GIROUX TORSION AND TWISTED COEFFICIENTS

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Abstract. We explain the effect of applying a full Lutz twist along a pre-Lagrangian torus in a contact 3-manifold, on the contact invariant in Heegaard Floer homology with twisted coefficients.

Let \((M, \xi)\) be a contact 3-manifold and \(T \subset M\) be a pre-Lagrangian torus, i.e., an embedded torus whose characteristic foliation \(\xi_L = \xi \cap TL\) is linear. By slightly perturbing \(T\), we may assume that it is linearly foliated by closed orbits, and, by choosing a suitable identification \(T \sim \mathbb{R}^2/\mathbb{Z}^2\), we may assume that the orbits have slope \(\infty\). We say that \((M, \xi')\) is obtained from \((M, \xi)\) by a full Lutz twist along \(T\) if we cut \((M, \xi)\) along \(T\) and insert \((T^2 \times [0, 1], \eta_{2\pi})\), where \((x, y, t)\) are coordinates on \(T^2 \times [0, 1] \cong \mathbb{R}^2/\mathbb{Z}^2 \times [0, 1]\) and \(\eta_{2\pi} = \ker(\cos(2\pi t)dx - \sin(2\pi t)dy)\). A contact manifold \((M, \xi)\) has \(2\pi n\)-torsion along \(T\) if there exists a thickened torus \((T^2 \times [0, 1], \eta_{2\pi n})\) which embeds into \((M, \xi)\), so that \(T^2 \times \{t\}\) are isotopic to \(T\) and \(\eta_{2\pi n}\) is obtained by stacking \(n\) copies of \(\eta_{2\pi}\). Also \((M, \xi)\) has finite torsion along \(T\) if there is a positive integer \(n\) so that \((M, \xi)\) has \(2\pi n\)-torsion along \(T\) but does not have \(2\pi(n + 1)\)-torsion along \(T\).

In this paper, we assume that our 3-manifolds are compact and oriented, and our contact structures are cooriented, unless stated otherwise. In a previous paper [GHV], the authors and Van Horn-Morris proved the following:

Theorem 1 (Vanishing Theorem). Suppose the coefficient ring of the Heegaard Floer homology groups is \(\mathbb{Z}\). If a closed, oriented contact 3-manifold \((M, \xi')\) is obtained from \((M, \xi)\) by a full Lutz twist along a pre-Lagrangian torus \(T\), then its contact invariant \(c(M, \xi'; \mathbb{Z})\) in \(\hat{HF}(-M; \mathbb{Z})/\{\pm 1\}\) vanishes.

Theorem 1 was first conjectured in [Gh2, Conjecture 8.3], and partial results were obtained by [Gh1, Gh2], and [LS1]. The corresponding vanishing result for the contact class in monopole Floer homology is due to Gay [Ga], using results of Mrowka and Rollin [MR]. Theorem 1 together with a non-vanishing result of the contact invariant proved by Ozsváth and Szabó [OSz3, Theorem 4.2], implies that a contact manifold with \(2\pi\)-torsion is not strongly symplectically fillable. This non-fillability result was conjectured by Eliashberg, and first proved by Gay [Ga].

The goal of the present paper is go further and explain what happens when we use twisted coefficients. Consider the group ring \(\mathbb{L} = \mathbb{Z}[H_2(M; \mathbb{Z})]\). If \(a \in H_2(M; \mathbb{Z})\), we denote by \(e^a\) the corresponding element in \(\mathbb{L} = \mathbb{Z}[H_2(M; \mathbb{Z})]\). If \(\mathbb{M}\) is a \(\mathbb{Z}\)-algebra and \(\mathbb{L} \rightarrow \mathbb{M}\) is a \(\mathbb{Z}\)-algebra
homomorphism which induces an \( \mathbb{L} \)-module structure on \( \mathbb{M} \), then the contact invariant \( c(M, \xi; \mathbb{M}) \) is an element of \( \widehat{HF}(-M; \mathbb{M})/\mathbb{M}^\times \), where \( \mathbb{M}^\times \) denotes the group of units of \( \mathbb{M} \). In this paper we follow the usual conventions and underline to indicate that there is a (presumably) nontrivial \( \mathbb{L} \)-action on the Heegaard Floer homology or sutured Floer homology group.

The following is our main theorem:

**Theorem 2.** There exists a Laurent polynomial \( p(t) = t - 1 \in \mathbb{Z}[t, t^{-1}] \) such that the following holds: For any closed, oriented contact 3-manifold \( (M, \xi) \) and a pre-Lagrangian torus \( T \subset M \), if \( (M, \xi') \) is obtained from \( (M, \xi) \) by a full Lutz twist along \( T \), then
\[
(1) \quad c(M, \xi'; \mathbb{M}) = p(e^{[T]}) \cdot c(M, \xi; \mathbb{M}).
\]

Here there is a \( \mathbb{Z} \)-algebra homomorphism \( \mathbb{L} \rightarrow \mathbb{M} \) which induces an \( \mathbb{L} \)-module structure on \( \mathbb{M} \), and \( c(M, \xi'; \mathbb{M}), c(M, \xi; \mathbb{M}) \) are elements in \( \widehat{HF}(-M; \mathbb{M})/\mathbb{M}^\times \).

Theorem 2 was partly inspired by the work of Hutchings-Sullivan [HS] on the calculation of invariants of contact structures on the 3-torus in embedded contact homology.

Observe that, for the applications below, it is only necessary to know that \( p(t) \) is divisible by \( t - 1 \). Let us write \( \xi_n, n \in \mathbb{Z}^{>0} \), for the contact structure obtained from \( (M, \xi) \) by applying \( n \) full Lutz twists along a pre-Lagrangian torus \( T \subset M \).

**Corollary 3.** Let \( \mathbb{L} \rightarrow \mathbb{M} \) be a \( \mathbb{Z} \)-algebra homomorphism. If \( e^{[T]} \) acts trivially on \( \mathbb{M} \), i.e., as the identity, then \( c(M, \xi_n; \mathbb{M}) = 0 \) in \( \widehat{HF}(-M; \mathbb{M})/\mathbb{M}^\times \) if \( n > 0 \).

In particular, we have the following:

**Corollary 4.** If \( T \) is a separating pre-Lagrangian torus in \( (M, \xi) \), then for \( n > 0 \):
\begin{enumerate}
  \item \( c(M, \xi_n; \mathbb{M}) = 0 \) in \( \widehat{HF}(-M; \mathbb{M})/\mathbb{M}^\times \).
  \item \( (M, \xi_n) \) is not weakly symplectically fillable.
\end{enumerate}

**Proof.** For (1), simply observe that \( [T] = 0 \) if \( T \) is separating. (2) follows from a result of Ozsváth-Szabó on the nonvanishing of the contact invariant for a weakly symplectically fillable contact structure. The precise statement is given as Theorem 8 in Section 11. \( \square \)

On the other hand, Colin [Co] and Honda-Kazez-Matić [HKM1] have proven that there exist infinitely many nonisomorphic universally tight contact structures on a toroidal \( M \) with a separating torus \( T \), of type \( (M, \xi_n) \). This gives large infinite families of universally tight contact structures which are not weakly symplectically fillable. Our results generalize prior examples of Ghiggini [Gh2].

**Corollary 5.** Let \( \mathbb{L} = \mathbb{Z}[H_2(M; \mathbb{Z})] \). If \( c(M, \xi; \mathbb{L}) \in \widehat{HF}(-M; \mathbb{L})/\mathbb{L}^\times \) is nontrivial and nontorsion, i.e., no nonzero element of \( \mathbb{L} \) annihilates \( c(M, \xi; \mathbb{L}) \), and \( [T] \neq 0 \in H_2(M; \mathbb{Z}) \), then
\begin{enumerate}
  \item \( \xi_n \) has finite torsion along \( T \) for \( n \geq 0 \);
  \item \( \xi_n \) and \( \xi_m \) are pairwise nonisotopic for \( n \neq m \).
\end{enumerate}
By the ascending chain property, the chain stabilizes at some point, i.e., localization. Now consider the ascending chain of
\[ c \]
Hence Corollary 6.

1) equation, it is immediate that equality holds if and only if
\[ \sum \]
which emphasizes twisted coefficients.)

