EXACT CONTROLLABILITY RESULTS FOR A CLASS OF ABSTRACT NONLOCAL CAUCHY PROBLEM WITH IMPULSIVE CONDITIONS

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Abstract. This paper deals with exact controllability of a class of abstract nonlocal Cauchy problem with impulsive conditions in Banach spaces. By using Sadovskii fixed point theorem and Mönch fixed point theorem, exact controllability results are obtained without assuming the compactness and Lipschitz conditions for nonlocal functions. An example is given to illustrate the main results.

1. Introduction. Impulsive dynamical systems reveal the various evolutionary processes, including those in engineering, biology and population dynamics which undergo abrupt changes in their state between intervals of continuous evolutions. In many evolution processes, such as optimal control models in economics, stimulated neural networks, frequency-modulated systems and some motions of missiles or aircrafts are characterized by the behavior of impulsive dynamical systems. In recent years, the analysis of impulsive systems are increasing due to their impact both in the theory and applications. We refer the reader for more facts of impulsive systems to [1, 2, 3, 4, 10, 28, 29].

Control theory deals with the behavior of dynamical systems. It is one of the basic concepts in mathematical control theory. By using various fixed point theorems, controllability of differential systems in Banach spaces under the assumption of compactness and noncompactness of the operator in semigroups has been studied by many authors [6, 11, 15, 16, 17, 22, 25, 30, 32, 33, 34, 35, 36].

The concept of nonlocal conditions can be applied in physics with improved effect than the usual initial condition $x(0) = x_0$. Nonlocal condition was first initiated by Byszewski [8] and he investigated the existence and uniqueness of mild and classical solution of nonlocal Cauchy problems. In [9], he studied the existence and

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uniqueness of solutions of abstract functional differential equations with nonlocal conditions of the form
\[
\begin{cases}
  x'(t) = f(t, x(t), x(a(t))), & t \in J, \\
x(t_0) + \sum_{k=1}^{m} c_k x(t_k) = x_0,
\end{cases}
\]
where \( J = [t_0, t_0 + b] \), \( b > 0 \) is constant, \( t_0 < t_1 < \cdots < t_m < t_0 + b \), \( f : J \times X \times X \to X \) and \( a : J \to J \) are given functions, \( X \) is a Banach space, \( x_0 \in X \), \( c_k \in \mathbb{R} \), \( c_k \neq 0 \), \( k = 1, 2, \cdots, m \), \( m \in \mathbb{N} \). The author pointed out that if \( c_k \neq 0 \), then the results of [9] can be applied to kinematics to determine the local evolution \( t \to x(t) \) of a physical object for which we do not know the positions \( x(t_1), \cdots, x(t_m) \), but the nonlocal conditions in (1) hold. We refer the reader for more facts of nonlocal systems to [19, 20, 27].

Recently, Chen et al. [12] considered the existence and uniqueness of strong solutions for semilinear evolution equations with nonlocal conditions. Liang et al. [21] studied the controllability of fractional integrodifferential evolution equations with nonlocal conditions.

Motivated by the above works, this paper establishes sufficient conditions for the exact controllability results for abstract neutral impulsive differential evolution equations with nonlocal initial conditions in Banach spaces of the form
\[
\begin{cases}
  \frac{d}{dt} [x(t) - g(t, x(t))] = A(t)x(t) + f(t, x(t)) + (Bu)(t), & t \in J = [0, b], \\
x(0) = \sum_{k=1}^{m} c_k x(t_k), \\
\Delta x(t_i) = I_i(x(t_i))\delta(t - t_i), & i = 1, 2, \cdots, n,
\end{cases}
\]
where \( A(t) : D(A(t)) \subset X \to X \) generates \( C_0 \) semigroup \( T(t)(t \geq 0) \) in a Banach space \( X \); \( g, f : J \times X \to X \) and \( I_i : X \to X \), \( i = 1, 2, \cdots, n \) are appropriate functions; the points \( 0 < t_0 < t_1 < \cdots < t_m < b \) are given and the symbol \( \Delta x(t_i) \) represents the jump of the function \( x \) at \( t_i \), which is defined by \( \Delta x(t_i) = x(t_i^+) - x(t_i^-) \), where \( x(t_i^+) \), \( x(t_i^-) \) represent right and left hand limit of \( x(t) \) at \( t = t_i \) respectively; the control function \( u(\cdot) \) is considered in the space \( L^2(J, V) \), where \( V \) is a Banach space of control and \( B : V \to X \) is a bounded linear operator.

We divided our work as follows. In the following section, we first introduce some notations and preliminaries which are used throughout this paper. In Section 3, exact controllability of abstract nonlocal Cauchy problem with impulsive conditions is established. In the last section, an example is given to demonstrate the application of the main results.

2. Preliminaries. In this section, we recall some definitions, notations and results that we need in this paper. Throughout this paper, \( (X, \| \cdot \|) \) is a Banach space. We denote \( C([0, b], X) \) the space of all \( X \)-valued functions on \( [0, b] \) with norm \( \| x \| = \sup \{ \| x(t) \| : t \in [0, b] \} \). A function \( u : [\sigma, \tau] \to X \) is a normalized piecewise continuous function on \( [\sigma, \tau] \). \( PC([\sigma, \tau]; X) \) denotes the space of normalized piecewise continuous functions from \( [\sigma, \tau] \) into \( X \). In particular, we denote the space \( PC \) formed by all function \( u : [0, b] \to X \) such that \( u \) is continuous at \( t \neq t_i \), \( u(t_i^+) = u(t_i^-) \) and \( u(t_i^+) \) exist, for all \( i = 1, 2, \cdots, n \). It is easy to see that \( PC \) is a Banach space with the norm \( \| x \|_{PC} = \sup_{s \in [0, b]} \| x(s) \| \).

Let \( \{ A(t) : t \in J \} \) generates an evolution operator and let us assume the following hypotheses:
(A1) The domain $D(A(t))$ of $A(t)$ is dense in $X$ and independent of $t$.

(A2) For each $t \in J$, the resolvent $R(\lambda : A(t))$ of $A(t)$ exists for all $\lambda$ with $\text{Re}\lambda \leq 0$ and there exists a constant $M > 0$ such that $\|R(\lambda : A(t))\| \leq M(1 + |\lambda|)^{-1}$.

