ON \( k \)-DECOMPOSABILITY OF POSITIVE MAPS

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Abstract. We extend the theory of decomposable maps by giving a detailed description of \( k \)-positive maps. A relation between transposition and modular theory is established. The structure of positive maps in terms of modular theory (the generalized Tomita-Takesaki scheme) is examined.

1. Definitions, notations and stating the problem

For any \( C^* \)-algebra \( A \) let \( A^+ \) denote the set of all positive elements in \( A \). A state on a unital \( C^* \)-algebra \( A \) is a linear functional \( \omega : A \to \mathbb{C} \) such that \( \omega(a) \geq 0 \) for every \( a \in A^+ \) and \( \omega(1) = 1 \) where \( 1 \) is the unit of \( A \). By \( \mathcal{S}(A) \) we will denote the set of all states on \( A \). For any Hilbert space \( H \) we denote by \( \mathcal{B}(H) \) the set of all bounded linear operators on \( H \).

A linear map \( \varphi : A \to B \) between \( C^* \)-algebras is called positive if \( \varphi(A^+) \subset B^+ \).

For \( k \in \mathbb{N} \) we consider a map \( \varphi_k : M_k(A) \to M_k(B) \) where \( M_k(A) \) and \( M_k(B) \) are the algebras of \( k \times k \) matrices with coefficients from \( A \) and \( B \) respectively, and \( \varphi_k(a_{ij}) = [\varphi(a_{ij})] \). We say that \( \varphi \) is \( k \)-positive if the map \( \varphi_k \) is positive. The map \( \varphi \) is said to be completely positive when it is \( k \)-positive for every \( k \in \mathbb{N} \).

A Jordan morphism between \( C^* \)-algebras \( A \) and \( B \) is a linear map \( J : A \to B \) which respects the Jordan structures of algebras \( A \) and \( B \), i.e. \( J(ab + ba) = J(a)J(b) + J(b)J(a) \) for every \( a, b \in A \). Let us recall that every Jordan morphism is a positive map but it need not be a completely positive one (in fact it need not even be 2-positive). It is commonly known (\cite{26}) that every Jordan morphism \( J : A \to \mathcal{B}(H) \) is a sum of a \( * \)-morphism and a \( * \)-antimorphism.

The Stinespring theorem states that every completely positive map \( \varphi : A \to \mathcal{B}(H) \) has the form \( \varphi(a) = W^* \pi(a) W \), where \( \pi \) is a \( * \)-representation of \( A \) on some Hilbert space \( K \), and \( W \) is a bounded operator from \( H \) to \( K \).

Following Størmer (\cite{25}) we say that a map \( \varphi : A \to \mathcal{B}(H) \) is decomposable if there are a Hilbert space \( K \), a Jordan morphism \( J : A \to \mathcal{B}(K) \), and a bounded linear operator \( W \) from \( H \) to \( K \) such that \( \varphi(a) = W^* J(a) W \) for every \( a \in A \).

Let \( (e_i) \) be a fixed orthonormal basis in some Hilbert space \( H \). Define a conjugation \( J_e \) associated with this basis by the formula \( J_e \left( \sum_i \lambda_i e_i \right) = \sum_i \overline{\lambda_i} e_i \). The map \( J_e \) has the following properties: (i) \( J_e \) is an antilinear isomorphism of \( H \); (ii) \( J_e^2 = \mathbb{I} \); (iii) \( \langle J_e \xi, J_e \eta \rangle = \langle \eta, \xi \rangle \) for every \( \xi, \eta \in H \); (iv) the map \( a \mapsto J_e a J_e \) is a \( * \)-automorphism of the algebra \( \mathcal{B}(H) \). For every \( a \in \mathcal{B}(H) \) we denote by \( a^t \) the element \( J_e a^* J_e \) and we call it a transposition of the element \( a \). From the

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above properties (i) – (iv) it follows that the \textit{transposition map} \( a \mapsto a^t \) is a linear \(^*\)-antiautomorphism of \( B(H) \).

We say that a linear map \( \varphi : A \rightarrow B(H) \) is \( k \)-copositive (resp. completely copositive) if the map \( a \mapsto \varphi(a)^t \) is \( k \)-positive (resp. completely positive). The following theorem characterizes decomposable maps in the spirit of Stinespring’s theorem:

\textbf{Theorem 1.1} (\cite{28}). Let \( \varphi : A \rightarrow B(H) \) be a linear map. Then the following conditions are equivalent:

(i) \( \varphi \) is decomposable;
(ii) for every natural number \( k \) and for every matrix \( [a_{ij}] \in M_k(A) \) such that both \( [a_{ij}] \) and \( [a_{ji}] \) belong to \( M_k(A)^+ \) the matrix \( [\varphi(a_{ij})] \) is in \( M_k(B(H))^+ \);
(iii) there are maps \( \varphi_1, \varphi_2 : A \rightarrow B(H) \) such that \( \varphi_1 \) is completely positive and \( \varphi_2 \) completely copositive, with \( \varphi = \varphi_1 + \varphi_2 \).

In spite of the enormous efforts, the classification of decomposable maps is still not complete even in the case when \( A \) and \( H \) are finite dimensional, i.e. \( A = B(C^m) \) and \( H = C^n \). The most important step was done by Størmer \cite{28}, Choi \cite{6,7} and Woronowicz \cite{32}. Størmer and Woronowicz proved that if \( m = n = 2 \) or \( m = 2, n = 3 \) then every positive map is decomposable. The first examples of nondecomposable maps was given by Choi (in the case \( m = n = 3 \)) and Woronowicz (in the case \( m = 2, n = 4 \)). It seems that the main difficulty in carrying out the classification of positive maps is the question of the canonical form of non-decomposable maps. As far as we know there are only special examples of maps from that class which are scattered across the literature \cite{32,7,14,10,15,24,27}.

In fact it seems that in the infinite dimensional case all known examples of non-decomposable maps rely on deep structure theory of the underlying algebras. (See for example \cite{27}.) On the other hand, it seems that very general positive maps (so not of the CP class) and hence possibly non-decomposable ones, are crucial for an analysis of nontrivial quantum correlations, i.e. for an analysis of genuine quantum maps \cite{31,23,12,19,20}. Having that motivation in mind we wish to present a step toward a canonical prescription for the construction of decomposable and non-decomposable maps. Namely, we study the notion of \( k \)-decomposability and prove an analog of Theorem 1.1. The basic strategy of the paper is to employ two dual pictures: one given in terms of operator algebras while the second one will use the space of states. Thus, it can be said that we are using the equivalence of the Schrödinger and Heisenberg pictures in the sense of Kadison \cite{13}, Connes \cite{8} and Alfsen, Shultz \cite{1}.

The paper is organized as follows. In section 2 we recall the techniques used in \cite{21} and compare it with results from \cite{17}. In section 3 we formulate our main result concerning the notion of \( k \)-decomposability. Section 4 is devoted to a modification of Tomita-Takesaki theory. Section 5, based on the previous Section, presents a description of \( k \)-decomposability at the Hilbert space level. Section 6 provides new results on partial transposition which are used to complete the description of \( k \)-decomposability.

\section{2. Dual construction}

Let us recall the construction of Choi \cite{6} (see also \cite{21}) which establishes a one-to-one correspondence between elements of \( B(C^m) \otimes B(C^n) \) and linear maps from
$\mathcal{B}(\mathbb{C}^m)$ to $\mathcal{B}(\mathbb{C}^n)$. Fix some orthonormal basis $e_1, e_2, \ldots, e_m$ (resp. $f_1, f_2, \ldots, f_n$) in $\mathbb{C}^m$ (resp. $\mathbb{C}^n$) and by $E_{ij}$ (resp. $F_{kl}$) denote the the matrix units in $\mathcal{B}(\mathbb{C}^m)$ (resp. $\mathcal{B}(\mathbb{C}^n)$). For any $x \in \mathbb{C}^m$ define the linear operator $V_x : \mathbb{C}^n \to \mathbb{C}^m \otimes \mathbb{C}^n$ by $V_x = x \otimes y$ where $y \in \mathbb{C}^n$. For simplicity, we write $V_i$ instead of $V_{e_i}$ for every $i = 1, \ldots, m$. Observe that for any $h \in \mathcal{B}(\mathbb{C}^m) \otimes \mathcal{B}(\mathbb{C}^n)$ we have

\begin{equation}
 h = \sum_{i,j=1}^{m} E_{ij} \otimes V_i^* h V_j.
\end{equation}

Consequently, for every $h$ one can define the map $\varphi_h : \mathcal{B}(\mathbb{C}^m) \to \mathcal{B}(\mathbb{C}^n)$ by

\[ \varphi_h(E_{ij}) = V_i^* h V_j, \quad i, j = 1, 2, \ldots, m. \]

On the other hand following (2.1) given a linear map $\varphi : \mathcal{B}(\mathbb{C}^m) \to \mathcal{B}(\mathbb{C}^n)$ one can reconstruct $h$ by the formula

\begin{equation}
 h = \sum_{i,j=1}^{m} E_{ij} \otimes \varphi(E_{ij}) = (\text{id} \otimes \varphi) \left( \sum_{i,j=1}^{m} E_{ij} \otimes E_{ij} \right).
\end{equation}

The main properties of this correspondence we summarize in the following

**Theorem 2.1** ([6, 21]). Let $h^* = h$. Then:

(i) The map $\varphi_h$ is completely positive if and only if $h$ is a positive operator, i.e.

\[ \langle z, h z \rangle \geq 0 \]

for every $z \in \mathbb{C}^m \otimes \mathbb{C}^n$;

(ii) The map $\varphi_h$ is positive if and only if

\begin{equation}
 \langle x \otimes y, h(x \otimes y) \rangle \geq 0
\end{equation}

for every $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$.

(iii) The map $\varphi_h$ is decomposable if and only if $\omega(h) \geq 0$ for each state $\omega$ on $\mathcal{B}(\mathbb{C}^m) \otimes \mathcal{B}(\mathbb{C}^n)$ such that $\omega \circ (t \otimes \text{id})$ is also a state.

If the operator $h$ fulfills the property (2.3) we will call it a **block-positive** operator.

In this section we compare Theorem 2.1 with the results presented in [17]. For the reader’s convenience we recall the main theorem from this paper.

**Theorem 2.2.** A linear map $\varphi : \mathcal{B}(\mathbb{C}^m) \to \mathcal{B}(\mathbb{C}^n)$ is positive if and only if it is of the form

\[ \varphi(a) = \sum_{k,l=1}^{n} \text{Tr}(ag_{kl}) F_{kl}, \quad a \in \mathcal{B}(\mathbb{C}^m) \]

where $g_{kl} \in \mathcal{B}(\mathbb{C}^m)$, $k, l = 1, \ldots, n$, satisfy the following condition: for every $x \in \mathbb{C}^m$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$

\begin{equation}
 \sum_{k,l=1}^{n} \lambda_k \overline{\lambda_l} \langle x, g_{kl} x \rangle \geq 0.
\end{equation}

In fact the condition (2.4) coincides with (2.3).

