About some possible blow-up conditions for the 3-D Navier-Stokes equations

Haroune Houamed*

April 30, 2019

Abstract

In this paper, we study some conditions related to the question of the possible blow-up of regular solutions to the 3D Navier-Stokes equations. In particular, up to a modification in a proof of a very recent result from [6], we prove that if one component of the velocity remains small enough in a sub-space of $\dot{H}^{\frac{3}{2}}$ "almost" scaling invariant, then the 3D Navier Stokes is globally wellposed. In a second time, we investigate the same question under some conditions on one component of the vorticity and unidirectional derivative of one component of the velocity in some critical Besov spaces of the form $L^p_T(\dot{B}^{\frac{2}{p}+\frac{3}{q}-\frac{2}{q}}_{2,q})$ or $L^p_T(\dot{B}^{\frac{2}{p}+\frac{1}{q}}_{2,\infty})$.

Keywords: Incompressible Navier-Stokes Equations, Anisotropic Littlewood-Paley Theory, Blow-up criteria.
AMS Subject Classification (2010): 35Q30, 76D03

1 Introduction

In this work we are interested in the study of the possible blow-up for regular solutions to the 3D incompressible Navier stokes equations

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \Delta u + \nabla P &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div} u &= 0 \\
\left. u \right|_{t=0} &= u_0
\end{aligned}
\]

where the unkowns of the equations $u = (u^1, u^2, u^3)$, $P$ are respectively, the velocity and the pressure of the fluid. We recall that the set of the solutions to (NS) is invariant under the transformation:

$$u_{0,\lambda}(x) \overset{\text{def}}{=} \lambda u_0(\lambda x), \quad u_{\lambda}(t, x) \overset{\text{def}}{=} \lambda u_{\lambda}(\lambda^2 t, \lambda x)$$

That is if $u(t, x)$ is a solution to (NS) on $[0, T] \times \mathbb{R}^3$ associated to the initial data $u_0$, then, for all $\lambda > 0$, $u_{\lambda}(t, x)$ is a solution to (NS) on $[0, \lambda^{-2}T] \times \mathbb{R}^3$ associated to the initial data $u_{0,\lambda}$.

It is well known that system (NS) has a global weak solution with finite energy

$$\|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(t')\|_{L^2}^2 \, dt' \leq \|u_0\|_{L^2}^2 \leq (1)$$
This result was proved first by J.Leray in [18]. In dimension three, uniqueness for such solutions stands to be an open problem. J.Leray proved also in his famous paper [18] that, for more regular initial data, namely for \( u_0 \in H^1(\mathbb{R}^3) \), \((NS)\) has a unique local smooth solution, that is there exists \( T^* > 0 \) and a unique maximal solution \( u \in L^\infty_t(H^1(\mathbb{R}^3)) \cap L^2_T \cdot (H^2(\mathbb{R}^3)) \).

The question of the behaviour of this solution after \( T^* \) remains to be also an open problem. In order to give a "formaly" large picture, let us define the set

\[
\chi_T \overset{\text{def}}{=} \left( (L^\infty_T L^2 \cap L^2_T H^1) \cdot (L^\infty_T L^2 \cap L^2_T H^1) \right)
\]

where

\[
(L^\infty_T L^2 \cap L^2_T H^1) \cdot (L^\infty_T L^2 \cap L^2_T H^1) \overset{\text{def}}{=} \{ uv : (u, v) \in (L^\infty_T L^2 \cap L^2_T H^1) \times (L^\infty_T L^2 \cap L^2_T H^1) \}
\]

Multiplying \((NS)\) by \(-\Delta u\), and integrating by parts yield

\[
\frac{d}{dt} \| \nabla u \|^2_{L^2} + \| \nabla u \|^2_{H^1} = - \int_{\mathbb{R}^d} (\nabla u \cdot \nabla u) |\nabla u|
\]

If we suppose that \( \nabla u \) is already bounded in \( G_T \) some sub-space of \( \chi_T \), then one may prove that \( \nabla u \) is bounded in \( L^\infty_T L^2 \cap L^2_T H^1 \). This is the case in dimension two where we get, for free, by the \( L^2 \)-energy estimate (1) a uniform bound of \( \nabla u \) in \( L^2_T L^2 \subset (L^4_T L^4 \cdot L^4_T L^4)' \subset \chi_T \).

In the case of dimension three, several works have been done in this direction, establishing a global wellposedeness of \((NS)\) under assumptions of the type \( \nabla u \in G_T \). We can set as an example of these results the well known Prodi-Serrin type criterions, saying that, if \( u \in L^p([0, T], L^q(\mathbb{R}^d)) \), with \( \frac{2}{p} + \frac{3}{q} = 1 \) and \( q \leq 3, \infty \), then \((NS)\) is globally wellposed. The limit case where \( q = 3 \) was proved recently by L. Escauriaza, G. Seregin and V. Sverák in [10] proving that: if \( T^* \overset{\text{def}}{=} T^*(u_0) \) denotes the life span of a regular solution \( u \) associated to the initial data \( u_0 \) then

\[
T^* < \infty \implies \limsup_{t \to T^*} \| u(t) \|_{L^3(\mathbb{R}^3)} = \infty
\]

This was extended to the full limit in time in \( H^{\frac{3}{2}}(\mathbb{R}^3) \) by G. Seregin in [21].

In another hand, one may notice that the divergence free condition can provide us another type of conditions for the global regularity (let us say anisotropic ones) under conditions on some components of the velocity or its gradiant. Several works have been done in this direction, one may see for instance [19, 3, 4, 11, 13, 17, 20, 22, 23, 26] for examples in some scaling invariant spaces or not of Serrin-type regularity criterions, or equivalently proving that, if \( T^* \) is finite then

\[
\int_0^{T^*} \| u^3(t, \cdot) \|^p_{L^q} = \infty \quad \text{or} \quad \int_0^{T^*} \| \partial_j u^3(t, \cdot) \|^p_{L^q} = \infty
\]

The first result in a scaling invariant space under only one component of the velocity has been proved by J.-Y Chemin and P.Zhang in [7] for \( p \in ]4, 6[ \) and a little bit later by the same authors together with Z.Zhang in [9] for \( p \in ]4, \infty[ \). The case \( p = 2 \) has been treated very recently by J.-Y Chemin, I.Gallagher and P.Zhang in [6]. As mentionned in [6] such a result in the case of \( p = \infty \), assuming it is true, seems to be the out of reach for the time being, however the authors in [6] proved some results for \( p = \infty \). Mainly they proved that if there is a blow-up at some time \( T^* > 0 \), then it is not possible for one component of the velocity to tend to 0 too fast. More precisely they proved the following blow-up condition

\[
\forall \sigma \in \mathbb{S}^2, \forall t < T^*, \quad \sup_{t' \in [t, T^*]} \left\| u(t') \cdot \sigma \right\|_{H^{\frac{3}{2}}} \geq c_0 \log^{-\sigma} \left( e + \frac{\| u(t) \|^4_{L^2}}{T^* - t} \right)
\]

2
The last result proved in their paper needs reinforcing slightly the $\dot{H}^{\frac{1}{2}}$ norm in some directions. Mainly, without loss of generality, their result can be stated as the following

**Theorem 1.** There exists a positive constant $c_0$ such that if $u$ is a maximal solution of (NS) in $C([0, T^*[: H^1])$, then for all positive real number $E$ we have:

$$T^* < \infty \implies \limsup_{y \to T^*} \| u^3 \|_{\dot{H}^{\frac{1}{2}}_{\log, E}} \geq c_0.$$  

where

$$\| a \|^2_{\dot{H}_{\log, E}^{\frac{1}{2}}} \stackrel{def}{=} \int_{\mathbb{R}^3} |\xi| \log(\xi|\xi_h| + e) |\hat{a}(\xi)|^2 d\xi < \infty$$

Motivated by this result, we aim to show that, up to a small modification in the proof of Theorem 1, we can obtain the same blow-up condition in the case $p = \infty$, by slightly reinforcing the $\dot{H}^{\frac{1}{2}}$ norm in the vertical direction instead of the horizontal one. More precisely, we define

**Definition 1.** Let $E$ be a positive real number. We define $\dot{H}_{\log, E}^{\frac{1}{2}}$ to be the sub space of $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ such that:

$$a \in \dot{H}_{\log, E}^{\frac{1}{2}} \iff \| a \|^2_{\dot{H}_{\log, E}^{\frac{1}{2}}} \stackrel{def}{=} \int_{\mathbb{R}^3} |\xi| \log(\xi|\xi_v| + e) |\hat{a}(\xi)|^2 d\xi < \infty$$

We will prove

**Theorem 2.** There exists a positive constant $c_0$ such that if $u$ is a maximal solution of (NS) in $C([0, T^*[: H^1])$, then for all positive real number $E$ we have

$$T^* < \infty \implies \limsup_{y \to T^*} \| u^3 \|_{\dot{H}^{\frac{1}{2}}_{\log, E}} \geq c_0.$$  

