Cofibrancy of operadic constructions in positive symmetric spectra

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Abstract

We show that for the underlying positive model structure on symmetric spectra one obtains cofibrancy conditions for operadic constructions while needing much milder assumptions than one would for general categories. Our main result provides such an analysis for a key operation, the “relative composition product” \( \cdot _{\mathcal{O}} \cdot \) between right and left \( \mathcal{O} \)-modules for \( \mathcal{O} \) a spectral operad. As a consequence we recover (and in some cases strengthen) previously known results establishing the Quillen invariance of model structures on categories of algebras via w.e.s of operads, compatibility with cofibrations of the forgetful functors from algebras to spectra and Reedy cofibrancy of bar constructions.

Key to the results above are novel cofibrancy results for \( n \)-fold wedge powers of positive cofibrant spectra (along with the correct analogue for maps). Roughly speaking, we show that such \( n \)-fold powers satisfy a (new) type of \( \Sigma_n \)-cofibrancy which can be thought of as “lax \( \Sigma_n \)-free/projective cofibrancy” in that it defines a larger class of cofibrations still satisfying the key technical properties one would expect from “true \( \Sigma_n \)-free/projective cofibrancy”.

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1 Introduction

Operads provide a convenient way to codify many types of algebraic structures on a category $\mathcal{C}$, such as monoids, commutative monoids, Lie algebras, among others. Indeed, any of these types of structures can be identified with the algebras in $\mathcal{C}$ over a specific operad.

When $\mathcal{C}$ is additionally a suitable model category it is then natural to ask whether the category of algebras over a fixed operad $\mathcal{O}$, denoted $\text{Alg}_{\mathcal{O}}(\mathcal{C})$, inherits a model structure from $\mathcal{C}$ and, moreover, just how compatible such a model structure on $\text{Alg}_{\mathcal{O}}(\mathcal{C})$ is with the underlying model structure on $\mathcal{C}$. Technical reasons then make it highly desirable for $\mathcal{C}$ and $\text{Alg}_{\mathcal{O}}(\mathcal{C})$ to be cofibrantly generated model categories, and one quickly finds that the biggest obstacle to tackling the questions above is the fact that general colimits in $\text{Alg}_{\mathcal{O}}(\mathcal{C})$ are not computed underlyingly in $\mathcal{C}$, so that proving properties of the (intended) cofibrations in $\text{Alg}_{\mathcal{O}}(\mathcal{C})$ requires a substantial amount of work.

More generally, related problems occur when studying other natural operadic constructions. Indeed, one of the most compact ways of describing operads is as the monoids over a certain monoidal structure $\circ$, the composition product, and many operadic constructions are then derived from $\circ$. However, $\circ$ is an unusual monoidal structure which behaves quite differently with respect to each of its variables, in particular preserving colimits in the first variable but not in the second, and one then finds that studying operadic constructions in a model category context naturally requires answering the non obvious question of which cofibrations are actually preserved by $\circ$, and when.

When dealing with a general model category $\mathcal{C}$ answering the questions above seems to require mild to severe cofibrancy conditions on the operad $\mathcal{O}$ itself. The main goal of this paper is to prove that for the category $\text{Sp}^\Sigma$ of symmetric spectra, however, these questions can be answered while making minimal to

\footnote{Briefly, this means that the classes of all cofibrations/all trivial cofibrations can be built from small nice generating subsets by using certain colimit constructions.}
none cofibrancy assumptions on \( O \), at least provided one uses the positive \( S \) model structure as the underlying model structure on \( \mathcal{S}p^{\Sigma} \).

Our main result in this paper is then Theorem 1.5 below, which provides a quite thorough control of the way \( \circ \) (or more generally, its relative version \( \circ_O \)) interacts with cofibrancy conditions. While this result is quite technical in its statement, it immediately implies a range of other results which aren’t, and indeed all four of Theorems 1.1, 1.2, 1.3, 1.4 below are essentially corollaries of Theorem 1.5. Of these Theorems 1.1, 1.3, 1.4 strengthen previously results in the literature (see Section 1.2 for a full discussion), while Theorem 1.2 seems to be novel.

Technically speaking, the proof of Theorem 1.5 follows by combining ideas and techniques previously found in the literature with two key new ingredients. The first of these is found in Proposition 5.17, which roughly speaking extends previously known results about filtrations of certain pushouts in \( Alg_{\mathcal{O}}(C) \) by also providing such filtrations when a relative composition product \( \circ_O \) is involved. This part of the discussion is quite formal and does not require that \( C \) be the category \( \mathcal{S}p^{\Sigma} \), and should hence be relevant even to those interested in a more general setting.

The second key idea is a more thorough characterization of what makes the positive model structures so useful. In the literature the author is aware of, the key lemma about positive model structures used is the fact that for \( A \) any spectrum with a \( \Sigma_n \) action and \( X \) a positive \( S \) cofibrant spectrum, then

\[
A \wedge \Sigma_n X^\wedge n \sim A \wedge h\Sigma_n X^\wedge n,
\]

so that \( A \wedge \Sigma_n X^\wedge n \) is homotopically well defined up to stable equivalences. However, the fact that the conclusion of this lemma makes no reference to cofibrancy conditions makes it somewhat cumbersome to use, with additional ad hoc arguments usually being required, making the kind of results we are aiming to prove in this paper hard to access. The key is then to improve this lemma to a result concerning cofibrations, and the most natural way of doing this would be to show that for \( X \) positive cofibrant then \( X^\wedge n \) is “build out of free \( \Sigma_n \) cells” or, in model category jargon, that \( X^\wedge n \) is \( \Sigma_n \)-projective cofibrant. Unfortunately this turns out to be false, though fortunately only subtly so, with \( X^\wedge n \) in fact satisfying a slightly laxer but formally no worse form of \( \Sigma_n \) cofibrancy, as shown by the key Theorems 1.6 and 1.7 below. Both of these results are somewhat technical, but roughly speaking, they satisfy complementary roles, with Theorem 1.6 showing that \( X^\wedge n \) indeed satisfies such a laxer \( \Sigma_n \) cofibrancy condition (together with a relative statement for cofibrations \( X \to Y \)), and Theorem 1.7 showing that the laxer \( \Sigma_n \) cofibrancy has no worse formal properties than “true” \( \Sigma_n \)-projective cofibrancy. Put together these two results provide a very thorough control of how the \( S \) positive model structure behaves with respect to cofibrations.

Much of this paper (Sections 3 and 4) is devoted to the somewhat technical tasks of both developing the new “laxer \( \Sigma_n \)-projective” model structure needed to state Theorems 1.6 and 1.7 and then proving those results.

\footnote{In fact, using certain cofibrancy claims made in by Harper in [7] as a base case, the current author previously “proved” such a result in the appendix to his thesis. However, as later pointed out by Pavlov and Scholbach those base cofibrancy claims were in fact incorrect.}
However, the reader interested only in the main result Theorem 1.5, or any of its “corollaries” Theorems 1.1, 1.2, 1.3 and 1.4 should be able to read Section 5 without having to read either of Sections 3 or 4, at least provided he is willing to accept Theorems 1.6 and 1.7 (and the auxiliary Propositions 4.1 and 4.2) as given.

1.1 Main results

We now list the main results of this paper concerning operadic constructions.

The $S$ and positive $S$ stable model structures on $Sp^\Sigma$ are defined in Section 2.4 while the eponymous model structures on $Sym$ are defined in Section 5.3.

The definition of projective (as well as injective) model structures can be found in Section 2.3.

The proofs of Theorems 1.1, 1.2, 1.3, 1.4 and 1.5 can be found in subsection 5.4.

Theorem 1.1. Let $O$ be any operad in $Sp^\Sigma$, and let $Sp^\Sigma$, $Sym$ be equipped with the respective positive $S$ stable model structures.

Then the respective induced projective model structures on $Alg_O$, $Mod^L_O$ exist, and these are simplicial model structures.

Further, if $O \to O'$ is a w.e. of operads then the induce-forget adjunctions

$$O \circ_O : Alg_O \rightleftarrows Alg_O : fgt, \quad O \circ_O : Mod^L_O \rightleftarrows Mod^L_O : fgt$$

are Quillen equivalences.

We will refer to the model structures in Theorem 1.1 as the projective positive $S$ stable model structures.

Theorem 1.2. Suppose $X$ is projective $S$ positive cofibrant in $Alg_O$, or more generally in $Mod^L_O$. Then the functor

$$Mod^R_O \overset{\circ_O X}{\longrightarrow} Sp^\Sigma, \quad or \text{more generally} \quad Mod^R_O \overset{\circ_O X}{\longrightarrow} Sym$$

preserves homotopy fiber sequences.

Theorem 1.3. Let $O$ be an operad in $Sp^\Sigma$ which is underlyingly $S$ cofibrant in $Sym$. Then, equipping the source categories with their respective projective positive $S$ stable model structures and the target categories with their respective $S$ stable model structures, the forgetful functors

$$fgt : Alg_O \to Sp^\Sigma, \quad fgt : Mod^L_O \to Sym$$

send cofibrations between cofibrant objects to cofibrations between cofibrant objects.

Theorem 1.4. Suppose $Sym$ is equipped with the positive $S$ stable model structure and consider an operad $O$ in $Sp^\Sigma$, right $O$-module $R$ and a left $O$-module $L$ such that the unit map $i : I \to O$, respectively $R$ and $L$, is an underlying cofibration, respectively are cofibrant objects, in $Sym$. Then the bar construction

$$B_n(R, O, L) = R \circ O^\ast n \circ L$$

is Reedy cofibrant with respect to the model structure on $Sym$. 4
Theorem 1.5. Let $\mathcal{O}$ be an operad in $\text{Sp}^\Sigma$ and consider the relative composition product

$$\mod^R_{\mathcal{O}} \times \mod^L_{\mathcal{O}} \rightarrow \text{Sym}.$$ 

Regard $\mod^L_{\mathcal{O}}$ as equipped with the projective positive $S$ stable model structure and $\text{Sym}$ as equipped with the $S$ stable model structure.

Suppose $f_2$ is a cofibration between cofibrant objects in $\mod^L_{\mathcal{O}}$. Then if the map $f_1$ in $\mod^R_{\mathcal{O}}$ is an underlying cofibration (respectively monomorphism) in $\text{Sym}$, then so is the pushout product

$$f_1 \circ^\mathcal{O} f_2$$

with respect to $\cdot \circ^\mathcal{O} \cdot$. Further, this map is also a w.e. if either $f_1$ or $f_2$ are.

The following are our key theorems concerning the positive $S$ model structure on $\text{Sp}^\Sigma$.

The $S \Sigma\text{-inj } \Sigma_n\text{-proj}$ stable model structure is defined in Section 3 and the name choice discussed in subsection 2.3.

The proof of Theorem 1.6 can be found in subsection 4.2 and the proof of Theorem 1.7 in subsection 4.1.

Theorem 1.6. Let $\text{Sp}^\Sigma$ be equipped with the positive $S$ stable model structure and $(\text{Sp}^\Sigma)^{\Sigma_n}$ with the $S \Sigma\text{-inj } \Sigma_n\text{-proj}$ stable model structure.

Then for $f: A \rightarrow B$ a cofibration in $\text{Sp}^\Sigma$ its $n$-fold pushout product

$$f^\otimes_n: Q_{n-1}^n(f) \rightarrow B^\otimes_n$$

is a cofibration in $(\text{Sp}^\Sigma)^{\Sigma_n}$, which is a w.e. when $f$ is.

Furthermore, if $A$ is cofibrant in $\text{Sp}^\Sigma$ then $Q_{n-1}^n(f)$ (resp. $A^\otimes_n \rightarrow B^\otimes_n$) is cofibrant (resp. cofibration between cofibrant objects) in $(\text{Sp}^\Sigma)^{\Sigma_n}$.

Theorem 1.7. Consider the functor

$$(\text{Sp}^\Sigma)^G \times (\text{Sp}^\Sigma)^G \rightarrow \text{Sp}^\Sigma,$$

where the first copy of $(\text{Sp}^\Sigma)^G$ is regarded as equipped with the $S \Sigma\text{-inj } G\text{-proj}$ stable model structure.

Then $\cdot \wedge_G \cdot$ is a left Quillen bifunctor if either:

(a) Both the second $(\text{Sp}^\Sigma)^G$ and the target $\text{Sp}^\Sigma$ are equipped with the respective monomorphism stable model structures;

(b) Both the second $(\text{Sp}^\Sigma)^G$ and the target $\text{Sp}^\Sigma$ are equipped with the respective $S$ stable model structures.

1.2 Relationship to previous work

Positive model structures on spectra were first introduced by Mandell, May, Schwede and Shipley in [14] and soon after used by Shipley in [19] to establish the existence of a projective model structure of symmetric ring spectra as well

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5 Note that we do not equip $\mod^R_{\mathcal{O}}$ with a model structure in this statement.
as compatibility between cofibrations and the forgetful functor from symmetric ring spectra to spectra.

The first of Shipley’s results was improved on by Elmendorf and Mandell in [2] to hold for algebras over any simplicial operad together with an additional Quillen equivalence result for weakly equivalent simplicial operads. These results were in turn further improved on by Harper in [7] to hold for any spectral operad, as well as for left modules. Theorem 1.1 is a slight improvement of these results in that it slightly extends the class of underlying cofibrations used.

The second of Shipley’s results was improved on by Harper and Hess in [9] who proved compatibility between cofibrations and forgetful functors on algebras and left modules over operads, under some cofibrancy conditions. They likewise also proved Reedy cofibrancy results for bar constructions, again under mild cofibrancy conditions. Theorems 1.3 and 1.4 improve on those results both by using more general cofibrancy assumptions but also by using more consistent cofibrancy assumptions across theorems (including Theorem 1.1), making it easier to use these results in tandem.

As discussed in the introduction, Theorem 1.5 effectively unifies the a priori disparate results above, and in turn leads to new results of interest such as Theorem 1.2.

The key technical lemma pertaining to the positive model structure used by Shipley in [19] (and subsequently by Elmendorf and Mandell in [2]) was the w.e. $A \wedge_{\Sigma_n} X \wedge_n \sim A \wedge_{A\Sigma_n} X \wedge_n$ referred to in the introduction. Harper improved this result in [7] to a result involving some cofibrancy conditions in Proposition 4.28 of that paper, although as was later pointed out by Pavlov and Scholbach that that result was subtly wrong. Theorems 1.6 and 1.7 improve on (and in the second case, correct) both of these by providing a much more thorough control of cofibrancy conditions.

1.3 Directions for future work

One of the main motivations for developing the cofibration results in the current paper comes from upcoming joint work between the author and Nick Kuhn where we study certain filtrations built using $L \circ \cdot$ type functors. A key point in that work is that one needs to be able to iterate such functors while still obtaining homotopically meaningful constructions, and this in turn requires understanding how those functors behave with respect to cofibrations.

Additionally, there are also two natural directions in which to directly generalize the results in the current paper.

The first direction would be to prove analogues of Theorems 1.1, 1.2, 1.3, 1.4 and 1.5 for the categories of algebras over a multicategory/colored operad, thereby fully extending the treatment of Elmendorf and Mandell [2], who in fact prove their version of Theorem 1.1 in that context.

The second direction would be to generalize our results for operads in other categories. Given that our treatment in Section 5 is essentially formal, one would expect versions of our main results to hold if one were studying another category satisfying appropriate analogues of Theorems 1.6 and 1.7 and the

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In fact, due to this our proofs of Theorems 1.3 and 1.4 effectively also correct the proofs of the corresponding results in [9] mentioned above, as the incorrect Proposition 4.28 of [7] was necessary for the proofs of those specific results.
main candidate for this is the category of genuine $G$-spectra or, to be precise, a suitable model. A particularly hopeful candidate is then the model for genuine $G$-spectra as $G$-invariant symmetric spectra currently being developed by Markus Hausmann, both because that work uses the same underlying category as we do in this paper and because the key notion of $S \Sigma$-inj $G$-proj cofibrancy introduced here is formally very close to the notions of cofibrancy that appear in that work.

1.4 Outline of the paper

Section 2 introduces some of the basic notation and terminology we will need. Of particular interest is subsection 2.3 which recalls the notions of injective and projective model structures and clarifies the naming conventions followed in this paper when iterating such constructions.

Section 3 defines and proves the existence (Theorem 3.5) of the $S \Sigma$-inj $G$-proj stable model structures necessary to formulate Theorems 1.6 and 1.7.

Section 4 then proves the key properties of the $S \Sigma$-inj $G$-proj stable model structures, namely Theorems 1.6 and 1.7 and the somewhat minor but convenient Propositions 4.1 and 4.2.

Section 5 deals with proving the main Theorems 1.1, 1.2, 1.3, 1.4 and 1.5. Key to this are subsection 5.2 and in particular Proposition 5.17, a filtration result which both reinterprets and extends filtrations previously found in the literature. This result is at the heart of the proof of Theorem 1.5 from which the other four theorems follow.

Appendix A introduces the auxiliary model structures used to formulate Theorem 1.7, the monomorphism stable and $S$ stable model structures on $(S p^\Sigma)^G$ which, while strictly speaking somewhat novel, are direct adaptations of the eponymous model structures in $S p^\Sigma$ used elsewhere in the literature.

