Ground State Instabilities and Entanglement in the Spin-Boson Model

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Ground state instabilities of the spin-boson model is studied in this work. The existence of sequential ground state instabilities is shown analytically for arbitrary detuning in the two-spin system. In this model, extra discontinuities of concurrence(entanglement measure) are found in the finite system, which do not appear in the on-resonant model. The above results remain intact by including extra boson modes. Moreover, by including extra modes, it is found that ground state entanglement can be obtained and enhanced even in the weak coupling regime.

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Entanglement has been recognized as the essential element of quantum information science. This is due to the fact that the nonlocal quantum coherent nature of entanglement can be used as a resource implementing quantum information protocols. Recently, the concept of entanglement has also been introduced to the investigation of quantum phase transition(QPT) which is induced by quantum fluctuations and therefore can occur even at zero temperature. More precisely, QPT can be identified as the appearance of non-analyticity in the ground state energy. For the case of spin-boson model treated in this work, a phenomenon closes to QPT known as Ground State Instabilities(GSI) also occurs in finite system. One will see that non-analyticity arises due to level crossing which indicates the instability of ground state. Similar problems of QPT in the weak coupling and thermodynamical limit have been studied by many authors. In addition, the related problem of entanglement in the so-called Dicke model(DM) has also attracted much attention recently. One of the interesting results of the Dicke model is the sequence of GSI in arbitrary finite-atom system which has been overlooked in the thermodynamical limit. More interestingly, at these infinite sequential instabilities, Bužek et al. show that there are corresponding discontinuities appearing in the ground-state entanglement of the reduced atomic system. However, it has been pointed out by K. Rzazewski and K. Wódkiewicz that in gauge invariance is spoiled by including the $A^2$ term of the minimal coupling hamiltonian. Furthermore, they also pointed out that without the $A^2$ contribution the hamiltonian is unbounded from below as the coupling goes to infinity. It is obvious that any two-level atomic system is isomorphic to a spin-1/2 system. Therefore the Dicke model can be identified as a spin-boson interacting system. Certainly for the spin-boson system, there is no requirement of gauge invariance. Furthermore, by keeping finite coupling it seems that the unbounded problem can be avoided. However, by requiring finite coupling, one can only allow the investigation of finite number of ground state transitions instead of the infinite transitions in $S$. Even though with such restriction, the correlation between entanglement and GSI can still be addressed. In order to understand the relation between ground state instability and concurrence, exact analytical results are needed for gaining insight. Here, we discuss the generic spin-boson model by introducing a parameter $r$ which is the detuning of the boson mode frequency. Moreover, for more realistic consideration, we also study the ground state instabilities in the multi-mode model. In this work, we rigorously show the existence of sequential ground state instabilities for arbitrary detuning in the two-spin system. In contrast to the resonant case where GIS and concurrence are strongly correlated, such detuning effect leads to the disconnection of ground state instability and concurrence. By including extra modes and tuning the frequency, it is found that the ground state can become entangled even in the weak coupling regime and the entanglement is enhanced comparing with the mono-mode on-resonant spin-boson model(SBM, from now on, the term SBM denotes the mono-mode on-resonant model). The plan of the paper starts by introducing the single-mode spin-boson model, and the exact spectrum is then presented for two-spin system. In the next section we will show that ground state instability is a generic phenomenon of spin-boson model. Section III provides the analysis of ground state entanglement by calculating the concurrence. In this section, we establish the fact that ground state instability is not directly correlated with the analyticity of concurrence. The study of ground state instabilities of the two-mode model is presented in section IV. By including extra modes, for two spins, it is found that there exists a region of detuning where enhanced ground state entanglement can be obtained even in the weak coupling regime. Finally, a brief summary is given in the last section.

I. THE SINGLE-MODE MODEL AND ITS SPECTRUM

To begin with, we discuss the general method to solve the $N$-spin model. The system is $N$ spins interacting
with a mono-mode boson field. The Hamiltonian of the total system in the interaction hamiltonian is given by 
\( H = \omega_0 J_z + \omega a^\dagger a + g J^+_a + g^* J^-a^\dagger \) (1)

where \( J_\alpha = \frac{1}{2} \sum_{j=1}^N \sigma^\alpha_j \), \( \alpha \) can either be \( \{+, -\} \) for raising and lowering operations or \( \{x, y, z\} \), \( a(a^\dagger) \) is the boson annihilation(creation) operator. \( \sigma^\alpha \) are the Pauli matrices. \( \omega_0 \) is the level spacing of the spin and \( \omega \) indicates the frequency of the boson mode. We have assumed that these spins couple to the boson mode with the same strength \( g \). This detuned spin-boson model (DSBM) can in principle be solved exactly [10]. In this work, we extend the method of Swain [11] to diagonalize DSBM. The Hamiltonian can be separated by \( H = H_0 + H_I \):

\[
H_0 = J_z + a^\dagger a \quad (2)
H_I = ra^\dagger a + \kappa J^+_a + \kappa^* J^-a^\dagger \quad (3)
\]

where \( r \equiv \omega/\omega_0 - 1 \) and \( \kappa \equiv g/\omega_0 \). The parameter \( r (-1 < r < \infty) \) is related to the detuning which is usually defined by \( \omega - \omega_0 \) in quantum optics. \( H_0 \) is the so-called excitation operator [2]. To obtain the spectrum, one uses the fact that \( \{H, H_0, H_I, J^2\} \) form a maximally compatible set, where \( J^2 = J^2_0 + J^2_1 + J^2_2 \). For a \( N \)-spin system, we focus on \( j = N/2 \) which is relevant to the ground state discussions. Due to the commutative \( J^2 \), the matrix of \( H \) is automatically block diagonal by each \( j \) in the basis of \( H_0 \) which is denoted by \( \{j, m\}_A|n\}_p \). \( |j, m\}_A \) and \( |n\}_p \) are the spin states and photon number states respectively. \( \{\lambda = m + n\} \) are the eigenvalues of \( \{j, m\}_A|n\}_p \). It is noted that by excluding \( ra^\dagger a \) our \( H_0 = J_z + a^\dagger a \) is a parameter free operator and so does its eigenvalues. This approach helps to ease the counting of degeneracy of \( H_0 \). Hence, for fixed \( \lambda \), there exists degenerate subspace such that the diagonalization of \( H \) reduces to diagonalize finite matrices of \( H_I \). The eigenstates of \( H \) are denoted by \( |j, \lambda, h\}_ \). Then

\[
H|j, \lambda, h\}_ = E_{\lambda h}|j, \lambda, h\}_ \quad (4a)
H_I|j, \lambda, h\}_ = h|j, \lambda, h\}_ \quad (4b)
|j, \lambda, h\}_ = \sum_{i=\lambda-j}^{\lambda+j} A^{j, \lambda, h}_i |j, \lambda-i\}_A |i\}_p \quad (4c)
E_{\lambda h} = \lambda + h. \quad (4d)
\]

