Hedging with Neural Networks

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Abstract

We study neural networks as nonparametric estimation tools for the hedging of options. To this end, we design a network, named \textit{HedgeNet}, that directly outputs a hedging strategy. This network is trained to minimise the hedging error instead of the pricing error. Applied to end-of-day and tick prices of S&P 500 and Euro Stoxx 50 options, the network is able to reduce the mean squared hedging error of the Black-Scholes benchmark significantly. We illustrate, however, that a similar benefit arises by simple linear regressions that incorporate the leverage effect. Finally, we show how a faulty training/test data split, possibly along with an additional ‘tagging’ of data, leads to a significant overestimation of the outperformance of neural networks.

Keywords: Benchmarking; Black-Scholes; Data leakage; Delta-vega hedging; Hedging error; Linear regression; Neural network; Statistical hedging

1 Introduction

Beginning with Hutchinson et al. [1994] and Malliaris and Salchenberger [1993], \textit{artificial neural networks} (ANNs) are being proposed as a nonparametric tool for the risk management of options. Since then about 150 papers have been published that apply ANNs to price and hedge options.\textsuperscript{1} We show that for the estimation of the optimal hedge ANNs do \textit{not} outperform simple linear regressions that use only standard option sensitivities. Despite their simplicity such linear regressions have not been suggested previously in the literature. Moreover, by means of an experiment, we argue that a lack of \textit{data hygiene}, in particular, faulty data cleaning procedures and violations of pseudo real-time, cause significant overconfidence about the performance of ANNs.

We study a specific and well defined risk management application, namely the variance reduction of the hedging error in daily options’ trading.\textsuperscript{2} More precisely, we consider a one-period model and imagine an operator who is short an option. To reduce the variance of her portfolio she is allowed to buy or sell the underlying. Today, the operator sells the option, say at price $C_0$. She is now allowed to buy $\delta$ shares of the underlying at price $S_0$ and $C_0 - \delta S_0$ units of the risk-free asset. Then today’s portfolio value equals $V_0 = 0$. Tomorrow, her portfolio has value

$$V_1^\delta = \delta S_1 + (1 + r_{\text{onr}} \Delta t)(C_0 - \delta S_0) - C_1,$$

where $S_1$ and $C_1$ denote tomorrow’s prices of the underlying and the option, respectively, $r_{\text{onr}}$ is the over-night rate at which the operator can borrow / lend money, and $\Delta t = 1/253$. The operator’ goal is to choose $\delta$ in such a way that the variance of tomorrow’s wealth, $\text{Var}[V_1^\delta]$ is minimised.

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\textsuperscript{1}We refer to the review Ruf and Wang [2020] for an overview of these papers.

\textsuperscript{2}We have in mind a financial entity, say on the ‘buy-side,’ that is short an option and tries to hedge its risk by trading the underlying. Such an entity might be interested in ‘selling volatility’ as a carry-trade and might not be bound by regulatory requirements, which would require it to provide a specific model as an interim step. The marking to market accounting convention requires the entity to have a a good understanding of the hedging error for short periods, even when considering long-dated options.
To make headway, since $\Delta t$ is small, we are allowed to approximate the variance by the expected squared mean.\(^3\) Then the operator’s objective is to minimise the mean squared hedging error (MSHE)

$$E \left[ (V_1^t)^2 \right] = E \left[ (\delta S_1 + (1 + r_{\text{ann}} \Delta t) (C_0 - \delta S_0) - C_1)^2 \right]. \quad (2)$$

Let us assume for the moment that the option is a European call. Then a standard and simple choice is using the practitioner’s Black-Scholes Delta (BS-Delta)

$$\delta_{\text{BS}} = N(d_1), \quad (3)$$

where $N$ denotes the cumulative normal distribution function and

$$d_1 = \frac{1}{\sigma_{\text{impl}} \sqrt{T}} \ln \left( \frac{S_0}{K} \right) + \left( r + \frac{1}{2} \sigma_{\text{impl}}^2 \right) \tau. \quad (4)$$

Here, $\tau = T \Delta t$ is the time-to-maturity in year fraction, $\sigma_{\text{impl}}$ the annualised implied volatility of the option, $K$ denotes the strike price, and $r$ the risk-free interest rate corresponding to the option’s maturity $T$. The operator would choose $\delta = \delta_{\text{BS}}$; if the option was a put then she would choose $\delta = \delta_{\text{BS}} - 1$ in line with put-call parity. Since the interest rate $r$ is negligible, we assume for the moment that it is zero. Then the BS-Delta can be written as a function of two variables, namely the moneyness $M = S_0 / K$ and the time-proportional implied volatility $\sigma_{\text{impl}} \sqrt{T}$. Thus, we get the functional representation

$$\delta_{\text{BS}} = f_{\text{BS}} (M, \sigma_{\text{impl}} \sqrt{T}).$$

It is now reasonable to study other functionals.\(^4\) We shall replace $f_{\text{BS}}$ by an ANN $f_{\text{NN}}$ with the two input features $M$ and $\sigma_{\text{impl}} \sqrt{T}$, trained to minimise the expression in (2). That corresponds to a nonparametric estimation of the optimal Delta that minimises the variance of the hedging error. We will provide more details on the implementation in Section 3.

To benchmark the hedging performance of the ANN, we introduce linear regression models that lead to hedging ratios that are linear in several option sensitivities. They are motivated by the leverage effect, which describes the negative correlation between an underlying’s price and its volatility.\(^5\) To illustrate how this matters, consider a call and assume it is hedged with the BS-Delta $\delta_{\text{BS}} > 0$. If now the underlying’s price goes up so do the call price and the hedging position. Due to the leverage effect, the underlying (implied) volatility tends to go down simultaneously, thus having a negative effect on the option price. Indeed, everything else equal, both call and put prices go up as (implied) volatility increases – their ‘Vega’ is positive. The BS-Delta $\delta_{\text{BS}}$ does not take into consideration this additional effect. As we only allow hedging with the underlying the obvious change is to hedge only partially, i.e., use the hedging ratio $\delta_{LR} = a \delta_{BS}$, where $a$ is estimated (in a training set). Here, LR stands for linear regression. For the moment it suffices to note that these arguments let us expect $a > 1$ for puts and $a < 1$ for calls.\(^6\) We shall discuss such simple modifications of the BS-Delta in Section 4, all based on statistical hedging models involving various option sensitivities.

The performance of the ANN and the benchmarks is tested on several different datasets: (a) data simulated from the standard Black-Scholes model; (b) data simulated from Heston’s model; (c) daily end-of-day mid-prices obtained from OptionMetrics; (d) tick data provided by Deutsche Börse. These data are described in more detail in Section 2. We also vary the length $\Delta t$ of the hedging period from 1 hour to 2 days. All in all, the ANN performs well in terms of MSHE relative to the BS-Delta, even when the latter is being used with contract-specific implied volatility. However, using the linear regression hedging ratios $\delta_{LR}$ performs roughly as well or at times better than $\delta_{NN}$. For a summary of the results, see Section 5.

An interpretation of these observations is that the option sensitivities already encapsulate all relevant nonlinearities in the data necessary for the hedging task. Hence, the ANN seems to be able to learn the leverage effect, but cannot not improve on a simple linear regression involving the relevant option sensitivities. What have we learned? Initially we were satisfied about the outperformance of the ANN relative to the BS-Delta on real datasets. When investigating what the ANN is learning, the linear regression models appeared as a natural competitor. These statistical models are extremely simple – for the easiest such model one only replaces the BS-Delta by a multiple of it.

\(^3\) If the expected return on the risk-free asset happens to be equal to the risk-free return then the expected value $E[V_1^t]$ does not depend on $\delta$ at all.

\(^4\) Indeed, even when data are simulated from the Black-Scholes model and continuous hedging is not allowed, using the BS-Delta is not optimal; see Subsection 4.1.

\(^5\) This observation is credited to Black [1976].

\(^6\) It turns out that hedging with $a \delta_{BS}$, where $a = 0.9$ for calls and $a = 1.1$ for puts works extremely well on real datasets; see Subsection 5.5.
Nevertheless, as far as we know, these models have not been used in the literature to benchmark more complicated models.

Last but not least, Section 6 discusses the issue of data leakage when estimating good hedging ratios. We argue that it is very hard to completely avoid data leakage when working with real datasets. Nevertheless, it is relatively easy to reduce data leakage far more than often done in the literature. We then quantify with simulation and real datasets the significant bias in the estimation of the MSHE when data are not properly split into in-sample out-of-sample sets, and additionally certain days are accidentally ‘tagged.’

We proceed as follows. Section 2 describes the datasets and the experimental setup. Section 3 introduces the HedgeNet architecture and implementation. This section also discusses the advantage of outputting directly the hedging ratio instead of option prices and then using a sensitivity as hedging ratio. Section 4 describes how the leverage effect motivates various benchmark models to be compared with ANNs. Section 5 presents the experimental results. Section 6 reflects on the potential data leakage from two sources, either from disregarding the data’s time series structure or from cleaning the datasets. Section 7 summarises the main findings. Several appendices provide further details on the various sections.

2 Datasets, data preparation, and setup of experiments

This section presents the data used. Subsection 2.1 explains the data-generating mechanism for the simulated data. Subsections 2.2 and 2.3 describe the two real datasets containing options on the S&P 500 and Euro Stoxx 50. Subsection 2.4 concludes by discussing the experimental setup. Appendix A contains additional details on the datasets.

2.1 Simulated data: Black-Scholes and Heston

For the simulation study two data-generating mechanisms are considered. In the first one, the underlying’s price process is simulated from the Black-Scholes stochastic integral equation

\[
S_t = 2000 + \mu \int_0^t S_u du + \sigma \int_0^t S_u dW_u,
\]

with annualised rate of return \(\mu = 0.1\), and annualised volatility \(\sigma = 0.2\). In the second example the underlying’s price process is simulated from the Heston [1993] model given by

\[
S_t = 2000 + \int_0^t \sqrt{V_u} S_u dW_u; \\
V_t = V_0 + \kappa \int_0^t (\theta - V_u) du + \sigma \sqrt{V_u} \int_0^t \sqrt{V_u} d\tilde{W}_u; \\
\text{Cov}(W_t, \tilde{W}_t) = \rho t,
\]

with initial and long-term variance \(V_0 = \theta = 0.04\), rate of mean reversion \(\kappa = 5\), volatility of variance \(\sigma = 0.3\), and correlation \(\rho = -0.8\). Here the volatility \(\sqrt{V_t}\) of the underlying is stochastic and modelled as the square root of a process mean-reverting to 0.04. Thanks to Feller’s test of explosions, the volatility is always strictly positive. We intentionally omit the drift to focus on the role that stochastic volatility plays.

