Classification of arbitrary-dimensional multipartite pure states under stochastic local operations and classical communication using the rank of coefficient matrix

Shuhao Wang¹, Yao Lu¹, Ming Gao¹, Jianlian Cui² and Junlin Li¹,³

¹ State Key Laboratory of Low-Dimensional Quantum Physics, Department of Physics, Tsinghua University, Beijing 100084, People's Republic of China
² Department of Mathematical Sciences, Tsinghua University, Beijing 100084, People’s Republic of China
³ Tsinghua National Laboratory for Information Science and Technology, Beijing 100084, People’s Republic of China

E-mail: center@mail.tsinghua.edu.cn

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Abstract
We study multipartite entanglement under stochastic local operations and classical communication (SLOCC) and propose the entanglement classification under SLOCC for arbitrary-dimensional multipartite (n-qudit) pure states via the rank of coefficient matrix, together with the permutation of qudits. The ranks of the coefficient matrices have been proved to be entanglement monotones. The entanglement classification of the 2⊗2⊗2⊗4 system is discussed in terms of the generalized method, and 22 different SLOCC families are found.

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(Some figures may appear in colour only in the online journal)

Entanglement plays a vital role in quantum information processing, which includes quantum teleportation, quantum cryptography, quantum computation, etc [1]. Classification of different types of multipartite entanglement has been one of the main tasks in quantum information theory. Many studies on multipartite entanglement classification under different restrictions, such as local operations and classical communication (LOCC) and stochastic LOCC (SLOCC) [2, 3], have been conducted in recent years. The difference between LOCC and SLOCC can be interpreted as follows: if two states can be made equivalent up to LOCC with some non-zero probability, they are said to be SLOCC equivalent [3]. Suppose that two n-qudit pure states \(|\psi\rangle\) and \(|\phi\rangle\) are in the n-partite Hilbert space \(\mathcal{H}^n = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n\), where \(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n\) have the dimensions \(d_1, d_2, \ldots, d_n\), respectively. In mathematics, if \(|\psi\rangle\) and \(|\phi\rangle\) are LOCC equivalent if there exists local unitary operators \(U_{(1)}, U_{(2)}, \ldots, U_{(n)}\) in \(U(d_1, \mathbb{C}), U(d_2, \mathbb{C}), \ldots, U(d_n, \mathbb{C})\), respectively, such that [3]

\[
|\psi\rangle = U_{(1)} \otimes U_{(2)} \otimes \cdots \otimes U_{(n)}|\phi\rangle.
\]
If $|\psi\rangle$ and $|\phi\rangle$ are SLOCC equivalent, then they can be expressed as [4]

$$|\psi\rangle = F_{(1)} \otimes F_{(2)} \otimes \cdots \otimes F_{(n)} |\phi\rangle,$$

where $F_{(1)}, F_{(2)}, \ldots, F_{(n)}$ are invertible local operators (ILOs) in $GL(d_1, \mathbb{C}), GL(d_2, \mathbb{C}), \ldots, GL(d_n, \mathbb{C})$, respectively. In this paper, we concentrate on the entanglement classification under SLOCC.

It has been shown that two pure states that are equivalent under SLOCC can perform the same quantum information tasks [4]. The main idea of entanglement classification is to find an invariant preserved under SLOCC, and considerable research has been conducted on the entanglement classification of three [4], four [5–10] and $n$-qubit pure states [11–14] under SLOCC since the beginning of this century. Recently, Li and Li have proposed a simpler and more efficient approach to the SLOCC classification of general $n$-qubit pure states in [15]. A general $n$-qubit pure state can be expanded as $|\psi\rangle = \sum_{i=0}^{2^n-1} a_i |i\rangle$, where $a_i$ are the coefficients and $|i\rangle$ are the binary basis states. The coefficient matrix is constructed as follows:

$$M(|\psi\rangle) = \begin{pmatrix}
    a_{00\cdots0} & \cdots & a_{00\cdots1} \\
    a_{10\cdots0} & \cdots & a_{10\cdots1} \\
    \vdots & \ddots & \vdots \\
    a_{1\cdots0} & \cdots & a_{1\cdots1}
\end{pmatrix},$$

where the subscripts of the coefficients are written in binary form. For two $n$-qubit pure states connected by SLOCC, Li and Li proved that the ranks of the coefficient matrices are equal whether or not the permutation of qubits is fulfilled on both states. This theorem provides a way of partitioning all the $n$-qubit states into different families.

