Gravitational energy in a small region for the modified Einstein and Landau–Lifshitz pseudotensors

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Abstract
The purpose of the classical Einstein and Landau–Lifshitz and other pseudotensors is to determine the gravitational energy. Neither of them can guarantee a positive energy in holonomic frames. In the small sphere approximation, it has been required that the quasilocal expression for the gravitational energy–momentum density should be proportional to the Bel–Robinson tensor $B_{\alpha\beta\mu\nu}$. However, we propose a new tensor $V_{\alpha\beta\mu\nu}$ that is the sum of certain tensors $S_{\alpha\beta\mu\nu}$ and $K_{\alpha\beta\mu\nu}$; it has certain properties so that it gives the same gravitational ‘energy–momentum’ density content as $B_{\alpha\beta\mu\nu}$ does. In comparison, the main difference is that $B_{\alpha\beta\mu\nu}$ fulfils the dominant energy condition while $V_{\alpha\beta\mu\nu}$ does not. Moreover, we show that a modified Einstein pseudotensor turns out to be a generalization of one of the Chen–Nester quasilocal expressions, while the modified Landau–Lifshitz pseudotensor becomes the Papapetrou pseudotensor; these two modified pseudotensors have positive gravitational energy in a small region.

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1. Introduction
Gravitational energy should be positive; the proof for a general region is not so easy. Let us just consider a small region. To get positive energy in the small sphere approximation, it has been required that the quasilocal expression for the gravitational energy–momentum density should be proportional to the Bel–Robinson tensor [1]. In the past (e.g., [2–4]) and also recently (e.g., [5–7]) there have been many studies of this tensor. It is the best tensor to assure positivity in the small region limit. The Bel–Robinson tensor also has many other nice properties, for instance the dominant energy condition, and it is completely symmetric, trace free and divergence free.

A pseudotensor is used for determining the gravitational energy. Unfortunately, a pseudotensor is not a tensorial object, which means that it is frame dependent. In particular,
soon after Einstein invented his pseudotensor in 1916, Schrödinger [8] showed that the energy density is zero in Schwarzschild coordinate. Moreover, Bauer [9] pointed out that it can give a non-zero energy density even in Minkowski coordinate. From this simple survey of the history in the past, it may seem that studying pseudotensors is absolutely useless and a waste of time. However, the situation is really not so bad. Indeed, there are some advantages in studying the classical pseudotensors.

First of all, a pseudotensor is the only way to define an integral energy–momentum which can fulfil the conservation law. Moreover, if the quasilocal gravitational expression is positive on a large scale that certainly guarantees that it is also positive in the small; i.e., in the quasilocal small sphere approximation. Conversely, a negative gravitational energy expression on the small scale certainly implies non-positive on the large scale. Therefore, the small sphere limit approximation can play a useful role in testing whether the gravitational expression has the opportunity to be good (positive energy) or definitely bad (negative energy).

As there is a successful proof on a large scale [10], we expect that there should exist at least one good small–scale gravitational expression in holonomic frames (i.e., coordinate frames). A good quasilocal expression should satisfy several requirements: the interior mass density, the ADM mass [11] at spatial infinity and a positive small sphere gravitational energy like the Bel–Robinson tensor \( B^{\alpha \beta \mu \nu} \) [1]. This tensor is desired because it provides a non-negative gravitational ‘energy’. In vacuum the quasilocal value for the gravitational energy should be a multiple of \( \frac{4}{3} \pi r^3 g_{\mu \nu} t^\mu t^\nu \), where \( g_{\mu \nu} \) is the Euclidean volume with radius \( r \) and \( t^\mu \) is a timelike unit normal [12].

The present paper considers two different types of gravitational quasilocal expressions in holonomic frames. One is a modified Einstein pseudotensor which turns out to be a generalization of one of the Chen–Nester four quasilocal expressions [13]. The other is a modified Landau–Lifshitz pseudotensor which is equivalent to the Papapetrou pseudotensor. Moreover, we propose a new tensor \( V^{\alpha \beta \mu \nu} \) which is a sum of certain tensors, \( S^{\alpha \beta \mu \nu} + K^{\alpha \beta \mu \nu} \), that contributes the same gravitational ‘energy–momentum’ density value as \( B^{\alpha \beta \mu \nu} \) does. Indeed \( V^{\alpha \beta \mu \nu} \) stands on the same equal footing as \( B^{\alpha \beta \mu \nu} \) as far as the small region limit is concerned [14]. Additionally, we found that \( V^{\alpha \beta \mu \nu} \) and \( B^{\alpha \beta \mu \nu} \) are not the same. In particular, \( V^{\alpha \beta \mu \nu} \) is not completely symmetric and it does not fulfil the dominant energy condition, these are the main differences compared with the Bel–Robinson tensor.

