SECON D MOMENT ESTIMATOR FOR AN AR(1) MODEL DRIVEN BY A 
LONG MEMORY GAUSSIAN NOISE

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Abstract. In this paper, we consider an inference problem for the first order autoregressive 
process driven by a long memory stationary Gaussian process. Suppose that the covariance 
function of the noise can be expressed as $|k|^{2H-2}$ times a function slowly varying at infinity. 
The fractional Gaussian noise and the fractional ARIMA model and some others Gaussian 
noise are special examples that satisfy this assumption. We propose a second moment esti-
mator and prove the strong consistency, the asymptotic normality and the almost sure central 
limit theorem. Moreover, we give the upper Berry-Esseen bound by means of Fourth moment 
th eorem.

Keywords: Gaussian process; asymptotic normality; almost sure central limit theorem; 
Berry-Esseen bound; Breuer-Major theorem; Fourth moment theorem.

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1. Introduction

For the first order autoregressive model $(X_t, t \in \mathbb{N})$ driven by a given noise sequence $\xi = 
(\xi_t, t \in \mathbb{Z})$:

$$X_t = \theta X_{t-1} + \xi_t, \quad t \in \mathbb{N}$$

(1.1)

with $X_0 = 0$, the inference problem regarding the parameter $\theta$ has been extensively studied 
in probability and statistics literatures. For when $\xi$ is independent identical distribution or a 
martingale difference sequence, this problem has been widely studied over the past decades (see 
[1, 2] and the references therein). In the heavy-tailed noise case, the least square estimator (LSE) 
of AR(p) models was studied in [3]. The maximum likelihood estimator (MLE) of AR(p) was 
investigated in [4] for regular stationary Gaussian noise, in which they transform the observation 
model into an “equivalent” model with Gaussian white noise. In [4], it is pointed out that for 
the strongly dependent noises the LSE is generally not consistent. Very recently, in case of long-
memory noise, the detection of a change of the above parameter $\theta$ is studied by means of the 
likelihood ratio test [5].

In this paper, we will discuss the long-range dependence Gaussian noise case and propose a 
second moment estimator. First, we find that it is very convenient to construct a second moment 
estimator when we restrict the domain of the parameter $\theta$ in

$$\Theta = \{ \theta \in \mathbb{R} \mid 0 < \theta < 1 \}.$$
It seems that this restriction is very reasonable for real-world context sometimes. In fact, $|\theta| < 1$ is an assumption to ensure the model (1.1) to have a stationary solution. We rule out the case of $-1 < \theta < 0$ in which the series tends to oscillate rapidly. We also rule out the case of $\theta = 0$ in which $X_t$ is not an autoregressive model any more.

Next, we assume that the stationary Gaussian noise $\xi$ satisfies the following Hypothesis 1.1:

**Hypothesis 1.1.** The covariance function $\rho(k) = \mathbb{E}(\xi_0 \xi_k)$ for any $k \in \mathbb{Z}$ satisfies
\[
\rho(k) = L(k)|k|^{2H-2}, \quad H \in \left(\frac{1}{2}, 1\right)
\]
with $L : (0, \infty) \to (0, \infty)$ is slowly varying at infinity in Zygmund’s sense and $\tilde{L}(\lambda) := L(\frac{1}{\lambda})$ is of bounded variation on $(a, \pi)$ for any $a > 0$. Moreover, $\rho(0) = 1$.

It is well known that Eq. (1.2) is equivalent to the spectral density of $\xi$ satisfying
\[
h_\xi(\lambda) \sim C_H L(\lambda^{-1})|\lambda|^{-2H}, \quad \text{as } \lambda \to 0
\]
with $C_H = \pi^{-1}\Gamma(2H - 1)\sin(\pi - \pi H)$. Please refer to [6] or Lemma 2.3 below.

We will see that the fractional Gaussian noise, the fractional ARIMA model driven by Gaussian white noise and some other long memory Gaussian processes are special examples satisfying Hypothesis 1.1.

When $|\theta| < 1$, the stationary solution to the model (1.1) is
\[
Y_t = \sum_{j=0}^{\infty} \theta^j \xi_{t-j},
\]
and the solution with initial value $X_0 = 0$ can be represented as:
\[
X_t = Y_t + \theta^t \zeta, \quad (1.3)
\]
where $\zeta$ is a normal random variable with zero mean. It is clear that the second moment of $Y_t$ is:
\[
f(\theta) := \mathbb{E}(Y_t^2) = \sum_{i,j=0}^{\infty} \theta^{i+j} \rho(i-j).
\]
If $0 < \theta < 1$ then $f(\theta)$ is positive and strictly increasing. Denote $f^{-1}(\cdot)$ is the inverse function of $f(\cdot)$. We propose a second moment estimator of $\theta$ as:
\[
\tilde{\theta}_n = f^{-1}\left(\frac{1}{n} \sum_{t=1}^{n} X_t^2\right), \quad (1.4)
\]

