Semisimple corings

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Abstract

This paper states the basic essentials for a theory of semisimple corings.

Introduction

M. Sweedler \[13\] introduced the notion of coring as a generalization of the concept of coalgebra in order to study the set of intermediate division rings for an extension of division rings. It turns out that this formalism embodies several kinds of relative module categories. Thus, graded modules, Doi-Hopf modules and, more generally, entwined modules are instances of comodules over suitable corings (see \[3\] and its references). From this point of view, an interesting question is to characterize those corings encoding the simplest type of category of comodules. Since, under certain hypotheses, the category of comodules is abelian, the simple objects play a relevant role in its structure. In the most favorable case all comodules are direct sums of simple comodules (that is, the category of comodules is semisimple). In the classical theory of modules over rings, the study of semisimple rings precedes the development of the entire theory. This paper states the basic essentials for a theory of semisimple corings.

Throughout this paper the word ring will refer to an associative algebra over a commutative ring \(K\), and the term subring is then understood as subalgebra. The category of all left modules over a ring \(R\) will be denoted by \(_{R}M\), being \(M_{R}\) the notation for the category of all right \(R\)-modules. An agile introduction to abelian categories is contained in \[12\]. The notation \(X \in A\) for a category \(A\) means that \(X\) is an object of \(A\), and the identity morphism attached to any object \(X\) will be represented by the object itself.

1 Corings and comodules

We discuss under which conditions the category of (right) comodules over a coring is abelian. We first recall from \[13\] the notion of a coring.

Let \(A\) denote a ring. An \(A\)-coring is a three-tuple \((\mathcal{C}, \Delta, \epsilon)\) consisting of an \(A\)-bimodule \(\mathcal{C}\) and two \(A\)-bimodule maps

\[
\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes_{A} \mathcal{C}, \quad \epsilon : \mathcal{C} \rightarrow A
\]
such that the diagrams

\[
\begin{array}{c}
\varepsilon \\
\downarrow \\
\varepsilon \otimes_A \varepsilon \\
\downarrow \\
\varepsilon \otimes_A \varepsilon
\end{array}
\xrightarrow{\Delta} 
\begin{array}{c}
\Delta \\
\downarrow \\
\Delta \otimes_A \varepsilon \\
\downarrow \\
\varepsilon \otimes_A \Delta
\end{array}
\xrightarrow{\varepsilon \otimes \varepsilon}
\begin{array}{c}
\varepsilon \\
\downarrow \\
\varepsilon \otimes_A \varepsilon \\
\downarrow \\
\varepsilon \otimes_A \varepsilon
\end{array}
\]

and

\[
\begin{array}{c}
\varepsilon \\
\downarrow \\
\varepsilon \otimes_A \varepsilon \\
\downarrow \\
\varepsilon \otimes_A \varepsilon
\end{array}
\xrightarrow{\Delta} 
\begin{array}{c}
\Delta \\
\downarrow \\
\Delta \otimes_A \varepsilon \\
\downarrow \\
\varepsilon \otimes_A \Delta
\end{array}
\xrightarrow{\varepsilon \otimes \varepsilon}
\begin{array}{c}
\varepsilon \\
\downarrow \\
\varepsilon \otimes_A \varepsilon \\
\downarrow \\
\varepsilon \otimes_A \varepsilon
\end{array}
\]

commute.

A left \( \mathcal{C} \)-comodule is a pair \((M, \lambda_M)\) consisting of a left \( A \)-module \( M \) and an \( A \)-linear map \( \lambda_M : M \to \mathcal{C} \otimes_A M \) such that the diagrams

\[
\begin{array}{c}
M \\
\downarrow \lambda_M \\
\mathcal{C} \otimes_A M
\end{array}
\xrightarrow{\lambda_M} 
\begin{array}{c}
\mathcal{C} \otimes_A M \\
\downarrow \Delta \otimes_A M \\
\mathcal{C} \otimes_A \mathcal{C} \otimes_A M
\end{array}
\]

commute. Right \( \mathcal{C} \)-comodules are similarly defined; we use the notation \( \rho_M \) for their structure maps. A morphism of left \( \mathcal{C} \)-comodules \((M, \lambda_M)\) and \((N, \lambda_N)\) is an \( A \)-linear map \( f : M \to N \) such that the following diagram is commutative

\[
\begin{array}{c}
M \\
\downarrow \lambda_M \\
\mathcal{C} \otimes_A M
\end{array}
\xrightarrow{f} 
\begin{array}{c}
N \\
\downarrow \lambda_N \\
\mathcal{C} \otimes_A N
\end{array}
\]

The \( K \)-module of all left \( \mathcal{C} \)-comodule morphisms from \( M \) to \( N \) is denoted by \( \text{Hom}_{\mathcal{C}}(M, N) \). The category of all left \( \mathcal{C} \)-comodules will be denoted by \( \mathcal{CM} \). Analogously, we can consider the category of all right \( \mathcal{C} \)-comodules \( \mathcal{CM}^\mathcal{C} \). Every valid statement about left comodules entails a correct assertion for right comodules, which will be implicitly understood.

Coproducts and cokernels in \( \mathcal{CM} \) do exist, and they can be already computed in \( _A \mathcal{M} \). Therefore, \( \mathcal{CM} \) has arbitrary inductive limits. If \( \mathcal{C}_A \) is a flat module, then \( \mathcal{CM} \) is easily proved to be an abelian category. The converse is not true, as the following example shows.

**Example 1.1.** Let \( A = \begin{pmatrix} R & B \\ 0 & S \end{pmatrix} \) be a generalized triangular matrix ring with \( B \) an \((R, S)\)-bimodule over the rings \( R \) and \( S \). It is well-known that the right \( A \)-modules are given by three-tuples \( M = (M', M'', \mu) \) consisting of a right \( R \)-module \( M' \), a right \( S \)-module \( M'' \) and an \( S \)-module map \( \mu : M' \otimes_R B \to M'' \). A homomorphism of right
$A$–modules is then given by a pair $f = (f', f'') : (M', M'', \mu) \to (N', N'', \nu)$ consisting of an $R$–module map $f' : M' \to N'$ and a right $S$–module map $f'' : M'' \to N''$ such that $f'' \mu = \nu(f' \otimes R B)$. Consider the ideal $I = \begin{pmatrix} R & B \\ 0 & 0 \end{pmatrix}$ of $A$ which, as a right $A$–module, corresponds to $(R, B, \mu)$, where $\mu : R \otimes R B \to B$ is the canonical isomorphism. Now, $M \otimes_A I = (M', M' \otimes_R B, id)$ and the multiplication map $M \otimes_A I \to M$ is given by $(id, \mu) : (M', M' \otimes_R B, id) \to (M', M'', \mu)$. The $A$–bimodule $I$ is an $A$–coring with comultiplication given by the isomorphism $I \cong I \otimes_A I$, and counit given by the inclusion $I \subseteq A$. It can be easily shown that a right $A$–linear coaction $\rho_M = (\rho', \rho'') : M \to M \otimes_A I$ is an $I$–module structure if and only if $\rho' = id_M$, and $\mu$ is an isomorphism with inverse $\rho''$. Therefore, the category $\mathcal{M}^I$ of all right $I$–comodules can be identified with the category of all right $A$–modules $(M', M'', \mu)$ such that $\mu$ is an isomorphism. Now the functor $F : \mathcal{M}^I \to \mathcal{M}_R$ given by $F(M', M'', \mu) = M'$ is easily shown to be an equivalence of categories. In particular, $\mathcal{M}^I$ is a Grothendieck category and $A I$ is not flat unless $R B$ is.

