TOWERING PHENOMENA FOR THE YAMABE EQUATION ON
SYMMETRIC MANIFOLDS

FILIPPO MORABITO, ANGELA PISTOIA, AND GIUSI VAIRA

Abstract. Let \((M, g)\) be a compact smooth connected Riemannian manifold (without boundary) of dimension \(N \geq 7\). Assume \(M\) is symmetric with respect to a point \(\xi_0\) with non-vanishing Weyl's tensor. We consider the linear perturbation of the Yamabe problem

\[ (P_\varepsilon) \quad -\mathcal{L}_g u + \varepsilon u = u^{\frac{4N}{N-2}} \text{ in } (M, g). \]

We prove that for any \(k \in \mathbb{N}\), there exists \(\varepsilon_k > 0\) such that for all \(\varepsilon \in (0, \varepsilon_k)\) the problem \((P_\varepsilon)\) has a symmetric solution \(u_\varepsilon\), which looks like the superposition of \(k\) positive bubbles centered at the point \(\xi_0\) as \(\varepsilon \to 0\). In particular, \(\xi_0\) is a towering blow-up point.

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1. Introduction

Let \((M, g)\) be a compact smooth connected Riemannian manifold (without boundary) of dimension \(N \geq 3\). The Yamabe conjecture claims that the conformal class of the metric \(g\) contains a metric with constant scalar curvature. From a PDE’s point of view, it turns to be equivalent to state that the critical problem

\[ \mathcal{L}_g u + \kappa u^{\frac{N+2}{N-2}} = 0 \text{ in } M, \tag{1.1} \]

has a positive solution for some constant \(\kappa\). Here \(\mathcal{L}_g u := \Delta_g u - \frac{N-2}{4(N-1)} R_g u\) is the conformal laplacian, \(\Delta_g\) is the Laplace-Beltrami operator and \(R_g\) is the scalar curvature of the manifold. Indeed, the scalar curvature of the metric \(\tilde{g} = u^{\frac{4}{N-2}} g\) (which belongs to the conformal class of \(g\)) is equal to the constant \(4\frac{(N-1)}{N-2}\kappa\).

The Yamabe conjecture has been proved through the works of Yamabe [36], Aubin [1], Trudinger [35] and Schoen [30]. Different proofs in low dimension, i.e. \(N = 3, 4, 5\) and in the case \((M, g)\) is locally conformally flat are given by Bahri and Brezis [3] and Bahri [2].

Once the question of existence was settled, it is natural to address the problem of uniqueness of the solution. Actually the solution is unique in the case of negative scalar curvature and it is unique (up to a constant factor) in the case of zero scalar curvature, while in the case of positive scalar curvature the uniqueness does not hold true anymore as it was showed by Schoen in [31] and Pollack in [26] where examples of manifolds with a large number of high energy solutions with high Morse index were built. That is why a relevant part of the research work has been devoted to understand the structure of the set of the solutions.

In particular, Schoen in his topics course at Stanford (see [32]) conjectured that the set of solutions (in the positive case) is compact. It is important to note that in the case of the round sphere \((S^N, g_0)\) the set of solutions is not compact as proved by Obata in [24]. Schoen’s conjecture turns out to be true when the dimension of the manifold satisfies \(3 \leq N \leq 24\) as it was shown by Khuri, Marques and Schoen [16] (previous results were obtained by Schoen [33], Schoen and Zhang [34], Li and Zhu [22], Li and Zhang [21], Marques [23] and Druet [11]), while it is false when \(N \geq 25\) thanks to the examples built by Brendle [5] and Brendle and
Marques [6].

From a PDE’s point of view, the compactness issue is equivalent to establishing a priori estimates for solutions to the equation (1.1). Therefore, to study the compactness of solutions to the Yamabe equation, it is crucial to establish sharp estimates of blowing-up solutions and in particular to find out their right asymptotic profile near a blow-up point. In particular, when the compactness holds all the possible blow-up points of a sequence of solutions to (1.1) must be isolated and simple, i.e. around each blow-up point \(\xi_0\) the solution can be approximated by a so called standard bubble

\[
\hat{u}_n(x) \sim \alpha_N \frac{\mu_n^{\frac{N-2}{2}}}{(\mu_n^2 + (d_g(x,\xi_n))^2)^{\frac{N-2}{2}}} \quad \text{for some } \xi_n \to \xi_0 \text{ and } \mu_n \to 0.
\]

Let us be more precise. Let \(u_n\) be a sequence of solutions to problem (1.1). We say that \(u_n\) blows-up at a point \(\xi_0 \in M\) if there exists \(\xi_n \in M\) such that \(\xi_n \to \xi_0\) and \(u_n(\xi_n) \to +\infty\). \(\xi_0\) is said to be a blow-up point for \(u_n\). Blow-up points can be classified according to the definitions introduced by Schoen in [32]. \(\xi_0 \in M\) is an isolated blow-up point for \(u_n\) if there exists \(\xi_n \in M\) such that \(\xi_n\) is a local maximum of \(u_n\), \(\xi_n \to \xi_0\), \(u_n(\xi_n) \to +\infty\) and there exist \(c > 0\) and \(R > 0\) such that

\[
0 < u_n(x) \leq c \frac{1}{(d_g(x,\xi_n))^{\frac{N-2}{2}}} \quad \text{for any } x \in B(\xi_0, R).
\]

Moreover, \(\xi_0 \in M\) is an isolated and simple blow-up point for \(u_n\) if the function

\[
\hat{u}_n(r) := r^{\frac{N-2}{2}} \frac{1}{|\partial B(\xi_n, r)|_{g}} \int_{\partial B(\xi_n, r)} u_n d\sigma_g
\]

has a exactly one critical point in \((0, R)\).

Motivated by the previous consideration, it is natural to ask if the linear perturbation of the Yamabe problem

\[
-L_g u + \epsilon u = u^{\frac{N+2}{N-2}}, \quad u > 0, \quad \text{in } (M, g) \tag{1.2}
\]

(i) has solutions with one or more blow-up points as \(\epsilon \to 0\),

(ii) has blowing-up solutions whose blow-up points are not isolated, i.e. clustering blow-up points,

(iii) has blowing-up solutions whose blow-up points are not neither isolated nor simple, i.e. towering blow-up points.

Here we assume that the first eigenvalue of \(-L_g\) is positive and \(\epsilon\) is a small parameter. Concerning question (i), Druet in [11] proved that equation (1.2) does not have any blowing-up solution when \(\epsilon < 0\) and \(N = 3, 4, 5\) (except when the manifold is conformally equivalent to the round sphere). It is completely open the case when the dimension is \(N \geq 6\). The situation is completely different when \(\epsilon > 0\). Indeed, if \(N = 3\) no blowing-up solutions exist as proved by Li-Zhu [22], while if \(m \geq 4\) blowing-up solutions do exist as shown by Esposito, Pistoia and Vetois in [13]. In particular, if the dimension \(N \geq 6\) and the manifold is not locally conformally flat, Esposito, Pistoia and Vetois build solutions which blow-up at non-vanishing stable critical points \(\xi_0\) of the Weyl’s tensor, i.e. \(|\text{Weyl}_g(\xi_0)|_g \neq 0\). Recently, Pistoia and Vaira in [25] showed that \(\xi_0\) is a clustering blow-up point as soon as it is a non-degenerate minimum point of the Weyl’s tensor with non-vanishing Weyl’s tensor. This result gives a positive answer to question (ii). We also quote the paper [29], where Robert and Vetois built solutions having clustering blow-up points for a special class of perturbed Yamabe type equation.

In this paper, we address question (iii) and we prove that, under some symmetry assumptions, it is possible to build solutions to equation (1.2) with towering blow-up points. More precisely, our result reads as follows.
Theorem 1.1. Assume \((M, g)\) is symmetric with respect to a point \(\xi_0\) and \(|\text{Weyl}_g(\xi_0)| \neq 0\)
Assume \(N \geq 7\). For any \(k \in \mathbb{N}\), there exists \(\varepsilon_k > 0\) such that for all \(\varepsilon \in (0, \varepsilon_k)\) the problem
\[
(1.2) \text{ has a symmetric solution } u_\varepsilon, \text{ which looks like the superposition of } k \text{ positive bubbles}
\text{centered at the point } \xi_0 \text{ as } \varepsilon \to 0. \text{ In particular, } \xi_0 \text{ is a towering blow-up point.}
\]

This result is new and it is in sharp contrast with what happens in the euclidean case. Indeed, let us consider the classical Brezis-Nirenberg problem [9]
\[
\begin{cases}
-\Delta u + \varepsilon u = \frac{\varepsilon \Delta u + u^{\frac{N+2}{N-2}}}{\varepsilon} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
where \(\Omega \subset \mathbb{R}^N, \ N \geq 3\) is an open and bounded smooth domain. It is well known that it possesses blowing-up solutions when \(\varepsilon < 0\) is small enough and \(N \geq 4\) (see Han [14], Rey [27, 28] and Musso and Pistoia [18]). Actually, all the possible blow-up points of solutions to (1.3), when \(\varepsilon < 0\) and small enough, are isolated and simple, namely clustering and towering blow-up points are forbidden, as it was showed by Cerqueti in [10] using the ideas of Li [20].

The proof of our result relies on a delicate finite dimensional Ljapunov-Schmidt reduction. As usual, we need to find a good approximation of the solution and this is carried out in Section 3. The second step consists in finding the remainder term and here a lot of technic-
alities are required because we need to split the error term into the sum of remainder terms of different orders. Finally, we estimate the reduced energy and again we need to be extreme careful because the leading terms appears at different orders. All the proofs of the results are postponed to the Appendix 5, while the main steps of the reduction and the proof of Theorem 1.1 are given in Section 4. Section 2 is devoted to exhibit examples of symmetric manifolds with non-vanishing Weyl’s tensor.

Finally, we conjecture that Theorem 1.1 is true even if we drop the symmetry assumption provided that \(\xi_0\) is a non-degenerate critical point of Weyl’s tensor with non-vanishing Weyl’s tensor. The fact that the manifold is symmetric with respect to the point \(\xi_0\) simplifies considerably the proof. Indeed, we are lead to build solutions which are symmetric with respect to the point \(\xi_0\), so that in the reduction argument we only need to take care of the concentration parameters (all the bubbles are centered at the same point \(\xi_0\)). We point out that our proof cannot be adapted to the general case because the presence of different points where the bubbles are centered would not allow to split the error into the sum of terms with the required properties (in particular, property (i) of Proposition 4.3 would not be true anymore).

2. Examples of compact symmetric manifolds with non-vanishing Weyl tensor
2.1. Riemannian manifolds which are symmetric with respect to a point. We recall that if \(M\) is a compact Riemannian manifold then it is complete. Consequently for any \(p \in M\), the exponential map \(\exp_p\) is defined on the entire tangent space \(T_pM\) and any geodesic curve is defined on \(R\). Furthermore, for any point \(q \in M\), the distance of \(q\) to \(p\) equals the length of a piece of the unique geodesic curve joining \(p\) and \(q\).

Definition 2.1. A Riemannian manifold \(M\) is symmetric with respect to a point \(p\) if there exists an isometry \(H : M \to M\), such that \(H(p) = p\) and \(dH_p : T_pM \to T_pM\) satisfies
\[
dH_p = -id_{T_pM}.
\]

We observe that a geodesic curve \(\gamma : R \to M\), with \(\gamma(0) = p\) and initial velocity vector \(v \in T_pM\), can be written as \(\gamma(t) = \exp_p(tv)\). If \(H\) is an isometry, then it always holds true that
\[
H(\gamma(t)) = H(\exp_p(tv)) = \exp_{H(p)}(dH_p(tv)).
\]
If in addition $M$ is symmetric with respect to $p$ and $H$ satisfies the conditions of previous definition, then

$$H(\gamma(t)) = \exp_p(-tv) = \gamma(-t).$$

Consequently, an equivalent definition is the following.

**Definition 2.2.** $M$ is symmetric with respect to a point $p \in M$ if there exists an isometry $H : M \to M$, such that $H(\gamma(t)) = \gamma(-t)$ for any geodesic curve $\gamma : R \to M$ such that $\gamma(0) = p$. In other terms the isometry $H$ reverses the geodesic curves passing by the point $p$.

If we set $t = 1$ in (2.1) then we get that the image of $\exp_p(v)$ under the action of $H$ is $H(\exp_p(v)) = \exp_p(-v)$, for any $v \in T_p M$. Since any isometry preserves the length of curves and $M$ is complete, then $d_g(p, \exp_p(v)) = d_g(p, \exp_p(-v))$, where $d_g$ denotes the distance with respect to the metric $g$.

An example of compact manifold which is symmetric with respect to a point is the unit sphere $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}, \sum_{i=1}^{n+1} x_i^2 = 1\}$ equipped with the standard metric. $S^n$ is symmetric with respect to any point $p \in S^n$. We show that holds true in the case where $p$ coincides with the south pole $S$, the point having coordinates $(0, \ldots, 0, -1)$.

We define a map $H : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ setting $H(x_1, \ldots, x_{n+1}) = (-x_1, -x_2, \ldots, -x_{n}, x_{n+1})$. It is immediate to check this map is an isometry of $\mathbb{R}^{n+1}$. Consequently the restriction $h$ of this map to the sphere is an isometry of $S^n$ as well and it fixes the south pole $S$. Furthermore its differential satisfies $dh_S = -id_{TS^n}$.

**2.2. Riemannian manifolds with non-vanishing Weyl tensor.** It is known that a $n$-dimensional Riemannian manifold is locally conformally flat if and only if the Cotton tensor vanishes identically in the case $n = 3$ and if and only if the Weyl tensor vanishes identically in the case $n \geq 4$.

Any space having constant sectional curvature is known to be locally conformally flat: $S^n(c)$ (the sphere of radius $\frac{1}{\sqrt{c}}$) and the hyperbolic space $H^n(-c)$, $c > 0$, have sectional curvature equal to $c, -c$ respectively.

A useful procedure to produce examples of Riemannian manifolds which are not locally conformally flat consists in considering the product or more generally the warped product of (eventually locally conformally flat) manifolds.

We start by recalling the definition of warped product of two Riemannian manifolds $(B, g_B)$ and $(F, g_F)$.

The **warped product** $B \times_f F$ is the Riemannian manifold $(B \times F, g)$, where $g = g_B \otimes f^2 g_F$ and $f : B \to \mathbb{R}$ is a positive function called warping function.

Theorem 1 in [7] provides the classification of the warped products which are locally conformally flat Riemannian manifolds.

**Theorem 2.3.** We set $M := B \times_f F$.

1. If $\dim(B) = 1$, then $M$ is locally conformally flat if and only if $(F, g_F)$ is a space of constant sectional curvature.

2. If $\dim(B) > 1$, $\dim(F) > 1$, then $M$ is locally conformally flat if and only if the two following conditions are satisfied:
   - $(F, g_F)$ is a space of constant curvature;
   - the warping function $f$ defines a conformal deformation on $B$, such that $(B, \frac{1}{f^2} g_B)$ has constant sectional curvature equal to $-c_F$.

3. If $\dim(F) = 1$, then $M$ is locally conformally flat if and only if the warping function $f$ defines a conformal deformation on $B$, such that $(B, \frac{1}{f^2} g_B)$ has constant sectional curvature.

If in the definition of warped product we allow the warping function $f$ to be defined on the whole set $B \times F$, then we get the definition of **twisted product** of $(B, g_B)$ and $(F, g_F)$.

A necessary condition for a twisted product to be locally conformally flat is provided by the following theorem (Theorem 6 in [7]):
Theorem 2.4. Suppose \( \text{dim}(B) > 1, \text{dim}(F) > 1 \). If the twisted product \( B \times_f F \) is a locally conformally flat manifold then it can be expressed as warped product.

It is easy to check that a twisted product can be regarded as a warped product if and only if \( f \) is the product of two functions \( f_1, f_2 \), the first being defined on \( B \), the second being defined on \( F \). If such a condition is not satisfied the twisted product is not locally conformally flat.

The third class of manifolds we consider is the one which consists in multiply warped products.

Given the Riemannian manifolds \((B, g_B), (F_i, g_{F_i})\), with \( i \in \{1, \ldots, k\}, k \geq 2 \), and \( g_R \) the euclidean metric, then their multiply warped product \( B \times_{f_1} F_1 \times_{f_2} F_2 \times \cdots \times_{f_k} F_k \) is the Riemannian manifold \((B \times F_1 \times F_2 \times \cdots \times F_k, g)\), with \( g = g_R \otimes f_1^2 g_{F_1} \otimes \cdots \otimes f_k^2 g_{F_k} \) and \( f_i : F_i \to \mathbb{R} \) is a positive function.

2.3. Compact symmetric manifolds without boundary and non-vanishing Weyl tensor. In this subsection we explain how to use warped products in order to produce examples of compact Riemannian manifolds of dimension at least 4, without boundary, which are symmetric with respect to a point and have non-vanishing Weyl tensor.

First we explain under which conditions a warped product \( M \times_f N \) is symmetric with respect to a point if \( M, N \) are. Let \( h_M, h_N \) denote the isometries which satisfy the conditions of Definition 2.1.

Lemma 2.6. We suppose \((M, g)\) is a Riemannian manifold which is symmetric with respect to \( p \in M \), and \((N, \tilde{g})\) is a Riemannian manifold which is symmetric with respect to \( q \in N \). Then the warped product \((M \times N, G)\), with \( G = g \otimes f^2 \tilde{g} \), is symmetric with respect to the point \((p, q) \in M \times N\), if the warping function \( f : M \to \mathbb{R}^+ \) satisfies \( f \circ h_M = f \). In other terms \( f \) is invariant under the action of the isometry \( h_M \).

Proof. The map \( h : M \times N \to M \times N \), defined as \( \pi_M \circ h = h_M, \pi_N \circ h = h_N \), where \( \pi_M : M \times N \to M, \pi_N : M \times N \to N \), denotes the projections, is an isometry of \( M \times N \).

In order to show that, we assume that
- \( r \in M, s \in N \).
- \( V, W \in T_{(r, s)}M \times N \cong T_r M \oplus T_s N \).
- \( V_1, V_2 \) are the projections of \( V \) on \( T_r M \) and \( T_s N \).
- \( W_1, W_2 \) are the projections of \( W \) on \( T_r M \) and \( T_s N \).

\( h \) is an isometry if it is a diffeomorphism (the proof of this is immediate) and

\[
G_{h(r, s)}(dh_{(r, s)}(V), dh_{(r, s)}(W)) = G_{(r, s)}(V, W).
\]

By definition of the metric \( G \), the right hand side equals

\[
gh_{(r)}((dh_M)_r(V_1), (dh_M)_r(W_1)) + [f(h_M(r))]^2 gh_{N(s)}((dh_N)_s(V_2), (dh_N)_s(W_2)).
\]

Using the fact that \( h_M \) and \( h_N \) are isometries and \( f \circ h_M = f \), we can write that as:

\[
g_r(V_1, W_1) + f^2(r)g_s(V_2, W_2) = G_{(r, s)}(V, W).
\]

It remains to show that \( dh_{(p, q)} \) coincides with the antipodal map on \( T_{(p, q)}M \times N \cong T_p M \oplus T_q N \). This follows from: \( dh_{(p, q)} = ((dh_M)_p, (dh_N)_q) = (-id_{T_p M}, -id_{T_q N}) = -id_{T_p M \oplus T_q N}. \)
The $n$-spheres $S^n$ are examples of compact manifolds which are symmetric with respect to a point, but their Weyl tensor vanishes identically because they are locally conformally flat manifolds. We can obtain manifolds which are not locally conformally flat if we take the product of at least two spheres.

