0 Introduction

Given a symplectomorphism $f$ of a symplectic manifold $X$, one can form the ‘symplectic mapping cylinder’

$$X_f = X \times \mathbb{R} \times S^1/\mathbb{Z}$$

where the $\mathbb{Z}$ action is generated by $(x, s, \theta) \mapsto (f(x), s + 1, \theta)$. In this paper we compute the Gromov invariants of the manifolds $X_f$ and of fiber sums of the $X_f$ with other symplectic manifolds. This is done by expressing the Gromov invariants in terms of the Lefschetz zeta function of $f$ and, in special cases, in terms of the Alexander polynomials of knots. The result is a large set of interesting non-Kähler symplectic manifolds with computational ways of distinguishing them. In particular, this gives a simple symplectic construction of the ‘exotic’ elliptic surfaces recently discovered by Fintushel and Stern and of related ‘exotic’ symplectic 6-manifolds.

A closed symplectic manifold $(X, \omega)$ has three ‘classical invariants’:

(i) the diffeomorphism type of $X$,
(ii) the DeRham cohomology class of $\omega$, and
(iii) the homotopy class $[J]$ of the almost complex structures $J$ compatible with $\omega$.

$([J]$ determines the canonical class $\kappa \in H^2(X; \mathbb{Z})$). For many purposes these invariants are inadequate for distinguishing symplectic manifolds. In 1985 Gromov proposed constructing new symplectic invariants by introducing a $J$, considering the $J$-holomorphic curves in $X$, and mimicking the construction of the enumerative invariants of algebraic geometry. After
much development, there now exist four or five related but dis
tinct versions of these “Gromov invariants” ([MS], [RT], [LT], [T3]).

In section 1 we define the “degree zero Gromov invariants” that we will use. These are
built from the invariants of Ruan and Tian, which count perturbed holomorphic curves ([RT]).
More specifically, for each class $A \in H_2(X)$, we assemble Ruan-Tian invariants into a ‘partial
Gromov series’ $Gr^A(X)$; this is a power series in a variable $t_A$ whose coefficients count, in a
rather non-obvious way, the perturbed holomorphic curves representing multiples of $A$. The
full degree zero Gromov series $Gr(X)$ is the product of the $Gr^A(X)$ over all primitive classes
$A$. These invariants are defined in all dimensions. In dimension four they can sometimes be
related to the Seiberg-Witten invariants by applying the Theorem of Taubes [T1] and a result
of the authors [IP1].

Now fix a symplectomorphism $f$ of a closed symplectic manifold $X$, and let $f_*k$ denote
the induced map on $H_k(X; \mathbb{Q})$. Note that $X_f$ fibers over the torus $T^2$ with fiber $X$. If
det $(I - f_*1) = \pm 1$ then there is a well-defined section class $T$ (Corollary 2.3). Our first main
result computes the Gromov invariants of the multiples of this section class.

**Theorem 0.1** If $\det (I - f_*1) = \pm 1$, the partial Gromov series of $X_f$ for the section class $T$
is given by the Lefschetz zeta function of $f$ in the variable $t = t_T$:

$$Gr^T(X_f) = \zeta_f(t) = \prod_{k \text{ odd}} \frac{\det(I - tf_{sk})}{\prod_{k \text{ even}} \det(I - tf_{sk})}.$$  \hfill (0.2)

The proof occupies the middle part of this paper. Section 3 reviews the zeta function,
including the well-known second equality in (0.2). We also hint at the first equality in (0.2)
by showing that the zeta function has a factorization that parallels the factorization of Taubes’
Gromov invariant. The difficult analysis is done in section 4, where we use a degeneration and
gluing argument to relate the Ruan-Tian invariants to the zeta function.

We can elaborate on this construction by fiber summing $X_f$ with other manifolds. Suppose
that $(Z, \omega)$ is a symplectic 4-manifold and $F \subset Z$ is a symplectically embedded torus with
trivial normal bundle. Since $T \subset X_f$ is represented by a symplectically embedded torus, we
can identify $T$ with $F \subset Z$ and form the fiber sum

$$Z(f) = Z \#_{F=T} X_f.$$  \hfill (0.3)

When this construction is done carefully, $Z(f)$ is a symplectic manifold ([G], [MW]).

**Theorem 0.2** The partial Gromov series of $Z(f)$ for the fiber class $F$ is

$$Gr^F(Z(f)) = Gr^F(Z) \cdot \zeta_f(t_F) \cdot (1 - t_F)^2.$$  \hfill (0.4)

The righthand side of (0.4) depends on $f$ only through its induced homology map. It is
therefore easy to produce explicit examples. In the last two sections we give such examples in
dimensions four and six.
When $X_f$ is a four-manifold, a wealth of examples arise from knots. Associated to each fibered knot $K$ in $S^3$ is a Riemann surface $\Sigma$ and a monodromy diffeomorphism $f_K$ of $\Sigma$. Taking $f = f_K$ gives symplectic 4-manifolds $X_K$ of the homology type of $S^2 \times T^2$ with

$$Gr(X_K) = \frac{A_K(t_T)}{(1 - t_T)^2}$$

(0.5)

where $A_K(t) = \det(I - tf_{s1})$ is the Alexander polynomial of $K$ and $T$ is the section class.

More interesting examples are constructed by fiber summing with elliptic surfaces. Let $E(n)$ be the simply-connected minimal elliptic surface with fiber $F$ and canonical divisor $\kappa = (n - 2)F$. Thus $E(1)$ is the rational elliptic surface and $K3 = E(2)$; we will also write $E(0) = S^2 \times T^2$, even though this is not simply-connected. Forming the fiber sum as in (0.3), we obtain a manifold

$$E(n, K) = E(n)\#_{F=T}X_K,$$

that is symplectic and homeomorphic to $E(n)$. In fact, for fibered knots $K$, $K'$ of the same genus there is a homeomorphism between $E(n, K)$ and $E(n, K')$ preserving the periods of $\omega$ and the canonical class $\kappa$. These manifolds are, however, distinguished by their Gromov invariants. In this case we can compute the full series.

**Theorem 0.3** For $n \neq 1$, the Gromov invariant $E(n, K)$ is

$$Gr(E(n, K)) = A_K(t_F) (1 - t_F)^{n-2}$$

and for $n > 1$ this is also the Seiberg-Witten series of $E(n, K)$.

(See section 5 for comments on $E(0, K)$ and $E(1, K)$.) Thus fibered knots with distinct Alexander polynomials give rise to symplectic manifolds $E(n, K)$ which are homeomorphic but not diffeomorphic. In particular, there are infinitely many distinct symplectic 4-manifolds homeomorphic to $E(n)$.

Moving to dimension six, we can consider the symplectic manifolds $E(n, K) \times S^2$. Using surgery theory, one can show that for $n \geq 1$ fixed, the $E(n, K) \times S^2$ are diffeomorphic. In fact, for knots of the same genus these manifolds have the same classical invariants (Lemma 6.2). On the other hand, in section 6 we compute their Gromov series and obtain the following.

**Theorem 0.4** For each $n \geq 1$, the smooth 6-manifolds $E(n) \times S^2$ admit infinitely many distinct symplectic structures, all with the same classical invariants. In particular, this is true for $K3 \times S^2$.

Here, ‘distinct’ means that their symplectic forms are not deformation equivalent. This simplifies and extends a result of Ruan and Tian ([RT], Proposition 5.5). It is a striking example of the “stabilization phenomenon” suggested by Donaldson, namely that homeomorphic but non-diffeomorphic symplectic 4-manifolds should give rise to diffeomorphic but deformation inequivalent symplectic 6-manifolds.

R. Fintushel and R. Stern have recently used knot theory and Seiberg-Witten theory to prove a result equivalent to Theorem 0.3 above ([FS]). This article arose from our efforts to fit their results into a purely symplectic context and extend it to higher dimensions. We thank both Rons for their help and encouragement.
1 Gromov invariants and Symplectic Sums

The symplectic invariants we will use are combinations of the invariants defined by Ruan and Tian [RT]. To define them, we will briefly recall the construction of the Ruan-Tian invariants, specialize to “stabilized degree 0” RT invariants, and then explain how to assemble these into the single symplectic invariant that we will call the ‘Gromov series’.

Let \((X, \omega)\) be a closed symplectic manifold of dimension \(2n\). Choose a compatible almost complex structure \(J\); this defines the canonical class \(\kappa \in H^2(X)\) of \((X,\omega)\). Given a genus \(g\) Riemann surface \(\Sigma\) with \(k\) marked points \(x_1, \ldots, x_k\) and a homology class \(A \in H_2(X,\mathbb{Z})\), consider pairs \((j,f)\) where \(j\) is a complex structure on \(\Sigma\) and \(f\) is a map \(\Sigma \to X\) satisfying

\[
\overline{\partial}_j f = \nu
\]

where \(\overline{\partial}_j = df \circ j - J \circ df\) and where \(\nu\) is a fixed, appropriately defined 1-form. The moduli space of such data \((j,x_1,\ldots,x_k,f)\) has dimension \(d\), where

\[
d = 2(n-3)(1-g) - 2\kappa \cdot A + 2k. \tag{1.1}
\]

We can reduce this dimension by requiring that \(f\) take the marked points into fixed constraint surfaces \(\alpha_i \subset X\). For appropriately chosen constraints, the moduli space will be reduced to finitely many points. Counting with orientation, we get an invariant

\[
RT_{A,g,k}(\alpha_1,\ldots,\alpha_k)
\]

that counts the total number of (perturbed) holomorphic genus \(g\) curves with homology class \(A\) passing through the constraints \(\alpha_1,\ldots,\alpha_k\) (for generic almost complex structure \(J\) and perturbation \(\nu\)). This count depends only on the deformation class of the symplectic structure ([RT] and [LT]).

We will be interested only in the invariants for unconstrained curves, i.e. for those pairs \((A,g)\) whose \(k = 0\) moduli space has dimension 0. Thus we define the “degree zero” invariants by

\[
RT^0(A) = \sum_g RT_{A,g,0} \tag{1.2}
\]

where the sum is over all \(g\) such that \(\kappa \cdot A = (n-3)(1-g)\) (cf. (1.1)). Note that there is at most one such \(g\) when \(n \neq 3\), while for \(n = 3\) there is none unless \(\kappa \cdot A = 0\).

