Higher spin quasinormal modes and one-loop determinants in the BTZ black hole

Shouvik Datta, Justin R. David
Centre for High Energy Physics, Indian Institute of Science,
C.V. Raman Avenue, Bangalore 560012, India.
shouvik, justin@cts.iisc.ernet.in

Abstract: We solve the wave equations of arbitrary integer spin fields in the BTZ black hole background and obtain exact expressions for their quasinormal modes. We show that these quasinormal modes precisely agree with the location of the poles of the corresponding two point function in the dual conformal field theory as predicted by the AdS/CFT correspondence. We then use these quasinormal modes to construct the one-loop determinant of the higher spin field in the thermal BTZ background. This is shown to agree with that obtained from the corresponding heat kernel constructed recently by group theoretic methods.
1. Introduction

Among the gauge/gravity dualities the $AdS_3/CFT_2$ correspondence is one of the most well studied. In string theory, this duality naturally occurs in the context of the D1-D5 system [1]. Till recently this example and related systems were the only examples with a well defined conformal field theory to study the $AdS_3/CFT_2$ duality. The proposal put forward by Gaberdiel and Gopakumar [2] provides a new
and interesting example to explore this version of the gauge/gravity duality. In some respects this example might be more tractable since the bulk theory does not contain the full string spectrum but a set of massless higher spin fields. All these examples contain interacting higher spin fields. The D1-D5 system and its cousins in general contain interacting higher spin massive fields, while the $AdS_3$ duals of minimal models put forward by [2] contain higher spin massless fields. In this paper we will focus on the more general situation when the higher spin field is massive. $AdS_3/CFT_2$ duality relates a field of spin $s$ propagating in $AdS_3$ to a operator $O$ in the dual conformal field theory characterized by conformal weights $(h_L, h_R)$ with

$$ h_R - h_L = \pm s. \quad (1.1) $$

The mass of the propagating field $m$ is related to the conformal dimension of the operator $O$ which is given by

$$ h_R + h_L = \hat{\Delta}. \quad (1.2) $$

An immediate consequence of this correspondence is that when the CFT is at finite temperature, the poles in the retarded point function of the operator $O$ is given by quasinormal modes of the spin $s$ field [1, 2, 3]. In $CFT_2$, two point functions of primary operators are determined entirely by conformal invariance and one can read out the poles of the retarded Green’s in the complex frequency plane. These are given by

$$ \omega_L = k - 4\pi i T_L (n + h_L), \quad \omega_R = -k - 4\pi i T_R (n + h_R). \quad (1.3) $$

Here $\omega$ and $k$ refers to the frequency and momentum respectively. $L, R$ are subscripts to denote the left and right moving poles. $T_L, T_R$ are the left and right moving temperatures of the conformal field theory. Thus, the $AdS_3/CFT_2$ correspondence predicts the quasinormal frequencies of fields with spin $s$. One of the aims of this paper is to verify this prediction for the case of arbitrary integer spins.

Quasinormal modes of various fields in the black hole background play an important role in the AdS/CFT correspondence. Their role for black holes in anti-de Sitter spaces were first investigated in [7]. The low lying quasinormal modes provide important information regarding transport properties of the dual field theory. A recent review of quasinormal modes in various backgrounds, their properties and a complete list of references is [8]. In most of the situations the complete spectrum of quasinormal modes can only be obtained numerically. Integrability of the string propagation in the BTZ background [9] suggests that it might be possible to obtain the quasinormal modes of arbitrary spin fields exactly. Indeed for the case of spins $s = 0, s = 1/2$ and $s = 1$ the corresponding wave equation were solved exactly [10, 11]. This confirmed the prediction of the AdS/CFT duality given in (1.3) [4]. These reasons and the simplicity of the expressions for the quasinormal spectrum in (1.3) are the motivations for deriving the quasinormal modes for higher spins. Another interesting property of quasinormal modes discovered recently is that
the 1-loop determinant of the corresponding field can be constructed by considering suitable products of the quasinormal modes [12]. The one-loop determinants for arbitrary spin fields in thermal AdS$_3$ was recently constructed in [13] using group theoretic methods. In this paper we show that these one-loop determinants can be written as product over higher spin quasinormal modes.

The organization of the paper is as follows. We will begin with some preliminaries where will we introduce the BTZ background and its properties, the equations satisfied by massive higher spin fields and the AdS/CFT prediction for the quasinormal modes from the CFT. In section 3, as a simple demonstration of the general method we develop in this paper we analyze the case of spin 2. We show the equations of motion for the spin 2 field can be simplified and its solutions can be found in closed form as hypergeometric functions. We then extract the quasinormal modes and show it agrees with that given in (1.3). In section 4 we generalize this analysis for integer higher spins. It turns out that the wave equations for these cases also can be solved in terms of hypergeometric functions. In section 5, we write down the one-loop determinant for the higher spin field in terms of products over the corresponding quasinormal modes and show that it agrees with that evaluated by group theoretic methods. Section 6 contains our conclusions. Appendix A and Appendix B contains the details of the quasinormal modes for spin 1 and spin 3 respectively. Appendix C contains the proofs of identities which are required in our analysis for the solutions of higher spin wave equations in section 4.

2. Preliminaries

In this section we will describe the geometry of the BTZ black hole and its properties. This will help us introduce the conventions and notations we will use in this paper. We will also describe the key properties of the BTZ background which will enable us to simplify the equations of motion for the higher spin fields in this background. We then review the properties of the higher spin field equations. Finally we recall the AdS/CFT dictionary for these fields and write down the conformal dimensions of the operator corresponding to these fields and their two point functions. We then extract the poles of the these two point functions. By the AdS/CFT correspondence, these poles must coincide with the quasinormal modes of the higher spin field in the bulk. In the subsequent sections we will focus on extracting these quasinormal modes by explicitly solving the equations of motion of the higher spin fields and we will demonstrate the quasinormal modes do indeed coincide with these poles.

2.1 The black hole geometry

The BTZ black hole is a solution of Einstein’s gravity with a negative cosmological constant in (2+1) dimensions [14]. In general it describes a spinning black hole which
asymptotes to $AdS_3$. Its metric is conventionally written as

$$\begin{align*}
    ds^2 &= -\frac{\Delta^2}{r^2}dt^2 + \frac{r^2}{\Delta^2}dr^2 + r^2 \left( d\phi - \frac{r_+ r_-}{r^2}dt \right)^2, \\
    \Delta^2 &= (r^2 - r_+^2)(r^2 - r_-^2).
\end{align*}$$

(2.1)

Here $r_+$ and $r_-$ are the radii of the inner and outer horizons respectively, $r$ is the radial distance and $t$ labels the time. The angular coordinate $\phi$ has the period of $2\pi$. The radii $r_+$ and $r_-$ are related to $M$, the mass of black hole and $J$, its angular momentum by the following expression

$$r_{\pm} = \frac{Ml^2}{2} \left( 1 \pm \sqrt{1 - \frac{J^2}{M^2l^2}} \right).$$

(2.2)

where $l$ is the radius of anti-de Sitter space. From this point onwards we will set the radius of $AdS_3$ to unity. The left and right temperatures are defined as

$$T_L = \frac{1}{2\pi}(r_+ - r_-), \quad T_R = \frac{1}{2\pi}(r_+ + r_-).$$

(2.3)

The two temperatures capture the fact this system has two intrinsic thermodynamical variables: the thermal temperature and the angular potential. From the boundary conformal field theory point of view they define the two temperatures of the left movers and right movers.

A convenient coordinate system for our analysis was discovered by [4]. We first define the coordinates

$$\begin{align*}
    z &= \tanh^2 \xi = \frac{r^2 - r_+^2}{r^2 - r_-^2}, \\
    x^+ &= r_+ t - r_- \phi, \\
    x^- &= r_+ \phi - r_- t.
\end{align*}$$

(2.4)

Note that in these coordinates, the range of $r$ from $r_+$ to $\infty$ is mapped to $z = 0$ or $\xi = 0$ to $z = 1$ or $\xi = \infty$ respectively. In these coordinates, the BTZ metric given in (2.1) reduces to the following diagonal metric

$$ds^2 = d\xi^2 - \sinh^2 \xi dx^+_2 + \cosh^2 \xi dx^-_2.$$  

(2.5)

This form of the metric will prove to be useful in our calculations and we will briefly list various properties of this metric which will be repeatedly used. The non-vanishing Christoffel symbols of the metric in (2.5) are given by

$$\begin{align*}
    \Gamma^\xi_{++} &= \cosh \xi \sinh \xi = \frac{\sqrt{z}}{1 - z}, \\
    \Gamma^\xi_{--} &= -\cosh \xi \sinh \xi = -\frac{\sqrt{z}}{1 - z}, \\
    \Gamma^\xi_{\xi+} &= \coth \xi = \frac{1}{\sqrt{z}}, \\
    \Gamma^\xi_{-\xi} &= \tanh \xi = \sqrt{z}.
\end{align*}$$
The metric and its Christoffel symbols obey the following identities which will be useful in simplifying the higher spin equations in the next sections

\[ \sqrt{-g} = \cosh \xi \sinh \xi = \frac{\sqrt{z}}{1 - z}, \]
\[ \frac{g_{++}}{\sqrt{-g}} = \tanh \xi = -\sqrt{z}, \]
\[ \frac{g_{--}}{\sqrt{-g}} = \coth \xi = \frac{1}{\sqrt{z}}, \]
\[ \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-gg^{\mu \nu} \Gamma_{\nu \rho}^\alpha}) = \frac{1}{\sqrt{-g}} \partial_\xi (\sqrt{-gg^{\xi \xi} \Gamma_{\xi \rho}^\alpha}) = 2 \hat{\delta}_\rho^\alpha \]
\[ g^{\pm \pm} \Gamma_{\pm \pm}^{\pm} + \Gamma_{\pm}^{\pm} = 0, \]
\[ g^{++} \Gamma_{++}^{\xi} = -\coth \xi = -\frac{1}{\sqrt{z}}, \]
\[ g^{--} \Gamma_{--}^{\xi} = -\tanh \xi = -\sqrt{z}. \]

where \( \hat{\delta}_\rho^\alpha \) is defined as

\[ \begin{cases} 
1 & \text{for } \rho, \sigma = \pm \text{ and } \rho = \sigma, \\
0 & \text{otherwise}. 
\end{cases} \]

The BTZ black hole is obtained by identifications of \( \text{AdS}_3 \) \[15\]. Thus it is locally \( \text{AdS}_3 \) and therefore its curvature obey the following relations

\[ R_{\alpha \beta \gamma \delta} = g_{\alpha \delta} g_{\beta \gamma} - g_{\alpha \gamma} g_{\beta \delta}, \]
\[ R_{\mu \nu} = -2g_{\mu \nu}, \quad G_{\mu \nu} = 4g_{\mu \nu}. \]

In 3 dimensions, the Riemann tensor further obeys the following relation

\[ R_{\alpha \beta \gamma \delta} = \epsilon_{\alpha \beta \rho} \epsilon_{\gamma \delta \sigma} (R^{\rho \sigma} - \frac{1}{2} Rg^{\rho \sigma}), \]
\[ = \epsilon_{\alpha \beta \rho} \epsilon_{\gamma \delta \sigma} G^{\rho \sigma}. \]

Here \( G^{\rho \sigma} \) is the Einstein tensor and the epsilon tensor is defined as

\[ \epsilon^{\alpha \beta \gamma} = \frac{\tilde{\epsilon}^{\alpha \beta \gamma}}{\sqrt{-g}}, \quad \tilde{\epsilon}^{+} = 1. \]

where, \( \tilde{\epsilon}^{\alpha \beta \gamma} \) is the completely antisymmetric Levi-Civita symbol. Finally, we also need the fact that the epsilon tensor in 3 dimensions satisfies the relation

\[ \epsilon_{\beta \rho}^{\alpha} \epsilon_{\alpha \delta \sigma} = -(g_{\beta \delta} g_{\rho \sigma} - g_{\beta \sigma} g_{\rho \delta}). \]
2.2 The description of higher spin fields

Massive integer spin $s$ fields in $AdS$ spaces are realized by totally symmetric tensors of rank $s$ satisfying the following equations \[16, 17\]

\[
\nabla^2 - m^2_s \Phi_{\mu_1 \mu_2 \cdots \mu_s} = 0, \quad (2.19)
\]

\[
\nabla^\mu \Phi_{\mu \mu_2 \cdots \mu_s} = 0, \quad (2.20)
\]

\[
g^{\mu \nu} \Phi_{\mu \nu \mu_3 \cdots \mu_s} = 0. \quad (2.21)
\]

Here $\Phi_{\mu_1 \mu_2 \cdots \mu_s}$ is a totally symmetric rank $s$ tensor and for $AdS_3$ the mass is given by

\[
m^2_s = s(s - 3) + M^2. \quad (2.22)
\]

The first term is the natural mass that exists due to the curvature of $AdS_3$. Note that it also exists when $M = 0$.

A fact that will be crucial in our analysis is that for massive spin $s$ field in $AdS_3$ the set of equations in (2.19) is equivalent the following first order equation \[18\]

\[
\epsilon^{\alpha \beta} \nabla_\alpha \Phi_{\beta \nu \nu_2 \cdots \nu_s} = -m \Phi_{\mu \nu \nu_2 \cdots \nu_s}. \quad (2.23)
\]

Such an equation is familiar for the massive spin-1 field in $AdS_3$. It is well known that for the spin-1 field, the massive Chern-Simons equation is equivalent to the massive Maxwell field. We will now demonstrate that the equation in (2.23) in a locally $AdS_3$ space is equivalent to the second order equation of motion (2.19) and the gauge constraint. To verify the gauge constraint we take the covariant derivative on both sides of (2.23). This results in

\[
-m \nabla^\mu \Phi_{\mu \nu_2 \cdots \nu_s} = \epsilon^{\mu \alpha \beta} \nabla_\mu \nabla_\nu \Phi_{\beta \nu \nu_2 \cdots \nu_s} = \frac{1}{2} \epsilon^{\mu \alpha \beta} [\nabla_\mu, \nabla_\alpha] \Phi_{\beta \nu \nu_2 \cdots \nu_s} \quad (2.24)
\]

\[
= \epsilon^{\mu \alpha \beta} \frac{\partial}{\partial \nu^\rho} R_{\beta \rho \mu \alpha} \Phi_{\rho \nu_2 \nu_3 \cdots \nu_s} + \epsilon^{\mu \alpha \beta} \frac{\partial}{\partial \nu^\rho} R_{\nu_2 \rho \mu \alpha} \Phi_{\beta \nu \nu_3 \cdots \nu_s} + \cdots
\]

\[
= \epsilon^{\mu \alpha \beta} \epsilon_{\beta \rho \delta} \epsilon_{\mu \rho \sigma} G^{\delta \rho} \Phi_{\rho \nu_2 \nu_3 \cdots \nu_s} + \epsilon^{\mu \alpha \beta} \epsilon_{\nu_2 \rho \sigma} \epsilon_{\mu \rho \delta} G^{\delta \rho} \Phi_{\beta \nu \nu_3 \cdots \nu_s} + \cdots
\]

\[
= -2 \epsilon_{\beta \rho \delta} G^{\delta \rho} \Phi_{\beta \rho \nu_2 \nu_3 \cdots \nu_s} - 2 \epsilon_{\nu_2 \rho \sigma} G^{\delta \rho} \Phi_{\beta \nu \nu_3 \cdots \nu_s} + \cdots
\]

\[
= -8 \epsilon^{\rho \beta} \Phi_{\beta \rho \nu_3 \cdots \nu_s} + \cdots
\]

\[
= 0.
\]

Here we have used (2.16) and (2.18) to arrive at the third and fourth line of the above equation. In the last but one line we have used (2.14) which results from the fact the the space is locally $AdS_3$. The last line results from the traceless condition of the spin $s$ field. Thus the gauge constraint is satisfied once the symmetric traceless field satisfies (2.23).

