Analysis of viscoelastic flow with a generalized memory and its exponential convergence to steady state

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Abstract. We investigate a viscoelastic flow model with a generalized memory, in which a weak singular component is introduced in the exponential convolution kernel of classical viscoelastic flow equations that remains untreated in the literature. We prove the well-posedness and regularity of the solutions, based on which we prove the exponential convergence of the solutions to the steady state. The proposed model serves as an extension of classical viscoelastic flow equations by adding a dimension characterized by the power of the weak singular kernel, and the derived results provide theoretical supports for designing numerical methods for both the considered equation and its steady state.

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1. Introduction

Viscoelastic flows arise in widely applications and have attracted extensive attentions. A typical governing equation is derived from the combination of the Oldroyd’s model, the continuity equation and the momentum conservation equation as follows

$$
\dot{u} - \mu \Delta u + (u \cdot \nabla)u - \rho \int_{0}^{t} e^{-\delta(t-s)} \Delta u ds + \nabla p = f.
$$

Here $u = u(x, t) = (u_1(x, t), u_2(x, t))$ is the velocity vector, $p = p(x, t)$ is the pressure of the fluid, $f = (f_1(x, t), f_2(x, t))$ is the prescribed external force, $x \in \Omega$ for some two-dimensional convex polygonal domain $\Omega$ with the boundary $\partial \Omega$, $\mu$ is the solvent viscosity, $\delta = 1/\lambda_1$ where $\lambda_1$ refers to the relaxation time and $\rho = (\mu/\lambda_1)(\lambda_1/\lambda_2 - 1) > 0$ where $\lambda_2$ stands for the retardation time satisfying the restriction $0 \leq \lambda_2 \leq \lambda_1$.

Extensive mathematical and numerical analysis for model (1) and related problems have been carried out, see, e.g., [3,5–10,21,22,24,25,29]. In particular, the convergence of (1) to a steady state in an exponential rate with respect to time was rigorously proved in [14,19,30,36]. A main application of this result lies in approximating the steady state of (1) with small viscosity $\mu$ via computing (1) [12]. To be specific, solving the steady state of (1) requires certain iterative method due to its strong nonlinear term $(u \cdot \nabla)u$. However, it was shown in [12,13] that the uniqueness condition of the Stokes iterative, which is
an efficient iterative method with a constant coefficient matrix compared with other iterative methods, may be lost for small viscosity \( \mu \). Thus, one may alternatively solve (1) to approximate its steady state, which requires the convergence estimates of (1) to its steady state.

Compared with classical Naiver–Stokes equations, which corresponds to \( \rho = 0 \) in (1), a mainly overcame difficulty in the investigation of model (1) in the literature stems from the newly encountered convolution term, which introduces nonlocal features in time by an exponential kernel to model the memory effects of the fluid and thus the dynamics. In the past decades, increasingly experimental evidences show that the power function kernels in, e.g., the fractional calculus, adequately describe the creep and relaxation of viscoelastic materials [4,18,26,28] and thus provide more accurate modeling for the memory effects in the dynamics of viscoelastic fluids [2,16,31,32,34,38–40]. Nevertheless, the corresponding viscoelastic flow equation like (1) involving the power function memory kernel is meagerly studied in the literature.

Motivated by these discussions, we consider the following viscoelastic flow model with a generalized memory kernel combining the exponential component as model (1) and the newly added power-law decay factor \( t^{-\beta} \) for \( 0 \leq \beta < 1 \)

\[
\begin{align*}
\{ u_t - \mu \Delta u + (u \cdot \nabla)u - \rho \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \Delta u ds + \nabla p &= f, \\
(x,t) &\in \Omega \times [0, \infty); \quad \text{div} \ u = 0, (x,t) \in \Omega \times [0, \infty); \\
u(x,0) &= u_0(x), \ x \in \Omega; \quad u(x,t) |_{\partial \Omega} = 0, \ t \in [0, \infty). 
\end{align*}
\]

(2)

The combined memory kernel \( t^{-\beta} e^{-\delta t} \) appears in many applications, see, e.g., [37, Equation 1.4] and [23, Equation 1.22]. Compared with (1), the additional factor \( t^{-\beta} \) introduces initial singularities and the convolution kernel is no longer continuously differentiable as that in (1). For this reason, the analysis needs careful treatments near the initial time. Furthermore, the indefinite integral of this combined kernel could not be evaluated into a closed form as its components \( e^{\delta t} \) and \( t^{-\beta} \), which significantly complicates the analysis.

In this paper, we address the aforementioned concerns to analyze model (2) and prove its exponential convergence to the following steady state

\[
\begin{align*}
\{- \left( \mu + \frac{\rho \Gamma(1-\beta)}{\delta^{1-\beta}} \right) \Delta \bar{u} + (\bar{u} \cdot \nabla)\bar{u} + \nabla \bar{p} &= \bar{f}, \ x \in \Omega, \\
\text{div} \ \bar{u} &= 0, \ x \in \Omega; \quad \bar{u}(x) |_{\partial \Omega} = 0 
\end{align*}
\]

(3)

which provide theoretical supports for designing numerical methods for both the considered Eq. (2) and its steady state (3). Note that when \( \beta \) tends to 0, this equation approaches to the steady state of the traditional viscoelastic flow equation (1) proposed in, e.g., [14, Equation 1.3]. That is, the current work mathematically generalizes the existing models and results by introducing an extra dimension characterized by the parameter \( \beta \).

The rest of the paper is organized as follows: In Sect. 2, we introduce notations and preliminary results to be used subsequently. In Sect. 3, we prove well-posedness and regularity results for model (2), based on which we prove exponential convergence of viscoelastic flow Eq. (2) to its steady state (3) in Sects. 4–5. We address concluding remarks in the last section.
2. Preliminaries

2.1. Spaces and notations

We follow [1, 11, 15, 35] to introduce useful spaces and notations. Define the following Hilbert spaces

\[ X = H^1_0(\Omega)^2, \quad Y = L^2(\Omega)^2, \quad M = L^2_0(\Omega) = \left\{ q \in L^2(\Omega) ; \int\limits_\Omega qdx = 0 \right\}, \]

where \( L^2(\Omega)^d \) \((d = 1, 2)\) is equipped with the usual \( L^2\)-scalar product \((\cdot, \cdot)\) and \( L^2\)-norm \( \| \cdot \|_0 \), and \( H^1_0(\Omega) \) and \( X \) are equipped with the following scalar product and equivalent norm

\[ (\nabla \cdot, \nabla \cdot), \quad | |_1 = \| \nabla \cdot \|. \]

Here, \( \| \cdot \|_i \) and \( | \cdot |_i \) denote the usual norm and semi norm of the Sobolev space \( H^i(\Omega)^d \), for \( i = 0, 1, 2 \). Define the closed subset \( V \) of \( X \) and the closed subset \( H \) of \( Y \) as follows

\[ V = \{ v \in X; \text{div} \ v = 0 \}, \quad H = \{ v \in Y; \text{div} \ v = 0, \ v \cdot n |_{\partial \Omega} = 0 \}. \]

Furthermore, let \( P \) be the \( L^2\)-orthogonal projection of \( Y \) onto \( H \) and \( A \) be the Stokes operator defined by \( A = -P \Delta \) with \( D(A) := H^2(\Omega)^2 \cap V \), which satisfies [1, 12, 15, 20]

\[ \| v \|^2_2 \leq c \| Av \|^2_0, \quad v \in D(A), \quad (4) \]

\[ \gamma_0 \| v \|^2_0 \leq | v |^2_1, \quad v \in X, \quad \gamma_0 | v |^2_1 \leq \| Av \|^2_0, \quad v \in D(A), \quad (5) \]

where \( \gamma_0, c > 0 \) are generic constants depending on \( \Omega \). In particular, \( A^{1/2} \) satisfies \( \| A^{1/2} v \|_0 = | v |_1 \) for \( v \in V \).

