The Physics of Instantons
in the Pseudoscalar and Vector Meson Mixing

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Abstract

When the theory of Quantum Chromodynamics was introduced, it was to explain the observed phenomena of quark confinement and scaling. It was then discovered that the emergence of instantons is an essential consequence of this theory. This led to unanticipated explanations not only for the anomalously high masses of the \( \eta \) and the \( \eta' \) particles, but also for the remarkable differences that have been observed in the mixing angles for the pseudoscalar mesons and the vector mesons.
1. Introduction.

The discovery of “The Eightfold Way”\(^1\) in 1961 implied that all observed mesons could be placed in 8 or 1 representations of the group $SU(3)$, which later became\(^2\) the flavor group $SU(3)^{\text{flavor}}$, with the quarks $u$, $d$ and $s$ forming the fundamental 3 representation. It was clear, however, that $SU(3)^{\text{flavor}}$ is broken, and consequently, mixing should take place between the eighth members of the octets and singlet states. Later in the sixties, this became a hot topic, when it appeared that this mixing for the pseudoscalar mesons is very different from what happens with the vector mesons. In terms of their quark components, we write the mesonic wave functions $|\phi_1\rangle$ and $|\phi_2\rangle$ as

$$|\phi_1\rangle = \cos \theta \left( \frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{\sqrt{6}} \right) + \sin \theta \left( \frac{u\bar{u} + d\bar{d} + s\bar{s}}{\sqrt{3}} \right),$$

$$|\phi_2\rangle = -\sin \theta \left( \frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{\sqrt{6}} \right) + \cos \theta \left( \frac{u\bar{u} + d\bar{d} + s\bar{s}}{\sqrt{3}} \right),$$

(1.1)

Great experimental efforts went into precisely determining these mixing angles. Indeed, an experimental set-up, especially designed to study the radiative decays of vector and pseudoscalar mesons, was designed and built at CERN\(^3\). Not only the $(\omega - \phi)$ and the $(\eta' - \omega)$ mixings\(^5\) were determined but also, by measuring its $2\gamma$ decay\(^6\), a meson that at that time was called the $X^0$, could be identified as being the ninth pseudoscalar meson; it was renamed $\eta'$. The outcome of these measurements was indeed remarkable. In the pseudoscalar case, the mixing angle turned out to be

$$\theta_{PS} \approx 10^\circ,$$

(1.2)

whereas the vector mesons mix with an angle

$$\theta_V \approx 51^\circ.$$

(1.3)

This gives

$$J^{PC} = 0^{-+} \left\{ \begin{array}{l}
\eta(\approx 550 \text{ MeV}) \approx .50 (u\bar{u} + d\bar{d}) - .70 (s\bar{s}), \\
\eta'(\approx 960 \text{ MeV}) \approx .49 (u\bar{u} + d\bar{d}) + .71 (s\bar{s}),
\end{array} \right.$$  

(1.4)

and

$$J^{PC} = 1^{--} \left\{ \begin{array}{l}
\omega(\approx 780 \text{ MeV}) \approx .71 (u\bar{u} + d\bar{d}) - .06 (s\bar{s}), \\
\phi(\approx 1020 \text{ MeV}) \approx .04 (u\bar{u} + d\bar{d}) + 1.00 (s\bar{s}),
\end{array} \right.$$  

(1.5)

so we see that $\eta$ and $\eta'$ are divided to a large extent as dictated by $SU(3)$, whereas $\omega$ and $\phi$ divide mainly in accordance with their quark composition.

These values for the mixing angles cannot be accidental, but should be explained. Many attempts were made to obtain some insights. Then, in the early seventies, it was
realized that meson dynamics can be understood as being described by a non-Abelian gauge theory with gauge group \( SU(3)_c \) (the “color group”), and fermions in the fundamental 3 representation of this group (the “quarks”). Since the input parameters of this theory, now called “Quantum Chromodynamics” (QCD), appeared to consist only of the color gauge coupling parameter \( g \) (or, equivalently, the parameter \( \Lambda_{QCD} \)) and the quark masses \( m_f \), the mixing angles should, in principle, be predictable. This, however, only added to the mystery: why are pseudoscalar mesons so much different from the vectors?

Actually, there were other, even more distressing problems associated to the pseudoscalars. The successes of low-energy current algebra considerations such as CVC (the conserved isovector vector current)\(^8\) and PCAC (the partially conserved isovector axial vector current)\(^9\), strongly indicated that meson physics has an approximate chiral \( SU(2) \otimes SU(2) \) symmetry. The pions, with their anomalously low mass values, can then be regarded as being the Goldstone bosons associated with this symmetry. It is easy to incorporate this symmetry in the QCD lagrangian, simply by postulating that the \( u \) and the \( d \) quarks must have very small mass terms here. The problem one then encounters is that, if this were the case, QCD should actually have an even larger symmetry: \( U(2) \otimes U(2) \), which differs from the observed symmetries by an extra chiral \( U(1) \) component, and this should be reflected in a (partially) conserved isoscalar axial vector current, \( J^A_\mu(x) \). Thus, the symmetry held responsible for the relatively small value of the pion masses, should necessarily induce another symmetry in the model that would strongly reduce the mass of yet another particle: the pions should have had a pseudoscalar partner, somewhat like the \( \eta \), but composed predominantly of \( u\bar{u} \) and \( d\bar{d} \) quarks, in the combination \( (u\bar{u} + d\bar{d})/\sqrt{2} \) (which we will refer to as the state \( \pi^0_o \)). Adding the strange quark \( s \), should then only result in having an extra pseudoscalar meson made of pure \( s\bar{s} \), and, since its mass would be constrained by the same terms that produce the kaon mass (the strange quark mass term), the \( s\bar{s} \) pseudoscalar meson could not be much heavier than the kaon. It appeared that the kaon mass times \( \sqrt{2} \), or 700 MeV, should be an upper limit.*

In the early days, it was therefore suspected that QCD requires explicit correction terms; after all, its ability to keep quarks permanently confined inside hadronic configurations was also not yet explained.\(^10\)

Surprisingly, no such correction terms are needed at all. Both confinement and the absence of the chiral \( U(1) \) symmetry can now be adequately explained as being special features of QCD alone. Both are due to special topological aspects of the system. Confinement is due to the existence of color-magnetic charges that undergo Bose condensation\(^11\), and the absence of chiral \( U(1) \) is due to instantons\(^12\).

