ZERO–CYCLES ON SELF–PRODUCTS OF SURFACES: SOME NEW EXAMPLES VERIFYING VOISIN’S CONJECTURE

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ABSTRACT. An old conjecture of Voisin describes how 0–cycles of a surface $S$ should behave when pulled–back to the self–product $S^m$ for $m > p_g(S)$. We exhibit some surfaces with large $p_g$ that verify Voisin’s conjecture.

1. INTRODUCTION

Let $X$ be a smooth projective variety over $\mathbb{C}$, and let $A^i(X)_{\mathbb{Z}} := CH^i(X)$ denote the Chow groups of $X$ (i.e. the groups of codimension $i$ algebraic cycles on $X$ with $\mathbb{Z}$–coefficients, modulo rational equivalence [22]). Let $A^i_{\text{hom}}(X)_{\mathbb{Z}}$ (and $A^i_{\text{AJ}}(X)_{\mathbb{Z}}$) denote the subgroup of homologically trivial (resp. Abel–Jacobi trivial) cycles.

The Bloch–Beilinson–Murre conjectures present a beautiful and coherent dream–world in which Chow groups are determined by cohomology and the coniveau filtration [29], [30], [47], [32], [48], [64]. The following particular instance of this dream–world was first formulated by Voisin:

Conjecture 1.1 (Voisin 1993 [63]). Let $S$ be a smooth projective surface. Let $m$ be an integer larger than the geometric genus $p_g(S)$. Then for any 0–cycles $a_1, \ldots, a_m \in A^2_{\text{AJ}}(S)_{\mathbb{Z}}$, one has

$$\sum_{\sigma \in S_m} \text{sgn}(\sigma) a_{\sigma(1)} \times \cdots \times a_{\sigma(m)} = 0 \quad \text{in} \quad A^{2m}(S^m)_{\mathbb{Z}}.$$  

(Here $S_m$ is the symmetric group on $m$ elements, and sgn($\sigma$) is the sign of the permutation $\sigma$.)

For surfaces of geometric genus 0, Conjecture 1.1 reduces to Bloch’s conjecture [9]. For surfaces $S$ of geometric genus 1, Conjecture 1.1 takes on a particularly simple form: in this case, the conjecture stipulates that any $a_1, a_2 \in A^2_{\text{AJ}}(S)_{\mathbb{Z}}$ should verify the equality

$$a_1 \times a_2 = a_2 \times a_1 \quad \text{in} \quad A^4(S \times S)_{\mathbb{Z}}.$$  

This conjecture is still open for a general $K3$ surface; examples of surfaces of geometric genus 1 verifying this conjecture are given in [63], [36], [38], [40]. One can also formulate versions of Conjecture 1.1 for higher–dimensional varieties; this is studied in [63], [37], [42], [43], [8], [45], [61].

On a historical note, it is interesting to observe that Voisin’s Conjecture 1.1 antedates Kimura’s conjecture “all varieties have finite–dimensional motive” [32]. Both conjectures have a similar
flavour. Chow groups of a surface $S$ should have controlled behaviour when pulled–back to the self–product $S^m$, for large $m$. The difference between Voisin’s conjecture and Kimura’s conjecture lies in the index $m$ which is much lower in Voisin’s conjecture. In fact (as explained in [8]), Voisin’s conjecture follows from a combination of Kimura’s conjecture with a strong form of the generalized Hodge conjecture.

The goal of the present note is to collect some (easy) examples of surfaces with geometric genus larger than 1 verifying Voisin’s conjecture.

**Theorem** (=Corollaries 2.6, 2.7, 3.2, 4.2 and 5.3). The following surfaces verify Conjecture 1.1:

(i) generalized Burniat type surfaces in the family $S_{16}$ of [2] ($p_g(S) = 3$);

(ii) the hypersurfaces $S \subset A/\iota$ considered in [44], where $A$ is an abelian threefold and $\iota$ is the $−1$-involution ($p_g(S) = 3$);

(iii) minimal surfaces $S$ of general type with $p_g(S) = q(S) = 3$ and $K_S^2 = 6$;

(iv) the double cover of certain cubic surfaces (among which the Fermat cubic) branched along the Hessian ($p_g(S) = 4$);

(v) the Fano surface of lines in a smooth cubic threefold ($p_g(S) = 10$);

(vi) the quotient $S = F/\iota$, where $F$ is the Fano surface of conics in a Verra threefold and $\iota$ is a certain involution ($p_g(S) = 36$);

(vii) the surface of bitangents $S$ of a general quartic in $\mathbb{P}^3$ ($p_g(S) = 45$);

(viii) the singular locus $S$ of a general EPW sextic ($p_g(S) = 45$).