In this paper, we will show the strong consistency and give the asymptotic distribution for that estimator. Moreover, we also give the Berry-Esséen bound when the limit distribution is Gaussian. These results are stated in the following theorems:

**Theorem 1.2.** Under Hypothesis 1.1, the estimator $\tilde{\theta}_n$ is strongly consistent, i.e.,
\[
\lim_{n \to \infty} \tilde{\theta}_n = \theta \quad \text{a.s.}
\]
Theorem 1.3. Under Hypothesis 1.1 and suppose $H \in (\frac{1}{2}, \frac{3}{4})$, we have the following asymptotic distribution of $\hat{\theta}_n$ as $n \to \infty$:

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\text{law}} N(0, \frac{\sigma_H^2}{f'(\theta)^2})$$

where $\sigma_H^2 = 2 \sum_{k \in \mathbb{Z}} R^2(k)$ and $f'(\theta)$ is the derivative of $f(\theta)$.

Remark 1.4. The case of $H \in [\frac{3}{4}, 1)$ will be treated in a separate paper.

Theorem 1.5. Let $Z$ be a standard Gaussian random variable. Under Hypothesis 1.1 and suppose $H \in (\frac{1}{2}, \frac{3}{4})$, then

$$G_n := f'(\theta)\sqrt{n}(\hat{\theta}_n - \theta)$$

satisfies an almost sure central limit theorem (ASCLT). In other words, almost surely, for all $z \in \mathbb{R}$,

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} 1\{G_k \leq z\} \to P(Z \leq z) \quad \text{as} \quad n \to \infty,$$

or, equivalently, almost surely, for all continuous and bounded functions $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(G_k) \to \mathbb{E}[\varphi(Z)] \quad \text{as} \quad n \to \infty.$$

Theorem 1.6. Let $Z$ be a standard Gaussian random variable. Under Hypothesis 1.1 and suppose $H \in (\frac{1}{2}, \frac{3}{4})$, there exists a constant $C_{H,\sigma} > 0$ such that when $n$ is large enough,

$$\sup_{z \in \mathbb{R}} \left| P \left\{ \frac{f'(\theta)\sqrt{n}(\hat{\theta}_n - \theta)}{\sigma_H} \leq z \right\} - P\{Z \leq z\} \right| \leq C_{H,\sigma} \varphi(n), \quad (1.6)$$

where $\sigma_H$ is given in Theorem 1.3 and

$$\varphi(n) = \begin{cases} \frac{1}{n^\frac{3}{2}}, & \text{if } H \in (\frac{1}{2}, \frac{3}{4}); \\ \frac{1}{n^\frac{1}{2}}, & \text{if } H \in [\frac{3}{4}, \frac{5}{4}). \end{cases}$$

Here $\frac{1}{n^\epsilon}$ means that $\frac{1}{n} - \epsilon$ for any $\epsilon > 0$.

Remark 1.7. We do not know how to improve the upper bound $1/n^\frac{3}{2}$ to $1/\sqrt{n}$ when $H \in (\frac{1}{2}, \frac{5}{4})$.

Next, we give some sequences that satisfy Hypothesis 1.1.

Example 1.8. The fractional Gaussian noise with covariance function

$$\rho(k) = \frac{1}{2} (|k + 1|^{2H} + |k - 1|^{2H} - 2|k|^{2H}), \quad k \in \mathbb{Z}$$

satisfies Hypothesis 1.1 when $H > \frac{1}{2}$. It is well-known that $\rho(k) \sim H(2H - 1)|k|^{2H - 2}$ as $k \to \infty$.

Example 1.9. The fractional ARIMA$(0,d,0)$ model driven by a Gaussian white noisesatisfies Hypothesis 1.1 when $d \in (0, \frac{1}{2})$. It is well-known that it’s the spectral density satisfies

$$h_\xi(\lambda) \sim \frac{1}{2\pi} |\lambda|^{-2d}, \quad \lambda \to 0$$
Example 1.10. $\xi_t$ is linear stationary sequence defined by

$$\xi_t = \sum_{k \leq t} b_t - k \epsilon_t, \quad t \in \mathbb{Z}.$$  

Here $\{\epsilon_t, t \in \mathbb{Z}\}$ is a Gaussian white noise, and the weights $\{b_t, t \in \mathbb{N}^+\}$ satisfy that $\sum b_t^2 < \infty$.

