A Consistency Relation in Cosmology

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Abstract

We provide a consistency relation between cosmological observables in general relativity without relying on the equation of state of dark energy. The consistency relation should be satisfied if general relativity is the correct theory of gravity and dark energy clustering is negligible. As an extension, we also provide the DGP counterpart of the relation.

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I. INTRODUCTION

General relativity has passed the experimental tests on the scales of the solar system with flying colors. Probing general relativity on cosmological scales \([1]\) will be the next target for gravitational physics. In this paper, we take a small step toward this aim. We understand that such a program suffers from fundamental degeneracy between gravity theories and properties of dark energy\([2]\). Rather than emphasizing the degeneracy, however, we focus on a consistency relation which tests the conventional theoretical framework in cosmology: general relativistic CDM model with (almost homogeneous) dark energy. Breaking the relation would be the signature of the breakdown of general relativity at the cosmological scale or non-negligible dark energy clustering.

A consistency test has recently been proposed by Knox et al.\([3]\) and Ishak et al.\([4]\) (see also \([5]\)). They look at a consistency between the expansion rate determined by the distance-redshift relation and that by the growth rate of large scale structure. An inconsistency would arise within the dark energy parameter space if the underlying gravity theory is different from general relativity. Our work is partly motivated by these study, and we shall provide an explicit consistency relation (without referring to the equation of state of dark energy) in general relativity which relates the two cosmological observables: the (luminosity) distance and the density perturbation. The basic idea is very simple. The expansion rate is determined both from the distance-redshift relation \([6, 7]\) and from density perturbations \([7]\). Equating them then gives a consistency relation. It should hold if general relativity is the correct theory of gravity in the universe and if dark energy clustering is negligible. As an extension of the program we also provide a consistency relation in the DGP model.

II. RECONSTRUCTING THE EXPANSION RATE OF THE UNIVERSE

A. From Standard Candles

The observations of type Ia supernovae, for example, yield the luminosity distance \(d_L(z)\) through \(m - M = 5 \log_{10}(d_L(z)/\text{Mpc}) + 25\), with \(m\) being the apparent magnitude and \(M\) being the absolute magnitude. The luminosity distance is then related to the Hubble parameter \(H(z)\) kinematically (assuming the energy conservation of photons) as

\[
d_L(z) = \frac{(1 + z)}{H_0 \sqrt{|\Omega_K|}} \sin_K \left( H_0 \sqrt{|\Omega_K|} \int_0^z \frac{dz}{H(z)} \right),
\]

(1) where \(\Omega_K = -K/a_0^2 H_0^2\) and \(\sin_K(x) = \sin(x)\ (K = 1), x\ (K = 0), \sinh(x)\ (K = -1)\). In terms of \(r(z) = d_L(z)/(1 + z)\), the Hubble parameter is rewritten as \([6]\)

\[
\left( \frac{H(z)}{H_0} \right)^2 = 1 + r(z)^2 H_0^2 \Omega_K \frac{dr(z)/dz}{H_0^2 (dr(z)/dz)^2}.
\]

(2) This is the first expression of \(H(z)\) in terms of observables. Since it is purely kinematical relation, it holds in any metric theories of gravity (again assuming photon energy conservation).
B. From Density Perturbation: Consistency Relation in General Relativity

The measurements of weak gravitational lensing (cosmic shear) give the information of linear density perturbations (or linear growth rate).

In general relativity, a linear density perturbation $\delta(z)$ at scales much smaller than the Hubble radius obeys the following differential equation

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_M\delta = 0,$$

(3)

where a dot denotes the derivative with respect to the cosmic time and $\rho_M$ is the matter density. Here we have assumed the perturbative properties of dark energy to some extent: dark energy does not have unusual sound speed and has negligible anisotropic stress and has negligible interaction with dark matter. According to Starobinsky [7], we rewrite Eq.(3) by changing the argument from the cosmic time to the scale factor $a$:

$$\frac{dH(a)^2}{da} + \left(\frac{3}{a} + \frac{d^2\delta/da^2}{d\delta/da}\right)H(a)^2 = \frac{3\Omega_{M0}H_0^2a_0^2\delta}{a^5(d\delta/da)},$$

(4)

where $\Omega_{M0}$ is the matter density parameter of today. By regarding the above equation as the differential equation for $H(a)^2$, the solution is obtained as [7]

$$\left(\frac{H}{H_0}\right)^2 = \frac{3\Omega_{M0}a_0^3}{a^5(d\delta/da)^2} \int_0^a a\delta d\delta/da = 3\Omega_{M0} \frac{(1 + z)^2}{\delta'(z)^2} \int_0^z \frac{\delta}{1 + z}(-\delta')dz,$$

