On convergence of neural network methods for solving elliptic interface problems

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Abstract

With the remarkable empirical success of neural networks across diverse scientific disciplines, rigorous error and convergence analysis are also being developed and enriched. However, most existing theoretical works fail to explain the performance of neural networks in solving interface problems. In this paper, we perform a convergence analysis of the neural network-based method for solving second-order elliptic interface problems. Here, we consider the domain decomposition method with gradient-enhanced on the boundary and interface and Lipschitz regularized loss. It is shown that the neural network sequence obtained by minimizing the regularization loss function converges to the unique solution to the interface problem in $H^2$ as the number of samples increases. This result serves the theoretical understanding of deep learning-based solvers for interface problems.

Key words. Elliptic interface problems; Generalization error; Convergence analysis; Neural networks;

1 Introduction

Deep learning in the form of deep neural networks (DNNs) has been effectively used in diverse scientific disciplines beyond its traditional applications. In particular, thanks to their potential nonlinear approximation power, DNNs are being exploited to construct alternative approaches for solving partial differential equations (PDEs), e.g., the deep Ritz method (DRM) and physics-informed neural networks (PINNs). The key idea of these methods is to reformulate the solution to a PDE with a closed-form expression in the form of a neural network, and the parameters of which are obtained by minimizing the physics-informed loss given by the corresponding PDE. The original works on the use of neural networks to solve PDEs were proposed in the 1990s and revisited recently with the renaissance of neural networks and the development of deep learning techniques; see e.g. and references therein.

Faced with the remarkable empirical achievements of these generic methods, the associated rigorous error and convergence analysis are also being developed and enriched. In previous work, the Hölder continuity constant was used to obtain the generalization analysis of PINNs in the case of linear second-order elliptic and parabolic type PDEs. used quadrature points in the formulation of the loss and carried out an a-posteriori-type generalization error analysis of PINNs for both forward and inverse problems. studied linear PDEs and proved both a priori and a posterior estimates for PINNs and variational PINNs in Sobolev spaces. provided
a theoretical understanding on the generalization abilities of PINNs and Extended PINNs (XPINNs) [15]. [24] derived an a priori generalization estimate for a class of second-order linear PDEs in the context of two-layer neural networks by assuming that the exact solutions of PDEs belong to a Barron-type space [2]. [16] provided a nonasymptotic convergence rate of PINNs with ReLU networks for the second linear elliptic equations. For high-dimensional PDEs, [23] derived a priori and dimension explicit generalization error estimates for the DRM under the assumption that the solutions of the PDEs lie in the spectral Barron space, and [3] provided an analysis of the generalization error for linear Kolmogorov equations by using tools from statistical learning theory and covering number estimates of neural network hypothesis classes.

Most existing theoretical works focus solely on basic equations and vanilla methods, however, fail to explain the performance of neural networks in solving interface problems, where the interaction at the interface and the application of the domain decomposition method (DDM) introduce additional analytical challenges. Interface problems are a widespread class of problems in scientific computing with many applications across diverse fields; see e.g. [19, 28, 13, 22, 11, 21] and references therein. Meanwhile, many efforts have been made to use neural networks to solve interface problems. In particular, the use of multiple neural networks based on the domain decomposition method (DDM) to solve interface problems has attracted increasing attention [15, 20, 10]. In the context of DDM-based deep learning methods, the computational domain is decomposed into several disjoint subdomains according to the interface, and the solution to the interface problem is the combination and ensemble of all local networks, where each of them is used to approximate the solution in one subdomain.

The main contribution of this work is to provide the convergence theory for solving elliptic interface problems using neural networks based on DDM. Our work is inspired by recent works [32, 36, 16]. First, we construct Lipschitz regularized empirical loss based on the domain decomposition method and gradient-enhanced approach [36] on the boundary and interface. Furthermore, following [32], we adopt probabilistic space filling arguments to quantify the generalization error of neural networks and derive an upper bound of the expected unregularized loss for the interface problems. With these results, we show that the sequence of minimizers of the designed regularized loss function converges to the solution to the elliptic interface problem in $H^2$. To the best of our knowledge, this is the first theoretical work that proves the convergence of solving interface problems using neural networks.

The rest of the paper is organized as follows. In Section 2 we briefly introduce the mathematical setup and the deep learning-based PDE solver for solving elliptic interface problems. In Section 3, we present the convergence analysis. The proof of the convergence analysis will be presented in Section 4. Finally, we conclude the paper in Section 5.

2 Preliminaries

We first introduce some notations. Let $x = (x_1, \ldots, x_d)$ be a point in $\mathbb{R}^d$ and $\mathcal{U} \subset \mathbb{R}^d$ be a open set. Let $C(\mathcal{U}) = \{ f : \mathcal{U} \to \mathbb{R}^d | f \text{ is continuous} \}$ denote the space of continuous functions. Let $\mathbb{Z}_+^d$ denote the lattice of $d$-dimensional nonnegative integers. For $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d$, we set $|\mathbf{m}| := m_1 + \cdots + m_d$, and

$$D^{\mathbf{m}} = \frac{\partial |\mathbf{m}|}{\partial x_1^{m_1} \cdots \partial x_d^{m_d}}.$$  

For a positive integer $m$, we define

$$C^m(\mathcal{U}) := \{ f : D^k f \in C(\mathcal{U}) \text{ for all } |k| \leq m \}.$$  

In particular, we define that for $f : \mathbb{R}^2 \to \mathbb{R}$ if $f \in C^2(\mathbb{R}^2)$,

$$Df = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right), \quad D^2 f = \left( \frac{\partial^2 f}{\partial x_1^2}, \frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial^2 f}{\partial x_2^2} \right).$$
2.1 Neural networks

First, we introduce the employed network architecture, i.e., fully connected neural network (FCNN), in this paper. Mathematically, a $N$-layer FCNN is a nested composition of sequential linear functions and nonlinear activation functions, which takes the form

$$s_i = f_i(s_{i-1}) := \sigma(W_is_{i-1} + b_i), \quad \text{for} \quad i = 1, \cdots, N-1,$$

$$s_N = f_N(s_{N-1}) := W_Ns_{N-1} + b_N,$$

where $s_0 = x \in \mathbb{R}^{d_0}$ is the input variable, $s_i \in \mathbb{R}^{d_i}$ denotes the output of the $i$-th hidden layer, $s_N \in \mathbb{R}^{d_{out}}$ is the corresponding output. Here, weights $W_i \in \mathbb{R}^{d_{i+1} \times d_i}$ and bias $b_i \in \mathbb{R}^{d_{i+1}}$ are trainable parameters. $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is the nonlinear activation function applied element-wise to a vector. Popular examples include rectified linear unit (ReLU) $\text{ReLU}(z) = \max(0, z)$, sigmoid $\text{Sig}(z) = 1/(1 + e^{-z})$, and tanh$(z)$. Given an input $x$, the output of a FCNN, denoted by $\mathcal{N}\mathcal{N}(x)$, can be expressed as

$$\mathcal{N}\mathcal{N}(x) = f_N \circ \cdots \circ f_1(x).$$

Furthermore, we denote all the trainable parameters (e.g., $W_i, b_i$) in (2) as $\theta \in \Theta$, where $\theta$ is a high-dimensional vector and $\Theta$ is the space of $\theta$. Given a network architecture $\overrightarrow{n}$ (e.g., the number of layers and the width of each hidden layer), we denote the set of all expressible functions by the FCNN $\mathcal{N}\mathcal{N}$ as

$$\mathcal{H}^{\mathcal{N}\mathcal{N}} = \{(\cdot, \theta) : \mathbb{R}^{d_0} \rightarrow \mathbb{R}^{d_{out}} | \theta \in \Theta\}.$$ (3)

2.2 Problem setup

Let $\Omega = \Omega_1 \cup \Omega_2$ be a bounded domain in $\mathbb{R}^2(\mathbb{R}^3)$ with smooth boundary $\partial \Omega$, $\Omega_1 \subset \Omega$ be an open domain with smooth boundary $\Gamma = \partial \Omega_1 \subset \Omega$ and $\Omega_2 = \Omega \setminus \Omega_1$ (see Fig. 1 for an illustration). We consider the following elliptic interface problem

$$-\nabla \cdot (a_i \nabla u) + b_i u = f_i, \quad \text{in} \ \Omega_i, \ i = 1, 2,$$

$$[a \nabla u \cdot n] = \psi, \quad \text{on} \ \Gamma,$$

$$[u] = \varphi, \quad \text{on} \ \Gamma,$$

$$u = g, \quad \text{on} \ \partial \Omega.$$ (4a) (4b) (4c) (4d)

where $[\mu] := \mu|_{\Omega_2} - \mu|_{\Omega_1}$ denotes the jump of a quantity $\mu$ across $\Gamma$, and $n$ denotes the unit outward normal of $\Omega_1$. For the ease of the exposition, we assume that the coefficient functions

$$a(x) = \begin{cases} a_1(x), & \text{if} \ x \in \Omega_1 \\ a_2(x), & \text{if} \ x \in \Omega_2 \end{cases}, \quad b(x) = \begin{cases} b_1(x), & \text{if} \ x \in \Omega_1 \\ b_2(x), & \text{if} \ x \in \Omega_2 \end{cases}$$

are positive and piecewise spatial functions. The unknown part (or exact solution) of this problem is $u$, while others are given in advance.
This paper focuses on a neural network method based on the domain decomposition method (DDM) for solving elliptic interface problems [4]. The DDM-based neural network solver is an extension of PINNs. The methodology is to decompose the whole domain $\Omega$ into several separate subdomains according to the interface, and train several sub-PINNs optimization problems. The continuity between sub-nets is maintained via the interface loss function, and the final solution is the combination and ensemble of all sub-nets, where each of them is responsible for prediction in one subdomain. Specifically, FCNNs [3], denoted as $\left( u(x, \theta_1) \big|_{\Omega_1}, u(x, \theta_2) \big|_{\Omega_2} \right)$ with parameter set $\theta = (\theta_1, \theta_2)$, are applied to parametrize the solution space for the interface problem [4].