* This result can easily be seen from the equations for the masses in the appendix, by substituting \( \kappa = 0 \) and \( M_\eta = M_\pi \) in Eqs (A.4)–(A.6), realizing that \( B > 0 \) because of (A.9) and (A.10). The limit is reached if \( F_1 \gg |F_3 - F_1| \).
2. Instantons.

The first topological structures in gauge theories were the Abrikosov vortices in superconducting material. They can be viewed as soliton solutions in 2 space-dimensions. When particle physicists\textsuperscript{13} began thinking of quarks being held together by stringlike structures, the stringlike nature of these vortices caught their attention. It was realised that all one needs is an Abelian Higgs theory\textsuperscript{14}, and the existence of such vortices is guaranteed.

When this vortex for the non-Abelian case was examined more closely, it was found to be unstable, and this implied the existence of other topologically stable objects, but in 3 rather than 2 dimensions. These 3-dimensional objects had to be magnetic monopoles\textsuperscript{15}. Bose condensation of color-magnetic monopoles is now the favored explanation of quark confinement\textsuperscript{11}.

This subsequently raised another question: are there topologically stable objects in more than 3 dimensions? What about 4 dimensions, and what would their physical interpretation be? A localised object in 4 dimensions describes an event rather than a particle, and so we devised the name “instanton” for such objects\textsuperscript{16}. The first example in gauge theory had been described by Belavin et al\textsuperscript{17} in 1975. In their paper, Minkowski space had been replaced by Euclidean space. In this space, they found localised solutions of the classical gauge field equations. This raised questions such as: what kind of events do these instantons correspond to, and why do they exist only in Euclidean space?

Euclidean spacetime is obtained upon analytic continuation of physical amplitudes for imaginary time: we replace $t$ by $i x_4$ with now $x_4$ a real coordinate. This is exactly what one needs to do if one wishes to compute a tunnelling amplitude, replacing the usual perturbation expansion by a BKW expansion. The exponential suppression factor in the amplitude is obtained by solving the classical equations with time being replaced by an imaginary parameter. Therefore, instantons are to be interpreted as tunnelling events. Indeed, their contributions to physical amplitudes are proportional to $\exp(-8\pi^2/g^2)$, an exponential suppression typical for tunnelling. But this is not all. The tunnelling event in question violates a conservation law that would be respected by any ordinary perturbative effect. Which conservation law? Belavin et al had noted in passing that their solution has

\begin{equation}
\int d^4 x F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} = \pm \frac{32\pi^2}{g^2},
\end{equation}

(2.1)

where $\tilde{F}^a_{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$, and the sign refers to instantons and anti-instantons, respectively. But, according to Adler\textsuperscript{18}, and Bell and Jackiw\textsuperscript{19},

\begin{equation}
\partial_\mu J^A_\mu = \frac{g^2}{16\pi^2} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu}.
\end{equation}

(2.2)

This is the well known axial triangle diagram anomaly. One-loop enormalisation effects cause an apparently tiny (proportional to $g^2$) violation of axial current conservation. We observe that the instanton would give rise to a transition between states with different values for the axial charge $Q_5(t) = \int d^3 x J^A_0(x, t)$:

\begin{equation}
\Delta Q_5 = Q_5(T) - Q_5(-T) = \int_{-T}^{T} dt \int d^3 x \partial_0 J^A_0(x) = \pm 2.
\end{equation}

(2.3)
Note that, although the tunnelling amplitude is computed using analytic continuation to Euclidean time, the actual event takes place in Minkowski space-time. Being topological equations, Eqs. (2.1) — (2.3) hold both in Euclidean and in Minkowski space-time, since they are independent of the metric $g_{\mu\nu}$. This is why it is permitted to use the real charge density $J_0(x,t)$ in Eq. (2.3).

According to a theorem by Adler\textsuperscript{18} and Bardeen\textsuperscript{20}, the anomaly equation (2.2) is essentially not affected by any renormalisation beyond the one-loop level. Apparently, the number 2 in Eq. (2.3) will not be affected by higher order corrections.\textsuperscript{†} The meaning of this is clear. One left-handed polarised quark (contributing +1 unit to the axial charge) is turned into a right-handed one (with $Q_5 = -1$), or vice versa. It is important to note that in a theory with $N_f$ quark flavors, Eq. (2.3) holds for each flavor separately. In total, one therefore has

$$\Delta Q_5 = \pm 2N_f. \quad (2.4)$$

There are several ways to understand, mathematically as well physically, why and how such transitions take place\textsuperscript{12}. Here we will limit ourselves to the explanation that, in a properly regularised and renormalised theory, the total number of Dirac levels for fermions, in a given volume, is precisely specified. The instanton causes exactly one such level to make a transition from positive to negative energy, or vice-versa, thus crossing the Fermi level of the vacuum. This way, one quark with one helicity may be materialised from the Dirac sea, while another, with opposite helicity, submerges into this sea. More precise and complete explanations have been given elsewhere\textsuperscript{12, 21}.

