A Candidate for $1^{--}$ Strangeonium Hybrid

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Abstract

We propose that the recently observed structure at 2175MeV by the Babar Collaboration is a $1^{--}$ strangeonium hybrid, and we investigate this interpretation from both the flux tube model and the constituent gluon model. The decay patterns and decay width in the flux tube model and the constituent gluon model (for the "gluon excited" hybrid) are very similar. The tetraquark hypothesis is not favored by the available experimental data. The crucial test of our scenario is suggested, furthermore, the promising channels which can discriminate the hybrid interpretation from the tetraquark are also suggested.

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I. INTRODUCTION

Very recently the Babar Collaboration observed a structure at 2175 MeV in the $f_0(980)$ recoil mass of the process $e^+e^- \rightarrow \gamma_{ISR}f_0(980)$ [1]. Its mass is $m = 2.175 \pm 0.010 \pm 0.015$ GeV/$c^2$ and width is narrow $\Gamma = 58 \pm 16 \pm 20$ MeV. It is claimed as a isospin singlet, and its spin-parity is determined to be $J^{PC} = 1^{--}$. Henceforth, this structure is denoted by $Y(2175)$. There are no known meson resonances with $I = 0$ near this mass, therefore it may not be a standard meson but rather an exotic.

Since both this structure and $Y(4260)$ [2] are observed in $e^+e^-$ annihilation through initial state radiation, it may have an analogical structure of $Y(4260)$ with $c\bar{c}$ replaced by $s\bar{s}$. $Y(4260)$ has been interpreted as $c\bar{c}g$ hybrid [3, 4], a $cs\bar{s}$ tetraquark state [5] or others [6], therefore $Y(2175)$ maybe a $s\bar{s}g$ hybrid or a $ss\bar{s}\bar{s}$ tetraquark state. In Ref. [7], the mass of $Y(2175)$ which is taken as a four quark state has been calculated from QCD sum rule. In the tetraquark picture, although $Y(2175)$ can not decay into $\eta\eta$, $\eta\eta'$, $\eta'\eta'$ and $\phi\phi$ due to the $C$–parity and $G$–parity, no symmetry forbids the decay $Y(2175) \rightarrow \eta'(\eta')\phi$, which can occur through the so-called "fall-apart" mechanism [8]. With enough phase space, the decay width is generally large. Moreover, $\eta'(\eta')\phi$ should be one of the dominant decay modes, which is also against the experimental observation. In addition we would like to mention that from the view of quark correlation the tetraquark hypothesis prefer to have the diquark-antidiquark picture, i.e. $\{ss\}\{s\bar{s}\}$, where the diquark(or antidiquark) is the so called "bad" diquark ("good" and "bad" diquarks in Jaffe’s terminology [9]). In short, four quark interpretation of $Y(2175)$ faces great challenge. However, more experimental investigation is needed to completely exclude the possibility of $Y(2175)$ being a tetraquark, although the available data already disfavors the tetraquark hypothesis.

Hybrid state is one of the most promising new species of hadrons, extensive investigations in searching of the hybrid states have been pursued, especially in the light hadrons. Although there is amounting evidence for hybrid consisting of light quarks, they still await confirmation, e.g., both $\rho_1(1450)$ and $\pi_1(1600)$ are good hybrid candidates. Hybrid states have been studied in various approaches such as the flux tube model [10, 11, 12, 13, 14], the constituent gluon model [15, 16, 17], the QCD sum rule [18] and the lattice QCD [19]. In the following, we will argue the mysterious $Y(2175)$ could be a $1^{--}$ strangeonium hybrid ($s\bar{s}g$) from the flux tube model and the constituent gluon model. 
In the flux tube model, a hybrid state is a quark-antiquark pair moving on an adiabatic surface generated by an excited gluonic flux-tube, and a hybrid meson would decay because of phenomenological pair production described by $^3P_0$ model coupled with a flux tube overlap. Isgur et al., estimated that the lowest lying strangeonium hybrid($s\bar{s}g$) could have a mass about 1.9GeV\[10\]. Close et al., improved the previous treatments using a Hamiltonian Monte Carlo algorithm, they predicted that the mass of the lightest $s\bar{s}g$ hybrid is 2.1-2.2GeV\[11\], which is consistent with experimental value of $Y(2175)(m = 2.175 \pm 0.010 \pm 0.015\text{GeV})$. In the constituent gluon model, a hybrid state is a quark-antiquark state with an additional constituent gluon, such a meson would decay though gluon dissociation into a $q\bar{q}$ pair\[15, 16, 17\]. The constituent gluon is expected to add $0.7 \sim 1\text{GeV}$ to the corresponding quarkonia and the excitation energy is about $0.4\text{ GeV}$ for the excited hybrid state, naively the mass of the $s\bar{s}g$ would be about 2.12-2.42GeV, which is also consistent experimental data of $Y(2175)$. So the predictions for mass of lightest strangeonium hybrid in both models supports that $Y(2175)$ could be a $s\bar{s}g$ hybrid state.

Since the quark pair are created from gluon in the constituent gluon model, this model belong to the $^3S_1$ decay model\[20\]. Although $^3P_0$ model is more successful than the $^3S_1$ decay model in conventional meson and baryon decays\[21\], we don’t know whether the same is also true in the hybrid decay. Moreover the constituent gluon model is widely used to discuss the decay of the hybrid meson, it is regarded as a useful theoretical tool to study hybrid hadron, so we would like to investigate the hybrid hypothesis from both the flux tube model and the constituent gluon model. If the results in two models are consistent within the uncertainties of both models, it will be a strong support to our picture.

Extensive and thorough analyse has been done for the $1^{--}$ system, the existence for $1^{--}$ hybrids in both isovector and isoscalar channels is required in $e^+e^-$ annihilation and $\tau$ decay\[22, 23\]. We claim that $Y(2175)$ is the isoscalar $s\bar{s}g$ hybrid state of this $1^{--}$ multiplet, and other candidates in this multiplet, such as $q\bar{q}g$, $s\bar{q}g(q = u, d)$ are expected to be observed in future. In this work, we will investigate the $1^{--}$ strangeonium hybrid from both the flux tube model and constituent gluon model in details. The outline of the paper is as follows. In section II, we study the $1^{--}$ strangeonium hybrid from the flux tube model, and in section III, this hybrid state is investigated in the constituent gluon model. A brief summary and discussions are given in Section IV.
II. 1−− STRANGEONIUM HYBRID FROM THE FLUX TUBE MODEL

