Applications of tangent transformations to the linearization problem of fourth-order ordinary differential equations

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1 Introduction

The problem of linearization of a nonlinear ordinary differential equation has attracted a lot of attention of scientists. The first linearization problem for ordinary differential equations was solved by S.Lie \cite{1}. He found the general form of all ordinary differential equations of second-order which can be reduced to a linear equation by changing the independent and dependent variables. He showed that any linearizable second-order ordinary differential equation should be at most cubic in the first-order derivative, and provided a linearization test in terms of its coefficients.

It should be noted that the linearization problem is a particular case of an equivalence problem, where one has to map one equation into another. A different approach for tackling an equivalence problem of second-order ordinary differential equations was developed by E.Cartan \cite{2}. The idea of his approach was to associate with every differential equation a uniquely defined geometric structure of a certain form.

These two approaches were also applied to third- and fourth-
order ordinary differential equations. The linearization problem of a third-order ordinary differential equation with respect to point transformations was studied in [3]. Complete criteria for linearization of third-order were obtained in [4,5] and fourth-order in [6].

S.Lie also noted that all second-order ordinary differential equations can be transformed to each other by means of contact transformations. Because this is not so for ordinary differential equations of higher order, contact transformations became interesting for applying them to a linearization problem. For third-order ordinary differential equations contact transformations were studied in [7,8,9,10,11]. The complete solution of the linearization problem was given in [12] and in explicit form the criteria for linearization is presented in [5]. For equations of fourth-order the linearization problem via contact transformations was considered in [13] and a complete solution is given in [14].

It is known that point and contact transformations are exceptional types of tangent transformations

\[ t = \varphi(x, y, y', ..., y^{(s)}), \quad u = \psi(x, y, y', ..., y^{(s)}). \] (1)

Here for the purpose of the present paper a single independent variable \( x \) and a single dependent variable \( y \) are used. Notice that the requirement to satisfy the tangent conditions defines the transformations of the derivatives through the functions \( \varphi(x, y, y', ..., y^{(s)}) \) and \( \psi(x, y, y', ..., y^{(s)}) \):

\[ u^{(j)} = \frac{Du^{(j-1)}}{D\varphi}, \quad (j \geq 1), \]

where \( u^{(0)} = \psi \), and \( D \) is the operator of the total derivative with respect to \( x \):

\[ D = \partial_x + \sum_{j=0} y^{(j+1)} \partial_{y^{(j)}}, \]

with \( y^{(0)} = y \). Remind also that if the function \( y = y_o(x) \) is given, then the transformed function \( u_o(t) \) is defined by the following
two steps. On the first step one has to solve with respect to \( x \) the equation

\[
t = \varphi(x, y_o(x), y'_o(x), \ldots, y^{(s)}_o(x)).
\]

Let say \( x = g(t) \) is a solution of this equation. The transformed function is determined by the formula

\[
u_o(t) = \varphi(g(t), y_o(g(t)), y'_o(g(t)), \ldots, y^{(s)}_o(g(t))).
\]

The inverse transformation is more complicated. If one has the function \( u_o(t) \), then for finding the function \( y_o(x) \) one has to solve the ordinary differential equation

\[
u_o(\varphi(x, y_o(x), y'_o(x), \ldots, y^{(s)}_o(x))) = \psi(x, y_o(x), y'_o(x), \ldots, y^{(s)}_o(x)).
\]

The set of point transformations corresponds to \( s = 0 \). In this case for the forward and inverse transformations one needs to overcome similar difficulties. For the set of contact transformations \( s = 1 \), and it is also required that \( u' \) do not depend on second-order derivatives \( y'' \). In this case the forward transformation is simpler than the inverse transformation since for the forward transformation one has just to use the differentiation, whereas for the inverse transformation one needs to solve the first-order ordinary differential equation

\[
u_o(\varphi(x, y_o(x), y'_o(x))) = \psi(x, y_o(x), y'_o(x)). \tag{2}
\]

The sets of point and contact transformations are exceptional because the transformed derivative \( u^{(j)} \) depends on derivatives of order not higher than \( j \). Moreover, Bäcklund [15] proved that the tangent transformations closed in a finite-dimensional space of the independent, dependent variables and derivatives of finite order (say order \( s \)) are comprised by point and contact transformations. It has to be mentioned here the following.

First, the Bäcklund theorem was proven with the assumption that there are no restrictions on derivatives except satisfying the

* This statement is called the Bäcklund non-existence theorem [16][17][18].
tangent conditions, whereas it is not so for a set of solutions of differential equations. This circumstance gives possibilities for the existence of tangent transformations of finite order which act on the set of solutions.

Second, for a linearization problem a change of order of an equation is not crucial. There are attempts to tackle the linearization problem by increasing order of an equation [20,21,22], reducing order [23] or their combinations [24].

Third, difficulties in applications of the inverse transformation of contact transformations and tangent transformations with $s = 1$ are similar. This allows one to assume that tangent transformations which are not contact could be also useful for a linearization problem.

Notice also that through involving derivatives in a transformation, the notion of a Lie group of point transformations is generalized. The limitations dictated by the Bäcklund theorem can be overcome by considering the admitted transformations or extending the space of the derivatives involved in the transformations up to infinity. The second alternative leads to Lie-Bäcklund transformations [17].

The present paper deals with tangent transformations (1) of first-order ($s = 1$) which are not point or contact transformations. It is worth to notice that the classical reduction of order and the Riccati substitution can be also considered as a particular case of tangent transformations of the form (1):

$$t = \varphi(x, y, p), \quad u = \psi(x, y, p),$$

where $p = y'$. Indeed, for the classical reduction of order the transformation is

$$t = y, \quad u = p,$$

**Details of this discussion can be found in [19].

* Examples of such transformations one can find in [19].
and for the Ricatti substitution

\[ t = x, \quad u = p/y. \]

For a linearization problem these transformations were applied in [23] and in [22].

The main goal of the present paper is to demonstrate possibilities of applications tangent transformations for a linearization problem. In the paper tangent transformations are applied for linearizing fourth-order ordinary differential equations. Complete study of fiber preserving transformations \((\varphi_p = \varphi_y = 0)\) mapping equations to the trivial third-order ordinary differential equation \(y^{(3)} = 0\) is given in the paper.

