Generalizations of 2-dimensional diagonal quantum channels with constant Frobenius norm

Ivan Sergeev
ETH Zürich, Rämistrasse 101, 8092 Zürich
e-mail: i.i.sergeyev@gmail.com

March 21, 2019

Abstract

We introduce the set of quantum channels with constant Frobenius norm, the set of diagonal channels and the notion of equivalence of one-parameter families of channels. First, we show that all diagonal 2-dimensional channels with constant Frobenius norm are equivalent. Next, we generalize four one-parameter families of 2-dimensional diagonal channels with constant Frobenius norm to an arbitrary dimension $n$. Finally, we prove that the generalizations are not equivalent in any dimension $n \geq 3$.

Keywords: quantum channels; Frobenius norm.

1 Introduction

Quantum information channels were first introduced by A. S. Holevo in 1972 [1]. The definition utilizes the notion of a completely positive map, introduced by W. F. Stinespring in 1955 [2] and studied by M.-D. Choi [3], K. Kraus [4] and others. Although quantum channels have been studied since the 1970s, the description of the set of all quantum channels in an arbitrary dimension has not yet been obtained. The advances toward solving this problem include, for example, a complete description in dimension 2 [5] and some generalizations in higher dimensions [6].

Many works study specific families of channels and their properties in various contexts, for example, the depolarizing channel [7, 8, 9, 10, 11], the transpose-depolarizing channel [12] (special cases can be found in [13, 14, 15, 16, 17]), the sets of ergodic and mixing channels [18], Weyl channels [9, 19] etc. In addition, various functions of the output of quantum channels are often considered, such as the entropy [19, 7, 8, 20, 8, 19] and the noncommutative $\ell_p$-norms [21, 22], especially in relation to the capacity of quantum channels [21, 20, 23, 24, 10].

Since it is difficult to describe the set of all quantum channels, we limit the scope of our work to two narrow sets of channels: the set of quantum channels
with constant output Frobenius norm and the set of diagonal quantum channels. More specifically, we prove that every 2-dimensional channel with constant Frobenius norm is either a completely depolarizing channel or is equivalent to the depolarizing channel (i.e. differs from it only by a change of basis in the input and output state spaces). Next, we introduce the notion of diagonal quantum channels and use it to generalize the four one-parameter families of channels that emerge from the 2-dimensional classification. In addition, we prove that the four generalizations not equivalent in any dimension $n \geq 3$. In the future, we hope to generalize the results to broader sets of channels. Although there are papers that deal with functions of the output of quantum channels (such as those mentioned above), the author is not aware of other works that consider either the sets of channels with constant output Frobenius norm or diagonal quantum channels as defined in this paper.

This paper is organized as follows. First, we provide basic definitions in Section 2. Next, in Section 3 we introduce the notion of equivalence of channels, following [24], and classify all 2-dimensional quantum channels with constant Frobenius norm up to equivalence. Then we move on to the $n$-dimensional case. Section 4 is dedicated to the set of diagonal quantum channels: we define diagonal channels in Section 4.1 and then prove the criterion for them to be channels with constant Frobenius norm in Section 4.2. In Section 5.1, we use the notion of diagonal channels to generalize the four families of channels with constant Frobenius norm that emerge from the 2-dimensional case. Then, in Section 5.2, the generalizations are shown to be inequivalent in any dimension $n \geq 3$ (in contrast to the 2-dimensional case, where all four channels are equivalent). Finally, Section 6 contains discussion of the results and states open questions for further investigation.

## 2 Basic definitions

In this section, we recall the definitions of quantum channels (following [19, 25, 26, 27]) and of the Frobenius norm and introduce the set of channels with constant function of the output for pure input states.

We begin by recalling the notions of adjoint operators and completely positive maps, which are essential in the definition of quantum channels. Denote the underlying Hilbert spaces for the input and the output of the channel as $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, the corresponding spaces of quantum states as $\mathcal{S}(\mathcal{H}_1)$ and $\mathcal{S}(\mathcal{H}_2)$, and the corresponding spaces of observables as $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$.

**Definition 1.** Let $\Phi : \mathcal{S}(\mathcal{H}_1) \to \mathcal{S}(\mathcal{H}_2)$. The adjoint operator $\Phi^* : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$ satisfies the following identity:

$$\text{Tr} (S \Phi^* (A)) = \text{Tr} (\Phi (S) A) \quad \forall S \in \mathcal{S}(\mathcal{H}_1), \forall A \in \mathcal{B}(\mathcal{H}_2).$$

In other words, $\Phi^*$ is the adjoint of $\Phi$ with respect to the bilinear form $B(A_1, A_2) = \text{Tr} (A_1 A_2)$. 

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Definition 2. For every \( k \in \mathbb{N} \) denote \( \text{id}_k : \mathcal{M}_k \to \mathcal{M}_k \) the identity operator on the space of linear operators in \( \mathbb{C}^k \). The map \( \Phi^* \) defined above is called \( k \)-positive if \( \Phi^* \otimes \text{id}_k \) is positive, i.e. maps positive elements in \( \mathcal{B}(\mathcal{H}_2) \otimes \mathcal{M}_k \) to positive elements in \( \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M}_k \). The map \( \Phi^* \) is called completely positive if \( \Phi^* \) is a \( k \)-positive map for every \( k \in \mathbb{N} \).

Using these two notions, we can now formulate the definition of a quantum channel.

Definition 3. A linear map \( \Phi : \mathcal{S}(\mathcal{H}_1) \to \mathcal{S}(\mathcal{H}_2) \) is called a quantum channel if its adjoint \( \Phi^* \) is a completely positive map.

The following two theorems give alternative characterizations of completely positive maps.

Theorem 1 (Choi, see [3]). Let \( \Phi : \mathcal{M}_n \to \mathcal{M}_m \). Then \( \Phi \) is a completely positive map if and only if the following matrix is positive:

\[
C_{\Phi} = \sum_{i,j=1}^{n} E_{ij} \otimes \Phi(E_{ij}),
\]

where the matrix \( E_{ij} \in \mathbb{C}^{n \times n} \) has a 1 in the \( ij \)-th entry and 0s everywhere else. We refer to \( C_{\Phi} \) as the Choi matrix of \( \Phi \).

Theorem 2 (Kraus, see [4]). Let \( \Phi : \mathcal{M}_n \to \mathcal{M}_m \). Then \( \Phi \) is a completely positive map if and only if there exists a set of operators \( \{ V_i \} \subseteq \mathbb{C}^{m \times n} \) such that

\[
\Phi(S) = \sum_i V_i S V_i^*. 
\]

The operators \( V_i \) are then called the Kraus operators of \( \Phi \).

Next, we define quantum channels with constant function of the output for pure input states.

Definition 4. Let \( \Phi : \mathcal{S}(\mathcal{H}_1) \to \mathcal{S}(\mathcal{H}_2) \) be a quantum channel and \( f : \mathcal{S}(\mathcal{H}_2) \to \mathbb{R} \) be a function of the output state. We say that \( \Phi \) is a channel with constant function \( f \) if for all pure input states \( S \in \mathcal{S}(\mathcal{H}_1) \) the value \( f(\Phi(S)) \) is the same.

Note that lifting the restriction of the input to pure states makes the resulting set of channels scarcer. For example, by the above definition, the depolarizing and the transpose-depolarizing channels are channels with constant Frobenius norm, but they do not have the same output Frobenius norm for mixed input states as for pure input states. Since we would like to include these channels into consideration, we restrict the constant output function condition to pure input states.

