6–DIMENSIONAL FJRW THEORIES OF THE SIMPLE–ELLIPTIC SINGULARITIES

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Abstract. We give explicitly in the closed formulae the genus zero primary potentials of the three 6–dimensional FJRW theories of the simple–elliptic singularity \(E_7\) with the non–maximal symmetry groups. For each of these FJRW theories we establish the CY/LG correspondence to the Gromov–Witten theory of the orbifold \(\mathbb{P}^{1}_{2,2,2,2}\). Namely, we give explicitly the R and S–group elements of Givental, whose action on the partition function of the Gromov–Witten theory of \(\mathbb{P}^{1}_{2,2,2,2}\) give the partition functions, coinciding with that of the FJRW theories we consider. We also show that by only using the axioms of Fan–Jarvis–Ruan the genus zero potential can only be reconstructed up to a scaling.

1. Introduction

Associated to a quasi–homogeneous polynomial \(W\), having an isolated critical point at the origin, and a group \(G\) of diagonal symmetries of \(W\), H.Fan, T.Jarvis an Y.Ruan constructed in \([FJR]\) the certain moduli space together with a virtual fundamental cycle, that are now known under the name of FJRW theory. Main application of this moduli space was first in the conjecture of Witten — this was Witten, who proposed existence of the moduli space, making the so–called Witten’s equation mathematically reasonable. Moreover it was shown in \([FJR]\) that for \(W\) defining ADE singularities, and certain symmetry groups of them, the partition function of the intersection numbers on this moduli space is a tau–function of the Kac–Wakimoto hierarchy. From this point of view this new moduli space of the pair \((W,G)\) was generalizing the moduli space of the \(r\)–spin curves.

Another important application of the FJRW theories lies in the area of mirror symmetry, where the A–side model of the Landau–Ginzburg orbifold \((W,G)\) is provided by the FJRW theory. Several mirror symmetry results about the FJRW theories were published in \([CR, MR, MS, KS, LLSS, SZ2, PS, BP]\). The explicit use of the FJRW theory virtual cycle appeared to be hard. To our knowledge, in all the examples known, FJRW theory is not computed by using the virtual fundamental cycle of Fan–Jarvis–Ruan itself, but only utilizing the certain properties, it satisfies. These properties were derived already in \([FJR]\), and called there “axioms”.

These axioms appeared to be powerful enough for the mirror symmetry purposes, where usually there is no need to compute the theory completely. For all mirror symmetry results

\[\text{Date: October 25, 2016.}\]
above just some small list of correlators was computed on the FJRW theory side. In particular up to now there is no closed formula even for the genus zero correlators of any FJRW theory except one particular case in [BP]. At the same time even in the computation of the certain correlators, only the most extreme possible symmetry groups $G$ are considered up to now, except one particular case in [SZ2].

The results of this paper come in two groups.

**FJRW theory.** In this paper we take the “axioms” of [FJR] as a definition of the FJRW theory. Namely, we consider the FJRW theory as a Cohomological field theory, satisfying certain additional list of axioms. We consider the simple–elliptic singularity $\tilde{E}_7$ represented by $W := x^4 + y^4 + z^2$ with the three symmetry groups:

$$
G_1 := \langle a_1, b_1, c_1 \rangle :
\begin{align*}
    a_1(x, y, z) &:= (\sqrt{-1}x, \sqrt{-1}y, z), \\
    b_1(x, y, z) &:= (x, -y, z), \\
    c_1(x, y, z) &:= (x, y, -z),
\end{align*}
$$

$$
G_2 := \langle a_2, b_2 \rangle :
\begin{align*}
    a_2(x, y, z) &:= (\sqrt{-1}x, \sqrt{-1}y, -z), \\
    b_2(x, y, z) &:= (x, -y, z),
\end{align*}
$$

$$
G_3 := \langle a_3, b_3 \rangle :
\begin{align*}
    a_3(x, y, z) &:= (\sqrt{-1}x, \sqrt{-1}y, z), \\
    b_3(x, y, z) &:= (x, y, -z),
\end{align*}
$$

All these groups are not maximal for $W$, and this is the first novelty of this paper. All three FJRW theories of $(\tilde{E}_7, G_k)$ are 6–dimensional. By using the “axioms” of [FJR] only, we reconstruct the genus zero potentials of these FJRW theories up to the scaling of the variables. We give the closed formulae for the three genus zero potentials. It turns out that two of these genus zero potentials can be reconstructed from the axioms only up to the scaling. This shows in particular that for the questions, where the particular values of the correlators are important, it’s not enough to consider the axioms of FJRW theory only. It turns out also that the third genus zero potential we compute has irrational coefficients. This potential can be written in $\mathbb{Q}[[t]]$ only after a rescaling of the variables.

**CY/LG correspondence.** Currently working with the non–maximal symmetry groups on the FJRW theory side makes it hard to speak about the mirror symmetry. This is because the B side should be considered with the non–trivial symmetry group then and an orbifolded Saito theory is not yet constructed (see [BTW1, BTW2]). However one could anyway consider one mirror symmetry conjecture in this setting too — the CY/LG correspondence conjecture. It suggests that the partition functions of the two different A–side models, being both mirror dual to the same B–model, are connected by a Givental’s action (acting on the space of all partition functions).

In this paper for the three FJRW theories of the pairs $(\tilde{E}_7, G_k)$ as above we establish also the CY/LG correspondence. Namely, we provide explicitly the R–matrices of Givental, s.t. up to the certain S–action of Givental the partition function of the FJRW theory is obtained by applying the Givental’s action to the partition function of the Gromov–Witten theory of the orbifold $\mathbb{P}^1_{2,2,2,2}$. 
Theorem (Theorem 5.2 in the text). Up to the certain different Givental’s $ S $–actions $ S^{(k)} $ the partition functions of the FJRW theories $ (\tilde{E}_7, G_k) $, $ k = 1, 2, 3 $ are connected to the partition function of the Gromov–Witten theory of $ \mathbb{P}^{1,2,2,2,2} $ by the same Givental’s $ R $–action of:

$$ R^{\sigma'} := \exp \left( \begin{pmatrix} 0 & \ldots & \sigma' \\ \vdots & 0 & \vdots \\ 0 & \ldots & 0 \end{pmatrix} z \right), \quad \text{for} \quad \sigma' = -\frac{1}{2\pi^2} \left( \frac{3}{4} \right)^4, $$

so that holds:

$$ Z^{(\tilde{E}_7, G_k)} = \hat{R}^{\sigma'} \cdot \hat{S}^{(k)} \cdot Z^{\mathbb{P}^{1,2,2,2,2}}, \quad k = 1, 2, 3. $$

The $ S $–actions are usually considered to be of little importance because they only stand for the shift of coordinates and a basis choice (in the Chen–Ruan cohomology ring in our case), and hence do not affect “the geometry” of the Cohomological field theory. However the $ S $–actions are very important for the explicit computations and are reconstructed explicitly in our paper.

For the simple–elliptic singularities CY/LG correspondence was also considered in [SZ2] in a beautiful manner. It was explained there in terms of a natural operation (Cayley transform) on the space of quasi–modular forms. However [SZ2] didn’t derive an $ R $–action of Givental giving this CY/LG correspondence. It was first [BP], where the explicit $ R $–action was given for the simple–elliptic singularities, but with the maximal symmetry group only.

The proof of the theorem uses extensively the explicit formulae for the genus zero potentials of $ \mathbb{P}^{1,4,2,4,2} $ Gromov–Witten theories and explicitly computed FJRW theories of $ (\tilde{E}_7, G_k) $. We utilize the fact that both Gromov–Witten theories can be written via the quasi–modular forms. At the same time, even missing the orbifolded Saito theory, we consider the certain $ \text{SL}(2, \mathbb{C}) $–action on the space of WDVV equation solutions, that allows us to connect the genus zero partition functions of $ \mathbb{P}^{1,2,2,2,2} $ and $ (\tilde{E}_7, G_k) $. This action was proposed in [BT] as a model for the primitive form change for the Saito theory and was shown to be represented by the particular Givental’s action in [B].

Organization of the paper. In Section 2 we define the FJRW theory as a CohFT, subject to the certain list of additional axioms. Gromov–Witten theory of elliptic orbifolds is reviewed in Section 3. We make certain preparations there, needed to perform the computations. In Section 4 we define the group action on the space of CohFTs. Section 5 is devoted to the CY/LG correspondence, where we give the proof of the main theorem with the help of computations, performed in Section 6. This is the last section too, where we give explicit formulae for the primary potentials of the FJRW theories of $ (\tilde{E}_7, G_k) $, $ k = 1, 2, 3 $ as above — see Propositions 6.1, 6.4 and 6.6. Certain useful formulae are given in Appendix.

Acknowledgement. The work of A.B. was partially supported by the DFG grant He2287/4–1 (SISYPH). The author is also grateful to Nathan Priddis, Amanda Francis and Yefeng Shen for the useful discussions and email correspondence.
2. FJRW theory

In this section we define the FJRW theory axiomatically as a Cohomological field theory $\Lambda^{(W,G)}$, satisfying some additional system of axioms, as given in Theorem 4.1.8 of [FJR]. In this way all our conclusions hold true for the FJRW theories of $(W,G)$, defined through the virtual fundamental cycle. At the same time it’s important to note that to our knowledge almost all computations done up to now in FJRW theories only use these “axioms” of [FJR].

2.1. Notations. Throughout this paper let $W = W(x) = W(x_1, \ldots, x_N)$ be a quasi–homogeneous polynomial. Namely there are integers $d, w_1, \ldots, w_N$, gcd$(w_1, \ldots, w_N) = 1$, s.t. for any $\lambda \in \mathbb{C}^*$ holds $W(\lambda^{w_1}x_1, \ldots, \lambda^{w_N}x_N) = \lambda^dW(x_1, \ldots, x_N)$. Denote $q_k := a_k/d$ for $k = 1, \ldots, N$. Assume also $0 \in \mathbb{C}^N$ to be an isolated critical point of $W$ and the weight set to be unique.

Denote $e[\alpha] := \exp(2\pi\sqrt{-1}\alpha)$ for any $\alpha \in \mathbb{Q}$. Let $G_W := \{\alpha \in (\mathbb{C}^*)^N \mid W(\alpha \cdot x) = W(x)\}$ be the so–called maximal group of symmetries of $W$ (also sometimes denoted by $G_{\text{max}}$). It’s non–empty because for $J := (e[q_1], \ldots, e[q_N])$, the group $\langle J \rangle$ is a non–empty subgroup of $G_W$.

The group $G \subseteq G_W$ is called admissible if $\langle J \rangle \subseteq G$. In what follows, we will assume $d$, the degree of $W$, to be also the exponent of $G_W$, i.e. for each $h \in G_W$, $h^d = \text{id}$. This is not the case in general, but holds in our examples.

2.2. Cohomological field theories. Let $(V, \eta)$ be a finite–dimensional vector space with a non–degenerate pairing. Consider a system of linear maps

$$\Lambda_{g,n} : V^\otimes n \rightarrow H^*(\overline{M}_{g,n}),$$

defined for all $g, n$ such that $\overline{M}_{g,n}$ exists and is non–empty. The set $\Lambda_{g,n}$ is called a cohomological field theory on $(V, \eta)$, or CohFT, if it satisfies the following axioms.

CohFT 1. $\Lambda_{g,n}$ is equivariant with respect to the $S_n$–action, permuting the factors in the tensor product and the numbering of marked points in $\overline{M}_{g,n}$.

CohFT 2. For the gluing morphism $\rho : \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \rightarrow \overline{M}_{g_1+g_2,n_1+n_2}$ we have:

$$\rho^*\Lambda_{g_1+g_2,n_1+n_2} = (\Lambda_{g_1,n_1+1} \cdot \Lambda_{g_2,n_2+1}, \eta^{-1}),$$

where we contract with $\eta^{-1}$ the factors of $V$ that correspond to the node in the preimage of $\rho$.

CohFT 3. For the gluing morphism $\sigma : \overline{M}_{g,n+2} \rightarrow \overline{M}_{g+1,n}$ we have:

$$\sigma^*\Lambda_{g+1,n} = (\Lambda_{g,n+2}, \eta^{-1}),$$

where we contract with $\eta^{-1}$ the factors of $V$ that correspond to the node in the preimage of $\sigma$.

In this paper we further assume the CohFT $\Lambda_{g,n}$ to be unital — i.e. there is a fixed vector $1 \in V$ called the unit such that the following axioms are satisfied.

U 1. For every $\alpha_1, \alpha_2 \in V$ we have: $\eta(\alpha_1, \alpha_2) = \Lambda_{0,3}(1 \otimes \alpha_1 \otimes \alpha_2)$. 
Let \( \pi : \overline{M}_{g,n+1} \rightarrow \overline{M}_{g,n} \) be the map forgetting the last marking, then:
\[
\pi^* \Lambda_{g,n}(\alpha_1 \otimes \cdots \otimes \alpha_n) = \Lambda_{g,n+1}(\alpha_1 \otimes \cdots \otimes \alpha_n \otimes 1).
\]

A CohFT \( \Lambda_{g,n} \) on \((V, \eta)\) is called quasihomogeneous if the vector space \( V \) is graded by a linear map \( \deg : V \rightarrow \mathbb{Q} \) and there is a number \( \delta \), such that for any \( \alpha_1, \ldots, \alpha_n \in V \) holds:
\[
((g - 1)\delta + n) \Lambda_{g,n}(\alpha_1, \ldots, \alpha_n) = \left( \frac{1}{2} \deg_{\text{coh}} + \sum_{k} \deg(\alpha_k) \right) \Lambda_{g,n}(\alpha_1, \ldots, \alpha_n),
\]
where \( \deg_{\text{coh}} \) is the (real) cohomology class degree in \( H^*(\overline{M}_{g,n}) \).

Let \( \psi_i \in H^*(\overline{M}_{g,n}) \), \( 1 \leq i \leq n \) be the so-called \( \psi \)-classes. The genus \( g \), \( n \)-point correlators of the CohFT are the following numbers:
\[
\langle \tau_{a_1}(e_{\alpha_1}) \cdots \tau_{a_n}(e_{\alpha_n}) \rangle_{\Lambda_{g,n}} := \int_{\overline{M}_{g,n}} \Lambda_{g,n}(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}) \psi_1^{a_1} \cdots \psi_n^{a_n}.
\]

Denote by \( F_g \) the generating function of the genus \( g \) correlators, called genus \( g \) potential of the CohFT:
\[
F_g := \sum_{\alpha, a} \frac{\langle \tau_{a_1}(e_{\alpha_1}) \cdots \tau_{a_n}(e_{\alpha_n}) \rangle_{g,n}}{\text{Aut}(\{\alpha, a\})} t_{a_1, a_1} \cdots t_{a_n, a_n}.
\]

It is useful to assemble the correlators into a generating function called partition function of the CohFT \( Z := \exp \left( \sum_{g \geq 0} \hbar^{g-1} F_g \right) \). We will also make use of the so-called primary genus \( g \) potential that is a function of the variables \( t^\alpha := t^{0, \alpha} \) defined as follows:
\[
F_g := F_g |_{t^\alpha := t^{0, \alpha}, \ t^{\ell, \alpha} = 0, \forall \ell \geq 1}
\]

what is also sometimes called a restriction to the small phase space.