Not only instantons violate chiral $U(1)$ invariance, but also the quark mass terms. This implies that a new phase angle, $\theta_{\text{inst}}$, emerges in the description of interference between these two symmetry breaking effects. In the early days of QCD, it had not been realized that QCD possesses two fundamental parameters, the gauge coupling $g$ and the instanton angle $\theta_{\text{inst}}$. In the QCD lagrangian, the effect of this angle can be described by adding a term

$$i\theta_{\text{inst}} \cdot \frac{g^2}{32\pi^2} F_{\mu\nu} \tilde{F}_{\mu\nu}. \quad (2.5)$$

In perturbation expansion, this term seems to give no effect at all because it can be written as a pure derivative:

$$F_{\mu\nu} \tilde{F}_{\mu\nu} = \frac{16\pi^2}{g^2} \partial_\mu K_\mu = \partial_\mu (2\varepsilon_{\mu\nu\alpha\beta} A_\alpha^a (\partial_\alpha A_\beta^a + \frac{g}{3} f^{abc} A_\alpha^b A_\beta^c)). \quad (2.6)$$

$K_\mu$ is the Chern-Simons current. Because of this equation, all Feynman diagrams with this vertex in them carry a factor $\sum_{i} p_{i\mu}^{(i)}$, the sum of all external momenta, and hence they vanish. Instantons nevertheless produce non-trivial physical effects depending on $\theta_{\text{inst}}$, only because $K_\mu$ is not gauge-invariant. Since instantons are the only stable objects\textsuperscript{†} Nor is there need to worry about the fact that $g$ is a running coupling strength. In fact, the fields $F_{\mu\nu}$ in Eqs. (2.1) and (2.2) should be replaced by $\mathcal{F}_{\mu\nu} = gF_{\mu\nu}$, so that $g$ no longer appears explicitly.
with a non-vanishing value of the integral (2.1), they are the only structures that can yield \( \theta_{\text{inst}} \)-dependent effects.

3. Instantons in QCD.

Thus, instantons produce a new kind of interaction in all non-Abelian gauge theories, and in particular in QCD. It is known since 1976 that this interaction can be mimicked by an ‘effective interaction lagrangian’ of the form:

\[
L^{\text{inst}}(x) = \kappa e^{i\theta_{\text{inst}}} \det \{-\bar{\psi}_R(x)\psi_L(x)\} + \text{h.c.} \quad (3.1)
\]

Here, \( \kappa \) is a constant that should be in principle computable, and it contains the factor \( e^{-8\pi^2/g^2} \). The subscripts \( L \) and \( R \) refer to the left- and right handed helicities, obtained by means of the projection operators \( \frac{1}{2}(1 \pm \gamma^5) \). The determinant is the determinant of the matrix \( \bar{\psi}_R^a \psi_L^b \), where \( a \) and \( b \) are flavor indices only. The color indices and the Dirac spin indices can be arranged in several ways (this can be computed explicitly \(^1^6\)), but for simplicity we will ignore these, since this effective lagrangian must be seen as active in an effective hadron model, and we limit ourselves to colorless scalar and pseudoscalar mesons. Note that, in any case, vector mesons consist of quark-antiquark pairs that are either both left-handed or both right handed, so that the determinant (3.1) will have no effect on them (apart from higher orders). We see that the interaction (3.1) has exactly the right quantum numbers for absorbing \( N_{\text{flavor}} \) left helicity fermions and creating an equal number of right handed ones (or vice-versa). In particular, the determinant is easily seen to be the simplest possible interaction that conserves \( SU(N_f) \otimes SU(N_f) \) symmetry, while breaking \( U(N_f) \otimes U(N_f) \).

The cases \( N_f = 0 \) and \( N_f = 1 \) are rather special. If \( N_f = 0 \) while \( \theta_{\text{inst}} \neq 0 \), the interaction (2.5), which is even under charge conjugation \(^\dagger\), but odd under parity, implies an explicit \( P \) and \( CP \) violation. One can show \(^2^2\) that the color-magnetic monopoles obtain fractional electric charges, proportional to \( \theta_{\text{inst}} \). This should have physically observable effects. This is not a purely academic statement, because at very low energies, in QCD, one may regard the up and down quarks as being heavy. If \( N_f \) were equal to one, the interaction (3.1) would blend with the quark mass term. In this case, no symmetry would protect the single quark flavor from getting a mass induced by QCD interactions.

Of particular interest are QCD models with \( N_f \geq 2 \). In this case, there is a global chiral \( SU(N_f) \otimes SU(N_f) \) symmetry that is not affected by instantons. In all such cases, the effective interaction (3.1) would be non-renormalisable. This means that no perturbative interaction of this sort may be admitted. The (non-perturbative) interaction shows up only at low energies. Indeed, at higher energies, one must substitute the running value of \( g^2 \) in \( \kappa \), so that the effective strength of this coupling rapidly decreases with energy.

