A KOHNO–DRINFELD THEOREM FOR QUANTUM WEYL GROUPS

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ABSTRACT. Let \( g \) be a complex, simple Lie algebra with Cartan subalgebra \( h \) and Weyl group \( W \). In [MTL], we introduced a new, \( W \)–equivariant flat connection on \( h \) with simple poles along the root hyperplanes and values in any finite–dimensional \( g \)–module \( V \). It was conjectured in [TL] that its monodromy is equivalent to the quantum Weyl group action of the generalised braid group of type \( g \) on \( V \) obtained by regarding the latter as a module over the quantum group \( U_\hbar g \). In this paper, we prove this conjecture for \( g = \mathfrak{sl}_n \).

1. Introduction

One of the many virtues of quantum groups is their ability to describe the monodromy of certain first order systems of Fuchsian PDEs. If \( U_\hbar g \) is the Drinfeld–Jimbo quantum group of the complex, simple Lie algebra \( g \), the universal \( R \)–matrix of \( U_\hbar g \) yields a representation of Artin’s braid group on \( n \) strings \( B_n \) on the \( n \)–fold tensor product \( V^\otimes n \) of any finite–dimensional \( U_\hbar g \)–module \( V \). A fundamental, and paradigmatic result of Kohno and Drinfeld establishes the equivalence of this representation with the monodromy of the Knizhnik–Zamolodchikov equations for \( g \) with values in \( V^\otimes n \) [Dr3, Dr4, Dr5, Ko1]. Lusztig, and independently Kirillov–Reshetikhin and Soibelman realised that \( U_\hbar g \) also yields representations of another braid group, namely the generalised braid group \( B_g \) of Lie type \( g \) [Lu1, KR, So]. Whereas the \( R \)–matrix representation is a deformation of the natural action of the symmetric group \( S_n \) on \( n \)–fold tensor products, these representations of \( B_g \) deform the action of (a finite extension of) the Weyl group \( W \) of \( g \) on any finite–dimensional \( g \)–module \( V \).

The aim of this paper is to show that these quantum Weyl group representations describe the monodromy of the flat connection introduced in [MTL] and, independently, in [FMTV]. More precisely, realise \( B_g \) as the fundamental group of the orbit space \( h_{\text{reg}}/W \) of the set of regular elements of a Cartan subalgebra \( h \) of \( g \) under the action of \( W \) [Br]. Then, one can define a flat vector bundle \( (V, \nabla_\kappa) \) with fibre \( V \) over \( h_{\text{reg}}/W \) [MTL]. The connection \( \nabla_\kappa \) depends upon a parameter \( \hbar \in \mathbb{C} \) and it was conjectured in [TL] that, when \( \hbar \) is regarded as a formal variable, its monodromy is equivalent to the quantum Weyl group action of \( B_g \) on \( V \). This conjecture was checked in [TL] for a number of pairs \((g, V)\) including vector representations of classical Lie algebras and adjoint representations of simple Lie algebras.

In the present paper, we prove this conjecture for \( g = \mathfrak{sl}_n \), so that \( B_g = B_n \). The proof relies on the Kohno–Drinfeld theorem for \( U_\hbar \mathfrak{sl}_k \) via the use of the dual pair \((\mathfrak{gl}_k, \mathfrak{gl}_n)\). Our main observation is that the duality between \( \mathfrak{gl}_k \) and \( \mathfrak{gl}_n \) derived from their joint action on the space \( \mathcal{M}_{k,n} \) of \( k \times n \) matrices exchanges \( \nabla_\kappa \) for \( \mathfrak{sl}_k \) and the Knizhnik–Zamolodchikov connection for \( \mathfrak{sl}_k \), thus acting as a simple–minded integral transform. This shows the equivalence of the monodromy representation of \( \nabla_\kappa \) for \( \mathfrak{sl}_n \) with a suitable \( R \)–matrix representation for \( U_\hbar \mathfrak{sl}_k \). The proof is completed by noting that the duality between \( U_\hbar \mathfrak{gl}_k \) and \( U_\hbar \mathfrak{gl}_n \) exchanges the \( R \)–matrix representation of \( U_\hbar \mathfrak{gl}_k \) with the quantum Weyl group representation of \( U_\hbar \mathfrak{sl}_n \).

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This may be schematically summarised by the following diagram

\[
\begin{array}{c}
\nabla_{KZ}, \mathfrak{sl}_k \quad \mathcal{M}_{k,n} \quad \nabla_\kappa, \mathfrak{sl}_n \\
KD \\
R^\vee, U_h \mathfrak{sl}_k \quad \mathcal{M}^h_{k,n} \quad W_h, U_h \mathfrak{sl}_n
\end{array}
\]

The structure of the paper is as follows. In section 2, we give the construction of the connection \( \nabla_\kappa \) following [MTL]. We show in section 3 that the duality between \( \mathfrak{gl}_k \) and \( \mathfrak{gl}_n \) identifies the Knizhnik–Zamolodchikov connection for \( n \)-fold tensor products of symmetric powers of the vector representation of \( \mathfrak{sl}_k \) and the connection \( \nabla_\kappa \) for \( \mathfrak{sl}_n \). In section 4 we recall the definition of the Drinfeld–Jimbo quantum groups \( U_h \mathfrak{gl}_k \) and \( U_h \mathfrak{gl}_n \) and, in section 5, show how they jointly act on the quantum \( k \times n \) matrix space \( S_h(\mathcal{M}^*_k) \). The corresponding \( R \)-matrix and quantum Weyl group representations of \( B_n \) on \( S_h(\mathcal{M}^*_k) \) are shown to coincide in section 6. Section 7 contains our main result.

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2. Flat connections on \( \mathfrak{h}_{\text{reg}} \)

The results in this section are due to J. Millson and the author [MTL]. They were obtained independently by De Concini around 1995 (unpublished). Let \( \mathfrak{g} \) be a complex, simple Lie algebra with Cartan subalgebra \( \mathfrak{h} \) and root system \( R \subset \mathfrak{h}^* \). Let \( \mathfrak{h}_{\text{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha \in R} \text{Ker}(\alpha) \) be the set of regular elements in \( \mathfrak{h} \) and \( V \) a finite-dimensional \( \mathfrak{g} \)-module. We shall presently define a flat connection on the topologically trivial vector bundle \( \mathfrak{h}_{\text{reg}} \times V \) over \( \mathfrak{h}_{\text{reg}} \). We need for this purpose the following simple flatness criterion due to Kohno [Ko2]. Let \( B \) be a complex, finite-dimensional vector space and \( \mathcal{A} = \{ H_i \}_{i \in I} \) a finite collection of hyperplanes in \( B \) determined by the linear forms \( \phi_i \in B^*, i \in I \).

**Lemma 2.1.** Let \( F \) be a finite-dimensional vector space and \( \{ r_i \} \subset \text{End}(F) \) a family indexed by \( I \). Then,

\[
\nabla = d - \sum_{i \in I} \frac{d\phi_i}{\phi_i} r_i \tag{2.1}
\]

defines a flat connection on \( (B \setminus \mathcal{A}) \times F \) iff, for any subset \( J \subseteq I \) maximal for the property that \( \bigcap_{j \in J} H_j \) is of codimension 2, the following relations hold for any \( j \in J \)

\[
[r_j, \sum_{j' \in J} r_{j'}] = 0 \tag{2.2}
\]
For any $\alpha \in R$, choose root vectors $e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha = \alpha^\vee$ and let

$$\kappa_\alpha = \frac{\langle \alpha, \alpha \rangle}{2} (e_\alpha f_\alpha + f_\alpha e_\alpha) \in U\mathfrak{g}$$

be the truncated Casimir operator of the $\mathfrak{sl}_2(\mathbb{C})$–subalgebra of $\mathfrak{g}$ spanned by $e_\alpha, h_\alpha, f_\alpha$. Note that $\kappa_\alpha$ does not depend upon the particular choice of $e_\alpha$ and $f_\alpha$ and that $\kappa_{-\alpha} = \kappa_\alpha$.

**Theorem 2.2.** The one–form

$$\nabla^h_\kappa = d - \hbar \sum_{\alpha > 0} \frac{d\alpha}{\alpha} \kappa_\alpha = d - \frac{\hbar}{2} \sum_{\alpha \in R} \frac{d\alpha}{\alpha} \kappa_\alpha$$

defines, for any $h \in \mathbb{C}$, a flat connection on $\mathfrak{h}_{\text{reg}} \times V$.

**Proof.** By lemma 2.1, we must prove that for any rank 2 subsystem $R_0 \subseteq R$, the following holds for any $\alpha \in R_0^+ = R_0 \cap R^+$

$$[\kappa_\alpha, \sum_{\beta \in R_0^+} \kappa_\beta] = 0$$

This may be proved by an explicit computation by considering in turn the cases where $R_0$ is of type $A_1 \times A_1, A_2, B_2$ or $G_2$ but is more easily settled by the following elegant observation of A. Knutson [Kn]. Let $\mathfrak{g}_0 \subseteq \mathfrak{g}$ be the semi–simple Lie algebra with root system $R_0$, $\mathfrak{h}_0 \subset \mathfrak{h}$ its Cartan subalgebra and $\mathfrak{c}_0 \subset Z(U\mathfrak{g}_0)$ its Casimir operator. Then, $\sum_{\beta \in R_0^+} \kappa_\beta - \mathfrak{c}_0$ lies in $U\mathfrak{h}_0$ so that (2.5) holds since $\kappa_\alpha$ commutes with $\mathfrak{h}_0$.

Let $G$ be the complex, connected and simply–connected Lie group with Lie algebra $\mathfrak{g}$, $T$ its torus with Lie algebra $\mathfrak{h}$, $N(T) \subset G$ the normaliser of $T$ and $W = N(T)/T$ the Weyl group of $G$. Let $B_\mathfrak{g} = \pi_1(\mathfrak{h}_{\text{reg}}/W)$ be the generalised braid group of type $\mathfrak{g}$ and $\sigma : B_\mathfrak{g} \to N(T)$ a homomorphism compatible with

$$B_\mathfrak{g} \xrightarrow{\sigma} N(T) \xrightarrow{\gamma} \mathfrak{h}_{\text{reg}} \leftarrow \mathfrak{h}_{\text{reg}}/W$$

We regard $B_\mathfrak{g}$ as acting on $V$ via $\sigma$. Let $\tilde{\mathfrak{h}}_{\text{reg}} \rightarrow \mathfrak{h}_{\text{reg}}$ be the universal cover of $\mathfrak{h}_{\text{reg}}$ and $\mathfrak{h}_{\text{reg}}/W$.

**Proposition 2.3.** The one–form $p^*\nabla^h_\kappa$ defines a $B_\mathfrak{g}$–equivariant flat connection on $\tilde{\mathfrak{h}}_{\text{reg}} \times V = p^*(\mathfrak{h}_{\text{reg}} \times V)$. It therefore descends to a flat connection on the vector bundle

$$V \rightarrow \tilde{\mathfrak{h}}_{\text{reg}} \times B_\mathfrak{g} \times V$$

$$\rightarrow \mathfrak{h}_{\text{reg}}/W$$

which is reducible with respect to the weight space decomposition of $V$ and unitary if $h \in i\mathbb{R}$.

**Proof.** The action of $B_\mathfrak{g}$ on $\Omega^*(\tilde{\mathfrak{h}}_{\text{reg}} \times V) = \Omega^*(\tilde{\mathfrak{h}}_{\text{reg}}) \otimes V$ is given by $\gamma \mapsto (\gamma^{-1})^* \otimes \sigma(\gamma)$. Thus, if $\gamma \in B_\mathfrak{g}$ projects onto $w \in W$, we get using $p \cdot \gamma^{-1} = w^{-1} \cdot p$,

$$\gamma^* p^* \nabla^h_\kappa \gamma^{-1} = d - \frac{\hbar}{2} \sum_{\alpha \in R} dp^* w\alpha/p^* w\alpha \otimes \sigma(\gamma)\kappa_\alpha \sigma(\gamma)^{-1}$$
Since $\kappa_\alpha = \frac{(\alpha, \alpha)}{2}(e_\alpha f_\alpha + f_\alpha e_\alpha)$ is independent of the choice of the root vectors $e_\alpha, f_\alpha$, Ad($\sigma(\gamma)$)$\kappa_\alpha = \kappa_{\gamma}$ and (2.8) is equal to $p^* \nabla^h$ as claimed. $p^* \nabla^h$ is flat by theorem 2.2, commutes with the fibrewise action of $\mathfrak{h}$ because each $\kappa_\alpha$ is of weight 0 and is unitary because the $\kappa_\alpha$ are self-adjoint. 