Due to the topology of the space \( \overline{M}_{0,n} \) the small phase space potential of a CohFT on \((V, \eta)\) satisfies the so-called WDVV equation. For any four fixed \( 1 \leq i, j, k, l \leq \dim(V) \) holds:
\[
\sum_{p,q} \frac{\partial^3 F_0}{\partial t^i \partial t^j \partial t^p} \eta^{p,q} \frac{\partial^3 F_0}{\partial t^k \partial t^l \partial t^q} = \sum_{p,q} \frac{\partial^3 F_0}{\partial t^i \partial t^k \partial t^p} \eta^{p,q} \frac{\partial^3 F_0}{\partial t^j \partial t^l \partial t^q}.
\]

It’s important to note that the function \( F_0 \) is reconstructed unambiguously from \( F_0 \) due to the topological recursion relation in genus zero.

2.3. Moduli of W-curves. An \( n \)-pointed orbifold curve \( C \) is a 1-dimensional Deligne-Mumford stack with at worst nodal singularities with orbifold structure only at the marked points and the nodes. Moreover the orbifold structure is required to be balanced at the nodes.

A \( d \)-stable curve is a proper connected orbifold curve \( C \) of genus \( g \) with \( n \) distinct smooth markings \( p_1, \ldots, p_n \) such that the \( n \)-pointed underlying coarse curve is stable, and all the stabilizers at nodes and markings have order \( d \). The moduli stack \( \overline{M}_{g,n,d} \) parameterizing...
such curves is proper, smooth and has dimension $3g - 3 + n$. It differs from the moduli space of curves only because of the stabilizers over the normal crossings.

Let $W$ be written as

$$W = \sum_{i=1}^{M} c_i \prod_{k=1}^{N} x_k^{a_{ik}}, \quad a_{ik} \in \mathbb{N}, \ c_i \in \mathbb{C}.$$ 

Given line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_N$ on the $d$–stable curve $\mathcal{C}$, we define the line bundle

$$W_i(\mathcal{L}_1, \ldots, \mathcal{L}_N) := \bigotimes_{k=1}^{N} \mathcal{L}_k^{a_{ik}}, \quad 1 \leq i \leq M.$$ 

**Definition 2.1.** A $W$–structure is the data $(\mathcal{C}, p_1, \ldots, p_n, \mathcal{L}_1, \ldots, \mathcal{L}_N, \varphi_1, \ldots \varphi_N)$, where $\mathcal{C}$ is an $n$–pointed $d$–stable curve, the $\mathcal{L}_k$ are line bundles on $\mathcal{C}$ satisfying

$$W_i(\mathcal{L}_1, \ldots, \mathcal{L}_N) \cong \omega_{\log} = \omega(p_1 + \cdots + p_n),$$

and for each $k$, $\varphi_k : \mathcal{L}_k^{\otimes d} \to \omega_{\log}^{w_k}$ is an isomorphism of line bundles.

One can consider the map $st$ forgetting the data $\mathcal{L}_k$ and $\varphi_k$ of a $W$–structure. It gives a morphism $st : W_{g,n} \to \overline{M}_{g,n}$.

**Theorem 2.2** (Fan–Jarvis–Ruan, [FJR]). There exists a moduli stack of $W$–structures, denoted by $W_{g,n}$, possessing also the suitable virtual fundamental cycle $[W_{g,n}]_{vir}$, defining the CohFT of the pair $(W, G)$ by the morphism $st : W_{g,n} \to \overline{M}_{g,n}$.

### 2.4. FJRW CohFT of a simple–elliptic singularity.

Denote $\Omega_W := \Omega_{\mathbb{C}^N, 0} / \left( dW \wedge d_{\mathbb{C}^N, 0}^{N-1} \right)$. It’s a finite dimensional rank one module over the Milnor ring of $W$ in case when it defines an isolated singularity. It’s equipped with the non–degenerate bilinear form $\langle \cdot, \cdot \rangle_W$ — the Poincaré residue pairing.

For any $h \in G$ denote by $\text{Fix}(h) \subseteq \mathbb{C}^N$ the fixed locus of $h$ and $N_h := \dim(\text{Fix}(h))$. Define $W^h := W \mid_{\text{Fix}(h)} : \mathbb{C}^{N_h} \to \mathbb{C}$. Let also $G_{\text{nar}} := \{ h \in G \mid N_h = 0 \}$ be the set of so-called narrow sector group elements. Denote further $\Omega_h := (\Omega_{W^h})^G$ — the $G$–invariant subspace of $\Omega_{W^h}$. Note that $\text{Fix}(h) = \text{Fix}(h^{-1})$. Let $\psi_h$ be an isomorphism $\Omega_h \cong \Omega_{h^{-1}}$.

Denote also by $\hat{c} = \sum_{k=1}^{N} (1 - 2q_k)$ the central charge of $W$.

**Definition 2.3.** We call the CohFT $\Lambda = \Lambda_{g,n}^{(W, G)}$ a FJRW CohFT of $(W, G)$ if it satisfies the following list of axioms 2.4.1 – 2.4.3.

#### 2.4.1. State space.

$\mathcal{H}_{W, G} := \oplus_{h \in G} \mathcal{H}_h$, where as a vector space each $\mathcal{H}_h \cong \Omega_h$. Let $\mathcal{H}_{W, G}$ be equipped with the pairing $\langle \cdot, \cdot \rangle_{W, G} := \oplus_{h \in G} \langle \cdot, \cdot \rangle_h$, for $\langle \cdot, \cdot \rangle_h : \mathcal{H}_h \times \mathcal{H}_{h^{-1}} \to \mathbb{C}$ defined by $\langle \cdot, \cdot \rangle_h := \langle \cdot, \psi_h(\cdot) \rangle_{W^h}$. 

In what follows $\forall h \in G$ by an $H_{W,G}$ element $\alpha_h$ we will always assume a vector, belonging to $H_h \subset H_{W,G}$. Let $H_{W,G}$ be graded by $\deg_W : H_{W,G} \to \mathbb{Q}$:

$$\deg_W(\alpha_h) := Nh + 2\iota(g), \quad \alpha_h \in H_h,$$

where the degree shifting number $\iota(h)$ is defined as follows. For any $h \in G$, let the numbers $\Theta_k^h \in \mathbb{Q} \cap [0,1)$ be s.t. $h$ is represented by $(e^{[\Theta_1^h]}, \ldots, e^{[\Theta_N^h]}) \in \text{SL}(2, \mathbb{C})$, then

$$\iota(h) := \sum_{k=1}^N (\Theta_k^h - q_k).$$

2.4.2. **Degree.** The class $\Lambda_{g,n}(\alpha_{h_1}, \ldots, \alpha_{h_n})$ vanishes if $\hat{c}(g - 1) + \sum_i \iota_{h_i} \notin \mathbb{Z}$. Otherwise it has the following degree

$$2 \left( (\hat{c} - 3)(1 - g) + n - \sum_{i=1}^n \iota(h_i) - \sum_{i=1}^n \frac{Nh_i}{2} \right).$$

2.4.3. **Selection rule.** The class $\Lambda_{g,n}(\alpha_{h_1}, \ldots, \alpha_{h_n})$ is zero unless for all $1 \leq k \leq N$ holds:

$$q_k(2g - 2 + n) - \sum_{i=1}^n \Theta_{k_i}^h \in \mathbb{Z}.$$

2.4.4. **$G_W$–invariance.** Consider the action of $G_W$ on each $\Omega_h$, $\forall h \in G$. Extending this action to $H_{W,G}$ we require the CohFT $\Lambda_{g,n}^{(W,G)}$ (considered as a system of linear maps) to be invariant under this action.

2.4.5. **Concavity.** Suppose that $h_i \in G^{\text{nar}}$ for all $i = 1, \ldots, n$. Let $\pi$ be the projection from the universal curve of the moduli space and $L_1, \ldots, L_N$ be the universal $W$–structure. If $\pi_\ast \left( \bigoplus_{k=1}^N L_k \right) = 0$, then holds:

$$\Lambda_{g,n}(\alpha_{h_1}, \ldots, \alpha_{h_n}) = \left[ \frac{G!}{\deg(st)} \right]^g \text{PD} \text{st}_{\ast} c_{\text{top}} \left( R_1^{\pi_{\ast}} \bigoplus_{k=1}^N L_k \right) \psi.$$ 

The subspace of $H_{W,G}$, generated by $\alpha_{h_1}, \ldots, \alpha_{h_n}$ is called **concave**.

2.5. **Remarks on the axioms.** The list of properties that hold by a virtual fundamental cycle of FJRW is much longer (Theorem 4.1.8 in [FJR]). The choice of axioms we made above is the minimal one, needed for our purposes.

The state space axiom is usually introduced via the so–called *Lefschetz thimbles* of $W^h$. However they are only used as the generators of the vector spaces, that are isomorphic to those we used — $\Omega_h$.

Degree axiom we formulate is exactly Degree axiom of Fan–Jarvis–Ruan, however there is a notational difference because we give only the degree of a cohomology class in $\overline{M}_{g,n}$ while in [FJR] the state space degrees are counted too.
It’s immediate to note that the CohFT $\Lambda^{(W,G)}$ is quasi–homogeneous with $\delta := 3 - \hat{c}$ and grading $\deg_W$ on $\mathcal{H}_{W,G}$.

2.6. **FJRW theory of a simple–elliptic singularity.** To write down the primary potential of a FJRW theory we make use of the following notation.

Let $h \in G$ be s.t. $N_h \neq 0$. Fixing the basis $\{ \phi_k^{(h)}(x) d^{N_h} x \}$ of $\mathcal{H}_{W,G}$ we will consider the basis $\{ [h, \phi_k^{(h)}(x)] \}_{h,k}$ of $\mathcal{H}_{W,G}$. Associate also to any $[h, \phi_k^{(h)}(x)]$ the variable $t_{\phi_k^{(h)}(x), h}$.

For $h \in G$, s.t. $N_h = 0$ we denote $\alpha_h \in \mathcal{H}_h \subset \mathcal{H}_{W,G}$ by $[h,1]$ and associate to it the variable $t_h$.

In the case of simple–elliptic singularities concavity axiom is in particular powerful.

**Proposition 2.4.** Let $W = x_1^4 + x_2^4 + x_3^2$ define a simple–elliptic singularity and $G$ be any admissible group of its symmetries. Then for any $h_1, \ldots, h_n$, s.t. $N_{h_k} = 0$ for all $1 \leq k \leq n$ the subspace generated by $\alpha_{h_1}, \ldots, \alpha_{h_n}$ is concave.

**Proof.** The proof copies proof of Proposition 1.6 in [PS]. It’s enough to count the line bundle degrees of $L_k$. Because $\sum_{k=1}^3 q_k = 1$ and $q_k < 1$ for a point $(C, p_1, \ldots, p_n, L_1, L_2, L_3, \phi_1, \phi_2, \phi_3)$ on each irreducible component $C_v$ of $C$ holds

$$\deg( |L_k|_{C_v} ) \leq q_k \left( \# \text{nodes}(C_v) - 2 \right) < \# \text{nodes}(C_v) - 1,$$

where $|L_k|$ denotes the pushforward of $L_k$ to the underlying curve of $C$. The inequality obtained finally shows that $|L_k|$ has no section. $\square$

**Corollary 2.5.** For a simple–elliptic singularity $W$ let $F_0^{(W,G)}$ and $F_0^{(W,G_W)}$ be the genus zero primary FJRW potentials of $(W,G)$ and $(W,G_{W})$ respectively. Then holds:

$$F_0^{(W,G)} \mid_{t_{\phi,h} = 0, \ h \notin G_{\text{nar}}} = F_0^{(W,G_W)} \mid_{t_{\phi,h} = 0, \ h \notin G_{\text{nar}}}$$

**Proof.** As the vector space $\mathcal{H}_{W,G}$ is defined as the direct sum over all $G$ elements, if $\alpha_h \in \mathcal{H}_{W,G}$, then there is a vector $\alpha'_h \in \mathcal{H}_h \subset \mathcal{H}_{W,G_{W}}$. These two vectors can be identified because $\Omega_h \cong \mathbb{C}$. The rest follows from Concavity axiom because the formula for the correlators of $\Lambda_{0,n}^{(W,G)}$ and $\Lambda_{0,n}^{(W,G_W)}$ is literally the same. $\square$

3. **Gromov–Witten theory of elliptic orbifolds**

We skip completely the definition of the Gromov–Witten theory here, referencing an interested reader to [A]. For the cases we are interested in — of the elliptic orbifolds, we define the Gromov–Witten theory in genus zero by giving explicitly the CohFT potentials, found in [ST] [BP] [SZ].

For the so–called elliptic orbifolds $X_2 := \mathbb{P}^1_{2,2,2,2}$ and $X_4 := \mathbb{P}^1_{4,4,2}$ fix the bases of the Chen–Ruan cohomology $H^*_{or}(X_k)$ as follows.
Let $\Delta_0, \Delta_{-1}$ be the degree 0 and degree 2 generators of $H^*(\mathbb{P}^1)$ respectively, viewed as untwisted sector of $H^*_{orb}(\mathcal{X}_k)$. Let $\Delta_{i,j}$ be the twisted sector generators, corresponding to the $i$-th point with a non-trivial isotropy group. We have:

$$H^*_{orb}(\mathcal{X}_2) \cong \mathbb{Q}\Delta_0 \oplus \mathbb{Q}\Delta_{-1} \bigoplus_{i=1}^{4} \mathbb{Q}\Delta_{i,1}, \quad H^*_{orb}(\mathcal{X}_4) \cong \mathbb{Q}\Delta_0 \oplus \mathbb{Q}\Delta_{-1} \bigoplus_{j=1}^{3} \mathbb{Q}\Delta_{1,j} \bigoplus_{j=1}^{3} \mathbb{Q}\Delta_{2,j} \bigoplus \mathbb{Q}\Delta_{3,1}.$$ 

The ring $H^*_{orb}(\mathcal{X}_k)$ is also endowed with the pairing $\eta$, an analogue of the Poincaré pairing.

Gromov–Witten theory of $\mathcal{X}_k$ expresses the intersection theory of the moduli space of the stable orbifold maps to $\mathcal{X}_k$. We will be only working with the CohFT it defines on the moduli space of stable curves. Also for brevity we will abbreviate Gromov–Witten theory just by GW in what follows.

The genus 0 potential of the Gromov–Witten theory of $\mathcal{X}_k$ is a function of the variables $t$, being dual to the basis element fixed, and also of the formal Novikov variable $q_{\text{formal}}$. We will fix the variables $t$ differently in what follows, but we always keep $t_0, t_{-1}$ to correspond to the basis elements $\Delta_0, \Delta_{-1}$ respectively.

