QUASI $BPS$ WILSON LOOPS, LOCALIZATION OF LOOP EQUATION BY HOMOLOGY AND EXACT BETA FUNCTION IN THE LARGE-$N$ LIMIT OF $SU(N)$ YANG-MILLS THEORY

M. Bochicchio

INFN Sezione di Roma
Dipartimento di Fisica, Universita’ di Roma ‘La Sapienza’
Piazzale Aldo Moro 2, 00185 Roma
e-mail: marco.bochicchio@roma1.infn.it

ABSTRACT

We localize the loop equation of large-$N$ $YM$ theory in the $ASD$ variables on a critical equation for an effective action by means of homological methods as opposed to the cohomological localization of equivariantly closed forms in local field theory. Our localization occurs for some special simple quasi $BPS$ Wilson loops, that have no perimeter divergence and no cusp anomaly for backtracking cusps, in a partial Eguchi-Kawai reduction from four to two dimensions of the non-commutative theory in the limit of infinite non-commutativity and in a lattice regularization in which the $ASD$ integration variables live at the points of the lattice, thus implying an embedding of parabolic Higgs bundles in the $YM$ functional integral. Homological localization is based on an analogy with cohomological localization. The analog of the invariance of the cohomological class of a closed form for the addition of a co-boundary is the zig-zag symmetry, i.e. the invariance of the holonomy class of a quasi $BPS$ Wilson loop for the addition of the boundary of
a tiny strip, of size of the cut-off. The analog of the action being a closed form is the invariance of the $R\!G$ flow of the Wilsonian renormalized effective action in the $ASD$ variables, $\Gamma_q$, under the local conformal transformation that generates the holonomically trivial deformation of the quasi $BPS$ Wilson loops. Finally, the analog of the rescaling by a divergent factor of the cohomological trivial deformation that localizes the exponential of a closed form is the local conformal rescaling that maps any (lattice) marked point of a quasi $BPS$ Wilson loop to a cusp at infinity. The homological reason for localization is that the contact term that occurs at the marked points of the loop, that is the obstruction to localization in the loop equation, vanishes at the cusps because of the absence of cusp anomaly for the backtracking cusps of the quasi $BPS$ Wilson loop and the manifest zig-zag invariant regularization of the loop equation in the $ASD$ variables. Yet, a posteriori, we find a simple reason for homological localization to occur: $\Gamma_q$ flows to the ultraviolet by the localizing conformal transformation and thus, since it is $AF$, to vanishing coupling. We find that the beta function of $\Gamma_q$ is saturated by the non-commutative $ASD$ vortices of the $E\Omega$ reduction. An exact canonical beta function of $NSVZ$ type that reproduces the universal first and second perturbative coefficients follows by the localization of $\Gamma_q$ on vortices. Finally we argue that a scheme can be found in which the canonical coupling coincides with the physical charge between static quark sources in the large-$N$ limit and we compare our theoretical calculation with some numerical lattice result.
1 Introduction

In this paper we revisit and refine our result [1] on the exact beta function in the large-$N$ limit of the pure Yang-Mills theory. From a computational point of view we can summarize our result as follows.

There exists a renormalization scheme in which the large-$N$ canonical beta function of the pure $YM$ theory is given by:

\[
\frac{\partial g_c}{\partial \log \Lambda} = \frac{-\beta_0 g_c^3 + \frac{\beta_J}{4} g_c^3 \frac{\partial \log Z}{\partial \log \Lambda}}{1 - \beta_J g_c^2} \tag{1}
\]

with:

\[
\beta_0 = \frac{1}{(4\pi)^2} \frac{11}{3}, \quad \beta_J = \frac{4}{(4\pi)^2} \tag{2}
\]

where $g_c$ is the 't Hooft canonical coupling constant and $\frac{\partial \log Z}{\partial \log \Lambda}$ is computed to all orders in the 't Hooft Wilsonian coupling constant, $g_W$, by:

\[
\frac{\partial \log Z}{\partial \log \Lambda} = \frac{1}{(4\pi)^2} \frac{10}{3} g_W^2 \frac{1 + cg_W^2}{1 + c g_W^2} \tag{3}
\]

with $c$ a scheme dependent arbitrary constant. At the same time, the beta function for the 't Hooft Wilsonian coupling is exactly one loop:

\[
\frac{\partial g_W}{\partial \log \Lambda} = -\beta_0 g_W^3 \tag{4}
\]

Once the result for $\frac{\partial \log Z}{\partial \log \Lambda}$ to the lowest order in the canonical coupling

\[
\frac{\partial \log Z}{\partial \log \Lambda} = \frac{1}{(4\pi)^2} \frac{10}{3} g_c^2 + \ldots \tag{5}
\]

is inserted in Eq.(1), it implies the correct value of the first and second perturbative coefficients of the beta function [2, 3, 4, 5]:

\[
\frac{\partial g_c}{\partial \log \Lambda} = -\beta_0 g_c^3 + \left( \frac{\beta_J}{4} \frac{1}{(4\pi)^2} \frac{10}{3} - \beta_0 \beta_J \right) g_c^5 + \ldots
\]
which are known to be universal, i.e. scheme independent.

We present in this paper a more general and simplified computation of the effective action that leads to the exact beta function.

In addition we argue that there is a scheme in which the canonical coupling coincides with a certain definition of the physical effective charge in the inter-quark potential. In this scheme the beta function is given by:

\[
\frac{\partial g_{\text{phys}}}{\partial \log r} = \beta_0 \frac{g_{\text{phys}}^3}{1 - \beta_J g_{\text{phys}}^2 \log(r\Lambda_W)} \log(r\Lambda_W) - \frac{15}{121}
\]

with \( \Lambda_W \) the RG invariant scale in the Wilsonian scheme.

But overall we furnish a mathematical and a physical explanation for the exactness of the beta function.

On the mathematical side our result for the beta function is based on localizing on a saddle-point equation for an effective action a version of the large-\( N \) loop equation for certain quasi \( BPS \) Wilson loops, in which the functional integration is, by a change of variables, on the anti-self-dual (\( ASD \)) part of the curvature of the connection.

The mentioned quasi \( BPS \) Wilson loops of the large-\( N \) pure \( YM \) theory enjoy non-renormalization properties analogous to the ones of \( BPS \) Wilson loops of theories with extended supersymmetry [6]. In particular we find that they do not have perimeter and cusp divergences for backtracking cusps.

We may wonder how this localization can be possible. In fact there are examples of non-trivial quantum field theories in which some special observables can be evaluated exactly at a saddle-point. The rationale behind these examples is a version of the localization described in [7], introduced in quantum field theories in [8]. Some of the most interesting cases are the cohomology ring of \( d = 2 \) \( YM \) theory [8], the prepotential of \( d = 4 \mathcal{N} = 2 \) \( SUSY \) gauge theory [9] and more recently some circular Wilson loops in
$d = 4 \mathcal{N} = 4,2$ SUSY gauge YM theories [10]. Localization in quantum field theory is based on cohomological concepts [7, 8] that involve the action of a nilpotent BRST differential on equivariantly closed forms. The proof of localization is obtained noticing that we can add freely to an exponentiated equivariantly closed form the BRST variation of any globally defined form without changing its integral. Then the added BRST differential is rescaled by a divergent factor to get localization by a semi-classical argument.

In all the mentioned four-dimensional cases the BRST charge on which localization is based is the supercharge of a kind of twisted supersymmetry of the action and of the observables. Thus localization, when it applies, applies only to special observables and cannot be extended to all the observables of the theory.

The localization in the large-$N$ pure YM theory that we refer to is furnished by a certain version of the large-$N$ loop equation and it is not based at all on a form of twisted SUSY, that of course is absent in the YM theory.

Rather, and perhaps not surprisingly, the localization of the loop equation on a saddle point for an effective action becomes possible only after exploiting a stringy character of the loop equation for the quasi BPS Wilson loops in which the zig-zag symmetry [11] of the Wilson loops plays a key role.

The zig-zag symmetry is the invariance of our quasi BPS Wilson loops for the addition of a backtracking arc. By means of the zig-zag symmetry we can add freely to the Wilson loop backtracking arcs of arbitrary length that end into cusps. In a regularized version we can substitute the backtracking arc with the boundary of a tiny strip, of size of the cut-off.

Thus the localization that we describe in the loop equation is a homological concept, rather than cohomological, and its proof is therefore obtained drawing a certain (weighted) arc family on the loop of the loop equation, that at topological level obeys the axioms of topological closed/open string theory [12].

It is of the utmost importance that the zig-zag symmetry be explicitly non-anomalous, i.e. that it holds without fine-tuning of the renormalization scheme, for localization to apply in our version of the loop equation. This is the case for the quasi BPS Wilson loops that we refer to, because of the
absence of cusp anomaly for backtracking cusps and of perimeter divergence. Since the zig-zag symmetry is the homological equivalent of the cohomological invariance of the cohomology class of a closed form by the addition of the BRST differential of a globally defined form, alone it is by far too general to suffice to localize the loop equation, as it is its corresponding notion in cohomology. Indeed we will see that there exist renormalization schemes in which the zig-zag symmetry holds for all the usual unitary Wilson loops, that form a complete set of the theory. It is very unlikely that any form of localization will ever apply to all these observables.  

Another, more dynamical, crucial ingredient is necessary. In fact, following the analogy with cohomological localization, we need the analog of the property that the action of the theory is an equivariantly closed form. We can rephrase this property by saying that there is a BRST symmetry of the action that generates the cohomologically trivial deformations. In our homological language there is no such symmetry of the action, but there is a symmetry of the RG flow of the Wilsonian renormalized effective action. In fact for the quasi BPS Wilson loops the two-dimensional local conformal transformation that generates the holonomically trivial deformation of the loop [13] can be lifted to a local conformal transformation of the four-dimensional theory, because of the quasi BPS constraint (see sect.2 and sect.3). The essential reason for this lifting is that a quasi BPS Wilson loop lives on a two-dimensional Riemann surface Σ that is diagonally embedded in the four-dimensional theory.

This conformal transformation moves the Wilsonian renormalized action along the RG trajectory since it is equivalent to add to it, as a local counter-term, the conformal anomaly. The conformal anomaly is computed exactly a posteriori from the exact beta function of the localized effective action.

From this point of view the reason a posteriori for homological localization

\footnote{Formally the computations of this paper apply, mutatis mutandis, also to some diagonally embedded Wilson loops whose curvature is, following our conventions, of SD type. The associated connection is, in the notation of sect.2, $B^{phys} = (A_z + D_a)dz + (A_{\bar{z}} - D_{\bar{a}})d\bar{z}$ and thus it is hermitean. These Wilson loops are physical in the sense that they have perimeter divergence and cusp anomaly and they imply a non-trivial inter-quark potential. Thus, for them, the zig-zag symmetry holds only by fine-tuning of the renormalization scheme. In addition, for them, the $EK$ reduction does not lead to the parabolic Higgs bundles of Hitchin type of sect.3.}
is that the Wilsonian renormalized effective action flows to the ultraviolet by the local conformal map and thus, being $AF$, to vanishing coupling. Then a semi-classical argument implies localization as in the cohomological case.

However we can find a purely homological reason for localization to apply in the loop equation in the $ASD$ variables. As we have anticipated, it turns out that the loop equation can be localized on a critical equation only on a weighted graph [12] whose spine, obtained collapsing all the tiny strips to strings, is a Mandelstam graph [14] on which backtracking strings that end into cusps are drawn. On such a graph each string meets transversely the Wilson loop. In addition each string starts and ends into a cusp in such a way that the Wilson loop separates the cusps pairwise into two regions. The effective action on the weighted graph has a stringy character in the sense that its local part is in fact bi-local, since it is localized on pairs of cusps having a string in common. The homological reason for which the loop equation in the $ASD$ variables localizes on the weighted graph is that the usual obstruction to localization in the loop equation, that is the contact term, vanishes at the backtracking cusps of the Mandelstam graph, because of the zig-zag symmetry in our (regularized) version of the loop equation and because of the absence of cusp anomaly for the quasi $BPS$ Wilson loop.

The cusps carry the local degrees of freedom of the theory and correspond to the parabolic points of a dense embedding in the functional integral of twisted parabolic Higgs bundles (of a non-commutative version of the large-$N$ $YM$ theory). From a stringy point of view the cusps are the $D$-branes at which open strings can end [12]. This also explains a posteriori why, to get localization in the loop equation, a change of variables is needed in which the functional integral is defined at the points rather than at the links of a lattice. This means, in continuum language, that in our version of the loop equation we change variables from the connection to (the $ASD$ part of) the curvature.

Parabolic Higgs bundles have been introduced in the functional integral of pure $YM$ theory in [15] and more recently in the functional integral of $\mathcal{N}=4$ SUSY $YM$ theory in [16] in their study of the ramified Langlands conjecture. We find that the exact beta function arises localizing the quasi $BPS$ Wilson loops on the moduli of twisted parabolic Higgs bundles of a special type that correspond to the $Z_N$ non-abelian vortices of a partial non-
commutative large-$N$ Eguchi-Kawai ($EK$) reduction [17, 18, 19, 20, 21].

On the physical side our result for the beta function resembles the computation of the exact $\mathcal{N} = 1$ SUSY YM beta function from the evaluation of a very special sector of the theory, the chiral sector, by the saddle-point method via instantons [22].

We describe below some features of our approach from this physical point of view. While summing all the large-$N$ diagrams of the pure Yang-Mills theory is a very difficult task, perhaps outside the limits of our present techniques, restricting to diagrams that contain only charge renormalization might result in a much simpler problem. Indeed experience with supersymmetric gauge theories, even with only $\mathcal{N} = 1$ supersymmetry, suggests that the scheme dependence of the beta function may be exploited in such a way to arrive at exact results, such as the $NSVZ$ exact beta function of $\mathcal{N} = 1$ SUSY Yang-Mills theory. The original derivation of the $NSVZ$ exact beta function relies crucially on instantons computations and on the cancellations between bosonic and fermionic non-zero modes due to the $\mathcal{N} = 1$ supersymmetry [22].

