Canonical components of character varieties of arithmetic two-bridge link complements

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Abstract
The desingularizations of the canonical components of $\text{SL}_2(\mathbb{C})$-character varieties of arithmetic two-bridge link groups are determined.

Keywords $\text{SL}_2(\mathbb{C})$-character variety · Arithmetic 3-manifold · Two-bridge link complement

Mathematics Subject Classification 57M25 · 14J26

1 Introduction

The $\text{SL}_2(\mathbb{C})$-character variety of a hyperbolic 3-manifold is one of the central topics in the study of hyperbolic geometry. However little is known about the algebro-geometric properties of the character variety of a hyperbolic 3-manifold as an algebraic variety. (Now a computer program for computing the defining polynomials of the $\text{SL}_2(\mathbb{C})$-character varieties is available. See [1].) In [13] the structure of the $\text{SL}_2(\mathbb{C})$-character varieties of torus knot groups was explicitly determined. In [11] Macasieb, Petersen and van Luijk studied properties of the $\text{SL}_2(\mathbb{C})$-character varieties of a certain family of two-bridge knots (called the double-twist knots) which contains the twist knots. In fact they showed that the canonical components of the $\text{SL}_2(\mathbb{C})$-character varieties of the twist knots are hyperelliptic curves. Petersen and Reid studied a relation between the genus and the gonality of canonical components in [14]. Chen determined the $\text{SL}_2(\mathbb{C})$-character varieties of odd classical pretzel knot complements in [4].
In the link case, Landes studied in [9] the canonical component of the Whitehead link and some other link complements. The cases of double-twist link complements and the \((−2, 2m + 1, 2n)\)-pretzel link complements were investigated by Petersen and Tran in [15] and Tran in [19].

The Whitehead link complement is one of the examples of arithmetic two-bridge links. In determining the canonical component of the character variety of the above link complements it was crucial that it can be considered as a (singular) conic bundle over the projective line \(\mathbb{P}^1 := \mathbb{P}^1_{\mathbb{C}}\) in a specific projective space, which made it easy to obtain an explicit minimal model of the canonical component as an algebraic surface. It is already seen in other examples computed by Landes that the canonical components of hyperbolic two-bridge links are not necessarily conic bundles over \(\mathbb{P}^1\) in general.

It is known [6] that there are only finitely many arithmetic two-bridge links in the 3-sphere \(S^3\). In fact, there are only four such links, the figure 8 knot \(4_1 = (5/3)\), the Whitehead link \(5_1^2 = (8/3), 6_2^2 = (10/3)\) and \(6_3^2 = (12/5)\) in the Rolfsen’s table. The canonical component of the character variety of the figure 8 knot complement is well known, which is an elliptic curve (for instance, see [10, Corollary 4.1]). In this note we study the canonical components of the \(\text{SL}_2(\mathbb{C})\)-character varieties of the other three arithmetic two-bridge links by a different approach from that in [9, 15]. We can see that those also are (singular) conic bundles over \(\mathbb{P}^1\). Hence we can characterize their desingularizations by following the same method as in [9]. Namely, we will study the minimal model of desingularization of each case by checking explicitly the \((-1)\)-curves on it, avoiding the use of [9, Corollary 1] (since we are not sure the minimal model of a conic bundle over \(\mathbb{P}^1\) is always \(\mathbb{P}^2\)). The main result in this note is as follows:

**Theorem** The desingularizations of the canonical components of the \(\text{SL}_2(\mathbb{C})\)-character varieties of \(5_1^2, 6_2^2\) and \(6_3^2\) are conic bundles over the projective line \(\mathbb{P}^1\) which are isomorphic to the surface obtained from \(\mathbb{P}^1 \times \mathbb{P}^1\) by repeating a one-point blow-up \(9, 12\) and \(9\) times (or equivalently obtained from \(\mathbb{P}^2\) by repeating a one-point blow-up \(10, 13\) and \(10\) times), respectively.

Here we explain the outline of this note. In Sect. 2 we show the explicit defining equations of the natural models (\(\text{SL}_2(\mathbb{C})\)-character varieties) of the arithmetic two-bridge links and study their irreducible components. In particular we identify their canonical components. In Sect. 3 we describe the singular points of certain projective models of the canonical components of the natural models which are equipped with the conic bundle structure over \(\mathbb{P}^1\). We also compute explicitly the degenerate fibers of them, which is useful for the determination of minimal models of the desingularizations of those projective models. In Sect. 4 we determine minimal models of the desingularization of the projective models by employing intersection theory of surfaces. In Sect. 5 we characterize the desingularizations in terms of the number of blow-ups from the minimal models obtained in Sect. 4 by computing the Euler characteristics of the projective models.
2 Natural models

The $\text{SL}_2(\mathbb{C})$-character variety of a manifold $M$ is the set of characters of $\text{SL}_2(\mathbb{C})$-representations of the fundamental group $\pi_1(M)$ of $M$, which is known to be an affine algebraic set. For basics and applications of $\text{SL}_2(\mathbb{C})$-character varieties, see Culler and Shalen’s original paper [5] or Shalen’s survey paper [17]. It is known in general that we can compute the defining polynomials of $\text{SL}_2(\mathbb{C})$-character varieties of finitely generated groups explicitly from their group presentations [7, Theorem 3.2]. However in this note we only consider two-bridge link groups. In this case we can compute the defining polynomials by Riley’s method ([16, Section 2], or see [9, Section 4]) by which we can compute the defining polynomials with less computation.

Here we only show the result of computation of the defining polynomials of the $\text{SL}_2(\mathbb{C})$-character varieties for the arithmetic two-bridge link groups and show which irreducible component is the canonical component (that is, the irreducible component containing a point corresponding to the holonomy representation). We also include the Whitehead link case for the convenience of the reader. For the detailed way of the computation of the defining polynomials, see [9, Section 4].

2.1 Notation

Here we summarize some basic results on group presentations of the fundamental group of two-bridge link groups.

Let $L$ be a two-bridge link in the 3-sphere. Then it is well known (cf. [3, Chapter 12, G, E 12.1]) that its fundamental group has the following group presentation:

$$\pi_1(S^3 \setminus L) \cong \langle a, b \mid awAW = 1 \rangle,$$

where $A$ and $B$ mean the inverses $a^{-1}$ and $b^{-1}$, respectively. When $L$ is represented by the Schubert normal form $(\alpha/\beta)$, the word $w$ is defined by

$$w := b^{\epsilon_1} a^{\epsilon_2} b^{\epsilon_3} \cdots a^{\epsilon_{a-2}} b^{\epsilon_{a-1}},$$

where $\epsilon_i := (-1)^{[i\beta/\alpha]}$. Here, for a real number $r$, $[r]$ is the maximal integer not greater than $r$.

The $\text{SL}_2(\mathbb{C})$-character variety $X(M)$ of a manifold $M$ is the set of $\text{SL}_2(\mathbb{C})$-characters of $\pi_1(M)$, i.e.,

$$X(M) := \{ \chi_\rho := \text{Tr}(\rho) : \pi_1(M) \to \mathbb{C} \mid \rho : \pi_1(M) \to \text{SL}_2(\mathbb{C}) \}.$$
It is known (cf. [17, Theorem 4.5.1]) that the canonical component of the SL\(_2(\mathbb{C})\)-character variety of an \(n\)-component hyperbolic link complement has dimension \(n\). Specifically, the canonical component of the SL\(_2(\mathbb{C})\)-character variety of a hyperbolic two-bridge link complement is an irreducible affine surface over \(\mathbb{C}\).

Any SL\(_2(\mathbb{C})\)-character \(\chi\) of \(\pi_1(S^3 \setminus L)\) is determined by the values \(\chi(a), \chi(b), \chi(ab)\). Thus we have a canonical injection \(X(S^3 \setminus L) \to \mathbb{A}^3 := \mathbb{C}^3\) defined by

\[
\chi_{\rho} \mapsto (x, y, z) := (\chi_{\rho}(a), \chi_{\rho}(b), \chi_{\rho}(ab)).
\]

Put \(q := x^2+y^2+z^2-xyz-4\). It is known (cf. [12, Lemma 1.2.3]) that a representation \(\pi_1(S^3 \setminus L) \to \text{SL}_2(\mathbb{C})\) is reducible if and only if \(q(x, y, z) = 0\). In particular the points corresponding to abelian characters are contained in the algebraic set \(V(q)\).