We first simulate 1.25 years of the underlying’s price from the Black-Scholes and Heston model, respectively. Then, along the simulated spot path, options are created following the Chicago Board Options Exchange (CBOE) rules. We price the options on each trading day using the Black-Scholes formula and the standard pricing formulas available for Heston, respectively; see, for example, Albrecher et al. [2007]. Here, we set the dividend

7Details of these rules are provided on [http://www.cboe.com/products/stock-index-options-spx-rut-msci-ftse/s-p-500-index-options/s-p-500-options-with-a-m-settlement-spx/spx-options-specs](http://www.cboe.com/products/stock-index-options-spx-rut-msci-ftse/s-p-500-index-options/s-p-500-options-with-a-m-settlement-spx/spx-options-specs). The idea is the following. The option expiration date is always the fourth Friday of its expiration month. The expiration months are the 12 immediate calendar months, plus some additional long-term months (we do not generate options for those long-term months). At each expiration date new options are created, so that the market still trades options with 12 expiration months. In general, the strike price step is set to 5 dollars. The two strike prices closest to the current underlying’s price are initially listed. If the underlying’s price is close to any one of the two strikes, a third strike will be included to cover the larger range. New series are generally added when the underlying’s price trades through the highest or lowest available strike price for each expiration.

8In the Heston case, we fix this pricing measure, under which \(\tilde{W}\) is also Brownian motion.
and interest rate to zero. These 1.25 years of simulated data correspond to the in-sample data (training and validation), on which the benchmarks and ANNs are trained. To estimate the MSHE, more data are simulated; however those data are only used to estimate the out-of-sample performance of the different statistical models.  

After computing the option prices and the sensitivities necessary for the statistical models, the data are organised in a table so that each row corresponds to exactly one observation, i.e., one option at one trading day (along with tomorrow’s price for training). Finally, samples with option price less than 0.01 (the tick size) or moneyness \( M \) outside of the interval \([0.8, 1.5]\) are removed. This means that if an option has a time-to-maturity of 90 trading days, it might appear, for example, 85 times in the dataset. The option might have a moneyness outside of the interval or a too small price for the other four trading days.

2.2 S&P 500 end-of-day midprices

We obtained daily closing bid and ask prices on calls and puts written on the S&P 500 between January 2010 and June 2019 from OptionMetrics\(^{10}\). We interpreted the midprice as the true market price. Figure 1 displays a sample of the obtained options, namely those puts with price quotes in the first three months of 2010 or 2015. Sensitivities are provided for the majority of options and are filled in for missing values.\(^{1112}\)

![Figure 1: A sample of the obtained put options along with the underlying’s (S&P 500) price process in blue. Only options that have a trading volume of more than 1000 on some trading day are included. Each red (black) line segment represents a put option that had price quotes within the first quarter of 2010 (2015). The corresponding strike is indicated as the value on the \(y\)-axis. Small random vertical shifts are added to increase the visibility of the options.](image)

We again arrange the data so that each row corresponds to exactly one option at one day. In the cleaning process, we remove the following samples:

- Samples with negative time value.
- Samples with time-to-maturity less than 1 day.
- Samples where the moneyness is outside the interval \([0.8, 1.5]\).
- Samples with an implied volatility higher than 100% or smaller than 1%.
- Samples with zero trading volume.

\(^{9}\)Choosing a time length of 1.25 years is done for the following reason. As explained below, when training the ANNs for the real datasets, we split the data up in training, validation, and out-of-sample (test) data using the ratio 4:1:1. For the simulated datasets we keep this ratio and choose the training set to be one year long. This then yields 1.25 years of training and validation data. Simulating options according to the CBOE rules yields roughly the same magnitude of training data as available in each time window of the real datasets.

\(^{10}\)See [https://optionmetrics.com/](https://optionmetrics.com/).

\(^{11}\)The required interest rates were interpolated from the rates provided by OptionMetrics. For maturities less than one week (in which case OptionMetrics does not provide the corresponding rates), we used the Overnight Libor Rates from Bloomberg.

\(^{12}\)The results presented below are robust to whether we use computed sensitivities for all options or the sensitivities provided by OptionMetrics where available.
• Samples where the ask is at least twice the bid.
• Samples with bid less than \(0.05\).
• Samples that do not have available next trade prices.

2.3 Euro Stoxx 50 tick data

We are grateful to Deutsche Börse, who provided us with tick data\(^\text{13}\) of Euro Stoxx 50 index options and futures between January 2016 and July 2018.

We now briefly outline how we process these data. If several trades are executed at exactly the same time stamp we aggregate these orders and consider the volume-weighted average price. We match each option transaction with the most recent tick price of the future with the shortest maturity (again, volume-weighted if several trades happen simultaneously). These futures, which are the most liquid ones, shall be used to hedge the option position. The computation of the option sensitivities requires a risk-free rate. We use interpolated Euro LIBOR rates from Thomson Reuters’DataStream.

To train and measure the hedging performance we require the option price after \(\Delta t\) (1 hour, 1 day, 2 days, etc.). There might not be a trade exactly after this time period. Hence we allow a matching tolerance window of 6 minutes, equivalent to 0.1 hours. Hence, for example, if \(\Delta t\) is a business day and we have a trade on Monday, say at 2.12pm, then we match it with the first price observation of this option on Tuesday after 2.12pm. If there is no transaction before 2:18 pm, this sample gets discarded. (We refer to Subsection 6.2 for a discussion of potential data leakage introduced in this step.)

In the cleaning process, the following samples are removed:
• Samples with negative time value.
• Samples with time-to-maturity less than 1 day.
• Samples where the moneyness is outside the interval \([0.8, 1.5]\).
• Samples with an implied volatility higher than 100% or smaller than 1%.
• Samples on expiry dates of a future.
• Samples that cannot be matched to a next trade (within the matching tolerance window of 6 minutes).
• Samples that are traded in the first or last half an hour of each trading day.

2.4 Data preparation and experimental setup

As discussed in the introduction, our goal is to determine the hedging ratio \(\delta\) as a function of observable quantities to minimise the variance over one period of the hedged portfolio

\[ V_1^d = \delta S_1 + (1 + r_{\text{on}} \Delta t)(C_0 - \delta S_0) - C_1. \] (5)

Here \(S_0\) and \(S_1\) denote the prices of the hedging instrument at the beginning and end of the period and \(C_0\) and \(C_1\) denote the prices of the call or put. We study how well an ANN performs in this task on simulated data (Black-Scholes and Heston – see Subsection 2.1), on end-of-day midprices (see Subsection 2.2), and on tick data (see Subsection 2.3). We benchmark these results with linear regression models for the hedging ratio \(\delta\).

Each of the datasets is split up into in-sample and out-of-sample (‘test’) data. Both the ANN and the benchmark models are trained to (estimated by) the in-sample dataset only. The variance of the hedged portfolio is approximated by the MSHE. The performance of each of the methods is measured on the out-of-sample dataset as follows:

\[ \text{Var}(V_1^d) \approx \text{MSHE} = \frac{1}{N_{\text{test}}} \sum_{i,j}^{N_{\text{test}}} \left(100 \frac{V_{i+1,j}^d}{S_i} \right)^2, \] (6)

where \(\delta\) is either modelled by an ANN or by a linear regression. Both the indexing and the normalisation by \(S_t/100\) need explanation.

\(^{13}\)See https://datashop.deutsche-boerse.com/samples-dbag/File_Description_Eurex_Tick.pdf for a description of the data.
First of all, the indexing has changed from (5) to (6). Indeed, each traded option yields a series of samples, one for each trading period. Moreover, several options corresponding to different strikes (indexed by \( j \)) are being priced in any specific period (e.g., a day). To emphasise this point, the samples are double indexed in (6). Next, (6) normalises the value of the hedging portfolio by dividing it by \( S_t/100 \). This normalisation ‘removes the units’ and allows to compare errors across the different datasets, and arguably more importantly, across time. Equivalently, at any point of time \( t \), instead of replicating a full option we replicate the fraction \( 100/S_t \) of this option.

We now provide more details on how we prepare each dataset. First we store each dataset in a dataframe as in Table 1. We then remove all in-the-money samples. That is, if at one specific date an option was in the money, we discard this specific date for the corresponding option.

| Index | Date      | Features        | Additional information | Target |
|-------|-----------|------------------|------------------------|--------|
|       |           | \( \sigma_{\text{impl}} \sqrt{T} \) | \( M \) | \( \delta_{\text{BS}} \) | \( V_{\text{BS}} \) | \( S_0 \) | \( S_1 \) | \( C_0 \) | \( r_{\text{out}} \) | CP flag | \( C_1 \) |
| 0     | 2018/07/02| 0.047            | 1.003                 | 0.531  | 9.357 | 100    | 98.223  | 2.002  | 1.0      | 0       | 1.130  |
|       |           |                  |                        |        |       |        |         |        |          |         |        |

Table 1: This table presents a (simplified) preview of one of the four processed datasets. The ‘Features’ columns are used as inputs for the ANN and the linear regressions. The labels \( \sigma_{\text{impl}} \sqrt{T} \) and \( M \) denote the time-proportional implied volatility and moneyness of the option. The labels \( \delta_{\text{BS}} \) and \( V_{\text{BS}} \) are the BS-Delta and Vega. The CP flag indicates whether the corresponding option is a call or a put. Prices and sensitivities are all normalised.

The two real datasets are broken up in several overlapping time windows in order to understand whether the comparisons between the ANN and the linear regressions are consistent across time. The S&P 500 dataset consists of 14 overlapping time windows of length 3 years. The Euro Stoxx 50 dataset consists of 5 overlapping time windows of length 1.5 years. The first 900 (450) days form the in-sample set, the last 180 (90) days are used for the out-of-sample set, yielding a ratio 5:1. For the training of the ANN, the 900 (450) days are furthermore split into 720 (360) days of training and 180 (90) days of validation yielding a ratio 4:1:1. We roll the time windows forward by 180 (90) days, so that sample appears maximally once in an out-of-sample set.

The Black-Scholes and Heston datasets consist both of a single time window of 1.5 years. The first 450 days form the in-sample set. For the ANN, the 450 days are furthermore split into 360 (training) and 90 (validation) days. To get a more precise estimate of the MSHE, twenty out-of-sample sets of 90 days are simulated, as illustrated in Figure 2.

Figure 2: The single simulated price path on which options are created for the in-sample set, and the multiple paths on which options are created for the out-of-sample sets.
3 HedgeNet

We now construct an ANN that maps an option’s relevant features (e.g., moneyness and time-proportional implied volatility) to a hedging ratio $\delta_{NN}$. In Subsection 3.1 we provide details about the architecture, implementation, and training of such an ANN. Subsection 3.2 provides some additional motivation why the ANN is designed to output directly the hedging ratio instead of the option price.

3.1 Architecture of HedgeNet, its implementation and training

An ANN is a composition of simple elements called neurons, which maps input features to outputs. Such an ANN then forms a directed, weighted graph.

As we shall discuss below in Subsection 3.2 it is not satisfactory to compute or estimate option prices and then use their sensitivities as hedging ratios. It is better to obtain the hedging ratio, our quantity of interest, directly. Hence, we desire that the ANN returns a hedging ratio and not a price. However, when training such an ANN what should it be trained to? Optimal hedging ratios are not provided in the data. For this reason, we design an ANN, named HedgeNet, to have two parts, as illustrated in Figure 3.