The manuscript is organized as follows. In Section 2, the necessary conditions of linearization of a fourth-order ordinary differential equation in the general case of tangent transformations with \(s = 1\) are presented. It is shown that linearizing equations are separated in two classes of equations. Further study is devoted to one of these classes. In particular, in Section 3, sufficient conditions for linearizing fiber preserving tangent transformations are obtained. In Section 4, we state the theorems that yield criteria for a fourth-order ordinary differential equation to be linearizable via fiber preserving tangent transformations. Relations between coefficients of a linearizable equation and tangent transformations reducing this equation into the equation \(y^{(3)} = 0\) are presented in this section. These relations are necessary for formulating the linearization theorems which are given in Section 4. For the sake of simplicity of reading, cumbersome formulae are moved into Appendix. Examples which illustrate the procedure of using the linearization theorems are presented in Section 5.
2 Necessary conditions of linearization

We begin with obtaining necessary conditions for linearizable equations. Recall that according to the Laguerre theorem a linear third-order ordinary differential equation has the form

$$u''' + \alpha(t)u = 0. \quad (3)$$

The necessary form of a linearizable fourth-order ordinary differential equation

$$y^{(4)} = f(x, y, y', y'', y''') \quad (4)$$

is obtained by the substitution of transformed derivatives into the Laguerre form (3). Using the tangent conditions, the transformation

$$t = \varphi(x, y, p), \quad u = \psi(x, y, p), \quad (5)$$

defines the change of derivatives

$$u' = \frac{D\psi}{D\varphi}, \quad u'' = \frac{Du'}{D\varphi}, \quad u''' = \frac{Du''}{D\varphi}. \quad (6)$$

Substituting the resulting expressions of the derivatives into linear equation (3), one arrives at the following equation

$$\left(\varphi_p(\psi_x + p\psi_y) - \psi_p(\varphi_x + p\varphi_y)\right)y^{(4)} + H(x, y, p, y'', y''') = 0, \quad (7)$$

with some function $H(x, y, y', y'', y''')$. Since for

$$\psi_p(\varphi_x + p\varphi_y) = \varphi_p(\psi_x + p\psi_y) \quad (8)$$

the tangent transformation is a contact transformation, the case $\varphi_p(\psi_x + p\psi_y) - \psi_p(\varphi_x + p\varphi_y) \neq 0$ is considered in the paper. The study of the function $H(x, y, y', y'', y''')$ in (7) leads to the property that the transformations (5) with $\varphi_p = 0$ and $\varphi_p \neq 0$, provide two distinctly different candidates for the linearizable

* Since the study requires a huge amount of analytical calculations, it is necessary to use a computer for these calculations. A brief review of computer systems of symbolic manipulations can be found, for example, in [25]. In our calculations the system REDUCE [26] was used.

* Complete study of this case is given in [12] and [5].
equations:
(a) if $\varphi_p = 0$, then the linearizable equation has the form

$$y^{(4)} + (A_1 y'' + A_0) y''' + B_3 y''^3 + B_2 y''^2 + B_1 y'' + B_0 = 0, \quad (9)$$

(b) if $\varphi_p \neq 0$, one arrives to the equation

$$y^{(4)} + \frac{1}{y'' + r} \left[ -3y''^2 + (C_2 y''^2 + C_1 y'' + C_0) y'''' + D_5 y''^5 + D_4 y''^4 + D_3 y''^3 + D_2 y''^2 + D_1 y'' + D_0 \right] = 0. \quad (10)$$

Here $A_i = A_i(x, y, p), B_i = B_i(x, y, p), C_i = C_i(x, y, p)$ and $D_i = D_i(x, y, p)$ are some functions of $x, y, p$. Expressions of the coefficients $A_i = A_i(x, y, p), B_j = B_j(x, y, p), (i = 0, 1; j = 0, 1, 2, 3)$ in the case $\alpha(t) = 0$ are presented in Appendix.

Thus, it is shown that every fourth-order ordinary differential equation linearizable by a tangent transformation of the form (5) belongs either to the class of (9) or to the class of (10).

3 Sufficient conditions of linearization for the case (9)

For obtaining sufficient conditions, one has to solve the compatibility problem considering the representations of the coefficients $A_i(x, y, p), B_j(x, y, p)$ through the unknown functions $\varphi, \psi$ and $\alpha$ with the given coefficients $A_i(x, y, p), B_j(x, y, p)$. For example, in the case $\alpha = 0$, these equations are (A.1)-(A.6).

For simplicity of calculations the function $k = \varphi_x + p\varphi_y \neq 0$ is introduced. From this definition one gets the derivative $\varphi_x = k - p\varphi_y$. Notice that the function $k$ is a linear function with respect to $p$ with $k_p = \varphi_y$.

From equations (A.1), (A.2) and (A.4) one can find the derivatives $\psi_{pp}, \psi_{px}$ and $\psi_{py}$, respectively. Substituting them into equation (A.3), it becomes

$$\varphi_y = k\lambda, \quad (11)$$
where
\[ 5\lambda^2 + A_1\lambda - (3A_1p + A_1^2 - 9B_3) = 0. \] (12)
The discriminant of this quadratic equation has to be not negative
\[ \Delta \equiv 20A_1p + 7A_1^2 - 60B_3 \geq 0. \] (13)
Defining the function \( q(x, y, p) \) such that
\[ q^2 = 3\Delta \] (14)
one finds
\[ \lambda = -(A_1 + q)/10. \]
Differentiating equation (11) with respect to \( p \), one finds that
\[ \lambda_p + \lambda^2 = 0. \] (15)
Comparing the mixed derivatives \( (\varphi_y)_x = (\varphi_x)_y \), one gets
\[ \lambda k_x + (p\lambda - 1)k_y + (\lambda_x + p\lambda_y)k = 0. \] (16)

Because of the representation of equations (15) and (16), further analysis of the compatibility depends on value of \( \lambda \). In the present paper a complete solution of the case where \( \lambda = 0 \) is given. In this case \( \varphi = \varphi(x) \). Transformations of the class where \( \varphi = \varphi(x) \) are called fiber preserving transformations and they are actively applied to a linearization problem [3,7,8].

Assuming that \( \lambda = 0 \), from equation (A.5) one finds the derivative \( \psi_{xy} \). From the relation \( (\psi_{yy})_x - (\psi_{px})_y = 0 \), one obtains the derivative \( \psi_{yy} \). The equation \( (\psi_{px})_p - (\psi_{pp})_x = 0 \) provides the relation
\[ \mu_2\psi_p - A_1\psi_y = 0, \] (17)
where
\[ \mu_1 = (3A_0p + A_0A_1 - 3B_2)/3, \]
\[ \mu_2 = 3A_{1x} + 3A_{1y}p + A_0A_1 - 3B_2 + 9\mu_1. \]

Further analysis of the compatibility depends on value of \( A_1 \): it is separated into two cases \( A_1 = 0 \) and \( A_1 \neq 0 \). Notice also that
equation (A.1) becomes
\[ \psi_{pp} = A_1 \psi_p / 3. \]  
(18)

