ON SOME LIOUVILLE TYPE THEOREMS FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. We prove several Liouville type results for stationary solutions of the $d$-dimensional compressible Navier-Stokes equations. In particular, we show that when the dimension $d \geq 4$, the natural requirements $\rho \in L^\infty(\mathbb{R}^d)$, $v \in H^1(\mathbb{R}^d)$ suffice to guarantee that the solution is trivial. For dimensions $d = 2, 3$, we assume the extra condition $v \in L^3\mathbb{R}^d$. This improves a recent result of Chae [1].

1. Introduction

In this paper we consider the stationary barotropic compressible Navier-Stokes equations on $\mathbb{R}^d$, $d \geq 2$

$$\nabla \cdot (\rho v) = 0,$$
$$\nabla \cdot (\rho v \otimes v) = -\nabla P + \mu \Delta v + (\mu + \lambda) \nabla (\nabla \cdot v),$$
$$P(\rho) = A\rho^\gamma, \quad \gamma > 1.$$

(1)  
(2)  
(3)

Here $\rho, v, P$ are the density, velocity and pressure of the fluid respectively. The coefficients $\lambda, \mu, A$ satisfy $\mu > 0, \lambda + 2\mu > 0, A > 0$. This system is a reduction of the compressible Navier-Stokes equations by assuming that the volumetric entropy $s$ remains constant. For more discussions on the compressible Navier-Stokes equations and its various related models, see e.g. [2, 4, 5].

For the system (1–3), it turns out that, if we put suitable global integrability conditions on $\rho$ and/or $v$, then the only possible solution is the trivial one $v = 0, \rho =$-constant. For example, if we assume $\rho \in L^\infty \cap C^\infty, v \in C^\infty_c$, we can then multiply the momentum equation (2) by $v$ and integrate over space:

$$\int_{\mathbb{R}^d} [\nabla \cdot (\rho v \otimes v)] \cdot v dx = -\int_{\mathbb{R}^d} \nabla P \cdot v dx + \mu \int_{\mathbb{R}^d} (\Delta v) \cdot v dx + (\mu + \lambda) \int_{\mathbb{R}^d} [\nabla (\nabla \cdot v)] \cdot v dx. \quad (4)$$

Integration by parts gives

$$\int_{\mathbb{R}^d} [\nabla \cdot (\rho v \otimes v)] \cdot v dx = \int_{\mathbb{R}^d} ([\rho v \cdot \nabla v]) \cdot v dx = \int_{\mathbb{R}^d} \rho v \cdot \nabla \left( \frac{v^2}{2} \right) dx = -\int_{\mathbb{R}^d} \nabla (\rho v) \frac{v^2}{2} dx = 0, \quad (5)$$

and

$$\int_{\mathbb{R}^d} \nabla P \cdot v dx = \int_{\mathbb{R}^d} (A\rho^\gamma) \cdot v dx = A \int_{\mathbb{R}^d} \rho v \cdot \nabla \rho^\gamma^{-1} dx = -A \int_{\mathbb{R}^d} \nabla (\rho v) \rho^\gamma^{-1} dx = 0. \quad (6)$$

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These lead to
\[ \mu \int_{\mathbb{R}^d} |\nabla v|^2 \, dx + (\mu + \lambda) \int_{\mathbb{R}^d} (\nabla \cdot v)^2 \, dx = 0 \implies v \equiv \text{constant}. \tag{7} \]

As \( v \in C^\infty_v \) we obtain \( v \equiv 0 \). Since \( \nabla v = 0 \) we also get from (2) that \( \nabla (A \rho) = \nabla P = 0 \). Hence \( \rho \equiv \text{constant} \) too.

Thus we have easily shown that, if \( \rho \in L^\infty \cap C^\infty, v \in C^\infty_c \), then the only solution to (11) is the trivial one. However such a simple argument does not work any more once we relax the compact support condition on \( v \) and weaken the regularity assumptions. Very recently, Chae [1] has shown that the above conclusion, \( v = 0, \rho = \text{constant} \), still holds as long as
\[ \|\rho\|_{L^\infty(\mathbb{R}^d)} + \|\nabla v\|_{L^2(\mathbb{R}^d)} + \|v\|_{L^\frac{6d}{3d-d+2}(\mathbb{R}^d)} < \infty \quad \text{when} \quad d \leq 6, \tag{8} \]
\[ \|\rho\|_{L^\infty(\mathbb{R}^d)} + \|\nabla v\|_{L^2(\mathbb{R}^d)} + \|v\|_{L^\frac{12d}{3d-d+2}(\mathbb{R}^d)} < \infty \quad \text{when} \quad d \geq 7. \tag{9} \]

The condition
\[ \|\rho\|_{L^\infty(\mathbb{R}^d)} + \|\nabla v\|_{L^2(\mathbb{R}^d)} < \infty \]
is very natural as most physical flows have bounded density and finite enstrophy. On the other hand, the integrability condition \( v \in L^\frac{3d}{3d-1}(\mathbb{R}^d) \) is a fairly strong assumption since physical flows with finite energy need not belong to this class. In this paper we will remove this assumption and weaken further the integrability conditions on the velocity field \( v \). Our main results are the following.

**Theorem 1.** Let the dimension \( d \geq 2 \). Suppose \((\rho, v)\) is a smooth solution to (11) satisfying
\[ \|\rho\|_{L^\infty(\mathbb{R}^d)} < \infty, \quad v \in \dot{H}^1(\mathbb{R}^d), \quad \text{if} \quad d \geq 4; \tag{10} \]
\[ \|\rho\|_{L^\infty(\mathbb{R}^d)} + \|\nabla v\|_{L^2(\mathbb{R}^d)} + \|v\|_{L^2(\mathbb{R}^d)} < \infty, \quad \text{if} \quad d = 3; \tag{11} \]
\[ \|\rho\|_{L^\infty(\mathbb{R}^d)} + \|\nabla v\|_{L^2(\mathbb{R}^d)} + \|v\|_{L^2(\mathbb{R}^d)} < \infty, \quad \text{if} \quad d = 2. \tag{12} \]

then \( v = 0 \) and \( \rho = \text{constant} \).

**Remark 1.** The space \( \dot{H}^1(\mathbb{R}^d) \) in (10) is defined in the usual way as the completion of \( C^\infty_0(\mathbb{R}^d) \) under the norm \( \|\nabla f\|_{L^2(\mathbb{R}^d)} \). In essence, (10) is the condition \( \|\nabla v\|_{L^2(\mathbb{R}^d)} < \infty \) together with the requirement that \( v \) decays to 0 at infinity. Such decay is necessary to conclude \( v = 0 \) as only derivatives of \( v \) appear in (11), while \( v \) itself doesn’t. Also note that conditions (10), (12) are weaker than (9) in the sense that \( v \) can decay more slowly at infinity.