The first part, a multilayer fully-connected feed-forward neural network (FCNN), transforms features into a hedging position, which is then turned by the second part into the replication value $\hat{C}_1 = V_1 + C_1$. This output of HedgeNet can then be trained to the observed option prices $C_1$ at the end of each period by minimising the sum of squared differences.

The FCNN has two hidden layers with 30 nodes each, connected by ReLU activation. The output of the FCNN is provided by a linear node (with truncation at zero and one) and corresponds to the hedging ratio $\delta_{NN}$. As illustrated in Figure 4, the non-trainable transformation module turns the hedging ratio $\delta_{NN}$ into the replication value $C_1$ by following (1). As the data includes both puts and calls, this module also requires an option type flag, which is set to 1 in the case of a put and to 0 in the case of a call. If the sample is a put, the module replaces $\delta_{NN}$ by $\delta_{NN} - 1$ in line with put-call parity. The non-trainable transformation module consists of a series of affine transformations, and hence does not affect the universal approximation property, discussed for example in Yarotsky [2017].

All numerical experiments are run on a standard desktop with GPU accelerated computation. We use Python as programming language. The ANN is implemented with the deep learning framework Tensorflow along with Keras. The inputs to the trainable part of HedgeNet are standardised. The weights of the ANN are initialised via the ‘Xavier’ initialiser (Glorot and Bengio [2010]) and the ‘Adam’ optimiser (Kingma and Ba [2015]) is applied for training the ANN. Appendix B contains details on the choice of additional hyperparameters.

For each dataset we consider two different feature sets for the trainable part of HedgeNet:

- ANN$(M; \sigma_{\text{impl}} \sqrt{\tau})$: The first one is already indicated in Figure 4. It uses moneyness $M$, time-proportional implied volatility $\sigma_{\text{impl}} \sqrt{\tau}$, and a flag to indicate whether the option is a call or a put. It is worth pointing

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14 The benefits of using ReLU activation are addressed in Glorot et al. [2011] and Section 3.1 of Krizhevsky et al. [2012].
15 We tried different architectures, for example 100 nodes in each hidden layer, or three (instead of two) hidden layers with 30 nodes each. Motivated by the representation of the BS-Delta in (3), we also tried the cumulative distribution function $N$ of a standard normally distributed random variable as output function instead of the linear output function. None of these modifications changed the overall conclusions below. We also tried a modification, where we interpret the output not as the hedging ratio but as the ‘bias’ term $\delta - \delta_{BS}$, which corrects the BS-Delta. Such change did not help the performance of the ANN – a similar observation as in Chen and Sutcliffe [2012].
16 Specification: GTX 1060 6GB GPU.
out that using moneyness instead of the underlying’s price and the strike price separately offers a better generalisation performance. The most important reason for its better performance is that moneyness resembles more a stationary feature compared to the underlying’s price and strike price separately. Indeed, options are created and traded only for a certain range of moneyness values. Ghysels et al. [1998], Garcia and Gençay [2000], and Ruf and Wang [2020] provide more comments on the advantage of using moneyness. The choice of time-proportional implied volatility is motivated by the fact that volatility squares with the square root of time; see also the expression for $\delta_{BS}$ in (3)&(4).

• ANN($\Delta_{BS}$; $V_{BS}$; $\tau$): Motivated by the leverage effect discussed in Section 4 below, we also consider a second set of features consisting of $\delta_{BS}$, $V_{BS}$, $1/\sqrt{\tau}$, and the put-call flag. Here $V_{BS}$ denotes Vega, the sensitivity of the option price with respect to the implied volatility.

3.2 Digression: Why outputting the hedging ratio instead of computing price sensitivities?

Most ANNs constructed in the literature for the risk management of options first learn the pricing function. Then in a second step hedging strategy is computed as the sensitivity of the option price with respect to the underlying’s price; see Ruf and Wang [2020] for an overview of the literature. In contrast, HedgeNet allows to predict the hedging position directly. In this way, the hedging strategy is no longer interpreted as a sensitivity.

From a risk-management point of view the hedging ratio is the main quantity of interest. It has been recommended, see for example Bengio [1997] or Claeskens and Hjort [2003], to estimate relevant quantities directly. This is in line with the important observation made in Lyons [1995] that different models might yield similar option prices but completely different hedging strategies. Obtaining directly the hedging ratio also avoids the otherwise necessary step to differentiate, possibly numerically, the trained option prices.

There is at least one more important advantage of outputting directly the hedging ratio. Computing sensitivities usually does not take into consideration that other model parameters also might change, in line with the underlying. Hence, such sensitivities tend to be not optimal for reducing the MSHE. Theoretical results supporting this observation are ample; see for example Denkl et al. [2013]. This discussion is continued in Subsection 4.2 below.

At this point, let us also mention a different approach to use ANNs in the context of option pricing, namely as computational tools to replace expensive PDE solvers or Monte-Carlo simulations. Indeed the risk management
of ‘sell-side institutions’ is subject to regulatory purposes. In particular, their options’ hedging is supposed to be derived from specific parametric models. ANNs are used to estimate (‘calibrate’) these model parameters. For references using this approach, see Ruf and Wang [2020]. Here, however, we do not intent to study the question how well models can be calibrated by the use of ANNs. Instead, we show the limitations and benefits of ANNs for estimating the optimal hedging ratio when not being restricted by a specific parametric model.

4 Linear regression models as benchmarks

We now discuss how we benchmark the hedging performance of the ANN. Although not very reasonable, one benchmark could be not hedging at all, i.e., \( \delta = 0 \). In this case the variance of the hedging error is just the variance of the change in the option price. More reasonable is to use the BS-Delta, obtained from the Black-Scholes formula, as discussed in Subsection 4.1. Subsections 4.2 and 4.3 introduce some further simple statistical hedging models.

4.1 Black-Scholes benchmark

Hedging via the BS-Delta is a standard benchmark. That is, for each option and for each date the corresponding implied volatility is used to obtain the hedge in (3), namely the partial derivative of the Black-Scholes option price with respect to the price of the underlying. Black-Scholes performs best if implied volatility is plugged in. In the literature, other volatilities, such as historical volatility estimates or GARCH predicted volatilities have been used. We refer to Ruf and Wang [2020] for an overview.

Since here we hedge only discretely, using the BS-Delta leads to an error even if the data are simulated from the Black-Scholes model. The performance of discrete-time hedging has been extensively studied; some pointers to the literature include Boyle and Emanuel [1980], Bertsimas et al. [2000], and Tankov and Voltchkova [2009], who provide an asymptotic analysis of hedging errors.

4.2 Delta hedging other sensitivities

The leverage effect, first discussed in Black [1976], describes the negative correlation of observed returns and their volatilities in equity markets. This effect has been confirmed in many follow-up studies which also consider implied volatilities. For example, Cont and Da Fonseca [2002] claim that the leverage effect is due to the general level of the implied volatility surface and not due to relative movements, that is, changes in the shape of the implied volatility surface. The non-zero correlation of returns and the implied option volatilities indicates that the BS-Delta can usually be outperformed by some relatively simple adjustments.

In this spirit, Vähämaa [2004] and Crépey [2004] use the observed smile in option implied volatilities to improve on the hedging performance of the BS-Delta. These ideas are developed further in several papers; see for example, Alexander et al. [2012].

The central idea is to note that a first-order Taylor series expansion of option prices yields

\[
dC \approx \delta_{BS} dS + \nu_{BS} d\sigma_{impl} = \delta_{BS} dS + \nu_{BS} \frac{d\sigma_{impl}}{dS} dS + \nu_{BS} dS \perp,
\]

where \( S \perp \) is orthogonal to \( S \). In words, the change in the option price is approximately the BS-Delta times the change in the underlying’s price plus Vega times the change in the implied volatility. The second term can be written in terms of changes in the underlying’s price and changes in the implied volatility that are uncorrelated with the changes in the underlying’s price. These observations lead us to consider a statistical model of the form:

\[
\delta = a \delta_{BS} + b \nu_{BS}.
\]

This statistical model replaces the BS-Delta by a multiple \( a \) of it plus a multiple \( b \) of Vega \( \nu_{BS} \). Here, \( a \) and \( b \) are estimated in the in-sample set, separately for puts and calls.\(^{17}\)

Next, a Taylor series expansion of the BS-Delta yields

\[
d\delta \approx \Gamma_{BS} dS + \nu_{BS} d\sigma_{impl}.
\]

\(^{17}\)Estimating \( a \) and \( b \) is equivalent to running a linear regression with two independent variables and no intercept on the in-sample set. Indeed, we minimise the expression in (6), where each summand can be written as the square of

\[
a \left( \delta_{BS, t, j} x_t \right) + b \left( \nu_{BS, t, j} x_t \right) - y_{t, j},
\]

where \( x_t = 100(S_{t+1}/S_t - (1 + r_{norm} \Delta t)) \) and \( y_{t, j} = 100/S_t (C_{t+1, j} - (1 + r_{norm} \Delta t) C_{t, j}) \).
Here, $\Gamma_{BS}$ denotes *Gamma*, namely the sensitivity of the BS-Delta to changes in the underlying’s price; $\text{Va}_{BS}$ denotes *Vanna*, namely the sensitivity of the BS-Delta to changes in the implied volatility.

Combining these two expansions we obtain the linear regression model

$$
\delta_{LR} = a \delta_{BS} + b \text{ Va}_{BS} + c \text{ Va}_{BS} + d \Gamma_{BS}. 
$$

(7)

Again, $a, b, c, d$ are estimated for puts and calls separately on each in-sample set. We also consider nested models; in this case, we force either $a$ to be one or one (or more) of the other coefficients to be zero and estimate the remaining coefficients. The Vega and Gamma sensitivities are large for options when the strike is close to the underlying’s current price. Thus, including these sensitivities allow the statistical model to make adjustments to the hedging ratio depending on whether an option is at-the-money or out-of-the-money. Using both two sensitivities helps, moreover, to make additional adjustments depending on the option’s time-to-maturity. Finally, Vanna for an out-of-the money option is largest when the option is somehow out-of-the-money but not too much. This allows the model to make the corresponding additional adjustments. We have also experimented with an additional intercept term in (7). Including it does not change the conclusions below; we hence only report the results without this additional term.

Furthermore, we include below the proposed hedging ratio of Hull and White [2017], given by

$$
\delta_{HW} = \delta_{BS} + \frac{\text{ Va}_{BS}}{\sqrt{\tau S}} (a + b \delta_{BS} + c \delta_{BS}^2).
$$

(8)

Here, $\tau$ is the time-to-maturity and $a, b, c$ are again estimated for puts and calls separately on each in-sample set. Hull and White [2017] obtain this model from a careful analysis of S&P 500 options and observe its excellent hedging performance on options written on the S&P 500 and other indices. We furthermore include a ‘Relaxed Hull-White’ model, where the coefficient in front of $\delta$ is not restricted to one.