3.1 Case \( A_1 \neq 0 \)

Using equation (17), one can exclude the derivative \( \psi_y \) from other equations. Comparing \((\psi_y)_y\) with the found derivative \( \psi_{yy} \), and composing the relations \((\psi_y)_p = \psi_y p\) and \((\psi_{py})_p = (\psi_{pp})_y\), one has the conditions
\[ \mu_{2y} = (A_{0y} A_1^2 + A_{1y} \mu_2 - 3 \mu_{1x} A_1^2 - 3 \mu_{1y} A_1^2 p) / A_1, \]  
(19)
\[ \mu_{1p} = A_{1y} / 3, \]  
(20)
\[ B_{2p} = (6 A_{1y} A_1 + 9 B_{3x} A_1 + 9 B_{3y} A_1 p + 3 A_0 A_1 B_3 - A_1^2 B_2 + 6 A_1^2 \mu_1 - 3 B_3 \mu_2) / (3 A_1). \]  
(21)
The equation \((\psi_{px})_y = (\psi_y)_xp\) can be solved with respect to \( \varphi_{xxx} \):
\[ \varphi_{xxx} = (3 \varphi_{xx}^2 + 2 \varphi_{x}^2 \mu_3) / (2 \varphi_x), \]  
(22)
where
\[ \mu_3 = (-9 A_{0y} A_1 + 3 B_{1p} A_1 - 3 B_{2x} A_1 - 3 B_{2y} A_1 p + 9 \mu_{1x} A_1 + 9 \mu_{1y} A_1 p - A_0 A_1 B_2 - 6 A_0 A_1 \mu_1 + A_1^2 B_1 + B_2 \mu_2 - 3 \mu_1 \mu_2) / (2 A_1^2). \]
Since \( \varphi = \varphi(x) \), one has
\[ \mu_{3y} = 0, \quad \mu_{3p} = 0. \]  
(23)
The relation \((\psi_y)_x = \psi_{xy}\) gives the condition
\[ \mu_{2x} = (-3 A_{0x} A_1^2 - 9 A_{0y} A_1^2 p - 6 A_{1y} A_1 p + 18 \mu_{1x} A_1^2 p + 18 \mu_{1y} A_1^2 p^2 - A_1^2 A_1^2 - 3 A_0 A_1 \mu_2 + 3 A_1^2 B_1 + 6 A_1^2 \mu_3 \]  
+6 B_2 \mu_2 - 18 \mu_1 \mu_2 + 4 \mu_3^2) / (6 A_1). \]  
(24)
Equation (A.6) can be solved with respect to

\[ \psi_{xxx} = (\psi_{p} A_{1})/3, \tag{26} \]

\[ \psi_{px} = (3 \varphi_{xx} \psi_{p} A_{1} + \varphi_{x} \psi_{p} (A_{0} A_{1} - 3 A_{1} \mu_{1} p - \mu_{2}))/3 \varphi_{x} A_{1}, \tag{27} \]

\[ \psi_{y} = (\psi_{p} \mu_{2})/A_{1}, \tag{28} \]

and satisfying the condition \((\psi_{py})_{x} = (\psi_{xy})_{p}\), is involutive.

3.2 Case \( A_{1} = 0 \)

From equation (17), one obtains the condition

\[ \mu_{2} = 0, \tag{29} \]

which means that

\[ \mu_{1} = B_{2}/3, \tag{30} \]
Substituting $A_1$ and $\mu_1$ into (12) and the definition of $\mu_1$, one arrives at the conditions

$$B_3 = 0, \quad A_{0p} = 4B_2/3. \quad (31)$$

Equating the mixed derivatives $(\psi_{py})_p - (\psi_{pp})_y = 0$, one gets $B_2 = B_2(x, y)$. Hence, one finds that

$$A_0 = 4pB_2/3 + A_{00}, \quad (32)$$

where $A_{00}(x, y)$ is a function of the integration. Equating the mixed derivatives $(\psi_{px})_y = (\psi_{xy})_p$, and integrating it with respect to $p$, one derives

$$B_1 = (9A_{00y}p + 6B_{2y}p^2 + 3A_{00}B_2p + 2B_2^2p^2)/3 + B_{10}, \quad (33)$$

where $B_{10} = B_{10}(x, y)$.

Since $\psi_{pp} = 0$, then

$$\psi = \psi_1p + \psi_0, \quad (34)$$

where $\psi_0(x, y)$ and $\psi_1(x, y) \neq 0$ are some functions. Substituting (34) into the relation for $\psi_{py}$, $\psi_{px}$ and $\psi_{xy}$, one finds that

$$\psi_{1y} = B_2\psi_1/3, \quad (35)$$

and the derivatives $\psi_{0y}$ and $\varphi_{xxx}$, respectively.

Equation (A.6) becomes a polynomial of degree 3 with respect to $p$ with coefficients not depending on $p$. One can split this equation. Equating the coefficient with respect to $p$, and integrating it, one obtains the relation:

$$B_0 = (27A_{00yy}p^3 + 18A_{00y}B_2p^3 + 9B_{2yy}p^4 + 9B_{2y}A_{00}p^3 + 9B_{2y}B_2p^4 + 3A_{00}B_2^2p^3 + B_2^3p^4)/27 + B_{00} + B_{01}p + B_{02}p^2, \quad (36)$$

where $B_{00}(x, y)$, $B_{01}(x, y)$ and $B_{02}(x, y)$ are some functions. From the other three coefficients one can find $\psi_{0xxx}$, $\psi_{1xxx}$ and

$$B_{10y} = (3B_{02} - B_{10}B_2)/3. \quad (37)$$
Comparing the mixed derivatives $(\psi_{yy})_x = (\psi_{xy})_y$, one arrives at the equation
\[ \mu_5(\varphi_{xx}\psi_1 - \varphi_x\psi_{1x} + \varphi_x\mu_6\psi_1) = 0, \] (38)

where
\[ \mu_5 = 3(-3A_{00y}+4B_{2x}), \quad \mu_5x + \mu_5\mu_6 = 6B_{2xx}+3B_{2x}A_{00}-3B_{02}+B_{10}B_2. \]

3.2.1 Case $\mu_5 \neq 0$

From equation (38), one finds
\[ \varphi_{xx} = (\varphi_x\psi_{1x} - \varphi_x\mu_6\psi_1)/\psi_1. \] (39)

Differentiating this equation with respect to $y$ and $x$, one gets
\[ 3\mu_{6y} - B_{2x} = 0, \] (40)
\[ \psi_{1xx} = (3\psi_{1x}^2 - 2\psi_{1x}\mu_6\psi_1 + \psi_1^2(-3A_{00x}+8\mu_6x-3A_{00}\mu_6+B_{10}+7\mu_6^2))/(2\psi_1). \] (41)

Substituting $\psi_{1xx}$ into the found earlier expression for $\psi_{1xxx}$, one obtains the condition
\[ \mu_{6xx} = (3A_{00xx} - 6A_{00x}A_{00} + 21A_{00x}\mu_6 + B_{10x} + 15\mu_6xA_{00} - 60\mu_6x\mu_6 - 6A_{00}^2\mu_6 + 2A_{00}B_{10} + 30A_{00}\mu_6^2 - 2B_{01} - 4B_{10}\mu_6 - 40\mu_6^3)/10. \] (42)

Comparing the mixed derivative $(\psi_{1xxx})_y = (\psi_{1y})_{xxx}$, one arrives at the condition
\[ \mu_{5xx} = (-36A_{00x}\mu_6y + 6A_{00x}\mu_5 - 18B_{01y} + 12B_{02x} - 4B_{10x}B_2 + \mu_{5x}A_{00} - 8\mu_{5x}\mu_6 - 14\mu_6x\mu_5 - 9\mu_6yA_{00}^2 + 24\mu_6yB_{10} + 3A_{00}B_{02} - A_{00}B_{10}B_2 + 7A_{00}\mu_5\mu_6 - 2B_{10}\mu_5 - 18\mu_5\mu_6^2)/2. \] (43)
The equation \((\psi_{0xxx})_y = (\psi_{0y})_{xxx}\) yields the condition

\[
-9A_{00xx}A_{00} - 18A_{00xx}\mu_6 + 72A_{00x}^2 - 432A_{00x}\mu_6x - 18A_{00x}A_{00}^2 \\
+ 189A_{00x}A_{00}\mu_6 - 24A_{00x}B_{10} - 432A_{00x}\mu_6^2 + 60B_{00y} - 36B_{01x} \\
+ 18B_{10xx} + 9B_{10x}A_{00} + 36B_{10x}\mu_6 + 540\mu_6^2 + 27\mu_6xA_{00}^2 \\
- 540\mu_6xA_{00}\mu_6 + 108\mu_6xB_{10} + 1080\mu_6x\mu_6^2 - 18A_{00}\mu_6 + 6A_{00}^2B_{10} \\
+ 162A_{00}\mu_6^2 - 6A_{00}B_{01} - 36A_{00}B_{10}\mu_6 - 540A_{00}\mu_6^3 + 20B_{00}B_2 \\
- 36B_{01}\mu_6 + 108B_{10}\mu_6^2 + 540\mu_6^4 - 6A_{00xx} = 0.
\]

