Quantum q-Field Theory: q-Schrödinger and q-Klein-Gordon Fields

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Abstract

We show how to deal with the generalized q-Schrödinger and q-Klein-Gordon fields in a variety of scenarios. These q-fields are meaningful at very high energies (TeVs) for $q = 1.15$, high ones (GeVs) for $q = 1.001$, and low energies (MeVs) for $q = 1.000001$ [Nucl. Phys. A 948 (2016) 19; Nucl. Phys. A 955 (2016) 16]. (See the Alice experiment of LHC). We develop here the quantum field theory (QFT) for the q-Schrödinger and q-Klein-Gordon fields, showing that both reduce to the customary Schrödinger and Klein-Gordon QFTs for $q$ close to unity. Further, we analyze the q-Klein-Gordon field for $q \geq 1.15$. In this case for $2q - 1 = n$ (n integer $\geq 2$) and analytically compute the self-energy and the propagator up to second order.

Keywords: Non-linear Klein-Gordon equation; Non-linear Schrödinger equation; Classical Field Theory, Quantum Field Theory.

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1 Introduction

Classical fields theories (CFT) associated to Tsallis’ q-scenarios have been intensely studied recently \[10\, 9\, 24\]. Associated quantum (QFT) treatments have also been discussed \[24\]. In this paper we show how to treat the q-Schrödinger and q-Klein-Gordon fields in a variety of cases. It has been shown in \[19\, 20\] that q-fields emerge at1) very high energies (TeV) for \(q = 1.15\), 2) High (GeV) for \(q = 1.001\), and 3) low (MeV) for \(q = 1.000001\). LHC-Alice experiments show that Tsallis q-effects manifest themselves \[21\] at TeV energies.

In this effort we develop QFTs associated to q-Schrödinger and q-Klein-Gordon fields. Moreover, we study the q-KG field in the case \(2q - 1 = n\), \(n\) integer \(\geq 2\). Here we evaluate the selfenergy and propagator up to second order, thus generalizing results of \[24\]. In this respect, note also recent work on Proca-de Broglies’ classical field theory \[22\].

Motivations for nonlinear quantum evolution equations can be divided up into two types, namely, (A) as basic equations governing phenomena at the frontiers of quantum mechanics, mainly at the boundary between quantum and gravitational physics (see, \[1\, 2\] and references therein). The other possibility is (B) regard nonlinear-Schrödinger-like equations (NLSE) as effective, single particle mean field descriptions of involved quantum many-body systems. A paradigmatic illustration is that of \[3\]. In earlier applications of nonlinear Schrödinger equations, one encounters situations involving a cubic nonlinearity in the wave function.

Referring to (A), our present NLSE can be used for a description of dark matter components, since the associated variational principle (the one that leads to the NLSE) is seen to describe particles that can not interact with the electromagnetic field \[4\]. With reference to (B), we remark that the NLSE displays strong similarity with the Schrödinger equation linked to a particle endowed with a time-position dependent effective mass \[5\, 6\, 7\, 8\], involving particles moving in nonlocal potentials, reminiscent of the energy density functional quantum many-body problem’s approach \[9\].

During the last years, the search for insight into a number of complex phenomena produced interesting proposals involving localized solutions attached to non linear Klein-Gordon and Schrödinger equations, i.e., non linear generalizations of these equations \[1\, 11\]. Following \[4\], we extend these generalizations here by developing quantum field theories (QFT) associated to the q-Schrödinger and q-Klein-Gordon equations \[10\].
Here, we develop first the classical field theory (CFT) associated to that q-Schrödinger equation deduced in [11] from the hypergeometric differential equation. We define the corresponding physical fields via an analogy with treatments in string theory [13] for defining physical states of the bosonic string. Our ensuing theory reduces to the conventional Schrödinger field theory for \( q \to 1 \).

Secondly, we develop the QFT for that very q-Schrödinger equation (see also [12]). This equation is similar but not identical to that advanced in [10]. Its treatment is however much simpler than that employed in [11].

In the third place, we develop the QFT for the q-K-G Field in several scenarios, generalizing results of [24] and showing that the ensuing q-K-G field reduces to the customary K-G field for \( q \to 1 \).