Contact structure [OSz3]. The reader is referred to [OSz2] for the definition and properties of the Heegaard Floer homology groups with twisted coefficients. (Also see [IM] for a good summary which emphasizes twisted coefficients.)
Let \( M \) be a closed 3-manifold and \([\omega] \in H^2(M; \mathbb{R})\). Then \([\omega]\) induces an evaluation map (a group homomorphism):

\[
\int : H_2(M; \mathbb{Z}) \to \mathbb{R}, \quad [A] \mapsto \int_A \omega,
\]
and we have an induced ring homomorphism of group rings:

\[
\mathbb{Z}[H_2(M; \mathbb{Z})] \to \mathbb{Z}[\mathbb{R}],
\]
which makes \(\mathbb{Z}[\mathbb{R}]\) into an \(\mathbb{L} = \mathbb{Z}[H_2(M; \mathbb{Z})]\)-module. We write \(\mathcal{M}_\omega\) to indicate \(\mathbb{Z}[\mathbb{R}]\) with this \(\mathbb{L}\)-module structure.

Given \(t \in \text{Spin}^c(M)\), we can define \(HF^\infty(M, t; \mathcal{M}_\omega)\) for any flavor of Heegaard Floer homology. The definition for \(HF^\infty\) is as follows (and the other \(HF^\infty\) are analogous): Let \((\Sigma, \alpha, \beta, z)\) be an admissible pointed Heegaard diagram for \(M\) and let \(A\) be a surjective additive assignment for the Heegaard diagram which takes values in \(H_2(M; \mathbb{Z})\), instead of the usual \(H^1(M; \mathbb{Z})\). Letting \(\bar{CF}^\infty(M, t; \mathcal{M}_\omega)\) be the free \(\mathcal{M}_\omega\)-module generated by pairs \([x, i]\) with \(x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta\) representing \(t\) and \(i \in \mathbb{Z}\), the differential is given by:

\[
\partial^\infty([x, i]) = \sum_{y \in \mathbb{T}_\alpha \cup \mathbb{T}_\beta} \sum_{\phi \in \pi_2(x, y)} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot t^{\int_A \omega}[y, i - n_z(\phi)].
\]

Next let \(X\) be a 4-dimensional cobordism between 3-manifolds \(M_0\) and \(M_1\), \(\omega\) be a closed 2-form on \(X\) defining the cohomology class \([\omega] \in H^2(X; \mathbb{R})\), and \(s\) be a Spin\(^c\) structure on \(X\). Then we have maps

\[
F^\circ_{X, s; \mathcal{M}_\omega} : HF^\infty(M_0, s|_{M_0}; \mathcal{M}_\omega|_{M_0}) \to HF^\infty(M_1, s|_{M_1}; \mathcal{M}_\omega|_{M_1}).
\]

In order to define the map for \(HF^\infty\), we need to introduce some notation. Let \((\Sigma, \alpha, \beta, \gamma, z)\) be an admissible pointed triple Heegaard diagram for \(X\) so that \((\Sigma, \alpha, \beta, z)\) is a Heegaard diagram for \(M_0\), \((\Sigma, \alpha, \gamma, z)\) is a Heegaard diagram for \(M_1\), and \((\Sigma, \beta, \gamma, z)\) is a Heegaard diagram for a connected sum of \((S^1 \times S^2)\)'s. There is a canonical intersection point \(\Theta \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma\), defined by choosing the intersection point with lowest relative Maslov index for any pair of parallel curves \(\beta_i\) and \(\gamma_i\). We denote by \(\pi_2(x, \Theta, y)\) the homotopy classes of Whitney triangles connecting the intersection points \(x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta\), \(\Theta \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma\), and \(y \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma\). Next, let \(A_i, i = 0, 1\), be a surjective additive assignment for the Heegaard diagram for \(M_i\). Let \(\psi_s \in \pi_2(x, \Theta, y)\) be a fixed representative of \(s\). If \(\psi \in \pi_2(x', \Theta, y')\) represents the same Spin\(^c\) structure \(s\), then there are Whitney disks \(\phi_{x', x}\) and \(\phi_{y, y'}\) so that \(\psi = \psi_s * \phi_{x', x} * \phi_{y, y'}\), where * denotes concatenation. (See [OSz1 Proposition 8.5].) Then define

\[
A_X(\psi) = \delta(-A_0(\phi_{x', x}) + A_1(\phi_{y, y'})),
\]

where

\[
\delta : H^1(\partial X) \to H^2(X, \partial X)
\]

is the coboundary map of the long exact sequence of \((X, \partial X)\).

\footnote{Our definition of the map looks slightly different from that given on pp. 323–324 of [OSz3], but is equivalent to it.}
We can now define the map $E_{X,s,M_ω}^∞$ in Equation 2 as follows:

$$E_{X,s,M_ω}^∞([x,i]) = \sum_{y \in T_α \cap T_γ} \sum_{\psi \in π_2(x, Θ, y)} \#M(ψ) \cdot tI_{X_ω}(ψ)[y, i - n_ε(ψ)].$$

Here the map $E_{X,s,M_ω}^∞$ depends on the choice of the reference triangle $ψ_ω$, and changing this choice has the effect of pre-composing (and post-composing) $E_{X,s,M_ω}^∞$ by an element of $H^1(M_0)$ (and an element of $H^1(M_1)$). The definitions for the other $E_{X,s,M_ω}^∞$ are analogous.

Let $t_i \in \text{Spin}^c(M_i)$ for $i = 0, 1$. Let $S(t_0, t_1)$ be the set of all Spin$^c$ structures $s \in \text{Spin}^c(X)$ which restrict to $t_i$ on $M_i$. Choose a reference Spin$^c$-structure $s_0 \in S(t_0, t_1)$ and choose $ψ_ω \in π_2(x, Θ, y)$ for all $s \in S(t_0, t_1)$, where $x$ and $y$ are the same for all $s$. When summing over $S(t_0, t_1)$ we form:

$$E_{X,S(t_0,t_1);M_ω}^∑(s) E_{X,s,M_ω}^∞ tD(ψ_ω-ψ_{s_0}) \omega.$$

Here $D(ψ_ω-ψ_{s_0})$ is a 2-cycle in $X$ corresponding to the triply-periodic domain $ψ_ω-ψ_{s_0}$.

Now, the composition law [OSz6, Theorem 3.9] can be stated as follows, for $M_ω$-coefficients:

**Theorem 7** (Composition Law). Let $X = X_1 \cup M_1 \cup X_2$ be a composition of cobordisms $X_1$ from $M_0$ to $M_1$ and $X_2$ from $M_1$ to $M_2$. If $ω$ is a closed 2-form on $X$, then

$$E_{X_2,s_2;M_ω}^∞ \circ E_{X_1,s_1;M_ω}^∞ = \sum_{s \in \text{Spin}^c(X)} E_{X,s,M_ω}^∞ tD(ψ_ω-ψ_{s_0}) \omega$$

$$= \sum_{s \in \text{Spin}^c(X)} E_{X,s,M_ω}^∞ t[D(ψ_ω-ψ_{s_0})∪(s-s_0), [X]].$$

where $s_0$ is a reference Spin$^c$ structure on $X$ which restricts to $s_1$ on $X_1$ and $s_2$ on $X_2$.

**Proof.** The proof follows immediately from [OSz6], after the following consideration: Suppose $ψ_s, ψ_{s_0} \in π_2(x, Θ, y)$ correspond to Spin$^c$-structures $s, s_0 \in \text{Spin}^c(X)$. Then

$$∫_{D(ψ_ω-ψ_{s_0})} ω = ⟨[ω], PD(s-s_0)⟩ = ⟨[ω] \cup (s-s_0), [X]⟩,$$

by the argument in the proof of [OSz1, Proposition 8.5].

The following result is proved in [OSz3, Theorem 4.2], using considerations in the above paragraphs:

**Theorem 8** (Ozsváth-Szabó). Let $(X, ω)$ be a weak symplectic filling of a contact 3-manifold $(M, ξ)$. Then the contact invariant $c(M, ξ; M_ω)$ is nontrivial and non-torsion over $M_ω$.  

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3A related formula is given on p. 325 of [OSz3], but the term $c_1(s)$ which appears there should be replaced by $s-s_0$. 