In this paper, we assume that both underlying Hilbert spaces are \( n \)-dimensional, i.e. \( \mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^n \). In this case, the spaces of quantum states \( \mathcal{S}(\mathcal{H}_1) \) and \( \mathcal{S}(\mathcal{H}_2) \) are isomorphic to the space of \( n \)-dimensional positive semidefinite Hermitian operators of trace 1 and the spaces of observables \( \mathcal{B}(\mathcal{H}_1) \) and \( \mathcal{B}(\mathcal{H}_2) \) are isomorphic to the space of linear operators \( \mathcal{M}_n \). Among the functions of the output state \( f : \mathcal{M}_n \to \mathbb{R} \) we consider only the Frobenius norm.
**Definition 5.** The Frobenius norm $||·||_F : \mathcal{M}_n \to \mathbb{R}$ is a matrix norm defined by the following formula:

$$||S||_F = \sqrt{\sum_{j=1}^{n} s_j^2},$$

where $s_1, \ldots, s_n$ are singular values of $S \in \mathcal{M}_n$. If $S$ is a quantum state, then $s_j$ are eigenvalues of $S$.

Instead of the Frobenius norm, we could use other functions of the output state, for example, the Schatten norm, the Ky Fan norm (see [28], section IV.2), the noncommutative $\ell_p$ norm (see [22]) or the von Neumann entropy. However, these and other functions that are invariant with respect to basis changes in the underlying Hilbert spaces are defined using the eigenvalues of the output state. Therefore, to compute these functions in general case, one needs to find the roots of the characteristic polynomial of an $n \times n$ matrix, i.e. to solve a polynomial equation of degree $n$. On the other hand, one can compute the Frobenius norm without finding the eigenvalues by using the following identities:

$$||S||_F = \sqrt{\sum_{i,j=1}^{n} |s_{ij}|^2} = \sqrt{\text{Tr} (S^* S)},$$

where $s_{ij}$ denote the entries of $S$ and $S^*$ is the adjoint of $S$. For this reason, we choose to only consider the Frobenius norm in this work.

### 3 Dimension 2

In this section, we introduce the notion of equivalence of quantum channels and classify all 2-dimensional quantum channels with constant Frobenius norm up to equivalence.

King and Ruskai [24] showed that every 2-dimensional channel admits the following representation:

$$\Phi (S) = U_2 \Lambda (U_1 S U_1^*) U_2^*, \quad (1)$$

where $U_1$ and $U_2$ are unitary operators and $\Lambda$ has the following representation in the Pauli basis $\{ I, \sigma_x, \sigma_y, \sigma_z \}$:

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_x & \lambda_x & 0 & 0 \\ t_y & 0 & \lambda_y & 0 \\ t_z & 0 & 0 & \lambda_z \end{pmatrix},$$

where $t_\alpha, \lambda_\alpha \in \mathbb{R}$ for every $\alpha \in \{ x, y, z \}$.

As a direct generalization of this representation, we introduce the following definition.
**Theorem 3.** Every 2-dimensional quantum channel with constant Frobenius norm is either a completely depolarizing channel or is equivalent to the depolarizing channel

\[ \Phi_d(p, S) = pS + \frac{1-p}{2} \text{Tr} S. \]

**Proof.** Let \( S \) be an arbitrary input state. Then the state \( S' = U_1SU_1^* \) and its image \( \Lambda(S') \) can be written in the following form:

\[
S' = \frac{1}{2} \left( I + \alpha_x \sigma_x + \alpha_y \sigma_y + \alpha_z \sigma_z \right) = \frac{1}{2} \left( \begin{array}{cc} 1 + \alpha_z & \alpha_x - i\alpha_y \\ \alpha_x + i\alpha_y & 1 - \alpha_z \end{array} \right),
\]

\[
\Lambda(S') = \frac{1}{2} \left( I + \frac{1}{2} \left( \sum_{\sigma=x,y,z} (t_i + a_i \lambda_i) \sigma_i \right) \right).
\]

Here \( \alpha_x, \alpha_y, \alpha_z \in \mathbb{R} \) are the Stokes parameters. Note that the condition \( \text{Tr} S' = 1 \) is already satisfied and the condition \( \det S' \geq 0 \) results in the following inequality:

\[ a_x^2 + a_y^2 + a_z^2 \leq 1. \]

Since the Frobenius norm is unitarily invariant,

\[
||\Phi(S)||_F^2 = ||U_2 \Lambda(U_1SU_1^*) U_2^*||_F^2 = ||\Lambda(U_1SU_1^*)||_F^2 = \frac{1}{2} \left( 1 + (t_x + a_x \lambda_x)^2 + (t_y + a_y \lambda_y)^2 + (t_z + a_z \lambda_z)^2 \right).
\]

Therefore, \( \Phi \) is a channel with constant Frobenius norm if and only if (2) is a constant polynomial of \( \alpha_x, \alpha_y \) and \( \alpha_z \), i.e. all combinations of \( \alpha_x, \alpha_y \) and \( \alpha_z \) in (2) have zero coefficients. For a pure input state \( S \), we can see that \( S' \) is also a pure state and thus

\[ a_x^2 + a_y^2 + a_z^2 = 1. \]

It follows that (2) is a constant polynomial in only two cases.
1. \( \lambda_x = \lambda_y = \lambda_z = 0 \) and \( t_x, t_y, \) and \( t_z \) satisfy the following inequality:
\[
 t_x^2 + t_y^2 + t_z^2 \leq 1.
\]
In this case, \( \Phi \) is a completely depolarizing channel.

2. \( t_x = t_y = t_z = 0 \) and \( |\lambda_x| = |\lambda_y| = |\lambda_z| = p \). In this case, \( \Phi \) is a diagonal channel. Without loss of generality, assume \( \lambda_z = p \). Then there are four variants:
- (a) \( \lambda_x = \lambda_y = p \),
- (b) \( \lambda_x = p \) and \( \lambda_y = -p \),
- (c) \( \lambda_x = \lambda_y = -p \),
- (d) \( \lambda_x = -p \) and \( \lambda_y = p \).

Denote the corresponding channels \( \Phi_1 - \Phi_4 \). Note that \( \Phi_1 \) is in fact the depolarizing channel:
\[
\Phi_1 (p, S) = \frac{1}{2} (I + pax_x + pay_y + paz_z) = \Phi_d (S).
\]
The following identities show that the other three channels are equivalent to the depolarizing channel:
\[
\begin{align*}
\sigma_y \Phi_2 (-p, S) \sigma_y &= \frac{1}{2} (\sigma_y \sigma_y - pa_x \sigma_y \sigma_x \sigma_y + pa_y \sigma_y \sigma_y \sigma_y - pa_z \sigma_y \sigma_z \sigma_y) = \\
&= \frac{1}{2} (I + pa_x \sigma_x + pa_y \sigma_y + pa_z \sigma_z) = \Phi_d (p, S),
\end{align*}
\]
\[
\begin{align*}
\sigma_z \Phi_3 (p, S) \sigma_z &= \frac{1}{2} (\sigma_z \sigma_z - pa_x \sigma_z \sigma_z \sigma_z - pa_y \sigma_z \sigma_y \sigma_z + pa_z \sigma_z \sigma_z \sigma_z) = \\
&= \frac{1}{2} (I + pa_x \sigma_x + pa_y \sigma_y + pa_z \sigma_z) = \Phi_d (p, S),
\end{align*}
\]
\[
\begin{align*}
\sigma_x \Phi_4 (-p, S) \sigma_x &= \frac{1}{2} (\sigma_x \sigma_x + pa_x \sigma_x \sigma_x \sigma_x - pa_y \sigma_x \sigma_y \sigma_x - pa_z \sigma_x \sigma_z \sigma_x) = \\
&= \frac{1}{2} (I + pa_x \sigma_x + pa_y \sigma_y + pa_z \sigma_z) = \Phi_d (p, S).
\end{align*}
\]

Remark 1. Note that the theorem also holds for channels with constant von Neumann entropy, Schatten \( p \)-norm (for \( 1 < p \leq \infty \)) and Ky Fan 1-norm, since the same proof works if the Frobenius norm is replaced with any injective function.