**Remark 1.** The blow-up condition stated in Theorem 2 above can be generalized to the following one

$$\forall \sigma \in S^2, \quad T^* < \infty \implies \limsup_{y \to T^*} \| \sigma \cdot u \|_{\dot{H}^{\frac{1}{2}}_{\log, E}} \geq c_0,$$

where

$$\| a \|^2_{\dot{H}_{\log, E}^{\frac{1}{2}}} \stackrel{def}{=} \int_{\mathbb{R}^3} |\xi| \log(\xi|\xi_v| + e) |\hat{a}(\xi)|^2 d\xi < \infty, \quad \text{with } \xi_v \stackrel{def}{=} (\xi \cdot \sigma) \sigma$$

The other two results that we will prove in this paper can be seen as some blow-up criterions under scaling invariant conditions on one component of the velocity and one component of the vorticity, whether in some anisotropic Besov spaces of the form $L^p((B_{2,\infty}^{s_p})h(B_{2,\infty}^{s_p-\alpha})), \text{for } \alpha \in [0, s_p]$, or $L^p(B_{q,p})$, where

$$s_p \stackrel{def}{=} \frac{2}{p} - \frac{1}{2}, \text{ and } B_{q,p} \stackrel{def}{=} B_{q,\infty}^{s_p+\frac{2}{p}-2}. \quad (3)$$

We will prove

**Theorem 3.** Let $u$ be a maximal solution of (NS) in $C([0, T^*[: H^1])$. If $T^* < \infty$, then

$$\forall p, m \in [2, 4], \quad \forall \alpha \in \left[0, \frac{2}{p} - \frac{1}{2}\right], \quad \forall \beta \in \left[0, \frac{2}{m} - \frac{1}{2}\right], \quad \text{we have:}$$

$$\int_0^{T^*} \| \partial_3 u^3(t') \|^p_{B_{2,\infty}^{s_p-\alpha}} dt' + \int_0^{T^*} \| \omega^3(t') \|^m_{B_{2,\infty}^{s_m-\beta}} dt' = \infty$$
Theorem 4. Let \( u \) be a maximal solution of \( (NS) \) in \( C([0,T^*];H^1) \). If \( T^* < \infty \), then for all \( q_1, q_2 \in [3,\infty[ \), for all \( p_1, p_2 \) satisfying
\[
\frac{3}{q_i} + \frac{2}{p_i} \in [1,2[, \quad i \in \{1,2\}
\]
we have
\[
\int_0^{T^*} \left\| \partial_3 u^3(t') \right\|_{B^{q_1}_{p_1}}^{p_1} \, dt' + \int_0^{T^*} \left\| \omega^3(t') \right\|_{B^{q_2}_{p_2}}^{p_2} \, dt' = \infty
\]

Remark 2. \( \text{1. All the spaces stated in Theorems 3 and Theorem 4 above are scaling invariant spaces under the natural 3-D Navier-Stokes scaling.} \)

\( \text{2. The regularity of the spaces stated in the blow-up conditions in Theorem 4 is negative, more precisely under assumption (\textbf{A}), } \frac{3}{q_i} + \frac{2}{p_i} - 2 \in ]-1,0[. \) Moreover, the integrability asked for in the associated Besov spaces is always higher than 3, which make these spaces larger than \( L^p_T(H^{\frac{2}{3} - \frac{1}{2}}) \).

\( \text{3. Taking in mind the embedding } L^p_T(H^{\frac{2}{3} - \frac{1}{2}}) \hookrightarrow L^p_T(B^{\frac{1}{2} - \frac{1}{2}}_{q_2,p_2}), \) for all \( \alpha \in [0, \frac{1}{p} - \frac{1}{2}) \) (see lemma A.2.3), it is obvious then that the blow-up conditions stated in Theorem 3 implies the ones in \( L^p_T(H^{\frac{2}{3} - \frac{1}{2}}). \)

\( \text{4. In the case } p = 4 \) (resp. \( m = 4 \)) in Theorem 3, \( \alpha \) (resp. \( \beta \)) is necessary zero, this means that the anisotropic space above is nothing but \( L^4_T(B^0_{2,\infty}) \), which is still larger than \( L^4_T(L^2) \). The proof in this case can be done without any use of anisotropic technics.

The structure of the paper is the following: In section 2, we reduce the proof of the Theorems to the proofs of three lemmas. In Section 3, we should present the proofs of these three lemmas, where we will use some results which will be recalled/proved in Section 4 ‘Appendix’ together with the definition and the properties of the functional spaces used in this work.

Notations If \( A \) and \( B \) are two real quantities, the notaion \( A \lesssim B \) means \( A \leq CB \) for some universal constant \( C \) which is independent on varying parameters of the problem. \((c_q)_{q \in \mathbb{Z}}\) (resp. \((d_q)_{q \in \mathbb{Z}}\)) will be a sequence satisfying \( \sum_{q \in \mathbb{Z}} c_q^2 \leq 1 \) (resp. \( \sum_{q \in \mathbb{Z}} d_q \leq 1 \)), which is allowed to differ from a line to another one.

Sometimes, we will use the notations
\[
L^p_T(L^q_h L^l_v) \overset{\text{def}}{=} L^p((0,T); L^q((\mathbb{R}^3_h); L^l((\mathbb{R}^3_v)))), \quad \dot{H}^s_h(\dot{H}^s_v) \overset{\text{def}}{=} \dot{H}^{s,1}(\mathbb{R}^3), \quad ||| \dot{H}^{s,t}_h(\dot{H}^{s,t}_v) ||| \overset{\text{def}}{=} \||| \dot{B}^s_{p,q}(\mathbb{R}^3) \|
\]

2 Proof of the Theorems

Denoting by:
\[
J_{\ell t}(u, u^3) \overset{\text{def}}{=} \int_{\mathbb{R}^3} \partial_3 u^3 \partial_3 u^\ell \partial_\ell u^t
\]
The proof of Theorem 3 is then based on the following lemma

Lemma 1. There exists \( C > 0 \) such that, for any \( E > 0 \), we have:
\[
\left| J_{\ell t}(u, u^3) \right| \leq \left( \frac{1}{10} + C \left\| u^3 \right\|_{L^\infty_{H^3,0,E}} \right) \left\| \nabla_h u \right\|_{H^1}^2 + C \left\| u^3 \right\|_{H^\frac{1}{2}}^2 \left\| \partial_3 u^h \right\|_{L^2}^2 \frac{E^2}{E^2}
\]
While, the proofs of Theorem 3 and Theorem 4 are essentially based on the following ones

**Lemma 2.** For all $q \in ]2,4[$, for all $\alpha \in ]0,s_p[$, where $s_p = 2 \cdot \frac{q}{q} - \frac{1}{2}$, we have:

$$|(fg)|_{L^2} \leq \frac{1}{10} \|h\|_{H^1(\mathbb{R}^3)}^2 + C \|g\|_{B_{q,p}^{s_p-\alpha}} \|g\|_{L^2(\mathbb{R}^3)}^2$$

**Lemma 3.** For any $p,q \in [1,\infty]$ satisfying $\frac{2}{q} + \frac{2}{p} \in ]1,2[$ we have

$$|(fg)|_{L^2} \leq \frac{1}{10} \|g\|_{H^1(\mathbb{R}^3)}^2 + C \|f\|_{B_{p,q}^p} \|g\|_{L^2(\mathbb{R}^3)}^2$$

As mentionned above, let us assume that lemmas 1, 2 and 3 hold true, which we will prove in the next section, and let us prove Theorems 2, 3 and 4.

### 2.1 Proof of Theorem 2

Following the idea of [1] we begin by establishing a bound of $\nabla_h u$ in $L_\infty^\infty(L^2) \cap L_\infty^2(\dot{H}^1)$, then we use this estimate to prove a bound of $\partial_3 u$ in $L_\infty^\infty(L^2) \cap L_\infty^2(\dot{H}^1)$. To do so we multiply (NS) by $-\Delta_h u$, usual calculation leads then to:

$$\frac{d}{2dt} \|\nabla_h u\|_{L^2}^2 + \|\nabla_h u\|_{H^1}^2 = \sum_{i=1}^4 \mathcal{E}_i(u) \text{ with:}$$

\[
\mathcal{E}_1(u) \overset{\text{def}}{=} -\sum_{i=1}^2 (\partial_1 u^h \cdot \nabla_h u^h | \partial_i u^h)_{L^2} \\
\mathcal{E}_2(u) \overset{\text{def}}{=} -\sum_{i=1}^2 (\partial_1 u^h \cdot \nabla_h u^3 | \partial_i u^3)_{L^2} \\
\mathcal{E}_3(u) \overset{\text{def}}{=} -\sum_{i=1}^2 (\partial_1 u^3 \partial_3 u^h | \partial_i u^h)_{L^2} \\
\mathcal{E}_4(u) \overset{\text{def}}{=} -\sum_{i=1}^2 (\partial_1 u^3 \partial_3 u^h | \partial_i u^h)_{L^2}
\]

A direct computation shows that $\mathcal{E}_1(u), \mathcal{E}_2(u)$ and $\mathcal{E}_4(u)$ can be expressed as a sum of terms of the form

$$I(u) \overset{\text{def}}{=} \int_{\mathbb{R}^3} \partial_1 u^3 \partial_j u^k \partial_\ell u^m$$

where: $(j,\ell) \in \{1,2\}^2$ and $(i,k,m) \in \{1,2,3\}^3$.

