Finding large and small dense subgraphs

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February 1, 2008

Abstract

We consider two optimization problems related to finding dense subgraphs, which are induced subgraphs with high average degree. The densest at-least-k-subgraph problem (DalkS) is to find an induced subgraph of highest average degree among all subgraphs with at least \( k \) vertices, and the densest at-most-k-subgraph problem (DamkS) is defined similarly. These problems are related to the well-known densest \( k \)-subgraph problem (DkS), which is to find the densest subgraph on exactly \( k \) vertices. Our main result is that DalkS can be approximated efficiently, while DamkS is nearly as hard to approximate as the densest \( k \)-subgraph problem. We give two algorithms for DalkS, a 3-approximation algorithm that runs in time \( O(m + n \log n) \), and a 2-approximation algorithm that runs in polynomial time. In contrast, we show that if there exists a polynomial time approximation algorithm for DamkS with ratio \( \gamma \), then there is a polynomial time approximation algorithm for DkS with ratio \( 4(\gamma^2 + \gamma) \).

1 Introduction

The density of an induced subgraph is the total weight of its edges divided by the size of its vertex set, or half its average degree. The problem of finding the densest subgraph of a given graph, and various related problems, have been studied extensively. In the past decade, identifying subgraphs with high density has become an important task in the analysis of large networks [14, 10].

There are a variety of efficient algorithms for finding the densest subgraph of a given graph. The densest subgraph can be identified in polynomial
time by solving a maximum flow problem [11] [9]. Charikar [5] gave a greedy algorithm that produces a 2-approximation of the densest subgraph in linear time. Kannan and Vinay [12] gave a spectral approximation algorithm for a related notion of density. Both of these approximation algorithms are fast enough to run on extremely large graphs.

In contrast, no practical algorithms are known for finding the densest subgraph on exactly $k$ vertices. If $k$ is specified as part of the input, and is allowed to vary with the graph size $n$, the best polynomial time algorithm known has approximation ratio $n^{\delta}$, where $\delta$ is slightly less than 1/3. This algorithm is due to Feige, Peleg, and Korsarz [7]. The densest $k$-subgraph problem is known to be $NP$-complete, but there is a large gap between this approximation ratio and the strongest known hardness result.

In many of the graphs we would like to analyze (for example, graphs arising from sponsored search auctions, or from links between blogs), the densest subgraph is extremely small relative to the size of the graph. When this is the case, we would like to find a subgraph that is both large and dense, without solving the seemingly intractable densest $k$-subgraph problem. To address this concern, we introduce the densest at-least-$k$-subgraph problem, which is to find the densest subgraph on at least $k$ vertices.

In this paper, we show that the densest at-least-$k$-subgraph problem can be solved nearly as efficiently as the densest subgraph problem. In fact, we show it can be solved by a careful application of the same techniques. We give a greedy 3-approximation algorithm for DalkS that runs in time $O(m + n \log n)$ in a weighted graph, and time $O(m)$ in an unweighted graph. This algorithm is an extension of Charikar’s algorithm for densest subgraph problem. We also give a 2-approximation algorithm for DalkS that runs in polynomial time, and can be computed by solving a single parametric flow problem. This is an extension of the algorithm of Gallo, Grigoriadis, and Tarjan [9] for the densest subgraph problem.

We also show that finding a dense subgraph with at most $k$ vertices is nearly as hard as finding the densest subgraph with exactly $k$ vertices. In particular, we prove that a polynomial time $\gamma$-approximation algorithm for the densest at-most-$k$-subgraph problem would imply a polynomial time $4(\gamma^2 + \gamma)$-approximation algorithm for the densest $k$-subgraph problem. More generally, if there exists a polynomial time algorithm that approximates DmS in a weak sense, returning a set of at most $\beta k$ vertices with density at least $1/\gamma$ times the density of the densest subgraph on at most $k$
vertices, then there is a polynomial time approximation algorithm for DkS with ratio $4(\gamma^2 + \gamma/\beta)$.

Our algorithms for DalkS can find subgraphs with nearly optimal density in extremely large graphs, while providing considerable control over the sizes of those subgraphs. Our reduction of DkS to DamkS gives additional insight into when DkS is hard, and suggests a possible approach for improving the approximation ratio for DkS.