With the above definitions, the bilinear forms \( a(\cdot, \cdot) \) and \( d(\cdot, \cdot) \) on \( X \times X \) and \( X \times M \) are, respectively, defined by

\[ a(u, v) = (\nabla u, \nabla v), \quad d(v, q) = -(v, \nabla q) = (q, \text{div} \ v), \quad u, v \in X, \quad q \in M. \]

The bilinear form \( a(\cdot, \cdot) \) is continuous and coercive on \( X \times X \), and \( d(\cdot, \cdot) \) is continuous on \( X \times M \) and satisfies the well-known inf–sup condition: there exist a positive constant \( C_0 > 0 \) such that for all \( q \in M \)

\[ \sup_{v \in X} \frac{| d(v, q) |}{| v |_1} \geq C_0 \| q \|_0. \quad (6) \]

In addition, the trilinear form \( a_1(\cdot, \cdot, \cdot) \) on \( X \times X \times X \) is defined by

\[ a_1(u, v, w) = ( (u \cdot \nabla) v, w )_{x, x} \quad u, v, w \in X, \]

which is continuous on \( X \times X \times X \) and satisfies [15, 35]

\[ a_1(u, v, w) = -a_1(u, w, v), \quad \forall u \in V, v, w \in X, \quad (7) \]

\[ | a_1(u, v, w) | \leq N | u |_1 | v |_1 | w |_1, \quad \forall u, v, w \in X, \quad (8) \]

\[ | a_1(u, v, w) | \leq c_0 \| u \|^\frac{3}{2} \| u \|^\frac{1}{2} | v |_1 | w |_0, \quad \forall u \in X, v \in D(A), w \in Y, \quad (9) \]

\[ | a_1(u, v, w) | \leq c_0 \| v \|^\frac{3}{2} \| Av \|^\frac{1}{2} | u |_1 | w |_0, \quad \forall u \in X, v \in D(A), w \in Y, \quad (10) \]

where \( c_0 \) is a positive constant depending only on \( \Omega \) and \( N \) is defined in terms of

\[ b(u, v, w) := ( (u \cdot \nabla) v, w ) + \frac{1}{2} ( (\text{div} u) v, w ) \]

\[ = \frac{1}{2} a_1(u, v, w) - \frac{1}{2} a_1(u, w, v), \quad u, v, w \in X \quad (11) \]
by

\[ N := \sup_{u,v,w \in X} \frac{b(u,v,w)}{|u||v||w|}. \]

**Lemma 1.** [11,15,17,35] There exists a unique solution \((v, q) \in (X, M)\) to the steady Stokes problem

\[-\Delta v + \nabla q = g, \quad \text{div} v = 0 \quad \text{in} \ \Omega, \quad v|_{\partial \Omega} = 0,\]

for any prescribed \(g \in Y\) and

\[ \|v\|_2 + \|q\|_1 \leq c\|g\|_0, \]

where \(c > 0\) is a generic constant depending on \(\Omega\).

**Lemma 2.** [27,33] For any \(\alpha, t^* > 0\) and \(\phi \in L^2(0, t^*)\), the following property holds

\[ \int_0^{t^*} \int_0^t (t-s)^{-\beta} e^{-\alpha(t-s)} \phi(s)ds \phi(t)dt \geq 0. \]

We then introduce the Gamma function \(\Gamma(\cdot)\) defined by

\[ \Gamma(z) := \int_0^{\infty} s^{z-1}e^{-s}ds, \quad z > 0 \]

that will be frequently used in the future. Note that by a simple transformation, the above equation implies

\[ \int_0^{\infty} s^{z-1}e^{-vs}ds = \frac{1}{v^z} \int_0^{\infty} s^{z-1}e^{-s}ds = \frac{\Gamma(z)}{v^z}, \quad z, v > 0. \] \hfill (12)

We finally introduce the Gronwall inequality to support the analysis.

**Lemma 3.** [5] If \(g, h, y, G\) are nonnegative locally integrable functions on the time interval \([0, \infty)\) such that for all \(t \geq 0\) and for some \(C \geq 0\)

\[ y(t) + G(t) \leq C + \int_0^t h(s)ds + \int_0^t g(s)y(s)ds, \]

then for \(t \geq 0\)

\[ y(t) + G(t) \leq \left( C + \int_0^t h(s)ds \right) \exp \left( \int_0^t g(s)ds \right). \]

### 2.2. Regularity of solutions to the steady state

By \((7)\) and \((11)\), the variational formulation of \((3)\) reads: find \((\tilde{u}, \tilde{p}) \in (X, M)\) such that for any \((v, q) \in (X, M)\)

\[ \left( \mu + \frac{\rho \Gamma(1-\beta)}{\delta^{1-\beta}} \right) a(\tilde{u}, v) - d(v, \tilde{p}) + d(\tilde{u}, q) + a_1(\tilde{u}, \tilde{u}, v) = (\tilde{f}, v). \] \hfill (13)

Then we follow [12] to assume that the solution \(\tilde{u}\) of \((13)\) satisfies

\[ \mu a(v, v) + b(v, \tilde{u}, v) \geq \mu_0|v|^2, \quad \forall \ v \in X, \] \hfill (14)
for some $0 < \mu_0 < \mu$. We show in the next theorem that this assumption could ensure the uniqueness of solutions to model (13).

**Remark 1.** In the literature, the commonly used assumption for proving the uniqueness of the solution pair $(\bar{u}, \bar{p})$ to problem (3) is

$$
\frac{N}{(\mu + \rho\Gamma(1-\beta)\delta^{1-\beta})^2} \|f\|_{-1} < 1, \quad \|f\|_{-1} = \sup_{v \in X} \frac{\langle f, v \rangle}{|v|_1},
$$

which arises from the application of the fixed point method, see Girault and Raviart [11] and Temam [35]. The assumption (14) indeed requires the coercivity of the sum of the bilinear term and part of the trilinear term, and could be derived from the traditional assumption (15) [12, Lemma 2.4], that is, (14) could be justified if (15) holds true. The weaker assumption (14) not only avoids the application of the fixed point method mentioned above when proving the uniqueness, but will facilitate the estimates in subsequent sections. If it is not verified, the proof of the uniqueness of the solutions could be intricate.