(A3) There exist constants $K > 0$ and $0 < \mu \leq 1$ such that $\|A(t)A(s)A^{-1}(\tau)\| \leq (t - s)^\mu$ for $t, s, \tau \in J$.

(A4) For each $t \in J$ and some $\lambda \in \rho(A(t))$, $R(\lambda, A(t))$ is a compact operator.

Under the assumptions (A1) – (A4), the family $\{A(t) : t \in J\}$ generates a unique evolution system $\{U(t, s) : 0 \leq s \leq t \leq b\}$ satisfying:

(a) There exists a positive constant $M$ such that $\|U(t, s)\| \leq M$ for $0 \leq s \leq t \leq b$.

(b) For $0 \leq s \leq t \leq b$, $U(t, s) : X \to D$ and $t \to U(t, s)$ is strongly differentiable in $X$. The derivative $\frac{\partial}{\partial t} U(t, s) \in L(X)$ and it is strongly continuous $0 \leq s \leq t \leq b$. Moreover

$$\frac{\partial}{\partial t} U(t, s) + A(t) U(t, s) = 0 \text{ for } 0 \leq s \leq t \leq b,$$

and

$$\|\frac{\partial}{\partial t} U(t, s)\| = \|A(t) U(t, s)\| \leq \frac{M}{t - s},$$

and

$$\|A(t) U(t, s) A^{-1}(s)\| \leq M \text{ for } 0 \leq s \leq t \leq b.$$

(c) For every $v \in D(A(t))$ and $t \in J$, $U(t, s)v$ is differential with respect to $s$ on $0 \leq s \leq t \leq b$.

$$\frac{\partial}{\partial s} U(t, s)v = U(t, s)A(s)v.$$

**Definition 2.1.** A two parameter family of bounded linear operators $U(t, s)$, $0 \leq s \leq t \leq b$ on $X$ is called an evolution system if the following two conditions are satisfied:

(i) $U(s, s) = I$, $U(t, r)U(r, s) = U(t, s)$ for $0 \leq s \leq t \leq b$;

(ii) $(t, s) \to U(t, s)$ is strongly continuous for $0 \leq s \leq t \leq b$.

More details about evolution system can be found in Pazy [26].

**Lemma 2.2.** [5] Let $E^+$ be the positive cone of an order Banach space $(E, \leq)$. A function $\Phi$ defined on the set of all bounded subsets of the Banach space $X$ with values in $E^+$ is called a measure of noncompactness (MNC) on $X$ if $\beta(\overline{\sigma\Omega}) = \Phi(\Omega)$ for all bounded subsets $\Omega \subseteq X$, where $\overline{\sigma\Omega}$ stands for the closed convex hull of $\Omega$.

The MNC $\Phi$ is said to be,

(1) Monotone if for all bounded subsets $\Omega_1, \Omega_2$ of $X$ we have,

$$\Omega_1 \subseteq \Omega_2 \Rightarrow (\Phi(\Omega_1) \leq \Phi(\Omega_2));$$

(2) Nonsingular if $\Phi(\{a\} \cup \Omega) = \Phi(\Omega)$ for every $a \in X$, $\Omega \subset X$;

(3) Regular if $\Phi(\Omega) = 0$ if and only if $\Omega$ is relatively compact in $X$;

One of the examples of MNC is the noncompactness measure of Hausdorff $\beta$ defined on each bounded subset $\Omega$ of $X$ by $\beta(\Omega) = \inf\{\epsilon > 0 ; \Omega \text{ has a finite } \epsilon\text{-net in } X\}$. It is well known that MNC $\beta$ enjoys the above properties and other properties (see [5, 18]), for all bounded subset $\Omega$, $\Omega_1$, $\Omega_2$ of $X$.

(4) $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$, where $\Omega_1 + \Omega_2 = \{x + y : x \in \Omega_1, y \in \Omega_2\}$;

(5) $\beta(\Omega_1 \cup \Omega_2) \leq \max\{\beta(\Omega_1), \beta(\Omega_2)\}$;

(6) $\beta(\lambda \Omega) \leq |\lambda| \beta(\Omega)$, for any $\lambda \in \mathbb{R}$;

(7) If the map $F : D(F) \subseteq X \to Y$ is Lipschitz continuous with constant $k$, then $\beta_Y(F \Omega) \leq k \beta(\Omega)$ for any bounded subset $\Omega \subseteq D(F)$, where $Y$ is a Banach space.
Lemma 2.3. [13] Let $B \subseteq \mathcal{PC}$ be bounded and equicontinuous. Then $\beta(B(t))$ is continuous on $J$ and $\beta_{\mathcal{PC}}(B) = \max_{t \in J} \beta(B(t))$.

Lemma 2.4. [11] Let $B_0 = \{x_n\} \subseteq \mathcal{PC}$ be countable. If there exists $\psi(t) \in L([0,b], \mathbb{R}^+)$ such that $\|x_n(t)\| \leq \psi(t)$ a.e. $t \in J = [0,b]$, $n = 1, 2, \cdots, N$, then $\beta(B_0(t))$ is Lebesgue integrable on $J$ and

$$\beta\left(\int_J x_n(t)dt : n \in N\right) \leq 2 \int_J \beta(B_0(t))dt.$$

To prove the main results, for $h \in C(J, X)$, we first consider the evolution equation

$$\begin{aligned}
&x'(t) = A(t)x(t) + h(t), \ t \in J, \\
&x(0) = \sum_{k=1}^{m} c_k x(t_k). \\
\end{aligned} \quad (3)
$$

For the problem (3), let us assume:

Assumption $(H_0)$.

$$\left\| \sum_{k=1}^{m} c_k U(t_k, 0) \right\| \leq \sum_{k=1}^{m} |c_k| \|U(t_k, 0)\| < 1.$$

Lemma 2.5. Assume that the condition $(H_0)$ holds. Then (3) has a unique mild solution $x \in C(J, X)$ given by

$$x(t) = \sum_{k=1}^{m} c_k U(t, 0)P \int_0^{t_k} U(t_k, s)h(s)ds + \int_0^{t} U(t, s)h(s)ds,$$

where $P := \left(I - \sum_{k=1}^{m} c_k U(t_k, 0)\right)^{-1}$.

