AROUND A MULTIVARIATE SCHMIDT-SPITZER THEOREM

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Abstract. Given an arbitrary complex-valued infinite matrix \( A = (a_{ij}), i = 1, \ldots, \infty; j = 1, \ldots, \infty \) and a positive integer \( n \) we introduce a naturally associated polynomial basis \( \mathcal{B}_A \) of \( \mathbb{C}[x_0, \ldots, x_n] \). We discuss some properties of the locus of common zeros of all polynomials in \( \mathcal{B}_A \) having a given degree \( m \); the latter locus can be interpreted as the spectrum of the \( m \times (m + n) \)-submatrix of \( A \) formed by its \( m \) first rows and by the \( (m + n) \) first columns. We initiate the study of the asymptotics of these spectra when \( m \to \infty \) in the case when \( A \) is a banded Toeplitz matrix. In particular, we present and partially prove a conjectural multivariate analog of the well-known Schmidt-Spitzer theorem which describes the spectral asymptotics for the sequence of principal minors of an arbitrary banded Toeplitz matrix. Finally, we discuss relations between polynomial bases \( \mathcal{B}_A \) and multivariate orthogonal polynomials.

1. Introduction

The approach of this paper is motivated by the modern interpretation of the Heine-Stieltjes multiparameter spectral problem as presented in [13] and [14]. Let us recall some relevant results in the matrix set-up.

Given integers \( m > 0 \) and \( n \geq 0 \) consider the space \( \text{Mat}(m, m + n) \) of complex-valued \( m \times (m + n) \)-matrices. For \( s = 0, \ldots, n \) define the \( s \)-th unit matrix
\[
I_s := (\delta_{s+i-j}) \in \text{Mat}(m, m + n).
\]
(In what follows the sizes of matrices can be infinite.)

Definition 1 (see [14]). Given a matrix \( A \in \text{Mat}(m, m + n) \) define its eigenvalue locus \( \mathcal{E}_A \) as
\[
\mathcal{E}_A := \left\{ (x_0, x_1, \ldots, x_n) \in \mathbb{C}^{n+1}: \text{rank} \left( A - \sum_{s=0}^{n} x_s I_s \right) < m \right\}.
\]

For \( n = 0 \), \( \mathcal{E}_A \) coincides with the usual set of eigenvalues of a square matrix \( A \).

Proposition 2 (Lemma 1 of [14]). For arbitrary \( A \in \text{Mat}(m, m + n) \) the eigenvalue locus \( \mathcal{E}_A \) consists of \( \binom{m+n}{n+1} \) points counting multiplicities. In other words, counting multiplicities there exist \( \binom{m+n}{n+1} \) eigenvalue tuples \( (x_0, x_1, \ldots, x_n) \) such that \( A - \sum_{s=0}^{n} x_s I_s \) has rank smaller than \( m \).

Remark 3. Notice that for \( n > 0 \), the locus \( \mathcal{E}_A \) is not a complete intersection (we need more polynomials than expected to define the ideal) since it is given by the vanishing of all maximal minors of \( A \). (A similar phenomenon can be observed for common zeros of multivariate Schur polynomials, since Schur polynomials are given by determinant formulas.)

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Notation 4. Given an infinite matrix $\mathcal{A} = (a_{ij})$, $i = 1, \ldots, \infty; j = 1, \ldots, \infty$, an integer $n \geq 0$, and an $m$-tuple of positive integers $I = (i_1, i_2, \ldots, i_m)$ satisfying $1 \leq i_1 < i_2 < \cdots < i_m \leq m + n$, consider the submatrix $A_I$ of $\mathcal{A} - \sum_{s=0}^{n} x_s I_s$ formed by the first $m$ rows and by the $m$ columns indexed by $I$. Define

\begin{equation}
P^I_A(x_0, x_1, \ldots, x_n) := \det A_I.
\end{equation}

Evidently, $P^I_A(x_0, x_1, \ldots, x_n)$ is a maximal minor of the principal $m \times (m + n)$ submatrix of $\mathcal{A} - \sum_{s=0}^{n} x_s I_s$ formed by its first $m$ rows and $m + n$ first columns. Therefore $P^I_A(x_0, x_1, \ldots, x_n)$ is a polynomial in $x_0, \ldots, x_n$.