2. Technical background

Using a Taylor series expansion, the metric tensor can be written as

\[
g_{\alpha \beta}(x) = g_{\alpha \beta}(0) + \partial_\mu g_{\alpha \beta}(0) x^\mu + \frac{1}{2} \partial_\mu \partial_\nu g_{\alpha \beta}(0) x^\mu x^\nu + \cdots.
\]

At the origin in Riemann normal coordinates

\[
g_{\alpha \beta}(0) = \eta_{\alpha \beta}, \quad \partial_\mu g_{\alpha \beta}(0) = 0,
\]

\[
-3 \partial_\mu \partial_\nu g_{\alpha \beta}(0) = R_{\mu \nu \beta \alpha} + R_{\mu \nu \alpha \beta}, \quad -3 \partial_\alpha \Gamma^\alpha_{\mu \nu}(0) = R^\mu_{\alpha \nu} + R^\mu_{\nu \alpha}.
\]

In vacuum the Bel–Robinson tensor \( B^{\alpha \beta \mu \nu} \), and the tensors \( S^{\alpha \beta \mu \nu} \) [15] and \( K^{\alpha \beta \mu \nu} \) [16] are defined as follows:

\[
B^{\alpha \beta \mu \nu} := R_{\lambda \mu \sigma} R_\beta^{\lambda \nu} + R_{\alpha \lambda \nu} R_\beta^{\lambda \mu} - \frac{1}{4} g_{\alpha \beta} g_{\mu \nu} R_{\rho \tau \xi \kappa} R^{\rho \tau \xi \kappa},
\]

\[
S^{\alpha \beta \mu \nu} := R_{\lambda \mu \sigma} R_\beta^{\lambda \nu} + R_{\alpha \lambda \nu} R_\beta^{\lambda \mu} + \frac{1}{4} g_{\alpha \beta} g_{\mu \nu} R_{\rho \tau \xi \kappa} R^{\rho \tau \xi \kappa},
\]

\[
K^{\alpha \beta \mu \nu} := R_{\lambda \mu \sigma} R_\beta^{\lambda \nu} + R_{\alpha \lambda \nu} R_\beta^{\lambda \mu} - \frac{1}{4} g_{\alpha \beta} g_{\mu \nu} R_{\rho \tau \xi \kappa} R^{\rho \tau \xi \kappa}.
\]
In order to extract the vacuum ‘energy–momentum’ density from the above three tensors, one can use the analog of the ‘electric’ $E_{ab}$ and ‘magnetic’ $H_{ab}$ parts of the Weyl tensor [17],

$$E_{ab} := C_{0a0b}, \quad H_{ab} := *C_{0a0b},$$

where $C_{\alpha\beta\mu\nu}$ is the Weyl conformal tensor and $*C_{\alpha\beta\mu\nu}$ is its dual,

$$*C_{\alpha\beta\mu\nu} = \frac{1}{2} \epsilon_{\alpha\beta\lambda\sigma} C^{\lambda\sigma}_{\mu\nu}.$$

(Here $\epsilon_{\alpha\beta\mu\nu} = \epsilon_{[\alpha\beta\mu\nu]}$ with $\epsilon_{0123} = \sqrt{-g}$ is the totally anti-symmetric Levi-Civita tensor, see [15], in particular equation 8.10 and example 8.3.) In vacuum using the Riemann tensor

$$E_{ab} = R_{0a0b}, \quad H_{ab} = *R_{0a0b}.$$  

In particular, the Riemann squared tensor can then be written as

$$R_{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} = 8(E_{ab} E_{ab} - H_{ab} H_{ab}),$$

Consequently, there is an identity between the components of $B_{\alpha\beta\mu\nu}$, $S_{\alpha\beta\mu\nu}$ and $K_{\alpha\beta\mu\nu}$:

$$B_{\mu000} = B_{\mu0l} = S_{\alpha\beta0} = K_{\alpha\beta0} = 0.$$  

This means that it is not necessary to obtain the Bel–Robinson tensor $B_{0000}, B_{00l}$ or $B_{\mu0l}$ for the positive ‘energy’ requirement, the sum $S_{0000} + K_{0000}$ or $S_{00l} + K_{00l}$ can fulfil the same task.

Based on the above argument, we propose a new tensor $V_{\alpha\beta\mu\nu}$ defined as follows:

$$V_{\alpha\beta\mu\nu} := S_{\alpha\beta\mu\nu} + K_{\alpha\beta\mu\nu}$$

This tensor has some nice properties but lacks some features of the Bel–Robinson tensor, as will be discussed further in section 3. Some detailed properties for $S_{\alpha\beta\mu\nu}$, $K_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ in vacuum are

$$S_{\alpha\beta\mu\nu} \equiv S_{(\alpha\beta)(\mu\nu)} \equiv S_{(\mu\nu)(\alpha\beta)}, \quad S_{\alpha\mu\nu} = 0, \quad S_{\alpha\mu0} = 0,$$

$$K_{\alpha\beta\mu\nu} \equiv K_{(\alpha\beta)(\mu\nu)} \equiv K_{(\mu\nu)(\alpha\beta)}, \quad K_{\alpha\mu\nu} = 0, \quad K_{\alpha\mu0} = 0,$$

where we have used the well-known vacuum identity [3]

$$R_{\alpha\beta\lambda\sigma} R_{\rho\lambda\sigma} = \frac{1}{2} g_{\alpha\beta} R_{\rho\lambda\sigma} R^{\rho\lambda\sigma}.$$
3. Differences and similarities of $V_{αβμν}$ and $B_{αβμν}$

It is known that the Bel–Robinson tensor has some very nice properties, such as being completely symmetric, completely trace free and divergence free, and it satisfies the dominant energy condition [6, 18–21]. The tensor $V_{αβμν}$ does not have all of these important properties, but it is totally trace free as indicated in (21) and gives a positive gravitational energy in a small sphere. The detailed discussion is as follows.