Proof. From $(H_0)$, we obtain

$$\left\| \sum_{k=1}^{m} c_k U(t_k, 0) \right\| \leq \sum_{k=1}^{m} |c_k| \|U(t_k, 0)\| < 1.$$

By operator spectrum theorem, the operator $P := \left(I - \sum_{k=1}^{m} c_k U(t_k, 0)\right)^{-1}$ exists and

$$||P|| = \sum_{n=0}^{\infty} \left\| \sum_{k=1}^{m} c_k U(t_k, 0) \right\|^n \leq \frac{1}{1 - M \sum_{k=1}^{m} |c_k|}. \quad (4)$$

\hfill \Box

Definition 2.6. A function $x := x(\cdot; u) \in \mathcal{PC}$ is said to be a mild solution of the system (2) if for any $u \in L^2(J, U)$, $x$ satisfies the following integral equation

$$\begin{aligned}
x(t) &= U(t, 0)P \sum_{k=1}^{m} c_k \left\{ g(t_k, x(t_k)) - U(t_k, 0)g(0, x(0)) \\
&+ \int_0^{t_k} U(t_k, s)[A(s)g(s, x(s)) \\
&+ f(x, x(s)) + Bu(s)]ds + \sum_{i=1}^{n} U(t_k, t_i)I_i(x(t_i))\delta(t - t_i) \right\} + g(t, x(t))
\end{aligned}$$
\[-U(t,0)g(0,x(0)) + \int_0^t U(t,s)[A(s)g(s,x(s)) + f(s,x(s)) + Bu(s)]ds \]
\[+ \sum_{i=1}^n U(t,t_i)I_i(x(t_i))\delta(t-t_i), \ t \in J.\]

**Definition 2.7.** (Exact controllability [7]) The system (2) is said to be controllable on the interval \( J \) if and only if for every \( x_0, x_1 \in X \), there exists a control \( u \in L^2(J,U) \) such that the mild solution \( x(t) \) of (2) satisfies \( x(0) = x_0 \) and \( x(b) = x_1 \).

**Remark 1.** There is a difficulty in considering exact controllable for functional differential equations since the value \( x(b) = x_T \) is often taken for that.

Consider the following linear control system

\[
\begin{align*}
x'(t) &= A(t)x(t) + Bu(t), \ t \in J, \\
x(0) &= x_0 \in X.
\end{align*}
\]

We introduce the controllability operator associated with linear control system (5) by

\[\Gamma^b_0 = \int_0^b U(b,s)BB^*U^*(b,s)ds,\]

where \( B^* \) and \( U^*(t,s) \) denote the adjoint of \( B \) and \( U(t,s) \) respectively. It is clear that \( \Gamma^b_0 \) is a linear bounded operator.

**Lemma 2.8.** [23] The linear control system (5) is exactly controllable on the interval \( J \) if and only if there exists a constant \( \mu > 0 \) such that \( \langle \Gamma^b_0 x, x \rangle \geq \mu \|x\|^2 \), \( \forall x \in X \). It follows that \( \| (\Gamma^b_0)^{-1} \| \leq \mu^{-1} \).

**Lemma 2.9.** (Sadovskii fixed point theorem) Let \( X \) be a Banach space and \( D \) be a nonempty, bounded, convex and closed subset in \( X \). If \( \Phi : D \to D \) is a condensing mapping, then \( \Phi \) has at least one fixed point in \( D \).

**Lemma 2.10.** ([24, Theorem 2.2]) Let \( \Omega \) be a closed convex subset of a Banach space \( X \) and \( 0 \in \Omega \). Assume that \( Q : \Omega \to \Omega \) is a continuous map, which satisfies Mönch’s condition, i.e., for \( D \subset \Omega \) is countable and \( D \subset \overline{m}(\{0\} \cup Q(D)) \to \mathcal{D} \) is compact. Then \( Q \) has at least one fixed point in \( \Omega \).

### 3. Exact controllability.
In order to establish the result, we need the following hypothesis:

**\( H_1 \)** \( A(t) \) is a family of linear operators, \( A(t) : D(A) \to X \) not depending on \( t \) and dense subset of \( X \), generating an equicontinuous evolution system \( \{U(t,s) : (t,s) \in J \times J\} \), i.e., \( (t,s) \to \{U(t,s)x : x \in E\} \) is equicontinuous for \( t > 0 \) and for all bounded subsets \( E \).

**\( H_2 \)** There exists a constant \( M_1 > 0 \) such that \( \|A(t)A(0)^{-1}\| \leq M_1 \) for \( t \in J \).

**\( H_3 \)** The function \( g : J \times X \to X \) is continuous and there exist constants \( L, L_1 > 0 \) such that

\[\|A(0)g(t,x(t))\| \leq L(\|x\| + 1),\]

for every \( (t,x) \in [0,b] \times X \), and the inequality

\[\|A(0)g(s_1,x_1) - A(0)g(s_2,x_2)\| \leq L_1(|s_1 - s_2| + \sup \|x_1 - x_2\|),\]

for every \( 0 \leq s_1, s_2 \leq b, \ x_1, x_2 \in X \).

**\( H_4 \)** The function \( f : J \times X \to X \) satisfies the following conditions:
Suppose that the hypotheses
\begin{equation}
(2)
\end{equation}
is exactly controllable on \( (J, \mathbb{R}^+) \) such that
\[
\sup_{||x|| \leq r} ||f(t, x(t))|| \leq f_r(t), \ t \in J, \ x \in X,
\]
and there exists a constant \( \gamma > 0 \) such that
\[
\lim_{r \to \infty} \inf_{t \in J} \frac{1}{r} \int_0^t f_r(s) ds \leq \gamma.
\]

\( (H_5) \) (i) The maps \( I_i : X \to X \) is continuous and there exists a nondecreasing continuous function \( l_i : [0, \infty) \to [0, \infty) \) such that \( ||I_i(x)|| \leq l_i(||x||), \) and
\[
\lim_{r \to \infty} \inf_{t \in J} \frac{l_i(r)}{r} = \lambda_i < \infty, \text{ where } \sum_{i=1}^n \lambda_i = \lambda.
\]

(ii) There exists a positive constant \( \gamma_i \) such that
\[
||I_i(x_1) - I_i(x_2)|| \delta(t - t_i) \leq \gamma_i ||x_1 - x_2||, \text{ for all } x_1, x_2 \in X.
\]

\( (H_6) \) The linear control system \((5)\) is exactly controllable.

