AN INTEGRAL LIFT OF THE $\Gamma$-GENUS

JACK MORAVA

Abstract. The Hirzebruch genus of complex-oriented manifolds associated to Euler’s $\Gamma$-function lifts to a homomorphism of ring-spectra associated to a family of deformations of the Dirac operator, parametrized by the homogeneous space $\text{Sp}/U$.

Introduction

Kontsevich, in his early work on deformation quantization [12 §4.6], drew attention to interesting formal properties of Euler’s $\Gamma$-function, regarded as defining something like a Hirzebruch genus. This note presents that idea in the language of cobordism and formal groups, following [16]. The formalism of multiplicative power series defines a homomorphism

$$\chi_\infty : M U_* \to \mathbb{C}[v]$$

(of graded rings, with a book-keeping indeterminate $v$) having no very immediate integrality properties, but classical function theory [§2.3.1] shows it to take values in the ring $\mathbb{Q}[\tilde{\zeta}(\text{odd})]$ generated over the rationals by normalized zeta-values, usually expected to be transcendental. The principal result here [§3.1] is that a topologically reasonable homomorphism

$$MU \xrightarrow{\Gamma} M U \wedge_{M \text{Sp}} KO \xrightarrow{\simeq \frac{1}{2}} \text{Sp}/U \wedge KO[\frac{1}{2}]$$

of ring-spectra provides a lift of $\chi_\infty$, via the composition

$$KO_*(\text{Sp}/U) \xrightarrow{\text{ch}} H_*(\text{Sp}/U, \mathbb{Q}[\sqrt{v}]) \to H_*(BU, \mathbb{Q}[\sqrt{v}]) \to \mathbb{C}[\sqrt{v}]$$

which sends primitive generators of $H_*(\text{Sp}/U, \mathbb{Q})$ to odd $\zeta$-values.

It was the appearance of these periods (and their relation to the theory of mixed Tate motives in algebraic geometry) that precipitated much of the interest in the $\Gamma$-genus. They appear in the lift as generic parameters for a family of deformations of a Dirac operator over the homogeneous space $\text{Sp}/U$. This seems to have interesting connections with [10] and [17].

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1. Coigns of vantage

1.0 It’s useful to distinguish a coordinate $z$ at a point $x_0$ of a space $X$ from the corresponding parametrization of a neighborhood $U \ni x_0$: the former is a nice function $X \ni U \xrightarrow{z} A$ sending $x_0$ to 0 in some commutative ring $A$, while the latter is the map $z : \text{Spec } A \to U \subset X$ it defines (assuming we’re in a context where this makes sense).

1.1 For example, at the point $x_0 = [1 : 1]$ of the projective line, we have a coordinate $[u:1] \mapsto u - 1 := z$ which defines the parametrization $z \mapsto [1 + z : 1]$ of a neighborhood of $[1 : 1]$. Similarly, $[q:1] \mapsto q^{-1} := z$ is a coordinate at $[1 : 0] = \infty \in P_1$, while $[x:1] \mapsto x := z$ is a coordinate at $[0 : 1] = 0$.

1.2 An abelian group germ $G$ at $x_0 \in X$ is the germ of a function $G : U \times U, x_0 \times x_0 \to U, x_0$ satisfying identities such as $G(x, G(y, z)) = G(G(x, y), z), G(x, x_0) = G(x_0, x) = x, \&c$; if $G$ is suitably analytic, then a coordinate $z$ at $x_0$ associates to $G$, the formal group law $(z \circ G)(z \times z) := z_0 + G z_1 \in A[[z_0, z_1]]$.

For example, the additive group germ $G_a(x, y) = x + y$ at $[0 : 1] \in P_1$ defines $z_0, z_1 \mapsto z_0 + z_1$, while the multiplicative group germ $G_m(u, v) = uv$ at $[1 : 1]$ defines $z_0 + G_m z_1 = z_0 + z_1 + z_0 z_1$. 

(with coordinates as above). Different choices of coordinate (for fixed \(G\) and \(x_0\)) define, in general, distinct (but isomorphic) formal group laws: for example, if \(t \in A^\times\) then \(z = t^{-1}(u - 1)\) associates the formal group law
\[
z_0, z_1 \mapsto z_0 + z_1 + t z_0 z_1.
\]
to the multiplicative group at \([1 : 1]\).