For our purposes, we are interested in the contact structures \((T^3, \xi_n)\), \(n \in \mathbb{Z}^\geq 0\), defined as follows: Let \(T^3 \cong \mathbb{R}^3 / \mathbb{Z}^3\) with coordinates \(x, y, z\), and let
\[
\xi_n = \ker(dz + \varepsilon (\cos(2\pi nz)dx - \sin(2\pi nz)dy)),
\]
for \(\varepsilon > 0\) small. The contact structures \((T^3, \xi_n)\) can be weakly filled by \(X = D^2 \times T^2\) with the product symplectic structure \(\omega = dD^2 + dx \wedge dy\). Here \(\partial D^2\) is parametrized by the \(z\)-coordinate of \(T^3\). Since the pullback of \(\omega\) to \(T^3\) is \(dx \wedge dy\), it follows that the image of \(\phi[\omega]\) in \(\mathbb{Z}[\mathbb{R}]\) is isomorphic to \(\mathbb{M} = \mathbb{Z}[t, t^{-1}]\), where \([T]\) is the homology class of the torus \(dz = const\) and \(t = e^{[T]}\). Hence:

**Proposition 9.** The contact invariant \(\xi(T^3, \xi_n; \mathbb{M}) \in \widehat{HF}(-T^3; \mathbb{M})\) is nonzero and non-torsion over \(\mathbb{M}\).

We note that there is a slight difference between \(\mathbb{M}\) and \(\mathcal{M}_\omega\). There are two ways around this: either assume \(\lceil \omega \rceil\) lives in \(H^2(M; \mathbb{Q})\) after perturbation, or observe that \(\mathcal{M}_\omega\) is a free \(\mathbb{M}\)-module. In the latter case, \(\xi(T^3, \xi_n; \mathcal{M}_\omega)\) is the image of \(\xi(T^3, \xi_n; \mathbb{M}) \otimes 1\) under the tensor product map
\[
\widehat{HF}(-T^3; \mathbb{M}) \otimes_{\mathbb{M}} \mathcal{M}_\omega \to \widehat{HF}(-T^3; \mathcal{M}_\omega),
\]
and the nonzero/non-torsion properties of \(\xi(T^3, \xi_n; \mathcal{M}_\omega)\) imply the corresponding properties for \(\xi(T^3, \xi_n; \mathbb{M})\).

1.2. **Change of coefficients.** We now briefly review the change-of-coefficients spectral sequence, which will be used extensively throughout this paper. Let \(\mathbb{L}\) be a ring, \(\mathbb{M}\) be an \(\mathbb{L}\)-module, and \((C_*, \partial)\) be a complex of \(\mathbb{L}\)-modules. For technical reasons we will assume that each \(C_i\) is a free \(\mathbb{L}\)-module and there are only finitely many degrees \(i\) for which \(C_i\) is nonzero.

The relationship between the homology of \(C = (C_*, \partial)\) and the homology of \(C \otimes \mathbb{M} = (C_* \otimes \mathbb{L}, \partial \otimes 1)\) is given by the change-of-coefficients spectral sequence. Consider a free resolution of \(\mathbb{M}\):
\[
\ldots \xrightarrow{f_{n+1}} F_n \xrightarrow{f_n} \ldots \xrightarrow{f_1} F_0 \to \mathbb{M} \to 0.
\]
Then the double complex \((C_* \otimes \mathbb{L}, F_*, 1 \otimes f_*, \partial \otimes 1)\) gives rise to the spectral sequence
\[
E^2_{i,j} = \text{Tor}^1_i(H_j(C), \mathbb{M}) \Rightarrow H_{i+j}(C \otimes \mathbb{M}),
\]
where its differentials map \(d_k: E^k_{i,j} \to E^k_{i-k,j+k-1}\). The convergence of the spectral sequence must be interpreted in the sense that \(\bigoplus_{i+j=n} E^\infty_{i,j}\) is the graded module associated to the filtration on \(H_n(C \otimes \mathbb{M})\) induced by the double complex. For details, we refer the reader to [Mc].

**Example.** Suppose \(\mathbb{L}\) is a principal ideal domain (PID). Then any finitely generated \(\mathbb{L}\)-module \(\mathbb{M}\) is a direct sum whose summands are of the form \(\mathbb{L}/(p)\), \(p \in \mathbb{L}\). Hence each \(\mathbb{L}/(p)\) admits a free resolution
\[
0 \to \mathbb{L} \xrightarrow{p} \mathbb{L} \to \mathbb{L}/(p) \to 0,
\]
and \(\text{Tor}^1_i(H_j(C), \mathbb{M}) = 0\) for all \(i \geq 2\). Hence the \(E^2\)-term of the spectral sequence consists only of two adjacent nonzero columns, i.e., the \(0\)th and \(1\)st. Since \(d_k\) decreases \(i\) by \(k\), all differentials \(d_k\) with \(k \geq 2\) are trivial and \(E^2_{i,j} \cong E^\infty_{i,j}\). In this case the convergence of the spectral sequence means that there is an exact sequence
\[
0 \to H_n(C) \otimes \mathbb{M} \to H_n(C \otimes \mathbb{M}) \to \text{Tor}^1_l(H_{n-1}(C), \mathbb{M}) \to 0.
\]
Returning to the general discussion, observe that there is a natural map:

\[ \psi : H_i(C) \otimes \mathbb{M} \to H_i(C \otimes \mathbb{M}), \]

\[ [a] \otimes m \mapsto [a \otimes m]. \]

**Lemma 10.** Let \( \mathbb{M} \) be a ring, whose \( \mathbb{L} \)-module structure is induced by a ring homomorphism \( \mathbb{L} \to \mathbb{M} \). If \((M, \xi)\) is a contact manifold, then the contact invariant \( \hat{c}(\xi; \mathbb{M}) \in \widehat{HF}(-M; \mathbb{M})/\mathbb{M}^\times \) is mapped to the contact invariant \( \hat{c}(\xi; \mathbb{L}) \in \widehat{HF}(-M; \mathbb{L})/\mathbb{L}^\times \) by the natural map

\[ \widehat{HF}(-M; \mathbb{L}) \to \widehat{HF}(-M; \mathbb{M}), \]

\[ [a] \mapsto \psi([a] \otimes 1). \]

**Proof.** The contact invariant is represented by the same intersection point, regardless of the coefficient system, and is a cycle for any coefficient system because there are no holomorphic strips emanating from it.

Similar results hold for \( \widehat{HF}^+ \) and for sutured Floer homology \( SFH \).

2. Sutured Floer Homology and Twisted Coefficients

For details on sutured Floer homology and the contact invariant in sutured Floer homology, the reader is referred to \[Ju1\] \[Ju2\] \[HKM2\] \[HKM3\].

Let \((M, \Gamma)\) be a balanced sutured manifold. A contact structure \( \xi \) on \( M \) with convex boundary and dividing set \( \Gamma \) on \( \partial M \) will be denoted \((M, \Gamma, \xi)\). The contact invariant of \((M, \Gamma, \xi)\) will be written as \( c(M, \Gamma, \xi; \mathbb{Z}) \in SFH(-M, -\Gamma; \mathbb{Z}) \). Next let \((M', \Gamma') \subset (M, \Gamma)\) be an inclusion; in particular, \( M' \subset int(M) \). If a connected component \( N \) of \( M - int(M') \) has boundary which is not part of \( \partial M' \), then we say \( N \) is not isolated. Otherwise \( N \) is isolated.

The main result of \[HKM3\] is the following:

**Theorem 11 (Gluing Map).** Let \((M', \Gamma') \subset (M, \Gamma)\) be an inclusion, and let \( \xi \) be a contact structure on \( M - int(M') \) with convex boundary and dividing set \( \Gamma \) on \( \partial M \) and \( \Gamma' \) on \( \partial M' \). If \( M - int(M') \) has \( m \) isolated components, then \( \xi \) induces a natural map:

\[ \Phi_\xi : SFH(-M', -\Gamma'; \mathbb{Z}) \to SFH(-M, -\Gamma; \mathbb{Z}) \otimes_{\mathbb{Z}} V^\otimes m, \]

so that \( \Phi_\xi(c(M', \Gamma', \xi'; \mathbb{Z})) = c(M, \Gamma, \xi' \cup \xi; \mathbb{Z}) \otimes (x \otimes \cdots \otimes x) \), where \( x \) is the contact class of the standard tight contact structure on \( S^1 \times S^2 \) and \( \xi' \) is any contact structure on \( M' \) with boundary condition \( \Gamma' \). Here \( V \cong \widehat{HF}(S^1 \times S^2; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \) is a \( \mathbb{Z} \)-graded vector space where the two summands have grading which differ by one, say \( 0 \) and \( 1 \).

