Anchored Discrete Factor Analysis

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Abstract

We present a semi-supervised learning algorithm for learning discrete factor analysis models with arbitrary structure on the latent variables. Our algorithm assumes that every latent variable has an "anchor", an observed variable with only that latent variable as its parent. Given such anchors, we show that it is possible to consistently recover moments of the latent variables and use these moments to learn complete models. We also introduce a new technique for improving the robustness of method-of-moment algorithms by optimizing over the marginal polytope or its relaxations. We evaluate our algorithm using two real-world tasks, tag prediction on questions from the Stack Overflow website and medical diagnosis in an emergency department.

1 Introduction

Estimating the parameters and structure of Bayesian networks from incomplete data is a fundamental task in many of the social and natural sciences. We consider the setting of latent variable models, where certain variables of interest are never directly observed, but their effects are discernible in the interactions of other directly observable variables. For example, in the clinical setting, diseases themselves are rarely observable, but their presence and absence are inferred from the combined evidence of patient narratives, lab tests, and other measurements. A full understanding of the relationships between diseases, how they interact with each other and how they present in individuals is still largely unknown. Another example is from the social sciences, where we may wish to know how opinions and beliefs interact or change over time. Surveys, textual analysis, and actions (e.g., voting registration, campaign donations) provide a noisy and incomplete view of people’s true beliefs, and we would like to build models to describe and analyze the underlying world-views.

Estimating models involving latent variables is challenging for two reasons. First, without some constraints on the model, the model may be non-identifiable, meaning that there may exist multiple parameter settings that cannot be distinguished based on the observed data. This non-identifiability severely diminishes the interpretability of the learned models, and has been the subject of many critiques of factor analysis and similar techniques. Second, even in the identifiable setting, finding the model that best describes the data is often computationally intractable, even for seemingly simple models. For example, without making assumptions on the data generation process, finding the maximum likelihood latent Dirichlet allocation model is NP-hard (Arora \textit{et al.} [2012]).

We present a new efficient algorithm for discrete factor analysis, learning models involving binary latent variables and non-linear relationships to binary observed variables (Martin & VanLehn [1995], Singliar & Hauskrecht [2006], Wood \textit{et al.} [2006], Jernite \textit{et al.} [2013a]). Our algorithm is semi-supervised, requiring that a domain expert identify anchors, observed variables that are constrained to have only a single parent among the latent variables. Given these anchors, we show how to learn factor analysis models with arbitrary structure in the latent variables. The anchor conditions we present are closely related to the exclusive views conditions used by Chaganty & Liang [2014], which enable learning model parameters provided that each clique has a set of observations that satisfy certain independence criteria. However, identifying exclusive views requires that the structure of the latent variable model be known in advance. This is a significant shortcoming in real-world
The contributions of this paper are as follows:

- We define anchored factor analysis, a subclass of factor analysis models, where the structure of the latent variables can be recovered from data.
- We derive a method-of-moments algorithm for recovering both the latent structure and the factor loadings, including an efficient method for the case of tree-structured latent variables.
- We show that moment recovery can be improved by using constrained optimization with increasingly tight outer bounds on the marginal polytope, suggesting a method of increasing robustness of method-of-moments approaches in the presence of model misspecification.
- We present experimental results, learning interpretable factor analysis models on two real-world multi-label text corpora, outperforming discriminative baselines.

Road map: The paper proceeds according to this high level structure:

- Section 2 formally defines anchor variables.
- Section 3 reviews how anchor variables can be used to recover moments of the latent variables.
- Section 4 describes a family of latent variable models that can be learned using the recovered latent moments, and outlines the learning procedure, which consists of two parts: learning a structure for the latent variables and learning connections between the latent variables and observed variables.
- Section 5 compares the models learned using the method described here to other standard baselines on real-world prediction tasks.
2 The anchored factor analysis task

Notation: We use uppercase letters to refer to random variables (e.g., \(Y_i, X_j\)) and corresponding lowercase letters to refer to the values. For compactness, we use superscript parenthetical values to denote a variable taking a particular value (e.g., \(P(x_j^{(0)})\) is equivalent to \(P(X_j = 0))\).

Anchored factor analysis: In this paper, we will discuss learning binary latent variable models. We divide the variables of our model into two classes: observed variables \(\mathcal{X} = \{X_1, \ldots, X_n\}\) and unobserved or latent variables \(\mathcal{Y} = \{Y_1, \ldots, Y_m\}\), where all variables have binary states 0 or 1.

As in other factor analysis models, we assume that all dependencies between the observations are explained by the latent variables. This implies that the observed distribution has the following factorized form: \(P(X_1, \ldots, X_n) = \sum_y P(y) \prod_{j=1}^n P(x_j | y)\).

We focus on a particular class of latent variable models, that we term anchored models. A latent variable model is anchored if for every \(Y_i \in \mathcal{Y}\), there is a corresponding observed variable \(A_{Y_i} \in \mathcal{X}\) that fits the following definition:

**Definition 1. Anchor variables:** An observed variable \(X_j \in \mathcal{X}\) is called an anchor for a latent variable \(Y_i \in \mathcal{Y}\) if \(X_j \not\perp Y_i\) and \(X_j \perp Y_k\) for all other \(Y_k \neq i\), \(k \in \mathcal{Y}\).

Using the language of directed probabilistic graphical models, we say that \(X_j\) is an anchor of \(Y_i\) if \(Y_i\) is the sole parent of \(X_j\). We use the notation \(A_{Y_i}\) to refer to the anchor of \(Y_i\) or \(A_{Z}\) to refer to the set of anchors of a set of latent variables \(Z \subseteq \mathcal{Y}\).

In this work, we assume that anchors are identified as an input to the learning algorithm. Specifying anchors can be a way for experts to inject domain knowledge into a learning task. In section 5 we show that even anchors obtained from simple rules can perform well in modeling.

3 Recovering moments of latent variables

3.1 Previous work – exclusive views

For the purpose of exposition, we assume that every anchor has a conditional distribution (i.e., \(P(A_i | Y_i)\)) which can be estimated consistently, and leave the discussion of estimating these conditional distributions from data to a later section (Section 3.3).

For a set of latent variables \(Z \subseteq \mathcal{Y}\), Chaganty & Liang (2014) show how to recover the marginal probabilities \(P(Z)\) using only the observed anchors, \(A_{Z}\). Eq. 1 relates the observed distribution \(P(A_{Z})\) to the unobserved moments \(P(Z)\):

\[
P(A_{Z}) = \sum_z P(z) P(A_{Z} | z) = \sum_z P(z) \prod_{i=1}^{|Z|} P(A_i | z_i). \tag{1}
\]

The first equality simply comes from marginalizing the latent variables \(Z\). The second uses the assumption that an anchor is independent of all other variables conditioned on its parent. Since we assume that \(P(A_i | Z_i)\) can be estimated, Eq. 1 gives \(|Z|\) linear equations for \(|Z|\) unknowns. Since the linear equations can be shown to be independent (see supplementary materials), the distribution of the latent variables \(P(Z)\) can be estimated from the observed distribution \(P(A_{Z})\). This can be trivially extended to a case where \(Z\) includes both latent and observed variables by noticing that every observed variable can act as its own anchor.

