Volume Dependence of the Axial Charge of the Nucleon

N. L. Hall,¹,² A. W. Thomas,¹,² R. D. Young,¹,² and J. M. Zanotti²

¹ARC Centre of Excellence for Particle Physics at the Terascale, School of Chemistry and Physics, University of Adelaide, Adelaide SA 5005, Australia
²CSSM, School of Chemistry and Physics, University of Adelaide, Adelaide SA 5005, Australia

It is shown that the strong volume-dependence of the axial charge of the nucleon seen in lattice QCD calculations can be explained quantitatively in terms of the pion-induced interactions between neighbouring nucleons. The associated wave function renormalization leads to an increased suppression of the axial charge as the strength of the interaction increases, either because of a decrease in lattice size or in pion mass.

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The axial form factor of the nucleon, $G_A(Q^2)$, encodes details of the nucleon's non-perturbative structure and plays a key role in its properties under chiral symmetry [1]. While the axial charge, $g_A \equiv G_A(0)$, has been known accurately for many years through measurements of neutron β-decay, and the shape is well described by a dipole form with mass parameter around 1 GeV, the calculation of these properties within QCD still presents significant challenges. In particular, the mean square radius of the axial form factor in modern lattice QCD simulations is almost a factor two too small compared with experiment [2] and $g_A$ itself exhibits a remarkable dependence on the size of the space-time lattice used [3–9].

The resolution of these challenges is made especially urgent by the recent contributions of lattice QCD towards unravelling the origin of the spin of the proton [2] [10–12]. For example, the isovector combination of the orbital angular momentum carried by the quarks, $L^u - L^d$, is dramatically different in several widely used models [13–15] and it can be deduced from a lattice calculation of $J^u - J^d$ by subtracting the value of $g_A$ implicit in that same calculation. While such estimates have already been published, one cannot assess their reliability without a better understanding of the systematic error associated with the finite lattice volume [16].

In this Letter we explain the origin of the dramatic volume dependence of $g_A$ in terms of the pion exchange force between neighbouring nucleons on the periodic lattice used in lattice QCD simulations. This quantitative, model independent explanation provides a framework within which to fully assess the systematic errors associated with the determination of the quark spin and orbital angular momentum as discussed above.

One of the earliest attempts to understand why $g_A$ might decrease rapidly at low pion mass as the lattice size decreases relied on the fact that the axial current has a contribution proportional to $\nabla \phi$, with $\phi$ the pion field. In a finite box one may expect that the axial charge, given as the volume integral of the matrix element of the axial current would receive a contribution from a surface integral of the pion field, arising through Green’s Theorem [17]. Promising as this idea seems, it has been proven that with the periodic boundary condition imposed on the quark fields, this surface contribution must vanish and hence cannot be the source of the problem [18].

Next one might think of the volume dependence of the chiral loops which renormalize the axial charge [19], with wave function renormalization reducing $g_A$ and the vertex renormalization, especially that associated with the axial current acting on an intermediate Δ baryon, tending to increase it [20–21]. These two contributions tend to effectively cancel each other over a wide range of bag radii, so that the renormalized and un-renormalized axial charges are very close to each other. Given that the volume dependence of the wave function and vertex renormalization is very similar for pions which do not travel outside the lattice volume, this is also not the source of the rapid variation of $g_A$. Similar conclusions have been reached within other approaches [22–23].

What has not previously been recognised, however, is the effect associated with the fact that on a finite lattice each nucleon must interact with its neighbours through pion exchange. (Alternatively, the pion travels outside the lattice volume or “round the world”.) In terms of its effect on the axial charge this may be treated in an analogous way to the renormalization of an individual nucleon. Once again the wave function renormalization, that is the probability to find a bare nucleon rather than one which has emitted a pion that will be absorbed by one of its neighbours, reduces $g_A$, while the vertex renormalization acts to increase it. In this case, however, the $N \rightarrow \Delta \pi$ transition is suppressed by an extra factor of $\exp[-\delta L]$, with $\delta = m_\Delta - m_N$ and $L$ the lattice size, which is the same as the separation between nearest neighbours on the lattice. As a result, at all but the very smallest lattices (where the neighbouring nucleons would overlap anyway), this contribution is not effective at countering the effect of wave function renormalization. Similarly, the vertex renormalization involving $N\pi$ intermediate states is small for a single nucleon (namely that the interchange of $\pi \tau \bar{\tau}$ with the $\sigma \tau$ factors in the pion loop generates a factor of 1/9 compared with the
wave function renormalization). Hence the wave function renormalization dominates and $g_A$ is suppressed as the factor $\exp[-m_\pi L]$ increases, or $m_\pi L$ decreases.