Thus, for any homomorphism $\sigma : B_\mathfrak{g} \to N(T)$ compatible with (2.9), proposition 2.3 yields a monodromy representation $\rho^h_\alpha : B_\mathfrak{g} \to GL(V)$ which permutes the weight spaces compatibly with $W$. By standard ODE theory, $\rho^h_\alpha$ depends analytically on the complex parameter $h$ and, when $h = 0$, is equal to the action of $B_\mathfrak{g}$ on $V$ given by $\sigma$. We record for later use the following elementary

**Proposition 2.4.** Let $\gamma \in B_\mathfrak{g} = \pi_1(\mathfrak{h}_{\text{reg}}/W)$ and $\tilde{\gamma} : [0, 1] \to \mathfrak{h}_{\text{reg}}$ be a lift of $\gamma$. Then,

$$\rho^h_\alpha(\gamma) = \sigma(\gamma) \mathcal{P}(\tilde{\gamma})$$

(2.9)

where $\mathcal{P}(\tilde{\gamma}) \in GL(V)$ is the parallel transport along $\tilde{\gamma}$ for the connection $\nabla^h$ on $\mathfrak{h}_{\text{reg}} \times V$.

**Proof.** Let $\tilde{\gamma} : [0, 1] \to \mathfrak{h}_{\text{reg}}$ be a lift of $\gamma$ and $\tilde{\gamma}$ so that $\tilde{\gamma}(1) = \gamma^{-1}\tilde{\gamma}(0)$. Then, since the connection on $p^*(\mathfrak{h}_{\text{reg}} \times V)$ is the pull–back of $\nabla^h$, and that on $(p^*(\mathfrak{h}_{\text{reg}} \times V))/B_\mathfrak{g}$ the quotient of $p^* \nabla^h$, we find

$$\rho^h_\alpha(\gamma) = \mathcal{P}(\gamma) = \sigma(\gamma) \mathcal{P}(\tilde{\gamma}) = \sigma(\gamma) \mathcal{P}(\tilde{\gamma})$$

(2.10)

By [Br], $B_\mathfrak{g}$ is presented on generators $T_i$, $i = 1 \ldots n$ labelled by a choice of simple roots $\alpha_i$ of $R$ with relations

$$T_i T_j T_i \cdots = T_j T_i T_j \cdots$$

(2.11)

for any $1 \leq i < j \leq n$ where each side of (2.11) has a number of factors equal to the order of $s_i s_j$ in $W$ and $s_i \in W$ is the orthogonal reflection across the hyperplane Ker($\alpha_k$). $T_i$ projects onto $s_i \in W$. An explicit choice of representatives of $T_1, \ldots, T_n$ in $\pi_1(\mathfrak{h}_{\text{reg}}/W)$ may be given as follows. Let $t \in \mathfrak{h}_{\text{reg}}$ lie in the fundamental Weyl chamber so that $\langle t, \alpha \rangle > 0$ for any $\alpha \in R_+$. Note that for any simple root $\alpha_i$, the intersection $t_{\alpha_i} = t - \frac{1}{2}(t, \alpha_i)\alpha_i$ of the affine line $t + C \cdot \alpha_i$ with Ker($\alpha_i$) does not lie in any other root hyperplane Ker($\beta$), $\beta \in R \setminus \{\alpha_i\}$. Indeed, if $\langle t_{\alpha_i}, \beta \rangle = 0$ then

$$2\langle t, \beta \rangle = \langle t, \alpha_i \rangle \langle \alpha_i, \beta \rangle = \langle t, \beta - s_i \beta \rangle$$

(2.12)

whence $\langle t, \beta \rangle = -\langle t, s_i \beta \rangle$, a contradiction since $s_i$ permutes positive roots different from $\alpha_i$. Let now $D$ be an open disc in $t + C \cdot \alpha_i$ of center $t_{\alpha_i}$ such that its closure $\overline{D}$ does not intersect any root hyperplane other than Ker($\alpha_i$). Consider the path $\gamma_i : [0, 1] \to t + C \cdot \alpha_i$ from $t$ to $s_i t$ determined by $\gamma_i(0, 1/3] \cup [2/3, 1]$ is affine and lies in $t + R \cdot \alpha_i \subset D$, $\gamma_i(1/3)$, $\gamma_i(2/3) \in \partial D$ and $\gamma_i(1/3, 2/3]$ is a semicircular arc in $\partial D$, positively oriented with respect to the natural orientation of $t + C \cdot \alpha_i$. Then, the image of $\gamma_i$ in $\mathfrak{h}_{\text{reg}}/W$ is a representative of $T_i$ in $\pi_1(\mathfrak{h}_{\text{reg}}/W, Wt)$.

3. Knizhnik–Zamolodchikov equations and dual pairs

We show in this section that the joint action of $\mathfrak{gl}_k$ and $\mathfrak{gl}_n$ on the space $\mathcal{M}_{k,n}$ of $k \times n$ matrices identifies the connection $\nabla^h$ for $\mathfrak{g} = \mathfrak{sl}_n$ and the Knizhnik–Zamolodchikov connection for $\mathfrak{sl}_k$. 

Let \( S(\mathcal{M}_{k,n}^*) = \mathbb{C}[x_{11}, \ldots, x_{kn}] \) be the algebra of polynomial functions on \( \mathcal{M}_{k,n} \). The group \( GL_k \times GL_n \) acts on \( S(\mathcal{M}_{k,n}^*) \) by
\[
(g_k, g_n) p(x) = p(g_k x g_n)
\]
and leaves the homogeneous components \( S^d(\mathcal{M}_{k,n}^*) \) of \( S(\mathcal{M}_{k,n}^*) \), \( d \in \mathbb{N} \), invariant. The decomposition of \( S(\mathcal{M}_{k,n}^*) \) under \( GL_k \times GL_n \) is well–known (see e.g., [Zh], §132) which we follow closely or [Mc, §1.4]. Let \( N_k, N_n \) be the groups of \( k \times k \) and \( n \times n \) upper triangular unipotent matrices respectively.

**Lemma 3.1.**
\[
S(\mathcal{M}_{k,n}^*)^N_k \times N_n = \mathbb{C}[\Delta_1, \ldots, \Delta_{\min(k,n)}]
\]
where \( \Delta_l(x) = \det(x_{ij})_{1 \leq i,j \leq l} \) is the \( l \)th principal minor of the matrix \( x \).

**Proof.** Assume for simplicity that \( k \leq n \). Let \( D \subset S(\mathcal{M}_{k,n}^*) \) be the subset of matrices \( x \) such that \( \Delta_i(x) \neq 0 \) for \( i = 1 \ldots k \). By the Gauss decomposition, any \( x \in D \) is conjugate under \( N_k^c \times N_n^c \) to a unique \( k \times n \) matrix \( d(x) \) with the same principal minors as \( x \), diagonal principal \( k \times k \) block and the remaining columns equal to zero. Consider now the \( k \times n \) matrix
\[
m(x) = \begin{pmatrix} \Delta_1(x) & \Delta_2(x) & \cdots & \Delta_{k-1}(x) & \Delta_k(x) & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 \end{pmatrix}
\]
Since \( \Delta_i(m(x)) = \Delta_i(x) \), \( 1 \leq i \leq k \), \( m(x) \) is also conjugate to \( d(x) \), and therefore to \( x \), under \( N_k^c \times N_n^c \). Thus, by density of \( D \), a polynomial \( p \in S(\mathcal{M}_{k,n}^*) \) is invariant under \( N_k \times N_n \) if and only if it is a polynomial in \( \Delta_1, \ldots, \Delta_k \). \( \square \)

Let \( Y_p \subset \mathbb{N}^p \) be the set of Young diagrams with at most \( p \) rows. For \( \lambda \in Y_p \), set \( |\lambda| = \sum_{i=1}^p \lambda_i \) and let \( V^{(p)}_\lambda \) be the irreducible representation of \( GL_p(\mathbb{C}) \) of highest weight \( \lambda \).

**Theorem 3.2.** As \( GL_k \times GL_n \)–modules,
\[
S^d(\mathcal{M}_{k,n}^*) \cong \bigoplus_{\lambda \in Y_{\min(k,n)}, |\lambda|=d} V^{(k)}_\lambda \otimes V^{(n)}_\lambda
\]

**Proof.** Assume again \( k \leq n \) for simplicity. By lemma 3.1, the highest weight vectors for the action of \( GL_k(\mathbb{C}) \times GL_n(\mathbb{C}) \) on \( S(\mathcal{M}_{k,n}^*) \) are the polynomials in \( \Delta_1, \ldots, \Delta_k \) which are eigenvectors for the torus of \( GL_k \times GL_n \). Since \( \Delta_l \) is of weight \( \varpi_l^{(k)} \oplus \varpi_l^{(n)} \), where \( \varpi_l^{(p)} \) is the \( l \)th fundamental weight of \( GL_p \), the highest weight vectors are the monomials \( \Delta_1^{m_1} \cdots \Delta_k^{m_k} \) with corresponding pair of Young diagrams \( (\lambda, \lambda) \) where
\[
\lambda = (m_1 + \cdots + m_k, m_2 + \cdots + m_k, \ldots, m_k)
\]
Thus, 3.4 holds since \( \Delta_l \) is a homogeneous function of degree \( l \). \( \square \)

As a \( \mathfrak{gl}_k \)–module,
\[
S(\mathcal{M}_{k,n}^*) = \mathbb{C}[x_{11}, \ldots, x_{kn}] \otimes \cdots \otimes \mathbb{C}[x_{1n}, \ldots, x_{kn}]
\]
and is therefore acted upon by the \( \mathfrak{gl}_k \)-intertwiners \( \tilde{\Omega}^{(k)}_{ij} \), \( 1 \leq i < j \leq n \), defined by
\[
\tilde{\Omega}^{(k)}_{ij} = \sum_a 1^{\otimes(i-1)} \otimes X_a \otimes 1^{\otimes(j-i-1)} \otimes X^a \otimes 1^{\otimes(n-j)}
\] (3.7)
where \( \{X_a\}, \{X^a\} \) are dual basis of \( \mathfrak{gl}_k \) with respect to the pairing \( \langle X, Y \rangle = \text{tr}(XY) \). On the other hand, given that the \( \mathfrak{sl}_n \)–module, \( \mathcal{S}(\mathcal{M}^*_{k,n}) \) is acted upon by the operators \( \kappa^{(n)}_{ij} \), \( 1 \leq i < j \leq n \), where
\[
\kappa^{(n)}_{ij} = e_a f_a + f_a e_a
\] (3.8)
is the truncated Casimir operator of the \( \mathfrak{sl}_n \)–subalgebra of \( \mathfrak{gl}_n \) corresponding to the root \( \alpha = \theta_i - \theta_j \). Let \( e_1, \ldots, e_p \) be the canonical basis of \( \mathbb{C}^p \) and \( E^{(p)}_{ab} e_c = \delta_{bc} e_a \), \( 1 \leq a, b \leq p \) the corresponding basis of \( \mathfrak{gl}_p \) with dual basis \( E^{(p)}_{ba} \). Let \( 1 \leq i < j \leq n \), then

**Proposition 3.3.** The following holds on \( \mathcal{S}(\mathcal{M}^*_{k,n}) \)
\[
2\tilde{\Omega}^{(k)}_{ij} = \kappa^{(n)}_{ij} - E^{(n)}_{ii} - E^{(n)}_{jj}
\] (3.9)

**Proof.** By (3.7), \( \tilde{\Omega}^{(k)}_{ij} \) acts on \( \mathcal{S}(\mathcal{M}^*_{k,n}) \) as
\[
\tilde{\Omega}^{(k)}_{ij} = \sum_{1 \leq a, b \leq k} x_{ai} \partial_{bi} x_{bj} \partial_{aj}
\] (3.10)
where \( x_{rc} \) and \( \partial_{rc} \) are the operators of multiplication by and derivation with respect to \( x_{rc} \). On the other hand, given that the \( \mathfrak{sl}_2(\mathbb{C}) \)–triple \( \{e_\alpha, h_\alpha, f_\alpha\} \) corresponding to the root \( \alpha = \theta_i - \theta_j \) of \( \mathfrak{sl}_n \) is \( \{E^{(n)}_{ij}, E^{(n)}_{ii} - E^{(n)}_{jj}, E^{(n)}_{ji}\} \), the following holds on \( \mathcal{S}(\mathcal{M}^*_{k,n}) \)
\[
\kappa^{(n)}_{ij} = \sum_{1 \leq a, b \leq k} x_{ai} \partial_{aj} x_{bj} \partial_{bi} + x_{bj} \partial_{bi} x_{ai} \partial_{aj}
\] (3.11)
Substracting, we find
\[
2\tilde{\Omega}^{(k)}_{ij} - \kappa^{(n)}_{ij} = - \sum_{1 \leq a, b \leq k} \delta_{ab} x_{ai} \partial_{bi} + \delta_{ab} x_{bj} \partial_{aj} = - E^{(n)}_{ii} - E^{(n)}_{jj}
\] (3.12)
as claimed \( \blacksquare \)

Let \( \lambda \in \mathbb{V}_{\min(k,n)} \) and \( V^{(n)}_{\lambda} \) the corresponding simple \( GL_n \)–module. By theorem 3.2, \( V^{(n)}_{\lambda} \) may be identified with the subspace of vectors of highest weight \( \lambda \) for the action of \( \mathfrak{gl}_k \) on \( \mathcal{S}(\mathcal{M}^*_{k,n}) \). Denote by \( \iota : V^{(n)}_{\lambda} \rightarrow \mathcal{S}(\mathcal{M}^*_{k,n}) \) the corresponding \( \mathfrak{gl}_n \)–equivariant embedding and let \( \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n \) be a weight of \( V^{(n)}_{\lambda} \).