The following result clarifies the situation created by our example. Recall from [8, Proposition 3.1] that the functor $\mathcal{C} \otimes_A : \mathcal{A} \mathcal{M} \to \mathcal{E} \mathcal{M}$ is right adjoint to the forgetful functor $U : \mathcal{E} \mathcal{M} \to \mathcal{A} \mathcal{M}$.

**Proposition 1.2.** Let $\mathcal{C}$ be an $A$–coring. Then the following are equivalent

(i) $\mathcal{C}_A$ is flat.

(ii) $\mathcal{E} \mathcal{M}$ is an abelian category and the functor $U$ is left exact.

(iii) $\mathcal{E} \mathcal{M}$ is a Grothendieck category and the functor $U$ is left exact.

**Proof.** (i) $\Rightarrow$ (iii) The exactness of the functor $\mathcal{C} \otimes_A : \mathcal{A} \mathcal{M} \to \mathcal{A} \mathcal{M}$ entails that kernels in $\mathcal{E} \mathcal{M}$ can be already computed in $\mathcal{A} \mathcal{M}$. This gives that $\mathcal{E} \mathcal{M}$ is a complete and co-complete abelian category with exact direct limits. We need to find a generator for $\mathcal{E} \mathcal{M}$. For this, we proceed as in the proof of [14, 13.13]. Let $M \in \mathcal{E} \mathcal{M}$, with coaction $\lambda_M$, and $A(l) \to M \to 0$ the free presentation of $M$ in $\mathcal{A} \mathcal{M}$, we have

$$
\begin{array}{c}
\mathcal{C}(l) \cong \mathcal{C} \otimes_A A(l) \xrightarrow{g} \mathcal{C} \otimes_A M \xrightarrow{\lambda_M} 0 \\
\downarrow \quad \quad \downarrow \\
g^{-1}(M) \xrightarrow{g^{-1}} M \xrightarrow{\lambda_M} 0
\end{array}
$$

now it is clear that

$$
g^l = \oplus \{L | L \text{ is left subcomodule of } \mathcal{C}^k, k \in \mathbb{N}\}
$$

is a generator of $\mathcal{E} \mathcal{M}$. Obviously, the forgetful functor $U : \mathcal{E} \mathcal{M} \to \mathcal{A} \mathcal{M}$ is exact in this case.

(iii) $\Rightarrow$ (ii) This is obvious.

(ii) $\Rightarrow$ (i) By [11] Corollary 3.2.3, $\mathcal{C} \otimes_A : \mathcal{A} \mathcal{M} \to \mathcal{E} \mathcal{M}$ is left exact and, thus, $U \circ (\mathcal{C} \otimes_A) : \mathcal{A} \mathcal{M} \to \mathcal{A} \mathcal{M}$ is a left exact too. Therefore, $\mathcal{C}_A$ is a flat module. \qed
As a consequence of the proof of Proposition 1.2 we obtain:

**Corollary 1.3.** If $\mathfrak{C}_A$ is flat then every left $\mathfrak{C}$–comodule is isomorphic to a subcomodule of a $\mathfrak{C}$–generated left comodule.

**Example 1.4.** ([13, Example 1.2]). Let $B \to A$ a ring extension. Then $\mathfrak{C} = A \otimes_B A$ is an $A$–coring with the coproduct $\mathfrak{C} \to \mathfrak{C} \otimes_A \mathfrak{C}$ given by $a \otimes_B a' \mapsto a \otimes a_1 \otimes_B a_2$ and the counit $A \otimes_B A \to A$ is the multiplication map.

**Example 1.5.** ([3, Proposition 2.2]) Let $(\mathcal{C}, \Delta, \epsilon)$ be a $K$–coalgebra and assume that the canonical left $A$–module $\mathcal{C} = A \otimes C$ has a right $A$–module structure which makes $\mathcal{C}$ an $A$–bimodule. Define the $K$–linear map $\psi : \mathcal{C} \otimes A \to \mathcal{C} \otimes C$ given by $\psi(c \otimes a) = (1 \otimes c)a$. Consider the left $A$–module maps $\Delta : \mathcal{C} \to \mathcal{C} \otimes A \mathcal{C} \simeq A \otimes \mathcal{C} \otimes C$, $\epsilon : A \otimes \mathcal{C} \simeq A \otimes \epsilon$. Then $(\mathcal{C}, \Delta, \epsilon)$ is an $A$–coring if and only if $(A, \mathcal{C})_\psi$ is an entwining structure (see [4]). Moreover, the category of comodules $\mathcal{M}_\mathcal{C}$ is isomorphic to the category of entwined modules $\mathcal{M}_\mathcal{C}^{\psi}$. Examples of categories of entwined modules are Doi-Koppinen modules, introduced in [5] and [9] (cf. [2, Example 3.1(3)]).

## 2 Rational modules and comodules

We state a formal framework (the notion of rational pairing) which reduces the study of some categories of comodules to the investigation of certain subcategories of categories of modules. The development is adapted from the given in [7] and [1] for coalgebras over commutative rings. A somewhat different approach is [15].