Here, for polynomials \(f_1, \ldots, f_r \in \mathbb{C}[x, y, z]\) let \(V(f_1, \ldots, f_r)\) be the set of common zeros of \(f_1, \ldots, f_r\), i.e.

\[
V(f_1, \ldots, f_r) := \{(x, y, z) \in \mathbb{A}^3 \mid f_i(x, y, z) = 0, 1 \leq i \leq r\}.
\]

### 2.2 Whitehead link \(5_1^2\) case

The fundamental group \(\pi_1(S^3 \setminus 5_1^2)\) of the Whitehead link complement \(5_1^2 = (8/3)\) has \(w := babABab\). Then SL\(_2(\mathbb{C})\)-character variety of \(S^3 \setminus 5_1^2\) is defined by the following two polynomials:

\[
f_0 := p_0q, \quad g_0 := p_0(y - 2)(y + 2),
\]

where \(p_0 := z^3 - xyz^2 + (x^2 + y^2 - 2)z - xy\) and \(q\) are irreducible in \(\mathbb{C}[x, y, z]\) (for instance, if we assume there is a factorization of \(p_0\), immediately we have a contradiction by comparing degrees of monomials on both sides. We can show that \(q\) is irreducible in the same manner). Hence we have the decomposition

\[
X(S^3 \setminus 5_1^2) = V(p_0) \cup V(q, y - 2) \cup V(q, y + 2).
\]

Here \(V(q, y - 2)\) and \(V(q, y + 2)\) are affine lines \(\mathbb{A}^1\). Now the affine algebraic set \(V(p_0)\) defined by \(p_0\) is the unique 2-dimensional component of the character variety of \(5_1^2\). Hence that is the canonical component \(X_0(S^3 \setminus 5_1^2)\) of \(5_1^2\). The points of the algebraic set defined by the polynomial \(q\) correspond to the reducible representations of \(\pi_1(S^3 \setminus 5_1^2)\). We summarize that the natural model \(X(S^3 \setminus 5_1^2)\) consists of three irreducible algebraic sets \(V(p_0), V(q, y - 2)\) and \(V(q, y + 2)\). The canonical component \(X_0(S^3 \setminus 5_1^2) = V(p_0)\) is the unique irreducible algebraic subset of \(X(S^3 \setminus 5_1^2)\) of dimension 2. The other two components consist of points corresponding to the reducible SL\(_2(\mathbb{C})\)-characters of \(\pi_1(S^3 \setminus 5_1^2)\).
2.3 $6^2_2$ case

The fundamental group of the arithmetic two-bridge link $6^2_2 = (10/3)$ has $w := babABAbab$. Then the $SL_2(\mathbb{C})$-character variety of $6^2_2$ is defined by the following two polynomials:

$$f_1 := p_1 q, \quad g_1 := p_1 (y - 2)(y + 2),$$

where $p_1 := z^4 - xyz^3 + (x^2 + y^2 - 3)z^2 - xzy + 1$. Note that $p_1$ is irreducible in $\mathbb{C}[x, y, z]$ by a similar argument as in the Whitehead link case. The $SL_2(\mathbb{C})$-character variety $X(S^3 \setminus 6^2_2)$ consists of three algebraic sets

$$X(S^3 \setminus 6^2_2) = V(p_1) \cup V(q, y - 2) \cup V(q, y + 2).$$

Here $V(q, y - 2)$ and $V(q, y + 2)$ are affine lines $\mathbb{A}^1$. Now the affine algebraic set $V(p_1)$ defined by $p_1$ is the unique 2-dimensional component of the character variety of $6^2_2$. Hence that is the canonical component of $6^2_2$.

Thus the natural model $X(S^3 \setminus 6^2_2)$ consists of three irreducible components, the canonical component $X_0(S^3 \setminus 6^2_2) = V(p_1)$ and two components $V(q, y - 2)$ and $V(q, y + 2)$ which correspond to $SL_2(\mathbb{C})$-reducible characters of $\pi_1(S^3 \setminus 6^2_2)$.

2.4 $6^2_3$ case

The fundamental group of the arithmetic two-bridge link $6^2_3 = (12/5)$ has $w := baBAbabABab$.

Then $SL_2(\mathbb{C})$-character variety of $6^2_3$ is defined by the following two polynomials:

$$f_2 := p_2 qr, \quad g_2 := p_2 (y - 2)(y + 2),$$

where $p_2 := z^3 - xyz^2 + (x^2 + y^2 - 1)z - xy$ and $r := x^2 + y^2 + z^2 - xzy - 3$ are irreducible polynomials in $\mathbb{C}[x, y, z]$ by a similar argument as in the Whitehead link case. Thus we have the decomposition

$$X(S^3 \setminus 6^2_3) = V(p_2) \cup V(r, (y - 2)(y + 2)).$$

Here $V(q, (y - 2)(y + 2))$ is the union of two affine lines $\mathbb{A}^1$. There are two irreducible components of dimension 2 in this case. By considering the hyperbolicity equations of $S^3 \setminus 6^2_3$ and the fact that the images of meridians of holonomy representations are parabolic elements, we can compute the holonomy representation concretely. In fact, the holonomy representation of $S^3 \setminus 6^2_3$ is defined by

$$a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix},$$

where $\alpha = \frac{-1 + \sqrt{-7}}{2}$ up to conjugation. Then the point corresponding to the holonomy representation is $(x, y, z) = (2, 2, 2 + \alpha)$, which is a zero of the polynomial $p_2$. Hence the canonical
component of $6^2_3$ is the irreducible component $V(p_2)$. The other irreducible component
of dimension 2, $V(r)$ is a smooth affine cubic surface. Moreover, we see that its natural
homogenization $V_+(R) \subset \mathbb{P}^3$ defined by $R := (x^2 + y^2 + z^2)w + xyz - 3w^3$ is a
smooth projective cubic surface. Cubic surfaces in $\mathbb{P}^3$ are well-studied objects. It is a
Del Pezzo surface of degree 3, which is isomorphic to $\mathbb{P}^2$ with six points blown up (or
$\mathbb{P}^1 \times \mathbb{P}^1$ with five points blown up since $\mathbb{P}^2$ with two points blown up is isomorphic to
$\mathbb{P}^1 \times \mathbb{P}^1$ with one point blown up, cf. [8, V, Remark 4.7.1]).

Thus the natural model $X(S^3 \setminus 6^2_3)$ consists of four irreducible components, two
of which are the canonical component $X_0(S^3 \setminus 6^2_3) = V(p_2)$ and a smooth affine
surface $V(r)$. The other two components correspond to reducible $SL_2(\mathbb{C})$-characters
of $\pi_1(S^3 \setminus 6^2_3)$.

3 Projective models

Let $p_0$, $p_1$ and $p_2$ be the polynomials in the previous section which define the canonical
component of the $SL_2(\mathbb{C})$-character variety of the Whitehead link, $6^2_2$ and $6^2_3$ link
respectively. Then the Jacobian criterion shows that $V(p_i)$ is a smooth affine surface
for any $i$. The projective surfaces in $\mathbb{P}^3$ obtained by homogenizing $p_i$ naturally have
infinitely many singularities. Thus we consider another compactification, namely a
compactification in $\mathbb{P}^2 \times \mathbb{P}^1$ to obtain a projective surface having fewer singularities.
We follow the method introduced in [9] and [11]. After reviewing $A_n$-singularities in
Sect. 3.1 we study the homogenizations $S_i$ of $V(p_i)$ in $\mathbb{P}^2 \times \mathbb{P}^1$.

3.1 $A_n$-singularities

The du Val singularities (or rational double points) are one kind of isolated singularity
of a complex surface whose exceptional curve consists of a tree of rational smooth
curves. They are the unique rational singularity for hypersurfaces in $\mathbb{A}^3$. It is classified
into three types ($A$-$D$-$E$ singularities). Here we only explain the $A_n$-singularity. The
$A_n$-singularity is one type of the du Val singularity characterized by the singular point
$(0, 0, 0)$ of the equation $x^2 + y^2 + z^{n+1} = 0$. The exceptional curve of $x^2 + y^2 + z^{n+1} = 0$ at the singular point
$(0, 0, 0)$ obtained by blowing up a number of times consists of $n$
smooth projective irreducible curves (isomorphic to $\mathbb{P}^1$) with self-intersection number
$-2$, which intersect each other transversally as described in Fig. 1.

Fig. 1 Exceptional curve of $A_n$-singularity
The left and right most curves each intersect only one other curve, and the intersection is one point. The other curves meet with two other curves transversally. It is also expressed by the Dynkin diagram of $A_n$-singularity

\[
\begin{array}{c}
\circ \quad \circ \quad \circ - - \circ \quad \circ \\
\end{array}
\]

We have a relation between the topological Euler characteristics of a (singular) surface and its desingularization as follows.

**Lemma 3.1** (cf. [9, Proposition 2]) Let $S$ be an irreducible smooth projective surface over $\mathbb{C}$ and $p$ a point of $S$. Let $\widetilde{S}$ be the blow-up of $S$ at $p$. Then $\chi(\widetilde{S}) = \chi(S) + 1$.

**Lemma 3.2** Let $S$ be an irreducible projective surface over $\mathbb{C}$, $p \in S$ an $A_n$-singular point and let $\epsilon : \widetilde{S} \rightarrow S$ be the desingularization of $S$ at the point $p$. Then we have $\chi(\widetilde{S}) = \chi(S) + n$.

**Proof** Note that the fiber of $\epsilon : \widetilde{S} \rightarrow S$ at the point $p$ consists of $n$ projective lines which intersect with each other as in Fig. 1. Since $\chi(\mathbb{P}^1) = 2$ and $\chi($point$) = 1$, we have

\[
\chi(\widetilde{S}) = \chi(\widetilde{S}\setminus \epsilon^{-1}(p)) + \chi(\epsilon^{-1}(p)) = \chi(S\setminus \{p\}) + (n\chi(\mathbb{P}^1) - (n - 1))
\]

\[= \chi(S) - 1 + n + 1 = \chi(S) + n. \quad \square \]

**3.2 Projective models of the canonical components**

Let

\[\mathbb{P}^2 \times \mathbb{P}^1 := \{(x : y : u, z : w) \mid (x : y : u) \in \mathbb{P}^2, (z : w) \in \mathbb{P}^1\}\]

be the product of $\mathbb{P}^2$ and $\mathbb{P}^1$, and let

\[
F_0 := u^2z^3 - xyz^2w + (x^2 + y^2 - 2u^2)zw^2 - xyw^3,
\]

\[
F_1 := u^2z^4 - xyz^3w + (x^2 + y^2 - 3u^2)z^2w^2 - xyzw^3 + u^2w^4,
\]

\[
F_2 := u^2z^3 - xyz^2w + (x^2 + y^2 - u^2)zw^2 - xyw^3
\]

be the homogenization of $p_0$, $p_1$ and $p_2$ in $\mathbb{P}^2 \times \mathbb{P}^1$. Consider the algebraic set

\[S_i := V(F_i) := \{(x : y : u, z : w) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid F_i(x, y, u, z, w) = 0\}\]

defined by $F_i$ in $\mathbb{P}^2 \times \mathbb{P}^1$. Since $\mathbb{A}^3$ is naturally embedded in $\mathbb{P}^2 \times \mathbb{P}^1$ as

\[
\{(x : y : 1, z : 1) \mid (x, y) \in \mathbb{A}^2, z \in \mathbb{A}^1\},
\]

$V(p_i)$ is embedded in $S_i$ birationally.
Let $\phi : \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection which is defined by $(x : y : u, z : w) \mapsto (z : w)$ and define $\phi_i$ by the restriction of $\phi$ on $S_i$. We note that all the fibers of $S_i$ except finitely many points are smooth conics in $\mathbb{P}^2$. Hence $\phi_i$ defines a (singular) conic bundle structure on $S_i$. In the following subsections we show the explicit description of the singular (degenerate) fibers of $\phi_i$ and compute the Euler characteristic $\chi(S_i)$ in terms of $\chi(S_0)$.

