WEAKLY NONLOCAL FLUID MECHANICS - THE SCHRÖDINGER EQUATION

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ABSTRACT. A weakly nonlocal extension of ideal fluid dynamics is derived from the Second Law of thermodynamics. It is proved that in the reversible limit the additional pressure term can be derived from a potential. The requirement of the additivity of the specific entropy function determines the quantum potential uniquely. The relation to other known derivations of Schrödinger equation (stochastic, Fisher information, exact uncertainty) is clarified.

1. Introduction

Weakly nonlocal or coarse grained or gradient are attributes of theories from different fields of physics indicating that in contradistinction the traditional treatments, the governing equations of the theory depend on higher order gradients of the state variables. Weakly nonlocal is a nomination in continuum physics dealing with internal structures [1], coarse grained appears in statistically motivated thermodynamics [2], and gradient is frequently used in mechanics [3, 4]). The simplest way to find weakly nonlocal equations can be exemplified by the Ginzburg-Landau equation which can be considered as a first weakly nonlocal extension of a homogeneous relaxation equation of an internal variable. The traditional derivation of the Ginzburg-Landau equation is based on a characteristic mixing of variational and thermodynamic considerations. One applies a variational principle for the static part and the functional derivatives are introduced as thermodynamic forces into a relaxation type equation. A clear variational derivation to obtain a first order differential equation is impossible without any further ado (e.g. without introducing new variables to avoid the first order time derivative, which is not a symmetric operator) [5]. One can apply these kinds of arguments in continuum theories in general, preserving the doubled theoretical framework separating a reversible and an irreversible parts of the equations [6, 7]. However, there are also other systematic attempts to unify the two parts with different additional hypotheses [8, 9] and to eliminate this inconsistency of the traditional approach.

The ultimate aim is to find a unified and predictive theoretical framework that involves higher order gradients in the governing equations of physics. All the approaches mentioned above are systematic in the sense that the gradient dependent terms are not introduced in an ad-hoc way, the Second Law of thermodynamics plays an important role in each of them. An analysis of the involved thermodynamics shows that in case of the Ginzburg-Landau equation one can determine the variational part purely from the requirement of a nonnegative entropy production, without any additional assumptions [9]. This way of thinking shows why and

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in what sense and under what conditions is the Ginzburg-Landau form distinguished among the possible other weakly nonlocal equations of a single internal variable (called order parameter in special cases).

However, the Ginzburg-Landau equation is only a prototypical example and several different weakly nonlocal differential equations can appear in continuum physics. It is well-known that quantum mechanics is a nonlocal theory. However, nonlocality is understood in different ways in the different treatments. A clear interpretation and appearance of nonlocality in quantum mechanics is that the Bohmian quantum potential depends on the derivatives of the quantum probability density \[10, 11\]. In this sense, quantum mechanics is a special weakly nonlocal (or coarse grained or gradient) fluid mechanics.

In this paper we investigate all possible weakly nonlocal extensions of fluid mechanics that are compatible with the Second Law. A straightforward application of Liu’s procedure gives the surprising consequence that a weakly nonlocal extension of traditional fluid mechanics in the density leads to a generalization of the Euler equation that incorporates the hydrodynamic model of quantum mechanics. Moreover, we will show that this model is distinguished among the different possible weakly nonlocal fluids in the sense that the characteristic Fisher information like form of the gradient dependent part of the entropy density function is a unique consequence of its isotropic and additivity properties. After these considerations we treat shortly some interpretational and conceptual problems, too. We argue that our investigations lead to a derivation of quantum mechanics that is preferable according to the principle of Occam’s razor because it is based on a minimal number of assumptions.

The starting point of our investigations is fluid mechanics. However, one should not consider it as a phenomenological theory with a rich and definite microscopic-molecular background, but rather as an empty bottle of general physical principles that are valid in any continuum (field) theory (e.g. balance of momentum, conservation of mass, Second Law, ...). The approach of non-equilibrium thermodynamics fills this bottle with content using a step-by-step approach. Therefore, arriving at the so-called hydrodynamic model of quantum mechanics does not prevent us from transforming it to a wave function form and enjoying the advantages of a linear equation. However, we should know the physical content of the linearity and be aware of the price. For example it is well-known that the Schrödinger equation is not a reference frame independent, objective equation. Moreover, it cannot be written in a frame independent form because of its energetic origin [12]. Similarly, canonical quantization is also frame/observer dependent (see e.g. [13]). However, the theories based on a momentum balance (e.g. the hydrodynamic model) are frame independent and objective [14].

2. FLUID MECHANICS IN GENERAL

The basic state space of one-component fluid mechanics is spanned by the density \( \rho \) and the velocity \( \mathbf{v} \) of the fluid. Hydrodynamics is based on the balance of mass and the balance of momentum [15]. Classical fluid mechanics is the theory where the constitutive space, the domain of the constitutive functions, is spanned by the basic state space \( (\rho, \mathbf{v}) \) and the gradient of the velocity \( \nabla \mathbf{v} \). The pressure/stress tensor is the only constitutive quantity in the theory. The pressure function defines the material in continuum physics. For example, it determines whether it is a fluid
or a solid, and whether it is a material in local equilibrium without memory or not, etc. Introducing higher order gradients of the basic variables into the constitutive space one can obtain weakly nonlocal extensions regarding the density and the velocity. In this paper we investigate the weakly nonlocal extension of classical fluid dynamics in the density.