Definition (1.2) must be clarified for the cases \(g = 0\) and 1. In [RT], the invariants \(RT_{A,g,k}\) are defined only for the “stable range” \(2g + k \geq 3\). However, the definition can be extended to the unstable cases by choosing a class \(\beta \in H_{2n-2}(X)\) with \(A \cdot \beta \neq 0\) and setting

\[
RT_{A,1,0} = \frac{1}{A \cdot \beta} RT_{A,1,1}(\beta) \quad \text{and} \quad RT_{A,0,0} = \frac{1}{(A \cdot \beta)^3} RT_{A,0,3}(\beta,\beta,\beta) \tag{1.3}
\]

These invariants are independent of \(\beta\) and still count perturbed holomorphic curves (cf [IP1]).
It is convenient to combine the degree zero invariants $RT^0(A)$ into a single quantity associated with $X$. To do this, we introduce formal symbols $t_A$ for $A \in H_2(X; \mathbb{Z})$ with relations $t_{A+B} = t_At_B$ and construct a generating function. Our generating function involves the Möbius function $\mu$, which is defined by $\mu(1) = 1$ and for $m > 1$

$$\mu(m) = \begin{cases} (-1)^k & \text{if } m \text{ is a product of } k \text{ distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

**Definition 1.1** The degree zero Gromov series is

$$Gr^0(X) = \exp \left[ \sum_A RT^0(A) \phi(t_A) \right] \quad (1.4)$$

where $\phi(t) = \sum_{k=1}^{\infty} \mu(k)t^k$ is the power series with coefficients given by the Möbius function. For $A \in H_2(X)$, the Gromov invariant $Gr^0_X(A)$ is defined by the expansion $Gr^0(X) = \sum Gr^0_X(A)t_A$.

Of course, different weighting functions $\phi$ give different symplectic invariants. For the simple choice $\phi(t) = t$, $Gr^0_X(A)$ counts the number of ways we can represent the homology class $A$ as the image of a perturbed degree zero holomorphic map whose domain is a disjoint union of (stable) Riemann surfaces. While it initially seems awkward to introduce the Möbius function into (1.4), this choice eliminates some “overcounting”, giving a simpler generating function which contains the same information.

**Remark 1.2** One can arrive at (1.4) as follows. Fix a function $F$, and assign a factor $F(t_A)$ to each curve that contributes $+1$ to the count $RT^0(A)$, and a factor $1/F(t_A)$ to each curve that contributes $-1$, obtaining

$$Gr^0(X) = \prod_{A \in H_2(X)} F(t_A)^{RT^0(A)}, \quad (1.5)$$

Again, the natural choice $F(t) = e^t$ overcounts; it is more efficient to choose $F$ so that

$$\prod_{k=1}^{\infty} F(t^k) = e^t. \quad (1.6)$$

This holds for

$$F(t) = \exp \left( \sum_{m \geq 1} \mu(m)t^m \right), \quad (1.7)$$

as can be verified by writing $\ell = mk$ and using the basic fact that $\sum_{m|\ell} \mu(m) = 0$ unless $\ell = 1$, in which case the sum equals to 1.

We will usually omit the superscript from $RT^0(A)$, $Gr^0(X)$, and $Gr^0_X(A)$, it being understood that we are always computing degree 0 invariants. We will often be interested in the
invariants $Gr^0_X(mA)$ for multiples of a given class $A$; it is then convenient to work with the partial Gromov series

$$Gr^A(X) = \prod_{m \geq 1} F(m^A)RT^0(mA)$$

and then

$$Gr(X) = \prod_{A \text{ primitive}} Gr^A(X).$$

The Gromov invariant $[L3]$ has several nice properties. In particular, it behaves well under the ‘symplectic normal sum’ construction described by Gompf [G] and McCarthy-Wolfson [MW]. A general formula for the invariants of the sum is given in [IP2]. Here we will state and use the formula only in the special case of an ordinary fiber sum.

Fix a $2n$-dimensional symplectic manifold $X$, not necessarily connected. By a symplectic fiber in $X$ we mean a codimension 2 symplectic submanifold with trivial normal bundle. Suppose that $F_1$ and $F_2$ are disjoint symplectic fibers in $X$ which are symplectomorphic. Following [G] and [MW], we can construct an orientation reversing symplectomorphism

$$\phi : N'F_1 \rightarrow N'F_2$$

(1.9)

where $N'F = N_{\varepsilon}F \setminus F$ denotes a $2\varepsilon$-tubular neighborhood (for some metric) with its core removed. Removing $\varepsilon$-neighborhoods of $F_1$ and $F_2$ and identifying the boundaries via $\phi$, yields the symplectic fiber sum $\#_{\phi}X$. This is a symplectic manifold, and the deformation type of its symplectic structure depends on the deformation class of $\phi$. We will usually fix an deformation class of $\phi$ and denote $\#_{\phi}X$ by $\#_{F_1=F_2}X$, or even $\#_{F}X$. When $X$ is the disjoint union of manifolds $X_1$ and $X_2$ and $F_i \subset X_i$ as above, we denote the corresponding fiber sum by

$$X_1 \#_{F_1=F_2}X_2 \quad \text{or} \quad X_1 \#_F X_2.$$

Note that in the sum, the homology classes $[F_1]$ and $[F_2]$ become a single ‘fiber class’ $[F]$.

**Theorem 1.3** (cf. [IP2]) Let $X_1$ and $X_2$ be closed symplectic 4-manifolds containing symplectic fibers $F_i \subset X_i$ as above. Then the partial Gromov invariant for the fiber class $[F]$ is

$$Gr^{[F]}_{X_1\#_F X_2} = (1 - t_F)^2 \cdot Gr^{[F_1]}_{X_1} \cdot Gr^{[F_2]}_{X_2}. \quad (1.10)$$

As an application, let us compute the Gromov series of $E(n)$. By an observation of Donaldson [D], a generic Kähler structure $J$ on $E(2) = K3$ admits no holomorphic curves whatsoever. By a limiting argument, this implies that $RT(A) = 0$ for each non-trivial class $A \in H_2(K3)$. Specifically, if $RT(A) \neq 0$, we could choose a sequence of generic $(J, \nu)$ converging to some $(J_0, 0)$ with $J_0$ generic and Kähler and a sequence of $(J, \nu)$-holomorphic A-curves. These curves
would limit to a bubble tree of $J_0$-holomorphic curves (cf. [PW], [P], and [RT]), contradicting Donaldson’s observation. Thus

$$Gr(K3) = 1. \tag{1.11}$$

Now, as symplectic manifolds

$$E(n) \# F E(1) = E(n + 1) \tag{1.12}$$

where this fiber sum glues a fiber of $E(n)$ to a fiber of $E(1)$. Taking $n = 1$ and applying (1.10) yields $Gr^F(E(1)) = 1/(1 - t)$; putting this into (1.12) and (1.10) inductively gives

$$Gr^F(E(n)) = (1 - t)^{n-2} \tag{1.13}$$

where the variable $t$ corresponds to the fiber class. In fact, we will show in Lemma 5.4 that for $n \geq 2$, this formula gives the full Gromov series of $E(n)$.

2 The Symplectic Mapping Torus

Fix a closed symplectic manifold $(X, \omega)$. Each symplectic diffeomorphism $f : X \to X$ induces a symplectomorphism $F$ of $X \times \mathbb{R} \times S^1$ (with the product symplectic structure) by $F(x, s, \theta) = (f(x), s + 1, \theta)$. The quotient

$$X_f = X \times \mathbb{R} \times S^1 / \mathbb{Z} \tag{2.1}$$

is a closed symplectic manifold — the symplectic mapping torus of $f$. As a differential manifold, $X_f$ is the product $M_f \times S^1$, where $M_f$ is the usual mapping torus. Note that $X_f$ fibers over the torus $T^2$ with fiber $X$, and that the $S^1$ factor gives a symplectic circle action on $X_f$.

We can also construct $X_f$ as a symplectic fiber sum. Start with $S^2 \times X$ and fix two points $a, b \in S^2$. Let $X_a = \{ a \} \times X$ and $X_b = \{ b \} \times X$. Extend $f : X_a \to X_b$ to a symplectic map between the $\varepsilon$-tubular neighborhoods of $X_a$ and $X_b$, and form the symplectic fiber sum. The result is $X_f$:

$$\#_{f(X_a) = X_b} S^2 \times X = X_f. \tag{2.2}$$

To see this equivalence, put a linear circle action on $S^2 \subset \mathbb{R}^3$ and take $X_a$ and $X_b$ to be the fibers at the poles, and let $S^2_\varepsilon$ be the sphere with $\varepsilon$-disks around the poles removed. While the usual symplectic structures on $S^2_\varepsilon$ and $[0, 1] \times S^1$ are not symplectomorphic, they are isotopic, and hence (2.1) and (2.2) are isotopic symplectic manifolds.

Example 2.1 Let $f$ be the symplectomorphism of the standard 2-torus induced by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

7
acting on the torus $\mathbb{R}^2/\mathbb{Z}^2$. Then $\text{cok}(I - f_*) = \text{cok}(I - A) = \mathbb{Z}$, so by Lemma 2.2 below, $X_f$ has first betti number $b_1 = 3$. This is Thurston’s famous example [Th] of a symplectic manifold with no Kähler structure (compact Kähler manifolds have $b_1$ even).

We can always assume that $f$ has a fixed point $p$ (if not, replace $f$ by its composition with a Hamiltonian flow taking $f(p)$ back to $p$; this changes $X_f$ to an isotopic symplectic manifold). The differential $df_p$ at the fixed point lies in the connected group $\text{Sp}(T_pX)$. Consequently, \( \{ p \} \times \mathbb{R} \times S^1/\mathbb{Z} \) defines a section $T_p$

\[
\begin{array}{c}
X_f \\
\downarrow^p T_p \\
T^2
\end{array}
\]  

whose image is a torus with trivial normal bundle. Let $T = [T_p]$ denote this ‘section class’ in $H_2(X_f; \mathbb{Z})$ (in general, $t$ depends on the choice of the fixed point $p$).

It is straightforward to calculate the fundamental group of $X_f = M_f \times S^1$.