The flux tube model is extracted from the strong coupling limit of the QCD lattice Hamiltonian, and decay occurs when the flux-tube breaks at any point along its length, producing a $q\bar{q}$ pair in a relative $J^{PC} = 0^{++}$ state. Since we have to consider the dynamics of the flux tube, the flux tube overlap has to be included in addition to the color, flavor, spin and spatial overlap. Hybrid decay has been studied carefully in the flux tube model and its extended version for both the light flavor and heavy flavors\cite{10, 12, 13}. In the following, the strangeonium hybrid decay will be considered using simple harmonic oscillation approximation, we will follow the formalism of Close et al.,\cite{12}. This is typical of decay calculation and it has been demonstrated that using Coulomb+linear wavefunctions from the relativized quark model of Godfrey and Isgur does not change the results significantly\cite{24, 25}. In the narrow width approximation, the partial decay width for the process $A \rightarrow BC$ is,

$$\Gamma_{LJ}(A \rightarrow BC) = \frac{p_B}{(2J_A + 1)\pi} \frac{\tilde{M}_B \tilde{M}_C}{M_A^{LJ}} |M_{LJ}(A \rightarrow BC)|^2$$  \hspace{1cm} (1)

where the phase space normalization of Kokoski and Isgur is employed\cite{24, 25}, $M_A, M_B, M_C$ are respectively the 'mock meson' masses of $A, B, C$. $M_{LJ}(A \rightarrow BC)$ is the partial wave amplitude for the decay process, $L$ is the relative angular momentum between $B$ and $C$, $J$ is the total angular momentum of $B$ and $C$, and $p_B$ is $B$’s momentum in the rest frame of the particle $A$. The hybrid wavefunction is taken as:

$$\psi_A(r) = \sqrt{\frac{3\beta_A^{3+2\delta}}{2\pi(3/2 + \delta)}} r^{\delta} D_{M_A}^{1}(\phi, \theta, -\phi) \exp\left(-\beta_A^2 r^2/2\right)$$  \hspace{1cm} (2)

Here $\delta = 0.62$, $M_A^L$ is the projection of the orbital angular momentum along $z$ axis, and $\Lambda$ is the flux tube angular momentum along the quark-antiquark axis in the hybrid. In our case $M_A^L = 0, \pm 1$ and $\Lambda = \pm 1$. The S.H.O. wavefunctions for the angular momentum $L = 0$ and $L = 1$ ordinary mesons respectively are:

$$\psi_S(r) = \frac{\beta^{3/2}}{\pi^{3/4}} \exp(-\beta^2 r^2/2), \quad \psi_P(r) = \frac{2\sqrt{2}\beta^{5/2}}{\sqrt{3}\pi^{1/4}} r Y_{1M}(\hat{r}) \exp(-\beta^2 r^2/2)$$  \hspace{1cm} (3)

where $\beta$ is the harmonic oscillation parameter, which can be different for various mesons. The amplitude for the hybrid state decaying into $S + P$-wave meson pair can be calculated analytically using the S.H.O. wavefunction. The $1^{--}$ $s\bar{s}g$ hybrid can decay into $K_2^*(1430)K$ with the relative angular momentum between the final states being 2, and it can also decay
into $K_1(1270)K$ and $K_1(1400)K$ in S-wave and D-wave. Under the first order approximation $\beta_B = \beta_C$, the corresponding decay amplitudes are in the followings\cite{12}:

\[
M_{S1}((s\bar{s}g) \rightarrow K_1(1270)K) = \varrho \, F((s\bar{s}g) \rightarrow K_1(1270)K) \frac{1}{3} (-3h_0 + g_1 - 4h_2)
\]

\[
M_{S1}((s\bar{s}g) \rightarrow K_1(1400)K) = \varrho \, F((s\bar{s}g) \rightarrow K_1(1400)K) \frac{\sqrt{2}}{3} (-3h_0 + g_1 - 4h_2)
\]

\[
M_{D1}((s\bar{s}g) \rightarrow K_1(1270)K) = \varrho \, F((s\bar{s}g) \rightarrow K_1(1270)K) \frac{\sqrt{3}}{6} (g_1 + 5h_2)
\]

\[
M_{D1}((s\bar{s}g) \rightarrow K_1(1400)K) = \varrho \, F((s\bar{s}g) \rightarrow K_1(1400)K) \frac{1}{3} (g_1 + 5h_2)
\]

\[
M_{D2}((s\bar{s}g) \rightarrow K_2^*(1430)K) = \varrho \, F((s\bar{s}g) \rightarrow K_2^*(1430)K) \frac{1}{\sqrt{2}} (g_1 + 5h_2)
\]

where $\varrho = \left( \frac{\alpha c}{9\sqrt{3}} A_0 \left( \frac{\pi}{1 + fb/(2\beta^2)} \right)^2 \sqrt{\frac{2\pi}{3\Gamma(3/2 + \delta)}} \frac{\beta_A^{3/2 + \delta}}{\beta_a} \right)$ with $\bar{\beta}^2 \equiv (\beta_B^2 + \beta_C^2)/2 = \beta_B^2 = \beta_C^2$. $F((s\bar{s}g) \rightarrow K_1(1270)K), F((s\bar{s}g) \rightarrow K_1(1400)K), F((s\bar{s}g) \rightarrow K_2^*(1430)K)$ are the flavor factors in the corresponding decay processes, in our case $F((s\bar{s}g) \rightarrow K_1(1270)K) = F((s\bar{s}g) \rightarrow K_1(1400)K) = F((s\bar{s}g) \rightarrow K_2^*(1430)K) = 2$. The analytical expressions for $h_0, g_1, h_2$ are listed in the Appendix A. To derive this result, $K_1(1270)$ and $K_1(1400)$ are taken to be linear combinations of $^1P_1$ and $^3P_1$ states\cite{13,24},

\[
|K_1(1270)\rangle = \sqrt{\frac{2}{3}} |^1P_1\rangle + \sqrt{\frac{1}{3}} |^3P_1\rangle
\]

\[
|K_1(1400)\rangle = -\sqrt{\frac{1}{3}} |^1P_1\rangle + \sqrt{\frac{2}{3}} |^3P_1\rangle
\]

It is well-known that the lowest lying hybrid does not decay to identical mesons, but prefers to decay into $S + P$-wave meson pair\cite{10,26}. This rule has been shown to be more general than specific models\cite{27}. Although decaying into $S + S$-wave meson pair is not forbidden if the internal structures\cite{i.e.,} of the two S-wave meson differ, the width is generally proportional to $(\beta_B^2 - \beta_C^2)^2$, which is usually small.

