Superboune and Loop Quantum Ekpyrotic Cosmologies from Modified Gravity: $F(R)$, $F(G)$ and $F(T)$ Theories

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ABSTRACT: We investigate the realization of two bouncing paradigms, namely of the superbounce and the loop quantum cosmological ekpyrosis, in the framework of various modified gravities. In particular, we focus on the $F(R)$, $F(G)$ and $F(T)$ gravities, and we reconstruct their specific subclasses which lead to such universe evolutions. These subclasses constitute from power laws, polynomials, or hypergeometric ansatzes, which can be approximated by power laws. The qualitative similarity of different effective gravities which realize the above two bouncing cosmologies, indicates to some universality lying behind such a bounce. Finally, performing a linear perturbation analysis, we show that the obtained solutions are conditionally or fully stable.

KEYWORDS: F(R) gravity, F(G) gravity, F(T) gravity, bounce cosmology, Loop Quantum Cosmology
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1 Introduction

According to many theoretical arguments and observational indications, the universe exhibited an early accelerating phase called inflation [1, 2]. The inflationary paradigm, amongst others, is very efficient in solving the flatness, horizon and monopole problem, and offers a consistent mechanism for the generation of the primordial fluctuations or primordial gravitational waves [3, 4] (see also [5, 6]). Hence, current observational research [7, 8] is focused on revealing the Universe’s evolution during all eras and theoretical research is focused on embedding all the striking new results in a consistent theoretical framework, and if possible under the same theoretical framework.

One alternative to the standard inflation description of the early acceleration is provided by bouncing cosmologies [9, 10], in which one avoids the appearance of the initial singularity. Some of the most appealing representative theories in this paradigm, are the ones that use scalar fields [9–22], the ones that use various models of modified gravity [23–35], and the ones that use the Loop Quantum Cosmology [36–44] (LQC) theoretical framework. In most of these scenarios, the bounce realization relies on the existence of matter fields, with an equation of state of suitable form in order to achieve the necessary violation of the null energy condition [9–12]. Bouncing cosmologies may have primordial instabilities, which make the contracting phase a rather problematic era [13, 14], however one can solve this problem in many ways, as for instance in the framework of ekpyrotic scenario [11, 12]. For relevant work on bouncing cosmologies see [45, 46]. Finally, note that there might be observable signatures of the bouncing phase [47–49].

In the present work we are interested in investigating the superbounce and the LQC ekpyrotic scenarios in the framework of modified gravity [50–52]. Since modified gravity is one of the two main directions in which one can achieve also the late-time acceleration (with the other one being dark energy), the above incorporation of bouncing solutions in modified gravity becomes more important under a unified picture, namely to be able to obtain the bouncing-initial phase of the universe as well as its late-time acceleration, simultaneously. In particular, we will investigate the above bouncing behaviors in the context of $F(R)$ [53–69] (for recent reviews see [70, 71]), $F(G)$ [72–84] and $F(T)$ [85–111] theories. The techniques that we shall use in order to provide a modified gravity description of the bouncing cosmologies are quite renowned [63, 64, 112] (see also [65–67]). For related works on bounce cosmology reconstruction from $F(R)$ theories see [113, 114], and for $F(R)$ ekpyrotic cosmology see [115]. Finally, for completeness, we investigate whether the obtained solutions are stable under linear perturbations.

The manuscript is organized as follows: In section 2, using very well known reconstruction techniques, we investigate the $F(R)$ modified gravity, and in particular we extract the $F(R)$ forms that give rise superbounce and LQC ekpyrotic evolution. For the superbounce case, we provide a complete analytic solution in closed form, while the case of LQC ekpyrotic is difficult to tackle, so we address this issue by finding the large and small curvature limits. In Section 3 we perform the same analysis for the case of $F(G)$ gravity, while in Section 4 for $F(T)$ gravitational paradigm. In Section 5 we investigate the stability of the obtained solutions $F(R)$, $F(G)$ and $F(T)$ solutions. With regards to the $F(R)$ and $F(G)$,
the stability analysis is focused on examining the dynamical system formed by resulting FRW equations, under linear perturbations. Also, the resulting conditions that ensure stability are thoroughly investigated. Finally, Section 6 is devoted to the conclusions.

2 Superbounce and loop quantum ekpyrotic cosmology from \( F(R) \) gravity

In this Section we will reconstruct the suitable \( F(R) \) form that can lead to the superbounce and loop quantum ekpyrotic cosmology realizations. Before investigating the reconstruction procedure, we briefly review in the following subsection the basic features of \( F(R) \) gravity in the metric formalism and we describe the geometric background we shall use in its cosmological application. For a detailed account on these issues, see [70, 71] and references therein.

2.1 \( F(R) \) gravity and cosmology

Let us review briefly the \( F(R) \) gravitational modification and its cosmological application. Throughout the article we assume that the spacetime manifold is pseudo-Riemannian, locally described by a Lorentz metric. Moreover, we consider a torsion-less, symmetric, and metric compatible affine connection, namely the Levi-Civita one. The Christoffel symbols write as

\[
\Gamma^k_{\mu\nu} = \frac{1}{2} g^{k\lambda} \left( \partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \right)
\]  

while the Ricci scalar becomes

\[
R = g^{\mu\nu} \left( \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} - \Gamma^\sigma_{\rho\nu} \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\rho} g^{\mu\nu} \Gamma^\sigma_{\rho\sigma} \right).
\]

The four dimensional action that describes \( F(R) \) theories then writes as

\[
S = \frac{1}{2\kappa^2} \int \sqrt{-g} F(R) + S_m, \tag{2.3}
\]

with \( \kappa^2 = 8\pi G \) is the gravitational constant, and where we have also considered the action of the matter sector \( S_m \). In the context of the metric formalism, by varying action (2.3) with respect to the metric tensor \( g_{\mu\nu} \) we obtain the equations of motion:

\[
F'(R)R_{\mu\nu}(g) - \frac{1}{2} F(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F'(R) + g_{\mu\nu} \square F'(R) = \kappa^2 T^m_{\mu\nu}. \tag{2.4}
\]

In the above expression and throughout the paper, the prime is assumed to denote differentiation with respect to the corresponding argument (hence in the present case we have \( F'(R) = \partial F(R)/\partial R \)). In addition, \( T^m_{\mu\nu} \) stands for the matter energy-momentum tensor arising from \( S_m \).

In order to investigate the cosmological implications of the above gravitational theory, we consider as usual a flat Friedmann-Lemaitre-Robertson-Walker (FRW) metric, with line element

\[
ds^2 = -dt^2 + a^2(t) \sum_i dx_i^2. \tag{2.5}
\]
In this case, the Ricci scalar is calculated to be
\[ R = 6(2H^2 + \dot{H}), \tag{2.6} \]
where \( H = \dot{a}/a \) stands for the Hubble parameter and dots indicate differentiation with respect to the cosmic time \( t \). Hence, the field equations (2.4), in the case of FRW geometry give rise to the Friedmann equation, namely
\[-18 \left[ 4H(t)^2 \dot{H}(t) + 2H(t)\ddot{H}(t) \right] F''(R) + 3 \left[ H^2(t) + \dot{H}(t) \right] F'(R) - \frac{F(R)}{2} + \kappa^2 \rho_{tot} = 0, \tag{2.7} \]
with \( \rho_{tot} \) the total energy density of the matter fields, and where the Ricci scalar is given in (2.6).

### 2.2 Superbounce Reconstruction from \( F(R) \) Gravity

The superbounce cosmological scenario was studied in [11], both in a supergravity and non-supersymmetric framework, with the most appealing attribute of this scenario being the ekpyrotic contraction phase. The ekpyrotic contraction was firstly introduced and studied in [12], along with a thorough study on cosmological perturbations. The purpose of this Section is to provide an \( F(R) \) description of the superbounce cosmological scenario, and in order to achieve this we will make use of the reconstruction technique firstly developed in [63].