**Lemma 2.2** Let $f_\#$ and $f_*$ be, respectively, the induced maps on $\pi_1(X)$ and $H_1(X, \mathbb{Z})$. Then

(a) $\pi_1 X_f = G \times \mathbb{Z}$ where $G = \{ (\pi_1X, \tau) \mid \tau^{-1}x\tau = f_\# x \ \forall x \in \pi_1X \}$, and

(b) $H_1(X_f; \mathbb{Z}) = \text{cok}(I - f_*) \oplus \mathbb{Z} \oplus \mathbb{Z}$.

**Proof.** A fixed point $p$ in $X$ determines a path $\tilde{\tau} = \{ p \} \times [0, 1]$ in $X \times [0, 1]$ that projects to a loop $\tau$ in $M_f$. Then $\pi_1 M_f$ is generated by $\pi_1 X$ and $\tau$ and the only relations are those obtained by conjugating by $\tau$; namely, for each $x \in \pi_1(X; p)$ the loops $\tilde{\tau}(f_{\#} \times \{ 1 \})\tilde{\tau}^{-1}$ and $x \times \{ 0 \}$ are homotopic rel basepoint in $X \times \mathbb{R}$, so $\tau(f_{\#} x)\tau^{-1} = x$ in $\pi_1 M_f$. This gives (a).

Abelianizing, we have $H_1(X_f) = K \oplus \mathbb{Z} \oplus \mathbb{Z}$ where $K$ is the quotient of $H_1(X; \mathbb{Z})$ by the relation $x = f_* x$. This gives (b). \( \square \)

The proof of Lemma 2.2 also implies the following simplification.

**Corollary 2.3** If $\det(I - f_*) = \pm 1$ then the section class $T = [T_p] \in H_2(X_f; \mathbb{Z})$ is well-defined, independent of the fixed point $p$.

We now focus on the case where $X$ is a Riemann surface of genus $g$. Each orientation-preserving diffeomorphism $f$ of $X$ preserves the cohomology class of the symplectic form. By Moser’s Theorem [M] we can assume (after an isotopy of $f$) that $f$ is a symplectic diffeomorphism, and hence determines a symplectic mapping torus $X_f$. Write $f_*$ for the induced map on $H_1(X, \mathbb{Z})$ and set

\[ A(t) = \det(I - tf_*) \]  

(2.4)

**Lemma 2.4** (a) $A(t)$ is a monic polynomial with integer coefficients with $A(t^{-1}) = t^{-2g}A(t)$.

(b) $X_f$ is a homology $S^2 \times T^2$ if and only if $A(1) = \pm 1$. 

8
Proof. The induced map $f_{*}1$ is symplectic by Poincaré duality. After fixing a basis of $H_{1}(X,\mathbb{Z})$ the matrix $A = f_{*}1$ lies in $Sp(2g,\mathbb{Z})$, so det $A = 1$. Then the first two properties listed in (a) are obvious. For the third, note that $A(t)$ is invariant under conjugation so it suffices to verify $A(t^{-1}) = t^{-2g}A(t)$ for matrices in the maximal torus of $Sp(2g,\mathbb{C})$. But that is clear since such matrices have the form diag$(\lambda_{1},\lambda_{1}^{-1},\ldots,\lambda_{g},\lambda_{g}^{-1})$.

Finally, (b) holds because $X_{f} = M_{f} \times S^{1}$ and, by Lemma 2.2(b) and Poincaré duality, $M_{f}$ is a homology $S^{2} \times S^{1}$ if and only if $I - f_{*}1$ is invertible over $\mathbb{Z}$. □

Fibered knots provide one source of diffeomorphisms of Riemann surfaces. Recall that a knot $K \subset S^{3}$ is genus $g$ fibered if there is an oriented fibration $\pi_{K} : S^{3} \setminus K \to S^{1}$ whose fiber $X_{0}$ is the 2-manifold of genus $g$ with one point removed. After a 0-surgery, this becomes an $X$-fibration over $S^{1}$. This new fibration is the mapping torus $M_{fK}$ of the monodromy diffeomorphism $f_{K} : X \to X$ along the knot, $M_{fK}$ is a homology $S^{2} \times S^{1}$ and

$$S^{3} \setminus K = M_{fK} \setminus \{\tau\}$$

Moreover, the Alexander polynomial of $K$ is the characteristic polynomial of the map induced by $f_{K}$ on $H_{1}(X)$ exactly as in (2.4).

Example 2.5 The monodromy matrices $f_{*}$ of the trefoil and figure 8 knots are, respectively,

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and the Alexander polynomials are $t^{2} - t - 1$ and $t^{2} - 3t + 1$.

Looking the other way, Burde [B] proved that any polynomial $A$ as in Lemma 2.4(a) with $A(1) = \pm 1$ arises as the Alexander polynomial of a knot in $S^{3}$. Summarizing,

Lemma 2.6 For each genus $g$ fibered knot in a homology 3-sphere, there is a symplectic 4-manifold

$$X_{K} = X_{fK}$$

This is a homology $S^{2} \times T^{2}$ and det$(I - t(f_{K})_{*}1) = A_{K}(t)$ is a monic polynomial with integer coefficients satisfying $A_{K}(1) = \pm 1$ and $A_{K}(t^{-1}) = t^{-2g}A_{K}(t)$. Conversely, each such $A_{K}(t)$ arises as det$(I - t(f_{K})_{*}1)$ for some such $K$.

3 Factorizing the Lefschetz Zeta Function

The Lefschetz number $L(f)$ of $f : X \to X$ is the homological intersection of the graph of $f$ with the diagonal in $X \times X$. When this intersection is transversal

$$L(f) = \sum_{f(x) = x} L(f, x) \quad \text{where} \quad L(f, x) = \text{sgn det}(df_{x} - I)$$

(3.1)
The Lefschetz trace formula asserts that

\[ L(f) = \sum_{k=0}^{\dim X} (-1)^k \text{tr}(f^k) \]  

(3.2)

where \( f^k \) is the induced endomorphism of \( H_k(X; \mathbb{Q}) \). Often, as in dynamics, one considers the Lefschetz numbers \( L(f^n) \) of the iterates of \( f \). These are neatly encoded in the Lefschetz zeta function

\[ \zeta_f(t) = \exp \left( \sum_{n=1}^{\infty} t^n L(f^n) \right) . \]  

(3.3)

Substituting in (3.2), one sees that this sum converges to a continuous function for small \( t \).

Starting from formulas (3.1)–(3.3), we will obtain two formulas for the zeta function. The first — the homological expression (3.4) — is well-known. The second expresses \( \zeta_f \) as a product of generating functions. The arguments required for this are completely elementary, but the resulting “factorization formula” does not seem to be in the literature.

**Lemma 3.1** \( \zeta_f \) extends to a rational function of \( t \in \mathbb{C} \). In fact,

\[ \zeta_f(t) = \prod_{k \text{ odd}} \frac{\det(I - tf_k)}{\det(I - tf_k)}. \]  

(3.4)

**Proof.** For any \( n \times n \) complex matrix \( A \) and \( t << |A| \) we have the identity

\[ \exp \left( -\sum_{n=1}^{\infty} \frac{t^n}{n} \text{tr}A^n \right) = \det(I - tA) . \]  

(3.5)

In the simplest case, when \( A \) is a \( 1 \times 1 \) matrix, (3.5) is the Taylor series of \( \ln(1 - tA) \). The general case follows from three observations: (i) the functions on both sides of (3.5) satisfy \( f(A \oplus B) = f(A)f(B) \), so (3.5) holds for diagonal matrices, (ii) the functions on both sides are invariant under conjugation, so (3.5) holds for diagonalizable matrices, and (iii) by Jordan normal form any matrix is a limit of diagonalizable matrices, and both sides of (3.5) are continuous.

Equation (3.4) is easily obtained from (3.2), (3.3) and (3.5). \( \square \)

A second expression for \( \zeta_f \) arises by substituting (3.1) into (3.3). At a fixed point \( p \) of a symplectic diffeomorphism \( f \), the differential \( df_p : T_p X \to T_p X \) is real and symplectic, and contributes \( L(f, x) = \text{sgn } \det (df_p - I) = \pm 1 \) to the Lefschetz number. After examining the structure of the symplectic group, we will be able to compute these signs for the powers of \( f \). This will yield an expression for the Lefschetz zeta function as a product of generating functions associated to the finite orbits of \( f \).
Let $A \in Sp(2n, \mathbb{R})$, i.e. $A^t J A = J$ where $J$ is the $2n \times 2n$ matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. When $n = 1$ such $A$ are either elliptic or hyperbolic type — in some basis they have one of the forms

$$E = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad H = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$ 

In general, $A \in Sp(2n, \mathbb{R})$ can be put into one of three block forms (cf. [R2]):

$$\begin{pmatrix} E & 0 \\ 0 & S \end{pmatrix}, \quad \begin{pmatrix} H & 0 \\ 0 & S \end{pmatrix}, \quad \begin{pmatrix} E & * & * \\ 0 & E' & 0 \\ 0 & * & S \end{pmatrix}.$$ (3.6)

In the first two cases $E$ and $H$ are as above and $S \in Sp(2n-2, \mathbb{R})$, while in the last case $E, E'$ are elliptic with eigenvalues $\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}$ and $S \in Sp(2n-4, \mathbb{R})$. Thus (3.6) inductively defines a ‘normal form’ for symplectic matrices.

The matrices that have $1$ as an eigenvalue define a codimension-one algebraic subset $W_1$ of $Sp(2n, \mathbb{R})$. Each $A \notin W_1$ has a certain number $R(A)$ of real eigenvalues, $R^+(A)$ of which are positive. It is easy to see that $Sp(2n, \mathbb{R}) \setminus W_1$ has three open components:

$$\begin{cases} 
\text{type } E : & \text{those with } R(A) \text{ even}, \\
\text{type } H : & \text{those with } R(A) \text{ odd but } R^+(A) \text{ even}, \\
\text{type } H' : & \text{those with } R(A) \text{ and } R^+(A) \text{ both odd}. 
\end{cases}$$ (3.7)

In the simplest case, a $2 \times 2$ matrix $A \in Sp(2, \mathbb{R})$ has type $E$, $H$, or $H'$ when it is, respectively, elliptic, hyperbolic with positive eigenvalues, or hyperbolic with negative eigenvalues.

