bigoplus_{n \leq s} F_n$ in [3,4].

Proof. As $\Delta$ is of the second order with respect to the multiplication, it is sufficient to consider the case of a product of two cyclic words

$$(a_{\rho_1} \ldots a_{\rho_r}) (a_{\tau_1} \ldots a_{\tau_t})^c,$$

which corresponds under $\mu_F$ to

$$\text{Tr}(A_{\rho_1} \ldots A_{\rho_r}) \text{Tr}(A_{\tau_1} \ldots A_{\tau_t}).$$

Applying

$$\Delta = \sum_{v,k;\theta,\beta} (-1)^{\varepsilon_{l,v,k}} \frac{\partial^2}{2 \partial A_{v,\theta} \partial A_{k,\beta}}$$

I get three terms. First, there is the term

$$\sum_{p,q;\theta,\beta} (-1)^{\varepsilon_{p,k}} l_{p\tau q} \left( \sum_{\alpha_1,\ldots,\alpha_r; \alpha_{p-1}=\beta, \alpha_{p}=\theta} (-1)^{\varepsilon_{p,k}} A_{\rho_1,\alpha_1} \ldots \overline{A_{\rho_{p-1},\theta}} \ldots A_{\rho_r,\alpha_r}, \ldots A_{\rho_{p-1},\alpha_{p-1}} \ldots A_{\rho_r,\alpha_r} \right),$$

where $\{\alpha\} = \{\alpha_1, \ldots, \hat{\alpha}_{p-1}, \hat{\alpha}_{p}, \ldots \alpha_r\}$, $\{\gamma\} = \{\gamma_1, \ldots, \hat{\gamma}_{q-1}, \hat{\gamma}_q, \ldots \gamma_t\}$, which is

$$\sum_{p,q} (-1)^{\varepsilon_{l,r}} l_{p\tau q} \text{Tr}(A_{\rho_1} \ldots A_{\rho_{p-1}} A_{\rho_{p-1} \tau_{q+1} \ldots A_{\rho_{p-1},\theta}} A_{\rho_{p+1} \tau_{q-1},\theta} A_{\rho_{p+1} \tau_{q-1},\theta} \ldots A_{\rho_r,\alpha_r}).$$

This is the term corresponding to the first term in the definition of the noncommutative Batalin–Vilkovisky operator acting on the product of the two cyclic words. The matching of signs is verified by using the general formalism of the Koszul rule,
via multiplying the odd elements by associated odd parameters and verifying that the signs match in the even degree case. The second term is

$$\sum_{p<q; \theta, \beta} (-1)^{\epsilon K} l_{\rho_p \rho_q} \left( \sum_{\alpha_1, \ldots, \alpha_r} (-1)^{\epsilon p, K} A_{\rho_1, \alpha_1}^{\alpha_r} \cdots \overline{A_{\rho_p, \theta}^{\beta}} \cdots \overline{A_{\rho_q, \beta}^{\theta}} \cdots \overline{A_{\rho_r, \alpha_r}} \right)$$

$$\cdot \left( \sum_{\gamma_1, \ldots, \gamma_t} (-1)^{\epsilon t, K} A_{\tau_1, \gamma_1}^{\gamma_t} \cdots A_{\tau_t, \gamma_t}^{\gamma_{t-1}} \right).$$

Assume first that the erased terms $\overline{A_{\rho_p, \theta}}^\beta$ and $\overline{A_{\rho_q, \beta}}^\theta$ are not sitting next to each other, i.e., $p+1 < q$. Then, rewriting this expression so that the pairs of terms with the same repeating lower and upper indexes follow one after the other, gives

$$\sum_{p+1 <q} (-1)^{\epsilon 2} l_{\rho_p \rho_q}$$

$$\cdot \text{Tr}(A_{\rho_1} \cdots A_{\rho_{p-1}} A_{\rho_{q+1}} \cdots A_{\rho_r}) \text{Tr}(A_{\rho_{p+1}} \cdots A_{\rho_{q-1}}) \text{Tr}(A_{\tau_1} \cdots A_{\tau_t}).$$

This corresponds to the second term in the formula for the noncommutative Batalin–Vilkovisky operator above. However, if $p+1 = q$, then I get instead

$$\sum_p (-1)^{\tilde{\epsilon} K} l_{\rho_p \rho_{p+1}} \text{Tr}(A_{\rho_1} \cdots A_{\rho_{p-1}} A_{\rho_{p+2}} \cdots A_{\rho_r}) \text{Tr}(Id) \text{Tr}(A_{\tau_1} \cdots A_{\tau_t}),$$

where $\text{Tr}(Id) = \sum_{\alpha} (-1)^{\alpha}$, which is equal to zero precisely because the even and odd parts of $U$ are of the same dimension

$$\dim_k U_0 = \dim_k U_1. \quad (12)$$

The third term is similar to the second one and it gives

$$\sum_{p+1 < q} (-1)^{\epsilon 3} l_{\tau_p \tau_q}$$

$$\cdot \text{Tr}(A_{\rho_1} \cdots A_{\rho_r}) \text{Tr}(A_{\tau_1} \cdots A_{\tau_{p-1}} A_{\tau_{q+1}} \cdots A_{\tau_t}) \text{Tr}(A_{\tau_{p+1}} \cdots A_{\tau_{q-1}})$$

corresponding to the third term. So I get the three terms corresponding exactly to the noncommutative Batalin–Vilkovisky operator defined in ([3,4]).

Since the map $\mu_F$ respects the multiplicative structure, the similar result concerning the odd symplectic bracket follows immediately.

**PROPOSITION 5.** The odd symplectic bracket

$$F_n \otimes F_n' \to F_{n+n'-2}$$
coincides with the odd symplectic bracket on the \( GL(N|N) \)-invariant subspaces for sufficiently big \( N > n + n' \)

\[
S^n (gl(N|N) \otimes V)^{GL(N|N)} \otimes S^{n'} (gl(N|N) \otimes V)^{GL(N|N)} \\
\rightarrow S^{n+n'-2} (gl(N|N) \otimes V)^{GL(N|N)}.
\]

Remark 6. The bracket on the subspace \( F_1 \), linearly generated by cyclic words, coincides with the symplectic bracket from [12]. As explained in [3], the noncommutative symplectic geometry from [12] can be viewed as the quasiclassical or, equivalently, tree-level approximation of the noncommutative Batalin–Vilkovisky geometry described in [3].

