An Efficient Polyphase Filter Based Resampling Method for Unifying the PRFs in SAR Data

Yoangel Torres, Student Member, IEEE,
Kamal Premaratne, Senior Member, IEEE, Falk Amelung,
and Shimon Wdowinski

Abstract

As current airborne and spaceborne synthetic aperture radar (SAR) systems aim to produce higher resolution and wider area products, their associated complexities call for handling stricter requirements. Variable and higher pulse repetition frequencies (PRFs) are increasingly being used to achieve these demanding requirements in modern radar systems. This paper presents a resampling scheme capable of unifying and downsampling variable PRFs within a single look complex (SLC) SAR acquisition and across a repeat pass sequence of acquisitions down to an effective lower PRF through the use of polyphase filters. To evaluate the performance of this resampling scheme, we use airborne SAR raw data with variable PRFs. The data were processed with and without the proposed resampling method as part of the flow of the imaging algorithm. Significant improvement in the point spread function (PSF) measurement and the visible image quality after rate conversion and normalization justify the theoretical basis of the proposed method and the benefits it can provide in application scenarios.
Index Terms

Synthetic aperture radar (SAR), interferometric synthetic aperture radar (InSAR), pulse repetition frequency (PRF), polyphase implementation.

I. INTRODUCTION

Synthetic aperture radar (SAR) and interferometric SAR (InSAR) have long been employed for geophysical and environmental remote sensing applications. However, as sensor technology advances, the complexity of how the data is collected also increases. Variations in pulse repetition frequency (PRF) are just one of these complexities introduced by modern sensors.

Some methods have been presented that make use of displaced phase centers (DPCs) in order to recover the unambiguous Doppler spectrum from non-uniform spatial sampling of the synthetic aperture \cite{1}–\cite{3}. In \cite{4} an innovative frequency domain algorithm is proposed which enables the unambiguous recovery of the Doppler spectrum in the case of a single channel. Not much attention appears to have been paid for the case of single channel non-uniform reconstruction in time domain. An exception is the work in \cite{5} which presents a computationally efficient technique to handle single channel non-uniform oversampled SAR data generated from a platform accelerating along-track by resampling the data in the slow-time domain. However, the method in \cite{5} assumes a constant PRF as the platform moves along track; the spatial non-uniformity is solely due to small changes in velocity arising from uncontrolled acceleration of the platform. This nonuniformity in along-track spatial sampling is typically much smaller than what could be generated from variable PRFs.

The approach being proposed in this paper resolves the issue of nonuniformity along-track arising from different PRFs that occur within a single look complex (SLC) SAR collection as well. The objective of the paper is to take in demodulated SAR data for different acquisitions, which are collected and oversampled at variable PRFs, and deliver resampled data at a lower, constant PRF within each acquisition, and uniformly sampled in the spatial frequency domain ($k$-space) \cite{6}, \cite{7}. The implementation of the proposed method is carried out in-place; by employing a polyphase implementation, the digital filtering operations are carried out at the lowest possible rate, viz., the effective output PRF rate. This enables real-time on-board processing and cuts down on down-link data, thus reducing bottlenecks. The proposed method approximately reconstructs the collected data on a uniformly spaced grid along the synthetic aperture, while preserving the
resolution and Nyquist constraint within the cross-range extent of interest. The same method can of course be employed to address the issue dealt with in [5] (viz., compensating for the effects of uncontrolled platform acceleration) which have been verified on simulated data. We use real SAR data to verify our proposed method.

The paper is structured as follows: Section II provides a brief summary of the technical background. Section III provides the main idea about our proposed resampling scheme; a more detailed description of the various stages appears in Sections IV and V. The results generated from the application of our method and a discussion of these results appear in Sections VI and VII respectively. Finally, concluding remarks appear in Section VIII. For convenience of reference, the notation used throughout the paper is summarized in Table I.

II. TECHNICAL BACKGROUND

A. Preliminaries

As in [8]–[10], we use a Cartesian coordinate system where the origin is at the center of the scene being imaged, $x$-axis is along-track and parallel to the SAR platform velocity vector $V_p$ (m/s), $y$-axis is along boresight, and $z$-axis denotes the altitude.

The SAR sensor acquires its information by transmitting pulses which get convolved with the scene over an angular region inversely proportional to the required azimuth resolution. The time interval it takes to cover this region is the coherent aperture time interval. Slow-time temporal domain quantities, when normalized by the platform velocity $V_p$, yield their corresponding spatial domain quantities [11]. The spatial quantity corresponding to the coherent aperture time interval is the synthetic aperture length $D$ (m).

**Resolutions in Range and Azimuth.** In spotlight mode SAR,

$$\delta_r = \frac{c}{2} \left( K_r / BW \right); \quad \delta_{cr} = \frac{\lambda_c}{2} \left( K_{cr} / \Delta \beta \right),$$

(1)

express the resolutions in range and azimuth, respectively [8]–[10]. Here, $K_r$ and $K_{cr}$ are broadening factors associated with pulse weighting in range and aperture weighting in azimuth, respectively, for sidelobe suppression; $\lambda_c$ is the center radar transmit wavelength (m); $BW$ is the transmit chirp bandwidth (Hz); and $\Delta \beta$ is the SAR integration angle (rad).

**PRF and Sampling.** In spotlight mode SAR, the Doppler bandwidth generated by scatterers
## TABLE I

### Notation

| Notation       | Description                                                                                                                                 |
|----------------|-------------------------------------------------------------------------------------------------------------------------------------------|
| \( R, N \)     | Real numbers and integers, respectively.                                                                                                    |
| \( c, V_p \)   | Speed of light (m/s) and radar platform velocity (m/s), respectively.                                                                            |
| \( \lambda_{\text{min}}, \lambda_c, \lambda_{\text{max}}, BW \) | Minimum, center, and maximum radar transmit wavelength (m) and transmit chirp bandwidth (Hz), respectively.                                      |
| \( D, D_{pad}, R \) | Synthetic aperture length (m), padded synthetic aperture length (m), and slant range between the radar and the scene (m), respectively. |
| \( \beta, \Delta \beta \) | Squint angle (rad) and SAR angle (rad), respectively.                                                                                           |
| \( \theta_{\text{min}}, \theta_{\text{max}} \) | Minimum and maximum look angles, respectively, seen by the synthetic aperture (rad).                                                            |
| \( t, f, \nu \) | Temporal dimension (s), temporal frequency (Hz), and its rotational counterpart (rad/s) respectively. \( \nu = 2\pi f \).                           |
| \( u, h, \omega \) | Spatial dimension (m), spatial frequency (1/m), and its rotational counterpart (rad/m), respectively. \( \omega = 2\pi h \).                       |
| \( \omega_r, \omega_{cr} \) | Wavenumber in range (rad/m) and cross-range (rad/m), respectively. \( \omega_r = \nu_r/c, \omega_{cr} = \nu_{cr}/V_p \).                   |
| \( \delta_r, \delta_{cr} \) | Resolution in range and azimuth, respectively.                                                                                                  |
| \( K_r, K_{cr} \) | Broadening factor associated with pulse weighting in range and aperture weighting in azimuth (or cross-range), respectively.                  |
| \((x, y, z)\)   | (cross-range, range, height) coordinates of the Cartesian coordinate system (m); scene center is at \((0, 0, 0)\).                               |
| \((X_{\text{out}}, Y_{\text{out}})\) | Azimuth and range dimensions, respectively, of the imaged scene (m).                                                                            |
| \((X_n, Y_n, Z_n)\) | Coordinates of the \(n\)-th point scatterer within the imaged scene (m).                                                                       |
| \( u, u_k, u_{\text{min}}, u_{\text{mid}}, u_{\text{max}} \) | Continuous-time \(x\)-coordinate, discrete-time \(x\)-coordinate, minimum, mid, and maximum \(x\)-coordinates, respectively, of the synthetic aperture (m). |
| \( \Delta u_{\text{in}}, \Delta u_{\text{out}} \) | Along-track sample and resample spacings at boresight (m), respectively.                                                                         |
| \( N_{\text{FFT}} \) | FFT size used for spatial FFT computation (after zero-padding).                                                                                 |
| \( N_{\text{XAC}}, N_D \) | Number of discrete points with and without boundary regions, respectively, needed to represent the synthetic aperture length without aliasing. |
| \( p_d \) | Spatial frequency spectrum oversampling factor.                                                                                                 |
| \( \text{PRF}_{\text{in}}, \text{PRF}_{\text{out}} \) | Input and output pulse repetition frequencies, respectively (1/m).                                                                           |

