THE DIMENSION OF AFFINE DELIGNE-LUSZTIG VARIETIES IN THE AFFINE GRASSMANNIAN OF UNRAMIFIED GROUPS

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1. Introduction

Let $k$ be a finite field of characteristic $p$ and let $\overline{k}$ be an algebraic closure of $k$. We consider a connected reductive group $G$ over $k$. By a theorem of Steinberg, $G$ is quasi-split. Let $k'$ be a finite subfield of $\overline{k}$ such that $G_{k'}$ is split. We fix $S \subset T \subset B \subset G$, where $S$ is a maximal split torus, $T$ a maximal torus which splits over $k'$ and $B$ a Borel subgroup of $G$. We denote $K = G(\overline{k}[t])$ and by $\mathcal{G}$ the affine Grassmannian of $G$.

Denote by $F = k((t))$, $E = k'((t))$ and $L = \overline{F}(t))$ the Laurent series fields. We identify the Galois groups $\text{Gal}(k'/k) = \text{Gal}(E/F) =: I$. Let $\sigma$ denote the Frobenius element of $\text{Gal}(\overline{k}/k)$ and also of $\text{Aut}_F(L)$.

For $\mu \in X_*(T)_{\text{dom}}$ and $b \in G(L)$ the affine Deligne-Lusztig variety is the locally closed subset

$$X_\mu(b)(\overline{F}) = \{ g \cdot K \in \mathcal{G}(\overline{F}) ; g^{-1}b\sigma(g) \in K \mu(t)K \}.$$

We equip $X_\mu(b)$ with reduced structure, making it a variety.

It is an important fact that the isomorphism class of $X_\mu(b)$ does not depend on $b$ itself but only on its $\sigma$-conjugacy class in $G(L)$. Indeed, if $b' = h^{-1}b\sigma(h)$ then $g \cdot K \rightarrow h^{-1}g \cdot K$ induces an isomorphism $X_\mu(b) \cong X_\mu(b')$.

Denote by $\pi_1(G)$ the fundamental group of $G$, i.e. the quotient of $X_*(T)$ by the coroot lattice. By a result of Kottwitz ([Kot85]), a $\sigma$-conjugacy class inside $G(L)$ is uniquely given by two invariants $\nu \in X_*(S)_{\text{dom}}$ and $\kappa \in \pi_1(G)_I$, which are called the Newton point and the Kottwitz point of the $\sigma$-conjugacy class. We will also speak of the Newton point and the Kottwitz point of an element $b \in G(L)$ meaning the invariant associated to the $\sigma$-conjugacy class of $b$.

We denote by $J_b$ the algebraic group whose $R$-valued points for any $F$-algebra $R$ are given by

$$J_b(R) = \{ g \in G(R \otimes_F L) ; g^{-1}b\sigma(g) = b \},$$

which is an inner form of the centralizer of the Newton point of $b$ in $G_F$ ([Kot85], §5.2). Then $J_b(F)$ acts on $X_\mu(b)$ by multiplication on the left. We define the defect of $b$ to be the integer $\text{def}_G(b) := \text{rk}_FG - \text{rk}_FJ_b$.

The aim of this paper is the following theorem:

**Theorem 1.1.** Assume that $X_\mu(b)$ is nonempty. Let $\nu \in X_*(S)_{\text{dom}}$ be the Newton point of $b$. Then

$$\dim X_\mu(b) = \langle \rho, \mu - \nu \rangle - \text{def}_G(b),$$

where $\rho$ denotes the half-sum of all (absolute) roots of $G$.

It is known that $X_\mu(b)$ is non-empty if and only if the Mazur inequality holds. Many authors have worked on this conjecture, the result for unramified groups was proven by Kottwitz and Gashi ([Kot03], §4.3; [Gas10] Thm. 5.2).

The assertion of Theorem 1.1 is already known in the case where $G$ is split. It is proven in the papers of Götz, Haines, Kottwitz and Reuman [GHKR06] and Viehmann [Vie06]. In [GHKR06]...
the assertion is reduced to the case where $G = \text{GL}_h$ and $b$ is superbasic, i.e. no $\sigma$-conjugate of $b$ is contained in a proper Levi subgroup of $G$. This case is considered in [Vie06], where the dimension is calculated.

The proof of Theorem 1.1 is a generalization of the proof in the split case. In section 2 we reduce the theorem to the case where $G = \text{Res}_{k'/k} \text{GL}_h$ and $b$ is superbasic. The reduction step is almost literally the same as in [GHKR06], we give an outline of the proof and explain how one has to modify the proof of [GHKR06]. The rest of the paper then focuses on proving the theorem in this special case. For this we generalize the proof of Viehmann in [Vie06]. We decompose the affine Deligne-Lusztig variety using combinatorial invariants called extended EL-charts, which generalize the notion of extended semi-modules considered in [Vie06] for $G = \text{GL}_h$, and calculate the dimension of each part by generalising the computations in the $GL_h$-case. As another application of this decomposition study the $J_b(F)$-action on the irreducible components of $X_\mu(b)$ in the superbasic case and give a conjecture on the number of orbits in the case where $\mu$ in minuscule.

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2. REDUCTION TO THE SUPERBASIC CASE

The aim of this section is to prove the following assertion.

Theorem 2.1. Assume Theorem 1.1 is true for each affine Deligne-Lusztig variety $X_\mu(b)$ with $G \cong \text{Res}_{k'/k} \text{GL}_h$ and $b \in G(L)$ superbasic. Then it is true in general.

As mentioned in the introduction, we follow the proof given in [GHKR06] for split groups. First we have to fix some more notation. Let

- $P = MN$ be a parabolic subgroup of $G$ containing $B$. We denote by $M$ the corresponding Levi subgroup containing $T$ and by $N$ the unipotent radical of $P$.
- $\mathcal{G}_r, \mathcal{G}_r P, \mathcal{G}_r M$ denote the affine Grassmannians of $G, P$ and $M$ respectively.
- $\mathcal{G}_r^{\omega}, \mathcal{G}_r M^{\omega}$ denote the geometric connected component of $\mathcal{G}_r$ resp. $\mathcal{G}_r M$ corresponding to $\omega \in \pi_1(G)$ resp. $\omega \in \pi_1(M)$. (cf. [PR08] Thm. 0.1)
- $x_\lambda$ denote the image of $\lambda(t)$ in $\mathcal{G}_r(k)$ for $\lambda \in \mathcal{X}_s(T)$. For $g \in G(L)$ we write $gx_0$ for the translate of $x_0$ w.r.t. the obvious $G(L)$-action on $\mathcal{G}_r(k)$.
- $X_s(T)_{\text{dom}}$ be the subset of $X_s(T)$ of cocharacters which are dominant w.r.t. $T \subset B \subset G$.
- $X_s(T)_{M-\text{dom}}$ be the subset of $X_s(T)$ of cocharacters which are dominant w.r.t. $T \subset B \cap M \subset M$.
- $R_N$ denote the set of roots of $T$ in $\text{Lie} N$.
- $\rho$ denote the half-sum of all positive roots in $G$.
- $\rho_N$ denote the half-sum of all elements of $R_N$.
- $\rho_M = \rho - \rho_N$ denote the half-sum of all positive roots in $M$.

We consider the $\mathcal{G}_r M(k)$ as subset of $\mathcal{G}_r(k)$ via the obvious embedding. Furthermore the canonical morphisms $P \hookrightarrow G$ and $P \twoheadrightarrow M$ induce morphisms of ind-schemes

$$
\begin{array}{ccc}
\mathcal{G}_r P & \xrightarrow{\pi} & \mathcal{G}_r M \\
\downarrow \iota & & \downarrow \\
\mathcal{G}_r G & \xrightarrow{\iota} & \mathcal{G}_r G
\end{array}
$$
The idea of the proof for Theorem 2.1 is to consider the image of an affine Deligne-Lusztig variety $X_{\mu}(b)$ in $\mathcal{G}_M$ under the above correspondence, assuming that $b \in M(L)$. We want to show that the image is a union of affine Deligne-Lusztig varieties, which we will later assume to be superbasic and relate the dimension of $X_{\mu}(b)$ to the dimension of its image.

Let us study the diagram more thoroughly. Certainly $\pi$ is surjective and $\iota$ is bijective on geometric points by the Iwasawa decomposition of $G$. Thus $\iota$ is a decomposition of $\mathcal{G}$ into locally closed subsets by Lemma 2.2 below. In particular we see that $X_{\mu}^{PG}(b) := \iota^{-1}(X_{\mu}(b))$ is also locally of finite type and has the same dimension as $X_{\mu}(b)$.

**Lemma 2.2.** Let $i : I \hookrightarrow H$ be a closed embedding of connected algebraic groups. Then the induced map on the identity components of the affine Grassmannians $i_G : \mathcal{G}_I^0 \rightarrow \mathcal{G}_H^0$ is an immersion.

**Proof.** First recall the following result in the proof of Thm. 4.5.1 of [BD] (see also [Gör10], Lemma 2.12): In the case where $H/I$ is quasi-affine (resp. affine), the induced morphism $\mathcal{G}_I \rightarrow \mathcal{G}_H$ is an immersion (resp. closed immersion). So we want to replace $I$ by a suitable closed subgroup $I'$ which is small enough such that $H/I'$ is quasi-affine, yet big enough such that the immersion $\mathcal{G}_I^0 \hookrightarrow \mathcal{G}_H^0$ is surjective.

Now let

$$0 \rightarrow R(I)_u \rightarrow I \rightarrow I_1 \rightarrow 0$$

be the decomposition of $I$ into a unipotent and a reductive group. We denote by $I_1^{\text{der}}$ the derived group of $I_1$ and by $R(I_1)$ its radical. As $I_1/I_1^{\text{der}}$ is affine, the canonical morphism $\mathcal{G}_I^{0} \rightarrow \mathcal{G}_{I_1}^{0}$ is a closed immersion. Using that $I_1 = R(I_1) \cdot I_1^{\text{der}}$, we see that it is also surjective.

We denote $I' := I \times_{I_1} I_1^{\text{der}}$. As $\pi_1(I) = \pi_1(I_1)$, the canonical morphism $\mathcal{G}_{I'}^0 \rightarrow \mathcal{G}_{I_1}^0$ is the pullback of $\mathcal{G}_{I_1}^0 \rightarrow \mathcal{G}_I^0$ and hence also a surjective immersion. Furthermore $I'$ has no non-trivial homomorphisms to $\mathbb{G}_m$, hence the quotient $H/I'$ is quasi-affine and $\mathcal{G}_{I'}^0 \rightarrow \mathcal{G}_H^0$ is an immersion. Altogether we have

$$\mathcal{G}_{I'}^0 \xrightarrow{i_{\mathcal{G}}} \mathcal{G}_I^0 \xrightarrow{i_{\mathcal{G}}} \mathcal{G}_H^0,$$

which proves that $i_{\mathcal{G}}$ is an immersion. \qed

In order to determine the dimension of $X_{\mu}^{PG}(b)$, we want to calculate the dimension of its fibres under $\pi$ and its image. For this we need a few auxiliary results. We note that the reasoning below still works if we replace $\mathbb{F}$ by a bigger algebraically closed field.

We fix a dominant, regular, $\sigma$-stable coweight $\lambda_0 \in X_*(T)$. We denote for $m \in \mathbb{Z}$

$$N(m) := \lambda_0(t)^m N(\mathbb{F}(t)) \lambda_0(t)^{-m}$$

Then we have a chain of inclusions $\cdots \supseteq N(-1) \supseteq N(0) \supseteq N(1) \supseteq \cdots$ and moreover $N(L) = \bigcup_{i \in \mathbb{Z}} N(i)$. Furthermore, we note that $N(-m)/N(n)$ has a canonical structure of a variety for $m, n > 0$.

**Definition 2.3.** (1) A subset $Y$ of $N(L)$ is called admissible if there exist $m, n > 0$ such that $Y \subseteq N(-m)$ and it is the preimage of a locally closed subset of $N(-m)/N(n)$ under
the canonical projection \( N(-m) \to N(-m)/N(n) \). For admissible \( Y \subset N(L) \) we define the dimension of \( Y \) by
\[
\dim Y = \dim Y/N(n) - \dim N(0)/N(n).
\]
(2) A subset \( Y \) of \( N(L) \) is called ind-admissible if \( Y \cap N(-m) \) is admissible for every \( m > 0 \). For any ind-admissible \( Y \subset N(L) \) we define
\[
\dim Y = \sup \dim( Y \cap N(-m) ) .
\]

Lemma 2.4. Let \( m \in M(L) \) and \( \nu \in X_*(S) \) be its Newton point. We denote \( f_m : N(L) \to N(L) \), \( n \mapsto n^{-1} m^{-1} \sigma(n) m \). Then for any admissible subset \( Y \) of \( N(L) \) the preimage \( f_m^{-1} Y \) is ind-admissible and
\[
\dim f_m^{-1} Y = \dim Y = \langle \rho, \nu - \nu_{\text{dom}} \rangle.
\]
Moreover, \( f_m \) is surjective.

