Maxwell electrodynamics modified by a $CPT$-odd dimension-5 higher-derivative term

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In this paper, we consider an electrodynamics of higher derivatives coupled to a Lorentz-violating background tensor. Specifically, we are interested in a dimension-5 term of the $CPT$-odd sector of the nonminimal Standard-Model Extension. By a particular choice of the operator $kAF$, we obtain a higher-derivative version of the Carroll-Field-Jackiw (CFJ) term, \( \frac{1}{2} \epsilon^{\kappa\lambda\mu\nu} A_\lambda D_\kappa \square F_{\mu\nu} \), with a Lorentz-violating background vector $D_\kappa$. This modification is subject to being investigated. We calculate the propagator of the theory and from its poles, we analyse the dispersion relations of the isotropic and anisotropic sectors. We verify that classical causality is valid for all parameter choices, but that unitarity of the theory is generally not assured. The latter is found to break down for certain configurations of the background field and momentum. In an analog way, we also study a dimension-5 anisotropic higher-derivative CFJ term, written as \( \epsilon^{\kappa\lambda\mu\nu} A_\lambda T_\kappa (T \cdot \partial)^2 F_{\mu\nu} \). Within the second model, purely timelike and spacelike $T_\kappa$ are considered. For the timelike choice, one mode is causal, whereas the other is noncausal. Unitarity is conserved, in general, as long as the energy stays real — even for the noncausal mode. For the spacelike scenario, causality is violated when the propagation direction lies within certain regimes. However, there are particular configurations preserving unitarity and strong numerical indications exist that unitarity is guaranteed for all purely spacelike configurations. The results improve our understanding of nonminimal $CPT$-odd extensions of the electromagnetic sector.

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I. INTRODUCTION

The minimal Standard-Model Extension (SME), which was proposed by V.A. Kostelecký and S. Samuel in 1998 \cite{Kostelecky1998, Samuel1998}, shares various established properties with the Standard Model (SM) such as power-counting renormalizability, energy-momentum conservation, and gauge invariance. However, it does not preserve Lorentz symmetry and, beyond that, it can violate CPT symmetry \cite{CPT}. The SME is an effective field-theory framework obtained from the SM including additional terms composed of observer Lorentz-invariant contractions of the physical SM fields and fixed background tensors. Studies of the SME have been carried out to look for Lorentz-violating (LV) effects and to develop a precision program that may allow us to examine the limitation of Lorentz symmetry in various physical interactions. In this sense, a large number of investigations has been realized in the context of the fermion sector \cite{CPT-violation-fermions, CPT-violation-fermions-2}, CPT symmetry violation \cite{CPT-violation-1}, the electromagnetic CPT-odd sector \cite{CPT-violation-electromagnetic, CPT-violation-electromagnetic-2}, the electromagnetic CPT-even sector \cite{CPT-violation-electromagnetic-3, CPT-violation-electromagnetic-4}, photon-fermion interactions \cite{CPT-violation-photon-fermion-1, CPT-violation-photon-fermion-2, CPT-violation-photon-fermion-3}, and radiative corrections \cite{CPT-violation-radiative-corrections-1, CPT-violation-radiative-corrections-2, CPT-violation-radiative-corrections-3}. Phenomenological and theoretical developments focusing on LV contributions of mass dimensions 3 and 4 have been continuously undertaken in the latest years. As a result, there is now a large number of tight constraints on Lorentz violation, mainly in the photon and lepton sector \cite{CPT-violation-photons-1}.

An alternative route of investigation in electrodynamics includes the possibility of higher derivatives. This idea was first proposed in 1942 by Podolsky \cite{Podolsky1942} who initially studied the Lorentz- and gauge-invariant dimension-6 term, $\theta^2 \partial_\alpha F^{\alpha\beta} \partial_\lambda F^{\lambda\beta}$, with the Podolsky parameter $\theta$ of mass dimension $-1$. This theory exhibits two dispersion relations, the usual one of Maxwell theory and a massive mode that renders the self-energy of a pointlike charge finite. However, at the quantum level, the massive mode produces ghosts \cite{Podolsky-ghosts}. The gauge-fixing condition in this extension can be adapted to be compatible with the 2 degrees of freedom of the photon and the 3 additional ones connected to the massive mode \cite{Podolsky-ghosts-2}. Further developments \cite{Podolsky-ghosts-3, Podolsky-ghosts-4} in Podolsky’s theory deserve to be mentioned.

Another relevant extension of Maxwell theory with higher derivatives is Lee-Wick electrodynamics, described by the dimension-6 term $F^{\mu\nu} \partial_\alpha \partial_\beta F_{\mu\nu}$ \cite{Lee-Wick}. This theory also implies a finite self-energy for a pointlike charge in $(1 + 3)$ spacetime dimensions. It provides a bilinear contribution to the Maxwell Lagrangian that is similar to the usual one of Maxwell theory and a massive mode that makes the self-energy of a pointlike charge finite. However, at the quantum level, the massive mode produces ghosts \cite{Lee-Wick-ghosts}. The gauge-fixing condition in this extension can be adapted to be compatible with the 2 degrees of freedom of the photon and the 3 additional ones connected to the massive mode \cite{Lee-Wick-ghosts-2}. Further developments \cite{Lee-Wick-ghosts-3, Lee-Wick-ghosts-4} in the Lee-Wick theory deserve to be mentioned.

In the latest years, an interplay between Lorentz-violating theories and electrodynamics endowed with higher derivatives has taken place. Indeed, Lorentz violation can incorporate operators of higher mass dimensions, which may include higher-derivative terms. The number of such contributions is infinite in contrast to the minimal LV theory. In this context, general investigations of LV extensions are found in \cite{CPT-violation-general} as well as applications in the self-energy and interaction of pointlike and spatially extended sources \cite{CPT-violation-photons-2, CPT-violation-photons-3, CPT-violation-photons-4, CPT-violation-photons-5}.

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The minimal SME can be interpreted as a zeroth-order correction to the SM. The higher-derivative corrections may become relevant for increasing energy scales. Thus, the nonminimal SME coefficients may be useful for investigating fundamental properties such as causality, stability, and unitarity at higher energies where they can become dominant.

It is important to mention that particular choices of nonminimal coefficients were proposed and investigated in other modifications of electrodynamics including higher-dimensional operators, such as \cite{CPT-violation-higher-dim, CPT-violation-higher-dim-2}. Nonminimal theories containing higher-dimensional couplings can also be constructed without introducing higher derivatives. Such
couplings were considered and constrained initially in [37, 38] and have been proposed recently in broader scenarios [39, 40], also including photon-photon scattering [41], electroweak interactions [42], nuclear chiral interactions [43], and scalar electrodynamics [44].

Lately, modified higher-derivative LV terms have been generated in other ways via quantum corrections to the photon effective action in a scenario with a nonminimal coupling between fermions and photons [45, 46] as well as in supersymmetric theories [47, 48]. Not so long ago, we addressed a CPT-even, dimension-6, higher-derivative electrodynamics, composed of an anisotropic Podolsky term, $\partial_\mu F^{\sigma \beta} \partial_\lambda F^{\lambda \alpha} D_{\beta \alpha}$, and an anisotropic Lee-Wick term, $F_{\mu \nu} \partial_\alpha \partial_\beta F^{\mu \nu} D^{\alpha \beta}$, respectively, with $D^{\alpha \beta}$ representing a LV rank-2 background field. Both models can be mapped onto the nonminimal dimension-6 structure of Eq. (2b). We obtained the associated gauge propagators and examined the dispersion relations for several background tensor configurations. We found that the model exhibits both causal and noncausal as well as both unitary and nonunitary modes [49].

To improve our understanding of higher-derivative LV violating extensions of the photon sector, it is now reasonable to pursue similar investigations of a CPT-odd theory. Therefore, in the present work, we study Maxwell electrodynamics modified by a CPT-odd, dimension-5 nonminimal SME term. In Sec. II, we consider a configuration composed of a fixed background field $D_\kappa$ where the additional four-derivatives are contracted with the metric tensor, i.e., $\epsilon^{\kappa \lambda \mu \nu} A_\lambda D_\kappa \square F_{\mu \nu}$. The gauge propagator is derived and the dispersion relations are obtained from its pole structure. Causality and unitarity of the modes is analysed subsequently. In Sec. III, we consider the dimension-5 term composed of a fixed background field $T_\kappa$ partially contracted with additional four-derivatives, that is, $\epsilon^{\kappa \lambda \mu \nu} A_\lambda T_\kappa (T \cdot \partial)^2 F_{\mu \nu}$. Similar investigations are performed for this more sophisticated modification. Finally, we conclude on our findings in Sec. IV. Studies that are not directly connected to the modifications proposed are relegated to App. A. Natural units will be used with $\hbar = c = 1$, unless otherwise stated.

