Primordial non-Gaussianities from inflation

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Abstract

These notes present a detailed introduction to Maldacena’s calculation [1] of the cubic terms in the inflationary action. These interactions are important since they produce the most readily observable evidence for a non-Gaussian component in the pattern of primordial fluctuations produced by inflation. In the simplest class of inflationary theories, those with only a single scalar field participating in the inflationary era, these non-Gaussianities are predicted to be extremely small, as will be reviewed here.

Any cosmological model that seeks a deeper or a more complete explanation of the universe must be able to account for the origin of the great variety of structures that appear in it today. Most of this structural complexity in the universe appears to have arisen through processes that are largely understood and from material that was once in a much simpler state than it is today. Stars and galaxies, for example, which now stand out so clearly from much emptier surroundings, could have grown through the slow gravitational collapse of gases which were initially distributed far more uniformly. This basic picture suggests that the universe in its primeval stages was once extremely homogeneous, a prediction that matches well with the observations made so far. However, this mechanism also requires that, even at the very earliest times, some spatial variation must always have existed. In a perfectly homogeneous and isotropic universe, the process of collapse and growth, and the development of ever more complex structures, would never have begun.

The theory of inflation provides one possible mechanism for generating the initial spatial variations in the universe. In the inflationary picture, space-time itself fluctuates quantum mechanically about a background that is expanding at an accelerating rate. This extreme expansion spreads the fluctuations, which begin with a tiny spatial extent, throughout a vast region of the universe, where they eventually become small classical fluctuations in the space-time curvature—or equivalently, small spatial variations in the strength of gravity. Since everything in the universe feels the influence of gravity, these fluctuations in the gravitational field are transferred to the matter and radiation fields, creating slightly overdense and underdense regions. The resulting matter fluctuations then become the ‘initial conditions’ that start the process of collapse which forms the stars and galaxies of later epochs.

To test whether this picture is correct, it is necessary to describe very accurately the properties of the pattern generated by inflation for the original, primordial fluctuations in space-time which can then be compared with what is inferred from observations. Before entering into a detailed calculation of these primordial perturbations, it is instructive to explain first from a more general perspective how the fluctuations of a quantum

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field lead to a classical pattern of perturbations. Since a quantum field is never measured directly, it is necessary also to determine how and in what form its information is available.

To begin, consider a quantum field, which will be written as $\zeta(t, \vec{x})$. Later this field will be connected with the fluctuations of the background space-time, but for now its exact physical meaning will be left a little vague. Although $\zeta(t, \vec{x})$ is a quantum field, its influence is inferred by how it affects classical things, so the information contained in $\zeta(t, \vec{x})$ needs to be converted into a classical function, or rather, into a set of classical functions. One way to do so is by taking the expectation values of products of the field $\zeta(t, \vec{x})$ at different places. If the field is in a particular, time-dependent, quantum state, $|0(t)\rangle$, then the functions,

$$\langle 0(t)|\zeta(t, \vec{x}_1)\zeta(t, \vec{x}_2)\cdots\zeta(t, \vec{x}_n)|0(t)\rangle,$$

(1)

tell the extent to which fluctuations in different places are correlated with each other in that state, and they are accordingly called n-point correlation functions. Usually, most of the emphasis on deriving the inflationary prediction for the initial fluctuations focuses on the field, but it is equally important to understand the state too. Because of the dramatic expansion during inflation, the detailed assumptions made about the properties of this state—even at seemingly infinitesimal distances—can have an important influence on the predictions for the pattern of the fluctuations.

Each n-point function potentially contains unique information not found in any of the others. If the fluctuations in the very early stages of the universe are small compared with the value of the background, then it should be possible to evaluate these correlation functions perturbatively. This fact implies further that the higher order n-point functions will be progressively more suppressed by this inherent smallness of $\zeta(t, \vec{x})$. So given a limited experimental accuracy, only the lowest few correlators will be observable in practice. If the spatially independent background has been chosen correctly, then the one-point function vanishes,

$$\langle 0(t)|\zeta(t, \vec{x})|0(t)\rangle = 0,$$

(2)

meaning that the first observable measure of the primordial fluctuations is provided by the two-point function,

$$\langle 0(t)|\zeta(t, \vec{x})\zeta(t, \vec{y})|0(t)\rangle,$$

(3)

followed next by the three-point function,

$$\langle 0(t)|\zeta(t, \vec{x})\zeta(t, \vec{y})\zeta(t, \vec{z})|0(t)\rangle,$$

(4)

and so on.

To calculate these correlation functions requires knowing something about the dynamics governing the field $\zeta(t, \vec{x})$. Suppose that these are determined by an action $S$ with an associated Lagrange density $\mathcal{L}[\zeta]$,

$$S = \int d^4x \mathcal{L}[\zeta(x)].$$

(5)

Since $\zeta$ is small, the Lagrangian can be expanded as a series in powers of the field,

$$S = \int d^4x \left\{ \mathcal{L}_0 + \mathcal{L}^{(2)}[\zeta(x)] + \mathcal{L}^{(3)}[\zeta(x)] + \mathcal{L}^{(4)}[\zeta(x)] + \cdots \right\},$$

(6)

$^2$Sometimes rather contrived potentials, with dramatic sudden changes are studied, although the state is assumed to be otherwise quite tame. Sometimes too only a finite amount of inflation is assumed to have occurred, and what came before this era might have left some imprint in the state as well.
where $\mathcal{L}^{(2)}$ is quadratic in the field, $\mathcal{L}^{(3)}$ is cubic, etc. The zeroth piece fixes the time evolution of the background; and since this background is always implicitly a solution to the equations of motion, the $\mathcal{L}^{(1)}$ term vanishes. Associating a vertex with each of these terms, with one leg for each factor of the field,

$$
\mathcal{L}^{(2)} \sim \begin{array}{c}
\text{vertex},
\end{array}
\mathcal{L}^{(3)} \sim \begin{array}{c}
\text{vertex},
\end{array}
\mathcal{L}^{(4)} \sim \begin{array}{c}
\text{vertex},
\end{array}
\text{etc.,} \quad (7)
$$

the first few contributions to the two-point function, in a perturbative expansion, can be diagrammatically represented by

$$
\langle 0(t) | \zeta(x) \zeta(y) | 0(t) \rangle = \begin{array}{c}
\text{vertex },
\end{array} + \begin{array}{c}
\text{vertex },
\end{array} + \ldots \quad (8)
$$

while the three-point function is

$$
\langle 0(t) | \zeta(x) \zeta(y) \zeta(z) | 0(t) \rangle = \begin{array}{c}
\text{vertex},
\end{array} + \ldots \quad (9)
$$

plus further higher order corrections which have not been written explicitly. In these expressions, the locations of the individual fields have been abbreviated as $x = (t, \vec{x})$, $y = (t, \vec{y})$ and $z = (t, \vec{z})$. Beyond cubic order, the leading contributions to the correlation functions are made up in part by powers of the lower order correlators. At quartic order, for instance, the leading behavior receives contributions both from the three possible pairings of two-point functions as well as from the connected graph associated with the $\mathcal{L}^{(4)}$ vertex,

$$
\begin{array}{c}
\text{vertex },
\end{array} + \begin{array}{c}
\text{vertex },
\end{array} + \begin{array}{c}
\text{vertex },
\end{array} + \begin{array}{c}
\text{vertex },
\end{array} + \ldots \quad (10)
$$

which is equivalently written as

$$
\langle 0(t) | \zeta(x_1) \zeta(x_2) \zeta(x_3) \zeta(x_4) | 0(t) \rangle = \langle 0 | \zeta(x_1) \zeta(x_2) | 0 \rangle \langle 0 | \zeta(x_3) \zeta(x_4) | 0 \rangle
+ \langle 0 | \zeta(x_1) \zeta(x_3) | 0 \rangle \langle 0 | \zeta(x_2) \zeta(x_4) | 0 \rangle
+ \langle 0 | \zeta(x_1) \zeta(x_4) | 0 \rangle \langle 0 | \zeta(x_2) \zeta(x_3) | 0 \rangle
+ \langle 0(t) | \zeta(x_1) \zeta(x_2) \zeta(x_3) \zeta(x_4) | 0(t) \rangle \big|_{\text{connected}}
+ \ldots \quad (11)
$$

A scenario that has no higher order interactions beyond the quadratic ones,

$$
\mathcal{L}[\zeta(x)] = \mathcal{L}_0 + \mathcal{L}^{(2)}[\zeta(x)], \quad \text{(exactly)} \quad (12)
$$

is said to be a *Gaussian* theory. In such theories, all the odd-point functions vanish (since diagrammatically it is quite obvious that it is impossible to divide an odd number
of points into pairs connected by $L^{(2)}$ without always leaving one point unpaired—and this one-point function vanishes), while all the even-point functions can be decomposed entirely into products of the two-point function. In the Gaussian case, the four-point function would then be equal to just the first three terms shown above, without the connected part. There is no reason to suppose that the primordial pattern of fluctuations in the actual universe forms a perfectly Gaussian pattern. But inflationary models do typically have cubic (and higher) interactions which are suppressed, so they usually predict a pattern that is still largely Gaussian in its character. One test of the inflationary picture then is to see whether the actual primordial fluctuations have a non-Gaussian component that is large or small. To do so, it is important to know the explicit form of the three-point function expected by inflation.

These notes will derive the standard inflationary source for the most readily observed signal of a non-Gaussian component within the pattern of primordial fluctuations, that generated by the cubic terms in the action, $L^{(3)}$. This set of terms will moreover be calculated here in the simplest possible setting, where there is only a single scalar field that is participating in the inflation.

This derivation will be developed in several stages, each of a successively greater complexity. The first will analyze how the background evolves during inflation, and it will also introduce a few dimensionless parameters associated with this background, before proceeding to a calculation of the quadratic action for the fluctuations and the resulting behavior for the two-point function. This calculation of the two-point function will also define a set of coordinates and a general method that will be used in these notes to describe the quantum fluctuations about the background. Only once the method that is used to describe the fluctuations has been illustrated for the quadratic terms will the full calculation of the cubic terms finally begin.

While deriving this standard prediction, along the way it will be necessary to make various assumptions about the properties of the universe during inflation and about what constitutes a ‘well behaved’ theoretical description of a quantum field theory in an expanding background. As long as these assumptions are part of a sound and consistent framework, there is nothing inherently wrong about adopting them—but they are assumptions all the same, and they might not necessarily hold in the actual universe. It is therefore always a good practice to be a little wary about what ideas are believed to hold during an era so far removed from what can be tested directly; at the very least, any such assumptions ought to be stated openly and clearly whenever they are made.

I. THE BACKGROUND

To begin, consider a simple inflationary model with a scalar field $\phi$ which is responsible for the accelerated expansion. If this field has a potential energy described by $V(\phi)$, then the total action, including a part both for the dynamics of $\phi$ and for the dynamics of the space-time itself, is

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} M_{pl}^2 R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right\}, \quad (13)$$

where $M_{pl} = (8\pi G)^{-1/2}$ is the reduced Planck mass, $G$ being Newton’s constant. $R$ is the scalar curvature associated with the space-time, whose geometry is described by the metric $g_{\mu\nu}$. All of the fields can be placed on the same footing by choosing units where $M_{pl} = 1$; since $M_{pl}$ only accompanies the gravitational field and not $\phi$, it is usually a simple matter to restore it whenever it might be needed.
Varying the action with respect to a small change in the metric, $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$ produces Einstein’s equation, which relates how the presence of matter and energy—in this case, that produced by the scalar field $\phi$—affect the geometry of the space-time where they exist,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}. \quad (14)$$

$T_{\mu\nu}$ is the energy-momentum tensor of the field $\phi$,

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}g_{\mu\nu}g^{\lambda\sigma} \partial_\lambda \phi \partial_\sigma \phi + g_{\mu\nu}V. \quad (15)$$

Varying with respect to the scalar field $\phi \rightarrow \phi + \delta \phi$ yields the equation of motion for the scalar field,

$$\nabla^2 \phi + \frac{\delta V}{\delta \phi} = 0. \quad (16)$$

Before attempting to explain the structure of the quantum fluctuations produced by the inflationary era, it is good to begin by following how the classical background evolves. Empirically, at very large scales or at very early times—at least at those that are visible—the universe appears to be quite uniform spatially. Inflation assumes this spatial uniformity holds over a large enough patch of the universe, from the beginning of inflation to its end, that the classical part of the metric can be treated as though it depended only on the time coordinate. Such a metric can be put into the following standard form\footnote{This choice of the metric implicitly assumes that space-time is spatially flat. Empirically the early universe does indeed appear to be spatially flat—or is at least very nearly so—so throughout these notes such a background will be assumed.}

$$ds^2 = dt^2 - e^{2\rho(t)} \delta_{ij} dx^i dx^j. \quad (17)$$

The rate at which $\rho(t)$ is changing corresponds to the Hubble scale,

$$H(t) = \rho(t) = \frac{d\rho}{dt}, \quad (18)$$

which is the characteristic energy scale associated with the gravitational evolution. Evaluating the components of the Einstein equation for this metric yields

$$3\dot{\rho}^2 = \frac{1}{2}\dot{\phi}^2 + V$$

$$-2\ddot{\rho} - 3\dot{\rho}^2 = \frac{1}{2}\dot{\phi}^2 - V, \quad (19)$$

while the equation for the field itself is

$$\ddot{\phi} + 3\dot{\phi} + \frac{\delta V}{\delta \phi} = 0. \quad (20)$$

Thus far, other than restricting to the case of a single scalar field and a particular assumption about the approximate homogeneity of the metric, the setting has not been otherwise constrained. Most especially, the potential energy of the field $V(\phi)$ has been left unspecified. Most inflationary models, if they are to produce a sufficient amount of expansion, must be in a ‘slowly rolling’ phase where the value of the field is not changing too rapidly—that is, $\dot{\phi}$ should be ‘small’. Further, to maintain this condition
over a sufficiently long period to produce a necessary amount of inflation, the field’s acceleration ($\dot{\phi}$) must also be tiny. These two conditions can be put a little more precisely by requiring the following dimensionless parameters to be small,

$$\varepsilon = \frac{d}{dt} \frac{1}{H} = -\frac{\ddot{\rho}}{\rho^2}, \quad \delta = \frac{1}{H} \frac{\ddot{\phi}}{\dot{\phi}^2}.$$  (21)

In the limit where both $\varepsilon \ll 1$ and $\delta \ll 1$, the derivatives of these parameters are still smaller, $\dot{\varepsilon}, \dot{\delta} \ll H \varepsilon, H \delta$, so $\varepsilon$ and $\delta$ will usually be treated as though they were constants. By applying the background equations of motion to solve for $\ddot{\rho} = -\frac{1}{2} \dot{\phi}^2$, $\varepsilon$ can also be expressed as

$$\varepsilon = \frac{1}{2} \frac{\dot{\phi}^2}{\rho^2};$$  (22)

This form for $\varepsilon$ makes what is meant by a ‘small’ value for $\dot{\phi}$ more precise: the value of the field should change slowly when compared with the rate at which the background itself is changing, $\dot{\rho} = H$. Restoring $M_{\text{pl}}$ for a moment, the value for $\varepsilon$ is more genuinely

$$\varepsilon = \frac{1}{2} \frac{1}{M_{\text{pl}}^2} \frac{\dot{\phi}^2}{\rho^2};$$  (23)

which shows that it is indeed dimensionless.