Since the Schatten 1-norm and the Ky Fan 2-norm in dimension 2 give the trace of the matrix, channels with constant Schatten 1-norm and Ky Fan 2-norm constitute the whole set of 2-dimensional channels, which was fully described in [5].
4 Diagonal channels

In this section, we introduce the set of diagonal quantum channels and prove the criterion for diagonal channels to be channels with constant Frobenius norm, which is used in Section 5.1 to define generalizations of the four one-parameter families of channels that appear in the proof of Theorem 3.

4.1 Definition

In essence, diagonal channels are defined as channels that have diagonal form in a specific basis in $\mathcal{M}_n$. The basis we use consists of Hermitian $n \times n$ matrices and is orthonormal with respect to the bilinear form $B(A_1, A_2) = \text{Tr}(A_1^* A_2)$. There are two reasons why we choose a basis with these properties. Firstly, the space of quantum states $\mathcal{S}(\mathbb{C}^n)$ is isomporhic space of $n$-dimensional positive semidefinite Hermitian operators of trace 1, hence the Hermitian requirement. Secondly, we would like all decompositions of quantum states over our basis to have real coefficients, hence the orthonormality condition.

Before defining an orthonormal basis of Hermitian matrices in $\mathcal{M}_n$, let us consider the 2-dimensional case. Recall that the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ together with the identity matrix $I$ form an orthogonal basis in the space of $2 \times 2$ Hermitian matrices. Therefore, we introduce an intuitive $n$-dimensional generalization of the Pauli matrices as a starting point for defining our basis in $\mathcal{M}_n$.

Definition 7. Denote $N = \frac{n(n-1)}{2}$. We define generalized Pauli matrices in dimension $n$ as follows:

$$
\sigma_{0,1} = I = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix};
$$

$$
\sigma_{x,1} = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}, \quad \sigma_{x,2} = \begin{pmatrix}
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad \sigma_{x,N} = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix};
$$

$$
\sigma_{y,1} = \begin{pmatrix}
0 & -i & 0 & \cdots & 0 \\
i & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad \sigma_{y,2} = \begin{pmatrix}
0 & 0 & -i & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
i & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad \sigma_{y,N} = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & -i
\end{pmatrix};
$$

$$
\sigma_{z,1} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}, \quad \sigma_{z,2} = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad \sigma_{z,N} = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & -1
\end{pmatrix}.
$$

Note that by construction, the generalized Pauli matrices are Hermitian and orthogonal with respect to the bilinear form $B(\cdot, \cdot)$ defined above. However, for $n \geq 3$ the matrices $\sigma_{x,1}, \ldots, \sigma_{z,N}$ are not linearly independent. Therefore, in the definition of our basis, we replace them with other matrices.
Definition 8. Denote
\[
\mathcal{E} = \left\{ \frac{I}{\sqrt{n}} \sigma_{x,1}, \sigma_{x,1}, \sigma_{x,N}, \sigma_{x,N}, \frac{\sigma_{y,1}}{\sqrt{2}}, \sigma_{y,1}, \frac{M_{z,1}}{\sqrt{2}}, \sigma_{y,N}, \frac{M_{z,2}}{\sqrt{2}}, \ldots, \frac{M_{z,n-1}}{\sqrt{(n-1)n}} \right\},
\]
where \( N = \frac{n(n-1)}{2} \) as before, \( \{\sigma_{x,i}\}_{i=1}^{N} \) and \( \{\sigma_{y,i}\}_{i=1}^{N} \) are generalized Pauli matrices, and
\[
M_{z,1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad M_{z,2} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ldots,
\]
\[
M_{z,n-1} = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & -(n-1) \end{pmatrix}.
\]

Note that, by construction, \( \mathcal{E} \) is a basis in \( \mathcal{M}_n \) over \( \mathbb{C} \), consists of Hermitian matrices, and is orthonormal with respect to the bilinear form \( B(\cdot, \cdot) \). However, there are also some disadvantages to using the basis \( \mathcal{E} \). For example, a permutation of basis vectors of the underlying Hilbert space \( \mathbb{C}^n \) results in a permutation of the generalized Pauli matrices, but the matrices \( e_{z,1}, \ldots, e_{z,n-1} \) are mapped to nontrivial linear combinations of the elements of the basis \( \mathcal{E} \).

Now, using the basis \( \mathcal{E} \), we define the set of diagonal channels.

Definition 9. We say that \( \Phi: \mathcal{S}(\mathbb{C}^n) \to \mathcal{S}(\mathbb{C}^n) \) is a diagonal channel if \( \Phi \) has diagonal form in the basis \( \mathcal{E} = \{e_i\}_{i=0}^{n^2-1} \), i.e.
\[
\Phi = \text{diag}\left(1, t_1, \ldots, t_{n^2-1}\right).
\]  

(3)

Note that since \( \mathcal{E} \) is an orthonormal basis of Hermitian matrices, the adjoint operator \( \Phi^* \) of a diagonal channel \( \Phi \) has the same form in \( \mathcal{E} \):
\[
\Phi^* = \text{diag}\left(1, t_1, \ldots, t_{n^2-1}\right).
\]

In this sense, diagonal channels can be said to be self-adjoint.

When appropriate, we also use the following notation for the basis elements and diagonal channels:
\[
\mathcal{E} = \{e_{0,1}, e_{x,1}, \ldots, e_{x,N}, e_{y,1}, \ldots, e_{y,N}, e_{z,1}, \ldots, e_{z,n-1}\},
\]
\[
\Phi = \text{diag}\left(1, t_{x,1}, \ldots, t_{x,N}, t_{y,1}, \ldots, t_{y,N}, t_{z,1}, \ldots, t_{z,n-1}\right).
\]

4.2 Diagonal channels with constant Frobenius norm

The following theorem gives a criterion for diagonal channels to be channels with constant Frobenius norm.
Theorem 4. Let $\Phi : S(\mathbb{C}^n) \to S(\mathbb{C}^n)$ be a diagonal channel. Then $\Phi$ is a channel with constant Frobenius norm if and only if $|t_i| = |t_j|$ for all $i, j \in 1, n^2 - 1$.

Proof. First, let us calculate the Frobenius norm of an arbitrary state $S \in S(\mathbb{C}^n)$ and its image $\Phi(S)$. Note that $S$ can be represented as

$$S = \frac{1}{\sqrt{n}} e_{0,1} + \sum_{i=1}^{n^2 - 1} a_i e_i,$$

where $e_i \in E$ are basis matrices. If $\Phi$ is a diagonal channel of the form (3), then

$$||S||^2_F = \frac{1}{n} + \sum_{i=1}^{n^2 - 1} a_i^2, \quad ||\Phi(S)||^2_F = \frac{1}{n} + \sum_{i=1}^{n^2 - 1} a_i^2 t_i^2.$$ 

Additionally, if $S$ is a pure state, then

$$||S||^2_F = \frac{1}{n} + \sum_{i=1}^{n^2 - 1} a_i^2 = 1.$$

Now, suppose $\Phi$ is a diagonal channel with $|t_i| = |t_j|$ for all $i, j \in 1, n^2 - 1$. Then for any pure state $S$ we have

$$||\Phi(S)||^2_F = \frac{1}{n} + t_i^2 \left(1 - \frac{1}{n}\right) = \text{const.}$$

Thus $\Phi$ is a channel with constant Frobenius norm.