**Proposition 2.3.** Let $A \in \mathcal{B}(\mathbb{C}^m) \otimes \mathcal{B}(\mathbb{C}^n)$. Then the following conditions are equivalent:
(i) for every \( x \in \mathbb{C}^m \) and \( y \in \mathbb{C}^n \)
\[
\langle x \otimes y, Ax \otimes y \rangle \geq 0;
\]
(ii) for every \( x \in \mathbb{C}^m \) and \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \)
\[
\sum_{k,l=1}^{n} \lambda_k \overline{\lambda_l} \langle x, A_{kl} x \rangle \geq 0
\]
where \( A_{kl} \) are unique elements of \( \mathcal{B}(\mathbb{C}^m) \) such that \( A = \sum_{k,l} A_{kl} \otimes F_{kl} \);
(iii) for every \( y \in \mathbb{C}^n \) and \( \mu_1, \ldots, \mu_m \in \mathbb{C} \)
\[
\sum_{i,j=1}^{m} \mu_i \overline{\mu_j} \langle y, A'_{ij} y \rangle \geq 0
\]
where \( A'_{ij} \) are unique elements of \( \mathcal{B}(\mathbb{C}^n) \) such that \( A = \sum_{i,j} E_{ij} \otimes A'_{ij} \).

Proof. (i) \( \iff \) (ii) Let the \( \lambda \)'s be coefficients of the expansion of \( y \) in the basis \( \{f_k\} \), i.e. \( y = \sum \lambda_s f_s \). Then we have
\[
\langle x \otimes y, Ax \otimes y \rangle = \sum_{s,t} \lambda_s \overline{\lambda_t} \langle x \otimes f_t, Ax \otimes f_s \rangle = \sum_{s,t} \sum_{k,l} \lambda_s \overline{\lambda_t} \langle x, A_{kl} x \rangle \langle f_t, F_{kl} f_s \rangle = \sum_{k,l} \lambda_k \overline{\lambda_l} \langle x, A_{kl} x \rangle.
\]
This proves the equivalence.

(i) \( \iff \) (iii) This follows by the same method. \( \square \)

The next proposition establishes the connection between the two constructions

**Proposition 2.4.** Let \( \varphi : \mathcal{B}(\mathbb{C}^m) \to \mathcal{B}(\mathbb{C}^n) \) be a linear map. If
\[
g = \sum_{k,l} g_{kl} \otimes F_{kl}
\]
where \( \{g_{kl}\} \) are operators described in Theorem 2.2 and \( h \) is the operator defined in (2.2) then \( h = g^t \).

Proof. Define the sesquilinear form \( (\cdot, \cdot) \) on \( \mathcal{B}(\mathbb{C}^m) \) by \( (a, b) = \text{Tr}(a^*b) \) for \( a, b \in \mathcal{B}(\mathbb{C}^m) \). Then \( \mathcal{B}(\mathbb{C}^m) \) becomes a Hilbert space and \( \{E_{ij}\} \) forms an orthonormal basis. From the definitions of \( h \) and \( g \) we get
\[
h = \sum_{i,j} E_{ij} \otimes \varphi(E_{ij}) = \sum_{i,j} \text{Tr}(E_{ij} B_{lk}) E_{ij} \otimes F_{kl}
\]
\[
= \sum_{k,l} \left( \sum_{i,j} (E_{ji}, g_{lk} E_{ji}) \right)^T \otimes F_{kl} = \sum_{kl} g_{lk}^T \otimes F_{lk} = g^t
\]
\( \square \)
3. $k$-DECOMPOSABILITY

The following theorem characterizes $k$-positivity of a map $\varphi$ in terms of the properties of the operators $g$ and $h$ and constitutes a generalization of Theorems 2.1 and 2.2.

**Theorem 3.1.** Let $\varphi : B(\mathbb{C}^m) \to B(\mathbb{C}^n)$ be a linear map. Then the following conditions are equivalent:

(i) $\varphi$ is $k$-positive;

(ii) for every $y_1, \ldots, y_m \in \mathbb{C}^n$ such that $\dim \text{span}\{y_1, \ldots, y_m\} \leq k$ we have

$$\sum_{i,j=1}^{n} \langle y_j, h_{ij} y_i \rangle \geq 0$$

where $h_{ij} \in B(\mathbb{C}^n)$ are such that $h = \sum_{i,j} E_{ij} \otimes h_{ij}$, i.e. $h_{ij} = \varphi(E_{ij})$;

(iii) for every $x_1, \ldots, x_n \in \mathbb{C}^m$ such that $\dim \text{span}\{x_1, \ldots, x_n\} \leq k$ we have

$$\sum_{k,l=1}^{n} \langle x_k, g_{kl} x_l \rangle \geq 0.$$

**Proof.** (i) $\iff$ (ii) Denote by $\{k_{\alpha}\}_{\alpha=1}^k$ and $\{K_{\alpha\beta}\}_{\alpha,\beta=1}^k$ the standard orthonormal basis in $\mathbb{C}^k$ and the standard system of matrix units in $M_k$ respectively. By Theorem 2.1 the map $\varphi_k = \text{id} \otimes \varphi : M_k \otimes B(\mathbb{C}^m) \to M_k \otimes B(\mathbb{C}^n)$ is positive if and only if

$$\langle x^{(k)} \otimes y^{(k)}, h^{(k)} x^{(k)} \otimes y^{(k)} \rangle \geq 0$$

for every $x^{(k)} \in \mathbb{C}^k \otimes \mathbb{C}^m$ and $y^{(k)} \in \mathbb{C}^k \otimes \mathbb{C}^n$, where

$$h^{(k)} = \sum_{\alpha,\beta=1}^{k} \sum_{i,j=1}^{m} K_{\alpha\beta} \otimes E_{ij} \otimes \varphi_k(K_{\alpha\beta} \otimes E_{ij}) = \sum_{\alpha,\beta=1}^{k} \sum_{i,j=1}^{m} K_{\alpha\beta} \otimes E_{ij} \otimes K_{\alpha\beta} \otimes h_{ij}.$$

Let $x^{(k)} \in \mathbb{C}^k \otimes \mathbb{C}^m$ and $y^{(k)} \in \mathbb{C}^k \otimes \mathbb{C}^n$, and let $x_1, \ldots, x_k \in \mathbb{C}^m$, $y_1, \ldots, y_k \in \mathbb{C}^n$ be such that

$$x^{(k)} = \sum_{\rho} k_\rho \otimes x_\rho, \quad y^{(k)} = \sum_{\sigma} k_\sigma \otimes y_\sigma.$$

Then

$$\langle x^{(k)} \otimes y^{(k)}, h^{(k)} x^{(k)} \otimes y^{(k)} \rangle =$$

$$= \sum_{\rho,\sigma,\rho',\sigma'} \langle k_\rho \otimes x_\rho \otimes k_\sigma \otimes y_\sigma, h^{(k)} k_{\rho'} \otimes x_{\rho'} \otimes k_{\sigma'} \otimes y_{\sigma'} \rangle$$

$$= \sum_{\rho,\sigma,\rho',\sigma'} \sum_{\alpha,\beta} \langle k_\rho, K_{\alpha\beta} k_{\rho'} \rangle \langle x_\rho, E_{ij} x_{\rho'} \rangle \langle k_\sigma, K_{\alpha\beta} k_{\sigma'} \rangle \langle y_\sigma, h_{ij} y_{\sigma'} \rangle$$

$$= \sum_{\alpha,\beta} \sum_{i,j} \langle x_\beta, E_{ij} x_\alpha \rangle \langle y_\beta, h_{ij} y_\alpha \rangle$$

$$= \sum_{\alpha,\beta} \sum_{i,j} \langle e_i, x_\alpha \rangle \langle x_\beta, e_j \rangle \langle y_\beta, h_{ij} y_\alpha \rangle$$

$$= \sum_{i,j} \sum_{\beta} \langle e_j, x_\beta \rangle y_\beta, h_{ij} \sum_{\alpha} \langle e_i, x_\alpha \rangle y_\alpha \rangle$$

Let $y_i = \sum_{\alpha} \langle e_i, x_\alpha \rangle y_\alpha$ for $i = 1, \ldots, m$. Then, the equivalence is obvious.
Theorem 3.3. Let $k$ be a Hilbert space (not necessarily finite dimensional) and $\varphi : A \to \mathcal{B}(H)$ a linear map. Then the following conditions are equivalent:

(i) $\varphi$ is $k$-positive;

(ii) for every $y_1, \ldots, y_m \in \mathbb{C}^n$ such that $\dim \text{span}\{y_1, \ldots, y_m\} \leq k$ we have

$$\sum_{i,j=1}^{n} \langle y_i, h_{ij} y_j \rangle \geq 0;$$

(iii) for every $x_1, \ldots, x_n \in \mathbb{C}^m$ such that $\dim \text{span}\{x_1, \ldots, x_n\} \leq k$ we have

$$\sum_{k,l=1}^{n} \langle x_k, g_{kl} x_l \rangle \geq 0.$$

Proof. With $t$ denoting the transposition map $a \to a^t$, we let $h'$ and $g'$ denote the operators corresponding to the map $\varphi \circ t$ in the construction described in Theorems 2.1 and 2.2 for $\varphi$. Then, it is easy to show that $h_i' = h_{ij}$ for every $i, j = 1, \ldots, m$ and $g_{kl}' = g_{kl}'$ for $k, l = 1, \ldots, n$. Thus, the theorem follows. \qed

Now, we can generalise this result to the general case. If $H$ is a Hilbert space let $\text{Proj}_k(H) = \{p \in \mathcal{B}(H) : p^* = p = p^2, \text{Tr}p \leq k\}$. Then we have

Theorem 3.3. Let $A$ be a $C^*$-algebra, $H$ a Hilbert space (not necessarily finite dimensional) and $\varphi : A \to \mathcal{B}(H)$ a linear map. Then the following conditions are equivalent:

(i) $\varphi$ is $k$-positive;

(ii) for every $n \in \mathbb{N}$, every set of vectors $\xi_1, \xi_2, \ldots, \xi_n \in H$ such that

$$\dim \text{span}\{\xi_1, \xi_2, \ldots, \xi_n\} \leq k,$$
and every \( [a_{ij}] \in M_n(A)^+ \), we have
\[
\sum_{i,j=1}^{n} \langle \xi_i, \varphi(a_{ij})\xi_j \rangle \geq 0;
\]
(iii) for every \( p \in \text{Proj}_k(H) \) the map \( A \ni a \mapsto p\varphi(a)p \in \mathcal{B}(H) \) is completely positive.