Let $P, Q$ be $A$–bimodules. Any balanced bilinear form $\langle -, - \rangle : P \times Q \to A$

provides natural transformations $\alpha : Q \otimes_A - \to \text{Hom}_A(\text{A}P, -)$ and $\beta : - \otimes_A P \to \text{Hom}_A(Q\text{A}, -)$ given by

$$\alpha_M : Q \otimes_A M \to \text{Hom}_A(\text{A}P, \text{A}M)$$

$$q \otimes_A m \mapsto [p \mapsto \langle p, q \rangle m]$$

$$\beta_N : N \otimes_A P \to \text{Hom}_A(Q\text{A}, N\text{A})$$

$$n \otimes_A p \mapsto [q \mapsto \langle p, q \rangle].$$

Moreover, if $M$ is an $A$–bimodule then $\alpha_M$ and $\beta_M$ are bimodule homomorphisms. The canonical isomorphisms provide two bimodule maps

$$\alpha_A : Q \to \text{Hom}_A(\text{A}P, \text{A}A) = ^*P$$

$$q \mapsto [p \mapsto \langle p, q \rangle]$$

$$\beta_A : P \to \text{Hom}_A(Q\text{A}, A\text{A}) = Q^*$$

$$p \mapsto [q \mapsto \langle p, q \rangle]$$

which are bimodule homomorphisms. So we can recover the balanced bilinear form if one of the natural transformations is given.
**Definition 2.1.** The data \( T = (P, Q, \langle -, - \rangle) \) are called a *left rational system* if \( \alpha_M \) is injective for each left \( A \)-module \( M \), and a *right rational system* if \( \beta_N \) is injective for every right \( A \)-module \( N \).

**Remark 2.2.** ([4], Remark 2.4). Let \((P, Q, \langle -, - \rangle)\) be a left rational system. Let \( M \in A \mathcal{M} \) and \( N \) be submodule of \( M \) with the canonical injection \( i_N \). Consider the following commutative diagram

\[
\begin{array}{ccc}
Q \otimes_A N & \xrightarrow{\alpha_N} & \text{Hom}_A(P, N) \\
Q \otimes_A i_N & \downarrow & \downarrow i \\
Q \otimes_A M & \xrightarrow{\alpha_M} & \text{Hom}_A(P, M).
\end{array}
\]

Hence \( Q \otimes_A i_N \) is injective. Since \( M \) was an arbitrary left \( A \)-module, we conclude that \( Q \) should be a flat \( A \)-module. Analogously, if \((P, Q, \langle -, - \rangle)\) is a right rational system, then we get that \( A P \) is flat.

Let \( \langle -, - \rangle : P \times Q \to A \) and \( [-, -] : P' \times Q' \to A \) be two balanced bilinear forms with natural transformations \( \alpha, \beta \) and \( \alpha', \beta' \) respectively. We can define a new balanced bilinear form

\[
\{ -, - \} : (P \otimes_A P') \times (Q' \otimes_A Q) \to A
\]

\[
(p \otimes_A p', q' \otimes_A q) \mapsto \{ p \otimes_A p', q' \otimes_A q \} = \langle p, [p', q] \rangle = \langle p[p', q'], q \rangle.
\]

The natural transformations associated to \( \{ -, - \} \) are given by the compositions

\[
\begin{array}{ccc}
Q' \otimes_A Q \otimes_A M & \xrightarrow{\alpha'_{Q \otimes_A M}} & \text{Hom}_A(P \otimes_A P', M) \\
\downarrow & \downarrow i_M & \downarrow \text{Hom}_A(A P', \text{Hom}_A(A P, M)), \\
\text{Hom}_A(A P', Q \otimes_A M) & \xrightarrow{(\alpha_M)^*} & \text{Hom}_A(A P', \text{Hom}_A(A P, M)),
\end{array}
\]

\[
\begin{array}{ccc}
N \otimes_A P \otimes_A P' \xrightarrow{\beta'_{N \otimes_A P}} & \text{Hom}_A(Q' \otimes_A Q, N) \\
\downarrow i_N & \downarrow \text{Hom}_A(Q'_A, \text{Hom}_A(Q_A, M)), \\
\text{Hom}_A(Q'_A, N \otimes_A P) & \xrightarrow{(\beta_N)^*} & \text{Hom}_A(Q'_A, \text{Hom}_A(Q_A, M)).
\end{array}
\]

The following proposition, which is now clear, replaces [4], Proposition 2.2] in order to show that the canonical comodule structure over a rational module is pseudocoassociative.

**Proposition 2.3.** Let \((P, Q, \langle -, - \rangle)\) and \((P', Q', [-, -])\) be two left (resp. right) rational systems. Then \((P \otimes_A P', Q' \otimes_A Q, \{ -, - \})\) is also a left (resp. right) rational system.

Let \((\mathcal{C}, \Delta, \epsilon)\) be an \( A \)-coring. Recall that \( \mathcal{C}^* = \text{Hom}_A(\mathcal{C}_A, A) \) (resp. \( ^* \mathcal{C} = \text{Hom}_A(A \mathcal{C}, A) \)) is a ring extension of \( A^{op} \) with multiplication \( gf = f \circ (g \otimes_A \mathcal{C}) \circ \Delta \) (resp. \( gf = g \circ (\mathcal{C} \otimes_A f) \circ \Delta \)). Both units are \( \epsilon \). See [13, Proposition 3.2] for details.
Definition 2.4. A left rational pairing is a left rational system \((B, \mathcal{C}, \langle -, - \rangle)\) such that \(B\) is a ring extension of \(A\), \(\mathcal{C}\) is an \(A\)-coring, and \(\beta : B \to \mathcal{C}^*\) is a ring antimorphism. If \(\Delta (c) = \sum_i c_i \otimes_A d_i\) then

\[
\langle ab, c \rangle = \langle a, \sum_i \langle b, c_i \rangle d_i \rangle, \quad \forall a, b \in B \quad \text{and} \quad \epsilon = \langle 1, - \rangle.
\]

Analogously, a right rational pairing is a right rational system \((\mathcal{C}, B', [-, -])\) such that \(B'\) is a ring extension of \(A\), \(\mathcal{C}\) is an \(A\)-coring and \(\alpha : B' \to \mathcal{C}^*\) is a ring antimorphism. If \(\Delta (c) = \sum_i c_i \otimes_A d_i\) then

\[
[c, ab] = \sum_i c_i [d_i, a], b, \quad \forall a, b \in B \quad \text{and} \quad \epsilon = [1, -].
\]

Example 2.5. Let \(\mathcal{C}\) be an \(A\)-coring such that \(\mathcal{C}\) is projective as a right \(A\)-module. By using any dual basis associated with the projectivity of \(\mathcal{C}_A\), we prove that the canonical balanced bilinear form \(\langle -, - \rangle : \mathcal{C}^* \times \mathcal{C} \to A\) gives a left rational pairing \(T = (\mathcal{C}^* \times \mathcal{C}, \langle -, -, \rangle)\). Analogously, if \(\mathcal{C}\) is an \(A\)-coring such that \(A\mathcal{C}\) is a projective module, then \(T' = (\mathcal{C}, \mathcal{C}^* \times [-, -])\) is a right rational pairing.

