GLOBAL STABILITY
IN THE 2D RICKER EQUATION REVISITED

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Abstract. We offer two improvements to prior results concerning global sta-
ibility of the 2D Ricker Equation. We also offer some new methods of approach
for the more pathological cases and prove some miscellaneous results including
a special nontrivial case in which the mapping is conjugate to the 1D Ricker
map along an invariant line and a proof of the non-existence of period-2 points.

1. Introduction. The 1D Ricker map $T_1(x) = xe^{p-x}$ has a long history, Ricker
[13], Sharkovsky et. al. [18], Devaney [5], Elaydi [6], [7], Cull [4], Liz [10] and is well
understood even in the periodic case [15]. The fixed point $x = p$ is locally stable if
and only if $0 < p < 2$, in fact it is not hard to show it is globally attracting with
respect to the positive $x$-axis in this case.

The 2D Ricker map consists of two 1D Ricker maps coupled together. That is, the
map $T : [0, \infty) \times [0, \infty) \to [0, \infty) \times [0, \infty)$ given by $T(x,y) = (xe^{p-x-ay}, ye^{q-y-bx})$.
Although the 1D Ricker map is straightforward, the coupled map has posed a
rather stiff challenge. The map has up to four fixed points: $(0,0)$, $(p,0)$, $(0,q)$,
\[
\left( \frac{p-aq}{1-ab}, \frac{q-bp}{1-ab} \right)
\]
and we consider the case when all four are present. We are mainly
interested in the question, does local stability of the coexistence fixed point imply
global stability with respect to the first quadrant in the 2D Ricker map? Though the
question has been addressed affirmatively under certain restrictions on the param-
eters, the question in its most general form remains open.

Ricker maps with coupling are also discussed in Ackleh and Salceanu [1], Jiang
and Rogers [8], Mira [11], Li [9], Sacker, et. al. [16], [17], For the general theory of
Difference Equations or Discrete Dynamical Systems see Elaydi [6], and Pötzsche,
[12].

In Section 2 we summarize past work on the 2D Ricker map. Then in Section
3 we provide some improvements to the bounds on the parameters for which these
results apply and also show that whenever the coexistence fixed point $(x^*, y^*)$ is in

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the open first quadrant it is asymptotically stable without further assumptions. In
Section 4, we consider the non-homeomorphism case and using methods that are
partly numerical we show that global stability of the coexistence fixed point still
follows. Finally in Section 5 we consider the special case of $p = q$ and show there
is an invariant line on which the 2D Ricker map is conjugate to a 1D Ricker map.
We also show that whenever $p, q \in (0, 2)$ and $a, b \in (0, 1)$ there can be no 2-periodic
points of the 2D Ricker map.

2. Past work on the 2D Ricker map. We quickly summarize the 1D Ricker
map $T_1(x) = xe^{p-x}$ for $0 < p < 2$. As $T'_1(x) = (1 - x)e^{p-x}$, we know the maximum
of $T_1(x)$ is at $x = 1$ with a value of $e^{p-1}$. Thus $T_1((0, \infty)) \subset (0, e^{p-1})$. We split
our discussion into $0 < p < 1$ and $1 < p < 2$. For $0 < p < 1$, any $x \in (0, p)$
monotonically increases to $p$ and any $x \in (p, e^{p-1})$ monotonically decreases to $p$.
For $1 < p < 2$, points in $(0, 1]$ map monotonically forward until they are in the
interval $(1, e^{p-1})$. Then, points alternate between the intervals $(1, p)$ and $(p, e^{p-1})$
on each iteration, but move closer to $p$ each time. For $p > 2$, the fixed point $x = p$
is locally unstable and period 2 orbits are born.

Now we turn to the 2D Ricker map

$$T(x, y) = (xe^{p-x-ay}, ye^{q-y-bx}) .$$

(1)

Throughout the paper, we take $p, q, a, b$ to be positive constants. In the case $0 < 
p < 1$ and $0 < q < 1$ the map is monotone and global stability follows by methods on
monotone maps by Smith [19] and Baigent and Hou [2]. The case when $1 < p < 2$
and $1 < q < 2$ is more difficult.

One fruitful method of approach for the case when $1 < p < 2$ and $1 < q < 2$ was
pioneered by Balreira, Elaydi, and Luis [3]. They noted that the Jacobian of the
map vanishes along a hyperbola given by $y = \frac{1 - x}{1 - (1 - ab)x}$ which consists of two
branches, which we denote by $C$ (the lower branch) and $C^+$ (the upper branch).
That is,

$$C = \{ (x, y) \in [0, 1] \times [0, 1] \mid y = \frac{1 - x}{1 - (1 - ab)x} \} $$

(2)

and

$$C^+ = \{ (x, y) \in (\frac{1}{1 - ab}, \infty) \times (\frac{1}{1 - ab}, \infty) \mid y = \frac{1 - x}{1 - (1 - ab)x} \} . $$

(3)

They proved the following Theorem (note in the Theorem that the notation
“<” refers to the ordering of curves in the first quadrant). Here $x^* = \frac{p - aq}{1 - ab}$ and
$y^* = \frac{q - bp}{1 - ab}$ refer to the coordinates of the coexistence fixed point. We also define
the set of points on the line $p = x + ay$ as $L_p$ and the set of points on the line
$q = bx + y$ as $L_q$ (these are the nontrivial isoclines of the 2D Ricker map).

**Theorem 2.1.** [3] Let $T$ be the 2D Ricker map (1) with $p, q \in (1, 2)$ and $ab < 1$
with $a, b > 0$. Suppose that the coexistence fixed point $(x^*, y^*)$ is in the first quadrant
and is locally asymptotically stable. Assume the following conditions:

1. The critical curve $C$ (2) satisfies $C < L_p$ and $C < L_q$.
2. For all $m \neq n$, $T^n(C) \cap T^m(C) = \emptyset$. 


Figure 1. The curves $T^k(C)$ for $k = 0, 1, 2, 3$ are shown, as well as the unstable manifolds (thicker lines) from $(p, 0)$ and $(0, q)$. The unstable manifolds intersect at the coexistence fixed point. The curves, ordered from bottom left to top right, go $C, T^2(C), T^3(C)$, and then finally $T(C)$.

Then $(x^*, y^*)$ is globally asymptotically stable with respect to the interior of the first quadrant.

The idea behind the proof is as follows. Since the curves are disjoint and the endpoints on the $x$ and $y$ axis behave the same as the 1D Ricker map, the curves are nested in the sense that $T^{2k}(C) < T^{2k+2}(C)$ and $T^{2k+1}(C) > T^{2k+3}(C)$ for $k = 0, 1, 2, ...$. As the fixed points $(p, 0)$ and $(0, q)$ are saddles, they each have an unstable manifold. They argue that these nested sequences converge to the union of these manifolds and that the manifolds “end” at the coexistence fixed point. See Figure 1.