Let \(R_{Z}\) be the Kronecker product of the conditional anchor distributions, \(R_{Z} = \otimes_{k=1}^{|Z|} P(A_k | Z_k)\), and \(\mu_{Z}\) be a vector of dimension \(|Z|\) of probabilities for \(P(Z)\). To recover \(P(Z)\), we seek to find a distribution that minimizes the divergence between the expected marginal vector of the anchors, \(R_{Z} \mu_{Z}\), and the observed marginal vector of the anchors, \(\mu_{A_{Z}}\). We solve this as a constrained optimization problem, where \(D\) is any Bregman divergence and \(\Delta\) is the probability simplex:

\[
\hat{\mu}_{Z} = \arg\min_{\mu \in \Delta} D(\mu_{A_{Z}}, R_{Z} \mu). \tag{2}
\]

Chaganty & Liang (2014) show the consistency of this estimator using both L2 distance and KL divergence for \(D\). The consistency of the estimator means that the recovered marginals \(\hat{\mu}_{Z}\) will converge to the true values, \(\mu_{Z}\), assuming that the anchor assumption is correct.
3.2 Robust moment recovery – connection to variational inference

Our algorithm’s theoretical guarantees, as with other method-of-moments approaches, assume that the data is drawn from a distribution in the model family. In our setting, this means that the anchors must perfectly satisfy Definition 1. We introduce a new approach to improve the robustness of method-of-moment algorithms to model misspecification. Our approach can be used as a drop-in replacement for parameter recovery in the exclusive views framework of [Chaganty & Liang, 2014]. For simplicity, we describe it here in the context of anchors.

Consider, for example, the case of learning a tree-structured distribution on the latent variables. Before running the Chow-Liu algorithm, we would need to estimate the edge marginals \( \hat{\mu}_{ij}(y_i, y_j) \) for every pair of random variables \( Y_i, Y_j \). Chaganty & Liang [2014]’s approach is to solve (\( \binom{n}{2} \)) independent optimization problems of the form given in Eq. 2 resulting in a set of estimates \( \hat{\mu}_{ij}(y_i, y_j) \). Our key insight is that since the true edge marginals \( \{ \hat{\mu}_{ij}(y_i, y_j) : i, j \in \mathcal{Y} \} \) all derive from \( P(\mathcal{Y}) \), i.e., \( \hat{\mu}_{ij}(y_i, y_j) = \sum_{y_{i,j}} P(y) \), there are additional constraints that they must satisfy. For example, the true edge marginals must satisfy the local consistency constraints:

\[
\sum_{y_i} \hat{\mu}_{ij}(y_i, y_j) = \sum_{y_k} \hat{\mu}_{jk}(y_j, y_k) \quad \forall i, j, k \in \mathcal{Y} \text{ and } y_j.
\]

More generally, \( \mu \) must lie in the marginal polytope, \( \mathcal{M} \), consisting of the space of all possible marginal distribution vectors that can arise from any distribution [Wainwright & Jordan, 2008]. Note that there exist vectors \( \mu \) which satisfy the local consistency constraints but are not in the marginal polytope. The marginal polytope has been widely studied in the context of probabilistic inference in graphical models. MAP inference corresponds to optimizing a linear objective over the marginal polytope, and computing the log-partition function can be shown to be equivalent to optimizing a non-linear objective over the marginal polytope [Wainwright & Jordan, 2008]. Optimizing over the marginal polytope is NP-hard, and so approximate inference algorithms such as belief propagation instead optimize over the outer bound given by the local consistency constraints. There has been a substantial amount of work on characterizing tighter relaxations of the marginal polytope, all of which immediately applies to our setting (e.g., Sontag & Jaakkola [2007]).

Putting these together, we obtain the following optimization problem for robust recovery of the true moments of the latent variables from noisy observations of the anchors:

\[
\hat{\mu} = \arg\min_{\mu \in \mathcal{P}} \sum_{\mathcal{Z} \subseteq \mathcal{Y}, |\mathcal{Z}| \leq K} D_{KL}(\mu, A_{\mathcal{Z}}),
\]

where \( K \) is the size of the moments needed within the structure learning algorithm (e.g., \( K = 2 \) for a tree-structured distribution) and \( \mathcal{P} \) denotes a relaxation of the marginal polytope.

We use the conditional gradient, or Frank-Wolfe, method to minimize 4. Frank-Wolfe solves this convex optimization problem by repeatedly solving linear programs over \( \mathcal{P} \). When \( \mathcal{P} \) corresponds to the local consistency constraints or the cycle relaxation (Sontag & Jaakkola, 2007), these linear programs can be solved easily using off-the-shelf LP solvers. Alternatively, when there are sufficiently few variables, one can optimize over the marginal polytope itself. For this, we use the observation of [Belanger et al., 2013] that optimizing a linear function over the marginal polytope can be performed by solving an integer linear program with local consistency constraints. In the experimental section we show that constrained optimization improves the robustness of the moment-recovery step compared to unconstrained optimization and that using increasingly tight approximations of the marginal polytope within the conditional gradient procedure yields increasingly improved results.

Constrained optimization has been used previously to improve the robustness of method-of-moments results [Shaban et al., 2015]. Our work differs in that the constrained space naturally coincides with the marginal polytope, which allows us to leverage the Frank-Wolfe algorithm for interior-point optimization and relaxations of the marginal polytope that have been studied in the context of variational inference.
3.3 Estimating anchor noise rates

Throughout we have assumed that the conditional probabilities of the anchors can be estimated from the observed data. This task of estimating label noise is the subject of a rich literature. In this section we describe four settings where that is a reasonable assumption.

1. **Two or more anchors for \( Y_i \) are provided**: If two anchors are provided, then the conditional distributions for both can be estimated using a multi-view tensor decomposition method [Berge, 1991; Anandkumar et al., 2012], where the third view is obtained using any other observed variable that is correlated with the anchors (see supplementary materials).

2. **Anchors are positive-only**: In the setting where anchors are positive-only (i.e. \( P(Y_i|A_i) = 1 \)), we have the setting known as Positive and Unlabeled (PU) learning. In this setting, the noise rates of these noisy labels can be estimated using the predictions of a classifier trained to separate the positive from unlabeled cases [Elkan & Noto, 2008].

3. **Mutually irreducible distributions**: This setting, discussed in Scott et al. (2013) can be described as requiring that in the data distribution there exists at least one unambiguous positive and negative case. Estimators for the noise rates under this setting are described in [Menon et al., 2015].

4. **Some (partially) labeled data is available**: In the medical diagnosis task that we consider in the experiments, we asked a physician a small number of questions about each patient as part of their regular workflow. This gave us singly-labeled data, where each data point observed \( X_i, Y_i \) for a single \( i \). Using a Chernoff bound, one can show that a small number of such observations suffices to accurately estimate an anchor’s conditional distribution.

4 Model learning

4.1 Learning \( P(Y) \)

**Structure learning background**: Approaches for Bayesian network structure learning typically follow two basic strategies: they either search over structures \( G \) that maximize the likelihood of the observed data (score-based methods), or they test for conditional independencies and use these to constrain the space of possible structures. A popular scoring function is the BIC score [Lam & Bacchus, 1994; Heckerman et al., 1995]:

\[
\text{Score}_{BIC}(G) = \sum_{i=1}^{m} N \hat{I}(Y_i; Y_{Pa(i); G}) - N \hat{H}(Y_i) - \log(N)2|Pa(i); G|
\]

where \( N \) is the number of samples and \( \hat{I}, \hat{H} \) are the empirical mutual information and entropy respectively. \( Pa(i; G) \) denotes the *parents* of node \( i \), in graph \( G \). The last term is a complexity penalty that biases toward structures with fewer parameters. Once the optimal graph structure is determined, the conditional probabilities, \( \theta \), that parametrize the Bayesian network are estimated from the empirical counts for a maximum likelihood estimate.

BIC is known to be *consistent*, meaning that if the data is drawn from a distribution which has precisely the same conditional independencies as a graph \( G^* \), once there is sufficient data it will recover \( G^* \), i.e. \( G^* = \arg\max_B \text{Score}_{BIC}(B) \). In general, finding a maximum scoring Bayesian network structure is NP-hard (Chickering, 1996). However, approaches such as integer linear programming [Jaakkola et al., 2010; Cussens & Bartlett, 2013], branch-and-bound [Fan et al., 2014], and greedy hill-climbing (Teyssier & Koller, 2005) can be remarkably effective at finding globally optimal or near-optimal solutions. Restricting the search to tree-structures, we can find the highest-scoring network efficiently using a maximum spanning tree algorithm [Chow & Liu, 1968].