The sutured Floer homology of \((M, \Gamma)\) can be defined over the twisted coefficient system \( \mathbb{L}_r = \mathbb{Z}[H_2(M; \mathbb{Z})] \) — its definition is completely analogous to the closed case. If \( \mathbb{L} \to \mathbb{M} \) is a \( \mathbb{Z} \)-algebra homomorphism, then the invariant for a compact contact 3-manifold \((M, \Gamma, \xi)\) is denoted by \( c(M, \Gamma, \xi; \mathbb{M}) \in SFH(-M, -\Gamma; \mathbb{M})/\mathbb{M}^\times \). If \( \partial M = \emptyset \), then the invariant will be denoted by \( \hat{c}(M, \xi; \mathbb{M}) \in \widehat{HF}(-M; \mathbb{M})/\mathbb{M}^\times \). (Assuming \( \partial M = \emptyset \) and \( M \) is connected, we view \( M \) as the sutured manifold \((M - B^3, S^1)\), where \( B^3 \) is a small 3-ball and \( S^1 \) is the suture on \( \partial B^3 = S^2 \).)

The version of Theorem 11 with respect to twisted coefficients is the following:
Theorem 12 (Gluing Map, Twisted Coefficients Version). Let \((M', \Gamma') \subset (M, \Gamma)\) be an inclusion, and let \(\xi\) be a contact structure on \(M - \text{int}(M')\) with convex boundary and dividing set \(\Gamma\) on \(\partial M\) and \(\Gamma'\) on \(\partial M'\). Then there is a natural map

\[
\Phi_\xi : \text{SFH}(-M', -\Gamma'; \Z[H_2(M')]) \rightarrow \text{SFH}(-M, -\Gamma; \Z[H_2(M)]),
\]

so that \(\Phi_\xi(\xi(M', \Gamma', \xi'; \Z[H_2(M')]) = \xi(M, \Gamma, \xi' \cup \xi; \Z[H_2(M)]),\) where \(\xi'\) is a contact structure on \(M'\) with boundary condition \(\Gamma'\).

Here, if \((M', \Gamma') = (M'_1, \Gamma'_1) \cup (M'_2, \Gamma'_2),\) then \(\text{SFH}(-M', -\Gamma'; \Z[H_2(M')])\) is isomorphic to \(\text{SFH}(-M'_1, -\Gamma'_1; \Z[H_2(M'_1)]) \otimes \Z \text{SFH}(-M'_2, -\Gamma'_2; \Z[H_2(M'_2)]).\)

Sketch of Proof. We briefly explain the modifications needed for the proof in the twisted coefficients case.

Without loss of generality, consider the situation where we glue \((M'_1, \Gamma'_1)\) and \((M'_2, \Gamma'_2)\) along a common closed, oriented, connected surface \(T_{ij}\) (so that the sutures match) to obtain \((M, \Gamma)\). More precisely, suppose the following holds: \((M', \Gamma') = (M'_1, \Gamma'_1) \cup (M'_2, \Gamma'_2),\) \(M'_1, M'_2\) are connected, and each \(M'_i\) has more than one boundary component. Moreover, \(M' \subset \text{int}(M)\) so that \(M - \text{int}(M')\) consists of components \(T_i \times [0, 1], i = 0, 1, \ldots, k,\) where \(T_i\) are closed, oriented, connected surfaces and the contact structures on \(T_i \times [0, 1]\) are \([0, 1]\)-invariant and compatible with the dividing set on \(\partial M \cup \partial M'.\) The component \(T_0 \times [0, 1]\) has one boundary component \(T_0 \times \{0\} \subset \partial M'_1\) and the other boundary component \(T_0 \times \{1\} \subset \partial M'_2.\) Each \(T_i \times [0, 1], i = 1, \ldots, k,\) has one boundary component \(\subset \partial M\) and the other boundary component \(\subset \partial M'_j\) for some \(j.\)

Let \(\Sigma'\) be a compatible Heegaard surface for \((M', \Gamma')\), and \(\Sigma\) be an extension to a Heegaard surface for \((M, \Gamma)\), as given by [HKM3]. In particular, \(\Sigma\) is contact-compatible on \(T_i \times [0, 1], i = 0, \ldots, k,\) since there is one isolated component \(T_0 \times [0, 1]\), the sets of \(\alpha'-\text{curves}\) and \(\beta'-\text{curves}\) for \(\Sigma'\) cannot be extended to a complete set of \(\alpha-\text{curves}\) and \(\beta-\text{curves}\) for \(\Sigma\). In order to remedy this problem, we take a connected sum of \(M\) with \(S^1 \times S^2\). More precisely, on the contact-compatible portion, \(\Sigma\) is (locally) of the form \(\partial(S \times [0, 1]),\) where \(S\) is a surface with boundary (i.e., a page of a very partial open book) which may possibly be disconnected. Then we attach a 2-dimensional 1-handle to \(S\) so as to connect \(T_0 \times [0, 1]\) to some other \(T_i \times [0, 1]\) adjacent to \(M'_2.\) (On the level of \(\Sigma,\) we remove two disks and glue their boundaries together.) This gives rise to a Heegaard decomposition \((\Sigma'', \alpha'', \beta'')\) for \(M'' = M\#(S^1 \times S^2).\)

The inclusion \(M' \hookrightarrow M''\) gives rise to a group homomorphism \(H_2(M') \rightarrow H_2(M'')\) and also to an algebra homomorphism \(\Z[H_2(M')] \rightarrow \Z[H_2(M'')].\) Here

\[
\Z[H_2(M')] = \Z[H_2(M'_1) + H_2(M'_2)] \cong \Z[H_2(M'_1)] \otimes \Z \Z[H_2(M'_2)].
\]

Therefore, the group \(\text{SFH}(-M'', -\Gamma; \Z[H_2(M'')])\) has the structure of a \(\Z[H_2(M'')]-\text{module}\) and also of a \(\Z[H_2(M'_1)] \otimes \Z \Z[H_2(M'_2)]-\text{module}.\)

By the above two paragraphs, the inclusion of \((\Sigma', \alpha', \beta')\) into \((\Sigma'', \alpha'', \beta'')\) gives rise to the map

\[
\Phi_\xi : \text{SFH}(-M', -\Gamma'; \Z[H_2(M')]) \rightarrow \text{SFH}(-M'', -\Gamma; \Z[H_2(M'')]),
\]

obtained by tensoring with the contact class in the contact-compatible portion. The fact that \(\Phi_\xi\) is independent of the choices is proved in the same way as in Theorem 11 and will be omitted.

Finally, we claim that:

\[
\text{SFH}(-M'', -\Gamma; \Z[H_2(M'')]) \cong \text{SFH}(-M, -\Gamma; \Z[H_2(M)]),
\]


where the isomorphism is a $\mathbb{Z}[H_2(M'')]$-module homomorphism, given by the projection map

$$\pi_1: H_2(M'') \cong H_2(M) \oplus H_2(S^1 \times S^2) \to H_2(M)$$

onto the first factor. Here $e^{[\{pt\} \times S^2]}$ acts trivially. This is a simple generalization of the calculation of

$$\widehat{HF}(S^1 \times S^2; \mathbb{Z}[H_2(S^1 \times S^2)]) \cong \widehat{HF}(S^1 \times S^2; \mathbb{Z}[t, t^{-1}]),$$

where $t$ is the exponential of the homology class of $\{pt\} \times S^2$ (or, equivalently, $\{\{pt\} \times S^2\}$, viewed multiplicatively). The chain complex is generated by two generators $x, y$, and $\partial x = 0$, $\partial y = (t - 1)x$. Therefore,

$$\widehat{HF}(S^1 \times S^2; \mathbb{Z}[H_2(S^1 \times S^2)]) \cong \mathbb{Z}[t, t^{-1}]/(t - 1) \cong \mathbb{Z}.$$

The homology group is generated by $x$, and $t$ acts trivially (i.e., by the identity) on $x$.

**Example.** Consider $M_i' = T^2 \times [0, 1]$ and $M_j' = T^2 \times [1, 2]$. Let $t_i$ be the exponential of the generator of $H_2(M_i')$, $i = 1, 2$. If we glue to obtain $M = T^2 \times [0, 2]$, then the map

$$\Phi: SFH(-M_i', -\Gamma_i^1; \mathbb{Z}[H_2(M_i')]) \otimes_{\mathbb{Z}} SFH(-M_j', -\Gamma_i^2; \mathbb{Z}[H_2(M_j')]) \to SFH(-M_i', -\Gamma_i^1; \mathbb{Z}[H_2(M_i')]) \cong SFH(-M, -\Gamma; \mathbb{Z}[H_2(M)]),$$

is a $\mathbb{Z}[t_1, t_1^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[t_2, t_2^{-1}]$-module homomorphism. Since $e^{[\{pt\} \times S^2]}$ acts trivially, it follows that the multiplication by $t_1$ and $t_2$ are the same. Hence the above becomes a homomorphism in the category of $\mathbb{Z}[t, t^{-1}]$-modules:

$$SFH(-M_i', -\Gamma_i^1; \mathbb{Z}[t, t^{-1}]) \otimes_{\mathbb{Z}} SFH(-M_j', -\Gamma_i^2; \mathbb{Z}[t, t^{-1}]) \to SFH(-M, -\Gamma; \mathbb{Z}[t, t^{-1}]).$$

### 3. Proof of Theorem 2

In this section we prove Theorem 2 without precisely determining the Laurent polynomial $p(t) \in \mathbb{Z}[t, t^{-1}]$. The precise polynomial will be determined in Section 4.