Finally, Appendix B recalls some basic notions and results about the localization theory of left proper cellular categories of [10] which are needed at several points in this paper when stabilizing levelwise w.e. spectra model categories, most notably when proving Theorems 5.5 and 1.7.

2 Basic definitions and notation

The majority of the material in subsections 2.1 and 2.2 is adapted from [7], which should be consulted for further details, as we cover here only the bare minimum necessary to establish notation along with some basic results which may be somewhat less standard.

Subsection 2.3 recalls the concepts of injective and projective model structures and details our naming conventions with regards to iterating such constructions.

2.1 $G$-invariant pointed spaces

Throughout this paper we will let $(S^*, \wedge, S^0)$ denote the closed monoidal category of pointed simplicial spaces together with its monoidal structure $\wedge$ and unit $S^0$.

We will make use the following standard notation:
• \(\Delta^k\), \(\partial\Delta^k\) and \(\Lambda^k\) denote the standard, boundary and horn (unpointed) simplicial sets;
• \(X_+\) denotes the pointed simplicial obtained by adding a disjoint base point to the (unpointed) simplicial set \(X\);
• \(S^n = (\Delta^1/\partial\Delta^1)^\wedge n\) denotes the pointed simplicial \(n\)-sphere.

**Definition 2.1.** Let \(G\) be a finite group. The category \(S^G_\ast\) of \(G\)-pointed spaces is the category of functors \(G \to S_\ast\).

Given \(G\)-pointed spaces \(X, Y\) note that \(X \wedge Y\) has a diagonal \(G\)-action, and equipping \(S^0\) with the trivial action this operation becomes a monoidal structure in \(S^G_\ast\). In fact, we have the following result.

**Proposition 2.2.** \((S^G_\ast, \wedge, S^0)\) form a closed symmetric monoidal category.

Further, both the left and right adjoint in the trivial-fixed point adjunction

\[
\text{triv}: S_\ast \rightleftarrows S^G_\ast: (\cdot)^G
\]

are monoidal functors.

**Proof.** It is obvious that \(\text{triv}\) is monoidal. For the right adjoint \((\cdot)^G\), recall that we have a pushout diagram

\[
\begin{array}{ccc}
X \vee Y & \rightarrow & X \times Y \\
\downarrow & & \downarrow \\
* & \rightarrow & X \wedge Y
\end{array}
\]

so the result now follows from the fact that \((\cdot)^G\) commutes with the cartesian product \(\times\) and by Proposition 2.1 bellow.

**Remark 2.3.** By the general theory of enriched categories (see for example Chapter 3 of [10]) the previous result formally implies that \(S^G_\ast\) is a \(S_\ast\)-enriched category that is also tensored and cotensored, and by extension also simplicially enriched, tensored and cotensored.

Rather than spelling out what this means in full detail, we list only the few facts we will need:

• given \(K\) a simplicial set and \(A \in S^G_\ast\) their tensor product is

\[
K \otimes A = K_+ \wedge A;
\]

• given \(K\) a simplicial set and \(X \in S^G_\ast\) their co-tensor product is

\[
X^K = \text{Hom}^G_\ast(K_+, X),
\]

where \(\text{Hom}^G_\ast(K_+, X)\) denotes the simplicial space of pointed maps from \(K_+\) to \(X\), together with the \(G\)-action induced from that on \(X\);

• given \(A, B \in S^G_\ast\) their simplicial mapping space is

\[
\text{Hom}(A, B) = (\text{Hom}^G_\ast(A, B))^G,
\]

where \(\text{Hom}^G_\ast(A, B)\) denotes the simplicial space of pointed maps from \(A\) to \(B\), together with the induced \(G\)-action by conjugation.
We will also make use of the following basic fact about fixed points.

**Proposition 2.4.** Consider any pushout diagram in $S^G_*$

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow^f & & \downarrow \\
B & \longrightarrow & Y
\end{array}
\]

such that $f$ is a monomorphism. Then the diagram remains a pushout diagram after applying $(\cdot)^G$.

**Proof.** Any monomorphism of $G$-simplicial spaces can be decomposed into a transfinite composition of maps adding the orbit of a single simplex, meaning that we can assume $f$ has the form $(\cdot \Delta^k)_+ \times G/H \rightarrow (\Delta^k)_+ \times G/H$ (where $H$ is the isotropy subgroup of a simplex). The result is clear for such maps. \qed

### 2.2 $G$-invariant spectra

Denote by $\Sigma$ the canonical skeleton of the category of finite sets and bijections, i.e., the category whose objects are the sets $m = \{1, 2, \ldots, m\}$ for $m \geq 0$ and whose morphisms are the bijections.

**Definition 2.5.** The category of symmetric sequences in pointed simplicial sets is the category $S^\Sigma_*$ of functors from $\Sigma$ to $S_*$.  

**Remark 2.6.** Unpacking the previous definition we see that a symmetric sequence $X$ consists of a sequence $X_m, m \geq 0$, of pointed simplicial spaces together with actions of the symmetric groups $\Sigma_m$ on $X_m$.

It is then clear that one has natural inclusions of categories $S^\Sigma_m \hookrightarrow S^\Sigma_*$ and we will hence simplify notation by omitting such inclusions (and we will do likewise when in the presence of additional $G$-actions).

**Definition 2.7.** The tensor product of two symmetric sequences $X$ and $Y$ is the symmetric sequence $X \otimes Y$ given by

\[
(X \otimes Y)_m = \bigvee_{i+j=m} \Sigma_m \wedge \Sigma_i \times \Sigma_j X_i \wedge Y_j
\]

together with the obvious $\Sigma_m$ actions.

The following is proven in Section 2.2 of [13].

**Proposition 2.8.** $(S^\Sigma_*, \otimes, \mathbb{1})$ form a symmetric monoidal category where the unit $\mathbb{1}$ is the sequence such that $\mathbb{1}_0 = S^0$ and $\mathbb{1}_m = *$ for $m > 0$.

It is straightforward to check that the symmetric sequence $S$, the sphere spectrum, defined by $S_m = S^m$ forms a symmetric monoid with respect to $\otimes$.

It then follows by the abstract theory of symmetric monoidal categories that modules over $S$ in $S^\Sigma_*$ form a symmetric monoidal category themselves.

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* A note on notation: in this paper we will be interested in spectra which have a $\Sigma_n$-action, so that $X_m$ will actually be acted on by $\Sigma_n \times \Sigma_m$. To avoid confusion we will reserve the letter $n$ for the symmetric group acting on the spectrum and use $m$ as the structure index.
Definition 2.9. The category $\mathsf{Sp}^\Sigma$ of symmetric spectra is the category of modules over $S$ in $S^\Sigma_\ast$.

The smash product $X \wedge Y$ of two symmetric spectra $X$ and $Y$ is the (reflexive) coequalizer

$$X \otimes S \otimes Y \rightrightarrows X \otimes Y \to X \wedge Y.$$  

Definition 2.10. Let $G$ be a finite group. The category $(\mathsf{Sp}^\Sigma)^G$ of $G$-spectra is the category of functors $G \to \mathsf{Sp}^\Sigma$.

Just as before in the case of pointed spaces, the smash product $X \wedge Y$ of two spectra $X, Y \in (\mathsf{Sp}^\Sigma)^G$ has a diagonal $G$-action. One then has the following basic result.

Proposition 2.11. Both $(\mathsf{Sp}^\Sigma, \wedge, S)$ and $(\mathsf{Sp}^\Sigma)^G, \wedge, S^G$ form closed symmetric monoidal categories.

Further, both the left and right adjoints in all the following adjunctions are monoidal except for $(\cdot)^G$

$$S \otimes \cdot : S_* \rightleftarrows \mathsf{Sp}^\Sigma: (\cdot)_0, \quad S \otimes \cdot : S^G_* \rightleftarrows (\mathsf{Sp}^\Sigma)^G: (\cdot)_0,$$

$$\text{triv} : \mathsf{Sp}^\Sigma \rightleftarrows (\mathsf{Sp}^\Sigma)^G: (\cdot)^G.$$  

Proof. All of these follow by a direct calculation (where in the case of the two maps denoted $(\cdot)_0$ one needs to use that $S_0 = S^G_0$).

Remark 2.12. By the general theory of enriched categories (see for example Chapter 3 of [10]) the previous result formally implies that $(\mathsf{Sp}^\Sigma)^G$ is enriched, tensored and cotensored over both $S_\ast$ and $\mathsf{Sp}^\Sigma_\ast$, and by extension also simplicially enriched, tensored and cotensored.

We will require the following particular consequences of this

- given $K$ a simplicial set and $A \in (\mathsf{Sp}^\Sigma)^G$ their tensor product is

  $$K \otimes A = (S \otimes K_\ast) \wedge A;$$

- given $K$ a simplicial set and $X \in (\mathsf{Sp}^\Sigma)^G$ their co-tensor product satisfies

  $$(X^K)_m = X^K_m$$

  where $X^K_m$ is the simplicial space of maps between the simplicial sets $K$ and $X^K_m$;

- given $A, B \in (\mathsf{Sp}^\Sigma)^G$ their simplicial mapping space is

  $$\mathsf{Hom}(A, B) = (\mathsf{Hom}^G(A, B))^G,$$

  where $\mathsf{Hom}^G(A, B)$ denotes the simplicial space of maps between the underlying spectra $A$ and $B$, together with the induced $G$-action by conjugation.

2.3 Injective and projective model structures

Definition 2.13. Let $C$ be a model category, $M$ be a monad on $C$ and $\mathsf{Alg}_M(C)$ the category of algebras over $M$.

The injective model structure on $\mathsf{Alg}_M(C)$, if it exists, is the model category structure such that both cofibrations and w.e.s are defined underlyingly on $C$.

The projective model structure on $\mathsf{Alg}_M(C)$, if it exists, is the model category structure such that both fibrations and w.e.s are defined underlyingly on $C$. 

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Remark 2.14. Generally speaking, since most model categories one is usually interested in are cofibrantly generated, it is much easier to build projective model structures than it is to build injective ones, since in the former case candidate generating cofibrations and trivial cofibrations are obtained automatically (see for example Lemma 2.3 in [17]).

The previous remark notwithstanding, for the specific categories and monads of interest in in this paper we will be able to not only often form both injective and projective model structures but also to iterate such constructions.

A good first simple example of this is given in Proposition 3.3, where what we call the $\Sigma_m$-inj $G$-proj model structure in $S\Sigma^G\hat{\times}\Sigma^m$ is in fact the $\Sigma_m$-injective model structure over the $G$-projective model structure over the standard model structure in $S\hat{\times}$. It is crucial to note that this is not the same as what we would call the $G$-proj $\Sigma_m$-inj model structure on $S\hat{\times}$, which is the model structure obtained by reversing the order of the constructions above and could be built by replacing the condition $H \cap G \times * = *$ in Proposition 3.1 with the condition $H \subset \star \times \Sigma_m$. The mistake in [7] alluded to in the introduction was precisely that of failing to distinguish between these two model structures, and it is precisely this distinction which leads to the need for the new $S\Sigma$-inj $G$-proj in $(Sp^\Sigma)^G$ introduced in this paper.

For a more representative example consider now the category $(Sp^\Sigma)^G$, which one can think of as built by starting first with the category $S\hat{\times}$ of sequences of pointed spaces and sequentially building algebras over a monad $\Sigma$, then over a monad $S$ and finally over a monad $G$. Note however that, abusing notation somewhat, these monads are interchangeable to some extent. Indeed, using the trivial $G$ action on $S\hat{\times}$ as in Proposition 2.11 one can interchange the $S$ and $G$ monads, and one can likewise interchange the $G$ and $\Sigma$ monads. One hence has identifications

$$(Sp^\Sigma)^G \cong \text{Alg}_G \text{Alg}_S \text{Alg}_\Sigma S\hat{\times} \cong \text{Alg}_G \text{Alg}_\Sigma \text{Alg}_S S\hat{\times} \cong \text{Alg}_S \text{Alg}_G \text{Alg}_\Sigma S\hat{\times},$$

and one might hence ask for model structures on $(Sp^\Sigma)^G$ built iterating injective or projective constructions using any of these monad orders, and then name them using strings of adjectives such as “$\Sigma$-inj”, “$S$-proj”, “$G$-proj”, etc. Although it is not the case that we will need all such model structures, we will nonetheless need to understand enough of them that it is worthwhile to introduce some convenient naming conventions, particularly since in Chapter 5 we further introduce the category $\text{Sym}^G$, built from $(Sp^\Sigma)^G$ using yet another $\Sigma$ monad. The conventions used follow.

- We will never use the adjective “$S$-inj” when naming model structures. This is because the only $S$-injective cofibrations we will need are those which are injective with respect to all monads, so that they are simply the underlying monomorphisms, and in keeping with tradition (e.g. [13]) we will refer to these as the “monomorphism” model structures. We prove the existence of such model structures in Appendix A.1 by slightly generalizing arguments in [13].

6Explicitly, cofibrations in these model structures are those injections $A \xrightarrow{j} B$ such that the “allowed” isotropies $H$ on $B - i(A)$ satisfy the given conditions. In the $\Sigma_m$-inj $G$-proj case this is simply the condition that the action on $B - i(A)$ is $G$ free after forgetting the $\Sigma_m$ action, but the $G$-proj $\Sigma_m$-inj case seems to lack an analogous interpretation.
• In keeping with the previous point we will abbreviate the adjective “S-proj” to simply “S”. This will also typically be last monad being applied.

• The maps in $\text{Sp}^\Sigma$ that one might call $S$-proj $\Sigma$-inj cofibrations are referred to simply as $S$-cofibrations in $[13]$ (and indeed this is our notation subsection 2.4), the idea being that these cofibrations are generated by the maps of the form

$$S \otimes \{\text{monomorphisms}\}.$$

In keeping with this same idea we will refer to the $S$-proj $G$-inj $\Sigma$-inj cofibrations in $(\text{Sp}^\Sigma)^G$ simply as $S$-cofibrations. The stable model structures using these cofibrations are built in Appendix A.2.

Note further that this naming convention is particularly justified in light of Corollary A.11, which says $S$-proj $G$-inj $\Sigma$-inj cofibrations are the same as $G$-inj $S$-proj $\Sigma$-inj cofibrations or, more explicitly, that $S$-cofibrations in $(\text{Sp}^\Sigma)^G$ are the maps which forget to $S$-cofibrations in $\text{Sp}^\Sigma$.

**Remark 2.15.** When dealing with $(\text{Sp}^\Sigma)^G$ merely following the constructions above would lead to model categories with levelwise w.e.s rather than stable equivalences. To correct this we will use the left Bousfield localization machinery of Hirschhorn in [10], which we briefly describe in Appendix B.1. The naming conventions will be kept when localizing except with the adjective “levelwise” replaced with “stable”.

In accordance with the conventions above the $S$ $\Sigma$-inj $\Sigma$-proj stable model structure featured in the statements of Theorems 1.6 and 1.7 is built as the (stabilization of) the $S$-projective model structure over the $\Sigma$-injective model structure over the $\Sigma$-projective model structure over the levelwise model structure in $\text{Sp}^\Sigma$. The apparent lack of a simpler conceptual description for this crucial model structure is what motivated the complex naming conventions.

Special mention should also be made about the $S$ $\Sigma_n$-proj $\Sigma$-inj stable model structure on $(\text{Sp}^\Sigma)^{\Sigma_n}$, which we shall also refer to by its “long name” while proving Theorem 3.5. We will do so despite the fact that this model structure does admit simpler names, such as “$\Sigma_n$-projective flat stable” if following [7] or “$\Sigma_n$-projective $S$-stable” if following [13], so as to emphasize the relation between this model structure and that in the previous paragraph, as the need for much of the current paper comes precisely from the subtle difference between them.

### 2.4 $S$ stable and positive $S$ stable model structures in $\text{Sp}^\Sigma$

We now recall the definition of the $S$ stable and positive $S$ stable model structures in $\text{Sp}^\Sigma$. A more detailed discussion can be found, for example, on [13], [19] and Appendix A.2.

**Definition 2.16.** The $S$ stable model structure (resp. positive $S$ stable model structure) on $\text{Sp}^\Sigma$ is the cofibrantly generated model structure such that

- the generating cofibrations are the maps

$$S \otimes \partial \Delta^k \times \Sigma_m / H \rightarrow S \otimes \Delta^k \times \Sigma_m / H$$

for $m \geq 0$ and any $H$ (resp. $m \geq 1$ and any $H$);
Remark 2.17 (S cofibrations vs. flat cofibrations). As indicated in the previous section we are following [13, 19] in naming the cofibrations in the model structures above rather than for example [15 and 17] which call these “flat cofibrations”, mainly due to the difficulty of finding a suitable name for the $S\Sigma$-inj $G$-proj stable model structure on $(\text{Sp}^\Sigma)^G$ when following the latter convention.

Additionally, we note that we will never use what are called simply “cofibrations” in those references, which are the class of maps obtained by forcing $H = *$ in the definition above. Indeed, while one could still obtain analogue versions of our main Theorems by restricting the cofibrancy conditions on the “hypothesis side” to the “non-$S$/non-flat” case, this does not not seem to improve the cofibrancy conditions on the “conclusion side” in an obvious way (the key example being Theorem 1.6).