The paper is organized as follows. We first consider the DalkS problem, presenting the greedy 3-approximation in Section 3 and the polynomial time 2-approximation in Section 4. We consider the DamkS problem in Section 5. In Section 6, we discuss the possibility of finding a good approximation algorithm for DamkS.

### 1.1 Related work

We will briefly survey a few results on the complexity of the densest $k$-subgraph problem. The best approximation algorithm known for the general problem (when $k$ is specified as part of the input) is the algorithm of Feige, Peleg, and Kortsarz [7], which has ratio $O(n^\delta)$ for some $\delta < 1/3$. For any particular value of $k$, the greedy algorithm of Asahiro et al. [4] gives the ratio $O(n/k)$. Algorithms based on linear programming and semidefinite programming have produced approximation ratios better than $O(n/k)$ for certain values of $k$, but have not improved the approximation ratio of $n^\delta$ for the general case [8, 6].

Feige and Seltser [8] showed the densest $k$-subgraph problem is $NP$-complete when restricted to bipartite graphs of maximum degree 3, by a reduction from max-clique. This reduction does not produce a hardness of approximation result for DkS. In fact, they showed that if a graph contains a $k$-clique, a subgraph with $k$ vertices and $(1 - \epsilon)\binom{k}{2}$ edges can be found in subexponential time. Khot [13] proved there can be no PTAS for the densest $k$-subgraph problem, under a standard complexity assumption.

Arora, Karger, and Karpinski [2] gave a PTAS for the special case $k = \Omega(n)$ and $m = \Omega(n^2)$. Asahiro, Hassin, and Iwama [3] showed that the problem is still $NP$-complete in very sparse graphs.
2 Definitions

Let \( G = (V, E) \) be an undirected graph with a weight function \( w : E \rightarrow \mathbb{R}_+ \) which assigns a positive weight to each edge. The weighted degree \( w(v, G) \) is the sum of the weights of the edges incident with \( v \). The total weight \( W(G) \) is the sum of the weights of the edges in \( G \).

**Definition 1.** For any induced subgraph \( H \) of \( G \), we define the density of \( H \) to be

\[
d(H) = \frac{W(H)}{|H|}.
\]

**Definition 2.** For an undirected graph \( G \), we define the following quantities.

\[
dal(G, k) := \text{the maximum density of an induced subgraph on at least } k \text{ vertices.}
\]

\[
dam(G, k) := \text{the maximum density of an induced subgraph on at most } k \text{ vertices.}
\]

\[
dex(G, k) := \text{the maximum density of an induced subgraph on exactly } k \text{ vertices.}
\]

\[
dmax(G) := \text{the maximum density of any induced subgraph.}
\]

The densest at-least-\( k \)-subgraph problem (DalkS) is to find an induced subgraph on at least \( k \) vertices achieving density \( dal(G, k) \). Similarly, the densest at-most-\( k \)-subgraph problem (DamkS) is to find an induced subgraph on at most \( k \) vertices achieving density \( dam(G, k) \). The densest \( k \)-subgraph problem (DkS) is to find an induced subgraph on exactly \( k \) vertices achieving \( dex(G, k) \), and the densest subgraph problem is to find an induced subgraph of any size achieving \( dmax(G) \).

We now define formally what it means to be an approximation algorithm for DalkS. Approximation algorithms for Damks, DkS, and the densest subgraph problem, are defined similarly.

**Definition 3.** An algorithm \( A(G, k) \) is a \( \gamma \)-approximation algorithm for the densest at-least-\( k \)-subgraph problem if for any graph \( G \) and integer \( k \), it returns an induced subgraph \( H \) on at least \( k \) vertices of \( G \) with density \( d(H) \geq \frac{dal(G, k)}{\gamma} \).

3 The densest at-least-\( k \)-subgraph problem

In this section, we give 3-approximation algorithm for the densest at-least-\( k \)-subgraph problem that runs in time \( O(m + n \log n) \) in a weighted graph, and
time $O(m)$ in an unweighted graph. The algorithm is a simple extension of Charikar’s greedy algorithm for the densest subgraph problem. To analyze the algorithm, we relate the density of a graph to the size of its $w$-cores, which are subgraphs with minimum weighted degree at least $w$.