\( (H_7) \) There is a function \( \eta \in L^1(J, \mathbb{R}^+) \) such that
\[
\beta(f(t, D)) \leq \eta(t) \beta(D), \ t \in J \text{ and any bounded subset } D \subset X.
\]

\( (H_8) \)
\[
K_1 = N \left( M_0 L_1 + M M_0 L_1 + M M_1 L_1 b + M \sum_{i=1}^m \gamma_i \right.
\]
\[
+ M^2 M_B^2 \mu^{-1} \left[ 1 + M_0 L_1 + M \sum_{k=1}^m |c_k| + M M_0 L_1 + M M_1 L_1 b + M \sum_{i=1}^m \gamma_i \right] \left. \right),
\]
\[
K_2 = 2 MN \eta \left( 1 + 2 M^2 M_B^2 \mu^{-1} b \right),
\]
\[
N = \frac{1}{1 - M \sum_{k=1}^m |c_k|}, \quad M_B = ||B||, \quad M_0 = \sup ||A(0)^{-1}|| \text{ and } \eta = \sup_{t \in J} \int_0^t \eta(s) ds,
\]
and \( K_1 + K_2 < 1. \)

**Theorem 3.1.** Suppose that the hypotheses \( (H_0)-(H_6) \) are satisfied. Then system \( (2) \) is exactly controllable on \( J \) provided that
\[
N \left( 1 + M^2 M_B^2 \mu^{-1} b \right) \left( M_0 L + M \left( M_0 L + M_1 L b + \gamma + \lambda \right) \right) < 1,
\]
and \( K_1 < 1. \)

**Proof.** Let \( B_r := \{ x \in \mathcal{PC} : ||x||_{\mathcal{PC}} \leq r \} \) for any \( r > 0. \) The \( B_r \) is is a bounded, closed and convex subset in \( \mathcal{PC}. \) For any \( x(\cdot) \in \mathcal{PC}, \) we introduce a control \( u(t) := u(t, x) \) by
\[
u(t; x) = B^* U^* (\Gamma_0) \left( x(b; u) - g(b, x(b)) - U(b, 0) \sum_{k=1}^m c_k x(t_k) - g(0, x(0)) \right)
\]
\[
- \int_0^b U(b, s) [A(s)g(s, x(s)) + f(s, x(s))] ds
\]
\[
+ \sum_{i=1}^n U(b, t_i) I_i(x(t_i)) \delta(t - t_i), \ t \in J.
\]
Define an operator \( \Phi : B_r \to B_r \) by
\[
(\Phi x)(t) = U(t, 0)P \sum_{k=1}^{m} c_k \left\{ g(t_k, x(t_k)) - U(t_k, 0)g(0, x(0)) + \int_0^{t_k} U(t_k, s) \left[ \begin{array}{c} A(s)g(s, x(s)) + f(s, x(s)) + Bu(s) \end{array} \right] ds + \sum_{i=1}^{n_i} U(t_k, t_i)I_i(x(t_i))\delta(t - t_i) \right\}
+ g(t, x(t)) - U(t, 0)g(0, x(0)) + \int_0^{t} U(t, s) \left[ A(s)g(s, x(s)) + f(s, x(s)) + Bu(s) \right] ds + \sum_{i=1}^{n} U(t(t_i))I_i(x(t_i))\delta(t - t_i).
\]

By \((H_2) - (H_4)\), we have
\[
\|u\|_{L^2} \leq MM_B\mu^{-1} \left\{ \|x(b; u)\| + M_0 L(1 + \|x\|) + M \left[ \sum_{k=1}^{m} |c_k|\|x(t_k)\| + M_0 L(1 + \|x\|) \right] + MM_1 L(1 + \|x\|)b + M \int_0^{b} f_r(s)ds + M \sum_{k=1}^{m} \lambda_k \right\}. \tag{8}
\]

To prove the exact controllability of the control system in (2), we shall use the Lemma 2.9 to show that the operator \( \Phi \) has a fixed point in \( B_r \). For this, we divide the proof into two steps.

**Step 1.** \( \Phi : B_r \to B_r \) is continuous. First we prove that \( \Phi(B_r) \subseteq B_r \) for some \( r > 0 \). Suppose that, for each positive integer \( r \), there exists \( x \in B_r \) such that \( \|(\Phi x)(t)\| > r \) for some \( t \in J \).

\[
r < \|(\Phi x)(t)\|
< \left\| U(t, 0)P \sum_{k=1}^{m} c_k \left\{ g(t_k, x(t_k)) - U(t_k, 0)g(0, x(0)) + \int_0^{t_k} U(t_k, s) \left[ A(s)g(s, x(s)) + f(s, x(s)) + Bu(s) \right] ds + \sum_{i=1}^{n} U(t_k, t_i)I_i(x(t_i))\delta(t - t_i) \right\}
+ g(t, x(t)) - U(t, 0)g(0, x(0)) + \int_0^{t} U(t, s) \left[ A(s)g(s, x(s)) + f(s, x(s)) + Bu(s) \right] ds + \sum_{i=1}^{n} U(t(t_i))I_i(x(t_i))\delta(t - t_i) \right\}
\]

\[
< \frac{M}{1 - M} \sum_{k=1}^{m} |c_k| \left\{ M_0 L(1 + r) + MM_0 L(1 + r) + MM_1 L(1 + r)b \right\}
+ M \int_0^{b} f_r(s)ds + MM_B\|u\|b + M \sum_{i=1}^{n} l_i(r) \right\} + M_0 L(1 + r)
+ MM_0 L(1 + r) + MM_1 L(1 + r)b + M \int_0^{b} f_r(s)ds + MM_B\|u\|b + M \sum_{i=1}^{n} l_i(r).
\]

Dividing on both sides by \( r \) and taking limit as \( r \to \infty \) and by the hypotheses \((H_4) - (H_5)\) and the equation (8) \( 1 \leq N \left( 1 + M^2 M_B^{-1}b \right) \left( M_0 L + M(M_0 L + \)
where the operators $\Phi$.

