Plactic monoids satisfy nontrivial identities*

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Abstract

It is shown that the plactic monoid $P_n$ of any fixed rank $n$ satisfies a nontrivial identity.

1 Introduction

The plactic monoid of rank $n \geq 1$ is the monoid, denoted by $P_n$, generated by the finite set $\{1, \ldots, n\}$ and subject to the defining relations:

\[ zxy = xzy \quad \text{for all } 1 \leq x \leq y < z \leq n \]
\[ yzx = yxz \quad \text{for all } 1 \leq x < y \leq z \leq n \]

referred to as the Knuth relations. Because of its deep connection to the Young tableau, it has attracted a lot of attention and it became a powerful and important tool in several areas of mathematics, especially in representation theory and in algebraic combinatorics. In the early stages of the development of the theory, the combinatorics of $P_n$ was extensively studied. We refer to [7] and [15] for classical results on applications of the plactic monoid. Then, it was also applied in the contexts of representations of Lie algebras, quantum groups, and other combinatorial problems, for example see [8], [14]. And more recently, it has been used as a key tool in the theory of crystal bases, [1], [2].

It has been an open problem whether plactic monoids of any rank satisfy nontrivial semigroup identities, see [10], Problem 8.2, and [12]. The purpose of this paper is to solve this problem. Our main result reads as follows.

**Theorem 1.1** For every $n \geq 1$ the plactic monoid $P_n$ of rank $n$ satisfies a nontrivial identity.

It is known that $P_n$ satisfies a nontrivial identity if $n = 1, 2$ or 3. For $n = 3$ this was accomplished in [12]. Later, it was noticed that $P_3$ embeds into $U_3(\mathbb{T}) \times U_3(\mathbb{T})$, where $U_n(\mathbb{T})$ denotes the semigroup of so called upper triangular $n \times n$ tropical matrices $U_n(\mathbb{T})$, see [10] and [3]. Since, for every $n \geq 1$, $U_n(\mathbb{T})$ satisfies a nontrivial semigroup identity, see [9] and [16] for a

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different very elementary proof, this provided another proof of existence of a nontrivial identity in \( P_3 \). However, it remains an open problem whether every \( P_n \) admits a faithful representation in some \( U_m(T) \).

The structure of the plactic algebra \( K[P_n] \) over a field \( K \) has also been studied, including ordinary linear representations. However, no consequences have been found in the context of our motivating problem.

On the other hand, it is known that there is no identity that can be satisfied simultaneously by all plactic monoids \( P_n, \ n \geq 1 \). It is worth mentioning that this is in contrast to the case of a related class of monoids, introduced in \( \cite{4} \), called Chinese monoids. Namely, there exists a nontrivial identity satisfied by Chinese monoids of all ranks. \( \cite{11} \).

## 2 Necessary background

We start with recalling some basic properties of \( P_n \). If \( w = x_{i_1} \cdots x_{i_k} \) is a word in the free monoid \( X^* \) generated by \( x_1, \ldots, x_n \) then by \( |w| \) we denote the length \( k \) of \( w \). Also, \( \deg_{x_i}(w) \) stands for the degree of \( w \) in \( x_i \). We say that \( w \) is a subword of a word \( v \) if \( v = z_1x_{i_1}z_2x_{i_2} \cdots z_kx_{i_k}z_{k+1} \) for some words \( z_1, \ldots, z_{k+1} \). The generators of \( P_n \) will also be denoted by \( x_1, \ldots, x_{n} \) and elements of \( P_n \) will be treated as words in \( x_1, \ldots, x_{n} \), with the agreement that \( x_1 < \cdots < x_{n} \).

