One example in concern with extension and separate analyticity properties of meromorphic mappings

S. Ivashkovich
March 1994

Abstract
We construct a compact complex manifold of dimension three, such that every meromorphic map from two-dimensional domain into this manifold extends meromorphically onto the envelope of holomorphy of this domain, but there is a meromorphic map of punctured three-dimensional ball into our manifold, which doesn’t extend to origin. A Theorem describing the obstructions occuring here is given.

0. Introduction.

The purpose of this paper is to present an example of compact complex three-fold having “strange” behavior with respect to a meromorphic mappings into it and give certain positive results in entitled direction.

All complex spaces, which we consider in this paper are supposed to be reduced and normal.

Let $D$ and $X$ be complex spaces. Recall that a meromorphic mapping $f$ from $D$ to $X$ is defined by its graph $\Gamma_f$, which is an analytic subset of the product $D \times X$, satisfying the following conditions:

(i) $\Gamma_f$ is locally irreducible analytic subset of $D \times X$.

(ii) The restriction $\pi|_{\Gamma_f} : \Gamma_f \to D$ of the natural projection $\pi : D \times X \to D$ to $\Gamma_f$ is proper, surjective and generically one to one.

This notion of meromorphicity is due to Remmert, see [Re], and is based on the observation that meromorphic functions on $D$ are precisely the meromorphic mappings, in the sense just defined into $X = \mathbb{CP}^1$. Denote by

$$H^n(r) = \{(z_1,\ldots,z_n) \in \mathbb{C}^n : \|z\| < r, |z_n| < 1 \text{ or } \|z\| < 1, 1 - r < |z_n| < 1\}$$

(0.1)
a $n$-dimensional Hartogs figure. Here $z' = (z_1,\ldots,z_{n-1})$, $0 < r < 1$ and $\|\cdot\|$ stands for the polydisk norm in $\mathbb{C}^n$.

Definition. Recall, that the complex space $X$ possesses a meromorphic (holomorphic) extension property in dimension $n$ if any meromorphic (holomorphic) mapping $f : H^n(r) \to X$ extends meromorphically (holomorphically) onto the unit polydisk $\Delta^n$.

When one studies an extension properties of holomorphis mappings one finds the following statements, proved correspondingly in [Sh-1] and [Iv-1], usefull:

AMS subject classification: 32 D 15, Key words: meromorphic map, Hartogs extension theorem, Lelong number, separate meromorphicity, Rothstein-type extension theorem.
1. If a complex space $X$ possesses a holomorphic extension property in dimension $n$ then for every domain $D$ over a Stein manifold $\Omega$ of dimension $n$ any holomorphic mapping $f : D \to X$ extends to a holomorphic mapping $\hat{f} : \hat{D} \to X$ from the envelope of holomorphy $\hat{D}$ of $D$ into $X$.

2. If a complex space $X$ possesses a holomorphic extension property in dimension 2 then $X$ possesses a holomorphic extension property in all dimensions $n \geq 2$.

The same proof as was given by B. Shiffman in [Sh-1] shows that the first statement remains valid for the meromorphic mappings. While, and this is our starting point here, the second fails to be true!

Namely in §1 we shall construct the following

Example. There exists a compact complex three-fold $X$ such that:

(a) For every domain $D$ in $\mathbb{C}^2$ every meromorphic mapping $f : D \to X$ extends to a meromorphic mapping $\hat{f} : \hat{D} \to X$. Here $\hat{D}$ stands for the envelope of holomorphy of $D$.

(b) But there exists a meromorphic mapping $F : \mathbb{B}^3 \setminus \{0\} \to X$ from punctured three-ball into $X$ which does not extend to the origin.

This example shows the essential difference between extension properties of holomorphic and meromorphic mappings. In fact this phenomena backwards several difficulties in studying the meromorphic maps. That's why we devote the "positive" part of this paper to the explanations what are the obstructions for passing from extension of mappings of two-dimensional domains to three-dimensional ones. We shall prove in this direction the following

Theorem. Let $f : H^3(r) \to X$ be a meromorphic mapping of 3-dimensional Hartogs figure into a complex space $X$, which possesses a meromorphic extension property in dimension two. Then:

(i) $f$ extends to a meromorphic mapping of $\Delta^3 \setminus S$, where $S$ has a form $S = S_1 \times S_2 \times S_3$, and $S_j$ are closed subsets in $\Delta$ of harmonic measure zero.

(ii) Let $s_0 \in S$ be a point. Suppose that the Lelong numbers $\Theta(f^* w_h, s_0)$ and $\Theta((f^* w_h)^2, s_0)$ are finite. Then there exists a regular modification $\pi : \hat{\Delta}^3 \to \Delta^3$, such that $f \circ \pi$ holomorphically extends onto some neighborhood of $\pi^{-1}(s_0)$.

Remark. In this theorem we suppose that the space $X$ is equipped with some Hermitian metric $h$, and $w_h$ its associated (1,1)-form. Currents $T_k = (f^* w)^k$ are defined on $\Delta^3 \setminus S$. We define the Lelong numbers of $(k,k)$-current $R$ in $s_0$ as

$$\Theta(R, s_0) = \lim_{\varepsilon \to 0} \sup_{\varepsilon > 0} 1/\varepsilon^{2(n-k)} \int_{B_{s_0}(\varepsilon) \setminus S} R \wedge (dd^c \|z\|^2)^{2(n-k)}. \quad (0.2)$$

$\Theta(R, s_0)$ can take also infinite values. Remark only that finiteness or infiniteness of those numbers in $s_0$ doesn't depend on the particular choice of metric $h$ on $X$.