A by–product of the proof is that these surfaces all have finite–dimensional motive, in the sense of Kimura [32] (this appears to be a new observation for cases (vi)–(viii)). Also, certain instances of the generalized Hodge conjecture are verified:

**Corollary** (=Corollary 2.7). Let $S$ be any of the above surfaces, and let $m > p_g(S)$. Then the sub–Hodge structure

$$\Lambda^m H^2(S, \mathbb{Q}) \subset H^{2m}(S^m, \mathbb{Q})$$

is supported on a divisor.

The surfaces considered in this note have an interesting feature in common (which makes it easy to prove Conjecture 1.1 for them): for many of them, intersection product induces a surjection

$$A^1_{\text{hom}}(S) \otimes A^1_{\text{hom}}(S) \to A^2_{\text{AJ}}(S).$$

In the other cases (cases (ii), (iv), (vi)–(viii), which have $q(S) = 0$), the surface $S$ is dominated by a surface $T$ with the property that the intersection product map

$$A^1_{\text{hom}}(T) \otimes A^1_{\text{hom}}(T) \to A^2_{\text{AJ}}(T)$$

surjects onto $\text{Im}(A^2_{\text{AJ}}(S) \to A^2_{\text{AJ}}(T))$.

Using this feature, to prove Conjecture 1.1 for these surfaces one is reduced to a problem concerning 0–cycles on abelian varieties. This last problem has recently been solved by Vial [61], using a strong version of the generalized Hodge conjecture for generic abelian varieties.

**Conventions.** In this note, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.
Unless indicated otherwise, all Chow groups will be with rational coefficients: we will denote by \( A_j(X) \) the Chow group of \( j \)-dimensional cycles on \( X \) with \( \mathbb{Q} \)-coefficients (and by \( A_{\mathbb{Z}}(X) \) the Chow groups with \( \mathbb{Z} \)-coefficients); for \( X \) smooth of dimension \( n \) the notations \( A_j(X) \) and \( A^{n-j}(X) \) are used interchangeably.

The notations \( A^j_{\text{hom}}(X) \), \( A^j_{A,\text{J}}(X) \) will be used to indicate the subgroups of homologically trivial, resp. Abel–Jacobi trivial cycles. The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [57], [48]) will be denoted \( M_{\text{rat}} \).

We will write \( H^j(X, \mathbb{Q}) \) to indicate singular cohomology.

2. Generalized Burniat Type Surfaces with \( p_g = 3 \)

**Definition 2.1** ([2]). Let \( A = E_1 \times E_2 \times E_3 \) be a product of elliptic curves. A generalized Burniat type surface (or “GBT surface”) is a quotient \( S = Y/G \), where \( Y \subset A \) is a smooth hypersurface corresponding to the square of a principal polarization, and \( G \cong \mathbb{Z}_2^2 \) acts freely.

**Remark 2.2.** GBT surfaces are minimal surfaces of general type with \( p_g(S) = q(S) \) ranging from 0 to 3. There are 16 irreducible families of GBT surfaces, labelled \( S_1, \ldots, S_{16} \) in [2]. The families \( S_1, S_2 \) have moduli–dimension 4, the other families are 3–dimensional.

**Theorem 2.3** (Peters [53]). Let \( S \) be a GBT surface with \( p_g(S) = 3 \) (i.e., \( S \) is in the family labelled \( S_{16} \) in [2]), and let \( A \) be the abelian threefold as in definition 2.1.

(i) \( S \) has finite–dimensional motive, and there are natural isomorphisms

\[
A^2_{\text{hom}}(A) \xrightarrow{\cong} A^2_{A,\text{J}}(S) \xrightarrow{\cong} A^3_{(2)}(A).
\]

(Here \( A^*_{\text{hom}}(A) \) refers to Beauville’s decomposition [5].)

(ii) Intersection product induces a surjection

\[
A^1_{\text{hom}}(S) \otimes A^1_{\text{hom}}(S) \rightarrow A^2_{A,\text{J}}(S).
\]

**Proof.** Part (i) is [53, Theorem 4.2].

Part (ii) follows from (i), in view of the fact that intersection product induces a surjection

\[
A^1_{\text{hom}}(A) \otimes A^1_{\text{hom}}(A) \twoheadrightarrow A^2_{(2)}(A)
\]

[5, Proposition 4].

Property (ii) of Theorem 2.3 is relevant to Conjecture 1.1.

**Proposition 2.4.** Let \( S \) be a smooth projective surface, and assume that intersection product induces a surjection

\[
A^1_{\text{hom}}(S) \otimes A^1_{\text{hom}}(S) \rightarrow A^2_{A,\text{J}}(S).
\]

Then \( S \) has finite–dimensional motive.