At a slant range \( R \) and extending over a cross-range extent \( X_{\text{in}} \) is \[8\]

\[
\Delta f_{cr} = (2/\lambda_c) (X_{\text{in}}/R) V_p \sin \beta,
\]

where \( \beta \) is the squint angle (rad). Thus, unambiguous recovery of the cross-range extent \( X_{\text{in}} \) at boresight (where \( \beta = \pi/2 \)) requires a minimum temporal PRF of \((2/\lambda_c)(V_p/R)X_{\text{in}} (1/s)\). Normalizing by \( V_p \) \[11\], and using \( \Delta u_{\text{in}} \) to denote the along-track sample spacing (m), we see that the spatial domain PRF must satisfy

\[
\text{PRF}_{\text{in}} = 1/\Delta u_{\text{in}} \geq (2/\lambda_c) (X_{\text{in}}/R).
\]
B. Variable PRFs in SAR Acquisitions

Conventional radar system operation relies on a constant PRF, say, \( PRF_0 \). However, as technology has advanced, it has allowed for newer radar modes of operation to be formulated.

For example, the innovative multi-channel SAR allows for high-resolution, wide-swath imaging, while enabling a shorter revisit time for frequent global mapping. In wide-swath imaging, the antenna length limitation which typically restricts the achievable swath width, is overcome by a novel technique which is based on a single azimuth channel with the system operating with a continuously varied PRF \[12\]. In principle, this allows for arbitrary wide swaths and distributes the discrete blind ranges according to the applied PRF span of values. In the end, continuous coverage is achieved at the cost of partial blockage (i.e., loss of some pulses for every target). This PRF variation manifests itself as nonuniform sampling of the slow-time domain along the synthetic aperture \[12\], thus requiring additional processing. For example, one may have to apply interpolation techniques to resample the signal to a regular azimuth grid \[12\].

Another method that exploits the advantages offered by high-resolution, ultra-wide swath SAR imaging is multiple elevation beam (MEB) SAR based on variable PRF \[13\]. This technique employs digital beamforming (DBF) with a reflector antenna to effectively improve SNR and suppress the range ambiguities. It also uses the notion of linear variation of the pulse repetition interval (PRI) to effectively overcome the blind range problem of conventional MEB SAR \[14\].

Both these new techniques of high-resolution, wide-swath imaging modes \[12\]–\[14\] come at the cost of nonuniform sampling of the slow-time along-track Doppler phase. Spatial DFT processing of such nonuniformly spaced data can introduce undesirable artifacts (such as, smearing, defocusing, and echoing) into the final image (e.g., see Figs 6(a) and (b)).

III. Proposed Resampling Scheme

SAR imagery calls for coherent processing of two-dimensional sampled radar data. In spotlight mode SAR, one can take the underlying complex-valued signal of interest to be modeled as \[11\]

\[
\begin{align*}
    s(\nu_r, u) &= S_p(\nu_r) \sum_N \sigma_n e^{(-j2\omega_t \sqrt{(X_n-u)^2+(Y_n-Y_c)^2+(Z_n-Z_c)^2})}.
\end{align*}
\]

Here, \( \nu_r \) and \( \omega_t \) are the rotational frequency (rad/s) and wave number (rad/m) in the range dimension, respectively; \( S_p(\nu_r) \) is the discrete-time (DT) Fourier transform (FT) of the transmitted signal \( p(t) \) (in the range dimension); and \( \sigma_n \) and \( (X_n, Y_n, Z_n) \) are the reflectivity and
coordinates of the $n$-th scatterer in the scene, respectively. The instantaneous coordinates of the radar are taken as $(u, Y_c, Z_c)$, where $u \in [u_{\text{min}}, u_{\text{max}}]$ is the along-track platform position (m) and, for simplicity of the model, the ground range and altitude coordinates $(Y_c, Z_c)$ are taken to be constant.

A. Uniformly Sampled Radar Signal

Consider uniformly sampling the radar signal $s(v_r, u)$ in the interval $[u_{\text{min}}, u_{\text{max}}]$ with a spatial PRF of $PRF_{\text{in}}$ (1/m). The DT signal thus generated is

$$s'(k) = s(v_r, u_k), \quad \text{where} \quad u_{k+1} - u_k = 1/PRF_{\text{in}},$$

for $k \in 0, K - 1$, with $u_{\text{min}} \leq u_0 \leq \cdots \leq u_{K-1} \leq u_{\text{max}}$. See Fig. 1.

From (5), it is clear that, unambiguous recovery of a cross-range extent of $X_{\text{in}} = R \Delta \beta$, where $\Delta \beta$ is the SAR angle (rad), calls for $PRF_{\text{in}} \geq PRF_{\text{in,min}} \equiv (2/\lambda_c) \Delta \beta$. In spotlight mode SAR, $R \Delta \beta$ denotes the maximum cross-range extent illuminated by the radar beam; in strip mode SAR, it is denoted by $X_{\text{in}} = R \psi$ where $\psi$ is the real aperture beamwidth. Fig. 2(a) shows the frequency response $S_{s'}(\omega_{cr})$ of the signal $s'(u_k)$ when $PRF_{\text{in}} = PRF_{\text{in,min}}$. Note that, to be consistent with the notation employed in digital filter design literature [15], [16] and also for convenience, in Fig. 2 and then on, we will use $\omega$ in place of $\omega_{cr}$.

(a) A smaller PRF $PRF_{\text{out}} < PRF_{\text{in,min}}$ causes aliasing, and the cross-range extent that can
Fig. 2. Spatial Doppler bandwidths of: (a) $s'(k)$ which is at a PRF of $PRF_{in} = PRF_{in.min} = (2/\lambda c) \Delta \beta$ (1/m); (b) $s(n)$ which is at a PRF of $L \cdot PRF_{out}$ (1/m); (c) $r(k)$ which is at a PRF of $L \cdot PRF_{out}$ (1/m); (d) $v(k)$ which is at a PRF of $L \cdot PRF_{out}$ (1/m); (e) $y(k)$ which is at a PRF of $PRF_{out}$ (1/m). The digital filter $F(z)$ has its passband and stopband edges at $\gamma \pi/L$, $\gamma < 1$, and $\pi/L$, respectively; $\rho = PRF_{in}/PRF_{out} \geq 1$ denotes the ratio between the input and output PRFs; $N_{save}$ denotes the number of bins guaranteed to be retained in the final image without distortion, and $\gamma = N_{save}/N_{FFT} < 1$. In (e), the axes corresponding to the spatial frequency values (with respect to the PRF $PRF_{out}$) and the FFT size $N_{FFT}$ are also shown.

be unambiguously recovered is reduced:

$$X_{out} = (\lambda c/2) R \cdot PRF_{out} < R \Delta \beta.$$  \hfill (6)

(b) Uniformly sampling $s(\nu_r, u)$ with a much higher PRF $L \cdot PRF_{out}$, $L \in \mathbb{N}_+$, $L \cdot PRF_{out} >> PRF_{in.min} = (2/\lambda c) \Delta \beta$, yields the DT signal

$$s(n) = s(\nu_r, u) \big|_{u=n/(L \cdot PRF_{out})}.$$  \hfill (7)

Its spectrum $S_s(\omega)$ would be as in Fig. 2(b). See also Fig. 1.
B. Nonuniformly Sampled Radar Signal

Typically, the signal $s'(k)$ generated from the radar acquisition process is a potentially nonuniformly sampled DT version of $s(ν_r,u)$. The SAR collection receives this signal $s'(k)$ whose variable PRF $PRF_{in}$ is assumed to be high enough to sample the available Doppler support for an illuminated cross-range extent of $R Δβ$ with no aliasing.