Further more by $(7)$, we have

$$0 \leq \left( \Phi t \right) + b x_t \leq \left( \Phi t \right) + b x_t$$

which is a contradicts to $(6)$. Hence for some positive $r$, $\Phi(B_r) \subseteq B_r$.

Step 2. $\Phi : B_r \rightarrow B_r$ is a condensing mapping. We decompose $\Phi$ into $\Phi = \Phi_1 + \Phi_2$, where the operators $\Phi_1, \Phi_2$ are defined on $B_r$ as

$$(\Phi_1 x)(t) = U(t, 0) P \sum_{k=1}^{m} c_k \left\{ g(t_k, x(t_k)) - U(t_k, 0) g(0, x(0)) + \int_{0}^{t_k} U(t_k, s) \right\}$$

$$\times \left[ A(s) g(s, x(s)) + BB^* U^*(b, s)(\Gamma ^b_0)^{-1} \left( x(b; u) - g(b, x(b)) \right) \right.$$ 

$$- U(b, 0) \left( \sum_{k=1}^{m} c_k x(t_k) - g(0, x(0)) \right) - \int_{0}^{b} U(b, \tau) A(\tau) g(\tau, x(\tau)) d\tau$$

$$+ \sum_{i=1}^{n} \left[ U(b, t_i) I_i(x(t_i)) \delta(t - t_i) \right] ds + \sum_{i=1}^{n} \left[ U(t_k, t_i) I_i(x(t_i)) \delta(t - t_i) \right]$$

$$+ g(t, x(t)) - U(t, 0) g(0, x(0)) + \int_{0}^{t} U(t, s) \left[ A(s) g(s, x(s)) \right.$$ 

$$+ BB^* U^*(b, s)(\Gamma ^b_0)^{-1} \times \left( x(b; u) - g(b, x(b)) \right) - U(b, 0) \left( \sum_{k=1}^{m} c_k x(t_k) \right.$$ 

$$- g(0, x(0)) \right] - \int_{0}^{b} U(b, \tau) A(\tau) g(\tau, x(\tau)) d\tau$$

$$+ \sum_{i=1}^{n} \left[ U(b, t_i) I_i(x(t_i)) \delta(t - t_i) \right] ds + \sum_{i=1}^{n} \left[ U(t, t_i) I_i(x(t_i)) \delta(t - t_i) \right],$$

and

$$(\Phi_2 x)(t) = U(t, 0) P \sum_{k=1}^{m} c_k \left\{ \int_{0}^{t_k} U(t_k, s) \left[ f(s, x(s)) - BB^* U^*(b, s)(\Gamma ^b_0)^{-1} \right.$$ 

$$\times \int_{0}^{b} U(b, \tau) f(\tau, x(\tau)) d\tau \right] ds \right\} + \int_{0}^{t} U(t, s) \left[ f(s, x(s)) \right.$$ 

$$- BB^* U^*(b, s)(\Gamma ^b_0)^{-1} \int_{0}^{b} U(b, \tau) f(\tau, x(\tau)) d\tau \right] ds,$$

$0 \leq t \leq b$, we will verify that $\Phi_1$ is contraction and $\Phi_2$ is compact operator.

To prove $\Phi_1$ is contraction, let $x_1, x_2 \in B_r$. For each $t \in [0, b]$ and by $(H_2)$ and $(7)$, we have

$$\| (\Phi_1 x_1)(t) - (\Phi_1 x_2)(t) \|$$

$$\leq \left\| U(t, 0) P \sum_{k=1}^{m} c_k \left\{ \left( g(t_k, x_1(t_k)) - g(t_k, x_2(t_k)) \right) - U(t_k, 0) \left( g(0, x_1(0)) - g(0, x_2(0)) \right) \right. \right.$$ 

$$+ \left. \int_{0}^{t_k} U(t_k, s) \left[ A(s) g(s, x_1(s)) - g(s, x_2(s)) \right] + BB^* U^*(b, s)(\Gamma ^b_0)^{-1} \left( x_1(b; u) - x_2(b; u) \right) - \left( g(b, x(b)) \right) \right.$$ 

$$- g(b, x_2(b)) \right\|.$$
\[-U(b,0) \left[ \sum_{k=1}^{m} c_k (x_1(t_k) - x_2(t_k)) - (g(0, x_1(0)) - g(0, x_2(0))) \right] \]
\[-\int_0^b U(b, \tau) A(\tau) (g(\tau, x_1(\tau)) - g(\tau, x_2(\tau))) d\tau \]
\[+ \sum_{i=1}^{n} U(b, t_i) \{ I_i(x_1(t_i)) - I_i(x_2(t_i)) \} \delta(t-t_i) \] 
\[ds + \sum_{i=1}^{n} U(t, t_i) \{ I_i(x_1(t_i)) - I_i(x_2(t_i)) \} \delta(t-t_i) \] 
\[= U(t, 0)(g(0, x_1(0)) - g(0, x_2(0))) \]
\[+ \left( g(t, x_1(t)) - g(t, x_2(t)) \right) + \int_0^t U(t, s) \left[ A(s)(g(s, x_1(s)) - g(s, x_2(s))) \right] \]
\[+ B B^* U^*(b, s)(\Gamma_0^b)^{-1} \left( x_1(b; u) - x_2(b; u) \right) \]
\[- (g(b, x_1(b)) - g(b, x_2(b))) - U(b, 0) \left[ \sum_{k=1}^{m} c_k (x_1(t_k) - x_2(t_k)) \right] \]
\[= (g(0, x_1(0)) - g(0, x_2(0))) - \int_0^b U(b, \tau) A(\tau) (g(\tau, x_1(\tau)) - g(\tau, x_2(\tau))) d\tau \]
\[+ \sum_{i=1}^{n} U(b, t_i) \{ I_i(x_1(t_i)) - I_i(x_2(t_i)) \} \delta(t-t_i) \] 
\[ds + \sum_{i=1}^{n} U(t, t_i) \{ I_i(x_1(t_i)) - I_i(x_2(t_i)) \} \delta(t-t_i) \] 
\[\leq N \left( M_0 L_1 + MM_0 L_1 + MM_1 L_1 b + M \sum_{i=1}^{m} \gamma_i \right) \]
\[\times \left[ 1 + M_0 L_1 + M \sum_{k=1}^{m} |c_k| + MM_0 L_1 + MM_1 L_1 b + M \sum_{i=1}^{m} \gamma_i \right] \sup_{0 \leq t \leq b} \| x_1 - x_2 \|, \]
i.e., \[\| (\Phi_1 x_1)(t) - (\Phi_1 x_2)(t) \| \leq K_1 \| x_1 - x_2 \|, \]
which shows that \( \Phi_1 \) is contraction.