A different approach is to use conditional independence tests. Under the assumption that every variable has at most \( k \) neighbors, these can be used to give polynomial time algorithms for structure learning [Pearl & Verma, 1991; Spirtes et al., 2001], which are also provably consistent.

We utilize the fact that both score-based and conditional independence based structure learning algorithms can be run using derived statistics rather than the raw data. Suppose \( P(Y) \) is a Bayesian
network with maximum indegree of $k$. Then, to estimate the mutual information $\hat{I}$ and entropy $\hat{H}$ needed to evaluate the BIC score for all possible such graphs, we only need to estimate all moments of size at most $k+1$. For example, to search over tree-structured Bayesian networks, we would need to estimate $p(y_i, y_j)$ for every $i, j$. Alternatively, we could use the polynomial-time algorithms based on conditional independence tests. These need to perform independence tests using conditioning sets with up to $k$ variables, requiring us to estimate all moments of order $k + 2$.

We showed in Section 3 how to recover these moments of the latent variables from moments of the anchor variables. Even though these latent variables are completely unobserved, we are able to use the recovered moments in structural learning algorithms designed for fully observed data. Together, this gives us a polynomial-time algorithm for learning the Bayesian network structure for the latent variables (Step 2 of Algorithm 1) as stated in Theorem 1.

**Theorem 1.** Let $G(\mathcal{Y})$ be the graph structure of a Bayesian network describing the probability distribution, $P(Y)$, of the latent variables in an anchored discrete factor analysis model. Using the low-order moments recovered in Equation (2) in a structure learning algorithm which is consistent for fully-observed data, is a consistent structural estimator. That is, as the number of samples, $N \to \infty$, the structure recovered, $G'(\mathcal{Y})$, is Markov equivalent to $G(\mathcal{Y})$.

While conditional independence tests give the theoretical result of a polynomial time recovery algorithm, in our experimental results we use the score-based approach, optimizing the BIC score which is efficient for trees using the Chow-Liu algorithm (Chow & Liu, 1968) and for higher-order graphs using a cutting-plane algorithm (Cussens & Bartlett, 2013).

### 4.2 Learning factor loadings, $P(X|Y)$

In this section we describe a moments-based approach to recovering the factor loadings in the Anchored Discrete Factor Analysis model (step 3 of Algorithm 1). We show how to make use of the recovered low-order moments and the structure learned in Section 4.1 to estimate the conditional distributions of the observations conditioned on the latent variables.

In this work, we use a noisy-or parametrization, a widely-used parametrization for Bayesian networks in medical diagnosis (e.g. Miller et al., 1986; Wood et al., 2006) and binary factor analysis (Singliar & Hauskrecht, 2006). In particular, we assume that the distribution of any observation conditioned on the latent variables has the form: $P(x_j = 0|y_1, ..., y_m) = (1 - l_j) \prod_i f_{i,j}^{y_i}$. The parameters $f_{i,j}$ are known as failure probabilities and the $l_j$ parameters are known as leak probabilities. Intuitively, the generative model can be described as follows: For every observation, $X_j$, each parent, $Y_i$, that is “on” has a chance to turn it on and fails with probability $f_{i,j}$. The leak probability $l_j$ represents the probability of $X_j$ being turned on without any of its parents succeeding.

We use $G(\mathcal{Y})$ to denote the Bayesian network structure over the latent variables that describes $P(Y)$. This graph structure is learned in section 4.1.

To properly estimate a failure probability $f_{i,j}$ from low order moments, we need to separate the direct effect of $Y_i$ in turning on $X_j$ and all other indirect effects (e.g. Conditioning on $Y_i = 1$ changes the likelihood of another latent variable $Y_i'$ being on, which in turn affects $X_j$).

**Estimation using Markov blanket conditioning:** The first method of separating direct and indirect effects uses the entire Markov blanket of each latent variable. Denote the Markov blanket of $Y_i$ in $G(\mathcal{Y})$ as $B_i \subset \mathcal{Y}$. For any setting of $B_i = b$, the following is a consistent estimator of $f_{i,j}$:

$$
\hat{f}_{i,j}^{\text{blanket}} = \frac{\hat{P}(x_j^{(0)}|y_i^{(1)}, b)}{\hat{P}(x_j^{(0)}|y_i^{(0)}, b)},
$$

as shown in the supplementary materials. The simplest application of this estimator is in models where the latent variables are assumed to be independent (e.g. Miller et al., 1986; Jernite et al., 2013a), where the Markov blanket is empty and the estimator is simply

$$
\hat{f}_{i,j}^{\text{direct}} = \frac{\hat{P}(x_j^{(0)}|y_i^{(1)})}{\hat{P}(x_j^{(0)}|y_i^{(0)})}.
$$
Figure 2: $Y_i$ has a direct effect on the distribution of $X_j$ as well as indirect effects that pass through its two neighbors. Correction factors are introduced to cancel the indirect effects through each neighbor, leaving only the direct effect which is modeled with the parameter $f_{i,j}$.

Unfortunately, for general graphs, in order to estimate these moments, we have to form moments that are of the same order as the Markov blanket of each latent variable which can potentially be quite large, even in simple tree models, giving poor computational and statistical efficiency.

**Improved estimation for tree-structured models:** When the $G(Y)$ is a tree, it is possible to correct for indirect effects serially, only requiring conditioning on two latent variables at a time. Each of the correction factors essentially cancels out the effect of an entire subtree in determining the likelihood that $X_j = 1$. Let $N_i$ denote the neighbors of $Y_i$ in $G(Y)$. When calculating $f_{i,j}^{tree}$, we introduce correction factors, $c_{i,j,k}$, to correct for indirect effects that flow through $Y_k \in N_i$ (Figure 2).

By correcting for each neighbor in serial we get the following estimator:

$$\hat{f}_{i,j}^{tree} = \left( \prod_{k \in N_i} \frac{1}{c_{i,j,k}} \frac{\hat{P}(x_j^{(0)}|y_i^{(1)})}{\hat{P}(x_j^{(0)}|y_i^{(0)})} \right),$$

where $c_{i,j,k}$ has the form:

$$c_{i,j,k} = \sum_{y_k} \frac{P(y_k|y_i^{(1)})P(x_j^{(0)}|y_i^{(0)}, y_k)}{P(x_j^{(0)}|y_i^{(0)})},$$

and can be estimated from the low-order moments recovered in Equation 2. We note that for this method we only need to make assumptions about the graph of the latent variables, requiring no assumptions on the in-degree of the observations. For the correction factors to be defined, we do require that it is possible to condition on $(y_i = 0, y_k = \{0, 1\})$, so we require that $P(Y_k|Y_i)$ is not deterministic for all pairs of latent variables, $(Y_i, Y_k)$. A full derivation is provided in the supplementary materials.

### 4.3 Estimating leak parameters

Leak parameters can be estimated by comparing the empirical estimate of $\hat{P}(x_j^{(0)})$ to the one predicted by the model without leak, denoted as $P_{-l_j}(x_j^{(0)})$.

$$l_j = 1 - \frac{P_{-l_j}(x_j^{(0)})}{P(x_j^{(0)})}.$$ 

If the latent variables are independent, the Quickscore algorithm [Heckerman, 1990] gives an efficient method of computing marginal probabilities:

$$P_{-l_j}(x_j^{(0)}) = \prod_i \left( P(y_i^{(0)}) + P(y_i^{(1)}f_{i,j}) \right).$$
Table 1: Complexity of learning different model classes. After performing the momenttransformations (step 1 in Algorithm 1), the complexity of learning the models with latent variables is no harder than learning with fully observed moments.

| Model Class   | Complexity of learning $G(Y)$ | Factor loadings $f$ |
|---------------|--------------------------------|---------------------|
| independent   | None                           | $f_{direct}$ (Eq. 7) |
| tree          | Chow-Liu: $O(n^2)$             | $f_{tree}$ (Eq. 8)   |
| degree-K      | Independence tests: $O(n^2 \sum_{i=1}^{K} \binom{n}{i} 2^K)$ | $f_{blanket}$ (Eq. 6) |
| indegree-K    | Score-based: NP-hard worst case | $f_{blanket}$ (Eq. 6) |

For trees, this value can be computed efficiently using belief propagation and in more complex models it can be estimated by forward sampling.