Note that, with arbitrary detuning, the energy eigenvalues depend not only on \( \kappa \), but also on \( r \) and one should expect some new results due to these parameters dependence. The detail form of \( H_I \) for arbitrary \( N \) are given in Appendix A. Since the dimension of \( H_I \) becomes bigger as the excitation number increases, most spectrum can only be obtained numerically for \( N \geq 3 \). However, for two-spin, the full spectrum with \( j = 1 \) can be obtained (we further neglect the index \( h \) since only the eigenvalue of \( H_I \) which is a decreasing function of \( \kappa \) is needed for fixed \( \lambda \)):

\[
E_1 = -1 \quad (5a)
E_0 = \frac{1}{2}(r - \sqrt{8|\kappa|^2 + r^2}) \quad (5b)
E_\lambda = \lambda + \lambda r - \frac{2}{3}\sqrt{3\kappa^2 \cos \frac{\pi}{3} - \frac{\varphi_\lambda}{3}} \quad (5c)
\]

\[
\alpha_\lambda \equiv (4\lambda + 2)|\kappa|^2 + r^2
\varphi_\lambda \equiv \cos^{-1}\left(\frac{3\sqrt{\kappa^2 \pi}}{\sqrt{\alpha_\lambda}}\right)
\]

where \( \lambda \) runs from 1 to infinity.

II. SEQUENTIAL GROUND STATE INSTABILITIES IN DSBM

Usually, the eigenenergies of a quantum system are analytic functions of the coupling constant \( \kappa \). However, there is a possibility that when \( H(\kappa) = H_0 + \kappa H_I \) and \( [H_0, H_I] = 0 \) such that \( H_0 \) and \( H_I \) can be simultaneously diagonalized and therefore the eigenfunctions are independent of \( \kappa \) even though the eigenvalues vary linearly with \( \kappa \). As a result, when one of the excited state is crossing with the ground state at some critical value \( \kappa = \kappa_c \), non-analyticity appears in the ground state energy. Such level-crossing phenomenon is called ground state instability (GSI) which also happens in the system considered in this work. The level crossing of SBM can be illustrated easily from the eigenstates with 0 and 1 excitations. From Eq.(A3), it is clear that as \( \kappa < \sqrt{(1 + r)/N} \) the energy \( E_{\lambda+1} \) is less than \( E_{\lambda+1} \), so the ground state is the one with zero excitation. However, for \( \kappa > \sqrt{(1 + r)/N} \), the ground state is replaced by the one with 1 excitation since \( E_{\lambda+1} < E_{\lambda+1} \). At the critical value \( \kappa_{\lambda+1} = \sqrt{(1 + r)/N} \), the excitation number changes discontinuously from \( \lambda = -N \) to \( \lambda = -N + 1 \). Obviously, when \( r = 0, \kappa_{\lambda+1} \) reduces to the on-resonant result in Ref. [3]. Buzek et al. have shown numerically that the ground state energy is non-analytic and the level crossing occurs in sequence: \( \{E_1 \rightarrow E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \ldots\} \). Due to the fact that the spectrum for two-spin can be obtained in closed form, we can provide an analytic proof for these sequential ground state transitions for all \( r \) if the following conditions are satisfied: \( \lambda \geq 0, -1 < r < \infty \) and \( \lambda \geq 1 \): (i) \( \{E_\lambda\} \) are monotonic decreasing functions, except for \( \lambda = -1 \). (ii) For all \( \lambda, f(\kappa, r, \lambda) = E_{\lambda+1} - E_\lambda \), \( f(\kappa, r, \lambda) \) is a monotonic decreasing function with opposite signs at small and large \( \kappa \). With \( \kappa_{\lambda+1} \) denoted the value of level crossing which is determined by \( E_{\lambda+1} = E_\lambda \), we have \( \{E_{\lambda+2} > E_{\lambda+1}\}, \). Due to the absolute sign of \( \kappa \) in Eq.(5), one may choose \( \kappa \geq 0 \) without losing generality. The first condition guarantees the eigenenergies of different \( \lambda \) involved in the ground state level crossing at different coupling strength. The second condition ensures that there is only one crossing between...
$E_\lambda$ and $E_{\lambda+1}$. If $E_{\lambda+2} < E_\lambda$, then the crossing $\tilde{\kappa}_{\lambda+1}$ which is determined by the equation $E_{\lambda+1} = E_{\lambda+2}$ must be larger than $\tilde{\kappa}_\lambda$. Therefore, these three conditions together ensure GSI occur in sequence. The detail proof is given in Appendix B.

It is interesting to point out that by adjusting the detuning parameter $r$, it is possible to have GSI in the small coupling regime. We recall the fact that, for on-resonant states is $r > \sqrt{\omega_0}$, the concurrence in-\(\frac{-1}{2}\)dependence is due to the fact that the energy eigen-\(\frac{-1}{2}\)is larger than $\omega_0$, then one needs a strong coupling strength to obtain GSI. Let $\tilde{\kappa}_i^-$, $\tilde{\kappa}_i^0$ and $\tilde{\kappa}_i^+$ be the $i$th GSI critical couplings for $-1 < r < 0$, $r = 0$ and $r > 0$ respectively. The above discussion on $\tilde{\kappa}_i$ can also be extended to all other cases. One can deduce that for $-1 < r < 0$, $\tilde{\kappa}_i^- < \tilde{\kappa}_i^0$, while for $r > 0$, $\tilde{\kappa}_i^+ > \tilde{\kappa}_i^0$. These results are numerically shown in Fig.(1) and the proof of these general results is the content of Appendix C. By introducing detuned frequency, one might control the system ground state entanglement by GSI as shown in the next section.

### III. GSI V.S. ENTANGLEMENT IN DSBM: TWO-SPIN CASE

Due to the fact that spins are coupled to the boson field, such interaction induces quantum correlation among spins. Hence, the spin system (by tracing out all boson states) is in general entangled. For $2 \times 2$ bipartite system, to quantify entanglement, it has been proposed by Wootters$^{12}$, by using the concurrence of the system density matrix $\rho$, which is defined by $C(\rho) = Max\{0, \xi_1 - \xi_2 - \xi_3 - \xi_4\}$ where $\xi_i$ are the square root of the eigenvalues of spin flow matrix $R$ defined by $R = \rho(\sigma_y^A \otimes \sigma_y^B)\rho^* (\sigma_y^A \otimes \sigma_y^B)$, subtracting in decreasing order. One should note that, after partially tracing out the boson degree of freedom, the spin density matrix belongs to the class of the generalized Werner state$^{13}$ which is defined as

$$\rho_A = Tr_p\{|j, \lambda, h\rangle\langle j, \lambda, h|\} = \begin{pmatrix} \rho_{11} & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} \\ 0 & \rho_{32} & \rho_{33} \end{pmatrix}.$$

