ABSTRACT: We prove that for an open system, in the Markovian regime, it is always possible to construct an infinite number of non-trivial sets of histories that exactly satisfy the probability sum rules. In spite of being perfectly consistent, these sets manifest a very non-classical behavior: they are quite unstable under the addition of an extra instant to the list of times defining the history. To eliminate this feature—which implies implications for the interpretation of the formalism we discuss—and to achieve the stability that characterizes the quasiclassical domain, it is necessary to separate the instants which define the history by time intervals significantly larger than the typical decoherence time. In this case environment induced superselection is very effective and the quasiclassical domain is characterized by histories constructed with “pointer projectors”.

1. INTRODUCTION

*Environment induced superselection* \(^{Zurek82}\) explains why macroscopic objects are not observed in the majority of the quantum states admissible in their Hilbert spaces. The basic idea is that these objects are impossible to isolate from their surroundings \(^{Zeh71}\). The continuous interaction with this environment results in a process of *decoherence* \(^{Zurekpt}\) which destroys, on a very short *decoherence timescale*, the purity of nearly all of the initial superpositions thus erasing quantum coherence between states which result in a different evolution of their quantum environment. Only the preferred set of relatively stable states (or the associated sets of observables) will exhibit the key attribute of “classical reality,” which is characterized by the predictive power of the associated records \(^{Zurek81,EP R}\). In order to study decoherence, the analysis of the evolution of the “reduced density matrix” for the system (obtained from the full density matrix by tracing out the environment variables) is often the most convenient strategy \(^{Zurekpt,Zurekmazag}\). Nevertheless the role of the *records* accessible to the observers as well as the *correlations* between these records and the rest of the Universe must be recognised in the discussion of the *existential interpretation* of quantum mechanics suggested by the decoherence process \(^{Zurekmazag,Zurekptnew}\).

The *consistent histories* formulation (proposed by Griffiths \(^{Griffiths}\) and developed by Omnès \(^{Omnès}\) and by Gell–Mann and Hartle \(^{GMHSfi}\)) is based in the study of properties of sets of quantum histories, which are represented by time ordered sequences of projection operators. Although quantum mechanics prevents one from assigning probabilities to arbitrary sets of quantum histories, it is still possible to do so in certain cases. The validity of the probability sum rules –absence of quantum interference– for sets of histories, which is one of the properties of the classical domain, can be ascertained by analyzing if consistency conditions are satisfied \(^{Griffiths}\). These conditions, which were originally expressed as commutation relations between the projectors which define historical events
Griffiths, Omnès, can be related to the properties of an object called “decoherence functional” $GMHsf \text{i}$.

The aim of this paper is to make contact between these two approaches and study also some interesting aspects of the consistent histories formulation. For this purpose, we will apply it to a situation which is usually considered when discussing the environment induced superselection approach. The Universe is divided into a “system of interest” $S$ and the “environment” $E$. The histories of interest refer to the system $S$ only (i.e., histories will be made of projectors $P = I_E \otimes P_S$, where $I_E$ is the identity in the Hilbert space of the environment and $P_S$ is a projector in the Hilbert space of the system).

As the first step in our study we will address a technical point and examine under what conditions the decoherence functional for the above histories can be constructed entirely from the reduced density matrix of the system. The conclusion of our analysis is not unexpected but is still worth mentioning: this construction is possible only in the Markovian regime in which the reduced density matrix satisfies a master equation which is local in time. It is of course of interest to enquire how accurately can one assess consistency of histories by investigating the evolution of the density matrix when the evolution is, say, approximately Markovian. We shall not discuss this issue here: in the rest of the paper we will restrict to consider situations in which the Markovian approximation is exactly valid.

As a second step, we will show how to construct sets of histories that exactly satisfy the probability sum rules. In particular we will analyze histories for which the projectors are associated with the eigenstates of the reduced density matrix. As these states form the so-called “Schmidt basis” Albrecht, the above histories are going to be called “Schmidt histories” (although this may be an extrapolation of the existing terminology). We will show that, in the Markovian regime, sets of Schmidt histories are always consistent. We will also analyze some of their intriguing and –from the point of view of devising an interpretation– somewhat worrisome properties. In particular, we will show that the
events that form Schmidt histories are themselves history-dependent: the set of projectors defining Schmidt events in the “next time” \( t_n \) depends on which projectors were applied in the past. These projectors are imposed “from the outside” of the Universe—that is, they do not depend just on the initial conditions or on the Hamiltonian but primarily on arbitrary selections of, for example, the time sequence. This implies that histories are influenced by an “unphysical” act of deciding what questions (which sets of projectors) are going to be posed and when will they be posed. This is to be contrasted with the events which occur within the Universe and may decide, for example, existence or non-existence of certain systems, thus making certain questions interesting or natural.

In the original proposal of Griffiths, quantum histories were defined as chains of branch independent projectors \( P_{\alpha_k}(t_k) \) (the superindex \( k \) labels the set of projectors and the subindex \( \alpha_k \) enumerates the different events within the complete set). A complete set of histories was therefore characterized by a fixed choice of a set of projectors \( P^{(k)}(t_k) \) at every time. By contrast, branch dependent histories are chains of the form \( C_{\alpha} = P_{\alpha_1}(t_1) \ldots P_{\alpha_n}^{(n,\alpha_1,\ldots,\alpha_{n-1})}(t_n) \) where the set of projectors used at time \( t_k \) depends on the alternatives \( \alpha_1, \ldots, \alpha_{k-1} \) that were realized in the past. Such branch dependent histories were first considered by Omnès (who called them histories of type II). Their use has been recently advocated by Gell–Mann and Hartle.

We will show that the “branch dependence” of Schmidt histories is a consequence of an inevitable property of the reduced dynamics which we shall call—in absence of a more concise description—“environment induced noncommutativity” (EINC). Two initially commuting density matrices describing an open system (and corresponding, for example, to two alternative events) will in general evolve into two non-commuting operators as a result of the interaction with the environment. Therefore, at some later time they will not be simultaneously diagonalizable and will give rise to distinct sets of Schmidt states.