**Lemma 3.4.** The embedding \( \iota \) maps the subspace \( V^{(n)}_{\lambda}[\mu] \subset V^{(n)}_{\lambda} \) of weight \( \mu \) onto the subspace \( M^\mu_{\lambda} \) of vectors of highest weight \( \lambda \) for the action of \( \mathfrak{gl}_k \) on
\[
S^\mu \mathbb{C}^k = S^{\mu_1} \mathbb{C}^k \otimes \cdots \otimes S^{\mu_n} \mathbb{C}^k \subset \mathbb{C}[x_{11}, \ldots, x_{k1}] \otimes \cdots \otimes \mathbb{C}[x_{1n}, \ldots, x_{kn}]
\] (3.13)
where \( S^{\mu_j} \mathbb{C}^k \) is the space of polynomials in \( x_{1j}, \ldots, x_{kj} \) which are homogeneous of degree \( \mu_j \). The corresponding isomorphism
\[
\bigoplus_{\nu \in \mathfrak{S}_n, \mu} V^{(n)}_{\lambda}[\nu] \cong \bigoplus_{\nu \in \mathfrak{S}_n, \mu} M^\nu_{\lambda}
\] (3.14)
is equivariant with respect to \( \mathfrak{S}_n \) which acts on \( \bigoplus_{\nu \in \mathfrak{S}_n, \mu} S^\nu \mathbb{C}^k \) by permuting the tensor factors and on \( V_{\lambda} \) by regarding \( \mathfrak{S}_n \) as the subgroup of permutation matrices of \( GL_n(\mathbb{C}) \).
PROOF. The equality $\iota(V^{(n)}_\lambda)[\mu] = M^\mu_\lambda$ holds because $S^\mu C^k$ is the subspace of $S(M^*_k, n)$ of weight $\mu$ for the $\mathfrak{gl}_n$–action since $E_i^{(m)} x_j = \delta_{ij} m x_{ij}^{(m)}$. The $\mathfrak{S}_n$–equivariance stems from the fact that the permutation of the tensor factors in $S^\bullet C^k \otimes \cdots \otimes S^\bullet C^k \cong S^\bullet (C^k \otimes C^n)$ is given by the action of $\mathfrak{S}_n \subset GL_n(C)$ action on $C^n$.

Let $D_n = \{(z_1, \ldots, z_n) \in C^n | z_i = z_j \text{ for some } 1 \leq i < j \leq n \}$ and $X_n = C^n \setminus D_n$. Regard $C_0^n = \{(z_1, \ldots, z_n) \in C^n | \sum_{j=1}^n z_j = 0 \}$ as the Cartan subalgebra of diagonal matrices in $\mathfrak{sl}_n$ and $X_0^n = C_0^n \setminus D_n$ as the set of its regular elements. Since the inclusion $X_0^n \subset X_n$ is a homotopy equivalence, $\pi_1(X_n) \cong \pi_1(X_0^n) = B_n$ are generated by $T_1 \ldots T_{n-1}$ with

$$T_i T_j = T_j T_i \quad \text{if } |i - j| \geq 2$$

(15)

$$T_{i+1} T_i = T_{i+1} T_{i+1} - 1 \quad \text{if } i = 1 \ldots n - 1$$

(16)

Define $\Omega_{ij}^{(k)} \in \text{End}_{\mathfrak{gl}_n}(S^\mu C^k)$ by $\text{(3.7)}$ where now $\{X_a, \{X^a\}$ are dual basis of $\mathfrak{sl}_n$ and extend the connection $\text{(2.8)}$ to $X_n$ in the obvious way. The following is the main result of this section.

**Theorem 3.5.** $f : X_n \to M^\mu_\lambda \subset S^\mu C^k \otimes \cdots \otimes S^\mu C^k$ is a horizontal section of the Knizhnik–Zamolodchikov connection

$$\nabla^\mu_{KZ} = d - \bar{h} \sum_{1 \leq i < j \leq n} \frac{dz_i - dz_j}{z_i - z_j} \Omega_{ij}^{(k)}$$

(17)

iff the $V^{(n)}_\lambda[\mu]$–valued function $g = f \cdot \prod_{1 \leq i < j \leq n} (z_i - z_j)^{h(\mu_i + \mu_j + 2 \mu_i \mu_j/k)}$ is a horizontal section of

$$\nabla^\mu_\kappa = d - h \sum_{1 \leq i < j \leq n} \frac{dz_i - dz_j}{z_i - z_j} \kappa_{ij}^{(n)}$$

(18)

where $\bar{h} = 2h$.

PROOF. Let $1^{(k)} = \sum_{i=1}^k E_i^{(k)}$ be the generator of the centre of $\mathfrak{gl}_n$ so that, in obvious notation, $\bar{\Omega}^{(k)}_{ij} = \Omega^{(k)}_{ij} + \frac{1}{k} \pi_1^{(k)}(\mathfrak{gl}_n) \pi_1(1^{(k)})$. The operators $2\Omega^{(k)}_{ij}$ and $\kappa_{ij}^{(n)}$ both act on $M^\mu_\lambda \cong V^{(n)}_\lambda[\mu]$ and, by proposition $\text{3.3}$, their restrictions differ by $-\mu_i - \mu_j$. The claim follows since, for any $1 \leq l \leq n$, $\pi_l(1^{(k)})$ acts on $S^\mu C^k$ as multiplication by $\mu_l$.

**Remark.** When $k = 2$ and $\lambda$ is of the form $(|\mu|/2, |\mu|/2, 0, \ldots, 0)$, where $|\mu| = \sum_{i=1}^n \mu_i$, theorem $\text{3.3}$ is a representation–theoretic analogue of the coincidence between the Kapovich–Millson bending flows on the space of $n$–gons in $\mathbb{R}^3$ with side lengths $\mu_1, \ldots, \mu_n$ [KM] and the Gel’fand–Cetlin flows on the Grassmannian $Gr_2(C^n)$ [GS] observed by Hausmann and Knutson in the context of Gel’fand–McPherson duality [HK]. I am grateful to J. Millson for a careful explanation of this coincidence.

**Remark.** An interesting relation between $\nabla_\kappa$ and the Knizhnik–Zamolodchikov connection was recently noted by Felder, Markov, Tarasov and Varchenko in [FMTV], where a variant of the connection $\text{(2.4)}$ is independently introduced and studied. One of the main results of [FMTV] is that, for any simple Lie algebra $\mathfrak{g}$, the connection $\nabla_\kappa$ with values in a tensor product $V_1 \otimes \cdots \otimes V_n$ of $n$ simple $\mathfrak{g}$–modules is, when supplemented by suitable dynamical parameters, bispectral to (i.e., commutes with) the Knizhnik–Zamolodchikov connection for $\mathfrak{g}$ with values in the same $n$–fold tensor product. An analogous result is obtained in [TV] for a difference analogue of the connection $\nabla_\kappa$. By comparison, theorem $\text{3.3}$ can only hold for $\mathfrak{g} = \mathfrak{sl}_n$, since it relies on the ‘coincidence’ of the regular Cartan of $\mathfrak{gl}_n$ with the configuration...
space of \( n \) ordered points in \( \mathbb{C} \), and asserts the \textit{equality} of the two connections.

To relate the monodromy representations of \( B_n \) corresponding to \( \nabla^h_{KZ} \) and \( \nabla^h_{\kappa} \), we need to specify how these induce flat connections on \( X_n/\mathfrak{S}_n \) and \( X_n^0/\mathfrak{S}_n \) respectively. For \( \nabla^h_{KZ} \), we let \( \mathfrak{S}_n \) act on the fibre

\[
\bigoplus_{\nu \in \mathfrak{S}_n, \mu} M^\nu_{\mu} \subset \bigoplus_{\nu \in \mathfrak{S}_n, \mu} S^\nu \mathbb{C}^k
\]

by permuting the tensor factors and take the quotient connection. For \( \nabla^h_{\kappa} \), we use the construction of proposition \ref{prop:construction} and the homomorphism \( \sigma : B_n \to SL_n(\mathbb{C}) \) given by

\[
T_j \mapsto \exp(E_{j,j+1}^{(n)}) \exp(-E_{j+1,j}^{(n)}) \exp(E_{j,j+1}^{(n)}) = \begin{pmatrix}
1 & & & \\
& \ddots & & \\
& & 1 & \\
& & & -1 \end{pmatrix}
\]

where the off–diagonal terms are the \((j, j + 1)\) and \((j + 1, j)\) entries. A direct computation, or \cite[thm. 3.3]{T}, show that the assignment \eqref{eq:assignment} does indeed extend to a homomorphism \( B_n \to SL_n(\mathbb{C}) \). Choose the generators \( T_1 \ldots T_{n-1} \) of \( B_n \) as at the end of section \ref{sec:corollary}.

\begin{corollary}
Let \( \mu \) be a weight of \( V_\lambda \) and 
\[
\pi^h_{\kappa} : B_n \to GL( \bigoplus_{\nu \in \mathfrak{S}_n, \mu} V_\lambda[v] ), \quad \pi^h_{KZ} : B_n \to GL( \bigoplus_{\nu \in \mathfrak{S}_n, \mu} M^\nu_{\mu} )
\]

the monodromy representations of the braid group \( B_n \) corresponding to the connections \eqref{eq:KZ} and \eqref{eq:kappa} respectively. Then, for any \( j = 1 \ldots n - 1 \),

\[
\pi^{2h}_{KZ}(T_j) = \pi^h_{KZ}(T_j) \cdot e^{-\pi ih \left( E_{jj}^{(n)} + E_{j+1,j+1}^{(n)} + 2E_{jj}^{(n)} E_{j+1,j+1}^{(n)} / k \right)} \cdot e^{i\pi E_{jj}^{(n)}}
\]

\end{corollary}

\textbf{Proof.} Let \( s_j = SL_n(\mathbb{C}) \) be the right–hand side of \eqref{eq:assignment} so that \( s_j = (j, j + 1) \cdot e^{i\pi E_{jj}^{(n)}} \) in \( GL_n(\mathbb{C}) \). Let \( \mathcal{P}^h_{KZ}, \mathcal{P}^h_{\kappa} \) denote parallel transport for \( \nabla^h_{KZ} \) and \( \nabla^h_{\kappa} \) respectively. Then, by theorem \ref{thm:corollary} and proposition \ref{prop:connection}, the following holds on \( M^\nu_{\mu} \cong V_\nu[h] \),

\[
\pi^{2h}_{KZ}(T_j) = (j,j + 1) \mathcal{P}^{2h}_{KZ}(T_j)
\]

\[
= (j,j + 1) e^{-\pi ih \left( \nu_j + \nu_{j+1} + 2\nu_j/\nu_{j+1} / k \right)} e^{i\pi E_{jj}^{(n)} E_{j+1,j+1}^{(n)} / k} \mathcal{P}^h_{\kappa}(T_j)
\]

\[
= s_j \mathcal{P}^h_{\kappa}(T_j) e^{i\pi E_{jj}^{(n)} E_{j+1,j+1}^{(n)} / k} e^{-\pi ih \left( E_{jj}^{(n)} + E_{j+1,j+1}^{(n)} \right)} e^{i\pi E_{jj}^{(n)}}
\]

as claimed. \hspace{1cm} \Box

4. The Quantum Group \( U_{gl_\mathfrak{p}} \)

In this, and the following sections, we work over the ring \( \mathbb{C}[h] \) of formal power series in the variable \( h \). All tensor products of \( \mathbb{C}[h] \)-modules are understood to be completed in the \( h \)-adic topology. For \( p \in \mathbb{N} \), let \( a_{ij} = 2\delta_{ij} - \delta_{|i-j| = 1}, 1 \leq i, j \leq p \), be the entries of the Cartan matrix
of type $A_{p-1}$ and let $U_h\mathfrak{g}l_p$ be the corresponding Drinfeld–Jimbo quantum group \cite{Dr1, Dr2} i.e., the algebra over $\mathbb{C}[\hbar]$ topologically generated by elements $E_i, F_i, i = 1 \ldots p - 1$ and $D_i, i = 1 \ldots p$ subject to the $q$–Serre relations

$$[D_i, D_j] = 0$$

(4.1)

$$[D_i, E_j] = (\delta_{ij} - \delta_{ij+1})E_j \quad [D_i, F_j] = -(\delta_{ij} - \delta_{ij+1})F_j$$