Let \(T = (B, \mathcal{C}, \langle -, - \rangle)\) be a left rational pairing. An element \(m\) in a left \(B\)-module \(M\) is called rational if there exists a set of left rational parameters \(\{(c_i, m_i)\} \subseteq \mathcal{C} \times M\) such that \(bm = \sum_i \langle b, c_i \rangle m_i\) for all \(b \in B\). The set of rational elements in \(M\) is denoted by \(\text{Rat}^T(M)\). The proofs detailed in [7, Section 2] can be adapted in a straightforward way in order to get that \(\text{Rat}^T(M)\) is a \(B\)-submodule of \(M\) and that the assignment \(M \mapsto \text{Rat}^T(M)\) defines a functor

\[
\text{Rat}^T : B\mathcal{M} \to B\mathcal{M},
\]

which is in fact a left exact preradical. Therefore, the full subcategory \(\text{Rat}^T(B\mathcal{M})\) of \(B\mathcal{M}\) whose objects are those \(B\)-modules \(M\) such that \(\text{Rat}^T(M) = M\) is a closed reflective subcategory [8, p. 395] and, in particular, it is a Grothendieck category. The modules in the subcategory \(\text{Rat}^T(B\mathcal{M})\) will be called rational left \(B\)-modules (with respect to \(T\)). Now it turns out that every rational left \(B\)-module is a left \(\mathcal{C}\)-comodule with structure map \(\lambda_M : M \to \mathcal{C} \otimes A M\) given by \(\lambda_M(m) = \sum c_i \otimes_A m_i\), where \(\{(c_i, m_i)\}\) is any set of rational parameters for \(m \in M\) ([7, Proposition 3.5] for a proof which can be adapted to the present setting). This leads to a functor

\[
\varepsilon(-) : \text{Rat}^T(B\mathcal{M}) \longrightarrow \mathcal{E}\mathcal{M}
\]

which can be shown to be an isomorphism of categories with the guide of [8, Section 3]. It can be also deduced that \(B\mathcal{C}\) becomes a subgenerator for \(\text{Rat}^T(\mathcal{C})\). Therefore, we can state

Theorem 2.6. Let \(T = (B, \mathcal{C}, \langle -, - \rangle)\) be a left rational pairing. The functor \(\varepsilon(-) : \text{Rat}^T(B\mathcal{M}) \to \mathcal{E}\mathcal{M}\) is an isomorphism of categories. Moreover, every left \(\mathcal{C}\)-comodule is isomorphic to a \(B\)-submodule of a \(B\mathcal{C}\)-generated \(B\)-module.
This theorem, when applied to the rational pairing \( T = (\mathcal{E}^{\ast \circ \circ}, \mathcal{E}, \langle -, - \rangle) \) given in Example 2.3 leads to

**Corollary 2.7.** Let \( \mathcal{E} \) an \( A \)-coring. If \( \mathcal{C}_A \) is projective, then the functor \((-)^{\mathcal{C}} : \text{Rat}^T(\mathcal{M}_{\mathcal{C}}) \to \mathcal{C} \mathcal{M} \) is an isomorphism of categories. Moreover, every left \( \mathcal{C} \)-comodule is isomorphic to a \( \mathcal{C}^\ast \)-submodule of a \( \mathcal{C}_{\mathcal{C}} \)-generated \( \mathcal{C}^\ast \)-module.

Theorem 2.6 has a right analogue. If \( T' = (\mathcal{C}, B', [-,-]) \) is a right rational pairing, then we can define functors \((-)^{\mathcal{C}} : \text{Rat}^{T'}(\mathcal{M}_{\mathcal{C}}) \to \mathcal{M}^{\mathcal{C}} \) and \((-)^{\mathcal{C}}_B : \mathcal{M}^{\mathcal{C}} \to \text{Rat}^{T'}(\mathcal{M}_{B'}) \). These functors lead to the following theorem

**Theorem 2.6'.** Let \( T' = (\mathcal{C}, B', [-,-]) \) be a right rational pairing. The functor \((-)^{\mathcal{C}} : \text{Rat}^{T'}(\mathcal{M}_{\mathcal{C}}) \to \mathcal{M}^{\mathcal{C}} \) is an isomorphism of categories. Moreover, every right \( \mathcal{C} \)-comodule is isomorphic to a \( B' \)-submodule of a \( \mathcal{C}_{B'} \)-generated \( B' \)-module.

Finally, we state a useful consequence of the former development.

**Proposition 2.8.** Let \( T = (B, \mathcal{C}, \langle -, - \rangle) \) be a left (resp. \( T' = (\mathcal{C}, B', [-,-]) \) right) rational pairing. Let \( M \in \mathcal{C} \mathcal{M} \). Then \( M \) is a finitely generated left (resp. right) \( \mathcal{C} \)-comodule if and only if \( M \) is finitely generated left (resp. right) \( A \)-module.

Recall that a \( \mathcal{C} \)-bicomodule is an \( A \)-bimodule \( M \) endowed with a right \( A \)-linear left \( \mathcal{C} \)-comodule structure \( \lambda_M : M \to \mathcal{C} \otimes_A M \) and a left \( A \)-linear right \( \mathcal{C} \)-comodule structure \( \rho_M : M \to M \otimes_A \mathcal{C} \) such that

\[
(\lambda_M \otimes_A \mathcal{C}) \rho_M = (\mathcal{C} \otimes_A \rho_M) \lambda_M.
\]

The \( \mathcal{C} \)-bicomodules are the objects of a category \( \mathcal{C} \mathcal{M} \) whose morphisms are those \( A \)-bimodule homomorphisms which are left and right \( \mathcal{C} \)-colinear.

Let \( T = (B, \mathcal{C}, \langle -, - \rangle) \) (resp. \( T' = (\mathcal{C}, B', [-,-]) \)) be a left (resp. right) rational pairing.

**Lemma 2.9.** Let \( M \) be an \( A \)-bimodule with a left \( \mathcal{C} \)-comodule structure \( \lambda_M : M \to \mathcal{C} \otimes_A M \) and a right \( \mathcal{C} \)-structure map \( \rho_M : M \to M \otimes_A \mathcal{C} \). Then \( M \) is a \( \mathcal{C} \)-bicomodule if and only if \( M \) is a \( (B, B') \)-bimodule.

