Pie Generalizations-Locally Closed Sets

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Abstract

The aim of this paper is to continue the study of generalizations of locally closed sets and investigate the classes of \( \pi \text{gl} \)-continuous functions in a topological space.

Key words: \( \pi \text{glc} \), \( \pi g \)-open set, \( \theta \)-locally closed set, \( \pi \text{glc}^* \) and \( \pi \text{glc}^{**} \)

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Introduction

The initiation of the study of generalized closed sets was done by Levine\(^6\) in 1970. The notion of \( \pi g \)-closed sets as a weak form of generalized closed sets was introduced by Dontchev and Noiri\(^4\) in 2000. The notion of locally closed sets in a topological space was introduced by Bourbaki\(^3\). Ganster and Reilly\(^5\) further studied the properties of locally closed sets and defined the LC-continuity and LC-irresoluteness. Balachandran \( et \ al.\)^\(^2\) introduced the concepts of generalized locally closed sets and GLC-continuous functions and investigated some of their properties. In 1997, Arockiarani \( et \ al.\)^\(^1\) studied regular generalized locally closed sets and RGL-continuous functions in a topological space.

The aim of this chapter is to continue the study of generalizations of locally closed sets and investigate the classes of \( \pi \text{gl} \)-continuous functions and \( \pi \text{gl} \)-irresolute functions in a topological space.

A set \( A \subseteq (X, \tau) \) is called \( \theta \)-closed \([65]\) if \( A = \text{cl}_\theta(A) \), where \( \text{cl}_\theta(A) = \{ x \in X : \text{cl}(U) \cap A = \emptyset , U \in \tau \) and \( x \in U \} \). The complement of a \( \theta \)-open set is called \( \theta \)-closed. Before entering into our work, we recall the following definitions which are prerequisite for this paper.
Main Results

Definition 2.1
A subset $S$ of $(X, \tau)$ is said to be $\pi_g$-locally closed ($\pi_{glc}$) if $S = G \cap F$ where $G$ is $\pi_g$-open and $F$ is $\pi_g$-closed in $(X, \tau)$.

Definition 2.2
A subset $S$ of $(X, \tau)$ is called $\pi_{glc}^*$ if there exists a $\pi_g$-open set $G$ and a $\pi_g$-closed set $F$ of $(X, \tau)$ such that $S = G \cap F$.

Definition 2.3:
A subset $B$ of $(X, \tau)$ is called $\pi_{glc}^{**}$ if there exists an open set $G$ and a $\pi_g$-closed set $F$ of $(X, \tau)$ such that $B = G \cap F$.

The collection of all $\pi_g$-locally closed (resp. $\pi_{glc}^*$, $\pi_{glc}^{**}$) sets of a space $(X, \tau)$ will be denoted by $\pi_{GLC}(X, \tau)$ (resp. $\pi_{GLC}^*(X, \tau)$, $\pi_{GLC}^{**}(X, \tau)$).

From the above definitions we have the following remark.

Remark 2.4:
1. Every locally closed set is $\pi_{glc}$.
2. Every $\theta$-locally closed set is $\pi_{glc}$.
3. Every $\theta_{glc}$-set is $\pi_{glc}$.
4. Every $\pi_{glc}^*$-set or $\pi_{glc}^{**}$ is $\pi_{glc}$.
5. Every glc-set is $\pi_{glc}$.
6. Every $\theta_{lc}$-set is $\pi_{glc}^*$ or $\pi_{glc}^{**}$.
7. Every glc$^*$-set is $\pi_{glc}^*$.
8. Every $\theta_{lc}^*$-set is $\pi_{glc}^*$.
9. Every $\theta_{lc}^{**}$-set is $\pi_{glc}^{**}$.
10. Every $\theta_{glc}^*$-set is $\pi_{glc}^*$.
11. Every locally closed set is $\pi_{glc}^*$ and $\pi_{glc}^{**}$.

However the converses of the above are not true may be seen by the following Examples.

Example 2.5
Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then locally closed sets are $\emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}$ and $\pi_{glc}$-sets are $P(X)$. It is clear that $\{a, c\}$ is $\pi_{glc}$-set but it is not locally closed.