For the $1^{-}$ strangeonium hybrid decaying into two S-wave mesons, the strangeonium hybrid can decay into $K^*(892)K, \phi\eta, \phi\eta'$ in relative $P$-wave, the amplitude is of the following form\cite{12},

\[
M_{P1}((s\bar{s}g) \rightarrow BC) = i\left( \frac{\alpha c}{9\sqrt{3}2} A_0 \left( \frac{\pi}{1 + fb/(2\bar{\beta}^2)} \right)^2 \sqrt{\frac{2\pi}{3\Gamma(3/2 + \delta)}} \frac{\beta_A^{3/2 + \delta}}{\beta_a} \right) \Delta \sqrt{\frac{\pi}{3\Gamma(3/2 + \delta)}} \frac{\beta_B^{3/2 + \delta} (\beta_B \beta_C)^{3/2}}{\beta^5}
\]

\[
\times 3F((s\bar{s}g) \rightarrow BC)P
\]

where $\Delta \equiv \beta_B^2 - \beta_C^2$, $\bar{\beta}^2 \equiv (\beta_B^2 + \beta_C^2)/2$, and $F((s\bar{s}g) \rightarrow BC)$ is the flavor factor in the process $(s\bar{s}g) \rightarrow BC$. $P$ is the overlap integral which can be integrated out using the technique of
Appendix A.

\[ P = \int_0^\infty dr r^{2+\delta} j_1(\frac{M_{PB}}{m + M}) \exp(-2\beta A^2 + \bar{\beta}^2 - (\frac{\Delta}{2\beta})^2 r^2) \]

\[ = \frac{M_{PB}}{m + M} \frac{2^{3+\delta} \Gamma(2 + \frac{\delta}{2})}{3[2\beta A^2 + \bar{\beta}^2 - (\frac{\Delta}{2\beta})^2]^{3+\delta/2}} \Gamma(3 + \frac{\delta}{2}) \frac{\pi}{\sqrt{2\pi}} F_1(2 + \frac{\delta}{2}, \frac{5}{2}, \frac{2}{m + M} \frac{M^2 p_B^2}{[2\beta A^2 + \bar{\beta}^2 - (\frac{\Delta}{2\beta})^2]^{3+\delta/2}}) \]  \( (7) \)

\( \eta \) and \( \eta' \) are taken to be "perfect mixing": \( \eta = \frac{1}{2}(u\bar{u} + d\bar{d}) - \frac{1}{\sqrt{2}} s\bar{s}, \eta' = \frac{1}{2}(u\bar{u} + d\bar{d}) + \frac{1}{\sqrt{2}} s\bar{s}. \) The mixing angle is consistent with the one obtained from the PDG[28]. But the decay mode \( K(1460) \) is forbidden by the "spin selection rule"[13], since \( K(1460) \) is a \( 2^1 S_0 \) strange quarkonium[28].

The overlapping integral \( P' \) is,

\[ P' = \int_0^\infty dr r^{2+\delta} j_1(\frac{M_{PB}}{m + M}) \{ r^2 \Delta(\Delta - 2\bar{\beta}^2)^2 (bf + 2\bar{\beta}^2)(\Delta + 2\bar{\beta}^2) + 8\bar{\beta}^2 [bf \Delta^2 + 2\Delta(-4bf + 5\Delta) \bar{\beta}^2 - 8bf \bar{\beta}^4 - 16\bar{\beta}^6] \}

\[ = \frac{M_{PB}}{3(m + M)} \{ \Delta(\Delta - 2\bar{\beta}^2)^2 (bf + 2\bar{\beta}^2)(\Delta + 2\bar{\beta}^2) - \frac{2^{5+\delta} \Gamma(3 + \frac{\delta}{2})}{2\beta A^2 + \bar{\beta}^2 - (\frac{\Delta}{2\beta})^2]^{3+\delta/2}} F_1(3 + \frac{\delta}{2}, \frac{5}{2}, \frac{2}{m + M} \frac{M^2 p_B^2}{[2\beta A^2 + \bar{\beta}^2 - (\frac{\Delta}{2\beta})^2]^{3+\delta/2}}) \]

\[ \times \frac{2^{6+\delta} \Gamma(2 + \frac{\delta}{2})}{[2\beta A^2 + \bar{\beta}^2 - (\frac{\Delta}{2\beta})^2]^{3+\delta/2}} F_1(2 + \frac{\delta}{2}, \frac{5}{2}, \frac{2}{m + M} \frac{M^2 p_B^2}{[2\beta A^2 + \bar{\beta}^2 - (\frac{\Delta}{2\beta})^2]^{3+\delta/2}}) \] \( (9) \)

We take the string tension \( b = 0.18\text{GeV}^2 \), and the constituent quark masses \( m_u = m_d = 0.33\text{GeV}, m_s = 0.55\text{GeV} \), the masses of the mesons will be taken from PDG[23]. A detailed discussion about other quantities such as \( f, \kappa, A_{00}^0 \) may be found in the Appendix A of Ref.[24] and Ref.[29]. The common factor \( \frac{\sqrt{6}}{9\sqrt{3}} \frac{A_{00}^0}{\sqrt{\pi}} \) is taken to be 0.64 which gives a best fit to the decay of conventional mesons in the flux tube model[22,24], and the estimated values \( f = 1.1, \kappa = 0.9 \) is used in this work. The oscillation parameter \( \beta \) is chosen as that of
Ref. [12], for the strangeonium hybrid \( s\bar{s}g \) we choose \( \beta_A = 0.30\text{GeV} \). \( \tilde{\beta} = 0.40\text{GeV} \) is used in the case of hybrid decaying into \( S+P \)–wave mesons pair. For the \( S+S \)–wave mesons final states, the effective \( \beta \) in Ref. [24] is used to determine the width, e.g. \( \beta_{K^*(1410)} = 0.41\text{GeV} \), \( \beta_{K^*} = 0.48\text{GeV} \), \( \beta_K = 0.71\text{GeV} \), \( \beta_{\eta'} = 0.74\text{GeV} \) and \( \beta_\phi = 0.51\text{GeV} \). In the following table, we present the dominant decay modes for the \( 1^{--} \) strangeness hybrid decay in various partial waves.

### TABLE I: The decay modes of \( 1^{--} \) strangeonium hybrid \( s\bar{s}g \) in the flux tube model

| Decay Channels | \( K^*_2(1430)K \) | \( K_1(1270)K \) | \( K_1(1400)K \) | \( K^*(1410)K \) | \( K^*(892)K \) | \( \phi\eta \) | \( \phi\eta' \) | Total |
|---------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------|-----------|-------|
| \( \Gamma \text{(MeV)} \) | D 15.0 | S 30.8 | S 65.8 | P 23.0 | P 3.7 | P 1.2 | P 0.4 | 148.7 |
| D 4.5 | D 4.3 | P  | | |

From the above table, we can see the "S+P" selection rule is satisfied for the strangeonium hybrid in a good manner, and the hybrid decaying into \( 2S + 1S \) final state is usually not suppressed. The \( 1^{--} \) strangeonium hybrid mainly decays into \( K_1(1400)K \), \( K_1(1270)K \), \( K^*(1410)K \), \( K^*_2(1430)K \). However, \( Y(2175) \) mainly decays into \( \phi\eta \) and \( \phi\eta' \) in the \( s\bar{s}s\bar{s} \) tetraquark scenario, thus a search for the latter channels, or the limit on its coupling, could be a significant discriminator for the nature of \( Y(2175) \). Furthermore, the \( 1^{--} \) strangeonium hybrid can decay to \( \phi\pi\pi \) by the cascade decay mechanism \( (s\bar{s}g) \rightarrow (s\bar{s})(gg) \rightarrow \phi + \pi + \pi \) [30], and \( (s\bar{s}g) \rightarrow \phi f_0(980) \) may make an significant contribution to the process. So the search for \( \phi KK \) decay modes are also merited.