Proposition 5. In the above notation the following holds:

(i) for any multiindex $I$ with $|I| = m$, $\deg P^I_A(x_0, x_1, \ldots, x_n) = m$; (ii) all $\binom{m+n}{m}$ polynomials $P^I_A(x_0, x_1, \ldots, x_n) \in \mathbb{C}[x_0, \ldots, x_n]$ with $|I| = m$ are linearly independent, which implies that the totality of all polynomials $P^I_A(x_0, x_1, \ldots, x_n)$ is a linear basis of $\mathbb{C}[x_0, \ldots, x_n]$; (iii) the set $E^{(m)}_A$ of common zeros of all $P^I_A(x_0, x_1, \ldots, x_n)$ with $|I| = m$ is a finite subset of $\mathbb{C}^{n+1}$ of cardinality $\binom{m+n}{n+1}$ counting multiplicities. Note that $E^{(m)}_A$ coincide with that in Definition 1.

Remark 6. Notice that for $\binom{m+n}{m}$ randomly chosen polynomials in $\mathbb{C}[x_0, x_1, \ldots, x_n]$ of degree $m$, the set of their common zeros is typically empty.

Proposition 5 together with our numerical experiments motivate the following question.

Given an arbitrary infinite matrix $\mathcal{A}$ as above, associate to each $E^{(m)}_A$ its “root-counting” measure $\mu^{(m)}_A$ supported on $E^{(m)}_A \subset \mathbb{C}^{n+1}$ by assigning to every point $p \in E^{(m)}_A$ the point mass $\kappa(p) / \binom{m+n}{n+1}$ where $\kappa(p)$ is the multiplicity of $p$. (Obviously, $\mu^{(m)}_A$ is a discrete probability measure.)

Main Problem. Under which assumptions on $\mathcal{A}$ does the weak limit $\mu_A = \lim_{m \to \infty} \mu^{(m)}_A$ exist? In case $\mu_A$ exist, is it possible to describe the support and density of the measure?

In the classical case $n = 0$, the above problem was intensively studied by many authors. The main focus has been when $\mathcal{A}$ is either a Jacobi or a Toeplitz matrix (or their generalizations such as block-Toeplitz matrices etc.), see e.g. [4, 3, 15, 16], and the more recent development [7, 5, 6].

The main goal of this note is to present a multivariate analogue of the well-known theorem by P. Schmidt and F. Spitzer [12], who describe $\mu_A$ for an arbitrary banded Toeplitz matrix $\mathcal{A}$ in the case $n = 0$.

Namely, let $\mathcal{A} = (a_{i-j})$, with $i, j = 1, 2, \ldots$ be an infinite, banded Toeplitz matrix, where $c_i = 0$ if $i < -k$ or $i > h$. Fixing $n \geq 0$ as above, we obtain for each positive integer $m$ the eigenvalue locus $E^{(m)}_A$ of the principal $m \times (m + n)$ submatrix $A^{(m)}$ of $\mathcal{A}$.

Define the limit set $B_A$ of eigenvalue loci as

\begin{equation}
B_A = \left\{ x \in \mathbb{C}^{n+1} : x = \lim_{m \to \infty} x_m, x_m \in E^{(m)}_A \right\}, \quad x = (x_0, \ldots, x_n).
\end{equation}

In other words, $B_A$ is the set of limit points of the sequence $\{E^{(m)}_A \}$. Thus $B_A$ is the support of the limiting measure $\mu_A$ if it exists. (For a general infinite matrix $\mathcal{A}$ as above, its limit set $B_A$ might be empty.)
Set

\[ Q(t, x) = t^k \left( \sum_{j=0}^{h} c_j t^j - \sum_{j=0}^{n} x_j t^j \right), \]

and let \( \alpha_1(x), \alpha_2(x), \ldots, \alpha_{k+h}(x) \) be the roots of \( Q(t, x) = 0 \), ordered according to their absolute values, i.e. \( |\alpha_i(x)| \leq |\alpha_{i+1}(x)| \) for all \( 0 < i < k + h \). Let \( C_A \) be the real semi-algebraic set given by the condition:

\[ C_A = \{ x \in \mathbb{C}^{n+1} : |\alpha_k(x)| = |\alpha_{k+1}(x)| = \cdots = |\alpha_{k+n+1}(x)| \}. \]

Our main conjecture is as follows.