Proof. This assertion is the analogue of Prop. 5.3.1 in [GHKR06]. Note that \( R_N \) is \( \sigma \)-stable and thus the sets \( N[i] \) defined in the proof of Prop. 5.3.2 in [GHKR06] are \( \sigma \)-stable. \( \square \)

We denote by \( p_M : X_*(T) \to \pi_1(M) \) the canonical projection.

Definition 2.5. (1) For \( \mu \in X_*(T)_{\text{dom}} \) let
\[
S_M(\mu) := \{ \mu_M \in X_*(T)_{M-\text{dom}} ; N(L)x_{\mu_M} \cap Kx_\mu \neq \emptyset \}.
\]
(2) For \( \mu \in X_*(T)_{\text{dom}} \), \( \kappa \in \pi_1(M)_I \) let
\[
S_M(\mu, \kappa) := \{ \mu_M \in S_M(\mu) ; \text{the image of } p_M(\mu_M) \text{ in } \pi_1(M) \text{ is } \kappa \}
\]
(3) For \( \mu \in X_*(T)_{\text{dom}} \) let
\[
\Sigma(\mu) := \{ \mu' \in X_*(T) ; \mu'_{\text{dom}} = \mu \}
\]
\[
\Sigma(\mu)_{M-\text{dom}} := \Sigma(\mu) \cap X_*(T)_{M-\text{dom}}
\]
We denote by \( \Sigma(\mu)_{M-\text{max}} \) the set of maximal elements in \( \Sigma(\mu)_{M-\text{dom}} \) w.r.t. the Bruhat order corresponding to \( M \).

Lemma 2.6. For any \( \mu \in X_*(T)_{\text{dom}} \) we have inclusions
\[
\Sigma(\mu)_{M-\text{max}} \subset S_M(\mu) \subset \Sigma(\mu)_{M-\text{dom}}.
\]
Moreover these sets have the same image in \( \pi_1(M)_I \). In particular, \( S_M(\mu, \kappa) \) is nonempty if and only if \( \kappa \) lies in the image of \( \Sigma(\mu)_{M-\text{dom}} \).

Proof. This is (a slightly weaker version of) Lemma 5.4.1 of [GHKR06] applied to \( G_{\kappa'} \). \( \square \)

Definition 2.7. Let \( \mu \in X_*(T)_{\text{dom}} \) and \( \mu_M \in \Sigma(\mu) \). We write
\[
d(\mu, \mu_M) := \dim(N(L)x_{\mu_M} \cap Kx_\mu).
\]

We can extend the definition above to arbitrary elements of \( \mathcal{G}_M(\mathfrak{g}) \). Multiplication by an element \( k_M \in K_M \) induces an isomorphism \( N(L)x_{\mu_M} \cap Kx_\mu \to N(L)k_Mx_{\mu_M} \cap Kx_\mu \), thus we have for each \( m \in K_M \)
\[
dim(N(L)x_0 \cap Kx_\mu) = d(\mu, \mu_M).
\]

Lemma 2.8. Let \( \mu \in X_*(T)_{\text{dom}} \). Then for all \( \mu_M \in S_M(\mu) \) we have
\[
d(\mu, \mu_M) \leq \langle \rho, \mu + \mu_M \rangle - 2\langle \rho_M, \mu_M \rangle
\]
If \( \mu_M \in \Sigma(\mu)_{M-\text{max}} \) this is an equality.

Proof. This is Cor. 5.4.4 of [GHKR06] applied to \( G_{\kappa'} \). \( \square \)
For $b \in M(L)$, $\mu_M \in X_*(T)_\text{dom}$ we denote by $X^M_{\mu_M}(b)$ the corresponding affine Deligne-Lusztig variety in the affine Grassmannian of $M$. On the contrary $X^M_{\mu_M}(b)$ still denotes the affine Deligne-Lusztig variety in $\mathcal{G}_T$, assuming that $\mu_M \in X_*(T)_\text{dom}.$

**Proposition 2.9.** Let $b \in M(L)$ be basic, i.e. its Newton point is central in $M$. We denote by $\kappa \in \pi_1(M)$ its Kottwitz point and by $v \in X_*(S)_{Q,M-\text{dom}}$ its Newton point.

1. The image of $X^PC_G(b)$ under $\pi$ is contained in
   $$\bigcup_{\mu_M \in S_M(\mu,\kappa)} X^M_{\mu_M}(b).$$
   Denote by $\beta : X^PC_G(b) \to \bigcup_{\mu_M \in S_M(\mu,\kappa)} X^M_{\mu_M}(b)$ the restriction of $\pi$.

2. For $\mu_M \in S_M(\mu,\kappa)$ and every geometric point $x$ of $X^M_{\mu_M}(b)$ the set $\beta^{-1}(x)$ is nonempty and ind-admissible. We have
   $$\dim \beta^{-1}(x) = d(\mu,\mu_M) + \langle \rho, \nu - \nu_{\text{dom}} \rangle - \langle 2\rho_N, \nu \rangle.$$

3. For all $\mu_M \in S_M(\mu,\kappa)$ and all $\omega \in \pi_1(M)$ the set $\beta^{-1}(X^M_{\mu_M}(b))$ is locally closed in $X_\mu(b)$ and
   $$\dim \beta^{-1}(X^M_{\mu_M}(b)) = \dim X^M_{\mu_M}(b) + d(\mu,\mu_M) + \langle \rho, \nu \rangle - \langle \nu_{\text{dom}}, \nu \rangle - \langle 2\rho_N, \nu \rangle.$$

4. If $X_\mu(b)$ is nonempty it has dimension
   $$\sup \{ \dim X^M_{\mu_M}(b) + d(\mu,\mu_M); \mu_M \in S_M(\mu,\kappa) \} + \langle \rho, \nu \rangle.$$}

**Proof.** This is the analogue of [GHKR06], Prop. 5.6.1. The proof of (1)-(3) is the same as in [GHKR06], as this is the centerpiece of this section, we give a sketch of the proof for the readers convenience. Let $x = gx_0 \in X_\mu(b)$. We write $g = mn$ with $m \in M(L)$, $n \in N(L)$. Then

$$(2.1) \quad n^{-1}m^{-1}b\sigma(m)\sigma(n) = g^{-1}b\sigma(g) \in K\mu(t)K.$$}

As $N(L) \subset P(L)$ is a normal subgroup, this implies

$$N(L) \cdot (m^{-1}b\sigma(m)) \cap K\mu(t)K \neq \emptyset.$$}

Thus $m^{-1}b\sigma(m) \in K\mu(t)K$ for a unique $\mu_M \in S_M(\mu)$, i.e. $\alpha(x) \in X^M_{\mu_M}(b)$ proving (1).

Now let $x = mx_0 \in X^M_{\mu_M}(b)$ and $b' = m^{-1}b\sigma(m)$. Then $\beta^{-1}(x)$ is the set of all $mnx_0$ satisfying (2.1), which is equivalent to

$$(n^{-1}b'\sigma(n)b^{-1})b' \in K\mu(t)K.$$}

Thus

$$\beta^{-1}(x) \cong f_b^{-1}(K\mu(t)Kb^{-1} \cap N(L))/N(0).$$}

Hence we get

$$\dim \beta^{-1}(x) \leq \dim(K\mu(t)Kb^{-1} \cap N(L)) - \langle \rho, \nu - \nu_{\text{dom}} \rangle$$

$$= (N(L)b'x_0 \cap Kx_\mu) + \dim(b'N(0)b'^{-1}) - \langle \rho, \nu - \nu_{\text{dom}} \rangle$$

$$= d(\mu,\mu_M) - (2\rho_N, \nu) + \langle \rho, \nu \rangle - \langle \nu_{\text{dom}} \rangle,$$

where the second equality is true because $N(L)b'x_0 \cap Kx_\mu \cong (K\mu(t)Kb^{-1} \cap N(L))/b'N(0)b'^{-1}$. This gives (2). Now (3) follows from (2) because source and target of $\beta$ are locally of finite type over $\overline{K}$.

Finally we prove (4). Since

$$X^P_{\mu G}(b) = \bigcup_{\mu_M \in S_M(\mu,\kappa)} \beta^{-1}(X^M_{\mu_M}(b))$$

is a decomposition into locally closed subsets, we have

$$\dim X_\mu(b) = \dim X^P_{\mu G}(b) = \sup \{ \dim X^M_{\mu_M}(b); \mu_M \in S_M(\mu,\kappa) \}.$$}

Applying (3) to this formula finishes the proof. □
Now the main part of Theorem 2.1 follows:

**Proposition 2.10.** Let $b \in M(L)$ be basic. Assume that Theorem 2.2 is true for $X^{M}_{\mu}(b)$ for every $\mu \in S_{M}(\mu, \kappa)$. Then it is also true for $X_{\mu}(b)$. 

**Proof.** This is a consequence of Lemma 2.8 and Proposition 2.9. Its proof is literally the same as the proof of its analogue Prop. 5.8.1 in [GHKR06]. 

In particular, replacing $b$ by a $\sigma$-conjugate if necessary, we may choose $M$ such that $b$ is superbasic in $M$. Thus we have reduced Theorem 1.3 to the case where $b$ is superbasic. Now it is only left to show that we may assume $G = \text{Res}_{k'/k} \text{GL}_{h}$. 

For this we show that it suffices to prove Theorem 1.3 for the adjoint group $G^{ad}$. We denote by subscript "ad" the image of elements of $G(L)$ resp. $X_{*}(T)$ resp. $\pi_{1}(G)$ in $G^{ad}(L)$ resp. $X_{*}(T^{ad})$ resp. $\pi_{1}(G^{ad})$. For $\omega \in \pi_{1}(G)$ we write $X_{\mu}(b)^{\omega} := X_{\mu}(b) \cap G^{\omega}$. Then it is easy to see that if $X_{\mu}(b)^{\omega}$ is non-empty, 

$$X_{\mu}(b)^{\omega} \cong X_{\mu(\text{ad})}(b_{\text{ad}})^{\omega_{\text{ad}}}.$$ 

Now in Lemma 2.1.2 of [CKV] it is proven that if $G$ is of adjoint type and contains a superbasic element $b \in G(L)$, then 

$$G \cong \prod_{i=1}^{r} \text{Res}_{k_{i}/k} \text{PGL}_{h_{i}},$$

where the $k_{i}$ are finite field extensions of $k$. As 

$$X_{(\mu_{i})_{i=1}^{r}}((b_{i})_{i=1}^{r}) \cong \prod_{i=1}^{r} X_{\mu_{i}}(b_{i})$$

it suffices to prove Theorem 1.3 for $b$ superbasic and $G \cong \text{Res}_{k'/k} \text{PGL}_{h}$. Using the isomorphism (2.2) again, we may also assume $G \cong \text{Res}_{k'/k} \text{GL}_{h}$, which finishes the proof of Theorem 2.1.

### 3. The superbasic case: Notation and conventions

Let us first fix some basic notation. Let $X$ be a set and $v \in X^{n}$ with $n$ some positive integer. We then write $v_{i}$ for the $i$-th component of $v$. Moreover, if $v, w \in X^{n}$ and $X \subset \mathbb{R}$ we write $v \leq w$ if $v_{i} \leq w_{i}$ for all $i$. For any real number $a$ we denote by $\{a\} := a - \lfloor a \rfloor$ its fractional part. 

Let $d := |k'| : |k| = |E : F|$, then $I \cong \mathbb{Z}/d \cdot \mathbb{Z}$. We choose the isomorphism such that $\sigma$ is mapped to 1. Let $G = \text{Res}_{k'/k} \text{GL}_{h}$ and $S \subset T \subset B \subset G$ where $S$ and $T$ are the maximal split resp. maximal torus which are diagonal and $B$ is the Borel subgroup of lower triangular matrices in $G$. 