### 3.3 Whitehead link case

Let $F_0 := u^2z^3 - xyz^2w + (x^2 + y^2 - 2u^2)zw^2 - xyw^3$ be the homogenization of the polynomial $p_0 = z^3 - xyz^2 + (x^2 + y^2 - 2)z - xy$ in the projective space $\mathbb{P}^2 \times \mathbb{P}^1$. Let $S_0 := V(F_0)$ be the algebraic set defined by $F_0$ in $\mathbb{P}^2 \times \mathbb{P}^1$. It is shown in [9, Section 4] that $S$ has only four singular points

$$(1:0:0, 1:0), (0:1:0, 1:0), (1:1:0, 1:1), (1:-1:0, 1:-1).$$

These four points are $A_1$ singularities (we can resolve the singularity by blowing up once) and the exceptional curves at the singular points are isomorphic to the projective line $\mathbb{P}^1$. Thus we have the following relation on the topological Euler characteristic of $S_0$ and the desingularization $\widetilde{S}_0$ by Lemma 3.2:

$$\chi(\widetilde{S}_0) = \chi(S_0 \setminus S_0,_{\text{sing}}) + 4\chi(\mathbb{P}^1)$$

$$= \chi(S_0) - 4 + 4\chi(\mathbb{P}^1) = \chi(S_0) + 4.$$

Here $S_{0,_{\text{sing}}}$ is the set of singular points of $S_0$. In Sect. 5 we compute the topological Euler characteristic $\chi(S_0)$ and determine $\widetilde{S}_0$ in terms of the number of one-point blow-ups from a minimal model of $\widetilde{S}_0$.

Note that we can consider the surface $S_0$ (hence $\widetilde{S}_0$) as a (singular) conic bundle over $\mathbb{P}^1$ by the projection $\phi_0 : S_0 \to \mathbb{P}^1$ which is defined by $(x : y : u, z : w) \mapsto (z : w)$. It has six degenerate fibers at $(1:0), (0:1), (1:\pm 1), (1: \pm \frac{1}{\sqrt{2}})$. In fact, the degenerate fibers of $\phi_0 : S_0 \to \mathbb{P}^1$ are expressed as follows:

$$\phi_0^{-1}(1:0) \cong \{(x : y : u) \in \mathbb{P}^2 \mid u^2 = 0\},$$

$$\phi_0^{-1}(0:1) \cong \{(x : y : u) \in \mathbb{P}^2 \mid xy = 0\},$$

$$\phi_0^{-1}(1: \pm \frac{1}{\sqrt{2}}) \cong \{(x : y : u) \in \mathbb{P}^2 \mid \frac{1}{2}(x \mp \sqrt{2}y)(x \mp \frac{1}{\sqrt{2}}y) = 0\},$$

$$\phi_0^{-1}(1: \pm 1) \cong \{(x : y : u) \in \mathbb{P}^2 \mid (x \mp y) - u((x \mp y) + u) = 0\}.$$

Note that the fiber $\phi_0^{-1}(1:0)$ contains the singular points $(1:0:0, 1:0), (0:1:0, 1:0)$ of the surface $S_0$. The fiber $\phi_0^{-1}(1: \pm 1)$ contains the singular point $(1: \pm 1:0, 1: \pm 1)$ of the surface $S_0$ respectively.
3.4 $6^3_2$ case

Let $F_1 := u^2 z^4 - xyz^3 w + (x^2 + y^2 - 3u^2) z^2 w^2 - xyz w^3 + u^2 w^4$ be the homogenization of $p_1 = z^4 - xyz^3 + (x^2 + y^2 - 3) z^2 - xyz + 1$ in $\mathbb{P}^2 \times \mathbb{P}^1$ with coordinates $x$, $y$, $u$ and $z$, $w$, and let

$$S_1 := V(F_1) := \{(x : y : u, z : w) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid F_1(x, y, u, z, w) = 0\}$$

be the algebraic set defined by $F_1$ in $\mathbb{P}^2 \times \mathbb{P}^1$. (This is symmetric on $x$, $y$ and $z$, $w$.) This projective surface has only finitely many singular points. In fact, its singularities are only the following six points:

$$(1 : 0 : 0, 1 : 0), \ (0 : 1 : 0, 1 : 0), \ (1 : 0 : 0, 0 : 1),$$
$$(0 : 1 : 0, 0 : 1), \ (1 : 1 : 0, 1 : 1), \ (1 : -1 : 0, 1 : -1),$$

which are $A_1$ singularities. Specifically they are resolved by one blow-up at each point. Let $\widetilde{S}_1 \to S_1$ be the desingularization of $S_1$ blown up at these six points. Then the exceptional curve at each singular point is isomorphic to $\mathbb{P}^1$. Thus we have the following relation on the topological Euler characteristic of $S_1$ and $\widetilde{S}_1$ by Lemma 3.2:

$$\chi(\widetilde{S}_1) = \chi(S_1 \setminus S_1, \text{sing}) + 6\chi(\mathbb{P}^1) = \chi(S_1) + 6 + 6\chi(\mathbb{P}^1) = \chi(S_1) + 6.$$

As a (singular) conic bundle over $\mathbb{P}^1$ by the projection $\phi_1 : S_1 \to \mathbb{P}^1$ the surface $S_1$ (hence $\widetilde{S}_1$) has eight degenerate fibers at $(1 : 0), \ (0 : 1), \ (1 : \pm 1), \ (1 : \pm \sqrt{5} \pm 1)$. In fact, they are written as follows:

$$\phi_1^{-1}(1 : 0) \cong \{(x : y : u) \in \mathbb{P}^2 \mid u^2 = 0\},$$
$$\phi_1^{-1}(0 : 1) \cong \{(x : y : u) \in \mathbb{P}^2 \mid u^2 = 0\},$$
$$\phi_1^{-1}(1 : \sqrt{5} \pm 1) \cong \{(x : y : u) \in \mathbb{P}^2 \mid \frac{3 \pm \sqrt{5}}{2}(x - \sqrt{5} \pm y)(x - \sqrt{5} \mp y) = 0\},$$
$$\phi_1^{-1}(1 : -\sqrt{5} \pm 1) \cong \{(x : y : u) \in \mathbb{P}^2 \mid \frac{3 \pm \sqrt{5}}{2}(x + \sqrt{5} \pm y)(x + \sqrt{5} \mp y) = 0\},$$
$$\phi_1^{-1}(1 : \pm 1) \cong \{(x : y : u) \in \mathbb{P}^2 \mid ((x \mp y) - u)((x \mp y) + u) = 0\}.$$

We remark that the fiber $\phi_1^{-1}(1 : 0)$ (resp. $\phi_1^{-1}(0 : 1)$) contains the singular points $(1 : 0 : 0, 1 : 0), \ (0 : 1 : 0, 1 : 0) \ (0 : 1 : 0, 0 : 1)$, and that each fiber $\phi_1^{-1}(1 : \pm 1)$ contains the singular point $(1 : \pm 1 : 0, 1 : \pm 1)$ of $S_1$. In the next section we compute a minimal model of the desingularization of the surface $S_1$.