Let $\Gamma$ be the following suture/dividing set on the boundary of $N = T^2 \times [0, 1]: \#\Gamma_{T_0} = \#\Gamma_{T_1} = 2$, $\Gamma_{T_0}$ and $\Gamma_{T_1}$ have no homotopically trivial components, slope($\Gamma_{T_0}$) = 0, and slope($\Gamma_{T_1}$) = $\infty$. Here $\#$ denotes the number of connected components, $T_i = T^2 \times \{i\}$, and the orientation of $T_i$ is inherited from that of $T^2$. (Hence $\partial N = T_1 \cup -T_0$.)

Next, let $[T] \in H_2(N)$ be the generator representing $T^2 \times \{pt\}$. Also let $L = \mathbb{Z}[H_2(N)] = \mathbb{Z}[t, t^{-1}]$, where $t = e^{[T]}$. Then we have the following:

**Lemma 13.** If $L \to M$ is a $\mathbb{Z}$-algebra homomorphism, then

(i) $SFH(N, \Gamma; M) \cong M \oplus M \oplus M \oplus M$, where each direct summand represents a distinct $Spin^c$ structure;

(ii) If $(N, \Gamma, \xi)$ is a basic slice, then $\xi(N, \Gamma, \xi; M)$ generates the appropriate $M$.

**Proof.** This follows from [HKM2, Section 5, Example 4], as well as [HKM2, Figure 15]. There are four intersection points in the Heegaard diagram given in that figure, and no holomorphic disks between any two. Hence each intersection point generates an $M$-direct summand. Moreover, one of the intersection points corresponds to the contact invariant for a basic slice. \qed
Lemma 14. If \((N, \Gamma, \xi)\) is a basic slice, then \((N, \Gamma, \xi')\), obtained from \(\xi\) via a full Lutz twist along a pre-Lagrangian torus parallel to \(T^2 \times \{pt\}\), satisfies:

\[
\xi(N, \Gamma; \xi'; \mathbb{L}) = p(t) \cdot \xi(N, \Gamma; \xi; \mathbb{L}),
\]

where \(p(t)\) is a nonzero element in \(\mathbb{L}\) which satisfies \(p(1) = 0\).

Remark. Such a pre-Lagrangian torus exists by [H1] Corollary 4.8.

Proof. Since \(\xi(N, \Gamma, \xi; \mathbb{L})\) generates \(\mathbb{L}\), and \(\xi\) and \(\xi'\) are homotopic (hence are in the same Spin\(^c\) structure), it follows that there is some element \(p(t)\) of \(\mathbb{L}\) so that Equation (10) holds.

Next we apply Lemma 10, i.e., the naturality of the contact invariant with respect to change of coefficients. Consider the algebra homomorphism \( \pi: SFH(-N, -\Gamma; \mathbb{L}) \rightarrow SFH(-N, -\Gamma; \mathbb{Z}) \) sends \(c(N, \Gamma, \xi; \mathbb{L})\) to \(c(N, \Gamma, \xi; \mathbb{Z})\), and \(c(N, \Gamma, \xi'; \mathbb{L})\) to \(c(N, \Gamma, \xi'; \mathbb{Z})\). Now, since \(c(N, \Gamma, \xi; \mathbb{Z}) \neq 0\) and we know from [GHV] that \(c(N, \Gamma, \xi'; \mathbb{Z}) = 0\), it follows that \(p(1) = 0\).

To prove that \(p(t)\) is nonzero, we show that \(\xi(N, \Gamma, \xi'; \mathbb{L}) \neq 0\). In fact, there is an inclusion of \((N, \Gamma)\) into \(T^3\) which sends \(\xi'\) to \(\xi_2\), the double cover of the standard Stein fillable contact structure on \(T^3\). We have a corresponding inclusion map

\[
\Phi: SFH(-N, -\Gamma; \mathbb{L}) \rightarrow SFH(-N, -\Gamma; \mathbb{Z}) \rightarrow H^3(-T^3; \mathbb{Z}[H^2(T^3)]).
\]

Taking a basis \(\{T, T', T''\}\) for \(H_2(T^3)\) where \(T\) comes from \(H_2(N)\), and setting \(T' = T'' = 0\), we have a projection of \(H_2(T^3)\) to \(\mathbb{Z}\), generated by \(T\). This gives rise to:

\[
\Phi: SFH(-N, -\Gamma; \mathbb{L}) \rightarrow H^3(-T^3; \mathbb{Z}),
\]

which sends \(\xi(N, \Gamma, \xi'; \mathbb{L})\) to \(\xi(T^3, \xi_2; \mathbb{L})\). Now, \(\xi(T^3, \xi_2; \mathbb{L})\) is nonzero by Proposition 9. This implies that \(\xi(N, \Gamma, \xi'; \mathbb{L})\) is also nonzero.

Now, \(p(1) = 0\) means that \(p(t)\) is divisible by \(t - 1\). This function \(p(t)\) is now the universal Laurent polynomial which is multiplied whenever \(2\pi\)-torsion is added.

Completion of proof of Theorem 2 without determining \(p(t)\). Let \(T\) be the pre-Lagrangian torus, along which the full Lutz twist will be applied. Then there exists a basic slice \((N_1 = T^2 \times [0, 1]; \xi|_{N_1}) \subset (M, \xi)\) so that \(T \subset N_1\) (and \(T\) is parallel to \(T^2 \times \{t\}\)). Decompose \(M\) into \(N_1\) and \(N_2 = M - N_1\). Moreover, \(\xi(M, \xi; \mathbb{Z}[H_2(M)])\) is obtained by taking the tensor product of \(\xi(N_i, \xi|_{N_i}; \mathbb{Z}[H_2(N_i)])\), \(i = 1, 2\). Now, applying a full Lutz twist is equivalent to changing \(\xi|_{N_1}\) to \(\xi'|_{N_1}\) as in Lemma 14. This has the effect of multiplying \(\xi(N_1, \xi|_{N_1}; \mathbb{L})\) by \(p(t)\). The theorem now follows from linearity.

4. Determination of \(p(t)\)

The goal of this section is to prove the following theorem:

Theorem 15. \(p(t) = t - 1\).

To pin down the polynomial we compute an example. A natural candidate is \(T^3\), whose tight contact structures \(\xi_n\) are all obtained by applying \((n - 1)\) full Lutz twists to the Stein fillable contact structure \(\xi_1\).
4.1. Calculation of $HF^+(T^3; \mathbb{M})$. Fix a primitive cohomology class $[\omega] \in H^2(T^3; \mathbb{Z})$ and define the $\mathbb{Z}[H_2(T^3; \mathbb{Z})]$-module $\mathbb{M}$ as $\mathbb{Z}[t, t^{-1}]$ endowed with the $H_2(T^3; \mathbb{Z})$-action $c \cdot t = t^{(\omega, c)}$.

In this subsection we calculate $HF^+(T^3; \mathbb{M})$ and determine the location of the contact invariant $\zeta(T^3, \xi; \mathbb{M})$. The computation in this subsection is similar to that of [OSz4, Proposition 8.4].

We will only be interested in the Spin$^c$ structure $s_0$ which satisfies $c_1(s_0) = 0$. The Heegaard Floer homology classes for the other Spin$^c$ structures vanish by the adjunction inequality.