### 3.1. Novikov variable

The Novikov variable $q = q_{\text{formal}}$ is used to keep track of the homology class — it appears in the potential as $q^\beta$, where $\beta \in H_2(X)$. In our case $\dim(H_2(\mathcal{X}_k)) = 1$ and by using Divisor equation (of the GW theory) the Novikov variable $q$ can be identified with $\exp(t_{-1})$ (cf. [SZ1, Section 1.2]). The correlation functions of the genus 0 potentials after such an identification appear to coincide with the Fourier expansions of the certain functions. However it’s useful to work with the functions itself rather than the Fourier expansions of them. To do this we make another identification of the Novikov variable that depends on the orbifold in question:

$$q_{\text{formal}} = \exp(t_{-1}) = \exp\left(\frac{2\pi \sqrt{-1}\tau}{k}\right) =: q_k, \quad \text{for the orbifold } \mathcal{X}_k. \quad (1)$$

This identification also affects the term of the partition function, fixed by the pairing by Axiom U1. Because of this we can’t just take the change of the variables $t_{-1} = 2\pi \sqrt{-1}\tau/k$ and will treat this identification carefully.

At the same time only after making an identification of the formal variable we get the clear holomorphicity properties of the genus zero potential and are able to introduce suitable group action we use later in the text. For this purpose we introduce new functions — analytic potentials of $\mathbb{P}^1_{2,2,2}$ and $\mathbb{P}^1_{4,4,2}$ GW theories in order to make the statements about the genuine genus zero potentials. One can do the same for the remaining two elliptic orbifolds $\mathbb{P}^1_{3,3,3}$ and $\mathbb{P}^1_{6,3,2}$ as well.

### 3.2. Gromov–Witten theory of $\mathbb{P}^1_{2,2,2}$

The genus zero potential of this GW theory was found explicitly by Satake–Takahashi in [ST]. We present their result here in a slightly modified form that will be useful for us in what follows.
Let the variables \{t_0, t_{-1}, t_1, t_2, t_3, t_4\} be dual to the following basis of \(H^*_{orb}(\mathbb{P}^1_{2,2,2})\) (recall the notation above)
\[
\left\{ \Delta_0, \Delta_{-1}, \frac{1}{\sqrt{2}}(\Delta_{2,1} - \Delta_{4,1}), \frac{1}{\sqrt{2}}(\Delta_{2,1} + \Delta_{4,1}), \frac{1}{\sqrt{2}}(\Delta_{1,1} - \Delta_{3,1}), \frac{1}{\sqrt{2}}(\Delta_{1,1} + \Delta_{3,1}) \right\}.
\]

Consider the functions \(\psi_k\), defined by the following formal series in \(q\):
\[
\psi_2(q) := \frac{1}{2} + 2 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2nq^n}{1 - q^n}, \quad \psi_3(q) := 2 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2nq^{n/2}}{1 - q^n}, \quad \psi_4(q) := -2 \sum_{n=1}^{\infty} \frac{2nq^{n/2}}{1 - q^n}.
\]

In the basis fixed the primary genus zero potential of the GW theory in question assumes the following form:
\[
F_0^{2,2,2} = \frac{1}{2} t_0^2 t_{-1} + \frac{1}{4} t_0 \sum_{k=2}^{5} t_k^2 - \frac{1}{16} (t_3^2 t_4^2 + t_1^2 t_2^2) \psi_4(q^2) - \frac{1}{16} (t_1^2 t_3^2 + t_2^2 t_4^2) \psi_2(q^2)
\]
\[
- \frac{1}{16} (t_2^2 t_3^2 + t_1^2 t_4^2) \psi_3(q^2) - \frac{1}{96} \left( \sum_{k=2}^{5} t_k^4 \right) \left( \sum_{k=2}^{4} \psi_k(q^2) \right), \quad q = \exp(t_{-1}).
\]

WDVV equation on this genus zero potential is equivalent to the following system of PDE's on the functions \(\{X_2(q), X_3(q), X_4(q)\}\), satisfied by the triple \(\{\psi_2(q^2), \psi_3(q^2), \psi_4(q^2)\}\):
\[
q \frac{\partial}{\partial q} X_2(q) = X_2(q) (X_3(q) + X_4(q)) - X_3(q) X_4(q),
\]
\[
q \frac{\partial}{\partial q} X_3(q) = X_3(q) (X_2(q) + X_4(q)) - X_2(q) X_4(q),
\]
\[
q \frac{\partial}{\partial q} X_4(q) = X_4(q) (X_2(q) + X_3(q)) - X_2(q) X_3(q),
\]
that we call a Halphen's system of equations.

Note that in all the steps above we didn't use the relation between \(q\) and \(t_{-1}\). For all \(\tau \in \mathbb{H}\) consider the Jacobi theta constants \(\vartheta_k(\tau)\) to be the holomorphic functions on \(\mathbb{H}\) given by the following Fourier series:
\[
\vartheta_2(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi \sqrt{-1} (n-1/2)^2}, \quad \vartheta_3(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi \sqrt{-1} n^2}, \quad \vartheta_4(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi \sqrt{-1} n^2}.
\]
The function \(\vartheta_1(\tau)\) is skipped because it vanishes identically. Consider the functions:
\[
X_k^\infty(\tau) := 2 \partial_\tau \log \vartheta_k(\tau), \quad X_k^{\infty}(q) := \frac{1}{\pi \sqrt{-1}} X_k^\infty \left( \frac{\tau}{\pi \sqrt{-1}} \right), \quad k = 2, 3, 4.
\]
Then the triple \( \{X_2^\infty(\tau), X_3^\infty(\tau), X_4^\infty(\tau)\} \) is a solution of Halphen’s system of equations:

\[
\begin{align*}
\frac{\partial}{\partial \tau} X_2(\tau) &= X_2(\tau) (X_3(\tau) + X_4(\tau)) - X_3(\tau) X_4(\tau), \\
\frac{\partial}{\partial \tau} X_3(\tau) &= X_3(\tau) (X_2(\tau) + X_4(\tau)) - X_2(\tau) X_4(\tau), \\
\frac{\partial}{\partial \tau} X_4(\tau) &= X_4(\tau) (X_2(\tau) + X_3(\tau)) - X_2(\tau) X_3(\tau),
\end{align*}
\]

(3)

and \( \{X_2^\infty(q), X_3^\infty(q), X_4^\infty(q)\} \) give solution to Eq. (2). We have the equality:

\[ \pi \sqrt{1 - \psi_k(q)} = X_k^\infty(\tau). \]

**Notation 3.1.** In what follows we denote by \( F_{an}^{p_{1,2,2,2}} \) the analytic potential of \( \mathbb{P}^4_{1,2,2,2} \):

\[
F_{an}^{p_{1,2,2,2}} = \left\{ \begin{array}{ll}
\frac{1}{2} t^2_2 \tau + \frac{1}{4} t_0 \sum_{k=2}^5 t_k^2 - \frac{1}{16} \left(t_3^2 t_4^2 + t_2^2 t_5^2\right) X_4^\infty(\tau) - \frac{1}{16} \left(t_2^3 t_4^2 + t_2^2 t_4^2\right) X_2^\infty(\tau) \\
- \frac{1}{16} \left(t_3^2 t_4^2 + t_2^2 t_5^2\right) X_3^\infty(\tau) - \frac{1}{96} \left(\sum_{k=2}^5 t_k^2\right) \left(\sum_{k=2}^4 X_k^\infty(\tau)\right)
\end{array} \right.
\]

**Proposition 3.2.** The function \( F_{an}^{p_{1,2,2,2}} \) is holomorphic on \( \mathbb{C}^5 \times \mathbb{H} \) and is solution to the WDVV equation.

**Proof.** This is straightforward by using the definition of the function \( X_k^\infty(\tau) \), Eq. (2) and the properties of \( F_{an}^{p_{1,2,2,2}} \). \( \square \)

The connection between the functions \( F_{0}^{p_{1,2,2,2}} \) and \( F_{an}^{p_{1,2,2,2}} \) is obvious — we have applied the relation \( q_{formal} = q_k(\tau) \), however in order to obtain the function, that is solution to the WDVV equation, we had to make an additional rescaling. In what follows we are going to use the second function (having only an indirect connection to the GW theory) in order to make statement about the first function (being indeed a true potential of the GW theory).

Comparing to the functions \( \psi_k(q) \) and \( X_k^\infty(q) \), big advantage of the functions \( X_k^\infty(\tau) \) is that they are holomorphic in \( \mathbb{H} \). Apart from the holomorphicity property, the functions \( X_k^\infty(\tau) \) enjoy another major advantage — there is a \( \text{SL}(2, \mathbb{C}) \) group action on the space of solutions to the Halphen’s system Eq. (3) written in \( \tau \), but not on that of Eq. (2).

### 3.3. Gromov–Witten theory of \( \mathbb{P}^4_{1,4,2} \)

We write this GW theory in the basis \( \Delta_{i,j} \), we have considered at the start of the section. Let also the coordinates \( t_{i,j} \) be corresponding to this basis elements. The genus 0 potential of this orbifold is written completely via the functions \( x(q) \), \( y(q) \), \( z(q) \) and \( w(q) \), defined by:

\[
\frac{1}{4} x(q) := \langle \Delta_{1,1}, \Delta_{1,1}, \Delta_{1,2} \rangle_{0,3}, \quad \frac{1}{4} y(q) := \langle \Delta_{1,2}, \Delta_{2,1}, \Delta_{2,1} \rangle_{0,3}
\]
The functions \( x(q), y(q), z(q), w(q) \) have the following expression:

\[
x(q) = (\vartheta_3(q^8))^2, \quad y(q) = (\vartheta_2(q^8))^2, \quad z(q) = (\vartheta_2(q^4))^2, \quad w(q) = \frac{1}{3} \left( f(q^4) - 2f(q^8) + 4f(q^{16}) \right)
\]

for the functions \( \vartheta_k(q) \) as above and \( f(q) := 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1 - q^k} \).

**Proposition 3.3** (Appendix A in [BP] and Section 3.2.3 in [SZ1]). The potential \( P_{\text{poly}}^{4,4,2} \) has an explicit form via the functions defined above. Namely there exists the polynomial

\[
P_{\text{poly}}^{4,4,2} = P_{\text{poly}}^{4,4,2}(t_0, t_{-1}, t_{i,j}, x, y, z, w) \in \mathbb{Q}[t_0, t_{-1}, t_{i,j}, x, y, z, w],
\]

where

\[
P_{\text{poly}}^{4,4,2}(t_0, t_{-1}, t_{i,j}, x, y, z, w) = P_{\text{poly}}^{4,4,2}(t_0, t_{-1}, t_{i,j}, x(y(q), z(q), w(q))),
\]

for \( x(q), y(q), z(q) \) and \( w(q) \) as above. Moreover the following homogeneity property holds:

\[
P_{\text{poly}}^{4,4,2}(t_0, t_{-1}, t_{i,j}, x, y, z, w) = \frac{1}{\alpha^2} P_{\text{poly}}^{4,4,2}(t_0, \alpha^2 \cdot t_{-1}, \alpha \cdot t_{i,j}, \frac{x}{\alpha^2}, \frac{y}{\alpha}, \frac{z}{\alpha^2}, \frac{w}{\alpha^2}),
\]

for any \( \alpha \in \mathbb{C}^* \).

In what follows the function \( z(q) \) will be sometimes skipped because the following identity holds:

\[
z(q)^2 = 4x(q)y(q).
\]

It was found by Shen–Zhou [SZ1] that WDVV equation on this genus 0 potential is equivalent to the following system (written in the Novikov variable)

\[
\begin{align*}
 q \frac{\partial}{\partial q} x(q) &= 2x(q)y(q)^2 - x(q)(x(q)^2 - w(q)), \\
 q \frac{\partial}{\partial q} y(q) &= 2x(q)^2y(q) - y(q)(x(q)^2 - w(q)), \\
 q \frac{\partial}{\partial q} w(q) &= w(q)^2 - x(q)^4.
\end{align*}
\]

The functions \( \vartheta_k(q) \) and \( \psi_k(q) \) are connected by the certain equalities (see Appendix [A]). Using also double argument formulae for \( \vartheta_k \) it’s not hard to see by comparing the formal series expansions that we have:

\[
\begin{align*}
x(q) &= \frac{1}{2} \left( \sqrt{2\psi_2(q^4) - 2\psi_4(q^4)} + \sqrt{2\psi_2(q^4) - 2\psi_3(q^4)} \right), \\
y(q) &= \frac{1}{2} \left( \sqrt{2\psi_2(q^4) - 2\psi_4(q^4)} - \sqrt{2\psi_2(q^4) - 2\psi_3(q^4)} \right), \\
w(q) &= \psi_2(q^4) + \frac{1}{2} \psi_3(q^4) + \frac{1}{2} \psi_4(q^4) + \sqrt{(\psi_2(q^4) - \psi_3(q^4))(\psi_2(q^4) - \psi_4(q^4))}.
\end{align*}
\]
Proposition 3.4. WDVV equation on the genus 0 GW potential of \( \mathbb{P}^1_{4,4,2} \) is equivalent to the Halphen’s system of equations.

Proof. This is an easy computation by using Eq. (4) and Eq. (5). □

This is not a subject of this paper, however there is a strong evidence to conjecture that WDVV equation for the genus zero potentials of GW theory of all elliptic orbifolds (namely, for \( \mathbb{P}^1_{3,3,3} \) and \( \mathbb{P}^1_{6,3,2} \) too) is also equivalent to Halphen’s system of equations — not just four different systems of PDEs, as derived in [SZI].

Notation 3.5. Fixing some branch of the square root, denote \( \kappa := \sqrt{2\pi \sqrt{-1}/4} \). For \( q(\tau) = \exp \left( \frac{2\pi \sqrt{-1}}{4} \tau \right) \) introduce the functions:

\[
x^\infty(\tau) = \kappa \cdot x(q(\tau)), \quad y^\infty(\tau) = \kappa \cdot y(q(\tau)), \quad z^\infty(\tau) = \kappa \cdot z(q(\tau)), \quad w^\infty(\tau) = \kappa^2 \cdot w(q(\tau)).
\]

Recall Proposition 3.3. We call the function \( F_{\text{an}}^{\mathbb{P}^1_{4,4,2}} \) the analytic potential of \( \mathbb{P}^1_{4,4,2} \):

\[
F_{\text{an}}^{\mathbb{P}^1_{4,4,2}}(t_0, \tau, t_{i,j}) := \mathbb{P}^1_{\text{poly}}(t_0, \tau, t_{i,j}, x^\infty(\tau), y^\infty(\tau), z^\infty(\tau), w^\infty(\tau)).
\]

Namely \( F_{\text{an}}^{\mathbb{P}^1_{4,4,2}}(t_0, \tau, t_{i,j}) \) is obtained by substituting \( t_{-1} = \tau \), \( x^\infty(\tau) \) instead of \( x(q) \) and so on.

Proposition 3.6. The function \( F_{\text{an}}^{\mathbb{P}^1_{4,4,2}}(\tau) \) is holomorphic on \( \mathbb{C}^8 \times \mathbb{H} \) and is a solution to WDVV equation.