The important consequence of the Theorem 4 is that the cohomology of the differential \( \Delta \) acting on \( F \) is zero. This follows from the standard identification of the Batalin–Vilkovisky differential on affine space with the de Rham differential.

**Proposition 7.** The matrix Batalin–Vilkovisky complex \( (S(gl(N|N) \otimes V), \Delta) \) is naturally isomorphic to the de Rham complex of the affine space \( (gl(N|N) \otimes \Pi V)_0 \)

Proof. Let us identify \( S(gl(N|N) \otimes V) \) with algebra of polynomial functions on \( \text{Hom}(V, gl(N|N)) \), if \( x_i \in V_0, x_{\pi j} \in V_1 \) is a basis in which the odd scalar product has the standard form \( l(x_i, x_{\pi j}) = \delta_{ij} \) then, in terms of the corresponding matrix elements \( X^\beta_{i,\alpha} \), generating the algebra of polynomial functions on \( \text{Hom}(V, gl(N|N)) \), the isomorphism with the de Rham complex is the standard isomorphism between polyvector fields and forms, the “odd Fourier transform”. For any set of odd elements \( X^\beta_{j,\alpha}, j + \alpha + \beta = 1 \), it sends their product to the differential form

\[
\prod X^\beta_{j,\alpha} \mapsto \left( \prod l \left( \frac{\partial}{\partial X^\alpha_{\pi j,\beta}} \right) \right) \Omega, \tag{13}
\]

where \( X^\alpha_{\pi j,\beta} \) are the even elements and

\[
\Omega = \prod_{j + \alpha + \beta = 0} d(X^\beta_{j,\alpha})
\]

is the canonical constant volume form on \( (gl(N|N) \otimes \Pi V)_0 \), and \( i(v) \) is the standard contraction with the vector field \( v \).

**Theorem 8.** The cohomology of the Batalin–Vilkovisky differential acting on \( F \) is trivial: \( H^*(\bigoplus_{n=0}^{\infty} F_n, \Delta) = 0 \).
The (algebraic) de Rham complex on the affine space \((gl(N|N) \otimes \Pi V)_0\) has an extra grading \(\deg(p(x) \Pi dx^a) = \deg(p(x))\), so that, for \(N > n + 2\),

\[
F_n \simeq \left( \bigoplus_{i=0}^{n} \Omega^{2rN^2-i,n-i}_{(gl(N|N) \otimes \Pi V)_0} \right)^{GL(N|N)}
\]

\[
d_{\text{DR}} : \Omega^j_{(gl(N|N) \otimes \Pi V)_0} \to \Omega^{j+1,i-1}_{(gl(N|N) \otimes \Pi V)_0}
\]

since the constant volume form \(\Omega\) is \(gl(N|N)\)-invariant. The cohomology of the de Rham differential is trivial on every bi-graded piece \(\Omega^{2rN^2-i,n-i}_{(gl(N|N) \otimes \Pi V)_0}\). The standard arguments, see, e.g., [11,13], show that the cohomology is concentrated on the \(GL(N|N)\)-invariant subspace. It follows that \(\ker \Delta_{F_n} = \text{im} \Delta_{F_{n+2}}\).

Another consequence of the Theorem 4 is a version of Morita equivalence, i.e., the action by tensor multiplication by \((gl(k|\bar{k}), \text{tr})\) on solutions to the noncommutative BV-equation. Recall, see [3,4], that the noncommutative Batalin–Vilkovisky equation is the equation

\[
h\Delta S + \frac{1}{2} \{S, S\} = 0 \iff \Delta \left( \exp \frac{1}{\hbar} S \right) = 0
\]

for a series of products of cyclic words

\[
S = \sum_{i,g \geq 0} h^{2g-i-1} S_{i,g}, \quad S_{i,g} \in F_i.
\]

**THEOREM 9.** Let

\[
(\tilde{V}, \tilde{l}) = (V, l) \otimes (gl(k|\bar{k}), \text{tr})
\]

and let \(M : F(V) \to F(\tilde{V})\) be the algebra map defined on generators by

\[
(a_{\rho_1}, \ldots, a_{\rho_r})^c \to \text{tr}(A_{\rho_1}, \ldots, A_{\rho_r})^c.
\]

Then for any solution \(S\) to the noncommutative Batalin–Vilkovisky equation in \(F(V)\), \(M(S)\) is a solution in \(F(\tilde{V})\). These solutions have extra \(gl(k|\bar{k})\)-symmetry.

**Proof.** Notice that as vector spaces with scalar products

\[
(gl(N|N), \text{tr}) \otimes (gl(k|\bar{k}), \text{tr}) \simeq (gl(\tilde{N}, \tilde{N}), \text{tr}),
\]

where \(\tilde{N} = N(k + \bar{k})\), and, therefore,

\[
gl(\tilde{N}|\tilde{N}) \otimes \Pi V \simeq gl(N|N) \otimes \Pi \tilde{V}
\]

as affine spaces with constant BV structures. The \(GL(\tilde{N}|\tilde{N})\)-invariant function \(\mu_{F(V)}(S)\) on \(gl(\tilde{N}|\tilde{N}) \otimes \Pi V\), coincides with \(GL(N|N)\)-invariant function on \(gl(N|N) \otimes \Pi \tilde{V}\) corresponding to \(\mu_{F(\tilde{V})}M(S)\). Therefore, \(\mu_{F(\tilde{V})}M(S)\) satisfies the Batalin–Vilkovisky quantum master equation and so does \(M(S)\).
3. Modular Operad Structure on \( k[S_n] \)

In [3] the operations on a collection of spaces \( \{ k[S_n] \}_{n \geq 1} \) giving rise to the modular operad structure were defined. The subspace of cyclic permutations corresponds to the cyclic operad of associative algebras with scalar product. The relation with \( GL(U) \)-invariant tensors on the matrix spaces allows to give a straightforward definition for this modular operad structure.