**Remark 2.** We stress that in dimensions \( d = 2, 3 \), the extra condition \( v \in L^\frac{3d}{3d-1} \) is very mild. For example, if \( v \in H^1(\mathbb{R}^d) \), then by Sobolev embedding we have \( v \in L^\frac{3d}{3d-1} \) for \( d = 2, 3 \). Thus Theorem 1 asserts that there are no nontrivial stationary solutions with the velocity field in the natural energy class.

If we are willing to put more integrability condition on \( v \), then the condition on \( \nabla v \) can even be dropped.

**Theorem 2.** Let \( d \geq 2 \). Suppose \((\rho, v)\) is a smooth solution to (11) satisfying
\[ \|\rho\|_{L^\infty(\mathbb{R}^d)} + \|v\|_{L^\frac{6d}{3d-d+2}(\mathbb{R}^d)} + \|v\|_{L^\frac{12d}{3d-d+2}(\mathbb{R}^d)} < \infty, \tag{13} \]
then \( v = 0 \) and \( \rho = \text{constant} \) on \( \mathbb{R}^d \).
Remark 3. It is not difficult to check that the condition (13) is weaker than (8) \(-1\) for all \(d \geq 2\).

Finally, it is also possible to put the integrability condition on \(\nabla \cdot v\) instead of \(\nabla v\).

Theorem 3. Let \(d \geq 2\). Suppose \((\rho, v)\) is a smooth solution to (1–3) satisfying
\[
\|\rho\|_{L^\infty(\mathbb{R}^d)} + \|\nabla \cdot v\|_{L^2(\mathbb{R}^d)} + \|v\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} < \infty, \quad \text{if } d \leq 3; \tag{14}
\]
\[
\|\rho\|_{L^\infty(\mathbb{R}^d)} + \|\nabla \cdot v\|_{L^2(\mathbb{R}^d)} + \|v\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} < \infty, \quad \text{if } d \geq 4; \tag{15}
\]
then \(v = 0\) and \(\rho = \text{constant}\) on \(\mathbb{R}^d\).

Remark 4. One can also check that the condition (14) \(-1\) is weaker than (8) \(-1\) for all \(d \geq 2\).

It is an interesting question whether the assumption on \(\|\nabla \cdot v\|_2\) can be dropped.

The following sections are devoted to the proofs of Theorems 1, 2, 3.

We conclude the introduction by setting up some notations.

Notations.

- For any two quantities \(X\) and \(Y\), we denote \(X \lesssim Y\) if \(X \leq CY\) for some harmless constant \(C > 0\). Similarly \(X \gtrsim Y\) if \(X \geq CY\) for some \(C > 0\). We denote \(X \sim Y\) if \(X \leq CY\) and \(Y \leq X\). We shall write \(X \lesssim_{Z_1, Z_2, \ldots, Z_k} Y\) if \(X \leq CY\) and we want to stress that the constant \(C\) depends on the quantities \((Z_1, \ldots, Z_k)\). Similarly we define \(\gtrsim_{Z_1, \ldots, Z_k}\).

- We shall denote by \(\|f\|_p = \|f\|_{L^p} = \|f\|_{L^p(\mathbb{R}^d)}\) the usual Lebesgue norm of a scalar or vector valued function \(f\) on \(\mathbb{R}^d\). For any Lebesgue measurable set \(A \subset \mathbb{R}^d\), we denote by \(|A|\) the Lebesgue measure of \(A\).

- For any two scalar functions \(f, g\) on \(\mathbb{R}^d\), we use the notation \(f * g\) to denote the standard convolution
\[
(f * g)(x) = \int_{\mathbb{R}^d} f(x - y)g(y)dy,
\]
whenever the above integral is well-defined.

2. Integrability Lemmas

In the assumptions of Theorems 1, 2, 3 we only require \(\rho \in L^\infty(\mathbb{R}^d)\) which gives \(P \in L^\infty(\mathbb{R}^d)\) through the constitutive relation (3). However, to rigorously carry out the integration by parts argument, we need more integrability on the pressure \(P\) and the density \(\rho\). A natural idea is to make use of the momentum equation (2) and upgrade the integrability of \((P, \rho)\) through the integrability of the velocity field \(v\). To this end, we re-write (2) as
\[
\nabla P = -\nabla \cdot (pv + \rho v) + \mu \Delta v + (\mu + \lambda) \nabla (\nabla \cdot v). \tag{16}
\]
After removing the gradient on both sides of the above equation, it clear that integrability of \(v\) may imply some integrability of \(P\) (modulo some constants), which in turn could lead to integrability of \(\rho\) through (3). We present the precise relations in the following two lemmas. They will play a key role in the proofs of our main theorems.

Lemma 1. Let \(P \in L^\infty(\mathbb{R}^d)\), \(p_1 \in L^{r_1}(\mathbb{R}^d)\), \(p_2 \in L^{r_2}(\mathbb{R}^d)\) with \(1 \leq r_1, r_2 < \infty\). Suppose \(P - p_1 - p_2\) is weakly harmonic, that is
\[
\triangle (P - p_1 - p_2) = 0
\]
Lemma 2. Suppose \( P - p_1 - p_2 = c \quad \text{a.e.} \quad x \in \mathbb{R}^d. \)

If furthermore \( P(x) \geq 0 \) a.e., then we also have \( c \geq 0. \)

Proof. This is fairly standard but we include the details here for the sake of completeness. By Weyl’s lemma (see e.g. [3]), the function \( P - p_1 - p_2 \) coincides almost everywhere with a strongly harmonic function which we denote by \( \phi(x) \). Take any \( x_1 \in \mathbb{R}^d \) and \( R > 0 \), denote by \( B_R = B_R(x_1) \) the ball centered at \( x_1 \) with radius \( R \). By using the mean-value property of harmonic functions, we have

\[
|\phi(0) - \phi(x_1)| = \frac{1}{|B_R|} \left| \int_{|y| < R} \phi(y) \, dy - \int_{|y-x| < R} \phi(y) \, dy \right| \leq \frac{1}{|B_R|} \int_{\Omega_R} |\phi(y)| \, dy
\]

where

\[
\Omega_R := \left( \{ y : |y| < R \} \setminus \{ y : |y-x_1| < R \} \right) \cup \left( \{ y : |y-x_1| < R \} \setminus \{ y : |y| < R \} \right).
\]

Now as \( \phi = P - p_1 - p_2 \) a.e., we get

\[
\frac{1}{|B_R|} \int_{\Omega_R} |\phi(y)| \, dy \leq \frac{1}{|B_R|} \left[ \int_{\Omega_R} |P| \, dy + \int_{\Omega_R} |p_1| \, dy + \int_{\Omega_R} |p_2| \, dy \right]
\]

\[
= \frac{1}{|B_R|} \left[ \|P\|_{L_\infty} |\Omega_R| + \|p_1\|_{L^{r_1}} |\Omega_R|^{\frac{r_1-1}{r_1}} + \|p_2\|_{L^{r_2}} |\Omega_R|^{\frac{r_2-1}{r_2}} \right].
\]

As \( |\Omega_R| \leq CR^{d-1} \) for some constant \( C \) depending on \( x_1 \) only, we see that the right hand side \( \to 0 \) as \( R \to \infty \). Consequently \( \phi(0) = \phi(x_1) \) and \( \phi \equiv \text{constant} \).