**Model of the Nonuniformly Sampled Signal.** We view this nonuniformly sampled signal $s'(k) ↔ S_s'(ω)$ as being the uniformly densely sampled signal $s(n) ↔ S_s(ω)$ but with ‘missing’ samples. Here, $↔$ denotes a DT FT pair (in the deterministic case) or a PSD (power spectral density) pair (in the stochastic case). Using this viewpoint, with appropriate scaling and grid alignment procedures in place (see Fig. 1), the signal $r(n) ↔ S_r(ω)$ in Fig. 1 can be viewed as a ‘gated’ version of $s(n)$, i.e., $r(n) = g(n) s(n)$, where the gating function $g(n) ↔ S_g(ω)$ is taken as a realization of an i.i.d. Bernoulli random process with parameter $p$, i.e.,

$$Pr(g(n) = 1) = p; \quad Pr(g(n) = 0) = 1 - p, \quad ∀N.$$  

(8)

Using $S_x(ω)$ to denote the PSD of the w.s.s. random process $x(•)$, the PSD of $g(n)$ is

$$S_g(ω) = p(1 - p) + (2πp^2) \sum_{k=−∞}^{+∞} δ_D(ω - 2πk),$$

(9)

where $δ_D(ω)$ denotes the Dirac delta function. Then, the PSD of $r(n)$ can be expressed as

$$S_r(ω) = \frac{1}{2π} (S_g(ω) * S_s(ω)) = \frac{1}{2π} p(1 - p) \int_{ω=−π}^{+π} S_s(ω) dω + p^2 S_s(ω),$$

(10)

i.e., the PSD $S_r(ω)$ is a scaled and ‘biased’ version of $S_s(ω)$. See Fig. 2(c).

**Recovery of Spatial Doppler Bandwidth at the Output PRF.** Suppose that we are interested in an image signal from which a cross-range extent of $X_{out}$ can be recovered with a sampling rate of $PRF_{out} (1/m)$. In Claim 2 in Appendix A, we demonstrate that such a signal, which approximates a downsampled version of $s(n)$ (which is at the constant PRF $L \cdot PRF_{out}$), can be generated from the resampling scheme in Fig. 1 by implementing the following operations:

- Generate the signal $v(n)$ by filtering $r(n)$ by a digital filter $f(n) ↔ F(z)$ possessing the following narrowband magnitude response: with $L >> 1$,

$$|F(ω)| = \begin{cases} 1, & \text{in the passband } |ω| \leq γ π/L, \gamma < 1; \\ 0, & \text{in the stopband } π/L \leq |ω|. \end{cases}$$

(11)

- Normalize $v(n)$ by $f(n) * g(n)$ to get the signal $v'(n)$.
• L-fold decimate $v'(n)$ to get the required output $y(n)$.

Then, provided that $p \gg \rho/(L+\rho)$, where $\rho = PRF_{in}/PRF_{out} \geq 1$, the PSD of the output $y(n)$ approximates the PSD of an $L$-fold decimated version of the uniformly densely sampled signal $s(n)$ within the frequency band $[0, \gamma \pi]$. So, to recover a spatial Doppler bandwidth corresponding to $N_{\text{save}}$ bins, we must have $\gamma = N_{\text{save}}/N_{FFT}$. Note that $N_{\text{save}}$ can be thought of as the number of bins guaranteed to be retained in the final image without distortion.

With the main concept behind the proposed resampling scheme in place, in the next several sections, we provide a more detailed description of the various stages of our scheme in Fig. 1.

Output Grid Spacing Design. First, we describe how the slow-time output grid spacing $\Delta u_{out}$ (m) is selected so that it conforms to a given spatial PRF $PRF_{out}$, preserves the resolution in the image domain, and avoids aliasing in both the spatial slow-time and the image domains. This output grid spacing $\Delta u_{out}$ is then used to view the raw data $s'(k)$ that the SAR system receives (at the variable PRF $PRF_{in}$) as being embedded in a uniformly densely sampled signal $r(n)$ (at the constant PRF $L \cdot PRF_{out}$).

Filter Design and Implementation. As Claim 2 in Appendix A shows, the proposed resampling scheme requires the design and implementation of a narrowband digital filter. The filter design and its implementation are described here.

Normalization/Decimation. Finally, normalization and decimation operations are applied to obtain the required signal $y(n)$.

IV. OUTPUT GRID SPACING DESIGN

A. Design Steps

FFT Size. Following the common practice in most SAR processing implementations, our design begins with the selection of the FFT size $N_{FFT}$ so that read access memory (RAM) and SAR processor power/speed limitations can be met.

# of Points for Representing the Synthetic Aperture. We next determine the number of discrete points $N_D$ necessary to represent the synthetic aperture without any aliasing. Note that $N_{FFT}$ is the next power-of-two FFT size obtained from $N_D$. We may then express the slow-time spectrum oversampling factor as

$$p_d = \frac{N_{FFT} \Delta u_{out}}{N_D \Delta u_{out}} K_{cr} = \frac{N_{FFT}}{N_D} K_{cr},$$

(12)
where \( \Delta u_{\text{out}} \) is the slow-time output grid spacing; as in (I), \( K_{cr} \) compensates for broadening due to aperture weighting [10]. Note that \( N_D \Delta u_{\text{out}} \) and \( N_{FFT} \Delta u_{\text{out}} \) denote the slow-time acquisition intervals \( D \), and \( D_{pad} \) generated by \( N_D \) and \( N_{FFT} \) number of points, respectively. Since \( N_D \leq N_{FFT} \), this implies that \( K_{cr} \leq p_d \). For our work, we used \( p_d = 1.5 \) which generated adequate oversampling to create a visually more pleasing image; oversampling also facilitates the application of certain image processing procedures. With an appropriate choice of \( p_d \), (12) can be used to obtain \( N_D \).

With \( N_{pr} \) denoting the order of the prototype filter \( H_{pr}(z) \) (see Section V-A), in Section V-C we show that \( (N_{pr} - 1)/2 \) and \( (N_{pr} + 1)/2 \) points must be appended on the left- and right-hand sides of the \( N_D \) number of points laid out along the synthetic aperture. This results in a total number of points of \( N_{XAC} = N_D + N_{pr} \).

**Synthetic Aperture Length.** The spatial slow-time output grid spacing \( \Delta u_{\text{out}} \) is selected to match range and azimuth resolutions to create square radar resolution cells in the oversampled image domain. Using \( \delta_r = \delta_{cr} \) in (I), this yields the SAR integration angle \( \Delta \beta = (\lambda_c B/W/c)(K_{cr}/K_r) \), which is then used to find the required synthetic aperture length \( D \) [11].

**Slow-Time Output Grid Spacing and Output.** The slow-time output grid spacing \( \Delta u_{\text{out}} \) of the points describing the synthetic aperture \( D \) and the corresponding spatial sampling frequency \( PRF_{\text{out}} \) of the resampled data are then given by

\[
\Delta u_{\text{out}} = D/N_D \iff PRF_{\text{out}} = 1/\Delta u_{\text{out}}. \tag{13}
\]

**Cross-Range Extent.** We can now use (6) to get the corresponding cross-range extent \( X_{\text{out}} \). In essence, for the selected FFT size \( N_{FFT} \), we are simply attempting to fit as much cross-range extent \( X_{\text{out}} \) as possible so that PRF conversion can be accomplished efficiently. This strategy is more parallel processing friendly, and aid the SAR processor to partition the image into smaller patches of cross-range extent so that one may run smaller FFT sizes faster on parallel nodes.

**B. Grid Alignment**

Now we describe how the nonuniformly sampled signal is embedded in a uniformly densely sampled grid.

**Scaling.** The first step involves a scaling operation which utilizes a linear transformation to map the input pulses spatial information into the output spatial grid. To explain, let \( s(\nu_r, \alpha) = \)
Here, \( \Delta u_{\text{out}} = (u_{\text{max}} - u_{\text{min}})/N_{XAC} \) is the output spacing along-track and \( N_D = N_{XAC} - N_{pr} \) is the number of output points along the acquisition interval \( D \). This transformation in (14) maps \( u = \{u_{\text{min}}, u_{\text{mid}}, u_{\text{max}}\} \) to \( \alpha = \{-(N_{pr} - 1)/2, (N_D + 1)/2, N_{XAC} - (N_{pr} - 1)/2\} \), and it transforms the sequence \( s'(k) \) to the sequence \( s''(\alpha_k) \), where \( \alpha_k = f(u_k), k \in \{0, K\} \), and

\[
\quad s''(k) \equiv s(\nu_r, u_k) |_{u_k = f^{-1}(\alpha_k)} \equiv s'(k) |_{u_k = f^{-1}(\alpha_k)}. \tag{15}
\]

Note that, just as \( s'(k) \), this scaled sequence \( s''(k) \) is also potentially nonuniformly sampled.