ChALK($G, k$) :
Input: a graph $G$ with $n$ vertices, and an integer $k$.
Output: an induced subgraph of $G$ with at least $k$ vertices.

1. Let $H_n = G$ and repeat the following step for $i = n, \ldots, 1$:
   (a) Let $r_i$ be the minimum weighted degree of any vertex in $H_i$.
   (b) Let $v_i$ be a vertex where $w(v_i, H_i) = r_i$.
   (c) Remove $v_i$ from $H_i$ to form the induced subgraph $H_{i-1}$.

2. Compute the density of $d(H_i)$ for each $i \in [1, n]$.

3. Output the induced subgraph $H_i$ maximizing $\max_{i \geq k} d(H_i)$.

**Theorem 1.** ChALK($G, k$) is a 3-approximation algorithm for the densest at-least-$k$-subgraph problem.

We will prove Theorem 1 in the following subsection. The implementation of step 1 described by Charikar (see [5]) gives us the following bound on the running time of ChALK.

**Theorem 2** (Charikar). The running time of ChALK($G, k$) is $O(m)$ in an unweighted graph, and $O(m + n \log n)$ in a weighted graph.

### 3.1 Analysis of ChALK

The ChALK algorithm is easy to understand if we consider the relationship between induced subgraphs of $G$ with high average degree (dense subgraphs) and induced subgraphs of $G$ with high minimum degree ($w$-cores).

**Definition 4.** Given a graph $G$ and a weight $w \in \mathbb{R}$, the $w$-core $C_w(G)$ is the unique largest induced subgraph of $G$ with minimum weighted degree at least $w$.

Here is an outline of how we will proceed. We first prove that the ChALK algorithm computes all the $w$-cores of $G$ (Lemma 1). We then prove that
for any induced subgraph $H$ of $G$ with density $d$, the $(2d/3)$-core of $G$ has total weight at least $W(H)/3$ (Lemma 2). We will prove Theorem 1 using these two lemmas.

**Lemma 1.** Let $\{H_1, \ldots, H_n\}, \{v_1, \ldots, v_n\}$, and $\{r_1, \ldots, r_n\}$ be the induced subgraphs, vertices, and weighted degrees determined by ChALK on the input graph $G$. For any $w \in \mathbb{R}$, if $I(w)$ is the largest index such that $r(v_{I(w)}) \geq w$, then $H_{I(w)} = C_w(G)$.

*Proof.* Fix a value of $w$. It easy to prove by induction that none of the vertices $v_n \ldots v_{I(w)+1}$ that were removed before $v_{I(w)}$ is contained in any induced subgraph with minimum degree at least $w$. That implies $C_w(G) \subseteq H_{I(w)}$. On the other hand, the minimum degree of $H_{I(w)}$ is at least $w$, so $H_{I(w)} \subseteq C_w(G)$. Therefore, $H_{I(w)} = C_w(G)$. $\square$

**Lemma 2.** For any graph $G$ with total weight $W$ and density $d = W/|G|$, the $d$-core of $G$ is nonempty. Furthermore, for any $\alpha \in [0, 1]$, the total weight of the $(\alpha d)$-core of $G$ is strictly greater than $(1 - \alpha)W$.

*Proof.* Let $\{H_1, \ldots, H_n\}$ be the induced subgraphs determined by ChALK on the input graph $G$. Fix a value of $w$, let $I(w)$ be the largest index such that $r(v_{I(w)}) \geq w$, and recall that $H_{I(w)} = C_w(G)$ by Lemma 1. Since each edge in $G$ is removed exactly once during the course of the algorithm,

$$W = \sum_{i=1}^{|G|} r(i)$$
$$= \sum_{i=1}^{I(w)} r(i) + \sum_{i=I(w)+1}^{|G|} r(i)$$
$$< W(H_{I(w)}) + w \cdot (|G| - I(w))$$
$$\leq W(C_w(G)) + w|G|.$$

Therefore,

$$W(C_w(G)) > W - w|G|.$$

Taking $w = d = W/|G|$ in the equation above, we learn that $W(C_d(G)) > 0$. Taking $w = \alpha d = \alpha W/|G|$, we learn that $W(C_{\alpha d}(G)) > (1 - \alpha)W$. $\square$
Proof of Theorem 1. Let \( \{H_1, \ldots, H_n\} \) be the induced subgraphs determined by the ChALK algorithm on the input graph \( G \). It suffices to show that for any \( k \), there is an integer \( I \in [k, n] \) satisfying \( d(H_I) \geq \text{dal}(G,k)/3 \).