The balance of mass in local substantial form can be written as

$$\dot{\rho} + \rho \nabla \cdot v = \sigma_m,$$

where $\rho$ is the density, $v$ is the velocity, $\sigma_m$ is mass production, and the dot above the quantities denotes the substantial (comoving) derivative.

The balance of momentum, i.e. the Cauchy equation, is

$$\rho \dot{v} + \nabla \cdot P = \rho f,$$

where $P$ is the pressure and $f$ is the force density. For discussing the characteristics of continuum materials the effects given by the source terms do not play any role.

The Second Law requires that the production of the entropy is nonnegative in insulated and source-free systems. Particularly in our case if there is no production of mass ($\sigma_m = 0$) and there are no external forces ($\rho f = 0$), then the production of entropy must be nonnegative

$$\rho \dot{s} + \nabla \cdot j_s \geq 0.$$

The constitutive quantities in the above equations are the pressure $P$, the specific entropy $s$, and the conductive current $j_s$ of the entropy. According to the Coleman-Mizel form of the Second Law, the entropy inequality is a constitutive requirement [16], that is, we are looking for constitutive functions that solve the inequality. Thus the entropy inequality should be a pure material property, it is required to be valid independently of the initial conditions. In the exploitation of the inequality Liu’s theorem plays an important technical role. Details of the different state spaces, more detailed description of thermodynamic concepts, and the applied mathematical methods (especially the Liu procedure) can be found in [17]; regarding the weakly nonlocal extension see [18][9].

Let us remark that, in fluid systems, the balances of different (kinetic, potential, internal and total) energies should all be considered [19]. However, in our treatment, dealing with nonlocal extensions only in the density, the internal energy changes do not play role, therefore we avoided the computational complications coming from additional governing equations. In case of moving fluids it is better to regard the function $s$ as a generalized (kinetic) potential [20]. However, avoiding the unnecessary extension of the terminology and emphasizing the very thermodynamic point of view of the argumentation, we call it entropy.

It is important that, in case of moving continua, the substantial time derivatives appear in the treatment, because these time derivatives express the time changes of the corresponding physical quantities in a frame fixed to the moving continuum. These time changes are independent of any external reference frames as one can see clearly in a frame independent treatment of continuum mechanics [21][22].

3. Nonlocal fluid mechanics - the Schrödinger-Madelung equation

Fluid mechanics is treated in the case when the basic state space is that of classical hydrodynamics ($\rho, v$) and the constitutive space contains gradients of the density
\( \rho \) in addition to the classical case and is spanned by the variables \((\rho, \nabla \rho, \mathbf{v}, \nabla \mathbf{v}, \nabla^2 \rho)\). Here \(\nabla^2 \rho\) denotes the second space derivative of \(\rho\) (sometimes written as \(\nabla \circ \nabla \rho\), where \(\circ\) is the traditional notation of the tensorial/dyadic product in hydrodynamics). The space of independent variables is spanned by the next time and space derivatives of the constitutive variables, that is, by \((\dot{\rho}, \nabla \dot{\rho}, \nabla \mathbf{v}, \nabla^2 \dot{\rho}, \nabla^2 \mathbf{v}, \nabla^3 \rho)\), as a consequence of the entropy inequality.

In applying the Liu procedure, the constraints are the balance of mass \((1)\) and the balance of momentum \((2)\). Moreover, because of the higher derivatives of density in the constitutive space, the space derivative of the mass balance is also to be considered as a constraint,

\[ \nabla \dot{\rho} + \nabla \rho \nabla \cdot \mathbf{v} + \rho \nabla \nabla \cdot \mathbf{v} = 0. \]

This situation is similar to what happen in case of the thermodynamic derivations of the Ginzburg-Landau equation \([9]\), or in relativistic constitutive theories \([24]\). On the other hand, higher order space derivatives of equation \((3)\) cannot be constraints, because those contain derivatives that have already been included in the space of independent variables.