Remark 2.18 (S cofibrations vs. positive $S$ cofibrations). Analysing the proof of Proposition 2.10 one sees that a $S$ cofibration $A \rightarrow B$ will also be a positive $S$ cofibration iff the 0-th level map $A_0 \rightarrow B_0$ is an isomorphism, with similar remarks applying to the positive $S$ stable model structure on $\text{Sym}$ defined in subsection 5.3 (or even positive analogues of the model structures we define on $(\text{Sp}^\Sigma)^G$, $\text{Sym}^G$).

As such we have chosen to simplify the statements of our main theorems by using positivity conditions as little as possible, these being used on the “hypothesis side” only when necessary, and almost never on the “conclusions side” of the results. The reader interested in additional positivity conditions on the “conclusions side” will then often find that they can obtained as a corollary of our results by simply adding additional positivity conditions on the “hypothesis side” together with the remark in the previous paragraph.

3 $S\Sigma$-inj $G$-proj stable model structure on $(\text{Sp}^\Sigma)^G$

Our goal in this section is to prove Theorem 3.5 which defines and proves the existence of the $S\Sigma$-inj $G$-proj stable model structures featured in Theorems 1.6 and 1.7. The reader interested only on the definition of this model structure should consult Remark 3.6 where the generating cofibrations are made explicit.

We build the model structure in stages, with subsection 3.1 first building the auxiliary $\Sigma_m$-$\text{inj}$ $G$-proj model structure on $S^G_{\Sigma_m}$, and subsection 3.2 then finishing the work.

3.1 $\Sigma$-$\text{inj}$ $G$-proj model structure on $S^G_{\Sigma_m}$

We will define the $\Sigma$-$\text{inj}$ $G$-proj model structure on $S^G_{\Sigma_m}$ in Proposition 3.5 (see subsection 2.3 for an explanation of the name) by left Bousfield localization of the auxiliary model structure in the following proposition.

Proposition 3.1. For $G$ any finite group there exists a cofibrantly generated model structure on $S^G_{\Sigma_m}$ such that

- w.e.s are the maps $A \rightarrow B$ such that $A^H \rightarrow B^H$ is a w.e. for any $H \subset G \times \Sigma_m$ s.t. $H \cap G \times \{\ast\} = \ast$. 

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• fibrations are the maps $A \to B$ such that $A^H \to B^H$ is a fibration for any $H \subseteq G \times \Sigma_m$ s.t. $H \cap G \times \{*\} = \ast$.

Further, this is a left proper cellular simplicial model category.

Proof. We’ll apply Theorem 2.1.19 in [11]. The sets $I$ and $J$ are immediately built from those in $S_\ast$ using left adjoints to the fixed point functors $(\cdot)^H$. Explicitly, $I$ is the set of maps of the form

$$(\Delta^k)_+ \times (G \times \Sigma_m)/H \to (\Delta^k)_+ \times (G \times \Sigma_m)/H, \quad H \cap G \times \{*\} = \ast$$

and $J$ is the set of maps of the form

$$(\Delta^k)_+ \times (G \times \Sigma_m)/H \to (\Delta^k)_+ \times (G \times \Sigma_m)/H, \quad H \cap G \times \{*\} = \ast.$$ 

Arguing by adjointness all the conditions in Theorem 2.1.19 in [11] turn out to be obvious except for the fact that all maps in $J$-cell are weak equivalences. This can be shown by first checking it for the maps in $J$ by direct computation, then for pushouts of those using Proposition 2.4, and finally for transfinite compositions of the latter by noting that $(\cdot)^H$ commutes with transfinite compositions of monomorphisms.

Left properness follows immediately by combining Proposition 2.4 with the left properness of $S_\ast$.

Cellularity is a particular instance of Corollary B.6.

The definition of the simplicial structure can be found in Remark 2.3. That the simplicial model structure axioms are satisfied is then clear by considering the cotensoring and using the analogous axioms in $S_\ast$, since one has

$$(X^K)^H = (X^H)^K.$$ 

Remark 3.2. As should be apparent from the proof one can construct analogous model structures using more general conditions on $H$, such as families of subgroups.

Proposition 3.3. Let $G$ be any finite group.

There exists a cofibrantly generated model structure on $S_\ast^{G \times \Sigma_m}$, which we call the $\Sigma_m$-inj $G$-proj model structure, such that

• cofibrations are as in the model structure in Proposition 3.1;

• w.e.s are the maps $A \to B$ such that the underlying map is a w.e. in $S_\ast$.

Further, this is a left proper cellular simplicial model category.

Proof. All the statements follow directly from Theorem B.2 provided one indicates an appropriate set of maps $S$ with respect to which to localize. We claim that the maps of the form

$$(G \times \Sigma_m) \times_H EH)_+ \to ((G \times \Sigma_m)/H)_, \quad H \cap G \times \{*\} = \ast$$

are enough, so that all that is left to show is that the $S$-local equivalences are precisely the underlying w.e.s.
That the $\mathcal{S}$-local equivalences are underlying w.e.s follows by applying Proposition B.3 to the forget-free power adjunction

$$fgt: \mathcal{S}_G^{G \times \Sigma_m} \cong \mathcal{S}: (\cdot)^{G \times \Sigma_m}$$

since the maps in $\mathcal{S}$ are underlying w.e.s (note that since the maps in $\mathcal{S}$ already have cofibrant domains and codomains there is no need for cofibrant replacements).

We now turn to the converse. It suffices to show that between $\mathcal{S}$-local objects any levelwise weak equivalence is in fact one of the “old” weak equivalences from the model structure in Proposition 3.1. But for an “old” fibrant object $X$ to be $\mathcal{S}$-local we must have induced w.e.s

$$X^H = \text{Map}_G (((G \times \Sigma_m)/H)_+, X) \cong \text{Map}_G (((G \times \Sigma_m) \times_H \text{EH})_+, X) = X^{hh},$$

so that the result now follows since $X^{hh}$ is invariant under underlying equivalences on $X$ (or, more explicitly, because $((G \times \Sigma_m) \times_H \text{EH})_+$ is cofibrant in the $G \times \Sigma_m$-projective model structure on $\mathcal{S}_G^{G \times \Sigma_m}$).

3.2 Existence of the $\mathcal{S} \Sigma$-inj $G$-proj stable model structure

We now use the $\Sigma_m$-inj $G$-proj model structures from the previous section to prove the existence theorem for the $\mathcal{S} \Sigma$-inj $G$-proj stable model structure. As usual we start by first building a levelwise w.e. analogue.

Proposition 3.4. Let $G$ be any finite group.

There exists a cofibrantly generated model structure on $(\text{Sp}^\Sigma)^G$, which we call the $\mathcal{S} \Sigma$-inj $G$-proj levelwise model structure, such that

- w.e.s are the maps $X \to Y$ such $X_m \to Y_m, m \geq 0$ are underlying w.e.s in $\mathcal{S}_\mathbb{B}$

- fibrations are the maps $X \to Y$ such that $X_m \to Y_m, m \geq 0$ are underlying fibrations in the $\Sigma_m$-inj $G$-proj model structure on $\mathcal{S}_\mathbb{B}^{G \times \Sigma_m}$.

Further, this is a left proper cellular simplicial model category.

Proof. Clearly there exists a cofibrantly generated positive $\Sigma$-inj $G$-proj model structure on $G$-symmetric sequences $\mathcal{S}_\mathbb{B}^{G \times G}$ which is obtained by using the $\Sigma_m$-inj $G$-proj model structures at each level. Letting $I_m, J_m$ denote the generating cofibrations and generating trivial cofibrations for those level model structures (regarded as maps in $\mathcal{S}_\mathbb{B}^{G \times G}$) we claim that

$$I = \bigcup_{m \geq 0} S \otimes I_m, \quad J = \bigcup_{m \geq 0} S \otimes J_m$$

form sets of generating cofibrations and generating trivial cofibrations for our the desired model structure.

As before all conditions in Theorem 2.1.19 in [11] follow immediately by adjunction arguments, except whether the maps in $J$-cell are w.e.s. Since pushouts in $(\text{Sp}^\Sigma)^G$ are just underlying pushouts of symmetric sequences, it suffices to check that for $S \otimes (A_m \to B_m)$ an element of $J$ (where $A_m \to B_m$
is in $J_m$) each of its levels is an underlying trivial cofibration in $S_*$, and since $(S \otimes (A_m \to B_m))_{m}$ is the map

$$
\Sigma \hat{m} \times \Sigma \hat{m} \cdot m \times \Sigma m \cdot \Sigma \hat{m} \cdot m \wedge A_m \to \Sigma \hat{m} \times \Sigma \hat{m} \cdot m \times \Sigma m \cdot \Sigma \hat{m} \cdot m \wedge B_m
$$

this follows immediately from the fact that $A_m \to B_m$ is an underlying trivial cofibration in $S_*$.

Left properness likewise follows from the remark about pushouts and the computation found in the previous paragraph.

Cellularity is a particular instance of Corollary B.6.

The definition of the simplicial structures can be found in Remark 2.12. That the simplicial model structure axioms are satisfied is clear by looking at the co-tensoring (which is computed in a levelwise manner) and using the analogous axioms for the $\Sigma_m$-inj $G$-proj model structures on $S^G$. \[Q.E.D.\]

**Theorem 3.5.** Let $G$ be any finite group.

There exists a cofibrantly generated model structure on $(\mathcal{S}^\Sigma)^G$, which we call the $S$ $\Sigma$-inj $G$-proj stable model structure, such that

- cofibrations are as in the model structure in Proposition 3.4;
- w.e.s are the maps $X \to Y$ which are underlying stable equivalences in $\mathcal{S}^\Sigma$.

Further, this is a left proper cellular simplicial model category.

**Proof.** Again all statements follow directly from Theorem B.2 once we have a suitable set of maps with respect to which to left Bousfield localize. We claim that the set $S_G$ of maps

$$
S \otimes ((S^1)_+ \times G \times \Sigma_{m+1}) \to S \otimes ((S^0)_+ \times G \times \Sigma_m), \quad m \geq 0
$$

suffices, i.e. that the $S_G$-equivalences are precisely the stable equivalences.

When $G = *$ this is well known, as (dropping the $G$ form the notation) the $S$ $\Sigma$-inj stable model structure on $\mathcal{S}^\Sigma$ is just the model structure from Theorem 2.4 in [19].

The case for general $G$ will follow from the case $G = *$.

To see this, start by considering the $G$-proj $\Sigma$-inj $S$ stable model structure on $(\mathcal{S}^\Sigma)^G$, which is literally the $G$-projective model structure over the model structure on $\mathcal{S}^\Sigma$ from the previous paragraph. We claim that this model structure can alternatively be built by first building its levelwise version, and then localizing with respect to the set $S_G$. By the discussion in Definition B.1, this will follow provided both procedures produce the same fibrant objects and this in turn follows from the fact that $S_G = G \times S_*$. To relate this back to our desired model structure, consider now the identity Quillen equivalence

$$
id: (\mathcal{S}^\Sigma)^G \rightleftarrows (\mathcal{S}^\Sigma)^G: id
$$

where we think of the left hand $(\mathcal{S}^\Sigma)^G$ as equipped with the $S$ $\Sigma$-inj $G$-proj levelwise model structure and the right hand one as equipped with the $S$ $G$-proj

\[Q.E.D.\]

\[\text{Note that the domain is a free spectrum generated at degree } m + 1 \text{ while the codomain is generated at degree } n.\]
\(\Sigma\text{-inj}\) levelwise model structure. We will be done if we can show that the \(\mathcal{S}_G\)-local equivalences are the same in both categories. But this is clear by analysing Definition 3.1, since one is free to choose the cofibrant replacement functor so as to land in \(\mathcal{S}_G\)-proj \(\Sigma\text{-inj}\) cofibrant objects and \(\mathcal{S}_\Sigma\text{-inj}\) \(\mathcal{G}\text{-proj}\) local objects suffice to detect \(\mathcal{S}_G\)-local equivalences, finishing the proof.

**Remark 3.6.** Tracking through the proofs of Propositions 3.1, 3.3, 3.4 and Theorem 3.5 we see that the generating cofibrations for the \(\Sigma\text{-inj}\) \(\mathcal{G}\text{-proj}\) stable model structure are those maps of the form

\[
S \otimes ((\hat{\partial}\Delta^k)_+ \times (G \times \Sigma_m)/H) \to S \otimes ((\Delta^k)_+ \times (G \times \Sigma_m)/H)
\]

for \(m \geq 0, H \cap G \times \{\ast\} = \ast\).

4 **Properties of the \(\Sigma\text{-inj}\) \(\mathcal{G}\text{-proj}\) stable model structure**

In this section we prove the key properties of the \(\Sigma\text{-inj}\) \(\mathcal{G}\text{-proj}\) stable model structure, most notably Theorems 1.6 and 1.7.

In subsection 4.1 we first prove what we call “\(\mathcal{G}\text{-proj}\) type properties”. These are those properties which one would formally expect if dealing instead with the truly \(\mathcal{G}\text{-projective}\) analogue, the \(\mathcal{S}_G\text{-proj}\) \(\Sigma\text{-inj}\) stable model structure (see subsection 2.3), and these include not only Theorem 1.7 but also Propositions 4.1 and 4.2.

Subsection 4.2, which in a sense is the technical heart of this paper, deals with the somewhat lengthier proof of Theorem 1.6.

4.1 **\(\mathcal{G}\text{-proj}\) type properties**

**Proposition 4.1.** Suppose all categories are equipped with their respective \(\Sigma\text{-inj}\) \(\mathcal{G}\text{-proj}\) stable model structure. Then the functor

\[
(\text{Sp}^\Sigma)^G \times (\text{Sp}^\Sigma)^G \xrightarrow{\wedge} (\text{Sp}^\Sigma)^{G \times G}
\]

is a left Quillen bifunctor.

**Proof.** The existence of the two necessary right adjoint bifunctors is a formal consequence of \((\text{Sp}^\Sigma, \wedge, S)\) forming a closed monoidal category.

We are hence left with verifying the axioms concerning the pushout product of two cofibrations.

First we need to show that the pushout product of any two cofibrations is a cofibration, and by general properties of the pushout product it suffices to do so for generating cofibrations. Let the chosen generating cofibrations be

\[
S \otimes ((\hat{\partial}\Delta^k)_+ \times (G \times \Sigma_m)/H) \to (\Delta^k)_+ \times (G \times \Sigma_m)/H, \quad m \geq 0, H \cap G \times \{\ast\} = \ast,
\]

\[
S \otimes ((\hat{\partial}\Delta^\bar{k})_+ \times (\hat{G} \times \Sigma_m)/\bar{H}) \to (\Delta^\bar{k})_+ \times (\hat{G} \times \Sigma_m)/\bar{H}, \quad \bar{m} \geq 0, \bar{H} \cap \hat{G} \times \{\ast\} = \ast.
\]
Then their pushout product (using the identification $H \times \bar{H} \subset G \times \Sigma_m \times \bar{G} \times \Sigma_m \subset G \times \bar{G} \times \Sigma_{m+m}$) is

$$S \otimes \left( \left( (\partial(\Delta^k \times \Delta^k))_+ \to (\Delta^k \times \Delta^k)_+ \right) \times (G \times \bar{G} \times \Sigma_{m+m}) / (H \times \bar{H}) \right)$$

which is a cofibration since $H \times \bar{H} \cap G \times \bar{G} \times \{\ast\} = \ast$.

That the pushout product of two cofibrations is a w.e. if either of them is follows by forgetting the $G$-action, since one is then computing the pushout product of two $S$-cofibrations in the sense of Definition 5.3.6 in [13] and Theorem 5.3.7 (part 5) in that paper applies.

**Proposition 4.2.** Let $\tilde{G} \subset G$ be finite groups, and suppose each category is equipped with the respective $S \Sigma$-inj $G$-proj stable model structure. Then both adjunctions

$$fgt : (\mathcal{S}p^{\Sigma})^G \rightleftarrows (\mathcal{S}p^{\Sigma})^\bar{G} : \mathcal{Q}$$

are Quillen adjunctions.

**Proof.** For the first adjunction the result follows due to the forgetful functor preserving all w.e.s together with the fact that forgetting a free $G$-action to a $\bar{G}$-action results in a free $\bar{G}$-action.

For the second adjunction, consider any generating cofibration

$$S \otimes \left( (\partial(\Delta^k))_+ \times (G \times \Sigma_m) / H \to (\Delta^k)_+ \times (\bar{G} \times \Sigma_m) / H \right), \quad m \geq 0, H \cap G \times \{\ast\} = \ast.$$

Applying $G \times \bar{G}$ to it yields the cofibration

$$S \otimes \left( (\partial(\Delta^k))_+ \times (G \times \Sigma_m) / H \to (\Delta^k)_+ \times (G \times \Sigma_m) / H \right), \quad m \geq 0, H \cap G \times \{\ast\} = \ast,$$

hence we conclude $G \times \bar{G}$ preserves all cofibrations.

That $G \times \bar{G}$ applied to a trivial cofibration yields a w.e. is clear by forgetting the actions since then $G \times \bar{G}$ is just a wedge over the cosets $G/\bar{G}$.