We want to point out here the following. If one wishes to use the Bishops extension theorem for analytic sets in order to extend the graph $\Gamma_f$ from $(\Delta^3 \setminus S) \times X$ onto $\Delta^3 \times X$, one
needs to estimate the volume of $\Gamma_f$ in the neighbourhood of $S$, say in the neighbourhood of some point $s_0 \in S$. For this one needs to estimate besides

$$\int_{B_{s_0}(\varepsilon)} f^* w \wedge (dd^c \|z\|^2)^2$$

and

$$\int_{B_{s_0}(\varepsilon)} (f^* w)^2 \wedge dd^c \|z\|^2$$

also the integral $\int_{B_{s_0}(\varepsilon)} (f^* w)^3$. The theorem stated above makes possible for us to get rid of the last integral.

Of course the finitness of Lelong numbers of the current $f^* w$ and its powers are nesessary for the extension of $f$. Finally, holomorphic extension of $f \circ \pi$ to the neighborhood of $\pi^{-1}(s_0)$ implies a meromorphic extension of $f$ to the neighborhood of $s_0$.

Methods which are developed here we apply to the questions of separate analyticity of meromorphic mappings with values in general complex spaces, see 2.5 for details.

I would also like to give my thanks to the referee for numerous remarks and corrections.

Table of content.

0. Introduction. .......................................................... pp. 1-3
1. Counterexample. ........................................................ pp. 3 - 9
   1.1. Construction of an Example. 1.2. Construction of the
   universal cover and nonextendable map. 1.3. Two-dimensional
   extension property of $X$.
2. Meromorphic polydisks. ............................................. pp. 9 - 20
   2.1. Generalities on pluripotential theory. 2.2. Passing to
   higher dimensions: holomorphic case. 2.3. Sequences
   of analytic sets of bounded volume. 2.4. Families of
   meromorphic polydisks. 2.5. Application: Rothstein-type theorem
   and separate meromorphy.
3. Estimates of Lelong numbers from below. ...................... pp. 20 - 29
   3.1. Generalities on blowings-up. 3.2. Estimates. 3.3. Proof of
   the Theorem.
4. References. ........................................................... pp. 29 - 31

1. Counterexample.

We shall start with the construction of the example announced in the Introduction.

1.1. Construction of an example.

Take $\mathbb{C}^3$ with coordinates $z_1, z_2, z_3$. Denote by $l_0$ the line $\{z_3 = z_2 = 0\}$. $l_0$ has a natural coordinate $z_1$. By $X_0$ denote the ball of radii $1/2$ centered at zero in $\mathbb{C}^3$.

Consider now a blowing up $\hat{\mathbb{C}}^3$ of $\mathbb{C}^3$ along $l_0$. The origin in $\mathbb{C}^3$ we denote by $0_0$. Let $\pi_0 : \hat{\mathbb{C}}^3 \to \mathbb{C}^3$ be our modification, $E_0 = \pi_0^{-1}(l_0)$ - the exceptional divisor. By $0_1$ we denote the point of intersection of the strict transform $\pi_0^{-1}([z_3])$ of $z_3$ - axis with $E_0$. In the affine neighborhood of $0_1$ we introduce the standard coordinate system $u_1, u_2, u_3$ such that modification $\pi_0 : \hat{\mathbb{C}}^3 \to \mathbb{C}^3$ in those coordinates is given by
The preimage of 0 under the modification $\pi_0$ we denote by $l'_1$. Put $\pi_0^{-1}(X_0) = X_1$. Now we blow up $X_1$ along $l'_1$. Denote by $\pi_1 : X_2 \rightarrow X_1$ this modification. Let $E_1$ be the exceptional divisor of $\pi_1$ and $l'_1$ let be the intersection of $E_1$ and $E_0$ (more precisely with the strict transform of $E_0$ under $\pi_1$). The proper preimage of 0 under $\pi_1$ we denote by $l''_1$. Put $\pi^{-1}_0(X_0) = X_1$. Now we blow up $X_1$ along $l''_1$. Denote by $\pi_1 : X_2 \rightarrow X_1$ this modification. Let $E_1$ be the exceptional divisor of $\pi_1$ and $l'_1$ let be the intersection of $E_1$ and $E_0$ (more precisely with the strict transform of $E_0$ under $\pi_1$). The proper preimage of 0 under $\pi_1$ we denote by $l''_1$. The point of intersection of $l'_1$ with $E_0$ denote by 0. In the affine neighborhood of 0 we introduce coordinate system $v_1, v_2, v_3$ (standard one for the modification $\pi_1$) in which our modification $\pi_1$ is given by