We will now demonstrate that the second equation in (2.19) is implied once the the field satisfies (2.23). Substituting for $\Phi_{\beta \nu \nu_2 \nu_3 \cdots \nu_s}$ on the left hand side of (2.23) by using the equation itself we obtain

\[
\epsilon^{\alpha \beta} \epsilon_{\beta \rho \sigma} \nabla_\alpha \nabla_\rho \Phi_{\sigma \nu_2 \nu_3 \cdots \nu_s} = m^2 \Phi_{\mu \nu \nu_2 \cdots \nu_s}. \quad (2.25)
\]
Using (2.18) and rearranging the equation we obtain,

\[(\nabla^2 - m^2)\Phi_{\mu_2\nu_3\cdots\nu_s} = \nabla^\alpha \nabla_\mu \Phi_{\sigma_2\nu_3\cdots\nu_s} \]
\[= g^\sigma{}^\rho \nabla_\rho \nabla_\mu \Phi_{\sigma_2\nu_3\cdots\nu_s} \]
\[= g^\sigma{}^\rho g^\delta{}^\epsilon (R_{\sigma_2\delta_\mu} \Phi_{\nu_3\nu_4\cdots\nu_s} + R_{\nu_2\delta\mu} \Phi_{\sigma_3\nu_3\cdots\nu_s} \]
\[+ R_{\nu_3\delta\mu} \Phi_{\sigma_2\nu_4\cdots\nu_s} + \cdots + R_{\nu_s\delta\mu} \Phi_{\sigma_2\nu_3\cdots\nu_{s-1}\nu}) \]
\[= -(s+1)\Phi_{\mu_2\nu_3\cdots\nu_s}. \quad (2.26)\]

To obtain the equality on the first line we have used the gauge constraint and finally to obtain the last line we have used the form of the Riemann tensor given in (2.16). Thus we obtain the following second order equations of motion of the spin $s$ field

\[(\nabla^2 - m^2 + (s+1))\Phi_{\mu_2\nu_3\cdots\nu_s} = 0. \quad (2.27)\]

We are now in a position to relate the mass, $m$ appearing in the first order equation (2.23) to the actual mass, $M$ of the higher spin field. Comparing, (2.27) with (2.19) we get,

\[m^2 = M^2 + (s-1)^2. \quad (2.28)\]

As a consistency check we can use the above gauge condition and the tracelessness condition of the spin $s$ field to show that the the LHS of the equation (2.23) is symmetric the indices $\mu$ and $\nu_2$. To do this we must show that the anti-symmetry combination of the LHS of (2.23) vanishes. This is equivalent to showing the following vanishes

\[\epsilon^{\mu_2\nu_2} \epsilon^{\alpha_3} \nabla_\alpha \Phi_{\beta_2\cdots\nu_s} = -(g^{\nu_2\alpha} g^{\rho_\beta} - g^{\mu_2\rho} g^{\beta_\alpha}) \nabla_\alpha \Phi_{\beta_2\nu_3\cdots\nu_s} \]
\[= -g^{\rho_\beta} \nabla_\alpha \Phi_{\beta\nu_3\cdots\nu_s} + g^{\rho_\alpha} \nabla_\alpha g^{\nu_\beta} \Phi_{\beta_2\nu_3\cdots\nu_s} \]
\[= 0. \quad (2.29)\]

Here we have used (2.18) to obtain the first equation. Then the gauge condition and the tracelessness condition implies that the antisymmetric component of the LHS of (2.23) vanishes.

Finally, it is instructive to count the number of degrees of freedom of the spin $s$ field in 3 dimensions which satisfies the gauge constraint and the tracelessness constraint in (2.19). The number of independent components of a totally symmetric rank $s$ tensor in 3 dimensions which is traceless is given by

\[D(s) = \frac{(s+2)(s+1)}{2} - \frac{s(s-1)}{2} = 2s + 1. \quad (2.30)\]

The number of independent equations in the gauge constraint is given by

\[C(s) = \frac{(s+1)s}{2} - \frac{(s-1)(s-2)}{2} = 2s - 1. \quad (2.31)\]
Here again the subtraction is due to the traceless condition. Thus the total number of degrees of freedom of a rank $s$ symmetric traceless field which satisfies the gauge constraint is

$$D(s) - C(s) = 2. \quad (2.32)$$

### 2.3 Higher spin fields in $AdS_3/CFT_2$

We now recall the relation between a rank $s$ symmetric tensor which satisfies the equations (2.19) in $AdS_3$ and the corresponding operator on the boundary. We first find the relation between the mass of the spin $s$ field and the dimension of the operator. For this we must investigate the behaviour of the solutions to (2.19) close to the boundary. Consider Euclidean $AdS_3$ in Poincare coordinates with the metric

$$ds^2 = \frac{1}{\tilde{z}^2} \left( d\tilde{z}^2 + (dx^1)^2 + (dx^2)^2 \right). \quad (2.33)$$

Then the spin $s$ Laplacian can be written as [19]

$$\nabla^2 \Phi_{\mu_1 \cdots \mu_s} = \tilde{z}^2 \left[ (\partial_{\tilde{z}} + \frac{s-1}{\tilde{z}})(\partial_{\tilde{z}} + \frac{s}{\tilde{z}}) + \tilde{z}^2 \partial_{i} \partial_{i} - s \right] \Phi_{\mu_1 \cdots \mu_s} \quad (2.34)$$

$$- 2s \tilde{z} \partial_{(\mu_1} \Phi_{\mu_2 \cdots \mu_s)\tilde{z}} + s(s-1)\eta_{(\mu_1 \mu_2} \Phi_{\mu_3 \cdots \mu_s)\tilde{z}\tilde{z}}$$

$$- s(2s-1)\delta_{\mu_2} \Phi_{\mu_3 \cdots \mu_s)\tilde{z}} + 2s \tilde{z} \partial_{(\mu_1} \delta_{\mu_2} \Phi_{\mu_3 \cdots \mu_s)\rho}.$$

while the gauge condition reduces to

$$(\partial_{\tilde{z}} - \frac{1}{\tilde{z}}) \Phi_{\tilde{z}\mu_2 \cdots \mu_s} + \partial_i \Phi_{i\mu_1 \cdots \mu_s} = 0. \quad (2.35)$$

To obtain this equation the traceless property of the spin $s$ tensor was used. We can now solve for the behaviour of these fields close to the boundary. Let us make the ansatz that

$$\Phi_{i_1 \cdots i_s}(x, \tilde{z}) \sim z^\delta, \quad \tilde{z} \to 0. \quad (2.36)$$

where $x$ and $i_k$ denote the coordinates along the boundary. Then from the gauge condition (2.34) we see that

$$\Phi_{\tilde{z}\mu_2 \cdots \mu_s} \sim z^{\delta+1}. \quad (2.37)$$

This implies that the leading terms in the Laplacian and hence in the equations of motion near the boundary $\tilde{z} \to 0$ are given by

$$\left( \tilde{z}^2(\partial_{\tilde{z}} + \frac{s-1}{\tilde{z}})(\partial_{\tilde{z}} + \frac{s}{\tilde{z}}) - s - m_s^2 \right) \Phi_{i_1 i_2 \cdots i_s} = 0. \quad (2.38)$$

Substituting the ansatz in (2.36) we can easily solve for $\delta$ which is given by the solution of the following equation

$$(\delta + s)(\delta + s - 2) - s = m_s^2. \quad (2.39)$$
Solving this quadratic equation one obtains the following two behaviours of the spin $s$ field at the boundary

$$\delta = -(s - 1) \pm \sqrt{s + 1 + m_s^2}, \quad (2.40)$$

$$= -(s - 1) \pm \sqrt{(s - 1)^2 + M^2},$$

$$= -(s - 1) \pm |m|.$$  

To obtain the second line we have substituted the value of $m_s^2$ from (2.22). The last line is obtained by using the relation (2.28). It is important to note that this asymptotic analysis also holds for the case of the BTZ black hole. This is because asymptotically the BTZ black hole reduces to $AdS_3$ as can be seen from the metric in (2.1) with the identification

$$\hat{\bar{z}} = \frac{1}{r}. \quad (2.41)$$

The conformal dimension $\Delta$ of the dual operator can then be obtained from the coupling of the boundary value of the spin $s$ field and a spin $s$ current which is given by

$$\int d^2x J^{i_1 \cdots i_s} \Phi_{i_1 \cdots i_s}. \quad (2.42)$$

From conformal invariance we obtain the following expression for the conformal dimension of the dual operator,

$$\hat{\Delta} = 2 - \delta - s = 1 + \sqrt{(s - 1)^2 + M^2};$$

$$= 1 + |m|. \quad (2.43)$$

Here we have chosen the negative branch of for $\delta$. Thus the conformal weights $(h_L, h_R)$ of the dual operator are given by relations

$$\hat{\Delta} = h_L + h_R, \quad h_R - h_L = \pm s. \quad (2.44)$$

The second relation can be understood as follows. The rank $s$ traceless symmetric tensor has only two independent degrees of freedom. In terms of the conformal group, this is just the spin of the boundary operator, which can be $+s$ or $-s$. For definiteness we will assume that $m$ is positive and as we will subsequently see this will lead to the situation $h_R - h_L = s$.

**The two point function and its poles**

The two point function of an conformal primary with weight $(h_L, h_R)$ with the conformal field theory at right and left temperatures $T_L, T_R$ is given by

$$G(t, x) = \langle \mathcal{O}(t, x) \mathcal{O}(0, 0) \rangle_T = \frac{C_0}{g^{(2h_L + 2h_R)}} \left( \frac{\pi T_L}{\sinh \pi T_L x^+} \right)^{2h_L} \left( \frac{\pi T_R}{\sinh \pi T_R x^-} \right)^{2h_R}. \quad (2.45)$$
where \( x^\pm = t \pm x \). The two point function is entirely determined by conformal invariance. We have followed the notations of [20] in writing down the above expression. The retarded Green’s function is then given by

\[
D^{\text{ret}}(x, x') = i\theta(t - t')\langle [\mathcal{O}(x), \mathcal{O}(x')] \rangle_T,
\]

(2.46)

where the commutator is evaluated in the canonical ensemble. From the definition given in (2.45) and (2.46) we obtain the relation

\[
D^{\text{ret}}(x, 0) = i\theta(t) \bar{D}(x, 0),
\]

(2.47)

\[
\bar{D}(x, 0) = \mathcal{G}(t + \imath \epsilon, x) - \mathcal{G}(t - \imath \epsilon).
\]

(2.48)

The Fourier transform of \( \bar{D}(x, 0) \) was performed in [20] and the result is given by

\[
\int d^2 x e^{-ip \cdot x} \bar{D}(x, 0) = C \frac{(2\pi T_L)^{2h_L} (2\pi T_R)^{2h_R} \epsilon^{\beta p/2} - (-1)^{2h_L + 2h_R} \epsilon^{-\beta p/2}}{\Gamma(2h_L) \Gamma(2h_R)} 2 | \Gamma \left( h_L + i \frac{p_+}{2\pi T_L} \right) \Gamma \left( h_R + i \frac{p_-}{2\pi T_R} \right) |^2.
\]

(2.49)

where \( p \cdot x = \omega t - kx \), \( p_\pm = \frac{1}{2}(\omega \mp k) \). From this we see that the poles of the retarded Green’s function are the poles that lie in the lower half plane. The two set of poles are given by

\[
\omega_L = k + 2\pi T_L(n + h_L), \quad \omega_R = -k + 2\pi T_R(n + h_R),
\]

\[
= k - 2\pi i T_L(2n + \hat{\Delta} - s), \quad = -k - 2\pi i T_R(2n + \hat{\Delta} + s).
\]

(2.49)

To obtain the second line from the first we have used the relations in (2.44) with the positive sign for \( s \). Note that the function in (2.48) has poles also in the upper half plane. The reason these poles are selected out in the retarded Green’s function is that the response has to die off as \( t \to +\infty \). Therefore the pole has to lie on the lower half plane. For spin \( s = 0, 1 \) and \( 1/2 \) it has been shown that these poles coincide with the quasinormal modes of the corresponding spin field in the BTZ background [4]. One of the aims of this paper is to generalize this result for the case of arbitrary spins.

In the next section, as a simple demonstration of the general method for the case of arbitrary integer spins, we will solve the wave equations of the massive spin 2 field in the BTZ background and extract out the quasinormal modes. We will demonstrate that the poles coincide with that given in (2.49). The experience gained by this analysis will enable us to generalize the result for the spin \( s \) case.
3. The spin 2 case

In this section we will evaluate the quasinormal modes of the \( s = 2 \) field. This will involve reduction of the spin \( s \) Laplacian to a scalar Laplacian by taking appropriate linear combinations of the various components of the spin 2 field. The spin 2 case will serve as an demonstration of the method we will develop for determining the quasinormal modes for fields of arbitrary spin \( s \). We begin by writing down the first order equations (2.23) satisfied by the spin 2 field

\[
e_{\mu}^{\alpha\beta} \nabla_{\alpha} h_{\beta\nu} = -mh_{\mu\nu}. \tag{3.1}
\]

An important fact that we will use for our analysis is that near the boundary components which have the radial coordinate that is \( h_{\xi_+}, h_{\xi_-}, h_{\xi\xi} \) are all determined from the components along the boundary and are suppressed. This is clear from our analysis around (2.37). quasinormal modes are obtained by imposing vanishing Dirichlet boundary conditions which demand the function to vanish at the boundary, Thus it is sufficient to examine the equations satisfied by the components \( h_{++}, h_{--}, h_{+-} \). From (3.1) we obtain the following equations satisfied by these components

\[
g_{++}(\nabla_{\xi} h_{++} - \nabla_{-} h_{\xi+}) = -mh_{++}, \tag{3.2}
\]

\[
g_{--}(\nabla_{+} h_{\xi-} - \nabla_{\xi} h_{++}) = -mh_{--}, \tag{3.3}
\]

\[
g_{+-}(\nabla_{\xi} h_{--} - \nabla_{-} h_{\xi-}) = -mh_{+-}, \tag{3.4}
\]

\[
g_{-+}(\nabla_{+} h_{++} - \nabla_{\xi} h_{+-}) = -mh_{-+}. \tag{3.5}
\]

The spin-2 field is also traceless, explicitly this condition is expressed by the following equation

\[
h_{\xi\xi} - \text{cosech}^2 2\xi h_{++} + \text{sech}^2 2\xi h_{--} = 0. \tag{3.6}
\]

3.1 Reduction of the spin-2 Laplacian to the scalar Laplacian

The first step in the analysis is to show that the spin 2 Laplacian on the components reduces \( h_{++}, h_{--} \) and \( h_{+-} \) to the scalar Laplacian together with a mixing ‘mass matrix’. We start by analyzing the action of the Laplacian on \( h_{\mu\nu} \)

\[
\nabla^2 h_{\mu\nu} = \Delta h_{\mu\nu} - \frac{1}{\sqrt{-g}} \partial_{\alpha}(\sqrt{-g}g^{\alpha\beta}\Gamma^\omega_{\beta\mu})h_{\sigma\nu} - g^{\alpha\beta}\Gamma^\omega_{\beta\mu}\partial_{\alpha}h_{\sigma\nu}
\]

\[- \frac{1}{\sqrt{-g}} \partial_{\alpha}(\sqrt{-g}g^{\alpha\beta}\Gamma^\omega_{\beta\nu})h_{\sigma\mu} - g^{\alpha\beta}\Gamma^\omega_{\beta\nu}\partial_{\alpha}h_{\sigma\mu}
\]

\[- \Gamma^\rho_{\alpha\mu}g^{\alpha\beta}\nabla_{\beta}h_{\rho\nu} - \Gamma^\rho_{\alpha\nu}g^{\alpha\beta}\nabla_{\beta}h_{\rho\mu} - g^{\alpha\beta}\Gamma^\sigma_{\beta\mu}h_{\rho\nu} - g^{\alpha\beta}\Gamma^\sigma_{\beta\nu}h_{\rho\mu} - 2g^{\alpha\beta}\Gamma^\sigma_{\alpha\nu}\Gamma^\rho_{\beta\mu}h_{\rho\sigma}. \tag{3.7}
\]
where
\[ \Delta h_{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\alpha (g^{\alpha\beta} \partial_\beta h_{\mu\nu}). \] (3.8)

To obtain the last line in (3.7) we have used (2.9) and rewritten the ordinary derivatives in terms of covariant ones along with terms involving the Christoffel symbols. The next step is to look at the (++) , (−−) and (+−) components of the above equation explicitly and repeatedly use the equations (3.2) to (3.5) and (3.6) to reduce the terms involving the Christoffel symbols in (3.7) to constants which will become the mixing ‘mass matrix’.

(++) component

Examining the (++) component of (3.7), we obtain
\[
\nabla^2 h^{++} = \Delta h^{++} - 4h^{++} - 4 \Gamma^{\xi}_{++} g^{++} \nabla_+ h^{+\xi} - 4g^{++} \Gamma^{\xi}_{+\xi} \nabla_\xi h^{++} - 2g^{++} \Gamma^{\xi}_{+\xi} \nabla_\xi h^{++} - 2g^{++} \Gamma^{\xi}_{+\xi} \nabla_\xi h^{++} - 2g^{++} \Gamma^{\xi}_{+\xi} \nabla_\xi h^{++}.
\]

The terms on the third line vanish because of the identity (2.10), therefore we obtain
\[
\nabla^2 h^{++} = \Delta h^{++} - 4h^{++} + 4 \coth \xi (\nabla_+ h^{+\xi} - \nabla_\xi h^{++}) + 2 \cosh^2 \xi h^{++} - 2 \coth^2 \xi h^{++}.
\]

(−−) component

\[
\nabla^2 h^{−−} = \Delta h^{−−} - 4h^{−−} - 4 \Gamma^{\xi}_{−−} g^{−−} \nabla_− h^{−\xi} - 4g^{−−} \Gamma^{\xi}_{−\xi} \nabla_\xi h^{−−} - 2g^{−−} \Gamma^{\xi}_{−\xi} \nabla_\xi h^{−−} - 2g^{−−} \Gamma^{\xi}_{−\xi} \nabla_\xi h^{−−} - 2g^{−−} \Gamma^{\xi}_{−\xi} \nabla_\xi h^{−−} - 2g^{−−} \Gamma^{\xi}_{−\xi} \nabla_\xi h^{−−}.
\]

The manipulations here are similar to the (++) case. The terms on the third line vanish because of the identity (2.10)
\[
\nabla^2 h^{−−} = \Delta h^{−−} - 4h^{−−} + 4 \tanh \xi (\nabla_− h^{−\xi} - \nabla_\xi h^{−−}) - 2 \sinh^2 \xi h^{−−} - 2 \tanh^2 \xi h^{−−}.
\]

To obtain the last line we have used (3.4) and the tracelessness condition (3.6).
On performing this linear coordinate transformation on the metric components when we make the coordinate transformation for the spin-1 case is a matrix appearing in (3.18). The linear combination which decouples the equation is

\[
\nabla^2 h_{++} = \Delta h_{++} - 4h_{++} - 2\Gamma^+_{++} g^{++} \nabla_+ h_{++} - 2\Gamma^+_{+-} g^{++} \nabla_\xi h_{++} - 2\Gamma^+_{-+} g^{++} \nabla_\xi h_{++} - g^{++} \Gamma^+_{++} h_{++} + g^{++} \Gamma^+_{+-} h_{++} - g^{++} \Gamma^+_{-+} h_{++} - 2g^{++} \Gamma^+_{++} \xi h_{++}. \tag{3.13}
\]

The terms on the fourth and fifth lines vanish because of the identity (2.10)

\[
\nabla^2 h_{+-} = \Delta h_{+-} - 4h_{+-} + 2 \coth \xi (\nabla_+ h_{+-} - \nabla_\xi h_{+-}) + 2 \tanh \xi (\nabla_- h_{+-} - \nabla_\xi h_{+-}) - 2h_{+-} = \Delta h_{+-} - 6h_{+-} - 2m(h_{++} + h_{--}). \tag{3.14}
\]

We have used equations (3.2), (3.3) to obtain the last equality.

