Intrinsic time gravity and the Lichnerowicz–York equation

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Received 2 January 2013, in final form 19 March 2013
Published 19 April 2013
Online at stacks.iop.org/CQG/30/095016

Abstract
We investigate the effect on the Hamiltonian structure of general relativity of choosing an intrinsic time to fix the time slicing. 3-covariance with momentum constraint is maintained, but the Hamiltonian constraint is replaced by a algebraic equation for the trace of the momentum. This reveals a very simple structure with a local reduced Hamiltonian. The theory is easily generalized; in particular, the square of the Cotton–York tensor density can be added as an extra part of the potential while at the same time maintaining the classic 2 + 2 degrees of freedom. Initial data construction is simple in the extended intrinsic time formulation; we get a generalized Lichnerowicz–York equation with nice existence and uniqueness properties. Adding standard matter fields is quite straightforward.

PACS numbers: 04.20.Cv, 04.20.Ex

1. Introduction

The Hamiltonian theory for general relativity was most clearly laid out by Arnowitt, Deser, and Misner [1] more than 50 years ago. The phase space consists of a pair ($g_{ij}, \pi^{ij}$), where $g_{ij}$ is a Riemannian 3-metric and $\pi^{ij}$ is the conjugate momentum. These cannot be freely chosen because they must satisfy the constraints

$$-gR + \pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2 = 0; \quad \text{and} \quad \nabla_i\pi^{ij} = 0,$$

where $\pi = g_{ij}\pi^{ij}$ is the trace of $\pi^{ij}$. These conditions are known respectively as the Hamiltonian and momentum constraints. We must also choose a scalar and a vector ($N, N^i$). These are the ‘lapse’ and ‘shift’. The total Hamiltonian, in the compact without boundary case, is

$$H = \int \left[ g^{-1/2}N \left( -gR + \pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2 \right) - 2N_j\nabla^i\pi^{ij} \right] d^3x.$$
Therefore \((N, N')\) are the Lagrange multipliers of the constraints. The first of Hamilton’s equations gives the relationship between \(\pi^j\) and the time derivative of \(g_{ij}\)

\[
\frac{\partial g_{ij}}{\partial t} = 2Ng^{-1/2} \left( \pi_{ij} - \frac{1}{2} g_{ij} \pi \right) + \nabla_i N_j + \nabla_j N_i.
\]

(3)

Since the definition of the extrinsic curvature, \(K_{ij}\), is

\[
\frac{\partial g_{ij}}{\partial t} = 2NK_{ij} + \nabla_i N_j + \nabla_j N_i,
\]

(4)

we immediately get \(g^{-1/2}(\pi_{ij} - \frac{1}{2} g_{ij} \pi) = K_{ij}\).

A major difficulty with the canonical quantization of gravity program is this freedom to choose \(N\) [2]. Each choice of \(N\) gives a different slicing of spacetime, and reflects the 4-covariance of the Einstein equations. We want to break this covariance, choose a natural time variable, and compute the emergent lapse. York in [3] pointed out that the local volume, \(\sqrt{g}\), and \(\pi / \sqrt{g}\) are canonically conjugate, and suggested that \(\pi / \sqrt{g}\) is a natural time, and that \(\sqrt{g}\) is the local energy. DeWitt, in [4], introduced the convention of calling time choices which are defined in terms of the intrinsic geometry ‘intrinsic time’ and time choices based on the extrinsic geometry ‘extrinsic time’. The York choice is obviously an ‘extrinsic’ time because \(\pi / \sqrt{g} = -2K\), where \(K\) is the trace of the extrinsic curvature of the slice. With York’s choice, since \(\sqrt{g}\) is the associated energy density, the total volume of the slice is the Hamiltonian. Two years earlier Charles Misner [5] made the opposite choice, albeit in a minisuperspace context. He picked the local volume as his time, thus an ‘intrinsic time’, and \(\pi\) is then the energy density. This is where he introduced the ‘mixmaster universe’. The great advantage of Misner’s choice is that a local reduced Hamiltonian appears. Recently, two of us [2] presented a theory of gravity passing from classical to quantum regimes with a paradigm shift from 4-covariance to 3-covariance. Its framework revealed the primacy of a local reduced Hamiltonian and intrinsic time proportional to \(\ln g^{1/3}\). The present article is an expansion and extension of part of that work.

