Abstract. Our work is concerned with the family $\nu : \text{Hess}(m) \rightarrow g$ of Hessenberg varieties associated to a complex semisimple Lie algebra $g$ and its standard Hessenberg subspace $m \subseteq g$. This family includes the Peterson variety and certain smooth projective toric varieties, and it features prominently in Bălibanu’s recent article [5]. Bălibanu considers a Kostant section $S \subseteq g$ and a partially compactified universal centralizer $Z_S \rightarrow S$, identifying the latter with $\nu^{-1}(S)$ as Poisson varieties over $S$. The Hessenberg variety over each $x \in S$ is thereby embedded into $G$, the wonderful compactification of the adjoint group $G$.

We examine a distinguished class of Poisson transversals in log symplectic geometry — the so-called Poisson slices. This equips us to meaningfully study one such slice in the log cotangent bundle $T^*\overline{G}(\log(D))$. The slice in question is a log symplectic Hamiltonian $G$-variety $\pi : G \times \overline{S} \rightarrow g$ that fibrewise compactifies the symplectic Hamiltonian $G$-variety $\mu : G \times S \rightarrow g$. Our main result is then a canonical isomorphism $\text{Hess}(m) \cong G \times \overline{S}$ of Poisson varieties over $g$. The pullback of our isomorphism to $S$ is shown to be Bălibanu’s Poisson isomorphism, and we obtain a canonical closed embedding $\nu^{-1}(x) \hookrightarrow G$ for every $x \in g$.

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2020 Mathematics Subject Classification. 14L30 (primary); 53D20, 14M17 (secondary).
Key words and phrases. Hessenberg variety, log symplectic variety, Poisson slice, wonderful compactification.
1. Introduction

1.1. Context. Hessenberg varieties arise as a natural generalization of Grothendieck–Springer fibres, and their study is central to modern research in algebraic geometry [3, 11, 15, 24, 28], combinatorics [1, 4, 20, 26, 29], representation theory [5–7], and symplectic geometry [2, 22]. One fixes a complex semisimple algebraic group $G$ of adjoint type with Lie algebra $\mathfrak{g}$, as well as a Borel subgroup $B \subseteq G$ with Lie algebra $\mathfrak{b} \subseteq \mathfrak{g}$. These data give rise to the notion of a Hessenberg subspace, i.e. any $B$-invariant vector subspace $H \subseteq \mathfrak{g}$ that contains $\mathfrak{b}$. Each Hessenberg subspace $H \subseteq \mathfrak{g}$ determines a $G$-equivariant vector bundle $G \times_B H \rightarrow G/B$, and the total space of this bundle is Poisson. The $G$-action on $G \times_B H$ is then Hamiltonian and admits an explicit moment map $\nu_H : G \times_B H \rightarrow \mathfrak{g}$. One calls $\text{Hess}(x, H) := \nu^{-1}_H(x)$ the Hessenberg variety associated to $H$ and $x \in \mathfrak{g}$, and regards $\nu_H$ as the family of all Hessenberg varieties associated to $H$.

The so-called standard Hessenberg subspace is the annihilator of $[u, u]$ under the Killing form, where $u$ is the nilradical of $\mathfrak{b}$. The resulting family $\nu := \nu_m : G \times_B \mathfrak{m} \rightarrow \mathfrak{g}$ has received considerable attention in the research literature. One reason is that the fibres of $\nu$ appear in interesting contexts; $\text{Hess}(x, \mathfrak{m})$ is isomorphic to the Peterson variety if $x \in \mathfrak{g}$ is regular and nilpotent, while $\text{Hess}(x, \mathfrak{m})$ is a well-studied smooth projective toric variety if $x$ is regular and semisimple. A second reason is elucidated in Bălibanu’s recent work [5], and it concerns the De Concini–Procesi wonderful compactification $\overline{G}$ of $G$ [14]. Bălibanu fixes a Kostant section $S \subseteq \mathfrak{g}$ and considers its universal centralizer $Z_g \rightarrow S$. She takes an appropriate Kostant–Whittaker reduction of the log cotangent bundle $T^*G(\log(D))$ and obtains a log symplectic fibrewise compactification $\overline{Z}_g \rightarrow S$ of the universal centralizer. The variety $\nu^{-1}(S)$ is subsequently shown to be Poisson, and to be isomorphic to $\overline{Z}_g$ as a Poisson variety over $S$. One thereby obtains an isomorphism $\text{Hess}(x, \mathfrak{m}) \cong \overline{G}_x$ for all $x \in S$, where $G_x$ is the $G$-stabilizer of $x$ and $\overline{G}_x$ denotes its closure in $\overline{G}$.

It is natural to seek a global version of Bălibanu’s Poisson isomorphism, by which we mean the following. One should find a log symplectic variety $X$ over $\mathfrak{g}$, together with an isomorphism

$$X \xrightarrow{\cong} G \times_B \mathfrak{m} \xleftarrow{\nu} \mathfrak{g} \xleftarrow{\mu} \mathfrak{g} \xrightarrow{\nu^{-1}} \nu^{-1}(S)$$

of Poisson varieties over $\mathfrak{g}$. The pullback of this triangle along the inclusion $S \hookrightarrow \mathfrak{g}$ should coincide with Bălibanu’s isomorphism

$$\overline{Z}_g \xrightarrow{\cong} \nu^{-1}(S) \xleftarrow{\nu^{-1}} \nu^{-1}(x) = \text{Hess}(x, \mathfrak{m})$$

A final requirement is that the fibrewise isomorphism $\mu^{-1}(x) \cong \nu^{-1}(x) = \text{Hess}(x, \mathfrak{m})$ somehow induce a closed embedding $\text{Hess}(x, \mathfrak{m}) \hookrightarrow \overline{G}$ for all $x \in \mathfrak{g}$.

1.2. Summary of results. We construct a global version of Bălibanu’s Poisson isomorphism in the sense outlined above, developing a self-contained theory of Poisson slices in the process. The following is a more detailed summary of our results. We work exclusively over $\mathbb{C}$ and take all Poisson varieties to be smooth. Where advantageous, we use the left trivialization and Killing form to identify $T^*G$ with $G \times \mathfrak{g}$. 

The Killing form induces a $G$-module isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$, and the canonical Poisson structure on $\mathfrak{g}^*$ thereby renders $\mathfrak{g}$ a Poisson variety. Now let $\tau = (\xi, h, \eta)$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ and consider the associated Slodowy slice

$$S_\tau := \xi + \mathfrak{g}_\eta \subseteq \mathfrak{g}.$$ 

This slice is known to be a Poisson transversal in $\mathfrak{g}$. It immediately follows that $\mu^{-1}(S_\tau)$ is a Poisson transversal in $X$ for every Poisson variety $X$ equipped with a Hamiltonian $G$-action and moment map $\mu : X \to \mathfrak{g}$. One thereby obtains a Poisson structure on $\mu^{-1}(S_\tau)$, and we call this Poisson variety the Poisson slice in $X$ determined by $\tau$. This leads to some first results.

**Proposition 1.1.** Let $X$ be a Poisson variety endowed with a Hamiltonian $G$-action and moment map $\mu : X \to \mathfrak{g}$. Suppose that $\tau = (\xi, h, \eta)$ is any $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. The following statements hold.

(i) The Poisson slice $\mu^{-1}(S_\tau)$ is transverse to the $G$-orbits in $X$.

(ii) There are canonical Poisson variety isomorphisms

$$(X \times (G \times S_\tau)) \sslash G \cong \mu^{-1}(S_\tau) \cong X \sslash _{\xi} \mu_\tau.$$

The Hamiltonian $G$-space structure on $G \times S_\tau$ and meaning of the unipotent subgroup $U_\tau \subseteq G$ are given in Section 3.4.

We also consider some special cases of the Poisson slice construction, including the following well-known result.

**Observation 1.2.** Let $X$ be a symplectic variety endowed with a Hamiltonian action of $G$ and a moment map $\mu : X \to \mathfrak{g}$. Suppose that $\tau$ is any $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. The Poisson structure on $\mu^{-1}(S_\tau)$ makes it a symplectic subvariety of $X$.

Now suppose that the above-mentioned Poisson variety $X$ is log symplectic [19], by which the following is meant: $X$ has a unique open dense symplectic leaf, whose complement is a normal crossing divisor on which the top exterior power of the Poisson bivector vanishes to first order. We establish the following log symplectic counterpart of Observation 1.2.

**Proposition 1.3.** Let $X$ be a log symplectic variety endowed with a Hamiltonian $G$-action and moment map $\mu : X \to \mathfrak{g}$. Suppose that $\tau$ is any $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Each irreducible component of $\mu^{-1}(S_\tau)$ is then a Poisson subvariety of $\mu^{-1}(S_\tau)$. The resulting Poisson structure on each component makes the component a log symplectic subvariety of $X$.

We next consider the De Concini–Procesi wonderful compactification $\overline{G}$ and divisor $D := \overline{G} \setminus G$. The data $(\overline{G}, D)$ determine a log cotangent bundle $T^*\overline{G}(\log(D))$, which is known to have a canonical log symplectic structure. Its unique open dense symplectic leaf is $T^*G$, and the canonical Hamiltonian $(G \times G)$-action on $T^*G$ extends to such an action on $T^*\overline{G}(\log(D))$. The moment maps

$$\mu = (\mu_L, \mu_R) : T^*G \to \mathfrak{g} \oplus \mathfrak{g} \quad \text{and} \quad \overline{\mu} = (\overline{\mu}_L, \overline{\mu}_R) : T^*\overline{G}(\log(D)) \to \mathfrak{g} \oplus \mathfrak{g}$$

can be written in explicit terms.

Now fix a principal $\mathfrak{sl}_2$-triple $\tau = (\xi, h, \eta)$, i.e. $\tau$ is an $\mathfrak{sl}_2$-triple consisting of regular elements in $\mathfrak{g}$. Set $\mathcal{S} := S_\tau$ and note that $S \times \mathcal{S} \subseteq \mathfrak{g} \oplus \mathfrak{g}$ is the Slodowy slice determined by a principal $\mathfrak{sl}_2$-triple in $\mathfrak{g} \oplus \mathfrak{g}$. The following is straightforward, and its proof uses Observation 1.2 and Proposition 1.3.

**Observation 1.4.** We have

$$\mu^{-1}(S \times S) = \mathcal{Z}_\mathfrak{g}, \quad \overline{\mu}^{-1}(S \times S) = \overline{\mathcal{Z}_\mathfrak{g}}, \quad \text{and} \quad \mu_R^{-1}(\mathcal{S}) = G \times \mathcal{S}.$$ 

The first and third Poisson slices are symplectic, while the second is log symplectic.
In light of this observation, we consider the log symplectic variety
\[ G \times S := \mu_R^{-1}(S). \]

The Poisson slices \( G \times S \) and \( \overline{G \times S} \) carry residual Hamiltonian actions of \( G = G \times \{ e \} \subseteq G \times G \), and the respective moment maps are
\[ \mu_S := \mu_L|_{G \times S} \quad \text{and} \quad \overline{\mu}_S := \overline{\mu_L}|_{G \times S}. \]

One has a commutative diagram
\[
\begin{array}{ccc}
G \times S & \xrightarrow{\mu_S} & \overline{G \times S} \\
\downarrow \quad \mu_S & & \downarrow \quad \overline{\mu}_S \\
\mathfrak{g} & & \overline{\mathfrak{g}}
\end{array}
\]

where the horizontal arrow is inclusion. This diagram realizes \( \overline{G \times S} \) as a fibrewise compactification of \( G \times S \), and restricting to the Poisson slices
\[ Z_\mathfrak{g} = \mu_S^{-1}(S) \quad \text{and} \quad \overline{Z}_\mathfrak{g} = \overline{\mu}_S^{-1}(S) \]
yields
\[
\begin{array}{ccc}
Z_\mathfrak{g} & \xrightarrow{\phi^{-1}} & \overline{Z}_\mathfrak{g} \\
\downarrow \quad \mathfrak{g} & & \downarrow \quad \overline{\mathfrak{g}} \\
S & & \overline{S}
\end{array}
\]

The role of \( \overline{Z}_\mathfrak{g} \) in Bălibanu’s Poisson isomorphism is analogous to that of \( \overline{G \times S} \) in our main result, stated below.

**Theorem 1.5.** There is a canonical isomorphism
\[
\begin{array}{ccc}
\overline{G \times S} & \xrightarrow{\phi^{-1}} & G \times_B \mathfrak{m} \\
\downarrow \quad \overline{\mu}_S & & \downarrow \quad \nu \\
\mathfrak{g} & & \mathfrak{g}
\end{array}
\]

of Poisson varieties over \( \mathfrak{g} \). Restricting to the Poisson slices
\[ \overline{Z}_\mathfrak{g} = \overline{\mu}_S^{-1}(S) \quad \text{and} \quad \nu^{-1}(S) \]
(i.e. pulling this triangle back along the inclusion \( S \hookrightarrow \mathfrak{g} \)) yields Bălibanu’s Poisson isomorphism (1.1).

Observe that \( \phi^{-1} \) necessarily restricts to an isomorphism
\[ \text{Hess}(x, \mathfrak{m}) = \nu^{-1}(x) \xrightarrow{\cong} \overline{\mu}_S^{-1}(x) \]
for each \( x \in \mathfrak{g} \). This gives context for the following straightforward corollary.

**Corollary 1.6.** Retain the notation of Theorem 1.5 and let \( \pi : T^*G(\log(D)) \rightarrow \overline{G} \) be the bundle projection map. If \( x \in \mathfrak{g} \), then the composite map
\[ \text{Hess}(x, \mathfrak{m}) \xrightarrow{\phi^{-1}} \overline{\mu}_S^{-1}(x) \xrightarrow{\pi} \overline{G} \]
is a \( G_x \)-equivariant closed embedding.

A more concise statement is that every Hessenberg variety in the family \( \nu : G \times_B \mathfrak{m} \rightarrow \mathfrak{g} \) admits a canonical equivariant closed embedding into \( \overline{G} \).
1.3. **Organization.** Section 2 assembles some of the Lie-theoretic facts, conventions, and notation underlying this paper. We devote Section 3 to the general theory of Poisson slices, as well as the proofs of Propositions 1.1 and 1.3. The heart of our paper begins in Section 4, which is principally concerned with the log symplectic variety \( G \times S \) and its properties. Section 5 expands our discussion to include implications for Hessenberg varieties. This section contains the proofs of Theorem 1.5 and Corollary 1.6, as well as the requisite machinery. A brief list of recurring notation appears after Section 5.

**Acknowledgements.** We gratefully acknowledge Ana Bălibanu for constructive conversations and suggestions. The first author is supported by an NSERC Postdoctoral Fellowship [PDF–516638].

2. Some preliminaries

This section gathers some of the notation, conventions, and standard facts used throughout our paper. Our principal objective is to outline the Lie-theoretic constructions relevant to later sections.

2.1. **Fundamental conventions.** This paper works exclusively over \( \mathbb{C} \). We understand “group action” as meaning “left group action”. The dimension of an algebraic variety is the supremum of the dimensions of its irreducible components. We use the term smooth variety in reference to a pure-dimensional algebraic variety \( X \) satisfying \( \dim(T_x X) = \dim X \) for all \( x \in X \).

2.2. **Lie theory.** Let \( \mathfrak{g} \) be a finite-dimensional, rank-\( \ell \), semisimple Lie algebra over \( \mathbb{C} \), and denote its adjoint group by \( G \). We write
\[
\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g}), \quad g \mapsto \text{Ad}_g, \quad g \in G
\]
for the adjoint representation of \( G \) on \( \mathfrak{g} \), and
\[
\text{ad} : \mathfrak{g} \rightarrow \text{gl}(\mathfrak{g}), \quad x \mapsto \text{ad}_x, \quad x \in \mathfrak{g}
\]
for the adjoint representation of \( \mathfrak{g} \) on itself. Each \( x \in \mathfrak{g} \) admits a \( G \)-stabilizer
\[
G_x := \{ g \in G : \text{Ad}_g(x) = x \}
\]
and \( \mathfrak{g} \)-centralizer
\[
\mathfrak{g}_x := \ker(\text{ad}_x) = \{ y \in \mathfrak{g} : [x, y] = 0 \}.
\]
One also has the \( G \)-invariant, open, dense subvariety
\[
\mathfrak{g}^r := \{ x \in \mathfrak{g} : \dim(\mathfrak{g}_x) = \ell \}
\]
of all regular elements in \( \mathfrak{g} \).

One calls \( x \in \mathfrak{g} \) semisimple (resp. nilpotent) if \( \text{ad}_x \in \text{gl}(\mathfrak{g}) \) is diagonalizable (resp. nilpotent) as a vector space endomorphism. Let \( \mathfrak{g}^{rs} \) denote the open, dense, \( G \)-invariant subvariety of regular semisimple elements in \( \mathfrak{g} \). We also set
\[
V^r := V \cap \mathfrak{g}^r \quad \text{and} \quad V^{rs} := V \cap \mathfrak{g}^{rs}
\]
for any subset \( V \subseteq \mathfrak{g} \).

Let \( \langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathbb{C} \mathfrak{g} \rightarrow \mathbb{C} \) denote the Killing form, and write \( V^\perp \subseteq \mathfrak{g} \) for the annihilator of a subspace \( V \subseteq \mathfrak{g} \) under this form. The Killing form is non-degenerate and \( G \)-invariant, implying that
\[
\mathfrak{g} \rightarrow \mathfrak{g}^*, \quad x \mapsto \langle x, \cdot \rangle, \quad x \in \mathfrak{g}
\]
defines a \( G \)-module isomorphism. We use this isomorphism to freely identify \( \mathfrak{g} \) and \( \mathfrak{g}^* \) throughout the paper.
The canonical Poisson structure on $\mathfrak{g}^*$ renders $\mathfrak{g}$ a Poisson variety. This endows the coordinate algebra $\mathbb{C}[\mathfrak{g}] = \text{Sym}(\mathfrak{g}^*)$ with a Poisson bracket, defined as follows:

$$ \{f_1, f_2\}(x) = \langle x, [(df_1)_x, (df_2)_x] \rangle $$

for all $f_1, f_2 \in \mathbb{C}[\mathfrak{g}]$ and $x \in \mathfrak{g}$, where $(df_1)_x, (df_2)_x \in \mathfrak{g}^*$ are regarded as elements of $\mathfrak{g}$ via (2.1). We also note that the symplectic leaves of $\mathfrak{g}$ are precisely the adjoint orbits $Gx := \{\text{Ad}_g(x) : g \in G\}, \quad x \in \mathfrak{g}$.