As a measure of the reliability of these predictions, the mass dependence of the partial decay width and total width are respectively shown in Fig.1 and Fig.2. We also show the harmonic parameter \( \beta_A \) dependence of the partial decay width and total width in Fig.3 and Fig.4. We don’t show the partial width of the modes \( K^*(892)K \), \( \phi\eta \) and \( \phi\eta' \), since they are small enough to be negligible. From these figures, we can see the total decay width decreases with the decrease of its mass. For a 2-2.2 GeV \( 1^{--} \) strangeonium hybrid, its width is about 120-150 MeV, which is consistent with the experimental observation about \( Y(2175) \) (\( \Gamma = 58 \pm 16 \pm 20 \text{MeV} \)) within the uncertainties of the flux tube model. On all accounts, it is reasonable to identify \( Y(2175) \) as a \( 1^{--} \) strangeonium hybrid from the flux tube model.
In order to provide further support to our proposal, we would like to study the $1^{−−}$ strangeonium hybrid from the constituent gluon model. The constituent gluon model is a generalization of the quark model with constituent gluon, the decay of hybrid mesons has been studied widely in the constituent gluon model\[16, 17, 26, 31\]. The reasonableness of $Y(4260)$ as a $1^{−−}$ charmonium hybrid has been investigated from this model\[4, 32\]. Here we will follow the notation of Ref.\[16\]. From the view of the constituent gluon model,
strangeonium hybrid is a bound state of \( s \bar{s} \) and a gluon. Defining \( L_g \) as the relative orbital angular momentum between the gluon and the \( s \bar{s} \) center of mass, \( L_s \) as the relative angular momentum between the strange quark and the anti-strange quark, \( S_{s \bar{s}} \) as the total spin of \( s \bar{s} \). Denoting the total gluon angular momentum by \( J_g \), \( J_g \equiv L_g + 1 \), and \( L \equiv L_{s \bar{s}} + J_g \). The parity and charge conjugation of the hybrid are given by:

\[
P = (-1)^{L_{s \bar{s}} + L_g}, \quad C = (-1)^{L_{s \bar{s}} + S_{s \bar{s}} + 1}
\]

(10)

The quantum number \( J^{PC} = 1^{--} \) implies that \( L_{s \bar{s}} = S_{s \bar{s}} = 1 \), \( L_g = 0 \) or \( L_{s \bar{s}} = S_{s \bar{s}} = 0 \), \( L_g = 1 \), the former is usually is referred as the "quark excited" hybrid, and the latter referred as the "gluon excited" hybrid. To lowest order, the decay is described by the matrix element of the QCD interaction between the initial hybrid wave function and final two mesons wavefunctions, and the interaction Hamiltonian is:

\[
H_I = g_s \int d^3x \bar{\psi}(x) \gamma_\mu \frac{\lambda^a}{2} \psi(x) A_\mu^a(x)
\]

(11)

The operator relevant to the decay can be expressed in terms of the creation and annihilation operators:

\[
H_I = g_s \sum_{s,s',\lambda,\ell,\ell',c,c',c_0} \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{\sqrt{2\omega(2\pi)^3}} (2\pi)^3 \delta^3(p - p' - k) \bar{u}_{psc} \gamma_\mu \frac{\lambda^a}{2} v_{c's'c''} b^\dagger_{psc} d^{\dagger p's'c'} a^{c_0}_{c'\lambda} \varepsilon^a(k, \lambda)
\]

(12)

here the flavor index of the quark has been omitted, \( c_a = 1, 2, \cdots 8 \), and

\[
\{b_{psc}, b^\dagger_{p's'c'}\} = \{d_{psc}, d^{\dagger p's'c'}\} = (2\pi)^3 \delta^3(p - p') \delta_{ss'} \delta_{\ell\ell'}
\]

\[
[a^c_{k\lambda}, a^c_{k'\lambda'}] = (2\pi)^3 \delta^3(k - k') \delta_{cc'} \delta_{\lambda\lambda'}
\]

(13)

under the non-relativistic approximation, the hybrid and the meson states are described as follows:

\[
|A(L_g, L_{s \bar{s}}; J_A, M_{J_A})\rangle = \sum \frac{f_{p_1} f_{p_2} d^3k}{(2\pi)^9} \langle L_g, M_{L_g}; 1, \lambda_g|J_g, M_{J_g}\rangle \langle L_{s \bar{s}}, M_{L_{s \bar{s}}}; J_g, M_{J_g}|L, m'\rangle \langle L, m'; S_{s \bar{s}}, M_{S_{s \bar{s}}}; J_A, M_{J_A}\rangle (2\pi)^3 \delta^3(p_1 + p_2 + k - p_A) \chi^1_{S_{s \bar{s}}, M_{s \bar{s}}} \varphi^1_A \frac{\lambda^a_{c_0 c}}{4}
\]

\[
\psi_{L_{s \bar{s}} M_{L_{s \bar{s}}} M_{L_g} M_{J_g}} (\frac{p_1 - p_2}{2}) \psi_{L_A M_{L_A}} \langle \frac{2M - m_g(p_1 + p_2)}{2M + m_g} \rangle b^\dagger_{p_{s'c_1} c_0} d^{\dagger p_{s'c_2} c_0} (L, m)|0\rangle
\]

(14)

The subscript 1 and 2 refer to the strange quark and the anti-strange quark within the hybrid meson. \( \chi^1_{S_{s \bar{s}}, M_{s \bar{s}}} \), \( \varphi^1_A \) are respectively the spin wavefunction and flavor wavefunction.
of $s\bar{s}$, $\psi_{LsM_{Ls}}(p_1-p_2)$, $\psi_{LsM_{Ls}}(2Mk-m_g(p_1+p_2))$ are respectively the spatial wavefunction of the $s\bar{s}$ and the constituent gluon, which usually is taken as the simple harmonic oscillator wavefunction.