When $\varepsilon = 0$ exactly, then $\dot{\phi} = 0$; in this case, both $V(\phi)$ and $\dot{\rho} = H$ are constants as well. A space-time with a constant, positive ($V > 0$) vacuum energy density is called de Sitter space. Therefore, $\varepsilon$ also characterizes by how much an inflationary space-time departs from a purely de Sitter background.

**II. THE QUADRATIC ACTION**

One very appealing property of inflation is that some tiny amount of spatial dependence is inevitable. These tiny, primordial variations in the space-time, which must be present in inflation, provide the initial inhomogeneities that are needed to explain the beginnings of the structures that are observed today. Their origin in the theory lies in the quantum behavior of both the field and the space-time. Therefore, in addition to the classical quantities $\phi(t)$ and $g_{\mu\nu}$ associated with the background, consider a quantum correction to each as well,

$$g_{\mu\nu}(t) \rightarrow g_{\mu\nu}(t) + \delta g_{\mu\nu}(t, x), \quad \phi(t) \rightarrow \phi(t) + \delta \phi(t, x).$$  (24)

Since $\delta g_{\mu\nu}$ and $\delta \phi$ are quantum mechanical, they are always fluctuating, and they thereby introduce some spatial dependence into what would be an otherwise featureless background. The extreme expansion during inflation in turn stretches these fluctuations, which would otherwise remain of a tiny spatial extent, to vast sizes. This process happens so rapidly that the fluctuations are soon frozen into the space-time and remain beyond any further causal influence while the inflationary stage lasts. It is this basic mechanism that fills the universe with a pattern of tiny primordial perturbations according to inflation. This section of the notes will derive the leading, quadratic component of this pattern as a preliminary step to analyzing the leading source of the non-Gaussian component.
One difficulty in describing these fluctuations is that general relativity contains a fair amount of redundancy. Although a coordinate system must be chosen to be able to compare a prediction with what is observed, nothing that depends in detail on this choice corresponds to a genuine physical effect. These notes will only be considering the scalar fluctuations, which are ultimately meant to be responsible for the density and the temperature fluctuations in the matter of the early universe. A scalar fluctuation is one that transforms as a scalar quantity under the unbroken spatial rotations and translations rather than as a scalar under a general four-dimensional coordinate transformation. To count the number of distinct scalar functions that are possible in a general perturbation to the classical background, \( \delta g_{\mu\nu}(t, \vec{x}) \), first divide the fluctuation in the metric into blocks,

\[
\delta g_{\mu\nu}(t, \vec{x}) = \begin{pmatrix} \delta g_{00}(t, \vec{x}) & \delta g_{0i}(t, \vec{x}) \\ \delta g_{i0}(t, \vec{x}) & \delta g_{ij}(t, \vec{x}) \end{pmatrix}.
\]  

(25)

\( \delta g_{00} \) itself provides one scalar field. Another is generated by making a three-vector from a scalar field by taking its spatial derivative; thus \( \delta g_{i0} \sim \partial_i \phi \) also contains one scalar field. \( \delta g_{ij} \) contains two scalar fields—specifically, its trace and one more generated by taking two derivatives of a scalar function, \( \partial_i \partial_j \phi \). Including a final fluctuation for the actual scalar field \( \delta \phi \), all told there are five separate scalar fields. Most of these fields have no physical meaning. Two of them, for example, are related to how the coordinates were chosen. Dividing a general coordinate transformation, \( x^\mu \rightarrow x^\mu + \delta x^\mu \), into its temporal and spatial pieces, there is one scalar function in \( \delta x^0 \) and it is again possible to form a spatial vector by taking a spatial derivative of a second, \( \delta x^i \sim \delta^i j \partial_j f \). Additionally, it will soon become apparent that two more are non-propagating degrees of freedom fixed by two constraints. Thus, the five potential scalar fields are reduced by four, leaving but a single physical field. This section will show how this reduction proceeds before isolating this remaining physical scalar field and analyzing how its dynamics lead to a prediction for the two-point function.

To analyze the fluctuations about the simple, spatially invariant inflationary background, write the metric in the following general form,

\[
ds^2 = N^2 \left( d t^2 - h_{ij} (N^i \, dt + dx^i) (N^j \, dt + dx^j) \right)
= (N^2 - N_i N^i) \, dt^2 - 2N_i \, dt \, dx^i - h_{ij} \, dx^i \, dx^j.
\]  

(26)

This metric was originally introduced by Arnowitt, Deser and Misner [2] to analyze gravity from a Hamiltonian perspective. In this framework, \( N(t, \vec{x}) \) is called the lapse function while \( N_i(t, \vec{x}) \) is the shift vector. Note that \( N_i \) is defined to be \( N_i \equiv h_{ij} N^j \). The components of the inverse metric are

\[
g^{00} = \frac{1}{N^2}, \quad g^{0i} = -\frac{1}{N^2} N^i, \quad g^{ij} = -\frac{1}{N^2} \left[ N^2 h^{ij} - N_i N^j \right].
\]  

(27)

The spatial components, \( h_{ij} \), can be used to define a metric for the three-dimensional spatial hypersurfaces of the full space-time. To distinguish the curvature and covariant derivatives calculated using this metric, \( h_{ij} \), from those evaluated with the full space-time metric, \( g_{\mu\nu} \), the former will be written with a caret—as, for example, \( \hat{V}_i \) or \( \hat{R} \). The full action of the theory, rewritten in terms of this metric, then becomes

\[
S = \frac{1}{2} \int d^4x \sqrt{h} \left\{ N \hat{R} + \frac{1}{N} (E_{ij} E^{ij} - E^2) + \frac{1}{N} (\hat{\phi} - N^i \partial_i \phi)^2 - N h^{ij} \partial_i \phi \partial_j \phi - 2N V(\phi) \right\}.
\]  

(28)
In this expression, a new spatial tensor $E_{ij}$ was introduced, which is defined by

$$E_{ij} = \frac{1}{2}[h_{ij} - \hat{\nabla}_j N_i - \hat{\nabla}_i N_j]$$

$$= \frac{1}{2}[h_{ij} - h_{ik} \partial_k N^k - h_{jk} \partial_i N^k - N^k \partial_k h_{ij}],$$

(29)

and which is closely related to the extrinsic curvature associated with how the spatial surfaces are embedded in the full space-time. Once again, the spatial indices are implicitly contracted using the metric $h_{ij}$, so that

$$E_{ij} E^{ij} - E^2 = [h^{ik} h^{jl} - h^{ij} h^{kl}] E_{ij} E_{kl}.$$  

(30)

One advantage of this approach is that the fields $N$ and $N^i$ are both Lagrange multipliers, with no underlying dynamics, so their equations of motion produce two constraints that reduce the number of independent scalar degrees of freedom by two, as was mentioned earlier. Varying with respect to $N \to N + \delta N$ yields

$$\tilde{R} - N^{-2}(E_{ij} E^{ij} - E^2) - N^{-2}(\delta N \delta t \delta \phi)^2 - h^{ij} \delta \phi \delta \phi - 2V = 0,$$

(31)

while $N^i \to N^i + \delta N^i$ gives

$$\hat{\nabla}_i [N^{-1}(E_i^j - \delta N^j)] = N^{-1}(\delta N^j \delta \phi) \delta \phi.$$  

(32)

Since the behavior of the gravitational part of the action is so complicated in general, and since the actual quantum fluctuations of the field and the metric are small when compared with the classical background values, the fluctuations can be studied by expanding the action only to the necessary order in the fluctuations for the particular quantity that is being analyzed. At first, when calculating the two-point function, it will be only necessary to keep those terms in the action that are quadratic in the fluctuations; but later, when evaluating the three-point function, the leading signal of a non-Gaussian pattern, the cubic terms will be kept as well.

A general parameterization of the scalar fluctuations in the metric is provided by

$$N = 1 + 2\Phi(t, \vec{x}), \quad N^i = \delta^{ij} \partial_j B(t, \vec{x}), \quad h_{ij} = e^{2\Phi(t)} [(1 + 2\zeta(t, \vec{x})) \delta_{ij} + \partial_i \partial_j \xi],$$

(33)

and in the field’s fluctuations by

$$\phi = \phi_0(t) + \delta \phi(t, \vec{x}).$$  

(34)

This particular parameterization is not left unchanged under a small change of the coordinates; but since the calculation here will use the freedom to choose a particular set of coordinates that simplifies certain parts of the analysis, it is not necessary to write an explicitly coordinate-invariant form for the fluctuations from the start. One final raising and lowering convention that will be followed is that the indices of derivatives acting on any one of the scalar fluctuations—$\zeta$, $\Phi$, $B$, or $\xi$—will always be implicitly raised or contracted with a Kronecker $\delta^{ij}$, as was done above in the definition of $N^i$ as $\delta^{ij} \partial_j B$.

By redefining the time coordinate and shifting the spatial coordinate through the derivative of a scalar function, $x^t \to x^t + \delta^{ij} \partial_j f(t, \vec{x})$, two of the five scalar functions in the general parameterization can be removed. This freedom will be used to choose the coordinates so that the fluctuations of the scalar field vanish, $\delta \phi = 0$, and so that $\xi = 0$, leaving

$$N = 1 + 2\Phi(t, \vec{x}), \quad N^i = \delta^{ij} \partial_j B(t, \vec{x}), \quad h_{ij} = e^{2\Phi(t) + 2\zeta(t, \vec{x})} \delta_{ij}, \quad \phi = \phi(t).$$  

(35)
Notice that $\zeta$ has been slightly redefined so that it now appears in the exponent, though to first order the coordinates are the same as the initial definition above; however, this form will be much more useful when analyzing the three-point function later. This choice for the coordinates simplifies the constraint equations quite a bit, since $\phi$ will then have no spatial derivatives,

$$\dot{\mathcal{R}} - N^{-2}(E_{ij}E^{ij} - E^2) - N^{-2}\dot{\phi}^2 - 2V = 0$$

$$\dot{\mathcal{V}}_j [N^{-1}(E_i^j - \delta_i^j E)] = 0.$$  \hspace{1cm} (36)

Notice that the potential energy $V(\phi)$ has still been left quite general; but it can be removed entirely from the first constraint equation by applying the $tt$ component of the Einstein equation,

$$\dot{\mathcal{R}} - N^{-2}(E_{ij}E^{ij} - E^2) - 6\dot{\rho}^2 + (1 - N^{-2})\dot{\phi}^2 = 0.$$ \hspace{1cm} (37)

Now solve these constraints to first order in the coordinates that were chosen. The scalar curvature $\mathcal{R}$ associated with the spatial metric $h_{ij}$ is

$$\mathcal{R} = e^{-2\rho - 2\zeta} \left[-4\partial_k \partial^k \zeta - 2\partial_k \zeta \partial^k \zeta \right] = -4e^{-2\rho} \partial_k \partial^k \zeta + O(\zeta^2),$$ \hspace{1cm} (38)

while $E_{ij}$ is

$$E_{ij} = e^{2\rho} \left[\dot{\rho}(1 + 2\zeta) \delta_{ij} + \dot{\zeta} \delta_{ij} - \partial_i \partial_j B \right] + \cdots$$ \hspace{1cm} (39)

so that

$$E_{ij}E^{ij} - E^2 = -6\dot{\rho}^2 - 12\dot{\rho} \dot{\zeta} + 4\dot{\phi} \partial_i \partial_j B + \cdots.$$ \hspace{1cm} (40)

The $N$ constraint equation then becomes—again, to first order in the fluctuations—

$$-3\dot{\rho} \left[2\rho \Phi - \dot{\zeta} \right] - \partial_k \partial^k \left[\rho B + e^{-2\rho} \dot{\zeta} \right] + \dot{\phi}^2 \Phi = 0.$$ \hspace{1cm} (41)

Similarly expanding the $N^i$ constraint to first order yields,

$$2\partial_i \left[2\rho \Phi - \dot{\zeta} \right] = 0.$$ \hspace{1cm} (42)

This equation removes one of the scalar degrees of freedom by fixing $\Phi$,

$$\Phi = \frac{1}{2} \frac{\dot{\zeta}}{\dot{\rho}}.$$ \hspace{1cm} (43)

Although this constraint would have seemingly allowed the addition of an arbitrary constant as well, it was chosen to be zero so that the original background metric would be restored when the fluctuations are removed. When this result is inserted into the $N$ constraint, it similarly completely fixes another of the scalar fields, $B$,

$$B = -\frac{e^{-2\rho}}{\rho} \zeta + \chi \quad \text{with} \quad \partial_k \partial^k \chi = \frac{1}{2} \frac{\dot{\phi}^2}{\rho^2} \dot{\zeta}.$$ \hspace{1cm} (44)

Having solved the constraint equations and thereby eliminated two of the scalar fields, $\Phi$ and $B$, by expressing them in terms of the one remaining field $\zeta$, it is time to determine the quadratic action for this remaining scalar field. First, again substitute the
background equation, $V = 3\rho^2 - \frac{1}{2}\dot{\phi}^2$, into the action, still using the coordinate system where the fluctuation of the field $\delta \phi = 0$ vanishes, to obtain

$$S = \frac{1}{2} \int d^4x \sqrt{\hat{h}} \left\{ N\hat{R} + N^{-1}(E_{ij}E^{ij} - E^2) - 6N\dot{\rho}^2 + (N+N^{-1})\dot{\phi}^2 \right\}.$$  (45)