Now we apply Liu procedure, with the method of Lagrange-Farkas multipliers (see \([9, 24]\)) as

\[
\begin{align*}
\rho D_1 s \dot{\rho} + \rho D_2 s \cdot (\nabla \rho) + \rho D_3 s \cdot \dot{\mathbf{v}} + \rho D_4 s : (\nabla \mathbf{v}) + \rho D_5 s : (\nabla^2 \rho) + \\
D_1 j_s \cdot \nabla \rho + D_2 j_s : \nabla^2 \rho + D_3 j_s : \nabla \mathbf{v} + D_4 j_s : \nabla^2 \mathbf{v} + D_5 j_s : \nabla^3 \rho + \\
\Gamma_1 (\dot{\rho}) + \Gamma_2 (\nabla \dot{\rho} + \nabla \rho \nabla \cdot \mathbf{v} + \rho \nabla \nabla \cdot \mathbf{v}) + \Gamma_3 (\rho \nabla \mathbf{v} + \nabla \mathbf{P}) \geq 0.
\end{align*}
\]

Here the multipliers \(\Gamma_1, \Gamma_2, \) and \(\Gamma_3\) were introduced for the constraints \((1), (2)\) and \((3)\), respectively. The subscript numbers denote derivation according to the corresponding variable in the constitutive space, \((\rho, \nabla \rho, \mathbf{v}, \nabla \mathbf{v}, \nabla^2 \rho)\), e.g. \(D_1 s = \frac{\partial}{\partial \rho}\). The source terms in the balances have been considered as zero. For what follows, it is important to observe that the substantial time derivative does not commute with the space derivative (gradient), instead, the following identities are to be applied:

\[
\begin{align*}
(\nabla a)^{\dot{}} &= \nabla \dot{a} - \nabla \mathbf{v} \cdot \nabla a, \\
(\nabla^2 a)^{\dot{}} &= \nabla^2 \dot{a} - \nabla^2 \mathbf{v} \cdot \nabla a - 2 \nabla \mathbf{v} \cdot \nabla^2 a.
\end{align*}
\]

For the sake of easier applicability we give these equations with indices, too:

\[
\begin{align*}
(\partial_i a)^{\dot{}} &= \partial_i \dot{a} - \partial_i v_j \partial_j a, \\
(\partial_i \partial_j a)^{\dot{}} &= \partial_i \partial_j \dot{a} - \partial_i \partial_j v_k \partial_k a - 2 \partial_i v_k \partial_j \partial_k a.
\end{align*}
\]

Here \(i, j, k = 1, 2, 3\) denote the Cartesian coordinates. Using these identities the terms in the above inequality can be rearranged as

\[
\begin{align*}
(\rho D_1 s - \Gamma_1) \dot{\rho} + (\rho D_2 s - \Gamma_2) \nabla \dot{\rho} + \rho (D_3 s - \Gamma_3) \cdot \dot{\mathbf{v}} + \rho D_4 s : \nabla \mathbf{v} + \\
\rho D_5 s : \nabla^2 \dot{\rho} + (D_3 j_s - \Gamma_3 \cdot D_5 \mathbf{P}) : \nabla^2 \rho + (D_4 j_s - \Gamma_3 \cdot D_4 \mathbf{P}) : \nabla^2 \mathbf{v} - \\
\rho \Gamma_2 \cdot \nabla \nabla \cdot \mathbf{v} - \rho D_5 s : (\nabla^2 \mathbf{v} \cdot \nabla \rho + 2 \nabla \mathbf{v} \cdot \nabla^2 \rho) - \rho D_4 s : (\nabla \mathbf{v} \cdot \nabla \mathbf{v}) + \\
(D_1 j_s - \rho D_2 s) \cdot \nabla \mathbf{v} - \Gamma_2 \nabla \cdot \mathbf{v} - \Gamma_3 \cdot D_1 \mathbf{P} \cdot \nabla \rho + (D_2 j_s - \Gamma_3 \cdot D_2 \mathbf{P}) : \nabla^2 \rho + \\
(D_3 j_s - \rho \Gamma_1 \mathbf{I}) - \Gamma_3 \cdot D_3 \mathbf{P} : \nabla \mathbf{v} \geq 0
\end{align*}
\]
The multipliers of the independent variables are the Liu equations, respectively:

\begin{align*}
(4) \quad \rho D_1 s &= \Gamma_1, \\
(5) \quad \rho D_2 s &= \Gamma_2, \\
(6) \quad D_3 s &= \Gamma_3, \\
(7) \quad D_4 s &= 0, \\
(8) \quad D_5 s &= 0, \\
(9) \quad (D_4 s)_{jk} = (\Gamma_3)^j_i (D_2 P)_{i}^{jk} - (\Gamma_2)_{i}^{jk} \rho (\delta_i^j \delta_j^k + \delta_j^i \delta_k^j) / 2 &= 0, \\
(10) \quad (D_5 s)_{jk} = (\Gamma_4)^i_i (\partial_i P)_{i}^{jk} &= 0.
\end{align*}

Here, the last two equations are given with indices to avoid misunderstanding. Equation (10) is symmetric in every tensorial component (for all permutations of j, k and l), and (9) is symmetric in j and k because of the symmetry of the corresponding independent variables. In the following we are to solve Liu’s equations (4)-(9).