The final $B$ meson’s state is given by:

$$|B(L_B,S_B,J_B,M_{J_B})\rangle = \sum_{M_{L_B},M_{S_B}} \int \frac{d^3p_1 d^3p_3}{(2\pi)^6} (L_B, M_{L_B}; S_B, M_{S_B}|J_B, M_{J_B})(2\pi)^3 \delta^3(p_1 + p_3 - p_B) \chi^{13}_{S_B M_{S_B}} \varphi^{13}_B \omega^{13}_B \psi_{L_B M_{L_B}}(\frac{m p_1 - M p_3}{M + m}) b^\dagger_{p_1 s_1} d_{p_3 s_3} |0\rangle$$

(15)

Here $\omega^{13}_B$ is the color wavefunction of $B$ meson. Another final state $C$ meson’s wavefunction can be write out analogously in terms of the quark, antiquark creation operators. It is straightforward to get the matrix element $\langle BC|H_1|A\rangle = g_s (2\pi)^3 \delta^3(p_A - p_B - p_C) M_{\ell,J}(A \to BC)$, so the partial decay width is:

$$\Gamma_{\ell J}(A \to BC) = \frac{\alpha_s C_B E_B E_C}{M_A} |M_{\ell J}(A \to BC)|^2$$

(16)

where $M_{\ell J}(A \to BC)$ is the partial wave amplitude, and $\Gamma_{\ell J}$ is the partial width of the corresponding partial wave. As usual, we would like to use the S.H.O basis wavefunctions, thereby enabling analytic studies that reveal the relationship among amplitudes. The partial wave amplitude $M_{\ell J}$ and the spatial overlap for various final states are presented in the Appendix B.

In our calculation, we take the following set of parameters: $\beta_B = \beta_C = \beta_g = 0.4\text{GeV}$, $\beta_{s\bar{s}} = 0.3\text{GeV}$, $m_s = 0.55\text{GeV}$, $m_u = m_d = 0.33\text{GeV}$ and $m_g = 0.8\text{GeV}$, which are often used in the constituent gluon model calculations. In the Ref. the authors used the oscillation parameters $R_B, R_g$ and so on, the parameters $\beta_B, \beta_g$ etc in this work are related to $R_B, R_g$ etc by the relations $\beta_B = 1/R_B, \beta_g = 1/R_g$.

In the case of the "quark excited" hybrid, the allowed decay channels and the partial decay width are shown in Table II. In this table, $S$ or $P$ indicate that the relative angular momentum between the two final states is $S$ or $P$ wave, and the symbols 0,1,2 denote the total angular momentum of two final states. From this table, we can see the quark excited hybrid mainly decays into $KK$, $K^*K$, $\phi\eta$, $\phi\eta'$, $K^*(892)K^*(892)$, $K_1(1270)K$, $K_1(1400)K$. The decay width is very large, which is approximately $307.4\text{MeV}$, $342.7\text{MeV}$, $464.2\text{MeV}$ respectively for the $L = 0, 1, 2$ quark excited hybrid.

In the case of the "gluon excited" hybrid, the allowed decay channels and the partial decay width are shown in Table III. It mainly decays into $K_1(1270)K$, $K_1(1400)K$ and
TABLE II: Decay of the $1^{--}$ quark excited strangeonium hybrid $s\bar{s}g$ in the constituent gluon model, width is in MeV×$\alpha_s$ for the channels. The QCD coupling constant $\alpha_s$ is of order 1 in this nonperturbative region.

|         | L=0    | L=1    | L=2    |
|---------|--------|--------|--------|
| $KK$    | P 0 14.3 | P 0 43.0 | P 0 71.7 |
| $K^*K$  | P 1 63.2 | P 1 47.4 | P 1 79.0 |
| $\phi\eta$ | P 1 30.6 | P 1 22.9 | P 1 38.2 |
| $\phi\eta'$ | P 1 12.0 | P 1 9.0  | P 1 15.0 |
| $K^*(892)K^*(892)$ | P 0 4.5  | P 0 13.4 | P 0 22.3 |
|         | P 1 0  | P 1 0   | P 1 0   |
|         | P 2 89.3 | P 2 67.0 | P 2 4.5 |
| $K_1(1270)K$ | S 1 28.9 | S 1 54.2 | S 1 91.0 |
|         | D 1 4.8 | D 1 12.3 | D 1 20.6 |
| $K_1(1400)K$ | S 1 46.2 | S 1 55.5 | S 1 92.0 |
|         | D 1 1.8 | D 1 3.3  | D 1 5.4 |
| $K_2^*(1430)K$ | D 2 3.1  | D 2 2.3  | D 2 3.9 |
| $K(1460)K$  | P 0 2.6 | P 0 7.8  | P 0 13.0 |
| $K^*(1410)K$ | P 1 6.1 | P 1 4.6  | P 1 7.6 |
| Tot      | 307.4  | 342.7  | 464.2  |

$K^*(1410)K$. Due to the selection rule, it can not decay to two pseudoscalars. The decay width is around 122.7MeV which is consistent with the flux tube model’s prediction 148.7MeV within the uncertainties of the models. Compared with the results in the flux tube model, the $1^{--}$ strangeonium hybrid can not decay to $^1P_1$ state and $\Gamma((s\bar{s}g)\rightarrow K(1400)K)\approx 2\Gamma((s\bar{s}g)\rightarrow K(1410)K)$ in both models. The decay width $\Gamma((s\bar{s}g)\rightarrow K_2^*(1430)K)$ is about 0 in the constituent gluon model, $\Gamma((s\bar{s}g)\rightarrow K_2^*(1430)K)\approx 15$MeV in the flux tube model, which is smaller than $\Gamma((s\bar{s}g)\rightarrow K_1(1270)K)$ and $\Gamma((s\bar{s}g)\rightarrow K_1(1400)K)$. So the decay pattern of the gluon excited hybrid is very similar to that in the flux tube model, and we conclude that $Y(2175)$ could mainly be the ”gluon excited” hybrid from the view of the constituent gluon model. The strong dependence of the decay width on the hybrid mass is displayed in Fig.5.
TABLE III: Decay of the $1^{--}$ gluon excited strangeonium hybrid $s\bar{s}g$ in the constituent gluon model, width is in MeV$\times\alpha_s$ for the channels. The QCD coupling constant $\alpha_s$ is of order 1 in this nonperturbative region.

| Channel          | Width (MeV) |
|------------------|-------------|
| $K_1(1270)K$     | 36.2        |
| $K_1(1400)K$     | 73.2        |
| $K^*(1410)K$     | 5.8         |
| Total            | 115.2       |

and we show the partial width for the $1^{--}$ gluon excited strangeonium hybrid decaying to $K_1(1270)K$, $K_1(1400)K$, $K^*(1410)K$ as a function of the hybrid mass in Fig.6. The $\beta_{s\bar{s}}$ dependence of total decay width and partial decay width are shown respectively in Fig.7 and Fig.8 in the constituent gluon model.

FIG. 5: The decay width for the $1^{--}$ strangeonium hybrid at various hybrid mass.

FIG. 6: The partial width for the $1^{--}$ gluon excited strangeonium hybrid decaying to $K_1(1270)K$, $K_1(1400)K$, $K^*(1410)K$ at various hybrid mass.