Conversely, suppose $\Phi$ is a diagonal channel with constant Frobenius norm. We prove that $|t_i| = |t_j|$ for all $i, j \in 1, n^2 - 1$ in three steps.

1. First, we prove that $|t_{z,i}| = |t_{z,j}| =: |t_z|$ for all $i, j \in 1, n - 1$. To show this, we introduce the following set of unit vectors in $\mathbb{C}^n$:

$$|\psi_0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad |\psi_1\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad |\psi_{n-1}\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$ 

Using these vectors, we define a set of pure states $S_k = |\psi_k\rangle \langle \psi_k|$ with the following decompositions in $E$:

$$S_k = \frac{1}{\sqrt{n}} e_{0,1} + \sum_{i=1}^{n-1} B(e_{z,i}, S_k) e_{z,i}.$$ 

For every $k \in 1, n - 1$ we have

$$B(e_{z,i}, S_k) = \frac{B(M_{z,i}, S_k)}{\sqrt{i(i+1)}} = \frac{1}{\sqrt{i(i+1)}} \begin{cases} 0, & i < k \\ -i, & i = k \\ 1, & i > k \end{cases}.$$
Since $\Phi$ is a channel with constant Frobenius norm, for every $k \in \overline{1, n-1}$ we have
\[
0 = ||\Phi(S_k)||_F^2 - ||\Phi(S_{k+1})||_F^2 = t_{z,k}^2 \frac{k}{k+1} - t_{z,k+1}^2 \frac{k}{k+1}.
\]
Therefore, $|t_{z,k}| = |t_{z,k+1}|$ for every $k \in \overline{1, n-1}$. Thus $|t_{z,i}| = |t_{z,j}| = |t_z|$ for all $i, j \in \overline{1, n-1}$.

2. Second, we prove that $|t_{x,i}| = |t_{y,i}|$ for all $i \in \overline{1, N}$. To show this, we introduce another two sets of unit vectors in $\mathbb{C}^n$:
\[
|\xi_i\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad |\eta_i\rangle = \begin{pmatrix} i \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \quad \ldots, \quad |\xi_N\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \quad |\eta_N\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\]
Using these vectors, we define two sets of pure states $S_k = |\xi_k\rangle \langle \xi_k|$ and $S'_k = |\eta_k\rangle \langle \eta_k|$ with the following decompositions in $E$:
\[
S_k = \frac{1}{\sqrt{n}} e_{0,1} + \sum_{i=1}^{n-1} B(e_{z,i}, S_k) e_{z,i} + \frac{1}{\sqrt{2}} e_{y,k},
\]
\[
S'_k = \frac{1}{\sqrt{n}} e_{0,1} + \sum_{i=1}^{n-1} B(e_{z,i}, S'_k) e_{z,i} + \frac{1}{\sqrt{2}} e_{y,k}.
\]
For every $k \in \overline{1, N}$ and every $i \in \overline{1, n-1}$, we have
\[
B(e_{z,i}, S_k) = B(e_{z,i}, S'_k).
\]
Since $\Phi$ is a channel with constant Frobenius norm, for every $k \in \overline{1, N}$ we have
\[
0 = \left| \left| \Phi\left(\hat{S}_k\right)\right|_F^2 - \left| \left| \Phi\left(S'_k\right)\right|_F^2 \right| = t_{x,k}^2 \left( \frac{1}{\sqrt{2}} \right)^2 - t_{y,k}^2 \left( \frac{1}{\sqrt{2}} \right)^2.
\]
Therefore, $|t_{x,k}| = |t_{y,k}|$ for all $k \in \overline{1, N}$.

3. Finally, we prove that $|t_{x,i}| = |t_z|$ for all $i \in \overline{1, N}$. To show this, we use the pure states $S_k$ and $S_l$ defined above.

(a) For an arbitrary index $l \in \overline{1, N}$, there exist indices $k_1, k_2 \in \overline{0, n-1}$ such that
\[
\hat{S}_l = \frac{1}{\sqrt{2}} e_{x,l} + \frac{1}{2} (S_{k_1} + S_{k_2}).
\]
The linear property of the form $B(\cdot, \cdot)$ implies
\[
\Phi\left(\hat{S}_l\right) = \frac{1}{\sqrt{n}} e_{0,1} + \frac{1}{\sqrt{2}} t_{x,l} e_{x,l} + \sum_{i=1}^{n-1} \frac{1}{2} B(e_{z,i}, S_{k_1} + S_{k_2}) t_{z,i} e_{z,i}
\]
and
\[ 2 = B (S_{k1} + S_{k2}, S_{k1} + S_{k2}) = \]
\[ = B \left( \frac{2}{\sqrt{n}} e_{0,1} + \sum_{i=1}^{n-1} B (e_{z,i}, S_{k1} + S_{k2}), S_{k1} + S_{k2} \right) = \]
\[ = \frac{4}{n} + \sum_{i=1}^{n-1} (B (e_{z,i}, S_{k1} + S_{k2}))^2. \]

Therefore, the Frobenius norm of \( \Phi (\tilde{S}_l) \) can be written in the following form:
\[ \| \Phi (\tilde{S}_l) \|_F^2 = \frac{1}{n} + t_{x,l}^2 \left( \frac{1}{\sqrt{2}} \right)^2 + \frac{t_z^2}{4} \sum_{i=1}^{n-1} (B (e_{z,i}, S_{k1}))^2 = \]
\[ = \frac{1}{n} + t_{x,l}^2 + t_z^2 \left( 2 - \frac{4}{n} \right). \quad (4) \]

(b) Let \( k \in 0, n-1 \) be an arbitrary index. The linear property of the form \( B (\cdot, \cdot) \) implies
\[ 1 = B (S_k, S_k) = \]
\[ = B \left( \frac{1}{\sqrt{n}} e_{0,1} + \sum_{i=1}^{n-1} B (e_{z,i}, S_k) e_{z,i}, S_k \right) = \]
\[ = \frac{1}{n} + \sum_{i=1}^{n-1} (B (e_{z,i}, S_k))^2. \]

Therefore, the Frobenius norm of \( \Phi (S_k) \) can be written in the following form:
\[ \| \Phi (S_k) \|_F^2 = \frac{1}{n} + t_z^2 \sum_{i=1}^{n} (B (e_{z,i}, S_k))^2 = \frac{1}{n} + t_z^2 \left( 1 - \frac{1}{n} \right). \quad (5) \]

(c) Since \( \Phi \) is a map with constant Frobenius norm, from (4) and (5) it follows that
\[ 0 = \left\| \Phi (\tilde{S}_l) \right\|_F^2 - \left\| \Phi (S_k) \right\|_F^2 = \frac{1}{2} t_{x,l}^2 - \frac{1}{2} t_z^2. \]

Therefore, \( |t_{x,l}| = |t_z| \) for every \( l \in \{1, N\}. \)

The identities above prove that \( |t_i| = |t_j| \) for every \( i, j \in \{1, n^2 - 1\}. \) \( \Box \)

5 Generalizations in dimension \( n \)

In this section, we generalize four one-parameter families of 2-dimensional channels that appear in the proof of Theorem 3 and prove that the generalizations are inequivalent in any dimension \( n \geq 3. \)
5.1 Definition