The Riemannian manifold $(P, g_P) = (S^n \times S^m, g_{S^n} \otimes f^2 g_{S^m})$ with $n, m \geq 2$ is not locally conformally flat for any choice of the warping function $f$ which does not satisfy the hypotheses of Theorem 2.3, part (2). In particular when $f$ is a constant function.

Such manifolds are also symmetric with respect to a point provided $f$ satisfies the condition of Lemma 2.6.

Alternatively, we can consider the twisted product of two spheres.

Products of an higher number of spheres can be shown to be not locally conformally flat, writing it as a product of two manifolds and using induction. For example $S^1 \times S^n \times S^m$ equipped with the metric $g_{S^1} \otimes g_{S^n} \otimes g_{S^m}$ is not locally conformally flat, because we can write it as product of $S^1$ and the manifold $P$ constructed above, with $f \equiv 1$. Now we can use again Theorem 2.3, because $(P, g_P)$ has non-constant sectional curvature. The last assertion follows from the fact that if $P$ was a manifold with constant sectional curvature then it would be locally conformally flat.

Similarly, the multiply warped product $(M_k, g_k) = (S^n \times S^{m_1} \times \cdots S^{m_k}, g_{S^n} \otimes f_1^2 g_{S^{m_1}} \otimes \cdots f_k^2 g_{S^{m_k}})$, $n, m_i \geq 2$, $k \geq n + 3$, is not locally conformally flat for any choice of the warping functions $f_i$, according to Lemma 2.5.

In order to study the symmetry, we observe that by Lemma 2.6 we can show the symmetry $(M_k, g_k)$ arguing by induction. The product $M_1$ is symmetric if $f_1 \circ h_2 = f_1$, where $h_2$ is the isometry of $S^n$. The product $M_{i+1}$ is symmetric if $f_{i+1} \circ h_i = f_{i+1}$, where $h_i$ is the isometry of $M_i$, with $i \in \{1, \ldots, k - 1\}$.

Examples with same structure are those ones we get if we replace the spheres by other compact manifolds. Let us consider the $n$-dimensional ellipsoids, $n \geq 2$, centered at the origin,

$$\left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}, \sum_{i=1}^{n+1} \frac{x_i^2}{a_i^2} = 1 \right\},$$

with $a_i > 0$, endowed with the metric induced by the euclidean one. Note that if the semi-axis length $a_i = r$ for any $i \in \{1, \ldots, n + 1\}$, then we get the $n$-sphere of radius $r$.

A direct computation shows that an $n$-dimensional ellipsoid in $\mathbb{R}^{n+1}$ equipped with the metric induced by the euclidean one, is not locally conformally flat if $n \geq 3$ and at least three of the semi-axis lengths $a_i$ are different. See also the Proposition by Cartan, Schouten in [17].

Using $m$-dimensional ellipsoids, $m \geq 2$, for which at least two of the semi-axis lengths are different (this hypothesis ensures that their curvature is not constant), then, in view of Theorem 2.3, we can construct warped products which are not locally conformally flat. As in the case of product of spheres, we can show the symmetry of these examples using the symmetry of each ellipsoid with respect to one of its vertices.

Of course there are plenty of other examples, the ones we presented here are relatively easy to describe.

### 2.4. Examples of symmetric manifolds with nowhere vanishing Weyl tensor.

In view of previous considerations, we already know that the product of sphere is symmetric with respect to a point. We finish the section by showing that a product of spheres is is example of compact Riemannian manifold which has nowhere vanishing Weyl tensor.

The sphere $S^m$ equipped with the standard metric enjoys the following property: Isometries of $S^m$ act transitively, that is for each fixed pair of distinct points $p, q \in S^m$, there exists an isometry $H : S^m \to S^m$, such that $H(p) = q$. This property is clearly inherited by any product $S^{m_1} \times S^{m_2}$.

Now we assume $m_1, m_2 \geq 2$. Since the Weyl tensor is preserved by isometries, if the Weyl tensor vanishes at $(p_1, p_2) \in S^{m_1} \times S^{m_2}$ then it vanishes also at the point $H(p_1, p_2)$, where
$H$ is an isometry from $S^{m_1} \times S^{m_2}$ onto itself. Since $H(p_1, p_2)$ can be chosen arbitrarily this would show that the Weyl tensor vanishes at each point. That says $S^{m_1} \times S^{m_2}$ would be locally conformally flat, and that contradicts Theorem 2.3, part (2), with $f \equiv 1$.

The same proof applies to multiple products of spheres.

3. The ansatz

3.1. Preliminaries. We will assume that $M$ is symmetric with respect to a point $\xi$ with $|\text{Weyl}_g(\xi)|_g \neq 0$. We will also assume that $M$ has dimension $N \geq 7$.

The main ingredient in our construction are the euclidean bubbles

$$U_{\mu, y}(x) = \mu^{-\frac{N-2}{4}} U \left( \frac{x-y}{\mu} \right), \; x, y \in \mathbb{R}^N, \; \mu > 0,$$

where

$$U(x) := \alpha_N \frac{1}{(1 + |x|^2)^{\frac{N-2}{2}}}. \quad (3.1)$$

Here $\alpha_N := N(N-2)^{\frac{N-4}{2}}$. They are all the solutions to the critical equation in the Euclidean space

$$-\Delta U = U^{\frac{N+2}{N-2}} \text{ in } \mathbb{R}^N. \quad (3.2)$$

Let us consider the euclidean bubble $U_{\mu, 0}$, centered around the origin (see (3.1)), which via a geodesic normal coordinate system around the point $\xi \in M$ read as

$$U_{\mu, \xi}(z) = U_{\mu, 0} \left( \exp^{-1}_\xi(z) \right) = \mu^{-\frac{N-2}{2}} U \left( \frac{\exp^{-1}_\xi(z) \mu}{\mu} \right) \text{ if } d_g(\xi, z) \text{ is small enough.}$$

A comparison between the conformal laplacian with the euclidean laplacian of the bubble shows that there is an error, which at main order looks like

$$L_g U_{\mu, \xi} - \Delta U_{\mu, \xi} \sim -\frac{1}{3} \sum_{a,b,i,j=1}^N R_{iabj}(\xi)x_a x_b \partial^2_{ij} U_{\mu, 0} + \sum_{i,l,k=1}^N \partial_i \Gamma_{ij}^k(\xi)x_l \partial_k U_{\mu, 0} + \beta_N R_g(\xi) U_{\mu, 0}. \quad (3.3)$$

Here $\beta_N := \frac{N-2}{4(N-1)}$, $R_{iabj}$ denotes the Riemann curvature tensor, $\Gamma_{ij}^k$ the Christoffel’s symbols and $R_g$ the scalar curvature. This easily follows by standard properties of the exponential map, which imply

$$-\Delta_g u = -\Delta u - (g^{ij} - \delta^{ij}) \partial^2_{ij} u + g^{ij} \Gamma_{ij}^k \partial_k u, \quad (3.4)$$

with

$$g^{ij}(x) = \delta^{ij}(x) - \frac{1}{3} R_{iabj}(\xi)x_a x_b + O(|x|^3) \text{ and } g^{ij}(x) \Gamma_{ij}^k(x) = \partial_l \Gamma_{iji}^l(x) x_l + O(|x|^2). \quad (3.5)$$

To build our solution it shall be necessary to kill the R.H.S of (3.3) by adding to the bubble an higher order term $V$ whose existence has been established in [12]. To be more precise, we need to remind (see [4]) that the all the solutions to the linear problem

$$-\Delta v = p U^p u \text{ in } \mathbb{R}^N, \quad (3.6)$$

are linear combinations of the functions

$$\psi^0(x) = x \cdot \nabla U(x) + \frac{N-2}{2} U(x) \text{ and } \psi^i(x) = \partial_i U(x), \; i = 1, \ldots, N. \quad (3.7)$$

The correction term $V$ is built in the following proposition (see Section 2.2 in [12]).

**Proposition 3.1.** There exist $\nu(\xi) \in \mathbb{R}$ and a function $V \in D^{1,2}(\mathbb{R}^N)$ solution to

$$-\Delta V - f'(U)V = -\sum_{a,b,i,j=1}^N \frac{1}{3} R_{iabj}(\xi)x_a x_b \partial^2_{ij} U + \sum_{i,l,k=1}^N \partial_l \Gamma_{ili}^l(\xi)x_l \partial_k U + \beta_N R_g(\xi) U + \nu(\xi) \psi^0 \text{ in } \mathbb{R}^N, \quad (3.8)$$
with
\[ \int_{\mathbb{R}^N} V(x)\psi^i(x)dx = 0, \quad i = 0, 1, \ldots, N \]
and
\[ |V(x)| + |x| |\partial_k V(x)| + |x|^2 |\partial^2_{ij} V(x)| = O \left( \frac{1}{(1 + |x|^2)^{N-1}} \right), \quad x \in \mathbb{R}^N. \quad (3.9) \]

3.2. The tower. Let \( r_0 \) be a positive real number less than the injectivity radius of \( M \), and \( \chi \) be a smooth cutoff function such that \( 0 \leq \chi \leq 1 \) in \( \mathbb{R} \), \( \chi \equiv 1 \) in \([-r_0/2, r_0/2] \), and \( \chi \equiv 0 \) out \([-r_0, r_0]\). For any positive real number \( \mu_j \), we define \( W_j \) by
\[ W_j(z) := \chi(d_g(z, \xi))\mu_j^{-\frac{N-2}{2}} U \left( \frac{\exp^{-1}(z)}{\mu_j} \right) + \mu_j^2 \chi(d_g(z, \xi))\mu_j^{-\frac{N-2}{2}} V \left( \frac{\exp^{-1}(z)}{\mu_j} \right), \quad z \in M \]
(3.10)
where the functions \( U \) and \( V \) are defined, respectively, in (3.1) and (3.8).

We look for symmetric solutions of (1.2), according to the following definition.

Definition 3.2. We say that a function \( u : M \to \mathbb{R} \) is symmetric if \( u(H(x)) = u(x) \) for any \( x \in M \). \( H \) is the isometry introduced in Definition (2.2).

More precisely, we look for symmetric solutions of (1.2) of the form
\[ u_\varepsilon(z) := \sum_{j=1}^k W_j(z) + \Phi_\varepsilon(z) \quad (3.11) \]
where each term \( W_j \) is defined in (3.10), and for any \( j = 1, \ldots, k \) the concentration parameter \( \mu_j \) satisfies
\[ \mu_j = d_j\varepsilon^{\gamma_j} \quad \text{with} \quad d_1, \ldots, d_k \in (0, +\infty) \quad \text{and} \quad \gamma_j := \left( \frac{N-2}{N-6} \right) j^{-1} - \frac{1}{2}. \quad (3.12) \]
We point out that the choice the concentration rate for \( \mu_j \) is motivated by the fact that (see the expansion of the reduced energy in (4.16))
\[ \mu_1^4 \sim \varepsilon \mu_1^4 \quad \text{and} \quad \left( \frac{\mu_j}{\mu_{j-1}} \right)^{\frac{N-2}{4}} \sim \varepsilon \mu_j^2 \text{ for any } j \geq 2. \]

The remainder term \( \Phi_\varepsilon \) shall be splitted into the sum of \( k \) terms of different order
\[ \Phi_\varepsilon(z) := \sum_{\ell=1}^k \phi_{\ell, \varepsilon}(z), \quad z \in M \]
(3.13)
where each remainder term \( \phi_{\ell, \varepsilon} \) only depends on \( d_1, \ldots, d_\ell \), it is symmetric according to Definition 3.2 and it belongs to the space \( K^\perp_\ell \) defined in (3.18).

3.3. Setting of the problem. We provide the Sobolev space \( H^1_g(M) \) with the scalar product
\[ \langle u, v \rangle = \int_M \langle \nabla u, \nabla v \rangle_g \, dv_g + \beta_N \int_M R_g uv \, dv_g \quad (3.14) \]
where \( dv_g \) is the volume element of the manifold. We let \( \| \cdot \| \) be the norm induced by \( \langle \cdot, \cdot \rangle \). Moreover, for any function \( u \) in \( L^q(M) \) and for any \( A \subset M \), we let \( |u|_{q, A} = \left( \int_A |u|^q \, dv_g \right)^{1/q} \).

We let \( \iota^* : L^{2N/(N+2)}(M) \to H^1_g(M) \) be the adjoint operator of the embedding \( 1 : H^1_g(M) \hookrightarrow L^{2N/(N+2)}(M) \), i.e. for any \( w \) in \( L^{2N/(N+2)}(M) \), the function \( u = \iota^*(w) \) in \( H^1_g(M) \) is the unique solution...
of the equation \(-L_g u = w\) in \(M\). By the continuity of the embedding of \(H^1_g(M)\) into \(L^{2^*}(M)\), we get
\[
\|i^* (w)\| \leq C \|w\| \frac{2^*}{2^* + 2} \tag{3.15}
\]
for some positive constant \(C\) independent of \(w\). We rewrite problem (1.2) as
\[
u = i^* [f(u) - \varepsilon u], \quad u \in H^1_g(M) \tag{3.16}
\]
where we set \(f(u) := (u^+)^p\) with \(p = \frac{N+2}{N-2}\).

For any \(j, \varepsilon\) we set
\[
Z_j^0(z) := \chi(d_g(z, \xi))\mu_j \frac{N+2}{2} \psi^0 \left( \frac{\exp_\xi^{-1}(z)}{\mu_j} \right), \quad z \in M
\]
where the function \(\psi^0\) is defined in (3.7) and for any integer \(\ell = 1, \ldots, k\), we define the subspaces
\[
K_\ell := \text{Span} \{i^* (Z_j^0), \ j = 1, \ldots, \ell\} \\
K_\ell^\perp := \{ \phi \in H^1_g(M) : \phi \text{ is symmetric and } \langle \phi, i^* (Z_j^0) \rangle = 0, \ j = 1, \ldots, \ell\}. \tag{3.18}
\]

We also define \(\Pi_\ell\) and \(\Pi_\ell^\perp\) the projections of the Sobolev space \(H^1_g(M)\) onto the respective subspaces \(K_\ell\) and \(K_\ell^\perp\).

In order to solve equation (3.16), we shall solve the system
\[
\begin{align*}
\Pi_k \{u_\varepsilon - i^* [f(u_\varepsilon) - \varepsilon u_\varepsilon]\} &= 0, \tag{3.19} \\
\Pi_k \{u_\varepsilon - i^* [f(u_\varepsilon) - \varepsilon u_\varepsilon]\} &= 0 \tag{3.20}
\end{align*}
\]
where \(u_\varepsilon\) is given in (3.11).

4. The Ljapunov-Schmidt Procedure

4.1. The remainder term: solving equation (3.19). In order to find the remainder term \(\Phi_\varepsilon\), we shall find functions \(\phi_{j, \varepsilon}\) for any \(j = 1, \ldots, k\), which solve the following system of \(k\) equations
\[
\begin{align*}
\mathcal{E}_1 + \mathcal{S}_1 (\phi_{1, \varepsilon}) + \mathcal{N}_1 (\phi_{1, \varepsilon}) &= 0 \\
\mathcal{E}_2 + \mathcal{S}_2 (\phi_{2, \varepsilon}) + \mathcal{N}_2 (\phi_{1, \varepsilon}, \phi_{2, \varepsilon}) &= 0 \\
&\vdots \\
\mathcal{E}_k + \mathcal{S}_k (\phi_{k, \varepsilon}) + \mathcal{N}_k (\phi_{1, \varepsilon}, \ldots, \phi_{k, \varepsilon}) &= 0. \tag{4.1}
\end{align*}
\]
The error terms \(\mathcal{E}_\ell\) are defined by
\[
\mathcal{E}_1 := \Pi_\ell^\perp \{W_1 - i^* [f(W_1) - \varepsilon W_1]\} \tag{4.2}
\]
and
\[
\mathcal{E}_\ell := \Pi_\ell^\perp \left\{ W_\ell - i^* \left[ f \left( \sum_{j=1}^{\ell} W_j \right) - f \left( \sum_{j=1}^{\ell-1} W_j \right) - \varepsilon W_\ell \right] \right\}, \quad \ell \geq 2. \tag{4.3}
\]
The linear operators \(\mathcal{S}_\ell\) are defined by for \(\ell = 1, \ldots, k\)
\[
\mathcal{S}_\ell (\phi_{\ell, \varepsilon}) := \Pi_\ell^\perp \left\{ \phi_{\ell, \varepsilon} - i^* \left[ f' \left( \sum_{j=1}^{\ell} W_j \right) \phi_{\ell, \varepsilon} - \varepsilon \phi_{\ell, \varepsilon} \right] \right\}. \tag{4.4}
\]
The higher order terms \(\mathcal{N}_\ell\) are defined by
\[
\mathcal{N}_1 (\phi_{1, \varepsilon}) := \Pi_1^\perp \left\{ -i^* \left[ f (W_1 + \phi_{1, \varepsilon}) - f (W_1) - f' (W_1) \phi_{1, \varepsilon} \right] \right\} \tag{4.5}
\]
and

$$
\mathcal{N}_\ell(\phi_1,\ldots,\phi_\ell,\varepsilon) := \Pi^{\perp}_\ell \left\{ -1^\ast \left[ f \left( \sum_{j=1}^\ell (W_j + \phi_{j,\varepsilon}) \right) - f \left( \sum_{j=1}^\ell W_j \right) \right] - f' \left( \sum_{j=1}^\ell W_j \right) \phi_{\ell,\varepsilon} - f' \left( \sum_{j=1}^{\ell-1} (W_j + \phi_{j,\varepsilon}) + f \left( \sum_{j=1}^{\ell-1} W_j \right) \right) \right\}, \quad \ell \geq 2.
$$

(4.6)

In order to solve system (4.1), first of all we need to evaluate the $H^1_0(M)$–norm of the error terms $\mathcal{E}_\ell$. This is done in the following lemma whose proof is postponed in Section 5.

**Lemma 4.1.** For any $\ell = 1,\ldots,k$ and for any compact subset $A_\ell \subset (0,+\infty)^\ell$ there exists a positive constant $C$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$ and for any $(d_1,\ldots,d_\ell) \in A_\ell$ there holds

$$
\| \mathcal{E}_\ell \| \leq C \begin{cases} 
\mu^\ell \varepsilon^2 + \varepsilon \mu^\ell N^{\frac{1}{2}} & \text{if } N = 7 \\
\mu^3 (\ln N)^{\frac{1}{2}} \varepsilon^2 + \varepsilon \mu^\ell N^{\frac{1}{2}} & \text{if } N = 8 \\
\mu^3 + \varepsilon \mu^\ell N^{\frac{1}{2}} & \text{if } N \geq 9,
\end{cases}
$$

where we agree that if $\ell = 1$ the interaction term $\frac{\mu}{\mu - 1}$ is zero. In particular, by the choice of $\mu$’s in (3.12) we deduce

$$
\| \mathcal{E}_1 \| = \begin{cases} 
O \left( \varepsilon^2 \right) & \text{if } N = 7, \\
O \left( \varepsilon^2 \ln \varepsilon \right) & \text{if } N = 8, \\
O \left( \varepsilon^2 \right) & \text{if } N \geq 9,
\end{cases}
$$

(4.7)

$$
\| \mathcal{E}_\ell \| = O \left( \varepsilon^{\theta_\ell} \right) \text{ if } \ell \geq 2,
$$

(4.8)

where

$$
\theta_\ell := (\gamma_\ell - \gamma_\ell - 1) \frac{N - 2}{2} + 1 + 2\gamma_\ell = 2 \left( \frac{N - 2}{N - 6} \right)^{\ell - 1}, \quad \ell \geq 2.
$$

(4.9)

Next, we need to understand the invertibility of the linear operators $S_\ell$. This is done in the following lemma whose proof can be carried out as in [18].