**Theorem 1.** Under the assumption (14), problem (13) admits a unique solution pair $(\bar{u}, \bar{p}) \in X \times M$ which satisfies

$$
|u|_1 \leq \left( \mu + \frac{\rho\Gamma(1-\beta)}{\delta^{1-\beta}} \right)^{-1} \|f\|_{-1}, \quad \|f\|_{-1} := \sup_{v \in X} \frac{\langle f, v \rangle}{|v|_1}. \quad (16)
$$

Moreover, if $f \in Y$, then the solution pair $(\bar{u}, \bar{p}) \in D(A) \times (H^1(\Omega) \cap M)$ satisfies

$$
\|A\bar{u}\|_0 + \left( \mu + \frac{\rho\Gamma(1-\beta)}{\delta^{1-\beta}} \right)^{-1} \|p\|_1 
\leq c \left( \mu + \frac{\rho\Gamma(1-\beta)}{\delta^{1-\beta}} \right)^{-1} \|f\|_0 \left( 1 + \left[ \mu + \frac{\rho\Gamma(1-\beta)}{\delta^{1-\beta}} \right]^{-4} \|f\|_0^2 \right). \quad (17)
$$

**Proof.** By Lemma 1, the proof could be carried out following those of Theorems 2.1, 2.2 and 2.4 in [12] and is thus omitted. \(\square\)

### 3. Analysis of viscoelastic flow

We investigate the well-posedness and regularity of the weak solution $u$ to the viscoelastic fluid (2) for $u_0 \in V$. In the rest of the paper, we may abbreviate a space-time-dependent function $g(x, t)$ as $g(t)$ for simplicity and use $\kappa$ to denote a generic positive constant that may assume different values at different occurrences. We introduce

$$
J(t; v, w) := \left( \rho \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \nabla v(s) \cdot \nabla w(t) \right), \quad \forall v(\cdot, t), w(\cdot, t) \in X
$$

such that the variational formulation of (2) could be formulated as

$$
(\bar{u}, v) + \mu(\bar{u}, v) + a_1(\bar{u}, u, v) - d(v, p) + d(u, q) + J(t; u, v) = \langle f, v \rangle, \quad (18)
$$

$$
u_0 = u_0 \in V, \quad \forall (v, q) \in X \times M.
$$

**Lemma 4.** For $r = 0, 1$, the following estimate holds

$$
|J(t; v, A^r w)| \leq \frac{\mu}{\varepsilon} \|A^{r+1} w\|_0^2 + \frac{\varepsilon \rho^2 \Gamma(1-\beta)}{4\mu \delta^{1-\beta}} \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \|A^{r+1} v\|_0^2 ds
$$

for any $v, w \in X$ where $\varepsilon > 0$ is a generic constant.
Proof. By Cauchy–Schwarz inequality, Minkowski’s integral inequality and noting \((A\mathbf{v}, \mathbf{w}) = (\nabla \mathbf{v}, \nabla \mathbf{w})\), we have

\[
|J(t; \mathbf{v}, A^r \mathbf{w})| = \left| \rho \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} A\mathbf{v}(s), A^r \mathbf{w}(t) \right| \\
\leq \left| \rho \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} A^{\frac{r+1}{2}} \mathbf{v}(s)ds \right| \|A^{\frac{r+1}{2}} \mathbf{w}(t)\|_0 \\
\leq \rho \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \|A^{\frac{r+1}{2}} \mathbf{v}(s)\|_0 ds \|A^{\frac{r+1}{2}} \mathbf{w}(t)\|_0 \\
\leq \frac{\mu}{\varepsilon} \left\|A^{\frac{r+1}{2}} \mathbf{w}(t)\right\|_0^2 + \frac{\varepsilon}{4\mu} \left( \rho \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \|A^{\frac{r+1}{2}} \mathbf{v}(s)\|_0 ds \right)^2 \\
= \frac{\mu}{\varepsilon} \left\|A^{\frac{r+1}{2}} \mathbf{w}(t)\right\|_0^2 + I_0.
\]

Using Holder inequality and (12) we bound \(I_0\) as

\[
I_0 \leq \frac{\varepsilon \rho^2}{4\mu} \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} ds \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \|A^{\frac{r+1}{2}} \mathbf{v}(s)\|_0^2 ds \\
\leq \frac{\varepsilon \rho^2}{4\mu} \frac{\Gamma(1-\beta)}{\delta^{1-\beta}} \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \|A^{\frac{r+1}{2}} \mathbf{v}(s)\|_0^2 ds,
\]

Combining the above two equations, we complete the proof. \(\Box\)

**Theorem 2.** Suppose \(u_0 \in H\) and \(f \in L^\infty(0, \infty; H)\). Then the problem (18) admits a unique solution \(u \in L^\infty(0, \infty; H) \cap L^2(0, t; V)\) for any \(t \geq 0\).

**Proof.** According to the positivity of the kernel in Lemma 2, the proof could be carried out following Theorem 3.1, Lemma 3.1 and Lemma 3.2 in [5] and is thus omitted. \(\Box\)

**Theorem 3.** Suppose \(u_0 \in V\), \(f \in L^\infty(0, \infty; H) \cap L^2(0, \infty; L^2(\Omega)^2)\) and satisfies

\[
\int_0^\infty e^{2\alpha s} \left\|f(s)\right\|_0^2 ds < \infty. \tag{19}
\]

If \((u, p)\) satisfies (18), then the following properties hold

\[
\left\|u\right\|_0^2 + \mu e^{-2\alpha t} \int_0^t e^{2\alpha s} \left|u\right|^2_1 ds \leq \kappa e^{-2\alpha t}, \quad \forall t \geq 0, \tag{20}
\]

\[
\left|u\right|^2_1 + \mu e^{-2\alpha t} \int_0^t e^{2\alpha s} \|Au\|^2_0 ds \leq \kappa e^{-2\alpha t}, \quad \forall t \geq 0, \tag{21}
\]

where \(\kappa > 0\) depends on the data and \(0 < \alpha < \frac{1}{2} \min\{\delta, \frac{\mu_0\gamma_0}{2}\}\).
Proof. Taking \((v, q) = e^{2\alpha t}(u, p)\) in (18), we have
\[
\frac{e^{2\alpha t}}{2} \frac{d}{dt} \|u\|_0^2 + e^{2\alpha t} \mu a(u, u) + e^{2\alpha t} a_1(u, u, u) + e^{2\alpha t} J(t; u, u) = e^{2\alpha t}(f(t), u).
\]  
(22)

Combining (22) with (5), and noting \(0 < \alpha \leq \frac{1}{2} \min\{\delta, \frac{\mu \rho_0}{2}\}\), we obtain
\[
e^{2\alpha t} \frac{d}{dt} \|u\|_0^2 + 2\alpha e^{2\alpha t} \|u\|_0^2 + \frac{3}{2} \mu e^{2\alpha t} \|u\|_1^2 + 2e^{2\alpha t} J(t; u, u) \leq 2e^{2\alpha t} (f(t), u).
\]  
(23)