We denote the number of training data as $m = (m_{r_1}, m_{r_2}, m_{\Gamma}, m_b)$, and denote the training data refer to points domains $\Omega_1, \Omega_2$, interface $\Gamma$, and boundary $\partial \Omega$ by $T_{r_1}^{m_{r_1}} = \{x_{r_1}^i\}_{i=1}^{m_{r_1}}$, $T_{r_2}^{m_{r_2}} = \{x_{r_2}^i\}_{i=1}^{m_{r_2}}$, $T_{\Gamma}^{m_{\Gamma}} = \{x_{\Gamma}^i\}_{i=1}^{m_{\Gamma}}$ and $T_b^{m_b} = \{x_b^i\}_{i=1}^{m_b}$, respectively. Moreover, we suppose these four types of data sets are independently and identically (iid) samples from probability distributions $\mu_{r_1}, \mu_{r_2}, \mu_{\Gamma}$ and $\mu_b$, respectively. In addition, we use a gradient-enhanced approach [36] on the boundary and interface, and adopt Lipschitz regularizations to construct the loss function:

$$
\text{Loss}_m(u_1, u_2; \lambda, \lambda^R) := \frac{\lambda_{r_1}}{m_{r_1}} \sum_{i=1}^{m_{r_1}} \left| \mathcal{L}_1[u_1](x_{r_1}^i) - f_1(x_{r_1}^i) \right|^2 + \lambda_{r_1}^R R_{r_1}(u_1) + \frac{\lambda_{r_2}}{m_{r_2}} \sum_{i=1}^{m_{r_2}} \left| \mathcal{L}_2[u_2](x_{r_2}^i) - f_2(x_{r_2}^i) \right|^2 + \lambda_{r_2}^R R_{r_2}(u_2) + \frac{\lambda_{\Gamma_D}}{m_{\Gamma}} \sum_{i=1}^{m_{\Gamma}} \left| \mathcal{I}_D[u_1, u_2](x_{\Gamma}^i) - \varphi(x_{\Gamma}^i) \right|^2 + \lambda_{\Gamma_D}^R R_{\Gamma_D}(u_1, u_2) + \frac{\lambda_{\Gamma_N}}{m_{\Gamma}} \sum_{i=1}^{m_{\Gamma}} \left| \mathcal{I}_N[u_1, u_2](x_{\Gamma}^i) - \psi(x_{\Gamma}^i) \right|^2 + \lambda_{\Gamma_N}^R R_{\Gamma_N}(u_1, u_2) + \frac{\lambda_b}{m_b} \sum_{i=1}^{m_b} \| \mathcal{B}[u_2](x_b^i) - g(x_b^i) \|^2 + \lambda_b^R R_b(u_2),
$$

where $\lambda = (\lambda_{r_1}, \lambda_{r_2}, \lambda_{\Gamma_D}, \lambda_{\Gamma_N}, \lambda_b) \geq 0$, $\lambda^R = (\lambda_{r_1}^R, \lambda_{r_2}^R, \lambda_{\Gamma_D}^R, \lambda_{\Gamma_N}^R, \lambda_b^R) \geq 0$ (element-wise inequality),

$$
\mathcal{L}_1[u_1] = -\nabla \cdot (a_1 \nabla u_1) + b_1 u_1, \quad i = 1, 2,
\mathcal{I}_D[u_1, u_2] = (u_2 - u_1, D(u_2 - u_1), D^2(u_2 - u_1)),
\mathcal{I}_N[u_1, u_2] = (a_2 \nabla u_2 \cdot n - a_1 \nabla u_1 \cdot n, D(a_2 \nabla u_2 \cdot n - a_1 \nabla u_1 \cdot n))
\mathcal{B}[u_2] = (u_2, D u_2, D^2 u_2),
$$

Figure 1: A schematic view of the geometry description.
and 
\[ \varphi = (\varphi, D\varphi, D^2\varphi), \quad \psi = (\psi, D\psi), \quad g = (g, Dg, D^2g). \]

Here, \( R_1, R_2, R_{\Gamma_D}, R_{\Gamma_N}, R_b \) are regularization functionals, more precisely,
\[
R_1(u_1) = \left[ \mathcal{L}_1[u_1] \right]_{\Omega_1}^2, \quad R_2(u_2) = \left[ \mathcal{L}_2[u_2] \right]_{\Omega_2}^2, \quad R_b(u_2) = \left[ \mathcal{B}[u_2] \right]_{\partial\Omega}^2,
\]
\[
R_{\Gamma_D}(u_1, u_2) = \left[ \mathcal{I}_D[u_1, u_2] \right]_{\Gamma_D}^2, \quad R_{\Gamma_N}(u_1, u_2) = \left[ \mathcal{I}_N[u_1, u_2] \right]_{\Gamma_N}^2,
\]
where \([\mu]_{\mathcal{U}}\) is the Lipschitz constant of \( \mu \) on \( \mathcal{U} \), i.e.,
\[
[\mu]_{\mathcal{U}} = \sup_{x,y \in \mathcal{U}, x \neq y} \frac{||\mu(x) - \mu(y)||_\infty}{||x - y||_\infty}.
\]

In order to distinguish from the \( k \)-th continuously continuously differentiable function space \( C^k \), we denote \( C^{k,L}(\mathcal{U}) \) to be the collection of functions whose \( k \)-th derivative exists and is Lipschitz continuous. Finally, the solution to the interface problem (4) can be obtained by minimizing physics-informed loss (5).

For the convenience, we denote the expected PINN losses of problem (4) (when \( \lambda^R = 0 \)) by \( \text{Loss}^{\text{PINN}}(u_1, u_2; \lambda) \) and \( \text{Loss}_m^{\text{PINN}}(u_1, u_2; \lambda) \), respectively. More precisely,
\[
\text{Loss}^{\text{PINN}}(u_1, u_2; \lambda) = \lambda r_1 \left\| \mathcal{L}_1[u_1] - f_1 \right\|_{L^2(\Omega_1; \mu_{r_1})}^2 + \lambda r_2 \left\| \mathcal{L}_2[u_2] - f_2 \right\|_{L^2(\Omega_2; \mu_{r_2})}^2
+ \lambda r_D \left\| \mathcal{I}_D[u_1, u_2] - \varphi \right\|_{L^2(\Gamma; \mu_{r_D})}^2 + \lambda r_N \left\| \mathcal{I}_N[u_1, u_2] - \psi \right\|_{L^2(\Gamma; \mu_{r_N})}^2
+ \lambda b \left\| \mathcal{B}[u_2] - g \right\|_{L^2(\partial\Omega; \mu_b)}^2,
\]
\[
\text{Loss}_m^{\text{PINN}}(u_1, u_2; \lambda) = \frac{\lambda r_1}{m_{r_1}} \sum_{i=1}^{m_{r_1}} \left\| \mathcal{L}_1[u_1](x_{r_1}^i) - f_1(x_{r_1}^i) \right\|^2 + \frac{\lambda r_2}{m_{r_2}} \sum_{i=1}^{m_{r_2}} \left\| \mathcal{L}_2[u_2](x_{r_2}^i) - f_2(x_{r_2}^i) \right\|^2
+ \frac{\lambda r_D}{m_{r_D}} \sum_{i=1}^{m_{r_D}} \left\| \mathcal{I}_D[u_1, u_2](x_{r_D}^i) - \varphi(x_{r_D}^i) \right\|^2 + \frac{\lambda r_N}{m_{r_N}} \sum_{i=1}^{m_{r_N}} \left\| \mathcal{I}_N[u_1, u_2](x_{r_N}^i) - \psi(x_{r_N}^i) \right\|^2
+ \frac{\lambda b}{m_b} \sum_{i=1}^{m_b} \left\| \mathcal{B}[u_2](x_b^i) - g(x_b^i) \right\|^2.
\]

Note that for well-defined the expected loss, it is required that \( \mathcal{L}_i[u_1] \) and \( f_i \) are in \( L^2(\Omega_i; \mu_{r_i}) \) with \( i = 1, 2 \), \( \mathcal{I}_D[u_1, u_2] \), \( \mathcal{I}_N[u_1, u_2] \), \( \varphi \) and \( \varphi \) are in \( L^2(\Gamma; \mu_{r_D}) \) and \( \mathcal{B}[u_2] \) and \( g \) are in \( L^2(\partial\Omega; \mu_b) \) for all \( (u_1, u_2) \in (\mathcal{H}_{r_1}^{\text{PINN}}, \mathcal{H}_{r_2}^{\text{PINN}}) \).

## 3 Main results

Throughout this section, our conclusion is based on the following two assumptions.

We first present assumptions on the training data distributions based on the probability space filling arguments [5] to guarantee that random samples drawn from probability distributions can fill up both the interior of the domains \( \Omega_1 \) and \( \Omega_2 \) as well as the boundary \( \partial\Omega \) and \( \Gamma \).

**Assumption 3.1 (Random sampling).** For the interface problem (4), let \( \mu_{r_1}, \mu_{r_2}, \mu_{r_D} \) and \( \mu_b \) be probability distributions defined on \( \Omega_1, \Omega_2, \Gamma \) and \( \partial\Omega \), respectively. Let \( \rho_{r_1}(\mu_{r_2}) \) be the probability density of \( \mu_{r_1}(\mu_{r_2}) \) with respect to \( d \)-dimensional Lebesgue measure on \( \Omega_1(\Omega_2) \). Let \( \rho_D(\mu_b) \) be the probability density of \( \mu_D(\mu_b) \) with respect to \( (d-1) \)-dimensional Hausdorff measure on \( \Gamma(\partial\Omega) \).
1. $\rho_{r_1}, \rho_{r_2}, \rho_{\Gamma}$ and $\rho_0$ are supported on $\overline{\Omega}_1$, $\overline{\Omega}_2$, $\Gamma$ and $\partial \Omega$, respectively. Also, $\inf_{\Omega_1} \rho_{r_1} > 0$, $\inf_{\Omega_2} \rho_{r_2} > 0$, $\inf_{\Gamma} \rho_{\Gamma} > 0$, and $\inf_{\partial \Omega} \rho_0 > 0$.

2. For $\epsilon > 0$, there exists partitions of $\Omega_1$, $\Omega_2$, $\Gamma$ and $\partial \Omega$, $\{ \Omega_{1,j}^\epsilon \}_{j=1}^{K_1}$, $\{ \Omega_{2,j}^\epsilon \}_{j=1}^{K_2}$, $\{ \Gamma_j^\epsilon \}_{j=1}^{K_3}$ and $\{ \partial \Omega_j^\epsilon \}_{j=1}^{K_4}$ that depend on $\epsilon$ such that for each $j$, there are cubes $H_\epsilon(z_{1,j}^\epsilon)$, $H_\epsilon(z_{2,j}^\epsilon)$, $H_\epsilon(z_{\Gamma,j}^\epsilon)$ and $H_\epsilon(z_{\partial \Omega,j}^\epsilon)$ of side length $\epsilon$ centered at $z_{1,j}^\epsilon \in \Omega_{1,j}^\epsilon$, $z_{2,j}^\epsilon \in \Omega_{2,j}^\epsilon$, $z_{\Gamma,j}^\epsilon \in \Gamma_j^\epsilon$ and $z_{\partial \Omega,j}^\epsilon \in \partial \Omega_j^\epsilon$, respectively, satisfying $\Omega_{1,j}^\epsilon \subset H_\epsilon(z_{1,j}^\epsilon)$, $\Omega_{2,j}^\epsilon \subset H_\epsilon(z_{2,j}^\epsilon)$, $\Gamma_j^\epsilon \subset H_\epsilon(z_{\Gamma,j}^\epsilon)$, $\partial \Omega_j^\epsilon \subset H_\epsilon(z_{\partial \Omega,j}^\epsilon)$.

