Coding with Noiseless Feedback over the Z-channel

Christian Deppe\(^1\), Vladimir Lebedev\(^2\), Georg Maringer\(^1\), and Nikita Polyanski\(i\)^\(^1\)

\(^1\) Institute for Communications Engineering
Technical University of Munich, Munich, Germany
\{christian.deppe,georg.maringer,nikita.polianski\}@tum.de
\(^2\) Kharkevich Institute for Information Transmission Problems
Russian Academy of Sciences, Moscow, Russia
lebedev37@mail.ru

Abstract. In this paper, we consider encoding strategies for the Z-channel with noiseless feedback. We analyze the asymptotic case where the maximal number of errors is proportional to the blocklength, which goes to infinity. Without feedback, the asymptotic rate of error-correcting codes for the error fraction \(\tau \geq 1/4\) is known to be zero. It was also proved that using the feedback a non-zero asymptotic rate can be achieved for the error fraction \(\tau < 1/2\). In this paper, we give an encoding strategy that achieves the asymptotic rate \((1+\tau)(1-h(\tau/(1+\tau)))\), which is positive for all \(\tau < 1\). Additionally, we state an upper bound on the maximal asymptotic rate of error-correcting codes for the Z-channel.

1 Introduction

In optical communications and other digital transmission systems the ratio between probability of errors of type \(1 \rightarrow 0\) and \(0 \rightarrow 1\) can be large \([1]\). Practically, one can assume that only one type of error can occur. These channels are called asymmetric. This paper addresses the problem of finding coding strategies for the Z-channel with feedback. The Z-channel depicted in Figure 1 is of asymmetric nature because it permits an error \(1 \rightarrow 0\), whereas it prohibits an error \(0 \rightarrow 1\). Transmission is referred to as being error-free if the output symbol matches the input symbol of the respective symbol transmission.

We are considering a combinatorial setting in this paper. In this setting, we limit the fraction of erroneous symbols by \(\tau = t/n\), where \(n\) denotes the blocklength and \(t\) the maximum number of errors within a block. This is in contrast to the probabilistic setting, in which the error probability of the channel is fixed. Feedback codes achieving the capacity of the Z-channel in the probabilistic setting are considered in \([2]\). The figure of merit examined in this work is the maximum asymptotic rate, written as \(R(\tau)\) and also called capacity error function \([3]\), which we define to be the maximum rate at which information can be communicated over a channel error-free as the blocklength \(n\) goes to infinity in the aforementioned combinatorial setting.
The problem of finding encoding strategies for the Z-channel using noiseless feedback is equivalent to a variation of Ulam’s game, the half-lie game. The first appearance of the half-lie game occurs in [4]. In this game for two players one player, referred to as Paul, tries to find an element $x \in \mathcal{M}$ by asking $n$ yes-no questions which are of the form: Is $x \in A$ for some $A \subseteq \mathcal{M}$? The other player, the responder Carole, is allowed to lie at most $t$ times if the correct answer to the question is yes. In comparison to the original Ulam game [5], Carole is not allowed to lie if the correct answer is no. Before Ulam proposed the game it was already described by Berlekamp [6] and by Renyi [7]. For a survey of results see [8]. It is known that for fixed $t$, the cardinality of the maximal set $\mathcal{M}$ is asymptotically $2^{n+t!n^{-t}}$ for Paul to win the half-lie game. First this was shown for $t = 1$ in [9] and later generalized in [10, 11] for arbitrary $t$. Due to the equivalence of the half-lie game and the coding problem with feedback for the Z-channel, the coding problem has been solved for an arbitrary but fixed number of errors for the asymptotic case when $n$ goes to infinity.

For coding without feedback Bassalygo has shown in [12] that the maximal asymptotic rate for the Z-channel is equal to the one of the binary symmetric channel (BSC). Notably the results presented there show that the maximum asymptotic rate is zero for $\tau \geq 1/4$. It is worth noticing that the maximal asymptotic rate of error-correcting codes for the BSC with feedback was completely characterized by Berlekamp [6] and Zigangirov [13]. Their results show that the rate is positive for $\tau < 1/3$. For the construction of error correcting codes for asymmetric channels without feedback we refer to the work of Kløve [14].

By random arguments, it was proved in [15] that $R(\tau) > 0$ for $\tau < 1/2$. In [16] a feedback strategy based on the rubber method [9] was introduced to find an encoding strategy achieving a positive asymptotic rate for any $\tau < 1/2$. The corresponding lower bound on $R(\tau)$ is plotted in green on Figure 2.

