Abstract

In this paper, we study scalable algorithms for influence maximization with general marketing strategies (IM-GMS), in which a marketing strategy mix is modeled as a vector \( \mathbf{x} = (x_1, \ldots, x_d) \) and could activate a node \( v \) in the social network with probability \( h_v(\mathbf{x}) \). The IM-GMS problem is to find the best strategy mix \( \mathbf{x}^* \) that maximize the influence spread due to influence propagation from the activated seeds, subject to the budget constraint that \( \sum_{j \in [d]} x_j \leq k \). We adapt the scalable reverse influence sampling (RIS) approach and design a scalable algorithm that provides a \((1 - 1/e - \varepsilon)\) approximate solution (for any \( \varepsilon > 0 \)), with running time near-linear in the network size. We further extend IM-GMS to allow partitioned budget constraint, and show that our scalable algorithm provides a \((1/2 - \varepsilon)\) solution in this case. Through extensive experiments, we demonstrate that our algorithm is several orders faster than the Monte Carlo simulation based hill-climbing algorithm, and also outperforms other baseline algorithms proposed in the literature.

1 Introduction

The classical influence maximization task is to find a small set of seed nodes to maximize the expected number of activated nodes from these seeds, referred to as the influence spread, based on certain diffusion process in a social network [20]. It models the viral marketing scenario in social networks and its variants also find applications in diffusion monitoring, rumor control, crime prevention, etc. (e.g., [21, 4, 17, 26]). Therefore, numerous studies on influence maximization have been conducted since its inception. One important direction is scalable influence maximization, which aims at design efficient approximation algorithms and heuristics for large social networks. Many diverse approaches including graph theoretic heuristics, sketching methods, and random sampling have been tried for scalable influence maximization (e.g., [7, 15, 32, 3, 31, 30, 9], see more discussion in Section 1.1). Other directions include competitive and complementary influence maximization [4, 17, 23], continuous-time influence maximization [13], topic-aware influence maximization [5], seed minimization [22, 14], and many more.
However, one direction that has been introduced in the original paper [20] receives much less attention and is left largely unexplored. This is the problem of influence maximization with general marketing strategies (IM-GMS). [20] consider viral marketing scenarios with a general marketing strategy mix of $d$ different strategies, with each strategy $j$ taking value $x_j$ (e.g., money put into strategy $j$). The combined strategy mix is a vector $\mathbf{x} = (x_1, x_2, \ldots, x_d)$. When applying the strategy mix $\mathbf{x}$ to the social network, each node $v$ in the social network has a probability of $h_v(\mathbf{x})$ to be activated as a seed. After the seeds are probabilistically activated by the marketing strategies, influence propagates from the seeds in the network as dictated by an influence diffusion model. The IM-GMS problem is to find the best strategy mix $\mathbf{x}^*$ that maximizes the influence spread subject to the budget constraint $\sum_{j \in [d]} x_j \leq k$.

Arguably the general marketing strategies model more realistic scenarios, since in practice companies often apply a mix of marketing strategies, such as coupons, direct mails, marketing events, and target at different segments of users. [20] outline the basic approach based on submodularity and greedy hill-climbing algorithm to solve the problem. This direction, however, has not been further investigated in the research community. The only relevant study we find is by [33], who investigate influence maximization with fractional or continuous discounts on users in the network, a special case of the IM-GMS problem.

In this paper, we provide a detailed study of the IM-GMS problem, in particular how to scale up its solutions. It is well known that the naive greedy approach for influence maximization is not scalable due to excessive Monte Carlo simulations. The problem could be even worse for IM-GMS when we have a large strategy space with complicated interactions with the social network. We tackle this problem by adapting the reverse influence sampling (RIS) approach [3, 31, 30], which is successful for the classical influence maximization problem. Due to the complicated interaction between strategies and network propagations, the adaptation is nontrivial. We first prove several important properties that enable the RIS approach in the IM-GMS setting. Then we identify a large class of IM-GMS problems in which each strategy could independently impact nodes in the social network. We show that this class of problem could cover many practical application scenarios including user segment marketing, personalize marketing, and repeated event marketing, etc. For this class of problems we carefully design a more efficient hill-climbing algorithm integrating with RIS, named IMM-GMS, and analyze its time complexity and prove that it provides $1 - 1/e - \varepsilon$ approximation to the IM-GMS problem, for any $\varepsilon > 0$.

Furthermore, we generalize IM-GMS originally proposed by [20] to accommodate partitioned budgets (denoted as the IM-GMS-PB problem), that is, the strategies are partitioned into groups and each group has a separate budget. This matches the practical scenario when marketing activities are coordinated by multiple parties, each of which focusing on different marketing channels with different marketing budgets. We connect the IM-GMS-PB problem with submodular maximization under matroid constraints, and show that a minor variant of the IMM-GMS algorithm achieves $1/2 - \varepsilon$ approximation.

Finally, we conduct extensive experiments of our algorithm on four real-world networks with two different type of marketing strategies. Our experimental results demonstrate that our algorithm IMM-GMS is several orders of magnitude faster than the Monte Carlo hill-climbing algorithm, and it also outperforms the algorithms proposed in [33] for personalized discount applications in both influence spread and running time.

In summary, we make the following contributions: (a) we propose a scalable algorithm to solve the IM-GMS problem with theoretical guarantees; (b) we extend the problem of IM-GMS to the
case of partitioned budgets, and show that our scalable algorithm can still provide constant approximation; and (c) we demonstrate through experiments that our algorithm is effective and efficient, and can be easily adapted between providing theoretical guarantee and practical performance.

1.1 Related Work

Influence maximization for viral marketing is first studied as a data mining task by [10, 25]. [20] are the first to formulate the problem as a discrete optimization problem. They propose the independent cascade (IC) model, linear threshold (LT) model, and their generalized versions, study their submodularity, and propose the greedy hill-climbing algorithm that gives $1 - 1/e - \varepsilon$ approximate solution for $\varepsilon > 0$. They also propose the IM-GMS problem and the hill-climbing approach to solve the problem.

Scalable influence maximization is an important direction receives many attention. Some early proposals rely on the properties of the IC and LT models as well efficient graph algorithms to design scalable heuristics [8, 32, 15, 18]. [3] propose the novel approach of reverse influence sampling (RIS), which is then improved in [31, 30]. The RIS approach is able to provide both theoretical guarantee and scalable performance in practice. One of the state-of-the-art RIS-based algorithms is IMM [30]. Our algorithm is based on the RIS and is adapted from the IMM algorithm. Other approaches such as sketching [9] are also explored.

Beyond scalable influence maximization, many other variants have been studies, such as competitive and complementary influence maximization [4, 17, 23], continuous-time influence maximization [13], topic-aware influence maximization [5], seed minimization [22, 14], etc. They are less relevant to our study, so we refer to a monograph [6] for more comprehensive coverage on influence maximization.