### 2.3. Slodowy Slices.

Recall that $\tau = (\xi, h, \eta) \in \mathfrak{g}^{\oplus 3}$ is called an $\mathfrak{sl}_2$-triple if the identities

$$ [h, \xi] = 2\xi, \quad [h, \eta] = -2\eta, \quad \text{and} \quad [\xi, \eta] = h $$

hold in $\mathfrak{g}$. One then has a unique Lie algebra morphism $\phi : \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{g}$ satisfying

$$ \phi \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) = \xi, \quad \phi \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) = h, \quad \text{and} \quad \phi \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) = \eta. \quad (2.2) $$

It follows that $\xi$ and $\eta$ are nilpotent, and that $h$ is semisimple. We also have the decomposition

$$ \mathfrak{g} = \text{image}(\text{ad}_\xi) \oplus \ker(\text{ad}_\eta) = [\mathfrak{g}, \xi] \oplus \mathfrak{g}_\eta = T_\xi(G\xi) \oplus \mathfrak{g}_\eta, \quad (2.3) $$

as follows from the representation theory of $\mathfrak{sl}_2(\mathbb{C})$.

The following well-known result considers the Slodowy slice $S_\tau := \xi + \mathfrak{g}_\eta \subseteq \mathfrak{g}$ determined by $\tau$.

**Lemma 2.1.** If $\tau$ is an $\mathfrak{sl}_2$-triple, then $S_\tau$ is transverse to every adjoint orbit.

**Proof.** The decomposition (2.3) amounts to $S_\tau$ and $G\xi$ being transverse at $\xi$. It follows that an open neighbourhood $U$ of $\xi$ in $S_\tau$ is transverse to every adjoint orbit.

Noting that $\text{SL}_2(\mathbb{C})$ is simply-connected, $\phi$ integrates to a Lie group morphism

$$ \tilde{\phi} : \text{SL}_2(\mathbb{C}) \rightarrow G. $$

Now consider the one-parameter subgroup

$$ \lambda : \mathbb{C}^\times \rightarrow G, \quad t \mapsto \tilde{\phi} \left( \begin{array}{cc} 0 & t \\ 0 & t^{-1} \end{array} \right), \quad t \in \mathbb{C}^\times, $$

and note that $\mathbb{C}^\times$ acts on $S_\tau$ via

$$ t \cdot x = t^{-2} \text{Ad}_{\lambda(t)}(x), \quad t \in \mathbb{C}^\times, \quad x \in S_\tau. \quad (2.4) $$

The same formula defines a linear action of $\mathbb{C}^\times$ on $\mathfrak{g}_\eta$, and the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ forces this linear action to have strictly negative weights. If $x \in S_\tau$, then the previous sentence and (2.4) allow one to find $t \in \mathbb{C}^\times$ satisfying $t \cdot x \in U$. We conclude that $S_\tau$ and the adjoint orbit $G(t \cdot x)$ are transverse at $t \cdot x$. Since $G(t \cdot x) = t^{-2}(Gx)$, this amounts to $S_\tau$ and $Gx$ being transverse at $x$. Our proof is therefore complete. \hfill \square
2.4. The adjoint quotient and Kostant section. Consider the subalgebra of \( \mathbb{C}[\mathfrak{g}] \) given by

\[
\mathbb{C}[\mathfrak{g}]^G := \{ f \in \mathbb{C}[\mathfrak{g}] : f(\text{Ad}_g(x)) = f(x) \text{ for all } g \in G \text{ and } x \in \mathfrak{g} \}.
\]

Denote by

\[
\chi : \mathfrak{g} \rightarrow \text{Spec}(\mathbb{C}[\mathfrak{g}]^G)
\]

the morphism of affine varieties corresponding to the inclusion \( \mathbb{C}[\mathfrak{g}]^G \subseteq \mathbb{C}[\mathfrak{g}] \). This morphism is called the adjoint quotient of \( \mathfrak{g} \).

Now let \( \tau = (\xi, h, \eta) \) be an \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g} \). One calls \( \tau \) a principal \( \mathfrak{sl}_2 \)-triple if \( \xi, h, \eta \in \mathfrak{g}^t \). The associated Slodowy slice \( S := S_\tau \) then consists of regular elements, and it is a fundamental domain for the adjoint action of \( G \) on \( \mathfrak{g}^t \). This slice is also called a Kostant section, reflecting the fact that

\[
\chi |_S : S \rightarrow \text{Spec}(\mathbb{C}[\mathfrak{g}]^G)
\]

is an isomorphism. We may therefore consider

\[
x_S := (\chi |_S)^{-1}(\chi(x))
\]

for each \( x \in \mathfrak{g} \), i.e. \( x_S \) is the unique point at which \( S \) meets \( \chi^{-1}(\chi(x)) \). It is known that

\[
\chi^{-1}(\chi(x)) = \overline{Gx_S} \quad \text{and} \quad \chi^{-1}(\chi(x)) \cap \mathfrak{g}^t = Gx_S
\]

for all \( x \in \mathfrak{g} \), and that

\[
\chi^{-1}(\chi(x)) = Gx = Gx_S
\]

for all \( x \in \mathfrak{g}^{rs} \).

3. Poisson slices

This section is concerned with the Poisson-geometric foundations of our work. The overarching objective is to elucidate a role for Poisson slices in each of the symplectic, log symplectic, and Poisson categories.

3.1. Poisson varieties. Let \( X \) be a smooth variety with structure sheaf \( \mathcal{O}_X \) and tangent bundle \( TX \). Suppose that \( P \) is a global section of \( \Lambda^2(TX) \), and consider the bracket operation defined by

\[
\{ f_1, f_2 \} := P(df_1 \wedge df_2) \in \mathcal{O}_X
\]

for all \( f_1, f_2 \in \mathcal{O}_X \). One calls \( P \) a Poisson bivector if this bracket renders \( \mathcal{O}_X \) a sheaf of Poisson algebras. We use the term Poisson variety in reference to a smooth variety \( X \) equipped with a Poisson bivector \( P \). In this case, \( \{.,.\} \) is called the Poisson bracket. Let us also recall that a variety morphism \( \phi : X_1 \rightarrow X_2 \) between Poisson varieties \( (X_1, P_1) \) and \( (X_2, P_2) \) is called a Poisson morphism if

\[
d\phi(P_1(\phi^*\alpha)) = P_2(\alpha)
\]

for all one-forms \( \alpha \) defined on any open subset of \( X_2 \).

Let \( (X, P) \) be a Poisson variety. Contracting the bivector with cotangent vectors allows one to view \( P \) as a bundle morphism

\[
P : T^*X \rightarrow TX,
\]

whose image is a holomorphic distribution on \( X \). One refers to the maximal integral submanifolds of this distribution as the symplectic leaves of \( X \). Each symplectic leaf \( Y \subseteq X \) carries a canonical symplectic form \( \omega_Y \), defined as follows on each tangent space \( T_yY = P(T^*_yX) \):

\[
(\omega_Y)_y(P_y((df_1)_y), P_y((df_2)_y)) = \{ f_1, f_2 \}(y)
\]

for all \( f_1, f_2 \in \mathcal{O}_X \) defined on an open neighbourhood of \( y \) in \( X \).
Recall that the Hamiltonian vector field associated with \( f \in \mathcal{O}_X \) is defined by

\[
    H_f := -P(df).
\]

This leads to the notion of a Hamiltonian action in the Poisson category, generalizing the more familiar version for symplectic varieties. To this end, let \( K \) be an algebraic group with Lie algebra \( \mathfrak{t} \). One calls an algebraic action of \( K \) on \( X \) Hamiltonian if \( K \) preserves \( P \) and there exists a \( K \)-equivariant morphism \( \mu : X \rightarrow \mathfrak{t}^* \) satisfying the following condition:

\[
    H_{\mu^y} = -V^y
\]

for all \( y \in \mathfrak{t} \), where \( \mu^y \in \mathcal{O}_X \) is defined by \( \mu^y(x) = (\mu(x))(y) \) for all \( x \in X \) and \( V^y \) denotes the fundamental vector field associated with \( y \). The triple \((X, P, \mu)\) is then called a Hamiltonian \( K \)-space, while \( \mu \) is called a moment map. One knows that \( \mu \) is a Poisson morphism with respect to the Lie–Poisson structure on \( \mathfrak{t}^* \) (e.g. [12, Proposition 7.1]).

Let \((X, P, \mu)\) be a Hamiltonian \( K \)-space. Suppose that \( \zeta \in \mathfrak{t}^* \) is fixed by the coadjoint action of \( K \), and that \( K \) acts freely on \( \mu^{-1}(\zeta) \). It follows that \( \zeta \) is a regular value of \( \mu \), implying that the closed subvariety \( \mu^{-1}(\zeta) \subseteq X \) is smooth. Now assume that the geometric quotient

\[
    \pi : \mu^{-1}(\zeta) \rightarrow \mu^{-1}(\zeta)/K
\]

exists in the category of algebraic varieties. The quotient variety

\[
    X \sslash \zeta := \mu^{-1}(\zeta)/K
\]

then inherits a Poisson bivector \( P_{X/\zeta K} : T^*(X \sslash \zeta K) \rightarrow T(X \sslash \zeta K) \). To describe it, suppose that \( x \in \mu^{-1}(\zeta) \). Define

\[
    (P_{X/\zeta K})_{\pi(x)} : T^*_{\pi(x)}(X \sslash \zeta K) \rightarrow T_{\pi(x)}(X \sslash \zeta K)
\]

by

\[
    (P_{X/\zeta K})_{\pi(x)}(\alpha) = d\pi_x(P_x(\tilde{\alpha}))
\]

for all \( \alpha \in T^*_{\pi(x)}(X \sslash \zeta K) \), where \( \tilde{\alpha} \in T^*_x X \) is any covector with the following property: the restriction of \( \tilde{\alpha} \) to \( T_x(\mu^{-1}(\zeta)) \) equals the image of \( \alpha \) under

\[
    (d\pi_x)^* : T^*_{\pi(x)}(X \sslash \zeta K) \rightarrow T^*_x(\mu^{-1}(\zeta)).
\]

Well-definedness is established via a straightforward calculation, and one calls \((X \sslash \zeta K, P_{X/\zeta K})\) the Hamiltonian reduction of \((X, P, \mu)\) at level \( \zeta \). In keeping with convention, we write \((X \sslash K, P_{X/\zeta K})\) for the Hamiltonian reduction of \((X, P, \mu)\) at level 0.

The preceding discussion generalizes to allow for Hamiltonian reduction at an arbitrary level \( \zeta \in \mathfrak{t}^* \). To this end, let \( K_\zeta \) denote the \( K \)-stabilizer of \( \zeta \) with respect to the coadjoint action. One simply sets

\[
    X \sslash \zeta K := \mu^{-1}(\zeta)/K_\zeta
\]

if \( K_\zeta \) acts freely on \( \mu^{-1}(\zeta) \) and the right-hand side exists as a geometric quotient in the category of algebraic varieties. The Poisson bivector \( P_{X/\zeta K} \) is then defined in a manner analogous to the one presented above.

We conclude by recalling the notion of a log symplectic variety. To this end, one calls a Poisson variety \((X, P)\) log symplectic if

1. \((X, P)\) has a unique open dense symplectic leaf \( X_0 \subseteq X \);
2. the complement \( D := X \setminus X_0 \) is a normal crossing divisor;
3. \( P^n \) vanishes to first order along \( D \), where \( 2n = \dim(X_0) \) and \( P^n \in H^0(X, \Lambda^{2n}(TX)) \) is the top exterior power of \( P \).

In this case, we call \( D \) the divisor of \((X, P)\).
Remark 3.1. Since symplectic leaves are connected, Condition (i) implies that log symplectic varieties are irreducible.

3.2. The symplectic and Poisson geometry of $T^*G$. Recall the Lie-theoretic notation and conventions established in Section [2]. In what follows, we use the Killing form and left trivialization to freely identify $T^*G$ and $G \times \mathfrak{g}$ as varieties. The canonical symplectic form on $T^*G$ thereby determines a symplectic form $\omega$ on $G \times \mathfrak{g}$, defined on each tangent space $T_{(g,x)}(G \times \mathfrak{g}) = T_g G \oplus \mathfrak{g}$ as follows:

$$\omega_{(g,x)} \left( ((dL_g)_e(y_1), z_1), ((dL_g)_e(y_2), z_2) \right) = \langle y_1, z_2 \rangle - \langle y_2, z_1 \rangle + \langle x, [y_1, y_2] \rangle$$

for all $y_1, y_2, z_1, z_2 \in \mathfrak{g}$, where $L_g : G \rightarrow G$ denotes left translation by $g$ and $(dL_g)_e : \mathfrak{g} \rightarrow T_g G$ is the differential of $L_g$ at $e \in G$ (see [23], Section 5, Equation (14L)). It is straightforward to verify that

$$(g_1, g_2) \cdot (h, x) = (g_1 h g_2^{-1}, \text{Ad}_{g_2}(x)), \quad (g_1, g_2) \in G \times G, \quad (h, x) \in G \times \mathfrak{g}$$

(3.3)
defines a Hamiltonian action of $G \times G$ on $G \times \mathfrak{g}$, and that

$$\mu = (\mu_L, \mu_R) : T^*G = G \times \mathfrak{g} \rightarrow \mathfrak{g} \oplus \mathfrak{g}, \quad (g, x) \mapsto (\text{Ad}_g(x), x), \quad (g, x) \in G \times \mathfrak{g}$$

(3.4)
is a moment map. We take this moment map to be $(\mathfrak{g} \oplus \mathfrak{g})$-valued via the isomorphism

$$\mathfrak{g} \oplus \mathfrak{g} \cong (\mathfrak{g} \oplus \mathfrak{g})^*$$

induced by the bilinear form $\langle \cdot, \cdot \rangle - \langle \cdot, \cdot \rangle$ on $\mathfrak{g} \oplus \mathfrak{g}$. Observe that $\mu_L$ (resp. $\mu_R$) is a moment map for the Hamiltonian action of $G = G \times \{e\} \subseteq G \times G$ (resp. $G = \{e\} \times G \subseteq G \times G$).

Now consider the identifications

$$T_{(e,x)}(G \times \mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g} \quad \text{and} \quad T_{(e,x)}^*(G \times \mathfrak{g}) = (\mathfrak{g} \oplus \mathfrak{g})^* = \mathfrak{g}^* \oplus \mathfrak{g}^*$$

for each $x \in \mathfrak{g}$. Write $P_\omega$ for the Poisson bivector on $G \times \mathfrak{g}$ determined by $\omega$, noting that $(P_\omega)_{(e,x)}$ is a vector space isomorphism

$$(P_\omega)_{(e,x)} : \mathfrak{g}^* \oplus \mathfrak{g}^* \cong \mathfrak{g} \oplus \mathfrak{g}$$

for each $x \in \mathfrak{g}$. To compute $(P_\omega)_{(e,x)}$, let

$$\kappa : \mathfrak{g}^* \cong \mathfrak{g}$$

denote the inverse of $(2.1)$ and consider the following lemma.

Lemma 3.2. If $x \in \mathfrak{g}$, then

$$(P_\omega)_{(e,x)}(\alpha, \beta) = (\kappa(\beta), [x, \kappa(\beta)] - \kappa(\alpha))$$

for all $(\alpha, \beta) \in \mathfrak{g}^* \oplus \mathfrak{g}^*$.

Proof. Write $P_\omega(\alpha, \beta) = (y, z) \in \mathfrak{g} \oplus \mathfrak{g}$ and note that

$$\alpha(v) + \beta(w) = \omega_{(e,x)}((P_\omega)_{(e,x)}(\alpha, \beta), (v, w))$$

$= \omega_{(e,x)}((y, z), (v, w))$

$= \langle y, w \rangle - \langle z, v \rangle + \langle x, [y, v] \rangle$

$= \langle y, w \rangle + \langle [x, y] - z, v \rangle.$

for all $v, w \in \mathfrak{g}$. It follows that

$$\kappa(\alpha) = [x, y] - z \quad \text{and} \quad \kappa(\beta) = y,$$

or equivalently

$$y = \kappa(\beta) \quad \text{and} \quad z = [x, \kappa(\beta)] - \kappa(\alpha).$$

$\square$
3.3. Poisson slices. Let \((X, P)\) be a Poisson variety. Given \(x \in X\) and a subspace \(V \subseteq T_x X\), we write \(V^\dagger\) for the annihilator of \(V\) in \(T^*_x X\). Our notation suppresses the dependence of \(V^\dagger\) on \(T_x X\), as the ambient tangent space will always be clear from context.

Recall that a smooth, locally closed subvariety \(Y \subseteq X\) is called a Poisson transversal (or cosymplectic subvariety) if
\[
T_y X = T_y Y \oplus P_y((T_y Y)^\dagger)
\] for all \(y \in Y\). This has the following straightforward implication for every symplectic leaf \(L \subseteq X\): \(L\) and \(Y\) have a transverse intersection in \(X\), and \(L \cap Y\) is a symplectic submanifold of \(L\).

The Poisson transversal \(Y\) inherits a Poisson bivector \(P_Y\) from \((X, P)\). To define it, note that the decomposition (3.5) gives rise to an inclusion \(T_y Y \subseteq T^*_y X\) for all \(y \in Y\). One can verify that
\[
P_y((T_y Y)^\dagger) \subseteq T_y Y,
\] and \(P_Y\) is then defined to be the restriction
\[
P_Y := P|_{T^*Y} : T^*Y \to TY.
\]
Note that \(Y\) need not be a Poisson subvariety of \(X\) in the usual sense; restricting functions need not define a morphism \(\mathcal{O}_X \to j_*\mathcal{O}_Y\) of sheaves of Poisson algebras, where \(j : Y \hookrightarrow X\) is the inclusion. This is particularly apparent if \(X\) is symplectic; the Poisson transversals are the symplectic subvarieties, while the Poisson subvarieties are the open subvarieties.

We record the following well-known fact for future reference (cf. [18, Example 4]).

**Lemma 3.3.** Let \(X\) be a symplectic variety. If \(Y \subseteq X\) is a Poisson transversal, then \(Y\) is a symplectic subvariety of \(X\). The resulting symplectic structure on \(Y\) coincides with the Poisson structure \(Y\) inherits as a transversal.

The following well-known result concerns the behaviour of Poisson transversals with respect to Poisson morphisms (cf. [18, Lemma 7]).

**Lemma 3.4.** Let \(\phi : X_1 \to X_2\) be a Poisson morphism between Poisson varieties \(X_1\) and \(X_2\). If \(Y \subseteq X_2\) is a Poisson transversal, then \(\phi^{-1}(Y)\) is a Poisson transversal in \(X_1\). The codimension of \(\phi^{-1}(Y)\) in \(X_1\) is equal to the codimension of \(Y\) in \(X_2\).