1.1 Our contribution

In this paper, we develop new encoding algorithms for the Z-channel with feedback. In particular, we provide a family of error-correcting codes with the asymptotic rate $(1+\tau)(1-h(\tau/(1+\tau)))$, which is positive for any $\tau < 1$ and improves the result from [16] in all but countable number of points. The corresponding lower bound on $R(\tau)$ is shown in blue in Figure 2. Additionally, we prove an upper bound on $R(\tau)$, which is depicted as the dashed line.
1.2 Outline

The remainder of the paper is organized as follows. In Section 2 we formally define the problem of coding with feedback over the Z-channel and introduce some auxiliary terminology. In Section 3 we provide our encoding algorithm achieving a positive asymptotic rate for any fraction of errors $\tau < 1$, which gives rise to our main result, Theorem 1. An upper bound on the asymptotic rate is proposed in Section 4. Finally, Section 5 concludes the paper.

![Fig. 2: Asymptotic rate of error-correcting codes for the Z-channel with noiseless feedback.](image)

2 Coding with feedback

A transmission scheme with feedback enables the sender to choose his encoding strategy in a way that makes use of the knowledge about previously received symbols at the receiver. This is shown in Figure 3. Let $\mathcal{M}$ denote the set of possible messages. The sender chooses one of them, say $m$, which he wants to send to the receiver.

An encoding algorithm for a feedback channel of blocklength $n$ is composed of a set of functions

$$c_i : \mathcal{M} \times \{0,1\}^{i-1} \to \{0,1\}, \quad i \in \{1,\ldots,n\}.$$  

The encoding algorithm is then constructed as

$$c(m, y^{n-1}) = (c_1(m), c_2(m,y_1), \ldots, c_n(m,y^{n-1})), \quad (1)$$
where $y_k := (y_1, \ldots, y_k)$ with $y_i$ being the $i^{th}$ received symbol. Moreover, the set of possible values for the received symbol $y_i$ conditioned on $c_i$ is defined by the channel, in our case the the Z-channel depicted in Figure 3.

Suppose that at most $t$ errors occur within a block of length $n$. For $m \in M$, we define the set of output sequences for an encoding strategy by

$$Y^n_t(m) := \{y^n \in \{0, 1\}^n : y_i \leq c_i(m, y^{i-1}), d_H(y^n, c(m, y^n-1) \leq t\},$$

where $d_H(a, b)$ denotes the Hamming distance between the sequences $a$ and $b$. Additionally, we denote the Hamming weight of a sequence $a$ by $w_H(a)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{channel.png}
\caption{Channel with feedback}
\end{figure}

**Definition 1.** An encoding strategy is called successful if $Y^n_t(m_1) \cap Y^n_t(m_2) = \emptyset$ for all $m_1, m_2 \in M$ with $m_1 \neq m_2$.

**Definition 2.** Let $M(n, t)$ denote the maximum number of messages in $M$ for which there exists a successful encoding strategy. Such a strategy is said to be optimal.

**Definition 3.** For any $\tau$ with $0 \leq \tau \leq 1$, we define the maximal asymptotic rate of an optimal encoding strategy to be

$$R(\tau) := \limsup_{n \to \infty} \frac{\log_2(M(n, \lceil \tau n \rceil))}{n}.$$

### 3 Lower Bound on $R(\tau)$

In this section we give a successful encoding strategy for the Z-channel, achieving an asymptotic rate $R(\tau)$ for any $\tau < 1$. This gives a lower bound on the maximal asymptotic rate of the Z-channel.

**Theorem 1.** For any $\tau$, $0 \leq \tau \leq 1$, we have

$$R(\tau) \geq \bar{R}(\tau) := (1 + \tau) - (1 + \tau) \log(1 + \tau) + \tau \log \tau.$$
At the start of the message transmission the receiver only knows the set of possible messages $\mathcal{M}$. The sender chooses a message $m \in \mathcal{M}$. The goal of an encoding strategy is to reduce the number of possible messages from the receiver’s viewpoint until only one message $m$ is left. The encoding algorithm we provide divides the number of channel uses $n$ into subblocks. Therefore, the encoding procedure is potentially divided into several steps. We denote the set of possible messages from the receiver’s perspective after the $i^{th}$ step as $\mathcal{M}_{i+1}$, the number of remaining channel uses as $n_{i+1}$ and the maximal number of possible errors $t_{i+1}$.

