DERIVED REVERSE MULTIPLES AND THEIR PALINOMIALS

BENJAMIN V. HOLT

Abstract. In this paper we consider several families of reverse multiples (also known as palintiples [1, 3]) whose carries themselves are digits of lower-base reverse multiples and give some methods for constructing them from fundamental reverse multiple types. We also continue the study of palinomials introduced by [3] by revealing a more direct relationship between the digits of certain reverse multiple types and the roots of their palinomials. We explore the consequences of this relationship for palinomials induced by reverse multiple families derived from lower-base reverse multiples. Finally, we pose some questions regarding Young graphs of derived reverse multiple families and consider the implications our general observations might have for relations between Young graph isomorphism classes.
1. Introduction

In a previous paper on reverse multiples [3] (also known as palintiples) it is noted that “the carries [of a reverse multiple]...play as critical a role as the digits themselves.” Indeed, the full measure of this statement is realized when one notices that the carries of a reverse multiple are often themselves the digits of a reverse multiple of a lower base. Consider the example of the (139, 10)-reverse multiple (28, 25, 108, 113, 2)_{139} which has carries given by \((r_4, r_3, r_2, r_1, r_0) = (8, 7, 1, 2, 0)\). One immediately notices that the nontrivial carries are digits of the well known (10, 4)-reverse multiple 8712.

Recent work on the problem, particularly the work of Sloane [8], translates the reverse multiple problem into graph-theoretical language by means of Young graphs which are a succinct visualization of reverse multiple structure showing how the possible carries generate the possible digits of a reverse multiple of arbitrary length. Young graphs are a modification of tree graphs introduced by Young [10, 11] which are a representation of an efficient reverse multiple search method with the possible carries represented as nodes and the potential digits being associated with the edges. We note that Hoey [1, 2] presented a similar idea using finite state machines. Representations of machines which recognize reverse multiples bear strong resemblance to Young graphs: compare the Young graph determined by (8, 5)-reverse multiples found in [8] to the machine which recognizes (8, 5)-reverse multiples [2].

Kendrick [4] extends the work of Sloane [8] by proving several of his conjectures, one of which includes one of Sloane’s main conjectures stating that a \((g, k)\)-reverse multiple Young graph is isomorphic to the 1089-graph if and only if \(k+1\) divides \(g\) and offers several conjectures of his own regarding other Young graph isomorphisms.

Other recent work includes that of [3] which establishes some general properties of reverse multiples of any base having an arbitrary number of digits using only elementary number theory. As with the work of [4, 8, 10, 11], the methods therein pay particular attention to the structure of the carries which naturally separates all reverse multiples into three mutually exclusive and exhaustive classes. Letting \(N = (a_n, a_{n-1}, \ldots, a_0)\) be a \((g, k)\)-reverse multiple with carries \(r_n, r_{n-1}, \ldots, r_0\), these classes are defined as follows: we say that \(N\) is symmetric if \(r_j = r_{n-j}\) for all \(0 \leq j \leq n\) and that \(N\) is shifted-symmetric if \(r_j = r_{n-j+1}\) for all \(0 \leq j \leq n\). A reverse multiple that is neither symmetric nor shifted-symmetric is called asymmetric. The \((10, 4)\)-reverse multiple seen \((8, 7, 1, 2)_{10}\) has carries \((r_3, r_2, r_1, r_0) = (0, 3, 3, 0)\) making it an example of a symmetric reverse multiple. For more examples, the reader is referred to [3]. (The reader already familiar with the literature will notice that this paper uses the formulation of reverse multiples used by [3] which defines the reverse multiple to be the number obtained after multiplication by \(k\) with the digits and carries being indexed from 0 rather than 1.)

Comparing the above mentioned classes to Young graph isomorphism classes, it is easily shown that \((g, k)\)-reverse multiples whose Young graph \(Y(g, k)\) is isomorphic to \(Y(10, 9)\), otherwise known as the 1089 graph [4, 8], are symmetric (see the last section for a discussion of this). Moreover, the work of [3] and [4] show that the class of shifted-symmetric reverse multiples is equal to the class of reverse multiples whose Young graph is complete. As for those mentioned by Kendrick [4] which are neither complete nor isomorphic to the 1089 graph, the asymmetric class is characterized by an astonishing plurality of Young graph isomorphism classes and admits many more subclassifications and isolated cases than previously thought.

The above underscores a primary aim of this paper to begin to understand more fully the asymmetric class. We shall describe several families of reverse multiples which can be constructed, or derived, from lower-base examples. In particular, we will outline some methods...
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for constructing new asymmetric reverse multiples using as carries “old” reverse multiples whose Young graph is isomorphic to either the 1089 graph or a complete graph in order to generate examples such as the one given in the first paragraph above.

Kendrick [4] mentions that it is still quite poorly understood how the number theoretical and graph theoretical aspects of the reverse multiple problem relate to one another. Our work, which relies mostly upon elementary number theoretic results, gives rise to some concrete questions as to how the reverse multiple families described here can be classified according to Young graph isomorphism as well as suggest how Young graph isomorphism classes might be generated from others.

Additionally, this paper further develops the topic of palinomials introduced in [3] revealing a more intimate relationship between the digits of ($g,k$)-reverse multiples and the roots of their palinomials when $Y(g,k)$ is isomorphic $Y(10,9)$. These results have implications for palinomials induced by reverse multiples derived from ($g,k$)-reverse multiples such that $Y(g,k)$ is isomorphic to either $Y(10,9)$ a complete graph $K_n$.

2. REVERSE MULTIPLES WHOSE CARRIES ARE REVERSE MULTIPLES OF A LOWER BASE

Henceforth we shall suppose that $N = (a_n, a_{n-1}, \ldots, a_0)$ is a ($g,k$)-reverse multiple with carries $r_n, r_{n-1}, \ldots, r_0$. It is well-established [3][8][10] how the digits of a reverse multiple are related to the carries:

\[ a_j = \frac{kr_{n-j+1} + gr_{j+1} - r_j}{k^2 - 1}. \] (2.1)

We pose the general question of when the carries of a reverse multiple are the digits of a reverse multiple of a lower base as in the example given in the introduction. There are two possibilities for when this may occur:

**Case 1:** We find conditions under which we can construct a new $n + 2$-digit ($\hat{g}, \hat{k}$)-reverse multiple $\hat{N}$ with carries ($\hat{r}_{n+1}, \hat{r}_n, \ldots, \hat{r}_0$) given by ($a_n, a_{n-1}, \ldots, a_0$) as in the example given in the introduction. Using Equation (2.1) the new digits $\hat{a}_j$ must satisfy

\[ \hat{a}_j = \frac{k\hat{g}\hat{r}_{n-j+2} - \hat{k}\hat{r}_{n-j+1} + \hat{g}\hat{r}_{j+1} - \hat{r}_j}{k^2 - 1}. \] (2.2)