**Grid Alignment.** The next step involves aligning the sampled values in \( s''(k) \) onto a dense grid corresponding to the rate \( L \cdot PRF_{\text{out}} \). This creates the sequence \( r(n) \), where

\[
\quad r(n) = s''(\alpha_k), \quad \text{when } n = \lfloor L \cdot \alpha_k \rfloor, \text{ and } r(n) = 0, \text{ otherwise.} \tag{16}
\]

This operation essentially aligns each sampled value in \( s''(k) \) (located at \( \alpha_k \) where \( \alpha_k \in \mathbb{R} \) in the interval \( [-(N_{pr} - 1)/2, N_{XAC} - (N_{pr} - 1)/2] \)) to a grid point (within the densely sampled grid with rate \( L \cdot PRF_{\text{out}} \)) which is closest and not higher than the location \( \alpha_k \). The remaining grid points (i.e., the ‘missing’ samples) are assigned value 0. So, one may view the signal \( r(n) \) as a ‘gated’ version of \( s(n) \), where only some samples of \( s(n) \) appear in \( r(n) \) while the others take...
values 0. This is exactly the representation mentioned in Section III-B and used in Appendix A.

V. FILTER DESIGN AND IMPLEMENTATION

A flexible SAR focusing processor (i.e., TerraSAR-X) must accommodate for the storage of a wide range of integer and non-integer resample ratios (which can vary with radar collection parameters, geometry, processed image resolution, processed scene size, and the required FFT size). Practicality dictates that what is preferable is a single filter which is flexible enough to handle a large range of resampling ratios. Another important aspect that must be taken into consideration is the computational efficiency in implementing the filtering operations.

A digital filter implementation using its polyphase representation addresses both these issues of flexibility and computational efficiency. Polyphase architectures, which constitute a critical component in multirate digital systems, are computationally more efficient because they operate at the lowest sampling rate of the structure [16].

A. Filter Design

With \( L >> 1 \), achieving the specifications in (11) is essentially a narrow bandwidth filter design.

**Step 1. Prototype Filter.** Use a standard finite impulse response (FIR) filter design technique (e.g., the Remez exchange/Parks-McClellan algorithm [15], [17]–[19]) to design a ‘prototype’ with its passband and stopband edge frequencies located \( L \)-times as much as the desired filter in (11):\[
F_{pr}(z) = \sum_{m=0}^{N_{pr}} f_{pr}(m) z^{-m}, \text{ where } F_{pr}(\omega) = \begin{cases} 1, & \text{for } \omega \in [0, \gamma \pi]; \\ 0, & \text{for } \omega = \pi. \end{cases}
\] (17)

We choose a Type-II FIR filter design so that the filter coefficients are symmetric and its order \( N_{pr} \) is odd. This allows us to exploit the non-integer delay in the polyphase subfilters [18] and the property that the samples of the original sequence are preserved at locations where no interpolation is carried out [20].

**Step 2. Shaping Filter.** As is typical in interpolated FIR filter design [21], [22], generate an \( L \)-fold upsampled version of \( f_{pr}(n) \) to get the ‘shaping’ filter \[
f_{be}(n) = \begin{cases} f_{pr}(n/L), & \text{for } n = 0, L, \ldots, N_{pr}L; \\ 0, & \text{otherwise.} \end{cases}
\] (18)
so that \( F_{be}(z) = F_{pr}(z^L) \). This \( N_{pr} L \)-order filter’s frequency response is the desired response in (11), except that spectral ‘images’ of this desired response now appear within the Nyquist interval. The filter order \( N_{pr} L \) enables a polyphase design consisting of \( L \) subfilters [15].

**Step 3. Image Suppression.** Conventional IFIR designs utilize a ‘masking’ filter to suppress these extra images [22]. But, this strategy increases the length of our overall impulse response and the filter order beyond \( N_{pr} L \). Consequently, we employ a direct least squared integral error (LSIE) FIR design [23], [24] to design an \( N_{pr} L \)-order filter to approximate \( F_{be}(\omega) \) in the frequency interval \( [0, \pi/L] \) but with support restricted to \( [0, N_{pr} L] \). So, the ‘ideal’ frequency response to be approximated is \( F_{id}(\omega) = F_{be}(\omega) F_{LPF}(\omega) \), where

\[
F_{LPF}(\omega) = \begin{cases} 
1, & \text{for } |\omega| \leq \pi/L; \\
0, & \text{otherwise}. 
\end{cases} \tag{19}
\]

Note that \( F_{LPF}(\omega) \leftrightarrow f_{LPF}(n) = (1/L) \text{sinc}(\pi n/L) \) and \( f_{id}(n) = f_{be}(n) * f_{LPF}(n) \). So,

\[
f_{id}(n) = \frac{1}{L} \sum_{m=0}^{N_{pr}} f_{pr}(m) \text{sinc}\left( (n - mL) \frac{\pi}{L} \right), \tag{20}
\]

and the filter with support in \( [0, N_{pr} L] \) which minimizes the LSIE with \( F_{id}(\omega) \) is [23], [24]

\[
f(n) = \begin{cases} 
\frac{1}{L} \sum_{m=0}^{N_{pr}} f_{pr}(m) \text{sinc}\left( (n - mL) \frac{\pi}{L} \right), & \text{for } n \in [0, N_{pr} L]; \\
0, & \text{otherwise.} 
\end{cases} \tag{21}
\]

Thus \( f(n) \) acts as an ‘interpolation’ of the samples \( f_{pr}(\cdot) \) of the prototype. At a sampling instant which is an integer multiple of \( L \), this interpolation reduces to \( f(KL) = f_{pr}(K)/L, K \in [0, N_{pr}] \).

**B. Filter Implementation**

Consider the \( L \)-fold polyphase representation of the \( N_{pr} L \)-order filter \( F(z) \) [15]:

\[
F(z) = \sum_{\ell=0}^{L-1} z^{-\ell} F_{\ell}(z^L), \quad \text{where } F_{\ell}(z) = \sum_{n=0}^{N_{pr}} f_{\ell}(n) z^{-n}, \tag{22}
\]

with \( f_{\ell}(n) = f(nL + \ell) \). See Fig. 4(a). Note that, \( F_0(z) \) is of order \( N_{pr} \); \( F_{\ell}(z) \), \( \ell \in [1, L - 1] \), is of order \( N_{pr} - 1 \).

Fig. 4(b) shows the efficient implementation of this polyphase representation. Using the
notation $n_{pr} = n - N_{pr}$, we get the output as $y(n) = y'(n)/(f \ast g)(nL)$, where
\begin{equation}
y'(n) = \sum_{m=n_{pr}}^{n} f_0(n-m) r(mL) + \sum_{\ell=1}^{L-1} \sum_{m=n_{pr}+1}^{n} f_\ell(n-m) r(mL - \ell)
\end{equation}
\begin{equation}
= f_0(n_{pr}) r(n_{pr} L) + \sum_{\ell=0}^{L-1} \sum_{m=n_{pr}+1}^{n} f_\ell(n-m) r(mL - \ell)
\end{equation}

The change of variables $mL - \ell \rightarrow \ell$ yields the expression
\begin{equation}
y'(n) = f_0(n_{pr}) r(n_{pr} L) + \sum_{m=n_{pr}+1}^{n} \sum_{\ell=mL-(L-1)}^{mL} f_{mL-\ell}(n-m) r(\ell).
\end{equation}

Clearly, computation of the one output sample $y'(n), n \in \mathbb{N}$, requires all input samples $r(\ell) \neq 0 : \ell \in \mathbb{N}$ s.t. $(n - N_{pr})L \leq \ell \leq nL$.\(^{(25)}\)

Conversely, the single input sample $r(\ell) \neq 0, \ell \in \mathbb{N}$, s.t. $(n - N_{pr})L \leq \ell \leq nL$, affects the computation of all output samples $y(n), \forall n \in \mathbb{N}$ s.t. $\ell \leq nL \leq \ell + N_{pr}L$, i.e.,
\begin{equation}
y(n) : n \in \mathbb{N} \text{ s.t. } \left\lfloor \frac{\ell - 1}{L} \right\rfloor + 1 \leq n \leq \left\lfloor \frac{\ell}{L} \right\rfloor + N_{pr}.
\end{equation}

We first establish the following result:

**Claim 1.** Consider a non-zero input sample $r(\ell) \neq 0 : \ell \in \mathbb{N}$ s.t. $(n - N_{pr})L \leq \ell \leq nL$. The only polyphase component that operates on $r(\ell)$ is $f_\ell(\cdot)$, where $x = mL - \ell$, with
\begin{equation}
m = \left\lfloor \frac{\ell - 1}{L} \right\rfloor + 1 \implies x = (L - \ell) + L \left\lfloor \frac{\ell - 1}{L} \right\rfloor.
\end{equation}
Here, $[x/L] = (x - (x)_L)/L$, where $(x)_L$ denotes the remainder when $x$ and $L$ are the dividend and divisor, respectively.