**Proof.** (i) \( \Rightarrow \) (iii) Observe that the map \( p\varphi p \) is \( k \)-positive as it is a composition of \( k \)-positive and completely positive maps. It maps \( A \) into \( p\mathcal{B}(H)p \), but the latter subalgebra is isomorphic with \( M_d \) where \( d = \text{Tr}p \leq k \). By the theorem of Tomiyama (R9) \( k \)-decomposability of \( p\varphi p \) implies its complete positivity.

(iii) \( \Rightarrow \) (ii) Let \( \xi_1, \xi_2, \ldots, \xi_n \in H \) and \( \dim \text{span}\{\xi_1, \xi_2, \ldots, \xi_n\} \leq k \). If \( p \) is a projection such that \( pH = \text{span}\{\xi_1, \xi_2, \ldots, \xi_n\} \), then \( p \in \text{Proj}_k(H) \) and hence \( p\varphi p \) is completely positive by assumption. So, for every \( [a_{ij}] \in M_n(A)^+ \) we have
\[
\sum_{i,j=1}^{n} \langle \xi_i, \varphi(a_{ij})\xi_j \rangle = \sum_{i,j=1}^{n} \langle p\xi_i, \varphi(a_{ij})p\xi_j \rangle = \sum_{i,j=1}^{n} \langle \xi_i, p\varphi(a_{ij})p\xi_j \rangle \geq 0
\]
(ii) \( \Rightarrow \) (i) Let \( [a_{ij}] \in M_k(A)^+ \). Then for every \( \xi_1, \xi_k, \ldots, \xi_k \in H \) we have
\[
\sum_{i,j=1}^{k} \langle \xi_i, \varphi(a_{ij})\xi_j \rangle \geq 0
\]
This condition is equivalent to the positivity of the matrix \( [\varphi(a_{ij})] \) in \( M_k(B(H)) \), which implies that \( \varphi \) is \( k \)-positive. \( \square \)

**Corollary 3.4.** A map \( \varphi : A \to \mathcal{B}(H) \) is completely positive if and only if \( p\varphi p \) is completely positive for every finite dimensional projector in \( \mathcal{B}(H) \).

Now we are ready to study the notion of \( k \)-decomposability.

**Definition 3.5.** Let \( \varphi : A \to \mathcal{B}(H) \) be a linear map.

1. We say that \( \varphi \) is \( k \)-decomposable if there are maps \( \varphi_1, \varphi_2 : A \to \mathcal{B}(H) \) such that \( \varphi_1 \) is \( k \)-positive, \( \varphi_2 \) is \( k \)-cpositive and \( \varphi = \varphi_1 + \varphi_2 \).
2. We say that \( \varphi \) is weakly \( k \)-decomposable if there is a \( C^* \)-algebra \( E \), a unital Jordan morphism \( \tilde{\varphi} : A \to E \), and a positive map \( \psi : E \to \mathcal{B}(H) \) such that \( \psi|\tilde{\varphi}(A) \) is \( k \)-positive and \( \varphi = \psi \circ \tilde{\varphi} \).

**Theorem 3.6.** For any linear map \( \varphi : A \to \mathcal{B}(H) \) consider the following conditions:

- (D) \( \varphi \) is \( k \)-decomposable;
- (W) \( \varphi \) is weakly \( k \)-decomposable;
- (S) for every matrix \( [a_{ij}] \in M_k(A) \) such that both \( [a_{ij}] \) and \( [a_{ji}] \) are in \( M_k(A)^+ \), the matrix \( [\varphi(a_{ij})] \) is positive in \( M_k(B(H)) \);
- (P) for every \( p \in \text{Proj}_k(H) \) the map \( p\varphi p \) is decomposable.

Then we have the following implications: (D) \( \Rightarrow \) (W) \( \iff \) (P) \( \iff \) (S).

**Proof.** (D) \( \Rightarrow \) (P) If \( \varphi = \varphi_1 + \varphi_2 \) with \( \varphi_1 \) is \( k \)-positive and \( \varphi_2 \) \( k \)-cpositive, then \( p\varphi p = p\varphi_1 p + p\varphi_2 p \). From Theorem (R9) \( p\varphi_1 p \) is a completely positive map. Observe that \( p' \in \text{Proj}_k(H) \) for every \( p \in \text{Proj}_k(H) \). Hence \( (p\varphi_2 p)^t = p'\varphi_2^t p' \) and \( (p\varphi_2 p)^t \) is completely positive. Thus \( p\varphi p \) is a sum of a completely positive and completely copositive map, and hence \( p\varphi p \) is decomposable.
(P_k) ⇒ (S_k) Let \([a_{ij}] \in M_k(A)\) be such that \([a_{ij}], [a_{ji}] \in M_k(A)^+\). Suppose that \(\xi_1, \xi_2, \ldots, \xi_k \in H\) and that \(p\) is a projector on \(H\) such that \(pH = \text{span}\{\xi_1, \xi_2, \ldots, \xi_k\}\). Then

\[
\sum_{i,j=1}^{k} \langle \xi_i, \varphi(a_{ij}) \xi_j \rangle = \sum_{i,j=1}^{k} \langle p\xi_i, \varphi(a_{ij}) p\xi_j \rangle = \sum_{i,j=1}^{k} \langle \xi_i, p\varphi(a_{ij}) p\xi_j \rangle \geq 0
\]

where in the last inequality we have used the fact that the matrix \([p\varphi(a_{ij})p]\) is positive by the theorem of Størmer. Hence the matrix \([\varphi(a_{ij})]\) is positive.

(S_k) ⇒ (P_k) Let \(p \in \text{Proj}_k(H)\) and \(d = \text{Tr}p\). One should show that for every \(n \in \mathbb{N}\) and every matrix \([a_{ij}] \in M_n(A)\) such that \([a_{ij}], [a_{ji}] \in M_n(A)^+\) the matrix \([p\varphi(a_{ij})p]\) is also positive. To this end we will show that for any vectors \(\xi_1, \xi_2, \ldots, \xi_n\) the inequality

\[
(3.1) \sum_{i,j=1}^{n} \langle \xi_i, p\varphi(a_{ij}) p\xi_j \rangle \geq 0
\]

holds. If \(n \leq k\) then we define vectors \(\eta_1, \eta_2, \ldots, \eta_k:\)

\[
\eta_i = \begin{cases} 
p\xi_i & \text{for } 1 \leq i \leq n, \\
0 & \text{for } n < i \leq k
\end{cases}
\]

and a matrix \([b_{ij}] \in M_k(A):\)

\[
b_{ij} = \begin{cases} 
a_{ij} & \text{for } 1 \leq i, j \leq n, \\
0 & \text{otherwise.}
\end{cases}
\]

Obviously both matrices \([b_{ij}]\) and \([b_{ji}]\) are positive in \(M_k(A)\). Thus

\[
\sum_{i,j=1}^{n} \langle \eta_i, p\varphi(a_{ij}) p\eta_j \rangle = \sum_{i,j=1}^{k} \langle \eta_i, \varphi(b_{ij}) \eta_j \rangle \geq 0
\]

by assumption. Now, let us assume that \(n = k + 1\). Define \(\eta_i = p\xi_i\) for \(i = 1, 2, \ldots, k + 1\). As \(\dim \text{span}\{\eta_1, \eta_2, \ldots, \eta_{k+1}\} \leq k\) then at least one of vectors \(\eta_1, \eta_2, \ldots, \eta_{k+1}\), say \(\eta_{k+1}\), is a linear combination of the others, i.e. \(\eta_{k+1} = \sum_{i=1}^{k} \alpha_i \eta_i\).
Then
\[
\begin{align*}
\sum_{i,j=1}^{k+1} \langle \xi_i, p\varphi(a_{ij})p\xi_j \rangle &= \sum_{i,j=1}^{k+1} \langle \eta_i, \varphi(a_{ij})\eta_j \rangle = \\
&= \sum_{i,j=1}^{k} \langle \eta_i, \varphi(a_{ij})\eta_j \rangle + \sum_{i=1}^{k} \langle \eta_i, \varphi(a_{i,k+1})\eta_{k+1} \rangle + \\
&\quad + \sum_{j=1}^{k} \langle \eta_{k+1}, \varphi(a_{k+1,j})\eta_j \rangle + \langle \eta_{k+1}, \varphi(a_{k+1,k+1})\eta_{k+1} \rangle = \\
&= \sum_{i,j=1}^{k} \langle \eta_i, \varphi(a_{ij})\eta_j \rangle + \sum_{i=1}^{k} \langle \eta_i, \alpha_j\varphi(a_{i,k+1})\eta_j \rangle + \\
&\quad + \sum_{j=1}^{k} \langle \alpha_i\eta_i, \varphi(a_{k+1,j})\eta_j \rangle + \sum_{i,j=1}^{k} \langle \alpha_i\eta_i, \alpha_j\varphi(a_{k+1,k+1})\eta_j \rangle = \\
&= \sum_{i,j=1}^{k} \langle \eta_i, \varphi(b_{ij})\eta_j \rangle
\end{align*}
\]
where \( b_{ij} = a_{ij} + \alpha_j a_{i,k+1} + \overline{\alpha_i} a_{k+1,j} + \overline{\alpha_j} a_{k+1,k+1} \) for \( i, j = 1, 2, \ldots, k \). The fact that both matrices \( [b_{ij}] \) and \( [b_{ji}] \) are positive in \( M_k(A) \), follows from the following matrix equality
\[
\begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1k} & 0 \\
b_{21} & b_{22} & \cdots & b_{2k} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
b_{k1} & b_{k2} & \cdots & b_{kk} & 0 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & \cdots & 0 & \overline{\alpha_1} \\
0 & 1 & \cdots & 0 & \overline{\alpha_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & \overline{\alpha_k} \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a_{ij} \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_k & 0
\end{bmatrix}
\]
Hence, by assumption inequality (8.1) holds. We may continue the proof for larger \( n \) by a similar inductive argument.