Due to the superposition of the triplet state with $j = 1$, one has $\rho_{22} = \rho_{33} = \rho_{23} = \rho_{32}. This density matrix has a simple formula for the concurrence $C(\rho_A) = 2Max\{\rho_{22} - \sqrt{\rho_{11}/4}\}$ which is determined by the competition of the populations between the entangled triplet and unentangled states. The form of this density matrix is invariant under time evolution$^{14}$ of $H$ given in Eq.(1). For $\kappa_1 \leq \kappa < \kappa_0$, the concurrence is

$$C_0 = \frac{(r + \sqrt{8\kappa^2 + r^2})^2}{2(8\kappa^2 + r(r + \sqrt{8\kappa^2 + r^2}))}.$$

For $\kappa_{\lambda-1} \leq \kappa < \kappa_\lambda$ with $\lambda \geq 1$,

$$C_\lambda = \frac{3\sqrt{3}\kappa^2(3(4\sqrt{1 + \lambda})\kappa^2 + \sqrt{3}r^2) - 4\sqrt{3}\alpha_\lambda(\sqrt{1 + \lambda} - \sqrt{\lambda})r\zeta_\lambda - 4\alpha_\lambda(2\sqrt{1 + \lambda} - \sqrt{\lambda})\zeta_\lambda^2 - 6\alpha_\lambda(2\sqrt{1 + \lambda} - \sqrt{\lambda})(\sqrt{r} + \sqrt{\alpha_\lambda\zeta_\lambda})}{9\kappa^2(2 + 2\lambda)(1 + \lambda)(\kappa^2 + \lambda r^2) - 12\sqrt{3}\alpha_\lambda\kappa^2 r\zeta_\lambda - 6\alpha_\lambda(2\lambda + \lambda)(\kappa^2 - r^2)\zeta_\lambda^2 + 8\alpha_\lambda^2 \zeta_\lambda^2(\sqrt{r} + \sqrt{\alpha_\lambda\zeta_\lambda})}$$

where $\zeta_\lambda = \cos\{\frac{\pi}{3} - \frac{\pi}{\lambda}\}$. One can easily check that, when $r = 0$, the concurrences become

$$C_\lambda(r = 0) = \frac{(\sqrt{1 + \lambda} - \sqrt{\lambda})^2}{2(1 + 2\lambda)}$$

which are positive and non-vanishing for all $\lambda$. Therefore, ground states for all $\lambda$ with $r = 0$ are entangled. However, it will be shown in below that this is not true for $r \neq 0$. Note that, for $r = 0$, the concurrence indeed has discontinuity whenever there is a ground state transition(quantum phase-like transition)$^{[8}$ for systems of finite number of spins(except for $N = 1$). Furthermore, in between GSI the concurrence is a constant, for example the concurrence for $\kappa_1 < \kappa < \kappa_0$ is $\frac{1}{2}$. The $\kappa$-independence is due to the fact that the energy eigenstates is $\kappa$-independent for $r = 0$. However, if two spins couple with an off-resonant mode, the characteristic of the concurrence is different from the on-resonant case. Since the eigenstates become $\kappa$-dependent, the concurrence is an explicit function of $\kappa$. Indeed this is clearly shown in Fig.(2) where the concurrence of the $r = 1$ case is plotted. In addition, it can be seen that for $\kappa_1 < \kappa < \kappa_0$, one has $C_0 > 0.5$ which indicates that the entanglement between spins are more enhanced than the on-resonant result. It is also shown in Fig.(2) that the concurrence is strongly suppressed for $C_{\lambda \geq 1}$. This is a general tendency which also holds for $r = 0$. Moreover, as shown in Table 1, $C_0$ becomes larger as $r$ increases. The suppression of $C_{\lambda \geq 1}$ and the enhancement of $C_0$ can be understood by considering the ground state eigenvectors. Up to a normalization constant, states which might become the ground state can be expressed as follows: we
FIG. 1: Regions of ground states with different $r$. The different shadow regions correspond to different ground states. For example, the darkest region denotes the system ground state as $|1\rangle$. The lowest curve is $\kappa_1 = \sqrt{1 + r^2}$ obtained by solving $E_0 = -1$.

FIG. 2: The concurrence with $r = 1$. Different shading regions correspond to different GSI. The maximum of entanglement appears in the interval $[\kappa_1, \kappa_0]$. However, for other GSI regions, the concurrence are strongly suppressed.

neglect the labelling of $j, h$ in the eigenkets):

$$
|0\rangle = a_0|0\rangle_p + |\bar{1}\rangle|1\rangle_p
$$

$$
a_0 = \frac{r + \sqrt{2r^2 + 8\kappa^2}}{2\sqrt{2\kappa}}
$$

$$
|\lambda\rangle = a_\lambda|1\rangle_p|\lambda - 1\rangle_p + b_\lambda|0\rangle_p|\bar{1}\rangle + |1\rangle_p|\lambda + 1\rangle_p
$$

$$
a_\lambda = \frac{\sqrt{1 + \lambda}}{\sqrt{\lambda}} + \frac{\sqrt{2\lambda + 3\alpha_\lambda \zeta_\lambda}}{2\sqrt{(1 + \lambda)\kappa^2}}(r + \frac{2}{3}\sqrt{3\alpha_\lambda \zeta_\lambda})
$$

$$
b_\lambda = \frac{1}{\sqrt{2(1 + \lambda)\kappa}}(r + \frac{2}{3}\sqrt{3\alpha_\lambda \zeta_\lambda})
$$

where $\lambda \geq 1$. The expression of $a_0$ in Eq.(9) indicates the entangled triplet state has a large amplitude as $r$ increases and as a result enhanced entanglement arises for $C_0$. This is due to the fact that it needs a stronger coupling strength to achieve GSI for large detuning (See Fig.(1)), therefore, stronger correlation (entanglement) exists. Note that this fact is also consistent with the results of Appendix C which requires, for large $r$, strong coupling constant for the occurrence of GSI. The eigenkets for $\lambda \geq 1$ (See Eq.(10)) are different from the $\lambda = 0$ state (Eq.(9)) by having an extra term, namely $|1\rangle_p|\lambda - 1\rangle_p$. Consequently, the existence of such term is the source of diluting the entanglement of the system. In passing, observe that, in Eq.(7) the numerator vanishes by cancellation in the large $\lambda$ limit. It can also be understood by noting that, at large $\lambda$, the resulting eigenket becomes:

$$
|\lambda \to \infty\rangle = (|1\rangle_p + \sqrt{2}|0\rangle_p + |\bar{1}\rangle)|\lambda\rangle_p.
$$

Therefore, the spin state inside the curly bracket is a separable state which implies $C = 0$.

One important point should be addressed is the connection between GSI and concurrence. It has recently been discussed in the literatures whether concurrence is a good measure to quantify QPT. For the case with $r = 0$, it has been shown that the discontinuity of concurrence is indeed associated with GSI even for finite system. However, by analyzing the $r = 1$ case
TABLE I: The \( r \)-dependence of concurrence for \( N = 2 \).  

| \( r \) | \( C_0^a \) | \( C_1 \) | \( C_2 \) |
|---|---|---|---|
| -0.9 | 0.0977 | 0.0425 | 0.0327 |
| -0.5 | 0.3613 | 0.0626 | 0.0273 |
| 0 | 0.5 | 0.0286 | 0.0101 |
| 0.5 | 0.5691 | 0.0124 | 0.0040 |
| 1 | 0.6667 | 0.0035 | 0.0008 |
| 1.2 | 0.6875 | 0.0010 | 0 |
| 1.3 | 0.6970 | 0 | 0 |

\(^a\)Generally, \( C_1 \) is \( \kappa \)-dependent, the data here are only showing the maximum values for each GSI region. This footnote also applies to the other tables when it is appropriate.