Next, by duality, product rules and then interpolation, for any $p \in [1,\infty]$, one may easily show that

\[
I(u) \lesssim \|\nabla_h u^3\|_{\dot{H}^{\frac{2}{p}+\frac{1}{2}}} \|\partial_j u^k \partial_\ell u^m\|_{\dot{H}^{\frac{2}{p}+\frac{1}{2}}} \\
\lesssim \|u^3\|_{\dot{H}^{\frac{2}{p}+\frac{1}{2}}} \|\nabla_h u\|_{\dot{H}^{1-\frac{1}{2}}}^2 \\
\lesssim \|u^3\|_{\dot{H}^{\frac{2}{p}+\frac{1}{2}}} \|\nabla_h u\|_{\dot{L}^2}^2 \|\nabla_h u\|_{\dot{H}^1}^{2-\frac{2}{p}}
\]

\[\text{Notice that } I(u) \text{ provides a globale bound if } u^3 \in L^p(\dot{H}^{\frac{2}{p}+\frac{1}{2}}) \text{ for some } p \in [1,\infty]. \text{ It is in fact the term } \mathcal{E}_3(u) \text{ which poses a problem, and this is why this methode doesn't give a complet answer to the regularity criteria under one component only in the case } p = 2 \text{ as mentionned in [6].} \]
In particular for \( p = \infty \) we have:

\[
I(u) \lesssim \|u^3\|_{H^2_\infty} \|\nabla h u\|_{H^1}^2
\]  

(6)

The term \( E_3(u) \), can be estimated by using lemma 1 to obtain

\[
E_3(u) \leq \left( \frac{1}{10} + C \|u^3\|_{H^2_{\log v,E}} \right) \|\nabla h u\|_{H^1}^2 + C \|u^3\|_{H^2_{\log v,E}}^2 \frac{\|\partial_3 u h\|_{L^2}}{E^2}
\]

We define then

\[
T_* \overset{\text{def}}{=} \sup \left\{ T \in [0, T^*] : \sup_{t \in [0, T]} \|u^3(t)\|_{H^2_{\log v,E}} \leq \frac{1}{4C} \right\}
\]

Therefore, for all \( t \leq T_* \), relation (5) together with estimate (6), lemma 1 and the classical \( L^2 \)-energy estimate lead to

\[
\|\nabla h u(t)\|_{L^2}^2 + \int_0^t \|\nabla h u(s)\|_{H^1}^2 \, ds \leq \|\nabla h u_0\|_{L^2}^2 + \frac{\|u_0\|_{L^2}^2}{E^2}
\]  

(7)

In the other hand, as explained in [6], multiplying \((NS)\) by \(-\partial_3^2 u\), integrating over \( \mathbb{R}^3 \), integration by parts together with the divergence free condition lead to

\[
\frac{d}{dt} \|\partial_3 u\|_{L^2}^2 + \|\partial_3 u\|_{H^1}^2 \lesssim \|\partial_3 u\|_{L^6} \|\nabla h u\|_{L^3} \|\partial_3 u\|_{L^2}
\]

\[
\lesssim \frac{1}{2} \|\partial_3 u\|_{H^1}^2 + C \|\nabla h u\|_{L^2} \|\nabla h u\|_{H^1} \|\partial_3 u\|_{L^2}^2
\]

(7) above leads then to a bound for \( u \) in \( L^\infty_T (\dot{H}^1) \).

Thus, by contraposition, if the quantity \( \|u(t)\|_{\dot{H}^1} \) blows-up at a finite time \( T^* > 0 \), then

\[
\forall t \in [0, T^*] : \sup_{s \in [0, t]} \|u^3(s)\|_{H^2_{\infty}} > c_0 \overset{\text{def}}{=} \frac{1}{4C}
\]

which gives the desired result by passing to the limit \( t \to T^* \).

Theorem 2 is proved. \( \square \)

### 2.2 Proof of Theorem 3

Following for example an idea from [27], we multiply \((NS)\) by \(-\Delta u\) and we integrate in space to obtain

\[
\frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 = \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \Delta u
\]

\[
= - \int_{\mathbb{R}^3} \sum_{i,j,k=1}^3 \partial_k u^j \partial_j u^i \partial_k u^i
\]

For the time being, we don’t know how to deal with the tri-linear term on the right hand-side above in order to obtain a global-estimate of \( u \) in \( L^\infty_T \dot{H}^1_x \cap L^2_T \dot{H}^2_x \), so to close the estimates the idea is similar to the one in Theorems [1] and [2] and it consists in looking at this term as a bilinear operator acting on \( (L^\infty_T \dot{H}^1_x \cap L^2_T \dot{H}^2_x)^2 \) after assuming a condition which allows to control some components of the matrix \( \partial_3 u^j \).
Let us recall the Biot-Savart law identity which allows to write the so-called div-curl decomposition of $u^h$ as
\[ u^h = \nabla_h^T \Delta_h^{-1}(\omega^3) - \Delta_h \nabla_h^{-1}(\partial_3 u^3) \] (8)

Identity (8) insures that, for $(i, j) \in \{1, 2\}^2$, $\partial_i u^j$ can be writing in terms of $\omega^3$ and $\partial_3 u^3$, modulo some anisotropic Fourier-multiplyers of order zero, more precisely we have, for $(i, j) \in \{1, 2\}^2$
\[ \partial_i u^j = \mathcal{R}_{i,j} \omega^3 + \tilde{\mathcal{R}}_{i,j} \partial_3 u^3 \]
where $\mathcal{R}_{i,j}$ and $\tilde{\mathcal{R}}_{i,j}$ are zero-order Fourier multiplyers bounded from $L^q$ into $L^q$ for all $q$ in $[1, \infty]$. In the other hand, the quantity $\partial_k u^j \partial_j u^i \partial_k u^i$ contains always, at least, one term of the form $\partial_i u^j$ with $(i, j) \in \{1, 2\}^2$ or $i = j = 3$, we infer that
\[ \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \lesssim \sum_{(l, k, m, n) \in \{1, 2, 3\}} \left| \int_{\mathbb{R}^3} (\mathcal{R}_{i,j} \omega^3 + \tilde{\mathcal{R}}_{i,j} \partial_3 u^3) \partial_k u^l \partial_m u^n \right| \] (9)

Lemma 3 gives then
\[ \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 \leq \frac{1}{10} \|\nabla u\|_{H^1}^2 + C \left( \|\partial_3 u_3\|_{B^{p, \alpha}_{\infty}} + \|\omega^3\|_{B^{m, \beta}_{\infty}} \right) \|\nabla u\|_{L^2} \]

Gronwall lemma leads then to
\[ \|\nabla u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(t')\|_{H^1}^2 \, dt' \lesssim \|\nabla u_0\|_{L^2} \exp \left[ C \int_0^t \left( \|\partial_3 u_3(t')\|_{B^{p, \alpha}_{\infty}} + \|\omega^3(t')\|_{B^{m, \beta}_{\infty}} \right) \, dt' \right] \] (10)

That is, if, for some $\alpha, \beta, p, m$ satisfying the hypothesis of Theorem 3, the quantity in the right hand side of (10) is finite, then $u$ is bounded in $L^2_T (H^1)$. By contraposition, if there is a blow-up of the $H^1$ norm at some finite $T^*$ then, for all $\alpha, \beta, p, m$
\[ \int_0^{T^*} \left( \|\partial_3 u_3(t')\|_{B^{p, \alpha}_{\infty}} + \|\omega^3(t')\|_{B^{m, \beta}_{\infty}} \right) \, dt' = \infty \]

Theorem 3 is proved.

2.3 Proof of Theorem 4

The proof of Theorem 4 doesn’t differ a lot from the previous one. We restart from (9), applying lemma 2 gives
\[ \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 \leq \frac{1}{10} \|\nabla u\|_{H^1}^2 + C \left( \|\partial_3 u_3\|_{B^{p_1}_{\infty}} + \|\omega^3\|_{B^{p_2}_{\infty}} \right) \|\nabla u\|_{L^2} \]

Next, integrating in time interval $[0, t]$, and applying Gronwall lemma gives
\[ \|\nabla u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(t')\|_{H^1}^2 \, dt' \lesssim \|\nabla u_0\|_{L^2} \exp \left[ C \int_0^t \left( \|\partial_3 u_3\|_{B^{p_1}_{\infty}} + \|\omega^3\|_{B^{p_2}_{\infty}} \right) \, dt' \right] \]

Same arguments as in the conclusion of the previous theorem lead to the desired result. Theorem 4 is proved.

\[ \text{Note that the case } q_i = \infty \text{ is included in the estimates proved in Lemma 3 however we did not say anything about this case in Theorem 4 due to the lack of continuity of Riesz operators } \mathcal{R}_{i,j} \text{ and } \tilde{\mathcal{R}}_{i,j} \text{ from } L^\infty \text{ into } L^\infty. \]
3 Proof of the three lemmas

3.1 Proof of lemma 7

Let us recall a definition from [6]. For $E \in \mathbb{R}^+$ and $a \in S'(\mathbb{R}^3)$:

\[
\begin{align*}
a_{h,E^{-1}} & \overset{\text{def}}{=} \mathcal{F}^{-1}(1_{B_h^c(0,E^{-1})} \hat{a}), & \quad a_{h,E} & \overset{\text{def}}{=} \mathcal{F}^{-1}(1_{B_h(0,E)} \hat{a})
\end{align*}
\]