(44)

The obtained system of equations for the functions \(\varphi(x), \psi_1(x, y)\) and \(\psi_0(x, y)\) consisting of the equations

\[
\psi_{0y} = \psi_1(A_{00} - 3\mu_6),
\]

(45)

\[
\psi_{0xxx} = (6\psi_{0xx}\psi_1\psi_1 - 6\psi_{0xx}\mu_6\psi_1^2 - 3\psi_{0x}\psi_1^2 + 6\psi_{0x}\psi_1\mu_6\psi_1 \\
+ \psi_{0x}\psi_1^2(-3A_{00}x + 6\mu_6x - 3A_{00}\mu_6 + B_{10} + 3\mu_6^2) + 2B_{00}\psi_1^3)/(2\psi_1^2),
\]

(46)

and satisfying the condition \((\psi)_yy = \psi_{yy}\), is involutive.

3.2.2 Case \(\mu_5 = 0\)

Equating the mixed derivatives \((\psi_{1xxx})_y = (\psi_{1y})_{xxx}\), one gets

\[
B_{01y} = (-12A_{00x}B_{2x} + 12B_{02x} - 4B_{10x}B_2 - 3B_{2x}A_{00}^2 + 8B_{2x}B_{10} \\
+ 3A_{00}B_{02} - A_{00}B_{10}B_2)/18.
\]

(47)

The equation \((\psi_{0xxx})_y - (\psi_{0y})_{xxx} = 0\) is a quadratic algebraic equation with respect to \(\psi_{1xx}\). Differentiating this equation with respect to \(y\) and \(x\), and excluding from them \(\psi_{1xx}^2\) using it, one
obtains the conditions

\[ 36B_{02xx} = (36A_{00xx}B_{2x} - 18A_{00x}B_{2x}A_{00} + 36A_{00x}B_{02} - 12A_{00x}B_{10}B_{2} \]
\[ + 216B_{00yy} + 72B_{00y}B_{2} - 18B_{02x}A_{00} + 12B_{10xx}B_{2} + 24B_{10x}B_{2x} \]
\[ + 6B_{10x}A_{00}B_{2} - 9B_{2x}A_{00}^3 + 36B_{2x}A_{00}B_{10} - 72B_{2x}B_{01} \]
\[ + 72B_{2y}B_{00} + 9A_{00}^2B_{02} - 3A_{00}B_{10}B_{2} - 30B_{02}B_{10} + 10B_{10}^2B_{2} \). \]

\[ (48) \]

\[ 2520\mu_7(2\psi_1\varphi_2^2\psi_1\psi_1 + 9\varphi_2^2\psi_1^2 - 20\varphi_2\varphi_1\psi_1\psi_1 + 8\varphi_2^2\psi_1^2) \]
\[ + 8\mu_8\psi_1\varphi_1(\psi_1\varphi_2 - \varphi_2\psi_1) + \mu_9\varphi_2^2\psi_1^2 = 0, \]

where

\[ \mu_7 = -4A_{00xx} - 6A_{00x}A_{00} + 8B_{10x} - A_{00}^3 + 4A_{00}B_{10} - 8B_{01}, \]

\[ \mu_8 = 3456A_{00x}^2 + 1728A_{00x}A_{00}^2 - 8448A_{00x}B_{10} - 38400B_{00y} + 15360B_{01x} \]
\[ - 3840B_{10xx} - 1920B_{10x}A_{00} - 1320\mu_7\psi_1 + 216A_{00}^4 - 2112A_{00}^2B_{10} \]
\[ + 3840A_{00}B_{01} + 855A_{00}\mu_7 - 12800B_{00}B_{2} + 3456B_{20}^2, \]

\[ \mu_9 = 1503A_{00x}\mu_7 - 360\mu_7\psi_1 + 1575\mu_7\psi_1A_{00} - \mu_8\psi_1 - 1278A_{00}\mu_7 \]
\[ + 2A_{00}\mu_8 - 792B_{10}\mu_7. \]

Considering equation (49), further study is separated in two cases related with value of \( \mu_7 \), i.e., \( \mu_7 \neq 0 \) and \( \mu_7 = 0 \).

3.2.2.1 Case \( \mu_7 \neq 0 \)

Substituting \( \psi_1\psi_1 \), found from equation (49), into \( \psi_1\psi_1 \), one gets
the equation

\[
8\varphi_{xx}\mu_{10}\psi_1 - 8\varphi_x\psi_1 x\mu_{10} + \varphi_x\psi_1 (-3333960A_{00x}A_{00}^2 + 7560A_{00x}\mu_7\mu_8 \\
+ 3810240B_{10x}\mu_7^2 - 2520\mu_7\mu_9 + 2520\mu_9\mu_7 + 952560A_{00}^3\mu_7^2 \\
- 793800A_{00}B_{10}\mu_7^2 - 2835A_{00}\mu_7\mu_9 - 2540160B_{01}\mu_7^2 + 2520B_{10}\mu_7\mu_8 \\
+ 952560\mu_7^3 + \mu_8\mu_9) = 0,
\]

(50)

where

\[
\mu_{10} = -6191640A_{00x}\mu_7^2 - 2520\mu_7\mu_8 + 2520\mu_8\mu_7 + 1131165A_{00}^2\mu_7^2 \\
- 1890A_{00}\mu_7\mu_8 + 952560B_{10}\mu_7^2 - 630\mu_7\mu_9 + \mu_8^2.
\]

Case $\mu_{10} \neq 0$

From equation (50), one arrives at

\[
\varphi_{xx} = (\varphi_x\psi_1 x + \varphi_x\mu_{11}\psi_1)/\psi_1,
\]

(51)

where

\[
8\mu_{10}\mu_{11} = (3333960A_{00x}A_{00}^2 + 7560A_{00x}\mu_7\mu_8 - 3810240B_{10x}\mu_7^2 + 2520\mu_7\mu_9 \\
- 2520\mu_9\mu_7 - 952560A_{00}^3\mu_7^2 + 793800A_{00}B_{10}\mu_7^2 + 2835A_{00}\mu_7\mu_9 \\
+ 2540160B_{01}\mu_7^2 - 2520B_{10}\mu_7\mu_8 - 952560\mu_7^3 - \mu_8\mu_9).
\]