The emission of a pion which is absorbed on a neighbouring nucleon is illustrated schematically in Fig. 1 for a nucleon on a periodic lattice. Indeed, when the nucleon under consideration emits a pion, every copy also emits a meson and the particular one in the box under consideration must absorb a pion emitted by the other neighbour. Looking at the contribution to the wave function renormalization coming from a single pair of neighbouring nucleons - as in Fig. 2 - one sees (in the heavy baryon approximation which should be perfectly adequate for this purpose) that

\[
\delta Z_2 = \langle N_{1\alpha}, N_{2\beta} \mid \frac{\tau_2 \tau_1 \pi}{16\pi^3} \left( \frac{g_A}{2f_\pi} \right)^2 \langle N_{1\alpha} N_{2\beta} \mid \delta \cdot \nabla + \frac{\nabla \cdot \delta}{i} \right\rangle, \quad (1)
\]

where $\alpha$, $\beta$, $\alpha'$ and $\beta'$ represent the nucleon spin states.

We suppose that the proton spin points up, along the $z$-axis, and consider first the interaction through the exchange of a $\pi^0$ with the nearest neighbour along the $y$-axis. This is illustrated by Fig. 2 (a), where the pion emission necessarily flips the spin of the proton. In this case $\langle \tau_2 \tau_1 \rangle = +1$ and the contribution to the wave function renormalization becomes

\[
\delta Z_2^{(a)} = \frac{1}{4\pi^2} \left( \frac{g_A}{2f_\pi} \right)^2 \langle N_{1\alpha} N_{2\beta} \mid \sigma_2 \sigma_1 \rangle \langle N_{1\alpha} N_{2\beta} \mid \delta \cdot \nabla \rangle \frac{\partial^2}{\partial L^2} \int_0^\infty \frac{dk k}{L} \sin(kL) \frac{\sin(kL)}{L(k^2 + m_\pi^2)^{3/2}}. \quad (2)
\]

Defining the usual spin raising and lowering operators

\[
\sigma_\pm = \frac{1}{2} (\sigma_x \pm i \sigma_y) \quad \sigma_0 = \sigma_z, \quad (3)
\]

we see that

\[
\sigma_1 \sigma_2 = 2 (\sigma_1 \sigma_2 - \sigma_1 - \sigma_2), \quad (4)
\]

so the spin matrix element gives a factor of 2 in this case. Finally, evaluating the momentum integral we find

\[
\delta Z_2^{(a)} = \frac{1}{2\pi^2} \left( \frac{g_A}{2f_\pi} \right)^2 \frac{\partial^2}{\partial L^2} K_0(m_\pi L)
\]

\[
= \frac{m_\pi^2}{4\pi^2} \left( \frac{g_A}{2f_\pi} \right)^2 (K_0(m_\pi L) + K_2(m_\pi L)), \quad (5)
\]

where $K_\nu$ are the modified Bessel functions (e.g., $K_0(z) \to (\pi/2)^{1/2} \exp(-z)$ as $z \to \infty$).

Because of the periodicity of the lattice one can also exchange a $\pi^+$ with the neighbouring nucleon along the $y$-axis. In this case the isospin factor is two, rather than one, and hence $\delta Z_2^{(a)} = 2\delta Z_2^{(a)}$. For the neighbouring nucleon along the $z$-axis the spin factor is unity. That is, the spin is not flipped. Thus $\delta Z_2^{(c)}$ and $\delta Z_2^{(d)}$ are each half as large as $\delta Z_2^{(a)}$ and $\delta Z_2^{(b)}$, respectively. This asymmetry between lattice sites perpendicular and parallel to the nucleon spin is a natural consequence of the tensor force generated by the exchange of a pseudoscalar meson.

The expressions above give the contribution to the wave function renormalization associated with pion exchange between a single pair of nucleons. To ensure that we represent what goes on on the lattice we need to include all neighbours which make significant contributions. There are four nearest neighbours in the $x-z$ plane and two along the $y$ direction. Thus the total contribution to the wave function renormalization from nearest neighbours, $\delta Z_2^{(n)}$, is

\[
\delta Z_2^{(n)} = 15 \left( \frac{g_A}{2f_\pi} \right)^2 \left( K_0(m_\pi L) + K_2(m_\pi L) \right) + 15 F(m_\pi^2, m_\pi L). \quad (6)
\]

As explained earlier, for $g_A$ the vertex renormalization is not very effective in countering the wave function renormalization when the $\Delta$ excitation is suppressed.
However, for a lattice which is not spherically symmetric the suppression is not simply 1/9 but differs from nearest (n1) to next-to-nearest (n2) to next-to-next-to-nearest neighbours (n3). It is a trivial calculation to show that for n1 the vertex renormalization is one fifth of $\delta Z_{2}^{nn}$. Thus the total relative correction to $g_{A}$ (i.e. $g_{A}(1 - \delta g_{A})$) from nearest neighbours is:

$$\delta g_{A}^{n1} = \frac{4}{5} \delta Z_{2}^{nn}$$  \hspace{1cm} (7)