I work in the category of \( \mathbb{Z}/2\mathbb{Z} \)-graded vector spaces and the modification of the modular operad notion needed here is defined as the algebra over triple, which is the functor on \( S \)-modules given by

\[
MV((n)) = \bigoplus_{G \in \Gamma((n))} V((G))_{\text{Aut}(G)}
\]

i.e. forgetting the extra \( \mathbb{Z} \)-grading, compared with definition from [10]. It is straightforward to see that forgetting the extra \( \mathbb{Z} \)-grading and orientation on the spaces of cycles, the formulas from (3), Section 9) define such modular operad, which I denote also by \( S \) as in [3].

Consider the endomorphism modular operad \( \mathcal{E}[\text{End}_k(U)] \), associated with the vector space \( \text{End}_k(U) \), \( \dim_k U = (N|N) \), equipped with the even scalar product defined by the super trace Equation (9). I have

\[
\mathcal{E}[\text{End}_k(U)]((n)) = \text{End}_k(U)^{\otimes n}
\]

and contractions along graphs are defined via contractions with the two-tensor corresponding to the super trace. The structure maps of \( \mathcal{E}[\text{End}_k(U)] \) are invariant under the \( GL(U) \)-action. Consider the \( GL(U) \)-invariant modular suboperad \( \mathcal{E}[\text{End}_k(U)]^{GL(U)} \). Because of Equation (3) its components for \( n < N \) are the same as the components of the operad \( S((n)) = k[S_n] \)

\[
\mathcal{E}[\text{End}_k(U)]((n))^{GL(U)} \cong S((n)).
\]

For the space \( U' = U \oplus k^{|I|} \), the natural maps \( \mathcal{E}[\text{End}_k(U)]((n))^{GL(U)} \to \mathcal{E}[\text{End}_k(U')](\mathcal{I})(n))^{GL(U')} \) are isomorphisms for small \( n < N \). Consider the modular operad \( \mathcal{E}[\text{End}_k]^{GL} \), which is the direct limit of \( \mathcal{E}[\text{End}_k(U_i)]^{GL(U_i)} \), \( \dim_k U_i = (N_i|N_i) \), \( N_i \to \infty \):

\[
\mathcal{E}[\text{End}_k]^{GL} = \lim_{\to} \mathcal{E}[\text{End}_k(U_i)]^{GL(U_i)}.
\]

Recall, see ([3], Section 9), that the basic contraction operators

\[
\mu_{j,j'}^S : S((I \sqcup \{ f, f' \})) \to S((I))
\]

are defined for the modular operad \( S \) as the linear maps

\[
k[\text{Aut}(I \sqcup \{ f, f' \})] \to k[\text{Aut}(I)].
\]
which act on permutations of the set \( I \cup \{ f, f' \} \) via

\[
(\rho_1 \ldots \rho_{p-1} f \rho_{p+1} \ldots \rho_r) \ldots (\tau_1 \ldots \tau_{q-1} f' \tau_{q+1} \ldots \tau_t) \\
\rightarrow (\rho_1 \ldots \rho_{p-1} \tau_{q+1} \ldots \tau_t \tau_1 \ldots \tau_{q-1} \rho_{p+1} \ldots \rho_r) \ldots ,
\]

if the elements \( f \) and \( f' \) are in the different cycles of the permutation, and via

\[
(\rho_1 \ldots \rho_{p-1} f \rho_{p+1} \ldots \rho_{q-1} f' \rho_{q+1} \ldots \rho_r) \ldots (\tau_1 \ldots \tau_t) \\
\rightarrow (\rho_1 \ldots \rho_{q-1} f' \rho_{q+1} \ldots \rho_r) (\rho_{p+1} \ldots \rho_{q-1}) (\tau_1 \ldots \tau_t),
\]

(15)

\[
(\rho_1 \ldots \rho_{p-1} f f' \rho_{p+1} \ldots \rho_r) \ldots (\tau_1 \ldots \tau_t) \rightarrow 0
\]

(16)

if the elements \( f \) and \( f' \) are in the same cycle of the permutation.

**Proposition 10.** The modular operad \( S \) is isomorphic to the modular operad \( \mathcal{E}[\text{End}_k]^{GL} \).

*Proof.* The calculations are very similar to the calculations from the proof of the Theorem 4. In particular the condition Equation (12) implies Equation (16). \( \square \)

4. Even Scalar Product

In the case of the even scalar product, the quantum master equation of the noncommutative Batalin–Vilkovisky geometry is defined on the space

\[
F = \text{Symm}(\bigoplus_{j=0}^{\infty} \Pi((\Pi V)^{\otimes j})^{\mathbb{Z}/2\mathbb{Z}})
\]

with components

\[
F_n = ((\Pi V)^{\otimes n} \otimes k[S_n])^{S_n},
\]

where \( k[S_n]' \) is the vector space with the basis indexed by elements \( (\sigma, \rho_\sigma) \),

where \( \sigma \in S_n \) is a permutation with \( i_\sigma \) cycles \( \sigma_\alpha \) and \( \rho_\sigma = \sigma_1 \wedge \ldots \wedge \sigma_{i_\sigma} \), \( \rho_\sigma \in \text{Det}(\text{Cycle}(\sigma)) \), \( \text{Det}(\text{Cycle}(\sigma)) = \text{Symm}^{|\sigma|} (k^{0|\sigma}) \), is one of the generators of the one-dimensional determinant of the set of cycles of \( \sigma \), i.e., \( \rho_\sigma \) is an order on the set of cycles defined up to even reordering, and \( (\sigma, -\rho_\sigma) = -(\sigma, \rho_\sigma) \).

If an even scalar product is fixed on the space \( V \), then \( F \) has canonical differential \( \Delta \), and it defines the Batalin–Vilkovisky algebra structure on \( F \), see [3,4].