Substituting $\varphi_{xx}$ into $\varphi_{xxx}$, one gets the condition

\[
3\mu_{11y} + B_{2x} = 0.
\]

(52)

Differentiating equation (51) with respect to $y$, one obtains the condition

\[
-7560A_{00x}\mu_7 + 7560A_{00}\mu_{11}\mu_7 + 2520B_{10}\mu_7 + 40320\mu_{11}^2\mu_7 \\
+ 8\mu_{11}\mu_8 + \mu_9 - 20160\mu_7\mu_{11x} = 0.
\]

(53)

The obtained system of equations for the functions $\varphi(x), \psi_1(x, y)$
and \( \psi_0(x, y) \) consisting of the equations

\[
\psi_{0y} = \psi_1(A_{00} + 3\mu_{11}), \tag{54}
\]

\[
\psi_{0xx} = (20160\psi_{0xx}\psi_{1x}\mu_7\psi_1 + 20160\psi_{0xx}\mu_{11}\mu_7\psi_1^2 - 10080\psi_{0x}\psi_{1x}\mu_7
- 20160\psi_{0x}\psi_{1x}\mu_{11}\mu_7\psi_1 + \psi_{0x}\psi_1^2(-2520A_{00}\mu_7
+ 2520A_{00}\mu_{11}\mu_7 + 840B_{10}\mu_7 - 30240\mu_{11}\mu_7 - 8\mu_{11}\mu_8
- \mu_9) + 6720B_{00}\mu_7\psi_1^3)/(6720\mu_7\psi_1^2), \tag{55}
\]

\[
\psi_{1xx} = (-22680\varphi_{xx}\mu_7\psi_1^2 + 50400\varphi_{xx}\varphi_{x}\psi_{1x}\mu_7\psi_1 - 8\varphi_{xx}\varphi_{x}\mu_{11}\varphi_1^2
- 20160\varphi_{xx}\varphi_{1x}\mu_7 + 8\varphi_{xx}\psi_{1x}\psi_1 - \varphi_{xx}\mu_{11}\varphi_1^2)/(5040\varphi_{xx}\mu_7\psi_1), \tag{56}
\]

and satisfying the conditions \((\psi_{1xx})_y = (\psi_{1y})_{xx}\), is involutive.

**Case** \( \mu_{10} = 0 \)

Substituting \( \mu_{10} = 0 \) into equation (50), one obtains the condition

\[
B_{10x} = (3333960A_{00x}A_{00}\mu_7^2 + 7560A_{00x}\mu_{11}\mu_7^2 + 2520\mu_{11}\mu_9 - 2520\mu_{11}\mu_7
- 952560A_{00x}\mu_7^2 + 793800A_{00x}B_{10}\mu_7^2 + 2835A_{00x}\mu_{11}\mu_9 + 2540160B_{01}\mu_7
- 2520B_{10}\mu_7\mu_8 - 952560\mu_7^3 - 8\mu_7\mu_8)/(3810240\mu_7^2) \tag{57}
\]

After all calculations the equations for the functions \( \varphi(x) \), \( \psi_1(x, y) \)
and \( \psi_0(x, y) \) consisting of the equations

\[
\psi_{0y} = (3\varphi_{xx}\psi_1 - 3\varphi_{xx}\psi_{1x} + \varphi_{xx}A_{00}\psi_1)/\varphi_{xx}, \tag{58}
\]

\[
\psi_{0xx} = (-30240\varphi_{xx}\psi_{0x}\mu_7\psi_1^2 + 20160\varphi_{xx}\varphi_{xx}\varphi_{0xx}\mu_7\psi_1^2 + 40320\varphi_{xx}\varphi_{xx}\psi_{0x}\psi_{1x}\mu_7\psi_1
+ 8\varphi_{xx}\varphi_{xx}\psi_{0x}\psi_1\psi_1(315A_{00}\mu_7 - \mu_8) - 20160\varphi_{xx}\psi_{0x}\psi_{1x}\mu_7
+ 8\varphi_{xx}\psi_{0x}\psi_{1x}\psi_1(-315A_{00}\mu_7 + \mu_8) + \varphi_{xx}\psi_{0x}\psi_1^2(-2520A_{00x}\mu_7
+ 840B_{10}\mu_7 - \mu_9) + 6720\varphi_{xx}B_{00}\mu_7\psi_1^3)/(6720\varphi_{xx}\mu_7\psi_1^2), \tag{59}
\]
\[ \varphi_{xx} = (-10080 \varphi_{xx}^2 \mu_7 \psi_1^2 + 40320 \varphi_{xx} \varphi_x \psi_1 \mu_7 \psi_1 + 8 \varphi_{xx} \varphi_x \psi_1^2 (315 A_{00} \mu_7 - \mu_8) \\
-20160 \varphi_x^2 \psi_1 \mu_7 + 8 \varphi_x^2 \psi_1 \psi_1 (-315 A_{00} \mu_7 + \mu_8) + \varphi_x^2 \psi_1^2 (-2520 A_{00} \mu_7 + 840 B_{10} \mu_7 - \mu_9) )/(6720 \varphi_x \mu_7 \psi_1^2), \]

(60)

\[ \psi_{1xx} = (-22680 \varphi_{xx}^2 \mu_7 \psi_1^2 + 50400 \varphi_{xx} \varphi_x \psi_1 \mu_7 \psi_1 - 8 \varphi_{xx} \varphi_x \mu_8 \psi_1^2 \\
-20160 \varphi_x^2 \psi_1 \mu_7 + 8 \varphi_x^2 \psi_1 \mu_8 \psi_1 - \varphi_x^2 \mu_8 \psi_1^2 )/(5040 \varphi_x \mu_7 \psi_1), \]

(61)

and satisfying the conditions \((\psi_{1xx})_y = (\psi_{1y})_{xx}\), is involutive.

3.2.2.2 Case \(\mu_7 = 0\)

Substituting \(\mu_7 = 0\) into equation (49), one gets

\[ 8 \varphi_{xx} \mu_8 \psi_1 - 8 \varphi_x \psi_1 \mu_8 + \varphi_x \psi_1 (-\mu_8 + 2 A_{00} \mu_8) = 0. \]

(62)

Further analysis of the compatibility depends upon the value of \(\mu_8\).