By generalizations of the four 2-dimensional channels from Theorem 3 we mean $n$-dimensional channels that are channels with constant Frobenius norm and in the case $n = 2$ coincide with the channels from Theorem 3. The idea is to consider the four 2-dimensional channels as special cases of $n$-dimensional diagonal channels, defined in Section 4.1, that multiply the $z$ set of basis matrices (i.e. $e_{z,1}, \ldots, e_{z,n-1} \in \mathcal{E}$) by $p$ and multiply the $x$ and $y$ sets of basis matrices (i.e. $e_{x,1}, \ldots, e_{x,N} \in \mathcal{E}$ and $e_{y,1}, \ldots, e_{y,N} \in \mathcal{E}$ respectively) by $\pm p$. Then Theorem 4 insures that the resulting $n$-dimensional channels are channels with constant Frobenius norm. Therefore, we consider the following four channels (defined by their form in the basis $\mathcal{E}$):

- the generalization of the case $\lambda_x = \lambda_y = p$:
  \[
  \Phi = \text{diag}
  \begin{pmatrix}
  1, p, \ldots, p, p, \ldots, p
  
  
  \end{pmatrix},
  \]  

- the generalization of the case $\lambda_x = -\lambda_y = p$:
  \[
  \Phi = \text{diag}
  \begin{pmatrix}
  1, p, \ldots, p, -p, \ldots, -p
  
  
  \end{pmatrix},
  \]  

- the generalization of the case $\lambda_x = \lambda_y = -p$:
  \[
  \Phi = \text{diag}
  \begin{pmatrix}
  1, -p, \ldots, -p, -p, \ldots, -p
  
  
  \end{pmatrix},
  \]  

- the generalization of the case $\lambda_x = -\lambda_y = -p$:
  \[
  \Phi = \text{diag}
  \begin{pmatrix}
  1, -p, \ldots, -p, p, \ldots, p
  
  
  \end{pmatrix}.
  \]

Plugging the explicit form of the basis matrices into the definitions (6)–(9), we find that

- the channel (6) is in fact the depolarizing channel:
  \[
  \Phi_d (p, S) = pS + \frac{1-p}{n} \text{Tr} S; \]

- the channel (7) is in fact the transpose-depolarizing channel:
  \[
  \Phi_t (p, S) = pS^T + \frac{1-p}{n} \text{Tr} S; \]
The channel (8) can be written in the following form:

$$\Phi_{dcq}(p, S) = -pS + \frac{1-p}{n} \text{Tr} S + 2p \sum_{i=1}^{n} \langle \psi_i | S | \psi_i \rangle \langle \psi_i | \psi_i \rangle,$$

(12)

which is why we refer to it as the hybrid depolarizing-classical-quantum channel (or the DCQ channel for short);

The channel (9) can be written in the following form:

$$\Phi_{tcq}(p, S) = -pS^T + \frac{1-p}{n} \text{Tr} S + 2p \sum_{i=1}^{n} \langle \psi_i | S | \psi_i \rangle \langle \psi_i | \psi_i \rangle,$$

(13)

which is why we refer to it as the hybrid transpose-depolarizing-classical-quantum channel (or the TCQ channel for short).

Although we refer to the maps (10)–(13) as channels, it is the case only for specific values of the parameter $p$, stated in the following theorem.

**Theorem 5.**

- The map $\Phi_d$ (defined in (10)) is a quantum channel if and only if the parameter $p$ satisfies the condition:

$$-\frac{1}{n^2 - 1} \leq p \leq 1.$$

(14)

- The map $\Phi_t$ (defined in (11)) is a quantum channel if and only if the parameter $p$ satisfies the condition:

$$-\frac{1}{n - 1} \leq p \leq \frac{1}{n + 1}.$$

(15)

- The map $\Phi_{dcq}$ (defined in (12)) is a quantum channel if and only if the parameter $p$ satisfies the condition:

$$-\frac{1}{2n - 1} \leq p \leq \frac{1}{(n - 1)^2}.$$

(16)

- The map $\Phi_{tcq}$ (defined in (13)) is a quantum channel if and only if the parameter $p$ satisfies the condition:

$$-\frac{1}{n - 1} \leq p \leq \frac{1}{n + 1}.$$

(17)

The following lemma, proved in Appendix A, is used to show that if the conditions (14)–(17) are satisfied, then the maps (10)–(13) are quantum channels. As stated in the proof of Theorem 5, the representations provided by the lemma are in fact Kraus representations of the four families of channels.
Lemma 1. The maps (10)–(13) can be expressed as

\[ \Phi (p, S) = c_0 S + c_x \sum_{i=1}^{N} \sigma_{x,i} S \sigma_{x,i} + c_y \sum_{i=1}^{N} \sigma_{y,i} S \sigma_{y,i} + c_z \sum_{i=1}^{N} \sigma_{z,i} S \sigma_{z,i} \]  

(18)

and

\[ \Phi (p, S) = \tilde{c}_0 e_{0,1} S e_{0,1} + \tilde{c}_x \sum_{i=1}^{N} e_{x,i} S e_{x,i} + \tilde{c}_y \sum_{i=1}^{N} e_{y,i} S e_{y,i} + \tilde{c}_z \sum_{j=1}^{n-1} e_{z,j} S e_{z,j} \]  

(19)

where \( \sigma_{\alpha,i} \) are the generalized Pauli matrices, \( e_{\alpha,i} \) are the elements of the basis \( \mathcal{E} \), and \( c_\alpha \) and \( \tilde{c}_\alpha \) are real coefficients for every \( \alpha \in \{0, x, y, z\} \). More specifically,

- for the depolarizing channel \( \Phi_d \),

\[
\begin{align*}
\tilde{c}_0 &= \frac{1 + (n^2 - 1)p}{n}, & c_0 &= \frac{1 + (n^2 - 1)p}{n^2}, & c_x &= \frac{1 - p}{2n}, & c_y &= \frac{1 - p}{2n}, & c_z &= \frac{1 - p}{n};
\end{align*}
\]

- for the transpose-depolarizing channel \( \Phi_t \),

\[
\begin{align*}
\tilde{c}_0 &= \frac{1 + (n^2 - 1)p}{n}, & c_0 &= \frac{1 + (n^2 - 1)p}{n}, & c_x &= \frac{1}{2n}, & c_y &= \frac{1}{2n}, & c_z &= \frac{1}{n};
\end{align*}
\]

- for the DCQ channel \( \Phi_{\text{dcq}} \),

\[
\begin{align*}
\tilde{c}_0 &= \frac{1 - (n-1)^2p}{n^2}, & c_0 &= \frac{1 - (n-1)^2p}{n^2}, & c_x &= \frac{1 - p}{2n}, & c_y &= \frac{1 + (2n-1)p}{n^2}, & c_z &= \frac{1}{n};
\end{align*}
\]

- for the TCQ channel \( \Phi_{\text{tcq}} \),

\[
\begin{align*}
\tilde{c}_0 &= \frac{1 - (n-1)^2p}{n^2}, & c_0 &= \frac{1 - (n-1)^2p}{n^2}, & c_x &= \frac{1}{2n}, & c_y &= \frac{1 + (2n-1)p}{n^2}, & c_z &= \frac{1}{n};
\end{align*}
\]

Proof of Theorem 5. Since (14) and (15) are well-known results (e.g. present in [10] and [12]), we omit the respective proofs, although one can use the method described below to prove (14) and (15) similarly to (16) and (17).

First, we prove sufficiency of the conditions (16) and (17) using Lemma 1 and Kraus’ theorem. Suppose that the condition (16) (respectively, (17)) is
satisfied. Then all coefficients in representations (18) and (19) are nonnegative. Therefore, (18) and (19) are Kraus representations, hence $\Phi_{dcq}$ (respectively, $\Phi_{tqc}$) is a quantum channel.