The existence of environment-induced noncommutativity, which can be shown to be
a very generic consequence of the openness of the system \( P \), does not seem to have been widely appreciated (some of its implications for consistent histories were discussed in Zurek et al.). As we are going to restrict our analysis to the Markovian regime, we will specifically address the importance of EINC in this context. We will show that this property of the evolution of the reduced density matrix is in fact predicted by generic Markovian master equations. Moreover, we will show that using such master equation one can study the consequences of EINC in physically relevant situations. In particular, we will consider a system with a finite dimensional Hilbert space interacting with a large environment. For concreteness, the system can be thought of as being formed by a set of atomic levels and the environment as the quantized electromagnetic field. The interaction between the system and the environment generates spontaneous transitions between the levels of the system. In the Markovian approximation, the evolution of the reduced density matrix can be modeled using the Bloch equations which are widely used in quantum optics. We will show that these equations predict the existence of environment induced non–commutativity and that this effect can appear in a physically relevant timescale.

Finally, we will discuss some of the consequences of environment–induced noncommutativity for the relationship between the process of decoherence and consistent histories approach. We will conclude that, when EINC is strong, Schmidt sets are not a good candidate to describe a quasiclassical domain. This is because they are highly unstable since the Schmidt projectors that form a perfectly consistent set at times \( t_1, t_2, \ldots, t_n \) are generally quite different from the ones that have to be used if the time sequence is \( t_2, \ldots, t_n \). This rather quantum mechanical property (which could be crudely described as “instability under observation”) disappears if one considers special sets of histories for which the minimal temporal separation \( t_i - t_{i-1} \) is larger than a certain quantity. This turns out to be crucial in achieving classicality. Thus, Schmidt histories are defined by a sequence of stable projectors if the differences \( (t_i - t_{i-1}) \) are sufficiently larger than the typical
decoherence time of the system (which is the time needed for an arbitrary initial state to become approximately diagonal in a fixed “pointer” basis). In this way, we conclude that sets of histories associated with “pointer states” Zurek$^8_1$ may be the basis for the description of our quasi–classical domain. In brief, histories expressed in terms of pointer states are stable and, when sampled on a timescale larger than the decoherence time, they are also approximately (although not exactly) consistent.

The paper is organized as follows: In section 2 we briefly review both the environment induced superselection and the consistent histories approaches. In section 3 we analyze the conditions under which it is possible to write the decoherence functional in terms of the reduced density matrix of the system and discuss the origin of environment induced non–commutativity. In section 4 we analyze the properties of the sets of Schmidt histories. Our conclusions are stated in section 5.

2. ENVIRONMENT INDUCED SUPERSELECTION AND CONSISTENT HISTORIES.

Here we shall provide a brief overview of these two approaches and of the relation between quantum and classical they suggest. More detailed reviews of the environment–induced superselection and the decoherence process are available elsewhere Zurek$^pt,Zurekmazag$. The paper of Griffiths$^Griffiths$ is still an excellent introduction to the consistent histories approach. In more recent publications, Gell–Mann and Hartle$^GMHsfi,GMHaspen$ introduce and discuss the decoherence functional using also the sum over histories formulation and discussing the relation with consistency. Omnès gives an useful overview in his most recent review article$^Omnès$.

2.1 Environment–Induced Superselection
Quantum formalism allows for the existence of many more states of the objects described by it than we seem to encounter. In particular, macroscopic objects appear to us in a small classical subset of a much larger quantum selection available in the Hilbert space. The purpose of environment–induced superselection is to “outlaw” the vast majority of such states by appealing to their instability in presence of the environment. The key point of this approach is simple: It starts with the realization that there is a basic difference between the consequences of quantum evolution for systems which are closed (isolated from their environments) and open (interacting with the “rest of the Universe”). Especially important is the fact that evolution of open quantum systems violates the “equal rights amendment” guaranteed for each and every state in the Hilbert space of a closed system by the quantum superposition principle. The process of decoherence affects different states differently. Some of such states evolve essentially unperturbed by the coupling to the outside. They form a “preferred set of states” in the Hilbert space of the system, known as “pointer basis” which can be, in principle, found by using the recently proposed predictability sieve \( Z_{urekmazag,zhp} \). By contrast, superpositions of such pointer states rapidly decay into density matrices which turn out to be –after the characteristic decoherence time has elapsed– given by mixtures approximately diagonal in the pointer basis.

Thus, decoherence results in a negative selection process which dynamically eliminates most of the superpositions. When monitored on a timescale larger than the decoherence time, the system appears to obey an effective environment–induced superselection rule which prevents it from existing in the vast majority of the states.

The distinguishing feature of classical observables, the essence of “classical reality”, is the persistence of properties of classical systems, which can exist in predictably evolving states and follow a trajectory which appears to be deterministic. Relative stability
or, more precisely, relative predictability of the evolution of the states of open quantum systems emerges as a useful criterion which can be employed to distinguish states which can persist (or deterministically evolve into other predictably evolving states). This emphasis on predictable existence gives rise to an existential interpretation of quantum mechanics. Only the states which can exist on a timescale accessible to the observers will be regarded by them as a part of the classical domain. Moreover, observers are also a part of the quantum Universe, and their perceptions are formed from their memories –records of the past measurements. These records are states of physical (and, generally, very open systems), which rapidly decohere and are subject to environment–induced superselection.

Thus, the observers accessing their own records will be restricted to perceiving their own memory in terms of sets of preferred “pointer states” which can exist for a long time. In this sense the dynamics of the observables of the open system “strikes twice”, on the one hand limiting the set of external observables to these which have the predictability characterizing the classical domain, while on the other hand, constraining the states of internal records accessible to the observer.

The ultimate focus of this approach is then the persistence of correlations between the states of two systems –the states of memory (the records) and the states of the measured system. Interaction with the environment is used to turn a non–separable quantum correlation between them, represented by a state of the form:

$$|\Psi_{AS} > = \sum_i \gamma_i |A_i > |\sigma_i >$$  \hspace{1cm} (corras)

where $|A_i >$ are the record states and $|\sigma_i >$ are the corresponding states of the system, into a classical correlation present in the density matrix:

$$\rho_{AS} = \sum_i |\gamma_i|^2 |A_i ><A_i| |\sigma_i ><\sigma_i|.$$ \hspace{1cm} (densmas)

The environment contributes to this transition by becoming correlated with the preferred
sets of states:

$$|\Psi_{AS} \rangle \otimes |\mathcal{E}_0 \rangle \rightarrow \sum_i \gamma_i |A_i \rangle \otimes |\sigma_i \rangle \otimes |\mathcal{E}_i \rangle,$$  \hspace{1cm} (pst)$$

where $|\mathcal{E}_i \rangle$ are orthonormal states of the environment. Tracing out of the environment results in a reduced density matrix $\rho_{AS}$ given by (densmas).

