Meson Regge trajectories in relativistic quantum mechanics

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Abstract

A model potential for two-particle relativistic systems is investigated in the framework of Poincare-invariant quantum mechanics (or relativistic Hamiltonian dynamics). The potential considered allows to reduce the main integro-differential equation of Poincare-invariant quantum mechanics to the equation analogous to the radial equation and have analytical solution for relativistic bound system. We discuss the possible choice of the parameters of potential and apply our model to the description of light meson Regge trajectories.
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A model potential for two-particle relativistic systems is investigated in the framework of Poincare-invariant quantum mechanics (or relativistic Hamiltonian dynamics). The potential considered allows to reduce the main integro-differential equation of Poincare-invariant quantum mechanics to the equation analogous to the radial equation and have analytical solution for relativistic bound system. We discuss the possible choice of the parameters of potential and apply our model to the description of light meson Regge trajectories.

1 Introduction
Relativistic few body problem has received a great attention in hadronic and nuclear physics. The most complete results here exist for the case of two particles. Description of the bound system in the relativistic quantum field theory is founded on the four-dimensional covariant Bethe-Salpeter equation \[1\]. However, this equation gives series of difficulties when the practical calculations are made.

There exist various reductions of the two-body Bethe-Salpeter equation. Different forms of this reduction were discussed Logunov and Tavkhelidze \[2\], Kadyshevsky \[3\], Todorov \[4\], Gross \[5\], Poluzou, Keister and Lev \[6\] and many others.

The relativistic two-body systems were analyzed by Weinberg \[9\], Frankfurt and Strikman \[10\], Kondratyuk and Terent’ev \[11\] in infinite-momentum frame. Some authors use diagrammatic approach i.e. they select leading diagrams and project them onto the three-dimensional space. Others make use effective Hamiltonians, Faddev equation. Different forms of the quasi-potential equation can be derived using this approach \[12\] \[13\].

In this work we apply the point form of relativistic quantum mechanics (RQM) \[6\] \[8\] to the description of meson Regge trajectories. RQM is also known in the literature as relativistic Hamiltonian dynamics or Poincare-invariant a quantum mechanics with direct interaction.

2 RQM formalism for quark-antiquark bound states
The formulation of relativistic quantum mechanics differs from non-relativistic quantum mechanics by the replacement of invariance under Galilean transformations with invariance under Poincare transformations. The dynamics of many-particle system in the RQM is specified by expressing the ten generators of the Poincare group \(\hat{M}_{\mu\nu}\) and \(\hat{P}_\mu\) in terms of dynamical variables. In the constructing generators for interacting systems it is customary to start with the generators of the corresponding non-interacting system (we shall write this operators without "\(\hat{a}\) hat") and then add interaction in the a way that is consistent with Poincare algebra. In the relativistic case it is necessary to add an interaction \(V\) to more than one generator in order to satisfy the commutation relations of the Poincare algebra. Dirac \[14\] observed, that there is no unique way of separating the generators into dynamical subset (the generators including interaction \(V\) and kinematic sub-
set. Kinematic subset must be associated with some subgroup of Poincare group, usually called stability group [14] or kinematic subgroup. Thus, the construction for two interacting particles proceeds as follows:

1. The two-particle Hilbert space of non-interacting system is defined as tensor product of two one-particle Hilbert spaces. A two-body unitary representation of Poincare group on the two-particle Hilbert space to linear representations of the non-interacting system are reducible. A basis in this space can be constructed from single-particle bases:

\[ |p_1 \lambda_1 \rangle |p_2 \lambda_2 \rangle \equiv |m_1 s_1; p_1 \lambda_1 \rangle \otimes |m_2 s_2; p_2 \lambda_2 \rangle \]  

with normalization

\[ \langle p_1' \lambda_1' | p_2' \lambda_2' \rangle | p_1 \lambda_1 \rangle | p_2 \lambda_2 \rangle = \delta_{\lambda_1' \lambda_1} \delta_{\lambda_2' \lambda_2} \delta(p_1' - p_1) \delta(p_2' - p_2), \]

where \( p_i, m_i, s_i, \lambda_i \) are momenta, masses, spins and projections of spin of particles, which will form bound system.