To prove \( \Phi_2 \) is compact, first we prove that \( \Phi_2 \) is continuous. By \((H_1) - (H_4)\) it is easy to prove that \( \Phi_2 \) is continuous. To prove that \( \Phi_2 \) is equicontinuous function on \( J \) for any \( 0 \leq \tau_1 \leq \tau_2 \leq b \) and \( x \in B_r \), denote

\[ T_1 = \left\| \left[ U(\tau_2, 0) - U(\tau_1, 0) \right] \left( \sum_{k=1}^{m} c_k \int_0^{t_k} U(t_k, s) \left[ f(s, x(s)) - BB^* U^*(b, t)(\Gamma_0^b)^{-1} \right] \right) \right\|, \]

\[ T_2 = \left\| \int_0^{\tau_2} \left[ U(\tau_2, s) - U(\tau_1, s) \right] \left[ f(s, x(s)) - BB^* U^*(b, t)(\Gamma_0^b)^{-1} \right] \right\|, \]
Then we have
\[ T_3 = \left\| \int_{\tau_1}^{\tau_2} U(\tau_2, s) \left[ f(s, x(s)) - BB^*U^*(b, t)(\Gamma_0^b)^{-1} \int_0^b U(b, \tau)f(\tau, x(\tau))d\tau \right] ds \right\| . \]

Then we have
\[ \| (\Phi_2 x)(\tau_2) - (\Phi_2 x)(\tau_1) \| \leq T_1 + T_2 + T_3. \]

By \((H_1)\), we can easily verify that \( T_j \to 0 (j = 1, 2, 3) \) as \( \tau_2 \to \tau_1 \).

Therefore \( \Phi_2 \) is equicontinuous.

It remains to prove that \( V(t) = \{ (\Phi_2 x)(t) : x \in B_r \} \) is relatively compact in \( X \), \( V(0) \) is relatively compact in \( X \). Let \( 0 < t \leq b \) be fixed, \( 0 < \epsilon < t \), for \( x \in B_r \), we define
\[
(\Phi_2, e^\epsilon)(t) = U(t, 0) P \sum_{k=1}^m c_k \left\{ \int_0^{t_k} U(t_k, s) \left[ f(s, x(s)) - BB^*U^*(b, t) \right. \right. \\
\left. \left. \times (t_k^b)^{-1} \int_0^b U(b, \tau)f(\tau, x(\tau))d\tau \right] ds \right\} + \int_0^{t-\epsilon} U(t, s) \left[ f(s, x(s)) \right. \\
\left. \left. - BB^*U^*(b, t)(t_0^b)^{-1} \int_0^b U(b, \tau)f(\tau, x(\tau))d\tau \right] ds \right.
\]
\[
= U(t, 0) P \sum_{k=1}^m c_k \left\{ \int_0^{t_k} U(t_k, s) \left[ f(s, x(s)) - BB^*U^*(b, t)(t_0^b)^{-1} \right. \right. \\
\left. \left. \times \int_0^b U(b, \tau)f(\tau, x(\tau))d\tau \right] ds \right\} + U(t, t - \epsilon) \int_0^{t-\epsilon} U(t - \epsilon, s) \left[ f(s, x(s)) \right. \\
\left. \left. - BB^*U^*(b, t)(t_0^b)^{-1} \int_0^b U(b, \tau)f(\tau, x(\tau))d\tau \right] ds. \right.
\]

Then from compactness of \( U(t, s) \), \( (t, s) > 0 \), we obtain that
\[
V_r(t) = \{ (\Phi_2, e^\epsilon)(t) : x \in B_r \},
\]
is relatively compact in \( X \) for every \( 0 < \epsilon < t \). Moreover \( x \in B_r \), we have
\[
\| (\Phi_2 x)(t) - (\Phi_2, e^\epsilon)(t) \| \leq \int_{t-\epsilon}^{t} U(t, s) \left[ f(s, x(s)) - BB^*U^*(b, t)(\Gamma_0^b)^{-1} \right. \right. \\
\left. \left. \times \int_0^b U(b, \tau)f(\tau, x(\tau))d\tau \right] ds \right.
\]
\[
\leq M \int_{t-\epsilon}^{t} \left[ f_r(s) + M^2M^2_B\mu^{-1} \right. \right. \\
\left. \left. \int_0^b f_r(\tau)d\tau \right] ds. \right.
\]

Therefore, there are relatively compact sets arbitrarily close to the set \( V(t) \) and hence \( V(t) \) is also relatively compact in \( X \). Thus by Arzela-Ascoli theorem \( \Phi_2 \) is compact operator. These arguments above enable us to conclude that \( \Phi = \Phi_1 + \Phi_2 \) is condense mapping on \( B_r \), and by Lemma 2.9 there exists a fixed point for \( \Phi \) on \( B_r \). Thus the system (2) is controllable on \( J \).

**Theorem 3.2.** Let the assumptions \((H_0)-(H_8)\) and \((6)\) are satisfied. Then system (2) is exactly controllable on \( J \).