While the study of the stability of correlations is the point of departure for the discussion of the interpretation $Zurekpt, Zurekmazag$, much can be learned by studying the reduced dynamics of individual open quantum systems such as an exactly solvable quantum harmonic oscillator immersed in a heat bath of other oscillators. Such studies demonstrate that the decoherence timescales are indeed very short for macroscopic quantum objects and that the form of the interaction between the system and the environment has crucial influence on the selection of the preferred set of states $zhp, UZ, phz$.

The division of the Universe into subsystems is –in addition to quantum theory– the only crucial input. While this assumption is far from trivial, one can argue that it is needed to formulate the very problem of the emergence of classicality which is being addressed by this approach $Zurekmazag$. In particular, the measurement problem cannot be stated without dividing the Universe into an apparatus and a measured system. Addition of an environment, if anything, makes the discussion more realistic.

2.2 Consistent Histories

The basic concept used in the consistent histories approach is obviously that of a “history” which is a sequence of events defined at various moments of time. An event in quantum mechanics can be thought of as corresponding to a projection operator. Therefore, histories are represented by time ordered sequences of projection operators. In the ordinary formulation of quantum mechanics it is always possible to assign probabilities to single events defined by projectors, and representing (for example) the alternative out-
comes of a measurement performed at an arbitrary time. Quantum interference prevents us from assigning probabilities to arbitrary histories but probabilities can be consistently assigned to special sets of histories. The basic idea in the consistent histories approach is to analyze sets of histories and establish the mathematical conditions that must be satisfied in order to be able to define a probability measure in such set.

A history is formally represented by a sequence of Heisenberg projectors of the form $C_\alpha = \{P_{\alpha_1}^{(1)}(t_1), \ldots, P_{\alpha_n}^{(n)}(t_n)\}$ (where we assume $t_1 < \ldots < t_n$). The superscript $(k)$ labels the set of projectors used at time $t_k$ and $\alpha_k$ denotes the particular alternative. We will consider the possibility that the set used at time $t_k$ depends on the previous alternatives $\alpha_j, j < k$. When necessary, we will make this dependence explicit by writing $P_{\alpha_k}^{(k,\alpha_1,\ldots,\alpha_{k-1})}(t_k)$ but we will try to avoid using this cumbersome notation when there is no danger of confusion. A set of histories is said to be exhaustive if it covers all possible alternatives at all of the different times. Technically this is expressed by the identity $\sum_\alpha C_\alpha = I$ (where $\alpha$ denotes the set of alternatives $\alpha = \{\alpha_1, \ldots, \alpha_n\}$).

In standard quantum mechanics the probability of a given event, associated with the projector $P_i(t)$, is computed using the formula $p(i) = Tr(P_i^\dagger(t)\rho(t_0)P_i(t))$ (where $\rho(t_0)$ is the initial density matrix). If we generalize this formula to histories, the natural candidate for the probability of the history $C_\alpha$ is

$$p(C_\alpha) = Tr(C_\alpha^\dagger\rho(t_0)C_\alpha) \quad (prhis)$$

The failure of the probability sum rules can be easily seen if we apply the formula (prhis) to the “coarse grained” history $C_\beta \equiv \{C_\alpha \text{ or } C_{\alpha'}\}$. As the logical operation “or” is represented by the sum of the respective operators, we have:

$$p(C_\alpha \text{ or } C_{\alpha'}) = p(C_\alpha) + p(C_{\alpha'}) + 2\text{Re}(Tr(C_\alpha^\dagger\rho(t_0)C_{\alpha'})) \quad (pror)$$

The last term in (pror) clearly violates the probability sum rules. Thus, these rules are satisfied in a complete set of histories if and only if the last term vanishes for every pair of
histories in the set. To express and analyze the validity of this condition it is convenient to define the decoherence functional $D(\alpha, \alpha')$ as

$$D(\alpha, \alpha') \equiv Tr(C^\dagger_\alpha \rho(t_0)C_{\alpha'})$$

$$= Tr\left[P_{n\alpha}(t_n) \ldots P_{1\alpha}(t_1) \rho(t_0)P_{1\alpha'}(t_1) \ldots P_{n\alpha'}(t_n)\right]$$

The necessary and sufficient condition to define the probability measure in the set of histories is now easily written as $Omnnes$:

$$\text{Re}(D(\alpha, \alpha')) = 0 \text{ for all } \alpha \neq \alpha'$$

When this condition is satisfied, the set is a consistent set of histories and the probabilities are given in terms of the diagonal elements of the decoherence functional. A more restrictive condition than (consomn) has been proposed by Gell–Mann and Hartle $GMHsfi$ which call for the cancellation of all the non diagonal elements of the decoherence functional and not only of its real part

$$D(\alpha, \alpha') = 0 \text{ for all } \alpha \neq \alpha'$$

(this is obviously a sufficient condition for consistency). A brief remark about terminology is in order here. The above consistency conditions are called “decoherence conditions” by Gell–Mann and Hartle $GMHsfi,GMHaspen$ who refer to (consomn) as “weak decoherence” and to the condition (consgmh) as “medium decoherence”. We prefer to use the word decoherence to describe a physical process outlined in the previous subsection. Therefore, following Griffiths and Omnès $Griffiths,Omnnes$ we will refer to the various conditions which assure the validity of the probability sum rules as to “consistency conditions”. In practice, we will always use the condition (consgmh) (“medium decoherence” in the terminology of Gell–Mann and Hartle).