We need the following refinement in the case of log symplectic varieties.

**Proposition 3.5.** Suppose that \((X, P)\) is a log symplectic variety with divisor \(D\). Let \(Y \subseteq X\) be an irreducible Poisson transversal, and write \(P_{\text{tr}}\) for the resulting Poisson bivector on \(Y\). The following statements hold.

(i) The Poisson variety \((Y, P_{\text{tr}})\) is log symplectic with divisor \(D \cap Y\).

(ii) If one equips \(Y \setminus D\) and \(X \setminus D\) with the symplectic structures inherited as symplectic leaves of \((Y, P_{\text{tr}})\) and \((X, P)\), respectively, then \(Y \setminus D\) is a symplectic subvariety of \(X \setminus D\).

**Proof.** We begin by proving that \(Y\) is a log symplectic subvariety of \(X\) in the sense of [19, Definition 7.16]. To this end, consider the unique open dense symplectic leaf \(X_0 := X \setminus D \subseteq X\). Since \(Y\) is a Poisson transversal in \(X\), Lemma 3.3 forces \(Y_0 := Y \cap X_0\) to be a symplectic subvariety of \(X_0\).

Now let \(D_1, \ldots, D_k\) be the irreducible components of \(D\), and set
\[
D_I := \bigcap_{i \in I} D_i
\] for each subset \(I \subseteq \{1, \ldots, k\}\). Each irreducible component of \(D\) is a union of symplectic leaves in \(X\) (cf. [25, Exercise 5.2]), implying that \(D_I\) is a union of symplectic leaves for each \(I \subseteq \{1, \ldots, k\}\). On the other hand, the Poisson transversal \(Y\) is necessarily transverse to the symplectic leaves in \(X\). These last two sentences imply that \(Y\) is transverse to \(D_I\) for all \(I \subseteq \{1, \ldots, k\}\).
The previous two paragraphs show $Y$ to be a log symplectic subvariety of $X$, and we let $P_{\log}$ denote the resulting Poisson bivector on $Y$. It follows that $Y_0$ is the unique open dense symplectic leaf of $(Y, P_{\log})$, and that its symplectic form is the pullback of the symplectic form on $X_0$. We also know that $P_{\text{tr}}$ is non-degenerate on $Y_0$, and that it coincides with the pullback of the symplectic structure from $X_0$ to $Y_0$ (see Lemma 3.3). One concludes that $P_{\log}$ and $P_{\text{tr}}$ coincide on $Y_0$. Since $Y_0$ is dense in $Y$, it follows that $P_{\log} = P_{\text{tr}}$. This establishes (i) and (ii).

We now consider a more concrete application of Lemma 3.4. To this end, recall the Lie-theoretic notation and setup established in Section 2.

**Lemma 3.6.** Suppose that $(X, P, \mu)$ is a Hamiltonian $G$-space with moment map $\mu : X \to \mathfrak{g}$. If $\tau = (\xi, h, \eta)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, then $\mu^{-1}(S_\tau)$ is a Poisson transversal in $X$. This transversal has codimension $\dim \mathfrak{g} - \dim(\mathfrak{g}_\eta)$ in $X$.

**Proof.** Since $\mu : X \to \mathfrak{g}$ is a morphism of Poisson varieties (e.g. [12, Proposition 7.1]), it suffices to show that $S_\tau \subseteq \mathfrak{g}$ is a Poisson transversal. To this end, let $P_\mathfrak{g}$ denote the Poisson bivector on $\mathfrak{g}$. Note that $(P_\mathfrak{g})_x$ is a linear map

$$(P_\mathfrak{g})_x : \mathfrak{g}^* \to \mathfrak{g}$$

for each $x \in \mathfrak{g}$. Using (2.4) to regard $(P_\mathfrak{g})_x$ as a linear map

$$(P_\mathfrak{g})_x : \mathfrak{g} \to \mathfrak{g},$$

one readily verifies that

$$(P_\mathfrak{g})_x = \text{ad}_x.$$ (3.6)

If $x \in S_\tau$, then we have $T_xS_\tau = \mathfrak{g}_\eta$. Equation (3.6) then makes our objective one of establishing that

$$\mathfrak{g} = \mathfrak{g}_\eta \oplus [\mathfrak{g}_\eta, x]$$

for all $x \in S_\tau$, where $\mathfrak{g}_\eta$ is the Killing annihilator of $\mathfrak{g}_\eta$ in $\mathfrak{g}$. Since the skew-symmetry of $\text{ad}_\eta$ under the Killing form gives

$$\mathfrak{g}_\eta^\perp = [\mathfrak{g}, \eta],$$

this amounts to the assertion that

$$\mathfrak{g} = \mathfrak{g}_\eta \oplus [[\mathfrak{g}, \eta], x]$$ (3.7)

for all $x \in S_\tau$.

The representation theory of $\mathfrak{sl}_2$ implies that

$$\mathfrak{g} = \mathfrak{g}_\eta \oplus [\mathfrak{g}, \xi] = \mathfrak{g}_\xi \oplus [\mathfrak{g}, \eta].$$

The second decomposition forces $[\mathfrak{g}, \xi] = [[\mathfrak{g}, \eta], \xi]$ to hold, and the first then gives

$$\mathfrak{g} = \mathfrak{g}_\eta \oplus [[\mathfrak{g}, \eta], \xi].$$

In other words, (3.7) holds at $x = \xi$. It follows that (3.7) holds for all $x$ in an open neighbourhood $U \subseteq S_\tau$ of $\xi$. On the other hand, consider the contracting $\mathbb{C}^\times$-action on $S_\tau$ defined in (2.4). Note that if $x \in S_\tau$ satisfies (3.7), then the same is true of $t \cdot x$ for all $t \in \mathbb{C}^\times$. A second observation is that every $x \in S_\tau$ admits a $t \in \mathbb{C}^\times$ for which $t \cdot x \in U$. These last two sentences force (3.7) to hold for all $x \in S_\tau$, showing that $S_\tau$ is a Poisson transversal in $\mathfrak{g}$. The statement about codimension is a direct consequence of Lemma 3.4. The proof is therefore complete.

Let $Z$ be an irreducible component of $\mu^{-1}(S_\tau)$. The transversal $\mu^{-1}(S_\tau)$ is smooth (see Proposition 3.5) and hence pure-dimensional, so that $P_{\tau}$ necessarily restricts to a Poisson bivector $P_{Z,\tau}$ on $Z$. This leads to the following observation.
Corollary 3.7. Suppose that \((X, P, \mu)\) is a Hamiltonian \(G\)-space with moment map \(\mu : X \to \mathfrak{g}\). Assume that \((X, P)\) is log symplectic with divisor \(D\), and let \(\tau\) be an \(\mathfrak{sl}_2\)-triple in \(\mathfrak{g}\). Let \(Z\) be an irreducible component of \(\mu^{-1}(\mathcal{S}_\tau)\).

(i) The Poisson variety \((Z, P_Z, \tau)\) is log symplectic with divisor \(Z \cap D\).

(ii) If one equips \(Z \setminus D\) and \(X \setminus D\) with the symplectic structures inherited as symplectic leaves of \((Z, P_Z, \tau)\) and \((X, P)\), respectively, then \(Z \setminus D\) is a symplectic subvariety of \(X \setminus D\).

(iii) If \((X, P)\) is symplectic, then \((\mu^{-1}(\mathcal{S}_\tau), P_\tau)\) is symplectic and the symplectic form on \((X, P)\) pulls back to the symplectic form on \((\mu^{-1}(\mathcal{S}_\tau), P_\tau)\).

Proof. This follows immediately from Lemma 3.3, Proposition 3.5, and Lemma 3.6.

The following additional fact is used extensively in later sections.

Corollary 3.8. If \(\tau\) is an \(\mathfrak{sl}_2\)-triple in \(\mathfrak{g}\), then \(G \times \mathcal{S}_\tau\) is a symplectic subvariety of \(G \times \mathfrak{g} \cong T^*G\).

Proof. Apply Corollary 3.7(iii) to \(X = T^*G\) with the Hamiltonian action of \(G = \{e\} \times G \subseteq G \times G\) (see (2.3)) and moment map \(\mu_R\) (see (3.4)).

One further consequence of Lemma 3.6 is that \(\mu^{-1}(\mathcal{S}_\tau)\) inherits a Poisson bivector \(P_\tau\) from \((X, P)\). This gives rise to our notion of a Poisson slice.

Definition 3.9. Suppose that \((X, P, \mu)\) is a Hamiltonian \(G\)-space with moment map \(\mu : X \to \mathfrak{g}\), and let \(\tau\) be an \(\mathfrak{sl}_2\)-triple in \(\mathfrak{g}\). We call \((\mu^{-1}(\mathcal{S}_\tau), P_\tau)\) the Poisson slice of \((X, P, \mu)\) with respect to \(\tau\).

The following proposition is a generalization of Lemma 2.1 and it explains why we call \((\mu^{-1}(\mathcal{S}_\tau), P_\tau)\) a Poisson slice; it is a slice for the \(G\)-action on \(X\) in the following sense.

Proposition 3.10. Let \((X, P, \mu)\) be a Hamiltonian \(G\)-space with moment map \(\mu : X \to \mathfrak{g}\). If \(\tau\) is an \(\mathfrak{sl}_2\)-triple in \(\mathfrak{g}\), then \(\mu^{-1}(\mathcal{S}_\tau)\) is transverse to the \(G\)-orbits in \(X\).

Proof. Fix \(x \in \mu^{-1}(\mathcal{S}_\tau)\) and set \(y := \mu(x) \in \mathcal{S}_\tau\). Consider the differential \(d\mu_x : T_x X \to \mathfrak{g}\) and its dual \(d\mu^*_x : \mathfrak{g}^* \to T^*_x X\), and let \(P_\mathfrak{g}\) be the Poisson bivector on \(\mathfrak{g}\). Since \(\mu\) is a morphism of Poisson varieties, we have

\[ (P_\mathfrak{g})_y = d\mu_x \circ P_x \circ d\mu^*_x. \]

We also know \(\mathcal{S}_\tau \subseteq \mathfrak{g}\) to be a Poisson transversal (e.g. by Lemma 3.6), so that

\[ \mathfrak{g} = T_y \mathcal{S}_\tau \oplus (P_\mathfrak{g})_y((T_y \mathcal{S}_\tau)^\dagger) = T_y \mathcal{S}_\tau \oplus d\mu_x(P_x(d\mu^*_x((T_y \mathcal{S}_\tau)^\dagger))). \]

One immediate conclusion is that \(\mu\) is transverse to \(\mathcal{S}_\tau\). We also conclude that

\[ T_x(\mu^{-1}(\mathcal{S}_\tau)) = \ker \left( \text{pr}_2 \circ d\mu_x : T_x X \to (P_\mathfrak{g})_y((T_y \mathcal{S}_\tau)^\dagger) \right), \]

where

\[ \text{pr}_2 : \mathfrak{g} = T_y \mathcal{S}_\tau \oplus (P_\mathfrak{g})_y((T_y \mathcal{S}_\tau)^\dagger) \to (P_\mathfrak{g})_y((T_y \mathcal{S}_\tau)^\dagger) \]

is the natural projection. It follows that

\[ T_x(\mu^{-1}(\mathcal{S}_\tau))^\dagger = \text{image} \left( d\mu^*_x \circ \text{pr}_2^* : (P_\mathfrak{g})_y((T_y \mathcal{S}_\tau)^\dagger)^* \to T^*_x X \right), \]

where

\[ \text{pr}_2^* : (P_\mathfrak{g})_y((T_y \mathcal{S}_\tau)^\dagger)^* \to \mathfrak{g}^* \]

is the dual of \(\text{pr}_2\). This amounts to the statement that

\[ T_x(\mu^{-1}(\mathcal{S}_\tau))^\dagger = (d\mu_x)^*(\mathfrak{g}_y^*), \]
while we know that the Killing form identifies $g^\perp_\eta \subseteq g^*$ with $g^\perp = [g, \eta] \subseteq g$. We conclude that

$$T_x(\mu^{-1}(S_\tau))^\perp = \text{span}\{(d\mu^{[n,b]})_x : b \in g\},$$

where $\mu^{[n,b]} : X \to \mathbb{C}$ is defined by

$$\mu^{[n,b]}(z) = \langle \mu(z), [\eta, b] \rangle, \quad z \in X.$$

Equations (3.1) and (3.2) now imply that

$$P_x(T_x(\mu^{-1}(S_\tau))^\perp) = \text{span}\{P_x((d\mu^{[n,b]})_x) : b \in g\} = \text{span}\{V_x^{[n,b]} : b \in g\} \subseteq T_x(Gx).$$

This combines with $\mu^{-1}(S_\tau)$ being a Poisson transversal to yield

$$T_xX = T_x(\mu^{-1}(S_\tau)) \oplus P_x(T_x(\mu^{-1}(S_\tau))^\perp) = T_x(\mu^{-1}(S_\tau)) + T_x(Gx),$$

completing the proof.

\[\square\]

3.4. Poisson slices via Hamiltonian reduction. Recall the Hamiltonian action of $G \times G$ on $T^*G \cong G \times g$ discussed in Section 3.2. Observe that the symplectic subvariety $G \times S_\tau = \mu_R^{-1}(S_\tau)$ is necessarily invariant under $G = G \times \{e\} \subseteq G \times G$, and that

$$\mu_L|_{G \times S_\tau} : G \times S_\tau \to g, \quad (g, x) \mapsto \text{Ad}_g(x), \quad (g, x) \in G \times S_\tau$$

is a corresponding moment map. Now let $(X, P, \mu_X)$ be a Hamiltonian $G$-space with moment map $\mu_X : X \to g$. Consider the product $X \times (G \times S_\tau)$ with Poisson structure given by $P_\tau = P \oplus (-Q_\tau)$, where $Q_\tau$ is the Poisson bivector induced by the symplectic structure on $G \times S_\tau$. The diagonal action of $G$ on $X \times (G \times S_\tau)$ is then Hamiltonian with moment map

$$\mu : X \times (G \times S_\tau) \to g, \quad (x, g, y) \mapsto \mu_X(x) - \text{Ad}_g(y), \quad (x, g, y) \in X \times (G \times S_\tau).$$

These considerations allow us to realize Poisson slices via Hamiltonian reduction.

Proposition 3.11. Let $(X, P, \mu_X)$ be a Hamiltonian $G$-space with moment map $\mu_X : X \to g$, and let $\tau$ be an $\mathfrak{sl}_2$-triple in $g$. If we endow $X \times (G \times S_\tau)$ with the Poisson structure and Hamiltonian $G$-action described above, then there is a canonical Poisson isomorphism

$$(X \times (G \times S_\tau)) \parallel G \cong \mu_X^{-1}(S_\tau).$$

Proof. Recall the moment map $\mu : X \times (G \times S_\tau) \to g$ defined above, and note that

$$\mu^{-1}(0) = \{(x, g, y) \in X \times (G \times S_\tau) : \mu_X(x) = \text{Ad}_g(y)\}$$

$$= \{(x, g, y) \in X \times (G \times S_\tau) : \mu_X(g^{-1} \cdot x) = y\}.$$

It follows that

$$J : X \times (G \times S_\tau) \to X, \quad (x, g, y) \mapsto g^{-1} \cdot x, \quad (x, g, y) \in X \times (G \times S_\tau)$$

satisfies $J(\mu^{-1}(0)) \subseteq \mu_X^{-1}(S_\tau)$, thereby inducing a map

$$\pi := J|_{\mu^{-1}(0)} : \mu^{-1}(0) \to \mu_X^{-1}(S_\tau).$$

One then verifies that

$$\pi^{-1}(x) = G \cdot (x, e, \mu_X(x)) \subseteq X \times (G \times S_\tau)$$

for all $x \in \mu_X^{-1}(S_\tau)$, where $G \cdot (x, e, \mu_X(x))$ is the $G$-orbit of $(x, e, \mu_X(x))$ in $X \times (G \times S_\tau)$. This forces $\pi$ to be the geometric quotient of $\mu^{-1}(0)$ by $G$ (e.g. by [27, Proposition 25.3.5]), i.e.

$$(X \times (G \times S_\tau)) / G = \mu_X^{-1}(S_\tau).$$
We now have two Poisson structures on $\mu_X^{-1}(\mathcal{S}_\tau)$: the Poisson structure $P_{\text{red}}$ from Hamiltonian reduction, and the structure $P_{\text{tr}}$ obtained from $\mu_X^{-1}(\mathcal{S}_\tau)$ being a Poisson transversal in $X$. It suffices to show that these Poisson structures coincide.

