ASYMPTOTIC DIMENSION AND SMALL SUBSETS
IN LOCALLY COMPACT TOPOLOGICAL GROUPS

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Abstract. We prove that for a coarse space $X$ the ideal $S(X)$ of small subsets of $X$ coincides with the ideal $\mathcal{D}_<(X) = \{A \subset X : \text{asdim}(A) < \text{asdim}(X)\}$ provided that $X$ is coarsely equivalent to an Euclidean space $\mathbb{R}^n$. Also we prove that for a locally compact Abelian group $X$, the equality $S(X) = \mathcal{D}_<(X)$ holds if and only if the group $X$ is compactly generated.

1. Introduction

In this paper we study the interplay between the ideal $S(X)$ of small subsets of a coarse space $X$ and the ideal $\mathcal{D}_<(X)$ of subsets of asymptotic dimension less than asdim($X$) in $X$. We show that these two ideals coincide in spaces that are coarsely equivalent to $\mathbb{R}^n$, in particular, they coincide in each compactly generated locally compact abelian group.

Let us recall that a coarse space is a pair $(X, \mathcal{E})$ consisting of a set $X$ and a coarse structure $\mathcal{E}$ on $X$, which is a family of subsets of $X \times X$ (called entourages) satisfying the following axioms:

(A) each $\varepsilon \in \mathcal{E}$ contains the diagonal $\Delta_X = \{(x, y) \in X^2 : x = y\}$ and is symmetric in the sense that $\varepsilon = \varepsilon^{-1} = \{(y, x) : (x, y) \in \varepsilon\}$;

(B) for any entourages $\varepsilon, \delta \in \mathcal{E}$ there is an entourage $\eta \in \mathcal{E}$ that contains the composition $\delta \circ \varepsilon = \{(x, z) \in X^2 : \exists y \in X \text{ with } (x, y) \in \varepsilon \text{ and } (y, z) \in \delta\}$;

(C) a subset $\delta \subset X^2$ belongs to $\mathcal{E}$ if $\Delta_X \subset \delta = \delta^{-1} \subset \varepsilon$ for some $\varepsilon \in \mathcal{E}$.

A subfamily $\mathcal{B} \subset \mathcal{E}$ is called a base of the coarse structure $\mathcal{E}$ if

$$\mathcal{E} = \{\varepsilon \subset X^2 : \exists \delta \in \mathcal{B} \text{ with } \Delta_X \subset \varepsilon = \varepsilon^{-1} \subset \delta\}.$$  

A family $\mathcal{B}$ of subsets of $X^2$ is a base of a (unique) coarse structure if and only if it satisfies the axioms (A),(B).

Each subset $A$ of a coarse space $(X, \mathcal{E})$ carries the induced coarse structure $\mathcal{E}_A = \{\varepsilon \cap A^2 : \varepsilon \in \mathcal{E}\}$. Endowed with this structure, the space $(A, \mathcal{E}_A)$ is called a subspace of $(X, \mathcal{E})$.

For an entourage $\varepsilon \subset X^2$, a point $x \in X$, and a subset $A \subset X$ let $B(x, \varepsilon) = \{y \in X : (x, y) \in \varepsilon\}$ be the $\varepsilon$-ball centered at $x$, $B(A, \varepsilon) = \bigcup_{a \in A} B(a, \varepsilon)$ be the $\varepsilon$-neighborhood of $A$ in $X$, and $\text{diam}(A) = A \times A$ be the diameter of $A$. For a family $\mathcal{U}$ of subsets of $X$ we put $\text{mesh}(\mathcal{U}) = \bigcup_{U \in \mathcal{U}} \text{diam}(U)$.

Now we consider two basic examples of coarse spaces. The first of them is any metric space $(X, d)$ carrying the 
metric coarse structure whose base consists of the entourages $\{(x, y) \in X^2 : d(x, y) < \varepsilon\}$ where $0 \leq \varepsilon < \infty$. A coarse space is metrizable if its coarse structure is generated by some metric.

The second basic example is a topological group $G$ endowed with the left coarse structure whose base consists of the entourages $\{(x, y) \in G^2 : x \in yK\}$ where $K = K^{-1}$ runs over compact symmetric subsets of $G$ that contain the identity element $1_G$ of $G$. Let us observe that the left coarse structure on $G$ coincides with the metric coarse structure generated by any left-invariant continuous metric $d$ on $G$ which is proper in the sense that each closed ball $B(\varepsilon, R) = \{x \in G : d(x, e) \leq \varepsilon\}$ is compact. In particular, the coarse structure on $\mathbb{R}^n$, generated by the Euclidean metric coincides with the left coarse structure of the Abelian topological group $\mathbb{R}^n$.

Now we recall the definitions of large and small sets in coarse spaces. Such sets were introduced in [1] and studied in [11] §11. A subset $A$ of a coarse space $(X, \mathcal{E})$ is called

- large if $B(A, \varepsilon) = X$ for some $\varepsilon \in \mathcal{E}$;
- small if for each large set $L \subset X$ the set $L \setminus A$ remains large in $X$.

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It follows that the family $\mathcal{S}(X)$ of small subsets of a coarse space $(X, \mathcal{E})$ is an ideal. A subfamily $\mathcal{I} \subset \mathcal{P}(X)$ of the power-set of a set $X$ is called an ideal if $\mathcal{I}$ is additive (in the sense that $A \cup B \in \mathcal{I}$ for all $A, B \in \mathcal{I}$) and downwards closed (which means that $A \cap B \in \mathcal{I}$ for all $A \in \mathcal{I}$ and $B \subset X$).

Small sets can be considered as coarse counterparts of nowhere dense subsets in topological spaces, see [2]. It is well-known [5, 7.4.18] that the ideal of nowhere dense subsets in an Euclidean space $\mathbb{R}^n$ coincides with the ideal generated by closed subsets of topological dimension $< n$ in $\mathbb{R}^n$. The aim of this paper is to prove a coarse counterpart of this fundamental fact.

For this we need to recall [14, 9.4] the definition of the asymptotic dimension $\text{asdim}(X)$ of a coarse space $X$.

**Definition 1.1.** The asymptotic dimension $\text{asdim}(X)$ of a coarse space $(X, \mathcal{E})$ is the smallest number $n \in \omega$ such that for each entourage $\varepsilon \in \mathcal{E}$ there is a cover $\mathcal{U}$ of $X$ such that $\text{mesh}(\mathcal{U}) \subset \delta$ for some $\delta \in \mathcal{E}$ and each $\varepsilon$-ball $B(x, \varepsilon), x \in X$, meets at most $n + 1$ sets $U \in \mathcal{U}$. If such a number $n \in \omega$ does not exist, then we put $\text{asdim}(X) = \infty$.

In Theorem [27] we shall prove that

$$\text{asdim}(A \cup B) \leq \max\{\text{asdim}(A), \text{asdim}(B)\}$$

for any subspaces $A, B$ of a coarse space $X$. This implies that for every number $n \in \omega \cup \{\infty\}$ the family $\{A \subset X : \text{asdim}(A) < n\}$ is an ideal in $\mathcal{P}(X)$. In particular, the family

$$D_<(X) = \{A \subset X : \text{asdim}(A) < \text{asdim}(X)\}$$

is an ideal in $\mathcal{P}(X)$. According to [5, 9.8.4], $\text{asdim}(\mathbb{R}^n) = n$ for every $n \in \omega$.

The main result of this paper is:

**Theorem 1.2.** For every $n \in \mathbb{N}$ the ideal $\mathcal{S}(X)$ of small subsets in the space $X = \mathbb{R}^n$ coincides with the ideal $D_<(X)$.

Theorem [12] will be proved in Section 5 with the help of some tools of combinatorial topology. In light of this theorem the following problem arises naturally:

**Problem 1.3.** Detect coarse spaces $X$ for which $\mathcal{S}(X) = D_<(X)$.

It should be mentioned that the class of coarse spaces $X$ with $\mathcal{S}(X) = D_<(X)$ is closed under coarse equivalences.