2 A non-linear q-Schrödinger Equation

2.1 Classical Theory

We develop here the CFT for that particular q-Schrödinger Equation advanced in [12] from the Hypergeometric Differential Equation. This NLSE is different from the pioneer one proposed in [10], but exhibits better qualitative features. One has

\[
i\hbar \frac{\partial}{\partial t} \psi(\vec{x}, t)^q = H\psi(\vec{x}, t).
\]

In the free particle instance one writes

\[
H_0 = -\frac{\hbar^2}{2m} \triangle,
\]

whose solution reads

\[
\psi(\vec{x}, t) = [1 + (1 - q)\frac{i}{\hbar} (\vec{p} \cdot \vec{x} - Et)]^{\frac{1}{1-q}}.
\]

Introduce now action

\[
S = \frac{1}{(4q - 2)V} \int_{-\infty}^{\infty} \int V \left( i\hbar \psi^i q^i \partial^i \phi^i - i\hbar \psi^q \partial^q \phi - \frac{\hbar^2}{2m} \nabla \psi \nabla \phi - \right)
\]
\[ \left( \frac{\hbar^2}{2m} \nabla \psi \nabla \phi \right) dt \, d^3x, \]  
\tag{2.4} \]

with \( V \) the Euclidian volume. Our action can be rewritten in the fashion

\[ S = \int_{-\infty}^{\infty} \int_{V} L(\psi, \psi^\dag, \partial_t \phi, \partial_t \phi^\dag, \nabla \psi, \nabla \psi^\dag, \nabla \phi, \nabla \phi^\dag) dt \, d^3x. \]  
\tag{2.5} \]

One obtains from (2.5) the field’s motion equations

\[ i\hbar \frac{\partial}{\partial t} \psi(\vec{x}, t) + \frac{\hbar^2}{2m} \Delta \psi(\vec{x}, t) = 0, \]  
\tag{2.6} \]

\[ i\hbar q \psi(\vec{x}, t) q^{-1} \frac{\partial}{\partial t} \phi(\vec{k}, t) - \frac{\hbar^2}{2m} \Delta \phi(\vec{x}, t) = 0. \]  
\tag{2.7} \]

whose solution is (2.3). Instead, that for (2.7) reads

\[ \phi(\vec{x}, t) = \left[ 1 + (1 - q)\frac{i}{\hbar} (\vec{p} \cdot \vec{x} - Et) \right]^{2q-1}. \]  
\tag{2.8} \]

If \( q \to 1 \), \( \phi \) becomes \( \psi^\dag \), the adjoint of \( \psi \). Now, the concomitant canonically conjugated momenta are

\[ \Pi_{\psi} = \frac{\partial L}{\partial (\partial_t \psi)} = 0; \quad \Pi_{\psi^\dag} = \frac{\partial L}{\partial (\partial_t \psi^\dag)} = 0, \]
\[
\Pi_\phi = \frac{\partial L}{\partial (\partial_t \phi)} = -\frac{i\hbar \psi^q}{(4q - 2)V}; \quad \Pi_{\phi^\dag} = \frac{\partial L}{\partial (\partial_t \phi^\dag)} = \frac{i\hbar \psi^{iq}}{(4q - 2)V}, \]  
\tag{2.9} \]

and the associated Hamiltonian is

\[ \mathcal{H} = \Pi_{\phi} \partial_t \phi + \Pi_{\phi^\dag} \partial_t \phi^\dag - L, \]  
\tag{2.10} \]

that we cast in terms of \( \psi - \phi \) as

\[ \mathcal{H} = \frac{\hbar^2}{(8q - 4)mV} (\nabla \psi \nabla \phi + \nabla \psi^\dag \nabla \phi^\dag). \]  
\tag{2.11} \]

The field energy is

\[ E = \int_{V} \mathcal{H} d^3x. \]  
\tag{2.12} \]
If we replace the solutions (2.3) and (2.8) into (2.12), one has
\[
E = \int_V \frac{\hbar^2}{(8q - 4)mV} (4q - 2) \frac{p^2}{\hbar^2} d^3x, \tag{2.13}
\]
or
\[
E = \frac{p^2}{2m}, \tag{2.14}
\]
that exactly correspond to the wave energy (2.3), as one should expected.