3. There exists positive constants $c_1, c_2, c_3, c_4$ such that for all $\epsilon > 0$, the partitions from the above satisfy $c_1 \epsilon^d \leq \mu_1(\Omega_{1,j}^\epsilon)$, $c_2 \epsilon^d \leq \mu_2(\Omega_{2,j}^\epsilon)$, $c_3 \epsilon^{d-1} \leq \mu(\Gamma_j^\epsilon)$ and $c_4 \epsilon^{d-1} \leq \mu(\partial \Omega_j^\epsilon)$ for all $j$.

In contrast to the traditional applications of deep learning, such as image classification tasks and natural language processing, where the data distributions are unknown and data sampling is very expensive, the aforementioned assumptions are mild and easy to satisfy when solving PDEs as the data distribution is known (e.g., the uniform probability distribution).

In addition, for well-defined loss function \[ \text{(5)} \], we have to make some assumptions about the interface problem \[ \text{(4)} \] and the hypothesis space of neural networks.

**Assumption 3.2 (Hypothesis space).** Let $\mathcal{H}_{1,m}$ and $\mathcal{H}_{2,m}$ be the class of neural networks defined on $\overline{\Omega}_1$ and $\overline{\Omega}_1$, respectively.

1. Let $f_1 \in C^{0,L}(\Omega_1)$, $f_2 \in C^{0,L}(\Omega_2)$, $\psi \in C^{1,L}(\Gamma)$, $\varphi \in C^{2,L}(\Gamma)$ and $g \in C^{2,L}(\partial \Omega)$, and the interface problem \[ \text{(4)} \] is well posed.

2. For each $m$, $\mathcal{H}_{1,m} \subset C^{2,L}(\overline{\Omega}_1)$, $\mathcal{H}_{2,m} \subset C^{2,L}(\overline{\Omega}_2)$ such that for any $(u_1, u_2) \in (\mathcal{H}_{1,m}, \mathcal{H}_{2,m})$, $\mathcal{L}_1[u_1] \in C^{0,L}(\Omega_1)$, $\mathcal{L}_2[u_2] \in C^{0,L}(\Omega_2)$, $\mathcal{I}_D[u_1, u_2] \in C^{0,L}(\Gamma)$, $\mathcal{I}_{\Gamma_N}[u_1, u_2] \in C^{0,L}(\Gamma)$ and $\mathcal{B}[u_2] \in C^{0,L}(\Gamma)$.

3. For each $m$, $\mathcal{H}_{1,m}(\mathcal{H}_{2,m})$ contains a network $\hat{u}_{1,m}(\hat{u}_{2,m})$ satisfying

$$Loss_{\mathcal{PNN}}^m(\hat{u}_{1,m}, \hat{u}_{2,m}; \lambda) = O(\max\{m_{r_1}^{1/3}, m_{r_2}^{1/3}, m_{\Gamma}^{-1/3}, m_{\partial \Omega}^{-1/3}\}^{-1/3});$$

4. and

$$\sup_m [\mathcal{L}_1[\hat{u}_{1,m}]]_{\Omega_1} < \infty, \quad \sup_m [\mathcal{L}_2[\hat{u}_{2,m}]]_{\Omega_2} < \infty, \quad \sup_m [\mathcal{B}[\hat{u}_{2,m}]]_{\partial \Omega} < \infty, \quad \sup_m [\mathcal{I}_{\Gamma_D}[\hat{u}_{1,m}, \hat{u}_{2,m}]]_\Gamma < \infty, \quad \sup_m [\mathcal{I}_{\Gamma_N}[\hat{u}_{1,m}, \hat{u}_{2,m}]]_\Gamma < \infty.$$

Popular activation functions, such as sigmoid $\text{Sig}(z) = 1/(1 + e^{-z})$ and $\tanh(z)$, could satisfy the Lipschitz condition. It is well known that FCNN can approximate continuous mappings \[ \text{(29)}, \text{(6)}, \text{(12)} \]. We recall the following Lemma \[ \text{[3.1]}, \text{[29]} \], which shows that FCNNs with enough neurons can simultaneously and uniformly approximate a continuous function and various of its partial derivatives. The network architecture $\overrightarrow{m}$ is expected to grow proportionally on the numbers of training samples $m$, the third term in Assumption \[ \text{[3.2]} \] can thus be attained. For the convenience, we rewrite $\mathcal{H}_m^{\overrightarrow{m}}$ as $\mathcal{H}_m$ simplicity.
Lemma 3.1. Let \( m \in \mathbb{Z}_+ \). Assume \( \sigma \in C^m(\mathbb{R}) \) and \( \sigma \) is not a polynomial. Then the hypothesis space of single hidden layer neural nets
\[
\mathcal{H}(\sigma) = \text{span}\{\sigma(wx+b) : w \in \mathbb{R}^d, b \in \mathbb{R}\}
\]
is dense in \( C^m(\mathbb{R}^d) \) i.e., for any \( f : \mathbb{R}^d \to \mathbb{R} \), any compact \( K \subset \mathbb{R}^d \), and any \( \varepsilon > 0 \), if \( f \in C^m(\mathbb{R}^d) \), there exists a \( g \in \mathcal{H}(\sigma) \) satisfying
\[
\max_{x \in K} |D^k f(x) - D^k g(x)| < \varepsilon,
\]
for all \( k \in \mathbb{Z}_+^d \) for which \( k \leq m_i \) for some \( i \).

Proof. The proof can be found in [29]. \( \Box \)

With these assumptions, the main result is presented as follows.

Theorem 3.2. Suppose Assumptions 3.1 and 3.2 hold. Let \( u^* \) be the unique solution to the interface problem (4). Let \( m_r, m_r', m_b \) and \( m_\Gamma \) be the number of iid samples from \( \mu_r, \mu_r', \mu_b \) and \( \mu_\Gamma \), respectively, and \( m_r' = \mathcal{O}(m_r) \), \( m_\Gamma = \mathcal{O}(m_r^{-\frac{1}{d}}) \), \( m_b = \mathcal{O}(m_r^{-\frac{1}{d}}) \). Let \( \lambda_{r,1}^R, \lambda_{r,2}^R, \lambda_{b,m}^R, \lambda_{D,m}^R, \lambda_{N,m}^R \) be a vector
\[
\lambda_{r,1}^R = \frac{3\lambda_1 d c_r^\frac{1}{3}}{C_m} \cdot m_r^{-\frac{1}{3}}, \quad \lambda_{r,2}^R = \frac{3\lambda_2 d c_r^\frac{1}{3}}{C_m} \cdot m_r^{-\frac{1}{2}},
\]
\[
\hat{\lambda}_{r,1}^R = \frac{3\lambda_1 d c_r^\frac{1}{3}}{C_m} \cdot m_r^{-\frac{1}{3}}, \quad \hat{\lambda}_{r,2}^R = \frac{3\lambda_2 d c_r^\frac{1}{3}}{C_m} \cdot m_r^{-\frac{1}{2}},
\]
\[
\lambda_{D,m}^R = \frac{3\lambda_{D} d c_b^\frac{1}{2}}{C_m} \cdot m_b^{-\frac{1}{2}},
\]
\[
\lambda_{N,m}^R = \frac{3\lambda_{N} d c_r^\frac{1}{2}}{C_m} \cdot m_r^{-\frac{1}{2}},
\]
\[
\lambda_{b,m}^R = \frac{3\lambda_{b} d c_b^\frac{1}{2}}{C_m} \cdot m_b^{-\frac{1}{2}}.
\]
Let
\[
C_m = 3 \max\{\kappa_1 d c_r^\frac{1}{d} m_r^{\frac{1}{3}}, \kappa_2 d c_r^\frac{1}{d} m_r^{\frac{1}{2}}, \kappa_b d c_r^\frac{1}{d} m_b^{\frac{1}{2}}, \kappa_\Gamma d c_r^\frac{1}{d} m_\Gamma^{\frac{1}{2}}, \kappa_\Gamma d c_r^\frac{1}{d} m_\Gamma^{\frac{1}{2}}\},
\]
where \( \kappa_1 = \frac{C_{r_1}}{c_{r_1}}, \kappa_2 = \frac{C_{r_2}}{c_{r_2}}, \kappa_b = \frac{C_{b}}{c_{b}}, \kappa_\Gamma = \frac{C_\Gamma}{c_\Gamma} \). Let \( \lambda_m^R \) be a vector satisfying
\[
\lambda_m^R \geq \hat{\lambda}_m^R, \quad ||\lambda_m^R||_\infty = \mathcal{O}(||\hat{\lambda}_m^R||_\infty).
\]
Let \( (u_{1,m}, u_{2,m}) \in (\mathcal{H}_{1,m}, \mathcal{H}_{2,m}) \) be a minimizer of the Lipschitz regularized loss \( \text{Loss}_m(\cdot; \lambda, \lambda_m^R) \) in (5). Then with probability 1 over iid samples, we have
\[
\lim_{m_r \to \infty} u_{1,m} = u^*, \quad \text{in } \mathcal{H}^2(\Omega_1), \quad \lim_{m_r \to \infty} u_{2,m} = u^*, \quad \text{in } \mathcal{H}^2(\Omega_2).
\]

Theorem 3.2 shows that the minimizers of the Lipschitz regularized empirical losses (5) converge to the unique solution to the interface problem (4). The proof is postponed to the next section.