4.4 Putting it all together

The full Anchored Factor Analysis model is a product of the latent distribution $P(Y)$, which is described by an arbitrary Bayesian network $G(Y; \theta)$ and the factor loadings which are described by noisy-or link functions. The computational complexity of learning the model depends on the choice of constraints for the Bayesian network, $G(Y)$. Table 1 outlines the different classes of models that can be learned and the associated computational complexities. We note that for some Bayesian network families, the complexity may be hard in a worst-case analysis, but practically cutting plane approaches that use integer linear programs can be successful in learning these models exactly and quickly (Cussens & Bartlett, 2013).

5 Experiments

We perform an empirical study of our learning algorithm using two real-world multi-tag datasets, testing its ability to recover the structure and parameters of the distribution of tags and observed variables.

5.1 Datasets

**Medical records:** We use a corpus of medical records, collected over 5 years in the emergency department of a Level I trauma center and tertiary teaching hospital in a major city. A physician collaborator manually specified 23 medical conditions that present in the emergency department and provided textual features that can be used as anchors as well as administrative billing codes that can be used for our purposes as ground truth. Features consist of binary indicators from processed medical text and medication records. Details of the processing and a selection of the physician-specified anchors are provided in the supplementary materials. Patients are filtered to exclude patients with fewer than two of the specified conditions, leaving 16,268 patients.

**Stack Overflow:** We also evaluate our methods on a large publicly available dataset using questions from Stack Overflow[^1] we treat user provided tags as latent variables. The observed vocabulary consists of the 1000 most common tokens in the questions, using a different vocabulary for the question header and body. Each question is described by a binary bag-of-words feature vector, denoting whether or not each word in the vocabulary occurs in the question text. We use a simple rule to define anchor variables: for each tag, the text of the tag appearing in the header of the question is used as an anchor. For example, for the tag `unix`, the anchor is a binary feature indicating whether the header of the text contains the word “unix”. We use the 50 most popular tags in our models.

5.2 Methods

In all of our experiments, we provide the algorithms and baselines with the empirical conditional probabilities of the anchors. Other methods of estimating these values exist and are discussed in

[^1]: [http://blog.stackoverflow.com/category/cc-wiki-dump/](http://blog.stackoverflow.com/category/cc-wiki-dump/) This data used was also used in a Kaggle competition: [https://www.kaggle.com/c/facebook-recruiting-iii-keyword-extraction](https://www.kaggle.com/c/facebook-recruiting-iii-keyword-extraction)
Section 3.3, but here we use the ground truth values for these noise rates in order to focus on the errors that arise from modeling error and finite data samples. Details on the optimization procedures and parameters used can be found in the supplementary materials.

5.3 Latent variable representation

In this section we evaluate the quality of the learned representations of the latent variables, \( P(Y) \). This task is interesting in its own right in settings where we care about understanding the interactions of the latent variables themselves for the purposes of knowledge discovery and understanding. Figure 3 shows a tree-structured graphical model learned to represent the distribution of the 23 latent variables in the Emergency dataset as well as small sub-graphs from a more complex model.

In the tree-structured model, highly correlated conditions are linked by an edge. For example, asthma and allergic reactions, or alcohol and suicidal ideation. This is significant, since the model learning procedure learns only using anchors and without access to the underlying conditions.

The insert shows subgraphs from a model learned with two parents per variable. This allows for more complex structures. For example: being HIV positive makes the patient more at risk for developing infections, such as cellulitis and pneumonia. Either one of these is capable of causing septic shock in a patient. The v-structure between cellulitis, septic shock, and pneumonia expresses the explaining away relationship: knowing that a patient has septic shock, raises the likelihood of cellulitis or pneumonia. Once one of those possible parents is discovered (e.g., the patient is known to have cellulitis), the septic shock is explained away and the the probability of having pneumonia is reduced. In the second example relationship, both asthma and urinary tract infections are associated with allergic reactions (asthma and allergic reactions are closely related and many allergic reactions in hospital occur in response to antibiotics administered for infections), but asthma and urinary tract infections are negatively correlated with each other since either one is sufficient to explain away the patient’s visit to the emergency department.

5.3.1 Robustness to model error

In practice, the anchors that we specify are never perfect anchors, i.e., they don’t fully satisfy the conditional independence conditions of Definition 1. We find that enforcing marginal or local polytope constraints (see section 3.2) during the moment recovery process provides moment estimates that are more robust to imperfect anchors, improving the overall quality of the learned models.
Figure 4: (Left) Per document held-out likelihood of learned tree structured models on Stack Overflow (50 variables) using held-out sets of 10K documents, each marker represents a different random train/test sample for the data. Solid lines represent the average of 8 runs. Different lines represent successively tight outer bounds to the marginal polytope constraints. (Right) When the Stack Overflow data is replaced by synthetic data drawn from a learned model, the difference between the constraints is much less pronounced.

Figure 5: Average accuracy on the last-tag prediction task for 5K held-out instances for Emergency and Stack Overflow datasets. Anchored Discrete Factor Analysis (ADFA) models are learned using a tree structured model for the latent variables and an independent model.

The weakest constraint we consider is using independent simplex constraints as in Chaganty & Liang (2014). In addition, we evaluate models learned with moments recovered using local consistency constraints, described in Equation 3 and the marginal polytope constraints described in Section 3.2. Figure 4 shows the held-out likelihood of the tags according to tree models learned from moments recovered with increasingly tight approximations to the marginal polytope constraints on the Stack Overflow corpora. The tighter constraints learn higher quality models. This effect persists even in the large sample regime, suggesting that the residual gaps between the methods are due to sensitivity to model error. On synthetically generated corpora, this effect disappears and all three constrained optimization approaches perform equally well.

Solving with tighter constraints does increase running time. For example, using our current solving procedures, learning a model with 50 latent variables requires 30 seconds using the simplex constraints, 519 seconds for the local polytope and 6,073 seconds for the marginal polytope. Unlike EM-based procedures for learning models with latent variables which iterate over training samples in an inner loop, the running time of these method-of-moments approaches depends on the number of samples only when reading the data for the first time.

5.4 Factor analysis – relating observations to latent variables

Table 2 shows a selection of learned factor loadings on the Stack Overflow dataset, learned using the tree-structured estimation procedure described in Section 4.2. The rest of the learned factors for both Stack Overflow and the Emergency datasets are found in the supplementary materials.
Table 2: A selection of the learned factor loadings in the Stack Overflow and emergency datasets. Highly weighted words are words with low failure probabilities in the noisy-or parametrization.

5.5 Comparison to related methods

We evaluate the full learned factor analysis model with a simulated last-tag prediction task where the model is presented with all but one of the positive tags (i.e., the latent variables) for a single document or patient and the goal is to predict the final tag. Our learned models use a tree-structured Bayesian network to represent the distribution $P(Y)$ and noisy-or gates to represent the conditional distributions $P(X|Y)$.

**Noise tolerant discriminative training:** As a baseline, we compare to the performance of a noise-tolerant learning procedure (Natarajan et al., 2013) using independent binary classifiers, trained using logistic regression with reweighted samples as described in their paper. To ensure a fair comparison with our learning algorithm which is provided with the anchor noise rates, we also provide the baselines with the exact values for the noise rates. Since the learning approach of (Natarajan et al., 2013) is not designed to use the noisy labels (anchors) at prediction time, we consider two variants: one ignores the anchors at test time, the other predicts only according to the noise rates of the anchors (ignoring all other observations), if the anchor is present. The baseline that we compare against is the best of the baselines described above.