Also, Conjecture 1.1 is true for \( S \) with \( m > \binom{q(S)}{2} \). (In particular, in case of equality \( p_g(S) = \binom{q(S)}{2} \) the full Conjecture 1.1 is true for \( S \).)
Proof. Let $\alpha: S \to A := \text{Alb}(S)$ be the Albanese map. There is a commutative diagram
\[
A^1_{\text{hom}}(S) \otimes A^1_{\text{hom}}(S) \to A^2_{AJ}(S)
\]
\[
\uparrow (\alpha^*, \alpha^*) \uparrow \alpha^*
\]
\[
A^1_{\text{hom}}(A) \otimes A^1_{\text{hom}}(A) \to A^2_{(2)}(A)
\]
(where horizontal maps are induced by intersection product, and $A^*_\text{	ext{rev}}(A)$ refers to the Beauville decomposition \cite{Beauville} of the Chow ring of any abelian variety). As the left vertical map is an isomorphism, the assumption implies that the right vertical map is surjective. In view of \cite[Theorem 3.11]{Lange}, this implies $S$ has finite–dimensional motive. (For an alternative proof of \cite[Theorem 3.11]{Lange} in terms of birational motives, cf. \cite[Theorem B.7]{Kollar}. For a similar result, cf. \cite[Proposition 2.1]{Kollar}.)

Next, let us consider Conjecture \ref{conj1} for $S$. Thanks to Rojtman’s result \cite{Rojtman}, it suffices to establish Conjecture \ref{conj1} for $0$–cycles with $\mathbb{Q}$–coefficients. Because $\alpha^*: A^2_{(2)}(A) \to A^2_{AJ}(S)$ is surjective, to prove Conjecture \ref{conj1} for $S$ it suffices to prove (a version of) Conjecture \ref{conj1} for elements $b_1, \ldots, b_m \in A^2_{(2)}(A)$. We now reduce to $0$–cycles on $A$: given $b_j \in A^2_{(2)}(A)$, let
\[
c_j := b_j \cdot h^{q-2} \in A^2_{(2)}(A), \quad j = 1, \ldots, m,
\]
be $0$–cycles, where $q := q(S)$ is the dimension of $A$ and $h \in A^1(A)$ is a symmetric ample divisor.

Let us consider the $\mathfrak{S}_m$–invariant ample divisor
\[
H := \sum_{j=1}^m (pr_j)^*(h) \in A^1(A^m).
\]

From K"unnemann’s hard Lefschetz result \cite{Kunnemann}, we know that the map
\[
\cdot H^{m(q-2)}: A^2_{(2m)}(A^m) \to A^m_{(2m)}(A^m)
\]
is an isomorphism. On the other hand,
\[
c_{\sigma(1)} \times \cdots \times c_{\sigma(m)} = (b_{\sigma(1)} \times \cdots \times b_{\sigma(m)}) \cdot (h^{q-2} \times \cdots \times h^{q-2})
\]
\[
= (b_{\sigma(1)} \times \cdots \times b_{\sigma(m)}) \cdot H^{m(q-2)} \in A^m_{(2m)}(A^m)
\]
(since intersecting $A^2(A)$ with a power $h^r$, $r > q - 2$ gives 0).

We are thus reduced to proving that for any $c_1, \ldots, c_m \in A^2_{(2)}(A)$, where $m > \binom{q}{2}$, there is equality
\[
\sum_{\sigma \in \mathfrak{S}_m} \text{sgn}(\sigma) c_{\sigma(1)} \times \cdots \times c_{\sigma(m)} = 0 \quad \text{in} \quad A^m_{(k)}(A^m).
\]

At this point, we can invoke the following general result on $0$–cycles on abelian varieties to conclude:

**Theorem 2.5** (Vial \cite{Vial}). Let $A$ be an abelian variety of dimension $g$, and let $c_1, \ldots, c_m \in A^g_{(k)}(A)$. 
If \( k \) is even and \( m > \binom{g_k}{k} \), there is vanishing
\[
\sum_{\sigma \in S_m} \text{sgn}(\sigma) c_{\sigma(1)} \times \cdots \times c_{\sigma(m)} = 0 \quad \text{in } A^{mg}(A^m) .
\]

If \( k \) is odd and \( m > \binom{g_k}{k} \), there is vanishing
\[
\sum_{\sigma \in S_m} c_{\sigma(1)} \times \cdots \times c_{\sigma(m)} = 0 \quad \text{in } A^{mg}(A^m) .
\]

**Proof.** This is [61, Theorem 4.1], whose proof uses the concept of “generically defined cycles on abelian varieties”, and a strong form of the generalized Hodge conjecture for powers of generic abelian varieties, due to Hazama [61, Theorem 2.12]. The case \( k = g \) was proven earlier (and differently) in [64, Example 4.40]. \( \square \)