Fix $x \in \mu_X^{-1}(\mathcal{S}_\tau)$ and $\alpha \in T^*_x(\mu_X^{-1}(\mathcal{S}_\tau))$. Since $\mu_X^{-1}(\mathcal{S}_\tau)$ is a Poisson transversal in $X$, there is a unique extension of $\alpha$ to an element

$$\tilde{\alpha} \in \left( P_x(T_x(\mu^{-1}(\mathcal{S}_\tau)))^\dagger \right) \subseteq T^*_x X.$$

The discussion of Poisson transversals in Section 3.3 then implies that

$$(P_{\text{tr}})_x(\alpha) = P_x(\tilde{\alpha}).$$

(3.10)

We also have

$$(P_{\text{red}})_x(\alpha) = d\pi_z((P_\tau)_x(\tilde{\alpha}')),\quad (3.11)$$

where $z = (x,e,\mu_X(x))$, $P_\tau = P \oplus (-Q_\tau)$ is the Poisson structure on $X \times (G \times \mathcal{S}_\tau)$,

$$\tilde{\alpha}' \in T^*_x(X \times (G \times \mathcal{S}_\tau))$$

is an arbitrary extension of $d\pi^*_z(\alpha)$, and

$$d\pi^*_z : T^*_x(\mu_X^{-1}(\mathcal{S}_\tau)) \rightarrow T^*_x(\mu^{-1}(0))$$

is the dual of

$$d\pi_z : T_x(\mu^{-1}(0)) \rightarrow T_x(\mu_X^{-1}(\mathcal{S}_\tau)).$$

Take

$$\tilde{\alpha}' := dJ^*_z(\tilde{\alpha})$$

and observe that

$$dJ_z(a,b,c) = a - (V^b)_x$$

for all $(a,b,c) \in T_z(X \times (G \times \mathcal{S}_\tau)) = T_zX \oplus g \oplus g_\eta$, where $V^b$ is the fundamental vector field on $X$ associated to $b \in g$. It follows that

$$(dJ^*_z(\tilde{\alpha}))(a,b,c) = \tilde{\alpha}(a) - \tilde{\alpha}((V^b)_x) = \tilde{\alpha}(a) - \tilde{\alpha}(P_x((d\mu^b_X)_x)) = \tilde{\alpha}(a) + (d\mu^b_X)_x(P_x(\tilde{\alpha})), $$

yielding

$$\tilde{\alpha}' = (\tilde{\alpha}, (d\mu_X)_x(P_x(\tilde{\alpha})), 0) \in T^*_x(X \times (G \times \mathcal{S}_\tau)) = T^*_x X \oplus g^* \oplus g_\eta^* = T^*_x X \oplus g \oplus g_\xi, \quad (3.12)$$

where we have made the identifications $g_\eta^* = (g/[g,\xi])^* = [g,\xi]^\perp = g_\xi$. Now set $w = (e,\mu_X(x)) \in G \times \mathcal{S}_\tau$ and note that Lemma 3.2 gives

$$(Q_\tau)_w((d\mu_X)_x(P_x(\tilde{\alpha})), 0) = (0, -(d\mu_X)_x(P_x(\tilde{\alpha}))).$$

This combines with (3.10), (3.11), and (3.12) to yield

$$(P_{\text{red}})_x(\alpha) = d\pi_z(P_x(\tilde{\alpha}), -(Q_\tau)_w((d\mu_X)_x(P_x(\tilde{\alpha})), 0)) = d\pi_z(P_x(\tilde{\alpha}), 0, (d\mu_X)_x(P_x(\tilde{\alpha}))) = P_x(\tilde{\alpha}) = (P_{\text{tr}})_x(\alpha),$$

as desired. \qed
Remark 3.12. In the special case $\tau = 0$, we have $S_\tau = g$ and $G \times S_\tau = G \times g = T^*G$. Now consider the action of $G = \{e\} \times G \subseteq G \times G$ on $T^*G$ coming from (3.3), and let $G$ act trivially on $X$. The resulting diagonal $G$-action on $X \times T^*G$ commutes with the one on $X \times T^*G$ discussed in the previous proof. The Hamiltonian reduction

$$(X \times T^*G) \sslash G$$

taken in Proposition 3.11 thereby carries a residual Hamiltonian $G$-action, with respect to which (3.9) is an isomorphism

$$(X \times T^*G) \sslash G \cong X$$
of Hamiltonian $G$-spaces. This well-known fact will be invoked at a later time.

Our next result is that Poisson slices can be realized via Hamiltonian reduction with respect to unipotent radicals of parabolic subgroups. To formulate this result, let $\tau = (\xi,h,\eta)$ be an $\frak{sl}_2$-triple in $g$ and write $g_\lambda \subseteq g$ for the eigenspace of $\text{ad}_h$ with eigenvalue $\lambda \in \mathbb{Z}$. The parabolic subalgebra

$$p_\tau := \bigoplus_{\lambda \leq 0} g_\lambda$$

then has

$$u_\tau := \bigoplus_{\lambda < 0} g_\lambda$$
as its nilradical. Now consider the identifications

$$u_\tau^* \cong g/u_\tau^\perp = g/p_\tau \cong u_\tau^- := \bigoplus_{\lambda > 0} g_\lambda,$$

and thereby regard $\xi \in u_\tau^-$ as an element of $u_\tau^*$. Write $U_\tau \subseteq G$ for the unipotent subgroup with Lie algebra $u_\tau$, and let $(U_\tau)_\xi$ be the $U_\tau$-stabilizer of $\xi$ under the coadjoint action.

Remark 3.13. The Lie algebra of $(U_\tau)_\xi$ is given by

$$(u_\tau)_\xi = \bigoplus_{\lambda \leq -2} g_\lambda.$$

It follows that $(U_\tau)_\xi = U_\tau$ if and only if $\tau$ is an even $\frak{sl}_2$-triple, i.e. $g_{-1} = \{0\}$. If $\tau$ is a principal triple, then $\tau$ is even and $(U_\tau)_\xi = U_\tau$ is a maximal unipotent subgroup of $G$.

Let $(X,P,\mu)$ be a Hamiltonian $G$-space with moment map $\mu : X \to g$. The action of $U_\tau$ is also Hamiltonian with moment map $\mu_\tau := p_\tau \circ \mu$, where

$$g = p_\tau \oplus u_\tau^- \xrightarrow{p_\tau} u_\tau^- = u_\tau^*$$
is the projection. One has

$$\mu_\tau^{-1}(\xi) = \mu^{-1}(\xi + p_\tau),$$
while the proof of [9] Lemma 3.2] shows $(U_\tau)_\xi$ to act freely on $\xi + p_\tau$. It follows that $(U_\tau)_\xi$ acts freely on $\mu_\tau^{-1}(\xi)$. This leads us to prove Proposition 3.15, i.e. that the geometric quotient

$$X \sslash_{\xi} U_\tau = \mu_\tau^{-1}(\xi)/(U_\tau)_\xi$$
(3.13)
extists and is Poisson-isomorphic to $\mu^{-1}(S_\tau)$.

Remark 3.14. The type of Hamiltonian reduction performed in (3.13) is particularly well-studied in the case of a principal triple $\tau$. In this case, one sometimes calls the Poisson variety $X \sslash_{\xi} U_\tau$ a Kostant–Whittaker reduction (e.g. [316]). The nomenclature reflects Kostant’s result [21] Theorem 1.2].
Proposition 3.15. Let \((X, P, \mu)\) be a Hamiltonian \(G\)-space with moment map \(\mu : X \to g\). If \(\tau = (\xi, h, \eta)\) is an \(sl_2\)-triple in \(g\), then there is a canonical isomorphism

\[
X \sslash_{\xi} U_\tau \cong \mu^{-1}(S_\tau)
\]

of Poisson varieties.

Proof. We begin by exhibiting \(\mu^{-1}(S_\tau)\) as the geometric quotient of \(\mu^{-1}(\xi)\) by \((U_\tau)_\xi\). To this end, the proof of \([9\text{, Lemma 3.2}]\) explains that

\[
(U_\tau)_\xi \times S_\tau \to \xi + p_\tau, \quad (u, x) \mapsto \text{Ad}_u(x), \quad (u, x) \in (U_\tau)_\xi \times S_\tau
\]
defines a variety isomorphism. Composing the inverse of this isomorphism with the projection

\[
(U_\tau)_\xi \times S_\tau \to (U_\tau)_\xi
\]
then yields a map

\[
\phi : \xi + p_\tau \to (U_\tau)_\xi.
\]

Note that for \(y \in \xi + p_\tau\), \(\phi(y)\) is the unique element of \((U_\tau)_\xi\) satisfying

\[
\text{Ad}_{\phi(y)^{-1}}(y) \in S_\tau.
\]

We may therefore define the map

\[
\mu^{-1}_\tau(\xi) = \mu^{-1}(\xi + p_\tau) \to \mu^{-1}(S_\tau), \quad \phi(\mu(x))^{-1} \cdot x, \quad x \in \mu^{-1}(\xi).
\]

One has

\[
\theta^{-1}(x) = (U_\tau)_\xi \cdot x
\]
for all \(x \in \mu^{-1}(\xi)\), and we deduce that \(\theta\) is the geometric quotient of \(\mu^{-1}_\tau(\xi)\) by \((U_\tau)_\xi\) (e.g. by \([27\text{, Proposition 25.3.5}]\)).

The previous paragraph establishes the following fact: Hamiltonian reductions of Hamiltonian \(G\)-spaces by \(U_\tau\) at level \(\xi\) always exist as geometric quotients. We implicitly use this observation in several places below.

To see that the Poisson structures on \(\mu^{-1}(S_\tau)\) and \(X \sslash_{\xi} U_\tau\) coincide, we argue as follows. One has a canonical isomorphism

\[
T^*G \sslash_{\xi} U_\tau \cong G \times S_\tau
\]
(3.14)
of symplectic varieties, where \(U_\tau\) acts on \(T^*G\) via \([3.3]\) as the subgroup \(U_\tau = \{e\} \times U_\tau \subseteq G \times G\) (see \([9\text{, Lemma 3.2}]\)). Note also that \(T^*G \sslash_{\xi} U_\tau\) and \(G \times S_\tau\) come with Hamiltonian actions of \(G\) induced by the action of \(G = G \times \{e\}\) on \(T^*G \cong G \times g\). One then readily verifies that (3.14) is an isomorphism of Hamiltonian \(G\)-spaces.

Proposition [3.11] gives a canonical isomorphism of Poisson varieties

\[
\mu^{-1}(S_\tau) \cong (X \times (G \times S_\tau)) \sslash G.
\]

The previous paragraph allows us to write this isomorphism as

\[
\mu^{-1}(S_\tau) \cong (X \times (T^*G \sslash_{\xi} U_\tau)) \sslash G = ((X \times T^*G) \sslash_{\xi} U_\tau) \sslash G,
\]

where \(U_\tau\) acts trivially on \(X\). Since the actions of \(G\) and \(U_\tau\) on \(X \times T^*G\) commute with one another, it follows that

\[
\mu^{-1}(S_\tau) \cong ((X \times T^*G) \sslash G) \sslash_{\xi} U_\tau.
\]

An application of Remark [3.12] then yields

\[
\mu^{-1}(S_\tau) \cong X \sslash_{\xi} U_\tau,
\]
completing the proof. \(\square\)
4. Poisson slices in the log cotangent bundle

In this section, we apply the Poisson slice construction to the log cotangent bundle of the wonderful compactification $\mathcal{G}$.

4.1. Additional conventions. Recall the Lie-theoretic objects, notation, and conventions introduced in Section 2. Fix a principal $\mathfrak{sl}_2$-triple $\tau = (\xi, h, \eta)$ in $\mathfrak{g}$, set

$$S := S_\tau = \xi + \mathfrak{g}_\eta,$$

and let $b \subseteq \mathfrak{g}$ be the unique Borel subalgebra that contains $\eta$. The Cartan subalgebra $t := \mathfrak{g}_h$ then satisfies $t \subseteq b$, and this gives rise to collections of roots $\Phi \subseteq t^*$, positive roots $\Phi^+ \subseteq \Phi$, negative roots $\Phi^- = -\Phi^+$, and simple roots $\Pi \subseteq \Phi^+$. Each subset $I \subseteq \Pi$ then determines parabolic subalgebras

$$p_I := b \oplus \bigoplus_{\alpha \in \Phi_I^-} \mathfrak{g}_\alpha \quad \text{and} \quad p_I^- = b^- \oplus \bigoplus_{\alpha \in \Phi_I^+} \mathfrak{g}_\alpha,$$

where $b^- \subseteq \mathfrak{g}$ is the Borel subalgebra opposite to $b$ with respect to $t$ and $\Phi_I^+$ (resp. $\Phi_I^-$) is the set of positive (resp. negative) roots in the $\mathbb{Z}$-span of $I$. Note that

$$l_I := p_I \cap p_I^-$$

is a Levi subalgebra of $\mathfrak{g}$. Let $u_I$ and $u_I^-$ denote the nilradicals of $p_I$ and $p_I^-$, respectively, observing that

$$p_I = l_I \oplus u_I \quad \text{and} \quad p_I^- = l_I \oplus u_I^-.$$

We have $p_\emptyset = b$, $b^- = p_\emptyset^-$, and $l_\emptyset = t$, and we adopt the notation

$$u := u_\emptyset \quad \text{and} \quad u^- := u_\emptyset^-.$$

4.2. The wonderful compactification of $G$. Observe that the adjoint action of $G \times G$ on $\mathfrak{g} \oplus \mathfrak{g}$ is given by

$$(g_1, g_2) \cdot (x, y) = (\text{Ad}_{g_1}(x), \text{Ad}_{g_2}(y)), \quad (g_1, g_2) \in G \times G, \ (x, y) \in \mathfrak{g} \oplus \mathfrak{g}.$$ 

This induces an action on the Grassmannian

$$\text{Gr}(n, \mathfrak{g} \oplus \mathfrak{g}),$$

where $n = \dim \mathfrak{g}$. Now consider the point

$$\mathfrak{g}_\Delta = \{(x, x) : x \in \mathfrak{g}\} \subseteq \text{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$$

and set

$$\gamma_g := (g, e) \cdot \mathfrak{g}_\Delta = \{(\text{Ad}_g(x), x) : x \in \mathfrak{g}\} \subseteq \text{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$$

for each $g \in G$. The map

$$\gamma : G \longrightarrow \text{Gr}(n, \mathfrak{g} \oplus \mathfrak{g}), \quad g \mapsto \gamma_g, \quad g \in G \quad (4.1)$$

is a $(G \times G)$-equivariant, locally closed embedding. The closure of its image is a $(G \times G)$-equivariant closed subvariety of $\text{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$, often denoted

$$\overline{\mathcal{G}} \subseteq \text{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$$

and called the wonderful compactification of $G$.

It is known that $\overline{\mathcal{G}}$ is smooth [17 Proposition 2.14], and that $D := \overline{\mathcal{G}} \setminus G$ is a normal crossing divisor [17 Theorem 2.22]. One also knows that $\overline{\mathcal{G}}$ is stratified into finitely many $(G \times G)$-orbits, indexed by the subsets $I \subseteq \Pi$ [17 Theorem 2.22 and Remark 3.9]. To describe the $(G \times G)$-orbit corresponding to $I \subseteq \Pi$, recall the Levi decompositions

$$p_I = l_I \oplus u_I \quad \text{and} \quad p_I^- = l_I \oplus u_I^-.$$
These allow us to define an $n$-dimensional subspace of $\mathfrak{g} \oplus \mathfrak{g}$ by
\[ p_I \times_{t I} p_I^- := \{(x, y) \in p_I \oplus p_I^- : x \text{ and } y \text{ have the same projection to } t I\}. \tag{4.2} \]
The $(G \times G)$-orbit corresponding to $I$ is then given by
\[ (G \times G)p_I \times_{t I} p_I^- \subseteq \mathcal{G}. \]
The following lemma implies that $p_I \times_{t I} p_I^- \in \mathcal{G}$, justifying the inclusion asserted above. This lemma features prominently in later sections.

**Lemma 4.1.** Let $\mathring{T} \subseteq G$ be a maximal torus with Lie algebra $\mathring{\mathfrak{t}} \subseteq \mathfrak{g}$, and consider a one-parameter subgroup $\lambda : \mathbb{C}^\times \longrightarrow \mathring{T}$. Let $p$ be the parabolic subalgebra spanned by $\mathring{t}$ and the root spaces $\mathfrak{g}_\alpha$ for all roots $\alpha$ of $(\mathfrak{g}, \mathring{\mathfrak{t}})$ satisfying $(\alpha, \lambda) \geq 0$, where $(\cdot, \cdot)$ is the pairing between weights and coweights. Let $\mathfrak{l} \subseteq \mathfrak{g}$ be the Levi subalgebra spanned by $\mathring{t}$ and all $\mathfrak{g}_\alpha$ with $(\alpha, \lambda) = 0$. Write $p^-$ for the opposite parabolic, spanned by $\mathring{t}$ and those root spaces $\mathfrak{g}_\alpha$ such that $(\alpha, \lambda) \leq 0$. We then have
\[ \lim_{t \to \infty} (\lambda(t), e) \cdot \mathfrak{g}_\Delta = p \times_1 p^-, \tag{4.2} \]
in $\text{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$, where the right-hand side is defined analogously to (4.2).

**Proof.** Choose a Borel subalgebra $\mathring{b} \subseteq \mathfrak{g}$ satisfying $\mathring{t} \subseteq \mathring{b} \subseteq p$, and write $\mathring{\Phi}$, $\mathring{\Phi}^+$, and $\mathring{H}$ for the associated sets of roots, positive roots, and simple roots, respectively. The subset $I := \{\alpha \in \mathring{H} : (\alpha, \lambda) = 0\}$ then corresponds to the standard parabolic subalgebra $p$. Let $\mathring{\Phi}_I^- \subseteq \mathring{\Phi}^+$ denote the set of positive roots in the $\mathring{\mathfrak{z}}$-span of $I$. We then have
\[ \mathring{\Phi}_I^- = \{\alpha \in \mathring{\Phi}^+ : (\alpha, \lambda) = 0\} \quad \text{and} \quad \mathring{\Phi}^+ \setminus \mathring{\Phi}_I^- = \{\alpha \in \mathring{\Phi}^+ : (\alpha, \lambda) > 0\}. \]

Choose a non-zero root vector $e_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \mathring{\Phi}$, and fix a basis $\{h_1, \ldots, h_\ell\}$ of $\mathring{t}$. It follows that
\[ \{(e_\alpha, e_\alpha)\}_{\alpha \in \mathring{\Phi}} \cup \{(h_i, h_i)\}_{i=1}^\ell \]
is a basis of $\mathfrak{g}_\Delta$. Now set
\[ E_\alpha := (e_\alpha, e_\alpha), \quad E_\alpha^1 := (e_\alpha, 0), \quad \text{and} \quad E_\alpha^2 := (0, e_\alpha) \]
for each $\alpha \in \mathring{\Phi}$, and also write
\[ H_i := (h_i, h_i) \]
for $i = 1, \ldots, \ell$. Observe that $(\lambda(t), e) \cdot \mathfrak{g}_\Delta$ then has a basis of
\[ \{t^{(\alpha, \lambda)}E_\alpha^1 + E_\alpha^2\}_{\alpha \in \mathring{\Phi}^+ \setminus \mathring{\Phi}_I^-} \cup \{t^{-(\alpha, \lambda)}E_{-\alpha}^1 + E_{-\alpha}^2\}_{\alpha \in \mathring{\Phi}^+ \setminus \mathring{\Phi}_I^-} \cup \{E_\beta\}_{\beta \in \mathring{\Phi}^+} \cup \{E_{-\beta}\}_{\beta \in \mathring{\Phi}^+_I} \cup \{H_i\}_{i=1}^\ell. \]
The image of $(\lambda(t), e) \cdot \mathfrak{g}_\Delta$ under the Plücker embedding
\[ \vartheta : \text{Gr}(n, \mathfrak{g} \oplus \mathfrak{g}) \longrightarrow \mathbb{P}(\Lambda^n (\mathfrak{g} \oplus \mathfrak{g})), \quad V \mapsto [\Lambda^n V], \quad V \in \text{Gr}(n, \mathfrak{g} \oplus \mathfrak{g}) \]
is therefore
\[ \vartheta((\lambda(t), e) \cdot \mathfrak{g}_\Delta) = \left[ \bigwedge_{\alpha \in \mathring{\Phi}^+ \setminus \mathring{\Phi}_I^-} (t^{(\alpha, \lambda)}E_\alpha^1 + E_\alpha^2) \wedge \bigwedge_{i=1}^\ell H_i \wedge \bigwedge_{\beta \in \mathring{\Phi}_I^+} (E_\beta \wedge E_{-\beta}) \wedge \bigwedge_{\alpha \in \mathring{\Phi}^+ \setminus \mathring{\Phi}_I^-} (t^{-(\alpha, \lambda)}E_{-\alpha}^1 + E_{-\alpha}^2) \right]. \]
We have
\[ \bigwedge_{\alpha \in \mathring{\Phi}^+ \setminus \mathring{\Phi}_I^-} (t^{(\alpha, \lambda)}E_\alpha^1 + E_\alpha^2) = t^{\sum_{\alpha \in \mathring{\Phi}^+ \setminus \mathring{\Phi}_I^+ (\alpha, \lambda)}} \left( \bigwedge_{\alpha \in \mathring{\Phi}^+ \setminus \mathring{\Phi}_I^+} (E_\alpha^1 + t^{-(\alpha, \lambda)}E_\alpha^2) \right), \]
and thus

\[ \vartheta((\lambda(t), e) \cdot g_{\Delta}) = \left[ \bigwedge_{\alpha \in \hat{\Phi}^+ \setminus \hat{\Phi}_I^+} (E^1_{\alpha} + t^{-(\alpha, \lambda)} E^2_{\alpha}) \wedge \bigwedge_{i=1}^{\ell} H_i \wedge \bigwedge_{\beta \in \hat{\Phi}_I^+} (E_{\beta} \wedge E_{-\beta}) \wedge \bigwedge_{\alpha \in \hat{\Phi}^+ \setminus \hat{\Phi}_I^+} (t^{-(\alpha, \lambda)} E^1_{-\alpha} + E^2_{-\alpha}) \right]. \]