Example 2.6
In the above Example 2.5, $\theta$-locally closed sets are $\emptyset, X$ and $\pi_{glc}$-sets are $P(X)$. It is clear that $\{a, b\}$ is $\pi_{glc}$-set but it is not $\theta$-locally closed set.

Example 2.7
In Example 2.5., $\theta_{glc}$-sets are $\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}$ and $\pi_{glc}$-sets are $P(X)$. It is clear that $\{b, c\}$ is $\pi_{glc}$-set but it is not $\theta_{glc}$-set.

Example 2.8
Let $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{b, e\}, \{c, e\}, \{d, e\}, \{a, b, e\}, \{a, c, d\}, \{a, c, e\}, \{a, d, e\}, \{b, c, d\}, \{c, d, e\}, \{a, b, c, d\}, \{a, c, d, e\}, \{b, c, d, e\}$ and $\pi_{glc}$-sets are $P(X)$. It is clear that $\{b,
c) is \( \pi \text{g} \text{lc} \)-set but it is not \( \pi \text{g} \text{lc}^* \)-set.

Example 2.9
In Example 2.5, \( \pi \text{g} \text{lc} \)-sets are \( P(X) \) and \( \text{gle} \)-sets are \( \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\} \). It is clear that \( \{b, c\} \) is \( \pi \text{g} \text{lc} \)-set but it is not \( \text{gle} \)-set.

Example 2.10
In Example 2.5, \( \theta \text{lc} \)-sets are \( \emptyset, X \) and \( \pi \text{g} \text{lc}^* \)-(or) \( \pi \text{g} \text{lc}^{**} \)-sets are \( P(X) \). It is clear that \( \{a, b\} \) is \( \pi \text{g} \text{lc}^{**} \)-set but it is not \( \theta \text{lc} \)-set.

Example 2.11
In Example 2.5, \( \text{gle}^* \)-sets are \( \emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\} \) and \( \pi \text{g} \text{lc}^* \)-sets are \( P(X) \). It is clear that \( \{b, c\} \) is \( \pi \text{g} \text{lc}^* \)-set but it is not \( \text{gle}^* \)-set.

Example 2.12
In Example 2.5, \( \theta \text{lc}^* \)-sets are \( \emptyset, X, \{c\}, \{d\}, \{c, d\} \) and \( \pi \text{g} \text{lc}^* \) sets are \( P(X) \). It is clear that \( \{a, d\} \) is \( \pi \text{g} \text{lc}^* \)-set but it is not \( \theta \text{lc}^* \)-set.

Example 2.13
In Example 2.5, \( \theta \text{glc}^* \)-sets are \( \emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{c, d\} \) and \( \pi \text{g} \text{lc}^* \)-sets are \( P(X) \). It is clear that \( \{b, c\} \) is \( \pi \text{g} \text{lc}^* \)-set but it is not \( \theta \text{glc}^* \)-set.

Example 2.14
In Example 2.5, \( \theta \text{glc}^{**} \)-sets are \( \emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{c, d\} \) and \( \pi \text{g} \text{lc}^{**} \)-sets are \( P(X) \). It is clear that \( \{a\} \) is \( \pi \text{g} \text{lc}^{**} \)-set but it is not \( \theta \text{glc}^{**} \)-set.

Example 2.15
In Example 2.5, locally closed sets are \( \emptyset, X, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\} \) and \( \pi \text{g} \text{lc}^* \) and \( \pi \text{g} \text{lc}^{**} \)-sets are \( P(X) \). It is clear that \( \{a, c\} \) is both \( \pi \text{g} \text{lc}^* \) and \( \pi \text{g} \text{lc}^{**} \)-set but it is not locally closed set.

Theorem 2.16
For a subset \( S \) of \((X, \tau)\) the following are equivalent:
1. \( S \in \pi \text{G} \text{lc}^* (X, \tau) \).
2. \( S = P \cap \text{cl}(S) \) for some \( \pi \text{g} \)-open set \( P \).
3. \( \text{cl}(S) - S \) is \( \pi \text{g} \)-closed.
4. \( S - (X - \text{cl}(S)) \) is \( \pi \text{g} \)-open.