However we can still look at the $NSVZ$ result as an ingenious way to exploit the scheme dependence of the beta function in order to sum all the loop diagrams that contribute to charge renormalization by means of what is a one-loop computation for the Wilsonian coupling and a (one-loop) rescaling anomaly for the canonical coupling of the $\mathcal{N} = 1$ SUSY gauge theory [23]. Yet in the pure $YM$ theory there is no supersymmetry. Thus our starting point, to simplify the problem, is the large-$N$ limit and the loop equation.

Solving the loop equation uniformly for all Wilson loops is equivalent to find the master field [24], yet a too difficult problem [1], equivalent to summing all the large-$N$ diagrams. Instead we want to restrict ourselves to charge renormalization only. Thus we want to find a Wilson loop whose only renormalization is charge renormalization in its internal loop diagrams. We thus require that the loop is simple, i.e. without self-intersections, and we also require that it has no perimeter divergence. In addition we require that it has no cusp anomaly for backtracking cusps since we want to use eventually the zig-zag symmetry (i.e. the invariance of the Wilson loop by adding a backtracking arc) in the loop equation.
Let us suppose that such a Wilson loop exists. We choose a simple loop and we insert it in the Migdal-Makeenko (MM) version of the loop equation [25, 26]. By loop equation we mean here the identity obtained requiring that the integral of the functional derivative of the Wilson loop times the exponential of the action vanishes, not its stringy version in terms of the loop operator [25, 26].

In the MM loop equation, when the equation of motion is inserted in front of the Wilson loop in the left hand side, a contact term is produced in the right hand side [25, 26]. This contact term is responsible of all quantum corrections in the large-\(N\) limit, in an iterative perturbative solution of the loop equation. The contact term gives rise to perimeter and cusp divergences in the right hand side of the loop equation [6]. Hence, because of its non-renormalization properties, we would expect that the right hand side of the MM loop equation is zero when the quasi BPS Wilson loop is inserted.

However this is not the case. The MM loop equation does not see any difference between the quasi BPS Wilson loop and an ordinary unitary Wilson loop. The quantum contact term is anyway generated and cancellations for the quasi BPS Wilson loop occur only in the iterative perturbative solution.

The situation gets even worse in presence of backtracking arcs ending into a cusp, because the arcs, having a length, contribute to both the perimeter divergence and the cusp anomaly in the right hand side of the loop equation. The question arises as to how we can reconcile the cusp anomaly with the zig-zag symmetry. For an ordinary unitary Wilson loop the way out is that the perimeter divergence in the right hand side of the loop equation has to be partially cancelled by an infinite cusp anomaly to recover the zig-zag symmetry, in such a way that the only perimeter divergence that remains is the one of the loop without the backtracking arcs. However this cancellation is highly regularization (i.e. scheme) dependent and essentially it is fixed assuming the zig-zag symmetry. Indeed, despite the zig-zag symmetry of the loop, the MM loop equation is not regularized in a manifestly zig-zag invariant way, since the perimeter and the cusp contribution in its right hand side are separate. Of course the same argument applies to the quasi BPS Wilson loop, but for the fact that in this case it should be possible to find a scheme in which the cusp anomaly cancels the perimeter divergence completely. In fact it should be possible, possibly by a change of variables,
to find a regularization scheme in which the right hand side of the loop equation in the new variables is actually zero because of the mentioned non-renormalization properties.

We may expect that, if it exists a manifestly zig-zag invariant regularization of the new loop equation (i.e. in a zig-zag invariant scheme), its right hand side be zero for a quasi $BPS$ Wilson loop. Indeed, since we already know that perimeter and cusp divergences have to mix together for backtracking cusps, in a regularization in which there cannot be a cusp anomaly for backtracking cusps because of the manifest zig-zag symmetry, there should not be a perimeter divergence either for the quasi $BPS$ Wilson loop. Thus in such a scheme the right hand side of the loop equation is zero for a quasi $BPS$ Wilson loop. But then, since charge renormalization occurs in the loop, it has to occur in the left hand side of the loop equation. This means that it occurs in the effective action, that gives rise to the equation of motion in the variables and regularization in which the new loop equation is written. This effective action will contain all the quantum corrections for such a loop, i.e. it will contain the sum of large-$N$ diagrams that renormalize the charge.

It turns out that a quasi $BPS$ Wilson loop exists in the large-$N$ limit of pure $YM$ theory and that the variables that admit a zig-zag invariant regularization of the loop equation are the $ASD$ components of the curvature of the connection. The zig-zag invariant regularization consists in analytically continuing the loop equation from Euclidean to Minkowskian space-time. It is perhaps at the heart of the modern Euclidean approach to quantum field theory that such analytic continuation be possible.

In fact, taking advantage of the mixing between the perimeter and cusp divergences, in our renormalization procedure every marked point of the loop is mapped into a backtracking cusp by a singular conformal transformation that changes the renormalized action by the conformal anomaly.

This is done by introducing a planar lattice, in a partial $EK$ non-commutative large-$N$ reduction of the theory from four to two dimensions, to which backtracking strings, that do not change the loop by the zig-zag symmetry, are attached by means of a singular change of the conformal structure [13]. This suffices to define the beta function, from the divergences of the Euclidean effective action, and the regularized loop equation, from the analytic contin-
uation to Minkowskian space-time. However, if we insist in keeping the ASD structure real, the analytic continuation of the Euclidean loop equation has actually to be performed to ultra-hyperbolic signature.

It turns out that the loop equation is saturated by the Euclidean ASD vortices of the non-commutative theory that, by consistency, admit analytic continuation to ultra-hyperbolic signature and that the (Euclidean) effective action computed on the vortices renormalizes with an exact beta function of NSVZ type, whose first two coefficients agree with large-$N$ perturbation theory.

We should perhaps stress the twofold role played by attaching backtrack- ing strings to the cusps, i.e. to the parabolic points of our lattice. At the level of loop equation the strings imply the absence of the contact term in the right hand side of loop equation because of the manifest zig-zag symmetry. At the level of the functional integral the strings, just because of their presence, allow the existence of a singular gauge in which the (hermitean) curvature of the vortices of ASD type can be diagonalized at each cusp. In this singular gauge the original order of $N^2$ non-abelian integration variables at each cusp appear as order of $N^2$ zero modes of the Jacobian of the change of variables from the connection to the ASD curvature. Thus the Wilsonian beta function is computed simply by counting zero modes as in the supersymmetric case, without actually performing the order of $N^2$ integrations over the vortices moduli space:

$$\frac{1}{2g_W^2(\tilde{a})} = \frac{1}{2g_W^2(a)} - \frac{1}{(4\pi)^2}(2 + \frac{5}{3}) \log(\frac{\tilde{a}}{a})$$  \hspace{1cm} (8)

where $a$ is a lattice cut-off and $\tilde{a}$ an infrared (lattice) scale. However a crucial difference arises in the pure Yang-Mills case as opposed to the $N = 1 $ SUSY case. In the SUSY case the whole Wilsonian beta function is saturated by instantons zero modes. In the pure YM case only part of the beta function (the term equal to 2) is accounted by the vortices zero modes. The remaining part (the term equal to $\frac{5}{3}$) appears as a standard one-loop divergence of the Jacobian of the change of variables. This divergence gives rise to a contribution of an anomalous dimension:

$$\frac{\partial \log Z}{\partial \log \Lambda} = \frac{1}{(4\pi)^2} \frac{10}{3} g_W^2$$

\hspace{1cm} (9)
Indeed in the $\mathcal{N} = 1$ SUSY case the Jacobian of the change to the ASD variables in the light-cone gauge cancels (but for the zero modes) against the gluinos determinant. In fact in the $\mathcal{N} = 1$ SUSY case and in the light-cone gauge our change of variables actually coincides with the Nicolai map of the supersymmetric theory [27, 28]. The loop equation beta function is then saturated by instantons satisfying certain conditions, thus reproducing the standard supersymmetric NSVZ beta function in the large-$N$ limit [29].

The canonical beta function is found computing its relation to the Wilsonian one, by rescaling the action into its canonical form. Indeed in the large-$N$ YM theory there is a new contribution from the zero modes for this rescaling:

$$\frac{1}{2g_W^2} = \frac{1}{2g_c^2} + \beta_J log g_c + \frac{\beta J}{4} log Z$$

(10)

that upon differentiation by $log(\frac{1}{a})$ leads to the canonical beta function.

A posteriori there is a way of looking at the homological localization purely in terms of the flow of the RG. In fact, remembering that the conformal map sends the marked points to the cusps and that thus adds the conformal anomaly to the Wilsonian effective action, the true renormalization of the Wilsonian coupling at the cusps reads:

$$\frac{1}{2g_W^2(\frac{a}{\sqrt{N_D}})} = \frac{1}{2g_W^2(a)} - \frac{1}{(4\pi)^2} \left(2 + \frac{5}{3}\right) log(\frac{a}{a\sqrt{N_D}})$$

(11)

where the large rescaling factor of $\sqrt{N_D}$ is due to the regularization of the singular conformal anomaly at the cusps. Thus the log never gets large at the cusps, since the theory flows back to the ultraviolet because of the conformal anomaly. The precise finite value of the log is just a matter of convention, i.e. a choice of the renormalization scheme at the cusps.

Thus at the cusps there is no infinite renormalization of the bare coupling. Localization then occurs because at the ultraviolet the bare coupling vanishes. Nevertheless, employing the RG improved formulae for the anomalous dimension and the canonical beta function obtained taking derivatives

We would like to thank Gabriele Veneziano for pointing out this feature to us.
with respect to $\log(\frac{1}{a})$, we get non-trivial formulae for the anomalous dimension and the canonical beta function.

This localization argument a posteriori is based only on $AF$ and not on the particular value of the (first) coefficient of the Wilsonian beta function. However this means that $AF$ is so difficult to produce that if it occurs in the Wilsonian effective action obtained by a change of variables, the resulting theory must necessarily be the $YM$ theory.

Coming back on the mathematical side, our result implies the existence, by explicit construction in a certain regularization and renormalization scheme, of the large-$N$ limit of the pure Yang Mills theory for the mentioned special quasi $BPS$ Wilson loops, essentially because of the super-renormalizability of the large-$N$ Wilsonian coupling, that turns out to be one-loop exact as in the $\mathcal{N} = 1$ supersymmetric case and that implies an all orders formula of NSVZ type for the canonical beta function [1].

We should perhaps specify in which sense we refer to the existence of the large-$N$ limit. The existence of the functional integral can be shown either by abstract mathematical means based on constructive quantum field theory or by direct construction of a solution. The constructive quantum field theory approach has a long tradition [30]. Instead the techniques presented in this paper lead naturally to the second point of view together with the idea that the functional integral is defined by the way of computing it, in this case by the solution of the large-$N$ loop equation.

We should perhaps mention that we construct a solution of the loop equation for observables that, just because of their non-renormalization properties, are physically almost trivial (at least in perturbation theory). It is precisely this almost triviality that allows us to reach a solution, because most of the vast information contained in the complete large-$N$ solution is lost. Nevertheless we are able to extract from our solution the exact beta function, that does not depend on a specific observable because of the universal character of the renormalization procedure.

The plan of the paper is as follows.

In sect.2 we show the existence of the quasi $BPS$ Wilson loop in large-$N$ pure $YM$ theory starting by analogy with the case of extended supersymme-
try. A few comments are in order. In the case of extended supersymmetry BPS Wilson loops are obtained adding to the gauge connection (scalar) Higgs fields with a factor of $i$ in front \[31\], in order to satisfy non-trivially the supersymmetric constraints.

Now already in $\mathcal{N} = 1 \ d = 4$ SUSY YM gauge theory there are no scalars to play with. However in such a theory we can construct a planar BPS Wilson loop adding to the gauge connection on the (01) plane the covariant derivatives along the orthogonal (23) plane, that transform as an Higgs field for gauge transformations \[1\]. In fact this holds more properly in a non-commutative version of the theory in which the derivative part of the covariant derivative can be absorbed into a gauge transformation, since in gauge theories on non-commutative space the gauge group contains the translations \[32, 33, 34\]. In turn this feature of absorbing translations into gauge transformations basically leads to the EK large-$N$ reduction \[32, 33, 34\].

Now, although there is no notion of supersymmetry, the very same operator exists also in pure non-commutative YM theory. Thus while in theories with $\mathcal{N} = 4, 2$ SUSY we obtain the Higgs field needed for the BPS property via dimensional reduction from the ten or the six dimensional $\mathcal{N} = 1$ SUSY theory that is actually used to construct the extended supersymmetry in $d = 4$, in $\mathcal{N} = 1, 0$ SUSY theories we obtain the Higgs fields via large-$N$ EK reduction by means of covariant derivatives.

Now, as we show in sect.2, if we limit ourselves to the perimeter and the cusp divergences, in the $\mathcal{N} = 1$ SUSY and in the pure YM theory case, the mentioned non-renormalization properties hold in the large-$N$ limit not because of the supersymmetry but because of the $O(4)$ ($O(2, 2)$ in ultra-hyperbolic signature) symmetry of the non-commutative theory in the limit of infinite non-commutativity.

In sect.3 we recall the loop equation of the pure YM theory in the ASD variables following \[1\]. We defer the study of the loop equation in the $\mathcal{N} = 1$ case to another paper \[29\], where we interpret our change of variables in the pure YM theory as the Nicolai map in $\mathcal{N} = 1$ SUSY \[3\]. In fact it turns out that in the $\mathcal{N} = 1$ case our loop equation leads to localization

\[3\] We would like to thank Gabriele Veneziano for pointing out this feature to us.
on the non-commutative (ultra-hyperbolic) instantons in the limit of infinite non-commutativity and to the already known exact NSVZ beta function in the large-$N$ limit [29].

In sect.4 we compute the beta function of the pure large-$N$ YM theory generalizing and simplifying in several ways the computation already presented in [1]. The basic fact is that the loop equation and the beta function are saturated by the non-commutative ASD (ultra-hyperbolic) vortices of the $E K$ reduction in the pure $Y M$ case as opposed to the non-commutative (ultra-hyperbolic) instantons of the $\mathcal{N} = 1$ case.

In sect.5 we write down rather explicitly our beta function and we exploit the residual scheme dependence to create a link with a certain definition of the physical effective charge [35] in the inter-quark potential in the large-$N$ limit. We also compare our large-$N$ beta function with the numerical results found for $SU(3)$ [35].