**Proof.** By Theorems 2.6 and 2.6', \( M \) is a rational left \( B \)-module and a rational right \( B' \)-module. We first prove that \( \lambda_M \) is right \( A \)-linear if and only if \( M \) is a \( (B, A) \)-bimodule as follows: for each \( m \in M \), write \( \lambda_M(m) = \sum c_i \otimes_A m_i \) for a set of left rational parameters \( \{c_i, m_i\} \). Thus, \( \lambda_M \) is right \( A \)-linear if and only if \( \{c_i, m_i, a\} \) is a set of rational parameters for \( ma \) for our generic \( m \) and every \( a \in A \). But this last condition is easily proved to be equivalent to require that \( M \) is a \( (B, A) \)-bimodule. Of course, \( \rho_M \) is left \( A \)-linear if and only if \( M \) is a \( (A, B') \)-bimodule. Thus we see that, in order to prove the Lemma, we can assume that \( M \) is a \( (B, A) \)-bimodule and an \( (A, B') \)-bimodule. Under this condition, \( M \) is a \( \mathcal{C} \)-bicomodule if and only if

\[
\sum c_i \otimes_A m_{ij} \otimes_A d_{ij} = \sum c_{ji} \otimes_A m_{ji} \otimes_A d_{j},
\]
where \((c_i, m_i)\) is a set of left rational parameters for \(m\), \((m_{ij}, d_{ij})\) is a set of right rational parameters for each \(m_{ij}\), \((m_j, d_j)\) is a set of right rational parameters for \(m\), and \((c_{ji}, m_{ji})\) is a set of left rational parameters for each \(m_j\). An easy computation gives
\[
(b, m) b' = \sum (b, c_i) m_{ij} [d_{ij}, b'], \quad \text{and} \quad b (m, b') = \sum (b, c_{ji}) m_{ji} [d_j, b'],
\]
for any \((b, b') \in B \times B'\).

Hence \(M\) is \((B, B')\)-bimodule if and only if
\[
\sum (b, c_i) m_{ij} [d_{ij}, b'] = \sum (b, c_{ji}) m_{ji} [d_j, b'], \quad \text{for any } (b, b') \in B \times B'.
\]
By Remark 2.2, the following map is injective
\[
\mathcal{C} \otimes_A M \otimes_A \mathcal{C} \xrightarrow{\varepsilon \otimes_A \beta_M \varepsilon} \mathcal{C} \otimes_A \text{Hom}_A(B'_A, M) \xrightarrow{\alpha_{\text{Hom}_A(B'_A, M)} \otimes_A} \text{Hom}_A(AB, \text{Hom}_A(B'_A, M)).
\]
Hence \(M\) is a \(\mathcal{C}\)-bicomodule if and only it is a \((B, B')\)-bimodule.

**Proposition 2.10.** Let \(T = (B, \mathcal{C}, \langle -, - \rangle)\) be and \(T' = (\mathcal{C}, B', \langle -, - \rangle)\) be rational pairings, and let \(\text{Rat}^{T,T'}(B_B \mathcal{M}_{B'})\) the full subcategory of the category \(B_B \mathcal{M}_{B'}\) whose objects are the \((B, B')\)-bimodules which are rational as \(B\)-modules and as \(B'\)-modules. Then there is an isomorphism of categories \(\text{Rat}^{T,T'}(B_B \mathcal{M}_{B'}) \cong \mathcal{C}_\mathcal{M}\).

**Proof.** If \(M\) is a \(\mathcal{C}\)-bicomodule, then, by Lemma 2.9, \(M\) is a \((B, B')\)-bimodule and, by Theorems 2.6 and 2.6', \(M\) is rational as a left \(B\)-module and as a right \(B\)-module. Conversely, every \((B, B')\)-bimodule \(M\) such that the modules \(B_B M\) and \(M_{B'}\) are rational is, by Lemma 2.3 and theorems 2.6 and 2.6', a \(\mathcal{C}\)-bicomodule.

**Corollary 2.11.** Let \(\mathcal{I}\) be an \(A\)-sub-bimodule of \(\mathcal{C}\).

1. \(\mathcal{I}\) is a sub-bicomodule of \(\mathcal{C}\) if and only if \(\mathcal{I}\) is a \((B, B')\)-sub-bimodule of \(\mathcal{C}\).

2. If \(\mathcal{I}\) is pure both as a left and a right \(A\)-submodule of \(\mathcal{C}\), then \(\mathcal{I}\) is a subcoring of \(\mathcal{C}\) if and only if it is a \((B, B')\)-sub-bimodule.

For a left \(\mathcal{C}\)-comodule \(M\) define \(C(M)\) as the sum of the images of all comodule homomorphisms from \(M\) to \(\mathcal{C}\). In presence of left and right rational pairings \(T = (B, \mathcal{C}, \langle -, - \rangle)\) and \(T' = (\mathcal{C}, B', \langle -, - \rangle)\), it is easy to prove that \(C(M)\) is a sub-bicomodule of \(\mathcal{C}\): by definition, it is a left \(B\)-submodule of the \((B, B')\)-bimodule \(\mathcal{C}\). Now, if \(b' \in B'\) and \(c = f(m)\) for some homomorphism of left \(\mathcal{C}\)-comodules, then \(cb' = (r_{B'} \circ f)(m)\), where \(r_{B'} : \mathcal{C} \to \mathcal{C}\) is the homomorphism of left comodules given by right multiplication by \(b'\). Therefore \(C(M)\) is a \((B, B')\)-sub-bimodule of \(\mathcal{C}\) and, by Corollary 2.11, it is a sub-bicomodule of \(\mathcal{C}\), which will be called bicomodule of coefficients of \(M\).

**Proposition 2.12.** Let \(\lambda_M : M \to \mathcal{C} \otimes_A M\) be a left comodule and assume there are rational pairings \(T = (B, \mathcal{C}, \langle -, - \rangle)\) and \(T' = (\mathcal{C}, B', \langle -, - \rangle)\) on the left and on the right, respectively.
1. If \( \tau : \mathcal{J} \to \mathcal{E} \) is a monomorphism of \( \mathcal{E} \)-bicomodule, such that \( \lambda_M(M) \subseteq (\tau \otimes_A M)(\mathcal{J} \otimes_A M) \), then \( C(M) \subseteq \mathcal{J} \).

2. If \( N \) is a subcomodule of \( M \), then \( C(N) \subseteq C(M) \) and \( C(M/N) \subseteq C(M) \).

3. If \( N \cong M \) is an isomorphism of comodules, then \( C(N) = C(M) \).

Proof. Let \( c = f(m) \in C(M) \), where \( f : M \to \mathcal{E} \) is a homomorphism of left comodules, and write \( \lambda_M(m) = \sum c_i \otimes m_i \), for some \( c_i \in \mathcal{J} \) and \( m_i \in M \). Since \( f \) is a comodule map, we have

\[
\Delta(c) = \Delta(f(m)) = (\mathcal{E} \otimes_A f)(\lambda_M)(m) = \sum c_i \otimes f(m_i),
\]

whence, by the counital property, \( c = \sum c_i \epsilon( f(m_i) ) \in \mathcal{J} \). This proves 1. Statements 2 and 3 are easy consequences of the definition of the bicomodule of coefficients.

\[ \square \]

3 Semisimple corings

We study the simplest kind of corings from the categorical point of view, namely, those corings having a semisimple category of comodules. We prove generalizations of known theorems for coalgebras and rings. In particular, we get a (unique) decomposition of any semisimple coring in terms of simple components. The structure of this simple components, which in the cases of rings and coalgebras over fields is described in terms of matrices, seems to be much more tangled in the present general setting. See, however, the last section, for a structure theorem for the case of simple semisimple corings having a grouplike element.