**Oracle comparisons:** We compare to two different oracle implementations to deconvolve different sources of error. The first, Oracle MoM, uses a method-of-moments approach to learning the joint model (as described in Sections 4.1 and 4.2), but uses oracle access to the marginal distributions that include latent variables (i.e., does not incur any error when recovering these distributions using the method described in Section 3). The second oracle method Oracle ML, uses a maximum likelihood approach to learning the joint models with full access to the latent variables for every instance (i.e., learning as though all data is fully observed.) Comparing the gap between oracle ML and oracle MoM shows the loss in accuracy that comes from choosing a method-of-moments approach over a maximum likelihood approach.

**Imputation:** We compare to an imputation-based learning method, which learns a maximum likelihood model (Tree structured $P(Y)$ and noisy-or gates for conditional distributions $P(X|Y)$) from the fully imputed data, using a single sample from the independent binary baseline classifiers (Natarajan et al., 2013) to impute the values of the latent variables.

**Models with independent latent variables:** Previous works (e.g., Jernite et al., 2013a) consider similar models where latent variables are assumed to be independent. In our work, we explicitly model dependencies between the latent variables. We test the advantage of having a structured representation of the latent variables compared to a model with independent latent variables.

**Maximizing data likelihood:** We could learn a model without explicitly using the anchors, attempting to maximize likelihood of the observations. This maximization would be a non-convex problem due to the latent variables, and marginalizing the latent variables to compute the likelihood is computationally difficult. However, an EM procedure (Dempster et al., 1977) can be applied here (more detail in the supplementary materials). We initialize the parameters using the ADFA-tree method and run EM to determine whether additional steps of likelihood optimization can improve beyond the solutions found with ADFA.

5.6 Results

Figure 5 shows the improvement over baseline on the heldout tag prediction task for the two real-world datasets. We observe that learning the factor analysis model does indeed help with prediction in both tasks and we believe that this is because the independent classifiers in the baseline cannot
Figure 6: (top) Effects of pure likelihood-based optimization on accuracy. A likelihood optimization procedure is initialized with the results of the anchor-based method-of-moments estimate for the Stack Overflow and Emergency datasets (ADFA-tree). As the likelihood optimization progresses, classification accuracy degrades. (bottom) The meaning of the latent variable associated with “headache” changes over the course of likelihood optimization. Before optimization the highly weighted words are associated with the headache exam (e.g., head, changes of vision, nausea). After a small number of EM-steps, the meaning of the latent variable has changed and is now associated with long negation scopes.

6 Conclusion

Learning interpretable models with latent variables is a difficult task. In this paper we present a fast and expressive method that allows us to learn models with complex interactions between the latent variables. The learned models are also interpretable due to the effect of the user-specified anchor observations. On real-world datasets, we show that modeling the correlations between latent variables is useful, and outperform competitive baseline procedures. We find that enforcing marginal polytope constraints is useful for improving robustness to model error, a technique we believe can...

| Before optimization | After 9 EM steps |
|----------------------|------------------|
| anchor:headache      | anchor:headache  |
| neg-changes          | neg-chest        |
| nausea               | neg-chest_pain   |
| a                    | neg-ed           |
| neg-                 | neg-breath       |
| neg-vision           | neg-imaging      |
| neg-or               | neg-shortness    |
| neg-weakness         | neg-shortness_breath |
| head                 | neg-course       |
| neg-of               | neg-interventions |
be more widely applied. Anchors have an interesting property that they make the structure and parameter estimation with latent variables as easy as learning with fully observed data for method-of-moments algorithms that require low order moments. In contrast, likelihood-based learning remains equally hard, to the best of our knowledge, even with anchors.

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### A R matrix is full rank

In this section we show that the matrix $R$, defined in Section 3.1, is invertible. $R$ is block-diagonal, so its determinant is equal to the product of the determinants of the blocks and will be non-zero as long as all of the blocks have non-zero determinants.

Each block, $R_{\mathcal{Z}}$, is a Kronecker product of conditional distributions: $R_{\mathcal{Z}} = \bigotimes_{k=1}^{2|\mathcal{Z}|} P(A_k|Z_k)$. The determinant of each of these $2 \times 2$ matrices is nonzero as long as $P(A_k|Y_k = 0) \neq P(A_k|Y_k = 1)$ which is assumed in the definition of anchored latent variable models since $A_{Y_{i_j}} \not\perp Y_{i_j}$. Thus, the determinant of the Kronecker product is also non-zero, and the $R$ matrix is full rank.

### B Estimating conditional probabilities with two anchors per latent variable

In section 3.3 we note that if a latent variable as two anchors, their conditional distributions can estimated. The derivation is as follows: Let $W_1$, $W_2$ be observed anchors of $Y_i$. We can choose any other observation $X_j$ that has positive mutual information with both $W_1$ and $W_2$. In that case, we have a *singly-coupled triplet* setting, as described in [Halpern & Sontag (2013)]. Marginalizing over all latent variables but $Y_i$, we have that the observed distribution $P(W_1, W_2, X_j)$ is a $2 \times 2 \times 2$ tensor with the following decomposition into a sum of rank-1 tensors:
\[ P(W_1, W_2, X_j) = P(Y_i = 0)P(W_1|Y_i = 0)P(W_2|Y_i = 0)P(X_j|Y_i = 0) \\
+ P(Y_i = 1)P(W_1|Y_i = 1)P(W_2|Y_i = 1)P(X_j|Y_i = 1) \]

This decomposition can be computed efficiently [Bergelson 1991], and the conditional probabilities can be recovered.

C Estimating failures with Markov blanket conditioning

The Markov Blanket estimator from Section 4.2 follows from the noisy-or parametrization of the model. Here we show that the estimator from Equation 8 is indeed a consistent estimator. We start with the estimator:

\[ f_{i,j}^{\text{blanket}} = \frac{P(X_j = 0|Y_i = 1, B = b)}{P(X = 0|Y_i = 0, B = b)} \]

Breaking the numerator and denominator of the RHS. Let \( C = B \cup Y_i \) represent the conditioned variables and \( U = Y \setminus C \) be the unconditioned variables. For the numerator, we have:

\[
P(X_j = 0|Y_i = 1, B = b) = (1 - l_j) f_{i,j} \prod_{i' \in B} f_{i',j}^{y_{i',j}} \left( \sum_{y_{B}} P(y_{B}|Y_i = 1, B = b) \prod_{k \in U} f_{k,j}^{y_k} \right)
\]

\[ = f_{i,j} \prod_{i' \in B} f_{i',j}^{y_{i',j}} \left( \sum_{y_{B}} P(y_{B}|B = b) \prod_{k \in U} f_{k,j}^{y_k} \right) \quad (13)\]

Similarly, for the denominator:

\[
P(X_j = 0|Y_i = 0, B = b) = (1 - l_j) \prod_{i' \in B} f_{i',j}^{y_{i',j}} \left( \sum_{y_{B}} P(y_{B}|Y_i = 0, B = b) \prod_{k \in U} f_{k,j}^{y_k} \right)
\]

\[ = \prod_{i' \in B} f_{i',j}^{y_{i',j}} \left( \sum_{y_{B}} P(y_{B}|B = b) \prod_{k \in U} f_{k,j}^{y_k} \right) \quad (14)\]

The second lines follow from the Markov blanket property. Thus, the ratio is equal to \( f_{i,j} \). As the number of samples approaches infinity, the empirical estimates, \( \hat{P}(X_j = 0|Y_i = 1, B = b) \) and \( \hat{P}(X_j = 0|Y_i = 1, B = b) \), respectively approach the values of the true probabilities, and thus the estimator is consistent, provided that, \( P(X_j = 0|Y_i = 0, B = b) > 0 \).