\( (W_k) \Leftrightarrow (S_k) \) We follow the proof of the Theorem in [28]. For the reader’s convenience we describe Størmer’s argument:

\( (W_k) \Rightarrow (S_k) \) If \( \mathcal{J} \) is a \( \ast \)-homomorphism (resp. \( \ast \)-antihomomorphism) and \([a_{ij}]\) (resp. \([a_{ji}]\)) is in \( M_k(A)^+ \) then \([\mathcal{J}(a_{ij})]\) belongs to \( M_k(E)^+ \). Since every Jordan morphism is a sum of a \( \ast \)-homomorphism and a \( \ast \)-antihomomorphism, if both \([a_{ij}]\) and \([a_{ji}]\) belong to \( M_k(A)^+ \) then \([\mathcal{J}(a_{ij})]\) \in \( M_k(B(H))^+ \). Applying \( \psi \) now yields the fact that \([\varphi(a_{ij})]\) \in \( M_k(B(H))^+ \).
Assume that $A \subset \mathcal{B}(L)$ for some Hilbert space $L$. Let
\[
V = \left\{ \begin{bmatrix} a & 0 \\ 0 & a' \end{bmatrix} \in M_2(\mathcal{B}(L)) : a \in A \right\}
\]
where $t'$ is the transposition map with respect to some orthonormal basis in $L$. Then $V$ is a selfadjoint subspace of $M_2(\mathcal{B}(L))$ containing the identity. One can observe that both $[a_{ij}]$ and $[a_{ji}]$ belong to $M_k(A)^+$ if and only if
\[
\begin{bmatrix}
  a_{11} & 0 & \cdots & a_{1k} \\
  0 & a'_{11} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{k1} & 0 & \cdots & a'_{kk}
\end{bmatrix}
\in M_k(V)^+.
\]

Thus the map $\psi : V \to \mathcal{B}(H)$ defined by
\[
\psi \left( \begin{bmatrix} a & 0 \\ 0 & a' \end{bmatrix} \right) = \phi(a)
\]
is $k$-positive. Now, take $E = M_2(\mathcal{B}(L))$ and define the Jordan morphism $\mathcal{J} : A \to M_2(\mathcal{B}(L))$ by
\[
\mathcal{J}(a) = \begin{bmatrix} a & 0 \\ 0 & a' \end{bmatrix}
\]
to prove the statement.

We end this section with the remark that it is still an open problem if conditions $(S_k)$, $(P_k)$ and $(W_k)$ are equivalent to $k$-decomposability. The main difficulty in proving the implication, say $(S_k) \Rightarrow (D_k)$, is to find a $k$-positive extension of the map $\psi$ constructed in (3.2) to the whole algebra $M_2(\mathcal{B}(L))$. So, one should answer the following question:

Given a $C^*$-algebra $A$ and a selfadjoint linear unital subspace $S$, find conditions for $k$-positive maps $\psi : S \to \mathcal{B}(H)$ which guarantee the existence of a $k$-positive extension of $\psi$ to whole algebra $A$.

In other words, the analog of Arveson’s extension theorem for completely positive maps should be proved (3.2, see also [29]). The results concerning this problem will be included in the forthcoming paper [10].

4. Tomita-Takesaki scheme for transposition

Let $H$ be a finite dimensional (say $n$-dimensional) Hilbert space. We are concerned with a strongly positive map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$, i.e. a map such that $\phi(a^*a) \geq \phi(a)^*\phi(a)$ for every $a \in \mathcal{B}(H)$ (also called a Schwarz map).

Define $\omega \in \mathcal{B}(H)_{+}^{\top} \setminus 1$ as $\omega(a) = \text{Tr}a\varrho$, where $\varrho$ is an invertible density matrix, i.e. the state $\omega$ is a faithful one. Denote by $(H_\pi, \pi, \Omega)$ the GNS triple associated with $(\mathcal{B}(H), \omega)$. Then, one has:

- $H_\pi$ is identified with $\mathcal{B}(H)$ where the inner product $\langle \cdot, \cdot \rangle$ is defined as $\langle a, b \rangle = \text{Tr}a^*b$, $a, b \in \mathcal{B}(H)$;
- With the above identification: $\Omega = \varrho^{1/2}$;
- $\pi(a)\Omega = a\Omega$;
- The modular conjugation $J_m$ is the hermitian involution: $J_m a\varrho^{1/2} = \varrho^{1/2}a^*$.
• The modular operator $\Delta$ is equal to the map $\varrho \cdot \varrho^{-1}$; We assume that $\omega$ is invariant with respect to $\varphi$, i.e. $\omega \circ \varphi = \omega$. Now, let us consider the operator $T_\varphi \in \mathcal{B}(H_\omega)$ defined by

$$T_\varphi(a\Omega) = \varphi(a)\Omega, \quad a \in \mathcal{B}(H).$$

Obviously $T_\varphi$ is a contraction due to the strong positivity of $\varphi$.

As a next step let us define two conjugations: $J_c$ on $H$ and $J$ on $H_\pi$. To this end we note that the eigenvectors $\{x_i\}$ of $\varrho = \sum_i \lambda_i |x_i\rangle\langle x_i|$ form an orthonormal basis in $H$ (due to the faithfulness of $\omega$). Hence we can define

$$(4.1) \quad J_c f = \sum_i \langle x_i, f \rangle x_i$$

for every $f \in H$. Due to the fact that $E_{ij} = |x_i\rangle\langle x_j|$ form an orthonormal basis in $H_\pi$ we can define in the similar way a conjugation $J$ on $H_\pi$

$$(4.2) \quad J_0 g^{1/2} = \sum_{ij} (E_{ij}, a g^{1/2}) E_{ij}$$

Obviously, $J_0 g^{1/2} = g^{1/2}$.

Now let us define a transposition on $\mathcal{B}(H)$ as the map $a \mapsto a^t \equiv J_c a^* J_c$ where $a \in \mathcal{B}(H)$. By $\tau$ we will denote the map induced on $H_\pi$ by the transposition, i.e.

$$(4.3) \quad \tau a g^{1/2} = a^t g^{1/2}$$

where $a \in \mathcal{B}(H)$. The main properties of the notions introduced above are the following

**Proposition 4.1.** Let $a \in \mathcal{B}(H)$ and $\xi \in H_\pi$. Then

$$a^t \xi = J a^* J \xi.$$

**Proof.** Let $\xi = b g^{1/2}$ for some $b \in \mathcal{B}(H)$. Then we can perform the following calculations

$$J a^* J b g^{1/2} =$$

$$= \sum_{ij} (E_{ij}, a^* J b g^{1/2}) E_{ij} = \sum_{ij} \sum_{kl} (E_{kl}, b g^{1/2})(E_{ij}, a^* E_{kl}) E_{ij}$$

$$= \sum_{ijk} \text{Tr}(E_{ik} b g^{1/2}) \text{Tr}(E_{ij} a^* E_{kl}) E_{ij} = \sum_{ij} \text{Tr}(E_{jk} b g^{1/2}) \text{Tr}(E_{ki} a^*) E_{ij}$$

$$= \sum_{ijk} \langle x_k, b g^{1/2} x_j, a^* x_k \rangle E_{ij} = \sum_{ijk} \langle J_c b g^{1/2} x_j, x_k \rangle \langle x_k, a x_i \rangle E_{ij}$$

$$= \sum_{ij} \langle J_c b g^{1/2} x_j, a x_i \rangle E_{ij} = \sum_{ij} \langle a^* J_c b g^{1/2} x_j, x_i \rangle E_{ij}$$

$$= \sum_{ij} \langle x_i, J_c a^* J_c b g^{1/2} x_j \rangle E_{ij} = \sum_{ij} \langle x_i, a^t b g^{1/2} x_j \rangle E_{ij}$$

$$= \sum_{ij} \text{Tr}(E_{ij} a^t b g^{1/2}) E_{ij} = \sum_{ij} (E_{ij}, a^t b g^{1/2}) E_{ij} = a^t b g^{1/2}$$

$\square$
As a next step let us consider the modular conjugation $J_m$ which has the form
(4.4) \[ J_m a^{1/2} = (a q^{1/2})^* = q^{1/2} a^* \]
Define also the unitary operator $U$ on $H_n$ by
(4.5) \[ U = \sum_{ij} |E_{ij}\rangle\langle E_{ij}| \]
Clearly, $UE_{ij} = E_{ji}$. We have the following

**Proposition 4.2.** Let $J$ and $J_m$ be the conjugations introduced above and $U$ be the unitary operator defined by (4.5). Then we have:
1. $U^2 = I$ and $U = U^*$
2. $J = U J_m$;
3. $J$, $J_m$ and $U$ mutually commute.

**Proof.**
1. We calculate
\[
\sum_{ijmn} |E_{ij}\rangle\langle E_{ji}| |E_{mn}\rangle\langle E_{nm}| = \sum_{ijmn} \text{Tr}(E_{ij} E_{mn}) |E_{ij}\rangle\langle E_{nm}| = \sum_{ij} |E_{ij}\rangle\langle E_{ij}| = I
\]
The rest is evident.

2. Let $b \in \mathcal{B}(H)$. Then
\[
U J_m b q^{1/2} = U q^{1/2} b^* = \sum_{ij} (E_{ij}, q^{1/2} b^*) E_{ij}
\]
\[
= \sum_{ij} \text{Tr}(E_{ij} q^{1/2} b^*) E_{ij} = \sum_{ij} \langle x_j, q^{1/2} b^* x_i \rangle E_{ij}
\]
\[
= \sum_{ij} \langle x_i, b q^{1/2} x_j \rangle E_{ij} = \sum_{ij} \text{Tr}(E_{ji} b q^{1/2}) E_{ij}
\]
\[
= \sum_{ij} \langle E_{ij}, b q^{1/2} \rangle E_{ij} = J b q^{1/2}
\]

3. $J$ is an involution, so by the previous point we have $U J_m U J_m = I$. It is equivalent to the equality $U J_m = J_m U$. Hence we obtain $U J_m = J = J_m U$ and consequently $U J = J_m = J U$ and $J M J = U = J J_m$ because both $U$ and $J_m$ are also involutions.

Now, we are ready to describe a polar decomposition of the map $\tau$.

**Theorem 4.3.** If $\tau$ is the map introduced in (4.3), then
\[ \tau = U \Delta^{1/2}. \]

**Proof.** Let $a \in \mathcal{B}(H)$. Then by Proposition 4.1 and Proposition 4.2) we have
\[ \tau a q^{1/2} = a^* q^{1/2} = J a^* J q^{1/2} = J J_m \Delta^{1/2} a q^{1/2} = U \Delta^{1/2} a q^{1/2}. \]

Now we wish to prove some properties of $U$ which are analogous to that of the modular conjugation $J_m$. To this end we firstly need the following

**Lemma 4.4.** $J$ commutes with $\Delta$
Proof. Let $a \in \mathcal{B}(H)$. Then by Propositions 4.1 and 4.2 and Theorem 4.3 we have

\[
\Delta^{1/2}Ja^q^{1/2} = \Delta^{1/2}Ja^{(a^* q)^{1/2}} = UU\Delta^{1/2}(a^*)^q a^{1/2} = Ua^*q^{1/2} = UJJa^*q^{1/2} = JUa^tq^{1/2} = JUU\Delta^{1/2}a^q^{1/2} = J\Delta^{1/2}a^q^{1/2}
\]

So, $\Delta^{1/2}J = J\Delta^{1/2}$ and consequently $\Delta J = \Delta^{1/2}J\Delta^{1/2} = J\Delta$.