\(^\dagger\) The charge conjugation operator \( C \) replaces the gluon field \( A_\mu \) by \( -A_\mu^\ast \), which is a non-trivial transformation already in the pure \( SU(3) \) gluon theory; for a purely gluonic \( SU(2)_c \), \( N_F = 0 \) theory, \( C \) would be indistinguishable from a gauge transformation, and therefore trivial.
Studying the case $N_f = 2$ gives us the physics of QCD if we allow ourselves to neglect the effects of the strange quarks. In Ref\textsuperscript{12}, a low energy effective meson model for QCD with instanton effects included is discussed at length. Here, we summarise its results. The effective meson fields $\phi_{ij}$ basically correspond to the composite operators $\bar{q}R_{j}q_{Li}$, and this $2 \times 2$ matrix is decomposed into eight real mesonic fields: a scalar isoscalar $\sigma$, a pseudoscalar isoscalar $\eta$, a scalar isovector $\vec{\alpha}$, and a pseudoscalar isovector $\vec{\pi}$. We write

$$\phi = \frac{1}{2}(\sigma + i\eta) + \frac{1}{2}(\vec{\alpha} + i\vec{\pi}) \cdot \vec{\tau},$$

where $\tau^{1,2,3}$ are the Pauli matrices. The interaction (3.1) now looks as

$$\mathcal{L}^{\text{inst}} = U + U^\dagger,$$

$$U = \kappa e^{i\theta_{\text{inst}}} \det(-\bar{q}RqL) = \kappa e^{i\theta_{\text{inst}}} \det \phi = \kappa e^{i\theta_{\text{inst}}} \left((\sigma + i\eta)^2 - (\vec{\alpha} + i\vec{\pi})^2\right).$$

Here, the parameter $\kappa$ differs from the $\kappa$ in (3.1) by some coefficient. This is because, in (3.1), the $\psi$ fields were defined such that the kinetic terms in the Lagrangian are normalised to $\bar{\psi}(\gamma D + m_f)\psi$, whereas in (3.3), we assume the kinetic terms for the mesons to be normalised to $\text{Tr} \partial_\mu \phi \partial_\mu \phi^\dagger = \frac{1}{2}((\partial_\mu \sigma)^2 + (\partial_\mu \eta)^2 + (\partial_\mu \vec{\alpha})^2 + (\partial_\mu \vec{\pi})^2)$.

In Ref\textsuperscript{12}, it is explained that if the $u$ and the $d$ quark masses may be neglected then $\theta_{\text{inst}}$ will be aligned to zero, and the effective coupling goes as

$$\mathcal{L}^{\text{inst}} \rightarrow 2\kappa(\sigma^2 + \vec{\pi}^2 - \eta^2 - \vec{\alpha}^2).$$

Now, since this is the only effect that splits the pion from the eta, and since the pion continues to behave as a massless Goldstone boson, one can deduce from (3.4) that the eta mass becomes

$$m_\eta^2 = 8\kappa$$

(both $\vec{\alpha}$ and $\sigma$ were already massive before the instantons were switched on, because of $U(2) \times U(2)$ invariant potential terms in the unperturbed Lagrangian, see Ref\textsuperscript{12}). The beauty of this simple analysis is that the instanton interaction bares exactly the quantum numbers required for the eta mass term\|\. Continuing the analysis furthermore shows that the operator $F_{\mu\nu} \tilde{F}_{\mu\nu}$ has the same quantum numbers as the eta field, and so, one expects a considerable mixture between the eta and pure gluonic matter.¶

Although, in principle, not only the pseudoscalars, but also the vector mesons could mix with gluonic matter, such a vector meson mixing is not directly associated to instantons, as mentioned when discussing the effective instanton action (3.1). In Sect. 5, we explain why it is much weaker than for the pseudoscalars.

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\textsuperscript{§} A few signs here are chosen to be different from Ref\textsuperscript{12}.

\textsuperscript{¶} Remember that we are still discussing the two-flavor case, so the effects of the strange quarks are ignored. Therefore, eta here stands for the pseudoscalar state $\pi_0 = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$.

\textsuperscript{¶} If one adds the third flavor $s$ here, this object will become a flavor octet instead of a singlet and hence not mix with glue but predominantly with $s\bar{s}$, whereas the privilege to mix with substantial amounts of pure glue will be reserved for the ninth pseudoscalar meson, $\eta'$.
The above, however, still ignored the presence of strange quarks. Adding the strange quark gives the effective interaction dimension 3 in terms of the mesonic fields:

$$\det (\phi) = \phi_{11}\phi_{22}\phi_{33} \pm \cdots .$$

(3.6)

This does not have the quantum numbers of the mass terms of either the $\eta$ or the $\eta'$ particles. Here, it is necessary to consider the consequences of chiral $U(3) \otimes U(3)$ breaking more carefully. As the $SU(3)$ case was not discussed in detail in Ref\textsuperscript{12}, we give a short review here.

4. A discussion of the 3 flavor case.

It is instructive to describe the scalar and the pseudoscalar mesons in terms of a simple model. The model of Ref\textsuperscript{12} can extended to the $N_f = 3$ case without any major changes. The meson fields are written in the form of a matrix field $\phi_{ij}$ which, as before, is assumed to have the quantum numbers of the quark-antiquark composite operator $^\ast - \bar{q}_R q_L$ . Under a chiral $U_L \otimes U_R$ transformation, it transforms as

$$\phi'_{ij} = U^L_{ik} \phi_{k\ell} U^{R\dagger}_{\ell j} .$$

(4.1)

The lagrangian is taken to be

$$\mathcal{L} = - \text{Tr} \partial_\mu \phi \partial^\mu \phi^\dagger - V(\phi) ,$$

(4.2)

where

$$V(\phi) = V_0 + V_m + V_{\text{inst}} ;$$

$$V_0 = -\mu^2 \text{Tr} \phi^\dagger \phi + A (\text{Tr} \phi^\dagger \phi )^2 + B \text{Tr} (\phi \phi^\dagger \phi \phi^\dagger) ,$$

(4.3)

$$V_m = -2m_u \text{Re} \phi_{11} - 2m_d \text{Re} \phi_{22} - 2m_s \text{Re} \phi_{33} ,$$

(4.4)

$$V_{\text{inst}} = -2\kappa \text{Re} (e^{i\theta_{\text{inst}} \text{det} \phi}) .$$

(4.5)

Here, $V_0$ has the complete $U(3) \otimes U(3)$ symmetry; $V_m$ represents the contributions of the quark mass terms, breaking the symmetry down to $U(2) \otimes U(2)$ if $m_u$ and $m_d$ are small (note that, here, the parameters $m_{u,d,s}$ are proportional to the current quark mass terms, but they do not carry the dimensions of a mass). $V_{\text{inst}}$ represents the instanton contribution, which, having the form of a determinant, breaks the symmetry into $SU(3) \otimes SU(3) \otimes U(1)^{\text{vector}}$ . The coefficient $\kappa$ contains the standard exponential term $^\dagger \exp(-8\pi^2/g^2)$.

