Gisin Nonlocality of the Doebner-Goldin 2-Particle Equation

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Abstract

Gisin’s argument against deterministic nonlinear Schrödinger equations is shown to be valid for every (formally) nonlinearizable case of the general Doebner-Goldin 2-particle equation in the following form:

The time-dependence of the position probability distribution of a particle ‘behind the moon’ may be instantaneously changed by an arbitrarily small instantaneous change of the potential ‘inside the laboratory’.

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1 Introduction

Several years ago N. Gisin pointed out that for every (deterministic, scalar) nonlinear 2-particle Schrödinger theory there is an initial wave function \( \Psi_0(\vec{x}_1, \vec{x}_2) \in L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3) \) and a self-adjoint operator \( \mathbf{A} \otimes 1 \) such that the ‘expectation value’ of \( \mathbf{A} \otimes 1 \) may be almost instantaneously influenced by performing measurements on particle 2 \([5, 6]\). As shown in \([7]\) the existence of such a Gisin effect does not depend on Gisin’s questionable assumptions concerning the measuring process. More precisely, the following holds:

There is an initial wave function \( \Psi_0(\vec{x}_1, \vec{x}_2) \in L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3) \) and a self-adjoint operator \( \mathbf{A} \otimes 1 \) such that

\[
\langle \Psi_t^V | \mathbf{A} \otimes 1 \Psi_t^V \rangle
\]
depends nontrivially on $V_t$, where $\Psi_t^V$ denotes the solution of the corresponding initial value problem for the nonlinear 2-particle Schrödinger equation

$$
i \partial_t \Psi_t(\vec{x}_1, \vec{x}_2) = \left( -\frac{1}{2} \Delta + V_t(\vec{x}_2) \right) \Psi_t(\vec{x}_1, \vec{x}_2) + F_t[\Psi_t](\vec{x}_1, \vec{x}_2), \quad \Delta = \Delta(\vec{x}_1, \vec{x}_2),$$

(1)
even if the latter is formally local in the sense that the nonlinearity $F_t$ is a local (non-linear) functional:

$$F_t[\Psi](\vec{x}_1, \vec{x}_2) = F_t[\Phi](\vec{x}_1, \vec{x}_2) \quad \forall (\vec{x}_1, \vec{x}_2) \notin \text{ supp } (\Psi - \Phi),$$

However, in a nonlinear quantum theory one cannot consider all linear self-adjoint operators as physical observables. Therefore such Gisin effects may be completely irrelevant as in the linearizable case of the general Doebner-Goldin equation [7]. In order to stress this we call a Gisin effect **relevant** if the corresponding operator $A \otimes 1$ is a physical observable. If $A$ is a multiplier in $L^2(\mathbb{R}^3)$ this is certainly the case due to the fundamental assumption of nonlinear quantum mechanics:

$$|\Psi_t(\vec{x}_1, \vec{x}_2)|^2 = \left\{ \begin{array}{l}
\text{probability density for localization of particle 1 around } \vec{x}_1 \\
\text{and particle 2 around } \vec{x}_2 \text{ at time } t.
\end{array} \right.$$

The purpose of the present paper is to show that there are relevant Gisin effects for every case, except $c_2 = -2c_5$, of the 2-particle **general Doebner-Goldin equation** [4, 5]. Using the short-hand notation

$$\begin{align*}
\rho_t &\equiv |\Psi|^2, \\
\vec{j}_t &\equiv \Im \left( \overline{\Psi}_t \vec{\nabla} \Psi_t \right), \\
\vec{\nabla} &\equiv \vec{\nabla}_{(\vec{x}_1, \vec{x}_2)},
\end{align*}$$

this nonlinear Schrödinger equation, up to some **nonlinear gauge transformation**

$$\Psi \mapsto e^{i\lambda \ln |\Psi|} \Psi, \quad \lambda \in \mathbb{R},$$

is given by (1) and

$$F[\Psi] = \left( c_1 \frac{\vec{\nabla} \cdot \vec{j}}{\rho} + c_2 \frac{\Delta \rho}{\rho} + c_3 \frac{\vec{j}^2}{\rho^2} + c_4 \frac{\vec{j} \cdot \vec{\nabla} \rho}{\rho^2} + c_5 \frac{(\vec{\nabla} \rho)^2}{\rho^2} \right) \Psi,$$

(2)

with real parameters $c_1, \ldots, c_5$ [3, 4].

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1. We use units in which $\hbar = 1$ and $m = 1$.
2. For simplicity we consider only the special case $V_t(\vec{x}_1, \vec{x}_2) = V_t(\vec{x}_2)$. 

2 Previous Results

We assume that there are sufficiently many, well-behaved, $V_t$-dependent solutions of (1), (2) – at least locally in time – which are physically acceptable in the following sense:

Switching on $V_t$ instantaneously does not cause an instantaneous change of the wave function.

