DISCRETE PRODUCT SYSTEMS AND TWISTED CROSSED PRODUCTS BY SEMIGROUPS

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Abstract. A product system $E$ over a semigroup $P$ is a family of Hilbert spaces $\{E_s : s \in P\}$ together with multiplications $E_s \times E_t \to E_{st}$. We view $E$ as a unitary-valued cocycle on $P$, and consider twisted crossed products $A \rtimes_{\beta,E} P$ involving $E$ and an action $\beta$ of $P$ by endomorphisms of a $C^*$-algebra $A$. When $P$ is quasi-lattice ordered in the sense of Nica, we isolate a class of covariant representations of $E$, and consider a twisted crossed product $B_P \rtimes_{\tau,E} P$ which is universal for covariant representations of $E$ when $E$ has finite-dimensional fibres, and in general is slightly larger. In particular, when $P = \mathbb{N}$ and $\dim E_1 = \infty$, our algebra $B_\mathbb{N} \rtimes_{\tau,E} \mathbb{N}$ is a new infinite analogue of the Toeplitz-Cuntz algebras $TO_n$. Our main theorem is a characterisation of the faithful representations of $B_P \rtimes_{\tau,E} P$.

Crossed products of $C^*$-algebras by semigroups of endomorphisms have been profitably used to model Toeplitz algebras [2, 1, 13] and the Hecke algebras arising in the Bost-Connes analysis of phase transitions in number theory [4, 1, 11]. There are two main ways of studying such a crossed product. First, one can try to embed it as a corner in a crossed product by an automorphic action of an enveloping group, and then apply the established theory. The algebra on which the group acts is typically a direct limit, and the success of this approach depends on being able to recognise the direct limit and the action on it [4, 23, 17]. Or, second, one can use the techniques developed in [4, 1, 13] to deal directly with the semigroup crossed product and its representation theory. Here the goal is a characterisation of the faithful representations of the crossed product, and such characterisations have given important information about a wide range of semigroup crossed products [2, 1, 4, 1].

For ordinary crossed products $A \rtimes_{\alpha} G$ (those involving an action $\alpha$ of $G$ by automorphisms of $A$), an important adjunct are the twisted crossed products $A \rtimes_{\alpha,\omega} G$, in which the multiplication of elements of $G$ has been twisted by a cocycle $\omega$. This cocycle might take values in the unitary groups of $A$, $M(A)$ or $ZM(A)$, but the most important are the scalar-valued cocycles $\omega : G \times G \to \mathbb{T}$. There is no obvious technical obstruction to developing a theory of twisted semigroup crossed products, and indeed this has already been done by Laca for scalar-valued cocycles on totally ordered groups [10]. Since scalar-valued cocycles on semigroups often extend to the enveloping group [2], one might expect this theory to be a routine combination of

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ideas involving semigroup crossed products and ordinary twisted crossed products.

In this paper we investigate a phenomenon which arises only for semigroups: crossed products twisted by unitary cocycles acting on Hilbert spaces of varying dimension. Such cocycles were introduced by Arveson under the name of product systems [4]. The idea is to associate to each element \( s \) of the semigroup \( S \) a Hilbert space \( E_s \), and then the cocycle describes a multiplication from \( E_s \times E_t \) to \( E_{st} \); a scalar cocycle \( \omega : S \times S \to \mathbb{T} \) determines such a system by taking \( E_s = \mathbb{C} \) for all \( s \) and using \( (w, z) \mapsto \omega(s, t) wz \) as the product from \( E_s \times E_t \) to \( E_{st} \). Because \( \dim E_{st} = \dim E_s \times \dim E_t \), product systems with fibres of dimension other than 1 cannot exist on groups (at least in a naive sense), so the possibility of twisting crossed products by product systems is appropriate only for actions of semigroups. It is not an entirely new idea: the crossed products of multiplicity \( n \) of Stacey [23] are twisted crossed products by actions of the semigroup \( \mathbb{N} \) in which the product system \( E \) has \( \dim E_1 = n \).

Of the various kinds of semigroups studied in the literature, we have chosen to work with the quasi-lattice ordered semigroups of Nica [19]; these include the totally ordered groups considered in [15, 2, 10], the direct sums \( \mathbb{N}^k \), and the free products considered in [13]. For a product system \( E \) over such a semigroup \( P \), one can define a natural notion of covariant representation generalising that of [19, 13]: loosely speaking, a representation \( \phi \) of \( E \) is a family of isometric maps \( \phi_s : E_s \to B(\mathcal{H}) \) such that each \( \phi_s(v) \) is an isometry and \( \phi_{st}(uv) = \phi_s(u) \phi_t(v) \), and \( \phi \) is covariant if the projections on the ranges \( \phi_s(E_s) \) are aligned in a manner compatible with the ordering on \( P \). The motivating example is the trivial product system on \( \mathbb{N}^2 \), where the representations are given by two commuting isometries and the covariant representations by two \( * \)-commuting isometries.

The main results of [13] and [13] concern the \( C^* \)-algebra, here denoted \( C^*_\text{cov}(P) \), which is universal for covariant isometric representations of the quasi-lattice ordered group \((G, P)\). In [13], \( C^*_\text{cov}(P) \) is viewed as a semigroup crossed product \( B_P \rtimes_{\tau} P \), where \( B_P \) is the \( C^* \)-subalgebra of \( \ell^\infty(P) \) spanned by the characteristic functions \( 1_x := \chi_{xP} \), and \( \tau_1(1_x) = 1_{tx} \). Here we aim to view the universal \( C^* \)-algebra \( C^*_\text{cov}(P, E) \) for covariant representations of \( E \) as a twisted crossed product \( B_P \rtimes_{\tau, E} P \), and use techniques like those of [13] to characterise their faithful representations. However, carrying out this program has raised some intriguing issues.

We shall construct suitable twisted crossed products \( B_P \rtimes_{\tau, E} P \), and show that the \( C^* \)-subalgebra of \( B_P \rtimes_{\tau, E} P \) generated by the canonical copy of \( E \) is universal for covariant representations of \( E \), and hence can reasonably be denoted \( C^*_\text{cov}(P, E) \). When the fibres of \( E \) are finite-dimensional, \( C^*_\text{cov}(P, E) \) is all of \( B_P \rtimes_{\tau, E} P \), but in general it may not be. This last phenomenon occurs, for example, when \( P = \mathbb{N} \) and \( E_1 \) is infinite-dimensional: \( C^*_\text{cov}(\mathbb{N}, E) \) is the Cuntz algebra \( \mathcal{O}_\infty \) generated by isometries \( \{V_k : k \in \mathbb{N}\} \) with orthogonal ranges, whereas \( B_{\mathbb{N}} \rtimes_{\tau, E} \mathbb{N} \) contains the projection \( 1 - \sum_{k=1}^{\infty} V_k V_k^* = 1 - \tau_1(1) \). This undermines the popular view that the Cuntz algebra \( \mathcal{O}_\infty \) coincides with the Toeplitz-Cuntz algebra \( \mathcal{T}_\mathcal{O}_\infty \), since \( B_{\mathbb{N}} \rtimes_{\tau, E} \mathbb{N} \) seems a logical candidate for the latter. Our main theorem characterises faithful
representations of $B_P \rtimes_{\tau, E} P$ rather than $C^*_{\text{cov}}(P, E)$, and thus achieves our goal only for systems with finite-dimensional fibres. We plan to return to the topic of systems with infinite-dimensional fibres in a sequel.

We have organised our work as follows. We begin with an introductory section on product systems and their representations, giving a variety of examples and constructions. General twisted crossed products are discussed only in §2: as in [13], we are mainly interested in the specific crossed products $B_P \rtimes_{\tau, E} P$ which capture the covariance condition on representations of $E$. The covariance condition itself is modelled on that of Nica, and only makes sense for product systems on quasi-lattice ordered semigroups. In §3 we discuss it and its connection with covariant representations of the system $(B_P, P, \tau, E)$. We can then prove that $C^*_{\text{cov}}(P, E)$ embeds naturally in the semigroup crossed product $B_P \rtimes_{\tau, E} P$ (Theorem 4.3).

Our main theorem is our characterisation of faithful representations of $B_P \rtimes_{\tau, E} P$. There are two main steps. First, under an amenability hypothesis, we follow the procedure pioneered by Cuntz, which reduces the problem to proving an estimate concerning the deletion of off-diagonal terms. The details are necessarily different, but the general plan of [13, §3] carries over under a spanning hypothesis on the product system which holds in the interesting examples. Second, we have to verify the amenability hypothesis in a reasonable number of situations. It is automatic, for example, if the enveloping group of $P$ is amenable, or if $P$ is a free product of such semigroups and the product system satisfies a modest-looking spanning condition. Both the spanning conditions we have mentioned are satisfied if $E$ has finite-dimensional fibres, so our main theorem applies to all such product systems on $\mathbb{N}^k$ or on free products of subsemigroups of amenable groups.

1. Product Systems and their Representations

Definition 1.1. Suppose $P$ is a semigroup with identity and $p : E \to P$ is a family of nontrivial complex Hilbert spaces whose fibre over the identity is one-dimensional. Write $E_t$ for the fibre $p^{-1}(t)$ over $t \in P$. We say that $E$ is a (discrete) product system over $P$ if $E$ is a semigroup, $p$ is a semigroup homomorphism, and for each $s, t \in P$ the map $(u, v) \in E_s \times E_t \mapsto uv \in E_{st}$ extends to a unitary isomorphism $U_{s,t}$ of $E_s \otimes E_t$ onto $E_{st}$.

Remark 1.2. The associativity of multiplication in the semigroup $E$ implies that the unitary operators $U_{s,t}$ satisfy

$$U_{r,s,t}(U_{r,s} \otimes I) = U_{r,st}(I \otimes U_{s,t})$$

for $r, s, t \in P$. Thus product systems over $P$ can be viewed as unitary 2-cocycles acting on a varying but coherent system of Hilbert spaces.

Lemma 1.3. Suppose $E$ is a product system over a semigroup $P$ with identity $e$. Then $E$ has an identity $\Omega$ such that $p(\Omega) = e$ and $\|\Omega\| = 1$.

Proof. Let $z$ be a unit vector in $E_e$. Then $z^2 \in E_e$ also, so $z^2 = \lambda z$ for some $\lambda \in \mathbb{C}$ such that $|\lambda|^2 = \langle \lambda z, \lambda z \rangle = \langle z^2, z^2 \rangle = \langle z, z \rangle \langle z, z \rangle = 1$. Suppose $x \in E$. Then $zx \in E_{p(x)}$, and for any $y \in E_{p(x)}$,

$$\langle zx, y \rangle = \langle z, z \rangle \langle zx, y \rangle = \langle (zx)x, zy \rangle = \langle \lambda z x, zy \rangle = \langle z, z \rangle \langle \lambda x, y \rangle = \langle \lambda x, y \rangle,$$
so $zx = \lambda x$. Similarly we have $xz = \lambda x$, and thus $\Omega = \overline{\lambda z}$ is an identity for $E$. We have $p(\Omega) = e$ because $z \in E_\epsilon$, and $||\Omega|| = 1$ because $z$ is a unit vector.

**Examples 1.4.** (E1) The trivial product system over $P$ is the trivial bundle $P \times \mathbb{C}$ with multiplication given by $(s, w)(t, z) = (st, wz)$.

(E2) (Lexicographic Product Systems) Given a product system $p : E \to P$ with $\dim E_t < \infty$ for each $t \in P$, the dimension function $d : t \mapsto \dim E_t$ is a semigroup homomorphism of $P$ into the multiplicative positive integers $\mathbb{N}^*$. Conversely, given $d \in \text{Hom}(P, \mathbb{N}^*)$, we can construct a product system over $P$ with dimension function $d$ as follows. Let $E = \bigsqcup_{t \in P} \{t\} \times \mathbb{C}^{d(t)}$, $p(t, v) = t$, and define multiplication in $E$ by $(s, u)(t, v) = (st, w)$ where

$$w_{i}(d(t) + j) = u_{i}v_{j}, \quad 1 \leq i \leq d(s), \quad 1 \leq j \leq d(t).$$

Since this construction is based on the lexicographic ordering of $\{1, 2, \ldots, d(s)\} \times \{1, 2, \ldots, d(t)\}$, we call $E$ the lexicographic product system over $P$ determined by $d$.