All exponents of \( t \) appearing in this expression are strictly negative, implying that

\[ \lim_{t \to \infty} \vartheta((\lambda(t), e) \cdot g_{\Delta}) = \left[ \bigwedge_{\alpha \in \hat{\Phi}^+ \setminus \hat{\Phi}_I^+} E^1_{\alpha} \wedge \bigwedge_{i=1}^{\ell} H_i \wedge \bigwedge_{\beta \in \hat{\Phi}_I^+} (E_{\beta} \wedge E_{-\beta}) \wedge \bigwedge_{\alpha \in \hat{\Phi}^+ \setminus \hat{\Phi}_I^+} E^2_{-\alpha} \right] = \vartheta(p \times p^\perp). \]

The desired conclusion now follows from the \((G \times G)\)-equivariance of the Plücker embedding \( \vartheta \). \( \square \)

4.3. **Kostant–Whittaker reduction of the log cotangent bundle of \( \overline{G} \).** One can consider the log cotangent bundle

\[ \pi : T^* \overline{G}(\log D) \to \overline{G}, \]

i.e. the vector bundle associated with the locally free sheaf of logarithmic one-forms on \((\overline{G}, D)\). It turns out that \( T^* \overline{G}(\log D) \) is the pullback of the tautological bundle \( T \to \text{Gr}(n, g \oplus g) \) to the subvariety \( \overline{G} \) (see [5, Section 3.1] or [10, Example 2.5]). In other words,

\[ T^* \overline{G}(\log D) = \{ (\gamma, (x, y)) \in \overline{G} \times (g \oplus g) : (x, y) \in \gamma \} \quad (4.3) \]

and

\[ \pi(\gamma, (x, y)) = \gamma \]

for all \((\gamma, (x, y)) \in T^* \overline{G}(\log D)\).

The variety \( T^* \overline{G}(\log D) \) carries a natural log symplectic structure, some aspects of which we now describe. To this end, recall the identification \( T^* G = G \times g \) made in Section 3.2. The map

\[ \Psi : G \times g \to T^* \overline{G}(\log D), \quad (g, x) \mapsto (\gamma_g, \text{Ad}_g(x), x), \quad (g, x) \in G \times g \quad (4.4) \]

then defines an open embedding of \( T^* G \) into \( T^* \overline{G}(\log D) \), and it fits into a pullback square

\[ \begin{array}{ccc} T^* G & \xrightarrow{\Psi} & T^* \overline{G}(\log D) \\ \downarrow & & \downarrow \\ G & \xrightarrow{\gamma} & \overline{G} \end{array}. \]

This embedding is known to define a symplectomorphism from \( T^* G \) to the unique open dense symplectic leaf in \( T^* \overline{G}(\log D) \) [5, Section 3.3]. On the other hand, the following defines a Hamiltonian action of \( G \times G \) on \( T^* \overline{G}(\log D) \):

\[ (g_1, g_2) \cdot (\gamma, (x, y)) := ((g_1, g_2) \cdot \gamma, (\text{Ad}_{g_1}(x), \text{Ad}_{g_2}(y))), \quad (g_1, g_2) \in G \times G, \quad (\gamma, (x, y)) \in T^* \overline{G}(\log D), \]

where \((g_1, g_2) \cdot \gamma\) refers to the action of \( G \times G \) on \( \overline{G} \). An associated moment map is given by

\[ \mu = (\mu_L, \mu_R) : T^* \overline{G}(\log D) \to g \oplus g, \quad (\gamma, (x, y)) \mapsto (x, y), \quad (\gamma, (x, y)) \in T^* \overline{G}(\log D), \]

where the moment map is taken to be \((g \oplus g)\)-valued via the isomorphism \( g \oplus g \cong (g \oplus g)^* \) induced by the non-degenerate form \( \langle \cdot, \cdot \rangle \) (see [5, Section 3.2] and [10, Example 2.5]). The open embedding \( \Psi \) is then \((G \times G)\)-equivariant and satisfies

\[ \Psi^* \mu = \mu, \]

where \( \mu : T^* G \to g \oplus g \) is the moment map defined in (3.4).
Recall that we have fixed a regular $\mathfrak{sl}_2$-triple $\tau = (\xi, h, \eta)$ and set $\mathcal{S} := \mathcal{S}_\tau$. In what follows, we study the Poisson slice

$$\underline{G \times \mathcal{S}} := \overline{\mu_{\mathcal{R}}^{-1}(\mathcal{S})} \subseteq T^*\overline{G}(\log D)$$

and its properties. The notation $\underline{G \times \mathcal{S}}$ is justified by the following proposition.

**Proposition 4.2.** The Poisson slice $\underline{G \times \mathcal{S}}$ is log symplectic, and $\Psi(G \times \mathcal{S})$ is its unique open dense symplectic leaf. In particular, $\underline{G \times \mathcal{S}}$ is the closure of $\Psi(G \times \mathcal{S})$ in $T^*\overline{G}(\log D)$.

**Proof.** Since $T^*\overline{G}(\log D)$ is log symplectic, Corollary 3.7 implies that $\underline{G \times \mathcal{S}}$ is also log symplectic. This corollary also implies that the open dense symplectic leaf in $\underline{G \times \mathcal{S}}$ is obtained by intersecting $\underline{G \times \mathcal{S}}$ with the open dense symplectic leaf in $T^*\overline{G}(\log D)$. The latter leaf is $\Psi(T^*G)$, as discussed above. It then remains only to observe that

$$\Psi(G \times \mathcal{S}) = \underline{G \times \mathcal{S}} \cap \Psi(T^*G).$$

□

Note that $\underline{G \times \mathcal{S}}$ acts on $\underline{G \times \mathcal{S}}$ via (4.5) as the subgroup $\underline{G} = G \times \{e\} \subseteq G \times G$. Let us also observe that $\overline{\mu_{\mathcal{L}}}: T^*\overline{G}(\log D) \rightarrow \mathfrak{g}$ restricts to a $G$-equivariant map

$$\underline{\mu_{\mathcal{S}}} := \overline{\mu_{\mathcal{L}}}|_{\underline{G \times \mathcal{S}}}: \underline{G \times \mathcal{S}} \rightarrow \mathfrak{g}.$$  

We then have a commutative diagram

$$\begin{array}{ccc}
G \times \mathcal{S} & \xrightarrow{\Psi|_{G \times \mathcal{S}}} & \underline{G \times \mathcal{S}} \\
& \downarrow \mu_{\mathcal{S}} & \downarrow \underline{\mu_{\mathcal{S}}} \\
\mathfrak{g} & \xleftarrow{\mu_{\mathcal{S}}} & \mathfrak{g} \\
\end{array}$$

of $G$-equivariant maps, where

$$\mu_{\mathcal{S}} := \mu_{\mathcal{L}}|_{G \times \mathcal{S}}$$

is the map defined in (3.8).

**Remark 4.3.** Recall the maximal nilpotent subalgebra $\mathfrak{u} \subseteq \mathfrak{g}$ defined in Section 4.1 and let $U \subseteq G$ be the closed unipotent subgroup with Lie algebra $U$. Consider the subgroup

$$U = \{e\} \times U \subseteq G \times G$$

and its Hamiltonian action on $T^*\overline{G}(\log D)$. Proposition 3.15 then yields a canonical isomorphism

$$T^*\overline{G}(\log D)/\sslash_{\xi} U \cong \underline{G \times \mathcal{S}}$$

of Poisson varieties.

**Proposition 4.4.** The map $\Psi|_{G \times \mathcal{S}}$ defines a $G$-equivariant symplectomorphism from $G \times \mathcal{S}$ to the unique open dense symplectic leaf in $\underline{G \times \mathcal{S}}$.

**Proof.** Recall that $\Psi : \overline{T^*G} \rightarrow T^*\overline{G}(\log D)$ defines a symplectomorphism from $\overline{T^*G}$ to the unique open dense symplectic leaf in $T^*\overline{G}(\log D)$. Corollary 3.8 then implies that $\Psi$ restricts to a symplectomorphism from $G \times \mathcal{S}$ to $\Psi(G \times \mathcal{S})$, where the symplectic form on $\Psi(G \times \mathcal{S})$ is the pullback of the symplectic form on the leaf in $T^*\overline{G}(\log D)$. On the other hand, $\Psi(G \times \mathcal{S})$ inherits a symplectic structure as a symplectic leaf of $\underline{G \times \mathcal{S}}$ (see Proposition 4.2). It now follows from Corollary 3.7(ii) that

$$\Psi|_{G \times \mathcal{S}} : G \times \mathcal{S} \rightarrow \Psi(G \times \mathcal{S})$$

...
is a symplectomorphism with respect to the symplectic structure on $\Psi(G \times S)$ discussed in the previous sentence.

**Corollary 4.5.** The action of $G$ on the Poisson slice $\overline{G \times S}$ is Hamiltonian with moment map $\overline{\mu_S}$.

**Proof.** This follows immediately from Proposition 4.4, the diagram (1.6), the and the fact that $G \times S$ is a Hamiltonian $G$-space with moment map $\mu_S$. □

An explicit description of $\overline{G \times S}$ is obtained as follows. One begins by noting that

$$\overline{G \times S} = \mu^{-1}_R(S) = \{ (\gamma, (x, y)) \in \overline{G} \times (\mathfrak{g} \oplus \mathfrak{g}) : (x, y) \in \gamma \text{ and } y \in S \}.$$ 

On the other hand, recall the adjoint quotient

$$\chi : \mathfrak{g} \to \text{Spec}(\mathbb{C}[\mathfrak{g}]^G)$$

and the associated concepts discussed in Section 2.4. The image of $\chi$ is known to be

$$\{ (x, y) \in \mathfrak{g} \oplus \mathfrak{g} : \chi(x) = \chi(y) \}$$

(see [5, Proposition 3.4]), and this implies the simplified description

$$\overline{G \times S} = \{ (\gamma, (x, x_S)) \in \overline{G} \times (\mathfrak{g} \times S) : (x, x_S) \in \gamma \}.$$ 

(4.7)

**Remark 4.6.** The description (4.7) allows one to define a closed embedding

$$\overline{G \times S} \to \overline{G} \times \mathfrak{g}, \quad (\gamma, (x, x_S)) \mapsto (\gamma, x), \quad (\gamma, (x, x_S)) \in \overline{G \times S}.$$ 

We thereby obtain a commutative diagram

$$\begin{array}{ccc}
\overline{G \times S} & \longrightarrow & \overline{G} \times \mathfrak{g} \\
\downarrow \overline{\mu_S} & & \downarrow \\
\mathfrak{g} & \longrightarrow & \mathfrak{g}
\end{array}$$

where $\overline{G} \times \mathfrak{g} \to \mathfrak{g}$ is projection to the second factor. One immediate consequence is that $\overline{\mu_S}$ has projective fibres, so that (1.6) realizes $\overline{\mu_S}$ as a fibrewise compactification of $\mu_S$. It also follows that

$$\overline{\mu_S}^{-1}(x) \to \{ \gamma \in \mathfrak{g} : (x, x_S) \in \gamma \}, \quad (\gamma, (x, x_S)) \mapsto \gamma, \quad (\gamma, (x, x_S)) \in \overline{\mu_S}^{-1}(x)$$

is a variety isomorphism for each $x \in \mathfrak{g}$.

**4.4. Relation to the universal centralizer and its fibrewise compactification.** Recall that the universal centralizer of $\mathfrak{g}$ is the closed subvariety of $T^*G = G \times \mathfrak{g}$ defined by

$$Z_\mathfrak{g} := \{ (g, x) \in G \times \mathfrak{g} : x \in S \text{ and } g \in G_x \},$$

where $G_x$ is the $G$-stabilizer of $x \in \mathfrak{g}$. At the same time, recall the Hamiltonian action of $G \times G$ on $T^*G$ and moment map $\mu : T^*G \to \mathfrak{g} \oplus \mathfrak{g}$ discussed in Section 3.2. Consider the product $S \times S \subseteq \mathfrak{g} \oplus \mathfrak{g}$ and observe that

$$Z_\mathfrak{g} = \mu^{-1}(S \times S).$$

Note also that $S \times S$ is the Slodowy associated to the $\mathfrak{sl}_2$-triple $((\xi, \xi), (h, h), (\eta, \eta))$. It follows that $Z_\mathfrak{g}$ is a Poisson slice in $T^*G$. Corollary 3.7(iii) then forces this Poisson slice to be a symplectic subvariety of $T^*G$.

**Remark 4.7.** Some papers realize the symplectic structure on $Z_\mathfrak{g}$ via a Kostant–Whittaker reduction of $T^*G$ (e.g. [5]). To this end, let $U \subseteq G$ be the unipotent subgroup with Lie algebra $\mathfrak{u}$. Proposition 3.15 then gives a canonical isomorphism

$$Z_\mathfrak{g} = \mu^{-1}(S \times S) \cong T^*G \sslash (\xi, \xi) U \times U$$

of symplectic varieties, where the symplectic structure on $Z_\mathfrak{g}$ is as defined in the previous paragraph.
One may replace $\mu : T^*G \to g \oplus g$ with $\mu : T^*G(\log D) = g \oplus g$ and proceed analogously. In the interest of being more precise, consider the Poisson slice

$$Z_g := \mu^{-1}(S \times S) = \{(\gamma, (x, x)) \in G \times (g \oplus g) : x \in S \text{ and } (x, x) \in \gamma\}$$

in $T^*G(\log D)$.

**Remark 4.8.** A counterpart of Remark 4.7 is that Proposition 3.15 gives a canonical isomorphism

$$Z_g \cong T^*G(\log D) \sslash (\xi, \xi) \cup \cup U$$

of Poisson varieties. This realization of $Z_g$ via Kostant–Whittaker reduction is used to great effect in [5].

Corollary 3.7 implies that the Poisson slice $Z_g$ is a log symplectic variety with $Z_g \cap \Psi(T^*G)$ as its unique open dense symplectic leaf. One then uses the definition of $\Psi$ to deduce that

$$\Psi|_{Z_g} : Z_g \to \Psi(Z_g)$$

is a symplectomorphism, where the symplectic structure on $\Psi(Z_g)$ comes from its being a symplectic leaf in $Z_g$.

We have a commutative diagram

$$\begin{array}{ccc}
Z_g & \xrightarrow{\Psi|_{Z_g}} & \overline{Z_g} \\
\downarrow{\pi_S} & & \downarrow{\overline{\pi_S}} \\
S & & S
\end{array}$$

(4.8)

where

$$\pi_S(g, x) = x \quad \text{and} \quad \overline{\pi_S}(\gamma, (x, x)) = x.$$ 

This diagram is readily seen to be the pullback of (4.6) along the inclusion $S \hookrightarrow g$. This amounts to the restriction of (4.6) to a morphism between the Poisson slices

$$Z_g = \mu^{-1}(S \times S) = \mu S^{-1}(S) \quad \text{and} \quad \overline{Z_g} = \mu^{-1}(S \times S) = \overline{\mu S^{-1}}(S).$$

This present section combines with Section 4.3 to yield the following comparisons between $(Z_g, \overline{Z_g})$ and $(G \times S, \overline{G \times S})$:

- $\overline{\pi_S}$ (resp. $\overline{\mu_S}$) is a fibrewise compactification of $\pi_S$ (resp. $\mu_S$);
- (4.8) is obtained by pulling (4.6) back along the inclusion $S \hookrightarrow g$;
- $Z_g$ and $G \times S$ are symplectic;
- $\overline{Z_g}$ and $G \times \overline{S}$ are log symplectic;
- $\Psi$ restricts to a symplectomorphism from $Z_g$ (resp. $G \times S$) to the unique open dense symplectic leaf in $Z_g$ (resp. $G \times \overline{S}$).

### 5. Relation to the standard family of Hessenberg varieties

#### 5.1. The standard family of Hessenberg varieties

Recall the notation and conventions established in Section 4.1 and let $B \subseteq G$ be the Borel subgroup with Lie algebra $b$. Suppose that $H \subseteq g$ is a Hessenberg subspace, i.e. a $B$-invariant subspace of $g$ that contains $b$. Let $G \times B$ act on $G \times H$ via

$$(g, b) \cdot (h, x) := (ghb^{-1}, \text{Ad}_b(x)), \quad (g, b) \in G \times B, \quad (h, x) \in G \times H,$$
and consider the resulting smooth $G$-variety

$$G \times_B H := (G \times H)/B.$$ 

Write $[g : x]$ for the equivalence class of $(g, x) \in G \times H$ in $G \times_B H$. The variety $G \times_B H$ is naturally Poisson, and its $G$-action is Hamiltonian with moment map $\nu_H : G \times_B H \rightarrow \mathfrak{g}$, $[g : x] \mapsto \text{Ad}_g(x)$, $[g : x] \in G \times_B H$ (see [5, Section 4]). Given any $x \in \mathfrak{g}$, one writes

$$\text{Hess}(x, H) := \nu_H^{-1}(x)$$

and calls this fibre the $Hessenberg$ variety associated to $H$ and $x$. The Poisson moment map $\nu_H$ is thereby called the family of Hessenberg varieties associated to $H$.