Then we have a relevant Gisin effect whenever there is a sufficiently well-behaved initial wave function $\Psi_0$ and some integer $k$ for which the function

$$\left( \left( \frac{\partial}{\partial t} \right)^k \int \rho_t(\vec{x}_1, \vec{x}_2) \, d\vec{x}_2 \right)_{t=0}$$

of $\vec{x}_1$ depends nontrivially on $V_t = V$. That such instantaneous Gisin effects exist unless

$$c_3 = c_1 + c_4 = 0 \tag{4}$$

was first shown by R. Werner \[13\]. Since Werner’s Ansatz, using entangled Gauß solutions for oscillator potentials, was too special the $V$-dependent part of (3) was calculated in \[8\] for general initial conditions and $k \leq 3$. This only confirmed Werners result. We will show, however, that (4) does not exclude nontrivial $V$-dependence of (3) for $k = 4$.

3 Additional Gisin Effects

Since (3) becomes very complicated for $k > 3$ we calculate $\left( \left( \frac{\partial}{\partial t} \right)^k \int \rho_t(\vec{x}) \, d\vec{x} \right)_{t=0}$, instead, assuming $\Psi_t$ to be sufficiently well behaved.

Using the continuity equation

$$\partial_t \rho_t + \vec{\nabla} \cdot \vec{j}_t = 0$$

we get by partial integration w.r.t. $x_1^1$ the first Ehrenfest relation

$$\partial_t \int x_1^1 \rho_t(\vec{x}) \, d\vec{x} = \mathcal{I} \int \Psi_t(\vec{x}) \partial_1 \Psi_t(\vec{x}) \, d\vec{x},$$

where

$$\partial_t \overset{\text{def}}{=} \frac{\partial}{\partial t}, \quad \partial_1 \overset{\text{def}}{=} \frac{\partial}{\partial x_1}, \quad \vec{x} \overset{\text{def}}{=} (\vec{x}_1, \vec{x}_2).$$

This assumption is well known to be fulfilled for the linear Schrödinger equation and should follow along the lines discussed in \[12\] for the Doebner-Goldin equation.
Further differentiation w.r.t. $t$ yields
\[
(\partial_t)^2 \int x^1 \rho_t(x) \, dx = - \int \rho_t(x) \, \partial_t R[\Psi_t](x) \, dx
\]
(note that since $\partial_1 V = 0$),
\[
(\partial_t)^3 \int x^1 \rho_t(x) \, dx = \int \nabla \cdot \mathbf{j}(x, t) \, \partial_t R[\Psi_t](x) \, dx - \int \rho_t(x) \, \partial_t \partial R[\Psi_t](x) \, dx,
\]
and finally
\[
(\partial_t)^4 \int x^1 \rho_t(x) \, dx = \int \left( \partial_t \nabla \cdot \mathbf{j}_t(x) \right) \, \partial_t R[\Psi_t](x) \, dx + 2 \int \left( \nabla \cdot \mathbf{j}_t(x) \right) \, \partial_t \partial R[\Psi_t](x) \, dx - \int \rho_t(x) \, \partial_t \partial_2 R[\Psi_t](x) \, dx,
\]
where
\[
F[\Psi_t](x) = R[\Psi_t](x) \Psi_t(x).
\]
Since the $V$-dependent part of $\left( \partial_t \nabla \cdot \mathbf{j}_t(x) \right)$ is\[4\]
\[
\text{ess} \left( \partial_t \nabla \cdot \mathbf{j}_t(x) \right)_{|t=0} = \nabla \cdot \Re \left( \Psi_0(x) \delta(V(x_2), \nabla \Psi_0(x)) \right)
= -\nabla \cdot \left( \rho_0(x) \nabla V(x_2) \right),
\]
(3) and (4) depend linearly on $R$ and therefore the contributions of the different terms in (2) may be checked separately.

Let us first consider the special case $c_4 = -c_1 \neq 0$, $c_\nu = 0$ else, i.e.
\[
R[\Psi](x) = c_1 \left( \frac{\nabla \cdot \mathbf{j}}{\rho} - \frac{\mathbf{j} \cdot \nabla \rho}{\rho^2} \right) = c_1 \Delta \arg (\Psi(x)) .
\]
Here
\[
\partial_t R[\Psi_t](x) = -c_1 \Delta \Re \left( \frac{i\partial_t \Psi_t(x)}{\Psi_t(x)} \right)
\]
and therefore
\[
\text{ess} \left( \partial_t R[\Psi_t](x) \right)_{|t=0} = -c_1 \Delta V(x_2) .
\]
Due to
\[
\partial_t \text{ess} \left( \partial_t^2 R[\Psi_t(x)] \right)_{|t=0} = \frac{c_1}{2} \partial_t \Delta \text{ess} \left( \Re \left( \frac{\partial_t \Delta \Psi_t(x)}{\Psi_t(x)} \right) \right)_{|t=0}
= \frac{c_1}{2} \partial_t \Delta \Im \left( \frac{[\Delta, V(x_2)] - \Psi_0(x)}{\Psi_0(x)} \right)
= -c_1 \partial_t \Delta \left( \nabla V(x_2) \cdot \nabla \arg (\Psi_0(x)) \right)
\]