We can also consider the walls $W_m = \{ A \mid \det (A^m - I) = 0 \}$ formed by those $A$ with some eigenvalue equal to a $m^{th}$ root of unity. Then the complement of $W = \bigcup_{m \neq 0} W_m$ is a Baire subset of $Sp(2n, \mathbb{R})$.

Define the sign of $A \in Sp(2n, \mathbb{R}) \setminus W_1$ by

$$\text{sgn } (A) = \text{sgn } \det (A - I).$$

**Lemma 3.2** If $A \in Sp(2n, \mathbb{R}) \setminus W$, then for each non-zero $m \in \mathbb{Z}$,

$$\text{sgn } (A^m) = \begin{cases} 
1 & \text{if } A \text{ has type } E, \\
-1 & \text{if } A \text{ has type } H, \\
(-1)^m & \text{if } A \text{ has type } H'. 
\end{cases}$$

**Proof.** From the normal form (3.6) we have sgn $A^m = \prod B_i$ sgn det $(B_i^m - I)$ where $B_i$ are $2 \times 2$ blocks along the diagonal. Each elliptic block has eigenvalues $\lambda, \bar{\lambda}$, so has det $(B^m - I) = |\lambda^m - 1|^2 > 0$ because $\lambda$ is not a root of unity. Similarly, for each hyperbolic block the sign of det $(B^m - I) = (\lambda^m - 1)(\lambda^{-m} - 1) = -|\lambda^m - 1|^2/\lambda^m$ is $-(\text{sgn } \lambda)^m$. The Lemma follows using the definitions (3.7). \(\square\)

To apply these sign counts, we use the following transversality result, which follows from a result of R.C. Robinson ([R1] Theorem 1Bi) and the fact that the group of symplectic diffeomorphisms is locally connected.
Lemma 3.3 Each symplectic diffeomorphism \( f_0 : X \to X \) is isotopic through symplectic diffeomorphisms to a symplectic diffeomorphism \( f \) for which \( \text{graph}(f^n) \) is transverse to the diagonal for all \( n \geq 1 \).

Since a symplectic isotopy changes neither the Lefschetz numbers nor the isotopy class of the symplectic manifold \( X_f \), we may henceforth assume that the powers of \( f \) are transverse as in Lemma 3.3. At a fixed point \( p \) of \( f \), this transversality condition says that \( \det (df^n_p - I) \neq 0 \). Thus the signs of \( df_p \) and \( df^n_p \) are related as in Lemma 3.2.

Now each orbit of \( f \) of minimal period \( k \) has exactly \( k \) points, all of the same type (3.7).

Set
\[
\begin{align*}
e_k & = \text{the number of type E orbits of } f \text{ of minimal period } k, \\
h_k & = \text{the number of type H orbits of } f \text{ of minimal period } k, \\
h'_k & = \text{the number of type H' orbits of } f \text{ of minimal period } k.
\end{align*}
\]

Then, because each fixed point of \( f^n \) has a minimal period \( k|n \), we have
\[
L(f^n) = \sum_{k|n} k \left[ e_k - h_k - (-1)^{n/k} h'_k \right].
\]

Substituting this into the Lefschetz zeta function (3.3), writing \( n = kl \), and rearranging the sum gives
\[
\zeta_f(t) = \exp \left[ \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[ e_k - h_k - (-1)^{n/k} h'_k \right] \frac{t^{kl}}{l} \right].
\]

Using the Taylor series \( \log(1 - t^k) = - \sum t^{kl}/l \), this reduces to
\[
\zeta_f(t) = \prod_k \left( \frac{1}{1 - t^k} \right)^{e_k - h_k} (1 + t^k)^{h'_k} \tag{3.8}
\]

In this we see the three “generating functions”
\[
f_E(t) = \frac{1}{1 - t} \quad f_H(t) = 1 - t, \quad f_{H'}(t) = 1 + t. \tag{3.9}
\]

Thus if we let \( \mathcal{O} \) be the set of finite-period orbits of \( f \), and assign to each \( \tau \in \mathcal{O} \) of order \( k = k(\tau) \) one of the generating function \( f_\tau(t) = f_E(t^k), f_H(t^k) \) or \( f_{H'}(t^k) \) according to the type of \( \tau \), we obtain
\[
\zeta_f(t) = \prod_{\tau \in \mathcal{O}} f_\tau(t^{k(\tau)}). \tag{3.10}
\]

Remark 3.4 Bifurcations in the dynamics of \( f \) give relations amongst the generating functions (3.9). In fact, for a one-parameter family \( \{ f_s \} \) of symplectic diffeomorphisms, two types of stable bifurcations can occur (cf. [AM] §8.6–8.7).

(i) At some \( s_0 \) a new pair of finite-order points is created, one of type \( E \) the other of type \( H \), both with the same minimal period \( k \). The invariance of the zeta function (3.10) then corresponds to the relation
\[
f_E(t^k)f_H(t^k) = 1. \tag{3.11}
\]
(ii) A more interesting bifurcation is called "subtle division": at some $s_0$ a type $H'$ orbit of minimal period $k$ disappears, and is replaced by a type $E$ orbit of minimal period $k$ and a type $H$ orbit of minimal period $2k$. The corresponding relation is

$$f_{H'}(t^k) = f_E(t^k)f_H(t^{2k}). \quad (3.12)$$

The generating functions (3.9) are identical to three of the generating functions Taubes uses to define his Gromov invariants [T3]. In fact, Taubes derives his generating functions using exactly the relations (3.11) and (3.12). Apparently there is a precise correspondence between the periodic orbits of $f$ and the $J$-holomorphic tori in $X_f$, with the orbits of types $E$, $H$, and $H'$ corresponding to tori of Taubes type $(+,0)$, $(-,0)$, and $(+,1)$ respectively (cf. [T3]). That is the content of our main result Theorem 0.1. For the proof, however, we must look away from the dynamics of $f$ and instead study the holomorphic curves in $X_f$.

4 The Gromov Series of $X_f$

In this section we will use the Gluing Theorem of [IP2] to establish the following remarkable formula linking the Gromov series of $X_f$ to the dynamics of the diffeomorphism $f$.

**Theorem 4.1** For $\dim X_f = 2n \neq 6$, the partial Gromov Series of $X_f$ with respect with $T$ is equal to the zeta function of $f$:

$$Gr^T_{X_f} = \zeta_f \quad (4.1)$$

When $n = 3$ this formula holds when the righthand side is the "$g = 1$ partial Gromov series" defined by (1.8) with $RT(kA)$ replaced by $RT_{kA,1,0}$, i.e. by the genus one term in (1.2).

This, together with the zeta function formula (3.4), yields Theorem 0.2 of the introduction.

To start, note that the canonical bundle of $X_f$ is trivial along the section defined in equation (2.3), so $\kappa \cdot T = 0$. The dimension formula (1.1) then shows that, when $n \neq 3$, the only curves in $X_f$ that contribute to the series (4.1) are tori; when $n = 3$ we will simply restrict attention to tori. To count these, we must stabilize as in (1.3). For this, we fix a point $c \in T^2$, let $X_c$ be the fiber of $X_f \to T^2$ over $c$, and consider the moduli space

$$\mathcal{M}_{mT}(T^2, X_f) \quad (4.2)$$

of perturbed holomorphic maps $\phi$ from the generic torus with a marked point $x_0$ to $X_f$ which represent $mT$ in homology and satisfy $\phi(x_0) \in X_c$. For generic $(J, \nu)$, this moduli space has

$$RT_m = RT_{mT,1,1}([X_c])$$

oriented points. Referring to (1.8) and (1.3), and noting that $X_c \cdot mT = m$,

$$Gr^T = \prod_{m=1}^{\infty} [F(t^m)]^{RT_m/m} \quad (4.3)$$
Thus we must calculate the invariant $RT_m$.

The key ingredient is a gluing theorem, which relates the moduli space $[4.3]$ of curves in the glued manifold $X_f$ to a moduli space of curves in the “unglued” manifold $S^2 \times X$. This result (Proposition $[4.3]$) is largely subsumed by the general gluing theorem given in [IP2]. With that in mind, we will outline the part of the argument that overlaps with [IP2], and give complete details on those points particular to the present case.

Recall that $X_f$, more properly denoted $X^\varepsilon_f$, is constructed from $S^2 \times X$ by fixing points $a, b \in S^2$, identifying the fibers $X_a, X_b$ via $f$, and symplectically gluing the $\varepsilon$-tubular neighborhoods $N_a(\varepsilon)$ and $N_b(\varepsilon)$. The degeneration formula describes what happens to holomorphic curves as $\varepsilon \to 0$ and explains how to compute the Gromov invariants from the limiting curves.

As $\varepsilon \to 0$, the spaces $X^\varepsilon_f$ converge to $X^0_f = S^2 \times X/\sim$, where $\sim$ is the identification of $X_a$ with $X_b$ via $f$. This limiting space is singular, but is a manifold away from $X_a = X_b$. Similarly, as $\varepsilon \to 0$, the curves in $X^\varepsilon_f$ representing a multiple $mT$ of the section class $T \in H_2(X^\varepsilon_f)$ converge to singular curves in $X^0_f$ representing $mT_0 \in H_2(X^0_f)$.

More precisely, let $f_\varepsilon : (T^2, j_\varepsilon) \to X$ be a sequence of $(J, \nu)$-holomorphic maps in $X^\varepsilon_f$ that take a marked point $x_0$ into a fixed fiber $X_c$ (as above, the domains must be tori). As $\varepsilon \to 0$, the sequence of complex structures $j_\varepsilon$ has a subsequence converging to some $j_0$ in the Deligne-Mumford compactification of the moduli space of complex structures on $(T^2, x_0)$. The Bubble Tree Convergence Theorem ([PW], [RT]) implies that $(f_\varepsilon, j_\varepsilon, x_0)$ has a subsequence that converges modulo diffeomorphisms to a map $f : B \to X^0_f$ where $B$ is a bubble tree.

The domain of a bubble tree $B$ is a connected union of three types of components (cf. Figure 4.1). The original torus limits to a “principal component” $(T^2, j_\infty, x_0)$, which is either a torus or a sphere with two special points besides $x_0$. There can be finitely many trees of “ordinary bubbles” attached to the principal component. Finally, if the principal component is singular, there can be “chains” of bubble spheres, each with exactly two special points, inserted between the two special points of the principal domain. The perturbation $\nu$ extends as 0 on all the bubbles.