Before closing this subsection let us make a remark on the definition of the decoherence functional given in (dfdef). In that expression we are obviously using the Heisenberg picture
and the projectors $P^k_{\alpha_k}(t_k)$ are defined in terms of the Schrödinger picture projectors as

$$P^k_{\alpha_k}(t_k) = U^\dagger(t_0, t_k) P^k_{\alpha_k}(t_0) U(t_0, t_k) \quad (projevol)$$

where $U(t_0, t_k)$ is the evolution operator that propagates the state vector from $t_0$ to $t_1$. Introducing (projevol) into (dfdef) we can obtain the following well known formula where the projectors are constant (Schrödinger) operators:

$$D(\alpha, \alpha') = Tr\left[P^n_{\alpha_n} U(t_{n-1}, t_n) \ldots P^1_{\alpha_1} \rho(t_1) P^1_{\alpha_1'} \ldots U^\dagger(t_{n-1}, t_n) P^n_{\alpha_n'}\right] \quad (dfsh1)$$

In the forthcoming discussion, and for reasons that will become evident later, it will be more convenient to use a different expression for $D(\alpha, \alpha')$ that can be easily derived from equation (dfsh1). Introducing the propagator of the density matrix (a superoperator acting in the space of operators Balescu), which we denote as $K_{t_i}^{t_f}$ and is defined as:

$$K_{t_i}^{t_f}[\rho(t_i)] = U(t_i, t_f)\rho(t_i)U^\dagger(t_i, t_f) = \rho(t_f), \quad (kdef)$$

the decoherence functional (dfsh1) can be rewritten as:

$$D(\alpha, \alpha') = Tr\left[P^n_{\alpha_n} K_{t_{n-1}}^{t_n} \ldots P^1_{\alpha_1} K_{t_0}^{t_1}[\rho(t_0)] P^1_{\alpha_1'} \ldots P^n_{\alpha_n'}\right] \quad (dfsh2)$$

2.3 Decoherence, Consistency, the Quantum and the Classical.

The purpose of this subsection is to provide a more specific motivation for the study of the relationship between the decoherence process and the consistency of histories. In particular, both of them aim to explain the emergence of the classical from the quantum substrate. The manner in which decoherence process and the resulting environment–induced superselection approach this goal is quite clear. It has been briefly explained in 2.1. When considered from the point of view of Everett’s “Many Worlds” point of view, decoherence defines branches. Its focus on the stable existence of the records allows one
to understand Bohr’s “Copenhagen Interpretation” as, in effect, an observers memory of one of the Everett’s branches, with the apparent collapses induced by the effective superselection rules Zurek.

The stated goals of consistent histories approach were initially somewhat different: consistency was invoked to discuss sequences of events in a closed evolving quantum system without the danger of logical contradictions Griffiths. However, this goal as well as the validity of the probability sum rules are also a precondition for the classical behavior. Thus, at least some of the aspects of the classical domain should be related to consistency.

However, consistency alone does not suffice to define classical behavior GMHsfi. For example, given a closed system, it is always possible to find a consistent set of histories which are defined simply by the projectors constructed from the evolved eigenstates of the initial density matrix. Thus, when

$$\rho_i = \sum_i p_i |i><i|,$$

events represented by projectors

$$\Pi_i(t_k) = |i><i|$$
or by their direct sums can be always used to construct consistent histories. However, when this simple algebraic algorithm is applied to the classic test cases (such as the measurement problem or a Schrödinger cat) it will result in extremely non–classical consistent histories with the events corresponding to superpositions of various outcomes of measurements, dead and alive cats, etc.

At the very least, consistency would need to be supplemented by extra ingredients (which, in the context of consistent histories interpretation, are yet to be identified) in order to become an effective tool in studying classicality. Moreover, it appears likely that exact consistency may be too strong a requirement, and will have to be relaxed in order to be relevant for the study of “classicality”.
On the other hand, as we will explicitly show in the next section, the decoherence process and the resulting environment–induced superselection rules enforce approximate validity of the probability sum rules for the subsystem of interest. From the perspective of the consistent histories approach, the process of tracing over the environment can be naturally related to a coarse grained class of histories which correspond to sequences of projectors $I_E \otimes P^{(k)}_{\alpha_k}(t_k)$ where $I_E$ is the identity in the Hilbert space of the environment and $P^{(k)}_{\alpha_k}(t_k)$ acts on the Hilbert space of the system.

In what follows, we will closely investigate the connection between the two formalisms and focus our attention on the possibility of constructing perfectly consistent histories for the system out of the eigenstates of the reduced density matrix. In this respect, it is worth remembering here that the reduced density matrix can always be instantaneously diagonalized. Its eigenstates, which are sometimes called Schmidt states $Albrecht$ are not necessarily identical (or even approximately the same) as the pointer states: The two sets of states can be expected to coincide only when the decoherence process has been effective which, in turn, implies restrictions on the timescales. Thus, the time at which the Schmidt states are calculated must be larger than the typical decoherence time scale of the problem. In that case the Schmidt states become independent of the details of the initial condition and coincide with the pointer projectors.

3. CONSISTENT HISTORIES FOR AN OPEN SYSTEM.

In this section we will first establish the conditions under which the decoherence functional can be constructed entirely in terms of the reduced density matrix of the system. This will be shown to be possible in the Markovian regime of the reduced evolution. In this case, we will analyze the importance of the “environment induced noncommutativity” in determining the properties of consistent histories. We should remark that our analysis
of induced noncommutativity will be restricted to the Markovian regime despite of the fact that EINC is a generic property of the evolution of an open system.

3.1 Decoherence Functional for an Open System.

When we evaluate the decoherence functional in the histories of the system, we have to use projectors of the form $I_\mathcal{E} \otimes P^k_{\alpha_k}(t_k)$ but we must remember that the evolution between intermediate times is entirely unitary, that $\rho_U$ is the full density matrix and that the final trace is over the whole Hilbert space. Thus, the decoherence functional is obtained by tracing over the environment at the final time while the reduced density matrix is defined by tracing over the environment at every moment of time. This indicates that the decoherence functional could be written in terms of the reduced density matrix only when taking the trace over the environment at the end is not very different from doing it at every time. This will be the case whenever the time evolution of the reduced density matrix does not depend upon the correlations that are created dynamically between the system and the environment. The demonstration of this simple observation can be easily done if we decompose the final trace

$$Tr_{\{S, \mathcal{E}\}} = Tr_S Tr_\mathcal{E}$$

and try to move the trace over the environment to the inside of the expression for the decoherence functional. Using equation (dfsh1) we obtain

$$D(\alpha, \alpha') = Tr_S \left[ P^n_{\alpha_n} Tr_\mathcal{E} \left[ O_n(t_n) \right] P^n_{\alpha_n'} \right] \quad (first\ step)$$

where we defined $O_0(t_0) = \rho_U(t_0)$ and

$$O_{k+1}(t_{k+1}) \equiv U(t_k, t_{k+1})(I_\mathcal{E} \otimes P^k_{\beta_k})O_k(t_k)(I_\mathcal{E} \otimes P^k_{\beta_k})U^\dagger(t_k, t_{k+1}) \quad 1 \leq k \leq n \quad (on)$$