Second, we prove necessity of the conditions (16) and (17) using Choi’s theorem.

Suppose that the map $\Phi_{dcq}$ is a quantum channel. Then $\Phi_{dcq}^*$ is a completely positive map and the Choi matrix

$$
C = \begin{pmatrix}
 p + \frac{1-p}{n} & 0 & \ldots & 0 & 0 & -p & \ldots & 0 & \ldots & 0 & 0 & \ldots & -p \\
p + \frac{1-p}{n} & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & \frac{1-p}{n} & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & p + \frac{1-p}{n} & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & \frac{1-p}{n} & \ldots & \ldots & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-p & 0 & \ldots & 0 & 0 & -p & \ldots & \ldots & \ldots & 0 & 0 & \ldots & \ldots \\
\end{pmatrix}
$$

is positive semidefinite by Choi’s theorem. In particular, the element $C_{2,2}$ is nonnegative: $\frac{1-p}{n} \geq 0$, hence the parameter $p$ satisfies $p \leq 1$. It remains to show that $p = \frac{1}{n^2-1}$. Since $C$ is positive semidefinite, we have $\det C \geq 0$. Expanding the determinant using the elements of the main diagonal, we get:

$$
\det C = \left(\frac{1-p}{n}\right)^{n(n-1)} \cdot \det \begin{pmatrix}
p + \frac{1-p}{n} & -p & \ldots & \ldots & \ldots & -p \\
p & p + \frac{1-p}{n} & \ldots & \ldots & \ldots & \ldots; \\
-p & \ldots & \ldots & \ldots & \ldots & \ldots \\
-p & \ldots & \ldots & \ldots & \ldots & \ldots \\
-p & \ldots & \ldots & \ldots & \ldots & \ldots \\
-p & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
$$

To compute the determinant on the right hand side, consider

$$
D(a, b) = \det \begin{pmatrix}
p + \frac{1-p}{n} & -p & \ldots & -p & -p & \ldots & -p \\
-p & p + \frac{1-p}{n} & \ldots & -p & -p & \ldots & -p \\
-p & -p & \ldots & -p & -p & \ldots & -p \\
-p & -p & \ldots & -p & -p & \ldots & -p \\
-p & -p & \ldots & -p & -p & \ldots & -p \\
-p & -p & \ldots & -p & -p & \ldots & -p \\
\end{pmatrix}
$$

Using the multilinearity property of the determinant, we obtain the following recurrence relation:

$$
D(a, b) = D(a - 1, b + 1) + \left(2p + \frac{1-p}{n}\right) D(a - 1, b).
$$
Solving this recurrence relation, we find

\[ D(n, 0) = \left(2p + \frac{1 - p}{n}\right)^{n-1} \cdot \frac{1 - (n - 1)^2p}{n}. \]

Since

\[ \det C = \left(\frac{1 - p}{n}\right)^{n(n-1)} \cdot D(n, 0), \]

the inequality \( \det C \geq 0 \) holds if and only if

\[ \left(\frac{1 + (2n-1)p}{n}\right)^{n-1} \cdot \frac{1 - (n - 1)^2p}{n} \geq 0. \] (20)

A similar calculation for the submatrix \((C_{i,j})_{i,j=1}^{n(n-1)}\) yields

\[ \left(\frac{1 + (2n-1)p}{n}\right)^{n-2} \cdot \frac{1 - (n - 1)^2p}{n} \geq 0. \] (21)

The inequalities (20) and (21) hold simultaneously if and only if the following conditions are satisfied:

\[ \frac{1 + (2n-1)p}{n} \geq 0, \quad \frac{1 - (n - 1)^2p}{n} \geq 0, \]

which is equivalent to the condition (14).

- Suppose that the map \( \Phi_{tq} \) is a quantum channel. Then \( \Phi_{tq}^* \) is a completely positive map. By Choi’s theorem, the Choi matrix

\[
C = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & -p & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -p
\end{pmatrix}
\]

is positive semidefinite. In particular,

\[ \det \left( \begin{array}{cc}
C_{2,2} & C_{2,n+1} \\
C_{n+1,2} & C_{n+1,n+1}
\end{array} \right) = \left(\frac{1}{n^2} - 1\right) p^2 - \frac{2}{n^2} p + \frac{1}{n^2} \geq 0. \]

Therefore,

\[ (1 + (n - 1)p) (1 - (n + 1)p) \geq 0, \]

which holds if and only if the parameter \( p \) satisfies the condition (17).
5.2 Inequivalence in dimension $n \geq 3$

In this section, we prove the following theorem.

**Theorem 6.** In dimension $n \geq 3$, the one-parameter families of channels $\Phi_d$, $\Phi_t$, $\Phi_{dcq}$, and $\Phi_{tcq}$ are not equivalent (unlike in dimension $n = 2$, as shown in Section 3).

To prove this theorem, we use the following two lemmas.

**Lemma 2.** Let $S_1$ and $S_2$ be two states with the same set of eigenvalues. Then $\Phi_d(p, S_1)$ and $\Phi_d(p, S_2)$, $\Phi_t(p, S_1)$ and $\Phi_t(p, S_2)$ have the same set of eigenvalues; $\Phi_{dcq}(p, S_1)$ and $\Phi_{dcq}(p, S_2)$, $\Phi_{tcq}(p, S_1)$ and $\Phi_{tcq}(p, S_2)$ may have different sets of eigenvalues, unless $p = 0$ or $n = 2$.

**Proof.** Denote $\{\lambda_k\}_{k=1}^n$ the eigenvalues of the states $S_1$ and $S_2$. From the definitions of the depolarizing channel and the transpose-depolarizing channel, we can see that the output states $\Phi_d(p, S_1)$, $\Phi_d(p, S_2)$, $\Phi_t(p, S_1)$ and $\Phi_t(p, S_2)$ all have the same eigenvalues $\{p\lambda_k + \frac{1-p}{n}\}_{k=1}^n$.

For the DCQ and the TCQ channels, consider two states $S_1$ and $S_2$ defined as follows:

$$|\eta\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad S_1 = |\eta\rangle \langle \eta| = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$|\zeta\rangle = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad S_2 = |\zeta\rangle \langle \zeta| = \frac{1}{n} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Since $S_1$ and $S_2$ are pure states, they have eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \cdots = \lambda_n = 0$. However, $\Phi_{dcq}(p, S_1)$ and $\Phi_{tcq}(p, S_1)$ have eigenvalues $\lambda_1 = p + \frac{1-p}{n}$ and $\lambda_2 = \cdots = \lambda_n = \frac{1-p}{n}$, while $\Phi_{dcq}(p, S_2)$ and $\Phi_{tcq}(p, S_2)$ have eigenvalues $\lambda_1 = -p + \frac{1-p}{n}$ and $\lambda_2 = \cdots = \lambda_n = -p + \frac{1+p}{n}$. These sets of eigenvalues are the same only in two cases: if $p = 0$, which is trivial, or if $n = 2$, which we already considered in Section 3.

**Lemma 3.** Suppose that for some $p$ and $\tilde{p}$ the channels $\Phi_d(p, S)$ and $\Phi_d(\tilde{p}, S)$ are equivalent. Let $\alpha \in \mathbb{R}$ be a multiplier such that $\alpha p$ and $\alpha \tilde{p}$ are in the valid parameter range for $\Phi_d$ and $\Phi_t$ respectively. Then the channels $\Phi_d(\alpha p, S)$ and $\Phi_t(\alpha \tilde{p}, S)$ are equivalent.