We invoke
\[
2|e^{2\alpha t}(f(t), u)| \leq \frac{1}{2} \mu e^{2\alpha t} \|u\|_0^2 + 2\mu^{-1} e^{2\alpha t} \|f(t)\|_1^2 - 1
\]
in (23) to obtain
\[
\frac{d}{dt} (e^{2\alpha t} \|u\|_0^2) + \mu e^{2\alpha t} \|u\|_1^2 + 2e^{2\alpha t} J(t; u, u) \leq 2\mu^{-1} e^{2\alpha t} \|f(t)\|_1^2,
\]  
(24)

Integrating (24) from 0 to \(t\), using Lemma 2 and the assumption (19), and multiplying the resulting equation by \(e^{-2\alpha t}\) on both sides, we obtain
\[
\|u\|_0^2 + \mu e^{-2\alpha t} \int_0^t e^{2\alpha s} \|u\|_1^2 ds \leq e^{-2\alpha t} \|u(0)\|_0^2 + 2\mu^{-1} e^{-2\alpha t} \int_0^t e^{2\alpha s} \|f(s)\|_1^2 ds
\]
\[
\leq \kappa e^{-2\alpha t}, \quad \forall t \geq 0.
\]  
(25)

Then we complete the proof of (20).

Next, taking \((v, q) = e^{2\alpha t}(Au, 0)\) in (18) leads to
\[
\frac{e^{2\alpha t}}{2} \frac{d}{dt} \|u\|_1^2 + \mu e^{2\alpha t} \|Au\|_0^2 + e^{2\alpha t} a_1(u, u, Au) + e^{2\alpha t} J(t; u, Au) = e^{2\alpha t}(f(t), Au).
\]  
(26)

By (10) and the Young’s inequality, we have
\[
e^{2\alpha t} |a_1(u, u, Au)| \leq c_0 e^{2\alpha t} \|u\|_0^2 \|u\|_1 \|Au\|_0^2 \|Au\|_0^2 \\
\leq \frac{\mu}{8} e^{2\alpha t} \|Au\|_0^2 + 8 c_0^4 \mu^{-3} e^{2\alpha t} \|u\|_0^2 \|u\|_1^4 \\
\leq \frac{\mu}{8} e^{2\alpha t} \|Au\|_0^2 + 8 c_0^4 \mu^{-3} e^{-4\alpha t} e^{\alpha t} \|u\|_1^2 \|u\|_1^2 e^{\alpha t} \|u\|_1^4,
\]
\[
|e^{2\alpha t}(f(t), Au)| \leq \frac{\mu}{8} e^{2\alpha t} \|Au\|_0^2 + 2\mu^{-1} e^{2\alpha t} \|f(t)\|_0^2.
\]

Combining these inequalities with (26) and using \(0 < \alpha \leq \min\{\delta, \frac{\mu \rho_0}{4}\}\) and (5), we get
\[
\frac{d}{dt} (e^{2\alpha t} \|u\|_1^2) + \mu e^{2\alpha t} \|Au\|_0^2 + 2e^{2\alpha t} J(t; u, Au) \leq \kappa e^{-4\alpha t} |e^{\alpha t} \|u\|_1^2| + 4\mu^{-1} e^{2\alpha t} \|f(t)\|_0^2.
\]  
(27)

Integrating (27) from 0 to \(t\), and noting \((Au, u) = (\nabla u, \nabla u)\), then using Lemma 2 yields
\[
e^{2\alpha t} \|u\|_1^2 + \mu \int_0^t e^{2\alpha s} \|u\|_0^2 ds \leq |u(0)|_1^2 + \kappa \int_0^t e^{-4\alpha s} |e^{\alpha s} \|u\|_0^2| e^{\alpha s} \|u\|_1^2 ds
\]
\[+ 4\mu^{-1} \int_0^t e^{2\alpha s} \|f(s)\|_0^2 ds, \quad \forall t \geq 0.
\]  
(28)
Invoking Lemma 3 and (25) in (28) and multiplying the resulting equation by $e^{-2\alpha t}$ on both sides, we have

$$|u|^2 + \mu e^{-2\alpha t}\int_0^t e^{2\alpha s}\|A u\|^2_0 ds$$

$$\leq e^{-2\alpha t}\left(|u(0)|_1^2 + 4\mu^{-1}\int_0^t e^{2\alpha s}\|f(s)\|^2_0 ds\right) \exp\left(\kappa \int_0^t e^{-4\alpha s}\|e^{\alpha s} u_1^2\|_0 ds\right)$$

$$\leq e^{-2\alpha t}\left(|u(0)|_1^2 + 4\mu^{-1}\int_0^t e^{2\alpha s}\|f(s)\|^2_0 ds\right) \exp\left(\kappa \int_0^t e^{-2\alpha s}\|u_1^2\|_0 ds\right)$$

$$\leq \kappa e^{-2\alpha t}\left(|u(0)|_1^2 + 4\mu^{-1}\int_0^t e^{2\alpha s}\|f(s)\|^2_0 ds\right).$$

Combining (29) with (19), we obtain (21) and thus complete the proof of this theorem.

**Theorem 4.** Suppose $u_0 \in D(A)$, $f \in L^\infty(0,\infty; H) \cap L^2(0,\infty; L^2(\Omega)^2)$, $f_t \in L^2(0,\infty; H^{-1}(\Omega)^2)$ and

$$\int_0^\infty e^{2\alpha s}\|f(s)\|^2_0 ds + \int_0^\infty e^{2\alpha s}\|f_s(s)\|^2_{-1} ds < \infty.$$  

(30)

If $(u, p)$ satisfies (18), then

$$\|u\|_0^2 + \mu e^{-2\alpha t}\int_0^t e^{2\alpha s}\|u_s\|_1^2 ds \leq \kappa e^{-2\alpha t}, \quad \|A u\|_0^2 \leq \kappa, \quad \forall t \geq 0,$$

(31)

where $\kappa > 0$ depends on the data, $0 < \alpha < \frac{1}{2} \min\{\delta, \frac{\mu a_{\infty}}{2}\}$, and $\|f_t(t)\|_{-1} := \sup_{0 \neq v \in X} \frac{\langle f_t(t), v \rangle}{\|v\|_1}$.

In particular, if further $\|f(t)\|_0 \leq c^* e^{-\alpha t}$ for $t \geq 0$ and for some constant $c^* \geq 0$, then the following exponential decay holds

$$\|u\|_2^2 \leq c \|A u\|_0^2 \leq \kappa e^{-2\alpha t}, \quad \forall t \geq 0$$

(32)

where $c$ is given in (4).