TABLE II: The \( r \)-dependence of concurrence for \( N = 3 \).  

| \( r \) | \( C_0 \) | \( C_1 \) | \( C_2 \) | \( C_3 \) | \( C_4 \) |
|---|---|---|---|---|---|
| 6 | 0.5833 | 0.2944 | 0.0029 | 0.0007 | 0.0002 |
| 7.2 | 0.5942 | 0.3126 | 0.0017 | 0.0002 | 0 |
| 8 | 0.6 | 0.3233 | 0.0011 | 0 | 0 |
| 10 | 0.6111 | 0.3460 | 0 | 0 | 0 |

Due to the limited spin space, after the \( N \)th transition, all ground states do not change qualitatively and hence the ability of creating entanglement is restricted even in the strong coupling regime. Here, we emphasize that the 1-1 correspondence between GSI and the discontinuities of concurrence seems just a special result for finite \( N \) SBM (or DM) and can not be extended in DSBM. Consequently, these results clearly establish the fact that 1-1 correspondence between GSI and concurrence can not be true in general for finite \( N \) DSBM. It is certainly interesting to see if the 1-1 correspondence remains valid as \( N \to \infty \).

IV. GSI WITH 1+1 MODES

In the previous sections, ground state instabilities of DSBM and its correlation with concurrence have been treated. It is interesting to see if the conclusions still hold for the case of multi-mode spin-boson model. In this section the case of two-mode (1+1 mode) model will be analyze by adding one off-resonant mode to the resonant SBM. The Hamiltonian is

\[
H = H_0 + H_I
\]

\[
H_0 = J_z + a^\dagger a + b^\dagger b
\]

\[
H_I = \gamma \{b^\dagger b + \kappa_a J^+_a + \kappa_b J^-_b + \kappa_a J^+_a \kappa_b J^-_b \}
\]

where \( \{b, b^\dagger\} \) are annihilation and creation boson operator of the off-resonant mode with \( \kappa_b \) being the coupling constant. Similar to the single mode model, it is easy to check \( [H_0, H_I] = 0 \) and one can choose the eigenstates of \( H_0 \) to represent operator \( H_I \):

\[
H_0 |j, m, m_a, m_b\rangle = \lambda |j, m, m_a, m_b\rangle
\]

\[
H_I |\lambda, h\rangle = h |\lambda, h\rangle
\]

\[
H |\lambda, h\rangle = E_{\lambda h} |\lambda, h\rangle
\]

where \( \lambda = n_a + m_a + m_b + E_{\lambda h} \) is the eigenvalue of \( H_I \) for \( \lambda = -1 \) and \( 3(\lambda + 1) \) for \( \lambda \geq 0 \). Similarly, we will omit the labelling, \( j \) and \( h \), of the eigenkets in the following discussions. GSI of the 1+1 model with \( r = 1 \) is shown in Figs.(3). For example, by fixing \( \kappa_b = 0.4 \) the...
sequential GSI, namely \{E_1 \rightarrow E_0 \rightarrow E_1 \rightarrow E_2 \ldots \} are clearly shown in Fig.(3a) (The result of keeping \(\kappa_a\) fixed at 0.4 is shown in Fig. (3b)). However, as \(\kappa_b\) increases, the state \(|1\rangle = |1\rangle|0_a, 0_b\rangle_p\) may no longer be the ground state. Therefore the sequence of GSI does not have to begin from \(\lambda = -1\). This result can be seen in Fig.(3c) which shows the sequence of GSI with \(\kappa_b = 1.1\) as \(\{E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \ldots \}\). This fact just reflects the result of GSI in DSBM discussed in Sec.III. Moreover, if \(\kappa_b\) is further increased, more lower spin sector eigenstates get kicked out of the GSI sequence, this is shown in Fig.(3d) with \(\kappa_b = 1.8\).

It is also interesting to see how the detuning parameter is related to the pattern of GSI. By adding one extra off-resonant mode to SBM, the results are shown in Table III which also includes the results of SBM for comparison. To keep things simple, the results are evaluated with \(\kappa_a = \kappa_b\) without losing generality. One can see that the critical values (\(\kappa_i\)) for GSI are increasing function of \(r\) but being bounded by the results of SBM (See the first line of the table). Similarly, for \(r\) closed to \(-1\), \(\kappa_i\) also approach the results of SBM which can be determined explicitly from Fig.(1). These interesting results can also be understood from the energy spectrum. For example, consider \(\kappa_1\) which indicates the crossing of the eigen-energies of \(|1\rangle\) and \(|0\rangle\). When \(r\) is large, the off-resonant mode has higher energy and its excitation costs more energy. Thus, \(|0\rangle\) involves dominantly the lower energy boson which is the on-resonant mode. As a consequence, the determination of \(\kappa_1\) is governed by SBM. Physically, what is happening is the effect that is well-known in most physical systems, namely, the result of decoupling effect of far off-resonant driving. On the other hand, at the limit of \(r \rightarrow -1\), the off-resonant mode with lower frequency is dominating. Therefore, \(|0\rangle\) can have more contributions from the off-resonant photon and consequently the determination of \(\kappa_1\) is dictated by DSBM. It is important to point out that with extra mode, \(\kappa_i\) can be reduced significantly. For example, as \(r = 1\), \(\kappa_1 = 0.5774\) which is smaller than 0.7071 and 1 of the critical couplings \(\kappa_1\) with \(r = 0\) and \(r = 1\) respectively (See Fig.(1)). Since for \(\kappa > \kappa_1\) the ground state is entangled, it is certainly important to obtain GSI in the weak coupling regime. However, having GSI at lower critical coupling is not enough for practical reasons. One important requirement for employing entanglement in quantum information science is to have strong enough entanglement or maximally entangled state. Hence, it is necessary to see if adding extra mode can either enhance or suppress entanglement.

The \(r\)-dependent results of few \(C_i\)’s are tabulated in Table IV which also contains the results of SBM (the first row). For practical aspect, we only concentrate on \(C_0\) which has higher entanglement. One can see that the maximum value is around \(r = 1.8\). However, at larger \(r\), \(C_0\) is decreasing toward the value of the SBM result. This result is the same decoupling effect discussed previously and once again the on-resonant mode determines \(C_0\). Note that the maximal value of \(C_0\) is higher than the one in SBM and the determination of the value of \(r\) with the maximal concurrence is a balanced result of the competition between two modes. The same effect happens for other \(C_i\) in Table IV. One should note that the entanglement obtained in the weak coupling region is distillable [19] and can be enhanced comparing with the on-resonant result (\(C = 0.5\)). Therefore, one can have an “entanglement switch” by controlling the first ground state transition and it seems that having extra mode can do just that. All in all, this result seems to suggest that 1+1 mode might be functioning better than mono-mode models. It is important to justify the above conjecture by studying a system with three cavity modes which will be reported elsewhere.