Perhaps the most important property for $V_{αβμν}$ is that it shares exactly the same gravitational energy–momentum of $B_{αβμν}$ in the small region limit. Referring to (17) and (18), we found the following relationship in vacuum:

$$V_{αβμν} u^α t^β t^μ t^ν \equiv B_{αβμν} u^α t^β t^μ t^ν,$$

where $t$ is a timelike unit normal vector and $u$ can be timelike or null. As $V_{αβμν}$ and $B_{αβμν}$ have some components equal, there is a natural suspicious question one may ask whether they are exactly the same? In fact, generally speaking, we found that they are not equal, i.e.,

$$V_{αβμν} \neq B_{αβμν}. \quad (23)$$

In particular, $B_{αβμν}$ is completely symmetric, completely divergence free, and it satisfies the dominant energy condition, but $V_{αβμν}$ does not have these three properties.

Before we give a proof, let us rewrite $V_{αβμν}$ in another representation [14]

$$V_{αβμν} := B_{αβμν} + W_{αβμν}, \quad (24)$$

where we defined

$$W_{αβμν} := \frac{1}{2} S_{αβμν} - \frac{5}{8} g_{αβ} g_{μν} R_{σρτ} R^{σρτ} + \frac{1}{8} (g_{αμ} g_{βν} + g_{αν} g_{βμ}) R_{σρτ} R^{σρτ}. \quad (25)$$

The algebra becomes crystal clear if we consider the following identity [5, 14]:

$$B_{αβμν} = -\frac{1}{2} S_{αβμν} + K_{αβμν} + \frac{5}{8} g_{αβ} g_{μν} R_{σρτ} R^{σρτ} - \frac{1}{8} (g_{αμ} g_{βν} + g_{αν} g_{βμ}) R_{σρτ} R^{σρτ}. \quad (26)$$

Some properties of $W_{αβμν}$ are

$$W_{αβμν} = W_{αβ(μν)} \equiv W_{(μν)(αβ)}, \quad W_{αβμ}^μ = 0 \equiv W_{αβμ}^μ, \quad (27)$$

$$W_{αβμν} + W_{αμβν} + W_{ανμβ} \equiv 0, \quad W_{αβμν} u^α t^β t^μ t^ν \equiv 0. \quad (28)$$

In order to compare $V_{αβμν}$ and $B_{αβμν}$, we consider the four main properties of the Bel–Robinson tensor as given in remarks on superenergy tensors by Senovilla [20]:

(i) Completely symmetric and traceless properties. It is known that the Bel–Robinson tensor fulfils these two properties

$$B_{αβμν} = B_{(αβμν)}, \quad B_{αβμ} = 0. \quad (29)$$

Especially, the completely symmetric condition is important [19]. However, both $V_{αβμν}$ and $W_{αβμν}$ only fulfil the completely traceless but not the totally symmetric property in general. The completely trace free property was already indicated in (21) and (27). For the totally symmetric, we found that both $V_{αβμν}$ and $W_{αβμν}$ do not fulfil this property

$$V_{αβμν} \neq V_{(αβμν)}, \quad W_{αβμν} \neq W_{(αβμν)}. \quad (30)$$

One can verify this by using Petrov type D [22]. In particular we found a case where

$$V_{0011} \neq V_{0101}, \quad W_{0011} \neq W_{0101}. \quad (31)$$
(ii) Positive energy density. Both $V_{\alpha\beta\mu\nu}$ and $B_{\alpha\beta\mu\nu}$ satisfy this condition which allows $V_{\alpha\beta\mu\nu}$ to become important and stands on the same equal footing with $B_{\alpha\beta\mu\nu}$ as far as the small sphere limit is concerned. Recall (22)

$$V_{\alpha\beta\mu\nu} t^\alpha t^\beta t^\mu t^\nu = B_{\alpha\beta\mu\nu} t^\alpha t^\beta t^\mu t^\nu = E_{ab}E^{ab} + H_{ab}H^{ab} \geq 0,$$

where $t$ is any timelike unit vector.

(iii) Dominant energy property. It is known that the Bel–Robinson tensor satisfies this property [6, 18, 21]; however, we found that both $V_{\alpha\beta\mu\nu}$ and $W_{\alpha\beta\mu\nu}$ do not fulfill this important condition. In particular, recall the requirement of the dominant super-energy condition from Senovilla (see lemma 4.1 of [6]): if a tensor $T_{\mu_1...\mu_s}$ satisfies the dominant super-energy property, then $T_{\mu_1...\mu_s} = |T_{\mu_1...\mu_s}|, \forall \mu_1, ..., \mu_s = 0, ..., n - 1$ in any orthonormal basis $\{\vec{e}_\nu\}$. In particular, again using Petrov type D we found a case where

$$V_{0000} \leq |V_{0011}|, \quad W_{0000} \leq |W_{0011}|.$$  

Indeed, we have verified that both $V_{\alpha\beta\mu\nu}$ and $W_{\alpha\beta\mu\nu}$ do not satisfy the dominant energy property. However, we found $V_{\alpha\beta\mu\nu}$ and $W_{\alpha\beta\mu\nu}$ which possess the following interesting inequality relation:

$$V_{\alpha\beta\mu\nu} u^\alpha_1 u^\beta_2 u^\mu_3 u^\nu_4 \geq W_{\alpha\beta\mu\nu} u^\alpha_1 u^\beta_2 u^\mu_3 u^\nu_4,$$

where $u_1$ to $u_4$ are any future-pointing causal vectors. The proof is simple. As $B_{\alpha\beta\mu\nu}$ has the dominant energy condition and rewriting (24)

$$V_{\alpha\beta\mu\nu} - W_{\alpha\beta\mu\nu} u^\alpha_1 u^\beta_2 u^\mu_3 u^\nu_4 = B_{\alpha\beta\mu\nu} u^\alpha_1 u^\beta_2 u^\mu_3 u^\nu_4 \geq 0.$$  

Hence the result follows.

(iv) Completely divergence free. It is known that the Bel–Robinson tensor satisfies this property (i.e., $\nabla_{\alpha}B^\alpha_{\beta\mu\nu} = 0$); however, both $V_{\alpha\beta\mu\nu}$ and $W_{\alpha\beta\mu\nu}$ do not. Checking the covariant derivative by using (24)

$$\nabla_{\alpha}V^{\alpha}_{\beta\mu\nu} = \nabla_{\alpha}W^{\alpha}_{\beta\mu\nu}$$

$$= -\frac{1}{2}(R_{\mu}^{\alpha\lambda\sigma}\nabla_{\alpha}R_{\beta\alpha\lambda\sigma} + R_{\alpha}^{\alpha\lambda\sigma}\nabla_{\lambda}R_{\beta\alpha\lambda\sigma}) + \frac{1}{8}(g_{\mu}^{\beta}\nabla_{\beta} + g_{\beta}^{\mu}\nabla_{\mu} + g_{\mu}^{\lambda}\nabla_{\lambda})R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau},$$

which does not vanish in particular for the Schwarzschild solution.

4. The interior, ADM and gravitational energy

Three physical regions of interest for the energy of a gravitating system in general relativity are the interior mass–energy density, the ADM mass–energy [11] at spatial infinity and the gravitational field energy–momentum in vacuum. Einstein described the gravitational energy–momentum density by the classical pseudotensor $t_{\alpha\mu}^{\nu}$ which follows from the Freud superpotential [23]

$$U_{\alpha}^{[\mu\nu]} = \sqrt{-g} (\delta^\nu_\lambda \Gamma^\mu_\kappa + \delta^\mu_\lambda \Gamma^\nu_\kappa + \Gamma^\mu\nu_\kappa) \delta_{\rho\sigma}^{\nu\mu},$$

in a way which guarantees conservation. (Note that the generalized Kronecker delta used here is defined as $\delta^{[\mu\nu]} = 2\delta_{\mu}^{[\mu\nu]} \delta^{[\nu]}_{\nu}$.) Such superpotentials cannot be uniquely defined, for example suppose

$$t_{\alpha}^{\mu} = \partial_{\alpha}U_{\alpha}^{[\mu\nu]},$$

then one can always introduce a new quantity $\tilde{U}_{\alpha}^{[\mu\nu]}$ and obtain a new pseudotensor such as

$$\tilde{t}_{\alpha}^{\mu} = t_{\alpha}^{\mu} + \partial_{\alpha}\tilde{U}_{\alpha}^{[\mu\nu]},$$

in a way which guarantees conservation.
because $\partial_\mu \tilde{t}_a^{\mu} = 0$ and $\partial_\mu t_a^{\mu} = 0$ they are both conserved densities. However, consideration of the interior mass–energy density and the ADM mass–energy provide some restrictions, so that one can have some more physical energy–momentum components. Consider the following generalization of the Freud superpotential:

$$U_{a\mu}^{[\mu \nu]} = \sqrt{-g} \left( k_1 \delta_\nu^\rho \Gamma^\lambda \_\_\_ \_ + k_2 \delta_\nu^\pi \Gamma^\lambda \_\_\_ \_ + k_3 \Gamma^{\pi \rho \_\_\_\_} \right) \delta_\mu^{\rho \pi}, \quad (40)$$

where $k_1$, $k_2$ and $k_3$ are extra added constants. Inside matter at the origin in Riemann normal coordinates to zeroth order (where $\kappa = 8\pi G/c^4$) we find

$$2\kappa t_a^{\mu}(0) = 2G_a^{\mu}(0) = 2\kappa T_a^{\mu}(0). \quad (41)$$

Just what we must have from the equivalence principle. In detail for the energy density inside matter at the origin, the zeroth order term is

$$\mathcal{E} = -t_a^{\mu}(0) = -\frac{G_0^{\mu}(0)}{\kappa} = -T_0^{\mu}(0) = \rho, \quad (45)$$

where $\rho$ is the mass–energy density. The momentum density is

$$\mathcal{P}_l = -t_l^{\mu}(0) = -\frac{G_l^{\mu}(0)}{\kappa} = -T_l^{\mu}, \quad (46)$$

