INCOMPRESSIBLE SURFACES IN LINK COMPLEMENTS

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Abstract. We generalize a theorem of Finkelstein and Moriah and show that if a link $L$ has a 2$n$-plat projection satisfying certain conditions, then its complement contains some closed essential surfaces. In most cases these surfaces remain essential after any totally nontrivial surgery on $L$.

A link $L$ in $S^3$ has a 2$n$-plat projection for some $n$, as shown in Figure 1, where a box on the $i$-th row and $j$-th column consists of 2 vertical strings with $a_{ij}$ left-hand half twist; in other words, it is a rational tangle of slope $1/a_{ij}$. See for example [BZ]. Let $n$ be the number of boxes in the even rows, so there are $n - 1$ boxes in the odd rows. Let $m$ be the number of rows in the diagram. It was shown by Finkelstein and Moriah [FM1, FM2] that if $n \geq 3$, $m \geq 5$, and if $|a_{ij}| \geq 3$ for all $i, j$, then the link exterior $E(L) = S^3 - \text{Int}N(L)$ contains some essential planar surfaces, which can be tubed on one side to obtain closed incompressible surfaces in $E(K)$. In this note we will prove a stronger version of this theorem, showing that $E(L)$ contains some essential surfaces if $n \geq 3$, the boxes on the two ends of the odd rows have $|a_{ij}| \geq 3$, and $a_{ij} \neq 0$ for the boxes which are not on the ends of the rows. We allow $a_{ij} = 0$ for boxes on the ends of the even rows, and there is no restriction on $m$, the number of rows in the diagram. The argument here provides a much simpler proof to the above theorem of Finkelstein and Moriah. In [FM2] that theorem was applied to show that if $L$ is a knot then all surgeries on $L$ contain essential surfaces. Corollary 2 below generalizes this to the case when $L$ has multiple components, with a mild restriction that each component of $L$ intersects some "allowable" spheres.

We first give some definitions. Let $\alpha = \alpha(a_1, \ldots, a_m)$ be an arc running monotonically from the top to the bottom of the 2$n$-plat, such that $\alpha$ is disjoint from the boxes, and on the $i$-th row there are $a_i$ boxes on the left of $\alpha$. See Figure 1 for the arc $\alpha(1, 1, 1, 2, 2)$. The arc $\alpha$ is an allowable path if (i) each row has at least one box on each side of $\alpha$, and (ii) $\alpha$ intersects $L$ at $m + 1$ points, (so $\alpha$ intersects $L$ once when passing from one row to another). Note that the leftmost allowable path is $\alpha(1, \ldots, 1)$, which has on its left one box from each row.

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Given an allowable path $\alpha = \alpha(a_1, \ldots, a_m)$, we can connect the two ends of $\alpha$ by an arc $\beta$ disjoint from the projection of $L$ to form a circle, then cap it off by two disks, one on each side of the projection plane, to get a sphere $S = S(a_1, \ldots, a_m)$, called an allowable sphere. $S$ cuts $(S^3, L)$ into two tangles $(B, T)$ and $(B', T')$, where $(B, T)$ denotes the one on the left hand side of $S$. Let $P = P(a_1, \ldots, a_m)$ be the planar surface $S \cap E(L)$, which cuts $E(L)$ into two pieces $X = X(a_1, \ldots, a_m)$ and $X' = X'(a_1, \ldots, a_m)$, with $X = B \cap E(L)$ the one on the left of $P$. Let $F = F(a_1, \ldots, a_m)$ be the surface obtained by tubing $P$ on the left hand side; in other words, $F$ is the component of $\partial X$ containing $P$, pushed slightly into the interior of $E(L)$. Similarly, denote by $F' = F'(a_1, \ldots, a_m)$ the surface obtained by tubing $P$ on the right hand side.

Recall that a properly embedded surface $F$ in a 3-manifold $M$ is an essential surface if it is incompressible, $\partial$-incompressible, and is not boundary parallel. We define a surface $F$ on the boundary of $M$ to be essential if it is incompressible, $M \neq F \times I$, and there is no compressing disk of $\partial M$ which intersects $F$ at a single essential arc in $F$. Thus if $F$ is properly embedded in $M$, then it is essential if and only if after cutting along $F$ the two copies of $F$ are essential in the resulting manifold. A 3-manifold $M$ is $\partial$-irreducible if $\partial M$ is incompressible in $M$. Given a set $A$ in $M$, denote by $N(A)$ a regular neighborhood of $A$ in $M$.