(4.2)

$$[E_i, F_j] = \delta_{ij} \frac{e^{\hbar H_i} - e^{-\hbar H_i}}{e^{h} - e^{-h}}$$

(4.3)

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1 - a_{ij}}{k} \right] E_i^k E_j E_i^{1-a_{ij}-k} = 0, \quad \forall i \neq j$$

(4.4)

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1 - a_{ij}}{k} \right] F_i^k F_j F_i^{1-a_{ij}-k} = 0, \quad \forall i \neq j$$

(4.5)

where $H_i = D_i - D_{i+1}$ and, for any $n \geq k \in \mathbb{N},$

$$[n] = \frac{e^{nh} - e^{-nh}}{e^{h} - e^{-h}}$$

(4.6)

$$[n]! = [n][n-1] \ldots [1]$$

(4.7)

$$\left[ \frac{n}{k} \right] = \frac{[n]!}{[k]![n-k]!}$$

(4.8)

$U_h\mathfrak{g}l_p$ is a topological Hopf algebra with coproduct $\Delta$ and counit $\varepsilon$ given by

$$\Delta(D_i) = D_i \otimes 1 + 1 \otimes D_i$$

(4.9)

$$\Delta(E_i) = E_i \otimes e^{\hbar H_i} + 1 \otimes E_i$$

(4.10)

$$\Delta(F_i) = F_i \otimes 1 + e^{-\hbar H_i} \otimes F_i$$

(4.11)

and

$$\varepsilon(E_i) = \varepsilon(F_i) = \varepsilon(D_i) = 0$$

(4.12)

Note that $I = D_1 + \cdots + D_p$ is central so that $U_h\mathfrak{g}l_p \cong U_h\mathfrak{s}l_p \otimes \mathbb{C}[I][\hbar]$ as Hopf algebras where the coproduct on $\mathbb{C}[I][\hbar]$ is given by $\Delta(I) = I \otimes 1 + 1 \otimes I$ and $U_h\mathfrak{s}l_p \subset U_h\mathfrak{g}l_p$ is the closed Hopf subalgebra generated by $E_i, F_i$ and $H_i, i = 1, \ldots, p - 1.$

By a finite–dimensional representation of $U_h\mathfrak{g}l_p$ we shall mean a $U_h\mathfrak{g}l_p$–module which is topologically free and finitely generated over $\mathbb{C}[\hbar]$ and on which $I$ acts semisimply with eigenvalues in $\mathbb{C}$. Choose an algebra isomorphism $\phi : U_h\mathfrak{g}l_p \to U\mathfrak{g}l_p[\hbar]$ mapping each $D_i$ onto $E_{ii}$ \cite{Dr2, prop. 4.3} and let $V$ be a finite–dimensional $\mathfrak{g}l_p$–module on which $1^{(p)} = \sum_{i=1}^{p} E_{ii}$ acts semisimply. Then, $U\mathfrak{g}l_p[\hbar]$ acts on $V[\hbar]$ and the latter becomes, via $\phi$, a finite–dimensional representation of $U_h\mathfrak{g}l_p$. Conversely,

**Proposition 4.1.** Let $V$ be a finite–dimensional representation of $U_h\mathfrak{g}l_p$ and $V = V/hV$ the corresponding $\mathfrak{g}l_p$–module. Then, as $U_h\mathfrak{g}l_p$–modules,

$$V \cong V[\hbar]$$

(4.13)

**Proof.** Since $I$ is diagonalisable on $V$ and commutes with $U_h\mathfrak{g}l_p$, we may assume that it acts on $V$ as multiplication by a scalar $\lambda \in \mathbb{C}$. Since $V$ is topologically free, $V \cong V[\hbar]$ as $\mathbb{C}[\hbar]$–modules so that $V$ is a deformation of the finite–dimensional $\mathfrak{s}l_p$–module $V$. Since $\mathfrak{s}l_p$ is
simple, \( H^1(\mathfrak{sl}_p, V) = 0 \) and \( V \) is isomorphic, as \( \mathfrak{sl}_p \), and therefore as \( \mathfrak{gl}_p \)-module to the trivial deformation of \( V \). Thus, \( V \cong V[h] \) as \( U \mathfrak{gl}_p[h] \), and therefore as \( U_h \mathfrak{gl}_p \)-modules.

**Corollary 4.2.** Let \( U, V \) be finite-dimensional \( \mathfrak{gl}_p \)-modules on which \( 1^{(p)} \) acts semisimply. If \( U \otimes V \) decomposes as

\[
U \otimes V \cong \bigoplus W \cdot N_W
\]

for some \( \mathfrak{gl}_p \)-modules \( W \) and multiplicities \( N_W \in \mathbb{N} \), then, as \( U_h \mathfrak{gl}_p \)-modules,

\[
U[h] \otimes V[h] \cong \bigoplus W \cdot N_W[h]
\]

**Proof.** By (4.14), both sides of (4.15) have the same specialisation at \( h = 0 \) and are therefore isomorphic by proposition 4.1.

5. **The dual pair** \((U_h \mathfrak{gl}_k, U_h \mathfrak{gl}_n)\)

We shall need the analogue of theorem 3.2 in the setting of the algebra \( S_h(\mathcal{M}^*_k, n) \) of functions on quantum \( k \times n \) matrix space. With the exception of theorems 5.4 and 5.5, this section follows [Ba, §1.5] (see also [Ga]). By definition, \( S_h(\mathcal{M}^*_k, n) \) is the algebra over \( \mathbb{C}[h] \) topologically generated by elements \( X_{ij}, 1 \leq i \leq k, 1 \leq j \leq n \) with relations

\[
X_{ij}X_{kl} = \begin{cases} 
X_{kl}X_{ij} & \text{if } k > i \text{ and } l < j \text{ or } k < i \text{ and } l > j \\
e^{-h}X_{kl}X_{ij} & \text{if } k > i \text{ and } l = j \text{ or } k = i \text{ and } l > j \\
X_{kl}X_{ij} - (e^h - e^{-h})X_{kj}X_{il} & \text{if } k > i \text{ and } l > j
\end{cases}
\]

\( S_h(\mathcal{M}^*_k, n) \) is \( \mathbb{N} \)-graded by decreeing that each \( X_{ij} \) is of degree 1 and we denote its homogeneous components by \( S^d_h(\mathcal{M}^*_k, n), d \in \mathbb{N} \). For any \( k \times n \) matrix \( m \) with entries \( m_{ij} \in \mathbb{N} \), set

\[
X^m = X_{11}^{m_{11}} \cdots X_{kn}^{m_{kn}} = X_{11}^{m_{11}} \cdots X_{1n}^{m_{1n}} \cdots X_{kn}^{m_{kn}}
\]

By the commutation relations (5.1), the \( X^m \), with \( m \in \mathcal{M}_k, n \) such that \( |m| = d \), span \( S^d_h(\mathcal{M}^*_k, n) \), where \( |m| = \sum_{i,j} m_{ij} \).

**Theorem 5.1** (Parshall–Wang). The monomials \( X^m, m \in \mathcal{M}_k, n, \) are linearly independent over \( \mathbb{C}[h] \). In particular, the set \( \{X^m\}_{|m| = d} \) is a \( \mathbb{C}[h] \)-basis of \( S^d_h(\mathcal{M}^*_k, n) \).

**Proof.** This is proved in [PW, thm. 3.5.1] for \( k = n \) and over the field \( \mathbb{C}(q) \) of rational functions of \( q = e^h \) rather than over \( \mathbb{C}[h] \). The proof however works equally well for \( k \neq n \) and, as remarked in [PW], over \( \mathbb{C}[h] \). \( \square \)

As in the classical case, \( S_h(\mathcal{M}^*_k, n) \) is a module algebra over \( U_h \mathfrak{gl}_k \otimes U_h \mathfrak{gl}_n \). This may be seen in the following way. For any \( l \in \mathbb{N} \), one readily checks that the assignement

\[
X_{ij} \rightarrow \sum_{l'=1}^l X_{il'} \otimes X_{l'j}
\]

(5.3)
Lemma 5.2. \( \Delta_{kln} : S_h(M_{k,n}^*) \to S_h(M_{k,l}^*) \otimes S_h(M_{l,n}^*) \) such that, for any \( l, m \in \mathbb{N} \), the following diagram commutes

\[
\begin{array}{ccc}
S_h(M_{k,n}^*) & \xrightarrow{\Delta_{kln}} & S_h(M_{k,l}^*) \otimes S_h(M_{l,n}^*) \\
\Delta_{knn} & & 1 \otimes \Delta_{lmn} \\
S_h(M_{k,m}^*) \otimes S_h(M_{m,n}^*) & \xrightarrow{\Delta_{klm} \otimes 1} & S_h(M_{k,l}^*) \otimes S_h(M_{l,m}^*) \otimes S_h(M_{m,n}^*)
\end{array}
\]

In particular, \( S_h(M_{k,k}^*) \) and \( S_h(M_{n,n}^*) \) are topological bialgebras with comultiplications \( \Delta_{kkk} \) and \( \Delta_{nnn} \) respectively and counit \( \varepsilon(X_{ij}) = \delta_{ij} \). Moreover, the maps \( \Delta_{kkn} \) and \( \Delta_{knm} \) give \( S_h(M_{k,n}^*) \) the structure of a \( S_h(M_{k,k}^*)_S(M_{n,n}^*) \) bicomodule algebra each homogeneous component of which is invariant under \( S_h(M_{k,k}^*) \) and \( S_h(M_{n,n}^*) \) since \( \Delta_{kln}(S_h(M_{k,k}^*)) \subset S_h(M_{k,k}^*) \otimes S_h(M_{l,l}^*) \).

We shall need a columnwise (resp. rowwise) description of the coaction of \( S_h(M_{k,k}^*) \) (resp. \( S_h(M_{n,n}^*) \)) on \( S_h(M_{k,n}^*) \). Consider the quantum \( k \) and \( n \)–dimensional planes i.e., the algebras \( S_h(M_{k,1}^*) \) and \( S_h(M_{1,n}^*) \). By the commutation relations (5.1) and theorem 5.1, these may be embedded as subalgebras of \( S_h(M_{k,n}^*) \) via the maps

\[
c_j : S_h(M_{k,n}^*) \to S_h(M_{k,n}^*), \quad c_j(X_{ij}) = X_{ij} \tag{5.5}
\]

\[
r_i : S_h(M_{1,n}^*) \to S_h(M_{k,n}^*), \quad r_i(X_{ij}) = X_{ij} \tag{5.6}
\]

with \( 1 \leq i \leq k, 1 \leq j \leq n \). By (5.4), \( S_h(M_{k,1}^*) \) is a left algebra comodule over \( S_h(M_{k,k}^*) \) and \( S_h(M_{1,n}^*) \) a right algebra comodule over \( S_h(M_{n,n}^*) \).

**Lemma 5.2.** As left, \( \mathbb{N} \)–graded \( S_h(M_{k,n}^*) \)–comodules,

\[
S_h(M_{k,n}^*) \cong S_h(M_{k,1}^*)^{\otimes n} \tag{5.7}
\]

via the map \( \Phi : p_1 \otimes \cdots \otimes p_n \to c_1(p_1) \cdots c_n(p_n) \). Similarly, as right, \( \mathbb{N} \)–graded \( S_h(M_{n,n}^*) \)–comodules,

\[
S_h(M_{k,n}^*) \cong S_h(M_{1,n}^*)^{\otimes k} \tag{5.8}
\]

via \( \Psi : q_1 \otimes \cdots \otimes q_k \to r_1(q_1) \cdots r_k(q_k) \).