To calculate the two-point function of $\zeta$, expand this integrand to second order in the small fluctuations. At this order,

$$\hat{R} = e^{-2\rho} \left\{ -4\partial_i \partial^k \zeta + 8\zeta \partial_k \partial^k \zeta - 2\partial_i \zeta \partial^k \zeta + \cdots \right\}$$  (46)

and

$$E_{ij}E^{ij} - E^2 = -6\dot{\rho}^2 - 12\rho \dot{\zeta} + 4\dot{\rho} \partial_i \partial^k \zeta - 6\dot{\zeta}^2 + 6\dot{\rho} \partial_i \partial^k \zeta - 6\dot{\zeta}^2 + 4\dot{\rho} \partial_i \partial^k \zeta B + 4\dot{\zeta} \partial_i \partial^k \zeta B + \cdots$$  (47)

and

$$N + \frac{1}{N} = 2 + \frac{\dot{\rho}^2}{\rho^2} + \cdots.$$  (48)

The first term in the action is responsible for the spatial part of the kinetic term for $\zeta$,

$$\sqrt{\hat{h}}N\hat{R} = 2\rho \partial_i \zeta \partial^k \zeta - \partial_i \left\{ 2\rho \frac{1}{\rho} \zeta \partial_k \zeta \right\}$$

$$-2\partial_i \partial_k \left\{ \left( 2 + \frac{\rho}{\rho^2} \partial_k \zeta + \frac{\dot{\zeta}}{\rho} \right) \partial^k \zeta - \frac{1}{\rho} \zeta \partial^k \zeta \right\} + \cdots;$$  (49)

using the background equations to replace $\rho = -\frac{1}{2} \dot{\phi}^2$, only one term remains

$$\sqrt{\hat{h}}N\hat{R} = -\rho \frac{\dot{\phi}^2}{\rho^2} \partial_i \zeta \partial^k \zeta + \cdots,$$  (50)

up to derivative terms which have no dynamical effect. Taken together, the rest of the terms to second order are

$$\sqrt{\hat{h}} \left\{ \frac{1}{N}(E_{ij}E^{ij} - E^2) - 6N\dot{\rho}^2 + \left( N + \frac{1}{N} \right)\dot{\phi}^2 \right\}$$

$$= e^{3\rho} \frac{\dot{\phi}^2}{\rho^2} \dot{\zeta}^2 + e^{3\rho} \left[ 2\dot{\rho} + \dot{\phi}^2 \right] \left[ 2 + 6\zeta + 9\zeta^2 \right]$$

$$-2\partial_0 \left\{ e^{3\rho} \rho \left[ 2 + 6\zeta + 9\zeta^2 \right] \right\}$$

$$+ e^{3\rho} \partial_i \left\{ (4\rho + 12\rho \zeta - 2\partial_i \partial^k \zeta \partial^k) \partial^k B + (\partial_i \partial^k) \partial^k B \right\} + \cdots.$$  (51)

Using the background equation $\dot{\phi}^2 = -2\rho$ once again leaves just one term that is not a total derivative,

$$\sqrt{\hat{h}} \left\{ \frac{1}{N}(E_{ij}E^{ij} - E^2) - 6N\dot{\rho}^2 + \left( N + \frac{1}{N} \right)\dot{\phi}^2 \right\} = e^{3\rho} \frac{\dot{\phi}^2}{\rho^2} \dot{\zeta}^2 + \cdots.$$  (52)

So what remains, after applying the background equations and ignoring total derivative terms, is a remarkably simple expression for the fluctuations,

$$S = \int dt \frac{\dot{\phi}^2}{\rho^2} \int d^3x e^{3\rho} \left\{ \frac{1}{8} \dot{\zeta}^2 - \frac{1}{2} e^{-2\rho} \partial_i \zeta \partial^k \zeta + \cdots \right\}.$$  (53)
Notice further that the action is directly proportional to $\dot{\phi}^2/\dot{\rho}^2$. Until now, the slowly rolling approximation has not been used at all; but replacing the background functions with the appropriate parameters introduced earlier, the fluctuations only survive as long as

$$\frac{\dot{\phi}^2}{\dot{\rho}^2} = 2\epsilon \neq 0.$$  \hfill (54)

In a purely de Sitter background, the second order action vanishes entirely and no primordial fluctuations are generated.

III. THE TWO-POINT FUNCTION

The fluctuations that are being considered are essentially quantum mechanical in their nature. The next step will therefore be to rewrite the field so that its action more closely resembles the standard form for a quantum field in a flat space-time. Once this has been done, some of the results of ordinary quantum field theory can be applied to this inflationary setting, though with a few important reservations and warnings. Flat space is a much tamer environment than the accelerating background of an inflating universe. The energy scale, in particular, that characterizes this expansion is often assumed to be close enough to the Planck scale, where this quantum description of gravitational fluctuations is no longer consistent, to be rather worrisome. It is entirely possible that some of the usual assumptions about the behavior of $\zeta$ in flat space no longer apply in this setting. Since these notes are intended to illustrate the standard calculations of the two- and three-point functions, they will accordingly make the usual assumptions along the way—but it is important to remember that they are still assumptions nonetheless.

Quantum field theory was originally established for a flat space-time, which is invariant under the ten-dimensional set of Poincaré transformations—translations, rotations and boosts. The canonical form of the action of a scalar field in flat space has a kinetic term which is weighted by a factor of one-half:

$$\frac{1}{2} \left( \dot{\phi}^2 - \partial_k \phi \partial_k \phi \right).$$

It will be easier to apply some of the results of quantum field theory if the quadratic action is first put into a form that resembles this one. To do so, rescale the fluctuation $\zeta$ by

$$\phi(t,\vec{x}) = e^{\rho} \frac{\dot{\phi}}{\dot{\rho}} \zeta(t,\vec{x}).$$ \hfill (55)

The time-derivative of $\zeta$, in terms of $\phi$ and the slow-roll parameters, is then

$$\dot{\zeta} = e^{-\rho} \frac{\dot{\rho}}{\dot{\phi}} \left[ \phi - \dot{\phi}(1 + \epsilon + \delta) \right].$$ \hfill (56)

The Lagrangian for the field $\phi$ then begins to resemble its canonical form,

$$\frac{1}{2} e^{3\rho} \frac{\dot{\phi}^2}{\dot{\rho}^2} \zeta^2 - \frac{1}{2} e^{\rho} \frac{\dot{\phi}^2}{\dot{\rho}^2} \partial_k \zeta \partial^k \zeta = \frac{1}{2} e^\rho \phi^2 - \frac{1}{2} e^{-\rho} \partial_k \phi \partial^k \phi - \frac{1}{2} e^{-\rho} m^2 \phi^2$$

$$- \frac{1}{2} \partial_0 \left[ e^\rho (1 + \epsilon + \delta) \phi^2 \right],$$ \hfill (57)

where the effective, time-dependent, mass is defined to be

$$m^2(t) = -e^{2\rho} \dot{\rho}^2 (2 + \delta)(1 + \epsilon + \delta) - e^{2\rho} \dot{\rho}(\dot{\epsilon} + \dot{\delta}).$$ \hfill (58)
So far, the definitions of the slow-roll parameters have been applied without actually assuming that they are small. But now looking in the slowly rolling limit, where $\varepsilon^2, \varepsilon \delta, \delta^2, \dot{\varepsilon}/\rho$, and $\dot{\delta}/\rho$ are all much smaller than $\varepsilon$ and $\delta$, the mass to leading order becomes more simply

$$m^2 = -e^{2\rho} \rho^2 (2 + 2\varepsilon + 3\delta) + \cdots.$$  \hspace{1cm} (59)

The time and space coordinates are still weighted with different factors; to put them on a more similar footing, introduce a conformal time coordinate, given by

$$\eta(t) \equiv \int dt e^{-\rho(t)}$$ \hspace{1cm} (60)

in terms of which the quadratic action for $\phi(\eta, \vec{x})$ becomes

$$S = \int d\eta d^3 \vec{x} \{ \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \vec{\nabla} \phi \cdot \vec{\nabla} \phi - \frac{1}{2} m^2 \phi^2 \}.$$ \hspace{1cm} (61)

Hereafter, a prime denotes a derivative with respect to the conformal time and the usual vector notation, where $\vec{a} \cdot \vec{b} = \delta_{ij} a^i b^j$, has been introduced. The conformal time coordinate will be assumed here to be negative, $\eta \in (-\infty, 0]$, since for this choice time runs forwards; the coordinate could equally have been chosen to be positive, but then as time advances the coordinate diminishes, which is why the negative branch is more often used.

The variation of $S$ determines the equation of motion for $\phi$,

$$\phi'' - \vec{\nabla} \cdot \vec{\nabla} \phi + m^2 \phi = 0.$$ \hspace{1cm} (62)

Superficially, this equation appears to be exactly that of a free massive field in flat space; however, it is not fully covariant under Poincaré transformations. The mass in particular is not a constant, but rather depends explicitly on the conformal time $\eta$,

$$m^2 = -\rho'^2 (2 + 2\varepsilon + 3\delta) + \cdots.$$ \hspace{1cm} (63)

The background, however, is still invariant under purely spatial transformations, so the field can be expanded, as usual, in operators that create or annihilate plane waves,

$$\phi(\eta, \vec{x}) = \int \frac{d^3 \vec{k}}{(2\pi)^3} \{ \phi_k(\eta) e^{i\vec{k} \cdot \vec{x}} a_k + \phi_k^*(\eta) e^{-i\vec{k} \cdot \vec{x}} a_k^\dagger \}.$$ \hspace{1cm} (64)

The time-dependent part of the eigenmodes are then the solutions to the equation

$$\phi_k'' + (k^2 + m^2) \phi_k = 0.$$ \hspace{1cm} (65)

To solve this equation requires knowing the behavior of $m(\eta)$—at least to leading order in the slowly rolling parameters. In terms of the conformal time, the $\varepsilon$ parameter is defined by

$$\varepsilon = \frac{d}{dt} \frac{1}{H} = e^{-\rho(\eta)} \frac{d}{d\eta} \frac{e^\rho(\eta)}{\rho'} = 1 + \frac{d}{d\eta} \frac{1}{\rho'} \Rightarrow d \left( \frac{1}{\rho'} \right) = -(1 - \varepsilon) d\eta.$$ \hspace{1cm} (66)

This equation can be easily integrated and, neglecting order $\varepsilon^2$ corrections, becomes

$$\rho' = -\frac{1 + \varepsilon}{\eta} + \cdots.$$ \hspace{1cm} (67)
The constant of integration was fixed so that the standard result, \( e^{-\rho(\eta)} = -H \eta \), is recovered in the de Sitter limit (\( \epsilon \to 0 \)), remembering that the Hubble scale is a constant in de Sitter space. This result allows \( \rho' \) in the expression for the mass to be replaced with its explicit time dependence,

\[
m^2 = -\frac{1}{\eta^2}(2 + 6\epsilon + 3\delta) + \cdots,
\]

which in turn determines the behavior of the modes

\[
\varphi'' + k^2 \left[ 1 - \frac{2 + 3(2\epsilon + \delta)}{k^2 \eta^2} \right] \varphi_k = 0,
\]

at least to leading order in \( \epsilon \) and \( \delta \).

The solution of this differential equation for \( \varphi \) is a sum of Bessel functions,

\[
\varphi_k(\eta) = N_k(-k\eta)^{1/2} \left[ H^{(1)}(k\eta) + \theta_k H^{(2)}(k\eta) \right],
\]

which, using a little hindsight, has been written in terms of Hankel functions. The index \( \nu \) of these Hankel functions is

\[
\nu = \frac{3}{2} \sqrt{1 + \frac{1}{3}(2\epsilon + \delta)} = \frac{3}{2} + 2\epsilon + \delta + \cdots.
\]

As a second order differential equation, any particular solution requires two further conditions which then determine the constants of integration, \( N_k \) and \( \theta_k \), if it is to be specified completely. One condition is automatically provided by quantum field theory. The canonical commutation relation between the field \( \varphi \) and its conjugate momentum \( \pi \) should be

\[
[\varphi(\eta,\vec{x}), \pi(\eta,\vec{y})] = i\delta^3(\vec{x} - \vec{y}) \quad \pi = \frac{\delta_{\varphi}}{\delta \varphi'} = \varphi',
\]

for a local, causal field. This relation is the field theoretical analogue of the quantum mechanical commutator between the position (\( \hat{X} \)) and momentum (\( \hat{P} \)) operators, \([\hat{X}, \hat{P}] = i \). In terms of the modes \( \varphi_k \), this relation implies that

\[
\varphi_k \varphi_k'^* - \varphi_k'^* \varphi_k = i,
\]

which fixes one of the constants of integration,

\[
N_k = -\frac{\sqrt{\pi}}{2\sqrt{k}} \frac{1}{\sqrt{1 - \theta_k \theta_k^*}}.
\]

Only one more condition is needed to fix \( \theta_k \).

The second property ordinarily assumed is that space-time becomes locally flat at arbitrarily small separations. One of the postulates of general relativity is that it is always possible to choose a locally flat frame for any space-time point. And since quantum field theory was developed for a globally flat background, it is very tempting to impose local flatness as a principle applicable in an arbitrary background. However,

\footnote{Note that this condition secretly contains yet another assumption—that the space-time has a point-like description down to arbitrarily small distances, \(|\vec{x} - \vec{y}| \).}
what is meant by a short distance in an inflationary background is not an absolute statement, for it depends on precisely when this condition is being imposed. Wavelengths that at one time might have been small compared with the curvature of the background will later no longer be so, having been stretched along with the expansion of the space.

Perhaps what is still more troubling is that the dynamical scale for the inflationary expansion, the Hubble scale $H$, is itself frequently chosen to be an appreciable fraction of the Planck scale. At distances smaller than this Planck threshold, a description of nature that simultaneously applies both the principles of quantum field theory and those of general relativity so far does not seem to be consistent. So there is always a danger that an incorrect prescription might be inadvertently chosen by defining the state by appealing to a particular background in a regime where it is not possible to describe space-time adequately and consistently.

It is important to keep these caveats in mind, even though the conventional calculation will still be followed here. Applying the usual assumption, that space-time looks flat at all scales, even those where quantum gravitational effects are strong, fixes the remaining constant of integration in $\phi_k(\eta)$. Put a little more precisely, this condition requires that the modes at very short distances—or equivalently at very large spatial momenta ($k \to \infty$)—should match the functional form of the standard flat-space modes.

