Note on deconfinement temperature with chemical potential from AdS/CFT

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Abstract

In this note we study the first-order Hawking-Page phase transition of R-charged black hole in truncated Anti-de Sitter space of various dimensions. This corresponds to confinement/deconfinement phase transition in the dual gauge field theory with fixed chemical potential. We demonstrate in general this critical temperature decreases with increasing charge density but with decreasing IR cutoff.

PACS numbers: 11.25Tq,25.75Nq

Keywords: Hawking-Page phase transition, R-charged black hole

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I. INTRODUCTION

Anti-de Sitter space-Conformal Field Theory (AdS/CFT) correspondence has been widely studied since the work by [1, 2, 3]. In particular, its application to relate the thermodynamics of $\mathcal{N} = 4$ super Yang-Mills (sYM) theory in four dimensions to the thermodynamics of Schwarzschild black holes in five-dimensional Anti-de Sitter space [4]. In this description, confinement/deconfinement phase transition of gauge theory on a sphere has its holographically gravitational description as the Hawking-Page phase transition [5]. Later, authors in [6, 7, 8] realized confinement by capping off the geometry (i.e. the Calabi-Yau cone) smoothly at the infrared tip. Though it has been known since then that Hawking-Page transition happens at zero temperature for gauge theory on a flat space, the papers [9, 10, 11] showed that a finite Hawking-Page transition temperature can still occur once an IR cutoff is introduced in AdS Poincaré patch. This transition can be understood as first-order confinement/deconfinement phase transition in the holographic gauge field theory in the Minkovski space. Transition with matter was recently studied in [12]. Transition up to $\mathcal{O}(\alpha'^3)$ correction was studied in [13]. Phase transition temperature was obtained in relation to chemical potentials for $\mathcal{N} = 4$ sYM on $S^3$ via direct evaluation of partition function [14].

In this note, we would like to extend this study to a more general background. To be specific, we will target at gauge theories with fixed chemical potential in spacetime dimension 3, 4, 5 and 6, via the study of a special kind of R-charged black hole in AdS space of one dimension higher according to the AdS/CFT correspondence [15].

We will also introduce an IR cutoff essential to realize the confinement/deconfinement phase transition and mimic a QCD-like model. There are hard wall and soft wall models proposed in the literature [16, 17, 18] and our result shows that hard wall model is enough to catch new features contributed from charge density and IR cut-off.

We begin in section II with brief review on the thermodynamics of these R-charged black holes in general. In section III we evaluate the Gibbs Euclidean action and obtain a implicit formula for the Hawking-Page transition temperature in various dimensions. In section IV we restrict to the case of single charge for simplicity and study the transition temperature in each specific dimension. We find that in most cases the Hawking-Page transition temperature decreases with increasing charge density, but with decreasing IR cut-off. We conclude with some comments in section V.
II. THERMODYNAMICS OF GENERAL R-CHARGED BLACK HOLES

Although the supergravity theories vary in different spacetime dimensions, here we consider a general bosonic action for gravity coupled to a set of scalars and vector fields given in the form,

\[ I = -\frac{1}{16\pi G_d} \int_{\mathcal{M}} d^d x \sqrt{-g} [R - \frac{1}{2} G_{ij}(\phi) \partial_{\mu} \phi^i \partial_{\nu} \phi^j - \frac{1}{4} G_{IJ} F_{\mu \nu}^I F^{\mu \nu J} - V(\phi)] + \frac{1}{8\pi G_d} \int_{\partial \mathcal{M}} d^{d-1} \sqrt{-h} \Theta. \]

This action is composed of a term integrated over \( d \)-dimensional spacetime \( \mathcal{M} \) and the other over a \( (d-1) \)-dimensional boundary \( \partial \mathcal{M} \). The latter is the Gibbon-Hawking surface term given in terms of trace of the extrinsic curvature \( \Theta_{\mu \nu} \), defined as

\[ \Theta_{\mu \nu} = -\nabla_{(\mu} n_{\nu)}, \]

where \( n^\mu \) is the outward-going normal on \( \partial \mathcal{M} \) and \( h_{\mu \nu} \) is the induced metric. Having in mind that \( \mathcal{M} \) is chosen to be asymptotic AdS_\( d \) space and \( \partial \mathcal{M} \) is its boundary with topology \( R_t \times S^{d-2} \), we are interested in spherically symmetric black holes carrying electric charge(s), i.e.