This ends the proof of Proposition 2.4. \( \square \)

We can now prove that surfaces in the family \( S_{16} \) verify Voisin’s conjecture:

**Corollary 2.6.** Let \( S \) be a GBT surface with \( p_g(S) = 3 \) (i.e., \( S \) is in the family labelled \( S_{16} \) in [2]). Then \( S \) verifies Conjecture [1.1] for any \( m > 3 \) and \( a_1, \ldots, a_m \in A^2_{AJ}(S) \), there is equality
\[
\sum_{\sigma \in S_m} \text{sgn}(\sigma) a_{\sigma(1)} \times \cdots \times a_{\sigma(m)} = 0 \quad \text{in } A^{2m}(S^m) .
\]

**Proof.** This follows from Proposition 2.4, in view of Theorem 2.3 plus the fact that \( q(S) = p_g(S) = 3 \). \( \square \)

We recall that the truth of Conjecture [1.1] implies a certain instance of the generalized Hodge conjecture:

**Corollary 2.7.** Let \( S \) be a surface verifying Conjecture [1.1] and let \( m > p_g(S) \). Then the sub–Hodge structure
\[
\wedge^m H^2(S, \mathbb{Q}) \subset H^{2m}(S^m, \mathbb{Q})
\]
is supported on a divisor.

**Proof.** This is already observed in [63]. Consider the Chow motive \( \wedge^m h^2(S) \) defined by the idempotent
\[
\Gamma := \left( \sum_{\sigma \in S_m} \text{sgn}(\sigma) \Gamma_\sigma \right) \circ \left( \pi^2_S \times \cdots \times \pi^2_S \right) \in A^{2m}(S^m \times S^m) .
\]

Conjecture [1.1] is equivalent to saying that \( A_0(\wedge^m h^2(S)) = 0 \).

Applying the Bloch–Srinivas argument [10] to \( \Gamma \), one obtains a rational equivalence
\[
\Gamma = \gamma \quad \text{in } A^{2m}(S^m \times S^m) ,
\]
where \( \gamma \) is a cycle supported on \( S^m \times D \) for some divisor \( D \subset S^m \). On the other hand, \( \Gamma \) acts on \( H^{2m}(S^m, \mathbb{Q}) \) as projector on \( \wedge^m H^2(S, \mathbb{Q}) \). It follows that \( \wedge^m H^2(S, \mathbb{Q}) \) is supported on \( D \). \( \square \)
3. A CRITERION

The approach of the last section can be conveniently rephrased as follows:

**Proposition 3.1.** Let \( S \) be a smooth projective surface. Assume that \( S \) has finite–dimensional motive, and that cup product induces an isomorphism
\[
C : \Lambda^2 H^1(S, \mathcal{O}_S) \xrightarrow{\cong} H^2(S, \mathcal{O}_S).
\]
Then Conjecture [7,1] is true for \( S \).

**Proof.** Surjectivity of \( C \), combined with finite–dimensionality of the motive of \( S \), ensures that intersection product induces a surjection
\[
A^1_{\text{hom}}(S) \otimes A^1_{\text{hom}}(S) \twoheadrightarrow A^2_{\text{AJ}}(S).
\]
[39]. The assumption that \( C \) is an isomorphism implies that \( p_g(S) = \binom{q(S)}{2} \). The result now follows from Proposition [2,4]. □

This takes care of two more cases announced in the introduction:

**Corollary 3.2.** Conjecture [7,1] is true for the following surfaces:
(i) minimal surfaces of general type with \( p_g(S) = q(S) = 3 \) and \( K^2 = 6 \);
(ii) the Fano surface of lines in a cubic threefold (\( p_g(S) = 10 \)).

**Proof.** In case (i), it is known that \( S \) is the symmetric square \( S = C^{(2)} \) where \( C \) is a genus 3 curve [11] (cf. also [3, Theorem 9]). Thus, the assumptions of Proposition [3,1] are clearly satisfied.

As for case (ii), it is well–known this satisfies the assumptions of Proposition [3,1] (finite–dimensionality is proven in [20] and [41]). Alternatively, one could apply Proposition [2,4] directly (the assumption of Proposition [2,4] is satisfied by the Fano surface thanks to [9]; an alternative proof is sketched in [58, Remark 20.8]). □

4. A VARIANT CRITERION

Let us now state a variant version of Proposition [2,4]

**Proposition 4.1.** Let \( S \) be a smooth projective surface. Assume that \( S = S'/\langle \iota \rangle \), where \( \iota \) is an automorphism of a surface \( S' \) such that intersection product induces a surjection
\[
A^1_{\text{hom}}(S') \otimes A^1_{\text{hom}}(S') \twoheadrightarrow A^2_{\text{AJ}}(S').
\]
Then \( S \) has finite–dimensional motive.