Expanding the solution for $\phi_k(\eta)$ in this limit, produces

$$
\lim_{k \to \infty} \phi_k(\eta) = \frac{1}{\sqrt{1 - \theta_k \theta_k^*}} \left[ e^{-i\frac{\pi}{2}(2\varepsilon + \delta)} \frac{e^{-i k \eta}}{\sqrt{2k}} + \theta_k e^{i\frac{\pi}{2}(2\varepsilon + \delta)} \frac{e^{i k \eta}}{\sqrt{2k}} \right] \quad (75)
$$

For a quantum field in flat space, the positive energy vacuum modes are those for which $\phi_k(\eta) = e^{-i k \eta} / \sqrt{2k}$, so that the assumption of local Lorentz invariance requires that $\theta_k = 0$. These two constraints on the state—the canonical commutation relation and the assumption of local Lorentz invariance at short distances—finally completely determine the momentum modes

$$
\phi_k(\eta) = -\frac{\sqrt{\pi}}{2} \sqrt{-\eta} H^{(1)}_\nu(-k\eta)
$$

which in turn defines the metric fluctuations, $\zeta(\eta, \vec{x})$.

With this exact expression for the modes, one can at last evaluate the two-point correlation function for the fluctuations that was mentioned back in the beginning,

$$
\langle 0(\eta)|\zeta(\eta, \vec{x})\zeta(\eta, \vec{y})|0(\eta)\rangle. \quad (77)
$$

Quite often this two-point function is expressed in terms of a power spectrum, which is essentially its Fourier transform together with a few additional factors extracted for convenience,

$$
\langle 0(\eta)|\zeta(\eta, \vec{x})\zeta(\eta, \vec{y})|0(\eta)\rangle = \int \frac{d^3 k}{(2\pi)^3} e^{i \vec{k} \cdot (\vec{x} - \vec{y})} \frac{2\pi^2}{k^3} P_k(\eta). \quad (78)
$$

Using the rescaling that connects $\zeta$ with $\phi$, the power spectrum for the fluctuation $\zeta$ is

$$
P_k(\eta) = e^{-2\rho} \frac{\dot{\rho}}{\dot{\phi}^2} k^3 \frac{\phi_k}{2\pi^2} \phi_k \phi_k^*.
$$

(79)
By further replacing $\frac{\rho^2}{\dot{\phi}^2}$ with $\frac{1}{2\epsilon}$ in this expression, while at the same time substituting the factor $e^{-2\rho(\eta)}$ with

$$e^{-2\rho} = \frac{(-H\eta)^2}{(1+\epsilon)^2}, \tag{80}$$

the leading form for the power spectrum is given by

$$P_k(\eta) = \frac{H^2}{M_{pl}^2} \frac{(-k\eta)^3}{16\pi} \frac{H_{v}^{(1)}(-k\eta)H_{v}^{(2)}(-k\eta)}{\epsilon(1+\epsilon)^2} + \cdots. \tag{81}$$

Since the interest here is to learn how the power spectrum changes with the scale $k$, rather than with the conformal time $\eta$, the time dependence of $\dot{H}(\eta)$ is not explicitly shown, since $H$ has no dependence on $k$ at all, unlike the other factors where it appears together with $k$ as $-k\eta$.

Inflation is intended to increase the size of a causally connected patch of the universe far beyond what it would have been in a purely radiation- or matter-dominated universe. The idea is that what is seen at large distance-scales today began as fluctuations with a very tiny spatial extent at the time of inflation. From the perspective of the more recent ages of the universe, the fluctuations laid down by inflation are, for a longer or shorter while, beyond the causal reach of any process until the horizon $1/\dot{\rho}$ catches up with the physical size of a fluctuation, which is also growing with the expansion of the universe, though not as fast as the horizon. During the inflationary era, this relation is reversed; during inflation, the horizon changes only very slowly, while the physical spatial sizes of the fluctuations are stretched very dramatically. In the de Sitter limit, the horizon size would in fact be constant; it is called a horizon since for any observer, anything at a distance greater than $1/\dot{\rho}$ is unobservable, and time-like Killing vectors tip over and become space-like as they are continued beyond the horizon, just as for the horizon of a black hole in the standard Schwarzschild coordinates. However, the de Sitter horizon is a little different—in de Sitter space, each observer sees his or her own horizon and it is not associated with any space-time singularity, and of course de Sitter space does not asymptotically approach Minkowski space as is the case far from a black hole horizon.

So for inflation, the physically important values of $k$ are those associated with modes that have been stretched well outside this Hubble horizon by the end of inflation. Writing the physical $k$, which is stretched by the scale factor over time, as $k_{phys} = e^{-\rho(\eta)}k$, the relevant modes are those for which

$$e^{-\rho(\eta)}k \ll H \quad \Rightarrow \quad \frac{H\eta}{1+\epsilon}k \ll H \quad \Rightarrow \quad -k\eta \ll 1. \tag{82}$$

Expanding the power spectrum in the appropriate limit, $-k\eta \to 0$, then yields

$$P_k(\eta) = \frac{1}{8\pi^2} \frac{1}{\epsilon} \frac{H^2}{M_{pl}^2} (-k\eta)^{-4\epsilon-2\delta} + \cdots. \tag{83}$$

This power spectrum can be equivalently written in the form

$$P_k(\eta) = \frac{1}{8\pi^2} \frac{1}{\epsilon} \frac{H^2}{M_{pl}^2} (-k_0\eta)^{-4\epsilon-2\delta} \left( \frac{k}{k_0} \right)^{-4\epsilon-2\delta} + \cdots. \tag{84}$$
by introducing a reference scale $k_0$, which is useful when comparing with experiments\footnote{In the WMAP experiment [3], for example, the data are normalized with respect to a $k_0$ of 2 Gpc$^{-1}$.}.

The inflationary picture has strongly influenced how experimental results are analyzed, so much so that fits to observations often choose from the start a form for the power spectrum that is compatible with the basic form predicted by inflation,

$$P_k(\eta) \equiv \Delta^2_\zeta(k_0) \left( \frac{k}{k_0} \right)^{n_s - 1}.$$  \hspace{1cm} (85)

$\Delta^2_\zeta$ is called the amplitude of the power spectrum and $n_s$ is its tilt. The predictions for these parameters for the simple inflationary model that has been studied here are

$$\Delta^2_\zeta(k_0) \approx \frac{1}{8\pi^2} \frac{1}{\epsilon} \frac{H^2}{M^2_{pl}} \quad \text{and} \quad n_s = 1 - 4\epsilon - 2\delta.$$ \hspace{1cm} (86)

IV. THE THREE-POINT FUNCTION

Just as the leading behavior of the two-point function is determined by the quadratic terms in the action—once it has been expanded in powers of $\zeta$—the three-point function is similarly determined by the cubic terms. The goal of these notes is to extract these cubic terms, using the same approach and coordinates as were used before; but before doing so, it is important to describe how the three-point function is itself calculated from these terms in the action.

A perturbative approach to quantum field theory in flat space, which is based on the inherent smallness of the strength of the interactions, is usually implemented in the interaction picture. The interaction picture is especially suited to situations where it is impossible to solve for the exact behavior of the full interacting theory, but where it is possible to solve for the behavior of the free field theory. The free theory is described by the quadratic terms in the action. The full solution is then built up through successive corrections to this free theory by adding the effects of the interactions perturbatively.

More formally, the interaction picture is implemented as follows. The action is first separated into two parts, \( S \rightarrow S_0 + S_I \). \hspace{1cm} (87)

The first represents the action for the free theory—the quadratic terms in the field $\zeta$—while the interacting part, $S_I$, contains everything else, in particular the cubic (and higher) order terms that are important for any non-Gaussianities in the primordial fluctuations. Operators, such as the field itself, evolve in accord with the free action while $S_I$ determines the time-evolution of the state. If the Hamiltonian associated with the interacting part is written as $H_I(\eta)$, then the state $|0(\eta)\rangle$ evolves so that it obeys Dyson’s equation,

$$\frac{d}{d\eta} |0(\eta)\rangle = -iH_I(\eta) |0(\eta)\rangle.$$ \hspace{1cm} (88)

The solution to this equation is

$$|0(\eta)\rangle = T e^{-i \int_0^\eta d\eta' H_I(\eta')} |0\rangle,$$ \hspace{1cm} (89)
where $|0\rangle$ is the ‘initial’ state of the field, in this case

$$|0\rangle \equiv |0(-\infty)\rangle,$$  \hfill (90)

though it could equally have been defined at some finite initial time instead, $|0(\eta_0)\rangle$. $T$ indicates that the operators in the exponential are to be time-ordered.

Here the initial state is defined, at least implicitly, in the infinitely far past. In ordinary quantum field theory in a flat space-time, this time is meant to be any sufficiently early time that the particular system being considered—two colliding particles, for example—is composed of sufficiently separated parts that they can be treated as though they are in a free, noninteracting state. In flat space, as long as this condition is met, the exact value of the initial time is not important since in this background energies and spatial distances remain conserved or fixed, independent of the time at which they are evaluated.

In an inflationary setting, there is no such time-invariance of the background. So although in flat space the assumptions made about the detailed spatial structure of the state of a field at distances shorter than a Planck length hardly affect observable measurements on much larger distances, in an inflationary background the expansion of the space-time will eventually stretch any such tiny, ‘Planckian’ structures in the state of a field at distances shorter than a Planck length at this infinitely remote ‘beginning’. Unfortunately, so far there is no complete and consistent theoretical framework that simultaneously reconciles the postulates of general relativity and those of quantum theory, and even an incomplete, effective treatment of gravity breaks down beyond the Planck scale. Even by restricting to an early but finite initial time $\eta_0$, only a fairly narrow window is available in which the theory is both perturbatively consistent and still relevant for explaining the origins of the primordial fluctuations.

Yet, for the standard choice of the initial state, which corresponds to choosing the mode functions $\phi_\eta(\eta)$ as in the last section, it is straightforward enough to evaluate the three-point function once the interactions $H_\eta(\eta)$ are known. Recall that the physically relevant field, $\zeta(t,\vec{x})$, is related to the canonically normalized one, $\phi(\eta,\vec{x})$, by

$$\phi(\eta,\vec{x}) = e^{\rho(t(\eta))} \frac{\phi}{\rho} \zeta(t(\eta),\vec{x}).$$ \hfill (91)

The ratio of these two is a classical function, which means that the $\zeta$ three-point function is also proportional to the $\phi$ three-point function,

$$\langle 0(t)|\zeta(t,\vec{x})\zeta(t,\vec{y})\zeta(t,\vec{z})|0(t)\rangle = e^{-3\rho(t(\eta))} \frac{\rho}{\rho} \langle 0(\eta)|\phi(\eta,\vec{x})\phi(\eta,\vec{y})\phi(\eta,\vec{z})|0(\eta)\rangle.$$ \hfill (92)

In the interaction picture, the three-point function is generated entirely by the time-evolution of the state,

$$\langle 0(t)|\zeta(t,\vec{x})\zeta(t,\vec{y})\zeta(t,\vec{z})|0(t)\rangle = e^{-3\rho} \frac{\rho^3}{\rho^3} \langle 0| \left( T e^{-i\int_{\eta_0}^{\eta} d\eta' H(\eta')} \right)^3 \phi(\eta,\vec{x})\phi(\eta,\vec{y})\phi(\eta,\vec{z}) \left( T e^{-i\int_{\eta}^{\infty} d\eta' H(\eta')} \right) |0\rangle.$$ \hfill (93)

As will be established in the next section, the effect of the cubic interactions is extremely small. Therefore, it is possible to evaluate the three-point function perturbatively, expanding the time-ordered exponentials, and keeping just the linear terms to
obtain the leading contribution,
\[
\langle 0(t)|\zeta(t,\vec{x})\zeta(t,\vec{y})\zeta(t,\vec{z})|0(t)\rangle = -ie^{-3\rho}\frac{\dot{\rho}}{\phi^3} \int_{-\infty}^{\eta} d\eta' \langle 0|\left[\phi(\eta,\vec{x})\phi(\eta,\vec{y})\phi(\eta,\vec{z}),H_I(\eta')\right]|0\rangle + \cdots. \tag{94}
\]

Since the three-point function is so small in this theory, it is sufficient to evaluate it in the limit where any corrections that are suppressed by additional factors of the slowly rolling parameters are neglected. In this limit, which is essentially the de Sitter background,
\[
\rho(\eta) = -\frac{1}{H\eta}, \tag{95}
\]
and using the fact that
\[
\varepsilon = \frac{1}{2} \frac{\dot{\phi}^2}{M_{pl}^2 \dot{\rho}^2}, \tag{96}
\]
the leading contribution to the three-point function is
\[
\langle 0(t)|\zeta(t,\vec{x})\zeta(t,\vec{y})\zeta(t,\vec{z})|0(t)\rangle = -i\frac{H^3}{M_{pl}^4} (-\eta)^3 \int_{-\infty}^{\eta} d\eta' \langle 0|\left[\phi(\eta,\vec{x})\phi(\eta,\vec{y})\phi(\eta,\vec{z}),H_I(\eta')\right]|0\rangle + \cdots, \tag{97}
\]
with the appropriate factors of $M_{pl}$ restored. The next section is devoted to calculating the cubic terms that occur in the inflationary action.

V. THE CUBIC ACTION

Much of the difficulty in extracting the cubic terms occurs because a straightforward expansion of the action to third order produces a set of interactions that obscures the true size of their physical effect, at least in the coordinates that have been chosen. Expressed in terms of the slowly rolling parameters, the terms that emerge at third-order do not appear to be suppressed at all, whereas in fact the three-point function is suppressed by a factor of $\varepsilon^2$ beyond the inherent smallness of $\zeta$. With another choice of the coordinates for the fluctuations, this suppression would have been more evident. Unfortunately, such coordinates would have in turn hidden other important properties of fluctuations—how they behave when they have been stretched beyond the horizon, for instance. This important property is much clearer in the coordinates that have been chosen here. Unfortunately, there is no perfect set of coordinates which is at once suited to all possible purposes.

Most of this section is devoted to showing how to convert the cubic interactions—those that emerge when the action is expanded directly—into a simpler form where the $\varepsilon^2$ suppression is made obvious. This latter form can then be used to calculate the inflationary prediction for the three-point function as was just outlined above. Going from one set of interactions to the other is in fact quite tedious, and many of the steps have been secretly guided by a foreknowledge of the final result. So much of the calculation will appear to be achieving nothing except to generate more and more terms. Only at the very end, when all of the separate calculations are put back together, will the goal of finding a simpler set of cubic interactions finally be attained.
The second set vanishes when the second-order equation of motion for $\zeta$ is applied,

$$\frac{d}{dt} \left\{ e^{3p} \frac{\phi^2}{\rho^2} \dot{\zeta} \right\} - e^p \frac{\phi^2}{\rho^2} \partial_k \partial^k \zeta = 0.$$  \hspace{1cm} (98)$$

The final set of terms are those that are sought, those that produce a genuine dynamical effect. This set corresponds to the true inflationary contribution to the primordial three-point function and is clearly small in the slowly rolling limit.