A function $f : X \to Y$ between two coarse spaces $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ is called

- **coarse** if for each $\delta_X \in \mathcal{E}_X$ there is $\varepsilon_Y \in \mathcal{E}_Y$ such that for any pair $(x, y) \in \delta_X$ we get $(f(x), f(y)) \in \varepsilon_Y$;
- a **coarse equivalence** if $f$ is coarse and there is a coarse map $g : Y \to X$ such that $\{(x, g \circ f(x)) : x \in X\} \subset \varepsilon_X$ and $\{(y, f \circ g(y)) : y \in Y\} \subset \varepsilon_Y$ for some entourages $\varepsilon_X \in \mathcal{E}_X$ and $\varepsilon_Y \in \mathcal{E}_Y$.

Two coarse spaces $X, Y$ are called coarsely equivalent if there is a coarse equivalence $f : X \to Y$.

**Proposition 1.4.** Assume that coarse spaces $X, Y$ be coarsely equivalent. Then

1. $\text{asdim}(X) = \text{asdim}(Y)$;
2. $D_<(X) = \mathcal{S}(X)$ if and only if $D_<(Y) = \mathcal{S}(Y)$.

This proposition will be proved in Section 4. Combined with Theorem 1.2 it implies:

**Corollary 1.5.** If a coarse space $X$ is coarsely equivalent to an Euclidean space $\mathbb{R}^n$, then $D_<(X) = \mathcal{S}(X)$.

**Problem 1.3** can be completely resolved for locally compact Abelian topological groups $G$, endowed with their left coarse structure. First we establish the following general fact:

**Theorem 1.6.** For each topological group $X$ endowed with its left coarse structure we get $D_<(X) \subset \mathcal{S}(X)$.

We recall that a topological group $G$ is compactly generated if $G$ is algebraically generated by some compact subset $K \subset G$.

**Theorem 1.7.** For an Abelian locally compact topological group $X$ the following conditions are equivalent:

1. $\mathcal{S}(X) = D_<(X)$;
2. $X$ is compactly generated;
3. $X$ is coarsely equivalent to the Euclidean space $\mathbb{R}^n$ for some $n \in \omega$. 
This theorem will be proved in Section 6.

Remark 1.8. Theorem 1.7 is not true for non-abelian groups. The simplest counterexample is the discrete free group \( F_2 \) with two generators. Any infinite cyclic subgroup \( Z \subset F_2 \) has infinite index in \( F_2 \) and hence is small, yet \( \text{asdim}(Z) = \text{asdim}(F_2) = 1 \).

A less trivial example is the wreath product \( A \wr Z \) of a non-trivial finite abelian group \( A \) and \( Z \). The group \( A \wr Z \) has asymptotic dimension 1 (see [9]) and the subgroup \( Z \) is small in \( A \wr Z \) and has \( \text{asdim}(Z) = 1 = \text{asdim}(A \wr Z) \).

Let us recall that the group \( A \wr Z \) consists of ordered pairs \( ((a_i, n), b) \in (\oplus Z)^A \times Z \) and the group operation on \( A \wr Z \) is defined by
\[
((a_i, n), b) \cdot ((b_j, m)) = ((a_i + m, a_i + b), n + m).
\]
The group \( A \wr Z \) is finitely-generated and meta-abelian but is not finitely presented, see [3].

Problem 1.9. Is \( S(X) = D_\prec(X) \) for each connected Lie group \( X \)? For each discrete polycyclic group \( X \)?

2. The asymptotic dimension of coarse spaces

In this section we present various characterizations of the asymptotic dimension of coarse spaces. First we fix some notation. Let \((X, \mathcal{E})\) be a coarse space, \( \varepsilon \in \mathcal{E} \) and \( A \subset X \). We shall say that \( A \) has diameter less than \( \varepsilon \) if \( \text{diam}(A) < \varepsilon \) where \( \text{diam}(A) = A \times A \). A sequence \( x_0, \ldots, x_m \in X \) is called an \( \varepsilon \)-chain if \( (x_i, x_{i+1}) \in \varepsilon \) for all \( i < m \). In this case the finite set \( C = \{x_0, \ldots, x_m\} \) also will be called an \( \varepsilon \)-chain. A set \( C \subset X \) is called \( \varepsilon \)-connected if any two points \( x, y \in C \) can be linked by an \( \varepsilon \)-chain \( x = x_0, \ldots, x_m = y \). The maximal \( \varepsilon \)-connected subset \( C(x, \varepsilon) \subset X \) containing a given point \( x \in X \) is called the \( \varepsilon \)-connected component of \( x \). It consists of all points \( y \in X \) that can be linked with \( x \) by an \( \varepsilon \)-chain \( x = x_0, \ldots, x_m = y \).

A family \( \mathcal{U} \) of subsets of \( X \) is called \( \varepsilon \)-disjoint if \( (U \times V) \cap \varepsilon = \emptyset \) for any distinct sets \( U, V \in \mathcal{U} \). Each natural number \( n \in \omega \) is identified with the set \( \{0, \ldots, n - 1\} \).

We shall study the interplay between the asymptotic dimension introduced in Definition 1.1 and the following its modification:

Definition 2.1. The colored asymptotic dimension \( \text{asdim}_{\text{col}}(X) \) of a coarse space \( (X, \mathcal{E}) \) is the smallest number \( n \in \omega \) such that for every entourage \( \varepsilon \in \mathcal{E} \) there is a cover \( \mathcal{U} \) of \( X \) such that \( \text{mesh}(\mathcal{U}) \subset \delta \) for some \( \delta \in \mathcal{E} \) and \( \mathcal{U} \) can be written as the union \( \mathcal{U} = \bigcup_{i \in n+1} \mathcal{U}_i \) of \( n + 1 \) many \( \varepsilon \)-disjoint subfamilies \( \mathcal{U}_i \). If such a number \( n \in \omega \) does not exist, then we put \( \text{asdim}_{\text{col}}(X) = \infty \).

Without lost of generality we can assume that the cover \( \mathcal{U} = \bigcup_{i \in n+1} \mathcal{U}_i \) in the above definition consists of pairwise disjoint sets. In this case we can consider the coloring \( \chi : X \to n + 1 = \{0, \ldots, n\} \) such that \( \chi^{-1}(i) = \mathcal{U}_i \) for every \( i \in n + 1 \). For this coloring every \( \chi \)-monochrome \( \varepsilon \)-connected subset \( C \subset X \) lies in some \( U \in \mathcal{U} \) and hence has diameter \( \text{diam}(C) \subset \text{diam}(U) \subset \text{mesh}(\mathcal{U}) \subset \delta \). A subset \( A \subset X \) is \( \chi \)-monochrome if \( \chi(A) \) is a singleton. Thus we arrive to the following useful characterization of the colored asymptotic dimension.

Proposition 2.2. A coarse space \( (X, \mathcal{E}) \) has \( \text{asdim}_{\text{col}}(X) \leq n \) for some number \( n \in \omega \) if and only if for any \( \varepsilon \in \mathcal{E} \) there is a coloring \( \chi : X \to n + 1 \) and an entourage \( \delta \in \mathcal{E} \) such that each \( \chi \)-monochrome \( \varepsilon \)-chain \( C \subset X \) has \( \text{diam}(C) \subset \delta \).

Proof. The “only if” part follows from the above discussion. To prove the “if” part, for every \( \varepsilon \in \mathcal{E} \) we need to construct a cover \( \mathcal{U} = \bigcup_{i \in n+1} \mathcal{U}_i \) such that \( \text{mesh}(\mathcal{U}) \in \mathcal{E} \) and each family \( \mathcal{U}_i \) is \( \varepsilon \)-disjoint. By our assumption, there is a coloring \( \chi : X \to n + 1 \) and an entourage \( \delta \in \mathcal{E} \) such that each \( \chi \)-monochrome \( \varepsilon \)-chain \( C \subset X \) has \( \text{diam}(C) \subset \delta \).

For each \( x \in X \) let \( C_\chi(x, \varepsilon) \) be the set of all points \( y \in X \) that can be linked with \( x \) by a \( \chi \)-monochrome \( \varepsilon \)-chain \( x = x_0, x_1, \ldots, x_m = y \). It follows that \( \text{diam}(C_\chi(x, \varepsilon)) \subset \delta \). For every \( i \in n + 1 \) consider the \( \varepsilon \)-disjoint family \( \mathcal{U}_i = \{C_\chi(x, \varepsilon) : x \in \chi^{-1}(i)\} \). It is clear that \( \mathcal{U} = \bigcup_{i \in n+1} \mathcal{U}_i \) is a cover with \( \text{mesh}(\mathcal{U}) \subset \delta \in \mathcal{E} \), witnessing that \( \text{asdim}_{\text{col}}(X) \leq n \).