As a consequence of (7) and (8), the specific entropy does not depend on the second gradient of \(\rho\) and on the gradient of \(v\). Hence, \(s(\rho, \nabla \rho, v, \nabla v, \nabla^2 \rho) = s(\rho, \nabla \rho, v)\). (4)-(6) give the Lagrange-Farkas multipliers as derivatives of the entropy. Therefore, from a thermodynamic point of view, they are a kind of generalized intensive variables in the theory \cite{25}. Let us treat the entropy as a primary physical quantity, that is we want to express the other constitutive functions with its help. Now, one can give a solution of (10) and (9) as

\[ j_s(\rho, \nabla \rho, v, \nabla v, \nabla^2 \rho) = \Gamma_3 \cdot P + \frac{1}{2} \rho \Gamma_2 \cdot \nabla v + j_0(\rho, \nabla \rho, v) = D_3 s \cdot P + \frac{1}{2} \rho^2 (D_2 s \cdot v + D_4 s \cdot v) + j_0(\rho, \nabla \rho, v), \]

where \(j_0\) is an arbitrary (differentiable) function. Thus Liu’s equations can be solved and yield the Lagrange-Farkas multipliers as well as restrictions for the entropy and the entropy current. Applying these solutions of the Liu equations, the dissipation inequality can be written as

\[ \rho \dot{s} + \nabla \cdot j_s = \nabla \cdot j_0 + (\nabla D_3 s) \cdot P + \left[ \frac{\rho^2}{2} (D_2 s \cdot I + \nabla D_2 s) - \rho^2 D_4 s I \right] : \nabla v \geq 0. \]

Here \(I\) denotes the second order unit tensor \((\delta_{ij})\). Let us now define a traditional fluid with a specific entropy of the following form

\[ s(\rho, \nabla \rho, v) = s_s(\rho, \nabla \rho) - \frac{v^2}{2}. \]

From this form it is clear that \(s_s\) corresponds to a static (equilibrium) specific entropy and \(s\) is a kind of general non-equilibrium entropy closely connected to the kinetic potential of Glansdorff and Prigogine \cite{20}, as it was mentioned in the introduction. However, here we exploited the entropic representation of the variables and \(s_s\) depends also on the gradient of the density function. Now it is reasonable to introduce a new notation of the derivatives as \(\partial_\rho := D_1\) and \(\partial_{\nabla \rho} := D_2\).

Assuming an ordinary entropy current with \(j_0 = 0\) we can write the dissipation inequality and the entropy current as

\[ -\nabla v : \left( P - \rho^2 \left( \frac{1}{2} \nabla \cdot \partial_{\nabla \rho} s_s - \partial_{\nabla \rho} s_s \right) I - \frac{\rho^2}{2} \nabla \partial_{\nabla \rho} s_s \right) \geq 0, \]
The advantage of the entropy (11) of a traditional fluid is that the inequality (12) is solvable, because it has the force-current form of irreversible thermodynamics and contains the pressure as a single dynamic constitutive function ($s_s$ is static and assumed to be known). We can define nonlocal reversible pressure as

$$P_r(\rho, \nabla \rho) := \rho^2 \langle (\nabla \cdot \partial_2 s_s - 2\partial_1 s_s) I + \nabla \partial_2 s_s \rangle.$$

If the total pressure is of this form then the entropy production is zero, there is no dissipation, the theory is reversible (conservative). If the entropy is local (independent of the gradient of the density) then we obtain

$$P_{\text{Euler}}(\rho) := \rho^2 \partial_\rho (\rho s_s I),$$

therefore, the corresponding equations are of the ideal Euler fluid, where $p(\rho) = \rho^2 \partial_\rho (\rho s_s)$ is the scalar pressure function. From this form we can identify the relation of our generalized potential and the traditional entropy.

Introducing the viscous pressure $P^v$ as usual, we can solve the dissipation inequality (12) and give the corresponding Onsagerian conductivity equation as

$$P^v := P - P^f = L \cdot \nabla \mathbf{v}.$$ 

Here $L$ is a nonnegative constitutive function. Let us recognize that if $s_s$ is independent of the gradient of the density, $L$ is constant, and $P^v$ is an isotropic function of only $\nabla \mathbf{v}$, then we obtain the traditional Navier-Stokes fluid.

One can prove easily that the reversible part of the pressure of a traditional fluid is potentializable, i.e., there is a scalar valued function $U$ such that

$$\nabla \cdot P^r = \rho \nabla U.$$

$U$ can be calculated from the entropy function as

$$U = \nabla \cdot (\rho \nabla \rho) - \partial_\rho (\rho s_s).$$

Therefore in case of reversible fluids the momentum balance can be written alternatively as

$$\rho \mathbf{\dot{v}} + \nabla \cdot P^r = 0 \iff \mathbf{\dot{v}} + \nabla U = 0.$$

Giving the form of $s_s$, we obtain some specific nonlocal fluids.