Let \( H_\ast \) be an induced subgraph of \( G \) with at least \( k \) vertices and with density \( d_\ast = W(H_\ast)/|H_\ast| = \text{dal}(G,k) \). We may apply Lemma 2 to \( H_\ast \) with \( \alpha = 2/3 \) to show that \( C(2d_\ast/3)(H_\ast) \) has total weight at least \( W(H_\ast)/3 \). This implies that \( C(2d_\ast/3)(G) \) has total weight at least \( W(H_\ast)/3 \).

The core \( C(2d_\ast/3)(G) \) has density at least \( d_\ast/3 \), because its minimum degree is at least \( 2d_\ast/3 \). Lemma 1 shows that \( C(2d_\ast/3)(G) = H_I \), for \( I = |C(2d_\ast/3)(G)| \). If \( I \geq k \), then \( H_I \) satisfies the requirements of the theorem. If \( I < k \), then \( C(2d_\ast/3)(G) = H_I \) is contained in \( H_k \), and the following calculation shows that \( H_k \) satisfies the requirements of the theorem.

\[
\frac{d(H_k)}{k} = \frac{W(H_k)}{k} \geq \frac{W(C(2d_\ast/3)(G))}{k} \geq \frac{W(H_\ast)/3}{k} = d_\ast/3.
\]

\( \blacksquare \)

Remark 1. Charikar proved that ChALK(\( G, 1 \)) is a 2-approximation algorithm for the densest subgraph problem. This can be derived from the fact that if \( w = \text{dmax}(G) \), the \( w \)-core of \( G \) is nonempty.

4 A 2-approximation algorithm for the densest at-least-\( k \)-subgraph problem

In this section, we will give a polynomial time 2-approximation algorithm for the densest at-least-\( k \) subgraph problem. The algorithm is based on the parametric flow algorithm of Gallo, Grigoriadis, and Tarjan [9]. It is well-known that the densest subgraph problem can be solved using similar techniques; Goldberg [11] showed that the densest subgraph can be found in polynomial time by solving a sequence of maximum flow problems, and Gallo, Grigoriadis, and Tarjan described how to find the densest subgraph using their parametric flow algorithm.

It is natural to ask whether there is a polynomial time algorithm for the densest at-least-\( k \)-subgraph problem. We do not know of such an algorithm, nor have we proved that DalkS is \( \mathcal{NP} \)-complete.

Theorem 3. There is a polynomial time 2-approximation algorithm for the densest at-least-\( k \)-subgraph problem.
Proof. The parametric flow algorithm of Gallo, Grigoriadis, and Tarjan can compute in polynomial time a collection \( \mathcal{H} \) of nested induced subgraphs of \( G \) such that for any value of \( \alpha \), the following expression is maximized by one of the subgraphs in \( \mathcal{H} \).

\[
\max_{H \subseteq G} |H| \left( d(H) - \alpha \right). \tag{1}
\]

Let \( \mathcal{H}' \) be the modified collection of subgraphs obtained by padding each subgraph in \( \mathcal{H} \) with arbitrary vertices until its size is at least \( k \). We will show that there is a set \( H \in \mathcal{H}' \) that satisfies \( d(H) \geq dal(G, k)/2 \). Thus, a polynomial time 2-approximation algorithm for DalkS can be obtained by computing \( \mathcal{H} \), padding some of the sets with arbitrary vertices to form \( \mathcal{H}' \), and returning the densest set in \( \mathcal{H}' \). The running time is dominated by the parametric flow algorithm.

