ON INSTANTON HOMOLOGY OF CORKS $W_n$

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ABSTRACT. We consider a family of corks, denoted $W_n$, constructed by Akbulut and Yasui. Each cork gives rise to an exotic structure on a smooth 4-manifold via a twist $\tau$ on its boundary $\Sigma_n = \partial W_n$. We compute the instanton Floer homology of $\Sigma_n$ and show that the map induced on the instanton Floer homology by $\tau : \Sigma_n \to \Sigma_n$ is non-trivial.

1. Introduction

In [4], Akbulut and Yasui defined a cork $C$ as a compact Stein 4-manifold with boundary together with an involution $\tau : \partial C \to \partial C$ which extends as a self-homeomorphism of $C$ but not as a self-diffeomorphism. In addition, $C \subset X$ is a cork of a smooth 4-manifold $X$ if cutting $C$ out and regluing it via $\tau$ changes the diffeomorphism type of $X$.

We will consider the family of corks $W_n$, $n \geq 1$, obtained by surgery on the link in Figure 1 where a positive integer $m$ in a box indicates $m$ right-handed half-twists. The boundary $\Sigma_n$ of $W_n$ is the integral homology 3-sphere with surgery description as in Figure 2. The involution $\tau : \Sigma_n \to \Sigma_n$ interchanges the two components of the link in Figure 2. It is best seen when the underlying link $L_n$ is drawn symmetrically, as in Figure 3. Note that the quotient manifold $\Sigma'_n = \Sigma_n/\tau$ is homeomorphic to $S^3$ so $\Sigma_n$ can be viewed as a double branched cover of $S^3$ with branch set $k_n$ as shown in Figure 4.

The goal of this paper is to study the instanton Floer homology $I_* (\Sigma_n)$ and the map $\tau_* : I_* (\Sigma_n) \to I_* (\Sigma_n)$ induced on it by $\tau$.

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Theorem. (1) For every integer \( n \geq 1 \), the instanton Floer homology group \( I_j(\Sigma_n) \), \( j = 0, \ldots, 7 \), is trivial if \( j \) is even, and is a free abelian group of rank \( n(n + 1)(n + 2)/6 \) if \( j \) is odd.

(2) The homomorphism \( \tau_* : I_*(\Sigma_n) \to I_*(\Sigma_n) \) is non-trivial for all \( n \geq 1 \).

The first example of an involution acting non-trivially on the instanton Floer homology of an irreducible homology 3-sphere was given in [1] and [16]; in fact, that example was exactly our \( \tau : \Sigma_1 \to \Sigma_1 \). The technique we use to show non-triviality of \( \tau_* \) is the same as the technique that was used in [15] to reprove the result of [16]: compare the Lefschetz number of \( \tau_* : I_*(\Sigma_n) \to I_*(\Sigma_n) \) with the Lefschetz number of the identity map. If the two are different, then the involution must be non-trivial. For any integral homology 3-sphere \( \Sigma \), the Lefschetz number of the identity equals the Euler characteristic of \( I_*(\Sigma) \), which by Taubes [18] is twice the Casson invariant \( \lambda(\Sigma) \). Ruberman and Saveliev [14] showed that the Lefschetz number of \( \tau_* \) equals twice the equivariant Casson invariant \( \lambda^\tau(\Sigma) \), defined in [7]. Therefore the non-triviality of \( \tau_* \) will follow as soon as we show that \( \lambda(\Sigma_n) \neq \lambda^\tau(\Sigma_n) \).

The calculation of \( I_*(\Sigma_n) \to I_*(\Sigma_n) \) is done using surgery techniques.

It should be noted that Akbulut and Karakurt proved a Heegaard Floer analogue of this result. In [2] they showed that the involution \( \tau : \Sigma_n \to \Sigma_n \) acts non-trivially on the Heegaard Floer homology group \( HF^+(\Sigma_n) \).

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2. The instanton Floer homology of \( \Sigma_n \)

Let \( \Sigma \) be an oriented integral homology 3-sphere. The instanton homology groups \( I_*(\Sigma) \) are eight abelian groups arising as the Floer homology of the Chern-Simons functional on the space of irreducible \( SU(2) \) connections on \( \Sigma \) modulo gauge equivalence. The Euler characteristic of \( I_*(\Sigma) \) is twice the Casson invariant \( \lambda(\Sigma) \), see Taubes [18].

Let \( \Sigma_n(p) \) be the integral homology sphere with surgery description obtained by replacing the 0-framing of the unknot on the left-hand side of
Figure 2 by a $p$-framing. In [16], Saveliev used the Floer exact triangle to show that the instanton homology groups of $I_*(\Sigma_1)(p)$ are independent of $p$. The same argument holds for $I_*(\Sigma_n)(p)$.