The superbounce scale factor is given by [11]
\[ a(t) \sim (-t + t_*)^{2/c^2}, \tag{2.8} \]
with \( t_* \) being the big crunch time, and \( c \) a parameter constrained to \( c > \sqrt{6} \) [11]. Thus, the corresponding Hubble parameter is
\[ H(t) = \frac{2}{c^2(t - t_*)}. \tag{2.9} \]

The reconstruction technique developed in [63] is an exact method based on the introduction of a new variable in place of the cosmological time \( t \), namely the e-folding number \( N \), related to the scale factor \( a(t) \) through
\[ e^{-N} = \frac{a_0}{a}. \tag{2.10} \]

Using the new variable \( N \), we can re-express the first FRW equation (2.7) in terms of the e-fold parameter \( N \) as
\[-18 \left[ 4H^3(N)H'(N) + H^2(N)H'(N)^2 + H^3(N)H''(N) \right] F''(R) + 3 \left[ H^2(N) + H(N)H'(N) \right] F'(R) - \frac{F(R)}{2} + \kappa^2 \rho = 0. \tag{2.11} \]
We mention that in relation (2.11), and in the rest of this subsection, the Hubble parameter is assumed to be a function of the e-folds \( N \) and in addition any derivative appearing in (2.11) is defined with respect to \( N \).
We now introduce the function \( G(N) = H^2(N) \), and therefore we have that
\[
R = 3G'(N) + 12G(N). \tag{2.12}
\]
This relation is very convenient, since it allows us to determine \( N(R) \). In particular, the Hubble parameter (2.9) can be written in terms of the scale factor as
\[
H = \frac{2}{c^2} a^{-\frac{c^2}{4}}, \tag{2.13}
\]
and thus eliminating \( a \) in favor of \( N \) using (2.10) we obtain
\[
G(N) = Ae^{-c^2N} \tag{2.14}
\]
where we have set \( A = \frac{4}{c^2}a_0^{-c^2} \). Hence, by combining relations (2.12) and (2.14), we can acquire the e-fold parameter \( N \) as a function of \( R \) as
\[
N = \frac{1}{c^2} \ln \left[ \frac{R}{3A(4 - c^2)} \right]. \tag{2.15}
\]
Inserting these in the Friedmann equation (2.11), we can re-express it as
\[
-9G(N(R)) \left[ 4G'(N(R)) + G''(N(R)) \right] F''(R)
+ \left[ 3G(N) + \frac{3}{2} G'(N(R)) \right] F'(R) - \frac{F(R)}{2} + \kappa^2 \rho_{\text{tot}} = 0, \tag{2.16}
\]
with \( G'(N) = \frac{dG(N)}{dN} \) and \( G''(N) = \frac{d^2G(N)}{dN^2} \). The last step is to express the total matter energy density \( \rho_{\text{tot}} \) in terms of \( N(R) \). Assuming the various matter components \( \rho_i \) to be independently conserved, i.e. \( \dot{\rho}_i + 3H(1 + w_i)\rho_i = 0 \), with \( w_i \) their corresponding equation-of-state parameters, we obtain
\[
\rho_{\text{tot}} = \sum_i \rho_0 a_0^{-3(1+w_i)} e^{-3N(R)(1+w_i)}. \tag{2.17}
\]
Hence, inserting all the above in the Friedmann equation (2.16), we obtain a differential equation for \( F(R) \), namely
\[
a_1 R^2 \frac{d^2F(R)}{dR^2} + a_2 R \frac{dF(R)}{dR} - \frac{F(R)}{2} + \sum_i S_i R^{3(1+w_i)} = 0, \tag{2.18}
\]
where we have set
\[
a_1 = \frac{c^2}{4 - c^2},
\]
\[
a_2 = \frac{2 - c^2}{2(4 - c^2)}, \tag{2.19}
\]
and
\[
S_i = \frac{\kappa^2 \rho_0 a_0^{-3(1+w_i)}}{[3A(4 - c^2)]^{\frac{3(1+w_i)}{c^2}}}. \tag{2.20}
\]
The solution of (2.18) provides the exact $F(R)$ form that produces the superbounce evolution (2.8).

Let us first consider pure $F(R)$ gravity, with no matter fluids present. In this case the differential equation (2.18) becomes a homogeneous Euler second-order differential equation, with solution

$$F(R) = c_1 R^{\rho_1} + c_2 R^{\rho_2},$$

(2.21)

where $c_1, c_2$ are arbitrary parameters, and $\rho_1$ and $\rho_2$ are equal to

$$\rho_1 = \frac{-(a_2 - a_1) + \sqrt{(a_2 - a_1)^2 + 2a_1}}{2a_1},$$

$$\rho_2 = \frac{-(a_2 - a_1) - \sqrt{(a_2 - a_1)^2 + 2a_1}}{2a_1}.$$  

(2.22)

Finally, it is worthy to examine if this $F(R)$ gravity can become Einstein gravity plus curvature terms under some parameter values. This indeed occurs when $c \gg 1$, in which case the $F(R)$ gravity takes the form,

$$F(R) \simeq c_1 R + c_2 R^{-1/2}.$$  

(2.23)

We now come to the more physically interesting case where matter fields are present. In this case, the general solution of (2.18) is easily found to be

$$F(R) = \left[ \frac{c_2 \rho_1}{\rho_2} + \frac{c_1 \rho_1}{\rho_2 (\rho_2 - \rho_1 + 1)} \right] R^{\rho_2 + 1} + \sum_i \left[ \frac{c_1 S_i}{\rho_2 (\delta_i + 2 + \rho_2 - \rho_1)} \right] R^{\delta_i + 2 + \rho_2} - \sum_i B_i c_2 R^{\delta_i + \rho_2} + c_1 R^{\rho_1} + c_2 R^{\rho_2}$$

(2.24)

where $c_1, c_2$ arbitrary parameters, and where we have set

$$\delta_i = \frac{3(1 + w_i) - 2c^2}{c^2} - \rho_2 + 2,$$

$$B_i = \frac{S_i}{\rho_2 \delta_i}.$$  

(2.25)

It is interesting to note that for $c \gg 1$ the above $F(R)$ gravity, responsible for the superbounce generalization in the present pf matter fields, becomes

$$F(R) \simeq R + \alpha R^2 + c_1 R^{-1/2} + \Lambda$$

(2.26)

where we have set $c_2 = 1$, $\alpha = \frac{c^2}{3} - 2 + 2 c_1 \sum_i \frac{S_i}{c^2}$ and $\Lambda = - \sum_i A_i$.

The $F(R)$ gravity (2.26) can describe both early and late time acceleration, since at early times is approximately equal to $R + \alpha R^2$ (Starobinsky inflation) and at late times $R - R^{-1/2} + \Lambda$. For a detailed study of such $F(R)$ forms, and the phenomenological determination of the involved parameters, see [68, 116]).

Let us close this subsection by a remark. In reference [112] a different reconstruction procedure was used in order to obtain, in the large and small curvature limits, the $F(R)$ gravity that produces the superbounce cosmological solution. The resulting $F(R)$ gravity
in the large $R$ limit is quite similar to the one we obtained here, namely of the form $R + aR^2$. This was expected to some extent, since the two reconstruction methods overlap at certain limits. This overlap however, is in some cases, model dependent and it is strongly affected by the particular form of the given Hubble parameter, and in particular from the limit of the Hubble parameter at large and small cosmological times. Later on we shall discuss on this issue in more detail.

### 2.3 Loop quantum cosmology ekpyrotic scenario reconstruction from $F(R)$ gravity

One quite appealing alternative to standard inflation is provided by the ekpyrotic scenario \[18–22\], in which scale invariant perturbations are generated before the Big-Bang phase. A recent refinement of the ekpyrotic scenario was presented in \[44\], in which case LQC modifications to the original ekpyrotic scenario are taken into account \[35–44\]. In the context of LQC, the ekpyrotic scenario is realized by using the scalar field potential \[44\]

$$V(\phi) = -\frac{V_0 e^{\sqrt{16\pi G \rho_c} \sqrt{\phi}}}{\left[1 + \frac{3V_0}{4\sqrt{16\pi G \rho_c}} e^{\sqrt{16\pi G \rho_c} \sqrt{\phi}}\right]^2},$$

(2.27)

with the parameter $\rho$ taking values $0 < \rho \ll 1$. As we shall explicitly show in the following Sections, the fact that $\rho \ll 1$ has particularly appealing consequences with respect to early and late time cosmological phenomenology, that are absent in other bouncing solutions, as for instance in the matter bounce scenario. Since this issue is very important, we shall thoroughly discuss it in the end of this Section.

In LQC ekpyrotic scenario the corresponding scale factor and the Hubble parameter are equal to

$$a(t) = \left(a_0 t^2 + 1\right)^{\rho/2}, \quad H(t) = \frac{2a_0 \rho t}{a_0 t^2 + 1},$$

(2.28)

with $\rho_c$ the critical density and $a_0 = \frac{8\pi G \rho_c}{3 \rho_c}$. It is the purpose of this Section to find the $F(R)$ gravity that generates the cosmology described by relations (2.28). Following closely the lines of the previous Section, in the case at hand the Hubble parameter as a function of the scale factor is given by

$$H^2 = 4\rho^2 a_0 \left(\frac{a}{a_0} - \frac{a_0}{a}\right),$$

(2.29)

and by using (2.10) and also the function $G(N) = H^2(N)$, we obtain

$$G(N) = A \left(\frac{4}{3} \rho c e^{\frac{6N}{\rho}} - e^{\frac{4N}{\rho}}\right),$$

(2.30)

where for simplicity we have set the initial scale factor equal to one and also $A = 4\rho^2 a_0$. Thus, using relations (2.12) and (2.30) we can express the e-fold number $N$ as a function of the Ricci scalar $R$ as

$$N = \frac{\rho}{2} \ln \left\{\frac{6A^2 (3 + 2\rho) \left[-32A(1 + \rho)^3 - 9R(3 + 2\rho)^2 + M_1\right]^{1/3}}{-82^{2/3} A^2 (1 + \rho)^2 + 4A^2 (1 + \rho) \left[-32A(1 + \rho)^3 - 9R(3 + 2\rho)^2 + M_1\right]^{1/3}}\right\},$$

(2.31)
with $M_1 = 3\sqrt{R\rho(3 + 2\rho)^2 [64A(1 + \rho)^3 + 9R\rho(3 + 2\rho)^2]}$.