Assume that the weights decay slowly hyperbolically:

$$b_k = L_0(k)|k|^{-H-\frac{3}{2}},$$

where $H \in (\frac{1}{2}, 1)$, and $L_0(\cdot)$ is a slowly varying function. Then Hypothesis 1.1 is valid if we take $L(k) \propto L_0^2(k)$ (see [7]).

In the remainder of this paper, $C$ and $c$ will be a generic positive constant independent of $n$ the value of which may differ from line to line.

2. Preliminary

In this section, we list the main definitions and theorems that is used to show our results. The following two definitions are cited from Definition 1.1 and 1.2 of [6] respectively.

Definition 2.1. A positive function $L(x)$ defined for $x > x_0$ is called a slowly varying at infinity function in Zygmund’s sense if, for any $\delta > 0$, $p_1(x) = x^\delta L(x)$ is an increasing, and $p_2(x) = x^{-\delta} L(x)$ is a decreasing, function of $x$ for $x$ large enough. Similarly, $L$ is called slowly varying at the origin if $\tilde{L}(x) = L(x^{-1})$ is slowly varying at infinity.

It is known that if $L(x)$ is slowly varying at infinity then

$$\lim_{x \to \infty} \frac{L(ux)}{L(x)} = 1$$

for every fixed $u > 0$, and even uniformly in every interval $a \leq u \leq \frac{1}{a}$, $0 < a < 1$ [8, p.186].

Definition 2.2. Let $\{\xi_t\}$ be a second-order stationary process with autocovariance function $\rho(k)(k \in \mathbb{Z})$ and spectral density

$$h_\xi(\lambda) = (2\pi)^{-1} \sum_{k=-\infty}^{\infty} \rho(k) \exp(-ik\lambda), \quad \lambda \in [-\pi, \pi].$$

Then $\{\xi_t\}$ is said to exhibit linear long-range dependence, if

$$h_\xi(\lambda) = L_h(\lambda)|\lambda|^{1-2H},$$

where $L_h(\lambda) > 0$ is a symmetric function that is slowly varying at zero and $H \in (\frac{1}{2}, 1)$.

The following theorem is well-known and is cited from Theorem 1.3 of [6]:

Theorem 2.3. Let $R(k)(k \in \mathbb{Z})$ and $h(\lambda) (\lambda \in [\pi, \pi])$ be the autocovariance function and spectral density respectively of a second-order stationary process $\{\xi_t\}$. Then the following holds:
If
\[ R(k) = L_R(k)|k|^{2H-2}, \quad k \in \mathbb{Z}, \]
where \( L_R(k) \) is slowly varying at infinity in Zygmund’s sense, and \( H \in \left( \frac{1}{2}, 1 \right) \), then
\[ h(\lambda) \sim L_h(\lambda)|\lambda|^{1-2H}, \quad \lambda \to 0, \]
where
\[ L_h(\lambda) = L_R(\lambda^{-1}) \pi^{-1} \Gamma(2H-1) \sin(\pi - \pi H). \]

If
\[ h(\lambda) = L_h(\lambda)|\lambda|^{1-2H}, \quad 0 < \lambda < \pi, \]
where \( H \in \left( \frac{1}{2}, 1 \right) \), and \( L_h(\lambda) \) is slowly varying at the origin in Zygmund’s sense and of bounded variation on \((a, \pi)\) for any \( a > 0 \), then
\[ R(k) \sim L_R(k)|k|^{2H-2}, \quad k \to \infty, \]
where
\[ L_R(k) = 2L_h(k^{-1}) \Gamma(2-2H) \sin \left( \pi H - \frac{1}{2} \pi \right). \]

The following Breuer-Major theorem is well-known, for example, see [9] and [10]:

**Theorem 2.4** (Breuer-Major theorem). \( Y = (Y_t, t \in \mathbb{Z}) \) is a centered stationary Gaussian sequence. Set \( R(k) = E(Y_0 Y_k) \). Define
\[ V_n := \frac{1}{\sqrt{n}} \sum_{t=1}^{n} [Y_t^2 - E(Y_t^2)], \quad n \geq 1. \quad (2.2) \]

If
\[ \sigma_H^2 := 2 \sum_{k \in \mathbb{Z}} R^2(k) < \infty, \]
then
\[ V_n = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} [Y_t^2 - E(Y_t^2)] \xrightarrow{law} \mathcal{N}(0, \sigma_H^2), \quad as \ n \to \infty. \]

The following theorem is taken form Theorem 7.3.1 of [10], which is a corollary of Fourth Moment Berry-Esséen bound.