To begin, start once more with the coordinates introduced during the derivation of the quadratic action,

$$N = 1 + \frac{\dot{\zeta}}{\rho} \quad N^i = \delta^{ij} \partial_j B, \quad h_{ij} = e^{2p+2\zeta} \delta_{ij}, \quad \phi = \phi(t),$$ \hspace{1cm} (99)

where one of the Lagrange constraints has been used to define $N$ in terms of $\zeta$. As before, the action for this system is

$$S = \frac{1}{2} \int d^4x \sqrt{h} \left\{ N \dot{\zeta}^2 + N^{-1} (E_{ij} E^{ij} - E^2) - 6N \rho^2 + \left( N + N^{-1} \right) \dot{\phi}^2 \right\}$$  \hspace{1cm} (100)$$

once the background equation has been imposed to remove the explicit appearance of the potential $V(\phi)$.

The exact expressions for $\dot{R}$ and $E_{ij}$ in these coordinates are

$$\dot{R} = -4e^{-2p} e^{-2\zeta} \partial_k \partial^k \zeta - 2e^{-2p} e^{-2\zeta} \partial_k \zeta \partial^k \zeta$$  \hspace{1cm} (101)$$

and

$$E_{ij} = e^{2p+2\zeta} \left\{ \left[ \dot{\zeta} - \delta_{ij} \partial_j B \right] \delta_{ij} - \partial_i \partial_j B \right\},$$  \hspace{1cm} (102)$$

so that

$$E_{ij} E^{ij} - E^2 = -6 (\rho + \dot{\zeta} - \partial_i \zeta \partial^i B)^2 + 4 (\rho + \dot{\zeta} - \partial_i \zeta \partial^i B) (\partial_i \partial^i B)$$

$$\left[ (\partial_i \partial_i B) (\partial_k \partial^k B) - (\partial_i \partial^i B) \right]^2. \hspace{1cm} (103)$$

The rest of the factors only need to be expanded to third order,

$$\sqrt{h} = e^{3p} e^{3\zeta} = e^{3p} \left\{ 1 + 3\zeta + \frac{9}{2} \zeta^2 + \frac{9}{2} \zeta^3 + \cdots \right\}$$

$$\frac{1}{N} = 1 - \frac{\dot{\zeta}}{\rho} + \frac{\dot{\zeta}^2}{\rho^2} - \frac{\dot{\zeta}^3}{\rho^3} + \cdots. \hspace{1cm} (104)$$

Expanding, for example, the first term in the Lagrangian produces the following set of third-order terms,

$$\sqrt{h} N \dot{R}^{(3)} = -2e^p \dot{\zeta}^2 \partial_k \partial^k \zeta - 2e^p \zeta \partial_i \zeta \partial^i \partial^i \zeta - 4e^p \frac{1}{\rho} \zeta \zeta \partial_i \partial_k \zeta \zeta - 2e^p \frac{1}{\rho} \zeta \zeta \partial_i \partial_k \zeta \zeta. \hspace{1cm} (105)$$
As was mentioned, the goal is to convert the action into a form where all of the leading contributions vanish, either once the equations of motion are imposed or because they are total derivatives. To start this process, convert this set of four terms into a single term by noticing that the last two terms resemble the first two, once they have been integrated by parts to remove the time derivatives from the $\zeta$ fields,

$$-4e^\rho \frac{1}{\rho} \zeta \frac{\partial k}{\partial k} \zeta - 2e^\rho \frac{1}{\rho} \zeta \frac{\partial k}{\partial k} \zeta$$

$$= 2e^\rho \frac{\partial}{\partial k} \zeta \frac{\partial k}{\partial k} \zeta + 2e^\rho \zeta \frac{\partial k}{\partial k} \zeta - 2e^\rho \frac{\partial}{\partial k} \zeta \frac{\partial k}{\partial k} \zeta - 2e^\rho \frac{\partial}{\partial k} \zeta \frac{\partial k}{\partial k} \zeta$$

$$-2 \frac{d}{dt} \left\{ e^\rho \frac{1}{\rho} \zeta \frac{\partial k}{\partial k} \zeta + e^\rho \frac{1}{\rho} \zeta \frac{\partial k}{\partial k} \zeta \right\} + \partial_k \left\{ 2e^\rho \frac{1}{\rho} \zeta \frac{\partial k}{\partial k} \zeta \right\}. \quad (106)$$

Inserted back into the previous equation—and integrating it by parts once more—yields a single term that is not a total derivative,

$$\sqrt{h}N\dot{R}^{(3)} = 2e^\rho \frac{\partial}{\partial k} \zeta \frac{\partial k}{\partial k} \zeta$$

$$- \frac{d}{dt} \left\{ e^\rho \frac{1}{\rho} \zeta \frac{\partial k}{\partial k} \zeta \right\} + \partial_k \left\{ 2e^\rho \frac{1}{\rho} \zeta \frac{\partial k}{\partial k} \zeta \right\}. \quad (107)$$

The background equation, $\ddot{\rho} = -\frac{1}{\rho} \dot{\phi}^2$, will be used frequently in this calculation, so that the first term in this expression can be rewritten as

$$\sqrt{h}N\ddot{R}^{(3)} = -e^\rho \frac{\partial}{\partial k} \zeta \frac{\partial k}{\partial k} \zeta + e^\rho \frac{1}{\rho} \left[ 2\ddot{\rho} + \dot{\phi}^2 \right] \zeta \frac{\partial k}{\partial k} \zeta$$

$$- \frac{d}{dt} \left\{ e^\rho \frac{1}{\rho} \zeta \frac{\partial k}{\partial k} \zeta \right\} + \partial_k \left\{ 2e^\rho \frac{1}{\rho} \zeta \frac{\partial k}{\partial k} \zeta \right\}. \quad (108)$$

This step is useful; it makes the scaling of the initial term with the slowly rolling parameter a little clearer, since the ratio $\dot{\phi}^3 / \rho^2$ is directly proportional to $\epsilon$.

The remaining terms in the Lagrangian contain the following cubic terms,

$$\sqrt{h} \left\{ \frac{1}{N} (E_{ij}E^{ij} - E^2) - 6N\rho^2 + \left( N + \frac{1}{N} \right) \dot{\phi}^2 \right\}^{(3)}$$

$$= e^{3\rho} \frac{\partial}{\partial \rho} \left( 3\zeta - \frac{\zeta}{\rho} \right) \zeta^2$$

$$+ e^{3\rho} \left[ \left( 3\zeta - \frac{\zeta}{\rho} \right) (\partial_k \partial_l B) (\partial_k \partial_l B) - \left( 3\zeta - \frac{\zeta}{\rho} \right) (\partial_k \partial_l B)^2 - 4 (\partial_k \zeta \partial_l B) (\partial_l \partial_k B) \right]$$

$$- \frac{d}{dt} \left[ 18e^{3\rho} \dot{\rho} \zeta^3 \right] + \partial_k \left[ 18e^{3\rho} \dot{\rho} \zeta^2 \partial_k B \right] + 9e^{3\rho} \left[ 2\ddot{\rho} + \dot{\phi}^2 \right] \zeta^3. \quad (109)$$

If the full Lagrangian for the cubic terms in the fluctuations is defined to be

$$2\mathcal{L}^{(3)} = \sqrt{h} \left\{ N\ddot{R} + \frac{1}{N} (E_{ij}E^{ij} - E^2) - 6N\rho^2 + \left( N + \frac{1}{N} \right) \dot{\phi}^2 \right\}^{(3)}, \quad (110)$$

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then
\[
2\mathcal{L}^{(3)} = e^{3\rho} \frac{\dot{\phi}^2}{\rho^2} \left( 3\zeta - \frac{\dot{\zeta}}{\rho} \right) \dot{\zeta}^2 - e^{\rho} \frac{\dot{\phi}^2}{\rho^2} \zeta \partial_\zeta \partial^k \zeta \\
+ e^{3\rho} \left\{ \left( 3\zeta - \frac{\dot{\zeta}}{\rho} \right) \left[ (\partial_k \partial_l B) (\partial^k \partial^l B) - (\partial_k \partial^k B) (\partial_l \partial^l B) \right] + 4(\partial_\zeta \partial^k B) (\partial_l \partial^l B) \right\} + \mathcal{D}_0,
\]
where all of the non-dynamical terms have been abbreviated by \( \mathcal{D}_0 \), given by
\[
\mathcal{D}_0 = -\frac{d}{dt} \left[ 18e^{3\rho} \rho^2 \zeta^3 + 2e^\rho \frac{1}{\rho} \zeta \partial_\zeta \left[ \zeta \partial^k \zeta \right] \right] \\
+ \partial_k \left[ 18e^{3\rho} \rho^2 \zeta^2 \partial^k B + 2e^\rho \frac{1}{\rho^2} \zeta^2 \left[ \rho \partial^k \zeta - \rho \partial^k \zeta \right] \right] \\
+ 9e^{3\rho} \left[ 2\dot{\rho} + \dot{\phi}^2 \right] \zeta^3 + e^\rho \frac{1}{\rho} \left[ 2\dot{\rho} + \dot{\phi}^2 \right] \zeta \partial_\zeta \partial^k \zeta. \tag{111}
\]

While this action is perfectly correct, it does not immediately convey the true size of the non-Gaussian component of the primordial perturbations. The terms on the first line are superficially suppressed by \( \epsilon \), while those of the second line are not obviously suppressed at all. As mentioned earlier, the size of these interactions is smaller than what either line would seem to imply, since the true suppression is \( \epsilon^2 \). To establish this property requires a much longer calculation.

To proceed, replace the \( B \) field with the \( \zeta \) field by imposing the second of the constraints derived earlier,
\[
B = -\frac{e^{-2\rho}}{\rho} \zeta + \chi \quad \text{with} \quad \partial_k \partial^k \chi = \frac{1}{2} \frac{\dot{\phi}^2}{\rho^2} \dot{\zeta}, \tag{113}
\]
and sort the resulting terms according to the power of their \( e^\rho \) prefactors,
\[
2\mathcal{L}^{(3)} = e^{3\rho} \frac{\dot{\phi}^2}{\rho^2} \left( 3\zeta - \frac{\dot{\zeta}}{\rho} \right) \dot{\zeta}^2 - \frac{1}{4} e^{3\rho} \frac{\dot{\phi}^4}{\rho^4} \left( 3\zeta - \frac{\dot{\zeta}}{\rho} \right) \dot{\zeta}^2 \\
- 2e^{3\rho} \frac{\dot{\phi}^2}{\rho^2} \zeta \partial_\zeta \partial^k \chi + e^{3\rho} \left( 3\zeta - \frac{\dot{\zeta}}{\rho} \right) \partial_k \partial_l \chi \partial^k \partial^l \chi \\
- e^\rho \frac{\dot{\phi}^2}{\rho^4} \left( \zeta - 2\frac{\dot{\zeta}}{\rho} \right) \partial_k \zeta \partial^k \zeta + e^\rho \frac{\dot{\phi}^2}{\rho^4} \left( 3\zeta - \frac{\dot{\zeta}}{\rho} \right) \dot{\zeta} \partial_\zeta \partial_k \zeta \\
- 2e^\rho \frac{1}{\rho} \left( 3\zeta - \frac{\dot{\zeta}}{\rho} \right) \partial_k \partial_l \zeta \partial^k \partial^l \chi + 4e^\rho \frac{1}{\rho} \partial_k \zeta \partial^k \chi \partial_l \partial^l \chi \\
+ e^{-\rho} \frac{1}{\rho^2} \left( 3\zeta - \frac{\dot{\zeta}}{\rho} \right) \left[ \partial_k \partial_l \zeta \partial^k \partial^l \zeta - (\partial_k \partial^k \zeta)^2 \right] - 4e^{-\rho} \frac{1}{\rho^2} \partial_k \zeta \partial^k \zeta \partial_l \partial^l \zeta \\
+ \mathcal{D}_0. \tag{114}
\]

This form for the cubic action is a function of the physical scalar field \( \zeta \), since \( \chi \) is determined by \( \zeta \). Both of the Lagrange constraints have been applied, and it can now be analyzed term by term. Because of the complexity of the conversion of this expression
for $\mathcal{L}^{(3)}$ into the one manifestly suppressed by $\varepsilon^2$, the calculation will be divided into four parts, one for each set of terms that share the same power for the $\varepsilon^p$ prefactor, except for one of the $\varepsilon^3p$ terms which will be treated by itself.

