On relationship of gauge transformation with Wigner's little group

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Wigner’s little group of a massless particle is ISO(2) which contains rotation and two translations. As well known, eigenvalues of the rotation are helicity. On the other hand, by S. Weinberg et al., it has been shown that two translations generate abelian gauge transformation by acting on polarization vectors. In this paper, we include unphysical modes and show abelian case result can be generalized to the case of non-abelian gauge transformation. By including the unphysical modes, we obtain Nakanishi-Lautrup physical state condition from the requirement of unitarity of the transformation. As a result, non-abelian gauge transformation is realized as the translation of the little group which acts on gauge group. We also obtain similar results for any spacetime dimensions.

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INTRODUCTION

In the context of relativistic quantum mechanics, a free single-particle state is defined as unitary irreducible representation of the Poincaré group. E. Wigner [1] has investigated that such representation is classified by the mass, spatial momenta and labels of the little group. To begin with, we review such classification briefly.

In particular, for massless particle the little group is ISO(2), namely two dimensional Euclidean group which contains one rotation and two translations. These operators act on field operators under the Lorentz transformation. The rotation generate $U(1)$ phase transformation corresponding to eigenvalues called helicity. On the other hand, two translations generate abelian gauge transformation by acting on transverse polarization vectors. This result has been shown by S. Weinberg et al. [2] and also generalized for the case of spin-2 or higher-spin fields. It is very interesting that these fact means abelian gauge symmetry is induced by only imposing Lorentz symmetry for the Lagrangian of massless integer spin fields.

We review and generalize this interesting result. First, we consider the case of including unphysical, namely, longitudinal and scalar modes. And then, we show that this result can be generalized to the case of non-abelian gauge transformation and for any spacetime dimensions.

I. A BRIEF REVIEW OF WIGNER’S LITTLE GROUP

In this section, we present representation of the Poincaré group and Wigner’s little group. A relativistic free single-particle state is defined as unitary irreducible representation of the Poincaré group which is generated by the Poincaré algebra;

$$[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\sigma} M_{\mu\rho}, \quad [M_{\mu\nu}, P_\rho] = \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu.$$

(1.1)

There are two Casimir operators of the Poincaré algebra;

$$P^2 := P_\mu P^\mu, \quad W^2 := W_\mu W^\mu,$$

(1.2)

where,

$$W_\mu := -\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}.$$

(1.3)

Therefore, we can classify the representation by the square of mass $p^2$; eigenvalues of $P^2$. Furthermore, $W_\mu$ generates several different algebras; called little group, depending on $p^\mu$ as in the following table:

| eigenvalues of $P^2$ | little group |
|----------------------|--------------|
| timelike $p^2 > 0$   | $SO(3)$      |
| null $p^2 = 0$       | $ISO(2)$ (2-dimensional Euclidean group) |
| spacelike $p^2 < 0$  | $SO(2,1)$    |
| no particles $p^\mu = 0$ | $SO(3,1)$   |
Furthermore, because of \([P_\mu, W_\nu] = 0\), labels of the representation of the little group are independent of \(p^\mu\). Consequently, we can classify completely the representation of Poincaré group by mass, spatial momenta and labels of the little group.

Next, we consider specific transformation law of such relativistic free single-particle states. Since the little group transformation does not change momenta, we can choose arbitrary reference frame. First, we consider massive \((p^2 > 0)\) particles and choose the static frame \(k^\mu = (m, 0, 0, 0)\). Hence, we introduce the little group transformation matrix \(W\) as follows.

\[
W^\mu_\nu(\Lambda, p)k^\nu = k^\mu .
\] (1.4)

This matrix can be rewritten by using two types of the Lorentz transformation; the generic transformation \(\Lambda\) and the standard transformation \(L(p)\) which satisfies \(\Lambda(p)^\mu_\nu k^\nu = p^\mu\). Then, we can find

\[
W(\Lambda, p) = L^{-1}(Ap)L(p) .
\] (1.5)

By using (1.5), we obtain the transformation law of free single-particle states \([p, \sigma]\). Now, \(\sigma\) is the index of internal symmetry, namely the little group \(SO(3)\), called spin. That is,

\[
U(\Lambda) [p, \sigma] = U(L(\Lambda p))U(W(\Lambda, p))U^{-1}(L(p))[p, \sigma] = U(L(\Lambda p))U(W(\Lambda, p))[0, \sigma] = D(\Lambda, p)_{\sigma\sigma'} [Ap, \sigma'] .
\] (1.6)

\(D(\Lambda, p)_{\sigma\sigma'}\) is some unitary representation of \(SO(3)\) called Wigner rotation. Consequently, the Lorentz transformation of free massive single-particle states is composed with both Lorentz boost of spatial momenta and the rotation of spin state.

After that, we consider massless \((p^2 = 0)\) particles and choose the reference frame \(k^\mu = (k, 0, 0, k)\). We can also use the relation (1.5). However, the little group is \(ISO(2)\); hence, we obtain

\[
[J, T_1] = iT_2, \quad [J, T_2] = -iT_1, \quad [T_1, T_2] = 0 .
\] (1.7)

\(J\) is generator of \(SO(2)\) but \(T_1\) and \(T_2\) are generator of translation. Therefore, states are labeled by one discrete index called helicity and two continuous indices. Since it is necessary to consider up to two-valued representation of \(SO(2)\), the helicity index take integer or half-integer same as ordinary spin index. Moreover, in the current reference frame \(J\) is the rotation generator of around z-axis. So, indeed, helicity is projected angular momentum to z-axis and regard as similar degree of freedom to spin. On the other hands, there is no physical quantity which corresponds to continuous indices called continuous spin. Thus, we define free single-particle states as only having the helicity index \(h\); \([p, h]\) and action of \(J, T_1\) and \(T_2\) as follows.

\[
J \langle p, h \rangle = h \langle p, h \rangle, \quad T_1 \langle p, h \rangle = T_2 \langle p, h \rangle = 0 .
\] (1.8)

Hence, we obtain

\[
U(\Lambda) \langle p, h \rangle = e^{i\theta(\Lambda, p)h} \langle Ap, h \rangle .
\] (1.9)

Consequently, the Lorentz transformation of free massless single-particle states is composed with both Lorentz boost of spatial momenta and the \(U(1)\) phase transformation differently from the case of massive particles.

### II. ABELIAN GAUGE TRANSFORMATION

In this section, we consider Lorentz transformation of massless spin-1,2 and any integer spin field operators and show \(U(1)\) gauge transformation, linearized general coordinate transformation and higher-spin gauge transformation are induced by the the Lorentz transformation, respectively.