We fix a superbasic element $b \in G(L)$ with Newton point $\nu \in X_{*}(S)_{\mathbb{Q}}$ and a cocharacter $\mu \in X_{*}(T)$. We have to show that if $X_{\mu}(b)$ is nonempty, we have 

$$\dim X_{\mu}(b) = \langle \rho, \mu - \nu \rangle - \frac{1}{2} \text{def}_{G}(b).$$

As $T$ splits over $k'$, the action of the absolute Galois group on $X_{*}(T)$ factorizes over $I$. We identify $X_{*}(T) = \prod_{\tau \in I} \mathbb{Z}^{h}$ with $I$ acting by cyclically permuting the factors. This yields an identification of $X_{*}(S) = X_{*}(T)^{I}$ with $\mathbb{Z}^{h}$ such that 

$$X_{*}(S) \hookrightarrow X_{*}(T), \nu' \mapsto (\nu')_{\tau \in I}.$$ 

Furthermore, we denote for an element $\mu' \in X_{*}(T)$ by $\mu' \in X_{*}(S)$ the sum of all $I$-translates of $\mu'$. We impose the same notation as above for $X_{*}(T)_{\mathbb{Q}} = \prod_{\tau \in I} \mathbb{Q}^{h}$ and $X_{*}(S)_{\mathbb{Q}} = \mathbb{Q}^{h}$. 

We note that an element $\nu' \in \mathbb{Q}^{h}$ is dominant if $\nu'_1 \leq \nu'_2 \leq \ldots \leq \nu'_{h}$ and $\mu' \in \prod_{\tau \in I} \mathbb{Q}^{h}$ is dominant if $\mu'_{\tau}$ is dominant for every $\tau \in I$.
The Bruhat order is defined on $X_*(S)_{\text{dom}}$ resp. $X_*(T)_{\text{dom}}$ such that an element $\mu''$ dominates $\mu'$ if and only if $\mu'' - \mu'$ is a non-negative linear combination of relative resp. absolute positive coroots. We write $\mu' \preceq \mu''$ in this case. This motivates the following definition. For $\nu', \nu'' \in \mathbb{Q}^h$ we write $\nu' \preceq \nu''$ if

$$\sum_{i=1}^j \nu_i' \geq \sum_{i=1}^j \nu_i'' \quad \text{for all } j < n$$

$$\sum_{i=1}^n \nu_i' = \sum_{i=1}^n \nu_i''.$$ 

For $\mu', \mu'' \in \prod_{\tau \in I} \mathbb{Q}^h$ we write $\mu' \preceq \mu''$ if $\mu_{\tau}' \preceq \mu_{\tau}''$ for every $\tau \in I$. If $\nu'$ and $\nu''$ resp. $\mu'$ and $\mu''$ are both dominant, this order coincides with the Bruhat order.

For every $k$-algebra $R$ the $R$-valued points of $G$ are given by $G(R) \cong \text{End}_{k' \otimes_k R}(k' \otimes_k R^h)$. We denote $N = k' \otimes_k L^h$, which is canonically isomorphic to the direct sum $\bigoplus_{\tau \in I} N_{\tau}$ of isomorphic copies of $L^h$. The Frobenius element $\sigma$ acts via the Galois action of $\text{Gal}(L/F)$ on $N$, for all $\tau \in I$ we have $\sigma : N_{\tau} \sim \sim N_{\tau+1}$. We fix a basis $(e_{\tau,i})_{i=1}^n$ of the $N_{\tau}$ such that $\varsigma(e_{\tau,i}) = e_{\tau,i}$ for all $\varsigma \in I$. For $\tau \in I, l \in \mathbb{Z}, i = 1, \ldots, h$ denote $e_{\tau,i+l} := t^l \cdot e_{\tau,i}$. Then each $v \in N_{\tau}$ can be written uniquely as an infinite sum

$$v = \sum_{n \geq -\infty} a_n \cdot e_{\tau,n}$$

with $a_n \in k$.

Now we denote by $M^0$ the $k[[t]]$-submodule of $N$ generated by the $e_{\tau,i}$ for $i \geq 0$. With respect to our choice of basis, $K$ is the stabilizer of $M^0$ in $G(L)$ and $g \mapsto gM^0$ defines a bijection

$$G_r(k) \cong \{ M = \prod_{\tau \in I} M_{\tau}; M_{\tau} \text{ is a lattice in } N_{\tau} \}.$$ 

Suppose we are given two lattices $M, M' \subset L^h$. By the elementary divisor theorem we find a basis $v_1, \ldots, v_n$ of $M$ and a unique tuple of integers $a_1 \leq \ldots \leq a_n$ such that $t^{a_i}v_1, \ldots, t^{a_n}v_n$ from a basis of $M'$. We define the cocharacter $\text{inv}(M, M') : S_m \rightarrow \text{GL}_{h, x} \rightarrow \text{diag}(x^{a_1}, \ldots, x^{a_n})$. If we write $M' = gM$ with $g \in \text{GL}_h(L)$ we may equivalently define $\text{inv}(M, M')$ to be the unique cocharacter of the diagonal torus which is dominant w.r.t. the Borel subgroup of lower triangular matrices and satisfies $g \in \text{GL}_h(k[[t]]) \Rightarrow (M, M')(t) \in \text{GL}_h(k[[t]])$.

In terms of the notation introduced above we have

$$X_*(\mu(b)\mathbb{F}) \cong \{ (M_{\tau} \subset N_{\tau} \text{ lattice})_{\tau \in I}; \text{inv}(M_{\tau}, b\sigma(M_{\tau-1})) = \mu_{\tau} \}.$$ 

**Definition 3.1.**

1. We call a tuple of lattices $(M_{\tau} \subset N_{\tau})_{\tau \in I}$ a $G$-lattice.
2. We define the volume of a $G$-lattice $M = gM^0$ to be the tuple

$$\text{vol}(M) = (\text{val det } g_{\tau})_{\tau \in I}.$$ 

Similarly, we define the volume of $M_{\tau}$ to be $\text{val det } g_{\tau}$. We call $M$ special if $\text{vol}(M) = (0)_{\tau \in I}$.

The assertion that $b$ is superbasic is by [CKV] equivalent to $\nu$ being of the form $(\frac{m_1}{d^h}, \frac{m_2}{d^h}, \ldots, \frac{m_h}{d^h})$ with $(m, d) = 1$. Then by [Kott93], Lemma 4.4 $X_*(b)$ is nonempty if and only if $\nu$ and $\mu$ have the same image in $\pi_1(G)_f$, which is equivalent to $\sum_{\tau \in I, i=1, \ldots, h} \mu_{\tau,i} = m$. We assume that this equality holds from now on.

Furthermore we have for each central cocharacter $\nu' \in X_*(S)$ the obvious isomorphism

$$X_*(b) \cong X_{\mu+\nu'}(\nu'(t) \cdot b).$$

So we may (and will) assume that $\mu \geq 0$, which amounts to saying that we have $b\sigma(M) \subset M$ for $G$-lattices $M \in X_*(b)(\mathbb{F})$. 

Since the affine Deligne-Lusztig varieties of two σ-conjugated elements are isomorphic, we can assume that b is the form $b(\epsilon_{r,i}) = \epsilon_{r,i+m}$, where $m_{\tau} = \sum_{i=1}^{h} \mu_{r,i}$. We could have chosen any tuple of integers $(m_{\tau})$ such that $\sum_{\tau \in J} m_{\tau} = m$ but this particular choice has the advantage that the components of any G-lattice in $X_\mu(b)$ have the same volume. In general,

$$\text{vol } M_{\tau} - \text{vol } M_{\tau-1} = (\text{vol } M_{\tau} - \text{vol } b \sigma(M_{\tau-1})) + (\text{vol } b \sigma(M_{\tau-1}) - \text{vol } M_{\tau-1})$$

$$= (\sum_{i=1}^{h} \mu_{r,i}) - m_{\tau}.$$

Recall that the geometric connected components of $\mathcal{G}_r$ are in bijection with $\pi_1(G) = \mathbb{Z}^J$. This bijection is given by mapping a $G$-lattice to its volume. Thus the subsets of lattices $\mathcal{G}_r$ resp. $X_\mu(b)$ obtained by restricting the value of the volume of the components is open and closed. Denote by $X_\mu(b)^i \subset X_\mu(b)$ the subset of all $G$-lattices $M$ such that $M_0$ (or equivalently every $M_{\tau}$) has volume $i$. Let $\tau \in J_0(F)$ be the element with $\pi(\epsilon_{r,i}) = \epsilon_{r,i+1}$ for all $\tau \in I, i \in \mathbb{Z}$. Then $g \cdot K \rightarrow \pi g \cdot K$ defines an isomorphism $X_\mu(b)^i \xrightarrow{\sim} X_\mu(b)^i$. Thus dim $X_\mu(b)$ is equal to the subset of special lattices.

4. Polygons

In this section we introduce our notion of polygons and reformulate the formula (3.1) in terms of this notation. For this we need to introduce some more notation.

We denote by $(\mathbb{Q}^h)^0$ resp. $(\prod_{\tau \in I} \mathbb{Q}^h)^0$ the subspaces of $\mathbb{Q}^h$ resp. $\prod_{\tau \in I} \mathbb{Q}^h$ generated by the relative resp. absolute coroots. Explicitly, these subspaces are given by

$$(\mathbb{Q}^h)^0 = \{ \nu' \in \mathbb{Q}^h; \nu'_1 + \ldots + \nu'_h = 0 \}$$

$$(\prod_{\tau \in I} \mathbb{Q}^h)^0 = \{ \mu' \in \prod_{\tau \in I} \mathbb{Q}^h; \mu'_\tau \in (\mathbb{Q}^h)^0 \text{ for every } \tau \in I \}.$$

We fix lifts $\omega_i$ and $\omega_{i,\tau}$ of relative resp. absolute fundamental weights of the derived group to $X_*(S)$ resp. $X_*(T)$. We thus have for $\nu' \in (\mathbb{Q}^h)^0$ and $\mu' \in (\prod_{\tau \in I} \mathbb{Q}^h)^0$

$$\langle \omega_i, \nu' \rangle = -\sum_{j=1}^{i} \nu'_j,$$

$$\langle \omega_{\tau,i}, \mu' \rangle = -\sum_{j=1}^{\tau} \mu'_{\tau,j}.$$

**Definition 4.1.**

1. For $\nu' \in \mathbb{Q}^h$ let

$$[\nu'] := \sum_{i=1}^{h-1} \lfloor \langle \nu', \omega_i \rangle \rfloor.$$

2. For $\nu', \nu'' \in \mathbb{Q}^h$ we define

$$\ell[\nu', \nu''] := [\nu'] + [-\nu''].$$

3. For $\mu', \mu'' \in (\prod_{\tau \in I} \mathbb{Q}^h)$ let

$$\ell_G[\mu', \mu''] := \ell[\mu'_\tau, \mu''_\tau].$$

Now we give a geometric interpretation of $\ell[\nu', \nu'']$ in terms of polygons in a special case that covers all applications in this paper. We start with the following observation.

**Lemma 4.2.** Let $\nu', \nu'' \in \mathbb{Q}^h$ with $\nu' \preceq \nu''$ and $\nu'' \in \mathbb{Z}^h$. Then

1. $\ell[\nu', \nu''] = [\nu'] - [\nu''] = [\nu' - \nu''].$
2. $\ell[\nu', \nu'']$ is independent of the choice of lifts of the fundamental weights.
Proof. (1) This is an easy consequence of the fact that \( \langle \nu', \omega_i \rangle \) is an integer for all \( i \).

(2) As \( \nu' - \nu'' \in (\mathbb{Q}^h)^0 \), the value \( \lceil \nu' - \nu'' \rceil \) is independent of our choice of lifts. Together with part (1) this proves the claim. □

Definition 4.3. To an element \( \nu' \in \mathbb{Q}^h \) we associate a polygon \( \mathcal{P}(\nu') \) which is defined over \([0, h]\) with starting point \((0, 0)\) and slope \( \nu'_i \) over \((i - 1, i)\). We also denote by \( \mathcal{P}(\nu') \) the corresponding piecewise linear function on \([0, h]\).

Let \( \nu' \) and \( \nu'' \) be as in Lemma 4.2. Now \( \nu' \preceq \nu'' \) amounts to saying that \( \mathcal{P}(\nu') \) is above \( \mathcal{P}(\nu'') \) and that these two polynomials have the same endpoint. It follows from the first assertion of the lemma that \( \ell[\nu', \nu''] \) is equal to the number of lattice points which are on or below \( \mathcal{P}(\nu') \) and above \( \mathcal{P}(\nu'') \).