VA. The $\varepsilon^{-p}$-terms

Since there are the fewest of them, the $\varepsilon^{-p}$ terms will be analyzed first.

$$2\mathcal{L}^{(3)}|_{\varepsilon^{-p}} = 3e^{-\rho} \frac{1}{\rho^2} \partial_k \partial_l \partial^k \partial^l \xi - 3e^{-\rho} \frac{1}{\rho^2} \xi (\partial_k \partial^k \xi) - 4e^{-\rho} \frac{1}{\rho^3} \partial_k \partial_l \partial^k \partial^l \xi$$

$$- e^{-\rho} \frac{1}{\rho^3} \partial_k \partial_l \partial^k \partial^l \xi + e^{-\rho} \frac{1}{\rho^3} \xi (\partial_k \partial^k \xi)^2. \quad (115)$$

The two terms on the second line differ from those on the first since they both have a time derivative of the $\zeta$ field. Integrating by parts as many times as is needed to remove all of the time derivatives from the $\zeta$’s eventually produces

$$- e^{-\rho} \frac{1}{\rho^3} \partial_k \partial_l \partial^k \partial^l \xi + e^{-\rho} \frac{1}{\rho^3} \xi (\partial_k \partial^k \xi)^2$$

$$= - \frac{1}{3} e^{-\rho} \frac{1}{\rho^2} \xi [\partial_k \partial_l \partial^k \partial^l \xi - (\partial_k \partial^k \xi)^2] + \frac{1}{2} e^{-\rho} \frac{\dot{\rho}}{\rho^3} \xi [\partial_k \partial_l \partial^k \partial^l \xi - (\partial_k \partial^k \xi)^2]$$

$$+ \frac{1}{3} \frac{d}{dt} \left\{ e^{-\rho} \frac{1}{\rho^3} \xi [(\partial_k \partial^k \xi)^2 - \partial_k \partial_l \partial^k \partial^l \xi] \right\}$$

$$+ \frac{2}{3} \partial_k \left\{ e^{-\rho} \frac{1}{\rho^3} \xi [\partial^k \partial^l \partial^k \partial^l \xi - \partial^l \partial^l \partial^k \partial^k \xi] - e^{-\rho} \frac{1}{\rho^3} \xi [\partial^k \partial^l \partial^k \partial^l \xi - \partial^l \partial^l \partial^k \partial^k \xi] \right\}$$

$$+ e^{-\rho} \frac{1}{\rho^3} \left[ \frac{1}{2} \dot{\zeta} + \frac{1}{2} \phi^2 \right] \xi [(\partial_k \partial^k \xi)^2 - \partial_k \partial_l \partial^k \partial^l \xi]. \quad (116)$$

The first of these terms—superficially the dominant one in the slowly rolling limit—when added to the rest of the order $\varepsilon^{-p}$ terms produces an expression that coalesces into a further set of total derivatives, leaving just one term remaining,

$$2\mathcal{L}^{(3)}|_{\varepsilon^{-p}} = \frac{1}{2} e^{-\rho} \frac{\dot{\rho}}{\rho^3} \xi [\partial_k \partial_l \partial^k \partial^l \xi - (\partial_k \partial^k \xi)^2] + \cdots, \quad (117)$$

where the unwritten terms are those that have no dynamical effect.

It might seem that what remains is only suppressed by $\varepsilon$ and not by $\varepsilon^2$ as was claimed. However, this term is actually a part of a larger set that is ultimately proportional to the $\zeta$ equation of motion. It will therefore be written as

$$\mathcal{F}_A = \frac{1}{2} e^{-\rho} \frac{\dot{\rho}}{\rho^3} \xi [\partial_k \partial_l \partial^k \partial^l \xi - (\partial_k \partial^k \xi)^2], \quad (118)$$

after a few more spatial integrations, in anticipation of things to come. In the end, the $\varepsilon^{-p}$ terms in the action have no real physical effect,

$$2\mathcal{L}^{(3)}|_{\varepsilon^{-p}} = \mathcal{F}_A + \mathcal{D}_A; \quad (119)$$

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\[ \mathcal{D}_A = \frac{1}{3} \frac{d}{dt} \left\{ -e^{-\rho} \frac{1}{\rho^3} \left[ \partial_k \partial_l \zeta \partial^k \partial^l - \partial_k \partial^k \zeta^2 \right] \right\} \]
\[ -e^{-\rho} \frac{1}{\rho^4} \left( \bar{\rho} + \frac{1}{2} \dot{\rho}^2 \right) \zeta \left[ \partial_k \partial_l \zeta \partial^k \partial^l \zeta - \partial_k \partial^k \zeta^2 \right] \]
\[ + \frac{2}{3} \partial_k \left\{ e^{-\rho} \frac{1}{\rho^3} \left[ \partial^k \zeta \partial_l \partial^l \zeta - \partial^k \zeta \partial_l \partial^l \zeta \right] - e^{-\rho} \frac{1}{\rho^3} \zeta \left[ \partial^k \zeta \partial_l \partial^l \zeta - \partial^k \zeta \partial_l \partial^l \zeta \right] \right\} \]
\[ + \frac{4}{3} \partial_k \left\{ e^{-\rho} \frac{1}{\rho^3} \left[ 2 \zeta \partial_k \partial^l \zeta \partial_l \partial^k \zeta - \partial^k \zeta \partial_l \partial^l \zeta \right] \right\} \]
\[ + \frac{1}{2} \partial_k \left\{ e^{-\rho} \frac{1}{\rho^3} \left[ 2 \zeta \partial_k \partial^l \zeta \partial_l \partial^k \zeta - \partial^k \zeta \partial_l \partial^l \zeta \right] \right\}. \quad (120) \]

\[ \text{V.B. The } e^\rho \text{-terms} \]

The next group to be examined is the set of \( e^\rho \) terms. This part of the action encompassed a total of seven interactions,
\[ 2\mathcal{L}^{(3)} \bigg|_{\rho} = -e^\rho \frac{\dot{\phi}^2}{\rho^3} \zeta \partial_k \zeta \partial^k \zeta + 2e^\rho \frac{\dot{\phi}^2}{\rho^3} \zeta \partial_k \zeta \partial^k \zeta + 3e^\rho \frac{\dot{\phi}^2}{\rho^3} \zeta \partial_k \zeta \partial^k \zeta - e^\rho \frac{\dot{\phi}^2}{\rho^3} \zeta \partial_k \zeta \partial^k \zeta \]
\[ -6e^\rho \frac{1}{\rho^2} \zeta \partial_k \partial_l \zeta \partial^k \partial^l \chi + 2e^\rho \frac{1}{\rho^2} \zeta \partial_k \partial_l \zeta \partial^k \partial^l \chi + 4e^\rho \frac{1}{\rho} \partial_k \zeta \partial^k \chi \partial_l \partial^l \zeta. \]
\[ (121) \]

Though the number of these terms is not much greater than that of the \( e^{-\rho} \) set just considered, their analysis is far more complicated.

The strategy for treating this set will be to concentrate on the second line first, the three terms that contain a \( \chi \) field. The middle term differs from the other two since it contains a time derivative acting on one of the \( \zeta \) fields, which will be integrated by parts. After doing so, some of the spatial derivatives will also be integrated, as the spatial derivatives are acting on different fields in the case of each of the terms. Only once the \( \chi \) terms have been thoroughly analyzed and put into a more useful form for later will the first line of the \( \mathcal{L}^{(3)} \bigg|_{\rho} \) Lagrangian be included among the rest.

So beginning with the \( \chi \) term that has the \( \dot{\zeta} \) factor, integrate it by parts until none of the terms with four spatial derivatives contain a factor of \( \dot{\zeta} \).
\[ 2e^\rho \frac{1}{\rho^2} \zeta \partial_k \partial_l \zeta \partial^k \partial^l \chi = -e^\rho \frac{\dot{\phi}^2}{\rho^3} \zeta \partial_k \partial_l \zeta \partial^k \partial^l \chi - e^\rho \frac{\dot{\phi}^2}{\rho^3} \zeta \partial_k \partial_l \zeta \partial^k \partial^l \chi - e^\rho \frac{\dot{\phi}^2}{\rho^3} \zeta \partial_k \partial_l \zeta \partial^k \partial^l \chi \]
\[ + \frac{1}{2} e^\rho \frac{\dot{\phi}^2}{\rho^3} \zeta \partial_k \partial_l \zeta \partial^k \partial^l \chi + \partial_k \left\{ e^\rho \frac{1}{\rho^2} \left[ \zeta \partial_k \partial_l \zeta \partial^k \partial^l \chi \right] \right\}. \quad (122) \]

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In deriving this expansion, the constraint condition,
\[ \partial_k \partial^k \chi = \frac{1}{2} \frac{\dot{\phi}^2}{\rho^2} \dot{\zeta}, \]  
has been used quite freely. The last term of the first line of this expansion can itself be rewritten as
\[ -e^\rho \frac{\dot{\phi}^2}{\rho^3} \partial_k \partial^k \partial^l \chi = -e^\rho \frac{\dot{\phi}^2}{\rho^3} \left[ \partial_k \partial^k \partial_l \chi \partial_l \partial^l \chi - \partial_k \partial_l (\partial^k \partial^l \chi) \right] - e^\rho \frac{\dot{\phi}^4}{\rho^5} \partial_k \partial^k \chi \]
\[ \partial_k \left\{ e^\rho \frac{\dot{\phi}^2}{\rho^3} \left[ \partial_l (\partial^k \partial^l \chi) + \partial^k \partial^l \partial^l \chi - \partial_l (\partial^l \partial^k \chi) \right] \right\}. \]  
(124)

Returning to the \( e^\rho \) Lagrangian, its last term can be put in a form that more closely resembles the others by reordering its spatial derivatives,
\[ 4e^\rho \frac{1}{\rho} \partial_k \partial^k \partial_l \chi \partial_l \partial^l \chi = 4e^\rho \frac{1}{\rho} \partial_k \partial^k \partial_l \chi \partial_l \partial^l \chi + 2e^\rho \frac{\dot{\phi}^2}{\rho^3} \partial_k \partial^k \partial^l \chi + e^\rho \frac{\dot{\phi}^2}{\rho^3} \partial_k \partial^k \partial^l \chi + e^\rho \frac{\dot{\phi}^2}{\rho^3} \partial_k \partial^k \partial^l \chi \]
\[ + \partial_k \left\{ 2e^\rho \frac{1}{\rho} \left[ 2\partial^k \partial_k \chi \partial^l \chi - \partial^k \partial_k \partial^l \chi - \partial^k \partial_k \partial^l \chi \right] \right\}. \]  
(125)

Assembling all the expanded versions of the \( \chi \)-terms found so far produces a rather lengthy expression,
\[ -6e^\rho \frac{1}{\rho} \partial_k \partial^k \partial_l \chi + 2e^\rho \frac{1}{\rho^2} \partial_k \partial^k \partial^l \chi + 4e^\rho \frac{1}{\rho} \partial_k \partial^k \partial^l \chi \]
\[ = -e^\rho \frac{1}{\rho^2} \partial_k \partial^k \partial^l \chi - 3e^\rho \frac{1}{\rho} \partial_k \partial^k \partial^l \chi \]
\[ - e^\rho \frac{\dot{\phi}^2}{\rho^3} \left[ \partial_k \partial^k \partial_l \chi \partial_l \partial^l \chi - \partial_k \partial_l (\partial^k \partial^l \chi) \right] \]
\[ + 2e^\rho \frac{\dot{\phi}^2}{\rho^3} \partial_k \partial^k \partial^l \chi + e^\rho \frac{\dot{\phi}^2}{\rho^3} \partial_k \partial^k \partial^l \chi - \frac{1}{2} e^\rho \frac{\dot{\phi}^4}{\rho^5} \partial_k \partial^k \partial^l \chi \]
\[ + \frac{\dot{d}}{dt} \left\{ e^\rho \frac{1}{\rho^2} \partial_k \partial^k \partial^l \chi \right\} + 2e^\rho \frac{1}{\rho^3} \left[ \dot{\rho} + \frac{1}{2} \dot{\phi}^2 \right] \partial_k \partial^k \partial^l \chi \]
\[ - \partial_k \left\{ e^\rho \frac{\dot{\phi}^2}{\rho^3} \left[ \partial_l (\partial^k \partial^l \chi) + \partial^k \partial^l \partial^l \chi - \partial_l (\partial^l \partial^k \chi) \right] \right\} \]
\[ + \partial_k \left\{ 2e^\rho \frac{1}{\rho} \left[ 2\partial^k \partial_k \chi \partial^l \chi - \partial^k \partial_k \partial^l \chi - \partial^k \partial_k \partial^l \chi \right] \right\} \]
\[ + \partial_k \left\{ e^\rho \frac{1}{\rho^3} \left[ \partial_l \partial^k \chi - \partial^k \partial^l \chi \right] \partial^k \partial^l \chi \right\}. \]  
(126)
Before including the rest of the order $e^\rho$ Lagrangian, integrate the terms on the first line of this equation by parts until they assume the following form,

$$-e^\rho \frac{1}{\rho^2} \zeta \partial_i \partial_j \partial_k \partial_l \chi - 3e^\rho \frac{1}{\rho^2} \zeta \partial_i \partial_j \partial_k \partial_l \chi$$

$$= -e^\rho \frac{1}{\rho^2} \zeta \partial_i \partial_j \partial_k \partial_l \chi + 3e^\rho \frac{1}{\rho^2} \zeta \partial_i \partial_j \partial_k \partial_l \chi$$

$$-e^\rho \frac{1}{\rho^2} \zeta \partial_i \partial_j \partial_k \partial_l \chi - 3e^\rho \frac{1}{\rho^2} \zeta \partial_i \partial_j \partial_k \partial_l \chi$$

$$-\partial_k \left\{ e^\rho \frac{1}{\rho^2} [\zeta \partial_i \zeta \partial_j \partial_k \partial_l \chi - \partial_k \zeta \partial_i \partial_j \partial_l \chi + \zeta \partial_l (\partial_k \partial_j \partial_l \chi)] \right\}$$

$$-\partial_k \left\{ 3e^\rho \frac{1}{\rho^2} [\zeta \partial_i \zeta \partial_j \partial_k \partial_l \chi - \partial_k \zeta \partial_i \partial_l \chi + \zeta \partial_l (\partial_k \partial_l \chi)] \right\}$$

$$+\partial_k \left\{ e^\rho \frac{1}{\rho^2} \zeta \partial_i \partial_j \partial_l \chi \right\} + \partial_k \left\{ \frac{3}{2} e^\rho \frac{2}{\rho^2} \zeta \partial_i \partial_j \partial_l \chi \right\}. \quad (127)$$

Again, the constraint equation has been applied to rewrite $\partial_k \partial^k \chi$ in terms of $\zeta$. Next, formally invert the spatial Laplacian operator, $\partial^{-2}(\partial_k \partial^k) = 1$, in the constraint so that it becomes

$$\chi = \frac{1}{2} \frac{\phi^2}{\rho^2} \partial^{-2} \zeta. \quad (128)$$

This form of the constraint allows the first line of the equation before the last to be written as

$$-e^\rho \frac{1}{\rho^2} \zeta \partial_i \zeta \partial^l \zeta \partial^l \zeta = e^\rho \frac{1}{\rho^2} \zeta \partial_i \zeta \partial^l \zeta + e^\rho \frac{1}{\rho^2} \zeta \partial_i \zeta \partial^l \zeta$$

$$= 1 e^{-2\rho} d \left\{ e^{3\rho} \frac{\phi^2}{\rho^2} \zeta \right\} [\partial_i \zeta \partial^k \zeta - \partial^k \partial_i (\partial_k \zeta \partial^l \zeta)] + \partial^{-1}\text{-terms}. \quad (129)$$

after formally integrating the $\partial^{-2}$ operator and applying the identity,

$$\left( \frac{d}{dt} + 3\rho \right) \left[ \frac{\phi^2}{\rho^2} \zeta \right] = e^{-3\rho} \frac{d}{dt} \left\{ e^{3\rho} \frac{\phi^2}{\rho^2} \zeta \right\}. \quad (130)$$