#### A. Massless fields of integer spin

We introduce creation and annihilation operators;

\[
\langle p, h \rangle = \sqrt{2E_p}a^\dagger_h(p) \langle 0 \rangle, \quad [a_h(p), a^\dagger_j(q)] = \delta_{hj}\delta^3(p - q) ,
\] (2.1)
and obtain the Lorentz transformation of these operators from \[19\]:

\[ U(\Lambda) a_h^\dagger(p) U^\dagger(\Lambda) = \sqrt{\frac{E_p}{E_\Lambda}} e^{i \theta_h} a_h^\dagger(\Lambda p), \quad U(\Lambda) a_h(p) U^\dagger(\Lambda) = \sqrt{\frac{E_p}{E_\Lambda}} e^{-i \theta_h} a_h(\Lambda p). \]  

(2.2)

To construct Lorentz covariant operators, we define two polarization vectors in the reference frame \( k^\mu = (k, 0, 0, k) \) as

\[ \epsilon_{\pm 1}^\mu := \frac{1}{\sqrt{2}} (0, 1, \pm 0), \]  

(2.3)

and find

\[ k_\mu \epsilon_{\pm 1}^\mu = (\epsilon_{\pm 1}^\mu)^2 = 0, \quad \epsilon_{\pm 1}^\* \epsilon_{\pm 1}^\mu = \epsilon_{\pm 1}^\* \epsilon_{\pm 1}^\mu = 1. \]  

(2.4)

Then, the little group \( ISO(2) \) generators acting polarization vectors are

\[ (T_1)^\mu_\nu = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (T_2)^\mu_\nu = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad (J)^\mu_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]  

(2.5)

and we obtain

\[ W^\mu_\nu \epsilon_{\pm 1}^\* = (e^{i \alpha_1 T_1 + i \alpha_2 T_2 + i \theta J})^\mu_\nu \epsilon_{\pm 1}^\* \simeq (1 \pm i \theta) \epsilon_{\pm 1}^\* + \frac{\alpha_1 \pm i \alpha_2}{\sqrt{2k}} k^\mu. \]  

(2.6)

By using the standard Lorentz transformation \( L(p) \), we define polarization vectors in general frame as

\[ \epsilon_{\pm 1}(p) := L^\mu_\nu(p) \epsilon_{\pm 1}^\* \]  

(2.7)

find equations similar to \[24\] by replacing \( \epsilon_{\pm 1}^\* \rightarrow \epsilon_{\pm 1}(p) \) and \( k^\mu \rightarrow p^\mu \), and obtain

\[ (1 \mp i \theta) \epsilon_{\pm 1}(p) \simeq (\Lambda^{-1})^\mu_\nu \epsilon_{\pm 1}(\Lambda p) + \frac{\alpha_1 \pm i \alpha_2}{\sqrt{2k}} p^\mu. \]  

(2.8)

from \[15\] and \[26\].

1. Spin-1 (Electromagnetic field)

We introduce the real vector field operator as follows.

\[ A^\mu(x) := \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} \sum_{h=\pm 1} [ \epsilon_h^\mu(p) a_h(p) e^{-ipx} + \epsilon_h^\*\mu(p) a_h^\dagger(p) e^{ipx} ] . \]  

(2.9)

We should note that \( A^\mu(x) \) is fixed to the Lorenz gauge \( \partial_\mu A^\mu(x) = 0 \). By using \[22\] and \[28\], we obtain infinite small Lorentz transformation of \[29\] is

\[ U(\Lambda) A^\mu(x) U^\dagger(\Lambda) \simeq \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} \sum_{h=\pm 1} \left[ (1 - i \theta_h) \epsilon_h^\mu(p) a_h(\Lambda p) e^{-ipx} + (1 + i \theta_h) \epsilon_h^\*\mu(p) a_h^\dagger(\Lambda p) e^{ipx} \right] \]

\[ = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} \sum_{h=\pm 1} \left[ (\Lambda^{-1})^\mu_\nu \epsilon_h^\nu(\Lambda p) + \frac{\alpha_1 + i \alpha_2}{\sqrt{2k}} p^\nu \right] a_h(\Lambda p) e^{-ipx} \]

\[ + \left( (\Lambda^{-1})^\mu_\nu \epsilon_h^\*\nu(\Lambda p) + \frac{\alpha_1 - i \alpha_2}{\sqrt{2k}} p^\nu \right) a_h^\dagger(\Lambda p) e^{ipx} \]

\[ = (\Lambda^{-1})^\mu_\nu (A^\nu(\Lambda x) + \partial^\nu \Theta(\Lambda x, \alpha_1, \alpha_2)) . \]  

(2.10)
Now, we define the local operator
\[
\Theta(x, \alpha_1, \alpha_2) := \frac{i}{\sqrt{2k}} \int \frac{d^3p}{(2\pi)^2 \sqrt{2E_p}} \sum_{h=\pm 1} \left[ \alpha_1(a_h(p)e^{-ipx} - a_h^\dagger(p)e^{ipx}) + i\hbar\alpha_2(a_h(p)e^{-ipx} + a_h^\dagger(p)e^{ipx}) \right]. \tag{2.11}
\]

The important point to be noted is that \(\Theta(x, \alpha_1, \alpha_2)\) is hermitian. Consequently, we find that abelian gauge transformation of gauge field is induced by the Lorentz transformation, and particularly, for factor of \(\alpha_1\) and \(\alpha_2\), generated by two translations of ISO(2). Lastly, we should note that the gauge fixing condition \(\partial_\mu A^\mu(x) = 0\) is invariant in this gauge transformation because of the Lorentz invariance \(^{\ast 1}\) of the gauge fixing condition. This result is consistent with gauge invariance of physical states.

2. Spin-2 (Linearized gravity)

We can obtain gauge transformation of linearized gravity from the Lorentz transformation by the similar way to spin-1 field. First, we introduce creation and annihilation operators of spin-2 field by taking symmetric tensor products of spin-1 operators:
\[
a_{2h}(p) := a_h(p) \otimes a_h(p). \tag{2.12}
\]
Hence, we can find
\[
U(\Lambda)a_{2h}(p)U^{\dagger}(\Lambda) = \sqrt{E_p/E_{Ap}}e^{-2i\theta_h}a_{2h}(Ap). \tag{2.13}
\]
Then, we also introduce two polarization tensors;
\[
\epsilon_{\pm 1}(p) := \epsilon_{\pm 1}^\mu(p) \otimes \epsilon_{\pm 1}^\nu(p). \tag{2.14}
\]
By using (2.13) and (2.14), we define the real rank-2 tensor field as follows.
\[
h^{\mu\nu}(x) := \int \frac{d^3p}{(2\pi)^2 \sqrt{2E_p}} \sum_{h=\pm 1} \left[ \epsilon_h^{\mu\nu}(p)a_{2h}(p)e^{-ipx} + \epsilon_h^{\nu\mu}(p)a_{2h}^\dagger(p)e^{ipx} \right], \tag{2.15}
\]
Obviously, \(h^{\mu\nu}(x)\) is a symmetric tensor and satisfy traceless transverse gauge condition from (2.4):
\[
\partial_\mu h^{\mu\nu}(x) = 0, \quad \eta_{\mu\nu}h^{\mu\nu}(x) = 0. \tag{2.16}
\]
We can find Lorentz transformation of \(h^{\mu\nu}(x)\) is
\[
U(\Lambda)h^{\mu\nu}(x)U^{\dagger}(\Lambda) \simeq (\Lambda^{-1})^\mu_\rho(\Lambda^{-1})^\nu_\sigma (h^{\rho\sigma}(Ax) + \partial^\rho \xi^\sigma(Ax) + \partial^\sigma \xi^\rho(Ax)), \tag{2.17}
\]
where,
\[
\xi^\mu(x) := \frac{i}{\sqrt{2k}} \int \frac{d^3p}{(2\pi)^2 \sqrt{2E_p}} \sum_{h=\pm 1} \left[ \alpha_2(\epsilon_h^{\mu\nu}(p)a_{2h}(q)e^{-ipx} - \epsilon_h^{\nu\mu}(p)a_{2h}^\dagger(q)e^{ipx}) + i\hbar\alpha_2(\epsilon_h^{\mu\nu}(p)a_{2h}(p)e^{-ipx} + \epsilon_h^{\nu\mu}(p)a_{2h}^\dagger(p)e^{ipx}) \right], \tag{2.18}
\]
\[
\xi^{\dagger\mu}(x) = \xi^\mu(x). \tag{2.19}
\]
Consequently, we can regard \(h^{\mu\nu}(x)\) as graviton and (2.17) as linearized general coordinate transformation.