By substituting expression (2.31) into equation (2.11), we acquire

$$A_1(\rho, R)\frac{d^2F(R)}{dR^2} + A_2(\rho, R)\frac{dF(R)}{dR} - \frac{F(R)}{2} + \rho_m = 0$$

(2.32)

with $\rho_m$ encompassing the contribution of all matter fluids to matter-energy density, and where the coefficients $A_i(\rho, R)$, $i = 1, 2$ read

$$A_1(\rho, R) = -\left[432A^4(3 + 2\rho)^5M_1^{10/3}\right]^{-1}\left[2^{11/3}A^2(1 + \rho)^2 - 4A(1 + \rho)M_2^{1/3} + 2^{1/3}M_2^{2/3}\right]^4 \times \left\{128A^4(1 + \rho)^4 - 12A(1 + \rho)A^2M_1 - 3A^2M_1(2M_2)^{1/3} + A^2(2M_2)^{1/3}\left[9R\rho(3 + 2\rho)^2 - 8(1 + \rho)^2(2M_2)^{1/3}\right] + 4A^3(1 + \rho)\left[9R\rho(3 + 2\rho)^2 + 8(1 + \rho)^2(2M_2)^{1/3}\right]\right\}$$

and

$$A_2(R, \rho) = \left[72A^2(3 + 2\rho)^3M_2^{5/3}\right]^{-1}\left[2^{11/3}A^2(1 + \rho)^2 - 4A(1 + \rho)M_2^{1/3} + 2^{1/3}M_2^{2/3}\right]^2 \times \left\{2A^3[12 + \rho(13 + 4\rho)][32A(1 + \rho)^3 + 9R\rho(3 + 2\rho)^2 - 3M_1] - 3(3 + \rho)A^2M_1(2M_2)^{1/3} + 32A^3(1 + \rho)^3(3 + \rho)(2M_2)^{1/3} + A^2(3 + \rho)(2M_2)^{1/3}\left[9R\rho(3 + 2\rho)^2 - 8(1 + \rho)^2(2M_2)^{1/3}\right]\right\}$$

(2.33)

(2.34)

with $M_1 = 3\sqrt{R\rho(3 + 2\rho)^2 [64A(1 + \rho)^3 + 9R\rho(3 + 2\rho)^2]}$ and $M_2 = 3A^2M_1 - 32A^3(1 + \rho)^3 - 9A^2R\rho(3 + 2\rho)^2$.

Solving explicitly the differential equation (2.32) is a challenging and rather formidable task, and thus in order to proceed we shall examine certain limits. Particularly, we shall study the large and small cosmological time limits of the scale factor (2.28), and by applying the same reconstruction technique we shall investigate which of the corresponding limits. Note that the large and small time limits correspond to small and large curvature limits respectively.

### 2.3.1 Large-time approximation of LQC ekpyrotic scenario

Let us start with the large-time limit. In this case the scale factor and the Hubble parameter of (2.28) become

$$a(t) \approx a_0t^\rho, \quad H(t) \approx \frac{\rho}{t}$$

(2.35)

and therefore the function $G(N) = H^2$ writes as

$$G(N) = \rho^2 e^{-\frac{2N}{\rho}}.$$  

(2.36)
Inserting (2.36) into equation (2.12) and solving with respect to \( R \) we obtain
\[
N = \frac{\rho}{3} \ln \left[ \frac{6\rho(2\rho - 1)}{R} \right].
\] (2.37)

Hence, substituting (2.37) into equation (2.16), we finally acquire the following differential equation:
\[
\left( \frac{1}{2\rho - 1} \right) R^2 F''(R) + \left( \frac{\rho - 1}{4\rho - 2} \right) RF'(R) - \frac{F(R)}{2} + \sum_i \rho_0 S_i R^\alpha_i = 0,
\] (2.38)
with
\[
S_i = \left[ 6a_0 \rho^2 (2\rho - 1) \right] \frac{3(1 + w_i)}{2},
\]
\[
\alpha_i = \frac{3(1 + w_i)\rho}{2}.
\] (2.39)

The differential equation (2.38) is the Euler non-homogeneous differential equation (2.18) which we came across in the previous subsection, thus in pure \( F(R) \) gravity (disregarding all matter fluids), we get the following solution:
\[
F(R) = c_1 R^{\rho_1} + c_2 R^{\rho_2},
\] (2.40)
with \( c_1, c_2 \) arbitrary parameters and with \( \rho_1 \) and \( \rho_2 \) given by
\[
\rho_1 = \frac{3 - \rho - \sqrt{1 + 10\rho + \rho^2}}{4}, \quad \rho_2 = \frac{3 - \rho + \sqrt{1 + 10\rho + \rho^2}}{4}.
\] (2.41)

Since by construction in the context of the LQC ekpyrotic scenario it is required \( 0 < \rho \ll 1 \), the solution (2.40) is approximated to
\[
F(R) = c_1 R + c_2 R^{1/2}.
\] (2.42)

By taking into account the presence of ordinary matter fields, and using the technique we employed in the previous subsection, the reconstructed \( F(R) \) gravity which is solution of (2.38) reads
\[
F(R) = c_1 \sum_i \frac{S_i}{\rho_2 (\alpha_i - \rho_2)} R^{\alpha_i} - \frac{c_1 \rho_1}{\rho_2 (\rho_2 - \rho_1 + 1)} R^{\rho_2 + 1} + c_1 R^{\rho_1} \]
\[- \sum_i \frac{S_i c_2}{\rho_2 (\alpha_i - \rho_2)} R^{\alpha_i} + \frac{\rho_1 c_2}{\rho_2} R^{\rho_2 + 1} + c_3 R^{\rho_2},
\] (2.43)
and therefore for \( 0 < \rho \ll 1 \) it becomes
\[
F(R) = c_1 R + \frac{\rho_1}{\rho_2} \left( \frac{3c_2 - 2c_1}{3} \right) R^2 + c_2 R^{1/2} + \Lambda,
\] (2.44)
with \( \Lambda = \sum_i S_i (c_2 - 2c_1) \). Thus, by looking relations (2.44) and (2.42), we deduce that the effect of the matter fluids is to introduce a cosmological constant plus subdominant curvature corrections. This result is quite similar to the reconstructed \( F(R) \) which produces the superbounce solution given in relation (2.26), but now the late-time behavior is different. However, note that relation (2.26) gives the superbounce generating \( F(R) \) gravity for all values of the Ricci curvature \( R \), whilst (2.44) is valid only for small values of the curvature (large cosmological times). Thereby, the result (2.26) has a larger validity region and moreover is phenomenologically more appealing.
2.3.2 Small-time approximation of LQC ekpyrotic scenario

In the small-time region the LQC ekpyrotic scale factor and Hubble parameter \((2.28)\) approximately becomes

\[
a(t) \approx 1 + \frac{a_0 \rho t^2}{2}, \quad H(t) \approx a_0 \rho t
\]

and therefore, using relations \((2.10)\) and \((2.12)\), the e-fold parameter \(N\) can be expressed as a function of the Ricci scalar \(R\) through

\[
N = \ln \left( \frac{24a_0 \rho + R}{30a_0 \rho} \right).
\]  

(2.46)

Therefore, the differential equation \((2.11)\), without taking into account any matter fluids, can be cast in the following way

\[
2 \left( 72a_0^2 - 18a_0 \rho R - R^3 \right) F''(R) + 3(4a_0 \rho + R)F'(R) - 5F(R) = 0.
\]  

(2.47)

This differential equation can be solved using hypergeometric functions, namely

\[
F(z) = \binom{2}{2} F_1(\alpha, \beta, \gamma; z)
\]

where

\[
z = \frac{6a_0 \rho - R}{30a_0 \rho}
\]

(2.49)

with \(\lambda_1 = 6a_0 \rho\) and \(\lambda_2 = -24a_0 \rho\), and where \(\alpha\), \(\beta\) and \(\gamma\) satisfy

\[
\alpha \beta = \frac{5}{2}, \quad \alpha + \beta + 1 = -\frac{3}{2}, \quad \gamma = -\frac{1}{2},
\]  

(2.50)

which accepts two sets of solutions, namely

\[
\alpha = \frac{1}{4} \left( -5 \pm \sqrt{65} \right), \quad \beta = \frac{1}{4} \left( -5 \mp \sqrt{65} \right).
\]  

(2.51)