This is a theoretical result as it requires checking an infinite number of topological conditions, namely that $T^n(C) \cap T^m(C) = \emptyset$. This result was later improved by the current authors to only checking a finite number of conditions [14]. There the other branch of the singular curve, denoted here by $C^+$ is used.

Theorem 2.2. [14] Let $T$ be the Ricker map (1) with $p, q \in (1, 2)$ and let $a, b \in (0, 1)$. Suppose that the coexistence fixed point $(x^*, y^*)$ is in the first quadrant and is locally asymptotically stable. Assume the curves $C, T(C), T^2(C), C^+$ satisfy the relation $C < T^2(C) < T(C) < C^+$. Then for all $m \neq n$, $T^n(C) \cap T^m(C) = \emptyset$ and $(x^*, y^*)$ is globally asymptotically stable with respect to the interior of the first quadrant.

The key idea of the improvement is that, under these conditions, one has a homeomorphism from the region bounded by $C$ and $T(C)$ to the region bounded by $T^2(C)$ and $T(C)$.
Sufficient conditions for the non-intersection of the four curves were also given; if \( p \in \left( \frac{a + 1 - 2a\sqrt{b}}{1 - ab}, a + 1 \right) \) and \( q \in \left( \frac{b + 1 - 2b\sqrt{a}}{1 - ab}, b + 1 \right) \), then it was shown that all of the above non-intersection conditions are satisfied except for possibly \( C < T^2(C) \). Thus in this regime only one topological condition needs to be checked. In this case the Theorem can be restated as follows:

**Theorem 2.3.** Let \( T \) be the Ricker map \([7]\) with \( p, q \in (1, 2) \) and let \( a, b \in (0, 1) \). Assume the following conditions:

1. The coexistence fixed point \((x^*, y^*)\) is in the first quadrant (equivalently, \( p > aq \) and \( q > bp \)),
2. \((x^*, y^*)\) is locally asymptotically stable,
3. The curves \( C(T^2) \), \( T^2(C) \), satisfy the relation \( C < T^2(C) \),
4. \( p > \frac{a + 1 - 2a\sqrt{b}}{1 - ab} \) and \( q > \frac{b + 1 - 2b\sqrt{a}}{1 - ab} \),
5. \( p < a + 1 \) and \( q < b + 1 \).

Then for all \( m \neq n \), \( T^n(C) \cap T^m(C) = \emptyset \) and \((x^*, y^*)\) is globally asymptotically stable with respect to the interior of the first quadrant.

In the next section, we will make two improvements to this Theorem. First, we will show that the condition that the coexistence fixed point \((x^*, y^*)\) be locally asymptotically stable is superfluous and is automatically satisfied. Second, we will significantly widen the bounds on \( p \) and \( q \) by extending the upper bound.

### 3. Improvements on prior results

We offer two new improvements to Theorem 2.3. First, we show that the condition that the coexistence fixed point be locally asymptotically stable (condition 2) is redundant, as condition 1 implies it.

**Lemma 3.1.** Let \( p, q \in (0, 2) \) and \( ab \in (0, 1) \) with \( a, b > 0 \). Suppose that the coexistence fixed point \((x^*, y^*)\) is in the open first quadrant. Then it is locally asymptotically stable.

**Proof of Lemma 3.1.** Assume that \((x^*, y^*)\) are in the first quadrant, so that \( p - aq > 0 \) and \( q - bp > 0 \).

It is sufficient to verify the following, see \([6]\) p. 188

\[
|\text{tr } DT(x^*, y^*)| < \det DT(x^*, y^*) + 1 < 2.
\]

As \( DT(x^*, y^*) = \begin{bmatrix} (1 - x^*) & -ax^* \\ -by^* & (1 - y^*) \end{bmatrix} \), the inequalities are equivalent to

\[
|2 - x^* - y^*| < 2 - x^* - y^* + (1 - ab)x^*y^* < 2
\]

First, we will verify the second inequality in \([4]\), or equivalently that \((1 - ab)x^*y^* - x^* - y^* < 0 \). Since \( x^* = \frac{p - aq}{1 - ab} \) and \( y^* = \frac{q - bp}{1 - ab} \), this is the same as showing \((p - aq)(q - bp) - (p - aq) - (q - bp) < 0 \). Thus we are trying to show that \( AB - A - B < 0 \) for \( A, B \in (0, 2) \). This follows from the observation that \( AB - A - B = AB - 1 \). Next we verify the first inequality in \([4]\). If \( 2 - x^* - y^* > 0 \) then this becomes \( 0 < (1 - ab)x^*y^* \) which is true from \( ab < 1 \). If \( 2 - x^* - y^* < 0 \) then the inequality becomes

\[
4 - 2x^* - 2y^* + (1 - ab)x^*y^* > 0
\]
or equivalently

\[4(1 - ab) - 2(p - aq) - 2(q - bp) + (q - bp)(p - aq) > 0.\]

To see that this is positive, first fix \(p \in (0, 2)\) and \(q \in (0, 2)\). We then wish to show the expression is positive for \(a \in \left(0, \frac{2}{q}\right)\) and \(b \in \left(0, \frac{2}{p}\right)\). Note that these upper bounds are imposed by the conditions \(p - aq > 0\), \(q - bp > 0\), and \(0 < ab < \frac{2}{q} \frac{2}{p} = 1\).

Viewing this as a rectangle in \(\mathbb{R}^2\), we find the absolute minimum of \(M(a, b) = 4(1 - ab) - 2(p - aq) - 2(q - bp) + (q - bp)(p - aq)\) on the closed rectangle \(a \in \left[0, \frac{2}{q}\right], b \in \left[0, \frac{2}{p}\right]\) and show that the absolute minimum of 0 only occurs on the boundary. We note that \(\frac{dM}{da} = -4b + 2q - q(b - bp) = b(pq - 4) + q(2 - q)\) and \(\frac{dM}{db} = a(pq - 4) + p(2 - p)\) so that the only critical point is \(\left(\frac{p(2-p)}{4pq}, \frac{q(2-q)}{4pq}\right)\). A calculation shows that the Hessian at this point is \(-pq)^2 < 0\) so that this critical point is a saddle point. Hence any extrema must occur on the boundary.