We will also use (cf. [2])

\[
V_\beta = \text{closure}\left\{\Delta^\beta a^q^{1/2} : a \geq 0, \beta \in \left[0, \frac{1}{2}\right]\right\}.
\]

Clearly, each $V_\beta$ is a pointed, generating cone in $H_\pi$ and

\[
V_\beta = \{\xi \in H_\pi : (\eta, \xi) \geq 0 \text{ for all } \eta \in V_{(1/2) - \beta}\}
\]

Recall that $V_{1/4}$ is nothing but the natural cone $\mathcal{P}$ associated with the pair $(\pi(\mathcal{B}(H)), \Omega)$ (see [3] Proposition 2.5.26(1)). Finally, let us define an automorphism $\alpha$ on $\mathcal{B}(H_\pi)$ by

\[
\alpha(a) = UaU^*, \quad a \in \mathcal{B}(H_\pi).
\]

Then we have

**Proposition 4.5.**

1. $U\Delta = \Delta^{-1}U$
2. $\alpha$ maps $\pi(\mathcal{B}(H))$ onto $\pi(\mathcal{B}(H))^\prime$;
3. For every $\beta \in [0, 1/2]$ the unitary $U$ maps $V_\beta$ onto $V_{(1/2) - \beta}$.

**Proof.**

1. By Proposition 4.2 and Lemma 4.4 we have

\[
U\Delta = JJ_m\Delta = J\Delta^{-1}J_m = \Delta^{-1}JJ_m.
\]

2. Let $a, b \in \mathcal{B}(H)$ and $\xi \in H_\pi$. Then Propositions 4.1 and 4.2 imply

\[
UaUb\xi = JJ_mJa_mJbJ\xi = JJ_mJa_m(b^*)^tJ\xi = J(b^*)^tJJ_mJa_m\xi
\]

and the proof is complete.

3. Let $a, b \in \mathcal{B}(H)^+$. Then by the point 1 and Theorem 4.3 we have

\[
(\Delta^\beta a^q^{1/2}, U\Delta^\beta a^q^{1/2}) = (\Delta^\beta a^q^{1/2}, (1/2)^{-\beta}U\Delta^{1/2}a^q^{1/2}) = (\Delta^\beta a^q^{1/2}, (1/2)^{-\beta}a^tq^{1/2})
\]

We recall that $a \mapsto a^t$ is a positive map on $\mathcal{B}(H)$ so by 1.3 the last expression is nonnegative. Hence $UV_\beta \subset V_{(1/2) - \beta}$ for every $\beta \in [0, 1/2]$. As $U$ is an involution, we get $V_{(1/2) - \beta} = U^2V_{(1/2) - \beta} \subset UV_\beta$ and the proof is complete.

**Corollary 4.6.** $U\Delta^{1/2}$ and $T_\omega U\Delta^{1/2}$ map $V_0$ into itself.

Summarizing, this section establishes a close relationship between the Tomita-Takesaki scheme and transposition. Moreover, we have the following:

**Proposition 4.7.** Let $\xi \mapsto \omega_\xi$ be the homeomorphism between the natural cone $\mathcal{P}$ and the set of normal states on $\pi(\mathcal{B}(H))$ described in [4] Theorem 2.5.31, i.e. such that

\[
\omega_\xi(a) = (\xi, a\xi), \quad a \in \mathcal{B}(H).
\]
For every state $\omega$ define $\omega^\tau(a) = \omega(a^\dagger)$ where $a \in B(H)$. If $\xi \in \mathcal{P}$ then the unique vector in $\mathcal{P}$ mapped into the state $\omega^\tau_\xi$ by the homeomorphism described above, is equal to $U\xi$

Proof. Let $\xi = \Delta^{1/4}a\Omega$ for some $a \in B(H)^+$. Then we have

\[
(U\Delta^{1/4}a\Omega, xU\Delta^{1/4}a\Omega) = (\Delta^{1/4}U\Delta^{1/4}a\Omega, x\Delta^{1/4}U\Delta^{1/4}a\Omega) = (\Delta^{1/4}a^\dagger\Omega, x\Delta^{1/4}a^\dagger\Omega) = (\Delta^{1/4}JaJ\Omega, x\Delta^{1/4}JaJ\Omega) = (x^*J\Delta^{1/4}a\Omega, J\Delta^{1/4}a\Omega) = (\Delta^{1/4}a\Omega, Jx^*J\Delta^{1/4}a\Omega)
\]

\[\square\]

5. \textit{k-decomposability at the Hilbert-space level}

The results of Section 4 strongly suggest that a more complete theory of $k$-decomposable maps may be obtained in Hilbert-space terms. To examine that question we will study the description of positivity in the dual approach to that given in Section 3, i.e. we will be concerned with the approach on the Hilbert space level.

Let $\mathcal{M} \subset B(H)$ be a concrete von Neumann algebra with a cyclic and separating vector $\Omega$. When used, $\omega$ will denote the vector state $\omega = (\Omega, \Omega)$. The natural cone (modular operator) associated with $(\mathcal{M}, \Omega)$ will be denoted by $\mathcal{P} (\Delta$ respectively).

By $\mathcal{P}_n$ we denote the natural cone for $(\mathcal{M} \otimes B(\mathbb{C}^n), \omega \otimes \omega_0)$ where $\omega_0$ is a faithful state on $B(\mathbb{C}^n)$ (as an example of $\omega_0$ one can take $\frac{1}{4}Tr$). For the same algebra, $\Delta_n = \Delta \otimes \Delta_0$ and $J_n$ being respectively the modular operator and modular conjugation for $M_n(\mathcal{M})$, are defined in terms of the vector $\Omega_n = \Omega \otimes \Omega_0$ (i.e. in terms of the state $\omega \otimes \omega_0$).

We will consider unital positive maps $\varphi$ on $\mathcal{M}$ which satisfy Detailed Balance II, i.e. there is another positive unital map $\varphi^\beta$ such $\omega(a^\ast \varphi(b)) = \omega(\varphi^\beta(a^\ast)b)$ (see [22]). Such maps induce bounded maps $T_{\varphi} = T$ on $H_\omega = H$ which commute strongly with $\Delta$ and which satisfy $T^+(\mathcal{P}) \subset \mathcal{P}$. Now under the above assumptions ([18], Lemma 4.10) assures us that this correspondence is actually 1-1. Partial transposition $(id \otimes \tau)$ on $M_n(\mathcal{M})$ also induces an operator at the Hilbert space level, but for the sake of simplicity we will where convenient retain the notation $(id \otimes \tau)$ for this operator.

In order to achieve the desired classification of positive maps we introduce the notion of the “transposed cone” $\mathcal{P}^\tau_n = (id \otimes U)\mathcal{P}_n$, where $\tau$ is transposition on $M_n(\mathbb{C})$ while the operator $U$ was defined in the previous Section (we have used the following identification: for the basis $\{e_i\}_i$ in $\mathbb{C}^n$ consisting of eigenvectors of $\varrho_{\omega_0}$ ($\omega_0(\cdot) = Tr\{\varrho_{\omega_0}\}$, we have the basis $\{E_{ij} \equiv |e_i > < e_j|\}_{ij}$ in the GNS Hilbert space associated with $(B(\mathbb{C}^n), \omega_0)$ with $U$ defined in terms of that basis). Note that in the same basis one has the identification $B(\mathbb{C}^n)$ with $M_n(\mathbb{C})$.

Now the natural cone $\mathcal{P}_n$ for $\mathcal{M} \otimes B(\mathbb{C}^n) = M_n(\mathcal{M})$ may be realised as

\[
\mathcal{P}_n = \Delta_n^{1/4}\{[a_{ij}]\Omega_n : [a_{ij}] \in M_n(\mathcal{M})^+\}
\]
Lemma 5.3. Let 
\[
(\lbrack \Delta_0^{1/4} \otimes I \rbrack \Omega_n : [a_{ij}] \in M_n(\mathcal{M})^+}
\]

Thus
\[
\mathcal{P}_n^+ = \Delta_0^{1/4} \{ [a_{ij}] \Omega_n : [a_{ij}] \in M_n(\mathcal{M})^+ \}.
\]

The task of describing the transposed cone will be addressed more adequately in the next section.

Lemma 5.1. The map \( \varphi : \mathcal{M} \to \mathcal{M} \) is \( k \)-positive (\( k \)-copositive) if and only if \((T_\varphi \otimes I)^* (\mathcal{P}_n) \subset \mathcal{P}_n \) (respectively \((T_\varphi \otimes I)^* (\mathcal{P}_n^+) \subset \mathcal{P}_n^+ \)) for every \( 1 \leq n \leq k \).

Proof. To prove \( k \)-positivity case it is enough to suitably adapt the proof of (\[9;\) Lemma 4.10), while to prove \( k \)-copositivity we observe that the “if” part of the hypothesis implies
\[
0 \leq (T_\varphi \otimes I)(\lbrack \Delta_0^{1/4} \otimes I \rbrack \mathcal{P}_n, \mathcal{P}_n).
\]

Thus
\[
(\Delta_0^{1/4} \lbrack T(a_{ij}) \rbrack \Omega_n, \Delta_0^{-1/4} \lbrack b_{kl} \rbrack \Omega_n) = (\lbrack T(a_{ij}) \rbrack \Omega_n, \lbrack b_{kl} \rbrack \Omega_n) \geq 0
\]
where \( [a_{ij}] \geq 0 \) is in the algebra \( M_n(\mathcal{M}) \), and \( [b_{kl}] \) in its commutant. This implies \( [T(a_{ij})] \geq 0 \) and the rest is again a suitable adaptation of the proof of (\[9;\) Lemma 4.10). \( \square \)

Lemma 5.2. For each \( n \), \( \mathcal{P}_n \cap \mathcal{P}_n^+ \) and \( \overline{\mathcal{P}_n \cup \mathcal{P}_n^+} \) are dual cones.

Proof. For any \( X \subset H \) we denote \( X^d = \{ \xi \in H : \langle \xi, \eta \rangle \geq 0 \text{ for any } \eta \in X \} \). To prove the lemma it is enough to observe that \( \mathcal{P}_n^d = \mathcal{P}_n \) and \( (\mathcal{P}_n^d)^d = \mathcal{P}_n^d \). \( \square \)

Lemma 5.3. Let \( n \) be given. For any \( [a_{ij}] \in M_n(\mathcal{M})^+ \), \( \Delta_0^{1/4} [a_{ij}] \Omega_n \in \mathcal{P}_n \cap \mathcal{P}_n^+ \) implies \( [a_{ij}] \in M_n(\mathcal{M})^+ \).