In terms of the IM-GMS problem, the only relevant study we are aware of is the continuous influence maximization problem by [33]. In their model, each user could receive a personalized discount, which is translated to the probability of the user being activated as a seed. This corresponds to the personalized marketing scenario in our setting. They propose a scalable heuristic algorithm based on coordinate decent to solve the problem. Comparing to their study, our algorithm has several advantages: (a) our algorithm provides theoretical guarantees on approximation ratio and running time; (b) we could solve a larger class of problems covering segment marketing, event marketing etc.; (c) with proper tuning our algorithm outperforms their algorithm in both running time and influence spread.

DR-submodular function maximization over lattices or continuous domain receives many attentions in recent years (e.g., [11, 27, 16, 19]). The main difference with our study is that we mainly focus on the specific DR-submodular function related to the influence maximization task to design scalable algorithms, while those studies focus on general DR-submodular functions. Another difference is that they often rely on gradient methods, which assume that the function is continuous and differentiable, but we do not rely on such assumptions.

2 Problem Definition

Influence propagation in social networks is modeled by a stochastic process. In this paper, for simplicity we use the independent cascade (IC) model, but all the results hold for the more general triggering model [20]. A social network is modeled as a directed graph $G = (V, E)$, where $V$ is the set of nodes representing individuals, and $E$ is the set of directed edges representing influence
relationships. We denote $n = |V|$ and $m = |E|$. In the IC model [20], every edge $(u,v) \in E$ has an influence probability $p(u,v)$. The propagation proceeds in discrete time steps $t = 0, 1, 2, \ldots$. Each node is either in the inactive state or the active state, and once active, the node stays active throughout the propagation process. At time $t = 0$, a seed set $S \subseteq V$ is given, such that all nodes in $S$ are activated at $t = 0$. For any time $t \geq 1$, each node $u$ activated at time $t - 1$ has one chance to activate each of its still inactive out-going neighbors $v$ independently with success probability $p(u,v)$. Node $v$ becomes active if at least one of its in-neighbors is successful in activating $v$. The process ends until there is no more newly activated node at a step. A key quantity is the influence spread of a seed set $S$, denoted as $\sigma(S)$, which is defined as the expected number of final active nodes for the propagation starting from $S$.

The classical influence maximization task is to select at most $k$ seed nodes to maximize the influence spread, i.e., to find $S^* = \operatorname{argmax}_{S \subseteq V, |S| \leq k} \sigma(S)$. The problem is NP hard, and [20] propose the greedy approximation algorithm, which is based on the submodularity of the influence model and guarantees $1 - 1/e - \varepsilon$ approximation for any small $\varepsilon > 0$. Extensive studies have been conducted on the scalability and many variants of influence maximization problems, as discussed in Section 1.1.

In this paper we study the extension of influence maximization with general marketing strategies, a version of which is originally proposed in [20]. A mix of marketing strategies is modeled as a $d$-dimensional vector $x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d$, where $\mathbb{R}_+$ is the set of nonnegative real numbers. Each dimension corresponds to a particular marketing strategy, e.g., direct mail to a particular segment of the user base. Under the marketing strategy mix $x$, each node $v \in V$ is independently activated as a seed with the probability given by the seed activation function $h(x)$. Then the set of activated seed nodes propagate the influence throughout the network following the influence diffusion model. We define the influence spread of a marketing strategy mix $x$ to be the expected number of nodes activated, and is denoted as $g(x)$. Based on the diffusion process described above, we have

$$g(x) = \mathbb{E}_S[\sigma(S)] = \sum_{S \subseteq V} \sigma(S) \cdot \prod_{u \in S} h_u(x) \cdot \prod_{v \notin S} (1 - h_v(x)).$$  \hspace{1cm} (1)

In this paper, we consider discretized marketing strategies with granularity parameter $\delta$, that is, each strategy $x_i$ takes discretized values $0, \delta, 2\delta, \ldots$. These set of vectors is referred to as a lattice, and is denoted as $\mathcal{X}$. We consider the marketing strategy mix $x$ has a total budget constraint $k$: $\sum_{i \in [d]} x_i \leq k$. Henceforth, we denote $|x| = \sum_{i \in [d]} x_i$. The above constraint can be thought as total monetary budget constraint, where $x_j$ is the monetary expense on strategy $j$, but other interpretations are also possible. We now formally define the IM-GMS problem.

**Definition 1 (Influence Maximization with General Marketing Strategies).** Given a social network $G = (V, E)$ with the IC model parameters $\{p(u,v)\}_{(u,v) \in E}$, given the seed node activation functions $\{h_v\}_{v \in V}$ and a total budget $k$, the task of influence maximization with general marketing strategies, denoted as IM-GMS, is to find an optimal strategy mix $x^*$ that achieves the largest influence spread within the budget constraint, that is

$$x^* = \operatorname{argmax}_{x \in \mathcal{X}, |x| \leq k} g(x).$$

Note that if $\mathcal{X} = \{0,1\}^n$ and $h_v(x) = x_v$, that is, $v$ is activated as a seed if and only if it is selected by strategy $x$, the IM-GMS problem becomes the classical influence maximization.
Algorithm 1 Hill Climbing Algorithm, HillClimbing\((f, k, \delta)\)

**Input:** monotone DR-submodular \(f\), budget \(k\), granularity \(\delta\)

**Output:** vector \(x\)

1: \(x = 0\)
2: for \(t = 1, 2, \ldots, k \cdot \delta^{-1}\) do
3: \(j^* = \text{argmax}_{j \in [d]} f(x + \delta e_j)\)
4: \(x = x + \delta e_{j^*}\)
5: end for
6: return \(x\)

problem. Therefore, IM-GMS is a more general problem, and inherits the NP-hardness of the classical problem.

To solve the IM-GMS problem, [20] propose the greedy algorithm based on the diminishing return property of \(g(x)\), commonly referred to as the DR-submodular property [28, 2]. For two vectors \(x, y \in \mathbb{R}^d\), we denote \(x \leq y\) if \(x_i \leq y_i\) for all \(i \in [d]\). Let \(e_i \in \mathbb{R}^d\) be the unit vector with the \(i\)-th dimension being 1 and all other dimensions being 0. For a vector function \(f : \mathcal{X} \rightarrow \mathbb{R}\), we say that \(f\) is DR-submodular if for all \(x, y \in \mathcal{X}\) with \(x \leq y\), for all \(i \in [d]\), \(f(x + \delta e_i) - f(x) \geq f(y + \delta e_i) - f(y)\); and we say that \(f\) is monotone (nondecreasing) if for all \(x \leq y\), \(f(x) \leq f(y)\). Vector function \(f\) is DR-supermodular (resp. monotone nonincreasing) if \(-f\) is DR-submodular (resp. monotone nondecreasing). Note that a set function \(f\) is monotone if \(f(S) \leq f(T)\) for all \(S \subseteq T\), and submodular if \(f(S \cup u) - f(S) \geq f(T \cup \{u\}) - f(T)\) for all \(S \subseteq T\) and \(u \not\in T\). It is clear that if we represent sets as binary vectors and take step size \(\delta = 1\), then it coincides with monotonicity and DR-submodularity of vector functions.