Proof.

(1) \( \Rightarrow \) (2):
Let \( S \in \pi \text{G} \text{lc}^* (X, \tau) \). Then there exists a \( \pi \text{g} \)-open set \( P \) and a closed set \( F \) such that \( S = P \cap F \). Since \( S \subseteq P \) and \( S \subseteq \text{cl}(S) \) we have \( S \subseteq P \cap \text{cl}(S) \).
Conversely, since \( \text{cl}(S) \subseteq F, P \cap \text{cl}(S) \subseteq P \cap F = S \) which implies that \( S \subseteq P \cap \text{cl}(S) \).

(2) \( \Rightarrow \) (1):
Since \( P \) is \( \pi \text{g} \)-open and \( \text{cl}(S) \) is closed
\( P \cap \text{cl}(S) \in \pi \text{G} \text{lc}^* (X, \tau) \).
Let $F = \text{cl}(S) \setminus S$. Then $F$ is $\pi g$-closed by the assumption and $X \setminus F = X \cap (\text{cl}(S) \setminus S) = S \cup (X \setminus \text{cl}(S))$. But $X \setminus F$ is $\pi g$-open. This shows that $S \cup (X \setminus \text{cl}(S))$ is $\pi g$-open.

Let $U = S \cup (X \setminus \text{cl}(S))$. Then $U$ is $\pi g$-open. This implies that $X \setminus U$ is $\pi g$-closed and $X \setminus U = X \setminus (S \cup (X \setminus \text{cl}(S))) = \text{cl}(S) \cap (X \setminus S) = \text{cl}(S) \setminus S$. Thus $\text{cl}(S) \setminus S$ is $\pi g$-closed.

Let $S = \phi$. Then $S \in \pi GLC^\ast(X, \tau)$. This proves that $X$ is $\pi g$-submaximal.

Remark 2.19.
It follows from definitions that if $(X, \tau)$ is $g$-submaximal, then it is $\pi g$-submaximal. But the converse is not true as seen by the following Example.

Example 2.20.
In Example 3., dense sets are $X$, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$. $g$-open sets are $\phi$, $X$, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{a, b, c\}$, $\{a, b, d\}$ and $\pi g$-open sets are $P(X)$. Then it is $\pi g$-submaximal but not $g$-submaximal.

Theorem 2.21
For a subset $S$ of $(X, \tau)$ if $S \in \pi GLC^\ast(X, \tau)$ then there exists an open set $P$ such that $S = P \cap \pi g \text{-cl}(S)$ where $\pi g \text{-cl}(S)$ is the $\pi g$-closure of $S$. 

Proof.
Let $S \in \pi GLC^{**}(X, \tau)$. Then there exists an open set $P$ and a $\pi g$-closed set $F$ such that $S = P \cap F$. Since $S \subseteq P$ and $S \subseteq \pi g$-cl$(S)$, we have $S \subseteq P \cap \pi g$-cl$(S)$. Conversely since $\pi g$-cl$(S) \subseteq F$, we have $P \cap \pi g$-cl$(S) \subseteq P \cap F = S$. Thus $S = P \cap \pi g$-cl$(S)$.

Theorem 2.22.
Let $A$ and $B$ be subsets of $(X, \tau)$. If $A \in \pi GLC^{*}(X, \tau)$ and $B \in \pi GLC^{*}(X, \tau)$ then $A \cap B \in \pi GLC^{*}(X, \tau)$.

Proof.
Let $A$ and $B \in \pi GLC^{*}(X, \tau)$. Then there exist $\pi g$-open sets $P$ and $Q$ such that $A = P \cap \text{cl}(A)$ and $B = Q \cap \text{cl}(B)$. Therefore $A \cap B = P \cap \text{cl}(A) \cap Q \cap \text{cl}(B) = P \cap Q \cap \text{cl}(A) \cap \text{cl}(B)$ where $P \cap Q$ is $\pi g$-open and $\text{cl}(A)$ and $\text{cl}(B)$ is closed. This shows that $A \cap B \in \pi GLC^{*}(X, \tau)$.