**Lemma 4.2.** For any $\ell = 1,\ldots,k$ and for any compact subset $A_\ell \subset (0,+\infty)^\ell$ there exists a positive constant $C$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$ and for any $(d_1,\ldots,d_\ell) \in A_\ell$ there holds

$$
\| \mathcal{L}_\ell(\phi_\ell) \| \geq C \| \phi_\ell \| \text{ for any } \phi_\ell \in K^1_\ell.
$$

(4.10)

Finally, we are able to solve system (4.1). This is done in the following proposition, whose proof is postponed in Section 5 and relies on a sophisticated contraction mapping argument.

**Proposition 4.3.** For any compact subset $A \subset (0,\infty)^k$ there exists a positive constant $C$ and $\varepsilon_0$ such that for $\varepsilon \in (0,\varepsilon_0)$, for any $(d_1,\ldots,d_k) \in A$ and for any $\ell = 1,\ldots,k$ there exists a unique function $\phi_{\ell,\varepsilon} \in K^1_\ell$ which solves the $\ell$–th equation in (4.1) such that

(i) $\phi_{\ell,\varepsilon}$ depends only on $d_1,\ldots,d_\ell$;

(ii) the map $(d_1,\ldots,d_\ell) \to \phi_{\ell,\varepsilon}(d_1,\ldots,d_\ell)$ is of class $C^1$ and

$$
\| \phi_{1,\varepsilon} \| = \begin{cases} 
O \left( \varepsilon^2 \right) & \text{if } N = 7, \\
O \left( \varepsilon^2 \ln \varepsilon \right) & \text{if } N = 8, \\
O \left( \varepsilon^2 \right) & \text{if } N \geq 9,
\end{cases}
$$

$$
\| \phi_{\ell,\varepsilon} \| = O \left( \varepsilon^{\theta_\ell} \right) \text{ if } \ell \geq 2, \quad (\theta_\ell \text{ is defined in (4.9))},
$$

(4.11)
Moreover,
\[ \|\nabla_{(d_1, \ldots, d_k)} \phi_{\ell, \varepsilon}\| = o(1). \tag{4.12} \]

(iii) there exists \( \rho > 0 \) such that
\[ \sup_{d_{\varepsilon}(x, \varepsilon) \leq \rho \mu_{\varepsilon}} |\phi_{\ell, \varepsilon}(x)| = O \left(\mu_{\ell}^{\frac{N-2}{2}}\right). \tag{4.13} \]

4.2. The reduced problem: proof of Theorem 1.1. Let us define the energy \( J_{\varepsilon} : H^1_{\phi}(M) \to \mathbb{R} \)
\[ J_{\varepsilon}(u) := \frac{1}{2} \int_M (|\nabla u|^2 + \beta_M \, R_M \, u^2 + c \, u^2) \, d\nu_M - \frac{1}{p+1} \int_M (u^+)^{p+1} \, d\nu_M, \tag{4.14} \]
whose critical points are solutions to the problem (1.2).

Let us introduce the reduced energy, defined if \( (d_1, \ldots, d_k) \in (0, +\infty)^k \) by
\[ \tilde{J}_{\varepsilon}(d_1, \ldots, d_k) := J_{\varepsilon} \left( \sum_{j=1}^k W_j + \Phi_{\varepsilon} \right), \tag{4.15} \]
where the remainder term \( \Phi_{\varepsilon} = \sum_{j=1}^k \phi_{j, \varepsilon} \) and the \( \phi_{j, \varepsilon} \)'s are defined in Proposition 4.3.

The following result allows as usual to reduce our problem to a finite dimensional one. The proof is quite involved and it is postponed in Section 5.

\textbf{Proposition 4.4.} \hspace{1em} (i) \( \sum_{j=1}^k W_j + \Phi_{\varepsilon} \) is a solution to (1.2) if and only if \( (d_1, \ldots, d_k) \in (0, +\infty)^k \) is a critical point of the reduced energy (4.15)
(ii) The following expansion holds true
\[ \tilde{J}_{\varepsilon}(d_1, \ldots, d_k) := D_N + \varepsilon^2 \left[ -A_N |\text{Weyl}_y(\xi)|^2 d_1^4 + B_N d_1^2 + \Upsilon_1 \right] \]
\[ + \sum_{\ell=2}^k \varepsilon^\theta_{\ell} \left[ -C_N \left( \frac{d_{\ell}}{d_{\ell-1}} \right)^{\frac{N-2}{2}} + B_N d_{\ell}^2 + \Upsilon_{\ell} \right] \tag{4.16} \]
as \( \varepsilon \to 0 \), uniformly with respect to \( (d_1, \ldots, d_k) \) in compact subsets of \( (0, +\infty)^k \). Here \( \theta_{\ell} \) is defined in (4.9), \( A_N, B_N, C_N, D_N \) are positive constants and the higher order terms \( \Upsilon_{\ell} = \Upsilon_{\ell}(d_1, \ldots, d_{\ell}) \) are smooth functions such that \( \vert \Upsilon_{\ell} \vert = o(1) \).

\textbf{Proof of Theorem 1.1.} By (i) of Proposition (4.4), it is sufficient to find a critical point of the reduced energy \( \tilde{J}_{\varepsilon} \). By (ii) of Proposition (4.4), it is sufficient to find a critical point of the function
\[ F_{\varepsilon}(d_1, \ldots, d_k) := \sum_{\ell=1}^k \varepsilon^\theta_{\ell} \left( G_{\ell}(d_1, \ldots, d_{\ell}) + o_{\ell}(1) \right) \tag{4.17} \]
where
\[ G_1(d_1) := -A_N |\text{Weyl}_y(\xi)|^2 d_1^4 + B_N d_1^2 \]
and
\[ G_{\ell}(d_1, \ldots, d_{\ell}) := -C_N \left( \frac{d_{\ell}}{d_{\ell-1}} \right)^{\frac{N-2}{2}} + B_N d_{\ell}^2 \text{ if } \ell = 2, \ldots, k. \]
Here \( o_{\ell}(1) \) only depends on \( d_1, \ldots, d_{\ell} \) and \( o_{\ell}(1) \to 0 \) as \( \varepsilon \to 0 \) uniformly with respect to \( (d_1, \ldots, d_{\ell}) \) in compact subsets of \( (0, +\infty)^{\ell} \) . We shall prove that \( F_{\varepsilon} \) has a maximum point. The claim will follow.
First, the function $G_1$ has a unique critical point $d_1^*$ which is a global maximum. In particular, given $\delta > 0$ there exists $\sigma_1 > 0$ such that
\[
G_1(d_1) \leq G_1(d_1^*) - \delta \text{ if } |d_1 - d_1^*| = \sigma_1.
\] (4.18)

Now, for any $\ell = 2, \ldots, k$ the function $d_\ell \rightarrow G_\ell(d_1^*, \ldots, d_\ell^*, d_\ell)$ has a unique critical point $d_\ell^*$ which is a global maximum. In particular, given $\delta > 0$ there exists $\sigma_\ell > 0$ such that
\[
G_\ell(d_1^*, \ldots, d_{\ell-1}^*, d_\ell) \leq G_\ell(d_1^*, \ldots, d_{\ell-1}^*, d_\ell^*) - \delta \text{ if } |d_\ell - d_\ell^*| = \sigma_\ell.
\] (4.19)

We consider the compact set $K := [d_1^* - \sigma_1, d_1^* + \sigma_1] \times \cdots \times [d_k^* - \sigma_k, d_k^* + \sigma_k]$. For any $\epsilon$ small enough, there exists a $(d_1^*, \ldots, d_k^*) \in K$ such that $F_{\epsilon}(d_1^*, \ldots, d_k^*) := \max_K F_{\epsilon}$. First of all, let us prove that
\[
\lim_{\epsilon \to 0} d_{\ell}^* = d_{\ell}^* \text{ for any } \ell = 1, \ldots, k.
\] (4.20)

Let us start with $\ell = 1$. We know that $F_{\epsilon}(d_1^*, \ldots, d_k^*) \geq F_{\epsilon}(d_1^*, d_2^*, \ldots, d_k^*)$, then by (4.17) we deduce that
\[
\epsilon^\theta_1 [G_1(d_1^*) - G_1(d_1^*) + o(1)] \geq 0,
\]
which implies
\[
G_1(d_1^*) \geq G_1(d_1^*) + o(1).
\]

On the other hand, since $d_1^*$ is the maximum of $G_1$ we also have
\[
G_1(d_1^*) \geq G_1(d_1^*).
\]

Combining the two inequalities and passing to the limit we get $\lim_{\epsilon \to 0} G_1(d_1^*) = G_1(d_1^*)$ and so (4.20) follows. Assume that (4.20) holds for $\ell = 1, \ldots, i-1$ and let us consider the case $\ell = i$. We know that
\[
F_{\epsilon}(d_1^*, \ldots, d_k^*) \geq F_{\epsilon}(d_1^*, d_i^*, d_{i+1}^*, \ldots, d_k^*),
\]
then by (4.17) we deduce that
\[
\epsilon^\theta_i [G_i(d_1^*, \ldots, d_i^*) - G_i(d_1^*, \ldots, d_i^*) + o(1)]
\]
\[
= \epsilon^\theta_i \left[ G_i(d_1^*, \ldots, d_i^*) - G_i(d_1^*, \ldots, d_{i-1}^*, d_i^*) + G_i(d_1^*, \ldots, d_{i-1}^*, d_i^*) - G_i(d_1^*, \ldots, d_i^*) + o(1) \right] \geq 0,
\]
which implies
\[
G_i(d_1^*, \ldots, d_{i-1}^*, d_i^*) - G_i(d_1^*, \ldots, d_i^*) + o(1) \geq 0.
\]

On the other hand, since $d_i^*$ is the maximum of $G_i(d_1^*, \ldots, d_{i-1}^*, \ldots)$ we also have
\[
G_i(d_1^*, \ldots, d_i^*) \geq G_i(d_1^*, \ldots, d_{i-1}^*, d_i^*).
\]

Combining the two inequalities and passing to the limit we get $\lim_{\epsilon \to 0} G_i(d_1^*, \ldots, d_{i-1}^*, d_i^*) = G_i(d_1^*, \ldots, d_{i-1}^*, d_i^*)$ and so (4.20) follows.

Now, let us prove that if $\epsilon$ is small enough $(d_1^*, \ldots, d_k^*) \not\in \partial K$. Assume, that $|d_\ell^* - d_\ell^*| = \sigma_i$. for some $i \geq 1$. We have
\[
F_{\epsilon}(d_1^*, \ldots, d_i^*, \ldots, d_k^*) \geq F_{\epsilon}(d_1^*, \ldots, d_i^*, \ldots, d_k^*).
\]

On the other hand, by (4.18) we deduce that
\[
F_{\epsilon}(d_1^*, \ldots, d_i^*, \ldots, d_k^*) - F_{\epsilon}(d_1^*, \ldots, d_i^*, \ldots, d_k^*)
\]
\[
= \epsilon^\theta_1 [G_i(d_1^*, \ldots, d_i^*) - G_i(d_1^*, \ldots, d_i^*) + o(1)]
\]
\[
= \epsilon^\theta_i \left[ G_i(d_1^*, \ldots, d_i^*) - G_i(d_1^*, \ldots, d_{i-1}^*, d_i^*) + G_i(d_1^*, \ldots, d_{i-1}^*, d_i^*) - G_i(d_1^*, \ldots, d_i^*) + o(1) \right] < 0
\]
and a contradiction arises.
5. Appendix

Proof of Lemma 4.1. When $\ell = 1$ we argue exactly as in Lemma 3.1 of [12]. Let us focus on the case $\ell \geq 2$.

It is useful to point out that by (3.9) in geodesic coordinate

\[ |W_j(x)| \leq c \frac{\mu_j^{\frac{N-2}{2}}}{(\mu_j^2 + |x|^2)^{\frac{N-2}{2}}}, \quad x \in B(0, r_0). \tag{5.1} \]

Since $\Pi^\perp \ell [i^* (\nu(\xi) Z^0_\ell)] = 0$ then we have

\[
\| E_\ell \| = \left\| \Pi^\perp \ell \left\{ W_\ell - i^* \left[ f \left( \sum_{i=1}^{\ell} W_i \right) - f \left( \sum_{i=1}^{\ell-1} W_i \right) - \nu(\xi) Z^0_\ell \right] \right\} \right\| \\
\leq c \left| -\mathcal{L}_g W_\ell + \nu(\xi) Z^0_\ell - f \left( \sum_{i=1}^{\ell} W_i \right) + f \left( \sum_{i=1}^{\ell-1} W_i \right) \right| \frac{N}{N+2} \tag{5.2}
\]

Arguing as in Lemma 3.1 of [12] we get that

\[
(I) \leq \begin{cases} 
\frac{\mu_\ell^2}{\mu_\ell^2 - 1} & \text{if } N = 7 \\
\mu_\ell^3 \ln \mu_\ell \frac{\mu_\ell^2}{\mu_\ell^2 - 1} + \varepsilon \mu_\ell^2 & \text{if } N = 8 \\
\mu_\ell^3 + \varepsilon \mu_\ell^2 & \text{if } N \geq 9. 
\end{cases} \tag{5.3}
\]

Now, let us prove that

\[
(II) = O \left( \left( \frac{\mu_\ell}{\mu_\ell - 1} \right)^{\frac{N+2}{N+2}} \right) \tag{5.4}
\]

For any $\ell = 1, \ldots, k$ we introduce the set of disjoint annuli

\[ A_h := B_\xi(\sqrt{\mu_{h-1}} \mu_h) \setminus B_\xi(\sqrt{\mu_h \mu_{h+1}}), \quad h = 1, \ldots, \ell \tag{5.5} \]

where we agree that $\mu_0 := \frac{\mu_2}{\mu_1}$ and $\mu_{\ell+1} := 0$. It is useful to point out that $B_\xi(r_0) = A_1 \cup \cdots \cup A_\ell$, so all the bubbles $W_i$ are supported in $B_\xi(r_0)$. Therefore we have

\[
(II) = \sum_{h=1}^{\ell} \left| f \left( \sum_{i=1}^{\ell} W_i \right) - f \left( \sum_{i=1}^{\ell-1} W_i \right) - f(W_\ell) \right| \frac{2N}{N+2}, A_h. 
\]

It is useful to remind that the choice of the $\mu_\ell$’s in (3.12) implies that

\[ \left( \frac{\mu_\ell}{\mu_{\ell-1}} \right)^{\frac{N+2}{2}} = O \left( \varepsilon^{\frac{N}{2}} \right). \]
If \( h = 1, \ldots, \ell - 1 \) by Lemma 5.1 we have

\[
|\langle II \rangle_{\frac{2N}{N+2} A_h} \rangle = \left| f \left( \sum_{i=1}^{\ell} W_i \right) - f \left( \sum_{i=1}^{\ell-1} W_i \right) - f(W_{\ell}) \right|_{\frac{2N}{N+2} A_h} 
\leq c \left| \sum_{i=1}^{\ell-1} W_i \right|_{\frac{2N}{N+2} A_h} |W_{\ell}|_{\frac{2N}{N+2} A_h} + c |W_{\ell}^{p-1}|_{\frac{2N}{N+2} A_h} + c |W_{\ell}|_{\frac{2N}{N+2} A_h} + c W_{\ell}^{p-1} W_i_{\frac{2N}{N+2} A_h} + c |W_{\ell}|_{\frac{2N}{N+2} A_h} 
= O \left( \left( \frac{\mu_{\ell}}{\mu_{\ell-1}} \right)^{\frac{N-2}{2}} \right),
\]

because, by (5.1) we get for any \( h = 1, \ldots, \ell - 1 \)

\[
|W_{\ell}|_{\frac{2N}{N+2} A_h} \leq c \left[ \int_{\mu_h \mu_{h+1} \leq |x| \leq \sqrt{\mu_{h-1} \mu_h}} \frac{\mu_{\ell}^N}{(\mu_{\ell}^2 + |x|^2)^{N}} \, dx \right]_{\frac{2N}{N+2}}^{\frac{N-2}{2}} 
= c \left[ \int_{\frac{\mu_h \mu_{h+1}}{\mu_{\ell}} \leq |x| \leq \sqrt{\mu_{h-1} \mu_h}} \frac{1}{(1 + |y|^2)^N} \, dy \right]_{\frac{2N}{N+2}}^{\frac{N-2}{2}} 
= O \left( \left( \frac{\mu_{\ell}}{\mu_h \mu_{h+1}} \right)^{\frac{N-2}{2}} \right) = O \left( \left( \frac{\mu_{\ell}}{\mu_{\ell-1}} \right)^{\frac{N-2}{2}} \right) \tag{5.6}
\]

and for any \( h = 1, \ldots, \ell - 1 \) and \( i = 1, \ldots, \ell - 1 \)

\[
|W_i|^{p-1} W_{\ell} \left|_{\frac{2N}{N+2} A_h} \right. \leq c \left\{ \int_{\mu_h \mu_{h+1} \leq |x| \leq \sqrt{\mu_{h-1} \mu_h}} \left[ \frac{\mu_i^2}{(\mu_i^2 + |x|^2)^{N}} \mu_{\ell}^\frac{N-2}{2} \mu_{\ell}^\frac{N-2}{2} \left( \mu_{\ell}^2 + |x|^2 \right)^\frac{N-2}{2} \right] \, dx \right\} \left( \frac{2N}{N+2} \right)^{\frac{N-2}{2}} 
\leq c \left\{ \int_{\frac{\mu_h \mu_{h+1}}{\mu_{\ell}} \leq |x| \leq \sqrt{\mu_{h-1} \mu_h}} \mu_{\ell}^\frac{N-2}{2} \mu_{\ell}^\frac{N-2}{2} \left[ \frac{\mu_i^2}{(1 + |y|^2)^{N-2}} \right] \, dy \right\} \left( \frac{2N}{N+2} \right)^{\frac{N-2}{2}} 
= O \left( \left( \frac{\mu_{\ell}}{\mu_i} \right)^2 \left( \frac{\mu_{\ell}}{\mu_h \mu_{h+1}} \right)^{\frac{N-6}{2}} \right) = O \left( \left( \frac{\mu_{\ell}}{\mu_{\ell-1}} \right)^{\frac{N+4}{2}} \right). \tag{5.7}
\]
If \( h = \ell \) by Lemma 5.1 we have
\[
|\langle II \rangle\bigg|_{\mathcal{A}_r} \leq c \left( \sum_{i=1}^{\ell} |W_i|^{p-1} W_i \right)_{\mathcal{A}_r} + c \left( \sum_{i=1}^{\ell-1} |W_i|^{p-1} W_i \right)_{\mathcal{A}_r} + c \left( \sum_{i=1}^{\ell-1} |W_i|^{p} W_i \right)_{\mathcal{A}_r}
\]
\[
= O \left( \left( \frac{\mu_{\ell}}{\mu_{\ell-1}} \right)^{\frac{N+2}{4}} \right),
\]
because by (5.1) we get for any \( i = 1, \ldots, \ell - 1 \)
\[
|W_i|_{\mathcal{A}_r} \leq c \left[ \int_{\sqrt{\mu_{i+1}} \leq \sqrt{\mu_i} \leq \sqrt{\mu_{i-1}}} \frac{\mu_i^N}{(\mu_i^2 + |x|^2)^{N}} \, dx \right]^{\frac{N-2}{2N}}
\]
\[
= c \left[ \int_{\sqrt{\mu_{i+1}} \leq \sqrt{\mu_i} \leq \sqrt{\mu_{i-1}}} \frac{1}{(1 + |y|^2)^{N}} \, dy \right]^{\frac{N-2}{2N}}
\]
\[
= O \left( \left( \frac{\sqrt{\mu_{i+1}} \mu_i}{\mu_i} \right)^{\frac{N-2}{2}} \right) = O \left( \left( \frac{\mu_{\ell}}{\mu_{\ell-1}} \right)^{\frac{N-2}{4}} \right)
\]
and
\[
|\langle W_i \rangle|_{\mathcal{A}_r} \leq c \left\{ \int_{\sqrt{\mu_{i+1}} \leq \sqrt{\mu_i} \leq \sqrt{\mu_{i-1}}} \left[ \frac{\mu_i^2}{(\mu_i^2 + |x|^2)^2} \right]^{\frac{N-2}{2}} \, dx \right\}^{\frac{N+2}{2N}}
\]
\[
\leq c \left\{ \int_{\sqrt{\mu_{i+1}} \leq \sqrt{\mu_i} \leq \sqrt{\mu_{i-1}}} \mu_i^N \left[ \frac{\mu_i^2 - \frac{\mu_i}{\mu_{i-1}}}{\mu_i^2 |y|^4} \right]^{\frac{N-2}{2}} \, dy \right\}^{\frac{N+2}{2N}}
\]
\[
= O \left( \left( \frac{\mu_{\ell}}{\mu_i} \right)^2 \left( \frac{\sqrt{\mu_{i+1}} \mu_i}{\mu_i} \right)^{\frac{N-6}{2}} \right) = O \left( \left( \frac{\mu_{\ell}}{\mu_{\ell-1}} \right)^{\frac{N+2}{4}} \right).
\]
\[
(5.9)
\]
The claim follows collecting all the previous estimates.