**Conjecture 7.** For any banded Toeplitz matrix \( A \), if \( B_A \) is defined by (2) and \( C_A \) defined by (4) then \( B_A = C_A \).

By Conjecture 7 the set \( B_A \) is a real semi-algebraic \((n+1)\)-dimensional subset of \( \mathbb{C}^{n+1} \). In the classical case \( n = 0 \), Conjecture 7 is settled by F. Schmidt and F. Spitzer in [12]. Another important case when Conjecture 7 has been proved follows from some known results on multivariate Chebyshev polynomials, which is is presented in Example 8 below. Namely, when \( k = 1 \) and \( h = n + 1 \) with \( c_{-1} \) and \( c_{n+1} \) non-zero, we may do an affine change of the variables and a scaling of \( A \). This reduces the latter case to \( c_{-1} = c_{n+1} = 1 \) and all other \( c_j = 0 \).

For these particular values, the family \( \{ P_A^k(x) \} \) becomes the multivariate Chebyshev polynomials of the second kind, see e.g. [8, 10, 2, 17]. These polynomials also have a close connection to another well-known family of polynomials, namely the Schur polynomials.

**Example 8.** For the above matrices corresponding to the multivariate Chebyshev polynomials the eigenlocus \( \Sigma_A^{(m)} \) can be described explicitly, see for example [9].

More precisely, the points in \( \Sigma_A^{(m)} \) lie on a real, \( n \)-dimensional surface \( C_A \subset \mathbb{C}^{n+1} \) which is naturally parametrized by an \((n+1)\)-dimensional torus \( T^{n+1} \). This parametrization is given by

\[ C_A = \{ x \in \mathbb{C}^{n+1} : x_j = -e_{j+1}(\exp(i\theta_1), \ldots, \exp(i\theta_{n+1}), \exp(i\theta_{n+2})) \} \]

where \( (\theta_1, \ldots, \theta_{n+1}) \) lie on \( T^{n+1} \), \( \sum_{j=1}^{n+2} \theta_j = 0 \), and \( e_j \) is the \( j \)-th elementary symmetric function in \( n + 2 \) variables.

Notice that for \( x \in C_A \),

\[ Q(t, x) = 1 + x_0 t + x_1 t^2 + \cdots + x_n t^{n+1} + t^{n+2} = \prod_j (t + e^{i\theta_j}) \]

by the Vieta formulas. Thus, for \( x \in C_A \), all roots of \( Q \) (as a polynomial in \( t \)) have absolute value equal to 1 when the \( x_j \) are parametrized as in (5).

Furthermore, the points in \( \Sigma_A^{(m)} \) are also expressed by (5), with the parameters \( (\theta_1, \ldots, \theta_{n+2}) \) being certain rational multiples of \( \pi \), distributed in a regular lattice. The mapping from the 2-torus to the eigenlocus is illustrated in Figure 1.

Another interesting aspect of Example 8 is that all the points \( x = (x_0, \ldots, x_n) \) in the sets \( \Sigma_A^{(m)} \) satisfy the conditions \( x_j = x_{n-j}, j = 0, 1, \ldots, n \), which explains why we can draw \( C_A \subset \mathbb{C}^2 \) in Figure 1a as a 2-dimensional set. For larger \( n \), \( C_A \) is an \((n+1)\)-dimensional analogue of the two-dimensional deltoid, shown in Figure 1a.
For general $\mathcal{A}$, we do not have the inclusion $\mathcal{E}_A^{(m)} \subseteq C_A$ for arbitrary finite $m$. However, if $\mathcal{A}$ has an additional extra symmetry, this seems to be case, as we will see below.

**Definition 9.** An $m \times (m+n)$ rectangular matrix $(a_{ij})$ is called centro-hermitian if its entries satisfy

$$a_{ij} = \overline{a_{m+1-i,m+n+1-j}}.$$

(These matrices appear in various context in linear algebra, see e.g., [11].)

One can show that for any such centro-hermitian matrix, we have that $(x_0, x_1, \ldots, x_n) \in \mathcal{E}_A^{(m)}$ if and only if $(\overline{x}_n, \overline{x}_{n-1}, \ldots, \overline{x}_0) \in \mathcal{E}_A^{(m)}$. (Professor Yuan Xu kindly informed us about this fact.) In other words, for centro-hermitian matrices, eigenvalues come in “complex conjugate” pairs where by a “complex conjugation” we mean the latter anti-holomorphic involution.