2. Clebsch-Gordan coefficients for Poincare group are constructed and used to reduce the unitary representation on the two-particle Hilbert space to linear superposition (direct integral) of irreducible representations. Poincare generators for irreducible representations of the non-interacting system are constructed, along with operators for the mass and spin and other operators. The result of this step is

\[ | \bar{P}_{12}, \mu, [J, k], (l s); [m_1 s_1; m_2 s_2] \rangle = \sum_{l_s} \sum_{\lambda_1 \lambda_2} \int d^3 k \frac{\omega_{m_1}(\vec{p}_1) \omega_{m_2}(\vec{p}_2) M_0}{\omega_{m_1}(\vec{k}) \omega_{m_2}(\vec{k}) \omega_{m_0}(\bar{P}_{12})} \]

\[ \sum_{m \lambda} \langle s_1 \nu_1, s_2 \nu_2 | s \lambda \rangle \langle l m, s \lambda | J \mu \rangle Y_{l m}(\theta, \phi) \]

\[ D_{\lambda_1 \nu_1}^{1/2}(\vec{n}(p_1, P_{12})) D_{\lambda_2 \nu_2}^{1/2}(\vec{n}(p_2, P_{12})) | p_1 \lambda_1 \rangle | p_2 \lambda_2 \rangle, \]

where \( \langle s_1 \nu_1, s_2 \nu_2 | s \lambda \rangle, \langle l m, s \lambda | J \mu \rangle \) are Clebsch-Gordan coefficients of \( SU(2) \)-group, \( Y_{l m}(\theta, \phi) \) are the spherical harmonics with spherical angle of \( \vec{k} \).

Also, in Eq. (2) \( D^{1/2}(\vec{n}) = 1 - i \vec{n} \cdot \vec{\sigma} / \sqrt{1 + \vec{n}^2} \) is D-function of Wigner rotation, which is determined by the vector-parameter \( \vec{n}(p_1, p_2) = \vec{u}_1 \times \vec{u}_2 / (1 - \vec{u}_1 \cdot \vec{u}_2) \) with \( \vec{u} = \vec{p} / (\omega_m(\vec{p}) + m) \). The three-momenta \( \vec{p}_1, \vec{p}_2 \) of the particles (quarks in our case) with the masses \( m_1 \) and \( m_2 \) of relativistic system are transformed to the total, \( \bar{P}_{12} \), and relative momenta \( \vec{k} \) to facilitate the separation of the center-of-mass motion:

\[ \bar{P}_{12} = \vec{p}_1 + \vec{p}_2, \]

\[ \vec{k} = \vec{p}_1 + \bar{P}_{12} \frac{\vec{p}_1 \cdot \vec{p}_1}{M_0 \bar{P}_{12} + M_0} + \omega_{m_1}(\vec{p}_1), \]

where

\[ M_0 = \omega_{m_1}(\vec{k}) + \omega_{m_2}(\vec{k}) \]

mass of non-interacting system and \( \omega_{m_1}(\vec{p}_1) = \sqrt{\vec{p}^2 + m^2} \).

The state vector \( | \bar{P}_{12}, \mu, [J, k], (l s); [m_1 s_1; m_2 s_2] \rangle \) is the eigenstate of operators \( \bar{P}_{12} \), \( J^2 \), \( J_3 \) and, also, \( L^2 \) and \( S^2 \), where \( L \) and \( S \) are relative orbital momentum and total spin momentum, respectively. This vector of the non-interacting \( Q \bar{q} \) system transforms irreducibly under Poincare transformations.

3. Following Bakamjian and Tomas, interactions are added to the mass operator in the irreducible free-particle representation. The new mass operator is defined by

\[ \bar{M} \equiv M_0 + \hat{V}. \]

If \( \hat{V} \) is any operator that satisfies the following conditions

\[ \bar{M} = \bar{M}^\dagger, \quad M > 0, \]

\[ [\bar{P}_{12}, \hat{V}]_\pm = [i \sqrt{\bar{p}_{12}}, \hat{V}]_\pm = [\bar{J}, \hat{V}]_\pm = 0 \]

then the similar set of interacting particles will satisfy the same commutation relations as the set of non-interacting system.

There are three forms of the dynamics in the relativistic quantum mechanics called "instant", "point", and "light-front" forms [4]. The description in the instant form implies that the operators of three-momentum and angular momentum do not depend on interactions, i.e. \( \hat{P} = \bar{P} \) and \( \hat{J} = \bar{J} \) (\( \bar{J} = (\bar{M}^{23}, \bar{M}^{31}, \bar{M}^{12}) \)) and interactions can
be presented in terms of operator $\hat{P}^0$ and generators of the Lorentz boosts $\hat{N} = (\hat{M}^{01}, \hat{M}^{02}, \hat{M}^{03})$. The description in the point form implies that the operators $\hat{M}^{\mu\nu}$ are the same as for non-interacting particles, i.e. $\hat{M}^{\mu\nu} = \hat{M}^{\mu\nu}$, and these interaction terms can be presented only in the form of the four-momentum operators $\hat{P}$. In the front form with the fixed $z$ axis we introduce the + and - components of the four-vectors as $p^+ = (p^0 + p^z)/\sqrt{2}$, $p^- = (p^0 - p^z)/\sqrt{2}$. We require that in the front form the operators $\hat{P}^+, \hat{P}^j, \hat{M}^{12}, \hat{M}^{++}, \hat{M}^{++} (j = 1, 2)$ are the same as the corresponding free operators, and interaction terms can be presented via the operators $\hat{M}^{-j}$ and $\hat{P}^-$.