When the vector function \(f\) on lattice \(\mathcal{X}\) is nonnegative, monotone and DR-submodular, the greedy hill-climbing algorithm as given in Algorithm 1 achieves \(1 - 1/e\) approximation [24, 20]. The hill-climbing algorithm searches the coordinate that gives the largest marginal return and moves one step of size \(\delta\) on that coordinate, until it exhausts the budget.

To apply the HillClimbing algorithm to the IM-GMS problem, [20] show that when \(h_v\)'s are monotone and DR-submodular with \(\sigma(S)\) being monotone and submodular, the influence spread \(g(x)\) given in Eq. (1) is also monotone and DR-submodular. Therefore, the HillClimbing algorithm can be applied to \(g(x)\). As it is \#P-hard to compute the influence spread \(\sigma(S)\) in the IC model [32], we could use Monte Carlo simulations to estimate \(g(x)\) to achieve \(1 - 1/e - \varepsilon\) approximation for any small \(\varepsilon > 0\).

In this paper, we further generalize the IM-GMS problem with partitioned budgets. More specifically, marketing strategies often belong to multiple categories, and each category may be assigned a separate budget. Formally, the strategy set \([d]\) is partitioned into \(\lambda\) categories \(C_1, \ldots, C_\lambda\), and each category \(C_j\) has a budget \(k_j\), i.e. \(\sum_{i \in C_j} x_i \leq k_j\). For convenience, we use \(x_{C_j}\) to denote the projection of vector \(x\) into index set \(C\). Then the above constraint is \(\|x_{C_j}\| \leq k_j\). The partitioned budget problem is formally defined below.

**Definition 2** (Influence Maximization with General Marketing Strategies and Partitioned Budgets). Given the same input as in the IM-GMS problem (Definition 1), except that total budget \(k\) is replaced by partitions \(\{C_j\}_{j \in [\lambda]}\) and partitioned budgets \(\{k_j\}_{j \in [\lambda]}\), the task of influence maximization with general marketing strategies and partitioned budgets, denoted as IM-GMS-PB, is to find an optimal strategy mix \(x^*\) that achieves the largest influence spread within the partitioned budget.
constraints, that is

\[ x^* = \arg\max_{x \in \mathcal{X}, |C_j| \leq k_j, \forall j \in [\lambda]} g(x). \]

Finally, we remark that for either IM-GMS or IM-GMS-PB problem, we can further restrict that each \( x_i \) has an upper bound \( b_i \). This would not change the problem, because we can extend the domain of \( x_i \) beyond \( b_i \) by redefining \( h_u(x) = h_v(x \wedge b) \), where \( x \wedge b \) denote coordinate-wise minimum operation. It is straightforward to verify that \( h_v \) preserves monotonicity and DR-submodularity of \( h_u \), and the hill-climbing algorithms introduced in the next section would not go beyond the bound \( b_i \) on dimension \( i \), because it would give zero marginal gain. In other words, bounds on individual strategies can be dismissed, and in the IM-GMS-PB problem, we thus require that \( |C_j| > 1 \) for all \( j \in [d] \). Henceforth, we ignore the constraints of upper bounding \( x_i \)'s.

3 Scalable Algorithm IMM-GMS

It is well known that the Monte Carlo hill-climbing algorithm is not scalable. In this section, we propose scalable solutions to the IM-GMS problem based on the seminal reverse influence sampling (RIS) approach [3, 31, 30]. In particular, we adapt the IMM (Influence Maximization with Martingales) algorithm of [30] to a new algorithm IMM-GMS to solve the IM-GMS problem.

3.1 Reverse Reachable Sets and Its Properties

The RIS approach is based on the key concept of the reverse reachable sets (RR sets), as defined below.

**Definition 3** (Reverse Reachable Set). Given a graph \( G = (V, E) \) with IC model parameter \( \{p(u, v)\}_{(u, v) \in E} \), a reverse reachable (RR) set rooted at a node \( v \), denoted \( R_v \), is the random set of nodes \( v \) reaches in one propagation instance, after we reverse all edge directions while keeping edge influence probabilities. An RR set \( R \) without specifying a root is an RR set with root \( v \) selected uniformly at random from \( V \).

Intuitively, RR sets rooted at \( v \) store nodes that are likely to influence \( v \) in the original IC model. Technically, it has the following important connection with the influence spread of a seed set \( S \): \( \sigma(S) = n \cdot \mathbb{E}_R[\mathbb{I}(S \cap R \neq \emptyset)] \), where \( \mathbb{I} \) is the indicator function [3, 30].

For our IM-GMS problem, we can extend the above property to the following result.

**Lemma 1.** For any strategy mix \( x \in \mathcal{X} \), we have

\[ g(x) = n \cdot \mathbb{E}_R \left[ 1 - \prod_{u \in R} (1 - h_u(x)) \right]. \]  

(2)

**Proof.** By Eq. (1), we have \( g(x) = \mathbb{E}_S[\sigma(S)] = n \cdot \mathbb{E}_{S,R}[\mathbb{I}(S \cap R \neq \emptyset)] = n \cdot \mathbb{E}_R[\text{Pr}_S\{S \cap R \neq \emptyset\}] \). Then \( \text{Pr}_S\{S \cap R \neq \emptyset\} \) is the probability that at least one node in \( R \) (now fixed) is activated as a seed under strategy mix \( x \), so it is \( 1 - \prod_{v \in R} (1 - h_v(x)) \).
Algorithm 2 General structure of IMM-GMS

**Input:** $G$: the social graph; $\{p(u, v)\}_{(u,v) \in E}$: IC model parameters; $\{h_v\}_{v \in V}$: seed activation functions (or $\{q_{v,j}\}_{v \in V,j \in S_v}$ for HillClimbingDelta); $k$: budget; $\delta$: granularity; $\epsilon$: accuracy; $\ell$: confidence

**Output:** $x \in \mathcal{X}$

1: $\mathcal{R} =$ Sampling($G$, $\{p(u, v)\}_{(u,v) \in E}$, $k$, $\delta$, $\epsilon$, $\ell$)
2: $x =$ HillClimbing($\hat{g}_R$, $k$, $\delta$)  
   // or HillClimbingDelta($\mathcal{R}$, $\{q_{v,j}\}_{v \in V,j \in S_v}$, $k$, $\delta$)
3: return $x$

3.2 General Structure of IMM-GMS

By Eq.(2), we can generate $\theta$ independent RR sets as a collection $\mathcal{R}$ to obtain

$$
\hat{g}_R(x) = \frac{n}{\theta} \sum_{R \in \mathcal{R}} \left( 1 - \prod_{v \in R} (1 - h_v(x)) \right)
$$

as an unbiased estimate of $g(x)$. Moreover, we have the following property for $\hat{g}_R(x)$.