Based on this decomposition, we write

\[
J_{i\ell}(u,u^3) = J_E^1 + J_E^2
\]

where

\[
\begin{align*}
J_E^1 & \overset{\text{def}}{=} \int_{\mathbb{R}^3} \partial_i u^3 \partial_3 u_{\ell,E^{-1}}\partial_i \ell, \quad \text{and} \quad J_E^2 \overset{\text{def}}{=} \int_{\mathbb{R}^3} \partial_i u^3 \partial_3 u_{\ell,E} \partial_i \ell \\
\end{align*}
\]

The main point consists in estimating $J_E^3$. Using Bony’s decomposition with respect to the horizontal variables, to write

\[
J_E^3 = J_{E,1}^3 + J_{E,2}^3
\]

with

\[
\begin{align*}
J_{E,1}^3 & \overset{\text{def}}{=} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \partial_i u^3(x_h,x_3) \Delta_h \partial_3 u_{\ell,E^{-1}}(x_h,x_3) dx_h \right) dx_3 \\
J_{E,2}^3 & \overset{\text{def}}{=} \int_{\mathbb{R}^3} \left( \sum_{k < \ell} \int_{\mathbb{R}^3} \Delta_k \partial_i u^3(x_h,x_3) \Delta_k \partial_3 u_{\ell,E^{-1}}(x_h,x_3) dx_h \right) dx_3.
\end{align*}
\]

$J_{E,1}^3$ can be estimated by duality then by using some product laws (lemma A.2.2), we obtain

\[
J_{E,1}^3 \lesssim \| \nabla_h u \|_{L^\infty(\mathcal{H}_h^\frac{1}{4})} \left\| \Delta_h \partial_3 u_{\ell,E^{-1}} \right\|_{L^1(\mathcal{H}_h^\frac{1}{2})} \lesssim \| \nabla_h u \|_{L^\infty(\mathcal{H}_h^\frac{1}{4})} \left\| \nabla_h u^3 \right\|_{L^2(\mathcal{H}_h^\frac{1}{2})} \| \partial_3 u \|_{L^2(\mathcal{H}_h^1)}
\]

Using then the inequality: $\| \nabla_h u \|_{L^\infty(\mathcal{H}_h^\frac{1}{4})} \lesssim \| \nabla_h u \|_{H^1(\mathbb{R}^3)}$ (see lemma A.2.4), we infer that

\[
J_{E,1}^3 \lesssim \| u^3 \|_{H^\frac{1}{2}(\mathbb{R}^3)} \| \nabla_h u \|_{H^1(\mathbb{R}^3)}^2 \tag{11}
\]

In order to estimate $J_{E,2}^3$ we split it into a sum of a good term $J_{E,2}^{G}$ and a bad one $J_{E,2}^{B}$ based on the dominated frequencies of $\partial_3 u_{\ell}$

\[
\begin{align*}
\partial_3 u_{\ell,E^{-1}} & = \partial_3 u_{\ell,E}^G + \partial_3 u_{\ell,E}^B \quad \text{with} \quad \partial_3 u_{\ell,E}^G \overset{\text{def}}{=} \sum_{q < k} \Delta_k \nabla_q \partial_3 u_{\ell,E^{-1}} \quad \text{and} \quad \partial_3 u_{\ell,E}^B \overset{\text{def}}{=} \sum_{k < q} \Delta_k \nabla_q \partial_3 u_{\ell,E^{-1}}
\end{align*}
\]

The good term can be easily estimated without using the fact that $u_{\ell,E^{-1}}$ contains only the high horizontal frequencies, but only providing that the horizontal frequencies control the vertical ones. We proceed as follows, by using the product lemma A.2.2 we find:

\[
J_{E,2}^{G} \lesssim \| \nabla_h u^3 \|_{H_h^\frac{1}{2}(\mathcal{L}_h^2)} \left\| \partial_3 u_{\ell,E}^G \right\|_{H_h^\frac{1}{2}(\mathcal{H}_h^\frac{1}{4})} \| \nabla_h u \|_{H_h^\frac{1}{2}(\mathcal{H}_h^\frac{1}{4})}^2
\]

