Chaotic analysis for a buckled curved beam resting on nonlinear elastic foundations

Xiaohua Zhang* and Huaiyuan Ma
Department of Mathematics, Jiangsu Vocational Institute of Commerce
zhangcy11@163.com

Abstract. The chaotic behaviors of a buckled curved beam around postbuckling configuration are investigated. The Melnikov's method is introduced to find the necessary conditions for chaotic behaviours of the beam. We get the maximal Lyapunov exponent which is greater than zero and we can see and the system is sensitive to small differences in initial values. Associated the numerical simulations, we can see the chaotic behaviours can occur and the numerical analysis verifies the theoretical analytical predictions.

1. Introduction
As a fundamental structural element, beam is extensively utilized in many fields, for example, civil engineering, aerospace, marine and so on. So many researchers have studied the beams. Among the main research directions of beams, the forced vibrations of buckled beams have been studied for more than 50 years since Eisley [1] in 1964.

A lot of researchers set their sights on the forced vibration of beams in the vicinity of the buckled configurations. In recent years, Emam et al. [2] investigated a clamped-clamped buckled beam and got the nonlinear response of the beam with a primary-resonance excitation. Tomasiello et al. [3] studied a hinged-hinged buckled beam and got the nonlinear planar response with a primary-resonance excitation. Emam et al.[4] presented the exact solution of composite beams. Emam et al.[5] studied a geometrically imperfect composite beam and obtained the static and dynamic response. Yuan et al. [6] performed the buckling and post-buckling analysis of the extensible beam-columns numerically. Ghayesh et al. [7] used numerical method to study the coupled longitudinal-transverse behaviour of an imperfect microbeam. Li et al.[8] investigated the static and dynamic characteristics of electrically actuated viscoelastic bistable microbeam.

Mohamed et al.[9] studied the free and forced nonlinear vibrations of a buckled curved beam in the vicinity of postbuckling configuration for the first time. Herein, we intend to extend their nonlinear forced vibration analysis to investigate chaotic behaviours of the beams by the Melnikov's method which can be used to detect Smale's chaos analytically. The method is simple, elegant, alternative to characterize the chaotic dynamics.

The structure of the paper is arranged as follows: Section 2 presents some dynamical propertied of the unperturbed system. In section 3, the chaotic behaviours are derived. Numerical simulations are illustrated in section 4. A short summary is given in section 5.
2. Problem formulation

Consider a clamped-clamped beam whose length is \( L \) (see Figure 1). The beam is initially curved in its lateral direction and resting on nonlinear elastic foundations. It is subjected to an axial load \( \bar{F} \) and distributed harmonic excitation \( F_\Omega \), and \( \bar{\delta}_0 \) is the initial imperfection.

![Fig. 1. Schematic of the curved beam.](image)

By adopting the Galerkin approximation method, the nonlinear ordinary differential equation of the beam was established as follows

\[
\ddot{q} + \mu \dot{q} + \omega_0^2 q + \alpha q^2 + \beta q^3 = f \cos (\Omega t)
\]

where \( q \) is the amplitude of the mode and \( \mu \) is damping coefficient. Overdot denotes derivative with respect to time. All the parameters can be found in paper [9]. Because eigenvalues of the linearization matrix of Equation (1) satisfy the resonance condition of odd order, the normal form of Equation (1) consists of only nonlinear term of odd orders according to the normal form theory. So we consider \( \alpha = 0 \), Equation (1) is

\[
\ddot{q} + \mu \dot{q} + \omega_0^2 q + \beta q^3 = f \cos (\Omega t)
\]

Assume \( \mu \) and \( f \) are small, setting \( \mu \rightarrow \varepsilon \mu, f \rightarrow \varepsilon f \) (\( \varepsilon \) is a small parameter), Equation (2) is depicted by

\[
\ddot{q} + \varepsilon \mu \dot{q} + \omega_0^2 q + \beta q^3 = \varepsilon f \cos (\Omega t)
\]

Let \((q, \dot{q}) = (u, v)\), Equation (3) is presented as

\[
\begin{align*}
\dot{u} &= v \\
\dot{v} &= -\omega_0^2 u - \beta u^3 - \varepsilon \mu v + \varepsilon f \cos (\Omega t)
\end{align*}
\]

Now we consider the unperturbed system of Equation (4). Let \( \varepsilon = 0 \), Equation (4) is portrayed as

\[
\begin{align*}
\dot{u} &= v \\
\dot{v} &= -\omega_0^2 u - \beta u^3
\end{align*}
\]

Equation (5) is a Hamiltonian. The Hamilton function is

\[
H(u, v) = \frac{1}{2} v^2 + \frac{\omega_0^2}{2} u^2 + \frac{\beta}{4} u^4
\]

Under the condition of \( \omega_0^2 > 0 \) and \( \beta < 0 \), we derive the system (5) has a center \((0,0)\) and two saddles \((\pm \sqrt{-\beta/4}, 0)\). The heteroclinic orbit connecting the two saddles \((\pm \sqrt{-\beta}, 0)\) is

\[
\begin{align*}
u_0 &= \pm \frac{\omega_0}{\sqrt{-\beta}} \tanh \left( \frac{\sqrt{-\beta}}{2\sqrt{2}} t \right) \\
v_0 &= \pm \frac{\omega_0}{2\sqrt{2}} \sec \left( \frac{\sqrt{-\beta}}{2\sqrt{2}} t \right)
\end{align*}
\]

The figure of heteroclinic orbit is as follows
3 Analysis of chaotic behaviors

In the section, we analyze the chaotic behaviors of system (4). Equation (5) is a unperturbed system and the heteroclinic orbit is closed. Equation (4) is the perturbed system and the closed heteroclinic orbit may break. If the broken heteroclinic orbit is transversal, the system (4) may be chaotic according to Smale-Birkhoff Theorem [10]. If the heteroclinic orbit breaks, the distance between the broken trajectories can be measured by Melnikov method [11].