The same holds for the DCQ and the TCQ channels: if for some $p$ and $\tilde{p}$ we have $\Phi_{dcq}(p, S) \sim \Phi_{tcq}(\tilde{p}, S)$, then $\Phi_{dcq}(\alpha p, S) \sim \Phi_{tcq}(\alpha \tilde{p}, S)$ for every $\alpha \in \mathbb{R}$ such that $\alpha p$ and $\alpha \tilde{p}$ are in the valid parameter range for $\Phi_{dcq}$ and $\Phi_{tcq}$ respectively.
Proof. Suppose that for some \( p \) and \( \tilde{p} \) we have \( \Phi_d(p, S) \sim \Phi_t(\tilde{p}, S) \). Then there exist unitary operators \( U_1 \) and \( U_2 \) such that for every quantum state \( S \) the following identity holds:

\[
\Phi_d(p, S) = U_2 \Phi_t(\tilde{p}, U_1 SU_1^*) U_2^*.
\]

Let \( \alpha \in \mathbb{R} \) be a multiplier such that \( \alpha p \) and \( \alpha \tilde{p} \) are in the valid parameter range for \( \Phi_d \) and \( \Phi_t \) respectively. Note that \( \Phi_d(\alpha p, S) = \alpha \Phi_d(p, S) + \frac{1 - \alpha}{n} \text{Tr} S \), \( \Phi_t(\alpha p, S) = \alpha \Phi_t(p, S) + \frac{1 - \alpha}{n} \text{Tr} S \).

Therefore,

\[
\Phi_d(\alpha p, S) = \alpha U_2 \Phi_t(\tilde{p}, U_1 SU_1^*) U_2^* = U_2 \left\{ \alpha \Phi_t(\tilde{p}, U_1 SU_1^*) + \frac{1 - \alpha}{n} \text{Tr} S \right\} U_2^* = U_2 \Phi_t(\alpha \tilde{p}, U_1 SU_1^*) U_2^*.
\]

Thus, equivalence \( \Phi_d(\alpha p, S) \sim \Phi_t(\alpha \tilde{p}, S) \) holds. We get the proof for the DCQ and the TCQ channels by replacing \( \Phi_d \) with \( \Phi_{dcq} \) and \( \Phi_t \) with \( \Phi_{tcq} \).

Proof of Theorem 6. Since unitary evolutions do not change the set of eigenvalues, the following inequivalences hold by Lemma 2 (for \( n \geq 3 \)): \( \Phi_d \not\sim \Phi_{dcq} \), \( \Phi_d \not\sim \Phi_{tcq} \), \( \Phi_t \not\sim \Phi_{dcq} \), \( \Phi_t \not\sim \Phi_{tcq} \). Thus it remains to prove that \( \Phi_d \not\sim \Phi_t \) and \( \Phi_{dcq} \not\sim \Phi_{tcq} \) in any dimension \( n \geq 3 \).

Suppose that \( \Phi_d \sim \Phi_t \). Note that the case when \( p = 0 \) and \( \tilde{p} \neq 0 \) is impossible, since a completely depolarizing channel cannot be equivalent to a channel that is not a constant mapping. Similarly, the case when \( p \neq 0 \) and \( \tilde{p} = 0 \) is also impossible. Therefore, without loss of generality we may assume that for some \( p \neq 0 \) and \( \tilde{p} \neq 0 \) we have \( \Phi_d(p, S) \sim \Phi_t(\tilde{p}, S) \). Let \( \alpha \in \mathbb{R} \) be a multiplier such that \( \alpha p \) and \( \alpha \tilde{p} \) are in the valid parameter range for \( \Phi_d \) and \( \Phi_t \) respectively. By Lemma 3 the channels \( \Phi_d(\alpha p, S) \) and \( \Phi_t(\alpha \tilde{p}, S) \) are equivalent for suitable values of \( \alpha \). On the one hand, \( \Phi_d(\alpha p, S) \) is required to be a channel, so \( \alpha \) lies between \(-\frac{1}{(n^2-1)p}\) and \(\frac{1}{p} \). On the other hand, \( \Phi_t(\alpha p, S) \) is required to be a channel, so \( \alpha \) lies between \(-\frac{1}{(n-1)p}\) and \(\frac{1}{(n+1)p} \). Since, a unitary evolution of a quantum channel is a quantum channel, the upper and the lower bound for \( \alpha \) we obtained in two ways are the same. Now, consider the following two cases.

1. The parameters \( p \) and \( \tilde{p} \) are of the same sign. The equality of the two upper and the two lower bound for \( \alpha \) yeilds the following system:

\[
\begin{align*}
\frac{1}{(n^2-1)p} &= -\frac{1}{(n-1)p}, \\
\frac{1}{p} &= \frac{1}{(n+1)p}.
\end{align*}
\]

These equalities hold only if either \( n = 0 \) or \( n = -2 \), which is impossible.

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2. The parameters $p$ and $\bar{p}$ are of different signs. Then

\[
\begin{align*}
\left\{ \begin{array}{l}
-\frac{1}{(n^2-1)p} = \frac{1}{(n+1)p} \\
\frac{1}{p} = -\frac{1}{(n-1)\bar{p}}.
\end{array} \right.
\]

These equalities hold only if either $n = 0$, which is impossible, or $n = 2$, which we already considered in Section 3.

Thus, $\Phi_d \not\sim \Phi_t$ in any dimension $n \geq 3$.

The same reasoning can be used for the DCQ and the TCQ channels.

1. If the parameters $p$ and $\bar{p}$ are of the same sign, then the equality of the two upper and the two lower bound for $\alpha$ yeilds the following system:

\[
\begin{align*}
\left\{ \begin{array}{l}
-\frac{1}{(2n-1)\bar{p}} = \frac{1}{(n-1)\bar{p}} \\
\frac{1}{(n-1)^2p} = \frac{1}{(n+1)p}.
\end{array} \right.
\]

These equalities hold only if either $n = 0$ or $n = \frac{5+\sqrt{17}}{2}$, which is impossible.

2. If the parameters $p$ and $\bar{p}$ are of different signs, then

\[
\begin{align*}
\left\{ \begin{array}{l}
-\frac{1}{(2n-1)p} = \frac{1}{(n+1)p} \\
\frac{1}{(n-1)^2p} = -\frac{1}{(n-1)p}.
\end{array} \right.
\]

These equalities hold only if either $n = 0$, which is impossible, or $n = 2$, which we already considered in Section 3.

Thus, $\Phi_{dcq} \not\sim \Phi_{tcq}$ in any dimension $n \geq 3$.

\[\square\]

6 Concluding remarks

The main result of this paper are the examples of channels with constant Frobenius norm — namely, $\Phi_{dcq}$ and $\Phi_{tcq}$ — that are equivalent neither to the depolarizing channel $\Phi_d$ nor to the transpose-depolarizing channel $\Phi_t$ in dimension $n \geq 3$, although all four channels are equivalent in dimension $n = 2$. It is also worth noting that the intuitive generalization of Pauli matrices and the basis $E$ in $\mathcal{M}_n$, which were used to define diagonal channels, are different from the generalized Pauli matrices and the basis used in [3].

Although some progress has been made in this work, there are many open questions about diagonal channels and sets of channels with constant functions.