\[
J_{E,2}^{B} \lesssim \| \nabla_h u^3 \|_{H_h^\frac{1}{2}(\mathcal{L}_h^2)} \left\| \partial_3 u_{\ell,E}^B \right\|_{H_h^\frac{1}{2}(\mathcal{H}_h^\frac{1}{4})} \| \nabla_h u \|_{H_h^\frac{1}{2}(\mathcal{H}_h^\frac{1}{4})}
\]
Lemma \[\text{A.2.6}\] in Appendix gives then
\[J_{E}^{\ell,2,G} \lesssim \| \nabla_{h} u \|^2_{L^\frac{3}{2}(L^\infty)} \| \nabla_{h} u \|^2_{L^\frac{3}{4}(L^\infty)}\]
which yields finally, by using lemma \[\text{A.2.3}\]
\[J_{E}^{\ell,2,G} \lesssim \| u \|^3_{L^\frac{3}{4}(L^\infty)} \| \nabla_{h} u \|^2_{L^1} \quad (12)\]
In order to estimate the bad term \[J_{E}^{\ell,2,B}\], we use the Bony’s decomposition with respect to vertical variables to infer that
\[J_{E}^{\ell,2,B} \lesssim \sum_{k \in \mathbb{Z}} \left\| \Delta_{k}^{h} \Delta_{q}^{v} \nabla_{h} u \right\|_{L^2(\mathbb{R}^3)} \left( I_{k,q}^{(1)} + I_{k,q}^{(2)} + I_{k,q}^{(3)} \right) \quad (13)\]
where
\[I_{k,q}^{(1)} \overset{\text{def}}{=} \left\| S_{k-1}^{h} S_{q-1}^{v} (\partial_{3} u \ell_{E}^{B,1}) \right\|_{L^\infty(L^\infty)} \left\| \Delta_{k}^{h} \Delta_{q}^{v} \nabla_{h} u \right\|_{L^2(\mathbb{R}^3)}\]
\[I_{k,q}^{(2)} \overset{\text{def}}{=} \left\| S_{k-1}^{h} \Delta_{q}^{v} (\partial_{3} u \ell_{E}^{B,1}) \right\|_{L^\infty(L^\infty)} \left\| \Delta_{k}^{h} S_{q-1}^{v} \nabla_{h} u \right\|_{L^2(\mathbb{R}^3)}\]
\[I_{k,q}^{(3)} \overset{\text{def}}{=} 2^\frac{k}{2} \sum_{j \geq q - N_0} \left\| S_{j-1}^{h} \Delta_{j}^{v} (\partial_{3} u \ell_{E}^{B,1}) \right\|_{L^2(L^\infty)} \left\| \Delta_{k}^{h} \Delta_{j}^{v} \nabla_{h} u \right\|_{L^2(\mathbb{R}^3)}\]
The estimates of these terms is based on lemma \[\text{A.2.7}\] proved in Appendix, by taking \(f_{\ell_{E}^{B,1}} = \partial_{3} u \ell_{E}^{B,1}\)
We use inequality \[\text{(30)}\] from lemma \[\text{A.2.7}\] to estimate \(I_{k,q}^{(1)}\), which gives
\[I_{k,q}^{(1)} \lesssim (\text{log}(E^{2q} + e))^{\frac{1}{2}} c_q 2^q \| \nabla_{h} \partial_{3} u \|_{L^2(\mathbb{R}^3)} 2^k \| \Delta_{k}^{h} \Delta_{q}^{v} u \|^3_{L^2(\mathbb{R}^3)} \]
\[\lesssim c_q 2^q 2^\frac{k}{2} 2^\frac{q}{2} \| \nabla_{h} \partial_{3} u \|_{L^2(\mathbb{R}^3)} \| u \|^3_{L^\frac{3}{4}(L^\infty)} \| u \|^2_{L^1} \]
finally we obtain
\[I_{k,q}^{(1)} \lesssim c_k 2^\frac{q}{2} 2^\frac{k}{2} \| \nabla_{h} \partial_{3} u \|_{L^2(\mathbb{R}^3)} \| u \|^3_{L^\frac{3}{4}(L^\infty)} \| u \|^2_{L^1} \quad (14)\]
In order to estimate \(I_{k,q}^{(2)}\) we use inequality \[\text{(29)}\], we infer that
\[I_{k,q}^{(2)} \lesssim (\text{log}(E^{2q} + e))^{\frac{1}{2}} c_q 2^q \| \nabla_{h} \partial_{3} u \|_{L^2(\mathbb{R}^3)} 2^k \| \Delta_{k}^{h} S_{q-1}^{v} u \|^3_{L^\infty(L^\infty)} \]
\[\lesssim c_q 2^k 2^\frac{q}{2} \| \nabla_{h} \partial_{3} u \|_{L^2(\mathbb{R}^3)} (\text{log}(E^{2q} + e))^{\frac{1}{2}} \Delta_{k}^{h} S_{q-1}^{v} \| u \|^3_{L^\infty(L^\infty)} \quad (15)\]
Next, we use the following estimate
\[2^{-\frac{q}{2}} (\text{log}(E^{2q} + e))^{\frac{1}{2}} 2^k \| \Delta_{k}^{h} S_{q-1}^{v} u \|^3_{L^\infty(L^\infty)} \lesssim \sum_{m \leq q} (2^{(m-q)} \text{log}(E^{2q} + e))^{\frac{1}{2}} 2^k \| \Delta_{k}^{h} \Delta_{m}^{v} u \|^3_{L^2(\mathbb{R}^3)}\]
together with the fact that
\[\text{log}(E^{2q} + e) \leq \text{log}(2^{q-m}(E^{2m} + e)), \quad \forall m \leq q\]
\[\leq \text{log}(E^{2m} + e) + (q - m)\]
\[\lesssim \text{log}(E^{2m} + e)(1 + (q - m))\]
this leads to
\[
2^{-\frac{j}{2}}(\log(E2^j + e))^{\frac{j}{2}}2^j \left\| \Delta_k^h S_q^u u^3 \right\|_{L^\infty(L_c^1)} \lesssim \sum_{m \leq q} (\kappa_{q-m} c_m) c_k \left\| u^3 \right\|_{H^\frac{k}{2} L_{logy,E}^3}
\]
where \( \kappa_j \equiv \sqrt{\frac{1}{j} + j} \in \ell^1(J) \). By using convolution inequality, we deduce that
\[
\mathcal{I}_{k,q}^{(2)} \lesssim c_k 2^\frac{k}{2} 2^j \left\| \nabla_h \partial_3 u \right\|_{L^2(\mathbb{R}^3)} \left\| u^3 \right\|_{H^\frac{k}{2} L_{logy,E}^3}
\]
(17)
Finally, in order to estimate \( \mathcal{I}_{k,q}^{(3)} \), we use again inequality (30) from lemma \( \text{A.2.7} \) below, we obtain
\[
\mathcal{I}_{k,q}^{(3)} \lesssim 2^\frac{k}{2} 2^j \left( \sum_{j \geq q - N_0} (\log(E2^j + e))^{\frac{j}{2}} c_j 2^j \left\| \Delta_k^h \Delta_q^u u^3 \right\|_{L^2(\mathbb{R}^3)} \right) \left\| \nabla_h \partial_3 u \right\|_{L^2(\mathbb{R}^3)}
\]
\[
\lesssim c_k 2^\frac{k}{2} 2^j \left\| \nabla_h \partial_3 u \right\|_{L^2(\mathbb{R}^3)} \left\| u^3 \right\|_{H^\frac{k}{2} L_{logy,E}^3}
\]
Together with (14) and (17) yield
\[
\sum_{i \in \{1,2,3\}} \mathcal{I}_{k,q}^{(i)} \lesssim c_k 2^\frac{k}{2} 2^j \left\| \nabla_h u \right\|_{H^1(\mathbb{R}^3)} \left\| u^3 \right\|_{H^\frac{k}{2} L_{logy,E}^3}
\]
Plugging this last one into (13) gives
\[
J_{E,B}^{(2,2)} \lesssim \left( \sum_{k, q \in \mathbb{Z}} c_k 2^\frac{k}{2} 2^j \left\| \Delta_k^h \Delta_q^u \nabla_h u \right\|_{L^2(\mathbb{R}^3)} \right) \left\| \nabla_h u \right\|_{H^1(\mathbb{R}^3)} \left\| u^3 \right\|_{H^\frac{k}{2} L_{logy,E}^3}
\]
\[
\lesssim \left\| \nabla_h u \right\|_{H^\frac{k}{2} (\bar{B}^{\frac{1}{2}}_{2,1})_v} \left\| \nabla_h u \right\|_{H^1(\mathbb{R}^3)} \left\| u^3 \right\|_{H^\frac{k}{2} L_{logy,E}^3}
\]
lemma \( \text{A.2.3} \) then gives
\[
J_{E,B}^{(2,2)} \lesssim \left\| \nabla_h u \right\|_{H^1(\mathbb{R}^3)} \left\| u^3 \right\|_{H^\frac{k}{2} L_{logy,E}^3}
\]
(18)
From (11), (12) and (18) we deduce
\[
J_{E}^2 \lesssim \left\| \nabla_h u \right\|_{H^1(\mathbb{R}^3)} \left\| u^3 \right\|_{H^\frac{k}{2} L_{logy,E}^3}
\]
\( J_E^2 \) can be estimated along the same lines as in [6], by using the product law \( (\bar{B}^{\frac{1}{2}}_{2,1})_h \times H^\frac{k}{2} \subset \dot{H}^\frac{k}{2} \), together with the embedding \( \dot{H}^1(\mathbb{R}^3) \hookrightarrow L^\infty(\dot{H}^\frac{k}{2}) \) (see lemma 5 in Appendix), we infer that
\[
J_{E}^2 \lesssim \left\| \nabla_h u^3 \right\|_{L^2(\dot{H}^{-\frac{k}{2}})h} \left\| \partial_3 u^3_{\Delta k} \right\|_{L^2(\dot{H}^{\frac{k}{2}})h} \left\| \Delta_k^h S_q^u u^3 \right\|_{L^\infty(L_c^1)}
\]
\[
\lesssim \left\| u^3 \right\|_{H^\frac{k}{2}} \left\| \partial_3 u^3_{\Delta k} \right\|_{L^2(\bar{B}^{\frac{1}{2}}_{2,1})_h} \left\| \nabla_h u \right\|_{L^\infty(\dot{H}^\frac{k}{2})h}
\]
\[
\lesssim \left\| u^3 \right\|_{H^\frac{k}{2}} \left\| \nabla_h u \right\|_{H^1} \left\| \partial_3 u \right\|_{L^2}
\]
\[
\lesssim \frac{1}{100} \left\| \nabla_h u \right\|_{H^1} + C \left\| u^3 \right\|_{H^\frac{k}{2}} \left\| \partial_3 u \right\|_{L^2_E}^2
\]
Lemma 1 is then proved. □
3.2 Proof of lemma 2

Let \( p \in [2, 4] \) and \( \alpha \in \left[0, \frac{2}{p} - \frac{1}{2}\right] \). We define \( q \) and \( \theta \) such that

\[
\frac{2}{q} \overset{\text{def}}{=} 1 - \frac{1}{p} \tag{19}
\]
\[
\theta \overset{\text{def}}{=} \frac{1}{2} + \frac{\alpha}{2} - \frac{1}{p} \tag{20}
\]

One may check that

\[
q \in ]2, \infty[ \quad \text{and} \quad \theta \in \left[0, \frac{2}{q}\right] \cap \left[0, \frac{1}{2}\right]
\]

which allow us to use the following embedding, due to lemmas \ref{A.2.3} and \ref{A.2.5}

\[
L^\infty_t (L^2(\mathbb{R}^3)) \cap L^2_t (\dot{H}^1(\mathbb{R}^3)) \hookrightarrow L^q_t (\dot{B}_{2,1}^\theta (\mathbb{R}^3)) \hookrightarrow L^q_t ((\dot{B}_{2,1}^{\frac{2}{p} - \theta})_h (\dot{B}_{2,1}^\theta)_v)
\]

Thus, by using lemma \ref{A.2.3} if \( g \in (\dot{B}_{2,1}^{\frac{2}{p} - \theta})_h (\dot{B}_{2,1}^\theta)_v \) then \( g \cdot g \in (\dot{B}_{2,1}^{\frac{2}{p} - \theta - 1})_h (\dot{B}_{2,1}^{2\theta - \frac{1}{2}})_v \).

By virtue of (19), (20) and embedding (21), we infer that

\[
\|g \cdot g\|_{(\dot{B}_{2,1}^{\frac{2}{p} - \theta - 1})_h (\dot{B}_{2,1}^{2\theta - \frac{1}{2}})_v} \lesssim \|g\|^2_{(\dot{B}_{2,1}^{\frac{2}{p} - \theta})_h (\dot{B}_{2,1}^\theta)_v}
\]

which gives by duality, embedding (21) and lemma \ref{A.2.5}

\[
|\langle fg \rangle\rangle_{L^2} \lesssim \|f\|_{(\dot{B}_{2,1}^{\frac{2}{p} - \theta - 1})_h (\dot{B}_{2,1}^{\frac{2}{p} - \theta})_v} \|g\|_{(\dot{B}_{2,1}^{\frac{2}{p} - \theta - 1})_h (\dot{B}_{2,1}^{2\theta - \frac{1}{2}})_v} \lesssim \|f\|_{(\dot{B}_{2,1}^{\frac{2}{p} - \theta})_h (\dot{B}_{2,1}^{\frac{2}{p} - \theta})_v} \|g\|_{(\dot{B}_{2,1}^{\frac{2}{p} - \theta})_h (\dot{B}_{2,1}^{2\theta - \frac{1}{2}})_v} \g_{(\dot{B}_{2,1}^{\frac{2}{p} - \theta})_h (\dot{B}_{2,1}^{\frac{2}{p} - \theta})_v}
\]

Finally we obtain

\[
|\langle fg \rangle\rangle_{L^2} \leq \frac{1}{10} \|g\|_{H^1}^2 + C \|f\|_{(\dot{B}_{2,1}^{\frac{2}{p} - \theta})_h (\dot{B}_{2,1}^{\frac{2}{p} - \theta})_v} \|g\|_{L^2}^2
\]

Lemma 2 is proved. \( \square \)

3.3 Proof of lemma 3

According to lemma \ref{A.2.5} in Appendix, in particular inequality (20) gives

\[
\|g(t, .)\|_{\dot{B}_{2,1}^{\frac{2}{p}}(\mathbb{R}^3)} \lesssim \|g(t, .)\|_{\dot{H}^1(\mathbb{R}^3)}^\frac{2}{p} \|g(t, .)\|_{L^2(\mathbb{R}^3)}^{1 - \frac{2}{p}}, \quad \forall m \in ]2, \infty[
\]

We use then the Bony’s decomposition to study the product \( g \cdot g \).