Since we are in the large curvature limit, we can expand the above general solution in terms of \(1/R\), obtaining

\[
F(R) \simeq R^{-\alpha} \left( 30a_0 \rho \right)^\alpha \left\{ \frac{R \Gamma[\beta - \alpha] \Gamma[\gamma] - \alpha \Gamma[\beta - \alpha - 1] \Gamma[\gamma]}{R \Gamma[\beta] \Gamma[\gamma - \alpha]} \right. \\
\left. + 36a_0^2 \rho^2 \left\{ \alpha(1 + \alpha) \Gamma[-\alpha + \beta] \Gamma[\gamma] \right\} \times \left[ (1 - \beta + \gamma)(\gamma - \beta) - 8(1 + \alpha - \gamma)(\gamma - \beta) + 16(1 + \alpha - \gamma)(2 + \alpha - \gamma) \right] \\
\right\} \frac{1}{2(1 + \alpha - \beta)(2 + \alpha - \beta) \Gamma[\beta] \Gamma[\gamma - \alpha] R^2} + O \left( \frac{1}{R^3} \right).
\]

(2.52)

Hence, given the values of \(\alpha\) and \(\beta\) from \((2.51)\), we deduce that the most dominant term is the first one, namely

\[
F(R) \simeq R^{-\alpha} \left( 30a_0 \rho \right)^\alpha \left\{ \frac{\Gamma[\beta - \alpha] \Gamma[\gamma]}{\Gamma[\beta] \Gamma[\gamma - \alpha]} \right\}.
\]

(2.53)
Let us comment here that in [112] it was shown that the large-$R$ reconstructed gravity which generates the LQC ekpyrotic scenario is of a different form compared to the above relation. The source of the difference is that in the present work we use a more accurate limiting behavior for the small-time scale factor (namely relation (2.45)) comparing to [112]. It is worth discussing here in brief this issue before proceeding. Apart from the method we used above, there is another well-known reconstruction, firstly developed in [64], where the reconstruction is achieved using an auxiliary scalar field. We refer to this technique as the “auxiliary-field reconstruction” hereafter, while we refer to the method of the present work as “non-auxiliary-field reconstruction”.

In reference [112] the auxiliary-field reconstruction was used in order to face the same issue, namely to extract $F(R)$ gravities produce the superbounce and LQC ekpyrotic scenario. Concerning the superbounce with ordinary matter fluids present, both in the large and small curvature limits, the results are the same to the one obtained in this work. On the other hand, concerning the LQC ekpyrotic expansion, in the small-curvature limit the two methods give the same results, while in the large-curvature limits a difference appears. Let us pinpoint the reason why there is this difference between the two approaches and explain it in detail. The reason behind this difference is traced in the particular form of the scale factor and Hubble parameter and also their limits when $t$ tends to zero, a case which corresponds to large curvatures. Specifically, the scale factor of the LQC ekpyrotic scenario given in relation (2.28), when $t$ tends to zero is actually equal to one, that is,

$$\lim_{t \to 0} a(t) = 1$$  \hspace{1cm} (2.54)

and also the limit of the Hubble parameter (2.28) is zero, that is,

$$\lim_{t \to 0} H(t) = 0$$  \hspace{1cm} (2.55)

This is the result that is taken into account in the auxiliary field reconstruction technique and is materialized explicitly in reference [112], resulting to a different $F(R)$ gravity in the large curvature limit (see also [117] for similar results).

In the approach we adopted in this article however, if we take that the limits for the scale factor and Hubble parameter are those given in relations (2.54) and (2.55), then the reconstruction technique yields trivial results or simply does not work, because $G(N)$ would be identically equal to zero. So we needed an exact functional dependence of the scale factor and of the Hubble parameter, with respect to cosmic time $t$, in order the method works perfectly. This is why we expanded the scale factor in power series of $t$ and we kept the leading order terms, so that in the end, the scale factor and Hubble parameter read,

$$a(t) \sim 1 + \frac{a_0 \rho t^2}{2}, \quad H(t) \sim a_0 \rho t$$  \hspace{1cm} (2.56)

In effect, we could say that in this particular case the auxiliary field method is a bit more accurate, to leading order at least. This artificial difference is traced on the particular form of the scale factor and of the Hubble parameter, but if someone is able to solve the differential equation without taking any limiting cases (a rather formidable task), then
there would be no difference between the two approaches. An exemplification to support this argument is the superbounce case which we were able to solve analytically, in which case we had almost absolute concordance. In conclusion, both reconstruction techniques lead to the same results, but in the case that the present technique is used, extra caution is required if the differential equation that yield the reconstructed $F(R)$ gravity, namely equation (2.16), is difficult to solve analytically. In such a case, both methods must be used in order the absolute reliability of the results is ensured.

3 Superbounce and loop quantum ekpyrotic cosmology from $F(G)$ gravity

In this Section we will reconstruct the $F(G)$ form that can lead to the superbounce and loop quantum ekpyrotic cosmology realizations. Before analyzing the reconstruction procedure, in the following subsection we briefly review the basic features of $F(G)$ gravity and we describe the geometric background we will use in its cosmological application. For a detailed account on these issues, see [52] and references therein.

3.1 $F(G)$ gravity and cosmology

Let us review briefly the $F(G)$ gravitational modification and its cosmological application. This class of modified gravity is based on the use of the Gauss-Bonnet combination $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, with $R_{\mu\nu\rho\sigma}$ and $R_{\mu\nu}$ the Riemann and Ricci tensors respectively [72–76]. In particular, one can construct gravitational modifications using an arbitrary function $F(G)$, which prove to lead to interesting cosmological behavior [77–84].

The $F(G)$ action takes the form [77, 78]

$$S = \frac{1}{2\kappa^2} \int dx^4 \sqrt{-g} [R + F(G)] + S_m,$$

with $\kappa^2 = 8\pi G$ is the gravitational constant, and where we have also considered the action of the matter sector $S_m$. The corresponding field equations for a general metric write as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}F(G) + \left(2RR_{\mu\nu} - 4R_{\mu\rho}R^\rho_\nu + 2R_{\mu\nu\rho\sigma}R^{\rho\sigma} - 4g^{\alpha\rho}g^{\beta\sigma}R_{\mu\lambda\nu\beta}R_{\rho\sigma}\right)F'(G)$$

$$+ 4 \left[\nabla_\rho \nabla_\sigma F'(G)\right] R^\rho_\mu - 4g_{\mu\nu} \left[\nabla_\rho \nabla_\sigma F'(G)\right] R^{\rho\sigma} + 4 \left[\nabla_\rho \nabla_\sigma F'(G)\right] g^{\alpha\rho}g^{\beta\sigma}R_{\mu\lambda\nu\beta}$$

$$- 2 \left[\nabla_\rho \nabla_\sigma F'(G)\right] R + 2g_{\mu\nu} \left[\Box F'(G)\right] R$$

$$- 4 \left[\Box F'(G)\right] R_{\mu\nu} + 4 \left[\nabla_\mu \nabla_\nu F'(G)\right] R^\rho_\nu = \kappa^2 T^m_{\mu\nu},$$

(3.2)

where $T^m_{\mu\nu}$ is the matter energy-momentum tensor arising from $S_m$. In the case of the FRW metric (2.5), the above equations give rise to the two Friedmann equations

$$6H^2 + F(G) - GF'(G) + 24H^3 \dot{G}F''(G) = 2\kappa^2 \rho_m$$

(3.3)

$$4\dot{H} + 6H^2 + F(G) - GF'(G) + 16H\dot{G} \left(\dot{H} + H^2\right) F''(G)$$

$$+ 8H^2 \dot{G} F''(G) + 8H^2 \dot{G} F''(G) = -2\kappa^2 p_m,$$

(3.4)
where $\rho_m$ and $p_m$ are the matter energy density and pressure respectively. Note that in FRW geometry, the Ricci scalar is given by (2.6), namely $R = 6(2H^2 + \dot{H})$, while $G$ reads

$$G = 24H^2 \left( \dot{H} + H^2 \right).$$

### 3.2 Superbounce Reconstruction from $F(G)$ Gravity

Let us now reconstruct the $F(G)$ form that leads to the superbounce realization. It proves convenient to use the well-known reconstruction technique [82, 83], which is based on the introduction of an auxiliary scalar field $\phi$. This scalar field can be identified with the cosmic time, so from now we assume that $\phi = t$. In this approach, one embeds the information of $F(G)$ into two suitable functions of $t$, namely $P(t)$ and $Q(t)$, as

$$F(G) = P(t)G + Q(t),$$

and hence the gravitational action (3.1) is written as

$$S = \frac{1}{2\kappa^2} \int dx^4 \sqrt{-g} \left[ R + P(t)G + Q(t) \right] + S_m.$$

Variation of this action with respect to $t$ yields

$$\frac{dP(t)}{dt} G + \frac{dQ(t)}{dt} = 0,$$

which can be solved with respect to $t = t(G)$. Thus, substituting the resulting expression to (3.6) we obtain the reconstructed $F(G)$ gravity.