Finally if \( P \) is nonnegative, we need to show \( c \geq 0 \). Take a nonnegative \( \psi \in C_c^\infty(\mathbb{R}^d) \) with \( \|\psi\|_{L^1(\mathbb{R}^d)} \neq 0 \). Taking convolution with \( \psi \) on both sides of (17), we have

\[
c \|\psi\|_{L^1} = c * \psi = (P * \psi)(x) - (p_1 * \psi)(x) - (p_2 * \psi)(x)
\]

\[
\geq - (p_1 * \psi)(x) - (p_2 * \psi)(x),
\]

where we have used the positivity of \( P \) and \( \psi \). Since by assumption \( p_1 \in L^{r_1} \) and \( p_2 \in L^{r_2} \) with \( 1 < r_1, r_2 < \infty \), we have \( p_1 * \psi \to 0, p_2 * \psi \to 0 \) as \( |x| \to \infty \). Hence \( c \geq 0. \)

In the proof of Theorem [2] we need the following variant of Lemma [1].

Lemma 2. Suppose \( P \in L^\infty(\mathbb{R}^d), p_1 \in L^{r_1}(\mathbb{R}^d), p_2 \in L^{r_2}(\mathbb{R}^d) \) with \( 1 < r_1, r_2 < \infty \). Assume \( \nabla \cdot p_2 \in L^1_{\text{loc}}(\mathbb{R}^d) \). If \( P - p_1 - \nabla \cdot p_2 \) is weakly harmonic, that is

\[
\Delta (P - p_1 - \nabla \cdot p_2) = 0
\]

in the sense of distributions, then there is a constant \( c \) such that

\[
P - p_1 - \nabla \cdot p_2 = c \quad \text{a.e.} \quad x \in \mathbb{R}^d.
\]

If furthermore \( P(x) \geq 0 \) a.e. \( x \in \mathbb{R}^d \), then we also have \( c \geq 0. \)

Proof. Note that Weyl’s lemma still applies here since \( P - p_1 - \nabla \cdot p_2 \) is locally integrable. Thus we have \( \phi \in C^\infty \) strongly harmonic such that

\[
\phi = P - p_1 - \nabla \cdot p_2 \quad \text{a.e.} \quad x \in \mathbb{R}^d.
\]
Take a nonnegative $\psi \in C^\infty_c(\mathbb{R}^d)$ with $\psi(x) = 1$ for $|x| \leq 1$ and $\|\psi\|_{L^1} = 1$. Define the standard mollifier $\psi_\varepsilon(x) = \varepsilon^{-d}\psi(x/\varepsilon)$ for $\varepsilon > 0$. Also denote $f_\varepsilon = \psi_\varepsilon * f$ for any function $f$. Taking convolution with $\psi_\varepsilon$ on both sides of (22), we get

$$\phi * \psi_\varepsilon = P_\varepsilon - p_{1\varepsilon} - \nabla \cdot p_{2\varepsilon}.$$  

Since $\phi$ is harmonic, by using the mean value property, we have $\phi * \psi_\varepsilon = \phi$ and hence

$$\phi = P_\varepsilon - p_{1\varepsilon} - \nabla \cdot p_{2\varepsilon}.$$  

By the same argument as in Lemma 1 (see (19)), we get

$$P_\varepsilon - p_{1\varepsilon} - \nabla \cdot p_{2\varepsilon} = c,$$  

where $c$ is a constant depending on $\phi$ (and hence independent of $\varepsilon$). Since by assumption $p_{1 \varepsilon} \in L^r$, $\nabla \cdot p_{2 \varepsilon} \in L^1_{loc}$, we have

$$p_{1 \varepsilon} \rightarrow p_1, \quad a.e. \ x \in \mathbb{R}^d,$$

$$\nabla \cdot p_{2 \varepsilon} = (\nabla \cdot p_2)_\varepsilon \rightarrow \nabla \cdot p_2, \quad a.e. \ x \in \mathbb{R}^d.$$  

Therefore

$$P_1 - p_1 - \nabla \cdot p_2 = c, \quad a.e. \ x \in \mathbb{R}^d.$$  

Finally if $P \geq 0$, then $P_\varepsilon \geq 0$. By (23), this yields $c \geq 0$. □

3. Proof of Theorem 1

3.1. Upgrading the integrability of $P$ and $\rho$. To prepare for later estimates, we first study the integrability of $P$ and $\rho$. Taking divergence on both sides of (2), we get

$$\triangle (P - p_1 - p_2) = 0,$$  

where

$$P_1 := (-\triangle)^{-1} \left\{ \sum_{i,j=1}^d \partial_i \partial_j (\rho v_i v_j) \right\}, \quad p_2 := (2\mu + \lambda) \nabla \cdot v.$$  

By assumption we have $p_2 \in L^2(\mathbb{R}^d)$. For $p_1$, using Sobolev embedding $H^1 \hookrightarrow L^{2d/(d-2)}$ when $d \geq 4$ together with the assumptions (10–12) we have

$$p_1 \in \begin{cases} L^{\frac{2d}{d-2}}(\mathbb{R}^d) & d \geq 4 \\ L^{9/4}(\mathbb{R}^d) & d = 3 \\ L^3(\mathbb{R}^d) & d = 2 \end{cases}.$$  

Since $P := A\rho_\gamma \geq 0$, by Lemma 1 we get

$$A\rho_\gamma = P = c + p_1 + p_2$$  

for some constant $c \geq 0$. Now consider the function

$$P_1 := \rho_\gamma^{-1} - \left( \frac{c}{A} \right)^{\frac{\gamma-1}{\gamma}}$$  

$$= \left( \frac{c + p_1 + p_2}{A} \right)^{\frac{\gamma-1}{\gamma}} - \left( \frac{c}{A} \right)^{\frac{\gamma-1}{\gamma}}.$$  

This function will be needed later in the integration by parts argument (see (47)).
Consider first \( c > 0 \). By the mean value theorem, we have

\[
|P_1| \lesssim c, A \begin{cases} |p_1 + p_2| & \text{if } |p_1 + p_2| < c/10, \\ \frac{c}{|p_1 + p_2|} & \text{if } |p_1 + p_2| \geq c/10. \end{cases}
\]

(29)

Thus for \( c > 0 \) we have the pointwise estimate

\[
|P_1| \lesssim c, A |p_1| + |p_2|.
\]

For \( c = 0 \), note that by (27), we also have

\[
P_1 \rho = \rho^\gamma \lesssim A |p_1| + |p_2|.
\]

Therefore for all \( c \geq 0 \), we have

\[
|P_1 \rho| \lesssim c, A, \|\rho\|_\infty |p_1| + |p_2|.
\]

(30)

This important pointwise estimate will be used later.