(5)

where a prime denotes the derivative with respect to $z$. In the above solution, a homogeneous is disregarded by taking only growing solution. Putting $z = 0$ in the solution, we obtain

$$1 = \frac{3\Omega_{M0}}{\delta'(0)^2} \int_0^\infty \frac{\delta}{1 + z}(-\delta')dz,$$

(6)

which expresses $\Omega_{M0}$ in terms of $\delta$. $\delta(z)$ at higher redshift ($z \gtrsim 5$) may not be determined from observations. However, in practice $\delta(z)$ can be well approximated as $\delta \propto 1/(1 + z)$ there. Using Eq.(6), Eq.(5) can then be alternatively written as

$$\left(\frac{H}{H_0}\right)^2 = \frac{(1 + z)^2\delta'(0)^2}{\delta'(z)^2} \left[ 1 - \frac{\int_0^z \frac{\delta}{1 + z}(-\delta')dz}{\int_0^\infty \frac{\delta}{1 + z}(-\delta')dz} \right],$$

(7)

Equating Eq.(7) with Eq.(2) gives the consistency relation between observables:

$$\frac{1 + r(z)^2H_0^2\Omega_{K0}}{H_0^2(dr(z)/dz)^2} = \frac{(1 + z)^2\delta'(0)^2}{\delta'(z)^2} \left[ 1 - \frac{\int_0^z \frac{\delta}{1 + z}(-\delta')dz}{\int_0^\infty \frac{\delta}{1 + z}(-\delta')dz} \right],$$

(8)

which should hold if general relativity is the correct theory of gravity in the universe. It should be noted that in deriving Eq.(8), we do not assume the Friedmann equation and hence

1 If we allow either of them, the program will fail: both general relativity and DGP can have the same expansion rate and growth rate [2].
we do not specify the equation of state of dark energy but specify the perturbative properties of dark energy (sound speed and anisotropic stress). Eq. (8) thus tests the underlying gravitational theory modulo these assumptions of dark energy.\(^2\)

The degeneracy with the curvature parameter \(K\) may be broken by using the CMB shift parameter \([9]\):

\[
R = \sqrt{\frac{\Omega_{M0}}{\Omega_{K0}}} \sin_K \left( H_0 |\Omega_{K0}| \int_0^{z_{\text{LSS}}} \frac{dz}{H(z)} \right),
\]

where \(z_{\text{LSS}} = 1089\), which is measured to be \(R = 1.70 \pm 0.03\)\([10]\).

\[C. \text{ Modified Gravity: DGP}\]

If the consistency relation Eq. (8) would not hold, then dark energy has anisotropic pressure or interaction with dark matter \([2]\), or general relativity is not the correct theory of gravity on cosmological scales. In this section, we focus on the latter possibility and look for another relation in gravitational theories other than general relativity. While Eq. (2) holds in any theories of gravity as mentioned earlier, the modification of gravity theories affects the gravitational instability and hence modifies the third term in Eq. (3): the self-gravity of density perturbations.

For example, in scalar-tensor theories of gravity, the growth of density perturbations is modified simply as \([11]\)

\[
\ddot{\delta} + 2H \dot{\delta} - 4\pi G_{\text{eff}} \rho_M \delta = 0,
\]

where \(G_{\text{eff}}\) is the effective local gravitational “constant” measured by Cavendish-type experiment and is time dependent. The modified evolution equation of density perturbations in general may be written as

\[
\ddot{\delta} + 2H \dot{\delta} - 4\pi G \rho_M \left( 1 + \frac{1}{3\beta} \right) \delta = 0,
\]

where \(\beta\) in general depends on time and is determined once we specify the modified gravity theory.

As a concrete example, we consider DGP(Dvali-Gabadadze-Poratti) model \([12]\). DGP model is a model of brane world in which three-dimensional brane is embedded in an infinite five-dimensional spacetime (bulk). The action for the five-dimensional theory is

\[
S = \frac{1}{2} M_5^3 \int d^4xdy \sqrt{-g_{(5)} R_{(5)}} + \frac{1}{2} M_4^2 \int d^4x \sqrt{-g_{(4)} R_{(4)}} + S_m,
\]

where the subscripts 4 and 5 denote the quantities on the brane and in the bulk, respectively, \(M_4(M_5)\) is the four(five)-dimensional reduced Planck mass, and \(S_m\) is the action for matter on the brane.