D Estimating correction coefficients serially in trees

In this section we derive the correction terms for estimating the failure probabilities \( f_{i,j} \) in tree models described in Section 4.2 (Equation 9). While the directionality of the tree is not important, it is easier notationally to consider a rooted tree, and without loss of generality, when estimating the failure term \( f_{i,j} \), we can consider the tree as rooted at \( Y_i \).

We introduce the notation \( P_{Y_i}(e) \) to denote the likelihood of an event \( e \) in the graphical model where all nodes but \( Y_i \) and its non-descendants are removed (i.e., all that remains is the subtree rooted at \( Y_i \)).

Conditioning on the variable \( Y_i \) taking the value \( y_i \), the likelihood of \( X_j = 0 \) is written as:

\[
P(X_j = 0|y_i) = (1 - l_j) P_{Y_i}(X_j = 0|y_i) \\
= (1 - l_j) f_{i,j}^{y_{i,j}} \prod_{k \in \text{child}(Y_i)} \sum_{y_k} P(y_k|y_i) P_{Y_i}(X_j = 0|y_k). \quad (15)\]

Correction terms are defined as

\[
c_{i,j,k} = \frac{\sum_{y_k} P(y_k|y_i^{(1)}) P_{Y_i}(X_j = 0|y_k)}{\sum_{y_k} P(y_k|y_i^{(0)}) P_{Y_i}(X_j = 0|y_k)}. \quad (16)\]
Using Equation (18), it is easy to see that:

$$
\frac{P(X_j = 0 | y^{(1)}_i)}{P(X_j = 0 | y^{(0)}_i)} \prod_{k \in \text{child}(Y_j)} \frac{1}{c_{i,j,k}} = f_{i,j}. \tag{17}
$$

It remains to show that $c_{i,j,k}$ can be estimated from low order moments as in Equation (9):

$$
c_{i,j,k} = \sum_{y_k} P(y_k | y^{(1)}_i) P(x^{(0)}_j | y^{(0)}_i, y_k) \frac{P(x^{(0)}_j | y^{(0)}_i)}{P(x^{(0)}_j | y^{(0)}_i)}. \tag{18}
$$

We can expand $P(x^{(0)}_j | y^{(0)}_i, y_k)$ as follows:

$$
P(x^{(0)}_j | y^{(0)}_i, y_k) = (1 - l_j) f^{y_k}_{k,j} \prod_{k' \in \text{child}(Y_j) \setminus k} \sum_{y_{k'}} P(y_{k'} | y^{(0)}_i, y_k) P_{T(Y_{k'})}(X_j = 0 | y_{k'}) \times \prod_{k' \in \text{child}(Y_k) \setminus k} \sum_{y_{k'}} P(y_{k'} | y^{(0)}_i) P_{T(Y_{k'})}(X_j = 0 | y_{k'})
= (1 - l_j) f^{y_k}_{k,j} \prod_{k' \in \text{child}(Y_j) \setminus k} \sum_{y_{k'}} P(y_{k'} | y^{(0)}_i) P_{T(Y_{k'})}(X_j = 0 | y_{k'}) \times \prod_{k' \in \text{child}(Y_k) \setminus k} \sum_{y_{k'}} P(y_{k'} | y^{(0)}_i) P_{T(Y_{k'})}(X_j = 0 | y_{k'}) \tag{19}
$$

Where the second equality comes from the conditional independence properties that conditioning on $(Y_i, Y_k)$ makes children of $Y_k$ independent of $Y_i$ and children of $Y_i$ (other than $Y_k$) independent of $Y_k$.

We now substitute Eq. (19) into Eq. (18) and treat the numerator (num) and denominator (denom) separately:

$$
\text{num} = (1 - l_j) \sum_{y_k} P(y_k | y^{(1)}_i) f^{y_k}_{k,j} \prod_{k' \in \text{child}(Y_k) \setminus k} \sum_{y_{k'}} P(y_{k'} | y^{(0)}_i) P_{T(Y_{k'})}(X_j = 0 | y_{k'}) \times \prod_{k' \in \text{child}(Y_i) \setminus k} \sum_{y_{k'}} P(y_{k'} | y^{(0)}_i) P_{T(Y_{k'})}(X_j = 0 | y_{k'})
\times \sum_{y_k} P(y_k | y^{(1)}_i) f^{y_k}_{k,j} \prod_{k' \in \text{child}(Y_k) \setminus k} \sum_{y_{k'}} P(y_{k'} | y^{(0)}_i) P_{T(Y_{k'})}(X_j = 0 | y_{k'})
= (1 - l_j) \prod_{k' \in \text{child}(Y_j) \setminus k} \sum_{y_{k'}} P(y_{k'} | y^{(0)}_i) P_{T(Y_{k'})}(X_j = 0 | y_{k'})
$$

The product over children of $Y_i$ does not depend on $y_k$ and can be pulled out of the sum.

$$
\text{num} = (1 - l_j) \prod_{k' \in \text{child}(Y_j) \setminus k} \sum_{y_{k'}} P(y_{k'} | y^{(0)}_i) P_{T(Y_{k'})}(X_j = 0 | y_{k'})
\times \sum_{y_k} P(y_k | y^{(1)}_i) f^{y_k}_{k,j} \prod_{k' \in \text{child}(Y_k) \setminus k} \sum_{y_{k'}} P(y_{k'} | y^{(0)}_i) P_{T(Y_{k'})}(X_j = 0 | y_{k'})
= (1 - l_j) \prod_{k' \in \text{child}(Y_j) \setminus k} \sum_{y_{k'}} P(y_{k'} | y^{(0)}_i) P_{T(Y_{k'})}(X_j = 0 | y_{k'})
\times \sum_{y_k} P(y_k | y^{(1)}_i) P_{T(Y_k)}(X_j = 0 | y_k) \tag{20}
$$

Expanding the denominator using Equation (18):

$$
\text{denom} = P(x^{(0)}_j | y^{(0)}_i) = (1 - l_j) \prod_{k' \in \text{child}(Y_i) \setminus y_k} \sum_{y_{k'}} P(y_{k'} | y^{(0)}_i) P_{T(Y_{k'})}(X_j = 0 | y_{k'}) \tag{22}
$$

Canceling the leak term and the product over all children of $Y_i$ except for $Y_k$ from both the numerator and denominator, we are left with:

$$
\frac{\text{num}}{\text{denom}} = \frac{\sum_{y_k} P(y_k | y^{(1)}_i) P_{T(Y_k)}(X_j = 0 | y_k)}{\sum_{y_k} P(y_k | y^{(0)}_i) P_{T(Y_k)}(X_j = 0 | y_k)}
$$

as desired.
E Dataset preparation

In this section we provide additional details about the preparation of the two real-world datasets used in the experimental results.

**Emergency**: The emergency dataset consists of the following fields from patients’ medical records: Current medications (medication reconciliation) and Administered medications (pxis records) mapped to GSN (generic sequence number) codes; Free text concatenation of chief complaint, triage assessment and MD comments; age binned by decade; sex; and administrative ICD9 billing codes (used to establish ground truth but not visible during learning or testing). We apply negation detection to the free-text section using “negex” rules Chapman et al. (2001) with some manual adaptations appropriate for Emergency department notes Jerme et al. (2013b), and replace common bigrams with a single token (e.g. ‘chest pain’ is replaced by ‘chest pain’). We reduce the dataset from 273,174 patients to 16,268 by filtering all patients that have fewer than 2 of the manually specified conditions. We filter words to remove those that appear in more than 50% of the dataset and take the 1000 most common words after that filtering. Table 3 lists the concepts that are used and a selection of their anchors specified by a physician research collaborator. In the feature vector, anchors are replaced by a single feature which represents a union of the anchors (i.e. whether any of the anchors appear in the patient record).