Theorem 2.23.
If $A \in \pi GLC^{**}(X, \tau)$ and $B$ is open, then $A \cap B \in \pi GLC^{**}(X, \tau)$.

Proof.
Let $A \in \pi GLC^{**}(X, \tau)$. Then there exists an open set $G$ and a $\pi g$-closed set $F$ such that $A = G \cap F$. So $A \cap B = G \cap F \cap B = G \cap B \cap F$. This proves that $A \cap B \in \pi GLC^{**}(X, \tau)$.

Theorem 2.24.
If $A \in \pi GLC^{*}(X, \tau)$ and $B$ is $\pi g$-open, then $A \cap B \in \pi GLC^{*}(X, \tau)$.

Proof.
Let $A \in \pi GLC^{*}(X, \tau)$. Then $A = G \cap F$ where $G$ is $\pi g$-open and $F$ is $\pi g$-closed. So $A \cap B = G \cap F \cap B = G \cap B \cap F$. This implies that $A \cap B \in \pi GLC^{*}(X, \tau)$.

Theorem 2.25.
If $A \in \pi GLC^{*}(X, \tau)$ and $B$ is $\pi g$-closed $\pi$-open subset of $X$, then $A \cap B \in \pi GLC^{*}(X, \tau)$.

Proof.
Let $A \in \pi GLC^{*}(X, \tau)$. Then $A = G \cap F$ where $G$ is $\pi g$-open and $F$ is closed. So $A \cap B = G \cap (F \cap B)$ where $G$ is $\pi g$-open and $F \cap B$ is closed. Hence $A \cap B \in \pi GLC^{*}(X, \tau)$.

Theorem 2.26.
Let $A$ and $Z$ be subsets of $(X, \tau)$ and let $A \subseteq Z$. If $Z$ is $\pi g$-open in $(X, \tau)$ and $A \in \pi GLC^{*}(Z, \tau Z)$, then $A \in \pi GLC^{*}(X, \tau)$.

Proof.
Suppose $A$ is $\pi g$-glc $^*$-set, then there exists a $\pi g$-open set $G$ of $(Z, \tau Z)$ such that $A = G \cap \text{cl}_Z(A)$. But $\text{cl}_Z(A) = Z \cap \text{cl}(A)$. Therefore, $A = G \cap Z \cap \text{cl}(A)$ where $G \cap Z$ is $\pi g$-open. Thus $A \in \pi GLC^{*}(X, \tau)$.

Example 2.27
Let $X = \{a, b, c, d, e\}$, $\tau = \{\emptyset, X, \{a\}, \{e\}, \{a, e\}, \{c, d\}, \{a, c, d\}, \{c, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}\}$. Let $V$
be the collection of all \( \pi g \)-open sets of \((X, \tau)\). Then \( V = \{ \phi, X, \{a\}, \{c\}, \{d\}, \{e\}, \{a, c\}, \{a, d\}, \{a, e\}, \{c, d\}, \{c, e\}, \{d, e\}, \{a, c, d\}, \{a, c, e\}, \{a, d, e\}, \{c, d, e\}, \{a, c, d, e\} \}. \) Put \( Z = A = \{a, b, c\} \). Then \( Z \) is not \( \pi g \)-open and \( A \in \pi glc^*(Z, \tau/Z) \).

However \( A \in \pi glc^*(X, \tau) \).

**Theorem 2.28.**

If \( Z \) is \( \pi g \)-closed, \( \pi g \)-open set in \((X, \tau)\) and \( A \in \pi GLC^*(Z, \tau/Z) \) then \( A \cap \pi GLC^*(X, \tau) \).

**Proof.**

Let \( A \in \pi GLC^*(Z, \tau/Z) \). Then \( A = G \cap F \) where \( G \) is \( \pi g \)-open and \( F \) is closed in \((Z, \tau/Z) \). Since \( F \) is closed in \((Z, \tau/Z) \), \( F = B \cap Z \) for some closed set \( B \) of \((X, \tau) \). Therefore \( A = G \cap B \cap Z \). Then \( B \cap Z \) is closed. Hence \( A \in \pi GLC^*(X, \tau) \).