We recall the following useful lemma.

**Lemma 5.1.** For any \( a > 0 \) and \( b \in \mathbb{R} \) we have
\[
|a + b|^\beta - a^\beta \leq \begin{cases} 
  c(\beta) \min\{|b|^\beta, a^{\beta-1}|b|\} & \text{if } 0 < \beta < 1 \\
  c(\beta) \left(|b|^\beta + a^{\beta-1}|b|\right) & \text{if } \beta > 1
\end{cases}
\]
and
\[ |a + b|^\beta (a + b) - a^{\beta + 1} - (1 + \beta)a^\beta b | \le \begin{cases} c(\beta) \min\{|b|^\beta + 1, a^{\beta - 1}b^2\} & \text{if } 0 < \beta < 1 \\ c(\beta) \max\{|b|^\beta + 1, a^{\beta - 1}b^2\} & \text{if } \beta > 1 \end{cases} \]

**Proof of Proposition 4.3.** *Step 1: The case \( \ell = 1 \)*

(i) is trivial and (ii) can be proved arguing exactly as in Proposition 3.1 of [12].

Let us prove that (iii) holds. The function \( \phi_{1, \varepsilon} \) weakly solves the first equation in (4.1), namely
\[ \mathcal{E}_1 + S_1(\phi_{1, \varepsilon}) + \mathcal{N}_1(\phi_{1, \varepsilon}) = 0. \]

Then, if \( \varepsilon > 0 \) is small enough, there exists a constant \( \lambda_\varepsilon \) (depending only on \( d_1 \)) such that \( \phi_{1, \varepsilon} \) weakly solves
\[ -\mathcal{L}_g(W_1 + \phi_{1, \varepsilon}) + \varepsilon(W_1 + \phi_{1, \varepsilon}) - f(W_1 + \phi_{1, \varepsilon}) = \lambda_\varepsilon (\mathcal{L}_g Z_1^0) \tag{5.10} \]

Let us first show that \( \lambda_\varepsilon = o(1) \) as \( \varepsilon \to 0 \).

We test the equation (5.10) by \( Z_1^0 \). We use the fact that \( \phi_{1, \varepsilon} \in K_1^+ \) and we get
\[ \int_M [-\mathcal{L}_g W_1 + \varepsilon W_1 - f(W_1)] Z_1^0 \, d\nu_g + \int_M (-\mathcal{L}_g \phi_{1, \varepsilon}) Z_1^0 \, d\nu_g + \varepsilon \int_M \phi_{1, \varepsilon} Z_1^0 \, d\nu_g = \frac{\lambda_\varepsilon}{\phi_{1, \varepsilon}} \]

\[ - \int M [f(W_1 + \phi_{1, \varepsilon}) - f(W_1)] Z_1^0 \, d\nu_g = \lambda_\varepsilon \int M (-\mathcal{L}_g Z_1^0) Z_1^0 \, d\nu_g \]

Let us estimate each term in (5.11). By (4.11), (5.34) and Lemma (5.1) we deduce
\[ \left| \int M \left[ \mathcal{L}_g(W_1) + \varepsilon W_1 - f(W_1) \right] Z_1^0 \, d\nu_g \right| \le c \int M \frac{\mu_1^{\frac{N-2}{2}}}{(\mu_1^2 + d_g(z, \xi)^2)^{\frac{N-2}{2}}} Z_1^0 \, d\nu_g \le C\mu_1^2. \]

\[ \varepsilon \left| \int M \phi_{1, \varepsilon} Z_1^0 \, d\nu_g \right| \le c\varepsilon \|\phi_{1, \varepsilon}\| Z_1^0 \le C\varepsilon \mu_1^2 \|\phi_{1, \varepsilon}\| \]

and
\[ \left| \int M [f(W_1 + \phi_{1, \varepsilon}) - f(W_1)] Z_1^0 \, d\nu_g \right| \le \int M |W_1^{p-1} \phi_{1, \varepsilon} Z_1^0| \, d\nu_g + \int M |\phi_{1, \varepsilon} Z_1^0| \, d\nu_g \le c\|\phi_{1, \varepsilon}\|. \]

Moreover, by (3.4) and (3.5) we deduce that for any \( j = 1, \ldots, k \)
\[ \mathcal{E}_j^0 := -\mathcal{L}_g Z_j^0 - f'(W_j) Z_j^0 \quad \text{and} \quad |\mathcal{E}_j^0| = O \left( \frac{\mu_j^{\frac{N-2}{2}}}{(\mu_j^2 + d_g(z, \xi)^2)^{\frac{N-2}{2}}} \right). \tag{5.12} \]

Therefore an easy computation leads to
\[ \int_M (-\mathcal{L}_g Z_1^0) Z_1^0 \, d\nu_g = \int_M f'(W_1) (Z_1^0)^2 \, d\nu_g + \int_M \mathcal{E}_j^0 Z_1^0 \, d\nu_g \]
\[ = \int_{\mathbb{R}^N} f'(U) (\psi^0)^2 \, dx + O(\mu_1^2). \]

Collecting all the estimates in (5.11), we deduce that \( \lambda_\varepsilon = o(1) \).

Let us set \( \hat{u}_1 := W_1 + \phi_{1, \varepsilon} \). By (3.4) and (5.12), equation (5.10) in geodesic coordinates can be written as
\[ -\Delta \hat{u}_1 - (g^{ij} - \delta^{ij}) \partial_{ij} \hat{u}_1 - g^{ij} \Gamma_{ij}^k \partial_k \hat{u}_1 + (\beta_N R_g + \varepsilon) \hat{u}_1 - f(\hat{u}_1) = \lambda_\varepsilon \left[ f'(W_1) Z_1^0 + \mathcal{E}_1^0 \right] \text{ in } B(0, r_0). \tag{5.13} \]
Therefore, if we take \( r = \rho \mu_1 \) and we scale \( \hat{v}_1(y) := \mu_1^{-\frac{N-2}{2}} \hat{u}_1 \circ \exp_\xi(\mu_1 y) \), the function \( \hat{v}_1 \) (taking into account (5.12)) solves

\[
\begin{align*}
- \Delta \hat{v}_1 & - \left( g^{ij}(\mu_1 y) - \delta^{ij}(\mu_1 y) \right) \partial_{ij}^2 \hat{v}_1 - \mu_1 g^{ij}(\mu_1 y) \Gamma^k_{ij}(\mu_1 y) \partial_k \hat{v}_1 \\
&= a_{ij}(y) \hat{v}_1 - \mu_1 g^{ij}(\mu_1 y) \Gamma^k_{ij}(\mu_1 y) \partial_k \hat{v}_1 \\
&= a(y) \\
&+ \mu_1^2 (\beta_N R g(\mu_1 y) + \varepsilon) \hat{v}_1 - \hat{v}_1^p = \lambda_{\varepsilon} \left( f'(U) \psi^0 + O(\mu_1^2) \right) \text{ in } B(0, \rho).
\end{align*}
\]

(5.14)

By (3.5)

\[
\sup_{y \in B(0, \rho)} |a_{ij}(y)| = O(\rho), \quad \sup_{y \in B(0, \rho)} (|\nabla a_{ij}(y)| + |b_k(y)|) = O(\mu_1), \quad \sup_{y \in B(0, \rho)} |c(y)| = O(\mu_1^2),
\]

\[
\sup_{y \in B(0, \rho)} |h(y)| = O(\lambda_{\varepsilon}) = o(1).
\]

We are in position to apply Proposition 5.2, which implies that there exists \( c > 0 \) such that

\[
\sup_{B(0, \rho)} |\hat{v}_1| \leq c.
\]

Therefore

\[
|\hat{v}_1(y)| = |U(y) + \mu_1^2 V(y) + \mu_1^{\frac{N-2}{2}} \phi_{1,\varepsilon}(\exp_\xi(\mu_1 y))| \leq c, \quad y \in B(0, \rho)
\]

and finally

\[
|\phi_{1,\varepsilon}(z)| \leq c \mu_1^{-\frac{N-2}{2}}, \quad z \in B_\varepsilon(\rho \mu_1).
\]

Step 2: The case \( \ell \geq 2 \).

Let us suppose that the first \((\ell - 1)\)-th equations of (4.1) have the solutions \( \phi_{j,\varepsilon} \) with \( j = 1, \ldots, \ell - 1 \) with the all the properties (i), (ii) and (iii) and let us consider the \( \ell \)-th equation of (4.1).

- **Proof of (i) and (ii): existence and the uniform estimate.**

  By Proposition 4.2 we can rewrite the equation \( \mathcal{E}_\ell + \mathcal{S}_\ell(\phi_{\ell,\varepsilon}) + \mathcal{N}_\ell(\phi_{1,\varepsilon}, \ldots, \phi_{\ell,\varepsilon}) = 0 \) as

  \[
  \phi_{\ell,\varepsilon} = \mathcal{S}_\ell^{-1}(\mathcal{E}_\ell + \mathcal{N}_\ell(\phi_{1,\varepsilon}, \ldots, \phi_{\ell,\varepsilon})) := \mathcal{T}_\ell(\phi_{\ell,\varepsilon}).
  \]

  As usual, we shall show that if \( \varepsilon \) is small enough, \( \mathcal{T}_\ell : \mathcal{B}_\ell \rightarrow \mathcal{B}_\ell \) is a contraction mapping, where \( \mathcal{B}_\ell := \{ \phi \in H^2_1(M) : \|\phi\| \leq R \varepsilon^{\frac{2}{\theta_\ell}} \} \) for some \( R > 0 \).

  First, by Proposition 4.2 we get

  \[
  \|\mathcal{T}_\ell(\phi_{\ell,\varepsilon})\| \leq \|\mathcal{S}_\ell^{-1}\| (\|\mathcal{E}_\ell\| + \|\mathcal{N}_\ell(\phi_{1,\varepsilon}, \ldots, \phi_{\ell,\varepsilon})\|) \leq c (\|\mathcal{E}_\ell\| + \|\mathcal{N}_\ell(\phi_{1,\varepsilon}, \ldots, \phi_{\ell,\varepsilon})\|)
  \]

  and by Proposition 4.1 we get

  \[
  \|\mathcal{E}_\ell\| \leq c \varepsilon^{\frac{2}{\theta_\ell}}.
  \]

  We shall prove that in the ball \( \mathcal{B}_\ell \) there hold true that

  \[
  \|\mathcal{N}_\ell(\phi_{1,\varepsilon}, \ldots, \phi_{\ell,\varepsilon})\| \leq c\|\phi_{\ell,\varepsilon}\|^p + c\|\phi_{1,\varepsilon}\|^{p-1}\|\phi_{\ell,\varepsilon}\| + c\varepsilon^{\frac{2}{\theta_\ell}}
  \]

  (5.15)

  and

  \[
  \|\mathcal{N}_\ell(\phi_{1,\varepsilon}, \ldots, \phi_{\ell,\varepsilon}) - \mathcal{N}_\ell(\phi_{1,\varepsilon}, \ldots, \tilde{\phi}_{\ell,\varepsilon})\| \leq L\|\phi_{\ell,\varepsilon} - \tilde{\phi}_{\ell,\varepsilon}\| \text{ for some } L \in (0, 1).
  \]

  (5.16)

  Then the claim will follow.
We introduce the set of annuli defined in (5.5) and we get
\[
\|\mathcal{N}_\ell(\phi_{1,\varepsilon}, \ldots, \phi_{\ell,\varepsilon})\| \leq C \left| f \left( \sum_{j=1}^{\ell} (W_j + \phi_{j,\varepsilon}) \right) - f \left( \sum_{j=1}^{\ell} W_j \right) - f' \left( \sum_{j=1}^{\ell} W_j \right) \phi_{\ell,\varepsilon} \right| \leq \sum_{h=1}^{\ell} \left[ \int_{A_h} \left| f \left( \sum_{j=1}^{\ell} W_j + \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right) - f \left( \sum_{j=1}^{\ell} W_j + \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right) \phi_{\ell,\varepsilon} \right| d\nu_g \right] \leq c \|\phi_{\ell,\varepsilon}\|^p + c \sum_{j=1}^{\ell-1} \|\phi_{j,\varepsilon}\|^{p-1} \|\phi_{\ell,\varepsilon}\| + c \left( \int_{A_h} |W_{\ell}|^{p+1} d\nu_g \right)^{\frac{N+2}{2N}}.
\]
If \( h = 1, \ldots, \ell - 1 \) by Lemma 5.1 we deduce
\[
\left| \int_{A_h} \left| f \left( \sum_{j=1}^{\ell-1} W_j + \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right) - f \left( \sum_{j=1}^{\ell-1} W_j + \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right) \phi_{\ell,\varepsilon} \right| d\nu_g \right| \leq c \|\phi_{\ell,\varepsilon}\|^p + c \sum_{j=1}^{\ell-1} \|\phi_{j,\varepsilon}\|^{p-1} \|\phi_{\ell,\varepsilon}\| + c \left( \int_{A_h} |W_{\ell}|^{p+1} d\nu_g \right)^{\frac{N+2}{2N}}.
\]
because of (5.6), (5.7) and the following new estimate
\[
\|\phi_{j}^{p-1} W_{\ell} \|^{\frac{2N}{N+2}}_{A_h} = O \left( \left( \frac{\mu_{\ell}}{\mu_{\ell-1}} \right)^{\frac{N+2}{4}} \right), \quad j, h = 1, \ldots, \ell - 1.
\]
Let us prove (5.18). We have to distinguish three cases \( j \geq h + 1, j \leq h - 1 \) and \( j = h \).
If \( j \leq h - 1 \) by (4.13) we deduce that \( \phi_j = O \left( \mu_j^{N-3} \right) \) in \( A_h \subset B(\xi, r \mu_j) \) and we have

\[
||\phi_j||^{p-1}W_{\ell} \leq c \frac{1}{\mu_j^2} |W_{\ell}|_{2N+2, \cdot, A_h} \leq c \frac{1}{\mu_j^2} (\text{meas } A_h)^{\frac{2N}{N-2}} |W_{\ell}|_{2N, \cdot, A_h}
\]

\[
\leq c \frac{\mu_h \mu_{h+1}}{\mu_j^2} \left( \frac{\mu_{\ell}}{\mu_{\ell-1}} \right)^{\frac{N-2}{4}} = O \left( \left( \frac{\mu_{\ell}}{\mu_{\ell-1}} \right)^{\frac{N-2}{4}} \right).
\]

(5.19)

If \( h \leq j \leq \ell - 1 \) and \( h + 1 \leq \ell - 1 \) then by (5.6) and (4.11) we deduce

\[
||\phi_j||^{p-1}W_{\ell} \leq c ||\phi_j||^{p-1} |W_{\ell}|_{2N+2, \cdot, A_h} \leq c ||\phi_1||^{p-1} \left( \frac{\mu_{\ell}}{\sqrt{\mu_h \mu_{h+1}}} \right)^{\frac{N-2}{2}}
\]

\[
\leq c ||\phi_1||^{p-1} \left( \frac{\mu_{\ell}}{\mu_{\ell-1}} \right)^{\frac{N-2}{4}} = O \left( \left( \frac{\mu_{\ell}}{\mu_{\ell-1}} \right)^{\frac{N-2}{4}} \right).
\]

(5.20)

If \( j = h = \ell - 1 \) we split the annulus

\[
\mathcal{A}_{\ell-1} = \{ \sqrt{\mu_{\ell-1}} \mu \leq d_g(x, \xi) \leq \mu_{\ell-1} \} \cup \{ \mu_{\ell-1} \leq d_g(x, \xi) \leq \sqrt{\mu_{\ell-1}} \mu_{\ell-2} \}
\]

\[
\mathcal{A}_{\ell-1}' = \{ \sqrt{\mu_{\ell-1}} \mu \leq d_g(x, \xi) \leq \mu_{\ell-1} \} \cup \{ \mu_{\ell-1} \leq d_g(x, \xi) \leq \sqrt{\mu_{\ell-1}} \mu_{\ell-2} \}
\]

and we get

\[
||\phi_{\ell-1}||^{p-1}W_{\ell} \leq c ||\phi_{\ell-1}||^{p-1} |W_{\ell}|_{2N+2, \cdot, A_{\ell-1}} + ||\phi_{\ell-1}||^{p-1} |W_{\ell}|_{2N+2, \cdot, A_{\ell-1}'}
\]

\[
\leq c \frac{1}{\mu_{\ell-1}^2} |W_{\ell}|_{2N+2, \cdot, A_{\ell-1}} + c ||\phi_{\ell-1}||^{p-1} |W_{\ell}|_{2N+2, \cdot, A_{\ell-1}'}
\]

\[
= O \left( \left( \frac{\mu_{\ell}}{\mu_{\ell-1}} \right)^{\frac{N-2}{4}} \right).
\]