On the boundary line \(a = 0\), we have \(M(0, b) = 4 - 2p + (p - 2)(q - bp) = (2 - p)(2 - q + bp) > 0\). Similarly, \(M(a, 0) = (2 - q)(2 - p + aq) > 0\). On the boundary line \(a = \frac{2}{q}\), we have \(M(p/q, b) = 4(1 - bq/q) - 2(q - bp) = \frac{2}{q}(2 - q)(q - bp) > 0\). Similarly, \(M(a, q/b) = \frac{2}{p}(2 - p)(p - aq) \geq 0\), with equality occurring at the point \((p/q, q/p)\). As there are no local extrema in the interior, the expression \(M(a, b)\) must be strictly positive in the interior. Since this holds for any \(p, q \in (0, 2)\) the result follows.

Our other improvement concerns the bounds on \(p\) and \(q\). Condition 4 in Theorem 2.3 implies the topological condition \(C < T(C)\) and condition 5 implies \(T(C) < C^+\). The bounds on Condition 5 can actually be significantly improved to cover a wider range of \(p\) and \(q\).

**Theorem 3.2.** Let \(T\) be the Ricker map \([4]\). Let \(a, b \in (0, 1)\). Let

\[
p < \frac{1 + a - a \sqrt{ab}}{1 - ab} - \ln(\sqrt{ab} + a) + \gamma
\]

and

\[
q < \frac{1 + b - b \sqrt{ab}}{1 - ab} - \ln(\sqrt{ab} + b) + \gamma
\]

where

\[
\gamma = -\frac{\sqrt{ab}}{1 - ab} + \ln\left(\frac{2 \sqrt{ab}(1 + \sqrt{ab})}{1 - \sqrt{ab}}\right)
\]

Then \(T(C) < C^+\), where \(C\) is given by \([2]\) and \(C^+\) is given by \([3]\).

In the proof, we make use of the following lemma from \([14]\).

**Lemma 3.3.** \([14]\) Let \(T\) be the Ricker map \([1]\). Let \(a, b \in (0, 1)\) and \(p, q \in (1, 2)\). Then the curves \(C^\#\) and \(T(C)\) are simple Jordan arcs that can be represented as functions that are monotonically decreasing in \(x\) and concave.

**Proof of Theorem.** First, we note that it suffices to show the result for \(p = \frac{1 + a - a \sqrt{ab}}{1 - ab} - \ln(\sqrt{ab} + a) + \gamma\) and \(q = \frac{1 + b - b \sqrt{ab}}{1 - ab} - \ln(\sqrt{ab} + b) + \gamma\), as this curve for \(T(C)\) acts as a bounding curve for all smaller values of \(p, q\), and the outer branch \(C^+\) does not depend on \(p, q\). Henceforth in the proof \(p\) and \(q\) will refer to these specific values.
we wish to show

We will now find the absolute maximum of

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for \(-1 \leq t \leq -ab\).

Letting \(u = \frac{ab}{t}\), the curve \(T(C)\) is then parametrized by

\[
(\alpha(1+t)e^{p-\alpha(1+t)-\alpha(1+u)}, \alpha(1+u)e^{q-\alpha(1+t)-\alpha(1+u)})
\]

\(T(C)\) and \(C^+\) being concave and convex respectively, a sufficient condition for \(T(C) < C^+\) is that \(T(C)\) lies beneath the tangent line of \(C^+\) at the point \((\frac{1}{1-\sqrt{ab}}, \frac{1}{1-\sqrt{ab}})\). The equation of this line is 

\[
x + y = \frac{2}{1 - \sqrt{ab}}.
\]

As the sum of the \(x\) and \(y\) coordinates of \(T(C)\) is given by

\[
\alpha(1+t)e^{p-\alpha(1+t)-\alpha(1+u)} + \alpha(1+u)e^{q-\alpha(1+t)-\alpha(1+u)},
\]

we wish to show

\[
\alpha(1+t)e^{p-\alpha(1+t)-\alpha(1+u)} + \alpha(1+u)e^{q-\alpha(1+t)-\alpha(1+u)} < \frac{2}{1 - \sqrt{ab}}.
\]

Rewriting \(\frac{2}{1-\sqrt{ab}} = 2\alpha(1+\sqrt{ab})\), this is equivalent to showing

\[
(1+t)e^{p-\alpha(1+t)-\alpha(1+u)} + (1+u)e^{q-\alpha(1+t)-\alpha(1+u)} < 2(1+\sqrt{ab}).
\]

Using the \(p\) and \(q\) values above, the left hand side that we define to be \(H(t)\) becomes

\[
H(t) = \frac{1}{\sqrt{ab} + a} (1 + (1+t)\left(-\alpha + \frac{\alpha a^2 b}{t^2}\right)) e^{-\alpha a \sqrt{ab} - at - au + \gamma} + \frac{1}{\sqrt{ab} + b} (1 + u) e^{-b \alpha \sqrt{ab} - bat - au + \gamma}.
\]

We will now find the absolute maximum of \(H(t)\) for \(-1 \leq t \leq -ab\).

We compute, noting \(\frac{du}{dt} = -\frac{ab}{t^2}\), that

\[
H'(t) = \frac{1}{\sqrt{ab} + a} \left(1 + (1+t)\left(-\alpha + \frac{\alpha a^2 b}{t^2}\right)\right) e^{-\alpha a \sqrt{ab} - at - au + \gamma} + \frac{1}{\sqrt{ab} + b} \left(-\frac{ab}{t^2} + (1+u)\left(-b \alpha + \frac{\alpha ab}{t^2}\right)\right) e^{-b \alpha \sqrt{ab} - bat - au + \gamma}.
\]

We note that when \(t = -\sqrt{ab}\), \(u = -\sqrt{ab}\), so that

\[
H'(-\sqrt{ab}) = \frac{1}{\sqrt{ab} + a} \left(1 + (1-\sqrt{ab})(-\alpha + \alpha a)\right) e^{-\alpha a \sqrt{ab} + a \sqrt{ab} + \alpha a \sqrt{ab} + \gamma} + \frac{1}{\sqrt{ab} + b} \left(-1 + (1-\sqrt{ab})(-b \alpha + \alpha)\right) e^{-b \alpha \sqrt{ab} + b \alpha \sqrt{ab} + \alpha \sqrt{ab} + \gamma} = \frac{1}{\sqrt{ab} + a} \left(1 - \frac{1}{1 + \sqrt{ab}}(1-a)\right) e^{\alpha \sqrt{ab} + \gamma} + \frac{1}{\sqrt{ab} + b} \left(-1 + \frac{1}{1 + \sqrt{ab}}(-b+1)\right) e^{\alpha \sqrt{ab} + \gamma}.
\]
Figure 2. The figure shows the upper bounds implied by Theorems 2.3 and Conjecture 1, respectively. In the left column, we have the upper bounds for $p$ and in the right column we have the upper bounds for $q$. We have capped the upper bounds at 2 for the plots since the fixed point loses stability past $p,q = 2$.