3.5 $6^2_3$ case

Let $p_2 := z^3 - xyz^2 + (x^2 + y^2 - 1) z - xy$. Let $F_2 = u^2 z^3 - xyz^2 w + (x^2 + y^2 - u^2) z w^2 - xy w^3$ be the homogenization of $p_2$ in $\mathbb{P}^2 \times \mathbb{P}^1$ with the coordinates $(x : y : u, z : w)$. The corresponding projective surface $S_2 = V(F_2) \subset \mathbb{P}^2 \times \mathbb{P}^1$ has the
following four singular points:

$$(1 : 0 : 0, 1 : 0), \ (0 : 1 : 0, 1 : 0), \ (1 : 1 : 0, 1 : 1), \ (1 : -1 : 0, 1 : -1).$$

We remark that the first two points are $A_1$ singularities and the other two points are $A_3$ singularities. Hence we can resolve the first two singularities by blowing up once at each point but we have to blow up twice for the latter two points. Let $\tilde{S}_i$ be the smooth projective surface obtained by blowing up $S_i$ at these four singular points. The exceptional curves at the singular points $(1 : 0 : 0, 1 : 0)$ and $(0 : 1 : 0, 1 : 0)$ are isomorphic to $\mathbb{P}^1$, and the exceptional curve at $(1 : 1 : 0, 1 : 1)$ (resp. $(1 : -1 : 0, 1 : -1)$) is the union of three curves $E_1^+, E_2^+$ and $E_3^+$ (resp. $E_1^-, E_2^-$ and $E_3^-$) respectively. Here $E_i^\pm$ are smooth projective curves isomorphic to $\mathbb{P}^1$ with self-intersection number $-2$. The curve $E_2^\pm$ intersects with $E_1^\pm$ and $E_3^\pm$ at one point respectively and $E_1^\pm$ and $E_3^\pm$ do not intersect each other (see Fig. 1). Thus we can express the Euler characteristic of $\tilde{S}_2$ in terms of that of $S_2$ by Lemma 3.2:

$$\chi(\tilde{S}_2) = \chi(S_2 \setminus S_2,\text{sing}) + 2\chi(\mathbb{P}^1) + 2\chi(E_1^+ \vee E_2^+ \vee E_3^+)$$

$$= \chi(S_2) - 4 + 4 + 8 = \chi(S_2) + 8.$$

We can consider the surface $S_2$ (hence $\tilde{S}_2$) as a conic bundle over $\mathbb{P}^1$ by the projection $\phi_2 : S_2 \to \mathbb{P}^1$. It has four degenerate fibers at $(1 : 0), (0 : 1), (1 : \pm 1)$. In fact, the degenerate fibers of $\phi_2 : S_2 \to \mathbb{P}^1$ are expressed as follows:

$$\phi_2^{-1}(1 : 0) \sim \{(x : y : u) \in \mathbb{P}^2 | u^2 = 0\},$$

$$\phi_2^{-1}(0 : 1) \sim \{(x : y : u) \in \mathbb{P}^2 | xy = 0\},$$

$$\phi_2^{-1}(1 : \pm 1) \sim \{(x : y : u) \in \mathbb{P}^2 | (x \mp y)^2 = 0\}.$$

We note that the fiber $\phi_2^{-1}(1 : 0)$ contains the singular points $(1 : 0 : 0, 1 : 0)$, $(0 : 1 : 0, 1 : 0)$ of $S_2$. The fiber $\phi_2^{-1}(1 : \pm 1)$ contains the singular point $(1 : \pm 1 : 0, 1 : \pm 1)$ of the surface $S_2$ respectively.

### 4 Minimal models

Since all the three surfaces $S_0, S_1, S_2$ are rational surfaces, their minimal models are either the projective plane $\mathbb{P}^2$ or the Hirzebruch surfaces $\mathbb{F}_n$ $(n \geq 0, n \neq 1)$ (cf. [2, Theorem V.10]). Here we compute a minimal model of the surface $S_i$ for each $i$, which is obtained naturally from its fibered surface structure. The purpose of this section is to prove the following two lemmas.

**Lemma 4.1** For each $\tilde{S}_i$ we can blow down $\tilde{S}_i$ over $\mathbb{P}^1$ a number of times so that it becomes a geometrically ruled surface $T_i$ over $\mathbb{P}^1$, namely all the fibers are isomorphic to $\mathbb{P}^1$.

**Lemma 4.2** $T_i$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. 

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In Sect. 4.1 we review some terminology on algebraic surfaces and some basic facts on the intersection theory of algebraic surfaces and minimal models. In Sects. 4.2 and 4.3 we show Lemmas 4.1 and 4.2.

4.1 Preliminaries on algebraic surfaces

4.1.1 Basic properties of intersection theory of surfaces

Here we summarize some basic properties of the intersection theory of algebraic surfaces. In particular we include some results on the intersection numbers of divisors of fibered surfaces, which are necessary for the computation of minimal models obtained by blowing down the conic bundles over \( \mathbb{P}^1 \) that appeared in the previous section. For more details, see [2, 8] or [18, III, Sections 7, 8].

In Sects. 4.1.1 and 4.1.2, a curve or a surface always means a smooth projective irreducible curve or surface unless otherwise mentioned.

Let \( S \) be a surface over the field of complex numbers \( \mathbb{C} \) and \( \mathbb{C}(S) \) the function field of \( S \). Let \( \text{Div}(S) \) be the group of all the divisors of \( S \) and \( \text{div} : \mathbb{C}(S)^\times \to \text{Div}(S) \) the divisor function. We say that two divisors \( D, D' \in \text{Div}(S) \) are linearly equivalent if \( D' = D + \text{div}(f) \) for some \( f \in \mathbb{C}(S)^\times \). Then there is a unique symmetric bilinear pairing

\[
(\cdot, \cdot) : \text{Div}(S) \times \text{Div}(S) \to \mathbb{Z}
\]

which satisfies the following two properties:

1. For curves \( C_1, C_2 \) on \( S \) which meet everywhere transversally, it holds \( (C_1, C_2) = \#(C_1 \cap C_2) \), where \( \#(C_1 \cap C_2) \) is the number of points of \( C_1 \cap C_2 \).
2. If \( D, D_1, D_2 \in \text{Div}(S) \) are divisors and \( D_1, D_2 \) are linearly equivalent, then \( (D, D_1) = (D, D_2) \). Hence the pairing \((\cdot, \cdot)\) induces the pairing \( \text{Pic}(S) \times \text{Pic}(S) \to \mathbb{Z} \) on the Picard group of \( S \) (the quotient group of \( \text{Div}(S) \) by the image of \( \text{div} \)).

We will also write \( D \cdot D' \) in place of \( (D, D') \) for two divisors \( D, D' \in \text{Div}(S) \). For any divisor \( D \in \text{Div}(S) \) we call \( D^2 := D \cdot D \) the self-intersection number of \( D \).

We say that a surface \( S \) is a fibered surface over a curve \( C \) if there is a surjective morphism \( \pi : S \to C \). The fiber \( \pi^{-1}(t) \) for a point \( t \in C \) is a smooth projective curve for all but finitely many points of \( C \). For any curve \( D \) on a fibered surface \( S \) over \( C \) its image \( \pi(D) \) is either a point or \( C \). A curve \( D \) on \( S \) is called fibral (or vertical) if the image is a point, and is called horizontal if the image is the curve \( C \). A divisor \( D \in \text{Div}(S) \) is called vertical (horizontal) if all the curves appearing as the components of \( D \) are vertical (horizontal). The structure morphism \( \pi : S \to C \) of a fibered surface induces the homomorphism

\[
\pi^* : \text{Div}(C) \to \text{Div}(S)
\]
defined by
\[
\sum_{t \in C} n_t \mathcal{I} \leftrightarrow \sum_{t \in C} n_t \sum_{\Gamma \subset \pi^{-1}(t)} \text{ord}_\Gamma(u_t \circ \pi)[\Gamma],
\]
where \( u_t \) is a uniformizer of the function field \( \mathbb{C}(C) \) of \( C \) at \( t \) and \( \Gamma \) runs through all the curves in \( \pi^{-1}(t) \), and \( \text{ord}_\Gamma(u_t \circ \pi) \) is the order of \( u_t \circ \pi \) in the function field \( \mathbb{C}(S) \) of \( S \) by the discrete valuation \( \text{ord}_\Gamma \) defined by \( \Gamma \). (For the definition of the inverse image of a divisor in general, see, e.g. [2, I] or [8, II.6].)

**Proposition 4.3** (cf. [2, Proposition I.8])

1. Let \( \pi : S \to C \) be a fibered surface over a curve \( C \). If \( F \) is a fiber of \( \phi \) (that is, \( F = \pi^*[t] \) for some \( t \in C \)), then \( F^2 = 0 \).
2. Let \( S, S' \) be surfaces and \( g : S' \to S \) a generically finite morphism of degree \( d \). If \( D, D' \) are divisors on \( S \), then \( g^*D \cdot g^*D' = d(D \cdot D') \).

**Lemma 4.4** ([18, Chapter III, Lemma 8.1]) Let \( \pi : S \to C \) be a fibered surface and \( \delta \in \text{Div}(C) \). If \( D \in \text{Div}(S) \) is a vertical divisor then \( D \cdot \pi^*(\delta) = 0 \).

### 4.1.2 Minimal models of surfaces

Here we summarize some basic facts on minimal models we need in this note.

Let \( S \) be a surface over \( \mathbb{C} \) and \( E \) a curve on \( S \). A curve \( E \) on \( S \) is called an exceptional curve if it is obtained as a component of the fiber of a point by blowing up a (possibly singular) surface.

**Proposition 4.5** (cf. [2, II, 1]) Let \( S \) be a smooth surface and \( p \) a point on \( S \). Let \( \epsilon : \tilde{S} \to S \) be the blow-up morphism of \( S \) at the point \( p \). Then \( \epsilon^{-1}(p) \) is an irreducible curve on \( S \) isomorphic to \( \mathbb{P}^1 \).

We say that a curve \( E \) is a \((-1)-curve\) if it is isomorphic to \( \mathbb{P}^1 \) and its self-intersection number is \(-1\), that is, \( E^2 = -1 \).

**Proposition 4.6** (cf. [2, Lemma II, 2, Proposition II, 3, (i), (ii)]) Let \( S \) be a surface and \( p \) a point on \( S \). Let \( \epsilon : \tilde{S} \to S \) be the blow-up at \( p \) and \( E \) the exceptional curve of \( p \).

1. There is an isomorphism \( \text{Pic } S \oplus \mathbb{Z} \sim \text{Pic } \tilde{S} \) defined by \( (D, n) \mapsto \epsilon^*D + nE \).
2. If \( D, D' \in \text{Div}(S) \), then \( \epsilon^*D \cdot \epsilon^*D' = D \cdot D', E \cdot \epsilon^*D = 0 \) and \( E^2 = -1 \).
3. If \( C \) is an irreducible projective curve on \( S \) which passes through the point \( p \) with multiplicity \( m \), then \( \epsilon^*C = \tilde{C} + mE \) (\( \tilde{C} \) is the strict transform of \( C \), namely the closure of \( \epsilon^{-1}(C - p) \) in \( \tilde{S} \)).