Proof: First, consider the case $(n - N_{pr})L + 1 \leq \ell \leq nL$ which corresponds to the summation term in (24). The polyphase components which operate on $r(\ell)$ are $f_x(\ell)$, where $x \equiv mL - \ell \in 0, L - 1$ with $m \in n - N_{pr} + 1, n$. Then $mL - \ell = x$ iff $(m - 1)L + (L - 1 - x) = \ell - 1$. Since $(L - 1 - x) \in 0, L - 1$, we conclude that $L - 1 - x = (\ell - 1)_L$. This yields $x = (L - 1) - (\ell - 1)_L = (L - \ell) + L \lfloor (\ell - 1)/L \rfloor$. The claim then follows for $(n - N_{pr})L + 1 \leq \ell \leq nL$.

Next, consider the case $\ell = (n - N_{pr})L$ which appears in the first term in (24), viz., $f_0(N_{pr})r((n - N_{pr})L)$. When $\ell = (n - N_{pr})L$ is substituted in the claimed expressions for $x$ and $m$, we get $x = 0$ and $m = n_{pr}$, which are consistent with $f_0(N_{pr})r((n - N_{pr})L)$.

C. Computational Aspects

For our implementation, instead of $\tilde{y}'(n)$, it is more convenient to work with

$$
\tilde{y}'(n) = y'\left(n + \frac{N_{pr} + 1}{2}\right),
$$

because the associated inequalities are more symmetric: corresponding to (25) we get

$$
\left(n - \frac{N_{pr} - 1}{2}\right)L \leq \ell \leq \left(n + \frac{N_{pr} + 1}{2}\right)L;
$$

and corresponding to (26) we get

$$
\left\lfloor \frac{\ell - 1}{L} \right\rfloor - \left(\frac{N_{pr} - 1}{2}\right) \leq n \leq \left\lfloor \frac{\ell}{L} \right\rfloor + \left(\frac{N_{pr} - 1}{2}\right).
$$

We can now develop a procedure to efficiently compute the output signal vector $\tilde{y}'(N_1:N_2) = [\tilde{y}'(N_1), \tilde{y}'(N_1+1), \ldots, \tilde{y}'(N_2)]^T$ via an input-centered convolution [9] scheme. See Algorithm 1.

Note that, the computation of $\tilde{y}'(N_1:N_2)$ requires all the non-zero input samples $r(\ell) \neq 0,$

$$
\forall \ell \in \mathbb{N} : \left(N_1 - \frac{N_{pr} - 1}{2}\right)L \leq \ell \leq \left(N_2 + \frac{N_{pr} + 1}{2}\right)L.
$$

The output $\tilde{y}'(N_1 : N_2)$ from Algorithm 1 yields $y'(N_1 + (N_{pr} + 1)/2 : N_2 + (N_{pr} + 1)/2)$.

Also note that the grid alignment procedure described in Section IV-B can be incorporated directly into this computation without the need for accounting for it at an earlier stage. Indeed, as and when a new non-zero signal pulse $s'(k) = s(\nu_r, \alpha_k)$ is received, one can generate $r(n)$ in (16) and proceed with the computation in Algorithm 1. In this manner, the computation is carried out ‘on-the-fly’ without the need to buffer the input signal pulses. The implementation
Algorithm 1 Input-Centered Convolution Algorithm to Compute $\tilde{y}'(N_1:N_2)$

Initialize: $\tilde{y}'(N_1:N_2) = 0$

for $r(\ell) \neq 0$, s.t. $\ell \in \mathbb{N}$ satisfies (30), do

$m = \lfloor (\ell - 1)/L \rfloor + 1$

for $n \in \mathbb{N}$, s.t. $n$ satisfies (29), do

$\tilde{y}'(n) = \tilde{y}'(n) + f_{mL-\ell}(n + (N_{pr} + 1)/2 - m) \cdot r(\ell)$

end for

end for

waits for non-zero pulses only and processes them one pulse at a time as it is received. One may even allow the input pulses to come in out-of-order and still be correctly processed.

D. Normalization

The final output $y(n)$ is created by normalizing $y'(n)$ by $(f * g)(nL)$ to get $y(n) = y'(n)/(f * g)(nL)$. See Fig. [4]b. One may interpret this ‘normalized’ convolution operation as a way to associate to each signal a ‘certainty’ value that captures the level of confidence one can place on the received signal [25]–[27]. With missing samples, the certainty associated with each missing sample is zero. Accordingly, the certainty associated with the received signal $r(n)$ is modeled via $(f * g)(n)$, where $g(n)$ is the gating function previously mentioned in Section III-B. Note that the normalization factor $(f * g)(nL)$ is nothing more than the sum of the filter coefficients that contribute to the output computation. During the implementation, the computation of $(f * g)(nL)$ is carried out by simply maintaining a separate buffer as an accumulator of the sum of the coefficients used at each output location.

Fig. 5 places the proposed resampling scheme within the image formation process.
TABLE II
ACQUISITION AND PROCESSING PARAMETERS CORRESPONDING TO DATASETS A AND B

| Dataset | Acquisition Parameters: $BW = 240$ (MHz); $\Delta \beta = 1.6$ (deg); $\lambda_c = 0.031$ (m) |
|---------|--------------------------------------------------------------------------------------------------|
|         | $D$      | $PRF_{in}$ | Nominal PRF | Range | $R \Delta \beta$ | Squint | $V_p$ |
| A       | 3395.29  | 3.34, 3.36, 3.41, 3.43, 3.57, 3.78, 3.99, 4.03, 4.21, 4.49 | 3.99 | 118320 | 3304.12 | 13.8 | 142.68 |
| B       | 3821.59  | 3.51, 3.88, 4.25, 4.54, 4.55, 4.61, 4.72 | 4.54 | 135205 | 3762.85 | 10.75 | 136.59 |

| Processing Parameters: $K_{cr} = 1.2$; $L = 64$; $N_{FFT} = 16384$; $p_d = 1.5$ |
|---------|--------------------------------------------------------------------------------------------------|
|         | $PRF_{out}$ | $X_{out}$ | $Y_{out}$ |
| A       | 3.86 | 7086.94 | 3360 |
| B       | 3.43 | 7196.15 | 3360 |

VI. RESULTS

Data Sets. For verification purposes, we employ two data sets both of which were acquired with stretch waveforms (using the technique of deramp-on-receive) in SAR’s spotlight mode. Table II provides the relevant acquisition and processing parameters. Fig. 6 shows the results of our algorithm.

Results. Fig. 6 shows the results. The first column of figures (viz., (6a) and (c)) refer to Dataset A; the second column of figures (viz., (6b) and (d)) refer to Dataset B. In each data set, the first row shows images formed with the spatial FT taken on data nonuniformly sampled along-track (without the resampler block on Fig. 5); the second row shows images formed after taking the discrete spatial FT on data that has undergone the resampling system in the spatial slow-time domain (with the resampler block on Fig. 5). These second row images have been cropped in azimuth with exactly $N_{save} = (2/3) N_{FFT}$ pixels (which is the FIR filter passband).

It should be noted that the output PRF $PRF_{out}$ for both data sets was selected to unambiguously sample a cross-range extent of $X_{out}$ which is slightly greater than the illuminated cross-range extent $R \Delta \beta$. This was possible with a 16384-point FFT which allows for the output rate to be closely comparable in magnitude to the variable input rates for easiness in comparison of the image with and without re-sampling. The filter on the other hand only passed 2/3 of the allowable cross-range extent $X_{out}$ without any distortion.

Resolution Improvement. To quantify the improvement in resolution offered by the proposed
algorithm, we used point spread function (PSF) measurements. For this purpose, we utilized the well established method of measuring how resolvable point scatters are in azimuth [28]. In particular, we employed a quadratic fit of the log magnitude of pixels adjacent to the main lobe response of point scatters, and then recorded the $-3$ (dB) width as an indication of resolution. Table III indicates the average improvement corresponding to 8 point scatterers.