3.2. Choice of the test function. Let \( \phi \in C_\infty (\mathbb{R}) \) be an even function such that

\[
\phi(s) = \begin{cases} 1 & |s| \leq \frac{1}{2} \\ 0 & |s| \geq 1 \end{cases} \quad \in [0, 1], \quad |s| \in \left(\frac{1}{2}, 1\right).
\]

(31)

Let \( \psi \in C_\infty (\mathbb{R}^d) \) be a radial function such that

\[
\psi(|x|) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0 & |x| \geq 1 \end{cases} \quad \in [0, 1], \quad |x| \in \left(\frac{1}{2}, 1\right).
\]

(32)

Now we take our test function as

\[
w(x) = v(x) \phi \left( \frac{|v(x)|^2}{R} \right) |x|^{\frac{d}{2}} \psi \left( \frac{x}{R_1} \right)
\]

(33)

where \( R, R_1 > 0 \) will be taken to infinity in an appropriate order later. The "Lebesgue" cut-off \( \phi(|v|^2/R) \) is used to chop off high values of \( v \).

Note that we have

\[
|w(x)| \leq R \quad \text{for all } x \in \mathbb{R}^d, \quad w(x) = 0 \quad \text{for all } |x| > R_1,
\]

(34)

and

\[
w(x) = v(x) \quad \text{for all } x \in \left\{ \tilde{x} : |v(\tilde{x})| \leq \left( \frac{R_1}{2} \right)^{1/2}, \quad |\tilde{x}| \leq \frac{R_1}{2} \right\}.
\]

(35)
3.3. **Proof of Theorem 1** Taking the inner product of \( w(x) \) on both sides of (2), we obtain

\[
0 = -\frac{1}{2} \int_{\mathbb{R}^d} (\rho v \cdot \nabla) \left( |v|^2 \right) \phi \left( \frac{v^2}{R} \right) \psi \left( \frac{x}{R_1} \right) dx
\]

\[
- \int_{\mathbb{R}^d} \nabla P \cdot v \phi \left( \frac{v^2}{R} \right) \psi \left( \frac{x}{R_1} \right) dx
\]

\[
+ \mu \int_{\mathbb{R}^d} \Delta v \cdot v \phi \left( \frac{v^2}{R} \right) \psi \left( \frac{x}{R_1} \right) dx
\]

\[
+ (\mu + \lambda) \int_{\mathbb{R}^d} (\nabla \cdot v) \cdot v \phi \left( \frac{v^2}{R} \right) \psi \left( \frac{x}{R_1} \right) dx
\]

\[
=: I_1 + I_2 + I_3 + I_4. \tag{36}
\]

In the following we will take \( R_1 \nearrow \infty \) first and then \( R \nearrow \infty \), and show that the above becomes

\[
0 = \mu \int_{\mathbb{R}^d} |\nabla v|^2 dx + (\mu + \lambda) \int_{\mathbb{R}^d} (\nabla \cdot v)^2 dx. \tag{37}
\]

This will imply \( v = \text{constant} \). Combined with (2) and the assumptions (10–12) we conclude that \( v = 0 \) and \( \nabla P = 0 \). The latter then yields \( \rho = \text{constant} \) and ends the proof.

We estimate \( I_1 - I_4 \) one by one. From now on we will adopt the following notations to simplify the presentation of the proofs.

\[
\Omega_R := \text{supp} \left( \phi \left( \frac{|v|^2}{R} \right) \right) \subseteq \left\{ x : |v|^2 \leq R \right\}, \tag{38}
\]

\[
C_R := \text{supp} \left( \phi' \left( \frac{|v|^2}{R} \right) \right) \subseteq \left\{ x : \frac{R}{2} \leq |v|^2 \leq R \right\}, \tag{39}
\]

\[
C'_{R_1} := \text{supp} \left( (\nabla \psi) \left( \frac{x}{R_1} \right) \right) \subseteq \left\{ x : \frac{R_1}{2} \leq |x| \leq R_1 \right\}. \tag{40}
\]

It is easy to see that

\[
\|v\|_{L^\infty(\Omega_R)}, \quad \|v\|_{L^\infty(C_R)} \leq R^{1/2}. \tag{41}
\]

**Estimate of** \( I_1 \). Let \( \hat{\phi}(z) := \int_0^z \phi(s/R) ds \). Clearly \( |\hat{\phi}(z)| \leq |z| \).

We notice that

\[
I_1 = -\frac{1}{2} \int_{\mathbb{R}^d} (\rho v \cdot \nabla) \left( |v|^2 \right) \phi \left( \frac{v^2}{R} \right) \psi \left( \frac{x}{R_1} \right) dx
\]

\[
= -\frac{1}{2} \int_{\Omega_R} \rho v \cdot \nabla \hat{\phi} \left( |v|^2 \right) \psi \left( \frac{x}{R_1} \right) dx
\]

\[
= \frac{1}{2} \int_{\Omega_R} \left[ \rho v \hat{\phi} \left( |v|^2 \right) \right] \cdot \nabla \left[ \psi \left( \frac{x}{R_1} \right) \right] dx
\]

\[
= \frac{1}{2} \int_{\Omega_R \cap C'_{R_1}} \left[ \rho v \hat{\phi} \left( |v|^2 \right) \right] \cdot \left[ \frac{1}{R_1} (\nabla \psi) \left( \frac{x}{R_1} \right) \right] dx. \tag{42}
\]
Now using Hölder’s inequality and the fact that $0 \leq \tilde{\phi} \left( |v|^2 \right) \leq |v|^2$, we have

$$|I_1| \leq \frac{1}{2} \|\rho\|_{L^\infty(\mathbb{R}^d)} \|v\|_{L^{3d} \left( \Omega_R \cap C_{R_1} \right)}^3 \left\| \frac{1}{R_1} \left( \nabla \psi \right) \left( \frac{x}{R_1} \right) \right\|_{L^d(\mathbb{R}^d)}. \quad (43)$$

Since

$$\left\| \frac{1}{R_1} \left( \nabla \psi \right) \left( \frac{x}{R_1} \right) \right\|_{L^d(\mathbb{R}^d)} = \|\nabla \psi\|_{L^d(\mathbb{R}^d)} \quad (44)$$

is independent of $R, R_1$, all we need to show is for any fixed $R$, $\|v\|_{L^{3d} \left( \Omega_R \cap C_{R_1} \right)} \to 0$ as $R_1 \to \infty$. It is clear that this is true if

$$v \in L^{\frac{3d}{d-2}} (\Omega_R) \quad (45)$$

over $\Omega_R$ as defined in (38).

- $d \geq 4$. In this case we use Sobolev embedding $\dot{H}^1 (\mathbb{R}^d) \hookrightarrow L^{\frac{3d}{d-2}} (\mathbb{R}^d)$ and standard interpolation. Since $\frac{3d}{d-2} \leq \frac{3d}{d-1}$ and $v \in L^\infty (\Omega_R)$, we have

$$v \in L^{\frac{3d}{d-2}} (\Omega_R) \cap L^\infty (\Omega_R) \implies v \in L^{\frac{3d}{d-1}} (\Omega_R). \quad (46)$$

- $d = 2, 3$. In these cases we have $\frac{3d}{d-1} = \frac{9}{2}, 6$ respectively. So $v \in L^{\frac{3d}{d-1}} (\Omega_R)$ follows immediately from the assumptions (11, 12).