\(^{\ast 1}\) Also we can check this invariance by specific calculation using \(p^2 = 0\).
3. Spin-n (Higher-spin gauge field)

From generalization of the case of spin-2 field, we can obtain gauge transformation of free higher-spin gauge theory. Free integer-spin massless field theory in 4-dimensional spacetimes has been found by Fronsdal [3] and known this theory has abelian gauge symmetry which is like generalization of linearized general coordinate transformation.

Similarly to the case of spin-2, we introduce creation and annihilation operators of spin-n field by taking symmetric tensor products of spin-1 operators;

$$a_{n\hbar}(p) := (a_{\hbar}(p))^\otimes n.$$  \hspace{1cm} (2.20)

Then, we also introduce two polarization tensors;

$$\epsilon_{\pm\ldots\pm}^{\mu_1\ldots\mu_n}(p) := \epsilon_{\pm}^{\mu_1}(p) \otimes \cdots \otimes \epsilon_{\pm}^{\mu_n}(p).$$  \hspace{1cm} (2.21)

($\mu_1 \cdot \cdot \cdot \mu_n$) means to take totally-symmetric product. By using (2.20) and (2.21), we define the real totally-symmetric rank-n tensor field as follows.

$$\phi^{\mu_1\ldots\mu_n}(x) := \int \frac{d^3p}{(2\pi)^2} \sqrt{2E_p} \sum_{h=\pm 1} \left[ \epsilon_h^{\mu_1\ldots\mu_n}(p)a_{nh}(p)e^{-ipx} + \epsilon_h^{\mu_1\ldots\mu_n}(p)a_{n\hbar}^\dagger(p)e^{ipx} \right].$$  \hspace{1cm} (2.22)

From (2.4), $\phi^{\mu_1\ldots\mu_n}(x)$ satisfies the gauge fixing condition [4];

$$\partial_\nu \phi^{\mu_2\ldots\mu_n}(x) = 0, \hspace{1cm} \eta_{\nu_1\nu_2} \phi^{\mu_1\mu_2\mu_3\ldots\mu_n} = 0.$$  \hspace{1cm} (2.23)

We can also find Lorentz transformation of $\phi^{\mu_1\ldots\mu_n}(x)$ is

$$U(\Lambda)\phi^{\mu_1\ldots\mu_n}(x)U^\dagger(\Lambda) = (\Lambda^{-1})^{\mu_1}_{\nu_1} \cdots (\Lambda^{-1})^{\mu_n}_{\nu_n} \left( \phi^{\nu_1\ldots\nu_n}(\Lambda x) + \partial(\nu_1 \xi^{\nu_2\ldots\nu_n})(\Lambda x) \right),$$  \hspace{1cm} (2.24)

where,

$$\xi^{\mu_1\ldots\mu_n-1}(x) := \frac{i}{\sqrt{2k}} \int \frac{d^3p}{(2\pi)^2} \sqrt{2E_p} \sum_{h=\pm 1} \left[ \alpha_1(\epsilon_h^{\mu_1\ldots\mu_n-1}(p)a_{nh}(p)e^{-ipx} - \epsilon_h^{\mu_1\ldots\mu_n-1}(p)a_{n\hbar}^\dagger(p)e^{ipx}) \right. \nonumber $$

$$+i\alpha_2(\epsilon_h^{\mu_1\ldots\mu_n-1}(p)a_{nh}(p)e^{-ipx} + \epsilon_h^{\mu_1\ldots\mu_n-1}(p)a_{n\hbar}^\dagger(p)e^{ipx}) \right],$$  \hspace{1cm} (2.25)

$$\xi^{\dagger\mu_1\ldots\mu_n-1}(x) = \xi^{\mu_1\ldots\mu_n-1}(x), \hspace{1cm} \eta_{\nu_1\nu_2} \xi^{\nu_1\nu_2\mu_3\ldots\mu_n-1}(x) = 0.$$  \hspace{1cm} (2.26)

We should note that this traceless condition of gauge parameter $\xi^{\dagger\mu_1\ldots\mu_n-1}(x)$ has been required in [3]. Consequently, we can regard $\phi^{\mu_1\ldots\mu_n}(x)$ as higher-spin gauge field and (2.24) as linearized general coordinate transformation.

B. Including unphysical modes and representation of the little group

In previous section, we consider the massless vector field operator composed of only physical, namely, two transverse modes In this section, on the other hand, we consider not only physical modes but also two unphysical modes, namely, longitudinal and scalar mode. $A^\mu(x)$ has, apparently, four components, but only two degree of freedom are physical because of gauge fixing $\partial_\mu A^\mu(x) = 0$. Now, we consider remaining longitudinal and scalar mode in reference frame

$$\epsilon_L^\mu := \frac{1}{\sqrt{2ik}}(k, 0, 0, k), \hspace{1cm} \epsilon_S^\mu := \frac{i}{\sqrt{2k}}(k, 0, 0, -k),$$  \hspace{1cm} (2.27)

and find

$$k_\mu \epsilon_L^\mu = (\epsilon_L^\mu)^2 = (\epsilon_S^\mu)^2 = 0, \hspace{1cm} k_\mu \epsilon_L^\mu = \sqrt{2ik}, \hspace{1cm} \epsilon_{L\mu} \epsilon_S^\mu = 1,$$  \hspace{1cm} (2.28)

$$W^\mu_{\nu\rho} \epsilon_L^\nu = \epsilon_L^\rho, \hspace{1cm} W^\mu_{\nu\rho} \epsilon_S^\nu \simeq \epsilon_S^\rho + \alpha_1(\epsilon_+^\rho + \epsilon_-^\rho) + \alpha_2 \frac{1}{\ell}(\epsilon_+^\rho - \epsilon_-^\rho).$$  \hspace{1cm} (2.29)
Moreover, we should note that physical and unphysical modes satisfy the completeness relation:

\[ \sum_{h=\pm 1} \epsilon^\mu_h \epsilon^\nu_h + \epsilon^\mu_L \epsilon^\nu_L + \epsilon^\mu_S \epsilon^\nu_S = -\eta^{\mu\nu}. \]  \hspace{1cm} (2.30)

Similarly to (2.7), we define unphysical mode in general frame. Then, we obtain similar relations to (2.28) and (2.30), and

\[ \epsilon^\nu_L(p) = (\Lambda^{-1})^\mu_\nu \epsilon^\mu_L(\Lambda p), \]  \hspace{1cm} (2.31)

\[ \epsilon^\nu_S(p) \simeq (\Lambda^{-1})^\mu_\nu (\epsilon^\mu_S(\Lambda p) + i\alpha_1 \omega^\nu(\Lambda p) + \alpha_2 \tilde{\omega}^\nu(\Lambda p)), \]  \hspace{1cm} (2.32)

where,

\[ \omega^\mu(p) := \epsilon^\mu_+ + \epsilon^\mu_-, \quad \tilde{\omega}^\mu(p) := \frac{1}{i} (\epsilon^\mu_+ - \epsilon^\mu_-). \]  \hspace{1cm} (2.33)

Furthermore, we introduce creation and annihilation operators of unphysical modes as \(^{\ast 2}\)

\[ [a_L(p), a_L^\dagger(q)] = [a_S(p), a_S^\dagger(q)] = 0, \quad [a_L(p), a_S^\dagger(q)] = \delta^3(p - q). \]  \hspace{1cm} (2.34)

Then, we introduce the real vector field operator including unphysical modes as follows.

\[ A^\mu(x) := \int \frac{d^3p}{(2\pi)^2 \sqrt{2E_p}} \sum_{h=\pm 1, L, S} \left[ \epsilon^\mu_h(p)a_h(p)e^{-ipx} + \epsilon^\mu_h(p)a_h^\dagger(p)e^{ipx} \right]. \]  \hspace{1cm} (2.35)

We should note that \( A^\mu(x) \) is fixed by general Lorentz invariant gauge condition

\[ \partial^\mu A^\mu(x) = \sqrt{2ik} \int \frac{d^3p}{(2\pi)^2 \sqrt{2E_p}} \left[ a_S(p)e^{-ipx} + a_S^\dagger(p)e^{ipx} \right], \]  \hspace{1cm} (2.36)

and the right hand side of (2.36) is the Nakanishi-Lautrup auxiliary field \( B(x) \) \(^{[3]}\).