If $Y(2175)$ is a $1^{--}$ strangeonium hybrid, in the constituent gluon model, the mechanism which generates the decay $Y(2175) \to \phi f_0(980) \to \phi \pi \pi$ could be the following: a gluon is emitted from the strange quark or the anti-strange quark, both the created gluon and the constituent gluon dissociate into a pair of quark-antiquark, then all the quarks and antiquarks combine to form $\phi$ and $f_0(980)$. The corresponding diagrams are shown in Fig.9 where only the diagrams in which gluon is emitted by the strange quark are plotted, and $f_0(980)$ is assumed as a four-quark state.
FIG. 7: The $\beta_{s\bar{s}}$ dependence of the total decay width for the $L=0,1,2$ quark excited hybrid and the gluon excited hybrid.

FIG. 8: The partial width of the $1^{--}$ gluon excited strangeonium hybrid decaying to $K_1(1270)K$, $K_1(1400)K$, $K^*(1410)K$ at various $\beta_{s\bar{s}}$.

FIG. 9: Decay diagrams for $Y(2175) \rightarrow \phi f_0(980)$ under the assumption that $Y(2175)$ is a $1^{--}$ strangeonium hybrid and $f_0(980)$ is a four quark state.

IV. CONCLUSION AND DISCUSSION

We propose that the recently observed structure $Y(2175)$ is a strangeonium hybrid, the reasonableness of this hypothesis has been studied from both the flux tube model and the constituent gluon model. The estimate for the mass of $1^{--}$ hybrid state $s\bar{s}g$ in both the flux tube model and the constituent gluon model are consistent with the experimental data. In the constituent gluon model, there exist the "quark excited" hybrids ($L = 0, 1, 2; L_{s\bar{s}} = S_{s\bar{s}} = 1; L_g=0$) and the "gluon excited" one ($L = 1; L_g=0; L_{s\bar{s}} = S_{s\bar{s}} = 0$) for the $1^{--}$ hybrid state. The decay width of the quark excited hybrid is very large, it is around 250-500MeV, and the decay pattern and decay width of the gluon excited hybrid is similar to the result of the flux tube model. Therefore we argue that $Y(2175)$ could be mainly a "gluon excited" strangeonium hybrid from the view of constituent gluon. Furthermore, both models predict the decay width is about 100-150MeV with some theoretical uncertainties, which is
consistent with the present data within errors.

Both the flux tube model and the constituent gluon model (for the "gluon excited" hybrid) predict that the $1^{--}$ hybrid $s\bar{s}g$ dominantly decays into $K_1(1400)K$ and $K_1(1270)K$, which is consistent with the well-known selection rule of hybrid decay. So the experimental search of channels $Y(2175) \rightarrow K_1(1400)K \rightarrow \pi K^*(892)K$, $Y(2175) \rightarrow K_1(1270)K \rightarrow \rho KK$ and $Y(2175) \rightarrow K_1(1270)K \rightarrow \pi K_0^*(1430)K$ are suggested, which is a important test of our scenario. However, $Y(2175)$ dominantly decays into $\phi\eta$ and $\phi\eta'$ in the tetraquark picture, the search for $Y(2175) \rightarrow \phi\eta$ and $Y(2175) \rightarrow \phi\eta'$ is also merited to discriminate the two interpretations, although the tetraquark hypothesis is not favored by the available experimental information.

Although both $Y(2175)$ and $Y(4260)$ have been found in the ISR $e^+e^-$ annihilation by the Babar collaboration, their decay property is different. In the hybrid picture, they both prefer to decay into the final states with a P-wave meson because of the "S+P" selection rule, i.e., $K_1(1400)K$ and $K_1(1270)K$ are the dominant decay modes as we have shown above, and $Y(4260)$ has strong coupling to $D\bar{T}^{**}$ and $\bar{T}D^{**}$. However, $Y(4260)$ are slightly larger than the threshold of $D\bar{T}^{**}$ and $\bar{T}D^{**}$, so those decay modes are suppressed. The $K^+K^-\pi^+\pi^-$ final state in the ISR $e^+e^-$ annihilation has been analyzed by the Babar collaboration, it is essential to note that a broad structure in the region of the $K_1(1270)$ and $K_1(1400)$ is claimed by performing the three-body mass combination, which is shown in Fig. 19(c) in that paper, this signal seems to support our hybrid picture of $Y(2175)$. It is essential to perform the mass combination of $K_1(1270)K$, $K_2^*(1430)K$ and $K_1(1400)K$, perform the partial wave analysis, and measure the branching ratios etc. in order to verify the hybrid hypothesis of $Y(2175)$.

No matter $Y(2175)$ is a hybrid or tetraquark quark state, it can mix with other conventional mesons, which have the same quantum numbers as $Y(2175)$. However, the nearest $1^{--}$ isospin singlet to $Y(2175)$ is $\phi(1680)$, and quantum mechanics tells us that the mixing strength is related to the inverse of the energy difference of the two states, so the mixing effect is expected to be rather small.

To confirm $Y(2175)$ being a hybrid, further deep theoretical understanding and predictions about the properties of $1^{--}(s\bar{s}g)$ hybrid state are needed to be confronted with the experimental data. It is essential to investigate if $Y(2175)$ could be a conventional strange quarkonium, e.g. $2^3D_1$ or $3^3S_1$ $s\bar{s}$ quarkonium. The decay of $3^3S_1$ $s\bar{s}$ quarkonium has
been studied in $^3P_0$ model by T.Barnes et al.\cite{23}, and this state is predicted to be a rather broad state, $\Gamma \approx 380$ MeV, so $Y(2175)$ should not be $^3S_1 s\bar{s}$ state. However, the decay of $2^3D_1 s\bar{s}$ quarkonium has not been considered so far, it is urgent and interesting to study $2^3D_1 s\bar{s}$ state from quark model\cite{33}. Experimentally, confirmation and dictated studies of $Y(2175)$ at BES and CLEO is valuable.

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APPENDIX A: THE ANALYTICAL EXPRESSIONS FOR $h_0, g_1, h_2$

$h_0, g_1, h_2$ are as follows:

$$h_0 = \beta^2 \int_0^\infty dr r^{3+\delta} j_0\left(\frac{M_{PB}r}{m+M}\right) \exp\left(-\left(2\beta_A^2 + \beta^2\right)\frac{r^2}{4}\right)$$  \hspace{1cm} (A1)

$$g_1 = \frac{2mp_B}{m+M} \int_0^\infty dr r^{2+\delta} j_1\left(\frac{M_{PB}r}{m+M}\right) \exp\left(-\left(2\beta_A^2 + \beta^2\right)\frac{r^2}{4}\right)$$  \hspace{1cm} (A2)

$$h_2 = \beta^2 \int_0^\infty dr r^{3+\delta} j_2\left(\frac{M_{PB}r}{m+M}\right) \exp\left(-\left(2\beta_A^2 + \beta^2\right)\frac{r^2}{4}\right)$$  \hspace{1cm} (A3)