**Estimate of $I_2$.**

Observe

$$\nabla (\rho^\gamma) = \frac{\gamma}{\gamma - 1} \rho \nabla (\rho^{\gamma - 1}) = \frac{\gamma}{\gamma - 1} \rho \nabla P_1, \quad (47)$$

where $P_1$ was defined in (28).

We have

$$I_2 = -A \frac{\gamma}{\gamma - 1} \int_{\mathbb{R}^d} \rho \nabla P_1 \cdot v \phi \left( \frac{|v|^2}{R} \right) \psi \left( \frac{x}{R_1} \right) \, dx$$

$$= \frac{A \gamma}{\gamma - 1} \int_{\mathbb{R}^d} P_1 \left( \rho v \right) \cdot \left( \nabla \psi \left( \frac{x}{R_1} \right) \right) \frac{1}{R_1} \phi \left( \frac{|v|^2}{R} \right) \, dx$$

$$+ \frac{A \gamma}{\gamma - 1} \int_{\mathbb{R}^d} P_1 \rho \frac{2}{R} \phi' \left( \frac{|v|^2}{R} \right) \psi \left( \frac{x}{R_1} \right) \sum_{j,k=1}^d v_j v_k \partial_j v_k \, dx$$

$$=: I_{2a} + I_{2b}. \quad (48)$$
For $I_{2a}$ by using (30), we have

$$|I_{2a}| = \left| \frac{A_T}{\gamma - 1} \int_{C_{R_1}} P_1 (\rho v) \cdot \left( \nabla \psi \left( \frac{x}{R_1} \right) \right) \frac{1}{R_1} \phi \left( \frac{|v|^2}{R} \right) d x \right|$$

$$\leq A_T \int_{C_{R_1}} |P_1 \rho| |v| \phi \left( \frac{|v|^2}{R} \right) \left[ \left( \nabla \psi \left( \frac{x}{R_1} \right) \right) \frac{1}{R_1} \right] d x$$

$$\leq A_T \left\| P_1 \rho v \phi \left( \frac{|v|^2}{R} \right) \right\|_{L^{\frac{2}{\gamma - 1}} (C_{R_1})} \left\| \left( \nabla \psi \left( \frac{x}{R_1} \right) \right) \frac{1}{R_1} \right\|_{L^d}$$

$$\lesssim A_T \| \rho \|_{\infty} \| p_1 v \|_{L^{\frac{2}{\gamma - 1}} (\Omega_R \cap C_{R_1})} + \| p_2 v \|_{L^{\frac{2}{\gamma - 1}} (\Omega_R \cap C_{R_1})}.$$ (49)

Here we have used the fact that the $\psi$ term is a constant depending only on $\psi$. Now it suffices to show

$$p_1 v, \quad p_2 v \in L^{\frac{2}{\gamma - 1}} (\Omega_R).$$ (50)

For the first term we recall

$$p_1 \in \begin{cases} L^{\frac{2d}{2d - 1}} (\Omega_R) & d \geq 4 \\ L^{9/4} (\Omega_R) & d = 3 \\ L^3 (\Omega_R) & d = 2 \end{cases},$$ (51)

and, as $R$ is fixed now,

$$v \in L^{\infty} (\Omega_R) \cap \begin{cases} L^{\frac{2d}{2d - 1}} (\Omega_R) & d \geq 4 \\ L^{9/2} (\Omega_R) & d = 3 \\ L^6 (\Omega_R) & d = 2 \end{cases} \implies v \in L^q (\Omega_R)$$

for any \( q \geq \frac{2d}{d - 2} \) \( d \geq 4 \) \( q \geq \frac{9}{8} \) \( d = 3 \) \( q \geq \frac{3}{2} \) \( d = 2 \). (52)

by interpolation. With these integrability properties we now proceed as follows.

\begin{itemize}
  \item $d \geq 4$. In this case we have $d \geq \frac{2d}{2d - 1}$. Thus following (51), (52)

$$p_1 \in L^{\frac{2d}{2d - 1}} (\Omega_R), \quad v \in L^d (\Omega_R) \implies p_1 v \in L^{\frac{2d}{2d - 1}} (\Omega_R);$$ (53)

  Furthermore

$$p_2 \in L^2 (\Omega_R), \quad v \in L^{\frac{2d}{2d - 1}} (\Omega_R) \implies p_2 v \in L^{\frac{2d}{2d - 1}} (\Omega_R).$$ (54)

  \item $d = 3$. In this case $\frac{d}{d - 1} = \frac{3}{2}$. We have

$$p_1 \in L^{9/4} (\Omega_R), \quad v \in L^{9/2} (\Omega_R) \implies p_1 v \in L^{\frac{2d}{2d - 1}} (\Omega_R);$$ (55)

$$p_2 \in L^2 (\Omega_R), \quad v \in L^6 (\Omega_R) \implies p_2 v \in L^{\frac{2d}{2d - 1}} (\Omega_R).$$ (56)

  \item $d = 2$. In this case $\frac{d}{d - 1} = 2$. We have

$$p_1 \in L^3 (\Omega_R), \quad v \in L^6 (\Omega_R) \implies p_1 v \in L^{\frac{2d}{2d - 1}} (\Omega_R);$$ (57)

$$p_2 \in L^2 (\Omega_R), \quad v \in L^{\infty} (\Omega_R) \implies p_1 v \in L^{\frac{2d}{2d - 1}} (\Omega_R).$$ (58)

\end{itemize}
Next we estimate $I_{2b}$. By the definitions of $C_R, \phi, \psi, P_1$ (see (30), (31), (32), (28)) and Hölder’s inequality we have

$$|I_{2b}| = \left| \frac{A_{\gamma}}{\gamma - 1} \int_{\mathbb{R}^d} P_1 \phi \frac{2}{R_1} \left( \frac{|v|^2}{R} \right) \psi \left( \frac{x}{R_1} \right) \sum_{j,k=1}^d v_j v_k \partial_j \partial_k v \, dx \right|$$

$$\lesssim |A_{\gamma} \int_{C_R} |P_1 \phi \frac{|v|^2}{R} |\nabla v| |dx$$

$$\lesssim |A_{\gamma} \int_{C_R} \left| \left( \rho^{\gamma-1} - \left( \frac{C}{A} \right)^{\gamma-1} \right) \rho \right| |\nabla v| |dx$$

$$\lesssim_{c,A,\gamma,||\rho||_\infty} \|1\|_{L^2(C_R)} \|\nabla v\|_{L^2}$$

$$= |C_R|^\frac{1}{p} \|\nabla v\|_{L^p}.$$

Thus we only need to show $|C_R|$ goes to 0 as $R \to \infty$. Thanks to Chebyshev’s inequality,

$$|C_R| \leq |\{ x : |v(x)| \geq R/2 \} \lesssim R^{-p} \|v\|_p^p,$$  

(59)

it suffices to check $v \in L^p(\mathbb{R}^d)$ for some $p > 0$. Verification of this is quite straightforward:

- $d \geq 4$. $v \in L^\frac{2d}{d-2}(\mathbb{R}^d)$ thanks to the Sobolev embedding $H^1(\mathbb{R}^d) \hookrightarrow L^\frac{2d}{d-2}(\mathbb{R}^d)$;
- $d = 3$. $v \in L^{\frac{6}{3}}(\mathbb{R}^d)$ by assumptions of the theorem;
- $d = 2$. $v \in L^6(\mathbb{R}^d)$ by assumptions of the theorem.