II. MAXWELL ELECTRODYNAMICS MODIFIED BY A CPT-ODD DIMENSION-FIVE HIGHER-DERIVATIVE TERM: A SIMPLE MODEL

In this work, we are interested in investigating the CPT-odd, dimension-5 extension of the electromagnetic sector, specifically represented by the Carroll-Field-Jackiw-like (CFJ-like) term of the Lagrange density given by

$$\mathcal{L} = \frac{1}{4} \epsilon^{\kappa \lambda \mu \nu} A_\lambda (\hat{k}_{AF})_\kappa F_{\mu \nu},$$

(3)

where the background field is a third-rank observer Lorentz tensor, whose general structure is

$$(\hat{k}_{AF})_\kappa = (k_{AF})_\kappa^{(5)} \alpha_1 \alpha_2 \partial_{\alpha_1} \partial_{\alpha_2},$$

(4)

As a first investigation, we consider the special case

$$(\hat{k}_{AF})_\kappa = \hat{D}_\kappa X^{\alpha_1 \alpha_2} \partial_{\alpha_1} \partial_{\alpha_2},$$

(5)

with a Lorentz-violating four-vector $\hat{D}_\kappa$. The tensor structure is chosen such that the vector properties of $(k_{AF})_\kappa$ are described by the preferred spacetime direction $\hat{D}_\kappa$, whereas the nonminimal sector is separately parameterized by the symmetric tensor $X^{\mu \nu}$. Thus, the Lagrange density of the theory to be considered is

$$\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{4} \epsilon^{\kappa \lambda \mu \nu} A_\lambda \hat{D}_\kappa X^{\alpha_1 \alpha_2} \partial_{\alpha_1} \partial_{\alpha_2} F_{\mu \nu} + \frac{1}{2\xi} (\partial_{\mu} A^{\mu})^2,$$

(6)

with gauge fixing parameter $\xi$, i.e., the final term is included to fix the gauge. We can consider an observer frame where $X^{\mu \nu}$ is diagonal. In this context, the simplest case is that of a tensor $X^{\mu \nu}$ with equal spacelike coefficients. Then $X^{\mu \nu}$ is composed of a traceless part $\Xi^{\mu \nu}$ and a part with nonvanishing trace that must be proportional to the Minkowski metric tensor:

$$X^{\mu \nu} = \alpha_{tr} \Xi^{\mu \nu} + \alpha_{tr} \eta^{\mu \nu},$$

(7a)

$$\Xi^{\mu \nu} = \text{diag}(1, 1/3, 1/3, 1/3)^{\mu \nu} = \frac{1}{3} [4\xi^{\mu} \xi^{\nu} - \eta^{\mu \nu}],$$

(7b)

with the preferred purely timelike direction $(\xi^{\mu}) = (1, 0, 0, 0)$ and parameters $\alpha_{tr}, \alpha$ suitably chosen.\(^1\) As the traceless part involves two preferred directions $\xi^{\mu}$ and $D^{\mu}$, its investigation is probably more complicated than that of the

\(^1\) Here we simply adopt the notation used in the CPT-even photon sector where the single coefficient $\kappa_{tr}$ is linked to a symmetric, traceless matrix, as well.
contribution proportional to the trace. Therefore, we leave the analysis of the traceless part for the future and choose
\( \alpha_{tr} = 0 \) so that \( \tilde{\kappa} D_{\kappa} \nabla_\alpha X^{\alpha_1 \alpha_2} \partial_{\alpha_1} \partial_{\alpha_2} = D_{\kappa} \nabla_\kappa \), with \( \nabla \equiv \partial_\mu \partial^\mu \) and the redefined background vector \( D_{\kappa} \equiv \alpha \tilde{D}_{\kappa} \). Thus, the LV background is

\[
(k_{AF})_{\kappa} = \tilde{D}_{\kappa} \nabla_\kappa, \tag{7c}
\]

and the Lagrange density to be studied has the form

\[
\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F_{\mu \nu} + \frac{1}{2} \xi \epsilon_{\kappa \lambda \mu \nu} D_\kappa A_\lambda \nabla_\mu \nabla_\nu + \frac{1}{2 \xi} (\partial_\mu A^\mu)^2. \tag{8}
\]

By performing suitable partial integrations and neglecting boundary terms, the latter can be written as

\[
\mathcal{L} = \frac{1}{2} A^\mu O_{\mu \nu} A_\nu, \tag{9a}
\]

with the differential operator

\[
O_{\mu \nu} = \nabla_\mu \left( \Theta_{\mu \nu} - 2 L_{\mu \nu} - \frac{1}{\xi} \Omega_{\mu \nu} \right), \tag{9b}
\]

sandwiched in between two vector fields. Here we introduced the symmetric transversal and longitudinal projectors, \( \Theta_{\mu \nu} \) and \( \Omega_{\mu \nu} \), respectively:

\[
\Theta_{\mu \nu} \equiv \eta_{\mu \nu} - \Omega_{\mu \nu}, \quad \Omega_{\mu \nu} \equiv \frac{\partial_\mu \partial_\nu}{\nabla}, \tag{10}
\]

while the Lorentz-violating part is described by the antisymmetric and dimensionless operator,

\[
L_{\mu \nu} \equiv \epsilon_{\mu \nu \kappa \lambda} D_\kappa \partial_\lambda. \tag{11}
\]

Now we intend to evaluate the propagator of the theory, i.e., we should find the Green’s function \( \Delta_{\alpha \beta} \), which is the inverse of the differential operator \( O_{\mu \nu} \), from the condition

\[
O_{\mu \sigma} \Delta^\nu_\sigma = \eta_{\mu \nu}. \tag{12}
\]

For the inverse to be found, we use a suitable basis of tensor operators, a procedure that was successful in several theoretical scenarios \[50, 51\]. Thus, we propose the following Ansatz:

\[
\Delta^\nu_\sigma = a \Theta^\nu_\sigma + b L^\nu_\sigma + c \Omega^\nu_\sigma + d D^\nu D_\sigma + e (D^\sigma \partial_\nu + D_\nu \partial^\sigma), \tag{13}
\]

where the parameters \( a \ldots e \) are expected to be scalar operators. The algebra of the individual tensor operators is displayed in Table I. We also use the definition

\[
\rho \equiv D^\mu \partial_\mu, \tag{14}
\]

and, for brevity, define the tensor

\[
\Gamma_{\mu \sigma} \equiv L_{\mu \nu} L^\nu_\sigma = (D_\mu \partial_\sigma + D_\sigma \partial_\mu) \rho - D_\mu D_\sigma \nabla - (D^2 \nabla - \rho^2) \Theta_{\mu \sigma} - \rho^2 \Omega_{\mu \sigma}, \tag{15}
\]

which originates from the contraction of two Levi-Civita symbols. Starting from the tensor equation \( (12) \), after performing some simplifications, we have

\[
\Theta_{\mu \sigma} + \Omega_{\mu \sigma} \overset{!}{=} \nabla \left\{ \left[ a b - 4 (D^2 \nabla - \rho^2) \right] \Theta_{\mu \sigma} + 2 (b - a) L_{\mu \sigma} - \left[ -4 \rho^2 b + \frac{c}{\xi} + \left( \frac{1}{\xi} \right) \rho e \right] \Omega_{\mu \sigma} + \left[ d + 4 \nabla b \right] D_\mu D_\sigma + (e - 4 \rho b) D_\mu \partial_\sigma - \left[ 4 \rho \left( \frac{1}{\xi} \right) + \frac{1}{\xi} \right] \rho \right\} \quad \nabla_\sigma \partial_\mu \right\}. \tag{16}
\]

By comparing both sides of Eq. \( (16) \) to each other, the following differential operators are obtained:

\[
a = \frac{b}{2} = \frac{1}{\Xi}. \tag{17a}
\]
versions of the Barnes-Rivers spin operator basis \[56–58\]. Finally, note that Υ(\(p\)) by a spontaneous violation of Lorentz symmetry. The latter is induced by the bumblebee field \[54, 55\] using generalized coefficients of the fermionic sector, such quantities are expected to appear. The structure of Υ(\(p\)) space) \[53\]. In Euclidean space, this quantity corresponds to the Gramian of the two vectors.