4 Proofs

Throughout this section, we assume that Assumptions 3.1 and 3.2 hold. Among crucial technical tools used here are some Sobolev inequality and probability space filling arguments [5]. We start with the following auxiliary lemma:
Lemma 4.1. Suppose Assumption 3.1 holds. For training data $\mathcal{T}^{m_1}_{r_1} = \{x_i^{m_1}_{r_1}\}_{i=1}^{m_1}$, $\mathcal{T}^{m_2}_{r_2} = \{x_i^{m_2}_{r_2}\}_{i=1}^{m_2}$, $\mathcal{T}^m_b = \{x_i^m_b\}_{i=1}^{m_b}$, and $\mathcal{T}^m_{\Gamma} = \{x_i^m_{\Gamma}\}_{i=1}^{m_{\Gamma}}$, if $m_{r_1}$, $m_{r_2}$, $m_b$ and $m_{\Gamma}$ are large enough to satisfy that there exists $x'_i \in \mathcal{T}_{r_1}$, $x'_i \in \mathcal{T}_{r_2}$, $x'_b \in \mathcal{T}_b$ and $x'_i \in \mathcal{T}_\Gamma$ such that $\|x_{r_1} - x'_{r_1}\| \leq \epsilon_{r_1}$, $\|x_{r_2} - x'_{r_2}\| \leq \epsilon_{r_2}$, $\|x_b - x'_{b}\| \leq \epsilon_b$ and $\|x_{\Gamma} - x'_{\Gamma}\| \leq \epsilon_{\Gamma}$ for any $x_{r_1} \in \Omega_1$, $x_{r_2} \in \Omega_2$, $x_b \in \partial \Omega$ and $x_{\Gamma} \in \Gamma$, then, we have

\[
\text{Loss}_{\text{PINN}}(u_1, u_2; \lambda) \leq C_m \cdot \text{Loss}_{\text{PINN}}^m (u_1, u_2; \lambda)
\]

\[
+ 3\lambda_{r_1} \epsilon_{r_1}^2 [\mathcal{L}_1[u_1]]_\Omega^2 + 3\lambda_{r_2} \epsilon_{r_2}^2 [\mathcal{L}_2[u_2]]_{\Omega_2}^2 + 3\lambda_b \epsilon_b^2 [\mathcal{B}[u_2]]_{\partial \Omega}^2
\]

\[
+ 3\lambda_{r_1} \epsilon_{r_1}^2 [f_1]_\Omega^2 + 3\lambda_{r_2} \epsilon_{r_2}^2 [f_2]_{\Omega_2}^2 + 3\lambda_b \epsilon_b^2 [\mathcal{G}]_{\partial \Omega}^2 + 3\epsilon_{\Gamma} \left( \lambda_{\Gamma_D} [\varphi]_{\Gamma}^2 + \lambda_{\Gamma_N} [\psi]_{\Gamma}^2 \right),
\]

where $C_{r_1}, C_{r_2}, C_b, C_{\Gamma}$ are those defined in Assumption 3.1 and

\[
C_m = 3 \max\{C_{r_1} m_{r_1} \epsilon_{r_1}^d, C_{r_2} m_{r_2} \epsilon_{r_2}^d, C_b m_b \epsilon_b^{d-1}, C_{\Gamma} m_{\Gamma} \epsilon_{\Gamma}^{d-1}\}.
\]

Proof. As a consequence of Cauchy’s inequality, i.e., $\|x + y + z\|^2 \leq 3(\|x\|^2 + \|y\|^2 + \|z\|^2)$ for any three vectors $x, y, z$, we deduce that for $x_{r_1}, x'_{r_1} \in \Omega_i, i = 1, 2$,

\[
\|\mathcal{L}_i[u_1](x_{r_1}) - f_i(x_{r_1})\|^2 \leq 3 \left(\|\mathcal{L}_i[u_1](x_{r_1}) - \mathcal{L}_i[u_1](x'_{r_1})\|^2 + \|\mathcal{L}_i[u_1](x'_{r_1}) - f_i(x'_{r_1})\|^2 + \|f_i(x'_{r_1}) - f_i(x_{r_1})\|^2\right).
\]

Similarly, for $x_b, x'_{b} \in \partial \Omega$, we have

\[
\|\mathcal{B}[u_2](x_b) - \mathcal{G}(x_b)\|^2 \leq 3 \left(\|\mathcal{B}[u_2](x_b) - \mathcal{B}[u_2](x'_{b})\|^2 + \|\mathcal{B}[u_2](x'_{b}) - \mathcal{G}(x'_{b})\|^2 + \|\mathcal{G}(x'_{b}) - \mathcal{G}(x_b)\|^2\right),
\]

and for $x_{\Gamma}, x'_{\Gamma} \in \Gamma$, we have

\[
\|\mathcal{I}_D[u_1, u_2](x_{\Gamma}) - \varphi(x_{\Gamma})\|^2 \leq 3 \left(\|\mathcal{I}_D[u_1, u_2](x_{\Gamma}) - \mathcal{I}_D[u_1, u_2](x'_{\Gamma})\|^2 + \|\mathcal{I}_D[u_1, u_2](x'_{\Gamma}) - \varphi(x'_{\Gamma})\|^2 \right)
\]

\[
+ \|\varphi(x_{\Gamma}) - \varphi(x'_{\Gamma})\|^2,
\]

\[
\|\mathcal{I}_N[u_1, u_2](x_{\Gamma}) - \psi(x_{\Gamma})\|^2 \leq 3 \left(\|\mathcal{I}_N[u_1, u_2](x_{\Gamma}) - \mathcal{I}_N[u_1, u_2](x'_{\Gamma})\|^2 + \|\mathcal{I}_N[u_1, u_2](x'_{\Gamma}) - \psi(x'_{\Gamma})\|^2 \right)
\]

\[
+ \|\psi(x_{\Gamma}) - \psi(x'_{\Gamma})\|^2.
\]

In addition, by the conditions, for $\forall x_{r_1} \in \Omega_1, \forall x_{r_2} \in \Omega_2, \forall x_b \in \partial \Omega$ and $\forall x_{\Gamma} \in \Gamma$, there exists $x'_{r_1} \in \mathcal{T}^{m_{r_1}}_{r_1}, x'_{r_2} \in \mathcal{T}^{m_{r_2}}_{r_2}, x'_b \in \mathcal{T}^m_b$ and $x'_i \in \mathcal{T}^m_{\Gamma}$ such that $\|x_{r_1} - x'_{r_1}\| \leq \epsilon_{r_1}, \|x_{r_2} - x'_{r_2}\| \leq \epsilon_{r_2}, \|x_b - x'_{b}\| \leq \epsilon_b$ and $\|x_{\Gamma} - x'_{\Gamma}\| \leq \epsilon_{\Gamma}$. Taking

\[
\mathbf{L}(x_{r_1}, x_{r_2}, x_b, x_{\Gamma}; u_1, u_2, \lambda, 0)
\]= $\lambda_{r_1} \left(\|\mathcal{L}_1[u_1](x_{r_1}) - f_1(x_{r_1})\|^2 + \lambda_{r_2} \|\mathcal{L}_2[u_2](x_{r_2}) - f_2(x_{r_2})\|^2 + \lambda_b \|\mathcal{B}[u_2](x_b) - \mathcal{G}(x_b)\|^2 \right)$

\[
+ \lambda_{\Gamma_D} \|\mathcal{I}_D[u_1, u_2](x_{\Gamma}) - \varphi(x_{\Gamma})\|^2 + \lambda_{\Gamma_N} \|\mathcal{I}_N[u_1, u_2](x_{\Gamma}) - \psi(x_{\Gamma})\|^2,
\]

\]

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we have that
\[
\mathbf{L}(x_{r_1}, x_{r_2}, x_b, x_{\Gamma}; u_1, u_2, \lambda, 0) \\
\leq 3\mathbf{L}(x'_{r_1}, x'_{r_2}, x'_b, x'_{\Gamma}; u_1, u_2, \lambda, 0) \\
+ 3\lambda_1 \left( \| \mathcal{L}_1[u_1](x_{r_1}) - \mathcal{L}_1[u_1](x'_{r_1}) \| + \| f_1(x_{r_1}) - f_1(x'_{r_1}) \| \right) \\
+ 3\lambda_2 \left( \| \mathcal{L}_2[u_2](x_{r_2}) - \mathcal{L}_2[u_2](x'_{r_2}) \| + \| f_2(x_{r_2}) - f_2(x'_{r_2}) \| \right) \\
+ 3\lambda_D \left( \| \mathcal{I}_D[u_1, u_2](x_{r_1}) - \mathcal{I}_D[u_1, u_2](x'_{r_1}) \| + \| \varphi(x_{r_1}) - \varphi(x'_{r_1}) \| \right) \\
+ 3\lambda_N \left( \| \mathcal{I}_N[u_1, u_2](x_{r_2}) - \mathcal{I}_N[u_1, u_2](x'_{r_2}) \| + \| \psi(x_{r_2}) - \psi(x'_{r_2}) \| \right) \\
+ 3\lambda_0 \left( \| \mathcal{B}[u_2](x_b) - \mathcal{B}[u_2](x'_b) \|^2 + \| \mathbf{g}(x'_b) - \mathbf{g}(x_b) \|^2 \right)
\]

For \( x^i_{r_1} \in \mathcal{T}_1^{m_r}, x^i_{r_2} \in \mathcal{T}_2^{m_r}, x^i_b \in \mathcal{T}_b^{m_b} \) and \( x^i_{\Gamma} \in \mathcal{T}_{\Gamma}^{m_{\Gamma}} \), we denote the Voronoi cell associated with \( x^i_{r_1}, x^i_{r_2}, x^i_b, x^i_{\Gamma} \) as \( A_{x^i_{r_1}}, A_{x^i_{r_2}}, A_{x^i_b} \) and \( A_{x^i_{\Gamma}} \), respectively, i.e.,

\[
A_{x^i_{r_1}} = \{ x \in \Omega | \| x - x^i_{r_1} \| = \min_{x' \in \mathcal{T}_1^{m_r}} \| x - x' \| \}, \quad A_{x^i_{r_2}} = \{ x \in \Omega | \| x - x^i_{r_2} \| = \min_{x' \in \mathcal{T}_2^{m_r}} \| x - x' \| \}, \\
A_{x^i_b} = \{ x \in \partial \Omega | \| x - x^i_b \| = \min_{x' \in \mathcal{T}_b^{m_b}} \| x - x' \| \}, \quad A_{x^i_{\Gamma}} = \{ x \in \Gamma | \| x - x^i_{\Gamma} \| = \min_{x' \in \mathcal{T}_{\Gamma}^{m_{\Gamma}}} \| x - x' \| \},
\]

and let \( \omega^i_{r_1} = \mu_{r_1}(A_{x^i_{r_1}}), \omega^i_{r_2} = \mu_{r_2}(A_{x^i_{r_2}}), \omega^i_b = \mu_b(A_{x^i_b}) \) and \( \omega^i_{\Gamma} = \mu_{\Gamma}(A_{x^i_{\Gamma}}) \). By taking the expectation with respect to \( (x_{r_1}, x_{r_2}, x_b, x_{\Gamma}) \sim \mu = \mu_{r_1} \times \mu_{r_2} \times \mu_b \times \mu_{\Gamma} \), we obtain that