Therefore $|C_R| \to 0$ as $R \to \infty$ in all cases. Consequently $I_{2b} \to 0$ as desired.

**Estimate of $I_3$.**

We have

$$I_3 = \mu \int_{\mathbb{R}^d} \nabla v \cdot \nabla \psi \left( \frac{\nu^2}{R} \right) \psi \left( \frac{x}{R_1} \right) \, dx$$

$$= -\mu \int_{\mathbb{R}^d} |\nabla v|^2 \phi \left( \frac{\nu^2}{R} \right) \psi \left( \frac{x}{R_1} \right) \, dx$$

$$- \mu \int_{\mathbb{R}^d} (\nabla v) v \cdot \nabla \left( \phi \left( \frac{\nu^2}{R} \right) \right) \psi \left( \frac{x}{R_1} \right) \, dx$$

$$- \mu \int_{\mathbb{R}^d} (\nabla v) v \cdot \nabla \left( \psi \left( \frac{x}{R_1} \right) \right) \phi \left( \frac{\nu^2}{R} \right) \, dx$$

$$= : I_{3a} + I_{3b} + I_{3c}.$$  

(60)

First consider $I_{3a}$. As $\nabla v \in L^2(\mathbb{R}^d)$ it is clear from Lebesgue dominated convergence that

$$I_{3a} \to -\mu \int_{\mathbb{R}^d} |\nabla v|^2 \, dx \quad \text{as } R_1 \to \infty \text{ followed by } R \to \infty.$$  

(61)

For $I_{3b}$, we have by dominated convergence

$$|I_{3b}| \lesssim \int_{\mathbb{R}^d} |\nabla v|^2 \left| \frac{|v|^2}{R} \phi' \left( \frac{|v|^2}{R} \right) \right| \, dx$$

$$\lesssim \int_{C_R} |\nabla v|^2 \, dx \to 0 \quad \text{as } R_1 \to \infty \text{ followed by } R \to \infty.$$  

(62)
For $I_{3c}$, we have

$$|I_{3c}| \lesssim \int_{C_{R_1}'} |\nabla v| |v| \phi \left( \frac{v^2}{R} \right) \left| \frac{1}{R_1} \nabla \psi \right| \, dx$$

$$\lesssim \| \nabla v \|_{L^2} \| |v| \|_{L^{\frac{2d}{d-2}}(C_{R_1}', \Omega_R)} \left\| \frac{1}{R_1} \nabla \psi \right\|_{L^d(\mathbb{R}^d)} \rightarrow 0 \quad \text{as } R_1 \nearrow \infty \quad (63)$$

since $v \in L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ by Sobolev embedding when $d \geq 3$. When $d = 2$ note that as $R$ is fixed, $v \in L^\infty(\Omega_R)$ so the above still holds.

**Estimate of $I_4$.**

This is similar to the estimate of $I_3$ and hence we omit the details.

Collecting the estimates, we have

$$0 = I_1 + \cdots + I_4 \rightarrow -\mu \int_{\mathbb{R}^d} |\nabla v|^2 \, dx - (\mu + \lambda) \int_{\mathbb{R}^d} (\nabla \cdot v)^2 \, dx \quad (64)$$

which gives $\nabla v = 0$ and therefore $v = \text{constant}$. Together with the assumptions (10–12) we have $v = 0$. From the momentum equation (2) and the relation (3) we also obtain $\rho \equiv \text{constant}$. This ends the proof of Theorem 1.

4. **Proofs of Theorems 2 and 3**

4.1. **Proof of Theorem 2**

Taking the dot product with $v$ on both sides of (2), we get

$$\rho v \cdot \nabla \left( \frac{|v|^2}{2} \right) = -A \frac{\gamma}{\gamma - 1} \rho v \cdot \nabla \left( \rho^{\gamma - 1} \right) + \mu v \cdot \Delta v + (\mu + \lambda) \rho \nabla \cdot \nabla \cdot v. \quad (65)$$

Using $\nabla \cdot (\rho v) = 0$ and the identities

$$\Delta \left( |v|^2 \right) = 2v \cdot \Delta v + 2|\nabla v|^2, \quad (66)$$

$$\nabla \cdot (v \nabla \cdot v) = v \cdot \nabla (\nabla \cdot v) + |\nabla \cdot v|^2, \quad (67)$$

we can rewrite (65) as

$$\frac{1}{2} \nabla \cdot (\rho v |v|^2) = -\frac{\gamma}{\gamma - 1} A \nabla \cdot (\rho^{\gamma - 1}) + \mu \left( \frac{1}{2} \Delta (|v|^2) - |\nabla v|^2 \right)$$

$$+ (\mu + \lambda) \left( \nabla \cdot (v \nabla \cdot v) - |\nabla \cdot v|^2 \right)$$

$$= -\frac{\gamma}{\gamma - 1} \nabla \cdot (P v) + \mu \left( \frac{1}{2} \Delta (v^2) - |\nabla v|^2 \right)$$

$$+ (\mu + \lambda) \left( \nabla \cdot (v \nabla \cdot v) - |\nabla \cdot v|^2 \right). \quad (68)$$
Now let $\psi$ be the cut-off function as defined in (32). Multiplying (68) by $\psi\left(\frac{x}{R_1}\right)$ and integrating by parts, we obtain

$$0 = \frac{1}{2} \int_{\mathbb{R}^d} \rho |v|^2 v \cdot (\nabla \psi) \left( \frac{x}{R_1} \right) \frac{1}{R_1} \, dx$$

$$+ \frac{\gamma}{\gamma - 1} \int_{\mathbb{R}^d} P v \cdot (\nabla \psi) \left( \frac{x}{R_1} \right) \frac{1}{R_1} \, dx$$

$$+ \frac{\mu}{2} \int_{\mathbb{R}^d} \frac{1}{R_1} \left( \Delta \psi \right) \left( \frac{x}{R_1} \right) \, dx$$

$$- \mu \int_{\mathbb{R}^d} |\nabla v|^2 \psi \left( \frac{x}{R_1} \right) \, dx - (\mu + \lambda) \int_{\mathbb{R}^d} |\nabla \cdot v|^2 \psi \left( \frac{x}{R_1} \right) \, dx$$

$$- (\mu + \lambda) \int_{\mathbb{R}^d} (\nabla \cdot v) v \cdot (\nabla \psi) \left( \frac{x}{R_1} \right) \frac{1}{R_1} \, dx$$

$$=: I_1 + \cdots + I_5. \quad (69)$$

We shall estimate $I_1, \cdots, I_5$ one by one.