**Proof.** Differentiating (18) with respect to time and noting

$$J_t(t; u, v) = (\rho t^{-\beta} e^{-\delta t} \nabla u(0), \nabla v) + \left(\rho \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \nabla u_t(s) ds, \nabla v\right)$$

we have

$$= \rho t^{-\beta} e^{-\delta t} a(u(0), v) + J(t; u_t, v),$$

we have

$$(u_{tt}, v) + \mu a(u_t, v) + a_1(u, u, v) + a_1(u, u, v) - d(v, p_t)$$

$$+ d(u_t, q) + J(t; u_t, v) = -\rho t^{-\beta} e^{-\delta t} a(u(0), v) + \langle f_t(t), v \rangle.$$  

(33)

Taking $(v, q) = e^{2\alpha t}(u_t, p_t)$ in (33), we get

$$\frac{d}{dt}\left(\frac{e^{2\alpha t}}{2}\|u\|_0^2 + \mu e^{2\alpha t}\|u_t\|_1^2 + e^{2\alpha t} a_1(u_t, u, u_t) + e^{2\alpha t} J(t; u_t, u_t)\right)$$

$$= -\rho t^{-\beta} e^{-\delta - 2\alpha t} a(u(0), u_t) + e^{2\alpha t}(f_t(t), u_t).$$

(34)
In view of (9) and \( u_0 \in D(A) \), we obtain
\[
\rho t^{-\beta} e^{-(\delta - 2\alpha)t} - a(u(0), u_t) \leq \rho t^{-\beta} e^{-(\delta - 2\alpha)t} \|Au(0)\|_0 \|u_t\|_0 \\
\leq \kappa t^{-\beta} e^{-(\delta - 2\alpha)t} \|u_t\|_0,
\]
\[
e^{2\alpha t} |a_1 (u_t, u, u_t)| \leq c_0 e^{2\alpha t} \|u_t\|_1 \|Au\|_0 \|u_t\|_0 \\
\leq \frac{\mu}{8} e^{2\alpha t} \|u_t\|_1^2 + 2c_0^2 \mu^{-1} e^{2\alpha t} \|Au\|_0^2 \|u_t\|_0^2,
\]
\[
e^{2\alpha t} (f_t(t), u_t) \leq \frac{\mu}{8} e^{2\alpha t} \|u_t\|_1^2 + 2\mu^{-1} e^{2\alpha t} \|f_t(t)\|_1^2.
\]
Combining these inequalities with (34) and noting \( 0 < \alpha \leq \min\{ \frac{\delta}{2}, \frac{\mu_{\text{min}}}{4} \} \) yield
\[
e^{2\alpha t} \frac{d}{ds} \|u_t\|_0^2 + 2\alpha e^{2\alpha t} \|u_t\|_1^2 + \mu e^{2\alpha t} \|u_t\|_0^2 + 2\alpha \mu^{-1} e^{2\alpha t} J(t; u_t, u_t) \\
\leq 2c_0^2 \mu^{-1} e^{2\alpha t} \|Au\|_0^2 \|u_t\|_0^2 + \kappa t^{-\beta} e^{-(\delta - 2\alpha)t} \|u_t\|_0 + 4\mu^{-1} e^{2\alpha t} \|f_t(t)\|_1^2.
\]
(35)
We integrate (35) from 0 to \( t \), use \( \|u_0(0)\|_0^2 \leq c(\|Au(0)\|_0^2 + \|f(0)\|_1^2) \leq c \) and \((Au, u_t) = (\nabla u_t, \nabla u_t)\) as well as Lemma 2 and (30) to obtain
\[
e^{2\alpha t} \|u_t\|_0^2 + \mu \int_0^t e^{2\alpha s} \|u_s\|_1^2 ds \leq \|u_t(0)\|_0^2 + 2c_0^2 \mu^{-1} \int_0^t e^{2\alpha s} \|Au\|_0^2 \|u_s\|_0^2 ds \\
+ \kappa \int_0^t \|u_s\|_0 ds + 4\mu^{-1} \int_0^t e^{2\alpha s} \|f_s(s)\|_1^2 ds \\
\leq \kappa + 2c_0^2 \mu^{-1} \int_0^t \|Au\|_1^2 e^{2\alpha s} \|u_s\|_0^2 ds \\
+ \kappa \int_0^t \|u_s\|_0 ds.
\]
(36)
Using Lemma 3 and (21) for (36), we have
\[
e^{2\alpha t} \|u_t\|_0^2 + \mu \int_0^t e^{2\alpha s} \|u_s\|_1^2 ds \leq \kappa \exp \left( \kappa \int_0^t \|Au\|_0^2 ds + \kappa \int_0^t e^{-\beta(-\delta - \alpha)s} ds \right) \leq \kappa,
\]
which proves the first equation of (31).

Then we take \((v, q) = (Au, 0)\) in (18) to get
\[
(u_t, Au) + \mu \|Au\|_0^2 + a_1(u, u, Au) + J(t; u, Au) = (f(t), Au).
\]
(37)
Due to (9), we have
\[
|a_1(u, u, Au)| \leq c_0 \|u\|_0^2 \|u_t\|_1^2 \|u\|_1^2 \|Au\|_0^2 \\
\leq \frac{\mu}{6} \|Au\|_0^2 + 6^3 c_0^4 \mu^{-3} \|u\|_0^2 \|u_t\|_1^2 \|u\|_1^2,
\]
\[
|(f(t), Au)| \leq \frac{\mu}{6} \|Au\|_0^2 + \frac{3}{2} \mu^{-1} \|f(t)\|_0^2.
\]
Combining these inequalities with (37) and using (20) and (21) yield
\[
2(u_t, Au) + \frac{4}{3} \mu \|Au\|_0^2 \leq \kappa \|u_t\|_1^2 + \frac{3}{2} \mu^{-1} \|f(t)\|_0^2 - 2J(t; u, Au).
\]
(38)
Using Lemma 4, we get

$$|J(t; u, Au)| \leq \frac{\mu}{12} \|Au\|_0^2 + \frac{3\rho^2}{\mu} \Gamma(1 - \beta) \frac{1}{\delta^{1-\beta}} t \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \|Au\|_0^2 ds.$$  \hspace{1cm} (39)

We incorporate

$$2|\langle u_t, Au \rangle| \leq \frac{\mu}{6} \|Au\|_0^2 + 6\mu^{-1} \|u_t\|_0^2$$

with (38) and use (39) to obtain

$$\mu \|Au\|_0^2 \leq \kappa \|u\|_1^2 + 6\mu^{-1} \|u_t\|_0^2 + \frac{3}{2} \mu^{-1} \|f(t)\|_0^2 + 6\rho^2 \Gamma(1 - \beta) \frac{1}{\delta^{1-\beta}} t \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \|Au\|_0^2 ds.$$  \hspace{1cm} (40)

Applying (31), (21) and (4) in (40), we get

$$\mu \|Au\|_0^2 \leq \kappa (e^{-2\alpha t} + \|f(t)\|_0^2) + \frac{6\rho^2}{\mu} \Gamma(1 - \beta) \frac{1}{\delta^{1-\beta}} t \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \|Au\|_0^2 ds.$$  \hspace{1cm} (41)