Finally, for comparison, the results of both cavity modes being off-resonant with \(r_a = 1.2\) and \(r_b\) are given in Table V. The result is quite interesting since the value of \(C_0\) can be higher than the corresponding results \((r = 1.2\) with \(C_0 = 0.6875\)) of SBM in Table I. For example, when \(r_b = 1.5\) one has \(C_0 = 0.6998\) which is larger than 0.6875 given in Table I. Furthermore, as \(r_b\) becomes very large, one can see from this table that \(C_0\) approach the value in SBM. Again this is just the effect of decoupling. Further support of this fact is revealed the

![Fig. 3: The eigenenergy of \(\lambda = -1, 0, 1, 2, 3\) for \(r = 1\). (a) \(\kappa_b = 0.4\); (b) \(\kappa_a = 0.4\); (c) \(\kappa_b = 1.1\); (d) \(\kappa_b = 1.8\). The black arrows in (c) and (d) indicate the eigenstate with \(\lambda = 0\) and \(\lambda = 1\) respectively.](image-url)
values of $C_1$ and $C_2$ in Table V. Hence, for enhancing entanglement, it is likely that off-resonant multi-mode model is a better candidate and deserve further analysis. Similarly, the absence of discontinuity of the concurrence with GSI can also be found in the multi-mode system (See Table V). Moreover, we find that the extra discontinuities of concurrence also appear in two off-resonant modes of spin-boson model. These facts show more supports for disconnecting GSI and discontinuities of concurrence in spin-boson model.

| $r_0$  | $C_0$ | $C_1$ | $C_2$ |
|-------|-------|-------|-------|
| 1     | 0.6764| 0.0023| 0.0004|
| 1.1   | 0.6823| 0.0016| 0.0002|
| 1.3   | 0.6921| 0.0005| 0     |
| 1.5   | 0.6998| 0     | 0     |
| 100   | 0.6920| 0.0043| 0.0035|
| 10000 | 0.6875| 0.0011| 0     |

TABLE V: The results of 1+1 cavity modes with $r_a = 1.2$ and $\kappa_a = \kappa_b$.

V. CONCLUSION

In this work, we show explicitly that GSI and entanglement are not necessarily connected. This is shown by a rigorous analysis of the spin-boson model(SBM) with two spins. By knowing the full spectrum, it is shown that the sequential quantum phase transitions occur in this system and the closed form expression of concurrence is obtained. Employing these results we are able to clarify the relation between GSI and entanglement. Contrary to the results in literatures which are concentrated on spin-spin interacting systems, it is shown clearly that in the detuned spin-boson model, concurrence is not a good measure for quantifying GSI. This is shown by realizing that not all the discontinuity of concurrence are associated with the ones appearing in GSI and on the contrary the system having GSI can be corresponding to a continuous concurrence. Even though the above results are obtained for $N = 2$ system, we have also obtained numerical results for $N = 3$ which also support our conclusion. Furthermore, the 1+1 mode model is analyzed and interesting results are obtained. It is seen that the effects of extra mode are two folds. First of all, GSI can happen at weak critical couplings which are important for having entangled ground state. Secondly, the entangled ground state with extra off-resonant mode have a higher concurrence comparing to the results of SBM and DSBM. These results may be useful in the context of quantum information science. Moreover, we also confirmed that GSI and the discontinuities of concurrence remain uncorrelated even extra mode is included. There are several directions for further study along this work. It is interesting to obtain results for adding more modes to justify the effects of extra modes as obtained here. It is also necessary to analyzed the $N$ atoms case. Furthermore, results for more than two-level system are very important. For example, for 3-level systems, it is interesting to analyze either the $\Lambda$ system or V system to see if the results obtained in this work remain valid, since these systems are also quite common in atomic physics. These problems will be pursuit in the future.

APPENDIX A

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In this appendix, we provide the finite matrices of $H_I$ for given $\lambda$ with $j = \frac{N}{2}$. For $\lambda < \frac{N}{2}$, the general matrix form of $H_I$ for arbitrary $N$-atom system is: (with excitation number $\lambda = -\frac{N}{2} + \nu$ and $\nu < N$)

\[
\begin{pmatrix}
0 & \kappa\sqrt{R_{\nu}} & & & \\
\kappa\sqrt{R_{\nu}} & 0 & \kappa\sqrt{2R_{\nu-1}} & & \\
0 & \kappa\sqrt{2R_{\nu-1}} & 2r & \kappa\sqrt{3R_{\nu-2}} & \\
0 & & & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & & & & \kappa\sqrt{R_{\nu-k-1}} & kr & \kappa\sqrt{R_{\nu-k}} & \\
0 & & & & & \kappa\sqrt{R_{\nu-k}} & (\nu - 1)r & (\nu - 1)r \\
0 & & & & & & \kappa\sqrt{R_{\nu-k}} & \nu r \\
\end{pmatrix}
\]

(A1)

where $R_x \equiv x(N+1-x)$ and $R_x \leq 0 = 0$. The non-vanishing off-diagonal elements only exist next to the diagonal on each row. For $\lambda \geq \frac{N}{2}$, the dimension of the matrix is fixed which is $(N+1) \times (N+1)$. The general matrix form of
where \( \lambda = \frac{N}{2} + \nu \) and \( \nu \geq 0 \). For the first ground state transition, the critical coupling for arbitrary \( N \) and detuning \( r \) is

\[
\kappa_{\frac{N}{2}} = \sqrt{\frac{1+r}{N}} \tag{A3}
\]

which can be obtained by solving the equation of \( E_{\frac{N}{2}} = E_{\frac{N}{2}+1} \) where

\[
E_{\frac{N}{2}} = -\frac{N}{2} \quad \text{and} \quad E_{\frac{N}{2}+1} = -\frac{N}{2} + 1 + \frac{1}{2}(r - \sqrt{4N|\kappa|^2 + r^2}) \tag{A4a}
\]

\[
E_{\frac{N}{2}+1} = -\frac{N}{2} + 1 + \frac{1}{2}(r - \sqrt{4N|\kappa|^2 + r^2}) \tag{A4b}
\]