In our work we use the point form of RQM because:

(1) The meson wave function (see below) being Lorentz invariant, as (in a point form of dynamics) the operator of a boost does not contain interaction. In the instant form of RQM this property of wave functions does not take a place.

(2) The relativistic impulse approximation in an instant form and dynamics on light front automatically breaks down the Poincare-invariance of models. In the point form of RQM such a violation does not happen.

Given such an interaction $\hat{V}$ it is useful to define two other related interactions $\hat{V}'$:

$$\hat{U} = \hat{M}^2 - M_0^2 = \hat{V}^2 + M_0 \hat{V} + \hat{V} M_0$$

$$\hat{W} = \frac{1}{4} \left[ (\hat{M}^2 - M_0^2) + (m_1^2 - m_2^2) + \left( \frac{1}{M^2} - \frac{1}{M_0^2} \right) \right]$$

The eigenvalue problem for the mass of $Q\bar{q}$ system can be expressed in the three equivalent forms [3]:

$$\hat{M} | \Psi > = (M_0 + \hat{V}) | \Psi >= M \mid \Psi >,$$

$$M_0^2 + \hat{U} | \Psi >= M^2 | \Psi >,$$

$$(k^2 + \hat{W}) | \Psi >= \eta | \Psi >,$$

where the mass $M$ of bound $Q\bar{q}$ system and $\eta$ have relationship,

$$M^2 = 2\eta + m_1^2 + m_2^2 +$$

$$+ 2\sqrt{\eta(\eta + m_1^2 + m_2^2)} + m_1^2 m_2^2.$$  \hspace{1cm} (11)

When $m_1 = m_2 = m$, the equation (14) reduces to

$$M^2 = 4 \left( \eta + m^2 \right).$$

(12)

Solution of any of the above eigenvalue problems in the point form of RQM leads to the eigenfunctions of the form

$$\langle \tilde{V}_{12}, J, \mu, k, (ls) | \tilde{V}, J', \mu', M \rangle =$$

$$\delta_{JJ'}\delta_{\mu\mu'}\delta(\tilde{V} - \tilde{V}_{12})\Psi^{J\mu}(kls)$$ \hspace{1cm} (13)

with the velocities of the bound system $\tilde{V} = \tilde{P}/M$ and non-interacting system $\tilde{V}_{12} = \tilde{P}_{12}/M_0$. The function $\Psi^{J\mu}(kls)$ satisfies (in the point form) the following equation [7]:

$$\sum_{l' s'} \int_0^\infty <k' (l's') \mid W^J \mid k' (l's') > \Psi^{J}(k' (l's')) k'^2 dk' +$$

$$k^2 \Psi^{J}(kls) = \eta \Psi^{J}(kls)$$ \hspace{1cm} (14)

with reduced matrix element of operator $\hat{W}$

$$\langle \tilde{V}_{12}, J, \mu, k, (ls) \mid W \mid \tilde{V}_{12}, J', \mu', k', (l's') \rangle =$$

$$\delta_{JJ'}\delta_{\mu\mu'}\delta(\tilde{V}_{12} - \tilde{V}_{12}')(k, (ls)) \mid W^J \mid k', (l's') \rangle.$$ \hspace{1cm} (15)

Equation (14) is a radial equation. Therefore we have used Poincare group properties to separate the radial and angular dependencies.