\[
\mathbb{E}_\mu[\mathbf{L}(x_{r_1}, x_{r_2}, x_b, x_{\Gamma}; u_1, u_2, \lambda, 0)] \\
= \sum_{i=1}^{m_r} \sum_{j=1}^{m_b} \sum_{k=1}^{m_{\Gamma}} \int_{A_{x^i_{r_1}}} \int_{A_{x^i_{r_2}}} \int_{A_{x^i_b}} \int_{A_{x^i_{\Gamma}}} \mathbf{L}(x_{r_1}, x_{r_2}, x_b, x_{\Gamma}; u_1, u_2, \lambda, 0) d\mu \\
\leq 3 \sum_{i=1}^{m_r} \sum_{j=1}^{m_b} \sum_{k=1}^{m_{\Gamma}} \omega^i_{r_1} \omega^i_{r_2} \omega^i_b \omega^i_{\Gamma} \mathbf{L}(x^k_{r_1}, x^i_{r_2}, x^j_b, x^i_{\Gamma}; u_1, u_2, \lambda, 0) \\
+ 3\lambda_1 \epsilon^2_r \left( \| \mathcal{L}_1[u_1] \|_{\Omega_1}^2 + \| f_1 \|_{\Omega_1}^2 \right) + 3\lambda_2 \epsilon^2_r \left( \| \mathcal{L}_2[u_2] \|_{\Omega_2}^2 + \| f_2 \|_{\Omega_2}^2 \right) \\
+ 3\lambda_D \epsilon^2_r \left( \| \mathcal{I}_D[u_1, u_2] \|_{\Gamma}^2 + \| \varphi \|_{\Gamma}^2 \right) + 3\lambda_N \epsilon^2_r \left( \| \mathcal{I}_N[u_1, u_2] \|_{\Gamma}^2 + \| \psi \|_{\Gamma}^2 \right) \\
+ 3\lambda_0 \left( \| \mathcal{B}[u_2] \|^2_{\Omega_b} + \| \mathbf{g} \|^2_{\Omega_b} \right),
\]

where we have used the fact that \( \sum_{i=1}^{m_r} \omega^i_{r_1} = 1, \sum_{i=1}^{m_r} \omega^i_{r_2} = 1, \sum_{i=1}^{m_b} \omega^i_b = 1 \) and \( \sum_{i=1}^{m_{\Gamma}} \omega^i_{\Gamma} = 1 \).

Next, we give the estimation of the first term on the right. Taking \( \omega^{m_r}_{r_1} = \max_i \omega^i_{r_1}, \omega^{m_r}_{r_2} = \max_i \omega^i_{r_2} \),
\( \omega_{m_r}^{m_r} = \max_i \omega_i^{m_r} \) and \( \omega_{m_r}^{m_r} = \max_i \omega_i^{m_r} \), yields that

\[
3 \sum_{i=1}^{m_r} \sum_{j=1}^{m_r} \sum_{k=1}^{m_r} \omega_i^{m_r} \omega_j^{m_r} \omega_k^{m_r} L(x_{r_1}^{k}, x_{r_2}^{j}, x_{r_3}^{i}; u_1, u_2, \lambda, 0) \\
\leq 3m_r \omega_r^{m_r} \cdot \frac{\lambda_i}{m_r} \sum_{i=1}^{m_r} ||L_1[u_1](x_{r_1}^{i}) - f_1(x_{r_1}^{i})||^2 + 3m_r \omega_r^{m_r} \cdot \frac{\lambda_r}{m_r} \sum_{r_2}^{m_r} ||L_2[u_2](x_{r_2}^{i}) - f_2(x_{r_2}^{i})||^2 \\
+ 3m_r \omega_r^{m_r} \cdot \left( \frac{\lambda_D}{m_r} \sum_{i=1}^{m_r} ||x_{r_1}^{i} - x_{r_2}^{i}||^2 \right) + 3m_r \omega_r^{m_r} \cdot \left( \frac{\lambda_N}{m_r} \sum_{i=1}^{m_r} ||x_{r_1}^{i} - x_{r_2}^{i}||^2 \right) \\
+ 3m_r \omega_r^{m_r} \cdot \left( \frac{\lambda_D}{m_r} \sum_{i=1}^{m_r} ||x_{r_2}^{i} - g(x_{r_2}^{i})||^2 \right)
\]

(6)

where we have used the fact that \( m_r \omega_r^{m_r}, m_r \omega_r^{m_r}, m_r \omega_r^{m_r}, m_r \omega_r^{m_r}, m_r \omega_r^{m_r} \geq 1 \). Let \( B_\epsilon(x) \) be a closed ball centered at \( x \) with radius \( \epsilon \). Let \( P^{d_1}_{r_1} = \max_{x \in \Omega} \mu_r(B_{\epsilon_1}(x) \cap \Omega) \), \( P^{d_2}_{r_2} = \max_{x \in \Omega} \mu_r(B_{\epsilon_2}(x) \cap \Omega) \), \( P^{d_3}_{b} = \max_{x \in \Omega} \mu_r(B_{\epsilon_3}(x) \cap \Omega) \) and \( P^{d_4}_{r} = \max_{x \in \Omega} \mu_r(B_{\epsilon_4}(x) \cap \Omega) \). Then for any \( x_{r_1} \in \Omega_1, x_{r_2} \in \Omega_2, x_{b} \in \partial \Omega \) and \( x_{r_3} \in \Gamma \), there exists \( x_{r_1}^{*}, x_{r_2}^{*} \in T_r, x_{b}^{*} \in T_b \) and \( x_{r_3}^{*} \in T_f \) such that \( ||x_{r_1} - x_{r_1}^{*}|| \leq \epsilon_1, ||x_{r_2} - x_{r_2}^{*}|| \leq \epsilon_2, ||x_{b} - x_{b}^{*}|| \leq \epsilon_3 \) and \( ||x_{r_3} - x_{r_3}^{*}|| \leq \epsilon_4 \) for each \( i \), there are closed balls \( B_{\epsilon_1}, B_{\epsilon_2}, B_{\epsilon_3} \) and \( B_{\epsilon_4} \), that include \( A_{r_1}, A_{r_2}, A_{b} \) and \( A_{r_3} \), respectively. These facts, together with Assumption 3.1 implies that

\[
\omega_{m_r}^{m_r} \leq P_{r_1}^{d_1}, \omega_{m_r}^{m_r} \leq P_{r_2}^{d_2}, \omega_{b}^{m_r} \leq P_{b}^{d_3}, \omega_{m_r}^{m_r} \leq P_{r}^{d_4}. \]

(7)

With the estimations (6) and (7), we obtain that

\[
\mathbb{E}_\mu[L(x_{r_1}, x_{r_2}, x_{b}, x_{r_3}; u_1, u_2, \lambda, 0)] \\
\leq 3C_1 m_r \epsilon_1^{d_1} \cdot \frac{\lambda_i}{m_r} \sum_{i=1}^{m_r} ||L_1[u_1](x_{r_1}^{i}) - f_1(x_{r_1}^{i})||^2 + 3C_2 m_r \epsilon_2^{d_2} \cdot \frac{\lambda_r}{m_r} \sum_{r_2}^{m_r} ||L_2[u_2](x_{r_2}^{i}) - f_2(x_{r_2}^{i})||^2 \\
+ 3C_3 m_r \epsilon_3^{d_3} \cdot \left( \frac{\lambda_D}{m_r} \sum_{i=1}^{m_r} ||x_{r_1}^{i} - x_{r_2}^{i}||^2 \right) + 3C_4 m_r \epsilon_4^{d_4} \cdot \left( \frac{\lambda_N}{m_r} \sum_{i=1}^{m_r} ||x_{r_1}^{i} - x_{r_2}^{i}||^2 \right)
\]

Finally, we conclude the proof by taking \( C_m = 3 \max \{C_1 m_r \epsilon_1^{d_1}, C_2 m_r \epsilon_2^{d_2}, C_3 m_r \epsilon_3^{d_3}, C_4 m_r \epsilon_4^{d_4} \} \). \( \square \)

With Lemma 4.1 and Assumption 3.1, we are able to quantify the generalization error and provide an upper bound of the expected unregularized PINN loss (2.2).

**Lemma 4.2.** Suppose Assumption 3.1 holds. Suppose that \( u_1, u_2 \) satisfy

\[
R_{r_1}(u_1) < \infty, R_{r_2}(u_2) < \infty, R_b(u_2) < \infty, R_{r_3}(u_1, u_2) < \infty, R_{r_4}(u_1, u_2) < \infty,
\]

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and $f_1, f_2, \psi, \varphi, g$ satisfy

\[
[f_1]_{\Omega_1}, [f_2]_{\Omega_2}, [g]_{\partial \Omega}, [\varphi]_{\Gamma}, [\psi]_{\Gamma} < \infty.
\]

Let $m_{r_1}$, $m_{r_2}$, $m_b$ and $m_\Gamma$ be the number of iid samples from $\mu_{r_1}$, $\mu_{r_2}$, $\mu_b$ and $\mu_\Gamma$, respectively. Let $\lambda = (\lambda_{r_1}, \lambda_{r_2}, \lambda_b, \lambda_{\Gamma_D}, \lambda_{\Gamma_N})$ be a fixed vector. Then, with probability at least,

\[
P = P(m_{r_1})P(m_{r_2})P(m_b)P(m_\Gamma), \quad \text{where } P(m) = (1 - \sqrt{m}(1 - 1/\sqrt{m})^m),
\]

we have

\[
\text{Loss}^{\text{PINN}}(u_1, u_2; \lambda) \leq C_m \cdot \text{Loss}_m(u_1, u_2; \lambda, \hat{\lambda}^R_m) + C'(m_{r_1}^{\frac{-1}{2}} + m_{r_2}^{\frac{-1}{2}} + m_b^{\frac{-1}{2}} + m_\Gamma^{\frac{-1}{2}}).
\]

Here, $\hat{\lambda}^R_m = (\hat{\lambda}^R_{r_1,m}, \hat{\lambda}^R_{r_2,m}, \hat{\lambda}^R_{b,m}, \hat{\lambda}^R_{\Gamma_D,m}, \hat{\lambda}^R_{\Gamma_N,m})$ is a vector where

\[
\hat{\lambda}^R_{r_1,m} = \frac{3\lambda_{r_1} dc_{r_1}^2}{C_m} \cdot m_1^{-\frac{1}{2}}, \quad \hat{\lambda}^R_{r_2,m} = \frac{3\lambda_{r_2} dc_{r_2}^2}{C_m} \cdot m_2^{-\frac{1}{2}},
\]

\[
\hat{\lambda}^R_{\Gamma_D,m} = \frac{3\lambda_{\Gamma_D} dc_{\Gamma_D}^2}{C_m} \cdot m_\Gamma^{-\frac{1}{2}}, \quad \hat{\lambda}^R_{\Gamma_N,m} = \frac{3\lambda_{\Gamma_N} dc_{\Gamma_N}^2}{C_m} \cdot m_\Gamma^{-\frac{1}{2}},
\]

\[
\hat{\lambda}^R_{b,m} = \frac{3\lambda_b dc_b^2}{C_m} \cdot m_b^{-\frac{1}{2}};
\]

\[
C_m = 3 \max\left\{\sqrt{d} m_1^{-\frac{1}{2}}, \sqrt{d} m_2^{-\frac{1}{2}}, \sqrt{d} m_b^{-\frac{1}{2}}, \sqrt{d} m_\Gamma^{-\frac{1}{2}}\right\} \text{ where } \kappa_{r_1} = \frac{C_{r_1}}{c_{r_1}}, \kappa_{r_2} = \frac{C_{r_2}}{c_{r_2}}, \kappa_b = \frac{C_b}{c_b}, \kappa_\Gamma = \frac{C_\Gamma}{c_\Gamma}. \text{ And } C' \text{ is a constant that depends only on } \lambda, d, c_{r_1}, c_{r_2}, c_b, c_\Gamma, f_1, f_2, g, \varphi, \psi.
\]