From this expression we can now notice that the trace can be moved one more step towards the center only if $Tr_\mathcal{E}(O_n(t_n))$ is a function of $Tr_\mathcal{E}(O_{n-1}(t_{n-1}))$, which is the trace of an
operator defined at $t_{n-1}$. This is not possible in general since $\text{Tr}_E(O_n(t_n))$ may depend on the full operator $O_{n-1}(t_{n-1})$ and not only upon its partial traces (this is precisely what happens when the correlations between the system and its environment affect the reduced dynamics of the system and produce non Markovian effects). We will restrict our future considerations to those cases where this is true, or equivalently, when there is a well defined reduced evolution operator, denoted as $\hat{K}_{t_{n-1}}^{t_n}$, acting in the following way

$$\text{Tr}_E(O_n(t_n)) = \hat{K}_{t_{n-1}}^{t_n} \left[ P_{\alpha_{n-1}}^{n-1} \text{Tr}_E(O_{n-1}(t_{n-1})) P_{\alpha'_{n-1}}^{n-1} \right]$$

If this operator exists the decoherence functional can be written as:

$$D(\alpha, \alpha') = \text{Tr}_S \left[ P_{\alpha_n}^n \hat{K}_{t_{n-1}}^{t_n} \left[ \ldots P_{\alpha_1}^1 \hat{K}_{t_0}^{t_1} [\rho_r(t_0)] P_{\alpha'_1}^1 \ldots \right] P_{\alpha''_n}^n \right] \quad (\text{dfred})$$

This is the main equation we will use in the next section. It is worth noting the similarity between the expression (dfsh2) (which is the decoherence functional for a closed system) and (dfred), in which all the characters are members of the “reduced theory”. In the previous section we showed that the decoherence functional for a closed system can be written in three equivalent ways given by equations (dfdef), (dfsh1) and (dfsh2). However we can prove that it is not possible to write the “reduced” decoherence functional in a way resembling Eqs. (dfdef) or (dfsh1). In this sense, equation (dfred) is unique. In fact, only if the evolution is unitary (as it is for the full density matrix) it is true that to propagate the density matrix $\rho$ we just have to multiply it from left and right with two operators that act on the same Hilbert space as $\rho$. This is the crucial property (see eq. (kdef)) allowing us to show the equivalence between equations (dfdef), (dfsh1) and (dfsh2). In the case of the reduced density matrix this property is no longer valid since two operators $A$ and $B$ satisfying

$$K_{t_i}^{t_f} [\rho_r(t_i)] = A \times \rho_r(t_i) \times B = \rho_r(t_f) \quad (\text{nokred})$$

do not exist: Existence of such operators would imply that initial pure states would remain pure forever, which is in contradiction with well known properties of the reduced dynamics.
Summarizing, we conclude that when the reduced evolution operator \( \hat{K} \) exists, the decoherence functional can be written as \( (\text{dfred}) \) in terms of “reduced objects”. Although this operator does not exist in general, it is also clear that there are very important cases for which the existence of \( \hat{K} \) is guaranteed. We will discuss those cases in the next subsection.

3.2 Reduced Dynamics and Environment Induced Non–Commutativity

The existence of the reduced evolution operator \( \hat{K} \) is a strong requirement. For example, it implies that the reduced density matrix satisfies a purely differential equation (the evolution cannot have memory). Such a Markovian equation does not exist in general since the exact master equation (the equation for the reduced density matrix) is typically nonlocal in time. Moreover, the existence of a local master equation is a necessary but not a sufficient condition for the existence of \( \hat{K} \). In fact, there are cases for which a local but explicitly time dependent master equation exists and the evolution is still (weakly) non-Markovian. In those cases the reduced evolution operator may not exist because the correlations still play some role in the reduced dynamics. Technically, the condition that guarantees the existence of \( \hat{K} \) is the locality of the Feynman–Vernon influence functional \( FV \) which enters in the path integral representation of the decoherence functional. This implies that the master equation is local in time and, for most realistic examples, also has time independent coefficients. Thus, we will restrict ourselves to consider cases for which the influence functional is local and the master equation is time independent. This strong assumption will still allow us to study realistic and relevant situations. Let us now mention two important physical examples in which the existence of \( \hat{K} \) constitute a sound approximation.

The first example is the well known linear quantum Brownian motion (QBM). In this case the system is a particle which interacts with an environment formed by a collection
of harmonic oscillators. Assuming that the initial state does not contain correlations between the particle and its environment, the model can be fully characterized by the spectral distribution and by the initial state of the environment (usually taken as thermal equilibrium at some temperature $T$). In such a rather general case, the existence of a local master equation was recently proved $HuPZ1$. It was shown that the master equation for the linear QBM is always of second order in partial derivatives and has time dependent coefficients that vary with temperature and with the spectral density of the environment (time dependence in the coefficients is responsible for all the non–Markovian effects). A particularly important case is that of an ohmic environment (linear spectral density) at high temperatures ($k_B T \gg \hbar \Lambda \gg \hbar \Omega_R$, where $\Lambda$ is the high frequency cutoff of the environment and $\Omega_R$ is the renormalized frequency of the system). In that case, after a short transient whose duration is determined by $\Lambda$, the master equation for the reduced density matrix $\rho$ reads

$$\dot{\rho} = -\frac{i}{\hbar} [H_R, \rho] - \frac{\gamma}{2\hbar} ([p, x], \rho) - \frac{i\gamma}{\hbar} ([x, \rho p] - [p, \rho x]) - \frac{2m\gamma k_B T}{\hbar^2} [x, [x, \rho]]$$  \hspace{1cm} (master eq)

Above, $H_R$ denotes the renormalized Hamiltonian of the system and $\gamma$ is a constant that fixes the relaxation rate. Although this equation is not valid for low temperatures, it has been also shown $UZ,phz$ that in that regime (i.e., $k_B T < \hbar \Lambda$) the high temperature approximation remains rather accurate since the coefficients approach their asymptotic values very fast.