### 2. Intrinsic time gravity

We start, following Misner [5], by choosing the intrinsic time function \(T = \ln g^{1/3}\). We choose an arbitrary 3-slice and call it the \(T = 0\) slice. The change in intrinsic time is given by the change in \(\delta \ln g^{1/3}\). Although \(g\) is a scalar density (one of York’s reservations [12] on Misner’s choice), \(\delta \ln g^{1/3} = \frac{1}{3} g^{1/3} \delta g_{ij}\) is a scalar under spatial diffeomorphisms. It is time interval, and not absolute time since \(\sqrt{g} = e^{-\frac{2}{3} \sqrt{g}}\pi\). The symplectic 1-form becomes

\[
\int \pi^i j \delta g_{ij} = \int \frac{\pi^i j \delta g_{ij}}{g^{1/3}} + \pi \delta \ln g^{1/3}.
\]

(6)

We see immediately that \((\delta g_{ij}, \pi^i j)\) and \((\ln g^{1/3}, \pi)\) form conjugate pairs, which is clearly the generalization of the \(P_i \delta x^i + P_0 \delta t\) one gets in the case of a simple particle. With the choice of the change of time as \(\delta \ln g^{1/3}\) (which varies from \(-\infty\) to \(+\infty\), instead of 0 to \(\infty\)), we can see that \(\pi\) is the direct analogue of \(P_0 = -E\).
We now substitute the decomposition of \((g_{ij}, \pi^{ij})\) into the Hamiltonian constraint, equation (1), to give
\[
-gR + \bar{g}_{ab}\bar{g}_{ij}\bar{\pi}^{ij}\bar{\pi}^{kl} - \beta^2 \pi^2 = 0.
\]
(7)
where \(\beta^2 = \frac{1}{3}\) for GR. It is a free positive parameter for the extended theory which we discuss later. We know that the Hamiltonian is the generator of time translations, is conjugate to time, and should equal the energy, and therefore the true Hamiltonian density with this choice of intrinsic time should be \(-\pi\). We then just solve the Hamiltonian constraint, which is equivalent to \((\pi - \bar{H}/\sqrt{\beta^2}) (\pi + \bar{H}/\sqrt{\beta^2}) = 0\), to find the reduced Hamiltonian
\[
-\pi = \frac{\bar{H}}{\beta} = \frac{1}{\beta} \sqrt{\bar{g}_{ab}\bar{g}_{ij}\bar{\pi}^{ij}\bar{\pi}^{kl} - gR}; \quad \beta = \pm \sqrt{\beta^2}.
\]
(8)
A simple toy model for this process is given by the relativistic particle, which satisfies a constraint
\[
-(p^0)^2 + \vec{p} \cdot \vec{p} + m^2 = 0.
\]
(9)
\(p^0\) is the energy, and therefore the physical Hamiltonian is
\[
E = p^0 = -P_0 = H = \sqrt{\vec{p} \cdot \vec{p} + m^2}.
\]
(10) Hamilton’s equations give the equations of motion for \(\vec{p}\). We then add the algebraic condition \(p^0 = H\) as a dynamical equation to determine \(p^0\) at each instant of time.

The Hamiltonian \(\bar{H}/\beta\) of equation (8) generates \(ln g^{1/3}(x,t)\) translations, and gives equations of motion for \(\bar{g}_{ij}\) and \(\bar{\pi}^{ij}\) with respect to this intrinsic time variable. We need to write \(gR\) in terms of \(\bar{g}_{ij}\) and \(ln g^{1/3}\). This is quite straightforward. We do not have an equation for \(ln g\), since it is the time, but we do need a dynamical equation for \(\pi\) which is given by
\[
\pi + \frac{1}{\beta} \bar{H}(\bar{g}_{ij}, \bar{\pi}^{ij}), ln g^{1/3} = 0.
\]
(11)
This is a rewriting of the Hamiltonian constraint, but it is no longer to be viewed as a constraint; rather, it is the evolution equation for \(\pi\), in the sense that it is an algebraic equation (or it can be regarded as one of the dynamical equations) that allows us to compute \(\pi\) on each slice. This is the fundamental equation for intrinsic time gravity; and it can be interpreted respectively as the Hamilton–Jacobi equation and Schrödinger equation in the semi-classical and quantum regimes [2].