By Lemma 3, we have

$$\mu \|Au\|_0^2 \leq \kappa (e^{-2\alpha t} + \|f(t)\|_0^2) \exp \left( \frac{6\rho^2}{\mu} \Gamma(1 - \beta) \frac{1}{\delta^{1-\beta}} t \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} ds \right),$$

which, together with (12) and $f \in L^\infty(0, \infty; H)$, leads to the second estimate of (31). We finally use (41) and $\|f(t)\|_0 \leq c^* e^{-\alpha t}$ to obtain (32) and thus complete the proof. \hfill \Box

4. Exponential convergence: basic results

We prove basic asymptotic results on the solutions between (2) and (3). Let $z := u - \bar{u}$, $\eta := p - \bar{p}$ and

$$J_1(t; \bar{u}, v) := \frac{\rho}{\delta^{1-\beta}} \int_0^\infty s^{-\beta} e^{-s} ds a(\bar{u}, v(t)), \quad \forall v(\cdot, t) \in X.$$

Lemma 5. The following estimates hold for $r = 0, 1$

$$e^{2\alpha t} |J_1(t; \bar{u}, A^r v)| \leq c_2 e^{-(\delta - 2\alpha) t} \|A^{\frac{r+1}{2}} \bar{u}\|_0^2 + \frac{\mu}{4\varepsilon} e^{2\alpha t} \|A^{\frac{r+1}{2}} v\|_0^2, \quad \forall v \in X,$$

where $\varepsilon > 0$ is a general constant, $c_2 > 0$ depends only on the data and $\bar{u} \in X$.

Proof. Since

$$\frac{\rho}{\delta^{1-\beta}} \int_0^\infty s^{-\beta} e^{-s} ds \leq \frac{\rho}{\delta^{1-\beta}} e^{-\frac{\delta}{\varepsilon}} \int_0^\infty s^{-\beta} e^{-\frac{s}{\varepsilon}} ds \leq c_1 e^{-\frac{\delta}{\varepsilon}},$$  \hspace{1cm} (42)

where $c_1 > 0$ is a constant depends only on the data $\rho, \delta, \beta$, noting $(A\bar{u}, v) = (\nabla \bar{u}, \nabla v)$, we have for $v \in X$

$$|e^{2\alpha t} J_1(t; \bar{u}, A^r v)| = e^{2\alpha t} \frac{\rho}{\delta^{1-\beta}} \int_0^\infty s^{-\beta} e^{-s} ds |(A\bar{u}, A^r v)|$$
Then the solution $u$ of (2) exponentially converges to $\bar{u}$ in the following sense

$$\|z\|_0^2 + \mu e^{-2\alpha t} \int_0^t e^{2\alpha s} |z|_1^2 ds \leq \kappa e^{-2\alpha t}, \quad \forall t \geq 0,$$

$$|z|_1^2 + \mu e^{-2\alpha t} \int_0^t e^{2\alpha s} \|A\|_2^2 ds \leq \kappa e^{-2\alpha t}, \quad \forall t \geq 0,$$

where $\kappa > 0$ depends on the data and $0 < \alpha < \frac{1}{2} \min\{\delta, \frac{\mu \gamma_0}{2}\}$.

Proof. By (12), Eq. (3) could be reformulated as follows

$$\dot{u}_t - \mu \Delta u + (\bar{u} \cdot \nabla)u + \nabla \rho - \rho \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \Delta v ds = f + \frac{\rho}{\delta^{1-\beta}} \int_0^\infty s^{-\beta} e^{-s} \Delta v ds.$$

Subtracting (46) from (2), we obtain that

$$z_t - \mu \Delta z + (\bar{z} \cdot \nabla)z + (\bar{u} \cdot \nabla)z + (z \cdot \nabla)\bar{u} - \rho \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \Delta z ds$$

$$+ \nabla \eta = -\frac{\rho}{\delta^{1-\beta}} \int_0^\infty s^{-\beta} e^{-s} \Delta v ds + f(t) - \bar{f},$$

$$z(t) \in V, \quad \eta(t) \in M, \quad t \geq 0, \quad z(0) = u_0 - \bar{u} \in V,$$

for all $(x, t) \in \Omega \times (0, \infty)$. Then the variational formulation of (47) is

$$(z_t, v) + \mu a(z, v) + a_1(z, z, v) + a_1(\bar{u}, z, v) + a_1(z, \bar{u}, v)$$

$$- d(v, \eta) + d(z, q) + J(t; z, v) = J_1(t; \bar{u}, z, v) + (f(t) - \bar{f}, v),$$

$$z(0) = u_0 - \bar{u} \in V,$$

for all $(v, q) \in (X \times M)$. Taking $(v, q) = e^{2\alpha t}(z, \eta)$ in (49), using (7) and noting $\text{div} z = 0$ from (48), we have

$$\frac{e^{2\alpha t}}{2} \frac{d}{dt} \|z\|_0^2 + e^{2\alpha t} \mu a(z, z) + e^{2\alpha t} b(z, z) + e^{2\alpha t} J(t; z, z)$$

$$= e^{2\alpha t} J_1(t; \bar{u}, z) + e^{2\alpha t} (f(t) - \bar{f}, z).$$

Combining the assumption (14), (5) and noting $0 < \alpha < \frac{1}{2} \min\{\delta, \frac{\mu \gamma_0}{2}\}$, we obtain from (50) that

$$e^{2\alpha t} \frac{d}{dt} \|z\|_0^2 + 2\alpha e^{2\alpha t} \|z\|_0^2 + \frac{\mu \alpha e^{2\alpha t}}{2} \|z\|_1^2 + 2e^{2\alpha t} J(t; z, z)$$

which completes the proof. \qed
\[ \leq 2e^{2\alpha t}J_1(t; \bar{u}, z) + 2e^{2\alpha t}(f(t) - \bar{f}, z). \] (51)

By Lemma 5, we have
\[ 2e^{2\alpha t}|J_1(t; \bar{u}, z)| \leq c_2e^{-(\delta-2\alpha)t}|\bar{u}|^2 + \frac{\mu}{4}e^{2\alpha t}|z|^2, \]
\[ 2|e^{2\alpha t}(f(t) - \bar{f}, z)| \leq \frac{\mu}{4}e^{2\alpha t}|z|^2 + 4\mu^{-1}e^{2\alpha t}|f(t) - \bar{f}|^2, \]
which, together with (51), yield
\[ \frac{d}{dt}(e^{2\alpha t}\|z\|_0^2) + \mu e^{2\alpha t}|z|_1^2 + 2e^{2\alpha t}J(t; z, z) \leq c_2e^{-(\delta-2\alpha)t}|\bar{u}|^2 + 4\mu^{-1}e^{2\alpha t}|f(t) - \bar{f}|^2. \] (52)

Integrating (52) from 0 to t and using Lemma 2, (43), we deduce
\[ \|z\|_0^2 + \mu e^{-2\alpha t} \int_0^t e^{2\alpha s}|z|_1^2\,ds \leq e^{-2\alpha t}\|z(0)\|_0^2 + 2c_2|\bar{u}|^2e^{-2\alpha t} \int_0^t e^{-(\delta-2\alpha)s}\,ds \]
\[ + \frac{4e^{-2\alpha t}}{\mu} \int_0^t e^{2\alpha s}\|f(s) - \bar{f}\|^2\,ds \]
\[ \leq e^{-2\alpha t}\|z(0)\|_0^2 + \kappa e^{-2\alpha t} \leq \kappa e^{-2\alpha t}, \quad \forall t \geq 0, \]
which proves (44).