Several remarks are in order. First, in App. A we obtain the propagator of Maxwell-Carroll-Field-Jackiw (MCFJ) theory in Lorenz gauge, showing that the latter and Eq. (19) are linked to each other by a simple replacement. Second, the inverse operator is

\[
\Delta_{\sigma\nu} = \frac{1}{\Xi} \left\{ \eta_{\sigma\nu} + 2L_{\sigma\nu} - \left( \frac{\Xi}{\square} + 1 + 4\bar{g}^2 \right) \Omega_{\sigma\nu} - 4\square D_{\sigma} D_{\nu} + 4\bar{g}[D_{\sigma}\partial_{\nu} + D_{\nu}\partial_{\sigma}] \right\}.
\]

The form of the propagator in momentum space follows from the latter result by carrying out the substitution \(\partial_{\mu} = -ip_{\mu}\) with the four-momentum \(p_{\mu}\), so that:

\[
\Delta_{\mu\sigma}(p) = \frac{-1}{p^2(1 + 4\Upsilon(p))} \left\{ \eta_{\mu\sigma} - 2i \epsilon_{\mu\sigma\nu\lambda} D_{\nu} p_{\lambda} - \left[ 1 - 4(D \cdot p)^2 + \xi (1 + 4\Upsilon(p)) \right] \frac{p_{\mu} p_{\sigma}}{p^2} 
+ 4p^2 D_{\mu} D_{\sigma} - 4(D \cdot p)[D_{\mu} p_{\sigma} + D_{\sigma} p_{\mu}] \right\},
\]

with

\[
\Upsilon(p) = D^2 p^2 - (D \cdot p)^2.
\]

A prefactor of \(i\) has been added to match the conventions for the photon propagator for zero Lorentz violation in \[52\]. Several remarks are in order. First, in App. A we obtain the propagator of Maxwell-Carroll-Field-Jackiw (MCFJ) theory in Lorenz gauge, showing that the latter and Eq. (19) are linked to each other by a simple replacement. Second, the propagator (19) is transverse, except for a piece that depends on the gauge-fixing parameter:

\[
\Delta_{\mu\sigma}(p)p^\tau = i\xi \frac{p_{\mu}}{p^2}.
\]

Third, note that contributions of the form \(\Upsilon(p)\) (with the four-momentum replaced by the four-velocity \(u^\mu\)) appear in the context of certain classical Lagrangians associated with the SME, especially in those that arise for the \(b\) coefficients (connected to Finsler \(b\) space) \[53\]. In Euclidean space, this quantity corresponds to the Gramian of the two vectors that appear in the expression. As there is a connection between the \(CPT\)-odd electromagnetic sector and the \(b\) coefficients of the fermionic sector, such quantities are expected to appear. The structure of \(\Upsilon(p)\) was also observed in the graviton propagator evaluated in the context of the linearized Einstein-Hilbert gravity (without torsion) modified by a spontaneous violation of Lorentz symmetry. The latter is induced by the bumblebee field \[54, 55\] using generalized versions of the Barnes-Rivers spin operator basis \[56–58\]. Finally, note that \(\Upsilon(p)\) is a dimensionless function, as the mass dimension of \(p^\mu\) cancels the inverse mass dimension of \(D^\mu\).

Fourth, it is interesting to recall that the propagator (19a) has an antisymmetric term, proportional to the projector \(L_{\mu\sigma}\) while the other pieces are symmetric with respect to the interchanges \(\mu \rightarrow \sigma, \sigma \rightarrow \mu\). The Feynman propagator is

\[\text{Note that we did not determine the Feynman propagator here, as the latter requires the definition of a suitable pole structure. The Feynman propagator is mentioned to emphasize the symmetries that must be same as those of the Green’s function.}\]
defined by the vacuum expectation value of the time-ordered product of field operators evaluated at distinct spacetime points,
\[ i(D_F)_{\alpha\beta}(x-y) \equiv \langle 0|T(A_\alpha(x)A_\beta(y))|0 \rangle. \] (21)
The Fourier transform of the latter is symmetric with respect to the combination of interchanging its indices and \( p^\mu \mapsto -p^\mu \). The antisymmetric piece of the propagator (19a) appears in CPT-odd electrodynamics, e.g., in the MCFJ model or Chern-Simons theory in (1+2) dimensions. But this piece does not mean that the propagator loses its symmetry. Indeed, the propagator continues being symmetric with regards to the two simultaneous operations mentioned before.

A. Dispersion relations

The poles of the propagator provide two dispersion equations for this model,
\[ p^2 = 0, \] (22a)
\[ 1 + 4 \left[ D^2 p^2 - (D \cdot p)^2 \right] = 0, \] (22b)
as usual in theories with higher-dimensional operators. The first one corresponds to the typical Maxwell pole, which also appears in the Podolsky and Lee-Wick models as well as in the corresponding anisotropic LV versions [49]. The second equation contains information on the higher-derivative dimension-5 term. It is reasonable to compare it to the dispersion equation obtained for the dimension-4 MCFJ theory (see Eq. (25) in the first paper of [7]), given in terms of the CFJ background vector \((k_{AF})^\mu\):
\[ p^4 + p^2 k^2_{AF} - (k_{AF} \cdot p)^2 = 0. \] (23)
The latter is a quartic-order dispersion equation, whereas Eq. (22b) is simpler (only of second order). Such a comparison reveals that the present dimension-5 MCFJ-like theory is totally distinct from the dimension-3 MCFJ model. The less involved dispersion equation can be ascribed to the simple structure of the background tensor that we have chosen in Eq. (5).

We will analyse the dispersion relations (DRs) for several configurations of the LV background where it makes sense to distinguish between purely timelike and spacelike preferred directions. For a purely timelike background, \( D_\gamma = (D_0, 0)_\gamma \), we have
\[ p^2 = \frac{D_0^2}{4}, \] (24)
which does not correspond to a propagating mode. It is a nonphysical DR, as it does not represent a relation between energy and momentum. Thus, there is no propagating mode associated with a timelike background vector. This property is an important difference between the dimension-5 model under consideration and MCFJ theory. The latter exhibits a DR associated with a timelike background vector. However, this timelike sector is plagued by consistency problems [7].

Now, for a purely spacelike background, \( D_\gamma = (0, D)_\gamma \), the corresponding DR is
\[ p_0 = \frac{1}{|D|} \sqrt{\frac{1}{4} + D^2 p^2 - (D \cdot p)^2} = \frac{1}{|D|} \sqrt{\frac{1}{4} + |D \times p|^2}, \] (25)
which can also be written as
\[ p_0 = \frac{1}{|D|} \sqrt{\frac{1}{4} + D^2 p^2 \sin^2 \alpha}, \] (26a)
with the angle \( \alpha \) enclosed by \( D \) and \( p \):
\[ D \cdot p = |D||p| \cos \alpha. \] (26b)
This is a DR that is compatible with the propagation of signals, whose properties need to be examined. In the current section, we are especially interested in classical causality that is characterized by the behavior of the group and front velocity \( u_{gr} \) and \( u_{fr} \), respectively, where [59]
\[ u_{gr} = \frac{\partial p_0}{\partial p}, \quad u_{fr} = \lim_{|p| \to \infty} \frac{p_0}{|p|}. \] (27)
FIG. 1: Magnitude of the group velocity of Eq. (30) for the spacelike case with $\alpha = 0$ (black, plain), $\alpha = \pi/40$ (red, dashed), $\alpha = \pi/10$ (blue, dotted), $\alpha = \pi/4$ (green, dashed-dotted), and $\alpha = \pi/2$ (orange, long dashes).