In the following the algorithm depicted in Figure 4 is described. At every step, the sender (as well as the receiver) checks the following two properties

$$t_i = 0 \quad \text{(2)}$$

and

$$|\mathcal{M}_i| \leq n_i - t_i + 1. \quad \text{(3)}$$

Depending on which of them hold the sender chooses one out of three algorithms for encoding. If both conditions (2)-(3) do not hold, then the sender makes use of

![Fig. 4: Encoding algorithm for transmission over the Z-channel](image-url)
Partitioning Algorithm. This strategy tries to limit the set of possible messages by dividing the message space into subsets and sending the index of the subset containing the message. After this subblock transmission the sender and the receiver examine the conditions $2, 3$ and check whether the encoding and decoding strategies have to be adjusted for the remainder of the block.

If the property $2$ is failed and the property $3$ holds, then the sender uses the Weight Algorithm for information transmission in the remaining channel uses.

If the condition $2$ is true, then the sender applies the Uncoded Algorithm for information transmission in the remaining channel uses. Below we describe the three algorithms required for our encoding strategy.

Partitioning Algorithm: This algorithm relies on the specific choice of positive integers $\delta$ and $p$ with $\delta > p$. We partition the message space before the $i$th step $M_i$ into $\binom{\delta}{p}$ subsets $M_{i,k}$ of almost equal sizes

$$M_i = \bigcup_{k=1}^{\binom{\delta}{p}} M_{i,k}.$$  

The size of each group is either \(\left\lceil \frac{M_i}{\binom{\delta}{p}} \right\rceil\), or \(\left\lfloor \frac{M_i}{\binom{\delta}{p}} \right\rfloor\). The exact way in which the message space is to be partitioned is to be agreed between the sender and the receiver before the data transmission. Then the sender finds the index of the group containing the message and transmit this index using a subblock of length $\delta$ containing $p$ ones. In this way the receiver can determine the number of errors inflicted by the channel within this subblock by counting the number of ones. There are $p + 1$ possible cases depending on the number of errors $e_i$ within the respective subblock of the $i$th step.

If $e_i = p$ errors occur, the message space consistent with the outcome of the channel is not changed and the receiver obtains the information that $M_{i+1} = M_i$, $n_{i+1} = n_i - \delta$ and $t_{i+1} = t_i - p$.

When $e_i < p$ errors occur, there are $\binom{\delta - p + e_i}{e_i}$ subsets of messages $M_{i,k}$ that are consistent with the outcome of the Z-channel. $M_{i+1}$ is then equal to the union of these subsets. Therefore, the set of possible messages in accordance with the received $\delta$ symbols is reduced and we have $M_{i+1} \leq \left\lceil \frac{M_i}{\binom{\delta}{p}} \right\rceil \binom{\delta - p + e_i}{e_i}$. Moreover, the receiver and the sender obtain $n_{i+1} = n_i - \delta$ and $t_{i+1} = t_i - e_i$.

Weight Algorithm: The sender would like to transmit a message $m'$ out of a given set $\mathcal{M}'$ using the channel $n'$ times. We order the messages within this set by enumerating them: $\mathcal{M}' = \{m_0, m_1, \ldots, m_{|\mathcal{M}'| - 1}\}$. The message $m'$ then corresponds to one of the indices, say $k$. The sender transmits the symbol 1 over the channel until it has been received exactly $k$ times. This happens at some point if a sufficient amount of channel uses is considered because the number of errors is limited. We denote this limit as $t'$. After that, the sender transmits 0-symbols which cannot be disturbed by the Z-channel. The receiver finds the Hamming weight $w$ of the received sequence and outputs the message $m_w$. This strategy is successful, i.e., $m_k = m_w$, if $|\mathcal{M}'| \leq n' - t' + 1$.  

Uncoded Algorithm: We denote the ordered set of possible messages as $\mathcal{M}' = \{m_0, m_1, \ldots, m_{|\mathcal{M}'|-1}\}$. The senders task is to send one of the messages, say $m_k$ to the receiver by using the channel $n'$ times. In order to do so, it sends the (standard) binary representation of the index $k$ over the channel. This strategy is successful if the sender is allowed to use the channel at least $\lceil \log_2 |\mathcal{M}'| \rceil$ times.

Now we prove that for any proper choice of integers $\delta$, $p$, $k$ and $t$ and real $\varepsilon > 0$, the sender can have the message set $\mathcal{M}$ of size at least

$$\left\lceil \frac{(\delta)^k}{\varepsilon} \right\rceil (1 - \varepsilon)^k \frac{\delta^k}{t + p} - 1$$

for $n = \left\lceil \frac{(\delta)^k}{\varepsilon} \right\rceil + \delta k$ channel uses, having at most $t$ errors. More formally, it is shown in Lemma 1, which we prove in Appendix.