In sect.6 we recall our conclusions.

2 Quasi $BPS$ Wilson loops

We recall in this section the properties of (locally) $BPS$ Wilson loops [31] originally introduced in the study of $\mathcal{N} = 4$ SUSY YM/AdS string duality in [36]. We use the notation and some of the arguments of [6] about non-renormalization properties of locally $BPS$ Wilson loops.

It has been argued in [6] that a locally $BPS$ Wilson loop in the four-dimensional $\mathcal{N} = 4$ SUSY gauge theory:

$$Tr \Psi(BPS) = Tr P \exp i \int_C (A_a dx_a(s) + i \phi_b dy_b(s))$$  \hspace{1cm}(12)$$

has no perimeter divergence to all orders in perturbation theory because of the local $BPS$ constraint:

$$\sum_a \dot{x}_a^2(s) - \sum_b \dot{y}_b^2(s) = 0$$  \hspace{1cm}(13)$$

Indeed at lowest order of perturbation theory this constraint assures the cancellation of the contribution to the perimeter divergence of the gauge
propagator versus the scalar propagator, because of the factor of $i^2$ in front of the scalar propagator at that order. As far as the perimeter divergence is concerned it is argued in [6] that this cancellation occurs to all orders in perturbation theory, when the locally $BPS$ Wilson loop is seen as the dimensional reduction to four dimensions of the ten-dimensional Wilson loop of the ten-dimensional $\mathcal{N} = 1$ SUSY YM theory from which the four-dimensional $\mathcal{N} = 4$ SUSY theory is obtained.

We report here only the ten-dimensional version of the argument in [6], in which $SUSY$ plays no role. In fact the argument is based only on $O(10)$ rotational symmetry, as we show momentarily. In ten dimensions the coefficient of the perimeter divergence of an ordinary unitary Wilson loop at any order in perturbation theory must necessarily contain as a factor a polynomial in the $O(10)$ invariant quantity $\sum_\alpha \dot{x}_\alpha^2(s)$ without a constant term. Indeed at any order in perturbation theory a generic contribution to the Wilson loop contains a correlator of gauge fields, i.e. a Green function, with tensor indices contracted with a product of monomials in $\dot{x}_\alpha(s)$ at generic insertion points on the loop, labeled by $s$.

The perimeter divergence arises when all insertion points coincide in such a way that all the arguments of the Green function vanish. In this case the Green function provides a factor that by $O(10)$ rotational invariance must be a polynomial in ten-dimensional Kronecker delta, since all the difference vectors are zero at coinciding points and thus no other tensorial structure can be produced.

This combines with the factors of $\dot{x}_\alpha(s)$ to produce an invariant polynomial in $\sum_\alpha \dot{x}_\alpha^2(s)$ with no constant term since the lowest order contribution is zero by direct computation. But $\sum_\alpha \dot{x}_\alpha^2(s)$ is zero for a $BPS$ Wilson loop because of the $BPS$ constraint. A naive application of this argument to the four-dimensional $BPS$ Wilson loop would imply the absence of the perimeter divergence for this loop on the basis of the $O(10)$ rotational invariance of the theory before the dimensional reduction. Independently of subtleties eventually needed to apply the argument to the dimensionally reduced $d = 4$ Wilson loop in the $\mathcal{N} = 4$ theory we show now that in $\mathcal{N} = 1$ and $\mathcal{N} = 0$ theories Wilson loops exist for which no perimeter divergence occurs because of the $O(4)$ rotational symmetry in the large-$N$ limit. We will show momentarily why in absence of $SUSY$ the large-$N$ limit is needed.
The connection that we look for is the following one:

\[ B = A + D = (A_z + D_u)dz + (A_{\bar{z}} + D_{\bar{u}})d\bar{z} \tag{14} \]

where \( z, \bar{z}, u, \bar{u} \) are the four-dimensional complex coordinates and \( D_u = \partial_u + iA_u \) the \( u \) component of the covariant derivative. From a two-dimensional point of view \( B \) is a non-hermitean connection defined as the sum of a hermitean two-dimensional connection and a Higgs field. Indeed covariant derivatives transform as a Higgs field for gauge transformations. Let us notice also that in the \( N = 1 \) theory \( B \) gives rise to a locally \( BPS \) Wilson loop because of the constraint that is implicit in its definition:

\[
\begin{align*}
   dz(s) &= du(s) \\
   d\bar{z}(s) &= d\bar{u}(s)
\end{align*} \tag{15}
\]

The factor of \( i \) in front of the connection in the covariant derivative plays the same role as the factor of \( i \) in front of the Higgs field in locally \( BPS \) Wilson loops of theories with extended \( SUSY \). If the derivative term were absent in the covariant derivatives we could invoke the \( O(4) \) symmetry of the theory and the local \( BPS \) constraint to imply the absence of the perimeter divergence as in the \( O(10) \) \( SUSY \) case. Yet, there are derivatives.

In fact if we make the space non-commutative in the \( u, \bar{u} \) directions, we can get rid of the derivatives by a gauge transformation because the translations can be absorbed into the gauge transformations in a non-commutative theory [37] and more generally in the large-\( N \) limit [32].

However non-commutativity breaks the \( O(4) \) symmetry in such a way that the property that we are looking for is lost.

Yet, since the limit of infinite non-commutativity is equivalent to the ordinary large-\( N \) limit [37], as can be seen for example from the loop equation of the non-commutative theory, and in the ordinary commutative large-\( N \) limit certainly \( O(4) \) symmetry holds, then in the limit of infinite non-commutativity our Wilson loop has no perimeter divergence.

This holds in the pure \( YM \) theory and in the \( N = 1 \) \( YM \) theory without using \( SUSY \). We now show, by an argument based on the \( MM \) loop equation, that the perimeter divergence and the cusp anomaly for backtracking
cusps have to mix for a zig-zag invariant Wilson loop in such a way that the cusp anomaly cancels the part of the perimeter divergence due to the backtracking arcs. An argument of this kind has been anticipated in [6] in their discussion of the zig-zag symmetry on the stringy side of the correspondence $AdS/CFT$.

A byproduct of this argument is that a zig-zag invariant loop that has no perimeter divergence has no cusp divergence for backtracking cusps, because the zig-zag symmetry requires cancellation between the two contributions and since one of them vanishes the other one has to vanish too.

Introducing the $MM$ loop equation at this stage will allow us to prepare our arguments on charge renormalization too. We can write the $MM$ loop equation for unitary Wilson loops in the large-$N$ $YM$ theory as:

$$
\int_{C(x,x)} dx_\alpha \tau \left( \frac{1}{2g^2} \delta S \frac{\delta A_\alpha}{\delta A_\alpha(x)} \Psi(x,x;A) \right) = \int_{C(x,x)} dx_\alpha \int_{C(x,x)} dy_\alpha \delta^{(4)}(x-y) \tau(\Psi(x,y;A)) \tau(\Psi(y,x;A))
$$

(16)

where the normalized trace, $\tau$, is the combination of the normalized v.e.v. with the normalized colour trace in the fundamental representation (see for example [1]) and

$$
\Psi(x,y;A) = P \exp i \int_{C(x,y)} A_\alpha dx_\alpha
$$

(17)

In the case of loops without self-intersections but with cusps the $MM$ loop equation reduces to:

$$
\int_{C(x,x)} dx_\alpha \tau \left( \frac{1}{2g^2} \delta S \frac{\delta A_\alpha}{\delta A_\alpha(x)} \Psi(x,x;A) \right) = i \int_{C(x,x)} dx_\alpha \int_{C(x,x)} dy_\alpha \delta^{(4)}(x-y) \tau(\Psi(x,x;A))
$$

(18)

Performing the two contour integrations along the loop in the right hand side, we get:

$$
\int_{C(x,x)} dx_\alpha \tau \left( \frac{1}{2g^2} \delta S \frac{\delta A_\alpha}{\delta A_\alpha(x)} \Psi(x,x;A) \right) =
\int_{C(x,x)} dx_\alpha \tau \left( \frac{1}{2g^2} \delta S \frac{\delta A_\alpha}{\delta A_\alpha(x)} \Psi(x,x;A) \right) = i(La^{-3} + \sum_{cusp} \frac{\cos \Omega_{\text{cusp}}}{\sin \Omega_{\text{cusp}}} a^{-2}) \tau(\Psi(x,x;A))
$$

(19)
where $L$ is the perimeter of the loop and $\Omega_{\text{cusp}}$ the cusp angle at a cusp. The perimeter divergence arises by the double integration of the four-dimensional delta function, i.e. of the contact term, along the loop. However integrating the contact term in a neighborhood of each cusp gives rise to a sub-leading quadratic divergence, since around a cusp we get two independent integrations instead of one, due to the two sides of the cusp. The coefficient of the cusp contribution is proportional to the ratio:

$$\frac{\cos \Omega_{\text{cusp}}}{\sin \Omega_{\text{cusp}}} \quad (20)$$

The numerator arises from the scalar product, the denominator from the two independent integrations of the two-dimensional delta function. In the limit in which the cusp angle $\Omega_{\text{cusp}}$ reaches $\pi$, i.e. the cusp backtracks, the cusp contribution in the contact term of the ordinary loop equation is negative and divergent.

If the Wilson loop is zig-zag invariant and the theory is regularized in a zig-zag invariant way this divergence must be fine-tuned to cancel part of the perimeter divergence due to the length of the cusps, in such a way that the only remaining perimeter divergence is the one of the loop without backtracking cusps, since these cusps are irrelevant because of the zig-zag symmetry. Thus we conclude that if a loop is zig-zag symmetric and it has no perimeter divergence it cannot have a cusp anomaly either for backtracking cusps. We can write the $MM$ loop equation also for a planar quasi $BPS$ Wilson loop:

$$\int_{C(x,x)} d\bar{z} \tau(\frac{1}{2g^2} \delta S_{\bar{z}}(x) \Psi(x,x;B)) =$$

$$i\delta^{(2)}(0) \int_{C(x,x)} d\bar{z} \int_{C(x,x)} dz \delta^{(2)}(z-x) \tau(\Psi(x,z;B)) \tau(\Psi(z,x;B)) \quad (21)$$

For quasi $BPS$ Wilson loops the contact term in the loop equation is the same as for unitary Wilson loops. Cancellations occur only in the solution.

Hence the $MM$ loop equation cannot localize, not even for a quasi $BPS$ Wilson loop, because it cannot be regularized in a way that does implement the zig-zag symmetry explicitly. In the next section we write a new loop equation for quasi $BPS$ Wilson loops in which the zig-zag symmetry can be implemented explicitly.
3 Localization by homology of the loop equation in the ASD variables

Cohomological localization in quantum field theory [8] is based on the fact that the integral of an equivariantly closed differential form depends only on its cohomology class and not on a particular representative:

$$\int (\omega + Q\alpha) = \int \omega$$

(22)

Thus we have the freedom to add to the exponential of a closed form $\omega$, the action of our quantum field theory, the BRST differential of any globally defined form, with an arbitrary coupling $t$, without changing the integral:

$$\int \exp(\omega + tQ\alpha)$$

(23)

as it is seen differentiating with respect to $t$ and using Eq.(22). In this way the saddle-point method applies exactly to the modified action in the limit $t \to \infty$. Localization of the functional integral on the critical points of $Q\alpha$ then follows. From this argument it is clear that localization is not a property of all the observables of the theory but of only those that are equivariantly closed as the action is.

We would like to find an analog of cohomological localization for the pure YM theory. Unfortunately $YM$ theory has no (twisted) SUSY and thus it is unlikely that this theory will ever admit a cohomological localization of any kind.

As a possible way out we look at the large-$N$ loop equation, the $MM$ equation. In the $MM$ equation the contact term is the obstruction to localization, since in its absence the $MM$ equation would reduce to the insertion in front of the Wilson loop of the classical equation of motion of the theory, that is the saddle-point equation for the action. Of course it is precisely this obstruction that makes the $MM$ equation non-trivial and interesting. By localization of the loop equation we mean a transformation of the loop equation into the new form:

$$\tau(\frac{\delta \Gamma}{\delta \mu(x)}\Psi(x; x; B)) = 0$$

(24)
for an effective action, $\Gamma_q$, for a Wilson loop of a special kind, $B$, and a trace, $\tau$, involving some new, yet unknown, integration variable, $\mu$. Thus the transformation that we look for must eliminate the contact term, after the change of variables from the gauge connection to the yet unspecified field $\mu$.

We can think of the $MM$ equation as an equation defined on loops rather than on points. Thus we will attempt to localize the loop equation by homological rather than cohomological methods.

Our basic homological property will be the zig-zag symmetry, i.e. the freedom to add to a Wilson loop a backtracking arc without changing its holonomy [11]. This symmetry follows by the definition of the path-ordered exponential. The zig-zag symmetry is the homological analog in our context of the cohomological identity in Eq.(22).

For ordinary Wilson loops the regularization and renormalization procedure may spoil the zig-zag symmetry. In fact we have already seen that the perimeter and cusp divergences of a smooth loop and of the same loop with the addition of the boundary of a tiny strip ending into a cusp may not coincide. A limit case is dimensional regularization, in which there is no perimeter divergence, since there are no linear divergences in $YM$ theory in this regularization, but there is a logarithmic cusp divergence. We saw however that we can still implement the zig-zag symmetry in another regularization by fine-tuning the perimeter and cusp divergences in such a way that they partially cancel each other in order to maintain the zig-zag symmetry. In any case we saw that for the quasi $BPS$ Wilson loops introduced in the previous section there is no perimeter divergence and no cusp anomaly for backtracking cusps. Thus the basic homological identity holds for them without worrying about regularization and renormalization.

In fact the zig-zag symmetry is by far too general to lead alone to homological localization. The next (dynamical) ingredient, needed to get homological localization, is a symmetry of the theory that generates the holonomically trivial deformation of a quasi $BPS$ Wilson loop. This is analogous in cohomology to a symmetry of the action that generates a deformation by a co-boundary, i.e. to the action being a closed form. As we will see below, in the classical $YM$ theory there is such a symmetry, it is the conformal symmetry. However just because it is a symmetry of the classical theory, it will
be of no use for our homological localization. What we need is a symmetry of the \(RG\) flow of the (Wilsonian) renormalized effective action. This symmetry arises as follows. There is an action of the two-dimensional conformal group that adds to the quasi \(BPS\) Wilson loop the boundary of a tiny strip [13], that lifts to a four-dimensional conformal rescaling because of the diagonal embedding of the planar quasi \(BPS\) Wilson loop in the four-dimensional theory.