A second example in which the use of a local master equation is a reasonable approximation can be found in the domain of atomic physics and quantum optics. In that case we consider the system to be an atom and the environment to be formed by the infinite number of modes of the quantized electromagnetic field. When the interaction between the system and the environment is taken into account, a local master equation known as Bloch equation can be derived under a number of approximations. The essential ones are the
following: absence of initial correlations between the system and the environment, Markovian behavior (very short life time of correlations in the environment), weak coupling (the equations are valid to second order in an expansion in the coupling constant) and rotating wave approximation (by which rapidly varying terms —counter rotating— are supposed to average to zero). If we denote with $|n>$ the eigenstates of $H_0$, the hamiltonian of the isolated atom, Bloch equations (in the interaction picture associated with $H_0$) read:

$$\dot{\rho}_{nm} = -\frac{i}{\hbar}[H_d, \rho]_{nm} + \delta_{nm} \sum_k w_{nk} \rho_{kk} - \Gamma_{nm} \rho_{nm} \quad (\text{bloch})$$

Here, the driving hamiltonian $H_d$ accounts for the coherent effects associated with the interaction between the atom and the electromagnetic field (such as coherent driving producing Rabi oscillations). The constants $w_{nk}$ are transition rates that, in the absence of driving, determine the evolution of the diagonal elements of $\rho$ (populations) while $\Gamma_{nm}$ are related to decay rates that affect the evolution of the nondiagonal elements (coherences). These constants can be, in principle, expressed in terms of some microscopic model and cannot be thought of as being independent of each other because of the fluctuation–dissipation relations (and the conservation of probability which implies that $\Gamma_{nn} = \sum_k w_{kn}$).

Let us now discuss one of the most remarkable features of the reduced dynamic associated with the above master equations: the existence of “environment induced non–commutativity” (EINC): The existence of EINC is a consequence of the non–unitarity of the reduced dynamics which does not necessarily preserve the commutation relations. Two initial states that satisfy

$$[\rho_a(0), \rho_b(0)] = 0$$

may evolve in such a way that

$$[\rho_a(t), \rho_b(t)] \neq 0$$

This is EINC, a property with important consequences in determining the qualitative nature of some interesting sets of consistent histories that we will consider in the next section.
It is worth noting that this effect takes place on a rather special timescale. Commutativity — rather than non-commutativity — is induced on a timescale which is very much shorter than the time needed to approach equilibrium. This is the decoherence timescale. Thus, on that timescale every initial state will become approximately diagonal in the same pointer basis (and therefore the final states will always commute). Therefore, the timescale on which EINC is most important is shorter than the decoherence timescale.

The existence of EINC is a prediction of both master equations (mastereq) and (bloch). In particular, we can show that in the linear QBM the commutator changes as

$$\frac{d}{dt} [\rho_a(t), \rho_b(t)] \propto \frac{2m\gamma k_B T}{\hbar^2} [\rho_a(t), x], [\rho_b(t), x]]. \quad (eincqbm)$$

Similarly, it is simple to show that Bloch equations also predict the existence of EINC with the only restriction that the dimension of the system’s Hilbert space be greater than two (no EINC for a spin 1/2 system). Finally, we should stress that in order to be really sure about the physical nature of the EINC predicted from the master equations (mastereq) and (bloch), we still need to show that the effect occurs on a timescale which is compatible with the ones used to obtain those equations. We will illustrate that this is indeed the case using a specific example in the next section.

4. CONSISTENCY, DECOHERENCE AND CLASSICALITY

4.1 Schmidt Histories Are Consistent

Using equation (dfred) for the reduced decoherence functional, it is very simple to find a systematic way of constructing an infinite number of consistent sets of histories. The method gives a clear prescription for choosing the sets of projectors $P_{\alpha_k}^k$ that guarantee
perfect consistency. Given a time sequence $t_1, t_2, \ldots, t_n$, the decoherence functional for the histories of the system

$$D(\alpha, \alpha') = Tr_S \left[ P_{\alpha n}^{t_n} \hat{K}_{t_{n-1}}^{t_n} \left[ \cdots P_{\alpha_1}^{t_1} \hat{K}_{t_0}^{t_1} [\rho_r(t_0)] P_{\alpha_1}^{t_1} \cdots \right] P_{\alpha_n}^{t_n} \right]$$

is automatically diagonal in its first index if we choose the projectors $P_{\alpha_1}^{t_1}$ in such a way that they commute with the reduced density matrix at time $t_1$. These projectors (if chosen to be one dimensional) are associated with the instantaneous eigenstates of the reduced density matrix (the so-called Schmidt basis). Doing so, the decoherence functional reads

$$D(\alpha, \alpha') = \delta_{\alpha_1, \alpha_1'} Tr_S \left[ P_{\alpha n}^{t_n} \hat{K}_{t_{n-1}}^{t_n} \left[ \cdots P_{\alpha_2}^{t_2} \hat{K}_{t_1}^{t_2} [P_{\alpha_1}^{t_1} \rho_r(t_1) P_{\alpha_1}^{t_1}] P_{\alpha_2}^{t_2} \cdots \right] P_{\alpha_n}^{t_n} \right]$$

Analogously, to achieve diagonality in the second index of the decoherence functional, we should choose the projectors $P_{\alpha_2}^{t_2}$ in such a way that they commute with the path projected reduced density matrix $\hat{K}_{t_1}^{t_2} [P_{\alpha_1}^{t_1} \rho_r(t_1) P_{\alpha_1}^{t_1}]$. However, due to the existence of environment induced non-commutativity, the eigenstates of the path projected density matrix will generally depend on the alternative $\alpha_1$. Therefore, the set of projectors chosen at time $t_2$ will generally depend on the previous alternatives and the history will be branch dependent.