We now prove Theorem 1.7 which roughly says that the $S \Sigma$-inj $G$-proj stable model structure behaves “$G$-projectively” with respect to taking $G$ fixed points.

The two auxiliary monomorphism stable and $S$ stable model structures on $(\mathcal{S}p^{\Sigma})^G$ are discussed in Appendixes A.1 and A.2. Their relevance is as follows: in the first every object is cofibrant; in the second the cofibrations are the underlying $S$ cofibrations in $\mathcal{S}p^{\Sigma}$, which is the largest (known) class of cofibrations making $\mathcal{S}p^{\Sigma}$ into a monoidal model category.

**Proof of Theorem 1.7** The existence of the two required right adjoints is formal.

We will carry the proof of both parts in parallel.

The first step is to prove the result with the stable model structures replaced by their levelwise equivalence counterparts throughout. Since the generating cofibrations and generating trivial cofibrations in the $S \Sigma$-inj $G$-proj levelwise
model structure are all of the form $S \otimes f$ for some map $f$ of symmetric sequences, this first step reduces to proving the analogous results for the bifunctor

$$S^G \times \Sigma \times (\text{Sp}^\Sigma)^G \overset{\otimes^G}{\longrightarrow} \text{Sp}^\Sigma,$$

where $S^G \times \Sigma$ is given the $\Sigma$-inj $G$-proj model structure.

To prove the monomorphism case of the first step, consider chosen $A_m \overset{i}{\to} B_m$ a cofibration in $S^G \times \Sigma$ (concentrated at degree $m$) and $C \overset{f}{\to} D$ any monomorphism in $(\text{Sp}^\Sigma)^G$. One then has

$$(i \circ \otimes f)_m = \Sigma_m \times \Sigma_m \times \Sigma_{m-m} i \circ^\wedge f_{m-m}.$$

Forgetting the $\Sigma_m$-action, but not the $G$-action, one sees that $(i \circ \otimes f)_m$ is a wedge of copies of the map $i \circ^\wedge f_{m-m}$, and since $i$ is $G$-projective cofibrant in $S^G$, it indeed follows that after taking $G$-orbits these maps are monomorphisms which are also w.e.s if either $i$ or all the $f_{m-m}$ are.

For the $S$ case, notice first that by the previous case we need no longer worry about trivial cofibrations. For the regular cofibration case, consider generating cofibrations

$$\partial \Delta^k \times (G \times \Sigma_m)/H \to \Delta^k \times (G \times \Sigma_m)/H, \quad m \geq 0, H \cap G \times \{\ast\} = \ast$$

and

$$S \otimes \partial \Delta^k \times (G \times \Sigma_m)/\tilde{H} \to S \otimes \Delta^k \times (G \times \Sigma_m)/\tilde{H}, \quad m \geq 0, \text{ any } \tilde{H}$$

whose pushout product over after taking $G$-orbits is

$$S \otimes \left( (\partial (\Delta^k \times \Delta^k) \to \Delta^k \times \Delta^k) \times G/(G \times G \times \Sigma_{m+1}) / (H \times \tilde{H}) \right)$$

which is indeed a $S$ cofibration.

We now turn to the second step of showing that $\cdot \wedge_G \cdot$ remains a left Quillen bifunctor after left Bousfield localizing all model categories from their levelwise versions to their stable versions. Since cofibrations do not change after left Bousfield localization we only need to deal with the pushout product axiom when one map is a cofibration and the other a trivial cofibration. This means that for each of the four conditions left to check (i.e. for both parts (a) and (b) and depending on whether the trivial cofibration is in the first or second category) we only ever need to stabilize a single of the source categories to verify that condition. Since in all the categories we are dealing with the generating cofibrations have cofibrant domains and codomains we need only verify the condition in Corollary B.3.4 for each case (note that no cofibrant replacements are necessary). But in all four cases the set of maps being localized is the set $S_G$ defined in the proof of Theorem B.3.5 and the $G$-freeness of the maps in $S_G$ immediately implies

$$S_G \wedge_G X \cong S_\ast \wedge X,$$

reducing all four cases to their version when $G = \ast$. Since the case $G = \ast$ was already known from Theorem 5.3.7 (part 5) in [13] this finishes the proof.
4.2 $\Sigma_n$ cofibrancy of $n$-fold pushout products of positive cofibrations

**Note:** Throughout this subsection we set $G = \Sigma_n$.

Our goal in this subsection is to prove Theorem 1.6. Roughly speaking the proof follows by induction on a decomposition of $f$ as a retract of a transfinite composition of pushouts of generating cofibrations.

We start by proving Theorem 1.6 while assuming three auxiliary result, namely Lemma 4.5, which is necessary to handle pushouts, and Lemma 4.7 and Corollary 4.9 needed to handle compositions. The rest of the subsection is then dedicated to proving those auxiliary results.

**Proof of Theorem 1.6.** Note first that since w.e.s ignore the $\Sigma_n$ action and the $S$ stable model structure on $\text{Sp}^\Sigma$ is monoidal, we need only prove the result in the case of regular (non trivial) cofibrations.

We hence assume $f$ is a positive $S$ cofibration in $\text{Sp}^\Sigma$ and argue by induction on a decomposition of $f$ as a retract of a transfinite composition of pushouts of generating cofibrations.

The base case is that of a generating cofibration, i.e. a map of the form

$$S \otimes (\hat{\partial} \Delta^k \times \Sigma_m/H \to \Delta^k \times \Sigma_m/H), \quad m \geq 1,$$

whose $n$-fold pushout product is (using the identifications $H^n \subset (\Sigma_m)^n \subset \Sigma_{nm}$)

$$S \otimes (\hat{\partial} (\Delta^k)^n \times (\Sigma_{nm})/H^n \to (\Delta^k)^n \times (\Sigma_{nm})/H^n), \quad m \geq 1,$$

which is a $S$-$\Sigma$-$\Sigma_n$-$\Sigma$proj cofibration since the map

$$(\hat{\partial} (\Delta^k)^n \times (\Sigma_{nm})/H^n \to (\Delta^k)^n \times (\Sigma_{nm})/H^n$$

is built by adding only free $\Sigma_n$ simplices due to it being $m \geq 1$.

We are now left with the induction cases. For the transfinite composition argument, let

$$A = A_0 \overset{i_0}{\to} A_1 \overset{i_1}{\to} A_2 \overset{i_2}{\to} \ldots \to A_\kappa = \lim_{\beta < \kappa} A_\beta$$

be a transfinite composition of pushouts of the generating cofibrations, with $i_\kappa$ denoting the full composition $A_0 \to A_\kappa$.

Combining Lemma 4.5 applied to each $i_\beta$ with Corollary 4.9 applied to each square in the diagram below (note that this also covers the leftmost map) we conclude that the vertical maps in the diagram (minus the terminal one)

$\xymatrix{ Q_{n-1}^n(i_0) \ar[r] \ar[d] & Q_{n-1}^n(i_1 i_0) \ar[r] \ar[d] & Q_{n-1}^n(i_2 i_1 i_0) \ar[r] \ar[d] & \ldots \ar[r] & Q_{n-1}^n(i_\kappa) \ar[d] \\ A_1^n \ar[r] & A_2^n \ar[r] & A_3^n \ar[r] & \ldots \ar[r] & A_\kappa^n }$

form a $\kappa$-projective cofibration between $\kappa$-diagrams with respect to the underlying $S$-$\Sigma$-$\Sigma_n$-$\Sigma$proj model structure. It hence follows that their colimit too is a $S$-$\Sigma$-$\Sigma_n$-$\Sigma$proj cofibration, which is the desired result.

---

$^{10}$Recall that this means that the leftmost vertical map is an underlying cofibration and so are the “pushout corner maps” for each square.

$^{11}$Note that the $Q_{n-1}^n$ functors commute with filtered colimits, as follows from the fact that so do the $(\cdot)^n$ functors.
For the extra claims, note that applying the first part of the result to the map 
\( * \to A \) says that \( A^n, 0 \leq n \leq n \) is \( S \Sigma \text{-inj} \Sigma_n \text{-proj} \) cofibrant (as then the \( Q^n_{n-1} \) construction is just *). The additional conditions in Lemma 4.7 and Corollary 4.9 are now satisfied so the strengthened conclusions allow us to deduce the \( \kappa \)-cofibrancy for \( 0 \leq n < n \) of the vertical natural transformations

\[
\begin{array}{cccccc}
Q^n_n(i_0) & \rightarrow & Q^n_n(i_1i_0) & \rightarrow & Q^n_n(i_2i_1i_0) & \rightarrow & \ldots & \rightarrow & Q^n_{n+1}(i_\kappa) \\
\downarrow & & \downarrow & & \downarrow & & \ldots & & \downarrow \\
Q^n_{n+1}(i_0) & \rightarrow & Q^n_{n+1}(i_1i_0) & \rightarrow & Q^n_{n+1}(i_2i_1i_0) & \rightarrow & \ldots & \rightarrow & Q^n_{n+1}(i_\kappa),
\end{array}
\]

hence showing \( Q^n_n(i_\kappa) \to Q^n_{n+1}(i_\kappa) \) is a \( S \Sigma \text{-inj} \Sigma_n \text{-proj} \) cofibration. This finishes the proof since \( Q^n_0(i_\kappa) = A^n \).

We will now deal with the induction cases necessary to complete the proof above. We start with some terminology.

**Definition 4.3.** Let \( i : I \to \text{Sp}^\Sigma \) be any diagram.

We will let \( i^\wedge n \) denote the “cubical” diagram

\[ i^\wedge n : I^\times n \xrightarrow{\times n} (\text{Sp}^\Sigma)^\times n \xrightarrow{c} \text{Sp}^\Sigma. \]

Further, for \( T \subset I^\times n \) any subset symmetric with respect to the obvious \( \Sigma_n \) action we denote

\[ Q^n_T(i) = \lim_{T'}(i^\wedge n). \]

Note that \( Q^n_T(i) \) then comes equipped with a natural \( \Sigma_n \)-action.

**Remark 4.4.** Note that what is denoted by \( Q^n_T(i) \) in \( \square \) gets reinterpreted in this notation as \( Q^n_T(i) \) where \( i = X \to Y \) is viewed as a functor \( (0 \to 1)^\times n \to \text{Sp}^\Sigma \) and \( T_i \) is the set of objects of \( (0 \to 1)^\times n \) of objects with at most \( t \) 1-coordinates.

We can now tackle the easier case of pushouts.

**Lemma 4.5.** Consider a pushout diagram

\[
\begin{array}{ccc}
A & \rightarrow & C \\
\downarrow & \downarrow & \downarrow \\
B & \rightarrow & D
\end{array}
\]

(4.6)

If \( f^\times n \) is a cofibration or trivial cofibration in \( (\text{Sp}^\Sigma)^\Sigma_n \) for the \( S \Sigma \text{-inj} \Sigma_n \text{-proj} \) stable model structure then so is \( g^\times n \).

**Proof.** This follows immediately if we show that the diagram

\[
\begin{array}{cccc}
Q^n_{n-1}(f) & \rightarrow & Q^n_{n-1}(g) \\
f^\times n & \downarrow & \downarrow g^\times n \\
B^\times n & \rightarrow & D^\times n
\end{array}
\]

hence showing \( Q^n_{n-1}(f) \to Q^n_{n-1}(g) \) is a \( S \Sigma \text{-inj} \Sigma_n \text{-proj} \) cofibration. This finishes the proof since \( Q^n_0(g) = B^n \).
is itself a pushout diagram. This in turn will follow if we show that the \((n+1)\)-cube \(X_i = ((A \to B) \to ((C \to D) \to (\to))^{n})\) is cocartesian.\(^{13}\)

Now consider the \(2n\)-cube \(X = ((A \to B) \to (C \to D))^{n}\) which we regard as a \(n\)-cube of \(n\)-cubes, were the outer cube directions are those induced by the horizontal maps in diagram (118) and the inner cube directions those induced the vertical maps. Since diagram (118) is cocartesian we see that the “outer direction edges” (themselves \((n+1)\)-cubes) of the \(2n\)-cube are also cocartesian.

The proof is then finished by noticing that \(X_i\) is the “outer direction diagonal” of \(X\), hence a composition of “outer direction edges”.

We now turn to the case of two-fold compositions, the key being the following lemma. The proof of this result is essentially lifted from the appendix to the author’s thesis, where the same result for the \(\Sigma\)-\(\Sigma\)-proj\(-\)inj stable model structure was proven.\(^{14}\) A similar result, with modified hypotheses and conclusions but using some of the key ideas in the proof, was proven independently by David White in \cite{20}.

Lemma 4.7. Let \(i: (0 \to 1 \to 2) \to \text{Sp}^{\Sigma}\) be a diagram \(Z_0 \xrightarrow{f_1} Z_1 \xrightarrow{f_2} Z_2\) such that

\[
f_i^\cap : Q_0^{\cap_{i}} \to Z_i^\cap, \quad 0 \leq i \leq n, i = 1, 2
\]

are cofibrations in \((\text{Sp}^{\Sigma})^{\Sigma_n}\) for the \(\Sigma\)-\(\Sigma\)-proj\()-inj\) stable model structure.

Suppose further \(T \subset \hat{T} \subset (0 \to 1 \to 2) \to \Sigma\)-\(\Sigma\)-proj stable subsets containing any point that has at least one \(0\) coordinate. Then the map

\[
Q_T^n(i) \to Q_T^\cap (i)
\]

is a cofibration in the \(\Sigma\)-\(\Sigma\)-proj\()-inj\) stable model structure.

Additionally, if one also knows that \(Z_0^\cap, \quad 0 \leq i \leq n\) is \(\Sigma\)-\(\Sigma\)-inj\)-\(\Sigma\)-proj\) cofibrant then the conclusion above holds for any downward closed \(T \subset \hat{T}\).

Proof. We will deal with the main and additional cases simultaneously.

Without loss of generality we assume \(\hat{T}\) is obtained from \(T\) by adding the orbit of a single point \(e \in \{0\}^{x_{n_0}} \times \{1\}^{x_{n_1}} \times \{2\}^{x_{n_2}}\).

Now letting \(T_e\) be the subset of points under (i.e. mapping to) and different from \(e\) (note \(T_e \subset \hat{T}\) by our hypotheses), one has a pushout diagram

\[
\begin{array}{ccc}
\Sigma_n^{\times} & \times & Q_T^n(i) \\
\Sigma_n^{\times} \times \Sigma_{n_0} \times \Sigma_{n_1} \times \Sigma_{n_2} & \xrightarrow{Q_T^n(i)} & Q_T^\cap (i) \\
\Sigma_n^{\times} \times \Sigma_{n_0} \times \Sigma_{n_1} \times \Sigma_{n_2} & \xrightarrow{Z_0^{\cap_0} \land Z_1^{\cap_1} \land Z_2^{\cap_2}} & Q_T^\cap (i).
\end{array}
\]

We need hence only show that the left hand map is a \(\Sigma\)-\(\Sigma\)-inj\)-\(\Sigma\)-proj\) cofibration, and by Proposition\(^{14}\) this will follow if \(Q_T^n(i) \to Z_0^{\cap_0} \land Z_1^{\cap_1} \land Z_2^{\cap_2}\) is a \(\Sigma\)-\(\Sigma\)-inj\)-\(\Sigma\)-\(\Sigma\)-proj\) cofibration.

\(^{12}\)Recall that a cubical diagram is called cocartesian if its terminal object is the colimit of the rest of the diagram.

\(^{13}\)A fact which is however ultimately not useful due to the \(\Sigma\)-\(\Sigma\)-proj\)\()-inj\) analogue of Theorem\(^{11}\) not being true.

\(^{14}\)We say that \(T\) is downward closed if for any object \(x \in T\) and map \(y \to x\), so is \(y \in T\).
Now notice that $T_e = T^2_e \cup T^2_1$, with $T^2_e$ the subset of points where one “reduces” at least one of the 2-coordinates of $e$, $T^2_1$ the subset where one “reduces” at least one of the 1-coordinates, and $T^2_{e^1}$ the intersection, where one performs both “reductions”. Since these “reductions” depend on different coordinates of $e$, and $\wedge$ preserves colimits in each variable, it follows that

\[
\begin{align*}
Q^n_{T^2_e}(i) &= Z^n_0 \wedge Z^n_{1} \wedge Q^n_{2}(f_2), \\
Q^n_{T^2_1}(i) &= Z^n_0 \wedge Q^n_{n_1-1}(f_1) \wedge Z^n_{2}, \\
Q^n_{T^2_{e^1}}(i) &= Z^n_0 \wedge Q^n_{n_1-1}(f_1) \wedge Q^n_{n_2-1}(f_2).
\end{align*}
\]

We now see that the map $Q^n_{T^2_1}(i) \to Z^n_0 \wedge Z^n_{1} \wedge Z^n_{2}$ is given by smashing $Z^n_0$ with the pushout product of the maps $Q^{n_1-1}_{n_1}(f_1) \to Z^n_{1}$ and $Q^{n_2-1}_{n_2}(f_2) \to Z^n_{2}$.