The fundamental result says that every element of \( P_n \) has a unique presentation in the canonical tableau form. Namely, by a row in \( P_n \) we mean an element of the form \( x_{i_1} \cdots x_{i_k} \), where \( r \geq 1 \) and \( i_1 \leq \cdots \leq i_r \). A column in \( P_n \) is defined as an element \( x_{j_1} \cdots x_{j_k} \), where \( s \geq 1 \) and \( j_1 < \cdots < j_s \). We say that a row \( v = x_{i_1} \cdots x_{i_r} \) dominates a row \( w = x_{j_1} \cdots x_{j_s} \), if \( r \leq s \) and \( i_k > j_k \) for every \( k = 1, \ldots, r \). Similarly, we say that a column \( v = x_{i_1} \cdots x_{i_r} \) dominates a column \( w = x_{j_1} \cdots x_{j_k} \) if \( r \geq s \) and \( i_k \leq j_k \) for all \( k = 1, \ldots, s \). We write \( v \triangleright w \) in both cases. A tableau is a word \( w = w_1 \cdots w_t \) such that all \( w_i \) are rows and \( w_1 \triangleright \cdots \triangleright w_t \). Then every element \( 1 \neq w \in P_n \) is equal in \( P_n \) to a unique tableau (see \( \cite{13} \)), referred to as the semistandard Young tableaux. For example

\[
w = x_5 \cdot x_3 x_4 x_4 \cdot x_2 x_3 x_3 x_3 \cdot x_1 x_1 x_2 x_2 x_2 x_3
\]

is a tableau with the subsequent rows

\[
w_1 = x_5, \quad w_2 = x_3 x_4 x_4, \quad w_3 = x_2 x_3 x_3 x_3, \quad w_4 = x_1 x_1 x_2 x_2 x_2 x_3.
\]

Such a tableau can be presented as a planar object

```
\[
\begin{array}{cccc}
  & x_5 \\
  & x_3 & x_4 & x_4 \\
  x_2 & x_3 & x_3 & x_3 \\
 x_1 & x_1 & x_2 & x_2 & x_2 & x_3
\end{array}
\]
```

Moreover, the subsequent columns of this array are

\[
v_1 = a_5 x_3 x_2 x_1, \quad v_2 = x_4 x_3 x_1, \quad v_3 = x_4 x_3 x_2, \quad v_4 = x_3 x_2, \quad v_5 = x_2, \quad v_6 = x_3,
\]
and then we have \( w = v_1 \cdots v_6 \) in \( P_n \), as well as \( v_1 \triangleright \cdots \triangleright v_6 \). So the row reading of the array (left to right and row-by-row) agrees in the monoid \( P_n \) with the so-called column reading (going down through the consecutive columns, from left to right).

The first ingredient of our approach is based on the so-called Greene’s invariants that determine the shape of the tableau associated to \( w \in P_n \) in terms of non-decreasing subsequences of the word \( w \), see \([13]\). Namely, for a word \( w \in P_n \) let \( l_i(w) \) denote the maximal sum of the lengths of \( i \) disjoint non-decreasing subsequences of \( w \). Then \( l_1(w) \) is the length of the last row of the tableau form of \( w \) in \( P_n \), while \( l_j(w) \) is the sum of the lengths of the last \( j \) rows of \( w \), for \( j = 1, \ldots, n \). In other words, to every word \( w \in X^* \) we can associate (not uniquely, in general) a sequence \( (w_1, \ldots, w_n) \) of disjoint non-decreasing subwords of \( w \) such that \( |w_1| \geq \cdots \geq |w_n| \) and \( l_i = |w_1| + \cdots + |w_i|, w = \sum_{i=1}^n |w_i| \). (Some of \( w_i \) might be the empty words.)

The second ingredient of our approach is based on the tropical semiring is \( T = \cup \{ -\infty \}, \) with addition and multiplication operations \( \oplus \) and \( \odot \) defined by \( x \oplus y = \max \{ x, y \} \) and \( x \odot y = x + y \). The set of \( n \times n \) upper-triangular tropical matrices is denoted by \( U_n(T) \). (An upper-triangular tropical matrix has all entries below the main diagonal equal to \( -\infty \), since this is the additive identity of \( T \).)