**Proof.** The map \( \Phi \) clearly preserves the grading and, by theorem 5.1, restricts to a \( \mathbb{C}[h] \)–linear isomorphism of homogeneous components since it bijectively maps the monomial basis of \( S_h(M_{k,1}^*)^{\otimes n} \) onto the basis \( X^m \) of \( S_h(M_{k,n}^*) \). The fact that \( \Phi \) itself is an isomorphism follows easily because any element of \( S_h(M_{1,n}^*)^{\otimes k} \) or \( S_h(M_{k,n}^*) \) is the convergent sum of its homogeneous components. As readily checked, the diagram

\[
\begin{array}{ccc}
S_h(M_{k,k}^*) \otimes S_h(M_{k,1}^*) & \xrightarrow{1 \otimes c_j} & S_h(M_{k,k}^*) \otimes S_h(M_{k,n}^*) \\
\Delta_{kk1} & & \Delta_{kkn} \\
S_h(M_{k,1}^*) & \xrightarrow{c_j} & S_h(M_{k,n}^*)
\end{array}
\]

extends uniquely to an algebra homomorphism \( \Delta_{kln} : S_h(M_{k,n}^*) \to S_h(M_{k,l}^*) \otimes S_h(M_{l,n}^*) \) such that, for any \( l, m \in \mathbb{N} \), the following diagram commutes

\[
\begin{array}{ccc}
S_h(M_{k,n}^*) & \xrightarrow{\Delta_{kln}} & S_h(M_{k,l}^*) \otimes S_h(M_{l,n}^*) \\
\Delta_{knn} & & 1 \otimes \Delta_{lmn} \\
S_h(M_{k,m}^*) \otimes S_h(M_{m,n}^*) & \xrightarrow{\Delta_{klm} \otimes 1} & S_h(M_{k,l}^*) \otimes S_h(M_{l,m}^*) \otimes S_h(M_{m,n}^*)
\end{array}
\]
is commutative for any \(1 \leq j \leq n\) and therefore so is

\[
(S_h(M_{k,k}^*) \otimes S_h(M_{k,1}^*))^n \xrightarrow{\mu^{(n)} \otimes \Phi} S_h(M_{k,k}^*) \otimes S_h(M_{k,n}^*)
\]

\[
\Delta_{kkl} \xrightarrow{\otimes} \Delta_{knn}
\]

where \(\mu^{(n)} : S_h(M_{k,k}^*)^n \rightarrow S_h(M_{k,n}^*)\) is the \(n\)-fold multiplication. This proves \((5.8)\). The proof of \((5.8)\) is identical. \(\blacksquare\)

We turn now to the action of \(U_{h\mathfrak{gl}_k} \otimes U_{h\mathfrak{gl}_n}\) on \(S_h(M_{k,n}^*)\). For \(p = k, n\), consider the vector representation of \(U_{h\mathfrak{gl}_p}\), i.e., the module \(V = \mathbb{C}^p[h]\) with basis \(e_1, \ldots, e_p\) and action given by

\[
D_i = E_{ii}, \quad E_i = E_{i+1}, \quad F_i = E_{i+1}
\]

(5.11)

where \(E_{ab}e_c = \delta_{bc}e_a\). Let \(e^1, \ldots, e^p \in V^*\) be the dual basis of \(e_1, \ldots, e_p\), \(U_{h\mathfrak{gl}_p}^\ast\), the restricted dual of \(U_{h\mathfrak{gl}_p}\) and \(t_{ij} \in U_{h\mathfrak{gl}_p}^\ast\) the matrix coefficient defined by

\[
t_{ij}(x) = \langle e^i, xe_j \rangle
\]

(5.12)

**Proposition 5.3.** The assignment \(X_{ij} \rightarrow t_{ij}\) extends uniquely to a bialgebra morphism \(\kappa_p : S_h(M_{k,p}^*) \rightarrow U_{h\mathfrak{gl}_p}^\ast\).

**Proof.** We need to check that the \(t_{ij}\) satisfy the relations \((5.1)\), i.e., that when evaluated on \(\Delta(x), x \in U_{h\mathfrak{gl}_p}\),

\[
t_{ij} \otimes t_{kl} = \begin{cases} 
  t_{kl} \otimes t_{ij} & \text{if } k > i \text{ and } l < j \text{ or } k < i \text{ and } l > j \\
  e^{-h}t_{kl} \otimes t_{ij} & \text{if } k > i \text{ and } l = j \text{ or } k = i \text{ and } l > j \\
  t_{kl} \otimes t_{ij} - (e^h - e^{-h})t_{kj} \otimes t_{il} & \text{if } k > i \text{ and } l > j
  
\end{cases}
\]

(5.13)

Let \(R' = R \in \text{End}(V \otimes V)\) where \(\sigma \in \text{GL}(V \otimes V)\) is the flip and \(R\) is the universal \(R\)-matrix of \(U_{h\mathfrak{gl}_p}\) acting on \(V \otimes V\). Then \([12], [CP], \S 8.3.G]\)

\[
R = \left( e^h \sum_{i=1}^{p} E_{ii} \otimes E_{ii} + \sum_{1 \leq i \neq j \leq p} E_{ii} \otimes E_{jj} + (e^h - e^{-h}) \sum_{1 \leq i < j \leq p} E_{ij} \otimes E_{ji} \right)
\]

(5.14)

so that the matrix entries of \(R'\) are

\[
R'_{ik,jl} = \begin{cases} 
  e^{\delta_{ij} h} & \text{if } i = l \text{ and } k = j \\
  e^h - e^{-h} & \text{if } i = j, k = l \text{ and } j > l \\
  0 & \text{otherwise}
\end{cases}
\]

(5.15)

From \((5.15)\), one readily checks that both sides of \((5.13)\) coincide when evaluated on any \(A \in \text{End}(V \otimes V)\) commuting with \(R'\). Since \(R'\) is a \(U_{h\mathfrak{gl}_p}^\ast\)-intertwiner, \(\kappa\) extends to an algebra morphism which respects the counit and coproduct since \(\Delta(t_{ij}) = \sum_{q=1}^{p} t_{iq} \otimes t_{qj}\). \(\blacksquare\)

**Theorem 5.4.**

1. The maps \(\kappa_p, p = k, n\) of proposition \(5.3\) give \(S_h(M_{k,n}^*)\) the structure of an algebra module over \(U_{h\mathfrak{gl}_k} \otimes U_{h\mathfrak{gl}_n}\) with invariant homogeneous components \(S_h^d(M_{k,n}^*)\), \(d \in \mathbb{N}\).
2. The maps $\Phi, \Psi$ of lemma $5.3$ yield isomorphisms

$$S_h(\mathcal{M}^*_k, n) \cong S_h(\mathcal{M}^*_k, 1)^{\otimes n} \quad \text{and} \quad S_h(\mathcal{M}^*_n, k) \cong S_h(\mathcal{M}^*_1, 1)^{\otimes k}$$

(5.16)

as $\mathbb{N}$–graded $U_h\mathfrak{gl}_k$ and $U_h\mathfrak{gl}_n$–modules respectively.

3. The action of the generators $E_q^{(p)}, F_q^{(p)}, p = 1 \ldots p - 1$ and $D_q^{(p)}, q = 1 \ldots p$ of $U_h\mathfrak{gl}_p$, $p = k, n$ in the monomial basis $X^m, m \in \mathcal{M}_{k,n}$, is given by

$$D_i^{(k)} X^m = \sum_{j=1}^n m_{ij} X^m$$

(5.17)

$$E_i^{(k)} X^m = \sum_{j=1}^n m_{ij+1} \prod_{j'=j+1}^n e^{\hbar (m_{ij'} - m_{i+1,j'})} X^{m+\epsilon_{ij} - \epsilon_{i+1,j}}$$

(5.18)

$$F_i^{(k)} X^m = \sum_{j=1}^n m_{ij} \prod_{j'=j+1}^{j-1} e^{-\hbar (m_{ij'} - m_{i,j+1})} X^{m-\epsilon_{ij} + \epsilon_{i,j+1}}$$

(5.19)

where $(\epsilon_{ab})_{cd} = \delta_{ac} \delta_{bd}$, and

$$D_j^{(n)} X^m = \sum_{i=1}^k m_{ij} X^m$$

(5.20)

$$E_j^{(n)} X^m = \sum_{i=1}^k m_{ij+1} \prod_{i'=i+1}^k e^{\hbar (m_{ij'} - m_{i+1,j})} X^{m+\epsilon_{ij} - \epsilon_{i+1,j}}$$

(5.21)

$$F_j^{(n)} X^m = \sum_{i=1}^k m_{ij} \prod_{i'=i+1}^{i-1} e^{-\hbar (m_{ij'} - m_{i,j+1})} X^{m-\epsilon_{ij} + \epsilon_{i,j+1}}$$

(5.22)

Proof. Using the transposition anti–involution $\tau$ on $S_h(\mathcal{M}^*_k, n)$ given by $\tau(X_{ij}) = X_{ji}$, we may regard $S_h(\mathcal{M}^*_k, n)$ as a right algebra module over $S_h(\mathcal{M}^*_k, 1) \otimes S_h(\mathcal{M}^*_n, 1)$ and therefore, via the pairings $\langle \cdot, \cdot \rangle : S_h(\mathcal{M}^*_m, n) \otimes U_h\mathfrak{gl}_m \to \mathbb{C}[[\hbar]]$, $m = k, n$, given by proposition $5.3$ as a left algebra module over $U_h\mathfrak{gl}_k \otimes U_h\mathfrak{gl}_n$. This proves (i) and (ii). Explicitly, for $x^{(m)} \in U_h\mathfrak{gl}_m$, $m = k, n$ and $p \in S_h(\mathcal{M}^*_k, n)$

$$x^{(k)} p = (x^{(k)} \otimes 1, \tau \otimes 1 \cdot \Delta_{kkn}(p))$$

(5.23)

$$x^{(n)} p = (1 \otimes x^{(n)}, \Delta_{knn}(p))$$

(5.24)

Using (5.23) and (5.11), one gets

$$D_i^{(k)} X_{i'}^j = \delta_{ii'} X_{ij}$$

(5.25)

$$E_i^{(k)} X_{i'}^j = \delta_{i+1,i'} X_{ij}$$

(5.26)

$$F_i^{(k)} X_{i'}^j = \delta_{i,i'} X_{i+1,j}$$

(5.27)

Using the algebra module property $x^{(pq)} = \mu(\Delta(x)p \otimes q)$ where $x \in U_h\mathfrak{gl}_k$, $p, q \in S_h(\mathcal{M}^*_k, n)$ and $\mu : S_h(\mathcal{M}^*_k, n)^{\otimes 2} \to S_h(\mathcal{M}^*_k, n)$ is multiplication, (4.9)–(4.11) and the commutation relations (5.1) shows by induction on $m \in \mathbb{N}$ that

$$D_i^{(k)} X_{i'}^j = \delta_{ii'} m X_{ij}^m$$

(5.28)

$$E_i^{(k)} X_{i'}^j = \delta_{i+1,i'} [m] X_{ij} X_{i+1,j}^{m-1}$$

(5.29)

$$F_i^{(k)} X_{i'}^j = \delta_{i,i'} [m] X_{ij}^{m-1} X_{i+1,j}$$

(5.30)
Let $\Delta^{(a)} : U_h \mathfrak{gl}_k \rightarrow U_h \mathfrak{gl}_k^{\otimes d}$, $a \in \mathbb{N}^*$ be recursively defined by $\Delta^{(1)} = \text{id}$, $\Delta^{(a+1)} = \Delta \otimes \text{id}^{(a)} \cdot \Delta^{(a)}$. Then, by (4.3)–(4.7)

$$\Delta^{(a)} D_i^{(k)} = \sum_{b=1}^{a} 1^{(b)} \otimes D_i^{(k)} \otimes 1^{(a-b)}$$

$$\Delta^{(a)} E_i^{(k)} = \sum_{b=1}^{a} 1^{(b)} \otimes E_i^{(k)} \otimes (e^{hH_i^{(k)}})^{0(b-a)}$$

$$\Delta^{(a)} F_i^{(k)} = \sum_{b=1}^{a} (e^{-hH_i^{(k)}})^{0(b-1)} \otimes F_i^{(k)} \otimes 1^{0(a-b)}$$

The formulae (5.17)–(5.19) now follow from the algebra module property and (5.31)–(5.33). The proof of (5.20)–(5.22) is similar.

The following result is proved in [Ba] and [Ga] for the quantum groups $U_q \mathfrak{gl}_k, U_q \mathfrak{gl}_n$ by a different method.