**Lemma 2.** If $h_v$ is monotone and DR-submodular for all $v \in V$, then $\hat{g}_R$ is also monotone and DR-submodular.

**Proof.** We rely on the very useful technical Lemma 3. It is easy to see that $1 - h_v(x)$ is nonnegative, monotone nonincreasing, and DR-supermodular, then by repeatedly applying Lemma 3 we know that $\prod_{v \in R} (1 - h_v(x))$ is monotone nonincreasing and DR-supermodular for every RR set $R$. Therefore, $\hat{g}_R$ is monotone nondecreasing and DR-supermodular.

**Lemma 3.** If $f_1$ and $f_2$ are nonnegative, monotone nonincreasing and DR-supermodular, then $f(x) = f_1(x) f_2(x)$ is also monotone nonincreasing and DR-supermodular.

The proof of Lemma 3 is included Appendix A. With Lemma 2, we can apply the HillClimbing algorithm on $\hat{g}_R$. Let $\hat{x}^o =$ HillClimbing($\hat{g}_R$, $k$, $\delta$) be the output of the HillClimbing algorithm on $\hat{g}_R$. When $\theta = |\mathcal{R}|$ is large enough, $\hat{g}_R$ would be very close to $g$, and we could show that $\hat{x}^o$ is a $1 - 1/e - \epsilon$ approximate solution for the original function $g$.

This leads to the general structure of the IMM-GMS algorithm as given in Algorithm 2, similar to the IMM algorithm in [30]. The algorithm takes the input as listed in Algorithm 2 and outputs $x$ such that $x$ is a $1 - 1/e - \epsilon$ approximate solution to the IM-GMS problem with probability at least $1 - n^\ell$. The algorithm contains two phases. In the first phase, the Sampling procedure determines the number of RR sets needed and generates these RR sets; in the second phase, a hill-climbing algorithm on these RR sets are used to find the resulting strategy vector $x$. We first discuss the second phase, which requires major changes from the original IMM algorithm, and then introduce the first phase.

3.3 Efficient Hill-Climbing on RR Sets

If the seed activation function $h_v(\cdot)$’s are given as black boxes, we have to compute $h_v(x)$ from scratch. Suppose that the running time cost for computing $h_v(x)$ is $O(T_{h_v})$. Then it is
fails to activate strategies that could activate strategy \( j \) of the hill-climbing algorithm. In particular, we consider a large class of functions where each strategy independently acts on each node \( v \) trying to activate \( v \) as a seed. Suppose that the set of strategies that could activate \( v \) is \( S_v \subseteq [d] \), and the probability that strategy \( j \) with amount \( x_j \) fails to activate \( v \) as a seed is \( q_{v,j}(x_j) \). Then we have
\[
h_v(x) = 1 - \prod_{j \in S_v} q_{v,j}(x_j). \tag{4}\]

The above equation enables efficient updates in the hill-climbing algorithm, but before discussing the efficiency, we first state the following property about the DR-submodularity of \( h_v \).

**Lemma 5.** If function \( q_{v,j}(x) \) is non-increasing and convex for every \( j \in S_v \), then \( h_v(x) \) is monotone and DR-submodular.

**Proof (Sketch).** The proof also uses Lemma 3, we only need to notice that one-dimensional convexity is a special case of DR-supermodularity. \(\square\)

With Eq. (4), instead of always computing \( \hat{g}_R(x + \delta e_j) \) from scratch in HillClimbing(\( \hat{g}, k, \delta \)), we compute \( \Delta_j(x) = \hat{g}_R(x + \delta e_j) - \hat{g}_R(x) \), which is given below.
\[
\Delta_j(x) = \frac{n}{\theta} \sum_{R \in \mathcal{R}} \left( \prod_{v \in R} \prod_{j' \in S_v} q_{v,j'}(x_{j'}) \right) \cdot \left(1 - \frac{\prod_{v \in R: j \in S_v} q_{v,j}(x_j + \delta)}{\prod_{v \in R: j \in S_v} q_{v,j}(x_j)} \right). \tag{5}\]

The advantage of Eq. (5) is that it allows efficient updates of \( \hat{g}_R(x + \delta e_i) \) without recomputation from scratch. Specifically, the term within the first parentheses is the same across all strategies, so its computation can be shared. Moreover, since it is often the case that each user is only exposed to a small subset of strategies (i.e. \( |S_v| \) is smaller than \( d \)), we carefully maintain a data structure to improve the efficiency when \( |S_v| < d \). Algorithm 3 presents the detailed hill-climbing update procedure HillClimbingDelta, which replaces HillClimbing(\( \hat{g}, k, \delta \)) when Eq. (4) holds.

In Algorithm 3, we use \( s_i \) to store the term \( \prod_{v \in R_i} \prod_{j' \in S_v} q_{v,j'}(x_{j'}) \) in Eq. (5) shared across different strategies \( j \). We use ratio to store the ratio term \( \prod_{v \in R, i \in S_v} q_{v,i}(x_i + \delta)q_{v,i}(x_i)^{-1} \) in Eq. (5). The \( \text{List}_j \) is a linked list for strategy \( j \), and it stores the pair \((i, v)\), which means RR set \( R_i \) contains node \( v \) that can be affected by strategy \( j \). The list is ordered by RR set index \( i \) first and then by node index \( v \). In each round \( t \) of the hill-climbing, the algorithm iterates through all strategies \( j \) (lines 7–20) to compute \( \Delta_j(x) \) for the current \( x \). In particular, for each strategy \( j \), the algorithm traverses the \( \text{List}_j \) (lines 9–16), and for the segment with the same RR set index \( i \), it updates ratio, and when it reaches a new RR set index \( i \neq \text{prev} \), it cumulates \( \Delta_j \) as given in Eq. (5) for the corresponding RR set. The reason we maintain \( \text{List}_j \) of pairs instead of simply looping through all RR set indices \( i \) and then all nodes within \( R_i \) is that RR sets are usually not very large, and it is likely that no node in RR set \( R_i \) is affected by strategy \( j \), and thus not looping through all RR sets save time. After computing \( \Delta_j = \Delta_j(x) \), we find the strategy \( j^* \) with the
Algorithm 3 HillClimbingDelta: Efficient hill-climbing implementation on RR sets

Input: \( R = \{R_1, \ldots, R_\theta\} \): RR sets; \( \{q_{v,j}\}_{v,j \in S_v} \): strategy-node activation functions; \( k \): budget; \( \delta \): granularity

Output: \( x \in \mathcal{X} \)

