Low dimensional models for stick-slip vibration of drill-strings

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Abstract. Effective reduction of drill-string vibration is still a major problem in drilling industry and therefore robust predictive tools need to be developed. In this paper we study two low dimensional nonlinear models. The first is a 1-DOF torsional model of the bottom-hole assembly (BHA). The second model is a 3-DOF torsional system having in addition to the BHA a rotary table, which allows simulation of interactions for which there is experimental evidence. Three different friction models with increasing levels of complexity are applied to determine their influence in the dynamical responses. Comparison between the dynamic responses for three friction models shows that the dangerous stick-slip limit-cycles do not change qualitatively. Simulations show that, if appropriately controlled, large amplitude stick-slip limit-cycles can change to small amplitude limit-cycles in Model 2. In Model 1, with constant velocity of the rotary table, it goes from a large amplitude stick-slip limit-cycle to a fixed point. Bifurcation diagrams confirm the existence of a set of parameters in which the system operates without stick-slip vibration.

1. Introduction
Drilling a borehole for oil and gas extraction is done by means of rotary motion of a drill-bit against the rock. The rotary motion is transmitted to the bit from a motor, usually at the surface of the well, by the use of a drill-string. This drill-string is made of tube sections with threaded connections. It can reach lengths of various kilometers, making it a very slender structure. Torsional vibration is one of the most important vibration modes in drill-string, and the main phenomenon related to it is stick-slip vibrations.

The lower part of the drill-string, comprised of drill-collars, stabilizers, vibration absorbers, MWD tools and the drill-bit is called the bottom-hole assembly (BHA). The dynamic behaviour of the BHA is greatly responsible for the most harmful vibrations, represented by the torsional, axial and lateral modes and the coupling mechanisms between them. The drill-string vibrations can either be induced by drill-bit—formation and drill-string—bore-hole interactions, or be self-induced [1]. Although the coupling mechanisms are highly nonlinear, the study of each of these modes separately is important to establish the relevance and influence that they have over the full dynamics of the system. The complexity of the drill-string dynamics comes in part from the coupling mechanisms, which converts energy from the different vibration modes. The drill-bit, which relates the axial force to torque is an important player in these mechanisms.
For this study, low dimensional models are presented that focus on torsional vibrations of the drill-string. Although not being able to reproduce fully the complex dynamics of the drilling system, they permit the study of the main characteristics of the stick-slip vibration effect. The assumption that the drill-string behaves as a torsional spring, meaning it has a dominant fundamental mode, is valid for a high ratio of mass moment of inertia of the BHA relative to the drill-string \[5, 6\].

2. Modelling stick-slip vibrations

2.1. Friction models for bit-rock interaction

The torque-on-bit \(T_b\) is a friction law, a function that defines the resistive torque on the bit and models the interaction between the drill-bit and the rock. It can be defined in various ways, following the form:

\[
T_{b1} = \begin{cases} 
    T_{st} & \text{if } \dot{\phi}_1 = 0, \\
    T_{sl} & \text{if } \dot{\phi}_1 \neq 0.
\end{cases}
\] (1)

The simplest model for \(T_b\) is a piecewise Coulomb-like friction law (Friction Model 1), with a value for static friction \(T_{st}\) and another, lower, value for sliding (or dynamic) friction \(T_{sl}\), which in this case is independent of the velocity \(\dot{\phi}_1\). More realistic friction laws account for dependency on the velocity and weight-on-bit (WOB), and are usually discontinuous at null velocity. A comprehensive survey of various friction models is given by Wojewoda et al. \[2\].

A more complex friction model (Friction Model 2) accounts for dependency of the dynamic coefficients, \(\mu_0\) and \(\mu_1\), on the contact velocity, and is given by \[3\]:

\[
T_{b2} = \text{sgn}(\dot{\phi}_1) W_b \left( \frac{\mu_0 - \mu_1}{1 + \lambda_1 |\dot{\phi}_1|} + \mu_1 + \lambda_2 \dot{\phi}_1 \right),
\] (2)

where \(\lambda_1\) and \(\lambda_2\) are dry friction constants and \(W_b\) is the WOB.

A third friction model (Friction Model 3) incorporates an exponential dependency on the contact velocity, allowing better accuracy near null velocity, and is given by \[4\]:

\[
T_{b3} = T_{sl} + (T_{st} - T_{sl}) e^{-|\dot{\phi}_1/\omega_{sl}|^\delta_{sl}} + b_l |\omega_l|,
\] (3)

where \(T_{st}\) and \(T_{sl}\) are the static and dynamic friction torques and \(\omega_{sl}, \delta_{sl}\) and \(b_l\) are dry friction constants.