Now we are ready to prove the equivalence of two definitions of asymptotic dimension. For metrizable coarse spaces this equivalence was proved in [5, 9.3.7].

Proposition 2.3. Each coarse space \( (X, \mathcal{E}) \) has \( \text{asdim}(X) = \text{asdim}_{\text{col}}(X) \).
Proof. To prove that $\text{asd}(X) \leq \text{asd}(Col(X))$, put $n = \text{asd}(Col(X))$ and take any entourage $\varepsilon \in \mathcal{E}$. By Definition 2.14 for the entourage $\varepsilon \in \mathcal{E}$ we can find a cover $\mathcal{U} = \bigcup_{i=0}^{n} \mathcal{U}_i$ with $\text{mesh}(\mathcal{U}) \in \mathcal{E}$ such that each family $\mathcal{U}_i$ is $\varepsilon \circ \varepsilon$-disjoint.

We claim that $\text{mesh}(\mathcal{U}) \leq \mathcal{U}_i$ for all $i$. The claim follows from the fact that each $\varepsilon$-ball $B(x, \varepsilon)$, $x \in X$, meets at most one set of each family $\mathcal{U}_i$. Assuming that $B(x, \varepsilon)$ meets two distinct sets $U, V \in \mathcal{U}_i$, we can find points $u \in U$ and $v \in V$ with $(u, v) \in \varepsilon$ and conclude that $(u, v) \in \varepsilon \circ \varepsilon$, which is not possible as $\mathcal{U}_i$ is $\varepsilon \circ \varepsilon$-disjoint. Now we see that the ball $B(x, \varepsilon)$ meets at most $n + 1$ element of the cover $\mathcal{U}$ and hence $\text{asd}(X) \leq n$.

The proof of the inequality $\text{asd}(Col(X)) \leq \text{asd}(X)$ is a bit longer. If the dimension $n = \text{asd}(X)$ is infinite, then there is nothing to prove. So, we assume that $n \in \omega$. To prove that $\text{asd}(Col(X)) \leq n$, fix an entourage $\varepsilon \in \mathcal{E}$. Let $\varepsilon^0 = \Delta_X$ and $\varepsilon^{k+1} = \varepsilon^{k} \circ \varepsilon$ for $k \in \omega$. Since $\text{asd}(X) \leq n$, for the entourage $\varepsilon^{n+1}$ we can find a cover $\mathcal{U}$ of $X$ such that $\delta = \text{mesh}(\mathcal{U}) \in \mathcal{E}$ and each $\varepsilon^{n+1}$-ball $B(x, \varepsilon^{n+1})$ meets at most $n + 1$ many sets $U \in \mathcal{U}$. For every $i \leq n + 1$ and $x \in X$ consider the subfamily $\mathcal{U}_i(x) = \{ U \in \mathcal{U} : B(x, \varepsilon^i) \cap U \neq \emptyset \}$ of $\mathcal{U}$. It follows that $1 \leq |\mathcal{U}_i(x)| \leq |\mathcal{U}_i(x)| \leq n + 1$ for every $0 \leq i \leq n$. Consequently, $|\mathcal{U}_i(x)| = i$ for some $0 \leq i \leq n + 1$. Let $\chi(x)$ be the maximal number $k \leq n$ such that $|\mathcal{U}_i(x)| = k + 1$. In such a way we have defined a coloring $\chi : X \to n + 1 = \{0, \ldots, n\}$.

To finish the proof it suffices to show that any $\chi$-monochrome $\varepsilon$-chain $C = \{x_0, \ldots, x_m\} \subseteq X$ has $\text{diam}(C) \subseteq \varepsilon^0 \circ \varepsilon^{n+1}$. Let $k = \chi(x_0)$ be the color of the chain $C$. It follows that $|\mathcal{U}_i(x), \varepsilon^{k+1}| = k + 1$ for all $x_i \in C$. We claim that $\mathcal{U}_i(x_i, \varepsilon^{k+1}) = \mathcal{U}_i(x_{i+1}, \varepsilon^{k+1})$ for all $i < m$. Assuming the converse, we would get that $|\mathcal{U}_i(x_i, \varepsilon^{k+1}) \cup \mathcal{U}_i(x_{i+1}, \varepsilon^{k+1})| \geq k + 3$ and then the family $\mathcal{U}_i(x_i, \varepsilon^{k+2}) \cup \mathcal{U}_i(x_i, \varepsilon^{k+1}) \cup \mathcal{U}_i(x_{i+1}, \varepsilon^{k+1})$ has cardinality $|\mathcal{U}_i(x_i, \varepsilon^{k+2})| \geq k + 3$, which implies that $|\mathcal{U}_i(x_i, \varepsilon^i)| = i$ for some $k + 3 \leq i \leq n + 1$. But this contradicts the definition of $k = \chi(x_i)$. Hence $\mathcal{U}_i(x_i, \varepsilon^{k+1}) = \mathcal{U}_i(x_0, \varepsilon^{k+1})$ for all $i \leq m$ and then $C \subseteq B(U, \varepsilon^{k+1})$ for every $U \in \mathcal{U}_i(x_0, \varepsilon^{k+1})$. Now we see that $\text{diam}(C) \subseteq \text{diam}(U) \circ \varepsilon^{k+1} \subseteq \varepsilon \circ \varepsilon^{k+1}$.

Proposition 2.2 and 2.3 imply:

Corollary 2.4. A coarse space $(X, \mathcal{E})$ has asymptotic dimension $\text{asd}(X) \leq n$ for some $n \in \omega$ if and only if for any $\varepsilon \in \mathcal{E}$ there is $\delta \in \mathcal{E}$ and a coloring $\chi : X \to n + 1$ such that $\text{diam}(\chi(C)) \subseteq \delta$.

This corollary can be generalized as follows (cf. [6]).

Proposition 2.5. A coarse space $(X, \mathcal{E})$ has $\text{asd}(X) \leq n$ for some $n \in \omega$ if and only if for any entourage $\varepsilon \in \mathcal{E}$ there is an entourage $\delta \in \mathcal{E}$ such that for any finite set $F \subseteq X$ there is a coloring $\chi : F \to n + 1$ such that each $\chi$-monochrome $\varepsilon$-chain $C \subseteq F$ has $\text{diam}(\chi(C)) \subseteq \delta$.

Proof. This proposition will follow from Corollary 2.4 as soon as for any $\varepsilon \in \mathcal{E}$ we find $\delta \in \mathcal{E}$ and a coloring $\chi : X \to n + 1$ such that each $\chi$-monochrome $\varepsilon$-chain in $X$ has diameter less that $\delta$.

By our assumption, there is an entourage $\delta \in \mathcal{E}$ such that for every finite subset $F \subseteq X$ there is a coloring $\chi_F : F \to n + 1$ such that each $\chi_F$-monochrome $\varepsilon$-chain in $F$ has diameter less that $\delta$. Extend $\chi_F$ to a coloring $\check{\chi}_F : X \to n + 1$.

Let $\mathcal{F}$ denote the family of all finite subsets of $X$, partially ordered by the inclusion relation $\subseteq$. The colorings $\check{\chi}_F$, $F \in \mathcal{F}$, can be considered as elements of the compact Hausdorff space $K = \{0, \ldots, n\}^X$ endowed with the Tychonov product topology. The compactness of $K$ implies that the net $\{\check{\chi}_F\}_{F \in \mathcal{F}}$ has a cluster point $\check{\chi} \in K$, see [8] 3.1.23. The latter means that for each finite set $F_0 \in \mathcal{F}$ and a neighborhood $O(\check{\chi}) \subseteq K$ there is a finite set $F \in \mathcal{F}$ such that $F \supseteq F_0$ and $\check{\chi}_F \in O(\chi)$.

We claim that the coloring $\check{\chi} : X \to n + 1$ has the required property: each $\chi$-monochrome $\varepsilon$-chain $C \subseteq X$ has $\text{diam}(X) \subseteq \delta$. Observe that the finite set $C$ determines a neighborhood $O_C(\check{\chi}) = \{ \check{\varepsilon} \in K : \check{\varepsilon}[C] = \check{\varepsilon}[C] \}$, which contains a coloring $\check{\chi}_F$ for some finite set $F \supseteq C$. The choice of the coloring $\check{\chi}_F = \check{\chi}_F|F$ guarantees that the set $C \subseteq F$ has $\text{diam}(C) \subseteq \delta$.