1. \( x = (x_1, \cdots, x_d) = 0 \)
2. // Lines 3–5 can be done while generating RR sets
3. \( s = (s_0, s_1, \cdots, s_\theta) \) with \( s_0 = 0, s_i = \prod_{v \in R_i} \prod_{j \in S_v} q_{v,j}(x_j) \)
4. \( \forall j \in [d], List_j = \emptyset \)
5. \( \forall R_i \in R, \forall v \in R_i, \forall j \in S_v, \) append \((i, v)\) to \( List_j \)
6. for \( t = 1, 2, \cdots, k \cdot \delta^{-1} \) do
7. for \( j \in [d] \) do
8. \( \Delta_j = 0, \) \( prev = 0, ratio = 1 \)
9. for \( (i, v) \in List_j \) do
10. if \( i \neq prev \) then
11. \( \Delta_j := \Delta_j + s_{prev} \cdot (1 - ratio) \)
12. \( ratio = 1 \)
13. \( prev = i \)
14. end if
15. \( ratio := ratio \cdot \frac{q_{v,j}(x_j + \delta)}{q_{v,j}(x_j)} \)
16. end for
17. if \( prev \neq 0 \) then
18. \( \Delta_j := \Delta_j + s_{prev} \cdot (1 - ratio) \)
19. end if
20. end for
21. \( j^* = \arg \max_{j \in [d]} \Delta_j \)
22. \( x := x + \delta e_{j^*} \)
23. \( \forall i \in [\theta], s_i := s_i \cdot \prod_{v \in R_{i,j^*} \in S_v} q_{v,j^*}(x_j^* + \delta) \cdot q_{v,j^*}(x_j^*)^{-1} \)
24. end for
25. return \( x \)

largest \( \Delta_j \) (line 21), move along the direction of \( j^* \) for one step (line 22), and then update all shared terms \( s_i \)'s (line 23).

Let \( T_q \) be the running time cost for computing \( q_{v,j}(x_j) \). The following lemma summarizes the time complexity of HillClimbingDelta.

Lemma 6. The time complexity of HillClimbingDelta is \( O(k \cdot \delta^{-1} \cdot (T_q \cdot \sum_{R \in R} \sum_{v \in R} |S_v| + d)) \).

Proof (Sketch). The algorithm has totally \( k \delta^{-1} \) rounds of hill climbing. In each round, it effectively enumerates all tuples \((i, v, j)\) for RR set \( R_i \), node \( v \in R_i \) and strategy \( j \in S_v \), and for each tuple it has a constant number of calls to function \( q_{v,j} \), so the running time in one round \( t \) is \( O(T_q \cdot \sum_{R \in R} \sum_{v \in R} |S_v|) \). The reason we add \( d \) is because (a) we loop through all strategies \( j \in [d] \) (line 9) even if \( j \) does not appear in the tuple at all, and (b) finding the maximum among \( \Delta_j \)'s take \( O(d) \) time.

Notice that if we compute \( \hat{g}(x + \delta e_j) \) directly instead of \( \Delta_j(x) \), then we have \( T_{h_v} = O(d \cdot T_q) \). By Lemma 4, the time complexity is \( O(d \cdot T_q^* \sum_{R \in R} \sum_{v \in R} |S_v|) \), which is worse than HillClimbingDelta.
Finally, remaining RR sets are generated so that \( |V_j \cap \{ x \} | = 1 \). In this scenario, the time complexity of HillClimbingDelta is \( O(k \cdot \delta^{-1} \cdot (T_q \cdot \sum_{R \in R} |R| + d)) \).

The second scenario is personalized marketing, where each user is targeted with a personalized strategy. The personalized discount strategies studied in [33] is an instance of this scenario. Technically, this scenario is a special case of the above segment marketing scenario, where the user segments \( V_j \)'s are all singletons, and thus \( d = n \). Thus the time complexity of HillClimbingDelta is \( O(k \cdot \delta^{-1} \cdot (T_q \cdot \sum_{R \in R} |R| + n)) \).

The third scenario is repeated marketing such as multi-event marketing. For example, each strategy \( j \) is a type of events, and \( x_j \) is the number of events of type \( j \). Suppose that for each event of type \( j \), a user \( v \) targeted by this event has an independent probability \( r_{v,j} \) to be activated as a seed, then \( q_{v,j}(x_j) = (1 - r_{v,j})^{x_j} \). This is a concrete example where \( q_{v,j}(x) \) is nonincreasing and convex, and thus by Lemma 5 \( h(x) \) is monotone and DR-submodular. The time complexity of computing \( q_{v,j}(x_j) \) is \( O(\log x_j) = O(\log k) \), and thus, the time complexity of HillClimbingDelta in this case is \( O(k \cdot \delta^{-1} \cdot (\log k \cdot \sum_{R \in R} \sum_{v \in R} |S_v| + d)) \).

### 3.4 The First Phase Sampling Procedure

The Sampling procedure in the first phase is to generate enough RR sets \( R \) to provide the theoretical guarantee on the approximation ratio. This procedure follows the structure of IMM in [30] and is presented in Algorithm 4. We now briefly explain the key points of the algorithm. To guarantee the approximation ratio, [30] show that the number of RR sets \( \theta = \Theta(n \log n / OPT) \) is enough, where \( OPT \) is the optimal solution. Since \( OPT \) is not known, the Sampling procedure tries to estimate its lower bound \( LB \). The estimate (lines 4–16) is done by trying \( y = \frac{n}{2}, \frac{n}{4}, \frac{n}{8}, \ldots \) as lower bounds, and generate increasingly more RR sets, and then call the HillClimbingDelta procedure (line 11) to find an approximate solution \( x \), and verify that if this approximate solution is indeed lower bounded by \( y \) with an error factor (lines 12–15). If the verification passes, then we have a lower bound; otherwise, we lower the estimate \( y \) by half and continue to the next iteration. After the lower bound \( LB \) is found, the total number of RR sets is set to \( \theta = \lambda^*/LB \) (line 17), where \( \lambda^* \) is given below.

\[
\lambda^* = 2n \cdot (1 - 1/e) \cdot (\alpha + \beta)^2 \cdot \varepsilon^{-2}, \tag{6}
\]

\[
\alpha = \sqrt{\ell \ln n + \ln 2}, \quad \beta = \sqrt{(1 - 1/e) \cdot (k \delta^{-1} \ln d + \ell \ln n + \ln 2)}.
\]

Finally, remaining RR sets are generated so that \( R \) contains totally \( \theta \) RR sets.

Comparing to the original IMM algorithm [30], in the definition of \( \lambda' \) and \( \lambda^* \), we replace \( \ln \binom{n}{k} \) with \( k \delta^{-1} \ln d \), because \( k \delta^{-1} \) is the upper bound on the number of vectors satisfying the constraint \( |x| \leq k \). Note that Algorithm 4 is used for the case when Eq. (4) holds. For general \( h_v \) functions, we could simply replace HillClimbingDelta in line 11 with HillClimbing(\( g_{R}, k, \delta \)).