We consider Lorentz transformation of (2.35), but we do not determine similarly the Lorentz transformation of these operators to the procedures of section I because \( a_L^\dagger(q) \) and \( a_S^\dagger(q) \) are creation operators corresponded to unphysical states. So, we set these transformation for not changing gauge fixing condition (2.36) because of the Lorentz invariance. Furthermore, by considering (2.31) and (2.32), we assume that little group should not act unphysical states. Then, we set

\[ U(\Lambda)a_L(p)U^\dagger(\Lambda) = \sqrt{\frac{E_{\Lambda p}}{E_p}} a_L(\Lambda p), \quad U(\Lambda)a_S(p)U^\dagger(\Lambda) = \sqrt{\frac{E_{\Lambda p}}{E_p}} a_S(\Lambda p). \]  \hspace{1cm} (2.37)

By using (2.2), (2.8), (2.31), (2.32) and (2.37) consequently, we obtain

\[ U(\Lambda)A^\mu(x)U^\dagger(\Lambda) \simeq (\Lambda^{-1})^\mu_\nu (A^\nu(\Lambda x) + \partial^\nu \Theta(\Lambda x, \alpha_1, \alpha_2) + \Omega^\nu(\Lambda x, \alpha_1, \alpha_2)), \]  \hspace{1cm} (2.38)

where,

\[ \Omega^\mu(x, \alpha_1, \alpha_2) := i \int \frac{d^3p}{(2\pi)^2 \sqrt{2E_p}} \left[ (\alpha_1 \omega^\mu(p) + \alpha_2 \tilde{\omega}^\mu(p)) a_S(p)e^{-ipx} - (\alpha_1 \omega^\mu(p) + \alpha_2 \tilde{\omega}^\mu(p)) a_S^\dagger(p)e^{ipx} \right]. \]  \hspace{1cm} (2.39)

It is obvious that (2.38) is not gauge transformation. So, we improve the transformation law of physical modes as follows

\[ \tilde{U}(\Lambda)a_h(p)\tilde{U}^{-1}(\Lambda) \simeq \sqrt{\frac{E_{\Lambda p}}{E_p}} [(1 - i\theta h)a_h(\Lambda p) - i(\alpha_1 - i\alpha_2)a_S(\Lambda p)], \]

\(^{\ast 2}\) By using (2.31), (2.34) and (2.30), and defining \( a^\mu(p) := \sum_{h=\pm 1, L, S} \epsilon^\mu_h(p)a_h(p) \) we obtain \[ [a^\mu(p), a^\nu^\dagger(q)] = -\eta^{\mu\nu}\delta^3(p - q). \]
\[
\hat{U}(\Lambda)a_h^\dagger(p)\hat{U}^{-1}(\Lambda) \simeq \sqrt{\frac{E_{\Lambda p}}{E_p}}[(1 + i\theta h)a_h^\dagger(\Lambda p) + i(\alpha_1 + i\alpha_2)a_h^\dagger(\Lambda p)],
\]
and can find
\[
\hat{U}(\Lambda')\hat{U}(\Lambda)a_h(p)\hat{U}^{-1}(\Lambda)\hat{U}^{-1}(\Lambda')
\simeq \sqrt{\frac{E_{\Lambda'\Lambda p}}{E_p}}[(1 - i(\theta' + \theta)h)a_h(\Lambda'\Lambda p) - i((\alpha_1' + \alpha_1) - ih(\alpha_2' + \alpha_2))a_S(\Lambda'\Lambda p)].
\]
(2.41)

Then, we obtain
\[
\hat{U}(\Lambda)A^\mu(x)\hat{U}^{-1}(\Lambda) \simeq (\Lambda^{-1})^\mu_\nu(A^\nu(\Lambda x) + \partial^\nu\Theta(\Lambda x, \alpha_1, \alpha_2)).
\]
(2.42)

However, (2.40) brakes the commutation relation \([a_{L}(p), a_{h}^\dagger(q)] = 0\). Accordingly, we should also improve the transformation law of the longitudinal mode as follows.
\[
\hat{U}(\Lambda)a_{L}(p)\hat{U}^{-1}(\Lambda) \simeq \sqrt{\frac{E_{\Lambda p}}{E_p}}[a_{L}(\Lambda p) - i \sum_{h = \pm 1} (\alpha_1 + i\alpha_2)a_h(\Lambda p)],
\]
(2.43)
\[
\hat{U}(\Lambda)a_{h}^\dagger(p)\hat{U}^{-1}(\Lambda) \simeq \sqrt{\frac{E_{\Lambda p}}{E_p}}[a_{h}^\dagger(\Lambda p) + i \sum_{h = \pm 1} (\alpha_1 - i\alpha_2)a_h^\dagger(\Lambda p)],
\]
(2.44)

In fact, additional terms of (2.43) and (2.44) generate inverse gauge transformation \(-\partial^\nu\Theta\). Consequently, we obtain
\[
\hat{U}(\Lambda)A^\mu(x)\hat{U}^{-1}(\Lambda) = (\Lambda^{-1})^\mu_\nu A^\nu(\Lambda x).
\]
(2.45)

The important point to note is that additional terms of the improved transformation which are proportional to \(\alpha_1\) and \(\alpha_2\) are representation of two translations of ISO(2). We define hermitian operators \(T_1, T_2\) and \(J\) as follows.
\[
[T_1, a_h(p)] = -a_S(p), \quad [T_2, a_h(p)] = iha_S(p), \quad [J, a_h(p)] = -ha_h(p),
\]
\[
[T_1, a_h^\dagger(p)] = a_S^\dagger(p), \quad [T_2, a_h^\dagger(p)] = iha_S^\dagger(p), \quad [J, a_h^\dagger(p)] = ha_h^\dagger(p),
\]
(2.46)
\[
[T_1, a_L(p)] = -\sum_{h = \pm 1} a_h(p), \quad [T_2, a_L(p)] = -i \sum_{h = \pm 1} ha_h(p),
\]
\[
[T_1, a_L^\dagger(p)] = \sum_{h = \pm 1} a_h^\dagger(p), \quad [T_2, a_L^\dagger(p)] = -i \sum_{h = \pm 1} ha_h^\dagger(p).
\]
(2.47)

Then, we obtain
\[
[[J, T_1], a_h(p)] = i[T_2, a_h(p)], \quad [[J, T_2], a_h(p)] = -i[T_1, a_h(p)], \quad [[T_1, T_2], a_h(p)] = 0,
\]
(2.48)
and similar relations for \(a_L(p), a_h^\dagger(p)\) and \(a_L^\dagger(p)\), while all commutators of \(a_S(p)\) and \(a_S^\dagger(p)\) vanish. Namely, \(T_1, T_2\) and \(J\) satisfy the algebra of ISO(2) \(\{1, 7\}\).