Figure 1. Geometric interpretation of \( \ell[\nu', \nu''] = 5 \) for \( \nu' = \left(\frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}\right) \), \( \nu'' = (0, 0, 0, 0, 0, 1, 2) \).

Proposition 4.4. The formula (3.1) is equivalent to
\[
\dim X_\mu(b) = \ell_G[\mu, \nu].
\]

In order to prove this proposition, we need the following lemmata.

First we need the following fact from Bruhat-Tits theory, which holds in greater generality than just our specific situation. For a reductive group \( H \) over a quasi-local field we denote by \( \mathcal{B}T(H) \) its Bruhat-Tits building.

Lemma 4.5. Assume that \( F \) is a quasi-local field and \( L \) the completion of its maximal unramified extension. Let \( H \) be a reductive group over \( F \) and \( A \) a maximal split torus of \( H_L \) defined over \( F \). If its apartment \( a \) contains a \( \Gal(F_{nr}/F) \)-stable alcove \( C \), then \( A \) contains a maximal split torus of \( H \).

Proof. We identify \( \mathcal{B}T(H) \) the \( \Gal(F_{nr}/F) \)-fixed points of \( \mathcal{B}T(H_L) \). Then \( \mathcal{B}T(H) \cap C \) is a nonempty open subset of \( \mathcal{B}T(H) \). In particular we have
\[
\dim \mathcal{O}\mathcal{G}(\mathcal{B}T(H)) \succeq \dim \mathcal{O}\mathcal{G}(\mathcal{B}T(H_L)) = \dim \mathcal{B}T(H) = \rk H.
\]

Thus \( A \) contains a maximal split torus of \( H \). □

Lemma 4.6.
\[
\frac{1}{2} \def_G(b) = \sum_{i=1}^{h-1} \langle \omega_i, \nu \rangle
\]

Proof. The analogous assertion for split reductive groups with simply connected derived group was proven by Kottwitz in [Kot06]. We modify his proof in order to get the result in the case of \( \Res_{k'/k} \GL_h \).

We consider the groups \( T' \subset B' \subset \GL_h \) where \( T' \) is the diagonal torus and \( B' \) the Borel subgroup of upper triangular matrices. Denote by \( W \) resp. \( \bar{W} \) the Weyl group resp. extended
We denote by $w_{n}$ the image of $\tilde{w}_{n}$ w.r.t. the canonical projection $\tilde{W} \to W$. Decompose $b = (b_{r})_{r \in I}$ according to $\text{Res}_{k'/k} \text{GL}_{h}(L) \cong \prod_{r \in I} \text{GL}_{h}(L)$. Then $b_{r}$ is the generalized permutation matrix representing $\tilde{w}_{m_{r}}$ in $\text{GL}_{h}(L)$. Denote by $a = \prod_{r \in I} a_{r} \cong \prod_{r \in I} \mathbb{R}^{h}$ the apartment of $G_{L}$ corresponding to $T$. Now $B^{T}(J_{h})$ is canonically isomorphic to the fixed points of the Bruhat-Tits building $B^{T}(G_{L})$ of $G_{L}$ of $\sigma' := \text{Int}(b) \circ \sigma = (w_{m_{r}})_{r \in I} \cdot \sigma$. Since the standard Iwahori is $\sigma'$-stable, we get by Lemma 4.7.

$$
\text{dim} \ a_{0} - \text{dim} \ a_{0}^{w_{m}} = \text{dim} \ a_{0} - \text{dim} \ a_{0}^{w_{m}} = \sum_{i=1}^{h-1} \left\{ \langle \omega_{i}, \left( \frac{m}{h}, \ldots, \frac{m}{h} \right) \rangle \right\} = \sum_{i=1}^{h-1} \left\{ \langle \omega_{i}, \underline{\nu} \rangle \right\}
$$

Proof of Proposition 4.4. Using the lemma above, we get that

$$
\langle \rho, \mu - \nu \rangle - \frac{1}{2} \cdot \text{def}_{G}(b) = \sum_{i=1, \ldots, h-1}^{h} \langle \omega_{r,i}, \mu - \nu \rangle - \sum_{i=1}^{h-1} \left\{ \langle \omega_{i}, \underline{\nu} \rangle \right\}
$$

$$
= \sum_{i=1}^{h-1} \left( \sum_{r \in I} \langle \omega_{r,i}, \mu - \nu \rangle \right) - \sum_{i=1}^{h-1} \left( \langle \omega_{i}, \underline{\nu} \rangle \right)
$$

$$
= \sum_{i=1}^{h-1} \left( \langle \omega_{i}, \underline{\nu} \rangle - \langle \omega_{i}, \underline{\nu} \rangle \right) - \sum_{i=1}^{h-1} \left( \langle \omega_{i}, \nu \rangle \right)
$$

$$
= \ell_{G}[\nu', \mu]
$$

Remark. Rapoport conjectures in [Rap05], footnote no. (10) a formula for the dimension of an affine Deligne-Lusztig variety. One easily checks that the conjecture and our theorem give different formulas if $d > 1$. However, if one multiplies all cocharacters in Rapoport’s formula by $d$ one gets the correct result for $G = \text{Res}_{k'/k} \text{GL}_{h}$, $b \in G(L)$ superbasic.

Finally we prove two lemmas which we will use in section 7. The reader may skip the rest of this section for the moment.

Lemma 4.7. Let $\nu' \in \mathbb{Z}^{h}$. Then

$$
\ell[\nu', \nu'_{\text{dom}}] = \sum_{1 \leq i, j \leq h} \max\{\nu'_{i} - \nu'_{j}, 0\}.
$$
Proof. The assertion follows from the following observation. If \( \nu'' \in \mathbb{Z}^h \) with \( \nu''_i > \nu''_{i+1} \) and we swap these coordinates, then \( [\nu''] \) is reduced by the difference of these two values. Now \( \nu'_{dom} \) is obtained from \( \nu' \) by carrying out the above transposition repeatedly until the coordinates are in increasing order. Since we have swapped the coordinates \( \nu'_i \) and \( \nu'_j \) during this construction if and only if \( i < j \) and \( \nu'_i > \nu'_j \) we get the above formula. \( \square \)

**Lemma 4.8.** Let \( \nu' \in \mathbb{Z}^h \) be dominant, \( 1 \leq i \leq j \leq h, \beta \in \mathbb{Z}_{>0} \) and
\[
\nu'' := (\nu'_1, \ldots, \nu'_{i-1}, \nu'_i - \beta, \nu'_{i+1}, \ldots, \nu'_{j-1}, \nu'_j + \beta, \nu'_{j+1}, \ldots, \nu'_h).
\]
Then
\[
\ell[\nu', \nu'_{dom}] = \left( \sum_{k=1}^{\beta} \sum_{l=\nu'_k - \beta}^{\nu'_k - 1} [\{n; \nu'_n = k + l\}] \right) - \beta.
\]

Proof. Obviously we have
\[
\ell[\nu', \nu''] = (j - i) \cdot \beta = \left( \sum_{j \leq n \leq j} \beta \right) - \beta
\]
and by the previous lemma
\[
\ell[\nu'', \nu'_{dom}] = \sum_{n < j} (\nu'_n - (\nu'_i - \beta)) + \sum_{n > j} (\nu'_n + \beta - \nu'_j).
\]
Using \( \ell[\nu', \nu'_{dom}] = \ell[\nu', \nu''] + \ell[\nu'', \nu'_{dom}] \) one easily deduces the above assertion. \( \square \)

**Corollary 4.9.** Let \( \nu', \nu'' \in \mathbb{Z}^d \) be dominant with \( \nu' \preceq \nu'' \) such that the multiset of their coordinates differs by only two elements. Say \( n_2, n_3 \) in the multiset of coordinates of \( \nu' \) are replaced by \( n_1, n_4 \) in the multiset of coordinates of \( \nu'' \) with \( n_1 \leq n_2 \leq n_3 \leq n_4 \). Then
\[
\ell[\nu', \nu''] = \sum_{k=0}^{n_4-n_3-1} \sum_{l=0}^{n_4-n_2-1} [\{n; \nu'_n = n_4 - k - l - 1\}].
\]

Proof. The assertion is just a reformulation of the previous lemma. \( \square \)

5. Extended EL-charts

In order to calculate the dimension of the affine Deligne-Lusztig variety, we decompose \( X^0_\mu(b) \) as follows. Denote
\[
\mathcal{I}_\tau : N_\tau \setminus \{0\} \to \mathbb{Z}
\]
\[
\sum_{n \geq 0} a_n \cdot e_{\tau,n} \mapsto \min\{n \in \mathbb{Z}; a_n \neq 0\}.
\]
Note that \( \mathcal{I}_\tau \) satisfies the strong triangle inequality for every \( \tau \). We denote \( N_{hom} := \prod_{\tau \in \ell}(N_\tau \setminus \{0\}) \), analogously \( M_{hom} \), and define the index map
\[
\mathcal{I} := \sqcup \mathcal{I}_\tau : N_{hom} \to \prod_{\tau \in \ell} \mathbb{Z}
\]
For \( M \in X_\mu(b)_0(\mathbb{F}) \), we define
\[
A(M) := \mathcal{I}(M_{hom})
\]
and a map \( \varphi(M) : \prod_{\tau \in \ell} \mathbb{Z} \to \mathbb{Z} \) by
\[
\varphi(M)(a) = \max\{n \in \mathbb{N}_0; \exists v \in M_{hom} \text{ with } \mathcal{I}_\tau(v) = a, t^{-n} b \sigma(v) \in M_{hom}\}.
\]
Now we decompose \( X^0_\mu(b) \) such that \( (A(M), \varphi(M)) \) is constant on each component. We will discuss the properties of this decomposition in section 6. In this section we give a description of the invariants \( (A, \varphi) \).
Definition 5.1. Let \( \mathbb{Z}^{(d)} := \prod_{\tau \in I} \mathbb{Z}(\tau) \) be the disjoint union of \( d \) isomorphic copies of \( \mathbb{Z} \). For \( a \in \mathbb{Z} \) we denote by \( a(\tau) \) the corresponding element of \( \mathbb{Z}(\tau) \) and write \( |a(\tau)| := a \). We equip \( \mathbb{Z}^{(d)} \) with a partial order “\( \leq \)” defined by
\[
a(\tau) \leq c(\zeta) :\Leftrightarrow a \leq c \text{ and } \tau = \zeta
\]
and a \( \mathbb{Z} \)-action given by
\[
a(\tau) + n = (a + n)(\tau).
\]
Furthermore we define a function \( f : \mathbb{Z}^{(d)} \to \mathbb{Z}^{(d)}, a(\tau) \mapsto (a + m_{\tau+1})(\tau+1) \).

We impose the notation that for any subset \( A \subset \mathbb{Z}^{(d)} \) we write \( A(\tau) := A \cap \mathbb{Z}(\tau) \).

Definition 5.2. (1) Let \( d, h \) be positive integers and \( (m_{\tau})_{\tau \in I} \in \mathbb{Z}^{d} \) such that \( m := \sum_{\tau \in I} m_{\tau} \) and \( h \) are coprime, let \( f \) be defined as above. An EL-chart for \( (\mathbb{Z}^{(d)}, f, h) \) is a nonempty subset \( A \subset \mathbb{Z}^{(d)} \) which is bounded from below, stable under \( f \) and satisfies \( A + h \subset A \).

(2) Let \( A \) be an EL-chart and \( B = A \setminus (A + h) \). We say that \( A \) is normalized if \( \sum_{b(0) \in B(0)} b = \frac{h(h-1)}{2} \).

Our next aim is to give a characterization of EL-charts. For this let \( A \) be an EL-chart and \( B := A \setminus (A + h) \). Obviously \( |B| = d \cdot h \). We define a sequence \( b_0, \ldots, b_{d-1} \) of distinct elements of \( B \) as follows. Denote by \( b_0 \) the minimal element of \( B(0) \). If \( b_1 \) is already defined, we denote by \( b_{i+1} \) the unique element which can be written as
\[
b_{i+1} = f(b_i) - \mu_i' \cdot h
\]
for some \( \mu_i' \in \mathbb{Z} \). These elements are indeed distinct: If \( b_i = b_j \) then obviously \( i \equiv j \mod d \) and then \( b_{i+d} = b_i + k \cdot m \mod h \) implies that \( i = j \mod m \) and \( h \) are coprime. This reasoning also shows that if we define \( b_{d-1} \) according to the recursion formula above, we get \( b_{d-1} = b_0 \). Therefore we will consider the index set of the \( b_i \) and \( \mu_i' \) as \( \mathbb{Z}/dh\mathbb{Z} \). We define
\[
succ(b_i) := b_{i+1}
\]
and call \( \mu' = (\mu_i')_{i=1,\ldots,d-1} \) the type of \( A \).