$$
= \frac{1}{\sqrt{ab} + a} \left( \frac{a + \sqrt{ab}}{1 + \sqrt{ab}} \right) e^{\alpha \sqrt{ab} + \gamma} + \frac{1}{\sqrt{ab} + b} \left( -\sqrt{ab} - b \right) e^{\alpha \sqrt{ab} + \gamma} \\
= \left( \frac{1}{1 + \sqrt{ab}} \right) e^{\alpha \sqrt{ab} + \gamma} + \left( \frac{-1}{1 + \sqrt{ab}} \right) e^{\alpha \sqrt{ab} + \gamma} \\
= 0. 
$$

Now since the curve $T(C)$ can be written as $y = \Phi(x)$ for some $\Phi$ with $\Phi''(x) < 0$ by Lemma 3.3 we know the sum of the $x$ and $y$ coordinates has negative second derivative and it follows that $H''(t) < 0$. Combining this fact with the above calculation with $H'(-\sqrt{ab})$, we know the absolute maximum of $H(t)$ on the interval occurs at $t = -\sqrt{ab}$.

Now

$$
H(-\sqrt{ab}) = \frac{1}{\sqrt{ab} + a} (1 - \sqrt{ab}) e^{-a\alpha \sqrt{ab} + a\sqrt{ab} + a\alpha \sqrt{ab} + \gamma} \\
+ \frac{1}{\sqrt{ab} + b} (1 - \sqrt{ab}) e^{-b\alpha \sqrt{ab} + b\alpha \sqrt{ab} + \alpha \sqrt{ab} + \gamma} 
$$

Simplifying,

$$
H(-\sqrt{ab}) = \left( \frac{1}{\sqrt{ab} + a} + \frac{1}{\sqrt{ab} + b} \right) (1 - \sqrt{ab}) e^{\alpha \sqrt{ab} + \gamma} 
$$

This is equal to $2(1 + \sqrt{ab})$ for the value of $\gamma$. 

\hfill \square
Though these bounds are a significant improvement (see Figure 2), they come at the expense of elegance. Furthermore, they are no longer invertible bounds: Given \( p \) and \( q \), it is impractical to determine the range of \( a \) and \( b \) for which the Theorem applies. As such, we also provide the following simpler bounds \( p < 2a + 1 \) and \( q < 2b + 1 \) obtained from a homotopy argument, but only stated as a conjecture with strong numerical supporting evidence. Typically these bounds are worse than those found in Theorem 3.2, but for some extreme values of \( a \) and \( b \) they cover parameter cases not covered by 3.2. In Figure 2 we show the upper bounds on \( p \) and \( q \) obtained from the two theorems plus a conjecture.

While the following is stated as a conjecture, there is strong numerical evidence supporting the statement.

**Conjecture 1.** Let \( T \) be the Ricker map \([7]\). Suppose \( a, b \in (0, 1) \) and let \( p \leq 2a + 1 \) and \( q \leq 2b + 1 \). Then \( T(C) < C^+ \), where \( C \) is given by \([4]\).

We begin our argument as follows: As in Theorem 3.2, it suffices to show \( T(C) < C^+ \) for the extreme bounds of \( p = 2a + 1 \) and \( q = 2b + 1 \). First, we parametrize \( C \) by \( (t, \phi(t)) \) with \( 0 \leq t \leq 1 \) where \( \phi(t) = \frac{1 - t}{1 - (1 - ab)t} \). Then the curve \( T(C) \) is parametrized by

\[
(x(t), y(t)) = \left( t e^{2a+1-t-a\phi(t)}, \phi(t) e^{2b+1-\phi(t)-bt} \right).
\]

As the upper branch \( C^+ \) is parametrized by \((t, \phi(t))\) with \( \frac{1}{1 - ab} < t < \infty \), the curve \( T(C) \) intersects \( C^+ \) if for some value of \( t \in (0, 1) \), \( y(t) = \phi(x(t)) \). Defining \( P(t) = 2a + 1 - t - a\phi(t) \) and \( Q(t) = 2b + 1 - \phi(t) - bt \), an intersection occurs if

\[
\frac{1 - t}{1 - (1 - ab)t} e^{Q(t)} = \frac{1 - t e^{P(t)}}{1 - (1 - ab)t e^{P(t)}}.
\]

To show this never occurs, we wish to show

\[
\frac{1 - (1 - ab)te^{P(t)}}{1 - (1 - ab)t} e^{Q(t)} > \frac{1 - t e^{P(t)}}{1 - t}, \quad 0 < t < 1.
\]  

(5)

To that end let us smoothly deform the left side to the right as follows: Define the linear function of \( \sigma \),

\[
\mathcal{L}(\sigma) = e^{Q} - \frac{e^{Q} - 1}{ab} [\sigma - (1 - ab)], \quad 1 - ab \leq \sigma \leq 1,
\]

where for convenience we will set \( Q = Q(t) \) and \( P = P(t) \) in what follows. Then define

\[
H(\sigma) = \frac{1 - \sigma e^{P}}{1 - \sigma t} \mathcal{L}(\sigma),
\]

so that (5) reduces to \( H(1 - ab) > H(1) \). To accomplish this we argue, partially numerically, that \( \int_{1-ab}^{1} \frac{dH(\sigma)}{d\sigma} d\sigma < 0 \) for all \( t \in (0, 1) \).

Computing,

\[
\frac{d}{d\sigma} H(\sigma) = \frac{-(1 - \sigma)te^{P} + (1 - \sigma e^{P})t}{(1 - \sigma t)^2} \mathcal{L}(\sigma) + \frac{1 - \sigma e^{P}}{1 - \sigma t} \frac{1 - e^{Q}}{ab}.
\]

(6)
Multiplying numerator and denominator of \( A \) by \( ab \) and \( B \) by \( (1 - \sigma t) \) and temporarily ignoring the (positive) common denominator, we have

\[
H' = \left[ -(1 - \sigma t)te^P + (1 - \sigma te^P)t \right] [abe^Q] \\
+ (1 - e^Q)(\sigma - (1 - ab)) + (1 - \sigma t)(1 - \sigma te^P)(1 - e^Q).
\]

The first term in brackets can be simplified so that (7) becomes

\[
\left[ (1 - e^P)t \right] [abe^Q + (1 - e^Q)(\sigma - (1 - ab))] + (1 - \sigma t)(1 - \sigma te^P)(1 - e^Q).
\]

This is a quadratic in \( \sigma \):

\[
(1 - e^Q)t^2e^P\sigma^2 + 2te^P(e^Q - 1)\sigma + t(1 - e^P)\left[ abe^Q - (1 - ab)(1 - e^Q) \right] + 1 - e^Q.
\]