Proof. The proof is straightforward. □

It’s important to note that we can write the function \( F_{\text{an}}^{\mathbb{P}^1_{4,4,2}} \) via the functions \( X_k^\infty(\tau) \) too by using the following formulae.

\[
\begin{align*}
x^\infty(\tau) &= \frac{1}{2} \left( \sqrt{(X_2^\infty(\tau) - X_1^\infty(\tau))} + \sqrt{(X_2^\infty(\tau) - X_1^\infty(\tau))} \right), \\
y^\infty(\tau) &= \frac{1}{2} \left( \sqrt{(X_2^\infty(\tau) - X_1^\infty(\tau))} - \sqrt{(X_2^\infty(\tau) - X_1^\infty(\tau))} \right), \\
z^\infty(\tau) &= \sqrt{(X_3^\infty(\tau) - X_1^\infty(\tau))}, \\
w^\infty(\tau) &= \frac{1}{4} \left( 2X_2^\infty(\tau) + X_3^\infty(\tau) + X_4^\infty(\tau) + 2\sqrt{(X_2^\infty(\tau) - X_3^\infty(\tau))(X_2^\infty(\tau) - X_4^\infty(\tau))} \right),
\end{align*}
\]

where we choose the square root branch as for \( x(q), y(q), z(q), w(q) \) in Eq. (5) (where it’s unambiguous because the operation of taking the square root is well-defined for the formal power series in \( q \)).
4. Group actions of the space of genus CohFT potentials

It was observed by A. Givental in [G] that the space of all partition functions of the CohFTs on \((V, \eta)\) possesses two group actions. These are the R–action of the \textit{upper–triangular} group \( \{ R \in \text{End}(V)[[z]] \mid R(z)R(−z)^T = 1 \} \) and the S–action of the \textit{lower–triangular} group \( \{ S \in \text{End}(V)[[z^{-1}]] \mid S(z)S(−z)^T = 1 \} \). For any element \( X \) from one of these groups and a partition function \( Z \) of a CohFT, A. Givental introduces another function \( Z' := \hat{X} \cdot Z \), that is a partition function of a (in general different one) CohFT too.

These actions of Givental appeared to be a powerful tool in working with the CohFTs last decades. However it’s usually hard to compute. Due to this fact in what follows we will work with the other actions, that are not that general as Givental’s action, but powerful enough for the purposes of this paper. However we will formulate our results in a final form in terms of Givental’s action, as playing de facto the role of a canonical group action on the space of CohFT partition functions.

4.1. SL(2, \mathbb{C})–group action on the potentials of elliptic orbifolds. Let \( F_0(t) \), the primary genus 0 potential of a unital CohFT on \((V, \eta)\), with \( n = \text{dim}(V) \), be written in coordinates as:

\[
F_0(t_1, \ldots, t_n) = \frac{t_2 t_n}{2} + t_1 \sum_{1 < \alpha \leq \beta < n} \eta_{\alpha, \beta} \frac{t_\alpha t_\beta}{|\text{Aut}(\alpha, \beta)|} + H(t_2, \ldots, t_n),
\]

where \( |\text{Aut}(\alpha, \beta)| = 2 \) if \( \alpha = \beta \) and 1 otherwise.

For any \( A \in \text{SL}(2, \mathbb{C}) \) consider another function \( F_0^A(t_1, \ldots, t_n) \):

\[
F_0^A(t_1, \ldots, t_n) := \frac{t_2 t_n}{2} + t_1 \sum_{1 < \alpha \leq \beta < n} \eta_{\alpha, \beta} \frac{t_\alpha t_\beta}{|\text{Aut}(\alpha, \beta)|} + \frac{c \left( \sum_{1 < \alpha \leq \beta < n} \eta_{\alpha, \beta} \frac{t_\alpha t_\beta}{|\text{Aut}(\alpha, \beta)|} \right)^2}{2(ct_n + d)}
\]

\[
+ (ct_n + d)^2 H \left( \frac{t_2}{ct_n + d}, \ldots, \frac{t_{n-1}}{ct_n + d}, \frac{at_n + b}{ct_n + d} \right) \quad \text{for} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

It’s not hard to see that \( F_0^A \) is solution to WDVV equation and hence a genus 0 primary potential of some CohFT.

It was shown in [B] that the \( \text{SL}(2, \mathbb{C}) \)–action \( F_0 \rightarrow F_0^A \) can be written via the Givental’s R–action. In what follows for any CohFT partition function \( Z \) and any Givental’s upper– or lower–triangular group element \( X \) we use the notation

\[
\hat{X} \cdot F_0 := \text{res}_\hbar \left( \hat{X} \cdot Z \right)
\]

where \( F_0 = \text{res}_\hbar(Z) \). This notation can also be supported by the fact that only genus zero correlators of the initial CohFT contribute to the genus zero correlators of the Givental–transformed CohFT.
Theorem 4.1 (Theorem 1 and Corollary 4.3 in [B]). Fix some $A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{C})$. Let the genus zero primary CohFT potential $F$ be analytic at $t = (0, \ldots, 0, A \cdot \tau)$. Denote by $F^A_\tau$ and $F_{A \cdot \tau}$ the local expansions of $F^A$ and $F^\tau$ at the points $(0, \ldots, 0, \tau)$ and $(0, \ldots, 0, A \cdot \tau)$ respectively. For $\sigma := -c(\sigma + d)$, $\sigma' := -c/(c\tau + d)$ holds:

\[
F^A_\tau = \left( \hat{S}^A_0 \right)^{-1} \cdot \hat{R}^\sigma \cdot F_{A \cdot \tau},
\]
\[
F^A_\tau = \hat{R}^{\sigma'} \cdot \left( \hat{S}^A_0 \right)^{-1} \cdot F_{A \cdot \tau},
\]

where

\[
R^\sigma(z) := \exp \left( \begin{array}{c} 0 & \ldots & \sigma \\ \vdots & 0 & \vdots \\ 0 & \ldots & 0 \end{array} \right) z), \quad S^A_0 := \left( \begin{array}{cccc} 1 & \ldots & 0 \\ \vdots & (c\tau + d)I_{n-2} & \vdots \\ 0 & \ldots & (c\tau + d)^2 \end{array} \right).
\]

The theorem above has an extension to the full partition functions of a CohFT (Theorem 2 in [B]), we just don’t give it here because at the moment it doesn’t play a role. Note that the expansion of the potential at some point can be viewed as an $S$–action of Givental.

4.2. SL$(2, \mathbb{C})$–action on the space of Halphen’s system solutions. Note that for any $A \in \text{SL}(2, \mathbb{C})$ the triple of functions $\{X^A_2(\tau), X^A_3(\tau), X^A_4(\tau)\}$ defined as follows is also a solution to the Halphen’s system of equations (8).

\[
X^A_k(\tau) := \frac{1}{(c\tau + d)^2} X^\infty_k \left( \frac{a\tau + b}{c\tau + d} \right) + \frac{c}{c\tau + d}, \quad A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right).
\]

Recall that the analytic genus zero GW potentials of $\mathbb{P}^1_{4,4,2}$ and $\mathbb{P}^1_{2,2,2,2}$ are written via the functions $X^\infty_k(\tau)$, and the WDVV equation on them is equivalent to the Halphen’s system of equations. Consider the new functions:

\[
A \cdot F^p_{an}^{1,2,2,2} := F^p_{an}^{1,2,2,2} \left| \{X^\infty_2, X^\infty_3, X^\infty_4\} \to \{X^A_2, X^A_3, X^A_4\} \right|,
\]
\[
A \cdot F^p_{an}^{1,4,4,2} := F^p_{an}^{1,4,4,2} \left| \{X^\infty_2, X^\infty_3, X^\infty_4\} \to \{X^A_2, X^A_3, X^A_4\} \right|,
\]

obtained by substituting one solution to the Halphen’s system $\{X^\infty_2, X^\infty_3, X^\infty_4\}$ by the other $\{X^A_2, X^A_3, X^A_4\}$. These functions will also be solutions to the WDVV equation and define the same pairing as the previous two.

Proposition 4.2. For any $A \in \text{SL}(2, \mathbb{C})$, the action of it on $F^p_{an}^{1,2,2,2}$ and $F^p_{an}^{1,4,4,2}$ via Eq. (7) is equivalent to the action of $A$ on the triple $\{X^\infty_2, X^\infty_3, X^\infty_4\}$ as is Eq. (8):

\[
\left( F^p_{an}^{1,2,2,2} \right)^A = F^p_{an}^{1,2,2,2} \left| \{X^\infty_2, X^\infty_3, X^\infty_4\} \to \{X^A_2, X^A_3, X^A_4\} \right|,
\]
\[
\left( F^p_{an}^{1,4,4,2} \right)^A = F^p_{an}^{1,4,4,2} \left| \{X^\infty_2, X^\infty_3, X^\infty_4\} \to \{X^A_2, X^A_3, X^A_4\} \right|.
\]
\[
\left( F^{p_{1.4.2}}_{an} \right)^A = F^{p_{1.4.2}}_{an} \left[ (x^\infty_2, y^\infty_3, z^\infty_4) \mapsto (x^A_2, y^A_3, z^A_4) \right]
\]

**Proof.** This is easy to see from the explicit form of the potential \( F^{p_{1.4.2}}_0 \) (see Appendix A in [BP], Eq. (5) and Proposition 3.3).

In particular for the first step we see that the functions \( x^\infty, y^\infty, z^\infty \) only get the factor of \((c\tau + d)^{-1}\) if one substitutes \( X_k^\infty \) to \( X^A_k \) while the function \( w^\infty \) gets indeed an additional summand of \( c/(c\tau + d) \). For the second step we note that the functions \( x, y, z \) come to the potential so that the factor of \((c\tau + d)^{-1}\) matches the formula of Eq. (7) by Proposition 3.3. And for the last step we note that this is only the function \( w^\infty \), that appears with the factor of \( t_i t_j t_k t_l \) s.t. \( \eta(\partial_{t_k}, \partial_{t_l}) \eta(\partial_{t_i}, \partial_{t_j}) \neq 0 \). Hence the additional summand it gets corresponds exactly to the additional summand of Eq. (7). \( \Box \)

Due to this proposition we will use the notations \( A \cdot F \) and \( F^A \) without making difference between them.

**Notation 4.3.** For any \( A \in \text{SL}(2, \mathbb{C}) \) denote by \( x^A(\tau), y^A(\tau), z^A(\tau) \) and \( w^A(\tau) \) the functions obtained from \( x^\infty(\tau), y^\infty(\tau), z^\infty(\tau) \) and \( w^\infty(\tau) \) by the substitution of the proposition above as in Eq. (6).

The following proposition makes the connection between the \( \text{SL}(2, \mathbb{C}) \)–actions on \( F^{X_k}_{an} \) and \( F^{X_k}_0 \) (see also Proposition 4.6 in [BP]).

**Proposition 4.4.** For any \( A \in \text{SL}(2, \mathbb{C}) \) consider the genus zero potential \( F^{X_k}_0 = F^{X_k}_0(t) \) of \( X_k \) written in the formal variables \( t \) and the analytic potential \( F^{X_k}_{an}(\tau) \). Let some square root branch \( \kappa = \sqrt{2\pi \sqrt{-1}/k} \) the following relation holds:

\[
A \cdot F^{X_k}_{an}(\tau) = \left( A' \cdot F^{X_k}_0(t) \right) \bigg|_{t-1=\tau}.
\]

where for \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), we set \( A' := \begin{pmatrix} a\kappa & b\kappa \\ c\kappa^{-1} & d\kappa^{-1} \end{pmatrix} \).

**Proof.** This follows immediately from the explicit form of the action and Proposition 4.2 above. \( \Box \)

### 4.3. The action of \( A^{(\tau_0, \omega_0)} \).

In what follows we will be in particular interested in the action of the \( \text{SL}(2, \mathbb{C}) \) elements of the particular form. For any fixed \( \tau_0 \in \mathbb{H}, \omega_0 \in \mathbb{C}^* \) define:

\[
A^{(\tau_0, \omega_0)} := \begin{pmatrix} \sqrt{-1}\tau_0 \\ 2\omega_0 \text{Im}(\tau_0) \end{pmatrix}, \begin{pmatrix} \omega_0 \tau_0 \\ \sqrt{-1} \end{pmatrix} \in \text{SL}(2, \mathbb{C}).
\]
This special choice of a SL(2, C) element comes from singularity theory assumptions and was first proposed in \cite{BT} (note however that in the reference given this element was introduced to have det = \(1/(2\pi \sqrt{-1})\) for any \(\tau_0\) and \(\omega_0\). We rescale it here because we want to work with the SL(2, C) element). We will comment on it later.

**Notation 4.5.** For any fixed \(\tau_0 \in \mathbb{H}, \omega_0 \in \mathbb{C}^*\) by using Eq. (8) denote:

\[
X_k^{(\tau_0, \omega_0)}(t) := (X_k^\infty(t))^{A^{(\tau_0, \omega_0)}}, \quad 2 \leq k \leq 4.
\]

It’s easy to see that these functions are holomorphic in \(\{t \in \mathbb{C} \mid |t| < |2\omega_0 \text{Im}(\tau_0)|\}\).

5. **CY/LG correspondence**

The idea of CY/LG correspondence came from global Mirror symmetry conjecture. In its framework both FJRW theory and GW theory appear to be the A–side models. The B–model of the global mirror symmetry is given by a singularity with a symmetry group fixed. However it should be understand globally, as varying in a family, given by the different choices of an additional structure — *primitive form* of the singularity. On the B–side different choices of the primitive form should give (generally) different CohFTs.

The A–model is said to the mirror to the B–model if the partition function of the A–model CohFT coincides up to an S–action of Givental with the partition function of the B–model with some primitive form choice. Two A–models can appear to be mirror to the same B–model. Hence two mirror B–model partition functions differ by a primitive form change. This led to the conjecture, that there should be a R–action of Givental, connecting two B–model CohFTs of the same singularity with the different primitive form choice, or, up to a mirror symmetry equivalently, there should be a R–action of Givental, connecting two A–models, that a mirror to the same global B–model.

In \cite{BT} the action of \(A^{(\tau_0, \omega_0)}\) was considered as a *model* for the primitive form change for elliptic singularities. One can use this action in a more general context, even when we don’t have orbifolded B–model.

Another important aspect of the global mirror symmetry is the symmetry group, that should be present on both A and B sides.

5.1. **Simple–elliptic singularities with the maximal symmetry group.** The global mirror symmetry program conjectures that for the B–model with the trivial symmetry group, the symmetry group of the A–model should be maximal — \(G_{\text{max}}\). In this case the B–model is given by the so–called Saito–Givental CohFT and several different mirror symmetry results were proven (see \cite{CR, MS, MR, KS, LLSS, SZ2, PS, BP}).

From this variety of mirror symmetry results, in this paper the most important for us is the following \(G_{\text{max}}—\text{CY/LG correspondence theorem}\).
\textbf{Theorem 5.1} (Theorem 4.1 and Lemma 4.9 in \cite{BP}). Consider the FJRW theory of the pair \((\tilde{E}_7, G_{\text{max}})\) and the GW–theory of \(\mathbb{P}^1_{4,4,2}\) written in the basis as in Section 3. Then we have:

\[ F_0^{(\tilde{E}_7, G_{\text{max}})}(\tilde{t}) = \mathcal{A}^{(\tau_0, \omega_0)} \cdot F_{\text{an}}^{\mathbb{P}^1_{4,4,2}}(\tilde{t}), \]

for \(\tau_0 = \sqrt{-1}, \omega_0 = \kappa \sqrt{2\pi} / (\Gamma(3/4))^2\) and the certain linear change of variables \(\tilde{t} = \tilde{t}(t)\). Moreover for the Givental’s \(R\)–matrix \(R^{\sigma'}\):

\[ R^{\sigma'} := \exp \left( \begin{pmatrix} 0 & \ldots & \sigma' \\ \vdots & 0 & \vdots \\ 0 & \ldots & 0 \end{pmatrix} z \right), \quad \text{for} \quad \sigma' = -\frac{1}{2\pi^2} \left( \Gamma\left(\frac{3}{4}\right) \right)^4, \]

up to the certain \(S\)–action holds:

\[ F_0^{(\tilde{E}_7, G_{\text{max}})} = R^{\sigma} \cdot \hat{S} \cdot F_{\text{an}}^{\mathbb{P}^1_{4,4,2}}. \]

The change of the variables \(\tilde{t}(t)\) and the \(S\)–action used are written explicitly in \cite{BP}.