Along the heteroclinic orbits (6), the Melnikov function of system (2) is obtained:

\[ M(t_0) = \int_{-\infty}^{+\infty} \left( -\frac{\mu \omega_L^2}{8} \text{sech} \left( \frac{\sqrt{-\beta}}{2\sqrt{2}} (t-t_0) \right) \right) + f \left( \pm \frac{\omega_L}{2\sqrt{2}} \text{sech} \left( \frac{\sqrt{-\beta}}{2\sqrt{2}} (t-t_0) \right) \cos (\Omega t) \right) dt \]

For convenience, we let

\[ M(t_0) = I_1 + I_2 \]

where the infinite integration

\[ I_1 = \int_{-\infty}^{+\infty} \left( -\frac{\mu \omega_L^2}{8} \text{sech} \left( \frac{\sqrt{-\beta}}{2\sqrt{2}} (t-t_0) \right) \right) dt \]

\[ I_2 = \int_{-\infty}^{+\infty} f \left( \pm \frac{\omega_L}{2\sqrt{2}} \text{sech} \left( \frac{\sqrt{-\beta}}{2\sqrt{2}} (t-t_0) \right) \cos (\Omega t) \right) dt \]

Integration \( I_2 \) can be obtained with the residue theorem,

\[ I_2 = \pm \frac{2\sqrt{2} \pi \omega_L \Omega}{\beta \sinh (\pi \Omega \sqrt{-\beta})} \cos (\Omega t_0) \]

So the Melnikov function is illustrated as

\[ M(t_0) = -\frac{\sqrt{2} \mu \omega_L^2}{3\sqrt{-\beta}} \pm \frac{2\sqrt{2} \pi \omega_L \Omega}{\beta \sinh (\pi \Omega \sqrt{-\beta})} \cos (\Omega t_0) \]

With Melnikov method, the condition for the existence of transverse heteroclinic orbits is derived as follows

\[ \left| \frac{\omega_L \Omega^*}{f} \right| < \frac{6\pi \Omega^*}{\sqrt{2} \sinh (\pi \Omega^* \sqrt{-\beta})} \]

Letting \( \pi \Omega^* \sqrt{-\beta} \) is X-axis and \( \frac{\omega_L}{f} \) is Y-axis, we get the region of chaos in Figure.3.
4 Numerical simulations

In the section, the numerical simulations are given to verify analytical results.

In system (4), the parameters are chosen as $\omega = 5, \beta = -10, \varepsilon = 0.02, \mu = 0.06, f = 0.05, \Omega = 3.5$. We can see the inequation (14) is not satisfied. When the initial values are chosen as $(0.04,0.02)$, the phase portrait, the waveforms, the Poincare section and the maximal Lyapunov exponent are presented in Figure 4(a)(b)(c)(d). From the figures, we can see that the motion is periodic and the maximal Lyapunov exponent is zero so there is no chaotic motion in the system.

In system (4), the parameters are chosen as $\omega = 5, \beta = -10, \varepsilon = 0.02, \mu = 0.06, f = 113, \Omega = 3.5$, the inequation (14) is satisfied. When the initial values as $(0.04,0.02)$, the phase portrait, the waveforms, the Poincare section and the maximal Lyapunov exponent are presented in Figure 5(a)(b)(c)(d). From the figures, we can see that the motion has no period and the maximal Lyapunov exponent is $15.9859 > 0$, so there are chaotic attractors in the system. The numerical simulations agree with the theoretical analysis perfectly. Keep the same parameters and just change the initial values as $(0.045,0.025)$. The initial values have a slight change. Let’s compare the waveforms with different initial values in Figure 6(a)(b). We can see system (4) is sensitive to small differences in initial values, but the motion remains bounded and the motion has no period. This confirms that the motion is chaotic.
Fig. 4. The period motion of the beam exists when $f = 0.05$: (a) the phase portrait; (b) the Poincare section; (c) the waveform on plane $(t, u(t))$; (d) the maximal Lyapunov exponent.

Fig. 5. The chaotic motion of the beam exists when $f = 113$: (a) the phase portrait; (b) the Poincare section; (c) the waveform on plane $(t, u(t))$; (d) the maximal Lyapunov exponent.
Fig. 6. The chaotic motion of the beam: the waveform on plane \((t, u(t))\) when \((u(0), v(0)) = (0.04, 0.02)\) and \((u(0), v(0)) = (0.045, 0.025)\).

5 Conclusions

The chaotic behaviors of the buckled curved beam around postbuckling configuration are investigated. The necessary condition for chaotic motions is derived with Melnikov’s method. The numerical simulations agree with theoretical analysis results perfectly. This result may provide some guidance for the parameter design of the buckled curved beam.

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