**TABLE III**

| Dataset A: Before (m) | 1.51 | 1.64 | 1.78 | 1.89 | 0.92 | 0.98 | 0.89 | 0.82 |
|-----------------------|------|------|------|------|------|------|------|------|
| After (m)             | 0.86 | 0.84 | 1.05 | 1.20 | 0.72 | 0.74 | 0.73 | 0.71 |
| Dataset B: Before (m) | 1.48 | 1.52 | 1.45 | 1.31 | 1.23 | 0.88 | 0.91 | 0.85 |
| After (m)             | 0.88 | 0.92 | 0.82 | 0.83 | 0.76 | 0.72 | 0.74 | 0.73 |
VII. DISCUSSION

**Design Considerations.** Several design considerations were taken into account in our work:

**FFT Size.** While it is common to select the FFT size so that it can accommodate a required cross-range extent, in this paper, we employed a different strategy: as detailed in Section IV, we first selected an FFT size $N_{FFT}$ that can comfortably be implemented on a given system and used the cross-range extent $X_{out}$ which can be accommodated with this chosen value of $N_{FFT}$. With this strategy, if the need arises, the processor is able to partition an image into smaller patches and still operate on the same FFT size. We used $N_{FFT} = 2^{14} = 16,384$.

**Frequency Response of the Prototype Digital Filter.** Given the cross-range extent that can be accommodated with the selected FFT size, the response of the FIR prototype filter was designed to pass only a portion (we used $\gamma = N_{save}/N_{FFT} = 2/3$) of the spectrum (see Fig. 2). This was necessary to arrive at a low order filter design which in turn was crucial to maintain a reasonable number of filtering operations.

**Type of Digital Filter.** An FIR digital filter of low order (we used $N_{pr} = 5$) was adequate for our purposes. In addition to the absence of stability issues, the FIR design allowed us to compute only every $L$-th output sample that is affected by an incoming input sample. This property was critical for implementing the downsampling portion of our system.

**Oversampling the Final Image.** Oversampling the final image generated a final image that was more pleasing to the eyes. We used an oversampling factor of $p_d = 1.5$ to describe each square radar resolution cell of $\delta_r \times \delta_{cr}$.

**Convolution Computation.** Input-centered convolution was utilized to arrive at a real-time, in-place implementation (see Algorithm 1). It allowed us to process an input sample one-by-one even if it were to arrive out-of-order. The output buffer could simply be updated when the next input sample is received.

**Computational Complexity.**

**Arithmetic Operations.** The polyphase implementation utilizes $L$ subfilters, each containing $N_{pr}$ real-valued coefficients (where we ignore the fact that one subfilter contains $N_{pr} + 1$ real-valued coefficients). Each complex-valued input sample gets operated on by one subfilter (see Claim 1) and is used to update $N_{pr}$ output points in its vicinity (see (29)). So, each complex-valued input sample entails $2N_{pr}$ real multiplications and $N_{pr}$ complex additions. Noting that the maximum look angle as seen by the synthetic aperture is $\Pi$ (where we ignore the swath
range extent \( Y_{\text{out}} \)

\[
\theta_{\text{max}} = \tan^{-1}\left(\frac{X_{\text{out}}/2 + D/2}{Y_c}\right) = \tan^{-1}\left(\frac{X_{\text{out}} + D}{2Y_c}\right),
\]

one may estimate of the total number of input pulses collected by the radar to fulfill the Nyquist criterion (at boresight) and to achieve the desired spatial resolution: from \([11]\), we have

\[
\Delta u_{\text{in}} \leq \frac{2\pi}{4\omega_{r_{\text{max}}} \sin \theta_{\text{max}}},
\]

where \( \omega_{r_{\text{max}}} = 2\pi/\lambda_{\text{min}} \) with \( \lambda_{\text{min}} \) being the minimum wavelength of transmitted signal. So,

\[
N_{D,\text{in}} = \frac{D}{\Delta u_{\text{in}}} K_{cr} \geq \frac{4D \sin \theta_{\text{max}}}{\lambda_{\text{min}}} K_{cr},
\]

where \( K_{cr} \) compensates for broadening due to aperture weighting.

**Buffer Requirements.** The implementation occurs in-place, and only requires a buffer of \( N_D \) complex floats for the output to be stored, a buffer of \( (N_{pr}L + 1) \) real floats for the filter coefficients to be stored, and one complex float for the current input to be temporarily stored for processing (and later overwritten by the next input).

**Real-Time Application.** The proposed algorithm can be executed in real-time, i.e., it can process the input pulses one-by-one as they are received; they can even be out-of-order (provided they are spatially correctly stamped of course). Moreover, if the processing is done on-board, only resampled data would occupy the down-link, thus leading to a significant improvement in throughput when compared to the down-link data transfer requirement if off-board processing were warranted. From \([12]\), we know that

\[
N_D = \frac{N_{FFT}}{p_d} K_{cr}.
\]

Thus, the parameters \( N_{FFT}, p_d, \) and/or \( K_{cr} \) can be used to significantly lower the number of output pulses. However, lowering the FFT size narrows the unambiguously sampled cross-range extent being imaged; lowering \( K_{cr} \) broadens the resolution; and increasing \( p_d \) increases the spectrum oversampling factor.

**Error Analysis.** Considering the floor operation associated with the grid alignment process in \([16]\), we may upper bound the error in the realignment of the samples by

\[
\Delta u_{\text{error}} \leq \frac{\Delta u_{\text{out}}}{L} = \frac{1}{L} \frac{D}{N_D} = \frac{D}{L} \frac{p_d}{N_{FFT}} \frac{1}{K_{cr}}.
\]

So, larger FFT sizes and larger upsampling factors can reduce this maximum error in the spatial interpolation tremendously. However, these strategies come at a higher computational load, viz.,
large $N_{FFT}$ means higher FFT computational load, and larger $L$ means higher number of subfilters in the polyphase representation. However the computational load on the resampler will remain the same because each input sample selects only one polyphase component and filtering is performed at the output rate.

**Benefits Yielded Elsewhere.** The method proposed can in general be exploited in situations where effects of nonuniform sampling across the aperture have to be compensated for, e.g., missing data, flight path deviation, imaging while in turn, and acceleration and deceleration.

**Dual-Aperture SAR Processing Algorithms.** Traditionally, coherent change detection (CCD) and ground moving target indicator (GMTI) algorithms, both of which are based on dual-aperture SAR processing algorithms, remove clutter by subtracting the SAR images formed within each aperture [29]. If the sampling rates are not uniform across apertures, then the performance of clutter cancellation algorithms can be significantly diminished [30]. The proposed algorithm provides an effective solution for compensating this sampling rate nonuniformity.

**TerraSAR-X Acquisition Modes and Possibly Other Satellites.** Another very important potential use of the proposed resampling algorithm is to account for different PRFs across different acquisitions in repeat-pass Interferometric SAR (InSAR). This is a challenge that has been identified in the spotlight (SL), high resolution spotlight (HS), and the most recently added staring spotlight SAR modes of TerraSAR-X. These modes have been known to be unable to collect data from the same scene at the same PRF [31], which constitutes a serious difficulty in interferometry. Our proposed algorithm can be employed to unify the different PRFs across different (SLCs), so that interferometry can be applied with equivalent spectrum widths.

Fig. 7 illustrates this effect on real data from TerraSAR-X. Fig. 7(a) shows the effect of this phenomenon on TerraSAR-X spotlight interferometry; Fig. 7(b) shows the absence of this effect when the same PRF is being used.

The widely used solution to unify different SLCs to a common grid space and Doppler spectrum width is via the process of co-registration in two steps. In the first step, the slave image is resampled in the image domain to the geometry of the master image information and low-resolution digital elevation model. In the second step, residual shifts in range and azimuth are estimated within sub-pixel accuracy by analysis of point-like scatterers common to both images [32]. The success of this strategy depends on a high persistence of the scatterers in order to remove all residual shifts. The second step becomes more challenging at higher resolutions.
Fig. 7. Georeferenced interferograms of TerraSAR-X HS mode formed with SLCs. (a) Different PRFs \{8300, 8200\}. (b) Same PRFs \{8300, 8300\}. The phase has been wrapped to be within \([-\pi, +\pi]\).

when sub-pixel accuracy by analysis of point-like scatterers has to be done at resolutions cells smaller than 1 (m). Our proposed resampling method resolves this issue directly from the raw data and interferometry of identical Doppler bandwidths would minimize the need for such meticulous co-registration techniques.