**Theorem 1.** Suppose $L$ has a $2n$-plat projection such that (i) $n \geq 3$; (ii) $a_{ij} \neq 0$ for $j \neq 0, n$; and (iii) $|a_{ij}| \geq 3$ for $i$ odd and $j = 0$ or $n - 1$. Let $S = S(a_1, \ldots, a_m)$ be an allowable sphere. Then $E(L)$ is irreducible, and the surfaces $F = F(a_1, \ldots, a_m)$ and $F' = F'(a_1, \ldots, a_m)$ are essential in $E(L)$.

Let $L = L_1 \cup \ldots L_k$ be a $k$ component link, let $r = (r_1, \ldots, r_k)$ be a set of slopes on $\partial N(L)$, with $r_i$ a slope on $\partial N(L_i)$. Then $L(r)$ denotes the $r$-Dehn surgery on $L$, which is the manifold obtained by gluing $k$ solid tori $V_1, \ldots, V_k$ to $E(L)$ so that each $r_i$ is identified with a meridian disk of $V_i$. The surgery and the slope $r$ are totally nontrivial if no $r_i$ is the meridian slope of $L_i$.

**Corollary 2.** Let $L$ be as in Theorem 1. If each component of $L$ intersects some allowable sphere, then $L(r)$ is a Haken manifold for all totally nontrivial $r$, and the
Remark. (1) It is easy to see that \( F = F(a_1, \ldots, a_m) \) being incompressible implies that \( P = P(a_1, \ldots, a_m) \) is an essential planar surface in \( E(L) \). With a similar proof to that of Theorem 1 one can show that \( P \) is essential even if the condition \( |a_{ij}| \geq 3 \) in (iii) of Theorem 1 is replaced by \( |a_{ij}| \geq 2 \). This generalizes the main theorem of [FM1].

(2) When \( n \leq 2 \), the link is a 2-bridge link, so by [HT] \( E(L) \) contains no closed essential surface. Hence the assumption \( n \geq 3 \) in Theorem 1 is necessary.

(3) By definition of \( 2n \)-plait projection, the number of rows \( m \) is odd. If \( m = 1 \) the link is a composite link, and our assumption implies that it is nonsplit. In this case \( E(L) \) is irreducible, and the surfaces in the theorem are swallow-follow tori, which are essential. Therefore the theorem is true for \( m = 1 \). We may thus assume that \( m \geq 3 \) in the proof of Theorem 1.

(4) In Corollary 2, each component of \( L \) intersects some allowable sphere if and only if no component of \( L \) is on the left of \( S(1, \ldots, 1) \) or the right of \( S(n-2, n-1, \ldots, n-2) \), which is equivalent to that \( a_{i1} \) and \( a_{j,n-1} \) are odd for some odd \( i, j \).

(5) The results remain true if we replace the twist tangles with rational tangles of slopes \( p_{ij}/a_{ij} \) with \( a_{ij} \) satisfying the conditions in the theorem, or certain kinds of more complicated tangles. However in this case the link diagram would not be in \( 2n \)-plait form.

A \( p/q \) rational tangle is a pair \((B,T)\), where \( B \) is a “pillow case” in \( \mathbb{R}^3 \) with corner points \((0, \pm 1, \pm 1)\), and \( T \) is obtained by taking 2 arcs of slope \( p/q \) on \( \partial B \) connecting the four corner points of the pillow case, then pushing the interior of the arcs into the interior of \( B \). The \( xz \)-plane intersects \( \partial B \) in a circle \( C \) of slope \( \infty \), called a vertical circle on \( \partial B \). Each component of \( \partial B - C \) contains two points of \( \partial T \). We need the following result about rational tangles.

**Lemma 3.** Suppose \((B,T)\) is a \( p/q \) rational tangle, and \( C \) a vertical circle on \( \partial B \). Let \( X = B - \text{Int} N(T) \), and let \( P \) be a component of \((\partial B \cap X) - C\).

(i) if \( q \geq 1 \) then \( P \) is incompressible in \( X \);

(ii) if \( q \geq 2 \), then \( \partial X - C \) is incompressible in \( X \);

(iii) if \( q \geq 3 \), then any compressing disk of \( \partial X \) intersects \( P \) at least twice.

**Proof.** (ii) Notice that when attaching a 2-handle to \( X \) along the curve \( C \), the manifold \( X_C \) is the exterior of a 2-bridge link associated to the rational number \( p/q \), which is nontrivial and nonsplit when \( q \geq 2 \). In particular, \( \partial X_C \) is incompressible. If \( D \) is a compressing disk of \( \partial X \) disjoint from \( C \), then since \( X \) is a handlebody of genus 2, we can find a nonseparating compressing disk \( D' \) which is still disjoint from \( C \). But then \( D' \) would remain a compressing disk in \( X_C \), a contradiction.