**Lemma 1.** Let $\delta$ and $p$ be positive integers such that $p < \delta$ and let $t \geq 0$. Let $\varepsilon > 0$ be fixed such that

$$\gamma_e := (1 - \varepsilon) \frac{\binom{\delta}{p}}{\frac{\delta^p}{\varepsilon}} > 1, \quad \forall e \in \{0, \ldots, p - 1\}.$$ 

We define $\gamma_p := 1$, $M_0 := \left\lceil \frac{(\delta)^k}{\varepsilon} \right\rceil$ and the set $S_{(k,t,p)} := \left\{(k_0, k_1, \ldots, k_p) \in \mathbb{N}_0^{p+1} : \sum_{i=0}^p k_i = k, \sum_{i=0}^p ik_i \leq t + p \right\}$.

Then for any non-negative integers $t$ and $k$ such that $\delta k \geq t$, we have

$$M(M_0 + \delta k, t) \geq M_0 \min_{S_{(k,t,p)}} \left\lceil \prod_{e=0}^p \gamma_e^{k_e} \right\rceil.$$  \hspace{1cm} (4)

In particular, it follows that

$$M(M_0 + \delta k, t) \geq M_0 \frac{(1 - \varepsilon)^k \binom{\delta}{p}^k}{(\delta^k + t + p) / \varepsilon} - 1.$$  \hspace{1cm} (5)

**Remark 1.** The set $S_{(k,t,p)}$ includes $(k,0,\ldots,0)$. Thus, the minimization in (4) is well defined.

**Proof of Theorem 1.** Let us fix some positive $\tau < 1$. For any $\varepsilon_R > 0$ and small enough $\varepsilon_\tau > 0$, we shall prove the existence of a code of an arbitrary large blocklength and code rate at least $R(\tau) - \varepsilon_R$ capable of correcting $\tau - \varepsilon_\tau$ fraction of errors.
In what follows, we vary positive integers \( k \) and \( \delta \) with \( k > \delta \). Define \( t = \lceil \tau \delta \rceil \), \( p = \lfloor \delta (1/2 + \tau/2) \rfloor \), \( M_0 = \lceil \delta/p \rceil /\varepsilon \) and \( n = M_0 + \delta k \), where the real parameter \( \varepsilon \) is fixed and satisfies
\[
0 < \varepsilon < \frac{1 - \tau}{2} \leq 1 - p/\delta,
\]
\[
0 < R(\tau) + \log(1 - \varepsilon) - 3\varepsilon \tau.
\] (6)

Let \( \delta_0 \) be such that for any \( \delta \geq \delta_0(\varepsilon, \tau) \) and \( k \geq \delta \), we have
\[
\left( \frac{\delta}{p} \right) \geq 2^{\left( h\left( \frac{1 + \tau}{2} \right) - \varepsilon \tau \right)}
\] (7)

and
\[
\left( \frac{\delta k - pk + t + p}{t + p} \right) \leq 2^{\delta k + \frac{1}{2}(h\left( \frac{1}{2} + \tau \right) + \varepsilon \tau)},
\] (8)

where the binary entropy function \( h(x) := -x \log(x) - (1 - x) \log(1 - x) \). To prove the existence of such \( \delta_0 \), we note that
\[
\lim_{\delta \to \infty} \frac{p}{\delta} = \lim_{\delta \to \infty} \frac{\lfloor \delta (1/2 + \tau/2) \rfloor}{\delta} = \frac{1 + \tau}{2},
\]
and for \( k \geq \delta \),
\[
\lim_{\delta \to \infty} \frac{t + p}{\delta k - pk + t + p} = \frac{2\tau}{1 + \tau},
\]
and for any integers \( u > v \geq 1 \), the binomial coefficient \( \binom{u}{v} \) satisfies
\[
\sqrt{\frac{u}{8v(u - v)}} 2^{uh(v/u)} \leq \binom{u}{v} \leq \sqrt{\frac{u}{2\pi(v(u - v))}} 2^{uh(v/u)}.
\] (9)