For a non-identity element \( w \in X^* \) and any pair \( p, q \in \{ 1, \ldots, n \} \) we define:

\[
\begin{align*}
    w_{pq} &= \begin{cases} 
        \text{the maximal length of a non-decreasing subsequence in } w \text{ with entries in the interval } [x_p, x_q] & \text{if } p \leq q, \\
        -\infty & \text{if } p > q.
    \end{cases}
\end{align*}
\]

To the identity of \( X^* \) we associate numbers \( w_{pq} = 0 \) if \( p = q \) and \( -\infty \) otherwise. The following observation is well known and easy to verify, \([3], [10]\). It indicates that the tropical matrices provide a natural tool in the study of the plactic monoid.

**Lemma 2.1** The rule \( \phi_n(w) = (w_{pq}) \) defines a homomorphism \( \phi_n : P_n \longrightarrow U_n(T) \).

We will use the fact that \( U_n(T) \) satisfies a nontrivial identity \( u(x, y) = v(x, y) \), where \( u, v \) are words in \( x, y \) such that \deg_{x}(u) = \deg_{y}(v) \) and \deg_{y}(u) = \deg_{y}(v)\), \([9], [10]\). For example, let

\[
    u_1 = u_1(x, y) = xyxxyyx, \quad v_1 = v_1(x, y) = xyyxxyyx
\]

and for \( j \geq 1 \) define

\[
    u_{j+1} = u_1(u_j(x, y), v_j(x, y)), \quad v_{j+1} = v_1(u_j(x, y), v_j(x, y)).
\]

Then the identity \( u_{n-1} = v_{n-1} \) holds in \( U_n(T) \).

The last ingredient of our inductive proof relies on the fact that the natural projection

\[
    \pi_n : P_n \longrightarrow \langle x_1, \ldots, x_{n-1} \rangle \cong P_{n-1},
\]

3
defined on generators by the rule: \( \pi_n(x_i) = x_i \) for \( i = 1, \ldots, n-1 \) and \( \pi_n(x_n) = 1 \), is easily seen to be a homomorphism. Moreover, erasing all occurrences of \( x_n \) from the tableau form of \( w \in P_n \) one gets the tableau form of \( \pi_n(w) \).

### 3 Proof of the main theorem

We are now ready to prove our main result, Theorem 1.1.

We proceed by induction on \( n \). As mentioned before, the result is known if \( n \leq 3 \). So, we may assume that \( n > 3 \) and \( P_{n-1} \) satisfies a nontrivial identity in two variables \( s_0, t_0 \) of the form \( p_1(s, t) = p_2(s, t) \), where \( p_1, p_2 \) are some words (of equal length) in \( s = s_0t_0, t = t_0s_0 \). We know that the natural projection \( \pi_n : P_n \rightarrow P_{n-1} \) is a homomorphism. Therefore, if \( s_0, t_0 \in P_n \), then \( x = p_1(s_0t_0, t_0s_0), y = p_2(s_0t_0, t_0s_0) \) satisfy \( \pi_n(x) = \pi_n(y) \). So, the tableau form of \( x, y \) can differ only in the location of \( x_n \) and also \( \deg_{x_n}(x) = \deg_{x_n}(y) \).

Let \( \phi_n : P_n \rightarrow U_n(\mathbb{T}) \) be the tropical representation used in Lemma 2.1. Since, \( U_n(\mathbb{T}) \) satisfies an identity, we also know that \( \phi_n(u(x, y)) = \phi_n(v(x, y)) \) for words \( u_{n-1} = u(x, y), v_{n-1} = v(x, y) \) in \( x, y \) described after Lemma 2.1 such that \( \deg_{x_n}(u) = \deg_{x_n}(v) \) and \( \deg_{y_n}(u) = \deg_{y_n}(v) \).