\begin{align*}
  u_1 &= v_1 \\
  u_2 &= v_2 \\
  u_3 &= v_3
\end{align*}

Next, consider the blowing up of $X_2$ along $l'_2$. $\pi_2 : X_3 \rightarrow X_2$. By $E_2$ denote the exceptional divisor of $\pi_2$. Put $l'_2 = E_2 \cap E_1$, $l'_3 = E_2 \cap E_0$. 03 let be the point of intersection $E_0 \cap E_1 \cap E_2$. Here again by $E_0, E_1$ we understand the appropriate strict transforms. We introduce a coordinates $w_1, w_2, w_3$ in affine neighborhood of 0 in such a way that $\pi_2 : X_3 \rightarrow X_2$ is given in those coordinates by

\begin{align*}
  v_1 &= w_1 w_2 \\
  v_2 &= w_2 \\
  v_3 &= w_3
\end{align*}

Take now the composition $\pi = \pi_0 \circ \pi_1 \circ \pi_2 : X_3 \rightarrow X_0$, which in coordinates can be written as

\begin{align*}
  z_1 &= w_1 w_2 \\
  z_2 &= w_1 w_2^2 w_3 \\
  z_3 &= w_1 w_2 w_3
\end{align*}

Take now two copies of $X_0$, denote them by $X_0^{(1)}$ and $X_0^{(2)}$. On $X_0^{(i)}$ let us fix initial coordinates $z_1^{(i)}, z_2^{(i)}, z_3^{(i)}$. On the affine neighborhood of $0_3^{(i)} \in X_3^{(i)}$ we fix coordinates $w_1^{(i)}, w_2^{(i)}, w_3^{(i)}$ so as where introduced by (1.1.1), (1.1.2), (1.1.3). Consider now a holomorphic mapping $\phi : X_0^{(1)} \rightarrow X_3^{(2)}$, which sends zero $0_0^{(1)}$ from $X_0^{(1)}$ to $0_3^{(2)}$ and in coordinates $z^{(1)}$ and $w^{(2)}$ is defined by

\begin{align*}
  w_1^{(2)} &= z_1^{(1)}
\end{align*}

4
\[ w_2^{(2)} = z_2^{(1)} \]  
\[ w_3^{(2)} = z_3^{(1)} \]  
(1.1.5)
i.e. \( \phi \) is identity in \( z^{(1)}, w^{(2)} \).

Now blow up \( l_0^{(1)} \) in \( X_0^{(1)} \) and \( l_3^{(2)'} \) in \( X_3^{(2)} \). Denote the manifolds obtained by \( \hat{X}_0^{(1)} \) and \( \hat{X}_3^{(2)} \). \( \hat{X}_0^{(1)} = X_1^{(1)} \) in our notations. Proper preimage of \( l_3^{(2)'} \) denote by \( E_3 \). Lift up \( \phi \) to a biholomorphism \( \hat{\phi} : \hat{X}_0^{(1)} \to \hat{\phi}(\hat{X}_0^{(1)}) \subset \hat{X}_3^{(2)} \). The fact that \( \phi \) lifts to a biholomorphism, not just a bimeromorphism, of blowings up follows from the observation that \( l_3^{(2)'} \) in coordinates \( w^{(2)} \) is given by \( \{w_2^{(2)} = w_3^{(2)} = 0\} \). Recall, that \( l_0^{(1)} \) is given by \( \{z_2^{(1)} = z_3^{(1)} = 0\} \). So from (1.1.5) we see that \( \phi \) sends the center of blown up \( l_0^{(1)} \) into another one \( l_3^{(2)'} \), and thus the lifting \( \hat{\phi} \) is holomorphic.

Now take small \( \varepsilon > 0 \). By \( X_j^{\pm \varepsilon} \) denote the ball in \( \mathbb{C}^3 \) of radii \( \frac{1}{2} \pm \varepsilon \). By \( X_j^{\pm \varepsilon}, j = 1, 2, 3 \) denote manifolds obtained by our blowings up from \( X_0^{\pm \varepsilon} \) as \( X_j \) from \( X_0 \). By \( X_j^{(i)\pm \varepsilon} \) denote the corresponding manifolds obtained from \( X_j^{(i)} \), \( i = 1, 2 \).

Take \( U^\varepsilon = \hat{X}_3^{(2)\varepsilon} \setminus \hat{\phi}(\hat{X}_0^{(1)\varepsilon}) \) - domain in \( \hat{X}_3^{(2)\varepsilon} \). Here \( (2)^\varepsilon \) and \( (1)^{-\varepsilon} \) one should understand as indices, not as numbers. By \( \pi_0^{(1)} : \hat{X}_0^{(1)} \to X_0^{(1)} \), \( \pi_i^{(2)} : X_i^{(2)} \to X_i^{(2)}, i = 0, 1, 2, \pi_3^{(2)} = \pi_0^{(2)} \circ \pi_1^{(2)} \circ \pi_2^{(2)} \) we denote the corresponding modifications.