In addition, we can write down explicitly these ISO(2) generators by using creation and annihilation operators as follows.
\[
J = \int d^3p \sum_{h = \pm 1} ha_h^\dagger(p)a_h(p)
\]
(2.49)
\[
T_\lambda = \sqrt{2}i \int d^3p \left(\hat{a}_\lambda^\dagger(p)a_S(p) - a_S^\dagger(p)\hat{a}_\lambda(p)\right)
\]
(2.50)
where,
\[
\hat{a}_1 := \frac{i}{\sqrt{2}} \sum_{h = \pm 1} a_h(p), \quad \hat{a}_2 := -\frac{1}{\sqrt{2}} \sum_{h = \pm 1} ha_h(p),
\]
(2.51)
\[ [\hat{a}_A(p), \hat{a}_B(q)] = [\hat{a}_A(\mathbf{p}), \hat{a}_B^\dagger(q)] = 0, \quad [\hat{a}_A(p), \hat{a}_B(p)] = \delta_{AB} \delta^{3}(p - q). \] (2.52)

Furthermore, these operators vanish acting on vacuum:

\[ J|0\rangle = T_A |0\rangle = 0. \] (2.53)

This result are consistent with Lorentz invariance of vacuum.

Finally, we should note it is obvious that \( U(A) \) does not act as the unitary operator on the Fock space despite hermiticity of \( T_1 \) and \( T_2 \). Unitarity of physical modes is broken by excited states of longitudinal mode because of \( J \). So, we should restrict the Fock space to the physical subspace which do not contain longitudinal mode such as,

\[ \{|\text{phys}\rangle : a_S(p)|\text{phys}\rangle = 0\}. \] (2.54)

On the other hand, this physical subspace still contain excited states of scaler mode. Since \( \hat{a}_A(p) \hat{a}_B(p) \) are zero norm. Consequently, continuous spin states generated by \( T_A \) are unphysical or zero norm state. Furthermore, if we take expectation values of (2.36) and (2.38) with \( a \), we obtain \( \partial_\mu A^\mu = 0 \) and gauge transformation (2.10). Moreover, since \( a_S(p) \) is the annihilation operator of \( B(x) \), the condition (2.54) is identical with the case of Nakanishi-Lautrup formalism.\[2\].

III. NON-ABELIAN GAUGE TRANSFORMATION

In this section, we generalize the previous result that “abelian gauge transformation is induced by the Lorentz transformation” to the case of non-abelian gauge group.

We introduce internal degree of freedom to massless spin-1 field. Single-particle states degenerate into multiplet of gauge group \( G \). Now, we assume that \( G \) is the compact semisimple Lie group and massless spin-1 field is the adjoint representation of \( G \). So, we should replace creation and annihilation operators so that \( a_h(p) \rightarrow a_h^\dagger(p) \) and commutation relations to

\[ [a_h^\dagger(p), a_j^b(q)] = [a_h^b(p), a_j^b(q)] = 0, \quad [a_h^a(p), a_j^b(q)] = \delta^{ab} \delta_{hj} \delta^{3}(p - q). \] (3.1)

Then, from the results of section II, we obtain abelian gauge transformation

\[ U(A)A^\mu(x)U^\dagger(A) \simeq (\Lambda^{-1})^{\mu}_{\nu}(A^\mu(\Lambda x) + \partial^\nu \Theta^a(\Lambda x, \alpha_1, \alpha_2)). \] (3.2)

This results is natural since we have consider the Lorentz transformation of free single-particle states. In other worlds, self interaction terms of non-abelian gauge field should be ignored in our formulation.

Nevertheless, we can reproduce non-abelian gauge transformation from the \textit{deformed Lorentz transformation} as follows.

\[ U(\Lambda, x)a_h^a(\mathbf{p})U^\dagger(\Lambda, x) \simeq \sqrt{\frac{E_{\Lambda^{\mathbf{p}}}}{E_p}} [(1 - i\theta h)a_h^a(\Lambda \mathbf{p}) - g \{ f^{abc} \Theta^b(\Lambda x, \alpha_1, \alpha_2)a_h^a(\Lambda \mathbf{p}) - [a_h^a(\Lambda \mathbf{p}), f^{bcd} \Theta^b(\Lambda x, \alpha_1, \alpha_2)]N^{cd} \}], \] (3.3)

\[ U(\Lambda, x)a_h^a(\mathbf{p})U^\dagger(\Lambda, x) \simeq \sqrt{\frac{E_{\Lambda^{\mathbf{p}}}}{E_p}} [(1 + i\theta h)a_h^a(\Lambda \mathbf{p}) - g \{ f^{abc} \Theta^b(\Lambda x, \alpha_1, \alpha_2)a_h^a(\Lambda \mathbf{p}) - [a_h^a(\Lambda \mathbf{p}), f^{bcd} \Theta^b(\Lambda x, \alpha_1, \alpha_2)]N^{cd} \}], \] (3.4)

and can find

\[ U(\Lambda, x)U(\Lambda, x)a_h(\mathbf{p})U^\dagger(\Lambda, x)U^\dagger(\Lambda', x) \]

\[ \simeq \sqrt{\frac{E_{\Lambda' \Lambda^{\mathbf{p}}}}{E_p}} [(1 - i(\theta' + \theta) h)a_h^a(\Lambda' \Lambda \mathbf{p}) - g \{ f^{abc} \Theta^b(\Lambda' \Lambda x, \alpha_1' + \alpha_1, \alpha_2' + \alpha_2)a_h^a(\Lambda' \Lambda \mathbf{p}) \}], \]
Now, \( N^{ab} := \int d^3p \sum_{h=\pm 1} a_{h}^{a\dagger}(p)a_{h}^{b}(p) = \int d^3p \sum_{A=1,2} \tilde{a}_{A}^{a\dagger}(p)\tilde{a}_{A}^{b}(p) \),

(3.6)

g is gauge coupling constant and \( f^{abc} \) is structure constant of \( G \) which is totally anti-symmetric. The last term of (3.3) and (3.4) is needed for preserving commutation relations of \( a_{h}^{a\dagger} \) and \( a_{h}^{b}(p) \).

The important point to note is the transformation of (3.3) and (3.4) depend on coordinates \( x \), namely, the local Lorentz transformation. Then, we obtain

\[ U(\Lambda, x)A^{\mu a}(x)U^{\dagger}(\Lambda, x) \simeq (\Lambda^{-1})_{\mu}^{\nu} (A^{\nu a}(x) + \partial^{\nu} \Theta^{a}(x, \alpha_{1}, \alpha_{2}) + g f^{abc} A^{b\nu}(x)\Theta^{c}(x, \alpha_{1}, \alpha_{2})) \, . \]

(3.7)

Proportional terms to \( N^{ab} \) vanish by the following reasons: The form of these terms are

\[ f^{abc}N^{bc} \int \frac{d^3p}{(2\pi)^2} E_{p} (\alpha_{1}\omega_{\mu}(p) + \alpha_{2}\tilde{\omega}_{\mu}(p)) \, . \]

(3.8)

Now, \( \alpha_{1}\omega_{\mu}(p) + \alpha_{2}\tilde{\omega}_{\mu}(p) = L(p^{0}, p)(0, \sqrt{2}a_{1}, \sqrt{2}a_{2}, 0) = L(p^{0}, -p)(0, -\sqrt{2}a_{1}, -\sqrt{2}a_{2}, 0) \). Therefore, because of the invariance on \( p \rightarrow -p \), (3.3) vanish.