The field-momentum density reads
\[
\vec{P} = -\frac{\partial L}{\partial (\partial_t \psi)} \nabla \psi - \frac{\partial L}{\partial (\partial_t \phi)} \nabla \phi - \frac{\partial L}{\partial (\partial_t \psi^\dagger)} \nabla \psi^\dagger - \frac{\partial L}{\partial (\partial_t \phi^\dagger)} \nabla \phi^\dagger, \tag{2.15}
\]
or
\[
\vec{P} = \frac{i\hbar}{(4q - 2)V} (\psi^q \nabla \phi - \psi^i q \nabla \phi^\dagger), \tag{2.16}
\]
the field-momentum becoming
\[
\vec{P} = \int_V \vec{P} d^3x. \tag{2.17}
\]

Employing (2.3) and (2.8), one finds for the momentum
\[
\vec{p} = \frac{i\hbar}{(4q - 2)V} \int_V \frac{4q - 2}{i\hbar} \vec{p} d^3x, \tag{2.18}
\]
or
\[
\vec{p} = \vec{p}. \tag{2.19}
\]
The probability density is now
\[
\rho = \frac{1}{2V} [\psi^q \phi + \psi^i q \phi^\dagger], \tag{2.20}
\]
verifying
\[
\frac{\partial}{\partial t} \rho + \nabla \cdot \vec{j} = K, \tag{2.21}
\]
where
\[
\vec{j} = \frac{\hbar^2}{8mVq_i} \left[ \phi \nabla \psi - \psi \nabla \phi + \psi^i \nabla \phi^\dagger - \phi^i \nabla \psi^\dagger \right]. \tag{2.22}
\]
is the probability-current. \( K \) reads

\[
K = \frac{\hbar^2(q - 1)}{8mVq_i} \left[ \psi^\dagger \Delta \phi^\dagger + \phi^\dagger \Delta \psi^\dagger - \phi \Delta \psi - \psi \Delta \phi \right], \tag{2.23}
\]

that vanishes at \( q = 1 \). However, the physical fields are those for which \( K = 0 \). For example, one lists as physical the solutions (2.3) and (2.8), since for them probability is indeed conserved.

### 2.2 Quantum Theory

We start with the action

\[
S = -\int \left( i\hbar \psi^\dagger q \partial_t \phi - i\hbar \psi^{\dagger q} \partial_t \phi^\dagger + \frac{\hbar^2}{2m} \nabla \psi \nabla \phi + \frac{\hbar^2}{2m} \nabla \psi^{\dagger q} \nabla \phi^\dagger \right) dt \, d^3x. \tag{2.24}
\]

We develop first a theory for 1) \( q \) close to unity and 2) weak fields \( \psi \). In these conditions one appeals to the approximation

\[
\psi^q \simeq \psi + (q - 1)\psi \ln \psi, \tag{2.25}
\]

and since \( \psi \) is a weak field

\[
\psi \simeq I + (q - 1)\eta. \tag{2.26}
\]

Consequently, the action (2.24) becomes

\[
S = -(q - 1) \int \left( i\hbar \eta \partial_t \phi - i\hbar \eta^{\dagger} \partial_t \phi^\dagger + \frac{\hbar^2}{2m} \nabla \eta \nabla \phi + \frac{\hbar^2}{2m} \nabla \eta^{\dagger} \nabla \phi^\dagger \right) dt \, d^3x, \tag{2.27}
\]

where we used

\[
\int \eta(\bar{x}, t) dt d^3x = \int \phi(\bar{x}, t) dt d^3x = 0, \tag{2.28}
\]

since the fields are

\[
\eta(\bar{x}, t) = \frac{1}{(2\pi\hbar)^{3/2}} \int a(\vec{p}) e^{i(\vec{p} \cdot \bar{x} - \omega t)} d^3p, \tag{2.29}
\]
η†(⃗x, t) = \frac{1}{(2\pi \hbar)^{\frac{3}{2}}} \int a†(⃗p)e^{-\frac{i}{\hbar}(⃗p \cdot ⃗x - Et)}d^3p \quad (2.30)

φ(⃗x, t) = \frac{1}{(2\pi \hbar)^{\frac{3}{2}}} \int b(⃗p)e^{\frac{i}{\hbar}(⃗p \cdot ⃗x - Et)}d^3p, \quad (2.31)

and

φ†(⃗x, t) = \frac{1}{(2\pi \hbar)^{\frac{3}{2}}} \int b†(⃗p)e^{-\frac{i}{\hbar}(⃗p \cdot ⃗x - Et)}d^3p. \quad (2.32)

Surprisingly enough, the q-Schrödinger field (qSF) reduce to the usual SF of low energies!