![FIG. 3. Illustration of the convergence of the correction to $g_{A}$ calculated on a lattice 3.5 fm on a side as we include the nearest neighbour (n1) up to next-to-next-to-next-to-nearest neighbour (n4), for three values of $m_{\pi} L$.](image)

Turning to the effect of the next-to-nearest neighbours, we note that there are more of them (12 instead of 6) but that their distance from the nucleon under study is $\sqrt{2} L$. Thus, provided $m_{\pi} L$ is relatively large, we expect that the correction to $g_{A}$ should converge relatively quickly. In the present work we consider the corrections out to and including next-to-next-to-nearest neighbours, or a distance $\sqrt{3} L$. In these cases, the factor one fifth for n1 becomes one sixth for n2 and one ninth for n3. In summary, the total corrections to $g_{A}$ from next-to-nearest and next-to-next-to-nearest neighbours are

$$\delta g_{A}^{n2} = \frac{5}{6} 24 \mathcal{F}(m_{\pi}^{2}, \sqrt{2} m_{\pi} L),$$

$$\delta g_{A}^{n3} = \frac{8}{9} 12 \mathcal{F}(m_{\pi}^{2}, \sqrt{3} m_{\pi} L).$$

The degree of convergence for three relatively light quark masses is illustrated in Fig. 3 for a 3.5 fm lattice. However we point out that the convergence of the correction is solely dependent on $m_{\pi} L$, and the figure displays the convergence down to $m_{\pi} L \approx 2.5$. Here we also show the contribution from the next (2L) term which justifies our truncation at a distance of $\sqrt{3} L$ in the remainder of this letter.

The total reduction in $g_{A}$ to this order is illustrated in Fig. 4 — note that for each volume, the curves are plotted only down to a minimum reliable pion mass estimated by $m_{\pi} L = 2.5$. This figure makes it clear that these corrections are substantial. Modern lattice simulations of $g_{A}$ are now available at masses as low as 0.3 GeV, but even there the correction is almost 20% on a 2 fm box and 6% at 2.5 fm. Even more significant, we see that as the pion mass approaches the physical value the correction is as large as 10% on a 4 fm lattice. This makes very clear the challenge of a brute force determination of this quantity at the physical quark mass.

We show the effect of this correction applied to a simple fit linear in $m_{\pi}^{2}$, $g_{A}(1 - \delta g_{A}) + B m_{\pi}^{2}$, with two fit parameters ($g_{A}^{0}, B$) applied to recent lattice data from the QCDSF collaboration in Fig. 5 and the RBC collaboration in Fig. 6. Note that we have made no attempt here to include any non-analytic effects in the pion mass dependence of $g_{A}$, although in this case, these effects are anticipated to be small \cite{19}. The very dramatic reductions in $g_{A}$ at small lattice sizes and low pion mass are very clearly seen there, and we find remarkable agreement between the present calculation and those data.

Earlier work has suggested that similarly large finite-volume effects can be described by the discretisation of standard one-loop corrections \cite{4}. The enhancement seen there can by understood noting that both the input $g_{\Delta N}$ coupling and the fitted $g_{\Delta \Delta}$ coupling are about 1.5 times larger than those predicted by SU(6) relations. In the second panel of Fig. 5 the standard contributions using SU(6) estimates for the couplings are shown for the two relevant volumes. We see that the effects are insignificant compared to the exchange effects investigated in the present work.

The calculations reported here show clearly that a significant part of the hitherto unexplained quenching of $g_{A}$ on small lattice volumes originates with the pion exchange force between neighbouring nucleons on a periodic lattice. In retrospect, this explanation seems to be very
FIG. 5. The axial charge $g_A$, with the finite volume correction associated with pion exchange between neighbouring nucleons included, is plotted for several different values of the lattice size, $L$ (in fm). Also plotted are lattice calculations of $g_A$ from QCDSF [4, 9] with three different $\beta$’s (lattice spacings), $\beta = 5.25 (a = 0.076\text{ fm})$, $\beta = 5.29 (a = 0.072\text{ fm})$, $\beta = 5.40 (a = 0.060\text{ fm})$, on multiple volumes. For comparison, in the middle figure we also show (dot-dash line) the predicted finite size effects from [4], using the SU(6) $\Delta$ couplings [22].

FIG. 6. Curves as described in Fig. 5 plotted with lattice calculations of $g_A$ by Yamazaki et al. [9].

natural, although the key role of wave function renormalization was somewhat unexpected.

With this fascinating issue resolved, one may ask what other problems being addressed by lattice QCD calculations may be similarly affected. The study of the fraction of the nucleon spin carried by its quarks is certainly a topical example, as is the orbital angular momentum of the quarks and gluons and hadron magnetic moments. The tensor nature of the pion exchange force suggests that special care may be required in those cases where spin related deformation is important [24], for example in the study of the E2/M1 ratio for the $\Delta$.

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