At this point we can get localization in the loop equation simply by drawing a picture. We must add to the Wilson loop a family of weighted arcs [12] in such a way that all the marked points of the loop are mapped into backtracking cusps [13]. This modification amounts to a (singular) change of the conformal structure around the marked points [13]. In the four-dimensional \(YM\) theory this generates a conformal anomaly in the effective action. Since the cusps are backtracking their contribution in the right hand side of a new loop equation in new variables will vanish, provided the new loop equation can be regularized in a way compatible with the zig-zag symmetry. Homological localization follows.

The family of arcs is not arbitrary, since it must be compatible with the gluing properties of the functional integral. This leads to the axioms of the arc complex of topological closed/open strings [12] as we will see below.

The argument that we have presented hides a subtle but crucial point that will lead us to the right change of variables. We can choose the marked point of a loop arbitrarily. This implies that we would need uncountably many arcs and cusps. Zig-zag symmetry applies instead to a family finitely generated. To get a finitely generated family of arcs we can introduce a finite lattice of cusps and then take the continuum limit as in lattice gauge theories. But in lattice gauge theories the integration variables live on the links of a lattice rather than on the points. This is the crucial point. We must change variables from the connection to the curvature for homological localization to apply. The curvature lives on the plaquettes, but they are dual to the points in two dimensions. Thus we need a partial \(EK\) reduction from four to two dimensions.

In addition the curvature has too many components for a change of variables to exist in four dimensions, since the gauge connection has four compo-
nents and the curvature six. The ASD part of the curvature has only three components. Thus we need the choice of an axial gauge to kill one component of the connection. In four dimensions there are reason to prefer the light-cone gauge [38]. In the light-cone gauge the number of components matches. In fact this change of variables in the SUSY YM theory is the Nicolai map. But the light-cone gauge exists only in Minkoskian space-time. This suggests that the localization of the loop equation of quasi BPS Wilson loops occurs only if the loop equation, that is gauge invariant, is written in the ASD variables in a way that admits analytic continuation to Minkowskian space-time and if the theory is regularized on a lattice.

We now write the formulae that correspond to our arguments. We can describe the chain of changes of variables and transformations that lead to large-\(N\) homological localization as follows. We start with the large-\(N\) YM theory defined on \(R^2 \times R_\theta^2\) in the limit of infinite non-commutativity \(\theta\), that is known to reproduce the ordinary commutative large-\(N\) limit:

\[
Z = \int \exp\left(-\frac{N}{2g^2} \sum_{\alpha \neq \beta} \int Tr_f(F_{\alpha \beta}^2)d^4x\right)DA
\]

\[
= \int \exp\left(-\frac{N8\pi^2}{g^2}Q - \frac{N}{4g^2} \sum_{\alpha \neq \beta} \int Tr_f(F_{\alpha \beta}^{-2})d^4x\right)DA \quad (25)
\]

In the second line the classical action is conveniently rewritten as the sum of a topological and a purely ASD term. The topological term \(Q\) is the second Chern class, given by:

\[
Q = \frac{1}{16\pi^2} \sum_{\alpha \neq \beta} \int Tr_f(F_{\alpha \beta}\tilde{F}_{\alpha \beta})d^4x \quad (26)
\]

while the ASD curvature \(F_{\alpha \beta}^{-}\) is defined by:

\[
F_{\alpha \beta}^{-} = F_{\alpha \beta} - \tilde{F}_{\alpha \beta}
\]

\[
\tilde{F}_{\alpha \beta} = \frac{1}{2} \epsilon_{\alpha \beta \gamma \delta} F_{\gamma \delta} \quad (27)
\]

Introducing the projectors, \(P^{-}\) and \(P^{+}\), the curvature can be decomposed into its ASD and SD components:

\[
F_{\alpha \beta} = P^{-}F_{\alpha \beta} + P^{+}F_{\alpha \beta}
\]

\(^{4}\text{We thank Gabriele Veneziano for pointing out this feature to us.}\)
Notice that the coefficient of the classical action as a functional of the projected ASD curvature is twice the coefficient of the classical action as a functional of the total curvature. This will be important when we will compute the beta function. The generators in the fundamental representation are normalized as:

\[ \text{Tr}_f(T^a T^b) = \frac{1}{2} \delta_{ab} \]

\[ \sum_a (T^a)_f^2 = \frac{N^2 - 1}{2N} 1_f \]  \hspace{1cm} (29)

We change variables from the connection to the ASD curvature, introducing in the functional integral the appropriate resolution of identity:

\[ 1 = \int \delta(F_{\alpha\beta} - \mu_{\alpha\beta}) D\mu_{\alpha\beta} \]  \hspace{1cm} (30)

The partition function thus becomes:

\[ Z = \int \exp(\frac{-N8\pi^2}{g^2} Q - \frac{N}{4g^2} \sum_{\alpha \neq \beta} \int Tr(\mu^{-2}_{\alpha\beta}) d^4 x) \times \delta(F_{\alpha\beta} - \mu_{\alpha\beta}) D\mu_{\alpha\beta} DA \]  \hspace{1cm} (31)

We can write the partition function in the new form:

\[ Z = \int \exp(\frac{-N8\pi^2}{g^2} Q - \frac{N}{4g^2} \sum_{\alpha \neq \beta} \int Tr(\mu^{-2}_{\alpha\beta}) d^4 x) \times \text{Det }^{\frac{1}{2}}(-\Delta_A \delta_{\alpha\beta} + D_\alpha D_\beta + iad_{\mu_{\alpha\beta}}) D\mu_{\alpha\beta} \]  \hspace{1cm} (32)

where the integral over the gauge connection of the delta function has been now explicitly performed:

\[ \int DA_\alpha \delta(F_{\alpha\beta} - \mu_{\alpha\beta}) = | \text{Det }^{\frac{-1}{2}}(P^{-1} d_A \wedge)| \]

\[ = \text{Det }^{\frac{-1}{2}}((P^{-1} d_A \wedge)^* (P^{-1} d_A \wedge)) \]

\[ = \text{Det }^{\frac{-1}{2}}(-\Delta_A \delta_{\alpha\beta} + D_\alpha D_\beta + iad_{F_{\alpha\beta}}) \]  \hspace{1cm} (33)
and, by an abuse of notation, the connection $A$ in the determinants denotes the solution of the equation $F_{a\beta}^- - \mu_{a\beta}^- = 0$. The $'$ superscript requires projecting away from the determinants the zero modes due to gauge invariance, since gauge fixing is not yet implied, though it may be understood if we like to.

We refer to the determinant in the preceding equation as to the localization determinant, because it arises localizing the gauge connection on a given level, $\mu_{a\beta}^-$, of the ASD curvature. Let us notice the unusual spin term $i \text{ad}_{F_{a\beta}^-}$ as opposed to the one that arises in the background field method $2i \text{ad}_{F_{a\beta}}$. In the background field method the quadratic form:

$$\frac{1}{2g^2} \int d^4x Tr (F_{a\beta})^2$$

is expanded around a solution of the equation of motion. This produces the factor of two in the spin term. The localization determinant arises instead from the expansion of the quadratic form:

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int d^4x Tr (F_{a\beta}^- - \mu_{a\beta}^-)^2$$

that arises in a definition of the delta function around the background:

$$F_{a\beta}^- = \mu_{a\beta}^-$$

The corresponding shift in the curvature explains why the coefficient of the spin term is one half of the one in the background field method. The occurrence of $F_{a\beta}^-$ is due instead to the projector on the ASD part of the curvature.

After this change of variables the functional integral is defined on the plaquettes rather than on the links of a lattice, in a lattice regularization of the theory that will be needed later.

We need another change of variables to a holomorphic gauge, in order to write the loop equation in the ASD variables in a way convenient to our further developments. This introduces a holomorphic anomaly in the functional integral as a Jacobian. The choice of a holomorphic gauge is required for the following reason. We want to reduce the $MM$ loop equation,
that is obviously defined on loops, to a critical equation defined on points. There is a canonical way to associate to a loop a point, via the evaluation of a residue. The change of variables to the holomorphic gauge is meant to produce the Cauchy kernel in the loop equation, that in turn can be evaluated as a regularized residue in a proper regularization. In fact the very idea of localization in the loop equation has a holographic interpretation [1] in which to a loop (with a marked point), on which the one dimensional quantum theory on the bulk lives, it is associated a critical equation at the marked point, on which the zero dimensional localized theory on the boundary lives.

We describe the holomorphic gauge as follows. We can interpret the ASD relations:

\[ F_{\alpha\beta} - \mu_{\alpha\beta} = 0 \]  

(37)

as an equation for the curvature of the non-Hermitean connection \( B = A + D = (A_z + D_u)dz + (A_{\bar{z}} + D_{\bar{u}})d\bar{z} \) and a harmonic constraint for the Higgs field \( \Psi = -iD = -i(D_u dz + D_{\bar{u}} d\bar{z}) = \psi + \bar{\psi} \):

\[
\begin{align*}
F_B - \mu &= 0 \\
\bar{F}_B - \bar{\mu} &= 0 \\
d^* A \Psi - \nu &= 0
\end{align*}
\]  

(38)

that can also be written as:

\[
\begin{align*}
F_B - \mu &= 0 \\
\bar{\partial}_A \psi - n &= 0 \\
\partial_A \bar{\psi} - \bar{n} &= 0
\end{align*}
\]  

(39)

where the fields \( \mu, \nu, n \) are suitable linear combinations of the ASD components \( \mu_{\alpha\beta} \). The resolution of identity in the functional integral then reads:

\[
1 = \int \delta(F_B - \mu)\delta(\bar{\partial}_A \psi - n)\delta(\partial_A \bar{\psi} - \bar{n})D\mu Dn D\bar{n}
\]  

(40)

where the measure \( D\mu \) is interpreted in the sense of holomorphic matrix models [40], employed in the study of the chiral ring of \( N = 1 \) SUSY gauge theories [39].
This interpretation of the measure $D\mu$ seems to be needed to get the correct counting, in order to reproduce the perturbative beta function, of (complex) zero modes in the effective action. The holomorphic gauge is defined as the change of variables for the connection $B$, in which the curvature of $B$ is given by the field $\mu'$, obtained from the equation:

$$F_B - \mu = 0 \quad (41)$$

by means of a complexified gauge transformation $G(x; B)$ that puts $B = b + \bar{b}$ in the gauge $\bar{b} = 0$:

$$\bar{\partial} b_z = -i\mu' \quad (42)$$

where $\mu' = G\mu G^{-1}$ (the factor of $i$ occurs because Eq. (42) is written in complex coordinates).

Employing Eq. (40) as a resolution of identity in the functional integral, the partition function becomes:

$$Z = \int \delta(F_B - \mu)\delta(\bar{\partial}_A\psi - n)\delta(\partial_A\bar{\psi} - \bar{n}) \exp(-\frac{N}{2g^2}S_{YM})$$

$$\times \frac{D\mu}{D\mu'}Db\bar{b}D\mu'DnD\bar{n} \quad (43)$$

The integral over $b, \bar{b}$ is the same as the integral over the four $A_A$. The resulting functional determinants, together with the Jacobian of the change of variables to the holomorphic gauge, are absorbed into the definition of $\Gamma$.

$\Gamma$ plays here the role of a classical action, since it must be still integrated over the fields $\mu', n, \bar{n}$. We call $\Gamma$ the classical ASD action, as opposed to the quantum ASD effective action, $\Gamma_q$. $\Gamma$ is given by:

$$\Gamma = \frac{N8\pi^2}{g^2}Q + \frac{N}{g^2} \int Tr_f(\bar{F}_{01}^2 + \bar{F}_{02}^2 + \bar{F}_{03}^2)d^4x$$

$$+ \log Det^{-\frac{1}{2}}(-\Delta_A\delta_{\alpha\beta} + D\alpha D\beta + iad_{\mu_{\alpha\beta}}) - \log \frac{D\mu}{D\mu'} \quad (44)$$

with:

$$\mu^0 = F_{01}^-$$

$$n + \bar{n} = F_{02}^-$$

$$i(n - \bar{n}) = F_{03}^- \quad (45)$$
Although $\Gamma$ is the classical action in the ASD variables it contains already quantum corrections because of the Jacobian of the change of variables. It turns out that its divergent part coincides with the divergent part of the Wilsonian localized quantum effective action, after the inclusion of zero modes. Until now the theory is still four dimensional. Taking functional derivatives with respect to the ASD field we get, for a planar quasi BPS loop, the loop equation:

$$0 = \int D\mu' Tr \frac{\delta}{\delta \mu'(w,0)} (\exp(-\Gamma)\Psi(x,x;b))$$

$$= \int D\mu' \exp(-\Gamma) (Tr \frac{\delta \Gamma}{\delta \mu'(w,0)} \Psi(x,x;b))$$

$$- \int_{C(x,x)} dy \left[ \frac{1}{2} \delta^{(2)}(0) \bar{\partial}^{-1}(w-y) Tr (\lambda^a \Psi(x,y;b) \lambda^a \Psi(y,x;b)) \right]$$

$$= \int D\mu' \exp(-\Gamma) (Tr \frac{\delta \Gamma}{\delta \mu'(w,0)} \Psi(x,x;b))$$

$$- \int_{C(x,x)} dy \left[ \frac{1}{2} \delta^{(2)}(0) \bar{\partial}^{-1}(w-y) (Tr(\Psi(x,y;b)) Tr(\Psi(y,x;b))) \right]$$

$$- \frac{1}{N} Tr(\Psi(x,y;b) \Psi(y,x;b)))$$ (46)

where in our notation we have omitted the integrations $DnD\bar{n}$ since they are irrelevant in the loop equation because the curvature of $B$ depends only on $\mu$. In the large-$N$ limit it reduces to:

$$\tau(\frac{\delta \Gamma}{\delta \mu'(w,0)} \Psi(x,x;b)) =$$

$$\int_{C(x,x)} dy \left[ \frac{1}{2} \delta^{(2)}(0) \bar{\partial}^{-1}(w-y) \tau(\Psi(x,y;b)) \tau(\Psi(y,x;b)) \right]$$ (47)

The quadratically divergent factor $\delta^{(2)}(0)$ arises as follows. The functional derivative at a point produces a factor of $\delta^{(4)} = \delta^{(2)} \times \delta^{(2)}$ at that point because the fields are four dimensional. One factor of $\delta^{(2)}$ is convoluted with the kernel of $\bar{\partial}^{-1}$, while the other factor produces $\delta^{(2)}(0)$ since the loop is assumed to lie on a plane.