Consider again the \( \mathbb{Z}/2\mathbb{Z} \)-graded vector space \( U \), \( \dim U_0 = \dim U_1 = N \). Let \( p \), \( p^2 = 1 \), denote an odd involution acting on \( U \). It acts by interchanging isomorphically \( U_0 \) with \( U_1 \). The Bernstein–Leites algebra is the subalgebra of \( \text{End}(U) \) of operators commuting with \( p \):

\[
q(U) = \{ G \in \text{End}(U) | [G, p] = 0 \}.
\]
It looks as follows in the standard block decomposition of supermatrices:

\[
G = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}
\]

in the base in which \( p = \begin{pmatrix} 0 & 1_N \\ 1_N & 0 \end{pmatrix} \). As a \( \mathbb{Z}/2\mathbb{Z} \)-graded vector space \( q(U) \) is isomorphic to \( \Pi T \text{End}(U_0) \):

\[
q(U) = \text{End}(U_0) \oplus \Pi \text{End}(U_0).
\]

The algebra structure is different, however, from the standard multiplication on \( \text{End}(U_0) \otimes k[\xi]/\{\xi^2 = 0\} \). The algebra \( q(U) \) is isomorphic to the tensor product of \( \text{End}(U_0) \) with the Clifford algebra \( Cl(1) \):

\[
q(U) = \text{End}(U_0) \otimes Cl(1), Cl(1) = k[\xi]/\{\xi^2 = 1\}.
\]

The property of \( q(U) \) of the main interest here is that \( q(U) \) has an odd analog of the super trace functional:

\[
\text{otr}(G) = \frac{1}{2} \text{tr}(Gp) = (-1)^G \frac{1}{2} \text{tr}(pG) = \text{tr}Y,
\]

\[
\text{otr}([G, G']) = 0
\]

which gives a canonical odd invariant scalar product on \( q(U) \):

\[
(G, G') \mapsto \text{otr}(GG') = \text{tr}(XY') + \text{tr}(YX').
\]

This odd scalar product on \( q(U) \) together with an even scalar product on \( V \) defines the natural odd symmetric scalar product on the tensor product \( \mathbb{Z}/2\mathbb{Z} \)-graded vector space

\[
\text{Hom}(q(U), \Pi V).
\]

Therefore, as in the previous case, this space is an affine space with constant odd symplectic structure and, therefore, its algebra of symmetric tensors

\[
\bigoplus_{n=0}^{\infty} S^n \text{Hom}(q(U), \Pi V)
\]

has the natural structure of Batalin–Vilkovisky algebra.

The subgroup \( GQ(U) \subset GL(U) \), preserving the odd involution \( p \):

\[
GQ(U) = \{ g \in GL(U) | gp g^{-1} = p \}.
\]

is the super group, which acts on the Lie algebra \( q(U) \) via the adjoint representation. The supergroup \( GQ \) is an odd analog of the general linear group \( GL \). In order to describe the quantum master equation of noncommutative Batalin–Vilkovisky geometry in the even scalar product case, I’ve introduced in [3] the twisted group algebras \( k[S_n]' \). The next proposition shows that, as it follows from
the results of [14], taking invariants in the tensor powers of the coadjoint representation of \( q(U) \) gives precisely this twisted group algebra \( k[S_n]' \), in complete analogy with the result for \( gl(U) \). The description of \( q(U) \)-invariants in terms of products of odd traces based on this seems to be new.

**PROPOSITION 11.**

\[
\text{Hom}(q(U)^\otimes n, k)^{GQ(U)} = k[S_n]' \tag{18}
\]

for \( \dim U = (N|N) \) sufficiently big (\( N > n \)). The \( GQ(U) \)-invariants in \( \text{Hom}(q(U)^\otimes n, k) \) are spanned linearly by the products of odd traces:

\[
\text{otr}(A_{\rho_1} \ldots A_{\rho_r}) \cdot \ldots \cdot \text{otr}(A_{\tau_1} \ldots A_{\tau_t}), \quad A_i \in q(U).
\]

**Proof.** See [14] for the proof of the first statement. Here is a sketch of other arguments. The basis for invariants of the \( GQ(U) \) action on \( \text{Hom}(\text{End}(U), k) \) is \( \text{tr}(G) = \sum_a e_a \otimes e^a \) and \( \text{tr}(pG) = \sum_a (pe_a) \otimes e^a \). Therefore, on the space of tensors \( \text{Hom}(\text{End}(U), k)^{\otimes n} \), the space of \( GQ(U) \)-invariants is spanned by all possible combinations of these two elements of the form

\[
\sum_{a_1, \ldots, a_n} (e_{\sigma(1)} \otimes e^{a_1}) \otimes \ldots \otimes (pe_{\sigma(i)} \otimes e^{a_i}) \otimes \ldots,
\]

where \( \sigma \in S_n \). Such element corresponds to an arbitrary permutation \( \sigma \in S_n \) and the marking, which associates one of the two types of tensors \( (e_{\sigma(i)} \otimes e^{a_i}) \) or \( (pe_{\sigma(i)} \otimes e^{a_i}) \) to every \( i \in \{1, \ldots, n\} \). If the cycle decomposition of \( \sigma \) is denoted by \( (\rho_1 \ldots \rho_r) \ldots (\tau_1 \ldots \tau_t) \), then such an element gives the linear functional on \( \text{End}(U)^{\otimes n} \) of the following type:

\[
\text{tr}(A_{\rho_1} \ldots A_{\rho_{i-1}} pA_{\rho_i} \ldots A_{\rho_r}) \cdot \ldots \cdot \text{tr}(A_{\tau_1} \ldots A_{\tau_{j-1}} pA_{\tau_j} \ldots A_{\tau_t})
\]