**Schrödinger-Madelung fluid.** Here the entropy is defined as

$$s_{\text{SchM}}(\rho, \nabla \rho, \mathbf{v}) = -\frac{\nu_{\text{SchM}}}{2} \left( \frac{\nabla \rho}{2\rho} \right)^2 - \frac{\mathbf{v}^2}{2} = -\frac{\nu_{\text{SchM}}}{8} (\nabla \ln \rho)^2 - \frac{\mathbf{v}^2}{2},$$

where $\nu_{\text{SchM}}$ is a constant scalar. The corresponding reversible pressure is

$$P_{\text{SchM}} = -\frac{\nu_{\text{SchM}}}{8} \left( \Delta \rho I + \nabla^2 \rho - \frac{2\nabla \rho \otimes \nabla \rho}{\rho} \right),$$

where $\otimes$ denotes the tensorial/dyadic product, as mentioned before. The potential is

$$U_{\text{SchM}} = -\frac{\nu_{\text{SchM}}}{4\rho} \left( \Delta \rho - \frac{(\nabla \rho)^2}{2\rho} \right) = -\frac{\nu_{\text{SchM}}}{2} \frac{\Delta R}{R}.$$
where we introduced $R = \sqrt{\rho}$ to show more clearly that \cite{21} is the quantum potential in the de Broglie-Bohm version of quantum mechanics (if $\nu_{\text{SchM}} = \hbar^2 / m^2$) \cite{10} \cite{26}.

Further, the entropy current of the Schrödinger-Madelung fluid is

\begin{equation}
\mathbf{j}_{\text{SchM}} = -\mathbf{v} \cdot \mathbf{P}_{\text{SchM}} - \frac{\nu_{\text{SchM}}}{8} (\nabla \rho \nabla \cdot \mathbf{v} + \nabla \rho \cdot \nabla \mathbf{v}).
\end{equation}

**Remark 3.1.** In the hydrodynamic model of quantum mechanics the pressure cannot be determined uniquely, because one concludes from the Schrödinger equation that a neutral quantum fluid preserves the vorticity. Therefore in the hydrodynamic model the pressure is calculated from the potentializability condition \cite{10} (from the potential) and thus one can add a curl of any function of the density and its gradient without changing the physics (Bernoulli equation). For example, replacing $\Delta \rho \mathbf{I} + \nabla^2 \rho$ with $2 \nabla^2 \rho$ in \cite{20} one obtains the Jánossy-Ziegler pressure \cite{21} \cite{28}. This is a (gauge) freedom in case of neutral fluids but it can be important when vorticity is not zero. In our thermodynamic derivation the pressure is the primary constitutive quantity, it is determined uniquely and the potentializability is the consequence of the thermodynamic structure.

**Landau fluid.** The simplest globally concave and isotropic entropy depends on the square of the density gradient. Because of the similarity to the Ginzburg-Landau free energy density, we will call Ginzburg-Landau fluid the material that is defined by the following form of the entropy

\begin{equation}
s_{\text{Lan}}(\rho, \nabla \rho, \mathbf{v}) = -\nu_{\text{Lan}} \frac{(\nabla \rho)^2}{2} - \frac{\mathbf{v}^2}{2},
\end{equation}

where $\nu_{\text{Lan}}$ is a constant coefficient. The corresponding reversible pressure function is

\begin{equation}
P_{\text{Lan}}^r = -\nu_{\text{Lan}} \frac{\rho^2}{2} (\Delta \rho \mathbf{I} + \nabla^2 \rho).
\end{equation}

The potential is

\begin{equation}
U_{\text{Lan}} = -\nu_{\text{Lan}} \frac{1}{2} \left( \rho \Delta \rho + \Delta \frac{\rho^2}{2} \right).
\end{equation}

The entropy current can be also given, as

\begin{equation}
\mathbf{j}_{\text{Lan}} = -\mathbf{v} \cdot \mathbf{P}_{\text{Lan}}^r - \nu_{\text{Lan}} \frac{\rho^2}{2} (\nabla \rho \nabla \cdot \mathbf{v} + \nabla \rho \cdot \nabla \mathbf{v}).
\end{equation}

**Alternative fluid.** The potential has the simplest form if the entropy is written as

\begin{equation}
s_{\text{Alt}}(\rho, \nabla \rho, \mathbf{v}) = -\nu_{\text{Alt}} \frac{(\nabla \rho)^2}{4\rho} - \frac{\mathbf{v}^2}{2},
\end{equation}

where $\nu_{\text{Alt}}$ is a constant coefficient. In this case the reversible pressure is

\begin{equation}
P_{\text{Alt}}^r = -\nu_{\text{Alt}} \frac{1}{4} \left( \rho (\Delta \rho \mathbf{I} + \nabla^2 \rho) - \nabla \rho \circ \nabla \rho \right).
\end{equation}

and the nonlocal term in the potential is simply

\begin{equation}
U_{\text{Alt}} = -\nu_{\text{Alt}} \frac{1}{2} \Delta \rho.
\end{equation}

Finally, the entropy current can be written as

\begin{equation}
\mathbf{j}_{\text{Alt}} = -\mathbf{v} \cdot \mathbf{P}_{\text{Alt}}^r - \nu_{\text{Alt}} \frac{\rho^2}{4} (\nabla \rho \nabla \cdot \mathbf{v} + \nabla \rho \cdot \nabla \mathbf{v}).
\end{equation}
4. The Origin of Quantum Potential

