The Braided Quantum 2-Sphere

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Abstract

In a recent paper the quantum 2-sphere $S^2_q$ was described as a quantum complex manifold. Here we consider several copies of $S^2_q$ and derive their braiding commutation relations. The braiding is extended to the differential and to the integral calculus on the spheres. A quantum analogue of the classical anharmonic ratio of four points on the sphere is given, which is invariant under the coaction of $SU_q(2)$.

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1 Introduction

In a recent paper\cite{1} we considered the quantum sphere $S^2_q$ in the sense of Podleś \cite{2, 3, 4, 5} for the particular value $c = 0$ of his parameter $c$. We showed that the quantum sphere can be described in terms of stereographic variables $z$ and $\bar{z}$ and we developed the differential and integral calculus on $S^2_q$. The entire formalism is covariant under the coaction of the quantum group $SU_q(2)$ which is described by quantum fractional transformations on the variables $z$ and $\bar{z}$, see Eq.(3) below. The quantum sphere appears as a q-deformation of the classical sphere considered as a complex Kähler manifold. A $\ast$-conjugation antihomomorphism exists.

In this letter we consider the braiding of several copies of $S^2_q$. There exists a general formulation \cite{6} for obtaining the braiding of quantum spaces in terms of the universal R matrix of the quantum group which coacts on the quantum space. Using this formulation the braiding commutation relations are obtained in Section 3 directly for the variables $z$ and $\bar{z}$. (An alternative derivation of the same braiding relations proceeds by first computing the braiding of two copies of the complex quantum plane on which $SU_q(2)$ coacts and then using the expressions of the stereographic variables $z$ and $\bar{z}$ in terms of the coordinates $x, y$ of the quantum plane

$$z = xy^{-1}, \quad \bar{z} = \bar{y}^{-1}\bar{x}.$$ 

The braiding can be extended to the differentials $dz$ and $d\bar{z}$. In Section 4 the braiding property of the $SU_q(2)$ invariant integral on the sphere is given. It is shown that it can be used to compute the integral.

The braiding of two quantum spheres is not symmetric with respect to the exchange of the two spheres. It can be extended to the case of an arbitrary number of spheres given in a certain order. For the case of four spheres one can construct a quantum analogue of the classical anharmonic ratio (cross ratio) of four points on a sphere. This quantity, which belongs to the braided algebra of the four spheres, is invariant under the coaction of
as realized by the quantum fractional transformations on the stereographic variables. It commutes with its $*$-conjugate. This is described in Section 5. The existence of the invariant anharmonic ratio seems very remarkable.

The results of this letter and those of \[1\] can be extended to suitably defined quantum deformations of complex projective space and complex Grassmann manifolds. These appear therefore as examples of quantum complex Kähler manifolds. We are planning to describe all this in a forthcoming publication.

2 Braiding for Quantum Group Comodules

Let $\mathcal{A}$ be the algebra of functions on a quantum group and $\mathcal{V}$ an algebra on which $\mathcal{A}$ coacts on the left:

$$\Delta_L : \mathcal{V} \rightarrow \mathcal{A} \otimes \mathcal{V}$$
$$v \mapsto v^{(1')} \otimes v^{(2)},$$

where we have used the Sweedler-like notation for $\Delta_L(v)$.

Let $\mathcal{W}$ be another left $\mathcal{A}$-comodule algebra,

$$\Delta_L : \mathcal{W} \rightarrow \mathcal{A} \otimes \mathcal{W}$$
$$w \mapsto w^{(1')} \otimes w^{(2)}.$$

It is known\[6\] that one can put $\mathcal{V}$ and $\mathcal{W}$ into a single left $\mathcal{A}$-comodule algebra with the multiplication between elements of $\mathcal{V}$ and $\mathcal{W}$ given by

$$vw = \mathcal{R}(w^{(1')}, v^{(1')})w^{(2)}v^{(2)}. \quad (1)$$

Here $\mathcal{R} \in \mathcal{U} \otimes \mathcal{U}$ is the universal R matrix for the quantum enveloping algebra $\mathcal{U}$ dual to $\mathcal{A}$ (with respect to the pairing $\langle \cdot, \cdot \rangle$) and

$$\mathcal{R}(a, b) = \langle \mathcal{R}, a \otimes b \rangle.$$
It satisfies:

\[ \mathcal{R}(f g, h) = \mathcal{R}(f, h(1)) \mathcal{R}(g, h(2)), \quad (2) \]
\[ \mathcal{R}(f, gh) = \mathcal{R}(f(1), h) \mathcal{R}(f(2), g), \quad (3) \]
\[ \mathcal{R}(1, f) = \mathcal{R}(f, 1) = \epsilon(f). \quad (4) \]