Then $\hat{g}a_0 \equiv \hat{k}a_n \mod (k^2 - 1)$ when $j = 0$. Therefore, in order to find a suitable higher base $\hat{g}$, it must be that $\gcd(a_0, k^2 - 1)$ divides $a_n$ in which case we have that $\hat{g} = s + \alpha \frac{k^2 - 1}{\gcd(a_0, k^2 - 1)}$ where $s$ is the least non-negative solution of the above congruence and $\alpha \geq 1$. The above then becomes

**Case 2:** We now ask when we may construct a new $n + 3$-digit ($\hat{g}, \hat{k}$)-reverse multiple $\hat{N}$ with carries ($\hat{r}_{n+2}, \hat{r}_{n+1}, \ldots, \hat{r}_0$) given by ($0, a_n, a_{n-1}, \ldots, a_0$). For $n + 3$-digits we have

\[ \hat{a}_j = \frac{k\hat{g}\hat{r}_{n-j+3} - \hat{k}\hat{r}_{n-j+2} + \hat{g}\hat{r}_{j+1} - \hat{r}_j}{k^2 - 1}. \] (2.3)

Then $\hat{g}a_0 \equiv 0 \mod (k^2 - 1)$ when $j = 0$ so that $\hat{g} = \alpha \frac{k^2 - 1}{\gcd(a_0, k^2 - 1)}$. It follows that

\[ \hat{a}_j = \frac{\alpha}{\gcd(a_0, k^2 - 1)}(\hat{k}a_{n-j+2} + a_j) - \frac{\hat{k}a_{n-j+1} + a_{j-1}}{k^2 - 1}. \] (2.4)
In order to simplify exposition, we will say that a reverse multiple constructed from a lower-base reverse multiple is a derived reverse multiple. In particular, reverse multiples derived in the manner described in Case 1 and Case 2 above will respectively be called singly-derived and doubly-derived reverse multiples.

3. Reverse Multiples Derived from 1089 Reverse Multiples

Any \((g, k)\)-reverse multiple for which \(Y(g, k) \equiv Y(10, 9)\) shall for the remainder of this article be called a 1089 reverse multiple. Moreover, the family of reverse multiples derived from 1089 reverse multiples shall, for the purpose of less cumbersome exposition, be called Hoey reverse multiples.

By the work of Kendrick [4] we may suppose that \(k + 1\) divides \(g\) with quotient \(b\) and that \(r_j \equiv 0 \mod (k - 1)\). Then by (4), \(r_{n-j} = r_j\) (that is, \(N\) is a symmetric reverse multiple) so that by Equation 2.1 \(a_j = kbq_{n-j+1} + bq_{j+1} - q_j\) for all \(0 \leq j \leq n\) where \(q_n, q_{n-1}, \ldots, q_0\) is a palindromic binary sequence such that \(q_1 = q_{n-1} = 1\) and there are no isolated zeros or ones except \(q_0 = q_n = 0\). Since \(a_0 = b\) and \(a_n = kb\) we have that \(\gcd(a_0, \hat{k}^2 - 1)\) divides \(a_n\) and that \(\hat{g} = k\hat{k} + a_0\). Equation 2.3 then yields

\[
\hat{a}_j = \frac{k^2 b q_{j-2} + k^2 \hat{k} b - k \hat{k} + k - b}{k^2 - 1} q_j + \frac{k^2 \hat{k} b - k \hat{k}^2 - k \hat{k} + 1}{k^2 - 1} q_j \cdot q_j + \frac{b q_{j+1} + (k b - \hat{k}) q_j + \hat{k} b q_j - 1}{\gcd(b, \hat{k}^2 - 1)}.
\]

To ensure that each term in the above is an integer for all \(\alpha > 1\), we must have both that \(\gcd(b, \hat{k}^2 - 1) = 1\) and \((k - 1)\hat{k} \equiv (k^2 - 1) \mod (k^2 - 1)\). A moment’s reflection reveals that \(g = b(k + 1)\) is the only value for \(\hat{k}\) which makes the congruence statement true (it also simultaneously ensures that \(b\) and \(\hat{k}^2 - 1\) are relatively prime). Therefore, we have \(\hat{g} = k \hat{g}\) with

\[
\hat{a}_j = k^2 b q_{j-2} + k^2 b - q_{j-2} + a_0\left(\frac{b q_{j+1} + (b g - 1) q_j - b q_{j-1} + b g q_{j-2}}{\gcd(b, \hat{k}^2 - 1)}\right).
\]

Since \(a_j < g = \hat{g}\), each \(\hat{a}_j\) is less than \(\hat{g}\), and since there are no singleton ones or zeros in \(q_n, q_{n-1}, \ldots, q_0\) (except \(q_0 = q_n\)), \(\hat{a}_j\) cannot be negative so that each of these is a base-\(\hat{g}\) digit. Additionally, since every \(\hat{a}_j\) and \(a_j\) satisfy Equation 2.2 it follows from a routine calculation that \((\hat{a}_n, \hat{a}_{n-1}, \ldots, \hat{a}_0)\) is \(g(\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{n+1})\) where \(a_{j-1}\) is the \(j\)th carry for \(0 < j \leq n + 1\). Thus, we have the following:

**Theorem 3.1.** Suppose \(N = (a_n, a_{n-1}, \ldots, a_0)\) is a \((g, k)\)-reverse multiple for which \(Y(g, k) \equiv Y(10, 9)\). Then for every \(\hat{g} > k g\) such that \(\hat{g} \equiv k g \mod (g^2 - 1)\) there exists an asymmetric \(n + 2\)-digit \((\hat{g}, \hat{g})\)-reverse multiple with carries \((\hat{r}_{n+1}, \hat{r}_n, \ldots, \hat{r}_0)\) given by \((a_n, a_{n-1}, \ldots, a_0)\).

**Example 3.2.** The table below contains several examples of Hoey reverse multiples constructed from the \((3, 2)\)-reverse multiple \((2, 1, 2, 0, 1)\) including the general form obtained from the arguments establishing Theorem 3.1.

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1In honor of D. J. Hoey, to whose memory we dedicate this work.
Theorem 3.3. No doubly-derived reverse multiple can be derived from a 1089 reverse multiple.