At the origin in vacuum, the zeroth and first derivatives are

$$t_a^{\mu}(0) = 0 = \partial_\mu t_a^{\mu}(0). \quad (47)$$

To check that we get the ADM mass at the spatial infinity, with the assumption that $GM/r \ll 1$, let us use the isotropic Schwarzschild metric in Cartesian coordinates:

$$ds^2 = -\left( 1 - \frac{2GM}{r} \right)^2 dt^2 + \left( 1 + \frac{2GM}{r} \right)^2 (dx^2 + dy^2 + dz^2) \sim -\left( 1 - \frac{2GM}{r} \right) dt^2 + \left( 1 + \frac{2GM}{r} \right) (dx^2 + dy^2 + dz^2). \quad (48)$$

The latter expression shows the dominant asymptotic terms which are sufficient for our calculation. The energy–momentum can be calculated at spatial infinity from the surface integral

$$2\kappa P_a = -\frac{1}{2} \int U_{a\mu}^{[\mu \nu]} dS_{\mu \nu}, \quad (49)$$

where the 2-form $dS_{\mu \nu} = \frac{1}{2\kappa} \epsilon_{\mu \nu \lambda \sigma} dx^\lambda \wedge dx^\sigma$. The associated ADM mass–energy term is

$$M = -\int \frac{1}{4\kappa} U_0^{[\mu \nu]} dS_{\mu \nu} = \frac{1}{2} (k_1 + k_3) M. \quad (50)$$
which gives one more constraint
\[ k_1 + k_3 = 2. \tag{51} \]
Considering (42), (43) and (51), the unique solution is
\[ k_1 = k_2 = k_3 = 1. \tag{52} \]
Therefore, the above constraint indicates that the Einstein pseudotensor (or any other which is asymptotically equivalent such as the Landau–Lifshitz pseudotensor) has this property.

5. Gravitational energy for the modified pseudotensors

Before we study the gravitational energy in general relativity, recall the Newtonian gravity theory. As the gravitational potential has no lower bound, one can construct a system that has the negative energy, which includes the rest mass–energy associated with the gravitational energy. This means that the gravitational energy can be negative without any restriction [24]. For example, the Newtonian gravitational energy for a sphere [25] with mass \( m \) and radius \( a \) is
\[ E = -\frac{3Gm^2}{5a} \leq 0. \tag{53} \]
However, this does not apply to general relativity simply because mass is equivalent to energy. Indeed gravitational energy must be non-negative. Positive energy is needed to ensure stability [24], and the Bel–Robinson tensor guarantees this positivity requirement in the small region vacuum limit.

5.1. Modification of the Einstein pseudotensor

The modified quasilocal expression in holonomic frames [16] is summarized as
\[ 2\kappa B_{c_1c_2}(N) = 2\kappa B_p(N) + c_1 i N \Delta \Gamma^a_b \wedge \Delta \eta^{a}_b - c_2 \Delta \Gamma^a_b \wedge i N \Delta \eta^{a}_b \]
\[ = -\frac{N^a}{2} \left\{ E U_{a}^{\mu\nu} + c_1 \sqrt{-g} h^{\lambda\pi} \Gamma_{\sigma}^{\mu\nu} \delta^{a}_{\sigma} + c_2 \sqrt{-g} h^{\beta\sigma} \Gamma_{\lambda\beta}^{a} \delta^{\mu\nu}_{\sigma} \right\} \epsilon^{\mu\nu}, \tag{55} \]
where \( c_1, c_2 \) are real numbers, \( h_{ab} := g_{ab} - \eta_{ab} \) and the generalized Kronecker delta \( \delta^{a}_{\sigma} := 3! \delta^{a}_{\lambda\sigma} \). When \( (c_1, c_2) = (0, 0), (0, 1), (1, 0) \) and \( (1, 1) \), this recovers the original Chen–Nester four holonomic expressions (note that here we take \( \Gamma^{a}_{\beta\mu} = 0 \) so \( \Delta \Gamma^a_b = \Gamma^a_b \)). The superpotential can be extracted from (55)
\[ U_{a}^{\mu\nu} = E U_{a}^{\mu\nu} + c_1 \sqrt{-g} h^{\lambda\pi} \Gamma_{\sigma}^{\mu\nu} \delta^{a}_{\sigma} + c_2 \sqrt{-g} h^{\beta\sigma} \Gamma_{\lambda\beta}^{a} \delta^{\mu\nu}_{\sigma}, \tag{56} \]
where \( E U_{a}^{\mu\nu} = -\sqrt{-g} h^{\mu\nu} \delta^{a}_{\lambda\beta} \delta^{\mu\nu}_{\sigma} \) which is the Freud superpotential, consequently this expression gives the ADM mass at spatial infinity. Note that the \( h \Gamma \) terms do not affect the results inside matter and at spatial infinity, but only the second-order vacuum value. The small region results of the modified Chen–Nester expressions in compact form in Riemann normal coordinates is [16]
\[ 2\kappa t_{a}^{\beta} = 2G_{a}^{\beta} + \frac{1}{18} \left\{ (4 + c_1 - 5c_2) B_{a}^{\beta} \xi_k + (1 - 2c_1 + c_2) S_{a}^{\beta} \xi_k + (c_1 - 3c_2) K_{a}^{\beta} \xi_k \right\} x^k + \mathcal{O}({\text{Ricci, } x}) + \mathcal{O}(x^3), \tag{57} \]
(where $O$ denotes terms which are of the indicated order). This expression satisfies the interior mass density requirement which was explained previously in section 4, but now we need to examine the gravitational energy in vacuum where $G_{\alpha\beta} = 0$. Rewrite (57) only concerning the second-order components in terms of $V_{\alpha\beta\mu\nu}$ which was defined in (18):