Next, taking \((v, q) = e^{2\alpha t}(Az, 0)\) in (49) we get
\[ \mu e^{2\alpha t}\|Az\|_0^2 + \frac{e^{2\alpha t}d}{2}\|z\|_1^2 + e^{2\alpha t}J(t; z, Az) \]
\[ + e^{2\alpha t}a_1(z, Az) + e^{2\alpha t}a_1(\bar{u}, z, Az) + e^{2\alpha t}a_1(z, \bar{u}, Az) \]
\[ = e^{2\alpha t}J_1(t; \bar{u}, Az) + e^{2\alpha t}(f(t) - \bar{f}, Az). \] (53)

Due to (9) and Lemma 5, we have
\[ e^{2\alpha t}|a_1(z, Az)| \leq c_0e^{2\alpha t}\|z\|_0^2 \|A\|_1^\frac{1}{2} \|Az\|_0^\frac{1}{2}, \]
\[ e^{2\alpha t}|a_1(\bar{u}, Az)| \leq c_0e^{2\alpha t}\|\bar{u}\|_0^\frac{1}{2} \|A\|_1^\frac{1}{2} \|Az\|_0^\frac{1}{2}, \]
\[ e^{2\alpha t}|J_1(t; u, Az)| \leq c_2e^{-(\delta-2\alpha)t}|Au|_0^2 + \|Au\|^2_0 + \frac{\mu}{12}e^{2\alpha t}\|Az\|^2_0, \]
\[ |e^{2\alpha t}(f(t) - \bar{f}, Az)| \leq \frac{\mu}{12}e^{2\alpha t}\|Az\|^2_0 + 3\mu^{-1}e^{2\alpha t}|f(t) - \bar{f}|^2. \]

Invoking these inequalities and (5), (17), (44), \(0 < \alpha < \frac{1}{2}\) \(\min\{\delta, \frac{\mu\gamma_0}{2}\}\) as well as \(\|z\|_1 \leq \|\bar{u}\|_1 + |u(t)|_1 \leq \kappa\) proved by (16) and (21) in (53), we get
\[ \frac{d}{dt}(e^{2\alpha t}\|z\|_0^2) + \mu e^{2\alpha t}\|Az\|_0^2 + 2e^{2\alpha t}J(t; z, Az) \]
\[ \leq \kappa e^{2\alpha t}\|z\|_1^2 + \|Au\|^2_0 + 6\mu^{-1}e^{2\alpha t}|f(t) - \bar{f}|^2. \] (54)

Integrating (54) from 0 to t and noting \((Az, z) = (\nabla z, \nabla z)\), then using Lemma 2 we reach
\[ |z|^2 + \mu e^{-2\alpha t} \int_0^t e^{2\alpha s}\|Az\|^2_0\,ds \leq e^{-2\alpha t}|z(0)|^2_1 + \kappa e^{-2\alpha t} \int_0^t e^{2\alpha s}|z|^2_1\,ds + \kappa e^{-2\alpha t} \int_0^t e^{2\alpha s}\|Az\|^2_0\,ds \]
which, together with (43) and (44), leads to (45).

□

Theorem 6. Under the assumptions of the Theorem 5, \( p \) converges to \( \bar{p} \) in an exponential rate

\[
\|p(t) - \bar{p}\|_0^2 \leq \kappa e^{-\alpha t}, \quad \forall t \geq 0,
\]

where \( \kappa > 0 \) depends on the data and \( 0 < \alpha < \frac{1}{2} \ min\{\delta, \frac{\mu_0\gamma_0}{2}\} \).

Proof. By \( \bar{z}_t = \bar{u}_t \) and the estimate of \( \bar{u}_t \) in Theorem 4, we have

\[
\|\bar{z}_t\|_0^2 \leq \kappa e^{-\alpha t}, \quad \forall t \geq 0.
\]

Using (8), (49) and (6), we obtain

\[
\|\eta(t)\|_0 \leq C_0^{-1}\|f(t) - \bar{f}\|_0 + C_0^{-1}\|\bar{z}_t\|_0 + C_0^{-1}\|z_1\|
\]
\[
+ C_0^{-1}(|z_1| + |\bar{u}_1|)|z_1| + C_0^{-1}\int_0^t (t-s)^{-\beta} e^{-\delta(t-s)}|z_1|ds
\]
\[
+ C_0^{-1}\frac{\rho}{\delta^{1+\beta}}\int_0^\infty s^{-\beta} e^{-s}|\bar{u}_1|ds.
\]

By (45), (42), (16) and \( 0 < \alpha < \frac{1}{2} \ min\{\delta, \frac{\mu_0\gamma_0}{2}\} \), we get

\[
\int_0^t (t-s)^{-\beta} e^{-\delta(t-s)}|z_1|ds \leq e^{-\alpha t} \int_0^t (t-s)^{-\beta} e^{-(\delta-\alpha)(t-s)} e^{\alpha s}|z_1|ds
\]
\[
\leq e^{-\alpha t} \int_0^t (t-s)^{-\beta} e^{-(\delta-\alpha)(t-s)} ds \leq \kappa e^{-\alpha t},
\]

\[
\frac{\rho}{\delta^{1+\beta}}\int_0^\infty s^{-\beta} e^{-s}|\bar{u}_1|ds \leq \kappa e^{-\frac{\alpha}{2}} \leq \kappa e^{-\alpha t}.
\]

Combining (57) with (58)-(59) and using (56), (45), (43) and (16), we prove (55).

\[
\square
\]

5. Exponential convergence in stronger norms

Theorem 7. Under the assumptions of Theorem 5, the solution \( (u, p) \) of (2) converges to the solution \( (\bar{u}, \bar{p}) \) of (3) in an exponential rate

\[
\|u(t) - \bar{u}\|_2^2 \leq \kappa e^{-2\alpha t}, \quad \forall t \geq 0,
\]

\[
\|p(t) - \bar{p}\|_1^2 \leq \kappa e^{-2\alpha t}, \quad \forall t \geq 0,
\]

where \( \kappa > 0 \) depends on the data and \( 0 < \alpha < \frac{1}{2} \ min\{\delta, \frac{\mu_0\gamma_0}{2}\} \).