* The minus sign was chosen here so as to achieve $\langle \phi \rangle > 0$ while keeping the sign convention for the mass terms of Eq. (4.4).

† Note that, in this term, $g$ is a running coupling strenth. In an accurate analysis\textsuperscript{15} , this matches the non-trivial canonical dimension of this interaction. Therefore, the exponential coefficient ends up to be of order one in units $\Lambda_{QCD}$.
The signs in the definition of $\phi_{ij}$ and in Eqs (4.3)–(4.5), were chosen in such a way that the vacuum expectation values will be positive. $\mu$, $A$ and $B$ are parameters of the model that must obey

$$\mu^2 > 0, \quad A + B > 0, \quad 3A + B > 0.$$ 

We take

$$\phi = \begin{pmatrix} F_1 & 0 & 0 \\ 0 & F_2 & 0 \\ 0 & 0 & F_3 \end{pmatrix} + \tilde{\phi}, \quad (4.6)$$

where $F_i$ are the vacuum expectation values. Imposing that the terms linear in the quantum fields $\tilde{\phi}$ cancel out, gives us the equations for the $F_i$. If $\theta_{\text{inst}} \neq 0$, these numbers in general will be complex, and mixture occurs between the scalars and the pseudoscalars, which makes the computations very lengthy. In experimental observations the value of $\theta_{\text{inst}}$ is found to be very close to zero, or in other words, there is no observed mixing between scalars and pseudoscalars, since otherwise there would have been substantial parity violation in the strong interactions. For simplicity, we will therefore now take $\theta_{\text{inst}}$ to be zero. Redefining

$$R = 2A(F_1^2 + F_2^2 + F_3^2) - \mu^2, \quad (4.7)$$

we get:

$$R + 2B F_1^2 = \frac{m_u + \kappa F_2 F_3}{F_1^2}, \quad (4.8)$$

and its permutations, replacing $m_u$ by $m_d$ and $m_s$.

Just as in Eq. (3.2), the real components are scalar fields, and the imaginary parts are pseudoscalars. We will denote the scalars by $S$ and the pseudoscalars as $P$.

It is now worth-while to compute the masses and mixing angles in this model. Writing

$$\tilde{\phi} = \begin{pmatrix} S_1 & S_{12} & S_{13} \\ *S_{12}^* & S_2 & S_{23} \\ *S_{13}^* & *S_{23}^* & S_3 \end{pmatrix} + i \begin{pmatrix} P_1 & P_{12} & P_{13} \\ *P_{12}^* & P_2 & P_{23} \\ *P_{13}^* & *P_{23}^* & P_3 \end{pmatrix}, \quad (4.9)$$

we expand the potential $V(\phi)$ up to the terms quadratic in $S$ or $P$:

$$V(\phi) = V(F) + S_1^2 \left( R + (4A + 6B)F_1^2 \right) 
+ 2S_1S_2 \left( 4AF_1F_2 - \kappa F_3 \right) 
+ 2|S_{12}|^2 \left( R + 2B F_1 F_2 + \kappa F_3 + 2B(F_1^2 + F_2^2) \right) 
+ P_1^2 \left( R + 2B F_1^2 \right) 
+ 2P_1P_2 \left( \kappa F_3 \right) 
+ 2|P_{12}|^2 \left( R - 2B F_1 F_2 - \kappa F_3 + 2B(F_1^2 + F_2^2) \right) 
+ \text{the two cyclic permutations}. \quad (4.10)$$
Using Eq. (4.8), the part depending on the fields \( P_1, P_2 \) and \( P_3 \) (the neutral pseudoscalars) can be simplified into
\[
\kappa F_1 F_2 F_3 \left( \frac{P_1}{F_1} + \frac{P_2}{F_2} + \frac{P_3}{F_3} \right)^2 + \frac{m_u}{F_1} P_1^2 + \frac{m_d}{F_2} P_2^2 + \frac{m_s}{F_3} P_3^2, \tag{4.11}
\]
which shows that, in the chiral limit \( (m_i = 0) \), where all \( F_i \) are equal, only the ninth component of the pseudoscalars gets a mass, which is proportional to the instanton coefficient \( \kappa \). The scalar mesons \( S \) will always have masses due to the regular interactions (4.3).