Now it is natural to perform a partial EK reduction from four to two dimensions. Let us describe what in fact the partial EK reduction means in
this context. We already observed that we can absorb the translations into a
gauge transformation in the four-dimensional non-commutative theory along
the two non-commutative directions [37], that are the one transverse to the
plane over which the loop lies. As a result the classical action looks two
dimensional, in the sense that the space-time dependence of the fields is
two dimensional, despite the fact that the theory is truly four dimensional.
The four-dimensional information is hidden in the central extension $H = \frac{1}{\theta}$
that shows up in the curvature of the connection because of the non-
commutativity of the partial derivatives in the non-commutative directions
[37]. However the loop equation, being gauge invariant, is unchanged by
this special gauge choice if the gauge is chosen after taking the functional
derivatives. This means that the quadratically divergent factor $\delta^{(2)}(0)$ occurs
in the four-dimensional loop equation even in a gauge in which the fields are
constant in the non-commutative directions.

If however we were to write the loop equation for the theory already
reduced to two dimensions we would miss the $\delta^{(2)}(0)$ divergent factor in
the right hand side, because we would get just one factor of $\delta^{(2)}$ by taking
functional derivatives, instead of the factor of $\delta^{(4)}$.

Now we use the fact that in the non-commutative theory the integral over
the non-commutative directions can be represented as a (colour) trace [37]:

$$\int d^2x_T = \frac{2\pi}{H} Tr$$

Hence the non-commutative classical action of the reduced theory gets a
volume factor of $V_2 = 2\pi\theta = \frac{2\pi}{H}$ because of the gauge choice. The equation
of motion of the reduced theory is therefore multiplied by this volume factor.
We can divide both sides of the loop equation by this volume factor in the
reduced theory in such a way that the equation of motion is normalized as
in the four-dimensional theory. Then the inverse volume will appear in the
right hand side instead of the factor $\delta^{(2)}(0)$. We can compensate this fact by
rescaling the classical action by a factor of $N_2^{-1}$ [32], with $N_2 = V_2\delta^{(2)}(0)$, in
such a way that the factor of $\frac{1}{N_2}$ in the reduced classical action produces the
factor of $\delta^{(2)}(0) = \frac{N_2}{V_2}$ once carried to the right hand side of the loop equation.
Of course in all this discussion we are implicitly assuming that the trace of
the reduced theory includes now the non-commutative degrees of freedom.
Thus the reduced classical action of the twisted $EK$ reduced theory in the $ASD$ variables is given by:

$$
\Gamma = \frac{N8\pi^2}{N_2g^2}Q + \frac{N}{g^2 N_2 H} \int Tr_f(F_{01}^{-2} + F_{02}^{-2} + F_{03}^{-2})d^2x \\
+ \log \text{Det}^{\nu^{-\frac{1}{2}}}(-\Delta_A \delta_{\alpha\beta} + D_\alpha D_\beta + i\lambda d_{\mu\alpha \beta}) - \log \frac{D\mu}{D\mu'}
$$

(49)

where the trace in the functional determinants has to be interpreted coherently with the partial $EK$ reduction.

The reduced loop equation is then:

$$
\tau\left(\frac{\delta \Gamma}{\delta \mu'(w)} \Psi(x,x;b)\right) = \\
\int_{C(x,x)} dy \frac{1}{2} \partial^{-1}(w-y)\tau(\Psi(x,y;b))\tau(\Psi(y,x;b))
$$

(50)

It is interesting to observe that the reducing $EK$ factor cancels exactly the quadratic divergence obtained evaluating the action on parabolic Higgs bundles, once it is assumed that the transverse and longitudinal cut-off are equal. But this is necessary to keep our $O(4)$ symmetry. The partial $EK$ reduction is not strictly needed for our arguments and for computing the beta function, but it is a convenient technical tool. It allows us to avoid overall infinite factors to appear in our formulae and by this very reason explains how apparently singular objects as the parabolic Higgs bundles survive in the large-$N$ limit of the functional integral. In this respect we should mention a different point of view between our treatment of parabolic Higgs bundles and the one in [16]. In fact in the mathematical literature we can think of parabolic bundles in two slightly different ways. We can think that they are defined on a compact surface with a divisor and a parabolic structure that belongs to the surface. This is our point of view. Or we can think that they arise as boundary conditions on a surface with boundary. This is the point of view in [16].

After the $EK$ reduction from four to two dimensions we perform a conformal compactification in such a way that the theory now is defined over a two-dimensional sphere. In the four-dimensional Euclidean theory this amounts to a compactification from $R^4$ to $S^2 \times S^2$. In the four-dimensional theory
in ultra-hyperbolic signature the conformal compactification is instead from the Minkowski space-time $M$ to $S^2 \times S^2 \mathbb{Z}_2$ [47]. This adds to the local part of the effective action at most a finite conformal anomaly, that can be ignored.

It is clear that the contour integration in the quantum term of the loop equation includes the pole of the Cauchy kernel. We need therefore a gauge invariant regularization. The natural choice consists in analytically continuing the loop equation from Euclidean to Minkowskian space-time. Thus $z \to i(x_+ + i\epsilon)$. It is at the heart of the modern Euclidean approach to quantum field theory that this analytic continuation be in fact possible. This regularization has the great virtue of being manifestly gauge invariant. In addition this regularization is not loop dependent.

The result of the $i\epsilon$ regularization of the Cauchy kernel is the sum of two distributions, the principal part plus a one-dimensional delta function (for simplicity we omit the underscript of $x_+$ in the following):

$$
\frac{1}{2} \bar{\delta}^{-1}(w_x - y_x + i\epsilon) = (2\pi)^{-1}(P(w_x - y_x)^{-1} - i\pi \delta(w_x - y_x))
$$

The loop equation thus regularized looks like:

$$
\tau \bigg( \frac{\delta \Gamma}{\delta \mu'(w)} \Psi(x,x;b) \bigg) = 
\int_{C(x,x)} dy_x (2\pi)^{-1}(P(w_x - y_x)^{-1} - i\pi \delta(w_x - y_x)) 
\times \tau(\Psi(x,y;b)) \tau(\Psi(y,x;b))
$$

The right hand side of the loop equation contains now two contributions. A delta-like one dimensional contact term, that is supported on closed loops and a principal part distribution that is supported on open loops. Since by gauge invariance it is consistent to assume that the expectation value of open loops vanishes, the principal part does not contribute and the loop equation reduces to:

$$
\tau \bigg( \frac{\delta \Gamma}{\delta \mu'(w)} \Psi(x,x;b) \bigg) =
\int_{C(x,x)} dy_x \frac{i}{2} \delta(w_x - y_x) \tau(\Psi(x,y;b)) \tau(\Psi(y,x;b))
$$

Taking \( w = x \) and using the transformation properties of the holonomy of \( b \) and of \( \mu(x)' \), the preceding equation can be rewritten in terms of the connection, \( B \), and the curvature, \( \mu \):

\[
\tau \left( \frac{\delta \Gamma}{\delta \mu(x)} \Psi(x, x; B) \right) = \\
\int_{C(x, x)} \! dy_{x} \frac{i}{2} \delta(x_{x} - y_{x}) \tau(\Psi(x, y; B)) \tau(\Psi(y, x; B)) \tag{54}
\]

where we have used the condition that the trace of open loops vanishes to substitute the \( b \) holonomy with the \( B \) holonomy.

Our argument about the vanishing of the contribution of the principal part becomes tricky in the non-exactly gauge invariant regularization of the loop equation that was considered in [6]. Indeed that regularization allows the contribution of quasi-closed loops, for which of course the principal part contributes too. Yet we will see that our final argument about reducing the loop equation to a saddle-point on the weighted graph is not affected, not even if the contribution of the principal part is allowed using the regularization in [6]. Indeed the zig-zag symmetry works for both the contact term and the principal part.

We need a lattice version of the continuum loop equation to implement our localization argument. Thus we write the loop equation in the ASD variables on a lattice in the partially EK reduced theory. On a dense set in the functional integral (in the sense of distributions), the equations:

\[
F_{z\bar{z}} - [D_{u}, D_{\bar{u}}] = i \left( \sum_{p} \mu_{p}^{0} \delta^{(2)}(x - x_{p}) - H1 \right) \\
\bar{\partial}_{A}(D_{u}) = i \sum_{p} n_{p} \delta^{(2)}(x - x_{p}) \\
\partial_{A}(D_{\bar{u}}) = i \sum_{p} \bar{n}_{p} \delta^{(2)}(x - x_{p}) \tag{55}
\]

define an infinite-dimensional twisted local system or, what is the same, a twisted parabolic Higgs bundle on a sphere. The curvature equation involves a central term, \( H = \frac{1}{7} \), that we can display explicitly since \([D_{u}, D_{\bar{u}}] = F_{u\bar{u}} + iH1\). Correspondingly, given that the gauge connection has to vanish at infinity also the right hand side has been shifted by \( H1 \) by deforming the
resolution of identity on the ASD part of the curvature in the functional integral, as follows by non-commutativity. $H$ vanishes in the large-$N$ limit, that coincides with the limit of infinite non-commutativity. In the case $n = \bar{n} = 0$, that will be the most relevant for us, we may interpret the preceding equations as vortices equations. In the partially $EK$ reduced theory the vortices live at the lattice points where the ASD curvature is singular. This means that in the original four-dimensional theory they form two-dimensional vortices sheets [15, 16]. The loop equation on our lattice now reads:

$$
\tau \left( \frac{\delta \Gamma}{\delta \mu(x_p)} \Psi(x_p, x_p; B) \right) = 
\int_{C(x_p, x_p)} dy_x \frac{1}{2} \delta^{-1}(x_p - y) \tau(\Psi(x_p, y; B)) \tau(\Psi(y, x_p; B)) 
$$

(56)

and correspondingly for the analytic continuation to Minkowskian spacetime:

$$
\tau \left( \frac{\delta \Gamma}{\delta \mu'(w_x)} \Psi(x_q, x_q; B) \right) = 
\int_{C(x_q, x_q)} dy_x (2\pi)^{-1} (P(w_x - y)^{-1} - i\pi \delta(w_x - y_x)) 
\times \tau(\Psi(x_q, y; B)) \tau(\Psi(y, x_q; B)) 
$$

(57)

Notice that a smooth marked point gives a non-trivial contribution to the right hand side of the loop equation in the ASD variables either in the continuum or on the lattice, that is finite for the contact term and logarithmically divergent for the principal part (in the regularization in [6]). However around a backtracking cusp the contributions of the two sides of the asymptotes to the cusp cancel each other for the contact term:

$$
\int_{C(x_q, x_q)} dy_x(s) \delta(w_{x_q}(s_{\text{cusp}}) - y_x(s)) = \frac{1}{2} \left| \frac{\dot{w}_{x_q}(s_{\text{cusp}})}{\dot{w}_{x_q}(s_{\text{cusp}})} \right| + \left| \frac{\dot{w}_{x_q}(s_{\text{cusp}})}{\dot{w}_{x_q}(s_{\text{cusp}})} \right| 
$$

(58)

because of the opposite sign of $\dot{w}_{x_q}(s_{\text{cusp}}^+)$ and $\dot{w}_{x_q}(s_{\text{cusp}}^-)$ on the two sides of the backtracking cusp. For the principal part the same argument applies because of the opposite orientations of the asymptotes and because both the cusp asymptotes are approached either from below or from above:

$$
| \int_{C(x_q, x_q)} dy_x(s) P(w_{x_q}(s_{\text{cusp}}) - y_x(s))^{-1} | = 
$$
Thus if every marked point of the loop can be transformed into a backtracking cusp we can complete our argument about localization. But this is precisely the effect of our lattice, since marked points of the loop contribute to the loop equation in the lattice theory only if they coincide with the lattice points. Thus we can simply draw our backtracking strings from the loop to the lattice points in order to transform all the marked points into cusps.

Hence we may say that open strings solve the $YM$ loop equation for the quasi $BPS$ Wilson loops, in the sense that they localize the loop equation on a saddle point for an effective action.

We now show that the cusps must be paired by the backtracking strings. Independently on the string gluing axioms we can understand directly from the gluing properties of the functional integral why it must be so.

We can represent the partition function on a sphere by gluing the two cups with the Wilson loop and the marked point on the loop in common. To apply our vanishing result for the contact term to the loop equation on each cup we must connect the same marked point on each cup with a backtracking string ending into a cusp. If we now glue the two cups, the cusps come in pairs on the glued surface and are linked by a string transverse to the loop that intersects the loop at the marked point. These are precisely the string gluing axioms for arc families at topological level [12]. Thus we come to the conclusion that if we combine the gluing properties of the functional integral with the vanishing requirement for the contact term, i.e. the requirement of localization, we get the string gluing axioms at topological level. A subtle point arises as follows. When we add a backtracking arc to a marked point we create a loop self-intersection that may contribute extra terms to the loop equation. However the self-intersection may be regularized adding instead tiny strips having a common asymptote to the two cusps. In this case the loop remains simple and the cusps are still pairwise identified [12]. Thus the proper graph to get localization is a weighted graph [12] whose spine is a Mandelstam graph. The weights in our language are the sizes of the strips.

One more comment is in order. It is one of the cornerstones of the Euclidean field theory that the analytic continuation to Minkowskian space-time

\[
\frac{1}{2} \int ds^+ \frac{\dot{y}_x(s^+)}{|w_{x}(s_{\text{cusp}}) - y_x(s^+)|} + \int ds^- \frac{\dot{y}_x(s^-)}{|w_{x}(s_{\text{cusp}}) - y_x(s^-)|}
\]

Thus if every marked point of the loop can be transformed into a backtracking cusp we can complete our argument about localization. But this is precisely the effect of our lattice, since marked points of the loop contribute to the loop equation in the lattice theory only if they coincide with the lattice points. Thus we can simply draw our backtracking strings from the loop to the lattice points in order to transform all the marked points into cusps.