We will consider words of the form

\[
    w = zuz, w' = zvz,
\]

where \( z = uw = u(x, y)v(x, y) \). Clearly we may assume that \( u \) and \( v \) are of length exceeding \( r \) in \( x, y \), for a fixed sufficiently big integer \( r > n \). The choice of \( r \) will be explained later. The aim is to show that \( w = w' \) is an identity (in variables \( s_0, t_0 \)) satisfied in \( P_n \). Notice that \( \phi_n(u) = \phi_n(w') \) and \( \pi_n(w) = \pi_n(w') \) because \( \phi_n, \pi_n \) are homomorphisms and \( \phi_n(u) = \phi_n(v), \pi_n(u) = \pi_n(v) \). Hence, the elements \( w, w' \) (written in the tableau form) only can differ in the location of \( x_n \). From the definition of \( \phi_n \) (actually, from the shape of the first row of the matrix \( \phi_n(w) \)) it follows also that the last row (in the tableau form) of \( w \) is equal to the last row of \( w' \). Using Greene’s invariants \( l_i(w), l_i(w') \) of the words \( w, w' \), respectively, it is then sufficient to show that \( l_i(w) = l_i(w') \) for \( i = 2, 3, \ldots, n \). Clearly, we may assume that \( x \) and \( y \) involve all generators, because otherwise \( x = y \) by the definition of \( x, y \), whence also \( w = w' \).

So, we fix some \( s_0, t_0 \in P_n \) and consider the resulting words \( w, w' \). Notice that \( \deg_{x_n}(x) = \deg_{x_n}(y) \) because of our definition of \( x, y \). Choose a sequence \( (w_1, \ldots, w_n) \) of disjoint non-decreasing subwords of \( w \) such that \( |w_1| \geq \cdots \geq |w_n| \) and \( l_i = |w_1| + \cdots + |w_i| \) (viewed as words in \( x_1, \ldots, x_n \)). And choose a similar sequence \( (w'_1, \ldots, w'_n) \) for the word \( w' \). So we know that \( |w_1| = l_1(w) = l_1(w') = |w'_1| \). Write \( w_1 = x_1^{a_1} \cdots x_n^{a_n} \) for some nonnegative integers \( a_i \). On the other hand, we may also treat \( w \) as a word in \( x, y \). Then we may write \( w_1 = z_1 \cdots z_{5r} \), where \( r \) is equal to the total degree of \( w \) in \( x, y \) and \( z_1, \ldots, z_{5r} \) (treated as words in \( x_1, \ldots, x_n \)) are subwords of the corresponding consecutive factors \( x, y \) of \( w \). So, \( z_1 \cdots z_{2r} \) is a subword of \( uv \), then \( z_{2r+1} \cdots z_{3r} \) is a subword of \( u \), and finally \( z_{3r+1} \cdots z_{5r} \) is a subword of \( uv \).
Let \( k_i = \deg_{x_i}(x) \) for \( i = 1, \ldots, n \). It is clear that if \( \deg_{x_i}(z_{\gamma}) = 0 \) or \( \deg_{x_i}(z_i) = k_n \) for some \( \gamma \) then \( z_{\gamma} \) is a subword of both \( x \) and \( y \). So, there exists at most one \( z_\delta \) which does not have this property. This will be used several times, without further comment.

We will prove the following claim.

**Claim.** There exists a non-decreasing word \( w'' = x_1^{b_1} \cdots x_n^{b_n} \) that is a subword of the word \( w \) of the same length as \( w_1 \) and there exists \( k \) such that \( x_1^{b_1} \cdots x_{k-1}^{b_{k-1}} \) is a subword of \( z \) and \( x_k^{b_k} \cdots x_n^{b_n} \) is a subword of \( z \). So, \( z_{2r+1} \cdots z_{3r} = (x_k)^{n-r} \) and \( w'' \) is also a subword of \( w' \).