Classical causality is established as long as both $u_{gr} \equiv |u_{gr}| \leq 1$ and $u_{fr} \leq 1$. We now evaluate these characteristic velocities for DR (25). The front velocity is

$$u_{fr} = \lim_{|p| \to \infty} \sqrt{\frac{1}{4D^2p^2} + \sin^2 \alpha} = \sin \alpha,$$

(28)
as $\alpha \in [0, \pi]$. Furthermore, we investigate the behavior of the group velocity:

$$u_{gr} = \frac{D^2p - D(D \cdot p)}{|D| \sqrt{1/4 + |D \times p|^2}},$$

(29)whose magnitude is

$$u_{gr} = \frac{|D \times p|}{\sqrt{1/4 + |D \times p|^2}} = \frac{\sin \alpha}{\sqrt{1/(4x^2) + \sin^2 \alpha}},$$

(30)where $x \equiv |D||p|$ is a dimensionless parameter. Large momenta correspond to large $x$. Hence,

$$\lim_{|p| \to \infty} u_{gr} = \lim_{x \to \infty} u_{gr} = 1,$$

(31)independently of the angle $\alpha$. As $u_{gr} \leq 1$ and $u_{fr} \leq 1$, classical causality is established for the whole range of LV coefficients and momenta. Figure 1 presents the behavior of the magnitude of the group velocity. The graph shows a monotonically increasing group velocity that reaches the asymptotic value 1, which is a behavior in accordance with causality. It shares this property with the spacelike sector of MCFJ theory where classical causality is guaranteed, as well (see fifth paper of [7]). The mode obtained here is spurious in the sense that it does not propagate when Lorentz violation goes to zero. Hence, it does not approach the standard DR in this limit. The mode must be understood as a high-energy effect that propagates in a well-behaved manner for large momenta.

**B. Unitarity**

The next step is to study unitarity at tree-level, which is performed by means of the saturated propagator $SP$ [60]. The latter is a scalar quantity that is implemented by contracting the propagator with external physical currents $J^\mu$ as follows:

$$SP \equiv J^{\mu}i\Delta_{\mu\nu}J^{\nu}.$$ 

(32)
The current $J^\mu$ satisfies the conservation law $\partial^\mu J_\mu = 0$, which in momentum space reads $p^\mu J^\mu = 0$. In accordance with this method, unitarity is assured whenever the imaginary part of the residue of the saturation $SP$ (evaluated at the poles of the propagator) is nonnegative. A way of carrying out the calculation consists in determining the eigenvalues of the propagator matrix, evaluated at their own poles with current conservation taken into account. For the propagator found in Eq. (19), the saturation is

$$SP = -i \left\{ \frac{J^2 + 4p^2 (J \cdot D)^2}{p^2[1 + 4(D^2p^2 - (D \cdot p)^2)]} \right\}, \quad (33)$$

where $J^\mu L_{\mu\alpha} J^\alpha = -i J^\mu \epsilon_{\mu\nu\alpha\beta} J^\nu D^\alpha p^\beta = 0$. Contracting the propagator with conserved currents corresponds to getting rid of all gauge-dependent (and therefore, unphysical) contributions. These usually are terms that involve four-momenta with free indices corresponding to the indices of the Green’s function. Although the term $L_{\mu\alpha}$ does not have this structure, it vanishes when coupled to two external conserved currents due to its antisymmetry. Hence, the only Lorentz-violating contributions of the propagator that have an impact on physics are the denominators and the symmetric term, $D_\mu D_\nu$, formed from a combination of two preferred spacetime directions.

Now, for the Maxwell pole, $p^2 = 0$, the residue of the saturation is

$$\text{Res}(SP)|_{p^2=0} = -i \left\{ \frac{J^2}{1 - 4(D \cdot p)^2} \right\}|_{p^2=0}. \quad (34)$$

At the pole $p^2 = 0$ it holds that $p_0^2 = p^2$. From the law of current conservation we obtain $p_0 J_0 = p \cdot J$. Therefore, we can cast the corresponding imaginary part into the form

$$\text{Im}[\text{Res}(SP)|_{p^2=0}] = \frac{1}{1 - 4(D_0|p| - D \cdot p)^2} \frac{|p \times J|^2}{|p|^2} \geq 0. \quad (35)$$

For any background, $D_\mu = (D_0, D)_\mu$, the imaginary part of the saturation (35) is nonnegative for small momenta, but becomes negative as the momentum increases. For a timelike configuration, $D_\mu = (D_0, 0)_\mu$, and for a spacelike configuration, $D_\mu = (0, D)_\mu$, the quantity (35) becomes negative for $1/4 < D_0^2|p|^2$ and $1/4 < (D \cdot p)^2$, respectively. So, unitarity is not assured at the pole $p^2 = 0$ for all configurations possible.

For the second DR, Eq. (22b), it only makes sense to examine the spacelike configuration, $D_\mu = (0, D)_\mu$, where

$$p^2 = \frac{1 - 4(D \cdot p)^2}{4D^2}. \quad (36)$$

It is reasonable to write the saturation in the form

$$SP = -i \left\{ \frac{J^2 + 4p^2(D \cdot J)^2}{p^2[1 - 4(D^2p^2 - (D \cdot p)^2)]} \right\} = \frac{i}{p^2 - \frac{1 - 4(D \cdot p)^2}{4D^2}} \left[ \frac{J^2}{4D^2p^2} + \frac{(D \cdot J)^2}{D^2} \right]. \quad (37)$$

Its residue at this pole is

$$\text{Res}(SP)|_{p^2=\frac{1 - 4(D \cdot p)^2}{4D^2}} = i \left[ \frac{J^2}{1 - 4(D \cdot p)^2} + \frac{(D \cdot J)^2}{D^2} \right]. \quad (38)$$

Due to current conservation, $J_0 = (p \cdot J)/p_0$, the four-current squared can be cast into the form

$$J^2 = -J^2[1 + 4(D \times p)^2] - 4D^2(J \cdot p)^2 = -J^2[1 - 4(D \cdot p)^2] + 4D^2[J^2p^2 - (J \cdot p)^2]. \quad (39)$$

In the standard case, current conservation implies that $J^2 < 0$, i.e., any physical four-current is spacelike. However, this property does not necessarily hold in Lorentz-violating theories, anymore — as shown by Eq. (39). Inserting this result into Eq. (38), leads to the residue

$$\text{Res}(SP)|_{p^2=\frac{1 - 4(D \cdot p)^2}{4D^2}} = i \left\{ -\frac{4D^4(J \times p)^2 + (1 - 4(D \cdot p)^2)[4D^2(J \cdot p)^2(1 + 4(D \cdot p)^2)]}{(1 - 4(D \cdot p)^2)[1 + 4(D \times p)^2]} \right\}. \quad (40)$$

There are configurations for which the imaginary part of the latter is positive. For example, we can choose $p$ parallel to $J$, whereby Eq. (40) reduces to

$$\text{Res}(SP)|_{p^2=\frac{1 - 4(D \cdot p)^2}{4D^2}, p \parallel J} = i \left[ -\frac{J^2}{1 + 4(D \times p)^2} + \frac{(D \cdot J)^2}{D^2} \right]. \quad (41)$$
The first (negative) term can be suppressed for large momenta as long as $D \parallel p$. As the second (positive) contribution does not depend on the momentum, the imaginary part of the residue can be positive for large enough momenta. Hence, there are configurations of background field, large momenta, and external current for which unitarity is valid. This behavior is in accordance with the previous interpretation that the mode is spurious and must be interpreted as a high-energy effect. In a more specific way, if $\theta$ is the angle between $D$ and $J$, we have $(D \cdot J)^2 = D^2 J^2 \cos^2 \theta$ and $(D \times p)^2 = D^2 p^2 \sin^2 \theta$. Thus, the residue (41) can be cast into the form

$$\text{Res}(SP)_{p^2 = -4(D \times p)^2} = i \left( -\frac{J^2}{1 + 4D^2 p^2 \sin^2 \theta} + J^2 \cos^2 \theta \right) = i J^2 \sin^2 \theta \left( \frac{4(D \cdot p)^2 - 1}{4(D \cdot p)^2 + 1} \right),$$

(42)

whose imaginary part is positive for

$$\langle D \cdot p \rangle^2 > \frac{1}{4}.$$  

(43)

The latter condition assures unitarity. A more general case to be investigated is $D \perp p$ with $p \parallel J$, for which $D \cdot p = 0$ and $(D \times p)^2 = D^2 p^2$. To examine it, we exploit observer rotation invariance and employ the following coordinate system:

$$D = |D| \hat{x}, \quad p = |p| \hat{y}, \quad J \cdot \hat{z} = |J| \cos \alpha,$$

(44)

in which

$$D \cdot J = |J||D| \sin \alpha \cos \phi,$$

(45a)

$$J \cdot p = |J||p| \sin \alpha \sin \phi,$$

(45b)

where $\phi$ is the angle between $x$ axis and the projection of the vector $J$ in the $x$-$y$ plane. For this configuration, the residue (40) reduces to

$$\text{Res}(SP)_{p^2 = -4(D \times p)^2} = -i J^2 \left( \frac{(1 + 4D^2 p^2) \cos^2 \alpha + \sin^2 \alpha \sin^2 \phi}{1 + 4D^2 p^2} \right).$$

(46)

The imaginary part of the latter is always negative, which demonstrates unitarity violation for any choice of the momentum and the angles. A similar investigation for $J \perp p$ yields a residue whose imaginary part can be either positive or negative, also providing unitarity violation in some situations.