Proof. Suppose there exists a doubly-derived \((\hat{g}, \hat{k})\)-reverse multiple \(\hat{N}\) constructed from a \((g, k)\)-reverse multiple \(N = (a_n, a_{n-1}, \ldots, a_0)\) with carries \(r_n, r_{n-1}, \ldots, r_0\) such that \(Y(g, k) \simeq Y(10, 9)\). Then by Equation 2.4 we have that \(b(a_{n-j+1} + a_{j-1}) \equiv 0 \pmod{(k - 1)}\). The cases \(j = 1\) and \(j = 2\) imply that \(2b \equiv 0 \pmod{(k - 1)}\). Thus, since it is well known that the carries of any reverse multiple must be less than the multiplier \([3, 8, 9, 10]\), we have \(b = a_0 = \hat{r}_1 \leq \hat{k} - 1\). Hence, either \(b = \frac{\hat{k} - 1}{2}\) or \(b = \hat{k} - 1\). In any case, \(\text{gcd}(b, \hat{k}^2 - 1)\) divides \(\hat{k} - 1\) so that \(\hat{k} + 1\) divides \(\hat{g}\). But this would imply by [3] that \(bk = a_n = a_0 = b\) which is impossible. \(\square\)

4. Reverse Multiples Derived From Complete Reverse Multiples

For the remainder of this article, any \((g, k)\)-reverse multiple such that \(Y(g, k)\) is complete (isomorphic to \(K_m\) for some \(m > 1\)) shall itself be referred to as complete. (We note that from [3] that this is equivalent to a reverse multiple having shifted symmetry, that is \(r_{n-j+1} = r_j\) for all \(0 \leq j \leq n\).)

We now consider singly-derived reverse multiples derived from complete reverse multiples. For brevity, such reverse multiples will be called Sutcliffe reverse multiples. We then suppose by arguments presented in [3] that \((g - k)r_j \equiv (kg - 1)r_j \equiv 0 \pmod{(k^2 - 1)}\) and \(a_j = \frac{(g-k)r_j + (kg-1)r_j}{k^2 - 1}\). Then by the reasoning of Case 1 we have \(\hat{g} = \frac{(kg-1)r_n}{g^k} \equiv \frac{k(1)(g-k)\alpha}{k^2 - 1} \equiv k\alpha \pmod{(k^2 - 1)}\). We shall suppose that \(s = \frac{k(1)(g-k)\alpha}{g^k}\) is an integer. Then, since \(r_j = r_{n-j+1}\) for all \(0 \leq j \leq n\) by definition, \(s\) is a particular solution for \(\hat{g}\) to the congruence above so that in general \(\hat{g} = \frac{k(1)(g-k)\alpha}{g^k} + \alpha\frac{k^2 - 1}{\text{gcd}(a_0, k^2 - 1)}\). Therefore, by Equation 2.3

\[
\hat{a}_j = \frac{s(kg - 1) - (g - k)}{(k - 1)(k^2 - 1)}r_j + \frac{kg - 1}{k^2 - 1}r_{j-1} + \frac{\alpha}{\text{gcd}(a_0, k^2 - 1)}\left(\frac{g - k}{k^2 - 1}r_{j-1} + (\hat{k} + 1)\frac{kg - 1}{k^2 - 1}r_j + \frac{\hat{k} - 1}{k^2 - 1}r_{j-1}\right).
\]

Supposing that \(\hat{k} > a_j\) guarantees by Equation 2.2 that \(0 \leq \hat{a}_j < \hat{g}\). Hence, Theorem 4.1. Suppose \((a_n, a_{n-1}, \ldots, a_0)\) is a complete \((g, k)\)-reverse multiple with carries \(r_n, r_{n-1}, \ldots, r_0\). If there exists a natural number \(\hat{k}\) such that \(s = \frac{k(1)(g-k)\alpha}{g^k}\) is an integer, \(\hat{k} > a_j\), and \(s\alpha\frac{k^2 - 1}{\text{gcd}(a_0, k^2 - 1)}\) \(\pmod{(g^k - k)}\) for all \(0 \leq j \leq n\), then for every \(\alpha \geq 1\) such that \(\text{gcd}(a_0, k^2 - 1)\) divides \(\alpha\frac{(g-k)r_j + (kg-1)r_j}{k^2 - 1}\) for all \(0 \leq j \leq n\), an asymmetric \(n + 2\)-digit \((\hat{g}, \hat{k})\)-reverse multiple exists with carries \((\hat{r}_n, \hat{r}_{n-1}, \ldots, \hat{r}_0)\) given by \((a_n, a_{n-1}, \ldots, a_0, 0)\) where \(\hat{g} = s + \alpha\frac{k^2 - 1}{\text{gcd}(a_0, k^2 - 1)}\).
By induction we must then have
\[(39x + 24) \leq 500 \] and in general
\[\hat{a}_j = \alpha(ga_{n-j+2} + a_j) - (kb_j + b_{j-1})\]

As argued previously, \(0 \leq \hat{a}_j < \hat{g}\) since each \(a_j < g = \hat{k}\). We therefore have the following.
Proof. The arguments leading up to Theorem 4.4 give us that a new 4-digit reverse multiple with carries \(r_n, r_{n-1}, \ldots, r_0\) and let \(D = \gcd(a_0, g^2 - 1)\). If \(k^2 - 1\) divides \(Dj\) with quotient \(b_j\) for all \(0 \leq j \leq n\), then for every \(\alpha \geq 1\) such that \(D\) divides \(\alpha(ga_{n-j+2} + a_j) - (kb_j + b_{j-1})\) for all \(0 \leq j \leq n\), a \(n + 3\)-digit asymmetric \((\hat{g}, g)\)-reverse multiple exists with carries \((\hat{r}_n+2, \hat{r}_{n+1}, \ldots, \hat{r}_0)\) given by \((0, a_n, a_{n-1}, a_0, 0)\) where \(\hat{g} = a \frac{g^{2-1}}{D}\).

The case \(n = 1\) gives us another condition which guarantees the existence of asymmetric reverse multiples.

Corollary 4.5. Suppose \((a_1, a_0)\) is a \((g, k)\)-reverse multiple with one non-zero carry \(r\) and \(D = \gcd(a_0, g^2 - 1)\). If \(k^2 - 1\) divides \(Dc\) with quotient \(b\) and \(\gcd(a_1, D)\) divides \(b\), then there exists an asymmetric \((\hat{g}, g)\)-reverse multiple where \(\hat{g} = \alpha \frac{g^{2-1}}{D}\).