In order to find the suitable functions $P(t)$ and $Q(t)$, we use relation (3.6) as well as the Friedmann equation (3.3), to obtain the following differential equation,

$$Q(t) = -6H^2(t) - 24H^3(t) \frac{dP}{dt}$$

which relates $Q(t)$ with $P(t)$. Thus, combining (3.6) and (3.9) we acquire

$$2H^2(t) \frac{d^2P}{dt^2} + 2H(t) \left[ 2\dot{H}(t) - H^2(t) \right] \frac{dP}{dt} + \dot{H}(t) = 0,$$

which provides the function $P(t)$, and then substituting into (3.9) we obtain $Q(t)$. Finally, from relation (3.8) we extract the function $t = t(G)$, and inserting everything into (3.6) we result to the reconstructed $F(G)$ gravity.

Let us apply the above method in order to extract the $F(G)$ form responsible for the superbounce\(^1\). We begin with the superbounce cosmological expansion, in which the Hubble parameter is of the form (2.9), namely

$$H(t) = \frac{2}{c^2(t - t_*)}.$$

\(^1\)For a relevant work on the production of bouncing cosmologies from $F(G)$ gravity see [84].
Therefore, the differential equation (3.10) becomes
\[
(t - t_*) \frac{d^2 P}{dt^2} - (2 + a) \frac{dP}{dt} - \frac{c^4}{8}(t - t_*) = 0
\]
which is solved as
\[
P(t) = \frac{c^4 t(2t_* - t)}{8(c^2 + 2)}.
\]
Thus, substituting into (3.9) we obtain \(Q(t)\) as
\[
Q(t) = \frac{32(c^2 - 1)}{c^4(c^2 + 2)(t - t_*)^2}.
\]
Combining relations (3.8), (3.13) and (3.14) we obtain the following two solutions of \(t\) as functions of \(G\):
\[
t_1(G) = t_* - \frac{\sqrt{2} [8(2 - 11c^2)G]^{1/4}}{c^2 \sqrt{G}}
\]
\[
t_2(G) = t_* + \frac{\sqrt{2} [8(2 - 11c^2)G]^{1/4}}{c^2 \sqrt{G}}.
\]
Therefore, we have two possible \(F(G)\) gravities, namely \(F_1(G)\) and \(F_2(G)\), that generate the same superbounce solution, and in order to find their explicit form we substitute \(t_1\) and \(t_2\) from (3.15) into (3.6), obtaining
\[
F_1(G) = \frac{c^4 t_*^2 G - 8 \sqrt{2(2 - 11c^2)G}}{8(c^2 + 2)}
\]
\[
F_2(G) = \frac{c^4 t_*^2 G - 4 \sqrt{2(2 - 11c^2)G(1 + G)}}{8(2c^2 + 2)G}.
\]

3.3 Loop quantum cosmology ekpyrotic scenario reconstruction from \(F(G)\) gravity

In this subsection we proceed to the reconstruction of the \(F(G)\) gravity responsible for the LQC ekpyrotic scenario. We shall focus our study in the late-time era, which corresponds to large cosmic time, since \(F(G)\) gravity provides a consistent description of late-time acceleration and dark energy [76]. In the large-time limit, the Hubble parameter (2.28) becomes \(H(t) = \frac{\rho}{t}\), and hence the differential equation (3.10) writes as
\[
2\rho^2 t \frac{d^2 P}{dt^2} - 2\rho^2(2 + \rho) \left[2\frac{\dot{H}(t)}{H^2(t)} - \frac{\dot{H}}{H}\right] \frac{dP}{dt} - \rho t = 0
\]
with solution
\[
P(t) = -\frac{t^2}{4\rho(1 + \rho)},
\]
Inserting (3.18) into (3.9) we get \(Q(t)\) as
\[
Q(t) = \frac{\rho^2}{t^2} + \frac{12\rho^2}{t^2(1 + \rho)},
\]
and using relations (3.8), (3.13) and (3.14) we obtain the following possible solutions:

\[
\begin{align*}
t_1(G) &= -\frac{\sqrt{2} \left[(\rho - 11)\rho^3\right]^{1/4}}{G^{1/4}}, \\
t_2(G) &= \frac{\sqrt{2} \left[(\rho - 11)\rho^3\right]^{1/4}}{G^{1/4}}.
\end{align*}
\] (3.20)

Hence, we result to two possible \( F(G) \) functions that generate the LQC ekpyrotic contraction, namely

\[
\begin{align*}
F_1(G) &= -\frac{\sqrt{G} \sqrt{(\rho - 11)}\rho^4}{\rho + \rho^2} \\
F_2(G) &= -\frac{(1 + G) \sqrt{(\rho - 11)}\rho^4}{2\sqrt{G}\rho(1 + \rho)}.
\end{align*}
\] (3.21)

4 Superbounce and loop quantum ekpyrotic cosmology from \( F(T) \) gravity

In this Section we will repeat the above reconstructions in the case of \( F(T) \) gravity, i.e. we desire to reconstruct the \( F(T) \) forms that can lead to the superbounce and loop quantum ekpyrotic cosmology realizations. We start our analysis by a brief introduction to \( F(T) \) gravity.

4.1 \( F(T) \) gravity and cosmology

In general, in the torsional formulations of gravity, one uses as dynamical fields the vierbeins \( e^A_\mu \) (we use Greek indices for the coordinate space and capital Latin indices for the tangent space-time), which at each point of space time form an orthonormal basis for the tangent space. The metric tensor is obtained from the dual vierbein through

\[
g_{\mu\nu}(x) = \eta_{AB} e^A_\mu(x) e^B_\nu(x),
\]

with \( \eta_{AB} = \text{diag}(1, -1, -1, -1) \). Contrary to General Relativity and its curvature-based extensions (\( F(R) \), \( F(G) \) etc), where one uses the torsionless Levi-Civita connection (2.1), in the present framework we use the curvatureless Weitzenböck connection defined as \( \Gamma^\lambda_{\nu\mu} \equiv e^A_\lambda \partial_\mu e^A_\nu \) [118]. Hence, all the gravitational information is embedded in the torsion tensor, written as

\[
T^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu} = e^A_\lambda (\partial_\mu e^A_\nu - \partial_\nu e^A_\mu).
\]

Similarly, the contorsion tensor reads as \( K^\mu^\nu_\rho \equiv -\frac{1}{2} \left(T^{\mu_\nu}_\rho - T^{\nu_\mu}_\rho - T^{\rho_\mu}_\nu\right) \), while we additionally define \( S^{\mu_\nu}_\rho \equiv \frac{1}{2} \left(K^{\mu_\nu}_\rho + \delta^\rho_\mu T^{\rho_\nu}_\alpha - \delta^\rho_\nu T^{\rho_\mu}_\alpha\right) \). One can now construct the torsion scalar through contractions of the torsion tensor (similarly to the construction of the Ricci scalar from the Riemann tensor) as [119–121]

\[
T = \frac{1}{4} T^{\rho\nu}_\mu T_{\rho\mu} + \frac{1}{2} T^{\rho_\mu_\nu} T^{\rho\nu}_\mu - T^{\rho\mu}_\rho T^{\rho_\nu}_\nu.
\]

\[– 15 –\]
In the simple, teleparallel equivalent of General Relativity, the Lagrangian is just \( T \).

However, inspired by the \( F(R) \) extensions, one can extend \( T \) to a function \( F(T) \), and thus the action in the \( F(T) \) modified gravity writes as:

\[
S = \frac{1}{2\kappa^2} \int d^4x e [F(T)] + S_m, \tag{4.4}
\]

where \( e = \det(e^A_\mu) = \sqrt{-g} \), \( \kappa^2 = 8\pi G \) is the gravitational constant, and we use units where the light speed is equal to 1. Note that we have also added the matter action \( S_m \). Therefore, varying the action (4.4) with respect to the vierbeins leads to the field equations:

\[
e^{-1} \partial_\mu (e^A_\mu S_\rho^\mu \nu) F_T + e^A_\lambda S_\rho^\mu \nu \partial_\mu (T F_T) - F_T e^A_\lambda T^\rho_\mu \lambda S_\rho^\nu \mu + \frac{1}{4} e^A_\lambda F(T) = \frac{\kappa^2}{2} e^A_\nu T^\mu_\nu, \tag{4.5}
\]

with \( F_T = \partial F/\partial T \), \( F_{TT} = \partial^2 F/\partial T^2 \), and \( T^\mu_\nu \) denoting the usual energy-momentum tensor.

Note the great advantage of \( F(T) \) gravity, namely that its field equations are of second order, while in \( F(R) \) gravity they are of fourth order.