If \( \deg_{x_n}(w_1) > 0 \) then let \( \gamma \in \{1, \ldots, 5r\} \) be minimal such that \( \deg_{x_n}(z_{\gamma}) > 0 \). We will assume that the factor \( z_{\gamma} \) of \( w_1 \) has been chosen as a subword of a factor \( x \) of \( w \). (The other case, when it was chosen from a factor \( y \), can be treated in a similar way.) If \( z_{2r+1} \cdots z_{5r} = (x_n)^{k_n-3r} \) then there is nothing to prove: \( w_1 \) is a subword of \( w \) and of \( w' \) and it has the desired form. So, we may assume that \( \gamma > 2r \) or \( \deg_{x_n}(w_1) > 0 \).

There exists \( i < n \) such that \( z_{\delta} = x_i^{k_i} \) for some \( \delta \). This is clear because \( u, v \) have degree \( > n \) in \( x, y \). Define the set

\[
Z = \{ i \mid i < n, z_{\delta} = x_i^{k_i} \text{ for some } \delta \in \{1, \ldots, 5r\} \}.
\]

We choose \( j < n \) such that \( k_j \) is maximal among all \( k_i, i \in Z \). Let \( \beta \) be minimal such that \( z_{\beta} = x_j^{k_j} \).

We will consider two cases.

**Case 1** \( k_n \geq k_i \) for every \( i \in Z \).

Choose any \( \alpha \) such that \( r+1 < \alpha \leq 2r \) and \( z_{\alpha} \) has been chosen as a subword of a factor \( x \) of the word \( w \). Write \( \gamma = 5r + 1 \) in the case when \( \deg_{x_n}(w_1) = 0 \).

First, consider the word

\[
w_1 x_n^{k_n(\gamma-\alpha)}.
\]

Next, in this word delete \( \gamma - \alpha \) factors, each of the form \( z_{\delta} = (x_i)^{k_i}, i < n \). This is possible, because on one hand the set of \( \gamma - 1 \geq 2r > 2n \) factors \( z_{\gamma-1} \) contains at least \( n - 2 \) factors \( z_{\delta} \) that are not of the form \( (x_i)^{k_i} \). While on the other hand \( \gamma - 1 - (\gamma - \alpha) = \alpha - 1 \geq r > n \). The new word \( L \), obtained in this way, is non-decreasing and from the construction it follows that it is a subword of \( w \). Moreover, \( |L| \geq |w_1| \). (Notice that the maximality of \( |w_1| \) as the length of a non-decreasing subword of \( w \) in fact implies in view of \( k_n \geq k_i \) for \( i \in Z \), that all the removed words \( (x_i)^{k_i} \) must satisfy \( k_i = k_n \), and clearly \( |L| = |w_1| \)).

It follows that we may replace \( w_1 \) by this new word \( L \). However, \( L \) contains \( (\gamma - \alpha) + (5r - \gamma) \geq 3r \) factors of the form \( (x_n)^{k_n} \), whence the claim follows in this case.

**Case 2** \( k_j > k_n \) (with the choice of \( j \) made after the definition of the set \( Z \)).

Suppose that for some \( m \neq j \), we have \( m \in Z \). Choose minimal \( \alpha \in \{1, \ldots, 5r\} \) with \( z_{\alpha} = (x_m)^{k_m} \). Let \( c \) be the maximal integer such that \( (x_m)^{k_m c} = z_{\alpha} \cdots z_{\alpha + c - 1} \). Then, for every \( d < c \) we may delete the factor \( z_{\alpha} \cdots z_{\alpha + d} \) from the word \( w_1 \), and insert the factor \( (x_j)^{k_m d} \) into the obtained word immediately after the factor \( z_{\beta} \).
By the choice of \( j \) and the assumptions on \( x \) and \( y \) we see that the word obtained in this way is a subword of \( w \), it is non-decreasing, and has the same length as \( w' \). (Notice that \( k_j \geq k_m \) in fact implies in view of the maximality of \( w_1 \) that \( k_m = k_j \).) But if we choose \( d = c - 1 \) then in this new word the exponent of \( x_m \) does not cover any of the factors \( x, y \) (meaning that in this new word there is no \( \delta \) such that \( z_\delta = (x_m)^{k_m} \)). So, we may replace the original word \( w_1 \) by a word such that \( m \notin Z \).