Proof. Let \( [a_{ij}] \in M_n(\mathcal{M})^+ \) be given and assume that \( \Delta_0^{1/4} [a_{ij}] \Omega_n \in \mathcal{P}_n \cap \mathcal{P}_n^+ \). We observe
\[
\Delta_0^{1/4} [a_{ij}] \Omega_n = (I \otimes U) \Delta_0^{1/4} [a_{ij}] \Omega_n \in (I \otimes U) (\mathcal{P}_n \cap \mathcal{P}_n^+) = \mathcal{P}_n \cap \mathcal{P}_n^+ \subset \mathcal{P}_n.
\]

But then the self-duality of \( \mathcal{P}_n \) alongside (\[9;\) 2.5.26) will ensure that
\[
0 \leq (\Delta_0^{1/4} [a_{ij}] \Omega_n, \Delta_0^{-1/4} [b_{kl}] \Omega_n) = ([a_{ij}] \Omega_n, [b_{kl}] \Omega_n)
\]
for each \( [b_{ij}] \in (M_n(\mathcal{M})^+ \). We may now conclude from (\[9;\) 2.5.1 or (\[9;\) 2.3.19) that \( [a_{ij}] \geq 0 \), as required. \( \square \)

Corollary 5.4. In the finite dimensional case \( \{ \Delta_0^{1/4} [a_{ij}] \Omega_n : [a_{ij}] \geq 0, [a_{ij}] \geq 0 \} = \mathcal{P}_n \cap \mathcal{P}_n^+ \).

Proof. First note that in this case \( \{ \Delta_0^{1/4} [a_{ij}] \Omega_n : [a_{ij}] \geq 0 \} = \mathcal{P}_n \) (cf. (\[9;\) Proposition 2.5.26)). Now apply the previous lemma. \( \square \)
Recall $\Delta_n^{-1/4}$ maps $\{[b_{ij}]\Omega_n : [b_{ij}] \in (M_n(\mathcal{M}))^+\}$ densely into $\mathcal{P}_n$ (see for example [4]). At least on a formal level one may therefore by analogy with [4, 2.5.26 & 2.5.27] expect to end up with a dense subset of the dual cone of $\overline{\sigma}(\mathcal{P}_n \cup \mathcal{P}_n^\tau)$ (ie. of $\mathcal{P}_n \cap P_n^\tau$) when applying $\Delta_n^{-1/4}$ to the set of all $\alpha$'s satisfying $([b_{ij}]\Omega_n, \alpha) \geq 0$ and $([b_{ij}]\Omega_n, (\mathbb{I} \otimes U)\alpha) \geq 0$ for each $[b_{ij}] \in (M_n(\mathcal{M}))^+$. If true such a fact would then put one in a position to try and show that in general $\mathcal{P}_n \cap P_n^\tau = \{\Delta_n^{-1/4}[a_{ij}]\Omega_n : [a_{ij}] \geq 0, [a_{ij}] \geq 0\}$.

Question. Is it generally true that

$$\{\Delta_n^{-1/4}[a_{ij}]\Omega_n : [a_{ij}] \geq 0, [a_{ij}] \geq 0\} = \mathcal{P}_n \cap P_n^\tau?$$

In the light of the following result the answer to this becomes important in an attempt to generalize the finite case to the infinite dimensional one.

**Theorem 5.5.** In general the property $(T_\varphi \otimes \mathbb{I})^*(\mathcal{P}_n) \subset \overline{\sigma}(\mathcal{P}_n \cup \mathcal{P}_n^\tau)$ for each $1 \leq n \leq k$ implies that $\varphi$ is weakly $k$-decomposable in the sense that for each $1 \leq n \leq k$, $\varphi(a_{ij}) \geq 0$ whenever $[a_{ij}], [a_{ij}] \in M_n(\mathcal{M})^+$.

If $[\Delta_n^{-1/4}[a_{ij}]\Omega_n : [a_{ij}] \geq 0, [a_{ij}] \geq 0] = \mathcal{P}_n \cap P_n^\tau$ for each $1 \leq n \leq k$, the converse implication also holds. In particular in the finite-dimensional case the two statements are equivalent. (Pending the answer to the aforementioned question, they may of course be equivalent in general.)

**Proof.** Suppose that $(T_\varphi \otimes \mathbb{I})^*(\mathcal{P}_n) \subset \overline{\sigma}(\mathcal{P}_n \cup \mathcal{P}_n^\tau)$ for each $1 \leq n \leq k$. Given $1 \leq n \leq k$ and $[a_{ij}] \in M_n(\mathcal{M})$ it now follows from [4, Proposition 2.3.19] and the strong commutation of $T_\varphi \otimes \mathbb{I}$ with $\Delta_k$, that $[\varphi(a_{ij})] \geq 0$ if and only if

$$0 \leq \varphi_n([a_{ij}]\Omega_n; [b_{ij}]\Omega_n) = ([a_{ij}]\Omega_n, (T_\varphi \otimes \mathbb{I})^*[b_{ij}]\Omega_n) = (\Delta_n^{-1/4}[a_{ij}]\Omega_n, (T_\varphi \otimes \mathbb{I})^*\Delta_n^{-1/4}[b_{ij}]\Omega_n)$$

for each $[b_{ij}] \in (M_n(\mathcal{M}))^+$.

Now if $[a_{ij}] \geq 0$ and $[a_{ij}] \geq 0$ then the fact that $\text{id} \otimes \tau$ commutes strongly with $\Delta_n$, surely ensures that $\Delta_n^{-1/4}[a_{ij}]\Omega_n \in \mathcal{P}_n \cap P_n^\tau$. Moreover for any $[b_{ij}] \in (M_n(\mathcal{M}))^+$, [4, Proposition 2.5.26] alongside the hypothesis ensures that $$(T_\varphi \otimes \mathbb{I})^*\Delta_n^{-1/4}[b_{ij}]\Omega_n \in \overline{\sigma}(\mathcal{P}_n \cup \mathcal{P}_n^\tau).$$

In this case it therefore follows from the duality of $\mathcal{P}_n \cap \mathcal{P}_n^\tau$ and $\overline{\sigma}(\mathcal{P}_n \cup \mathcal{P}_n^\tau)$ that $0 \leq (\Delta_n^{-1/4}[a_{ij}]\Omega_n, (T_\varphi \otimes \mathbb{I})^*\Delta_n^{-1/4}[b_{ij}]\Omega_n)$ for each $[b_{ij}] \in (M_n(\mathcal{M}))^+$, and hence that $[\varphi(a_{ij})] \geq 0$ as required.

For the converse suppose that $\{\Delta_n^{-1/4}[a_{ij}]\Omega_n : [a_{ij}] \geq 0, [a_{ij}] \geq 0\} = \mathcal{P}_n \cap P_n^\tau$ for each $1 \leq n \leq k$ and that for each $1 \leq n \leq k$ we have that $[\varphi(a_{ij})] \geq 0$ whenever $[a_{ij}], [a_{ij}] \in M_n(\mathcal{M})^+$. To see that then $(T_\varphi \otimes \mathbb{I})^*(\mathcal{P}_n) \subset \overline{\sigma}(\mathcal{P}_n \cup \mathcal{P}_n^\tau)$ for each $1 \leq n \leq k$, we need only show that $(T_\varphi \otimes \mathbb{I})^*(\Delta_n^{-1/4}[b_{ij}]\Omega_n) \subset \overline{\sigma}(\mathcal{P}_n \cup \mathcal{P}_n^\tau)$ for each $1 \leq n \leq k$ and each $[b_{ij}] \in (M_n(\mathcal{M}))^+$ ([4, Proposition 2.5.26]). To see that this is indeed the case, the duality of $\mathcal{P}_n \cap \mathcal{P}_n^\tau$ and $\overline{\sigma}(\mathcal{P}_n \cup \mathcal{P}_n^\tau)$ ensures that we need only show that

$$0 \leq (\eta, (T_\varphi \otimes \mathbb{I})^*\Delta_n^{-1/4}[b_{ij}]\Omega_n)$$
for each \( \eta \in \mathcal{P}_n \cap \mathcal{P}_n^* \). In the light of our assumption regarding \( \mathcal{P}_n \cap \mathcal{P}_n^* \), this in turn means that we need to show that

\[
0 \leq (\Delta_n^{1/4}[a_{ij}]\Omega_n, (T_\varphi \otimes \mathbb{I})^*\Delta_n^{-1/4}[b_{ij}]\Omega_n) \\
= ([a_{ij}]\Omega_n, (T_\varphi \otimes \mathbb{I})^*[b_{ij}]\Omega_n) \\
= (\varphi_n([a_{ij}])\Omega_n, [b_{ij}]\Omega_n)
\]

for each \([b_{ij}] \in (M_n(\mathcal{M}))^+\) and each \([a_{ij}] \in M_n(\mathcal{M})\) with \([a_{ij}] \geq 0, [a_{ji}] \geq 0\). Since by assumption \([\varphi(a_{ij})] \geq 0\) whenever \([a_{ij}] \geq 0, [a_{ji}] \geq 0\) \((1 \leq n \leq k)\), the claim therefore follows from [4, Proposition 2.3.19].

\[\square\]

6. Tomita-Takesaki approach for partial transposition

In order to obtain a more complete characterisation of \(k\)-decomposable maps, one should describe elements of the cone \(\mathcal{P}_k \cap \mathcal{P}_k^*\) (cf. Theorem 6.4). In this section we formulate the general scheme for this description.

Suppose that \(A\) is a \(C^*\)-algebra equipped with a faithful state \(\omega_A\). Let \(B = \mathcal{B}(K_B)\) for some Hilbert space \(K_B\), \(\varrho\) be an invertible density matrix in \(\mathcal{B}(K)\) and \(\omega_B\) be a state on \(B\) such that \(\omega_B(b) = \text{Tr}(\varrho b)\) for \(b \in B\). By \((H, \pi, \Omega), (H_A, \pi_A, \Omega_A)\) and \((H_B, \pi_B, \Omega_B)\) we denote the GNS representations of \((A \otimes B, \omega_A \otimes \omega_B)\), \((A, \omega_A)\) and \((B, \omega_B)\) respectively. We observe that we can make the following identifications:

1. \(H = H_A \otimes H_B\),
2. \(\pi = \pi_A \otimes \pi_B\),
3. \(\Omega = \Omega_A \otimes \Omega_B\).

With these identifications we have \(J_m = J_A \otimes J_B\) and \(\Delta = \Delta_A \otimes \Delta_B\) where \(J_m, J_A, J_B\) are modular conjugations and \(\Delta, \Delta_A, \Delta_B\) are modular operators for \((\pi(A \otimes B)'', \Omega), (\pi(A)'', \Omega_A), (\pi(B)'', \omega_B)\) respectively. Since \(\Omega_A\) and \(\Omega_B\) are separating vectors, we will write \(a\Omega_A\) and \(b\Omega_B\) instead of \(\pi_A(a)\Omega_A\) and \(\pi_B(b)\Omega_B\) for \(a \in A\) and \(b \in B\).

The natural cone \(\mathcal{P}\) for \((\pi(A \otimes B)'', \Omega)\) is defined (see [1] or [2]) as the closure of the set

\[
\left\{ \left( \sum_{k=1}^{n} a_k \otimes b_k \right) j_m \left( \sum_{l=1}^{n} a_l \otimes b_l \right) \Omega : n \in \mathbb{N}, a_1, \ldots, a_n \in A, b_1, \ldots, b_n \in B \right\}
\]

where \(j_m(\cdot) = J_m\), \(J_m\) is the modular morphism on \(\pi(A \otimes B)' = \pi_A(A)'' \otimes \pi_B(B)'\).