(E3) Suppose $p : E \to P$ is a product system over a semigroup $P$ with identity $e$, and $\mu$ is a multiplier on $P$; that is, $\mu : P \times P \to \mathbb{T}$ satisfies

- $\mu(t, e) = 1 = \mu(e, t)$ for each $t \in P$, and
- $\mu(r, s)\mu(rs, t) = \mu(s, t)\mu(r, st)$ for each $r, s, t \in P$;

alternatively, one might say $\mu$ is a 2-cocycle on $P$ with values in $\mathbb{T}$. Let $E^\mu = E$, $p^\mu = p$, and define multiplication by $(u, v) \mapsto \mu(p(u), p(v))uv$. Then $E^\mu$ is a product system over $P$; we say that $E^\mu$ is $E$ twisted by $\mu$.

If $\nu$ is another multiplier on $P$, then $E^\mu$ is isomorphic to $E^\nu$ iff $[\mu] = [\nu]$ as elements of the second cohomology group $H^2(P, \mathbb{T})$; the automorphism group of $E^\mu$ is $\text{Hom}(P, \mathbb{T})$.

(E4) For each $\lambda$, let $p : E^\lambda \to P^\lambda$ be a product system. Then there is a product system $*E^\lambda$ over the free product $*P^\lambda$: for a reduced word $s = s_1 \cdots s_n \in *P^\lambda$, say $s_i \in P^\lambda_i$, we take $(*E^\lambda)_s := E_{s_1}^\lambda \otimes \cdots \otimes E_{s_n}^\lambda$, and if also $t = t_1 \cdots t_m \in *P^\lambda$, say $t_i \in P^{\mu_i}$, we define

$$(w_1 \otimes \cdots \otimes w_n)(v_1 \otimes \cdots \otimes v_m) := \begin{cases} w_1 \otimes \cdots \otimes w_nv_1 \otimes \cdots \otimes v_m & \text{if } \lambda_n = \mu_1 \\ w_1 \otimes \cdots \otimes w_n \otimes v_1 \otimes \cdots \otimes v_m & \text{otherwise.} \end{cases}$$

Product systems over $\mathbb{N}$ are particularly easy to describe:

**Proposition 1.5.** Suppose $E$ and $F$ are product systems over $\mathbb{N}$. Then $E$ and $F$ are isomorphic iff $E_1 \cong F_1$.

**Proof.** If $U$ is a unitary isomorphism of $E_1$ onto $F_1$, then the unitary operators $U^{\otimes n} : E_1^{\otimes n} \to F_1^{\otimes n}$ induce a family of unitaries $\psi_n : E_n \to F_n$ such that

$$\psi_n(u_1u_2 \cdots u_n) = (Uu_1)(Uu_2) \cdots (Uu_n), \quad u_1, \ldots, u_n \in E_1,$$

and these combine to give an isomorphism of product systems. \qed

**Corollary 1.6.** For each $d \in \{1, 2, \ldots, n_0\}$ there is, up to isomorphism, a unique product system $E^d$ over $\mathbb{N}$ whose fibre over 1 is $d$-dimensional.
Proof. Let $d \in \{1, 2, \ldots, \aleph_0\}$, fix a $d$-dimensional Hilbert space $\mathcal{H}$, let $E^d = \bigsqcup_{n=0}^{\infty} \{n\} \times \mathcal{H}^{\otimes n}$, and define $(m, u)(n, v) := (m + n, u \otimes v)$. \hfill \Box

**Definition 1.10** (The Left Regular Representation). A representation of a product system $p : E \to P$ on a Hilbert space $\mathcal{H}$ is a map $\phi : E \to \mathcal{B}(\mathcal{H})$ such that

1. $\phi(uv) = \phi(u)\phi(v)$ for every $u, v \in E$, and
2. $\phi(v)^*\phi(u) = \langle u, v \rangle I$ whenever $p(u) = p(v)$.

**Remarks 1.8.** (1) Condition (2) implies that every operator in the range of $\phi$ is a multiple of an isometry, and that $\phi$ is linear on the fibres of $p$; see [1], p.8. It also implies that $\phi$ is isometric, hence injective; thus each vector space $\phi(E_t)$ has a Hilbert space structure in which the inner product is given by $\langle S, T \rangle I = T^*S$, and the corresponding Hilbert space norm on $\phi(E_t)$ agrees with the operator norm.

(2) Condition (1) implies that $\phi(\Omega)$ is an idempotent, and condition (2) that it is an isometry. Thus $\phi(\Omega)$ is the identity operator $I$.

**Examples 1.9.** (1) A representation of the trivial product system $P \times \mathbb{C}$ on $\mathcal{H}$ is a homomorphism of $P$ into the semigroup of isometries on $\mathcal{H}$. If $P \times \mathbb{C}$ is twisted by a multiplier $\mu$, the representations are $\mu$-twisted representations of $P$ by isometries.

(2) Suppose $E$ is a product system over $\mathbb{N}$ with $\dim E_1 = d$. A representation $\phi$ of $E$ will map an orthonormal basis $\{e_i\}$ for $E_1$ to a family of $d$ isometries $S_i = \phi(e_i)$ whose ranges are mutually orthogonal, and each such family $\{S_i\}$ determines a representation of $E$. We call $\{S_i\}$ a Toeplitz-Cuntz family.

(3) Let $E$ be the lexicographic product system over $\mathbb{N} \oplus \mathbb{N}$ determined by the homomorphism $d : (m, n) \in \mathbb{N} \oplus \mathbb{N} \mapsto 2^m3^n \in \mathbb{N}^*$. Representations of $E$ are in one-one correspondence with pairs of Toeplitz-Cuntz families $\{U_1, U_2\}, \{V_1, V_2, V_3\}$ satisfying the following commutation relations:

\begin{equation}
\begin{aligned}
U_1V_1 &= V_1U_1, & U_2V_1 &= V_2U_2, \\
U_1V_2 &= V_1U_2, & U_2V_2 &= V_3U_1, \\
U_1V_3 &= V_2U_1, & U_2V_3 &= V_3U_2.
\end{aligned}
\end{equation}

**Lemma 1.10** (The Left Regular Representation). Suppose $p : E \to P$ is a product system and $P$ is left-cancellative. Let $S(E) = \bigoplus_{t \in P} E_t$. Then there is a unique representation $l : E \to \mathcal{B}(S(E))$ such that $l(v)w = vw$ for $v, w \in E$.

**Proof.** Suppose $v \in E$ and $w = \bigoplus_{t \in P} w_t \in S(E)$. Since $p(vw_s) = p(vw_t)$ only when $s = t$, the infinite series $\sum_{t \in P} vw_t$ converges in norm to a vector $l(v)w$ of norm $\|v\|\|w\|$, thus defining a bounded linear operator $l(v)$ on $S(E)$. The associativity of the product in $E$ implies that $l : E \to \mathcal{B}(S(E))$ is multiplicative, and if $u, v \in E$ have $p(u) = p(v)$, then for any $w, z \in E$

$$
\langle l(v)^*l(u)w, z \rangle = \langle uw, vz \rangle = \begin{cases} 
\langle u, v \rangle \langle w, z \rangle & \text{if } p(u) = p(v) \\
0 & \text{otherwise}
\end{cases}
$$

so that $l(v)^*l(u) = \langle u, v \rangle I$. Thus $l$ is a representation of $E$. \hfill \Box
The following two propositions introduce concepts and notation which will be used throughout the remainder of this paper. The first is merely a translation of ([H], Proposition 2.7) to our setting, so we omit the proof.

**Proposition 1.11.** Suppose \( E \) is a product system over \( P \) and \( \phi : E \to B(\mathcal{H}) \) is a representation. For each \( t \in P \) there is a unique normal \( ^* \)-endomorphism \( \alpha_t^\phi \) of \( B(\mathcal{H}) \) such that
\[
\phi(E_t) = \{ T \in B(\mathcal{H}) : \alpha_t^\phi(A)T = TA \quad \text{for each} \ A \in B(\mathcal{H}) \};
\]
the map \( t \mapsto \alpha_t^\phi \) is a semigroup homomorphism. If \( B \) is an orthonormal basis for \( E_t \), then \( \alpha_t^\phi \) is given by the strongly convergent sum
\[
\alpha_t^\phi(A) = \sum_{u \in B} \phi(u)A\phi(u)^*.
\]

**Proposition 1.12.** Suppose \( E \) is a product system over \( P \) and \( \phi : E \to B(\mathcal{H}) \) is a representation.

1. For each \( t \in P \) there is a unique faithful normal \( ^* \)-homomorphism \( \rho_t^\phi : B(E_t) \to B(\mathcal{H}) \) such that
\[
\rho_t^\phi(u \otimes v) = \phi(u)\phi(v)^* \quad \text{for} \ u,v \in E_t,
\]
where \( u \otimes v \) denotes the rank-one operator \( w \mapsto \langle w,v \rangle u \) on \( E_t \).

2. If \( Q \) is a nonzero projection on \( \mathcal{H} \) and \( t \in P \), then the map \( T \mapsto \alpha_t^\phi(Q)\rho_t^\phi(T) \) is a faithful normal \( ^* \)-homomorphism.

**Proof.** (1) Let \( B \) be an orthonormal basis for \( E_t \). Since \( \{ u \otimes v : u,v \in B \} \) is a self-adjoint system of matrix units which generate \( B(E_t) \) and \( \{ \phi(u)\phi(v)^* : u,v \in B \} \) is also a self-adjoint system of nonzero matrix units, the map \( u \otimes v \mapsto \phi(u)\phi(v)^* \) extends to the desired homomorphism \( \rho_t^\phi \).

(2) For any \( u,v \in B \), note that \( \alpha_t^\phi(Q)\phi(u)\phi(v)^* = \phi(u)Q\phi(v)^* \). Since \( \{ \phi(u)Q\phi(v)^* : u,v \in B \} \) is a self-adjoint system of nonzero matrix units, the map \( u \otimes v \mapsto \phi(u)Q\phi(v)^* \) extends as claimed. \( \square \)

2. Twisted Semigroup Crossed Products

In this section we discuss how to twist semigroup crossed products by product systems. We consider **twisted systems** \( (A,P,\beta,E) \) in which \( A \) is a unital \( C^* \)-algebra, \( P \) is a semigroup with identity, \( \beta \) is an action of \( P \) on \( A \) by endomorphisms, and \( E \) is a product system over \( P \). We emphasise that the endomorphisms \( \beta_s \) need not be unital.

**Definition 2.1.** A **covariant representation** of \( (A,P,\beta,E) \) on a Hilbert space \( \mathcal{H} \) is a pair \( (\pi,\phi) \) consisting of a unital representation \( \pi : A \to B(\mathcal{H}) \) and a representation \( \phi : E \to B(\mathcal{H}) \) such that \( \pi \circ \beta_s = \alpha_s^\phi \circ \pi \) for \( s \in P \); by **Proposition 1.11** this is equivalent to choosing an orthonormal basis \( B \) for \( E_s \) and asking that
\[
\pi(\beta_s(a)) = \sum_{v \in B} \phi(v)\pi(a)\phi(v)^* \quad \text{for} \ s \in P, a \in A.
\]

A **crossed product** for \( (A,P,\beta,E) \) is a triple \( (B,i_A,i_E) \) consisting of a \( C^* \)-algebra \( B \), a unital \( ^* \)-homomorphism \( i_A : A \to B \), and a \( C^* \)-morphism \( i_E : E \to B \) such that
(a) there is a faithful unital representation \(\sigma\) of \(B\) such that \((\sigma \circ i_A, \sigma \circ i_E)\) is a covariant representation of \((A, P, \beta, E)\);
(b) for every covariant representation \((\pi, \phi)\) of \((A, P, \beta, E)\), there is a unital representation \(\pi \times \phi\) of \(B\) such that \((\pi \times \phi) \circ i_A = \pi\) and \((\pi \times \phi) \circ i_E = \phi\); and
(c) the \(C^*\)-algebra \(B\) is generated by \(i_A(A) \cup i_E(E)\).