However, for centro-hermitian Toeplitz matrices, all the eigenvalues seem to be “real” with respect to the above “complex conjugation”. We pose the following conjecture.

**Conjecture 10.** If $\mathcal{A}$ is Toeplitz and centro-hermitian, then each point $(x_0, x_1, \ldots, x_n) \in \mathcal{E}_A^{(m)}$ satisfies $x_j = \overline{x}_{n-j}$ for $j = 0, 1, \ldots, n$.

Conjecture 10 obviously holds for the case $n = 0$, as it reduces to the fact that hermitian matrices have real eigenvalues. It is also straightforward to check that Conjecture 10 is true for the Chebyshev case of Example 8 above.

We have extensive numerical evidence for this conjecture. Another strong indication supporting Conjecture 10 is that if $\mathcal{A}$ is Toeplitz and centro-hermitian, then every point $x \in C_A$ (which by Conjecture 7 is in the limit eigenlocus) satisfies the required symmetry $x_j = \overline{x}_{n-j}$ for $j = 0, 1, \ldots, n$.

![Figure 1. The eigenvalue locus $\mathcal{E}_A^{(20)}$ and its pull-back to $T^2$. The torus $T^2$ is covered with a hexagon, where each triangle is mapped to the eigenlocus. The 6-fold symmetry is due to the $S_3$-action by permutation of the arguments $\theta_1, \theta_2, \theta_3$ in (5).](image-url)
Figure 2. 5-edged star, when $d = 2$ and 7-edged star, when $d = 3$

The next group of examples are bivariate analogues of special univariate cases originally studied in [12], and later in [3], where they are referred to as “star-shaped curves”:

**Example 11.** The bivariate case when $Q(t, x) = 1 + t^d x_0 + t^{d+1} x_1 + t^{2d+1}, d \geq 1$ gives sets in $\mathbb{C}^2$ where $x_0 = \bar{x}_1$, by Conjecture 10. They correspond to Toeplitz matrices of the form

$$
\begin{pmatrix}
  x_0 & x_1 & 1 & 0 & 0 & \cdots \\
  1 & x_0 & x_1 & 1 & 0 & \cdots \\
  0 & 1 & x_0 & x_1 & 1 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \ddots 
\end{pmatrix}
$$

The above two matrices represent $d = 1$ and $d = 2$.

Figures 2 and 3 present the distributions of $x_0 \in \mathbb{C}$, for $d = 2, 3, 4$. (Recall that $x_1 = \bar{x}_0$.) The points shown on these figures belong to $E_A^{(m)}$ for $m = 13, 14, 15$, and the curves are certain hypocycloids, parametrizing the boundary of $C_A$. More explicitly, for a given integer $d \geq 1$ the hypocycloid boundary for $x_0 \in \mathbb{C}$ is given by

$$
x_0 = (-1)^d e^{-i(d+2)\theta} \left( (d + 2)e^{i(2d+3)\theta} + d + 1 \right) \text{ where } \theta \in [0, 2\pi],
$$

which is one of the implications of Conjecture 7.

Finally, the main result of this note is as follows.

**Theorem 12.** For any banded Toeplitz matrix $A$, where $B_A$ is defined by (2) and $C_A$ is defined by (4), one has $B_A \subseteq C_A$.

2. Proofs

**Proof of Proposition 5.** We shall prove items (i) and (ii) simultaneously by calculating the leading homogeneous part of $P_A^I(x_0, ..., x_n)$. Let us order the set of all admissible indices $I = (1 \leq i_1 < \ldots < i_m \leq m + n)$ lexicographically. We can also order lexicographically all monomials of degree $m$ in $x_0, \ldots, x_n$. By equation (1) $P_A^I(x_0, ..., x_n) = \det A_I$ where the columns of $A_I$ are indexed by
I. Let $\tilde{P}_A^I(x_0, \ldots, x_n)$ be the homogeneous part of $P_A^I(x_0, \ldots, x_n)$ of degree $m$. One can easily see that the product of all entries on the main diagonal of $A_I$ contains the monomial $m_I$ of degree $m$ given by $m_I = \prod_{j=1}^m x_{i_j-j+1}$. Moreover it is straight-forward that $\tilde{P}_A^I(x_0, \ldots, x_n) = m_I + \ldots$ where $\ldots$ stands for the linear combination of monomials $m_{I'}$ of degree $m$ obtained from other indices $I'$ which are lexicographically smaller than $I$. In other words, the matrix formed by $\tilde{P}_A^I(x_0, \ldots, x_n)$ against monomials is triangular in the lexicographic ordering with unitary main diagonal, which proves items (i) and (ii).