In DGP model, the Friedmann equation is modified as \([13]\)

\[
H^2 + \frac{K}{a^2} = \frac{8\pi G}{3} \left( \sqrt{\rho_M + \rho_{\text{vac}}} \right)^2,
\]

\[^2\text{Linder introduced the gravitational growth index \(\gamma\) defined by } d\ln(\delta/a)/d\ln a = \Omega_M(a)^\gamma - 1 \text{ and found that it is insensitive to the equation of state of dark energy.} \]
where \( \rho_{r_c} = 3/(32\pi G r_c^2) \). \( r_c \) is related to the four dimensional Planck mass \( M_4 \) and the five dimensional Planck mass \( M_5 \) as \( r_c = M_4^2/2M_5^3 \). The evolution of density perturbations is also modified and \( \beta \) in Eq. (11) is given by (14):

\[
\beta = 1 - 2 r_c H \left( 1 + \frac{\dot{H}}{3H^2} \right). \tag{14}
\]

If we use the Friedmann equation Eq. (13), \( \beta \) can be written as

\[
1 + \frac{1}{3\beta} = \frac{4\Omega_M(a)^2 - 4(1 - \Omega_K(a))^2 + 2\sqrt{1 - \Omega_K(a)(3 - 4\Omega_K(a) + 2\Omega_M(a)\Omega_K(a) + \Omega_K(a)^2)}}{3\Omega_M(a)^2 - 3(1 - \Omega_K(a))^2 + 2\sqrt{1 - \Omega_K(a)(3 - 4\Omega_K(a) + 2\Omega_M(a)\Omega_K(a) + \Omega_K(a)^2)}} \tag{15}
\]

where

\[
\Omega_M(a) = \frac{8\pi G \rho_M}{3H(a)^2} = \frac{\rho_M}{\sqrt{\rho_{r_c}} + \sqrt{\rho_{r_c} + \rho_{M}}^2} - \frac{3K}{8\pi G a^2} \frac{\Omega_M(a/a_0)^{-3}}{\Omega_M(a/a_0)^{-3}} \
= \frac{(1 - \Omega_{M0} - \Omega_{K0})}{2\sqrt{1 - \Omega_{K0}}} + \frac{(1 - \Omega_{M0} - \Omega_{K0})^2}{4(1 - \Omega_{K0})} + \Omega_{M0}(a/a_0)^{-3} + \Omega_{K0}(a/a_0)^{-2} \tag{16}
\]

where we have used \( \Omega_c = 8\pi \rho_{r_c}/3H_0^2 = (1 - \Omega_{M0} - \Omega_{K0})^2/4(1 - \Omega_{K0}) \). \( \Omega_K(a) = -K/a^2 H(a)^2 \). Thus the evolution of density perturbations is determined once we specify two parameters: \( \Omega_{M0} \) and \( \Omega_{K0} \). Henceforth, for simplicity of presentation, we restrict ourselves to a flat universe. But the analysis is easily extended to non-flat universes straightforwardly. In a flat universe Eq. (15) is further simplified

\[
1 + \frac{1}{3\beta} = \frac{2 + 4\Omega_M(a)^2}{3(1 + \Omega_M(a)^2)}. \tag{17}
\]

As in the case of general relativity, we rewrite Eq. (11) by changing the argument from the cosmic time to the scale factor \( a \):

\[
\frac{dH(a)^2}{da} + 2 \left( \frac{3}{a} + \frac{d^2\delta/da^2}{d\delta/da} \right) H(a)^2 = \frac{2\Omega_{M0}H_0^2 a_0^3 \delta}{a^3(d\delta/da)} \left( 1 + 2\Omega_M(a)^2 \right). \tag{18}
\]

Putting \( \Omega_M(a) = 1 \) (or \( \Omega_M(a) + \Omega_K(a) = 1 \) for non-flat universes) recovers the equation in general relativity. Quite similar to the case of general relativity, we solve the above equation for \( H(a)^2 \) to obtain

\[
\left( \frac{H}{H_0} \right)^2 = 2\Omega_{M0} \frac{(1 + z)^2}{\delta'(z)^2} \int_z^\infty \frac{\delta}{1 + z} \left( -\delta' \right) \left( 1 + 2\Omega_M(z)^2 \right) dz. \tag{19}
\]

\footnote{The use of the Friedmann equation (theory) may be out of the spirit of the program of the consistency relation: determine \( H(a) \) only from observations. However, this does not imply that we have in advance specified the equation of state of dark energy. In fact, \( \rho_{r_c} \) in Eq. (13) is not unknown function: the scale-factor dependence is known and \( r_c \) is written in terms of \( \Omega_{K0} \) and \( \Omega_{M0} \). Hence in DGP the equation of state of dark energy is already specified.
Putting $z = 0$ gives an implicit equation for $\Omega_M$ (or $\Omega_M$ and $\Omega_K$ for non-flat universes)

$$1 = \frac{2\Omega_M}{\delta'(0)^2} \int_0^\infty \frac{\delta}{1+z}(\delta' \left( \frac{1 + 2\Omega_M(z)^2}{1 + \Omega_M(z)^2} \right) dz. \quad (20)$$

Then we can rewrite the solution

$$\left( \frac{H}{H_0} \right)^2 = \frac{\delta'(0)^2}{\delta'(z)^2} - 2\Omega_M \frac{(1+z)^2}{\delta'(z)^2} \int_0^z \frac{\delta}{1+z}(\delta' \left( \frac{1 + 2\Omega_M(z)^2}{1 + \Omega_M(z)^2} \right) dz, \quad (21)$$

which is the DGP counterpart of Eq.(7). Here $\Omega_M(z)$ is defined by Eq.(11).