Let \((q, p) \in [1, \infty]^2\) satisfying

\[
q \in [3, \infty] \quad \text{and} \quad \frac{3}{q} + \frac{2}{p} \in ]1, 2[\]

Let \((m_1, m_2)\) be in \([2, \infty] \times ]2, \infty[\), given by

\[
\frac{2}{3m_1} \overset{\text{def}}{=} \frac{1}{q} \quad \text{and} \quad 2\left(1 - \frac{1}{m_2}\right) \overset{\text{def}}{=} \frac{3}{q} + \frac{2}{p} \in ]1, 2[ \iff m_2 \in ]2, \infty[\]

(23)
Let us defined the real number \( N_{m_1} \) associated to the embedding \( \dot{H}^{\frac{m_1}{2}}(\mathbb{R}^3) \) in \( L^{N_{m_1}}(\mathbb{R}^3) \)

\[
\frac{1}{N_{m_1}} \overset{\text{def}}{=} \frac{1}{2} - \frac{2}{3m_1} \in \left[ \frac{1}{6}, \frac{1}{2} \right]
\]

Let us also define \( r \) to be the conjugate of \( q \), that is

\[
\frac{1}{r} \overset{\text{def}}{=} 1 - \frac{1}{q} \in \left[ \frac{2}{3}, 1 \right]
\]

We write

\[
\Delta_j(g \cdot g) = 2\Delta_j T_j(g) + \Delta_j R(g, g)
\]

where \( T \) and \( R \) are the operators associated to the Bony’s decomposition, defined in the Appendix.

We turn now to estimate the two parts of \( \Delta_j(g \cdot g) \). We have

\[
\|\Delta_j T_j(g)\|_{L^r} \lesssim \|S_{j-1} g\|_{L^{N_{m_1}}} \|\Delta_j g\|_{L^2} \lesssim \|g\|_{L^{N_{m_1}}} d_j 2^{-j} \frac{q}{m_2} \|g\|_{\dot{B}^{\frac{m_2}{2}}_{2,1}}
\]

using then the embedding

\[
\|g\|_{L^{N_{m_1}}} \lesssim \|g\|_{\dot{H}^{\frac{m_1}{2}}}
\]

(24)

together with the interpolation inequality (22) gives

\[
\|\Delta_j T_j(g)\|_{\dot{B}^{\frac{m_2}{2}}_{r,1}} \lesssim 2^{-j} \frac{q}{m_2} d_j \|g\|_{\dot{H}^{\frac{m_1}{2}}} \|g\|_{L^2} \|g\|_{\dot{B}^{\frac{m_2}{2}}_{2,1}}
\]

(25)

For the reminder term, we proceed almost similary

\[
\|\Delta_j R(g, g)\|_{L^r} \lesssim \sum_{j \geq j-5} \|\tilde{\Delta}_j g\|_{L^{N_{m_1}}} \|\Delta_j g\|_{L^2} \lesssim 2^{-j} \frac{q}{m_2} \sum_{j' \geq j-5} (d_j 2^{-(j'-j)} \frac{q}{m_2}) \|g\|_{L^{N_{m_1}}} \|g\|_{\dot{B}^{\frac{m_2}{2}}_{2,1}}
\]

Where \( \tilde{\Delta}_j \overset{\text{def}}{=} \sum_{i \in \{-1,0,1\}} \Delta_j + i \). By convolution inequatlity, interpolation inequality (22) and the embedding one (24), we get

\[
\|\Delta_j R(g, g)\|_{L^r} \lesssim 2^{-j} \frac{q}{m_2} d_j \|g\|_{\dot{H}^{\frac{m_1}{2}}} \|g\|_{L^2} \|g\|_{\dot{B}^{\frac{m_2}{2}}_{2,1}}
\]

which gives, together with (25)

\[
\|\Delta_j (g \cdot g)\|_{L^r} \lesssim 2^{-j} \frac{q}{m_2} d_j \|g\|_{\dot{H}^{\frac{m_1}{2}}} \|g\|_{L^2} \|g\|_{\dot{B}^{\frac{m_2}{2}}_{2,1}}
\]

In the other hand, by duality, we get

\[
\|\left<f \right>_{L^2} \|_{L^2} \lesssim \sum_{j \in \mathbb{Z}} \|\Delta_j f\|_{L^2} \|\Delta_j (g \cdot g)\|_{L^r}
\]

\[
\lesssim \sum_{j \in \mathbb{Z}} (2^{-j} \frac{q}{m_2} \|\Delta_j f\|_{L^2} d_j) \|g\|_{\dot{H}^{\frac{m_1}{2}}} \|g\|_{L^2} \|g\|_{\dot{B}^{\frac{m_2}{2}}_{2,1}}
\]

\[
\lesssim \|f\|_{\dot{B}^{\frac{m_1}{2}}_{q,\infty}} \|g\|_{\dot{H}^{\frac{m_1}{2}}} \|g\|_{L^2} \|g\|_{\dot{B}^{\frac{m_2}{2}}_{2,1}}
\]

12
By virtue of (23) we have
\[ 1 - \left( \frac{1}{m_1} + \frac{1}{m_2} \right) = \frac{1}{p} \quad \text{and} \quad - \frac{2}{m_2} = \frac{3}{q} + \frac{2}{p} - 2 \]
This gives
\[ |(fg|g)_{L^2}| \leq \frac{1}{10} \|g\|_{L^2}^2 + C \|f\|_{B_{q,p}} \|g\|_{L^2}^2 \]
Lemma 3 is proved. \(\square\)

A Appendix

A.1 Functional framework

In this part we recall some notions/definitions used in the previous sections. Let us first recall some notions of the Littlewood-Paley theory, the anisotropic Besov spaces used in this paper and some of their properties. For more details one may see for instance [2]. Let \((\psi, \varphi)\) be a couple of smooth functions with value in \([0, 1]\) satisfying:
\[ \text{Supp } \psi \subset \{ \xi \in \mathbb{R} : |\xi| \leq \frac{4}{3} \}, \quad \text{Supp } \varphi \subset \{ \xi \in \mathbb{R} : \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \} \]
\[ \psi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q} \xi) = 1 \ \forall \xi \in \mathbb{R}, \quad \sum_{q \in \mathbb{Z}} \varphi(2^{-q} \xi) = 1 \ \forall \xi \in \mathbb{R}\setminus\{0\}. \]

Let \(a\) be a tempered distribution, \(\hat{a} = F(a)\) its Fourier transform and \(F^{-1}\) denotes the inverse of \(F\). We define the homogeneous dyadic blocks \(\Delta_q\) by setting
\[ \Delta_q^\nu a \overset{def}{=} F^{-1}(\varphi(2^{-q}|\xi_3|\hat{a})) \quad \forall q \in \mathbb{Z}, \quad \Delta_j^h a \overset{def}{=} F^{-1}(\varphi(2^{-j}|\xi_3|\hat{a})) \quad \forall j \in \mathbb{Z}. \]
\[ S_q^\nu \overset{def}{=} \sum_{q' < q} \Delta_{q'}^\nu, \forall q \in \mathbb{Z}, \quad S_j^h \overset{def}{=} \sum_{j' < j} \Delta_{j'}^h, \forall j \in \mathbb{Z}. \]

Moreover, in all the situations, i.e. for \(\Delta, S\) with the same index of direction (horizontal or vertical) it holds:
\[ \Delta_m \Delta_{m'} a = 0 \text{ if } |m - m'| \geq 2 \]
\[ \Delta_m (S_{m'-1}a \Delta_{m'}a) = 0 \text{ if } |m - m'| \geq 5 \]
\[ \Delta_m \sum_{i \in \{0, 1, -1\}} \sum_{m' \in \mathbb{Z}} (\Delta_{m'+i}a \Delta_{m'}a) = \Delta_m \sum_{i \in \{0, 1, -1\}} \sum_{m' \geq m-5} (\Delta_{m'+i}a \Delta_{m'}a), \]
We should recall the so-called Bony decomposition (see [2]):
\[ ab = T_a(b) + T_b(a) + R(a, b), \]
\[ T_a(b) \overset{def}{=} \sum_{q \in \mathbb{Z}} S_{q-1}a \Delta_q b, \quad R(a, b) \overset{def}{=} \sum_{i \in \{0, 1, -1\}} \sum_{q \in \mathbb{Z}} \Delta_{q+i}a \Delta_q b. \]
Here again all the situations may be considered however particular cases must be precised by using the adequate notations. For instance if we consider the version for the vertical variable, we have to add the exponent \(v\) in all the operators \(T_a, T_b, R, S_q\) and \(\Delta_q\).
Definition A.1.1. Let \( s, t \) be two real numbers and let \( p_1, p_2, q_1, q_2 \) be in \([1, +\infty]\), we define the space \( (\dot{B}^{t}_{p_1,q_1})_h(\dot{B}^{s}_{p_2,q_2})_v \) as the space of tempered distributions \( u \) such that

\[
\left\| u \right\| (\dot{B}^{t}_{p_1,q_1})_h(\dot{B}^{s}_{p_2,q_2})_v \overset{\text{def}}{=} \left\| 2^{kt}2^{js} \right\| \Delta_k^h \Delta_j^v u \left\|_{L^p_h L^q_v} \right\| < \infty
\]