We choose the positive root for \(\bar{H}\) when we take the square-root. In mini-superspace models, negative values of \(\pi\) correspond, in agreement with current observations, to an expanding universe. The reduced Hamiltonian can be constructed before solving for \(\pi\). Since the system is 3-covariant, the evolution equations will propagate the momentum constraint. This is the only constraint left since the Hamiltonian constraint is gone because we have a unique choice of (intrinsic) time. Thus, we are left, modulo 3-covariance, with a system that has the expected 2 + 2 degrees of freedom.

The symplectic potential of the conjugate pair \((ln g^{1/3}, \pi)\) contributes
\[
\int \int (\pi \frac{\partial ln g^{1/3}}{\partial x}) d^3xd\beta = -\int \int \left[ \bar{H} \frac{\partial ln g^{1/3}}{\partial x} d^3x \right] d\beta, \text{ to the action; and it is thus clear that }
\int \int \frac{\partial ln g^{1/3}}{\partial x} d^3x \text{ contributes to the total Hamiltonian generating } t\text{-translations, where } t \text{ is the }
\text{ADM time parameter. By subtracting the tangential change generated by the momentum constraint, the rate of change of the normal component of } ln g^{1/3} \text{ is}
\]
\[
f = \lim_{\beta \to 0} \delta ln g^{1/3} = \frac{\partial ln g^{1/3}}{\partial x} = \frac{\partial ln g^{1/3}}{\partial t} = -\frac{2}{3} \nabla N'.
\]
(12)
The classical evolution of \((\bar{g}_{ij}, \bar{\pi}^{ij})\) w.r.t. the ADM time variable \(t\) can equivalently be obtained from the effective Hamiltonian (where \(H_f\) is the momentum constraint and \(N^t\) is the shift)

\[
H_f = \int \left[ -\frac{3}{2} \pi + N^t H_i \right] d^3x = \int \left[ f \sqrt{\bar{g}} \bar{g}^{ij} \bar{\pi}^{kl} - gR - 2N_l (g^{-\frac{1}{2}} \nabla_j \bar{\pi}^{ij}) \right] d^3x, \tag{13}
\]

where \(f\) is just as given in equation (12) and we replaced the \(\pi\) using equation (11), including the \(\pi\) in the momentum constraint. \(f\) is not a Lagrange multiplier, it is shorthand for expression (12) and when we vary \(H_f\) by the shift, including the \(N^t\) in \(f\) we recover, the momentum constraint. On the other hand, when we vary \(H_f\) with respect to the dynamical variables \((\bar{g}_{ij}, \bar{\pi}^{ij})\) we have no contribution from \(f\) because \(\nabla_j N_l = \delta_i (\sqrt{\bar{g}} N^l) / \sqrt{\bar{g}}\).

Let us compare equation (5) with equation (12). They obviously agree if

\[
N = -\frac{3f \sqrt{\bar{g}}}{\pi}. \tag{14}
\]

It is a straightforward exercise to show that the evolution equations arising from \(H_f\) agree with those from the ADM evolution equations if one uses the emergent lapse given by equation (14). Therefore the reduced Hamiltonian \(H_f\) generates vacuum solutions of the Einstein equations with a preferred foliation. We continue to deal with standard GR.