As the vector of irreducible basis for a system with interaction and without its satisfies to conditions of a normalization and completeness, the wave functions satisfy to the following normalization condition,

$$\sum_{ls} \int_0^\infty dk k^2 \mid \Psi^{J}(kls) \mid^2 = 1.$$ \hspace{1cm} (16)

3 Model potential and solving main equation.

In this section we apply the formalism developed above to calculate mass spectra of mesons containing $u$, $d$ and $s$ quarks. To choose appropriate interquark potential we use a well-known experimental fact that light hadrons populate approximately
linear Regge trajectories, i.e., $M^2 \simeq \beta J + \text{const}$, with the same slope, $\beta \simeq 1.2 \text{ GeV}^2$, for all trajectories (see, for example, [16]). Therefore, we can take the effective model potential $W$ in the oscillator form

$$\hat{W}(r) = W_0 \delta(\hat{r}) + \beta^4 r^2,$$  \hspace{1cm} (17)

where $W_0$ and $\beta$ are free parameters. As follows from calculations of meson spectra done by Godfrey and Isgur [17], spin-dependent corrections are important for $1S$, $2S$, $1P_2$ and $1\bar{3}P_2$ states. However for other states spin-dependent corrections are small and they can be neglected in Eq. (17). Using standard relationships for operator $\hat{r}$ and orbital momentum operator $\hat{L}$

$$\langle k | \hat{r} | k' \rangle = -i \vec{\nabla}_k \delta(k - k'),$$

$$\hat{L}^2 |\tilde{V}_{12}, J, \mu, k, (l s) \rangle = l(l + 1) |\tilde{V}_{12}, J, \mu, k, (l s) \rangle$$

we reduce main integro-differential equation of RQM to the ordinary quantum-mechanical radial equation with the oscillator potential (only impulse representation),

$$\left[ \frac{d^2}{dk^2} + \frac{2}{k} \frac{d}{dk} - \frac{l(l+1)}{k^2} - \frac{k^2}{\beta^4} \right] \Psi(kls) =$$

$$= \frac{W_0 - \eta}{\beta^4} \Psi(kls).$$  \hspace{1cm} (18)

The eigenfunctions of Eq. (18) are

$$\Psi(kls) = N_{nl} \exp \left(-\frac{k^2}{2\beta^2}\right) *$$

$$\left(\frac{k}{\beta}\right)^l F \left(-n, l + \frac{3}{2}, \frac{k^2}{\beta^2}\right)$$  \hspace{1cm} (19)

with

$$N_{nl} = \frac{\beta^{-l-3/2}}{\Gamma(l + 3/2)} \sqrt{\frac{\Gamma(n + l + 3/2)}{\Gamma(n + 1)}},$$

where $n, l = 0, 1, 2, \ldots$, $F(a, b, z)$ is the hypergeometric function, $\Gamma(n)$ is the Gamma function. Note that the wave function of the ground state $(n, l = 0)$ has the Gaussian form, which is used in many relativistic models of hadrons. This wave functions gives an excellent description of electromagnetic formfactors at small transfers $Q^2$, $Q^2 \ll$ a few GeV.

Quantization condition is defined by

$$\eta = W_0 + 2\beta^2 \left(2n + l + 3/2\right).$$  \hspace{1cm} (20)

The spectra of mesons composed of quarks with equal masses $m_1 = m_2 \equiv m$, for example, $u$ and $d$ quarks, are given by (see Eq. (12) and Eq. (20))

$$M^2 = 4(m^2 + W_0) + 8\beta^2 \left(2n + l + \frac{3}{2}\right).$$  \hspace{1cm} (21)

Thus we reproduce the linear dependence $M^2(l)$ in the framework of the two-body relativistic equation (14). The parameters $(m^2 + W_0)$ and $\beta$ have been found from the fitting the Regge trajectories (see Fig.1),

$$W_0 = -0.31 \text{ GeV}^2,$$ (22)

If we take the same masses of $u$ and $d$ quark as in Ref. [17], i.e.,

$$m_u = m_d \equiv m = 0.22 \text{ GeV},$$ (22)

then we obtain

$$W_0 = -0.31 \text{ GeV}^2.$$
There are eight meson Regge trajectories populated by $u-d$ bound states (for each isospin $I$ and angular momenta $J = l \pm 1, J = l$ and total spin $S$ of $q\bar{q}$ system, $S = 0, 1$). Some of the trajectories are plotted in Fig. 2-4.

We observe that all experimental data are in good agreement with spectrum given by Eq. (21) for $l \geq 1$ and $S = 1$ (Fig.2). As in the case of bound states with $S = 0$, the agreement between our theoretical predictions and the existing experimental data is not good (see Fig.3-4). Such deviations can be explained by absence of spin-dependent terms and short-distance term of the potential.

