Introduction

Level structures play an important role in the definition of moduli spaces, because they offer a possibility to rigidify moduli problems. For example level $N$ structures of elliptic curves over $\mathbb{C}$ are defined by an isomorphism of $\mathbb{Z}/N \times \mathbb{Z}/N$ and $N$-torsion points. If one replaces the base $\mathbb{C}$ by an arbitrary scheme then one is forced to regard the $N$-torsion points as a group scheme which possibly has at some points of the base connected components. So in this case the concept of an isomorphism of a constant group scheme and a the group scheme of $N$-division points does not work any more. After Drinfeld one weakens this isomorphism condition to a morphism which matches the
corresponding Cartier divisors properly. This idea leads to the notion of generators of a level structure [KM85, (3.1)], respectively in a more general setup to the notion of $A$-structures and $A$-generators [KM85, Ch. 1].

A similar situation occurs in the theory of Drinfeld modules, which can be seen as an analogue of elliptic curves in characteristic $p$. As it is discussed in loc. cit. one runs into a small difficulty in the definition of level structures as a condition for all prime divisors of $N$ versus a condition for $N$ itself which is discussed in loc. cit. in general [KM85, Prop. 1.11.3, Rk. 1.11.4] and in the case of elliptic curves [KM85, Th. 5.5.7]. The aim of this article is to prove the analogous result in the case of Drinfeld modules.

An $I$-level structure of a Drinfeld module $(E, e)$ of rank $d$ over a base scheme $S$ is defined as an $A$ module homomorphism

$$\iota : \left( I^{-1}/A \right)^d \longrightarrow E(S)$$

such that for all prime ideals $p \supseteq I$ we have an equality

$$E[p] = \sum_{x \in (p^{-1}/A)^d} \iota(x)$$

of relative Cartier divisors [Dri76]. In [Leh09] 4, prop. 3.3, it is proved that if $\iota$ is an $I$-level structure then

$$E[I] = \sum_{x \in (I^{-1}/A)^d} \iota(x)$$

is an equality of relative Cartier divisors too.

If one defines an $I$-level structure by the equality

$$E[I] = \sum_{x \in (I^{-1}/A)^d} \iota(x)$$

of relative Cartier divisors does it follow that

$$E[p] = \sum_{x \in (p^{-1}/A)^d} \iota(x)$$

is an equality of relative Cartier divisors for every prime ideal $p \supseteq I$ too? This question was solved in the authors PhD-Thesis [Wie04] if the base scheme $S$ is reduced. A careful reading of the arguments used in [Leh09] gives the general result for an arbitrary base scheme $S$. 
1 Definitions

1.1 Drinfeld modules

Let \( X \) be a geometrically connected smooth algebraic curve over the finite field \( \mathbb{F}_q \), let \( \infty \in X \) be a closed point and let \( A := \Gamma(X \setminus \infty, \mathcal{O}_X) \) be the ring of regular functions outside \( \infty \). In this case \( A \) is a Dedekind ring.

Let \( S/\mathbb{F}_q \) be a scheme, \( \mathcal{L} \) a line bundle over \( S \) and let \( \mathbb{G}_a/\mathcal{L} \) be the additive group scheme corresponding to the line bundle \( \mathcal{L} \). For all open subsets \( U \subset S \) the group scheme is defined by

\[
\mathbb{G}_a/\mathcal{L}(U) = \mathcal{L}(U).
\]

The (additive) groups \( \mathcal{L}(U) \) are in a canonical way \( \mathbb{F}_q \)-vector spaces.

**Definition 1.1 ([Dri76])**

Let \( \text{char} : S \longrightarrow \text{spec} A \) be a morphism over \( \mathbb{F}_q \). A Drinfeld module \( E := (\mathbb{G}_a/\mathcal{L}, e) \) consists of an additive group scheme \( \mathbb{G}_a/\mathcal{L} \) and a ring homomorphism

\[
e : A \longrightarrow \text{End}_{\mathbb{F}_q}(\mathbb{G}_a/\mathcal{L})
\]

such that:

1) The morphism \( e(a) \) is finite for all \( a \in A \) and for all points \( s \in S \) there exists an element \( a \in A \) such that locally in \( s \) the rank of the morphism \( e(a) \) is bigger than 1.