Item (iii) is just a reformulation of Proposition 2 above. □

Throughout the rest of the paper, we put $\alpha = (\alpha_1, \ldots, \alpha_{h+k})$. We will also assume that $c_h = 1$, which corresponds to a rescaling of the original matrix $A$. This is equivalent to the assumption that $Q(t, x)$ is monic. By shifting the variables in $x$, we may also assume, without loss of generality, that $c_0 = c_1 = \cdots = c_n = 0$ in $A$.

In the above notation, it is convenient to work with the roots of $Q(t, x)$. This motivates the following definitions. Let $\Gamma_j \subset \mathbb{C}^{h+k}$, $j = k, \ldots, k+n$ denote the real semi-algebraic hypersurface consisting of all $\alpha = (\alpha_1, \ldots, \alpha_{h+k})$ such that when the $\alpha_j$ are ordered with increasing moduli, $|\alpha_j| = |\alpha_{j+1}|$. Similarly, let $G_j$ be defined as the real semi-algebraic set

$$\{x \in \mathbb{C}^{n+1}: Q(t, x) = (t-\alpha_1) \cdots (t-\alpha_{h+k}) \text{ where } \alpha \in \Gamma_j \}.$$ 

Then, by definition, $C_A = \bigcap_{j=k}^{k+n} G_j$.

**Proposition 13.** For any banded Toeplitz matrix $A$ and any non-negative $n < h$, the set $C_A$ defined by (3)-(4) is compact.

**Proof.** As discussed above, we may without loss of generality assume that $c_h = 1$ and $c_0 = c_1 = \cdots = c_n = 0$. Since $Q$ may be assumed to be monic, we have $c_j = e_{h-j}(-\alpha)$ for $-k \leq j < 0$ and
n < j ≤ h, and x_j = -e_h - j(-\alpha) when 0 ≤ j ≤ n. Thus, it suffices to show that the set of \alpha ∈ \mathbb{C}^{h+k} that satisfies the conditions (3)-(4), is compact. It is also evident that the set \partial C_{A} is closed, so we only need to show that it is bounded. We show this fact by contradiction.

Assume we have a sequence of roots \{\alpha^m\}_{m=1}^{\infty} of (3) such that \|\alpha^m\| → ∞ where (4) is satisfied for each \alpha^m. We assume that the modulus of the roots are always ordered increasingly. There are two cases to consider.

Case 1: Assume that for some 0 ≤ b < k, a sequence of individual roots satisfies the condition |\alpha_{b+1}^m| → ∞ but |\alpha_j^m| are bounded for all m and j ≤ b. Then consider \varepsilon_{b+k-b}(\alpha). Since \varepsilon < k, in our notation \varepsilon_{b+k-b}(\alpha) equals the coefficient \varepsilon_{b-k}. Notice that \varepsilon_{b+k-b} contains the term \alpha_{b+1}\alpha_{b+2} \cdots \alpha_{b+k} which grows quicker than all other terms in the expansion of \varepsilon_{b+k-b}(\alpha). This contradicts the assumption \varepsilon_{b+k-b}(\alpha) = \varepsilon_{b-k}.

Case 2: Assume that for some b with \varepsilon < b < h + k, we have a sequence of individual roots |\alpha_{b+1}^m| → ∞ but |\alpha_j^m| are bounded for all m and j ≤ b. Consider

\[ e_b(\alpha) = e_b(\alpha_1, \ldots, \alpha_{b+k}) = \sum_{\sigma \in \{b+1\}} e_0 \frac{e_0}{\alpha_{\sigma_1} \alpha_{\sigma_2} \cdots \alpha_{\sigma_{n}}} \]

This contains an expression with the denominator \alpha_1\alpha_2 \cdots \alpha_{b}, i.e. the product of all bounded roots. Now, since \varepsilon + k - b roots among all \varepsilon + k roots grow in absolute value, and the product \alpha_1 \cdots \alpha_{\varepsilon+k} equals \varepsilon_h, it follows that |\alpha_1\alpha_2 \cdots \alpha_{b}| → 0 as m → ∞, and this term converges to 0 quicker than any other product \alpha_{\sigma_1}\alpha_{\sigma_2} \cdots \alpha_{\sigma_n}. Thus, \varepsilon_b should grow. This contradicts the assumption \varepsilon_b(\alpha) = \varepsilon_{b-k}.