Also, Conjecture [7,1] is true for \( S \) with \( m > \binom{q(S')}{2} \). (In particular, if \( p_g(S) = \binom{q(S')}{2} \) the full Conjecture [7,1] is true for \( S \).)

**Proof.** This is proven just as Proposition [2,4] □

This takes care of several more cases announced in the introduction:

**Corollary 4.2.** Conjecture [7,1] is true for the following surfaces:
(i) surfaces \( S = T/\langle \iota \rangle \), where \( T \) is a smooth divisor in the linear system \( |2\Theta| \) on a principally polarized abelian threefold, and \( \iota \) is the \((-1)\)–involution (\( p_g(S) = 3 \));
(ii) the quotient $S = F/\iota$, where $F$ is the Fano surface of conics in a general Verra threefold and $\iota$ is a certain involution ($p_9(S) = 36$);
(iii) the surface of bitangents $S$ of a general quartic in $\mathbb{P}^3$ ($p_9(S) = 45$);
(iv) the surface $S$ that is the singular locus of a general EPW sextic ($p_9(S) = 45$).

Proof.
(i) The surface $S$ verifies the assumptions of Proposition 4.1 with $S' = T$, according to [44 Subsection 7.2].

(iii) More generally, one may consider the surface $S$ studied by Welters [65] and defined as follows. Let $Y$ be a quartic double solid, i.e., $Y \to \mathbb{P}^3$ is a double cover branched along a smooth quartic $Q$. Let $T$ be the surface of conics contained in $Y$, and let $\iota \in \text{Aut}(T)$ be the involution induced by the covering involution of $Y$. Then the surface $S := T/\langle \iota \rangle$ is a smooth surface of general type with $p_9(S) = 45$. (The generic quartic $K3$ surface $Q$ does not contain a line. In this case, as explained in [21 (cf. also [6, Example 3.5] and [23, Remark 8.5]), the surface $S$ is (isomorphic to) the so-called “surface of bitangents”, which is the fixed locus of Beauville’s anti–symplectic involution

$$Q^{[2]} \to Q^{[2]}$$

first considered in [4]. As noted in [6, Example 3.5], this gives another proof of the fact that $p_9(S) = 45$.)

Voisin has proven [62, Corollaire 3.2(b)] (cf. also [62, Remarque 3.4]) that intersection product induces a surjection

$$A^1_{\text{hom}}(T) \otimes A^1_{\text{hom}}(T) \twoheadrightarrow A^2_{AJ}(T)^\iota = A^2_{AJ}(S).$$

Since $p_9(S) = 45$ and $q(T) = 10$ [65], the assumptions of Proposition 4.1 are met with.

(ii) A Verra threefold $Y$ is a divisor of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ (these varieties were introduced in [59]). Let $F$ be the Fano surface of conics of bidegree $(1,1)$ contained in $Y$. As observed in [28, Section 5], $F$ admits an involution $\iota$ such that $(F, \iota)$ enters into the set–up of Voisin’s work [62]. Thus, [62, Corollaire 3.2(b)] implies that intersection product induces a surjection

$$A^1_{\text{hom}}(F) \otimes A^1_{\text{hom}}(F) \twoheadrightarrow A^2_{AJ}(F)^\iota = A^2_{AJ}(S).$$

Since $q(F) = 9$ and $p_9(S) = 36$ [28, Proposition 5.1], the assumptions of Proposition 4.1 are again met with.

(iv) Let $Y$ be a transverse intersection of the Grassmannian $\text{Gr}(2,5) \subset \mathbb{P}^9$ with a codimension 2 linear subspace and a quadric (i.e., $Y$ is an ordinary Gushel–Mukai threefold, in the language of [15, 14]). For generic $Y$, the surface $F$ of conics contained in $Y$ is smooth and irreducible. There exists a birational involution $\iota \in \text{Bir}(F)$, such that intersection product induces a surjection

$$A^1_{\text{hom}}(F) \otimes A^1_{\text{hom}}(F) \twoheadrightarrow A^2_{AJ}(F)^\iota.$$

[62, Corollaire 3.2(b)]. The surface $F$ and the birational involution $\iota$ are also studied in [46 and [12]. There exists a (geometrically meaningful) birational morphism $F \to F_m$, where $F_m$ is smooth and such that $\iota$ extends to a morphism $\iota_m$ on $F_m$ [46, 12, Section 6], [27, Section 5.1]. For $Y$ generic, the quotient $S := F_m/\langle \iota_m \rangle$ is smooth, and it is isomorphic to the singular
locus of the EPW sextic associated to \( Y \). (This is contained in [46], [12]. The double cover \( F_m \to S \) is also described in [17, Theorem 5.2(2)].)