1.3.1 The introduction of such a variable \(t\) suggests the consideration of families, or deformations, of group laws:
\[
\mathbf{u}, \mathbf{v} \mapsto \mathbf{uv},
\]
at \([1 : 1]\) (easily checked, e.g., for nilpotent \(t\), to satisfy the axioms) is an interesting example. With coordinate as above, the associated group law
\[
z_0, z_1 \mapsto \frac{z_0 + z_1 + (1 + t)z_0 z_1}{1 - t z_0 z_1};
\]
is (strictly) isomorphic to \(+_{G_m}\), under the coordinate change
\[
z \mapsto (1 + t)^{-1}\log \left[ \frac{t}{-1} \right](z) \in \mathbb{Q}[[t]][[z]];
\]

1.3.2 Similarly, \(\exp_A(z) := 2 \sinh \frac{z}{2}\) defines
\[
z_0 +_A z_1 = z_0(1 + \frac{1}{4}z_1^{1/2}) + z_1(1 + \frac{1}{4}z_0^{1/2}) \in \mathbb{Z}[[z_0, z_1]],
\]
which is a specialization (at \(\delta = -\frac{1}{8}, \epsilon = 0\)) of the formal group law
\[
z_0 +_E z_1 = \frac{z_0 R(z_1) + z_1 R(z_0)}{1 - \epsilon z_0^2 z_1^2}
\]
defined by Jacobi’s quartic \(Y^2 = R(X)^2 := 1 - 2\delta X^2 + \epsilon X^4\).

1.4 The focus of this note is the group germ
\[
G_\infty : [q_0 : 1], [q_1 : 1] \mapsto [\Gamma(\log_\infty(q_0^{-1}) + \log_\infty(q_1^{-1})) : 1]
\]
at \(\infty \in P_1(\mathbb{R})\) defined by the expansion
\[
\exp_\infty(z) := z \exp(\gamma z - \sum_{k \geq 2} \zeta(k) k (-z)^k) \in \mathbb{R}[[z]]
\]
of the entire function \(\Gamma(z)^{-1}\) near 0 (with \(\log_\infty(z)\) denoting its formal composition inverse): thus
\[
z_0 +_{\infty} z_1 = \Gamma(\log_\infty(z_0) + \log_\infty(z_1))^{-1} = z_0 + z_1 + 2\gamma z_0 z_1 + \cdots \in \mathbb{R}[[z_0, z_1]]
\]
with \(z_k = q_k^{-1}\). Ohm’s law for parallel resistors, in comparison, defines a group germ
\[
[q_0 : 1], [q_1 : 1] \mapsto [1 : q_0^{-1} + q_1^{-1}]
\]
\(^1\text{a.k.a. the harmonic mean of Archytas of Tarentum}\)
at ∞, which (because $\frac{x^y}{x+y}$ is not differentiable at $(0,0)$) is not analytic.

2. Characteristic classes and Hirzebruch genera

2.1 A complex line bundle $\lambda \in H^1(X, \mathbb{C}^\times)$ has an associated class

$$\lambda^{-1}d\lambda \mapsto 2\pi i[\lambda] : H^1(X, \mathbb{Z}(1)) \to H^2(X, 2\pi i\mathbb{Z})$$

corresponding to the coordinate [1 §2.3, 19 §5.10]

$$z = vx \in H^\text{even}(X, \mathbb{Z}[v^{\pm 1}])$$
on the Picard group of topological complex line bundles. Interpreting $v$ as the product of the Bott class with Deligne’s motive $2\pi i$ reconciles some conventions of algebraic geometry with those of algebraic topology: for example

$$\frac{\pi[\lambda]}{\sin \pi[\lambda]} \mapsto \frac{vx/2}{\sinh vx/2}.$$When the grading is of background interest, I’ll set $v$ equal to 1.