Case \(\mu_8 \neq 0\)

Solving equation (62) with respect to \(\varphi_{xx}\), and substituting it into \(\varphi_{xxx}\), one obtains

\[ \psi_{1xx} = (192 \psi_1^2 \mu_8^2 + 16 \psi_1 \mu_8 \psi_1 (\mu_8 - 2 A_{00} \mu_8) + \psi_1^2 (-64 A_{00} \mu_8^2 - 64 \mu_8 \mu_8 \mu_8 \\
+ 71 \mu_8^2 - 4 \mu_8 A_{00} \mu_8 - 20 A_{00}^2 \mu_8^2 + 64 B_{10} \mu_8^2))/(128 \mu_8^2 \psi_1), \]

(63)

Substituting \(\psi_{1xx}\) into the found earlier expression for \(\psi_{1xxx}\), one arrives at the condition

\[ \mu_{8xxx} = (48 A_{00} \mu_8 \mu_8 \mu_8^2 + 96 A_{00} A_{00} \mu_8^3 - 128 B_{10} \mu_8^3 + 300 \mu_8 \mu_8 \mu_8 \mu_8 \mu_8 \\
- 225 \mu_8^3 + 12 \mu_8 \mu_8 A_{00}^2 \mu_8^2 - 32 \mu_8 B_{10} \mu_8^2 + 24 A_{00} \mu_8^3 \\
- 96 A_{00} B_{10} \mu_8^3 + 192 B_{01} \mu_8^3)/(80 \mu_8^2), \]

(64)

The obtained system of equations for the functions \(\varphi(x), \psi_1(x, y)\)

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and \( \psi_0(x, y) \) consisting of the equations

\[
\psi_{0y} = (\psi_1(3\mu_{8x} + 2A_{00}\mu_8))/(8\mu_8), \tag{65}
\]

\[
\psi_{0xx} = (384\psi_{0xx}\mu_8^2\psi_1 + 48\psi_{0xx}\mu_8\psi_1^2(\mu_{8x} - 2A_{00}\mu_8) - 192\psi_{0x}\mu_8^2\mu_{8x}^2 + 48\psi_{0x}\mu_8\psi_1(-\mu_{8x} + 2A_{00}\mu_8) + \psi_{0x}\psi_1^2(-96A_{00}\mu_8^2 - 48\mu_{8x}\mu_8 + 51\mu_{8x}^2 + 12\mu_{8x}A_{00}\mu_8 - 36\mu_{00}^2\mu_8^2 + 64B_{10}\mu_8^2) + 128B_{00}\mu_8^2\psi_1^3)/(128\mu_8^2\psi_1^3), \tag{66}
\]

\[
\varphi_{xx} = (8\varphi_x\mu_8 + \varphi_x(\mu_{8x} - 2A_{00}\mu_8))/(8\mu_8\psi_1) \tag{67}
\]
is involutive.

\textit{Case} \( \mu_8 = 0 \)

Since \( \mu_8 = 0 \), then equation \((62)\) is satisfied. The obtained system of equations for the functions \( \varphi(x) \), \( \psi_1(x, y) \) and \( \psi_0(x, y) \) consisting of the equations

\[
\psi_{0y} = (3\varphi_{xx}\psi_1 - 3\varphi_x\psi_{1x} + \varphi_xA_{00}\psi_1)/\varphi_x, \tag{68}
\]

\[
\psi_{0xx} = (-9\varphi_{xx}^2\psi_0\psi_1 + 24\varphi_{xx}\varphi_x\varphi_0\psi_1 - 12\varphi_{xx}\varphi_x\psi_0\psi_{1x} + 3\varphi_{xx}\varphi_x\psi_0\psi_1A_{00} + 6\varphi_x^2\psi_0\psi_{1xx} - 3\varphi_x^2\psi_0\psi_{1x}A_{00}) \tag{69}
\]

\[
\psi_{1xx} = (90\varphi_{xx}^2\psi_1^2 - 540\varphi_{xx}\varphi_x\psi_{1xx}\psi_1 + 135\varphi_{xx}^2\varphi_xA_{00}\psi_1^2 + 420\varphi_{xx}\varphi_x^2\psi_{1xx}\psi_1 + 240\varphi_{xx}\varphi_x^2\psi_1^2 - 330\varphi_{xx}\varphi_x^2\psi_{1xx}A_{00}\psi_1 + 3\varphi_{xx}\varphi_x^2\psi_1^2(-14A_{00x} + 19A_{00} - 14B_{10}) - 120\varphi_x^3\psi_{1xx}\psi_1 + 90\varphi_x^3\psi_{1xx}A_{00}\psi_1 + 60\varphi_x^3\psi_1^2A_{00} + \varphi_x^3\psi_1(12A_{00x} - 57A_{00}^2 + 52B_{10}) + \varphi_x^3\psi_1(-21A_{00x}A_{00} + 24B_{10} + 6A_{00}^3 - 5A_{00}B_{10} - 16B_{01}))/(80\varphi_x^3\psi_1), \tag{70}
\]
\[ \varphi_{xxx} = (15\varphi_{xx}^2\psi_1 - 12\varphi_{xx}\varphi_x\psi_{1x} + 3\varphi_{xx}\varphi_x A_{00}\psi_1 + 6\varphi_x^2\psi_{1xx} - 3\varphi_x^2\psi_{1x} A_{00} + \varphi_x^2 \psi_1(-3A_{00}x + B_{10}))/(8\varphi_x\psi_1) \]  

(71)

is involutive.

4 Linearization theorems

All obtained results can be summarized in the following theorems.

**Theorem 1** Any fourth-order ordinary differential equation obtained from a linear equation (3) by a tangent transformation (5) has to be either to the form (9) or (10).

**Theorem 2** Sufficient conditions for equation (9) to be linearizable via a tangent transformation with \( \varphi_p = \varphi_y = 0 \) and \( \alpha = 0 \) are as follows.

(a) If \( A_1 \neq 0 \), then the conditions are (19), (20), (21), (23), (24), (A.7) and (A.8).

(b) If \( A_1 = 0 \) and \( \mu_5 \neq 0 \), then the conditions are (29), (30), (31), (32), (33), (36), (37), (40), (42), (43) and (44).

(c) If \( A_1 = 0, \mu_5 = 0, \mu_7 \neq 0 \) and \( \mu_{10} \neq 0 \), then the conditions are (29), (30), (31), (32), (33), (36), (37), (47), (48), (52) and (53).

(d) If \( A_1 = 0, \mu_5 = 0, \mu_7 \neq 0 \) and \( \mu_{10} = 0 \), then the conditions are (29), (30), (31), (32), (33), (36), (37), (47), (48) and (57).

(e) If \( A_1 = 0, \mu_5 = 0, \mu_7 = 0 \) and \( \mu_8 \neq 0 \), then the conditions are (29), (30), (31), (32), (33), (36), (37), (47), (48), and (64).

(f) If \( A_1 = 0, \mu_5 = 0, \mu_7 = 0 \) and \( \mu_8 = 0 \), then the conditions are (29), (30), (31), (32), (33), (36), (37), (47) and (48).

**Theorem 3** Provided that the sufficient conditions in Theorem 2 are satisfied, the transformation (5) mapping equation (9) to a linear equation \( u''' = 0 \) is obtained by solving the following com-
compatible system of equations for the functions $\varphi(x)$ and $\psi(x,y,p)$:

(a) (22), (25), (26), (27) and (28),
(b) (34), (35), (39), (41), (45) and (46),
(c) (34), (35), (51), (54), (55) and (56),
(d) (34), (35), (58), (59), (60) and (61),
(e) (34), (35), (63), (65), (66) and (67),
(f) (34), (35), (68), (69), (70) and (71).