**APPENDIX B**

This appendix is to prove that not only the existence of GSI of two-atom system with arbitrary detuning but also these GSI occur in sequence. In order to prove these results, it is necessary to determine the ranges of \( \varphi_\lambda \) and \( \theta_\lambda \). Denoting \( x \equiv \left( \frac{r}{\tau} \right)^2 \) and recalling the definition of \( \varphi_\lambda \), one has:

\[
\varphi_\lambda = \cos^{-1}\left\{ \frac{\sqrt{3\kappa^2}r}{\sqrt{\alpha_\lambda^3}} \right\}
\]

\[
= \cos^{-1}\left\{ \frac{\text{sgn}(r)3\sqrt{x}}{\sqrt{1 + 6\tau_\lambda x + 12\tau_\lambda^2 x^2 + 8\tau_\lambda^3 x^3}} \right\} \tag{B1}
\]

where \( \tau_\lambda \equiv 1 + 2\lambda \) and \( \text{sgn}(r) \) is the sign function. Since \( 0 \leq \kappa < \infty \) and \( -1 < r < \infty \), one has \( 0 \leq x < \infty \). The ranges of \( \varphi_\lambda \) and \( \theta_\lambda \) are:

\[
\cos^{-1}\left( \frac{1}{\tau_\lambda} \right) \leq \varphi_\lambda \leq \cos^{-1}\left( \frac{1}{\tau_\lambda} \right)
\]

\[
\theta_\lambda^- \leq \theta_\lambda \leq \theta_\lambda^+
\]

\[
\theta_\lambda^\pm = \frac{1}{3}(\pi \pm \cos^{-1}\left( \frac{\pm 1}{\tau_\lambda} \right)). \tag{B4}
\]

Although the maximal value of \( \theta_\lambda \) is \( \lambda \) dependent, it is easy to check that \( \theta_\lambda \) is bounded as follows:

\[
\theta_\lambda^- \leq \theta_\lambda \leq \theta_\lambda^+ \tag{B5}
\]

By knowing the ranges of the angles, we are now in the position of showing the conditions \( (i) \), \( (ii) \) and \( (iii) \).

For \( (i) \), it is easy to see that \( E_0 \) is a monotonic decreasing function by directly checking \( \partial_\kappa E_0 \):

\[
\partial_\kappa E_0 = -\frac{4\kappa}{\sqrt{8\kappa^2 + r^2}} \tag{B6}
\]

which is negative definite for \( \kappa > 0 \) and \( -1 < r < \infty \). For \( E_\lambda \) given by Eq.\((5c)\) with \( \lambda \geq 1 \), the partial \( \kappa \)-derivative of \( E_\lambda \) is:

\[
\partial_\kappa E_\lambda = -\Omega_\lambda \lambda \tag{B7}
\]

\[
\Omega_\lambda = \frac{4\kappa}{\sqrt{3\alpha_\lambda \sqrt{\alpha_\lambda^3 - 2\tau_\lambda^4 r^2}}}
\]

\[
\chi_\lambda = \tau_\lambda^2 \lambda \sqrt{\alpha_\lambda^3 - 2\tau_\lambda^4 r^2} + \sqrt{3}(\tau_\lambda \kappa^2 - r^2) \sin \theta_\lambda.
\]

Note that, \( \alpha_\lambda^3 > 27\kappa^4 r^2 \) and \( \alpha_\lambda \geq 0 \), then \( \Omega_\lambda \geq 0 \) for all \( \kappa \) and \( r \). Moreover, by using the ranges of the angles, one has

\[
0.804 \sim \cos \theta_\lambda^- \leq \cos \theta_\lambda \leq \cos \theta_\lambda^+ \sim 0.917
\]

\[
0.399 \sim \sin \theta_\lambda^- \leq \sin \theta_\lambda \leq \sin \theta_\lambda^+ \sim 0.595
\]

Therefore, \( \cos \theta_\lambda > \sin \theta_\lambda > 0 \). For \( r \geq 0 \) and \( \tau_\lambda \geq 3 \), one has \( \chi_\lambda > \chi_A - \chi_B \) with

\[
\chi_A = \sqrt{3(\alpha_\lambda^3 - 27\kappa^4 r^2)} \cos^2 \theta_\lambda + 3\kappa^2 r \sin \theta_\lambda
\]

\[
\chi_B = \sqrt{3r^2 \sin^2 \theta_\lambda},
\]

By expanding out \( \alpha_\lambda \) in \( \chi_A \) and regrouping terms one obtains

\[
\chi_A = \sqrt{3r^2 \cos^2 \theta_\lambda + \Delta + 3\kappa^2 r \sin \theta_\lambda}
\]

where \( \Delta \) denotes the remaining positive part inside the square root. Obviously, \( \cos^2 \theta_\lambda > \sin^2 \theta_\lambda \), one concludes \( \chi_A > \chi_B \Rightarrow \chi_\lambda > 0 \) for all \( \kappa \) and \( r \geq 0 \). Similarly, it is easy to check, for \( -1 < r < 0 \), \( \chi_\lambda \geq 0 \) for all \( \kappa \). Therefore, \( E_\lambda \) are monotonic decreasing function for \( \lambda \geq 0 \).

To show \( f(\kappa, r, \lambda) = E_{\lambda+1} - E_\lambda \) is a monotonic decreasing function of \( \kappa \) for any \( r \) and \( \lambda \), it is necessary to break down the proof for different regions of \( \lambda \). For the \( \lambda = -1 \) case, by Eq.\((5)\),

\[
\partial_\kappa f(\kappa, r, 1) = \partial_\kappa (E_0 - E_1) = \partial_\kappa E_0
\]

which is monotonic decreasing as proved in the criterion \( (i) \). One can easily check \( f(\kappa, r, 1) \) approaches 1 at small
\( \kappa \), while becomes \(-\infty \) at larger \( \kappa \). Therefore, the critical coupling \( \tilde{\kappa}_1 \) determined by \( f(\kappa, r, 1) = 0 \) uniquely exists. For \( \lambda \geq 1 \), one should show that the \( \kappa \) derivative of \( f \) does not change sign for all \( \kappa \). Alternatively, it is equivalent to show the function \( g(\kappa, r, \lambda) \equiv \partial_\kappa E_\lambda \) being a monotonic decreasing function in \( \lambda \), such that it ensures \( g(\kappa, r, \lambda + 1) - g(\kappa, r, \lambda) = \partial_\kappa E_{\lambda+1} - \partial_\kappa E_\lambda = \partial_\kappa f < 0 \). By using Eq.(B7),

\[
\partial_\lambda g(\kappa, \lambda, r) = \partial_\lambda \partial_\kappa E_\lambda = -\frac{4\kappa y_1^3}{3y_1^3 \kappa \alpha^4 - 27\kappa^3 + 4} \times (3(\kappa^2 + 1)) \cos \theta - 3(\kappa^2 - y_1^3) \sin \theta \lambda
\]

where

\[
y_1^3 = 8\tau^3 + 3(16\lambda^2 + 16\lambda - 5)\kappa^4 + 6\kappa^2 r^4 + 6\kappa^2 r^4 + r^6
\]

\[y_2^3 = 4\tau^3 + 2(16\lambda^2 + 16\lambda - 5)\kappa^4 + 5\kappa^2 r^4 + r^6.
\]