The constant (in \( \sigma \)) term simplifies so that we obtain

\[
(1 - e^Q)t^2e^P\sigma^2 + 2te^P(e^Q - 1)\sigma + t(1 - e^P)\left[ e^Q - (1 - ab) \right] + 1 - e^Q. \tag{8}
\]

Next define

\[
z = \frac{e^Q - (1 - ab)}{e^Q - 1} > 1, \tag{9}
\]

and in (8) factor out \( (1 - e^Q)e^P < 0 \) to obtain (after re-introducing the denominator),

\[
\frac{d}{d\sigma} H(\sigma) = (1 - e^Q)e^P \frac{W(\sigma)}{ab(1 - \sigma t)^2}, \tag{10}
\]

where \( W \) is a quadratic in \( \sigma \) and

\[
V(\sigma) = \frac{W(\sigma)}{ab(1 - \sigma t)^2} = \frac{1}{ab} \frac{t^2 \sigma^2 - 2t\sigma + t(1 - e^{-P})z + e^{-P}}{(1 - \sigma t)^2}
\]

\[
= \frac{1}{ab} \frac{(t\sigma - 1)^2 - (1 - tz)(1 - e^{-P})}{(1 - \sigma t)^2}
\]

\[
= \frac{1}{ab} \left[ 1 - \frac{u}{(1 - \sigma t)^2} \right], \tag{11}
\]

where

\[
u = (1 - tz)(1 - e^{-P}).
\]

That \( 0 < u < 1 \) will follow from Lemma 3.4. The final expression in (11) defines a graph with a vertical asymptote at \( \sigma = \frac{1}{t} > 1 \) and each branch is easily seen to be concave. It is the left branch in which we are interested and it decreases from \( \frac{1}{ab} \) at \( \sigma = -\infty \) to \( -\infty \) at \( \sigma = \frac{1}{t} \). A typical graph of \( V \) is shown in Figure 3 over a portion or \( \mathbb{R} \) containing \([1 - ab, 1]\).

**Lemma 3.4.** \( 0 < tz < 1, \ 0 < 1 - e^{-P} < 1 \) and \( 1 - tz = \frac{\phi(t)e^Q - 1}{e^Q - 1} (1 - (1 - ab)t) > 0. \)

**Proof of Lemma 3.4.** Since \( e^Q > 1 \) it is clear from (9) that \( tz > 0 \) and \( e^P > 1 \implies 0 < 1 - e^{-P} < 1. \) For the remaining inequalities note that

\[
1 - tz = 1 - t \frac{e^Q - (1 - ab)}{e^Q - 1} = \frac{e^Q - 1 - t(e^Q - (1 - ab))}{e^Q - 1}
\]

\[
= \frac{(1 - t)e^Q - (1 - (1 - ab)t)}{e^Q - 1} = \frac{\phi(t)e^Q - 1}{e^Q - 1} (1 - (1 - ab)t). \tag{12}
\]
Since $C^+$ resides in the region $\{ x > \frac{1}{1-ab}, y > \frac{1}{1-ab} \}$ then at the parameter value $t$ at which $T(C)$ intersects $C^+$, we must have
\[ te^p > \frac{1}{1-ab} \quad \text{and} \quad \phi(t)e^Q > \frac{1}{1-ab}, \tag{13} \]
where as usual $P = P(t)$ and $Q = Q(t)$. The second inequality in (13) implies the right side of (12) is positive. \hfill \Box

Continuing with the justification of 1, we will show the integral of $V(\sigma)$ is positive as follows. We shall show numerically this expression is positive by a partially numerically argument. Thus,
\[ \mathcal{I} = \int_{1-ab}^{1} V(\sigma)d\sigma = \frac{1}{ab} \int_{1-ab}^{1} \left\{ 1 - \frac{u}{(1-\sigma t)^2} \right\}d\sigma = 1 - \frac{u}{(1-t)(1-(1-ab)t)}. \tag{14} \]
We shall show numerically this expression is positive as follows.

Divide the interval $[0,1]$ into 30,000 equal subintervals and let $\mathcal{P}$ denote the collection of endpoints of these subintervals lying in $(0,1)$. Then for each $(a,b) \in \mathcal{P} \times \mathcal{P}$, minimize $\mathcal{I}_{(ab)}$, for those $t \in \mathcal{P}$ that satisfy (13) and record this value. Finally, taking the smallest of these minima we obtain
\[ \mathcal{I} = 1 - \frac{u}{(1-t)(1-(1-ab)t)} \geq 0.2432(ab). \tag{15} \]
The minima were obtained by simply taking the smallest number of a finite set. This concludes the justification of Conjecture 1. \hfill \Box

The values $p = 2a + 1$ and $q = 2b + 1$ were discovered by graphing the curves $T(C)$ and $C^+$. Replacing $p$ by $2.00001a + 1$ causes the right side of (15) to be negative for some values of $a, b$ and $t$.

To summarize, we now have the following Theorem.

**Theorem 3.5.** Let $T$ be the Ricker map (1) with $p, q \in (1,2)$ and let $a, b \in (0,1)$. Assume the following conditions:
1. The coexistence fixed point $(x^*, y^*)$ is in the first quadrant (equivalently, $p > aq$ and $q > bp$).
2. The curves $C$ on $T^2(C)$, satisfy the relation $C < T^2(C)$.
3. $p > \frac{a+1-2\sqrt{a}}{1-ab}$ and $q > \frac{b+1-2\sqrt{b}}{1-ab}$.
4. $p < \frac{1+a-\sqrt{ab}}{1-ab} - \ln(\sqrt{ab} + a) + \gamma$ and $q < \frac{1+b-\sqrt{ab}}{1-ab} - \ln(\sqrt{ab} + b) + \gamma$ where $\gamma = -\frac{\sqrt{ab}}{1-ab} + \ln \left( \frac{2\sqrt{ab}(\sqrt{ab} + a)}{1-\sqrt{ab}} \right)$.

Then for all $m \neq n$, $T^m(C) \cap T^n(C) = \emptyset$ and $(x^*, y^*)$ is globally asymptotically stable with respect to the interior of the first quadrant.

As a corollary to Conjecture 1 one has the following:

**Corollary 1.** Let $T$ be the Ricker map (1) with $p, q \in (1,2)$ and let $a, b \in (0,1)$. Assume the following conditions:
1. The coexistence fixed point $(x^*, y^*)$ is in the first quadrant (equivalently, $p > aq$ and $q > bp$).
2. The curves $C$ on $T^2(C)$, satisfy the relation $C < T^2(C)$.
3. $p > \frac{a+1-2\sqrt{a}}{1-ab}$ and $q > \frac{b+1-2\sqrt{b}}{1-ab}$.
4. $p < 2a + 1$ and $q < 2b + 1$. 