Then we take \( k_0 = k_0(\delta, \tau, \varepsilon_\tau) \) such that for any \( k \geq k_0 \), the fraction of errors
\[
\frac{t}{n} = \frac{t}{M_0 + \delta k} \geq \tau - \varepsilon_\tau
\]
and the blocklength
\[
n = M_0 + \delta k \leq \delta k (1 + \varepsilon_\tau)
\]
and
\[
\frac{(1 - \varepsilon)^k \left( \frac{\delta}{p} \right)^k}{2^{\delta k - pk + t + p}} \geq 2.
\]
The latter can be achieved because of the choice of \( \varepsilon \) in (8) and large enough \( \delta \) in (7)-(8). By Lemma 1, there exists a feedback error-correcting code of blocklength \( n = M_0 + \delta k \) and of size
\[
M \geq M_0 \left( 1 - \varepsilon \right)^k \left( \frac{\delta}{p} \right)^k - 1 \geq \frac{(1 - \varepsilon)^k \left( \frac{\delta}{p} \right)^k}{2^{\delta k - pk + t + p}},
\] (10)
capable of correcting $t$ errors when transmitted through the Z-channel. Thus, combining (7)-(10) yields

$$R(\tau - \varepsilon) \geq \frac{\log M}{M_0 + \delta k}$$

$$\geq k \log(1 - \varepsilon) + \delta k \left( h(\frac{\varepsilon}{\tau}) - \varepsilon - \frac{2\varepsilon}{\tau + \tau} - \varepsilon \right) - 1$$

$$= (1 + \tau) \log \left( \frac{2}{1 + \tau} \right) + \tau \log \tau - \varepsilon_R,$$

where

$$\varepsilon_R \leq -\log(1 - \varepsilon) + O(1).$$

As $\varepsilon$ and $\varepsilon_R$ can be taken as small as needed and $\delta$ and $k$ can be arbitrary large, the statement of Theorem 1 follows.

4 Upper Bound on $R(\tau)$

In this section we establish an upper bound on the rate $R(\tau)$. This upper bound is close to our lower bound for small values of $\tau$. We make use of an approach similar to the one in [11]. We take an encoding strategy and consider only messages $m \in M$ such that any output sequence in $\mathcal{Y}_{n,t}(m)$ has a relatively large Hamming weight. For those messages, it is possible to derive a good lower bound on the size of $\mathcal{Y}_{n,t}(m)$. The upper bound on the set of possible messages is then obtained by a sphere-packing argument.

Theorem 2. For any $\tau$, $0 < \tau < 1$, we have

$$R(\tau) \leq R(\tau) := \min_{0 \leq \tau' \leq \tau} \max_{0 \leq r \leq 1} \frac{r}{h(v) \leq 1} \left( \min_{0 \leq \tau' \leq \tau} \frac{v}{1 - \tau} \right)$$

where $v = v(r, \tau')$ is a real number such that $0 \leq v \leq 1/2$ and $h(v)(1 - \tau') = r$. 

Proof. We fix $\tau$ and $\tau'$ fulfilling the inequalities $0 \leq \tau' < \tau \leq 1$ and define $t := \tau n$ and $t' := \tau' n$. Denote $R(\tau)$ by $R$. Next we fix some $\varepsilon > 0$. We define $\tau \in [0,1/2]$ as the unique real number that satisfies $h(\tau)(1 - \tau') = \tau - \varepsilon$. We define the set of output sequences of the encoding strategy when the encoder would like to transmit the message $m$ and the channel output is zero for the first $t'$ symbols to be

$$\mathcal{Y}_{n,t}(m) := \{ y^n \in \mathcal{Y}_{n,t}(m) : y_i = 0 \text{ for } i \in [t'] \}.$$ 

For any real $v$ with $0 \leq v \leq 1$, let $W(n,t',v)$ denote the set of all binary words $x^n$ that have $x_i = 0$ for all $i \leq t'$ and the Hamming weight at most $v(n-t')$. For $n \to \infty$, we have that the cardinality of $W(n,t',\tau - \varepsilon)$ is

$$|W(n,t',\tau - \varepsilon)| = \binom{n-t'}{i} \leq 2^{(n-t')(h(\tau - \varepsilon) + o(1))}.$$
As sets of output sequences are mutually disjoint, we conclude with

\[ \{ \min_{1 \leq i \leq n - t'} \phi(i) \} \]

with \(1 \leq n \leq n_0\) such that we have an encoding function \(\Phi\) for a set of messages \(\mathcal{M}\) with \(|\mathcal{M}| \geq 2^n(\tau - \varepsilon)\). For simplicity of notation, we assume that \((n - t')(\tau - \varepsilon)\) is an integer and equal to \(n'\). Define the set of \textit{good} messages, written as \(\mathcal{M}_{\text{good}}\), that consists of \(m \in \mathcal{M}\) such that the Hamming weight of any \(y^m \in Y_{t',n'}(m)\) is at least \(n'\). Since \(n \geq n_0\), we obtain that \(|\mathcal{M}_{\text{good}}| \geq |\mathcal{M}| - 2^{(n - t')(\tau - \varepsilon)} \geq 2^{(n - t')(\tau - \varepsilon)} - 1\), where we used the fact \(h(\tau)(1 - \tau') = \tau - \varepsilon\). Now we prove that for any message \(m \in \mathcal{M}_{\text{good}}\), the size of \(Y_{t',n'}(m)\) is uniformly bounded from below as follows

\[ |Y_{t',n'}(m)| \geq \max_{0 \leq i \leq \min(t - t', n')} \binom{n'}{i}. \]