Recall (see Section 4) that \(H_B\) is the closure of the set \(\{b^1/2 : b \in B\}\) and \(\Omega_B\) can be identified with \(\varrho^{1/2}\). Let \(U_B\) be the unitary operator on \(H_B\) described in Section 4. Then we have

**Lemma 6.1.** \((1 \otimes U_B)\mathcal{P}\) is the closure of the set

\[
\left\{ \left( \sum_{k=1}^{n} a_k \otimes \alpha(b_k) \right) j_m \left( \sum_{l=1}^{n} a_l \otimes \alpha(b_l) \right) \Omega : n \in \mathbb{N}, a_1, \ldots, a_n \in A, b_1, \ldots, b_n \in B \right\},
\]
Proof. Using the Tomita-Takesaki approach one has
\[
(\mathbb{I} \otimes U_B) \left( \sum_k a_k \otimes b_k \right) j_m \left( \sum_l a_l \otimes b_l \right) \Omega =
\]
\[
= \sum_{kl} a_k j_A(a_l) \Omega_A \otimes U_B b_k J_B b_l J_B \Omega_B
\]
\[
= \sum_{kl} a_k j_A(a_l) \Omega_A \otimes U_B b_k U_B J_B b_l J_B \Omega_B
\]
\[
= \sum_{kl} a_k j_A(a_l) \Omega_A \otimes U_B b_k U_B J_B J_B b_l J_B U_B \Omega_B
\]
\[
= \left( \sum_k a_k \otimes \alpha(b_k) \right) j_m \left( \sum_l a_l \otimes \alpha(b_l) \right)
\]
In the third equality we used the fact that \( U_B \) commutes with \( J_B \). \( \square \)

This leads us to:

**Theorem 6.2.** Suppose that \( K \) is a finite dimensional Hilbert space. Then \( (\mathbb{I} \otimes U_B) \mathcal{P} = \mathcal{P}' \) where \( \mathcal{P}' \) is the natural cone associated with \( (\pi_A(A) \otimes \pi_B(B)' , \Omega) \).

**Proof.** We just proved, that \((\mathbb{I} \otimes U_B)\mathcal{P}\) is the closure of the set
\[
\left\{ \left( \sum_{k=1}^n a_k \otimes \alpha(b_k) \right) j_m \left( \sum_{l=1}^n a_l \otimes \alpha(b_l) \right) \Omega : n \in \mathbb{N}, a_1, \ldots, a_n \in A, b_1, \ldots, b_n \in B \right\}.
\]
By Proposition 4.5.2, \( \alpha \) maps \( \pi_B(B)' \) onto \( \pi_B(B)' \), so the assertion is obvious. \( \square \)

Consequently, \( \mathcal{P}_k \cap \mathcal{P}_k' \) is nothing else but \( \mathcal{P}_k \cap \mathcal{P}_k' \).

In the sequel we will assume that \( A = B(K_A) \) for some finite dimensional Hilbert space \( K_A \) and that \( \omega_A \) is determined by some density matrix \( \varrho_A \) in \( B(K_A) \).

**Remark 6.3.** The operator \( \mathbb{I} \otimes U_B \) is a symmetry in \( B(H_A \otimes H_B) \) in the sense of [1] (see the paragraph preceding Lemma 6.33). Obviously, \( \mathbb{I} \otimes U_B \) has a spectral decomposition of the form \( \mathbb{I} \otimes U_B = P - Q \) where \( P \) and \( Q \) are mutually orthogonal projections in \( B(H_A \otimes H_B) \) such that \( P + Q = \mathbb{I} \).

Moreover, if \( \mathcal{S}(B(H_A \otimes H_B)) \) denotes the set of states on \( B(H_A \otimes H_B) \) and \( F \) and \( G \) are norm closed faces in \( \mathcal{S}(B(H_A \otimes H_B)) \) associated with \( P \) and \( Q \) respectively, then \( F \) and \( G \) are antipodal and affinely independent faces in \( \mathcal{S}(B(H_A \otimes H_B)) \) forming a generalized axis \( (F,G) \) of \( \mathcal{S}(B(H_A \otimes H_B)) \).

Furthermore, the symmetry \( \mathbb{I} \otimes U_B \) provides the one parameter group \( (\alpha_t')_{t \in \mathbb{R}} \) (where \( \alpha_t(\cdot) = \exp \left( \frac{it}{\mathbb{I} \otimes U_B, \cdot} \right) \) for \( t \in \mathbb{R} \) which is the generalised rotation of \( \mathcal{S}(B(H_A \otimes H_B)) \) about \( (F,G) \) (cf. [1] Chapter 6). On the other hand (see again [1]) in the algebra \( B(K_A \otimes K_B) \) there are canonical symmetries associated to \( 2 \times 2 \)-matrix units \( \{e_{ij}\} \). Moreover these symmetries can be extended to a Cartesian triple of symmetries of \( B(K_A \otimes K_B) \); a fact which is the basic ingredient of the definition of orientation of \( B(K_A \otimes K_B) \). By contrast partial transposition yields a symmetry \( \mathbb{I} \otimes U_B \) in the larger algebra \( B(H_A \otimes H_B) \supset B(K_A \otimes K_B) \) and it would seem that in general this symmetry tends to “spoil” the orientation structure of the algebra of interest, i.e. \( B(K_A \otimes K_B) \).

More precisely: one can repeat the above arguments for \( U_B \), so \( U_B \) is the symmetry of \( B(H_B) \supset B(K_B) \). The operator \( \mathbb{I} \) is a symmetry of the first factor, being
an element of the smaller algebra $\mathcal{B}(H_A) \supset \mathcal{B}(K_A)$. Clearly, this symmetry does not change the orientation of the algebra of the first factor. As $1 \otimes U_B$ is the tensor product of $1$ and $U_B$, we “translated” the basic feature of partial transposition – tensor product of morphism and antimorphism.

As a clarification of the role of the symmetry $I \otimes U_B$ in the structure of orientation of $\mathcal{B}(K_A \otimes K_B)$ is an open question, we wish to collect some properties of $I \otimes U_B$ in the rest of that section. To this end assume that $(e_i)$ and $(f_k)$ are orthonormal bases in $K_A$ and $K_B$ respectively consisted of eigenvectors of $\varrho_A$ and $\varrho_B$ respectively; by $(E_{ij})$ and $(F_{kl})$ we denote matrix units associated with $(e_i)$ and $(f_k)$ respectively.

Each element $a$ of $\mathcal{B}(K_A) \otimes \mathcal{B}(K_B)$ can be uniquely written in the form $a = \sum_{ij} a_{ij} \otimes F_{ij}$ for some elements $a_{ij} \in \mathcal{B}(K_A)$.

Let $\tilde{U} = I \otimes U_B$. Observe that projections $P$ and $Q$ are of the form

$$P = \frac{1}{2}(1 + \tilde{U}), \quad Q = \frac{1}{2}(1 - \tilde{U}).$$

At first, we formulate a step towards an eventual characterisation of the cone $P \cap \tilde{U}P$.

**Proposition 6.4.**

1. $\mathcal{P} \cap \tilde{U}\mathcal{P}$ is a maximal subcone of $\mathcal{P}$ which is globally invariant with respect to $\tilde{U}$.

2. $\mathcal{P}_A \otimes \mathcal{P}_B \subset \tilde{U}\mathcal{P}$, where $\mathcal{P}_A \subset H_A$ and $\mathcal{P}_B \subset H_B$ are the natural cones respectively corresponding to the algebras $A$ and $B$.

3. Let $U_A$ denote the unitary operator on $H_A$ introduced in section 4 and $P_A$, $Q_A$ (resp. $P_B$, $Q_B$) be spectral projections of $U_A$ (resp. $U_B$). For $\xi \in H$ the following are equivalent
   (a) $\xi \in \mathcal{P} \cap \tilde{U}\mathcal{P}$;
   (b) for every $\eta \in \mathcal{P}$ we have
       $$(\eta, Q\xi) \leq (\eta, P\xi);$$
   (c) for every $\eta \in \mathcal{P}$ we have
       $$(\eta, \xi) \geq 0 \quad \text{and} \quad 2(\eta, Q\xi) \leq (\eta, \xi);$$
   (d) for every $\eta \in \mathcal{P}$ we have
       $$(\eta, (P_A \otimes P_B)\xi) + (\eta, (Q_A \otimes P_B)\xi)$$
       $$\geq (\eta, (P_A \otimes Q_B)\xi) + (\eta, (Q_A \otimes Q_B)\xi);$$
   (e) for every $\eta \in \mathcal{P}$ we have
       $$(\eta, (P_A \otimes P_B)\xi) - (\eta, (Q_A \otimes P_B)\xi)$$
       $$\geq - (\eta, (P_A \otimes Q_B)\xi) + (\eta, (Q_A \otimes Q_B)\xi).$$

4. If $\xi \in \mathcal{P} \cap \tilde{U}\mathcal{P}$, then $\|Q\xi\| \leq \|P\xi\|$.

5. $\xi \in \mathcal{P} \cap \tilde{U}\mathcal{P}$ implies that for every $\eta \in \mathcal{P}$

   $$2(\eta, Q_A \otimes Q_B\xi) \leq (\eta, P^{\text{tot}}\xi)$$

   where $P^{\text{tot}} = \frac{1}{2}(1 + U_A \otimes U_B)$. 
Proof. Properties (1) and (2) follow from easy observations. In order to prove (3) observe that both \( \xi \) and \( \tilde{\U} \xi \) are in \( \mathcal{P} \), so the selfduality of \( \mathcal{P} \) implies that for every \( \eta \in \mathcal{P} \) we have

\[
0 \leq (\eta, \xi) = (\eta, P \xi) + (\eta, Q \xi),
\]
\[
0 \leq (\eta, \tilde{\U} \xi) = (\eta, P \xi) - (\eta, Q \xi).
\]

Thus we have the equivalence of (a) and (b). The equivalence of (a) and (c) follows from the fact that \( \tilde{\U} \xi \in \mathcal{P} \) is equivalent to the following inequality: for every \( \eta \in \mathcal{P} \)

\[
0 \leq (\eta, \tilde{\U} \xi) = (\eta, P \xi) - (\eta, Q \xi) = (\eta, \xi) - 2(\eta, Q \xi).
\]

The rest of (3) can be checked by simple calculations. To prove (4) assume that \( \xi \in \mathcal{P} \cap \mathcal{P} \) and \( \eta, \eta' \in \mathcal{P} \). From (3) we have

\[
- (\eta, P \xi) \leq (\eta, Q \xi) \leq (\eta, P \xi)
\]
\[
- (\eta', P \xi) \leq - (\eta', Q \xi) \leq (\eta', P \xi)
\]

and consequently

\[
| (\eta - \eta', Q \xi) | \leq (\eta + \eta', P \xi).
\]

To see that (4) holds, we merely need to apply the above inequality to the case \( \eta = \frac{1}{2} \xi \) and \( \eta' = \frac{1}{2} \tilde{\U} \xi \).