2) The diagram

\[
\begin{array}{ccc}
A & \xrightarrow{e} & \text{End}_{\mathbb{F}_q}(\mathbb{G}_a/\mathcal{L}) \\
\downarrow{\text{char}} & & \downarrow{\partial} \\
\mathcal{O}_S(S) & & \\
\end{array}
\]

commutes.

If \( S = \text{spec} R \) is affine und if \( \mathcal{L} \) is trivial, then we will simply write \( E = (R, e) \).

**Proposition 1.2**

Let \( S \) be a connected scheme. Then there exists a natural number \( d > 0 \), such that for all \( 0 \neq a \in A \) we have \( \text{rk}(e(a)) = q^{-d \deg(\infty) \infty(a)} \). The number \( d \) is called the rank of the Drinfeld module.
If \( S = \text{spec}(R) \) is an affine scheme and if \( \mathcal{L} \) is trivial, then we can show that

\[
\text{End}_{\mathcal{P}}(\mathbb{G}_a/\mathcal{L}) \cong \text{AddPol}_q(R)
\]

where \( \text{AddPol}_q(R) \) is the ring of \( \mathbb{F}_q \)-linear polynomials, i.e. every polynomial \( f(X) \in \text{AddPol}_q(R) \) is of the form

\[
f(X) = \sum_{i=0}^{n} \lambda_i X^{q^i}.
\]

In the affine situation a Drinfeld module is therefore given by a non trivial ring homomorphism \( e \) and a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{e} & \text{AddPol}_q(R) \\
\downarrow\text{char} & & \downarrow\partial \\
R & & 
\end{array}
\]

If we define \( e_a(X) := e(a) \) and if \( e_a(X) = \sum_{i=0}^{n} \lambda_i X^{q^i} \) then \( \partial(e_a(X)) = \lambda_0 \), the coefficient \( \lambda_{-d \deg(\infty) \infty(a)} \) is a unit in \( R \) and \( \lambda_i \) is nilpotent for \( i > -d \deg(\infty) \infty(a) \). If in this case \( \lambda_i = 0 \) for all \( a \in A \) then the Drinfeld module is called standard. One can show, that every Drinfeld module is isomorphic to a Drinfeld module in standard form.

By abuse of language the image of the map \( \text{char} : S \longrightarrow \text{spec } A \) is called the characteristic of \( E \). If it consists only of the zero ideal then we say \( E \) has general characteristic.

### 1.2 Division points and level structures

Let \( E \) be a Drinfeld module of rank \( d \) over a base scheme \( S \) and let \( 0 \neq I \subsetneq A \) be an ideal.

**Definition 1.3**

The contravariant functor \( E[I] \) on the category of schemes over \( S \) with image in the category of \( A/I \) modules defined by

\[
T/S \mapsto \{ x \in E(T) \mid Ix = 0 \} = \text{Hom}_A(A/I, E(T))
\]

for all schemes \( T/S \) is called the scheme of \( I \)-division points.
Properties 1.4
1) $E[I] \subseteq E$ is a closed, finite and flat (sub-)group scheme over $S$ of rank $|A/I|^d$. If $I = (a_1, \ldots, a_n)$ for appropriate elements $a_1, \ldots, a_n \in A$ then it is
$$E[I] = \ker(E \xrightarrow{e_{a_1}, \ldots, e_{a_n}} E \times_S \cdots \times_S E).$$
In the affine case $S = \text{spec } R$ we have
$$E[I] = \text{spec } R[X]/(e_{a_1}(X), \ldots, e_{a_n}(X)).$$

2) If $I, J$ are coprime ideals in $A$, then
$$E[IJ] \cong E[I] \times_S E[J].$$

3) If $I$ is coprime to the characteristic of the Drinfeld module $E$ then $E[I]$ is étale over $S$.