5 Examples

Example 1. Consider the nonlinear ordinary differential equation

$$xy y^{(4)} + (4xy' + 3y) y''' + 3xy''^2 + 9y'y'' = 0.$$  (72)

It is an equation of the form (9) with the coefficients

$$A_1 = 0, \quad A_0 = \frac{4px+3y}{xy}, \quad B_3 = 0,$$
$$B_2 = \frac{3}{y}, \quad B_1 = \frac{9p}{xy}, \quad B_0 = 0, \quad q = 0, \quad \lambda = 0.$$

The other functions corresponding to the case where $A_1 = 0$ are

$$\mu_1 = \frac{1}{y}, \quad \mu_2 = 0, \quad A_{00} = \frac{3}{x}, \quad B_{10} = 0, \quad B_{00} = 0, \quad B_{01} = 0, \quad B_{02} = 0,$$
$$\mu_5 = 0, \quad \mu_7 = \frac{3}{x^3}, \quad \mu_8 = \frac{21519}{x^4}, \quad \mu_9 = \frac{111672}{x^5},$$
$$\mu_{10} = -\frac{17915904}{x^8}, \quad \mu_{11} = -\frac{1}{x}.$$

Since $\mu_5 = 0$, $\mu_7 \neq 0$ and $\mu_{10} \neq 0$, and all these coefficients obey the conditions (29), (30), (31), (32), (33), (36), (37), (47), (48), (52) and (53), one concludes that equation (72) is linearizable. Applying Theorem 3, the linearizing transformation is found by solving the following equations. For the function $\psi_1(x, y)$ one has

$$\psi_{1y} = \frac{\psi_1}{y}, \quad \psi_{1xx} = \frac{3\psi_{1x}^2x^2 - 2\psi_{1x}\psi_1x - \psi_1^2}{2\psi_{1x}^2}. \quad (73)$$
The function $\psi_1 = xy$ is a particular solution of equations (73). The equations for the function $\psi_0(x, y)$ become

$$\psi_{0y} = 0, \quad \psi_{0xx} = 0.$$  

One can choose the simplest solution of these equations: $\psi_0 = 0$. For finding the function $\varphi(x)$ one has to solve the equation

$$\varphi_{xx} = 0.$$  

Choosing the particular solution $\varphi = x$, one obtains the linearizing transformation

$$t = x, \quad u = xyp.$$  

Thus, the nonlinear equation (72) can be mapped by transformation (75) into the linear equation $u^{(3)} = 0$.

**Example 2.** Consider the nonlinear ordinary differential equation

$$y^3 y^{(4)} - 4y^2 y' y''' - 3y^2 y''^2 + 12yy''^2 y'' - 6y'^4 = 0.$$  

(76)

It is an equation of form (9) with the coefficients

$$A_1 = 0, \quad A_0 = -\frac{4p}{y}, \quad B_3 = 0,$$

$$B_2 = -\frac{3}{y}, \quad B_1 = \frac{12p^2}{y^2}, \quad B_0 = -\frac{6p^4}{y^3}, \quad q = 0, \quad \lambda = 0.$$  

The other coefficients applying in the case $A_1 = 0$ are

$$\mu_1 = -\frac{1}{y}, \quad \mu_2 = 0, \quad A_{00} = 0, \quad B_{10} = 0, \quad B_{00} = 0,$$

$$B_{01} = 0, \quad B_{02} = 0, \quad \mu_5 = 0, \quad \mu_7 = 0, \quad \mu_8 = 0, \quad \mu_9 = 0.$$  

Since $\mu_5 = 0, \mu_7 = 0$ and $\mu_8 = 0$, one can check that these coefficients obey the conditions (29), (30), (31), (32), (33), (36), (37), (47) and (48). Thus, equation (76) is linearizable. The linearizing transformation is found as follows. According to Theorem 3, one obtains that the function $\psi_1$ satisfies the equation

$$\psi_{1y} = -\frac{\psi_1}{y}.$$  

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One can take the solution \( \psi_1 = \frac{1}{y} \) of this equation. For the function \( \varphi(x) \) the equations are

\[
\varphi_{xxx} = \frac{15\varphi_x^2}{8\varphi_x}, \quad \varphi_{xx}^3 = 0. \tag{77}
\]

The function \( \varphi = x \) is a particular solution of equations (77). For finding the function \( \psi_0(x, y) \), one has to solve the system of equations

\[
\psi_{0y} = 0, \quad \psi_{0xxx} = 0. \tag{78}
\]

A particular solution of these equations is \( \psi_0 = 0 \). Hence one obtains the linearizing transformation

\[
t = x, \quad u = p/y. \tag{79}
\]

which maps nonlinear equation (76) into the linear equation \( u^{(3)} = 0 \).

6 Acknowledgement

This research was financially supported by the Discovery Based Development Grant (P-10-11295), National Science and Technology Development Agency, Thailand and Naresuan University. The research of SVM was partially supported by the Office of the Higher Education Commission under NRU project (SUT).

A Appendix

For proving theorems one needs relations between \( \varphi(x, y, p) \) and \( \psi(x, y, p) \) and coefficients of equation (9). These relations are presented here.

\[
A_1 = (3\psi_{pp} - 4\psi_{pp})\varphi_y + 3\varphi_x\psi_{pp})/((\varphi_x + \varphi_yp)\psi_{pp}), \quad (A.1)
\]
$$A_0 = -((3\phi_{xx} + \phi_{yy})p^2 + 2\phi_{xy}p)\psi_{pp} - (3\psi_{py}p + \psi_y + 3\psi_{px})\phi_x$$
$$- (3\psi_{py}p^2 - \psi_x + 3\psi_{px}p)\varphi_y)/((\phi_x + \phi_y)p)\psi_{pp}, \quad (A.2)$$

$$B_3 = (((2\psi_{ppp} - 3\psi_{pp})\varphi_y + \varphi_x\psi_{ppp})\varphi_x - (3(\psi_{pp} - \psi_{pp})$$
$$- \psi_{ppp}p^2)\varphi_y^2)/((\varphi_x + \varphi_y)p^2\psi_{pp}), \quad (A.3)$$

$$B_2 = -3(((\psi_{pp} - 2\psi_{pp})\varphi_y + \varphi_x\psi_{pp})\varphi_{xx} + (\psi_{pp} - \psi_{pp})\varphi_{yy}\varphi_y p^2$$
$$+ ((2\psi_{pp} + \psi_{pp})\varphi_x + (2\psi_{pp} - 3\psi_{pp})\varphi_{xy} - (3(\psi_{pp} - \psi_{pp})\varphi_x$$
$$- ((2\psi_{pp}p^2 - \psi_y + 2\psi_{px}p - 2\psi_{px}p)\varphi_y - (\psi_{pp}p + \psi_{pp})\varphi_{yy}p)\varphi_x$$
$$- (3(\psi_{pp}p^3 + \psi_x + \psi_{pp}p^2 - \psi_{pp}p^2 - 2\psi_{px}p)\varphi_y^2))/((\varphi_x + \varphi_y)p^2\psi_{pp}),$$
$$A.4)$$