**Lemma 16.** $HF^\infty_j(T^3; \mathbb{M}) \cong \mathbb{Z}^2$ for all half-integers $j$.

**Proof.** In [OSz2, Theorem 10.12], it was shown that

$$HF^\infty(T^3; \mathbb{Z}[H_2(T^3; \mathbb{Z})]) \cong \mathbb{Z}[U, U^{-1}],$$

where $U$ has degree $-2$. Also, from the computation of $HF^+(T^3; \mathbb{Z}[H_2(T^3; \mathbb{Z})])$ in [OSz4, Section 8.4], one easily sees that the nonzero elements sit in degrees $j \equiv \frac{1}{2} \mod 2$. In order to prove our lemma, we apply the change-of-coefficients spectral sequence

$$\text{Tor}_{j\mathbb{Z}[H_2(T^3; \mathbb{Z})]}^i(HF^\infty_j(T^3; \mathbb{Z}[H_2(T^3; \mathbb{Z})]), \mathbb{M}) \Rightarrow HF^\infty_{i+j}(T^3; \mathbb{M}).$$

Choose coordinates on $H_2(T^3; \mathbb{Z})$ so that the group algebra $\mathbb{Z}[H_2(T^3; \mathbb{Z})]$ is identified with $\mathbb{L} = \mathbb{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}, t_3, t_3^{-1}]$ and $\mathbb{M}$ is identified with $\mathbb{Z}[t_3, t_3^{-1}]$ on which $t_1$ and $t_2$ act trivially. In order to compute the $E^2$-term of the spectral sequence, we take a free resolution of $\mathbb{Z}[t_3, t_3^{-1}]$:

$$0 \longrightarrow \mathbb{L} \xrightarrow{f} \mathbb{L} \oplus \mathbb{L} \xrightarrow{g} \mathbb{L} \longrightarrow \mathbb{Z}[t_3, t_3^{-1}] \longrightarrow 0,$$

where

$$f(1) = (t_2 - 1, t_1 - 1), \quad g(1, 0) = t_1 - 1, \quad \text{and} \quad g(0, 1) = t_2 - 1.$$

Observe that $\text{Im}(f) = \text{Ker}(g)$ follows from the fact that $\mathbb{L}$ is a unique factorization domain.

If we truncate the last term in Equation (11) and tensor with $\mathbb{Z}$ over $\mathbb{L}$, then we obtain the complex

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where all maps are trivial. Thus

$$\text{Tor}_{j\mathbb{Z}[H_2(T^3; \mathbb{Z})]}^i(HF^\infty_j(T^3; \mathbb{Z}[H_2(T^3; \mathbb{Z})]), \mathbb{M}) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}^2, & \text{if } i = 1; \\ \mathbb{Z}, & \text{if } i = 2 \end{cases}$$

if $j \equiv \frac{1}{2} \mod 2$, and 0 otherwise.

Therefore the $E^2$-term of the spectral sequence has the form

$$
\begin{array}{ccccccc}
\vdots & \vdots & \vdots \\
\mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{array}$$
and all the higher differentials are trivial for degree reasons, so $E^2 \cong E^\infty$. For $i$ odd and $j \equiv \frac{1}{2} \mod 2$ we have $HF^-_{\infty,j}(T^3; \mathbb{M}) \cong \mathbb{Z}^2$, and for $i$ even and $j \equiv \frac{1}{2} \mod 2$ the $E^\infty$-term gives an exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow HF^-_{\infty,j}(T^3; \mathbb{M}) \rightarrow \mathbb{Z} \rightarrow 0,$$

so $HF^-_{\infty,j}(T^3; \mathbb{M}) \cong \mathbb{Z}^2$ in this case also, since $\mathbb{Z}$ is a free Abelian group. \hfill \Box

Let $M\{a, b, c\}$ denote the 3-manifold obtained by surgery on the Borromean rings with surgery coefficients $a$, $b$, and $c$. The 3-torus $T^3$ is homeomorphic to $M\{0, 0, 0\}$.

**Lemma 17.** $HF^+_j(T^3; \mathbb{M}) \cong HF^-_{\infty,j}(T^3; \mathbb{M})$ for all $j \geq \frac{1}{2}$, and $HF^+_j(T^3; \mathbb{M}) = 0$ for all $j \leq -\frac{3}{2}$.

**Proof.** From [OSz2, Theorem 9.11], we have the exact sequence:

$$\ldots \rightarrow \hat{HF}(M\{0, 0, \infty\}; \mathbb{Z})[t, t^{-1}] \rightarrow \hat{HF}(M\{0, 0, 0\}; \mathbb{M}) \rightarrow$$

$$\hat{HF}(M\{0, 0, 1\}; \mathbb{Z})[t, t^{-1}] \rightarrow \ldots,$$

where the two central maps decrease the degree by $\frac{1}{2}$ (see [OSz4, Lemma 3.1]). Now, according to the proof of [OSz4, Proposition 8.4],

$$\hat{HF}(M\{0, 0, 1\}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^2, & \text{if } j = -1, 0; \\ \mathbb{Z}, & \text{otherwise.} \end{cases}$$

Also $M\{0, 0, \infty\} = (S^1 \times S^2)\#(S^1 \times S^2)$, so

$$\hat{HF}_j(M\{0, 0, \infty\}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } j = 1; \\ \mathbb{Z}^2, & \text{if } j = 0; \\ \mathbb{Z}, & \text{if } j = -1; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, by Equation (12) $\hat{HF}(T^3; \mathbb{M})$ is supported in degrees $\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$.

We now claim that $\hat{HF}_{-\frac{3}{2}}(T^3; \mathbb{M}) = 0$. Suppose on the contrary that $\hat{HF}_{-\frac{3}{2}}(T^3; \mathbb{M}) \neq 0$. If we forget the action of $t$ and view $\hat{CF}(T^3; \mathbb{Z}[t, t^{-1}])$ as a $\mathbb{Z}$-module, then it is easy to see that there is some prime $p$ for which $\hat{HF}_{-\frac{3}{2}}(T^3; \mathbb{F}_p[t, t^{-1}]) \neq 0$, where $\mathbb{F}_p$ is the field of $p$ elements. The advantage of $\mathbb{F}_p[t, t^{-1}]$ is that it is a PID. Now, if $K$ is an $\mathbb{F}_p[t, t^{-1}]$-module, then we can apply Equation (5) to obtain

$$\hat{HF}_{-\frac{3}{2}}(T^3; K) \cong \hat{HF}_{-\frac{3}{2}}(T^3; \mathbb{F}_p[t, t^{-1}]) \otimes_{\mathbb{F}_p[t, t^{-1}]} K,$$

in view of the fact that $\hat{HF}_{-\frac{3}{2}}(T^3; \mathbb{F}_p[t, t^{-1}]) = 0$. If $K$ is a field, then the existence of an orientation-reversing diffeomorphism of $T^3$ forces $\hat{HF}_{-\frac{3}{2}}(T^3; K) \cong \hat{HF}_{-\frac{3}{2}}(T^3; \mathbb{K}) = 0$ (see [OSz6, Proposition 7.11]). Since $\mathbb{F}_p[t, t^{-1}]$ is a PID, we can decompose

$$\hat{HF}_{-\frac{3}{2}}(T^3; \mathbb{F}_p[t, t^{-1}]) \cong \mathbb{F}_p[t, t^{-1}]^{n_0} \oplus \mathbb{F}_p[t, t^{-1}]/(f_1^{n_1}) \oplus \ldots \oplus \mathbb{F}_p[t, t^{-1}]/(f_k^{n_k}),$$

where the $f_i$ are irreducible. In particular, if we take $K = \mathbb{F}_p[t, t^{-1}]/(f_1)$, then we have

$$\hat{HF}_{-\frac{3}{2}}(T^3; \mathbb{F}_p[t, t^{-1}]) \otimes_{\mathbb{F}_p[t, t^{-1}]} K \neq 0,$$

which is a contradiction. This proves the claim.
Next, from the exact sequence
\[ \ldots \rightarrow \hat{HF}_{k+2}(T^3; M) \rightarrow \hat{HF}_{k+2}^+(T^3; M) \rightarrow U \rightarrow \hat{HF}_{k+1}^+(T^3; M) \rightarrow \ldots, \]
we see that \( U \) is an isomorphism if \( k \geq \frac{1}{2} \) or if \( k \leq -\frac{7}{2} \). Since \( HF^+ \) is isomorphic to \( HF^\infty \) in sufficiently high degrees and zero in sufficiently low degrees, the lemma follows.