In order to apply \( F(T) \) gravity in a cosmological framework, we have to consider an ansatz for the vierbeins that corresponds to the FRW metric (4.5), namely

\[
e^A_\mu = \text{diag}(1, a, a, a). \tag{4.6}
\]

In this case, the field equations (4.5) give the two Friedmann equations as:

\[
\frac{T F_T}{3} - \frac{F}{6} + \frac{\kappa^2}{3} \rho_m = 0 \tag{4.7}
\]

\[
\dot{H}(F_T + 2T F_{TT}) = -\frac{\kappa^2}{2} (\rho_m + p_m), \tag{4.8}
\]

with \( \rho_m \) and \( p_m \) the matter energy density and pressure respectively, and where

\[
T = -6H^2, \tag{4.9}
\]

as it arises from (4.3) for the FRW ansatz (4.6). Hence, the General Relativity equations are obtained for \( F(T) = T \).

**4.2 Superbounce Reconstruction from \( F(T) \) Gravity**

Let us now suitable reconstruct the \( F(T) \) form that can give rise to the superbounce realization [11], i.e. to the scale factor (2.8), namely

\[
a(t) \sim (-t + t_*)^{2/c^2}, \tag{4.10}
\]

with \( t_* \) the big crunch time, and \( c > \sqrt{6} \) the ansatz parameter. As usual we use the e-folding number \( N \) through \( e^{-N} = \frac{a_0}{a} \), and thus for the Hubble parameter we obtain \( H = 2a^{-c^2/2}/2 \). The great advantage of \( F(T) \) reconstruction, comparing to \( F(R) \) reconstruction of subsection 2.2, is that the torsion scalar \( T \) is related straightaway with the Hubble parameter through (4.9), while the Ricci scalar is related to both the Hubble parameter
and its derivative. Hence, in the present case we can immediately express $N$ as a function of $T$ as

\[ N = -\frac{1}{c^2} \ln \left( \frac{T}{A} \right), \]  

(4.11)

with

\[ \tilde{A} = -\frac{24}{c^2} a_0 c^2. \]  

(4.12)

Additionally, using the matter energy density conservation, we can express $\rho_m$ as

\[ \rho_m = \sum_i \rho_i a_0^{-3(1+w_i)} e^{-3N(T)(1+w_i)}. \]  

(4.13)

Inserting these in the first Friedmann equation (4.7) we obtain

\[ T \frac{dF(T)}{dT} - \frac{F(T)}{2} + \sum_i S_i T^{\frac{3(1+w_i)}{c^2}} = 0, \]  

(4.14)

with

\[ S_i = \frac{\kappa^2 \rho_i a_0^{-3(1+w_i)} \tilde{A}^{-\frac{3(1+w_i)}{c^2}}}{c^2}. \]  

(4.15)

The solution of (4.14) provides the exact $F(T)$ form that produces the superbounce evolution.

We can immediately see that (4.14) is much more simple than the corresponding equation (2.18) of the $F(R)$ case. In particular, in the case of absence of matter fields, the solution reads

\[ F(T) = c_1 \left(-T\right)^{\frac{1}{2}}, \]  

(4.16)

where $c_1$ is an arbitrary parameter (note that the presence of the minus sign in front of $T$ was expected, since $T = -6H^2 < 0$ in the conventions used in this work). Hence, simple teleparallel equivalent of General Relativity, without matter fields, cannot give rise to the superbounce evolution. However, when matter fields are present, the corresponding $F(T)$ is found to be

\[ F(T) = c_1 \left(-T\right)^{\frac{1}{2}} - \sum_i Q_i \left(-T\right)^{\frac{3(1+w_i)}{c^2}}, \]  

(4.17)

where

\[ Q_i = \frac{2\kappa^2 c^2 \rho_i a_0^{-3(1+w_i)}}{6(1+w_i) - c^2}. \]  

(4.18)

As we observe, for $c = 3(1+w_i)$ ($c = 3$ for simple dust matter) we find that the superbounce is realized in the case of standard teleparallel equivalent of General Relativity, i.e. standard gravity, plus corrections. Finally, note that the above $F(T)$ form is in general different than the one required for matter bounce realization in $F(T)$ gravity [122].
4.3 Loop quantum cosmology ekpyrotic scenario reconstruction from \(F(T)\) gravity

Let us now reconstruct the \(F(T)\) form that gives rise to the LQC ekpyrotic scenario (see also [123] for a different approach). As we mentioned in subsection 2.3, the desirable scale factor and Hubble parameter should have the form (2.28), namely

\[
a(t) = (a_0 t^2 + 1)^{\rho/2}, \quad H(t) = \frac{2a_0 \rho t}{a_0 t^2 + 1},
\]

with \(0 < \rho \ll 1\), and where \(\rho_c\) is the critical density and \(a_0 = \frac{8\pi G \rho_c}{3\rho}\). The Hubble parameter can then be expressed as

\[
H^2 = 4\rho^2 a_0 \left( a^\frac{6}{\rho} - a^\frac{4}{\rho} \right),
\]

and since in \(F(T)\) gravity \(T = -6H^2\) (relation (4.9)), we can immediately find that

\[
T = -24\rho^2 a_0 \left( a^\frac{6}{\rho} - a^\frac{4}{\rho} \right).
\]

Therefore, introducing the e-folding number \(N = \ln(a/a_0)\) and eliminating \(a\) we can express \(N\) as a function of \(T\) through

\[
N(T) = \ln \left\{ \frac{3\rho^2}{a_0} \left[ 1 + \left( \frac{2}{M_2} \right)^{\frac{1}{3}} + \left( \frac{M_2}{2} \right)^{\frac{1}{3}} \right]\right\},
\]

with \(M_2 = 2 - \frac{9T}{8\rho^2 a_0} + \frac{3\sqrt{3} \rho}{\rho^2 a_0} \sqrt{\frac{3}{3} T^2 - \frac{\rho^2 a_0 T}{t}}\). Finally, inserting \(N(T)\) from (4.22) into \(\rho_m\)-equation (4.13) and then into the Friedmann equation (4.7) we obtain

\[
T \frac{dF(T)}{dT} - \frac{F(T)}{2} + \sum_i X_i \left[ 1 + \left( \frac{2}{M_2} \right)^{\frac{1}{3}} + \left( \frac{M_2}{2} \right)^{\frac{1}{3}} \right]^{-\frac{3\rho(1+w_i)}{2}} = 0,
\]

with

\[
X_i = 3\frac{2^{\rho(1+w_i)}}{2} \kappa^2 \rho_{00}.
\]

The solution of the above differential equation will give the \(F(T)\) form that generates the LQC ekpyrotic scenario. Unfortunately, this equation cannot be solved analytically, and hence, similarly to subsection 2.3, we solve it in the large and small time phases separately.

4.3.1 Large-time approximation of LQC ekpyrotic scenario

At large times the required scale factor (4.19) becomes \(a(t) \approx a_0 t^\rho\), and thus \(H^2 = \rho^2 (a_0/a)^{2/\rho}\). Thus, the \(N(T)\) expression is significantly simplified, and it reads

\[
N(T) = \frac{\rho}{2} \ln \left( \frac{-6\rho^2}{T} \right).
\]

Hence, inserting \(N(T)\) from (4.25) into \(\rho_m\)-equation (4.13) and then into the Friedmann equation (4.7) we obtain

\[
T \frac{dF(T)}{dT} - \frac{F(T)}{2} + \sum_i Y_i (-T)^{\alpha_i} = 0,
\]

with
with
\[
Y_i = \kappa^2 \rho_0 a_0^{-3(1+w_i)} (6\rho^2)^{-\frac{3(1+w_i)}{2}} \\
\alpha_i = \frac{3(1+w_i)}{2}. \tag{4.27}
\]

Equation (4.26) can be easily solved as:
\[
F(T) = c_1 (-T)^{\frac{1}{2}} - \sum_i \frac{2Y_i}{2\alpha_i - 1} (-T)^{\alpha_i}, \tag{4.28}
\]
which is the form of \( F(T) \) gravity that can generate a LQC ekpyrotic scenario at large times.