As it is easy to see, our method for bound systems with different masses of quarks does not require special processing of solving main equation of RQM, as it was made in Ref. [16] (introduction of additional parameter). When the masses of the quark and antiquark are different we obtain

$$M^2 = 2W_0 + 2\beta^2 (2n + l + 3/2) + m_1^2 + m_2^2 +$$

$$+ 2\sqrt{(W_0 + 2\beta^2 (2n + l + 3/2))} *$$

$$* \sqrt{(W_0 + 2\beta^2 (2n + l + 3/2) + m_1^2 + m_2^2) + m_1^2 m_2^2}$$

(23)

We see that the dependence (23) is, also, linear, but asymptotically at large $l$. We apply (23) to the description of strange meson trajectories. For the mass of the $u(d)$—quark and for mass strange quark we used the same value as in Ref. [17], i.e., $m_1 \equiv m_u = 0.22 GeV$ (see (22)), $m_2 \equiv m_s = 0.419 GeV$.

As in the case of meson with hidden strangeness, the agreement between our model predictions and existing experimental data [18] is good (see Fig.5).

Also we can predict all the strange meson trajectories (see Fig. 6-9). The model describes all $s-\bar{q}(q-\bar{s})$ meson Regge trajectories with $S = 1$ in quite satisfactory way (see Fig. 6-8). But for mesons with $S = 0$, the agreement between model and the experimental data is not good (see Fig. 9), as in the case of $u-d$ mesons.

In summary we shall mark that, the analytical solution of a main equation a RQM (19) can be used as zero approximation for problem solving with more realistic potentials. The model adequately describes, with only four free parameters, $m_u = m_d, m_s, W_0, \beta$, orbitally excited meson Regge trajectories.

References

[1] N.Nakanichi, Progr. Theor. Phys. Suppl., N43 (1969); E.E.Salpeter,H.A.Bethe, Phys. Rev. 84 , p.232 (1951).
Figure 4: $\pi$-meson Regge trajectory

[2] A.A.Logunov, A.N.Tavkhelidze, Nuovo Cimento. 29, p.380 (1963).

[3] V.G.Kadyshevsky, Nucl.Phys. B, 6 p.125 (1968).

[4] I.T.Todorov, Phys.Rev. D, 10, p.2351 (1971).

[5] F.Gross, Phys.Rev., 186, p.1448 (1969); Phys. Rev. D, 10, p.223 (1974).

[6] B.D.Keister, W.N.Polyzou in Advances in Nuclear Physics, edited J.W.Negele and E.Vogt (Plenum, New York, 1991).

[7] W.N. Polyzou Annals of Physics 193, p.367 (1989).

[8] F.M.Lev Annals of Physics 237, p.355 (1995).

[9] S.Weinberg, Phys.Rev. 150, p.1313 (1966).

[10] L.L.Frankfurt, M.I.Strikman Phys.Rep., 76, p.215 (1981).

[11] L.A.Kondratyuk, M.V.Terent’ev Yad.Fiz., 31, p.1087 (1980).

[12] V.G.Kadyshevsky, Zh. Eksp. Teor. Fiz., 46, p.654, p.872 (1964); V.G.Kadyshevsky, R.M.Mir-Kasimov, N.B.Skachkov Yad. Fiz., 9, p.219, p.462 (1969).

[13] V.G.Kadyshevsky, R.M.Mir-Kasimov, N.B.Skachkov Part. Nucl., 2, p.635 (1972).

[14] P.A.M. Dirac Rev.Mod.Phys., 21, p.392 (1949)

[15] H. Leutwyler and J. Stern, Annals of Phys. 112, p.94 (1978).

[16] E. Di Salvo, L.Kondratyuk, P. Saracco Z.Phys. C69.P.149.(1995)

[17] S.Godfrey and N.Isgur Phys.Rev. D 32 p.189 (1985).

[18] Review of Particle Properties, Phys. Rev.D 54, N1 (1996).

Figure 5: $s - \bar{s}$-meson Regge trajectory

[13] V.G.Kadyshevsky, R.M.Mir-Kasimov, N.B.Skachkov Part. Nucl., 2, p.635 (1972).

[14] P.A.M. Dirac Rev.Mod.Phys., 21, p.392 (1949)

[15] H. Leutwyler and J. Stern, Annals of Phys. 112, p.94 (1978).

[16] E. Di Salvo, L.Kondratyuk ,P. Saracco Z.Phys. C69.P.149.(1995)

[17] S.Godfrey and N.Isgur Phys.Rev. D 32 p.189 (1985).

[18] Review of Particle Properties, Phys. Rev.D 54, N1 (1996).
Figure 6: $K$-meson Regge trajectory 1

Figure 7: $K$-meson Regge trajectory 2

Figure 8: $K$-meson Regge trajectory 3

Figure 9: $K$-meson Regge trajectory 4
$I=1/2$, $L=J+1$, $S=1$

$M^2, \text{GeV}^2$

$L$

$K^*(1430)$

$K^*(1680)$

$K^*(1980)$