Example 4.6. A family of Pudwell reverse multiples may be constructed from the \((55, 6)\)-reverse multiple \((47, 7)_{55}\) with carries \((r, 0) = (5, 0)\). The conditions of Corollary 4.5 are satisfied and we have that \((\hat{a}_3, \hat{a}_2, \hat{a}_1, \hat{a}_0)\) given by (55\(\alpha\), 2585\(\alpha^{-1}\), 47\(\alpha^{-1}\), 232\(\alpha^{-1}\)) is an \((432\alpha, 55)\)-reverse multiple with carries \((\hat{r}_3, \hat{r}_2, \hat{r}_1, \hat{r}_0) = (0, 47, 7, 0)\) where \(\alpha\) is any natural number congruent to 4 modulo 7.

5. Reverse Multiples Derived from Reversals of Reverse Multiples

Digit reversals of reverse multiples also appear in the carries of higher-base reverse multiples. Therefore, we now construct asymmetric reverse multiples from digit-reversals of reverse multiples. We will not present the amount of detail as in the previous section as the arguments are essentially the same for each case. However, we will highlight points which deserve additional explanation. We shall consider both \(n + 2\)-digit \((\hat{g}, \hat{k})\)-reverse multiples with carries \((\hat{r}_{n+1}, \hat{r}_{n}, \ldots, \hat{r}_0)\) of the form \((a_0, a_1, \ldots, a_n, 0)\) and \(n + 3\)-digit \((\hat{g}, \hat{k})\)-reverse multiples with carries \((\hat{r}_{n+2}, \hat{r}_{n+1}, \ldots, \hat{r}_0)\) of the form \((0, a_0, a_1, \ldots, a_n, 0)\). Such reverse multiples will be called singly-\(p\)-derived and doubly-\(p\)-derived.

5.1. Reverse Multiples Derived from Reversals of 1089 Reverse Multiples. We shall now consider families of singly-\(p\)-derived reverse multiples constructed from 1089 reverse multiples. These shall be called reflected-Hoey reverse multiples. In a manner similar to the arguments leading to Equation 2.3 it must be that \(\hat{g}a_n \equiv ka_0 \mod (k^2 - 1)\), or \(\hat{g}kb \equiv kb \mod (k^2 - 1)\). Thus, \(\gcd(kb, k^2 - 1)\) must divide \(b\) so that \(k\) and \(k^2 - 1\) must be relatively prime. We then have that \(\hat{g} = m\hat{k} + \alpha \frac{k^2 - 1}{\gcd(b, k^2 - 1)}\) where \(m\) is the multiplicative inverse of \(k\) modulo \(k^2 - 1\). Re-parameterizing, we let \(\ell\) be the least non-negative residue of \(mk\) modulo \(k^2 - 1\) so that \(\hat{g} = \ell + \alpha \frac{k^2 - 1}{\gcd(b, k^2 - 1)}\) for \(\alpha \geq 1\). Then

\[
\hat{a}_j = \frac{(\ell k - \hat{k})br_{j+1} + (\hat{k} - \ell - k\hat{k} + \hat{k}b)r_j + (1 - \ell k - \hat{k}kb + \hat{k}b)r_{j-1} + (\ell \hat{k}k - 1)br_{j-2}}{k^2 - 1} + \alpha \frac{kbr_{j+1} + (\hat{k}b - 1)r_j + (b - \hat{k})r_{j-1} + \hat{k}kr_{j-2}}{\gcd(b, k^2 - 1)}.
\]
Since \( \ell k - \hat{k} \equiv \ell \hat{k} - 1 \equiv 0 \mod (k^2 - 1) \), in order to ensure that the above is an integer we shall require that \( \hat{k} = \ell - k b + \ell k b \equiv 1 - \ell k - k b + \ell b \equiv 0 \mod (k^2 - 1) \) which is equivalent to \( (k - 1) \hat{k} \equiv (k^2 - 1) b \mod (k^2 - 1) \). Therefore, as before, \( \hat{k} = g = b(k + 1) \) so that \( \gcd(b, k^2 - 1) = \gcd(b, g^2 - 1) = 1 \). Thus,
\[
\hat{a}_j = \frac{(\ell k - g) b r_{j+1} + (g - \ell - k b + \ell b) r_j + (1 - g \ell - g k b + \ell b) r_{j-1} + (\ell g - 1) b r_{j-2}}{g^2 - 1} + \alpha(k b r_{j+1} + (g b - 1) r_j + (b - g) r_{j-1} + g k b r_{j-2})
\]

The above gives us the following compliment to Theorem 3.1.

**Theorem 5.1.** Suppose \( N = (a_n, a_{n-1}, \ldots, a_0)_g \) \((g, k)\)-reverse multiple such that \( Y(g, k) \simeq Y(10, 9) \) where \( g^2 - 1 \) and \( k \) are relatively prime and \( m \) is the the multiplicative inverse of \( k \) modulo \( g^2 - 1 \). Furthermore, let \( \ell \) be the least non-negative residue of \( m g \) modulo \( g^2 - 1 \). Then for every \( \hat{g} > \ell \) such that \( \hat{g} \equiv \ell \mod (g^2 - 1) \) there exists an asymmetric \( n + 2 \)-digit \((\hat{g}, g)\)-reverse multiple with carries \( (\hat{r}_{n+1}, \hat{r}_n, \ldots, \hat{r}_0) \) given by \((a_0, a_1, \ldots, a_n, 0)\).

**Example 5.2.** Applying the arguments for Theorem 5.1 to the well known \((10, 4)\)-reverse multiple \((8, 7, 1, 2)_{10}\) with carries \((r_3, r_2, r_1, r_0) = (0, 3, 3, 0)\), we have \( \ell = 52 \) which gives rise to the family of \((52 + 99 \alpha, 10)\)-reverse multiples with digits \((a_4, a_3, a_2, a_1, a_0)\) given by \((42 + 80 \alpha, 37 + 72 \alpha, 5 + 11 \alpha, 14 + 27 \alpha, 4 + 8 \alpha)_{52 + 99 \alpha}\) all with carries \((\hat{r}_4, \hat{r}_3, \hat{r}_2, \hat{r}_1, \hat{r}_0) = (2, 1, 7, 8, 0)\) for all \( \alpha \geq 1 \).

**Remark 5.3.** Although it is not always the case, the example above also yields a reverse multiple for \( \alpha = 0 \).

As shown by an argument nearly identical to that of Theorem 3.3 no doubly-\( \rho \)-derived reverse multiples can be constructed from 1089 reverse multiples.