\[ ds^2 = -e^{2(d-3)B(r)} f(r) dt^2 + e^{2B(r)} \left( \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2 \right), \]

\[ \phi^i = \phi^i(r), \quad A^I_t = A^I_t(r). \]

For a black hole solution, we request finite \( B(r) \) and vanishing \( f(r) \) at the event horizon \( r = r_+ \), which is the largest real root satisfying equation \( f(r) = 0 \). The asymptotic geometry is obtained by sending \( r \to \infty \) so that

\[ \lim_{r \to \infty} B(r) = 0, \quad \lim_{r \to \infty} f(r) = g^2 r^2, \]

where \( g \) is the inverse of AdS radius. To our interests, a class of stationary R-charged black hole shares a common description \([15]\), i.e.

\[ e^{2(d-2)B(r)} \equiv \mathcal{H}(r) = \prod_{i=1}^{n} \left( 1 + \frac{q_I}{r^{d-3}} \right), \]

\[ f(r) = 1 - \frac{\mu}{r^{d-3}} + g^2 r^2 \mathcal{H}(r), \]

where \( n \) is the maximum number of \( U(1) \) which can be embedded inside the R-symmetry group for each specific dimension. This black hole can be viewed as Kaluza-Klein reduction.
of a rotating black brane on the sphere, after taking certain limit\[15, 19\]. The angular momentum in the full ten or eleven dimensions plays the role of R-charge, i.e. $U(1)$ charge to the gauge field $A^I$ in the $d$-dimensional effective theory.

Now we will follow \[20\] for the discussion of its thermodynamics, but also see \[19, 21\]. The appropriate ensemble applied to that with a nontrivial chemical potential is the grand-canonical ensemble, and the on-shell action (11) relates to the Gibbs free energy by $I = \beta \Omega$, where

$$\Omega = E - TS - \Phi^I Q_I.$$  \hfill (6)

This relation needs some explanation. $\beta$ is the periodicity of Euclidean time and inverse of Hawking temperature in thermal equilibrium, i.e.

$$\beta^{-1} = T = \frac{1}{4\pi} e^{-(d-2)B(r)} f'(r)|_{r+}.$$  \hfill (7)

The entropy can be obtained via the Bekenstein-Hawking relation,

$$S = \frac{\omega_{d-2}}{8\pi G_d} (2\pi e^{(d-2)B_r r^{d-2}})|_{r_+},$$  \hfill (8)

where $\omega_{d-2}$ is the volume of $S^{d-2}$. $E$ is the black hole mass, here given by

$$E = \langle T_{tt} \rangle = \frac{2}{\sqrt{-h}} \frac{\delta \Omega}{\delta h^{tt}} = \frac{\omega_{d-2}}{8\pi G_d} \sqrt{-h} (-\Theta^t + \Theta).$$  \hfill (9)

At last, the chemical potential $\Phi^I$ and the normalized charge $Q_I$ are given by

$$\Phi^I = A^I_t(\infty) - A^I_t(r_+),$$

$$Q_I = \frac{\beta \omega_{d-2} q_I}{16\pi G_d}.$$  \hfill (10)

The only subtlety is that the action (11) and mass (9) are still divergent though it has been regularized via the Gibbon-Hawking term. The paper \[20\] demonstrates that a holographic renormalization scheme naturally applies to render finite quantities but also points out that (6) is still valid regardless of renormalization. In this note we only concern the difference between free energy of black hole and that of thermal AdS, and expect those counterterms cancel each other. Therefore, the regularized action is enough to our purpose\[1\].