Since \( A_{\text{hom}}^1(), \ A_{\text{AJ}}^2() \) are birational invariants among smooth varieties, Voisin’s result implies there is also a surjection

\[
A_{\text{hom}}^1(F_m) \otimes A_{\text{hom}}^1(F_m) \to A_{\text{AJ}}^2(F_m)^{\otimes m} = A_{\text{AJ}}^2(S).
\]

It is known that \( q(F_m) = 10 \) [46] and \( p_g(S) = 45 \) [50] (this can also be deduced from [6]), and so Proposition 4.1 applies.

\[ \Box \]

Remark 4.3. In cases (ii), (iii) and (iv) of Corollary 4.2 the surface \( S \) is the fixed locus of an anti–symplectic involution of a hyperkähler fourfold. For the surface of bitangents, this is Beauville’s involution on the Hilbert square \( Q^2 \). For the singular locus \( S \) of a general EPW sextic, this is (isomorphic to) the fixed locus of the anti–symplectic involution of the associated double EPW sextic.

For the surface \( S \) of (ii), this is the anti–symplectic involution of the “double EPW quartic” (double EPW quartics form a 19–dimensional family of hyperkähler fourfolds, introduced in [28]).

Is this merely a coincidence, or is there something fundamental going on? Do other two–dimensional fixed loci of anti–symplectic involutions of hyperkähler fourfolds also enter in the set–up of Proposition 4.1?

Remark 4.4. Inspired by the famous results concerning the Fano surface of the cubic threefold, Voisin [62] systematically studies the Fano surface \( F \) of conics contained in Fano threefolds \( Y \). Under certain conditions, she is able to prove [62, Corollaire 3.2] that there is a birational involution \( \iota \) on \( F \), with the property that

\[
A_{\text{hom}}^1(F) \otimes A_{\text{hom}}^1(F) \to A_{\text{AJ}}^2(F)^{<\iota>}
\]

is surjective (and so one could hope to apply Proposition 7.1 to find more examples of surfaces verifying Conjecture 7.7).

Examples given in [62] (other than those mentioned in Corollary 4.2 above) include:

1. Fano threefolds \( Y \) of index 1, Picard number 1 and genus \( g \in \{7, 10\} \cup \{12\} \) [62, Section 2.4];
2. a general complete intersection of two quadrics in \( \mathbb{P}^5 \) [62, Section 2.7];
3. the intersection of the Grassmannian \( Gr(2, 5) \subset \mathbb{P}^9 \) with a general codimension 3 linear subspace [62, Section 2.7].

(In all these cases, \( \iota \) is actually the identity.)

In case (1), the surface of conics \( F \) is not very interesting. (for \( g = 12 \), \( F \cong \mathbb{P}^2 \) [35, Proposition B.4.1]; for \( g = 10 \), \( F \) is an abelian surface [35, Proposition B.5.5]; for \( g = 9 \), \( F \) is a \( \mathbb{P}^1 \)–bundle over a curve [35, Proposition 2.3.6]; for \( g = 8 \), \( F \) is isomorphic to the Fano surface of a cubic threefold [35, Proposition B.6.1]; for \( g = 7 \), \( F \) is the symmetric product of a curve of genus 7 [34]. These results are also discussed in [26, Section 3.1].)

The other two cases also turn out to reduce to known cases: Indeed, for case (2) the Fano surface of lines is isomorphic to the Jacobian of a genus 2 curve [19, Theorem 2]. For case (3), the Fano threefold \( Y \) is birational to a cubic threefold \( Y’ \), and the Fano surface of conics on \( Y \)
is birational to the Fano surface of lines on \( Y' \) [54, Theorem B and Section 6]. Since Conjecture \( [77] \) is obviously a birationally invariant statement, Conjecture \( [77] \) for the Fano surface of case (3) thus reduces to Corollary \( [12, \text{Section 5.2} \text{)} ii).

**Remark 4.5.** There are interesting relations between the surfaces of Corollary \( [72] \) and other Fano surfaces:

In case (ii), the general Verra threefold \( Y \) is birational to a one–nodal ordinary Gushel–Mukai threefold \( \bar{X} \), and there is an induced birational map between the Fano surface of lines \( F(Y) \) and the Fano surface of conics \( F(\bar{X}) \) [13, Section 5.4 and Proposition 6.6].

In case (iii), the general quartic double solid \( Y \) is known to be birational to a one–nodal ordinary degree 10 Fano threefold \( \bar{X} \), and there is an induced birational map between the Fano surface of lines \( F(Y) \) and the Fano surface of conics \( F(\bar{X}) \) [12, Proposition 5.2].