**Theorem 4.7** (Castelnuovo’s contractibility Theorem, cf. [2, II, Theorem 17]) Let \( S \) be a surface over \( \mathbb{C} \) and \( E \) a \((-1)-curve\) on \( S \). Then \( E \) is an exceptional curve on \( S \), namely there is a surface \( S' \) and a morphism \( \epsilon : S \to S' \) such that \( \epsilon \) is the blow-up of \( S' \) at a point \( p \) with \( E = \epsilon^{-1}(p) \).
A surface \( S \) is called \textit{relatively minimal} if there are no \((-1)\)-curves on \( S \). There are only finitely many \((-1)\)-curves on a surface. Therefore, for a surface \( S \) we can find a sequence of surfaces

\[ S \rightarrow S_1 \rightarrow S_2 \rightarrow \cdots \rightarrow S_n \]

such that \( S_n \) is relatively minimal. Such a surface \( S_n \) is called a \textit{relatively minimal model} (usually called a minimal model) of \( S \). Note that a minimal model is not necessarily unique for a given surface.

A \textit{rational surface} is an irreducible smooth projective surface which is birational to \( \mathbb{P}^2 \). It is known that, for any surface \( S \) except rational surfaces, there is a unique relatively minimal model (the \textit{minimal model} of \( S \)). For rational surfaces, the classification of relatively minimal models is known. Namely, there are two types of minimal models: the projective plane \( \mathbb{P}^2 \) and the Hirzebruch surfaces \( \mathbb{F}_n \) \((n \geq 0, n \neq 1)\) (cf. [2, Theorem V.10]). Here a Hirzebruch surface \( \mathbb{F}_n \) is a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^1 \) associated with the sheaf \( \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n) \) for \( n \geq 0 \), where \( \mathcal{O}_{\mathbb{P}^1} \) is the structure sheaf of \( \mathbb{P}^1 \) and \( \mathcal{O}_{\mathbb{P}^1}(-n) \) is the inverse of the \( n \) tensor product of the Serre twisting sheaf \( \mathcal{O}_{\mathbb{P}^1}(1) \).

A \textit{geometrically ruled surface} \( S \) over a curve \( C \) is an irreducible smooth projective surface together with a smooth morphism \( \pi : S \rightarrow C \) such that all the fibers are isomorphic to \( C \). It is known that geometrically ruled surfaces over \( \mathbb{P}^1 \) are only Hirzebruch surfaces \( \mathbb{F}_n \) (see [2, Proposition III 15]). Moreover, the surfaces \( \mathbb{F}_n \) are relatively minimal for any \( n \geq 0 \) except 1, and \( \mathbb{F}_1 \) is isomorphic to \( \mathbb{P}^2 \) blown up at one point (see [2, Proposition IV 1]).

### 4.2 Proof of Lemma 4.1

To show Lemma 4.1, here we compute the surface \( T_i \) obtained by blowing down all the \((-1)\)-curves on \( \tilde{S}_i \) which appear as the components of fibers of the morphism \( \tilde{S}_i \rightarrow S_i \xrightarrow{\phi_i} \mathbb{P}^1 \). Then we see that \( T_i \) is a geometrically ruled surface over \( \mathbb{P}^1 \). In Sect. 4.3 we show that \( T_i \) is a minimal model of \( \tilde{S}_i \) for each \( i \).

#### 4.2.1 Whitehead link case

As we have seen in Sect. 3, \( \phi_0 : S_0 \rightarrow \mathbb{P}^1 \) has six degenerate fibers. Thus, the composite morphism of \( \phi_0 \) and the blow-up morphism \( \tilde{\phi}_0 : \tilde{S}_0 \rightarrow S_0 \xrightarrow{\phi_0} \mathbb{P}^1 \) also has six degenerate fibers. Since the other fibers are conics in \( \mathbb{P}^2 \) they are isomorphic to \( \mathbb{P}^1 \). Here we show that we can blow down \( \tilde{S}_0 \) over \( \mathbb{P}^1 \) a number of times so that it becomes a geometrically ruled surface over \( \mathbb{P}^1 \).

First consider the \( \phi_0^{-1}(0 : 1) \) case. The fiber \( \phi_0^{-1}(0 : 1) \) consists of two \( \mathbb{P}^1 \) which intersect each other transversally. There are no singular points of \( S_0 \) on this fiber. Hence it is isomorphic to \( \tilde{\phi}_0^{-1}(0 : 1) \). Now we show that each curve \( C_i \) in the fiber has self-intersection number \(-1\). By Lemma 4.4, \( (C_1 + C_2) \cdot C_i = 0 \). Since \( (C_1 + C_2) \cdot C_i = C_i^2 + 1 \), we have \( C_i^2 = -1 \). Thus we can blow down one of these two \((-1)\)-curves in the fiber by Theorem 4.7, and the fiber becomes a curve \( \mathbb{C} \xrightarrow{\sim} \mathbb{P}^1 \) with self-intersection number \(-1\).
number 0 by Proposition 4.6 (2), (3). We can work on the two fibers \( \phi_0^{-1}(1: \pm \frac{1}{\sqrt{2}}) \) completely in the same way.

Next we consider the case \( \phi_0^{-1}(1:0) \). As we have seen in Sect. 3.3, the fiber \( \phi_0^{-1}(1:0) \) is a double \( \mathbb{P}^1 \) line \( C \) which contains two \( A_1 \) singular points of \( S_0 \). Then the divisor \( \widetilde{\phi}_0^*(1:0) \) in \( \text{Div} \widetilde{S}_0 \) is written as \( 2C' + E_1 + E_2 \), where \( \widetilde{\phi}_0: \widetilde{S}_0 \to S_0 \xrightarrow{\phi_0} \mathbb{P}^1 \), the curve \( C' \) is the strict transform of \( C \) in \( \widetilde{S}_0 \) and \( E_i \) are the exceptional curves which are isomorphic to \( \mathbb{P}^1 \) with \( E_i^2 = -2 \) and \( E_i \cdot C' = 1 \). Since \( \phi_0^*(1:0) \cdot C' = 0 \), we have \( C'^2 = -1 \). Hence we can blow down at \( C' \) and the fiber becomes \( E_1' + E_2' \) with \( E'^2 = -1 \) by Proposition 4.6 (2), (3). Thus we can blow down again and the fiber becomes \( \mathbb{P}^1 \). Finally we consider the case \( \phi_0^{-1}(1:1) \). We only have to consider the case \( \phi_0^{-1}(1:1) \) (we can work on \( \phi_0^{-1}(1:-1) \) similarly). It consists of two rational curves \( C_1 \) and \( C_2 \) which intersects each other transversally at one point \((1:1:0, 1:1)\), which is an \( A_1 \) singular point of \( S_0 \). Hence the divisor \( \widetilde{\phi}_0^*(1:1) \) is \( C'_1 + C'_2 + E \), where \( C'_i \) is the strict transform of \( C_i \) on \( \widetilde{S}_0 \) and \( E \) is the exceptional curve obtained by resolving the singularity at \((1:1:0, 1:1)\). The exceptional curve \( E \) intersects with \( C'_1, C'_2 \) transversally at one point respectively and \( C'_1 \) and \( C'_2 \) do not meet at any point. Hence \( E \cdot C'_i = 1 \) and \( C'_1 \cdot C'_2 = 0 \). Since \( E^2 = -2 \), we have \( C'^2_i = -1 \). Thus we can blow down at \( C'_1 \) and \( C'_2 \) and obtain one rational curve \( E'' \) with self-intersection number 0.

Therefore the surface \( T_0 \) obtained by blowing down all the degenerate fibers of \( \widetilde{S}_0 \) over \( \mathbb{P}^1 \) is a geometrically ruled surface over \( \mathbb{P}^1 \).
4.2.2 \( g_2^2 \) case

As we have seen in Sect. 3, \( \phi_1 : S_1 \to \mathbb{P}^1 \) has eight degenerate fibers. Thus the composite morphism \( \widetilde{\phi}_1 : \widetilde{S}_1 \to S_1 \xrightarrow{\phi_1} \mathbb{P}^1 \) also has eight degenerate fibers. Here we show that the surface \( T_1 \) obtained by blowing down all the degenerate fibers of \( \widetilde{S}_1 \) is also a geometrically ruled surface.

First note that, the four fibers \( \phi_1^{-1}(1 : \frac{\sqrt{5}+1}{2}) \) and \( \phi_1^{-1}(1 : -\frac{\sqrt{5}+1}{2}) \) are the unions of two \( \mathbb{P}^1 \)s intersecting each other transversally at one point, and they do not contain singular points of the surface \( S_1 \). Hence the situation is the same as the \( \phi_0^{-1}(0 : 1) \) case. By the same manner as in the \( \phi_0^{-1}(0 : 1) \) case, we can show that each curve in the four fibers have self-intersection number \(-1\) and can be blown down to obtain one \( \mathbb{P}^1 \).

The fibers \( \phi_1^{-1}(1 : 0) \) and \( \phi_1^{-1}(1 : \pm 1) \) have the same shape and singular points as \( \phi_0^{-1}(1 : 0) \) and \( \phi_0^{-1}(1 : \pm 1) \) which appeared in the Whitehead link case. Therefore we see that the fibers \( \widetilde{\phi}_1^{-1}(1 : 0) \) and \( \widetilde{\phi}_1^{-1}(1 : \pm 1) \) can be blown down and we have \( \mathbb{P}^1 \).