Remark 2.2. The semigroup crossed products considered in [2], [3] and [13] are recovered by taking \(E\) to be the trivial product system \(P \times \mathbb{C}\), and twisted semigroup crossed products \(A \times_{\beta, \mu} P\) involving a multiplier \(\mu\) by taking \(E = (P \times \mathbb{C})^\mu\), as in Examples [14] (E3). Stacey’s crossed products of multiplicity \(n\) [23] are recovered by taking \(E\) to be the essentially unique product system over \(\mathbb{N}\) with \(\dim E_1 = n\) (see Corollary [1,6]).

Remark 2.3. Instead of condition (a) in the definition of a crossed product, one might expect to see something more like:

- (a’) for every unital representation \(\sigma\) of \(B\), the pair \((\sigma \circ i_A, \sigma \circ i_E)\) is a covariant representation of \((A, P, \beta, E)\).

This condition would ensure that every unital representation of \(B\) came from a covariant representation of \((A, P, \beta, E)\), so that \(B\) would be truly universal for covariant representations. When the fibres of \(E\) are finite-dimensional, conditions (a) and (a’) are both equivalent to the following condition on \((i_A, i_E)\): if \(a \in A\), \(s \in P\) and \(\{v_1, \ldots, v_n\}\) is an orthonormal basis for \(E_s\), then

\[
(2.2) \quad i_A(\beta_s(a)) = \sum_{k=1}^{n} i_E(v_k) i_A(a) i_E(v_k^*)
\]

However, if \(E_s\) were infinite-dimensional, the sum on the right of (2.2) would have to be infinite, and because the isometries \(i_E(v_k)\) have orthogonal range projections such sums cannot possibly converge in the \(C^*\)-algebra \(B\). Indeed, for systems with infinite-dimensional fibres conditions (a) and (a’) need not coincide. The following example shows that condition (a’) is too much to hope for if there is to be a crossed product for every system with a covariant representation.

Example 2.4. Consider the system \((c, \mathbb{N}, \tau, E^{\mathbb{N}_0})\), where \(\tau\) is the action of \(\mathbb{N}\) by translation on the algebra \(c\) of convergent sequences and \(E^{\mathbb{N}_0}\) is the product system over \(\mathbb{N}\) with \(\dim E_1 = \aleph_0\) (see Corollary [16]). We shall show that a crossed product \(B\) for \((c, \mathbb{N}, \tau, E^{\mathbb{N}_0})\) does not satisfy condition (a’).

Let \(\{S_k : k \in \mathbb{N}\}\) be a countably-infinite collection of isometries on a Hilbert space \(\mathcal{H}\) such that \(\sum S_k S_k^* = I\), and let \(\{\delta_k : k \in \mathbb{N}\}\) be an orthonormal basis for \(E_1\). The formula \(\phi(\delta_k) = S_k\) extends uniquely to a representation \(\phi : E \to B(\mathcal{H})\). Define \(L : c \to B(\mathcal{H})\) by \(L(a) = (\lim_{k \to \infty} a_k) I\). Then \((L, \phi)\) is a covariant representation of \((c, \mathbb{N}, \tau, E^{\mathbb{N}_0})\), and \(L \times \phi(B) = C^*(\{S_k\})\).

Now let \(\{T_k : k \in \mathbb{N}\}\) be a family of isometries on a Hilbert space such that \(\sum T_k T_k^* < I\). By Cuntz’s theorem the map \(S_k \mapsto T_k\) extends to an isomorphism \(\pi\) of \(C^*(\{S_k\})\) onto \(C^*(\{T_k\})\). Let \(\sigma = \pi \circ (L \times \phi)\). The pair \((\sigma \circ i_c, \sigma \circ i_E) = (\pi \circ L, \pi \circ \phi)\) is not covariant since

\[
\sigma \circ i_c(\tau_1(1)) = \pi \circ L(\tau_1(1)) = \pi(I) = I
\]
whereas
\[
\alpha_1^{\sigma \circ i_c(1)}(\pi \circ L(1)) = \alpha_1^{\pi \circ \phi}(\pi \circ L(1)) = \sum_{k=1}^{\infty} \pi \circ \phi(\delta_k)\pi \circ \phi(\delta_k)^* = \sum_{k=1}^{\infty} T_k T_k^* < I.
\]

**Proposition 2.5.** If \((A,P,\beta,E)\) has a covariant representation, then it has a crossed product \((A \rtimes_{\beta,E}P, i_A, i_E)\) which is unique in the following sense: if \((B,i_A',i_E')\) is another crossed product for \((A,P,\beta,E)\), then there is an isomorphism \(\theta : A \rtimes_{\beta,E} P \to B\) such that \(\theta \circ i_A = i_A'\) and \(\theta \circ i_E = i_E'\).

**Proof.** Say that a covariant representation \((\pi,\phi)\) is cyclic if the \(C^*\)-algebra \(C^*(\pi,\phi)\) generated by \(\pi(A) \cup \phi(E)\) acts cyclically, i.e., has a cyclic vector. If \((\pi,\phi)\) is any covariant representation on \(\mathcal{H}\), the usual Zorn’s Lemma argument shows that \(\mathcal{H}\) is the direct sum of subspaces on which \(C^*(\pi,\phi)\) acts cyclically. These subspaces are then invariant for \(\pi\) and \(\phi\), and the projection \(Q\) onto such a subspace commutes with \(\pi(A)\) and \(*\)-commutes with \(\phi(E)\). Since compressing by \(Q\) preserves the strong operator convergence in \([2.1]\), the pair \((Q\pi,Q\phi)\) is covariant, and is cyclic because \(C^*(Q\pi,Q\phi) = QC^*(\pi,\phi)Q\) acts cyclically on \(Q\mathcal{H}\). Thus every covariant representation is a direct sum of cyclic representations.

Let \(S\) be a set of cyclic covariant representations with the property that every cyclic covariant representation of \((A,P,\beta,E)\) is unitarily equivalent to an element in \(S\). It can be shown that such a set \(S\) exists by fixing a Hilbert space \(\mathcal{H}\) of sufficiently large cardinality (depending on the cardinalities of \(A\) and \(E\) and considering only representations on \(\mathcal{H}\). Note that \(S\) is nonempty because the system has a covariant representation, which has a cyclic summand. Define \(i_A = \bigoplus_{(\pi,\phi) \in S} \pi\), \(i_E = \bigoplus_{(\pi,\phi) \in S} \phi\), and let \(A \rtimes_{\beta,E} P\) be the \(C^*\)-algebra generated by \(i_A(A) \cup i_E(E)\). Condition (a) for a crossed product is satisfied by taking \(\sigma\) to be the identity representation, condition (b) holds since every covariant representation decomposes as a direct sum of cyclic ones, and condition (c) was built into the definition of \(A \rtimes_{\beta,E} P\).

We now prove the uniqueness. Condition (a) allows us to realise \(A \rtimes_{\beta,E} P\) and \(B\) as \(C^*\)-subalgebras of \(\mathcal{B}(\mathcal{H})\) and \(\mathcal{B}(\mathcal{H}')\) in such a way that \((i_A, i_E)\) and \((i_A', i_E')\) become covariant representations of \((A,P,\beta,E)\). Condition (b) then gives a representation \(i_A' \times i_E' : A \rtimes_{\beta,E} P \to B\) whose range is contained in \(B\) because \((i_A' \times i_E') \circ i_A = i_A', (i_A' \times i_E') \circ i_E = i_E',\) and \(A \rtimes_{\beta,E} P\) is generated by \(i_A(A) \cup i_E(E)\). From (b) and (c) we see that \((i_A \times i_E) \circ (i_A' \times i_E')\) is the identity on \(A \rtimes_{\beta,E} P\), and similarly \((i_A' \times i_E') \circ (i_A \times i_E)\) is the identity on \(B\).

Hence \(\theta = i_A' \times i_E'\) is the desired isomorphism. \(\square\)

When \(P\) is a subsemigroup of a group \(G\), every twisted crossed product \(A \rtimes_{\beta,E} P\) carries a dual coaction of \(G\):

**Proposition 2.6.** Suppose \((A,P,\beta,E)\) is a twisted system which has a covariant representation. If \(P\) is a subsemigroup of a group \(G\), then there is an injective coaction
\[
\delta : A \rtimes_{\beta,E} P \to (A \rtimes_{\beta,E} P) \otimes_{\min} C^*(G)
\]
such that 
\[ \delta(i_A(a)) = i_A(a) \otimes 1 \quad \text{and} \quad \delta(i_E(v)) = i_E(v) \otimes i_G(p(v)). \]
If \( G \) is abelian, there is a strongly continuous action \( \hat{\beta} \) of \( \hat{G} \) on \( A \rtimes_{\beta,E} P \) such that 
\[ \hat{\beta}_\gamma(i_A(a)) = i_A(a) \quad \text{and} \quad \hat{\beta}_\gamma(i_E(v)) = \gamma(p(v))i_E(v). \]

Proof. Choose a faithful unital representation \( \sigma \) of \( A \rtimes_{\beta,E} P \) such that \( (\sigma \circ i_A, \sigma \circ i_E) \) is a covariant representation of \( (A, P, \tau, E) \), and a unitary representation \( U \) of \( G \) whose integrated form \( \pi_U \) is faithful on \( C^*(G) \). Then 
\[ ((\sigma \circ i_A) \otimes 1, (\sigma \circ i_E) \otimes (U \circ p)) \] is a covariant representation of \( (A, P, \beta, E) \), and hence there is a representation \( \rho \) of \( A \rtimes_{\beta,E} P \) such that 
\[ \rho(i_A(a)) = \sigma \circ i_A(a) \otimes I = (\sigma \otimes \pi_U)(i_A(a) \otimes 1) \]
and 
\[ \rho(i_E(v)) = \sigma \circ i_E(v) \otimes U(p(v)) = (\sigma \otimes \pi_U)(i_E(v) \otimes i_G(p(v))). \]
Since \( \sigma \) and \( \pi_U \) are faithful, \( \sigma \otimes \pi_U \) is faithful on \( (A \rtimes_{\beta,E} P) \otimes_{\text{min}} C^*(G) \), and we can define \( \delta := (\sigma \otimes \pi_U)^{-1} \circ \rho. \)

Next let \( \epsilon \) be the augmentation representation of \( C^*(G) \): \( \epsilon(i_G(s)) = 1 \) for all \( s \in G \). Then there is a representation \( \sigma \otimes \epsilon \) of \( (A \rtimes_{\beta,E} P) \otimes_{\text{min}} C^*(G) \) on \( \mathcal{H}_\sigma = \mathcal{H}_\sigma \otimes \mathbb{C} \), and checking on generators shows that \( (\sigma \otimes \epsilon) \circ \delta = \sigma. \) Thus \( \delta \) is injective. It is also easy to check on generators that \( (\text{id} \otimes \delta_G) \circ \delta = (\delta \otimes \text{id}) \circ \delta \) as homomorphisms of \( A \rtimes_{\beta,E} P \) into \( (A \rtimes_{\beta,E} P) \otimes C^*(G) \otimes C^*(G) \), so \( \delta \) is a coaction.

The last part follows because coactions of an abelian group \( G \) are in one-to-one correspondence with actions of \( \hat{G} \). Alternatively, one could use the uniqueness of the crossed product to obtain the automorphisms \( \hat{\beta}_\gamma \) directly, as in [13, Remark 3.6].

\[ \Box \]

3. Quasi-lattice Ordered Groups

Suppose \( P \) is a subsemigroup of a group \( G \) such that \( P \cap P^{-1} = \{e\} \). Then \( s \leq t \) iff \( s^{-1}t \in P \) defines a partial order on \( G \) which is left-invariant in the sense that \( s \leq t \) iff \( rs \leq rt \). Following [19] and [13], we say that \((G, P)\) is quasi-lattice ordered if every finite subset of \( G \) which has an upper bound in \( P \) has a least upper bound in \( P \). We shall occasionally write \( \sigma A \) for the least upper bound of a subset \( A \) of \( P \), and write \( \sigma A = \infty \) when \( A \) has no upper bound. Our main examples will be direct sums and free products of totally-ordered groups of the form \((\Gamma, \Gamma^+)\), where \( \Gamma \) is a countable subgroup of \( \mathbb{R} \) and \( \Gamma^+ = \Gamma \cap [0, \infty) \).