(i) If \( q \geq 2 \) this follows from (ii) and the fact that \( P \) is a subsurface of \( \partial X - C \) whose complement contains no disk components. If \( q = 1 \), \( X \) is a product \( P \times I \), and the result is obvious.

(iii) By (i) \( P \) is incompressible, which also implies that \( \partial X - P \) is incompressible because any simple loop on \( \partial X - P \) is isotopic to one in \( P \). By [Wu, Lemma 2.1] there is no compressing disk of \( X \) intersecting \( P \) at a single essential arc. □

The following lemma is well-known. The proof is an easy inner-most circle outer-most arc argument, and will be omitted.
Lemma 4. Let $F$ be an essential surface in a compact orientable 3-manifold $M$. If $M' = M - \text{Int}N(F)$ is irreducible, and no compressing disk of $\partial M'$ is disjoint from the two copies of $F$ on $\partial M'$, then $M$ is irreducible and $\partial$-irreducible.

We now proceed to prove Theorem 1. In the following, we will assume that $L$ is a link as in Theorem 1. By the remark above, we may assume $m \geq 3$.

Lemma 5. The manifold $X = X(1, \ldots, 1)$ is irreducible and $\partial$-irreducible.

Proof. Consider the tangle $(B, T)$ on the left of $S$. By an isotopy of $(B, T)$ we can untwist the boxes in $T$ which lie on the even rows of the projection of $L$, so the tangle $(B, T)$ is equivalent to the one shown in Figure 2, where each box corresponds to the first box on an odd row of the projection of $L$; hence there are $k = (m + 1)/2 \geq 2$ boxes, ($k = 3$ in Figure 2.) Let $D_1, \ldots, D_k$ be the disks represented by the dotted lines in Figure 2, which cuts $(B, T)$ into $k + 1$ subtangles $(B_0, T_0), \ldots, (B_k, T_k)$, where $(B_0, T_0)$ is the one in the middle, which intersects all the $D_i$. Let $P_i = D_i \cap X$ be the twice punctured disk in $X$ corresponding to $D_i$. They cut $X$ into $X_0, \ldots, X_k$, with $X_i = B_i - \text{Int}N(T_i)$ the tangle space of $(B_i, T_i)$.

We want to show that $\cup P_i$ is essential in $X$. Since each $(B_i, T_i)$, $i \geq 1$, is a twist tangle with at least 3 twists, by Lemma 3, the surface $P_i$ is essential in $X_i$. Now consider $X_0$. Put $Q = \partial B_0 - \cup D_i$. If $D$ is a compressing disk of $Q$ in $X_0$, then it is a disk in $B_0$ disjoint from $T_0 \cup (\cup D_i)$; but since $T_0 \cup (\cup D_i)$ is connected, this would imply that one side of $D$ is disjoint from all $D_i$, hence $\partial D$ is a trivial curve on $Q$, which is a contradiction. Therefore $Q$ is incompressible in $X_0$. Assume there is a disk $D$ in $X_0$ such that $\partial D \cap (\cup P_i)$ has only one component. Since each string of $T_0$ has ends on different $D_i$, we see that $\partial D \cap \partial N(T_0) = \emptyset$, so $\partial D \cap (\cup P_i)$ is either a proper arc in some $D_i$, which separates the two points of $T_0$ on $D_i$, or it is a circle bounding a disk on $D_i$ containing exactly one point of $T_0$, or $\partial D$ can be isotoped into $Q$. The first two cases are impossible because then $D$ would be a disk in $B_0$ disjoint from $T_0$ and yet each component of $\partial B_0 - \partial D$ contains an odd number of endpoints of $T_0$. The third case contradicts the incompressibility of $Q$. This completes the proof that $\cup P_i$ is an essential surface in $X$.

Notice that all $X_i$ are handlebodies, and hence irreducible. Since $Q$ is incompressible in $X_0$, and by Lemma 3, the surfaces $\partial X_i - P_i \subset \partial X_i - \partial D_i$ are incompressible in $X_i$ for $i \geq 1$, it follows from Lemma 4 that $X$ is irreducible and $\partial$-irreducible. □
Lemma 6. The manifold \( X = X(a_1, \ldots, a_m) \) associated to an allowable sphere \( S(a_1, \ldots, a_m) \) is irreducible and \( \partial \)-irreducible.