Given any \(t\)-foliation of spacetime, we can find a corresponding reduced Hamiltonian that generates it. This means that ‘many fingered time’ lives on in the intrinsic time picture but in a very different form by the way the emergent \(N\), as given by equation (14), depends on \(f\). For compact manifolds without boundary, Hodge decomposition yields \(\frac{gR}{\sqrt{\bar{g}}} = C + \nabla i W^i\) uniquely; wherein the harmonic 0-form \(C\) is constant (independent of spatial coordinates), and \(W^i\) can furthermore be completely gauged away by the shift vector \(N^t\). Thus, taking the simplest and gauge-invariant solution, \(f = C\), we get

\[
N = -\frac{3C \sqrt{\bar{g}}}{\pi} = \frac{3C}{2K}, \tag{15}
\]

where \(K\) is the trace of the extrinsic curvature. We can see the slicing we obtain is that generated by what the mathematicians call the ‘inverse mean curvature flow’. This is extremely well behaved. In particular, if the big bang is in any way reasonable (all we need is that it be a crushing singularity [6] and that the 3-volume goes to zero), then we get a foliation which strikes the singularity all at once and along which \(\pi\) smoothly approaches \(+\infty\) [7].

On quantizing, as noted earlier, equation (8) becomes a Schrödinger equation. One might worry about the square root in the reduced Hamiltonian. On shell, its square equals \(\sqrt{\bar{g}} \int d^3x \sqrt{\bar{g}} \bar{g}^{ij} \bar{\pi}^{kl} - gR - 2N_l (g^{-\frac{1}{2}} \nabla_j \bar{\pi}^{ij})\), which is equivalent to changing the \(\pi^2/6\) and so is obviously a positive operator. Further, since \(\pi^2/6\) is bounded away from zero, and blows up as we approach the initial singularity, there exists a open growing neighborhood of the classical solution in the phase space where the function remains positive.

We have expressed vacuum GR in a very simple form: we have only the effective Hamiltonian, equation (13), combined with the dynamical equation for the trace of the momentum, equation (11), and the momentum constraint. How can we generalize this structure, while maintaining the 2 + 2 degrees of freedom and the 3-covariance?

There are two changes we can make. We can multiply each of the three terms in the Hamiltonian constraint, as given by either equation (1) or equation (7), with arbitrary constants. This means deforming \(\bar{g}\) in the \(\pi^2\) term away from \(\frac{1}{2}\), which is equivalent to changing the constant in the DeWitt supermetric [4]. We really only need two constants because a constant rescaling of every term is equivalent to the identity. Thus we replace \(gR\) by \(\alpha^2gR\) and leave the \(\bar{g}_{ij} \bar{\pi}^{ij} \bar{\pi}^{kl}\) alone. More radically, we can replace the ‘potential’, \(-R\), by any scalar function \(V\) of the metric, and everything still works. In particular, \(V\) can be made positive semidefinite as in [2]. The set of equations now reads

\[
H_f = \int \left[ f \sqrt{\bar{g}} \bar{g}^{ij} \bar{\pi}^{kl} + gV - 2N_l (g^{-\frac{1}{2}} \nabla_j \bar{\pi}^{ij}) \right] d^3x, \tag{16}
\]
\[
\pi := -\frac{1}{\beta} \sqrt{\tilde{g} \tilde{g}_{ij} \tilde{\gamma}^{ij} \tilde{\gamma}^{kl} + gV},
\]
(17)
\[
f := \frac{\partial \ln g^{1/3}}{\partial t} - \frac{2}{3} \nabla_i N^i.
\]
(18)

We can freely pick the shift \(N^i\), and either \(f\) or \(\partial T/\partial t\) and reconstruct the other from equation (18). Another route is to pick \(\partial T/\partial t\) and use the Hodge decomposition to write it as a constant (equaling the average over the space of \(\partial T/\partial t\) together with the divergence of a vector (which can be gauged away by the shift). In every case we get a solution to the vacuum Einstein equations with emergent \(N\) given by equations (14) and (15).