(5.21)

because

\[
|W_{\ell}|_{2N+2, \cdot, A_{\ell-1}} \leq c \left[ \int_{\sqrt{\mu_{\ell-1}} \mu \leq |x| \leq \mu_{\ell-1}} \left( \frac{\mu_{\ell}^{N-2}}{\mu_{\ell}^2 + |x|^2} \right)^{\frac{2N}{N+2}} dx \right]^{\frac{N+2}{2N}}
\]

\[
\leq c \mu_{\ell}^2 \left[ \int_{\sqrt{\mu_{\ell-1}} \mu \leq |y| \leq \mu_{\ell-1}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{2N}{N+2}} dy \right]^{\frac{N+2}{2N}}
\]

\[
= O \left( \mu_{\ell}^2 \left( \frac{\mu_{\ell}}{\mu_{\ell-1}} \right)^{\frac{N-6}{4}} \right).
\]

(5.22)
and

\[
|W_\ell|_{N/2,A_{\ell-1}^\varepsilon} \leq c \left[ \int_{\mu_{\ell-1} \leq |x| \leq \sqrt{\mu_{\ell-1}^{-1} \mu_{\ell}^{-2}}} \left( \frac{\mu_{\ell}^{-2}}{(\mu_{\ell}^2 + |x|^2)^{N/2}} \right)^{2N \over N-2} \frac{\mu_{\ell}^{-2}}{N-2} \right]^{N-2 \over 2N}
\]

\[
\leq c \left[ \int_{\mu_{\ell-1} \leq |y| \leq \sqrt{\mu_{\ell-1}^{-1} \mu_{\ell}^{-2}}} \frac{1}{(1 + |y|^2)^N} dy \right]^{N-2 \over 2N} = O \left( \left( \frac{\mu_{\ell}}{\mu_{\ell-1}} \right)^{N-2 \over 2} \right)
\]

(5.23)

It remains to evaluate the last term \( h = \ell \) in (5.17). By Lemma (5.1) we get

\[
\left[ \int_{A_{\ell}} \left| \sum_{j=1}^{N+2 \over 2N} \sum_{j=1}^{N+2 \over 2N} \right| d\nu_g \right]^{N+2 \over 2N} \leq \left[ \int_{A_{\ell}} \left| f \left( \sum_{j=1}^{\ell} \left( W_j + \phi_{j,\varepsilon} \right) \right) - f \left( \sum_{j=1}^{\ell-1} W_j + \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right) \right| d\nu_g \right]^{N+2 \over 2N} + \left[ \int_{A_{\ell}} \left| \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right| d\nu_g \right]^{N+2 \over 2N}
\]

\[
= c \| \phi_{\ell,\varepsilon} \|^p + c \| \phi_{1,\varepsilon} \|^p (\| \phi_{1,\varepsilon} \|^{p-1} |\phi_{\ell,\varepsilon} - \phi_{1,\varepsilon}|) + c \sum_{j=1}^{\ell-1} \left( \int_{A_{\ell}} |\phi_{j,\varepsilon}|^{p+1} d\nu_g \right) + c \sum_{j=1}^{\ell-1} \left( \int_{A_{\ell}} |W_j|^{p+1} d\nu_g \right)
\]

\[
\leq c \left( \| \phi_{\ell,\varepsilon} \|^p + \| \phi_{1,\varepsilon} \|^p (\| \phi_{1,\varepsilon} \|^{p-1} |\phi_{\ell,\varepsilon} - \phi_{1,\varepsilon}|) + \varepsilon^{2p \theta_j} \right),
\]

because if \( j \leq \ell - 1 \) by (4.13) we deduce that \( \phi_j = O \left( \mu_j^{-N/2} \right) \) in \( A_{\ell} \subset B(\xi, r \mu_j) \) and we have

\[
\left( \int_{A_{\ell}} |\phi_{j,\varepsilon}|^{p+1} d\nu_g \right)^{N+2 \over 2N} \leq c \mu_j^{-N+2 \over 2} (\text{meas } A_{\ell})^{N+2 \over 2N} \leq c \left( \sqrt{\mu_{\ell-1}^{-1} \mu_j} \right)^{N+2 \over 2} = O \left( \left( \frac{\mu_{\ell}}{\mu_{\ell-1}} \right)^{N+2 \over 2} \right)
\]
and

\[
\left( \int_{\mathcal{A}_\ell} \left| W_\ell \right|^{p-1} \phi_{j,\varepsilon} \right|^{2N} \frac{2N}{2N} \, d\nu_g \right)^{\frac{N+2}{2N}} \leq c \mu_j \left( \int_{|\mu_{\ell+1}| \leq |x| \leq \mu_{\ell-1}} \left( \frac{\mu_{\ell}}{(\mu_{\ell}^2 + |x|^2)^{\frac{N+2}{2}}} \right)^{\frac{2N}{N+2}} \right)^{\frac{N+2}{2N}}
\]

\[
\leq c \mu_j^{\frac{N-2}{2}} \left( \int_{|\mu_{\ell+1}| \leq |x| \leq \mu_{\ell-1}} \frac{1}{|x|^{\frac{N+2}{2}}} \, dx \right)^{\frac{N+2}{2N}}
\]

\[
\leq c \mu_j^{\frac{N-2}{2}} (\mu_{\ell-1} \mu_{\ell})^{\frac{N+6}{4}} = O \left( \left( \frac{\mu_{\ell}}{\mu_{\ell-1}} \right)^{\frac{N+2}{N+2}} \right).
\]

That concludes the proof of (5.15). Now, let us prove (5.16). Again, by Lemma 5.1 we get

\[
\| N_\ell(\phi_{1,\varepsilon}, \ldots, \phi_{\ell,\varepsilon}) - N_\ell(\overline{\phi}_{1,\varepsilon}, \ldots, \overline{\phi}_{\ell,\varepsilon}) \| 
\]

\[
\leq c \left| f \left( \sum_{j=1}^{\ell} W_j + \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} + \phi_{\ell,\varepsilon} \right) - f \left( \sum_{j=1}^{\ell} W_j + \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} + \phi_{\ell,\varepsilon} \right) - f' \left( \sum_{j=1}^{\ell} W_j \right) \left( \phi_{\ell,\varepsilon} - \phi_{\ell,\varepsilon} \right) \right|^{\frac{2N}{N+2}}
\]

\[
= \left| f' \left( \sum_{j=1}^{\ell} W_j + \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} + t \phi_{\ell,\varepsilon} + (1-t) \overline{\phi}_{\ell,\varepsilon} \right) - f' \left( \sum_{j=1}^{\ell} W_j \right) \left( \phi_{\ell,\varepsilon} - \phi_{\ell,\varepsilon} \right) \right|^{\frac{2N}{N+2}}
\]

\[
\leq c \left| \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} + t \phi_{\ell,\varepsilon} + (1-t) \overline{\phi}_{\ell,\varepsilon} \right|^{p-1} \left( \phi_{\ell,\varepsilon} - \phi_{\ell,\varepsilon} \right) \right|^{\frac{2N}{N+2}}
\]

\[
\leq c \left( \sum_{j=1}^{\ell-1} \| \phi_{j,\varepsilon} \|^{p-1} + \| \phi_{\ell,\varepsilon} \|^{p-1} + \| \overline{\phi}_{\ell,\varepsilon} \|^{p-1} \right) \left( \phi_{\ell,\varepsilon} - \phi_{\ell,\varepsilon} \right) \leq L \| \phi_{\ell,\varepsilon} - \phi_{\ell,\varepsilon} \|
\]

for some $L \in (0, 1)$ provided $\varepsilon$ is small enough.

That concludes the proof.

- **Proof of (ii): the $C^1$-estimate.**
We apply the Implicit Function Theorem to the map $F_\varepsilon : (0, +\infty)^\ell \times H^1_g(M) \to H^1_g(M)$ defined by

$$F_\varepsilon(\bar{d}, u) := \phi + \pi(\bar{d}) \left\{ \sum_{i=1}^\ell W(d_i) - \iota^* \left[ f_\varepsilon \left( \sum_{i=1}^\ell W(d_i) + \pi(\bar{d})(u) \right) \right] \right\} = \phi + \sum_{i=1}^\ell W(d_i) - \iota^* \left[ f_\varepsilon \left( \sum_{i=1}^\ell W(d_i) + \pi(\bar{d})(u) \right) \right] - \sum_{i,j=1}^\ell \langle W(d_i), Z(d_j) \rangle Z(d_j) + \sum_{j=1}^\ell \left\langle \iota^* \left[ f_\varepsilon \left( \sum_{i=1}^\ell W(d_i) + \pi(\bar{d})(u) \right) \right], Z(d_j) \right\rangle Z(d_j)$$

where $\bar{d} := (d_1, \ldots, d_\ell) \in (0, +\infty)^\ell$, $\pi(\bar{d})(u) := u - \sum_{i=1}^\ell \langle u, Z(d_i) \rangle Z(d_i)$, $Z(d_i) := Z^0_i$ are defined in (3.17), $W(d_i) := W_i$ are defined in (3.10) and $f_\varepsilon(u) := f(u) - \varepsilon u$.

It is clear that $F_\varepsilon$ is a $C^1-$map. Moreover, by previous steps we deduce that for any $d_0 \in (0, +\infty)^\ell$ there exists $\phi_0 = \sum_{i=1}^\ell \phi_{i, \varepsilon} \in H^1_g$ such that $\pi(\bar{d}_0)(\phi_0) = \phi_0$ and (see (5.26)) $F_\varepsilon(\bar{d}_0, \phi_0) = 0$. We shall prove that

$$\sup_{u \neq 0} \frac{\|DF_\varepsilon(\bar{d}_0, \phi_0)[u]\|}{\|u\|} \geq c > 0 \quad (5.24)$$

and

$$\sup_{d \neq 0} \frac{\|DF_\varepsilon(\bar{d}_0, \phi_0)[d]\|}{\|d\|} = o(1) \quad \text{as} \quad \varepsilon \to 0, \quad (5.25)$$

uniformly with respect to $\bar{d}_0$ in compact sets of $(0, \infty)^\ell$. The Implicit Function Theorem will imply that the map $\bar{d}_0 \to \phi_0$ is a $C^1-$map and also that $|\nabla \bar{d}_0 \phi_0| = o(1)$.

First, we have

$$DF_\varepsilon(\bar{d}_0, \phi_0)[u] = u - \pi(\bar{d}_0) \left\{ \iota^* \left[ f_\varepsilon' \left( \sum_{i=1}^\ell W(d_i^0) + \phi_0 \right) \pi(\bar{d}_0)(u) \right] \right\}$$

and so

$$\|DF_\varepsilon(\bar{d}_0, \phi_0)[u]\| \geq c \|u - \pi(\bar{d}_0)(u)\| + c \left\| \pi(\bar{d}_0)(u) - \pi(\bar{d}_0) \left\{ \iota^* \left[ f_\varepsilon' \left( \sum_{i=1}^\ell W(d_i^0) + \phi_0 \right) \pi(\bar{d}_0)(u) \right] \right\} \right\|$$

$$\geq c \|u - \pi(\bar{d}_0)(u)\| + c \left\| \pi(\bar{d}_0)(u) - \pi(\bar{d}_0) \left\{ \iota^* \left[ f_\varepsilon' \left( \sum_{i=1}^\ell W(d_i^0) \right) \pi(\bar{d}_0)(u) - \varepsilon \pi(\bar{d}_0)(u) \right] \right\} \right\|$$

$$\geq c \|u - \pi(\bar{d}_0)(u)\| + c \pi(\bar{d}_0)(u) - O \left( \left\| f_\varepsilon' \left( \sum_{i=1}^\ell W(d_i^0) + \phi_0 \right) - f_\varepsilon' \left( \sum_{i=1}^\ell W(d_i^0) \right) \right\| \frac{2N}{N+2} \|u\| \right)$$

$$\geq c \|u\|$$
because of Lemma 4.2 and Lemma 5.1. Then (5.24) follows.

Now, we can compute

$$DF_\varepsilon(d_0, \phi_0)[d] = \sum_{i=1}^\ell W'(d_i^0)d_i$$

$$- \iota^* \left[ f_\varepsilon' \left( \sum_{i=1}^\ell W(d_i) + \phi_0 \right) \left( \sum_{i=1}^\ell W'(d_i^0)d_i - \sum_{i=1}^\ell \langle \phi_0, Z'(d_i^0) \rangle Z(d_i^0)d_i - \sum_{i=1}^\ell \langle \phi_0, Z(d_i^0) \rangle Z'(d_i^0)d_i \right) \right]$$

$$- \sum_{i,j=1}^\ell \left[ \langle W'(d_i^0), Z(d_j^0) \rangle Z(d_j^0)d_i + \langle W(d_i^0), Z'(d_j^0) \rangle Z(d_j^0)d_i + \langle W(d_i^0), Z(d_j^0) \rangle Z'(d_j^0)d_i \right]$$

$$+ \sum_{i,j=1}^\ell \left\{ \left[ i^* \left[ f_\varepsilon \left( \sum_{i=1}^\ell W(d_i^0) + \phi_0 \right) \right] , Z'(d_j^0) \right] Z(d_j^0)d_j + \left[ i^* \left[ f_\varepsilon \left( \sum_{i=1}^\ell W(d_i^0) + \phi_0 \right) \right] , Z(d_j^0) \right] Z'(d_j^0)d_j \right\}$$

$$= \sum_{i=1}^\ell \left[ W'(d_i^0) - \iota^* \left[ f_\varepsilon \left( \sum_{i=1}^\ell W(d_i) + \phi_0 \right) W'(d_i^0) \right] \right] d_i$$

$$- \iota^* \left[ f_\varepsilon' \left( \sum_{i=1}^\ell W(d_i) + \phi_0 \right) \left( - \sum_{i=1}^\ell \langle \phi_0, Z'(d_i^0) \rangle Z(d_i^0)d_i - \sum_{i=1}^\ell \langle \phi_0, Z(d_i^0) \rangle Z'(d_i^0)d_i \right) \right]$$

$$- \sum_{i,j=1}^\ell \left[ f_\varepsilon' \left( \sum_{i=1}^\ell W(d_i) + \phi_0 \right) W'(d_i^0) \right] Z(d_j^0)d_i$$

$$+ \sum_{i,j=1}^\ell \left\{ i^* \left[ f_\varepsilon \left( \sum_{i=1}^\ell W(d_i) + \phi_0 \right) \left( - \sum_{i=1}^\ell \langle \phi_0, Z'(d_i^0) \rangle Z(d_i^0)d_i - \sum_{i=1}^\ell \langle \phi_0, Z(d_i^0) \rangle Z'(d_i^0)d_i \right) \right] , Z(d_j^0) \right\}$$

$$- \sum_{j=1}^\ell \left[ \sum_{i=1}^\ell W(d_i^0) - \iota^* \left[ f_\varepsilon \left( \sum_{i=1}^\ell W(d_i^0) + \phi_0 \right) \right] Z(d_j^0)d_j \right]$$

$$- \sum_{j=1}^\ell \left[ \sum_{i=1}^\ell W(d_i^0) - \iota^* \left[ f_\varepsilon \left( \sum_{i=1}^\ell W(d_i^0) + \phi_0 \right) \right] Z(d_j^0)d_j \right]$$
and so
\begin{align*}
\|DF_\varepsilon(d_0, \phi_0)[d]\| &= O \left( \sum_i \left\| W'(d_i^0) - i^* \left[ f_\varepsilon \left( \sum_{i=1}^\ell W(d_i) + \phi_0 \right) W'(d_i^0) \right] \right\| \sum_i |d_i| \right) \\
&+ O \left( \sum_i W(d_i) + \phi_0 \right) \| \phi_0 \| \sum_i \| Z'(d_i^0) \| \sum_i \| Z(d_i^0) \| \sum_i |d_i| \\
&+ O \left( \sum_i W'(d_i^0) - i^* \left[ f_\varepsilon \left( \sum_{i=1}^\ell W(d_i) + \phi_0 \right) W'(d_i^0) \right] \right) \sum_i \| Z'(d_i^0) \| \sum_i \| Z(d_i^0) \| \sum_i |d_i| \\
&+ O \left( \sum_{i=1}^\ell W(d_i) + \phi_0 \right) \| \phi_0 \| \sum_i \| Z'(d_i^0) \| \sum_i \| Z(d_i^0) \| \sum_i |d_i| \\
&= o(1)|d|,
\end{align*}

because a straightforward computation shows that \( \| Z'(d_i^0) \| = O(1) \) and \( \| Z(d_i^0) \| = O(1) \). Moreover, taking into account Lemma 5.1, the estimate of the error in Lemma 4.1 and estimate in (5.12) we get
\begin{align*}
\left\| \sum_{i=1}^\ell W(d_i^0) - i^* \left[ f_\varepsilon \left( \sum_{i=1}^\ell W(d_i) + \phi_0 \right) \right] \right\| = o(1)
\end{align*}

and for any index \( i \)
\begin{align*}
\left\| W'(d_i^0) - i^* \left[ f_\varepsilon \left( \sum_{i=1}^\ell W(d_i) + \phi_0 \right) W'(d_i^0) \right] \right\| = o(1).
\end{align*}

Then (5.25) follows.

That concludes the proof.

• Proof of (iii): the pointwise estimate.