To summarize, while causality is assured for any configuration of the spacelike background configuration, unitarity can hold, but does not do so necessarily. In the next section we will examine a more involved version of this first higher-derivative model.

### III. Maxwell Electrodynamics Modified by a CPT-odd Dimension-5 Higher Derivative Term: A Second Model

In the last section, we have examined a CPT-odd, dimension-5, nonminimal extension of the electromagnetic sector, specifically represented by the CFJ-like Lagrange density

$$\frac{1}{2} \epsilon^{\lambda \mu \nu} A_\lambda (\hat{k}_{AF})_\kappa F_{\mu \nu},$$

(47)

where the background vector field $(\hat{k}_{AF})_\kappa$ was written according to Eq. (7c). As a second possibility, we propose the more sophisticated choice

$$(\hat{k}_{AF})_\kappa = (k_{AF})_\kappa \epsilon^{\alpha \beta \gamma} \partial_{\alpha 1} \partial_{\alpha 2} = T_\kappa T^{\alpha_1} T^{\alpha_2} \partial_{\alpha 1} \partial_{\alpha 2} = T_\kappa (T \cdot \partial)^2,$$

(48)

where $T_\kappa$ is a Lorentz-violating four-vector whose mass dimension is

$$[T_\kappa] = -1/3,$$

(49)

or equivalently $[T^3_\kappa] = -1$. In contrast to the theory studied before, the vectorlike background field is now also contracted with the additional derivatives. This structure is supposed to render the properties of the current theory...
more involved than those of the previously studied one. Modifying Maxwell’s theory by including this term into its Lagrange density, leads to a higher-derivative (dimension-5) anisotropic CFJ-like theory described by

\[
{\mathcal{L}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} T^{\kappa\lambda\mu\nu} A_\kappa T_\lambda (T \cdot \partial)^2 F_{\mu\nu} + \frac{1}{2\xi} (\partial_\mu A^\mu)^2, \tag{50}
\]

which can be also written as

\[
{\mathcal{L}} = \frac{1}{2} A^\mu \Xi_{\mu\nu} A^\nu, \tag{51a}
\]

with the operator

\[
\Xi_{\mu\nu} = \Box \Theta_{\mu\nu} - 2\tilde{L}_{\mu\nu} - \frac{1}{\xi} \Box \Omega_{\mu\nu}, \tag{51b}
\]

sandwiched between two gauge fields. Comparing the latter to Eq. (9b), it is no longer possible to extract a Laplacian of the operator. The projectors \(\Theta_{\mu\nu}, \Omega_{\mu\nu}\) are given as before by Eq. (10) and the Lorentz-violating antisymmetric operator now reads

\[
\tilde{L}_{\mu\nu} = \epsilon_{\mu\kappa\lambda\nu} T^\kappa (T \cdot \partial)^2 \partial^\lambda. \tag{52}
\]

To determine the propagator of theory given by Eq. (50), we propose the following Ansatz:

\[
\Delta^\mu_\alpha = a\Theta^\mu_\alpha + b\tilde{L}^\mu_\alpha + c\Omega^\mu_\alpha + eT^\mu T_\alpha + f(T^\mu \partial_\alpha + T_\alpha \partial^\mu), \tag{53}
\]

that must satisfy the identity

\[
\Xi_\nu\mu \Delta^\mu_\alpha = \eta_\nu\alpha. \tag{54}
\]

The algebra of the tensor operators \(\Theta_{\nu\mu}, \tilde{L}_{\nu\mu}, \Omega_{\nu\mu}, T_\nu T_\mu, T_\nu \partial_\mu,\) and \(T_\mu \partial_\nu\) is the same as that presented in Tab. I with the replacement \(D_\mu \to T_\mu\) to be performed. For brevity, we introduce

\[
\rho \equiv T^\mu \partial_\mu, \tag{55a}
\]

\[
\tilde{\Gamma}_{\alpha\nu} \equiv \tilde{L}_{\nu\mu}\tilde{L}^\mu_\alpha = -[T_\alpha T_\nu \Box - (T_\nu \partial_\alpha + T_\alpha \partial_\nu)\rho + (\rho^2 - T^2 \Box) \Theta_\nu\alpha + \Omega_\nu\alpha \rho^2] \rho^4. \tag{55b}
\]

To find the operators \(a \ldots f\), we start from the tensor equation (54) and employ the algebra of the operators as before. The calculation is completely analogous. After performing some algebraic simplifications, we obtain

\[
a = \Box \Lambda, \quad b = \frac{2}{\Lambda} a, \quad c = -\xi - 4 \frac{\rho^5}{\Box \Lambda}, \quad e = \frac{4 \rho^4}{\Lambda}, \quad f = 4 \frac{\rho^5}{\Box \Lambda}, \tag{56a}
\]

where

\[
\Lambda = \Box^2 + 4(\rho^2 - T^2 \Box) \rho^4. \tag{56b}
\]

The form of the propagator in momentum space is:

\[
\Delta_{\mu\alpha}(p) = \frac{-i}{p^2} \left\{ p^4 + 4(T^2 p^2 - (T \cdot p)^2) [(T \cdot p)^4 - 2p^2 (T \cdot p)^2 \varepsilon_{\mu\kappa\lambda\nu} T^\kappa p^\lambda] \right\}
\]

\[
+ \left\{ (1 + \xi) [4(T \cdot p)^6 - p^4] - 4\xi T^2 p^2 (T \cdot p)^4 \right\} \frac{p_\mu p_\alpha}{p^2}
\]

\[
+ 4p^2 (T \cdot p)^4 T_\mu T_\alpha - 4(T \cdot p)^5 (T_\mu p_\alpha + T_\alpha p_\mu) \right\}. \tag{57}
\]

As before, the parts of the propagator independent of the gauge-fixing parameter are transversal, i.e., the result of Eq. (20) can be carried over.
A. Dispersion relations

As observed in the first model, the poles of the propagator provide two dispersion equations, namely

\[ \rho^2 = 0, \]
\[ \rho^4 + 4 [T^2 \rho^2 - (T \cdot p)^2] (T \cdot p)^4 = 0. \]

The first corresponds to the usual Maxwell pole, while the second contains information from the higher-derivative dimension-5 term. We will analyse the second dispersion equation for two configurations of the LV background, which exhibit preferred directions in a spacetime modified by Lorentz violation. For a purely timelike background, \( T_\gamma = (T_0, 0)_\gamma \), we have

\[ \rho^2_0 = \frac{p^2 + 2T_0^3 |p|^3}{1 - 4T_0^6 p^2} = \frac{p^2}{1 + 2T_0^3 |p|}, \]
\[ \rho^\pm_0 = \frac{|p|}{\sqrt{1 + 2T_0^3 |p|}}. \]

In contrast to the first model, there are now two distinct modified modes. The notation \( \oplus / \ominus \) for the mode with the relative plus/minus sign is self-evident. The energy is well-defined for the mode \( \ominus \), but not for the mode \( \oplus \), for which it is only meaningful as long as \( |p| < 1/(2T_0^3) \). Furthermore, a sign change of the controlling coefficient \( T_0 \) simply interchanges the two DRs. Hence, without restriction of generality, we assume a nonnegative coefficient: \( T_0 \geq 0 \). Now, for a purely spacelike background, \( T_\gamma = (0, T)_\gamma \), the corresponding DR is obtained from

\[ \rho^2_0 = p^2 + 2(T \cdot p)^4 T^2 + 2(T \cdot p)^3 \sqrt{1 + (T \cdot p)^2 T^4}, \]

which leads to

\[ \rho^\pm_0 = |p| \left( 1 + 2|T|^6 |p|^2 \cos^4 \alpha \pm 2|T|^3 |p| \cos^3 \alpha \sqrt{|T|^6 |p|^2 \cos^2 \alpha + 1} \right)^{1/2}, \]

with the angle \( \alpha \) enclosed by \( T \) and \( p \):

\[ T \cdot p = |T||p| \cos \alpha. \]

Recall that \( (T \cdot p)^3 \), \( (T \cdot p)^4 T^2 \), have mass dimension equal to 2, while \( (T \cdot p)^4 T^2 \) is dimensionless. It is possible to show that the energy (61b) is real for any absolute value of the background vector and direction relative to \( p \), that is, \( \rho^\pm_0 > 0 \). Besides, the DR of the mode \( \oplus \) is mapped to that of the mode \( \ominus \) and vice versa when the direction of \( T \) is reversed, i.e., when \( T \rightarrow -T \) or \( \alpha \rightarrow \pi - \alpha \). When electromagnetic waves propagate along a direction perpendicular to \( T \), Lorentz violation does not have any effect and the DR is standard. In the suitable momentum range, Eqs. (60b) and (61b) represent DRs compatible with a propagation of signals, whose properties need to be examined.