The homology 3-sphere $\Sigma_n(2n + 2)$ was shown by Maruyama [13] to be homeomorphic to the Brieskorn homology sphere $\Sigma(2n + 1, 2n + 2, 2n + 3)$. In Theorem 10 of [17], Saveliev proved that the Floer homology group $I_j(\Sigma(2n + 1, 2n + 2, 2n + 3))$ is trivial when $j$ is even, and is isomorphic to a free abelian group of rank $n(n + 1)(n + 2)/6$ when $j$ is odd. This completes the calculation of $I_*(\Sigma_n)$. In particular, the Casson invariant of $\Sigma_n$ is given by

$$\lambda(\Sigma_n) = -\frac{1}{3}n(n + 1)(n + 2).$$
3. **The Casson invariant**

Although the instanton Floer homology groups of $\Sigma_n$ are known, and therefore so is its Casson invariant, we can compute $\lambda(\Sigma_n)$ using topological methods.

Let $L$ be a framed 2-component link in $S^3$, and assume that the 3-manifold $\Sigma$ resulting from surgery on $L$ is a homology 3-sphere. Boyer and Lines [5] compute $\lambda(\Sigma)$ as a sum of derivatives of the multivariable Alexander polynomial $\Delta_L$ of the underlying oriented link $L$ (and those of its sublinks), and Dedekind sums that only depend on the framings of the link components.

In our case, both components of $L_n$ are framed by zero, and the Boyer-Lines [5] formula for $\lambda(\Sigma_n)$ is simply

$$\lambda(\Sigma_n) = -\frac{1}{\det(B)} \frac{\partial^2 \Delta_{L_n}}{\partial x \partial y}(1,1),$$

where $B$ is the framing matrix for $\Sigma_n$. Thus to compute $\lambda(\Sigma_n)$ we will need only to compute $\Delta_{L_n}$.

### 3.1. The Alexander polynomial and Conway potential function.

Rather than computing $\Delta_L$ directly we will consider the related Conway potential function $\nabla_L$. Given an oriented link $L$ in $S^3$, normalize $\Delta_L(x,y)$ using the Conway potential function $\nabla_L(x,y)$ of Hartley [11] by requiring that

$$\Delta_L(x^2,y^2) = \nabla_L(x,y).$$  \hspace{1cm} (2)

Note that (2) implies that

$$\frac{\partial^2 \Delta_L}{\partial x \partial y}(1,1) = \frac{1}{4} \frac{\partial^2 \nabla_L}{\partial x \partial y}(1,1),$$

hence we only need to compute the Conway potential function of $L$ and its partial derivatives.

The Conway potential function $\nabla_L$ enjoys the replacement relation

$$\nabla_\ell + \nabla_r = (xy + x^{-1}y^{-1})\nabla_s,$$\hspace{1cm} (4)

where $\nabla_\ell$, $\nabla_r$, and $\nabla_s$ are Conway functions of links that differ only in a neighborhood of a single crossing as shown in Figure 5. Note that the arcs may belong to the same component of $L$ or to different components. If the
three links differ as in Figure 5 but with one of the arcs oppositely oriented, then we have the relation
\[ \nabla_\ell + \nabla_r = (xy^{-1} + x^{-1}y)\nabla_s. \] (5)

Also, we will note that the Conway potential function vanishes for split links and is equal to 1 for the right-handed Hopf link.

In order to calculate the Conway potential function of \( L_n \) we will use the replacement relations (4) and (5) to produce a linear recurrence which we will then solve.

3.2. The recurrence relation. Let \( f_n = \nabla_{L_n} \) and \( g_n = \nabla_{H_n} \), where \( H_n \) is the link in Figure 6. A straightforward calculation shows that
\[ g_n = \left( \frac{1}{r_2 - r_1} \right) r_1^n + \left( \frac{1}{r_1 - r_2} \right) r_2^n \]
with \( v = xy^{-1} + x^{-1}y \), \( r_1 = (v + \sqrt{v^2 - 4})/2 \), and \( r_2 = (v - \sqrt{v^2 - 4})/2 \). Using Hartley’s replacement relations (4), we change crossings of \( L_n \) two at a time, until we have undone the upper tangle in Figure 2. We obtain the recurrence \( f_{n+2} = -f_n + u f_{n+1} \) with initial conditions \( f_0 = -g_{n+1} + ug_n \) and \( f_1 = -ug_{n+1} + (u^2 - 1)g_n \), where \( u = xy + x^{-1}y^{-1} \).