Let us consider the \( j \)-th equation in (4.1) with \( j < \ell \). Then, for any \( \varepsilon > 0 \) sufficiently small, there exist constants \( \lambda_{j,0}^\varepsilon \) for \( j = 1, \ldots, \ell \) depending on \( d_j \) for \( j < \ell \) such that
\begin{align*}
- \mathcal{L}_g(W_j + \phi_{j,\varepsilon}) + \varepsilon(W_j + \phi_{j,\varepsilon}) - f \left( \sum_{m=1}^j (W_m + \phi_{m,\varepsilon}) \right) \\
+ f \left( \sum_{m=1}^{j-1} (W_m + \phi_{m,\varepsilon}) \right) = \sum_{m=1}^\ell \lambda_{m,\varepsilon}(-\mathcal{L}_g Z_{m,\varepsilon}).
\end{align*}

If we sum on \( j = 1, \ldots, \ell \) we get
\begin{align*}
- \mathcal{L}_g \left( \sum_{j=1}^\ell (W_j + \phi_{j,\varepsilon}) \right) + \varepsilon \left( \sum_{j=1}^\ell (W_j + \phi_{j,\varepsilon}) \right) - f \left( \sum_{j=1}^\ell (W_j + \phi_{j,\varepsilon}) \right) = \sum_{j=1}^\ell \lambda_{j,\varepsilon}(-\mathcal{L}_g Z_{j,\varepsilon})
\end{align*}

(5.26)
First, we prove that $\lambda_{j,\varepsilon} = o(1)$ as $\varepsilon \to 0$.
We test the equation (5.10) by $Z_i^0$ for $i = 1, \ldots, \ell$. We use the fact that each $\phi_{j,\varepsilon} \in \mathcal{K}_\varepsilon$ and we get

\[
\sum_{j=1}^\ell \int_M \left[ -\mathcal{L}_g W_j + \varepsilon W_j - f(W_j) \right] Z_i^0 \, d\nu_g + \sum_{j=1}^\ell \int_M \left( -\mathcal{L}_g \phi_{j,\varepsilon} \right) Z_i^0 \, d\nu_g + \varepsilon \sum_{j=1}^\ell \int_M \phi_{j,\varepsilon} Z_i^0 \, d\nu_g = \sum_{j=1}^\ell \lambda_{j,\varepsilon} \int_M ( -\mathcal{L}_g Z_j^0 ) Z_i^0 \, d\nu_g
\]

Let us estimate each term in (5.11). By (4.11) and (5.34) we deduce

\[
\left| \int_M [\mathcal{L}_g(W_j) + \varepsilon W_j - f(W_j)] Z_i^0 \, d\nu_g \right| \leq c \int_M \frac{\mu_j^{N-2}}{(\mu_j^2 + d_g(z, \xi)^2)^{\frac{N-2}{2}}} Z_i^0 \, d\nu_g \leq C \mu_i^2,
\]

\[
\varepsilon \left| \int_M \phi_{j,\varepsilon} Z_i^0 \, d\nu_g \right| \leq c\varepsilon \|\phi_{j,\varepsilon}\| \|Z_i^0\|_2 \leq c\varepsilon \mu_i^2 \|\phi_{1,\varepsilon}\|
\]

Moreover, we have

\[
\int_M \left[ f \left( \sum_{j=1}^\ell W_j + \phi_{j,\varepsilon} \right) - \sum_{j=1}^\ell f(W_j) \right] Z_i^0 \, d\nu_g = \int_M \left[ f \left( \sum_{j=1}^\ell W_j + \phi_{j,\varepsilon} \right) - f \left( \sum_{j=1}^\ell W_j \right) \right] Z_i^0 \, d\nu_g
\]

\[
+ \sum_{\kappa=1}^\ell \int_M \left[ f \left( \sum_{j=1}^\kappa W_j \right) - f \left( \sum_{j=1}^{\kappa-1} W_j \right) - f(W_\kappa) \right] Z_i^0 \, d\nu_g = o(1),
\]

because by Lemma (5.1)

\[
\left| \int_M \left[ f \left( \sum_{j=1}^\ell W_j + \phi_{j,\varepsilon} \right) - \sum_{j=1}^\ell f(W_j) \right] Z_i^0 \, d\nu_g \right| \leq \sum_{j,m=1}^\ell \int_M |W_j^{p-1} \phi_{m,\varepsilon} Z_i^0| \, d\nu_g + \sum_{\kappa=1}^\ell \int_M |\phi_{\kappa,\varepsilon} Z_i^0| \, d\nu_g \leq c\|\phi_{1,\varepsilon}\| = o(1)
\]

and by (5.4) described

\[
\int_M \left[ f \left( \sum_{j=1}^\kappa W_j \right) - f \left( \sum_{j=1}^{\kappa-1} W_j \right) - f(W_\kappa) \right] Z_i^0 \, d\nu_g \leq \left| \int_M \left[ f \left( \sum_{j=1}^\kappa W_j \right) - f \left( \sum_{j=1}^{\kappa-1} W_j \right) - f(W_\kappa) \right] Z_i^0 \, d\nu_g \right| \leq c(\delta_{ij} + o(1)) = o(1).
\]

Moreover, by (5.12) we deduce that

\[
\int_M (-\mathcal{L}_g Z_j^0) Z_i^0 \, d\nu_g = \int_M f'(W_j) Z_j^0 Z_i^0 \, d\nu_g + \int_M \mathcal{L}_j Z_i^0 \, d\nu_g = c_0 (\delta_{ij} + o(1))
\]
where $c_0$ is defined as follows. Indeed, by (5.12) and Holder inequality

$$
\int_M \mathcal{E}_j^0 Z_j^0 \, d\nu_g = O \left( \int_{B(0,r_0)} \frac{\mu_j^{\frac{N-2}{2}}}{\mu_j^2 + |x|^2} \frac{\mu_i^{\frac{N-2}{2}}}{(\mu_i^2 + |x|^2)^\frac{N-2}{2}} \, dx \right)
$$

$$
= O \left( \left| \frac{\mu_j^{\frac{N-2}{2}}}{\mu_j^2 + |x|^2} \right| \frac{\mu_i^{\frac{N-2}{2}}}{(\mu_i^2 + |x|^2)^\frac{N-2}{2}} \right) = O(\mu_i^2)
$$

and

$$
\int_M f'(W_j) \left( Z_j^0 \right)^2 \, d\nu_g = \int_{\mathbb{R}^N} f'(U)(\psi^0)^2 \, dx + o(1)
$$

and if $j \neq i$

$$
\int_M f'(W_j) Z_j^0 Z_i^0 \, d\nu_g = O \left( \int_{B(0,r_0)} \frac{\mu_j^{\frac{N+2}{2}}}{\mu_j^2 + |x|^2} \frac{\mu_i^{\frac{N-2}{2}}}{(\mu_i^2 + |x|^2)^\frac{N-2}{2}} \, dx \right)
$$

$$
= \left\{ 
\begin{align*}
&O \left( \int_{B(0,r_0)} \frac{\mu_j^{\frac{N+2}{2}}}{\mu_j^2 + |x|^2} \frac{\mu_i^{\frac{N-2}{2}}}{(\mu_i^2 + |x|^2)^\frac{N-2}{2}} \, dx \right) = O \left( \left( \frac{\mu_j}{\mu_i} \right)^\frac{N-2}{2} \right) \quad \text{if } j > i \\
&O \left( \int_{B(0,r_0)} \frac{\mu_j^{\frac{N+2}{2}}}{\mu_j^2 + |x|^2} \frac{\mu_i^{\frac{N-2}{2}}}{|x|^\frac{N-2}{2}} \, dx \right) = O \left( \left( \frac{\mu_i}{\mu_j} \right)^\frac{N-2}{2} \right) \quad \text{if } j < i.
\end{align*}
\right.
$$

Therefore an easy computation leads to

$$
\int_M (-\mathcal{L}_g Z_1^0) Z_1^0 \, d\nu_g = \int_M f'(W_1) (Z_1^0)^2 \, d\nu_g + \int_M \mathcal{E}_j^0 Z_1^0 \, d\nu_g = \int_{\mathbb{R}^N} f'(U)(\psi^0)^2 \, dx + O(\mu_i^2).
$$

Collecting all the previous estimates we get that each $\lambda_{i,\varepsilon} = o(1)$. That proves our first claim.

Now, let us set $\tilde{u}_\ell := \sum_{j=1}^\ell (W_j + \phi_{j,\varepsilon})$. By (3.4) and (5.12), equation (5.26) in geodesic coordinates can be written as

$$
- \Delta \tilde{u}_\ell - (g^{ij} - \delta^{ij}) \partial_j^2 \tilde{u}_\ell - g^{ij} \Gamma_{ij}^k \partial_k \tilde{u}_\ell + (\beta_N R_g + \varepsilon) \tilde{u}_\ell - f(\tilde{u}_\ell)
$$

$$
= \sum_{j=1}^\ell \lambda_{j,\varepsilon} \left[ f'(W_j) Z_j^0 + \mathcal{E}_j^0 \right] \quad \text{in } B(0,r_0).
$$

(5.29)
Therefore, if we take \( r = \rho \mu \ell \) and we scale \( \hat{v}_\ell(y) := \mu^{-\frac{N-2}{2}} \hat{u}_\ell \circ \exp_\xi(\mu \ell y) \), the function \( \hat{v}_\ell \) solves

\[
- \Delta \hat{v}_\ell - \left( g^{ij}(\mu \ell y) - \delta^{ij}(\mu \ell y) \right) \partial^2_{ij} \hat{v}_\ell - \mu \ell g^{ij}(\mu \ell y) \Gamma^{k}_{ij}(\mu \ell y) \partial_k \hat{v}_\ell \bigg|_{y \in \partial B(\rho \mu \ell)} = 0
\]

\[
+ \mu^2 \ell (\delta_N R_g(\mu \ell y) + \varepsilon) \hat{v}_\ell - \hat{v}_\ell^p
\]

\[
= \lambda_{\ell,\varepsilon} \left[ f'(U) \hat{v}^0 + O(\mu^2 \ell) \right] + \sum_{j=1}^{\ell-1} \lambda_{j,\varepsilon} \hat{E}_j^0 \quad \text{in} \ B(0, \rho),
\]

where by (5.12) we easily deduce that

\[
\hat{E}_j^0 = O \left( \left( \frac{\mu_{\ell}^{N+2} \mu_j^{N+2}}{\mu_j^2 + |\mu \ell y|^2} \right)^{\frac{N+2}{2}} \right) = O \left( \left( \frac{\mu_{\ell}^{N+2}}{\mu_j^2} \right)^{\frac{N+2}{2}} \right) = o(1).
\]

By (3.5)

\[
\sup_{y \in B(0, \rho)} |a_{ij}(y)| = O(\rho), \quad \sup_{y \in B(0, \rho)} (|\nabla a_{ij}(y)| + |b_k(y)|) = O(\mu_{\ell}), \quad \sup_{y \in B(0, \rho)} |c(y)| = O(\mu_{\ell}^2)
\]

\[
\sup_{y \in B(0, \rho)} |h(y)| = o(1).
\]

We are in position to apply Proposition 5.2, which implies that there exists \( c > 0 \) such that

\[
\sup_{B(0, \rho)} |\hat{v}_\ell| \leq c.
\]

Therefore

\[
|\hat{v}_\ell(y)| = \left| \sum_{j=1}^{\ell-1} \left( \hat{W}_j + \hat{\phi}_{j,\varepsilon} \right) + U(y) + \mu^2 \ell V(y) + \mu^2 \ell - \hat{\phi}_{\ell,\varepsilon}(\exp_\xi(\mu \ell y)) \right| \leq c \quad \text{if} \ y \in B(0, \rho)
\]

and this implies that

\[
|\hat{\phi}_{\ell,\varepsilon}(z)| \leq c \mu_{\ell}^{\frac{N+2}{2}} \quad \text{if} \ z \in B_{\xi}(\rho \mu \ell).
\]

**Proposition 5.2.** Let \( u \in W^{1,2}(B(0, r)) \) be a solution of

\[
- \Delta u + \sum_{i,j} a_{ij} \partial^2_{ij} u + \sum_{\ell} b_\ell \partial_\ell u + cu - u^p = h \quad \text{in} \ B(0, r).
\]

Assume that there exist \( \lambda \) positive and small enough and \( c > 0 \) such that

\[
\max_{i,j} |a_{ij}|_{\infty, B(0, r)} \leq \lambda; \quad |\nabla a_{ij}|_{\infty, B(0, r)} + |b_\ell|_{\infty, B(0, r)} \leq c; \quad |h|_{\infty, B(0, r)} \leq c.
\]

Then, if \( \rho < r/2 \)

\[
|u|_{\infty, B(0, \rho)} \leq C
\]

for some positive constant \( C \).

**Proof.** The proof relies on a boot-strap argument as in Lemma 6 of [14] together with standard elliptic estimates as in Theorem 8.17 of [15].
Proof of Proposition 4.4. Proof of (i).
Let us prove that if \((d_1, \ldots, d_k)\) is a critical point of \(\tilde{J}_\varepsilon\) then \(\sum_{\ell=1}^k (W_\ell + \phi_{\ell, \varepsilon})\) is a critical point of the functional \(J_\varepsilon\). We have

\[
0 = \partial_{dh} \tilde{J}_\varepsilon(d_1, \ldots, d_k) = \nabla J_\varepsilon \left( \sum_{\ell=1}^k (W_\ell + \phi_{\ell, \varepsilon}) \right) \left[ \partial_{dh} W_h + \partial_{dh} \sum_{\ell=1}^k \phi_\ell \right], \text{ for any } h = 1, \ldots, k.
\]

Since

\[
\nabla J_\varepsilon \left( \sum_{\ell=1}^k (W_\ell + \phi_{\ell, \varepsilon}) \right) = \sum_{j=1}^k \lambda_j \varepsilon Z_j^0,
\]

we get

\[
0 = \sum_{j=1}^k \lambda_j \varepsilon \left( Z_j^0, \partial_{dh} W_h + \partial_{dh} \sum_{\ell=1}^k \phi_\ell \right), \text{ for any } h = 1, \ldots, k.
\]

Now,

\[
\partial_{dh} W_h = \varepsilon^\gamma \frac{1}{\mu_h} (Z_h^0 + o(1)) = \frac{1}{d_h} (Z_h^0 + o(1)).
\]

Moreover, by (4.12) we get

\[
\langle Z_j^0, \partial_{dh} \phi_\ell \rangle = O \left( \| Z_j^0 \| : \| \partial_{dh} \phi_\ell \| \right) = o(1).
\]

Finally, by (5.28) we get

\[
\langle Z_j^0, Z_h^0 \rangle = o(1) \text{ if } j \neq h \text{ and } \langle Z_h^0, Z_h^0 \rangle = c_0 + o(1).
\]

Therefore, the matrix relative to the system of the \(\lambda_{j, \varepsilon}\)'s is diagonally dominant and so each \(\lambda_{j, \varepsilon}\) is equal to zero. That proves our claim.

Proof of (ii).

Step 1. Let us first show that

\[
J_\varepsilon \left( \sum_{\ell=1}^k (W_\ell + \phi_{\ell, \varepsilon}) \right) = J_\varepsilon \left( \sum_{\ell=1}^k W_\ell \right) + \varepsilon \theta_\varepsilon \theta_\varepsilon
\]

where \(\theta_\varepsilon = \theta_\varepsilon (d_1, \ldots, d_\ell)\) are smooth functions such that \(|\theta_\varepsilon| = o(1)\) for any \(\ell = 1, \ldots, k\). Indeed:

\[
J_\varepsilon \left( \sum_{\ell=1}^k (W_\ell + \phi_{\ell, \varepsilon}) \right) - J_\varepsilon \left( \sum_{\ell=1}^k W_\ell \right) = \frac{1}{2} \sum_{\ell=1}^k \int_M |\nabla \phi_{\ell, \varepsilon}|^2 d\nu_g + \sum_{\ell=1}^k \int_M \nabla g \phi_{\ell, \varepsilon} \nabla \phi_{m, \varepsilon} d\nu_g
\]

\[
+ \frac{1}{2} \sum_{\ell=1}^k \int_M (\beta_N R_g + \varepsilon) \phi_{\ell, \varepsilon}^2 d\nu_g + \sum_{\ell=1}^k \int_M (\beta_N R_g + \varepsilon) \phi_{\ell, \varepsilon} \phi_{m, \varepsilon} d\nu_g
\]

\[
+ \int_M \nabla g \left( \sum_{\ell=1}^k W_\ell \right) \nabla g \left( \sum_{m=1}^k \phi_{m, \varepsilon} \right) d\nu_g + \int_M (\beta_N R_g + \varepsilon) \left( \sum_{\ell=1}^k W_\ell \right) \left( \sum_{m=1}^k \phi_{m, \varepsilon} \right) d\nu_g
\]

\[
- \int_M \left[ F \left( \sum_{\ell=1}^k (W_\ell + \phi_{\ell, \varepsilon}) \right) - F \left( \sum_{\ell=1}^k W_\ell \right) \right] d\nu_g.
\]

Since each function \(\phi_{\ell, \varepsilon}\) solves the equation

\[
-\mathcal{L}_g (W_\ell + \phi_{\ell, \varepsilon}) + \varepsilon (W_\ell + \phi_{\ell, \varepsilon}) = f \left( \sum_{j=1}^\ell (W_j + \phi_{j, \varepsilon}) \right) - f \left( \sum_{j=1}^{\ell-1} (W_j + \phi_{j, \varepsilon}) \right) + \mathcal{L}_g \psi
\]

\[
(5.33)
\]
Therefore, (5.33) reads as

\[
\begin{align*}
J_\varepsilon \left( \sum_{\ell=1}^k (W_\ell + \phi_{\ell,\varepsilon}) \right) - J_\varepsilon \left( \sum_{\ell=1}^k W_\ell \right) &= -\frac{1}{2} \sum_{\ell=1}^k \|\phi_{\ell,\varepsilon}\|^2 - \frac{1}{2} \varepsilon \sum_{\ell=1}^k |\phi_{\ell,\varepsilon}|^2 \\
+ \sum_{\ell=1}^k \int_M f \left( \sum_{j=1}^{\ell} (W_j + \phi_{j,\varepsilon}) \right) \phi_{\ell,\varepsilon} \, d\nu_g - \frac{k}{2} \sum_{\ell=1}^k \int_M f \left( \sum_{j=1}^{\ell-1} (W_j + \phi_{j,\varepsilon}) \right) \phi_{\ell,\varepsilon} \, d\nu_g \\
+ \sum_{m=1}^k \int_M f \left( \sum_{j=1}^{m} (W_j + \phi_{j,\varepsilon}) \right) \phi_{m,\varepsilon} \, d\nu_g - \sum_{m=1}^k \int_M f \left( \sum_{j=1}^{m-1} (W_j + \phi_{j,\varepsilon}) \right) \phi_{m,\varepsilon} \, d\nu_g \\
+ \sum_{m=1}^k \int_M [\mathcal{L}_g(W_\ell) + \varepsilon W_\ell - f(W_\ell)] \phi_{m,\varepsilon} \, d\nu_g + \sum_{m=1}^k \int_M f(W_\ell) \phi_{m,\varepsilon} \, d\nu_g \\
- \int_M \left[ F \left( \sum_{\ell=1}^k (W_\ell + \phi_{\ell,\varepsilon}) \right) - F \left( \sum_{\ell=1}^k W_\ell \right) \right] \, d\nu_g
\end{align*}
\]

(1) (2) (3) (4) (5)

First, (3) only depends on \(d_1, \ldots, d_\ell\) and

\[
|{(3)}| \leq \left[ \int_M \left( \frac{\mu^2 \varepsilon^{1+\tau}}{(\mu^2 + d_g(z, \xi)^2)^{N+2}} \right)^{\frac{N+2}{N-2}} \, d\nu_g \right]^{\frac{2N}{N+2}} \|\phi_{1,\varepsilon}\| \leq c \mu^{N+2} \varepsilon^{1+\tau} = o(\varepsilon^\eta),
\]

because by (3.4) and (3.5) we easily deduce that

\[
| - \mathcal{L}_g(W_\ell) + \varepsilon W_\ell - f(W_\ell) | \leq c \mu^{N+2} \varepsilon^{1+\tau} (\mu^2 + d_g(z, \xi)^2)^{\frac{N+2}{N-2}}.
\]

(5.34)

Next, we remark that

\[
(2) := \sum_{\ell=2}^k \int_M f \left( \sum_{j=1}^{\ell-1} (W_j + \phi_{j,\varepsilon}) \right) \phi_{\ell,\varepsilon} \, d\nu_g \quad \text{and} \quad (4) := \sum_{\ell=2}^k \int_M f(W_\ell) \sum_{j=1}^{\ell-1} \phi_{\ell,\varepsilon} \, d\nu_g
\]
and so

\[(1) + (2) + (4) + (5)\]

\[-\int_M \left[ F(W_1 + \phi_{1,\varepsilon}) - F(W_1) - f(W_1 + \phi_{1,\varepsilon}) \phi_{1,\varepsilon} \right] \, d\nu_g\]

\[= - \sum_{\ell=2}^k \int_M \left[ F \left( \sum_{j=1}^\ell (W_j + \phi_{j,\varepsilon}) \right) - F \left( \sum_{j=1}^{\ell-1} (W_j + \phi_{j,\varepsilon}) \right) \right] \, d\nu_g\]

\[+ \sum_{\ell=2}^k \int_M \left[ F \left( \sum_{j=1}^\ell W_j \right) - F \left( \sum_{j=1}^{\ell-1} W_j \right) \right] \, d\nu_g + \sum_{\ell=2}^k \int_M f \left( \sum_{j=1}^\ell (W_j + \phi_{j,\varepsilon}) \right) \phi_{\ell,\varepsilon} \, d\nu_g\]

\[+ \sum_{\ell=2}^k \int_M f(W_\ell) \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \, d\nu_g\]

\[= \int_M a_1 \, d\nu_g + \sum_{\ell=2}^k \int_M a_\ell \, d\nu_g\]

where

\[a_1 := - \left[ F(W_1 + \phi_{1,\varepsilon}) - F(W_1) - f(W_1 + \phi_{1,\varepsilon}) \phi_{1,\varepsilon} \right]\]

and for any \(\ell = 2, \ldots, k\)

\[a_\ell := - \left[ F \left( \sum_{j=1}^\ell (W_j + \phi_{j,\varepsilon}) \right) - F \left( \sum_{j=1}^{\ell-1} (W_j + \phi_{j,\varepsilon}) \right) \right]\]

\[+ \left[ F \left( \sum_{j=1}^\ell W_j \right) - F \left( \sum_{j=1}^{\ell-1} W_j \right) \right] + f \left( \sum_{j=1}^\ell (W_j + \phi_{j,\varepsilon}) \right) \phi_{\ell,\varepsilon}\]

\[+ f(W_\ell) \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon}\]