As before, we investigate classical causality via the group and front velocity \( u_{gr} \) and \( u_{fr} \), respectively, defined in Eq. (27). We now evaluate these characteristic velocities for DRs originating from the dispersion equation (59). On the one hand, we obtain the front velocity for DR (60b). For the mode \( \ominus \), the latter is not defined, as the energy becomes complex for momenta beyond \( 1/(2T_0^3) \). For the mode \( \oplus \), we obtain

\[ u_{fr}^- = \lim_{|p| \rightarrow \infty} \frac{\rho^-_0}{|p|} = \lim_{|p| \rightarrow \infty} \frac{1}{\sqrt{1 - 2T_0^3 |p|}} = 0. \]

On the other hand, we evaluate the group velocity:

\[ u_{gr}^\pm = \frac{\partial \rho^\pm_0}{\partial p} = \frac{1 + T_0^3 |p|}{(1 + 2T_0^3 |p|)^{3/2}} \frac{p}{|p|}, \]

whose absolute values read

\[ u_{gr}^+ = \frac{1 - T_0^3 |p|}{(1 - 2T_0^3 |p|)^{3/2}} \bigg|_{|p| < 1/(2T_0^3)}, \quad u_{gr}^- = \frac{1 + T_0^3 |p|}{(1 + 2T_0^3 |p|)^{3/2}}, \]
FIG. 2: Group velocity (64) for the mode $\oplus$ (blue, plain) and the mode $\ominus$ (red, dashed).

where $u_{gr}^+$ only makes sense for the range of momenta $|p| < 1/(2T_0^3)$, which is the same range that assures real energies for this mode. The behavior of $u_{gr}^+$ is depicted in Fig. 2. The large-momentum limit of the group velocity for the mode $\ominus$ is simply given by

$$\lim_{|p| \to \infty} u_{gr}^- = 0.$$  \hfill (65)

The results for the front and group velocity indicate that signals do not propagate for large momenta. Furthermore, $u_{gr}^-$ is well-behaved for all momenta, whereas $u_{gr}^+$ exhibits a singularity at $|p| = 1/(2T_0^3)$. For $|p| \geq 1/(2T_0^3)$, classical causality breaks down for this mode.

The next step is to investigate the properties of DR (61b). The associated front velocity is calculated as

$$u_{fr}^\pm = \lim_{|p| \to \infty} \frac{\tilde{p}_0^\pm}{|p|} = \lim_{|p| \to \infty} \sqrt{1 + 2|T|^6|p|^2 \cos^4 \alpha \pm 2|T|^6|p|^2 \cos^4 \alpha},$$  \hfill (66)

which provides

$$u_{fr}^+ = \lim_{|p| \to \infty} 2\sqrt{|T|^6|p|^2 \cos^4 \alpha} = \infty, \quad u_{fr}^- = 1.$$  \hfill (67)

Hence, the front velocity is divergent for the mode $\oplus$, which again indicates a breakdown of classical causality for this mode. In contrast to that, the front velocity for the mode $\ominus$ is well-behaved. As before, we also study the properties of the group velocity:

$$u_{gr}^\pm = \frac{\partial p_0^\pm}{\partial p} = \frac{\sqrt{1 + (T \cdot p)^2 T^4 [p + 4(T \cdot p)^3 T^2 T] \pm (T \cdot p)^2 T [3 + 4(T \cdot p)^2 T^4]}}{\sqrt{1 + (T \cdot p)^2 T^4 [p^2 + 2(T \cdot p)^4 T^2 \pm 2(T \cdot p)^3 \sqrt{1 + (T \cdot p)^2 T^4}]}},$$  \hfill (68)

whose absolute value is

$$u_{gr}^\pm = \frac{\sqrt{2|4x^3 \cos^4 \alpha + 3x \cos^2 \alpha|^2 + (x^2/4) \sin^2(2\alpha) + 1 \pm U(x)}}{\sqrt{1 + x^4 \cos^4 \alpha \sqrt{2x^2 \cos^4 \alpha + 1 + 2x \cos^3 \alpha \sqrt{1 + x^2 \cos^2 \alpha}}}},$$  \hfill (69a)

where $|T|^3 |p| \equiv x$ and

$$U(x) = 2x \cos^3 \alpha [(2 + 4x^2 \cos^2 \alpha)^2 - 1] \sqrt{1 + x^2 \cos^2 \alpha}.$$  \hfill (69b)

The plots shown in Fig. 3 illustrate the behavior of the group velocity for the modes $\oplus$ and $\ominus$ for different angles. For the mode $\oplus$, the norm of the group velocity is equal 1 for $x = 0$ and can be either larger or smaller than 1 for
FIG. 3: Group velocity of Eq. (69) for the mode $\oplus$ for $\alpha = 0$ (black, plain), $\alpha = 2\pi/5$ (red, dashed), $\alpha = \pi/2$ (blue, dotted), $\alpha = 9\pi/10$ (green, dashed-dotted), and $\alpha = \pi$ (orange, long dashes) (a). Corresponding result for the mode $\ominus$ for $\alpha = 0$ (black, plain), $\alpha = \pi/10$ (red, dashed), $\alpha = \pi/2$ (blue, dotted), $\alpha = 3\pi/5$ (green, dashed-dotted), and $\alpha = 9\pi/10$ (orange, long dashes) (b).

$x > 0$. For $\alpha \in [0, \pi/2)$, it steadily becomes larger for $x > 0$, which implies causality violation. For $\alpha \in (\pi/2, \pi)$, it falls below 1 for $x > 0$. In this regime, it has a minimum for a certain value of $x$ and increases again whereupon it approaches 1 in the limit of large momenta. This behavior is compatible with classical causality. For $\alpha = \pi$, the group velocity decreases monotonically with $x$ until it reaches zero. Finally, the group velocity corresponds to the standard result $u_{gr} = 1$ when $\alpha = \pi/2$, as the dispersion relation is not modified in this case. The general behavior of the mode $\ominus$ is analogous when $\alpha$ is replaced by $\pi - \alpha$. Hence, the mode $\oplus$ cannot propagate for large momenta when the momentum points in a direction opposite to $T$ and a similar behavior occurs for the mode $\ominus$ when the momentum is parallel to $T$. Thus, these results seem to show that the mode $\oplus$ preserves classical causality for $T \cdot p \leq 0$, whereas $\ominus$ exhibits this property for $T \cdot p \geq 0$.

B. Unitarity

The analysis of unitarity at tree-level follows the same procedure applied in Sec. II B. For the propagator found in Eq. (57), the saturation is

$$SP = -i \left\{ \frac{p^2 J^2 + 4(T \cdot p)(T^2 p^2 - (T \cdot p)^2)}{p^4 + 4(T^2 p^2 - (T \cdot p)^2)^2} \right\}. \quad (70)$$

In contrast to the first model, the Maxwell pole cancels in the saturated propagator. For the timelike configuration, $T_\gamma = (T_0, 0)_\gamma$, the DR is given by Eq. (60b). Then, the saturated propagator is

$$SP_{\text{time-like}} = -i \left[ \frac{p^2 J^2 + 4T_0 \delta p_0^4 J_0^2}{(1 - 4T_0 \delta p^2)(p^2 - p_+^2)(p^2 - p_-^2)} \right], \quad (71a)$$

where

$$p_\pm^2 = p_0^2 \pm p^2 = \frac{p^2}{1 \mp 2T_0^3|p|} \mp p^2 = \pm \frac{2T_0^3|p|^3}{1 \mp 2T_0^3|p|}. \quad (71b)$$
We deduce that \( p^2_\pm \) only makes sense for \( |p| < 1/(2T_0^3) \) where it is larger than zero. Besides, we observe that \( p^2_\pm \leq 0 \) for all values of \( |p| \). Apart from that, we have the useful relation

\[
p^2_+ - p^2_- = \frac{4T_0^3|p|^3}{1 - 4T_0^2p^2}.
\]