2.2.1 A (one-dimensional) formal group law over a $\mathbb{Q}$-algebra $A$ can be written uniquely as

$$z_0 +_G z_1 = \exp_G(\log_G(z_0) + \log_G(z_1)) ;$$in that case let

$$H_G(z) := \frac{z}{\exp_G(z)} \in A[[z]]^\times$$
denote its Hirzebruch multiplicative series [8 §15.5]. The function

$$M \mapsto \left( \prod_{i=1}^{i=n} H_G(vx_i) \right)[M] \in A[v]$$
from (cobordism classes of) compact closed complex-oriented manifolds of real dimension $2n$, with Chern roots $x_i$, defines a homomorphism

$$\chi_G : MU_* \to A[v]$$of graded rings: the Hirzebruch genus associated to the group law $G$. By a theorem of Mishchenko,

$$\log_G(v) = \sum_{n \geq 1} \frac{\chi_G(P_{n-1}(\mathbb{C}))}{n} \in A[[v]] ;$$the deformation of the multiplicative group in §1.3.1, for instance, represents Hirzebruch’s genus $\chi_{-t}$ genus (defined on smooth projective complex varieties by

$$V \mapsto \sum (-1)^p (-t)^q \dim \mathcal{H}^{p,q}_{dg}(V) v^{\dim C} V .$$
The coordinate rescaling \( v \mapsto t^{-1/2} v \) sends its logarithm to
\[
\sum_{n \geq 1} [n](t) \frac{v^n}{n}
\]
(with Gaussian \( \frac{n^{1/2} - t^{-n/2}}{t^{1/2} - t^{-1/2}} = [n](t) \)), and its formal group law to
\[
X, Y \mapsto \frac{X + Y + (t^{1/2} + t^{-1/2})vXY}{1 - vXY}
\]
(which is symmetric under the involution \( t \mapsto 1/t \)).

2.2.2 I’ll refer below to \( MSO, MU, \) and \( MSp \) as the cobordism theories of \( R, C, \) and \( H \)-oriented manifolds, respectively.

The Pontryagin classes
\[
P_t^{SO}(V) = \sum_{k \geq 0} p_k^{SO}(V)t^{2k} := \sum_{k \geq 0} (-1)^k c_{2k}(V \otimes C)t^{2k}
\]
of a real vector bundle \( V \) are defined in terms of the Chern classes of its complexification; if \( V \) was complex to begin with, then
\[
c_t(V \otimes C) = \sum_{k \geq 0} c_k(V \otimes C)t^k = c_t(V) \cdot c_t(V)
\]
equals
\[
\prod (1 - x_i^2 t^2) = \sum (-1)^k e_k(x_i^2)t^{2k}
\]
which expresses the Pontryagin classes
\[
p_k^{SO}(V) = e_k(x_i^2)
\]
in terms of elementary symmetric functions of the Chern roots \( x_i \) of \( V \otimes C \).

If \( H_G(z) := \hat{H}_G(z^2) \) is an even power series, then the associated genus \( \chi_G \) of a \( C \)-oriented manifold \( M \) can be evaluated in terms of Pontryagin classes, since
\[
\prod \hat{H}_G(x_i^2) := H_G(p_k^{SO})
\]
for some polynomial \( H_G \); this factors \( \chi_G \) through a homomorphism
\[
MU \longrightarrow MSO \overset{\chi_G}{\longrightarrow} A[v].
\]
The complex vector bundle underlying a quaternionic vector bundle \( V \), on the other hand, can be decomposed as the sum of a complex bundle with its conjugate. In that case we have
\[
P_t^{SO}(V) = p_t^{SO}(W \oplus \overline{W}) = p_t^{SO}(W)^2
\]
(at least, with coefficients in a \( \mathbb{Z}[\frac{1}{2}] \)-algebra). The symplectic Pontryagin classes of \( V \) are defined by
\[
P_t^{Sp}(V) = \sum (-1)^k c_{2k}(V)t^{2k}
\]
so $p^\text{Sp}_t(V) = p^\text{SO}_t(W)$, hence $p^\text{SO}_t(V) = (p^\text{Sp}_t(W))^2$. Since $p^\text{SO}_t(V)$ can be expressed in terms of the power sums $\sum x_i^{2k} = s^\text{SO}_k$ of the Chern roots of $V \otimes \mathbb{C}$ as

$$\exp\left(\sum s^\text{SO}_k \frac{t^{2k}}{k}\right),$$

we have

$$s^\text{SO}_{2k} := s_{2k}(V \otimes \mathbb{C}) = 2s_{2k}(V) := 2s^\text{Sp}_{2k},$$

(in terms of the Chern roots of the complex structure underlying a quaternionic structure on $V$).