An important property of the Schrödinger-Madelung fluid is that if the motion of the fluid is vorticity free, $\nabla \times \mathbf{v} = 0$, then the mass and momentum balances can be transformed into and united in the Schrödinger equation. Hence the balance of momentum (2) can be derived from a Bernoulli equation (in a given inertial reference frame). Defining a scalar valued phase (velocity potential) by

$$\mathbf{v} = \frac{h}{m} \nabla S,$$

the Bernoulli equation is obtained from the second part of (18) written as

$$\frac{\hbar}{m} \frac{\partial S}{\partial t} = -\frac{v^2}{2} + U_{\text{SchM}}.$$

Then, introducing a single complex valued function $\psi := \text{Re} e^{iS}$ that unifies $R = \sqrt{\rho}$ and $S$, it is easy to find that sum of (11) multiplied by $i\hbar e^{iS}/(2R)$ and (31) multiplied by $m\text{Re} e^{iS}$ form together the Schrödinger equation for free particles

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi.$$

Therefore, what we have accomplished is a kind of derivation of the quantum mechanics in a very general framework. Moreover, we have obtained some more, different but related fluid models. At this point several questions can arise. What is the relation of this thermodynamic derivation to other approaches? Are there 'quantized' solutions of the Landau or alternative fluids under similar conditions that in quantum mechanics? Are there any distinctive properties of the Schrödinger-Madelung model among the other weakly nonlocal fluids?

First of all, to understand the results of Liu procedure from a different point of view, let us observe that equation (17) has an Euler-Lagrange form. Therefore, one can derive it from a traditional Hamiltonian variational principle with the Lagrangian density $L(\rho, \nabla \rho) = -\rho s_{\text{s}}(\rho, \nabla \rho)$!

Moreover, in case of rotational free motion, the mass and momentum balances can be calculated from a variational principle (see e.g. [29, 30]). In our thermodynamic train of thought the existence of a Hamiltonian variational principle was not a starting point, but a consequence of the Second Law in a special reversible case.

It is interesting to observe that the Schrödinger-Madelung model is not distinguished from the other two fluids in producing static "quantum" solutions. The static fluid dynamic equations give a generalized eigenvalue problem

$$\nabla \cdot \left( \rho \partial_{\rho} s_{\text{s}} \right) - \partial_{\rho} (\rho s_{\text{s}}) = U = \text{const.}$$

Because $U$ is given as a functional derivative of $\rho s_{\text{s}}$, the existence of multiple solutions is connected to the concavity properties of the entropy function. Among the previous three fluids only the Landau fluid preserves the global concavity of the entropy both in the density and in the gradient of the density. One can calculate easily, that the second derivatives of the entropies of the Schrödinger-Madelung fluid and of the alternative fluid are positive semidefinite. The Schrödinger-Madelung fluid has a rich structure of stationary solutions in case of simple conservative force fields (the same as for the Schrödinger equation). Since this property seems to be connected to the semidefinite property of the entropy [31], the existence of similar "quantized solutions" can be expected in the alternative fluid, too. This is an
interesting question both from a physical and from a mathematical points of view. Evidently, a large set of different generalized eigenvalue problems emerge.

Let us remark here that the concavity of the entropy (kinetic potential) is also connected to the stability properties of the equilibrium solutions of the fluid dynamic equations [31].

Finally, we turn our attention to the form of the static part of the specific entropy function. Apart from a constant multiplier, it can be written as \( s_{\text{SchM}}(\rho, \nabla \rho) = \frac{\nabla \rho^2}{\rho} \), which is the trace of the Fisher information tensor of the probability density \( \rho \) [32]. Here we will show shortly that this special form is a straightforward consequence of some general properties of the theory and of the entropy function \( s_s(\rho, \nabla \rho) \). In the following arguments we require some basic properties of point masses, encoded in a continuum theory.

First of all, the entropy is \textit{isotropic}. An isotropic function \( s \) of \( \rho \) and \( D\rho \) has the following property [33] [34]

\[
(34) \quad s_s(\rho, \nabla \rho) = \hat{s}(\rho, (\nabla \rho)^2).
\]

On the other hand, for independent particles one requires the separability of the governing equations, therefore the \textit{additivity} of the entropy function in a definite way. We postulate that, in case of independent particles, the governing equation should be additively separable and, at the same time the probability interpretation has to be preserved with considering the entropy of the system of particles. This is exactly the physical property why one requires the linearity of the Schrödinger equation and any of its dissipative extensions written in probability amplitude (wave function) variables.