Substituting the angles that have been measured, Eqs (1.4) and (1.5), and the masses of \( \pi^0, \eta \) and \( \eta' \), gives us the numbers \( \kappa F_i \) and \( m_i/F_i \):
\[
\kappa F_1 \approx \kappa F_2 \approx .22 \text{ GeV}^2, \quad \kappa F_3 \approx .28 \text{ GeV}^2; \tag{4.12}
\]
\[
m_u/F_1 + m_d/F_2 = 2M_\pi^2 = .0365 \text{ GeV}^2, \quad m_s/F_3 = .44 \text{ GeV}^2.
\]
This gives the ratio
\[
\frac{2m_s}{m_u + m_d} \approx 30.7. \tag{4.13}
\]

The \( \pi^\pm \) and the \( K \) masses are now computable. Since the input parameters had an exact isospin invariance, \( \pi^\pm \) are degenerate with the \( \pi^0 \), and \( K^\pm \) with \( K^0 \). The \( K \) mass-squared corresponds to the coefficient in front of \( |P_{13}|^2 \) and \( |P_{23}|^2 \) in (4.10), which is
\[
M_K^2 = R + 2B(F_1^2 + F_3^2 - F_1 F_3) - \kappa F_2, \tag{4.14}
\]
and since all numbers in here were already determined by the \( \eta \) and \( \eta' \) masses and mixing angles, the outcome is ‘predicted’, yielding
\[
M_K \approx 509 \text{ MeV}, \tag{4.15}
\]
no more than 3% away from the actual value. Although this beautiful agreement with the experimental value of the \( K \) mass may be accidental, this does indicate that the mechanism for chiral symmetry breaking described here is realistic. In fact, this just confirms the long known fact that the meson masses squared are approximately linearly proportional to the quark masses.

It is probably even better to use the observed value for the kaon mass, 495 MeV, to estimate the strange quark mass term, \( m_s \). One then gets, instead of (4.13):
\[
\frac{2m_s}{m_u + m_d} = \frac{M_K^2}{M_\pi^2} \left( 1 + \frac{F_3}{F_1} \right) - 1 \approx 29. \tag{4.16}
\]

The scalar mesons in this model are fairly heavy. We found the scalar pion to be in the range 1340 to 1580 MeV, and the scalar kaon to be about 150 MeV heavier than the scalar pions. The masses and mixing angles of the scalar counterparts of \( \eta \) and \( \eta' \), here called \( \sigma \) and \( \sigma' \), depend explicitly on the parameter \( A \), which was not yet determined (see the Appendix for further details). We observe, that our rather crude model of Eqs. (4.2)–(4.5) gives a quite realistic phenomenology for the pseudoscalar mesons.
5. A few words on \( \omega - \phi \) mixing and the \( \eta_c \).

The model of Sect. 4 is to be regarded as a low energy, effective theory. The scalar resonances predicted in the region of 1340 to 1580 MeV are expected to be quite wide. Their mass formulas are given in the Appendix. From the way the masses depend on the parameters \( A \) and \( B \) one deduces that the scalar masses are very much model dependent. Indeed, an alternative model can be constructed in which the fields \( \phi \) obey a non-linear unitarity constraint: \( \phi \phi^\dagger = I \). This model only contains pseudoscalars; the scalar masses were sent to infinity.

At high energies, QCD is more effectively described in terms of vortex dynamics; at higher energies still, the asymptotic perturbation expansion of asymptotically free QCD is the best. The question why \( \omega \) and \( \phi \) only mix rather weakly, and have nearly no gluon content, can be explained in the high-energy limit. Replacing these particles by the \( J/\psi \), we can make the following observation.

Whereas \( \eta_c \) couples to gluonic states via intermediate states with only two gluons, it is well-known that \( J/\psi \) needs a 3-gluon intermediate state to decay. This means that, in the limit \( m_c \to \infty \), \( J/\psi \) is coupled to gluonic matter much more weakly than \( \eta_c \). Intermediate gluonic states are the only way in which a \( cc \) bound state can couple to other flavor states such as \( \bar{u}u \) and \( \bar{d}d \). We see that \( J/\psi \) is shielded from these other states by a factor \( \alpha_{\text{strong}} \) relative to \( \eta_c \). Clearly then, \( J/\psi \) will not hardly mix as strongly to these other states as \( \eta_c \) will do. If we now replace the charmed quark by the strange quark, we may expect the same qualitative behaviour, although the precise numerical coefficients will be much harder to calculate. In any case, we should not be surprised to find that the vector state \( \bar{s}s \) hardly mixes with \( \bar{u}u \) and \( \bar{d}d \); the reason for this is that the mixing goes via an intermediate state of pure glue, and the coupling between the vector states and pure glue is suppressed as compared to the pseudoscalar particles.

This argument must be added to the observation made at the beginning of Sect. 3, that the operator associated with the creation and absorption of vector mesons, \( \bar{\psi} \gamma_\mu \psi \), contains either only left handed fermion-antifermion pairs or only right handed ones, and therefore it does not match the quantum numbers of an effective instanton interaction (as both the scalar and the pseudoscalar mesons do).

We now see that, in contrast, instantons give a fairly effective mixing between all diagonal pseudoscalar states. Indeed, the model of Sect. 4 could be used to study the charmed sector, in particular in an approximation where we ignore the strange quark.

An alternative way to understand the two-flavor model of Sect. 3, and Ref\(^1\), is to integrate first over all virtual strange-quark loops. A Feynman diagram containing the interaction (3.1), can then be seen to yield an amplitude proportional to a \( U(2) \) determinant that does not contain the strange quarks, see Fig. 1. In order to yield a non-vanishing contribution, the strange quark, in its closed loop, must switch its helicity, but this can happen due to the non-negligible value of the strange quark mass. Indeed, the effective instanton interaction is now proportional to \( m_s \). The case for more heavy flavors is a bit more subtle. Primarily, the effect of an instanton will carry as a factor the product of all "heavy" flavor masses, \( m_s \cdot m_c \cdot m_b \cdots \), but when they get heavier than \( \Lambda_{QCD} \) the
heavy flavors decouple, and the instanton behaves as if they were not there. This is why effects due to charm, bottom and top are usually not considered. These heavy flavors do not mix very much with the light ones, be it via instantons or other forms of glue.