Figure 3. The left branch of a typical graph of $V$ versus $\sigma$. For small $t$ the graph may lie completely above the $\sigma$-axis on the interval $[1 - ab, 1]$.

Then for all $m \neq n$, $T^n(C) \cap T^m(C) = \emptyset$ and $(x^*, y^*)$ is globally asymptotically stable with respect to the interior of the first quadrant.

4. The nonhomeomorphism case. The results of Theorem 3.2 cannot cover all possible $p, q, a, b$ where the coexistence fixed point is locally stable, as for some values of the parameters the topological constraints are not satisfied, so that the map is not a homeomorphism between $C$ and $T(C)$, a crucial ingredient to the proof of the Theorem. Here, we provide a method to approach such cases and some partial results which we have verified numerically for a number of parameters. These results would, if proven to be true, imply global stability.

As before, we define the set of points on the line $p = x + ay$ as $L_p$ and the set of points on the line $q = bx + y$ as $L_q$.

First, we examine when the $x$ coordinate maps closer to $x^* = \frac{p - aq}{1 - ab}$ under the mapping $T(x, y)$. To this end, we examine the function

$$F(x, y) = |x - x^*| - |x^* - xe^{p-x-ay}|.$$

As this function is continuous, to determine the regions in the plane where this is positive we only need to find where it is zero. This occurs either when $x - xe^{p-x-ay} = 0$ so that $p - x - ay = 0$ (that is, the isocline $L_p$) or when $x(1 + e^{p-x-ay}) - 2x^* = 0$. The latter occurs when $e^{p-x-ay} = \frac{2x^*}{x} - 1$ which is when $y = -\frac{1}{a} \ln \left(\frac{2x^*}{x} - 1\right) - \frac{x - p}{a}$. We will prove momentarily that this curve lies...
Figure 4. In the left figure, we show the isocline \( L_p \) and the curve \( y = -\frac{1}{a} \ln \left( \frac{2x^*}{x} - 1 \right) - \frac{x - p}{a} \) by solid lines. The shaded regions are where the function moves closer in the \( x \) coordinate. On the right, the isocline \( L_q \) and the curve \( x = -\frac{1}{b} \ln \left( \frac{2y^*}{y} - 1 \right) - \frac{y - q}{b} \) are shown as solid lines. The shaded regions are where the \( y \) coordinate moves closer. The union of the two regions is the entire plane.

below the isocline \( L_q \) for \( x < x^* \) and above it for \( x > x^* \), is strictly increasing, and intersects the isocline \( L_p \) at the fixed point.

Similarly, the boundary of the region where the \( y \) coordinate moves closer is given by the isocline \( L_q \) and the curve \( x = -\frac{1}{b} \ln \left( \frac{2y^*}{y} - 1 \right) - \frac{y - q}{b} \). These curves divide our plane into regions shown in Figure 4. We will now prove that for any \((x,y)\) in the first quadrant, at least one of the two coordinates is moving closer to the fixed point.

Lemma 4.1. Let \( h(x) = -\frac{1}{a} \ln \left( \frac{2x^*}{x} - 1 \right) - \frac{x - p}{a} \) and \( k(y) = -\frac{1}{b} \ln \left( \frac{2y^*}{y} - 1 \right) - \frac{y - q}{b} \). Then the curves \( y = h(x), x = k(y), L_p, \) and \( L_q \) divide the first quadrant into eight regions. On each of these regions, either \(|x - x^*| - |x^* - xe^{p-x-a}y| > 0\), \(|y - y^*| - |y^* - ye^{q-bx-y}| > 0\), or both are true.

Proof of Lemma 4.1. First we establish that these curves divide the first quadrant into 8 distinct regions. Observe first that all curves intersect at the fixed point \((x^*, y^*)\). The result will follow if we show that this is the only intersection. We note that \( y = h(x) \) has a vertical asymptote at \( x = 2x^* \) and is only well defined on \((0, 2x^*)\). Now

\[
h'(x) = -\frac{1}{a} \left( \frac{-2x^*}{x^2} \right) / \left( \frac{2x^*}{x} - 1 \right) - \frac{1}{a} = -\frac{1}{a} \left( 1 - \frac{2x^*}{2x^*x - x^2} \right).
\]
Since $x < 2x^*$, we have that
\[ 1 - \frac{2x^*}{2x^*x - x^2} = \frac{2x^*x - x^2 - 2x^*}{2x^*x - x^2} \]
is always negative since the denominator $x(2x^* - x)$ is always positive and the numerator $-(x - x^*)^2 + x^*(x^* - 2)$ is always negative since $x^* < p < 2$. Hence $y = h(x)$ only intersects the two isoclines at the fixed point. Similarly, one can show that $x = k(y)$ is strictly increasing. It remains to show that they intersect at a single point. Since their intersection occurs when $y = h(k(y))$, we consider the function $G(y) = h(k(y)) - y$. Its derivative $G'(y) = h'(k(y))k'(y) - 1$ is always positive whenever it is defined, so $G(y)$ has at most one root. As $G(y^*) = 0$ the curves only intersect at a single point. By taking limits, one sees that $x = k(y)$ lies “above” $y = h(x)$ for $x < x^*$ and $x = k(y)$ lies “below” $y = h(x)$ for $x > x^*$.

Now we argue that on each of these regions, either $|x - x^*| - |x^* - xe^{p-x-ay}| > 0$, $|y - y^*| - |y^* - ye^{q-bx-y}| > 0$, or both are true. This follows from the fact that the left hand side of each of these equations changes sign across their corresponding zero curves, so we can identify the regions where $|x - x^*| - |x^* - xe^{p-x-ay}| < 0$ and $|y - y^*| - |y^* - ye^{q-bx-y}| < 0$. Of the four regions separated by $y = h(x)$ and $L_p$, the regions where $|x - x^*| - |x^* - xe^{p-x-ay}| < 0$ is true must necessarily be the ones that contain the line $(x^*, y)$. Likewise, of the four regions separated by $x = k(y)$ and $L_q$, the ones where $|y - y^*| - |y^* - ye^{q-bx-y}| < 0$ are the ones that contain the line $(x, y^*)$.