**Proof.** Since $\mathcal{T}_r_1 = \{x_{r_1,i}\}_{i=1}^{m_{r_1}}$ is iid samples from $\mu_{r_1}$ on $\Omega_1$, $\mathcal{T}_r_2 = \{x_{r_2,i}\}_{i=1}^{m_{r_2}}$ is iid samples from $\mu_{r_2}$ on $\Omega_2$, $\mathcal{T}_b = \{x_{b,i}\}_{i=1}^{m_b}$ is iid samples from $\mu_b$ on $\partial \Omega$ and $\mathcal{T}_\Gamma = \{x_{\Gamma,i}\}_{i=1}^{m_\Gamma}$ is iid samples from $\mu_\Gamma$ on $\Gamma$, respectively, therefore, by Lemma B.2 in [32], with probability at least

\[
P = P(m_{r_1})P(m_{r_2})P(m_b)P(m_\Gamma), \quad \text{where } P(m) = (1 - \sqrt{m}(1 - 1/\sqrt{m})^m)
\]

\[
\forall x_{r_1} \in \Omega_1, \forall x_{r_2} \in \Omega_2, \forall x_b \in \partial \Omega \text{ and } \forall x_\Gamma \in \Gamma, \text{ there exists } x_{r'_1} \in \mathcal{T}_{r_1}^{m_{r_1}}, x_{r'_2} \in \mathcal{T}_{r_2}^{m_{r_2}}, x_b \in \mathcal{T}_b^{m_b} \text{ and } x_\Gamma \in \mathcal{T}_\Gamma^{m_\Gamma}
\]

such that $||x_{r_1} - x_{r'_1}|| \leq \sqrt{d} c_{r_1}^{-\frac{1}{2}} m_{r_1}^{-\frac{1}{2}}$, $||x_{r_2} - x_{r'_2}|| \leq \sqrt{d} c_{r_2}^{-\frac{1}{2}} m_{r_2}^{-\frac{1}{2}}$, $||x_b - x_b|| \leq \sqrt{d} c_b^{-\frac{1}{2}} m_b^{-\frac{1}{2}}$, $||x_\Gamma - x_\Gamma|| \leq \sqrt{d} c_\Gamma^{-\frac{1}{2}} m_\Gamma^{-\frac{1}{2}}$ and $\epsilon_{r_1} = \sqrt{d} c_{r_1}^{-\frac{1}{2}} m_{r_1}^{-\frac{1}{2}}$, $\epsilon_{r_2} = \sqrt{d} c_{r_2}^{-\frac{1}{2}} m_{r_2}^{-\frac{1}{2}}$, $\epsilon_b = \sqrt{d} c_b^{-\frac{1}{2}} m_b^{-\frac{1}{2}}$, $\epsilon_\Gamma = \sqrt{d} c_\Gamma^{-\frac{1}{2}} m_\Gamma^{-\frac{1}{2}}$, implies that with probability at least $[8],

\[
\text{Loss}^{\text{PINN}}(u_1, u_2; \lambda) \leq C_m \cdot \text{Loss}_m^{\text{PINN}}(u_1, u_2; \lambda) + Q
\]

\[
+ C_m \cdot \left[\hat{\lambda}^R_{r_1,m} \cdot [L_1[u_1]]_{\Omega_1}^2 + \hat{\lambda}^R_{r_2,m} \cdot [L_2[u_2]]_{\Omega_2}^2 + \hat{\lambda}^R_{b,m} \cdot [B[u_2]]_{\partial \Omega}^2\right]
\]

\[
+ C_m \cdot \left[\hat{\lambda}^R_{\Gamma_D,m} \cdot [L_D[u_1,u_2]]_{\Gamma}^2 + \hat{\lambda}^R_{\Gamma_N,m} \cdot [I_N[u_1,u_2]]_{\Gamma}^2\right],
\]

where

\[
Q = 3 \lambda_{r_1} dc_{r_1}^{\frac{2}{3}} m_{r_1}^{-\frac{1}{3}} [f_1]_{\Omega_1}^2 + 3 \lambda_{r_2} dc_{r_2}^{\frac{2}{3}} m_{r_2}^{-\frac{1}{3}} [f_2]_{\Omega_2}^2 + 3 \lambda_b dc_b^{-\frac{2}{3}} m_b^{-\frac{1}{3}} [g]_{\partial \Omega}^2
\]

\[
+ 3 dc_\Gamma^{-\frac{2}{3}} m_\Gamma^{-\frac{1}{3}} \left(\lambda_{\Gamma_D} [\varphi]_{\Gamma}^2 + \lambda_{\Gamma_N} [\psi]_{\Gamma}^2\right),
\]

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and

\[ C_m = 3 \max \left\{ \frac{C_r_1}{c_r_1} \sqrt{d} m_r_1, \frac{C_r_2}{c_r_2} \sqrt{d} m_r_2, \frac{C_b}{c_b} \sqrt{d} m_r_2, \frac{C_\Gamma}{c_\Gamma} \sqrt{d} m_\Gamma, \frac{C_r_1}{c_r_1} \right\}, \]

\[ \lambda_{r_1,m}^R = \frac{3 \lambda_r d c_r \sigma^2}{C_m} \cdot m_{r_1}^{-\frac{1}{2}}, \quad \lambda_{r_2,m}^R = \frac{3 \lambda_r d c_r \sigma^2}{C_m} \cdot m_{r_2}^{-\frac{1}{2}}, \]

\[ \lambda_{D,m}^R = \frac{3 \lambda_{D} d c_{D} \sigma^2}{C_m} \cdot m_{D}^{-\frac{1}{2}}, \quad \lambda_{N,m}^R = \frac{3 \lambda_{N} d c_{N} \sigma^2}{C_m} \cdot m_{N}^{-\frac{1}{2}}, \]

\[ \lambda_{b,m}^R = \frac{3 \lambda_b d c_b \sigma^2}{C_m} \cdot m_{b}^{-\frac{1}{2}}. \]

By taking

\[ C' = 3 \max \{ \lambda_r d c_r \sigma^2 [f_1]^2_{\Omega_1}, \lambda_r d c_r \sigma^2 [f_2]^2_{\Omega_2}, \lambda_b d c_b \sigma^2 \} \]

we conclude that

\[ \text{Loss}^{\text{PINN}}(u_1, u_2; \lambda) \leq C_m \cdot \text{Loss}_m(u_1, u_2; \lambda, \hat{\lambda}_m^R) + C'(m_{r_1}^{-\frac{1}{2}} + m_{r_2}^{-\frac{1}{2}} + m_{b}^{-\frac{1}{2}} + m_{\Gamma}^{-\frac{1}{2}}), \]

where \( \hat{\lambda}_m^R = (\lambda_{r_1,m}^R, \lambda_{r_2,m}^R, \lambda_{D,m}^R, \lambda_{N,m}^R, \lambda_{b,m}^R) \). The proof is completed.

Using Lemma 4.2 we will show that the expected PINN loss (2.2) at the minimizers of the Lipschitz regularized empirical loss (5) converges to zero according to Assumptions 3.2.

**Lemma 4.3.** Suppose Assumptions 3.1 and 3.2 hold. Let \( m_{r_1}, m_{r_2}, m_{\Gamma} \) and \( m_{b} \) be the number of iid samples from \( \mu_{r_1}, \mu_{r_2}, \mu_{\Gamma} \) and \( \mu_b \), respectively, and satisfy \( m_{r_1} = O(m_{r_1}), m_{r_2} = O(m_{r_1}^{-\frac{1}{2}}), m_{b} = O(m_{r_1}^{-\frac{1}{2}}) \). Let \( \lambda_m^R \) be a vector satisfying

\[ \lambda_m^R \geq \hat{\lambda}_m^R, \quad \| \lambda_m^R \|_\infty = O(\| \hat{\lambda}_m^R \|_\infty), \]

where \( \hat{\lambda}_m^R = (\hat{\lambda}_{r_1,m}^R, \hat{\lambda}_{r_2,m}^R, \hat{\lambda}_{D,m}^R, \hat{\lambda}_{N,m}^R, \hat{\lambda}_{b,m}^R) \) are defined in Lemma 4.2. Let \( (u_{1,m}, u_{2,m}) \in (\mathcal{H}_{1,m}, \mathcal{H}_{2,m}) \) be a minimizer of the Lipschitz regularized empirical loss \( \text{Loss}_m(\cdot; \lambda, \lambda_m^R) \). Then the following holds:

- **With probability at least**
  \[ P = P(m_{r_1}) P(m_{r_2}) P(m_{b}) P(m_{\Gamma}), \quad \text{where} \ P(m) = (1 - \sqrt{m}(1 - 1/\sqrt{m}))^m \]
  
  over iid samples,

  \[ \text{Loss}^{\text{PINN}}(u_{1,m}, u_{2,m}; \lambda) = O(m_{r_1}^{-\frac{1}{2}}). \]