$$2\kappa t^\alpha_\beta = \frac{1}{18} \left\{ (4 + c_1 - 5c_2)B^0_{\mu\nu} + (c_1 - 3c_2)V^0_{\mu\nu} + (c_1 + 2c_2 - 1)S^0_{\mu\nu} \right\} x^\mu x^\nu. \quad (58)$$

This can be compared with the known result for the Einstein pseudotensor \cite{2, 5, 15}, which is

$$2\kappa t^\alpha_\beta = \frac{1}{18} (4B^0_{\mu\nu} - S^0_{\mu\nu}) x^\mu x^\nu. \quad (59)$$

Consider (58), when $(c_1, c_2) = \left( \frac{3}{5}, \frac{1}{5} \right)$ we reproduce the pure Bel–Robinson tensor result \cite{16},

$$2\kappa t^\alpha_\beta = \frac{1}{5} B^0_{\mu\nu} x^\mu x^\nu. \quad (60)$$

However, instead of confining ourselves to the pure Bel–Robinson tensor, we found a more satisfactory result by introducing $V_{\alpha\beta\mu\nu}$ (in place of $K_{\alpha\beta\mu\nu}$). Using a calculation method similar to that in \cite{26}, the gravitational energy–momentum in the small sphere limit is

$$P_\mu = (-E, \vec{P}) = \frac{1}{2\kappa} \int \frac{1}{18} \left\{ (4 + c_1 - 5c_2)B^0_{\mu\nu} + (c_1 - 3c_2)V^0_{\mu\nu} + (c_1 + 2c_2 - 1)S^0_{\mu\nu} \right\} x^\mu x^\nu d^3x = -\frac{c_1}{180G} r^5 B_{\mu000}, \quad (61)$$

provided $c_1 > 0$ and we take the unique combination $c_1 + 2c_2 = 1$, which is the constraint from requiring the coefficients of $S_{\mu\nu\alpha\beta}$ and $K_{\mu\nu\alpha\beta}$ to be the same. In other words, this eliminates the uncertain energy sign of $S_{\mu\nu\alpha\beta}$. The result we found in (61) is desirable because it is proportional to $B_{\mu000}$ which ensures the positive gravitational energy. Additionally, it may need to be emphasized that the vector $P_\mu$ is future pointing and non-spacelike (i.e., the 4-momentum is inside the light cone such that $-P_0 \geq |\vec{P}|$).

Furthermore, there exists an infinite number of solutions because of the parameter $c_1$ which was mentioned in (61). Referring to (55), different solutions are associated with different boundary conditions that can be adjusted by $c_1$ and $c_2$ \cite{16}. However there is one solution with locally positive energy and a simple boundary condition, which is when $(c_1, c_2) = (1, 0)$. It reduces to one of the Chen–Nester holonomic expressions, $B_c(N)$. In detail

$$2\kappa B_c(N) = iN \Gamma^\alpha_\beta \wedge \Delta \eta_\alpha^\beta + \Gamma^\alpha_\beta \wedge iN \eta_\alpha^\beta, \quad (62)$$

the associated superpotential is

$$U_{\alpha}^{[\mu\nu]} = \eta U_{\alpha}^{[\mu\nu]} + \sqrt{-g} h^\lambda \eta \Gamma^\sigma_{\alpha\beta} \delta^{\mu\nu}_{\lambda\sigma}, \quad (63)$$

The corresponding gravitational energy–momentum in a small sphere vacuum region is

$$P_\mu = -\frac{r^5}{180G} B_{\mu000}. \quad (64)$$

5.2. Modification of the Landau–Lifshitz pseudotensor

The superpotential for the Papapetrou pseudotensor \cite{27} is

$$p H^{[\mu\nu][\alpha\beta]} = -\sqrt{-g} g^{\rho\xi} \eta^\tau \delta^{\mu\nu}_{\xi\rho} \delta^{\alpha\beta}_{\tau\sigma}. \quad (65)$$

In another form it is

$$p U_{\mu}^{[\nu\sigma]} = \partial_\mu \left( p H^{[\mu\nu][\alpha\beta]} \right) = B U_{\alpha}^{[\mu\nu]} + \sqrt{-g} \left( g^{\rho\xi} h^\pi \Gamma^\tau_{\rho\xi} + g^{\rho\sigma} h^\tau \Gamma^\pi_{\rho\sigma} \right) \delta^{\mu\nu}_{\xi\rho} \delta^{\lambda\alpha}_{\tau\sigma}. \quad (66)$$
where $U^{[\mu \nu]}_\alpha = \sqrt{-g} U^{[\mu \nu]}_\alpha = g^{\alpha \beta} \Gamma^\gamma_{\lambda \alpha} b^{\lambda \mu \nu}$, 'L' stands for Landau–Lifshitz and 'B' refers to Bergmann–Thomson (to the order of the interest here, the Bergmann–Thomson pseudotensor gives the same effective contribution as the Landau–Lifshitz pseudotensor, for more details see [28]). Once again, the $h \Gamma$ terms do not affect the results inside matter and at spatial infinity, they would contribute, however, to the second-order vacuum value. The pseudotensor can be obtained as