3. Initial data and extensions of the Lichnerowicz–York equation

It is not quite that easy, however; we need to find explicit initial data. There is no point in having a nice Hamiltonian system with no solutions. We need to find a pair \((g_{ij}, \pi^i)\) that satisfies both \(\beta \pi + H = 0\) and \(\nabla_i \pi^i = 0\). In the case of GR these are equivalent to the standard constraints. The only really successful general way of solving them is the conformal method, which was initiated by Lichnerowicz [8]. There is a very comprehensive account in [9], especially in chapter 7. The technique is to choose free data that consist of a base metric \(\hat{g}_{ij}\), a symmetric tensor density \(\hat{\pi}^{TT}\) that is both tracefree and divergence-free with respect to \(\tilde{g}\), and a scalar \(\hat{p}\). It is particularly simple if \(\hat{p}\) is a constant. This guarantees that the extrinsic curvature has constant trace, and thus we construct a ‘constant mean curvature’ (CMC) slice. We make a conformal transformation \(g_{ij} = \hat{g}^{ij} \hat{g}_{ij}, \pi^i = \hat{\phi}^{-\frac{2}{3}} \hat{\pi}^i\). It turns out that \(\hat{\pi}^{TT}\) is conformally invariant; it remains \(TT\) with respect to \(g_{ij}\). Thus we write \(\pi^{TT} = \hat{\pi}^{TT}\). We also set \(\pi = \sqrt{\hat{g}} \hat{p}\). We are guaranteed that \(\pi^i = (\nabla_i \pi + \sqrt{\hat{g}} \hat{p}/3)\) satisfies the momentum constraint for any conformal factor \(\hat{\phi}\). In particular, we can specialize to \(\hat{\phi}^4 = \hat{g}^{1/3}\) to get \((\hat{g}_{ij}, \hat{\pi}^i) = (\hat{g}_{ij}, \hat{\pi}^{TT}_i)\). We now seek an appropriate \(\hat{\phi}\) in order to solve the Hamiltonian constraint. In GR, this reduces to solving the Lichnerowicz–York (LY) equation

\[
8\hat{\nabla}^2 \hat{\phi} - \hat{R} \hat{\phi} + \hat{g}^{-1} \hat{\pi}^{TT}_i \hat{\pi}^{TT}_i \hat{\phi}^{-7} - \frac{1}{6} \hat{\phi}^2 \hat{\phi}^5 = 0,
\]
(19)

where \(\hat{g}\) is the determinant of \(\hat{g}_{ij}\). This is an extremely nice equation because it always has a unique, positive solution [10].

Let us now multiply the \(R\) and the \(\pi^2\) terms in the Hamiltonian constraint by arbitrary positive constants. The new LY equation becomes

\[
8\alpha^2 \hat{\nabla}^2 \hat{\phi} - \alpha^2 \hat{R} \hat{\phi} + \tilde{g}^{-1} \hat{\pi}^{TT}_i \hat{\pi}^{TT}_i \hat{\phi}^{-7} - \beta^2 \hat{\phi}^2 \hat{\phi}^5 = 0.
\]
(20)

This equation is just as nice as the original LY equation, equation (19), because it too always has a unique positive solution. Such a rescaling has been recently discussed in a different context in [11].

However, if we replace \(R\) by any other function of the metric, as we suggest above, we destroy the nice properties of the LY equation. For example, if we were to add an \(R^2\) term, the LY equation would pick up the term \((8\nabla^2 \phi - R \phi)^2\), which changes the nature of the LY equation completely and the existence and uniqueness results no longer hold.

There is one exception. York, in [12], rediscovered a conformally covariant metric tensor. This is now known as the Cotton–York tensor(density), i.e., \(\beta^i_j\). This transforms exactly as a TT tensor does under conformal transformations, so \(\beta^i_j\) is conformally invariant. Therefore we can pick a generalized Hamiltonian as

\[
-\alpha^2 gR + \tilde{g} \tilde{g}_{ij} \tilde{\pi}^{ij} \tilde{\pi}^{kl} + \gamma^2 \beta^i_j \beta^i_j - \beta^2 \pi^2 = 0,
\]
(21)
where $\gamma$ is another coupling constant. The generalized LY equation becomes

$$8\alpha^2\hat{\nabla}^2 \phi - \alpha^2 \hat{R}\phi + \hat{g}^{-1}(\hat{\pi}_{TT}^{ij} + \gamma^2 \hat{\beta}_j^i \hat{\beta}_i^j)\phi^{-7} - \hat{g}^{-1} \beta^2 \hat{p}^2 \phi^5 = 0. \quad (22)$$

This is just as well-behaved as the original LY equation: it always possesses a positive unique solution. This conformal factor maps the free data onto a solution both of the generalized Hamiltonian constraint, equation (21), and the momentum constraint.