We repeat this procedure for every \( m \neq j \), such that \( m \in Z \). This allows us to assume that every factor \( x_n^{a_n} \) (for \( m \neq j, n \)) of \( w_1 \) does not contain any \( z_\delta \).

Therefore, \( x_1^{a_1} \cdots x_{j-1}^{a_{j-1}} \) can overlap at most \( j - 1 \leq n \) consecutive factors \( x, y \) of \( w_1 \). Since we have chosen \( u \) of length \( r \gg n \) in \( x, y \), it follows that \( x_1^{a_1} \cdots x_{j-1}^{a_{j-1}} \) is a subword of \( z_1 \cdots z_r \). So \( \beta \leq r + 1 \).

Similarly, \( x_{j+1}^{a_{j+1}} \cdots x_{n-1}^{a_{n-1}} \) can overlap not more than \( n - j - 1 \leq n \) consecutive factors \( x, y \) of \( w_1 \).

Suppose \( \deg_{x_n}(z_1 \cdots z_{4r}) = 0 \). Then it follows that \( x_{j+1}^{a_{j+1}} \cdots x_n^{a_n} \) is a subword of \( z_{3r+1} \cdots z_{5r} \). Therefore \( z_{2r+1} \cdots z_{3r} = (x_j)^{k_j} \). Hence, the claim follows in this case. Notice also that the choice of \( r \gg n \) allows us to assume that the integers \( r \) used above satisfy \( c - e \leq d \leq c - 1 \) for any fixed sufficiently big natural number \( e < c \).

It remains to consider the case when \( \gamma \leq 4r \). Then choose any \( \epsilon \) such that \( \epsilon > 4r \) and the factor \( z_\epsilon \) of the word \( w_1 \) has been chosen as a subword of a factor \( x \) of the word \( w \). First, we create the word

\[
z_1 \cdots z_\beta (z_\beta)^{r-\gamma} z_{\beta+1} \cdots z_{5r}
\]

by inserting \((z_\beta)^{r-\gamma} = (x_j)^{k_j(\epsilon - \gamma)}\) into the word \( w_1 \). Then we delete the factor \( z_{5r-(\epsilon - \gamma)+1} \cdots z_{5r} = (x_n)^{k_n(\epsilon - \gamma)}\) from this latter word. Again, from the construction it is clear that this new word \( L \) is non-decreasing, and it is a subword of \( w \). However, it satisfies \(|L| > |w_1|\) because \( k_j > k_n \). This contradicts the choice of \( w_1 \), and this case cannot occur. This completes the proof of the claim.

Since \( w'' \) is also a subword of \( w' \), it is a non-decreasing subword of maximal length in \( w' \), because \( l_1(w) = l_1(w') \).

We claim that the above procedure can be extended in order to construct two sequences of disjoint non-decreasing subwords \((\overline{w}_1, \ldots, \overline{w}_n)\) of \( w \) and \((\overline{w}'_1, \ldots, \overline{w}'_n)\) of \( w' \) such that: \( \overline{w}_1 = w'' = \overline{w}'_1 \) and \( |\overline{w}_i| = |\overline{w}'_i| = |w_i| \) for \( i = 2, \ldots, n \). In other words, sequences that determine the shape of the tableau forms of \( w \) and \( w' \). It is enough to discuss the word \( w \) and the way \( \overline{w}_2, \ldots, \overline{w}_n \) can be obtained, since the argument for \( w' \) is the same.