Consider a map \( \psi : X_0^{(2)\varepsilon} \to X_0^{(1)\varepsilon} \) which in our coordinates \( z^{(i)} \) is given as identity:

\[ z_1^{(1)} = z_1^{(2)} \]
\[ z_2^{(1)} = z_2^{(2)} \]
\[ z_3^{(1)} = z_3^{(3)} \]  
(1.1.6)

Lift this map onto the blowings up to get

\[ \hat{\psi} : \hat{X}_0^{(2)\varepsilon} \to \hat{X}_0^{(1)\varepsilon} \]  
(1.1.7)

Now remark that further blowings up of \( \hat{X}_0^{(2)\varepsilon} = X_1^{(2)\varepsilon} \) do not effect the neighborhoods of the boundaries of \( X_0^{(i)} \). We shall denote this neighborhoods as \( V^{(i)\varepsilon} \), having in mind that they are copies in \( \hat{X}_0^{(i)} \) of \( V^\varepsilon = \hat{X}_0^{\varepsilon} \setminus cl(\hat{X}_0^{-\varepsilon}) \). So biholomorphism \( \hat{\psi} \), defined in (1.1.7), can be restricted to a biholomorphism (which we denote by the same letter) \( \hat{\psi} : V^{(2)\varepsilon} \to V^{(1)\varepsilon} \).

Now we can define a biholomorphism \( g = \hat{\phi} \circ \hat{\psi} \) between \( V^{(2)\varepsilon} \) and \( \hat{\phi}(V^{(1)\varepsilon}) \) - neighborhoods of two connected components of the boundary of \( U^{(2)} \) in \( U^{(2)\varepsilon} \). Here we defined \( U^{(i)} \) and \( U^{(i)\varepsilon} \) to be a copies in \( X_3^{(i)\varepsilon} \), of the open sets \( U = \hat{X}_3^{(2)\varepsilon} \setminus \hat{\phi}(\hat{X}_0^{(1)}) \) and \( U^\varepsilon = \hat{X}_3^{(2)\varepsilon} \setminus \hat{\phi}(\hat{X}_0^{(1)\varepsilon}) \).

Now we can glue the neighborhoods \( V^{(2)\varepsilon} \) and \( \hat{\phi}(V^{(1)\varepsilon}) \) by biholomorphism \( g \) to get from \( U^{(2)\varepsilon} \) a compact complex threefold \( X \).

Note that after gluing \( E_0 \) and \( E_3 \) constitute one divisor. We shall denote it both as \( E_0 \) or \( E_3 \).

1.2. Construction of the universal cover and nonextendable map.
We must prove two facts about $X$. First: that there exists a meromorphic mapping $F : B^3 \setminus \{0\} \to X$ which does not extend to the origin. Second: any meromorphic map $f : H^2(r) \to X$ extends to a meromorphic map $\hat{f} : \Delta^2 \to X$. Here $H^2(r) = \{(u_1, u_2) \in \Delta^2 : |u_1| < r \text{ or } 1 - r < |u_2| < 1 \}$ is a Hartogs figure of dimension two. To prove this we shall need the universal covering $\tilde{X}$ of $X$. In this paragraph we give the construction of $\tilde{X}$ together with nonextendable mapping $F : B^3 \setminus \{0\} \to X$.

We shall make this construction in 5 steps.

In the previous paragraph we had constructed a biholomorphic mapping $g : V^\varepsilon \to \hat{\phi}(V^\varepsilon)$, where both $V^\varepsilon$ and $\hat{\phi}(V^\varepsilon)$ are open subsets of $\tilde{X}_3^\varepsilon$, and actually are the neighborhoods of the connected components of the boundary of $U = \tilde{X}_3 \setminus \hat{\phi}(\tilde{X}_0)$. To simplify a bit our notations we denote the subset $\hat{\phi}(V^\varepsilon)$ of $\tilde{X}_3^\varepsilon$ simply as $V_0^\varepsilon$.

**Step 1. Construction of manifolds $X^{(i)}$ and $U_i^\varepsilon$, $i \geq 0$.**

Put $X^{(0)} = \tilde{X}_3^\varepsilon$. Domain $U^\varepsilon$ in $X^{(0)}$ denote as $U_0^\varepsilon$. Attach $X^{(0)}$ to $U^\varepsilon$ by $g : V^\varepsilon \to V_0^\varepsilon \subset U^\varepsilon$. Manifold obtained, denote as $X^{(1)}$.

Manifold $X^{(1)}$ has exactly one end, namely $V^\varepsilon$. Domain in $X^{(1)}$ which is $U^\varepsilon \sqcup U_0^\varepsilon /g$ we denote by $U_1^\varepsilon$.

We can repeat this construction with $X^{(1)}$ instead of $X^{(0)}$. Namely, take $X^{(1)}$ and attach it to $U^\varepsilon$ by $g : V^\varepsilon \to V_0^\varepsilon \subset U^\varepsilon$ to get $X^{(2)}$. Domain in $X^{(2)}$ which is $U_1^\varepsilon \sqcup U^\varepsilon /g$ we denote by $U_2^\varepsilon$.

Suppose that the manifold $X^{(i)}$ is constructed and it has one end $V^\varepsilon$. Attach $X^{(i)}$ to $U^\varepsilon$ by $g : V^\varepsilon \to U_i^\varepsilon \subset U^\varepsilon$ to get $X^{(i+1)}$, which again has one end $V^\varepsilon$. Domain in $X^{(i+1)}$ which is $U_i^\varepsilon \sqcup U^\varepsilon /g$ denote by $U_{i+1}^\varepsilon$.