With the development of quantum information theory, the importance of qudit is gradually recognized. Maximally entangled qudits have been shown to violate local realism more strongly and are less affected by noise than qubits [16–21]. Using entangled qudits can provide more secure schemes against eavesdropping attacks in quantum cryptography [22–25], and also offers advantages including greater channel capacity for quantum communication [26] as well as more reliable quantum processing [27]. Much effort has been put into the classification of bipartite and tripartite states with higher dimensions in systems such as $2 \otimes 2 \otimes n$ [29, 30], $2 \otimes n \otimes n$ [31], $2 \otimes m \otimes n$ [32–34] and $m \otimes n \otimes n$ [35].

In this paper, we generalize the concept of coefficient matrix to $n$-qudit pure states. A theorem is provided to show that the rank of the coefficient matrix is invariant under SLOCC. By calculating the rank of coefficient matrix along with the permutation of qudits, we successfully obtain the results of classification for $n$-qudit pure states under SLOCC. We have also proved that each of the ranks of the coefficient matrices is an entanglement monotone. We investigate several examples and interesting entanglement properties are discovered. Using our theorems, we discuss the entanglement classification of the $2 \otimes 2 \otimes 2 \otimes 4$ system, which we believe has never been studied before.

Suppose an $n$-qudit pure state $|\psi\rangle$ in the $n$-partite Hilbert space $\mathcal{H}^n = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$, where $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ have the dimensions $d_1, d_2, \ldots, d_n$, respectively, which can be expanded in the form

$$|\psi\rangle = \sum_{s_1s_2 \cdots s_n} a_s |s_1s_2 \cdots s_n\rangle,$$

where $s_i$ are integers from 0 to $d_i-1$, and $s = (s_1, s_2, \ldots, s_n)$.
where \( a_i \) are the coefficients and \(|s_1s_2\cdots s_n]\) are the basis states
\[
|s_1s_2\cdots s_n]\rangle = |s_1]\otimes |s_2]\otimes \cdots \otimes |s_n]\rangle
\] (5)
with \( s_k \in \{0, 1, \ldots, d_k - 1\}, k = 1, \ldots, n \). The coefficient matrix \( M(|\psi]\rangle) \) is constructed by arranging \( a_i (i = 0, \ldots, \prod_{k=1}^{n} d_k - 1) \) in lexicographical ascending order
\[
M(|\psi]\rangle) = \begin{pmatrix}
    a_{00}\cdots 0 & \cdots & a_{0d_{n-1}}0 & \cdots & a_{0d_{n-1}d_{n-2}}0 & \cdots & a_{0d_{n-1}d_{n-2}\cdots d_2}0 & \cdots & a_{0d_{n-1}d_{n-2}\cdots d_2d_1}0 \\
    a_{10}\cdots 0 & \cdots & a_{1d_{n-1}}0 & \cdots & a_{1d_{n-1}d_{n-2}}0 & \cdots & a_{1d_{n-1}d_{n-2}\cdots d_2}0 & \cdots & a_{1d_{n-1}d_{n-2}\cdots d_2d_1}0 \\
    \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
    a_{d_{n-1}}0 & \cdots & a_{d_{n-1}d_{n-2}}0 & \cdots & a_{d_{n-1}d_{n-2}\cdots d_2}0 & \cdots & a_{d_{n-1}d_{n-2}\cdots d_2d_1}0 \\
    \end{pmatrix}
\] (6)
where \( 1 \leq l \leq n - 1 \).

To illustrate, we consider the \( n \)-qudit GHZ state [36]
\[
|\text{GHZ}\rangle = \frac{1}{\sqrt{d}} |0\rangle^{\otimes n} + |1\rangle^{\otimes n} + \cdots + |d-1\rangle^{\otimes n}.
\] (7)
It can be calculated that all the coefficient matrices have the form of
\[
M(|\text{GHZ}\rangle) = \begin{pmatrix}
    1 & 0 & \cdots & 0 & 0 \\
    0 & \ddots & \cdots & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & 0 & 0 \\
    0 & 0 & \cdots & 0 & 1/\sqrt{d} \\
\end{pmatrix}
\] (8)
where the coefficient matrices are usually not square matrices, they have \( d \) diagonal element being non-zero, and the non-diagonal elements are all zero. A simple calculation shows that \( \text{rank}(|\text{GHZ}\rangle) = d \).