In the above formula, $M$ and $m$ are respectively the masses of the original quark and the created quark. The integrals can be evaluated in terms of the confluent hypergeometric functions using the general formula

$$\int_0^\infty dr r^n j_m(ar)e^{-br^2} = \sqrt{\pi} \frac{a^m b^\delta}{2^{m+2}\Gamma[m+3/2]} \frac{\Gamma[\delta]}{\Gamma[\delta+1]} F_1(\delta, m+\frac{3}{2}, -\frac{a^2}{4b^2})$$  \hspace{1cm} (A4)

where $\phi = (m+n+1)/2$, then

$$h_0 = \frac{2^{3+\delta}\beta^2}{(2\beta^2 + \beta^2)^{2+\delta/2}} \frac{\Gamma(2+\delta/2)}{\Gamma(2+\delta/2)} F_1(2+\delta/2, 3/2, 2, -\frac{1}{4})$$  \hspace{1cm} (A5)

$$g_1 = \frac{mM_{PB}^2}{(m+M)^2} \frac{2^{4+\delta}}{3(2\beta_A^2 + \beta^2)^{2+\delta/2}} \frac{\Gamma(2+\delta/2)}{\Gamma(2+\delta/2)} F_1(2+\delta/2, 5/2, 2, -\frac{1}{4})$$  \hspace{1cm} (A6)

$$h_2 = \frac{(M_{PB})^2}{(m+M)^2} \frac{2^{5+\delta}\beta^2}{15(2\beta_A^2 + \beta^2)^{3+\delta/2}} \frac{\Gamma(3+\delta/2)}{\Gamma(3+\delta/2)} F_1(3+\delta/2, 7/2, 2, -\frac{1}{4})$$  \hspace{1cm} (A7)
APPENDIX B: THE PARTIAL DECAY WIDTH $M_{\ell,J}$ IN THE CONSTITUENT
GLUON MODEL

Starting from the interaction Hamiltonian Eq. (12) and the non-relativistic wavefunction
Eq. (14) and Eq. (15), the matrix element $\langle BC | H_I | A \rangle = g_s(2\pi)^3\delta^3(p_A - p_B - p_C) M_{\ell,J}(A \to BC)$ can be easily obtained, and the partial wave amplitude $M_{\ell,J}$ has the following form:

$$M_{\ell,J}(A \to BC) = \sum_{M_L, \lambda_g, M_{L_0}, M_{L_{as}}, M_{L_B}, M_{S_B}, M_{L_{C}}, M_{S_C}, M_{J_B}, M_{J_C}, M_J, M_\ell} C \mathcal{F} S(M_{S_{as}}, \lambda_g, M_{S_B}, M_{S_C}) I(M_{L_{as}}, M_{L_0}, M_{L_B}, M_{L_{C}}, M_\ell) \langle L_9, M_{L_9}; 1, \lambda_g | J_g, M_{L_9} + \lambda_g \rangle \langle L_{ss}, M_{L_{as}}; J_g, M_{L_9} + \lambda_g | L, M_{L_9} + \lambda_g + M_{L_{as}} \rangle \langle L, M_{L_0} + \lambda_g + M_{L_{as}}; S_{ss}, M_{S_{as}} | A_M, M_{J_A} \rangle \langle L_B, M_{L_B}; S_B, M_{S_B} | J_B, M_{J_B} \rangle \langle L_C, M_{L_C}; S_C, M_{S_C} | J_C, M_{J_C} | J, M_J \rangle \langle \ell, M_\ell; J, M_J | J_A, M_{J_A} \rangle \tag{B1}$$

Here $C$, $\mathcal{F}$, $S(M_{S_{as}}, \lambda_g, M_{S_B}, M_{S_C})$ and $I(M_{L_{as}}, M_{L_0}, M_{L_B}, M_{L_{C}}, M_\ell)$ are respectively color, flavor, spin and spatial overlap factors with $C = \frac{2}{3}$. In the non-relativistic limit, the spin overlap factor is:

$$S(M_{S_{as}}, \lambda_g, M_{S_B}, M_{S_C}) = \sum_S \sqrt{6(2S_B + 1)(2S_C + 1)(2S_{ss} + 1)} \left\{ \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \\ S_B \\ \frac{1}{2} \\ \frac{1}{2} \\ S_C \\ S_{ss} \end{array} \right\} \langle S_{ss}, M_{S_{as}}; 1, S_C \rangle \lambda_g | S, M_{S_B} + M_{S_C} \rangle \langle S_B, M_{S_B}; S_C, M_{S_C} | S, M_{S_B} + M_{S_C} \rangle \tag{B2}$$

The flavor overlap factor $\mathcal{F}$ is:

$$\mathcal{F} = \sqrt{(2I_B + 1)(2I_C + 1)(2I_A + 1)} \left\{ \begin{array}{ccc} i_1 & i_3 & I_B \\ i_2 & i_4 & I_C \\ I_A & 0 & I_A \end{array} \right\} \eta \varepsilon \tag{B3}$$

where $\eta = 1$ if the gluon goes into strange quarks and $\eta = \sqrt{2}$ if it goes into non-strange ones. $\varepsilon$ is the number of the diagrams contributing to the decay. Finally the spatial overlap is given by:

$$I(M_{L_{as}}, M_{L_0}, M_{L_B}, M_{L_{C}}, M_\ell) = \int \int \frac{d^3p d^3k}{\sqrt{2\omega(2\pi)^6}} \psi_{L_{as}M_{L_{as}}} (p_B - p) \psi_{L_9M_{L_9}} (k) \psi^*_{L_{B}M_{L_{B}}} \left( \frac{m_{p_B}}{M + m} - p - \frac{k}{2} \right) d\Omega_B Y_{\ell M_\ell}^* (\Omega_B) \tag{B4}$$
The partial decay width is:

$$\Gamma_{\ell I}(A \to BC) = \frac{\alpha_s p_B E_B E_C}{M_A} |M_{\ell I}(A \to BC)|^2$$  \hspace{1cm} (B5)$$

where $M_{\ell I}(A \to BC)$ is the partial wave amplitude, and $\Gamma_{\ell I}$ is the partial width of that partial wave integral. As usual, we would like to use the S.H.O basis wavefunctions, thereby enabling analytic studies that reveal the relationship among amplitudes.

$$\psi_{L_s s^L_{s^L}}(p) = \frac{16\pi^3}{\Gamma\left(\frac32 + L_{ss}\right)\beta_{s s}^2 L_{ss} + 3} \frac{1}{\sqrt{2}} Y_{L_s}^{M_{L_s s}}(p) \exp\left[-\frac{p^2}{2\beta_{s s}^2}\right]$$  \hspace{1cm} (B6)$$

and

$$\psi_{L_s g}(p_g) = \frac{16\pi^3}{\Gamma\left(\frac32 + L_g\right)\beta_g^2 L_g + 3} \frac{1}{\sqrt{2}} Y_{L_g}^{M_{L_g}}(p_g) \exp\left[-\frac{p_g^2}{2\beta_g^2}\right]$$  \hspace{1cm} (B7)$$

where $Y_{L_s}^{M_{L_s s}}(p)$ is a solid harmonic polynomial, and analogously for $Y_{L_g}^{M_{L_g}}(p_g)$. The wavefunctions $\psi_{LBM_L}$ and $\psi_{LCM_L}$ are defined similarly. In the following, we will take $\beta_B = \beta_C$. If the final states are two $S$–wave mesons, i.e., $L_B = L_C = 0$, then the overlap integral Eq. (B4) can be integrated out analytically.