Notice that under our assumptions, the above cases cover all possible ways for a sequence of roots to diverge. Since both cases yield a contradiction, it follows that any sequence of roots of (3) satisfying (4) must be bounded. Thus, \partial C_{A} is compact. \□

The following result is a multivariate analog of a known fact in the case \varepsilon = 0, see [3, Prop. 11.8, Prop. 11.9].

**Proposition 14.** In the notation of (3)-(4), for any x belonging to the boundary \partial C_{A} of C_{A}, at least one of the following three conditions is satisfied:

(i) the discriminant of \varphi_{t,x} with respect to t vanishes, i.e. \varphi_{t,x} has (at least) a double root in t.

(ii) |\alpha_{k-1}(x)| = |\alpha_k(x)| = |\alpha_{k+1}(x)| = \cdots = |\alpha_{k+n+1}(x)|.

(iii) |\alpha_k(x)| = |\alpha_{k+1}(x)| = \cdots = |\alpha_{k+n+1}(x)| = |\alpha_{k+n+2}(x)|.

**Proof.** We need the following two simple statements.

**Lemma 15.** Let Pol_d be the set of all monic polynomials of degree d with complex coefficients. Let \Sigma_{p,q} ⊂ Pol_d be the subset of polynomials satisfying

\[ |\alpha_p| = |\alpha_{p+1}| = \cdots = |\alpha_q|, \]

where 1 ≤ p < q ≤ d and \alpha_1, \alpha_2, \ldots, \alpha_d are the roots of the polynomials ordered according to their increasing absolute values. Then \Sigma_{p,q} is a real semi-algebraic set of codimension q − p whose boundary is the union of three pieces: \Sigma_{p-1,q}, \Sigma_{p,q+1} and the intersection of \Sigma_{p,q} with the standard discriminant in Pol_d, i.e. the set of polynomials having multiple roots. (Notice that if p = 1 then \Sigma_{p-1,q} is empty, and if q = d then \Sigma_{p,q+1} is empty by definition.)
Proof. $\Sigma_{p,q}$ is obtained as the image under the Vieta map of an obvious semi-algebraic set $|\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_p| = |\alpha_{p+1}| = \cdots = |\alpha_q| \leq |\alpha_{q+1}| \leq \cdots \leq |\alpha_d|$. Notice that the Vieta map is a local diffeomorphism outside the preimage of the standard discriminant. Therefore the boundary of $\Sigma_{p,q}$ must either belong to the standard discriminant or to one of $\Sigma_{p-1,q}$ or $\Sigma_{p,q+1}$. The former is the common boundary between $\Sigma_{p,q}$ and $\Sigma_{p-1,q}$ and the latter is the common boundary between $\Sigma_{p,q}$ and $\Sigma_{p+1,q+1}$. $\square$

Given a closed Whitney stratified set $X$ (for example, semi-analytic) we say that $X$ is a $k$-dimensional stratified set without boundary if

(i) the top-dimensional strata of $X$ have dimension $k$;

(ii) for any point $x$ lying in any stratum of dimension $k-1$, one can choose an orientation of the (germs of) $k$-dimensional strata of a sufficiently small neighborhood of $x$ in $X$ so that $\partial X = 0$.

Lemma 16. The boundary of the intersection of any closed semi-algebraic set $\Gamma$ with any closed algebraic set $\Theta$ is included in the intersection of the boundary $\partial \Gamma$ with $\Theta$.

Proof. Observe that any real algebraic variety $X$ of dimension $k$ is a stratifiable set without boundary. Indeed, the fact we are proving is local, and it suffices to prove it for generic $x$ on $(k-1)$-dimensional strata.

Consider an embedding of $X$ in some high-dimensional linear space, take the Whitney stratification with $x$ on its stratum $Y \subset B$ of dimension $k-1$, and a transversal to $Y$ of codimension $k-1$ at $x$.