### 5.2. Reverse Multiples Derived from Reversals of Complete Reverse Multiples

Singly-\( \rho \)-derived reverse multiples constructed from complete reverse multiples (called reflected-Sutcliffe reverse multiples) yield an argument and theorem statement nearly identical to that of Theorem 5.1 with the exception that the roles of \( g - k \) and \( kg - 1 \) as well as \( a_0 \) and \( a_n \) are interchanged.

**Theorem 5.4.** Suppose \((a_n, a_{n-1}, \ldots, a_0)_g\) is a complete \((g, k)\)-reverse multiple with carries \((r_n, r_{n-1}, \ldots, r_0)\). If there exists a natural number \( \hat{k} \) such that \( s = \frac{k(g-k)}{kg-1} \) is an integer, \( \hat{k} > a_j \), and \( s \frac{(g-k)r_j}{k^2-1} \equiv \frac{(kg-1)r_j}{k^2-1} \mod (\hat{k} - 1) \) for all \( 0 \leq j \leq n \), then for every \( \alpha \geq 1 \) such that \( \gcd(a_n, \hat{k}^2 - 1) \) divides \( \alpha(kg-1)(r_{i+1} + kr_{i-1} + (k+1)(g-k)r_i) \) for all \( 0 \leq j \leq n \), an asymmetric \( n + 2 \)-digit \((\hat{g}, k\hat{k})\)-reverse multiple exists with carries \((\hat{r}_{n+1}, \hat{r}_n, \ldots, \hat{r}_0)\) given by \((a_0, a_1, \ldots, a_n, 0)\) where \( \hat{g} = s + \alpha \frac{\hat{k}^2-1}{\gcd(a_n, \hat{k}^2-1)} \).

**Corollary 5.5.** If \((a_1, a_0)_g\) is a \((g, k)\)-reverse multiple and there is an \( \hat{k} > \max\{a_0, a_1\} \) such that \( s = \frac{k(g-k)}{kg-1} \) is an integer and \( sa_0 \equiv a_1 \mod (\hat{k} - 1) \), then asymmetric \((\hat{g}, \hat{k})\)-reverse multiples exist where \( \hat{g} = s + \alpha \frac{\hat{k}^2-1}{\gcd(a_1, \hat{k}^2-1)} \) for any \( \alpha \geq 1 \).

**Example 5.6.** Corollary 5.3 applies to the \((5, 2)\)-reverse multiple \((3, 1)_5\) with one nontrivial carry \( c = 1 \) with \( \hat{k} = 9 \) satisfying its hypotheses and giving us the family of \((3 + 80 \alpha, 9)\)-reverse multiples \((1 + 27 \alpha, 10 \alpha, 3 \alpha)_{3 + 80 \alpha}\) for any \( \alpha \geq 1 \) each with carries \((\hat{r}_2, \hat{r}_1, \hat{r}_0) = (1, 3, 0)\).
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Considering doubly-$ρ$-derived reverse multiples constructed from complete reverse multiples (reflected-Pudwell), we have

**Theorem 5.7.** Suppose $N = (a_n, a_{n-1}, \ldots, a_0)_g$ is a complete $(g, k)$-reverse multiple with carries $r_n, r_{n-1}, \ldots, r_0$ and let $D = \gcd(a_n, g^2 - 1)$. If $k^2 - 1$ divides $Dr_j$ with quotient $b_j$ for all $0 \leq j \leq n$, then for every $α ≥ 1$ such that $D$ divides $α(a_{g-2} + a_{n-1}) - (b_j + kb_{j-1})$ for all $0 \leq j \leq n$, a $n + 3$-digit asymmetric $(g, g)$-reverse multiple exists with carries $(\hat{r}_{n+2}, \hat{r}_{n+1}, \ldots, \hat{r}_0)$ given by $(0, a_0, a_1, \ldots, a_n, 0)$ where $\hat{g} = αg^2 - 1$.

**Corollary 5.8.** Suppose $(a_1, a_0)_g$ is a $(g, k)$-reverse multiple with one non-zero carry $r$ and $D = \gcd(a_1, g^2 - 1)$. If $k^2 - 1$ divides $Dc$ with quotient $b$ and $\gcd(a_0, D)$ divides $b$, then there exists an asymmetric $(\hat{g}, g)$-reverse multiple where $\hat{g} = αg^2 - 1$.

**Example 5.9.** We again look at the $(5, 2)$-reverse multiple $(3, 1)_{g}$ with carry $r = 1$. The conditions of Corollary 5.8 are satisfied and we get the $(8α, 5)$-reverse multiple $(5α, \frac{5α - 2}{3}, \frac{α - 1}{3}, α)_{8α}$ with carries $(\hat{r}_3, \hat{r}_2, r_1, r_0) = (0, 1, 3, 0)$ where $α \equiv 1 \mod 3$.

6. Palinomials and Derived Reverse Multiples

We state a definition from [3]: the $(g, k)$-palinomial induced by a $(g, k)$-reverse multiple $(a_n, \cdots, a_0)_g$ is the polynomial

$$\text{Pal}(x) = \sum_{j=0}^{n} (a_j - kα_{n-j})x^j.$$ 

We note that the term “palinomial” comes from the alternative terminology for reverse multiples: “palintiples.”

**Theorem 6.1.** Palinomials induced by 1089 reverse multiples have at least one root on the unit circle.

**Proof.** By [3] $\text{Pal}(x) = (x - g)\sum_{j=1}^{n} r_j x^{j-1}$, and since $Y(g, k) \succ Y(10, 9)$, we have by [3] that $k + 1$ divides $g$. Hence, by arguments in [3] [4], $r_j = r_{n-j}$ is either 0 or $k - 1$. Results in [6] then establish the claim.

The next theorem reveals an even closer connection between the digits of 1089 and complete reverse multiples and the roots of their palinomials.

**Theorem 6.2.** Let $ξ \neq g$ be a non-zero root of the palinomial induced by a 1089 or complete reverse multiple $N = (a_n, a_{n-1}, \cdots a_0)_g$. Then $ξ$ is a root of both the digit and reverse-digit polynomials. That is,

$$\sum_{j=0}^{n} a_j ξ^j = \sum_{j=0}^{n} a_{n-j} ξ^j = 0.$$ 

**Proof.** By the theorem hypothesis, $\sum_{j=1}^{n} r_j x^{j-1}$ is a palindromic polynomial (by [3] $r_j = r_{n-j+1}$ for all complete reverse multiples). Therefore, $\text{Pal}(ξ) = \text{Pal}(\frac{1}{ξ}) = 0$. It follows that, $\sum_{j=0}^{n} a_j ξ^j = k\sum_{j=0}^{n} a_{n-j} ξ^j$ and $\sum_{j=0}^{n} a_j ξ^{-j} = k\sum_{j=0}^{n} a_{n-j} ξ^{-j}$. Multiplying the second equation by $kξ^α$ and re-indexing the sum we have $k\sum_{j=0}^{n} a_{n-j} ξ^j = k^2\sum_{j=0}^{n} a_j ξ^j$ from which the result follows. The reverse-digit case follows in the same manner. 