To summarize, the results of (3.10), (3.11) and (3.14) are the following equations.

\[
\begin{align*}
\nabla^2 h_{++} &= \Delta h_{++} - 4h_{++} - 4mh_{++} - 2h_{--}, \tag{3.15} \\
\nabla^2 h_{--} &= \Delta h_{--} - 4h_{--} - 4mh_{--} - 2h_{++}, \tag{3.16} \\
\nabla^2 h_{+-} &= \Delta h_{+-} - 6h_{+-} - 2m(h_{++} + h_{--}). \tag{3.17}
\end{align*}
\]

Note that we have reduced the action of the spin 2 Laplacian on these components to a scalar Laplacian together with a ‘mass matrix’. The fact that the spin 2 Laplacian in the BTZ background reduces to the scalar Laplacian was noticed earlier in [21, 22].

It will prove useful to cast these equations in the following matrix form

\[
\nabla^2 \begin{pmatrix} h_{++} \\ h_{+-} \\ h_{--} \end{pmatrix} = \Delta \begin{pmatrix} h_{++} \\ h_{+-} \\ h_{--} \end{pmatrix} + \begin{pmatrix} -4 & -4m & -2 \\ -2m & -6 & -2m \\ -2 & -4m & -4 \end{pmatrix} \begin{pmatrix} h_{++} \\ h_{+-} \\ h_{--} \end{pmatrix}. \tag{3.18}
\]

### 3.2 Solutions of the spin-2 components

We now need to decouple these equations. In other words, diagonalize the ‘mass matrix’ appearing in (3.18). The linear combination which decouples the equation for the spin-1 case is \( A_+ \pm A_- \). This is the same linear combination which results when we make the coordinate transformation

\[
x_1 = x_+ + x_-, \quad x_2 = x_+ - x_-. \]

On performing this linear coordinate transformation on the metric components \( h_{++}, h_{--} \) and \( h_{+-} \), we obtain the following transformation matrix

\[
\begin{pmatrix} h_{++} \\ h_{--} \\ h_{+-} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} h_{11} \\ h_{12} \\ h_{22} \end{pmatrix}. \tag{3.19}
\]
Note that the co-ordinate transformation in (3.19) is used just as a mathematical device to construct the linear combinations given in (3.19). It is important to emphasize that we do not replace the co-ordinates in the Laplacian or in the functional dependence of the metric components by the definition in (3.19). Substituting the linear combinations of (3.19) in (3.18) results in the following decoupled equation

\[
\nabla^2 \begin{pmatrix} h_{11} \\ h_{12} \\ h_{22} \end{pmatrix} = \begin{pmatrix} \Delta - 4m - 6 & 0 & 0 \\ 0 & \Delta - 2 & 0 \\ 0 & 0 & \Delta + 4m - 6 \end{pmatrix} \begin{pmatrix} h_{11} \\ h_{12} \\ h_{22} \end{pmatrix}.
\]

(3.20)

Thus we have diagonalized the action of the Laplacian by considering the linear combinations \(h_{11}, h_{12}\) and \(h_{22}\). The second order equations of motion of the spin 2 field which can be read out from (2.27) is given by

\[
(\nabla^2 - m^2 + 3)h_{\mu\nu} = 0.
\]

(3.21)

Thus the metric components \(h_{11}, h_{12}\) and \(h_{22}\) obey the following equations

\[
(\Delta - (m + 2)^2 + 1)h_{11} = 0,
\]

(3.22)

\[
(\Delta - m^2 + 1)h_{12} = 0,
\]

(3.23)

\[
(\Delta - (m - 2)^2 + 1)h_{22} = 0.
\]

(3.24)

Since these are the equations of a massive scalar, their solutions can be easily found. We first substitute the ansatz,

\[
h_{ij} = e^{-i(k^+ x^+ + k^- x^-)} R_{ij},
\]

(3.25)

where \(i, j \in \{1, 2\}\). Note that with the definition of \(x^+\) and \(x^-\) given in (2.4) we see that the frequency and the momenta of these solutions are related to \(k_+\) and \(k_-\) by the following equations

\[
(k_+ + k_-)(r_+ - r_-) = \omega - k, \quad (k_+ - k_-)(r_+ + r_-) = \omega + k.
\]

(3.26)

We then use the coordinate \(z = \tanh^2 \xi\) to write out the Laplacian for each of the equations in (3.22). This results in

\[
z(1 - z) \frac{d^2 R_{11}}{dz^2} + (1 - z) \frac{dR_{11}}{dz} + \left[ \frac{k_+^2}{4z} - \frac{k_-^2}{4} - \frac{(m + 2)^2 - 1}{4(1 - z)} \right] R_{11} = 0,
\]

(3.27)

\[
z(1 - z) \frac{d^2 R_{12}}{dz^2} + (1 - z) \frac{dR_{12}}{dz} + \left[ \frac{k_+^2}{4z} - \frac{k_-^2}{4} - \frac{m^2 - 1}{4(1 - z)} \right] R_{12} = 0,
\]

(3.28)

\[
z(1 - z) \frac{d^2 R_{22}}{dz^2} + (1 - z) \frac{dR_{22}}{dz} + \left[ \frac{k_+^2}{4z} - \frac{k_-^2}{4} - \frac{(m - 2)^2 - 1}{4(1 - z)} \right] R_{22} = 0.
\]

(3.29)
The solution that obeys in-going boundary conditions at the horizon is

\[ R_{11}(z) = e_{11} z^\alpha (1-z)^{\beta_{11}} F(a_{11}, b_{11}, c; z), \]
\[ R_{12}(z) = e_{12} z^\alpha (1-z)^{\beta_{12}} F(a_{12}, b_{12}, c; z), \]
\[ R_{22}(z) = e_{22} z^\alpha (1-z)^{\beta_{22}} F(a_{22}, b_{22}, c; z). \]

(3.30)

Here \( e_{ij} \)s are constants which we will call as the polarizations. The parameters of the functions defined in (3.30) are given by

\[ \alpha = \frac{-ik_+}{2}, \quad c = 1 + 2\alpha, \]  
\[ \beta_{11} = \frac{1}{2}(m + 3), \quad \beta_{12} = \frac{1}{2}(m + 1), \quad \beta_{22} = \frac{1}{2}(m - 1), \]
\[ a_{ij} = \frac{k_+ - k_-}{2i} + \beta_{ij}, \quad b_{ij} = \frac{k_+ + k_-}{2i} + \beta_{ij}. \]

Note that in each of the cases one can choose an alternate form of the solution due to the following identity obeyed by the hypergeometric function

\[ F(a, b, c; z) = (1-z)^{c-a-b} F(c-a, c-b, c; z). \]

(3.32)

**Determining the polarization constants**

Our next task is to fix the polarization constants or coefficients \( e_{ij} \) appearing in the solutions given in (3.30). Since these are constants through out space time, their values can be found by looking at the behaviour at any specific point or surface. We will see that these constants are easily determined by examining the solutions near the horizon \( z \to 1 \). This approach is different from the one followed by [11] for the case of vectors. There the values of the polarization constants were found using the recursion properties the of hypergeometric functions. We will see that the method developed here is also easily generalized to the case of arbitrary spin.

The near-horizon \((z \to 0)\) behaviour of these solutions is

\[ R_{ij} \to e_{ij} z^\alpha e^{-i(k_+ x^+ k_- x^-)}, \quad \text{as} \quad z \to 0. \]

(3.33)

Now from the relation (3.19) we see that the behaviour of of the solutions \( h_{++}, h_{+-}, h_{--} \) near the horizon is given by

\[ h_{\hat{\mu} \hat{\nu}} \to e_{\hat{\mu} \hat{\nu}} z^\alpha e^{-i(k_+ x^+ k_- x^-)}, \quad \text{as} \quad z \to 0. \]

(3.34)

where, \( \hat{\mu}, \hat{\nu} \in \{+, -\} \) and

\[
\begin{pmatrix}
  e_{++} \\
  e_{--} \\
  e_{+-}
\end{pmatrix}
= 
\begin{pmatrix}
  1 & 2 & 1 \\
  1 & 0 & -1 \\
  1 & -2 & 1
\end{pmatrix}
\begin{pmatrix}
  e_{11} \\
  e_{12} \\
  e_{22}
\end{pmatrix}.
\]

(3.35)
We will now determine the relations between the coefficients $e_{\mu\nu}$ by examining the equation (3.1) near the horizon. The $(\xi^+)$ and $(\xi^-)$ components of equation (3.1) gives rise to the following relations

\begin{align}
-m\sqrt{z\left(1-z\right)}h_{\xi^+} &= \partial_- h_{++} - \partial_+ h_{+-} + \frac{\sqrt{z}}{1-z} h_{\xi^-}, \\
-m\sqrt{z\left(1-z\right)}h_{\xi^-} &= \partial_- h_{+-} - \partial_+ h_{--} + \frac{\sqrt{z}}{1-z} h_{\xi^+}.
\end{align}

(3.36) (3.37)

Near the horizon the above equations can be used to determine the form of $h_{\xi^+}$ and $h_{\xi^-}$ in terms of $e_{\mu\nu}$. This can be written in terms of the following matrix equation

\begin{equation}
\begin{pmatrix}
m & 1 \\
1 & m
\end{pmatrix}
\begin{pmatrix}
h_{\xi^+} \\
h_{\xi^-}
\end{pmatrix}
= i
\begin{pmatrix}
k_- e_{++} - k_+ e_{+-} \\
k_- e_{+-} - k_+ e_{--}
\end{pmatrix}
\frac{1}{\sqrt{z}} e^{-\frac{i}{4} (k_+ x^+ + k_- x^-)}.
\end{equation}

(3.38)

Thus $h_{\xi^\pm}$ has the following behaviour

\begin{equation}
h_{\xi^\pm} \rightarrow e_{\xi^\pm} z^{\alpha - \frac{1}{2}} e^{-\frac{i}{4} (k_+ x^+ + k_- x^-)}, \quad \text{as } z \rightarrow 0.
\end{equation}

(3.39)

Let us now examine the equation (3.3) in detail

\begin{equation}
-mh_{--} = -\frac{1}{\sqrt{z}} \left(\partial_+ h_{+-} - \partial_- h_{++}\right) + h_{+-}
= -\frac{1}{\sqrt{z}} \left(2\sqrt{z}(1-z)\partial_+ h_{+-} - \partial_- h_{++}\right) + h_{+-}.
\end{equation}

Near the horizon, the first two terms on the right hand side have the behaviour $\sim z^{\alpha - 1}$ while other terms have the behaviour $\sim z^\alpha$. Thus to the leading order as $z \rightarrow 0$, we must have the following relation

\begin{equation}
e_{+-} = e_{\xi^-}.
\end{equation}

(3.40)

Similarly if we investigate the near-horizon behaviour of equation (3.5), we obtain the relation

\begin{equation}
e_{++} = e_{\xi^+}.
\end{equation}

(3.41)

Substituting these (3.40) and (3.41) in (3.38) we obtain the following homogeneous equation relating the three coefficients $e_{\mu\nu},$

\begin{equation}
\begin{pmatrix}
m - ik_- & 1 + ik_+ & 0 \\
1 & m - ik_- & ik_+
\end{pmatrix}
\begin{pmatrix}
e_{++} \\
e_{+-}
\end{pmatrix}
= 0.
\end{equation}

(3.42)

We can write this equation in in terms $e_{11}, e_{12}$ and $e_{22}$ using (3.19), this results in

\begin{equation}
\begin{pmatrix}
m - ik_- & 1 + ik_+ & 0 \\
1 & m - ik_- & ik_+
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 1 \\
1 & 0 & -1 \\
1 & -2 & 1
\end{pmatrix}
\begin{pmatrix}
e_{11} \\
e_{12} \\
e_{22}
\end{pmatrix}
= 0.
\end{equation}

(3.43)
The easiest way to write one of the polarization constants in terms of the other is to multiply the above equation by the spin-1 coefficient transformation matrix. Thus we have

\[
\begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
\begin{pmatrix}
m - i k_- & 1 + i k_+ & 0 \\
1 & m - i k_- & i k_+
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 1 \\
1 & 0 & -1 \\
1 & -2 & 1
\end{pmatrix}
\begin{pmatrix}
e_{11} \\
e_{12} \\
e_{22}
\end{pmatrix} =
\begin{pmatrix}
e_{11} \\
e_{12} \\
e_{22}
\end{pmatrix}
\]

(3.44)

\[
\begin{pmatrix}
m + 1 + i(k_+ - k_-) & m + 1 - i(k_+ + k_-) & 0 \\
0 & m - 1 + i(k_+ - k_-) & m - 1 - i(k_+ + k_-)
\end{pmatrix}
\begin{pmatrix}
e_{11} \\
e_{12} \\
e_{22}
\end{pmatrix} = 0.
\]

These result in the following recursion relations

\[
(m + 1 + i(k_+ - k_-))e_{11} = -(m + 1 - i(k_+ + k_-))e_{12},
\]

(3.45)

\[
(m - 1 + i(k_+ - k_-))e_{12} = -(m - 1 - i(k_+ + k_-))e_{22}.
\]

(3.46)

Now we can easily obtain the polarization constants \(e_{11}, e_{12}\) in terms of \(e_{22}\) easily. We therefore obtain the following relations

\[
e_{12} = \frac{(m + 1 - i(k_+ + k_-))}{(m + 1 + i(k_+ - k_-))}e_{22},
\]

(3.47)

\[
e_{11} = \frac{(m - 1 - i(k_+ + k_-))}{(m - 1 + i(k_+ - k_-))}e_{12},
\]

\[
e_{11} = \frac{(m - 1 + i(k_+ + k_-))(m + 1 + i(k_+ - k_-))}{(m - 1 + i(k_+ - k_-))(m + 1 + i(k_+ - k_-))}e_{22}.
\]

(3.48)

### 3.3 Quasinormal modes

Now we have all the ingredients necessary to obtain the quasinormal modes of the spin 2 field in the BTZ background. To do this we examine the behaviour of the solutions \((3.30)\) at the boundary \(z \to 1\). Let us consider first the case of the solution \(R_{11}\). Near the boundary the dominant behaviour for the case \(m > 0\) is given by

\[
R_{11}(z) \simeq e_{11}z^\alpha(1 - z)^{-\frac{m-1}{2}} \frac{\Gamma(c)\Gamma(a_{11} + b_{11} - c)}{\Gamma(a_{11})\Gamma(b_{11})}
\]

\[
= \frac{4e_{11}z^\alpha(1 - z)^{-\frac{m-1}{2}}\Gamma(c)\Gamma(a_{11} + b_{11} - c)}{(m - 1 - i(k_+ + k_-))(m + 1 - i(k_+ + k_-))\Gamma(a_{11})\Gamma(b_{11}) - 2)}
\]

\[
= \frac{4e_{22}z^\alpha(1 - z)^{-\frac{m-1}{2}}\Gamma(c)\Gamma(a_{11} + b_{11} - c)}{(m - 1 + i(k_+ - k_-))(m + 1 + i(k_+ - k_-))\Gamma(a_{11})\Gamma(b_{11}) - 2)}.
\]

(3.49)

Quasinormal modes are obtained by imposing vanishing Dirichlet conditions at the boundary. This implies we look for the condition in which \(R_{11}\) near the boundary vanishes. The zeros of the function in \((3.49)\) occur at the locations, \(a_{11} = -n\) and \(b_{11} - 2 = -n\) with \(n = 0, 1, 2, \cdots\), which in terms of the momenta are given by

\[
i(k_+ + k_-) = 2n + \hat{\Delta} - 2, \quad i(k_+ - k_-) = 2n + \hat{\Delta} + 2, \quad n = 0, 1, 2, \cdots \]