- First, it remains to describe all diagonal channels with constant Frobenius norm in dimensions $n \geq 3$, since we only fully covered the case $n = 2$. Intermediate results in dimension 3 show that more families of inequivalent diagonal channels emerge and that the parameter ranges become...
more complex (for example, the parameter range for one family of 3-dimensional diagonal channels is \( \frac{1}{26} (-1 - 3\sqrt{3}) \leq p \leq \frac{1}{26} (-1 + 3\sqrt{3}) \), while the bound of the parameter range for another family are the two largest roots of the polynomial \(-1 + 21x^2 + 7x^3\)). Since complexity of calculations grows as dimension \( n \) increases, using more advanced methods and theories, such as algebraic geometry, might become necessary in higher dimensions and in the general case.

- Second, it is unknown if the set of channels with constant Frobenius norm in an arbitrary dimension is limited to completely depolarizing and diagonal channels (as in the 2-dimensional case).

- Finally, almost nothing is known about sets of channels with constant von Neumann entropy and other matrix norms (outside the 2-dimensional case) and their relationship with the set of channels with constant Frobenius norm.

In the future, we hope to tackle these and other problems and generalize the results obtained in this paper.

**Acknowledgement**

I would like to thank my adviser professor Grigori G. Amosov for formulating the problem, fruitful discussions, and helpful advice.

**Appendices**

**A Proof of Lemma 1**

This section contains the proof of Lemma 1 (from Section 5.1), which gives two representations for each of the four generalizations of diagonal channels. The proof relies on the following technical lemma.

**Lemma 4.** Let \( S = (s_{ij})^n_{i,j=1} \in \mathcal{M}_n \) be an arbitrary matrix. Then the following
identities hold:

\[
\begin{align*}
\sum_{i=1}^{N} \sigma_{x,i} S \sigma_{x,i} & = S + \text{Tr} \ S - 2 \cdot \text{diag} \ (s_{11}, \ldots, s_{nn}), \\
\sum_{i=1}^{N} \sigma_{y,i} S \sigma_{y,i} & = \text{Tr} \ S - S, \\
\sum_{i=1}^{N} \sigma_{z,i} S \sigma_{z,i} & = n \cdot \text{diag} \ (s_{11}, \ldots, s_{nn}) - S^T, \\
\sum_{i=1}^{N} e_{x,i} S e_{x,i} & = \frac{1}{2} \left( S + \text{Tr} \ S - 2 \cdot \text{diag} \ (s_{11}, \ldots, s_{nn}) \right), \\
\sum_{i=1}^{N} e_{y,i} S e_{y,i} & = \frac{1}{2} \left( \text{Tr} \ S - S \right), \\
\sum_{i=1}^{n-1} e_{z,i} S e_{z,i} & = \frac{1}{n} \left( n \cdot \text{diag} \ (s_{11}, \ldots, s_{nn}) - S^T \right).
\end{align*}
\]

**Proof.** First, we calculate the following products:

\[
\sigma_{x,i} S \sigma_{x,i} = \begin{pmatrix}
    s_{22} & s_{21} & 0 & \cdots & 0 \\
    s_{21} & s_{11} & 0 & \cdots & 0 \\
    0 & 0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \\
\sigma_{x,N} S \sigma_{x,N} = \begin{pmatrix}
    0 & 0 & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{pmatrix};
\]

\[
\sigma_{y,i} S \sigma_{y,i} = \begin{pmatrix}
    s_{22} & -s_{21} & 0 & \cdots & 0 \\
    -s_{21} & s_{11} & 0 & \cdots & 0 \\
    0 & 0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \\
\sigma_{y,N} S \sigma_{y,N} = \begin{pmatrix}
    0 & 0 & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{pmatrix};
\]

\[
\sigma_{z,i} S \sigma_{z,i} = \begin{pmatrix}
    s_{22} & -s_{21} & 0 & \cdots & 0 \\
    -s_{21} & s_{11} & 0 & \cdots & 0 \\
    0 & 0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \\
\sigma_{z,N} S \sigma_{z,N} = \begin{pmatrix}
    0 & 0 & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Adding up these products, we obtain the required sums (22)–(24).

Since \( e_{x,i} = \frac{1}{\sqrt{2}} \sigma_{x,i} \) and \( e_{y,i} = \frac{1}{\sqrt{2}} \sigma_{y,i} \), we can use the results for \( \sigma_{x,i} \) and \( \sigma_{y,i} \) to obtain the sums (25) and (26). We prove (27) by calculating the following products, similar to those computed above:

\[
\begin{align*}
ee_{x,i} S e_{x,i} & = \frac{1}{\sqrt{2}} \begin{pmatrix}
    s_{11} & -s_{12} & 0 & \cdots & 0 \\
    -s_{12} & s_{21} & 0 & \cdots & 0 \\
    0 & 0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \\
\epsilon_{x,2} S e_{x,2} & = \frac{1}{\sqrt{2}} \begin{pmatrix}
    s_{11} & s_{12} & -2s_{13} & \cdots & 0 \\
    s_{21} & s_{22} & -2s_{23} & \cdots & 0 \\
    -2s_{31} & -2s_{32} & 4s_{33} & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \ldots,
\end{align*}
\]

\[
\begin{align*}
ee_{z,n-1} S e_{z,n-1} & = \frac{1}{\sqrt{2}} \begin{pmatrix}
    s_{1,1} & \ldots & s_{1,n-2} & s_{1,n-1} & -(n-1) s_{1,n} \\
    \ldots & \ldots & \ldots & \ldots & \ldots \\
    s_{n-2,1} & \ldots & s_{n-2,n-2} & s_{n-2,n-1} & -(n-1) s_{n-1,n} \\
    s_{n-1,1} & \ldots & s_{n-1,n-2} & s_{n-1,n-1} & -(n-1)^2 s_{n,n} \\
    -(n-1) s_{n,1} & \ldots & -(n-1) s_{n,n-2} & -(n-1)^2 s_{n,n-1} & -(n-1)^3 s_{n,n}
\end{pmatrix}.
\end{align*}
\]
Adding up the products and using the identity
\[
\sum_{i=1}^{l} \frac{1}{i(i+1)} = \frac{l}{l+1},
\]
which can be proved by induction, we find:
\[
\left( \sum_{i=1}^{n-1} e_{z,i} S e_{z,i} \right)_{k,k} = \left[ \frac{(k-1)^2}{(k-1)k} + \sum_{i=k}^{n-1} \frac{1}{i(i+1)} \right] s_{k,k} = \frac{n-1}{n} s_{k,k},
\]
\[
\left( \sum_{i=1}^{n-1} e_{z,i} S e_{z,i} \right)_{k,l} = \left[ -\frac{(l-1)(l-1)}{(l-1)l} + \sum_{i=l}^{n-1} \frac{1}{i(i+1)} \right] s_{k,l} = -\frac{1}{n} s_{k,l}.
\]

**Proof of Lemma** To prove the representation of the depolarizing channel with Pauli matrices, consider the channel of the form (18) with coefficients \( c_{\alpha,i} \) as in the statement of the theorem. Applying Lemma 4, we find:
\[
(\Phi (S))_{ii} = \frac{1 + (n^2 - 1)p}{n^2} s_{ii} + 2 \cdot \frac{1}{2n} (\text{Tr} S - s_{ii}) + \frac{1}{n} (n - 1) s_{ii} = ps_{ii} + \frac{1}{n} \text{Tr} S,
\]
\[
(\Phi (S))_{ij} = \frac{1 + (n^2 - 1)p}{n^2} s_{ij} + \frac{1}{2n} (s_{ji} - s_{ji}) - \frac{1}{n^2} s_{ij} = ps_{ij}.
\]
Therefore,
\[
\Phi (p, S) = pS + \frac{1 - p}{n} \text{Tr} S = \Phi_d (p, S),
\]
which proves the representation. For all other representations the proof follows the same plan.

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