Hence we may say that open strings solve the $YM$ loop equation for the quasi $BPS$ Wilson loops, in the sense that they localize the loop equation on a saddle point for an effective action.

We now show that the cusps must be paired by the backtracking strings. Independently on the string gluing axioms we can understand directly from the gluing properties of the functional integral why it must be so.

We can represent the partition function on a sphere by gluing the two cups with the Wilson loop and the marked point on the loop in common. To apply our vanishing result for the contact term to the loop equation on each cup we must connect the same marked point on each cup with a backtracking string ending into a cusp. If we now glue the two cups, the cusps come in pairs on the glued surface and are linked by a string transverse to the loop that intersects the loop at the marked point. These are precisely the string gluing axioms for arc families at topological level [12]. Thus we come to the conclusion that if we combine the gluing properties of the functional integral with the vanishing requirement for the contact term, i.e. the requirement of localization, we get the string gluing axioms at topological level. A subtle point arises as follows. When we add a backtracking arc to a marked point we create a loop self-intersection that may contribute extra terms to the loop equation. However the self-intersection may be regularized adding instead tiny strips having a common asymptote to the two cusps. In this case the loop remains simple and the cusps are still pairwise identified [12]. Thus the proper graph to get localization is a weighted graph [12] whose spine is a Mandelstam graph. The weights in our language are the sizes of the strips.

One more comment is in order. It is one of the cornerstones of the Euclidean field theory that the analytic continuation to Minkowskian space-time
can always be performed. In fact if we want to keep the ASD structure real we must continue to ultra-hyperbolic signature.

If we perform the compactification to $S^2 \times S^2$ in Euclidean signature we get the double cover of the conformal compactification in ultra-hyperbolic signature [47]. Thus if we require that the Euclidean equation of ASD type be continued to ultra-hyperbolic signature we must take into account this global constraint.

We are now ready to write the quantum effective action for our localized version of the loop equation. A subtle point arises about the cut-off of this effective action. We recall that the introduction of a lattice is essential for localization, since it allows us to transform every non-trivial marked lattice point into a backtracking cusp. Now, because all lattice points are pairwise linked by strings and the strings are transverse to the loop, there are the same number of lattice points inside and outside the loop.

Hence we need different cut-off scales to fix the two different areas of the cups in which the loop divides the sphere. Thus the topological axioms imply different cut-offs at the cusps of the two cups of the sphere in the quantum effective action.

In fact an equal cut-off would lead to overcounting in the normalization of the classical action, that has to be the same as the one for a unique marked point, up to terms vanishing with the ultraviolet cut-off. Hence the local part of the effective action, as a consequence of the stringy nature of localization in the loop equation, is in fact bi-local with two local fields (differing in fact only by a gauge transformation not connected to the identity since they are originally associated to the same marked point by gluing) living at different scales, one at the ultraviolet cut-off and one at the infrared cut-off. The field at the ultraviolet, but not the one at the infrared, affects the renormalization of the Wilsonian coupling constant, as it is expected from its very definition. Instead the field at the infrared together with the one at the ultraviolet affects the renormalization of the canonical coupling constant.

Finally we draw our weighted graph, that is made by two charts with the boundary loop in common. The charts are a conformal transformation of two topological disks with marked points. This introduces a conformal transformation in the $E_{K}$ two-dimensional reduced theory [12, 13, 14]. However this
transformation is four dimensional in the original theory because of the BPS constraint extended to a neighborhood of the marked points:

\[
dz = du \\
d\bar{z} = d\bar{u}
\]

Thus the four-dimensional metric changes conformally and the effective action changes by the appropriate conformal anomaly. This means that, in addition to the explicit cut-off dependence, the effective action on the weighted graph is related to the one on the sphere with marked points by the addition of a divergent conformal anomaly, because of the singularity of the conformal transformation. This divergent conformal anomaly plays a key role in our computation of the contribution of the anomalous dimension in the canonical beta function and more generally in the interpretation a posteriori of localization as a RG flow to the ultraviolet.

The loop equation on the weighted graph reduces to the localized form:

\[
0 = \tau(\frac{\delta \Gamma_q}{\delta \mu(x_p)} \Psi(x_p, x_p; B) )
\]

which we refer to as the master equation.

4 Effective action and exact beta function

In this section we compute the local divergent part of the quantum effective action for the purpose of obtaining the beta function. While the computation and the results are already essentially contained in [1], the calculations here are considerably simplified and several unnecessary constraints are removed.

In particular we compute the beta function in the sector of the YM theory in which the second parabolic Chern class, $Q$, is negligible with respect to the quadratic divergence induced by a non-trivial parabolic divisor in the four-dimensional theory. This is equivalent to require that the reduced topological term, $\frac{Q}{N_2}$, vanishes when $N_2 \to \infty$, the reduced $EK$ action being finite on the parabolic divisor. In [1] it was assumed instead the stronger constraint that the parabolic Chern class vanishes.
Moreover in [1] it was assumed that the structure of the fibration of parabolic Higgs bundles in the four-dimensional theory implies, as a consequence of the vanishing of the first and second parabolic Chern classes, the value \( \frac{2kN(2\pi)^2}{g^2} \) for the reduced \( EK \) classical action at leading \( \frac{1}{N} \) order, that corresponds to a superposition of \( k \) \( Z_N \) vortices and \( k \) anti-vortices of lowest vortex number. The anti-vortices occur in [1] because a Wilson loop in the adjoint representation was considered there, that factorizes in the large-\( N \) limit into the product of Wilson loops in the fundamental and in the conjugate representation.

Also this constraint is removed, since no four-dimensional structure of the parabolic fibration survives the \( EK \) reduction, because of the vanishing of \( QN^2 \). Yet, in this section, the correct normalization of the \( EK \) classical action, that reproduces the universal coefficients of the perturbative the beta function, is obtained automatically for vortices of any vortex number. Finally we find that our construction works directly for a quasi \( BPS \) Wilson loop in the fundamental representation without introducing the adjoint representation, unless we wish to do so.

The quantum effective action for a quasi \( BPS \) Wilson loop in the fundamental representation has the following structure:

\[
\exp(-\Gamma_q) = \int d(zero - modes) \exp(-\Gamma + ConformalAnomaly) \quad (62)
\]

In the evaluation of the local divergent part of \( \Gamma_q \) the global features of the weighted graph over which \( \Gamma_q \) is defined are irrelevant, since the divergences of \( \Gamma \), the conformal anomaly and even the zero modes are locally defined, as we will see momentarily. \( \Gamma \) is the classical effective action in the \( ASD \) variables, that includes the non-zero modes of the Jacobian of the change of variables. The local part of \( \Gamma \) differs by the quantum effective action on the weighted graph by a conformal anomaly, since the weighted graph is a conformal image of the sphere with the marked divisor over which \( \Gamma \) is defined.

Computationally the conformal anomaly plays an important role later, in the calculation of the beta function for the canonical coupling, but for the moment can be ignored. \( \Gamma_q \) may contain the contribution of zero modes that are not included in \( \Gamma \). \( \Gamma \) contains a globally defined part that arises
by the topological term in the classical $YM$ action. This topological term never contributes to the quantum effective action in every sector in which it is finite, since in the reduced theory it is divided by the divergent factor $N_2$. This observation makes irrelevant the whole discussion made in [1] of the four-dimensional constraints that have to be satisfied in order to make the parabolic Chern number vanishing in a way compatible with localization on vortices.

There is a precise relation between $N_2$ and the quadratic divergence at the parabolic points dictated by the requirement that the cut-off in the longitudinal $z$ plane and in the transverse $u$ plane be equal. In fact the $EK$ reduction requires the relation $N_2 = \frac{H}{2\pi}(\frac{\Lambda_T}{2\pi})^2$ where $\frac{\Lambda_T}{2\pi} = \frac{1}{a_T}$ is the ultraviolet cut-off in the transverse directions of the $EK$ partial reduction. But it must be $\Lambda_T = \Lambda_L$ to keep the O(4) symmetry of the large-$N$ theory, that in turn ensures the non-renormalization properties of our quasi $BPS$ Wilson loop.

In the reduced $EK$ theory $\Gamma$ is defined on the two-dimensional sphere with marked points, before mapping conformally to the weighted graph. Thus the lattice divisor that is the support of the local part of $\Gamma$ is not uniform in general, since this lattice has the same number of points in the interior and in the exterior of the Wilson loop. Indeed the weighted graph must have the same number of cusps in the interior and in the exterior of the Wilson loop. As a consequence the lattice cut-off differs for each pair of marked points (cusps on the weighted graph) linked by a backtracking string. One of these points has a cut-off in the ultraviolet and we call the set of all such points the ultraviolet divisor. The other one has a cut-off in the infrared and we call the set of all such points the infrared divisor.

Though $\Gamma$ is the classical action in the $ASD$ variables, $\Gamma$ is not finite because generically the non-zero modes of the Jacobian determinant of the change of variables plus the gauge-fixing ghost determinant introduce already some divergences. These divergences can be computed exactly, being only one loop. They correct the classical $YM$ action in $\Gamma$ by a $Z^{-1}$ factor that is explicitly displayed below:

$$
\left( \frac{N}{2g_W^2} - (2 - \frac{1}{3})\frac{N}{(4\pi)^2} \log\left( \frac{\Lambda}{\Lambda} \right) \right) \sum_{\alpha \neq \beta} \int d^4x Tr_f(2(\frac{1}{2}F_{\alpha\beta})^2)
$$
\[
\begin{align*}
&= \left( \frac{N}{2g_W^2} - \frac{5}{3} \frac{N}{(4\pi)^2} \log\left( \frac{\Lambda}{\Lambda} \right) \right) \sum_{\alpha \neq \beta} \int d^4x Tr_f\left( 2\left( \frac{1}{2} F_{\alpha\beta} \right)^2 \right) \\
&= \frac{N}{2g_W^2} Z^{-1} \sum_{\alpha \neq \beta} \int d^4x Tr_f\left( 2\left( \frac{1}{2} F_{\alpha\beta} \right)^2 \right) \\
\end{align*}
\]

(63)

where \( Z^{-1} \) is given by:

\[
Z^{-1} = 1 - \frac{10}{3} \frac{1}{(4\pi)^2} g_W^2 \log\left( \frac{\Lambda}{\Lambda} \right)
\]

(64)

and we have added to \( g \) the under-script \( W \) to stress that our computation here refers to the Wilsean coupling constant. This formula for \( Z \) is actually exact to all orders in the Wilsonian coupling constant, up to finite terms. The divergent contribution in \( \Gamma \) is due entirely to the non-zero modes of the localization determinant and of the ghost determinant. The contribution of the holomorphic anomaly in \( \Gamma \) vanishes in any gauge in which \( \mu \) can be triangularized. Indeed we can reach this gauge either in a unitary basis or in a holomorphic basis. The unitary and the holomorphic basis induce the same Vandermonde determinant of the eigenvalues of \( \mu \) in the functional measure and thus the holomorphic anomaly vanishes in this gauge.

\( \Gamma \) does not reproduce the perturbative one-loop beta function. This is due to the peculiar spin term that occurs in the localization determinant as opposed to the spin term of the gluons determinant in the background field calculation. The \( Z^{-1} \) factor is responsible in the pure \( YM \) case of the occurrence of an anomalous dimension in the formula for the canonical beta function as opposed to the \( \mathcal{N} = 1 \) SUSY \( YM \) case. The \( Z^{-1} \) factor is not present in the \( \mathcal{N} = 1 \) SUSY \( YM \) theory [29] because in that case the existence of the Nicolai map ensures the cancellation of the non-zero modes of the determinant of the change of variables versus the gluinos determinant in the light-cone gauge [27, 28].

Generically there are no normalizable zero modes because the gauge connection and the Higgs field are both singular at the parabolic points. The resulting pairing in the symplectic volume form in function space:

\[
\int d^4x Tr(\delta A \wedge \delta A)
\]

(65)

is divergent. Thus generically in function space and in absence of normalizable zero modes a less negative beta function than the correct one is obtained.
However there is a special locus of the Higgs field for which normalizable zero modes exist. This locus corresponds to zeros of the Higgs field, i.e. to vortices equations. The following volume form, \( \omega \wedge \bar{\omega} \), is then finite, because the singularity of the gauge connection is compensated by the zero of the Higgs field:

\[
\omega = \int d^4x Tr(\delta A_z \wedge \delta D_u) = \frac{2\pi}{H} \int d^2x Tr(\delta A_z \wedge \delta D_u) \tag{66}
\]

Since the Higgs field is smooth, for vortices the Hitchin equations of \( ASD \) type \([41, 42, 43, 44, 45, 46]\) reduce to the case \( n_p = 0 \). Correspondingly the eigenvalues of the \( ASD \) curvature are quantized in such a way that the local holonomy carries a \( Z_N \) charge in the fundamental representation. This quantization follows by the existence of the zeros of the Higgs field. At the same time the \( Z_N \) local holonomy fixes the normalization of the eigenvalues of the curvature (up to a shift by large gauge transformations with trivial local holonomy) and thus the value of the action (up to these shifts, see below) without any extra condition. For a \( Z_N \) vortex of charge \( k \) in a \( SU(N) \) orbit we get \( N - k \) eigenvalues of the curvature equal to \( \frac{2\pi k}{N} \) and \( k \) eigenvalues equal to \( \frac{2\pi(k - N)}{N} \). The trace of the eigenvalues of the \( ASD \) curvature in the fundamental representation is thus:

\[
(N - k)\left(\frac{2\pi k}{N}\right)^2 + k\left(\frac{2\pi(k - N)}{N}\right)^2 = (2\pi)^2 \frac{k(N - k)}{N} \tag{67}
\]