Algorithms 2, 3, and 4 together form the IMM-GMS algorithm. The following theorem summarizes the approximation and time complexity guarantee of the IMM-GMS algorithm. Its analysis is based on [30] and is given in Appendix B.
Algorithm 4 First phase Sampling procedure

**Input:** $G$: the social graph; $\{p(u,v)\}_{(u,v) \in E}$: IC model parameters; $\{q_{v,j}\}_{v \in V, j \in S_v}$: strategy-node activation functions; $k$: budget, $\delta$: granularity; $\epsilon$: accuracy; $\ell$: confidence

**Output:** A collection of RR sets $\mathcal{R}$

1. $\mathcal{R} = \emptyset$; $LB = 1$
2. $\ell = \ell + \ln 2/\ln n$
3. Let $\epsilon' = \sqrt{2} \cdot \epsilon$
4. for $i = 1, 2, \cdots, \log_2 n$ do
5.   Let $y = n/2^i$
6.   $\theta_i = \frac{\lambda'}{y}$, where $\lambda' = (\frac{2 + \frac{3}{2} \epsilon'}{2}) \cdot (k\delta^{-1} \ln d + \ell \ln n + \ln \log_2 n) \cdot n$
7.   while $|\mathcal{R}| \leq \theta_i$ do
8.     Select a node $v$ from $G$ uniformly at random
9.     Generate an RR set for $v$, and insert it into $\mathcal{R}$
10. end while
11. $x = \text{HillClimbingDelta}(R, \{q_{v,j}\}_{v \in V, j \in S_v}, k, \delta)$
12. if $\hat{g}_R(x) \geq (1 + \epsilon') \cdot y$ then
13.   $LB = \hat{g}_R(x)/(1 + \epsilon')$
14.   break
15. end if
16. end for
17. $\theta = \lambda^*/LB$, where $\lambda^*$ is defined in Eq. (6)
18. while $|\mathcal{R}| \leq \theta$ do
19.   Select a node $v$ from $G$ uniformly at random
20.   Generate an RR set for $v$, and insert it into $\mathcal{R}$$$
21. end while
22. return $\mathcal{R}$

**Theorem 1.** When Eq. (4) holds for nonincreasing and convex functions $q_{v,j}$’s, then the IMM-GMS algorithm composed of Algorithms 2, 3, 4 returns a $(1 - 1/e - \epsilon)$-approximate solution to the IM-GMS problem with at least $1 - 1/n^\ell$ probability. When $q_{v,j}$’s are such that the optimal solution is at least as good as the best single node influence spread, the IMM-GMS algorithm runs in $O(k\delta^{-1} T_q(\max_{v \in V} |S_v|)(k\delta^{-1} \log d + \ell \log n)(n + m)/\epsilon^2)$ expected time.

Note that the technical assumption above assuming the optimal solution is at least as good as the best single node influence spread is reasonable, since it means the budget and the functions $q_{v,j}$’s are at least good enough to activate one single best node. If it is not true, the entire marketing scheme is not very useful anyway. Comparing to the time complexity $O((k + \ell)(m + n) \log n/\epsilon^2)$ of IMM in [30], the time complexity of IMM-GMS bears several similarities, including the near-linear-time dependency ($O((m + n) \log n)$) on the graph size. Some new terms demand more explanations. First, the term $\max_{v \in V} |S_v|$ is the fan-in parameter specifying the maximum number of strategies that could impact one node. High fan-in means that one node can be affected by many strategies, and thus computing the optimal strategy mix could be more costly. Second, the $k\delta^{-1} \log d$ corresponds to the $k \log n$ term in IMM, both of which are logarithms of the number of feasible solutions, and are needed to ensure the high confidence of the returned solution. The leading term $k\delta^{-1}$ in the running time of IMM-GMS has no direct corresponding term in IMM.
This is due to the complexity of \textsc{HillClimbingDelta} algorithm (Lemma 6). In particular, because a strategy vector only probabilistically activate nodes as seeds, \textsc{HillClimbingDelta} has to process all RR sets in all hill-climbing steps. But in IMM, the deterministic seed selection permits that each RR set \( R \) is processed at most once, at the step when some node in \( R \) is selected as a seed. This key difference leads to the extra \( k\delta^{-1} \) factor for \textsc{HillClimbingDelta}.

4 Solving the IM-GMS-PB Problem

In this section we explain how to solve the partitioned budget constraint version IM-GMS-PB. Our method relies on the submodular maximization problem under the general matroid constraint. A matroid on a set of elements \( U \) is a collection of subsets of \( U \) called independent sets, which satisfy the following two properties: (a) If \( I \subseteq U \) is an independent set, then every subset of \( I \) is also an independent set; and (b) If \( I, I' \) are two independent sets with \( |I| < |I'| \), then there must be some element \( e \in I' \setminus I \) such that \( I \cup \{e\} \) is also an independent set. The simplest matroid is the uniform matroid, where for some parameter \( k \) all subsets \( I \) with \( |I| \leq k \) is an independent set. Classical influence maximization essentially uses the uniform matroid constraint. A partition matroid is such that, for a certain partition of \( U \) into disjoint sets \( A_1, \ldots, A_\lambda \), and for parameters \( k_1, \ldots, k_\lambda \), all subsets \( I \subseteq U \) satisfying \( |\bigcap A_i| \leq k_i \) for all \( i \in [\lambda] \) are independent sets. The classical result by [12] shows that the greedy hill-climbing algorithm on a general matroid could achieve \( 1/2 \) approximation ratio for nonnegative monotone and submodular set functions.

Through Lemmas 3.1 and 3 and the result in [20], we already know that our objective functions \( g(x) \) and \( \tilde{g}_R(x) \) are nonnegative, monotone, and DR-submodular, but they are vector functions. We now show how to translate them into equivalent set functions and then show that the IM-GMS-PB problem corresponds to a partitioned matroid constraint under the set representation. Let \( b \geq \sum_{j \in [\lambda]} k_j \cdot \delta^{-1} \) be a large enough integer. Construct the set of elements \( U = \{(j, s) \mid j \in [d], s \in [b]\} \). For any subset \( A \subseteq U \), denote \( A^{(j)} = A \cap \{(j, s) \mid s \in [b]\} \). We map \( A \) into a vector \( x^A = (x_1^A, \ldots, x_d^A) \) such that \( x_j^A = |A^{(j)}| \cdot \delta \). Conversely, for every vector \( x \in X \) satisfying the partitioned budget constraint, we map \( x \) to a set \( A^x = \{(j, s) \mid j \in [d], s \cdot \delta \leq x_j \} \). For every vector function \( f \), we define a set function \( f^U \) on \( U \) to be \( f^U(A) = f(x^A) \), for all \( A \subseteq U \). It is easy to see that the marginal \( f^U(A \cup \{(j, s)\}) - f^U(A) = f(x^A + \delta e_j) - f(x^A) \). Thus, one can verify that if \( f \) is monotone and DR-submodular, then \( f^U \) is monotone and submodular. Next, for the partitioned budget constraint \( |x_{C_i}| \leq k_i \) given partition \( C_1, \ldots, C_\lambda \) of \( [d] \) and budgets \( k_1, \ldots, k_\lambda \), it is equivalent to partition \( U \) to \( U_1, \ldots, U_\lambda \), with \( U_i = C_i \times [b] \), and enforce constraint \( |A \cap U_i| \leq k_i \cdot \delta^{-1} \) for all \( A \subseteq U \) and \( i \in [\lambda] \). Therefore, we translate the IM-GMS-PB problem of maximizing \( g(x) \) with the partitioned budget constraint to maximizing \( g^U(A) \) under the partition matroid constraint. Similarly we can translate \( \tilde{g}(x) \) to \( \tilde{g}^U(A) \). Therefore, we can conclude that the hill-climbing algorithms under the partitioned budget constraint could achieve \( 1/2 \) approximation.