Creation-destruction operators verify

\[ [a(⃗p), a†(⃗p′)] = [b(⃗p), b†(⃗p′)] = \delta(⃗p - ⃗p′). \quad (2.33) \]

The propagator for the field η is [23]

\[ \Delta_η(⃗x, t) = \left(\frac{m}{2\pi \hbar}\right)^{\frac{3}{2}} t^{\frac{3}{2}} e^{\frac{im|⃗x|^2}{2\hbar}}. \quad (2.34) \]

that, in terms of energy and momentum reads

\[ \hat{\Delta}_η(⃗p, E) = \frac{i\hbar}{E - \frac{p^2}{2m} + i0}. \quad (2.35) \]

These two representations are related via

\[ \Delta_η(⃗x, t) = \frac{1}{(2\pi \hbar)^{\frac{3}{2}}} \int \hat{\Delta}(⃗p, E)e^{\frac{i}{\hbar}(⃗p \cdot ⃗x - Et)}dEd^3p. \quad (2.36) \]

The convolution of this propagator with itself, with E and ⃗p as variables, is NOT finite. It can be calculated, however, by appeal to distributions’ theory using the relation

\[ \hat{f} \ast \hat{g} = (2\pi \hbar)^{\frac{4}{3}} \mathcal{F}(fg). \quad (2.37) \]

This is so because divergences in the convolution of two phase space functions derive from multiplication of distributions possessing singularities at the same configuration-space point. Keeping in mind that

\[ \Delta_η^2(⃗x, t) = \left(\frac{m}{2\pi i \hbar}\right)^{\frac{3}{2}} t^3 e^{\frac{i|⃗x|^2}{\hbar}}. \quad (2.38) \]
\[
\left(2.37\right) \text{ yields}
\]
\[
\frac{1}{\left(2\pi\hbar\right)^4} \left(\hat{\Delta}_\eta(\vec{p}, E) \ast \hat{\Delta}_\eta(\vec{p}, E)\right) = \int \left(\frac{m}{2\pi i\hbar}\right)^3 t_+^{-3} e^{\frac{i m \vec{x}^2}{\hbar}} e^{\frac{i}{\hbar}(\vec{p} \cdot \vec{x} - E t)} dtd^3x. \tag{2.39}
\]

The spatial integral is
\[
\int e^{\frac{i m \vec{x}^2}{\hbar}} e^{-\frac{i}{\hbar} \vec{p} \cdot \vec{x}} d^3x = \pi^\frac{3}{2} \left(\frac{i \hbar}{t}\right)^\frac{3}{2} e^{-\frac{i}{4\hbar m} \vec{p}^2 t}, \tag{2.40}
\]
so that the convolution becomes
\[
\hat{\Delta}_\eta(\vec{p}, E) \ast \hat{\Delta}_\eta(\vec{p}, E) = \frac{(2\pi\hbar)^4}{8} \left(\frac{m}{i\pi\hbar}\right)^\frac{3}{2} \int t_+^{-\frac{3}{2}} e^{\frac{i}{\hbar}(E - \frac{\vec{p}^2}{4m}) t} dt. \tag{2.41}
\]

Using the result below (see \[14\])
\[
\mathcal{F}[\chi^\lambda] = i e^{\frac{i \lambda}{2}} \Gamma(\lambda + 1)(k + i0)^{-\lambda - 1}, \tag{2.42}
\]
one finds
\[
\hat{\Delta}_\eta(\vec{p}, E) \ast \hat{\Delta}_\eta(\vec{p}, E) = 4\pi^2 \hbar^2 m^\frac{3}{2} \left(E - \frac{\vec{p}^2}{4m} + i0\right)^{\frac{1}{2}}. \tag{2.43}
\]

### 3 A non-linear q-Klein-Gordon Equation

The classical FT associated to the q-Klein-Gordon equation was developed in \[24\]. Here we tackle the quantum version, whose action is
\[
\mathcal{S} = \int \left\{ \partial_\mu \phi(x_\mu) \partial^\mu \bar{\psi}(x_\mu) + \partial_\mu \phi^\dagger(x_\mu) \partial^\mu \bar{\psi}^\dagger(x_\mu) \right\} d^4x.
\]  
This theory is 1) adequate for very energetic (TeVs) q-particles, according to CERN-Alice experiments, and 2) non re-normalizable for any q > 1. Thus, it cannot be dealt with neither with dimensional regularization nor with differential one. A way out is provided by the ultradistributions’ convolution of Bollini and Rocca \[15, 16, 17, 18\]. Ultradistribuions provides a general formalism to treat non-renormalizable theories and gives in the configuration space a general product in a ring with zero divisors (a product of distributions...
of exponential type). For example we can treat cases with $q \geq 1.15$ as we will do later.