2.3.1 Rewriting the logarithm of Weierstrass’s product formula for $\Gamma$, we have

$$\Gamma(1 + z) = \exp(-\gamma z + \sum_{k > 1} \frac{\zeta(k)}{k} (-z)^k);$$

from this, and the duplication formula

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z},$$

it follows that

$$\frac{x/2}{\sinh x/2} = \exp\left(\sum_{k \geq 1} \frac{\zeta(2k)}{(2\pi i)^{2k}} x^{2k}\right),$$

with rational coefficients

$$\frac{\zeta(2k)}{(2\pi i)^{2k}} = -\frac{B_{2k}}{2(2k)!}.$$

The $\hat{A}$-genus of an oriented manifold (corresponding to the group law in §1.3.2) can thus be calculated by evaluating

$$\prod \left(\frac{vx_i/2}{\sinh vx_i/2}\right) = \exp\left(-\sum \frac{B_{2k}}{4k!} s^\text{SO}_{2k} x_i^{2k}\right)$$

on its fundamental class. If the manifold is $\mathbb{H}$-oriented, this characteristic class equals the product

$$\prod \left(\frac{x_i/2}{\sinh x_i/2}\right)^{1/2}$$

(now taken over the Chern roots of the complex bundle underlying the $\mathbb{H}$-oriented structure).

**Proposition.** The genus of complex-oriented manifolds defined by the multiplicative series

$$H_{G_{\infty}}(x) = \Gamma(1 + [\lambda]) = \left(\frac{x/2}{\sinh x/2}\right)^{1/2} \exp\left(i \gamma \frac{\zeta(\text{odd})}{2\pi} x + \sum \frac{\zeta(\text{odd})}{(2\pi i)^{\text{odd}}} x^{\text{odd}}\right) \in \mathbb{C}[[x]]$$

agrees on the image of $M\text{Sp}$ in $MU$ with the $\hat{A}$-genus.

[Because the odd terms in the exponential cancel, for a bundle of the form $W \oplus \overline{W}$.]
2.3.2 Note that the Witten genus [14]

\[ H_W(x) = \frac{x/2}{\sinh x/2} \prod_{n \geq 1} [(1 - q^n e^x)(1 - q^n e^{-x})]^{-1} \]

can be written similarly, in terms of Eisenstein series, as

\[ \exp(\sum_k G_{2k}(q) \frac{x^{2k}}{2k}) ; \]

but this deformation of the \( \hat{A} \)-genus is an even function of \( x \).

2.4 The elementary symmetric functions \( e_n \) and the corresponding power sums \( s_n \) are related by

\[ e(z) = \sum_{n \geq 0} e_n z^n := \prod_{k \geq 1} (1 + x_k z) = \exp(- \sum_{n \geq 1} \frac{s_n (-z)^n}{n}) . \]

The assignment \( x_k \mapsto 1/k \) requires some care, but, suitably interpreted, sends \( s_k \) to \( \zeta(k) \) if \( k > 1 \), and \( s_1 \) to \( \gamma \). The formal power series

\[ \text{Exp}_\infty(z) = z \cdot e(z) \]

thus specializes to \( \text{exp}_\infty(z) \) under this mapping, defining a lift \( G_\infty \) of \( G_\infty \) to a formal group law over the polynomial algebra \( \mathbb{Z}[e_n \mid n \geq 1] \). Since its exponential is defined over \( \mathbb{Z} \), it is of additive type, and is in fact the universal such formal group law.

Similarly

\[ H_{G_\infty}(z) = \sum_{k \geq 0} h_k (-z)^k , \]

in terms of the complete symmetric functions \( h_k \).