4) The group scheme $E[I]$ is compatible with base change, that is for each scheme $T/S$ we have
$$E[I] \times_S T \cong (E \times_S T)[I].$$

Proof Cf. [Leh09], chapter 2, proposition 4.1, page 27 et seq. \[\Box\]

If $S = \text{spec } R$ is an affine Scheme and if $\mathcal{L}$ ist trivial we can use the following lemma to describe the group scheme $E[I]$ by a unique additive polynomial $h_I(X)$ of degree $|A/I|^d$.

Lemma 1.5
Let $R$ be an $\mathbb{F}_q$-algebra. Let $H \subseteq \mathbb{G}_{a/R}$ be a finite flat subgroup scheme of rank $n$ over $R$. Then there is a uniquely defined normalized additive polynomial $h \in R[X]$ of degree $n$ such that $H = V(h)$.

Proof Cf. [Leh09], chapter 1, lemma 3.3, page 9. \[\Box\]

If $S = \text{spec } L$ is a field then the characteristic of the Drinfeld module $(L, e)$ is a prime ideal of $A$. Thus we can define the height of $(L, e)$, denoted by $h$.

In the case of an algebraically closed field we have the following explicit description of the $I$-division points:

Satz 1.6
Let $0 \neq p \subset A$ be a prime ideal and let $I = p^n$ for an $n > 0$. Then we have
$$E[p^n](L) \cong \begin{cases} (p^{-n}/A)^d & \text{for } p \neq \text{char } E \smallskip \quad \quad \quad \quad \quad \quad \text{for } p = \text{char } E. \end{cases}$$
Proof  Cf. [Leh09], chapter 2, corollary 2.4, page 24.

This result motivates the following definition:

**Definition 1.7 ([Dri76])**

Let $E = (\mathbb{G}_a, e)$ be a Drinfeld module of rank $d$ over $S$ and let $0 \neq I \subseteq A$ be an ideal. A Level $I$ structure is an $A$-linear map

$$\iota : (I^{-1}/A)^d \to E(S),$$

such that for all prime ideals $p \supseteq I$ we have an identity of Cartier divisors

$$E[p] = \sum_{x \in (p^{-1}/A)^d} \iota(x).$$

**Remark 1.8**

1. If $I$ is coprime to the characteristic of $E$ then a level $I$ structure is an isomorphism of group schemes

$$(I^{-1}/A)^d_S \simeq E[I].$$

2. If $E = (R, e)$ the equality of Cartier divisors simply means an equality of the polynomials

$$h_p(X) = \prod_{x \in (p^{-1}/A)^d} (X - \iota(x)).$$

**2 Level Structures and Deformations**

In [Leh09], 3, prop. 3.3 we have the following result.

**Proposition 2.1**

If $(E, \iota)$ is a Drinfeld module equipped with a level $I$ structure $\iota$ then we have the identity of Cartier divisors

$$E[I] = \sum_{x \in (I^{-1}/A)^d} \iota(x).$$

The proof in loc. cit. is based on the construction of deformation spaces of Drinfeld modules, isogenies and level $I$ structures. In the following we will repeat the basic definitions and results.
Definition 2.2

1. Let $i : A \rightarrow O$ be a complete noetherian $A$-algebra with residue field $\ell$. Let $\mathcal{C}_O$ be the category of local artinian $O$-algebras with residue field $\ell$ and let $\hat{\mathcal{C}}_O$ be the category of noetherian complete local $O$-algebras with residue field $\ell$.