**Theorem 3.1.** Let \( \mathcal{E} \) be an \( A \)-coring. The following statements are equivalent:

(i) \( \mathcal{E} \) is semisimple as a left \( \mathcal{E} \)-comodule and \( \mathcal{E}_A \) is flat;

(ii) every left \( \mathcal{E} \)-comodule is semisimple and \( \mathcal{E}_A \) is flat;

(iii) \( \mathcal{E} \) is semisimple as a right \( \mathcal{E} \)-comodule and \( _A\mathcal{E} \) is flat;

(iv) every right \( \mathcal{E} \)-comodule is semisimple and \( _A\mathcal{E} \) is flat;

(v) every (left or right) \( \mathcal{E} \)-comodule is semisimple, and \( _A\mathcal{E} \) and \( \mathcal{E}_A \) are projectives;

(vi) \( \mathcal{E} \) is semisimple as a left \( \mathcal{E}^\ast \)-module, and as right \( \mathcal{E}^\ast \)-module, and \( _A\mathcal{E} \) and \( \mathcal{E}_A \) are projectives.

Proof. Since in (i) and (ii) \( \mathcal{E}_A \) is assumed to be flat, we know that \( \mathcal{M}^\mathcal{E} \) is a Grothendieck category and, therefore, the equivalence between (i) and (ii) is a consequence of Corollary 1.3. Now, let us show that (ii) does imply (vi). By Proposition 1.2, the forgetful functor \( U : \mathcal{E}\mathcal{M} \to _A\mathcal{M} \) is exact. Moreover, it has an exact right adjoint \( \mathcal{E} \otimes_A - \), which implies that it preserves projective objects. Therefore, every left \( \mathcal{E} \)-comodule is projective as a
left $A$–module, and, in particular, $A \mathcal{C}$ is projective. By Corollary 2.7, the category $\mathcal{M}^\mathcal{C}$ of all right $\mathcal{C}$–comodules is isomorphic to the category $\text{Rat}(\mathcal{C}^\mathcal{M})$ of rational left $\mathcal{C}$–modules. Moreover, since $\mathcal{C}$ is a semisimple object in the Grothendieck category $\mathcal{M}^\mathcal{C}$, it follows that $\mathcal{C}$ is semisimple as a left module over its endomorphism ring. Now, we know that this last ring is isomorphic to $^\mathcal{C} \mathcal{C}$. Therefore, $^\mathcal{C} \mathcal{C}$ is semisimple and, thus, $\mathcal{C}^\mathcal{C}$ is semisimple. Our coring satisfies now conditions symmetric to that in (ii) which entails that $\mathcal{C}^\mathcal{A}$ is also projective and that $^\mathcal{C} \mathcal{M}$ is isomorphic to $\text{Rat}(\mathcal{M}_{^\mathcal{C}})$. Of course, we deduce that $\mathcal{C}_{^\mathcal{C}}$ is semisimple, too, and we arrive at (vi). Using Corollary 2.7 we get $(vi) \Rightarrow (i)$. On the other hand, symmetric arguments are used to show that $(iii)$, $(iv)$, and $(vi)$ are equivalent. Finally, the equivalence $(vi) \Leftrightarrow (v)$ is consequence of Corollary 2.7.

**Definition 3.2.** An $A$–coring satisfying the equivalent conditions in Theorem 3.1 will be called a semisimple coring.

The always marvelous Wedderburn-Artin’s structure theorem for semisimple artinian rings reposes upon a unique decomposition of the ring as a direct sum of simple artinian rings. This ‘abstract’ part of that classical result holds in the present setting. We first define the natural notions of simple coring and semiartinian coring.

**Definition 3.3.** A coring is said to be simple if it does not contain non-trivial sub-bicomodules. Notice that if $\mathcal{C}$ is a semisimple coring, then it is simple if and only if it does not contain non-trivial sub-corings.

**Definition 3.4.** Assume that the category of all left comodules over an $A$–coring $\mathcal{C}$ is a Grothendieck category (see Proposition 1.2). The coring $\mathcal{C}$ is said to be left semiartinian if it is semiartinian as an object in $^\mathcal{C} \mathcal{M}$, namely, every factor comodule of $\mathcal{C}^\mathcal{C}$ contains a (nonzero) simple subcomodule.

**Theorem 3.5.** Let $\mathcal{C}$ be an $A$–coring such that the modules $A \mathcal{C}$ and $\mathcal{C}^A$ are projective. The following statements are equivalent:

(i) $\mathcal{C}$ is a simple left semi-artinian coring;

(ii) $\mathcal{C}$ is a simple coring and contains a simple left $\mathcal{C}$–subcomodule;

(iii) $\mathcal{C}$ is a semisimple coring with a unique type of simple left $\mathcal{C}$–comodule;

(iv) $\mathcal{C}$ is a simple right semi-artinian coring;

(v) $\mathcal{C}$ is a simple coring and contains a simple right $\mathcal{C}$–subcomodule;

(vi) $\mathcal{C}$ is a semisimple coring with a unique type of simple right $\mathcal{C}$–comodule.
Proof. (i) ⇒ (ii) is obvious.

(ii) ⇒ (iii) Let \( S \) be a simple left submodule of \( C \). By Corollary 2.7, \( S \) is a simple right \( C^* \)-submodule of \( C \). Now, \( \ast C S \) is a nonzero \( (\ast C, C^*) \)-bi-submodule of \( C \), which, by Corollary 2.11, is a nonzero sub-bicomodule. Hence, \( \ast C S = C \) and, therefore, \( C \) is a sum of homomorphic images of the simple right \( C^* \)-module \( S \). Apply Corollary 2.7.

(iii) ⇒ (i) Obviously, every semisimple coring is left semi-artinian. Let \( \mathcal{I} \) be a non-zero sub-bicomodule of \( C \). In particular, \( \mathcal{I} \) is a left \( C \)-subcomodule of \( C \), so that it contains a simple submodule \( S \). By the statements 1 and 2 of Proposition 2.12 we get that \( C(S) \subseteq C(\mathcal{I}) = \mathcal{I} \). Since \( C \) is isomorphic to a direct sum of copies of \( S \), we apply part 3 of Proposition 2.12 to obtain \( C(S) = C \). Hence, \( \mathcal{I} = C \) and \( C \) is simple.