Remark 3.1. We shall not use the full strength of this definition, so our results may be slightly more general than we have claimed. To see why, recall from [19] that a partially ordered group \((G, P)\) is quasi-lattice ordered if and only if

(QL1) whenever \( g \in G \) has an upper bound in \( P \), it has a least upper bound in \( P \), and
(QL2) whenever \( s, t \in P \) have a common upper bound they have a least common upper bound.
We make no use of condition (QL1). All the results in §3 apply to cancellative semigroups which satisfy (QL2). In §3 it is necessary to assume that $P$ can be embedded in a group, but it makes no difference what the group is. The amenability results in §3 can be restated in terms of a homomorphism $\theta : P \to \mathcal{P}$ into a subsemigroup of an amenable group.

Recall from [19] and [13] that a representation $V$ of $P$ by isometries on a Hilbert space is called \textit{covariant} if

$$V_sV_s^*V_{t}^* = \begin{cases} V_{s\vee t}V_{s\vee t}^* & \text{if } s \vee t < \infty \\ 0 & \text{otherwise.} \end{cases}$$

We believe the appropriate generalisation to product systems over $P$ to be:

**Definition 3.2.** Suppose $(G, P)$ is a quasi-lattice ordered group and $E$ is a product system over $P$. A representation $\phi : E \to \mathcal{B}(\mathcal{H})$ is \textit{covariant} if

$$\alpha_s^\phi(I)\alpha_t^\phi(I) = \begin{cases} \alpha_{s\vee t}^\phi(I) & \text{if } s \vee t < \infty \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 3.3.** If $(G, P)$ is totally ordered, then $s \leq t$ implies $\alpha_t^\phi(I) \leq \alpha_s^\phi(I)$, so every representation of $E$ is covariant.

**Proposition 3.4.** If $(G, P)$ is a quasi-lattice ordered group and $E$ is a product system over $P$, then the left regular representation $\mathcal{L}$ of $E$ is covariant.

For the proof we shall need some basic properties of $\mathcal{L}$.

**Lemma 3.5.** Suppose $P$ is a left-cancellative semigroup with identity, $E$ is a product system over $P$, $\mathcal{L} : E \to \mathcal{B}(S(E))$ is the left regular representation, $v, w \in E$, and $s \in P$. Then

1. $l(v)^*w$ is zero unless $p(w) \in p(v)P$.
2. If $p(w) = p(v)r$ for some $r \in P$, then $l(v)^*w \in E_r \subset S(E)$.
3. $\alpha_s^\mathcal{L}(I)$ is the orthogonal projection onto $\bigoplus_{t \in sP} E_t$.

**Proof.** (1) Suppose $p(w) \notin p(v)P$. Then for any $u \in E$ we have $p(w) \neq p(v)p(u) = p(vu)$, so $\langle l(v)^*w, u \rangle = \langle w, vu \rangle = 0$. Thus $l(v)^*w = 0$.

(2) If $u \in E_r$ and $z \in E_r$, then $l(v)^*(uz) = l(v)^*l(u)z = \langle u, v \rangle z \in E_r$.

Since vectors of the form $uz$ have dense linear span in $E_{p(w)}$, this gives (2).

(3) Let $\mathcal{B}$ be an orthonormal basis for $E_s$. By (1) above,

$$\alpha_s^\mathcal{L}(I)w = \sum_{e \in \mathcal{B}} l(e)l(e)^*w = 0$$

unless $p(w) \in sP$. If $w = uz$ with $u \in \mathcal{B}$, $z \in E$, then

$$\alpha_s^\mathcal{L}(I)uz = \sum_{e \in \mathcal{B}} l(e)l(e)^*l(u)z = \sum_{e \in \mathcal{B}} (u, e)l(e)z = uz.$$  

Since vectors of this form are total in $\bigoplus_{t \in sP} E_t$, this gives (3).

**Proof of Proposition 3.4.** From Lemma 3.5(3) we deduce that $\alpha_s^\mathcal{L}(I)\alpha_t^\mathcal{L}(I)$ is the projection onto $\bigoplus \{ E_r : r \in sP \cap tP \}$. But $r \in sP \cap tP \iff r \geq s$ and $r \geq t \iff r \geq s \vee t$, so this is precisely the range of $\alpha_{s\vee t}^\mathcal{L}(I)$.

$\square$
Since we shall be doing a lot of calculations with covariant representations, we shall give some basic properties, and an alternative characterisation.

**Lemma 3.6.** Suppose \((G, P)\) is a quasi-lattice ordered group, \(E\) is a product system over \(P\), \(\phi\) is a representation of \(E\) on \(H\), \(u \in E\) and \(s \in P\).

1. If \(p(u) \leq s\), then \(\alpha_s^\phi(A)\phi(u) = \phi(u)\alpha_s^\phi(A)\) for any \(A \in \mathcal{B}(H)\).
2. If \(\phi\) is covariant, then

\[
\alpha_s^\phi(I)\phi(u) = \begin{cases} 
\phi(u)\alpha_s^\phi(p(u)^{-1}(p(u)\lor s)) & \text{if } p(u) \lor s < \infty, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** Suppose \(p(u) \leq s\). Since \(\alpha_s^\phi(A)\phi(u) = \phi(u)A\) for each \(A \in \mathcal{B}(H)\),

\[
\alpha_s^\phi(A)\phi(u) = \alpha_s^\phi\left(\alpha_s^\phi(A)\right)\phi(u) = \phi(u)\alpha_s^\phi(A),
\]

giving (1). If \(\phi\) is covariant, then \(\alpha_s^\phi(I)\phi(u) = \alpha_s^\phi(I)\alpha_s^\phi(I)\phi(u)\) is zero unless \(p(u) \lor s < \infty\), in which case

\[
\alpha_s^\phi(I)\phi(u) = \alpha_s^\phi(I)\alpha_s^\phi(I)\phi(u) = \alpha_s^\phi(I)\phi(u) = \phi(u)\alpha_s^\phi(p(u)^{-1}(p(u)\lor s))I,
\]

giving (2).

\[\square\]

**Proposition 3.7.** Suppose \((G, P)\) is a quasi-lattice ordered group, \(E\) is a product system over \(P\) and \(\phi\) is a representation of \(E\) on \(H\).

1. Suppose \(v, w \in E\) satisfy \(p(v) \lor p(w) < \infty\), and \(\mathcal{B}, \mathcal{C}\) are orthonormal bases for \(E_{p(v)^{-1}(p(v)\lor p(w))}\) and \(E_{p(w)^{-1}(p(v)\lor p(w))}\), respectively. Then the series

\[
\sum_{f \in \mathcal{B}, g \in \mathcal{C}} \langle wg, vf \rangle \phi(f)\phi(g)^*,
\]

converges \(\sigma\)-weakly to a bounded operator on \(H\).

2. \(\phi\) is covariant if and only if for every \(v, w \in E\)

\[
\phi(v)^*\phi(w) = \begin{cases} 
\sum_{f, g} \langle wg, vf \rangle \phi(f)\phi(g)^* & \text{if } p(v) \lor p(w) < \infty, \\
0 & \text{otherwise}.
\end{cases}
\]

**Remark 3.8.** If \((G, P)\) is totally ordered, then either \(f\) or \(g\) disappears from the sum in (3.1), and thus the series is norm convergent.

**Proof of Proposition 3.7.** (1) It does no harm to assume that \(v\) and \(w\) are unit vectors. Then the series \(\sum_{f} vf \otimes \overline{vf}\) and \(\sum_{g} wg \otimes \overline{wg}\) converge strongly in the unit ball of \(B(E_{p(v)^{-1}(p(v)\lor p(w))})\), and thus the series

\[
\sum_{f, g} (vf \otimes \overline{vf})(wg \otimes \overline{wg}) = \sum_{f, g} \langle wg, vf \rangle vf \otimes \overline{wg}
\]

converges strongly to a bounded operator on \(E_{p(v)^{-1}(p(v)\lor p(w))}\). Since this convergence also occurs in the unit ball, the series converges \(\sigma\)-weakly. Applying the isomorphism \(\rho_{p(v)^{-1}(p(v)\lor p(w))}\) gives that the series \(\sum_{f, g} \langle wg, vf \rangle \phi(vf)\phi(wg)^*\)
converges $\sigma$-weakly, and multiplying on the left by $\phi(v)^*$ and on the right by $\phi(w)$ gives (1).

(2) If $\phi$ is covariant, then
\[
\phi(v)^*\phi(w) = \phi(v)^*\alpha^\phi_{p(v)}(I)\alpha^\phi_{p(w)}(I)\phi(w)
\]
is zero unless $p(v) \lor p(w) < \infty$, in which case
\[
\phi(v)^*\phi(w) = \alpha^\phi_{p(v)\lor p(w)}(I)\phi(w)
= \left(\sum_f \phi(f)\phi(f)^*\right)\phi(v)^*\phi(w)\left(\sum_g \phi(g)\phi(g)^*\right)
= \sum_{f,g} \phi(f)\phi(vf)^*\phi(wg)\phi(g)^*
= \sum_{f,g} \langle wg, vf \rangle\phi(f)\phi(g)^*.
\]

Conversely, suppose (3.1) holds for every $v, w \in E$. Let $s, t \in P$. Summing over $v, w$ in orthonormal bases for $E_s$ and $E_t$, respectively, we find that
\[
\alpha^\phi_s(I)\alpha^\phi_t(I) = \left(\sum_v \phi(v)\phi(v)^*\right)\left(\sum_w \phi(w)\phi(w)^*\right)
= \sum_{v,w} \phi(v)\phi(v)^*\phi(w)\phi(w)^*
\]
is zero unless $s \lor t < \infty$, in which case
\[
\alpha^\phi_s(I)\alpha^\phi_t(I) = \sum_{v,w} \phi(v)\left(\sum_{f,g} \langle wg, vf \rangle\phi(f)\phi(g)^*\right)\phi(w)^*
= \sum_{v,w} \sum_{f,g} \langle wg, vf \rangle\phi(vf)\phi(wg)^*
= \sum_{v,w} \sum_{f,g} \phi(vf)\phi(vf)^*\phi(wg)\phi(wg)^*
= \rho^\phi_{s\lor t}\left(\sum_{v,w} \sum_{f,g} (vf \otimes \overline{vf})(wg \otimes \overline{wg})\right)
= \rho^\phi_{s\lor t}(I)
= \alpha^\phi_{s\lor t}(I),
\]
as required. \qed

4. The system $(B_P, P, \tau, E)$

Suppose $(G, P)$ is a quasi-lattice ordered group and $E$ is a product system over $P$. For each $t \in P$ denote by $1_t$ the projection in $\ell^\infty(P)$ defined by
\[
1_t(s) = \begin{cases} 1 & \text{if } s \geq t \\ 0 & \text{otherwise.} \end{cases}
\]
The product $1_s1_t$ is $1_{s\lor t}$ if $s \lor t < \infty$ and 0 otherwise; it follows that span$\{1_t : t \in P\}$ is a $*$-algebra, whose closure is a $C^*$-subalgebra $B_P$ of
$\ell^\infty(P)$. The action of $P$ by left translation on $\ell^\infty(P)$ restricts to an action
$\tau$ of $P$ on $B_p$ such that $\tau_s(1_t) = 1_{st}$ for $s, t \in P$. We are interested in
the twisted system $(B_p, P, \tau, E)$ because its covariant representations are in
one-to-one correspondence with the covariant representations of $E$.

**Proposition 4.1.** Suppose $(G, P)$ is a quasi-lattice ordered group and $E$ is
a product system over $P$.

1. If $(\pi, \phi)$ is a covariant representation of $(B_p, P, \tau, E)$, then $\phi$ is a
covariant representation of $E$ and $\pi(1_s) = \alpha^\phi_s(I)$.

2. If $\phi$ is a covariant representation of $E$, then there is a representation
$\pi_\phi$ of $B_p$ such that $\pi_\phi(1_s) = \alpha^\phi_s(I)$; moreover, $(\pi_\phi, \phi)$ is then a covariant
representation of $(B_p, P, \tau, E)$.

3. $\pi_\phi$ is faithful iff $\prod_{k=1}^n (I - \alpha^\phi_{s_k}(I)) \neq 0$ whenever $s_1, \ldots, s_n \in P \setminus \{e\}$.