Note that the same argument as for \( \phi_1^{-1}(1 : 0) \) also works for \( \phi_1^{-1}(0 : 1) \) since they are symmetric.

Hence the surface \( T_1 \) obtained by blowing down \( \widetilde{S}_1 \) over \( \mathbb{P}^1 \) is a geometrically ruled surface over \( \mathbb{P}^1 \).

4.2.3 \( g_3^2 \) case

As we have seen in the previous section, \( \phi_2 : S_2 \to \mathbb{P}^1 \) has four degenerate fibers. We can work on the fiber \( \phi_2^{-1}(1 : 0) \) (resp. \( \phi_2^{-1}(0 : 1) \)) in the same way as in the \( \phi_0^{-1}(1 : 0) \) (resp. \( \phi_0^{-1}(0 : 1) \)) case.

Now we consider the \( \phi_2^{-1}(1 : 1) \) (resp. \( \phi_2^{-1}(1 : -1) \)) case. Since the point \( (1 : 1 : 0, 1 : 1) \) (resp. \( (1 : -1 : 0, 1 : -1) \)) is an \( A_3 \) singular point, the exceptional curve consists of three rational curves \( E_1, E_2 \) and \( E_3 \) with \( E_1^2 = -2, E_1 \cdot E_2 = E_3 \cdot E_2 = 1 \) and \( E_1 \cdot E_3 = 0 \). Then the divisor \( \widetilde{\phi}_2^{-1}(1 : 1) \) (resp. \( \widetilde{\phi}_2^{-1}(1 : -1) \)) is \( 2C' + E_1 + 2E_2 + E_3 \) (see Fig. 2), where \( C' \) is the strict transform on \( S_2 \) of the curve on \( S_2 \) defined by \( x - y = 0 \) (resp. \( x + y = 0 \)).

Since \( \widetilde{\phi}_2^{-1}(1 : 1)^2 = 0 \) (resp. \( \widetilde{\phi}_2^{-1}(1 : \pm 1)^2 = 0 \)), we have \( C'^2 = -1 \). Hence we can also blow down this fiber for \( C' \), and the divisor of fiber on the blow down is \( E_1 + 2E_2 + E_3 \) with \( E_2'^2 = -1 \) and \( E_1 \cdot E_2' = E_3 \cdot E_2' = 1 \). Now the situation is the same as the \( \phi_0^{-1}(1 : 0) \) case. We can blow down the fiber twice to have one \( \mathbb{P}^1 \). Thus we obtain a geometrically ruled surface \( T_2 \) over \( \mathbb{P}^1 \) by blowing down \( \widetilde{S}_2 \) repeatedly.
4.3 Proof of Lemma 4.2

In Sect. 4.2 we have shown that the surfaces $T_i$ obtained by blowing down the degenerate fibers of $\tilde{S}_i$ are geometrically ruled surfaces. Since Hirzebruch surfaces are the only rational surfaces which are geometrically ruled surfaces, each $T_i$ is isomorphic to a Hirzebruch surface $F_n$ for some $n \geq 0$. It remains to determine the number $n$. In fact we show that $T_i$ are the Hirzebruch surface $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Therefore $T_i$ are minimal models of $\tilde{S}_i$. Here we review a proposition on the Hirzebruch surfaces.

Proposition 4.8 (cf. [2, Proposition IV.1])

Let $F_n$ be a Hirzebruch surface ($n \geq 0$). Let $f \in \text{Pic } F_n$ be the element defined by a fiber of $F_n$ over $\mathbb{P}^1$ and let $h \in \text{Pic } F_n$ be the element corresponding to the tautological line bundle (that is, the invertible sheaf $O_{F_n}(-1)$).

1. \[ \text{Pic } F_n = \mathbb{Z}h \oplus \mathbb{Z}f \text{ with } f^2 = 0 \text{ and } h^2 = n. \]
2. When $n > 0$, there exists a unique irreducible projective curve $B$ on $F_n$ with $b = [B]^2 < 0$. $b$ is written as $b = h - nf$. Thus $b^2 = -n$.
3. If $n \neq m$, two surfaces $F_n$ and $F_m$ are not isomorphic. $F_n$ is a minimal model except if $n = 1$. $F_0$ is $\mathbb{P}^1 \times \mathbb{P}^1$ and $F_1$ is isomorphic to $\mathbb{P}^2$ with one point blown-up.

Note that we can take two sections $s_k : \mathbb{P}^1 \to S_i$ which are defined by

$$ s_1 : (z : w) \mapsto (z : w : 0, z : w), $$
$$ s_2 : (z : w) \mapsto (w : z : 0, z : w). $$

These two sections meet on $S_i$ at the two points $(1 : 1 : 0, 1 : 1)$ and $(1 : -1 : 0, 1 : -1)$. We see that their lifts $\tilde{s}_k$ on the desingularization $\tilde{S}_i$ (that is, the strict transforms of $s_k(\mathbb{P}^1)$) do not intersect on the exceptional curves at $(1 : 1 : 0, 1 : 1)$ and $(1 : -1 : 0, 1 : -1)$, which means they do not intersect on $\tilde{S}_i$. Considering the process of blowing downs in Sect. 4.2 (see the figures of $\phi_i^{-1}(1 : \pm 1)$), we know that their images in $T_i$ also do not intersect each other. Note that $s_1(\mathbb{P}^1) = V(xw - yz, u)$, $s_2(\mathbb{P}^1) = V(xz - yw, u)$ and

$$ F_0 = u^2(z^2 - 2w^2) + w(xz - yw)(xw - yz), $$
$$ F_1 = u^2(z^4 - 3z^2w^2 + w^4) + zw(xz - yw)(xw - yz), $$
$$ F_2 = u^2(z^2 - w^2) + w(xz - yw)(xw - yz). $$

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Then we can check that in \( \text{Div} (\tilde{S}_i) \)

\[
\text{div}(xw - yz) = [V(xw - yz, u)], \quad \text{div}(xz - yw) = [V(xz - yw, u)].
\]

Thus we see that \( \tilde{s}_k(\mathbb{P}^1) \) are linearly equivalent. Therefore we have

\[
\tilde{s}_1(\mathbb{P}^1)^2 = \tilde{s}_2(\mathbb{P}^1)^2 = \tilde{s}_1(\mathbb{P}^1) \cdot \tilde{s}_2(\mathbb{P}^1) = 0.
\]

The same is true for the images of \( \tilde{s}_k(\mathbb{P}^1) \) in \( T_i \).

We use the notation in Proposition 4.8 in the next lemma.

**Lemma 4.9** Let \( C \) be an irreducible projective curve on a Hirzebruch surface \( \mathbb{P}_n \) which is different from \( B \). If \( n > 0 \), then \( [C]^2 = 0 \) if and only if \( C \) is a fiber of \( \mathbb{P}_n \).

**Proof** Put \( b = [B] \) and \( c = [C] \). Since \( \text{Pic} \mathbb{P}_n = \mathbb{Z} h \oplus \mathbb{Z} f \), there exist \( \alpha, \beta \in \mathbb{Z} \) such that \( c = \alpha h + \beta f \). First note that \( c \cdot f > 0 \) and \( c \cdot b > 0 \) since \( B \neq C \). By Proposition 4.8, we know that \( f^2 = 0, h \cdot f = 1 \) and \( b = h - nf \). Hence \( \alpha \geq 0 \) and \( \beta \geq 0 \). Since \( c^2 = \alpha^2 + 2\alpha\beta \), we see that \( c^2 = 0 \) if and only if \( \alpha = 0 \). If \( C \) is not a fiber on \( \mathbb{P}_n \), then the restriction morphism \( C \hookrightarrow \mathbb{P}_n \to \mathbb{P}^1 \) is surjective. This means \( c \cdot f > 0 \). Therefore \( C \) is a fiber of \( \mathbb{P}_n \).

Recall that the above sections \( \tilde{s}_k \) are global sections. Hence we conclude that \( n = 0 \), namely the geometrically ruled surface \( T_i \) over \( \mathbb{P}^1 \) is \( \mathbb{P}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \), which is already a minimal model of \( \tilde{S}_i \).

**Remark 4.10** The set \( S_i \setminus V(p_i) \) of ideal points of the canonical component in \( \mathbb{P}^2 \times \mathbb{P}^1 \) consists of three ‘ideal curves’ for \( i = 0, 2 \), that is, one fiber \( \phi_i^{-1}(1:0) \) and two global sections \( s_k(\mathbb{P}^1) \). For \( S_1 \), there is an additional fiber. Namely \( S_1 \setminus V(p_1) \) consists of \( \phi_1^{-1}(1:0), \phi_1^{-1}(0:1) \) and two global sections \( s_k(\mathbb{P}^1) \).

### 5 Desingularization of the models

In this section we determine \( \tilde{S}_i \) in terms of the number of blow-ups from their minimal models we have computed in Sect. 4.

From the result in Sect. 4, the smooth surfaces \( \tilde{S}_i \) are isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \) with one point blown up a number of times. Suppose that \( \tilde{S}_i \) is obtained from \( \mathbb{P}^1 \times \mathbb{P}^1 \) by \( n \) one-point blow-ups. Since \( \chi(\mathbb{P}^1 \times \mathbb{P}^1) = 4 \) we have \( \chi(\tilde{S}_i) = \chi(\mathbb{P}^1 \times \mathbb{P}^1) + n = n + 4 \) by repeatedly using Lemma 3.1. To determine the number \( n \) we have to compute \( \chi(\tilde{S}_i) \). This is done by comparing the Euler characteristics of \( S_i \) and \( \tilde{S}_i \). For the computation of \( \chi(S_i) \) we follow the Landes’ method in [9, Section 4]. Here we introduce a rational map \( \phi : \mathbb{P}^2 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \) defined by \((x : y : u, z : w) \mapsto (x : y, z : w)\) and set \( \phi_i := \phi|_{S_i} \). This plays a crucial role for the computation of \( \chi(S_i) \) in this section.
5.1 Whitehead link case

The computation of $\chi(S_0)$ has already been done in [9, Section 4]. Here we review the computation for the completeness of this note. For more details, see op. cit. or the $6_2^3$ case.