**Lemma 18.** \( \hat{HF}^+_{\frac{3}{2}}(T^3; M) \cong \mathbb{Z}^2 \oplus \mathbb{Z}[t, t^{-1}] \).

**Proof.** First recall that \( HF^+(\#^2(S^1 \times S^2); \mathbb{Z}) \cong \Lambda^* H^1(\#^2(S^1 \times S^2); \mathbb{Z}) \otimes \mathbb{Z}[U^{-1}] \), where \( U \) has degree \(-2\). This means that
\[ HF^+_j(\#^2(S^1 \times S^2); \mathbb{Z}) = \begin{cases} \mathbb{Z}^2, & \text{if } j \geq 0; \\ \mathbb{Z}, & \text{if } j = -1; \\ 0, & \text{if } j < -1. \end{cases} \]

Next, we calculate that
\[ HF^+_j(M\{0, 0, 1\}; \mathbb{Z}) = \begin{cases} \mathbb{Z}^2, & \text{if } j \geq -1; \\ 0, & \text{if } j < -1. \end{cases} \]

This follows immediately from
\[ \hat{HF}_j(M\{0, 0, 1\}; \mathbb{Z}) = \begin{cases} \mathbb{Z}^2, & \text{if } j = 0, -1; \\ 0, & \text{otherwise.} \end{cases} \]

and \( HF^\infty_j(M\{0, 0, 1\}; \mathbb{Z}) \cong \mathbb{Z}^2 \) for all \( j \) ([OSz2, Theorem 10.1]).

Now we apply the surgery exact sequence for the triple \( M\{0, 0, \infty\}, M\{0, 0, 0\}, M\{0, 0, 1\} \) with twisted coefficients:
\[ \ldots \rightarrow (\mathbb{Z}[t, t^{-1}])^2 \overset{F_1^+}{\rightarrow} \hat{HF}^+_{\frac{3}{2}}(T^3; M) \overset{F_2^+}{\rightarrow} (\mathbb{Z}[t, t^{-1}])^2 \overset{F_3^+}{\rightarrow} \mathbb{Z}[t, t^{-1}] \overset{F_4^+}{\rightarrow} 0. \]

The image of \( F_1^+ \) is isomorphic to \( \mathbb{Z}^2 \) by \( U \)-equivariance. Indeed, for \( j \geq \frac{1}{2} \), the long exact sequence splits as:
\[ 0 \rightarrow (\mathbb{Z}[t, t^{-1}])^2 \rightarrow (\mathbb{Z}[t, t^{-1}])^2 \rightarrow \hat{HF}^+_{\frac{3}{2}}(T^3; M) \cong \mathbb{Z}^2 \rightarrow 0, \]
\[ 0 \rightarrow (\mathbb{Z}[t, t^{-1}])^2 \rightarrow (\mathbb{Z}[t, t^{-1}])^2 \rightarrow \hat{HF}^+_{\frac{3}{2}}(T^3; M) \rightarrow \ldots, \]
and we can apply \( U \) to the top sequence when \( j = \frac{3}{2} \). We now claim that \( \text{Im}(F_2^+) = \text{Ker}(F_3^+) \) is isomorphic to \( \mathbb{Z}[t, t^{-1}] \). Indeed, since \( F_3^+ \) is surjective, \( F_2^+ \) must map \((1, 0) \mapsto f(t), (0, 1) \mapsto g(t)\), where \( f, g \) are relatively prime and hence have no common factors. By unique factorization on \( \mathbb{Z}[t, t^{-1}] \), \( \text{Ker}(F_3^+) \) must be generated by \((f(t), f(t))\) and is free since there are no zero divisors. The sequence splits since \( \text{Im}(F_2^+) \) is free.

Putting together Lemmas [16][17] and [18] we obtain:
Proposition 19.

\[ HF_j^+(T^3; \mathbb{M}) \cong \begin{cases} 
\mathbb{Z}^2; & \text{if } j \geq \frac{1}{2}; \\
\mathbb{Z}^2 \oplus \mathbb{Z}[t, t^{-1}]; & \text{if } j = -\frac{1}{2}; \\
0; & \text{if } j \leq -\frac{1}{2}.
\end{cases} \]

Next we identify the location of the contact invariant for \( \xi_n \) on \( T^3 \).

Lemma 20. The contact invariant of \((T^3, \xi_n)\) has a non-zero component in the summand \( \mathbb{Z}[t, t^{-1}] \) of \( HF_+^+(T^3; \mathbb{M}) \).

Proof. We claim that \( HF_{red}(T^3; \mathbb{M}) \) can be identified with the summand of \( HF_+^+(T^3; \mathbb{M}) \) isomorphic to \( \mathbb{Z}[t, t^{-1}] \). For this we use the exact triangle

\[ \cdots \rightarrow HF_j^\infty(T^3; \mathbb{M}) \xrightarrow{a_j} HF_j^+(T^3; \mathbb{M}) \xrightarrow{b_j} HF_j^-(T^3; \mathbb{M}) \rightarrow \cdots \]

The left two terms have been computed, and we would like to compute the image of \( b_j \). For large \( j \), \( HF_j^\infty(T^3; \mathbb{M}) \cong HF_j^+(T^3; \mathbb{M}) \cong \mathbb{Z}^2 \). By Lemma 17, \( U : HF_j^+(T^3; \mathbb{M}) \rightarrow HF_j^+(T^3; \mathbb{M}) \) is an isomorphism for \( j \geq \frac{1}{2} \) and is injective for \( j = -\frac{1}{2} \). By \( U \)-invariance, it follows that \( HF_{\geq \frac{1}{2}}^+(T^3; \mathbb{M}) \) maps isomorphically onto the \( \mathbb{Z}^2 \)-summand of \( HF_{-\frac{1}{2}}^+(T^3; \mathbb{M}) \). Hence \( HF_{red}(T^3; \mathbb{M}) \cong \mathbb{Z}[t, t^{-1}] \).

Now, \( \xi_{red}(\xi_n; \mathbb{M}) \neq 0 \) by [OSZ3, Theorem 4.1], since \( \xi_n \) has a weak filling with \( b_+ > 0 \), whose symplectic form restricts to \( [\omega] \) on \( T^3 \). \( \square \)

4.2. Comparison of \( E(2) \) and \( E(3) \). We start with a few preparatory lemmas.

Lemma 21. There is a symplectic cobordism \( W_0 \) from \((T^3, \xi_2)\) to \((T^3, \xi_3)\) which is diffeomorphic to the elliptic surface \( E(1) \) with the tubular neighborhoods of two regular fibers removed.

By the above phrase “from \((T^3, \xi_2)\) to \((T^3, \xi_3)\)” we mean that \((T^3, \xi_2)\) is the convex boundary and \((T^3, \xi_3)\) is the concave boundary.

Proof. Choose an integral basis \( \langle [a], [b] \rangle \) of \( H_1(T^2; \mathbb{Z}) \), where \( a, b \) are simple closed curves, and let \( \tau_a, \tau_b \) be the positive Dehn twists around \( a \) and \( b \). Identify \( T^3 \cong \mathbb{R}^3/\mathbb{Z}^3 = \mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z}) \). Let \( L \) be the link which is the union of \( a \times \{ \frac{2i}{6} \} \) for \( i = 0, \ldots, 5 \), and \( b \times \{ \frac{i}{6} + \frac{1}{12} \} \) for \( i = 0, \ldots, 5 \). We will use the framing induced from \( T^2 \times \{ t \} \). It is well-known that \((\tau_a \tau_b)^6 = id \). This means that a \((-1)\)-surgery on \( T^3 \) along all the components of the link \( L \) yields \( T^3 \). The 4-dimensional cobordism associated to this surgery admits a Lefschetz fibration over the annulus with generic fiber \( F \) diffeomorphic to \( T^2 \) and twelve singular fibers corresponding to the twelve Dehn twists in \((\tau_a \tau_b)^6 \) — this is precisely \( E(1) \) minus the neighborhoods of two generic fibers.

If we consider the contact structure \( \xi_3 \) on \( T^3 \), we can make the link \( L \) Legendrian with twisting number 0 (i.e., make each component a Legendrian divide). Hence, by applying Legendrian \((-1)\)-surgery along all the components of \( L \), we can endow \( W_0 \) with the structure of a symplectic cobordism. By a direct computation (see [LS1]) we can show that the Legendrian surgery along \( L \) decreases the Giroux torsion by 1, so we obtain \( \xi_2 \) as the convex boundary of the cobordism. \( \square \)
Let us write the Ozsváth-Szabó 4-manifold invariant of a closed, oriented 4-manifold $X$ as:

$$\Phi_X = \sum_{s \in \text{Spin}^c(X)} \Phi_{X, s} t^{s-s_0},$$

where $s_0$ is a reference Spin$^c$-structure. Also given $[\omega] \in H^2(X; \mathbb{Z})$, we can form:

$$\Phi_{X; M_\omega} = \sum_{s \in \text{Spin}^c(X)} \Phi_{X, s} t^{(s-s_0)}[\omega,(s-s_0),\omega].$$

Here we take $s_0$ to be the canonical Spin$^c$-structure if $\omega$ is symplectic. We then have the following:

**Lemma 22.** Consider the elliptic surfaces $E(n)$, $n \geq 2$. If $F$ is a regular fiber, then

$$\Phi_{E(n)} = (T - 1)^{n-2},$$

where $T = t^{PD([F])}$. If $\omega$ is a symplectic form on $E(n)$ arising from the Lefschetz fibration, so that $[\omega] \in H^2(E(n); \mathbb{Z})$ and $[\omega]_F \in H^2(F; \mathbb{Z})$ is primitive, then

$$\Phi_{E(n); M_\omega} = (t - 1)^{n-2}.$$

This was proved for $E(2)$ by Ozsváth-Szabó [OSz5] and for $E(n)$ in general by Jabuka-Mark [JM, Section 1.4.1].