consisting of products of traces of compositions of the endomorphisms with arbitrary inclusions of the operator \( p \). The subspace \( q(U) \subset \text{End}(U) \) has complementary subspace, preserved by \( GQ(U) \), which consists of endomorphisms anticommuting with \( p \). Therefore, the restriction map from \( \text{Hom}(\text{End}(U)^{\otimes n}, k)^{GQ(U)} \) to \( \text{Hom}(q(U)^{\otimes n}, k)^{GQ(U)} \) is onto it. The operator \( p \) commutes with any \( A_{\rho_j} \in q(U) \). Therefore, on \( q(U)^{\otimes n} \) all inclusions of \( p \) inside the given trace cancel with each other, except for possibly one inclusion:

\[
\text{tr}(A_{\rho_1} \ldots A_{\rho_{i-1}} pA_{\rho_i} \ldots A_{\rho_r}) = \text{tr}(p^l A_{\rho_1} \ldots A_{\rho_r}),
\]

where \( l = 0 \) or \( l = 1 \) depending on the parity of the total number of inclusions of \( p \). Notice now that for any \( A \in q(U) \), \( \text{tr}(A) = 0 \). Therefore, the traces with even number of inclusions of \( p \) vanish on \( q(U)^{\otimes n} \). The trace with odd number of inclusions of \( p \) becomes the odd trace \( \text{otr} \) when restricted to \( q(U) \). Therefore, the \( GQ(U) \)-invariants in \( \text{Hom}(q(U)^{\otimes n}, k) \) are spanned by the products of odd traces:

\[
\text{otr}(A_{\rho_1} \ldots A_{\rho_r}) \cdot \ldots \cdot \text{otr}(A_{\tau_1} \ldots A_{\tau_t}), \quad A_i \in q(U).
\]
One can also deduce from the corresponding result for $gl$, that these products of odd traces are linearly independent for $N \geq n$.

The super group $GQ(U)$ preserves the odd trace $otr$ and; therefore, the invariants subspace $\bigoplus_{n=1}^{\infty} (S^n \text{Hom}(q(U), \Pi V))^{GQ(U)}$ inherits the natural Batalin–Vilkovisky algebra structure. I have now the following analogs of the Propositions 1, 5 and of the Theorem 4. The proofs are completely analogous to the proofs in the odd scalar product case.

**Proposition 12.** The vector space $F_n$ is canonically identified with $GQ(U)$-invariant subspace of $n$-th symmetric powers of the vector space $q(U) \otimes V$:

$$F_n \simeq \left(S^n \text{Hom}(q(U), \Pi V)\right)^{GQ(U)},$$

where $q(U) \subset GL(U)$ is the odd general linear algebra and $\dim_k U = N$, $N \geq n$.

Identify the symmetric algebra generated by $\text{Hom}(q(U), \Pi V)$ with polynomial functions on $\text{Hom}(\Pi V, q(U))$. For an element $a_i \in \Pi V$, let $A_i$ denote the corresponding $q(U)$-valued linear function on $\text{Hom}(\Pi V, q(U))$.

**Proposition 13.** The isomorphism Equation (19) sends $(a_{\rho_1} \ldots a_{\rho_r})^c \wedge \ldots \wedge (a_{\tau_1} \ldots a_{\tau_t})^c$ to the product of odd traces

$$otr(A_{\rho_1} \ldots A_{\rho_r}) \ldots \cdot otr(A_{\tau_1} \ldots A_{\tau_t}).$$

**Theorem 14.** The operator $\Delta$ defined on the $GQ(U)$-invariant subspace

$$\Delta : \left(S^n \text{Hom}(q(U), \Pi V)\right)^{GQ(U)} \rightarrow \left(S^{n-2} \text{Hom}(q(U), \Pi V)\right)^{GQ(U)},$$

where $\dim_k U = (N|N)$ is sufficiently big ($N \geq n$), coincides with the differential $\Delta : F_n \rightarrow F_{n-2}$ from [3,4].

**Proposition 15.** The odd symplectic bracket on the $GQ(U)$-invariant subspaces

$$\left(S^n \text{Hom}(q(U), \Pi V)\right)^{GQ(U)} \otimes \left(S^n' \text{Hom}(q(U), \Pi V)\right)^{GQ(U)}$$

$$\rightarrow S^{n+n'-2} \text{Hom}(q(U), \Pi V)^{GQ(U)}$$

coincides with the standard odd symplectic bracket ([3,4]):

$$F_n \otimes F_{n'} \rightarrow F_{n+n'-2}.$$

Next, in order to calculate the cohomology of $\Delta$ on $S(\text{Hom}(q(U), \Pi V))$, I identify it with the de Rham complex of the affine space $(q(U) \otimes \Pi V)_0$. For $U$ with a fixed basis $U \simeq k^N$, I use the standard notation $q(N)$. 

PROPOSITION 16. The matrix Batalin–Vilkovisky complex \( (S(\text{Hom}(q(N), \Pi V)), \Delta) \) is naturally isomorphic to the de Rham complex of the affine space \((q(N) \otimes \Pi V)_0\)

Proof. As above I consider \( S(\text{Hom}(q(N), \Pi V)) \) as the algebra of polynomial functions on the affine space \( \text{Hom}(\Pi V, q(N)) \). Let \( v_j \in \Pi V \) be a basis and let \((X^\beta_{j,\alpha}, Y^\gamma_{j,\alpha})\) denote the corresponding natural basis of linear functions on \( \text{Hom}(\Pi V, q(N)) \). Denote by \( \tilde{v}_j \) the dual basis satisfying \( l(\pi y_j, \pi \tilde{y}_j) = \delta_{jj'} \) and via \((\tilde{X}^\beta_{j,\alpha}, \tilde{Y}^\gamma_{j,\alpha})\) the corresponding basis of linear functions on \( \text{Hom}(\Pi V, q(N)) \). The standard isomorphism between polyvector fields and forms sends in this case the product of a set of odd elements \( X^\beta_{j,\alpha}, \tilde{y}^\gamma_j, \tilde{y}^\gamma_{j',\alpha} \) to the differential form

\[
\prod X^\beta_{j,\alpha} \prod Y^\gamma_{k,\varepsilon} \leftrightarrow \left( \prod i \left( \frac{\partial}{\partial y^\alpha_{j,\beta}} \right) \prod i \left( \frac{\partial}{\partial \tilde{X}^\gamma_{k,\epsilon}} \right) \right) \Omega,
\]

where

\[
\Omega = \prod_{j=0,\alpha,\beta} d(X^\beta_{j,\alpha}) \prod_{j=1,\alpha,\beta} d(Y^\gamma_{j,\alpha})
\]

is the constant volume form on \((q(N) \otimes \Pi V)_0\) and \( i(v) \) is the standard contraction with the vector field \( v \).