Solving the recurrence relation, we obtain a formula for \( f_n \) in terms of \( x, y, \) and \( n \),
\[ f_n = \left( f_0 + \frac{f_0 s_1 - f_1}{s_2 - s_1} \right) s_1^n + \left( \frac{f_1 - f_0 s_1}{s_2 - s_1} \right) s_2^n \]

where \( s_1 = (u + \sqrt{u^2 - 4})/2 \) and \( s_2 = (u - \sqrt{u^2 - 4})/2 \). We then find an explicit formula for \( \lambda(\Sigma_n) \) by having Maple differentiate \( f_n \) twice and putting together (1) and (3). The answer is

\[ \lambda(\Sigma_n) = -\frac{1}{3} n(n + 1)(n + 2). \]  

(6)

4. The Equivariant Casson Invariant

Let \( \tau : \Sigma \to \Sigma \) be an orientation preserving involution on a homology 3-sphere, and suppose that the fixed point set of \( \tau \) is non-empty. Then the quotient manifold \( \Sigma' = \Sigma/\tau \) is a homology 3-sphere, and the projection map \( \Sigma \to \Sigma' \) is a double branched cover with branch set a knot \( k \subset \Sigma' \). In [7], Collin and Saveliev computed the equivariant Casson invariant \( \lambda^\tau(\Sigma) \) in terms of \( \lambda(\Sigma') \) and the knot signature \( \sigma(k) \). When \( \Sigma' = S^3 \), we have simply

\[ \lambda^\tau(\Sigma) = \frac{1}{8} \sigma(k). \]

Since we know that \( \lambda(\Sigma_n) \) is decreasing as \( n \to \infty \), see (6), our strategy for showing that \( \lambda^\tau(\Sigma_n) \neq \lambda(\Sigma_n) \) will be to show that \( \sigma(k_n)/8 \) is bounded from below by a function strictly greater than \( \lambda(\Sigma_n) \), where \( k_n \) is the knot shown in Figure 4.

4.1. Bounding knot signatures. The knot signature of a knot \( k \subset S^3 \) may be bounded from below using the formula

\[ \sigma(k_r) \leq \sigma(k_\ell) \leq \sigma(k_r) + 2, \]

see Conway [8] or Giller [10]. Here \( k_r \) and \( k_\ell \) are knots that only differ in a neighborhood of a crossing as shown in Figure 7. Note that our sign convention is opposite of Giller’s. By [7], the signature of a knot \( k \) is bounded from below by negative twice the number of right-handed crossings that must be changed in order to undo the knot \( k \). Note that we may need to change some left handed crossings while undoing \( k \) but this will not contribute to our estimate.
We will now apply this observation to the knot $k_n$ shown in Figure 4. In order to see $k_n$ more clearly, we will isotope the link diagram in Figure 4 so that the 1-framed curve is interchanged with the branch set $k_n$ and then blow down the 1-framed curve. The blow down has the effect of a full left-handed twist on the $2n + 4$ strands passing through the 1-framed curve, see Figure 8.

Note that when $n$ is odd, Figure 8 is a diagram of $k_n$ where no right-handed crossings must be changed in order to undo the knot, hence $\sigma(k_n)$ must be non-negative. When $n$ is even, the left-most strand of the strands being twisted in Figure 8 is oriented oppositely of the other strands being twisted. In this case there are $2n + 3$ right-handed crossings that must be changed. We change them to arrive at a knot with only left-handed crossings that further need to be changed. Thus the signature of $k_n$ is bounded from below by $-4n - 6$, and we have shown that

$$\lambda(\Sigma_n) = -\frac{n(n+1)(n+2)}{3} < -\frac{4n+6}{8} \leq \lambda^7(\Sigma_n).$$
Lastly, we remark that for small $n$ the equivariant Casson invariant can be computed explicitly using the following technique. If we undo the upper left tangle in Figure 8 by changing crossings, then we will arrive at a knot that is isotopic to a torus knot. If $n$ is odd, then the corresponding torus knot is the $T(2n+4, 2n+3)$ torus knot. If $n$ is even, then the corresponding torus knot is $T(2n+2, 2n+1)$. If $n \leq 4$, then we need to change at most 3 crossings, and consequently the signature of $k_n$ differs from that of the corresponding torus knot by at most $\pm 6$.

For example, if $n = 1$ or $n = 2$, then the corresponding torus knot is $T(6, 5)$ and the signature $\sigma(T(6, 5))$ is 16. Since the signature of $k_n$ is divisible by 8, we have that $\sigma(k_1) = \sigma(k_2) = 16$. Similarly, if $n = 3$ or $n = 4$, then the corresponding torus knot is $T(10, 9)$ and

$$\sigma(k_3) = \sigma(k_4) = \sigma(T(10, 9)) = 48.$$

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