Let \(\binom{[a]}{b}\) denote the set of all possible subsets of \([a]\) of size \(b\). To show the above inequality, take an arbitrary \(i\) with \(0 \leq i \leq \min(t - t', n')\) and define the mapping \(\phi : \binom{[n']}{i} \to Y_{t',n'}(m)\) that takes an arbitrary subset \(\{i_1, \ldots, i_t\} \in \binom{[n']}{i}\) with \(1 \leq i_1 < i_2 < \ldots < i_t \leq n'\) and outputs \(y^n \in \{0, 1\}^n\) defined as

\[
y_i := \begin{cases} 0 & \text{for } i \in [t'], \\ c_i(m, y^{i-1}) & \text{for } i \in J, \\ 1 - c_i(m, y^{i-1}) & \text{o/w}, \end{cases}
\]

where \(J := \bigcup_{k=0}^{j} [j_k + 1, j_k + 1 - 1], j_0 := t', j_{i+1} := n + 1\) and for \(k \in [\hat{i}], j_k\) is the smallest \(j\) so that the Hamming weight \(w_H(y^{i-1}, c_j(m, y^{j-1})) = i_k\). One can easily see that this \(y^n\) belongs to \(Y_{t',n'}(m)\) and for distinct \(\{i_1, \ldots, i_t\} \neq \{s_1, \ldots, s_t\}\), the outputs \(\phi(\{i_1, \ldots, i_t\})\) and \(\phi(\{s_1, \ldots, s_t\})\) are different. As the sets of output sequences are mutually disjoint, we conclude with

\[ |\mathcal{M}_{\text{good}}| \max_{0 \leq i \leq \min(t - t', n')} \binom{n'}{i} \leq 2^{n - t'}. \]

As \(n\) can be taken arbitrary large, letting \(n \to \infty\) yields

\[ (n - t')h(\tau) + n'h \left( \min \left( \frac{t - t'}{n'}, \frac{1}{2} \right) \right) + o(n) \leq n - t'. \]

Recall that \(n' = (n - t')(\tau - \varepsilon)\). Since the above inequality is true for any \(\varepsilon > 0\), we have

\[ h(\tau) \leq 1 - \tau h \left( \min \left( \frac{\tau - \tau'}{\tau(1 - \tau')}, \frac{1}{2} \right) \right). \]

\[ \square \]
5 Conclusion

In this paper, we discussed a new family of error-correcting codes for the Z-channel with noiseless feedback in the combinatorial setting. By providing an explicit construction, we showed that the maximal asymptotic rate $R(\tau)$ is positive for any $\tau < 1$. We believe that the lower bound on $R(\tau)$ presented in Theorem 1 is tight for all $\tau$. Another natural question to ask is whether the channel capacity (probabilistic setting) of the Z-channel can be achieved by the same encoding algorithm.

6 Acknowledgment

Christian Deppe was supported by the Bundesministerium für Bildung und Forschung (BMBF) through Grant 16KIS1005. Vladimir Lebedev’s work was supported by the Russian Foundation for Basic Research (RFBR) under Grant No. 19-01-00364 and by RFBR and JSPS under Grant No. 20-51-50007. Georg Maringer’s work was supported by the German Research Foundation (Deutsche Forschungsgemeinschaft, DFG) under Grant No. WA3907/4-1. Nikita Polyaniskii’s research was supported in part by a German Israeli Project Cooperation (DIP) grant under grant no. KR3517/9-1.
References