It is clear that each \(a_\ell\)’s only depends on \(d_1, \ldots, d_\ell\). Moreover, by Lemma 5.1 it follows that

\[\int_M a_1 \, d\nu_g := - \int_M \left[ F(W_1 + \phi_{1,\varepsilon}) - F(W_1) - f(W_1 + \phi_{1,\varepsilon}) \phi_{1,\varepsilon} \right] \, d\nu_g\]

\[+ \int_M \left[ F'(W_1 + \phi_{1,\varepsilon}) - F'(W_1) \phi_{1,\varepsilon} \right] \, d\nu_g\]

\[= O \left( \|\phi_{1,\varepsilon}\|^2 \right) = o \left( \varepsilon^2 \right) .\]

Now, we shall prove that

\[\int_M a_\ell \, d\nu_g = o \left( \varepsilon^\delta_\ell \right) \text{ for any } \ell = 2, \ldots, k.\]

Let \(\ell\) be fixed and let us split \(M = \bigcup_{h=0}^\ell A_h\) where we agree that \(A_0 := M \setminus B_\ell(r_0)\) and the annuli \(A_1, \ldots, A_\ell\) are defined in (5.5). Then it is clear that

\[\int_M a_\ell \, d\nu_g = \sum_{h=0}^\ell \int_{A_h} a_\ell \, d\nu_g.\]
First, let us consider the case $h = \ell$. By Lemma 5.1 we get

\[
\int_{\mathcal{A}_\ell} a_\ell \, d\nu_g := - \int_{\mathcal{A}_\ell} \left[ F \left( \sum_{j=1}^{\ell} (W_j + \phi_{j,\varepsilon}) \right) - F \left( \sum_{j=1}^{\ell} W_j + \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right) - f \left( \sum_{j=1}^{\ell} W_j + \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right) \phi_{\ell,\varepsilon} \right] \, d\nu_g \\
- \int_{\mathcal{A}_\ell} \left[ F \left( \sum_{j=1}^{\ell-1} W_j + \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right) - F \left( \sum_{j=1}^{\ell-1} W_j \right) - f \left( \sum_{j=1}^{\ell-1} W_j \right) \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right] \, d\nu_g \\
+ \int_{\mathcal{A}_\ell} \left[ F \left( \sum_{j=1}^{\ell} (W_j + \phi_{j,\varepsilon}) \right) - F \left( \sum_{j=1}^{\ell} W_j \right) - f \left( \sum_{j=1}^{\ell} W_j + \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right) \right] \phi_{\ell,\varepsilon} \, d\nu_g \\
- \int_{\mathcal{A}_\ell} \left[ f \left( \sum_{j=1}^{\ell} W_j \right) - f \left( \sum_{j=1}^{\ell-1} W_j \right) - f(W_\ell) \right] \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \, d\nu_g \\
= O \left( \int_{\mathcal{A}_\ell} \phi_{\ell,\varepsilon}^{p+1} \, d\nu_g \right) + O \left( \int_{\mathcal{A}_\ell} \left( \sum_{j=1}^{\ell} W_j + \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right)^{p-1} \phi_{\ell,\varepsilon}^2 \, d\nu_g \right) \\
+ O \left( \int_{\mathcal{A}_\ell} \left( \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right)^{p+1} \, d\nu_g \right) \\
+ O \left( \int_{\mathcal{A}_\ell} \left( \sum_{j=1}^{\ell-1} W_j \right)^{p-1} \left( \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right)^2 \, d\nu_g \right) \\
+ O \left( \int_{\mathcal{A}_\ell} \left( \sum_{j=1}^{\ell-1} W_j \right)^p \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \, d\nu_g \right) + O \left( \int_{\mathcal{A}_\ell} \left( \sum_{j=1}^{\ell-1} W_j \right)^{p-1} \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \, d\nu_g \right) \\
= o \left( \varepsilon^{\theta_\ell} \right),
\]

because of the rate of the error term $\phi_{\ell,\varepsilon}$ given in (4.11) and the following four new estimates.

\[
\odot \text{ For any } j = 1, \ldots, \ell - 1, \text{ by the pointwise estimate of } \phi_{j,\varepsilon} \text{ in (4.13) we get}
\]

\[
|\text{(I)}| \leq c \int_{\mathcal{A}_\ell} |\phi_{j,\varepsilon}^{p+1}| \, d\nu_g \leq c \mu_j^{-N} (\text{meas } \mathcal{A}_\ell)^N \leq c \mu_j^{-N} (\mu_\ell \mu_{\ell-1})^N \leq c \left( \frac{\mu_\ell}{\mu_{\ell-1}} \right)^N.
\]
If \( j = 1, \ldots, \ell - 1 \) and \( m = 1, \ldots, \ell - 1 \) we get by (5.8) and (5.35)

\[
|\text{(II)}| \leq c \int_{A_\ell} |W_j^{p-1} \phi_{m,\varepsilon}^2| \, d\nu_g \leq c |W_j|^{p-1} \frac{2N}{N-2} A_\ell |\phi_{m,\varepsilon}|^2 \frac{2N}{N-2} A_\ell \\
\leq c \left( \frac{\sqrt{\mu_\ell \mu_{\ell-1}}}{\mu_j} \right)^2 \mu_m^{-N+2} (\mu_\ell \mu_{\ell-1})^{\frac{N-2}{2}} \leq c \left( \frac{\mu_\ell}{\mu_{\ell-1}} \right)^{N/2},
\]

while for \( j = \ell \) and \( m = 1, \ldots, \ell - 1 \), by the pointwise estimate of \( \phi_{m,\varepsilon} \) in (4.13) we get

\[
|\text{(II)}| \leq c \int_{A_\ell} |W_\ell^{p-1} \phi_{m,\varepsilon}^2| \, d\nu_g \leq c \mu_m^{-N+2} \mu_\ell^2 \int_{\sqrt{\mu_\ell \mu_{\ell-1}}}^{1} \frac{1}{|x|^{4}} \, dx
\leq c \mu_m^{-N+2} \mu_\ell^2 (\mu_\ell \mu_{\ell-1})^{\frac{N-4}{2}} \leq c \left( \frac{\mu_\ell}{\mu_{\ell-1}} \right)^{N/2}.
\]

If \( j = 1, \ldots, \ell - 1 \) and \( m = 1, \ldots, \ell - 1 \) by (5.8) and (5.35) we immediately get

\[
|\text{(III)}| \leq \int_{A_\ell} |W_j^p \phi_{m,\varepsilon}| \, d\nu_g \leq c |W_j|^{p} \frac{2N}{N-2} A_\ell |\phi_{m,\varepsilon}| \frac{2N}{N-2} A_\ell \leq c \left( \frac{\mu_\ell}{\mu_{\ell-1}} \right)^{N/2}.
\]

If \( j = 1, \ldots, \ell - 1 \) and \( m = 1, \ldots, \ell - 1 \) by (5.9) and (5.35) we immediately get

\[
|\text{(IV)}| \leq \int_{A_\ell} |W_j^{p-1} W_\ell W_j \phi_{m,\varepsilon}| \, d\nu_g \leq c |W_\ell|^{p-1} W_j \frac{2N}{N+2} A_\ell |\phi_{m,\varepsilon}| \frac{2N}{N-2} A_\ell \leq c \left( \frac{\mu_\ell}{\mu_{\ell-1}} \right)^{N/2}.
\]
Now, let us consider the case $h = 0, \ldots, \ell - 1$. By Lemma 5.1 we get

\[
\int_{A_h} a_\ell \, dv := - \int_{A_h} \left[ F \left( \sum_{j=1}^\ell (W_j + \phi_{j,\varepsilon}) \right) - F \left( \sum_{j=1}^{\ell-1} W_j + \sum_{j=1}^\ell \phi_{j,\varepsilon} \right) - f \left( \sum_{j=1}^\ell W_j + \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right) \phi_{\ell,\varepsilon} \right]
\]

\[
- \int_{A_h} \left[ F \left( \sum_{j=1}^\ell W_j + \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right) - F \left( \sum_{j=1}^{\ell-1} (W_j + \phi_{j,\varepsilon}) \right) - f \left( \sum_{j=1}^{\ell-1} (W_j + \phi_{j,\varepsilon}) \right) W_\ell \right]
\]

\[
+ \int_{A_h} \left[ F \left( \sum_{j=1}^\ell W_j \right) - F \left( \sum_{j=1}^{\ell-1} W_j \right) - f \left( \sum_{j=1}^{\ell-1} W_j \right) W_\ell \right]
\]

\[
+ \int_{A_h} \left[ f \left( \sum_{j=1}^\ell (W_j + \phi_{j,\varepsilon}) \right) - f \left( \sum_{j=1}^{\ell-1} W_j + \sum_{j=1}^\ell \phi_{j,\varepsilon} \right) \right] \phi_{\ell,\varepsilon} + \int_{A_h} f(W_\ell) \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon}
\]

\[
- \int_{A_h} f \left( \sum_{j=1}^{\ell-1} (W_j + \phi_{j,\varepsilon}) \right) W_\ell + \int_{A_h} f \left( \sum_{j=1}^\ell W_j \right) W_\ell
\]

\[
= O \left( \int_{A_h} \phi_{\ell,\varepsilon}^{p+1} \right) + O \left( \int_{A_h} \left( \sum_{j=1}^\ell W_j + \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right)^{p-1} \phi_{\ell,\varepsilon}^2 \right)
\]

\[
+ O \left( \int_{A_h} W_\ell^{p+1} \right) + O \left( \int_{A_h} \left( \sum_{j=1}^{\ell-1} (W_j + \phi_{j,\varepsilon}) \right)^{p-1} W_\ell^2 \right)
\]

\[
+ O \left( \int_{A_h} W_\ell^p \left( \sum_{j=1}^{\ell-1} \phi_{j,\varepsilon} \right) \right)_{{(III')}}
\]

\[
+ O \left( - \int_{A_h} f \left( \sum_{j=1}^{\ell-1} (W_j + \phi_{j,\varepsilon}) \right) W_\ell + \int_{A_h} f \left( \sum_{j=1}^\ell W_j \right) W_\ell \right)_{{(IV')}}
\]

= o(\varepsilon^{\theta_h}),
\]

(5.36)

because of the rate of the error term $\phi_{\ell,\varepsilon}$ given in (4.11) and the following four new estimates.

\(\diamond\) If $h = 1, \ldots, \ell - 1$ by (5.6) we immediately get

\[ |(I')| \leq c \left( \frac{\mu_\ell}{\mu_{\ell-1}} \right)^{\frac{N}{2}}. \]
If $h = 1, \ldots, \ell - 1$ and $j = 1, \ldots, \ell - 1$ by (5.6), (5.7) and (5.18) we immediately get

$$
|\langle II' \rangle| \leq c \int_{A_h} |W_j^{p-1}W^2_\ell| \, d\nu_g + c \int_{A_h} |\phi_j^{p-1}W^2_\ell| \, d\nu_g
$$

$$
\leq c \|W_j^{p-1}W^2_\ell\|_{L^{\frac{2N}{N + 2}}(\mathbb{R}^N, A_h)}^2 + c \|\phi_j^{p-1}W^2_\ell\|_{L^{\frac{2N}{N + 2}}(\mathbb{R}^N, A_h)}^2
$$

$$
\leq c \left( \frac{\mu_\ell}{\mu_{\ell - 1}} \right)^{\frac{N}{2}}.
$$

If $h = 1, \ldots, \ell - 1$ and $j = 1, \ldots, \ell - 1$ we have to estimate the term $\langle III' \rangle$, by distinguish three cases, namely $j \leq h - 1, j = h$ and $j \geq h + 1$.

If $j \geq h + 1$, by (5.6), by (3.12) and (4.11) we get

$$
\left| \int_{A_h} W_\ell^p \phi_{j, \varepsilon} \, d\nu_g \right| \leq c \|\phi_{j, \varepsilon}\| \|W_\ell^p\|_{L^{\frac{2N}{N + 2}}(\mathbb{R}^N, A_h)} \leq c \varepsilon^\frac{N + 2}{\mu_{h+1}} \frac{N + 2}{\mu_\ell} \frac{N + 2}{\mu_{h+1}} \frac{1}{(\varepsilon^\theta_\ell)^{\frac{N + 2}{4}}}
$$

$$
\leq c \varepsilon^\frac{N + 2}{\mu_{h+1}} \frac{N + 2}{\mu_\ell} \frac{N + 2}{\mu_{h+1}} \frac{1}{(\varepsilon^\theta_\ell)^{\frac{N + 2}{4}}} \leq c \varepsilon^\frac{N + 2}{\mu_{h+1}} \frac{N + 2}{\mu_\ell} \frac{N + 2}{\mu_{h+1}} \frac{1}{(\varepsilon^\theta_\ell)^{\frac{N + 2}{4}}} = O \left( \varepsilon^\theta_\ell \right)
$$

because $h \leq \ell - 1$ and so

$$
\frac{p}{2} \theta_{h+1} + \frac{N + 2}{2} \gamma_\ell - \frac{N + 2}{4} \gamma_h - \frac{N + 2}{4} \gamma_{h+1} = \frac{N + 2}{2} \gamma_\ell - \frac{N + 2}{2} \gamma_{\ell-1}.
$$

If $j \leq h - 1$, since $A_h \subset B_\varepsilon(\mu_j)$ we can use the pointwise estimate (4.13) for $\phi_{j, \varepsilon}$ and so

$$
\left| \int_{A_h} W_\ell^p \phi_{j, \varepsilon} \, d\nu_g \right| \leq c \mu_j^{-\frac{N + 2}{2}} \int_{A_h} |W_\ell^p| \, d\nu_g
$$

$$
\leq c \mu_j^{-\frac{N + 2}{2}} \frac{\mu_\ell}{\mu_{h+1}} \int_{\frac{\sqrt{\mu_h \mu_{h+1}}}{1 + |y|^2 ^{\frac{N + 2}{2}}} \leq y \leq \frac{\sqrt{\mu_h \mu_{h+1}}}{\mu_\ell}} \frac{1}{(1 + |y|^2 ^{\frac{N + 2}{2}}} \, dy
$$

$$
\leq c \mu_j^{-\frac{N + 2}{2}} \frac{\mu_\ell}{\mu_{h+1}} \int_{\frac{\sqrt{\mu_h \mu_{h+1}}}{1 + |y|^2 ^{\frac{N + 2}{2}}} \leq y \leq \frac{\sqrt{\mu_h \mu_{h+1}}}{\mu_\ell}} \frac{1}{(1 + |y|^2 ^{\frac{N + 2}{2}}} \, dy
$$

$$
\leq c \left( \frac{\mu_\ell}{\mu_{h+1}} \right)^{\frac{N}{2}} \left( \frac{\mu_{h+1}}{\mu_{h+1}} \right)^{\frac{N}{2}} .
$$

If $j = h$ we split the annulus

$$
A_h = \left\{ \sqrt{\mu_h \mu_{h+1}} \leq d_g(x, \xi) \leq \mu_h \right\} \cup \left\{ \mu_h \leq d_g(x, \xi) \leq \sqrt{\mu_h \mu_{h+1}} \right\}
$$

and we get combining the previous estimates

$$
\int_{A_h} W_\ell^p \phi_{h, \varepsilon} \, d\nu_g = \int_{A_h} W_\ell^p \phi_{h, \varepsilon} \, d\nu_g + \int_{A_h'} W_\ell^p \phi_{h, \varepsilon} \, d\nu_g
$$

$$
\leq c \mu_h^{-\frac{N + 2}{2}} \int_{A_h'} |W_\ell^p| \, d\nu_g + c \|\phi_{h, \varepsilon}\| \|W_\ell^p\|_{L^{\frac{2N}{N + 2}}(\mathbb{R}^N, A_{h}')} \leq c \mu_h^{-\frac{N + 2}{2}} \frac{\mu_\ell}{\mu_{h+1}} + c \varepsilon^\theta_\ell \left( \frac{\mu_\ell}{\mu_h} \right)^{\frac{N + 2}{2}}
$$

$$
\leq c \left( \frac{\mu_\ell}{\mu_{h+1}} \right)^{\frac{N}{2}} .
$$
We need to estimate the last term \( (IV') \). We have to distinguish two cases \( h = 0, \ldots, \ell - 2 \) and \( h = \ell - 1 \). By Lemma 5.1 we deduce that if \( h = 0, \ldots, \ell - 2 \) then

\[
|(IV')| \leq c \sum_{j=1}^{\ell-1} \int_{A_h} |W_j^p W_\ell| \, dv_y + c \sum_{j=1}^{\ell-1} \int_{A_h} ||\phi_{j,\varepsilon}||^p W_\ell | \, dv_y 
\]

(5.38)

while if \( h = \ell - 1 \) we get

\[
|(IV')| \leq c \sum_{j,m=1}^{\ell-1} \int_{A_{\ell-1}} |W_j|^{p-1} \phi_{m,\varepsilon} W_\ell | \, dv_y + \sum_{j=1}^{\ell-1} \int_{A_{\ell-1}} ||\phi_{j,\varepsilon}||^p W_\ell | \, dv_y .
\]

\[\diamond \diamond \diamond \] We estimate (i).

Let \( h = 1, \ldots, \ell - 2 \). If \( j = h \)

\[
\int_{A_h} |W_j^p W_\ell| \, dv_y \leq c \int_{B(\frac{\sqrt{p} h_{h-1}}{\lambda})} \frac{N-2}{\mu_h^{\frac{N-2}{2}}} \frac{N-2}{\mu_\ell^{\frac{N-2}{2}}} (1 + |y|^2)^{\frac{N-2}{2}} (\mu_\ell^2 + \mu_h^2 |y|^2)^{\frac{N-2}{2}} \, dy
\]

\[
\leq c \left( \frac{\mu_\ell}{\mu_h} \right)^{\frac{N-2}{2}} \mu_h^{\frac{N-2}{2}}
\]

if \( j \geq h + 1 \)

\[
\int_{A_h} |W_j^p W_\ell| \, dv_y \leq c \int_{B(\frac{\sqrt{p} h_{h-1}}{\lambda})} \frac{N-2}{\mu_h^{\frac{N-2}{2}}} \frac{N-2}{\mu_\ell^{\frac{N-2}{2}}} (1 + |y|^2)^{\frac{N-2}{2}} (\mu_\ell^2 + \mu_h^2 |y|^2)^{\frac{N-2}{2}} \, dy
\]

\[
\leq c \left( \frac{\mu_\ell}{\mu_h} \right)^{\frac{N-2}{2}} \mu_h^{\frac{N-2}{2}}
\]

and if \( j \leq h - 1 \)

\[
\int_{A_h} |W_j^p W_\ell| \, dv_y \leq c \int_{B(\frac{\sqrt{p} h_{h-1}}{\lambda})} \frac{N-2}{\mu_h^{\frac{N-2}{2}}} \frac{N-2}{\mu_\ell^{\frac{N-2}{2}}} (1 + |y|^2)^{\frac{N-2}{2}} (\mu_\ell^2 + \mu_h^2 |y|^2)^{\frac{N-2}{2}} \, dy
\]

\[
\leq c \left( \frac{\mu_\ell}{\mu_h} \right)^{\frac{N-2}{2}} \mu_h^{\frac{N-2}{2}}
\]

\[\diamond \diamond \diamond \] We estimate (ii).
We have to distinguish some cases. Let \( h = 1, \ldots, \ell - 1 \).