For \( \mathcal{A} = SU_q(2) \), it is

\[ \mathcal{R}(T^i_j, T^k_l) = q^{-1/2} \hat{R}^{ki}_{jl}, \]

where \( \hat{R} \) is the \( GL_q(2) \) R-matrix. For example,

\[ \mathcal{R}(a, T) = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}, \quad \mathcal{R}(b, T) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \]
\[ \mathcal{R}(d, T) = \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}, \quad \mathcal{R}(c, T) = \begin{pmatrix} 0 & \lambda q^{-1/2} \\ 0 & 0 \end{pmatrix}, \]

where \( \lambda = q - q^{-1} \).

3 The Braided Sphere

We first recall that the complex sphere \( S^2_q \) is described by coordinates \( z, \bar{z} \) which obey the commutation relation

\[ z\bar{z} = q^{-2}\bar{z}z + q^{-2} - 1 \quad (5) \]

and the \( * \)-structure \( z^* = \bar{z} \). It is covariant under the fractional transformation, with \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU_q(2) \),

\[ z \rightarrow (az + b)(cz + d)^{-1}, \quad \bar{z} \rightarrow -(c - d\bar{z})(a - b\bar{z})^{-1}, \quad (6) \]

where \( a, b, c \) and \( d \) commute with \( z \) and \( \bar{z} \).
One can extend $SU_q(2)$ by introducing $a^{-1}, d^{-1}$ satisfying

$$aa^{-1} = a^{-1}a = 1, \quad dd^{-1} = d^{-1}d = 1,$$

$$\epsilon(a^{-1}) = \epsilon(d^{-1}) = 1,$$

$$a^{-1*} = d^{-1}, \quad d^{-1*} = a^{-1}$$

and coproduct

$$\Delta(d^{-1}) = (c \otimes b + d \otimes d)^{-1}\sum_{n=0}^{\infty}(-1)^n(d^{-1}c)^n d^{-1} \otimes (d^{-1}b)^n d^{-1}.$$

The transformation for $z$ and the braided copy $z'$ can then be written as

$$z \rightarrow \sum_{n=0}^{n=\infty} f_n z^n, \quad z' \rightarrow \sum_{n=0}^{n=\infty} f_n z'^n,$$

where

$$f_0 = bd^{-1},$$

$$f_n = (-1)^{n-1}d^{-2}(cd^{-1})^{n-1}, \quad n \geq 1$$

and Eq.(1) gives

$$zz' = \sum_{n,m=0}^{n=\infty} R(f_m, f_n) z'^n z^n. \quad (7)$$

To calculate $R(f_m, f_n)$, we notice that, for example, Eqs.(2)-(4) give

$$R(a^{-1}, T) = \left(\frac{q^{-1/2}}{q^{1/2}}\right), \quad R(T, a^{-1}) = \left(\frac{q^{-1/2}}{q^{1/2}}\right),$$

$$R(d^{-1}, T) = \left(\frac{q^{1/2}}{q^{-1/2}}\right), \quad R(T, d^{-1}) = \left(\frac{q^{1/2}}{q^{-1/2}}\right).$$

4
It is not hard to prove that for any functions \( f, g \) of \( a^{\pm 1}, b, c, d^{\pm 1} \),

\[
R(bf, g) = 0, \quad R(f, cg) = 0,
\]

\[
R(cf, g) = 0, \quad \text{if } g \text{ has no } b,
\]

\[
R(g, bf) = 0, \quad \text{if } g \text{ has no } c
\]

and

\[
R(d^{\pm 1}, f(a, b, c, d)) = f(q^{\mp 1/2}, 0, 0, q^{\pm 1/2}) \quad (8)
\]

\[
R(f(a, b, c, d), d^{\pm 1}) = f(q^{\mp 1/2}, 0, 0, q^{\pm 1/2}) \quad (9)
\]

together with Eqs.(8),(9) with \( d^{\pm 1} \) replaced by \( a^{\mp 1} \).