This procedure can be implemented recursively for arbitrarily many steps. It will produce a set of branch dependent histories for which the decoherence functional is automatically diagonal. At time $t_k$, the projectors are associated with the eigenvectors of the path projected reduced density matrix $\hat{K}_{t_{k-1}}^{t_k} [\cdots \hat{K}_{t_1}^{t_2} [P_{\alpha_1}^{t_1} \rho_r(t_1) P_{\alpha_1}^{t_1}] \cdots]$. These projectors always exist and in general, due to EINC, depend upon the alternatives $\alpha_1, \ldots, \alpha_{k-1}$. As we mentioned above, we will refer to them as “Schmidt histories”. It is important to realize that by following the above procedure we can construct an infinite number of different sets of histories all of which are exactly consistent. Thus, we can obtain a different set just by choosing a different sequence of historical instants $\{t_k\}$. Moreover, by changing the time
sequence we may drastically change the sets of projectors. In this sense, these sets are highly unstable.

To illustrate this point and clarify the nature of the instability let us imagine that we follow the above procedure and construct a consistent set of histories specifying projectors at times $t_1, t_2, \ldots, t_n$. The histories belonging to this set are strings of the form $P^{(1)}_{\alpha_1}(t_1)P^{(\alpha_1,2)}_{\alpha_2}(t_2)\ldots P^{(n,\alpha_1,\ldots,\alpha_n-1)}_{\alpha_n}(t_n)$. Let us now construct another consistent set by using the same method but specifying histories at times $t_2, \ldots, t_n$. In this way we obtain histories which are strings of the form $\tilde{P}^{(2)}_{\alpha_2}(t_2)\ldots \tilde{P}^{(n,\alpha_2,\ldots,\alpha_n-1)}_{\alpha_n}(t_n)$. The sets are unstable because, due to the “environment induced non–commutativity”, the projectors $P^k_{\alpha_k}$ are different from the projectors $\tilde{P}^k_{\alpha_k}$. In fact, in the second case the set of projectors we must use at $t_2$ depends only upon the initial density matrix while in the first case may strongly depend on the alternatives $\alpha_1$. In the next subsection we will illustrate this fact with an example that demonstrates that the effect is real and can be rather large.

The natural question to ask is if there is some situation in which the above diagonalization procedure generates a unique (and stable) output. This is going to be the case only when the eigenbasis of the “path projected reduced density matrix ” at time $t_k$ (i.e., $\hat{K}^t_{t_k-1}[^{t_1}t_2[1^{\alpha_1,1}_{\alpha_1}\rho(t_1)P^1_{\alpha_1}]\ldots])$ is independent of the path projected reduced density matrix at time $t_{k-1}$. This requirement is satisfied when there exists a stable pointer basis Zurekmanag: We need the environment to help select a preferred (and stable) set of states. However, this can only happen if we wait long enough between the intermediate times for which we specify the history. Roughly speaking, the difference $\Delta t = t_i - t_{i-1}$ must be larger than the typical decoherence timescale $\tau_{\text{dec}}$ of the problem.

4.2 The Importance Of Environment
We will analyze here a particular example that illustrates the importance of EINC in producing consistent histories which may be highly unstable. For simplicity, we will use Bloch equations and consider a system with a low dimensional Hilbert space. As we have mentioned above, Bloch equations cannot result in EINC if the dimension of the system’s Hilbert space is equal to two. Thus, we need at least three dimensions. However, we are also interested in showing an example in which EINC takes place in a timescale for which Bloch equations are valid. This implies, roughly speaking, that the interesting effect should take place in a timescale longer than the lifetimes $\Gamma^{-1}_{nn}$. It is simple to show that this cannot be done in a three dimensional example. Thus, we will take our system to have a Hilbert space with four dimensions. We will consider the simplest situation in which all the levels are stable except one (say $|4>$) which can decay only to the ground state (say $|1>$). Thus, in this case we see—neglecting induced emission and absorption processes—that all the coefficients entering in the Bloch equation are either identically zero or given by:

$$\Gamma_{44} = 2\Gamma_{14} = 2\Gamma_{24} = 2\Gamma_{34} = w_{14} \equiv \Gamma.$$  \hfill (blochcoef)

It is interesting to note that the decoherence timescale of this system can be controlled in a very simple way. In the absence of external driving (i.e., $H_d = 0$) the nondiagonal elements $\rho_{k4}$ will disappear in a timescale related to $\Gamma^{-1}$. However, as there is only one dissipative channel, the density matrix will not become diagonal in the three dimensional subspace generated by $\{|k>, k = 1,2,3\}$. This situation can be changed if one introduces a coherent driving by coupling the system to intense laser fields which are in resonance with the transitions between state $|4>$ and other states. The intensities of the different laser fields control the frequencies of the Rabi oscillations and these frequencies control the decoherence timescale of the system. In our example we use this idea to make the deco-
herence timescale rather large (this allows us to observe EINC on a reasonable timescale). In particular, we consider the following simple driving terms:

\[ H_d = \Omega_{14} |1\rangle \langle 4| + \Omega_{13} |1\rangle \langle 3| + \Omega_{12} |1\rangle \langle 2| + \text{h.c.} \]  

\[ (hdriv) \]

Using this hamiltonian in Bloch equations (which were integrated numerically), we demonstrated the existence of EINC for a rather robust set of parameters and for relevant timescales. An example is displayed in figure 1 in which approximately 40% of the branches show significant degree of instability. We remark that the parameters we used are rather reasonable from the point of view of the systems for which Bloch equations are typically used in atomic physics.
Figure 1: Consistent Schmidt histories for a system with a four dimensional Hilbert space are represented by a branching diagram. This example corresponds to a system described by Bloch equations (bloch). The coefficients defining the driving Hamiltonian (hdriv) are $\Omega_{12} = 5$, $\Omega_{13} = 10$, $\Omega_{14} = 50$ and the decay rate (blochcoef) is $\Gamma = 3000$ (measured in units in which the time separation $T_3 - T_1 = T_3 - T_2$ is set to unity). The origin of the diagram corresponds to the state $|1\rangle$, which is the state of the system at time $T_1$ (we do not draw the fourth branch corresponding to the unstable state $|4\rangle$ since it has zero probability in our example). The tabulated $p_i$'s are the projection of the state defining each branch onto the basis $|i\rangle$, i.e., $p_i = |<i|\Psi>|^2$. When the histories are constructed at times $T_1, T_3$, there are three consistent branches corresponding to the states $|i\rangle$, $i = 1, 2, 3$ ($T_3 - T_1$ is larger than the decoherence time and the states $|i\rangle$ form a pointer basis). On the contrary, when the histories are constructed at times $T_1, T_2, T_3$, there are nine consistent branches. In the first six ones (which correspond to 40% of the total) the effect of environment induced noncommutativity is important: the states associated with the different consistent branches form a basis of Hilbert space which is different from the one formed by the $|i\rangle$ vectors. The quantum instability of the Schmidt histories is easily noticeable.