(ii) ⇒ (v) We have already proved that if \( C \) is simple and contains a simple left comodule, then \( C \) is semisimple. Thus, it contains a simple right \( C \)-comodule.

Finally, (iv), (v), (vi) are proved to be equivalent in an analogous way to the proof of the equivalence between (i), (ii), (iii); which allows to derive also that (v) implies (ii). This finishes the proof.

\[
\text{Remark 3.6.} \quad \text{Let} \ C \ \text{be a simple semi-artinian} \ A-\text{coring. By Theorem 3.3} \ \text{we have that} \ C \cong S(\Xi), \ \text{where} \ S \ \text{is a simple left} \ C-\text{comodule and} \ \Xi \ \text{is an index set. In contrast with} \ \text{the coalgebra or ring cases (i.e., when} \ C \ \text{is a coalgebra over a field or} \ C = A, \ \text{the set} \ \Xi \ \text{needs not to be finite. In fact, consider the} \ A-\text{coring} \ I \ \text{given in Example 3.1 with} \ R \ \text{a simple artinian ring,} \ B \ \text{the coproduct of} \ \Xi \ \text{copies of the unique simple left} \ R-\text{module} \ (\Xi \ \text{is any infinite set}) \ \text{and} \ S \ \text{is the endomorphism ring of the left} \ R-\text{module} \ B. \ \text{Then} \ I \ \text{is a simple semi-artinian ring and it is isomorphic to a direct sum of infinitely many copies of a simple left} \ I-\text{comodule (which is essentially the unique simple left} \ R-\text{module). This easy example also shows that the basis ring} \ A \ \text{needs not to be semisimple or even artinian for a semisimple} \ A-\text{coring.}
\]

We finish this section by showing that semisimple corings can be completely described in terms of simple semiartinian (or simple semisimple corings).

\textbf{Theorem 3.7.} An \( A-\text{coring} \ C \) is semisimple if and only if it decomposes as \( C = \oplus_{\alpha \in \Lambda} C_{\alpha} \), where \( C_{\alpha} \) is a simple semisimple \( A-\text{subcoring} \) for every \( \alpha \in \Lambda \). In such a case, the decomposition is unique.

\textit{Proof.} Assume that \( C \) is semisimple. Let \( \Lambda \) be a set of representatives of all simple right \( C \)-comodules. For each \( \alpha \in \Lambda \), define \( C_{\alpha} \) to be the \( \alpha \)-th isotypic component of \( C_{\Xi} \). Since \( C \) is right semisimple, it follows that \( C = \oplus_{\alpha \in \Lambda} C_{\alpha} \). We know from Corollary 2.7 that \( C_{\alpha} \) is a left \( \ast C \)-submodule of \( C \). Given \( c^* \in C^* \), its right multiplication map is a homomorphism of left \( \ast C \)-modules, and, thus, of right \( C \)-comodules. It follows that \( C_{\alpha} \) is a right \( C^* \)-submodule of \( C \) and, by Corollary 2.11, \( C_{\alpha} \) is a subcoring of \( C \). Obviously, \( C_{\alpha} \) is semisimple with a unique type of simple; by Theorem 3.3, \( C_{\alpha} \) is a simple semiartinian \( A-\text{coring}. \) Finally, the converse implication is easily deduced from the fact that, given the stated decomposition \( C = \oplus_{\alpha \in \Lambda} C_{\alpha} \), the right \( C \)-subcomodules of \( C \) are of the form \( \oplus_{\alpha \in \Lambda} M_{\alpha} \), where \( M_{\alpha} \) is a \( C_{\alpha} \)-comodule of \( C_{\Xi} \) for every \( \alpha \). The uniqueness comes from the observation that the \( C_{\alpha} \)'s are just the isotypic components of \( C_{\Xi} \).
4  Simple semiartinian corings with a grouplike element

A complete description of all semisimple corings over a given ring \( A \) would be obtained, in view of Theorem 3.7, throughout the knowledge of the structure of simple semiartinian \( A \)-corings. The structure of a general simple semiartinian coring seems to be quite intricate (see Example 3.6). It is possible, however, to recognize the simple semiartinian \( A \)-corings having a grouplike element as the canonical corings \( A \otimes_B A \), where \( B \) runs the set of simple artinian subrings of \( A \), as we will prove in this section.

Let \( \mathfrak{C} \) be an \( A \)-coring, a non-zero element \( g \in \mathfrak{C} \) such that \( \epsilon(g) = 1 \) and \( \Delta(g) = g \otimes_A 1 \) is called a *grouplike* element. An example of coring with such an element is \( A \otimes_B A \) cited in Example 1.4 taking \( g = 1 \otimes_B 1 \).

**Lemma 4.1.** [3, Lemma 5.1] Let \( \mathfrak{C} \) be an \( A \)-coring. Then \( A \) is a right \( \mathfrak{C} \)-comodule if and only if \( A \) is a left \( \mathfrak{C} \)-comodule if and only if there exists a grouplike element \( g \in \mathfrak{C} \). In that case the left and right coactions are given by

\[
\lambda_A : A \to \mathfrak{C}, \quad \rho_A : A \to \mathfrak{C},
\]

\[
a \mapsto ag \otimes_A 1, \quad a \mapsto 1 \otimes_A ga.
\]

Assume that \( \mathfrak{C} \) has a grouplike element \( g \), and consider the subring of coinvariants of \( A \) defined by

\[
A^{\text{co}\mathfrak{C}} = \{ a \in A | ag = ga \};
\]

this ring is isomorphic to \( \text{End}(A_{\mathfrak{C}}) \), and also to \( \text{End}(\mathfrak{C}A) \). Then we have a functor [3, Proposition 5.2] \( (-)^{\text{co}\mathfrak{C}} : \mathcal{M}_\mathfrak{C} \to \mathcal{M}_A^{\text{co}\mathfrak{C}} \) which assigns to every right \( \mathfrak{C} \)-comodule \( M \) the right \( A^{\text{co}\mathfrak{C}} \)-module of coinvariants

\[
M^{\text{co}\mathfrak{C}} = \{ m \in M | \rho_M(m) = m \otimes_A g \}.
\]

It is easily shown that this functor is naturally isomorphic to the functor \( \text{Hom}_\mathfrak{C}(A_{\mathfrak{C}}, -) \). The analogous discussion is pertinent for left \( \mathfrak{C} \)-comodules.