THEOREM 17. The cohomology of the Batalin–Vilkovisky differential acting on \( F \) is trivial: \( H^*(\oplus_{n=0}^{\infty} F_n, \Delta) = 0 \).

Proof. The proof is parallel to the case of the odd scalar product above and follows from the identification

\[
F_n \simeq \left( \bigoplus_{i=0}^n \Omega^r_{(q(N)\otimes \Pi V)_0} \right)^{GQ(N)},
\]

\[
d_{\text{DR}} : \Omega^{j,l}_{(q(N)\otimes \Pi V)_0} \to \Omega^{j+1,l-1}_{(q(N)\otimes \Pi V)_0},
\]

where \( \Omega^{j,l}_{(q(N)\otimes \Pi V)_0} \) is the de Rham complex with the extra grading by the polynomial degree, \( N > n + 2 \) and \( r = \dim_k V \). The cohomology of the de Rham differential is trivial on every bi-graded piece \( \Omega^r_{(q(N)\otimes \Pi V)_0} \) and the standard arguments, see, e.g., [11,13], imply that the cohomology is concentrated on the \( GQ(N) \)-invariant subspace. It follows that \( \ker \Delta|_{F_n} = \text{im} \Delta|_{F_{n+2}} \).

A version of the Morita equivalence holds also.

THEOREM 18. Let

\[
(\tilde{V}, \tilde{l}) = (V, l) \otimes (gl(k|k'), tr)
\]
and let \( M : F(V) \to F(\tilde{V}) \) be the map of the commutative algebras defined on generators by

\[
(a_{\rho_1}, \ldots, a_{\rho_r})^c \mapsto \text{tr}(A_{\rho_1}, \ldots, A_{\rho_r})^c.
\]

Then for any solution \( S \) to the noncommutative Batalin–Vilkovisky equation in \( F(V) \), \( M(S) \) is a solution in \( F(\tilde{V}) \). The class of such solutions is characterized by the extra \( gl(k|k') \)-symmetry.

**Proof.** This follows from the isomorphism of vector spaces with scalar products

\[
(q(N), \text{otr}) \otimes (gl(k|k'), \text{tr}) \simeq (q(\tilde{N}), \text{otr}),
\]

where \( \tilde{N} = N(k+k') \), which follows directly from the definition of \( q(N) \) and \( \text{otr} \). Therefore,

\[
quotient{q(\tilde{N}) \otimes \Pi V}{\text{Pi1}} \simeq \quotient{q(N) \otimes \Pi \tilde{V}}{\text{Pi1}}
\]

as affine spaces with constant BV structures. The \( GQ(\tilde{N}) \)-invariant function \( \mu_{F(V)}(S) \) on \( q(\tilde{N}) \otimes \Pi V \), coincides by definition with \( GQ(N) \)-invariant function on \( q(N) \otimes \Pi \tilde{V} \) corresponding to \( \mu_{F(\tilde{V})}M(S) \). Hence, \( \mu_{F(\tilde{V})}M(S) \), and therefore \( M(S) \), both satisfy the Batalin–Vilkovisky quantum master equations.

\[\square\]

4.1. SUPER MORITA EQUIVALENCE

It is remarkable that thanks to the supersymmetry there exists a superversion of Morita equivalence.

**THEOREM 19.** Let \((V, l)\) be a vector space with odd (even) scalar product, define the vector space with even (respectively,odd) scalar product

\[
(\tilde{V}, \tilde{l}) = (V, l) \otimes (q(1), \text{otr})
\]

and let \( M : F(V) \to F(\tilde{V}) \) be the map of the commutative algebras defined on generators by

\[
(a_{\rho_1}, \ldots, a_{\rho_r})^c \mapsto \text{otr}(A_{\rho_1}, \ldots, A_{\rho_r})^c.
\]

Then for any solution \( S \) to the noncommutative Batalin–Vilkovisky equation in \( F(V) \), \( M(S) \) is a solution in \( F(\tilde{V}) \). These solutions have extra \( GQ(1) \)-symmetry.

**Proof.** Let \( (V, l) \) be a vector space with odd scalar product, the other case is analogous. The statement follows from the isomorphism of vector spaces with scalar products

\[
(q(N), \text{otr}) \otimes (q(1), \text{otr}) \simeq (gl(N|N), \text{tr})
\]
which follows from the definition of $q(N)$ and otr. Therefore,

$$gl(N|N) \otimes \Pi V \simeq q(N) \otimes \Pi \tilde{V}$$

are as affine spaces with constant BV structures. The $GL(N|N)$-invariant function $\mu_{F(V)}(S)$ on $gl(N|N) \otimes \Pi V$, coincides by definition with $GQ(N)$-invariant function $\mu_{F(\tilde{V})}M(S)$ after identification Equation (21). Hence, $\mu_{F(\tilde{V})}M(S)$, and therefore $M(S)$, both satisfy the Batalin–Vilkovisky quantum master equations.

4.2. TWISTED MODULAR OPERAD ON $k[\mathbb{S}_n]'$

Using Equation (18) it is straightforward to prove the following statement

**PROPOSITION 20.** The twisted modular operad structure on the collection of spaces $\{k[\mathbb{S}_n]’\}$ described in [3] is isomorphic to the stable part, as $N \to \infty$, of the $GQ$-invariants suboperad of the standard tensor twisted modular operad, based on the vector space with odd scalar product $(q(N), otr)$

$$\mathcal{E}[q(N)]((n)) = q(N)^{\otimes n}$$

with contractions along graphs defined via contractions with the two-tensor corresponding to the odd trace.