\[4\]We always denote by $\text{ess} (X)$ the partial sum of $V$-dependent items of $X$.\]
the $V$-dependent part of (3) at $t = 0$ is
\[ \text{ess} \left( (\partial_t)^4 \int x^1 \rho_t(\vec{x}) \, d\vec{x} \right)_{|t=0} \]
\[ = -c_1 \int \left( \vec{\nabla} \cdot (\rho_0(\vec{x}) \vec{\nabla} V(\vec{x}_2)) \right) \Delta \partial_1 \arg(\Psi_0(\vec{x})) \, d\vec{x} \]
\[ - c_1 \int \rho_0(\vec{x}) \partial_1 \Delta \left( (\vec{\nabla} V(\vec{x}_2)) \cdot \vec{\nabla} \arg(\Psi_0(\vec{x})) \right) \, d\vec{x} \]
\[ = -c_1 \int \rho_0(\vec{x}) \left[ \Delta, \vec{\nabla} V(\vec{x}_2) \right]_- \cdot \vec{\nabla} \partial_1 \arg(\Psi_0(\vec{x})) \, d\vec{x}. \]

(9)

Obviously, (9) does not always vanish. Hence there are relevant Gisin effects for (3).

Next, let us consider the case $c_2 \neq 0$, $c_\nu = 0$ else, i.e.
\[ R[\Psi](\vec{x}) = c_2 \frac{\Delta \rho(\vec{x})}{\rho(\vec{x})}. \]

(10)

Then, since
\[ \text{ess} \left( (\partial_t)^2 \frac{\Delta \rho_t(\vec{x})}{\rho_t(\vec{x})} \right)_{|t=0} \]
\[ = \left( \frac{\Delta \rho_0(\vec{x})}{\rho_0(\vec{x})^2} - \frac{1}{\rho_0(\vec{x})} \Delta \right) \text{ess} \left( \partial_1 \vec{\nabla} \cdot \vec{j}_1(\vec{x}) \right)_{|t=0}, \]

the $V$-dependent part of (3) at $t = 0$ is
\[ \text{ess} \left( (\partial_t)^4 \int x^1 \rho_t(\vec{x}) \, d\vec{x} \right)_{|t=0} \]
\[ = -c_2 \int \left( \vec{\nabla} \cdot (\rho_0(\vec{x}) \vec{\nabla} V(\vec{x}_2)) \right) \partial_1 \frac{\Delta \rho_0(\vec{x})}{\rho_0(\vec{x})} \, d\vec{x} \]
\[ + c_2 \int \rho_0(\vec{x}) \partial_1 \left( \left( \frac{\Delta \rho_0(\vec{x})}{\rho_0(\vec{x})^2} - \frac{1}{\rho_0(\vec{x})} \Delta \right) \left( \vec{\nabla} \cdot (\rho_0(\vec{x}) \vec{\nabla} V(\vec{x}_2)) \right) \right) \, d\vec{x} \]
\[ = c_2 \int \left( \left[ \Delta, \frac{1}{\rho_0(\vec{x})} \right]_- \partial_1 \rho_0(\vec{x}) \right) \vec{\nabla} \cdot (\rho_0(\vec{x}) \vec{\nabla} V(\vec{x}_2)) \, d\vec{x}. \]

This is nonzero, for example, when
\[ V(\vec{x}_2) = (x_2^1)^3, \quad \rho_0(\vec{x}) = \exp \left( -\|\vec{x}\|^2 - x_1^2 \right). \]

Hence there are relevant Gisin effects for (10), too.

Finally, let us consider the case
\[ c_1 = c_3 = c_4 = 0 = c_2 + 2c_5, \]

(11)

i.e.
\[ R[\Psi](\vec{x}) = c_2 \left( \frac{\Delta \rho(\vec{x})}{\rho(\vec{x})} - \frac{1}{2} \left( \vec{\nabla} \rho(\vec{x}) \rho(\vec{x}) \right)^2 \right). \]

(12)

In this case (1),(2) fulfills also the second Ehrenfest relation [3]. Then already \((\partial_t)^2 \int x^1 \rho_t(\vec{x}) \, d\vec{x}\) vanishes for all $t$, hence (12) does not contribute to (3).

\[ ^5 \text{Note that (1),(2) is (formally) linearizable in this case (1).} \]
4 Conclusion