**Corollary 6.3.** Digit and reverse-digit polynomials of a 1089 reverse multiple have at least one root on the unit circle.
6.1. Additional Roots of Digit Polynomials of 1089 and Complete Reverse Multiples. Since every negative or purely complex root of a palinomial induced by a 1089 reverse multiple is also a root of the digit polynomial, the digit and reverse-digit polynomial of a 1089 reverse multiple has two roots which differ from its corresponding palinomial.

**Theorem 6.4.** Let $\text{Pal}(x)$ be the palinomial induced by a 1089 $(g,k)$-reverse multiple $N = (a_n, a_{n-1}, \cdots, a_0)_g$ and let $D$ and $\overline{D}$ respectively denote the digit and reverse-digit polynomials. Then

$$D(x) = (a_nx^2 - x + a_0) \frac{\text{Pal}(x)}{(k-1)(x-g)} \quad \text{and} \quad \overline{D}(x) = (a_0x^2 - x + a_n) \frac{\text{Pal}(x)}{(k-1)(x-g)}.$$  

**Proof.** Suppose $\text{Pal}(x) = (k-1)(x-g)\prod_{j=1}^{n-2}(x-\xi_j)$ is a palinomial induced by a symmetric $(g,k)$-reverse multiple. By Theorem 6.2 we may express these as $D(x) = a_n(x - \omega_1)(x - \omega_2)\prod_{j=1}^{n-2}(x - \xi_j)$ and $\overline{D}(x) = a_0(x - \frac{1}{\omega_1})(x - \frac{1}{\omega_2})\prod_{j=1}^{n-2}(x - \xi_j)$ where $\omega_1$ and $\omega_2$ are the two extra roots. Since $x = 1$ cannot be a root of any palinomial [3] and is clearly not a root of $D$ or $\overline{D}$, we have $a_0(1 - \frac{1}{\omega_1})(1 - \frac{1}{\omega_2}) = a_n(1 - \omega_1)(1 - \omega_2)$, or $\omega_1\omega_2 = \frac{1}{a_n}$. Now, $D(g) = nD(\xi)\omega_1\omega_2$ implies $(g - \omega_1)(g - \omega_2) = (g - \frac{1}{\omega_1})(g - \frac{1}{\omega_2})$ from which we have $\omega_1 + \omega_2 = \frac{i}{k\xi}$. Thus, $\omega_1$ and $\omega_2$ are the conjugate pair $\frac{1}{2ik}(1 \pm i\sqrt{4k^2g^2 - 1})$. The digit and reverse-digit polynomials may then be expressed as $D(x) = (a_nx^2 - x + a_0)\prod_{j=1}^{n-2}(x - \xi_j)$ and $\overline{D}(x) = (a_0x^2 - x + a_n)\prod_{j=1}^{n-2}(x - \xi_j)$. □

Digit and reverse-digit polynomials of complete $(g,k)$-reverse multiples have one more root than their corresponding palinomials.

**Theorem 6.5.** Let $\text{Pal}(x)$ be the palinomial induced by a complete $(g,k)$-reverse multiple $N = (a_n, a_{n-1}, \cdots, a_0)_g$ with carries $r_n, r_{n-1}, \cdots, r_1, r_0$ and let $D$ and $\overline{D}$ respectively denote the digit and reverse-digit polynomials. Then

$$D(x) = (a_nx + a_0) \frac{\text{Pal}(x)}{r_n(x-g)} \quad \text{and} \quad \overline{D}(x) = (a_0x + a_n) \frac{\text{Pal}(x)}{r_n(x-g)}.$$

**Proof.** Suppose $\text{Pal}(x) = r_n(x-g)\prod_{j=1}^{n-1}(x - \xi_j)$. Since the digit and reverse-digit polynomials have one root that differs from $\text{Pal}(x)$ we have $D(x) = a_n(x - \omega)\prod_{j=1}^{n-1}(x - \xi_j)$ and $\overline{D}(x) = a_0(x - \frac{1}{\omega})\prod_{j=1}^{n-1}(x - \xi_j)$. $D(1) = \overline{D}(1)$ then implies that $\omega = -\frac{g-k}{k\xi}$ so that $D(x) = (a_nx + a_0)\prod_{j=1}^{n-1}(x - \xi_j)$ and $\overline{D}(x) = (a_0x + a_n)\prod_{j=1}^{n-1}(x - \xi_j)$. □

**Corollary 6.6.** Let $\overline{\text{Pal}}(x)$ be the palinomial induced by a singly-derived or doubly-derived $(\hat{g}, \hat{k})$-reverse multiple $\hat{N}$ constructed from a $(g,k)$-reverse multiple $N = (a_n, a_{n-1}, \cdots, a_0)_g$ and let $\text{Pal}(x)$ be the palinomial induced by $N$. Then

$$\overline{\text{Pal}}(x) = (x - \hat{g})(a_nx^2 - x + a_0) \frac{\text{Pal}(x)}{(k-1)(x-g)}$$

if $N$ is a 1089 reverse multiple, and

$$\overline{\text{Pal}}(x) = (x - \hat{g})(a_nx + a_0) \frac{\text{Pal}(x)}{r_n(x-g)}$$

if $N$ is complete where $r_n$ is the $n$th carry of $N$.

**Proof.** Since $\hat{N}$ is singly-derived, its carries are $a_n, a_{n-1}, \cdots, a_0, 0$ so that by [3] we have $\overline{\text{Pal}}(x) = (x - \hat{g})\sum_{j=1}^{n+1} \hat{r}_jx^{j-1} = (x - \hat{g})\sum_{j=1}^{n+1} a_{j-1}x^{j-1} = (x - \hat{g})D(x)$. The doubly-derived case follows in a similar fashion. □
Corollary 6.7. Let $\overline{\text{Pal}}(x)$ be the palinomial induced by a singly-$\rho$-derived or doubly-$\rho$-derived $(\hat{g}, \hat{k})$-reverse multiple $\hat{N}$ constructed from a $(g, k)$-reverse multiple $N = (a_n, a_{n-1}, \ldots, a_0)_g$ and let $\text{Pal}(x)$ be the palinomial induced by $N$. Then

$$\overline{\text{Pal}}(x) = (x - \hat{g})(a_0 x^2 - x + a_n) \frac{\text{Pal}(x)}{(k - 1)(x - g)}$$

if $N$ is a 1089 reverse multiple, and

$$\overline{\text{Pal}}(x) = (x - \hat{g})(a_0 x + a_n) \frac{\text{Pal}(x)}{r_n(x - g)}$$

if $N$ is complete where $r_n$ is the $n$th carry of $N$.