In addition each \( Z_N \) vortex carries a number of zero modes of the localization determinant equal to the dimension of the adjoint orbit \( g\lambda g^{-1} \). We do not include zero modes associated to translations of the vortices since their contribution is sub-leading in \( \frac{1}{N} \). The complex dimension of an adjoint orbit for a generic parabolic bundle of rank \( N \) is given by:

\[
dim = \frac{1}{2}(N^2 - \sum_i m_i^2) \tag{68}
\]

where \( m_i \) are the multiplicities of the eigenvalues. For vortices this reduces to:

\[
dim = \frac{1}{2}(N^2 - k^2 - (N - k)^2) = k(N - k) \tag{69}
\]
The classical EK reduced action at one point of the ultraviolet divisor on vortices reads:

\[ \frac{N}{g_W^2} (2\pi)^2 \frac{k(N-k)}{2g_W^2} = \frac{8\pi^2}{2g_W^2} k(N-k) \]  

(70)

while at one point of the infrared divisor reads:

\[ \frac{a^2}{\tilde{a}^2} \frac{8\pi^2}{2g_W^2} (k(N-k) + N^2n^2) \]  

(71)

where the shift \( n \) represents the contribution to the eigenvalues of the curvature of a central large gauge transformation that does not affect the dimension of the vortex adjoint orbit. Notice that the action at the infrared divisor is suppressed by a power of the ultraviolet cut-off. We recall that the different cut-off scales arise because there are the same number of cusps internal and external to the Wilson loop and thus the different cut-off scales are the only way to measure different internal and external areas. In particular the external area has to go to infinity in the thermodynamic limit. The existence of two scales seems also necessary in non-perturbative definitions of the renormalization procedure [48]. The local action at the ultraviolet on vortices is renormalized by the \( Z^{-1} \) factor and now also by the vortices zero modes. Thus we get for the local part of \( \Gamma_q \) at a vortex of charge \( k \):

\[
\exp(-\Gamma_q(\text{one - vortex})) = \exp\left(-\frac{8\pi^2}{2g_W^2} k(N-k)(Ha^2)^{-\frac{k(N-k)}{2}}\right) \\
\exp\left(-\frac{a^2}{\tilde{a}^2} \frac{8\pi^2}{2g_W^2} (k(N-k) + N^2n^2)(H\tilde{a}^2)^{-\frac{k(N-k)}{2}}\right)
\]  

(72)

We omit the contribution of the Vandermonde determinant of the eigenvalues because it is finite. In addition we do not include it in the canonical effective action as we did in [1] since the canonical normalized action in this paper is not defined by the rescaling of the eigenvalues of the ASD curvature as in [1].

Notice that we have not included a \( Z^{-1} \) factor in the contribution to \( \Gamma \) at the infrared because there it is finite and it can be set equal to 1 by a convenient choice of the subtraction point. The factor of \( \tilde{H} \) in front of \( a^2 \) arises from the normalization of the symplectic volume form in the non-commutative theory. There is an analogous contribution at the infrared, that
is however finite. $a^{-1}$ is the Pauli-Villars regulator of vortices zero modes at
the ultraviolet [49]. The exponent of the Pauli-Villars regulator counts the
number of complex vortices zero modes rather than the number of real zero
modes according to our interpretation of the $D_{\mu}$ integral as a holomorphic
integral. We can avoid the holomorphic counting for the vortices moduli
space by noticing that after continuation to ultra-hyperbolic signature the
vortices equations are defined on the double cover of $S^2 \times S^2 \mathbb{Z}_2$ [47] and that we
can use this doubling to define a real pairing of the complex moduli at each
location of the vortices.

Alternatively we can consider a Wilson loop in the adjoint representation.
Then we get a factorized contribution from the fundamental and conjugate
representation in the large-$N$ limit. This implies vortices and anti-vortices in
the effective action and a real counting of zero modes by pairing the vortices
with the anti-vortices, as opposed to the holomorphic counting. Of course
the beta function is unchanged because everything gets doubled.

The renormalization of the Wilsonian coupling constant now follows im-
mediately from the effective action at the ultraviolet:

$$\frac{8\pi^2 k(N-k)}{2g_W^2(\tilde{a})} = \frac{8\pi^2 k(N-k)(1}{2g_W^2(a)} - \frac{1}{(4\pi)^2(2 + \frac{5}{3})\log(\frac{\tilde{a}}{a})}$$ (73)

We should notice that without doing the $EK$ reduction we would obtain the
same beta function once it is observed that the action and the number of
zero modes would have been multiplied by the common factor of $N_2$.

It is interesting to write the effective action before the $EK$ reduction since
it will be useful in the computation of the beta function for the canonical
coupling. The unreduced effective action reads:

$$\exp(-\Gamma_q(one \ - \ vortex)) = \exp(-(\frac{2\pi}{Ha^2} \frac{8\pi^2}{2g_W^2}(\frac{1}{(4\pi)^2(2 + \frac{5}{3})\log(\frac{a}{\tilde{a}})}$$

We should notice that without doing the $EK$ reduction we would obtain the
same beta function once it is observed that the action and the number of
zero modes would have been multiplied by the common factor of $N_2$.

It is interesting to write the effective action before the $EK$ reduction since
it will be useful in the computation of the beta function for the canonical
coupling. The unreduced effective action reads:

$$\exp(-\Gamma_q(one \ - \ vortex)) = \exp(-(\frac{2\pi}{Ha^2} \frac{8\pi^2}{2g_W^2}(\frac{1}{(4\pi)^2(2 + \frac{5}{3})\log(\frac{a}{\tilde{a}}}))$$ (74)
where the factor of $\frac{2\pi}{Ha_\tau}$ is the transverse measure over the zero modes two-dimensional sheet. Indeed, since in the $EK$ partially reduced theory the vortices live on points, in the four-dimensional theory they live on two-dimensional sheets. Because of rotational invariance we must have $a = a_T$ and thus the formula for the reduced action follows. Another way of formulating this condition is that the longitudinal measure on the size of vortices that we read from the classical action and the transverse measure on the vortices two-dimensional sheet must coincide.

If a quasi $BPS$ Wilson loop in the adjoint representation is considered as in [1] vortices and anti-vortices contribute to the effective action and it is possible to pair the holomorphic integral to the anti-holomorphic one. In this case the counting of zero modes corresponds to the real dimension of the orbit. For completeness we write the local part of the effective action for a Wilson loop in the adjoint representation in the notation of [1]:

$$\exp(-\Gamma_q) = \prod_p \sum_{k_p, e_p} \exp\left(-\frac{2\pi}{Ha^2} \frac{8\pi^2}{2g_W^2} Z^{-1} k_p (N - k_p) + c.c.\right)$$

$$\exp\left(-2\pi \frac{8\pi^2}{Ha^2} \frac{2g_W^2}{2g_W^2} (k_p (N - k_p) + N^2 e_p^2) + c.c.\right)$$

$$\exp\left(2\pi \frac{k_p (N - k_p)}{Ha_\tau^2} \log\left(\frac{1}{Ha^2}\right) + c.c.\right)$$

$$(75)$$

where $e_p$ denotes the shift of the local curvature by a central large gauge transformation valued in $Z_N$.

The analogous expression for a Wilson loop in the fundamental representation is:

$$\exp(-\Gamma_q) = \prod_p \sum_{k_p, n_p} \exp\left(-\frac{2\pi}{Ha^2} \frac{8\pi^2}{2g_W^2} Z^{-1} k_p (N - k_p) + c.c.\right)$$

$$\exp\left(-2\pi \frac{8\pi^2}{Ha^2} \frac{2g_W^2}{2g_W^2} (k_p (N - k_p) + N^2 n_p^2) + c.c.\right)$$

$$\exp\left(2\pi \frac{k_p (N - k_p)}{Ha_\tau^2} \log\left(\frac{1}{Ha^2}\right) + c.c.\right)$$

$$41$$
exp \left( \frac{2\pi k_p (N - k_p)}{H a_f^2} \log \left( \frac{1}{H a_f^2} \right) \right) \tag{76}

We now come to the canonical coupling. We start recalling how the difference between the Wilsonian and the canonical beta function can be understood in terms of a rescaling anomaly in the functional integral, in the $\mathcal{N} = 1$ supersymmetric case, following [23].

The canonical coupling constant can be related to the Wilsonian one taking into account an anomalous Jacobian that occurs in the functional integral:

\[
Z = \int \exp \left( - \frac{N}{2g_W^2} S_{YM}(A) \right) DA
= \int \exp \left( - \frac{N}{2g_W^2} S_{YM}(g_c A_c) \right) \frac{D(g_c A_c)}{D A_c} D A_c
= \int \exp \left( - \frac{N}{2g_c^2} S_{YM}(g_c A_c) + \log \frac{D(g_c A_c)}{D A_c} \right) D A_c
= \int \exp \left( - \frac{N}{2g_c^2} S_{YM}(g_c A_c) \right) D A_c \tag{77}
\]

From this relation it follows that:

\[
\frac{N}{2g_c^2} = \frac{N}{2g_W^2} - S_{YM}^{-1}(g_c A_c) \log \frac{D(g_c A_c)}{D A_c} \tag{78}
\]

We observe that the fields at the ultraviolet and at the infrared are not canonically normalized in the reduced $E K$ effective action. It is however difficult to canonically normalize the reduced action because the only possibility would be to rescale the vortices eigenvalues. However they are strongly rigid since they are quantized and thus we have not this freedom. A way out is to use the unreduced effective action. In the unreduced effective action we can canonically normalize the fields at the ultraviolet and at the infrared by writing $H = \frac{Z_c^{-1}}{g_c} H_c$ and $\tilde{a}^2 = Z \tilde{a}^2_c$ and by taking fixed these canonically defined scales. However because of rotational invariance also the canonically defined transverse cut-off must be taken fixed $a_T^2 = Z a_T^2 c$. Finally, we consistently set $a = a_T c$, since the longitudinal measure on the size of vortices that we read from the classical canonical action and the transverse measure on the vortices two-dimensional sheet must coincide as in the Wilsonian case. This
way of defining the canonical transverse cut-off $a_{Tc}$ implies that the rescaled transverse measure does not depend on the rescaling factor being $g_c^2$ or $g_c^2Z$, as it should be.

After a new $EK$ reduction in which this time $N_2 = \frac{2\pi}{H \cdot \alpha}$ and the unreduced action is divided by $N_2$ we get the following relation between the Wilsonian and the canonical coupling:

$$\frac{1}{2g_W^2} = \frac{1}{2g_c^2} + \beta_j \log g_c + \frac{\beta_f}{4} \log Z$$

(79)

Differentiating this relation we get for the canonical beta function [1]:

$$\frac{\partial g_c}{\partial \log \Lambda} = -\beta_0 g_c^3 + \frac{\beta_f}{4} g_c^3 \frac{\partial \log Z}{\partial \log \Lambda}$$

(80)

Now we come to the computation of the anomalous dimension. Since $Z$ is one-loop exact it leads to the anomalous dimension:

$$\frac{\partial \log Z}{\partial \log a} = -\frac{\frac{1}{(4\pi)^2} \frac{10}{3} g_W^2}{1 - g_c^2 \frac{\frac{10}{(4\pi)^2} \frac{10}{3} \log \left( \frac{\alpha}{\sqrt{N_D a}} \right)}{1 - \frac{\beta_f}{4} g_c^2}}$$

(81)

where now we have included the contribution of the conformal anomaly, that combines with the subtraction point to give a finite but arbitrary result for the higher order contributions to the anomalous dimension. Thus the $RG$ trajectory must be followed along the line $c = \frac{1}{(4\pi)^2} \frac{10}{3} \log \left( \frac{\alpha}{\sqrt{N_D a}} \right) = constant$. We observe that it is precisely the contribution of the conformal anomaly that makes the anomalous dimension a function of the Wilsonian coupling only, according with the $RG$. We conclude that the loop equation and the beta function for the quasi $BPS$ Wilson loops are saturated by the $Z_N$ vortices of the partial $EK$ reduction:

$$F_{zz} = [D_u, D_{\bar{u}}] = i\sum_p g_p \lambda_p g_p^{-1} \delta^{(2)}(z - z_p) - H1$$

$$\partial_A(D_u) = 0$$

$$\partial_A(D_{\bar{u}}) = 0$$

(82)
5 Physical effective charge

We exploit the residual scheme dependence left in our approach to find a link with the physical charge between static quark sources in the large-$N$ limit. The scheme dependence arises because the conformal anomaly in the Wilsonian effective action is fine-tuned along the $RG$ flow, in order to obtain a finite result for the higher order contributions to the anomalous dimension. These contributions are then finite but contain an arbitrary dimensionless parameter $c$. In turn the $c$ dependence of the anomalous dimension determines the $RG$ flow in the infrared of the canonical coupling.

We argue that there should exist in our approach a scheme in which the canonical coupling coincides with the physical effective charge, essentially because of the very definition of our canonical coupling via the loop equation. By physical effective charge we mean the coefficient of the Coulomb-like potential in the complete inter-quark potential, after having separated the linear confining term proportional to the string tension, by taking second derivatives of the potential as in [35]:

$$V(r) = -\frac{g_{\text{phys}}^2(r)}{4\pi r} + Kr$$  \hspace{1cm} (83)

where $K$ is the string tension. With this definition the physical charge may still possess, from a stringy point of view, non-perturbative string-like contributions [50] proportional, in the large distance limit, to inverse powers of $\sqrt{Kr}$. In fact it is known from the string model for the large distance inter-quark potential [50] that the universal infrared asymptotic value of $\frac{g_{\text{phys}}^2}{4\pi}$, equal to $\frac{\pi}{12}$, might get non-universal corrections in powers of $\sqrt{Kr}$. These corrections in principle could start at $\frac{1}{\sqrt{Kr}}$ order but the analysis in [50] shows that, assuming open/closed string duality, they in fact could start only at order of $\frac{1}{(\sqrt{Kr})^2}$. Another important point is that the beta function of the physical charge thus defined may possess an infrared fixed point without implying that the large-$N$ $YM$ theory be conformal in the infrared, because the conformal invariance is always broken by the confining linear term in the inter-quark potential, that shows up in physical Wilson loops.