5. CONCLUSIONS

Let us summarize our results. We analyzed first the conditions under which the de-
coherence functional can be written entirely in terms of “reduced” quantities and showed that this can be done when the correlations dynamically created between the system and the environment do not affect the future evolution of the reduced density matrix. As can be explicitly shown, this is the case for the high temperature limit of the Caldeira–Leggett model of ohmic dissipation and in any other case for which the Feynman–Vernon influence functional is local in time. Expressing the decoherence functional in terms of elements of the reduced dynamics and using the properties of the reduced evolution operator we proved that it is always possible to construct an infinite number of sets of perfectly consistent histories. We also showed that these sets are generally rather unstable under “observations” since by deleting one of the times at which the histories are defined, the projection operators defining the consistent histories may be substantially changed.

In discussing the classical limit using the consistent histories approach various concepts are usually brought together. The first one is consistency, which in this formulation is the primary criterion. Coarse graining is usually invoked as a necessary way of achieving consistency. This is even the case when a separation of relevant and irrelevant degrees of freedom is made. In fact, in discussions of Caldeira–Leggett type of models it is usually argued that to achieve consistency for histories of the Brownian particle one needs to introduce some degree of spatial coarse graining and consider histories defined by a sequence of “slits” characterized by some widths. In this case, the width of the slits is associated with the degree of coarse graining. However, as we explicitly showed here, such coarse grainings are not necessary to achieve consistency. In fact, the Schmidt projectors can be one dimensional and still define consistent histories. The problem with the usual argument in favor of coarse graining as a way to achieve consistency is that it uses an essential extra ingredient that remains hidden for the most part. Thus, a spatial coarse graining is needed only if one restricts to consider very special histories constructed with special classes of projection operators: position projections for example. But there is
nothing in the consistent histories approach telling us that we must like a set of projectors better than another! The extra ingredient that makes us think that is “natural” to describe the world around us using position projectors (or any other set of projectors we happen to like) has nothing to do with consistency. This extra ingredient is the crucial one in defining the quasi-classical domain.

As we discussed in section 2.3, there are very simple ways of achieving consistency for a closed system. The sets one constructs in this way are based on the use of projection operators that are blatantly non-classical. Our results show that for an open system (which interacts with an external environment) the situation is much the same. Consistency is achieved easily by means of the Schmidt histories. However, these histories have no relation with the one describing a sensible quasi-classical domain.

The Schmidt histories we discussed provide an interesting example that may help us to disentangle the many concepts that enter in the definition of a quasiclassical domain. On the one hand they are consistent but do not require any (spatial) coarse graining. On the other hand, although we cannot prove it rigorously, it is likely that this set will also satisfy other criteria that have been advanced so far (and in a less rigorous manner) to characterize the quasi–classical domain \( \text{GMH}_{\text{sf}}, \text{GMH}_{\text{singap}} \). In fact, the set of Schmidt histories is likely to be “full” since to every history of the system there should exists a projection operator in the complete Hilbert space. Schmidt histories are therefore an example of a set which is consistent, rather fine grained (for the system) and most likely full. Despite all these properties the set is still very non–classical! This is, of course, unless we require predictability (or stability of the set under addition of extra intermediate times) in which case we need to require the separation between time slices to be larger than the typical decoherence time. In that case, the set becomes stable and the projectors are determined (by the environment and not by us!) to be the ones associated with the pointer basis.
A further conclusion one can draw from our paper is the following: In working within the framework of the consistent histories approach one may be tempted to think that by looking for sets of histories that satisfy the consistency conditions (or other stronger versions like those invoked by Gell–Mann and Hartle) one is trying to find “real” histories that are “out there” in some vague way. In saying this, we do not claim that the original proponents of the formalism really made this assertion: we are just discussing what we believe is part of the informal folklore of the field. In some sense, consistent histories would be the “natural” way of looking at the system: finding the right projectors that, when used to describe the system, don’t “damage it” in any way. However, our example proves that this is certainly not the case. Consistent histories are not “real” in any sense. The projectors used in constructing them must be chosen from the outside, by us the physicists, and have a decisive effect on the consistent histories of the quantum system. In our example, we could decide to add an extra instant to the list of times defining the stroboscopic history and by this simple act we would have to change completely the description of the system! Reality is a subtle concept that has been debated over the years in many physics texts and does not have a clear definition. In Einstein’s view, an essential ingredient characterizing it is predictability. In that sense, histories could be considered to be real if, apart from the consistency condition, they are predictable behaving in a stable way. As we showed, this is the case if histories are constructed with pointer states.

Consistent histories interpretation was introduced by Griffiths and pursued by others to allow for a discussion of the quantum evolutions without reference to “measurements” or “collapses” of the wave function. The difficulties we have pointed out in our discussion appear to stem not from attempting to achieve this goal, but from retaining the key elements of the formal machinery of “measurements” (such as projection operators acting at well defined instants of time) which then tend to influence histories in a distinctly non–classical manner. Instead of introducing such formal constructs “from the
outside” of the investigated quantum Universe, one might search for the equivalents “on the inside”, in the structure of the correlations between the quantum systems. Instead of projection operators, one would then have “records” –relatively stable states, which, owing to the nature of the quantum dynamics, retain their correlations with the observables of other quantum systems. This program is, of course, embodied in the environment induced superselection approach.

Acknowledgements

Part of this work was done while we were participating in the workshop on Decoherence and the Physics of Information held at the Aspen Center for Physics during June of 1992. We would like to thank the Center for the hospitality and all the participants of the workshop for all their many comments. In particular we would like to thank Murray Gell–Mann and Jim Hartle for many interesting discussions.

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