**Proof.** (1) If $(\pi, \phi)$ is covariant, then $\alpha^\phi_s(I) = \pi(\tau_s(1)) = \pi(1_s)$, so the
covariance of $\phi$ follows from the identity $1_s 1_t = 1_{st}$.

(2) If $\phi$ is a covariant representation of $E$, then by [3, Proposition 1.3]
the map $1_s \mapsto \alpha^\phi_s(I)$ extends uniquely to a representation $\pi_\phi$ of $B_p$. Since
$\pi_\phi(\tau_s(1)) = \pi_\phi(1_{st}) = \alpha^\phi_s(I) = \alpha^\phi_s(\alpha^\phi_t(I)) = \alpha^\phi_s(\pi_\phi(1_t))$, $(\pi_\phi, \phi)$ is a covariant
representation of $(B_p, P, \tau, E)$.

(3) By [3, Proposition 1.3], it suffices to show that

$$\prod_{k=1}^n (\alpha^\phi_a(I) - \alpha^\phi_{z_k}(I)) \neq 0$$

 whenever $a, z_1, \ldots, z_n \in P$ and $a < z_k$ for $k = 1, \ldots, n$. But $a < z_k$ means
$z_k = a \cdot s_k$ for some $s_k \in P \setminus \{e\}$, and

$$\prod_{k=1}^n (\alpha^\phi_a(I) - \alpha^\phi_{z_k}(I)) = \alpha^\phi_a \left( \prod_{k=1}^n (I - \alpha^\phi_{s_k}(I)) \right),$$

so the injectivity of $\alpha^\phi_a$ implies that (4.1) is equivalent to (3). \qed

**Corollary 4.2.** If $(G, P)$ is a quasi-lattice ordered group and $E$ is a product
system over $P$, then the system $(B_p, P, \tau, E)$ has a covariant representation,
and $i_{B_p} : B_p \to B_p \rtimes_{\tau, E} P$ is faithful.

**Proof.** Since the left regular representation $l : E \to B(S(E))$ is covariant
(Proposition 3.4), the pair $(\pi_l, l)$ is a covariant representation of $(B_p, P, \tau, E)$. Lemma [3,3(3)] implies that the identity $\Omega$ of $E$, viewed as an element of
$E_e \subset S(E)$, is in the range of the projection $I - \alpha^\phi_s(I)$ whenever $s \in P \setminus \{e\}$,
and hence $\pi_l$ is faithful. Since $\pi_l$ factors through $i_{B_p}$, this in turn implies
that $i_{B_p}$ is faithful. \qed

**Theorem 4.3.** Suppose $(G, P)$ is a quasi-lattice ordered group and $E$ is a
product system over $P$. The $C^*$-subalgebra $A$ of $B_p \rtimes_{\tau, E} P$ generated by the
range of the canonical embedding $i_E$ is universal for covariant representations
of $E$, in the sense that:

- there is a faithful unital representation $\sigma$ of $A$ on Hilbert space such
that $\sigma \circ i_E$ is a covariant representation of $E$, and
(b) for every covariant representation $\phi$ of $E$ there is a unital representation $\pi$ of $A$ such that $\phi = \pi \circ i_E$.

If $E$ has finite-dimensional fibres, the algebra $A$ is all of $BP \rtimes_{\tau,E} P$, and

$$BP \rtimes_{\tau,E} P = \text{span}\{i_E(u)i_E(v)^* : u, v \in E\}.$$  \hfill (4.2)

**Remark 4.4.** Since the usual argument shows that there is at most one pair $(A,i_E)$ with these properties (see the proof of Proposition 3.7), we can reasonably write $C^*_{\text{cov}}(P,E)$ for $A := C^*(i_E(E))$.

**Remark 4.5.** As in [2] one would expect and prefer to be able to replace condition (a) by something like

(a') for every representation $\sigma$ of $A$, $\sigma \circ i_E$ is a covariant representation of $E$.

If the sum in (3.1) is always norm convergent, then Proposition 3.7 implies that conditions (a) and (a') are both equivalent to the following:

$$i_E(v)^*i_E(w) = \begin{cases} \sum_{f,g} \langle wg,vf \rangle i_E(f)i_E(g)^* & \text{if } p(v) \vee p(w) < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where the sum runs through orthonormal bases for $E_{p(v)^{-1}(p(v)\vee p(w))}$ and $E_{p(w)^{-1}(p(v)\vee p(w))}$. This is the case when the fibres of $E$ are finite-dimensional, or when $(G,P)$ is totally ordered (Remark 3.3). For this class of product systems $C^*_{\text{cov}}(P,E)$ is indeed universal. In a subsequent paper we will study a larger class of product systems for which (a') holds in $C^*_{\text{cov}}(P,E)$.

**Proof of Theorem 4.3.** We can represent $BP \rtimes_{\tau,E} P$ faithfully on a Hilbert space $K$ in such a way that $(i_B, i_E)$ becomes a covariant representation of $(BP, P, \tau, E)$, and then $i_E$ is a covariant representation of $E$ by Proposition 4.1(1). If $\phi$ is a covariant representation of $E$, then Proposition 4.1(2) gives us a covariant representation $(\pi_\phi, \phi)$ of $(BP, P, \tau, E)$, and hence a representation $\pi_\phi \times \phi$ of $BP \rtimes_{\tau,E} P$ such that $(\pi_\phi \times \phi) \circ i_E = \phi$. Restricting $\pi_\phi \times \phi$ to $A$ gives the required representation $\pi$.

Suppose now that $s \in P$ and that dim $E_s < \infty$. If $B$ is an orthonormal basis for $E_s$, then

$$i_B(1_s) = i_B(\tau_s(1)) = \alpha_s^{i_E}(i_B(1)) = \sum_{u \in B} i_E(u)i_E(u)^*$$

belongs to $C^*_{\text{cov}}(P,E)$. Thus if all the fibres of $E$ are finite-dimensional we have $C^*_{\text{cov}}(P,E) = BP \rtimes_{\tau,E} P$.

To establish (4.2), it suffices to show that span$\{i_E(u)i_E(v)^* : u, v \in E\}$ is closed under multiplication. But by Proposition 3.7 each product $i_E(u)i_E(v)^*i_E(w)i_E(z)^*$ is zero unless $p(v) \vee p(w) < \infty$, in which case it is a finite sum of operators of the form $i_E(u)f_iE(zg)^*$.

5. FAITHFUL REPRESENTATIONS

Our characterisation of faithful representations of $BP \rtimes_{\tau,E} P$ requires an amenability hypothesis, which we shall discuss shortly, and a spanning hypothesis, which says that

$$BP \rtimes_{\tau,E} P = \text{span}\{i_E(u)i_B(1_s)i_E(v)^* : u, v \in E, \ s \in P\}.$$  \hfill (5.1)
This spanning hypothesis is automatically satisfied if $E$ has finite-dimensional fibres (Theorem 4.3), or if $G$ is totally ordered (in which case we can simplify monomials using $i_{B_P}(1_s)i_E(u) = i_E(u)$ or $i_E(u)i_{B_P}(1_P(u^{-1}s))$, and the norm convergent expansion (3.1)).

When the enveloping group $G$ of $P$ is abelian, the system $(B_P, P, \tau, E)$ is amenable if averaging over the dual action $\hat{\tau}$ of $\hat{G}$ gives a faithful expectation onto the fixed-point algebra. In general we use the dual coaction $\delta$ of $B_P \rtimes_{\tau,E} P$ (Proposition 2.4), and the canonical trace $\rho$ on $C^*(G)$ extending $f \mapsto f(e) : \ell^1(G) \to \mathbb{C}$. Then $\Phi_\delta := (\text{id} \otimes \rho) \circ \delta$ is a positive linear map of norm one of $B := B_P \rtimes_{\tau,E} P$ onto the fixed-point algebra $B^\delta := \{b \in B : \delta(b) = b \otimes 1\}$ (see [13, 2.3] or [22, Lemma 1.3]). A quick look at the characterisation of the coaction $\delta$ on generators shows that

$$
\Phi_\delta(i_E(u)i_{B_P}(1_s)i_E(v)^*) = \begin{cases}
i_E(u)i_{B_P}(1_s)i_E(v)^* & \text{if } p(u) = p(v) \\
0 & \text{otherwise,}
\end{cases}
$$

and under the spanning hypothesis (5.1) this characterises $\Phi_\delta$. (This implies, incidentally, that the expectation $\Phi_\delta$ is independent of the choice of enveloping group $G$.) We say the system is amenable if $\Phi_\delta$ is faithful in the sense that $\Phi_\delta(b^*b) = 0$ implies $b = 0$. The argument of [13, Lemma 6.5] shows that if the enveloping group $G$ is amenable, then $(B_P, P, \tau, E)$ is amenable in our sense; in the next section we shall give further examples in which $P$ and $G$ are free products.

We can now state our main theorem.

**Theorem 5.1.** Suppose $(G, P)$ is a quasi-lattice ordered group, $(B_P, P, \tau, E)$ is an amenable twisted system which satisfies the spanning hypothesis (5.1), and $\phi$ is a covariant representation of $E$. Then $\pi_\phi \times \phi$ is a faithful representation of $B_P \rtimes_{\tau,E} P$ if and only if

$$
\prod_{k=1}^n (I - \alpha_{s_k}^\phi(I)) \neq 0 \quad \text{whenever } s_1, \ldots, s_n \in P \setminus \{e\}.
$$

One direction is trivial: if $\pi_\phi \times \phi$ is faithful, then by Corollary 4.2 so is $\pi_\phi = (\pi_\phi \times \phi) \circ i_{B_P}$, and then (5.2) follows from Proposition 4.1(3). For the other direction, we follow the strategy of [13, §3]. We show that for systems which satisfy the spanning hypothesis, faithfulness of $\pi_\phi$ is sufficient to construct a spatial version $\Phi_\phi$ of $\Phi_\delta$ such that

$$
\begin{array}{c}
B_P \rtimes_{\tau,E} P \xrightarrow{\pi_\phi \times \phi} \pi_\phi \times \phi(B_P \rtimes_{\tau,E} P) \\
\downarrow \Phi_\delta \quad \quad \quad \quad \quad \downarrow \Phi_\phi \\
(B_P \rtimes_{\tau,E} P)^\delta \xrightarrow{\pi_\phi \times \phi} \pi_\phi \times \phi((B_P \rtimes_{\tau,E} P)^\delta)
\end{array}
$$

commutes (Proposition 5.3). We also show that $\pi_\phi \times \phi$ is faithful on the fixed-point algebra (Proposition 5.4), and the amenability of the system completes the chain

$$
\pi_\phi \times \phi(b) = 0 \implies \Phi_\phi(\pi_\phi \times \phi(b^*b)) = 0
$$

$$
\iff \pi_\phi \times \phi(\Phi_\delta(b^*b)) = 0
$$

$$
\iff \Phi_\delta(b^*b) = 0
$$

$$
\implies b = 0.
$$
We begin by recalling some conventions from [13, Lemma 1.4]. Suppose $F$ is a finite subset of $P$. For each subset $A$ of $F$, define a projection $Q_A$ in $B_P$ by

$$Q_A = \begin{cases} 1_{\sigma A} \prod_{t \in F \setminus A} (1 - 1_t) & \text{if } \sigma A < \infty \\ 0 & \text{otherwise,} \end{cases}$$

with the convention that $\sigma \emptyset = e$.

Remark 5.2. It can be routinely verified that $Q_A(s) = 1$ iff $A = \{t \in F : t \leq s\}$. Thus $\{Q_A : A \subset F\}$ is a decomposition of the identity into mutually orthogonal projections, and $Q_A$ is nonzero iff $A$ is an initial segment of $F$ in the sense that $\sigma A < \infty$ and $A = \{t \in F : t \leq \sigma A\}$. In this case,

$$Q_A = \prod_{\{t \in F : \sigma A < \sigma A \lor t < \infty\}} (1_{\sigma A} - 1_{\sigma A \lor t}) = \tau_{\sigma A} \left( \prod_{\{t \in F : \sigma A < \sigma A \lor t < \infty\}} (1 - 1_{\sigma A^{-1}(\sigma A \lor t)}) \right)$$

Thus if $\phi$ is a covariant representation of $E$ and $A$ is an initial segment of $F$,

$$\pi_\phi(Q_A) = \alpha_{\sigma A}^{\phi} \left( \prod_{\{t \in F : \sigma A < \sigma A \lor t < \infty\}} (I - \alpha_{\sigma A^{-1}(\sigma A \lor t)}^{\phi}(I)) \right).$$

The following technical lemma will be used in the proofs of both Proposition 5.4 and Proposition 5.5.