(3.50)
Here, we have used the relation
\[ \hat{\Delta} = 1 + m. \]
which results from (2.43), note that we have assumed \( m > 0 \). Now the behaviour of
the function \( R_{12} \) near the boundary is given by
\[
R_{12}(z) \simeq e_{12}z^\alpha(1-z)^{-\frac{m+1}{2}} \frac{\Gamma(c)\Gamma(a_{12} + b_{12} - c)}{\Gamma(a_{11})\Gamma(b_{11})}
\] (3.51)
\[
= \frac{2e_{12}z^\alpha(1-z)^{-\frac{m+1}{2}} \Gamma(c)\Gamma(a_{12} + b_{12} - c)}{(m + 1 - i(k_+ + k_-))\Gamma(a_{11})\Gamma(b_{11} - 1)}
\] (3.52)
\[
= \frac{2e_{22}z^\alpha(1-z)^{-\frac{m+1}{2}} \Gamma(c)\Gamma(a_{12} + b_{12} - c)}{(m + 1 + i(k_+ - k_-))\Gamma(a_{11})\Gamma(b_{11} - 1)}. \] (3.53)
The zeros of \( R_{12} \) therefore occur at the locations, \( a_{12} = -n \) and \( b_{12} - 1 = -n \). In
the terms of the momenta they are at
\[
i(k_+ + k_-) = 2n + \hat{\Delta} - 2, \quad i(k_+ - k_-) = 2n + \hat{\Delta}, \quad n = 0, 1, 2, \ldots. \] (3.54)
Finally the component \( R_{22} \) near the boundary is given by
\[
R_{22}(z) \simeq e_{22}z^\alpha(1-z)^{-\frac{m+1}{2}} \frac{\Gamma(c)\Gamma(a_{22} + b_{22} - c)}{\Gamma(a_{22})\Gamma(b_{22})}.
\] (3.55)
The zeros of \( R_{22} \) thus occur at the locations, \( a_{22} = -n \) and \( b_{22} = -n \), which in terms
of momenta are given by
\[
i(k_+ + k_-) = 2n + \hat{\Delta} - 2, \quad i(k_+ - k_-) = 2n + \hat{\Delta} - 2, \quad n = 0, 1, 2, \ldots. \] (3.56)
Vanishing Dirichlet conditions at the boundary for all the components of spin-2 field
require all the functions in (3.49), (3.51) and (3.55) to vanish simultaneously as
\( z \to 1 \). The quasinormal modes are thus given by the common set of zeros of all the
components. These are
\[
i(k_+ + k_-) = 2n + \hat{\Delta} - 2, \quad i(k_+ - k_-) = 2n + \hat{\Delta} + 2, \quad n = 0, 1, 2, \ldots. \] (3.57)
Writing the above relations in terms of frequency, momenta and the temperatures
using (3.26), we obtain the following set of quasinormal modes
\[
\omega_L = k + 2\pi T_L(k_+ + k_-) \quad \omega_R = -k + 2\pi T_R(k_+ - k_-)
\]
\[
= k - 2\pi i T_L(2n + \hat{\Delta} - 2), \quad = -k - 2\pi i T_R(2n + \hat{\Delta} + 2). \] (3.58)
with \( n = 0, 1, 2, \ldots. \) Comparing this to equation (2.49) we see that the quasinormal
modes for the spin 2 field precisely coincide with the poles of the corresponding
retarded two point function. Notice that we have obtained \( h_R - h_L = +2 \). The case
\( h_R - h_L = -2 \) arises when we consider the situation with \( m < 0 \).
4. The spin s case

In this section we will generalize the same procedure which we developed for the spin 2 case for arbitrary spins. The first order equation which we will repeatedly use for the reduction of the spin s Laplacian to the scalar Laplacian is the following

\[
\epsilon_{\mu_1}^{\alpha \beta} \nabla_\alpha \Phi_{\mu_2 \mu_3 \cdots \mu_s} = -m \phi_{\mu_1 \mu_2 \mu_3 \cdots \mu_s}. \tag{4.1}
\]

The \((\pm \mu_2 \mu_3 \cdots \mu_s)\) components of the above equation can be written as

\[
\pm \frac{g_{\pm \pm}}{\sqrt{-g}} (\nabla_\xi \Phi_{\pm \mu_2 \mu_3 \cdots \mu_s} - \nabla_\mp \Phi_{\mu_2 \mu_3 \cdots \mu_s}) = -m \phi_{\pm \mu_2 \mu_3 \cdots \mu_s}. \tag{4.2}
\]

We will also require the use of the tracelessness condition which is

\[
\Phi_{\xi \xi \mu_3 \cdots \mu_s} - \text{cosech}^2 \xi \Phi_{++ \mu_3 \cdots \mu_s} + \text{sech}^2 \xi \Phi_{-- \mu_3 \cdots \mu_s} = 0. \tag{4.3}
\]

4.1 Reduction of the spin-s Laplacian to the scalar Laplacian

After some straightforward manipulations and the use of \((2.9)\), the action of the Laplacian on an arbitrary spin field, \(\Phi_{\mu_1 \mu_2 \mu_3 \cdots \mu_s}\) can be written as

\[
\nabla^2 \Phi_{\mu_1 \mu_2 \mu_3 \cdots \mu_s} = \Delta \Phi_{\mu_1 \mu_2 \mu_3 \cdots \mu_s}
\]

\[
- 2(\delta_1^{\sigma} \Phi_{\sigma \mu_2 \mu_3 \cdots \mu_s} + \delta_2^{\sigma} \Phi_{\mu_1 \sigma \mu_3 \cdots \mu_s} + \cdots + \delta_s^{\sigma} \Phi_{\mu_1 \mu_2 \cdots \mu_{s-1} \sigma})
\]

\[
- 2g^{\alpha \beta}(\Gamma_{\beta \mu_1}^\eta \nabla_\alpha \Phi_{\eta \mu_2 \mu_3 \cdots \mu_s} + \Gamma_{\beta \mu_2}^\eta \nabla_\alpha \Phi_{\mu_1 \eta \mu_3 \cdots \mu_s} + \cdots + \Gamma_{\beta \mu_s}^\eta \nabla_\alpha \Phi_{\mu_1 \cdots \mu_{s-1} \eta})
\]

\[
- g^{\alpha \beta} \Gamma_{\alpha \mu_1}^\eta (\Gamma_{\beta \mu_2}^\eta \Phi_{\eta \mu_3 \cdots \mu_s} + \Gamma_{\beta \mu_2}^\eta \Phi_{\mu_1 \sigma \mu_3 \cdots \mu_s} + \cdots + \Gamma_{\beta \mu_s}^\eta \Phi_{\mu_1 \cdots \mu_{s-1} \sigma})
\]

\[
- 2g^{\alpha \beta} \sum_{i,j, i \neq j} \Gamma_{\beta \mu_1}^\eta \Gamma_{\alpha \mu_j}^\sigma \Phi_{\eta \sigma \mu_i \cdots \mu_j \cdots \mu_s}.
\tag{4.4}
\]

Here, \(\Delta\) is the scalar Laplacian or

\[
\Delta \Phi_{\mu_1 \mu_2 \mu_3 \cdots \mu_s} = \frac{1}{\sqrt{-g}} \partial_\alpha (g^{\alpha \beta} \partial_\beta \Phi_{\mu_1 \mu_2 \mu_3 \cdots \mu_s}). \tag{4.5}
\]

In the last line of \((4.4)\), the \(\hat{\mu}_i, \hat{\mu}_j\) refer to the indices which are missing and \(i, j\) run over 1 to \(s\). Just as for the spin 2 case, to determine the quasinormal modes it is sufficient to focus on components which have only \(+s\) or \(\mp s\). This is because, the other components can be determined by them and can be shown by the same argument used for the spin 2 case to be suppressed near the boundary. Therefore, let us now examine the action of the Laplacian on the object, \(\Phi_{\mu_1 \cdots \mu_p \mu_1 \cdots \nu_q}\) with
\( \mu_1 \cdots \mu_p = + \cdots + \) and \( \nu_1 \cdots \nu_q = - \cdots - \). This is given by

\[
\nabla^2 \Phi_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_q} = \Delta \Phi_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_q} - 2s \Phi_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_q} - 2g_{\alpha \beta} (\sum_{i,j=1,i\neq j}^p \Gamma^\eta_{\beta \alpha} \Phi_{\eta \mu_1 \cdots \mu_p \nu_1 \cdots \nu_q} + \sum_{i=1,j=1}^{p+q} \Gamma^\eta_{\beta \alpha} \Phi_{\eta \mu_1 \cdots \mu_p \nu_1 \cdots \nu_q}) + q(q-1)g_{\alpha \beta} \Phi_{\eta \mu_1 \cdots \mu_p \nu_1 \cdots \nu_q}.
\]

(4.6)

Let’s now introduce some notation, by ‘(p)’ we mean \( p \) number of + indices and by ‘(q)’ we mean \( q (= s - p) \) number of – indices. Here’s an example, with a field of \( s = 5 \)

\[ \Phi_{(2)(3)} = \Phi_{+++--}. \]

For the case \( \mu_1, \mu_2, \ldots, \mu_p = + \) and \( \nu_1, \nu_2, \ldots, \nu_q = - \) and using the fact that \( \Phi_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_q} \) is a completely symmetric tensor we obtain

\[
\nabla^2 \Phi_{(p)(q)} = \Delta \Phi_{(p)(q)} - 2s \Phi_{(p)(q)} - 2g_{\alpha \beta} (p-1)g_{\beta \alpha} \Phi_{(p-1)(q)} + q(q-1)g_{\alpha \beta} \Phi_{(p-1)(q-2)}.
\]

(4.7)

Now writing out the terms that contribute we obtain

\[
\nabla^2 \Phi_{(p)(q)} = \Delta \Phi_{(p)(q)} - 2s \Phi_{(p)(q)}
- 2p(g^{\xi +} \Gamma^\xi_{\xi +} \nabla_\xi \Phi_{(p-1)(q)} + g^{\xi \xi} \nabla_\xi \Phi_{(p-1)(q)} - 1)
- 2q(g^{\xi -} \Gamma^\xi_{\xi -} \nabla_\xi \Phi_{(p-1)(q)} + g^{\xi \xi} \nabla_\xi \Phi_{(p-1)(q)} - 1)
- p(g^{\xi +} \Gamma^\xi_{\xi +} \Phi_{(p-1)(q)} + g^{\xi \xi} \Gamma^\xi_{\xi +} \Phi_{(p-1)(q)} - 1)
- q(g^{\xi -} \Gamma^\xi_{\xi -} \Phi_{(p-1)(q)} + g^{\xi \xi} \Gamma^\xi_{\xi -} \Phi_{(p-1)(q)} - 1)
- p(p-1)(g^{\xi +} \Gamma^\xi_{\xi +} \Phi_{(p-1)(q)} + g^{\xi \xi} \Gamma^\xi_{\xi +} \Phi_{(p-1)(q)} - 1)
- q(q-1)(g^{\xi -} \Gamma^\xi_{\xi -} \Phi_{(p-1)(q)} + g^{\xi \xi} \Gamma^\xi_{\xi -} \Phi_{(p-1)(q)} - 1).
\]

(4.8)
The terms in the fourth and fifth lines vanish because of (2.10). We now substitute the values of the Christoffel symbols and the relevant metric components to obtain

\[
\nabla^2 \Phi(p)(q) = \Delta \Phi(p)(q) - 2s\Phi(p)(q) + 2p \coth \xi (\nabla_+ \Phi_{\xi(p-1)(q)} - \nabla_\xi \Phi_{+(p-1)(q)}) \\
+ 2q \tanh \xi (\nabla_- \Phi_{\xi(p)(q-1)} - \nabla_\xi \Phi_{-(p)(q-1)}) \\
- p(p-1)(-\cosh^2 \xi \Phi_{\xi\xi(p-2)(q)} + \coth^2 \xi \Phi_{++(p-2)(q)}) \\
- q(q-1)(\sinh^2 \xi \Phi_{\xi\xi(p)(q-2)} + \tanh^2 \xi \Phi_{--(p)(q-2)}) \\
- 2pq \Phi_{+- (p-1)(q-1)}.
\]

(4.9)

Using the first order equation (4.2) and the tracelessness condition (4.3), we get

\[
\nabla^2 \Phi(p)(q) = \Delta \Phi(p)(q) - 2s\Phi(p)(q) - 2pm\Phi_{(p-1)(q+1)} - 2qm\Phi_{(p+1)(q-1)} \\
- p(p-1)\Phi_{(p-2)(q+2)} - q(q-1)\Phi_{(p+2)(q-2)} - 2pq\Phi(p)(q).
\]

(4.10)

Since \( p + q = s \), the equation can also be written as

\[
\nabla^2 \Phi(p)(s-p) = \Delta \Phi(p)(s-p) - 2s\Phi(p)(s-p) - 2pm\Phi_{(p-1)(s-p+1)} - 2(s-p)m\Phi_{(p+1)(s-p-1)} \\
- p(p-1)\Phi_{(p-2)(s-p+2)} - (s-p)(s-p-1)\Phi_{(p+2)(s-p-2)} - 2p(s-p)\Phi(p)(s-p).
\]

(4.11)

Note that we have now obtained a set of closed equations for the components of the metric with only the boundary indices + or -.

### 4.2 Solutions of the spin-s components

In this section we will solve the set of \((s+1)\) coupled equations which we have derived in (4.11). From now on we will suppress the second label ‘(q)’ for the number of \(-s\) as it is understood we are working with a rank \(s\) symmetric tensor. The set of equations in (4.11) can also be written in the form of a matrix as,

\[
\nabla^2 \Phi(p) = \Delta \Phi(p) + M_{pr}^{(s)} \Phi(r).
\]

(4.12)

Here, the ‘mass matrix’ \(M_{pq}^{(s)}\) is defined as

\[
M_{pq}^{(s)} = -(2s + 2p(s-p))\delta_{p,r} - 2mp\delta_{p-1,r} - 2m(s-p)\delta_{p+1,r} \\
- p(p-1)\delta_{p-2,r} - (s-p)(s-p-1)\delta_{p+2,r}.
\]

(4.13)
Written out in the explicit matrix form, $M^{(s)}$ looks like

$$
M^{(s)} = \begin{pmatrix}
-2s & -2sm & -s(s-1) & 0 & 0 & \ldots & \ldots & \ldots \\
-2m-(4s-2) & -2m(s-1) & -(s-1)(s-2) & 0 & \ldots & \ldots & \ldots \\
-2 & -4m & -(6s-8) & -2m(s-2) & -(s-2)(s-3) & \ldots & \ldots & \ldots \\
0 & -6 & -6m & -(8s-18) & -2m(s-3) & \ldots & \ldots & \ldots \\
0 & 0 & -8 & -8m & -(10s-32) & \ldots & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ldots & -(4s-2) & -2m \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \ldots & -2sm & -2s
\end{pmatrix}
$$

(4.14)

Note that this is a $(s+1) \times (s+1)$ matrix.

**Diagonalization of the mass matrix**

From our study of the $s = 2$ case we have seen that we could diagonalize $M_{pq}^{(2)}$ by considering the linear combinations of the components obtained using the coordinate transformation

$$
x_1 = x_+ + x_- \quad , \quad x_2 = x_+ - x_- .
$$

We will see that we can use the same transformation to diagonalize the ‘mass matrix’ but generalized for the arbitrary spin case. This motivates us to consider the following linear combination

$$
\hat{\Phi}_{[p]} = \sum_{a=0}^{p} \sum_{b=0}^{s-p} (-1)^b \begin{pmatrix} p \\ a \end{pmatrix} \begin{pmatrix} s-p \\ b \end{pmatrix} \Phi_{(a-a-b)} .
$$

(4.15)

where $\hat{\Phi}$ is a symmetric rank $s$ tensor with indices 1 or 2. By ‘$[p]$’ we mean, there are $p$ number of 1s in the component of $\hat{\Phi}$ we are considering. This automatically implies that the are $s-p$ number of 2s in the components. We will show that this transformation diagonalizes the matrix $M_{pq}^{(s)}$. Let us write $\Phi_{[p]}$ as

$$
\hat{\Phi}_{[p]} = \sum_{q=0}^{s} T_{pq}^{(s)} \Phi_{(q)} .
$$

(4.16)

We can then write a generating function for the coefficient $T_{pq}^{(s)}$ by proceeding as follows. Consider the polynomial

$$
P_{(s,p)}(x) = \sum_{q=0}^{s} T_{pq}^{(s)} x^q = \sum_{a=0}^{p} \sum_{b=0}^{s-p} (-1)^b \begin{pmatrix} p \\ a \end{pmatrix} \begin{pmatrix} s-p \\ b \end{pmatrix} x^{(s-a-b)} ,
$$

(4.17)

Thus $T_{pq}^{(s)}$ is the coefficient of $x^q$ of the function above. A formal expression for $T_{pq}^{(s)}$ can then be obtained by a Taylor series expansion and be expressed as a contour integral.

$$
T_{pq}^{(s)} = \frac{1}{2\pi i} \oint \frac{dx}{x^{q+1}} P_{(s,p)}(x) .
$$

(4.18)
It can be shown that this transformation matrix \(T^{(s)}\) obeys the following identities.

**Identity 1:**

\[
\sum_{q=0}^{s} T^{(s)}_{pq} T^{(s)}_{qr} = 2^s \delta_{qr}. \tag{4.19}
\]

**Identity 2:**

\[
(T^{(s)} M^{(s)} [T^{(s)}]^{-1})_{pq} = -((2p - s)^2 + 2m(2p - s) - s) \delta_{pq}, \tag{4.20}
\]

i.e., \(T^{(s)} M^{(s)} [T^{(s)}]^{-1}\) is diagonal.

The proofs of these identities are provided in Appendix C.