- **With probability 1 over iid samples,**

  \[ \lim_{m_{r_1} \to \infty} \mathcal{L}[u_{1,m}] = f_1 \text{ in } L^2(\Omega_1), \quad \lim_{m_{r_1} \to \infty} \mathcal{L}[u_{2,m}] = f_2 \text{ in } L^2(\Omega_2), \]

  \[ \lim_{m_{r_1} \to \infty} \mathcal{I}_D[u_{1,m}, u_{2,m}] = \varphi \text{ in } L^2(\Gamma), \quad \lim_{m_{r_1} \to \infty} \mathcal{I}_N[u_{1,m}, u_{2,m}] = \psi \text{ in } L^2(\Gamma), \]

  \[ \lim_{m_{r_1} \to \infty} B[u_{2,m}] = g \text{ in } L^2(\partial \Omega). \]
Proof. Since $m_{r_1} = O(m_{r_2}) = O(m_b^{d/4-1}) = O(m_1^{d/4-1})$, we have

$$\hat{\lambda}_{r_1,m}^R, \hat{\lambda}_{r_2,m}^R, \hat{\lambda}_{\Gamma_D,m}^R, \hat{\lambda}_{\Gamma_N,m}^R, \hat{\lambda}_{b,m}^R = O(m_{r_1}^{-\frac{1}{4}-\frac{3}{4}}),$$

where $\hat{\lambda}_{r_1,m}^R, \hat{\lambda}_{r_2,m}^R, \hat{\lambda}_{\Gamma_D,m}^R, \hat{\lambda}_{\Gamma_N,m}^R, \hat{\lambda}_{b,m}^R$ are defined in Lemma 4.2. Let $\lambda$ be a vector independent of $m$ and $\hat{\lambda}_m^R = (\hat{\lambda}_{r_1,m}^R, \hat{\lambda}_{r_2,m}^R, \hat{\lambda}_{\Gamma_D,m}^R, \hat{\lambda}_{\Gamma_N,m}^R, \hat{\lambda}_{b,m}^R)$ be a vector satisfying

$$\lambda_m^R \geq \hat{\lambda}_m^R, \quad \|\lambda_m^R\|_\infty = O(\|\hat{\lambda}_m^R\|_\infty),$$

where $\hat{\lambda}_m^R = (\hat{\lambda}_{r_1,m}^R, \hat{\lambda}_{r_2,m}^R, \hat{\lambda}_{\Gamma_D,m}^R, \hat{\lambda}_{\Gamma_N,m}^R, \hat{\lambda}_{b,m}^R)$. Let $(u_1,m, u_2,m) \in (\mathcal{H}_1,m, \mathcal{H}_2,m)$ minimizes the Lipschitz regularized loss $\text{Loss}_m(\cdot; \lambda, \lambda_m^R)$ \textcircled{5}. Let $(\hat{u}_1,m, \hat{u}_2,m)$ be the neural networks defined in the third term of Assumption 3.2 i.e., they satisfy $\text{Loss}_{PINN}^m(\hat{u}_1,m, \hat{u}_2,m; \lambda) = O(m_{r_1}^{-\frac{1}{2}-\frac{3}{4}})$.

Then, we have

$$\begin{align*}
\text{Loss}_m(u_1,m, u_2,m; \lambda, \lambda_m^R) \\
\leq & \text{Loss}_m(\hat{u}_1,m, \hat{u}_2,m; \lambda, \lambda_m^R) \\
\leq & \|\lambda_m^R\|_\infty (R_{r_1}(\hat{u}_1,m) + R_{r_2}(\hat{u}_2,m) + R_{\Gamma_D}(\hat{u}_2,m, \hat{u}_1,m) + R_{\Gamma_N}(\hat{u}_2,m, \hat{u}_1,m) + R_b(u_2,m)) \\
& + \text{Loss}_m(\hat{u}_1,m, \hat{u}_2,m; \lambda, 0).
\end{align*}$$

Let

$$\hat{R} = \sup_m (R_{r_1}(\hat{u}_1,m) + R_{r_2}(\hat{u}_2,m) + R_{\Gamma_D}(\hat{u}_1,m, \hat{u}_2,m) + R_{\Gamma_N}(\hat{u}_1,m, \hat{u}_2,m) + R_b(\hat{u}_2,m)).$$

By the last term in Assumption 3.2 we have $\hat{R} < \infty$. Therefore, $\text{Loss}_m(u_1,m, u_2,m; \lambda, \lambda_m^R) = O(m_{r_1}^{-\frac{1}{2}-\frac{3}{4}})$. Note that $\text{Loss}_{PINN}^m(\hat{u}_1,m, \hat{u}_2,m; \lambda) = \text{Loss}_m(\hat{u}_1,m, \hat{u}_2,m; \lambda, 0)$ and $\|\lambda_m^R\|_\infty = O(\|\lambda_m^R\|_\infty) = O(m_{r_1}^{-\frac{1}{2}-\frac{3}{4}})$.

According to Lemma 4.2, with probability at least

$$P = P(m_{r_1})P(m_{r_2})P(m_b)P(m_\Gamma), \quad \text{where } P(m) = (1 - \sqrt{m}(1 - 1/\sqrt{m})^m),$$

we have that

$$\text{Loss}_{PINN}^m(u_1,m, u_2,m; \lambda) \leq C_m \cdot \text{Loss}_m(u_1,m, u_2,m; \lambda, \lambda_m^R) + C'(m_{r_1}^{-\frac{1}{4}} + m_{r_2}^{-\frac{3}{4}} + m_b^{-\frac{1}{4}} + m_{\Gamma}^{-\frac{1}{4}}),$$

$$= O(m_{r_1}^{-\frac{1}{2}}),$$

which completes the first part of the proof. Here, we have used the fact that $m_{r_2} = O(m_{r_1})$, $m_\Gamma = O(m_{r_1}^{d-1})$, $m_b = O(m_{r_1}^{d-1})$ and $C_m = O(m_{r_1}^{d-1})$.

Since $m$ is dominated by $m_{r_1}$, we rewrite $m$ and $m_{r_1}$ as $m$ for simplicity. We have $\max_j(\lambda_m^R)_j = O(1)$ due to the selection of $\lambda_m^R$. Since

$$\min_j (\lambda_m^R)_j((R_{r_1}(u_1,m) + R_{r_2}(u_2,m) + R_{\Gamma_D}(u_1,m, u_2,m) + R_{\Gamma_N}(u_1,m, u_2,m) + R_b(u_2,m))$$

$$\leq \text{Loss}_m(u_1,m, u_2,m; \lambda, \lambda_m^R),$$

by \textcircled{9}, we obtain that for all $m$

$$R_{r_1}(u_1,m), R_{r_2}(u_2,m), R_{\Gamma_D}(u_1,m, u_2,m), R_{\Gamma_N}(u_1,m, u_2,m), R_b(u_2,m) \leq O(\hat{R}).$$
Since
\[ R_{r_1}(u_{1,m}) = \left[ \mathcal{L}_1[u_{1,m}] \right]_{\Omega_1}^2, \quad R_{r_2}(u_{2,m}) = \left[ \mathcal{L}_2[u_{2,m}] \right]_{\Omega_2}^2, \]
\[ R_{\Gamma_D}(u_{1,m}, u_{2,m}) = \left[ \mathcal{I}_{\Gamma_D}[u_{1,m}, u_{2,m}] \right]_{\Gamma}^2, \quad R_{\Gamma_N}(u_{1,m}, u_{2,m}) = \left[ \mathcal{I}_{\Gamma_N}[u_{1,m}, u_{2,m}] \right]_{\Gamma}^2, \]
\[ R_B(u_{2,m}) = \left[ \mathcal{B}[u_{2,m}] \right]_{\partial \Omega}^2, \]
the Lipschitz coefficients of \( \mathcal{L}_1[u_{1,m}], \mathcal{L}_2[u_{2,m}], \mathcal{I}_{\Gamma_D}[u_{1,m}, u_{2,m}], \mathcal{I}_{\Gamma_N}[u_{1,m}, u_{2,m}] \) and \( \mathcal{B}[u_{2,m}] \) are uniformly bounded above. With the second assumption of (3.2), \( \mathcal{L}_1[u_{1,m}], \mathcal{L}_2[u_{2,m}], \mathcal{I}_{\Gamma_D}[u_{1,m}, u_{2,m}], \mathcal{I}_{\Gamma_N}[u_{1,m}, u_{2,m}] \) and \( \mathcal{B}[u_{2,m}] \) are uniformly bounded and uniformly equicontinuous sequences of functions in \( C^{0,L}(\Omega_1), C^{0,L}(\Omega_2), C^{0,L}(\Gamma), C^{0,L}(\Gamma) \) and \( C^{0,L}(\partial \Omega) \), respectively. Then, the well-known Arzela-Ascoli Theorem indicates that there exists a subsequence \( u_{m_j} = (u_{1,m_j}, u_{2,m_j}) \) and functions \( \hat{f}_1 \in C^{0,L}(\Omega_1), \hat{f}_2 \in C^{0,L}(\Omega_2), \hat{\phi} \in C^{0,L}(\Gamma), \hat{\psi} \in C^{0,L}(\Gamma), \hat{g} \in C^{0,L}(\partial \Omega) \) such that as \( j \to \infty \),
\[ \mathcal{L}_1[u_{1,m_j}] \to \hat{f}_1 \text{ in } C^0(\Omega_1), \quad \mathcal{L}_2[u_{2,m_j}] \to \hat{f}_2 \text{ in } C^0(\Omega_2), \]
\[ \mathcal{I}_{\Gamma_D}[u_{1,m_j}, u_{2,m_j}] \to \hat{\phi} \text{ in } C^0(\Gamma), \quad \mathcal{I}_{\Gamma_N}[u_{1,m_j}, u_{2,m_j}] \to \hat{\psi} \text{ in } C^0(\Gamma), \]
\[ \mathcal{B}[u_{2,m_j}] \to \hat{g} \text{ in } C^0(\partial \Omega). \]
Thus, with probability one,
\[
0 = \lim_{j \to \infty} \text{Loss}(u_{1,m_j}, u_{2,m_j}; \lambda, 0)
= \lim_{j \to \infty} \left( \lambda_{r_1} \int_{\Omega_1} \| \mathcal{L}_1[u_{1,m_j}](x) - f_1(x) \|^2 \, d\mu_{r_1}(x) + \lambda_{r_2} \int_{\Omega_2} \| \mathcal{L}_2[u_{2,m_j}](x) - f_2(x) \|^2 \, d\mu_{r_2}(x) \right.
\]
\[ + \lambda_B \int_{\partial \Omega} \| \mathcal{B}[u_{2,m_j}](x) - g(x) \|^2 \, d\mu_B(x) + \lambda_{\Gamma_D} \int_{\Gamma} \| \mathcal{I}_{\Gamma_D}[u_{1,m_j}, u_{2,m_j}](x) - \varphi(x) \|^2 \, d\mu_{\Gamma_D}(x) \]
\[ + \lambda_{\Gamma_N} \int_{\Gamma} \| \mathcal{I}_{\Gamma_N}[u_{1,m_j}, u_{2,m_j}](x) - \psi(x) \|^2 \, d\mu_{\Gamma_N}(x) \bigg) \]
\[ = \lambda_{r_1} \int_{\Omega_1} \| \hat{f}_1(x) - f_1(x) \|^2 \, d\mu_{r_1}(x) + \lambda_{r_2} \int_{\Omega_2} \| \hat{f}_2(x) - f_2(x) \|^2 \, d\mu_{r_2}(x) \]
\[ + \lambda_B \int_{\partial \Omega} \| \hat{g}(x) - g(x) \|^2 \, d\mu_B(x) + \lambda_{\Gamma_D} \int_{\Gamma} \| \hat{\varphi}(x) - \varphi(x) \|^2 \, d\mu_{\Gamma_D}(x) \]
\[ + \lambda_{\Gamma_N} \int_{\Gamma} \| \hat{\psi}(x) - \psi(x) \|^2 \, d\mu_{\Gamma_N}(x), \]
which shows that \( \hat{f}_1 = f_1 \) in \( L^2(\Omega_1; \mu_{r_1}) \), \( \hat{f}_2 = f_2 \) in \( L^2(\Omega_2; \mu_{r_2}) \), \( \hat{g} = g \) in \( L^2(\partial \Omega; \mu_B) \), \( \hat{\phi} = \phi \) in \( L^2(\Gamma; \mu_{\Gamma_D}) \) and \( \hat{\psi} = \psi \) in \( L^2(\Gamma; \mu_{\Gamma_N}) \). Here, the first equality holds by the first part of the Lemma; the third equality holds by Lebesgue’s Dominated Convergence Theorem since \( \mathcal{L}_1[u_{1,m_j}], \mathcal{L}_2[u_{2,m_j}], \mathcal{B}[u_{2,m_j}], \mathcal{I}_{\Gamma_D}[u_{1,m_j}, u_{2,m_j}], \mathcal{I}_{\Gamma_N}[u_{1,m_j}, u_{2,m_j}] \) are uniformly bounded and uniformly converge to \( f_1, f_2, g, \phi \) and \( \psi \), respectively.