**Theorem 5.5.** For any $d \in \mathbb{N}$, the $U_h \mathfrak{gl}_k \otimes U_h \mathfrak{gl}_n$–module $S^d_h(\mathcal{M}_{k,n}^*)$ decomposes as

$$S^d_h(\mathcal{M}_{k,n}^*) \cong \bigoplus_{\lambda \in \mathcal{Y}_{\min}(k,n), |\lambda|=d} V_{\lambda}^{(k)} [h] \otimes V_{\lambda}^{(n)} [h]$$

**Proof.** By theorem 5.1, $S^d_h(\mathcal{M}_{k,n}^*)$ has no torsion, and is therefore a topologically free $\mathbb{C}[h]$–module. Moreover, by (5.17)–(5.22), $S^d_h(\mathcal{M}_{k,n}^*)/hS^d_h(\mathcal{M}_{k,n}^*)$ is the $\mathfrak{gl}_k \otimes \mathfrak{gl}_n$–module $S^d(\mathcal{M}_{k,n}^*)$. The conclusion follows from theorem 3.2 and proposition 4.1.

### 6. Braid group actions on quantum matrix space

We compare in this section two actions of the braid group $B_n$ on the algebra $S_h(\mathcal{M}_{k,n}^*)$ of functions of quantum $k \times n$ matrix space. The first is the $\tilde{R}$–representation obtained by regarding $S_h(\mathcal{M}_{k,n}^*)$ as the $U_h \mathfrak{gl}_k$–module $S_h(\mathcal{M}_{k,n}^*) \otimes \mathfrak{gl}_n$. The second is the quantum Weyl group action of $B_n$ on $S_h(\mathcal{M}_{k,n}^*)$ viewed as a $U_h \mathfrak{gl}_n$–module. We will show that these representations essentially coincide, thus extending to the $q$–setting the fact that the symmetric group $S_n$ acts on $(\mathcal{C}^{*C^k})^\otimes n \cong S(\mathcal{M}_{k,n}^*)$ via the permutation matrices in $GL_n(\mathbb{C})$.

More precisely, for any $1 \leq j \leq n$, let $R_j^\vee$ be the universal $R$–matrix of $U_h \mathfrak{gl}_k$ acting on the $j$ and $j+1$ tensor copies of $S_h(\mathcal{M}_{k,n}^*) \cong S_h(\mathcal{M}_{k,1}^*)^\otimes n$ and $S_j$ the quantum Weyl group element of $U_h \mathfrak{gl}_n$ corresponding to the simple root $\alpha_j = \theta_j - \theta_{j+1}$. We will show that

$$R_j^\vee = S_j \cdot e^{-h(D_j^{(n)}) + D_j^{(n)} D_{j+1}^{(n)/k}} \cdot e^{i\pi D_j^{(n)}}$$

where $D_1^{(n)}, \ldots, D_n^{(n)}$ are the generators of the Cartan subalgebra of $U_h \mathfrak{gl}_n$. The proof of (6.1) is based upon the following observation, which we owe to B. Feigin. Both sides of (6.1) only act upon the $j$ and $j+1$ tensor copies of $S_h(\mathcal{M}_{k,1}^*)^\otimes n$ so that its proof reduces to a computation in $S_h(\mathcal{M}_{k,1}^*)^\otimes 2 \cong S_h(\mathcal{M}_{k,2}^*)$. Since both sides intertwine the action of $U_h \mathfrak{gl}_k$ on $S_h(\mathcal{M}_{k,2}^*)$, it suffices to compare them on highest weight vectors. These, and the action of $R_j^\vee$ are computed in [6.1]. The action of $S_j$ is computed in [6.2].
Remark. It is easy to check that neither action of $B_n$ is compatible with the algebra structure of $S_h(M_{k,n})$, so that (5.1) cannot be proved by merely checking it on the generators $X_{ij}$. This stems from the fact that quantum Weyl group operators are not group–like.

6.1. $R$–matrix action on singular vectors. For any $d \in \mathbb{N}$, let $S_h^d \mathbb{C}^k$ be the homogeneous component of degree $d$ of $S_h(M_{k,k})$. By (5.17)–(5.19), $S_h^d \mathbb{C}^k$ is a deformation of the $d$–th symmetric power $S^d \mathbb{C}^k$ of the vector representation of $\mathfrak{gl}_k$. Let $\mu_1, \mu_2 \in \mathbb{N}$, then

Lemma 6.1. As $U_h\mathfrak{g}l_k$–modules,

$$S_{\mu_1} \mathbb{C}^k \otimes S_{\mu_2} \mathbb{C}^k \cong \bigoplus_{i=0}^{\min(\mu_1, \mu_2)} V^{(i)}_{\mu_1+\mu_2-i} [h] \quad (6.2)$$

where $V^{(i)}_{\mu_1+\mu_2-i}$ is the irreducible representation of $\mathfrak{gl}_k$ with highest weight $(a, b, 0, \ldots, 0)$. The corresponding highest weight vectors $v_{\mu_1, \mu_2}$ are given by

$$v_{\mu_1, \mu_2}^i = \sum_{a=0}^{\min(\mu_1, \mu_2)} (-1)^a \left[ \begin{array}{c} i \\ a \end{array} \right] e^{ha(\mu_2-a+1)} X_{11}^{\mu_1-i+a} X_{21}^{i-a} X_{12}^{\mu_2-a} X_{22}^{a} \quad (6.3)$$

Proof. The decomposition (6.2) follows from the Pieri rules for $\mathfrak{gl}_k$ and corollary 4.2. Fix $i \in \{0, \ldots, \min(\mu_1, \mu_2)\}$. By (5.17), any $v \in S_{\mu_1} \mathbb{C}^k \otimes S_{\mu_2} \mathbb{C}^k$ of weight $(\mu_1 + \mu_2 - i, i, 0, \ldots, 0)$ is of the form

$$v = \sum_{a=0}^{i} c_a X_{11}^{\mu_1-i+a} X_{21}^{i-a} X_{12}^{\mu_2-a} X_{22}^{a} \quad (6.4)$$

for some constants $c_a \in \mathbb{C}$. By (1.1) and (5.18), $\Delta(E_j)v = 0$ for any $j \geq 2$ so that $v$ is a highest weight vector iff

$$\Delta(E_1)v = \sum_{a=0}^{i} c_a [i-a] e^{h(\mu_2-a)} X_{11}^{\mu_1-i+a} X_{21}^{i-a} X_{12}^{\mu_2-a} X_{22}^{a}$$

$$+ \sum_{a=0}^{i} c_a [a] X_{11}^{\mu_1-i+a} X_{21}^{i-a} X_{12}^{\mu_2-a+1} X_{22}^{a-1} \quad (6.5)$$

is equal to zero. This yields $c_a = -c_{a-1} e^{h(\mu_2-a+2)} [i-a+1]/[a]$ and therefore

$$c_a = (-1)^a \left[ \begin{array}{c} i \\ a \end{array} \right] e^{ha(\mu_2-a+1)} c_0 \quad (6.6)$$

whence (6.3). \hfill \Box

Let $R$ be the universal $R$–matrix of $U_h\mathfrak{g}l_k$ and $R^\vee = \sigma \cdot R : S_{\mu_1} \mathbb{C}^k \otimes S_{\mu_2} \mathbb{C}^k \to S_{\mu_2} \mathbb{C}^k \otimes S_{\mu_1} \mathbb{C}^k$ the corresponding $U_h\mathfrak{g}l_k$–intertwiner, where $\sigma$ is the permutation of the tensor factors.

Proposition 6.2. The following holds on $S_{\mu_1} \mathbb{C}^k \otimes S_{\mu_2} \mathbb{C}^k \bigoplus S_{\mu_2} \mathbb{C}^k \otimes S_{\mu_1} \mathbb{C}^k$,

$$R^\vee v_{\mu_1, \mu_2}^i = (-1)^i e^{h((\mu_1-i)(\mu_2-i)-i-\mu_1\mu_2/k)} v_{\mu_2, \mu_1}^i \quad (6.7)$$

Proof. For any $1 \leq i \leq k - 1$, let $s_i = (i \ i + 1) \in \mathfrak{S}_k$ be the $i$th elementary transposition and let

$$w_0 = (1 \ k)(2 \ k - 1) \cdots (\left\lfloor \frac{k}{2} \right\rfloor \left\lceil \frac{k}{2} \right\rceil) \quad (6.8)$$
be the longest element of $\mathfrak{S}_k$. Consider the following reduced expression for $w_0$

$$w_0 = s_{k-1} \cdots s_1 s_{k-1} \cdots s_2 \cdots s_{k-1} s_{k-2} s_{k-1} s_{i_{k(k-1)/2}}$$

(6.9)

and let $\beta_j = s_{i_1} \cdots s_{i_{j-1}} (\theta_{i_j} - \theta_{i_{j+1}})$ be the associated enumeration of the positive roots of $\mathfrak{sl}_k$ so that

$$\beta_1 = \theta_{k-1} - \theta_k, \quad \beta_2 = \theta_{k-2} - \theta_k, \quad \cdots \quad \beta_{k-1} = \theta_1 - \theta_k,$$

$$\beta_k = \theta_{k-2} - \theta_{k-1}, \quad \cdots \quad \beta_{k-3} = \theta_1 - \theta_{k-1},$$

$$\beta_{k(k-1)/2} = \theta_1 - \theta_2.$$

(6.10)

Let $E_{\beta_j}, F_{\beta_j} \in U_h \mathfrak{sl}_k$, $1 \leq j \leq k(k - 1)/2$ be the corresponding quantum root vectors so that $E_{\beta_j} = E_i$ and $F_{\beta_j} = F_i$ whenever $\beta_j$ is the simple root $\alpha_i$. [Lu2, prop. 1.8]. Then, [KR, LS, RQ, CP, thm. 8.3.9]

$$R = \exp \left( h \sum_{i=1}^{k-1} H^i \otimes H_i \right) \prod_{j=1}^{k(k-1)/2} \exp_q \left( (q - q^{-1}) E_{\beta_j} \otimes F_{\beta_j} \right)$$

(6.11)

where $\{H^i\}_{i=1}^{k-1} \subset \mathfrak{h}$ is the basis of the Cartan subalgebra of $\mathfrak{sl}_k$ dual to $\{H_i\}_{i=1}^{k-1}$ with respect to the pairing $\langle X, Y \rangle = \text{tr}(XY)$, $q = e^{\hbar}$,

$$\exp_q(x) = \sum_{n \geq 0} q^{n(n-1)/2} \frac{x^n}{[n]!}$$

(6.12)

and the product in (6.11) is taken so that the factor $\exp_q \left( (q - q^{-1}) E_{\beta_j} \otimes F_{\beta_j} \right)$ is placed to the left of $\exp_q \left( (q - q^{-1}) E_{\beta_j'} \otimes F_{\beta_j'} \right)$ whenever $j > j'$. To compute $R v_i^{\mu_1, \mu_2}$, note that for any positive root $\beta \neq \theta_1 - \theta_2$ and $0 \leq a \leq \mu_1$,

$$E_{\beta} X_{11}^{\mu_1-a} X_{21}^{a} = 0$$

(6.13)

since, by (5.17), $(\mu_1 - a, a, 0, \ldots, 0) + \beta$ is not a weight of $S^\mu_\mu \mathbb{C}^k$. Thus, using (5.17)–(5.19)

$$R v_i^{\mu_1, \mu_2} = \exp \left( h \sum_{i=1}^{k-1} H^i \otimes H_i \right) \exp_q \left( (q - q^{-1}) E_1 \otimes F_1 \right) v_i^{\mu_1, \mu_2}$$

$$= \sum_{0 \leq a \leq i \leq \mu_1} \sum_{0 \leq n \leq i-a} (-1)^a e^{h a (\mu_2-a+1)} \frac{i}{a} e^{h ((\mu_1-i+a+n)(\mu_2-a-n) + (i-a-n)(a+n)+\mu_1\mu_2/k)}$$

$$\cdot \frac{e^{h n(n-1)/2}(e^{\hbar} - e^{-\hbar})^n}{[n]!} \cdot \frac{[i-a]!(\mu_2-a)!}{[i-a-n]!\mu_2-a-n)!}$$

$$\cdot X_{11}^{\mu_1-i+a+n} X_{21}^{i-a-n} \otimes X_{12}^{\mu_2-a-n} X_{22}^{a+n}$$

(6.14)

which, upon setting $\alpha = a + n$, yields

$$e^{h ((\mu_1-i)(\mu_2-i)-i+\mu_1\mu_2/k)} \sum_{\alpha=0}^i (-1)^\alpha \frac{i}{\alpha} e^{h (i-\alpha)(\mu_1-i+\alpha+1)} S^\mu_\alpha X_{11}^{\mu_1-i+\alpha} X_{21}^{i-a} \otimes X_{12}^{\mu_2-a} X_{22}^{a}$$

(6.15)

where

$$S^\mu_\alpha = e^{h a (\mu-a+1)} \sum_{n=0}^\alpha (-1)^n \frac{\alpha}{n} \frac{\mu-a+n+1)!}{\mu-a)!} e^{h (\alpha-n)(\mu-a-n+1)+hn(n-1)/2(e^{\hbar} - e^{-\hbar})^n}$$

(6.16)
We claim that $S_\alpha^\mu = 1$ for any $\alpha \leq \mu \in \mathbb{N}$ so that (6.7) holds. Indeed, using

$$
\begin{bmatrix} \alpha \\ a \end{bmatrix} = \delta_{\alpha > a} \begin{bmatrix} \alpha - 1 \\ a \end{bmatrix} e^{-ha} + \delta_{\alpha > 0} \begin{bmatrix} \alpha - 1 \\ a - 1 \end{bmatrix} e^{h(\alpha - a)}
$$

(6.17)

one readily finds

$$S_\alpha^\mu = e^{2h(\mu - \alpha + 1)} S_{\alpha - 1}^{\mu - 1} - (e^{2h(\mu - \alpha + 1)} - 1) S_{\alpha - 1}^\mu$$

(6.18)

whence $S_\alpha^\mu = 1$ by induction on $\alpha$ since $S_0^\mu = 1$ for any $\mu \in \mathbb{N}$.