Using the transformation \(T^{(s)}\), the Laplacian acting on \(\hat{\Phi}^{[p]}\) reduces to

\[
\nabla^2 \hat{\Phi}^{[p]} = (\Delta - (2p - s)^2 - 2m(2p - s) - s) \hat{\Phi}^{[p]}. \tag{4.21}
\]

Now including the spin dependent mass shift in the second order equations of motion given in (2.27) we obtain the following equations of motion for \(\hat{\Phi}^{[p]}\).

\[
(\Delta - (m + 2p - s)^2 + 1) \hat{\Phi}^{[p]} = 0. \tag{4.22}
\]

We make the ansatz,

\[
\Phi^{[p]} = e^{-i(k_+ x^+ + k_- x^-)} R^{[p]}(\xi). \tag{4.23}
\]

Substituting this ansatz and writing out the scalar Laplacian we obtain

\[
z(1 - z) \frac{d^2 R^{[p]}}{dz^2} + (1 - z) \frac{dR^{[p]}}{dz} + \left[ \frac{k_+^2}{4z} - \frac{k_-^2}{4} - \frac{(m + 2p - s)^2 - 1}{4(1 - z)} \right] R^{[p]} = 0. \tag{4.24}
\]

where \(z = \tanh^2 \xi\). The solution that obeys the ingoing boundary conditions at the horizon is given by

\[
R^{[p]}(z) = e^{[p] z^\alpha} (1 - z)^{\beta^{[p]}} F(a^{[p]}, b^{[p]}, c; z). \tag{4.25}
\]

where

\[
\alpha = \frac{-ik_+}{2}, \quad c = 1 + 2\alpha \tag{4.26}
\]

\[
\beta^{[p]} = \frac{1}{2} (1 + m + 2p - s)
\]

\[
a^{[p]} = \frac{k_+ - k_-}{2i} + \beta^{[p]}, \quad b^{[p]} = \frac{k_+ + k_-}{2i} + \beta^{[p]}. \]

As mentioned for the spin 2 case one can also work with the other choice of hypergeometric function given by the relation (3.32) which leads to equivalent results.
Determining the polarization constants $e_{[p]}$

We shall now find the coefficients or polarization constants, $e_{[p]}$ along the same lines of what we had done for the $s = 2$ case. Since these are constants we can determine them at any point in space. As we have done in the earlier section, we will see that it will be easy to determine them by examining the behaviour of the solutions near the horizon.

Near the horizon $z \to 0$, the behaviour of the solutions can be obtained by examining (4.25). Therefore we have

$$
\hat{\Phi}_{[p]}(z) \to e_{[p]} z^\alpha e^{i(k_+ x^+ + k_- x^-)}, \quad z \to 0.
$$

(4.27)

Since $\Phi_{(p)}$ is just a linear combination of $\hat{\Phi}_{[p]}(z)$ we also have

$$
\Phi_{(p)}(z) \to e_{(p)} z^\alpha e^{i(k_+ x^+ + k_- x^-)}, \quad z \to 0.
$$

(4.28)

where

$$
e_{(p)} = \frac{1}{2^s} \sum_{q=0}^{s} T_{pq}^{(s)} e_{[q]}.
$$

(4.29)

Here we have used (4.19) to write out the inverse transformation. Let us now examine the ‘$\xi_{(p)}(q)$’-component$^1$ of the equation (4.1) and see its consequences

$$
\epsilon_\xi^{\alpha\beta} \nabla_\alpha \Phi_{\beta(p)q} = -m \Phi_{\xi(p)q}.
$$

(4.30)

Expanding this equation and after some simple manipulations we obtain

$$
\partial_- \Phi_{(p+1)(q)} - \partial_+ \Phi_{(p)(q+1)} = -p \Gamma_{+++}^\xi \Phi_{\xi(p-1)(q+1)} - m \sqrt{-g} \Phi_{\xi(p)q} + q \Gamma_{--}^\xi \Phi_{\xi(p+1)(q-1)}.
$$

(4.31)

The near-horizon behaviour of the above equation is

$$
-ik_- \Phi_{(p+1)(q)} + ik_+ \Phi_{(p)(q+1)}
\simeq -\sqrt{z} (p \Phi_{\xi(p-1)(q+1)} + m \Phi_{\xi(p)q} - q \Phi_{\xi(p+1)(q-1)}).
$$

(4.32)

Thus, it’s evident that the near-horizon behaviour of $\Phi_{\xi(p)q}$ is given by

$$
\Phi_{\xi(p)q} \to e_{\xi(p)q} z^{\alpha - \frac{1}{2}} e^{-i(k_+ x^+ + k_- x^-)}, \quad z \to 0.
$$

(4.33)

Where $e_{\xi(p)q}$ is the polarization of $\Phi_{\xi(p)q}$. To proceed, we shall now prove the following identity

$$
e_{\xi(p)} = e_{(p+1)}.
$$

(4.34)

where, ‘$\xi(p)$’ in the subscript means, one of the indices is $\xi$, $p$ of the indices are + and the rest $s - p - 1$ indices are −. Let $q$ be the number of − indices we write. Now consider the ‘$-(p)q$’ component of the first order equation (4.1) which is given by

$$
\frac{g_{--}}{\sqrt{g}} (\nabla_+ \Phi_{\xi(p)q} - \nabla_\xi \Phi_{+(p)q}) = -m \Phi_{-(p)q}.
$$

(4.35)

$^1$It’s implied that $q = s - p - 1$. 
Expanding this equation and rearranging the terms we obtain
\[ \partial_+ \Phi_{\xi(p)(q)} - 2\sqrt{z}(1 - z)\partial_z \Phi_{+(p)(q)} + \sqrt{z}(p\Phi_{(p-1)(q+2)} - q\Phi_{(p+1)(q)}) = -m\sqrt{z}\Phi_{(p)(q+1)}. \]
(4.36)

In the simplifications we have also used the traceless condition (4.33). Near the horizon, \( z \to 0 \), the first two terms of (4.36) go as \( \sim z^{\alpha - \frac{1}{2}} \), while the other terms have the behaviour \( \sim z^{\alpha + \frac{1}{2}} \). Since the equation in (4.36) must hold in the leading order in \( z \) we must have the following equality
\[ \partial_+ \Phi_{\xi(p)(q)} = 2\sqrt{z}\partial_z \Phi_{+(p)(q)}. \]
(4.37)

Substituting the behaviour near the horizon we obtain
\[ -ik_+ c_{\xi(p)(q)} = 2\alpha c_{+(p)(q)}, \]
which implies
\[ c_{\xi(p)(q)} = c_{+(p)(q)}. \]
(4.39)

Substituting this result in (4.32), we obtain a recursion relation between the coefficients \( c_{(p)} \) as follows
\[ (s - p - 1)c_{(p+2)} + (m - ik_-)c_{(p+1)} + (p + ik_+)c_{(p)} = 0. \]
(4.40)

This is set of \( s \) equations in \( s + 1 \) variables. Defining the ‘recursion matrix’, \( C_{jl} \) as
\[ C_{jl}^{(s)} = (s - j - 1)\delta_{j+2,l} + (m - ik_-)\delta_{j+1,l} + (j + ik_+)\delta_{j,l}. \]
(4.41)

Here, \( j \) runs from \( s - 1 \) to \( 0 \) and \( l \) runs from \( s \) to \( 0 \). We can write, (4.40) as
\[ \sum_{l=0}^{s} C_{jl}^{(s)} e_{(l)} = 0 \quad \text{for} \quad j = 0, 1, 2, \ldots , s - 1. \]
(4.42)

This \( s \times (s + 1) \) matrix \( C^{(s)} \) in which the rows run from \( s - 1 \) to \( 0 \) and columns run from \( s \) to \( 0 \) can be written explicitly in matrix form as
\[
C^{(s)} = \begin{pmatrix}
(m - ik_- s - 1 + ik_+) & 0 & 0 & \cdots & \cdots & \cdots \\
1 & m - ik_- s - 2 + ik_+ & 0 & \cdots & \cdots & \cdots \\
0 & 2 & m - ik_- s - 3 + ik_+ & \cdots & \cdots & \cdots \\
& & & & & \\
& & & & & \\
& & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & m - ik_- 1 + ik_+ & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & s - 1 & m - ik_- ik_+
\end{pmatrix}
\]
(4.43)

We now want to write (4.42) in terms of the coefficients of the \( \Phi_{[p]} \). Using (4.29) in (4.42) we get
\[ \sum_{l=0}^{s} C_{jl}^{(s)} T_{ln}^{(s)} e_{[n]} = 0. \]
(4.44)
or, writing without the matrix labels
\[ C^{(s)} T^{(s)} e = 0, \quad (4.45) \]
where by ‘e’ we mean the column matrix containing the coefficients, \( e_{[p]} \). To simplify
the recursion relations we use the following identity which is proved in Appendix C.

**Identity 3:**
\[
\frac{1}{2s-1} (T^{(s-1)} C^{(s)} T^{(s)})_{j,l} = (2j - s + 1 + m + i(k_+ - k_-))\delta_{j+1,l} + (2j - s + 1 + m - i(k_+ + k_-))\delta_{j,l}. \quad (4.46)
\]

So multiplying (4.45) by \( T^{(s-1)} \) and using (4.46) we obtain
\[
(T^{(s-1)} C^{(s)} T^{(s)})_{j,l} e[l] = (2j - s + 1 + m + i(k_+ + k_-))e_{[j+1]} + (2j - s + 1 + m - i(k_+ - k_-))e_{[j]} = 0, \quad (4.47)
\]
or,
\[
e_{[j+1]} = - \frac{2j - s + 1 + m - i(k_+ + k_-)}{2j - s + 1 + m + i(k_+ - k_-)} e_{[j]}. \quad (4.48)
\]

Thus, all such coefficients can be written terms of \( e_{[0]} \) as
\[
e_{[p]} = (-1)^p \prod_{j=0}^{p-1} \frac{2j - s + 1 + m - i(k_+ + k_-)}{2j - s + 1 + m + i(k_+ - k_-)} e_{[0]} \quad (4.49)
\]

Substituting this in the solution (4.25) we get the required solutions.
\[
R_{[p]}(z) = e_{[0]}(-1)^p \prod_{j=0}^{p-1} \frac{2j - s + 1 + m - i(k_+ + k_-)}{2j - s + 1 + m + i(k_+ - k_-)} z^{\alpha} (1 - z)^{\beta_{[p]}} F(a_{[p]}, b_{[p]}, c; z). \quad (4.50)
\]

### 4.3 Quasinormal modes

To obtain the quasinormal modes we need to impose the vanishing Dirichlet condition
at the boundary \( z \to 1 \). For the case \( m > 0 \), the dominant behaviour of the solutions
near the boundary is given by
\[
R_{[p]}(z) \simeq e_{[0]}(-1)^p \prod_{j=0}^{p-1} \frac{2j - s + 1 + m - i(k_+ + k_-)}{2j - s + 1 + m + i(k_+ - k_-)} \times z^{\alpha} (1 - z)^{\beta_{[p]}} \frac{\Gamma(c) \Gamma(a_{[p]} + b_{[p]} - c)}{\Gamma(a_{[p]}) \Gamma(b_{[p]})}. \quad (4.51)
\]

To find the quasinormal modes we need obtain the zeros of this function. Notice that
\[
b_{[p]} = b_{[0]} + p. \quad (4.52)
\]
Therefore we have
\[
\Gamma(b_{[p]}) = \Gamma(b_{[0]}) \prod_{u=0}^{p-1} (b_{[0]} + u),
\]
\[
= \frac{\Gamma(b_{[0]})}{2^p} \prod_{u=0}^{p-1} [2u - s + 1 + m - i(k_+ + k_-)].
\]
(4.53)
Substituting the above expression for the gamma function in (4.51), we obtain
\[
R_{[p]}(z) \simeq \frac{e_{[0]}(-2)^p}{\prod_{j=0}^{p-1} [2j - s + 1 + m + i(k_+ - k_-)]} \frac{\Gamma(c)\Gamma(a_{[p]} + b_{[p]} - c)}{\Gamma(a_{[p]})\Gamma(b_{[0]})} z^\alpha (1 - z)^{\frac{1-m-2p+s}{2}}.
\]
(4.54)
It is understood here that the product in the denominator doesn’t occur at all for \( p = 0 \).

In order to impose vanishing Dirichlet conditions on the propagating components of the spin-\( s \) field at infinity, we require all the above functions which form the components of the spin-\( s \) field to vanish. The common set of zeros for all values of \( p \) is given by
\[
a_s = -n \quad \text{and} \quad b_{[0]} = -n, \quad n = 0, 1, 2, \ldots.
\]
(4.55)
In terms of \( k_+ \) and \( k_- \) these mean,
\[
i(k_+ + k_-) = 2n + \hat{\Delta} - s, \quad i(k_+ - k_-) = 2n + \hat{\Delta} + s.
\]
(4.56)
Hence, the quasinormal modes of an arbitrary spin-\( s \) field are
\[
\omega_L = k + 2\pi T_L(k_+ + k_-) \quad \omega_R = -k + 2\pi T_R(k_+ - k_-)
\]
\[
= k - 2\pi i T_L(2n + \hat{\Delta} - s), \quad = -k - 2\pi i T_R(2n + \hat{\Delta} + s).
\]
(4.57)
These coincide precisely with the poles of the corresponding two point function (2.49) for the corresponding spin \( s \) field as expected from the AdS/CFT correspondence. Reading out \( h_L \) and \( h_R \) we get \( h_R - h_L = s \), the case \( h_R - h_L = -s \) will arise when we carry out the same analysis but with \( m < 0 \).

5. Higher spin 1-loop determinants
The poles of the retarded Green’s function contain important physical information of the theory. As emphasized in [12], they can be used to construct the one-loop determinant of the corresponding field in the bulk. [12] constructed the one-loop determinant for scalars in asymptotically AdS black holes including the BTZ black hole using analyticity arguments and the information of the quasinormal modes. In this section we would like to use the quasinormal modes of the spin \( s \) fields along with the analyticity to construct the one loop determinant of the corresponding spin field. We then show that this determinant agrees with that constructed in [13] using group theoretic methods.
5.1 1-loop determinant from the quasinormal spectrum

Following [12] we consider the non-rotating BTZ black hole for which the metric is given by

$$ds^2 = -(r^2 - r_+^2)dt^2 + \frac{dr^2}{(r^2 - r_+^2)} + r^2d\phi^2 \quad (5.1)$$

We then continue the BTZ black hole to Euclidean time together with the identification

$$t = -i\tau, \quad \tau \sim \tau + \frac{1}{T}. \quad (5.2)$$

and

$$T = T_H = T_L = \frac{r_+}{2\pi}. \quad (5.3)$$

We are interested in evaluating the following one loop determinant of the spin $s$ Laplacian given by

$$Z_s(\Delta) = \frac{1}{2} \frac{1}{\det(-\nabla^2(s) + m_s^2)}. \quad (5.4)$$

The basic strategy is to identify the poles of the determinant in the complex $\Delta$ space where $\Delta$ is the conformal dimension of the corresponding dual operator. This occurs whenever the wave equation of the corresponding field has a zero mode and also obeys the periodicity (5.2). It is argued in [12] that these zero modes are precisely the black hole quasinormal modes. For the case of spin $s$ field in the non-rotating BTZ background, these modes are given by

$$\omega_L = p - 2\pi iT(2n + \Delta - s), \quad \omega_R = -p - 2\pi iT(2n + \Delta + s), \quad n = 0, 1, 2, \ldots \quad (5.5)$$

where $p$ is the momentum along the $\theta$. It is quantized and therefore takes values in the set of integers

$$p = 0, \pm 1, \pm 2, \ldots \quad (5.6)$$

To proceed further, let us define the following

$$z_L = p - 2\pi iT(2n + \Delta - s), \quad \bar{z}_L = p + 2\pi iT(2n + \Delta - s), \quad (5.7)$$

$$z_R = -p - 2\pi iT(2n + \Delta + s), \quad \bar{z}_R = -p + 2\pi iT(2n + \Delta + s).$$