Similar results were also obtained by Shen and Zhou in \cite{SZ}. However not as explicit as we state and without the explicit \(R\)–matrix. This explicitness will play a decisive role in the prove main theorem of this paper.

Recall that we can write the function \(\mathcal{A}^{(\tau_0, \omega_0)} \cdot F_{\text{an}}^{\mathbb{P}^1_{4,4,2}}\) (and hence \(F_0^{(\tilde{E}_7, G_{\text{max}})}\)) via the (holomorphic) functions \(X_k^{(\tau_0, \omega_0)}\) with \(k = 2, 3, 4\). For \(\tau_0\) and \(\omega_0\) as in theorem above the following series expansions hold:

\[ X_2^{(\tau_0, \omega_0)}(t) = \frac{1}{4} - \frac{t}{16} + \frac{t^2}{64} - \frac{t^3}{768} + \frac{t^4}{3072} - \frac{t^5}{20480} + \frac{t^6}{245760} - \frac{13t^7}{20643840} + \frac{t^8}{9175040} + O(t^9) \]

\[ X_3^{(\tau_0, \omega_0)}(t) = \frac{t}{16} - \frac{t^3}{768} + \frac{t^5}{20480} - \frac{13t^7}{20643840} + O(t^9), \]

\[ X_4^{(\tau_0, \omega_0)}(t) = -\frac{1}{4} + \frac{t}{16} - \frac{t^2}{64} - \frac{t^3}{768} + \frac{t^4}{3072} - \frac{t^5}{20480} + \frac{t^6}{245760} - \frac{13t^7}{20643840} - \frac{t^8}{9175040} + O(t^9). \]

Note that these functions have an explicit form via the logarithmic derivatives of the Jacobi theta constants, and are very useful for the explicit computations. Namely by the Halphen’s system of equations the series expansion of these functions can be reconstructed completely from the values of them at the origin \(t = 0\).

5.2. Simple–elliptic singularities with a non–maximal symmetry group. With the B–model missing we can try to resolve the A–model form CY/LG correspondence, in which all the objects are already defined. Consider the simple–elliptic singularity \(\tilde{E}_7\) written by \(W = x^4 + y^4 + z^2\) and the symmetry groups (recall the notation of Section 2):

\[ G_1 := \langle (1/4, 1/4, 0), (0, 1/2, 0), (0, 0, 1/2) \rangle, \quad G_2 := \langle (1/4, 1/4, 1/2), (0, 1/2, 0) \rangle, \quad G_3 := \langle (1/4, 1/4, 0), (0, 0, 1/2) \rangle. \]
Theorem 5.2. Up to the certain different Givental’s S–actions $S^{(k)}$ the partition functions of all three FJRW theories $(\tilde{E}_7, G_1)$, $(\tilde{E}_7, G_2)$ and $(\tilde{E}_7, G_3)$ are connected to the partition function of the Gromov–Witten theory of $\mathbb{P}^{1,2,2,2}$ by the same Givental’s $R$–action of:

$$R^\sigma := \exp\left(\begin{pmatrix} 0 & \ldots & \sigma' \\ \vdots & 0 & \vdots \\ 0 & \ldots & 0 \end{pmatrix} z\right), \quad \text{for } \sigma' = -\frac{1}{2\pi^2} \left(\Gamma\left(\frac{3}{4}\right)\right)^4,$$

so that holds:

$$Z^{(\tilde{E}_7, G_k)} = \hat{R}^\sigma \cdot \hat{S}^{(k)} \cdot Z^{\mathbb{P}^{1,2,2,2}}, \quad k = 1, 2, 3.$$

Proof. We show in Propositions 6.2, 6.5 and 6.7 of the next section that there are $A_k \in SL(2, \mathbb{C})$ for $k = 1, 2, 3$, s.t. $F_0^{(\tilde{E}_7, G_k)} = A_k \cdot F_0^{\mathbb{P}^{1,2,2,2}}$, acting as in (7). By using topological recursion relation in genus zero together with Theorem 4.1 we get an $R$–action of Givental, s.t. $F_0^{(\tilde{E}_7, G_k)} = \text{res}_\hbar \left(\hat{R} \cdot S^{(k)} \cdot Z_0^{\mathbb{P}^{1,2,2,2}}\right)$. It turns out that even though the matrices $A_k$ are not the same in all three cases, the $R$–action appears to be the same (however the $S$–actions needed are anyway different).

The FJRW theories of $(\tilde{E}_7, G_k)$ are all semisimple. One can show it for all three functions $F_0^{(\tilde{E}_7, G_k)}$ by using the explicit expressions of the potentials. In particular the point $t = 0$ is not semisimple, however the point in the neighborhood is semisimple, and this is enough because the property of being semisimple is open. It’s a computational exercise to see that the point $t = (0, 1, 2, 3, -1, 0)$ is semisimple for $A^{(\tau_0, \omega_0)} \cdot F_{an}^{\mathbb{P}^{1,2,2,2}}$. We can apply the reconstruction theorem of Teleman [T], that gives us that our genus zero equality extends to the higher genera too, what completes the proof.

Note that applying Theorem 4.1 we made a choice, in which order to apply the S and R–actions. In the equality of two partition functions this is equivalent to the choice, on which side to apply the S–action — on the FJRW, or on the GW side. The S–action used makes a shift of the coordinates, not only the linear change of the variables. Hence, in order to have the correlators and make the equality of the partition functions reasonable we should have some analyticity statement about the partition function, to which the S–action is applied. We know such a property only on the GW side, what supports the choice made.

6. Proof of the theorem

In this section we assume $\tau_0$ and $\omega_0$ to be fixed as in Theorem 5.1. Recall also Notation 4.3 for $x^A, y^A, z^A$ and $w^A$. In this section we keep:

$$x_0 := x^{A^{(\tau_0, \omega_0)}}(t), \quad y_0 := y^{A^{(\tau_0, \omega_0)}}(t), \quad z_0 := z^{A^{(\tau_0, \omega_0)}}(t), \quad w_0 := w^{A^{(\tau_0, \omega_0)}}(t).$$

We first reconstruct explicitly the genus primary potentials of the three FJRW theories in question. The reconstruction procedure is always the following. We compute the state space.
FJRW–theory by Corollary 2.5. All the rest unknown functions are further reconstructed by the WDVV equation. Note that on this step we need indeed to work with the full potential and not just certain correlators, that are enough to know to reconstruct all the other correlators.

The most amazing example of such a reconstruction is the last one, where the concave sector gives only one function we know explicitly out of the total 10 building up the potential.

We make use of the several technical lemmas that we give in Appendix B.

### 6.1. Case 1: 1-dimensional broad sector.

Consider \( W = x^4 + y^4 + z^2 \) and the symmetry group \( G_1 := \langle a, b, c \rangle \), where \( a = (1/4, 1/4, 0) \), \( b = (0, 1/2, 0) \) and \( c = (0, 0, 1/2) \). We have \( ac = J \in G_1 \) and \( a^2 J = J^{-1} \). The state space \( \mathcal{H} \) has the following basis:

\[
\mathcal{H} = \{ [J, 1], [aJ, 1], [bJ, 1], [a^2 bJ, 1], [c, xy], [a^2 J, 1] \}.
\]

By using the selection rule and degree axiom the genus 0 potential of the FJRW–theory \( (E_7, G_1) \) reads:

\[
F_{0}^{(E_7,G_1)} = \frac{1}{2} t_{a^2 J}^2 + t_J \left( \frac{t_{aJ}^2}{2} + t_{bJ} t_{a^2 bJ} + \frac{t_{c,xy}^2}{32} \right) + t_{c,xy}^4 g_1(t_{a^2 J}) + t_{bJ} t_{a^2 bJ} t_{c,xy}^2 g_2(t_{a^2 J})
\]

\[
+ t_{aJ} t_{a^2 bJ} t_{c,xy} g_3(t_{a^2 J}) + t_{aJ} t_{bJ} t_{c,xy} g_4(t_{a^2 J}) + t_{a^2 bJ} t_{c,xy}^2 g_5(t_{a^2 J}) + t_{a^2 bJ} t_{c,xy}^2 h_1(t_{a^2 J})
\]

\[
+ t_{aJ} t_{c,xy} h_2(t_{a^2 J}) + t_{aJ} t_{c,xy}^3 h_3(t_{a^2 J}) + t_{aJ} t_{bJ} t_{a^2 bJ} t_{c,xy} h_4(t_{a^2 J})
\]

\[
+ t_{aJ} t_{c,xy} h_5(t_{a^2 J}) + t_{bJ} t_{a^2 bJ} f_{0,1}(t_{a^2 J}) + t_{aJ} t_{a^2 bJ} f_{0,2}(t_{a^2 J}) + t_{aJ} t_{c,xy}^2 f_0,3(t_{a^2 J}) + t_{aJ} t_{c,xy}^2 f_{0,4}(t_{a^2 J})
\]

\[
+ t_{a^2 bJ} f_{1,1}(t_{a^2 J}) + t_{a^2 bJ} f_{1,2}^2(t_{a^2 J}) + t_{a^2 bJ} f_{1,3}(t_{a^2 J}) + t_{a^2 bJ} f_{1,4}(t_{a^2 J}) + t_{a^2 bJ} f_{1,5}(t_{a^2 J}).
\]

for some unknown functions \( g_k(t) \), \( h_k(t) \) and \( f_{1,k}(t) \). However from the selection rule we know that all functions \( g_k(t) \) are odd while the functions \( h_k(t) \) are even. The correlators of the \( (E_7, G_1) \) theory involving narrow insertions only are concave. Hence we can identify some of the functions above with those from \( (E_7, G_{max}) \)–theory. We have:

\[
f_{0,1}(t_{a^2 J}) = 0, \quad f_{0,2}(t_{a^2 J}) = 0,
\]

\[
f_{0,3}(t_{a^2 J}) = -\frac{x_0^2}{8} - \frac{x_0 y_0}{4} - \frac{y_0^2}{8}, \quad f_{0,4}(t_{a^2 J}) = -\frac{x_0^2}{8} - \frac{x_0 y_0}{4} - \frac{y_0^2}{8}, \quad f_{1,1}(t_{a^2 J}) = -\frac{x_0^2}{48} + \frac{x_0 y_0}{8} - \frac{y_0^2}{48},
\]

\[
f_{1,2}(t_{a^2 J}) = -\frac{w_0}{2} + \frac{3x_0^2}{8} - \frac{x_0 y_0}{4} - \frac{y_0^2}{8}, \quad f_{1,3}(t_{a^2 J}) = -\frac{x_0^2}{48} + \frac{x_0 y_0}{8} - \frac{y_0^2}{48},
\]

\[
f_{1,4}(t_{a^2 J}) = -\frac{w_0}{2} + \frac{x_0^2}{4} + \frac{x_0 y_0}{2} - \frac{y_0^2}{4}, \quad f_{1,5}(t_{a^2 J}) = -\frac{w_0}{8} + \frac{x_0^2}{12} - \frac{y_0^2}{24}.
\]
where the functions \(x_0 = x_0(t_{a_2})\), \(y_0 = y_0(t_{a_2})\), \(z_0 = z_0(t_{a_2})\), \(w_0 = w_0(t_{a_2})\) are those found in the \(G_{\text{max}}\) theory.

6.1.1. The WDVV equation. Writing the WDVV equation for \(F_0^{(E_7,G_1)}\) we get the following system:

\[
\begin{align*}
  w'_0(t) &= w_0^2 - x_0^4, \quad x'_0(t) = x_0 \left( w_0 - x_0^2 + 2y_0^2 \right), \quad y'_0(t) = y_0 \left( w_0 + x_0^2 \right), \\
  g_3(t_{a_2}) &= 0, \quad g_4(t_{a_2}) = 0, \quad h_3(t_{a_2}) = 0, \quad h_4(t_{a_2}) = 0, \quad h_5(t_{a_2}) = 0,
\end{align*}
\]

and also

\[
\begin{align*}
  g_1(t_{a_2}) &= \frac{x_0^2}{3072} - \frac{w_0}{2048} - \frac{y_0^2}{6144}, \quad g_2(t_{a_2}) = \frac{x_0^2}{64} + \frac{w_0 y_0}{32} - \frac{y_0^2}{64} - \frac{w_0}{32}, \\
  h_1(t_{a_2}) &= \frac{x_0^2}{128} + \frac{w_0 y_0}{64} + \frac{y_0^2}{128}, \quad h_2(t_{a_2}) = \frac{x_0^2}{128} + \frac{w_0 y_0}{64} + \frac{y_0^2}{128}.
\end{align*}
\]

The differential part of the system above involves only the functions we know already and the PDEs written are equivalent to the WDVV of the genus 0 GW potential of \(\mathbb{P}^4_{1,4,2}\) (see Section 3). Hence we do not have to solve the PDEs and we know all functions building up \(F_0^{(E_7,G_1)}\) explicitly. The potential of this FJRW theory reads:

\[
\begin{align*}
  F_0^{(E_7,G_1)} &= \frac{1}{2} t_{a_2}^2 + t_f \left( t_{a_2}^2 + t_{b_3} t_{a_2 b_3} + \frac{t_{c,xy}^2}{32} \right) + t_{a_1}^4 \left( \frac{x_0^2}{12} - \frac{y_0^2}{24} - \frac{w_0}{8} \right) - t_{b_1}^4 \left( \frac{x_0^2}{48} - \frac{x_0 y_0}{8} + \frac{y_0^2}{48} \right) \\
  &- t_{a_2 b_3}^2 \left( \frac{x_0^2}{48} - \frac{x_0 y_0}{8} + \frac{y_0^2}{48} \right) + t_{b_1} t_{a_2 b_3} t_{c,xy}^2 \left( \frac{x_0^2}{64} + \frac{x_0 y_0}{32} - \frac{y_0^2}{64} - \frac{w_0}{32} \right) + t_{a_1} \left( t_{b_1}^2 \left( \frac{x_0^2}{128} + \frac{x_0 y_0}{64} + \frac{y_0^2}{128} \right) + t_{c,xy}^2 \left( \frac{x_0^2}{3072} - \frac{y_0^2}{6144} - \frac{w_0}{2048} \right) + t_{a_2 b_3}^2 \left( \frac{x_0^2}{8} + \frac{x_0 y_0}{4} + \frac{y_0^2}{8} \right) \right) \\
  &+ t_{a_2 b_3}^2 \left( \frac{x_0^2}{8} + \frac{x_0 y_0}{4} + \frac{y_0^2}{8} \right) + t_{b_1} t_{a_2 b_3} \left( \frac{x_0^2}{4} + \frac{x_0 y_0}{2} - \frac{y_0^2}{4} - \frac{w_0}{2} \right) - t_{a_2 b_3}^2 \left( \frac{x_0^2}{8} + \frac{x_0 y_0}{4} + \frac{y_0^2}{8} \right) \right) \\
  &+ t_{b_1} \left( t_{a_2 b_3}^2 \left( \frac{3x_0^2}{8} - \frac{x_0 y_0}{2} - \frac{y_0^2}{8} \right) + t_{c,xy}^2 \left( \frac{x_0^2}{128} + \frac{x_0 y_0}{64} + \frac{y_0^2}{128} \right) \right) \right).
\end{align*}
\]