**Theorem 2.5.** Set \( Z \sim \mathcal{N}(0,1) \) and let \( R(k) \), \( V_n \) be given in Theorem 2.4. Set \( v_n^2 = E(V_n^2) \). Then, for all \( n \geq 1, \)
\[ d_{TV}(V_n/v_n, Z) \leq \frac{4\sqrt{2}}{v_n^2 \sqrt{n}} \left( \sum_{k=-n+1}^{n-1} |R(k)| \frac{1}{2} \right)^{\frac{3}{2}}. \]

The following theorem is rephrased from Theorem 5.1 and Proposition 5.2 of [11], which gives a sufficient condition to the almost sure central limit theorem of \( \tilde{V}_n = \frac{V_n}{v_n} \).

**Theorem 2.6.** Let \( V_n \) be given by (2.2) and \( Z \sim \mathcal{N}(0,1) \). Set \( v_n^2 = E(V_n^2) \). Assume that \( R(k) \sim |k|^{-\beta} L(|k|) \), as \( |k| \to \infty \), for some \( \beta > \frac{1}{2} \) and some slowly varying function \( L(\cdot) \). Then \( \left\{ \frac{V_n}{v_n} \right\} \) satisfies an almost sure central limit theorem.
3. Proofs of the Strong Consistency, The Asymptotic Normality and ASCLT

**Lemma 3.1.** Let \((Y_t)_{t \in \mathbb{N}}\) and \(\zeta\) be given by (1.3). Then for all \(\varepsilon > 0\) there exists a random variable \(C_\varepsilon\) such that
\[
\left| \zeta \sum_{t=1}^{n} \theta^t Y_t \right| \leq C_\varepsilon n^\varepsilon \quad \text{a.s.} \tag{3.1}
\]
for all \(n \in \mathbb{N}\), and moreover, \(E|C_\varepsilon|^p < \infty\) for all \(p \geq 1\).

**Proof.** Fix \(p \geq 1\) and denote by \(\| \cdot \|_p\) the \(L^p\)-norm. Since \(\sum_{t=1}^{n} \theta^t Y_t\) is Gaussian, we have
\[
\left\| \sum_{t=1}^{n} \theta^t Y_t \right\|_p \leq c_2 \left\| \sum_{t=1}^{n} \theta^t Y_t \right\|_2 = c_2 \left( \sum_{i,j=1}^{n} \theta^{i+j} E[Y_i Y_j] \right) \leq C_\theta,
\]
for all \(n \in \mathbb{N}\). The Hölder equality implies that
\[
\left\| \sum_{t=1}^{n} \theta^t Y_t \right\|_p \leq \left( \sum_{t=1}^{n} \theta^t Y_t \right)^{\frac{p}{2}} \leq c_1 \left( \sum_{t=1}^{n} \theta^t Y_t \right) \leq C_\theta, \tag{3.2}
\]
which implies (3.1) from Lemma 2.1 of [12]. In fact, it is easy to check that the conclusion of Lemma 2.1 of [12] is valid if its assumption \(\alpha > 0\) is changed to \(\alpha \geq 0\). \(\square\)

**Proof of Theorem 1.2.** It is obvious that the covariance function of \(Y_t\) is
\[
R(k) = \text{Cov}(Y_t, Y_{t+k}) = \sum_{i,j=0}^{\infty} \theta^{i+j} \rho(k - i + j).
\]
Since \(\rho(k) \to 0\) as \(k \to \infty\) and when \(0 < \theta < 1\),
\[
\sum_{i,j=0}^{\infty} \theta^{i+j} \rho(k - i + j) \leq \sum_{i,j=0}^{\infty} \theta^{i+j} < \infty,
\]
the dominated convergence theorem implies that
\[
\lim_{k \to \infty} R(k) = 0.
\]
Hence, the stationary Gaussian sequence \((Y_t)\) is ergodic. Since \(EY_t^2 = f(\theta)\), we obtain
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} Y_t^2 = f(\theta) \quad \text{a.s.}
\]
Lemma 3.1 implies that for \(0 < \varepsilon < 1\) there exists a random variable \(C_\varepsilon\) such that
\[
\left| \frac{1}{n} \zeta \sum_{t=1}^{n} \theta^t Y_t \right| \leq C_\varepsilon n^{-1+\varepsilon} \quad \text{a.s.}
\]
Hence, as \(n \to \infty\),
\[
\frac{1}{n} \zeta \sum_{t=1}^{n} \theta^t Y_t \to 0
\]
almost surely. Thus,
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} X_t^2 = f(\theta) \quad \text{a.s.}, \tag{3.3}
\]
which, together with the continuous mapping theorem, implies that as \( n \to \infty \),
\[
\tilde{\theta}_n \to \theta, \quad \text{a.s.}
\]
\[\Box\]