One then gets

\[
R(f_1, f_1) = q^2,
\]

\[
R(f_2, f_0) = -\lambda q,
\]

\[
R(f_m, f_n) = 0, \quad \text{all other } n, m.
\]

Therefore

\[
zz' = q^2 z' z - \lambda q z'^2. \quad (10)
\]

Similarly,

\[
\bar{z} \to \sum_{n=0}^{\infty} g_n \bar{z}^n
\]

\[
g_0 = -ca^{-1},
\]

\[
g_n = q^{-2(n-1)} a^{-2} \quad \text{for } n \geq 1
\]

and

\[
R(g_1, f_1) = q^{-2},
\]

\[
R(g_0, f_0) = -\lambda q^{-1},
\]

\[
R(g_m, f_n) = 0, \quad \text{all other } n, m.
\]
Therefore
\[ z\bar{z}' = \sum_{n,m=0}^{\infty} R(g_m, f_n)\bar{z}'^m z^n \] (11)
gives
\[ z\bar{z}' = q^{-2}\bar{z}' z - \lambda q^{-1}. \] (12)

The differential calculus can also be defined on the braided spheres by imposing the Leibniz rule on the exterior derivatives \( d \) and \( d' \) so that \( d' \) acts on \( z' \) and \( \bar{z}' \) in the same way \( d \) acts on \( z \) and \( \bar{z} \), and
\[ dz' = z'd, \quad d\bar{z}' = \bar{z}'d, \]
\[ d'z = zd', \quad d'\bar{z} = \bar{z}d' \]
together with
\[ dd' = -d'd. \]

Then Eqs.(11),(12) and their \( * \)-involution will imply commutation relations between functions and forms of different copies of the sphere. As a consequence, the area element of the second copy \( K' = dz'd\bar{z}'(1 + \bar{z}'z')^{-2} \) is central in the whole braided algebra, while \( K = dzd\bar{z}(1 + \bar{z}z)^{-2} \) is only central in the original copy \((z, \bar{z})\).

4 Remarks on the Integration

For symmetry with respect to the \( * \)-involution of the algebra the braiding order of \( z, \bar{z}, z', \bar{z}' \) has to be \( z < (z', \bar{z}') < \bar{z} \) after we have determined \( z < z' \) and \( z < \bar{z}' \) as in Eqs.(9) and (11). Because \( z' \) and \( \bar{z}' \) are always on the same side of the variables of their braided copy, \( z \) and \( \bar{z} \), the integration on \( z', \bar{z}' \), has the following property:

if
\[ f(z', \bar{z}')g(z, \bar{z}) = \sum_i g_i(z, \bar{z})f_i(z', \bar{z}') , \]
then

\[< f(z', \bar{z}') > g(z, \bar{z}) = \sum_i g_i(z, \bar{z}) < f_i(z', \bar{z}') >, \]  \tag{13}

where \(< \cdot >\) is the invariant integral on \(S^2_q\). This can be shown by using the invariance of the integral under the \(SU_q(2)\) coaction. However,

\[f(z', \bar{z}') < g(z, \bar{z}) > \neq \sum_i < g_i(z, \bar{z}) > f_i(z', \bar{z}').\]

The above property (13) can be used to derive explicit integral rules. For example, consider the case of \(f(z', \bar{z}') = \bar{z}' \rho'^{-n}\), where \(\rho' = 1 + \bar{z}' z'\) and \(g(z, \bar{z}) = z\). Since

\[
\bar{z}' \rho'^{-n} z = q^2 z \bar{z}' \rho'^{-n} + q^{1-2n} \lambda([n+1]_q - [n]_q \rho') \rho'^{-n}, \quad n \geq 0,
\]

where \([n]_q = \frac{q^{2n}-1}{q^2-1}\), using Eq.(13) and \(< \bar{z}' \rho'^{-n} >= 0\) we get the recursion relation:

\[
[n+1]_q < \rho'^{-n} >= [n]_q < \rho'^{-(n-1)}>, \quad n \geq 1. \tag{14}
\]

This adds one more to two other different ways of computing the invariant integrals. One way is to impose the invariance condition directly [1]:

\ [< \chi f > = 0,

for any generator \(\chi\) of the \(SU_q(2)\) quantum Lie algebra. The other way is to use the cyclic property\[\dagger\]

\ [< f(z, \bar{z}) g(z, \bar{z}) > = < g(z, \bar{z}) f(q^{-2} z, q^2 \bar{z}) > .

They all give the same recursion relation (14).

\[\dagger\] Similar cyclic properties have been found by H. Steinacker [7] for integrals over higher dimensional quantum spheres in quantum Euclidean space.
5 Anharmonic Ratios

Let us first review the classical case. The coordinates $x, y$ on a plane transform as

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(15)

by an $SU(2)$ matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The determinant-like object $xy' - yx'$ defined for $x, y$ together with the coordinates on a second plane $x', y'$ is invariant under the $SU(2)$ transformation. On each plane we define $z = x/y$ so that

$$z - z' = y^{-1}(xy' - yx')y'^{-1}.$$ 

It now follows that with $x_i, y_i$ for $i = 1, 2, 3, 4$ as coordinates on four copies of the planes,

$$(z_2 - z_1)(z_2 - z_4)^{-1}(z_3 - z_4)(z_3 - z_1)^{-1}$$

$$= (x_1y_2 - y_1x_2)(x_4y_2 - y_4x_2)^{-1}(x_4y_3 - y_4x_3)(x_1y_3 - y_1x_3)^{-1}$$

is invariant because all the factors $y_i^{-1}$ cancel and only the invariant parts $(x_iy_j - y_ix_j)$ survive. Therefore the anharmonic ratio is invariant.