Now we wish to consider the subset of the first quadrant where the Euclidean distance becomes closer to $(x^*, y^*)$ after 1 iteration. Clearly this is true for the overlap of the regions above. To see where else this is true, one needs to find the set.
We conjecture that, in the case when the map is not a homeomorphism, this is true for a rather large region below $T(C)$ and above $C$ that is away from the axes. A numerically computed example of the function $G(x,y) = (xe^{p-x-ay} - x^*)^2 + (ye^{q-bx-y} - y^*)^2 < (x-x^*)^2 + (y-y^*)^2$ is shown in Figure 5 for the parameter values $p = 1.8$, $q = 1.9$, $a = 0.2$, and $b = 0.3$ (Note that Theorem 3.2 only applies for $q < 1.72$ for this choice of $a, b$). Here the region consists of all points between $C$ and $T(C)$ such that $x$ and $y$ are both larger than 0.85, to ensure the points are not too close to the axes.

We observe that this function appears very convincingly to be concave in this region. To verify this, one can conceivably compute the Hessian of $G(x,y)$ and check that it is negative definite. This would require one to show that the top left entry $G_{xx}$ is negative and that the determinant is positive for all $x,y$ in the region. We show a plot for the same $p, q, a, b$ of $G_{xx}$ in Figure 6 and the determinant in Figure 7 with the $xy$ plane shown for emphasis. As each of these expressions contains sums of different exponential terms, a complete analytic solution is likely to require some clever insight with the equations. Nonetheless, the numerical graphs leave little doubt of the truth of this claim, and we leave it here as a conjecture. Note that, if it is true, global attraction to the fixed point follows as a consequence.

5. Miscellaneous results.

5.1. The case $p = q$. In the special case when the carrying capacities are equal, the 2D Ricker map takes the form $T(x,y) = (xe^{p-x-ay}, ye^{p-bx-y})$. In this map, we will prove there is an invariant line through the coexistence fixed point, along which the dynamics behave the same as the 1D Ricker map.
Figure 7. A graph of the determinant of the Hessian is shown for parameters \( p = 1.8, \ q = 1.9, \ a = 0.2, \ b = 0.3 \). The determinant is clearly positive.

Theorem 5.1. Define \( L_{a,b} = \{ (x, y) \mid y = \frac{1-b}{1-a} x \} \). If \( p = q \), then the line \( L_{a,b} \) contains the coexistence fixed point, is invariant, and \( T|_{L_{a,b}} \) is conjugate to \( T_1(x) = x e^{p-x} \).

Proof of Theorem 5.1. When \( p = q \), the coexistence fixed point is given by

\[
\left( \frac{p - ap}{1 - ab}, \frac{p - bp}{1 - ab} \right)
\]

and

\[
\frac{1 - b}{1 - a} \frac{p - ap}{1 - ab} = \frac{p - bp}{1 - ab}
\]

so

\[
\left( \frac{p - ap}{1 - ab}, \frac{p - bp}{1 - ab} \right) \in L_{a,b}.
\]

To see invariance, note that

\[
T \left( x, \frac{1 - b}{1 - a} x \right) = \left( x e^{p-x - a \frac{1 - b}{1 - a} x}, \frac{1 - b}{1 - a} x e^{p-bx - \frac{1 - b}{1 - a} x} \right) = e^{p-x - a \frac{1 - b}{1 - a} x} \left( x, \frac{1 - b}{1 - a} x \right).
\]

Now consider the map \( F : \mathbb{R} \to L_{a,b} \) given by \( F(x) = \left( \frac{1 - a}{1 - ab} x, \frac{1 - b}{1 - ab} x \right) \). We compute

\[
F \circ T_1(x) = \left( \frac{1 - a}{1 - ab} x e^{p-x}, \frac{1 - b}{1 - ab} x e^{p-x} \right)
\]

and

\[
T \circ F(x) = \left( \frac{1 - a}{1 - ab} x e^{p-x - a \frac{1 - b}{1 + ab} x}, \frac{1 - b}{1 - ab} x e^{p-bx - \frac{1 - b}{1 - ab} x} \right) = F \circ T_1
\]

so that \( F \circ T_1 = T \circ F \). \qed
We note that this proves, when \( p = q \), that the 2D Ricker map has periodic points when \( p > 2 \) as they are present in \( T_1(x) \) in this case.

Although the case when \( p = q \) is special, we suspect a curve with similar properties exists in the general case. Note that its existence alone would not establish global stability unless attraction to this curve could be globally established.

5.2. Periodic points in the 2D Ricker map. A very natural question is about the possibility of period 2 (or higher) points in the cases where the Theorems of global convergence do not apply. To address this possibility in the parameter space \( p, q \in (0, 2) \), we look at the nontrivial isoclines of the second iterate of \( T \). As

\[
T^2(x, y) = \left( x e^{p-x-ay} + p - x e^{p-x-ay} - a y e^{q-bx-y}, y e^{q-bx-y} + q - b x e^{p-x-ay} - y e^{q-bx-y} \right),
\]

the nontrivial isoclines of the map \( T^2 \) are given by

\[
I_p = \{ (x, y) \mid p - x - ay + p - x e^{p-x-ay} - a y e^{q-bx-y} = 0 \}
\]

and

\[
I_q = \{ (x, y) \mid q - bx - y + q - b x e^{p-x-ay} - y e^{q-bx-y} = 0 \}.
\]

A sample plot of the isoclines \( I_p, I_q \) are shown in Figure 9. The Figure also includes the straight line \( y = \frac{y^*}{x^*} x \). We notice that in this example \( I_p, I_q \) only intersect at the coexistence fixed point, and that this is the only point they cross through the line \( y = \frac{y^*}{x^*} x \). We will prove this is typical.

**Lemma 5.2.** Let \( p, q \in (0, 2), a, b \in (0, 1) \) and let \( p - a q > 0, q - b p > 0 \). Then the isoclines \( I_p \) and \( I_q \) only intersect the line \( y = \frac{y^*}{x^*} x \) at \( (x^*, y^*) \).
Figure 9. A plot of the isoclines of $T^2$ relative to the isoclines of $T$ as proved in Lemma 5.4, see also Figure 8. The straight dashed line is $y = \frac{y^*}{x^*} x$.

Proof of Lemma 5.2. We will prove this for the isocline $I_p$; the proof for $I_q$ is identical.