**Stack Overflow**: Questions were initially filtered to remove all questions that do not have at least two of the 200 most popular tags. We filter words to remove those that appear in more than 50% of the dataset and take the 1000 most common words after that filtering. Tag names that contain multiple words are treated as N-grams that are replaced by a single token in the text.

F Detailed methods

F.1 Regularization

We found it useful to introduce an additional regularization parameter to Equation 4 that encourages the recovered marginals to be close to independent unless there is strong evidence otherwise.

$$\mu_\gamma = \arg\min_{\mu \in \mathcal{M}} D_{KL}(\mu_A, R\mu) + \lambda D_{KL}(\mu_{indep}, \mu),$$  \hspace{1cm} (23)

Where $\mu_{indep}$ is a marginal vector constructed using the single-variable marginals in an independent distribution.

For recovery of marginals under local consistency and marginal polytope constraints, we use the the conditional gradient algorithm, as discussed in section 3.2, using a tolerance of 0.005 for the duality gap (as described in Jaggi (2013)) as a stopping criterion for Stack Overflow and 0.01 for medical records. When using simplex constraints, we use the more appropriate Exponentiated Gradient algorithm Kivinen & Warmuth (1995) for each marginal independently. We use a regularization parameter of $\lambda = 0.01$ to learn $P(Y)$ and $\lambda = 0.1$ to learn $P(X|Y)$. The moments required to learn $P(X|Y)$ are recovered using only simplex constraints.

Linear programs in the conditional gradient algorithm are solved using Gurobi (Gurobi Optimization [2014]) and integer linear programs are solved using Toulbar2 (Sanchez et al. [2008]). For structure learning we use the gobnilp package (Cussens & Bartlett, 2013) with a BIC score objective, though we note that any exact or approximate structure learner that takes a decomposable score as input could be used equally well.

F.2 Conditional gradient algorithm for moment recovery

Algorithm 2 describes the conditional gradient (Frank & Wolfe [1956]) algorithm that was used for moment recovery. Line 3 minimizes a linear objective over a compact convex set. In our setting, this is the marginal polytope or its relaxations. To minimize a linear objective over a compact convex set, it suffices to search over the vertices of the set. For the marginal polytope, these correspond to the integral vertices of the local consistency polytope. Thus, this step can be solved as an integer linear program with local consistency constraints.

We use a “fully corrective” variant of the conditional gradient procedure. If the minimization of line 3 returns a vertex that has previously been used, we perform an additional step, moving to the point that minimizes the objective over convex combinations of all previously seen vertices.

F.3 Monte Carlo EM

When running EM, we optimize the variational lower bound on the data likelihood:

$$\log P(X; \theta) \geq \mathcal{L}(q, \theta) = E_{y \sim q}[\log P(x, y; \theta) - \log P(y)]$$
Algorithm 2 Moment recovery (cond. gradient)

Minimize $f(\mu) = D_{KL}(\mu_A, R\mu)$ s.t. $\mu \in M$

1: initialize $\mu^0 \in M$ (e.g. uniform)
2: for $k = 0, 1, \ldots, M$ do
3: \hspace{1em} $s \leftarrow \arg\min_{s' \in M} \langle s', \nabla f(\mu^k) \rangle$
4: \hspace{1em} Compute search direction: $d \leftarrow s - \mu^k$
5: \hspace{1em} Determine stepsize, $\gamma \in [0, 1]$
6: \hspace{1em} Move in descent direction: $\mu^{k+1} \leftarrow \mu^k + \gamma d^k$
7: end for

Algorithm 3 Alternative generative model with auxiliary $A$ variables.

$Y \sim P(Y)$

for $j$ in $1..m$
\hspace{1em} $A_j \sim P(A_j = k|Y)$
\hspace{1em} $X_j = A_j \leq n$
end for

$P(A_j = k|Y) = \begin{cases} 
(1 - f_{k,j}) \prod_{i=0}^{k-1} f_{i,j}^{w_i} & k \leq n \\
1 - \prod_{i=0}^{k-1} f_{i,j}^{h_i} & k = n + 1 
\end{cases}$

The EM algorithm optimizes this bound with a coordinate ascent, alternating between E-steps which improve $q$ and M-steps which improve $\theta$. Usually both the E-step and M-steps are maximization steps, but incomplete M-steps which only improve $\theta$ in every M-step also leads to monotonic improvement of $\log(P(X; \theta))$ with every step. In this section, we describe a variant of Monte Carlo EM which is useful for this model using Gibbs sampling to approximate the E-step and a custom M-step which is guaranteed to improve the variational lower bound at every step. We hold the distribution $P(Y)$ fixed and only optimize the failure and leak probabilities. For these purposes, the leak probabilities can be treated as simply failure probabilities of an extra latent variable whose value is always 1.

F.3.1 Outer E-step

For the E-step, we use the Gibbs sampling procedure described in 4.3 of [Wang et al.] (2014).

F.3.2 Outer M-step

The M-step consists of a coordinate step guaranteed to improve $\theta$ for a fixed $q$. $P(x, y; \theta)$ is a fully observed model, but optimizing $\theta$ has no closed form. Instead we introduce $m$ auxiliary variables $A \in [0, n + 1]^m$, and adopt the generative model described in Algorithm 3 which equivalently describes the fully-observed noisy-or model (i.e. $\sum_a P(X, Y, a; \theta) = P_{noisy or}(X, Y; \theta)$). In this new expanded model, we can perform a single E-M step with respect to the latent variable $A$ in closed form.

inner E-step:

$P(A_j = k|X_j, Y) = \begin{cases} 
1 & X_j = 0 \cap k = n + 1 \\
0 & X_j = 0 \cap k \neq n + 1 \\
0 & X_j = 1 \cap k = n + 1 \\
0 & X_j = 1 \cap k \leq n \cap Y_k = 0 \\
\propto \prod_{i=0}^{k-1} f_{i,j}^{w_i} (1 - f_{k,j}) & X_j = 1 \cap k \leq n \cap Y_k = 1 
\end{cases}$

inner M-step:

$f_{i,j} = 1 - \frac{\text{count}(A_j = i)}{\text{count}(A_j \leq i)}$ (25)

G Learned models

G.1 Tree models

Figures 7 and 8 show models learned using second order recovered moments to learn maximal scoring tree structured models. Tables 3 and 4 show the factor loadings learned for these tree-structured models.
G.2 Bounded in-degree models

Figures 9 and 10 show models learned using third order recovered moments to learn maximal scoring graphs with bounded in-degree of two.
Figure 7: Tree model learned for Emergency data. Red and green edges represent positive and negative correlations between the variables, respectively.
Figure 8: Tree model learned for Stack Overflow data. Red and green edges represent positive and negative correlations between the variables, respectively.
Figure 9: Bounded in-degree model (≤ 2) learned for Emergency data. Red and green edges represent positive and negative correlations between the variables, respectively.
Figure 10: Bounded in-degree model (≤ 2) learned for Stack Overflow data. Red and green edges represent positive and negative correlations between the variables, respectively.
Table 3: Learned medical concepts. For each concept we display the top 10 weighted factors and one supplied anchor.