Proof. Taking \( (v, q) = (Az, 0) \) in (49), we get

\[
(z_t, Az) + \mu\|Az\|_0^2 + a_1(z, z, Az) + a_1(\bar{u}, z, Az) + a_1(z, \bar{u}, Az) + J(t; z, Az)
\]
\[
= J_1(t; \bar{u}, Az) + (\bar{f}(t) - \bar{f}, Az).
\]
By (9) and (10), we derive
\[ |a_1(z, z, Az)| \leq c_0 \|z\|_0^2 \|z^2_1\|_1^2 \|Az\|_0^2, \]
\[ \leq \frac{\mu}{8} \|Az\|_0^2 + 8^3 c_0^2 \mu^{-3} \|z\|_0^2 \|z^2_1\|_1^2, \]
\[ |a_1(\bar{u}, z, Az)| + |a_1(z, \bar{u}, Az)| \leq 2c_0 \|z_1\| \|A\bar{u}\|_0 \|Az\|_0, \]
\[ \leq \frac{\mu}{8} \|Az\|_0^2 + 8c_0^2 \mu^{-1} \|A\bar{u}\|_0^2 \|z^2_1\|_1^2, \]
\[ |(f(t) \mid - \bar{f}, Az)| \leq \frac{\mu}{8} \|Az\|_0^2 + 2\mu^{-1} \|f(t) - \bar{f}\|_0^2. \]

Combining these inequalities with (62) and using (17), (44) and (45), we obtain
\[ 2(z_t, Az) + \frac{5}{4} \mu \|Az\|_0^2 + 2J(t; z, Az) \leq \kappa \|z_1\|^2 + 2J_1(t; \bar{u}, Az) + 4\mu^{-1} \|f(t) - \bar{f}\|_0^2, \] (63)

Using Lemma 4, we get
\[ |J(t; z, Az)| \leq \frac{\mu}{24} \|Az\|_0^2 + \frac{6\rho^2}{\mu} \Gamma(1-\beta) \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \|Az\|_0^2 ds. \] (64)

We incorporate
\[ 2(z_t, Az) \leq \frac{\mu}{12} \|Az\|_0^2 + 12\mu^{-1} \|z_t\|_0^2, \]
\[ 2|J_1(t; \bar{u}, Az)| = \frac{2\rho}{\delta^{1-\beta}} \int_0^\infty s^{-\beta} e^{-s} ds \|A\bar{u}\|_0^2 \]
\[ \leq \frac{\mu}{12} \|Az\|_0^2 + \frac{12}{\mu} \left( \frac{\rho}{\delta^{1-\beta}} \int_0^\infty s^{-\beta} e^{-s} ds \right)^2 \|A\bar{u}\|_0^2. \]

with (63) and (64) to obtain
\[ \mu \|Az\|_0^2 \leq \kappa \|z_1\|^2 + 12\mu^{-1} \|z_t\|_0^2 + 4\mu^{-1} \|f(t) - \bar{f}\|_0^2 \]
\[ + 12\mu^{-1} \left( \frac{\rho}{\delta^{1-\beta}} \int_0^\infty s^{-\beta} e^{-s} ds \right)^2 \|A\bar{u}\|_0^2 \]
\[ + \frac{12\rho^2}{\mu} \Gamma(1-\beta) \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \|Az\|_0^2 ds. \] (65)

Invoking (45), (56), (43), (42) and (17) in (65) yields
\[ \mu \|Az\|_0^2 \leq \kappa(e^{-\delta t} + e^{-2\alpha t}) + \frac{12\rho^2}{\mu} \Gamma(1-\beta) \delta^{1-\beta} \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \|Az\|_0^2 ds; \] (66)

together with $0 < \alpha < \frac{1}{2} \min\{\delta, \frac{\mu\rho}{2}\}$ and Lemma 3, then (66) becomes
\[ \|Az\|_0^2 \leq \kappa(e^{-2\alpha t}) \exp \left( \frac{12\rho^2}{\mu} \Gamma(1-\beta) \delta^{1-\beta} \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} ds \right), \]

which, combine with (4) and (12), implies (60).
Finally, using (49) yields
\[ d(v, \eta) = (zt, v) + \mu a(z, v) + a_1(z, z, v) + a_1(\bar{u}, z, v) + a_1(z, \bar{u}, v) + d(z, q) + J(t; z, v) - J_1(t; \bar{u}, v) - (f(t) - \bar{f}, v), \quad \forall v \in X. \] (67)

Using (5), (9) and (10) in (67), we obtain
\[
\|\eta(t)\|_1 \leq (1 + \gamma_0^{-1})|\eta(t)|_1 \leq C_1 \sup_{v \in X} \frac{|d(v, q)|}{\|v\|_0} \\
\leq C_1 \|f(t) - \bar{f}\|_0 + C_1 \|z(t)\|_0 + C_1 |Az(t)|_0 + C_1 (\|Az(t)\|_0 + \|A\bar{u}\|_0) |z(t)|_1 \\
+ C_1 \int_0^{t} (t - s)^{-\beta} e^{-\delta (t-s)} \|Az\|_0 ds \\
+ \frac{\rho C_1}{\delta^{1-\beta}} \int_0^{\infty} s^{-\beta} e^{-s} \|A\bar{u}\|_0 ds,
\] (68)

where \( C_1 > 0 \) is a positive constant. We apply (66), (42) and (17) as well as \( 0 < \alpha < \frac{1}{2} \cdot \min\{\delta, \frac{\mu_0 \gamma_0}{2} \} \) to obtain
\[
\int_0^{t} (t - s)^{-\beta} e^{-\delta(t-s)} \|Az\|_0 ds \leq e^{-\alpha t} \int_0^{t} (t - s)^{-\beta} e^{-(\delta - \alpha)(t-s)} e^{\alpha s} \|Az\|_0 ds \\
\leq e^{-\alpha t} \int_0^{t} (t - s)^{-\beta} e^{-(\delta - \alpha)(t-s)} ds \leq \kappa e^{-\alpha t},
\]
\[
\frac{\rho}{\delta^{1-\beta}} \int_0^{\infty} s^{-\beta} e^{-s} \|A\bar{u}\|_0 ds \leq \kappa \frac{\rho}{\delta^{1-\beta}} \int_0^{\infty} s^{-\beta} e^{-s} ds \leq \kappa e^{-\frac{\delta t}{2}} \leq \kappa e^{-\alpha t}.
\]

Combining (68) with the above two equations and using (56), (45), (66) and (17), we prove (61) and thus complete the proof of the theorem. \( \square \)

### 6. Concluding remarks

We investigate the well-posedness and regularity of the solutions to a viscoelastic flow model with a generalized memory, as well as proving the exponential convergence of the solutions to the steady state. In particular, the exponential decay rates depend on the regularity of the data; for instance, in Theorem 3 the rate \( \alpha \) arises from the assumption (19) on \( f \). Therefore, the current exponential decay rates could be improved if we impose stronger assumptions on the data.

There are many possible improvements that deserve further investigation. For instance, as \( \beta \) tends to 0, the proposed model approaches to the traditional viscoelastic flow equation. How to characterize this approximation in terms of \( \beta \) is an important but challenging topic, as this may correspond to such inverse problem as the stability estimate of the solutions to the proposed model in terms of the parameter \( \beta \). We will study this interesting topic in the near future.
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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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