**Lemma 23.** Let $W_1$ be $E(2)$ with a 4-ball and a neighborhood $N(F)$ of a regular fiber $F$ removed. Then $b_2^+(W_1) > 1$.

**Proof.** We will produce two pairs of closed surfaces $A_i$ and $B_i$ in $E(2)$ for $i = 1, 2$ which are disjoint from some regular fiber $F$ and have intersection form

$$\begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}$$

for some integers $m, n$. Since both matrices have determinant $-1$, it follows that there is some linear combination of $A_1$ and $B_1$ and some combination of $A_2$ and $B_2$ with positive self-intersection.

Recall that $E(2)$ is obtained by gluing two copies of $E(1) - N(F)$. Take a basis of $H_1(\partial N(F)) = H_1(F \times S^1)$ of the form $\alpha, \beta, \gamma$, where $\alpha, \beta$ form a basis for $H_1(F)$ and $\gamma$ is null-homologous in $H_1(N(F))$. For $i = 1, 2$, choose $\alpha_i = \alpha \times \{\theta_i\}$ and $\beta_i = \beta \times \{\theta_i\}$, where $\theta_1 \neq \theta_2 \in S^1$.

By [GS, Lemma 3.1.10], $\alpha_1$ and $\beta_2$ bound disjoint embedded disks in $E(1) - N(F)$; if we think of the Lefschetz fibration picture, these disks are “thimbles” which correspond to vanishing cycles. Since we have disks on both copies of $E(1) - N(F)$, they glue to give two disjoint spheres which we call $B_1$ and $B_2$. On the other hand, let $A_1 = \beta \times \gamma$ and $A_2 = \alpha \times \gamma$. Then we have $A_1 \cdot B_1 = \pm 1$. Moreover, if $i \neq j$, then $A_i \cdot B_j = 0$. It is also clear that $A_i \cdot A_j = 0$ because $A_i$ can be pushed off of $N(F)$. Finally, the $A_i$ and $B_i$ can be made disjoint from some copy of $F$. 

We are now ready to prove Theorem [15].

**Proof of Theorem [15]** We compare the 4-dimensional Ozsváth–Szabó invariants of the elliptic surfaces $E(2)$ and $E(3)$.

We regard $E(2)$ as a cobordism $W = E(2) - B^4 - B^4$ from $S^3$ to $S^3$, and decompose $W$ into $W_1 = E(2) - B^4 - N(F)$ and $W_2 = N(F) - B^4$ along a 3-torus $T^3$. Let $\omega$ be a symplectic form.

\footnote{Note that we are using $s - s_0$ instead of $c_1(s)$. This means the statement of Lemma 22 looks slightly different from the results of [OSz5] and [JM].}
on \(E(2)\) arising from the Lefschetz fibration, so that \([\omega] \in H^2(E(2); \mathbb{Z})\) and \([\omega]|_{T^3} \in H^2(T^3; \mathbb{Z})\) is primitive.

Let \(\Theta_\ominus\) be the top degree generator of \(HF^{-}(S^3; \mathcal{M}_\omega)\). If \(s\) is a Spin\(^c\) structure on \(E(2)\), then, by the composition law (Theorem \(7\)), we have:

\[
\Phi_{E(2); \mathcal{M}_\omega} = \sum_{\eta \in H^4(T^3; \mathbb{Z})} \Phi_{E(2), s+\delta \eta} t^{\langle \omega, (s-s_0+\delta \eta), [E(2)] \rangle} = F^{+}_{W_2; s|W_2; \mathcal{M}_\omega} \circ F_{W_1; s|W_1; \mathcal{M}_\omega}^{mix}(\Theta_\ominus).
\]

Observe that \(F_{W_1; s|W_1; \mathcal{M}_\omega}^{mix}\) is defined, since \(b_2^+(W_1) > 1\) by Lemma \(23\). Here \(\delta: H^1(T^3) \to H^2(E(2))\) is the connecting homomorphism in the Mayer-Vietoris sequence for \(E(2)\) involving \(W_1\) and \(W_2\). The map \(\delta\) is equivalent, via Poincaré Duality, to the inclusion map \(i: H_2(T^3) \to H_2(E(2))\). This means that \(PD([F])\) is in the image of \(\delta\). Now, if we sum Equation \(13\) over a suitable collection of Spin\(^c\) structures, we obtain:

\[
\Phi_{E(2); \mathcal{M}_\omega} = F^{+}_{W_2; s|W_2; \mathcal{M}_\omega} \circ F_{W_1; s|W_1; \mathcal{M}_\omega}^{mix}(\Theta_\ominus) = 1,
\]

by Lemma \(22\).

Now consider the map

\[
F^{+}_{W_2; s|W_2; \mathcal{M}_\omega}: HF^+(T^3; \mathcal{M}_\omega) \to HF^+(S^3; \mathcal{M}_\omega) = HF^+(S^3; \mathbb{Z}) \otimes Z \mathcal{M}_\omega.
\]

Observe that all the summands of \(HF^+(S^3; \mathbb{Z}) \otimes Z \mathcal{M}_\omega\) are \([R]\), whereas only one summand of \(HF^+(T^3; \mathcal{M}_\omega)\) is \([R]\) by Lemma \(19\) and the rest have torsion. Since \(F^{+}_{W_2; s|W_2; \mathcal{M}_\omega}\) is \(t^2\)-invariant, the only summand of \(HF^+(T^3; \mathcal{M}_\omega)\) which maps nontrivially to \(HF^+(S^3; \mathcal{M}_\omega)\) is \([R]\). This means that \(F_{W_1; s|W_1; \mathcal{M}_\omega}^{mix}\) maps \(\Theta_\ominus\) to an element with nonzero projection to the summand \([R]\) of \(HF^+(S^3; \mathcal{M}_\omega)\), which, in turn, is mapped to the bottom generator \(\Theta_\ominus\) of \([R]\) by \(HF^+(T^3; \mathcal{M}_\omega)\). Recall that \(\zeta(T^3, \xi_\ominus; \mathcal{M}_\omega)\) also has a nontrivial projection to \([R]\) by \(HF^+(S^3; \mathcal{M}_\omega)\).

Next we regard \(E(3)\) as a cobordism \(W' = E(3) - B^4 - B^4\), which is decomposed into \(W_1\), \(W_0\), and \(W_2\). Orient the 3-torus \(M_1 = \partial W_1 \cap \partial W_0\) as \(\partial W_1\) and the 3-torus \(M_2 = \partial W_0 \cap \partial W_2\) as \(\partial W_0\). We view the cobordism \(W_0\) turned “upside-down” so that the induced map

\[
F^{+}_{W_0; s|W_0; \mathcal{M}_\omega}: HF^+(M_1; \mathcal{M}_\omega) \to HF^+(M_2; \mathcal{M}_\omega)
\]

sends

\[
\zeta(-M_1, \xi_2; \mathcal{M}_\omega) \mapsto \zeta(-M_2, \xi_3; \mathcal{M}_\omega).
\]

Restricted to the summand \([R]\), this means that

\[
F^{+}_{W_0; s|W_0; \mathcal{M}_\omega}: [R] \to [R]
\]

is multiplication by \(p(t)\). As in the \(E(2)\) case, we compute:

\[
\Phi_{E(3); \mathcal{M}_\omega} = F^{+}_{W_2; s|W_2; \mathcal{M}_\omega} \circ F^{+}_{W_0; s|W_0; \mathcal{M}_\omega} \circ F_{W_1; s|W_1; \mathcal{M}_\omega}^{mix}(\Theta_\ominus) = t - 1.
\]

Since the projection of \(F_{W_1; s|W_1; \mathcal{M}_\omega}^{mix}(\Theta_\ominus)\) to \([R]\) is nonzero, we have that \(p(t) = t - 1\). \(\Box\)

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