At some point, it may be helpful to distinguish the \( b_i \)'s and \( \mu_i' \)'s of different components. For this we may change the index set to \( I \times \{1, \ldots, h\} \) via
\[
b_{\tau,i} := b_{\tau+(i-1)d},
\mu_{\tau,i} := \begin{cases} \mu_{\tau+(i-1)d}' & \text{if } \tau \neq 0 \\ \mu_{id}' & \text{if } \tau = 0 \end{cases}.
\]

Here we choose that standard set of representatives \( \{0, \ldots, d-1\} \subset \mathbb{Z} \) for \( I \).

With the change of notation we have that \( b_{\tau,i} \in B(\tau) \) for all \( i, \tau \) and that \( b_{0,1} \) is the minimal element of \( B(0) \) and we have the recursion formula
\[
b_{\tau+1,i} = f(b_{\tau,i}) - \mu_{\tau+1,i}'h
\]
if \( \tau \neq 0 \), \( b_{0,1} = f(b_{d-1,1}) - \mu_{0,1}'h \).

Lemma 5.3. (1) For every EL-chart \( A \) there exists a unique integer \( n \) such that \( A + n \) is normalized.

(2) Mapping an EL-chart to its type induces a bijection between normalized EL-charts and the set \( \{ \mu' \in \prod_{\tau \in I} \mathbb{Z}^{h} : \mu' \leq \mathbb{u} \} \).

Proof. (1) In order to obtain a normalized EL-chart, we have to choose \( n = \frac{1}{h} \left( \frac{h(h-1)}{2} - \sum_{b(0) \in B(0)} b \right) \).

Since by definition every residue modulo \( h \) occurs exactly once in \( B(0) \), this is indeed an integer.

(2) Since an EL-chart \( A \) is uniquely determined by \( A \setminus (A + h) \) which is, up to \( \mathbb{Z} \)-action, uniquely
determined by the type of $A$, we know the type induces an injection on the set of normalized EL-charts into $\prod_{\tau \in I} \mathbb{Z}^h$. For $1 \leq k \leq h - 1$ we have

$$b_0 \leq b_{kd} \quad \iff \quad b_0 \leq b_0 + k \cdot d \cdot m - \sum_{i=1}^k (\mu'_i) \cdot h$$

$$\iff \quad \sum_{i=1}^k \mu'_i \leq k \cdot \frac{m}{h}$$

Similarly, one shows the equivalence of $b_0 = b_{nd}$ and $\sum_{i=1}^h \mu'_i = \sum_{i=1}^h \mu$. Thus if $\mu'$ is the type of an EL-chart, we have $\mu' \geq \nu$. On the other hand this also shows that any such $\mu'$ is the type of some EL-chart and thus by (1) also the type of a normalized EL-chart.

**Definition 5.4.** For $a \in A$ we call $\text{ht}(a) := \max\{n \in \mathbb{N}_0; a - n \cdot h \in A\}$ the height of $a$.

**Definition 5.5.**

1. An extended EL-chart is a pair $(A, \varphi)$ where $A$ is a normalized EL-chart and $\varphi : \mathbb{Z}^d \to \mathbb{N}_0 \cup \{-\infty\}$ such that the following conditions hold for every $a \in \mathbb{Z}^d$.
   - $\varphi(a) = -\infty$ if and only if $a \notin A$.
   - $\varphi(a + \tau) \geq \varphi(a) + 1$.
   - $\varphi(a) \leq \text{ht}(a)$ if $a \in A$ with equality if $\{c \in \mathbb{Z}^d; c \geq a\} \subset A$.
   - $\{|c \in \mathbb{Z}^d; c \geq a\} \cap \varphi^{-1}(\{n\}) | \leq |\{c \in \mathbb{Z}^d; c \geq a + \tau \cap \varphi^{-1}(\{n + 1\})| \text{ for all } n \in \mathbb{N}_0$.

2. An extended EL-chart is called cyclic if equality holds in (c) for every $a \in A$.

3. The Hodge-point of an extended EL-chart $(A, \varphi)$ is the dominant cocharacter $\mu'' \in \prod_{\tau \in I} \mathbb{Z}^h$ for which the coordinate $n$ occurs with multiplicity $|A(\tau) \cap \varphi^{-1}(\{n\})| - |A(\tau) \cap \varphi^{-1}(\{n + 1\})|$ in $\mu''$. We also say that $(A, \varphi)$ is an extended EL-chart for $\mu''$.

**Remark.** Because of condition (c) we have $|A(\tau) \cap \varphi^{-1}(\{n\})| = h$ for every $\tau \in I$ and sufficiently large $n$. Thus the Hodge point is indeed an element of $\prod_{\tau \in I} \mathbb{Z}^h$.

Except for condition (d) the definition of an EL-chart is obviously a generalization of Definition 3.4 in [Vie06]. As we will frequently refer to Viehmann’s paper we give an equivalent condition for (d) which is easily seen to be a generalization of condition (4) in her definition. However, we will not use this assertion in the sequel.

**Lemma 5.6.** For every $(A, \varphi)$ satisfying conditions (a)-(c) of Definition 5.5, (d) may equivalently be replaced with the following condition. For every $\tau$, we can write $A_{(\tau)} = \bigcup_{i=0}^h \{a_{\tau, i}^\tau\}_{j=0}$ with

- $\varphi(a_{\tau, i+1}^\tau) = \varphi(a_{\tau, i}^\tau) + 1$
- If $\varphi(a_{\tau, i}^\tau + h) = \varphi(a_{\tau, i}^\tau) + 1$, then $a_{\tau, i+1}^\tau = a_{\tau, i}^\tau + h$, otherwise $a_{\tau, i+1}^\tau > a_{\tau, i}^\tau + h$.

Then the Hodge point is the dominant cocharacter associated to $(a_{0, \tau}^\tau)_{\tau \in I, h}$.

**Proof.** If we have a decomposition of $A$ as above, it is obvious that (d) is true. Now let $(A, \varphi)$ be an extended EL-chart. We construct the sequences $(a_{\tau, i}^\tau)_{i \in \mathbb{N}}$ separately for each $\tau$. So fix $\tau \in I$. We construct the sequences by induction on the value of $\varphi$. Take every element of $A$ for which $\varphi$ has minimal value as initial element for some sequence. Now if we have sorted all elements $a \in A$ with $\varphi(a) \leq n$ in sequences $(a_{\tau, i}^\tau)_{i \in \mathbb{N}}$ we proceed as follows. Condition (d) of Definition 5.5 guarantees that we can continue all our sequences such that they satisfy (a) and (b). If there are still some $a \in A$ with $\varphi(a) = n + 1$ which are not already an element of a sequence, we take them as initial objects for some sequences. Since $|\varphi^{-1}(n) \cap A(\tau)| = h$ for $n \gg 0$, we get indeed $h$ sequences.

**Lemma 5.7.** Let $A$ be an EL-chart of type $\mu'$. There exists a unique $\varphi_0$ such that $(A, \varphi_0)$ is a cyclic extended EL-chart. The Hodge point of $(A, \varphi_0)$ is $\mu'_0$.\]
Proof. The function $\varphi_0 : \mathbb{Z}^{(d)} \to \mathbb{N} \cup \{-\infty\}$ is uniquely determined by equality in (c) and condition (a). For any $a \in A$ we get $\varphi_0(a + h) = \varphi_0(a) + 1$, which proves (b) and (d). The second assertion follows from $\mu_{i+1}' = \varphi_0(b_i)$. \hfill \square

The following construction will help us to deduce assertions for general extended EL-charts from the assertion in the cyclic case.

**Definition 5.8.** Let $(A, \varphi)$ be an extended EL-chart and $(A, \varphi_0)$ the cyclic extended EL-chart associated to $A$. For any $\tau \in I$ we denote

$$
\{x_{\tau,1}, \ldots, x_{\tau,n_{\tau}}\} = \{a \in A(\tau); \varphi(a + h) > \varphi(a) + 1\}
$$

where the $x_{\tau,i}$ are arranged in decreasing order. We write $n := (n_\tau)_{\tau \in I}$. For $0 \leq i \leq n$ let

$$
\varphi_1 = \begin{cases} 
-\infty & \text{if } a \notin A, \\
\varphi_0(a) & \text{if } a \notin A(\tau) \text{ and } i_\tau = 0 \\
\varphi(a) & \text{if } a \in A(\tau), i_\tau > 0 \text{ and } a \geq x_{\tau,i_\tau}, \\
\varphi(a + h) - 1 & \text{otherwise.}
\end{cases}
$$

We call the family $(A, \varphi_1)_{0 \leq i \leq n}$ the **canonical deformation** of $(A, \varphi)$.

One easily checks that the $(A, \varphi_1)$ are indeed extended EL-charts (the properties (a)-(d) of Definition 5.5 follow from the analogous properties of $(A, \varphi)$) and that $\varphi_1 = \varphi_0$ for $i = (0)_{\tau \in I}$ and $\varphi_1 = \varphi$ for $i = n$. Denote the Hodge-point of $(A, \varphi_1)$ by $\mu'$.

We note that one can define the $\varphi_1$ recursively. Let $\varsigma \in I$ and $0 \leq i \leq i' \leq n$ with $i'_\varsigma = i_\varsigma + 1$ and $i'_\tau = i_\tau$ for $\tau \neq \varsigma$. We denote $\alpha := \varphi(x_{\varsigma,i_\varsigma} + h) - (\varphi(x_{\varsigma,i_\varsigma}) + 1)$. Then

$$
\varphi_1(a) = \begin{cases} 
\varphi(a) - \alpha & \text{if } a = x_{\varsigma,i_\varsigma}, x_{\varsigma,i_\varsigma} - h, \ldots, x_{\varsigma,i_\varsigma} - \text{ht}(x_{\varsigma,i_\varsigma}) \cdot h, \\
\varphi_1(a) & \text{otherwise}
\end{cases}
$$

**Lemma 5.9.** Let $(A, \varphi)$ be an extended EL-chart of type $\mu$ with Hodge point $\mu$. Then $\mu_{\text{dom}}' \leq \mu$. Furthermore, we have $\mu_{\text{dom}}' = \mu$ if and only if $(A, \varphi)$ is cyclic.

**Proof.** We have already shown that $\mu = \mu_{\text{dom}}'$ if $(A, \varphi)$ is cyclic in Lemma 5.7. It suffices to show $\mu_1' < \mu'$ for all pairs $i,i'$ such that $i'_\varsigma = i_\varsigma + 1$ for some $\varsigma \in I$ and $i'_\tau = i_\tau$ for $\tau \neq \varsigma$. From the description of $\varphi_1$ above we see that we get $\mu'$ from $\mu_1'$ by replacing two coordinates in $\mu_1'$ and permuting its coordinates if necessary to get a dominant cocharacter. Using the same notation as above, we replace

$$
\{\varphi_1(x_{\varsigma,i_\varsigma}) - \text{ht}(x_{\varsigma,i_\varsigma}), \varphi_1(x_{\varsigma,i_\varsigma}) - \alpha + 1\}
$$

with

$$
\{\varphi_1(x_{\varsigma,i_\varsigma}) - \alpha - \text{ht}(x_{\varsigma,i_\varsigma}), \varphi_1(x_{\varsigma,i_\varsigma}) + 1\}.
$$

Since

$$
\varphi_1(x_{\varsigma,i_\varsigma}) - \text{ht}(x_{\varsigma,i_\varsigma}), \varphi_1(x_{\varsigma,i_\varsigma}) - \alpha + 1 \in \{\varphi_1(x_{\varsigma,i_\varsigma}) - \alpha - \text{ht}(x_{\varsigma,i_\varsigma}), \varphi_1(x_{\varsigma,i_\varsigma}) + 1\},
$$

we get $\mu_1' < \mu'$.

Deducing the following corollaries from Lemma 5.9 is literally the same as the proofs of Cor. 3.7 and Lemma 3.8 in [Vie06]. We give the proofs for the reader’s convenience.