**Step 2. Construction of manifolds $X^{(-i)}$ and $U_{-i}^\varepsilon$, $i \geq 0$.**

As $X^{(0)}$ we take again $\tilde{X}_3^\varepsilon$ with $U_0^\varepsilon = U^\varepsilon$.

To obtain $X^{(-1)}$ attach $\tilde{X}_3^\varepsilon$ to $U_0^\varepsilon$ by $g : V^\varepsilon \to V_0^\varepsilon \subset U_0^\varepsilon$. Domain in $X^{(-1)}$ which is $U_0^\varepsilon \sqcup U^\varepsilon /g$ denote by $U_{-1}^\varepsilon$.

Suppose that the manifold $X^{(-i)}$ and domain $U_{-i}$ in $X^{(-i)}$ are constructed. To get $X^{(-i-1)}$ attach $\tilde{X}_3^\varepsilon$ to $U_{-i}^\varepsilon$ by $g : V^\varepsilon \to V_0^\varepsilon \subset U_{-i}^\varepsilon$ and put $U_{-i-1}^\varepsilon = U_{-i}^\varepsilon \sqcup U^\varepsilon /g$.

**Step 3. Construction of the universal cover.**

Manifolds $U_i^\varepsilon$ and $U_{-i}^\varepsilon$ have the common part $U_0^\varepsilon$. So we can glue them by the identification in $U_i^\varepsilon \sqcup U_{-i}^\varepsilon$ of two copies of $U_0^\varepsilon$. The manifold obtained we denote by $U_{i,i}^\varepsilon$. In the same way we can consider $U_i^\varepsilon \sqcup U_{-j}^\varepsilon /g = U_{i,j}^\varepsilon$.

We have now an increasing sequence of complex manifolds $U_0^\varepsilon \subset U_1^\varepsilon \subset \ldots \subset U_i^\varepsilon \subset \ldots$. The union $\bigcup_{i \geq 0} U_{i,-i}^\varepsilon = U_{\infty}$ is a complex manifold, and we are going to prove that it is the universal cover of $X$. We first remark, that obviously $U_{\infty}$ is simply connected.

**Step 4. Construction of group of covering transformations.**

We start with the biholomorphic mapping $g : V^\varepsilon \to V_0^\varepsilon$ which sends one end of $U_0^\varepsilon$ to another. Recall that $U_0^\varepsilon /g \cong X$ is our compact threefold.

**Lemma 1.2.1** $g$ extends to biholomorphic automorphism of $U_{\infty}$. Moreover we have $X \cong U_{\infty} /\langle g^n \rangle_{n \in \mathbb{Z}}$. 


For any pair of integers \((i,j)\) such that \(i > j\) we consider a subdomain \(U_{i,j}^\varepsilon\) in \(U_\infty\) defined in a following way.

If \(i \geq 0 \geq j\) then \(U_{i,j}^\varepsilon\) is already defined.

If \(0 \geq i > j\) then \(U_{i,j}^\varepsilon = U_j^\varepsilon \setminus U_i^\varepsilon\).

If \(i > j \geq 0\) then \(U_{i,j}^\varepsilon = U_i^\varepsilon \setminus U_j^\varepsilon\).

Here domains \(U_{i,j}^\varepsilon\) are constructed from \(U^\varepsilon - \tilde{X}_3^{(2)} - \hat{\phi}(\tilde{X}_0^{(1)^\varepsilon})\) in a similar way as \(U_i^\varepsilon\) from \(U^\varepsilon\), and \(U_{i,j}^\varepsilon\) form \(U^{\varepsilon -}\) onto \(\hat{\phi}(\tilde{X}_0^{(1)^\varepsilon})\).

By attaching \(X_3^\varepsilon\) to \(U_{i,j}^\varepsilon\) by \(g\) we get manifolds \(X^{(i,j)}, \infty \geq i > j \geq -\infty\).

Our map \(g : V^\varepsilon \longrightarrow V_0^\varepsilon\) extends to a biholomorphic map of \(U_{i,j}^\varepsilon\) onto \(U_{-1,-\infty}^\varepsilon\). We are going to prove that \(g\) extends to a bimeromorphic map from \(U_{0,-\infty}^\varepsilon\) onto \(U_{-1,-\infty}^\varepsilon\).

For this purpose take any \(j < 0\). Remark that \(g\) maps the neighborhood \(V^\varepsilon\) of \(X(0,j)\) biholomorphically onto the neighborhood \(V_0^\varepsilon\) of the boundary of \(X(-1,j-1)\). Note that \(X(0,j)\) and \(X(-1,j-1)\) could be naturally blown down to get an \(X_0\) - the usual ball in \(\mathbb{C}^3\). So \(g\) became a bimeromorphism of the neighborhood of \(S^5\) onto itself. So it extends by the Hartogs extension theorem.

In the same way \(g^{-1}\) extends to a biholomorphic map of \(U_{\infty,-1}^\varepsilon\) onto \(U_{\infty,0}^\varepsilon\).

q.e.d.

Step 5. Construction of nonextendable map.

Our construction gives a natural bimeromorphic map \(\tilde{f}\) from \(X_0^\varepsilon \setminus \{0\} \cong \mathbb{B}_*^3\) onto \(U_{0,-\infty}^\varepsilon\). Then taking the composition \(p \circ \tilde{f} = F\) we get our nonextendable map.