At last it is possible to assemble all the rest of the order $e^\rho$ terms (those that did not contain the $\chi$ field) together with what was derived so far to find

$$2 \mathcal{L}^{(3)} \big|_{e^\rho} = -\frac{1}{2} e^{-2\rho} d \left\{ e^{3\rho} \frac{\phi^2}{\rho^2} \zeta \right\} [\partial_i \zeta \partial^k \zeta - \partial^k \partial_i (\partial_k \zeta \partial^l \zeta)]$$

$$-\frac{1}{2} e^{-2\rho} \left\{ \partial_i \zeta \partial^k \zeta \partial^l \zeta - \zeta \partial_i \partial_l (\partial^k \partial^l \chi) \right\}$$

$$-\frac{1}{2} e^{-2\rho} \left\{ \partial_i \zeta \partial^k \zeta + 3 \frac{\phi^2}{\rho^2} \zeta \partial_i \partial_k \zeta + 3 \frac{1}{2} e^\rho \frac{2}{\rho^2} \zeta \partial_i \partial_k \zeta \right\}$$

$$+2 e^\rho \frac{\phi^2}{\rho^2} \zeta \partial_i \zeta \partial_k \zeta - 2 e^\rho \frac{\phi^2}{\rho^2} \zeta \partial_i \partial_k \zeta - e^\rho \frac{1}{\rho^2} \zeta \partial_i \partial_k \zeta \partial^l \chi$$

$$-e^\rho \frac{\phi^2}{\rho^2} \zeta \partial_i \zeta \partial^l \zeta + \frac{1}{2} e^\rho \frac{\phi^2}{\rho^2} \zeta \partial_i \zeta \partial^l \zeta - \frac{1}{2} e^\rho \frac{\phi^2}{\rho^2} \zeta \partial_i \partial_k \zeta \partial^l \chi + \cdots. \quad (131)$$
V.C. The first $e^{3p}$ term

There remain only the $e^{3p}$ terms. This set will be broken in two, with one of the terms being treated by itself,

$$e^{3p} \frac{\dot{\phi}^2}{\rho^2} \left( 3\zeta - \frac{\dot{\zeta}}{\rho} \right) \dot{\zeta}^2.$$  

It produces the leading contribution to the three-point function that does not contain any spatial derivatives. Superficially it is proportional to $\epsilon$; but as before, this appearance is deceptive.
Start this time by integrating one of the $\zeta$ factors in the second term by parts,

$$e^{3p} \frac{\dot{\phi}^2}{\rho^2} \left( 3\zeta - \frac{\dot{\zeta}}{\rho} \right) \dot{\zeta}^2 = 3e^{3p} \frac{\dot{\phi}^2}{\rho^2} \dot{\zeta}^2 - e^{3p} \frac{\dot{\phi}^2}{\rho^3} \dot{\zeta} \frac{d}{dt} \zeta$$

$$= e^{3p} \frac{\dot{\phi}^2}{\rho^2} \left[ \frac{\dot{\dot{\phi}}}{\phi} \dot{\phi} + \frac{3}{4} \phi \frac{\dot{\phi}}{\rho} \right] \dot{\zeta}^2 - e^{3p} \frac{\dot{\phi}^2}{\rho^3} \left[ \frac{\dot{\phi}}{\rho} + \frac{1}{2} \phi^2 \right] \frac{d}{dt} \zeta.$$

The dynamical term looks a bit like the time-derivative piece of the $\zeta$ equation of motion,

$$e^{3p} \frac{\dot{\phi}^2}{\rho^2} \left[ \frac{\dot{\dot{\phi}}}{\phi} + \frac{3}{4} \phi \frac{\dot{\phi}}{\rho} + 3\rho \frac{\dot{\phi}}{\rho} \right] = \frac{d}{dt} \left\{ e^{3p} \frac{\dot{\phi}^2}{\rho^2} \right\} + 2e^{3p} \frac{\dot{\phi}^2}{\rho^3} \left[ \frac{\dot{\phi}}{\rho} + \frac{1}{2} \phi^2 \right] \dot{\zeta},$$

though with a few differences in some of the coefficients of the terms in the brackets. Adding and subtracting the necessary pieces yields

$$e^{3p} \frac{\dot{\phi}^2}{\rho^2} \left( 3\zeta - \frac{\dot{\zeta}}{\rho} \right) \dot{\zeta}^2$$

$$= \frac{d}{dt} \left\{ e^{3p} \frac{\dot{\phi}^2}{\rho^2} \dot{\zeta}^2 \right\} + 2e^{3p} \frac{\dot{\phi}^2}{\rho^3} \left[ \frac{\dot{\phi}}{\rho} + \frac{1}{2} \phi^2 \right] \dot{\zeta}^2 \zeta.$$

It might not be immediately obvious why the last term on the first line was explicitly extracted, since it is already proportional to one of the pieces in the second term, but it will produce a term that matches with one already present on the second line of the definition for $\mathcal{F}_B$, which contains the spatial-derivative part of the $\zeta$ equation of motion.

Notice that the second of the dynamical terms looks as though it is second order in the slowly rolling parameters, although it is actually of even higher order. To make this fact apparent, integrate a factor $\frac{\dot{\zeta}}{\rho}$ by parts, so that

$$-2e^{3p} \frac{\dot{\phi}^2}{\rho^2} \left[ \frac{\dot{\phi}}{\phi} + \frac{1}{2} \frac{\dot{\phi}}{\rho^2} \right] \dot{\zeta}^2 \zeta = -e^{3p} \frac{\dot{\phi}^2}{\rho^2} \zeta \left[ \frac{\dot{\phi}}{\phi} \zeta + \frac{1}{2} \frac{\dot{\phi}}{\rho^2} \right] \frac{d}{dt} \zeta$$

$$= e^{3p} \frac{\dot{\phi}^2}{\rho^2} \dot{\zeta} \frac{d}{dt} \zeta \left[ \frac{\dot{\phi}}{\phi} \zeta + \frac{1}{2} \frac{\dot{\phi}}{\rho^2} \right]$$

$$+ \frac{d}{dt} \left\{ e^{3p} \frac{\dot{\phi}^2}{\rho^2} \dot{\zeta} \right\} \left[ \frac{\dot{\phi}}{\phi} \zeta^2 + \frac{1}{2} \frac{\dot{\phi}}{\rho^2} \dot{\zeta}^2 \right]$$

$$- \frac{d}{dt} \left\{ e^{3p} \frac{\dot{\phi}^2}{\rho^2} \left[ \frac{\dot{\phi}}{\phi} \zeta^2 + \frac{1}{2} \frac{\dot{\phi}}{\rho^2} \dot{\zeta}^2 \right] \right\}.$$

What has thus been learned is that the first term in the cubic action can be rewritten as

$$e^{3p} \frac{\dot{\phi}^2}{\rho^2} \left( 3\zeta - \frac{\dot{\zeta}}{\rho} \right) \dot{\zeta}^2 = \frac{1}{2} e^{3p} \frac{\dot{\phi}^4}{\rho^4} \dot{\zeta}^2 \zeta + e^{3p} \frac{\dot{\phi}^2}{\rho^2} \zeta \frac{d}{dt} \left[ \frac{\dot{\phi}}{\phi} \zeta + \frac{1}{2} \frac{\dot{\phi}}{\rho^2} \dot{\zeta}^2 \right] \zeta + \mathcal{F}_C + \mathcal{F}_C,$$
where

$$\mathcal{D}_C = -\frac{d}{dt} \left\{ e^{3p} \frac{\dot{\phi}^2}{\rho^2} \left[ \frac{\dot{\phi}}{\rho} \right] + \frac{1}{2} \frac{\dot{\phi}^2}{\rho^2} \right\} \zeta \dot{\zeta}^2 + e^{3p} \frac{\dot{\phi}^2}{\rho^2} \left[ \rho + \frac{1}{2} \frac{\dot{\phi}^2}{\rho} \right] \dot{\zeta}^2 \zeta \right\}$$

(141)

is something that has no dynamical effect in an inflationary background, and where

$$\mathcal{F}_C = \frac{d}{dt} \left\{ e^{3p} \frac{\dot{\phi}^2}{\rho^2} \left[ \frac{\dot{\phi}}{\rho} \right] + \frac{1}{2} \frac{\dot{\phi}^2}{\rho^2} \right\} \left[ \frac{\dot{\phi}}{\rho} \right] \zeta \dot{\zeta}^2 + e^{3p} \frac{\dot{\phi}^2}{\rho^2} \left[ \rho + \frac{1}{2} \frac{\dot{\phi}^2}{\rho} \right] \dot{\zeta}^2 \zeta \right\}$$

(142)

will eventually contribute to a term that vanishes when the $\zeta$ equation of motion is imposed, once it has been combined with an appropriate term in $\mathcal{F}_B$.

**V.D. The remaining $e^{3p}$ terms**

There is left just one final set to examine,

$$2L^{(3)}_{\nu p} = 3e^{3p} \xi \partial_k \partial_l \chi \partial^k \partial^l \chi - \frac{e^{3p}}{\rho} \xi \partial_k \partial_l \chi \partial^k \partial^l \chi - \frac{3}{4} e^{3p} \frac{\dot{\phi}^4}{\rho^2} \zeta \dot{\zeta}^2 \zeta + \frac{1}{4} e^{3p} \frac{\dot{\phi}^4}{\rho^2} \zeta^2$$

(143)

those proportional to $e^{3p}$, aside from the one that was just evaluated. By now, the strategy for rearranging the terms should more or less be clear, and this same strategy will be applied to this set too.

Begin by integrating the second term by parts to remove the time derivative from the $\zeta$ and then integrate further some of the spatial derivatives to remove them from the term that contains a $\chi$ factor,

$$-e^{3p} \frac{1}{\rho} \xi \partial_k \partial_l \chi \partial^k \partial^l \chi = e^{3p} \frac{1}{\rho} \xi \partial_k \partial_l \chi \partial^k \partial^l \chi + 3e^{3p} \xi \partial_k \partial_l \chi \partial^k \partial^l \chi$$

(144)

$$-e^{3p} \frac{1}{\rho} \xi \partial_k \partial_l \chi \partial^k \partial^l \chi - 2e^{3p} \frac{1}{\rho} \xi \partial_k \partial^k \partial^l \chi$$

$$-\frac{d}{dt} \left\{ e^{3p} \frac{1}{\rho} \xi \partial_k \partial_l \chi \partial^k \partial^l \chi \right\} - e^{3p} \frac{1}{\rho} \left[ \rho + \frac{1}{2} \frac{\dot{\phi}^2}{\rho} \right] \xi \partial_k \partial_l \chi \partial^k \partial^l \chi$$

$$+ \partial_k \left\{ e^{3p} \frac{1}{\rho} \left[ \xi \partial_k \partial^k \partial^l \chi - \partial^k \xi \partial_k \partial^l \chi \right] + \chi \partial_l (\partial^l \xi \partial^k \chi) \right\}.$$  

The second term in this expression is the same as the first term in $L^{(3)}_{\nu p}$. Consider the first two terms of the order $e^{3p}$ Lagrangian together, and integrate the spatial derivatives on one of the $\chi$ fields by parts, to produce

$$3e^{3p} \xi \partial_k \partial_l \chi \partial^k \partial^l \chi - e^{3p} \frac{1}{\rho} \xi \partial_k \partial_l \chi \partial^k \partial^l \chi$$

$$= e^{3p} \frac{\dot{\phi}^4}{\rho^2} \xi \partial_k \partial_l \chi \partial^k \partial^l \chi + 2e^{3p} \frac{1}{\rho} \xi \partial_k \partial^k \partial^l \chi \partial^l \chi + 3\rho \chi$$

$$+ 2e^{3p} \frac{1}{\rho} \partial_k \xi \partial^k \chi \partial^l \chi \partial^l \chi - 2e^{3p} \frac{1}{\rho} \partial_k \xi \partial^k \chi \partial^l \chi = 0$$
Putting all the terms together, one obtains
\begin{align*}
&-\frac{d}{dt}\left\{ e^{3p} \frac{1}{\rho} \xi \partial_k \partial_l \partial^k \partial^l \chi \right\} - e^{3p} \frac{1}{\rho^2} \left[ \rho + \frac{1}{2} \phi^2 \right] \xi \partial_k \partial_l \partial^k \partial^l \chi \\
&+ \partial_k \left\{ 2 e^{3p} \frac{1}{\rho} \left[ \xi \partial_l \partial^k \partial^l \chi - \partial^k \xi \partial_l \partial^l \chi + \chi \partial_l (\partial^l \xi \partial^k \chi) \right] \right\} \\
&+ \partial_k \left\{ 6 e^{3p} \left[ \xi \partial_l \partial^k \partial^l \chi - \partial^k \xi \partial_l \partial^l \chi + \chi \partial_l (\partial^l \xi \partial^k \chi) \right] \right\} \\
&- \partial_k \left\{ 2 e^{3p} \frac{1}{\rho} \xi \partial^k \partial^l \partial^l [\chi + 3 \rho \chi] \right\}.
\end{align*}
(145)

Now apply the same trick that was used for the order $e^0$ terms, where some of the $\chi$ fields were replaced with
\begin{equation}
\chi = \frac{1}{2} \frac{\phi^2}{\rho^2} \partial^{-2} \zeta
\end{equation}
(146)
and then integrate to remove the $\partial^{-1}$ operators from the time derivative of the $\zeta$ field, to arrive at
\begin{align*}
3 e^{3p} \xi \partial_k \partial_l \partial^k \partial^l \chi - e^{3p} \frac{1}{\rho} \xi \partial_k \partial_l \partial^k \partial^l \chi &= \frac{1}{2} e^{3p} \frac{\phi^2}{\rho^2} \xi \partial_k \partial_l \partial^k \partial^l \chi + \frac{1}{2} \phi^2 \frac{d}{dt} \left\{ e^{3p} \frac{\phi^2}{\rho^2} \xi \right\} \\
&+ \frac{1}{\rho} \frac{d}{dt} \left\{ e^{3p} \frac{\phi^2}{\rho^2} \xi \right\} \left[ \partial_k \xi \partial_l \partial^k \partial^l \chi - \partial^{-2} \partial_k \partial_l (\partial^k \zeta \partial^l \chi) \right] \\
&+ \partial^{-1} \text{ terms and } \cdots.
\end{align*}
(147)