The models in (7) and (8) should be considered ‘statistical’ in contrast to ‘model-driven’ as the hedging ratio is derived purely from statistical considerations instead of being derived from stochastic models. In the language of Carr and Wu [2019], these models are ‘local’ and ‘decentralised,’ as only one period is considered instead of the option’s whole time horizon, and as each option contract is treated separately instead of finding an overall consistent valuation model. To the best of our knowledge, the model in (7) not been suggested in the literature before, despite its simplicity.18

4.3 Possible other benchmarks

One could consider hedging ratios derived from parametric models such as stochastic volatility models. Bakshi et al. [1997] observe that such models outperform the BS-Delta in the case of hedging out-of-the money options, but not necessarily in-the-money options. Viähämäa [2004] provides additional references that test the hedging performance of stochastic volatility models and concludes with the observation that “such models do not necessarily provide better hedging performance.” Hull and White [2017] note that the hedging ratio $\delta_{HW}$ of (8) leads to a better performance than stochastic volatility models.

We initially also investigated the following two (semi-)linear benchmarks:

$$
\bar{\delta}_1 = aM + b\sigma_{\text{impl}}\sqrt{\tau} + c; \quad \bar{\delta}_2 = N(aM + b\sigma_{\text{impl}}\sqrt{\tau} + c),
$$

where $M$ denotes moneyness, $\sigma_{\text{impl}}\sqrt{\tau}$ time-proportional implied volatility, and $N$ the cumulative normal distribution function. Here, the parameters $a, b, c$ were estimated again in each in-sample set. It turns out that these two linear regressions perform far worse than the BS-Delta $\delta_{BS}$; hence we will not present results on these two benchmarks. The underperformance of these two linear regressions also shows that the performance of the ANN is not entirely due to the hand-crafted features.

5 Results

We now present the results on the performance of the various statistical hedging models in terms of MSHE reduction. As a quick summary, the hedging ratios of the ANNs do not outperform the linear regression models. On the S&P 500 dataset, the Hull-White and Delta-Vega-Vanna regressions tend to perform the best, with Hull-White

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18In the context credit risk, Cont and Kan [2011] also provide a careful study of regression-based hedging. While here the hedging ratio is regressed on option sensitivities, they regress changes in the option price on changes in the underlying.
better on the one-day hedging period, and the Delta-Vega-Vanna regression better on the two-day period. On the Euro Stoxx 50 dataset, the Vega-only regression tends to perform the best.

In the next four subsections we discuss each of the datasets. We start with the real datasets (Subsections 5.1 and 5.2) and then briefly summarise the results on the simulated data (Subsections 5.3 and 5.4). In Subsection 5.5, we conclude with some general observations on these experiments.

Recall from Subsection 2.4 that each data sample is normalised so that the underlying’s price $S_0$ at time 0 is 100. This allows to compare the absolute hedging errors across different datasets. Recall also that we only consider out-of-the (and at-the)-money puts and calls.

5.1 S&P 500 end-of-day midprices

Table 2 gives an overview of the MSHEs across different hedging periods. The first two rows give the MSHEs for the zero hedge and the BS-Delta. The remaining rows give the relative improvement over the BS-Delta, i.e.,

$$\frac{\text{MSHE}(\delta_s) - \text{MSHE}(\delta_{BS})}{\text{MSHE}(\delta_{BS})},$$

(9)

All competing methods outperform the BS-Delta. Among them, the Delta-Vega-Vanna and (relaxed) Hull-White regressions perform the best, with Hull-White doing slightly better on the one-day hedging period while Delta-Vega-Vanna performing better on two-day hedging period. Indeed, Hull and White [2017] study the same dataset to create the Hull-White regression, so it is surprising how close the other regressions get. The major improvement in the regressions (apart from the Hull-White regression) comes from allowing the coefficient in front of Delta to be estimated, rather than equal to one. Regressions with the second-order sensitivities on its own (i.e., with the Delta coefficient fixed to one as in Hull-White) are not performing as well, and we have omitted them from Table 2. The two ANNs perform similarly to the regressions in case of the one-day period, but underperform for the two-day period.

| Regressions | Calls 1 day | Puts 1 day | Both 1 day | Calls 2 days | Puts 2 days | Both 2 days |
|-------------|-------------|------------|------------|-------------|------------|------------|
| Zero hedge  | -21.3       | -14.8      | -16.9      | -16.3       | -12.8      | -13.9      |
| BS-Delta    | -13.7       | -11.7      | -12.3      | -10.4       | -10.1      | -10.2      |
| Delta-only  | -15.5       | -10.1      | -11.8      | -14.5       | -11.2      | -12.2      |
| Vega-only   | -12.4       | -12.6      | -12.5      | -10.6       | -13.0      | -12.2      |
| Gamma-only  | -21.6       | -14.8      | -17.0      | -17.1       | -13.1      | -14.4      |
| Vanna-only  | -21.4       | -14.9      | -17.0      | -16.4       | -12.8      | -13.9      |
| Delta-Gamma | -22.6       | -16.6      | -18.5      | -17.7       | -15.4      | -16.1      |
| Delta-Vega  | -21.5       | -14.8      | -17.0      | -16.8       | -13.5      | -14.5      |
| Delta-Vanna | -23.0       | -16.6      | -18.7      | -18.1       | -15.4      | -16.2      |
| Delta-Gamma-Vanna | -22.6 | -16.6 | -18.5 | -17.7 | -15.2 | -16.0 |
| Delta-Vega-Gamma-Vanna | -22.9 | -16.4 | -18.5 | -17.4 | -14.9 | -15.7 |
| Hull-White  | -23.1       | -16.9      | -18.9      | -17.8       | -14.5      | -15.5      |
| Relaxed Hull-White | -23.2 | -16.9 | -18.9 | -18.3 | -14.6 | -15.8 |

| ANNs | Calls 1 day | Puts 1 day | Both 1 day | Calls 2 days | Puts 2 days | Both 2 days |
|------|-------------|------------|------------|-------------|------------|------------|
| $M; \sigma_{\text{impl}} \sqrt{\tau}$ | -22.3       | -15.6      | -17.7      | -17.1       | -10.9      | -12.8      |
| $\Delta_{BS}; \nu_{BS}; \tau$ | -23.4       | -16.9      | -18.9      | -17.1       | -11.9      | -13.5      |

Table 2: Performance of the linear regressions and ANNs on the S&P 500 dataset. The hedging periods $\Delta t$ are here either one day or two days. The columns ‘Both’ are the weighted average of the ‘Puts’ and ‘Calls’ columns. The row ‘Zero hedge’ corresponds to the MSHE when $\delta = 0$ is chosen; i.e., the mean squared changes in the option prices. The values in the top two rows are multiplied by 100 to improve readability. The regression and ANN rows correspond to the various statistical models including HedgeNet with two different feature sets. For these two sets of rows, the numbers are reported as relative improvements in MSHE over using the BS-Delta, i.e., (9). Numbers in bold represent the largest outperformance (in each column the best one is chosen along with the ones that are within 1% of the best).
Table 2 indicates that it is easier to outperform the BS-Delta when hedging out-of-the money calls than out-of-the money puts. However, note that the BS-Delta itself reduces the MSHE more for puts than for calls when using the zero hedge as baseline. To see this, let us have a closer look at the one-day period. For calls, hedging with the BS-Delta reduces the MSHE by $1 - 0.687/4.01 \approx 83\%$, while for puts, it reduces the MSHE by $1 - 0.655/4.78 \approx 88\%$. Using the Hull-White Delta reduces the MSHE for calls only by $1 - (1 - 0.231) \times 0.687/4.01 \approx 87\%$, but for puts by $1 - (1 - 0.169) \times 0.655/4.78 \approx 89\%$. Hence, the relative outperformance of the linear regressions and ANNs over the BS-Delta is higher exactly when the BS-Delta has a worse performance.\textsuperscript{19}

Recall from Section 2 that the S&P 500 dataset is been split in rolling windows, each time shifted by 180 days. This yields 14 out-of-sample sets. The samples in each out-of-sample set are evaluated with the model parameters estimated on its corresponding in-sample set. Figure 5 compares the MSHEs of different statistical models by time window. Consistent with Table 2, the blue dots corresponding to the BS-Delta are usually the largest. However sometimes, for example in the first time window, the competing models underperform relative to BS-Delta. Both Table 2 and Figure 5 show that for two-day hedging period, the MSHEs are about twice those for the one-day period. The only exceptions are the 7th and the 13th time window, when the errors are about 4 times and 3 times larger in the two-day period.

![Figure 5: MSHEs of four different statistical models for the hedging ratio across all 14 time windows in the S&P 500 dataset, for the one-day (left) and two-day (right) hedging period. Note that in the first time window the models lead to a higher MSHE than the BS-Delta. We try to give an explanation for this effect in Subsection 6.1.](image)

Figure 6 provides the coefficients (plus their standard errors) for the Delta-Vega-Vanna regression in the one-day period setting.\textsuperscript{20} The intervals are getting smaller for later time windows due to the fact that later time windows contain more samples as illustrated in Appendix A. Especially the Vanna coefficients for calls are very stable across time windows.

The Delta coefficients of calls being smaller than one implies that hedging a short position on a call, one would usually buy less of the underlying than implied by the BS-Delta. On the other hand, for hedging a short position on a put, one needs to short more of the underlying. This phenomenon is consistent with the leverage effect, discussed in Subsection 4.2. Note that Vanna is positive (negative) for out-of-the money calls (puts). Hence the Vanna term in the regression further contributes to holding an even smaller number of the underlying than only implied by the Delta term. Since Vanna is largest in absolute value for slightly out-of-the money options, this correction term is largest for such options. The Vega coefficients are negative for puts and most time windows also for calls, adding yet a third correction, most effective for long-dated at-the-money options.

Figure 5 shows that both the 7th and the 12th time window, whose out-of-sample data are the second half of 2015 and the first half of 2018, respectively,\textsuperscript{21} lead to an overall large MSHE. The corresponding samples are

\textsuperscript{19}These observations are not due to the asymmetric choice of moneyness (recall that we only consider out-of-the money options with moneyness $M = S_0/K$ between 0.8 and 1 for calls and between 1 and 1.5 for puts). Indeed the same results as outlined in this paragraph hold true when we allow moneyness to be between 0.6 and 1 for calls and restrict it to be between 1 and 1.2 for puts. When one does not remove samples with very small or very large moneyness in the cleaning process then the median moneyness in the S&P 500 dataset for out-of-the-money and at-the money calls (puts) is 0.97 (1.09). In this case, 95% of the out-of-the-money and at-the money calls (puts) satisfy 0.89 $\leq M (M \leq 1.51$).

\textsuperscript{20}The coefficient plots for the two-day hedging periods (not displayed here) look very similar; in particular the Vanna coefficients for calls are again stable. However the Vanna coefficients for puts and the Vega coefficients for calls and puts are slightly more fluctuating.