The physical four-current squared can be conveniently expressed in the form

\[
J^2 = -\frac{1}{p^2_0 \pm} [(J \times p)^2 + p^2_\pm J^2] .
\]

Note that \( p^2_\pm \leq 0 \) according to Eq. (71b), which does not render the four-current squared negative for all configurations possible. Now, we use Eq. (73) to write the residues for both poles as

\[
\text{Res}(SP)|_{p^2 = p^2_\pm} = \pm \frac{i}{p^2_0 \pm} \left[ p^2_\pm (J \times p)^2 + p^2_\pm J^2 - 4T_0^3p^4_\pm (J \cdot p)^2 \right].
\]

This expression is involved and can be better analyzed for some special configurations. For the particular case \( J \parallel p \), we obtain \((J \times p)^2 = 0\) and \( J \cdot p = |J||p|\). Thus, the residue (74) is

\[
\text{Res}(SP)|_{p^2 = p^2_\pm} = \pm p^2_\pm J^2 \left( \frac{p^2_\pm/|p|^3}{4T_0^3} \right) ,
\]

which leads to a vanishing result when we take into account that

\[
\frac{p^2_\pm}{p^2_0} = \pm 2T_0^2|p| .
\]

A vanishing residue means that the corresponding pole does not contribute to physical observables, which is a situation compatible with unitarity. Another particular configuration is \( J \perp p \) for which \((J \times p)^2 = J^2p^2\) and \( J \cdot p = 0\). In this case, the residue (74) reduces to

\[
\text{Res}(SP)|_{p^2 = p^2_\pm} = \pm p^2_\pm J^2 \left( \frac{p^2_\pm/|p|^3}{4T_0^3} \right) ,
\]

which can be simplified as

\[
\text{Res}(SP)|_{p^2 = p^2_\pm} = \frac{iJ^2}{2} \left( \frac{1}{1 \pm 2T_0^2|p|} \right) .
\]

The latter result confirms unitarity for both modes as far as the associated DRs are real, which requires \( |p| < 1/(2T_0^3) \). Alternatively, the residue (74) can be expressed as

\[
\text{Res}(SP)|_{p^2 = p^2_\pm} = \mp i \frac{p^2_\pm J^2}{4T_0^3|p|^3} \left[ (J^2)^2 + \frac{(J^1)^2}{2} \right] \left( \frac{1}{1 \mp 2T_0^2|p|} \right) ,
\]

which holds in an observer frame where the three-momentum points along the third axis: \( p = (0,0, p^3) \). Due to observer Lorentz invariance and isotropy of the theory considered, such a choice does not restrict generality, confirming the particular results obtained for \( J \parallel p \) and \( J \perp p \). For the mode \( \oplus \), the imaginary part of the residue is nonnegative for \( |p| < 1/(2T_0^3) \), assuring unitarity in the momentum range in which DR (60b) is real. But unitarity of this mode is violated for \( |p| \geq 1/(2T_0^3) \), when the energy associated becomes complex. This is an expected breakdown, therefore.

The imaginary part of the residue is manifestly nonnegative for the mode \( \ominus \), granting unitarity for the full momentum range. These properties are in accordance with the observations made in Sec. III A and Fig. 2. In contrast to \( \oplus \), the mode \( \ominus \) is well-behaved with respect to both classical causality and unitarity at tree-level. Furthermore, this mode is not spurious, as its limit for zero Lorentz violation corresponds to the standard DR. It does not propagate for large momenta, though. However, it is interesting to mention that a breakdown of classical causality does not necessarily imply unitarity violation, as can be seen for the mode \( \oplus \) where \( |p| \in [0,1/(2T_0^3)] \).

For the purely spacelike choice, \( T_\gamma = (0, T)|_\gamma \), the saturated propagator is

\[
SP|_{\text{spacelike}} = -i \frac{p^2 J^2 + 4(T \cdot p)^4(T \cdot J)^2}{(p^2 - p_\|^2)(p^2 - p_\perp^2)} ,
\]

(80a)
where in the same manner as before, we introduce the auxiliary quantities

$$\tilde{p}_\pm^2 \equiv \tilde{p}_0^2 - p^2 = 2(T \cdot p)^3 \left[ (T \cdot p)T^2 \pm \sqrt{1 + (T \cdot p)^2 T^4} \right]. \quad (80b)$$

The four-current squared can be written as in Eq. (73) with the replacements $p_{0\pm} \mapsto \tilde{p}_{0\pm}$ and $p_{\pm} \mapsto \tilde{p}_{\pm}$. We observe that $\tilde{p}_\pm^2 \leq 0$ for $T \cdot p \leq 0$ and $\tilde{p}_\pm^2 < 0$ for $T \cdot p > 0$. Therefore, the standard condition $J^2 < 0$ for a conserved current does again not necessarily hold in the Lorentz-violating context. Based on this result, we write the residues of the two poles in the form

$$\text{Res}(SP)|_{p^2 = \tilde{p}_\pm^2} = \pm \frac{i}{\tilde{p}_0^2} \left[ \tilde{p}_\pm^2 (J \times p)^2 + \tilde{p}_\pm^4 J^2 - 4\tilde{p}_0^2 (T \cdot p)^4 (T \cdot J)^2 \right] \times \frac{4(T \cdot p)^3}{4(T \cdot p)^3 \sqrt{1 + (T \cdot p)^2 T^4}} \quad (81)$$

The latter can also be cast into

$$\text{Res}(SP)|_{p^2 = \tilde{p}_\pm^2} = \frac{N_\pm}{2\tilde{p}_0^2 \sqrt{1 + (T \cdot p)^2 T^4}} \quad (82a)$$

with

$$N_\pm = \mp \left\{ 2(T \cdot p)(T \cdot J)^2 p^2 + 4(T \cdot p)^4 (T \cdot J)^2 \gamma_\pm - 2J^2 \gamma_\pm (T \cdot p)^4 (T \cdot J)^2 - \gamma_\pm (J \times p)^2 \right\}, \quad (82b)$$

$$N_\pm = \mp \left\{ 2(T \cdot p) [p^2 (T \cdot J)^2 - J^2 (T \cdot p)^2] - \gamma_\pm [4(T \cdot p)^4 (J \times T)^2 + (J \times p)^2] \right\}, \quad (82c)$$

and

$$\gamma_\pm = (T \cdot p)T^2 \pm \sqrt{1 + (T \cdot p)^2 T^4}. \quad (82d)$$

Although the residue of Eq. (82a) does not look that involved, an analysis of this expression turned out to be challenging. Based on the new quantities of Eq. (82d), the modified DRs can be expressed in a short manner via

$$\tilde{p}_\pm^2 = p^2 + 2(T \cdot p)^3 \gamma_\pm. \quad (83)$$

A further useful relation in this context is

$$\gamma_\pm^2 = 1 + 2(T \cdot p)T^2 \left[ (T \cdot p)T^2 \pm \sqrt{1 + (T \cdot p)^2 T^4} \right] = 1 + 2(T \cdot p)T^2 \gamma_\pm. \quad (84)$$

According to our criterion, unitarity is guaranteed as long as $N_\pm \geq 0$. The second contribution of $N_\pm$ is manifestly nonnegative due to $\pm \gamma_\pm > 0$. However, the first is not. For the configurations $J \parallel p$ and $T \perp p$, the first term simply vanishes, i.e., unitarity can be demonstrated quickly for these particular cases. In particular, the imaginary part of the residue for $J \parallel p$ is

$$\text{Im}[\text{Res}(SP)|_{p^2 = \tilde{p}_\pm^2}] = \pm \frac{2\gamma_\pm (T \cdot p)^4 (J \times T)^2}{\tilde{p}_0^2 \sqrt{1 + (T \cdot p)^2 T^4}} \geq 0. \quad (85)$$
Specifically for the situation $\mathbf{T} \perp \mathbf{p}$, DR (84) recovers the usual Maxwell DR and $\Upsilon_\pm = 1$, so that the residue simply yields
\[ \text{Im}\{\text{Res}(SP)|_{p^2=0}\} = \frac{\mathbf{J} \times \mathbf{p}}{2p^2} \geq 0. \quad (86) \]

The origin of the factor of 2 in the denominator is due to the existence of two distinct DRs for nonzero Lorentz violation and configurations other than $\mathbf{T} \perp \mathbf{p}$. When both merge, each of those contributes the above value to the residue providing its standard result.