Each permutation of qubits gives a permutation \([q_1, q_2, \ldots, q_n]\) of \([1, 2, \ldots, n]\). So in this case, the coefficient matrices \( M_{q_1\cdots q_n} \) (here we have omitted the column qudits) can be constructed by taking the corresponding permutation. The relation between all the reduced density matrices and the coefficient matrices is given by [37]
\[
\rho_{q_1\cdots q_n} = M_{q_1\cdots q_n}M_{q_1\cdots q_n}^\dagger,
\] (9)
where \( M_{q_1\cdots q_n}^\dagger \) is the conjugate transpose of \( M_{q_1\cdots q_n} \). It is obvious that \( \text{rank}(M_{q_1\cdots q_n}) = \text{rank}(\rho_{q_1\cdots q_n}) \). Therefore, when considering all the particles, the local ranks [4] are exactly the ranks of the coefficient matrices in the case where \( l = 1 \).

In the following context, in the case where \( l \geq 2 \), the permutations of qudits are included in the set
\[
[s] = [(r_1, c_1)(r_2, c_2)\cdots (r_k, c_k)]
\] (10)
where \( 1 \leq r_1 < r_2 < \cdots < r_k < l + (n \text{mod } 2) \), \( l < c_1 < c_2 < \cdots < c_k \leq n \), and \((r_i, c_i)\) represents the transposition of \( r_i \) and \( c_i \). The purpose of choosing the permutation form in equation (10) is to omit the permutations that end up exchanging rows or columns in the coefficient matrix. Letting \( k \) vary from 0 to \( l - (n \text{mod } 2) \), we obtain all the elements included in the set \([s] \). The case where \( k = 0 \) is defined as identical permutation, denoted by \( s_0 = I \). When \( l = 1 \), we choose \( s_k = (1, k + 1) \), \( k = 0, 1, \ldots, n - 1 \).
Theorem 1. According to equation (2), the coefficient matrices of $|\psi\rangle$ and $|\phi\rangle$ satisfy the relation

$$M(|\psi\rangle) = (F(1) \otimes \cdots \otimes F([n/2])) M(|\phi\rangle) (F([n/2]+1) \otimes \cdots \otimes F(n))^T.$$  \hspace{1cm} (11)

Applying permutation $\sigma$ to both sides of equation (11) gives

$$M^\sigma(|\psi\rangle) = (F(1)^\sigma \otimes \cdots \otimes F([n/2])^\sigma) M^\sigma(|\phi\rangle) (F([n/2]+1)^\sigma \otimes \cdots \otimes F(n)^\sigma)^T,$$  \hspace{1cm} (12)

which indicates that $M^\sigma(|\psi\rangle)$ and $M^\sigma(|\phi\rangle)$ have the same rank.

The detailed proof is given in appendix.

Therefore, the classification of entanglement via the rank of the coefficient matrix has the significant advantage of being independent of the dimension of state and permutation of qudits. Let $F_{n,1}$ represent the family of all $n$-qudit states with rank $r$. It is clear that all full separable states belong to $F_{n,1}$. With the help of permutation of qudits, the families $F_{n,r}$ can be further divided into subfamilies. Define $F^\sigma_r$ (here we have omitted the subscript $n$) as the subfamily whose coefficient matrix rank is $r$ with respect to permutation $\sigma$. The general expression of the subfamilies is

$$F_{r_1,r_2,\ldots,r_m} = F_{r_1} \cap \cdots \cap F_{r_m}.$$  \hspace{1cm} (13)

In order to maximize the number of families, the value of $l$ is given by

$$l = \text{argmax}\{P(l)\},$$  \hspace{1cm} (14)

where

$$P(l) = \prod_{\{q\}} \min \left\{ \prod_{k=1}^{l} d_{q_k}, \prod_{k=l+1}^{n} d_{q_k} \right\}$$  \hspace{1cm} (15)

with $d_{q_k}$ the dimension of the party corresponding to $q_k$. It is obvious that for states with each party of the same dimension, the family number is maximized when $l = \lfloor n/2 \rfloor$.