![Figure 4.1 — A general bubble tree domain](image)

To simplify the exposition, we first assume that the limit has no components entirely contained in the singular set $X_a = X_b$ of $X^0_f$. Then the limiting map has a special property:
the double points of the domain are exactly those points in the inverse image of the singular set [IP2]. Remove all these double points and lift each component $B_i$ of $B \setminus D$ to a map $B_i \to S^2 \times X$. By the Removable Singularity Theorem for $J$-holomorphic curves [PW], these extend to $J$-holomorphic maps $\overline{B_i} \to S^2 \times X$. Thus we may consider the limit as a map
\[ \bigsqcup B_i \to S^2 \times X \] (4.4)

Together with the constraint that the image intersects $X_a$ and $X_b$ in the same set of points (counted with multiplicities).

Since no homology class in $H_2(S^2 \times X)$ can intersect $X_a \cup X_b$ in exactly one point, there are no ordinary bubbles (each tree of ordinary bubbles has a top bubble with only one double point). It also means that the principal component is a sphere ($j = \infty$), since otherwise it is a torus with no double points. Thus the domain of the limit map is a “necklace” of spheres, one of them marked, and the others having exactly two double points each, one mapped into $X_a$ and the other mapped into $X_b$ (Figure 4.2).

![Figure 4.2 — A necklace map](image)

Assume the necklace has $k$ beads, the $i$-th bead representing $A_i$ in homology. The constraint that the image intersects $X_a$ and $X_b$ in the same set of points (counted with multiplicities) implies that each contact point of the necklace with $X_a$ and $X_b$ has multiplicity $d = A_i \cdot X_a = A_i \cdot X_b$, for the same $d$. Thus we can write $A_i = ds + \alpha_i \in H_2(S^2 \times X)$ where $s = [S^2 \times \text{pt}]$ and $\alpha_i = [\text{pt} \times \alpha'_i]$ for some $\alpha'_i \in H_2(X)$. Furthermore, the total lifted bubble tree represents an element of $\pi_*^{-1}(mT_0) = ms + \ker \pi_*$ in $H_2(S^2 \times X)$, where $\pi$ is the projection $S^2 \times X \to X_f^0$. Therefore we have
\[ m = kd \quad \text{and} \quad \sum \alpha'_i \in \ker \pi_* \] (4.5)

Note that only one bead of the necklace is a stable curve. To make the gluing construction well-defined, we stabilize the other $k - 1$ beads as in (1.3), using $\beta = X_c$. This process is not unique — each bead stabilizes to exactly $d = A_i \cdot X_c$ stable maps.

The following definition will help make precise which maps occur in a necklace.
Definition 4.2 Let $M^\nu_{d,A}$ denote the moduli space of perturbed maps $\phi : (S^2, x, y, u) \to S^2 \times X$ representing $A$ in homology and such that $\phi$ has a contact of order $d$ to $X_a$ at $x$, a contact of order $d$ to $X_b$ at $y$, and $\phi(u) \in X_c$ (where $d = A \cdot [X_a]$).

The moduli space $M^\nu_{d,A}$ comes with an evaluation map $ev : M^\nu_{d,A} \to X \times X$ by sending $(\phi, x, y)$ to $(p_X \phi(x), p_X \phi(y))$. This extends to an evaluation map $ev : M^\nu_{d,A_i} \times \ldots \times M^\nu_{d,A_k} \to (X \times X)^k$. The condition that each map $(\phi_i, x_i, y_i)$ in the necklace connects to the next is

$$f(p_X \phi_i(x_i)) = p_X \phi_{i+1}(y_{i+1})$$

(and similarly the last map connects to the first). These conditions say that $ev((\phi_i, x_i, y_i)_{i=1}^k)$ lies on the “cyclic graph” of $f$:

$$G_f = \{ (x_1, y_1, \ldots, x_k, y_k) \mid f(x_i) = y_{i+1} \text{ for } i = 1, \ldots, k-1 \text{ and } f(x_k) = y_1 \}.$$  

Thus the moduli space of stabilized necklaces is parameterized by

$$ev^{-1}(G_f) = \bigsqcup ev(M^\nu_{d,A_i} \times \ldots \times M^\nu_{d,A_k}) \cap G_f$$

where the union is over collections $A_1, \ldots, A_k$ of elements $A_i = ds + \alpha_i$ in $H_2(S^2 \times X)$ satisfying (4.5). Note that the evaluation map is orientation preserving on moduli spaces of spheres so pushing forward by the evaluation map (4.6) is the same as evaluating by pullback.

Thus far we have assumed that the bubble tree limit in $X^0_f$ has no components entirely in the singular set. To achieve such a condition in general it is necessary to renormalize in a slightly different way. Consider the original sequence of maps $f\epsilon$ into $X^\epsilon_f$. Instead of simply letting $\epsilon \to 0$, we choose $r$ and sequences $0 < \epsilon_1^\epsilon < \epsilon_2^\epsilon < \cdots < \epsilon_r^\epsilon$ all approaching 0 as $i \to \infty$, renormalize the annuli $A^\epsilon_a = N_a(\epsilon^{i+1}) \setminus N_a(\epsilon^i)$, and collapse their boundaries by a “symplectic cut” (cf. [IP2]). The resulting bubble tree limit goes not into $X^0_f$, but instead into the singular space

$$X^r_f = (S^2 \times X)_0 \cup (S^2 \times X)_1 \cup \cdots \cup (S^2 \times X)_r / \sim$$

where $\sim$ identifies $X_a$ on one copy of $S^2 \times X$ to $X_a$ on the next by the identity map, and identifies $X_b$ on last copy to $X_a$ on the first by the map $f$. This renormalization insures that the limit

(a) has no components entirely contained in the singular set of $X^r_f$, and
(b) the total homology class in each copy of $S^2 \times X$ besides the first one is not a multiple of $s$ (otherwise there was no need to renormalize).

We can then remove double points, lift, and remove singularities to obtain a map $\sqcup B_i \to S^2 \times X$ exactly as in (4.4). The rest of the discussion carries through. The necklaces
then lie in \( \text{ev}(G_f^r) \), where \( G_f^r \) is the cyclic graph of the \( r + 1 \) string \((id, id, \ldots, id, f)\), that is, the subset of \( (X \times X)^{(r+1)k} \) defined as in [1.8] where the identifications are in the order \((id, \ldots, id, f; \ldots; id, \ldots, id, f)\).

Thus the moduli space of stabilized necklaces is parameterized by the set analogous to (4.8) with \( G_f \) replaced by \( G_r^f \) for \( r = 0, 1, \ldots \). Because each unstable bead stabilizes in \( d \) ways, the “unstable” necklaces of Figure 4.2 are parameterized by the set

\[
\bigcup_{r, A} \frac{1}{d^{k-1}} \text{ev}(M_{d,A}^{\nu_1} \times \ldots \times M_{d,A}^{\nu_k}) \cap G_f^r,
\]

where the union is over all \( r \) and over the set \( A \) of all collections \( A_1, \ldots, A_k \) of elements \( A_i = ds + \alpha_i \) in \( H_2(S^2 \times X) \) satisfying (4.5) and (4.10b).

Going the other way, a gluing argument shows that each necklace in \( \text{ev}^{-1}(G_f^r) \) can be perturbed to exactly \( d^k \) approximate \((J, \nu)\)-holomorphic tori in \( X_f^r \), and that each of these can be uniquely corrected to a perturbed holomorphic map. The approximate maps are obtained by an essentially canonical “rounding off” procedure applied to each of the \( k \) points where the beads intersect. In general, whenever two stable maps have a contact of order \( d \), we obtain \( d \) approximate maps; these have the same image, but have distinct complex structures on their domains. The details of this construction can be found in [IP2].

Altogether, each element of \( \text{ev}^{-1}(G_f^r) \) corresponds to \( d^k/d^{k-1} = d \) distinct \((J, \nu)\)-holomorphic tori. This, combined with the main result of [IP2] therefore implies the following.

**Proposition 4.3** For generic \((J', \nu')\) on \( X_f \) and generic \((J, \nu)\) on \( S^2 \times X \), there is an oriented cobordism between the moduli space (4.2) defining \( RT_m \) for \((J', \nu')\) and a covering of the space of \((J, \nu)\)-necklaces:

\[
\mathcal{M}_m(T^2, X_f) \sim \bigcup_{r, A} \bigcup_{i=1}^d \text{ev}(M_{d,A}^{\nu_1} \times \ldots \times M_{d,A}^{\nu_k}) \cap G_f^r,
\]

where the last union is over \( d \) identical disjoint copies.

Thus the Gromov invariant we seek can be written as a sum of homological intersections:

\[
RT_m = \sum_{r, A} d \cdot \left[ \text{ev}(M_{d,A}^{\nu_1} \times \ldots \times M_{d,A}^{\nu_k}) \cap G_f^r \right].
\]

This formula expresses \( RT_m \) (an invariant of \( X_f \)) as a count of curves in \( S^2 \times X \). By construction, the righthand side counts perturbed holomorphic curves for a rather special set of pairs \((J, \nu)\), namely the ones for which the restrictions of \( J \) and \( \nu \) to \( X_a \) and \( X_b \) agree under \( f : X_a \to X_b \). But in fact the righthand side is a Ruan-Tian invariant on \( S^2 \times X \), and hence can be computed using any generic \((J, \nu)\) on \( S^2 \times X \). We will exploit this by fixing a product almost complex structure \( J_0 \) on \( S^2 \times X \), showing that the pair \((J_0, 0)\) is generic, and then explicitly computing the invariant for \((J_0, 0)\).

The first step is to show that the only necklaces which contribute to the invariant are those where every bead represents a multiple of \( s \) in homology. The proof hinges on the
observation that the symplectic form $\omega$ on $X$ pullback to a 2-form $\bar{\omega}$ on $S^2 \times X$; this satisfies $\bar{\omega}|_{X_a} = f^*\bar{\omega}|_{X_b}$, so descends to a form $\omega_f$ on $X_f^*$. These forms determine cohomology classes $[\omega], [\bar{\omega}]$ and $[\omega_f]$.