In the situation where \( q_1 = q_2 = q \) and \( p_1 = p_2 = p \), we use the notation \( \dot{B}^{t,s}_{p,q} = (\dot{B}^{t}_{p,q})_h(\dot{B}^{s}_{p,q})_v \). If \( p = q = 2 \) then this last space is equivalent to \( \dot{H}^{t,s} \). More precisely, we have:

\[
\left\| a \right\|_{\dot{B}^{t,s}_{2,2}} \overset{\text{def}}{=} \int_{\mathbb{R}^3} \left| \xi_h \right|^{2t} \left| \xi_v \right|^{2s} \left| \hat{a}(\xi) \right|^2 d\xi
\]

A.2 Technical lemmas

In this part we present seven lemmas used in the previous section, we will prove the three last ones and give references for the four first ones.

We Start by recalling a Bernstein type lemma from [7].

Lemma A.2.1. Let \( B_h \) (resp. \( B_v \)) be a ball of \( \mathbb{R}_h^2 \) (resp. \( \mathbb{R}_v^2 \)) and \( C_h \) (resp. \( C_v \)) a ring of \( \mathbb{R}_h^2 \) (resp. \( \mathbb{R}_v^2 \)). Let also \( a \) be a tempered distribution and \( \hat{a} \) its Fourier transform. Then for \( 1 \leq p_2 \leq p_1 \leq 1 \) and \( 1 \leq q_2 \leq q_1 \leq \infty \) we have:

\[
\text{Supp } \hat{a} \subset 2^k B_h \implies \left\| \partial_{x_h}^\alpha a \right\|_{L^p_h(L^q_v)} \leq 2^{k(\lvert \alpha \rvert + 2\left(\frac{1}{p_2} - \frac{1}{p_1}\right))} \left\| a \right\|_{L^p_v(L^q_h)}
\]

\[
\text{Supp } \hat{a} \subset 2^k B_v \implies \left\| \partial_{x_v}^\alpha a \right\|_{L^p_v(L^q_h)} \leq 2^{k(\lvert \alpha \rvert + 2\left(\frac{1}{q_2} - \frac{1}{q_1}\right))} \left\| a \right\|_{L^p_h(L^q_v)}
\]

\[
\text{Supp } \hat{a} \subset 2^k C_h \implies \left\| a \right\|_{L^p_h(L^q_v)} \leq 2^{-kn} \sup_{\lvert \alpha \rvert = N} \left\| \partial_{x_h}^\alpha a \right\|_{L^p_h(L^q_v)}
\]

\[
\text{Supp } \hat{a} \subset 2^k C_v \implies \left\| a \right\|_{L^p_v(L^q_h)} \leq 2^{-kn} \left\| \partial_{x_v}^\alpha a \right\|_{L^p_v(L^q_h)}
\]

Let us also recall an anisotropic version of the usual product laws in Besov spaces (see lemma 4.5 from [7]).

Lemma A.2.2. Let \( q \geq 1 \), \( p_1 \geq p_2 \geq 1 \) with \( \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \), and \( s_1 < \frac{2}{p_1}, \ s_2 < \frac{2}{p_2} \) (resp. \( s_1 \leq \frac{2}{p_1}, s_2 \leq \frac{2}{p_2} \) if \( q = 1 \)) with \( s_1 + s_2 > 0 \). Let \( \sigma_1 < \frac{1}{p_1}, \ \sigma_2 < \frac{1}{p_2} \) (resp. \( \sigma_1 \leq \frac{1}{p_1}, \ \sigma_2 \leq \frac{1}{p_2} \) if \( q = 1 \)) with \( \sigma_1 + \sigma_2 > 0 \). Then for \( a \) in \( B_{p_1,q}^{s_1,\sigma_1} \) and \( b \) in \( B_{p_2,q}^{s_2,\sigma_2} \), the product \( ab \) belongs to \( B_{p_1,q}^{s_1+s_2-\frac{1}{p_2},\sigma_1+\sigma_2-\frac{1}{p_2}} \) and we have

\[
\left\| ab \right\|_{B_{p_1,q}^{s_1+s_2-\frac{1}{p_2},\sigma_1+\sigma_2-\frac{1}{p_2}}} \leq \left\| a \right\|_{B_{p_1,q}^{s_1,\sigma_1}} \left\| b \right\|_{B_{p_2,q}^{s_2,\sigma_2}}
\]

A very useful lemma in the anisotropic context (lemma 4.3 from [7]), is the following

Lemma A.2.3. For any \( s \) positive, for all \( (p,q) \in [1, \infty] \) and any \( \theta \in [0,s] \), we have

\[
\left\| f \right\|_{(\dot{B}_p^{s-\theta})_h(\dot{B}_h^q)_v} \leq \left\| f \right\|_{\dot{B}^{s}_{p,q}}
\]

Finally, we recall lemma A.2 from [6].

Lemma A.2.4. For any function \( a \) in the space \( \dot{H}_h^{1+s}(\mathbb{R}^3) \) with \( \frac{1}{2} \leq s < 1 \), there holds

\[
\left\| a \right\|_{L^\infty(\dot{H}_h^s)} \leq \sqrt{2} \left\| a \right\|_{\dot{H}_h^{1+s}(\mathbb{R}^3)}
\]
Next, we will prove an interpolation version in space-time spaces

**Lemma A.2.5.** For all $p \in [2, \infty[$, there exists a constant $c_p > 0$, such that for all $u$ in $L^p_T(L^2(\mathbb{R}^3)) \cap L^{p}_T(H^1(\mathbb{R}^3))$ we have

$$
\|u\|_{L^p_T(B^{\frac{3}{2}}_{2,1}(\mathbb{R}^3))} \leq c_p \|u\|^\frac{2}{p}_{L^2_T(H^1(\mathbb{R}^3))} \|u\|^1_{L^{p}_T(L^2(\mathbb{R}^3))}
$$

**Proof** The proof is classical, we proceed as the following:

Let $N(t) > 0$ to be fixed later, we use lemma A.2.1 and Cauchy-Swartz inequality, to write

$$
\sum_{j \in \mathbb{Z}} 2^\frac{3}{2} \|\Delta_j u(t,.)\|_{L^2} \leq \sum_{j \leq N(t)} 2^\frac{3}{2} \|\Delta_j u(t,.)\|_{L^2} + \sum_{j > N(t)} 2^\frac{3}{2} \|\Delta_j \nabla u(t,.)\|_{L^2}
$$

$$
\leq 2^{\frac{3}{2}N(t)} \|u(t,.)\|_{L^2} + 2^{-\frac{3}{2}N(t)} \|\nabla u(t,.)\|_{L^2}
$$

The choice of $N(t)$ such that

$$
2^{N(t)} \overset{\text{def}}{=} \left(\frac{1 - \frac{9}{2}}{p}\right) \frac{\|\nabla u(t,.)\|_{L^2}}{\|u(t,.)\|_{L^2}}
$$

gives

$$
\|u(t,.)\|_{B^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)} \leq c_p \|u(t,.)\|^\frac{2}{p}_{L^2_T(H^1(\mathbb{R}^3))} \|u(t,.)\|^1_{L^{p}_T(L^2(\mathbb{R}^3))}
$$

(26)

The lemma follows by taking the $L^p$ norm in time. □

The following lemma can be used when the horizontal frequencies controle the vertical ones

**Lemma A.2.6.** Let $s, t$ be two real numbers, let $f$ be a regular function, we define $f^G$ as

$$
f^G \overset{\text{def}}{=} \sum_{q \leq k} \Delta^h_q \Delta^v_q f.
$$

Then we have:

$$
\|\partial_3 f^G\|_{H^{s,t}} \lesssim \|\nabla h f\|_{H^{s,t}}
$$

**Proof** Let us use Plancherel-Parseval identity to write:

$$
\|\partial_3 f^G\|^2_{H^{s,t}} \approx \int_{\mathbb{R}^3} |\xi_h|^{2s} |\xi_v|^{2t} \left| \sum_{q \leq k} |\xi_v| \varphi_h(\xi) \varphi_v^q(\xi) \right|^2 |\hat{f}(\xi)|^2 d\xi
$$