**Remark 5.1.** Note that $G \times_B H$ is the total space of a $G$-equivariant vector bundle over $G/B$, and that the associated bundle projection map is

$$\pi_H : G \times_B H \rightarrow G/B, \quad [g : x] \mapsto [g], \quad [g : x] \in G \times_B H.$$ \hfill (5.1)

The map $$(\pi_H, \nu_H) : G \times_B H \rightarrow G/B \times \mathfrak{g}$$ is then a closed embedding. We also have a commutative diagram

$$\begin{array}{ccc} G \times_B H & \xrightarrow{(\pi_H, \nu_H)} & G/B \times \mathfrak{g} \\ \downarrow_{\nu_H} & & \downarrow \mathfrak{g} \\ \mathfrak{g} & & \end{array},$$

where $G/B \times \mathfrak{g} \rightarrow \mathfrak{g}$ is projection to the second factor. It follows that the fibres of $\nu_H$ are projective, and that $\pi_H$ restricts to a closed embedding

$$\text{Hess}(x, H) \hookrightarrow G/B$$

for each $x \in \mathfrak{g}$. One may thereby regard Hessenberg varieties as closed subvarieties of $G/B$.

In what follows, we restrict our attention to the so-called standard family of Hessenberg varieties. This is defined to be the family $\nu := \nu_m : G \times_B \mathfrak{m} \rightarrow \mathfrak{g}$ associated to the standard Hessenberg subspace

$$\mathfrak{m} := [u, u]^\perp = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Pi} \mathfrak{g}_{-\alpha}.$$ \hfill (5.2)

To study this family in more detail, we note the following consequences of the setup in Section 4.1:

$$\xi = \sum_{\alpha \in \Pi} e_{-\alpha} \text{ and } \mathfrak{g}_q \subseteq u,$$

where

$$e_{-\alpha} \in \mathfrak{g}_{-\alpha} \setminus \{0\}$$

for all $\alpha \in \Pi$. These considerations imply that $\mathcal{S} \subseteq \mathfrak{m}$, allowing one to define the map

$$\rho : G \times \mathcal{S} \rightarrow G \times_B \mathfrak{m}, \quad (g, x) \mapsto [g : x], \quad (g, x) \in G \times \mathcal{S}.$$ 

Let us also consider the open, dense, $G$-invariant subvariety

$$G \times_B \mathfrak{m}^\times \subseteq G \times_B \mathfrak{m},$$
where $m^x \subseteq m$ is the open, $B$-invariant subvariety defined by
\[ m^x := b + \sum_{\alpha \in \Pi} (g_{-\alpha} \setminus \{0\}) := \left\{ x + \sum_{\alpha \in \Pi} c_\alpha e_{-\alpha} : x \in b \text{ and } c_\alpha \in \mathbb{C}^x \text{ for all } \alpha \in \Pi \right\}. \]

One then has the following consequence of [24 Theorem 41] and [15 Section 4.2 and Theorem 4.16].

**Proposition 5.2.** The Poisson variety $G \times_B m$ is log symplectic with $G \times_B m^x$ as its unique open dense symplectic leaf. The map $\rho$ is a symplectomorphism onto the leaf $G \times_B m^x$.

5.2. Some toric geometry. This section uses the techniques of toric geometry to compare the fibres $\nu^{-1}(x) = \text{Hess}(x, m)$ and $\overline{\mu_S}^{-1}(x)$ over regular semisimple elements $x \in \mathfrak{g}^{\text{rs}}$. Many of the underlying ideas appear in [5] and [17]. We therefore do not regard this section as containing any original material.

We begin by observing that $G_x$ acts on the fibres of $\nu$ and $\overline{\mu_S}$ over $x$ for all $x \in \mathfrak{g}$. This leads to the following two lemmas, parts of which are well-known. To this end, recall the notation and discussion from Section 2.4.

**Lemma 5.3.** Suppose that $x \in \mathfrak{g}^\ell$. The following statements hold.

(i) There is a unique open dense orbit of $G_x$ in $\text{Hess}(x, m)$.
(ii) The group $G_x$ acts freely on the above-mentioned orbit.
(iii) If $g \in G$ satisfies $x = \text{Ad}_g(x_S)$, then $[g : x_S]$ belongs to the above-mentioned $G_x$-orbit.
(iv) If $x \in \mathfrak{g}^{\text{rs}}$, then $\text{Hess}(x, m)$ is a smooth, projective, toric $G_x$-variety.

**Proof.** Let $g \in G$ be such that $x = \text{Ad}_g(x_S)$. Our first observation is that
\[ \nu([g : x_S]) = \text{Ad}_g(x_S) = x, \]
\[
\text{i.e. } [g : x_S] \in \text{Hess}(x, m). \]

At the same time, Corollaries 3 and 14 in [24] imply that $\text{Hess}(x, m)$ is irreducible and $\ell$-dimensional. Claims (i), (ii), and (iii) would therefore follow from our showing the $G_x$-stabilizer of $[g : x_S]$ to be trivial.

Suppose that $h \in G_x$ is such that $[hg : x_S] = [g : x_S]$. It follows that $(hgb^{-1}, \text{Ad}_b(x_S)) = (g, x_S)$ for some $b \in B$. Since the $B$-stabilizer of every point in $S$ is trivial (see [5 Lemma 4.9]), we must have $b = e$. This yields the identity $hg = g$, or equivalently $h = e$. In light of the conclusion reached in the previous paragraph, Claims (i), (ii), and (iii) hold.

Claim (iv) is well-known and follows from Theorems 6 and 11 in [15]. \hfill \Box

Let us also recall the notation $\gamma_g$ defined in Section 4.2.

**Lemma 5.4.** Suppose that $x \in \mathfrak{g}^\ell$. The following statements hold.

(i) There is a unique open dense orbit of $G_x$ in $\overline{\mu_S}^{-1}(x)$.
(ii) The group $G_x$ acts freely on the above-mentioned orbit.
(iii) If $g \in G$ satisfies $x = \text{Ad}_g(x_S)$, then $(\gamma_g, (x, x_S))$ belongs to the above-mentioned $G_x$-orbit.
(iv) If $x \in \mathfrak{g}^{\text{rs}}$, then $\overline{\mu_S}^{-1}(x)$ is a smooth, projective, toric $G_x$-variety.

**Proof.** The moment map $\overline{\mu_S}$ is $G$-equivariant, so that acting by the element $g^{-1}$ defines a variety isomorphism
\[ \overline{\mu_S}^{-1}(x) \xrightarrow{g^{-1}} \overline{\mu_S}^{-1}(x_S). \]

Since the latter variety is given by
\[ \overline{\mu_S}^{-1}(x_S) = \{(\gamma, (x_S, x_S)) : \gamma \in \mathcal{G} \text{ and } (x_S, x_S) \in \gamma\}, \]

we can use [5 Corollary 3.12] and conclude that $\overline{\mu_S}^{-1}(x_S)$ is irreducible and $\ell$-dimensional. It follows that $\overline{\mu_S}^{-1}(x)$ is irreducible and $\ell$-dimensional. As with the proof of our previous lemma,
Claims (i), (ii), and (iii) would now follow from knowing \((\gamma, (x, x_S))\) to have a trivial \(G_x\)-stabilizer. This is established via a straightforward calculation.

To verify Claim (iv), recall the isomorphism \([5,3]\). This isomorphism implies that \(\overline{\mu_S}^{-1}(x)\) is a smooth, projective, toric \(G_x\)-variety if and only if \(\overline{\mu_S}^{-1}(x_S)\) is a smooth, projective, toric \(g^{-1}G_xg = G_{x_S}\)-variety. The latter condition holds because of \([5,4]\), \([5]\) Corollary 3.12, and the fact that the closure of \(G_{x_S}\) in \(G\) is a smooth, projective, toric \(G_{x_S}\)-variety \([17, Remark 4.5]\). Our proof is therefore complete.

Our next two lemmas study the \(G_x\)-fixed point sets \(\text{Hess}(x, m)^{G_x} \subseteq \text{Hess}(x, m)\) and \(\overline{\mu_S}^{-1}(x)^{G_x} \subseteq \overline{\mu_S}^{-1}(x)\) for each \(x \in \mathfrak{g}_{rs}\). To formulate these results, let \(B\) denote the flag variety of all Borel subalgebras in \(\mathfrak{g}\) and set \(B_x := \{\tilde{b} \in B : x \in \tilde{b}\}\) for each \(x \in \mathfrak{g}\). Recall that \(G/B \rightarrow B, \ [g] \mapsto \text{Ad}_g(b), \ [g] \in G/B\) (5.5) defines a \(G\)-equivariant variety isomorphism. Let us also consider the map \(\pi_m : G \times_B \mathfrak{m} \rightarrow G/B, \ [g : x] \mapsto [g], \ [g : x] \in G \times_B \mathfrak{m}\) and its composition with (5.3), i.e. \(\varpi_m : G \times_B \mathfrak{m} \rightarrow B, \ [g : x] \mapsto \text{Ad}_g(b), \ [g : x] \in G \times_B \mathfrak{m}\).

**Lemma 5.5.** If \(x \in \mathfrak{g}_{rs}\), then there is a canonical bijection \(B_x \rightarrow \text{Hess}(x, m)^{G_x}, \ \tilde{b} \mapsto z(\tilde{b}), \ \tilde{b} \in B_x\) satisfying \(\tilde{b} = \varpi_m(z(\tilde{b}))\) for all \(\tilde{b} \in B_x\).

**Proof.** Recall that \(\pi_m\) restricts to a closed embedding \(\text{Hess}(x, m) \rightarrow G/B\) (see Remark [5,1]). Composing this embedding with (5.5), we deduce that \(\varpi_m\) restricts to a closed embedding \(\text{Hess}(x, m) \rightarrow B\).

This embedding is \(G_x\)-equivariant, implying that it restricts to an injection \(\text{Hess}(x, m)^{G_x} \rightarrow B^{G_x} = B_x\). (5.7)

We claim that (5.7) is also surjective. To this end, suppose that \(\tilde{b} \in B_x\). We then have \(\tilde{b} = \text{Ad}_g(b)\) for some \(g \in G\). It follows that \(x = \text{Ad}_g(y)\) for some \(y \in \mathfrak{b}\), so that we have a point \([g : y] \in \text{Hess}(x, m)\). The image of this point under (5.6) is \(\tilde{b}\). Noting that \(\tilde{b} \in B_x = B^{G_x}\) and that (5.6) is injective and \(G_x\)-equivariant, we conclude that \([g : y] \in \text{Hess}(x, m)^{G_x}\). This establishes that (5.7) is surjective, i.e. that it is a bijection. The bijection advertised in the statement of the lemma is obtained by inverting (5.7).
If \( x \in \mathfrak{g}^{rs} \), then the Cartan subalgebra \( \mathfrak{g}_x \) satisfies \( \mathfrak{g}_x \subseteq \mathfrak{b} \) for all \( \mathfrak{b} \in B_x \) (see [13, Lemma 3.1.4]).

This fact gives rise to an opposite Borel subalgebra \( \mathfrak{b}^- \in B_x \) for all \( \mathfrak{b} \in B_x \). We may therefore define

\[
\theta(\mathfrak{b}) := \mathfrak{b} \times_{\mathfrak{g}_x} \mathfrak{b}^{-}
\]

analogously to (4.2).

**Lemma 5.6.** Suppose that \( x \in \mathfrak{g}^{rs} \) and let \( g \in G \) be such that \( x = \text{Ad}_g(x_S) \). We then have a bijection defined by

\[
B_x \rightarrow \overline{\mu S^{-1}(x)^{G_x}}, \quad \mathfrak{b} \mapsto \left( (e, g^{-1}) \cdot \theta(\mathfrak{b}), (x, x_S) \right), \quad \mathfrak{b} \in B_x.
\]

**Proof.** Note that conjugation by \( g^{-1} \) defines a bijection \( \phi : B_x \rightarrow B_{x_S} \). At the same time, consider the automorphism of \( G \times S \) through which the element \( g^{-1} \) acts. This automorphism restricts to a bijection

\[
\overline{\mu S^{-1}(x)^{G_x}} \overset{\sim}{\rightarrow} \overline{\mu S^{-1}(x S)^{G_{x_S}}}.
\]

It also sends

\[
\left( (e, g^{-1}) \cdot \theta(\mathfrak{b}), (x, x_S) \right) \mapsto (\theta(\phi(\mathfrak{b})), (x_S, x_S))
\]

for all \( \mathfrak{b} \in B_x \), where the subspaces \( \{\theta(\mathfrak{b})\}_{\mathfrak{b} \in B_{x_S}} \) are defined analogously to the \( \{\theta(\mathfrak{b})\}_{\mathfrak{b} \in B_x} \). It therefore suffices to prove that

\[
B_{x_S} \rightarrow \overline{\mu S^{-1}(x S)^{G_{x_S}}}, \quad \mathfrak{b} \mapsto (\theta(\mathfrak{b}), (x_S, x_S)), \quad \mathfrak{b} \in B_{x_S}
\]

defines a bijection. In other words, it suffices to prove our lemma under the assumption that \( x \in S \cap \mathfrak{g}^{rs} \) and \( g = e \).

Let us make the assumption indicated above. The fibre \( \overline{\mu S^{-1}(x)} \) is then given by

\[
\overline{\mu S^{-1}(x)} = \{(\gamma, (x, x)) : \gamma \in G \text{ and } (x, x) \in \gamma\}.
\]

It now follows from [5, Corollary 3.12] that

\[
\overline{\text{G}_x} \rightarrow \overline{\mu S^{-1}(x)}, \quad \gamma \mapsto (\gamma, (x, x)), \quad \gamma \in \overline{\text{G}_x}
\]

defines an isomorphism of varieties, where \( \overline{\text{G}_x} \) denotes the closure of \( \text{G}_x \) in \( G \). This isomorphism is \( G_x \)-equivariant if one lets \( G_x \) act on \( \overline{\text{G}_x} \) by restricting the \( (G \times G) \)-action on \( G \) to \( G_x = G_x \times \{e\} \subseteq G \times G \). We therefore have

\[
\overline{\mu S^{-1}(x)^{G_x}} = \{(\gamma, (x, x)) : \gamma \in \overline{(\text{G}_x)^{G_x}}\}. \quad (5.8)
\]

At the same time, we will prove that

\[
(\overline{\text{G}_x})^{G_x} = \{\theta(\mathfrak{b}) : \mathfrak{b} \in B_x\} \quad (5.9)
\]

in Lemma 5.7. Our current lemma now follows from \( (5.8), (5.9) \), and the conclusion of the previous paragraph.

**Lemma 5.7.** We have

\[
(\overline{\text{G}_x})^{G_x} = \{\theta(\mathfrak{b}) : \mathfrak{b} \in B_x\}
\]

for all \( x \in \mathfrak{g}^{rs} \), where \( G_x \) acts on \( \overline{\text{G}_x} \) as the subgroup \( G_x = G_x \times \{e\} \subseteq G \times G \) via the \( (G \times G) \)-action on \( G \).
Proof. It suffices to assume that \(x \in \mathfrak{t}^\ast\), so that \(G_x = T\). Now suppose that \(\mathfrak{b} \in \mathfrak{b}_x\). Lemma 4.1 implies that
\[
\theta(\mathfrak{b}) = \lim_{t \to \infty} (\lambda(t), e) \cdot \mathfrak{g}_{\Delta}
\]
for a suitable one-parameter subgroup \(\lambda : \mathbb{C}^\ast \to T\), while we observe that
\[
(\lambda(t), e) \cdot \mathfrak{g}_{\Delta} \in \gamma(T) \subseteq \overline{\mathfrak{g}}
\]
for all \(t \in \mathbb{C}^\ast\). It follows that \(\theta(\mathfrak{b}) \in \mathcal{T}\). A direct calculation establishes that \(\theta(\mathfrak{b})\) is a \(T\)-fixed point, yielding the inclusion
\[
\{ \theta(\mathfrak{b}) : \mathfrak{b} \in \mathfrak{b}_x \} \subseteq \mathcal{T}^T.
\]
To establish the opposite inclusion, suppose that \(\gamma \in (\mathcal{T}^T)^T\). Using the \((\mathbb{G} \times \mathbb{G})\)-orbit decomposition of \(\overline{\mathfrak{g}}\) from Section 4.2, we may find \(I \subseteq \Pi\) and \(g_1, g_2 \in \mathbb{G}\) such that
\[
\gamma = (g_1, g_2) : \mathfrak{p}_I \times_I \mathfrak{p}_I^-.
\]
We will first show that \(g_1\) can be taken to lie in \(N_{\mathbb{G}}(T)\) without the loss of generality. Note that \(\mathfrak{b} := \text{Ad}_{g_1}(\mathfrak{b})\) will then be a Borel subalgebra containing \(x\). We will then explain that \(\gamma = \theta(\mathfrak{b})\), completing the proof. In what follows, we will need the following description of the \((\mathbb{G} \times \mathbb{G})\)-stabilizer of \(\mathfrak{p}_I \times_I \mathfrak{p}_I^-\):
\[
(G \times G)_I := \{(l_1 u, l_2 v) \in L_I U_I \times L_I U_I^- : l_1 l_2^{-1} \in Z(L_I)\},
\]
where \(L_I \subseteq \mathbb{G}\) is the Levi subgroup with Lie algebra \(\mathfrak{l}_I\), \(Z(L_I)\) is the centre of \(L_I\), and \(U_I \subseteq \mathbb{G}\) (resp. \(U_I^- \subseteq \mathbb{G}\)) is the unipotent subgroup with Lie algebra \(\mathfrak{u}_I\) (resp. \(\mathfrak{u}_I^-\)) [17 Proposition 2.25].