\textsuperscript{21}The in-sample sets for these two periods range from 2013 to the first half of 2015 and the second half of 2015 to 2017, respectively. The test data for the 7th time window fall exactly in the 2015-16 selloff. The test data for the 12th time window contain the first week of February.
Figure 6: The coefficients in the Delta-Vega-Vanna regression for each of the 14 time windows in the S&P 500 dataset. The top and bottom of each line segment are the point estimate plus/minus two standard errors. These numbers correspond to the one-day hedging period.

then part of the in-sample set for the following periods. And indeed, Figure 6 indicates a jump in some of the coefficients in the 8th and 13th time window.

Additional diagnostics are available in Appendices C and D.

We run two extra experiments to see whether the above conclusions depend on the chosen setup.

1. In the first modified experiment we remove all options that have a time-to-maturity of 14 calendar days or less from both the in-sample and out-of-sample sets. This yields an additional relative improvement of about 2% in the one-day experiment and about 3% in the two-day experiment for all methods presented in Table 2. We omit presenting the precise numbers here.

2. In the second modified experiment we abstain from splitting the dataset in 14 time windows. Instead of 14 experiments we hence only have one, but with a much larger number of samples. We keep the ratio 4:1:1, now across the whole dataset, leading to an in-sample set of length 2850 (2280 + 570) days and a test set of length 570 days (instead of 14 test in-sample sets of length 900 (720 + 180) days and an out-of-sample set of length 180 days; see Subsection 2.4).

We omit the detailed results of this experiment but only remark on the major commonalities with and differences to Table 2. The regression models and ANNs improve their relative performance by about 3% to 4% when using only one time window instead of 14 time windows. Again the ANNs do not outperform the linear regression models. Now, the Delta-Vanna regression performs slightly better than the Delta-Vega-Vanna one. One should not put too much emphasis on the improved relative performance as the out-of-sample data of the modified example is different from the original experimental setup. For example, in Table 2 for the one-day period, the BS-Delta reduces the overall MSHE by 85%, while on this modified experiment, the BS-Delta only reduces it by 82%, exactly 3% less.

2018, where the S&P 500 experienced a 10% drop; see also Figure 1.
5.2 Euro Stoxx 50 tick data

Table 3 shows the performance of all competing methods on the Euro Stoxx 50 dataset. Moreover, Figure 7 compares the MSHEs of different statistical models by time window. In contrast to the S&P 500 dataset, the results are slightly more inconclusive. Nevertheless, again we can conclude that the ANNs in general do not outperform the linear regressions. Indeed, the performance of the two ANNs is highly inconsistent. The ANN using moneyness $M$ and time-proportional implied volatility $\sigma_{\text{impl}}\sqrt{\tau}$ performs the best for the one-hour period, while the ANN using $\Delta_{\text{BS}}, \nu_{\text{BS}}, \tau$ as features performs well for the one-hour period, very well for the two-day period, but worse than all the linear regression for the one-day period.

|                | Calls   | Puts    | Both    | Calls   | Puts    | Both    | Calls   | Puts    | Both    |
|----------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| **Zero hedge** | 0.314   | 0.587   | 0.462   | 3.99    | 7.86    | 6.09    | 6.96    | 13.3    | 10.5    |
| **BS-Delta**   | 0.063   | 0.107   | 0.087   | 0.873   | 1.39    | 1.15    | 1.76    | 2.16    | 1.98    |
| **Delta-only** | -8.47   | -4.64   | -5.91   | -18.4   | -5.65   | -10.1   | -15.3   | -4.46   | -8.79   |
| **Vega-only**  | -13.9   | -7.52   | -9.64   | -17.6   | -11.7   | -13.7   | -17.7   | -8.8    | -12.3   |
| **Gamma-only** | -1.7    | 0.76    | -0.06   | -14.4   | -1.56   | -6.02   | -10.7   | -0.81   | -4.77   |
| **Vanna-only** | -9.39   | -1.81   | -4.33   | -10.9   | -11.4   | -11.2   | -11.6   | -5.99   | -8.21   |
| **Delta-Gamma**| -7.52   | -7.54   | -7.53   | -14.3   | -2.06   | -6.31   | -14.3   | -3.13   | -7.57   |
| **Delta-Vega** | -13.4   | -7.38   | -9.4    | -19.2   | -9.89   | -13.1   | -17.1   | -8.94   | -12.2   |
| **Delta-Vanna**| -9.86   | -4.14   | -6.05   | -17.4   | -6.34   | -10.2   | -15.1   | -6.27   | -9.76   |
| **Delta-Vega-Gamma** | -12.0   | -8.19   | -9.45   | -15.3   | -5.63   | -8.99   | -16.0   | -7.83   | -11.1   |
| **Delta-Vega-Vanna** | -12.7   | -7.13   | -9.0    | -17.2   | -8.62   | -11.6   | -15.9   | -9.08   | -11.8   |
| **Delta-Vega-Gamma-Vanna** | -10.8   | -7.69   | -8.74   | -14.3   | -8.0    | -10.2   | -15.1   | -7.31   | -10.4   |
| **Hull-White** | -12.2   | -8.1    | -9.46   | -14.5   | -8.38   | -10.5   | -16.0   | -8.08   | -11.2   |
| **Relaxed Hull-White** | -10.2   | -5.64   | -7.14   | -17.3   | -8.79   | -11.8   | -14.8   | -4.97   | -8.86   |

Table 3: Performance of the benchmarks and ANNs on the Euro Stoxx 50 dataset. See the caption of Table 2 for further explanations.

In contrast to the mixed performance of the ANNs, the Vega-only linear regression performs quite consistently the best among all regressions including the Hull-White model. To recall, the Vega-only regression uses as hedging ratio the sum of the BS-Delta $\delta_{\text{BS}}$ plus an estimated multiple of the option Vega; i.e., in (7), $a = 1, c = d = 0$, and only $b$ is estimated. We plot the coefficients of this one-factor regression in Figure 8. Since Vega is small for short-maturity options, this seems to indicate that the leverage effect has a relatively small impact on the optimal hedging of such options; see also Figure 14 in Appendix C. We see in Figure 8 a large jump in the coefficients going from time window 4 to 5. This is consistent with Figure 7, where the out-of-sample set in time window 4 displays an overall large MSHE. This out-of-sample set becomes part of the in-sample set for time window 5.

Surprisingly similar to the S&P 500 dataset, just using the BS-Delta reduces the overall MSHE by about 81%. This percentage is very stable across the three different hedging periods. However, using any of the statistical models has a slightly smaller improvement compared to the S&P 500 dataset. Again, this benefit is larger for calls than for puts.

Additional diagnostics are available in Appendices C and E.

As for the S&P 500 dataset, we run two additional experiments.

1. In the first one, we only consider options with a time-to-maturity of 14 calendar days or more. This yields an additional relative improvement of about 4% to 8%, in comparison with Table 3. We again omit the precise numbers here as the overall conclusions do not change.

2. In the second additional experiment, we aggregate the data to one time window (instead of five), again with a 4:1:1 ratio. This leads to 750 (600+150) days in the in-sample set and 150 days in the out-of-sample set. As for the S&P 500 dataset, the conclusions do not change. The ANNs still underperform relative to the
Figure 7: MSHEs of four different statistical models for the hedging ratio across all 5 time windows in the Euro Stoxx 50 dataset, for the one-hour (upper left), one-day (upper right), and two-day (bottom) hedging period. Figure 22 in Appendix E shows that the return distribution in the in-sample and out-of-sample sets of the fourth time window are very different. This is a likely explanation for the bad performance in that time window.

5.3 Simulated data from Black-Scholes

As reported in Table 4, in the one-day hedging period, the BS-Delta performs best (with the exception of the Vanna-only and Vega-only regressions). For the two-day hedging period, all regressions outperform the BS-Delta. Relative to the BS-Delta the regressions are about 2% to 3% better. At first glance, this seems surprising since the BS-Delta should be close to optimal for data generated from the Black-Scholes model. Indeed, in both hedging periods, using the BS-Delta instead of not hedging at all reduces the MSHE by about 99%.

What is happening? Recall that we do not hedge continuously but only once in each hedging period. During the hedging period, the underlying’s price changes, and thus, the BS-Delta chosen at the beginning of the hedging period is not optimal at other times during the hedging period. Gamma measures how fast the option’s Delta changes as the underlying moves. Since the underlying’s price path has been simulated with an annualised drift rate of 10% (see Subsection 2.1), in average the option’s Delta increases over the hedging period. The linear regressions are able to capture this effect. For example, in the Delta-only regression, the Delta coefficient is larger than one for out-of-the-money calls and smaller than one for out-of-the-money puts (in which case the BS-Delta is negative). This is in line with the observation that the option’s Delta increases over the hedging period in average.

For the one-day hedging period this drift effect is not strong enough for the linear regression models to outperform; they tend to slightly overfit to the in-sample data. For the two-day hedging period, however, this drift effect

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22Even with the drift being zero, such an effect would exist due to the convexity of option prices in the underlying.
Figure 8: The coefficients in the Vega-only regression for each of the 5 time windows in the Euro Stoxx 50 dataset for the different hedging periods. The top and bottom of each line segment are the point estimate plus/minus two standard errors.

is captured by the linear regressions, as can be seen in Table 4. The ANNs are not able to capture this effect, due to overfitting.

We have run another experiment, where we set the drift rate of the underlying’s price path to zero and leave all others parameters the same. In this case, the linear regressions underperform (overperform) relative to the BS-Delta by about 0.5% for the one-day (two-day) hedging period. Again, ANNs have the lowest performance among all considered models.

5.4 Simulated data from Heston

For the Heston dataset, we report the numbers in Table 5. Again the ANNs do not lead to a better performance than the regression models. Using the BS-Delta reduces the variance by about 93% for calls, and by about 91% for puts, both for the one-day and two-day hedging period. This is a larger improvement than for the real datasets. Note that we have roughly 3 times more put samples than call samples in the in-sample test as Appendix A explains. Let us report here the coefficients for the Delta-only regression for the one-day hedging period only: calls: 0.97; puts: 1.03; both with a standard deviation ± 0.001. Again, this is as expected due to the leverage effect.

5.5 Overall comments

In none of the four datasets do ANNs outperform the linear regression models. We conclude that the option sensitivities suffice to capture the nonlinearities in the data that are relevant for the hedging task. Additional drawbacks of ANNs are their computational demands and the necessary effort to tune their hyperparameters (see Appendix B).

Let us briefly mention some statistical properties of the MSHEs of the different methods. For the mean and median of the MSHEs we do not observe any consistent differences across the various statistical models. For the 5% and 95% value-at-risk, we observe the following. For the Euro Stoxx 50 dataset, the statistical methods decrease both value-at-risks by about 4% relative to the value-at-risks of the MSHE corresponding to the BS-Delta.
Table 4: Performance of the benchmarks and ANNs on the Black-Scholes simulated dataset. See the caption of Table 2 for further explanations.

For the S&P 500 dataset, the same holds true for the 95% value-at-risk, but the statistical models seem to increase the 5% value-at-risk seems by about 2% relative to the BS-Delta.