We can do slightly better than this. Let us assume that we would like to have a more general Hamiltonian, and add, say, an $R^2$ term to the potential. Thus, we would like initial data which satisfies

$$-\alpha^2 gR - \rho gR^2 + \bar{g}_{ab} \beta \bar{\pi}^{ij} \bar{\pi}^{kl} + \gamma^2 \beta_j^i \beta_i^j - \beta^2 \pi^2 = 0, \quad (23)$$

where $\rho$ is another parameter. The analogue of the LY equation will be

$$8\alpha^2\hat{\nabla}^2 \phi - \alpha^2 \hat{R}\phi + \hat{g}^{-1}(\hat{\pi}_{TT}^{ij} + \gamma^2 \hat{\beta}_j^i \hat{\beta}_i^j)\phi^{-7} - \hat{g}^{-1} \beta^2 \hat{p}^2 \phi^5 = \rho \phi^{-5} (8\hat{\nabla}^2 \phi - \hat{R}\phi)^2. \quad (24)$$

This, as mentioned earlier, is a deeply unpleasant equation. If we linearize this equation about $\rho = 0$, a combination of the Fredholm alternative and the implicit function theorem [13] shows that the nonlinear equation, equation (24), has a solution for a range of $\rho$s in a neighborhood of zero. Unfortunately, this technique gives us no estimate as to the size of this neighborhood. One can immediately see that this technique works for any choice of metric potential.

We construct CMC initial data, not because it plays any fundamental role in this theory, but simply out of convenience. The slices generated by the Hamiltonian, equation (16), will not stay CMC. We can use the same technique, i.e., the Fredholm alternative + the implicit function theorem, to slightly relax the condition that $\hat{p}$ is a constant. We can construct non-CMC initial data, but it is not clear how large the deviation from CMC we can allow.

We can add a whole range of matter fields to the system. We can add a cosmological constant, a massive or massless scalar field, a Maxwell or a Yang–Mills field, dust, or even neutrinos [14]. The reader should note that the coupling constant of the Cotton–York density term in the generalized Hamiltonian, $\gamma^2 \beta_j^i \beta_i^j$, is dimensionless in natural units (in equation (1) and elsewhere in this article we use units $2\kappa = 1$). In 3-covariant modifications of general relativity this term is essential to the perturbative power-counting renormalizability of the quantum theory [15, 2].

At early (intrinsic) times, the $R$ term in equation (17) is suppressed by the $e^{\ln \tilde{g}}$ factor, while the Cotton–York tensor density-squared term is conformally invariant and independent of $\tilde{g}$; and at late times the theory becomes more and more like GR. Thus, near the Big Bang, with $\ln \tilde{g} \to -\infty$, the Cotton–York density term should dominate. In particular, we expect that the BKL model [16], where the time variation of the gravitational field is expected to dominate over the spatial variations, will no longer hold in the presence of non-trivial amount of the Cotton–York tensor.

A key advantage the intrinsic time formalism has over the extrinsic time formalism is that the Hamiltonian constraint can be easily put in the form of an algebraic equation for $\pi$. If we wanted to use extrinsic time, we would need to solve the Hamiltonian constraint for $\sqrt{\tilde{g}}$. Therefore the reduced Hamiltonian in the extrinsic time gauge is a non-local object. Another sign of non-locality is that the lapse function for a CMC foliation is determined by

$$(\nabla^2 - \hat{g}^{-1} \pi^{ij} \pi_{ij}) \kappa = \text{spatial constant}. \quad (25)$$

This is obviously a nice elliptic equation, but clearly non-local as distinct from equation (14), the equation for the intrinsic time lapse.

It is clear that using an intrinsic time gives a very clean Hamiltonian structure in classical gravity. We expect that these good properties will carry over as one tries to implement the canonical quantization program.
Acknowledgments

This work has been supported in part by the National Science Council of Taiwan under grant nos NSC101-2112-M-006-007-MY3, the Academia Sinica and the National Center for Theoretical Sciences, Taiwan. We wish to thank Jörg Fraundeiner and Gerhard Huisken for helpful comments.

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