$$2 \kappa_t^{\mu \beta} = \partial_\mu U^{[\beta \mu]}_\alpha,$$

(67)

the expression in Riemann normal coordinates is

$$2 \kappa_t^{\mu \beta} = 2 G^{\mu \beta} + \frac{1}{9} \left( 4 B^{\mu \beta} x_\lambda - x_\lambda x_\mu x_\nu x_\kappa - K^{\mu \beta} x_\lambda \right) x_\mu x_\nu x_\kappa + \mathcal{O}(\text{Ricci}, x) + \mathcal{O}(x^3).$$

(68)

As discussed in section 5.1, we rewrite the above expression only considering the second-order vacuum components which concern the gravitational energy and introducing $V^{\alpha \beta}_{\mu \nu}$:

$$2 \kappa_t^{\mu \beta} = \frac{1}{9} \left( 4 B^{\mu \beta} x_\lambda - V^{\mu \nu}_{\alpha \beta} x_\mu x_\lambda \right) x_\mu x_\kappa,$$

(69)

whereas the known result for the Landau–Lifshitz pseudotensor $L^{\alpha \beta}$ [5] is

$$L^{\alpha \beta} = \frac{1}{18} \left( 7 B^{\alpha \beta} x_\lambda + \frac{1}{2} S^{\alpha \beta} x_\lambda \right) x_\lambda x_\kappa.$$

(70)

For the convenience of discussion, we define

$$\tilde{P}^{\sigma \tau \lambda \rho} := \frac{1}{4} \left( 4 B^{\sigma \tau \lambda \rho} - V^{\sigma \tau \lambda \rho} \right).$$

(71)

The result for the Papapetrou pseudotensor we obtained in (70) was found long ago by Yefremov [3]. Yefremov obtained the same result for the second approximation Papapetrou pseudotensor density in empty space in Riemann normal coordinates which can be expressed as follows:

$$2 \kappa_t^{\sigma \tau} = \frac{1}{3} P^{\sigma \tau \lambda \rho} x_\lambda x_\rho,$$

(72)

where $P^{\sigma \tau \lambda \rho}$ is defined as

$$P^{\sigma \tau \lambda \rho} := R^{\sigma \tau \lambda \rho} R^{\mu \lambda \rho \nu} + R^{\sigma \tau \lambda \rho} R^{\mu \rho \sigma \nu} - \frac{1}{8} R^{\sigma \tau \lambda \rho} R^{\mu \sigma \nu \rho}. $$

(73)

Indeed, compare (71) and (73), we recovered

$$\tilde{P}^{\sigma \tau \lambda \rho} \equiv P^{\sigma \tau \lambda \rho}.$$

(74)

Yefremov discovered some interesting properties of $P^{\sigma \tau \lambda \rho}$,

$$P^{\sigma \tau \lambda \rho} = P^{(\sigma \tau) (\lambda \rho)} = P^{(\lambda \rho) (\sigma \tau)},$$

(75)

and also found the result

$$P^{(\sigma \tau) (\lambda \rho)} = B^{\sigma \tau \lambda \rho}.$$

(76)

This is the main result Yefremov found in his paper. It is very nice to achieve the Bel–Robinson tensor after complete symmetrization. The equality in (76) is not difficult to verify. In fact, however, it is not necessary to use up all the four indices, we found that any three of them is sufficient. Explicitly

$$P^{(\sigma \tau) (\lambda \rho)} = B^{\sigma \tau \lambda \rho}.$$

(77)

It is easy to check the above relationship if we use the expression of $\tilde{P}^{\sigma \tau \lambda \rho}$ which is defined in (71) instead of $P^{\sigma \tau \lambda \rho}$ from (73). Because $B^{\alpha \beta \mu \nu}$ is already completely symmetric; using (5) and (6), it is easy to verify that

$$V^{(\sigma \tau) (\lambda \rho)} = \frac{1}{3} (V^{\sigma \tau \lambda \rho} + V^{\sigma \lambda \rho \tau} + V^{\sigma \rho \tau \lambda}) = B^{\sigma \tau \lambda \rho}.$$

(78)

Then the result follows

$$\tilde{P}^{(\sigma \tau) (\lambda \rho)} = B^{\sigma \tau \lambda \rho}.$$

(79)
At this point, it seems that using $V^{\alpha\beta\mu\nu}$ is more convenient as it possesses some symmetry properties \[29\], though not as many as $B^{\alpha\beta\mu\nu}$.

Moreover, the new small sphere region result we obtained is that we found that $V^{\alpha\beta\mu\nu}$ contributes exactly the same ‘energy–momentum’ content as $B^{\alpha\beta\mu\nu}$ offers. The associated gravitational energy–momentum in the small sphere vacuum approximation is

$$P_{\mu} = \frac{1}{2\kappa} \int \frac{1}{9} (4B^{0}_{\mu ij} - V^{0}_{\mu ij}) x^{i} x^{j} d^{3}x = - \frac{r^{5}}{180G} B_{\mu 000}.$$  \hspace{1cm} (80)

This is a desirable result which we found, it not only guarantees the gravitational energy positivity, but moreover, as mentioned previously, the vector $P_{\mu}$ is future pointing and non-spacelike.