Proposition 2.5 admits the following self-generalization.

Theorem 2.6. A coarse space $(X, \mathcal{E})$ has $\text{asd}(X) \leq n$ for some $n \in \omega$ if and only if for any entourage $\varepsilon \in \mathcal{E}$ there is an entourage $\delta \in \mathcal{E}$ such that for any finite $\varepsilon$-connected subset $F \subseteq X$ there is a coloring $\chi : F \to n + 1$ such that each $\chi$-monochrome $\varepsilon$-chain $C \subseteq F$ has $\text{diam}(C) \subseteq \delta$.

Finally, let us prove Addition Theorem for the asymptotic dimension. For metrizable spaces this theorem is well known; see [14, 9.13] or [5, 9.7.1].
Theorem 2.7. For any subspaces $A, B$ of a coarse space $(X, \mathcal{E})$ we get
\[
\text{asdim}(A \cup B) \leq \max\{\text{asdim}(A), \text{asdim}(B)\}.
\]

Proof. Only the case of finite $n = \max\{\text{asdim}(A), \text{asdim}(B)\}$ requires the proof. Without loss of generality the sets $A$ and $B$ are disjoint. To show that $\text{asdim}(A \cup B) \leq n$ we shall apply Corollary 2.4. Fix any entourage $\varepsilon \in \mathcal{E}$. Since $\text{asdim}(A) \leq n$ there are an entourage $\delta_A \in \mathcal{E}$ and a coloring $\chi_A : A \to \{0, \ldots, n\}$ such that each $\chi$-monochrome $\varepsilon$-chain in $A$ has diameter less that $\delta_A$. Since $\text{asdim}(B) \leq n$, for the entourage $\varepsilon_B = \varepsilon \circ \delta_A \circ \varepsilon$ there are an entourage $\delta_B \in \mathcal{E}$ and a coloring $\chi_B : A \to \{0, \ldots, n\}$ such that each $\chi$-monochrome $\varepsilon_B$-chain in $B$ has diameter less that $\delta_B$.

The union of the colorings $\chi_A$ and $\chi_B$ yields the coloring $\chi : A \cup B \to \{0, \ldots, n\}$ such that $\chi|A = \chi_A$ and $\chi|B = \chi_B$. We claim that each $\chi$-monochrome $\varepsilon$-chain $C = \{x_0, \ldots, x_m\} \subset A \cup B$ has diam(C) $\subset \delta$ where $\delta = \delta_A \circ \varepsilon \circ \delta_B \circ \varepsilon \circ \delta_A$. Without loss of generality, the points $x_0, \ldots, x_m$ of the chain $C$ are pairwise distinct.

If $C \subset A$, then $C'$ being a $\chi_A$-monochrome $\varepsilon$-chain in $A$ has diam(C) $\subset \delta_A \subset \delta$ and we are done. So, we assume that $C \not\subset A$. In this case $b = |C \cap B| \geq 1$ and we can choose a strictly increasing sequence $0 \leq k_1 < k_2 < \cdots < k_b \leq m$ such that $\{x_{k_1}, \ldots, x_{k_b}\} = C \cap B$. Then $\{x_0, \ldots, x_{k_1-1}\}$, being a $\chi_A$-monochrome $\varepsilon$-chain in $A$, has diameter less that $\delta_A$. Consequently, the $\varepsilon$-chain $\{x_0, \ldots, x_{k_1}\}$ has diameter less that $\delta_A \circ \varepsilon \subset \varepsilon_B$. By the same reason the $\varepsilon$-chain $\{x_{k_1}, \ldots, x_{k_2}\}$ has diameter less that $\varepsilon \circ \delta_A \subset \varepsilon_B$ and for every $1 \leq i < b$ the $\varepsilon$-chain $\{x_{k_i}, \ldots, x_{k_{i+1}}\} \subset \{x_{k_i}\} \cup A \cup \{x_{k_{i+1}}\}$ has diameter less than $\varepsilon \circ \delta_A \circ \varepsilon = \varepsilon_B$. Now we see that the $\varepsilon$-chain $C = \{x_{k_1}, \ldots, x_m\}$ has diam(C) $\subset \delta_A \circ \varepsilon \circ \delta_B \circ \varepsilon \circ \delta_A = \delta$. □

The characterization Theorem 2.6 will be applied to prove the following theorem which was known [7, 2.1] in the context of countable groups.

Theorem 2.8. If $G$ is a topological group endowed with its left coarse structure, then
\[
\text{asdim}(G) = \sup \{\text{asdim}(H) : H \text{ is a compactly generated subgroup of } G\}.
\]

Proof. Let $n = \sup \{\text{asdim}(H) : H \text{ is a compactly generated subgroup of } G\}$. It is clear that $n \leq \text{asdim}(G)$. The reverse inequality asdim$(G) \leq n$ is trivial if $n = \infty$. So, we assume that $n < \infty$. To prove that asdim$(G) \leq n$, we shall apply Theorem 2.6. Let $\mathcal{E}$ be the left coarse structure of the topological group $G$.

Given any entourage $\varepsilon \in \mathcal{E}$, we should find an entourage $\delta \in \mathcal{E}$ such that for each finite $\varepsilon$-connected subset $F \subset G$ there is a coloring $\chi : F \to n + 1$ such that for each $\chi$-monochrome $\varepsilon$-chain $C \subset F$ diam(C) $\subset \delta$.

By the definition of the coarse structure $\mathcal{E}$, for the entourage $\varepsilon \in \mathcal{E}$ there is a compact subset $K_\varepsilon = K_\varepsilon^{-1} \subset G$ such that $\varepsilon \subset \{(x, y) \in G^2 : x \in yK_\varepsilon\}$. Let $H$ be the subgroup of $G$ generated by the compact set $K_\varepsilon$, $\mathcal{E}_H$ be the left coarse structure of $H$, and $\varepsilon_H = \{(x, y) \in H^2 : x \in yK_\varepsilon\} \in \mathcal{E}_H$. Since asdim$_G(H) = \text{asdim}(H) \leq n$, by Proposition 2.2 there is a coloring $\chi_H : H \to n + 1$ and an entourage $\delta_H \in \mathcal{E}_H$ such that each $\chi_H$-monochrome $\varepsilon$-chain $C \subset H$ has diameter diam(C) $\subset \delta_H$. By the definition of the coarse structure $\mathcal{E}_H$, there is a compact subset $K_\delta = K_\delta^{-1} \subset 1_H$ of $H$ such that $\{(x, y) \in H \times H : x \in yK_\delta\}$.

We claim that the entourage $\delta = \{(x, y) \in G \times G : x \in yK_\delta\}$ satisfies our requirements. Let $F$ be a finite $\varepsilon$-connected subset of $G$. Then for each point $x_0 \in F$ we get $F$ $\subset x_0H$ and hence $x_0^{-1}F \subset H$. So, we can define a coloring $\chi : F \to n + 1$ letting $\chi(x) = \chi_H(x_0^{-1}x)$ for $x \in F$. If $C \subset F$ is a $\chi$-monochrome $\varepsilon$-chain, then $x_0^{-1}C$ is a $\chi_H$-monochrome $\varepsilon_H$-chain in $H$ and hence diam$(x_0^{-1}C) \subset \delta_H$. The latter means that for any points $c, c' \in C$ we get $(x_0^{-1}c, x_0^{-1}c') \in \delta_H \subset \{(x, y) \in H \times H : x \in yK_\delta\}$ and hence $x_0^{-1}c \in x_0c'K_\delta$ and $c \in c'K_\delta$, which means that $(c, c') \in \delta_H$ and hence diam(C) $\subset \delta_H$. □

3. Proof of Proposition 1.4

Let $f : X \to Y$ be a coarse equivalence between two coarse spaces $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$. Then there is a coarse map $g : Y \to X$ such that $\{(x, g \circ f(x)) : x \in X\} \subset \eta_X$ and $\{(y, f \circ g(y)) : y \in Y\} \subset \eta_Y$ for some entourages $\eta_X \in \mathcal{E}_X$ and $\eta_Y \in \mathcal{E}_Y$. It follows that $B(f(X), \eta_X) = Y$ and $B(g(Y), \eta_Y) = X$.