Requiring the quasinormal modes to obey the thermal periodicity conditions given in (5.2) results in the following equations [12]

$$2\pi iT(\tilde{n} + s) = z_L(\Delta), \quad \tilde{n} \geq 0, \quad (5.8)$$

$$2\pi iT(\tilde{n} - s) = \bar{z}_L(\Delta), \quad \tilde{n} < 0$$

$$2\pi iT(\tilde{n} - s) = z_R(\Delta), \quad \tilde{n} \geq 0,$$

$$2\pi iT(\tilde{n} + s) = \bar{z}_R(\Delta), \quad \tilde{n} < 0.$$
These equalities are taken to mean, that when $\hat{\Delta}$ is tuned to these integral values, the one loop determinant exhibits poles. The ranges of $\tilde{n}$ are chosen so that the quantities

$$p - 2\pi iT(2n + \hat{\Delta}), \quad \text{and} \quad p + 2\pi iT(2n + \hat{\Delta}),$$

when considered together take values $2\pi T\tilde{n}$ where $\tilde{n}$ assumes values in the set of integers. Similarly the range of $\tilde{n}$ for the case of the right-moving quasinormal modes is chosen so that the quantities,

$$-p - 2\pi iT(2n + \hat{\Delta}), \quad \text{and} \quad -p + 2\pi iT(2n + \hat{\Delta}),$$

when considered together takes the values $2\pi T\tilde{n}$ where $\tilde{n}$ assumes values in the set of integers. Another requirement satisfied by the ranges in (5.8) is that they reduce to that of the scalar when $s = 0$. Though we do not have a first principle justification of the choice of these ranges we will show that they do indeed lead to the answer evaluated using the group theoretic methods given in [13]. The function which is analytic in $\Delta$ and has poles at the locations (5.8) is given by

$$Z(s) = e^{\text{Pol}(\Delta)} \prod_{z_L, \bar{z}_L} \left[ \left( s + \frac{iz_L}{2\pi T} \right)^{-1} \left( -s - \frac{i\bar{z}_L}{2\pi T} \right)^{-1} \right] \times \prod_{n \geq s} \left( n + \frac{iz_L}{2\pi T} \right)^{-1} \left( n - \frac{i\bar{z}_L}{2\pi T} \right)^{-1} \prod_{n \geq -s} \left( n + \frac{iz_R}{2\pi T} \right)^{-1} \left( n - \frac{i\bar{z}_R}{2\pi T} \right)^{-1} \], \quad (5.9)$$

where Pol$(\Delta)$ is a non-singular holomorphic function of $\Delta$ and can be determined by examining the $\Delta \to \infty$ behaviour. The product over $z_L, \bar{z}_L, z_R, \bar{z}_R$ mean the product over $p, n, \tilde{n}$ occurring in the definition of these variables. Using the following relations

$$i(z_L + \bar{z}_R) = -2\pi T(2s), \quad i(\bar{z}_L + z_R) = 2\pi T(2s). \quad (5.10)$$

we can write the one loop determinant as

$$Z(s) = e^{\text{Pol}(\Delta)} \prod_{z_L, \bar{z}_L} \left[ \left( s + \frac{iz_L}{2\pi T} \right)^{1/2} \left( s - \frac{i\bar{z}_L}{2\pi T} \right)^{1/2} \left( -s + \frac{iz_R}{2\pi T} \right)^{1/2} \left( -s - \frac{i\bar{z}_R}{2\pi T} \right)^{1/2} \right] \times \prod_{n \geq s} \left( n + \frac{iz_L}{2\pi T} \right)^{-1} \left( n - \frac{i\bar{z}_L}{2\pi T} \right)^{-1} \times \prod_{n \geq -s} \left( n + \frac{iz_R}{2\pi T} \right)^{-1} \left( n - \frac{i\bar{z}_R}{2\pi T} \right)^{-1}. \quad (5.11)$$
Note that according to the analysis of [12] the expression in (5.11) is the partition function for a complex field and therefore one needs to take a square root for a real field. However for our higher spin \( s > 0 \), we have two modes, corresponding to \( h_R - h_L = \pm s \). Taking into account both these modes results in the expression given in (5.11). We now substitute in the values of the left and right quasinormal modes and perform the same manipulations as in [12] to obtain

\[
Z(s) = e^{\text{Pol}(\Delta)} \prod_{N \geq 0, p} \left[ \left( 2N + \hat{\Delta} \right)^2 + \frac{p^2}{(2\pi T)^2} \right]^{-2^{-}}. \tag{5.12}
\]

Taking logarithms on both sides of the equation lead to

\[
-\log Z(s) = -\text{Pol}(\Delta) + 2 \sum_{n \geq 0, N \geq 0, p} \log \left( (n + 2N + \hat{\Delta})^2 + \frac{p^2}{(2\pi T)^2} \right) \\
- \sum_{N \geq 0, p} \log \left( (2N + \hat{\Delta})^2 + \frac{p^2}{(2\pi T)^2} \right) \\
= -\text{Pol}(\Delta) + 2 \sum_{n \geq 0, N \geq 0, p} \log \left( (n + 2N + \hat{\Delta})^2 + \frac{p^2}{(2\pi T)^2} \right) \\
+ \sum_{N \geq 0, p} \log \left( (2N + \hat{\Delta})^2 + \frac{p^2}{(2\pi T)^2} \right) \\
= -\text{Pol}(\Delta) + \sum_{\kappa \geq 0, p} (\kappa + 1) \log \left( (\kappa + \hat{\Delta})^2 + \frac{p^2}{(2\pi T)^2} \right). \tag{5.13}
\]

Here the sums over \( n \) and \( N \) were combined and written as a sum over \( \kappa = n + 2N = 0, 1, \ldots \). The factor \( \kappa + 1 \) accounts for the multiplicity. We have also made use of \( \log(a + ib) + \log(a - ib) = \log(a^2 + b^2) \). We then extract out the divergent terms which are absorbed in Pol(\( \Delta \)) and make use of the identity \( \sum_{p \geq 1} \log \left( 1 + \frac{x^2}{p^2} \right) = \log \frac{\sinh \pi x}{\pi x} = \pi x - \log(\pi x) + \log(1 - e^{-2\pi x}) \) to obtain

\[
\log Z(s) = \text{Pol}(\Delta) + 2 \log \prod_{\kappa \geq 0} \left( 1 - q^{-\kappa + \hat{\Delta}} \right)^{-(\kappa + 1)}. \tag{5.14}
\]

Here

\[
q = e^{2\pi i \bar{\tau}}, \quad \bar{\tau} = 2\pi i T. \tag{5.15}
\]

Now we can use the same argument as in [12] to determine Pol(\( \Delta \)). Taking \( \hat{\Delta} \to \infty \), the partition function should reduce to that of the BTZ which is locally identical to that of AdS\(_3\). This determines Pol(\( \Delta \)) to be a function proportional to the volume of the Euclidean BTZ black hole. We will not write this explicitly since we do not require it in the subsequent discussion.
5.2 1-loop determinant from the heat kernel

We will now show that the term in the one loop partition function (5.14) determined from the quasinormal modes agrees with that constructed from the heat kernel of the spin $s$ field. The trace of the heat kernel for the spin $s$ Laplacian on thermal $AdS_3$ is given by [13],

$$\text{Tr}(e^{-t\nabla^2}) = K^{(s)}(\tau, \bar{\tau}; t) = \sum_{n=1}^{\infty} \frac{\tau_2}{\sqrt{4\pi t} \sin \frac{n\pi}{2}} \cos(sn\tau_1)e^{-\frac{n^2\beta^2}{4t}}e^{-(s+1)t}. \quad (5.16)$$

Note that in the above expression we have suppressed the term which is proportional to the volume of the $AdS_3$. Here $\tau$ is related to the temperature of the Euclidean non-rotating BTZ by,

$$\tau = \frac{i}{2\pi T}. \quad (5.17)$$

We will substitute this value of $\tau$ towards the end of our analysis. The 1-loop determinant is then given by

$$-\frac{1}{2} \log(\det(-\nabla^2 + m_s^2)) = \frac{1}{2} \int_0^{\infty} \frac{dt}{t} e^{-m_s^2 t} K^{(s)}(\tau, \bar{\tau}; t). \quad (5.18)$$

where $m_s^2$ is defined in (2.22). Substituting the values of $m_s$ and using the definition of $\Delta$ given in (2.43) we obtain

$$e^{-m_s^2 t} K^{(s)}(\tau, \bar{\tau}; t) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\tau_2}{\sqrt{4\pi t} \sin \frac{n\pi}{2}} \cos(sn\tau_1)e^{-\frac{n^2\beta^2}{4t} e^{-(\Delta-1)^2 t}}. \quad (5.19)$$

The integration over $t$ can be easily performed using

$$\frac{1}{\sqrt{4\pi}} \int_0^{\infty} \frac{dt}{t^{3/2}} e^{-\frac{\alpha^2}{4t} - \beta^2 t} = \frac{1}{\alpha} e^{-\alpha \beta}. \quad (5.20)$$

we then obtain

$$-\frac{1}{2} \log(\det(-\nabla^2 + m_s^2)) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\cos(sn\tau_1)}{n |\sin \frac{n\pi}{2}|} e^{-n \tau_2 (\Delta-1)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \frac{(q^n + \bar{q}^n)}{|1 - q^n|^2} q^{(\Delta-s)n}$$

$$= \sum_{n=1}^{\infty} \frac{2q^n}{n (1 - q^n)^2}$$

$$= -2 \log \prod_{m=0}^{\infty} (1 - q^{m+\Delta})^{m+1}. \quad (5.21)$$

where

$$q = e^{2\pi i r}. \quad (5.22)$$
In the third step we have used the fact that $\tau$ is purely imaginary for the case of the non-rotating BTZ black hole which results in $q = \bar{q}$. Therefore we obtain

$$\log Z_s = -\frac{1}{2} \log(\det(-\nabla^2 + m_s^2)) = -2 \log \prod_{m=0}^{\infty} (1 - q^{-m+\Delta})^{m+1}. \quad (5.23)$$

Comparing the (5.14) and (5.23) we see that the two expressions indeed agree on the performing the modular transformation,

$$\tilde{\tau} = -\frac{1}{\tau}. \quad (5.24)$$

which is the expected relation between the one-loop determinants on Euclidean BTZ and thermal $AdS_3$.

6. Conclusions

We have solved the wave equations for massive higher integer spin fields in the BTZ background. To obtain the quasinormal modes we have focused only on the ingoing modes at the horizon, but the analysis can easily be carried out for the outgoing modes. This will lead to the complete set of modes for the massive higher spin field which is the starting point to quantize the field in this background. It will be useful to carry out this analysis in detail to study quantum properties of the black hole like Hawking radiation. From our discussion of the wave equations, it seems that other quantities like the bulk to boundary propagator for massive higher spin fields in $AdS_3$ can also be solved. These are important tools to study the $AdS_3/CFT_2$ correspondence for massive higher spin fields and useful to obtain them. Another obvious direction to extend this work is to consider the case of the fermionic higher spin fields.

At present there are only a few examples of black hole backgrounds known in which the string world sheet theory can be quantized. The 2d black hole \cite{23,24} and the BTZ background with NS flux \cite{25} are the well known cases. The analysis of this paper and the fact the string motion in the BTZ black hole is integrable \cite{4} suggests that it might be possible to quantize the string in this background which is clearly an interesting direction to pursue.

Acknowledgments

We wish to thank Rajesh Gopakumar, Gautam Mandal, Shiraz Minwalla, Spenta Wadia and for useful discussions regarding the general properties of quasinormal modes in higher spin theories. We also thank B. Ananthanarayan, Kavita Jain and Sujit Nath for helpful discussions on some technical aspects of this problem. We thank the
the International Centre of Theoretical Sciences of the TIFR for organizing a stimulating meeting at IISc, Bangalore for facilitating this discussion. The work of J.R.D is partially supported by the Ramanujan fellowship DST-SR/S2/RJN-59/2009.

A. The vector case

The spin-1 case is a special case which does not fall in our general description of higher spin fields. This is because we can impose the tracelessness condition for fields with \( s \geq 2 \). Nevertheless, in this section we shall use the similar methods developed for the graviton and for higher spin cases to calculate the quasinormal modes of the \( s = 1 \) field. We use the Chern-Simons equations of motion for a massive gauge field

\[
\epsilon_{\mu}^{\alpha\beta} \partial_{\alpha} A_{\beta} = -mA_{\mu}. \tag{A.1}
\]

Using this we obtain the following equations for the components

\[
\frac{g_{--}}{\sqrt{-g}} (\partial_{+} A_{z} - \partial_{z} A_{+}) = -m A_{-}, \tag{A.2}
\]

\[
\frac{g_{++}}{\sqrt{-g}} (\partial_{z} A_{-} - \partial_{-} A_{z}) = -m A_{+}. \tag{A.3}
\]

It can be shown that the above equation is equivalent to second order massive spin equations for the vector field, with \( m = M \).

Reduction of the vector Laplacian to the scalar Laplacian

We will now show that the spin-1 Laplacian acting on components \( A_{+} \) and \( A_{-} \) can be reduced to the scalar Laplacian

\[
\nabla^{2} A_{+} = \frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} g^{\mu\nu} \partial_{\nu} A_{+}) - \frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} g^{\mu\nu} \Gamma^\sigma_{\nu} - 2 \Gamma^\sigma_{\mu+} g^{\mu\nu} \partial_{\nu} A_{\sigma} + \Gamma^\sigma_{\mu+} g^{\mu\nu} \Gamma^\alpha_{\nu} A_{\alpha})
\]

\[
= \frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} g^{\mu\nu} \partial_{\nu} A_{+}) - \frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} g^{\mu\nu} \Gamma^\sigma_{\nu} - 2 \Gamma^\sigma_{\mu+} g^{\mu\nu} \partial_{\nu} A_{\sigma} + \Gamma^\sigma_{\mu+} g^{\mu\nu} \Gamma^\alpha_{\nu} A_{\alpha})
\]

\[
= \Delta A_{+} - 2A_{+} + 2 \coth \mu (\partial_{+} A_{\xi} - \partial_{\xi} A_{+}) + (\Gamma^{\xi}_{++} + \Gamma^{\xi}_{\Xi+}) \Gamma^{\Xi+}_{\Xi} A_{+}
\]

\[
= \nabla^{2} A_{+} = \Delta A_{+} - 2A_{+} + 2m A_{-}.
\]

\[
\nabla^{2} A_{-} = \frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} g^{\mu\nu} \partial_{\nu} A_{-}) - \frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} g^{\mu\nu} \Gamma^\sigma_{\nu} - 2 \Gamma^\sigma_{\mu-} g^{\mu\nu} \partial_{\nu} A_{\sigma} + \Gamma^\sigma_{\mu-} g^{\mu\nu} \Gamma^\alpha_{\nu} A_{\alpha})
\]

\[
= \frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} g^{\mu\nu} \partial_{\nu} A_{-}) - \frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} g^{\mu\nu} \Gamma^\sigma_{\nu} - 2 \Gamma^\sigma_{\mu-} g^{\mu\nu} \partial_{\nu} A_{\sigma} + \Gamma^\sigma_{\mu-} g^{\mu\nu} \Gamma^\alpha_{\nu} A_{\alpha})
\]

\[
= \Delta A_{-} - 2A_{-} + 2 \tanh \mu (\partial_{-} A_{\xi} - \partial_{\xi} A_{-}) + (\Gamma^{\xi}_{--} + \Gamma^{\xi}_{\Xi-}) \Gamma^{\Xi-}_{\Xi} A_{-}
\]

\[
= \Delta A_{-} - 2A_{-} + 2m A_{+}.
\]
where we have used (A.2) and (2.10) to obtain the last equality. To summarize for the spin-1 case, the spin one Laplacian reduces to the scalar Laplacian as follows

\[ \nabla^2 A_+ = \Delta A_+ - 2mA_+ - 2A_+ , \quad (A.6) \]
\[ \nabla^2 A_- = \Delta A_- - 2A_- - 2mA_- . \]

with

\[ \Delta A_\alpha = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{gg}{}^{\mu\nu} \partial_\nu A_\alpha) . \quad (A.7) \]

**Solutions of the spin-1 components**

The equations in (A.6) can be decoupled by considering the combinations \( A_{1,2} = A_+ \pm A_- \). We then have

\[ \nabla^2 A_1 = (\Delta - 2m - 2)A_1 , \quad (A.8) \]
\[ \nabla^2 A_2 = (\Delta + 2m - 2)A_2 . \quad (A.9) \]

The second order equation of motion satisfied by the spin one field is given by

\[ (\nabla^2 - m^2 + 2)A_\mu = 0. \quad (A.10) \]

Using (A.8) and (A.9) we obtain the following equations of motion for the components \( A_1, A_2 \).

\[ (\Delta - (m + 1)^2 + 1)A_1 = 0 , \quad (A.11) \]
\[ (\Delta - (m - 1)^2 + 1)A_2 = 0 . \]

We now substitute the ansatz

\[ A_1 = e_1 e^{-i(k_+ x^+ + k_- x^-)} R_1(\xi) , \quad (A.12) \]
\[ A_2 = e_2 e^{-i(k_+ x^+ + k_- x^-)} R_2(\xi) . \]

into the equations (A.11). Expanding out the scalar Laplacian result in the following equations

\[ z(1 - z) \frac{d^2 R_1}{dz^2} + (1 - z) \frac{dR_1}{dz} + \left[ \frac{k_+^2}{4z} - \frac{k_-^2}{4} - \frac{(m + 1)^2 - 1}{4(1 - z)} \right] R_1 = 0 , \quad (A.13) \]
\[ z(1 - z) \frac{d^2 R_2}{dz^2} + (1 - z) \frac{dR_2}{dz} + \left[ \frac{k_+^2}{4z} - \frac{k_-^2}{4} - \frac{(m - 1)^2 - 1}{4(1 - z)} \right] R_2 = 0 . \]

where \( z = \tanh^2 \xi \). The solution that obeys ingoing boundary conditions at the horizon is given by

\[ R_1(z) = e_1 z^\alpha (1 - z)^{\beta_1} F(a_1, b_1, c; z) , \quad (A.14) \]
\[ R_2(z) = e_2 z^\alpha (1 - z)^{\beta_2} F(a_2, b_2, c; z) . \]
where
\[
\alpha = \frac{-ik_+}{2}, \quad c = 1 + 2\alpha, \quad (A.15)
\]
\[
\beta_1 = \frac{m}{2} + 1, \quad \beta_2 = \frac{m}{2},
\]
\[
a_{1,2} = \frac{k_+ - k_-}{2i} + \beta_{1,2}, \quad b_{1,2} = \frac{k_+ + k_-}{2i} + \beta_{1,2}.
\]

The polarization constants \(e_1, e_2\) are not independent by are determined by the first order equation (A.1). Since they are constants to determine it is sufficient to examine the behaviour of the solutions near the horizon \(z \to 0\). This is given by
\[
A_{1,2} \to e_{1,2}z^\alpha e^{-i(k_+ x^+ + k_- x^-)}, \quad z \to 0.
\]

Therefore we also have the relation,
\[
A_\pm \to (e_1 \pm e_2)z^\alpha e^{-i(k_+ x^+ + k_- x^-)}, \quad z \to 0.
\]