**Proof.** In order to apply Lemma 2.10, we define the control \( u \) and the function \( \Phi \) as in Theorem 3.1. By step 1 of Theorem 3.1, \( \Phi \) is continuous. Next to prove that \( \Phi \) satisfies Mönch condition. Let \( D \subset B_r \) be countable and \( D \subset \overline{\text{co}}(\{0\} \cup \Phi(D)) \).
We will show that $D$ is relatively compact. From the properties of MNC $\beta$, it is enough to prove that $\beta(D) = 0$.

First, we prove that $\Phi(D)$ is equicontinuous on $J$. For any $0 \leq \tau_1 \leq \tau_2 \leq b$ and $x \in B_r$, denote

$$N_1 = \left\| U(\tau_2, 0) - U(\tau_1, 0) \right\| P \sum_{k=1}^{m} c_k \left\{ \int_{0}^{t_k} U(t_k, s) \left[ A(s)g(s, x(s)) + f(s, x(s)) + Bu(s; x) \right] ds + \sum_{i=1}^{n} U(t_k, t_i)I_i(x(t_i))\delta(t - t_i) + g(t_k, x(t_k)) \right\} - U(t_k, 0)g(0, x(0)) \right\|,$$

$$N_2 = \left\| \int_{0}^{\tau_1} U(\tau_1, s) - U(\tau_2, s) \left[ A(s)g(s, x(s)) + f(s, x(s)) + Bu(s; x) \right] ds \right\|,$$

$$N_3 = \left\| \int_{\tau_1}^{\tau_2} U(\tau_2, s) \left[ A(s)g(s, x(s)) + f(s, x(s)) + Bu(s; x) \right] ds \right\|,$$

$$N_4 = \left\| U(\tau_1, s) - U(\tau_2, s) \right\| g(0, x(0)) \right\|,$$

$$N_5 = M_0 \left\| A(s)g(\tau_2, x(\tau_2)) - A(s)g(\tau_1, x(\tau_1)) \right\|,$$

$$N_6 = \left\| \sum_{i=1}^{n} U(\tau_1, s) - U(\tau_2, s) \right\| I_i(x(t_i))\delta(t - t_i) \right\|.$$

Then we have

$$\|\Phi_x(\tau_2) - (\Phi x)(\tau_1)\| \leq N_1 + N_2 + N_3 + N_4 + N_5 + N_6.$$

By (H1) and (H2), we can easily verify that $N_j \to 0$ ($j = 1, 2, \cdots, 6$) as $\tau_2 \to \tau_1$.

Therefore $\Phi$ is equicontinuous.

To estimate $\beta(\Phi(D))$, we decompose $\Phi$ as in Theorem 3.1.

Without loss of generality, we may suppose that $D = \{x_n\}_{n=1}^{\infty}$.

By (H6), we have

$$\beta(\Phi_2(D(t))(t)) = \beta \left( \{\Phi_2(x_n(t))\}_{n=1}^{\infty} \right)$$

$$\leq \beta \left( \left\{ U(t, 0)P \sum_{k=1}^{m} c_k \left\{ \int_{0}^{t_k} U(t_k, s) \left[ f(s, x_n(t)) - BB^*U^*(b, s)(\Gamma_0^b)^{-1} \right] ds + \int_{0}^{t} U(t, s) \left[ f(s, x_n(t)) - BB^*U^*(b, s)(\Gamma_0^b)^{-1} \right] ds \right\}_{n=1}^{\infty} \right\} \right)$$

$$\leq \frac{M}{1 - \sum_{k=1}^{m} |a_k|} 2M \int_{0}^{t_k} \beta \left( \left\{ f(s, D(s)) - BB^*U^*(b, s)(\Gamma_0^b)^{-1} \right\} ds + 2M \int_{0}^{t} \beta \left( \left\{ f(s, D(s)) - BB^*U^*(b, s)(\Gamma_0^b)^{-1} \right\} ds \right) \right).$$

(9)
Since \( \Phi_2D \) is equicontinuous on every \( J_i \), by proposition 7.3 of [14], we have

\[
\beta(\Phi_2D) = \max_{0 \leq i \leq s} \max_{t \in J_i} \beta((\Phi_2D)(t)) = \max_{0 \leq i \leq s} \max_{t \in J_i} \beta((\Phi_2D)(t)).
\]

Thus the system is controllable on \( J \) by Lemma 2.10, \( \Phi \) has at least one fixed point which is the mild solution of the system \( (2) \).

From Step 2 of Theorem 3.1,

\[
\|((\Phi_1x_1)(t) - (\Phi_1x_2)(t))\| \leq K_1\|x_1 - x_2\|,
\]

i.e., \( \Phi_1 \) is Lipschitz continuous with Lipschitz constant \( K_1 \).

Hence, by the property (7) of Lemma 2.2

\[
\beta(\Phi(D)) = \beta(\Phi_1(D)) + \beta(\Phi_2(D)) \leq (K_1 + K_2)\beta(D).
\]

Therefore

\[
\beta(\Phi(D)) \leq \beta(\sigma(0 \cup \Phi(D))) \leq \beta(\Phi(D)) \leq (K_1 + K_2)\beta(D).
\]

Since \( (K_1 + K_2) < 1 \), we obtain \( \beta(D) = 0 \). That is, \( D \) is relatively compact. Hence by Lemma 2.10, \( \Phi \) has at least one fixed point which is the mild solution of the system (2). Thus the system is controllable on \( J \). \( \square \)

4. Example. As an application of Theorem 3.1 we study the following impulsive partial functional differential system with nonlocal conditions

\[
\frac{\partial}{\partial t} \left[ z(t,x) - \int_0^\pi b(t,y,x) \left[ z(\sin t, y) + \frac{\partial z}{\partial y}(\sin t, y) \right] dy \right] + \frac{\partial^2 z(t,x)}{\partial x^2} + u(t,y) + h \left( t, z(\sin t, x), \frac{\partial z}{\partial x}(\sin t, x) \right), \ 0 \leq t \leq 1, \ 0 \leq x \leq \pi, \ t \neq t_k, \ k = 0, 1, 2, \ldots, m.
\]

\[
z(t,0) = z(t, \pi) = 0, \ 0 \leq t \leq 1,
\]

\[
z(t_k^+) - z(t_k^-) = I_k(z(t_k)), \ k = 0, 1, 2, \ldots, m
\]

\[
z(0, x) + \sum_{i=0}^p \int_0^\pi k_i(y,x)z(s_i, y)dy = z_0(x), \ 0 \leq x \leq \pi.
\]

where \( p \) is a positive integer, \( 0 < s_0 < s_1 < \cdots < s_p < 1 \), and \( 0 < t_1 < t_2 < \cdots < t_m < \cdots < 1 \); \( z_0(x) \in X = L^2([0, \pi]) \) is defined by \( Aw = w'' \) with the domain

\[
D(A) = H_0^2([0, \pi])
\]

\[
= \{ w(\cdot) \in X : w, w' \text{ are absolutely continuous, } w'' \in X, \ w(0) = w(\pi) = 0 \}.
\]

Then \( A \) generates a strongly continuous semigroup \( T(\cdot) \) which is compact, analytic and self-adjoint.