We already know from [13, 8] — or easily rederive from (5) — that for (2) there are relevant Gisin effects whenever Werner’s condition
\[ c_3 = c_1 + c_4 = 0 \]
is violated. If this condition is fulfilled, \( R[\Psi] = F[\Psi]/\Psi \) may be written in the form
\[
R = c_1 (R_1 - R_4) + c_2 R_2 + c_5 R_5 \\
= c_1 (R_1 - R_4) + (c_2 + 2c_5) R_2 + c_5 (-2R_2 + R_5).
\]
where
\[
R_1[\Psi] \equiv \frac{\vec{\nabla} \cdot \vec{j}}{\rho}, \quad R_2[\Psi] \equiv \frac{\Delta \rho}{\rho}, \quad R_4[\Psi] \equiv \frac{\vec{j} \cdot \vec{\nabla} \rho}{\rho^2}, \quad R_5[\Psi] \equiv \frac{(\vec{\nabla} \rho)^2}{\rho^2}.
\]

As shown in the previous section, the contributions to \( \text{ess} \left( \left( \partial_t \right)^4 \int x^1 \rho_t(x) \, d\vec{x} \right) \bigg|_{t=0} \) by \( R_1 - R_4 \) and \( R_2 \) are functionally independent while \(-2R_2 + R_5\) does not contribute. We conclude that there are relevant instantaneous Gisin effects for the general Doebner-Goldin equation (1),(2) whenever condition (11) is violated. Due to translation invariance this represents a serious locality problem:

The time-dependence of the position probability distribution of a particle ‘behind the moon’ may be instantaneously changed by an arbitrarily small instantaneous change of the potential ‘inside the laboratory’.

Since the change of the potential may be arbitrarily small, such superluminal effects are unacceptable in spite of the nonrelativistic character of the theory.

One might try to exclude relevant Gisin effects by changing the coupling of nonlinear quantum mechanical systems. However, the coupling should be

- the same for uncorrelated subsystems,
- invariant under nonlinear gauge transformations, and
- mathematically respectable for sufficiently many wave functions.

Unfortunately, no modification fulfilling these requirements is known up to now.

Of course, the results presented in this paper do not indicate any problem for the general Doebner-Goldin equation if it is interpreted as a 1-particle equation, as originally suggested [1, 2].

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References

[1] H.-D. Doebner and G.A. Goldin, Physics Letters A 162 (1992) 397–401.

[2] H.-D. Doebner and G.A. Goldin, Properties of nonlinear Schrödinger equations associated with diffeomorphism groups representations. J. Phys. A 27 (1994) 1771–1780.

[3] H.-D. Doebner and G.A. Goldin, Introducing nonlinear gauge transformations in a family of nonlinear Schrödinger equations. Phys. Rev. A 54 (1996) 3764–3771.

[4] H.-D. Doebner, G.A. Goldin, and P. Nattermann, Gauge transformations in quantum mechanics and the unification of nonlinear Schrödinger equations. ASI-TPA/21/96, quant-ph/9709030.

[5] N. Gisin, Weinberg’s non-linear quantum mechanics and superluminal communications. Physics Letters A 143 (1990) 1–2.

[6] N. Gisin, Relevant and irrelevant nonlinear Schrödinger equations. In: Nonlinear, deformed and irreversible quantum systems, eds. H.-D. Doebner, V.K. Dobrev, and P. Nattermann (World Scientific, 1995) p. 109–124.

[7] W. Lücke, Nonlinear Schrödinger dynamics and nonlinear observables. In: Nonlinear, deformed and irreversible quantum systems, eds. H.-D. Doebner, V.K. Dobrev, and P. Nattermann (World Scientific, 1995) p. 140–154.

[8] W. Lücke and P. Nattermann, Nonlinear quantum mechanics and locality, ASI-TPA/12/97, quant-ph/9707053. To appear in: B. Gruber and M. Ramek, editors, Symmetry in Science X, Plenum Press, New York, 1998.

[9] P. Nattermann, Struktur und Eigenschaften einer Familie nichtlinearer Schrödingergleichungen, Diplomarbeit, TU Clausthal (1993).

[10] P. Nattermann, Solutions of the general Doebner-Goldin equation via nonlinear transformations. In: Proceedings of the XXVI Symposium on Mathematical Physics, Toruń, December 7-10, 1993, p. 47.

[11] P. Nattermann, Dynamics in Borel-quantization: Nonlinear Schrödinger equations vs. master equations, Dissertation, TU Clausthal (1997).

[12] H. Teismann, The Cauchy Problem for the Doebner–Goldin Equation. In: Physical applications and mathematical aspects of geometry, groups and algebras, eds. H.-D. Doebner, P. Nattermann, and W. Scherer, (World Scientific, 1997) p. 433–438.

[13] R. Werner, private communication.