**Remark 3.2.** Since \( f(\theta), \theta \in (0, 1) \) is a strictly increasing function, the limit (3.3) implies that when \( n \) is large enough, \( \frac{1}{n} \sum_{t=1}^{n} X_t^2 \) belongs to the range of \( f(\theta) \). Hence, when \( n \) is large enough,
\[
(0, 1) \ni \tilde{\theta}_n = f^{-1}(\frac{1}{n} \sum_{t=1}^{n} X_t^2), \quad \text{a.s.}
\]

**Lemma 3.3.** Under Hypothesis 1.1, the stationary solution \( Y_t \) to the model (1.1) exhibits linear long-range dependence. Namely,
\[
h_Y(\lambda) \sim C_{\theta,H} L(\lambda^{-1}) \vert\lambda\vert^{1-2H}, \quad \text{as } \lambda \to 0,
\]
where the constant \( C_{\theta,H} > 0 \) depends on \( \theta \) and \( H \).

**Proof.** The spectral density of the stationary solution \( Y_t \) to the model (1.1) satisfies
\[
h_Y(\lambda) = \vert 1 - \theta e^{-i\lambda} \vert^{-2} h_\xi(\lambda).
\]
Under Hypothesis 1.1, Theorem 2.3(1) implies that as \( \lambda \to 0 \),
\[
h_\xi(\lambda) \sim C_H L(\lambda^{-1}) \vert\lambda\vert^{1-2H}.
\]
Hence, we obtain that as \( \lambda \to 0 \),
\[
h_Y(\lambda) \sim (1 - \theta)^{-2} h_\xi(\lambda) \sim C_{\theta,H} L(\lambda^{-1}) \vert\lambda\vert^{1-2H}.
\]
\[\Box\]

**Lemma 3.4.** Let \( Y_t \) be the stationary solution to the model (1.1) and \( R(k) = \mathbb{E}(Y_k Y_0) \). Let \( V_n \) be given in Theorem 2.4 and \( v_n^2 = \mathbb{E}(V_n^2) \). When \( H \in (\frac{1}{2}, \frac{3}{4}) \),
\[
\lim_{n \to \infty} v_n^2 = 2 \sum_{k \in \mathbb{Z}} R^2(k) < \infty. \tag{3.6}
\]

**Proof.** It is well known the following two identities hold:
\[
v_n^2 = \frac{2}{n} \sum_{k,l=1}^{n} R^2(k-l) = 2 \sum_{|k| < n} \left( 1 - \frac{|k|}{n} \right) R^2(k),
\]
\[
\lim_{n \to \infty} v_n^2 = 2 \sum_{k \in \mathbb{Z}} R^2(k), \quad \text{if } \sum_{k \in \mathbb{Z}} R^2(k) < \infty, \tag{3.7}
\]
see, for example, (7.2.6) of [10].

Thus, to check (3.6) for \( H \in (\frac{1}{2}, \frac{3}{4}) \), we need only to show the condition \( \sum_{k \in \mathbb{Z}} R^2(k) < \infty \) holds. In fact, Theorem 2.3 (2) and the identity (3.5) imply that the covariance function of \( Y_t \) satisfies that when \( \frac{1}{2} < H < 1 \),
\[
R(k) \sim C_{\theta,H} L(k) |k|^{2H-2}, \quad \text{as } k \to \infty. \tag{3.8}
\]
Recall that $L(k) < c |k|^\delta$ for any fixed $\delta > 0$ and $k$ large enough (see, for example, [13, p.277]).

Hence,

$$\sum_{k \in \mathbb{Z}} R^2(k) < \infty$$

if and only if

$$4H - 4 + 2\delta < -1.$$ 

Note that $\delta > 0$ is arbitrary, we obtain that when $1/2 < H < 3/4$, the condition $\sum_{k \in \mathbb{Z}} R^2(k) < \infty$ holds.

\[ \square \]

**Proof of Theorem 1.3.** Using the identity (1.3), we have that

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} (X_t^2 - f(\theta)) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (Y_t^2 - f(\theta)) + \frac{2\zeta}{\sqrt{n}} \sum_{t=1}^{n} \theta^t Y_t + \frac{\zeta^2}{\sqrt{n}} \sum_{t=1}^{n} \theta^{2t}. \tag{3.9}$$

Theorem 2.4 and Lemma 3.4 imply that when $1/2 < H < 3/4$, as $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} (Y_t^2 - f(\theta)) \overset{law}{\to} N(0, \sigma_H^2).$$

Lemma 3.1 implies that as $n \to \infty$,

$$\left| \frac{\zeta}{\sqrt{n}} \sum_{t=1}^{n} \theta^t Y_t \right| \leq C_\epsilon n^{\epsilon - 1/2} \to 0. \tag{3.10}$$