By using Eq. 63 and the definition of the \(A^{(\tau_0,\omega_0)}\)–action we get the following proposition.
Proposition 6.1. The genus zero primary potential of the FJRW theory of \((\tilde{E}_7, G_1)\) reads:

\[
F_0^{(\tilde{E}_7, G_1)} = \frac{1}{2} t_J^2 a_J^2 + t_J \left( \frac{t_{a_J}^2}{2} + t_b J a_J^2 b_J + \frac{t_{c_{,xy}}^2}{32} \right) - \left( \frac{t_{a_J}^4}{24} + \frac{t_{b_J}^4}{48} + \frac{t_{a_J}^2 b_J^2}{48} \right)
\]

\[
+ \frac{t_{c_{,xy}}^4}{6144} + \frac{t_{a_J}^2 b_J J a_J^2 b_J}{4} + \frac{t_{c_{,xy}}^2}{8} \left( t_{b_J} J a_J^2 b_J + \frac{1}{64} t_{b_J} J a_J^2 b_J J c_{,xy}^2 \right) + \left( \frac{1}{128} t_{b_J}^2 J a_J^2 b_J c_{,xy}^2 \right)
\]

\[
(9)
\]

\[
+ \left( \frac{1}{128} t_{b_J}^2 J a_J^2 b_J c_{,xy}^2 \right) - \left( \frac{t_{a_J}^4}{24} - \frac{t_{b_J}^4}{24} + \frac{t_{c_{,xy}}^2}{6144} \right) \left( \frac{t_{a_J}^2 b_J^2}{4} + \frac{t_{b_J}^2 J a_J^2 b_J}{64} + \frac{1}{64} t_{a_J}^2 J a_J^2 b_J \right) X_3^{(\tau_0, \omega_0)}.
\]

where \(X_k^{(\tau_0, \omega_0)} = X_k^{(\tau_0, \omega_0)}(t_{a_J})\) are as in Section 5.4.

6.1.2. CY/LG correspondence. Consider the change of the variables:

\[
t_J = t_0, \quad t_{a_J} = \tau
\]

\[
t_{a_J} = \frac{t_1}{\sqrt{2}}, \quad t_{b_J} = \frac{t_2}{2} + \frac{\sqrt{-1} t_4}{2}, \quad t_{a_J} = t_2 - \frac{\sqrt{-1} t_3}{2}, \quad t_{c_{,xy}} = 2\sqrt{2} t_4.
\]

By using Eq. (9) we get:

\[
F_0^{(\tilde{E}_7, G_1)} = \frac{1}{2} t_0^2 + \frac{1}{4} t_0 \sum_{k=1}^4 t_k^2 - \frac{1}{16} \left( t_3^2 t_4 + t_4^2 t_1 \right) + \frac{1}{64} \left( t_3^2 t_4 + t_4^2 t_1 \right) X_2^{(\tau_0, \omega_0)}(\tau) - \frac{1}{16} \left( t_1^2 t_3 + t_2^2 t_4 \right) X_4^{(\tau_0, \omega_0)}(\tau)
\]

\[
- \frac{1}{16} \left( t_2^2 t_3 + t_1^2 t_4 \right) X_3^{(\tau_0, \omega_0)}(\tau) - \frac{1}{96} \left( \sum_{k=1}^4 t_k^4 \right) \left( \sum_{k=2}^4 X_k^{(\tau_0, \omega_0)}(\tau) \right).
\]

It’s obvious that we get:

\[
F_0^{(\tilde{E}_7, G_1)}(t(t)) = A^{(\tau_0, \omega_0)} \cdot F_{an}^{p_{1,2,2,2}}.
\]

In order to derive the equality for the potential \(F_{0}^{p_{1,2,2,2}}\) we apply Proposition 4.4. We proved:

Proposition 6.2. For the linear change of the variables as above holds:

\[
F_0^{(\tilde{E}_7, G_1)}(t(t)) = A^{G_1} \cdot F_{0}^{p_{1,2,2,2}}, \quad A^{G_1} := \begin{pmatrix} 1 & -\pi \Theta \\ 0 & 2 \pi \Theta \end{pmatrix}
\]

for \(\Theta = \sqrt{2\pi} / (\Gamma(\frac{3}{4}))^2\).
6.2. Case 2: 2–dimensional broad sector. Consider \( W = x^4 + y^4 + z^2 \) and the symmetry group \( G_2 := \langle a, b \rangle \), where \( a = (1/4, 1/4, 1/2), b = (0, 1/2, 0) \). We have \( a = J \in G_2 \) and \( a^3 J = J^{-1} \). The state space \( \mathcal{H} \) has the following basis:

\[
\mathcal{H} = \{ [J, 1], [ab, 1], [a^3 b, 1], [a^2 b, xy], [b, x], [a^3, 1] \}.
\]

By using the selection rule, degree axiom and \( G_{max} \)-invariance axiom the genus 0 potential of the FJRW – theory \( (\tilde{E}_7, G_2) \) reads

\[
F_0(\tilde{E}_7, G_2) = \frac{1}{2} t_a^3 t_J^2 + t_J \left( t_{ab} t_{a^3 b} + \frac{t_{b,x}^2}{16} + \frac{1}{32} t_{a^2 b, xy}^2 \right) + t_{b,x}^2 g_1(t_{a^3}) + t_{b,x}^2 t_{a^2 b, xy}^2 g_2(t_{a^3}) + t_{a^2 b, xy}^2 g_3(t_{a^3})
\]

+ \( t_{ab} t_{a^3 b} t_{b,x}^2 g_4(t_{a^3}) + t_{ab} t_{a^3 b} t_{a^2 b, xy} g_5(t_{a^3}) + t_{a^3 b} t_{b,x}^2 h_1(t_{a^3}) + t_{a^3 b} t_{a^2 b, xy} h_2(t_{a^3}) + t_{ab} t_{b,x}^2 h_3(t_{a^3}) + t_{ab} t_{a^2 b, xy} h_4(t_{a^3}) + t_{ab} t_{a^3 b} f_{0,1}(t_{a^3}) + t_{ab} t_{a^3 b} f_{0,2}(t_{a^3}) + t_{a^3 b} f_{1,1}(t_{a^3}) + t_{a^3 b} f_{1,2}(t_{a^3}) + t_{ab} t_{a^3 b} f_{1,3}(t_{a^3}),
\)

for some unknown functions \( g_k(t), h_k(t) \) and \( f_{k,l}(t) \). However from the selection rule we know that all functions \( g_k(t) \) are odd while the functions \( h_k(t) \) are even. The correlators of the \( (\tilde{E}_7, G_2) \) theory involving narrow insertions only are concave. Hence we can identify some of the functions above with those from \( (\tilde{E}_7, G_{max}) \) – theory. We have:

\[
\begin{align*}
\frac{f_{0,1}(t_{a^3})}{2} &= 0, \quad \frac{f_{0,2}(t_{a^3})}{2} = 0, \quad f_{1,1}(t_{a^3}) = -\frac{x_0^2}{48} + \frac{x_0 y_0}{8} - \frac{y_0^2}{48}, \\
\frac{f_{1,2}(t_{a^3})}{2} &= -\frac{w_0}{2} + \frac{3x_0^2}{8} - \frac{x_0 y_0}{4} - \frac{y_0^2}{8}, \quad f_{1,3}(t_{a^3}) = -\frac{x_0^2}{48} + \frac{x_0 y_0}{8} - \frac{y_0^2}{48}.
\end{align*}
\]

6.2.1. The WDVV equation. Writing the WDVV equation for \( F_0(\tilde{E}_7, G_2) \) we get two cases. The first one is when \( h_2(t) \equiv 0 \) or \( h_4(t) \equiv 0 \). This case also concludes \( f_{1,1}(t) \equiv 0 \), what we know to be false. For the second case \( h_2(t) h_4(t) \neq 0 \) we get the following system:

\[
\begin{align*}
g_3'(t) &= \frac{1}{2} g_5(t)^2 + \frac{2}{3} h_2(t) h_4(t) - 64 g_3(t) g_5(t), \\
g_5'(t) &= -32 g_5(t)^2 + 128 h_2(t) h_4(t), \\
h_2'(t) &= 128 (g_5(t) - 96 g_3(t)) h_2(t), \\
h_4'(t) &= 128 (g_5(t) - 96 g_3(t)) h_4(t),
\end{align*}
\]

and also

\[
\begin{align*}
f_{1,1}(t) &= -\frac{h_2(t)}{h_4(t)} \left( 256 g_3(t) - 4 g_5(t) \right), \quad f_{1,2}(t) = 8 (192 g_3(t) - g_5(t)), \quad g_1(t) = 4 g_3(t), \\
g_2(t) &= 12 g_5(t) - \frac{1}{8} g_5(t), \quad g_4(t) = 2 g_5(t), \quad h_1(t) = -2 h_2(t), \quad h_3(t) = -2 h_4(t), \\
h_4(t) \neq 0, \quad h_4(t) \neq 0.
\end{align*}
\]
From the PDEs on $h_2$ and $h_4$ we see that $h_2(t) = ch_4(t)$ for some non–zero complex $c \in \mathbb{C}$. Hence we get an expression of $g_3(t)$ and $g_5(t)$ via the functions $X_k^{(\tau_0, \omega_0)}$ and the constant $c$.

$$g_3(t) = -\frac{1}{24576c} \left((3c+1)X_2^{(\tau_0, \omega_0)}(t) + 2(3c-1)X_3^{(\tau_0, \omega_0)}(4t) + (3c+1)X_4^{(\tau_0, \omega_0)}(t)\right),$$

$$g_5(t) = -\frac{1}{128c} \left((c+1)X_2^{(\tau_0, \omega_0)}(t) + 2(c-1)X_3^{(\tau_0, \omega_0)}(4t) + (c+1)X_4^{(\tau_0, \omega_0)}(t)\right).$$

However we also have two PDEs on $g_3(t)$ and $g_5(t)$ that give us the compatibility condition:

$$\frac{3}{2} \left(64g_3(t)g_5(t) - \frac{1}{2}g_5(t)^2 + g_3(t)\right) = \frac{1}{128} (32g_5(t)^2 + g_5(t)).$$

Putting the explicit expressions of $g_3(t)$ and $g_5(t)$ via the functions $X_k^{(\tau_0, \omega_0)}$ here we get that this condition is satisfied if and only if $c^2 = 1$. Knowing the functions $g_3(t)$ and $g_5(t)$ explicitly we resolve the function $h_2(t)$ as the square root.

We have got two different solutions to the WDVV equation and consider them both in what follows.

6.2.2. Positive solution. For $c = 1$ we get the following solution to this system:

$$f_1(t) = \frac{1}{48} \left(-X_2^{(\tau_0, \omega_0)} + 2X_3^{(\tau_0, \omega_0)} - X_4^{(\tau_0, \omega_0)}\right), \quad f_2(t) = -\frac{1}{8} \left(X_2^{(\tau_0, \omega_0)} + 2X_3^{(\tau_0, \omega_0)} + X_4^{(\tau_0, \omega_0)}\right),$$

$$g_1(t) = -\frac{1}{1536} \left(X_2^{(\tau_0, \omega_0)} + X_3^{(\tau_0, \omega_0)} + X_4^{(\tau_0, \omega_0)}\right), \quad g_2(t) = -\frac{1}{512} X_3^{(\tau_0, \omega_0)},$$

$$h_1(t) = h_3(t) = -\frac{1}{64} \sqrt{\left(X_2^{(\tau_0, \omega_0)} - X_4^{(\tau_0, \omega_0)}\right)^2}, \quad h_2(t) = h_4(t) = \frac{1}{128} \sqrt{\left(X_2^{(\tau_0, \omega_0)} - X_4^{(\tau_0, \omega_0)}\right)^2},$$

$$g_3(t) = -\frac{1}{6144} \left(X_2^{(\tau_0, \omega_0)} + X_3^{(\tau_0, \omega_0)} + X_4^{(\tau_0, \omega_0)}\right), \quad g_4(t) = -\frac{1}{32} \left(X_2^{(\tau_0, \omega_0)} + X_4^{(\tau_0, \omega_0)}\right),$$

$$g_5(t) = -\frac{1}{64} \left(X_2^{(\tau_0, \omega_0)} + X_4^{(\tau_0, \omega_0)}\right).$$

The square root appearing in the expression of the functions $h_k(t)$ above can be easily resolved (with the positive sign) because we know the explicit Taylor series of the functions $X_k^{(\tau_0, \omega_0)}$.

6.2.3. Negative solution. For $c = -1$ we get the following answer.

$$f_1(t) = \frac{1}{48} \left(2X_3^{(\tau_0, \omega_0)} - X_2^{(\tau_0, \omega_0)} - X_4^{(\tau_0, \omega_0)}\right), \quad f_2(t) = -\frac{1}{8} \left(X_2^{(\tau_0, \omega_0)} + 2X_3^{(\tau_0, \omega_0)} + X_4^{(\tau_0, \omega_0)}\right),$$

$$g_1(t) = -\frac{1}{3072} \left(X_2^{(\tau_0, \omega_0)} + 4X_3^{(\tau_0, \omega_0)} + X_4^{(\tau_0, \omega_0)}\right), \quad g_2(t) = -\frac{1}{1024} \left(X_2^{(\tau_0, \omega_0)} + X_4^{(\tau_0, \omega_0)}\right),$$

$$h_1(t) = -h_3(t) = 2h_4(t) = -2h_2(t) = \frac{1}{32} \sqrt{\left(X_2^{(\tau_0, \omega_0)} - X_3^{(\tau_0, \omega_0)}\right) \left(X_3^{(\tau_0, \omega_0)} - X_4^{(\tau_0, \omega_0)}\right),}$$
\[ g_3(t) = -\frac{1}{12288} \left( X_2^{(\tau_0,\omega_0)} + 4X_3^{(\tau_0,\omega_0)} + X_4^{(\tau_0,\omega_0)} \right), \quad g_4(t) = -\frac{1}{16}X_3^{(\tau_0,\omega_0)}, \quad g_5 = -\frac{1}{32}X_3^{(\tau_0,\omega_0)}. \]

6.2.4. Comparison of the two solutions. Let \( F_0^+ \) and \( F_0^- \) be the two primary genus zero potentials given by the “positive” and “negative” solutions to the WDVV above respectively. We establish the connection between them.