We want to conclude this section with a further sobering observation. Motivated by the reported results we try another ‘fixed’ hedging strategy. All calls are hedged by $0.9 \times \delta_{BS}$ and puts are hedged by $1.1 \times \delta_{BS}$. We have not run other such ‘fixed’ hedging strategies (hence, we have not optimised this 10% relative correction term). Table 6 shows the relative performance of this ‘fixed’ strategy with respect to BS-Delta on the S&P 500 and Euro Stoxx 50 datasets. The out-of-sample tests are the same ones that were used for Tables 2 and 3. On the S&P 500 dataset, this simple strategy does very well but underperforms the Delta-Vega-Vanna regression. However, on the Euro Stoxx 50 dataset it provides competitive results.

6 Data leakage

Data leakage occurs when trained model parameters (such as in a linear regression or in an ANN) are unintentionally allowed to depend on certain information that would not be available when using the model in real time. Hence the backtesting and comparison of different statistical models, as in this work, requires extra care.

In this section we provide some examples for data leakage in the context of the hedging problem and discuss its implications. More precisely, in Subsection 6.1 we illustrate how important it is to keep the time series structure of the data in mind. In Subsection 6.2 we illustrate how the data cleaning process can introduce data leakage and we argue that it is very difficult to avoid any data leakage due to missing observations.

6.1 Potential data leakage for time series

In this paper the one-period hedging problem is studied. Hence, when preparing the data as described in Subsection 2.4, the intrinsic time series structure of the data is not automatically preserved as each time series is broken up in many one-period samples.

As discussed in Subsection 2.4, here the data in each time window are separated chronologically into an in-sample and an out-of-sample set. In each time window roughly the first 83% (≈5/6) of days are assigned to the in-sample set (again chronologically split in a training and a validation set for ANNs) and the last 17% (≈1/6) of
Table 5: Performance of the benchmarks and ANNs on the Heston dataset. See the caption of Table 2 for further explanations.

| Regressions | 1 day | 2 days |
|-------------|-------|--------|
| | Calls | Puts | Both | Calls | Puts | Both |
| Zero hedge  | 29.7  | 15.2  | 19.5 | 63.1  | 32.1  | 40.0 |
| BS-Delta    | 2.12  | 1.34  | 1.55 | 4.74  | 3.00  | 3.43 |
| Delta-only  | -1.02 | -2.23 | -1.8 | -0.16 | -2.52 | -1.09 |
| Gamma-only  | -1.3  | -2.64 | -2.2 | -0.19 | -3.38 | -1.48 |
| Vega-only   | -0.7  | -1.74 | -1.39| -0.2  | -1.86 | -0.84 |
| Vanna-only  | -0.75 | -2.05 | -1.55| -0.31 | -2.61 | -1.19 |
| Delta-Gamma | -1.29 | -2.7  | -2.2 | -0.16 | -3.27 | -1.42 |
| Delta-Vega  | -0.74 | -2.2  | -1.63| -0.08 | -1.76 | -0.61 |
| Delta-Vanna | -0.88 | -2.69 | -1.94| -0.31 | -3.28 | -1.43 |
| Delta-Vega-Gamma | -0.84 | -2.54 | -1.91| -0.27 | -2.53 | -1.02 |
| Delta-Vega-Vanna | -0.82 | -2.84 | -1.97| -0.14 | -2.85 | -1.06 |
| Delta-Vega-Gamma-Vanna | -0.88 | -2.72 | -1.95| -0.28 | -3.06 | -1.31 |
| Hull-White  | -0.81 | -2.5  | -1.8 | 0.04  | -2.88 | -1.05 |
| Relaxed Hull-White | -0.78 | -2.5  | -1.78| 0.08  | -2.89 | -1.04 |

Table 6: Performance of the ‘fixed’ hedging strategy on the S&P 500 and Euro Stoxx 50 datasets. In the ‘fixed’ hedges strategy, calls (puts) are hedged by $0.9 \times \delta_{\text{BS}}$. See the caption of Table 2 for further explanations.

| | 1 hour | 1 day | 2 days |
| | Calls | Puts | Both | Calls | Puts | Both | Calls | Puts | Both |
| S&P 500 | - | - | - | -18.6 | -13.1 | -14.8 | -15.0 | -11.4 | -12.6 |
| Euro Stoxx 50 | -12.5 | -8.59 | -9.88 | -16.4 | -11.0 | -12.9 | -16.4 | -11.4 | -13.4 |

Days are assigned to the out-of-sample set.\textsuperscript{23}\textsuperscript{24}

Alternatively, we could have split the data randomly into in-sample and out-of-sample sets. (This approach has been taken in several research papers; see Ruf and Wang [2020] for a review.) In this approach, the in-sample and out-of-sample sets are also disjoint. However, we now argue that such an approach introduces significant data leakage. Indeed, as on each day several options are traded and hence we have several samples, the same day might show up both in in-sample and out-of-sample sets, with different options.

We illustrate with a series of experiments how such a wrong split in in-sample and out-of-sample sets may lead to wrong conclusions. We run these experiments both on the Black-Scholes simulated data and the S&P 500 data, both for the one-day hedging period. For each of these two datasets we simulate a ‘fake VIX’; i.e., we simulate daily samples from an Ornstein-Uhlenbeck process\textsuperscript{25} completely independent from either dataset. Clearly, adding this ‘fake VIX’ value as a feature should not help at all in reducing the MSHE, as the corresponding Ornstein-Uhlenbeck process is independently simulated.

The four experiments are the following.

1. The ‘Baseline’ experiment corresponds to the standard setup of Section 2.4. The dataset is separated chronologically in in-sample and out-of-sample sets. We consider ANN($\Delta_{\text{BS}}$; $\nu_{\text{BS}}$; $\tau$) and the Delta-Vega-Vanna linear regression.

\textsuperscript{23}Due to the growth of traded options (see Figure 11 in Appendix A), this actually corresponds to about 23% of samples in each time window being in the out-of-sample set.

\textsuperscript{24}For the two-day hedging period, we additionally make sure that the samples on the day separating the in-sample and out-of-sample sets are taken out. This avoids that the last day in the in-sample set overlaps with the first day in the out-of-sample set.

\textsuperscript{25}As parameters we use 1 for the rate of mean reversion, 25 for the volatility coefficient, 13 for the starting value, and 15 for the long-term mean.
2. The ‘VIX’ experiment takes the baseline setup, but adds the simulated ‘fake VIX’ variable as an additional feature to the linear regression and the ANN.

3. The ‘Permute’ experiment is done as follows. We compute the number of training, validation, and test samples. Then within each time window we permute the samples by randomly reassigning training, validation, and test labels to them. We do this in such a way that the numbers of training, validation, and test samples do not change. For the linear regression, the permuted training and validation sets are merged to be the in-sample set. Then, ANN(\(\Delta_{\text{BS}}; \nu_{\text{BS}}; \tau\)) and the Delta-Vega-Vanna regression are trained again on this permuted dataset. After each permutation, the Black-Scholes benchmark is recomputed since each permutation changes the constituents of the out-of-sample set.

4. The ‘Permute + VIX’ experiment is executed exactly as the ‘Permute’ experiment, but now with the ‘fake VIX’ variable as an additional feature.

The simulated and real data need slightly different treatments. Recall that the S&P 500 dataset is split into 14 time windows. We keep these 14 time windows, and run all four experiments for each of them. More precisely, for each time window, we run the third and fourth experiments five times as different permutations might lead to different results. For the Black-Scholes data, we run each experiment twenty times, on different out-of-sample sets but the same in-sample set, so that the ‘Baseline’ and ‘VIX’ experiments yield exactly the same trained ANN and regression coefficients.

Figures 9 and 10 summarise the results on the Black-Scholes and the S&P 500 datasets, respectively. The left panels show the the relative improvement over the BS-Delta, as given in (9), for each of the four experiments, averaged over time windows and the permutation sets, respectively. The right panels in Figures 9 and 10 show these results broken down by permutation set (Black-Scholes data) or time window (S&P 500 data). The time windows for the S&P 500 data are chronologically ordered; the permutation sets for the Black-Scholes data are ordered by performance of the ANN in the baseline experiment. Each of the presented numbers in the right panel corresponds to the additional relative improvement over the BS-Delta due to the permutation and ‘fake VIX’ feature. For example, a value of -20% for the ‘ANN (Permute + VIX)’ setup means that the ‘Permute + VIX’ experiment adds an extra 20% to the relative improvement of the ANN in the ‘Baseline’ experiment.

Let us summarise now our observations.

• In the ‘VIX’ experiment, both the linear regression and the ANN perform worse than in the ‘Baseline’ experiment. This effect is stronger for the ANN than for the linear regression, at least in the S&P 500 dataset. An explanation is easy. The additional feature is simulated completely independently from the data. Hence, it has no predictive power for the hedging ratio at all. Its inclusion adds additional noise, and the lower performance is due to an overfit of the training procedure, being more dramatic for the nonparametric ANN than for the three-parameter linear regression.

• Even without using the ‘fake VIX’ as feature, the permuted datasets lead to a better performance relative to the BS-Delta benchmark. This holds even in the case that the samples are generated by a time-homogeneous Black-Scholes model. There, instead of underperforming by about 0.2% for the linear regression and 2% for the ANN (see Table 4), the linear regression and ANN reduce the BS-Delta in the Black-Scholes model by about 3% after data permutation, with a larger relative improvement for the ANN.

• More striking are the results if the ‘fake VIX’ is included as an additional feature. Both statistical models improve, but most dramatically the ANN, which now outperforms the BS-Delta by about 7% in the Black-Scholes simulated data and by about 29% in the S&P 500 data. What is going on? By construction, different samples have the same ‘fake VIX’ value. Indeed, each day has several options (corresponding to different strikes) but only one ‘fake VIX’ value. The random permutation now allows samples from the same day to appear both in the training and in the test set. It is now possible for the ANN (and partially also for the linear regression models) to learn whether on one specific day the underlying’s price goes up or down (or, in case of the S&P 500 data, there is a shift in the implied volatility surface). Hence, the ‘fake VIX’ tags the different days and the models are able to pick up on it.

• The right panel of Figure 9 breaks the average value of the left panel up into the twenty repetitions of the experiment. The differences between the repetitions are different out-of-sample sets for the ‘Baseline’ setup, and different random permutations. A relative improvement of up to 20% can be observed. As the outperformance of the ANN in the ‘Baseline’ setup increases, the improvement through the permutations becomes less significant.
Figure 9: Illustration of data leakage when failing to take into account the time series structure of a simulated dataset. The left panel displays the relative reduction in MSHE over using the BS-Delta for each of the four experiments described in the main text. In the ‘Baseline’ experiment, neither the ANN nor the linear regression improve the MSHE relative to the BS-Delta. Adding the ‘fake VIX’ feature reduces furthermore their performance since this feature is simulated independently of the data, and thus, pure noise. However, when the in-sample and out-of-sample sets are randomly permuted, both the ANN and the linear regression outperform the BS-Delta. Moreover, now the ‘fake VIX’ feature reduces the error further, illustrating the data leakage induced by the random permutations.