Now note that in the main case $T$ already contains all points with a 0 coordinate so that it must be $n_0 = 0$, while in the additional case $n_0$ can take any value. In either case one then has the necessary cofibrancy conditions to apply Proposition \[4.1\] and finish the proof.

\[\square\]

\textbf{Remark 4.8.} While nothing in the previous proof relies specifically on using diagrams indexed by $0 \to 1 \to 2$, with similar arguments applying to finite linear indexing categories like $0 \to 1 \to 2 \to 3$ (or even to transfinite ones if one adds adequate transfinite arguments), this case turns out to be sufficient while keeping the notation reasonably simple, and we hence restrict ourselves to it.

\textbf{Corollary 4.9.} Let $Z_0 \xrightarrow{f_1} Z_1 \xrightarrow{f_2} Z_2$ be as in Lemma \[4.7\].

If one knows additionally that $Z^n_0$, $0 \leq n \leq n$ is $\Sigma$-$\text{proj}$, then the maps (where the $Q^n_k$ objects are defined in Remark \[4.4\])

\[
Q^n_k(f_2 f_1) \cup Q^n_{k+1}(f_1) \to Q^n_{k+1}(f_2 f_1), \\
0 \leq n \leq n
\]

are cofibrations in the $\Sigma$-$\text{proj}$ $\Sigma$-$\text{proj}$ stable model structure.

Further, in the absence of the additional condition the result above still holds when $\bar{n} = n - 1$.

\textbf{Proof.} This is a direct consequence of Lemma \[4.7\] by identifying all objects with $Q^n_k(i)$ for some $T$. For $Q^n_k(f_1)$ this is $T^1_k$, the subset of tuples with no 2-coordinates and at most $k$ 1-coordinates, while for $Q^n_k(f_2 f_1)$ it is $T^2_k$, the subset of tuples with at least $n - k$ 0 coordinates (or equivalently, at most $k$ 2-or-1-coordinates). The result then follows by noting that $T^2_k \wedge T^1_{k+1} = T^1_{k}$ and $T^2_k \cup T^1_{k+1} = T^2_{k+1}$.

\[\square\]

\section{Cofibrancy of operadic constructions}

Our goal in this section is to prove Theorems \[1.1\] and \[1.5\].

In subsection \[5.1\] we recall the necessary operadic terminology along with some basic results. Subsection \[5.2\] is dedicated to proving Proposition \[5.1.7\], a filtration result which is key to the proof of Theorem \[1.5\]. Subsection \[5.3\] introduces the necessary model structures in the category of $\text{Sym}$ of spectral symmetric sequences and proves for them analogues of the key results from Section \[1\]. Finally, subsection \[5.4\] proves the main results.

To obtain the $Q^n_{n_2-1}(f_2)$ factors one uses cofinality to show one can ignore those cases where a 2-coordinate is “reduced” to a 0-coordinate.
5.1 Definitions: operads, modules and algebras

In this section we recall some standard definitions concerning operads. We will present our definitions in terms of a general closed symmetric monoidal category \( \mathcal{C} \), a fact which is key in substantially reducing the amount of work needed. Indeed, as the proof of Theorem 1.5 shows, even when proving only the algebra’s version of the result, one is required to understand filtrations of both algebras and modules, and it is hence convenient to make use of Proposition 5.13 so as to unify those two types of filtration.

**Definition 5.1.** Let \( (\mathcal{C}, \otimes, \mathbf{1}) \) denote a closed symmetric monoidal category.

Then the category \( \text{Sym}(\mathcal{C}) \) of symmetric sequences in \( \mathcal{C} \) is the category of functors \( \Sigma \rightarrow \mathcal{C} \).

Further, for \( G \) a finite group the category \( \text{Sym}^G(\mathcal{C}) \) of \( G \)-symmetric sequences in \( \mathcal{C} \) is the category of functors \( G \rightarrow \text{Sym}(\mathcal{C}) \).

**Remark 5.2.** Note that a symmetric sequence \( X \) is formed by objects \( X^p \) each with a \( \Sigma^r \) action. To avoid confusing indexes when \( \mathcal{C} \) is a related category, we will reserve the letter \( r \) for this external index, while keeping \( m \) for the internal spectral index (so that \( X_m^r \) denotes the \( m \)-th space of \( X^r \)).

We now introduce the two usual monoidal structures on \( \text{Sym}(\mathcal{C}) \). Roughly speaking, the composition product \( \circ \) is the most important of the two, with \( \hat{\otimes} \) playing an auxiliary role.

**Definition 5.3.** Given \( X, Y \in \text{Sym}(\mathcal{C}) \) we define their tensor product to be

\[
(X \hat{\otimes} Y)(r) = \bigvee_{0 \leq r \leq r'} \Sigma_r \times \Sigma_r \times \Sigma_{r-r'} X^{\hat{r}} \land Y^{(r-r')}
\]

and their composition product to be

\[
(X \circ Y)(r) = \bigvee_{r \geq 0} X^{\hat{r}} \land \Sigma_r (Y^{\circ r})(r).
\] (5.4)

One has the following result (for a discussion of reflexive coequalizers see Definition 3.26 in [7] and the propositions immediately after it).

**Proposition 5.5.** Let \( (\mathcal{C}, \otimes, \mathbf{1}) \) be a closed symmetric monoidal category with initial object \( \emptyset \). Then

- \( (\text{Sym}, \otimes, \mathbf{1}) \) is a closed symmetric monoidal category, with unit
  \[
  \mathbf{1}(0) = \mathbf{1}, \quad \mathbf{1}(r) = \emptyset, r \geq 1;
  \]

- \( (\text{Sym}, \hat{\otimes}, I) \) is a (non-symmetric) monoidal category, with unit
  \[
  I(1) = \mathbf{1}, \quad I(r) = \emptyset, r \neq 1.
  \]

Further, \( \circ \) commutes with all colimits in the first variable and with filtered colimits and reflexive coequalizers in the second variable.

**Definition 5.6.** An operad \( \mathcal{O} \) in \( \mathcal{C} \) is a monoid object in \( \text{Sym}(\mathcal{C}) \) with respect to \( \circ \), i.e., a symmetric sequence \( \mathcal{O} \) together with multiplication and unit maps

\[
\mathcal{O} \circ \mathcal{O} \rightarrow \mathcal{O}, \quad I \rightarrow \mathcal{O}
\]
satisfying the usual associativity and unit conditions.
Definition 5.7. Let $O$ be an operad in $C$. A left module $L$ (resp. right module $R$) over $O$ is an object in $\text{Sym}(C)$ together with a map

$$O \circ L \to L \quad \text{(resp. } R \circ O \to R)$$

which satisfies the usual associativity and unit conditions. The category of left modules (resp right modules) over $O$ is denoted $\text{Mod}_O^L$ (resp. $\text{Mod}_O^R$).

Further, left modules $X$ over $O$ concentrated\footnote{I.e. such that $X(r) = \emptyset$ for $r \geq 1$.} in degree 0 are called algebras over $O$. The category of algebras over $O$ is denoted $\text{Alg}_O$.

Proposition 5.8. The categories $\text{Mod}_O^L$, $\text{Mod}_O^R$ and $\text{Alg}_O$ have all small limits and colimits.

Further, all limits and colimits in $\text{Mod}_O^L$ are calculated underlyingly, and all limits, filtered colimits and reflexive coequalizers are calculated underlyingly in both $\text{Mod}_O^L$ and $\text{Alg}_O$.

Remark 5.9. We will also be interested in the analogue of Definition 5.7 for the category $\text{Sym}^G(C)$ of symmetric sequences with a $G$-action for $G$ a finite group.

It is then formal to check that Proposition 5.5 is also satisfied for $\text{Sym}^G(C)$ using the same monoidal structures $\otimes$ and $\circ$ (now with a diagonal $G$-action) and units (now with a trivial $G$-action).

Operads and their left and right modules and algebras in $\text{Sym}^G(C)$ are defined just as above.

By iterating the $\text{Sym}$ construction we are able to use Proposition 5.13 to reduce the study of left modules to that of algebras.

Definition 5.10. The category $\text{BSym}(C)$ of bi-symmetric sequences in $C$ is the category $\text{Sym}(\text{Sym}(C))$ of symmetric sequences of symmetric sequences in $C$.

Remark 5.11. Note that an object $X \in \text{BSym}(C)$ is composed of objects $X(r, s) \in C$ with $\Sigma_r \times \Sigma_s$-actions, so that one has two different inclusions

$$(\cdot)^{1}: \text{Sym}(C) \to \text{BSym}(C), \quad (\cdot)^{2}: \text{Sym}(C) \to \text{Sym}(C)$$

defined by

$$X^{1}(r, s) = \begin{cases} X(r), & \text{if } s = 0 \\ \emptyset, & \text{if } s \neq 0 \end{cases}, \quad X^{2}(r, s) = \begin{cases} X(s), & \text{if } r = 0 \\ \emptyset, & \text{if } r \neq 0 \end{cases}.$$ 

Following Definition 5.3 one can then build two monoidal structures in $\text{BSym}(C)$, which we denote by $\hat{\otimes}$ and $\hat{\circ}$, respectively, where $\hat{\otimes}$ is built using $s$ as the “operadic” index. Note that while $\hat{\otimes}$ behaves symmetrically with respect to the indexes $r$ and $s$, $\hat{\circ}$ does not.

Both of the following propositions follow by a straightforward computation.

Proposition 5.12. Both functors $(\cdot)^{1}, (\cdot)^{2}$ are monoidal functors from to the symmetric monoidal structure $\hat{\otimes}$ to the symmetric monoidal structure $\hat{\circ}$.

$(\cdot)^{2}$ is a monoidal functor from the monoidal structure $\circ$ to the monoidal structure $\hat{\circ}$.

Proposition 5.13. Let $O$ be an operad in $C$. Then there is a natural isomorphism of categories

$$(\cdot)^{1}: \text{Mod}_O^{L}(C) \xrightarrow{\cong} \text{Alg}_O^{\text{Sym}(C)}.$$
5.2 Filtrations

Our main goal in this subsection is to prove Proposition 5.17, which provides filtrations necessary to prove Theorem 1.5. These filtrations are strongly motivated by previous filtrations found in the literature (such as in \([2]\) and \([7]\), among others), but the key claim that one can still get such filtrations after applying \(M \circ \cdot\) functors seems to be new. This extra claim is enabled by the following somewhat novel definition of the \(O_A\) sequences.

**Definition 5.14.** Let \(O\) be an operad in \(C\) and \(A \in \text{Alg}_O\) regarded as an element in \(\text{Mod}^L_O\).

We define

\[
O_A = O \coprod A,
\]

were the coproduct is taken in the category \(\text{Mod}^L_O\).

Additionally, for \(M \in \text{Mod}^R_O\) we define

\[
M_A = M \circ_O O_A.
\]

**Remark 5.15.** Note that there are adjunctions

\[
\iota: \text{Alg}_O \rightleftarrows \text{Mod}^L_O: (\cdot)_{0} \quad (\cdot)_{0}: \text{Mod}^L_O \rightleftarrows \text{Alg}_O: \hat{\cdot},
\]

where \(\iota\) is the inclusion and \(\hat{A}(0) = A, \hat{A}(r) = \ast\), for \(r \geq 1\).

This implies in particular that colimits in \(\text{Alg}_O\) can be computed after the inclusion into \(\text{Mod}^L_O\) and that \(O_A(0) = A\).

**Proposition 5.16.** Let \(A \in \text{Alg}_O(C)\) and \(X \in \text{Sym}(C)\). Then there is a natural isomorphism of functors

\[
O_A \circ X \simeq A \coprod (O \circ X),
\]

where the coproduct is calculated in \(\text{Alg}_O\).

Additionally, for any \(M \in \text{Mod}^R_O\) we have a natural isomorphism

\[
M_A \circ X \simeq M \circ_O (A \coprod (O \circ X)).
\]

**Proof.**

\[
O_A \circ X = (O \coprod A) \circ X \simeq O \circ X \coprod A \circ X = O \circ X \coprod A
\]

where in the second step we use the fact that \(\cdot \circ X\) commutes with colimits in \(\text{Mod}^L_O\) (this follows from the fact that any \(B \in \text{Mod}^L_O\) is canonically the coequalizer of \(O \circ O \circ B \rightrightarrows O \circ B\) and the fact that \(\cdot \circ X\) commutes with the free left module functor \(O \circ \cdot\) and the third step uses the equality \(A \circ X = A\), which holds due to \(A\) being concentrated in degree 0.

The additional claim involving a \(M \in \text{Mod}^R_O\) is obvious from the previous case.

The following is the key result in this subsection. For the definition of \(Q_{r-1}(f)\) check Remark 4.4.
Proposition 5.17. Consider any pushout in \( \text{Alg}_C(\mathbf{C}) \) of the form
\[
\begin{array}{ccc}
\mathcal{O} \circ X & \longrightarrow & A \\
\mathcal{O} \circ f & \downarrow & \\
\mathcal{O} \circ Y & \longrightarrow & B,
\end{array}
\]
and let \( M \in \text{Mod}_\mathcal{O}^R \).

Then, in the underlying category \( \mathbf{C} \),
\[
M \circ \mathcal{O} B \cong \underset{\longrightarrow}{\text{colim}}(A_0^M \rightarrow A_1^M \rightarrow A_2^M \rightarrow \cdots)
\]
(5.18)
where \( A_0^M = M \circ \mathcal{O} A \) and the \( A_r^M \) are built inductively from pushout diagrams
\[
\begin{array}{ccc}
M_A(r) \otimes Q_r^{-1}(f) & \longrightarrow & A_r^{M-1} \\
\downarrow & & \downarrow \\
M_A(r) \otimes Y^r & \longrightarrow & A_r^M.
\end{array}
\]
(5.19)

The following proof is essentially adapted from that of Proposition 4.20 in [7]. However, we include it here both for completeness and because our discussion here is quite significantly “repackaged”, making it much more categorical.

Proof. By general considerations one can describe \( B \) as a reflexive coequalizer
\[
B = \text{coeq} \left( \bigprod A \big( (\mathcal{O} \circ (X \coprod Y)) \rightrightarrows \bigprod \mathcal{O} \circ Y \right) \right),
\]
and hence using the fact that \( M \circ \mathcal{O} \cdot \) preserves reflexive coequalizers \( ^{17} \) combined with Proposition 5.16 we conclude that \( M \circ \mathcal{O} B \) is the reflexive coequalizer
\[
M \circ \mathcal{O} B = \text{coeq} \left( M_A \circ (X \coprod Y) \rightrightarrows M_A \circ Y \right).
\]
(5.20)

Now note that
\[
M_A \circ (X \coprod Y) = \bigvee_{r_x, r_y \geq 0} M_A(r_x + r_y) \otimes \Sigma_{r_x} \times \Sigma_{r_y} X^{r_x} \wedge Y^{r_y},
\]
with \( M_A \circ Y \) being naturally identified with the subobject formed by the wedge summands with \( r_x = 0 \), so that \( M \circ \mathcal{O} B \) is hence naturally identified as a quotient of the wedge sum above. Further, using the naturality in Proposition 5.16 one see that the maps being equalized in (5.20) both send wedge summands to wedge summands and we can hence rewrite
\[
M \circ \mathcal{O} B = \underset{\longrightarrow}{\text{colim}}_{D^a} F^a
\]
(5.21)
where \( D^a \) is a diagram category whose objects we think of as “abelian monomials” of the form \( X^{r_x} Y^{r_y} \) and whose only non identity morphisms are unique maps \( X^{r_x} Y^{r_y} \rightarrow Y^{r_x+r_y} \) and \( X^{r_x} Y^{r_y} \rightarrow Y^{r_y} \), with each type of maps thought of as induced by each of the maps being equalized in (5.20). \( F^a \) is defined by
\[
F^a(X^{r_x} Y^{r_y}) = M_A(r_x + r_y) \otimes \Sigma_{r_x} \times \Sigma_{r_y} X^{r_x} \otimes Y^{r_y}.
\]

\(^{17}\)This follows from the fact that \( \cdot \circ \cdot \) preserves reflexive coequalizers in the second variable.
At this point one would like to deduce the filtration in (5.18) from a filtration of $D^a$, but as it turns out $D^a$ lacks some necessary relations which can not be captured by simply enlarging $D^a$ due to its “abelian” nature.

The trick is to replace $D^a$ by a “non-abelian” analogue $D^{na}$ whose objects can be thought of as “non-abelian monomials”. To build $D^{na}$, start by letting $D$ be the Grothendieck construction of the functor

$$\text{Fin}^{op} \to \text{Cat}, \quad \underline{L} \mapsto (X \to Y)^{\underline{L}},$$

where $\text{Fin}$ denotes (a skeleton) of the category of finite sets and injections.

Explicitly, the objects of $D$ are pairs

$$(\underline{r}, m: \underline{r} \to \{X, Y\})$$

and an arrow $$(\underline{r}, m) \to (\underline{r}', m')$$ is a pair

$$(\iota: \underline{r}' \hookrightarrow \underline{r}, m \circ \iota \to m').$$

Intuitively, we think of the objects of $D$ as “non-abelian monomials” $m$ in the variables $X$ and $Y$, and of a map $(\iota, m \circ \iota \to m')$ as first “dropping” some of the terms in the monomial $m$ and reordering the remaining ones to get a monomial $m\circ\iota$, and then replacing (while now maintaining the order) some $X$’s by $Y$’s to obtain the monomial $m'$.