If the last two terms on the first line of $\mathcal{L}^{(3)} |_{e^0}$ are combined with the second term of this last expression, together they yield
\begin{align*}
&\frac{3}{4} e^{3p} \frac{\phi^4}{\rho^4} \zeta \partial^2 \zeta + \frac{1}{4} e^{3p} \frac{\phi^4}{\rho^3} \zeta^2 + \frac{1}{2} \phi^2 \frac{d}{dt} \left\{ e^{3p} \frac{\phi^2}{\rho^2} \zeta \right\} \\
&= \frac{1}{8} e^{3p} \frac{\phi^6}{\rho^6} \zeta \partial^2 \zeta + \frac{d}{dt} \left\{ \frac{1}{4} e^{3p} \frac{\phi^4}{\rho^3} \zeta \partial^2 \zeta \right\} + \frac{1}{4} e^{3p} \frac{\phi^4}{\rho^6} \left[ \rho + \frac{1}{2} \phi^2 \right] \zeta \partial^2 \zeta.
\end{align*}
(148)

Putting all the terms together, one obtains
\begin{equation}
2 \mathcal{L}^{(3)} |_{e^0} = -2 e^{3p} \frac{\phi^2}{\rho^2} \zeta \partial_k \xi \partial^k \chi + \frac{1}{2} e^{3p} \frac{\phi^2}{\rho^2} \zeta \partial_k \partial_l \partial^k \partial^l \chi - \frac{1}{8} e^{3p} \frac{\phi^6}{\rho^6} \zeta \partial^2 \zeta + \mathcal{F}_D + \mathcal{D}_D,
\end{equation}
(149)

with
\begin{equation}
\mathcal{F}_D = \frac{1}{\rho} \frac{d}{dt} \left\{ e^{3p} \frac{\phi^2}{\rho^2} \zeta \right\} \left[ \partial_k \xi \partial^k \chi - \partial^{-2} \partial_k \partial_l (\partial^k \zeta \partial^l \chi) \right]
\end{equation}
(150)

and
\begin{align*}
\mathcal{D}_D &= \frac{d}{dt} \left\{ \frac{1}{4} e^{3p} \frac{\phi^4}{\rho^3} \zeta \partial^2 \zeta \right\} + \frac{1}{4} e^{3p} \frac{\phi^4}{\rho^6} \left[ \rho + \frac{1}{2} \phi^2 \right] \zeta \partial^2 \zeta \\
&- \frac{d}{dt} \left\{ e^{3p} \frac{1}{\rho} \xi \partial_k \partial_l \partial^k \partial^l \chi \right\} - e^{3p} \frac{1}{\rho^2} \left[ \rho + \frac{1}{2} \phi^2 \right] \xi \partial_k \partial_l \partial^k \partial^l \chi.
\end{align*}
results of the four groupings of the cubic terms yields the following set of interactions

\[
\begin{align*}
+ \delta \left\{ 2e^{3p} \frac{1}{\rho} \left[ \xi \partial_i \xi \partial^i \chi - \partial^k \xi \partial_k \chi \partial^i \chi + \chi \partial_i (\partial^i \xi \partial^k \chi) \right] \right\} \\
+ \delta \left\{ 6e^{3p} \left[ \xi \partial_i \xi \partial^i \chi - \partial^k \xi \partial_i \chi \partial^i \chi + \chi \partial_i (\partial^i \xi \partial^k \chi) \right] \right\} \\
- \delta \left\{ 2e^{3p} \frac{1}{\rho^5} \xi \partial^k \chi \partial_i \partial^i [\dot{\chi} + 3\dot{\rho} \chi] \right\} + \partial^{-1}\text{-terms.}
\end{align*}
\]

Note that the second dynamical term was not altered at all from its form in the original interaction.

\textbf{V.E. Reassembling the cubic action}

Finally, the results for each of these separate calculations for the different pieces of the cubic action must be reassembled to see what is its true size in terms of the slowly rolling parameters. Most importantly, it is still necessary to show that when all of the \( \mathcal{F} \)-terms are combined, they too produce no dynamical effect. Adding together the results of the four groupings of the cubic terms yields the following set of interactions

\[
S^{(3)} = \frac{1}{2} \int d^4x \left\{ \frac{1}{2} e^{3p} \frac{\dot{\phi}^4}{\rho^4} \xi^2 \zeta + \frac{1}{2} e^{3p} \frac{\dot{\phi}^4}{\rho^4} \frac{\dot{\xi}}{\rho^2} \partial_k \xi \partial^k \zeta - 2e^{3p} \frac{\dot{\phi}^2}{\rho^2} \xi \partial_k \xi \partial^k \chi \\
+ 6e^{3p} \frac{\dot{\phi}^2}{\rho^2} \xi \partial^k \chi + \frac{1}{2} e^{3p} \frac{\dot{\phi}^6}{\rho^6} \xi^2 \zeta + \frac{1}{2} e^{3p} \frac{\dot{\phi}^2}{\rho^2} \partial_k \xi \partial^k \chi \partial^i \chi \\
+ \mathcal{F} + \mathcal{D} \right\},
\]

where \( \mathcal{D} \) represents the accumulated total derivative terms,

\[
\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_A + \mathcal{D}_B + \mathcal{D}_C + \mathcal{D}_D,
\]

together with the terms the vanish when \( \rho = -\frac{1}{2} \dot{\phi}^2 \).

Analogously, \( \mathcal{F} \) represents the sum terms that vanishes upon imposing the equation of motion for \( \zeta \),

\[
\frac{d}{dt} \left[ e^{3p} \frac{\dot{\phi}^2}{\rho^2} \xi \right] - e^{3p} \frac{\dot{\phi}^2}{\rho^2} \partial_k \partial^k \zeta = 0.
\]

Together, these terms are

\[
\mathcal{F} = \mathcal{F}_A + \mathcal{F}_B + \mathcal{F}_C + \mathcal{F}_D
\]

\[
= \left\{ \frac{d}{dt} \left[ e^{3p} \frac{\dot{\phi}^2}{\rho^2} \xi \right] - e^{3p} \frac{\dot{\phi}^2}{\rho^2} \partial_k \partial^k \zeta \right\} \left[ \frac{\dot{\phi}}{\rho} e^2 + \frac{1}{2} \frac{\dot{\phi}^2}{\rho^2} s^2 + 2 \frac{1}{\rho^3} \xi \right]
\]

\[
\frac{d}{dt} \left[ e^{3p} \frac{\dot{\phi}^2}{\rho^2} \xi \right] \left\{ \frac{1}{\rho} \left[ \partial_k \xi \partial^k \chi - \partial^{-2} \partial_k \partial_i (\partial^k \xi \partial^i \chi) \right] \\
- \frac{1}{2} \left[ e^{-2p} \partial_k \xi \partial^k \chi - \partial^{-2} \partial_k \partial_i (\partial^k \xi \partial^i \chi) \right] \right\}
\]

\[
- e^{3p} \frac{\dot{\phi}^2}{\rho^2} \partial_k \xi \partial^k \chi \partial^i \zeta - \xi \partial_k \partial_i (\partial^k \xi \partial^i \chi) \]

\[
+ \frac{1}{2} e^{3p} \frac{\dot{\phi}^2}{\rho^2} \partial_k \xi \partial^k \chi \partial^i \zeta - \xi \partial_k \partial_i (\partial^k \xi \partial^i \chi) \right\}.
\]
The first line is already proportional to the $\zeta$ equation of motion, but the last two lines do not yet quite match with the two preceding them; but they can be converted by taking two of the $\zeta$’s in them and inserting a factor of the identity operator in the form, $\partial^{-2}\partial_i \partial^k \zeta$, and then integrating by parts one last time so that these lines become,

$$\begin{align*}
&= \cdots \\
&\quad + \left\{ -e^o \dot{\phi}^2 \rho^2 \partial_i \partial^k \zeta \right\} \frac{1}{\rho} \left[ \partial_i \zeta \partial^j \chi - \partial^{-2} \partial_i \partial_l (\partial^j \zeta \partial^l \chi) \right] \\
&\quad - \left\{ -e^o \dot{\phi}^2 \rho^2 \partial_i \partial^k \zeta \right\} \frac{1}{2} \frac{e^{-2p}}{\rho^2} \left[ \partial_i \zeta \partial^j \chi - \partial^{-2} \partial_i \partial_l (\partial^j \zeta \partial^l \chi) \right].
\end{align*}$$

(156)

The expression for $\mathcal{F}$ then becomes,

$$\begin{align*}
\mathcal{F} &= \left\{ \frac{d}{dt} \left[ e^o \dot{\phi}^2 \rho^2 \zeta \right] - e^o \dot{\phi}^2 \rho^2 \partial_i \partial^k \zeta \right\} \left[ \frac{\dot{\phi}}{\rho} \zeta + \frac{1}{2} \frac{\dot{\phi}^2}{\rho^2} \zeta^2 + \frac{1}{2} \frac{\dot{\phi} \zeta}{\rho} \right] \\
&\quad + \left\{ \frac{d}{dt} \left[ e^p \dot{\phi}^2 \rho^2 \zeta \right] - e^p \dot{\phi}^2 \rho^2 \partial_i \partial^k \zeta \right\} \frac{1}{\rho} \left[ \partial_i \zeta \partial^j \chi - \partial^{-2} \partial_i \partial_l (\partial^j \zeta \partial^l \chi) \right] \\
&\quad - \left\{ \frac{d}{dt} \left[ e^p \dot{\phi}^2 \rho^2 \zeta \right] - e^p \dot{\phi}^2 \rho^2 \partial_i \partial^k \zeta \right\} \frac{1}{2} \frac{e^{-2p}}{\rho^2} \left[ \partial_i \zeta \partial^j \chi - \partial^{-2} \partial_i \partial_l (\partial^j \zeta \partial^l \chi) \right] \\
&\quad + \partial^{-1}\text{-terms.}
\end{align*}$$

(157)

In this form, $\mathcal{F}$ plainly has no dynamical effect.

What remains is just a set of cubic interactions which is small in the slowly rolling limit. Rewriting $S^{(3)}$ in terms of $\varepsilon$ and $\delta$, the cubic action becomes

$$S^{(3)} = \int d^4 x \left\{ \varepsilon^2 e^o \dot{\phi}^2 \zeta^2 + \varepsilon^2 e^p \dot{\phi} \zeta \partial_k \zeta - 2 \varepsilon e^o \dot{\phi} \zeta \partial_i \zeta \partial^k \chi \\
+ \varepsilon (\delta + \varepsilon) e^p \dot{\phi} \zeta^2 - \frac{1}{2} \varepsilon^2 e^o \dot{\phi} \zeta \partial_k \partial_l \partial^k \chi + \varepsilon e^p \partial_k \partial_l \partial^k \chi + \cdots \right\}.$$  

(158)

When counting the number of powers of $\varepsilon$ or $\delta$, it is important to remember that $\chi$ is itself of order $\varepsilon$,

$$\chi = \frac{1}{2} \frac{\dot{\phi}^2}{\rho^2} \zeta = \varepsilon \partial^{-2} \zeta.$$  

(159)

Thus, for $\varepsilon, \delta \ll 1$, the terms that generate the leading non-Gaussian component in the primordial fluctuations are just the following three interactions,

$$S^{(3)} = \int dt \varepsilon^2 e^o \int d^3 \vec{x} \left\{ \zeta^2 \dot{\zeta} + e^{-2p} \zeta \partial_k \zeta \partial^k \zeta - 2 \zeta \partial_k \zeta \partial^k (\partial^{-2} \zeta) + O(\varepsilon, \delta) \right\},$$  

(160)

an extremely simple result given the complexity of the intermediate steps needed to arrive here.

The last step is to convert this action, expressed in terms of the physical fluctuation $\zeta(t, \vec{x})$, into a corresponding interaction Hamiltonian, $H_I(\eta)$, for the canonically normalized field $\phi(\eta, \vec{x})$. The two fields, and their time derivatives, are related by classical, spatially independent functions,

$$\zeta = e^{-p} \frac{\dot{\phi}}{\phi} \phi$$

$$\dot{\zeta} = e^{-p} \frac{\dot{\phi}}{\phi} \left[ \phi - \rho (1 + \varepsilon + \delta) \phi \right].$$  

(161)
Noting that $\dot{\phi}/\rho = \sqrt{2\varepsilon}$ and converting the time coordinate into the conformal time coordinate on the right sides of these equations, the leading behavior in the slowly rolling limit for each is

$$\zeta = -H\eta \frac{1}{\sqrt{2\varepsilon}} \phi + \cdots$$
$$\dot{\zeta} = H^2\eta \frac{1}{\sqrt{2\varepsilon}} (\eta \phi)' + \cdots. \quad (162)$$

Substituting these expressions into the leading, order $\varepsilon^2$, terms of the cubic action produces

$$S^{(3)} = -\int \frac{d\eta}{\eta} \sqrt{\frac{\varepsilon}{8}} H \int d^3\vec{x} \left\{ (\eta \phi)'(\eta \phi)' \phi + \eta^2 \phi \partial_k \phi \partial^k \phi - 2(\eta \phi)' \partial_k \phi \partial^k [\partial^{-2}(\eta \phi)'] + O(\varepsilon, \delta) \right\}, \quad (163)$$

so that the resulting interaction Hamiltonian becomes

$$H_I(\eta) = \frac{\sqrt{\varepsilon} H}{\sqrt{8} \eta} \int d^3\vec{x} \left\{ (\eta \phi)'^2 \phi + \eta^2 \phi \partial_k \phi \partial^k \phi - 2(\eta \phi)' \partial_k \phi \partial^k [\partial^{-2}(\eta \phi)'] + \cdots \right\}, \quad (164)$$

up to further corrections which are suppressed in the slowly rolling limit.

**AFTERWORD**

A part of the material covered in these notes was originally presented to a few of the members of the Discovery Center at the Niels Bohr Institute. These notes are not meant to provide an introduction to the theory of inflation, for which many textbooks and reviews are available [4–5], nor to treat exhaustively the origin of non-Gaussianities in the inflationary picture, which is also thoroughly reviewed elsewhere [6]. Rather, they are intended just to provide a detailed, pedagogical introduction to Maldacena’s calculation of the cubic interactions in the inflationary action, which in ordinary circumstances are the most easily observed signal of a non-Gaussian component within the pattern of the primordial fluctuations produced by inflation. Therefore they provide an important and testable prediction of the inflationary picture.

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