The right panel displays by how much the relative reduction improves by permuting in-sample and out-of-sample sets. The relative reduction is improved more in the case of the ANN than in the case of the linear regression, and the ‘fake VIX’ helps both the linear regression and the ANN. The permutation sets are ordered from left to right by the relative reduction of the ANN in the baseline case; the permutation set on which the ANN performs the worst in the ‘Baseline’ experiment is on the left. The increasing trend hence illustrates that the relative improvement is the largest when the ANN has the least relative reduction.

• As indicated by Figure 5, in the first time window of the S&P 500 data, the statistical models underperform the BS-Delta. This is most likely due to the in-sample and out-of-sample sets being very different. Figure 10 seems to support this – it shows that in the first time window shuffling the in-sample and out-of-sample sets (the ‘Permutation’ experiment) leads to the largest benefit.

• As mentioned above for each of the 14 time windows in the S&P 500 dataset we did five repetitions of the experiment, their only difference being different random permutations. The five repetitions lead to very similar results. Figure 10 reports the average.

To conclude this subsection, let us summarise these observations. We show experimentally how random permutations of the in-sample and out-of-sample sets lead to a remarkable overestimation of the relative performance. This effect is especially strong for the ANN, but is also present for the linear regression. When adding an independent feature to the data, random permutations make this feature informative, leading to a further seemingly important improvement. For example, when using Black-Scholes data, such a random permutation leads to an outperformance of the ANN relative to the BS-Delta by about 7%. Of course, this additional feature by construction has nothing to do with finding a good hedging ratio. Thus, we have illustrated that a wrong split in in-sample and out-of-sample sets leads to significant data leakage, along with a wrong conclusion on the benefits of using an ANN over parametric models.

6.2 Potential data leakage through data cleaning

In this subsection we briefly discuss data leakage issues connected to the data cleaning process. One obvious mistake would be removing samples with wrong-way option price changes. An example is the removal of call option samples, whenever the underlying’s price increases but the call price decreases. Although a first thought might be that this is a data issue such samples are very well possible due to changes in the bid-ask spread or due to the leverage effect; see also Bakshi et al. [2000] and Pérignon [2006] for empirical evidence.

The availability of end-of-period prices is a more difficult issue to be resolved. Here, in our opinion, data leakage cannot be completely avoided since it is not clear at the beginning of a period whether prices can be
observed at its end. If those prices were missing at random, it would be fine to remove those samples during backtesting. However, for financial price data, such an assumption cannot be easily justified. Indeed, missing observations tend to be caused by missing market liquidity. Market liquidity and the implied volatility surface might very well depend on each other. Hence, removing missing observations could potentially lead to biased parameter estimations.

To understand whether data leakage through missing price observations appears in our experiments we ran robustness checks for both the S&P 500 and the Euro Stoxx 50 datasets.

We begin with the S&P 500 dataset. For these data, we have quoted prices for all options, along with trading volumes. For the results in Subsection 5.1, we remove all samples whose trading volume at the beginning of its period are zero. We keep those samples whose volume at the beginning is positive, but zero at the end of the period. As a robustness check we rerun the complete analysis with those samples removed whose trading volume is zero at the end of the period. This reduces the overall dataset by about 22% and increases the MSHE of the zero-hedge for puts (by more than 10%). An explanation for this increase is that this modified cleaning procedure removes especially deep out-of-the-money puts, thus increasing the average squared prices changes. However, the relative performance improvement of the models with respect to the BS-Delta does not change much; in particular, the conclusions of Subsection 5.1 seem to be robust with respect to this cleaning procedure.

Next, let us discuss the Euro Stoxx 50 dataset consisting of tick data. Using such tick data leads to several difficulties concerning missing price observations. First, the underlying’s prices (we use short-term futures on the Euro Stoxx 50) and option prices are not observed synchronously. This issue is relatively mild since futures are extremely liquid. For an option observation at some time $t$ we thus use the future’s price at the last transaction before $t$.

However, a major issue in the data cleaning process is to determine the price of the option at the end of a period. To illustrate, consider the one-hour period setup. If an option transaction in the dataset is observed at some time $t$, then we would like to know the option price at time $t+1$ hour to backtest the hedging performance of the different methods. It is very unlikely to find a trade at exactly this time. To handle this issue we introduced a matching tolerance window of 6 mins (see Subsection 2.3). That is, if at some time $t$ a transaction occurs then the sample’s end-of-period price is the first price observation after time $t+1$ hour, and the sample is discarded if this end-of-period transaction occurs later than $t+66$ minutes.

As discussed above, we have clearly introduced some data leakage by removing illiquid samples for which no end-of-period price is observed. Let us now do again a robustness check. To this end, we increase the matching tolerance window from 6 minutes to 30 minutes. In the one-day period situation, this increases the overall number of samples from 0.6 million to 1.4 million, a 133% increase. This modified set contains now many more illiquid options, reflected also in a smaller MSHE of the zero-hedge.

Let us first summarise how the Vega-only regression performs on this modified and enlarged dataset. For the two-day hedging period, the performance improves from about -12% to -13%, most of this improvement coming from a better performance for puts. For the one-day period, the longer matching tolerance window benefits only
puts but not calls, leaving the total performance of the Vega-only regression unchanged. For the one-hour hedging period, the overall performance is again unchanged, but now the longer matching tolerance window benefits calls and not puts. All in all, for the regression models, the conclusions of Subsection 5.2 are still valid. However, the longer matching tolerance window has a significantly negative effect for the ANNs. Now five out of six ANN setups produce worse results, up to even a 4% loss in outperformance. Overall, doubling the dataset by increasing the matching tolerance window does not change the regression results much, but significantly handicaps the training of the ANNs. A further test with a matching tolerance window of 60 minutes leads to the same conclusions.

7 Conclusion and discussion

In this work, we consider the problem of hedging an option over one period. We consider statistical, regression-type hedging ratios (in contrast to model-implied hedging ratios). To study whether the option sensitivities already capture the relevant nonlinearities we develop an ANN. Experiments involving both quoted prices (S&P 500 options) and high-frequency tick data (Euro Stoxx 50 options) show that the ANNs perform roughly as well (but not better) as the sensitivity-based linear regression models. However, the ANNs are not able to find additional nonlinear features. Hence option sensitivities by themselves (in particular, Delta, Vega, and Vanna) in combination with a linear regression are sufficient for a good hedging performance.

The linear regression models improve the hedging performance (in terms of MSHE) of the BS-Delta by about 15-20% in real datasets. An explanation is the leverage effect that allows the partial hedging of changes in the implied volatility by using the underlying. As a rule of thumb, historical data seem to imply that calls should be hedged with about $0.9\delta_{BS}$ and puts with about $1.1\delta_{BS}$.

We also show how data leakage in backtesting can lead to the wrong conclusions. Splitting data into in-sample and out-of-sample sets without paying attention to their time series structure can mislead researchers to conclude that ANNs (or, in general, complex statistical models) outperform. Moreover, even for linear regression models with few parameters, such a wrong split may lead to strongly overconfident estimates of their performance.

We have not performed a cross-sectional study where the hedging ratio is estimated not only from options written on the same underlying. It would be interesting to see whether the hedging ratios of the linear regression models can be further improved by using options written on different underlyings, e.g., the constituents of an index.

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Appendices

A Sizes of in-sample and out-of-sample sets

Recall that we only consider out-of-the-money and at-the-money options. Figure 11 shows the number of samples in each time window for the S&P 500 and Euro Stoxx 50 datasets. For the S&P 500 data (ranging from 2010 to 2019), the overall number of samples is 2.6 million. On average, there are 1144 samples per trading day. In each time window, the number of total samples grows continually. More puts than calls are traded, and the number of puts traded grows faster than that of calls traded. For the Euro Stoxx 50 data (ranging from 2016 to 2018), the number of samples overall is 0.62 million. On average, there are 988 samples per trading day. In each time window, the number of samples decrease slightly. Roughly the same number of puts and calls are traded.

Figure 11: Sample size of out-of-the-money and at-the-money calls and puts in training and validation sets. The left panel corresponds to the S&P 500 dataset, the right panel to the Euro Stoxx 50 dataset.

Figure 12 shows the distribution of moneyness in the S&P 500 and Euro Stoxx 50 datasets. As we only consider out-of-the-money and at-the-money options each sample with moneyness less than 1 corresponds to a call, and similarly, each sample with moneyness greater than 1 corresponds to a put. The distribution of moneyness for Euro Stoxx 50 data is more concentrated around a moneyness of 1. This difference is explained by the fact that the S&P 500 dataset consists of end-of-day quotations of all listed options, while the Euro Stoxx 50 dataset consists of tick prices of all traded options. Since close-to-the-money options are more frequently traded, the Euro Stoxx 50 dataset hence has relatively more such samples.

Figure 12: Histogram of moneyness in the S&P 500 (left panel) and the Euro Stoxx 50 (right panel) datasets. Samples with moneyness less than 1 correspond to calls, and samples with moneyness greater than 1 to puts.

Figure 13 shows the distribution of time-to-maturity for both datasets. The S&P 500 dataset has many more long-dated options than the Euro Stoxx 50 dataset.

Figure 13: Distribution of time-to-maturity for the S&P 500 (left panel) and the Euro Stoxx 50 (right panel) datasets.
We conclude by summarising that the in-sample dataset in the Black-Scholes dataset is 0.36 million and in the Heston dataset 0.25 million. As explained in Subsection 2.1, for the simulated datasets we created options according to the CBOE rules and then removed all in-the-money samples. Since the underlying tends to move upwards in the Black-Scholes dataset (the drift rate was set to 10%) we expect to have more out-of-the-money put samples than call samples. Indeed, an investigation of the Black-Scholes dataset yields that we have roughly 91k call samples and 277k put samples in the in-sample set. It turns out that the Heston in-sample dataset, just by chance (the simulated underlying’s path process moves from 2000 to about 2600) also has more put samples (187k) than call samples (62k).

B Additional hyperparameters of HedgeNet

We now add details on the implementation and training of HedgeNet (see Subsection 3.1).

Based on preliminary experiments on simulated data we set the learning rate to $10^{-4}$ and the batch size to 64. Usually we train each ANN for 300 epochs. Using a validation set, we apply early stopping by choosing the ANN with the smallest validation error.

The optimisation criterion is a Tikhonov regularised version of squared loss. We use an $L^2$ penalty term for the ANN weights. We also experimented with other regularisations, such as an $L^1$ penalty, a combined $L^1$-$L^2$ penalty, and dropout. They all lead to similar results and the same conclusions. The regularisation strength $\alpha$ is tuned for each dataset and hedging period. The larger $\alpha$ is the more the weights are pushed to zero. In case of the simulated data, $\alpha$ is tuned by using an independent dataset that is simulated from the same model but with a different random seed. Hence, the actual training and test datasets are different from the ones used for tuning. For the real datasets (S&P 500 / Euro Stoxx 50), we tune only using the first four / two time windows.