We now define two subcategories $\bar{D}^{na}$ and $D^{na}$, both with the same set of objects. The maps in $\bar{D}^{na}$ are characterized by the condition

$$\#m^{-1}(Y) = \#(m \circ \iota)^{-1}(Y)$$

while $D^{na}$ is the further subcategory such that for any non identity map either

$$(m \circ \iota)^{-1}(X) = \emptyset \text{ or } \iota \in \Sigma_{\underline{r}} \text{ and } (m')^{-1}(X) = \emptyset.$$  

Keeping with the intuitive description from before, $D^{na}$ is the subcategory of $D$ containing only those maps which never “drop” a $Y$ coordinate, while $\bar{D}^{na}$ is the further subcategory where non identity maps either always “drop” all $X$’s or they turn all $X$’s into $Y$’s.

It is now clear from the intuitive description above that one has a “forgetful” functor $\pi: D^{na} \to D$ which just “abelianizes” the monomials. We will now use this to replace the colimit in (5.21) by one over $\bar{D}^{na}$. For this we use the functor $F^{na}: D^{na} \to \mathcal{C}$ defined by

$$F^{na}(\underline{r}, m) = M_A(\underline{r}) \otimes m(X, Y),$$

where $m(X, Y)$ denotes the product in $\mathcal{C}$ suggested by the monomial $m$. To see what $F^{na}$ does to a map $(\iota, m \circ \iota \to m')$, note that using the map $\mathcal{O}X \to A$ to get rid of the first $X$ we get a map

$$M \circ \mathcal{O} (A \coprod \mathcal{O}(X) \coprod \mathcal{O}(Y)) \to M \circ \mathcal{O} (A \coprod \mathcal{O}(X) \coprod \mathcal{O}(X) \coprod \mathcal{O}(Y)),$$

and using equation (5.24) to reinterpret both sides we obtain the action of $F^{na}$ on the “$\iota$ component” of the map. The action of $F^{na}$ on the “$m \circ \iota \to m'$ component” follows from the obvious maps $(m \circ \iota)(X, Y) \to m'(X, Y)$. Associativity

\footnote{Note that since $(X \to Y)^{\underline{L}}$ is a poset there are no choice for the map $m \circ \iota \to m'$.}
can be checked by applying \[5.21\] to \(M \circ \mathcal{O}(A) \coprod \mathcal{O}(X) \coprod \mathcal{O}(X) \coprod \mathcal{O}(Y)\) and using the fact that “getting rid” of the first two \(X\)’s one at a time is the same as doing it at once.

We now claim that (where \(F^{na}\) is just the restriction of \(F^{na}\))

\[
M \circ \mathcal{O} B = \text{colim}_{D^{na}} F^{na} = \text{colim}_{D^{na}} F^{na} = \text{colim}_{D^{na}} F^{na}.
\]

The first new equality will follow if we show that \(F^{na} = L \text{Kan}_{\mathcal{C}} F^{na}\). This is clear when evaluated at a \(X^{r \times Y^{r'}}\) such that \(r_X \neq 0\), since then the slice category \(D^{na} \downarrow X^{r \times Y^{r'}}\) is a groupoid. When \(r_X = 0\) the categories \(D^{na} \downarrow Y^{r'}\) are more interesting, consisting of a special \(Y^{r'}\) object \(^2\) to which all other objects map, those being all the non isomorphisms. Since the automorphisms of \(Y^{r'}\) act transitively on any set \(\text{Hom}(\cdot, Y^{r'})\), it is then clear that those its automorphisms are final in \(D^{na} \downarrow Y^{r'}\), proving the Kan extension claim.

The second new equality follows from the easy claim that \(D^{na}\) is final \(^{21}\) in \(D^{na}\).

Now consider the full subcategories \(\bar{D}^{na}_{\leq r}, \bar{D}^{na}_r\) of \(D^{na}\) formed respectively by those objects of degrees \(\leq r\) and \(= r\). It formally follows that one has a filtration of the form \([5.13]\) by setting \(\bar{A}_r^{na} = \text{colim}_{D^{na}_{\leq r}} F^{na}\), and one needs hence only prove an inductive description as in \([5.19]\).

Letting \(\bar{D}^{na}_{\leq r} - Y^{r}\) be the full subcategory obtained by removing \(Y^{r}\) and noting that \(N(\bar{D}^{na}_{\leq r}) = N(\bar{D}^{na}_{\leq r} - Y^{r}) \cup N(\bar{D}^{na}_r)\) we obtain a pushout diagram

\[
\begin{array}{ccc}
\text{colim}_{\bar{D}^{na}_{\leq r} - Y^{r}} F^{na} & \longrightarrow & \text{colim}_{\bar{D}^{na}_{\leq r} - Y^{r}} F^{na} \\
\downarrow & & \downarrow \\
\text{colim}_{\bar{D}^{na}_{\leq r}} F^{na} & \longrightarrow & \text{colim}_{\bar{D}^{na}_{\leq r}} F^{na},
\end{array}
\]

which we want to identify with the diagram in \([5.19]\).

We need now identify this with the diagram in \([5.19]\). For the right bottom corner this is tautological and for the left hand vertical map it follows by noting that \(\bar{D}^{na}_{\leq r}\) is the Grothendieck construction of \((X \to Y)^{\leq r}\) (regarded as a functor \(\Sigma_{\leq r} \to \text{Cat}\)). Finally, the identification of the right top corner follows by noting that \(\bar{D}^{na}_{\leq r - 1}\) is terminal in \(\bar{D}^{na}_{\leq r} - Y^{r}\), finishing the proof.

\[
\square
\]

**Remark 5.22.** In analogy to Proposition 4.20 in \([7]\), which Proposition \([5.17]\) generalizes, we also want this result to cover the category \(\text{Mod}_{\mathcal{O}}(\mathcal{C})\), but this is now automatic by Proposition \([5.13]\).

This works mostly in the obvious way, with occurrences of the category \(\mathcal{C}\) replaced by \(\text{Sym}(\mathcal{C})\) and \(\otimes\) replaced by \(\otimes\), though identifying the replacement for \(M_A\) when \(A\) is now in \(\text{Mod}_{\mathcal{O}}^L\) requires some care, being given by

\[
M^L_A = M^L \otimes_{\mathcal{O}} (\mathcal{O}^{L} \coprod A'),
\]

where the coproduct is taken in \(\text{Mod}_{\mathcal{O}}^L(\text{Sym}(\mathcal{C}))\).

---

\(^{19}\)See Theorem 1.3.5 in \([10]\) for the slice category formula for Kan extensions.

\(^{20}\)Here we abuse notation by denoting the “non-abelian monomial” which is the “product” of \(r_Y Y\)’s by \(Y^{r'}\) as well.

\(^{21}\)See Lemma 8.3.4 in \([16]\) for a slice category characterization of final functors.
The following basic identification was used when proving Proposition 5.1 and will also be necessary in the proof of Theorem 1.5.

**Proposition 5.23.** Let $O$ be an operad in $C$. Then for $M \in \text{Mod}_O^R(C)$, $X \in C$ and $A \in \text{Alg}_O(C)$ there are isomorphisms

$$
(M \circ_O (A \coprod O \coprod O X))(r) \simeq \coprod_{r \geq 0} M_A(r + \bar{r}) \otimes_{\Sigma_r} X^r
$$

natural in all three variables.

**Proof.** This follows by formal manipulations using the $(\cdot)^r, (\cdot)^s$ functors.

Firstly, we identify the left hand side with

$$
M^s \circ_{O^s} \left( A' \coprod O^r \coprod O^s \circ^s X' \right),
$$

so that Proposition 5.10 applied to $\text{Sym}(C)$ allows us to identify this with

$$
M^s_{A' \coprod O^r} \circ^s X'.
$$

But now note that

$$
M^s_{A' \coprod O^r} = M^s \circ_{O^s} \left( A' \coprod O^r \coprod O^s \right) \simeq M^s \circ_{O^s} \left( (A') \coprod O^s \circ^s (I^s \coprod I') \right)
$$

$$
\simeq M^s \circ^s (I^s \coprod I') \simeq (M_A)^s \circ^s (I^s \coprod I')
$$

showing that $M^s_{A' \coprod O^r}(r, s) = M_A(r + s)$. Plugging this into (5.25) finishes the proof. $\square$

### 5.3 Model structures on Sym and Sym$^G$

**Note:** In this section and the following we abbreviate $\text{Sym}((\text{Sp}^\Sigma)^G)$ simply as $\text{Sym}$.

We now introduce the key model structures on $\text{Sym}$ we will need and prove for them analogues of the main results in Section 4.

**Definition 5.26.** The $S$ stable model structure on $\text{Sym}$ is obtained by combining the $S$ stable model structures in $((\text{Sp}^\Sigma)^G)^{\Sigma_r}$ in all degrees (see Appendix A.2 for the definition and properties).

**Definition 5.27.** The positive $S$ stable model structure on $\text{Sym}$ is the model structure obtained by combining the positive $S$ stable model structure in $\text{Sp}^\Sigma$ on degree $r = 0$ with the $S$ stable model structures in $((\text{Sp}^\Sigma)^G)^{\Sigma_r}$ in degrees $r \geq 1$ (see Appendix A.2 for the definition and properties).

**Remark 5.28.** To understand the choice of the word “positive” in the definition above, recall that a $X \in \text{Sym}$ is composed of pointed simplicial spaces $X_m(r)$, making it a “bigraded object”. One can then check that $\otimes$ is additive in both gradings, making it natural to consider $m + r$ to be a “total grading”, and the term “positive” hence refers to this “total grading”.

We will also want to have an analogue for $\text{Sym}^G$ of the $\Sigma$-inj $G$-proj $S$ stable model structure on $((\text{Sp}^\Sigma)^G)^G$.

---

22The isomorphism $M^s_{A'} = (M_A)^s$ follows from $A$ being an algebra, so that $A' = A^s$. 30
Definition 5.29. The \( S \Sigma \times \Sigma\)-inj G-proj stable model structure on \( \text{Sym}^G \) is the model structure obtained by combining the \( S \Sigma_r \times \Sigma\)-inj G-proj stable model structures for all degrees (see Remark A.12).

As a direct consequence of Propositions 4.1 and 4.2 and Theorems 1.7 and 1.6, we now obtain the analogous results for the operation \( \otimes \) in \( \text{Sym} \).

Proposition 5.30. Suppose all the categories are equipped with their respective \( S \Sigma \times \Sigma\)-inj G-proj stable model structure. Then the functor
\[
\text{Sym}^G \times \text{Sym}^G \xrightarrow{\otimes} \text{Sym}^{G \times G}
\]
is a left Quillen bifunctor.

Proof. The existence of the two necessary right adjoints is formal. Now recall that
\[
(X \otimes Y)(r) = \bigvee_{0 \leq r' \leq r} \Sigma_r \times \Sigma_r \times \Sigma_{r-r} X(\bar{r}) \wedge Y(r - \bar{r}).
\]
By injectiveness of the model structures (Corollary A.11), we can ignore all of the \( \Sigma_r, \Sigma_r \) and \( \Sigma_{r-r} \) actions, and we hence see that \( \otimes \) is a wedge of bifunctors to each of which Proposition 4.1 applies.

Proposition 5.31. Let \( \bar{G} \subset G \) be finite groups, and suppose each category is equipped with the respective \( S \Sigma \times \Sigma\)-inj G-proj stable model structure. Then both adjunctions
\[
fgt : \text{Sym}^G \rightleftarrows \text{Sym}^{G \times \bar{G}} : \text{Sp}^\Sigma^G \\
G \times_G \cdot : \text{Sp}^\Sigma^{\bar{G}} \rightleftarrows \text{Sp}^\Sigma^G : \text{fgt}
\]
are Quillen adjunctions.

Proof. Both claims are obvious from Proposition 4.2 since we are dealing with injective model structures (Corollary A.11).

Proposition 5.32. Consider the functor
\[
\text{Sym}^G \times \text{Sym}^G \xrightarrow{\cdot \Delta_G} \text{Sym},
\]
where the first copy of \( \text{Sym}^G \) is regarded as equipped with the \( S \Sigma \times \Sigma\)-inj G-proj stable model structure.

Then \( \cdot \Delta_G \) is a left Quillen bifunctor if either:

(a) Both the second \( \text{Sym}^G \) and the target \( \text{Sym} \) are equipped with the respective monomorphism stable model structures;

(b) Both the second \( \text{Sym}^G \) and the target \( \text{Sym} \) are equipped with the respective \( S \) stable model structures.

Proof. This follows immediately by combining the “wedge of bifunctors” argument from the proof of Proposition 5.30 with Theorem 1.7.
Proposition 5.33. Let Sym be equipped with the positive $S$ stable model structure and $\text{Sym}^\Sigma_n$ with the $S \times \Sigma$-$\text{inj}$ $\Sigma_n$-$\text{proj}$ stable model structure.

Then for $f: A \to B$ a cofibration in Sym its $n$-fold pushout product

$$f^{\otimes n}: Q_n(f) \to B$$

is a cofibration in $\text{Sym}^\Sigma_n$, which is a w.e. when $f$ is.

Furthermore, if $A$ is cofibrant in Sym then $Q_n(f)$ (resp. $A^{\otimes n} \xrightarrow{f^{\otimes n}} B^{\otimes n}$) is cofibrant (resp. cofibration between cofibrant objects) in $\text{Sym}^\Sigma_n$.

Proof. Computing $X_1 \otimes \cdots \otimes X_n$ iteratively and regrouping terms we get

$$(X_1 \otimes \cdots \otimes X_n)(r) = \bigvee_{\phi: \underline{\Sigma} \to \underline{n}} X_1(\phi^{-1}(1)) \wedge \cdots \wedge X_n(\phi^{-1}(n)).$$

Noting that the symmetric monoidal structure isomorphisms\(^{23}\) for $\otimes$ involve a $\Sigma_n$-action that interchanges wedge summands (and since we don’t care about the $\Sigma_n$ action for cofibrancy purposes) we can $\Sigma_n$ equivariantly decompose the functor above as

$$\bigvee_{(\phi) \in \{\phi: \underline{\Sigma} \to \underline{n}\}/\Sigma_n} \bigvee_{\phi(\bar{\phi})} X_1(\phi^{-1}(1)) \wedge \cdots \wedge X_n(\phi^{-1}(n)),$$

where the first wedge is over $\Sigma_n$ cosets of functions $\phi: \underline{\Sigma} \to \underline{n}$. It hence follows that it suffices to verify the conclusions of the theorem when considering only wedge summands over a single class $(\bar{\phi}) \in \{\phi: \underline{\Sigma} \to \underline{n}\}/\Sigma_n$.

Now consider a map $f: A \to B$ in Sym. Without loss of generality we can assume that the representative $\phi$ misses precisely the first $\bar{n}$ elements in $n$, implying that when computing $f^{\otimes n}$ the $\Sigma_n$ isotropy of the $\otimes$ wedge summand (i.e. the subgroup sending that summand to itself) is $\Sigma_{\bar{n}}$, and hence the wedge component of $f^{\otimes n}$ over $(\bar{\phi})$ can be rewritten as

$$\Sigma_{\bar{n}} \times \Sigma_n f(0)^{\otimes \bar{n}} \circ f(\phi^{-1}(\bar{n} + 1)) \circ \cdots \circ f(\phi^{-1}(n)).$$

We need to show that this is a $\Sigma$-$\text{inj}$ $\Sigma_{\bar{n}}$-$\text{proj}$ $S$ cofibration if $f$ is a positive $S$ cofibration. This follows by first applying Theorem 1.6 to $f(0)^{\otimes \bar{n}}$, then applying Proposition 5.30 to conclude $f(0)^{\otimes \bar{n}} \circ f(\phi^{-1}(\bar{n} + 1)) \circ \cdots \circ f(\phi^{-1}(n))$ is a $\Sigma$-$\text{inj}$ $\Sigma_{\bar{n}}$-$\text{proj}$ $S$ cofibration, and finishing by applying Proposition 1.2.

The additional claim assuming that $A$ is positive $S$ cofibrant follows by the same argument, but now using the additional statements in Theorem 1.6.

\[\square\]

Remark 5.34. All of this subsection immediately generalizes to the category $\text{BSym} = \text{Sym}(\text{Sym})$.

Indeed one can define $S$ stable and positive $S$ stable model structures in $\text{BSym} = \text{Sym}(\text{Sym})$ and $S \times \Sigma \times \Sigma$-$\text{inj}$ $\Sigma_n$ stable model structures in $\text{BSym}^\Sigma_n$ by simply repeating Definitions 5.30, 5.31 and 5.32 except replacing the initial model structures in $\text{Sp}^{\Sigma_n}$ by their analogues in $\text{Sym}$.

Analyzing the proofs of Propositions 5.30, 5.31 and 5.32 it is then clear that those results themselves then imply their $\text{BSym}$ versions as well.

\[^{23}\text{i.e. the structural isomorphisms } X_1 \otimes \cdots \otimes X_n \simeq X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(n)} \text{ for } \sigma \in \Sigma_n.\]
5.4 Proofs of Theorems 1.1, 1.2, 1.3, 1.4 and 1.5

We now prove Theorems 1.1, 1.2, 1.3, 1.4, and 1.5. We do so in that same order, although the first four results are essentially short corollaries of the more involved Theorem 1.5.