Equating Eq.(21) with Eq.(2) gives the consistency relation between observables in DGP in a flat universe:

$$\frac{1}{H_0^2(\delta'(z)/dz)^2} = \frac{(1+z)^2\delta'(0)^2}{\delta'(z)^2} - 2\Omega_M \frac{(1+z)^2}{\delta'(z)^2} \int_0^z \frac{\delta}{1+z}(\delta' \left( \frac{1 + 2\Omega_M(z)^2}{1 + \Omega_M(z)^2} \right) dz, \quad (22)$$

which should hold if DGP model is the correct theory of gravity in the universe.  

**D. Reconstructing $H(z)$ from $\delta$: DGP vs. GR**

In order to demonstrate how the breakdown of the consistency relation Eq.(8) occurs, we calculate $H(z)$ from $\delta$ using Eq.(7) when the correct theory of gravity is not general relativity and compare it with the modified Friedmann equation Eq.(13) which can be determined by the distance measurements Eq.(2). In short, we compare the right hand side and the left hand side of Eq.(8). More concretely, we consider the case when the true cosmology is a flat DGP model Eq.(13) and prepare $\delta$ (data) for $\Omega_M = 0.3, 0.2$ using Eq.(11). But we wrongly assume the true cosmology to be a flat FRW model in GR and determine $\Omega_M$ using Eq.(6) and calculate $H(z)$ from Eq.(7). We do not include the effects of observational uncertainties which will be considered in subsequent work.

The results are shown in Fig. 1 $H(z)^2$ calculated from $\delta$ using Eq.(7) (RHS) is compared with Eq.(13) (LHS). The solid (dashed) curve is for $\Omega_M = 0.3, 0.2$ showing that about 20% differences are expected. The curve deviates from unity for $z \lesssim 5$ and the difference is saturated beyond that since the universe is matted dominated then. The dotted line is the result of the case when $H(z)$ is determined by using the correct equation Eq.(21). $\Omega_M$ determined from Eq.(6) is 0.243(0.156) for $\Omega_M = 0.3(0.2)$, respectively, which also differs by about 20%. It gives another consistency test of the cosmological model.

According to the analysis of the simulated future weak lensing data (like LSST[15]) in [8], the distances would be measured within 1% out to $z \simeq 3$ (the error is similar for SNAP[16] but out to $z \simeq 2$) and the growth rates would be determined within 4% at $z \leq 1.2$. So, we expect the left hand side of Eq.(8) would be determined within $\sim 2 \times 1\%$, while the right hand side within $\sim 4 \times 4 = 16\%$ at $z \leq 1.2$. The detailed analysis is left as our future work.

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4 Again in non-flat universes, the CMB shift parameter may be used to break the degeneracy with $\Omega_K$. 

Fig. 1: The ratio of RHS to LHS of Eq. (8). We assume true cosmology to be a flat DGP model and reconstruct $H(z)$ using general relativity Eq. (7). The solid (dashed) curve is for $\Omega_{M0} = 0.3(0.2)$. The dotted line is the result of the case when $H(z)$ is reconstructed using DGP Eq. (21).

III. SUMMARY

In this paper, we have derived a consistency relation in general relativity which relates the distance and the density perturbation assuming perturbative properties of dark energy. Breaking of the consistency relation would be the signature of the breakdown of the assumptions: general relativity is not the correct theory of gravity at the cosmological scale or dark energy has unusual properties (unusual sound speed or anisotropic stress or interaction with dark matter).

Four cosmological observables (the amplitudes and the spectral indices of scalar/tensor perturbations) out of three inflationary parameters (the energy scale, two slow-roll parameters $\epsilon$ and $\eta$) gives a consistency relation for single-field inflation [17]. It represents an extremely distinctive signature of inflation. The verification of the relation would be the direct proof of (single-field) inflation and would be the milestone of inflationary cosmology, although it would be difficult to verify the relation in the foreseeable future.

Likewise, the proof or disproof of the consistency relation in cosmology would give us a
clue as to the nature of dark energy or the nature of gravity on cosmological scales which would not have been obtained by local experiments. As such, any observational methods to test it should be welcome. Twentieth century clouds should have a silver lining. The consistency relation would help to clear up the dark clouds of the cosmos.

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