Theorem 2. Each of the ranks of the coefficient matrices is an entanglement monotone.

Proof. It has been shown that the rank of the coefficient matrix $M_{q_1,q_2,\ldots,q_r}(|\psi\rangle)$, which is the direct generalization of the Schmidt rank of the bipartite pure states, cannot be increased by LOCC [38]. Therefore, $\text{rank}(M_{q_1,q_2,\ldots,q_r}(|\psi\rangle))$ is an entanglement monotone. \hfill {\Box}

The theorem has shown that the rank of coefficient matrix is closely connected with the degree of entanglement. As an application of the generalized method, consider the following state:

$$|l_1, l_2, n\rangle = \left( \frac{n!}{l_0!l_1!l_2!} \right)^{-\frac{1}{2}} \sum_k P_k \begin{pmatrix} 1, \ldots, 1, 2, \ldots, 2, 0, \ldots, 0 \end{pmatrix},$$  \hspace{1cm} (16)

where $|1\rangle$, $|2\rangle$ are the excitations, $|0\rangle$ represents the ground state and $l_0, l_1, l_2$ are the number of states $|0\rangle$, $|1\rangle$, $|2\rangle$, respectively, which satisfy $l_1 + l_2 \leq n - 1$. $P_k$ is the set that contains all permutations. We denote the states in equation (16) as $D^\sigma_n$ states.

For $D^\sigma_n$ states, states $|l_1, l_2, n\rangle$, $|l_2, l_1, n\rangle$, $|n - l_1 - l_2, l_1, n\rangle$, $|n - l_1 - l_2, l_2, n\rangle$, $|l_1, n - l_1 - l_2, n\rangle$ and $|l_2, n - l_1 - l_2, n\rangle$ can be transformed into each other under SLOCC, namely, they belong to the same family. In the following, we can arrange these states and denote them as $a(l_1, l_2, l_0)$, where $l_0 = n - l_1 - l_2$. We study the classification of entanglement of $D^\sigma_n$ states with respect to $l_1$, $l_2$ and $l_0$. The variance of $l_1$, $l_2$ and $l_0$ and the ranks of the coefficient matrices $M_{q_1,q_2,q_3}$ under different arrangements are shown in figure 1, which shows...
that the rank of the coefficient matrix increases with the decrease of the variance, and most of the $D_9^3$ states can be distinguished by the ranks of the coefficient matrices.

Physically speaking, states $|0\rangle$, $|1\rangle$ and $|2\rangle$ are on an equal footing. So the state is maximal entangled when $l_0$, $l_1$ and $l_2$ are close to each other, namely, the variance of $l_0$, $l_1$ and $l_2$ is as small as possible. According to theorem 2, figure 1 shows an inverse relationship between the variance and the rank of $M_{q_1q_2q_3q_4}$.

We then consider $D_n^4$ states, which are defined as

$$|l_1, l_2, l_3, n\rangle = \left(\frac{n!}{l_0! l_1! l_2! l_3!}\right)^{-\frac{1}{2}} \sum_{k} P_k \left(\frac{1, \ldots, 1, 2, \ldots, 2, 3, \ldots, 3, 0, \ldots, 0}{l_1, l_2, l_3, l_0}\right).$$

where $|1\rangle$, $|2\rangle$ and $|3\rangle$ are the excitations with $l_1$, $l_2$ and $l_3$ as their numbers, which satisfy $l_1 + l_2 + l_3 \leq n - 1$, whereas $|0\rangle$ is the ground state.

We study the classification of entanglement of $D_n^4$ states with respect to $l_1$, $l_2$, $l_3$ and $l_0$. The variance of $l_1$, $l_2$, $l_3$ and $l_0$ and the ranks of the coefficient matrices $M_{l_1l_2l_3l_0}$ under different arrangements are shown in figure 2. The rank of the coefficient matrices shows a contrasting trend with the decrease of the variance, the physical interpretation of this phenomenon is the same as the $D_n^3$ states, and we can distinguish most states in terms of the ranks of the coefficient matrices.