**Theorem 2.29**

If \( Z \) is closed and open in \((X, \tau)\) and \( A \in \pi GLC(Z, \tau/Z) \), then \( A \in \pi GLC(X, \tau) \).

**Proof.**

Let \( A \in \pi GLC(Z, \tau/Z) \). Then there exists a \( \pi g \)-open set \( G \) and a \( \pi g \)-closed set \( F \) of \((Z, \tau/Z) \) such that \( A = G \cap F \). Then by the above theorem \( A \in \pi GLC(X, \tau) \).

**Theorem 2.30**

If \( Z \) is \( \pi g \)-closed, \( \pi g \)-open subset of \((X, \tau)\) and \( A \in \pi GLC^*(Z, \tau/Z) \), then \( A \in \pi GLC^*(X, \tau) \).

**Proof.**

Let \( A \in \pi GLC^*(Z, \tau/Z) \). Then \( A = G \cap F \) where \( G \) is open and \( F \) is \( \pi g \)-closed in \((Z, \tau/Z) \). Since \( Z \) is \( \pi g \)-closed \( \pi g \)-open subset of \((X, \tau)\), then \( F \) is \( \pi g \)-closed in \((X, \tau) \). Therefore \( A \in \pi GLC^*(X, \tau) \).

**Theorem 2.31**

If \( A \) is \( \pi g \)-open and \( B \) is open, then \( A \cap B \) is \( \pi g \)-open.

**Proof.**

Let \( A \) be \( \pi g \)-open. Then \( \text{int}(A) \supseteq F \) whenever \( A \supseteq F \) and \( F \) is \( \pi g \)-closed set. Suppose \( A \cap B \supseteq F \), then we prove that \( \text{int}(A \cap B) \supseteq F \). Since \( B \) is open, \( \text{int}(B) = B \supseteq F \). Therefore by assumptions \( \text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B) \supseteq F \). This proves that \( A \cap B \) is \( \pi g \)-open.

**Theorem 2.32**

Suppose that the collection of all \( \pi g \)-open sets of \((X, \tau)\) is closed under finite unions. Let \( A \in \pi GLC^*(X, \tau) \) and \( B \in \pi GLC^*(X, \tau) \). If \( A \) and \( B \) are separated, then \( A \cap B \in \pi GLC^*(X, \tau) \).

**Proof.**

Let \( A, B \in \pi GLC^*(X, \tau) \). Then there exist \( \pi g \)-open sets \( G \) and \( S \) of \((X, \tau)\) such that \( A = G \cap \text{cl}(A) \) and \( B = S \cap \text{cl}(B) \). Put \( V = G \cap (X \setminus \text{cl}(B)) \) and \( W = S \cap (X \setminus \text{cl}(A)) \). Then \( V \) and \( W \) are \( \pi g \)-open sets and \( A = V \cap \text{cl}(A) \) and \( B = W \cap \text{cl}(B) \). Also \( V \cap \text{cl}(B) = \phi \) and \( W \cap \text{cl}(A) = \phi \). Hence it follows that \( V \) and \( W \) are \( \pi g \)-open sets of \((X, \tau)\). Therefore \( A \cap B = (V \cap \text{cl}(A)) \cup (W \cap \text{cl}(B)) = V \cup W \cap \text{cl}(A) \cup \text{cl}(B) \).
Here $V \cup W$ is $\pi g$-open by assumption. Thus $A \cup B \in \pi GLC^*(X, \tau)$.

**Example 2.33**

Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then $\{a, b\}$ and $\{a, d\} \in \pi GLC^*(X, \tau)$ but $\{a, b, d\} \notin \pi GLC^*(X, \tau)$, since they are not separated. For we have $\{a, b\} \cap cl(\{a, d\}) = \{a\} = \emptyset$ and $\{a, d\} \cap cl(\{a, b\}) = \{a, d\} = \emptyset$.

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