5. Equivariant Differential and Localization

In the previous sections I have identified the non-commutative Batalin–Vilkovisky operator $\Delta$, acting on the basic spaces $F$, with the de Rham differential acting on $GL(N|N)$ and $GQ(N)$ invariant subspaces in the de Rham complexes of affine spaces. The invariant subspaces of de Rham complexes appear also in the definition of the equivariant differential. So it is natural to look for the analogue of the remaining part of the equivariant differential acting on $F$. I’ll treat both even and odd cases simultaneously in this section and the next sections. Denote by $g$ and $G$ in the odd scalar product case the super Lie algebra $gl(N|N)$ and $GL(N|N)$, and in the even scalar product case the super Lie algebra $q(N)$ and the super group $GQ(N)$. Denote by $\langle \cdot, \cdot \rangle$ the odd symplectic structure on the affine space $g \otimes \Pi V$.

**PROPOSITION 21.** The adjoint action of the super Lie algebra $g$ on the vector space $g \otimes \Pi V$ preserves the odd symplectic structure. The Hamiltonian of the linear vector field corresponding to $\gamma \in g$, is the quadratic function

$$S_{2, \gamma} = \langle [\gamma, X], X \rangle.$$  \hspace{1cm} (22)

It satisfies

$$\Delta S_{2, \gamma} = 0.$$  \hspace{1cm} (23)
**Proof.** The first equation is the standard formula of symplectic geometry and follows from the definition of the Hamiltonian:

\[ dS_{2,Y}(Y) = (i_{\text{ad}(\gamma)} \langle \cdot, \cdot \rangle)(Y) \]

for any vector field \( Y \). The second formula is equivalent to

\[ \text{trace}(\text{ad}(\gamma)|_g) = 0 \]

satisfied for any \( \gamma \) from the algebras \( gl(N|N) \) and \( q(N) \).

\[\square\]

**PROPOSITION 22.** The Lie derivative by the linear vector fields \( L_{\text{ad}(\gamma)} \) satisfies on \( \mathcal{O}(g \otimes \Pi V) \) the Cartan homotopy formula

\[ L_{\text{ad}(\gamma)} = [\Delta, S_{2,Y}], \]

where slightly abusing notation, I denote here by \( S_{2,Y} \) the multiplication by the quadratic Hamiltonian Equation (22).

**Proof.** This is immediate from Equation (23) and the basic formula of Batalin–Vilkovisky geometry, expressing the bracket via the action of \( \Delta \).

\[\square\]

**PROPOSITION 23.** The equivariant differential \( d_{DR} + i_{\gamma} \) of the action of \( G_0 \) on \( \Omega_{DR}((g \otimes \Pi V)_0) \) corresponds to the differential on \( \mathcal{O}(g \otimes \Pi V) \):

\[ \Delta + S_{2,Y}. \]

In the next proposition I show that promoting the standard action of \( g_0 \) on \( \Omega_{DR}((g \otimes \Pi V)_0) \) to the action of the super Lie algebra \( g \) gives a natural construction of equivariantly closed differential forms for the action of one-parameter subgroups of \( G_0 \) generated by elements of the form \( [\gamma, \gamma] \), \( \gamma \in g_1 \).

**PROPOSITION 24.** For any \( G \)-invariant function \( \Psi \in \mathcal{O}(g \otimes \Pi V)^G \) the formula

\[ \gamma \in g \to \exp(S_{2,Y})\Psi \]

defines a \( G \)-invariant element from \( (\mathcal{O}(g) \otimes \mathcal{O}(g \otimes \Pi V))^G \), which corresponds under the odd Fourier transform to the equivariant differential form from \( (\mathcal{O}(g) \otimes \Omega_{DR}((g \otimes \Pi V)_0))^G \). If \( \Psi \) is \( \Delta \)-closed, then this element is closed under the equivariant differential

\[ (\Delta + S_{2,\zeta})[\exp(S_{2,Y})\Psi] = 0, \]

where \( \zeta = \frac{1}{2}[\gamma, \gamma] \).
Proof. By proposition 22 for any $G$-invariant function

$$[\Delta, S_{2,\gamma}]\Psi = 0.$$ 

Now the proof follows from the standard Cartan calculus formula:

$$\Delta[\exp(S_{2,\gamma})] = \exp(S_{2,\gamma})\left(\Delta + [\Delta, S_{2,\gamma}] + \frac{1}{2}S_{2,\gamma,\gamma}\right).$$

\[\square\]

**COROLLARY 25.** The Lagrangians of the matrix integrals

$$\int \mathcal{L} \exp \left( \frac{1}{\hbar} \left( -\frac{1}{2} \langle [\Xi, X], X \rangle + S_q(X) \right) \right) \, dX$$

and

$$\int \mathcal{L} \exp \left( \frac{1}{\hbar} \left( -\frac{1}{2} \langle [\Xi, X], X \rangle + S_{gl}(X) \right) \right) \, dX$$

constructed in [4], represent, after the odd Fourier transforms Equations (13, 20), equivariant differential forms from $\Omega_{DR}((g \otimes \Pi V)_0)$. These forms are closed under the equivariant differential $(\Delta + S_{2,\Xi_2}) = d_{DR} + i_{\Xi_2,\cdot}.$

**EXAMPLE 26.** In the case of the matrix integral Equation (25), with $\Xi = \begin{pmatrix} 0 & 1 & d \\ \Lambda_{01} & 0 \end{pmatrix}$, $\Lambda_{01} \in gl(N)$, this gives an equivariant differential form from $\Omega_{DR}((g \otimes \Pi V)_0)$ with respect to the block-diagonal action $\begin{pmatrix} \Lambda_{01} & 0 \\ 0 & \Lambda_{01} \end{pmatrix}$ of $gl(N) \subset gl(N|N)$.