In the end, we discuss the entanglement classification of the $2 \otimes 2 \otimes 2 \otimes 4$ system. For the cases where $l = 1$, $l = 2$ and $l = 3$, the values of $\mathcal{P}(l)$ are 4, 64 and 4, respectively. To maximize the family number, we consider the case where $l = 2$. The set of permutation consists of three elements: $\{\sigma\} = \{\sigma_0 = I, \sigma_1 = (1, 3), \sigma_2 = (1, 4)\}$. The classification results are shown in table 1. It needs to be noted that the entangled states ($|W\rangle$ and $|GHZ\rangle$ states) in $F_{2,2,2}^{0\sigma_0,\sigma_1,\sigma_2}$ have a similar Frobenius algebra structure [39]. The entanglement structure of the $2 \otimes 2 \otimes 2 \otimes 4$ system is illustrated by an entanglement pyramid in figure 3.

In summary, the rank invariance of the coefficient matrix under SLOCC has been proven to be valid in the $n$-qudit pure states regardless of the dimension of each partite and the permutation of qudits. It has also been proved that each of the ranks of the coefficient matrices is an entanglement monotone. Numerical results showed that this generalization can investigate the entanglement feature of quantum states with qudits. We have discussed the entanglement...
Theoretical and experimental results. We expect that our generalization could come up with further theoretical and experimental results.

![Figure 2](image)

**Figure 2.** Variance of $l_1$, $l_2$, $l_3$ and $l_4$ ranks of coefficient matrices $M_{(e_1 e_2 e_3 e_4)}$ under different arrangements (shown in the vertical axis) existing in $D_4^4$ states.

**Table 1.** SLOCC classification of the $2 \otimes 2 \otimes 2 \otimes 4$ system. The permutations are $\sigma_0 = I$, $\sigma_1 = (1,3)$, $\sigma_2 = (1,4)$.

| SLOCC family | Representative entangled states |
|--------------|---------------------------------|
| $\mathcal{F}_{\sigma_0,\sigma_1,\sigma_2}^{2,4,4}$ | $|0000\rangle + |0010\rangle + |0101\rangle + |0111\rangle + |1002\rangle + |1012\rangle + |1103\rangle + |1113\rangle$ |
| $\mathcal{F}_{\sigma_0,\sigma_1,\sigma_2}^{4,2,2}$ | $|0000\rangle + |0101\rangle + |0102\rangle + |1002\rangle + |0112\rangle + |0113\rangle$ |
| $\mathcal{F}_{\sigma_0,\sigma_1,\sigma_2}^{4,4,2}$ | $|0000\rangle + |0010\rangle + |0102\rangle + |1002\rangle + |0112\rangle + |1012\rangle + |0113\rangle$ |
| $\mathcal{F}_{\sigma_0,\sigma_1,\sigma_2}^{4,4,3}$ | $|0000\rangle + |0111\rangle + |0112\rangle + |1113\rangle$ |
| $\mathcal{F}_{\sigma_0,\sigma_1,\sigma_2}^{4,4,1}$ | $|0000\rangle + |0111\rangle + |0102\rangle + |1002\rangle + |1112\rangle + |1113\rangle$ |
| $\mathcal{F}_{\sigma_0,\sigma_1,\sigma_2}^{3,3,2}$ | $|0000\rangle + |1001\rangle + |1112\rangle$ |
| $\mathcal{F}_{\sigma_0,\sigma_1,\sigma_2}^{3,3,3}$ | $|0000\rangle + |0110\rangle + |1112\rangle + |1113\rangle$ |
| $\mathcal{F}_{\sigma_0,\sigma_1,\sigma_2}^{2,2,2}$ | $|0000\rangle + |0110\rangle + |1001\rangle + |1112\rangle$ |
| $\mathcal{F}_{\sigma_0,\sigma_1,\sigma_2}^{2,2,1}$ | $|0000\rangle + |0110\rangle + |1112\rangle + |1113\rangle$ |
| $\mathcal{F}_{\sigma_0,\sigma_1,\sigma_2}^{4,4,4}$ | $|0000\rangle + |0101\rangle + |0102\rangle + |0112\rangle + |0113\rangle + |0111\rangle$ |
| $\mathcal{F}_{\sigma_0,\sigma_1,\sigma_2}^{4,4,1}$ | $|0000\rangle + |0111\rangle + |0102\rangle + |0113\rangle + |1112\rangle + |1113\rangle$ |
| $\mathcal{F}_{\sigma_0,\sigma_1,\sigma_2}^{4,4,2}$ | $|0000\rangle + |0111\rangle + |0112\rangle + |1113\rangle + |1112\rangle + |1113\rangle$ |
| $\mathcal{F}_{\sigma_0,\sigma_1,\sigma_2}^{4,4,3}$ | $|0000\rangle + |0111\rangle + |0112\rangle + |1113\rangle + |1112\rangle + |1113\rangle$ |
| $\mathcal{F}_{\sigma_0,\sigma_1,\sigma_2}^{4,4,4}$ | $|0000\rangle + |0111\rangle + |0112\rangle + |1113\rangle + |1112\rangle + |1113\rangle$ |
| $\mathcal{F}_{\sigma_0,\sigma_1,\sigma_2}^{4,4,1}$ | $|0000\rangle + |0111\rangle + |0112\rangle + |1113\rangle + |1112\rangle + |1113\rangle$ |