VIII. CONCLUSION

In this paper, a computationally efficient method is developed to resample along-track SAR data in slow-time domain for a radar that operates at variable PRFs. The algorithm designs a spatial output grid along-track with a desired spatial sampling rate chosen by design to be efficient for FFT computations and the filtering operations. The algorithm upsamples this grid by a factor of \(L\) and aligns the nonuniform input samples with the new upsampled grid. This allows us to view the nonuniformly sampled signal as a subset of samples of a uniformly densely sampled underlying signal. The resulting signal is then low-pass filtered and normalized to get the missing sample values. Only the portion of interest from the spectrum is extracted in the frequency domain after taking the spatial FFT. The order \(N_{pr}\) of the digital filter and its implementation are critical factors affecting the computational complexity of the algorithm. The filter implementation was carried out as a polyphase network of \(L\) polyphase components, each having approximately \(N_{pr}\) time varying coefficients. The proposed algorithm was employed on two different data sets and significant improvement in image quality was apparent with each data set. We identify several application scenarios where the proposed method could be used. For
example, it can be employed to unify PRFs across a sequence of acquisitions taken at different PRFs within a InSAR time series in TerraSAR-X spotlight mode data. The proposed method can also be used to improve clutter cancellation in coherent change detection and ground moving target indication.

**APPENDIX**

**ANALYTICAL BASIS OF THE RESAMPLING SCHEME**

Here we provide justification for the proposed resampling scheme in Fig. 1. As argued in Sections III-B and IV-B, interpret the signal \( r(n) \) in Fig. 1 as a ‘gated’ version of the uniformly densely sampled radar signal \( s(n) \), i.e., \( r(n) = g(n) s(n) \), where the gating function \( g(n) \) is a realization of the i.i.d. Bernoulli random process with parameter \( p \) in (8). Let us denote the index set for which \( g(n) = 1 \) by \( \mathcal{N} \), i.e., \( \mathcal{N} = \{ n \in \mathbb{N} : g(n) = 1 \} \). So, \( r(n) = s(n) \), \( \forall n \in \mathcal{N} \), and \( r(n) = 0 \), \( \forall n \in \mathbb{N} \setminus \mathcal{N} \), i.e., \( \mathbb{N} \setminus \mathcal{N} \) indicates the index set for which the \( r(n) \) is ‘missing’ the samples of \( s(n) \). The mean and autocorrelation of the random process \( g(\cdot) \) are given by

\[
\mu_g = p, \forall n \in \mathbb{N}; \quad C_g(n) = p(1 - p) \delta(n) + p^2,
\]

where \( \delta(n) \) denotes the Kronecker delta function. The random process \( g(\cdot) \) being w.s.s., its autocorrelation \( C_g(n) \) and PSD \( S_g(\omega) \) form a DT FT pair:

\[
C_g(n) \leftrightarrow S_g(\omega) = p(1 - p) + (2\pi p^2) \sum_{k=-\infty}^{+\infty} \delta_D(\omega - 2\pi k).
\]

We notice the following relationships:

\[
r(n) = g(n) s(n); \quad v(n) = f(n) * r(n),
\]

where \( f(n) \) represents the IPR of the digital filter \( F(z) \) and \( f(n) \leftrightarrow F(\omega) \). In addition, the normalization of \( v(n) \) yields

\[
v'(n) = v(n)/(f(n) * g(n)) \implies v(n) = (f(n) * g(n)) v'(n).
\]

So, we may express the output as

\[
y(n) = v'(nL) = (f(n) * r(n))/(f(n) * g(n))|_{n \to nL}.
\]

In terms of PSDs, we can express (38) as

\[
S_r(\omega) = \frac{1}{2\pi} (S_g(\omega) * S_s(\omega)); \quad S_v(\omega) = |F(\omega)|^2 S_r(\omega).
\]
We can also express (39) and (40) as
\[ S_v(\omega) = \frac{1}{2\pi} ((|F(\omega)|^2 S_g(\omega)) * S_w(\omega)); \quad S_y(\omega) = \frac{1}{L} \sum_{\ell=0}^{L-1} S_{\omega\ell}(\omega_\ell), \quad \omega_\ell = \omega - \frac{2\pi \ell}{L}. \] (42)

Compare the expressions for \( S_v(\omega) \) in (41) and (42):
\[ S_v(\omega) = \frac{1}{2\pi} |F(\omega)|^2 (S_g(\omega) * S_w(\omega)) = \frac{1}{2\pi} (|F(\omega)|^2 S_g(\omega)) * S_w(\omega). \] (43)

Use (37) to substitute for \( S_g(\omega) \):
\[ S_v(\omega) = \frac{p(1-p)}{2\pi} |F(\omega)|^2 \int_{\theta=-\pi}^{+\pi} S_w(\omega - \theta) d\theta + \frac{2\pi p^2}{2\pi} |F(\omega)|^2 S_w(\omega) \] (44)
\[ = \frac{p(1-p)}{2\pi} \int_{\theta=-\pi}^{+\pi} |F(\omega)|^2 S_w^*(\omega - \theta) d\theta + \frac{2\pi p^2}{2\pi} |F(0)|^2 S_w^*(\omega). \] (45)

Claim 2. Suppose \(|F(\omega)| \) has support \([-\pi/L, +\pi/L]\), \( L >> 1 \), and satisfies (11). See Fig. 2(c). Then, for \( p >> \rho/(L + \rho) \), within \(|\omega| \leq \gamma \pi \), the PSD of the output \( y(n) \) approximates the PSD of an \( L \)-fold decimated version of the densely sampled input signal \( s(n) \).

Proof: With \( L >> 1 \), we note that
\[ \int_{\theta=-\pi}^{+\pi} |F(\omega)|^2 S_w^*(\omega - \theta) d\theta \leq \int_{\theta=-\pi}^{+\pi/L} S_w^*(\omega - \theta) d\theta \approx (2\pi/L) S_w^*(\omega). \]

Compare the two terms in the second expression for \( S_v(\omega) \) in (45): the first term is much smaller than the second term if
\[ \frac{p(1-p)}{2\pi} \frac{2\pi}{L} S_w(\omega) \ll \frac{2\pi p^2}{2\pi} |F(0)|^2 S_w(\omega) = \frac{2\pi p^2}{2\pi} S_{\omega\ell}(\omega) \iff p >> \frac{1}{L + 1}. \]

Thus, for \( p >> 1/(L + 1) \), (44) and (45) become
\[ S_v(\omega) = \frac{p(1-p)}{2\pi} |F(\omega)|^2 \int_{\theta=-\pi}^{+\pi} S_w(\omega - \theta) d\theta + \frac{2\pi p^2}{2\pi} |F(\omega)|^2 S_w(\omega) \approx \frac{2\pi p^2}{2\pi} S_{\omega\ell}(\omega). \]

Now, using the expression for \( S_y(\omega) \) in (42), consider
\[ \frac{1}{L} \sum_{\ell=0}^{L-1} S_{\omega\ell}(\omega_\ell) = \frac{p(1-p)}{2\pi L} \sum_{\ell=0}^{L-1} |F(\omega_\ell)|^2 \int_{\theta=-\pi}^{+\pi} S_{\omega\ell}(\omega_\ell - \theta/L) d\theta + \frac{2\pi p^2}{2\pi L} \sum_{\ell=0}^{L-1} |F(\omega_\ell)|^2 S_{\omega\ell}(\omega_\ell) \] (46)
\[ \approx \frac{2\pi p^2}{2\pi L} \sum_{\ell=0}^{L-1} S_{\omega\ell}(\omega_\ell) = \frac{2\pi p^2}{2\pi} S_y(\omega). \] (47)