\[1\] However, for the case of multiple charges or higher dimensions, we may need a counterterm at IR cut-off thanks to the truncated AdS space. We will deal with this subtlety later we face it.
III. HAWKING-PAGE ANALYSIS AND DECONFINEMENT PHASE TRANSITION

In this section, we are going to determine the phase transition temperature via the Hawking-Page analysis adopted in [10]. To proceed, we impose an infrared cut-off at $r = r_c$ as in the hard wall model proposed in [17] as well as an ultraviolet cut-off at $r = r_0$. The IR cut-off is essential for confinement. The UV cut-off is only for calculation convenience and will be sent to infinity at the end. We consider the following three on-shell actions:

$$I_1 = \frac{\beta \omega d-2}{8\pi G_d} [(3-d) r_0^{d-3} f(r_0) - \frac{1}{2} r_0^{d-2} f'(r_0) - r_0^{d-3} + r_+^{d-3}],$$  \hspace{1cm} (11)$$

for AdS black hole where $r_+ > r_c$.

$$I_2 = \frac{\beta \omega d-2}{8\pi G_d} [(3-d) r_0^{d-3} f(r_0) - \frac{1}{2} r_0^{d-2} f'(r_0) - r_0^{d-3} - r_c^{d-2} f(r_c) B'(r_c) - r_c^{d-3} (f(r_c) - 1)],$$  \hspace{1cm} (12)$$

for AdS black hole where $r_+ < r_c$. At last for thermal AdS with non-vanishing gauge field, we have

$$I_3 = \frac{\beta' \omega d-2}{8\pi G_d} [(3-d) r_0^{d-3} f_0(r_0) - \frac{1}{2} r_0^{d-2} f_0'(r_0) - r_0^{d-3} - r_c^{d-3} (f_0(r_c) - 1)],$$  \hspace{1cm} (13)$$

where

$$f_0(r) \equiv \lim_{\mu \to 0} f(r) = 1 + g^2 r^2 \mathcal{H}(r).$$  \hspace{1cm} (14)$$

We request asymptotically two geometries have the same periodicity along Euclidean time, say $\sqrt{f_0(r_0) \beta'} = \sqrt{f(r_0) \beta}$. Then the difference of Gibbs energy, after the UV cut-off removed, becomes

$$\Delta V_{13} \equiv - \lim_{r_0 \to \infty} \beta^{-1} (I_1 - I_3) = \frac{\omega_2}{8\pi G_d} \left[ \frac{\mu}{2} - r_+^{d-3} - g^2 r_c^{d-1} \mathcal{H}(r_c) \right].$$  \hspace{1cm} (15)$$

The indefinite sign of $\Delta V_{13}$ indicates a first-order Hawking-Page transition at $\Delta V_{13} = 0$. It is energetically preferred for thermal AdS with non-vanishing gauge field at low temperature, but R-charged black hole at high temperature. When the black hole is unaccessible in the truncated AdS, i.e. $\infty > r > r_c > r_+$, the difference becomes

$$\Delta V_{23} \equiv - \lim_{r_0 \to \infty} \beta^{-1} (I_2 - I_3) = \frac{\omega_2}{8\pi G_d} \left[ \frac{\mu}{2} + r_c^{d-2} f(r_c) B'(r_c) \right].$$  \hspace{1cm} (16)$$

The positivity of $\Delta V_{23}$ indicates that thermal AdS with non-vanishing gauge field is always energetically preferred whenever black hole is unaccessible in the truncated AdS space.
IV. DECONFINEMENT PHASE TRANSITION IN VARIOUS DIMENSIONS

Now we are ready to investigate the confinement/deconfinement phase transition in various dimensions via the above Hawking-Page analysis on R-charged black hole in the AdS space. For simplicity, we will restrict our discussion to the case of single R-charge, say $q_1 = q$ and $q_J = 0$ for $J \neq 1$. Equation (15) states that the Hawking-Page transition happens at

$$\frac{\mu}{2} - r_+^{d-3} - g^2 r_c^{d-1} \mathcal{H}(r_c) = 0,$$

and one can in principle inverse equation (7) to relate $r_+$ to the deconfinement transition temperature $T_c$, though the resulting polynomial equation may not be solved analytically.