We have classified the $2 \otimes 2 \otimes 2 \otimes 4$ system and found 22 different SLOCC families with respect to the generalized method. We expect that our generalization could come up with further theoretical and experimental results.
Figure 3. The entanglement pyramid of the $2 \otimes 2 \otimes 2 \otimes 4$ system, where we use $(i, j, k)$ to represent $\sigma_{i,j}^{\sigma_0,\sigma_1,\sigma_2}$.

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Appendix

Now we prove the following theorem:

Let $|\psi\rangle, |\phi\rangle$ be any states in the $n$-partite Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$, where $\mathcal{H}_i$ is of dimension $d_i$, $1 \leq i \leq n$. If there exist $A_i \in M_{d_i}(\mathbb{C})$ $(1 \leq i \leq n)$ such that

$$|\psi\rangle = A_1 \otimes A_2 \otimes \cdots \otimes A_n |\phi\rangle,$$

then, for any $1 \leq l < n$,

$$M(|\psi\rangle) = A_1 \otimes \cdots \otimes A_l M(|\phi\rangle) (A_{l+1} \otimes \cdots \otimes A_n)^T. \quad (A.2)$$

We will prove equation (A.2) by the induction method. Clearly, if $A_i = I_i$ (the identity matrix in $M_{d_i}(\mathbb{C})$) for every $1 \leq i \leq n$, then equation (11) holds.

Let $|\psi\rangle = \sum_{i=0}^{d_i-1} c_i |i\rangle$ and for $1 \leq r < n$,

$$|\psi\rangle = I_1 \otimes \cdots \otimes I_r \otimes A_{r+1} \otimes \cdots \otimes A_n |\phi\rangle. \quad (A.3)$$

For any $1 \leq l < n$, we assume that

$$M(|\psi\rangle) = I_1 \otimes \cdots \otimes I_r \otimes A_{r+1} \otimes \cdots \otimes A_l M(|\phi\rangle) (A_{l+1} \otimes \cdots \otimes A_n)^T,$$

when $r + 1 \leq l < n$;

$$M(|\psi\rangle) = I_1 \otimes \cdots \otimes I_r M(|\phi\rangle) (I_{r+1} \otimes \cdots \otimes I_n \otimes A_{r+1} \otimes \cdots \otimes A_n)^T,$$

when $1 \leq l < r < n$;

$$M(|\psi\rangle) = I_1 \otimes \cdots \otimes I_r M(|\phi\rangle) (A_{r+1} \otimes \cdots \otimes A_n)^T,$$

when $1 \leq l = r < n$. \quad (A.4)
Next, we will prove that when
\[ |\psi^\prime\rangle = I_1 \otimes \cdots \otimes I_{r-1} \otimes A_r \otimes \cdots \otimes A_n |\phi\rangle, \]  
(A.5)
there is
\[ M(|\psi^\prime\rangle) = I_1 \otimes \cdots \otimes I_{r-1} \otimes A_r \otimes \cdots \otimes A_n M(|\phi\rangle)(A_{j+1} \otimes \cdots \otimes A_l)^T, \]
when \( r + 1 \leq \ell < n; \)
\[ M(|\psi^\prime\rangle) = I_1 \otimes \cdots \otimes I_{r-1} \otimes A_r \otimes \cdots \otimes A_n |\phi\rangle, \]
when \( 1 \leq \ell < r < n; \)
\[ M(|\psi^\prime\rangle) = I_1 \otimes \cdots \otimes I_{r-1} \otimes A_n M(|\phi\rangle)(A_{j+1} \otimes \cdots \otimes A_l)^T, \]
when \( 1 \leq \ell = r < n. \)
(A.6)