6.2. Quantum Weyl group action on singular vectors. Let $E, F, H$ be the standard generators of $U_\hbar \mathfrak{sl}_2$.

Lemma 6.3. The following holds in $S_\hbar(\mathcal{M}^*_{k,2})$,

$$E \, v_i^{\mu_1,\mu_2} = [\mu_2 - i] \, v_i^{\mu_1+1,\mu_2-1}$$

(6.19)

$$F \, v_i^{\mu_1,\mu_2} = [\mu_1 - i] \, v_i^{\mu_1-1,\mu_2+1}$$

(6.20)

$$H \, v_i^{\mu_1,\mu_2} = (\mu_1 - \mu_2) \, v_i^{\mu_1,\mu_2}$$

(6.21)

Proof. By (5.21),

$$E \, v_i^{\mu_1,\mu_2} = \sum_{a=0}^{i} (-1)^a \begin{bmatrix} a \\ i \end{bmatrix} e^{ha(\mu_2-a+1)} [\mu_2-a] e^{h(i-2a)} X_{11}^{\mu_1-i+a} X_{21}^{i-a} X_{12}^{\mu_2-a-1} X_{22}^a$$

$$+ \sum_{a=0}^{i} (-1)^a \begin{bmatrix} a \\ i \end{bmatrix} e^{ha(\mu_2-a+1)} [a] X_{11}^{\mu_1-i+a} X_{21}^{i-1+a} X_{12}^{\mu_2-a-1} X_{22}$$

(6.22)

$$= \sum_{a=0}^{i} (-1)^a \begin{bmatrix} a \\ i \end{bmatrix} e^{ha(\mu_2-a+1)} [\mu_2-a] e^{h(i-2a)} X_{11}^{\mu_1-i+a+1} X_{21}^{i-a} X_{12}^{\mu_2-a-1} X_{22}$$

$$- \sum_{a=0}^{i} (-1)^a \begin{bmatrix} a \\ i \end{bmatrix} e^{h(a+1)(\mu_2-a)} [i-a] X_{11}^{\mu_1-i+a+1} X_{21}^{i-a} X_{12}^{\mu_2-a-1} X_{22}$$

$$= [\mu_2 - i] \sum_{a=0}^{i} (-1)^a \begin{bmatrix} a \\ i \end{bmatrix} e^{ha(\mu_2-a)} X_{11}^{\mu_1+1-i+a} X_{21}^{i-a} X_{12}^{\mu_2-1-a} X_{22}$$

$$= [\mu_2 - i] v_i^{\mu_1+1,\mu_2-1}$$
Similarly, by (5.22),
\[
F v_i^{\mu_1,\mu_2} = \sum_{a=0}^{i} (-1)^a \binom{i}{a} e^{h(\mu_2-a+1)} \left[ \mu_1 - i + a \right] X^{\mu_1-i+a-1} X^{\mu_2-a+1} + \sum_{a=0}^{i} (-1)^a \binom{i}{a} e^{h(\mu_2-a+1)} \left[ \mu_1 - i - a \right] e^{-h(\mu_1-\mu_2-2a)} X^{\mu_1-i+a} X^{\mu_2-a+1} X^{\mu_2-a+1} X^{\mu_2-a+1} X^{\mu_2-a+1} X^{\mu_2-a+1} \]
\[
= \sum_{a=0}^{i} (-1)^a \binom{i}{a} e^{h(\mu_2-a+1)} \left[ \mu_1 - i + a \right] X^{\mu_1-i+a-1} X^{\mu_2-a+1} X^{\mu_2-a+1} X^{\mu_2-a+1} X^{\mu_2-a+1} X^{\mu_2-a+1} \]
\[
= [\mu_1 - i] \sum_{a=0}^{i} (-1)^a \binom{i}{a} e^{h(\mu_2-a+1)} X^{\mu_1-i+a} X^{\mu_2-a+1} X^{\mu_2-a+1} X^{\mu_2-a+1} X^{\mu_2-a+1} X^{\mu_2-a+1} \]
\[
= [\mu_1 - i] v_i^{\mu_1-1,\mu_2+1} \quad (6.23)
\]

Finally, (6.21) follows from (5.20) □

Let now
\[
S = \exp_{q^{-1}}(q^{-1} Eq^{-1}) \exp_{q^{-1}}(-F) \exp_{q^{-1}}(qEq^{-1})q^{H(H+1)/2} \quad (6.24)
\]
be the generator of the quantum Weyl group of $U_q \mathfrak{sl}_2$ (Lus, KR, So), we use the form given in (Kla, Ser) where $q = e^{\hbar}$ and the $q$-exponential is defined by (6.12).

**Proposition 6.4.** The following holds in $S_H(M^*_k,2)$,
\[
S v_i^{\mu_1,\mu_2} = (-1)^{\mu_1-i} q^{(\mu_1-i)(\mu_2-i+1)} v_i^{\mu_2,\mu_1} \quad (6.25)
\]

**Proof.** Fix $\mu, i \in \mathbb{N}$ with $2i \leq \mu$. By lemma 6.3, the vectors $u_k = v_i^{\mu-i-k, i+k}$, with $0 \leq k \leq \mu - 2i$ satisfy
\[
E u_k = [k] u_{k-1} \quad (6.26)
\]
\[
F u_k = [\mu - 2i - k] u_{k+1} \quad (6.27)
\]
\[
H u_k = (\mu - 2i - 2k) u_k \quad (6.28)
\]
and therefore span the indecomposable $U_q \mathfrak{sl}_2$-module of dimension $\mu - 2i + 1$. Moreover,
\[
u_k = \frac{[\mu - 2i - k]!}{[\mu - 2i]!} F^{\mu-2i} u_0 = \frac{[\mu - 2i - k]!}{[\mu - 2i]!} E^{\mu-2i-k} u_{\mu-2i} \quad (6.29)
\]

Since $\text{Ad}(S) H = -H$, $S u_0$ is proportional to $u_{\mu-2i}$ and, by (6.24) is therefore equal to
\[
(-1)^{\mu-2i} q^{\mu-2i} \frac{F^{\mu-2i}}{[\mu - 2i]!} u_0 = (-1)^{\mu} q^{\mu-2i} u_{\mu-2i} \quad (6.30)
\]

Next, using $\text{Ad}(S) F = -q^{-H} E$ and (6.29), we find
\[
S u_k = \frac{[\mu - 2i - k]!}{[\mu - 2i]!} \text{Ad}(S) F^{k} S u_0 = (-1)^{\mu-k} q^{(k+1)(\mu-2i-k)} u_{\mu-2i-k} \quad (6.31)
\]
Thus, setting $\mu = \mu_1 + \mu_2$, so that $v_i^{\mu_1,\mu_2} = u_{\mu_2-i}$, we find
\[
S v_i^{\mu_1,\mu_2} = (-1)^{\mu_1-i} q^{(\mu_1-i)(\mu_2-i+1)} v_i^{\mu_2,\mu_1} \quad (6.32)
\]
as claimed ■

6.3. **Identification of R and quantum Weyl group actions.** Fix $1 \leq j \leq n$ and let $R^j$ be the universal $R$-matrix of $U_\hbar \mathfrak{g}_{k}$ acting on the $j$ and $j+1$ tensor copies of $S_h(\mathcal{M}_{k,1}^*)^{\otimes n}$. Let $S_j$ be the element of the quantum Weyl group of $U_\hbar \mathfrak{g}_{n}$ corresponding to the simple root $\theta_j - \theta_{j+1}$.

**Theorem 6.5.** The following holds on $S_h(\mathcal{M}_{k,1}^*)$:

$$R^j_j = S_j \cdot e^{-\hbar(D^{(n)}_j + D^{(n)}_{j+1}/k)} \cdot e^{i\pi D^{(n)}_j} \quad (6.33)$$

**Proof.** Let $U_\hbar \mathfrak{g}_{l,1} \subset U_\hbar \mathfrak{g}_{n}$ be the Hopf subalgebra generated by $E^{(n)}_j, F^{(n)}_j, D^{(n)}_j, D^{(n)}_{j+1}$. By (6.20)–(6.22), $U_\hbar \mathfrak{g}_{l,1}$ only acts upon the variables $X_{ij}, X_{ij+1}, 1 \leq i \leq k$. Thus, $U_\hbar \mathfrak{g}_{l,1}$ and therefore $S_j$ only act on the $j$ and $j+1$ tensor copies $S_h(\mathcal{M}_{k,1}^*)_j, S_h(\mathcal{M}_{k,1}^*)_{j+1}$ of $S_h(\mathcal{M}_{k,1}^*)^{\otimes n} \cong S_h(\mathcal{M}_{k,1}^*)$. Since this is also the case of $R^j$, the proof of (6.33) reduces to a computation on the $U_\hbar \mathfrak{g}_{l,1} \otimes U_\hbar \mathfrak{g}_{l,2}$-module $S_h(\mathcal{M}_{k,1}^*)_j \otimes S_h(\mathcal{M}_{k,1}^*)_{j+1} \cong S_h(\mathcal{M}_{k,2}^*)$. Both sides of (6.33) clearly commute with $U_\hbar \mathfrak{g}_{l,1}$, so it is sufficient to check their equality on the singular vectors of

$$S_h(\mathcal{M}_{k,1}^*)_j \otimes S_h(\mathcal{M}_{k,1}^*)_{j+1} = \bigoplus_{\mu_1, \mu_2 \in \mathbb{N}} S_h^{\mu_1} \mathbb{C}^k \otimes S_h^{\mu_2} \mathbb{C}^k \quad (6.34)$$

i.e., on the vectors $v_i^{\mu_1, \mu_2} \in S_h^{\mu_1} \mathbb{C}^k \otimes S_h^{\mu_2} \mathbb{C}^k$ of lemma 6.1. By propositions 6.2 and 6.4,

$$R^j_j v_i^{\mu_1, \mu_2} = S_j \cdot e^{-\hbar(\mu_1 + \mu_2)/k}(-1)^{\mu_1} v_i^{\mu_1, \mu_2} \quad (6.35)$$

whence (6.33) since, for any $1 \leq l \leq n$, $D^{(n)}_l$ gives the $\mathbb{N}$-grading on $S_h(\mathcal{M}_{k,1}^*)$ ■

**Remark.** The coincidence of the two representations of $B_n$ studied above was also noted by Baumann [Ba, prop. 12] in the special case when both actions are restricted to the subspace $S_h^{\mu_1} \mathbb{C}^k \otimes \cdots \otimes S_h^{\mu_n} \mathbb{C}^k$ of $S_h(\mathcal{M}_{k,n}^*)$ where $\mu_1 = \cdots = \mu_n = 1$.

7. **Monodromic realisation of quantum Weyl group operators**

The following is the main result of this paper. It was conjectured for any simple Lie algebra $\mathfrak{g}$ by De Concini around 1995 (unpublished) and, independently, in [TL]. We prove it here for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$.