We should note that since both of \( A^{\mu a}(x) \) and \( \Theta^{a}(x) \) are operators, there do not commute, but \( f^{abc}A^{b\mu}\Theta^{c} = f^{abc}\Theta^{a}A^{bc} \) because of the commutation relations (3.1). Consequently, we find that non-abelian gauge transformation of gauge field is induced by the local transformation of (3.3) and (3.4).

Similarly to abelian transformation (2.40), proportional terms to \( \alpha_{1} \) and \( \alpha_{2} \) are representation of two transformations of ISO(2). We define

\[ T_{A}^{ab}(x) := -f^{abc}ig \int \frac{d^3p}{(2\pi)^2} \sqrt{E_{p}} \left[ \tilde{a}_{A}(p)e^{-ipx} + \tilde{a}_{A}^{cl}(p)e^{ipx} \right] \, , \]

(3.9)

and find \((T_{A}^{ab}(x))^{\dagger} = -T_{A}^{ab}(x)\). Then, we can rewrite (3.3) and (3.4) for

\[ U(\Lambda, x)a_{h}^{a}(p)U^{\dagger}(\Lambda, x) \simeq E_{p} \left[ 1 - i\theta h a_{h}^{a}(\Lambda p) + \sum_{A=1,2} i\alpha_{A} [T_{A}^{ab}(x)a_{h}^{b}(\Lambda p) + [a_{h}^{a}(\Lambda p), T_{A}^{bc}(x)]N^{bc}] \right] \, , \]

(3.10)

\[ U(\Lambda, x)a_{h}^{a\dagger}(p)U^{\dagger}(\Lambda, x) \simeq E_{p} \left[ 1 + i\theta h a_{h}^{a\dagger}(\Lambda p) + \sum_{A=1,2} i\alpha_{A} [T_{A}^{ab}(x)a_{h}^{b\dagger}(\Lambda p) + [a_{h}^{a\dagger}(\Lambda p), T_{A}^{bc}(x)]N^{bc}] \right] \, . \]

(3.11)

Thus, similarly to (2.40), we define hermitian operators \( T_{1}(x), T_{2}(x) \) and \( J \) as follows.

\[ [T_{A}(x), a_{h}^{a}(p)] = T_{A}^{ab}(x)a_{h}^{b}(p) + [a_{h}^{a}(p), T_{A}^{bc}(x)]N^{bc} \, , \]

(3.12)

\[ [J, a_{h}^{a}(p)] = -ha_{h}^{a}(p) \, . \]

(3.13)

By using \([J, T_{1}^{bc}] = iT_{2}^{bc}, [J, T_{2}^{bc}] = -iT_{1}^{bc}, [a_{h}^{a}(T_{1}), T_{2}^{bc}] = ih[a_{h}^{a}, T_{2}^{bc}] \) and the commutation relation (3.1), then, we obtain

\[ [[J, T_{1}(x)], a_{h}^{a}(p)] = i[T_{2}(x), a_{h}^{a}(p)] \, , \] \[ [[J, T_{2}(x)], a_{h}^{a}(p)] = -i[T_{1}(x), a_{h}^{a}(p)] \, , \] \[ [[T_{1}(x), T_{2}(x)], a_{h}^{a}(p)] = 0 \, , \]

(3.14)

and similar relations for \( a_{h}^{a\dagger}(p) \). In addition, we can also write down explicitly \( T_{A} \) and \( J \) by using the creation and annihilation operators as follows

\[ T_{A}(x) = -\sum_{a,b} T_{A}^{ab}(x)N^{ab} \, , \]

(3.15)

\[ J = \int d^{3}p \sum_{h=\pm 1,a} ha_{h}^{a\dagger}(p)a_{h}^{a}(p) \, , \]

(3.16)
and also find
\[ J |0\rangle = T_A(x) |0\rangle = 0. \] (3.17)

Furthermore, in a similar way to section II B, we can introduce two unphysical modes which transform as follows.
\[ [T_1(x), a_h^a(p)] = T_1^a_{bc}(x) a_h^b(p) + [a_h^c(p), T_1^{bc}(x)] \tilde{N}^{bc} - a_h^a(p), \] (3.18)
\[ [T_2(x), a_h^a(p)] = T_2^a_{bc}(x) a_h^b(p) + [a_h^c(p), T_2^{bc}(x)] \tilde{N}^{bc} + i h a_h^a(p), \] (3.19)
\[ [T_1(x), a_L(p)] = T_1^a_{bc}(x) a_L^b(p) - \sum_{h=\pm 1} a_h^a(p), \quad [T_2(x), a_L(p)] = T_2^a_{bc}(x) a_L^b(p) - i \sum_{h=\pm 1} h a_h^a(p), \] (3.20)
\[ [T_1(x), a_S(p)] = T_1^a_{bc}(x) a_S^b(p), \quad [T_2(x), a_S(p)] = T_2^a_{bc}(x) a_S^b(p), \] (3.21)
where,
\[ \tilde{N}^{ab} := \int d^3p \left\{ \sum_{h=\pm 1} a_h^{a_1}(p) a_h^{b_1}(p) + a_L^{a_1}(p) a_S^{b_1}(p) + a_S^{a_1}(p) a_L^{b_1}(p) \right\}. \] (3.22)

Consequently, we obtain the local Lorentz transformation as follows.
\[ \tilde{U}(\Lambda, x) A^\mu(x) \tilde{U}^{-1}(\Lambda, x) = (\Lambda^{-1})^\mu_\rho A^\rho(x) + g f^{abc} A^{\nu b}(x) \Theta^c(\Lambda x, \alpha_1, \alpha_2) \] (3.23)

It seems that (3.23) is global gauge transformation. However, we cannot regard there as so because we can find the transformation at other point \( y \) as
\[ \tilde{U}(\Lambda, y) A^\mu(x) \tilde{U}^{-1}(\Lambda, y) = (\Lambda^{-1})^\mu_\rho A^\rho(x) + g f^{abc} A^{\nu b}(\Lambda x) \Theta^c(\Lambda y, \alpha_1, \alpha_2) \]
\[ + g f^{abc} \tilde{N}^{bc} F(\Lambda(x - y)), \] (3.24)
\[ F(x - y) := \int \frac{d^3p}{(2\pi)^3 2E_p} \left( \alpha_1 \omega^\mu(p) e^{-i p(x - y)} + \alpha_2 \tilde{\omega}^\mu(p) e^{i p(x - y)} \right), \] (3.25)
namely, the proportional term to \( \tilde{N}^{bc} \) is remaining.

### IV. IN OTHER DIMENSIONS

In this section, we show that gauge transformation is realized as representation of the little group in any spacetime dimensions.