1. M. Blaum, *Codes for detecting and correcting unidirectional errors*. IEEE Computer Society Press, 1994.
2. L. G. Tallini, S. Al-Bassam, and B. Bose, “Feedback codes achieving the capacity of the z-channel,” *IEEE Transactions on Information Theory*, vol. 54, no. 3, pp. 1357–1362, 2008.
3. R. Ahlswede, C. Deppe, and V. Lebedev, “Non-binary error correcting codes with noiseless feedback, localized errors, or both,” *Annals of European Academy of Sciences*, no. 1, pp. 285 – 309, 2005.
4. R. L. Rivest, A. R. Meyer, D. J. Kleitman, K. Winklmann, and J. Spencer, “Coping with errors in binary search procedures,” *Journal of Computer and System Sciences*, vol. 20, no. 3, pp. 396–404, 1980.
5. S. M. Ulam, *Adventures of a Mathematician*. Univ of California Press, 1991.
6. E. R. Berlekamp, “Block coding for the binary symmetric channel with noiseless, delayless feedback,” *Proc. Symposium on Error Correcting Codes*, 1968.
7. A. Renyi, “On a problem of information theory,” *MTA Mat. Kut. Int. Kozl.*, vol. 6, no. B, pp. 505–516, 1961.
8. F. Cicalese, *Fault-Tolerant Search Algorithms - Reliable Computation with Unreliable Information*, ser. Monographs in Theoretical Computer Science. An EATCS Series. Springer, 2013. [Online]. Available: https://doi.org/10.1007/978-3-642-17327-1
9. F. Cicalese and D. Mundici, “Optimal coding with one asymmetric error: below the sphere packing bound,” in *International Computing and Combinatorics Conference*. Springer, 2000, pp. 159–169.
10. I. Dumitriu and J. Spencer, “A halfliar’s game,” *Theoretical computer science*, vol. 313, no. 3, pp. 353–369, 2004.
11. J. Spencer and C. H. Yan, “The halflie problem,” *Journal of Combinatorial Theory, Series A*, vol. 103, no. 1, pp. 69–89, 2003.
12. L. A. Bassalygo, “New upper bounds for error correcting codes,” *Problems Information Transmission*, vol. 1, no. 4, pp. 32–35, 1965.
13. K. Zigangirov, “On the number of correctable errors for transmission over a binary symmetrical channel with feedback,” *Problems Information Transmission*, vol. 12, no. 2, pp. 85–97, 1976.
14. T. Kløve, *Error correcting codes for the asymmetric channel*. Department of Pure Mathematics, University of Bergen, 1981.
15. A. G. D’yachkov, “Upper bounds on the error probability for discrete memoryless channels with feedback,” *Problemy Peredachi Informatsii*, vol. 11, no. 4, pp. 13–28, 1975.
16. C. Deppe, V. Lebedev, and G. Maringer, “Bounds for the capacity error function for unidirectional channels with noiseless feedback,” *arXiv preprint arXiv:2001.09030*, will appear in *Proc. IEEE ISIT 2020*, 2020.
7 Appendix

Proof of Lemma 7. First we notice that the set $S_{(k,t,p)}$ includes $(k,0,\ldots,0)$. Thus, the minimization in (4) is well defined.

We shall prove this lemma by applying the encoding algorithm described above and using induction on the sum $k + t$ in the pair $(k,t)$. The base cases when $k = \lceil t/\delta \rceil$ or $t = 0$ follow from the description of Weight Algorithm and Uncoded Algorithm. Indeed, for $t = 0$, we get the minimum in (4) is at most $M_0^k \gamma_0 \leq \left(\delta \frac{p}{\delta}\right)^k \leq M_0 2^\delta k$. However, according to the encoding algorithm the sender has to use Uncoded Algorithm and can have the message set of size $2^{M_0 + \delta k}$ which is larger than $M_0 2^\delta k$. When $k = \lceil t/\delta \rceil$, then the minimum in (4) is equal to $M_0$ and is attained with $(k_0, k_1, \ldots, k_p) = (0, \ldots, 0, k)$ as

$$\sum_{i=0}^p k_i = k, \quad \sum_{i=0}^p ik_i = p\lceil t/\delta \rceil \leq pt/\delta + p \leq t + p.$$ 

Due to the above algorithm, the sender makes use of Weight Algorithm at this point. According to its description, we can transmit $M_0 + \delta k - t + 1$ which is larger than $M_0$ for $k = \lceil t/\delta \rceil$.

This concludes the base cases of the induction. In the following we show the inductive step.