5. **Double covers of cubic surfaces**

**Theorem 5.1** (Ikeda [25]). Let \( Y \subset \mathbb{P}^3 \) be a smooth cubic surface, and let \( \bar{S} \to Y \) be the double cover of \( Y \) branched along its Hessian. Let \( S \to \bar{S} \) be a minimal resolution of singularities. The surface \( S \) is a minimal surface of general type with \( p_g(S) = 4 \) and \( K^2 = 6 \).

**Remark 5.2.** The intersection of \( Y \) with its Hessian is smooth (and so \( S = \bar{S} \)) precisely when \( Y \) has no Eckardt points. In this case, the Picard rank of \( S \) is 28 [25, Theorem 6.1]. At the other extreme, if \( Y \) is the Fermat cubic (which is the only cubic surface attaining the maximal number of Eckardt points) the Picard rank of \( S \) is 44 [25, Theorem 6.6], and so in this case \( S \) is a \( \rho \)-maximal surface (in the sense of [7]). For more on Eckardt points of cubic surfaces, cf. [24, Chapter 2 Section 3.6].

Let us now prove Voisin’s conjecture for some of Ikeda’s double covers:

**Corollary 5.3.** Let \( Y \subset \mathbb{P}^3 \) be a smooth cubic surface, and let \( S \) be a double cover as in theorem 5.1. Assume that \( Y \) is in the pencil

\[ x_0^3 + x_1^3 + x_2^3 - 3\lambda x_0x_1x_2 + x_3^3 = 0. \]

Then \( S \) verifies Conjecture \( [77] \) for any \( m > 4 \) and \( a_1, \ldots, a_m \in A^2_{\text{hom}}(S)_{\mathbb{Z}} \), there is equality

\[ \sum_{\sigma \in S_m} \text{sgn}(\sigma)a_{\sigma(1)} \times \cdots \times a_{\sigma(m)} = 0 \text{ in } A^{2m}(S^m)_{\mathbb{Z}}. \]

**Proof.** A first part of the argument works for arbitrary smooth cubic surfaces \( Y \); only in the last step will we use that \( Y \) is of a specific type. Let us assume \( Y \subset \mathbb{P}^3 \) is any smooth cubic, defined by a cubic polynomial \( f(x_0, \ldots, x_3) \). Let \( Z \subset \mathbb{P}^4 \) be the smooth cubic threefold defined by

\[ f(x_0, \ldots, x_3) + x_4^3 = 0, \]

so \( Z \) has the structure of a triple cover

\[ \rho: Z \to \mathbb{P}^3 \]

branched along \( Y \). Let \( F(Z) \) denote the Fano surface of lines contained in \( Z \). Ikeda [25] shows that there is a dominant rational map of degree 3

\[ f: F(Z) \dashrightarrow S, \]
and an isomorphism
\[ f^*: H^2_{tr}(S, \mathbb{Q}) \xrightarrow{\cong} H^2_{tr}(F(Z), \mathbb{Q})^{\text{Gal}(\rho)}. \]
This implies that there is an isomorphism of homological motives
\[ t^!\Gamma_f: t(S) \cong t(F(Z))^{\text{Gal}(\rho)} := (F(Z), \frac{1}{3} \sum_{g \in \text{Gal}(\rho)} \Gamma_g \circ \pi_{tr}^2, 0) \text{ in } \mathcal{M}\text{hom}. \]
(Here for any surface \( T \), the motive \( t(T) := (T, \pi_{tr}^2, 0) \in \mathcal{M}_{\text{rat}} \) denotes the transcendental part of the motive as in [31].)

According to [20] and [41], the Fano surface \( F(Z) \) has finite-dimensional motive (in the sense of Kimura [32], [1], [30]). The surface \( S \), being rationally dominated by \( F(Z) \), also has finite-dimensional motive. Thus, one may upgrade (1) to an isomorphism of Chow motives
\[ t^!\Gamma_f: t(S) \cong t(F(Z))^{\text{Gal}(\rho)} \text{ in } \mathcal{M}_{\text{rat}}. \]
In particular, this implies that there is an isomorphism of Chow groups
\[ f^*: A^2_{\text{hom}}(S) = A^2_{\text{AJ}}(S) \cong A^2_{\text{AJ}}(F(Z))^{\text{Gal}(\rho)}. \]

Let \( A \) be the 5-dimensional Albanese variety of \( F(Z) \) (which is isomorphic to the intermediate Jacobian of \( Z \)). As observed in [20], the inclusion \( F(Z) \hookrightarrow A \) induces an isomorphism
\[ A^2_{(2)}(A) \cong A^2_{\text{AJ}}(F(Z)). \]
In particular, there is a restriction–induced isomorphism
\[ A^2_{(2)}(A)^{\text{Gal}(\rho)} \cong A^2_{\text{AJ}}(F(Z))^{\text{Gal}(\rho)}, \]
where we simply use the same letter \( \rho \) for the action induced by the triple cover \( \rho: Z \to \mathbb{P}^3 \).