Since \(\gamma\) is fixed by \(T\), we have
\[
(t, e) \cdot \gamma = \gamma
\]
for all \(t \in T\). It follows that
\[
(t g_1, g_2) : \mathfrak{p}_I \times_I \mathfrak{p}_I^- = (g_1, g_2) : \mathfrak{p}_I \times_I \mathfrak{p}_I^-,
\]
i.e.
\[
(g_1^{-1} t g_1, e) : \mathfrak{p}_I \times_I \mathfrak{p}_I^- = \mathfrak{p}_I \times_I \mathfrak{p}_I^-.
\]
We deduce that \((g_1^{-1} t g_1, e) \in (G \times G)_I\), which by (5.10) implies that
\[
g_1^{-1} t g_1 \in U_I Z(L_I)
\]
for all \(t \in T\). In other words,
\[
g_1^{-1} T g_1 \subseteq U_I Z(L_I) \subseteq P_I,
\]
where \(P_I \subseteq \mathbb{G}\) is the parabolic subgroup with Lie algebra \(\mathfrak{p}_I\). Noting that \(g_1^{-1} T g_1\) and \(T\) are maximal tori in \(P_I\), we can find a \(p \in P_I\) such that
\[
g_1^{-1} T g_1 = p T p^{-1}.
\]
This shows that \((g_1 p)^{-1} T g_1 p = T\), so that \(g_1 p \in N_{\mathbb{G}}(T)\). Now use the decomposition \(P_I = L_I U_I\) to write \(p = l u\) with \(l \in L_I\) and \(u \in U_I\). Equation (5.10) then tells us that \((p, l) \in (\mathbb{G} \times \mathbb{G})_I\), yielding
\[
\gamma = (g_1, g_2) : \mathfrak{p}_I \times_I \mathfrak{p}_I^- = (g_1 p, g_2 l) : \mathfrak{p}_I \times_I \mathfrak{p}_I^-.
\]
We may therefore take \(g_1 \in N_{\mathbb{G}}(T)\) without the loss of generality.

Now we show that \(I = \emptyset\). An argument given above establishes that
\[
T = g_1^{-1} T g_1 \subseteq U_I Z(L_I),
\]
i.e. \(T \subseteq U_I Z(L_I)\). Let \(t \in T\) be arbitrary and write \(t = v z\) with \(v \in U_I\) and \(z \in Z(L_I)\). Since \(T\) contains \(Z(L_I)\), we have \(z \in T\) and \(v = t z^{-1} \in T \cap U_I = \{e\}\). We conclude that \(v = e\) and...
Proof. Let \( t \in Z(L_I) \). This shows that \( T \subseteq Z(L_I) \), which can only happen if \( T = Z(L_I) \). One deduces that \( I = \emptyset \), and this yields \( p_I = b \), \( p_I^- = b^- \), \( I_t = t \), and

\[
\gamma = (g_1, g_2) \cdot b \times_t b^-.
\]

We now prove that \( g_2 = g_1 \in N_G(T) \) without the loss of generality. Our first observation is that

\[
(\text{Ad}_{g_1^{-1}}(x), \text{Ad}_{g_2^{-1}}(x)) \in b \times_t b^-,
\]

as \((x, x) \in \gamma\). Since \( g_1 \in N_G(T) \), we have \( \text{Ad}_{g_1^{-1}}(x) \in t \). These last two sentences then force

\[
\text{Ad}_{g_1^{-1}}(x) - \text{Ad}_{g_2^{-1}}(x) \in u^{-}
\]
to hold. We also note that \( \text{Ad}_{g_1^{-1}}(x) \in t \) is regular, which by \([13\text{ Lemma 3.1.44]}\] implies that \( \text{Ad}_{g_1^{-1}}(x) + u^- \) is a \( U^- \)-orbit. We can therefore find \( u_- \in U^- \) such that

\[
\text{Ad}_{g_2^{-1}}(x) = \text{Ad}_{u_-g_1^{-1}}(x),
\]
i.e. \((g_2u_-)g_1^{-1} \in G_x = T\). This shows that \( g_2u_- \in Tg_1 = g_1T, \) where we have used the fact that \( g_1 \in N_G(T) \). We may therefore write \( g_2u_- = g_1t \) for some \( t \in T \), i.e.

\[
g_1 = g_2u_-^{-1}.
\]

Equation (5.10) also implies that \((e, u_-^{-1}) \in (G \times G)_0 \), giving \((e, u_-^{-1}) \cdot b \times_t b^- = b \times_t b^- \). This in turn implies that

\[
\gamma = (g_1, g_2) \cdot b \times_t b^- = (g_1, g_2u_-^{-1}) \cdot b \times_t b^- = (g_1, g_1) \cdot b \times_t b^- = \theta(\bar{b}),
\]

where \( \bar{b} = \text{Ad}_{g_1}(b) \). Our proof is therefore complete.

The preceding results have implications for the toric geometries of \( \text{Hess}(x, m) \) and \( \overline{\nu S}^{-1}(x) \), where \( x \in g^{rs} \). The following elementary lemma is needed to realize these implications.

**Lemma 5.8.** Suppose that \( S \) is a complex torus, and that \( X \) is a smooth, projective, toric \( S \)-variety with finitely many \( S \)-fixed points. Let \( \lambda : \mathbb{C}^X \rightarrow S \) be a coweight of \( S \), and assume that \( (\alpha, \lambda) \neq 0 \) for all \( y \in X^S \) and all weights \( \alpha \) of the isotropy \( S \)-representation \( T_yX \). Assume that \( x \) belongs to the unique open dense orbit \( S \)-orbit in \( X \). We have

\[
\lim_{t \rightarrow \infty} \lambda(t) \cdot x = z
\]

for some \( z \in X^S \) satisfying \((\alpha, \lambda) < 0 \) for all \( S \)-weights \( \alpha \) of \( T_xX \).

**Proof.** Let \( \mathbb{C}^X \) act on \( X \) through \( \lambda \). If \( y \in X^S \), then \( T_yY \) is an isotropy representation of both \( S \) and \( \mathbb{C}^X \). Each weight of the \( \mathbb{C}^X \)-representation is given by \((\alpha, \lambda)\) for a suitable weight \( \alpha \) of the \( S \)-representation. Since \((\alpha, \lambda) \neq 0 \) for all \( y \in X^S \) and \( S \)-weights \( \alpha \) of \( T_yX \), the previous sentence implies that \( X^S = X^{\mathbb{C}^X} \).

Now note that \( \lim_{t \rightarrow \infty} \lambda(t) \cdot x \) exists and coincides with a point \( z \in X^{\mathbb{C}^X} = X^S \) (e.g. by \([13\text{ Lemma 2.4.1]}\) ). The previous paragraph explains that each \( \mathbb{C}^X \)-weight of \( T_xX \) takes the form \((\alpha, \lambda)\), where \( \alpha \) is an \( S \)-weight of \( T_xX \). General facts about Bialynicki-Birula decompositions (e.g. \([13\text{ Theorem 2.4.3]}\) ) now imply that \((\alpha, \lambda) < 0 \) for all \( S \)-weights \( \alpha \) of \( T_xX \).

Suppose that \( x \in g^{rs} \). Recall that a coweight \( \lambda : \mathbb{C}^X \rightarrow G_x \) is called \textit{regular} if \((\alpha, \lambda) \neq 0 \) for all roots \( \alpha \) of \((g, g_x)\). Let us also recall the notation adopted in Lemmas \[5.5\] and \[5.6\].
Lemma 5.9. Suppose that \( x \in g^\text{rs} \) and let \( g \in G \) be such that \( x = \text{Ad}_g(x_S) \). Given any regular coweight \( \lambda : \mathbb{C}^\times \rightarrow G_x \) and element \( \tilde{\mathfrak{b}} \in \mathcal{B}_x \), one has the following equivalence:

\[
\lim_{t \to \infty} \lambda(t) \cdot [g : x_S] = z(\tilde{\mathfrak{b}}) \iff \lim_{t \to \infty} \lambda(t) \cdot (\gamma_g, (x, x_S)) = \left( (e, g^{-1}) \cdot \theta(\tilde{\mathfrak{b}}), (x, x_S) \right),
\]

where the left- (resp. right-) hand side is computed in \( \text{Hess}(x, \mathfrak{m}) \) (resp. \( \mu_{G_S}^{-1}(x) \)).

Proof. Let us write \( \Pi(\mathfrak{b}) \subseteq \mathfrak{g}_x^* \) for the set of simple roots determined by \( \mathfrak{g}_x \) and \( \tilde{\mathfrak{b}} \in \mathcal{B}_x \). By [15 Lemma 7], the \( G_x \)-weights of the isotropy representation \( T_{z(\tilde{b})}\text{Hess}(x, \mathfrak{m}) \) form the set \( -\Pi(\tilde{\mathfrak{b}}) \). This has two implications for any regular coweight \( \lambda : \mathbb{C}^\times \rightarrow G_x \). One is that \((\alpha, \lambda) \neq 0 \) for all \( \tilde{\mathfrak{b}} \in \mathcal{B}_x \) and \( G_x \)-weights \( \alpha \) of \( T_{z(\tilde{b})}\text{Hess}(x, \mathfrak{m}) \). The second implication is the existence of a unique \( \tilde{\mathfrak{b}} \in \mathcal{B}_x \) such that \((\alpha, \lambda) < 0 \) for all \( G_x \)-weights \( \alpha \) of \( T_{z(\tilde{b})}\text{Hess}(x, \mathfrak{m}) \). These last few sentences and Lemma 5.8 imply the following about a given regular coweight \( \lambda \) and element \( \tilde{\mathfrak{b}} \in \mathcal{B}_x \):

\[
\lim_{t \to \infty} \lambda(t) \cdot [g : x_S] = z(\tilde{\mathfrak{b}}) \iff (\alpha, \lambda) < 0 \text{ for all } \alpha \in -\Pi(\tilde{\mathfrak{b}}),
\]
or equivalently

\[
\lim_{t \to \infty} \lambda(t) \cdot [g : x_S] = z(\tilde{\mathfrak{b}}) \iff (\alpha, \lambda) > 0 \text{ for all } \alpha \in \Pi(\tilde{\mathfrak{b}}).
\]

In light of the previous paragraph, we are reduced to proving that

\[
\lim_{t \to \infty} \lambda(t) \cdot (\gamma_g, (x, x_S)) = \left( (e, g^{-1}) \cdot \theta(\tilde{\mathfrak{b}}), (x, x_S) \right) \iff (\alpha, \lambda) > 0 \text{ for all } \alpha \in \Pi(\tilde{\mathfrak{b}}).
\] (5.11)

Consider the action of \( \lambda(t) \) indicated above, noting that it coincides with the action of \( (\lambda(t), e) \in G \times G \) on points in \( \mu_{G_S}^{-1}(x) \subseteq T^*G(\log D) \). At the same time, the actions of \( (\lambda(t), e) \) and \( (e, g) \) on \( T^*G(\log D) \) commute with one another for all \( t \in \mathbb{C}^\times \). Acting by \( (e, g) \) therefore allows us to reformulate (5.11) as

\[
\lim_{t \to \infty} \lambda(t) \cdot (\mathfrak{g}_\Delta, (x, x)) = (\theta(\tilde{\mathfrak{b}}), (x, x)) \iff (\alpha, \lambda) > 0 \text{ for all } \alpha \in \Pi(\tilde{\mathfrak{b}}),
\]
or equivalently

\[
\lim_{t \to \infty} (\lambda(t), e) \cdot \mathfrak{g}_\Delta = \theta(\tilde{\mathfrak{b}}) \iff (\alpha, \lambda) > 0 \text{ for all } \alpha \in \Pi(\tilde{\mathfrak{b}}).
\]

This last equivalence is a straightforward consequence of Lemma 4.1 and our proof is complete. \( \square \)

5.3. Proof of the main result. By means of the \( G \)-equivariant symplectomorphism in Proposition 4.4 we may identify \( G \times S \) with the unique open dense symplectic leaf in \( G \times \mathcal{S} \). The \( G \)-equivariant map \( \rho : G \times S \rightarrow G \times_B \mathfrak{m} \) in Section 5.1 is then given by

\[
\rho(\gamma_g, (\text{Ad}_g(x), x)) = [g : x]
\]

for all \((\gamma_g, (\text{Ad}_g(x), x)) \in G \times S \subseteq G \times \mathcal{S} \). One also has the commutative diagram

\[
\begin{array}{ccc}
G \times S & \xrightarrow{\rho} & G \times_B \mathfrak{m} \\
\mu_{G \times S}^{-1}(x) \downarrow & & \downarrow \nu \\
g & & \mathfrak{m}
\end{array}
\]

(5.12)

Now fix \( x \in g^\text{rs} \) and consider the fibres

\[
(\mu_{G \times S}^{-1}(x))^{-1}(x) = \mu_{G_S}^{-1}(x) \cap (G \times S) =: \mu_{G_S}^{-1}(x)
\]

and

\[
\nu^{-1}(x) = \text{Hess}(x, \mathfrak{m}).
\]
The commutative diagram (5.12) implies that $\rho$ restricts to a $G_x$-equivariant morphism

$$
\rho_x : \nu S^{-1}(x)^0 \to \text{Hess}(x, m).
$$

**Lemma 5.10.** If $x \in g^{rs}$, then there exists a unique $G_x$-equivariant variety isomorphism

$$
\overline{\rho}_x : \nu S^{-1}(x) \xrightarrow{\cong} \text{Hess}(x, m)
$$

that extends $\rho_x$.

**Proof.** Choose $g \in G$ such that $x = \text{Ad}_g(x_S)$. Lemma 5.14 combines with our description of $G \times S$ as a subset of $G \times \mathbb{S}$ to imply that $\nu S^{-1}(x)^0$ is the unique open dense $G_x$-orbit in $\nu S^{-1}(x)$. The uniqueness of $\overline{\rho}_x$ is then a consequence of $\nu S^{-1}(x)^0$ being dense in $\nu S^{-1}(x)$.

To establish existence, recall that $\nu S^{-1}(x)$ and $\text{Hess}(x, m)$ are smooth, projective, toric $G_x$-varieties with respective points

$$
(\gamma_g, (x, x_S)) \quad \text{and} \quad [g : x_S]
$$

in their open dense $G_x$-orbits (see Lemmas 5.3 and 5.4). Recall also that elements of $\nu S^{-1}(x)^{G_x}$ and $\text{Hess}(x, m)^{G_x}$ are in correspondence with elements of $\mathcal{B}_x$ (see Lemmas 5.5 and 5.6), and that the points (5.13) limit to corresponding $G_x$-fixed points under a given regular coweight (see Lemma 5.9). By these last two sentences, there exists a unique $G_x$-variety isomorphism

$$
\overline{\rho}_x : \nu S^{-1}(x) \xrightarrow{\cong} \text{Hess}(x, m)
$$

satisfying

$$
\overline{\rho}_x(\gamma_g, (x, x_S)) = [g : x_S].
$$

Note that $\rho_x$ is also $G_x$-equivariant and satisfies

$$
\rho_x(\gamma_g, (x, x_S)) = [g : x_S].
$$

Since $\overline{\rho}_x$ and $\rho_x$ are $G_x$-equivariant, these last two equations imply that $\overline{\rho}_x$ and $\rho_x$ coincide on

$$
G_x \cdot (\gamma_g, (x, x_S)) = \nu S^{-1}(x)^0.
$$

$\square$

**Remark 5.11.** It is worthwhile to take this proof and use it to examine the behaviour of $\overline{\rho}_x$ as $x$ varies in $g^{rs}$. Such an examination reveals that the isomorphisms $\overline{\rho}_x$ glue together to define a variety isomorphism $\nu S^{-1}(g^{rs}) \xrightarrow{\cong} \nu^{-1}(g^{rs})$. This observation is used in the proof of Theorem 5.16.

We now undertake a brief digression on transverse intersections in a $G$-variety. To this end, let $X$ be an arbitrary algebraic variety. Write $X_{\text{sing}}$ and $X_{\text{smooth}} := X \setminus X_{\text{sing}}$ for the singular and smooth loci of $X$, respectively. Given any closed subvariety $Y \subseteq X$, let

$$
\text{codim}_X(Y) := \dim X - \dim Y
$$

denote the codimension of $Y$ in $X$.

**Lemma 5.12.** Suppose that $X$ is a smooth $G$-variety containing a closed, $G$-invariant subvariety $Y \subseteq X$. Let $Z \subseteq X$ be a smooth, closed subvariety with the property of being transverse to each $G$-orbit in $X$. If $W$ is an irreducible component of $Y \cap Z$, then

$$
\text{codim}_X(W) \geq \text{codim}_X(Y).
$$

**Proof.** Set $Y_0 := Y$ and recursively define $Y_{j+1} := (Y_j)_{\text{sing}}$ for all $j \in \mathbb{Z}_{\geq 0}$. One then has a descending chain

$$
Y = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots
$$

of closed, $G$-invariant subvarieties in $Y$. The locally closed subvarieties $(Y_j)_{\text{smooth}} \subseteq Y$ are also $G$-invariant, and we observe that these subvarieties have a disjoint union equal to $Y$. 

Let $k$ be the smallest non-negative integer for which $(Y_k)_{\text{smooth}} \cap W \neq \emptyset$. It is then straightforward to verify that $(Y_k)_{\text{smooth}} \cap W = W \setminus Y_{k+1}$. We conclude that $(Y_k)_{\text{smooth}} \cap W$ is an open dense subset of $W$, and this implies that $W \subseteq Y_k$.

Now choose a point $w \in (Y_k)_{\text{smooth}} \cap W$. Since $(Y_k)_{\text{smooth}}$ is $G$-invariant and $Z$ is transverse to every $G$-orbit in $X$, the varieties $(Y_k)_{\text{smooth}}$ and $Z$ have a transverse intersection at $w$. It follows that $w$ is a smooth point of $Y_k \cap Z$, and that

$$\dim W \leq \dim(T_w(Y_k \cap Z)) = \dim(T_w Y_k) + \dim Z - \dim X \leq \dim Y + \dim Z - \dim X.$$  

This yields the conclusion

$$\text{codim}_Z(W) \geq \text{codim}_X(Y).$$

We will ultimately apply this result in a context relevant to our main theorem. To this end, use \[4.1\] to identify $T^*G$ with the unique open dense symplectic leaf in $T^*\overline{G}(\log D)$. Consider the open subvariety

$$T^*\overline{G}(\log D)^\circ := T^*G \cup \bar{\mu}_L^{-1}(\mathfrak{g}^{rs}) \subseteq T^*\overline{G}(\log D)$$

and its complement

$$T^*\overline{G}(\log D)' := T^*\overline{G}(\log D) \setminus T^*\overline{G}(\log D)^\circ,$$

where $\bar{\mu}_L : T^*\overline{G}(\log D) \rightarrow \mathfrak{g}$ is defined in Section 4.3.

**Lemma 5.13.** We have

$$\text{codim}_{T^*\overline{G}(\log D)}(T^*\overline{G}(\log D))' \geq 2.$$  

**Proof.** We begin by observing that

$$T^*\overline{G}(\log D)' = \pi^{-1}(D) \setminus \bar{\mu}_L^{-1}(\mathfrak{g}^{rs}),$$

where $\pi : T^*\overline{G}(\log D) \rightarrow \overline{G}$ is the bundle projection and $D = \overline{G} \setminus G$. It therefore suffices to prove that $\bar{\mu}_L^{-1}(\mathfrak{g}^{rs})$ meets each irreducible component of $\pi^{-1}(D)$.