Let $\varphi_0 : S_0 \mapsto \mathbb{P}^2 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the rational map defined by $(x : y : u, z : w) \mapsto (x : y, z : w)$. This is not defined at the three points $(0 : 0 : 1, 0 : 1)$ and $(0 : 0 : 1, 1 : \pm \frac{1}{\sqrt{2}})$. Let $P_0$ be the set of those three points and put $U_0 := S_0 \setminus P_0$. The image $\text{Im}(\varphi_0)$ of $U_0$ is $\mathbb{P}^1 \times \mathbb{P}^1 \setminus Q_0$, where

$$Q_0 = \mathbb{P}^1 \times \{(0 : 1) \setminus (1 : 0, 0 : 1), (0 : 1, 0 : 1)\}$$

$$\cup \mathbb{P}^1 \times \{(1 : \frac{1}{\sqrt{2}}) \setminus \{(1 : \sqrt{2}, 1 : \frac{1}{\sqrt{2}}), (1 : \frac{1}{\sqrt{2}}, 1 : \frac{1}{\sqrt{2}})\}\}

$$\cup \mathbb{P}^1 \times \{(1 : \frac{1}{\sqrt{2}}) \setminus \{(1 : -\sqrt{2}, 1 : -\frac{1}{\sqrt{2}}), (1 : -\frac{1}{\sqrt{2}}, 1 : -\frac{1}{\sqrt{2}})\}\}.

Hence $\chi(Q_0) = 0$. Let $L_0$ be the set of the six points below:

$$(1 : 0, 0 : 1), (0 : 1, 0 : 1), (1 : \sqrt{2}, 1 : \frac{1}{\sqrt{2}}),$$

$$(1 : \frac{1}{\sqrt{2}}, 1 : \frac{1}{\sqrt{2}}), (1 : -\sqrt{2}, 1 : -\frac{1}{\sqrt{2}}), (1 : -\frac{1}{\sqrt{2}}, 1 : -\frac{1}{\sqrt{2}}).$$

Let

$$F_0 = u^2z^3 - xyz^2w + (x^2 + y^2 - 2u^2)zw^2 - xyw^3 = G_0 + H_0u^2$$

be the decomposition of $F_0$ in terms of the power of $u$, where

$$G_0 = -xyz^2w + (x^2 + y^2)zw^2 - xyw^3, \quad H_0 = z(z^2 - 2w^2).$$

For $(z : w) \in \mathbb{P}^1$, we see that $H_0(z, w) = 0$ if and only if $(z : w) = (0 : 1), (1 : \pm \frac{1}{\sqrt{2}})$. Then we can check that the set $\{G_0 = H_0 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$ is equal to $L_0$. Therefore each point of $L_0$ has an infinite fiber isomorphic to the affine line $\mathbb{A}^1$. Hence we have $\chi(L_0) = 6$ and $\chi(\varphi_0^{-1}(L_0)) = 6$. Since $G_0 = w(xz - yw)(xw - yz)$, the set $B_0 := V(G_0) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is decomposed into the following three subsets:

$$B_{01} = V(w) = \mathbb{P}^1 \times \{(1 : 0)\} \subset \mathbb{P}^1 \times \mathbb{P}^1,$$

$$B_{02} = V(xz - yw) = \{(1 : y, y : 1), (0 : 1, 1 : 0)\} \cong \mathbb{P}^1,$$

$$B_{03} = V(xw - yz) = \{(1 : y, 1 : y), (0 : 1, 0 : 1)\} \cong \mathbb{P}^1.$$

Their relations are as follows:

$$B_{01} \cap B_{02} = \{(0 : 1, 1 : 0)\}, \quad B_{01} \cap B_{03} = \{(1 : 0, 1 : 0)\},$$

$$B_{02} \cap B_{03} = \{(1 : 1, 1 : 1), (1 : -1, -1 : 1)\}, \quad B_{01} \cap B_{02} \cap B_{03} = \emptyset.$$
Hence

\[
\chi(B_0) = \chi(B_{01} \cup B_{02} \cup B_{03}) \\
= \chi(B_{01} \cup B_{02}) + \chi(B_{03}) - \chi(B_{01} \cap B_{03} \cup B_{02} \cap B_{03}) \\
= \chi(B_{01}) + \chi(B_{02}) + \chi(B_{03}) - \chi(B_{01} \cap B_{02}) \\
\quad - \chi(B_{01} \cap B_{03}) - \chi(B_{02} \cap B_{03}) + \chi(B_{01} \cap B_{02} \cap B_{03}) \\
= 2 + 2 + 2 - 1 - 1 - 2 + 0 = 2.
\]

Thus we have

\[
\chi(U_0) = 2\chi(\mathbb{P}^1 \times \mathbb{P}^1 \setminus (B_0 \cup Q_0)) + \chi(B_0 \setminus L_0) + \chi(\varphi_0^{-1}(L_0)) \\
= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1) - \chi(B_0) - 2\chi(Q_0) - \chi(L_0) + \chi(\varphi_0^{-1}(L_0)) \\
= 2 \times 4 - 2 - 6 + 6 = 6, \\
\chi(S_0) = \chi(U_0) + \chi(P_0) = \chi(U_0) + 3 = 9.
\]

Thus \(\widetilde{S}_0 = \chi(S_0 \setminus S_0,\text{sing}) + 4\chi(\mathbb{P}^1) = \chi(S_0) - 4 + 8 = 13.\) Therefore \(\widetilde{S}_0\) is isomorphic to \(\mathbb{P}^1 \times \mathbb{P}^1\) blown up \(13 - 4 = 9\) times.

### 5.2 \(6^2\) case

Let \(\varphi_1 : S_1 \leftrightarrow \mathbb{P}^2 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1\) be the rational map defined by \((x : y : u, z : w) \mapsto (x : y, z : w)\). This is not defined at the following four points:

\[P_1 := \{(0 : 0 : 1, 1 : \pm \frac{1+\sqrt{5}}{2}\}\}.

The image \(\text{Im}(\varphi_1)\) of the open subset \(U_1 := S_1 \setminus P_1\) is \(\mathbb{P}^1 \times \mathbb{P}^1 \setminus Q_1\), where

\[Q_1 = \mathbb{P}^1 \times \left\{(1 : \frac{1+\sqrt{5}}{2}) \setminus \{(1 : \frac{1+\sqrt{5}}{2}, 1 : \frac{1+\sqrt{5}}{2})\}\right\} \\
\quad \cup \mathbb{P}^1 \times \left\{(1 : \frac{1-\sqrt{5}}{2}) \setminus \{(1 : \frac{1-\sqrt{5}}{2}, 1 : \frac{1-\sqrt{5}}{2})\}\right\} \\
\quad \cup \mathbb{P}^1 \times \left\{(1 : -\frac{1-\sqrt{5}}{2}) \setminus \{(1 : \frac{1+\sqrt{5}}{2}, 1 : -\frac{1+\sqrt{5}}{2})\}\right\} \\
\quad \cup \mathbb{P}^1 \times \left\{(1 : -\frac{1+\sqrt{5}}{2}) \setminus \{(1 : \frac{1-\sqrt{5}}{2}, 1 : -\frac{1-\sqrt{5}}{2})\}\right\}.
\]

Let \(L_1\) be the subset of \(\mathbb{P}^1 \times \mathbb{P}^1\) which consists of the eight points

\[(1 : \frac{1+\sqrt{5}}{2}, 1 : \frac{1+\sqrt{5}}{2}), \quad (1 : \frac{1-\sqrt{5}}{2}, 1 : \frac{1-\sqrt{5}}{2}), \]

\[(1 : \frac{-1+\sqrt{5}}{2}, 1 : \frac{-1+\sqrt{5}}{2}), \quad (1 : \frac{-1-\sqrt{5}}{2}, 1 : \frac{-1-\sqrt{5}}{2}).\]

Let

\[F_1 := u^2 z^4 - xyz^3 w + (x^2 + y^2 - 3u^2) z w^2 - xyz w^3 + u^2 w^4 = G_1 + H_1 u^2\]
be the decomposition of $F_1$ in terms of the power of $u$, where

$$G_1 = -xyz^3w + (x^2 + y^2)z^2w^2 - xyzw^3, \quad H_1 = z^4 - 3z^2w^2 + w^4.$$  

Then the image $\text{Im}(\varphi_1)$ of $\varphi_1$ is decomposed into three subsets

$$\text{Im}(\varphi_1) = \varphi_1(U_1) = \{G_1 = H_1 = 0\} \cup \{G_1 = 0, H_1 \neq 0\} \cup \{G_1 \neq 0, H_1 \neq 0\}.$$  