**Corollary 5.10.** If $\mu$ is minuscule, then all extended EL-charts for $\mu$ are cyclic.

**Proof.** Let $(A, \varphi)$ be an extended EL-chart for $\mu$ and let $\mu'$ be the type of $(A, \varphi)$. Since $\mu$ is minuscule, $\mu_{\text{dom}}' \leq \mu$ implies $\mu_{\text{dom}}' = \mu$. Hence the assertion follows from Lemma 5.9. \hfill \square

**Corollary 5.11.** There are only finitely many extended EL-charts for $\mu$. 

**Proof.**
Proof. As a consequence of Lemma 5.9 there are only finitely many possible types of extended EL-charts with Hodge point \( \mu \). If we fix such a type, the EL-chart \( A \) is uniquely determined. The value of the function \( \varphi \) is uniquely determined by \( A \) for all but finitely many elements and for each such element, \( \varphi \) can only take finitely many values by the inequality of Definition 5.5(c).

6. Decomposition of \( X_\mu(b) \)

We fix a lattice \( M \in X_\mu(b)^0 \). Let \( (A(M), \varphi(M)) \) be defined as above and \( B(M) := A(M) \setminus (A(M) + h) \).

**Lemma 6.1.** Let \( a_v \in A(M) \) such that \( c \in A(M) \) for every \( c \geq a_v \). Then \( \{ v \in N_\tau; \mathcal{I}_\tau(v) \geq a \} \subset M_\tau \).

*Proof.* We denote \( \tau \). Let \( M' := \{ v \in N_\tau; \mathcal{I}_\tau(v) \geq a \} \) and \( M'' := M' \cap M_\tau \). For \( b = a, \ldots, a + h - 1 \) choose \( v_b \in M'' \) with \( \mathcal{I}_\tau(v_b) = b \). Obviously we can write any element \( x \in M' \) in the form

\[
x = \sum_{b=a}^{a+h-1} \alpha_b \cdot v_b + x'
\]

with \( \alpha_b \in \mathbb{k} \) and \( x' \in t \cdot M' \). We choose an element \( v \) and (c), fix \( t \), \( M' \) satisfies property (d), we fix \( c \). Then

\[
\mathcal{I}(v) = \mathcal{I}(v) + \mathcal{I}(v_b) = \mathcal{I}((t \cdot v)_{v_b}) = \mathcal{I}(t \cdot v) \cap \mathcal{I}(v_b) \in A(M) \text{ which implies the inequality part of (c).}
\]

Let \( a \in A \) such that \( c \in A(M) \) for every \( c \geq a \). We choose an element \( v' \in M_\text{hom} \), such that \( \mathcal{I}(v') = f(a) - \mathcal{I}(f(a)) \). Then \( t^{-\mathcal{I}(f(a))} \cdot \mathcal{I}(v') = t^{-\mathcal{I}(f(a))} \cdot \mathcal{I}(v) \in A(M) \) which implies the inequality part of (c).

**Lemma 6.2.** Let \( M \in X_\mu(b)^0 \). Then \( (A(M), \varphi(M)) \) is an extended EL-chart for \( \mu \).

*Proof.* Let us first check that \( A(M) \) is a normalized EL-chart. It is stable under \( f \) and the addition of \( h \) since

\[
\mathcal{I}(t \cdot v) = \mathcal{I}(v) + h
\]

\[
\mathcal{I}(b \cdot v) = f(\mathcal{I}(v))
\]

and \( t \cdot M \subset M \) and \( b \sigma(M) \subset M \). The fact that \( A(M) \) is bounded from below is obvious. Let \( M = g M^0 \). We have

\[
0 = \text{ vol det } g_0 = |N^d \setminus A(M)_{(0)}| - |A(M)_{(0)} \setminus N^d|,
\]

hence

\[
\sum_{b \in B(M)_{(0)}} b = \sum_{i=0}^{h-1} i = \frac{b(h - 1)}{2}
\]

and thus \( A(M) \) is indeed a normalized EL-chart.

Now \( \varphi(M) \) satisfies property (a) of Definition 5.5 by definition. To see that it satisfies (b) and (c), fix \( a \in A \) and let \( v \in M_\text{hom} \), such that \( \mathcal{I}(v) = a \) and \( t^{-\mathcal{I}(f(a))} \cdot b \sigma(v) \in M \). Then \( t^{-\mathcal{I}(f(a))} \cdot b \sigma(v) \in \mathcal{I}(v) \) and \( f(a) - \mathcal{I}(f(a)) - \mathcal{I}(t^{-\mathcal{I}(f(a))} \cdot b \sigma(v)) \in A(M) \) which implies the inequality part of (c).

Let \( a \in A \) such that \( c \in A(M) \) for every \( c \geq a \). We choose an element \( v' \in M_{\text{hom}} \), such that \( \mathcal{I}(v') = f(a) - \mathcal{I}(f(a)) \). Then \( \mathcal{I}(v) = a \), thus \( v \in M \) and \( t^{-\mathcal{I}(f(a))} \cdot b \sigma(v) = v' \in M \). Thus \( \varphi(a) = \mathcal{I}(f(a)) \). To verify that \( \varphi \) has property (d), we fix \( \tau \in \mathcal{I} \) and define for \( a \in \mathbb{Z}(\tau) \cup \{-\infty\}, n \in \mathbb{N} \) \( \mathbb{k} \)-vector space

\[
V'_{a,n} := \{ v \in M_\tau; v = 0 \text{ or } \mathcal{I}(v) > a, t^{-\mathcal{I}(f(a))} \cdot b \sigma(v) \in M \}
\]

and \( V_{a,n} := V'_{a,n}/V'_{a,n+1} \). Now associate to every \( c \in \{ a' \in A(\tau); a' \geq a \} \cap \varphi^{-1}(\{ n \}) \) an element \( v_c \in M_\tau \) with \( \mathcal{I}(v_c) = c \) and \( t^{-\mathcal{I}(f(c))} \cdot v_c \in M \). Using the strong triangle inequality for \( \mathcal{I}_\tau \), we see that the images \( v_c \) in \( V_{a,n} \) are linearly independent. Thus \( \dim V_{a,n} \geq |\{ a' \in A(\tau); a' \geq a \} \cap \varphi(M)^{-1}(\{ n \})|. \) By counting dimensions in a suitable finite dimensional quotient of \( V_{a,0} \),
we see that this is in fact an equality. Now the images of the \( t \cdot v_t \) in \( V_{a+h,n+1} \) are also linearly independent, thus

\[
\{|a' \in A(\tau); a' \geq a\} \cap \varphi(M)^{-1}(\{n\})| \leq |\{a' \in A(\tau); a' \geq a+h\} \cap \varphi(M)^{-1}(\{n+1\})|.
\]

Now it remains to show that \((A(M), \varphi(M))\) has Hodge point \( \mu \). But

\[
|\{(i; \mu, i = n)\}| = \dim V_{-\infty,n} - \dim V_{-\infty,n-1} = |A(\tau) \cap \varphi^{-1}(\{n\})| - |A(\tau) \cap \varphi^{-1}(\{n-1\})|.
\]

This proof also shows that \( A(M) \) is an EL-chart for every \( G \)-lattice \( M \subset N \) and \( A(M) \) is normalized if and only if \( M \) is special. For any extended EL-chart \( (A, \varphi) \) for \( \mu \) we denote

\[
\mathcal{S}_{A,\varphi} = \{M \in G(\mathbb{K}); (A(M), \varphi(M)) = (A, \varphi)\}.
\]

Since the Hodge point of \( M \) and \((A(M), \varphi(M))\) coincide by the lemma above, we have indeed \( \mathcal{S}_{A,\varphi} \subset X_\mu(b)^0 \).

**Lemma 6.3.** The \( \mathcal{S}_{A,\varphi} \) define a decomposition of \( X_\mu(b)^0 \) into finitely many locally closed subsets. In particular, \( \dim X_\mu(b)^0 = \max_{(A,\varphi)} \dim \mathcal{S}_{A,\varphi} \).

**Proof.** By Lemma 6.2, \( X_\mu(b)^0 \) is the (disjoint) union of the \( \mathcal{S}_{A,\varphi} \) and by Corollary 5.11 this union is finite. It remains to show that \( \mathcal{S}_{A,\varphi} \) is locally closed. One shows that the condition \((A(M), \varphi(M)) = (A(\tau), \varphi(\tau))\) is locally closed analogously to the proof of Lemma 4.2 in [Vie06]. Then it follows that \( \mathcal{S}_{A,\varphi} \) is locally closed as it is the intersection of finitely many locally closed subsets. \( \square \)

**Definition 6.4.** Let \((A, \varphi)\) be an extended EL-chart for \( \mu \). We define

\[
\mathcal{V}(A, \varphi) = \{(a, c) \in A \times A; a < c, \varphi(a) > \varphi(c) > \varphi(a-h)\}
\]

**Proposition 6.5.** Let \((A, \varphi)\) be an extended EL-chart for \( \mu \). There exists an open subscheme \( U_{A,\varphi} \subseteq A^{\mathcal{V}(A,\varphi)} \) and a morphism \( U_{A,\varphi} \to \mathcal{S}_{A,\varphi} \) which is bijective on \( \mathbb{K} \)-valued points. In particular, \( \dim \mathcal{S}_{A,\varphi} = |\mathcal{V}(A, \varphi)| \).

**Proof.** The proof is almost the same as of Thm. 4.3 in [Vie06]. We give an outline of the proof and explain how to adapt the proof of Viehmann to our more general notion.

For any \( \mathbb{K} \)-algebra \( R \) and \( x \in R^{\mathcal{V}(A,\varphi)} = A^{\mathcal{V}(A,\varphi)}(R) \) we denote the coordinates of \( x \) by \( x_{a,c} \). We associate to every \( x \) a set of elements \( \{v(a) \in \mathcal{N}_{\text{hom}}; a \in A\} \) which satisfies the following equations.

If \( a = y := \max\{b \in B(0)\} \) then

\[
v(a) = e_a + \sum_{(a,c) \in \mathcal{V}(A, \varphi)} x_{a,c} \cdot v(c).
\]

For any other element \( a \in B \) we want

\[
v(a) = v' + \sum_{(a,c) \in \mathcal{V}(A, \varphi)} x_{a,c} \cdot v(c)
\]

where \( v' = t^{-\varphi(a')} \cdot b\sigma(v(a')) \) for \( a' \) being minimal satisfying \( f(a') - \varphi(a') \cdot h = a \). At last, if \( a \notin B \), we impose

\[
v(a) = t \cdot v(a-h) + \sum_{(a,c) \in \mathcal{V}(A, \varphi)} x_{a,c} \cdot v(c).
\]

**Claim 1.** The set \( \{v(a); a \in A\} \) is uniquely determined by the equations above.

Hence the rule \( x \mapsto M(x) := \{v(a); a \in A\}_{\mathbb{K}[x]} \) is well-defined and as it is obviously functorial, induces a morphism \( A^{\mathcal{V}(A,\varphi)} \to \mathcal{G}_C \). But the image of this morphism is in general not contained in \( \mathcal{S}_{A,\varphi} \), we only have the following assertions:
Theorem 7.2. Let $i$ where $(b, n) = \max\{(b, n) | b \in B, n \in N\}$. Now we construct an injective map from $A$.

Claim 2. For every $x \in A^{V(A, \varphi)}(\overline{\mathbb{K}})$ we have $\ell(M(x)) = A$ and $\varphi(M(x))(a) \geq \varphi(a)$ for every $a \in A$.

Claim 3. The preimage $U(A, \varphi)$ of $S_{A, \varphi}$ is nonempty and open in $A^{V(A, \varphi)}$.

Now the fact that the restriction $U(A, \varphi) \to S_{A, \varphi}$ of above morphism defines a bijection of $\overline{\mathbb{K}}$-valued points follows from the following assertion.

Claim 4. Let $M \subset N$ be a special $F$-lattice such that $(A(M), \varphi(M)) = (A, \varphi)$. Then there exists a unique set of elements $\{v(a); a \in A\} \subset M$ satisfying the equations above.

It remains to prove the four claims. But their proofs are literally the same as in [Vie06] if one replaces “$a + m$” and “$a + im$” respectively “$f(a)$”.