It is clear that as $n \to \infty$,

$$\frac{\zeta^2}{\sqrt{n}} \sum_{t=1}^{n} \theta^{2t} \to 0. \tag{3.11}$$

Substituting the above three limits into (3.9), we deduce from Slutsky’s theorem that as $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} (X_t^2 - f(\theta)) \overset{law}{\to} N(0, \sigma_H^2),$$

which implies the desired (1.5) from the delta method. \[ \square \]

**Proof of Theorem 1.5.** The ASCLT can be obtained by arguments similar to those of Theorem 4.3 in [14].

First, Eq. (3.8) and Theorem 2.6 imply that when $1/2 < H < 3/4$,

$$\left\{ \frac{V_n}{v_n} : n \geq 1 \right\}$$

satisfies the ASCLT.

Second, by (3.7), (3.9)-(3.11), we have that

$$\frac{1}{\sqrt{n} \sigma_H} \sum_{i=1}^{n} (X_i^2 - f(\theta)) = \frac{V_n}{v_n} \frac{v_n}{\sigma_H} + \frac{2\zeta}{\sqrt{n} \sigma_H} \sum_{i=1}^{n} \theta^t Y_t + \frac{\zeta^2}{\sqrt{n} \sigma_H} \sum_{i=1}^{n} \theta^{2t},$$

which, together with Theorems 3.1 and 3.2 of [14], implies that

$$\left\{ \frac{1}{\sqrt{n} \sigma_H} \sum_{i=1}^{n} (X_i^2 - f(\theta)) : n \geq 1 \right\}$$
satisfies the ASCL.

Third, the mean value theorem implies that

\[
\frac{f'(\theta)\sqrt{n}(\hat{\theta}_n - \theta)}{\sigma_H} = \frac{f'(\theta)}{f'(\eta_n)} \sqrt{n} \sum_{t=1}^{n} X_t^2 - f(\theta) = f'() \sqrt{n} \sum_{t=1}^{n} (X_t^2 - f(\theta))
\]

where \( \eta_n \) is a random variable between \( \frac{1}{n} \sum_{t=1}^{n} X_t^2 \) and \( f(\theta) \). The convergence (3.3) leads to

\[
\frac{f'() \sqrt{n}(\hat{\theta}_n - \theta)}{\sigma_H} : n \geq 1
\]

satisfies the ASCL.

\[\square\]

4. The Berry-Esséen Bound

The following Fourth Moment Berry-Esséen bound is similar to Corollary 7.4.3 of [10].

**Proposition 4.1.** Let \( V_n \) be given in Theorem 2.4 and \( v_n^2 = E(V_n^2) \). When \( H \in (\frac{1}{2}, \frac{3}{4}) \), there exists a constant \( c_H > 0 \) such that, for all \( n \geq 2 \):

\[
d_{TV}(V_n/v_n, Z) \leq c_H \varphi_1(n),
\]

where

\[
\varphi_1(n) = \begin{cases} 
  n^{\frac{1}{2}}, & \text{if } H \in (\frac{1}{2}, \frac{5}{8}) ; \\
  n^{-\frac{1}{2}}, & \text{if } H \in [\frac{5}{8}, \frac{3}{4}) .
\end{cases}
\]

**Remark 4.2.** When the function slowly varying at infinity \( L(k) \) degenerates to a positive constant as \( k \) large enough, we can improve the upper bound \( \frac{1}{n^{\frac{3}{2}H}} \) to \( \frac{1}{n^{\frac{3}{2}H}} \) in the case of \( H \in (\frac{5}{8}, \frac{3}{4}) \).

**Proof.** Since \( L(k) < c|k|^\delta \) for any fixed \( \delta > 0 \) and \( k \) large enough, we have that

\[
|R(k)|^\frac{1}{4} \leq C|k|^\frac{1}{4}((2H - 2 + \delta).)
\]

When \( H \in (\frac{1}{2}, \frac{5}{8}) \), we can take \( \delta > 0 \) small enough such that \( \frac{1}{4}((2H - 2 + \delta) < -1 \) and the following series converges,

\[
\sum_{k \in \mathbb{Z}} |R(k)|^\frac{1}{4} < \infty.
\]

Together with the limit (3.6), we obtain form Theorem 2.5 that the desired bound (4.1) holds.