We prove that the sender can transmit

$$M := \left[ M_0 \min_{S_{(k-1,t-\epsilon,p)}} \prod_{i=0}^p \gamma_i^{k_i} \right],$$

messages using $M_0 + k\delta$ channel uses when at most $t$ errors may occur. We note that the set $S_{(k,t,p)}$ includes $(k,0,\ldots,0)$. Thus, the minimization in (4) is well defined. According to the above algorithm, the sender check two conditions (2)-(3). If (2) is true, then we are in a base case. If (3) holds, then the sender makes use Weight Algorithm and successfully transmit the message. If both conditions are failed, then the sender uses Partitioning Algorithm. It remains to check that for any $e \in \{0, 1, \ldots, p\}$ (the number of errors in the block of $\delta$ bits), the sender would be able to transmit the message out of

$$\begin{cases} \left[ M \left(\frac{\delta-p+e}{e}\right) \right], & \text{for } e \in \{0, 1 \ldots, p-1\} \\ M, & \text{for } e = p \end{cases}$$

ones using remaining $M_0 + \delta(k-1)$ channel uses and having at most $t-e$ errors. By the inductive hypothesis we are able to transmit

$$\left[ M_0 \min_{S_{(k-1,t-\epsilon-\cdot,p)}} \prod_{i=0}^p \gamma_i^{k_i} \right].$$
messages for this set-up. Thus it remains to check the inequality

\[ \left\lceil \frac{M}{\binom{p}{e}} \right\rceil \left( \frac{\delta - p + e}{e} \right) \leq \min_{S_{(k-1,t-e,p)}} \prod_{i=0}^{p} \gamma_{i}^{k_{i}}, \quad \forall e \in \{0,1\ldots,p-1\}. \quad (11) \]

Let us elaborate on the left-hand side of the inequality

\[ \left( \frac{\delta - p + e}{e} \right) \leq \frac{M!}{\delta!e!} + \frac{(\delta - p + e)}{\delta!e!} \]

\[ \leq \frac{M!}{\delta!e!} + M_{0}\varepsilon \quad (b) \]

where we used the property \( M_{0}\varepsilon = \left\lceil \frac{\delta}{\varepsilon} \right\rceil \geq \frac{\delta - p + e}{e} \) for any \( e \in \{0,1\ldots,p\} \) in (a) and \( \gamma_{e} = (1 - \varepsilon) \frac{\delta!e!}{\delta!(\delta-p+e)!} \) in (b). We note that

\[ M \leq \left[ M_{0}\gamma_{e} \min_{S_{(k-1,t-e,p)}} \prod_{i=0}^{p} \gamma_{i}^{k_{i}} \right]. \quad (13) \]

This inequality holds because

\[ \gamma_{e} \min_{S_{(k-1,t-e,p)}} \prod_{i=0}^{p} \gamma_{i}^{k_{i}} \geq \min_{S_{(k,t,p)}} \prod_{i=0}^{p} \gamma_{i}^{k_{i}} \]

as we get the left-hand side of the inequality from the right-hand side by adding the additional constraint that \( k_{e} \geq 1 \) to the minimization. Thus, combining the inequalities (12)-(13), we get

\[ \left( \frac{\delta - p + e}{e} \right) \leq \frac{M!}{\delta!e!} + \frac{(\delta - p + e)}{\delta!e!} \]

\[ \leq \frac{M!}{\delta!e!} + M_{0}\varepsilon \quad (c) \]

To prove (c), we observe that \( \gamma_{i} \geq 1 \) for all \( i \). As we compare an integer in the left-hand side with a real number in right-hand side, we can apply the floor operation to the latter. This proves (11) and completes the proof of the inductive step.

It remains to show (5). Suppose that the minimum in (4) is attained with \( k_{0} = k_{0}', k_{1} = k_{1}', \ldots, k_{p} = k_{p}' \) such that

\[ \sum_{i=0}^{p} k_{i}' = k, \quad \sum_{i=0}^{p} ik_{i}' = t' \leq t + p. \]
Then we derive

\[ M = \left\lfloor M_0 \prod_{i=0}^{p} \frac{\gamma_i^{k_i}}{\gamma_i^{k_i} - 1} \right\rfloor \geq M_0 (1 - \varepsilon)^k \prod_{i=0}^{p} \frac{\left(\frac{\delta_{i}'}{p_i} \right)^{k_i'}}{\left(\frac{\delta_{i}'}{p_i} \right)^{k_i'} - 1} \]

\[ = M_0 \frac{(1 - \varepsilon)^k \left(\frac{\delta_{i}'}{p_i} \right)^k}{\prod_{i=0}^{p} \left(\frac{\delta_{i}'}{p_i} + 1\right)^{k_i'}} - 1 \geq M_0 \frac{(1 - \varepsilon)^k \left(\frac{\delta_{i}'}{p_i} \right)^k}{\left(\frac{\delta k - pk + t'}{t' + p} \right)^k} - 1 , \]

where the property \( \binom{u}{v} \binom{w}{z} \leq \binom{u+w}{v+z} \) yields \((d)\) and the monotonicity of the function \( \binom{u+x}{z} \) in \( x \) implies \((e)\). \( \blacksquare \)