2. Let $E_0$ be a Drinfeld module of rank $d$ over $\ell$, and let $B$ be an algebra in $\mathcal{C}_O$. A deformation of $E_0$ over $B$ is a Drinfeld module of rank $d$ over $\text{spec } B$ which specializes mod $m_B$ to $E_0$. Thus we obtain a functor:

\[
\text{Def}_{E_0} : \mathcal{C}_O \rightarrow \text{Sets} \\
B \mapsto \{\text{Isomorphyclasses of Deformations of } E_0\}
\]

3. Let $\varphi : E_0 \rightarrow F_0$ be an isogeny of Drinfeld modules of rank $d$ over $\ell$. A deformation of $\varphi_0$ over $B$ is an isogeny $\varphi : E \rightarrow F$ where $E, F$ are deformations of $E_0$ and $F_0$, such that $\varphi$ specializes mod $m_B$ to $\varphi_0$. We obtain a corresponding functor:

\[
\text{Def}_{\varphi_0} : \mathcal{C}_O \rightarrow \text{Sets} \\
B \mapsto \{\text{Isomorphyclasses of Deformations of } \varphi_0\}
\]

4. Let $(E_0, \iota_0)$ be a Drinfeld module of rank $d$ over $\ell$ equipped with a level $I$ structure $\iota_0$. A deformation is a Drinfeld module $(E, \iota)$ over $B$ of rank $d$ equipped with an level $I$ structure $\iota$ such that $E$ is a deformation of $E_0$ and $\iota$ specializes to $\iota_0$ mod $m_B$. We define the functor:

\[
\text{Def}_{(E_0, \iota_0)} : \mathcal{C}_O \rightarrow \text{Sets} \\
B \mapsto \{\text{Isomorphyclasses of Deformations of } (E_0, \iota_0)\}
\]

5. We will denote the tangent spaces of the functors above by

\[
T_{E_0} := \text{Def}_{E_0}(\ell[\varepsilon]), \ T_{\varphi_0} := \text{Def}_{\varphi_0}(\ell[\varepsilon]), \ T_{(E_0, \iota_0)} := \text{Def}_{(E_0, \iota_0)}(\ell[\varepsilon])
\]

where $\ell[\varepsilon]$ is the $\ell$-algebra with $\varepsilon^2 = 0$.

Results 2.3

1. The deformation functor $\text{Def}_{E_0}$ is pro-represented by the smooth $O$-algebra $R_0 := O[[T_1, \ldots, T_{d-1}]]$.

2. The deformation functor $\text{Def}_{\varphi_0}$ is pro-represented by an object in $\hat{\mathcal{C}}_O$.

3. The deformation functor $\text{Def}_{(E_0, \iota_0)}$ is pro-represented by an object in $\hat{\mathcal{C}}_O$. 
3 Equivalence of Definitions

The question is, what happens if we would change the definition 1.7 of level $I$ structures with:

**Definition 3.1**

Let $E = (\mathbb{G}_{a,C}, e)$ be a Drinfeld module of rank $d$ over $S$ and let $0 \neq I \subseteq A$ be an ideal. A Level $I$ structure is an $A$ linear map

$$\iota : (I^{-1}/A)^d \longrightarrow E(S),$$

such that we have an identity of Cartier divisors

$$E[I] = \sum_{x \in (I^{-1}/A)^d} \iota(x).$$

To distinguish the two definitions we will refer the original definition as $A$ and the one above as $B$.

**Proposition 3.2**

Definition $A$ and definition $B$ are equivalent.

As it is proved in [Leh09], 3, prop. 3.3. being a level $I$ structure in the sense of definition $A$ implies being on in the sense of definition $B$. One the other hand it is proved in [Wie04, Ch. 6, Prop. 6.7] by a simple counting argument that $B$ implies $A$ if the base scheme $S$ is reduced. On the other hand the result is clear if $I$ is away from the characteristic of the Drinfeld module. Thus for the proof of proposition 3.2 we are allowed to make the following assumptions:

1. $S = \text{spec } B$ where $B$ is the localization of a finitely generated $A$-algebra at a maximal prime Ideal, $p := A \cap m_B$ is not zero and $\ell := B/m_B$ is a finite extension of $A/p$.

2. The result is true if it is true for all quotients $B/m^n_B$. So we can assume, that $B$ is a local artinian ring with residue field $\ell$.