For $I_1$ we have

$$|I_1| \leq \frac{1}{2} \int_{C'_{R_1}} \rho |v|^3 \left| (\nabla \psi) \left( \frac{x}{R_1} \right) \right| \frac{1}{R_1} \, dx$$

$$\leq \frac{1}{2} \|\rho\|_{L^\infty} \left\| \frac{1}{R_1} \right\|_{L^1} \left\| \left( \nabla \psi \right) \left( \frac{x}{R_1} \right) \frac{1}{R_1} \right\|_{L^d}.$$ \quad (70)

Here $C'_{R_1}$ is the support of $\nabla \psi \left( \frac{x}{R_1} \right)$, as defined in (40). By assumption we have $v \in L^{\frac{3d}{d-1}}(\mathbb{R}^d)$, thus $\|v\|_{L^{\frac{3d}{d-2}}(C'_{R_1})} \to 0$ as $R_1 \to \infty$. So $I_1 \to 0$ as $R_1 \to \infty$.

For $I_2$ we have

$$|I_2| \leq \frac{\gamma}{\gamma - 1} \|P\|_{L^\infty} \left\| \frac{1}{R_1} \right\|_{L^{d/(d-1)}(C'_{R_1})} \left\| \frac{1}{R_1} \right\|_{L^d}.$$ \quad (71)

By assumption $v \in L^{d/(d-1)}(\mathbb{R}^d)$ so this term goes to 0 too.

For $I_3$ we notice that the assumption $v \in L^{\frac{3d}{d-1}}(\mathbb{R}^d) \cap L^{\frac{3d}{d-2}}(\mathbb{R}^d)$ leads to

$$v \in L^2(\mathbb{R}^d) \quad (72)$$

through interpolation (When $d = 2$ we already have $\frac{d}{d-1} = 2$). Thus

$$|I_3| \leq \frac{\mu}{2} \left\| \frac{1}{R_1} \right\|_{L^{(d)}(\mathbb{R}^d)} \left\| \left( \frac{x}{R_1} \right) \frac{1}{R_1} \right\|_{L^{\infty}(\mathbb{R}^d)}$$

$$\leq \|v\|_{L^2(\mathbb{R}^d)} R_1^{-2} \to 0, \quad \text{as } R_1 \to \infty. \quad (73)$$

For $I_4$ we notice that

$$|\nabla v|^2 \psi \left( \frac{x}{R_1} \right), \quad |\nabla \cdot v|^2 \psi \left( \frac{x}{R_1} \right) \geq 0 \quad (74)$$
and furthermore at every \( x \) are increasing with respect to \( R_1 \). An application of Lebesgue’s monotone convergence theorem then gives

\[
I_4 \rightarrow -\mu \int_{\mathbb{R}^d} |\nabla v|^2 \, dx - (\mu + \lambda) \int_{\mathbb{R}^d} |\nabla \cdot v|^2 \, dx
\]

as \( R_1 \not\rightarrow \infty \).

For \( I_5 \) we first recall \([24, 25]\), which are obtained from taking divergence of \([2]\):

\[
\triangle (P - p_1 - \nabla \cdot p_2) = 0, \quad p_1 := (-\triangle)^{-1} \left[ \sum_{i,j=1}^d \partial_i \partial_j (\rho v_i v_j) \right], \quad p_2 := (2\mu + \lambda) v.
\]

The assumptions of Theorem \([2]\) give

\[
P \in L^\infty (\mathbb{R}^d), \quad p_1 \in L^\frac{2d}{d-1}(\mathbb{R}^d), \quad v \in L^{\frac{2d}{d-1}}(\mathbb{R}^d).
\]

Applying Lemma \([2]\) we obtain

\[
\nabla \cdot v = \frac{1}{2\mu + \lambda} [P - p_1 - c] \in L^\infty (\mathbb{R}^d) + L^\frac{2d}{d-1} (\mathbb{R}^d).
\]

From this it is easy to see that \( I_5 \) enjoys the same estimates as \( I_1 + I_2 \). More precisely, we have

\[
|I_5| \lesssim \int_{C_{R_1}^c} |P - c| |v| \left| (\nabla \psi) \left( \frac{x}{R_1} \right) \right| \frac{1}{R_1} \, dx
\]

\[
+ \int_{C_{R_1}^c} |p_1| |v| \left| (\nabla \psi) \left( \frac{x}{R_1} \right) \right| \frac{1}{R_1} \, dx
\]

\[
\lesssim \|P - c\|_{L^\infty} \|v\|_{L^{4/(d-1)}(C_{R_1}^c)} \left\| (\nabla \psi) \left( \frac{x}{R_1} \right) \right\|_{L^d} \frac{1}{R_1} \|v\|_{L^{4/(d-1)}(C_{R_1})} \left\| (\nabla \psi) \left( \frac{x}{R_1} \right) \right\|_{L^d} \frac{1}{R_1} \rightarrow 0, \quad \text{as } R_1 \not\rightarrow \infty.
\]

Collecting the estimates of \( I_1 - I_5 \), we conclude that as \( R_1 \not\rightarrow \infty \), we have

\[
0 = I_1 + \cdots + I_5 \rightarrow -\mu \int_{\mathbb{R}^d} |\nabla v|^2 \, dx - (\mu + \lambda) \int_{\mathbb{R}^d} |\nabla \cdot v|^2 \, dx
\]

which gives \( \nabla v = 0 \) and the conclusions of Theorem \([2]\) follow.

### 4.2. Proof of Theorem \([3]\)

Application of Lemma \([1]\) gives

\[
A^\rho = P = p_1 + (2\mu + \lambda) \nabla \cdot v + c \quad \text{a.e. } x \in \mathbb{R}^d.
\]

Defining \( P_1 \) as in \([28]\) and following the estimate \([30]\), we have

\[
|P_1 \rho| \lesssim |p_1| + |\nabla \cdot v|.
\]

Recall that

\[
p_1 := (-\triangle)^{-1} \left[ \sum_{i,j=1}^d \partial_i \partial_j (\rho v_i v_j) \right] \in L^{\frac{2d}{d-1}}(\mathbb{R}^d)
\]

by the assumption on \( v \) and the boundedness of Riesz operators.