We next use a contradiction argument to complete proof. Now assume, for contradiction, that \( \mathcal{L}_1[u_{1,m}] \) does not converge to \( f_1 \) in \( L^2(\Omega_1; \mu_{r_1}) \). Then there exists a constant \( \delta \) and a subsequence \( u_{1,m_k} \) satisfying \( \| \mathcal{L}_1[u_{1,m_k}] - f_1 \|_{L^2(\Omega_1; \mu_{r_1})} > \delta \). Again using Arzela-Ascoli Theorem, there exists a subsequence of \( u_{1,m_k} \), denoted as \( u_{1,m_{k_j}} \), such that \( \mathcal{L}_1[u_{1,m_{k_j}}] \to f_1 \) in \( L^2(\Omega_1; \mu_{r_1}) \), which yields a contradiction. Similar arguments hold for \( \mathcal{L}_2[u_{2,m}], \mathcal{B}[u_{2,m}], \mathcal{I}_{\Gamma_D}[u_{1,m}, u_{2,m}], \mathcal{I}_{\Gamma_N}[u_{1,m}, u_{2,m}] \) and \( \mathcal{B}[u_{2,m}] \). Therefore, we conclude that \( \mathcal{L}_1[u_{1,m}] \to f_1 \) in \( L^2(\Omega_1; \mu_{r_1}) \), \( \mathcal{L}_2[u_{2,m}] \to f_2 \) in \( L^2(\Omega_2; \mu_{r_2}) \), \( \mathcal{B}[u_{2,m}] \to g \) in \( L^2(\partial \Omega; \mu_B) \), \( \mathcal{I}_{\Gamma_D}[u_{1,m}, u_{2,m}] \to \phi \) in \( L^2(\Gamma; \mu_{\Gamma_D}) \) and \( \mathcal{I}_{\Gamma_N}[u_{1,m}, u_{2,m}] \to \psi \) in \( L^2(\Gamma; \mu_{\Gamma_N}) \) as \( m \to \infty \).

Finally, to complete the proof, it is sufficient to present the following estimate for the interface problem (4).

For convenience, we define \( X = H^2(\Omega_1) \cap H^2(\Omega_2) \) and define
\[ \| u \|_X = \| u \|_{H^2(\Omega_1)} + \| u \|_{H^2(\Omega_2)}, \forall u \in X. \]
Lemma 4.4. Assume that $\varphi \in H^2(\Gamma)$, $\psi \in H^1(\Gamma)$, $g \in H^2(\partial\Omega)$, $f_i \in L^2(\Omega)$, $i = 1, 2$. Then the solution to problem (4), $u \in X$, satisfies the priori estimate:

$$\|u\|_X \leq C \left( \|f\|_{L^2(\Omega)} + \|g\|_{H^2(\partial\Omega)} + \|\varphi\|_{H^2(\Gamma)} + \|\psi\|_{H^1(\Gamma)} \right).$$

Proof. Let $\tilde{u}_1$ solve

$$-\Delta \tilde{u}_1 = 0, \text{ in } \Omega_1,$$

$$\tilde{u}_1 = \varphi \text{ on } \Gamma.$$  

We know $\tilde{u}_1$ exists and $\tilde{u}_1 \in H^2(\Omega_1)$ satisfying (cf. Grisvard [9])

$$\|\tilde{u}_1\|_{H^2(\Omega_1)} \leq C\|\varphi\|_{H^{3/2}(\Gamma)}.$$

Let $\tilde{u}_2$ solve

$$-\Delta^2 \tilde{u}_2 = 0, \text{ in } \Omega_2,$$

$$\tilde{u}_2 = 0, \ \frac{\partial \tilde{u}_2}{\partial n} = \frac{a_1}{a_2} \frac{\partial \tilde{u}_1}{\partial n} + \frac{\psi}{a_2} \text{ on } \Gamma,$$

$$\tilde{u}_2 = g, \ \frac{\partial \tilde{u}_2}{\partial \nu} = 0 \text{ on } \partial\Omega.$$  

We have (cf. Girault-Raviart [8], pp.15-17)

$$\|\tilde{u}_2\|_{H^2(\Omega_2)} \leq C \left( \|g\|_{H^{3/2}(\partial\Omega)} + \|\tilde{u}_1\|_{H^{1/2}(\Gamma)} + \|\psi\|_{H^{1/2}(\Gamma)} \right),$$

where $C$ is a universal constant that depends on $\Omega, a_1, a_2$. Let

$$\tilde{u}(x) = \begin{cases} 
\tilde{u}_1(x), & x \in \Omega_1, \\
\tilde{u}_2(x), & x \in \Omega_2.
\end{cases}$$

Then, $v = u - \tilde{u}$ solve the equation

$$-\nabla \cdot (a_i \nabla v) + b_i v = f_i + \nabla \cdot (a_i \nabla \tilde{u}_i) - b_i \tilde{u}_i, \ \text{ in } \Omega_i, \ i = 1, 2,$$

$$[a \nabla v \cdot \mathbf{n}] = 0, \ \text{ on } \Gamma,$$

$$[v] = 0, \ \text{ on } \Gamma,$$

$$v = 0, \ \text{ on } \partial\Omega.$$  

According to the estimation in [1][4], $v$ satisfies

$$\|v\|_X \leq C \left( \|f_1 + \nabla \cdot (a_1 \nabla \tilde{u}_1) - b_1 \tilde{u}_1\|_{L^2(\Omega_1)} + \|f_2 + \nabla \cdot (a_2 \nabla \tilde{u}_2) - b_2 \tilde{u}_2\|_{L^2(\Omega_2)} \right),$$

$$\leq C \left( \|f_1\|_{L^2(\Omega_1)} + \|f_2\|_{L^2(\Omega_2)} + \|\tilde{u}_1\|_{H^2(\Omega_1)} + \|\tilde{u}_2\|_{H^2(\Omega_2)} \right).$$

Therefore, we deduce that

$$\|u\|_X \leq \|v\|_X + \|\tilde{u}\|_X \leq C \left( \|f_1\|_{L^2(\Omega_1)} + \|f_2\|_{L^2(\Omega_2)} + \|\tilde{u}\|_X \right),$$

$$\leq C \left( \|f_1\|_{L^2(\Omega_1)} + \|f_2\|_{L^2(\Omega_2)} + \|g\|_{H^{3/2}(\partial\Omega)} + \|\varphi\|_{H^{1/2}(\Gamma)} + \|\psi\|_{H^{1/2}(\Gamma)} \right),$$

$$\leq C \left( \|f_1\|_{L^2(\Omega_1)} + \|f_2\|_{L^2(\Omega_2)} + \|g\|_{H^2(\partial\Omega)} + \|\varphi\|_{H^2(\Gamma)} + \|\psi\|_{H^1(\Gamma)} \right).$$

The proof is completed. \qed
proof of Theorem 3.2. For the sequence of the minimizers given in Lemma \(m\), \(\{(u_1,m, u_2,m)\}_m\). Lemma 4.3 indicates that \(\|L_1[u_1,m] - f_1\|_{L^2(\Omega_1)} \to 0, \|L_2[u_2,m] - f_2\|_{L^2(\Omega_2)} \to 0, \|u_2,m - u_1,m - \varphi\|_{H^2(\Gamma)} \to 0\) and \(\|a \nabla u_2,m \cdot n - a \nabla u_1,m \cdot n - \psi\|_{H^1(\Gamma)} \to 0\) as \(m_r \to \infty\). Here, we have used the definition of \(B, I_D\) and \(I_N\). In addition, Lemma 4.4 implies that

\[
\|u_1,m - u^*\|_{H^2(\Omega_1)} + \|u_2,m - u^*\|_{H^2(\Omega_2)} \leq C \left( \|L_1[u_1,m] - f_1\|_{L^2(\Omega_1)} + \|L_2[u_2,m] - f_2\|_{L^2(\Omega_2)} + \|u_2,m - u_1,m - \varphi\|_{H^2(\Gamma)} + \|a \nabla u_2,m \cdot n - a \nabla u_1,m \cdot n - \psi\|_{H^2(\Gamma)} \right)
\]

Therefore, we conclude that

\[
\lim_{m_r \to \infty} u_1,m = u^*, \quad \text{in} \ H^2(\Omega_1), \quad \lim_{m_r \to \infty} u_2,m = u^*, \quad \text{in} \ H^2(\Omega_2),
\]

which completes the proof.

5 Summary

The main contribution of this paper is to perform the convergence analysis of the neural network method for solving second-order elliptic interface problems. We prove that the neural network sequence converges to the unique solution to the interface problem in \(H^2\). This result advanced the mathematical foundations of the deep learning-based solver of PDEs.

To complete the proof, we first derive a Lipschitz regularized empirical loss from the probabilistic space filling arguments \([5]\) to bound the expected PINN loss and then show that the expected PINN loss at the minimizers of the Lipschitz regularized empirical loss converges to zero. Finally, we demonstrate that the minimizers of the Lipschitz regularized empirical losses converge to the solution to the interface problem uniformly as the number of training samples grows in \(H^2\) and conclude the main theorem.

One limitation of our work is that Theorem 3.2 only holds for the global minimizer. The landscape of non-convex objective functions and the optimization process by stochastic gradient still remains open. We would like to further investigate such problems and quantify the optimization error of solving elliptic interface problems using neural networks in the future.

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