Proof of Theorem 1.1. To prove the first part asserting the existence of the model structures is suffices, by Lemma 2.3 in [17], to show that, for \( J \) a set of generating trivial cofibrations in \( \text{Sp}^\Sigma/\text{Sym} \), then any transfinite composition of pushouts of maps in \( \mathcal{O} \circ J \) is a w.e.. Since the proof of Theorem 1.5 follows precisely by using such a decomposition of \( f_2 \) the result then follows from that proof by setting \( f_1 = * \to \mathcal{O} \) and noting that \( * = *_A \to \mathcal{O}_A \) is automatically a monomorphism even if not assuming a cofibrancy condition on \( A \).

For the second part concerning Quillen equivalences we see, after unpacking definitions, that it suffices to show that the unit of the adjunctions

\[ A \to \mathcal{O} \circ_A A \]

are w.e.s whenever \( A \) is cofibrant. Applying Theorem 1.5 we see that any functor of the form \( \cdot \circ A \) preserves all monomorphisms which are w.e.s (note that this uses as \( f_2 \) the map \( \mathcal{O}(0) \to A \), so one must use the fact that \( \mathcal{O}(0) \cong \mathcal{O} \circ \mathcal{O}(0) \to \mathcal{O} \circ \mathcal{O}(0) \cong \mathcal{O}(0) \) is a w.e.), and hence by Ken Brown’s lemma combined with Theorem A.4 it in fact preserves all w.e.s, finishing the proof.

Proof of Theorem 1.2. By the existence of the injective and projective model structures on \( \text{Mod}^P(\mathcal{O}) \) (the existence of the former is the content of Theorem A.4 and the existence of the latter is used when proving that same theorem) together with the fact that both limits and colimits are computed underlyingly, both homotopy fiber sequences and homotopy cofiber sequences in \( \text{Mod}^P(\mathcal{O}) \) are just under underlying homotopy fiber and homotopy cofiber sequences in \( \text{Sym} \). Since \( \text{Sym} \) is stable we conclude the two types of sequences match. But by the the argument in the second part of the proof of Theorem 1.1 above we know that \( \cdot \circ A \) preserves all homotopy cofiber sequences (since it is a left derived functor already), finishing the proof.

Proof of Theorem 1.3. This follows directly from Theorem 1.5 by using as \( f_1 \) the map \( * \to \mathcal{O} \) and as \( f_2 \) the intended cofibration between cofibrant objects.

Lemma 5.35. Consider positive \( S \)-cofibrations

\[ f_i: A_i \to B_i, \quad 1 \leq i \leq n, \]

in \( \text{Sym} \) all with positive \( S \) cofibrant domains.

Then their pushout product with respect to \( \circ \),

\[ \circ^\mathcal{O}(f_1, f_2, \ldots, f_n) \]

is a positive \( S \) cofibration in \( \text{Sym} \) between positive \( S \) cofibrant objects in \( \text{Sym} \), which is a w.e. if any of the \( f_i \) is.
Proof. The proof follows by induction on $n$.

The case $n = 2$ is a essentially a particular case of Theorem 1.5 with $O = S$, except with an extra claim about positiveness, and this claim follows by equation 5.4 which makes clear that $(X \circ Y)_0(0) = *$ if both $X_0(0) = *$ and $Y_0(0) = *$.

For the induction case, recalling that $\circ$ preserves all colimits in the first variable when the second is fixed it follows that

$$\circ^2(f_1, f_2, \cdots, f_n) = (\circ^2(f_1, f_2, \cdots, f_n - 1)) \circ f_n,$$

and the result now follows by combining the induction hypothesis with the $n = 2$ case.

Proof of Theorem 1.4. Recall that the degeneracies of $B_n(R, O, L)$ are formed using only the unit map $i: I \to O$. The result then follows from Lemma 5.35 since the maps whose cofibrancy must be verified are the maps

$$\circ^2(\ast \to R, i, \cdots, i, \ast \to L),$$

where $i$ is allowed to appear any number of times.

Proof of Theorem 1.5. To simplify notation somewhat, assume first that $f_2$ is actually a map in $Alg_O \subset Mod_G^2$. Using the fact that $\circ_O$ commutes with transfinite compositions in the second variable one reduces to the case where $f_2: A \to B$ is the pushout of a generating cofibration, such as in Proposition 6.14. Borrowing the notation from that proposition and setting $f_1: M \to N$ we see that it suffices to show that in each of the induced filtration diagrams

the “pushout corner map” $A_{r+1}^M \coprod A^N_{r+1} \to A^N_{r+1}$ is a $S$ cofibration (respectively monomorphism). Using the inductive construction this reduces to showing that the “pushout corner map” of the diagram

$$M \cup_A (O \coprod A) \to N \cup_O (O \coprod A)$$

is itself a $S$ cofibration (respectively a monomorphism). Applying Theorems 1.6 and 1.7 we hence reduce to showing that $M_A(r) \to N_A(r)$ is a $S$ cofibration (respectively a monomorphism). Recalling Definition 5.14 we hence want to show

$24$Note however, that since $\circ$ doesn’t preserve all colimits in the second variable, the analogous statement with bracketing on the right does not follow.

$25$To simplify notation we also include here the case $r = 0$, for which $A^M_0 = A^N_0 = \ast$. 34
is a $S$ cofibration (respectively monomorphism). Now note that this follows from the particular instance of the result we are trying to prove for $f_2: O \to \mathcal{O} \llbracket A \rrbracket$, which is now a map in $\text{Mod}_{\mathcal{O}}^G$. Since we are assuming $A$ is cofibrant, by writing it as a retract of a pushout of generating cofibrations we can assume

$$A = \text{colim}_{\beta < \kappa} A_\beta$$

where $A_\beta \to A_{\beta + 1}$ is a pushout of a generating positive $S$ cofibration $\mathcal{O}X_\beta \to \mathcal{O}Y_\beta$. We now prove that $A_\beta$ is a $S$ cofibration (respectively monomorphism) by proving it inductively for each $\beta \leq \kappa$. Repeating the first part of the proof this reduces to checking that the pushout corner maps in

$$\begin{array}{ccc}
\mathcal{O} \llbracket A_\beta \rrbracket^M_{s-1} & \longrightarrow & \mathcal{O} \llbracket A_\beta \rrbracket^M_s \\
\mathcal{O} \llbracket A_\beta \rrbracket^N_{s-1} & \longrightarrow & \mathcal{O} \llbracket A_\beta \rrbracket^N_s
\end{array}$$

are $S$ cofibrations (respectively monomorphisms), and this reduces to doing so for the pushout corner maps of

$$\begin{array}{ccc}
M_{\mathcal{O} \llbracket A_\beta \rrbracket}(s)^{\wedge} \Sigma_s Q^s_{s-1} & \longrightarrow & M_{\mathcal{O} \llbracket A_\beta \rrbracket}(s)^{\wedge} \Sigma_s Y^s \\
N_{\mathcal{O} \llbracket A_\beta \rrbracket}(r)^{\wedge} \Sigma_r Q^s_{s-1} & \longrightarrow & N_{\mathcal{O} \llbracket A_\beta \rrbracket}(r)^{\wedge} \Sigma_r Y^s.
\end{array}$$

We will hence be done if we can show that $M_{\mathcal{O} \llbracket A_\beta \rrbracket}(r, s) \to N_{\mathcal{O} \llbracket A_\beta \rrbracket}(r, s)$ is an $S$ cofibration, but this follows automatically by combining the induction hypothesis with the proof of Proposition 5.23 which identifies this map with $M_{A_\beta}(r+s) \to N_{A_\beta}(r+s)$.

Tracking through the steps above we also see that indeed $f_1 \circ f_2$ will be a w.e. if either $f_1$ or (the original) $f_2$ is.

We now explain what changes when $f_2$ is a general cofibration between cofibrant objects in $\text{Mod}_{\mathcal{O}}^G$. Using Proposition 5.13 all of the discussion above follows through unchanged except that one needs to replace using Theorems 1.6 and 1.7 by their Sym analogues Propositions 5.33 and 5.32. This comes with a small caveat, in that when running the factorization argument a second time (to deal with the map $\mathcal{O} \llbracket A \rrbracket \to N \circ \mathcal{O} \llbracket A \rrbracket$ we were in fact already using the Sym analogue results, and hence to deal the general case one further needs the $\text{BSym}$ versions of the results mentioned in Remark 5.34.

A Auxiliary stable model structures on $(\text{Sp}^\Sigma)^G$

The goal of this appendix is to establish the existence of the two auxiliary model structures on $(\text{Sp}^\Sigma)^G$ required for the formulation of Theorem 1.7.

\footnote{In what follows expressions like $(\mathcal{O} \llbracket A_\beta \rrbracket)^M_s$ and $M_{\mathcal{O} \llbracket A_\beta \rrbracket}$ need to be interpreted in light of Remark 5.22. We are abusing notation somewhat by omitting the necessary $(\cdot)^{\wedge},(\cdot)^{\ast}$ functors.}
A.1 The monomorphism stable model structure on \((\text{Sp}^\Sigma)^G\)

The goal of this subsection is to establish the existence of a cofibrantly generated stable model structure in \((\text{Sp}^\Sigma)^G\) for which the cofibrations are precisely the monomorphisms. The arguments presented here are mainly an adaptation of similar arguments in Section 5 of [13], where the result for the case \(G = \ast\) is asserted (with the proof left has an exercise), though the use of Hirchorn’s localization theory ([10]) simplifies some arguments substantially.

**Theorem A.1.** Let \(G\) be any finite group.

There exists a cofibrantly generated model structure on \((\text{Sp}^\Sigma)^G\), which we call the monomorphism levelwise model structure such that

- cofibrations are the maps \(X \rightarrow Y\) such that \(X_n \rightarrow Y_n\) is a monomorphism of pointed simplicial sets for each \(n \geq 0\).
- weak equivalences are the maps are the maps \(X \rightarrow Y\) such that \(X_n \rightarrow Y_n\) is a w.e. of pointed simplicial sets for each \(n \geq 0\).

Further, this is a left proper cellular simplicial model category.

**Proof.** We first need sets \(I\) and \(J\) of generating cofibrations and generating trivial cofibrations, and we define \(I\) to be the set of (representatives of isomorphism classes) of inclusions of countable \(G\)-spectra, and \(J\) to be the subset of those that also happen to be levelwise w.e.s. Note that the fact that \(I\)-cof is the class of all levelwise monomorphisms then follows from the fact that every simplex of a spectrum generates a countable subspectrum.

The proof is now an application of Theorem 2.1.19 in [12]. Conditions 1, 2 and 3 in that theorem are immediate.

Part 5 follows from the fact that \(J \subseteq I\) together with the fact that \(I\) contains all maps of the form \(S \otimes G \times \Sigma_m \times (\partial \Delta^k \rightarrow \Delta^k)\), so that the \(I\)-inj maps are levelwise trivial fibrations, and hence levelwise w.e.s.

Part 4 follows from the the fact that \(J \subseteq I\) together with the fact that colimits are computed in a levelwise manner.

Part 6 will follow once we show that \(J\)-cof is the class of maps that are levelwise underlying trivial cofibrations. That that class contains \(J\)-cof again follows from colimits being computed levelwisely.

To prove the converse, we will be done if we show that we can solve any lifting problem

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \\
B & \longrightarrow & Y
\end{array}
\]

where \(A \rightarrow B\) is a levelwise underlying trivial cofibration and \(X \rightarrow Y\) has the right lifting property with respect to all maps in \(J\). A standard application of Zorn’s Lemma shows that there must be a \(G\)-subspectrum \(\bar{A} \subseteq B\) such that \(\bar{A} \rightarrow B\) is a levelwise w.e. together with a lift \(\bar{A} \rightarrow X\) that is maximal (in the sense that the lift can not be extended to a larger \(G\)-subspectrum levelwise w.e. to \(B\)). We prove by contradiction that it must be \(\bar{A} = B\). If not we let be \(C\) be some countable \(G\)-subspectrum of \(B\) generated by some simplex of \(B\) not in \(A\). Using lemma [A,2] we then produce a countable \(G\)-subspectrum \(D\) such that \(C \subseteq D\) and both \(D \cap \bar{A} \rightarrow D\) and \(A \rightarrow A \cup D\) are levelwise trivial cofibrations.
The first of these conditions allows one to produce a “larger” compatible lift $\bar{A} \cup D \to X$, and the second shows $\bar{A} \cup D \to B$ is still a levelwise underlying $w. e.$, hence contradicting the assumed maximality of $\bar{A}$.

Left properness and the simplicial model category axioms follow immediately from the analogous properties for $S_w$, and cellularity follows by Corollary 3.6.

**Lemma A.2.** Let $f: A \to B$ be a level trivial cofibration in $(\text{Sp}^{\Sigma})^G$. For every countable $G$-subspectrum $C$ of $B$ there is a countable $G$-subspectrum $D$ of $B$ such that $C \subseteq D$ and $D \cap A \to D$ and $A \to A \cup D$ are both level trivial cofibration.

**Proof.** The result is a direct generalization of Lemma 5.1.7 in [13], and essentially the same exact proof applies, with the only difference being that when constructing the $FC$ subspectrum described in the proof one needs it to be a $G$-subspectrum, but this can easily be fixed by simply closing the $FC$ constructed in that proof with respect to the $G$-action.

**Theorem A.3.** Let $G$ be any finite group.

There exists a cofibrantly generated model structure on $(\text{Sp}^{\Sigma})^G$, which we call the **monomorphism stable model structure** such that

- cofibrations are the maps $X \to Y$ such that $X_n \to Y_n$ is a monomorphism of pointed simplicial sets for each $n \geq 0$.
- weak equivalences are the maps are the maps $X \to Y$ which are underlying stable equivalences of spectra.

Further, this is a left proper cellular simplicial model category.

**Proof.** This is another application of Theorem B.2, localizing by the exact same set $S_G$ of maps as in Theorem 3.5, with the proof that the $S_G$-local equivalences are the stable equivalences following exactly as in that theorem.

When proving Theorems 1.1 and 1.2 we needed the following result, the proof of which just repeats the ideas used above.

**Theorem A.4.** Let $O$ be any operad.

There exists a cofibrantly generated model structure on $\text{Mod}_O^R$, which we call the **monomorphism stable model structure** such that

- cofibrations are the maps $X \to Y$ such that $X_n \to Y_n$ is a monomorphism of pointed simplicial sets for each $n \geq 0$.
- weak equivalences are the maps are the maps $X \to Y$ which are underlying stable equivalences of spectra.

Further, this is a left proper cellular simplicial model category.

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Note that even though the levelwise trivial cofibrancy of $A \to A \cup D$ is not included in the statement in [13], that is a direct consequence of the levelwise trivial cofibrancy of $D \cap A \to D$
Proof. Let $\kappa$ be an infinite cardinal which is larger than the number of simplices in $\mathcal{O}$ (check the proof of Proposition B.5 for the meaning of this).

One first proves a levelwise analogue of the result, the proof following just as that of Theorem A.1 with $I$ (resp. $J$) now the levelwise monomorphisms (resp. levelwise trivial monomorphisms) of $\mathcal{O}$ right modules with less than $\kappa$ simplices. The only differences are that now when proving part 5 one uses the fact that the maps $(\Sigma_r \times S \otimes \Sigma_m \times (\partial \Delta^k \to \Delta^k)) \circ \mathcal{O}$ are in $I$ and that in part 6 one needs a “$\kappa$ analogue” of Lemma A.2 which again can be proven by simply repeating the proof of Lemma 5.1.7 in [13] while noting that when dealing with simplicial sets with less than $\kappa$ simplices so do the relative homotopy groups have less than $\kappa$ elements.

Left properness is automatic, cellularity follows from Corollary B.6, and the simplicial model structure is built formally, with the simplicial cotensoring being defined underlyingly, so that the simplicial model category axioms hence follow.

We now again use Theorem B.2 to localize the level model structure to a stable one. We claim the set $S = \cup_{r \geq 0} (\Sigma_r \times S) \circ \mathcal{O}$, where $S$ was defined in Theorem 3.5, suffices, so that we need only check that we get the correct w.e.s. This can be done by looking at the identity adjunction

$$id : \text{Mod}_G \xrightarrow{\simeq} \text{Mod}_G : id,$$

where in the left model structure we use $\mathcal{O}$-proj $\Sigma \times \Sigma$-proj cofibrations and in the right model structure we use monomorphism cofibrations. Repeating the arguments at the end of the proof of Theorem 3.5 one sees that both sides have the same w.e.s. after localizing with respect to $S$ and that the w.e.s. in the left side are just the stable equivalences since that is a $\mathcal{O}$-projective model structure, hence finishing the proof.

![proof](image)

A.2 The $S$ stable model structure on $(\text{Sp}^\Sigma)^G$

Our goal in this subsection is to establish the existence of a cofibrantly generated stable model structure on $(\text{Sp}^\Sigma)^G$ which is $G$-injective, i.e., such that the cofibrations are just the underlying $S$ cofibrations in $\text{Sp}^\Sigma$.