Corollary 6.8. Palinomials induced by Hoey reverse multiples and reflected-Hoey reverse multiples have at least one root on the unit circle.

Corollary 6.9. Palinomials induced by any two singly-derived reverse multiples constructed from a common reverse multiple which is either 1089 or complete, differ by only a linear factor.

The statement of Corollary 6.9 also holds for doubly-derived, singly-$\rho$-derived, and doubly-$\rho$-derived reverse multiples.

Example 6.10. The 7-digit, symmetric $(10, 4)$-reverse multiple $N = (8, 7, 9, 9, 9, 1, 2)_{10}$ induces the palinomial $\text{Pal}(x) = 3(x - 10)(x^4 + x^3 + x^2 + x + 1)$. The reader may also verify that, $D(x) = (8x^2 - x + 2)(x^4 + x^3 + x^2 + x + 1)$ and $\overline{D}(x) = (2x^2 - x + 8)(x^4 + x^3 + x^2 + x + 1)$. Moreover, constructing a new 8-digit reverse multiple from $N$ using Theorem 5.1 and its supporting arguments, we take the $(139, 10)$-reverse multiple $\hat{N} = (28, 25, 136, 138, 138, 110, 113, 2)_{139}$ as an example. The reader may verify that the palinomial induced by $\hat{N}$ can be expressed as

$$\overline{\text{Pal}}(x) = (x - 139)(8x^2 - x + 2)(x^4 + x^3 + x^2 + x + 1).$$

7. Open Questions and Future Work

Finding broader conditions for the existence of reverse multiples belonging to the families presented in this paper remains an open problem. It is still unknown if Theorems 3.1 and 5.1 and their arguments respectively give us all Hoey and reflected-Hoey reverse multiples. So far this seems to be the case. On the other hand, however, Sutcliffe and reflected-Sutcliffe reverse multiples exist under conditions for which Theorems 4.1 and 5.4 and their corollaries do not apply. We give as an example the $(129, 14)$-reverse multiple $(37, 89, 2)_{129}$ with carries $(9, 4, 0)$ derived from the $(14, 2)$-reverse multiple $(9, 4)_{14}$; for this particular case, $s = \frac{k(kg - 1)}{g-k}$ is not an integer.

We have already stated that it is unknown if there are Pudwell reverse multiples such that $\hat{k} \neq g$. However, we must point out that reflected-Pudwell reverse multiples do exist for values of $\hat{k}$ other than $g$. As an example we present the $(55, 34)$-reverse multiple $(34, 1, 0, 1)_{55}$ with carries $(0, 1, 21, 0)$ which is derived from the $(23, 11)$-reverse multiple $(21, 1)_{23}$. Moreover, for whatever reason, reflected-Pudwell reverse multiples, speaking in terms of lower bases, seem to occur much more frequently than their forward counterparts. Both forward and reflected-Pudwell reverse multiples so far have proven to be the least well understood.

Also it is still unknown if singly-derived or doubly-derived reverse multiples can be constructed from other asymmetric reverse multiples (neither 1089 nor complete). So far none have been found.
Another question relates to the patterns found in the carries of 1089 reverse multiples. Kendrick [4] showed that $Y(g, k) \simeq Y(10, 9)$ is equivalent $g$ being divisible by $k + 1$ and [3] showed the latter implies $r_j = r_{n-j}$ for all $0 \leq j \leq n$. We then ask if the following are equivalent for a $(g, k)$-reverse multiple $N$:

1. $N$ is symmetric (that is, $r_j = r_{n-j}$ for all $0 \leq j \leq n$)
2. $N$ is 1089
3. $r_j \equiv 0 \mod (k - 1)$
4. $k + 1$ divides $g$

If $N$ is 1089, the work of Kendrick [4] shows that any node of has the form $[0, 0], [0, k-1], [k-1, 0]$, or $[k-1, k-1]$ which establishes $2 \implies 3$. $3 \implies 4$ is easily established since by Equation 2.1, $a_0 = \frac{qr_1}{k-1}$ and $a_0 \neq 0$. $4 \implies 1$ is demonstrated in [3]. We leave whether or not $4 \implies 1$ holds as an open question (posed by [3]).

An obvious question is whether or not all asymmetric reverse multiples can be constructed in this way. Given the variety of Young graph isomorphism classes [4], the answer to this question is, not surprisingly, no. If we consider the example of the $(23, 4)$-reverse multiple $(6, 15, 1)$, with carries $(r_2, r_1, r_0) = (2, 1, 0)$, it is not difficult to show that no 2-digit reverse multiple has these carries as digits. We might then ask if there is a more general principle at work here; perhaps the carries are not the digits of a reverse multiple, but rather the digits of a permutiple. Indeed, one sees that for the above case that $(2, 1, 0)_4 = 2 \cdot (1, 0, 2)_4$. While such examples are promising, one can verify that for the $(17, 11)$-reverse multiple $(14, 12, 5, 1)$, there is no permutation, base, or multiplier for which the carries $(r_3, r_2, r_1, r_0) = (3, 8, 9, 0)$ are a non-trivial permutiple. However, we do point out that there do seem to be strong connections, and naturally so, between reverse multiples and the more general permutiple problem. Thus, in addition to being quite rich and worthy of study in its own right, a more developed understanding of permutiples may very well provide a much better understanding of reverse multiples.

7.1. Young Graph Isomorphism Classes of Derived Reverse Multiples. As we have seen, the carries of a reverse multiple can themselves be the digits of lower-base reverse multiple. If we elevate our perspective to entire reverse multiple families (e.g. Hoey reverse multiples derived from 1089 reverse multiples), an abundance of questions present themselves:

- Is it possible to construct a new Young graph using the old edges as nodes?
- Can we construct new Young graph isomorphism classes from old? If so, how?
- How do the families considered here “interact” with Young graph isomorphism classes?

Although we will not provide any complete answers to these questions, we will explore some suggestive examples which, as we shall see, give rise to other questions.