Another way to obtain \(\tilde{f}\) is to consider the composition \(i \circ \pi^{-1}\) of inclusion \(i : V^\varepsilon \longrightarrow U_0^\varepsilon\) with the restriction of blowing up \(\pi^{-1}\) onto \(X_0^\varepsilon \setminus \text{cl}(X_0^\varepsilon)\). Now, similarly to the step 4, one can easily observe that \(i \circ \pi^{-1}\) extends to a bimeromorphic map \(\tilde{f} : X_0^\varepsilon \setminus \{0\} \longrightarrow U_{0,-\infty}^\varepsilon\).

1.3. Two-dimensional extension property of \(X\).

In this paragraph we shall prove that every meromorphic map \(f : H^2(r) \longrightarrow X\) extends to a meromorphic map \(\tilde{f} : \Delta^2 \longrightarrow X\).

Take \(0 < \delta < r\) and denote by \(H_\delta(r) = \{(u_1,u_2) \in \Delta^2 : |u_1| < r - \delta, |u_2| < 1 - \delta\text{ or }|u_1| < 1 - \delta, |u_2| < 1 - \delta\} \) - shrunk Hartogs domain. It is sufficient to prove the extendability of \(f\) onto \(\Delta^2_{1-\delta} = \{(u_1,u_2) \in C^2 : |u_j| < 1 - \delta, j = 1,2\}\) for arbitrary \(\delta > 0\).

Using the fact that \(H^2(r)\) is simply connected we can lift \(f\) to a meromorphic map \(\tilde{f} : H^2(r) \longrightarrow U_\infty\). \(H_\delta(r) \subset \subset H^2(r)\) so \(\tilde{f}(H_\delta(r)) \subset \subset U_\infty\), the latter means that there exists \(i < \infty\) such that \(\tilde{f}(H_\delta(r)) \subset \subset U_i^\varepsilon\). Applying \(g^i\) to \(\tilde{f}\) we get a map \(f_1 = g^i \circ \tilde{f} : H_\delta(r) \longrightarrow U_{0,-2i}\). If we extend \(f_1\) as a meromorphic map \(\hat{f}_1\) from \(\Delta^2_{1-\delta}\) to \(U_{0,-\infty}^\varepsilon\) then \(p \circ g^{-i} \circ \hat{f}_1\) will give us a desired extension of \(f\).
Take a composition \( h = p \circ f_1 : H_\delta(r) \to X_0 \cong B^{3}_{1/2} \subset \mathbb{C}^{3} \). This is a holomorphic mapping of \( H_\delta(r) \) into \( X_0 \subset \mathbb{C}^{3} \). So it extends to a holomorphic map \( \hat{h} : \Delta^{2}_{1-\delta} \to X_0 \). By \( A \) denote the preimage of zero under \( \hat{h} \). There are three possibilities.

**Case 1.** \( A = \Delta^{2}_{1-\delta} \). That means that \( f(H_\delta(r)) \subset E \), where \( E \) is an exceptional divisor of \( X \). But all three components of \( E \) are bimeromorphic to \( \mathbb{CP}^{2} \). So in that case \( f \) obviously extends onto \( \Delta^{2}_{1-\delta} \).

**Case 2.** \( A \) has components of positive dimension.

Denote them by \( A_{1},...,A_{N} \). Let also \( a_{1},...,a_{M} \) are the isolated preimages of zero under \( \hat{h} \).

Because \( p : U_{0,-\infty}^{s} \setminus \left( \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{3} E_{j}^{1} \cup \bigcup_{j=1}^{3} E_{j}^{0} \right) \to X_{0} \setminus \{0\} \) is bimeromorphic we have an extension of \( f_{1} \) onto \( \Delta^{2}_{1-\delta} \). Here we denote by \( A_{j}^{+} = A_{j} \cap H_\delta(r) \), by \( A_{j}^{-} = A_{j} \setminus A_{j}^{+} \), and \( E_{j}^{i} \) denote the copies of \( E_{j} \) in \( U_{i,i-1}^{s} \), \( j = 0,1,2,3 \). Note that \( E_{3}^{j} \) is glued to \( E_{0}^{-1} \) and constitute a single divisor, which we shall denote either by \( E_{3}^{1} \) or \( E_{0}^{-1} \).

Consider now \( f_{1} \) as a meromorphic map from \( H_\delta(r) \) to \( U_{0,-2i}^{s} \subset X^{(-2i-L)} \). Because \( X^{(-2i-L)} \) is nothing but a blown up ball we can extend \( f_{1} \) as a map \( \hat{f}_{1} \) from \( \Delta^{2}_{1-\delta} \) to \( X^{(-2i-L)} \). Now \( \hat{f}_{1}(U_{i=1}^{N} A_{j}) \subset U_{k=0}^{-2i} U_{j=0}^{3} E_{j}^{i} \), because \( f_{1}(U_{j=1}^{N} A_{j}^{+}) \subset U_{k=0}^{-2i} U_{j=0}^{3} E_{j}^{i} \).