Let \( H_* \) be an induced subgraph of \( G \) with at least \( k \) vertices that has density \( d(H_*) = dal(G, k) \). Let \( \alpha = dal(G, k)/2 \), and let \( H \) be the set from \( \mathcal{H} \) that maximizes (1) for this value of \( \alpha \). In particular,

\[
|H| (d(H) - \alpha) \geq |H_*| (d(H_*) - \alpha) \geq |H_*| d(H_*)/2. \tag{2}
\]

This implies that \( H \) satisfies \( d(H) \geq \alpha = dal(G, k)/2 \). If \( |H| \geq k \), then we are done. If \( |H| < k \), then consider the set \( H' \) of size exactly \( k \) obtained by padding \( H \) with arbitrary vertices. We will show that \( d(H') \geq dal(G, k)/2 \), which will complete the proof. First, notice that (2) implies a lower bound on the size of \( H \).

\[
|H| \geq |H_*| \frac{d(H_*)}{2d(H)} = |H_*| \frac{dal(G, k)}{2d(H)}.
\]

We can then bound the density of the padded set \( H' \).

\[
d(H') \geq d(H) \left( \frac{|H|}{k} \right) \\
\quad \geq d(H) \left( \frac{|H_*| dal(G, k)}{k \, 2d(H)} \right) \\
\quad = \frac{dal(G, k) \, |H_*|}{2} \\
\quad \geq \frac{dal(G, k)}{2}.
\]

\( \square \)
5 The densest at-most-$k$-subgraph problem

In this section, we show that the densest at-most-$k$-subgraph problem is nearly as hard to approximate as the densest $k$-subgraph problem. We will show that if there exists a polynomial time algorithm that approximates $\text{DamkS}$ in a weak sense, returning a set of at most $\beta k$ vertices with density at least $1/\gamma$ times the density of the densest subgraph on at most $k$ vertices, then there exists a polynomial time approximation algorithm for $\text{DkS}$ with ratio $4(\gamma^2 + \gamma \beta)$. As an immediate consequence, a polynomial time $\gamma$-approximation algorithm for the densest at-most-$k$-subgraph problem would imply a polynomial time $4(\gamma^2 + \gamma)$-approximation algorithm for the densest $k$-subgraph problem.

**Definition 5.** An algorithm $A(G, k)$ is a $(\beta, \gamma)$-algorithm for the densest at-most-$k$-subgraph problem if for any input graph $G$ and integer $k$, it returns an induced subgraph of $G$ with at most $\beta k$ vertices and density at least $\text{dam}(G, k)/\gamma$.

**Theorem 4.** If there is a polynomial time $(\beta, \gamma)$-algorithm for the densest at-most-$k$-subgraph problem (where $\beta$ and $\gamma$ are at least 1), then there is a polynomial time $4(\gamma^2 + \gamma \beta)$-approximation algorithm for the densest $k$-subgraph problem.

**Proof.** Assume there exists a polynomial time algorithm $A(G, k)$ that is $(\beta, \gamma)$-algorithm for $\text{DamkS}$. We will now describe a polynomial time approximation algorithm for $\text{DkS}$ with ratio $4(\gamma^2 + \gamma \beta)$.

Given as input a graph $G$ and integer $k$, let $H_1 = G$, let $i = 1$, and repeat the following procedure. Let $H_i = A(G_i, k)$ be an induced subgraph of $G_i$ with at most $\beta k$ vertices and with density at least $\text{dam}(G_i, k)/\gamma$. Remove all the edges in $H_i$ from $G_i$ to form a new graph $G_{i+1}$ on the same vertex set as $G$. Repeat this procedure until all edges have been removed from $G$.

Let $n_i$ be the number of vertices in $H_i$, let $W_i = W(H_i)$, and let $d_i = d(H_i) = W_i/n_i$. Let $H_s$ be an induced subgraph of $G$ with exactly $k$ vertices and density $d_s = \text{dex}(G, k)$. Notice that if $(W_1 + \cdots + W_{i-1}) \leq W(H_s)/2$, then $d_i \geq d_s/2\gamma$. This is because $d_i$ is at least $1/\gamma$ times the density of the induced subgraph of $G_i$ on the vertex set of $H_s$, which is at least