For each dataset and each value $\alpha$ on a logarithmic grid, we run five iterations of the ANN training, each with a different (random) weight initialisation. For each dataset we then pick an $\alpha$ after inspecting the average and standard deviations of the test errors (on the independent dataset when using simulated data, and on the first few time windows when using real data). Table 7 summarises the chosen $L^2$ regularisation parameters.

| $M$; $\sigma_{\text{impl}}\sqrt{T}$ | S&P 500 | Euro Stoxx 50 | Black-Scholes | Heston |
|---|---|---|---|---|
| 1H | - | $10^{-5}$ | - | $10^{-4}$ |
| 1D | $10^{-7}$ | $10^{-2}$ | $10^{-4}$ | $10^{-4}$ |
| 2D | $10^{-3}$ | $10^{-2}$ | $10^{-4}$ | $10^{-4}$ |

| $\Delta_{\text{BS}}$; $V_{\text{BS}}$; $\tau$ | S&P 500 | Euro Stoxx 50 | Black-Scholes | Heston |
|---|---|---|---|---|
| 1H | - | $10^{-2}$ | - | - |
| 1D | $10^{-4}$ | $10^{-2}$ | $10^{-4}$ | $10^{-3}$ |
| 2D | $10^{-4}$ | $10^{-1}$ | $10^{-3}$ | $10^{-3}$ |

Table 7: Regularisation parameters used for the training of HedgeNet in the different experiments.

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26 We also apply visual inspections of the training / validation loss to confirm that the ANN is indeed trained.
Some heuristics on the leverage effect

To understand the leverage effect and its interaction with the coefficients of the linear regressions a bit better we do the following empirical study. For each option type (put or call) and for different time-to-maturities (namely $\tau$ smaller than 1 month, $\tau$ between 1 and 6 months, and $\tau$ greater than 6 months) we regress $\Delta \sigma_{\text{impl}}$ on $\Delta S$, without intercept. This yields a slope $b$. We then compute

$$LC = b \frac{1}{N_{\text{train}}} \sum_{t,j} \frac{V_{\text{BS},t,j} \Delta \sigma_{\text{impl}}}{\delta_{\text{BS},t,j}},$$

(10)

which we call leverage coefficient. These heuristics are motivated by how much we should adjust a hedge due to the leverage effect. Indeed, a change of $\Delta \sigma_{\text{impl}}$ leads roughly to a change of $V_{\text{BS}} \Delta \sigma_{\text{impl}}$ in the option price. A part $V_{\text{BS}} b \Delta S$ of this change can be explained by the change in the underlying’s price due to the correlation of implied volatilities and returns. Considering a multiplicative effect on the BS-Delta, we need to divide this number by $\delta_{\text{BS}}$.

Figure 14 shows the leverage coefficients for the different option categories for the one-day hedging period. The plots for the other hedging periods (for which $\Delta \sigma_{\text{impl}}$ and $\Delta S$ are different, yielding slightly different estimates for $b$ in (10)) look similar. The fact that the leverage coefficient tends to be negative for calls (positive for puts) reflects how the regression models replace the BS-Delta by a number smaller (larger) than one. Note the jumps of the leverage coefficient in the S&P 500 plot from period 4 to 5, 7 to 8, and 12 to 13. This is consistent with the change of the Delta coefficient in Delta-Vega-Vanna regression of Figure 6.

Figure 14: Leverage coefficients as given in (10) on the three categories of time-to-maturity in the S&P 500 (left) and Euro Stoxx 50 (right) dataset for the one-day hedging period. ‘Short’ means a time-to-maturity of less than 1 month, ‘middle’ means between 1 month and 6 months, and ‘long’ means more than 6 months.

Short-maturity options have a relatively higher leverage coefficient in the S&P 500 dataset than in the Euro Stoxx 50 dataset. This is consistent with the fact that in the Euro Stoxx 50 dataset the Vega-only regression outperforms the Delta-only regression, in contrast to the S&P 500 dataset. Indeed, in the Vega-only regression the hedging ratios for short-maturity options do not change much as their Vega is relatively small.

Additional diagnostics for the S&P 500 dataset

We use this appendix to provide some additional figures concerning the performance of the various statistical models on the S&P 500 dataset.

Figure 15 extends Figure 5 by including the MSHE of the zero hedge strategy. As we can see, the MSHE for any of the methods is large exactly when the MSHE of the unhedged portfolio is large. Figure 16 shows the ratio of the MSHEs of the same four statistical models to the zero hedge MSHE. The hedging performance gets worse in later periods. The MSHE corresponding to the BS-Delta minus the MSHE of one of the statistical models divided by the zero hedge MSHE is about 2%.
Figure 15: MSHEs for four different statistical models of the hedging ratio and the zero hedge across all 14 time windows in the S&P 500 dataset for the one-day (left) and two-day (right) hedging periods. The numbers of the statistical models correspond to the numbers in Figure 5, but are now presented on a logarithmic scale.

Figure 16: The ratio of the MSHEs of four statistical models to the hedging ratio and the zero hedge MSHE in the S&P 500 dataset for the one-day (left) and two-day (right) hedging.

Figure 17 shows the average logarithmic return and its standard deviation of the S&P 500 dataset in each time window. We see that the standard deviations in the out-of-sample sets tend to be large when the zero hedge MSHEs in Figure 15 are large.

Figure 18 scatterplots the hedging ratios corresponding to the different statistical models. Here, we provide only one such plot, namely comparing the Delta-Vega-Vanna hedging ratio with the hedging ratio of the ANN ($\Delta_{BS}; \nu_{BS}; \tau$) for the one-day hedging period. Each point is a sample in the test set. We do not directly plot the hedging ratios but $N^{-1}(\delta_{NN})$ against $N^{-1}(\delta_{LR})$, where $N$ denotes again the cumulative standard normal distribution. The ratios are very similar but different in the tails, where the ANN seems to overfit. We only provide the plots for two representative time windows. In window 1, the BS-Delta outperforms all regression models, while window 12 represents a more typical situation where the BS-Delta underperforms the regression model and the ANN.

Figure 19 plots the mean squared relative hedging error, i.e., the average of the hedging errors divided by the option prices, of the Delta-Vega-Vanna regression against time-to-maturity and Vega. The left panel shows an exponential decrease (due to the logarithmic scale) of the relative hedging error with respect to time to maturity. The right panel shows that the relative hedging errors decrease super-exponentially as Vega increases, i.e., as the options have a longer time-to-maturity and are less out-of-the-money.

E Additional diagnostics for the Euro Stoxx 50 dataset

Similar to Appendix D we now provide some additional figures for the Euro Stoxx 50 dataset.
Figure 17: The average annualised logarithmic one-day (left) and two-day (right) return of the S&P 500 in each of the 14 time windows. Each line segment shows the average logarithmic return plus/minus one standard error of the logarithmic returns for each time window. The lines tend to be longer, meaning a higher standard deviation, when the returns are smaller, illustrating the leverage effect.

Figure 20 extends Figure 7 by including the MSHE of the zero hedge strategy. Exactly as in the S&P 500 dataset, the MSHE for any of the statistical models is large exactly when the MSHE of the zero hedge strategy is large. Figure 21 shows the ratio of the MSHEs of the same four statistical models to the zero hedge MSHE. Across the five time windows, the BS-Delta and three regressions reduce the MSHE by more than 80%.

Different to Figure 17, which shows the logarithmic returns of the underlying, Figure 22 displays the logarithmic returns of the futures written on the Euro Stoxx 50, which are used as hedging instruments. We note the large difference of the returns in the in-sample and out-of-sample sets in the fourth time window, which we believe explains the large MSHE in this window, displayed in Figure 7.

Next, Figure 23 scatterplots the hedging ratios corresponding to the different statistical models. We refer to the caption of Figure 18 for explanations. Different to Figure 18 with the S&P 500 dataset, the hedging ratios of the ANN now look quite different from the linear regression model. Consistently with the prevalence of red points, for the one-day hedging period, the ANNs display a relatively bad performance (recall Table 3 and Figure 7).

Figure 24 plots the mean squared relative hedging error, i.e., the average of the hedging errors divided by the option prices, of the Vega-only regression against time-to-maturity and Vega. In comparison to the S&P 500 dataset (see Figure 19), the decrease seems to be a little bit smaller as time-to-maturity and Vega increase, respectively.
Figure 18: ANN($\Delta_{BS}$; $V_{BS}$; $\tau$) versus Delta-Vega-Vanna regression hedging ratios in the S&P 500 dataset. Each point represents a sample. We use transformed scales so that the $x$-value of each sample corresponds to $N^{-1}(\delta_{LR})$ and the $y$-value to $N^{-1}(\delta_{NN})$, where $N$ denotes the cumulative standard normal distribution. If the point is blue the MSHE corresponding to the ANN is smaller than the one corresponding to the linear regression. On the other hand, if the point is red the linear regression outperforms. Each row shows a time window; the one on the top is a window when the BS-Delta outperforms the statistical models; the one on the bottom is a more typical one, when the linear regressions and ANNs outperform the BS-Delta. Each column corresponds to a different set of maturities; namely less than one month (left); 1 month to 6 months (middle), and more than 6 months (right).

Figure 19: Mean squared relative hedging error of the Delta-Vega-Vanna regression on a logarithmic scale against time-to-maturity (left) and Vega (right) in the S&P 500 dataset for the one-day hedging period. Each line segment provides a point estimate plus/minus one standard error. Each interval has 10% of the overall samples, and the tick on the $x$-axis shows the average time-to-maturity and Vega, respectively, of the samples falling into the corresponding interval. Calls and puts may have different averages in each interval.
Figure 20: MSHEs for the hedging ratios of four different statistical models and the zero hedge across all 5 time windows in the Euro Stoxx 50 dataset for the one-hour (left), one-day (middle), and two-day (right) hedging periods. The numbers of the statistical models correspond to the numbers in Figure 7, but are now presented on a logarithmic scale.

Figure 21: The ratio of the MSHEs of four statistical models to the zero hedge MSHE in the Euro Stoxx 50 dataset for the one-hour (left), one-day (middle), and two-day (right) hedging period.

Figure 22: The average annualised logarithmic returns of Euro Stoxx 50 futures in each of the 5 time windows, for for the one-hour (left), one-day (middle), and two-day (right) hedging period. See the caption of Figure 17 for additional explanations.
Figure 23: $\text{ANN}(\Delta_{BS}; \nu_{BS}; \tau)$ versus Vega-only regression hedging ratios in the Euro Stoxx 50 dataset. The top (bottom) row shows the time window for which the regression models perform the worst (best) relative to the BS-Delta. See Figure 18 for additional explanations.

Figure 24: Mean squared relative hedging error of the Vega-only regression on a logarithmic scale against time-to-maturity (left) and Vega (right) in the Euro Stoxx 50 dataset for the one-day hedging period. See Figure 19 for additional explanations.