That means that \( \hat{f}_{1}(U_{i=1}^{N} A_{j}) \) do not intersect \( E_{2}^{-2i-L} \) for \( L \) big enough. This follows from Lemma 1.3.1 stated below and the fact that nontrivial holomorphic mapping from 1-dimensional space is always locally proper. So its image is locally contained in a curve by the Remmert proper mapping theorem. Before applying Lemma 1.3.1 one only should embed this curve into a (nonsmooth in general) hypersurface. So if we consider \( f_{1} \) as a map into \( U_{0,-\infty} \), then \( f_{1}(U_{j=1}^{N} A_{j}) \subset U_{0,-2i-2}^{s} \). So there is a neighborhood \( W \) of \( U_{j=1}^{N} A_{j} \) such that \( f_{1}(W) \subset U_{0,-2i-2}^{s} \). The latter means that \( p \circ g^{-1} \circ f_{1} \) extends onto \( W \). So our map \( f \) is extended onto \( \Delta^{2}_{1-\delta} \setminus \bigcup_{k=1}^{M} \{a_{k}\} \).

So the last case left.

**Case 3.** \( A \) is a finite set of points \( \{a_{1},...,a_{M}\} \).

Take a neighborhood \( B \) of \( a_{1} \). Because \( a_{1} \) is isolated preimage of zero under \( \hat{h} \), there is a neighborhood of \( a_{1} \), say \( B \) itself, such that \( \hat{h} \mid_{B} \) is proper. So \( \hat{h}(B) \) is contained in a germ of hypersurface \( P \) in \( X_{0} \), passing through origin. Note that this fact was not obvious from the beginning because of the well known Osgood example, see [4], p.155. If we shall prove that the lifting \( p^{-1}(P) \) of \( P \) onto \( U_{0,-\infty} \) is relatively compact in \( U_{0,-\infty}^{s} \) then we are done, because then \( f_{1} \) maps \( B \setminus \{a_{1}\} \) into some \( U_{0,L}^{s} \subset X_{1-L} \).

To prove that the lifting of \( P \) by \( p \) is relatively compact in \( U_{0,-\infty}^{s} \), the next lemma is sufficient. Denote by \( \Phi \) the transformation given by (1.1.4).

**Lemma 1.3.1** Let \( P(z) \) be a convergent series in variables \( z_{1},z_{2},z_{3} \). Then there exist an \( N \) such that \( P(\Phi^{-N}(z)) \) has the form \( z_{1}^{N_{1}} \cdot z_{2}^{N_{2}} \cdot z_{3}^{N_{3}} \cdot \left[ a + F(z) \right] \), where \( a \) is some nonzero constant, \( F \) is convergent series and \( F(0) = 0 \).

**Proof.** Consider a Newton polyhedron \( N_{P} \) of \( P(z) \) in \( \mathbb{R}^{3} \). Consider also the next unimod-
ular transformation
\[ A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \]
of \( \mathbb{R}^3 \). One can easily check the following properties of \( A \):
1. The real eigenvalue \( \mu \) of \( A \) satisfies \( \mu > 3 \).
2. Consequently the complex one \( \lambda \) satisfies \( |\lambda|^2 = 1/\mu < 1/3 \).
3. The eigenvector \( v \), which corresponds to \( \mu \), has positive coordinates.
4. The \( A \)-invariant plane \( \Pi \), which corresponds to \( \lambda \) intersects the positive octant by zero.
5. \( \Pi \) contains no vectors with integer coordinates.

The latter is true because \( A \) acts on \( \Pi \) as rotation together with contraction by \( \lambda \).

From 5 we see that there is a constant \( \alpha > 0 \), such that every edge of \( N_P \) has an angle at least \( \alpha \) with \( \Pi \). Now, because \( A \) acts as contraction on \( \Pi \) and as dilatation on \( [Rv] \), all edges of \( N_{PN} \) have angles with \( \Pi \) which are uniformly tending to the angle between \( v \) and \( \Pi \). Here by \( P^N \) we denoted \( P(\Phi \circ \ldots \circ \Phi(z)) \).

This proves the lemma.

q.e.d.

Remarks

1. The example, just constructed, has some common futures with an example of Hirschowitz, see [Hs], of weakly but not strongly meromorphic mapping. In fact the map \( F \), obtained in 1.2 is weakly meromorphic in the sense, that for any complex curve \( C \ni 0 \) the restriction \( F|_{C\setminus 0} \) extends holomorphically onto \( C \). But clearly the two dimensional extension property of our \( X \) is much stronger condition than just a weak meromorphicity of some given map, namely of \( F \), which is not strongly meromorphic.

2. Note also, that \( X \) contains some sort of ”spherical shell”. Namely a bimeromorphic image \( V^\varepsilon \) of the neighborhood of \( S^5 \) from \( \mathbb{C}^3 \) which doesn’t bound a domain in \( X \). We want to point out that this is different from the situation examined by M. Kato in [Ka-1], who studied so called ”global spherical shells” i.e. biholomorphic images of some neighborhoods of \( S^{2n-1} \) from \( \mathbb{C}^n \), which doesn’t bound. Our \( X \) doesn’t contain a GSS in the sense of Kato.