1. First we prove that asdim$(X) = \text{asdim}(Y)$. Actually, this fact is known [14, p.129] and we present a proof for the convenience of the reader. By the symmetry, it suffices to show that asdim$(X) \leq \text{asdim}(Y)$. This inequality is trivial if $n = \text{asdim}(Y)$ is infinite. So, assume that $n < \infty$. By Proposition 2.2 and 2.8 the inequality asdim$(X) \leq n$ will be proved as soon as for each $\varepsilon_X \in \mathcal{E}_X$ we find $\delta_X \in \mathcal{E}_X$ and a coloring $\chi_X : X \to n + 1$ such that each $\chi_X$-monochrome $\varepsilon_X$-chain $C \subset X$ has diameter diam(C) $\subset \delta_X$. 


Since the map \( f : X \to Y \) is coarse, for the entourage \( \varepsilon_X \) there is an entourage \( \varepsilon_Y \) such that \( \{(f(x), f(x')) : (x, x') \in \varepsilon_X\} \subset \varepsilon_Y \). Since \( \text{asdim}(Y) = n \), for the entourage \( \varepsilon_Y \) there is an entourage \( \delta_Y \in \varepsilon_Y \) and a coloring \( \chi_Y : Y \to n + 1 \) such that each \( \chi_Y \)-monochrome \( \varepsilon_Y \)-chain \( C_Y \subset Y \) has diameter \( \text{diam}(C_Y) \subset \delta_Y \).

Since the function \( g : Y \to X \) is coarse, for the entourage \( \delta_Y \) there is an entourage \( \delta_Y' \subset \delta_Y \). Put \( \delta_X = \eta_X \circ \delta_X \circ \eta_Y \) and consider the coloring \( \chi_X = \chi_Y \circ f : X \to n + 1 \) of \( X \). We claim that each \( \chi_X \)-monochrome \( \varepsilon_X \)-chain \( C_X \subset X \) has diameter \( \text{diam}(C_X) \subset \delta_X \). Then choice of \( \varepsilon_Y \) guarantees that the set \( C_Y = f(C_X) \) is an \( \varepsilon_Y \)-chain. Being \( \chi_X \)-monochrome, it has diameter \( \text{diam}(C_Y) \subset \delta_Y \). Then the set \( C_Y' = g(C_Y) \) has diameter \( \text{diam}(C_Y') \subset \delta_X' \). Now take any two points \( c, c' \in C_X \) and observe that the pairs \( (c, g \circ f(c)) \) and \( (c', g \circ f(c')) \) belong to the entourage \( \eta_X \). Consequently,

\[
(c, c') \in \{(c, g \circ f(c))\} \cup \{(g \circ f(c), g \circ f(c'))\} \subset \eta_X \circ \delta_X' \circ \eta_Y = \delta_X
\]

which means that the \( \varepsilon_X \)-chain \( C_X \) has diameter \( \text{diam}(C_X) \subset \delta_X \). So, \( \text{asdim}(X) \leq n \).

2. The second statement of Proposition \( [1.4] \) follows Claims \( 3.1 \) and \( 3.3 \) proved below.

**Claim 3.1.** A subset \( A \subset X \) and its image \( f(A) \subset Y \) have the same asymptotic dimension \( \text{asdim}(A) = \text{asdim}(f(A)) \).

**Proof.** This claim follows from Proposition \( [1.4(1)] \) proved above, since \( A \) and \( f(A) \) are coarsely equivalent. \( \square \)

**Claim 3.2.** A subset \( A \subset X \) is large in \( X \) if and only if its image \( f(A) \) is large in \( Y \).

**Proof.** If \( A \) is large in \( X \), then \( B(A, \varepsilon_X) = X \) for some \( \varepsilon_X \in \varepsilon_X \). Since \( f \) is coarse, there exists \( \varepsilon_Y \in \varepsilon_Y \) such that for each \( (x_0, x_1) \in \varepsilon_X \) we get \( f((x_0), f((x_1))) \in \varepsilon_Y \). It follows that \( B(f(A), \varepsilon_Y) \supset \varepsilon_Y \) and \( B(f(A), \varepsilon_Y \circ \eta_Y) = B(f(A), \varepsilon_Y), \eta_Y) \supset \varepsilon_Y \). This means that \( f(A) \) is large.

Now assume conversely that the set \( f(A) \) is large in \( Y \). Then \( g \circ f(A) \) is large in \( X \). Since \( g \circ f(A) \subset B(A, \eta_X) \), we conclude that \( A \) is large in \( X \). So, \( A \) is large in \( X \) if and only if \( f(A) \) is large in \( Y \). \( \square \)

**Claim 3.3.** A subset \( A \subset X \) is small if and only if for each entourage \( \varepsilon_X \in \varepsilon_X \) the set \( B(A, \varepsilon_X) \) is small.

**Proof.** The “if” part is trivial. To prove the “only if” part, assume that the set \( A \) is small in \( X \). To show that \( B(A, \varepsilon_X) \) is small in \( X \), it is necessary to check that for each large subset \( L \subset X \) the complement \( L \setminus B(A, \varepsilon_X) \) is large in \( X \). Consider the set \( L' = (L \setminus B(A, \varepsilon_X)) \cup A \) and observe that \( L \subset B(L', \varepsilon_X) \) and hence \( L' \) is large in \( X \). Since \( A \) is small, the set \( L' \setminus A = L \setminus B(A, \varepsilon_X) \) is large in \( X \). \( \square \)

**Claim 3.4.** A subset \( A \subset X \) is small in \( X \) if and only if its image \( f(A) \) is small in \( Y \).

**Proof.** Assume that \( A \) is small in \( X \). To prove that \( f(A) \) is small in \( Y \), we need to check that for any large subset \( L \subset X \) the complement \( L \setminus f(A) \) is large in \( Y \). Claim \( 3.2 \) implies that the set \( g(L) \) is large in \( X \). By Claim \( 3.3 \), the set \( B(A, \varepsilon_X) \) is small in \( X \) and hence the complement \( g(L) \setminus B(A, \varepsilon_X) \) remains large in \( X \). By Claim \( 3.2 \) \( g(L) \setminus B(A, \varepsilon_X) \) is large in \( Y \). We claim that \( f(g(L) \setminus B(A, \varepsilon_X)) \subset B(L \setminus f(A), \eta_Y) \). Indeed, given point \( y \in f(g(L) \setminus B(A, \varepsilon_X)) \), find a point \( x \in g(L) \setminus B(A, \varepsilon_X) \) such that \( y = f(x) \) and a point \( z \in L \) such that \( x = g(z) \). We claim that \( z \notin f(A) \). Assuming conversely that \( z \notin f(A) \), we get \( x = g(z) \in g \circ f(A) \subset B(A, \varepsilon_X) \), which contradicts the choice of \( x \). So, \( z \in L \setminus f(A) \) and \( y = f \circ g(z) \in B(z, \eta_Y) \subset B(L \setminus f(A), \eta_Y) \).

Taking into account that the set \( f(g(L) \setminus B(A, \varepsilon_X)) \subset B(L \setminus f(A), \eta_Y) \) is large in \( Y \), we conclude that the set \( L \setminus f(A) \) is large in \( Y \) and hence \( f(A) \) is small in \( Y \).

Now assume that the set \( f(X) \) is small in \( Y \). Then the set \( g \circ f(A) \) is small in \( X \) and so are the sets \( B(g \circ f(A), \eta_X) \subset A \). \( \square \)

### 4. Proof of Theorem \( 1.6 \)

Let \( G \) be a topological group and \( \mathcal{E} \) be its left coarse structure. The inclusion \( \mathcal{D}_G(G) \subset S(G) \) will follow as soon as we prove that each non-small subset \( A \subset G \) has asymptotic dimension \( \text{asdim}(A) = \text{asdim}(G) \). We divide the proof of this fact into 3 steps.

**Claim 4.1.** There is an entourage \( \varepsilon_A \in \mathcal{E} \) such that the set \( G \setminus B(A, \varepsilon_A) \) is not large in \( G \).