When \( 5/8 \leq H < 3/4 \), the inequality (4.2) implies that

\[
\left( \sum_{k=-n+1}^{n-1} |R(k)|^\frac{1}{4} \right)^4 \leq Cn^{4H - \frac{1}{2} + 25}.
\]

Again by (3.6) and Theorem 2.5, we have the desired bound (4.1) since \( \delta > 0 \) is arbitrary. \( \square \)

The following result is Lemma 2 of [15].
Lemma 4.3. For any random variable $\xi, \eta$ and real constant $a > 0$,
\[ \sup_{u \in \mathbb{R}} |P(\xi + \eta \leq u) - \Phi(u)| \leq \sup_{u \in \mathbb{R}} |P(\xi \leq u) - \Phi(u)| + P(|\eta| > a) + \frac{a}{\sqrt{2\pi}}, \] (4.3)
where $\Phi(u)$ stands for the standard normal distribution function.

Proof of Theorem 1.6. The Berry-Esseen bound (1.6) can be obtained by arguments similar to those of Theorem 3.2 in [16]. Denote
\[ A := P \left\{ \frac{f'(\theta)\sqrt{n}(\hat{\theta}_n - \theta)}{\sigma_H} \leq z \right\} - P\{Z \leq z\}. \]
Since $\hat{\theta}_n \in (0, 1)$ almost surely (see Remark 3.2), we shall suppose that $z \in D$:
\[ D := \left\{ z : -\frac{f'(\theta)\sqrt{n}}{\sigma_H} \theta < z < \frac{f'(\theta)\sqrt{n}}{\sigma_H} \left( \theta \wedge (1 - \theta) \right) \right\}. \] (4.4)
Otherwise, the upper-tail inequality for standard normal distribution
\[ P(Z \geq t) \leq e^{-\frac{t^2}{2}}, \quad t > 0 \]
yields
\[ |A| = P\{Z > |z|\} \leq \frac{C}{\sqrt{n}}. \]
Since $f(\theta)$ is strictly increasing and continuous, we have by (1.4) for the formula of $\hat{\theta}_n$
\[ A = P \left\{ \frac{f'(\theta)\sqrt{n}(\hat{\theta}_n - \theta)}{\sigma_H} \leq z \right\} - P\{Z \leq z\} \]
\[ = P \left\{ \hat{\theta}_n \leq \theta + \frac{\sigma_H}{f'(\theta)\sqrt{n}} z \right\} - P\{Z \leq z\} \]
\[ = P \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 \leq f \left( \theta + \frac{\sigma_H}{f'(\theta)\sqrt{n}} z \right) \right\} - P\{Z \leq z\} \]
\[ = P \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 - f(\theta) \leq f \left( \theta + \frac{\sigma_H}{f'(\theta)\sqrt{n}} z \right) - f(\theta) \right\} - P\{Z \leq z\}. \]
We take the short-hand notation
\[ u(z) = \frac{\sqrt{n}}{\sigma_H} \left[ f \left( \theta + \frac{\sigma_H}{f'(\theta)\sqrt{n}} z \right) - f(\theta) \right] \] (4.5)
and
\[ w = \frac{2\zeta}{\sigma_H\sqrt{n}} \sum_{i=1}^{n} \theta^i Y_i + \frac{\kappa^2}{\sigma_H\sqrt{n}} \sum_{i=1}^{n} \theta^{2i}. \]
By (1.3) the relationship between $X_i$ and $Y_i$ and (2.2) the formula of $V_n$, we have
\[ |A| = \left| P \left\{ \frac{V_n}{\sigma_H} + w \leq u \right\} - P\{Z \leq z\} \right| \]
\[ \leq \left| P \left\{ \frac{V_n}{\sigma_H} + w \leq u - \Phi(u) \right\} + \Phi(u) - P\{Z \leq z\} \right|. \]
The second term is bounded by $Cn^{-\frac{1}{2}}$ from Lemma 4.4 below. Hence, we need only show that
\[
\sup_{u \in \mathbb{R}} \left| P \left\{ \frac{V_n}{\sigma_H} + w \leq u \right\} - \Phi(u) \right| \leq C\varphi(n). \tag{4.6}
\]
In fact, Lemma 4.3 implies that for any $\gamma > 0$,
\[
\sup_{u \in \mathbb{R}} \left| P \left\{ \frac{V_n}{\sigma_H} + w \leq u \right\} - \Phi(u) \right| \leq \sup_{u \in \mathbb{R}} \left| P \left\{ \frac{V_n}{\sigma_H} \leq u \right\} - \Phi(u) \right| + P \{|w| > n^{-\gamma}\} + \frac{n^{-\gamma}}{\sqrt{2\pi}}. \tag{4.7}
\]
By Proposition 4.1, the Fourth moment Berry–Esséen bound, we have that
\[
\sup_{u \in \mathbb{R}} \left| P \left\{ \frac{V_n}{\sigma_H} \leq u \right\} - \Phi(u) \right| \leq C\varphi_1(n).
\]
Chebyshev’s inequality implies that
\[
P \{|w| > n^{-\gamma}\} \leq n^p |\mathbb{E}|w|^p|.
\]
The triangular inequality and the inequality (3.2) imply that
\[
\|w\|_p \leq \frac{2}{\sigma_H \sqrt{n}} \left\| \frac{n}{\sqrt{n}} \sum_{t=1}^{n} \theta^t Y_t \right\|_p + \frac{1}{\sigma_H \sqrt{n}} \left\| \sqrt{n} \sum_{t=1}^{n} \theta^t \right\|_p \leq C/\sqrt{n},
\]
from which we have that
\[
P \{|w| > n^{-\gamma}\} \leq Cn^{p(\gamma - \frac{1}{2})}.
\]
We take $\gamma < \frac{1}{2}$, $p$ large enough, and substitute the above inequalities into (4.7) to get (4.6). \qed