Lemma 5.3. Suppose $(G, P)$ is a quasi-lattice ordered group, $E$ is a product system over $P$, $\phi$ is a covariant representation of $E$, $F$ is a finite subset of $P$, $A$ is an initial segment of $F$, $u, v \in E$ and $s \in P$. Let $a = \sigma A$, so that $A = \{t \in F : t \leq a\}$.

1. If $p(u) = p(v)$, then the operator $\phi(u)\alpha_{s}^{\phi}(I)\phi(v)^*$ is in the commutant of $\pi_\phi(B_P)$. In particular, it commutes with $\pi_\phi(Q_A)$.
2. If $p(u)s, p(v)s \in F$, then

$$\pi_\phi(Q_A)\phi(u)\alpha_{s}^{\phi}(I)\phi(v)^*\pi_\phi(Q_A) = \begin{cases} \pi_\phi(Q_A)\phi(u)\alpha_{p(u)-1a}^{\phi}(I)\alpha_{p(v)-1a}^{\phi}(I)\phi(v)^*\pi_\phi(Q_A) & \text{if } p(u)s \leq a \text{ and } p(v)s \leq a \\ 0 & \text{otherwise}. \end{cases}$$

Proof. (1) Suppose $p(u) = p(v)$; it suffices to show that $\phi(u)\alpha_{s}^{\phi}(I)\phi(v)^*$ commutes with $\pi_\phi(1_t)$ for each $t \in P$. If $p(u)s$ and $t$ have no common upper bound, then by Lemma 3.6,

$$\pi_\phi(1_t)\phi(u)\alpha_{s}^{\phi}(I)\phi(v)^* = \alpha_{1}^{\phi}(I)\alpha_{p(u)-1a}^{\phi}(I)\phi(u)\phi(v)^* = 0 = \phi(u)\phi(v)^*\alpha_{p(v)-1a}^{\phi}(I)\phi(v)^*\pi_\phi(1_t).$$
Similarly, \( \tau(E) \) is a covariant representation of \( E \).

**Proposition 5.4.** Suppose \((G, P)\) is a quasi-lattice ordered group, \( E \) is a product system over \( P \) which satisfies the spanning hypothesis \((5.1)\), and \( \phi \) is a covariant representation of \( E \) which satisfies \((5.2)\). Then the representation \( \pi_\phi \times \phi \) of \( B_P \rtimes_{\tau, E} P \) is isometric on \( (B_P \rtimes_{\tau, E} P)^\delta \).
Proof. Let \( X \) be a nonzero element in \( B_P \times_{\tau,E} P \) of the form
\[
X = \sum_{(u,s,v) \in J} i_E(u) i_{B_P}(1_s) i_{E}(v)^*,
\]
where \( J \) is a finite subset of \( \{(u,s,v) \in E \times P \times E : p(u) = p(v)\} \). The spanning hypothesis (5.1) implies that elements such as \( X \) are dense in \( (B_P \times_{\tau,E} P)^{\beta} \), so it suffices to show that
\[
(5.6) \quad \|\pi_\phi \times \phi(X)\| = \|X\|.
\]
Let \( \sigma \) be a faithful representation of \( B_P \times_{\tau,E} P \) such that \( (\sigma \circ i_{B_P}, \sigma \circ i_{E}) \) is a covariant representation of \( (B_P, P, \tau, E) \). By Proposition 1.1, \( i := \sigma \circ i_{E} \) is a covariant representation of \( E \) and \( \sigma \circ i_{B_P} = \pi_i \); in particular \( \pi_i(1_s) = \alpha^\psi_1(I) \) for each \( s \in P \).

Let \( F = \{p(u)s : (u,s,v) \in J\} \). By Lemma 5.3, the operator \( \pi_i \times i(X) \) commutes with each \( \pi_i(Q_A) \). Since these projections form a decomposition of the identity, there is a subset \( A \subseteq F \) such that
\[
\|\pi_i(Q_A)\pi_i \times i(X)\| = \|\pi_i \times i(X)\| = \|X\|.
\]
Since \( X \neq 0 \), we have \( \pi_i(Q_A) \neq 0 \). From Remark 5.2 we deduce that \( a := \tau A < \infty \) and \( A \) is the initial segment \( \{t \in F : t \leq a\} \).

Let \( K := \{(u,s,v) \in J : p(u)s \leq a\} \), and define \( T \in \mathcal{B}(E_a) \) by
\[
T = \sum_{(u,s,v) \in K} \beta_{a,p(u)}(u \otimes \overline{v}).
\]
We claim that
\[
(5.7) \quad \|\pi_\phi \times \phi(X)\| \geq \|T\| = \|X\|,
\]
from which (5.6) is immediate. Suppose \( \psi \) is a covariant representation of \( E \); we shall later take \( \psi = i \) and \( \psi = \phi \). By Lemma 5.3 and (5.5),
\[
\pi_\psi(Q_A)\pi_\psi \times \psi(X) = \pi_\psi(Q_A) \sum_{(u,s,v) \in J} \psi(u) \alpha^\psi_s(I) \psi(v)^*
\]
\[
= \pi_\psi(Q_A) \sum_{(u,s,v) \in K} \psi(u) \alpha^\psi_{\rho(u)-1_s}(I) \psi(v)^*
\]
\[
= \pi_\psi(Q_A) \sum_{(u,s,v) \in K} \rho^\psi_a \left( \beta_{a,p(u)}(u \otimes \overline{v}) \right)
\]
\[
= \pi_\psi(Q_A) \rho^\psi_a(T).
\]
From (5.4) we see that \( \pi_\psi(Q_A) \) is in the range of \( \alpha^\psi_a \). If \( \pi_\psi(Q_A) \neq 0 \), Proposition 1.12 implies that \( S \mapsto \pi_\psi(Q_A) \rho^\psi_a(S) \) is a faithful representation of \( \mathcal{B}(E_a) \), so that
\[
\|\pi_\psi(Q_A)\pi_\psi \times \psi(X)\| = \|\pi_\psi(Q_A)\rho^\psi_a(T)\| = \|T\|.
\]
We have already seen that \( \pi_i(Q_A) \neq 0 \), and since (5.2) implies that \( \pi_\phi \) is faithful, we deduce that \( \pi_\phi(Q_A) \) is nonzero. Thus
\[
\|\pi_\phi \times \phi(X)\| \geq \|\pi_\phi(Q_A)\pi_\phi \times \phi(X)\| = \|T\| = \|\pi_i(Q_A)\pi_i \times i(X)\| = \|X\|,
\]
so that (5.7) holds as claimed. \( \square \)
Proposition 5.5. Suppose \((G, P)\) is a quasi-lattice ordered group, \(E\) is a product system over \(P\) which satisfies the spanning hypothesis (5.1), and \(\phi\) is a covariant representation of \(E\) which satisfies (5.2). Let \(\Delta = \{(u, s, v) \in E \times P \times E : p(u) = p(v)\}\). Then there is a linear map \(\Phi_\phi\) of norm one of \(\pi_\phi \times \phi(B_P \rtimes \tau, E^\phi)\) onto \(\pi_\phi \times \phi((B_P \rtimes \tau, E^\phi)^\delta)\) such that, for each finite subset \(J\) of \(E \times P \times E\),

\[
\Phi_\phi \left( \sum_{(u, s, v) \in J} \phi(u) \alpha_{s}^{\phi}(I) \phi(v)^* \right) = \sum_{(u, s, v) \in J \cap \Delta} \phi(u) \alpha_{s}^{\phi}(I) \phi(v)^*.
\]

Proof. Fix a finite subset \(J\) of \(E \times P \times E\) and let

\[
X = \sum_{(u, s, v) \in J} \phi(u) \alpha_{s}^{\phi}(I) \phi(v)^*, \quad X_\Delta = \sum_{(u, s, v) \in J \cap \Delta} \phi(u) \alpha_{s}^{\phi}(I) \phi(v)^*.
\]

We will show that \(\|X_\Delta\| \leq \|X\|\), so that \(\Phi_\phi\) is well-defined on finite sums such as \(X\) and extends to a projection of norm one on their closure, which by the spanning hypothesis (5.1) is all of \(\pi_\phi \times \phi((B_P \rtimes \tau, E^\phi)^\delta)\). Certainly we may assume that \(X_\Delta \neq 0\). Let

\[
F = \{p(u)s : (u, s, v) \in J\} \cup \{p(v)s : (u, s, v) \in J\}.
\]

By Lemma 5.3, \(X_\Delta\) commutes with each \(\pi_\phi(Q_A)\), and since the \(Q_A\) form a decomposition of the identity, there exists \(A \subseteq F\) such that

\[
\|X_\Delta\| = \|\pi_\phi(Q_A)X_\Delta\|.
\]

Because \(X_\Delta \neq 0\), we have \(\pi_\phi(Q_A) \neq 0\), so by Remark 5.2, we have \(\alpha_A < \infty\) and \(A = \{t \in F : t \leq a\}\). It follows from (5.4) that \(\pi_\phi(Q_A)\) is in the range of \(\alpha_a^\phi\).

Define \(T \in B(E_a)\) by

\[
T = \sum_{\{u, s, v\} \in J \cap \Delta : p(u) \leq a} \beta_{a, p(u)}(u \otimes v).
\]

Exactly as in the proof of Proposition 5.4 we have \(\pi_\phi(Q_A)X_\Delta = \pi_\phi(Q_A)\rho_a^\phi(T)\), so that by Proposition 1.12 we have

\[
\|X_\Delta\| = \|\pi_\phi(Q_A)X_\Delta\| = \|\pi_\phi(Q_A)\rho_a^\phi(T)\| = \|T\|.
\]

We will construct another nonzero projection \(Q\) in the range of \(\alpha_a^\phi\) with the property that \(QXQ = Q\rho_a^\phi(T)\). This will complete the proof, since from this another application of Proposition 1.12 gives

\[
\|X_\Delta\| = \|T\| = \|Q\rho_a^\phi(T)\| = \|QXQ\| \leq \|X\|.
\]

For each \(b, c \in A\) such that \(b \neq c\) and \(b^{-1}a \vee c^{-1}a < \infty\), define \(d_{b, c} \in P\) as in [13, Lemma 3.2]:

\[
d_{b, c} = \begin{cases} (b^{-1}a)^{-1}(b^{-1}a \vee c^{-1}a) & \text{if } b^{-1}a < b^{-1}a \vee c^{-1}a \\ (c^{-1}a)^{-1}(b^{-1}a \vee c^{-1}a) & \text{otherwise}, \end{cases}
\]

noting in particular that \(d_{b, c}\) is never the identity in \(P\). Define

\[
R' = \prod_{b \neq c \in A \atop b^{-1}a \vee c^{-1}a < \infty} (I - \alpha_{d_{b, c}}^\phi(I)),
\]
\[
Q' = \prod_{\substack{t \in E \ni \alpha \in \mathbb{N}^+ \land \alpha < a \land \alpha \lor t < \infty}} (I - \alpha^\phi_{p(u^{-1}a)}(I)) \prod_{\substack{b \neq c \in A \ni b^{-1}a \lor c^{-1}a < \infty}} (I - \alpha^\phi_{d_{b,c}}(I)),
\]

\(R = \alpha^\phi_R(Q')\) and \(Q = \alpha^\phi_Q(\mathbb{R})\), so that \(Q = \pi^\phi_\mathbb{R}(Q_A) R\). By condition (3.2), \(Q' \neq 0\), and thus \(Q\) is a nonzero projection in the range of \(\alpha^\phi_a\). We claim that \(Q\) is the desired projection satisfying \(Q X Q = Q \rho^\phi_a(T)\).