6. Conclusion

In the small sphere approximation, it has been argued that the quasilocal expression for the gravitational energy–momentum density should be proportional to the Bel–Robinson tensor $B^{\alpha\beta\mu\nu}$. In addition to its positivity property this tensor has many nice properties (e.g., completely symmetric, trace free and divergence free). Therefore, it can be considered as the best tensor for the gravitational ‘energy–momentum’ density expression. However, we here propose a new tensor $V^{\alpha\beta\mu\nu}$ with a certain explicit expression, the sum of the tensors $S^{\alpha\beta\mu\nu} + K^{\alpha\beta\mu\nu}$, which gives the same gravitational ‘energy–momentum’ density content as $B^{\alpha\beta\mu\nu}$ does. In other words, in the small region limit, $V^{\alpha\beta\mu\nu}$ is an alternative tensor which is on the same equal footing as $B^{\alpha\beta\mu\nu}$. Although $V^{\alpha\beta\mu\nu}$ shares the same gravitational energy–momentum of $B^{\alpha\beta\mu\nu}$; strictly speaking, they are not the same. In particular, $V^{\alpha\beta\mu\nu}$ does not fulfill the dominant energy condition and perhaps this is the most important difference with the Bel–Robinson tensor.

The purpose of the classical Einstein and Landau–Lifshitz pseudotensors is to define the gravitational energy. Neither of them can guarantee positive energy in holonomic frames in the small sphere limit, i.e., neither is proportional to $B^{\alpha\beta\mu\nu} t^{\alpha} t^{\beta} t^{\mu} t^{\nu}$ in vacuum to second order. In order to achieve the desired expression for the gravitational energy, one may consider the modification of these two pseudotensors. We found that the modified Einstein pseudotensor becomes a generalization of one of the Chen–Nester quasilocal expressions, while the modified Landau–Lifshitz pseudotensor is equivalent to the Papapetrou pseudotensor; these two modified pseudotensors have a positive gravitational energy which comes from $V^{\alpha\beta\mu\nu} t^{\alpha} t^{\beta} t^{\mu} t^{\nu} = B^{\alpha\beta\mu\nu} t^{\alpha} t^{\beta} t^{\mu} t^{\nu}$ in the small region vacuum limit.

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References

[1] Szabados L B 2004 Living Rev. Rel. 7 4
[2] Garecki J 1973 Acta Phys. Pol. B 4 347
[3] Yefremov A P 1975 Acta Phys. Pol. B 6 667
[4] Garecki J 1977 Acta Phys. Pol. B 8 159
[5] Deser S, Franklin J S and Seminara D 1999 Class. Quantum Grav. 16 2815
[6] Senovilla J M M 2000 Class. Quantum Grav. 17 2799
[7] Garecki J 2001 Ann. Phys. Lpz. 19 911
[8] Schrödinger E 1918 Phys. Z. 19 4
[9] Bauer H 1918 Phys. Z. 19 163
[10] Schoen R and Yau S T 1979 Phys. Rev. Lett. 43 1457
[11] Arnowitt R, Deser S and Misner C W 1961 Phys. Rev. 122 997
[12] So L L 2008 Class. Quantum Grav. 25 175012
[13] Chen C M and Nester J M 2001 Phys. Rev. 122 997
[14] So L L 2008 arXiv:0901.4828
[15] Misner C W, Thorne K S and Wheeler J A 1973 Gravitation (San Francisco, CA: Freeman)
[16] so L L 2007 Int. J. Mod. Phys. D 16 875
[17] Carmeli M 1982 Classical Fields General relativity and Gauge Theory (New York: Wiley)
[18] Penrose R and Rindler W 1984 Spinors and Spacetime vol 1 (Cambridge: Cambridge University Press)
[19] Bergqvist G and Lankinen P 2004 Class. Quantum Grav. 21 3499
[20] Senovilla J M M 1999 arXiv:9901019
[21] Bergqvist G and Senovilla J M M 2001 Class. Quantum Grav. 18 5299
[22] Gomez-Lobo A G P 2008 Class. Quantum. Grav. 25 015006
[23] Freud Ph 1939 Ann. Math. 40 417
[24] Horowitz G T 1984 The positive energy theorem and its extensions Asymptotic Behavior of Mass and Spacetime Geometry (Lecture Notes in Physics vol 203) ed F J Flaherty (Berlin: Springer) pp 1–20
[25] Landau L D and Lifshitz E M 1962 The Classical Theory of Fields 2nd edn (Reading, MA: Addison-Wesley)
[26] So L L and Nester J M 2009 Chin. J. Phys. 47 10
[27] Papapetrou A 1948 Einstein’s theory of gravitation and flat space Proc. R. Irish. Acad. A 52 11
[28] So L L, Nester J M and Chen H 2009 Class. Quantum. Grav. 26 085004
[29] So L L 2008 arXiv:0809.3868