Consequently, it suffices to prove a version of Conjecture [14] for cycles in \( A^2_{(2)}(A)^{\text{Gal}(\rho)} \). Also, using Künnemann’s hard Lefschetz theorem (for some \( \text{Gal}(\rho) \)–invariant ample divisor), one reduces to a statement for cycles in \( A^5_{(2)}(A)^{\text{Gal}(\rho)} \) (i.e., 0–cycles). This last statement can be proven, subject to some restrictions on the cubic surface \( Y \), thanks to the following result:

**Proposition 5.4** (Vial [61]). Let \( B \) be an abelian variety of dimension \( g \), and assume \( B \) is isogenous to \( E_1^g \times E_2^g \times E_3^g \), where the \( E_j \) are elliptic curves. Let \( \Gamma \in A^g(B \times B) \) be an idempotent which lies in the sub–algebra generated by symmetric divisors. Assume that \( \Gamma^*H^j(B) = 0 \) for all \( j \). Then also
\[ \Gamma^*A^g(B) = 0. \]

**Proof.** This is a special case of [61], Theorem 3.15, whose hypotheses are more general. \( \square \)

It remains to verify that Proposition 5.4 applies to our set–up. If the cubic threefold \( Z = Z_\lambda \) is in the pencil
\[ x_0^3 + x_1^3 + x_2^3 - 3\lambda x_0 x_1 x_2 + x_3^3 + x_4^3 = 0, \]
its intermediate Jacobian \( A \) is isogenous to \( E_0^g \times E_\lambda^g \), where \( E_\lambda \) is the elliptic curve
\[ x_0^3 + x_1^3 + x_2^3 - 3\lambda x_0 x_1 x_2 = 0. \]
We can apply Proposition 5.4 with $B := A^m$ and 
$$
\Gamma := \left( \sum_{g \in \text{Gal}(\rho)} \Gamma_g \times \cdots \times \Gamma_g \right) \circ \left( \sum_{\sigma \in \mathfrak{S}_m} \text{sgn}(\sigma) \Gamma_\sigma \right) \circ \left( \pi^8_A \times \cdots \times \pi^8_A \right) \in A^{5m}(A^m \times A^m).
$$

Here $\pi^8_A$ is part of the Chow–Künneth decomposition of $[18]$, with the property that 
$$
A^5_{(2)}(A) = (\pi^8_A)_* A^5(A).
$$

Since $g \in \text{Gal}(\rho)$ and $\sigma \in \mathfrak{S}_m$ are homomorphisms of abelian varieties, and the $\pi^8_A$ are symmetrically distinguished (in the sense of O’Sullivan [52]) and generically defined (in the sense of Vial [61]), the correspondence $\Gamma$ is in the sub–algebra generated by symmetric divisors [61, Proposition 3.11]. In particular, the correspondence $\Gamma$ is symmetrically distinguished, and so (since it is idempotent in cohomology) idempotent.

The correspondence $\Gamma^*$ acts on cohomology as projector on 
$$
\bigwedge^m \left( H^2(A)^{\text{Gal}(\rho)} \right).
$$

Since 
$$
\dim \text{Gr}^p_{H^2(A)^{\text{Gal}(\rho)}} = p_g(S) = 4,
$$
we have that $\Gamma^* = (\Gamma)_*$ is zero on $H^{1,0}(B)$ as soon as $m > 4$. Applying Proposition 5.4, we can prove Conjecture 1.1 for $A^5_{(2)}(A)^{\text{Gal}(\rho)}$ (and hence, as explained above, also for $A^5_{2,1}(S)$): let 
$$
b_1, \ldots, b_m \in A^5_{(2)}(A)^{\text{Gal}(\rho)}, \text{ where } m > 4.
$$

Then 
$$
\sum_{\sigma \in \mathfrak{S}_m} \text{sgn}(\sigma) b_{\sigma(1)} \times b_{\sigma(2)} \times \cdots \times b_{\sigma(m)} = \Gamma_*(b_1 \times b_2 \times \cdots \times b_m) = 0 \text{ in } A^{5m}(A^m).
$$

□

Remark 5.5. The argument of Corollary 5.3 also applies to double covers of some other cubic surfaces. For instance, let $Y$ be a cubic surface, let $S$ be the double cover as in theorem 5.7, and let $J(Z)$ be the intermediate Jacobian of the associated cubic threefold. If $J(Z)$ is $\rho$–maximal, then $S$ verifies conjecture 1.1. Indeed, $\rho$–maximality implies that $J(Z)$ is isogenous to $E^5$ for some elliptic curve $E$ [7, Proposition 3], and so Proposition 5.4 applies.

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