Indeed, in the proof of the Claim we remove certain factors \( z_\delta = x_i^{k_i} \) from \( w_1 \) and insert the same number of factors \( x_i^{k_i} \) or \( x_j^{k_j} \) for a fixed \( j \) such that \( k_j \) is maximal among all \( k_q = \deg_{x_q}(x) = \deg_{x_q}(y) \) such that \( x_q^{k_q} = z_{\alpha} \) for some \( \alpha \). This also means that these inserted factors have to be deleted from the subsequences \( w_2, \ldots, w_n \) (in order to obtain disjoint subwords of \( w \)). It is then sufficient to show that the factors removed from \( w_1 \) can be inserted in \( \overline{w}_2, \ldots, \overline{w}_n \) in the desired way.
Write \( w_i = z_{i,1} \cdots z_{i,5r} \), where as before \( r \) is equal to the total degree of \( w \) in \( x, y \) and \( z_{i,1}, \ldots, z_{i,5r} \) (treated as words in \( x_1, \ldots, x_n \)) are subwords of the corresponding successive factors \( x, y \) of \( w \). Let \( \gamma_i \) be smallest such that \( \deg_{x_n}(z_{i,\gamma_i}) > 0 \) if \( \deg_{x_n}(w_i) > 0 \). Suppose for example that \( z_{i,\gamma_i} \) was chosen from a factor \( x \) of \( w \). The only potential obstruction for inserting the desired number of factors \( x_1^{k_1} \) into \( w_i \) comes from the requirement that in the obtained word the first occurrence of \( x_n \) should also appear in a factor corresponding to \( x \) (and not to \( y \)). Because then we indeed create a subword of \( w \).

Notice that the words \( w, w' \) that we consider do not contain factors of the form \( x^3, y^3 \). Moreover, as indicated at the beginning of the proof, we may choose \( r \) as big as needed. It is then easy to see that the flexibility of the choice of the integer \( \alpha \) in Case 1 of the proof of the Claim, and of the choice of the integers \( d \) in Case 2, leads to the desired conclusion on the existence of the system \( (w_1, \ldots, w_n) \) of subwords of \( w \).

Now, erasing \( w'' \) from \( w \) and \( w' \), we come to two words of the form
\[
\overline{w} = \overline{x} \cdot u(x, y) \cdot \overline{x}, \quad \overline{w'} = \overline{x} \cdot v(x, y) \cdot \overline{x},
\]
where \( \overline{t} \) denotes the word obtained from \( t \) by erasing the corresponding subword of \( w'' \). Clearly, \( u(x, y) = u(\overline{x}, \overline{y}) \) and \( v(x, y) = v(\overline{x}, \overline{y}) \) are obtained by erasing all appearances of the letter \( x_k \) (as in the Claim) in each of the factors \( x, y \).

By the choice of the words \( u, v \) (they yield an identity in \( P_{n-1} \)) it follows that \( \overline{w} = \overline{w'} \). Hence \( \overline{w} = \overline{w'} \) treated as elements in the isomorphic copy of \( P_{n-1} \) obtained by deleting \( x_k \) from the set of generators of \( P_n \), and hence they are equal as elements of \( P_n \). This implies that \( \overline{w} = \overline{w'} \) in \( P_n \). It follows that \( l_i(\overline{w}) = l_i(\overline{w'}) \) in \( P_n \), for \( i = 1, \ldots, n-1 \). Since we also know that \( l_1(w) = l_1(w') \), it follows that the tableaux \( w \) and \( w' \) have the same shape. Since they potentially might differ only in the location of \( x_n \), it follows that \( w = w' \). This completes the proof of the theorem.

Notice that the above proof relies on identities that are satisfied in the semigroup \( U_n(T) \), described in Section [2]. Hence, a concrete form of an identity satisfied in \( P_n \) can be derived from this inductive proof.

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