Consider (46): for \(|\omega| \leq \gamma \pi \), \(|F(\omega_\ell)| = 1\). In addition, given that \( S_w(\omega) \) has the support \([-\rho\pi/L, +\rho\pi/L]\), where \( \rho = PRF_{in}/PRF_{out} \geq 1 \) (see Fig. 2(b)), we may write
\[ \int_{\theta=-\pi}^{+\pi} S_w(\omega_\ell - \theta/L) d\theta = \int_{\theta=-\rho\pi/L}^{+\rho\pi/L} S_w(\omega_\ell - \theta/L) d\theta \approx (2\rho\pi/L) S_w(\omega_\ell). \]
Thus, for $|\omega| \leq \gamma \pi$, and when $p >> \rho/(L + \rho)$, (46) becomes

$$1 \sum_{\ell=0}^{L-1} S_v(\omega_\ell) = p(1 - p) \frac{2\rho \pi}{2\pi L} \sum_{\ell=0}^{L-1} S_v(\omega_\ell) + \frac{2\pi p^2}{2\pi L} \sum_{\ell=0}^{L-1} S_s(\omega_\ell) \approx \frac{2\pi p^2}{2\pi L} \sum_{\ell=0}^{L-1} S_s(\omega_\ell).$$

(48)

Use (48) instead of (46) to express (46)-(47) as

$$1 \sum_{\ell=0}^{L-1} S_v(\omega_\ell) \approx \frac{2\pi p^2}{2\pi L} \sum_{\ell=0}^{L-1} S_s(\omega_\ell) \approx \frac{2\pi p^2}{2\pi L} S_y(\omega),$$

thus establishing the claim. ■

REFERENCES

[1] G. Krieger, N. Gebert, and A. Moreira, “SAR signal reconstruction from non-uniform displaced phase centre sampling,” in IEEE Int. Geosci. and Remote Sensing Symp. (IGARSS), vol. 3, Anchorage, AK, Sept. 2004, pp. 1763–1766.

[2] N. Gebert, G. Krieger, and A. Moreira, “SAR signal reconstruction from non-uniform displaced phase center sampling in the presence of perturbations,” in IEEE Int. Geosci. and Remote Sensing Symp. (IGARSS), vol. 2, Seoul, South Korea, July 2005, pp. 1034–1037.

[3] ——, “Multichannel azimuth processing in ScanSAR and TOPS mode operation,” IEEE Trans. on Geosci. and Remote Sensing, vol. 48, no. 7, pp. 2994–3008, July 2010.

[4] Y. Jiang, X. Wu, and B.-B. Zhang, “Study on spectrum reconstruction algorithm for high resolution and wide swath spaceborne SAR,” in Int. Symp. on Instrumentation and Measurement, Sensor Network and Automation (IMSNA), vol. 1, Sanya, China, Aug. 2012, pp. 200–204.

[5] G. H. Goldman, “Computationally efficient resampling of nonuniform oversampled SAR data,” in Proc. IEEE Radar Conf., Washington, DC, May 2010, pp. 70–74.

[6] J. L. Walker, “Range-Doppler imaging of rotating objects,” IEEE Trans. on Aerospace Electronic Syst., vol. 16, no. 1, pp. 23–52, Jan. 1980.

[7] M. Soumeck, Fourier Array Imaging. Prentice-Hall, 1994.

[8] D. R. Wehner, High Resolution Radar. Norwood, MA: Artech House, Inc., 1987.

[9] W. G. Carrara, R. S. Goodman, and R. M. Majewski, Spotlight Synthetic Aperture Radar: Signal Processing Algorithms. Boston, MA: Artech House, 1995.

[10] C. V. J. Jr., D. E. Wahl, P. H. Eichel, D. C. Ghiglia, and P. A. Thompson, Spotlight-Mode Synthetic Aperture Radar: A Signal Processing Approach. New York, NY: Springer, 1996.

[11] M. Soumeck, Synthetic Aperture Radar Signal Processing, with MATLAB Algorithms. John Wiley & Sons, 1999.

[12] N. Gebert and G. Krieger, “Ultra-wide swath SAR imaging with continuous PRF variation,” in Proc. European Conf. on Synthetic Aperture Radar (EUSAR), Aachen, Germany, June 2010, pp. 1–4.

[13] L. Yadong and C. Qian, “A novel ultra-wide swath SAR based on variable PRF and digital beamforming,” in Proc. IET Int. Radar Conf., Xi’an, P. R. China, Apr. 2013, pp. 1–5.

[14] X. Luo, R. Wang, W. Xu, Y. Deng, and L. Guo, “Modification of multichannel reconstruction algorithm on the SAR with linear variation of PRL,” IEEE J. of Selected Topics in Appl. Earth Observations and Remote Sensing, vol. 7, no. 7, pp. 3050–3059, July 2014.

[15] A. V. Oppenheim and R. Schafer, Digital Signal Processing. Englewood Cliffs, NJ: Prentice-Hall, 1975.
[16] R. E. Crochiere and L. R. Rabiner, “Interpolation and decimation of digital signals: A tutorial review,” *Proc. of the IEEE*, vol. 69, no. 3, pp. 300–331, Mar. 1981.

[17] T. W. Parks and J. H. McClellan, “Chebyshev approximation for nonrecursive digital filters with linear phase,” *IEEE Trans. on Circ. Theory*, vol. 19, no. 2, pp. 189–194, Mar. 1972.

[18] L. R. Rabiner, “The design of finite impulse response digital filters using linear programming techniques,” *The Bell Syst. Tech. J.*, vol. 51, no. 6, pp. 1177–1198, July-Aug. 1972.

[19] J. H. McClellan, T. W. Parks, and L. R. Rabiner, “A computer program for designing optimum FIR linear phase digital filters,” *IEEE Trans. on Audio and Electroacoustics*, vol. 21, no. 6, pp. 506–526, Dec. 1973.

[20] L. R. Rabiner and R. W. Schafer, “Recursive and nonrecursive realizations of digital filters designed by frequency sampling techniques,” *IEEE Trans. on Audio and Electroacoustics*, vol. AU-19, no. 3, pp. 200–207, Sept. 1971.

[21] T. Saramaki, Y. Neuvo, and S. K. Mitra, “Design of computational efficient interpolated FIR filters,” *IEEE Trans. on Circ. and Syst.*, vol. 35, no. 1, pp. 70–88, Jan. 1988.

[22] R. Lyons, “Interpolated narrowband lowpass FIR filters,” *IEEE Sig. Proc. Mag.*, vol. 20, no. 1, pp. 50–57, Jan. 2013.

[23] T. W. Parks and C. S. Burrus, *Digital Filter Design*, ser. Topics in Digital Signal Processing. New York, NY: John Wiley & Sons, 1987.

[24] T. I. Laakso, V. Valimaki, M. Karjalainen, and U. K. Laine, “Splitting the unit delay,” *IEEE Sig. Proc. Mag.*, vol. 13, no. 1, pp. 30–60, Jan. 1996.

[25] H. Knutsson and C.-F. Westin, “Normalized and differential convolution,” in *IEEE Comp. Soc. Conf. on Comp. Vision and Pattern Recognition (CVPR)*, New York, NY, June 1993, pp. 515–523.

[26] C.-F. Westin, K. Nordberg, and H. Knutsson, “On the equivalence of normalized convolution and normalized differential convolution,” in *IEEE Int. Conf. on Acoust., Speech, and Sig. Proc. (ICASSP)*, vol. V, Adelaide, Australia, Apr. 1994, pp. 457–460.

[27] K. Andersson and H. Knutsson, “Continuous normalized convolution,” in *IEEE Int. Conf. on Multimedia and Expo (ICME)*, vol. 1, Lausanne, Switzerland, Aug. 2002, pp. 725–728.

[28] J. Russ, *The Image Processing Handbook*. Boca Raton, FL: CRC Press, 2011.

[29] M. Soumekh, “Moving target detection in foliage using along track monopulse synthetic aperture radar imaging,” *IEEE Trans. on Image Proc.*, vol. 6, no. 8, pp. 1148–1163, Aug. 1997.

[30] M. I. Skolnik, *Introduction to Radar Systems*. New York, NY: McGraw-Hill, 1980.

[31] M. Eineder, N. Adam, R. Bamler, N. Yague-Martinez, and H. Breit, “Spaceborne spotlight SAR interferometry with TerraSAR-X,” *IEEE Trans. on Geosci. and Remote Sensing*, vol. 47, no. 5, pp. 1524–1535, May 2009.

[32] N. Adam, B. Kampes, M. Eineder, J. Worawattanamateekul, and M. Kircher, “The development of a scientific permanent scatterer system,” in *Proc. ISPRS/EARSeL Joint Workshop on High-Resolution Mapping from Space*, Hannover, Germany, 2003, pp. 1–6.