**Lemma 4.4.** Let $u(z)$ be given by (4.5) and the interval $D$ given by (4.4). Then there exists some positive number $C_{\theta, H}$ independent of $n$ such that
\[
\sup_{z \in D} |\Phi(u(z)) - \Phi(z)| \leq \frac{C_{\theta, H}}{\sqrt{n}}.
\]

**Proof.** We follow the line of the proof of Theorem 3.2 in [16]. By the mean value theorem, there exists some number $\eta \in (\theta, \theta + \frac{\sigma_H}{f'(\theta)\sqrt{n}}z)$ when $z > 0$ or $\eta \in (\theta + \frac{n \sigma_H}{f'(\theta)\sqrt{n}}z, \theta)$ when $z < 0$ such that
\[
u(z) = \frac{\sqrt{n}}{\sigma_H} \left[ f \left( \theta + \frac{\sigma_H}{f'(\theta)\sqrt{n}}z \right) - f(\theta) \right] = \frac{f'(\eta)}{f'(\theta)} z.
\]
Since $f(\theta)$ is convex, we have for any $z$, $\nu(z) \geq 0$. Hence,
\[
|\Phi(u) - \Phi(z)| = \frac{1}{\sqrt{2\pi}} \int_{z}^{\nu(u)} e^{-\frac{t^2}{2}} dt.
\]
Since the function
\[
f(x, z) = z^2 e^{-\frac{z^2}{2}} |x - 1|
\]
is uniformly bounded, we have when $z < 0$,
\[
\frac{1}{\sqrt{2\pi}} \int_{z}^{\nu(u)} e^{-\frac{t^2}{2}} dt \leq \frac{1}{\sqrt{2\pi}} |z| e^{-\frac{z^2}{2}} \left[ \frac{f'(\eta)}{f'(\theta)} \right]^2 |f'(\eta) - 1| \leq \frac{C_{\theta, H}}{|z|}.
\]
Thus, we obtain that
\[
\sup_{\frac{f'(\theta)\sqrt{n}}{2\sigma_H} \theta < z \leq \frac{f'(\theta)\sqrt{n}}{2\sigma_H} \theta} |\Phi(u) - \Phi(z)| \leq \frac{C_{\theta,H}}{\sqrt{n}}.
\]
When \(-\frac{f'(\theta)\sqrt{n}}{2\sigma_H} < z < \frac{f'(\theta)\sqrt{n}}{2\sigma_H} \theta \land (1 - \theta)\), using the mean value theorem and making the change of variable \(t = z^2s\), together with the fact that \(f_2(s, z) = z^2e^{-\frac{z^4}{2s}}\) is also uniformly bounded, we conclude that there exists a number \(\eta_1 \in (\theta, \eta) \subset [\theta, \theta + \frac{1}{2}\theta \land (1 - \theta)]\) such that
\[
\int_z^{f'(\theta)\sqrt{n}} e^{-\frac{t^2}{2s}} dt = \int_z^{f'(\theta)\sqrt{n}} z^2 e^{-\frac{z^4}{2s}} ds \\
\leq C_{\theta,H} \frac{1}{|z|} |f'(\eta) - f'(\theta)| \\
\leq C_{\theta,H} \frac{1}{|z|} |f''(\eta_1)| \frac{|z|}{\sqrt{n}} \\
\leq C_{\theta,H} \frac{1}{\sqrt{n}},
\]
since the second derivative function \(f''\) is bound on the close interval \([\theta, \theta + \frac{1}{2}\theta \land (1 - \theta)]\). When \(z \geq \frac{f'(\theta)\sqrt{n}}{2\sigma_H} \theta \land (1 - \theta)\),
\[
\frac{1}{\sqrt{2\pi}} \int_z^{f'(\theta)\sqrt{n}} e^{-\frac{t^2}{2s}} dt \leq \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-\frac{t^2}{2s}} dt \leq \frac{C_{\theta,H}}{\sqrt{n}}.
\]

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