If \( j \leq h - 1 \) then by the pointwise estimate (4.13) for \( \phi_{j,\varepsilon} \) in \( A_h \subset B_\varepsilon(\mu_j) \) we get

\[
\int_{A_h} |\phi_{j,\varepsilon}|^p W_\ell^\varepsilon \, d\nu_g \leq c \frac{\mu_j^{N/2}}{\mu_{h-1}^{N/2}} \int_{\sqrt{\mu_h \mu_{h-1}}}^{\mu_h \mu_{h-1}} \frac{1}{|x|^{N-2}} \, dx \leq c \frac{\mu_j^{N/2}}{\mu_{h-1}^{N/2}} (\mu_{h-1} \mu_h)
\]

\[
\leq c \left( \frac{\mu_j}{\mu_{h-1}} \right)^{N/2} \frac{\mu_h}{\mu_{h-1}} \leq c \left( \frac{\mu_j}{\mu_{h-1}} \right)^{N/2} \left( \frac{\mu_{\ell-1}}{\mu_{h-1}} \right)^{N/2} .
\]

If \( j \geq h + 1 \) by (4.11) and (5.6) we get

\[
\int_{A_h} |\phi_{j,\varepsilon}|^p W_\ell^\varepsilon \, d\nu_g \leq \|\phi_{j,\varepsilon}\|_{L^p(N-2\mu_j)} A_h \leq c \varepsilon^2 \frac{\mu_j^{N/2}}{(\mu_h \mu_{h-1})^{N/2}} \frac{\mu_j^{N/2}}{(\mu_h \mu_{h-1})^{N/2}}
\]

\[
\leq c \varepsilon^2 \left( \frac{\mu_j}{\mu_{h-1}} \right)^{N/2} \left( \frac{\mu_j}{\mu_{h-1}} \right)^{N/2} = o \left( \varepsilon^\theta_\ell \right)
\]

because by the choice of \( \mu_j \) in (3.12) and the definition of \( \theta_{h+1} \) in (4.9) we get

\[
\frac{p^2}{2} \theta_{h+1} + \gamma \frac{N - 2}{2} - (\gamma_h + \gamma_{h+1}) \frac{N - 2}{4} > (\gamma_\ell - \gamma_{\ell-1}) \frac{N - 2}{2},
\]

since \( h + 1 \leq j \leq \ell - 1 \).

If \( j = h \leq \ell - 1 \) we split the annulus

\[
A_h = \{ \sqrt{\mu_h \mu_{h+1}} \leq \|\phi_{j,\varepsilon}\|_{L^p(N-2\mu_j)} \} \cup \{ \|\phi_{j,\varepsilon}\|_{L^p(N-2\mu_j)} \leq \sqrt{\mu_h \mu_{h-1}} \}
\]

and we get combining the previous estimates

\[
\int_{A_h} |\phi_{j,\varepsilon}|^p W_\ell^\varepsilon \, d\nu_g = \int_{A_h'} |\phi_{j,\varepsilon}|^p W_\ell^\varepsilon \, d\nu_g + \int_{A_h''} |\phi_{j,\varepsilon}|^p W_\ell^\varepsilon \, d\nu_g
\]

\[
\leq c \frac{\mu_j^{N/2}}{\mu_{h-1}^{N/2}} \mu_h^2 + c \varepsilon^2 \frac{\mu_j^{N/2}}{\mu_{h-1}^{N/2}} \mu_h^2
\]

\[
= o \left( \varepsilon^\theta_\ell \right) \text{ if } h \leq \ell - 2.
\]

If \( h = \ell - 1 \) we need to change the estimate of the term

\[
\int_{A_{\ell-1}} |\phi_{\ell-1,\varepsilon}|^p W_\ell^\varepsilon \, d\nu_g \leq c \frac{1}{\mu_{\ell-1}^{N-2\mu}} |\phi_{\ell-1,\varepsilon}|_{L^p(N-2\mu_j)} A_{\ell-1}
\]

\[
\leq c \frac{1}{\mu_{\ell-1}^{N-2\mu}} \varepsilon^2 \theta_{\ell-1} \mu_{\ell}^2 \left( \frac{\mu_\ell}{\mu_{\ell-1}} \right)^{N/2} = o \left( \varepsilon^\theta_\ell \right)
\]

because by the definition of \( \theta_\ell \) in (4.9) we get

\[
\frac{p}{2} \theta_\ell + \frac{p}{2} \theta_{\ell-1} > \theta_\ell.
\]

\( \diamond \diamond \diamond \) We estimate (iii).
If \( m \leq \ell - 2 \)

\[
\int_{A_{\ell-1}} |W_j^{p-1}\phi_{m,c}W_\ell| \, d\nu_g \leq c \mu_m^{-\frac{N-2}{2}} \int_{A_{\ell-1}} |W_j^{p-1}W_\ell| \, d\nu_g
\]

\[
\leq c \mu_m^{-\frac{N-2}{2}} \mu_\ell^{-\frac{N-2}{2}} \int_B \left( \frac{\sqrt{\nu_j \mu_\ell^{-1}}}{\mu_j} \right) \frac{1}{(1 + |y|^2)^{\frac{N-2}{2}}} \frac{1}{\mu_\ell} \, dy
\]

\[
\leq \mu_m^{-\frac{N-2}{2}} \mu_\ell^{-\frac{N-2}{2}} \int_{\sqrt{\nu_j \mu_\ell^{-1}}} B \left( \frac{\sqrt{\nu_j \mu_\ell^{-1}}}{\mu_j} \right) \frac{1}{(1 + |y|^2)^{\frac{N-2}{2}}} \frac{1}{\mu_\ell} \, dy
\]

\[
\leq c \mu_m^{-\frac{N-2}{2}} \mu_\ell^{-\frac{N-2}{2}} \mu_\ell^{-2} \frac{\mu_\ell^{-1}}{\mu_\ell^{-2}} \left( \frac{\mu_\ell^{-1}}{\mu_\ell^{-2}} \right)^{\frac{N-4}{2}}.
\]

If \( m = \ell - 1 \) we split the annulus

\[
A_{\ell-1} = \left\{ \sqrt{\mu_\ell \mu_{\ell-1}} \leq d_g(x, \xi) \leq \mu_{\ell-1} \right\} \cup \left\{ \mu_{\ell-1} \leq d_g(x, \xi) \leq \sqrt{\mu_{\ell-1} \mu_{\ell-2}} \right\}
\]

and so

\[
\int_{A_{\ell-1}} |W_j^{p-1}\phi_{\ell-1,c}W_\ell| \, d\nu_g = \int_{A_{\ell-1}^{''}} |W_j^{p-1}\phi_{\ell-1,c}W_\ell| \, d\nu_g + \int_{A_{\ell-1}^{''}} |W_j^{p-1}\phi_{\ell-1,c}W_\ell| \, d\nu_g.
\]

If \( j \leq \ell - 2 \) then arguing as before

\[
\int_{A_{\ell-1}^{''}} |W_j^{p-1}\phi_{\ell-1,c}W_\ell| \, d\nu_g \leq c \mu_\ell^{-\frac{N-2}{2}} \mu_\ell^{-\frac{N-2}{2}} \mu_\ell^{-2} \mu_\ell^{-1} \left( \frac{\mu_\ell}{\mu_\ell^{-1}} \right)^{\frac{N-2}{2}} \left( \frac{\mu_\ell^{-1}}{\mu_{\ell-2}} \right)^2 = o \left( \varepsilon^{\theta_\ell} \right).
\]

If \( j = \ell - 1 \) then

\[
\int_{A_{\ell-1}^{''}} |W_j^{p-1}\phi_{\ell-1,c}W_\ell| \, d\nu_g \leq c \frac{1}{\mu_\ell^{-2}} \varepsilon^{\theta_\ell-1} \mu_\ell^2 \left( \frac{\mu_\ell}{\mu_{\ell-1}} \right)^{\frac{N-6}{6}} = o \left( \varepsilon^{\theta_\ell} \right),
\]

because of (5.39). If \( j \leq \ell - 1 \) by (5.7) and (4.11) we get

\[
\int_{A_{\ell-1}^{''}} ||W_j||^{p-1}\phi_{\ell-1,c}W_\ell| \, d\nu_g \leq c ||\phi_{\ell-1,c}|| ||W_j||^{p-1} W_\ell|_{N+2} A_{\ell-1}^{''} \leq c \varepsilon^{\theta_\ell-1} \left( \frac{\mu_\ell}{\mu_\ell^{-1}} \right)^{\frac{N-6}{6}} \left( \frac{\mu_\ell}{\mu_\ell^{-1}} \right)^{\frac{N-6}{4}} = o \left( \varepsilon^{\theta_\ell} \right),
\]

because of (5.39).

That concludes the proof.
Step 2. We shall write the expansion of $J_{\epsilon} \left( \sum_{j=1}^{k} W_j \right)$. We will split the manifold $M = \bigcup_{h=0}^{k} A_h$ where the annuli $A_h$ are defined in (5.5). We have

$$J_{\epsilon} \left( \sum_{j=1}^{k} W_j \right) = \sum_{j=1}^{k} J_{\epsilon}(W_j) + \sum_{i<j}^{k} \int_M \left( \nabla_{g} W_i \nabla_{g} W_j + (\beta_N R_{g} + \epsilon) W_i W_j \right)
$$

$$- \int_M \left[ F \left( \sum_{j=1}^{k} W_j \right) - \sum_{j=1}^{k} F(W_j) \right]
$$

$$= \sum_{j=1}^{k} J_{\epsilon}(W_j) + \sum_{i<j}^{k} \int_M \left( -\mathcal{L}_{g}(W_i) + \epsilon W_i - f(W_i) \right) W_j d\nu_g
$$

$$- \int_M \left[ F \left( \sum_{j=1}^{k} W_j \right) - \sum_{j=1}^{k} F(W_j) - \sum_{i,j=1}^{k} f(W_i) W_j \right] d\nu_g
$$

$$= \sum_{j=1}^{k} J_{\epsilon}(W_j) + \sum_{i=1}^{j-1} \int_M \left( -\mathcal{L}_{g}(W_i) + \epsilon W_i - f(W_i) \right) W_j d\nu_g
$$

$$- \sum_{\ell=2}^{k} \sum_{h=0}^{\ell-1} \int_{A_h} \left[ F \left( \sum_{j=1}^{\ell} W_j \right) - F \left( \sum_{j=1}^{\ell-1} W_j \right) - F(W_{\ell}) - \sum_{j=1}^{\ell-1} f(W_j) W_{\ell} \right] d\nu_g
$$

where each $a_{\ell}$ only depends on $d_1, \ldots, d_{\ell}$ and they are defined as

$$a_1 := J_{\epsilon}(W_1)
$$

and for any $\ell = 2, \ldots, k$

$$a_{\ell} := J_{\epsilon}(W_{\ell}) + \sum_{i=1}^{\ell-1} \int_M \left( -\mathcal{L}_{g}(W_i) + \epsilon W_i - f(W_i) \right) W_{\ell} d\nu_g
$$

$$- \sum_{h=0}^{\ell-1} \int_{A_h} \left[ F \left( \sum_{j=1}^{\ell} W_j \right) - F \left( \sum_{j=1}^{\ell-1} W_j \right) - F(W_{\ell}) - \sum_{j=1}^{\ell-1} f(W_j) W_{\ell} \right] d\nu_g.
$$

We shall prove that

$$a_1 = \frac{K_{N}^{-N}}{N} + \epsilon^2 \left( -A_N |\text{Weyl}_{g}(\xi)|_g^2 d_1^4 + B_N d_1^2 \right) + o(\epsilon^2)
$$

and

$$a_{\ell} = \frac{K_{N}^{-N}}{N} + \epsilon^2 \left( -C_N \left( \frac{d_{\ell}}{d_{\ell-1}} \right)^{\frac{N-2}{2}} + B_N d_{\ell}^2 \right) + o(\epsilon^2), \quad \text{if } \ell = 2, \ldots, k.
$$

Now in [12] it has been proved that

$$J_{\epsilon}(W_{\ell}) := \frac{K_{N}^{-N}}{N} - A_N |\text{Weyl}_{g}(\xi)|_g^2 \mu_{\ell}^4 + \epsilon B_N \mu_{\ell}^2 + O(\mu_{\ell}^5)
$$

where

$$A_N := \frac{K_{N}^{-N}}{24N(N-4)(N-6)}; \quad B_N := \frac{2(N-1)K_{N}^{-N}}{N(N-2)(N-4)}$$
and \( K_N := \sqrt{\frac{4}{N(N-2)e_N^2}} \) is the sharp constant for the embedding of \( D^{1,2}(\mathbb{R}^N) \) into \( L^{p+1}(\mathbb{R}^N) \).

By (5.34) we immediately get for any \( i = 1, \ldots, \ell - 1 \)

\[
\left| \int_M ( - L_g(W_i) + \varepsilon W_i - f(W_i) ) W_\ell \, dv_g \right| = O \left( \int_{B(0, r_0)} \left( \frac{\mu_i^{N-2}}{\mu_\ell^{N-2}} \frac{|x|^{2\frac{N-2}{2}}}{\mu_i^{\frac{N-2}{2}} |x|^{\frac{N-2}{2}}} \right) \, dx \right) = O \left( \int_{B(0, r_0)} \left( \frac{\mu_i^{N-2}}{\mu_\ell^{N-2}} \frac{|x|^{2\frac{N-2}{2}}}{\mu_i^{\frac{N-2}{2}} |x|^{\frac{N-2}{2}}} \right) \, dx \right) = O \left( \left( \frac{\mu_\ell}{\mu_i} \right)^{\frac{N-2}{2}} \right).
\]

Finally, it remains to estimate for \( h = 1, \ldots, \ell \)

\[
I_h := \int_{A_h} \left[ F \left( \sum_{j=1}^{\ell} W_j \right) - F \left( \sum_{j=1}^{\ell-1} W_j \right) - \sum_{j=1}^{\ell-1} f(W_j) W_j \right] \, dv_g.
\]

If \( h = \ell \) by Lemma 5.1

\[
I_\ell = \int_{A_\ell} \left[ F \left( \sum_{j=1}^{\ell} W_j \right) - F(W_\ell) - \sum_{j=1}^{\ell-1} W_j \right] \, dv_g - \int_{A_\ell} F \left( \sum_{j=1}^{\ell-1} W_j \right) \, dv_g
\]

\[
= O \left( \int_{A_\ell} \left( \sum_{j=1}^{\ell-1} W_j \right)^{p+1} \, dv_g \right) + O \left( \int_{A_\ell} \left( \sum_{j=1}^{\ell-1} W_j \right)^2 W_\ell^{p-1} \, dv_g \right)
\]

\[
= O \left( \sum_{j=1}^{\ell-1} |W_j|^{p+1} A_\ell \right) + O \left( \sum_{j=1}^{\ell-1} |W_j|^{2N-2} A_\ell \right)
\]

\[
= o (\varepsilon^{\theta_\ell}) \), \text{ because of (5.8) and (5.9)}
\]

If \( h = 0, \ldots, \ell - 1 \) get

\[
I_h = \int_{A_h} \left[ F \left( \sum_{j=1}^{\ell} W_j \right) - F \left( \sum_{j=1}^{\ell-1} W_j \right) - f \left( \sum_{j=1}^{\ell-1} W_j \right) W_\ell \right] \, dv_g - \int_{A_h} F \left( W_\ell \right) \, dv_g
\]

\[
- \int_{A_h} \left( \sum_{j=1}^{\ell-1} W_j \right) f(W_\ell) \, dv_g + \int_{A_h} f \left( \sum_{j=1}^{\ell-1} W_j \right) W_\ell \, dv_g.
\]

Now, (i) = \( o (\varepsilon^{\theta_\ell}) \) as in \((I')\) and \((II')\) in (5.36), (ii) = \( o (\varepsilon^{\theta_\ell}) \) as in \((III')\) in (5.36) (see (5.37)) and (iii) = \( o (\varepsilon^{\theta_\ell}) \) when \( h = 1, \ldots, \ell - 2 \) as in \((IV')\) in (5.38).

It only remains to estimate (iii) when \( h = \ell - 1 \), which contains the leading term given by the interaction of two consecutive bubbles. Indeed

\[
\int_{A_{\ell-1}} f \left( \sum_{j=1}^{\ell-1} W_j \right) W_\ell \, dv_g = \int_{A_{\ell-1}} \left[ f \left( \sum_{j=1}^{\ell-1} W_j \right) - f(W_{\mu_{\ell-1}}) \right] W_\ell \, dv_g + \int_{A_{\ell-1}} f(W_{\mu_{\ell-1}}) W_\ell \, dv_g,
\]
where the first term is estimated as in (IV') in (5.38) when \( h = \ell - 1 \)

\[
\left| \int_{\mathcal{A}_{\ell-1}} f \left( \sum_{j=1}^{\ell-1} W_j \right) W_\ell \, d\nu_g \right| = O \left( \sum_{j=1}^{\ell-2} \int_{\mathcal{A}_{\ell-1}} |W_{\ell-1}^{p-1} W_j W_\ell| \, d\nu_g \right) + O \left( \sum_{j=1}^{\ell-2} \int_{\mathcal{A}_{\ell-1}} |W_j^{p} W_\ell| \, d\nu_g \right)
\]

(5.42)

and the second term is the leading term:

\[
\int_{\mathcal{A}_{\ell-1}} f(W_{\mu_{\ell-1}}) W_\ell \, d\nu_g = \left( \frac{\mu_\ell}{\mu_{\ell-1}} \right)^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \left| \frac{U(y)}{|y|^{N-2}} \right| dy (1 + o(1))
\]

\[
= \left( \frac{\mu_\ell}{\mu_{\ell-1}} \right)^{\frac{N-2}{2}} \frac{2^{N-1} K_N^{-N} \omega_N^{N-1}}{N \omega_N} (1 + o(1)).
\]

(5.43)

That concludes the proof.

\[\square\]

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Filippo Morabito, KAIST 291 Daehak-ro Yuseong-gu, Daejeon, Republic of Korea, 305-701

E-mail address: morabitf@gmail.com

Angela Pistoia, Dipartimento di Scienze di Base e Applicate per l’Ingegneria, Sapienza Università di Roma, via Antonio Scarpa 16, 00161 Roma, Italy

E-mail address: angela.pistoia@uniroma1.it

Giusi Vaira, Dipartimento di Scienze di Base e Applicate per l’Ingegneria, Sapienza Università di Roma, via Antonio Scarpa 16, 00161 Roma, Italy

E-mail address: vaira.gius@uniroma1.it