| Latent variable name | Top weights                                                                 | anchors          |
|----------------------|------------------------------------------------------------------------------|------------------|
| abdominal pain       | pain, dispensed:ondansetron, nausea, neg:fevers, dispensed:morphine sulfate, vomiting, days, dispensed:hydromorphone (dilaudid), neg:vomiting | abdominal pain  |
| alcohol              | sex:m, sober, admits, found, drink, dispensed:thiamine, dispensed:folic acid, dispensed:diazepam, dispensed:multivitamins, dispensed:multivitamins | etoh             |
| allergic reaction    | disposition, initial vitals, pending, consults, interventions, trigger, imaging, ed, diagnosis, pain | allergy          |
| asthma-copd          | med-history:albuterol sulfate, sob, dispensed:methylprednisolone sodium succ, cough, nebs, med-history:spiriva with handihalet, home, days, dyspea | asthma           |
| back pain            | pain, neg:or, neg:pain, back, denies, lower back, dispensed:oxygenode-acetaminophen, neg:of, low back, neg:bowel | back pain        |
| cellulitis           | ed, admit, swelling, imaging, diagnosis, days, consults, iv, leg, interventions | cellulitis       |
| stroke               | age 80, admit, patient, head, ekg, weakness, ed, ct head, neuro, htn | stroke           |
| epistaxis            | nose, bleeding, blood, neg:bleeding, ekg, neg:with, ed, bleed, consults, today | epistaxis        |
| fall                 | pain, denies, neg:pain, p fall, neg:or, ed, imaging, consults, interventions, neg:loc | fall down        |
| gastrointestinal bleed | blood, dispensed: pantoprazole sodium, hct, gi, stool, admit, rectal, today, dispensed:lidocaine jelly 2% (urojet), | gi bleed         |
| headache             | head, nausea, today, neg:or, denies, neg:pain, days, dispensed:acetaminophen, pain, neck pain | headache         |
| hematuria            | sex:m, urine, urology, blood, Foley, neg:pain, days, neg:fevers, dysuria, bladder | hematuria         |
| hiv+                 | sex:m, med-history:truvada, cd, age 40, med-history:ritonavir, days, pcp, cough, fever, denies | hiv              |
| intracranial hemorrhage | osh, ct, fall, age 80, found, head ct, transfer, sex:m, repeat, small | ic                |
| kidney stone         | pain, flank pain, dispensed:ondansetron, dispensed:ketorolac, stone, nausea, urology, ct, dispensed:morphine sulfate | kidney stone     |
| liver failure (history) | sex:m, liver, age 50, med-history:lactulose, admit, ed, med-history:folic acid, med-history:fuosemide, med-history:multivitamin, med-history:lasix | cirrhosis        |
| motor vehicle collision | car, neg:loc, age 20, hit, neck, neg:head, driver, mph, neg:airbag, front | mvc              |
| pneumonia            | cxr, admit, sex:m, cough, ed, fever, diagnosis, ekg, med-history:icin, admit | pna              |
| severe sepsis        | dispensed: fentanyl citrate, found, icu, osh, vanc, age 80, imaging, consults, interventions, lactate | severe sepsis    |
| sexual assault       | patient, dispensed:ondansetron odt, evaluation, ceftriaxone, man, eval, age 30, home, plan, flagyl | sane nurse       |
| suicidal ideation    | depression, neg:hi, denies, states, neg:cs, plan, dispensed: lorazepam, eval, age 40, section | si               |
| syncope              | ekg, neg:pain, fall, head, fell, denies, neg:cp, ed, neg:of, loc | syncope          |
| urinary tract infection | ed, imaging, consults, interventions, diagnosis, ua, cipro, admit, pending, disposition | uti              |

Table 4: Learned concepts from Stack Overflow. For each concept we display the top weighted factors and one supplied anchor. Note that in the Stack Overflow dataset we use a simple rule to provide a single anchor for every latent variable.

| Latent variable name | Top weights                                                                 | anchors          |
|----------------------|------------------------------------------------------------------------------|------------------|
| osx                  | osx, i’m, running, i’ve, install, installed, os, code, managed, existing | osx              |
| ruby-on-rails-3      | parsing, executed, web-application, don’t, numbers, developed, resources, dynamic, named, rest | ruby-on-rails-3 |
| image                | image, code, size, upload, html, save, picture, width, page, display | image            |
| mysql                | mysql, query, rows, row, id, 1, tables, join, insert, column | mysql            |
| web-services         | web, service, web-services, client, java, services, things, sharepoint, don’t, true | web-services    |
| objective-c          | objective-c, i’m, code, don’t, xcode, dynamic, syntax, including, follow, trouble | objective-c     |
| linux                | linux, running, command, machine, system, server, directory, install, php, servers | linux            |
| query                | query, table, a, result, results, queries, tables, return, returns, select | query            |
| regex                | regex, match, expression, regular, pattern, i’m, replace, characters, html, extract | regex            |
| php                  | php, server, array, send, data, i’m, database, output, website, user | php              |
| java                 | java, string, program, client, xml, read, code, existing, send, environment | java            |
| ruby-on-rails        | ruby-on-rails, rails, i’m, database, user, controller, page, ruby, model, code | ruby-on-rails    |
| asp.net | asp.net, asp.net-mvc-3, controller, site, website, database, control, ajax, jquery, action | asp.net |
|----------|--------------------------------------------------------------------------------|---------|
| sql-server | sql-server, query, tsql, sql, sql-server-2005, server, stored, procedure, rows, columns | sql-server |
| forms | form, forms, page, code, data, html, jquery, user, button, submit | forms |
| python | python, i'm, a, program, string, list, module, output, don't, write | python |
| json | json, string, response, data, php, parse, code, format, ajax, javascript | json |
| html | html, tags, tag, page, website, code, links, show, webpage, text | html |
| iphone | iphone, app, view, device, sqlite, api, video, uitelaveimages, iOS5 | iphone |
| performance | performance, time, data, question, takes, slow, faster, run, seconds, running | performance |
| android | android, code, xml, java, activity, device, app, phone, screen, android-layout | android |
| multithreading | thread, multithreading, code, threads, application, method, data, main, run, managed | multithreading |
| xcode | xcode, project, build, app, 4, version, i've, running, debug, target | xcode |
| css | css, page, css3, jquery, style, javascript, chrome, works, width, browser | css |
| jquery | jquery, html, works, plugin, javascript, jquery-ui, css, jquery-ajax, elements, link | jquery |
| string | string, a, strings, array, characters, output, convert, json, character, split | string |
| c# | c#, function, code, methods, event, excel, click, written, program, call | c |
| javascript | javascript, jquery, js, script, works, ajax, php, button, browser, code | javascript |
| ios | ios, iphone, app, iOS5, device, i'm, ipad, 5, user, screen | ios |
| linq | linq, linq-to-sql, code, class, c#, error, xml, expression, property, collection | linq |
| xml | xml, parse, document, read, data, string, node, output, format, tag | xml |
| ajax | ajax, code, javascript, jquery-ajax, works, call, response, user, request, html | ajax |
| facebook | facebook, users, login, api, app, sdk, post, php, id, link | facebook |
| html5 | html5, html, browser, chrome, css, support, image, browsers, works, android | html5 |
| sql | sql, table, database, query, data, tsql, tables, statement, column, sqlserver | sql |
| wpf | wpf, control, window, property, binding, a, bind, controls, ui, items | wpf |
| asp.net-mvc | code, existing, browsers, query, faster, issues, row, flash, environment, program | asp.net-mvc |
| ruby | ruby, i'm, rails, running, string, install, installed, array, feed, executed | ruby |
| ipad | ipad, works, fine, screen, ios, page, device, problem, safari, app | ipad |
| c | c, program, a, write, c+, library, compile, string, array, functions | c |
| database | database, a, table, user, db, code, sql, data, store, created | database |
| arrays | arrays, i'm, 2, data, dynamic, results, saved, row, finally | arrays |
| vb.net | vb.net, string, project, work, database, code, developed, results, rest, existing | vb.net |
| .net | .net, framework, i've, windows, executed, managed, valid, don't, developed, existing | .net |
| eclipse | eclipse, project, build, installed, plugin, debug, running, folder, version, projects | eclipse |
| c++ | c++, code, i'm, managed, find, access, application, implementation, writing | c++ |
| windows | windows, windows-7, a, running, winapi, c#, machine, run, application, window | windows |
| winforms | winforms, control, user, i'm, windows, a, .net, controls, forms, don't | winforms |
| cocoa-touch | a, i'm, cocoa-touch, code, dynamic, results, follow, row, send, environment | cocoa-touch |
| sql-server-2008 | sql-server-2008, sql-server, sql, query, tsql, stored, 1, procedure, developed, resources | sql-server-2008 |