Therefore, we may now assume that the germ of $X$ near $x$ is topologically a product of a germ of an algebraic curve and a germ of a smooth manifold of dimension $k-1$. Furthermore, a germ of any real algebraic curve $\Gamma$ can always be oriented so that $\partial \Gamma = 0$, which follows from the existence of Puiseux series for an arbitrary branch of an algebraic curve. This argument shows that any point in the intersection $\Gamma \cap \Theta$ which does not belong to the boundary of $\Gamma$ can not lie on the boundary of this intersection, which settles Lemma 16. $\square$

Lemmas 15 and 16 immediately imply Proposition 14 since every $C_A$ is the intersection of an appropriate $\Sigma_{p,q}$ with an appropriate affine subspace in $\text{Pol}_{k+h}$.

Proof of Theorem 12. In our notation, let $D_{j}^{m}(x)$ be the determinant of the $m \times m$-matrix $A_{I}$ with $I = \{j+1, j+2, \ldots, j+m\}$ for $0 \leq j \leq n$. It is evident that $E_{A}^{(m)}$ is a subset of the set $\overline{E}_{A}^{(m)}$ of solutions to the system of polynomial equations

$$D_{0}^{m}(x) = D_{1}^{m}(x) = \cdots = D_{n}^{m}(x) = 0.$$  

(7)

We will show the stronger statement that, in notation of Theorem 12,

$$\lim_{m \to \infty} \overline{E}_{A}^{(m)} \subseteq C_{A},$$

which follows from Proposition 18 below.

Remark 17. Although each individual $\overline{E}_{A}^{(m)}$ (considered as a points set with multiplicities) is strictly bigger than $\overline{E}_{A}^{(m)}$, the limits $B_{A} = \lim_{m \to \infty} \overline{E}_{A}^{(m)}$ and $\lim_{m \to \infty} \overline{E}_{A}^{(m)}$ seem to coincide as infinite sets.
In Theorem 4 of [1] it was shown that each sequence of determinants \( \{D^m_j(x)\}_{m=1}^\infty \) as above satisfies a linear recurrence relation with coefficients depending on \( x \). The characteristic polynomial \( \chi_j(t) \) of the \( j \)-th recurrence can be factorized as
\[
\chi_j(t, x) = \prod_{\sigma} (t - r_{j\sigma}), \quad \text{where } r_{j\sigma} = (-1)^{k+j}(\alpha_{\sigma_1} \cdots \alpha_{\sigma_{k+j}})^{-1},
\]
and \( \sigma \) is a \( k + j \)-subset of \( 1, 2, \ldots, k + h \).

**Proposition 18.** Suppose that \( \{x_m\}_1^\infty \) is a sequence of points in \( \mathbb{C}^{n+1} \) satisfying the system of equations
\[
D^m_j(x_m) = 0 \quad \text{for } j = 0, 1, \ldots, n \text{ and } m = 1, 2, \ldots
\]
and such that the limit \( \lim_{m \to \infty} x_m =: x^* \) exists. Then for all \( j = 0, \ldots, n \) we have \( |\alpha_{k+j}(x^*)| = |\alpha_{k+j+1}(x^*)| \) when the \( \alpha_i \) are indexed with increasing order of their modulus.

**Proof.** Provided that all the roots of \( \chi_j(t, x) \) are distinct, by using a version of Widom’s formula, (see [1, 4]) we have
\[
D^m_j(x) = \sum_{\sigma} \prod_{l \in \sigma, i \notin \sigma} \left( 1 - \frac{\alpha_l(x)}{\alpha_i(x)} \right)^{-1} r_{j\sigma}(x)^m.
\]

We may assume that for \( x^* \) and fixed \( j \), the \( r_{j\sigma}(x^*) \) are ordered decreasingly with respect to their modulus (for some ordering \( \sigma = 1, 2, \ldots \)). The goal is to prove that \( |r_{j1}(x^*)| = |r_{j2}(x^*)| \) since this implies \( |\alpha_{k+j}(x^*)| = |\alpha_{k+j+1}(x^*)| \). We show this fact by contradiction.

Assume that \( |r_{j1}(x^*)| > |r_{j2}(x^*)| \geq \cdots \geq |r_{j_h}(x^*)| \), i.e. that the largest root is simple and has modulus strictly larger than any other root of the characteristic equation (8). By examining (10), it is evident that \( r_{j1}(x_m)^m \) is the dominating term for sufficiently large \( m \), that is, \( D^m_j(x_m)/r_{j1}(x_m)^m \to L \neq 0 \) as \( m \to \infty \).