$$B_1 = -((\varphi_{yy}\varphi_y - 3\phi_{yy}^2)p^4 - 3\phi_{xx}^2 - 12\phi_{xy}p^2 + (\varphi_{xx} + 3\phi_{xx}p)$$
$$+ 3\phi_{xy}p^2)(\varphi_x + \varphi_y p))\psi_{pp} + 3(2\psi_{py}p^2 - \psi_x + 2\psi_{px}p)\varphi_{yy}\varphi_y p^2$$
$$- 3(\psi_{xy} + \psi_{yy}p + \psi_{py}p^2 + \psi_{xx} + 2\psi_{px}y)p)\varphi_x^2 + 3(\psi_{yy} + \psi_{xx} - \psi_{py}p^3$$
$$- \psi_{px}p - 2\psi_{py}p^2)\varphi_y^2 + 3((2\psi_{py}p + \psi_y + 2\psi_{px}p)\varphi_x - 2\varphi_{yy}\psi_{pp}p^2$$
$$- (2\psi_x + \psi_{yy}p - 2\psi_{py}p^2 - 2\psi_{py}p^2)\psi_y)\varphi_{xx} + 3((\psi_{xx} + 3\psi_{yy}p + 4\psi_{yy}p^2$$
$$+ 4\psi_{px}p)\varphi_x - 4\varphi_{xx} + \varphi_{yy}p^2)\psi_{pp} - (3(\psi_x + \psi_{yy}p - 4\psi_{py}p^2 - 4\psi_{px}p)\varphi_y)\varphi_{xy}$$
$$+ (3(\psi_{xx} - \psi_{yy}p^2 - 2\psi_{py}p^3 - 2\psi_{px}p - 4\psi_{px}p)\varphi_y + (3(\psi_x + 2\psi_{yy}p$$
$$+ 2\psi_{py}p^2 + 2\psi_{px}p)\varphi_{yy} + \varphi_{yy}\psi_{pp}p^2)p)\varphi_x)])/((\varphi_x + \varphi_y)p^2\psi_{pp}),$$
$$A.5)$$

$$B_0 = -((\varphi_{yy}\varphi_y - 3\phi_{yy}^2)p^4 - 3\phi_{xx}^2 - 12\phi_{xy}p^2 + (\varphi_{xx} + 3\phi_{xx}p)$$
$$+ 3\phi_{xy}p^2)(\varphi_x + \varphi_y p))\psi_{pp} + 3(\psi_{xy} + \psi_{yy}p^2 + 2\psi_{xy}p)\varphi_{yy}\varphi_y p^3$$
$$+ 3((\psi_{xx} + \psi_{yy}p^2 + 2\psi_{xy}p)(\varphi_x + \varphi_y p) - 2(\psi_{xx} + \psi_{yy}p)\varphi_{yy}p^2)\varphi_{xx}$$
$$+ 6((\psi_{xx} + \psi_{yy}p^2 + 2\psi_{xy}p)(\varphi_x + \varphi_y p) - 2\varphi_{xx} + \varphi_{yy}p^2)\varphi_{xy}p$$
$$- ((3\psi_{xx} + \psi_{yy}p^2)p + \psi_{xx} + 3\psi_{xy}p^2)(\varphi_x^2 + \varphi_y p^2)$$
$$- (2((3\psi_{xx} + \psi_{yy}p^2)p + \psi_{xx} + 3\psi_{xy}p^2)\varphi_y - (3(\psi_{xx} + \psi_{yy}p^2$$
$$+ 2\psi_{xy}p)\varphi_{yy} + (\psi_x + \psi_{yy}p)\varphi_{yy}p)\varphi_{xx})])/((\varphi_x + \varphi_y)p^2\psi_{pp}).$$
$$A.6)$$
\[ A_{0xy} = (-18A_{0x}A_1\mu_1p^2 - 6A_{0y}A_0A_1p^2 + 18A_{0y}A_1\mu_1p^3 + 6A_{0y}\mu_2p^2 \\
+ 12B_{0p}A_1p + 6B_{1x}A_1p - 6B_{1y}A_1p^2 + 36\mu_{1xy}A_1p^3 \\
- 18\mu_{1x}A_0A_1p^2 + 72\mu_{1x}A_1\mu_1p^3 + 6\mu_{1yy}A_1p^4 + 12\mu_{1y}A_0A_1p^3 \\
- 18\mu_{1y}A_1\mu_1p^4 - 12\mu_{1y}\mu_2p^3 + 6\mu_{3x}A_1p - 15\mu_{4p}A_1p \\
- 6A_0^2A_1\mu_1p^2 + 2A_0A_1B_0p + 30A_0A_1\mu_1^2p^3 + 8A_0A_1\mu_3p \\
+ 6A_0\mu_1^2p^2 + 4A_1^2B_0p - 5A_1^2\mu_4p - 18A_1B_0 - 12A_1B_1\mu_1p^2 \\
- 30A_1\mu_1^3p^4 - 36A_1\mu_1\mu_3p^2 + 9A_1\mu_4 - 2B_1\mu_2p - 12\mu_1^2\mu_2p^3 \\
+ 4\mu_2\mu_3p)/(18A_1p^2), \]  
(A.7)

\[ \mu_{1yyy} = (12A_{0x}A_1\mu_3 + 24A_{0y}A_1\mu_3p - 2A_{1y}A_1B_0p + A_{1y}A_1\mu_4p - 6B_{0px}A_1 \\
- 12B_{0py}A_1p - 2B_{0p}A_0A_1 - 6B_{0p}A_1\mu_1p + 2B_{0p}\mu_2 - 2B_{0x}A_1^2 \\
- 4B_{0y}A_1^2p + 18B_{0y}A_1 + 6B_{1xy}A_1p + 6B_{1x}A_1\mu_1p + 6B_{1yy}A_1p^2 \\
+ 2B_{1y}A_0A_1p + 6B_{1y}A_1\mu_1p^2 - 2B_{1y}\mu_2p - 12\mu_{1xyy}A_1p^3 - 36\mu_{1xy}A_1\mu_1p^3 \\
- 36\mu_{1x}\mu_1yA_1p^3 + 6\mu_{1x}A_1B_1p - 36\mu_{1x}A_1\mu_1^2p^3 - 36\mu_{1x}A_1\mu_3p \\
- 4\mu_{1yy}A_0A_1p^3 - 12\mu_{1yy}A_1\mu_1p^4 + 4\mu_{1yy}\mu_2p^3 - 18\mu_{1y}^2A_1p^4 \\
- 12\mu_{1y}A_0A_1\mu_1p^3 + 6\mu_{1y}A_1B_1p^2 - 24\mu_{1y}A_1\mu_3p^2 + 12\mu_{1y}\mu_1\mu_2p^3 \\
- 6\mu_{3x}\mu_2 + 3\mu_{4px}A_1 + 6\mu_{4py}A_1p + \mu_{4p}A_0A_1 + 3\mu_{4p}A_1\mu_1p - \mu_{4p}\mu_2 \\
+ \mu_{4x}A_1^2 + 2\mu_{4y}A_1^2p + 4A_0^2A_1\mu_3 + 2A_0A_1B_1\mu_1p - 4A_0A_1\mu_3^2p^3 \\
- 4A_0\mu_2\mu_3 - 2A_1^2B_0\mu_1p + A_1^2\mu_1\mu_4p - 2A_1B_0B_2 + 24A_1B_0\mu_1 \\
- 12A_1B_1\mu_3 + A_1B_2\mu_4 + 6A_1\mu_4p^4 + 12A_1\mu_1^2\mu_3p^2 - 3A_1\mu_1\mu_4 \\
+ 24A_1\mu_3^2 - 2B_1\mu_1\mu_2p + 4\mu_3\mu_2p^3 + 12\mu_1\mu_2\mu_3p)/(6A_1p^4). \]  
(A.8)
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