3. The Real structure of \( MU \)

3.1 Proposition. In the homotopy-commutative diagram

\[
\begin{array}{ccccccc}
\text{MU} & \longrightarrow & S^0[BU_+] \wedge \text{HZ} & \longrightarrow & S^0[Sp/U_+ \wedge BSp_+] \wedge \text{HZ}[\frac{1}{2}] & \downarrow & \zeta(\text{even}) \\
\Gamma[\frac{1}{2}] & \downarrow & S^0[Sp/U_+] \wedge \text{MSp} & \longrightarrow & S^0[Sp/U_+] \wedge \text{KO}[\frac{1}{2}] & \longrightarrow & S^0[Sp/U_+] \wedge \text{HQ}[v^{\pm 1}] \\
\text{MSp} & \longrightarrow & \hat{A} & \longrightarrow & \text{KO} & \longrightarrow & \text{HC}[v^{\pm 1}] \\
\end{array}
\]

of spectra, the diagonal composition represents the \( \Gamma \)-genus.
3.2 Proof. Here $S^0[G_+]$ is the suspension ring-spectrum defined by an $H$-space $G$, such as the fiber $Sp/U \sim \Omega Sp \sim B(U/O)$ of the quaternionification map $BU \rightarrow BSp$. Note that the inclusion of the fiber into $BU$ makes $S^0[BU_+]$ (and hence $MU$) into $S^0[Sp/U_+]$-modules.

The two vertical maps at the lower left side of the diagram are the obvious smash products with the unit $S^0 \rightarrow S^0[Sp/U_+]$, while the horizontal maps across the middle of the diagram are smash products with the $\hat{A}$-genus, regarded as defined by the index of a Dirac operator on an $\mathbb{H}$-oriented manifold, followed by the Chern character on $KO$. The top left-hand map is just the total characteristic number homomorphisms of Boardman and Quillen, and can alternately be described as the composition

$$MU_* \rightarrow MU_* \otimes S_* \rightarrow \mathbb{Z} \otimes S_* = S_*$$

of the total Landweber-Novikov operation with Steenrod’s cycle map

$$1 \in H^0(BU, \mathbb{Z}) \rightarrow H^0(MU, \mathbb{Z}) = [MU, H\mathbb{Z}]_0.$$

The (related) upper left-hand vertical and upper right-hand horizontal maps are more interesting. An element of $MSp_*(Sp/U_+)$ can be interpreted as the bordism class of an $\mathbb{H}$-oriented manifold $M$, equipped with a map to $Sp/U$, and if we regard $M$ as merely complex-oriented, then the product composition

$$M \rightarrow Sp/U_+ \wedge BU_+ \rightarrow BU_+$$

defines a new complex orientation on $M$, and thus a ring homomorphism

$$MSp_*(Sp/U_+) \rightarrow MU_*.$$

By [3], this is in fact an isomorphism away from the prime $(2)$; similarly, the composition

$$Sp/U_+ \wedge BSp_+ \rightarrow Sp/U_+ \wedge BU_+ \rightarrow BU_+$$

defines an isomorphism

$$H_*(Sp/U, \mathbb{Z}[[\frac{1}{2}]]) \otimes_{\mathbb{Z}[[\frac{1}{2}]}} H_*(BSp, \mathbb{Z}[[\frac{1}{2}]]) \cong H_*(BU, \mathbb{Z}[[\frac{1}{2}]])$$

of Hopf algebras, which is the upper right-hand map.

Since the diagonal maps are defined by the diagram, only the right-hand vertical maps remain to be constructed, but that is the content of §2.4: the power-sum generators of $H_*(BU, \mathbb{Q})$ map to normalized zeta-values

$$s_k \mapsto \tilde{\zeta}(k) := (2\pi i)^{-k}\zeta(k) \text{ if } k > 1, \quad \mapsto -\frac{\gamma}{2\pi} \cdot i \text{ if } k = 1.$$

This is factored into two steps:

$$\zeta(\text{even}) : s_{2k} \mapsto \frac{B_{2k}}{4k(2k)!} \in \mathbb{Q}$$

can be interpreted as defining the $\hat{A}$-genus, while

$$\zeta(\text{odd}) : s_{2k+1} \mapsto (-1)^{k+1}(2\pi)^{-2k-1}\zeta(2k+1) \cdot i.$$
3.3 Complex conjugation on $MU$ is represented by the coordinate change $z \mapsto [-1](z)$ on the formal group, which corresponds to complex conjugation on the value group of the $\Gamma$-genus. In other words, the $\Gamma$-genus is naturally $\mathbb{Z}_2$-equivariant, with respect to the Galois action defined by the Real structure on complex cobordism.