The concomitant theory is tractable here for weak fields and for A) $q \sim 1$ or B) particular $q$-values. We analyze first the case $q \sim 1$, associated to energies smaller that 1 TeV. We can thus write

$$q m^2 \phi^{2q-1} = q m^2 \phi + 2(q - 1) m^2 \phi \ln \phi. \quad (3.2)$$

Since the field is weak we have

$$\phi \simeq 1 + (q - 1) \eta, \quad (3.3)$$

$$\ln \phi \simeq (q - 1) \eta, \quad (3.4)$$

Using (3.2), (3.3), and (3.4) the field’s action becomes

$$S = (q - 1) \int \left\{ \partial_\mu \eta(x_\mu) \partial^\mu \psi(x_\mu) + \partial_\mu \eta(x_\mu) \partial_\mu \psi(x_\mu) \right. \right.$$

$$\left. \left. -(3q - 2) m^2 \left[ \eta(x_\mu) \psi(x_\mu) + \eta(x_\mu) \psi(x_\mu) \right] \right\} d^4 x, \quad (3.5)$$

where we employed

$$\int \eta(x_\mu) d^4 x = \int \psi(x_\mu) d^4 x = 0. \quad (3.6)$$

Defining

$$3q - 2 \neq 0 \quad \mu^2 = (3q - 2) m^2. \quad (3.7)$$

one has

$$S = (q - 1) \int \left\{ \partial_\mu \eta(x_\mu) \partial^\mu \psi(x_\mu) + \partial_\mu \eta(x_\mu) \partial_\mu \psi(x_\mu) \right. \right.$$

$$\left. \left. -\mu^2 \left[ \eta(x_\mu) \psi(x_\mu) + \eta(x_\mu) \psi(x_\mu) \right] \right\} d^4 x. \quad (3.8)$$

The low energy field is just the usual Klein-Gordon one! For the fields we have

$$\eta(x_\mu) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left[ \frac{a(k)}{\sqrt{2}\omega} e^{-ik_\mu x_\mu} + \frac{b^\dagger(k)}{\sqrt{2}\omega} e^{ik_\mu x_\mu} \right] d^3 k, \quad (3.9)$$

$$\psi(x_\mu) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left[ \frac{c(k)}{\sqrt{2}\omega} e^{-ik_\mu x_\mu} + \frac{d^\dagger(k)}{\sqrt{2}\omega} e^{ik_\mu x_\mu} \right] d^3 k, \quad (3.10)$$
where \( k_0 = \omega = \sqrt{k^2 + \mu^2} \).

Field quantization proceeds then along familiar lines:

\[
[a(\vec{k}), a^\dagger(\vec{k}')] = [b(\vec{k}), b^\dagger(\vec{k}')] = [c(\vec{k}), c^\dagger(\vec{k}')] = [d(\vec{k}), d^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}').
\]

For \( 3q - 2 = 0 \), i.e., \( q = \frac{2}{3} \), the low energy theory is one for a null mass field

\[
S = -\frac{1}{3} \int \left[ \partial_\mu \eta(x_\mu) \partial^\mu \psi(x_\mu) + \partial_\mu \eta^\dagger(x_\mu) \partial^\mu \psi^\dagger(x_\mu) \right] d^4x,
\]

where \( k_0 = \omega = |\vec{k}| \).

We tackle now the q-KG theory for an integer \( n \) such that \( 2q - 1 = n \), for \( m \) small, where the action is

\[
S = \int \left\{ \partial_\mu \phi(x_\mu) \partial^\mu \psi(x_\mu) + \partial_\mu \phi^\dagger(x_\mu) \partial^\mu \psi^\dagger(x_\mu) \right. \\
\left. - \frac{n + 1}{2} m^2 \left[ \phi^n(x_\mu) \psi(x_\mu) + \phi^n(x_\mu) \psi^\dagger(x_\mu) \right] \right\} d^4x.
\]