Remark 27. The localization in equivariant cohomology reduces the integrals from Equations (24, 25) over $S^1$-invariant relative cycles from $H_{dR}^\xi(L, \text{Re} \frac{1}{\hbar}(S_2 + S) < -r)$, $r \to +\infty$, where $L$ is the $[\Xi_2, \cdot]$-invariant subspace of $(g \otimes \Pi V)_0$ and $S^1$ is the compact subgroup of the group generated by $[\Xi_2, \cdot]$-action, to the integrals over $[\Xi_2, \cdot]$-fixed points. The details of the computation will appear in [7].

**6. Noncommutative AKSZ Formalism**

The relation of nc-BV differential $\Delta$ with invariant integration with respect to the supergroups $GQ(N)$ and $GL(N|N)$, gives an interpretation to the Lagrangians from [4] as non-commutative super-equivariant analogues of the AKSZ $\sigma$-model. This and other non-commutative analogues of some standard Lagrangians are studied in [6] and [7].
I consider first the even scalar product case, the case of the odd scalar product is parallel. The initial step is to interpret the space $q(N) \otimes \Pi V$ as the space of morphisms

$$\text{Free}(\Pi V)_{\text{dual}} \to q(N)$$

from the free associative algebra generated by $(\Pi V)_{\text{dual}}$. This space can be interpreted as the functor of points of $\text{Spec}(\text{Free}(\Pi V)_{\text{dual}})$ over simple associative super algebra $q(N)$.

$$\text{Spec}(q(N)) \to \text{Spec}(\text{Free}(\Pi V)_{\text{dual}})$$

Next, interpret the even scalar product on $V$ as an even symplectic 2-form on $\text{Spec}(\text{Free}(\Pi V)_{\text{dual}})$, and interpret the odd trace $otr$ on $q(N)$ as a kind of integral with respect to the odd volume element. Therefore their tensor product, defining the odd symplectic structure on $q(N) \otimes \Pi V$, is the analogue of the odd symplectic structure on space of maps $f:\Sigma \to X$:

$$(u, v)_f = \int_{\Sigma} \omega_f(y)(u(y), v(y))dY,$$

where $u, v \in T_f\text{Maps}(\Sigma, X)$. Now, notice that the supergroup $GQ(N)$ acts on the space of morphisms $\text{Mor}(\text{Free}(\Pi V)_{\text{dual}}, q(N))$, preserving the odd symplectic structure. The Hamiltonians of the corresponding vector fields are the quadratic functions $S_2, \gamma$. Equation (22). Solution $S$ to the non-commutative BV-equation gives $GQ(N)$-invariant function $\mu_f(S)$ [denoted by $S_q$ in Equation (25)] on

$$\text{Mor}(\text{Free}(\Pi V)_{\text{dual}}, q(N)),$$

which corresponds also to a Hamiltonian vector field. Their sum $S_q + S_{2, \gamma}$ satisfies the equivariant quantum master equation

$$\hbar \Delta(S_q + S_{2, \gamma}) + \frac{1}{2}[S_q + S_{2, \gamma}, S_q + S_{2, \gamma}] + S_{2, \frac{1}{2}[\gamma, \gamma]} = 0.$$

Derivations of $\text{Free}(\Pi V)_{\text{dual}}$, preserving the even scalar product on $V$, correspond to cyclic Hochschild co-chains. They also act on $\text{Mor}(\text{Free}(\Pi V)_{\text{dual}}, q_N)$, preserving the odd symplectic structure and commuting with the $GQ(N)$ supergroup action. Derivations preserving $S_q$ correspond to closed cyclic co-chains. Their Hamiltonians can also be added to the Lagrangian and the resulting integrals depend in addition on the extra parameters given by the cyclic cohomology classes, which can also be viewed as natural observables of the theory.

**Acknowledgements**

I would like to thank the referees for useful comments.
References

1. Alexandrov, M., Schwarz, A., Zaboronsky, O., Kontsevich, M.: The geometry of the master equation and topological quantum field theory. Intern. J. Mod. Phys. A 12(7), 1405–1429 (1997)
2. Barannikov, S.: Quantum periods—I. Semi-infinite variations of Hodge structures. Preprint ENS DMA-00-19. Intern. Math. Res. Notices 23 (2001)
3. Barannikov, S.: Modular operads and non-commutative Batalin–Vilkovisky geometry. IMRN, vol. 2007, rnm075. Preprint (2007). Max Planck Institute for Mathematics 2006–48 (25/04/2006)
4. Barannikov, S.: Noncommutative Batalin–Vilkovisky geometry and matrix integrals. Preprint NI06043 (2006). Isaac Newton Institute for Mathematical Sciences, Cambridge University. Preprint hal-00102085 (09/2006). Comptes Rendus Mathématique, of the French Academy of Sciences, submitted on May 17 2009, vol. 348, issue no. 7–8, pp. 359–362
5. Barannikov, S.: Supersymmetry and cohomology of graph complexes. Preprint hal-00429963 (2009)
6. Barannikov, S.: Supersymmetric matrix integrals and sigma-model. Preprint hal-00443592 (2009)
7. Barannikov, S.: (2013)
8. Bernstein, J.N., Leites, D.A.: The superalgebra Q(n), the odd trace, and the odd determinant. Dokl. Bolg. Akad. Nauk 35(3), 285–286 (1982)
9. Feigin, B., Tsygan, B.: Additive K-theory. Springer, LNM, vol. 1289, pp. 97–209 (1987)
10. Getzler, E., Kapranov, M.: Modular operads. Compos. Math. 110(1), 65–126 (1998)
11. Goodman, R., Wallach, N.R.: Representations and invariants of the classical groups. Cambridge University Press, Cambridge (1998)
12. Kontsevich, M.: Formal (non)commutative symplectic geometry. The Gelfand Mathematical Seminars, vol. 1990–1992, pp. 173–187. Birkhauser, Boston (1993)
13. Loday, J.-L.: Cyclic homology. Springer, Berlin (1992)
14. Sergeev, A.: Tensornaya algebra tozhdestvennogo predstavleniya kak modul nad superalgebrami Lie gl(n, m) i q(n). Mat. Sbornik 123(165) v.3, 422–430 (1984)