For \( r \geq 0 \), \( \kappa^2 - y_1^3 \), one has \( \partial_\kappa g(\kappa, \lambda, r) < 0 \). Similarly, \( \partial_\kappa g(\kappa, \lambda, r) < 0 \) is still true for \(-1 < r < 0\) due to \( 3\kappa^2 |r| < y_1^3 \). Thus one has \( \partial_\kappa g < 0 \) for all \( \kappa, \lambda \) and \( r \). Therefore, we have shown \( g(\kappa, r, \lambda) \) is a monotonic decreasing function of \( \lambda \) and then \( f(\kappa, r, \lambda) \) is a strictly decreasing function of \( \kappa \). Furthermore, one can check that, for \( \lambda \geq 1 \),

\[
f(\kappa, r, \lambda)|_{\kappa \to 0} = 1 + \frac{r}{3} < 0
\]

\[
f(\kappa, r, \lambda)|_{\kappa \to \infty} = \sqrt{2 + 4\lambda} - \sqrt{6 + 4\lambda} < 0.
\]

As a result, the crossings \( \{ \tilde{\kappa}_i \} \) \( i \geq 1 \) has unique solution. To prove the remaining case with \( \lambda = 0 \), we express \( \partial_\kappa f(\kappa, r, 0) \) in terms of \( x \):

\[
\partial_\kappa f(\kappa, r, 0) = \partial_\kappa (E_1 - E_0) = \eta \Gamma
\]

\[
\eta = \frac{4}{\sqrt{(8x + 1)(6x + 1)}} \Gamma = \frac{\sqrt{x(6x + 1)} - \sqrt{3x(8x + 1)} \cos \theta_1}{-\operatorname{sgn}(r)(3x - 1) \sqrt{x(8x + 1)} \sin \theta_1}
\]

where \( \tilde{y} \equiv y_1^3 / r^6 \). If \( \Gamma \) is negative for all \( x \geq 0 \), then \( f(\kappa, r, 0) \) is monotonic decreasing. Let us start with \( \operatorname{sgn}(r) = + \). For \( x \geq \frac{1}{4} \), \( \sqrt{x(8x + 1)} < \sqrt{8x + 1} \) and \( \sqrt{3} \cos \theta_1 > 1 \), then \( \sqrt{x(6x + 1)} < \sqrt{x(8x + 1)} \sqrt{3} \cos \theta_1 \). Therefore, we have \( \Gamma < 0 \) for \( x < \frac{1}{4} \) and \( r \geq 0 \). Similarly one has \( \Gamma < 0 \) for \( x \in (0, \frac{1}{3}) \) with \( \operatorname{sgn}(r) = - \). However, for \( x \in [0, \frac{1}{4}] \) with \( \operatorname{sgn}(r) = + \), the last term of \( \Gamma \) is negative, thus it is not obvious that \( \Gamma \) is negative definite. Therefore a different approach is called for. To proceed further for \( x \in [0, \frac{1}{4}] \) with \( \operatorname{sgn}(r) = + \), let

\[
A \equiv \sqrt{x(6x + 1)}
\]

\[
B \equiv (1 - 3x) \sqrt{x(8x + 1)} \sin \theta_1
\]

\[
C \equiv \sqrt{3x(8x + 1)} \cos \theta_1.
\]

and then \( \Gamma = A + B - C \). Our logic to prove \( \Gamma \) is still negative is to show that there exists \( \delta \geq 0 \) such that \( \delta C \geq A \) and \( (1 - \delta)C \geq B \), then \( C \geq A + B \Rightarrow \Gamma \leq 0 \) for \( x \in (0, \frac{1}{4}) \). To begin with, one has

\[
\delta C \geq A \Rightarrow \delta \geq \frac{A}{C}
\]

\[
\delta \geq \frac{6x + 1}{3(8x + 1)} \sec \theta_1
\]

The other condition is

\[
(1 - \delta) \geq \frac{B}{C}
\]

\[
\delta \leq 1 - \frac{1 - 3x}{\sqrt{3} \tan \theta_1}
\]

Combining Eq.(B9) and (B10),

\[
1 - \frac{1 - 3x}{\sqrt{3} \tan \theta_1} \geq \delta \geq \frac{6x + 1}{3(8x + 1)} \sec \theta_1 \geq 0
\]

Therefore, \( \delta \) exists if the following condition is satisfied:

\[
1 - \left( \frac{1 - 3x}{\sqrt{3} \tan \theta_1} + \frac{6x + 1}{3(8x + 1)} \sec \theta_1 \right) \geq 0
\]

\( Z_1 \) and \( Z_2 \) are:

\[
0 \leq Z_1 \leq \frac{1}{3}
\]

\[
\frac{2}{3} \leq Z_2 \leq \frac{3}{11} \sec \theta_1^+ \]

Therefore

\[
\frac{2}{3} \leq Z_1 + Z_2 \leq \frac{1}{3} + \frac{3}{11} \sec \theta_1^+ < 1
\]

By the same approach, one can show that \( \Gamma \) is negative definite for \( r \geq \frac{1}{4} \) with \( \operatorname{sgn}(r) = - \). Furthermore, it is easy to show that \( f(\kappa, r, 0) \to 1 \) when \( \kappa \) is small and changes sign at large \( \kappa \). This completes the proof of showing \( \kappa_2 \) exists. For now, we have shown the crossings \( \{ \tilde{\kappa}_i \} \) between \( E_\lambda \) and \( E_{\lambda+1} \) exist even in the detuning two-atom system. In what following, we will show these crossings occur in sequence.

We start from \( \lambda = -1 \) case, with the solution of \( E_1 = E_0, \kappa_1 = \sqrt{1 + r} \), one has:

\[
\tilde{E}_1 - \tilde{E}_0 = (2 + r) - \frac{2}{\sqrt{3}} \sqrt{(2 + r)^2 - (1 + r) \cos \theta_1}
\]

where the tilde symbol denotes quantity at the appropriate critical \( \kappa \), for here it is \( \kappa = \kappa_1 \). For \( \lambda = 0 \), imposing \( E_1 = E_0 \), one obtains

\[
1 + r = \frac{2}{3} \sqrt{3 \kappa_1 \cos \theta_1 + r} - \frac{\sqrt{8\kappa_1^3 + r^2}}{2}
\]

(B13)
And
\[
\hat{E}_2 - \hat{E}_1 = 1 + r - \frac{2}{3}\{\sqrt{3\lambda_2 \cos \tilde{\theta}_2} - \sqrt{3\lambda_1 \cos \tilde{\theta}_1}\}
\]
\[
= \frac{r}{2} + \frac{4\sqrt{6\kappa^2 + r^2} \cos \tilde{\theta}_1}{3\sqrt{3}}\{1 + \frac{2}{3}|\sqrt{\frac{10\kappa_0^2 + r^2}{6\kappa_0^2 + r^2} \cos \tilde{\theta}_2} - \frac{1}{3}\}
\]
\[
- \frac{\sqrt{3}}{8} \sqrt{\frac{8\kappa_0^2 + r^2}{6\kappa_0^2 + r^2} \cos \tilde{\theta}_1}\}
\]
(B14)

where Eq.(B13) has been used in the second line and

\[
\cos \tilde{\theta}_1 = \cos\left(\frac{1}{3}(\pi - \cos^{-1}\frac{3\sqrt{3}r(1+r)}{2(3+3r+r^2)^{3/2}})\right)
\]
\[
\cos \tilde{\theta}_2 = \cos\left(\frac{1}{3}(\pi - \cos^{-1}\frac{3\sqrt{3}r(1+r)}{2(5+5r+r^2)^{3/2}})\right).
\]