**Lemma 4.4** For all $(J, \nu)$ close to $(J_0, 0)$, the only non-vanishing intersections in (4.11) are those with $r = 0$ and $A_i = d \cdot s$ for all $i$. Thus

$$RT_m = \sum_{m=dk} d \cdot \left[ \text{ev}(M^\nu_{d,ds} \times \ldots \times M^\nu_{d,ds}) \cap G_f \right]. \quad (4.12)$$

**Proof.** Suppose that for some choice $(A_1, \ldots, A_k) \in \mathcal{A}$ the intersection (4.11) is non-empty for all $(J, \nu)$ near $(J_0, 0)$. Choose a sequence $(J_j, \nu_j) \to (J_0, 0)$ and, for each $j$, a bubble tree $B_j = \bigcup B_{ij}$ in the intersection (4.11) with the component $B_{ij}$ representing $A_i$ in homology for each $j$. The Bubble Tree Convergence Theorem applies to each component, showing that a subsequence of $\{B_{ij}\}$ converges to a bubble tree limit $B_{i\infty}$ in $X^*_f$ and lifts to a bubble tree $\bar{B}_{i\infty}$ in $S^2 \times X$. Let $\{\bar{C}_{i,\ell}\}$ be the components of this limit and write the homology class $[\bar{C}_{i,\ell}]$ as $d_{i,\ell} s + \alpha'_{i,\ell}$ as we did before (4.15). Then for each $i$ we have $A_i = \sum_{\ell} [\bar{C}_{i,\ell}] = ds + \alpha'_{i,\ell}$. Furthermore, each $\bar{C}_{i,\ell}$ is $(J_0, 0)$-holomorphic for the product structure $J_0$, so the union

$$C = \bigcup_{i,\ell} \bar{C}_{i,\ell}$$

projects to a holomorphic curve $p(C)$ in $X$ representing $\sum_{i,\ell} \alpha'_{i,\ell} = \sum_i \alpha'_{i,\ell}$ with

$$\sum \text{Area}(p(\bar{C}_{i,\ell})) = \text{Area}(p(C)) = [\omega] \cdot [C] = [\bar{\omega}] \cdot [C].$$

On the other hand, $\bar{\omega}$ descends to $\omega_f$ under the projection $\pi$ from $S^2 \times X$ to $X_f^*$, and by (4.13) $\pi_*[C]$ represents a multiple of $T_0$ in $H_2(X^*_f)$. Thus

$$[\bar{\omega}] \cdot [C] = [\omega_f] \cdot \pi_*[C] = 0.$$ 

We conclude that $p(C)$ is trivial, so all the $\alpha'_{i,\ell}$ are zero. Therefore $A_i = ds$ for each $i$, and hence $r = 0$ by property (4.10(b)). \qed

**Lemma 4.5** The choice $(J, \nu) = (J_0, 0)$ is generic for the class $ms$ and for this choice there is an oriented identification

$$\text{ev}(M^0_{d,ds} \times \ldots \times M^0_{d,ds}) = \Delta^k$$

where $\Delta^k = \Delta \times \ldots \times \Delta$ is the multidiagonal in $(X \times X)^k$.

**Proof.** Modulo diffeomorphisms, we may assume that the marked points on $S^2$ are $a = x = 0$, $c = u = 1$ and $b = y = \infty$. Then, for the choice $(J_0, 0)$,

$$M^0_{d,ds} = \{ \phi : S^2 \to S^2 \times X \mid \phi(z) = (z^d, p), \text{ for some fixed } p \in X \},$$

so $M^0_{d,ds}$ is canonically identified with $X$. The linearization of the $J$-holomorphic map equation at $\phi \in M^0_{d,ds}$ is then the $\bar{\partial}$ operator on the pullback normal bundle to the image. Then $(J_0, 0)$ is generic because at $\phi \in M^0_{d,ds}$ the cokernel of the linearization vanishes:

$$\text{cok} \bar{\partial} = H^1(S^2, \phi^*TX) = H^{0,1}(\mathbb{P}^1, \mathcal{O}) \otimes T_pX = 0.$$
The moduli space is then oriented by the complex structure on
\[ T_\phi \mathcal{M}_{d,ds}^0 = \ker \overline{\partial} = H^0(S^2; \phi^*TX) = H^0(\mathbb{P}^1, \mathcal{O}) \otimes T_p X = T_p X. \]
Thus the identification \( \mathcal{M}_{d,ds}^0 \cong X \) respects orientation. Under the corresponding identification of \( \mathcal{M}_{d,ds}^0 \times \cdots \times \mathcal{M}_{d,ds}^0 \) with \( X \times \cdots \times X \), the evaluation map (4.6) becomes the map
\[ \text{ev}(p_1, \ldots, p_k) = (p_1, p_1, \ldots, p_k, p_k) \]
whose image is \( \Delta^k \). \( \square \)

By Lemma 4.5, the intersection in (4.12) for the product complex structure becomes
\[ \text{ev}(\mathcal{M}_{d,ds}^0 \times \cdots \times \mathcal{M}_{d,ds}^0) \cap G_f = \Delta^k \cap G_f \] (4.13)
for each \( d \) and \( k \). This is directly related to the Lefschetz number of \( f \).

**Lemma 4.6** The homological intersection \( \Delta^k \cap G_f \) is \( L(f^k) \).

**Proof.** After perturbing as in Lemma 4.3, \( L(f^k) \) is, by definition, the sum of the fixed points of \( f^k \), each oriented by the sign of \( \det (df^k - I) \). It is easy to see that there is a one-to-one correspondence between the fixed points of \( f^m \) and the points of \( \Delta^k \cap G_f \). To verify that the signs agree, consider the isomorphism \( B \) of \( TX^k \oplus TX^k \) given by the \( 2k \times 2k \) block matrix
\[ B = \begin{pmatrix} I & I \\ I & A \end{pmatrix} \]
where
\[ A = \begin{pmatrix} 0 & df & 0 & 0 \\ 0 & 0 & df & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ df & 0 & 0 & 0 \end{pmatrix}. \]
Then \( B \) takes the first factor \( TX^k \) into \( T\Delta^k \) and the second factor to \( TG_f \). The sign of an intersection point in \( \Delta^k \cap G_f \) is therefore the sign of \( \det B = \det (A - I) = \det (df^k - I) \) (to obtain the last equality expand along the top row). \( \square \)

Combining Lemma 4.6 with equations (4.12) and (4.13) gives
\[ R T_m = \sum_{m=kd} d \ L(f^k). \] (4.14)

Theorem 4.1 now follows by simple algebra. From equations (4.13) and (4.14)
\[ \log \text{Gr}^T(t) = \sum_{m=1}^\infty \frac{R T_m}{m} \log F(t^m) = \sum_{m=kd} \frac{d L(f^k)}{kd} \log F(t^{kd}) = \sum_{k=1}^\infty \frac{L(f^k)}{k} \sum_{d=1}^\infty \log F(t^{kd}), \]
Using relation (4.6) this simplifies to
\[ \log \text{Gr}^T(t) = \sum_{k=1}^\infty \frac{L(f^k)}{k} t^k, \]
which exponentiates to give \( \text{Gr}^T(t) = \zeta_f(t) \). This completes the proof of Theorem 4.1.
5 Gluing in $X_f$

Thus far, we have constructed the symplectic mapping tori $X_f$ and computed their Gromov invariants. In this section we will view the $X_f$ as operations on the set of symplectic four-manifolds. While the basic construction works in all dimensions, it is particularly interesting in dimension four, where the mapping tori are related to knots, and the Gromov invariants are related to the Seiberg-Witten invariants.

Taking $X$ a genus $g$ Riemann surface and $f$ a symplectomorphism of $X$ such that

$$\det(I - f_{*1}) = \pm 1 \quad (5.1)$$

gives symplectic 4-manifolds $X_f$ which is homology $S^2 \times T^2$ with a well-defined section class $T \subset X_f$ represented by a symplectically embedded torus, and a fiber class $F$ represented by a symplectically embedded genus $g$ surface. Both $T$ and $F$ have square zero, and $F \cdot T = 0$.

**Theorem 5.1** For $f$ as above, the Gromov series of $X_f$ is

$$\text{Gr}(X_f) = \frac{\det(I - tT \cdot f_{*1})}{(1 - t)^2}. \quad (5.2)$$

**Proof.** Theorem 4.1 shows that the partial Gromov series $\text{Gr}^T(X_f)$ has this form. Thus we need only show that $\text{Gr}^T(X_f) = \text{Gr}(X_f)$, that is, that any class $A \in H_2(X_f)$ with $RT(A) = 0$ is a multiple of $T$. Write such a class as $A = \alpha F + \beta T$. By choosing a sequence of generic pairs $(J_i, \nu_i)$ converging to $(J,0)$ and applying the Bubble Tree Convergence Theorem, we see that $A$ can be represented by a $J$-holomorphic curve, so $\alpha = A \cdot T \geq 0$. Then, by (1.2) and Lemma 5.5 below, we have $g - 1 = \kappa \cdot A = -2\alpha$. Thus $\alpha = 0$ and $A$ is a multiple of $T$. $\square$

Now suppose $(Z, \omega)$ is a symplectic 4-manifold and $F \subset Z$ is a symplectically embedded torus with trivial normal bundle (“symplectic torus fiber”). Then for each symplectomorphism $f$ of the Riemann surface $X$ we can form the symplectic fiber sum

$$Z(f) = Z \#_{F=T} X_f \quad (5.3)$$

by removing tubular neighborhoods of $T \subset X_f$ and $F \subset Z$ and gluing. Thus each symplectomorphism $f$ gives an operation $Z \mapsto Z(f)$ on the set of symplectic 4-manifolds with symplectic torus fibers. In particular, for each fibered knot $K$ we get an operation $Z \mapsto Z(K)$ by taking $f$ to be the monodromy of $K$.