Considering $(14, 3)$-reverse multiples, their nontrivial carries are $(3, 2)$-reverse multiple digits and every $(3, 2)$-reverse multiple is the nontrivial carry sequence of some $(14, 3)$-reverse multiple. In other words the Young graph describing $(14, 3)$-reverse multiple structure can be “derived” from the Young graph describing $(3, 2)$-reverse multiple structure. The figure below compares $Y(3, 2)$ and $Y(14, 3)$ where the “digit-edges” of the former become the “carry-nodes” of the latter. (We note that our Young graph representation reverses the order of the digit
pairs associated with the edges since the formulation of reverse multiples used in this article involves finding the number that is obtained after multiplying by \( k \).

We point out that the kind of correspondence which exists between \((3, 2)\) and \((14, 3)\)-reverse multiples does not always exist. In particular, a \((\hat{g}, \hat{k})\)-reverse multiple constructed from a \((g, k)\)-reverse multiple does not always guarantee that the carries of any \((\hat{g}, \hat{k})\)-reverse multiple will also be a \((g, k)\)-reverse multiple. For instance, the \((107, 9)\)-reverse multiple \((12, 40, 1)_{107}\) has carries \((3, 1, 0)\) whose nontrivial elements are the digits of the \((5, 2)\)-reverse multiple \((3, 1)_5\) as seen in an earlier example. However, the \((107, 9)\)-reverse multiple \((24, 80, 2)_{107}\) has carries \((6, 2, 0)\) which are not the digits of a \((5, 2)\)-reverse multiple.

On the other hand the family of \((39, 5)\)-reverse multiples can be constructed from \((5, 2)\)-reverse multiples. Consider the \((39, 5)\)-reverse multiple \((8, 29, 1)_{39}\) with carries \((3, 1, 0)\) whose nontrivial elements are again the digits of the \((5, 2)\)-reverse multiple \((3, 1)_5\). The nontrivial carries of any \((39, 5)\)-reverse multiple are the digits of a \((5, 2)\)-reverse multiple and every \((5, 2)\)-reverse multiple is a nontrivial carry sequence of a \((39, 5)\)-reverse multiple.

This is all to say that, in general, the correspondence between derived reverse multiples and their reverse multiple carries can break down when \( \hat{k} \neq g \). We therefore pose the question:

*Suppose is a \((\hat{g}, \hat{k})\)-reverse multiple constructed from a \((g, k)\)-reverse multiple. Under what conditions is it guaranteed that the carries of any \((\hat{g}, \hat{k})\)-reverse multiple will also be a \((g, k)\)-reverse multiple? Is \( \hat{k} = g \) such a condition?*

Considering Hoey reverse multiples, it appears that not only \( Y(14, 3) \), but also \( Y(22, 3) \), and in general \( Y(6 + 8\alpha, 3) \) (see the example in Section 3.2) for all \( \alpha \geq 1 \) can be constructed from \( Y(3, 2) \). Moreover, it appears, using Kendrick’s data [5], that every \( Y(6 + 8\alpha, 3) \) is isomorphic to \( Y(14, 3) \). In fact, not surprisingly, for every collection of symmetric \((g, k)\)-reverse multiples we have checked, the Young graph of its corresponding Hoey \((\hat{g}, g)\)-reverse multiples is isomorphic to \( Y(14, 3) \). In this way, the isomorphism class determined by the 1089-graph \([Y(9, 10)]\) in some sense “generates” \([Y(14, 3)]\).

We note that not every element of \([Y(14, 3)]\) is the Young graph of Hoey reverse multiples as Young graphs of reflected-Hoey reverse multiples also seem to be isomorphic to \( Y(14, 3) \). Furthermore, \([Y(14, 3)]\) contains elements which are not Young graphs of Hoey or reflected-Hoey reverse multiples. The \((14, 9)\)-reverse multiple \((11, 9, 1, 4, 1)_9\) with carries \((2, 1, 6, 7, 0)\)
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demonstrates this. These observations lead us to ask:

Are Young graphs of Hoey and reflected-Hoey reverse multiples always isomorphic to \( Y(14, 3) \)? Do elements of \([Y(14, 3)]\) for which the carries are not reverse multiple digits have any special properties?

Young graphs of Sutcliffe and reflected-Sutcliffe reverse multiples constructed from \( K_2, K_3, \) and \( K_4 \) complete reverse multiples all appear to be isomorphic to \( Y(7, 11) \). Of course, considering larger values of \( m \) and checking more cases may very well reveal other isomorphism classes. We therefore ask the following:

Are Young graphs of Sutcliffe and reflected-Sutcliffe reverse multiples always isomorphic to \( Y(11, 7) \)?

Additionally, for all cases we have checked, Young graphs of Pudwell and reflected-Pudwell reverse multiples derived from \( K_2, K_3, \) and \( K_4 \) complete reverse multiples, all seem to be isomorphic to \( Y(5, 8) \). Thus:

Are Young graphs of Pudwell and reflected-Pudwell reverse multiples always isomorphic to \( Y(8, 5) \)?

It is not entirely unexpected that Young graphs of Hoey, Sutcliffe, and Pudwell reverse multiples should be isomorphic to Young graphs of their respective reflected-derived counterparts. On the other hand, it is not entirely obvious that this should always hold. In all cases considered so far it seems to be true.

Are Young graphs of derived reverse multiples always isomorphic to their reflected-derived counterparts?

Finally, Young graphs of \((\hat{g}, \hat{k})\)-reverse multiples derived from \((g,k)\)-reverse multiples for which \( \hat{k} \neq g \) leave cases which have hardly yet been explored. We leave the reader to ponder the example of \((107, 9)\)-reverse multiples considered earlier whose Young graph is isomorphic to \( Y(59, 25) \). These reverse multiples are in some sense “partially” derived from \((5,2)\)-reverse multiples. We suspect that these nodes might make up a subgraph \( G \) which is isomorphic to \( Y(11, 7) \). The reader is likely to have noticed that other carries of \((9, 107)\)-reverse multiples are sometimes doubles of \((5,2)\)-reverse multiples. Thus, \( Y(107, 9) \) might contain another subgraph \( G' \) which is isomorphic to \( Y(11, 7) \) but with nodes double those of \( G \). Additional structure which may exist between these possible subgraphs is a matter of further inquiry and we leave these and other such questions to the inquisitive reader.

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It is worth noting that although \((2,1,6,7)_g \) is not a reverse multiple in any base \( g \), it is a base-9 permultiple: \((6,7,2,1)_9 = 4 \cdot (1,6,2,7)_9 \) and \((7,2,1,6)_9 = 4 \cdot (1,7,2,6)_9 \).
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Department of Mathematics, Humboldt State University, Arcata, California, 95521, U.S.A.
E-mail address: bvh6@humboldt.edu