**Proposition 6.3.** Let \( F_0^+ \) be written in coordinates \( t_{g,\phi(x)}^+ \) and \( F_0^- \) be written in coordinates \( t_{g,\phi(x)}^- \). Then they are connected by the following linear change of the variables:

\[
t_J = K^{-2}t_J^+, \quad t_{J-1} = K^2t_{J-1}^-,
\]

\[
t_{ab}^- = \frac{(1 - \sqrt{-1})K}{\sqrt{2}}t_{ab}^+, \quad t_{a^2b}^- = \frac{(1 + \sqrt{-1})K}{\sqrt{2}}t_{a^2b}^+,
\]

\[
t_{b,x}^- = Kt_{b,x}^+,
\]

where \( K = e^{\pi \sqrt{-1}/2} \).

**Proof.** It’s enough to compare the 4–point correlators, what in our case amounts to the comparison of the potentials with \( X_k^{(\tau_0,\omega_0)} \) evaluated at the point \( t = 0 \). The rest is straightforward.

**Proposition 6.4.** Up to a \( S \)-action of Givental, performing the scaling of the variables, the genus zero primary potential of the FJRW theory of \((\hat{E}_7, G_2)\) reads:

\[
F_0(\hat{E}_7, G_2) = \sum_{a,b} t_{ab}t_{a^2b}t_J + \frac{1}{2}t_{a^2b}t_J^2 + \frac{1}{16}t_Jt_{b,x} + \frac{1}{32}t_Jt_{a^2b,xy} - \left( \frac{t_{ab}^4}{48} + \frac{t_{a^2b}^4}{48} + \frac{t_{b,x}^4}{1536} \right)

+ \frac{t_{a^2b,xy}^4}{6144} + \frac{t_{ab}^2t_{a^2b}^2}{8} + \frac{1}{32}t_{ab}t_{a^2b}t_{b,x} + \frac{1}{64}t_{ab}t_{a^2b}t_{a^2b,xy} + \left( \frac{X_2^{(\tau_0,\omega_0)}}{X_3^{(\tau_0,\omega_0)}} + X_4^{(\tau_0,\omega_0)} \right)

+ \left( \frac{1}{64}t_{ab}^2t_{b,x}^2 + \frac{1}{64}t_{a^2b}^2t_{b,x}^2 - \frac{1}{128}t_{ab}t_{a^2b,xy}^2 + \frac{1}{128}t_{ab}t_{a^2b,xy}^2 \right) \left( X_4^{(\tau_0,\omega_0)} - X_2^{(\tau_0,\omega_0)} \right)

+ \left( \frac{t_{ab}^4}{24} + \frac{t_{a^2b}^4}{1536} + \frac{t_{b,x}^4}{6144} - \frac{t_{a^2b,xy}^4}{1536} - \frac{1}{4}t_{ab}^2t_{a^2b}^2 - \frac{1}{512}t_{b,x}^2t_{a^2b,xy}^2 \right) X_3^{(\tau_0,\omega_0)}
\]

where \( X_k^{(\tau_0,\omega_0)} = X_k^{(\tau_0,\omega_0)}(t_{a^3}) \) are as in Section 5.1.

**Proof.** It’s easy to see that Proposition 6.3 above performs the scaling \( X_k^{(\tau_0,\omega_0)}(t) \to \sqrt{-1} \cdot X_k^{(\tau_0,\omega_0)}(\sqrt{-1}t) \). This can be obviously realized as an \( S \)-action of Givental. Together with the previous section we get the proof.

6.2.5. CY/LG correspondence. By using explicit expression of all the functions coming to \( F_0^+ \) via \( X_k^{(\tau_0,\omega_0)}(t) \) and applying the following change of variables:

\[
t_J = t_0, \quad t_{a^3} = \tau.
\]
By using the selection rule, degree axiom and for some unknown functions $g$ group $G$ we get:

\[
t_{ab} = \frac{1}{2} (t_1 + \sqrt{-1} t_2), \quad t_{a^3b} = \frac{1}{2} (t_1 - \sqrt{-1} t_2), \quad t_{a^2b,xy} = 2\sqrt{2} t_3, \quad t_{b,x} = 2t_4.
\]

we get:

\[
F_0^{(E_7, G_2)} = \frac{1}{2} \Theta_0^2 + \frac{1}{4} \Theta_0 \sum_{k=2}^{5} \Theta_k - \frac{1}{16} (t_3^2 t_4^2 + t_1^2 t_2^2) X_3^{(\tau_0, \omega_0)}(t) - \frac{1}{16} (t_1^2 t_3^2 + t_2^2 t_4^2) X_4^{(\tau_0, \omega_0)}(t) - \frac{1}{16} (t_2^2 t_3^2 + t_1^2 t_4^2) X_2^{(\tau_0, \omega_0)}(t) - \frac{1}{96} \left( \sum_{k=2}^{5} t_k^4 \right) \left( \sum_{k=2}^{4} X_k^{(\tau_0, \omega_0)}(t) \right).
\]

It’s obvious that we get:

\[
F_0^{(E_7, G_2)} (\tilde{t}(t)) = A^{(\tau_0, \omega_0)} \cdot F_0^{E_7, G_2}.
\]

In order to derive the equality for the potential $F_0^{E_7, G_2}$ we apply Proposition 4.3. We get:

**Proposition 6.5.** Up to the linear change of the variables holds:

\[
F_0^{(E_7, G_2)} (\tilde{t}(t)) = A^{G_2} \cdot F_0^{E_7, G_2}, \quad A^{G_2} := \begin{pmatrix}
\frac{1}{\Theta} & \frac{-\pi \Theta}{2 \\ 2\pi \Theta & 2}
\end{pmatrix}
\]

for $\Theta = \sqrt{2/\pi} / (\Gamma(\frac{3}{2}))^2$.

**6.3. Case 3:** 3-dimensional broad sector. Consider $W = x^4 + y^4 + z^2$ and the symmetry group $G_3 := \langle a, b \rangle$, where $a = (1/4, 1/4, 0)$ and $b = (0, 0, 1/2)$. We have $ab = J \in G_3$ and $a^2 J = J^{-1}$. The state space $\mathcal{H}$ has the following basis:

\[
\mathcal{H} = \{ [J, 1], [aJ, 1], [b, x^2], [b, xy], [b, y^2], [a^2 J, 1] \}.
\]

By using the selection rule, degree axiom and $G_{max}$–invariance axiom the genus 0 potential of the FJRW – theory $(\tilde{E}_7, G_3)$ reads:

\[
F_0^{(E_7, G_3)} = \frac{1}{2} t_2 t_3 t_{a^2J} + t_J \left( \frac{t_{aJ}^2}{2} + \frac{t_{bxy}^2}{32} + \frac{1}{16} t_{b,x^2 t_{b,y}^2} + t_{b,y^2}^2 g_1(t_{a^2J}) + t_{b,x}^2 g_2(t_{a^2J}) + t_{b,x}^2 + t_{b,y^2}^2 g_3(t_{a^2J}) + t_{b,x}^2 t_{b,y^2}^2 g_4(t_{a^2J}) + t_{b,x}^2 g_5(t_{a^2J}) + t_{b,x}^2 t_{b,xy}^2 g_6(t_{a^2J}) + t_{aJ}^2 t_{b,xy}^2 g_7(t_{a^2J}) + t_{aJ} t_{b,xy} t_{b,y^2} h_1(t_{a^2J}) + t_{aJ} t_{b,x}^2 t_{b,xy}^2 h_2(t_{a^2J}) + t_{aJ} f_{1,1}(t_{a^2J}) \right),
\]

for some unknown functions $g_k(t), h_k(t)$ and $f_{1,1}(t)$. However from the selection rule we know that all functions $g_k(t)$ and also $f_{1,1}(t)$ are odd while the functions $h_k(t)$ are even.

Note that the correlators of $(\tilde{E}_7, G)$ involving the insertions of $[J, 1], [a^2 J, 1]$ and $[a J, 1]$ only are concave. Hence we have an explicit expression for the function $f_{1,1}$ that we have found in $(\tilde{E}_7, G_{max})$.

\[
f_{1,1}(t) = \frac{1}{4} \left( -\frac{w_0(t)}{8} + \frac{x_0(t)^2}{12} - \frac{y_0(t)^2}{24} \right).
\]
For simplicity we are going to rescale this function for what follows: \( f(t) := -16f_{1,1}(t) \). Then we get:

\[
    f(t) = \frac{2}{3}X_2^{(\tau_0,\omega_0)}(t) + \frac{2}{3}X_3^{(\tau_0,\omega_0)}(t) + \frac{2}{3}X_4^{(\tau_0,\omega_0)}(t).
\]

### 6.3.1. The WDVV equation.

Writing the WDVV equation of \( F_0^{(E_7,G_2)} \) we get two cases: when \( h_1(t)h_2(t) \equiv 0 \) and \( h_1(t)h_2(t) \not\equiv 0 \). The first case gives system of equations that can be integrated explicitly giving \( f_{1,1}(t) \) as a rational function. We know from Eq. (12) and the series expansion of \( X_k^{(\tau_0,\omega_0)}(t) \) that this is not true. The second case is equivalent to the following system of equations:

\[
    g_5'(t) = \frac{16}{3}h_2(t)^2 - 64g_5(t)g_7(t),
\]

\[
    g_7(t) = 512h_1(t)h_2(t) - 32g_7(t)^2,
\]

\[
    h_1'(t) = \frac{64h_1(t)(192g_5(t)h_1(t) - g_7(t)h_2(t))}{h_2(t)},
\]

\[
    h_2'(t) = 64(192g_5(t)h_1(t) - g_7(t)h_2(t)).
\]

and also:

\[
    g_1(t) = \left(\frac{h_1(t)}{h_2(t)}\right)^2g_5(t), \quad g_2(t) = \frac{1}{64}\left(g_7(t) - 128\frac{h_1(t)g_5(t)}{h_2(t)}\right), \quad g_3(t) = \frac{g_7(t)}{16},
\]

\[
    g_4(t) = \frac{g_7(t)}{16} - 6\frac{h_1(t)g_5(t)}{h_2(t)}, \quad g_6(t) = \frac{g_7(t)}{2}, \quad f(t) = \frac{8192g_5(t)h_1(t) - 64g_7(t)h_2(t)}{h_2(t)}.
\]

In particular it’s enough to consider the WDVV equations given by: \( \{t_{aJ},t_{aJ},t_{b,x^2},t_{b,x^2}\} \), \( \{t_{aJ},t_{aJ},t_{b,x^2},t_{b,y^2}\} \), \( \{t_{aJ},t_{aJ},t_{b,xy},t_{b,xy}\} \), \( \{t_{aJ},t_{aJ},t_{b,xy},t_{b,y^2}\} \), \( \{t_{aJ},t_{aJ},t_{b,y^2},t_{b,y^2}\} \) and \( \{t_{aJ},t_{aJ},t_{b,x^2},t_{a^2J}\} \), \( \{t_{aJ},t_{aJ},t_{b,xy},t_{a^2J}\} \).

### 6.3.2. Solving the WDVV equation.

From the first system above we conclude that \( h_1(t) = ch_2(t) \) for some non–zero constant \( c \).

We are going to use now the relation between the functions \( g_5(t), g_7(t), f(t) \) and explicitly known functions \( X_k^{(\tau_0,\omega_0)}(t) \). Due to the oddness of the functions \( g_5(t) \) and \( g_7(t) \) (and using also the WDVV equation) without loss of generality we can assume:

\[
    g_7(t) = \frac{1}{64}p(t) - \frac{X_3^{(\tau_0,\omega_0)}}{32}, \quad g_5(t) = \frac{1}{8192c}p(t) + \frac{1}{12288c}\left(X_2^{(\tau_0,\omega_0)} + X_4^{(\tau_0,\omega_0)} - 2X_3^{(\tau_0,\omega_0)}\right)
\]

for some odd function \( p(t) \). From the first two PDEs on \( g_5 \) and \( g_7 \) we get the compatibility condition:

\[
    \frac{3}{16}(g_5(t) + 64g_5(t)g_7(t)) = \frac{1}{512c}\left(g_7'(t) + 32g_7(t)^2\right),
\]
that gives us the expression of \( p'(t) \) via \( p(t) \) and \( X^k(\tau_0,\omega_0) \):

\[
p'(t) = p(t) \left( p(t) + 2 \left( X^2(\tau_0,\omega_0) - X^3(\tau_0,\omega_0) + X^4(\tau_0,\omega_0) \right) \right).
\]

From the PDE on \( g_\tau(t) \) we get the expression of \( h_2(t) \), that we put into the PDE of \( h_2(t) \) and get by using the formula for \( p(t) \) above:

\[
3p(t) \left( p(t) + 2 \left( X^2(\tau_0,\omega_0) - X^3(\tau_0,\omega_0) \right) \right) \left( p(t) + 2 \left( X^4(\tau_0,\omega_0) - X^3(\tau_0,\omega_0) \right) \right) = 0,
\]

from where we find the function \( p(t) \) explicitly to be one of the following three:

\[
p(t) = 0, \quad p(t) = -2 \left( X^2(\tau_0,\omega_0) - X^3(\tau_0,\omega_0) \right), \quad p(t) = 2 \left( X^4(\tau_0,\omega_0) - X^3(\tau_0,\omega_0) \right)
\]

giving the different solutions:

\[
g_\tau(t) = -\frac{1}{32} X^3(\tau_0,\omega_0), \quad g_5(t) = \frac{1}{12288c} \left( X^2(\tau_0,\omega_0) - 2X^3(\tau_0,\omega_0) + X^4(\tau_0,\omega_0) \right),
\]

\[
h_2(t) = \frac{1}{128} \sqrt{-\frac{1}{c} \left( X^2(\tau_0,\omega_0) - X^3(\tau_0,\omega_0) \right) \left( X^3(\tau_0,\omega_0) - X^4(\tau_0,\omega_0) \right)}
\]

\[
g_\tau(t) = -\frac{1}{32} X^2(\tau_0,\omega_0), \quad g_5(t) = \frac{1}{12288c} \left( -2X^2(\tau_0,\omega_0) + X^3(\tau_0,\omega_0) + X^4(\tau_0,\omega_0) \right),
\]

\[
h_2(t) = \frac{1}{128} \sqrt{-\frac{1}{c} \left( X^2(\tau_0,\omega_0) - X^3(\tau_0,\omega_0) \right) \left( X^3(\tau_0,\omega_0) - X^4(\tau_0,\omega_0) \right)}
\]

\[
g_\tau(t) = -\frac{1}{32} X^4(\tau_0,\omega_0), \quad g_5(t) = \frac{1}{12288c} \left( X^2(\tau_0,\omega_0) + X^3(\tau_0,\omega_0) - 2X^4(\tau_0,\omega_0) \right),
\]

\[
h_2(t) = \frac{1}{128} \sqrt{-\frac{1}{c} \left( X^2(\tau_0,\omega_0) - X^4(\tau_0,\omega_0) \right) \left( X^3(\tau_0,\omega_0) - X^4(\tau_0,\omega_0) \right)}
\]

Actually only one of them — Eq. (14a) is correct for the FJRW theory because \( g_\tau(t) \) is odd by the Selection rule and from the Taylor series expansions of \( X^k(\tau_0,\omega_0) \) we know that only \( X^3(\tau_0,\omega_0) \) is odd.