\section{Combinatorics for extended EL-charts}

Proposition 6.5 reduces the proof of the formula (3.1) to an estimation of $|V(A, \varphi)|$ for extended EL-charts $(A, \varphi)$ with Hodge point $\mu$. We start with the case where $(A, \varphi)$ is cyclic. In this case we have $\varphi(a + h) = \varphi(a)$ for all $a \in A$ and thus

$V(A, \varphi) = \{(a, c) \in B \times A; b < c, \varphi(b) > \varphi(c)\}$

**Proposition 7.1.** Let $(A, \varphi)$ be a cyclic extended EL-chart of type $\mu$. Then

$|V(A, \varphi)| \geq \ell_{G}[\nu, \mu]$

**Proof.** First we show that the right hand side of the inequality counts the number of positive integers $n$ such that $b_0 + n \not\in A_{(0)}$. Indeed, as $A_{(0)} + h \subset A_{(0)}$, we have

$|\{n \in \mathbb{N}; b_0 + n \not\in A_{(0)}\}| = \frac{1}{h} \sum_{j=1}^{h-1} (b_{j}d - b_0 - j) = \frac{1}{h} \sum_{j=1}^{h-1} \left(f^{j}d(b_0) - \left(\sum_{i=1}^{j} \mu_{i} \cdot h\right) - b_0 - j\right) = \frac{1}{h} \sum_{j=1}^{h-1} \left(b_0 + j \cdot m - \left(\sum_{i=1}^{j} \mu_{i} \cdot h\right) - b_0 - j\right).$

Now we construct an injective map from $\{n \in \mathbb{N}; b_0 + n \not\in A\}$ into $V(A, \varphi)$. For this we remark that $(b, b_{i} + n) \in V(A, \varphi)$ if and only if $b_{i} + n \in A$ and $b_{i+1} + n \not\in A$. Thus $n \mapsto (b, b_{i} + n)$ where $i = \max\{i = 1, \ldots, h - 1; b_{i} + n \in A\}$ gives us the injection we sought. Note this map is well-defined since for any $n \in \mathbb{N}$ and maximal element $b$ of $B$ we have $b + n \in A$.

**Theorem 7.2.** Let $(A, \varphi)$ be an extended EL-chart for $\mu$. Then $|V(A, \varphi)| \leq \ell_{G}[\nu, \mu]$.
Proof. We assume first that \((A, \varphi)\) is cyclic with type \(\mu'\). Then
\[
|\mathcal{V}(A, \varphi)| = |\{(b_i, a) \in B \times A; b_i < a, \varphi(a) < \varphi(b_i)\}| = \\
\sum_{(b_i, b_j); b_i < b_j, \mu'_{i+1} > \mu'_{j+1}} \mu'_i + 1 - \mu'_j + 1 + \left|\{(b_i, b_j + ah); a \in \mathbb{N}, b_i < b_j + ah, \mu'_{i+1} > \mu'_{j+1} + a\}\right|
\]
We refer to these two summands by \(S_1\) and \(S_2\).

For each \(\tau \in I\) denote by \((\tilde{\mu}_{\tau,1}, \tilde{\mu}_{\tau+1,1}), \ldots, (\tilde{\mu}_{\tau,h}, \tilde{\mu}_{\tau+1,h})\) the permutation of \((b_{\tau,1}, \mu_{\tau+1,1}), \ldots, (b_{\tau,h}, \mu_{\tau+1,h})\) such that the \((\tilde{\mu}_{\tau,i})_i\) are arranged in increasing order. From the ordering we obtain for any \(1 \leq j \leq h, \tau \in I\)
\[
\sum_{i=0, \ldots, j-1} \tilde{b}_{\tau,i} \leq \sum_{i=0, \ldots, j-1} \text{succ}(\tilde{b}_{\tau-1,i})
\]
and thus
\[
\sum_{i=1}^j |\tilde{b}_{\tau,i}| - |\tilde{b}_{\tau-1,i}| \leq j \cdot m_{\tau} - \sum_{i=1}^j \tilde{\mu}_{\tau,i} \cdot h.
\]
Adding these inequalities for all \(\tau \in I\) and rearranging the terms we obtain \(\sum_{i=1}^j \tilde{\mu}_{\tau,i} \leq j \cdot \frac{m_{\tau}}{h}\).
Thus \(\nu \ prem \leq \overline{\mu}\).

Using this notation we can simplify
\[
S_1 = \sum_{i<j, \tau \in I} \max\{\tilde{\mu}_i - \tilde{\mu}_j, 0\} = \sum_{\tau \in I} \ell[\tilde{\mu}_{\tau} + \tilde{\mu}_{\tau, \text{dom}}] = \ell_G[\tilde{\mu}, \mu],
\]
where the second line holds because of Lemma 4.7.

We have now reduced the claim to \(S_2 \leq \ell_G[\nu, \mu] - \ell_G[\tilde{\mu}, \mu]\), which is equivalent to \(S_2 \leq \ell_G[\nu, \tilde{\mu}]\).

Now
\[
S_2 = \sum_{i=2}^h \sum_{j=1}^i \sum_{\tau \in I} |\{(\alpha \in \mathbb{Z}; \tilde{b}_{\tau,i} < b_{\tau,j} + ah, \tilde{\mu}_{\tau+1,i} > \tilde{\mu}_{\tau+1,j} + \alpha)\}|
\]
\[
= \sum_{i=2}^h \sum_{j=1}^i \sum_{\tau \in I} |\{(\alpha \in \mathbb{Z}; \tilde{b}_{\tau,i} - \tilde{b}_{\tau,j} < ah < \tilde{\mu}_{\tau+1,i} - h - \tilde{\mu}_{\tau+1,j} + h)\}|
\]
\[
\leq \sum_{i=2}^h \sum_{j=1}^i \sum_{\tau \in I} |\{(\alpha \in \mathbb{Z}; 0 < ah < (\tilde{b}_{\tau,j} - \tilde{\mu}_{\tau+1,j} + h) - (\tilde{b}_{\tau,i} - \tilde{\mu}_{\tau+1,i} + h))\}|
\]
\[
= \sum_{i=2}^h \sum_{j=1}^i \sum_{\tau \in I} \max\left\{\frac{\text{succ}(\tilde{b}_{\tau,i}) - \text{succ}(\tilde{b}_{\tau,j})}{h}, 0\right\}
\]
Recall that \(\ell_G[\nu, \tilde{\mu}]\) counts the lattice points between the polynomials associated to \(\mu\) and \(\tilde{\mu}\).
So it is enough to construct a decreasing sequence (with respect to \(\preceq\)) of \(\tilde{\psi}^i \in Q^h\) for \(i = 1, \ldots, h\) with \(\psi^1 = \mu\) and \(\psi^h = \tilde{\mu}\) such that there are at least
\[
\sum_{i=1}^{h-1} \sum_{\tau \in I} \max\left\{\frac{\text{succ}(\tilde{b}_{\tau,i}) - \text{succ}(\tilde{b}_{\tau,j})}{h}, 0\right\}
\]
lattice points which are on or below \(P(\tilde{\psi}^{i-1})\) and above \(P(\tilde{\psi}^i)\).
We define a bijection $\text{succ}_i : B \to B$ as follows: For $j > i$, $\tau \in I$ let $\text{succ}_i(b_{j,\tau}) = \text{succ}(b_{j,\tau})$. For $j \leq i$ define $\text{succ}_i(b_{r,j})$ such that for every $\tau \in I$ the tuple $(\text{succ}_i(b_{r,j}))_{j=1}^i$ is the permutation of $(\text{succ}(b_{r,j}))_{j=1}^i$, which is arranged in increasing order. Let $\psi^i \in \prod_{\tau \in I} \mathbb{Q}^h$ be defined by $\text{succ}_i(b_{r,j}) = f(b_{r,j}) - \psi^i_{j+1,j} \cdot h$ and $\widehat{\psi} = \psi^i$. By definition we have

$$\psi^i_{j+1,j} = \frac{m_{j+1}}{h} - \frac{\text{succ}_i(b_j) - b_j}{h}$$

and thus $\widehat{\psi}^1 = \widehat{\mu}$ and $\widehat{\psi}^h = \mu$.

We have the following recursive construction of $\text{succ}_i$. Let $i_0 \leq i$ be minimal such that $\text{succ}_{i-1}(b_{r,i_0}) \leq \text{succ}(b_{r,i})$. Then

$$\text{succ}_i(b_{r,j}) = \begin{cases} 
\text{succ}_{i-1}(b_{r,j}) & \text{if } j < i_0 \\
\text{succ}(b_{r,i}) & \text{if } j = i_0 \\
\text{succ}_{i-1}(b_{r,j-1}) & \text{if } i_0 < j \leq i \\
\text{succ}_{i-1}(b_{r,j}) & \text{if } j > i
\end{cases}$$

Now

$$\mathcal{P}(\widehat{\psi}^i)(j) - \mathcal{P}(\widehat{\psi}^{i-1})(j) = \sum_{\tau \in I} \sum_{k=1}^j (\psi^i_{\tau,k} - \psi^{i-1}_{\tau,k})$$

$$= \sum_{\tau \in I} \sum_{k=1}^j \frac{1}{h} (\text{succ}_{i-1}(b_{r,k}) - \text{succ}_i(b_{r,k}))$$

$$= \sum_{\tau \in I} \frac{1}{h} \left( \sum_{k=1}^j \text{succ}_{i-1}(b_{r,k}) - \sum_{k=1}^j \text{succ}_i(b_{r,k}) \right).$$

By the recursive formula above the right hand side equals zero if $j \geq i$ and

$$\sum_{\tau \in I} \max\left\{0, \frac{\text{succ}_{i-1}(b_{r,j}) - \text{succ}(b_{r,j})}{h} \right\}$$

if $j < i$. Thus there are at least

$$\sum_{\tau \in I} \sum_{j<i} \max\left\{0, \frac{\text{succ}_{i-1}(b_{r,j}) - \text{succ}(b_{r,j})}{h} \right\} = \sum_{\tau \in I} \sum_{j<i} \max\left\{0, \frac{\text{succ}(b_{r,j}) - \text{succ}(b_{r,j})}{h} \right\}$$

lattice points which are above $\mathcal{P}(\widehat{\psi}^i)$ and on or below $\mathcal{P}(\widehat{\psi}^{i-1})$, which finishes the proof for a cyclic EL-chart.

For a nonecyclic EL-chart $(A, \varphi)$ consider the canonical deformation $(\{A, \varphi_i\})_i$ (see Definition [5.8]). The theorem is proven by induction on $i$. For $i = (0)_{\tau \in I}$ the extended EL-chart is cyclic and the claim is proven above. Now the induction step requires that we show that the claim remains true if increase a single coordinate of $i$ by one. Let $i' \leq n$ with $i'_\tau = i_\tau + 1$ for some $\varsigma \in I$ and $i'_\tau = i_\tau$ for $\tau \neq \varsigma$. For convenience, we introduce the notations

$$\alpha := \varphi(x_{c, i_\varsigma} + h) - (\varphi(x_{c, i_\varsigma}) + 1)$$

$$n := \text{ht}(x_{c, i_\varsigma})$$

$$\mu^i \ := \mu^i_\varsigma$$

Then the right hand sides of the formula (3.1) for $\mu^{i'}$ and $\mu^i$ differ by

$$\langle \rho, \mu^{i'} \rangle - \langle \rho, \mu^i \rangle = \ell_\alpha[\mu^{i'}, \mu^i].$$
Recall the explicit description of the difference between \( \varphi_1 \) and \( \varphi_\nu \) resp. \( \mu^l \) and \( \mu^k \) which we gave right before resp. in the proof of Lemma \[5,9\] Then Corollary \[1,9\] implies

\[
\ell_G[\mu^k, \mu^l] = \ell[\mu^k, \mu^l] = \left( \sum_{k=0}^{\alpha-1} \sum_{l=0}^{n} |\{j; \mu^l_j = \varphi_1(x_{c,i}) - k - l\}| \right) - \min\{\alpha, n+1\}.
\]

We denote this term by \( \Delta \). We have to show that \( \Delta \geq |V(A, \varphi_\nu)| - |V(A, \varphi_1)| \). Now

\[
V(A, \varphi_\nu) \setminus V(A, \varphi_1) = D_1 \cup D_3
\]

\[
V(A, \varphi_1) \setminus V(A, \varphi_\nu) = D_2
\]

where

\[
D_1 = \{(x_{c,i} + h, c) \in A \times A; c > x_{c,i} + h, \varphi_1(x_{c,i} + h) > \varphi_\nu(c) > \varphi_1(x_{c,i})\}
\]

\[
D_2 = \{(x_{c,i} + nh, c) \in A \times A; \varphi_1(x_{c,i} + nh) \leq \varphi_\nu(c)\}
\]

\[
D_3 = \{(b, x_{c,i} - \delta h) \in B \times A; b \neq x_{c,i} - nh, b < x_{c,i} - \delta h, \varphi_\nu(b) > \varphi_1(x_{c,i} - \delta h), \varphi_1(b) \leq \varphi_1(x_{c,i} - \delta h)\}
\]