**Theorem 5.2** Let $(Z, \omega)$ be a symplectic 4-manifold that contains a symplectically torus fiber $F$. If $f$ satisfies (5.7) then the partial Gromov invariant for the modified manifold (5.3) is

$$\text{Gr}^F_{Z(f)} = \text{Gr}^F_Z \cdot \det(I - tf_{*1}). \quad (5.4)$$

If (a) the linking circle of $F$ is homotopic to zero in $Z \setminus F$ and (b) $f$ is the monodromy of a fibered knot $K$ in a homotopy 3-sphere, with Alexander polynomial $A_K$ then $Z(K)$ is homotopic to $Z$

$$Z(K) \sim Z \quad (5.5)$$

20
via a homotopy that preserves $F$ and

$$Gr^F_{Z(K)} = Gr^F_Z \cdot A_K(t_F). \quad (5.6)$$

**Proof.** Combining (1.10) with (5.2) gives (5.4) and (5.6). If $X_f$ arises from a fibered knot then $X_f = M_f \times S^1$, where by (2.3) $\pi_1(M_f \setminus \{\tau\}) = \pi_1(S^3 \setminus K) = \mathbb{Z}$ is generated the linking circle of $\tau$. The homotopy equivalence (5.5) then follows from the Van Kampen Theorem, applied to the decompositions $(Z \setminus F) \cup (X_f \setminus T)$ and $(Z \setminus F) \cup (S^2 \times T \setminus T)$. \(\square\)

Note how knots enter this picture. For any symplectomorphism $f$ of Riemann surfaces that satisfies (5.1), we can form $X_f$ and modify $Z$ to $Z(f)$, and get formula (5.4) with $A(t) = \det(I - tf_*)$. But $Z$ and $Z(f)$ will have different homotopy types unless $\pi_1(M_f)$ is normally generated by $\tau$, and it is this condition that leads us to knots.

¿From (5.6) and Lemma 2.6 we obtain:

**Corollary 5.3** Let $Z$ be a symplectic 4-manifold with a symplectic torus fiber $F$ such that the linking circle of $F$ is homotopic to zero in $Z \setminus F$. Then $Z$ is homotopic to infinitely many distinct symplectic 4-manifolds $Z(K)$.

In the remainder of this section we will apply Theorem 5.2 to elliptic surfaces. Thus we take $Z = E(n)$ and consider the manifolds

$$E(n; K) = E(n)\#X_K.$$ 

For most of these, we can compute the full Gromov series. The following result appeared as Theorem 6.3 of the introduction.

**Theorem 5.4** For $n \neq 1$, the Gromov series of $E(n, K)$ is

$$Gr(E(n, K)) = (1 - t)^{n-2} A_K(t) \quad (5.7)$$

where the variable $t$ corresponds to the fiber class, and $A_K$ is the Alexander polynomial of the knot. For $n > 1$ the same formula gives the Seiberg-Witten series:

$$SW(E(n, K)) = (1 - t)^{n-2} A_K(t). \quad (5.8)$$

**Proof.** When $n = 0$ (5.7) is the same as (5.2), so take $n \geq 1$. Let $C \in H_2(E(n, K))$ be a class with $RT(C) \neq 0$. We will show that $C$ is a multiple of the fiber class $F$, and then (5.7) will follow from Theorem 5.2 and (1.13).

Since $n > 1$ we can write $E(n, K)$ as a symplectic fiber sum

$$E(n, K) = E(1)\#F_1 E(2)\#F_2 \cdots \#F_{k-1} E(2)\#F_kX_K$$

where there is at least one copy of $E(2)$ and at most one copy of $E(1)$. This fiber sum depends on parameters $\varepsilon_i$, the radii of the tubular neighborhoods of the fibers $F_i$ removed in its construction. Choose a sequence $\varepsilon_i \to 0$ and a sequence of pairs $(J_i, \nu_i)$ generic on $E(n, K)_{\varepsilon}$.
with $\nu_i \to 0$ and $J_i$ converging to a generic Kähler structure $J_0$ on each $E(2)$. Then fix a sequence $C_i$ of $(J_i, \nu_i)$-holomorphic curves representing $C$. In the limit, these converge to a bubble tree curve $C_0$ in the singular space obtained by identifying $E(1)$, $E(2)$, and $X_K$ along the fibers $F_i$ as above.

Since a generic Kähler structure on $E(2)$ admits no holomorphic curves, the curve $C_0$ is a disjoint union of curves $A \subset E(1)$ and $B \subset X_K$ which, moreover, do not intersect the fiber. Since the canonical divisor is a multiple of the fiber in both $E(1)$ and $X_K$ (Lemma 5.3 below), the adjunction inequality implies that $A \cdot A \geq 0$ unless at least one component of $A$ is an exceptional sphere, and that cannot be true since in $E(1)$ all exceptional spheres intersect the fiber. We similarly conclude that $B \cdot B = 0$.

We now have classes $A,F$ in $E(1)$ with $A \cdot A \geq 0$ and $F \cdot F = A \cdot F = 0$. Since $E(1)$ has $b^+ = 1$, the “Lightcone Lemma” ([LL]), implies that $A$ is a multiple of $F$. Since $b^+(X_K) = 1$ the same argument applies to $B$, finishing the proof of (5.7).

To connect this with the Seiberg-Witten series, note that the manifolds $E(n,K)$ have $b^+ > 1$ when $n > 1$. For such manifolds, Taubes proved that the Seiberg-Witten invariants are equal to the Taubes-Gromov invariants [T3] and that both of these invariants are nontrivial only for zero-dimensional moduli spaces (i.e. unconstrained curves) [T2]. On the other hand, we have just shown that the only nontrivial unconstrained Ruan-Tian invariants for $E(n,K)$ correspond to tori representing multiples of $T$. But for these, the Ruan-Tian series defined above agrees with the one defined in [IP1], equation (3.3), which in turn is equal to the Taubes-Gromov series by the main result in [IP1]. □

Because the Seiberg-Witten invariants depend only on the differentiable structure, Theorem 7.7 implies that $E(n,K)$ and $E(n,K')$ are distinct differentiable manifolds whenever $n > 1$ and $K$ and $K'$ have different Alexander polynomials. Thus by Burde’s Theorem there are infinitely many distinct symplectic 4-manifolds homeomorphic to $E(n)$.

**Remark** For the exceptional case $E(1,K)$, we still have that the partial Gromov series $Gr^T$ is given by $(1-t)^{n-2} A_K(t)$, but there are other classes — such as the exceptional curves — which contribute to the full Gromov invariant. On the other hand, Fintushel and Stern showed that the Seiberg-Witten series of $E(1,K)$ is $(1-t)^{n-2} A_K(t)$. Thus in this case the Gromov series defined in section 1 carries more information than the SW series.

The above proofs required explicit expressions for the canonical divisors. These are calculated in next lemma.

Let $F$ denote the ‘fiber class’ of $E(n;K) = E(n)\#X_K$, which can be thought of as either the fiber class of $E(n)$ or the section class $T$ of $X_K$. Then

**Lemma 5.5** The canonical divisor of $X_f$ is $\kappa = (2g-2)T$ and the canonical divisor of $E(n,K)$ is $\kappa = (2g-2+n)F$

**Proof.** For both parts we will use the adjunction formula: if $A \in H_2(X)$ is represented by a symplectically embedded curve of genus $g$ then

$$\kappa \cdot A = 2(g-1) - A \cdot A. \quad (5.9)$$

Recall that $H_2(X_f)$ is generated by classes $F$ and $T$, both represented by a symplectically embedded curves of square zero and genus $g$ and $1$ respectively, such that $F \cdot T = 1$. Then (5.9) uniquely determines $\kappa = (2g-2)T$. 

22
Next, fix a fiber $F_0$ and a holomorphic section $\sigma$ in $E(n)$; $\sigma$ is symplectic with genus 0, $\sigma \cdot \sigma = -n$, and $\sigma \cdot F_0 = 1$. Similarly, fix a section $T$ of $X_K \to T^2$. For each $p \in T$ there is a fiber $X_p$ (a symplectically embedded copy of $X$) with $X_p \cap T = \{p\}$ and with self-intersection zero. We can then identify $F_0$ with $T$ and form the fiber sum $E(n, K)$, simultaneously gluing $\sigma$ to the appropriate $X_p$, obtaining a symplectically embedded genus $g$ curve $s$ in $E(n, K)$ (cf. [G] Theorem 1.4) with $s \cdot s = -n$.

Now let $\kappa$ be the canonical divisor of $E(n, K)$. Then $s \cdot s = -n$, $s \cdot F = 1$, and $F \cdot F = 0$, so we can complete $\{s, F\}$ to a basis $\{s, F, a_j\}$ of $H_2(E(n, K))$ with $a_j \cdot F = 0$ for all $i$. But any class $a$ with $a \cdot F = 0$ can be represented by the disjoint union of cycles $C \subset E(n)$ and $C' \subset X_K$ with $C \cdot F = 0$ and $C' \cdot t = 0$. Along $C$, the canonical bundle of $E(n, K)$ is the same as the canonical bundle of $E(n)$, so $\kappa \cdot C = (n - 2)F \cdot C = 0$. Similarly, $\kappa \cdot C' = (2g - 2)T \cdot C = 0$. Thus $F$ and all the classes $a_i$ are perpendicular to $\kappa$. This and the equation $\kappa \cdot s = 2g - 2 + n$ (from the adjunction formula) uniquely determine $\kappa$ to be $(2g - 2 + n)F$. 

Finally, the following result verifies the assertion made before Theorem 0.3.

**Proposition 5.6** For any fibered knots $K, K'$ there is a homeomorphism between $E(n, K)$ and $E(n, K')$ preserving the fiber class, the periods of $\omega$, and the Stiefel-Whitney class $w_2$. When $K$ and $K'$ have the same genus this homeomorphism also preserves the canonical class.

**Proof.** Since the elliptic surfaces $E(n) \to \mathbb{P}^1$ admit sections, the linking circle of a fiber $F$ is contractible in $E(n) \setminus F$. Hence by Theorem 5.2 and the classification of simply-connected topological 4-manifolds there is a homeomorphism between $E(n, K')$ and $E(n, K)$ that preserves $F$. This homeomorphism preserves the canonical divisor $\kappa = (2g - 2 + n)F$ when $K$ and $K'$ have the same genus, and preserves $w_2$(the mod 2 reduction of $\kappa$) regardless of the genus. It remains to show that we can choose the symplectic structures so that the periods of $\omega$ match under this homeomorphism.