We can obtain the Casimir operators of the \( D(\geq 3) \)-dimensional Poincaré algebra by replacing
\[ W_\mu \rightarrow W_{\mu_1 \cdots \mu_{D-3}} := -\frac{1}{2} \varepsilon_{\mu_1 \cdots \mu_{D-3} \nu \rho \sigma} P^\nu M^{\rho \sigma}. \] (4.1)

Then, we can find that the little group of massless particle is the \( (D - 2) \)-dimensional Euclidean group \( ISO(D - 2) \);
\[ [T_A, T_B] = 0, \quad [J_{AB}, J_C] = i (\delta_{BC} P_A - \delta_{AC} P_B), \]
\[ [J_{AB}, J_{CD}] = -i (\delta_{AC} J_{BD} + \delta_{BD} J_{AC} - \delta_{AD} J_{BC} - \delta_{BC} J_{AD}). \] (4.2)
A. The 3-dimensions

The little group is \textit{ISO}(1), namely, \( \mathbb{R} \). Accordingly, there is just a one continuous degree of freedom. In other words, there is no degree of freedom corresponding to helicity. We define the translation generator of \( \mathbb{R} \) and the polarization vector as follows.

\[
(T)^{\mu}_{\nu} = \begin{pmatrix}
0 & -i & 0 & 0 \\
-i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \epsilon^{\mu} := (0, 1, 0, 0).
\]

(4.3)

It is obvious that we can obtain this matrix and polarization vector from the case of 4-dimension by ignoring the y-component. Therefore, we can find representation of the little group \( \text{ISO}(4) \) are each parameters of transformations of \( J^{(3)} \) acting on the Lorentz vectors as follows.

\[
\begin{align*}
\epsilon^{\mu}_{\pm 1} := (0, 1, 0, \pm i, 0), \quad &\epsilon^{\mu}_{0} := (J^{(3)})^{\mu}_{\nu} \epsilon^{\nu}_{\pm 1} = (0, 0, -i, 0, 0).
\end{align*}
\]

(4.6)

Then, we define three polarization vectors as eigenvectors of \( J \):

\[
\begin{align*}
&\epsilon^{\mu}_{\pm 1} := (0, 1, 0, \pm i, 0), \quad &\epsilon^{\mu}_{0} := (J^{(3)})^{\mu}_{\nu} \epsilon^{\nu}_{\pm 1} = (0, 0, -i, 0, 0).
\end{align*}
\]

(4.7)

Therefore, we obtain equations similar to (2.7) as follows.

\[
(1 \mp i(\theta))\epsilon^{\mu}_{\pm 1}(p) - i\eta_{\pm} \epsilon^{\mu}_{0}(p) \simeq \Lambda^{-1})_{\nu}^{\mu} \epsilon^{\nu}_{\pm 1}(\Lambda p) + \frac{\alpha_{1} \pm i\alpha_{3}}{\sqrt{2k}} p^{\mu},
\]

(4.8)

Now, \( \eta_{\pm} \) are each parameters of transformations of \( J^{(3)} \).

After that, we define actions of \textit{SO}(3) generators \( J^{(3)} \) as follows.

\[
\begin{align*}
[J, a_{\pm}(p)] = \mp a_{\pm}(p), \quad & [J_{+}, a_{+1}(p)] = [J_{-}, a_{-1}(p)] = 0, \quad [J_{+}, a_{-1}(p)] = [J_{-}, a_{+1}(p)] = -a_{0}(p),
\end{align*}
\]

(4.9)

\[
\begin{align*}
[J, a_{\pm}^\dagger(p)] = \pm a_{\pm}^\dagger(p), \quad & [J_{+}, a_{+1}(p)] = [J_{-}, a_{-1}(p)] = 0, \quad [J_{+}, a_{-1}(p)] = [J_{-}, a_{+1}(p)] = a_{0}^\dagger(p),
\end{align*}
\]

(4.10)

\[
\begin{align*}
[J, a_{0}(p)] = [J, a_{0}^\dagger(p)] = 0, \quad & [J_{\pm}, a_{0}(p)] = -a_{\pm 1}(p), \quad [J_{\pm}, a_{0}^\dagger(p)] = a_{\pm 1}^\dagger(p).
\end{align*}
\]

(4.11)

B. The 5-dimensions

The little group is \textit{ISO}(3). Accordingly, there are three continuous degrees of freedom. First, we define six generators of \textit{ISO}(3) acting on the Lorentz vectors as follows.

\[
\begin{align*}
&(T_{1})^{\mu}_{\nu} = \frac{1}{\sqrt{2}}(J_{12} \pm iJ_{23})^{\mu}_{\nu} = \frac{1}{\sqrt{2}}(J_{1}^{\mu}_{\nu} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad (T_{2})^{\mu}_{\nu} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad (T_{3})^{\mu}_{\nu} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\end{align*}
\]

(4.4)

Then, we define three polarization vectors as eigenvectors of \( J \):

\[
\epsilon^{\mu}_{\pm 1} := (0, 1, 0, \pm i, 0), \quad \epsilon^{\mu}_{0} := (J^{(3)})^{\mu}_{\nu} \epsilon^{\nu}_{\pm 1} = (0, 0, -i, 0, 0).
\]

(4.5)

Therefore, we obtain equations similar to (2.7) as follows.

\[
\begin{align*}
&\epsilon^{\mu}_{\pm 1}(p) - i\eta_{\pm} \epsilon^{\mu}_{0}(p) \simeq \Lambda^{-1})_{\nu}^{\mu} \epsilon^{\nu}_{\pm 1}(\Lambda p) + \frac{\alpha_{1} \pm i\alpha_{3}}{\sqrt{2k}} p^{\mu},
\end{align*}
\]

(4.6)

Now, \( \eta_{\pm} \) are each parameters of transformations of \( J^{(3)} \).
By using \((4.7)\) to \((4.11)\), we obtain the Lorentz transformation of vector field similar to \((2.10)\). However, the parameter of transformation is different from \((2.11)\):

\[
\Theta(x, \alpha_1, \alpha_2, \alpha_3) := \frac{i}{\sqrt{2k}} \int \frac{d^3p}{(2\pi)^2 \sqrt{2E_p}} \left[ \alpha_1 \sum_{h=\pm 1} \left( a_h(p)e^{-ipx} - a_h^+(p)e^{ipx} \right) - \sqrt{2} i(a_0(p)e^{-ipx} + a_0^+(p)e^{ipx}) + i\hbar \alpha_3 \sum_{h=\pm 1} \left( a_h(p)e^{-ipx} + a_h^+(p)e^{ipx} \right) \right]. \tag{4.12}
\]

Hence, we define three translation generators similar to \((3.15)\), but now,

\[
T_1^{ab}(x) := f^{abc} \frac{g}{k} \int \frac{d^3p}{(2\pi)^2 \sqrt{2E_p}} \sum_{h=\pm 1} \left[ a_h^c(p)e^{-ipx} - a_h^c(p)e^{ipx} \right], \tag{4.13}
\]

\[
T_2^{ab}(x) := -f^{abc} \frac{i g}{k} \int \frac{d^3p}{(2\pi)^2 \sqrt{2E_p}} \left[ a_0^c(p)e^{-ipx} + a_0^c(p)e^{ipx} \right], \tag{4.14}
\]

\[
T_3^{ab}(x) := f^{abc} \frac{i g}{k} \int \frac{d^3p}{(2\pi)^2 \sqrt{2E_p}} \sum_{h=\pm 1} \left[ a_h^c(p)e^{-ipx} + a_h^c(p)e^{ipx} \right], \tag{4.15}
\]

\[
N^{ab} := \int d^3p \sum_{h=\pm 1, 0} a_h^a(p)a_h^b(p). \tag{4.16}
\]

In the case of non-abelian extension, then, we can find that these operators satisfy \(ISO(3)\) algebra;

\[
[J, T_1(x)] = iT_3(x), \quad [J, T_3(x)] = -iT_1(x), \quad [J, T_2(x)] = [T_A(x), T_B(x)] = 0,
\]

\[
[J_\pm, T_1(x)] = -\frac{i}{\sqrt{2}} T_2(x), \quad [J_\pm, T_2(x)] = \frac{1}{\sqrt{2}} (i T_1(x) \pm T_2(x)), \quad [J_\pm, T_3(x)] = \mp \frac{1}{\sqrt{2}} T_2(x), \tag{4.17}
\]

and generate non-abelian gauge transformation similar to \(\text{(4.18)}\) after all.