2.2. 1-DOF torsional pendulum model - Model 1

As a first approximation, the drill-string can be modelled as a 1-DOF torsional pendulum. In this arrangement, the equivalent inertia of the system is represented by a mass at the bottom, the drill-pipes are represented by a torsional spring, the equivalent damping acting on the system is represented by a torsional damper, corresponding to the lower portion of figure 1. Also as a first approximation, the angular velocity of the rotary table is constant, as the amplitude of its oscillations are considerably smaller than the amplitude of oscillations of the velocity of the bit. The equation of motion for this system is:

\[
J_1 \ddot{\phi}_1 + c_1 \dot{\phi}_1 + k_1 (\dot{\phi}_1 - \dot{\phi}_2) - T_b = 0,
\] (4)
where $J_1$ is the equivalent mass moment of inertia of the drill-string, $c_1$ is the equivalent damping coefficient along the drill-string, $k_1$ is the equivalent torsional stiffness of the drill-pipes, $\phi_1$ is the angular position of the bit at the bottom of the drill-string, $\phi_2$ is the angular position of the rotary table at the top of the drill-string, $T_b$ is the torque-on-bit and a dot denotes differentiation over time.

In this problem, the interest is in the relative displacement between the top and the bottom, as the torque stored in the spring (representing the drill-pipes) depends on this quantity. With this in mind, the relative angular displacement between the rotary table and the bit is defined as:

$$\phi = \phi_2 - \phi_1,$$

then (4) can be written as:

$$J_1 \ddot{\phi} + c_1 \dot{\phi} + k_1 \phi - T_b = 0.$$

2.3. 1-DOF torsional pendulum with parametric excitation of the angular velocity of the rotary table – Model 2

As observed in field data [5–7], the velocity of the rotary table oscillates around the nominal velocity when the bit is experiencing stick-slip oscillations. The oscillations of the rotary table have smaller amplitude and higher frequency than the stick-slip oscillations. One simple manner of introducing this oscillation in the model is to apply a parametric excitation in the velocity of the rotary table, in the form of:

$$\phi_2 = \Omega_2 t + A \cos(\omega t),$$

where $A$ is the amplitude of oscillations and $\omega$ is the frequency of oscillations. It is possible then to tune the values of $A$ and $\omega$ to emulate the behaviour of the velocity of the rotary table. It is clear that nulling these two parameters makes this model similar to Model 1.
2.4. 3-DOF torsional pendulum model – Model 3

In order to include the dynamics of the drive system, a 3-DOF electromechanical model is used. In this model are included the characteristics of the motor and the rotary table. It is assumed that the connection between the rotary table and the motor is stiff. The equations of motion for this system are:

\[
\begin{align*}
J_1 \ddot{\phi}_1 + c_1 \dot{\phi}_1 + k_1 \phi_1 - T_b &= 0, \\
J_2 \ddot{\phi}_2 + c_2 \dot{\phi}_2 + k_1 \phi_2 - KnI &= 0, \\
LI + Kn\dot{\phi} + RI - V &= 0,
\end{align*}
\]

where \(J_2\) is the equivalent mass moment of inertia of the drive system, \(c_2\) is the equivalent damping coefficient of the drive system, \(I\) is the armature current, \(K\) is the electric motor constant, \(L\) is the armature inductance, \(n\) is the combined gear ratio of transmission, \(R\) is the armature resistance, \(V\) is the armature voltage and a dot denotes differentiation over time. Other parameters are as in the 1-DOF model.

3. Numerical results

3.1. Simulating discontinuous systems

The models considered are highly non-linear, exhibiting relaxation vibrations, and are also discontinuous because of the friction models used. Therefore, the investigations were performed with numerical simulations of the differential equations, using a 4th order Runge-Kutta solver in conjunction with the bisection method to improve accuracy.

Extensive work has been done in simulation of discontinuous systems, and specifically in system undergoing stick-slip vibrations [3, 8–11]. These simulations have to tackle the problem by separating the discontinuous phase space into a series of adjacent continuous regions. In the present case, two regions are necessary, one for the stick mode and one for the slip mode, with the stick region being a straight line, which is visible in the usual stick-slip limit-cycle. A switch
function is used to perform the transitions from stick to slip modes. This is done by monitoring the velocity and the torque built in the torsional spring to determine in which mode the system is, and selecting the equations of motion accordingly.

A total of six points were selected along the stick-slip limit-cycle in order to evaluate the response for the different models, as shown in figure 2. Points 1 and 5 are of great importance for the simulations as they enclose the limits of each mode. Therefore, the evolution of states depends on the accuracy of these two points. Points 2, 3 and 4 represent the region that encloses the states of the system during the cycle. Point 6 gives an indication of long-term convergence, as it is the final position after the whole simulation, which is 100 s in this case.

![Figure 3. Chosen points on a typical stick-slip limit-cycle](image)

The results of the convergence study shows that all 6 points chosen converge similarly for decreasing step size. The point showing more sensitivity to step size is point 6, as was expected. The point showing less sensitivity is point 5, although this value must be taken carefully as for higher step sizes this point on the cycle degenerates. As a good balance between accuracy and speed, the step size chosen for the simulations was 0.01, as for this value both points 1 and 5, which define the transition from stick to slip modes, are already stabilised and the other points are close to stabilisation.