In contrast, for most other choices, an evaluation of the inequality $N_\pm \geq 0$ seems to be highly involved. What can be done in the general case, is to employ observer rotational invariance and to choose a coordinate system such that the momentum points along the third axis and $\mathbf{T}, \mathbf{J}$ lie in the plane spanned by the first and third axes. So we consider
\[
\mathbf{p} = \begin{pmatrix} 0 \\ 0 \\ |\mathbf{p}| \end{pmatrix}, \quad \mathbf{T} = |\mathbf{T}| \begin{pmatrix} \sin \alpha \\ 0 \\ \cos \alpha \end{pmatrix}, \quad \mathbf{J} = |\mathbf{J}| \begin{pmatrix} \sin \theta \\ 0 \\ \cos \theta \end{pmatrix},
\]

with $\alpha \in [0, 2\pi]$ and $\theta \in [0, 2\pi]$. Inserting these representations into the left-hand side of $N_\pm \geq 0$, we obtain
\[
\frac{N_\pm}{J^2p^2} \equiv g_\pm(\xi, \theta, \alpha) \geq 0,
\]

\[
g_\pm(\xi, \theta, \alpha) = \pm 4\xi^3 (1 + \sin^2 \alpha)(\sin \theta - \sin \alpha \cos \theta)^2
\]
\[+ \left[4\xi^2(\sin \theta - \sin \alpha \cos \theta)^2 + \sin^2 \theta\right] \sqrt{1 + (1 + \sin^2 \alpha)^2 \xi^2}
\]
\[\pm \xi \left[(2 + \cos^2 \alpha) \sin^2 \theta - 2 \sin \alpha \sin(2\theta)\right], \quad (88b)
\]

with
\[
\xi \equiv |\mathbf{p}| |\mathbf{T}|^3. \quad (88c)
\]

The left-hand side of the new inequality are functions of $\xi \geq 0$ and the two angles $\alpha, \theta$. It is challenging to prove that $g_\pm(\xi, \theta, \alpha)$ is nonnegative for general $\xi$ and angles. Plots of these functions for a given choice of $\xi$ are presented in Fig. 4. According to the plots, both functions most probably do not provide negative values. Numerical investigations allow for determining the zeros of $g_\pm(1, \theta, \alpha)$; cf. Fig. 5. There are no isolated zeros, but they lie along certain lines. Evaluating the first and second derivatives at these points, indicates that they are local minima. Minima other than those were not found. Hence, there are strong numerical indications that $g_\pm(1, \theta, \alpha)$ are nonnegative. Analog results can be obtained for $\xi \neq 1$, granting unitarity for all configurations of the purely spacelike case.
IV. CONCLUSIONS AND FINAL REMARKS

We analyzed Maxwell’s electrodynamics modified by dimension-five and CPT-odd higher-derivative terms that are part of the photon sector of the nonminimal Standard-Model Extension. The first dimension-five term to be addressed was a kind of higher-derivative CFJ-like contribution. The propagator of the theory was computed by means of the algebra for the operators involved. Based on this propagator, modifications in the particle spectrum of the theory due to the presence Lorentz violation were investigated.

The analysis of the dispersion relations obtained from the propagator poles revealed that classical causality is preserved for any choice of background coefficients. Furthermore, an analysis of unitarity at tree-level was carried by contracting the propagator with conserved currents and studying the pole structure of the resulting expression. It was found that the dispersion relations describe nonunitary modes, in general. This propagator possesses some analogy with that of MCFJ theory. But the dispersion relations and the related physics are different between these two models.

The second dimension-five term examined was a type of anisotropic higher-derivative CFJ-like contribution. The propagator was again computed and the dispersion relations were obtained from it for a purely timelike and a purely spacelike background vector field. For the timelike background, one mode was found to be unphysical, whereas the second generally obeys classical causality and unitarity, as shown in Eq. (79). For the spacelike configuration, there are two modes, as well. These are causal only for certain special regimes of the propagation direction. However, numerical investigations indicate that unitarity is preserved for arbitrary configurations within this scenario (in spite of causality violation).

The results of this paper complement our findings for nonminimal CPT-even extensions of the electromagnetic sector considered in [49]. In general, the studies performed for the CPT-odd extensions were easier from a technical point of view in comparison to their CPT-even counterparts. Furthermore, our results reveal that a breakdown of classical causality does not necessarily imply unitarity violation at tree-level. The well-behaved modes can be suitable as a base for studying the quantization of nonminimal CPT-odd field theories in future works.

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Appendix A: Comparison to propagator of MCFJ theory

It would be interesting to compare the propagator (19) with that of MCFJ theory, which will be carried out in the current section. The MCFJ Lagrange density is

\[ \mathcal{L}_{\text{MCFJ}} = -\frac{1}{4} F_{\alpha \beta} F^{\alpha \beta} - \frac{1}{4} \varepsilon^{\beta \alpha \rho \varphi} V_\beta A_\alpha F_{\rho \varphi} + \frac{1}{2\xi} (\partial_{\mu} A^\mu)^2, \]  

(A1)

where \( V_\beta \) is the CFJ vector background and the last term is included to fix the gauge. Such a Lagrange density can be written as

\[ \mathcal{L}_{\text{MCFJ}} = \frac{1}{2} A^{\beta} \Box_{\beta \alpha} A^\alpha, \]  

(A2a)

\[ \Box_{\beta \alpha} = \Box \Theta_{\beta \alpha} - \frac{1}{\xi} \Omega_{\beta \alpha} + \Sigma_{\beta \alpha}, \quad \Sigma_{\beta \alpha} = \varepsilon^{\beta \alpha \rho \varphi} V_\rho \partial_\varphi, \]  

(A2b)

with the projectors of Eq. (10). Using an algebra similar to that of Tab. I, one obtains the following propagator:

\[ \Delta_{\alpha \nu} = \frac{1}{[\Box^2 - (V^2 \Box - \lambda^2)]} [\Box^2 \Theta_{\alpha \nu} - \{\xi \left[ \Box^2 - (V^2 \Box - \lambda^2) \right] + \lambda^2 \} \Omega_{\alpha \nu} \]  

\[-\Box (S_{\alpha \nu} + V_\alpha V_\nu) + \lambda (V_\alpha \partial_\nu + V_\nu \partial_\alpha)] , \]  

(A3)
with $\lambda \equiv V^\mu \partial_\mu$. In momentum space, the latter is

$$\Delta_{\alpha\nu}(p) = -\frac{i}{p^2} \left\{ \left( \frac{V^4 \Theta_{\alpha\nu}(p)}{p^2} \right) - \left\{ \xi \left( \frac{V^4 - (V \cdot p)^2}{p^4} \right) - \left( \frac{V \cdot p}{p^2} \right) \right\} \right\} \frac{p_{\alpha} p_{\nu}}{p^2} \right\}
+ \frac{p^2 S_{\alpha\nu}(p) + p^2 V_{\alpha} V_{\nu} - (V \cdot p)(V_{\alpha} p_{\nu} + V_{\nu} p_{\alpha})}{p^2} \right\}
\right\}, \quad \Delta_{\alpha\nu}(p) = -\frac{i}{p^2} \left\{ 1 - \frac{(V \cdot p)^2}{p^4} + \left( \frac{V^2}{p^4} \right) \right\}^{-1} \left\{ \eta_{\alpha\nu} - \left( 1 - \frac{(V \cdot p)^2}{p^4} + \frac{1}{p^4} \right) \left( \frac{V^2}{p^4} \right) \right\} \frac{p_{\alpha} p_{\nu}}{p^2} \right\}
\right\}.
\right\}
$$

The replacement

$$\frac{V^\mu}{p^2} \mapsto 2D^\mu, \quad \frac{S_{\alpha\nu}(p)}{p^2} \mapsto 2L^{\beta\gamma}(p),$$

implies

$$1 - \frac{(V \cdot p)^2}{p^4} + \frac{V^2}{p^4} \mapsto 1 - 4(D \cdot p)^2 + 4D^2 p^2,$$

which, inserted into Eq. (A4a), leads to the propagator (19) of the dimension-5 model examined. Furthermore, this propagator, as it stands, corresponds to Eq. (3.3) of the first paper of Ref. [8] for $V^\mu \mapsto m k_\mu$, $\xi \mapsto -\xi$ and $n^\mu \mapsto p^\mu$, with an appropriate choice of the prefactor. Here, $m$ is the Chern-Simons mass and $n^\mu$ a fixed four-vector used in their axial gauge fixing condition.

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