The fields in the interaction representation satisfy the equations of motion for free fields, corresponding to the action \( S_0 \). This is to satisfy the usual massless Klein-Gordon equation. As a consequence, we can cast the fields \( \phi \) and \( \psi \) in the fashion

\[
\phi(x_\mu) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left[ \frac{a(\vec{k})}{\sqrt{2\omega}} e^{ik_\mu x_\mu} + \frac{b^\dagger(\vec{k})}{\sqrt{2\omega}} e^{-ik_\mu x_\mu} \right] d^3k,
\]

\[
\psi(x_\mu) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left[ \frac{c(\vec{k})}{\sqrt{2\omega}} e^{ik_\mu x_\mu} + \frac{d^\dagger(\vec{k})}{\sqrt{2\omega}} e^{-ik_\mu x_\mu} \right] d^3k,
\]
where \( k_0 = \omega = |\vec{k}| \). The quantification of these two fields is i) immediately tractable and ii) the usual one, given by

\[
[a(\vec{k}), a^\dagger(\vec{k}')] = [b(\vec{k}), b^\dagger(\vec{k}')] = [c(\vec{k}), c^\dagger(\vec{k}')] = [d(\vec{k}), d^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}').
\] (3.18)

The naked propagator corresponding to both fields is the customary one, and it is just the Feynman propagator for massless fields

\[
\Delta_0(k_\mu) = \frac{i}{k^2 + i0}
\] (3.19)

where \( k^2 = k_0^2 - \vec{k}^2 \). The dressed propagator, which takes into account the interaction, is given by

\[
\Delta(k_\mu) = \frac{i}{k^2 + i0 - i\Sigma(k_\mu)},
\] (3.20)

where \( \Sigma(k_\mu) \) is the self-energy. Let us calculate the self-energy for the field \( \phi \) at second order in perturbation theory, for which the only non vanishing diagram corresponds to the convolution of \( n - 1 \) propagators for the field \( \phi \) and one propagator for the field \( \psi \). All remaining diagrams are null. (this is easily demonstrated using the regularization of Guelfand for integrals containing powers of \( x \) [14]).
Therefore, we have for the self-energy the expression

\[
\Sigma(k_\mu) = \frac{(n + 1)^2 m^4}{4} \left( \frac{i}{k^2 + i0} \ast \frac{i}{k^2 + i0} \ast \frac{i}{k^2 + i0} \cdots \frac{i}{k^2 + i0} \right).
\] (3.21)

The convolution of \( n \) Feynman’s propagators of zero mass is calculated directly using the theory of convolution of Ultradistributions [15]-[18]. Here, we just give the result, that turns out to be rather simple. A detailed demonstration lies beyond this paper’s scope. We arrive at

\[
\frac{i\pi^{2(n-1)} k^{2(n-2)}}{\Gamma(n)\Gamma(n-1)} \left[ \ln(k^2 + i0) + 2\lambda(1) - \lambda(n-1) - \lambda(n) \right],
\] (3.22)
where \( \lambda(z) = \frac{d \ln \Gamma(z)}{dz} \). The self-energy is then

\[
\Sigma(k_\mu) = \frac{(n + 1)^2 m^4 i \pi^{2(n-1)} k^{2(n-2)}}{4 \Gamma(n) \Gamma(n - 1)} \left[ \ln(k^2 + i0) + 2 \lambda(1) - \lambda(n-1) - \lambda(n) \right]
\]

For both fields \( \phi \) and \( \psi \), the self-energy and the dressed propagator coincide up to second order.

Note that the current of probability is given by

\[
J_\mu = \frac{i}{4m} [\psi \partial_\mu \phi - \phi \partial_\mu \psi + \phi^\dagger \partial_\mu \psi^\dagger - \psi^\dagger \partial_\mu \phi^\dagger].
\]

and it is verified that

\[
\partial_\mu J^\mu = 0
\]

This implies that the fields defined in the representation of interaction are physical fields.

4 Conclusions

We have here obtained some results that may be regarded as interesting.

1) We developed the CFT for the particular q-SE advanced in [12].
2) For this CFT we showed that the customary dispersion relations apply.
   We also introduced the physical fields i.e., those that the probability current is conserved. The physical states are introduced via analogy with bosonic string theory.
3) We developed the QFT associated to the q-SE of [12]. For weak fields, this q-QFT coincides with the ordinary SE-QFT. This result confirms our Nuclear Physics A results. These show that one needs energies of up to 1 TeV in order to clearly distinguish between q-theories and q=1, ordinary ones.
4) Using Distribution Theory [14] we discussed the convolution of two Schrödinger propagators obtaining a finite result.
5) We developed the QFT associated to the q-KGE, generalizing our result of [24].
6) For low energies and q close to 1 this theory coincides with the ordinary KG-QFT.
7) For particular q-values \( q = \frac{n+1}{2} \) n integer we develop the q-KG-QFT.
8) We calculate the convolution of n naked propagators, the corresponding self-energy up to second order and the dressed propagator. This was achieved appealing to Ultradistributions theory.
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