3.1.1. Bisection method In order to improve the accuracy of the values at the transition from stick mode to slip mode (point 1 in figure 2), the bisection method with 10 iterations was used. This means that the original interval in which the transition values were calculated is divided in 10 parts, and the values are narrowed in one of these parts. One comparison was made with Model 2 taking $\omega = 8\omega_d$ and $A = 0.02$, where $\omega_d$ is the natural frequency of the torsional pendulum, and using Friction model 1. The interval containing the transition angular displacement $\phi$ calculated without the bisection method is $[8.477, 8.533]$. This interval is narrowed to $[8.4566, 8.4568]$ using the bisection method as discussed. This comparison shows that the method gives a considerable gain in accuracy ($1E - 4$ compared to $1E - 1$), although the increase in computing time is negligible (0.37%).
3.1.2. Comparison between the friction models  

The comparison between friction models is carried out by selecting five representative points along the stick-slip limit-cycle, these being the first five points chosen on the stick-slip limit-cycle, shown in figure 2. Model 2 was used for this comparison, varying the parameters $A$, $\omega$, and $\Omega_2$.

![Phase-plane of stick-slip vibrations](image)

**Figure 4.** Phase-plane of stick-slip vibrations (a) Friction Model 1, (b) Friction Model 2, (c) Friction Model 3, (d)–(f) are zoom-up of the bottom left regions showing switching from slip to stick of respective figures (a)–(c)

Results of the comparison between the three friction models show small difference between a more simple friction model and a more complex one, confirming that the main aspect of the system in order to exhibit stick-slip vibrations is the difference between the static and dynamic friction characteristics. The small oscillations on the velocity of the top (Model 2) introduces quasi-periodicity to the system. The point on the limit-cycle most affected by this is the transition between the slip mode into stick mode. Bifurcation diagrams confirm that there is a region in parameter space that allows the system to operate without stick-slip vibrations. In figure 5 it can be observed that for values of $\Omega_2$ less than $\Omega_2^* = 5.227$ the system presents large amplitude stick-slip limit-cycle, while for values of $\Omega_2$ greater than $\Omega_2^*$ the system presents small amplitude limit-cycle. This region is related to a combination of high angular velocities of the rotary table and high damping coefficient of the drill-string. A control system aiming to suppress such vibrations should enclose the parameters in this region.

3.1.3. Sliding surface  

The limits of the sliding surface, which determine the conditions for stick mode, are given in this system by the torque in the drill-string, result of the twist. Expressing the torque in terms of $\phi$ results in the upper and lower limits of this state variable that represent the sliding surface [12, 13], which is shown in figure 5. It is possible to demonstrate that the
sliding surface does not depend on the state of the system while this is at stick mode. The upper and lower limits are calculated by:

\[ \phi_{upper} = -\frac{c_1 \dot{\phi}_1 + T_{f_{max}}}{k_1}, \]
\[ \phi_{lower} = -\frac{c_1 \dot{\phi}_1 - T_{f_{max}}}{k_1}, \]

where \( T_{f_{max}} \) is the maximum torque obtained with the friction model being used, usually being equal to \( T_{st} \). While at stick mode, \( \dot{\phi}_1 = 0 \). Moreover, \( T_{f_{max}} \) and \( k_1 \) do not depend on the system’s state, and \( \phi_2 \) depends only on \( A \) and \( \omega \). It follows that \( \phi_{lower} \) and \( \phi_{upper} \) will not change for varying \( \Omega_2 \), while the dependency on \( A \) and \( \omega \) clearly comes through the excitation present in \( \phi_2 \) in Model 2.

4. Conclusions
The comparison between the three friction models used shows small difference between more simple and more complex models, which confirms that the main aspect responsible for a system to exhibit stick-slip vibrations is the difference between the static and dynamic friction characteristics. The point on the limit-cycle most affected by oscillations of the velocity of the rotary table is the transition between the slip mode into stick mode, as can be seen in figure 3. It can also be seen that the system using Friction Model 3 has more pronounced difference. However, this small difference is attenuated during the stick phase and it does not propagate in the long term dynamics. Bifurcation diagrams, such as the one in figure 5, allowed to find \( \Omega_{2}^* \) above which the system operates without stick-slip vibrations. More importantly, \( \Omega_{2}^* \) does not change substantially with different friction models. Also, the sliding surface does not change for different friction models, as the maximum value for friction does not depend on the contact velocity. A combination of angular velocities and damping on the system can also avoid stick-slip vibrations. A control-system aiming to suppress such vibrations should enclose the parameters in this region. For further analysis, a small scale experimental setup is being constructed, which will enable the investigations of correlating the numerical simulations to experimental data.
Figure 6. Three-dimensional space of stick-slip vibrations showing the lower and upper limits of the sliding surface

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