For the sake of simplicity we restrict ourself for two particles. The generalization to finite number of independent particles is straightforward. For two distinct particles we can consider the two-particle probability density \( \rho(x_1, x_2) \) which is the product of the one particle probability densities \( \rho_1(x_1) \) and \( \rho_2(x_2) \). Then the gradient of \( \rho \) is meant in two variables, too. Thus, we have \( (\nabla \rho)(x_1, x_2) = \rho_2(x_2) \nabla x_1, \rho_1(x_1) \nabla x_2, \rho_2(x_2)) \) and - omitting the variables \( x_1 \) and \( x_2 \) - \( (\nabla \rho)^2 = (\rho_2 \nabla \rho_1)^2 + (\rho_1 \nabla \rho_2)^2 \). As a consequence, in case of isotropic entropies the additivity requirement can be written as:

\[
(35) \quad \hat{s}(\rho_1 \rho_2, (\rho_2 \nabla \rho_1)^2 + (\rho_1 \nabla \rho_2)^2) = \hat{s}(\rho_1, (\nabla \rho_1)^2) + \hat{s}(\rho_2, (\nabla \rho_2)^2).
\]

If \( \hat{s} \) is continuously differentiable, then differentiating the above equality by \( (\nabla \rho_1)^2 \) and \( (\nabla \rho_2)^2 \) respectively we have that

\[
\rho_2^2 D_2 \hat{s}(\rho_1 \rho_2, (\rho_2 \nabla \rho_1)^2 + (\rho_1 \nabla \rho_2)^2) = D_2 \hat{s}(\rho_1, (\nabla \rho_1)^2),
\]

\[
\rho_1^2 \partial_2 \hat{s}(\rho_1 \rho_2, (\rho_2 \nabla \rho_1)^2 + (\rho_1 \nabla \rho_2)^2) = D_2 \hat{s}(\rho_2, (\nabla \rho_2)^2).
\]

Here \( D_2 \) denotes the partial derivative of \( \hat{s} \) by its second argument. Therefore \( \rho^2 D_2 \hat{s}(\rho, (\nabla \rho)^2) = \nu = \text{const.} \), from which it follows, that

\[
(36) \quad \hat{s}(\rho, (\nabla \rho)^2) = \nu \frac{(\nabla \rho)^2}{\rho^2} + \hat{s}(\rho),
\]

where \( \hat{s} \) is an arbitrary function (the local part of the entropy). Repeating the above argument with the derivatives by the first argument of \( \hat{s} \), one finds that

\[
\rho \partial_1 \hat{s}(\rho) = k = \text{const.}
\]
Consequently, \( \dot{s}(\rho) = k\ln \rho + s_0 \), where \( s_0 \) is an arbitrary constant. Therefore

\[
(37) \quad \dot{s}(\rho, (\nabla \rho)^2) = \nu \frac{(\nabla \rho)^2}{\rho^2} + k\ln \rho + s_0.
\]

The first term has the form of a Fisher information and the second term has the form of a Shannon information measure. The solution is unique with the above requirements.

Let us consider now, that our probability continuum should be a theory of particles. Therefore we require the mass-scale invariance of the entropy, that is the entropy density is a first order homogeneous function of the density. In this case the specific entropy is scale invariant, a zeroth order homogeneous function of the density

\[
(38) \quad s_s(\lambda \rho, \nabla (\lambda \rho)) = s_s(\rho, \nabla \rho),
\]

for any real number \( \lambda \). This a necessary condition to have a particle interpretation of a continuum. In this case the quantum potential is independent of the mass-scale and thus the equations of free motion also have the same property. One can see this clearly from the particle equation form of the momentum balance (the second part of (18)). Considering this condition the mass, the measure of the inertia of the continuum acts uniformly, independently of the space coordinates. With using specific entropy, that depends only on the specific quantities (the density is the reciprocal value of the specific volume) we required a volume-scale invariance of the corresponding functions. The mass-scale invariance is something similar, it is a kind of mass-extensivity.

Mass-scale invariance requires that \( k = 0 \) in (37) excluding the logarithmic part.

It is interesting to note that a weakly nonlocal statistical theory is developing that investigates the foundations of equilibrium probability distributions and static nonlocal extensions of fundamental equations of mathematical physics from the point of view of Fisher information \([35, 36, 37]\). However, the relationship between Fisher information and thermodynamic entropy in non-equilibrium situations is not clear and especially the nonnegativity of the entropy production is questioned \([38, 39, 40]\). In the light of the recent investigations for the case of the Schrödinger equation the entropy inequality is identically fulfilled.

5. Conclusions

In this paper we have shown that the Second Law of thermodynamics restricts considerably the possible pressure functions of fluids that are weakly nonlocal in density. Several different traditional fluids were defined, where the non-equilibrium specific entropy is additively quadratic in the velocity \([41]\), by some simple possible forms of the nonlocal part of the entropy function. In the conservative limit, when the entropy production is zero, we have found that the entropy density is a Lagrangian of the gradient dependent potentials, therefore, in case of vorticity free motion it can be substituted for example into a Seliger-Whitham-type variational principle \([41, 29]\). However, in our thermodynamic approach, the Euler-Lagrange form was a consequence of the Second Law in the nondissipative limit. There was no need to refer to any variational principle at all.