6. Instantons and spontaneous chiral symmetry breaking.

There are various other aspects of QCD dynamics that are directly associated to instantons. One of these is the nature of the dynamical forces that cause the spontaneous breakdown of chiral $SU(N_F) \otimes SU(N_F)$ down to the vector flavor symmetry group $SU(N_F)$. Why should such a spontaneous symmetry breaking occur at all?

Several hand-waving arguments can be brought forward. We see spontaneous symmetry breaking happen explicitly in some model calculations. In $1+1$ dimensions, QCD can be solved exactly in the $N_c \to \infty$ limit\textsuperscript{23}. In this limit, we see the Goldstone pions emerge in the exact solution. They are also observed when QCD is solved on a lattice in the large coupling limit\textsuperscript{24}.

We would like to know whether QCD related theories can be constructed in which chiral symmetry is \emph{not} spontaneously broken, but realized explicitly in the Wigner mode. Such a theory could be employed to describe a new strong interaction regime for the weak interactions at ultra-high energies. It was attempted to construct a theory in which the presently elementary leptons and quarks are seen as bound states of a new kind of quarks at ultra-high energies. An ultra-strong color force should bind these new quarks. Such a theory, often referred to as “Technicolor”\textsuperscript{25}, however only works if some symmetry protects the ordinary quarks and leptons against developing too large mass terms. This symmetry can only be a chiral symmetry that is \emph{not} spontaneously broken. The attempts at constructing technicolor theories for the electro-weak forces were unsuccessful. The

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\textbf{Fig. 1.} In the 2-flavor case, the quantum numbers of the effective instanton interaction exactly match the quantum numbers of a mass term for the quark combination $\pi^0 \approx u\bar{u} + d\bar{d}$. In the 3-flavor case, there is an extra $s$-quark emitted and absorbed, with opposite helicities. This gives a contribution to the $\pi^0$ mass that is proportional to the strange quark mass. Then, because of the vacuum shift described by Eq. (4.6), this object mixes with $s\bar{s}$ and it becomes the physical $\eta$ particle, which is close to the $SU(3)^{\text{flavor}}$ octet state $\eta_8 = (1/\sqrt{6})(u\bar{u} + d\bar{d} - 2s\bar{s})$. The mass of the $\eta' \approx (1/\sqrt{3})(u\bar{u} + d\bar{d} + s\bar{s})$ is not limited by the strange quark mass, but arises as described in Sect. 4; i.e., its mass is proportional to the instanton action $\kappa$. 

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\(\pi^0 \quad \text{Inst} \quad \pi^0\)
demise of these theories was partly due to the following insight concerning QCD related theories.

Using a background of classical flavor gauge fields, one can derive the anomalies in the vacuum-to-vacuum amplitudes in given channels of these classical background fields. One does this by constructing instantons out of these background fields, and asking how many axial charges are generated by it. The QCD Lagrangian gives us explicit and exact answers. We derive which of the chiral charges associated with the background gauge fields are created or destroyed by these instantons, via the generalized version of Eq. (2.4). Now consider some effective theory describing the mesonic and baryonic bound states. Any effective model for these hadrons should reproduce the same answers, i.e., the same chiral charges $Q_5$, with the same quantum numbers, should be produced by our background instanton. This observation provides us with very strict constraints on the spectrum of states that should be introduced in the effective meson model. If now the chiral flavor symmetry is not spontaneously broken, the total amount of axial flavor charges involved in the anomaly, as dictated by Eq. (2.4), must be reproduced by the fermionic (mesonic and baryonic) objects described in the effective model. This is called the ‘anomaly matching condition’.

Explicit calculations then lead to a surprise: if the symmetry were realized in the Wigner mode, one finds that, more often than not, the bound states carry too large axial charges to allow us to match the anomalies using Eq. (2.4). This would force us to consider models in which the ‘number of mesonic or baryonic species’ is fractional. Numbers such as $1/9$ and $1/25$ emerge in $SU(3)^{\text{color}}$ and $SU(5)^{\text{color}}$ theories, which would be an absurdity. If however the effective degrees of freedom are assumed to realize the external symmetries in the Goldstone mode, then the constraints posed by the anomaly matching condition can always be realized. Thus we arrive at a contradiction if we assume the chirally symmetric Wigner mode to be realized in QCD. An exception is QCD with exactly two flavors. It is the strange quark that causes the first real problem in constructing a chirally symmetric spectrum. This leads one to conclude that, if chiral symmetry were not already spontaneously broken in the up + down sector of the flavor group, surely the strange quark would trigger spontaneous chiral symmetry breakdown.

Thus the anomaly matching requirement rules out many attempts to use a QCD related theory at the TeV scale and assume it to realize its chiral symmetries in some unconventional way.

7. Conclusion and remarks.

A simple polynomial lagrangian for the effective interactions between scalar and pseudoscalar mesons, with in addition the simplest polynomial that reflects the correct quantum numbers of an instantonic interaction, can reproduce the observed meson spectrum quite reasonably. The fact that the pseudoscalars tend to mix along the dividing lines of the $SU(3)$ octet and singlet representations, while the vectors $\omega$ and $\phi$ tend to mix in such a way that pure flavor bound states emerge, can be understood quite naturally. The pseudoscalar mesons can be addressed in a simple model. In addition to pseudoscalar
mesons, this model contains also scalar mesons, but no vectors. If one wishes to include vector mesons, one has to turn the effective model into an $SU(3)^{\text{flavor}}$ non-Abelian, spontaneously broken gauge theory. Such a model contains much more mesonic fields and freely adjustable parameters, and consequently it gives little further insight.

The experimental evidence that there is little PC violation in QCD indicates that $\theta_{\text{inst}}$ must be very small or zero. The sign of the instanton interactions (which is a free parameter since one may freely choose $\theta_{\text{inst}}$), is as indicated in the effective action term (3.1) (note the minus signs in Eqs. (4.2), (4.5) and in the definition $\phi = -\bar{q}_R q_L$), with $\theta_{\text{inst}} \approx 0$.