Away from (2), the Landweber-Novikov algebra of cobordism operations is an enveloping algebra of a $\mathbb{Z}_2$-graded Lie (NB not super-Lie) algebra. The odd part corresponds, in classical Lie theory, to the tangent space of the symmetric space associated to the complexification of a real Lie group; it acts transitively on $\text{Spec} \, H_*(Sp/U, \mathbb{Q})$, cf. [3, 17].

4. Closing remarks

4.1 The index map $MSp \to KO$ dates back to Conner and Floyd’s 1968 work on the relation of cobordism to $K$-theory, but seems to have received remarkably little attention: it is surely represented geometrically by a Dirac operator on $\mathbb{H}$-oriented manifolds, but the question of a nice construction seems not to have caught the differential geometers’ attention. In view of this, I have not tried to define an explicit family of deformations of such an operator over $Sp/U$.

4.2 R. Lu [8] has proposed an analytic interpretation of a variant of the $\Gamma$-genus of a complex-oriented $M$ as a $\mathbb{T}$-equivariant Euler class of its free loopspace, following Atiyah ([2]; see also [1]). Lu’s construction depends on a choice of polarization

$$
\begin{align*}
U/O \quad \overset{BGL_{\text{res}} \sim B(\mathbb{Z} \times BO)}{\longrightarrow} \quad \overset{LBU \sim B(LU) \sim B(\mathbb{Z} \times BU)}{\longrightarrow} \quad L\!M \quad \overset{BGL_{\text{res}} \sim B(\mathbb{Z} \times BO)}{\longrightarrow} \quad \overset{LBU \sim B(LU) \sim B(\mathbb{Z} \times BU)}{\longrightarrow} \quad \overset{\text{U/O} \sim \Omega(Sp/U)}{\longrightarrow} \quad \overset{\text{U/O} \sim \Omega(Sp/U)}{\longrightarrow} \quad U/O \sim \Omega(Sp/U)
\end{align*}
$$

of the tangent bundle of $LM$: that is, a lift of the map classifying its tangent bundle, to the restricted Grassmannian defined by writing loops in the tangent space as a sum of something like positive and negative-frequency components. Since $M$ is complex-oriented, such a lift exists, but is not in general unique: it can be twisted by a map

$$
LM \to U/O \sim \Omega(Sp/U)
$$

[6 §2, 7, 10]. The free loops on a map $\alpha : M \to Sp/U \in KO^2(X)$ thus define a twist

$$
L(\alpha) : LM \to L(Sp/U) \sim U/O \times Sp/U \to U/O ;
$$

its restriction to the subspace $M$ of constant loops defines a map to $Sp/U \sim \Omega Sp$ which acts naturally on $U/O \sim \Omega(Sp/U)$, and it seems reasonable to expect that Lu’s class for the polarized manifold $\langle M, L(\alpha) \rangle$ can be expressed
in terms of $\Gamma(M)$ evaluated at suitable values $s_{2k}(\alpha)$ of the deformation parameters.

4.3 I have also not tried to pin down the two-local properties of $\Gamma$, which seem quite interesting. Away from (2), $\text{Sp}/U$ is closely related [4] to $B\text{Sp}(\mathbb{Z})^+$, which is in turn related (via Siegel) to the $K$-theory spectrum of the symmetric monoidal category of Abelian varieties. This suggests that one might hope to see in the $\Gamma$-genus, some homotopy-theoretic residue of the intermediate Jacobians of complex projective manifolds.

4.4 Kontsevich’s original remarks were motivated by questions of quantization, and nothing in the discussion above says much about that: homotopy theory is often revealing about the bones of a subject, without resolving the surrounding analytical structures.

It is intriguing that the points 0, 1, $\infty$ on the projective line seem to have naturally associated genera and cohomology theories: the additive group at zero is related to de Rham theory, and the multiplicative group at one to $K$-theory. The association of the point at infinity with the Kontsevich genus suggests it might be related to a Galois theory of asymptotic expansions, along lines suggested by Cartier, Connes, Kreimer, Marcolli, and others.

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Department of Mathematics, Johns Hopkins University, Baltimore, Maryland 21218

E-mail address: jack@math.jhu.edu