### 4.3.2 Small-time approximation of LQC ekpyrotic scenario

At small times the required scale factor (4.19) becomes \( a(t) \approx 1 + \frac{a_0 \rho t}{2} \), and thus \( H(t) \sim a_0 \rho_{\text{t}} \). Thus, the \( N(T) \) expression becomes
\[
N(T) = \ln \left( \frac{1}{a_0} - \frac{T}{12\rho a_0^2} \right). \tag{4.29}
\]

Therefore, inserting \( N(T) \) from (4.29) into \( \rho_{\text{m}} \)-equation (4.13) and then into the Friedmann equation (4.7) we obtain
\[
T \frac{dF(T)}{dT} - \frac{F(T)}{2} + \sum_i Z_i \left( 1 - \frac{T}{12a_0 \rho} \right)^{\alpha_i} = 0, \tag{4.30}
\]
with
\[
Z_i = \kappa^2 \rho_0, \\
\alpha_i = -3(1 + w_i). \tag{4.31}
\]

Equation (4.30) has the solution
\[
F(T) = c_1 (-T)^{\frac{1}{2}} + 2 \sum_i Z_i \ 2F_1 \left( \frac{-1}{2}; -\alpha_i; \frac{1}{2}; \frac{T}{12a_0 \rho} \right), \tag{4.32}
\]
with \( 2F_1(a; b; c; z) \) the hypergeometric function. Hence, this is the form of \( F(T) \) gravity that can generate a LQC ekpyrotic scenario at small times. Finally, since at early times we are in the large torsion regime \( T \gg 1 \), we can approximate the above solution as
\[
F(T) \approx c_1 (-T)^{\frac{1}{2}} + 2 \sum_i Z_i \left\{ \sqrt{\frac{6\rho_{\text{m}}}{12a_0 \rho}} \frac{\Gamma[\frac{1}{2} - \alpha_i]}{\Gamma[-\alpha_i]} (-T)^{\frac{1}{2}} + \frac{1}{1 - 2\alpha_i} (-T)^{\alpha_i} \right\}. \tag{4.33}
\]
5 Stability of $F(R)$, $F(G)$ and $F(T)$ superbounce and LQC ekpyrosis

In the previous Sections we investigated the superbounce and LQC ekpyrosis in the framework of $F(R)$, $F(G)$ and $F(T)$ modified gravities. A crucial issue in any cosmological evolution is whether at the perturbation level it exhibits instabilities, which thus could constrain or exclude the scenario. In this Section, we perform such stability investigation, by linearly perturbing the above solutions. Our analysis for the $F(R)$ case, is performed in both pure gravity and gravity in the presence of matter fluids cases, while for the $F(G)$ and $F(T)$ cases, only the pure gravity case is considered, just to obtain a first picture of the resulting behavior.

5.1 Stability in the $F(R)$ reconstructions

In the context of the reconstruction method [63], the stability study was performed in detail in [124], the notation and formalism of which we adopt in this Section. The stability of the solutions which we obtained in the previous Sections, can be examined by performing linear perturbations of the $F(R)$ solution and specifically of the auxiliary function $G(N)$. We consider its perturbation to be of the form [124]

$$G(N) = g(N) + \delta g(N), \quad (5.1)$$

and we insert it in the first FRW equation (2.16). Since the background quantity $g(N)$ satisfies exactly equation (2.16), we easily acquire the equation for the perturbation $\delta g(N)$, namely

$$g(N) \frac{d^2 F(R)}{dR^2} \bigg|_{R=R_1} \delta'' g(N) + \left\{ 3g(N) \left[ 4g'(N) + g''(N) \right] \frac{d^3 F(R)}{dR^3} \bigg|_{R=R_1} + \left[ 3g(N) - \frac{1}{2} g'(N) \right] \frac{d^2 F(R)}{dR^2} \bigg|_{R=R_1} \right\} \delta' g(N) + \left\{ 12g(N) \left[ 4g'(N) + g''(N) \right] \frac{d^3 F(R)}{dR^3} \bigg|_{R=R_1} + \left[ -4g(N) + 2g'(N) + g''(N) \right] \frac{d^2 F(R)}{dR^2} \bigg|_{R=R_1} + \frac{1}{3} \frac{dF(R)}{dR} \bigg|_{R=R_1} \right\} \delta g(N) = 0, \quad (5.2)$$

with $R_1 = 3g'(N) + 12g(N)$. From this equation one may directly extract the stability conditions for the perturbations of $G(N)$. In particular, they read

$$J_1 = \frac{6[4g'(N) + g''(N)]F'''(R)}{F''(R)} + 6 - \frac{g'(N)}{g(N)} > 0 \quad (5.3)$$

and

$$J_2 = \frac{36[4g'(N) + g''(N)]F'''(R)}{F''(R)} - 12 + \frac{6g'(N)}{g(N)} + \frac{3g''(N)}{g(N)} + \frac{F'(R)}{g(N)F''(R)} > 0. \quad (5.4)$$

Let us now examine whether the superbounce and LQC ekpyrotic $F(R)$ forms found in Section 2 satisfy the above stability conditions. Considering the superbounce cosmological
solution (2.21), without the presence of matter field, the stability conditions (5.3) and (5.4) respectively become
\[
J_1 = 6 + c^2 - 2c^2 \left[ 3^{p_1} c_1 Q_0^p_1 (p_1 - 2)(p_1 - 1)q_1 + 3^{p_2} c_2 Q_0^p_2 (p_2 - 2)(p_2 - 1)q_2 \right]
\times \left[ 3^{p_1} c_1 Q_0^p_1 (p_1 - 1) 1 + 3^{p_2} c_2 Q_0^p_2 (p_2 - 1)q_2 \right] \geq 0, \quad (5.5)
\]
with \( Q_1 = -A(c^2 - 4)e^{-c^2 N}, \) and
\[
J_2 = -12 - 6c^2 + 3c^4 + e^{-2N} \left( c_1 Q_0^p_2 (p_1 - 1)q_1 + c_2 Q_0^p_2 (p_2 - 1)q_2 \right) A^{-1} Q_3^{-1}
- 12c^2 Q_2 \left[ c_1 Q_0^p_1 (p_1 - 2)(p_1 - 1)q_1 + c_2 Q_0^p_2 (p_2 - 2)(-1 + p_2)q_2 \right] Q_3^{-1} > 0 \quad (5.6)
\]
with \( Q_2 = 3Ae^{-c^2 N} (4 - c^2) \) and \( Q_3 = c_1 Q_0^p_1 (p_1 - 1)q_1 + c_2 Q_0^p_2 (p_2 - 1)q_2. \) The most interesting subcase that we found in Section 2 was when the parameter \( c \) is large, in which case the above stability conditions become
\[
J_1 = 4c^2 + 6 > 0 \quad (5.7)
J_2 = 3c^4 + 18c^2 - 12 > 0. \quad (5.8)
\]
Hence, in the interesting case of the large-\( c \) regime, the \( F(R) \) superboune is free of instabilities.

Considering now the superboune cosmological solution (2.21), with the presence of the matter field, i.e. the \( F(R) \) form (2.24), the stability conditions (5.3) and (5.4) respectively become
\[
J_1 = 6 + c^2 - 6c^2 Q_1 \left[ 3^{p_1} c_1 (p_1 - 2)(p_1 - 1)q_1 + 3^{p_2} c_2 (p_2 - 2)(p_2 - 1)q_2 \right]
\times \left[ 3^{p_1} c_1 (p_1 - 1)q_1 + 3^{p_2} c_2 (p_2 - 1)q_2 \right] \geq 0, \quad (5.9)
\]
with \( Q_1 = -A(c^2 - 4)e^{-c^2 N}, \) and
\[
J_2 = -12 - 6c^2 + 3c^4 + A (Q_3 + Q_4) \left\{ e^{-2N} \left[ c_1 Q_0^p_2 (p_1 - 1)q_1 + c_2 Q_0^p_2 (p_2 - 1)q_2 \right]
\times \left[ 3^{p_1} c_1 (p_1 - 1)q_1 + 3^{p_2} c_2 (p_2 - 1)q_2 \right] \right\}
+ 36Ae^{-c^2 N} (c^2 - 4) (Q_3 + Q_4)^{-1}
\times \left[ c_1 Q_0^p_1 (p_1 - 2)(p_1 - 1)q_1 + c_2 Q_0^p_2 (p_2 - 2)(p_2 - 1)q_2 \right]
\times \left[ 3^{p_1} c_1 (p_1 - 1)q_1 + 3^{p_2} c_2 (p_2 - 1)q_2 \right] \geq 0, \quad (5.10)
\]
with \( Q_2 = 3Ae^{-c^2N}(4 - c^2) \), \( Q_3 = c_1Q_2^{\alpha_1}(\rho_1 - 1)\rho_1 + c_2Q_2^{\alpha_2}(\rho_2 - 1)\rho_2 \) and \( Q_4 = Q_2^{\alpha_3 - 1}q_1\rho_2(1 + \rho_2) - Q_2^{\alpha_4 + \rho_2 - 2}q_2(\delta_i + \rho_2 - 1)(\delta_i + \rho_2) + Q_2^{\alpha_5 + \rho_2}q_2(1 + \delta_i + \rho_2)(2 + \delta_i + \rho_2) \). Thus, in the large-c regime they become

\[
J_1 \approx 6e^c \quad (5.11)
\]

\[
J_2 \approx c^d \quad (5.12)
\]

We mention that by large \( c \) we practically mean \( c \geq 10 \) and we also recall that \( c > \sqrt{6} \) in order a superbounce exists. From relations (5.11) and (5.12) it is obvious that \( J_1 > 0 \) and also \( J_2 > 0 \), and therefore the superbounce generating \( F(R) \) gravity, with or without the presence of matter, is stable under the perturbation (5.1).