**Proof.** Since \( A \) is not small, there is a large subset \( L \subset X \) such that the complement \( L \setminus A \) is not large. Since \( L \) is large in \( X \), there is an entourage \( \varepsilon_A \in \mathcal{E} \) such that \( B(L, \varepsilon_A) = G \). We claim that the set \( G \setminus B(A, \varepsilon_A) \) is not large. Assuming the opposite, we can find an entourage \( \delta \in \mathcal{E} \) such that \( B(G \setminus B(A, \varepsilon_A), \delta) = G \). Then for each \( x \in G \) the ball \( B(x, \delta) \) meets \( G \setminus B(A, \varepsilon_A) \) at some point \( y \). By the choice of \( \varepsilon_A \), the ball \( B(y, \varepsilon_A) \) meets
Claim 4.3. $\text{asdim}(B(A, \varepsilon_A)) = \text{asdim}(A)$.

Proof. Observe that the identity embedding $i : A \to B(A, \varepsilon_A)$ is a coarse equivalence. The coarse inverse $j : B(A, \varepsilon_A) \to A$ to $i$ can be defined by choosing a point $j(x) \in B(x, \varepsilon_A) \cap A$ for each $x \in B(A, \varepsilon_A)$. Now we equality $\text{asdim}(B(A, \varepsilon_A)) = \text{asdim}(A)$ follows from the invariance of the asymptotic dimension under coarse equivalences, see Proposition 1.3. □

Claim 4.2. $\text{asdim}(G) = \text{asdim}(A)$.

Proof. The inequality $\text{asdim}(A) \leq \text{asdim}(G)$ is trivial. So, it suffices to check that $\text{asdim}(A) \leq n$ where $n = \text{asdim}(A) = \text{asdim}(B(A, \varepsilon_A))$. If $n$ is infinite, then there is nothing to prove. So, we assume that $n \in \omega$.

For the proof of the inequality $\text{asdim}(G) \leq n$, we shall apply Theorem 2.6. Given any $\varepsilon \in \mathcal{E}$ we should find $\delta \in \mathcal{E}$ such that for each finite $\varepsilon$-connected subset $F \subset G$ there is a coloring $\chi : F \to n + 1$ such that each $\chi$-monochrome $\varepsilon$-chain $C \subset F$ has $\text{diam}(C) \leq \delta$. By the definition of the left coarse structure $\mathcal{E}$ we lose no generality assuming that $\varepsilon = \{(x, y) \in G \times G : x \in yK_\varepsilon\}$ for some compact subset $K_\varepsilon = K_\varepsilon^{-1} \subset G$ containing the neutral element $1_G$ of $G$. In this case the entourage $\varepsilon$ is left invariant in the sense that for each pair $(x, y) \in \varepsilon$ and each $z \in G$ the pair $(zx, zy)$ belongs to $\varepsilon$.

Since $\text{asdim}_{\varepsilon}(B(A, \varepsilon_A)) = \text{asdim}(B(A, \varepsilon_A)) \leq n$, for the entourage $\varepsilon \in \mathcal{E}$, there are an entourage $\delta \in \mathcal{E}$ and a coloring $\chi_A : B(A, \varepsilon_A) \to n + 1$ such that each $\chi$-monochrome $\varepsilon$-chain $C \subset B(A, \varepsilon_A)$ has $\text{diam}(C) \leq \delta$, see Proposition 2.2. By the definition of the left coarse structure $\mathcal{E}$, we lose no generality assuming that $\delta = \{(x, y) \in G \times G : x \in yK_\delta\}$ for some compact set $K_\delta = K_\delta^{-1} \supseteq 1_G$ of $G$, which implies that the entourage $\delta$ is left invariant.

Now take any finite $\varepsilon$-connected subset $F \subset G$. Replacing $F$ by $F \cup F^{-1} \cup \{1_G\}$ we can assume that $F = F^{-1} \supseteq 1_G$. Since the set $G \setminus B(A, \varepsilon_A)$ is not large, there is a point $z \notin (G \setminus B(A, \varepsilon_A))F$. Then $zF^{-1}$ is disjoint with $G \setminus B(A, \varepsilon_A)$ and hence $zF = zF^{-1} \subset B(A, \varepsilon_A)$. So, it is legal to define a coloring $\chi : F \to n + 1$ by the formula $\chi(x) = \chi_A(zx)$ for $x \in F$. Taking into account the left invariance of the entourages $\varepsilon$ and $\delta$, it is easy to see that each $\chi$-monochrome $\varepsilon$-chain $C \subset F$ has diameter $\text{diam}(C) \leq \delta$. By Propositions 2.2 and 2.3 $\text{asdim}(G) = \text{asdim}_{\varepsilon}(G) \leq n = \text{asdim}(A)$. □

5. Proof of Theorem 1.1.2.

We need to prove that a subset $A \subset \mathbb{R}^n$ is small if and only if it has asymptotic dimension $\text{asdim}(\mathbb{R}^n) = n$. The “if” part of this characterization follows from the inclusion $\mathcal{D}_<(\mathbb{R}^n) \subset S(\mathbb{R}^n)$ proved in Theorem 1.6. To prove the “only if” part, we need to recall some (standard) notions of Combinatorial Topology 12, 10.

On the Euclidean space $\mathbb{R}^n$ we shall consider the metric generated by the sup-norm $\|x\| = \max_{i \in \mathbb{N}} |x(i)|$.

By the standard $n$-dimensional simplex we understand the compact convex subset
$$\Delta = \{(x_0, \ldots, x_n) \in [0, 1]^{n+1} : \sum_{i=0}^{n} x_i = 1\} \subset \mathbb{R}^{n+1}$$
of the Euclidean space $\mathbb{R}^{n+1}$ endowed with the sup-norm. For each $i \leq n$ by $v_i : n + 1 \to \{0, 1\} \subset \mathbb{R}$ we denote the vertex of $\Delta$ defined by $v_i^{-1}(1) = \{i\}$. For each vertex $v_i$ of $\Delta$ consider its star
$$S\Delta(v_i) = \{x \in \Delta : x(i) > 0\}$$
and its barycentric star
$$S\Delta'(v_i) = \{x \in \Delta : x(i) = \max_{j \leq n} x(j)\} \subset S\Delta(v_i).$$

It is clear that $\bigcup_{i=0}^{n} S\Delta'(v_i) = \Delta$ while $\bigcap_{i=0}^{n} S\Delta'(v_i) = \{b_\Delta\}$ is the singleton containing the barycenter
$$b_\Delta = \frac{1}{n+1} \sum_{i=0}^{n} v_i$$
of the simplex $\Delta$.

Claim 5.1. $\bigcap_{i=0}^{n} B(S\Delta'(v_i), \varepsilon) \subset B(b_\Delta, n\varepsilon)$ for each positive real number $\varepsilon$. 


Proof. Given any vector \( x \in \bigcap_{i=0}^{n} B(St'_\Delta(v_i), \varepsilon) \), for every \( i \leq n \) we can find a vector \( y \in St'_\Delta(v_i) \) with \( \|x - y\| < \varepsilon \). Then \( |x_i - y_i| \leq \|x - y\| < \varepsilon \) and hence \( x_i > y_i - \varepsilon = \max_{j \leq n} y_j - \varepsilon \geq \frac{1}{n+1} - \varepsilon \). On the other hand, \( x_i = 1 - \sum_{j \neq i} x_j < 1 - \sum_{j \neq i} \left( \frac{1}{n+1} - \varepsilon \right) = 1 - \frac{n}{n+1} + n\varepsilon = \frac{1}{n+1} + n\varepsilon. \) So, \( \|x - b\Delta\| < n\varepsilon. \) \( \square \)

Now we are going to generalize Claim 5.1 to arbitrary simplexes. By an \( n \)-dimensional simplex in \( \mathbb{R}^n \) we understand the convex hull \( \sigma = \text{conv}(\{0\}) \) of an affinely independent subset \( \sigma^{(0)} \subset \mathbb{R}^n \) of cardinality \( |\sigma^{(0)}| = n+1 \). Each point \( v \in \sigma^{(0)} \) is called a vertex of the simplex \( \sigma \). The arithmetic mean of the vertices is called the baricenter of the simplex \( \sigma \). By \( \partial\sigma \) we denote the boundary of the simplex \( \sigma \) in \( \mathbb{R}^n \). Observe that the homothetic copy \( \frac{1}{b} \sigma + \frac{1}{2}\sigma = \{ \frac{1}{b} x + \frac{1}{2} x : x \in \sigma \} \) of \( \sigma \) is contained in the interior \( \sigma \setminus \partial\sigma \) of \( \sigma \). For each vertex \( v \in \sigma^{(0)} \) let

\[ St_\sigma(v) = \sigma \setminus \text{conv}(\sigma^{(0)} \setminus \{v\}) \]

be the star of \( v \) in \( \sigma \).