Proof. There is a sequence of allowable spheres \( S_1, \ldots, S_{k+1} \), such that \( S_1 = S(1, \ldots, 1) \), \( S_{k+1} = S(a_1, \ldots, a_m) \), and the non-common part of \( S_i \) and \( S_{i+1} \) bounds a single box in the projection of \( L \), that is, \( S_i \cup S_{i+1} - \text{Int}(S_i \cap S_{i+1}) = \partial B_i \) for some twist tangle \((B_i, T_i)\) with \( a \neq 0 \) left hand half-twists. Let \( (B_i, T_i) \) be the tangle on the left of \( S_i \), and let \( X_i = B_i - \text{Int}N(T_i) \). Similarly, let \( \tilde{X}_i = \tilde{B}_i - \text{Int}N(\tilde{T}_i) \).

Thus \( X = X_{k+1} = X_k \cup \partial \tilde{X}_k \), where \( P = X_k \cap \tilde{X}_k \) is a twice punctured disk. By Lemma 5, \( X_1 \) is irreducible and \( \partial \)-irreducible, and by induction on the length of the sequence we may assume that \( X_k \) is irreducible and \( \partial \)-irreducible. Clearly \( P \) is incompressible and \( \partial \)-incompressible on the \( X_k \) side. If \( |a| \geq 3 \) then, by Lemma 3, \( P \) is also incompressible and \( \partial \)-incompressible on the \( \tilde{X}_k \) side, hence \( P \) is an essential surface in \( X \). Since \( \partial \tilde{X}_k - P \) is also incompressible in \( \tilde{X}_k \), and since \( X_k \) and \( \tilde{X}_k \) are irreducible, it follows that \( X = X_k \cup \partial \tilde{X}_k \) is irreducible and \( \partial \)-irreducible. Also, if \( |a| = 1 \) then \( \tilde{X}_k \) is a product \( P \times I \), so \( X_{k+1} \cong X_k \), and the result follows.

It remains to prove the lemma for the case \( |a| = 2 \). In this case there is a disk \( D \) in \( \tilde{X}_k \) which intersects \( P \) in a single arc \( \gamma \), cutting \( \tilde{X}_k \) into a pair \((A \times I, A \times \partial I)\), where \( A \) is an annulus. Thus

\[
X = X_k \cup \partial \tilde{X}_k = (X_k \cup_{\gamma \times I} (D \times I)) \cup_{A \times \partial I} (A \times I) \cong X_k \cup_{A \times \partial I} (A \times I).
\]

Since a compressing disk of \( \partial(A \times I) \) intersects \( A \times \partial I \) at least twice, by the same argument as above, one can show that \( A \times \partial I \) is essential in \( X \), and \( X \) is irreducible and \( \partial \)-irreducible. \( \square \)

Proof of Theorem 1. Let \( F, F' \) be the surfaces in the theorem, isotoped slightly to be disjoint from each other. Then \( F \cup F' \) cuts \( E(L) \) into three parts: The component on the left of \( F \) is homeomorphic to \( X \), the one on the right of \( F' \) is homeomorphic to \( X' \), and the one \( X'' \) between \( F \) and \( F' \) is the union of \( P \times I \) and \( Q \times I \), where \( Q \) is the set of tori in \( \partial E(L) \) which intersect \( \partial P \). We have shown in Lemma 6 that \( X \) is irreducible and \( \partial \)-irreducible, and because of symmetry, so is \( X' \). Now \( X'' \) can be cut into \( F \times I \) along some (essential) meridional annuli in \( Q \times I \), hence by Lemma 4 it is irreducible and \( \partial \)-irreducible. Since \( F \) and \( F' \) have genus at least 2, they are not boundary parallel. It follows that \( F \cup F' \) is essential in \( X \), and \( X \) is irreducible. \( \square \)

Proof of Corollary 2. Let \( S_1, \ldots, S_k \) be a set of disjoint allowable spheres, so that \( S_1 = S(1, \ldots, 1) \), \( S_k = S(n-2, n-1, \ldots, n-2) \), and there is only one box of the projection of \( L \) between \( S_i \) and \( S_{i+1} \). These spheres are similar to those in the proof of Lemma 6, except that they are now mutually disjoint, so the manifold between \( S_i \) and \( S_{i+1} \) is a product \( S^2 \times I \).