To begin with, because \(Q \leq \pi^\phi_\mathbb{R}(Q_A)\), we can use Lemma 5.3 to rewrite
\[
(5.8) \quad Q X Q = Q \left( \sum_{(u,s,v) \in J, p(u)s \leq a} \phi(u) a^\phi_{p(u) - 1a}(I) \alpha^\phi_{p(v) - 1a}(I) \phi(v)^* \right) Q.
\]

Now suppose \((u,s,v) \in J, p(u)s \leq a, p(v)s \leq a\) and \(p(u) \neq p(v)\). If \(p(u)^{-1}a\) and \(p(v)^{-1}a\) have no common upper bound, then the corresponding term in the above sum is zero. On the other hand, if \(p(u)^{-1}a \lor p(v)^{-1}a < \infty\), then
\[
(p(u)s)^{-1}a \lor (p(v)s)^{-1}a = s^{-1}(p(u)^{-1}a \lor p(v)^{-1}a) < \infty.
\]

Let \(d = d_{p(u)s,p(v)s}\). The previous equation shows that \(d\) is either \((p(u)^{-1}a)^{-1}(p(u)^{-1}a \lor p(v)^{-1}a)\) or \((p(v)^{-1}a)^{-1}(p(u)^{-1}a \lor p(v)^{-1}a)\).

Then \(R \leq \alpha^\phi_a(I) - \alpha^\phi_{ad}(I)\), and
\[
(\alpha^\phi_a(I) - \alpha^\phi_{ad}(I)) \phi(u) a^\phi_{p(u) - 1a}(I) \alpha^\phi_{p(v) - 1a}(I) \phi(v)^* (\alpha^\phi_a(I) - \alpha^\phi_{ad}(I))
\]
\[
= \phi(u)(\alpha^\phi_{p(u) - 1a}(I) - \alpha^\phi_{p(u) - 1a}(I)) \alpha^\phi_{p(v) - 1a}(I)
\]
\[
= \alpha^\phi_{p(v) - 1a}(I)(\alpha^\phi_{p(v) - 1a}(I) - \alpha^\phi_{p(v) - 1a}(I)) \phi(v)^*
\]
\[
= 0,
\]

since either \(p(u)^{-1}ad\) or \(p(v)^{-1}ad\) is equal to \(p(u)^{-1}a \lor p(v)^{-1}a\). This shows that
\[
R(\phi(u) a^\phi_{p(u) - 1a}(I) \alpha^\phi_{p(v) - 1a}(I) \phi(v)^*) R = 0
\]
for each \((u,s,v) \in J\) satisfying \(p(u)s \leq a, p(v)s \leq a\) and \(p(u) \neq p(v)\). Equation (5.8) now simplifies to
\[
Q X Q = Q \left( \sum_{(u,s,v) \in J, p(u)s = p(v)s \leq a} \phi(u) a^\phi_{p(u) - 1a}(I) \phi(v)^* \right) Q = Q \rho^\phi_a(T),
\]

so this \(Q\) will suffice.

**Examples 5.6.** (1) Applying Theorem 5.1 to the trivial product system \(P \times \mathbb{R}\) gives \([13,\text{Theorem 3.7}]\). More generally, if \(\mu\) is a multiplier on \(P\), applying it to \((P \times \mathbb{C})^\mu\) gives a characterisation of the faithful representations of the universal \(C^*\)-algebra for covariant \(\mu\)-representations of \((G,P)\).

(2) If \(E\) is a product system over \(\mathbb{N}\) with \(\dim E_1 = n < \infty\), then \(C_{cov}^*(P,E)\) is the Toeplitz-Cuntz algebra \(T\mathcal{O}_n\). In this case, our Theorem 5.1 reduces to Cuntz’s Theorem: the representation of \(T\mathcal{O}_n\) corresponding to a Toeplitz-Cuntz family \(\{V_1, V_2, \ldots, V_n\}\) is faithful iff \(\sum V_k V_k^* < I\).

Since \((\mathbb{Z}, \mathbb{N})\) is totally ordered, the theorem still applies when \(\dim E_1 = \aleph_0\), and states that the representation of \(B_{\mathbb{N}} \times_{\tau,E} \mathbb{N}\) corresponding to an infinite
Proposition 5.7. Suppose \( \sum V_k V_k^* < I \); this implies in particular that \( B_N \rtimes \tau, E N \) is not simple. Since \( C^*_{\text{cov}}(N, E) \) is isomorphic to the simple \( C^* \)-algebra \( \mathcal{O}_\infty \), it is a proper subalgebra of \( B_N \rtimes \tau, E N \). The system considered in Example 2.4 is exactly \( (B_N, N, \tau, E) \).

(3) Consider the lexicographic product system \( E \) over \( N \oplus N \) determined by the homomorphism \( d : (m, n) \in N \oplus N \mapsto 2^m 3^n \in \mathbb{N}^* \). A representation \( \phi \) of \( E \) is determined by a pair of Toeplitz-Cuntz families \( \{U_1, U_2\}, \{V_1, V_2, V_3\} \) satisfying (5.9), and from Proposition 3.7 it is easy to see that \( \phi \) is covariant iff

\[
U_1^* V_1 = V_1 U_1^* + V_2 U_2^*, \quad U_2^* V_1 = 0, \\
U_1^* V_2 = V_3 U_1^*, \quad U_2^* V_2 = V_1 U_2^*, \\
U_3^* V_3 = 0, \quad U_2^* V_3 = V_2 U_1^* + V_3 U_2^*.
\]

Thus \( C^*_{\text{cov}}(N^2, E) \) is universal for pairs of Toeplitz-Cuntz families satisfying all of these relations, and Theorem 5.1 implies that \( \{U_1, U_2\} \) and \( \{V_1, V_2, V_3\} \) generate a faithful representation of \( C^*_{\text{cov}}(N^2, E) \) iff

\[
(I - U_1 U_1^* - U_2 U_2^*)(I - V_1 V_1^* - V_2 V_2^* - V_3 V_3^*) \neq 0.
\]

We conclude this section by showing that, under the spanning hypothesis (5.3), amenability of \( E \) (strictly speaking, amenability of \( (B_P, P, \tau, E) \)) is equivalent to faithfulness of the left regular representation.

Proposition 5.7. Suppose \( (G, P) \) is a quasi-lattice ordered group and \( E \) is a product system over \( P \). Let \( l : E \to B(S(E)) \) be the left regular representation of \( E \) and let

\[
X = l(u_1)\alpha_{s_1}^l(I)l(v_1)^* \cdots l(u_n)\alpha_{s_n}^l(I)l(v_n)^*.
\]

Then the map

\[
X \mapsto \begin{cases} 
X & \text{if } p(u_1)p(v_1)^{-1} \cdots p(u_n)p(v_n)^{-1} = e \\
0 & \text{otherwise}
\end{cases}
\]

extends to a projection \( \Phi_l \) of norm one on \( \pi_l \times l(B_P \rtimes \tau, E P) \) which is faithful on positive elements.

Proof. For each \( s \in P \) let \( Q_s \) be the orthogonal projection of \( S(E) \) onto \( E_s \). Since the \( Q_s \)'s are mutually orthogonal, the formula

\[
\Phi_l(T) = \sum_{s \in P} Q_s T Q_s, \quad T \in B(S(E)),
\]

defines a completely positive projection of norm one on \( B(S(E)) \) which is faithful on positive operators. We claim that the restriction of \( \Phi_l \) to \( \pi_l \times l(B_P \rtimes \tau, E P) \) satisfies (5.3).

Let \( r = p(u_1)p(v_1)^{-1} \cdots p(u_n)p(v_n)^{-1} \). For each \( s \in P \), Lemma 3.5 implies that \( X \) is zero on \( E_s \) unless \( rs \in P \), in which case \( X \) maps \( E_s \) into \( E_{rs} \). Thus if \( r \neq e \), \( Q_s X Q_s = 0 \) for every \( s \in P \), and \( \Phi_l(X) = 0 \). If on the other hand \( r = e \), then \( Q_s X Q_s = X Q_s \) for each \( s \in P \), and

\[
\Phi_l(X) = \sum_{s \in P} Q_s X Q_s = X \sum_{s \in P} Q_s = X.
\]

\( \square \)
Corollary 5.8. Suppose \((G, P)\) is a quasi-lattice ordered group and \(E\) is a product system over \(P\) satisfying (5.1). Then \(E\) is amenable if and only if \(\pi_l \times l\) is faithful.

Proof. Suppose \(\pi_l \times l\) is faithful. By Proposition 5.1, \((\pi_l \times l) \circ \Phi_\delta = \Phi_\delta \circ (\pi_l \times l)\) is faithful on positive elements, hence so is \(\Phi_\delta\); that is, \(E\) is amenable. If (5.1) is satisfied, then Proposition 5.4 implies that \(\pi_l \times l\) is faithful on \((B_P \rtimes_{\tau,E} P)^{\delta}\). If in addition \(E\) is amenable, then \(\Phi_l \circ (\pi_l \times l) = (\pi_l \times l) \circ \Phi_\delta\) is faithful on positive elements, from which faithfulness of \(\pi_l \times l\) follows.

6. Amenability

Suppose \((G, P)\) is a quasi-lattice ordered group and \(E\) is a product system over \(P\). In this section we give conditions which ensure that \(E\) is amenable; these conditions also ensure that \(E\) satisfies the spanning condition (5.1), so that Theorem 5.1 applies. Our argument follows those of \([13, \S 4]\) and \([8, \text{ Proposition 2.10}]\).

Theorem 6.1. Suppose \(\theta : (G, P) \rightarrow (\mathcal{G}, P)\) is a homomorphism of quasi-lattice ordered groups such that, whenever \(s \vee t < \infty\),

\[
\theta(s \vee t) = \theta(s) \vee \theta(t) \quad \text{and} \quad \theta(s) = \theta(t) \implies s = t,
\]

and suppose that \(\mathcal{G}\) is amenable. If \(E\) is a product system over \(P\) which satisfies

\[
i_E(v)^* i_E(w) \in \overline{\text{span}} \{i_E(f) i_E(g)^*: f \in E_{\pi_l \times l}^{-1}(\pi_l \times l); g \in E_{\pi_l \times l}^{-1}(\pi_l \times l)\},
\]

then \(E\) is amenable and the spanning hypothesis (5.1) holds.

Remark 6.2. (1) As in \([13, \text{ Proposition 4.3}]\), the main example of such a map \(\theta\) will be the canonical homomorphism of a free product of quasi-lattice ordered groups onto the corresponding direct sum. However, we could also take \(\theta\) to be the length function on the free group \(\mathbb{F}_n\) (the homomorphism into \(\mathbb{Z}\) which takes each generator to 1), and this example gives a good feel for both our constructions and those of \([13, \S 4]\).

Proof of Theorem 6.1. The homomorphism \(\theta : G \rightarrow \mathcal{G}\) induces a coaction \(\delta_\theta = (\text{id} \otimes \theta) \circ \delta\) of \(\mathcal{G}\) on \(B_P \rtimes_{\tau,E} P\), and hence a conditional expectation \(\Phi_{\delta_\theta}\) of \(B_P \rtimes_{\tau,E} P\) onto the fixed-point algebra \((B_P \rtimes_{\tau,E} P)^{\delta_\theta}\), such that

\[
\Phi_{\delta_\theta}(i_E(u) i_{B_P}(1) i_E(v)^*) = \begin{cases} i_E(u) i_{B_P}(1) i_E(v)^* & \text{if } \theta(p(u)) = \theta(p(v)) \\ 0 & \text{otherwise}. \end{cases}
\]

Since \(\mathcal{G}\) is amenable, \(\Phi_{\delta_\theta}\) is faithful on positive elements. We can recover the original expectation \(\Phi_\delta\) by first applying \(\Phi_{\delta_\theta}\), and then killing the terms with \(p(u) = \theta(p(v))^{-1} \in \ker \theta \setminus \{e\}\), which can be accomplished spatially by representing \((B_P \rtimes_{\tau,E} P)^{\delta_\theta}\) using the regular representation \(\pi_l \times l\), and compressing to the diagonal via the expectation \(\Phi_l\) of Proposition 5.4. Since \(\Phi_l\) is faithful on positive operators, this last step is faithful whenever \(\pi_l \times l\) is faithful on \((B_P \rtimes_{\tau,E} P)^{\delta_\theta}\).