**Proposition 4.2.** Let \( B \to A \) be a ring extension, and \( A \otimes_B A \) with the canonical \( A \)-coring structure defined in Example 1.4. Assume that \( B \) is a simple artinian ring. Then \( A \otimes_B A \) is a simple semisimple \( A \)-coring. Moreover, \( A^{\text{co}\otimes_B A} = B \), with respect to the grouplike element \( 1 \otimes_B 1 \).

**Proof.** We know that \( A_B \) and \( B_A \) are projective modules, and this implies that the coring \( \mathfrak{C} = A \otimes_B A \) is projective as a left and as a right \( A \)-module. By Corollary 2.7, the category \( A^{\otimes_B A} \mathcal{M} \) is isomorphic to the category of all rational right \( \mathfrak{C}^* \)-modules. Recall from [13, Example 3.3] that there is an anti-isomorphism of rings

\[
\begin{array}{ccc}
\mathfrak{C}^* & \longrightarrow & \text{End}(A_B) \\
g & \mapsto & [a \mapsto g(a \otimes_B 1)] \\
f \otimes_B A & \mapsto & f.
\end{array}
\]
Some straightforward computations show that the canonical left $\text{End}(A_B)$-module structure of $A$ corresponds to the structure of (rational) right $\mathcal{C}^*$-module, whenever the coactions of $A$ are derived from $1 \otimes_B 1$ (see Lemma 1.1). Since $A_B$ is a homogeneous semisimple right $B$-module it follows that $\text{End}(A_B)A$ is homogeneous semisimple, too. We conclude, by Lemma 1.1 and Theorem 2.4, that $A$ is direct sum of copies of a simple left $\mathcal{C}$-comodule. Now, let $a \in A$, and consider the following homomorphism of left $A$–modules

$$
\phi_a : A \rightarrow \mathcal{C},
$$

$$
a' \rightarrow a' \otimes_B a,
$$

which is, in fact, a homomorphism of left $\mathcal{C}$–comodules. It follows that $A$ generates $A \otimes_B A$ as a left comodule. In particular, we get that $A \otimes_B A$ is a sum of copies of a simple comodule which, in the light of Theorem 3.3, shows that $A \otimes_B A$ is a simple semisimple $A$–coring. Finally,

$$
A^{coA \otimes_B A} \cong \text{End}(A \otimes_B A) = \text{End}(A) = \text{End}(\text{End}_B(A)) = \text{BiEnd}(A_B),
$$

and $A_B$ is a balanced $B$–module (remember that $B$ is simple artinian). Hence, $B = A^{coA \otimes_B A}$.

Next theorem tell us that Proposition 4.2 gives all possible examples of simple semisimple corings with a grouplike element.

**Theorem 4.3.** Let $\mathcal{C}$ be an $A$–coring, and $g \in \mathcal{C}$ be a grouplike element. Assume that $\mathcal{C}$ is a simple semisimple $A$–coring. Then $A^{co\mathcal{C}}$ is a simple artinian ring and the canonical $A$–bimodule map $A \otimes_A^{co\mathcal{C}} A \rightarrow \mathcal{C}$ which sends $1 \otimes_A 1$ to $g$ is an isomorphism of $A$–corings.

**Proof.** Endow $A$ with the structure of right $\mathcal{C}$–comodule derived from $g$. Since $\mathcal{C}$ is assumed to be simple semisimple, $A$ is a direct sum of copies of the only simple right $\mathcal{C}$–comodule. Moreover, this direct sum, being of right $A$–submodules after all, is finite. Therefore, $\text{End}(A_{\mathcal{C}}) \cong A^{co\mathcal{C}}$ is a simple artinian ring. It is easily proved that the $A$–bimodule homomorphism $\varphi : A \otimes_A^{co\mathcal{C}} A \rightarrow \mathcal{C}$ which sends $a \otimes a'$ onto $aga'$ is an $A$–coring homomorphism. Since $A_{\mathcal{C}}$ is a finitely generated projective generator for $\mathcal{M}^{\mathcal{C}}$, a standard consequence of Gabriel-Popescu’s Theorem (see, e.g. [10, Corol 9.7]) says that $\text{Hom}_\mathcal{C}(A, -) : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_{\text{End}(A_{\mathcal{C}})}$ is an equivalence of categories. Now, $\text{Hom}_\mathcal{C}(A, -) \cong (-)^{co\mathcal{C}}$ naturally which implies, by [3, Theorem 5.6], that $\varphi$ is an isomorphism.

Following [3, Definition 5.3] we say that an $A$–coring with grouplike $g$ is Galois if the $A$–coring map which sends $1 \otimes_A^{co\mathcal{C}} 1$ to $g$ gives an isomorphism $\mathcal{C} \cong A \otimes_A^{co\mathcal{C}} A$. Thus, Theorem 4.3 says that every simple semisimple $A$–coring with a grouplike element is Galois. We have already more, as the following theorem, which collects the relevant information about simple semisimple corings with a grouplike element, shows.

**Theorem 4.4.** The following conditions are equivalent for an $A$–coring $\mathcal{C}$ with a grouplike element $g$. 

---
(i) \( \mathcal{C} \) is a simple simisimple \( A \)-coring;

(ii) \( \mathcal{C} \cong A \otimes_B A \) for some simple artinian subring \( B \) of \( A \);

(iii) \( \mathcal{C} \) is Galois and \( A^{\text{co} \mathcal{C}} \) is a simple artinian ring;

(iv) \( \mathcal{C}_A \) is flat, \( A \) is a projective generator in \( \mathcal{C} \mathcal{M} \), and \( A^{\text{co} \mathcal{C}} \) is a simple artinian ring;

(v) \( A \mathcal{C} \) is flat, \( A \) is a projective generator in \( \mathcal{M}^{\mathcal{C}} \), and \( A^{\text{co} \mathcal{C}} \) is a simple artinian ring.

Proof. (i) \( \Rightarrow \) (iii) This is Theorem 4.3.

(iii) \( \Rightarrow \) (ii) Obvious.

(ii) \( \Rightarrow \) (i) It follows from Theorem 4.2.

(i) \( \Rightarrow \) (iv) By Theorem 3.1, \( \mathcal{C}_A \) is in fact projective. Theorem 3.5 gives that every right left \( \mathcal{C} \)-comodule is a direct sum of copies of the unique simple comodule. Thus, every nonzero comodule is a projective generator for \( \mathcal{C} \mathcal{M} \). Finally, since (i) is equivalent to (iii), we know that \( A^{\text{co} \mathcal{C}} \) is a simple artinian ring.

(iv) \( \Rightarrow \) (iii) The proof of Theorem 4.3 is easily adapted to obtain this implication.

(v) \( \Leftrightarrow \) (iv) It follows by symmetry. \( \square \)

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