**Theorem 7.1.** Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, $V$ a finite-dimensional $\mathfrak{g}$-module, $\mu$ a weight of $V$ and

$$V^\mu = \bigoplus_{\nu \in W^\mu} V[\nu] \quad (7.1)$$

the direct sum of the weight spaces of $V$ corresponding to the Weyl group orbit of $\mu$. Let $\pi^h_\alpha : B_\hbar \rightarrow GL(V^\mu[\hbar])$ be the monodromy representation of the connection

$$d - \hbar \sum_{\alpha > 0} \frac{d\alpha}{\alpha} \pi_V(\kappa_\alpha) \quad (7.2)$$

obtained by regarding $\hbar$ as a formal variable. Let $\pi_W : B_\hbar \rightarrow GL(V^\mu[\hbar])$ be the quantum Weyl group action obtained by regarding $V[\hbar]$ as a $U_\hbar \mathfrak{g}$-module with $\hbar = 2\pi i\hbar$. Then, $\pi^h_\alpha$ and $\pi_W$ are equivalent.
PROOF. We may assume that $V$ is irreducible with highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ where $\lambda_i \in \mathbb{N}$. Regard $V$ as a $\mathfrak{gl}_n$-module by letting $1^{(n)} = \sum_{j=1}^n E_{jj}^{(n)}$ act as multiplication by $|\lambda| = \sum_{i=1}^n \lambda_i$ and fix some $k \geq n$. By lemma 3.4, $V[\nu]$ is isomorphic to the space $M^\nu_\lambda$ of singular vectors of weight $\lambda$ for the diagonal action of $\mathfrak{gl}_k$ on

$$S^\nu \mathcal{C}^k = S^{\nu_1} \mathcal{C}^k \otimes \cdots \otimes S^{\nu_n} \mathcal{C}^k \subset S(\mathcal{M}_{k,n}^\nu)$$

(7.3)

and, by corollary 3.3, the monodromy representation of the Knizhnik–Zamolodchikov connection (3.17) on $\mathfrak{gl}_n$ is related by

$$\pi^\nu_{KZ}(T_j) = \pi^\nu(T_j)e^{-\pi \hbar (E_{jj}^{(n)} + E_{j+1,j+1}^{(n)} + 2E_{j,j+1}^{(n)} - E_{j+1,j+1}/k)} e^{i \pi E_{jj}^{(n)}}$$

(7.4)

where $\hbar = 2h$.

We shall now use the Kohno–Drinfeld theorem to relate $\pi^\nu_{KZ}$ to the $R$-matrix representation of $B_\nu$ corresponding to the action of $U_h \mathfrak{gl}_k$ on $S^\nu \mathcal{C}^k[h]$. Let for this purpose $\phi : U_h \mathfrak{gl}_k \to U \mathfrak{gl}_k$ be an algebra isomorphism whose reduction mod $\hbar$ is the identity and which acts as the identity on the Cartan subalgebras i.e., $\phi(D_i^{(k)}) = E_{ii}^{(k)}$, $1 \leq i \leq k$ [Dr2, prop. 4.3]. Then, $U_h \mathfrak{gl}_k$ act on each $S^{\nu_j} \mathcal{C}^k[h]$ via $\phi$ and on

$$S^{\nu_1} \mathcal{C}^k[h] \otimes \cdots \otimes S^{\nu_n} \mathcal{C}^k[h] = S^{\nu_1} \mathcal{C}^k \otimes \cdots \otimes S^{\nu_n} \mathcal{C}^k[h] = S^\nu \mathcal{C}^k[h]$$

(7.5)

via the $n$-fold coproduct $\Delta^{(n)} : U_h \mathfrak{gl}_k \to U_h \mathfrak{gl}_k^{\otimes n}$ recursively defined by

$$\Delta^{(1)} = \text{id}$$

(7.6)

$$\Delta^{(a+1)} = \Delta \otimes \text{id}^{\otimes (a-1)} \circ \Delta^{(a)}, \quad a \geq 1$$

(7.7)

where $\Delta = \Delta^{(2)}$ is given by (1.9)–(4.11).

Let $\Delta_0$ be the standard, cocommutative coproduct on $U \mathfrak{gl}_k$ so that $U \mathfrak{gl}_k$ acts on $S^\nu \mathcal{C}^k$ via $\Delta_0^{(n)} : U \mathfrak{gl}_k \to U \mathfrak{gl}_k^{\otimes n}$ defined as in (7.8)–(7.11) with $\Delta$ replaced by $\Delta_0$. Since $\Delta = \Delta_0 + o(\hbar)$

and $H^1(s_1, s_2, s_3) = 0$, there exists a twist $F = 1 \otimes 1 + o(\hbar) \in U \mathfrak{gl}_k^{\otimes (2)}[h]$ such that, for any $x \in U_h \mathfrak{gl}_k$,

$$\phi \otimes \phi \circ \Delta(x) = F \Delta_0(\phi(x)) F^{-1}$$

(7.8)

It follows that the actions of $U_h \mathfrak{gl}_k$ and $U \mathfrak{gl}_k[h]$ on $S^\nu \mathcal{C}^k$ are related by

$$\phi^{\otimes (n)} \circ \Delta^{(n)}(x) = F^{(n)} \Delta_0^{(n)}(\phi(x)) F^{(n)-1}, \quad x \in U_h \mathfrak{gl}_k$$

(7.9)

where $F^{(n)} \in U \mathfrak{gl}_k^{\otimes n}[h]$ is recursively defined by

$$F^{(1)} = 1$$

(7.10)

$$F^{(a+1)} = F \otimes 1^{\otimes (a-1)} \cdot \Delta_0 \otimes \text{id}^{\otimes (a-1)}(F^{(a)}), \quad a \geq 1$$

(7.11)

Let now $M^\nu_\lambda \subset S^\nu \mathcal{C}^k$ and $M^\nu_{\lambda, \hbar} \subset S^\nu \mathcal{C}^k[h]$ be the subspaces of vectors of highest weight $\lambda$ for the actions of $\mathfrak{gl}_k$ and $U_h \mathfrak{gl}_k$ respectively. We shall need the following

**Lemma 7.2.** $F^{(n)} M^\nu_{\lambda, \hbar} \subset M^\nu_{\lambda, \hbar}$.

**Proof.** Let $S^\nu \mathcal{C}^k(\lambda) \subset S^\nu \mathcal{C}^k$ and $S^\nu \mathcal{C}^k[h](\lambda) \subset S^\nu \mathcal{C}^k[h]$ be the isotypical components of types $V_\lambda$ and $V_\lambda[h]$ for $\mathfrak{gl}_k$ and $U_h \mathfrak{gl}_k$ respectively. The subspace $F^{(n)} S^\nu \mathcal{C}^k(\lambda)[h]$ is invariant under $U_h \mathfrak{gl}_k$ by (7.3) and its reduction mod $\hbar$ is equal to $S^\nu \mathcal{C}^k(\lambda)$ since $F^{(n)} = 1^{\otimes (n)} + o(\hbar)$. Thus, by proposition 1.4, $F^{(n)} S^\nu \mathcal{C}^k(\lambda)[h]$ is isomorphic to $S^\nu \mathcal{C}^k(\lambda)[h]$ and therefore contained in $S^\nu \mathcal{C}^k[h](\lambda)$. Since this holds for any $\lambda$, the inclusion is an equality. Noting
now that $F^{(n)}$ is equivariant for the action of the Cartan subalgebras of $\mathfrak{gl}_k$ and $U_h\mathfrak{gl}_k$ since
\[ \Delta_0 \circ \phi(h) = \phi \circ \Delta(h) \text{ for any } h \in h, \]
we get that
\[ F^{(n)} M^\nu_\lambda[h] = F^{(n)} \left( S^\nu C^k(\lambda)[h] \right) \]
\[ = F^{(n)} \left( S^\nu C^k(\lambda)[h] \right)[\lambda] \]
\[ = S^\nu C^k[h](\lambda)[\lambda] \]
\[ = M^\nu_{h,\lambda} \] (7.12)
as claimed.

Summarising, we have an action of $B_n$ on $\bigoplus_{\nu \in \mathbb{N}_n} M^\nu_\lambda$ via the monodromy of the Knizhnik–Zamolodchikov connection and an action of $B_n$ on $\bigoplus_{\nu \in \mathbb{N}_n} M^\nu_{h,\lambda}$ via the $R$–matrix representation of $U_h\mathfrak{sl}_k$. Drinfeld’s theorem [Dr3, Dr4, Dr5] asserts that the twist $F$ may be chosen so that
\[ F^{(n)} : \bigoplus_{\nu \in \mathbb{N}_n} M^\nu_\lambda[h] \rightarrow \bigoplus_{\nu \in \mathbb{N}_n} M^\nu_{h,\lambda} \] (7.13)
is $B_n$–equivariant so that, for any $1 \leq j \leq n - 1$,
\[ \pi_W^j(T_j) = F^{(n)} R_j^\nu F^{(n)} \] (7.14)
where $R_j^\nu$ is the universal $R$–matrix for $U_h\mathfrak{sl}_k$ acting on the $j$ and $j + 1$ copies of $\bigoplus_{\nu \in \mathbb{N}_n} S^\nu C^k[h]$ and $h = \pi_i h$.

Let now $S_h(\mathcal{M}_{k,n}^*) \cong (S_h(\mathcal{M}_{1,k}^*))^\otimes n$ be the algebra of functions on quantum $k \times n$ matrix space defined in section 2 and, for any $\nu \in \mathbb{N}_n$, let
\[ S^\nu C^k \subseteq S^\nu C^k \otimes \cdots \otimes S^\nu C^k \] (7.15)
be the space of polynomials which are homogeneous of degree $\nu_j$ in the variables $X_{1j}, \ldots, X_{kj}$, for any $1 \leq j \leq n$. By (7.17)–(7.19), $S^\nu C^k/hS^\nu C^k$ is the $\mathfrak{sl}_k$–module $S^\nu C^k$ so that, by proposition 4.1, we may identify $S^\nu C^k$ with $S^\nu C^k[h]$ as $U_h\mathfrak{sl}_k$–modules.

By theorem 5.3 and the fact that the $\mathbb{N}$–grading on the $j$th tensor factor of $S_h(\mathcal{M}_{k,n}^*) \cong (S_h(\mathcal{M}_{1,k}^*))^\otimes n$ is given by the action of $D_j^{(n)}$, the space $M^\nu_{h,\lambda}$ is isomorphic to the subspace $V^{(n)}_\lambda[\nu][h] \subset V^{(n)}_\lambda[h]$ of weight $\nu$. Using now theorem 5.3, we find
\[ \pi_W^j(T_j) = F^{(n)} \cdot S_j \cdot e^{-h(D_j^{(n)} + (n)D_{j+1}^{(n)}/k)} \cdot e^{jD_j^{(n)}} \cdot F^{(n)} \]
\[ \cdot e^{\pi h(E_j^{(n)} + E_j^{(n)} + 2E_j^{(n)}E_{j+1}^{(n)}/k)} \cdot e^{\pi E_j^{(n)}} \] (7.16)
where $S_j = \pi_W(T_j)$ is the quantum Weyl group element of $U_h\mathfrak{sl}_n$ corresponding to the simple root $\alpha_j = \theta_j - \theta_{j+1}$. Since for any $l$, $E_l^{(n)} = D_l^{(n)}$ on $V^\mu \cong \bigoplus_{\nu \in \mathbb{N}_n} M^\nu_\lambda$ we get
\[ \pi_W^j(T_j) = F^{(n)} \cdot S_j e^{-h(D_j^{(n)} - (n)D_{j+1}^{(n)}/k)} F^{(n)} \]
\[ = F^{(n)} \cdot S_j e^{-hD_j^{(n)}/2} \cdot F^{(n)} \]
\[ = F^{(n)} \cdot e^{h\rho^{(n)}/2} S_j e^{-h\rho^{(n)/2}} F^{(n)} \] (7.17)
where $\rho^{(n)} = \frac{1}{2} \sum_{j=1}^n (n - 2j + 1) D_j^{(n)}$ is the half–sum of the positive (co)roots of $\mathfrak{sl}_n$. ■
Remark. Theorem 7.1 is proved in [TL] for the following pairs \((g, V)\) where \(g\) is a simple Lie algebra and \(V\) an irreducible, finite–dimensional representation

1. \(g = \mathfrak{sl}_2\) and \(V\) is any irreducible representation.
2. \(g = \mathfrak{sl}_n\) and \(V\) is a fundamental representation.
3. \(g = \mathfrak{so}_n\) and \(V\) is the vector or a spin representation.
4. \(g = \mathfrak{sp}_n\) and \(V\) is the defining vector representation.
5. \(g = e_6, e_7\) and \(V\) is a minuscule representation.
6. \(g = g_2\) and \(V\) is the 7–dimensional representation.
7. \(g\) is any simple Lie algebra and \(V \cong g\) its adjoint representation.

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