Permuting the indices in the above expression we may get other anharmonic ratios, but they are all functions of the one above. For example,

$$(z_2 - z_3)(z_2 - z_4)^{-1}(z_1 - z_4)(z_3 - z_1)^{-1} = (z_2 - z_1)(z_2 - z_4)^{-1}(z_3 - z_4)(z_3 - z_1)^{-1} - 1.$$ 

The $SU_q(2)$ covariant quantum plane obeys

$$xy = qyx$$

which is covariant under the transformation (15) with $T$ now being an $SU_q(2)$ matrix. Braided quantum planes can be introduced by using Eq.(16). Let $V$ be the $i$-th copy and $W$ be the $j$-th one, then we have for $i < j$,

$$x_iy_j = qy_jx_i + q\lambda x_jy_i,$$
\[ x_i x_j = q^2 x_j x_i, \]
\[ y_i y_j = q^2 y_j y_i, \]
\[ y_i x_j = q x_j y_i. \]

In the deformed case we have to be more careful about the ordering. Let the deformed determinant-like object be

\[(ij) := x_i y_j - q y_i x_j,\]

which is invariant under the \(SU_q(2)\) transformation, and let

\[[ij] := z_i - z_j = q^{-1}y_i^{-1}(ij)y_j^{-1},\]

where \(z_i := x_i y_i^{-1}\).

Using the relations

\[ y_i (ij) = q(ij) y_i, \]
\[ (ij) y_j = q y_j (ij) \]

for \(i < j\) and

\[ y_i (jk) = q^3 (jk) y_i, \]
\[ (ij) y_k = q^3 y_k (ij) \]

for \(i < j < k\), we can see that, for example,

\[ A := [12][24]^{-1}[34][13]^{-1} \]

is again invariant. Similarly, \(B := [12][23]^{-1}[34][14]^{-1}\) as well as a number of others are invariant.

To find out whether these invariants are independent of one another, we now discuss the algebra of the \([ij]\)'s.

Because \([ij] = [ik] + [kj]\) and \([ij] = -[ji]\) the algebra of \([ij]\) for \(i, j = 1, 2, 3, 4\) is generated by only three elements \([12], [23], [34]\). It is easy to prove that

\[ [ij][kl] = q^2[kl][ij], \]
if \( i < j \leq k < l \).

It follows that we have

\[
[i,j][ik][jk] = q^4[jk][ik][ij]
\]

for \( i < j < k \), and

\[
[12][34] + [14][23] = [12][24] + [24][23].
\]

Using these relations we can check the dependency between the different anharmonic ratios. For example, let \( C := [13][23]^{-1}[24][14]^{-1} \), and \( D := [14][13]^{-1}[23][24]^{-1} \), both invariant, then

\[
B^{-1}AC = [14][34]^{-1}[23][24]^{-1}([34][23]^{-1}[24])[14]^{-1}
\]

\[
= [14][34]^{-1}[23][24]^{-1}([24][23]^{-1}[34])[14]^{-1}
\]

\[
= 1,
\]

where we used the relation \([34][23]^{-1}[24] = [24][23]^{-1}[34] \) which follows Eq. (16), and

\[
q^2B - D^{-1} = ([12][34][23]^{-1} - [24][23]^{-1}[13])[14]^{-1}
\]

\[
= ([12][34][23]^{-1} - [24][23]^{-1}([12] + [23]))[14]^{-1}
\]

\[
= ([12][34] - [12][24] - [24][23])[23]^{-1}[14]^{-1}
\]

\[
= (-[14][23])[23]^{-1}[14]^{-1}
\]

\[
= -1.
\]

In this manner it can be checked that all products of four terms \([ij], [kl], [mn]^{-1}, [pr]^{-1} \) in arbitrary order, which are invariant, are functions of only one invariant, say, \( A \). Namely, all invariants are related and just like in the classical case, there is only one independent anharmonic ratio. It can be checked that the anharmonic ratio commutes with all the \( \bar{z}_i \)'s and so commutes with its \(*\)-complex conjugate, which is also an invariant.
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