We do this by first building the model structure in Theorem 3.5 by specifying its generating cofibrations, then showing it indeed has the desired cofibrations in Corollary 3.11.

Theorem A.5. Let $G$ be any finite group.

There exists a cofibrantly generated model structure on $(\text{Sp}^\Sigma)^G$, which we call the $S$ stable model structure, such that

- the generating cofibrations are the maps $S \otimes \partial \Delta^k \times (G \times \Sigma_m) / H \to S \otimes \Delta^k \times (G \times \Sigma_m) / H, \quad m \geq 0, \text{ any } H$;

- weak equivalences are the maps $X \to Y$ which are underlying stable equivalences of spectra.

Further, this is a left proper cellular simplicial model category.

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28 That such a levelwise model structure exists follows directly by Lemma 2.3 in [17].
Proof. This result is a slightly easier analogue of Theorem 3.5 (the model structures in the two theorems are identical except for the fact that in Theorem 3.5 the generating cofibrations have extra restrictions on \( m, H \), hence we give only an outline here.

One first proves a modified analogue of Proposition 3.1 except now with with weak equivalences (resp. fibrations) the maps which are weak equivalences (resp. fibrations) on all fixed points. This model structure is then localized as in Proposition 3.3 to a “mixed” one were the weak equivalences are just underlying one (the only thing that changes in the proof is that the set of localizing maps is enlarged by removing the restriction on \( H \)). One then uses this “mixed” model structure as the basis to build a levelwise equivalence model structure in \((Sp^\Sigma)^G\) analogous to that in Proposition 3.4. Finally, one localizes that model structure by the same maps as in Theorem 3.5 finishing the proof.

We now turn to the task of showing that the cofibrations in the \( S \) stable model structure in \((Sp^\Sigma)^G\) are indeed the underlying \( S \) cofibrations in \( Sp^\Sigma \).

The key to proving this is an inductive description of cofibrations, mirroring that in section 5.2 of [13].

Definition A.6. Let \( \hat{S} \) be the spectrum defined by
\[
\hat{S}_0 = *, \quad \hat{S}_m = S^m
\]
together with the obvious spectra structure maps and
\[
i: \hat{S} \to S
\]
be the obvious inclusion.

Then for a symmetric spectrum \( X \) we define the \( m \)-latching object to be
\[
L_m X = (X \wedge \hat{S})_m
\]
Note that there is a canonical map \( L_m X \to X_m \) induced by the inclusion \( i: \hat{S} \to S \).

Definition A.7. Given symmetric spectra \( A, B \) define a symmetric spectrum map up to degree \( m \) as a sequence of maps
\[
A_i \xrightarrow{L_i} B_i, \quad 0 \leq i \leq m
\]
compatible with the spectrum structure maps and \( \Sigma_i \)-actions bellow degree \( m \).

The relevance of the latching objects \( L_m X \) follows from the following lemma, which is implicit (though left unstated) in section 5.2 of [13].

Lemma A.8. A symmetric spectrum map up to degree \( m \) induces a unique map
\[
L_{m+1} A \to B_{m+1}.
\]
Further, if in a commutative diagram of symmetric spectra
\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
B & \xrightarrow{g} & Y
\end{array}
\]

The model structure thus obtained is actually just one of the “mixed” model structures described in Prop 1.3 of [19] in the case where the group has the form \( G \times \Sigma_m \).
one has specified “lifts up to degree $m - 1$".

$$B_i \xrightarrow{f_i} X_i, \quad 0 \leq i \leq m - 1,$$

then the compatible “lifts up to degree $m$” are in bijection with the lifts of the induced diagram of $\Sigma_m$-pointed spaces

$$A_m \vee_{L_m A} L_mB \xrightarrow{X_m} B_m \xrightarrow{Y_m} Y_m.$$

**Remark A.9.** Note that the uniqueness conditions in the previous lemma imply that the lemma immediately generalizes to $(S^\Sigma)^G$.

**Proposition A.10.** The cofibrations in the $S$ stable model structure on $(S^\Sigma)^G$ are precisely the maps $A \xrightarrow{f} B$ such that for all $m \geq 0$

$$(f \circ i)_m : A_m \vee_{L_m A} L_mB \xrightarrow{B_m} B_m$$

is a monomorphism.

**Proof.** Recall that in $S^\Sigma_{G} \times \Sigma_m$ the monomorphisms are the class of maps with the left lifting property with respect to the maps of $X \rightarrow Y$ such that $X^H \rightarrow Y^H$ is a trivial fibration of pointed spaces for any $H \subset \Sigma_m \times G$. Since the trivial fibrations in the $S$ stable model structure are precisely the maps $X \rightarrow Y$ of $G$-spectra such that $X^H \rightarrow Y^H$ is a trivial fibration of pointed spaces for any $H \subset G \times \Sigma_m$, Lemma A.8 says that a map of $G$-spectra $A \rightarrow B$ such that all maps $A_m \vee_{L_m A} L_mB \rightarrow B_m$ are monomorphisms is indeed a cofibration.

For the converse, we need to check $(f \circ i)_m$ is a monomorphism for $f$ any cofibration, and by general properties of the pushout product $\circ$ it suffices to do so when $f$ is a generating cofibration. But if $f$ is the map

$$S \otimes (\partial \Delta^k \times (G \times \Sigma_m)) / H \xrightarrow{\partial} \Delta^k \times (G \times \Sigma_m) / H$$

then $f \circ i = f \circ \otimes i$ where $\circ \otimes$ now denotes the pushout product with respect to the bifunctor $S^\Sigma \times S^\Sigma \otimes S^\Sigma$, so the result is now obvious.

**Corollary A.11.** The cofibrations in the $S$ stable model structure on $(S^\Sigma)^G$ are precisely the underlying $\Sigma$-inj $S$ cofibrations in $S^\Sigma$.

**Proof.** One simply applies Proposition A.10 twice, noting that if $A \rightarrow B$ is a map of $G$-spectra that proposition gives the same cofibrancy condition in $(S^\Sigma)^G$ as it does (after forgetting the $G$-action) in $S^\Sigma$.

**Remark A.12.** Analyzing the proof of Theorem A.5 it is clear that essentially the same proof would work for slightly modified generating cofibrations.

In particular, in subsection 5.3 we make use of the stable model structure on $(S^\Sigma)^G \times G$ with generating cofibrations

$$S \otimes \partial \Delta^k \times (G \times G \times \Sigma_m) / H \xrightarrow{\partial} S \otimes \Delta^k \times (G \times G \times \Sigma_m) / H, \quad m \geq 0, H \cap G \times \ast \times \ast = \ast.$$

We call this the $S \times \Sigma$-inj $G$-proj stable model structure, and repeating the argument in Corollary A.11 it is then clear that its cofibrations are just the underlying $S \times \Sigma$-inj $G$-proj cofibrations in $(S^\Sigma)^G$.

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301. I.e, maps that both lift the diagram and which form a symmetric spectrum map up to degree $n$. Note that we are also allowing for $m = 0$, in which case the set of maps is empty.
B  Left Bousfield localization

B.1 Definitions and existence

In this section we outline the localization results used in this paper.

Given that this is sufficient for our purposes, we will simplify our definitions substantially by assuming throughout that we are dealing with simplicial model categories and, further, that those model categories have functorial cofibrant replacement functors\(^{31}\), usually denoted by \(C\).

**Definition B.1.** Let \(\mathcal{C}\) be a simplicial model category, and \(\mathcal{S}\) a class of maps in \(\mathcal{C}\).

Then

- an object \(X \in \mathcal{C}\) is called \(\mathcal{S}\)-local if \(X\) is a fibrant object and further the induced maps
  \[
  \text{Map}(CB, X) \xrightarrow{\sim} \text{Map}(CA, X)
  \]
  on simplicial mapping spaces are w.e.s for any map \(A \rightarrow B\) in \(\mathcal{S}\).

- a map \(A \rightarrow B\) in \(\mathcal{C}\) is called a \(\mathcal{S}\)-local equivalence if for any \(\mathcal{S}\)-local object \(X\) the induced maps
  \[
  \text{Map}(CB, X) \xrightarrow{\sim} \text{Map}(CA, X)
  \]
  on simplicial mapping spaces are w.e.s.

The key result we will need is the following, which is a restricted version of Theorem 4.1.1 in \([10]\).

**Theorem B.2.** (Hirschhorn-simplicial version)

Let \(\mathcal{C}\) be a left proper cellular simplicial model category.

Then for any set \(\mathcal{S}\) of maps in \(\mathcal{C}\) the left Bousfield localization \(L_{\mathcal{S}}\mathcal{C}\) exists. I.e., there exists a model structure \(L_{\mathcal{S}}\mathcal{C}\) on the same category such that

- cofibrations are as in the original model structure on \(\mathcal{C}\);

- w.e.s are the \(\mathcal{S}\)-local equivalences.

Further, \(L_{\mathcal{S}}\mathcal{C}\) is itself again a left proper cellular simplicial model category.

The following is lifted from Proposition 3.3.18 in Hirschorn.

**Theorem B.3.** (Hirschhorn 3.3.18(1))

Suppose

\[
F : \mathcal{C} \rightleftarrows \mathcal{D} : G
\]

is a Quillen adjunction and that \(\mathcal{S}\) is a class of maps in \(\mathcal{C}\) such that the left Bousfield localization \(L_{\mathcal{S}}\mathcal{C}\) exists. Then

\[
F : L_{\mathcal{S}}\mathcal{C} \rightleftarrows \mathcal{D} : G
\]

is also a Quillen adjunction iff \(F(\mathcal{C}f)\) is a w.e. in \(\mathcal{D}\) for any map \(f\) in \(\mathcal{S}\).

\(^{31}\)Which is always the case for cofibrantly generated model categories.
The following corollary generalizes the previous result to biadjunctions, at least provided one has some additional hypothesis. We use this result in the proof of Theorem B.7.

**Corollary B.4.** Suppose

\[ F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E} \]

is a left Quillen bifunctor, that \( \mathcal{S} \) is a class of maps in \( \mathcal{C} \) such that the left Bousfield localization \( L_{\mathcal{S}}\mathcal{C} \) exists, and that \( \mathcal{D} \) is a cofibrantly generated model category for which the generating cofibrations have cofibrant domains and codomains.

Then

\[ F: L_{\mathcal{S}}\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E} \]

is also a left Quillen bifunctor iff \( F(Cf, D) \) is a w.e. in \( \mathcal{E} \) for any map \( f \) in \( \mathcal{S} \) and \( D \) any domain or codomain of a generating cofibration of \( \mathcal{D} \).

**Proof.** If \( D \) is cofibrant then \( F(\cdot, D) \) is a left Quillen functor with respect to the original model structure on \( \mathcal{C} \), so Theorem B.3 shows that the given condition is necessary and that if that condition holds we have \( F(g, D) \) a w.e. in \( \mathcal{E} \) for \( g: A \rightarrow B \) any trivial cofibration in \( L_{\mathcal{S}}\mathcal{C} \) and \( D \) any domain or codomain of a generating cofibration. But then if \( D \rightarrow \bar{D} \) is any generating cofibration of \( \mathcal{D} \) we have that in the diagram

\[
\begin{array}{ccc}
F(A, D) & \rightarrow & F(B, D) \\
\downarrow & & \downarrow \\
F(A, D) & \rightarrow & F(B, \bar{D})
\end{array}
\]

both horizontal maps are trivial cofibrations, and hence so is

\[ F(B, D) \prod_{F(A, D)} F(A, \bar{D}) \rightarrow F(B, \bar{D}) \]

a w.e., finishing the proof. \( \square \)

### B.2 Cellularity

In this short section we prove the necessary cellularity conditions used in this paper. For the full definition of a cellular category\(^{32}\) the reader should consult \[^{10}\] or, equivalently, the somewhat more condensed Appendix A to \[^{12}\].

**Proposition B.5.** Let \( \mathcal{C} \) be one of the categories \( \mathcal{S}\mathcal{G}_{\ast}^{\Sigma} \), \( (\mathcal{S}\mathcal{P}_{\Sigma}^{\Sigma})^{G} \).

Then any object \( X \) in \( \mathcal{C} \) is both compact relative to and small relative to any set \( I \) of monomorphisms in \( \mathcal{C} \).

**Proof.** For simplicity, we will say that the “number of simplices” of a simplicial set/symmetric spectrum is the cardinal of the union of all the sets that compose it\(^{33}\).

\[^{32}\] It is however worth keeping in mind that though quite technical in the case of a general category \( \mathcal{C} \), the conditions in the definition are quite straightforward for the categories \( \mathcal{S}\mathcal{G}_{\ast}^{\Sigma} \), \( (\mathcal{S}\mathcal{P}_{\Sigma}^{\Sigma})^{G} \) discussed in this paper.

\[^{33}\] So that in the case of simplicial sets this is a union over simplicial degrees, whereas for symmetric spectra it is a union over both simplicial and spectral degrees.
Choose $\kappa$ to be any initial ordinal which is larger the sizes of $X$ and all domains and codomains in $I$.

Since $X$ only has $\alpha < \kappa$ simplices, any map of the form $X \to \text{colim}_{\beta < \kappa} Y_{\beta}$ must factor through some $Y_{\beta}$, given that this is true for each simplex together with the fact that initially of $\kappa$ implies any subset of $\kappa$ of size $\alpha < \kappa$ must have a majorant. It hence follows that $X$ is small relative to $I$.

The argument for compactness is similar. If now $X \to \text{colim}_{\beta < \kappa} Y_{\beta}$ is a map from $X$ into a $I$-cell complex one immediately sees that the image lies in the union of a set $C^0$ of cells such that $|C^0| < \kappa$ cells. Since $\kappa$ is also larger than the number of simplices in the domains of $I$ we can increase $C^0$ to a set of $C^1$ of cells such that both $C^1 < \kappa$ and $C^1$ contains the “boundaries” of the cells in $C^0$. Repeating this process a countable number of times to get sets $C^n$ and taking the union $C^\infty = \bigcup_{n \in \mathbb{N}} C^n$ we finally find a subcell complex of size less than $\kappa$ and containing the image of the initial map, as desired.

**Corollary B.6.** Any cofibrantly generated model category on $S^G$ or $(Sp^\Sigma)^G$ (or additionally $\text{Mod}_R^O$ for an operad $O$) such that the cofibrations are monomorphisms is a cellular model category.

**Proof.** All conditions are immediate from Proposition B.5.

**References**

[1] M. Basterra and M. Mandell, Homology and Cohomology of E-infinity Ring Spectra, Math. Z. 249 (2005), no. 4, 903-944.

[2] Elmendorf and Mandell, Rings, modules, and algebras in infinite loop space theory, Adv. in Math. 205 (2006), no. 1, 163-228.

[3] Ezra Getzler and J.D.S. Jones, Operads, homotopy algebra and iterated integrals for double loop spaces, 1994.

[4] Thomas Goodwillie, Calculus I, The first derivative of pseudoisotopy theory, K-Theory 4 (1990), no. 1, 1–27.

[5] Thomas Goodwillie, Calculus II, Analytic functors, K-Theory 5 (1991/92), no. 4, 295–332.

[6] Thomas Goodwillie, Calculus III, Taylor series, Geometry and Topology 7 (2003) 645-711.

[7] John Harper, Homotopy theory of modules over operads in symmetric spectra, Algebr. Geom. Topol., 9(3):1637–1680, 2009.

[8] John Harper, Bar constructions and Quillen homology of modules over operads, Algebr. Geom. Topol., 10(1):87–136, 2010.

[9] John Harper and Kathryn Hess, Homotopy completion and topological Quillen homology of structured ring spectra, [arXiv:1102.1234], 2012.

[10] Philip S. Hirschhorn, Model Categories and Their Localizations, Mathematical Surveys and Monographs, 2003, Volume: 99, ISBN-10: 0-8218-4917-4

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[11] Mark Hovey, Model Categories, Volume 63 of Mathematical Surveys and Monographs, American Math. Soc., 1999 ISBN 0821813595, 9780821813591.

[12] Mark Hovey, Spectra and symmetric spectra in general model categories, J. Pure Appl. Alg. 165 (2001), 63-127.

[13] Mark Hovey, Brooke Shipley, Jeff Smith, Symmetric spectra, J. of the Amer. Math. Soc. 13 (2000), 149-209

[14] M. Mandell, J.P. May, S. Schwede, B. Shipley, Model categories of diagram spectra, Proc. London Math. Soc. 82 (2001), 441-512.

[15] C. Resk, S. Schwede and B. Shipley, Simplicial structures on model categories and functors, Amer. J. of Math. 123 (2001), 551-575

[16] Emily Riehl, Lecture notes for Math266x: Categorical Homotopy Theory, http://www.math.harvard.edu/~eriehl/266x/lectures.pdf

[17] Stefan Schwede and Brooke Shipley, Algebras and modules in monoidal model categories, Proc. London Math. Soc. 80 (2000), 491-511.

[18] Stefan Schwede, untitled book project on symmetric spectra, J. Pure Appl. Algebra 120 (1997), 77-104

[19] Brooke Shipley, A convenient model category for commutative ring spectra, Contemp. Math. 346 (2004), 473–483.

[20] , David white, Model structures on commutative monoids in general model categories, arXiv:1403.6759, 2014.