A direct gauge theoretic computation of the physical charge needs the
evaluation of physical Wilson loops. In our approach only quasi BPS Wilson loops can be computed directly. However there are physical Wilson loops that can be obtained by analytic continuation from quasi BPS Wilson loops. This will be considered elsewhere [51]. Yet it is natural to assume, because of the universality of the renormalization procedure, that the canonical coupling that arises in the loop equation for the quasi BPS Wilson loop coincides in some scheme with the physical charge. Since the canonical coupling is a local quantity that is determined only by the local part of the effective action, while the physical charge may get contributions also from the non-local part, if it does not vanish in the large-N limit, we can hope to identify the canonical coupling in some scheme with the effective charge only in the local limit for the effective action, that is the limit in which the effective action is actually computed in this paper.

Thus we find that if the effective action in the large-N limit implies non-local interactions between the vortices on which it is localized, then these interactions might imply the existence of string-like corrections to the physical effective charge. Otherwise the physical effective charge is completely accounted by the canonical coupling constant in some scheme.

But then this scheme is uniquely determined, as we will see momentarily. We find that the effective charge has a (non-conformal) infrared fixed point at the inverse RG invariant scale in the Wilsonian scheme, i.e. at the Landau pole of the Wilsonian coupling.

Given our formulae for the beta function of large-N YM:

$$\frac{\partial g_c}{\partial \log \Lambda} = \frac{-\beta_0 g_c^3 + \frac{\beta_1}{4} g_c^3 \frac{\partial \log Z}{\partial \log \Lambda}}{1 - \beta_2 g_c^2}$$  \hspace{1cm} (84)

$$\frac{\partial \log Z}{\partial \log \Lambda} = \frac{\gamma_0 g_W^2}{1 + c g_W^2}$$ \hspace{1cm} (85)

$$\gamma_0 = \frac{1}{(4\pi)^2} \frac{10}{3}$$ \hspace{1cm} (86)

$$\frac{\partial g_W}{\partial \log \Lambda} = -\beta_0 g_W^3$$ \hspace{1cm} (87)

45
we inquire as to whether we can find a value of $c$ for which the canonical coupling coincides with the gauge invariant physical charge between static quark sources in the large-$N$ limit.

The scheme of the physical charge is uniquely fixed (up perhaps to the mentioned non-local contributions) by the requirement that the physical charge be continuous and differentiable. Indeed we can study the behavior of the RG flow as a function of $c$.

For $c$ negative the anomalous dimension $\frac{\partial \log Z}{\partial \log \Lambda}$ has a pole at a finite value of $g_W$. Thus the canonical beta function has a zero in the numerator at a finite distance. However the derivative of the physical charge does not vanish at that zero. This means that the derivative is discontinuous, since on that point on the physical charge stays constant. Thus $c$ negative is not acceptable in the scheme of the physical charge.

For $c$ positive we must distinguish three cases. For $c > \frac{1}{(4\pi)^2} \beta_0$ the RG flow from the ultraviolet ends into a cusp, that is an infrared fixed point at the value $g_c^2 = \frac{(4\pi)^2}{4}$, where the beta function has a pole as in the $\mathcal{N} = 1$ SUSY YM theory. However a cusp is not acceptable as the end of the RG flow of the physical charge, because of the divergence of the derivative of the inter-quark potential.

For $\frac{1}{(4\pi)^2} \beta_0 > c$ the flow ends into an infrared fixed point, but the flow is continuously differentiable (with zero derivative) only at the critical value of $c = \frac{1}{(4\pi)^2} \beta_0$. From that point on the canonical coupling remains constant in this scheme. The scale at which this occurs is $\Lambda_W^{-1}$, the inverse RG invariant scale in the Wilsoanean scheme. We can write the formulae for the beta function in this scheme:

$$\frac{\partial g_{phys}}{\partial \log \Lambda} = -\beta_0 \frac{g_{phys}^3}{1 - \beta_J g_{phys}^2} \frac{1}{1 + \frac{1}{(4\pi)^2} \frac{10}{11} g_W^2}$$ (88)

$$\frac{\partial g_W}{\partial \log \Lambda} = -\beta_0 g_W^3$$ (89)

Eliminating $g_W$ as a function of $\Lambda_W$:

$$g_W^2 = \frac{1}{2\beta_0 \log(r\Lambda_W)}$$ (90)
we get for the physical beta function as a function of the inter-quark distance:

\[
\frac{\partial g_{\text{phys}}}{\partial \log r} = \beta_0 g_{\text{phys}}^3 1 - \frac{\beta_1 g_{\text{phys}}^2}{1 - \beta_1 g_{\text{phys}}^2} \log(r \Lambda_W) - \frac{15}{121}
\]  

(91)

It is then easy to see that the junction point with the constant flow is \( C^\infty \) and not only continuous and differentiable. Since \( YM \) theory contains only one arbitrary parameter, in our case \( \Lambda_W \), the value of the physical charge at the infrared fixed point is not arbitrary. However it cannot be found using only the formula for the beta function, but it needs a direct computation from physical Wilson loops. It is possible to perform this computation by analytic continuation from quasi \( BPS \) Wilson loops to physical Wilson loops [51] and by our Wilsonian effective action (the infrared fixed point of the canonical coupling is the Landau pole of the Wilsonian coupling).

Finally we should observe that the prediction of the existence of an infrared fixed point in the large-\( N \) limit for our definition of the physical charge compares favorably, within the expected order of \( \frac{1}{N} \) accuracy, with the plateau observed in [35] for the group \( SU(3) \). Of course a large-\( N \) lattice computation would be more than welcome.

6 Conclusions

We have shown the existence in the large-\( N \) limit of pure \( YM \) and of \( \mathcal{N} = 1 \) \( SUSY \) \( YM \) of quasi \( BPS \) Wilson loops that are protected by some of the usual renormalizations, since they have no perimeter and no cusp divergences for backtracking cusps, as in cases with extended \( SUSY \).

The existence of such objects is used to localize a version of the loop equation in the \( ASD \) variables on a saddle-point equation for an effective action. The proof of localization is obtained by homological methods as opposed to the cohomological localization in local field theory.

The crucial point is that the loop equation reduces to the insertion of the equation of motion for an effective action once the local degrees of freedom of the theory are holographically mapped by a local conformal transformation into backtracking arcs ending into cusps at infinity. Indeed in the new
variables and renormalization scheme the Wilsonian renormalized effective action flows to the ultraviolet by the conformal mapping and, being \( AF \), to vanishing coupling.

This effective action contains the whole information about the only remaining renormalization of quasi \( BPS \) Wilson loops, i.e. charge renormalization.

An explicit formula for the canonical beta function of large-\( N \) pure \( YM \) theory follows.

At the same time this shows the existence of the large-\( N \) limit of the pure \( YM \) theory for this (almost trivial, at least in perturbation theory) class of observables.

However one key point is that, since charge renormalization must be the same in all the sectors of the theory, essentially by the same token we get information on the physical charge between static quark sources in the large-\( N \) limit.

For the future the obvious interesting new game consists in trying to extend these techniques to those physical Wilson loops that can be obtained by analytic continuation of quasi \( BPS \) Wilson loops.

Another interesting aside consists in reproducing the known beta function in the \( \mathcal{N} = 1 \) SUSY case by the same technique.

Finally the present results suggest, in the light of the role that topological strings play in the localization of the loop equation on a saddle-point, that a purely stringy approach to the solution of the loop equation should exist, according to a long-standing conjecture. In particular the localization on non-commutative vortices suggests that the equations of motion in target space of the string theory dual to large-\( N \) \( YM \) should be the vortices equations of \( ASD \) type in presence of a \( B \) field.
7 Acknowledgments

We thank the Galileo Galilei Institute for Theoretical Physics and the INFN for the hospitality and the financial support during the completion of this work.

We thank the participants to the GGI workshop on ”Strongly Coupled Gauge Theories” for the fruitful atmosphere and in particular Emil Akhmedov, Adriano di Giacomo and Valentin Zakharov for several discussions about the existence of quasi BPS Wilson loops.

We thank Gabriele Veneziano for pointing out to us that the change of variables that we employ is actually, in the $\mathcal{N} = 1$ SUSY YM case, the Nicolai map.

We thank Martin Luscher for sending us a copy of his October 2004 talk at NBI on ”SU(N) Gauge Theories and the Bosonic String”.

We thank the organizers of the meeting of INFN networks on ”Theories of the Fundamental Interactions” (Villa Mondragone, June 2008), Massimo Bianchi, Loriano Bonora, Luciano Girardello, Kenichi Konishi, for the invitation to speak about the part of this paper on the $\mathcal{N} = 1$ SUSY YM.

We thank the organizers of the workshop on ”Strings and Strong Interactions: Holography and beyond” (LNF, September 2008), Stefano Bellucci, Massimo Bianchi, Maria Paola Lombardo, for the invitation to speak about the part of this paper on the homological localization.

References

[1] M. Bochicchio, JHEP 0709 (2007) 033, [hep-th/0705.0082].
[2] D. Gross, F. Wilczek, Phys. Rev. Lett. 30 (1973) 1345.
[3] H. Politzer, Phys. Rev. Lett. 30 (1973) 1346.
[4] W. E. Caswell, Phys. Rev. Lett. 33 (1974) 244.
[5] D. R. T. Jones, *Nucl. Phys.* **B 75** (1974) 531.

[6] D. Drukker, D. Gross, I. Ooguri, *Phys. Rev. D* **60** (1999) 125006, [hep-th/9904191].

[7] J.J. Duistermaat, G. J. Heckman, *Invent. Math.* **69** (1982) 259.

[8] E. Witten, *J. Geom. Phys.* **9** (1992) 303, [hep-th/9204083].

[9] N. A. Nekrasov, *Proceedings of the ICM, vol.3 p. 477*, Beijing (2003), [hep-th/0306211].

[10] V. Pestun, "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops", [hep-th/0712.2824].

[11] A. M. Polyakov, "The wall of the cave", *Int. J. Mod. Phys. A* **14** (1999) 645, [hep-th/9809057].

[12] R. M. Kaufmann, R. C. Penner, *Nucl. Phys B* **748**, [math.GT/0603485].

[13] R. C. Penner, "Decorated Teichmüller Theory of Bordered Surfaces", [math.GT/0210326].

[14] L. Hadasz, Z. Jaskolski, *Int. J. Mod. Phys. A* **18** (2003) 2609, [hep-th/0202051].

[15] M. Bochicchio, JHEP 9901 (1999) 006, [hep-th/9810015].

[16] S. Gukov, E. Witten, "Gauge theory, ramification and the geometric Langlands program", [hep-th/0612073].

[17] T. Eguchi, H. Kawai, *Phys. Rev. Lett.* **48** (1982) 1063.

[18] G. Bhanot, U. Heller, H. Neuberger, *Phys. Lett. B* **113** (1982) 47.

[19] A. Gonzales-Arroyo, C. P. Korthals-Altes, *Phys. Lett. B* **131** (1983) 396.

[20] A. Gonzales-Arroyo, M. Okawa, *Phys. Lett. B* **120** (1983) 174.

[21] T. Eguchi, R. Nakayama, *Phys. Lett. B* **122** (1983) 59.

[22] V. Novikov, M. Shifman, A. Vainshtein, V. Zakharov, *Phys. Lett. B* **217** (1989) 103.
[23] N. Arkani-Hamed, H. Murayama, "Holomorphy, Rescaling Anomalies and Exact beta Functions in Supersymmetric Gauge Theories", JHEP 0006 (2000) 030, [hep-th/9707133].

[24] E. Witten, Proc. Nato IAS, Plenum Press, New York (1980).

[25] Yu. M. Makeenko, A. A. Migdal, Phys. Lett. B 88 (1979) 135.

[26] Yu. M. Makeenko, A. A. Migdal, Nucl. Phys. B 188 (1981) 269.

[27] V. De Alfaro, S. Fubini, G. Furlan, G. Veneziano, Phys. Lett. B 142 (1984) 1.

[28] V. De Alfaro, S. Fubini, G. Furlan, G. Veneziano, Nucl. Phys. B 255 (1985) 399.

[29] M. Bochicchio, to appear.

[30] G. Gallavotti, "Constructive Quantum Field Theory", Encyclopedia of Mathematical Physics, Elsevier (2006), [math-ph/0510014].

[31] J. Maldacena, Phys. Rev. Lett. 80 (1998) 4859, [hep-th/9803002].

[32] G. C. Rossi, M. Testa, Phys. Lett. B 125 (1983) 476.

[33] Yu. Makeenko, Large-N gauge theories”, [hep-th/0001047].

[34] Yu. Makeenko, The first thirty year of large-N gauge theory”, [hep-th/0407028].

[35] M. Luscher, P. Weisz, JHEP 0207 (2002) 049, [hep-lat/0207003].

[36] J. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231.

[37] M. R. Douglas, N. A. Nekrasov, Rev. Mod. Phys. 73 (2001) 977, [hep-th/0106048].

[38] A. Bassetto, G. Heinrich, Z. Kunszt, W. Vogelsang, Phys. Rev D 58 (1997) 094020, [hep-th/9805283].

[39] R. Dijkgraaf, C. Vafa, "A perturbative window into non-perturbative physics", [hep-th/0208048].
[40] C. I. Lazaroiu, JHEP 0305 (2003) 044.

[41] N. J. Hitchin, *Proc. London Math. Soc.* **55** (1987) 59.

[42] C. Simpson, "The Hodge filtration on non Abelian cohomology", [alg-geom/9604005].

[43] C. Simpson, "Higgs bundles and local systems", *Publ. Math. IHES* **75** (1992) 5.

[44] T. Hausel, M. Thaddeus, "Mirror symmetry, Langlands duality and the Hitchin systems", [math.AG/0205236].

[45] J. Jost, J. Li, K. Zuo, "Harmonic bundles on quasi-compact Kahler manifolds", [math.AG/0108166].

[46] O. Biquard, P. Bloach, "Wild non-Abelian Hodge theory on curves", [math.DG/0111098].

[47] L. J. Mason, "Global anti-self-dual Yang-Mills fields in split signature and their scattering", [math-ph/0505039].

[48] M. Bonini, G. Marchesini, M. Simionato, *Nucl. Phys. B* **483** (1997) 475, [hep-th/9604114].

[49] M. Bianchi, S. Kovacs, G. C. Rossi, "Instantons and Supersymmetry", [hep-th/0703142].

[50] M. Luscher, P. Weisz, JHEP 0407 (2004) 014, [hep-th/0406205].

[51] M. Bochicchio, to appear.