The \(A_\xi\) component can be obtained from (A.1), which is given by
\[
A_\xi = \frac{1}{m \cosh \xi \sinh \xi} \partial_+ A_- | \quad (A.18)
\]
\[
= \frac{i(1 - z)}{2m \sqrt{z}} (k_- A_+ - k_+ A_-).
\]

The above equation leads to the following near horizon behaviour of \(A_\xi\)
\[
A_\xi \to \frac{i}{2m} ((k_- - k_+)e_1 + (k_- + k_+)e_2)z^{\alpha - \frac{1}{2}} e^{-i(k_+ x^+ + k_- x^-)}, \quad z \to 0,
\]
\[
= e_\xi z^{\alpha - \frac{1}{2}} e^{-i(k_+ x^+ + k_- x^-)}.
\]

Let’s now consider the (−) component of the first order equation
\[
-m A_- = \epsilon_{-\alpha\beta} \partial_\alpha A_\beta \quad (A.20)
\]
\[
= -2(1 - z)\partial_+ A_+ + \frac{1}{\sqrt{z}} \partial_\xi A_\xi.
\]

Examining the above equation near the horizon \(z \to 0\), the terms on the right tend to \(\sim z^{\alpha - 1}\). Thus they are dominant compared to that on the left (\(\sim z^\alpha\)). Substituting the behaviour of \(A_+\) and \(A_\xi\) near the horizon we obtain the relation
\[
-ik_+ e_\xi = \alpha (e_1 + e_2),
\]

which results in
\[
e_\xi = \frac{e_1 + e_2}{2}.
\]
Using the above equation in (A.19) we obtain
\[ \frac{e_1}{e_2} = \frac{i(k_+ + k_-) - m}{i(k_+ - k_-) + m}. \]  
(A.23)

Finally substituting this in the solution (A.14) we obtain the following solutions
\[ R_1(z) = e_2 \frac{i(k_+ + k_-) - m}{i(k_+ - k_-) + m} z^\alpha (1 - z)^{\beta_1} F(a_1, b_1, c; z), \]  
(A.24)
\[ R_2(z) = e_2 z^\alpha (1 - z)^{\beta_2} F(a_2, b_2, c; z). \]  
(A.25)

Quasinormal modes

Quasinormal modes are obtained by imposing vanishing Dirichlet boundary conditions at the boundary \( z \to 1 \). The leading behaviour of (A.24) at the boundary for \( m > 0 \) is given by
\[ R_1(z) \approx e_2 \frac{i(k_+ + k_-) - m \Gamma(c) \Gamma(a_1 + b_1 - c)}{i(k_+ - k_-) + m \Gamma(a_1) \Gamma(b_1)} (1 - z)^{-\frac{m}{2}}, \]  
(A.26)
\[ R_2(z) \approx e_2 (1 - z)^{1 - \frac{m}{2}} \frac{\Gamma(c) \Gamma(a_2 + b_2 - c)}{\Gamma(a_2) \Gamma(b_2)} (1 - z)^{1 - \frac{m}{2}}. \]  
(A.27)

The common set of zeros of the above functions are given by
\[ b_1 - 1 = b_2 = -n, \quad a_1 = -n, \quad n = 0, 1, 2, \ldots. \]  
(A.28)

In terms of \( k_+ \) and \( k_- \) these are
\[ i(k_+ + k_-) = 2n + \hat{\Delta} - 2, \quad i(k_+ - k_-) = 2n + \hat{\Delta} + 2. \]  
(A.29)

where \( \hat{\Delta} = 1 + m \). Expressing \( k_+ \) and \( k_- \) in terms of frequency and momenta we see that the quasinormal modes for the gauge field are given by
\[ \omega_L = k + 2\pi T_L (k_+ + k_-), \quad \omega_R = -k + 2\pi T_R (k_+ - k_-) \]  
(A.30)
\[ = k - 2\pi i T_L (2n + \hat{\Delta} - 1), \quad = -k - 2\pi i T_R (2n + \hat{\Delta} + 1). \]

These modes coincide with the poles of the two point function (2.49) for the \( s = 1 \) case. Note that we have obtained the situation with \( h_R - h_L = 1 \). If we perform the same analysis with \( m < 0 \) we will obtain \( h_R - h_L = -1 \).

B. The spin 3 case

In this section we shall find the quasinormal modes of the spin-3 field. This is to demonstrate the methods developed for the arbitrary-\( s \) calculations explicitly. The first order equation for this case is
\[ \epsilon^{\alpha\beta}_{\mu} \nabla_\alpha \phi_{\beta\nu\eta} = -m \phi_{\mu\nu\eta}. \]  
(B.1)
Reduction of the spin-3 Laplacian to the scalar Laplacian

From (4.11) we get the following expressions for the Laplacian acting on various components of \( \phi_{\mu \nu \eta} \) in terms of the scalar Laplacian. These are

\[
\nabla^2 \phi_{+++} = \Delta \phi_{+++} - 6 \phi_{+++} - 6m \phi_{++-} - 6 \phi_{++-}, \\
\nabla^2 \phi_{--=} = \Delta \phi_{--=} - 6 \phi_{--=} - 6m \phi_{+-} - 6 \phi_{+-}, \\
\nabla^2 \phi_{++-} = \Delta \phi_{++-} - 6 \phi_{++-} - 4m \phi_{--=} - 2m \phi_{+-} - 2 \phi_{--=} - 4 \phi_{+-}, \\
\nabla^2 \phi_{--=} = \Delta \phi_{--=} - 6 \phi_{--=} - 4m \phi_{++-} - 2m \phi_{++-} - 2 \phi_{++-} - 4 \phi_{--=}.
\]

(B.2) (B.3) (B.4) (B.5)

Rewriting the above equation in the form of a matrix as in (4.12) results in

\[
\nabla^2 \phi(p) = \Delta \phi(p) + M^{(3)}_{pq} \phi(q)
\]

(B.6)

\[
\nabla^2 \begin{pmatrix} \phi_{+++} \\ \phi_{++-} \\ \phi_{--=} \\ \phi_{++-} \end{pmatrix} = \Delta \begin{pmatrix} \phi_{+++} \\ \phi_{++-} \\ \phi_{--=} \\ \phi_{++-} \end{pmatrix} + \begin{pmatrix} -6 & -6m & -6 & 0 \\ -2m & -10 & -4m & -2 \\ -2 & -4m & -10 & -2m \\ 0 & -6 & -6m & -6 \end{pmatrix} \begin{pmatrix} \phi_{++-} \\ \phi_{--=} \end{pmatrix}.
\]

(B.7)

Diagonalization of the mass matrix

We now require to diagonalize the mass matrix. The transformation matrix that diagonalizes the \( M^{(3)} \) in (B.6) can be obtained from (4.15). This is given by,

\[
T^{(3)} = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix}.
\]

(B.8)

Then the linear combination for the components of the field for which the equations decoupled is

\[
\begin{pmatrix} \phi_{111} \\ \phi_{112} \\ \phi_{122} \\ \phi_{222} \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix} \begin{pmatrix} \phi_{+++} \\ \phi_{++-} \\ \phi_{--=} \\ \phi_{++-} \end{pmatrix}.
\]

(B.9)

It can then be seen by explicit matrix multiplication that

\[
T^{(3)} M^{(3)} (T^{(3)})^{-1} = \begin{pmatrix} -12 & -6m & 0 & 0 \\ 0 & -4 & -2m & 0 \\ 0 & 0 & -4 + 2m & 0 \\ 0 & 0 & 0 & -12 + 6m \end{pmatrix}.
\]

(B.10)

This verifies the identity (4.20) for \( s = 3 \). Therefore we obtain

\[
\nabla^2 \begin{pmatrix} \phi_{111} \\ \phi_{112} \\ \phi_{122} \\ \phi_{222} \end{pmatrix} = \begin{pmatrix} \Delta - 12 & -6m & 0 & 0 \\ 0 & \Delta - 4 & -2m & 0 \\ 0 & 0 & \Delta - 4 + 2m & 0 \\ 0 & 0 & 0 & \Delta - 12 + 6m \end{pmatrix} \begin{pmatrix} \phi_{111} \\ \phi_{112} \\ \phi_{122} \\ \phi_{222} \end{pmatrix}.
\]

(B.11)
The second order equation satisfied by the the spin 3 field can be read out from (using (2.27)). This is given by

\[(\nabla^2 - m^2 + 4)\phi_{\mu\nu\eta} = 0.\]  

(B.12)

Thus the component \(\phi_{111}, \phi_{112}, \phi_{122}\) and \(\phi_{222}\) satisfy the following equations

\[(\Delta - (m + 3)^2 + 1)\phi_{111} = 0,\]

(B.13)

\[(\Delta - (m + 1)^2 + 1)\phi_{112} = 0,\]

\[(\Delta - (m - 1)^2 + 1)\phi_{122} = 0,\]

\[(\Delta - (m - 3)^2 + 1)\phi_{111} = 0.\]

Written out explicitly in terms of the coordinates with \(z = \tanh^2 \xi\) and using the ansatz \(\phi_{\mu\nu\eta} = e^{-i(k_+ x^+ + k_- x^-)} R_{\mu\nu\eta}\), the above equations reduce to the following

\[z(1 - z) \frac{d^2 R_{111}}{dz^2} + (1 - z) \frac{dR_{111}}{dz} + \left[\frac{k_+^2}{4z} - \frac{k_-^2}{4} - \frac{(m + 3)^2 - 1}{4(1 - z)}\right] R_{111} = 0,\]

(B.14)

\[z(1 - z) \frac{d^2 R_{112}}{dz^2} + (1 - z) \frac{dR_{112}}{dz} + \left[\frac{k_+^2}{4z} - \frac{k_-^2}{4} - \frac{(m + 1)^2 - 1}{4(1 - z)}\right] R_{112} = 0,\]

\[z(1 - z) \frac{d^2 R_{122}}{dz^2} + (1 - z) \frac{dR_{122}}{dz} + \left[\frac{k_+^2}{4z} - \frac{k_-^2}{4} - \frac{(m - 1)^2 - 1}{4(1 - z)}\right] R_{122} = 0,\]

\[z(1 - z) \frac{d^2 R_{222}}{dz^2} + (1 - z) \frac{dR_{222}}{dz} + \left[\frac{k_+^2}{4z} - \frac{k_-^2}{4} - \frac{(m - 3)^2 - 1}{4(1 - z)}\right] R_{222} = 0.\]

**Solutions of the spin-3 components**

The solutions of (B.14) that obeys ingoing boundary conditions at the horizon are

\[R_{111}(z) = e^{111z^a(1 - z)^{\beta_{111}}} F(a_{111}, b_{111}, c; z),\]  

(B.15)

\[R_{112}(z) = e^{112z^a(1 - z)^{\beta_{112}}} F(a_{112}, b_{112}, c; z),\]  

(B.16)

\[R_{122}(z) = e^{122z^a(1 - z)^{\beta_{122}}} F(a_{122}, b_{122}, c; z),\]  

(B.17)

\[R_{222}(z) = e^{222z^a(1 - z)^{\beta_{222}}} F(a_{222}, b_{222}, c; z).\]  

(B.18)

where,

\[\alpha = \frac{-ik_+}{2}, \quad c = 1 + 2\alpha,\]  

(B.19)

\[\beta_{111} = \frac{1}{2}(m + 4), \quad \beta_{112} = \frac{1}{2}(m + 2), \quad \beta_{122} = \frac{1}{2}m, \quad \beta_{222} = \frac{1}{2}(m - 2)\]

\[a_{ijl} = k_+ - k_-^{i-j-l} + \beta_{ijl}, \quad b_{ijl} = k_+ + k_-^{i-j-l} + \beta_{ijl}.\]
Fixing the polarization constants

After investigating the behaviour near the horizon we obtain from (4.42) a set of recursion relations for the polarization constants. This is of the form

\[
\begin{pmatrix}
  m - ik_- & 2 + ik_- & 0 & 0 \\
  1 & m - ik_- & 1 + ik_- & 0 \\
  0 & 2 & m - ik_- ik_+ & 0 \\
\end{pmatrix}
\begin{pmatrix}
  e_{++} \\
  e_{++-} \\
  e_{+-} \\
  e_{--} \\
\end{pmatrix} = 0.
\]

(B.20)

It can then be verified that

\[
\frac{1}{2s-1} T^{(2)} C^{(3)} T^{(3)}
= \begin{pmatrix}
  2 + m + i(k_+ - k_-) & 2 + m - i(k_+ + k_-) & 0 & 0 \\
  0 & m + i(k_+ - k_-) & m - i(k_+ + k_-) & 0 \\
  0 & 0 & -2 + m + i(k_+ - k_-) & -2 + m - i(k_+ + k_-) \\
\end{pmatrix}.
\]

(B.21)

This verifies (4.46) for \( s = 3 \). The above relation then gives a simple recursion relation between two \( e_p \) from which we can obtain all coefficients in terms of \( e_0 \).

Using (4.49) we have

\[
e_{111} = -\prod_{j=0}^{2} \frac{2j - 2 + m - i(k_+ + k_-)}{2j - 2 + m + i(k_+ - k_-)} e_{222}
\]

(B.22)

\[
e_{112} = \prod_{j=0}^{1} \frac{2j - 2 + m - i(k_+ + k_-)}{2j - 2 + m + i(k_+ - k_-)} e_{222}
\]

(B.23)

\[
e_{122} = -\frac{-2 + m - i(k_+ + k_-)}{-2 + m + i(k_+ - k_-)} e_{222}
\]

(B.24)

Thus, the final form of the solutions are

\[
R_{111}(z) = -e_{222} \prod_{j=0}^{2} \frac{2j - 2 + m - i(k_+ + k_-)}{2j - 2 + m + i(k_+ - k_-)} z^\alpha (1 - z)^{\beta_{111}} F(a_{111}, b_{111}, c; z)
\]

(B.25)

\[
R_{112}(z) = e_{222} \prod_{j=0}^{1} \frac{2j - 2 + m - i(k_+ + k_-)}{2j - 2 + m + i(k_+ - k_-)} z^\alpha (1 - z)^{\beta_{112}} F(a_{112}, b_{112}, c; z)
\]

(B.26)

\[
R_{122}(z) = -e_{222} \frac{-2 + m - i(k_+ + k_-)}{-2 + m + i(k_+ - k_-)} z^\alpha (1 - z)^{\beta_{122}} F(a_{122}, b_{122}, c; z)
\]

(B.27)

\[
R_{222}(z) = e_{222} z^\alpha (1 - z)^{\beta_{222}} F(a_{222}, b_{222}, c; z)
\]

(B.28)

Quasinormal modes

We can now obtain the quasinormal modes by imposing vanishing Dirichlet conditions at the boundary. For \( m > 0 \) the leading behaviour of these solutions near the
boundary $z \to 1$ is given by

$$R_{111}(z) \simeq -e^{222} \prod_{j=0}^{2} \frac{2j - 2 + m - i(k_+ + k_-)}{2j - 2 + m + i(k_+ - k_-)} z^\alpha (1 - z)^{-m} \frac{1}{\Gamma(1 + \alpha)} \frac{\Gamma(c)\Gamma(a_{111} + b_{111} - c)}{\Gamma(a_{111})\Gamma(b_{111})}$$

$$= -\frac{8e^{222}}{\prod_{j=0}^{2}[2j - 2 + m + i(k_+ - k_-)]} z^\alpha (1 - z)^{-m} \frac{1}{\Gamma(1 + \alpha)} \frac{\Gamma(c)\Gamma(a_{111} + b_{111} - c)}{\Gamma(a_{111})\Gamma(b_{111} - 3)}, \quad (B.29)$$

$$R_{112}(z) \simeq e^{222} \prod_{j=0}^{1} \frac{2j - 2 + m - i(k_+ + k_-)}{2j - 2 + m + i(k_+ - k_-)} z^\alpha (1 - z)^{-m} \frac{1}{\Gamma(1 + \alpha)} \frac{\Gamma(c)\Gamma(a_{112} + b_{112} - c)}{\Gamma(a_{112})\Gamma(b_{112})}$$

$$= \frac{4e^{222}}{\prod_{j=0}^{1}[2j - 2 + m + i(k_+ - k_-)]} z^\alpha (1 - z)^{-m} \frac{1}{\Gamma(1 + \alpha)} \frac{\Gamma(c)\Gamma(a_{112} + b_{112} - c)}{\Gamma(a_{112})\Gamma(b_{112} - 2)}, \quad (B.30)$$

$$R_{122}(z) \simeq -e^{222} \frac{m - 2 - i(k_+ + k_-)}{m - 2 + i(k_+ - k_-)} z^\alpha (1 - z)^{-m} \frac{1}{\Gamma(1 + \alpha)} \frac{\Gamma(c)\Gamma(a_{122} + b_{122} - c)}{\Gamma(a_{122})\Gamma(b_{122})}$$

$$= -\frac{2e^{222}}{m - 2 + i(k_+ - k_-)} z^\alpha (1 - z)^{-m} \frac{1}{\Gamma(1 + \alpha)} \frac{\Gamma(c)\Gamma(a_{122} + b_{122} - c)}{\Gamma(a_{122})\Gamma(b_{122} - 1)}, \quad (B.31)$$

$$R_{222}(z) \simeq e^{222} z^\alpha (1 - z)^{-m} \frac{1}{\Gamma(1 + \alpha)} \frac{\Gamma(c)\Gamma(a_{222} + b_{222} - c)}{\Gamma(a_{222})\Gamma(b_{222})} \quad (B.32)$$