A. 4d black hole/thermal 3d super Yang-Mills

The R-charged black hole solution in four dimensions corresponds to a particular $U(1)^4$ truncation of maximal gauged supergravity in $d = 4$, which can be traced back to compactification of $d = 11$ supergravity. The four charges can be seen as near-extremal black M2 brane spinning along four transverse directions\[19\]. The corresponded field theory is the thermal $d = 3, \mathcal{N} = 2$ super Yang-Mills living on the boundary with topology $\mathbb{R}^t \times S^2$. The deconfinement temperature, as a function of cut-off and charge density, will not be detailed here for its complicated form, which comes from solving a third-order polynomial equation of two unknowns. In the figure 1(a) we observe that increasing IR cut-off raises transition temperature, however, increasing charge density lowers it. We find transition temperature $T_c = g/\pi$ for zero charge density and no IR cut-off.

B. 5d black hole/thermal 4d super Yang-Mills

The R-charged black hole solution in five dimensions corresponds to a particular $U(1)^3$ truncation of maximal gauged supergravity in $d = 5$, which can be traced back to compactification of $d = 10$ supergravity on the special geometry, so called STU model. Therefore the black hole may carry up to three charges, seen as a near-extremal black D3 brane spinning along three transverse directions\[19\]. The corresponded field theory is the thermal $d = 4, \mathcal{N} = 2$ super Yang-Mills living on the boundary with topology $\mathbb{R}^t \times S^3$. In the case
of single charge, the transition temperature can be found in a manageable form,

\[ g^{-1}T_c = \frac{3(1 + \Delta) - qq^2(4 + \Delta) + g^4(q^2 + 8qr^2 + 8r_c^4)}{\sqrt{2\pi}(1 + \Delta - qq^2)\sqrt{1 + \Delta + qq^2}}, \]

\[ \Delta \equiv \sqrt{1 - 2qq^2 + g^4(q^2 + 8qr^2 + 8r_c^4)}. \]  

(18)

It is interesting to compare with the result found in [22] at the limit \( q \to 0 \). In particular, we recover \( T_c = 3g/2\pi \) as IR cut-off is also removed. The regularity of \( \Delta \) and non-negativity of \( T_c \) impose constraints on admissible \( q \) and \( r_c \). In the figure 1(b) we also observe similar effects of IR cut-off and charge density on the deconfinement temperature.

C. 6d black hole/thermal 5d super Yang-Mills

The R-charged black hole solution in six dimensions corresponds to a particular \( U(1) \) subgroup of \( SU(2) \times U(1) \), \( \mathcal{N} = (1, 1) \) supergravity in \( d = 6 \). To be specific, we have instead two equal charges, i.e.

\[ \mathcal{H} = (1 + \frac{q}{r^3})^2. \]  

(19)

According to equation (15), phase transition is supposed to happen at

\[ \frac{\mu}{2} - r_+^3 - g^2 r_c^5 - 2qq^2 r_c^2 - \frac{g^2 g^2}{r_c} = 0. \]  

(20)

The last term in the above equation diverges at small \( r_c \) for finite charge density \( q \). Technically, we may set \( q = 0 \) before sending \( r_c \to 0 \) to avoid the divergence and render the usual case of zero charge density and no IR cut-off, where \( T_c = 2g/\pi \) is expected. In generic, however, we expect this divergent term will be cancelled via a counterterm at the IR cut-off as well as other finite terms might be modified. Naively dropping this divergent term, we fail to find another solution beside the trivial one mentioned above.