Let us find all $(x, y)$ points on the line $y = \frac{y^*}{x^*} x$ that also lie on the isocline $I_p$. Note that when $y = \frac{y^*}{x^*} x$, we have that

$$p - x - ay = p - x \left(1 + a \frac{y^*}{x^*}\right) = p - \frac{x}{x^*} (x^* + ay^*) = p \left(1 - \frac{x}{x^*}\right)$$

where we make use of $p - x^* - ay^* = 0$. Similarly, $q - bx - y = q \left(1 - \frac{x}{x^*}\right)$. Making this substitution into the $I_p$ isocline, we have that

$$p \left(1 - \frac{x}{x^*}\right) + p - xe^{p(1-x/x^*)} - ay e^{q(1-x/x^*)} = 0.$$

Now we make the change of variable $z = 1 - x/x^*$. The equation becomes

$$pz + p - (1-z)x^*e^{pz} - ay^*(1-z)e^{qz} = 0.$$

Using $p = x^* + ay^*$, we can rewrite this as

$$x^*(z + 1 - (1-z)e^{pz}) + ay^*(z + 1 - (1-z)e^{qz}).$$

Now $f(z) = z + 1 - (1-z)e^{pz}$ has derivative $f'(z) = 1 - (1-z)pe^{pz} + e^{pz}$ and we will argue that this is always positive, so that $f(z)$ has at most one root (the root $z = 0$). As $f''(z) = -(1-z)pe^{pz} + pe^{pz} + pe^{pz} = pe^{pz}(2 - (1-z)p)$, we note that $f'(z)$ has an absolute minimum when $2 - (1-z)p = 0$, that is $z = 1 - \frac{2}{p}$ or $1 - z = 2/p$. We compute $f'(1 - 2/p) = 1 - 2e^{p-2} + e^{p-2} = 1 - e^{p-2}$. As $p < 2$ this value is clearly positive.

Next, we consider the isoclines of $T$ given by $L_p$ and $L_q$. These, along with the line $y = \frac{y^*}{x^*} x$, divide the plane into six regions $H_n$: See Figure 8. The order of lines
shown in the Figure is typical for all parameters considered; the y intercept of \( L_q \) is \((0, q)\) while the y intercept of \( L_p \) is \((0, p/a)\), and \(aq < p\) by assumption.

Lemma 5.3. \( I_p \cap L_p = \{(x^*, y^*)\}, (p, 0)\) and \( I_q \cap L_q = \{(x^*, y^*), (q, 0)\}\)

Proof of Lemma 5.3 The points \((x, y) \in L_p\) satisfy \(p - x = ay\) so that the points in \( L_p \cap I_p\) satisfy

\[(p - x - ay) + p - x = axe^{-bx - y} = 0\]

so that

\[ay(1 - e^{-bx - y}) = 0\]

Either \(y = 0\) so that \(x = p\) or \(q - bx - y = 0\) which combined with \(p - x = ay\) gives \((x, y) = (x^*, y^*)\). The proof for \( I_q \cap L_q \) is similar.

Lemma 5.4. Let \( p, q \in (0, 2), a, b \in (0, 1)\) and let \( p - aq > 0, q - bp > 0\). Then the isocline \( I_p \in H_4\) for \(0 < y < y^*\) and \( I_p \in H_3\) for \(y > y^*\) and the isocline \( I_q \in H_1\) for \(0 < x < x^*\) and \( I_q \in H_6\) for \(x > x^*\).

Proof of Lemma 5.4 We compute \(\frac{dx}{dy}\) of \( I_p \) at the point \((p, 0)\). Using implicit differentiation, we obtain that \(\frac{dx}{dy}(p, 0) = -\frac{1 + e^{-bp} - p}{2 - p}\). As \(q - bp > 0\), we have that \(\frac{dx}{dy}(p, 0) < -a\) so that \( I_p \) begins beneath \( L_p \). As the curve only intersects \( L_p \) and the line \( y = \frac{p}{a}x \) at \((x^*, y^*)\) by the two preceding Lemmas, we know it must remain in the region \( H_4 \) until this point. To show that \( I_p \) lies in \( H_3 \) for \( y > y^*\), it suffices to show that \(\frac{dx}{dy}(x^*, y^*) < \frac{x^*}{y^*}\) and \(\frac{dx}{dy}(x^*, y^*) > -a\). Again using implicit differentiation, we find that \(\frac{dx}{dy}(x^*, y^*) = -a - \frac{2 - x^* - y^*}{2 - x^* - aby^*}\). We first claim that the denominator is always positive. We have that \(2 - x^* - aby^* = 2 - x^* - ay^* + ay^* - aby^* = 2 - p + ay^*(1 - b) > 0\) since \(p < 2\) and \(b < 1\).

We now show first that \(\frac{dx}{dy}(x^*, y^*) < \frac{x^*}{y^*}\). Cross multiplying this inequality using our value from the implicit differentiation, we obtain

\[-(2 - x^* - y^*)ay^* < 2x^* - (x^*)^2 - aby^*x^*\]

Using \(-ay^* = x^* - p\), the inequality is equivalent to showing that

\[(2 - x^* - y^*)x^* - p(2 - x^* - y^*) < 2x^* - (x^*)^2 - aby^*x^*\]

This is equivalent to showing

\[p(2 - x^* - y^*) + (1 - ab)y^*x^* > 0\]

This clearly holds if \(2 - x^* - y^* \geq 0\). If \(2 - x^* - y^* < 0\), we note that \(p(2 - x^* - y^*) + (1 - ab)y^*x^* > 2(2 - x^* - y^*) + (1 - ab)y^*x^*\). That \(2(2 - x^* - y^*) + (1 - ab)y^*x^* > 0\) was proved in Lemma 3.1

Now we show that \(\frac{dx}{dy}(x^*, y^*) > -a\). From the implicit differentiation, we wish to show that \(-a - \frac{2 - x^* - y^*}{2 - x^* - aby^*} > -a\). This follows from noting that \(ab < 1\) and that the denominator is positive.

The proof for \( I_q \) is similar, with computations of \(\frac{dy}{dx}\) at the points \((0, q)\) and \((x^*, y^*)\). Similar to above, the resulting inequalities rely on the assumptions \(q < 2\) and \(a < 1\). □
Putting all of these Lemmas together, we can conclude there are no intersections of $I_p$ and $I_q$ except at the fixed point $(x^*, y^*)$. Hence we have proved

**Theorem 5.5.** Let $p, q \in (0, 2)$, $a, b \in (0, 1)$ and $p - aq > 0$, $q - bp > 0$. Then the 2D Ricker map $T$ has no period 2 points.

6. **Conclusion.** Our main points can be summarized as follows. We offer two improvements to a previous Theorem on global stability in the two-dimensional Ricker map. We showed one of the necessary conditions follows from the others, and we also extended the bounds on the parameters to which the Theorem applies. We outline a potential method of approach to tackle the issue of global stability in the cases where prior Theorems do not apply and support these approaches with numerical evidence. Finally, we provided some miscellaneous results including (i) when $p = q$ the line $L$ through $(0, 0)$ and the coexistence fixed point is invariant and the 2D Ricker map restricted to $L$ is conjugate to a 1D Ricker map and (ii) there are no period-2 points for $p, q \in (0, 2)$ and $a, b \in (0, 1)$.

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