It therefore suffices to show that \(\pi_l \times l\) is faithful on \((B_P \rtimes_{\tau,E} P)^{\delta_\theta}\). Let \(\sigma\) be a faithful representation of \(B_P \rtimes_{\tau,E} P\) such that \((\sigma \circ i_{B_P}, \sigma \circ i_E)\) is a...
covariant representation of \((BP, P, \tau, E)\). By Proposition 4.1, \(i = \sigma \circ i_E\) is a covariant representation of \(E\) and \(\sigma \circ i_{BP} = \pi_i\); in particular, we have \(\pi_i(1_s) = \alpha_s^i(I)\) for each \(s \in P\).

Suppose \(S\) is a subset of \(\mathcal{P}\) for which \(q \lor r \in S\) whenever \(q, r \in S\) and \(q \lor r < \infty\). We claim that

\[
\mathcal{U}_S = \text{span}\{i_E(u)i_{BP}(1_s)i_E(v)^* : \theta(p(u)s) = \theta(p(v)s) \in S\}
\]

is a \(C^*\)-subalgebra of \(BP \rtimes \tau, E\). For this, suppose that \(u, v, w, z \in E\) and \(s, t \in P\) are such that \(\theta(p(u)s) = \theta(p(v)s) \in S\) and \(\theta(p(w)t) = \theta(p(z)t) \in S\). Then by Lemma 3.3

\[
i(u)\alpha_s^i(I)i(v)^*i(w)\alpha_t^i(I)i(z)^* = i(u)i(v)^*\alpha_p^i(p)\alpha_p^i(p)\alpha_p^i(p)\alpha_s^i(I)i(w)i(z)^*.
\]

By Proposition 3.7, this operator is is zero unless \(p(v)s \lor p(w)t < \infty\), in which case by (6.2) it can be approximated in norm by a finite sum of operators of the form

\[
i(u)\alpha_s^i(I)i(f)i(g)^*\alpha_t^i(I)i(z)^*;
\]

where \(p(f) = p(v)^{-1}(p(v) \lor p(w))\) and \(p(g) = p(w)^{-1}(p(v) \lor p(w))\). Again using Lemma 3.6, each operator (6.3) can be rewritten as

\[
i(u)f\alpha_p^i(p)^{-1}(p(v) \lor s)(I)\alpha_p^i(p)^{-1}(p(g) \lor t)(I)i(zg)^*.
\]

Now

\[
p(f)^{-1}(p(f) \lor s) = (p(v) \lor p(w))^{-1}(p(v)(p(f) \lor s)
\]

\[
= (p(v) \lor p(w))^{-1}(p(v)p(f) \lor p(v)s)
\]

\[
= (p(v) \lor p(w))^{-1}(p(v) \lor p(w) \lor p(v)s)
\]

\[
= (p(v) \lor p(w))^{-1}(p(v)s \lor p(w)),
\]

and similarly \(p(g)^{-1}(p(g) \lor t) = (p(v) \lor p(w))^{-1}(p(v) \lor p(w)t)\). Thus

\[
p(f)^{-1}(p(f) \lor s) \lor p(g)^{-1}(p(g) \lor t)
\]

\[
= (p(v) \lor p(w))^{-1}(p(v)s \lor p(w)) \lor (p(v) \lor p(w))^{-1}(p(v) \lor p(w)t)
\]

\[
= (p(v) \lor p(w))^{-1}(p(v)s \lor p(v) \lor p(w)t)
\]

\[
= (p(v) \lor p(w))^{-1}(p(v)s \lor p(w)t).
\]

Using this to simplify (6.4), we see that \(i(u)\alpha_s^i(I)i(v)^*i(w)\alpha_t^i(I)i(z)^*\) can be approximated in norm by a finite sum of operators of the form

\[
i(u)f\alpha_p^i(i)^{-1}(p(v) \lor p(w)t)(I)i(zg)^*.
\]

Now

\[
\theta(p(u)f)(p(v) \lor p(w))^{-1}(p(v)s \lor p(w)t)) = \theta(p(u)p(v)^{-1}(p(v)s \lor p(w)t))
\]

\[
= \theta(p(v)s \lor p(w)t),
\]

and similarly \(\theta(p(zg)(p(v) \lor p(w))^{-1}(p(v)s \lor p(w)t)) = \theta(p(v)s \lor p(w)t)\). Since

\[
\theta(p(v)s \lor p(w)t) = \theta(p(v)s) \lor \theta(p(w)t) \in S,
\]
this shows that \( i(u)\alpha_s^t(I)i(v)^*i(w)\alpha_t^z(I)i(z)^* \) is an element of \( \sigma(U_S) \), and hence that \( U_S \) is closed under multiplication. This proves that \( U_S \) is a \( C^* \)-algebra.

Minor revisions of the above argument show that \( \overline{\text{span}}\{i_E(u)i_{B_P}(1_s)i_E(v)^* : u, v \in E, s \in P \} \) is a \( C^* \)-algebra, so that (5.1) holds. Applying \( \Phi \delta_0 \) to both sides of (5.1) gives

\[
(B_P \times_{\tau,E} P)^{\delta_0} = \overline{\text{span}}\{i_E(u)i_{B_P}(1_s)i_E(v)^* : \theta(p(u)) = \theta(p(v))\}.
\]

Let \( \mathcal{F} \) be the set of all finite subsets of \( P \) which are closed under \( \vee \). As in [3, Lemma 4.1], \( \mathcal{F} \) is directed under set inclusion, so that

\[
(B_P \times_{\tau,E} P)^{\delta_0} = \bigcup_{F \in \mathcal{F}} U_F.
\]

By [2, Lemma 1.3], to prove that \( \pi_l \times l \) is faithful on \( (B_P \times_{\tau,E} P)^{\delta_0} \) it is enough to prove it is faithful on each of the subalgebras \( U_F \). We shall accomplish this by inducting on \( |F| \).

First suppose \( F = \{ r \} \) for some \( r \in P \), and write \( U_r \) for \( U_{\{ r \}} \). Let \( \phi \) be a covariant representation of \( E \), and suppose that \( x \) and \( y \) are unit vectors in \( E \) such that \( p(x) \neq p(y) \) and \( \theta(p(x)) = \theta(p(y)) = r \). Since \( \theta \) satisfies (5.2) we must have \( p(x) \vee p(y) = \infty \), and thus by Proposition 3.7 the isometries \( \phi(x) \) and \( \phi(y) \) have orthogonal ranges. Hence \( \phi \) extends to a bounded linear map on \( F_r := \bigoplus_{t \in \theta^{-1}(r)} E_t \), and the following analogues of Propositions 1.11 and 1.12 hold: \( \alpha_t^\phi := \sum_{t \in \theta^{-1}(r)} \alpha_t^\phi \) defines a normal \( * \)-endomorphism of \( \mathcal{B}(H_\phi) \), and there is a faithful normal \( * \)-representation \( \rho_t^\phi \) of \( \mathcal{B}(F_r) \) such that \( \rho_t^\phi(x \otimes y) = \phi(x)\phi(y)^* \) for \( x, y \in F_r \). Moreover, if \( Q \) is a nonzero projection on \( H_\phi \), then \( T \mapsto \alpha_t^\phi(Q)\rho_t^\phi(T) \) is also a faithful representation of \( \mathcal{B}(F_r) \).

There should be no confusion caused by our abuse of notation; just take note of whether the subscript is an element of \( P \) or \( \mathcal{P} \). A word of caution, however: although \( t \mapsto \alpha_t^\phi \) is a semigroup homomorphism, in general the map \( r \in \mathcal{P} \mapsto \alpha_r^\phi \) is not: the bundle \( \{ F_r : r \in \mathcal{P} \} \) is not a product system in the multiplication inherited from \( E \).

Suppose that \( J \) is a finite subset of \( \{ (u, s, v) \in E \times P \times E : \theta(p(u)s) = \theta(p(v)s) = r \} \), and let

\[
X = \sum_{(u,s,v) \in J} i_E(u)i_{B_P}(1_s)i_E(v)^*;
\]

to prove \( \pi_l \times l \) faithful on \( U_r \) we will show that \( \|\pi_l \times l(X)\| = \|X\| \). Define \( T \in \mathcal{B}(F_r) \) by

\[
T = \sum_{(u,s,v) \in J} \sum_f u f \otimes v f,
\]

where \( f \) ranges over an orthonormal basis for \( E_s \). It is routine to check that

\[
\rho_r^\phi(T) = \sum_{(u,s,v) \in J} l(u)\pi_l(1_s)l(v)^* = \pi_l \times l(X),
\]

and similarly \( \rho_r^\phi(T) = \pi_i \times i(X) = \sigma(X) \). Since \( \rho_r^\phi \), \( \rho_r^\phi \) and \( \sigma \) are isometric,

\[
\|\pi_l \times l(X)\| = \|\rho_r^\phi(T)\| = \|T\| = \|\rho_r^\phi(T)\| = \|\sigma(X)\| = \|X\|.
\]
For the inductive step, suppose $F \in \mathcal{F}$ and $\pi_l \times l$ is faithful on $U_{F'}$ whenever $F' \in \mathcal{F}$ and $|F'| < |F|$; we aim to prove that $\pi_l \times l$ is faithful on $U_F$. Since $F$ is finite it has a minimal element; that is, there exists $r_0 \in F$ such that $r_0 < r_0 \lor r$ for each $r \in F \setminus \{r_0\}$. Notice that if $u, v, w \in E$ and $s \in P$ are such that $\theta(p(u)s) = \theta(p(v)s) \in F$ and $\theta(p(w)) = r_0$, then by Lemma 3.3 the vector $\pi_l \times l(\ell_E(u) \ell_B(1_s) \ell_E(v)^*) w = l(u)\alpha(r_0)l(v)^* w$ is nonzero only when $p(v)s \leq p(w)$. Since this in turn implies that $\theta(p(v)s) \leq r_0$, the minimality of $r_0$ forces $\theta(p(v)s) = r_0$. Thus if we let $P_{r_0}$ denote the orthogonal projection of $S(E)$ onto $F_{r_0}$, then $\pi_l \times l(U_{F_{r_0}}) = \{0\}$ for each $r \in F \setminus \{r_0\}$.

On the other hand, we have already demonstrated that $\pi_l \times l$ maps $U_{F_{r_0}}$ into the range of $\rho_{r_0}$, and an easy calculation shows that $P_{r_0} = \alpha(r_0)Q_e$, where $Q_e$ is the orthogonal projection onto $E_e$. Since $T \mapsto \alpha(r_0)(Q(T))P_{r_0}$ is a faithful normal $*$-homomorphism, so is the map $X \mapsto \pi_l \times l(X)P_{r_0}$.

Now suppose $Y \in U_F$ and $\pi_l \times l(Y) = 0$. We will show that $Y \in U_{F \setminus \{r_0\}}$, from which the inductive hypothesis implies that $Y = 0$. Let $(Y_n)$ be a sequence in

$$\text{span}\{i_E(u)\ell_B(1_s)i_E(v)^* : \theta(p(u)s) = \theta(p(v)s) \in F\}$$

which converges in norm to $Y$, and express each $Y_n$ as a sum $\sum_{r \in F} Y_{n,r}$, where $Y_{n,r} \in U_r$. For each $n$, $\|\pi_l \times l(Y_{n,r})P_{r_0}\| = \|\pi_l \times l(Y_{n,r_0})P_{r_0}\| = \|Y_{n,r_0}\|$, and consequently $Y_{n,r_0} \to 0$. Thus $Y_n - Y_{n,r_0} \to 0$, which shows that $Y \in U_{F \setminus \{r_0\}}$, as claimed.

**Corollary 6.3.** Suppose $(G^\lambda, P^\lambda)$ is a quasi-lattice ordered group with $G^\lambda$ amenable for each $\lambda$ belonging to some index set $\Lambda$. Then any product system over $*P^\lambda$ which satisfies (6.2) is amenable. In particular, any product system over $*P^\lambda$ which has only finite-dimensional fibres is amenable.

**Proof.** The group $\bigoplus G^\lambda$ is amenable, and by [13, Proposition 4.3] the canonical map $\theta : *G^\lambda \to \bigoplus G^\lambda$ satisfies (6.1). It follows from Proposition 3.7 that any system with finite-dimensional fibres will satisfy (6.2).

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