The common set of zeros of the above functions are given by

$$b_{111} - 3 = b_{112} - 2 = b_{122} - 1 = b_{222} = -n, \quad (B.33)$$

$$a_{111} = -n, \quad n = 0, 1, 2, \ldots, \quad (B.34)$$

which implies

$$i(k_+ + k_-) = 2n + \hat{\Delta} - 3, \quad i(k_+ - k_-) = 2n + \hat{\Delta} + 3. \quad (B.35)$$

where, $\hat{\Delta} = 1 + m$. Thus the quasinormal modes for the spin-3 field are,

$$\omega_L = k + 2\pi T_L(k_+ + k_-), \quad \omega_R = -k + 2\pi T_R(k_+ - k_-)$$

$$= k - 2\pi i T_L(2n + \hat{\Delta} - 3) \quad = -k - 2\pi i T_R(2n + \hat{\Delta} + 3). \quad (B.36)$$

C. Proofs of the identities

In this section we shall be proving the identities (4.19), (4.20) and (4.46).
Proof of Identity 1 (4.19)

We require to prove

\[ \sum_{q=0}^{s} T_{pq}^{(s)} T_{qr}^{(s)} = 2^s \delta_{pr}, \]  

(C.1)

or equivalently,

\[ \sum_{q=0}^{s} \sum_{r=0}^{s} T_{pq}^{(s)} T_{qr}^{(s)} x^r = 2^s \sum_{r=0}^{s} \delta_{pr} x^r = 2^s x^p. \]  

(C.2)

Let’s start with the left hand side

\[
\sum_{q=0}^{s} \sum_{r=0}^{s} T_{pq}^{(s)} T_{qr}^{(s)} x^r = \sum_{q=0}^{s} T_{pq}^{(s)} (x+1)^q (x-1)^{s-q}
\]

\[
= (x-1)^s \sum_{q=0}^{s} T_{pq}^{(s)} \left( \frac{x+1}{x-1} \right)^q
\]

\[
= (x-1)^s \sum_{p=0}^{s} \sum_{a=0}^{p} \sum_{b=0}^{s-p} (-1)^b \binom{p}{a} \binom{s-p}{b} \left( \frac{x+1}{x-1} \right)^{s-a-b}
\]

\[
= (x+1)^s \sum_{p=0}^{s} \binom{p}{a} \left( \frac{x-1}{x+1} \right)^a \sum_{b=0}^{s-p} (-1)^b \binom{s-p}{b} \left( \frac{x-1}{x+1} \right)^b
\]

\[
= (x+1)^s \left( \frac{1 + \frac{x-1}{x+1}}{x+1} \right)^p \left( \frac{1 - \frac{x-1}{x+1}}{x+1} \right)^{s-p}
\]

\[
= (x+1)^s \frac{(2x)^p}{(x+1)^p} \frac{2^{s-p}}{(x+1)^{s-p}}
\]

\[= 2^s x^p. \quad \text{Q.E.D.} \]  

(C.3)

So we have proved

\[ T^{(s)} T^{(s)} = 2^s I. \]  

(C.4)

Proof of Identity 2 (4.20)

We require to prove

\[ T^{(s)} M^{(s)} [T^{(s)}]^{-1} = D^{(s)} \]  

(C.5)

\[ \Rightarrow M^{(s)} [T^{(s)}]^{-1} = [T^{(s)}]^{-1} D^{(s)} \]

\[ \Rightarrow M^{(s)} T^{(s)} = T^{(s)} D^{(s)}. \]  

(C.6)

where, \( D_{pq}^{(s)} = -\left((2p-s)^2 + 2m(2p-s) + s\right) \delta_{pq} \). This is equivalently

\[
\sum_{b=0}^{s} \sum_{c=0}^{s} M_{ab}^{(s)} T_{bc}^{(s)} x^c = \sum_{b=0}^{s} \sum_{c=0}^{s} T_{ab}^{(s)} D_{bc}^{(s)} x^c
\]

(C.7)
Let’s start with the LHS of (C.7). Often we shall be using the variable

\[ z = \frac{x + 1}{x - 1} \]  

(C.8)

Then we have

\[
\sum_{b=0}^{s} M_{ab}^{(s)} \sum_{c=0}^{s} T_{bc}^{(s)} x^c \\
= \sum_{b=0}^{s} M_{ab}^{(s)} (x + 1)^b (x - 1)^{s-b} \\
= (x - 1)^s \sum_{b=0}^{s} M_{ab}^{(s)} z^b \\
= (x - 1)^s \sum_{b=0}^{s} \left[ -(2s + 2a(s-a)) \delta_{a,b} - 2mp \delta_{a-1,b} - 2m(s-a) \delta_{a+1,b} \right] \frac{s - b}{s - a - 1} z^b \\
= (x - 1)^s \left[ -(2s + 2a(s-a)) z^a - 2maz^{a-1} - 2m(s-a)z^{a+1} \\
- a(a-1)z^{a-2} - (s-a)(s-a-1)z^{a+2} \right] \\
= (x - 1)^s z^a \left[ -(2s + 2a(s-a)) - 2mz^{a-1} - 2m(s-a)z^1 \\
- a(a-1)z^{a-2} - (s-a)(s-a-1)z^2 \right] \\
= -(x - 1)^s a^{a-2} \left[ (2s + 2a(s-a))(x - 1)^2(x + 1)^2 \\
+ 2ma(x - 1)^3(x + 1) + 2m(s-a)(x - 1)(x + 1)^3 \\
+ a(a-1)(x - 1)^4 + (s-a)(s-a-1)(x + 1)^4 \right] \\
= (x + 1)^{a-2} (x - 1)^{-a+s-2} \left[ -s(x + 1)^2 (x-8a + x - 6) + 2m (x^2 - 1) + 1 \\
- 8ax \left( 2ax - mx^2 + m + x^2 + 1 \right) - s^2(x+1)^4 \right].
\]

(C.9)

We now need to evaluate the RHS of (C.7). Before proceeding we shall list a few definitions and identities which we shall be using the following relations

\[
P_{(s,a)}(x) = \sum_{b=0}^{s} T_{ab}^{(s)} x^b = (x + 1)^{a}(x - 1)^{s-a}, \quad (C.10)
\]

\[
x \frac{dP_{(s,a)}(x)}{dx} = \sum_{b=0}^{s} b T_{ab}^{(s)} x^b, \quad (C.11)
\]

\[
x^2 \frac{d^2P_{(s,a)}(x)}{dx^2} = \sum_{b=0}^{s} b(b-1) T_{ab}^{(s)} x^b. \quad (C.12)
\]

– 42 –
For the RHS of (C.7) we have

$$\sum_{b=0}^{s} T_{ab}^{(s)} \sum_{c=0}^{s} D_{bc}^{(s)} x^{c}$$

$$= - \sum_{b=0}^{s} T_{ab}^{(s)} \sum_{c=0}^{s} [(2b - s)^2 + 2m(2b - s) + s] \delta_{bc} x^{c}$$

$$= - \sum_{b=0}^{s} T_{ab}^{(s)} [(2b - s)^2 + 2m(2b - s) + s] x^{b}$$

$$= - \sum_{b=0}^{s} T_{ab}^{(s)} [4(b - 1) + 4(1 - s + m)b + s(1 + s - 2m)] x^{b}$$

$$= - 4 \sum_{b=0}^{s} b(b - 1) T_{ab}^{(s)} x^{b} - 4(1 - s + m) \sum_{b=0}^{s} b T_{ab}^{(s)} x^{b} - s(1 + s - 2m) \sum_{b=0}^{s} T_{ab}^{(s)} x^{b}$$

$$= - 4x^2 \frac{d^2 P_{(s,a)}(x)}{dx^2} - 4(1 - s + m) x \frac{dP_{(s,a)}(x)}{dx} - s(1 + s - 2m) P_{(s,a)}(x)$$

$$(x + 1)^{a-2} (x - 1)^{-a+s-2} [-s(x + 1)^2 (x-8a + x - 6) + 2m (x^2 - 1) + 1 - 8ax (2ax - mx^2 + m + x^2 + 1)] - s^2 (x + 1)^4]. \quad (C.13)$$

This completes the proof of (4.20).

**Proof of Identity 3 (4.46)**

We require to prove

$$\frac{1}{2^{s-1}} T^{(s-1)} C^{(s)} T^{(s)} = C'^{(s)} \quad (C.14)$$

$$\Rightarrow C^{(s)} T^{(s)} = T^{(s-1)} C'^{(s)}. \quad (C.15)$$

where

$$C_{a,b}^{(s)} = (s - a - 1) \delta_{a+2,b} + (m - ik_-) \delta_{a+1,b} + (a + ik_+) \delta_{a,b}, \quad (C.16)$$

$$C_{a,b}'^{(s)} = (2a - s + 1 + m + i(k_+ - k_-)) \delta_{a+1,b} + (2a - s + 1 + m - i(k_+ + k_-)) \delta_{a,b}. \quad (C.17)$$

This is equivalent to proving

$$\sum_{b=0}^{s} \sum_{c=0}^{s} C_{ab}^{(s)} T_{bc}^{(s)} x^{c} = \sum_{b=0}^{s-1} \sum_{c=0}^{s} T_{ab}^{(s-1)} C_{bc}^{(s)} x^{c}. \quad (C.18)$$
The LHS of (C.18) is
\[ \sum_{b=0}^{s} C_{ab}^{(s)} \sum_{c=0}^{s} T_{bc}^{(s)} x^c \]
\[ = \sum_{b=0}^{s} C_{ab}^{(s)} (x+1)^b (x-1)^{s-b} \]
\[ = (x-1)^s \sum_{b=0}^{s} C_{ab}^{(s)} z^b \]  \hspace{1cm} (C.19)

Now let’s examine the RHS of (C.18),
\[ \sum_{b=0}^{s} T_{ab}^{(s-1)} \sum_{c=0}^{s} C_{bc}^{(s)} x^c \]
\[ = \sum_{b=0}^{s-1} T_{ab}^{(s-1)} \sum_{c=0}^{s} [(2b-s+1+m+i(k_+ - k_-))\delta_{b+1,c} \]
\[ + (2b-s+1+m-i(k_+ + k_-))\delta_{b,c}] x^c \]
\[ = \sum_{b=0}^{s-1} T_{ab}^{(s-1)} [(2b-s+1+m+i(k_+ - k_-))x^{b+1} \]
\[ + (2b-s+1+m-i(k_+ + k_-))x^b] \]
\[ = 2x \sum_{b=0}^{s-1} bT_{ab}^{(s-1)} x^b + (s+1+m+i(k_+ - k_-))x \sum_{b=0}^{s-1} T_{ab}^{(s-1)} x^b \]
\[ + 2 \sum_{b=0}^{s-1} bT_{ab}^{(s-1)} x^b + (s+1+m-i(k_+ + k_-)) \sum_{b=0}^{s-1} T_{ab}^{(s-1)} x^b \]
\[ = 2x^2 \frac{dP_{(s-1,a)}(x)}{dx} + (s+1+m+i(k_+ - k_-))xP_{(s-1,a)}(x) \]
\[ + 2x \frac{dP_{(s-1,a)}(x)}{dx} + (s+1+m-i(k_+ + k_-))P_{(s-1,a)}(x) \]
\[ = 2x(x+1) \frac{dP_{(s-1,a)}(x)}{dx} + [(-s+1+m-ik_-)(x+1)+ik_+(x-1)]P_{(s-1,a)}(x) \]
\[ = (x+1)^a(x-1)^{-a+s-2}[2x(-2a+s-1)-ik_- (x^2 - 1) + ik_+ (x-1)^2 \]
\[ + x^2(m+s-1) - m + s - 1] \]
\[(x - 1)^{s-a-2}(x + 1)^a [(s - a - 1)(x + 1)^2 + (m - ik_-)(x + 1)(x - 1) + (a + ik_+)(x - 1)^2].\]  
(C.20)

This completes the proof of (4.46).

References

[1] J. M. Maldacena and A. Strominger, \textit{AdS(3) black holes and a stringy exclusion principle}, \textit{JHEP} \textbf{9812} (1998) 005, [hep-th/9804085].

[2] M. R. Gaberdiel and R. Gopakumar, \textit{An AdS$_3$ Dual for Minimal Model CFTs}, \textit{Phys.Rev.} \textbf{D83} (2011) 066007, [arXiv:1011.2986].

[3] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, \textit{Large N field theories, string theory and gravity}, \textit{Phys.Rept.} \textbf{323} (2000) 183–386, [hep-th/9905111].

[4] D. Birmingham, I. Sachs, and S. N. Solodukhin, \textit{Conformal field theory interpretation of black hole quasinormal modes}, \textit{Phys.Rev.Lett.} \textbf{88} (2002) 151301, [hep-th/0112055].

[5] D. T. Son and A. O. Starinets, \textit{Minkowski space correlators in AdS / CFT correspondence: Recipe and applications}, \textit{JHEP} \textbf{0209} (2002) 042, [hep-th/0205051].

[6] P. K. Kovtun and A. O. Starinets, \textit{Quasinormal modes and holography}, \textit{Phys.Rev.} \textbf{D72} (2005) 086009, [hep-th/0506184].

[7] G. T. Horowitz and V. E. Hubeny, \textit{Quasinormal modes of AdS black holes and the approach to thermal equilibrium}, \textit{Phys.Rev.} \textbf{D62} (2000) 024027, [hep-th/9909056].

[8] E. Berti, V. Cardoso, and A. O. Starinets, \textit{Quasinormal modes of black holes and black branes}, \textit{Class.Quant.Grav.} \textbf{26} (2009) 163001, [arXiv:0905.2975].

[9] J. R. David and A. Sadhukhan, \textit{Classical integrability in the BTZ black hole}, \textit{JHEP} \textbf{1108} (2011) 079, [arXiv:1105.0480].

[10] D. Birmingham, \textit{Choptuik scaling and quasinormal modes in the AdS / CFT correspondence}, \textit{Phys.Rev.} \textbf{D64} (2001) 064024, [hep-th/0101194].

[11] S. Das and A. Dasgupta, \textit{Black hole emission rates and the AdS / CFT correspondence}, \textit{JHEP} \textbf{9910} (1999) 025, [hep-th/9907116].

[12] F. Denef, S. A. Hartnoll, and S. Sachdev, \textit{Black hole determinants and quasinormal modes}, \textit{Class.Quant.Grav.} \textbf{27} (2010) 125001, [arXiv:0908.2657].

[13] J. R. David, M. R. Gaberdiel, and R. Gopakumar, \textit{The Heat Kernel on AdS(3) and its Applications}, \textit{JHEP} \textbf{1004} (2010) 125, [arXiv:0911.5085].
[14] M. Banados, C. Teitelboim, and J. Zanelli, *The Black hole in three-dimensional space-time*, *Phys.Rev.Lett.* **69** (1992) 1849–1851, [hep-th/9204099](http://arxiv.org/abs/hep-th/9204099).

[15] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, *Geometry of the (2+1) black hole*, *Phys.Rev.* **D48** (1993) 1506–1525, [gr-qc/9302012](http://arxiv.org/abs/gr-qc/9302012).

[16] R. Metsaev, *Massive totally symmetric fields in AdS(d)*, *Phys.Lett.* **B590** (2004) 95–104, [hep-th/0312297](http://arxiv.org/abs/hep-th/0312297).

[17] I. Buchbinder, V. Krykhtin, and P. Lavrov, *Gauge invariant Lagrangian formulation of higher spin massive bosonic field theory in AdS space*, *Nucl.Phys.* **B762** (2007) 344–376, [hep-th/0608005](http://arxiv.org/abs/hep-th/0608005).

[18] I. Tyutin and M. A. Vasiliev, *Lagrangian formulation of irreducible massive fields of arbitrary spin in (2+1)-dimensions*, *Teor.Mat.Fiz.* **113N1** (1997) 45–57, [hep-th/9704132](http://arxiv.org/abs/hep-th/9704132).

[19] S. Giombi and X. Yin, *Higher Spin Gauge Theory and Holography: The Three-Point Functions*, *JHEP* **1009** (2010) 115, [arXiv:0912.3462](http://arxiv.org/abs/0912.3462).

[20] S. S. Gubser, *Absorption of photons and fermions by black holes in four-dimensions*, *Phys.Rev.* **D56** (1997) 7854–7868, [hep-th/9706100](http://arxiv.org/abs/hep-th/9706100).

[21] I. Sachs and S. N. Solodukhin, *Quasi-Normal Modes in Topologically Massive Gravity*, *JHEP* **0808** (2008) 003, [arXiv:0806.1788](http://arxiv.org/abs/0806.1788).

[22] B. Chen and J. Long, *Hidden Conformal Symmetry and Quasi-normal Modes*, *Phys.Rev.* **D82** (2010) 126013, [arXiv:1009.1010](http://arxiv.org/abs/1009.1010).

[23] G. Mandal, A. M. Sengupta, and S. R. Wadia, *Classical solutions of two-dimensional string theory*, *Mod.Phys.Lett.* **A6** (1991) 1685–1692.

[24] E. Witten, *On string theory and black holes*, *Phys.Rev.* **D44** (1991) 314–324.

[25] J. M. Maldacena, H. Ooguri, and J. Son, *Strings in AdS3 and the SL(2, R) WZW model. Part 2. Euclidean black hole*, *J.Math.Phys.* **42** (2001) 2961–2977, [hep-th/0005183](http://arxiv.org/abs/hep-th/0005183).