Let \( F_i \) be the essential surfaces corresponding to \( S_i \), as defined before Theorem 1, isotoped slightly so that they are disjoint from each other. Also, isotope \( F'_k \) to be disjoint from \( F_k \). Then the set of \( k + 1 \) surfaces \( F_1, F_2, \ldots, F_k, F'_k \) cuts \( E(L) \) into \( k + 2 \) components \( Y_0, \ldots, Y_{k+1} \), where \( Y_0 \) is the manifold \( X(1, \ldots, 1) \) on the left of \( F_1 \), \( Y_{k+1} = X'(n-2, n-1, \ldots, n-2) \) is the manifold on the right of \( F_{k+1} \), \( Y_k \) is between \( F_k \) and \( F'_k \), and for \( 1 \leq i \leq k - 1 \), \( Y_i \) is between \( F_i \) and \( F_{i+1} \). Since all the \( F_i \) and \( F'_k \) are essential, we see that \( Y_i \) are all irreducible and \( \partial \)-irreducible.

We need to show that the manifold \( \tilde{Y}_i \) obtained from \( Y_i \) by Dehn filling on its
toroidal boundary components (if any), with slopes the corresponding subset of \( r \), is still irreducible and \( \partial \)-irreducible. The result will then follow by gluing the pieces together along \( F_i \) and \( F'_i \).

Our assumption implies that \( Y_0 \) and \( Y_{k+1} \) are disjoint from \( \partial E(L) \), hence \( \hat{Y}_i = Y_i \) for \( i = 0, k + 1 \). Now \( Y_k \) is a regular neighborhood of \( P \cup Q \), where \( P = S_k \cap E(L) \), and \( Q \) is the set of tori in \( \partial E(L) \) which intersect \( P \). Since \( S \) is separating, each component \( Q_j \) of \( Q \) intersects \( \partial P \) at least twice, so there are two nonparallel essential annuli in \( Y_k \), each having a boundary component on \( Q_j \) with meridional slope. Applying Menasco’s theorem [Me] and Scharlemann’s theorem [Sch] on each component of \( Q \), we see that after any totally nontrivial Dehn filling on \( Q \) the manifold \( \hat{Y}_k \) is still irreducible and \( \partial \)-irreducible.

Now assume \( 1 \leq i \leq k - 1 \). Let \( (B'_i, T'_i) \) be the twist tangle between \( S_i \) and \( S_{i+1} \). Notice that if the twist number \( a \) of \( T'_i \) is odd then \( Y_i \) contains no component of \( \partial E(L) \), so \( \hat{Y}_i = Y_i \) and we are done. If \( a \) is even, then the tangle \( (B_{i+1}, T_{i+1}) \) on the left of \( S_{i+1} \) may contain a loop \( K \) intersecting the twist tangle \( (B'_i, T'_i) \), so \( Y_i \) may contain a single component \( Q \) of \( \partial E(L) \).

Let \( Y_i(m) \) be the manifold obtained by the trivial Dehn filling on \( Q \). Then \( F_i \) has a compressing disk \( D \) in \( Y_i(m) \) intersecting the core \( K \) of the Dehn filling solid torus only once, so \( K \) is not a cable knot in \( Y_i(m) \). See Figure 3. It follows from [Sch] that after surgery the manifold \( \hat{Y}_i \) is irreducible. Also, by [CGLS, Theorem 2.4.3] \( \hat{Y}_i \) is \( \partial \)-irreducible if the surgery slope \( r_j \) on the torus \( Q \) intersects the meridian slope \( m \) at least twice. Now if \( r_j \) intersects \( m \) only once, then \( m \) is a longitude after the surgery, hence the manifold \( \hat{Y}_i \) is homeomorphic to the one obtained by cutting \( Y_i \) along the annulus \( D \cap Y_i \), denoted by \( \hat{Y}_i \). Now there is an annulus \( A \) in \( B_{i+1} - \text{Int}B_i \) (\( B_i \) is the ball on the left of \( S_i \)) separating the twist tangle \( (B'_i, T'_i) \) from the other arcs of \( L \), which cuts \( \hat{Y}_i \) into \( \hat{X} \cong B'_i - \text{Int}N(T'_i) \) and some \( G \times I \), where \( G \) is a subsurface of \( F_i \) with one boundary component. Clearly \( A \) is essential in \( G \times I \). Since the twist number \( a \) is even, our assumption in Theorem 1 implies that \( |a| \geq 2 \). Hence by Lemma 3 the surface \( \partial \hat{X} - A \) is incompressible in \( \hat{X} \), which implies that \( A \) is essential in \( \hat{X} \). It follows that \( \hat{Y}_i \) is irreducible and \( \partial \)-irreducible. □

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