In fact, \( n \)-dimensional simplexes can be alternatively defined an images of the standard \( n \)-dimensional simplex \( \Delta \) under injective affine maps \( f : \Delta \to \mathbb{R}^n \).

A map \( f : \Delta \to \mathbb{R}^n \) is called affine if \( f(tx + (1-t)y) = tf(x) + (1-t)f(y) \) for any points \( x, y \in \Delta \) and a real number \( t \in [0,1] \). It is well-known that each affine function \( f : \Delta \to \mathbb{R}^n \) is uniquely defined by its restriction \( f|\Delta^{(0)} \) to the set \( \Delta^{(0)} = \{ v_i \}_{i=1}^n \) of vertices of \( \Delta \).

A map \( f : \Delta \to \mathbb{R}^n \) will be called \( b_\Delta \)-affine if for every \( i \leq n \) the restriction \( f|\text{conv}(\{b_\Delta \cup \Delta^{(0)} \setminus \{v_i\}) \) is affine. A \( b_\Delta \)-affine function \( f : \Delta \to \mathbb{R}^n \) is uniquely determined by its restriction \( f|\Delta^{(0)} \cup \{b_\Delta\} \).

A function \( f : X \to Y \) between metric spaces \((X, d_X)\) and \((Y, d_Y)\) is called Lipschitz if it its Lipschitz constant

\[ \text{Lip}(f) = \sup \left\{ \frac{d_Y(f(x), f(x'))}{d_X(x, x')} : x, x' \in X, \ x \neq x' \right\} \]

is finite. A bijective function \( f : X \to Y \) is bi-Lipschitz if \( f \) and \( f^{-1} \) are Lipschitz.

Claim 5.2. For any \( n \)-dimensional simplex \( \sigma \in \mathbb{R}^n \) there is a real constant \( L \) such that each \( b_\Delta \)-affine function \( f : \Delta \to \sigma \) with \( f(\Delta^{(0)}) = \sigma^{(0)} \) and \( f(b_\Delta) \in \frac{1}{b_\Delta} \sigma + \frac{1}{2} \sigma \) is bijective, bi-Lipschitz and has \( \text{Lip}(f) \cdot \text{Lip}(f^{-1}) \leq L \).

This claim can be easily derived from the fact that each \( b_\Delta \)-affine function \( f : \Delta \to \sigma \) with \( f(\Delta^{(0)}) = \sigma^{(0)} \) is Lipschitz and its Lipschitz constant \( \text{Lip}(f) \) depends continuously on \( f(b_\Delta) \).

Given an \( n \)-dimensional simplex \( \sigma \in \mathbb{R}^n \) and a point \( b' \in \sigma \setminus \partial\sigma \) in its interior, fix a \( b_\Delta \)-affine function \( f : \Delta \to \sigma \) such that \( f(\Delta^{(0)}) = \sigma^{(0)} \) and \( f(b_\Delta) = b' \). For each vertex \( v \in \sigma^{(0)} \) consider its \( b' \)-barycentric star

\[ St_{\sigma, b'}(v) = f(St'_\Delta(f^{-1}(v))) \subset St_\sigma(v). \]

It is easy to see that the set \( St_{\sigma, b'}(v) \) does not depend on the choice of the \( b_\Delta \)-affine function \( f \).

Claim 5.3. For any \( n \)-dimensional simplex \( \sigma \in \mathbb{R}^n \) there is a real constant \( L \) such that for each point \( b' \in \frac{1}{b_\Delta} \sigma + \frac{1}{2} \sigma \) and each \( \varepsilon > 0 \) we get \( \sigma \cap \bigcap_{v \in \sigma^{(0)}} B(St'_{\sigma, b'}(v), \varepsilon) \subset B(b', L\varepsilon) \).

Proof. By Claim 5.2 there is a real constant \( C \) such that each bijective \( b_\Delta \)-affine function \( f : \Delta \to \sigma \) with \( f(\Delta^{(0)}) = \sigma^{(0)} \) and \( f(b_\Delta) \in \frac{1}{b_\Delta} \sigma + \frac{1}{2} \sigma \) has \( \text{Lip}(f) \cdot \text{Lip}(f^{-1}) \leq C \). Put \( L = nC \). Given any point \( b' \in \frac{1}{b_\Delta} \sigma + \frac{1}{2} \sigma \), choose a bijective \( b_\Delta \)-affine function \( f : \Delta \to \sigma \) such that \( f(\Delta^{(0)}) = \sigma^{(0)} \) and \( f(b_\Delta) = b' \). The choice of \( C \) guarantees that \( \text{Lip}(f) \cdot \text{Lip}(f^{-1}) \leq C \). Now observe that

\[
\sigma \cap \bigcap_{v \in \sigma^{(0)}} B(St'_{\sigma, b'}(v), \varepsilon) = \bigcap_{v \in \sigma^{(0)}} f \circ f^{-1}(B(St'_{\sigma, b'}(v), \varepsilon)) \subset \bigcap_{v \in \sigma^{(0)}} f(B(f^{-1}(St'_{\sigma, b'}(v)), \text{Lip}(f^{-1})\varepsilon)) =
\]

\[
= \bigcap_{v \in \sigma^{(0)}} f(B(St'_\Delta(f^{-1}(v)), \text{Lip}(f^{-1})\varepsilon)) = f\left( \bigcap_{v \in \Delta^{(0)}} B(St'_\Delta(v), \text{Lip}(f^{-1})\varepsilon) \right) \subset f(B(b_\Delta, n\text{Lip}(f^{-1})\varepsilon)) \subset B(f(b_\Delta), \text{Lip}(f)\text{Lip}(f^{-1})n\varepsilon) = B(b', Cn\varepsilon) = B(b', L\varepsilon).
\]
Now consider the binary unit cube $K = \{0,1\}^n \subset \mathbb{R}^n$ endowed with the partial ordering $\leq$ defined by $x \leq y$ iff $x(i) \leq y(i)$ for all $i < n$. Given two vectors $x, y \in \{0,1\}^n$, we write $x < y$ if $x \leq y$ and $x \neq y$.

For every increasing chain $v_0 < v_1 < \ldots < v_n$ of points of the binary cube $K = \{0,1\}^n$, consider the simplex $\text{conv}\{v_0, \ldots, v_n\}$ and let $T_K$ be the (finite) set of these simplexes. Next, consider the family $T = \{\sigma + z : \sigma \in T_K, z \in \mathbb{Z}^n\}$ of translations of the simplexes from the family $T_K$, and observe that $\bigcup T = \mathbb{R}^n$. For each point $v \in \mathbb{Z}^n$ let

$$St_T(v) = \bigcup \{St_{\sigma}(v) : v \in \sigma \in T\}$$

be the $T$-star of $v$ in the triangulation $T$ of the space $\mathbb{R}^n$.

Now we are able to prove the “only if” part of Theorem 1.2. Assume that a subset $A \subset \mathbb{R}^n$ is small. Then there is a function $\varphi : (0, \infty) \to (0, \infty)$ such that for each $\delta \in (0, \infty)$ and a point $x \in \mathbb{R}^n$ there is a point $y \in \mathbb{R}^n$ with $B(y, \delta) \subset B(x, \varphi(\delta)) \setminus A$. The inequality $\text{asdim}(A) < n$ will follow as soon as given any $\delta < \infty$ we construct a cover $U$ of $A$ with finite mesh($U$) = $\sup_{U \in U} \text{diam}(U)$ such that each $\delta$-ball $B(a, \delta)$, $a \in A$, meets at most $n$ elements of the cover $U$.