D. 7d black hole/thermal 6d super Yang-Mills

The R-charged black hole solution in seven dimensions corresponds to a particular \( U(1)^2 \) truncation of \( SO(5) \) maximal gauged supergravity, which can be deduced from the \( S^4 \) reduction of \( d = 11 \) supergravity. Therefore the black hole may carry up to two charges, seen as a near-extremal black M5 brane spinning along two transverse directions [19]. The corresponded field theory is the thermal \( d = 6, \mathcal{N} = 2 \) super Yang-Mills living on the boundary.
with topology $R_t \times S^5$. The deconfinement temperature, as a function of cut-off and charge density, will not be detailed here for its complicated form, which comes from solving a six-order polynomial equation of two unknowns. In the figure [1(c)], we find that the regularity of $\Delta$ and non-negativity of $T_c$ impose more restrict constraints on admissible $q$ and $r_c$. Some values of q’s are not allowed for small $r_c$. However in general, we still observe similar effects of IR cut-off and charge density on deconfinement temperature, however, we remark that $T_c$ has minimum around $g^2 q \sim 0.6$ and $gr_c \sim 0.3$. We find transition temperature $T_c = 5g/2\pi$ for zero charge density and no IR cut-off.

V. CONCLUSION

In this note we study the confinement/deconfinement phase transition of gauge theory with fixed chemical potential via the Hawking-Page analysis of single R-charged black hole in truncated AdS space of various dimensions. In the limit of zero charge density and no IR cut-off, we recover the transition temperature,

$$T_c = (d - 2)\frac{g}{\pi},$$

(21)

for $d$-dimensional AdS black hole or gauge field on $R_t \times S^{d-2}$, where $4 \leq d \leq 7$. This is consistent with the known fact that $T_c \to 0$ as $g \to 0$. For nonzero charge density $q$ and IR cut-off $r_c$, we obtain $T_c$ as a function of $q$ and $r_c$ for each case except $d = 6$, where the difference of Gibbs free energy diverges at $r_c \to 0$. We expect that a counterterm constructed at this IR boundary in the way similar to those in [20] will render it finite, and then the improved function $T_c$ will make better sense. We leave a comprehensive treatment for future projects. In the dimension $d = 4, 5, 7$, $T_c$ in general decreases with increasing $q$ but decreasing $r_c$. Similar relation between $T_c$ and $q$ was also observed in [12] though their matter field came from a different source. This relation might be understood as follows: At first, high charge density makes quarks (charged under global $U(1)$) expel from each other, weakening the confining force and thus lowering the confinement/deconfinement transition temperature. Secondly, Hawking-Page transition happens whenever the black hole horizon $r_+$ is around IR cut-off $r_c$. From equation [7] we learn that temperature $T_c$ grows with $r_+$. Therefore we can deduce that $T_c$ decreases with decreasing $r_c$. However, we notice that in the case of $d = 7$, $T_c$ bounces back for high charge density. Though this seems a contradiction to
our previous statement, a possible resolution can be given as follows: Once \( T_c \) has reached its minimum, charge density also reaches its critical value of saturation. If one tries to put more quarks in the system, any pair of them is forced to recombine by expelling from the surrounded others. Each pair takes extra energy to be deconfined and therefore \( T_c \) raises up. As the last remark, the author in [10] showed that a more realistic meson spectrum can be produced via the soft wall model rather than the hard one, it might be interesting to repeat this Hawking-Page analysis in the soft wall model of the same R-charged black hole background.

Acknowledgments

The author would like to thank Furuuchi Kazuyuki for useful discussion. This work is supported in part by the Taiwan’s National Science Council under Grant No. NSC96-2811-M-002-018.

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(a) Hawking-Page analysis of AdS$_4$ R-charged black hole. The origin represents zero charge density without IR cut-off where $T_c = \frac{4}{g^2}$.

(b) Hawking-Page analysis of AdS$_5$ R-charged black hole. The origin represents zero charge density and no IR cut-off where $T_c = \frac{3g}{2g^2}$.

(c) Hawking-Page analysis of AdS$_7$ R-charged black hole. Only the region where $gr_c > 0.24, g^2 q > 0$ is shown. We remark $T_c = \frac{5g}{2\pi}$ for zero charge density and no IR cut-off.

FIG. 1: The contour plot of confinement/deconfinement phase transition temperature $T_c$ as a function of IR cut-off $r_c$ and charge density $q$. 

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