At the same time it’s clear that the rescaling of the variables \( t_{b,y^2} \rightarrow t_{b,y^2}/c \) and \( t_{b,x^2} \rightarrow ct_{b,x^2} \) preserves the pairing fixed by \( F_0^{(E_7,G_3)} \) and in the new coordinates this constant \( c \) doesn’t appear in the potential anymore.

Hence up to this rescaling we can set \( c = 1 \) and the WDVV equation has the unique solution. We get:
Proposition 6.6. Up to a scaling of the variables the primary FJRW potential reads:

\[
F_0^{(E_7,G_3)} = \frac{1}{2} t_{a_1}^2 t_{a_2} + \frac{1}{2} t_{a_2}^2 t_{a_1} + \frac{1}{32} t_J t_{b,y}^2 + \frac{1}{16} t_J t_{b,x} t_{b,y}^2 + \left( \frac{t_{b,x}^2}{12288} + \frac{t_{b,y}^2}{12288} - \frac{t_{b,x} t_{b,y}}{6144} - \frac{t_{a_1}^2}{24} - \frac{t_{b,x}^2 t_{b,y}^2}{2048} \right) \left( X_2^{(\tau_0,\omega_0)} + X_4^{(\tau_0,\omega_0)} \right)
\]

(15)

\[
+ \frac{1}{128} \left( t_{a_1} t_{b,x} t_{b,y}^2 + t_{a_2} t_{b,x}^2 t_{b,y}^2 \right) \sqrt{X_3^{(\tau_0,\omega_0)} - X_2^{(\tau_0,\omega_0)}} \left( X_3^{(\tau_0,\omega_0)} - X_4^{(\tau_0,\omega_0)} \right)
\]

\[- \left( \frac{1}{24} t_{a_1}^2 + \frac{t_{b,x}^2}{6144} + \frac{1}{64} t_{a_2} t_{b,x}^2 + \frac{t_{b,y}^2}{6144} + \frac{1}{32} t_{a_1} t_{b,x} t_{b,y}^2 + \frac{1}{512} t_{b,x} t_{b,y}^2 \right) X_3^{(\tau_0,\omega_0)}
\]

where \(X_k^{(\tau_0,\omega_0)} = X_k^{(\tau_0,\omega_0)}(t_{a_1})\) are as in Section 5.1. Moreover there is an \(S\)-action of Givental, performing the scaling of the variables, s.t. \(\hat{S} \cdot F_0^{(E_7,G_3)} \in \mathbb{Q}[t]\).

Proof. The first part follows immediately from the preceding sections.

From the explicit series expansions of the functions \(X_3^{(\tau_0,\omega_0)}\) we have:

\[
g_1(t), \ldots, g_7(t) \in \mathbb{Q}[t], \quad f_{1,1}(t) \in \mathbb{Q}[t],
\]

and

\[
h_1(t), h_2(t) \in \sqrt{-1}\mathbb{Q}[t^2].
\]

Hence we see that \(F_0^{(E_7,G_3)} \notin \mathbb{Q}[t]\).

Consider the rescaling \(X_a^{(\tau_0,\omega_0)}(t) \to \sqrt{-1}X_a^{(\tau_0,\omega_0)}(\sqrt{-1}t)\), that can be easily realized as a scaling of the variables, preserving the cubic terms. Note also that we have the relations \(\sqrt{-1}X_a^{(\tau_0,\omega_0)}(\sqrt{-1}t) = X_a^{(\tau_0,\omega_1)}(t)\) for \(\omega_1 := \exp(-\pi\sqrt{-1}/2)\omega_0\), that is equivalent to the rescaling discussed. We get:

\[
g_a(t) \to \sqrt{-1}g_a(\sqrt{-1}t) \in \mathbb{Q}[t], \quad f_{1,1}(t) \to \sqrt{-1}f_{1,1}(\sqrt{-1}t) \in \mathbb{Q}[t],
\]

because these functions are odd, and

\[
h_a(t) \to \sqrt{-1}h_a(\sqrt{-1}t) \in \mathbb{Q}[t^2],
\]

because these functions are even. \(\square\)

6.3.3. CY/LG correspondence. Note that all three solutions from Eq. [14] to the WDVV equation [13] differ just by the permutations of the functions \(X_a^{(\tau_0,\omega_0)}\). All three solutions give some genus zero primary CohFT potentials, but only one of them is indeed a FJRW–theory genus zero primary potential as we have shown above.
Denote the genus zero primary potential of the third WDVV solution — Eq. (14c) by $F_{0}^{\text{aux}}$. We identify this potential with the $\mathcal{A}^{(\tau_1, \omega_1)}$—transformed potential of $F_{0}^{\mathbb{P}^1_{2,2,2}}$. Then Lemma B.3 gives the CY/LG correspondence action.

6.3.4. Computation of $F_{0}^{\text{aux}}$. Comparing to the previously computed FJRW theories here we also make use of Lemma B.1. We get:

\[
32 \cdot g_7(t) = - (4X_4^2 (4t))^{(\tau_0, \omega_0)} = - \frac{1}{2} \left( X_3^{(\tau_1, \omega_1)} (t) + X_4^{(\tau_1, \omega_1)} (t) \right)
\]

\[
12288 \cdot g_5(t) = (4X_2^2 (4t))^{(\tau_0, \omega_0)} + (4X_3^2 (4t))^{(\tau_0, \omega_0)} - 2 (4X_4^2 (4t))^{(\tau_0, \omega_0)}
\]

\[
= 2X_2^{(\tau_1, \omega_1)} (t) - X_3^{(\tau_1, \omega_1)} (t) - X_4^{(\tau_1, \omega_1)} (t),
\]

\[
128\sqrt{c} \cdot h_2(t) = \sqrt{\left((4X_2^2 (4t))^{(\tau_0, \omega_0)} - (4X_4^2 (4t))^{(\tau_0, \omega_0)}\right)} \left((4X_3^2 (4t))^{(\tau_0, \omega_0)} - (4X_4^2 (4t))^{(\tau_0, \omega_0)}\right)
\]

\[
= \frac{1}{2} \left( X_3^{(\tau_1, \omega_1)} (t) - X_4^{(\tau_1, \omega_1)} (t) \right).
\]

where $\tau_1 = 2\tau_0$ and $\omega_1 = \omega_0 / \sqrt{2}$.

Applying the following linear change of variables:

\[
t_J = t_0, \quad t_{a2} = \tau
\]

\[
t_{aJ} = \frac{1}{2} (t_1 - t_3), \quad t_{b,xy} = 2t_2 + 2\sqrt{-1}t_4, \quad t_{b,xy} = 2(t_1 + t_3), \quad t_{b,xy} = 2t_2 - 2\sqrt{-1}t_4.
\]

to the potential $F_{0}^{\text{aux}}$ we get:

\[
F_{0}^{\text{aux}} (t) = \frac{1}{2} t_0^2 \tau + \frac{1}{4} t_0 \sum_{k=2}^{5} t_k^2 - \frac{1}{16} \left( t_3^2 t_4 + t_1^2 t_2 \right) X_4^{(\tau_1, \omega_1)} (\tau) - \frac{1}{16} \left( t_1^2 t_3 + t_2^2 t_4 \right) X_2^{(\tau_1, \omega_1)} (\tau)
\]

\[
- \frac{1}{16} \left( t_1^2 t_3 + t_2^2 t_4 \right) X_3^{(\tau_1, \omega_1)} (\tau) - \frac{1}{96} \left( \sum_{k=2}^{5} t_k^4 \right) \left( \sum_{l=2}^{4} X_l^{(\tau_1, \omega_1)} (\tau) \right).
\]

Therefore for $\tau_2 = 2\tau_0 + 1$ and $\omega_2 = \omega_0 / \sqrt{2}$ holds:

\[
F_{0}^{(\tilde{E}_7, G_3)} (\tilde{t}(t)) = \mathcal{A}^{(\tau_2, \omega_2)} \cdot F_{\mathbb{P}^1_{2,2,2}}^{\mathbb{P}^1_{2,2,2}}.
\]

In order to derive the equality for the potential $F_{0}^{\mathbb{P}^1_{2,2,2}}$ we apply Lemma B.2 and Proposition 4.4. We have got:
**Proposition 6.7.** Up to the linear change of the variables holds:

\[ F_0^{(\tilde{\mathcal{E}}_7, G_3)}(\tilde{t}(t)) = A_{G_3}^{P_{1,2,2,2}}, \quad A_{G_3} := \left( \frac{\sqrt{-1} - 2}{\sqrt{2}} \right) \Theta \left( \frac{\sqrt{-1} - \sqrt{2}}{2\sqrt{2}} \right) \]

for \( \Theta = \sqrt{2\pi} / (\Gamma(\frac{3}{4}))^2 \).

**Appendix A. Some formulae on the theta constants**

The Jacobi theta constants have the following connection to the Fourier series \( \psi_k(q) \), \( k = 2, 3, 4 \) of Section 3:

\[ (\vartheta_2(q))^4 = 2(\psi_3(q) - \psi_4(q)), \quad (\vartheta_3(q))^4 = 2(\psi_2(q) - \psi_4(q)), \quad (\vartheta_4(q))^4 = 2(\psi_2(q) - \psi_3(q)). \]

Note that these equalities are not enough to express \( \psi_k(q) \) via the theta constants.

Consider the double argument formulae for the theta constants:

\[ (\vartheta_2(q^2))^2 = \frac{1}{2} (\vartheta_3(q))^2 - (\vartheta_4(q))^2, \]

\[ (\vartheta_3(q^2))^2 = \frac{1}{2} (\vartheta_3(q))^2 + (\vartheta_4(q))^2, \]

\[ (\vartheta_4(q^2))^2 = \vartheta_3(q) \vartheta_4(q). \]

Combining these formulae with the definition of the functions \( X_k^\infty(q) \) we get:

\[ 2X_2^\infty(q^2) = \frac{X_3^\infty(q) (\vartheta_3(q))^2 - X_4^\infty(q) (\vartheta_4(q))^2}{(\vartheta_3(q))^2 - (\vartheta_4(q))^2}, \]

\[ 2X_3^\infty(q^2) = \frac{X_3^\infty(q) (\vartheta_3(q))^2 + X_4^\infty(q) (\vartheta_4(q))^2}{(\vartheta_3(q))^2 + (\vartheta_4(q))^2}, \]

\[ 2X_4^\infty(q^2) = \frac{1}{2} (X_3^\infty(q) + X_4^\infty(q)). \]

**Appendix B. Scaling lemmas**

The following lemma is only applicable to the scaling of \( \tau \) by 2 and uses double argument formulae of the theta constants.

**Lemma B.1.** For any \( A \in \text{SL}(2, \mathbb{C}) \) we have the following equalities:

\[ (2X_2(2\tau))^A = \frac{X_3^A(\tau) T_3^A(\tau) - X_4^A(\tau) T_4^A(\tau)}{T_3^A(\tau) - T_4^A(\tau)}, \]

\[ (2X_3(2\tau))^A = \frac{X_3^A(\tau) T_3^A(\tau) + X_4^A(\tau) T_4^A(\tau)}{T_3^A(\tau) + T_4^A(\tau)}, \]
\[(2X_4(2\tau))^A = \frac{1}{2}(X_3^A(\tau) + X_4^A(\tau)),\]

where

\[T_k^A(\tau) := \frac{1}{c\tau + d} \left( \vartheta_k \left( \frac{a\tau + b}{c\tau + d} \right) \right)^2, \quad k = 2, 3, 4.\]

**Proof.** First of all note that we can not apply \(A\) to the function \(X_k^\infty(2\tau)\) because the latter one doesn’t solve the Halphen’s system. Let’s apply it to \(2X_k^\infty(2\tau)\). We only do it in one example, while all the others are similar. Let:

\[A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C}), \quad \text{and} \quad \tau' := \frac{a\tau + b}{c\tau + d}.\]

Using the double argument formula for \(X_2^\infty\) (see Appendix A) we have:

\[(2X_2^\infty(2\tau))^A = \frac{1}{(c\tau + d)^2} \cdot 2X_2^\infty \left( \frac{2(a\tau + b)}{c\tau + d} \right) + \frac{c}{c\tau + d},\]

\[= \frac{1}{(c\tau + d)^2} \left( X_3^\infty(\tau')\vartheta_3^2(\tau') - X_4^\infty(\tau')\vartheta_4^2(\tau') \right) + \frac{c}{c\tau + d},\]

\[= \frac{[X_3^\infty(\tau') + c(c\tau + d)]\vartheta_3^2(\tau') - [X_4^\infty(\tau') + c(c\tau + d)]\vartheta_4^2(\tau')}{(c\tau + d)^2(\vartheta_3^2(\tau') - \vartheta_4^2(\tau'))}.\]

The other two cases are treated in the same way. \qed

For a more general scaling we have.

**Lemma B.2.** For any \(\tau_0 \in \mathbb{H}, \omega_0 \in \mathbb{C}^*\) and \(k \in \mathbb{Q}_>0\) holds:

\[(kX_a^\infty(k\tau))(\tau_0, \omega_0) = (X_a^\infty(\tau))(\tau_1, \omega_1), \quad 2 \leq a \leq 4,\]

where \(\tau_1 = k\tau_0, \omega_1 = \omega_0/\sqrt{k}.\)

**Proof.** First of all note that the formula given makes sense. Namely, the triple of functions \(kX_a^\infty(k\tau)\) is solution of the Halphen’s system too. The rest follows from the following equalities.

\[(kX_a^\infty(k\tau))(\tau_0, \omega_0) = A(\tau_0, \omega_0) \cdot (kX_a^\infty(k\tau))\]

\[= k \cdot \frac{(2\omega_0\text{Im}(\tau_0))^2}{(\sqrt{-1\tau + 2\omega_0^2\text{Im}(\tau_0)})^2} X_a^\infty \left( k \cdot \frac{\sqrt{-1}\tau_0 + \tau_0 \cdot 2\omega_0^2\text{Im}(\tau_0)}{\sqrt{-1\tau + 2\omega_0^2\text{Im}(\tau_0)}} \right) - \frac{1}{\tau - 2\sqrt{-1}\omega_0^2\text{Im}(\tau_0)},\]

\[= A(\tau_1, \omega_1) \cdot (X_a^\infty(\tau)).\] \qed
Lemma B.3. For any $\tau_0 \in \mathbb{H}$ and $\omega_0 \in \mathbb{C}^*$ holds:
\[
X_2^{(\tau_0, \omega_0)}(t) = X_2^{(\tau_1, \omega_0)}(t), \quad X_3^{(\tau_0, \omega_0)}(t) = X_4^{(\tau_1, \omega_0)}(t), \quad X_4^{(\tau_0, \omega_0)}(t) = X_3^{(\tau_1, \omega_0)}(t),
\]
for $\tau_1 := \tau_0 + 1$.

Proof. This follows immediately from the identities $X_2^\infty(t + 1) = X_2^\infty(t)$, $X_3^\infty(t + 1) = X_4^\infty(t)$, $X_4^\infty(t + 1) = X_3^\infty(t)$ and the definition of the the $A^{(\tau_0, \omega_0)}$–action. \(\square\)

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