**C. Even-dimensions**

We consider the case of \(D = 2N + N\). So, the little group is \(ISO(2N)\). We define generators of \(ISO(2N)\) acting on the Lorentz vectors as follows.

\[
(T_A)^\mu = \begin{pmatrix} 0 & -i\delta_{A\nu} & 0 \\ -i\delta_{A\mu} & 0 & i\delta_{A\nu} \\ 0 & -i\delta_{A\nu} & 0 \end{pmatrix}, \quad (J_{AB})^\mu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i(\delta_{A\mu}\delta_{B\nu} - \delta_{B\mu}\delta_{A\nu}) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.18}
\]

There are the \(N\) Cartan generators of \(SO(2N)\) which are \(2 \times 2\) block diagonalized matrices; \(H_n := J_{2n-1,2n}\). Therefore, we introduce \(2N\) polarization vectors as the eigenvectors of \(H_n\) similar to 4-dimensional case, namely, labeled by helicity \(h = \pm 1\) for each \(n\). Hence, if we only consider \(H_n\) about subalgebra of \(SO(2N)\), we can obtain representation of \(ISO(2N)\) as \(n\) copies of \(ISO(2)\) because each translation generators corresponding to two directions of \(H_n\) are commutable to the other Cartan generators. Then, the actions of other generators of \(SO(2N)\) are exchanging each \(n\) states. Thus, we can also find the representation of these generators similar to the case of \(ISO(2)\) with redefining the Cartan generators. Consequently, we can reproduce gauge transformation form Lorentz transformation by similar way to the 4-dimensional case, although we do not show this explicitly.
We consider the case of $D = 2N + 3$ and the little group $ISO(2N + 1)$. We define generators of $ISO(2N + 1)$ similar to (4.18). In contrast, there are $N + 1$ Cartan generators of $SO(2N + 1)$ which are $N - 1$, $2 \times 2$ and one $3 \times 3$ block diagonalized matrices. Therefore, we introduce polarization vectors which are $2N - 2$ vectors as the even-dimensional case and three vectors as the 5-dimensional case and then we can reproduce gauge transformation form Lorentz transformation by similar way to the even-dimensional case.

**SUMMARY**

In this paper we investigated the relations between gauge symmetry and the Lorentz symmetry, specifically the symmetry of little group. A Free single-particle state was classified by mass, spatial momentum, and representation of the little group. In particular, the little group of massless particles in $D$-dimensional spacetime is $ISO(D - 2)$, namely $D - 2$-dimensional Euclidean group which contains rotations and translations. Therefore, these states have continuous degree of freedom but there are unphysical. Hence, we have assumed that the action of translations are trivial. As a result, we realized abelian gauge transformation form the Lorentz transformation of the massless vector, also tensor, field because of the action of little group translations on polarization vectors. Moreover, by including the unphysical modes, we constructed the representation of little group as transforming physical mode to unphysical modes, and vice versa and we realized ordinary Lorentz transformation. Furthermore, we obtained the same physical state condition with the Nakanishi-Lautrup formalism for unitarity. After that, we extended these results to the case of non-abelian gauge transformation as Lorentz transformation was restricted to the local transformation and the representation of little group was realized as tensor operators acting on gauge group constructed from creation and annihilation operators. Finally, we showed that we can obtain gauge transformation by similar way in any spacetime dimensions.

We obtained non-abelian gauge transformation form Lorentz transformation of physical single-particle stats. Consequently, if we impose the Lorentz symmetry for Lagrangian of vector field, it should have also gauge symmetry. So, we conclude it is not necessarily to impose gauge symmetry explicitly. It is introduced automatically if we impose only the Lorentz symmetry.

Finally, we comment that the extension to curved backgrounds and general coordinate transformation. It is easy to extend the procedure of this paper to curved backgrounds by using vielbein, because we can regard the Lorentz transformation as local one. Then, by using connection 1-form field, we can consider whether nonlinear general coordinate transformation is reproduced by our procedure. Furthermore, as generalization of the case of connection 1-form, we can consider interacting higher-spin gauge theory. It is known to construct interacting higher-spin gauge theory is difficult, but it could be found by our procedure.

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**Appendix: “No-go theorem” for higher-spin fields with non-abelian gauge symmetry**

In section III, we have shown that non-abelian gauge transformation of spin-1 field are generated by translations of the little group $ISO(N)$. Thus, we can conjecture that for any integer spin field, generally, gauge transformation should be realized as Lorentz transformation. Indeed, as shown in section II A, abelian gauge transformation for any integer spin fields are identified actions of the little group. Therefore, we consider to construct non-abelian gauge transformation of more then spin-2 fields similarly to the case of abelian transformation. In this section, we consider only the 4-dimensional case for simplicity.

We introduce spin-n fields with indices of adjoint representation of $G$ from (2.22) by replacing $a_{nh}(p) \rightarrow a_{nh}(p)$:

$$\phi^{\mu_1 \cdots \mu_n a}(x) := \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2p}} \sum_{h = \pm 1} \left[ \epsilon_{h}^{\mu_1 \cdots \mu_n}(p) a_{nh}(p) e^{-ipx} + \epsilon^{*\mu_1 \cdots \mu_n}(p) a_{nh}^{\dagger}(p) e^{ipx} \right], \quad (A.1)$$
and consider trivial generalization of non-abelian gauge transformation (3.7) as follows.

\[
U(\Lambda)\phi^{\mu_1 \cdots \mu_n a}(x) U^\dagger(\Lambda) = (\Lambda^{-1})^{\mu_1}_{\nu_1} \cdots (\Lambda^{-1})^{\mu_n}_{\nu_n} \left( \phi^{\nu_1 \cdots \nu_n a}(Ax) + \partial^{(\nu_1} \xi^{\nu_2 \cdots \nu_n)}a(\Lambda x) \right),
\]

where,

\[
\Theta^{(n)a}(x, \alpha_1, \alpha_2) := \frac{i}{\sqrt{2}k} \int \frac{d^3p}{(2\pi)^3} \sqrt{2E_p} \sum_{h=\pm 1} \left[ \alpha_1(a_{nh}(p)e^{-ipx} - a_{nh}^\dagger(p)e^{ipx}) + \imath h\alpha_2(a_{nh}(p)e^{-ipx} + a_{nh}^\dagger(p)e^{ipx}) \right].
\]

(A.3)

It is easy to reproduce the transformation (A.2) similar to the spin-1 case by also replacing \(a_{nh}(p)\rightarrow a_{nh}^a(p)\) for \(J\) and \(T_A(x)\). However, such generators are not representation of ISO(2);

\[
[J, T_1(x)] = \imath n T_2(x), \quad [J, T_2(x)] = -\imath n T_1(x).
\]

(A.4)

In other words, the transformation (A.2) is not realized as the Lorentz transformation. Therefore, if the conjecture; 

\textit{gauge transformation should be realized as Lorentz transformation} is true, we can conclude that there is no higher-spin field with (internal) non-abelian gauge symmetry in flat spacetime. This conclusion is consistent with Coleman-Mandula theorem[7].

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