Thus we get \( |V(A, \varphi_\nu)| - |V(A, \varphi_1)| = S_1 - S_2 + S_3 \) with

\[
S_1 = \{|a \in A; a > x_{c,i} + h, \varphi_1(a) \in [\varphi_1(x_{c,i}) - \alpha + 1, \varphi_1(x_{c,i})]\|
\]

\[
S_2 = \{|a \in A; a > x_{c,i} - nh, \varphi_1(a) \in [\varphi_1(x_{c,i}) - \alpha - n, \varphi_1(x_{c,i}) - n - 1]\|
\]

\[
S_3 = \{|(b, \delta) \in B \times \{0, \ldots, n\}; b \neq x_{c,i} - nh, b < x_{c,i} - \delta h, \varphi_\nu(b) > \varphi_1(x_{c,i} - \delta h), \varphi_1(b) \leq \varphi_1(x_{c,i} - \delta h)\|
\]

Now let

\[
C_1 = \{|a \in A; a \leq x_{c,i} + h, \varphi_1(a) \in [\varphi_1(x_{c,i}) - \alpha + 1, \varphi_1(x_{c,i})]\|
\]

\[
C_2 = \{|a \in A; a \leq x_{c,i} - nh, \varphi_1(a) \in [\varphi_1(x_{c,i}) - \alpha - n, \varphi_1(x_{c,i}) - n - 1]\|
\]

As \( \varphi_1(x + h) = \varphi_1(x) + 1 \) for all \( x \in A \) with \( x \leq x_{c,i} \), we have \( C_2 + (n+1)h \subset C_1 \). We denote \( C_3 := C_1 \setminus (C_2 + (n+1)h) \). Then

\[
C_3 = \left\{ \begin{array}{l}
\{b + \delta h \in A; b \in B, \delta \in \{0, \ldots, n\}, b \leq x_{c,i} + h - \delta h, \\
\varphi_1(b) \in [\varphi_1(x_{c,i}) - \delta - \alpha + 1, \varphi_1(x_{c,i}) - \delta]\}
\end{array} \right\}
\]

\[
\cup \{\varphi_1(b) \in [\varphi_1(x_{c,i}) - \delta - \alpha + 1, \varphi_1(x_{c,i}) - \delta]\}
\]

In particular, we have \( |C_3| \geq S_3 + \min\{\alpha, n+1\} \).

Alltogether, we get

\[
\Delta = \left( \sum_{k=0}^{\alpha-1} \sum_{l=0}^{n} |\{j; \mu^l_j = \varphi_1(x_{c,i}) - k - l\}| \right) - \min\{\alpha, n+1\}
\]

\[
= \sum_{k=0}^{\alpha-1} \sum_{l=0}^{n} \left( |A_{(i)} \cap \varphi_1^{-1}(\{\varphi_1(x_{c,i}) - k - l\})| - |A_{(i)} \cap \varphi_1^{-1}(\{\varphi_1(x_{c,i}) - k - l - 1\})| \right)
\]

\[
- \min\{\alpha, n+1\}
\]

\[
= |A_{(i)} \cap \varphi_1^{-1}(\{\varphi_1(x_{c,i}) - \alpha + 1, \varphi_1(x_{c,i})\})| - |A_{(i)} \cap \varphi_1^{-1}(\{\varphi_1(x_{c,i}) - \alpha - n, \varphi_1(x_{c,i}) - n - 1\})|
\]

\[
- \min\{\alpha, n+1\}
\]

\[
= (S_1 + |C_1|) - (S_2 + |C_2|) - \min\{\alpha, n+1\}
\]

\[
= S_1 - S_2 + |C_3| - \min\{\alpha, n+1\}
\]

\[
\geq S_1 - S_2 + S_3.
\]
Proof of Theorem 7.1 We reduced the claim of the theorem to $G = \text{Res}_{k/k'} GL_h$ and superbasic $b \in G(L)$ in Theorem 4.4. As we have a bijection $\{\text{top-dimensional irreducible components of } X_{\mu}(b)\} \leftrightarrow \mathcal{M}_{\mu}$, the assertion is equivalent to $\dim X_{\mu}(b) = \ell G(\mu, \nu)$ by Proposition 5.3. As a consequence of Proposition 6.5 we get

$$\dim X_{\mu}(b) = \max\{|V(A, \varphi)|; (A, \varphi) \text{ is an extended EL-chart for } \mu\}.$$  

Now in Proposition 7.1 we showed that the maximum is at least $\ell G(\mu, \nu)$ and in Theorem 2.8 we showed that it is at most $\ell G(\mu, \nu)$, finishing the proof. \qed

8. Irreducible Components in the Superbasic Case

We know consider the $J_b(F)$-action on the irreducible components of $X_{\mu}(b)$ for superbasic $b$ and arbitrary $G$. Recall that the canonical projection $G \to G^{\text{ad}}$ induces isomorphisms

$$X_{\mu}(b)^{\omega} \cong X_{\mu^{\text{ad}}(b^{\text{ad}})}^{\omega^{\text{ad}}}$$

for all $X_{\mu}(b)^{\omega} \neq \emptyset$. As $J_b(F)$ acts transitively on the set of non-empty $X_{\mu}(b)^{\omega}$ (cf. [CKV] sect.3.3), this implies that the induced map on $J_b(F)$-orbits to $J_b^{\text{ad}}(F)$-orbits of the respective irreducible components is bijective. Using the same argument as in the end of section 2, the investigation of the set of irreducible components of $X_{\mu}(b)^{\omega}$ (resp. the set of $J_b(F)$-orbits of irreducible components of $X_{\mu}(b)$) can be reduced to the case $G = \text{Res}_{k/k'} GL_h$.

Lemma 8.1. Let $b \in G(L)$ be superbasic and $\omega \in \pi_1(G)$ such that $X_{\mu}(b)^{\omega}$ is non-empty. Then every $J_b(F)$-orbit of irreducible components of $X_{\mu}(b)$ contains a unique irreducible component of $X_{\mu}(b)^{\omega}$.

Proof. Denote by $J_b(F)^0$ the stabiliser of $X_{\mu}(b)^{\omega}$. Using the argument above, it suffices to show that $J_b(F)^0$ stabilises the irreducible components of $X_{\mu}(b)$ in the case $G = \text{Res}_{k/k'} GL_h$.

For this we consider the action of $J_b(F)$ on $N_{\text{hom}}$. Let $g \in J_b(F)$ and let $v_0 := g(e_0, 1), c(g) := \mathcal{I}(v_0)$. Now every element $e_{\tau,i}$ can be written in the form $e_{\tau,i} = \frac{1}{p^j}(b\sigma)^k(e_{0,1})$ for some integers $j, k$; then

$$g(e_{\tau,i}) = \frac{1}{p^j}(b\sigma)^k(v_0).$$

Hence $\text{val det}(g) = (c(g))_{\tau \in \ell}$, in particular we have $g \in J_b(F)^0$ if and only if $c(g) = 0$. In this case the above formula implies $\mathcal{I}(g(e_{\tau,i})) = i_{\tau}$ and thus $\mathcal{I}(g(v)) = \mathcal{I}(v)$ for all $v \in N_{\text{hom}}$.

We obtain $A(M) = A(g(M))$ and $\varphi(M) = \varphi(g(M))$ for all $M \in X_{\mu}(b)^0$. Thus $S_{A,\varphi}$ is $J_b(F)^0$-stable for every extended EL-chart $(A, \varphi)$ for $\mu$. As the $S_{A,\varphi}$ are irreducible, every irreducible component of $X_{\mu}(b)^0$ is of the form $S_{A,\varphi}$ and thus $J_b(F)^0$-stable. \qed

We denote by $\mathcal{M}_{\mu}$ the set of extended EL-Charts $(A, \varphi)$ for $\mu$ for which $|V(A, \varphi)|$ is maximal. As noted above we have a bijection

$$\mathcal{M}_{\mu} \leftrightarrow \{\text{top-dimensional irreducible components of } X_{\mu}(b)\}$$

$$(A, \varphi) \leftrightarrow S_{A,\varphi}.$$  

Rapoport conjectured in [Rap05] that $X_{\mu}(b)$ is equidimensional. If this holds true, the above bijection becomes

$$\mathcal{M}_{\mu} \leftrightarrow \{\text{irreducible components of } X_{\mu}(b)\}$$

$$\leftrightarrow \{J_b(F) - \text{orbits of irreducible components of } X_{\mu}(b)\}.$$  

It is known that $J_b(F)$ does not act transitively on the irreducible components of $X_{\mu}(b)$ in general, even in the case $G = GL_h$ ([Ve04] Ex. 6.2). The following lemma shows that we have transitive action of $J_b(F)$ only in a few degenerate cases.

Lemma 8.2. (1) Assume there exist $\tau_1, \tau_2 \in I$ such that $\mu_{\tau_1}$ and $\mu_{\tau_2}$ are not of the form $(a, a, \ldots, a)$ for some integer $a$. Then $\mathcal{M}_{\mu}$ contains more than one element.
(2) On the contrary, if there exists $\tau \in I$ such that $\mu_\tau = (a_{\nu}, a_{\nu}, \ldots, a_{\nu})$ for some integer $a_{\nu}$ for all $\nu \neq \tau$ then

$$|\mathcal{M}_\mu| = |\mathcal{M}_\mu|$$

Proof. (1) We assume without loss of generality that $\tau_1 \in [0, \tau_2)$. By Proposition 7.4 the cyclic EL-chart of type $\mu$ is contained in $\mathcal{M}_\mu$. The same reasoning shows that $\mathcal{M}_\mu$ also contains the cyclic EL-chart of type $(\mu_{\tau_1}', \mu_{\tau_2}', \ldots, \mu_{\tau_2-1}', \mu_{\tau_2-1}', \ldots, \mu_{\tau_1-1}')$ with $\mu_{\tau_1}' = (\mu_{\tau_1}, \mu_{\tau_2}, \ldots, \mu_{\tau_2-1}, \mu_{\tau_2-1})$. Note that our condition on $\mu_{\tau_1}$ implies $\mu_{\tau_1} \neq \mu_{\tau_1}'$ and thus $\mu \neq (\mu_{\tau_1}', \mu_{\tau_2}', \ldots, \mu_{\tau_2-1}', \mu_{\tau_2-1}')$.

(2) The claim holds as the bijection

$$\{\text{extended EL-charts for } \mu\} \leftrightarrow \{\text{extended semi-modules for } \mu\}$$

$$(A, \varphi) \leftrightarrow (A_{\tau}, \varphi_{\tau})$$

$$(\bigcup_{\tau \in I} B, \bigcup_{\tau} \varphi) \leftrightarrow (B, \varphi)$$

preserves $|\mathcal{V}|$. \qed

We now consider the case $\mu$ minuscule. De Jong and Oort showed that in this case, $|\mathcal{M}_\mu| = 1$ for (extended) semi-modules (i.e. the case $G = \text{GL}_n$). For EL-charts we have the following conjecture, which generalises \cite{JOO00} Rem. 6.16.

**Conjecture 8.3.** Let $\mu$ be minuscule. Then the construction of $\tilde{\mu}$ in the proof of Thm. 7.3 induces a bijection

$$\{(\text{extended EL-charts for } \mu) \rightarrow \{\tilde{\mu} \in W_{\mu}; \nu \leq \tilde{\mu}\}$$

where $W = (S_n)'$ denotes the absolute Weyl group of $G$. In particular

$$|\mathcal{M}_\mu| = \left|\{\tilde{\mu} \in W_{\mu}; \nu \leq \tilde{\mu}, \ell_G[\nu, \tilde{\mu}] = 0\}\right|$$

$$= \left|\{\tilde{\mu} \in W_{\mu}; \mu = \left(\frac{m}{h} 1, \ldots, \frac{m}{h} (\frac{m}{h})(1-\frac{m}{h}) \right), \frac{m}{h} \in \mathbb{Z}\}\right|.$$ 

Here $m$ is defined as in section 8 i.e. $\nu = \left(\frac{m}{h}, \ldots, \frac{m}{h}\right)$.

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