It remains to prove (5). Assume \( \xi \in \mathcal{P} \cap \mathcal{P} \). One can easily check that \((U_A \otimes U_B) \mathcal{P} = \mathcal{P}\). Hence, from (3) we have \(((U_A \otimes U_B) \eta, \xi) \geq 0\) and \(2((U_A \otimes U_B) \eta, Q \xi) \leq ((U_A \otimes U_B) \eta, \xi)\). Observe that

\[
((U_A \otimes U_B) \eta, Q \xi) = (\eta, (U_A \otimes U_B)(I \otimes Q B) \xi) = (\eta, (U_A \otimes (P_B - Q B)Q B) \xi) = - (\eta, (U_A \otimes Q B) \xi).
\]

Thus we have

\[
-2(\eta, (U_A \otimes Q B) \xi) \leq (\eta, (U_A \otimes U_B) \xi)
\]
\[
2(\eta, (I \otimes Q B) \xi) \leq (\eta, \xi)
\]

where the second inequality follows from (3). Consequently, we have

\[
2(\eta, (Q_A \otimes Q B) \xi) = (\eta, (I \otimes Q B) \xi) - (\eta, (U_A \otimes Q B) \xi) \leq \frac{1}{2}((\eta, \xi) + (\eta, (U_A \otimes U_B) \xi)) = (\eta, P^{tot} \xi)
\]

and the proof is ended. \( \square \)

Now, assume that \( \dim K_A = \dim K_B = 2 \). If \( \xi = \Delta^{1/4}[a_{ij}] \Omega \), then \( \tilde{\U} \xi = \Delta^{1/4}[a_{ji}] \Omega \), and consequently

\[
P \xi = \frac{1}{2} \Delta^{1/4} \begin{bmatrix} 2a_{11} & a_{12} + a_{21} \\ a_{12} + a_{21} & 2a_{22} \end{bmatrix} \Omega,
\]
\[
Q \xi = \frac{1}{2} \Delta^{1/4} \begin{bmatrix} 0 & a_{12} - a_{21} \\ a_{12} - a_{21} & 0 \end{bmatrix} \Omega.
\]

It is easy to observe that if \( \xi, \tilde{\U} \xi \in \mathcal{P} \), then \( P \xi \in \mathcal{P} \). Moreover, we have the following
Proposition 6.5. Let \( \xi \in \mathcal{P} \). Then the following are equivalent:

1. \( Q\xi \in \mathcal{P} \),
2. \( Q\xi = 0 \),
3. \( \xi \) is a fixed point of \( \tilde{U} \).

Proof. If \( \xi \in \mathcal{P} \) then \([a_{ij}]\) is positive in \( \mathcal{B}(H_A) \). Then \( a^*_{12} = a_{21} \). Let \( b = \frac{1}{2}(a_{12} - a^*_{12}) \). We have that \( b = ih \) for some selfadjoint element \( \mathcal{B}(K_A) \) and \( Q\xi = \Delta^{1/4} \begin{bmatrix} 0 & ih \\ -ih & 0 \end{bmatrix} \Omega \). Now if \( P_2\xi \in \mathcal{P} \), we necessarily have that \( \begin{bmatrix} 0 & ih \\ -ih & 0 \end{bmatrix} \) is positive if and only if \( h = 0 \), so \( 1 \) and \( 2 \) are equivalent. The equivalence of \( 2 \) and \( 3 \) is evident. \( \square \)

Hence, in general, \( Q\xi \) is not in \( \mathcal{P} \). However, (cf \[2\]), for each \( \zeta \in H \) there exists \( |\zeta| \in \mathcal{P} \) such that \( \zeta = u|\zeta| \) for some partial isometry \( u \). In the considered case we can calculate \( |Q\xi| \) explicitly. Namely we get

Proposition 6.6. Let \( i\hbar = v|\hbar| \) be the polar decomposition of element \( i\hbar \). Then \( Q\xi = \tilde{V}\xi_b \), where \( \tilde{V} = \Delta^{1/4} \begin{bmatrix} 0 & v \\ -v & 0 \end{bmatrix} \Delta^{-1/4} \) and \( \xi_b \in \mathcal{P} \).

Proof. Let \( B = \begin{bmatrix} 0 & ih \\ -ih & 0 \end{bmatrix} \) and \( V = \begin{bmatrix} 0 & v \\ -v & 0 \end{bmatrix} \). Then one can check that \( B = V \begin{bmatrix} |\hbar| & 0 \\ 0 & |\hbar| \end{bmatrix} \) is the polar decomposition of \( B \). Furthermore we have

\[
Q\xi = \Delta^{1/4} B\Omega = \Delta^{1/4} V|B|\Omega = \Delta^{1/4} V\Delta^{-1/4} \Delta^{1/4} B|\Omega = \tilde{V}\xi_b
\]

where \( \xi_b = \Delta^{1/4} |B|\Omega \) is an element of \( \mathcal{P} \). \( \square \)

Here is another way of writing \( Q\xi \). Namely, there is \( |Q\xi| \in \mathcal{P} \) such that \( Q\xi = u|P_2\xi| \) where \( u \) is a partial isometry such that \( u \in (A \otimes B)' \), \( uu^* = [(A \otimes B)'|Q\xi] \) and \( u^*u = [(A \otimes B)'|\xi] \) (cf. \[2\]). Moreover, one can check that

\[
Q\xi = \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -g_A^{1/4}b^{1/4}_A \end{bmatrix}
\]

where \( \alpha = \lambda_1^{1/4}\lambda_2^{-1/4} \) and \( \lambda_1, \lambda_2 \) are eigenvalues of \( g_B \) (\( b \) was defined in the proof of Proposition 6.5).

What is still lacking is an explicit description of the role of the symmetry \( \tilde{U} \) in terms of the algebra \( \mathcal{B}(K_A) \otimes \mathcal{B}(K_B) \). This will be done in the forthcoming paper \[10\].

References

1. E.M. Alfsen and W. Shultz, State spaces of operator algebras, Birkhauser, Boston, 2001.
2. H. Araki, Some properties of modular conjugation operator of a von Neumann algebra and a non-commutative Radon-Nikodym theorem with a chain rule, Pac. J. Math. 50 (1974), 309–354.
3. W. Arveson, Subalgebras of \( C^* \)-algebras, Acta Math. 123 (1969), 141–224.
4. O. Bratteli and D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics 1: Second Edition, Springer-Verlag, New York, 1987.
5. M.-D. Choi, A Schwarz inequality for positive linear maps on C*-algebras, Illinois J. Math. 18(4) (1974), 565–574.
6. M.-D. Choi, Completely positive maps on complex matrices, Lin. Alg. Appl. 10 (1975), 285–290.
7. M.-D. Choi, Positive semidefinite biquadratic forms, Lin. Alg. Appl. 12 (1975), 95–100.
8. A. Connes, Caractérisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von Neumann, Ann. Inst. Fourier, Grenoble 24 (1974), 121-155
9. J. Dixmier, Les C*-Algèbres et leurs Représentations, Gauthier-Villars, Paris, 1964
10. M.-H. Eom and S.-H. Kye, Duality for positive linear maps in matrix algebras, Math. Scand. 86 (2000), 130–142.
11. K.-C. Ha, Atomic positive linear maps in matrix algebras, Publ. RIMS, Kyoto Univ. 34 (1998), 591–599.
12. M. Horodecki, P. Horodecki, R. Horodecki, Separability of mixed states: necessary and sufficient conditions, Phys. Lett A 223, 1-8 (1996)
13. R. V. Kadison, Transformations of states in operator theory and dynamics, Topology 3 (1965) 177-198
14. H.-J. Kim and S.-H. Kye, Indecomposable extreme positive linear maps in matrix algebras, Bull. London Math. Soc. 26 (1994), 575–581.
15. A. Kossakowski, A family of positive linear maps in matrix algebras, submitted to Open Systems & Information Dynamics.
16. L. E. Labuschagne, W. A. Majewski and M. Marciniak, On decomposition of positive maps, in preparation.
17. W. A. Majewski, Transformations between quantum states, Rep. Math. Phys. 8 (1975), 295–307.
18. W.A. Majewski, Dynamical Semigroups in the Algebraic Formulation of Statistical Mechanics, Fortschr. Phys. 32(1984)1, 89–133.
19. W. A. Majewski, Separable and entangled states of composite quantum systems; Rigorous description, Open Systems & Information Dynamics, 6, 79-88 (1999)
20. W. A. Majewski, “Quantum Stochastic Dynamical Semigroup”, in Dynamics of Dissipations, eds P. Garbaczewski and R. Olkiewicz, Lecture Notes in Physics, vol. 597, pp. 305-316, Springer (2002).
21. W. A. Majewski and M. Marciniak, On a characterization of positive maps, J. Phys. A: Math. Gen. 34 (2001), 5863–5874.
22. W. A. Majewski and R. Streater, Detailed balance and quantum dynamical maps, J. Phys. A: Math. Gen. 31 (1998), 7981–7995.
23. A. Peres, Separability criterion for density matrices, Phys. Rev. Lett 77, 1413 (1996)
24. G. Robertson, Schwarz inequalities and the decomposition of positive maps on C*-algebras, Math. Proc. Camb. Phil. Soc. 94 (1983), 291–296.
25. E. Størmer, Positive linear maps of operator algebras, Acta Math. 110 (1963), 233–278.
26. E. Størmer, On the Jordan structure of C*-algebras, Trans. Amer. Math. Soc. 120 (1965), 438–447.
27. E. Størmer, Decomposition of positive projections on C*-algebras, Math. Annalen 247 (1980), 21 - 41.
28. E. Størmer, Decomposable positive maps on C*-algebras, Proc. Amer. Math. Soc. 86 (1982), 402–404.
29. E. Størmer, Extension of positive maps into B(H), J. Funct. Anal. 66, (1986), 235–254.
30. J. Tomiyama, On the difference of n-positivity and complete positivity in C*-algebras, J. Funct. Anal. 49 (1982), 1–9.
31. G. Wittstock, Ordered Normed Tensor Products in “Foundations of Quantum Mechanics and Ordered Linear Spaces” (Advanced Study Institute held in Marburg) A. Hartkämer and H. Neumann eds. Lecture Notes in Physics vol. 29, Springer Verlag 1974.
32. S. L. Woronowicz, Positive maps of low dimensional matrix algebras, Rep. Math. Phys. 10 (1976), 165-183.
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