Recall that $X_K$ is the symplectic mapping cylinder ([1.4]) of the monodromy maps $f$ of $K$. Then $H_2(X_K)$ is generated by the section class $T$ with $\omega(T) = 1$ and the dual class $F$ with $\omega(F) = \text{vol}(X)$ where $X$ is the fiber of the knot. Thus the periods of $\omega$ on $X_K$ and $X_{K'}$ match.

Now let $\{s, F, a_i\}$ be the basis of $H_2(E(n, K))$ used in Lemma 5.5. Since the classes $F$ and $a_i$ have representatives that are disjoint from the gluing region (a neighborhood of one fiber), they are preserved under the homeomorphism and have the same periods. The period of the remaining generator $s$ also matches since it is formed in $E(n, K)$ and $E(n, K')$ by applying the same construction to curves of equal volume. 

6 Sympetctic Structures in Dimension Six

It is difficult to distinguish differentiable structures on 4-manifolds, but in higher dimensions differentiable structures are well-understood via surgery theory. The simplest way to move to higher dimensions is to take the product with a 2-sphere. In this context, one can show that if $X$ and $Y$ are smooth oriented 1-connected 4-manifolds which are homeomorphic and have the same characteristic classes, the 6-manifolds $X \times S^2$ and $Y \times S^2$ are diffeomorphic. We will use
that fact to construct families of manifolds that admit infinitely many deformation classes of symplectic structures. Previously, Ruan and Tian [RT] have found such manifolds.

The examples described here are remarkable for their simplicity. Furthermore, the differences in the symplectic structures cannot be detected by any of the “classical invariants” mentioned in the introduction: the diffeomorphism type, the cohomology class of the symplectic form, and the homotopy class $[J]$.

**Theorem 6.1** The differentiable 6-manifolds $K3 \times S^2$ and, more generally, $E(n) \times S^2$ have infinitely many deformation classes of symplectic structures, all with the same $[J]$ and $[\omega] \in H^2_{DR}$.

We will prove this after giving three lemmas. The proof actually shows more: for each integer $g > 1$ there are infinitely many deformation classes of symplectic structures whose canonical divisor is

$$
\kappa = \kappa_{E(n) \times S^2} - 2gF \times [S^2].
$$

Theorem 6.1 is a result of comparing the symplectic manifolds $E(n, K) \times S^2$ for different knots $K$. The first lemma shows that these cannot be distinguished by the classical invariants. Let $F'$ denote the fiber class in $E(n, K)$ and let $F' = F \times \{\text{pt}\}$ be the corresponding class in $E(n, K) \times S^2$.

**Lemma 6.2** For any two fibered knots $K, K'$, there is a diffeomorphism $E(n, K) \times S^2 \rightarrow E(n, K') \times S^2$ that preserves $[\omega]$. When $K$ and $K'$ have the same genus this diffeomorphism also preserves $[J]$.

**Proof.** By Proposition 5.6 there is a homeomorphism $\phi : E(n, K) \rightarrow E(n, K')$ which preserves $[\omega]$ and $w_2$ (and also $\kappa$ when $g = \text{genus}(K')$ equals $g' = \text{genus}(K')$). It also preserves $p_1$ because for 4-manifolds $p_1$ is three times the signature and because signatures add under fiber sums, so

$$
\sigma(E(n, K)) = \sigma(E(n)) + \sigma(X_K) = \sigma(E(n)).
$$

Now $E(n, K) \times S^2$ and $E(n, K') \times S^2$ are homeomorphic by $\psi = \phi \times \text{id}$. Since the symplectic form, the Pontryagin class, and the canonical class all respect the product structure (e.g. $\kappa = \pi_1^* \kappa_{E(n, K)} + \pi_2^* \kappa_{S^2}$), $\psi$ preserves $[\omega]$ and $p_1$, and preserves $\kappa$ when $g = g'$. We can then apply the result mentioned above: if $X, Y$ are closed, 1-connected, oriented smooth 6-manifolds with torsion-free homology, and $X \rightarrow Y$ is a homeomorphism that preserves orientation, $w_2$ and $p_1$, then there is a diffeomorphism $X \rightarrow Y$ that induces the same map on homology (see [J]). Thus we get the desired diffeomorphism, and it preserves $[\omega]$ and preserves $\kappa$ when $g = g'$. But for 6-manifolds the homotopy class of the tamed almost complex structures is determined by $\kappa$ (see [R]), so $[J]$ is also preserved when $g = g'$. 

Lemma 6.2 shows that we can regard $E(n, K) \times S^2$ as a symplectic structure on $E(n) \times S^2$, and that two of these structures have the same classical invariants whenever the corresponding knots have the same genus. We will complete the proof of Theorem 6.1 by showing that these symplectic structures can be distinguished by Gromov invariants. Recall from section 1 that when $n = 3$ there is a separate Gromov series $Gr_g$ for each genus $g$. We will compute the genus 1 Gromov series and show that

$$
Gr_1(E(n, K) \times S^2) \neq Gr_1(E(n, K') \times S^2).$

The first observation is that the genus 1 Gromov series are well-behaved with respect to products.
Lemma 6.3 Let $X$ and $Y$ be symplectic manifolds and consider $Z = X \times Y$ with the product symplectic structure. Let $\chi(Y)$ be the euler characteristic of $Y$. Then

$$Gr^A_{\{pt\}}(X \times Y) = \left[Gr^A(X)\right]^{\chi(Y)}.$$ 

\textbf{Proof.} By the definition of the genus 1 Gromov series, this follows from the equality

$$RT_{A \times \{pt\},1,X \times Y}(\alpha_1 \times [Y],...,\alpha_k \times [Y]) = \chi(Y) \cdot RT_{A,1,X}(\alpha_1,...,\alpha_k).$$  \hfill (6.1)

for any $A \in H_2(X,Z)$ and constraints $\alpha_1,...,\alpha_k \in H_2(X,Z)$. Let $\mathcal{M}$ denote the moduli space corresponding to the lefthand side of (6.1), that is, the set of genus one perturbed holomorphic maps from a torus to $Z$ representing $A \times \{pt\}$ and passing through the given constraints. Similarly, the $\mathcal{M}'$ denote the moduli space corresponding to the righthand side of (6.1). Choose a generic $(J_0,\nu_0)$ on $X$, a generic $J_1$ on $Y$, and consider $(J_0 \oplus J_1,\nu_0)$ on $Z$; with this choice $\nu = 0$ in the tangent direction to $Y$.

The moduli space of $(J_0 \oplus J_1,\nu_0)$-holomorphic maps in $Z$ consists of maps $f: T^2 \to Z$ whose projection $\pi_X f$ on the $X$ factor is an element of $\mathcal{M}'$ while $\pi_Y f$ is a constant map. Thus $\mathcal{M}$ is canonically diffeomorphic to a product of copies of $Y$, one for each map in $\mathcal{M}$ and each with a sign. More precisely,

$$\mathcal{M} \cong \mathcal{Y} \sqcup \cdots \sqcup \mathcal{Y} \sqcup -\mathcal{Y} \sqcup \cdots \sqcup -\mathcal{Y},$$ \hfill (6.2)

with $RT_{A,1,X} = p-n$.

The choice $(J_0 \oplus J_1,\nu)$ is not generic, so the invariant must be calculated as the Euler class of the Taubes obstruction bundle over $\mathcal{M}$. The fiber of the obstruction over $f \in \mathcal{M}$ is the cokernel of the linearization $D_f$ of the perturbed holomorphic equation. In our case, because of the product structure and because $\pi_Y f$ is the constant map to some point $p \in Y$,

$$\text{coker } D_f = \text{coker } D_{\pi_X f} \oplus \text{coker } D_{\pi_Y f} = 0 \oplus H^0,1(T^2, (\pi_Y f)^*(TY)) \cong T_p Y$$

($D_{\pi_Y f}$ has no cokernel since $(J_0,\nu_0)$ is generic on $X$). Thus the obstruction bundle is canonically isomorphic to the tangent bundle of $Y$ over each copy of $Y$ in (6.2). Consequently, when we perturb to a generic $(J,\nu)$, each copy of $Y$ in (6.1) gives rise to exactly $\chi(Y)$ perturbed holomorphic maps with sign, for a total of $\chi(Y) \cdot RT_{A,1,X}$. \hfill \square

Lemma 6.4 If $X = E(n,K)$ with fiber class $F$, then

$$Gr^{F'}(X \times S^2) = A_K(t)^2(1-t)^{2n-4}$$ \hfill (6.3)

where $A_K$ is the Alexander polynomial and $t$ corresponds to the class $F'$ in $H_2(X \times S^2)$.

\textbf{Proof.} We first show that any class $A \in H_2(X \times S^2)$ with $RT(A) \neq 0$ is a multiple of $F'$. Fix such a class $A = \alpha \times \{pt\} + b\{pt\} \times [S^2]$. Choose a sequence of $(J_k,\nu_k)$-holomorphic maps $f_k$ with $\nu_k \to 0$ and $J_k$ converging to a product structure $J \times j$. Then the projections $p \circ f_k$ converge to a $J$-holomorphic bubble tree in $X$ representing $\alpha$. But by Theorem 5.4 any
A $J$-holomorphic curve in $X$ is a multiple of the fiber class $F$. The remark following (1.2) then shows that $RT(A) = 0$ unless $0 = \kappa_{X \times S^2} \cdot A$. Using Lemma 5.3, this condition is

$$0 = \kappa_X \cdot \alpha - b \kappa_{S^2} \cdot S^2 = (2g - 2 + n)F \cdot mF - 2b = -2b.$$  

Thus $A = mF \times \{ \text{pt} \} = mF'$ for some $m$. With this, Lemma 6.3 and Theorem 5.4 give

$$\text{Gr}(X \times S^2) = \text{Gr}^J(X \times S^2) = \left[ \text{Gr}^F(X) \right]^2 = \left[ A(t)(1-t)^{n-2} \right]^2 \quad \square$$

Theorem 6.1 now follows easily. When $g > 1$, there are infinitely many monic, integer coefficients, symmetric polynomial of degree $2g$ s.t. $A(1) = \pm 1$. For any such polynomial Burde [B] proved that there is a fibered knot $K$ that has $A$ as its Alexander polynomial. This gives infinitely many distinct Alexander polynomials for each $g$. The corresponding spaces $E(n, K) \times S^2$ then have distinct Gromov series by Lemma 6.4, so define distinct symplectic structures.

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