By Claim 5.3 there is a constant $L$ such that for any simplex $\sigma \in T$, each point $b' \in \frac{1}{2}b_\sigma + \frac{1}{2}\sigma$ and each $\varepsilon > 0$ we get $\sigma \cap \bigcap_{v \in \sigma(0)} B(St'_{\sigma, b'}(v), \varepsilon) \subset B(b', Le)$. Given any $\delta < \infty$, choose $\varepsilon > 0$ so small that for any simplex $\sigma \in T$ the following conditions hold:

1. $B(b_\sigma, \varphi(L\delta)) \subset \frac{1}{2}b_\sigma + \frac{1}{2}\sigma$;
2. for any $b' \in \frac{1}{2}b_\sigma + \frac{1}{2}\sigma$ and any vertex $v \in \sigma(0)$ the $2\varepsilon\delta$-neighborhood $B(St'_{\sigma, b'}(v), 2\varepsilon\delta)$ lies in the $T$-star $St_T(v)$ of $v$.

Now consider the closed cover

$\tilde{T} = \{\varepsilon^{-1}\sigma : \sigma \in T\}$

of the space $\mathbb{R}^n$ and observe that for each simplex $\sigma \in \tilde{T}$ we get

1. $B(b_\sigma, \varphi(L\delta)) \subset \frac{1}{2}b_\sigma + \frac{1}{2}\sigma$;
2. for any $b' \in \frac{1}{2}b_\sigma + \frac{1}{2}\sigma$ and any vertex $v \in \sigma(0)$ the $2\delta$-neighborhood $B(St_{t, b'}(v), 2\delta)$ lies in the $\tilde{T}$-star $St'_{\tilde{T}}(v)$ of $v$.

By the choice of the function $\varphi$, for each simplex $\sigma \in \tilde{T}$, there is a point $b'_\sigma \in \mathbb{R}^n$ such that $B(b'_\sigma, L\delta) \subset B(b_\sigma, \varphi(L\delta)) \setminus A$. The condition (1) guarantees that

$$b'_\sigma \in B(b_\sigma, \varphi(L\delta)) \subset \frac{1}{2}b_\sigma + \frac{1}{2}\sigma.$$ 

For every point $v \in \frac{1}{2}\mathbb{Z}^n$ consider the set

$$St'(v) = \bigcup \{St_{\sigma, b'_\sigma}(v) : \sigma \in \tilde{T}, v \in \sigma(0)\} \subset St'_{\tilde{T}}(v)$$

and observe that $U = \{St'(v) : v \in \varepsilon^{-1}\mathbb{Z}^n\}$ is a cover of the Euclidean space $\mathbb{R}^n$. It follows that

$$\text{mesh}(U) = \sup_{v \in \varepsilon^{-1}\mathbb{Z}^n} \text{diam}(St'(v)) \leq 2 \sup_{v \in \varepsilon^{-1}\mathbb{Z}^n} \text{diam}(\sigma) \leq 2\varepsilon^{-1}\text{diam}(0,1] < \infty.$$ 

It remains to check that each ball $B(a, \delta)$, $a \in A$, meets at most $n$ sets $U \in U$.

Assume conversely that there are a point $a \in A$ and a set $V \subset \varepsilon^{-1}\mathbb{Z}^n$ of cardinality $|V| = n + 1$ such that $B(a, \delta) \cap St'(v) \neq \emptyset$ for each $v \in V$. Then $a \in \bigcap_{v \in V} B(St'(v), \delta)$. It follows from $a \in \bigcap_{v \in V} B(St'(v), \delta) \subset \bigcap_{v \in V} St_T(v)$ that $V$ coincides with the set $\sigma(0)$ of vertices of some simplex $\sigma \in \tilde{T}$ and $a$ lies in the interior of the simplex $\sigma$.

Next, we show that $a \in B(St'_{\sigma, b'_\sigma}(v), \delta)$ for each $v \in V$. In the opposite case, $a \in B(St'_{\sigma, b'_\sigma}(v), \delta) \subset B(\tau, \delta)$ for some simplex $\tau \in \tilde{T} \setminus \{\sigma\}$ such that $v \in \tau(0) \setminus \sigma(0)$. Choose a vertex $u \in \sigma(0) \setminus \tau(0)$ and observe that the condition (2) implies that $a \in B(St'_{\sigma, b'_\sigma}, \delta) \cap B(\tau, \delta) = \emptyset$, which is a contradiction.

Finally, the choice of $L$ and $b'_\sigma$ yields the desired contradiction

$$a \in \sigma \cap \bigcap_{v \in \sigma(0)} B(St'_{\sigma, b'_\sigma}(v), \delta) \subset B(b'_\sigma, L\delta) \subset \mathbb{R}^n \setminus A,$$

completing the proof of the theorem.
6. Proof of Theorem 1.7

Given an Abelian locally compact topological group $G$ endowed with its left coarse structure, we need to prove the equivalence of the following statements:

(1) $S(G) = D_\infty(G)$;
(2) $G$ is compactly generated;
(3) $G$ is coarsely equivalent to an Euclidean space $\mathbb{R}^n$ for some $n \in \omega$.

We shall prove the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). The implication (3) $\Rightarrow$ (1) follows from Corollary 1.5.

To prove that (2) $\Rightarrow$ (3), assume that the group $G$ is compactly generated. By Theorem 24 [11, p.85], $G$ is topologically isomorphic to the direct sum $\mathbb{R}^n \times \mathbb{Z}^m \times K$ for some $n, m \in \omega$ and a compact subgroup $K \subset G$. Since the projection $\mathbb{R}^n \times \mathbb{Z}^m \times K \to \mathbb{R}^n \times \mathbb{Z}^m$ and the embedding $\mathbb{Z}^n \times \mathbb{Z}^m \to \mathbb{R}^n \times \mathbb{Z}^m$ are coarse equivalences, we conclude that $G$ is coarsely equivalent to $\mathbb{Z}^{n+m}$ and to $\mathbb{R}^{n+m}$.

To prove that (1) $\Rightarrow$ (2), assume that $S(G) = D_\infty(G)$. First we prove that $G$ has finite asymptotic dimension. By the Principal Structure Theorem 25 [11, p.26], $G$ contains an open subgroup $G_0$ that is topologically isomorphic to $\mathbb{R}^n \times K$ for some $n \in \omega$ and some compact subgroup $K$ of $G_0$. The subgroup $G_0$ has asymptotic dimension $\text{asdim}(G_0) = \text{asdim}(\mathbb{R}^n) = n < \infty$. If $\text{asdim}(G) = \infty$, then the quotient group $G/G_0$ has infinite asymptotic dimension and hence has infinite free rank. Then the group $G/G_0$ contains a subgroup isomorphic to the free abelian group $\oplus \omega \mathbb{Z}$ with countably many generators. It follows that $G$ also contains a discrete subgroup $H$ isomorphic to $\oplus \omega \mathbb{Z}$. Replacing $H$ by a smaller subgroup, if necessary, we can assume that $H$ has infinite index in $G$ and hence is small in $G$. Since $\text{asdim}(H) = \infty = \text{asdim}(G_0)$, we conclude that $S(G) \neq D_\infty(G)$, which is a desired contradiction showing that $\text{asdim}(G) < \infty$.

By Theorem 25, there is a compactly generated subgroup $H \subset G$ with $\text{asdim}(H) = \text{asdim}(G)$. Since $H \notin D_\infty(G) = S(G)$, the subset $H$ is not small in $G$. Repeating the proof of Claim 1.1 we can show that the set $G \setminus B(H, \varepsilon)$ is not large for some entourage $\varepsilon \in \mathcal{E}$. By the definition of the left coarse structure $\mathcal{E}$, there is a compact subset $K \subset G$ such that $B(H, \varepsilon) \subset HK$. We claim that $K^{-1}HK = G$. Assuming the opposite, we can find a point $x \in G \setminus K^{-1}HK$ and consider the finite set $F = \{x, x^{-1}, xx^{-1}\} = F^{-1}$. Since the set $G \setminus HK$ is not large, there is a point $z \in (G \setminus HK)F$. For this point $z$ we get $zF \cap (G \setminus HK) = \emptyset$ and hence $z \in zF \subset HK$. Then $x \in z^{-1}zF \subset z^{-1}HK \subset K^{-1}HK = K^{-1}HK$, which is a contradiction. Now the compact generacy of the subgroup $H$ implies the compact generacy of the group $G = K^{-1}HK$.

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