3. We will fix an element $\varpi_p \in A$, such that $(\varpi_p) = pJ$ with $p \nmid J$. Then we have

$$A \hookrightarrow \hat{A}_p \cong A/p[[\varpi_p]] \hookrightarrow \ell[[\varpi_p]] =: O$$

such that $m_O \cap A = p$ and $pO = m_O$.

4. As $B$ is artinian, and therefore complete, there is a unique lift of the coefficient field $\ell$ to $B$ and we can consider $B$ as an object of $\mathcal{C}_O$. 

5. We fix a Drinfeld module $E_0 = (e^{(0)}, \ell)$ and a level $p^n$ structure $\iota_0$. As $\ell$ is a field there is no difference between the definitions $A$ and $B$.

6. Let $E$ be the universal deformation of $E_0$ and $O$ as above. Then $E$ is defined over $R_0 := O[[T_1, \ldots, T_{d-1}]] \cong \ell[[T_0, \ldots, T_{d-1}]]$ for $T_0 := \varpi_p$. It is a complete regular local ring of dimension $d$. In especially $R_0$ is integral and the map $\text{char} : A \rightarrow R_0$ is injective and the Drinfeld module has general characteristic.

To prove the proposition we will follow the arguments of [Leh09, 3.3.1]. The main difference is now to use $p^n$ instead of $p$.

Let $p_{p^n} : E \longrightarrow E/E[p^n]$ the canonical quotient isogeny of Drinfeld modules with kernel $E[p^n]$. The corresponding polynomial $p_{p^n} = h_{p^n} \in R_0[X]$ is an additive, normalized and separable polynomial of degree $|A/p^n|^d$. We define $L$ to be a splitting field of $h_{p^n}$ over the field of quotients $\text{Quot}(R_0)$. Using the zeros $V(h_{p^n})$ of $h_{p^n}$ in $L$ we define the $R_0$-algebra $R_{h_{p^n}} := R_0[V(h_{p^n})]$ inside $L$. It is an integral and finite extension of $R_0$ because $h_{p^n}$ is normalized over $R_0$.

Along the lines of [Leh09] we will prove

$$R_{h_{p^n}} \cong R_n$$

where $R_n$ is the base ring of the universal Drinfeld module of deformations of $E_0$ and a level $p^n$ structure over $\ell$. We will use induction on $n$.

If $n = 0, 1$ then nothing is to prove. For $n > 1$ we have:

$$R_n := R_{n-1}[[S_1, \ldots, S_d]]/\mathfrak{a}$$

where $\mathfrak{a}$ is the ideal generated by elements of the form $e^{\varpi_p^x}(S_i) - \iota_p(x_i) + e^{\varpi_p^x}(\tilde{y}_i)$. These elements are normalized polynomials in $S_i$, so

$$R_n := R_{n-1}[S_1, \ldots, S_d]/\mathfrak{a}$$

We can find elements $\tilde{x}_1, \ldots, \tilde{x}_d$ in $R_{h_{p^n}}$ such that $e^{\varpi_p^x}(\tilde{x}_i) = x_i$ and $\tilde{x}_1, \ldots, \tilde{x}_d$ generates $V(h_{p^n})$ as an $A/p^n$-module. We define a map of $R_{n-1}$-algebras:

$$R_n \longrightarrow R_{h_{p^n}}$$

$$S_i \longmapsto \tilde{x}_i - \tilde{y}_i$$

By assumption the map is well defined and surjective. As both rings have dimension $d$, it is an isomorphism.
Corollary 3.3
\( R_{h^n} \) is a regular algebra in \( \hat{\mathcal{C}}_O \).

Now we can use the setup of [Leh09], proof of proposition 3.3.1, to show that \( E_{p^n} := E \otimes_{R_0} R_{h^n} \) and a corresponding lift of \( t_0 \) is the universal deformation of level \( p^n \). This is true if we can prove [Leh09], lemma 3.3.2, with \( p \) replaced by \( p^n \), but there is no obstruction to do so.

Now we are done, because \( R_{h^n} \) is an integral ring and definition \( A \) and \( B \) coincide on the level of the universal deformation.

4 References

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