We can characterize these three subsets as follows: For any point $(x : y, z : w) \in \text{Im}(\varphi_1)$, the fiber of $\varphi_1$ at $(x : y, z : w)$ is an infinite set if and only if $G_1(x, y, z, w) = H_1(x, y, z, w) = 0$; the fiber of $\varphi_1$ at $(x : y, z : w)$ consists of one point if and only if $G_1(x, y, z, w) = 0$ and $H_1(x, y, z, w) \neq 0$; the fiber of $\varphi_1$ at $(x : y, z : w)$ consists of two points if and only if $G_1(x, y, z, w) \neq 0$ and $H_1(x, y, z, w) \neq 0$. For $(z : w) \in \mathbb{P}^1$, we see that $H_1(z, w) = 0$ if and only if $(z : w) = (1 : \pm \sqrt{2})$. Then it is easy to see that $L_1$ is equal to the set of points satisfying $G_1(x, y, z, w) = H_1(x, y, z, w) = 0$. This means each point of $L_1$ has an infinite fiber which is isomorphic to the affine line $\mathbb{A}^1$. Hence $\chi(L_1) = 8$ and $\chi(\varphi_1^{-1}(L_1)) = 8$. Since $G_1 = zw(xz - yw)(xw - yz)$, the set $B_1 := V(g_1) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is decomposed into four subsets

$$B_{11} = V(z) = \mathbb{P}^1 \times \{(0 : 1)\} \subset \mathbb{P}^1 \times \mathbb{P}^1,$$

$$B_{12} = V(w) = \mathbb{P}^1 \times \{(1 : 0)\},$$

$$B_{13} = V(xz - yw) = \{(1 : y, y : 1), (0 : 1, 1 : 0)\} \sim \mathbb{P}^1,$$

$$B_{14} = V(xw - yz) = \{(1 : y, 1 : y), (0 : 1, 0 : 1)\} \sim \mathbb{P}^1.$$  

Immediately we have

$$B_{11} \cap B_{12} = \emptyset, \quad B_{11} \cap B_{13} = \{(1 : 0, 0 : 1)\}, \quad B_{11} \cap B_{14} = \{(0 : 1, 0 : 1)\},$$

$$B_{12} \cap B_{13} = \{(0 : 1, 1 : 0)\}, \quad B_{12} \cap B_{14} = \{(1 : 0, 1 : 0)\},$$

$$B_{13} \cap B_{14} = \{(1 : 1, 1 : 1), (1 : -1, -1 : 1)\},$$

$$B_{11} \cap B_{12} \cap B_{13} = B_{11} \cap B_{12} \cap B_{14} = B_{11} \cap B_{13} \cap B_{14} = \emptyset,$$

$$B_{12} \cap B_{13} \cap B_{14} = \emptyset, \quad B_{11} \cap B_{12} \cap B_{13} \cap B_{14} = \emptyset.$$  

Hence we can compute the Euler characteristic $\chi(B_1)$:

$$\chi(B_1) = \chi(B_{11} \cup B_{12} \cup B_{13} \cup B_{14})$$

$$= \chi(B_{11}) + \chi(B_{12}) + \chi(B_{13}) + \chi(B_{14})$$

$$- \chi((B_{11} \cap B_{12}) \cup (B_{11} \cap B_{13}) \cup (B_{11} \cap B_{14}))$$

$$= \chi(B_{11}) + \chi(B_{12}) + \chi(B_{13}) + \chi(B_{14})$$

$$- \chi(B_{11} \cap B_{12}) - \chi(B_{11} \cap B_{13}) - \chi(B_{11} \cap B_{14})$$

$$- \chi(B_{12} \cap B_{13}) - \chi(B_{12} \cap B_{14}) - \chi(B_{13} \cap B_{14})$$

$$= 2 + 2 + 2 + 2 - 0 - 1 - 1 - 1 - 1 - 2 = 2.$$  

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Thus we have

\[
\chi(U_1) = 2\chi((\mathbb{P}^1 \times \mathbb{P}^1) \setminus (B_1 \cup Q_1)) + \chi(B_1 \setminus L_1) + \chi(\varphi_1^{-1}(L_1)) \\
= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1) - \chi(B_1) - 2\chi(Q_1) - \chi(L_1) + \chi(\varphi_1^{-1}(L_1)) \\
= (2 \times 4) - 2 - (2 \times 0) - 8 + 8 = 6,
\]

\[
\chi(S_1) = \chi(U_1) + \chi(P_1) = \chi(U_1) + 4 = 10.
\]

Let \(\tilde{S}_1\) be the desingularization of \(S_1\) by blowing up at six singular points. Each fiber is a smooth conic curve inside \(\tilde{S}_1\), which is isomorphic to \(\mathbb{P}^1\). Thus we have

\[
\chi(\tilde{S}_1) = \chi(S_1 \setminus S_{1,\text{sing}}) + 6\chi(\mathbb{P}^1) = 10 - 6 + 12 = 16.
\]

Recall that \(\tilde{S}_1\) is isomorphic to \(\mathbb{P}^1 \times \mathbb{P}^1\) blown up \(n\) times. That means \(\chi(\tilde{S}_1) = \chi(\mathbb{P}^1 \times \mathbb{P}^1) + n = n + 4\), which implies \(n = 12\).

### 5.3 \(6^2_3\) case

We can compute \(\chi(S_2)\) in the same way as in the \(6^2_2\) case, therefore we omit the details. Let \(\varphi_2: S_2 \leftrightarrow \mathbb{P}^2 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1\) be the rational map defined by \((x : y : u, z : w) \mapsto (x : y, z : w)\). This is not defined at three points \((0 : 0 : 1, 0 : 1), (0 : 0 : 1, 1 : 1)\) and \((0 : 0 : 1, 1 : -1)\). Let \(P_2\) be the set of those three points and put \(U_2 := S_2 \setminus P_2\). The image \(\text{Im}(\varphi_2)\) of \(U_2\) is \(\mathbb{P}^1 \times \mathbb{P}^1 \setminus Q_2\), where

\[
Q_2 = \mathbb{P}^1 \times \{(0 : 1)\} \setminus \{(1 : 0, 0 : 1), (0 : 1, 0 : 1)\} \\
\quad \cup \mathbb{P}^1 \times \{(1 : 1)\} \setminus \{(1 : 1, 1 : 1)\} \\
\quad \cup \mathbb{P}^1 \times \{(1 : -1)\} \setminus \{(1 : -1, 1 : -1)\}.
\]

Hence \(\chi(Q_2) = 2\). Let

\[
F_2 = u^2z^3 - xyz^2w + (x^2 + y^2 - u^2)zw^2 - xyw^3 = G_2 + H_2u^2
\]

be the decomposition of \(F_2\) in terms of the power of \(u\), where

\[
G_2 = -xyz^2w + (x^2 + y^2)zw^2 - xyw^3, \quad H_2 = z(z^2 - w^2).
\]

For \((z : w) \in \mathbb{P}^1\), it is easy to check that \(H_2(z, w) = 0\) if and only if \((z : w) = (0 : 1), (1 : 1)\) or \((1 : -1)\). Let \(L_2\) be the subset of \(\mathbb{P}^1 \times \mathbb{P}^1\) which consists of the following four points:

\[
(1 : 0, 0 : 1), (0 : 1, 0 : 1), (1 : 1, 1 : 1), (1 : -1, 1 : -1).
\]

Then we see that \(L_2 = \{G_2 = H_2 = 0\}\) as in the \(6^2_2\) case. Therefore each point of \(L_2\) has an infinite fiber isomorphic to the affine line \(\mathbb{A}^1\). Hence we have \(\chi(L_2) = 4\) and...
\( \chi(\varphi_2^{-1}(L_2)) = 4. \) Since \( G_2 = w(xz - yw)(xw - yz) \), the set \( B_2 := V(g_2) \subset \mathbb{P}^1 \times \mathbb{P}^1 \) is decomposed into the following three subsets:

\[
\begin{align*}
B_{21} &= V(w) = \mathbb{P}^1 \times \{(1:0)\} \subset \mathbb{P}^1 \times \mathbb{P}^1, \\
B_{22} &= V(xz - yw) = \{(1:y, y:1), (0:1, 1:0)\} \sim \mathbb{P}^1, \\
B_{23} &= V(xw - yz) = \{(1:y, 1:y), (0:1, 0:1)\} \sim \mathbb{P}^1, \\
B_{21} \cap B_{22} &= \{(0:1, 1:0)\}, \quad B_{21} \cap B_{23} = \{(1:0, 1:0)\}, \\
B_{22} \cap B_{23} &= \{(1:1, 1:1), (1:-1, -1:1)\}, \quad B_{21} \cap B_{22} \cap B_{23} = \emptyset.
\end{align*}
\]

Hence we have

\[
\chi(B_2) = \chi(B_{21} \cup B_{22} \cup B_{23}) = \chi(B_{21}) + \chi(B_{22}) + \chi(B_{23}) - \chi(B_{21} \cap B_{22}) - \chi(B_{21} \cap B_{23}) - \chi(B_{22} \cap B_{23}) + \chi(B_{21} \cap B_{22} \cap B_{23}) = 2 + 2 + 2 - 1 - 1 - 2 + 0 = 2.
\]

Thus we can compute \( \chi(S_2) \) as follows:

\[
\chi(U_2) = 2\chi(\mathbb{P}^1 \times \mathbb{P}^1 \setminus (B_2 \cup Q_2)) + \chi(B_2 \setminus L_2) + \chi(\varphi_2^{-1}(L_2)) = 2\chi(\mathbb{P}^1 \times \mathbb{P}^1) - \chi(B_2) - 2\chi(Q_2) - \chi(L_2) + \chi(\varphi_2^{-1}(L_2)) = 2 \times 4 - 2 - (2 \times 2) - 4 + 4 = 2.
\]

\[
\chi(S_2) = \chi(U_2) + \chi(P_2) = \chi(U_2) + 3 = 5.
\]

We have already seen in Sect. 3 that \( \tilde{\chi}(S_2) = \chi(S_2) + 8 = 5 + 8 = 13. \) Hence \( \tilde{S}_2 \) is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \) blown up \( 13 - 4 = 9 \) times.

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