By standard properties of linear recurrences, this holds even when there are multiple zeros among the smaller roots; remember that our assumption was that \( r_{j1}(x_m) \) is a simple zero of (8) when \( m \) is large enough.

Hence, for sufficiently large \( m \), \( D^m_j(x_m) \approx L r_{j1}(x_m)^m \), which is non-zero for sufficiently large \( m \). This contradicts the condition that \( x_m \) satisfies (9). Consequently, \( |r_{j1}(x^*)| = |r_{j2}(x^*)| \) for \( j = 0, 1, \ldots, n \) and this implies Proposition 18. \( \square \)

Proposition 18 implies that \( x \) lies in \( B_\mathcal{A} \) only if \( x \) is a limit of solutions to (9), but such limit \( x \) must satisfy that \( |\alpha_k(x)| = |\alpha_{k+1}(x)| = \cdots = |\alpha_{k+n+1}(x)| \). Therefore, \( B_\mathcal{A} \subseteq C_\mathcal{A} \). \( \square \)

**3. Further directions**

**1.** It seems relatively easy to describe the stratified structure of \( C_\mathcal{A} \) at least in the case of generic \( \mathcal{A} \). In particular, in the Chebyshev case of Example 8 the set \( C_\mathcal{A} \) has the same stratification as a simplex of corresponding dimension. One can also understand the stratified structure of the sets \( \Sigma_{p,q} \) introduced in Lemma 15. Since each \( C_\mathcal{A} \) is obtained from a corresponding \( \Sigma_{p,q} \) by intersecting it with an affine subspace, the stratified structure of the former for generic \( \mathcal{A} \) is also describable. On the other hand, our Example 11 seems to show more complicated stratified structure due to the presence of additional symmetry.

**2.** We say that an (infinite) complex-valued matrix \( \mathcal{A} \) has a weak univariate orthogonality property if the sequence of characteristic polynomials of its principal minors obeys the standard 3-term...
recurrence relation with complex coefficients. There is a straightforward version of this notion for finite square matrices. Obviously, any Jacobi matrix has this property. However, it seems that for any \( m \geq 3 \) the set \( \text{WO}_m \subset \text{Mat}(m,m) \) of all \( m \times m \)-matrices with the latter property has a bigger dimension than the set \( \text{Jac}_m \subset \text{Mat}(m,m) \) of all Jacobi \( m \times m \)-matrices.

**Problem 19.** Find the dimension of \( \text{WO}_m \).

3. Analogously, given a non-negative integer \( n \), we say that an (infinite) complex-valued matrix \( \mathcal{A} \) has a weak \( n \)-variate orthogonality property if the above family \( \{ P_\mathcal{A}(x_0, x_1, \ldots, x_n) \} \) (see Definition 4) satisfies the 3-term recurrence relation (2.2) of Theorem 2.1 of [17] with complex coefficients.

There are many similarities between families \( \{ P_\mathcal{A}(x_0, x_1, \ldots, x_n) \} \) and families of multivariate orthogonal polynomials which by one of the standard definitions of such polynomials also satisfy (2.2) of Theorem 2.1 of [17] with real coefficients.

Our computer experiments show that in this aspect the case \( n > 0 \) is quite different from the classical case \( n = 0 \). In particular, we believe that the following conjecture holds.

**Conjecture 20.** Given \( n > 0 \), a banded matrix \( \mathcal{A} \) has a weak \( n \)-variate orthogonality property if it is of the form

\[
\mathcal{A} = \begin{pmatrix}
a_0 & a_1 & a_2 & \ldots & a_{n+1} & 0 & 0 & 0 & \ldots \\
d_{-1} & d_0 & d_1 & \ldots & d_n & d_{n+1} & 0 & 0 & \ldots \\
0 & d_{-1} & d_0 & \ldots & d_{n-1} & d_n & d_{n+1} & 0 & \ldots \\
0 & 0 & d_{-1} & \ldots & d_{n-2} & d_{n-1} & d_n & d_{n+1} & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix},
\]

where \( a_0, \ldots, a_{n+1}, d_{-1}, \ldots, d_{n+1} \in \mathbb{C} \).

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