Now note that

$$D = \bigcup_I (G \times G)p_I \times_{l_I} p_I^{-}$$

is the decomposition of $D$ into irreducible components, where $I$ ranges over all subsets of the form $\Pi \setminus \{\alpha\}$, $\alpha \in \Pi$ [3 Section 3.1]. We conclude that

$$\pi^{-1}(D) = \bigcup_I \pi^{-1}(G \times G)p_I \times_{l_I} p_I^{-}$$

is the decomposition of $\pi^{-1}(D)$ into irreducible components. It therefore suffices to prove that $\mu^{-1}(\mathfrak{g}^{rs})$ meets $\pi^{-1}((G \times G)p_I \times_{l_I} p_I^{-})$ for all $I \subseteq \Pi$ of the form $I = \Pi \setminus \{\alpha\}$, $\alpha \in \Pi$.

Choose $x \in t^*$ and an element $g \in G$ satisfying $x = \text{Ad}_g(x_S)$. We then have

$$(x, x_S) \in (e, g^{-1}) \cdot p_I \times_{l_I} p_I^{-}$$

for all $I \subseteq \Pi$. We also observe that

$$\left((e, g^{-1}) \cdot p_I \times_{l_I} p_I^{-}, (x, x_S)\right) \in \bar{\mu}_L^{-1}(\mathfrak{g}^{rs}) \cap \pi^{-1}((G \times G)p_I \times_{l_I} p_I^{-})$$

for all $I \subseteq \Pi$. This completes the proof. \[\square\]
Now consider the open subvariety
\[
\overline{G \times \mathcal{S}^0} := (G \times \mathcal{S}) \cup \overline{\mu_\mathcal{S}^{-1}(\mathfrak{g}^\mathcal{S})} \subseteq G \times \mathcal{S}
\]
and its complement
\[
\overline{G \times \mathcal{S}} := G \times \mathcal{S} \setminus \overline{G \times \mathcal{S}^0}.
\]

**Lemma 5.14.** We have
\[
\text{codim}_{G \times \mathcal{S}}(\overline{G \times \mathcal{S}}) \geq 2.
\]

**Proof.** Our task is to prove that each irreducible component of $\overline{G \times \mathcal{S}}$ has codimension at least two in $G \times \mathcal{S}$. We begin by showing ourselves to be in the situation of Lemma 5.12. To this end, consider $X := T^*G(\log D)$ and its action of $G = \{e\} \times G \subseteq G \times G$. Let $Y$ be the subvariety $T^*G(\log D)' \subseteq T^*G(\log D)$ considered in Lemma 5.13 and set $Z := \overline{G \times \mathcal{S}}$. Since $Z = \overline{\mu_R^{-1}(\mathcal{S})}$, Proposition 3.10 implies that $Z$ is transverse to every $G$-orbit in $X$. We also note that
\[
Y \cap Z = \overline{G \times \mathcal{S}} \setminus (T^*G \cup \overline{\mu_\mathcal{L}^{-1}(\mathfrak{g}^\mathcal{S})}) = \overline{G \times \mathcal{S}} \setminus ((G \times \mathcal{S}) \cup \overline{\mu_\mathcal{S}^{-1}(\mathfrak{g}^\mathcal{S})}) = \overline{G \times \mathcal{S}},
\]
and that
\[
\text{codim}_X(Y) = \text{codim}_{T^*G(\log D)'}(T^*G(\log D)') \geq 2
\]
by Lemma 5.13. The desired conclusion now follows immediately from Lemma 5.12. \[\square\]

We require one additional lemma prior to proving our main result. To this end, recall the notation and content of Section 5.1. Consider the open subvariety
\[
(G \times_B m)^0 := (G \times_B m^\times) \cup \nu^{-1}(\mathfrak{g}^\mathcal{S}) \subseteq G \times_B m
\]
and its complement
\[
(G \times_B m)' := G \times_B m \setminus (G \times_B m)^0.
\]

**Lemma 5.15.** We have
\[
\text{codim}_{G \times_B m}(G \times_B m)' \geq 2.
\]

**Proof.** It suffices to prove that the open subvariety $\nu^{-1}(\mathfrak{g}^\mathcal{S}) \subseteq G \times_B m$ meets each irreducible component of $(G \times_B m) \setminus (G \times_B m^\times)$. To this end, suppose that $\beta \in \Pi$ is a simple root and define
\[
m^\times_\beta := b + \sum_{\alpha \in \Pi \setminus \{\beta\}} (\mathfrak{g} - \mathfrak{a}) \setminus \{0\} := \left\{ x + \sum_{\alpha \in \Pi \setminus \{\beta\}} c_\alpha e_\alpha : x \in b \text{ and } c_\alpha \in \mathbb{C}^\times \text{ for all } \alpha \in \Pi \setminus \{\beta\} \right\}.
\]
The irreducible components of $(G \times_B m) \setminus (G \times_B m^\times)$ are then the subvarieties
\[
G \times_B m^\times_\beta \subseteq G \times_B m, \quad \beta \in \Pi.
\]

Fix $\beta \in \Pi$ and choose an element $x \in e^\mathfrak{t}$. By [13] Lemma 3.1.44, the elements
\[
x \quad \text{and} \quad x_\beta := x + \sum_{\alpha \in \Pi \setminus \{\beta\}} e_\alpha
\]
are in the same adjoint $G$-orbit. It follows that $x_\beta \in \mathfrak{g}^\mathcal{S}$, while we also observe that $x_\beta \in m^\times_\beta$. These considerations imply that
\[
[e : x_\beta] \in (G \times_B m^\times_\beta) \cap \nu^{-1}(\mathfrak{g}^\mathcal{S}),
\]
forcing
\[
(G \times_B m^\times_\beta) \cap \nu^{-1}(\mathfrak{g}^\mathcal{S}) \neq \emptyset
\]
to hold. In light of the previous paragraph, our proof is complete. \[\square\]

We are now equipped to state and prove our main result.
Theorem 5.16. There exists a unique $G$-equivariant isomorphism of Poisson varieties

$$\overline{\rho} : G \times S \xrightarrow{\cong} G \times_B m$$

that extends $\rho : G \times S \rightarrow G \times_B m$ and makes

$$\begin{array}{ccc}
G \times S & \xrightarrow{\tau} & G \times_B m \\
\downarrow{\overline{\mu}_S} & & \downarrow{\nu} \\
G & \simeq & Bm
\end{array}$$

(5.14)

commute.

Proof. Uniqueness is an immediate consequence of $G \times S$ being dense in $G \times S$. To establish existence, recall the variety isomorphism

$$\mu_S^{-1}(g^\text{rs}) \xrightarrow{\cong} \nu^{-1}(g^\text{rs}) \subseteq G \times_B m$$

(5.15)
discussed in Remark 5.11. Let us also use Proposition 5.2 and regard $\rho$ as a map to its image, i.e. $\rho$ is an isomorphism

$$\rho : G \times S \xrightarrow{\cong} G \times_B m^\times \subseteq G \times_B m.$$ 

Lemma 5.10 implies that the isomorphism (5.15) coincides with $\rho$ on the overlap $(G \times S) \cap \overline{\mu}_S^{-1}(g^\text{rs})$. It follows that $\rho$ extends to a variety isomorphism

$$G \times S^\circ = (G \times S) \cup \overline{\mu}_S^{-1}(g^\text{rs}) \xrightarrow{\rho'} (G \times_B m^\times) \cup \nu^{-1}(g^\text{rs}) = (G \times_B m)^\circ.$$ 

The diagram

$$\begin{array}{ccc}
G \times S^\circ & \xrightarrow{\rho'} & (G \times_B m)^\circ \\
\downarrow{\overline{\mu}_S} & & \downarrow{\nu} \\
G \times S & \simeq & G \times_B m^\circ
\end{array}$$

is then readily seen to commute, where $\overline{\mu}_S^\circ$ (resp. $\nu^\circ$) is the restriction of $\overline{\mu}_S$ (resp. $\nu$) to $G \times S^\circ$ (resp. $(G \times_B m)^\circ$). We also know that the complements

$$G \times S \setminus G \times S^\circ \quad \text{and} \quad (G \times_B m) \setminus (G \times_B m^\circ)$$

have codimensions at least two in $G \times S$ and $G \times_B m$, respectively, as follows from Lemmas 5.14 and 5.15. One further consideration is that $\overline{\mu}_S$ and $\nu$ have projective fibres, as discussed in Remarks 4.6 and 5.1. These last three sentences show us to be in a situation analogous to one encountered in the proof of [5, Proposition 4.8]. We proceed analogously, deducing that $\rho'$ extends to a variety isomorphism

$$\overline{\rho} : G \times S \xrightarrow{\cong} G \times_B m$$

for which (5.14) commutes. By construction, $\overline{\rho}$ extends $\rho$.

It remains to establish that $\overline{\rho}$ is $G$-equivariant and Poisson. The former is a straightforward consequence of $\rho$ being $G$-equivariant and $G \times S$ being dense in $G \times S$. The latter follows from the fact that $\rho$ defines a symplectomorphism between the open dense symplectic leaves $G \times S \subseteq G \times S$ and $G \times_B m^\times \subseteq G \times_B m$ (see Proposition 5.2). Our proof is therefore complete. \hfill \Box

Remark 5.17. Consider the pullback of (5.14) along the inclusion $S \hookrightarrow g$, i.e.

$$\begin{array}{ccc}
\overline{\mu}_S^{-1}(S) & \xrightarrow{\cong} & \nu^{-1}(S) \\
\downarrow{\overline{\mu}_S} & & \downarrow{\nu} \\
S & \simeq & g
\end{array}$$
Theorem 5.16 implies that the horizontal arrow is a Poisson variety isomorphism between the Poisson slices $\mu^{-1}(S) \subseteq G \times S$ and $\nu^{-1}(S) \subseteq G \times m$. On the other hand, Section 4.4 explains that $\mu^{-1}(S) \rightarrow S$ is Balibănu’s fibrewise compactified universal centralizer $\pi_S : \mathcal{Z}_0 \rightarrow S$.

Our pullback diagram thereby becomes

$$
\begin{array}{ccc}
\mathcal{Z}_0 & \xrightarrow{\cong} & \nu^{-1}(S) \\
\downarrow{\pi_S} & & \downarrow{\nu_S} \\
S & & S
\end{array}
$$

where $\nu_S$ is the restriction of $\nu$ to a map $\nu^{-1}(S) \rightarrow S$. The Poisson variety isomorphism $\mathcal{Z}_0 \cong \nu^{-1}(S)$ is exactly the one obtained in [5, Proposition 4.8].

5.4. Applications to Hessenberg varieties. We now consider the implications of Theorem 5.16 for the geometry of Hessenberg varieties. To this end, consider the bundle projection $\pi : T^*G(\log D) \rightarrow \mathcal{C}$ and composite map

$$\psi := \pi \circ \rho^{-1} : G \times m \rightarrow \overline{G}.$$ 

Let us write $\psi_x := \psi|_{\text{Hess}(x,m)} : \text{Hess}(x,m) \rightarrow \overline{G}$ for the restriction of $\psi$ to a Hessenberg variety $\text{Hess}(x,m) \subseteq G \times m$.

**Corollary 5.18.** If $x \in \mathfrak{g}$, then

$$\psi_x : \text{Hess}(x,m) \rightarrow \overline{G}$$

is a closed embedding. This embedding is $G_x$-equivariant with respect to the action of $G_x = G \times \{e\} \subseteq G \times G$ on $\overline{G}$.

**Proof.** Theorem 5.16 tells us that $\rho$ restricts to an isomorphism

$$\rho_x : \mu^{-1}(x) \xrightarrow{\cong} \nu^{-1}(x) = \text{Hess}(x,m).$$

At the same time, Remark 4.6 implies that $\pi$ restricts to a closed embedding

$$\pi_x : \mu^{-1}(x) \rightarrow \mathcal{C}, \quad (\gamma, (x,x_S)) \mapsto \gamma, \quad (\gamma, (x,x_S)) \in \mu^{-1}(x).$$

The composite map

$$\psi_x = \pi_x \circ (\rho_x)^{-1} : \text{Hess}(x,m) \rightarrow \overline{G}$$

is therefore a closed embedding. The claim about equivariance follows from the $G$-equivariance of $\rho$, together with the fact that $\pi$ is $(G \times G)$-equivariant. \(\square\)

One can be reasonably explicit about the image of $\psi_x$, especially if $x \in \mathfrak{g}^r$. To elaborate on this, recall the $(G \times G)$-equivariant locally closed embedding

$$\gamma : G \rightarrow \overline{C} \subseteq \text{Gr}(n,\mathfrak{g} \oplus \mathfrak{g})$$

defined in (4.11). Given any $x \in \mathfrak{g}$, let us write $\overline{\mathcal{C}}_x$ for the closure of $\gamma(G_x)$ in $\overline{G}$.

**Proposition 5.19.** Suppose that $x \in \mathfrak{g}$. The following statements hold.

- (i) We have

$$\text{image}(\psi_x) = \{\gamma \in \overline{C} : (x,x_S) \in \gamma\}.$$
(ii) If \( x \in g^r \), then
\[
\text{image}(\psi_x) = (e, g^{-1}) \cdot G_x
\]
for any \( g \in G \) satisfying \( x = \text{Ad}_g(x_S) \).

Proof. To prove (i), recall the notation used in the proof of Corollary 5.18. The image of \( \psi_x \) coincides with that of \( \pi_x \), and the latter is
\[
\{ \gamma \in G : (x, x_S) \in \gamma \}.
\]
We now verify (ii). Note that \( (x, x_S) = (\text{Ad}_h g(x_S), x_S) \in \gamma_h \) for all \( h \in G_x \). This combines with (i) to imply that \( \gamma_h \in \text{image}(\psi_x) \) for all \( h \in G_x \), i.e.
\[
\gamma(G_x g) \subseteq \text{image}(\psi_x).
\]
Since \( \text{image}(\psi_x) \) is closed in \( G \), it must therefore contain the closure \( \overline{\gamma(G_x g)} \) of \( \gamma(G_x g) \) in \( G \). We also note that \( \text{image}(\phi_x) \cong \text{Hess}(x, m) \) is \( \ell \)-dimensional and irreducible (see [24, Corollaries 3 and 14]), and that
\[
\dim(\overline{\gamma(G_x g)}) = \dim(G_x) = \ell.
\]
It follows that
\[
\overline{\gamma(G_x g)} = \text{image}(\psi_x).
\]
It remains only to invoke the \((G \times G)\)-equivariance of \( \gamma \) and conclude that
\[
\overline{\gamma(G_x g)} = (e, g^{-1}) \cdot \overline{\gamma(G_x)} = (e, g^{-1}) \cdot G_x.
\]

\[\square\]

Remark 5.20. Suppose that \( x \in S \). One may apply Proposition 5.19(ii) with \( g = e \) and conclude that \( \text{image}(\psi_x) = G_x \). Corollary 5.18 therefore yields a \( G_x \)-equivariant isomorphism
\[
\text{Hess}(x, m) \xrightarrow{\cong} G_x.
\]
This is precisely the isomorphism obtained in [5, Corollary 4.10].

Notation

- \( g \) — finite-dimensional complex semisimple Lie algebra
- \( n \) — dimension of \( g \)
- \( \ell \) — rank of \( g \)
- \( \langle \cdot, \cdot \rangle \) — Killing form on \( g \)
- \( V^\perp \) — annihilator of \( V \subseteq g \) with respect to \( \langle \cdot, \cdot \rangle \)
- \( V^\dagger \) — annihilator of \( V \) in \( W^* \), where \( W \) is an ambient vector space containing \( V \)
- \( V^r \) — subset of regular elements in \( V \subseteq g \)
- \( V^{rs} \) — subset of regular semisimple elements in \( V \subseteq g \)
- \( G \) — adjoint group of \( g \)
- \( G_x \) — \( G \)-stabilizer of \( x \in g \) with respect to the adjoint action
- \( \mu = (\mu_L, \mu_R) : T^*G \to g \oplus g \) — moment map for the \((G \times G)\)-action on \( T^*G \)
- \( \mu_S : G \times S \to g \) — restriction of \( \mu_L \) to \( G \times S \)
- \( \tau \) — \( sl_2 \)-triple in \( g \)
- \( S_\tau \) — Slodowy slice associated to \( \tau \)
- \( S \) — Slodowy slice associated to a fixed principal \( sl_2 \)-triple
- \( \chi : g \to \text{Spec}(\mathbb{C}[g]^G) \) — adjoint quotient
- \( x_S \) — unique point at which \( S \) meets the fibre \( \chi^{-1}(\chi(x)) \)
• $\overline{G} \to$ De Concini–Procesi wonderful compactification of $G$

• $D$ — the divisor $\overline{G} \setminus G$

• $\mathfrak{g}_\Delta$ — diagonal in $\mathfrak{g} \oplus \mathfrak{g}$

• $\gamma_g$ — the point $(g,e) \cdot \mathfrak{g}_\Delta \in \overline{G}$

• $\gamma : G \to \overline{G}$ — open embedding defined by $g \mapsto \gamma_g$

• $\overline{G}_\gamma$ — closure of $\gamma(G_x)$ in $\overline{G}$

• $T^*\overline{G}(\log(D))$ — log cotangent bundle of $(\overline{G}, D)$

• $\overline{\mu} = (\overline{\mu}_L, \overline{\mu}_R) : T^*\overline{G}(\log(D)) \to \mathfrak{g} \oplus \mathfrak{g}$ — moment map for the $(G \times G)$-action on $T^*\overline{G}(\log(D))$

• $\overline{G} \times \mathcal{S}$ — the Poisson slice $\overline{\mu}_R^{-1}(\mathcal{S})$

• $\overline{G}_\mu$ — universal centralizer of $\mathfrak{g}$

• $\overline{G}_B$ — Balibanu’s fibrewise compactification of $\overline{G}_\mu$

• $\mathcal{B}$ — abstract flag variety of all Borel subalgebras in $\mathfrak{g}$

• $\mathfrak{m}$ — standard Hessenberg subspace

• $\nu : G \times \mathcal{B} \mathfrak{m} \to \mathfrak{g}$ — moment map for the $G$-action on $G \times \mathcal{B} \mathfrak{m}$.

• $\text{Hess}(x, \mathfrak{m})$ — fibre of $\nu$ over $x \in \mathfrak{g}$

• $\Pi$ — set of simple roots

• $\mathfrak{p}_I, \mathfrak{p}_I^-$ — standard parabolic, opposite parabolic associated to $I \subseteq \Pi$

• $l_I$ — standard Levi subalgebra $\mathfrak{p}_I \cap \mathfrak{p}_I^-$ associated with $I \subseteq \Pi$.

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