The actual hill-climbing algorithm is straightforward. For example, in line 21 of Algorithm 3, instead of taking argmax among all possible \( j \in [d] \), we only search for \( j \) such that \( x + \delta e_j \) still satisfies the partitioned budget constraint. The hill-climbing terminates until the partitioned budgets are exhausted. We refer this version of \textsc{HillClimbingDelta} as \textsc{HillClimbingDeltaPB}, and the full algorithm as IMM-GMS-PB. By a similar analysis, we can obtain the following theorem for IMM-GMS-PB.

\textbf{Theorem 2.} When Eq. (4) holds for nonincreasing and convex functions \( q_{e,j} \)'s, the IMM-GMS-PB algorithm composed of Algorithms 2, 4 and \textsc{HillClimbingDeltaPB} returns a \((1/2 - \varepsilon)\)-
Table 1: Dataset Statistics

| Network     | n     | m     | Average Degree |
|-------------|-------|-------|----------------|
| DM          | 679   | 3,374 | 4.96           |
| NetHEPT     | 15,233| 62,752| 4.12           |
| Flixster    | 29,357| 425,228| 14.48         |
| DBLP        | 654,628| 3,980,318| 6.08         |

approximate solution to the IM-GMS-PB problem with at least \(1 - 1/n^\ell\) probability, and runs in \(O(k\delta^{-1}T_q(\max_{v \in V} |S_v|)(k\delta^{-1}\log d + \ell \log n)(n + m)/\varepsilon^2)\) expected time, where \(k = \sum_{i \in [\lambda]} k_i\).

5 Experiments

5.1 Experiment Setup

Datasets. We ran our experiments on 4 real-world networks, with statistics summarized in Table 1. Three of them, denoted DM, NetHEPT, and DBLP, are collaboration networks: every node is an author and every edge means the two authors collaborated on a paper. DM network is a network of data mining researchers extracted from the ArnetMiner archive (arnetminer.org) [29], NetHEPT is a network extracted from the high energy physics section of arxiv.org, while DBLP is extracted from the computer science bibliography database dblp.org [32]. Their sizes are small (679 nodes), medium (15K nodes), and large (654K) nodes, respectively. We include the small DM dataset mainly to suit the slow Monte Carlo hill-climbing algorithm. The last dataset is Flixster, which is a user network of the movie rating site flixster.com. Every node is a user and a directed edge from \(u\) to \(v\) means that \(v\) has rated some movie(s) that \(u\) rated earlier [1]. The IC model parameters of NetHEPT and DBLP are synthetically set using the weighted cascade method [20]: edge \(p(u, v) = 1/d_v\), where \(d_v\) is the in-degree of node \(v\). For the DM and Flixster networks, we obtain learned edge parameters from the authors of [29, 1] respectively.

Application scenarios. We test two application scenarios, as explained at the end of Section 3.3. The first is the personalized marketing scenario tested in [33]. In this scenario, each user \(v\) has one unique strategy \(x_v\) such as the personalized discount to \(v\), \(h_v(x)\) only depends on \(x_v\). We set \(h_v(x) = 2x_v - x_v^2\) following the same setting in [33]. The second one is the segmented event marketing scenario, which is not covered by previous studies. In this case, each strategy \(j\) is targeting at a disjoint subset of users \(V_j\), and \(x_j\) is the number of marketing events of type \(j\) for user group \(V_j\). In our experiments, we set \(d = 200\) for each dataset. Moreover, we choose top \(\min\{n, 2000\}\) nodes \(V^*\) with the highest degrees from \(V\). For every node \(v \in V^*\), we generate \(i_v\) from \([d]\) uniformly at random and generate \(r_{v,i_v}\) from \([0, 0.3]\) uniformly at random. Then for every \(v \in V^*\), we set \(S_v = \{i_v\}\) and \(h_v(x) = 1 - (1 - r_{v,i_v})x_v\); for every \(v \in V \setminus V^*\), we set \(S_v = \emptyset\). This simulates the scenario where marketing efforts are focused on top connected nodes in the network.

Algorithms in Comparison. We test the following algorithms.

- IMM-GMS. For our algorithm, we set \(\ell = 1\), \(\varepsilon = 0.5, 1, 2\). Note that when \(\varepsilon = 1\) or 2, IMM-GMS no longer has the approximation guarantee, but it is still a valid heuristic algorithm, since all other baselines are heuristic algorithms.
• UD. UD is proposed in [33] for **personalized marketing**. For each discount $c \in \{0, 0.1, 0.2, \ldots, 1\}$, it will return a vector $x$ s.t. $x_i = 0$ or $x_i = c$ ($i \in [d]$). Then they run an exhaustive search of $c$ to find a best $c$.

• CD. CD is also proposed in [33]. CD uses the output of UD as the initial value and runs a coordinate decent algorithm to achieve better result.

• HD. HD (heuristics degree) is a heuristic baseline, where we choose top $M$ nodes with the largest degrees from $V$ and then distribute the budget to those $M$ nodes proportional to their degrees. We set $M = 100$ and $M = 200$ in our experiments.

• MCHC. This is HillClimbing algorithm (Algorithm 1) with Monte Carlo simulations to estimate influence spread $g(x)$. We use 10,000 simulations for each estimation of $g(x)$.

For the personalized marketing scenario, we test all algorithms with granularity $\delta = 0.1$. For the segmented event marketing scenario, we do not test UD, CD, and HD, since they are all designed for the personalized marketing scenarios. In this case, $\delta = 1$ as required by the scenario. For all cases, we test total budget $k$ from 5 to 50. We do not test the IMM for seed set optimization, since it is designed on a different solution space and [33] already demonstrates that IMM is inferior to UD and CD.

All our tests are run on a Ubuntu 14.04.5 LTS server with 3.3GHz and 125GB memory. All algorithms are coded in C++ and compiled by g++.