---

\footnote{We can take a subsequence in \( R_1 \) if necessary so that the function \( \psi(x/R_1) \) is increasing.}
We rewrite (65) as
\[ \frac{1}{2} \nabla \cdot \left( \rho v |v|^2 \right) \]
\[ = - \frac{A_\gamma}{\gamma - 1} \nabla \cdot (P_1 \rho v) + \mu \left( \frac{1}{2} \Delta (|v|^2) - |\nabla v|^2 \right) + (\mu + \lambda) \left( \nabla \cdot (v \nabla v) - |\nabla \cdot v|^2 \right). \]  
(84)

and pick the cut-off function \( \psi \) as before with a parameter \( R_1 > 0 \) which will tend to \( \infty \) later.

Multiplying (81) by \( \psi \left( \frac{x}{R_1} \right) \) and integrating by parts, we reach
\[ 0 = \frac{1}{2} \int_{\mathbb{R}^d} \rho |v|^2 v \cdot (\nabla \psi) \left( \frac{x}{R_1} \right) \frac{1}{R_1} dx \]
\[ + \frac{A}{\gamma - 1} \int_{\mathbb{R}^d} P_1 \rho v \cdot (\nabla \psi) \left( \frac{x}{R_1} \right) \frac{1}{R_1} dx \]
\[ + \frac{\mu}{2} \int_{\mathbb{R}^d} |v|^2 \frac{1}{R_1^2} (\Delta \psi) \left( \frac{x}{R_1} \right) dx \]
\[ - \mu \int_{\mathbb{R}^d} |\nabla v|^2 \psi \left( \frac{x}{R_1} \right) dx - (\mu + \lambda) \int_{\mathbb{R}^d} |\nabla \cdot v|^2 \psi \left( \frac{x}{R_1} \right) dx \]
\[ = : I_1 + \cdots + I_5. \]
(85)

Now we estimate them one by one.

For \( I_1 \), we have
\[ |I_1| \lesssim \|\rho\|_{L^\infty(\mathbb{R}^d)} \|v\|_{L^\frac{3d}{d-1}(\mathbb{R}^d)}^3 \left\| \nabla \psi \right\|_{L^d(\mathbb{R}^d)} \left( \frac{x}{R_1} \right) \frac{1}{R_1} \rightarrow 0 \text{ as } R_1 \nearrow \infty \]  
(86)
because \( v \in L^{\frac{3d}{d-1}}(\mathbb{R}^d) \) by assumption.

For \( I_2 \) we apply the estimate (82) to obtain
\[ |I_2| \lesssim \int_{\mathbb{R}^d} |p_1| |v| \left\| \nabla \psi \right\|_{L^d(\mathbb{R}^d)} \left( \frac{x}{R_1} \right) \frac{1}{R_1} dx + \int_{\mathbb{R}^d} |\nabla \cdot v| |v| \left\| \nabla \psi \right\|_{L^d(\mathbb{R}^d)} \left( \frac{x}{R_1} \right) \frac{1}{R_1} dx \]
\[ = : I_{2a} + I_{2b}. \]
(87)

For \( I_{2a} \) we have
\[ I_{2a} \lesssim \|p_1\|_{L^\frac{3d}{d-1}(\mathbb{R}^d)} \|v\|_{L^\frac{3d}{d-1}(\mathbb{R}^d)} \left\| \nabla \psi \right\|_{L^d(\mathbb{R}^d)} \left( \frac{x}{R_1} \right) \frac{1}{R_1} \rightarrow 0 \text{ as } R_1 \nearrow \infty. \]
(88)

For \( I_{2b} \) we consider two cases:

- \( d \geq 4 \). We estimate it as
\[ I_{2b} \lesssim \|\nabla \cdot v\|_{L^2(\mathbb{R}^d)} \|v\|_{L^\frac{3d}{d-1}(\mathbb{R}^d)} \left\| \nabla \psi \right\|_{L^d(\mathbb{R}^d)} \left( \frac{x}{R_1} \right) \frac{1}{R_1} \rightarrow 0 \text{ as } R_1 \nearrow \infty. \]
(89)

- \( d \leq 3 \). In this case we have
\[ I_{2b} \lesssim \|\nabla \cdot v\|_{L^2(\mathbb{R}^d)} \left\| \nabla \psi \right\|_{L^{\frac{3d}{d+2}}(\mathbb{R}^d)} \left( \frac{x}{R_1} \right) \frac{1}{R_1} \rightarrow 0 \text{ as } R_1 \nearrow \infty \]
(90)
since when $d \leq 3$ we have $\frac{6d}{d+2} > d$ and therefore
\[
\left\| (\nabla \psi) \left( \frac{x}{R_1} \right) \frac{1}{R_1} \right\|_{L^{d/(d+2)}(\mathbb{R}^d)} \to 0 \text{ as } R_1 \not\to \infty.
\] (91)

For $I_3$ we estimate it as follows:

- $d \geq 5$.
\[
|I_3| \leq \left\| |v|^2 \right\|_{L^{4/(d-2)}(\text{C}^d_{R_1})} \left\| \frac{1}{R_1^2} (\triangle \psi) \left( \frac{x}{R_1} \right) \right\|_{L^{d/(d-2)}(\mathbb{R}^d)}
= \left\| |v|^2 \right\|_{L^{4/(d-2)}(\text{C}^d_{R_1})} \left\| \frac{1}{R_1^2} (\triangle \psi) \left( \frac{x}{R_1} \right) \right\|_{L^{d/(d-2)}(\mathbb{R}^d)} \to 0 \text{ as } R_1 \not\to \infty.
\] (92)

- $d \leq 4$.
\[
|I_3| \leq \left\| |v|^2 \right\|_{L^{4/(d-1)}(\mathbb{R}^d)} \left\| \frac{1}{R_1^2} (\triangle \psi) \left( \frac{x}{R_1} \right) \right\|_{L^{d/(d+2)}(\mathbb{R}^d)}
= \left\| |v|^2 \right\|_{L^{4/(d-1)}(\text{C}^d_{R_1})} \left\| \frac{1}{R_1^2} (\triangle \psi) \left( \frac{x}{R_1} \right) \right\|_{L^{d/(d+2)}(\mathbb{R}^d)}.
\] (93)

When $d \leq 4$ we have $\frac{6d}{d+2} \geq \frac{4}{2} = 2$, so $I_3 \to 0$.

For $I_4$ the same argument for $I_4$ in the proof of Theorem 2 works, and we have
\[
I_4 \to -\mu \int_{\mathbb{R}^d} |\nabla v|^2 \, dx - (\mu + \lambda) \int_{\mathbb{R}^d} |\nabla \cdot v|^2 \, dx \text{ as } R_1 \not\to \infty.
\] (94)

Finally, since $I_5$ is essentially the same as $I_{2b}$, we have
\[
I_5 \to 0 \text{ as } R_1 \not\to \infty.
\] (95)

Collecting all the estimates, we have
\[
0 = I_1 + \cdots + I_5 \to -\mu \int_{\mathbb{R}^d} |\nabla v|^2 \, dx - (\mu + \lambda) \int_{\mathbb{R}^d} |\nabla \cdot v|^2 \, dx.
\] (96)

Consequently $\nabla v = 0$ and the conclusions of Theorem 3 follow.

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