Within the paradigm of QCD, in the absence of weak interactions, it is not unnatural to put $\theta_{\text{inst}} = 0$, since only weak interactions, with their explicit CP violation effects by having a phase angle in the Kobayashi-Maskawa matrix, can send $\theta_{\text{inst}}$ away from zero. Note however that, by rotating the field $\phi$, one can transport the $\theta_{\text{inst}}$ angle from the instanton term (4.5) to one of the quark mass terms in Eq. (4.4). Since $m_u$ is the tiniest mass term, it is most natural to put $\theta_{\text{inst}}$ as a phase in the $u$ quark mass term, which then becomes $\text{Re}(e^{-i\theta_{\text{inst}} m_u \phi_1})$. As $m_u$ runs towards smaller values at very high energy scales, its phase will be affected by the weak interactions. As the TeV scale is reached, it becomes difficult to see why such effects should stay extremely small. From the observed absence of scalar-pseudoscalar mixing one must deduce however that the $\theta_{\text{inst}}$ is extremely small. How this fine-tuning can be explained is as yet an unresolved problem. We suspect that new physics at the TeV scale must be responsible.

The question whether instantons also lead to observable effects at high energies is more difficult to answer. At high energies, where the running coupling parameter $g_{\text{strong}}$ tends to become small, instantons are very efficiently screened, as their amplitudes vary as $\exp(-8\pi^2/g_{\text{strong}}^2)$, so we do not expect direct instanton effects at high energy. Instantons are very soft objects.

When the fields $\phi$ are coupled to the electro-weak gauge field via their currents (defined by their weak transformation rules), we hit upon a serious shortcoming of this simple model: it does not reproduce the experimentally well-confirmed $\Delta I = \frac{1}{2}$ rule. Indeed, neutral kaons tend to decay only into charged pions; the $2\pi^0$ decay is suppressed! This is because the GIM mechanism prevents the decay through neutral vector bosons. This deficiency impedes attempts to investigate the $\varepsilon'/\varepsilon$ problem using models of this sort. To reproduce the $\Delta I = \frac{1}{2}$ rule one must take renormalization group effects in QCD into account.

Appendix A. Mass formulae for scalar and pseudoscalar mesons.

The model of Sect. 4 appears to generate fairly decent estimates for mesonic mass relations. We here give some of the formulae. The derivations are straightforward. We start from the effective action described by Eqs. (4.2) – (4.5). The masses are determined by Eqs. (4.10). Because of isospin symmetry, and because electromagnetism was ignored, we keep $m_u = m_d$, a number that in more refined theories should be replaced by $\frac{1}{2}(m_u + m_d)$. The two independent vacuum expectation values $F_1$ and $F_3$ are determined by Eq. (4.8),
which can be written as
\[
F_1 ((4A + 2B) F_1^2 + 2A F_3^2 - \mu^2 - \kappa F_3) = m_u, \quad (A.1)
\]
\[
F_3 (4A F_1^2 + (2A + 2B) F_3^2 - \mu^2) - \kappa F_1^2 = m_s. \quad (A.2)
\]
The pseudoscalar masses (after some algebraic manipulations) are then described by
\[
M_K^2 = M_{\pi}^2 + (2B F_3 + \kappa)(F_3 - F_1) = \frac{m_s + m_u}{F_1 + F_3}; \quad (A.3)
\]
\[
M_{\eta}^2 - 2M_{\pi}^2 = 2B(F_3^2 - F_1^2) + 3\kappa F_3; \quad (A.4)
\]
\[
\left(\frac{M_{\eta}^2 - M_{\pi}^2}{M_K^2 - M_{\pi}^2}\right) = 2\kappa (F_1 + F_3); \quad (A.5)
\]
\[
(M_{\eta}^2 - M_{\pi}^2) \sin 2\theta_P = \frac{2\sqrt{2}}{3} \left(2B(F_3^2 - F_1^2) - \kappa (F_3 - F_1)\right); \quad (A.6)
\]
\[
(M_{\eta}^2 - M_{\pi}^2) \cos 2\theta_P = \frac{1}{3} (8\kappa F_1 + \kappa F_3 + 2B(F_1^2 - F_3^2)). \quad (A.7)
\]

For completeness, we list here the formulae for the scalar mesons \(\pi_S, K_S, \sigma\) and \(\sigma'\), which are like \(\pi, K, \eta\) and \(\eta'\) but with \(J^{PC} = 0^{++}\):
\[
M_{\pi S}^2 = M_{\pi}^2 + 2\kappa F_3 + 4B F_1^2; \quad (A.9)
\]
\[
M_{K S}^2 = M_K^2 + 2\kappa F_1 + 4B F_1 F_3; \quad (A.10)
\]
\[
M_{\sigma}^2 + 2M_{\pi S}^2 = (4A + 6B) F_3^2 + (8A - 6B) F_1^2 - 3\kappa F_3; \quad (A.11)
\]
\[
(M_{\sigma}^2 - M_{\pi}^2) \sin 2\theta_P = \frac{2\sqrt{2}}{3} (F_3 - F_1) \left((4A + 6B) F_3 + (8A + 6B) F_1 + \kappa\right); \quad (A.12)
\]
\[
(M_{\sigma}^2 - M_{\pi}^2) \cos 2\theta_P = \frac{1}{3} ((8A + 6B) F_1^2 - (4A + 6B) F_3^2 + 32A F_1 F_3 - \kappa (8F_1 + F_3)). \quad (A.13)
\]

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