In order to estimate the value of Eq.(B14), By using the inequalities (B3) and (B5) one has:

\[
\frac{\cos \theta_{\tilde{2}}}{\cos \theta_{\tilde{1}}} \leq \frac{\cos \tilde{\theta}_2}{\cos \tilde{\theta}_1} \leq \frac{\cos \theta_{\tilde{2}}}{\cos \theta_{\tilde{1}}},
\]
\[
\frac{1}{\cos \tilde{\theta}_1} \leq \frac{1}{\cos \tilde{\theta}_1} \leq \frac{1}{\cos \tilde{\theta}_1}.
\]

For sgn(r) = +, it is easy to determine the minimal value of the part in the bracket:

\[
\{\ldots\} > 1 - \frac{1}{2}\frac{10\kappa_0^2 + r^2 \cos \theta_{\tilde{2}}}{6\kappa_0^2 + r^2 \cos \theta_{\tilde{1}}}
\]
\[
- \frac{\sqrt{3}}{8} \sqrt{\frac{8\kappa_0^2 + r^2}{6\kappa_0^2 + r^2} \cos \tilde{\theta}_1}\}
\]
\[
> 1 - \frac{1}{2}\frac{5 \cos \theta_{\tilde{2}}}{3 \cos \tilde{\theta}_1} - \frac{\sqrt{3}}{8} \frac{1}{3 \cos \tilde{\theta}_1}\}
\]
(B15)

Then, \(\hat{E}_2 > \hat{E}_1\) at \(\kappa_0\) for \(r \geq 0\). For \(\lambda \geq 1\) cases, from \(E_{\Lambda} = E_{\Lambda+1}\),

\[
1 + r = \frac{2}{3}\{\sqrt{3\lambda_{\Lambda+1} \cos \tilde{\theta}_{\Lambda+1}} - \sqrt{3\lambda_\Lambda \cos \tilde{\theta}_\Lambda}\}. \quad (B16)
\]

Therefore,

\[
\{E_{\Lambda+2} - E_{\Lambda+1}\}_{\kappa_0}
\]
\[
= (r + 1) - \frac{2}{3}\{\sqrt{3\lambda_{\Lambda+2} \cos \tilde{\theta}_{\Lambda+2}} - \sqrt{3\lambda_{\Lambda+1} \cos \tilde{\theta}_{\Lambda+1}}\}
\]
\[
= \frac{2}{3}\{2\sqrt{3\lambda_{\Lambda+1} \cos \tilde{\theta}_{\Lambda+1}} - \sqrt{3\lambda_\Lambda \cos \tilde{\theta}_\Lambda}
\]
\[
- \sqrt{3\lambda_{\Lambda+2} \cos \tilde{\theta}_{\Lambda+2}}\}
\]
(B17)

where Eq.(B14) has been used in the second line. For the ease of discussion, we denote \(\xi(\lambda) = \sqrt{3\lambda_\Lambda \cos \tilde{\theta}_\Lambda}\). If the curvature of \(\xi(\lambda)\) is negative, then we have \(2\xi(\lambda + 1) > \xi(\lambda) + \xi(\lambda + 2)\) which is just the condition of Eq.(B15) > 0. By taking second derivative with \(\lambda\) directly, we obtain

\[
\partial^2_{\Lambda}\xi(\lambda) = -\frac{\lambda_\Lambda}{y_\Lambda^2} \frac{4\kappa^2}{r^6} \times \{
\sqrt{\lambda_\Lambda \cos \tilde{\theta}_\Lambda} + 27\kappa^2 \sin \tilde{\theta}_\Lambda\} < 0. \quad (B18)
\]

Therefore, the crossing \{\(\kappa_\Lambda\)\}_{\kappa_0 \geq 1} actually occur in sequence for all \(-1 < r < \infty\). This complete the proof for having sequential GSI.

APPENDIX C

In order to prove \(\kappa^{<}_{\Lambda} < \kappa^{0}_{\Lambda} < \kappa^{>}_{\Lambda}\) where \(\kappa^{<,0,>}_{\Lambda}\) are for \(r^{<,=,>}_{\Lambda}\) respectively, the Hamiltonian of the system can be rearranged as:

\[
H^r = H^0 + ra^\dagger a \quad (C1)
\]

where \(H^0\) denotes the on-resonance Hamiltonian and \(H^r\) is the detuned Hamiltonian. \(a^\dagger a\) is a positive-valued operator which is positive-valued. For \(r \geq 0\), given \(\kappa = \kappa^{>}_{\Lambda}\), the ground state of the system is \(|g^{>}_{\Lambda+1}\rangle\) such that \(H^r|g^{>}_{\Lambda+1}\rangle = E^r|g^{>}_{\Lambda+1}\rangle\). The expectation value of \(H^r\) in \(|g^{>}_{\Lambda+1}\rangle\) can be written as:

\[
E^r = \langle g^{>}_{\Lambda+1}\rangle H^0|g^{>}_{\Lambda+1}\rangle + r\langle a^\dagger a\rangle|g^{>}_{\Lambda+1}\rangle
\]
\[
> \langle g^0|H^0|g^0\rangle|\kappa = \kappa^{>}_{\Lambda}| + r\langle a^\dagger a\rangle|g^{>}_{\Lambda+1}\rangle
\]
\[
= E^0|\kappa = \kappa^{>}_{\Lambda}|. \quad (C2)
\]

Where \(|g^0\rangle\) is the ground state of \(H^0\). The inequality occur by noting that \(|g^{>}_{\Lambda+1}\rangle\) is a trial state for \(H^0\). Moreover, due to the fact of sequential GSI proven in appendix B, Eq.(C2) implies \(\kappa^{<}_{\Lambda} < \kappa^{>}_{\Lambda}\). Similarly, for \(-1 < r < 0\), instead of using \(|g^{<}_{\Lambda+1}\rangle\), we evaluate the expectation value of \(H^r\) with the ground state \(|g^{<}_{\Lambda+1}\rangle\) of \(H^0\) at \(\kappa = \kappa^{<}_{\Lambda}\) and obtain the desired result:

\[
E^0 = \langle g^{<}_{\Lambda+1}\rangle H^r|g^{<}_{\Lambda+1}\rangle + r\langle a^\dagger a\rangle|g^<_{\Lambda+1}\rangle
\]
\[
> E^r|\kappa = \kappa^{<}_{\Lambda}| + r\langle a^\dagger a\rangle|g^{<}_{\Lambda+1}\rangle
\]
\[
= E^r|\kappa = \kappa^{<}_{\Lambda}| < E^0. \quad (C3)
\]

Again the property of trial state has been used in Eq.(C3) which implies \(\kappa^{<}_{\Lambda} < \kappa^{<}_{\Lambda}\).

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