$$I(M_{L_s s}, M_{L_g}, 0, 0, M_e) = \frac{1}{\sqrt{\pi \omega}} \frac{1}{\beta_B^2} \frac{1}{\Gamma\left(L_{ss} + \frac32\right)\Gamma\left(L_g + \frac32\right)\beta_{s s}^2 L_{ss} + 3\beta_g^2 L_g + 3} \frac{1}{\sqrt{2}} \frac{2\pi^2 \beta_{s s}^2 \beta_{g g}^2 \beta_B^4}{\left(\beta_B^2 + 2\beta_{s s}^2\right)\left(\beta_B^2 + \beta_g^2/2\right)} \delta_{L_{L_s s}, L_{L_g}} \delta_{M_e, M_{L_s s}} \delta_{M_g, 0} \delta_{M_B, 0}$$  \hspace{1cm} (B8)$$

The above equation implies that the ”gluon excited” hybrid doesn’t decay into two ground states mesons, which has been shown to be true in both the flux tube model and the constituent gluon model. Particularly for the quark excited hybrid $L_g = 0, L_{ss} = 1$,

$$I(M_{L_s s}, 0, 0, 0, M_e) = \frac{\sqrt{\pi}}{3\omega} \frac{\beta_{s s}^5 \beta_{g g}^2 \beta_B^3}{\left(\beta_B^2/2 + \beta_{s s}^2\right)^{3/2}} \frac{2M p_B}{(M + m)^2 \beta_B^2 + 2\beta_{s s}^2} \exp\left[-\frac{M^2}{M + m} \frac{p_B^2}{\beta_B^2 + 2\beta_{s s}^2}\right] \delta_{L_{L_s s}} \delta_{M_{L_s s}}$$  \hspace{1cm} (B9)$$

If the final states are $S + P$–wave meson pairs, the overlap integral Eq. (B4) can be exactly integrated out likewise. For the gluon excited hybrid $L_{ss} = 0, L_g = 1$, the relevant overlap integral is:

$$I(0, M_{L_g}, M_{L_B}, 0, M_e) = -\frac{\sqrt{\pi}}{2\omega} \frac{\beta_{s s}^{3/2} \beta_{g g}^{5/2} \beta_B^4}{\left(\beta_B^2/2 + \beta_{s s}^2\right)^{3/2} \left(\beta_B^2/2 + \beta_{g g}^2/4\right)^{5/2}} \frac{2M p_B}{(M + m)^2 \beta_B^2 + 2\beta_{s s}^2} \exp\left[-\frac{M^2}{M + m} \frac{p_B^2}{\beta_B^2 + 2\beta_{s s}^2}\right] \delta_{M_{L_g}, M_{L_B}} \delta_{L_{L_s s}} \delta_{M_e, 0}$$  \hspace{1cm} (B10)$$

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where the \( \delta_{\ell,0} \) term above tells us that the \( 1^- \) hybrid decay into \( S + P \)-wave final states in relative \( S \)-wave, so the \( 1^- \) strangeonium hybrid can not decay to \( K_2^*(1430)K \) due to conservation of angular momentum. And the corresponding overlap integral for the quark excited hybrid is:

\[
I(M_{Ls}, 0, M_{LB}, 0, M) = \sqrt{\frac{2\pi}{\omega}} \frac{\beta_g^{5/2} \beta_B^{3/2} \beta_B^4}{\beta_s^{5/2} \beta_g^{3/2} \beta_B^4} \exp[- \frac{M^2}{(M + m)^2} \frac{p_B^2}{\beta_B^2 + 2\beta_s^2}] \\
\times [\delta_{M_{Ls},M_{LB}} \delta_{\ell,0} \delta_{M_b,0} - \frac{2}{3} \sqrt{2\ell + 1} \left( \frac{M}{M + m} \right)^2 \frac{p_B^2}{\beta_B^2 + 2\beta_s^2} \langle 1, M_{LB}; \ell, M_b | 1, M_{Ls} \rangle \langle 1, 0; \ell, 0 | 1, 0 \rangle]
\]

Both the quark excited hybrid and the gluon excited hybrid can decay into \( 2S + 1S \) final states, if these processes are not forbidden by the phase space. In order to distinguish the notations from those for hybrid decaying into \( S + S \) and \( S + P \) final states, the overlap integral for hybrid decaying into \( 2S + 1S \) final states is denoted as \( I'(M_{Ls}, M_{Lg}, 0, 0, M) \), which is of the following form,

\[
I'(M_{Ls}, 0, 0, 0, M) = -\sqrt{\frac{\pi}{\omega}} \frac{\beta_g^{5/2} \beta_B^{3/2} \beta_B^3}{\beta_s^{5/2} \beta_g^{3/2} \beta_B^4} \frac{M p_B}{3(M + m)^3} \exp[- \frac{M^2}{(M + m)^2} \frac{p_B^2}{\beta_B^2 + 2\beta_s^2}] \\
\times [\{m^2(\beta_B^2 + 2\beta_s^2)(5\beta_B^4 + \beta_B^2\beta_g^2 - 3\beta_g^2\beta_s^2) - 2mM(\beta_B^2 + 2\beta_s^2)(5\beta_B^4 + \beta_B^2\beta_g^2 - 3\beta_g^2\beta_s^2) \}
+ M\{p_B^2 \beta_B^2(2\beta_B^2 + \beta_g^2) - (\beta_B^2 + 2\beta_s^2)(5\beta_B^4 + \beta_B^2\beta_g^2 - 3\beta_g^2\beta_s^2) \}] \delta_{L_s,0} \delta_{\ell,L_s} \delta_{M_b,M_{Ls}}
\]

and:

\[
I'(0, M_{Lg}, 0, 0, M) = -\sqrt{\frac{\pi}{\omega}} \frac{\beta_g^{3/2} \beta_B^{5/2} \beta_B^5}{\beta_s^{5/2} \beta_g^{3/2} \beta_B^4} \frac{4Mp_B}{3(M + m)^3} \exp[- \frac{M^2}{(M + m)^2} \frac{p_B^2}{\beta_B^2 + 2\beta_s^2}] \delta_{L_s,0} \delta_{\ell,L_g} \delta_{M_b,M_{Lg}}
\]

The first result Eq. (B11) corresponds to the ”quark excited” hybrid, and the second Eq. (B12) corresponds to the ”gluon excited” hybrid.