We have seen that the Schrödinger-Madelung fluid, the hydrodynamic model of quantum mechanics plays a distinguished role among the thermodynamically possible weakly nonlocal fluids. It is the only model where the nonlocal part of
the entropy function is isotropic, additive and mass-scale invariant. All these three properties have a clear physical meaning. We need an additive entropy to obtain independent equations for independent free particles. Additivity requires isotropy. We need a mass-scale invariant entropy to have a mass that measures the inertia of the particle in our continuum theory. Only in this case will the inertia of the "fluid" be uniform and proportional to the force.

From a fluid mechanical point of view, the Schrödinger equation appears as a complex formulation of the coupled Bernoulli equation and the mass balance and has the great advantage of being linear. On the other hand, we should emphasize that our derivation of the hydrodynamic model of quantum mechanics was completely independent of the Schrödinger equation, and quantum mechanics appeared only as a special fluid with remarkable properties. There are several examples of the applicability of generalized fluid models. Generalized Schrödinger-type equations appear as structure forming equations in several fields without any connection to quantum mechanics [42]. The possibility to transform generalized weakly nonlocal fluids into a well known linear complex form is an important advantage in several investigations as, e.g., in cosmology [43, 44, 45].

Let us give some remarks on other derivations of the basic equation of quantum mechanics. As it is well known, Schrödinger himself did not derive the equation, in a strict sense, his suggestion is based on analogies and is justified by its consequences. The de Broglie-Bohm form or the hydrodynamic model of Madelung all start from the Schrödinger equation, so they are not derivations only different points of view given for interpretational purposes (nevertheless important and thought provoking ones).

The stochastic model [46, 47, 11, 48] provides a kind of derivation. Here one assumes a background random field, the active role of "vacuum fluctuations" as an origin of the quantum potential, but one should not forget about the special assumed properties. The introduced stochastic velocity has a particular form, being proportional to the gradient of $\rho$, which form is motivated by the analogy (!) with diffusion. Here nonlocality is disguised as a kind of velocity, hence requires a special structure and is similar - results in - the mass-scale invariance of the entropy.

Nonlocal kinetic theories (e.g. that of Kaniadakis [49]) all assume a definite microscopic background. In the previous section we have mentioned other approaches based on the Fisher information measure. These start from a variational principle and postulate the form of quantum term (as minimum Fisher information e.g., [30]) and interpret it with information theoretical ideas based on the measurement process [50]. Such an approach is thus less a derivation but rather a different interpretation (nevertheless important and thought provoking).

The only remarkable exception is the approach of Hall and Reginatto [51, 52] who give arguments on the form of the entropy (in their formulation it is an additional term in the Lagrangian due to fluctuations) with a reasoning that is similar to ours. They require additivity and isotropy (implicitly), but instead of mass-scale invariance they use a different principle what they call "exact uncertainty". Nevertheless with these requirements they arrive at the same Fisherian form of the quantum part of the entropy function. On the other hand they start from a variational approach which is a consequence of the Second Law in our considerations.
We have to emphasize again that our derivation here is based on general principles (Second Law) and requirements (additivity) and therefore is independent on any kind of interpretational issues of quantum mechanics. It contains a minimal set of assumptions that one can gain on a phenomenological level about one component fluid systems. It is surprising that we have reached almost everything. The only thing in the equations that one should determine from microscopic considerations (from experiments) is the value of the Planck constant.

Our approach is similar that of Jaynes to equilibrium statistical physics both on the dynamic and on the static level [53]. On the dynamic level we used the entropy inequality - a part of the Second Law - and the basic balances as constraints to extract the maximum amount of information regarding the structure of a definite physical system. We have derived serious restrictions on the constitutive functions. Moreover, the reasoning is predictive in the Jaynesian sense. E.g. in the dynamic case requiring nonnegative entropy production one can give definite predictions on the structure of nonlocal equations of multicomponent fluids, too. Considering phase separation as a constraint, we can construct promising new models for granular and porous media [54, 55].

Moreover, regarding the static part, our reasoning is completely analogous to the Jaynesian, phenomenological approach of statistical physics. Jaynes’s arguments are based on the uniqueness of Shannon’s information measure with some expected properties that are common with the required properties of the entropy function (extensivity, additivity). However, Shannon’s proof was related to a discrete probability space and exploited that the entropy is a composite (local) function. Here, we can regard our approach as a kind of nonlocal extension of the Jaynes-Shannon argumentation. The form (37) of the entropy is unique with the given requirements, therefore one can use it as a starting point. The emergent structure is independent of the microscopic background. Every reasonable microscopic approach (kinetic or stochastic) corresponds to the formulated general requirements.

Moreover, in (37) one can recognize the central formula of the Extreme Physical Information (EPI) principle of Frieden [50] and our argumentation gives a kind of foundation of the principle (uniqueness) and also some limitations (conditions of validity as mass-scale invariance) and a different interpretation (there is no need of the measurement-information arguments, objectivity of the theory is reconstructed). Here the predictivity of the approach is even more evident as one can see from the increasing number of applications of statistical physics based on Fisher information.

We think that the recent point of view and results can be generalized far beyond quantum mechanics, can be used as a general approach to non-equilibrium and weakly nonlocal statistical physics.

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