Considering the LQC ekpyrotic \( F(R) \) gravity that we reconstructed in subsection 2.3, we mention that we extracted only approximate solutions in the large and small time regimes. Although the consistent perturbation analysis should be performed in the full expressions, for completeness we examine the stability of the approximate \( F(R) \) form given in (2.53). In this case the stability conditions (5.5) and (5.6) become

\[
J_1 = 1 - 2\alpha + \frac{1}{1 - e^N} + 8(2 + \alpha) > 0 \quad (5.13)
\]

and

\[
J_2 = -12 + \frac{9e^N}{e^N - 1} - \frac{60e^N(2 + \alpha)}{5e^N - 4} - 3e^N + 12\left(e^N - 1\right)\left(1 + \alpha\right) < 0. \quad (5.14)
\]

As we can see, for small e-folding values \( N \) these conditions are not satisfied and thus the approximate solution exhibits instabilities. However, the numerical investigation of the exact solution is necessary before we conclude in the stability of the \( F(R) \) LQC ekpyrotic scenario.

### 5.2 Stability in the \( F(G) \) reconstructions

We now examine the stability in the \( F(G) \) reconstructions of Section 3, repeating the steps of the previous subsection. For brevity we shall study only the superbounce case. The general \( F(G) \) stability analysis was performed in [84], so we adopt the notation of that work. Inserting the perturbation (5.1) into the Friedmann equation (3.3) in the absence of matter fields, we extract the following stability conditions:

\[
\frac{J_2}{J_1} > 0, \quad \frac{J_3}{J_1} > 0 \quad (5.15)
\]

where \( J_1 \) stands for,

\[
J_1 = 288g(N)^3F''(G) \quad (5.16)
\]

\[
J_2 = 432g(N)^2\left\{ (2g(N) + g'(N))F''(G)
+ 8g(N)\left[g'(N)^2 + g(N)(4g'(N) + g''(N))\right]\right\} F''(G) \quad (5.17)
\]

\[
J_3 = 6\left\{ 1 + 24g(N)\left\{ -8g(N)^2 + 3g'(N)^2 + 6g(N)[3g'(N) + g''(N)]\right\}\right\} F''(G)
+ 24g(N)[4g(N) + g'(N)]\left\{ g'(N)^2 + g(N)[4g(N) + g''(N)]\right\} F''(G). \quad (5.18)
\]
In the case of the superbounce $F(G)$ gravity, we found two solutions $F_i(G)$, $i = 1, 2$, given in relation (3.16). Thus, for the function $F_1(G)$ we calculate:

$$\frac{J_2}{J_1} = \frac{3}{2} (c^2 - 2) \left(16A^2c^2e^{-2c^2N} - 1\right) > 0 \quad (5.19)$$

and

$$\frac{J_3}{J_1} = \left(\frac{-99}{11c^6} + 108c^4 + 26c^2 - 8\right) + \frac{8Ac^2e^{-2c^2N}}{2 - 11c^2} + \frac{Ac^2e^{-2c^2N}(11c^2 - 46c^4 + 30c^2 - 4)}{2 - 11c^2} - \sqrt{6(4 - c^4)} \sqrt{\frac{c^2 - 2}{11c^2 - 2}} > 0, \quad (5.20)$$

with $A = \frac{4c_0e^{-c^2}}{c^2}$. Hence the $F_1(G)$ solution is conditionally stable (note however that in the large-$c$ region the first condition is not satisfied and thus instabilities appear).

Similarly, for the case $F_2(G)$ the stability conditions (5.19) and (5.20) write as

$$\frac{J_2}{J_1} = \frac{3}{2} (c^2 - 2) \left(16A^2c^2e^{-2c^2N} - 1\right) \quad (5.21)$$

and

$$\frac{J_3}{J_1} = \frac{4A^2(16 - 60c^2 - 190c^4 + 207c^6)}{Q_a} + \frac{4A^3c^2e^{-2c^2N}(-32 + 60c^2 - 152c^4 + 122c^6 - 68c^8 + 11c^10)}{Q_a} + \frac{Ac^2e^{-2c^2N}(16 - 92c^2 + 30c^4 - 46c^6 + 11c^8)}{Q_a} + \frac{e^{2c^2N}(-8 + 26c^2 + 117c^4 - 99c^6)}{Q_a} + \frac{16\sqrt{6}A^2(2 + c^2)(4 - 4c^2 + c^4)\sqrt{(2 - 11c^2)(c^2 - 2)}}{c^2Q_a}, \quad (5.22)$$

where $Q_a = (2 - 11c^2) \left[4A^2(c^2 - 2) + e^{2c^2N}\right]$. Hence the $F_1(G)$ solution is conditionally stable (note however that in the large-$c$ region the first condition is not satisfied and thus instabilities appear).

### 5.3 Stability in the $F(T)$ reconstructions

Let us now discuss on the stability of the obtained superbounce and LQC ekpyrosis in $F(T)$ gravity. As we have mentioned, a crucial difference, and a main advantage, of $F(T)$ gravity, comparing to $F(R)$, $F(G)$ and other curvature gravitational modifications, is that its equations of motion are of second and not of fourth order. This can be immediately seen in the Friedmann equation (4.7) of $F(T)$ gravity, in which only $H$ and not $\dot{H}$ or $\ddot{H}$ appears, and compare it with the Friedmann equations of $F(R)$ and $F(G)$ cases, equation (2.7) and (3.3) respectively, where all $H$, $\dot{H}$ and $\ddot{H}$ are present. Hence, in order to study the stability under linear perturbations in $F(T)$ gravity solutions, we cannot apply the method we used in the $F(R)$ and $F(G)$ cases, which was based on perturbing the auxiliary function $G(N) \equiv H^2(N)$ as in (5.1), since in this case the results will be trivial. The method used in the $F(R)$
and \( F(G) \) cases is basically speaking a method of perturbations of the dynamical system formed by the FRW equations. In the \( F(T) \) CASE, the stability examination requires to perform the complete perturbation analysis, starting from perturbing the vierbeins, which is easier in this case, compared to the other two. This analysis has been performed in detail in \[87, 88, 102\], and therefore we do not repeat it here. However, using those results, we can easily deduce that the simple power-law \( F(T) \) forms of the superbounce ((4.16) and (4.17)), and LQC ekpyrotic ((4.28) and (4.33)) solutions are stable (power-law solutions and especially the square root ansatz are very common in \( F(T) \) gravity \[85–111\]).

6 Conclusions

In this work we investigated the superbounce and the loop quantum cosmological ekpyrosis, in the framework of various modified gravities. Bouncing solutions can be an alternative to inflation and thus it is both necessary and interesting to see whether it can be obtained naturally, for suitably chosen classes of gravitational modification. We focused our investigation in the \( F(R) \), \( F(G) \) and \( F(T) \) modified gravities and in each case we reconstructed the forms of modification that give rise to superbounce and LQC ekpyrotic evolutions. The reconstruction methods that were used in this work are not the only ones that one could follow. Indeed, depending on the modified gravity that is used, one could use different techniques, which usually lead to the same exact results, or similar results when approximations are used. We will perform a careful comparison and discussion of the various reconstruction methods in a separate publication.

Concerning \( F(R) \) gravity, we found that the superbounce evolution at the large-curvature regime is obtained from \( R + \alpha R^2 \), while at small curvatures from \( R + c_1 R^{-1/2} + \Lambda \), which agrees with the results obtained in \[112\] from a different approach. Similarly, the LQC ekpyrosis is realized from hypergeometric functions that can be approximated by power-law \( F(R) \) forms. Concerning \( F(G) \) gravity, we found that the superbounce and LQC ekpyrosis are obtained from \( F(G) \) forms having \( G, \sqrt{G} \) and \( 1/\sqrt{G} \) factors. Furthermore, in the case of \( F(T) \) gravity, we found that the superbounce is generated by power-law \( F(T) \) forms, while the LQC ekpyrosis from hypergeometric \( F(T) \) functions that can be approximated to power laws. Moreover, performing a linear perturbation analysis, we showed that the obtained solutions are conditionally or fully stable.

It is expected that bouncing cosmology should occur due to quantum gravity effects. In this respect, the important step in it’s realization is the construction of realistic bouncing cosmology within a specific effective gravity theory. In fact, it looks that very similar classes of different effective gravities, i.e. power-law, polynomials and hypergeometric functions, naturally lead to realization of proposed bounce universe. This indicates that indeed quantum gravity, which may give the origin to specific effective \( F(R) \), \( F(G) \) or \( F(T) \) gravity theories, maybe responsible for the occurrence of bouncing universe.

Finally, we mention that a crucial issue is to analyze in detail the processing of perturbations through the bouncing phase in these scenarios, which would lead to extract observable signatures than can be confronted with observations and in particular with the Planck data. This is the main test that would lead to constraining or excluding the sce-
narios at hand, similarly to the various inflationary models. In addition, a very compelling task related to the perturbations analysis is the deviation from scale-invariance is the power spectrum of the perturbations, which would result to a blue or red tilt in the spectrum of tensor perturbations. This necessary analysis lies beyond the scope of the present work, and is left for a future investigation.

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