$$\frac{W(H_s) - (W_1 + \cdots + W_{i-1})}{k} \geq \frac{W(H_s)}{2k} = \frac{d_s}{2}.$$
Let $T$ be the smallest integer such that $(W_1 + \cdots + W_T) \geq W(H_*)/2$, and let $U_T$ be the induced subgraph on the union of the vertex sets of $H_1, \ldots, H_T$. The total weight $W(U_T)$ is at least $W(H_*)/2$. The density of $U_T$ is

$$d(U_T) = \frac{W(U_T)}{|U_T|} \geq \frac{W_1 + \cdots + W_T}{n_1 + \cdots + n_T} \geq \min_{1 \leq t \leq T} \frac{W_t}{n_t} \geq \frac{d_*}{2\gamma}.$$ 

To bound the number of vertices in $U_T$, notice that $(n_1 + \cdots + n_{T-1}) \leq \gamma k$, because

$$d_* = \frac{W(H_*)}{2} \geq \sum_{i=1}^{T-1} W_i = \sum_{i=1}^{T-1} n_i d_i \geq \frac{d_*}{2\gamma} \sum_{i=1}^{T-1} n_i.$$

Since $n_T$ is at most $\beta k$, we have $|U_T| \leq (n_1 + \cdots + n_T) \leq (\gamma + \beta)k$.

There are now two cases to consider. If $|U_T| \leq k$, we add vertices to $U_T$ arbitrarily to form a set $U'_T$ of size exactly $k$. The set $U'_T$ is more than dense enough to prove the theorem,

$$d(U'_T) \geq \frac{W(H_*)/2}{k} = \frac{d_*}{2}.$$

If $|U_T| > k$, then we employ a simple greedy procedure to reduce the number of vertices. We begin with the induced subgraph $U_T$, greedily remove the vertex with smallest degree to obtain a smaller subgraph, and repeat until exactly $k$ vertices remain. The resulting subgraph $U''_T$ has density at least $d(U_T)(k/2|U_T|)$ by the method of conditional expectations (see also [7]). The set $U''_T$ is sufficiently dense,

$$d(U''_T) \geq d(U_T) \frac{k}{2|U_T|} \geq \left(\frac{d_*}{2\gamma}\right) \left(\frac{k}{2(\gamma + \beta)k}\right) = \frac{d_*}{4(\gamma^2 + \gamma\beta)}.$$ 

Remark 2. The argument from Theorem 4 proves a slightly more general statement: if there is a polynomial time algorithm for DamkS that is a $(\beta, \gamma)$-algorithm for certain values of $k$, then there is a polynomial time algorithm for DkS that is a $4(\gamma^2 + \gamma\beta)$-approximation algorithm for those same values of $k$.

We remark that the densest at-most-$k$-subgraph is easily seen to be $NP$-complete, since a subgraph of size at most $k$ has density at least $(k-1)/2$ if and only if it is a $k$-clique. As mentioned previously, Feige and Seltser [8] proved that the densest $k$-subgraph problem remains $NP$-complete when restricted to graphs with maximum degree 3, and their proof shows that the same statement is true for the densest at-most-$k$-subgraph problem.
6 Conclusion

In this section, we discuss the possibility of improving the approximation ratio for DkS via an approximation algorithm for DamkS. One possible approach is to develop a local algorithm for DamkS, analogous to the recently developed local algorithms for graph partitioning [15, 1]. For any partition separating \(k\) vertices, these algorithms can produce a partition separating \(O(k)\) vertices that is nearly as good (in terms of conductance).

We conjecture that there is a local algorithm for the densest subgraph problem that finds a subgraph of density at least \(\theta / \log n\) on at most \(O(k^{1+\delta})\) vertices, whenever there exists a subgraph of density \(\theta\) on \(k\) vertices. This would be a \((\log n, k^\delta)\)-approximation algorithm for DamkS, which would lead to an approximation algorithm for the densest \(k\)-subgraph problem with ratio \(O(k^3 \log^2 n)\). An algorithm with \(\delta = 1\) would not be helpful for approximating DkS, since an approximation ratio of \(O(k)\) can be obtained trivially. At the other extreme, an algorithm with \(\delta = 0\) would produce an \(O(\log^2 n)\) approximation algorithm for DkS, which seems unlikely.

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