(a) Also \( A \) has a discrete spectrum representation

\[
Aw_n = \sum_{n=1}^\infty (-n^2) \langle w_n, w \rangle w_n, \ w \in D(A), n \in N;
\]
where \( w_n(x) = \frac{\sqrt{2}}{2} \sin(nx); n = 1, 2, \cdots \) is the orthogonal set of eigenvector of \( A \). The eigen values are \(-n^2, n \in \mathbb{N}\).

(b) The operator \( A^{1/2} \) is given by \( A^{1/2}w = \sum_{n=0}^{\infty} n\langle w, w_n \rangle w_n \) on the space

\[
D(A^{1/2}) = \left\{ w(\cdot) \in X; \sum_{n=1}^{\infty} nn < w, \ w_n < w_n \in X \right\}.
\]

The control operator \( B : L^2(J, X) \rightarrow X \) is defined by \( (Bu)(t)(y) = z(t, y); y \in (0, \pi) \) which satisfies condition \((H_0)\). Here \( B \) is an identity operator and the control function \( u(\cdot) \) is given in \( L^2([0, \pi], U)\).

We assume that the following conditions hold:

(i) The function \( b \) is measurable and \( \sup_{0 \leq t \leq 1} \int_0^\pi \int_0^\pi b^2(t, y, x) dy dx < \infty. \)

(ii) The function \( \frac{\partial b(t, y, x)}{\partial x^2} \) is measurable, \( b(t, y, 0) = b(t, y, \pi) = 0 \) and

\[
n_1 = \sup_{0 \leq t \leq 1} \left[ \int_0^\pi \int_0^\pi \left( \frac{\partial b(t, y, x)}{\partial x^2} \right)^2 dy dx \right]^{1/2} < \infty.
\]

(iii) For the function \( h : [0, 1] \times R \times R \rightarrow R \) the following three conditions are satisfied:

1. For each \( t \in [0, 1], h(t, \cdot, \cdot) \) is continuous.
2. For each \( z \in X_{1/2}, h(\cdot, z, z') \) is measurable.
3. There is a positive number \( c_1 \) such that

\[
\|g(t, z, z')\| \leq C_1 \|Z\|, \text{ for all } (t, z) \in [0, 1] \times X_{1/2}.
\]

(iv) \( I_k \in C(X_{1/2}, X_{1/2}), k = 1, 2, \cdots, m \) and there exists constants \( d_k, k = 1, 2, \cdots, m \), such that

\[
\|I_k(z)\|_{1/2} \leq d_k, z \in X_{1/2}.
\]

Here we choose \( \alpha = \beta = \frac{1}{2} \). According to paper \([31]\) we know that, if \( z \in X_{1/2}, \) then \( z \) is absolutely continuous, \( z' \in X \) and \( z(0) = z(\pi) = 0 \). In view of this result, for \( (t, z) \in [0, 1] \times X_{1/2}, w \in X \), we can define respectively that

\[
F(t, z)(x) = \int_0^\pi b(t, y, x)[z(y) + z'(y)] dy,
\]

\[
G(t, z)(x) = h(t, z(x), z'(x)), \text{ and}
\]

\[
g(w(t)) = \sum_{i=0}^p K_i w_1(s_i), \ w \in \Omega,
\]

where \( K_i : X_{1/2} \rightarrow X_{1/2} \) is completely continuous such that \( K_i(z)(x) = \int_0^\pi k_i(y, x) [z(y)] dy \) and \( G : [0, 1] \times X_{1/2} \rightarrow X \). It is easy to see that \( F : [0, 1] \times X_{1/2} \rightarrow X_{1/2}, \)

\( A^{1/2}F : [0, 1] \times X_{1/2} \rightarrow X_{1/2} \). In fact for each \( t \in [0, 1], \) we have

\[
\langle F(t, z), w_n \rangle = \frac{1}{n} \sqrt{\frac{2}{\pi}} \left\langle \int_0^\pi \frac{\partial b(t, y, x)}{\partial x}[z(y) + z'(y)] dy, \cos(nx) \right\rangle,
\]

also

\[
\langle F(t, z), w_n \rangle = -\frac{1}{n^2} \sqrt{\frac{2}{\pi}} \left\langle \int_0^\pi \frac{\partial^2 b(t, y, x)}{\partial x^2}[z(y) + z'(y)] dy, \sin(nx) \right\rangle.
\]
This shows that $F$ and $A^{1/2}F$ both take values in $X_{1/2}$ in terms of properties $(a)$ and $(b)$ and therefore the function $g$. Since, for any $x_1, x_2 \in X_{1/2}$

$$\|x_2 - x_1\|^2 = \sum_{n=0}^{\infty} (x_2 - x_1, z_n)^2$$

$$\leq \sum_{n=0}^{\infty} n^2 (x_2 - x_1, z_n)^2$$

$$\leq \|x_2 - x_1\|_{1/2}^2.$$

This inequality along with condition (ii) says that $(H_3)$ is satisfied. Also $f$ satisfies $(H_4)$ and $g$ satisfies $(H_3)$. Therefore by (i), $F(t, z)$ is bounded linear operator on $X$. Thus $(H_1), (H_2), (H_3), (H_4), (H_5)$ are satisfied and the system (11) is exactly controllable on $[0, 1]$.

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