Write \[ |\psi^\prime\rangle = \sum_{m=0}^{d-1} b_m |\phi\rangle \]
and
\[ A_r = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d_r} \\ a_{21} & a_{22} & \cdots & a_{2d_r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d_r1} & a_{d_r2} & \cdots & a_{dd_r} \end{pmatrix}. \]  
(A.7)

Since
\[ |\psi^\prime\rangle = I_1 \otimes \cdots \otimes I_{r-1} \otimes A_r \otimes I_{r+1} \otimes \cdots \otimes I_n |\psi\rangle, \]  
(A.8)
we need only to prove that
\[ M(|\psi^\prime\rangle) = I_1 \otimes \cdots \otimes I_{r-1} \otimes A_r \otimes I_{r+1} \otimes \cdots \otimes I_n M(|\phi\rangle), \]
when \( r + 1 \leq \ell < n; \)
\[ M(|\psi^\prime\rangle) = I_{r-1} \otimes I_{r+1} \otimes \cdots \otimes I_n M(|\phi\rangle)(A_{j+1} \otimes \cdots \otimes A_l)^T, \]
when \( 1 \leq \ell < r; \)
\[ M(|\psi^\prime\rangle) = I_1 \otimes \cdots \otimes I_{r-1} \otimes A_n M(|\phi\rangle), \]
when \( 1 \leq \ell = r < n. \)
(A.9)

From equation (A.8), it can be computed that
\[ b_{khl} = a_{1k} c_{hl} + a_{2k} c_{hl} + \cdots + a_{dk} c_{hl}, \]  
(A.10)
where \( t, 1, 2, \ldots, d_2; k, 0, 1, \ldots, d_1 \cdots d_{r-1} - 1; s, 0, 1, \ldots, d - 1; h = d_{r+1} \cdots d_n. \) If \( r + 1 \leq l < n, \) write
\[ M(|\psi\rangle) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{d_1-1} \\ c_{d_1} & c_{d_1+1} & \cdots & c_{2d_1-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{d_1 \cdots d_{r-1} \cdots d_n} & c_{d_1 \cdots d_{r-1} d_{r+1} \cdots d_n} & \cdots & c_{d_1 \cdots d_{r-1} \cdots d_n} \end{pmatrix}, \]  
(A.11)
if \( 1 \leq \ell < r < n, \) write
\[ M(|\psi\rangle) = \begin{pmatrix} c_0 & c_1 & \cdots & c_h & \cdots & c_{d_1+1} & \cdots & c_{d_1 \cdots d_n} \\ c_{d_1} & c_{d_1+1} & \cdots & c_{h+1} & \cdots & c_{d_1+1+1} & \cdots & c_{d_1 \cdots d_n+1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{d_1 \cdots d_{r-1} \cdots d_n} & c_{d_1 \cdots d_{r-1} d_{r+1} \cdots d_n} & \cdots & c_{d_1 \cdots d_{r-1} \cdots d_n} \end{pmatrix}, \]  
(A.12)
if 1 ≤ l = r < n, write

\[ M(|\psi\rangle) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{h-1} \\ c_d & c_{d+1} & \cdots & c_{2h-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{(d-1)d+b} & c_{(d-1)d+b+1} & \cdots & c_{(d-1)d+b+h-1} \end{pmatrix}, \]  

(A.13)

then it follows from equation (A.10) that equations (A.9) holds.

Finally, we consider the permutation of qudits. Applying the permutation \( \sigma \) defined in equation (10) to both sides of equation (A.2), we have

\[ M^\sigma(|\psi\rangle) = A_{\sigma}^1 \otimes \cdots \otimes A_{\sigma}^N (|\phi\rangle)(A_{\sigma}^1 \otimes \cdots \otimes A_{\sigma}^N)^T. \]  

(A.14)

When \( A_1, \ldots, A_n \) are ILOs, it can be directly concluded from equation (A.14) that \( M^\sigma(|\psi\rangle) \) and \( M^\sigma(|\phi\rangle) \) have the same rank. Thus, two SLOCC equivalent states have the same rank with respect to every permutation of qudits.

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