### 5.2 Experimental Results

![Influence spread in personalized marketing scenario.](image1)

![Running time in the personalized marketing scenario.](image2)

We first look at the results for personalized marketing. Figure 1 shows the influence spread result and Figure 2 shows the running time result. When comparing to MCHC algorithm (only
run on DM), our IMM-GMS algorithm shows clear advantage: its running time is three to four orders of magnitude faster than MCHC while its influence spread is also clearly better than MCHC. The reason for MCHC’s low influence spread is because 10,000 Monte Carlo simulations are not enough for \( \hat{g}(x) \) estimation in this scenario. A larger number of simulations would be better, but its running time would further increase. Even in the current setup, MCHC is already too slow and can only run on the DM dataset.

When comparing to UD and CD heuristics of [33], we see that our IMM-GMS algorithm with \( \varepsilon = 0.5, 1, 2 \) consistently performs better than UD and CD in influence spread. In terms of running time, our algorithm with \( \varepsilon = 2 \) is in general faster than UD and CD. Although \( \varepsilon = 2 \) would not have approximation guarantee, it is still a valid heuristic, and our results show that it has the best running time while also being among the best in influence spread. Therefore, by tuning the \( \varepsilon \) parameter, our IMM-GMS algorithm could either provide strong theoretical guarantee (with small \( \varepsilon \)) or competitive empirical performance (with a larger \( \varepsilon \)). But UD and CD algorithms cannot provide theoretical guarantee, nor a superior empirical performance.

For the baseline heuristic HD, the result shows that its influence spread is significantly lower than others (e.g. in DM and Flixster), and thus it is not a competitive heuristic. The results on segmented event marketing are shown in Figures 3 and 4. We only compare IMM-GMS with MCHC since other heuristics are not designed for this case. MCHC is too slow so we only run it on the smaller DM and NetHEPT datasets. Our results show that our IMM-GMS in all settings are at least 3 orders of magnitude faster than MCHC. In terms of the influence spread, MCHC does perform a little better than IMM-GMS in this case. However, MCHC is not scalable and cannot run on larger datasets, making it less useful than IMM-GMS.

![Figure 3: Influence spread in the segmented event marketing scenario.](image)

![Figure 4: Running time in the segmented event marketing scenario.](image)
6 Future Work

There are several future directions to this study. One direction is to study continuous domain, and see how gradient methods can be incorporated together with RIS-like scalable approach to solve the IM-GMS in the continuous domain. The other is to further explore the interaction between the strategies and seed activation to improve the scalability of the algorithm. It may also be interesting to study general marketing strategies in other influence propagation setting such as competitive influence maximization.

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Appendix

A  Proof of Lemma 3

Proof. The monotonicity is straightforward. For DR-supermodularity, for any $x \leq y$, we have

\[
\begin{align*}
(f(x + \delta e_i) - f(x)) - (f(y + \delta e_i) - f(y)) &= f_1(x + \delta e_i)f_2(x + \delta e_i) - f_1(x)f_2(x) \\
&- (f_1(y + \delta e_i)f_2(y + \delta e_i) - f_1(y)f_2(y)) \\
&= f_1(x + \delta e_i)(f_2(x + \delta e_i) - f_2(x)) + f_2(x)(f_1(x + \delta e_i) - f_1(x)) \\
&- (f_1(y + \delta e_i)(f_2(y + \delta e_i) - f_2(y)) + f_2(y)(f_1(y + \delta e_i) - f_1(y)) \\
&\leq f_1(x + \delta e_i)(f_2(y + \delta e_i) - f_2(y)) + f_2(x)(f_1(x + \delta e_i) - f_1(x)) \\
&- (f_1(y + \delta e_i)(f_2(y + \delta e_i) - f_2(y)) + f_2(y)(f_1(y + \delta e_i) - f_1(y)) \\
&= (f_1(x + \delta e_i) - f_1(y + \delta e_i))(f_2(x + \delta e_i) - f_2(y)) \\
&+ (f_2(x) - f_2(y))(f_1(y + \delta e_i) - f_1(y)) \\
&\leq 0,
\end{align*}
\]

where the first inequality is due to the DR-supermodular and nonnegative conditions, and the second inequality is due to the monotone nonincreasing property.

B  Proof of Theorem 1

This proof is directly modified from the proof of Theorem 4 in [30].

Lemma 7. Given $k, \delta, d, n, l$, Algorithm 3 returns a $(1 - 1/e - \varepsilon)$-approximation with at least $1 - 1/n^l$ probability if the size of $R$ 0 is at least $\lambda^*/OPT$, where $\lambda^*$ is defined in equation 6.

Proof. Denote $x^*$ as the solution of Algorithm 3 and $x^\circ$ as the optimal solution of IM-GMS problem. Through replacing the number of possible $k$-seed set $\binom{n}{k}$ of Lemma 3 and 4 in [30] by the number of possible allocations $d^{k-1}$ in our problem, we can derive that with $1 - n^l$ probability,

\[
\hat{g}(x^\circ) \geq \left(1 - \varepsilon \cdot \frac{\alpha}{(1 - 1/e) \cdot \alpha + \beta}\right) \cdot OPT
\]

and

\[
\hat{g}(x^*) \leq g(x^*) + \left(\varepsilon - \frac{(1 - 1/e)\varepsilon \alpha}{(1 - 1/e)\alpha + \beta}\right) \cdot OPT
\]

Then by combining the greedy property that $\hat{g}(x^*) \geq (1 - 1/e)\hat{g}(x^\circ)$, we have,

\[
g(x^*) \geq (1 - 1/e - \varepsilon) \cdot OPT.
\]

Lemma 8. With at least $1 - 1/2n^l$ probability, Algorithm 4 returns a set $R$ of RR sets with $|R| \geq \lambda^*/OPT$, where $\lambda^*$ is as defined in equation 6.
Proof. Through replacing the number of possible $k$-seed set $\binom{n}{k}$ of Lemma 6 and 7 in [30] by the number of possible allocations $d^{k-1}$ in our problem, we can easily get the result of this lemma. It’s $1 - 1/2n^\ell$ rather than $1 - 1/n^\ell$ because we let $\ell = \ell + \ln 2/\ln n$ in Algorithm 4.

Combining Lemma 7 and Lemma 8, we have the IMM-GMS algorithm composed of Algorithms 2, 3, 4 returns a $(1 - 1/e - \varepsilon)$-approximate solution to the IM-GMS problem with at least $1 - 1/n^\ell$ probability.

When $q_v,j$’s are such that the optimal solution is at least as good as the best single node influence spread, i.e. $n \cdot EPT \leq m \cdot OPT$. Then we can derive that the expected time is $O(k\delta^{-1}T_q(\max_{v \in V} |S_v|)(k\delta^{-1} \log d + \ell \log n)(n + m)/\varepsilon^2$) by the same techniques in [30].