(27)

where: $\varphi_h \overset{\text{def}}{=} \varphi(2^{-k}|\xi_h|)$, $\varphi_v^q \overset{\text{def}}{=} \varphi(2^{-q}|\xi_v|)$, and $\varphi$ is the function defined at the begining of the Appendix part. Thus, using the support properties of $\varphi_h$, $\varphi_v^q$, and the condition $q < k$, we infer that, for all $\xi = (\xi_h, \xi_v) \in \text{Supp}(\varphi_h) \times \text{supp}(\varphi_v^q)$

$$
|\xi_v| \lesssim 2^q \lesssim 2^k \lesssim |\xi_h|
$$

(28)

plugging (28) into (27) concludes the proof of the lemma. □

The last lemma that we will prove is ulpse to estimate some parts of the anisotropic Bony’s decomposition for functions having a dominated vertical frequencies compared to the horizontal ones, and which are supported away from zero horizontaly in Fourier side.
Lemma A.2.7. Let $f$ be regular function, and $E > 0$. We define $f^B_{\ast,E^{-1}}$ as

$$f^B_{\ast,E^{-1}} \overset{\text{def}}{=} \sum_{k<q} \Delta_k^h \Delta_q^v f_{\ast,E^{-1}}$$

where

$$f_{\ast,E^{-1}} \overset{\text{def}}{=} F^{-1}(1_{B^c_h(0,E^{-1})} \hat{f})$$

Then we have the following estimates

$$\left\| \Delta_q^v S_{j-1}^h (f^B_{\ast,E^{-1}}) \right\|_{L^2_h(L^\infty)} \lesssim (\log(E2^q + e))^{\frac{3}{2}} c_q \left\| \nabla_h f \right\|_{L^2(\mathbb{R}^3)} \quad (29)$$

$$\left\| S_{q-1}^h S_{j-1}^h (f^B_{\ast,E^{-1}}) \right\|_{L^\infty_h(L^\infty)} \lesssim (\log(E2^q + e))^{\frac{3}{2}} c_q 2^\frac{3}{2} \left\| \nabla_h f \right\|_{L^2(\mathbb{R}^3)} \quad (30)$$

Proof According to the support properties we have

$$\Delta_q^v S_{j-1}^h (f^B_{\ast,E^{-1}}) = \left( \Delta_q^v S_{j-1}^h \sum_{i\in\{-1,0,1\}} S_{q-1+i}^h \Delta_q^v \right) f_{\ast,E^{-1}}$$

therefore, Bernstein’s inequality, we can write

$$\left\| \Delta_q^v S_{j-1}^h (f^B_{\ast,E^{-1}}) \right\|_{L^2_h(L^\infty)} \lesssim \sum_{E^{-1} < 2^{k-q} < 2^q} 2^k \left\| \Delta_k^h \Delta_q^v f \right\|_{L^2(\mathbb{R}^3)}$$

$$\lesssim \left( \sum_{E^{-1} < 2^{k-q} < 2^q} c_k \right) c_q \left\| \nabla_h f \right\|_{L^2(\mathbb{R}^3)}$$

$$\lesssim (\log(E2^q + e))^{\frac{3}{2}} c_q \left\| \nabla_h f \right\|_{L^2(\mathbb{R}^3)}$$

Thus the first inequality is proved. For the second, one we first write

$$\left\| S_{q-1}^h S_{j-1}^h (f^B_{\ast,E^{-1}}) \right\|_{L^\infty_h(L^\infty)} \lesssim \sum_{m\leq q} 2^\frac{3}{2} \left\| \Delta_m S_{j-1}^h (f^B_{\ast,E^{-1}}) \right\|_{L^2_h(L^\infty)}$$

inequality (29) gives then

$$2^{-\frac{q}{2}} \left\| S_{q-1}^h S_{j-1}^h (f^B_{\ast,E^{-1}}) \right\|_{L^\infty_h(L^\infty)} \lesssim \left( \sum_{m\leq q} 2^\frac{1}{2}(m-q) \left( \log(E2^m + e) \right)^{\frac{3}{2}} c_m \right) \left\| \nabla_h f \right\|_{L^2(\mathbb{R}^3)}$$

$$\lesssim (\log(E2^q + e))^{\frac{3}{2}} c_q \left\| \nabla_h f \right\|_{L^2(\mathbb{R}^3)}$$

Inequality (30) follows. \hfill \Box

References

[1] D. Adhikari, C. Cao, J. Wu: Global regularity results for the 2D Boussinesq equations with vertical dissipation, Journal of Differential Equations 251, 1637-1655, (2011).

[2] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Springer-Verlag, Berlin-Heidelberg-Newyork, 2011.

[3] C. Cao and E. S. Titi, Regularity criteria for the three-dimensional Navier-Stokes equations, Indiana University Mathematics Journal, 57, 2008, pages 2643-2661.
C. Cao and E. S. Titi, *Global regularity criterion for the 3D Navier-Stokes equations involving one entry of the velocity gradient tensor*, Archiv for Rational Mechanics and Analysis, 202, 2011, pages 919-932.

J.-Y. Chemin, B. Desardins, I. Gallagher and E. Grenier: *Fluid with anisotropic viscosity*, M2AN Math. Model. Numer. A. al., 34, pp. 315-335, 2000.

J.-Y. Chemin, I. Gallagher, P. Zhang: *Some remarks about the possible blow-up for the Navier-Stokes equations*, [https://arxiv.org/abs/1807.09939](https://arxiv.org/abs/1807.09939)

J.-Y. Chemin, P. Zhang: *On the critical one component regularity for 3-D Navier-Stokes system*, Z. Arch Rational Mech Anal (2017) 224: 871.

J.-Y. Chemin and P. Zhang, *On the global wellposedness to the 3-D incompressible anisotropic Navier-Stokes equations*, Communications in Mathematical Physics, 272, 2007, pages 529-566.

J.-Y. Chemin and P. Zhang: *On the critical one component regularity for 3-D Navier-Stokes system: General case*, Archiv for Rational Mechanics and Analysis, 224, 2017, pages 871-905.

L. Escauriaza, G. Seregin and V. Sverák, *$L^3;1$-solutions of Navier-Stokes equations and backward uniqueness*, (Russian) Uspekhi Mat. Nauk, 58, 2003, no. 2(350), pages 3-44; translation in Russian Math. Surveys, 58 (2), 2003, pages 211-250.

I. Gallagher, G. Koch and F. Planchon, *A profile decomposition approach to the $L^\infty_t(L^3_x)$ Navier-Stokes regularity criterion*, Mathematische Annalen, 355, 2013, pages 1527-559.

H. Houamed, P. Dreyfuss *Uniqueness result for the 3-D Navier-Stokes-Boussinesq equations with horizontal dissipation*, [https://arxiv.org/abs/1904.00437](https://arxiv.org/abs/1904.00437)

C. He, *Regularity for solutions to the Navier-Stokes equations with one velocity component regular*, Electronic Journal of Differential Equations, 29, 2002, pages 1-13.

D. Iftimie: *The 3D Navier-Stokes equations seen as a perturbation of the 2D Navier-Stokes equations*, Bulletin de la S. M. F., tome 127, n°4 (1999), p. 473-517.

D. Iftimie: *The resolution of the Navier-Stokes equations in anisotropic spaces*, Revista Mathematica Iberoamericana, vol. 15, n°1 (1999)

D. Iftimie: *A uniqueness result for the Navier-Stokes equations with vanishing vertical viscosity*, SIAM J. Math. Analysis, Vol. 33, No. 6 : pp. 1483-1493, 2002.

I. Kukavica and M. Ziane, *One component regularity for the Navier-Stokes equations*, Nonlinearity, 19 (2), 2006, pages 453-469.

J. Leray, *Essai sur le mouvement d’un liquide visqueux emplissant l’espace*, Acta Mathematica, 63, 1933, pages 193-248.

J. Neustupa and P. Penel, *Regularity of a suitable weak solution to the Navier-Stokes equations as a consequence of regularity of one velocity component*, Applied Nonlinear Analysis, 391-402, Kluwer/Plenum, New York, 1999.

J. Neustupa, A. Novotny and P. Penel, *An interior regularity of a weak solution to the Navier-Stokes equations in dependence on one component of velocity*, Topics in mathematical fluid mechanics, 163-183, Quad. Mat., 10, Dept. Math., Seconda Univ. Napoli, Caserta, 2002.

M. Paicu, *Equation anisotrope de Navier-Stokes dans des espaces critiques*, Revista Mathematica Iberoamericana 21 (2005), n°1, 179-235.
[22] P. Penel and M. Pokorny, *Some new regularity criteria for the Navier-Stokes equations containing gradient of the velocity*, Applications of Mathematics, 49, 2004, pages 483-493.

[23] M. Pokorny, *On the result of He concerning the smoothness of solutions to the Navier-Stokes equations*, Electronic Journal Differential Equations, 11, 2003, pages 1-8.

[24] G. Seregin, *A certain necessary condition of potential blow up for Navier-Stokes equations*, Communications in Mathematical Physics, 312, 2012, pages 833-845.

[25] Z. Zhang, Z.-A. Yao, M. Lu, L. Ni, *Some Serrin-type regularity criteria for weak solutions to the Navier-Stokes equations*, Math. Phys. 52 (2011), 053103.

[26] Y. Zhou and M. Pokorný, *On the regularity of the solutions of the Navier-Stokes equations via one velocity component*, Nonlinearity, 23, 2010, pages 1097-1107.

[27] Zujin Zhang, Ganzhou *Serrin-type regularity criterion for the Navier-Stokes equations involving one velocity and one vorticity component*. Czechoslovak Mathematical Journal, 68 (143) (2018), 219225.