2. Meromorphic polydisks.

2.1. Generalities on pluripotential theory.

We start with some well known facts from pluripotential theory. Let \( D \) be an open subset of \( \mathbb{C}^n \) and \( S \) subset of \( D \). Consider the next class of functions

\[ U(S,D) = \{ u \in \mathcal{P}_-(D) : u|_S \geq 1 \} \]  \hspace{1cm} (2.1.1)

where by \( \mathcal{P}_-(D) \) we denote the class of nonnegative plurisuperharmonic functions in \( D \).

Definition 2.1.1. The lower regularization \( w_* \) of the function

\[ w(\zeta,S,D) = \inf\{ u(\zeta) : u \in U(S,D) \} \]  \hspace{1cm} (2.1.2),
is called a $P$-measure of $S$ in $D$, i.e.

$$w_*(z, S, D) = \lim_{\zeta \to z} \inf w(\zeta, S, D) \quad (2.1.3)$$

Note that $w_*$ is plurisuperharmonic in $D$.

**Definition 2.1.2.** A point $s_0 \in S$ is called a locally regular point of $S$ if $w_*(s_0, S \cap \Delta(s_0, \varepsilon), \Delta(s_0, \varepsilon)) = 1$ for all $\varepsilon > 0$.

We shall also say that the set $S$ is locally regular at $s_0$.

The next statement is known as the two constants theorem; see for ex. [Sa]:

let $v$ be plurisubharmonic in $D$ such that $v|_S \leq M_0$ and $v|_D \leq M_1$. Then for $z \in D$ one has

$$v \leq M_0 \cdot w_*(z) + M_1 \cdot [1 - w_*(z)] \quad (2.1.4).$$

The following lemma can be proved by Taylor expansion and using (2.1.4).

**Lemma 2.1.1.** Suppose that the function $f$ is holomorphic and bounded by modulus with constant $M$ in $\Delta^n \times \Omega$, where $\Omega$ is a subdomain in $\Delta^q$. Let for $s \in S$ $f_s = f(s, \cdot)$ extends holomorphically onto $\Delta^q = \{s\} \times \Delta^q$. Suppose that all this extensions are also bounded by modulus with $M$. If $s_0 \in S$ is a locally regular point then for every $0 < b < 1$ there is an $r > 0$ such that $f$ holomorphically extends onto $\Delta^n(s_0, r) \times \Delta^q(b)$ and is bounded by modulus with $M \cdot \frac{1+b}{1-b}$.

Recall that a compact subset $S \subset D$ is called pluripolar if there exists a plurisubharmonic function $u$ in $D, u \neq -\infty$, such that $u|_S \equiv -\infty$. $S$ is complete pluripolar if $S = \{z : u(z) = -\infty\}$. We shall repeatedly use the following statement:

if subset $S \subset D$ (D is now pseudoconvex) is not locally regular at all its points then $S$ is pluripolar,

see [B-T],[Sa]. We shall use the following immediate corollary from the famous Josefson theorem, see [Kl]:

Let $\Omega$ be a pseudoconvex set in $\mathbb{C}^n$, and let $S_n$ be a sequence of subsets of $\Omega$ such that:

1) $S_1$ is closed and pluripolar in $\Omega$;
2) $S_{n+1} \subset \Omega \setminus S_n$ is closed in $\Omega \setminus S_n$ pluripolar;

Then $S := \bigcup_{n=1}^{\infty} S_n$ is pluripolar in $\Omega$.

**2.2. Passing to higher dimensions: holomorphic case.**

We want to show now that if a complex space $X$ possesses a holomorphic extension property in dimension two than $X$ possesses this property in all higher dimensions. Let $n \geq 3$. Put

$$E^n(r) = (\Delta^{n-2}_r \times \Delta_r \times \Delta) \cup (\Delta^{n-2}_r \times \Delta \times \Delta_{1-r,1}) = \Delta^{n-2}_r \times H^2(r)$$

**Lemma 2.2.1.** If any meromorphic (holomorphic) map $f : E^n(r) \to X$ extends to a meromorphic (holomorphic) map $\hat{f} : \Delta^{n-2}_r \times \Delta^2 \to X$ then the space $X$ possesses a meromorphic (holomorphic) extension property in dimension $n$. 

10
Proof. Let \( f : H^n(r) \to X \) be a meromorphic (holomorphic) map. Because \( E^n(r) \subset H^n(r) \)
the map \( f \) can be extended onto \( \Delta^n \times \Delta^2 \). We shall prove by induction on \( j = 2, \ldots, n \)
that \( f \) can be extended onto \( \Delta^{n-j} \times \Delta^j \). For \( j = n \) we shall get the statement of Lemma.

Suppose it is proved for \( j \). Let \( z_1, \ldots, z_n \) be the coordinates in \( \mathbb{C}^n \). For a point \( z' = (z_{n-j+1}, \ldots, z_n) \in \Delta^{n-j}_1 \) consider a domain \( E_r(z') \) depending on \( \Delta^j \)

\[
E^m_{z_0}(r) = [\Delta^m \times \Delta_r \times \Delta^{j-1}_r(z'_0)] \cup [\Delta^m \times \Delta \times \Delta^{j-1}_r(z'_0)] \times A_{1-r,1}.
\]

Here by \( \Delta^{j-1}_r(z'_0) \) we denote the \( (j-1) \)-disk of radii \( r \) centered at \( z'_0 \). Because \( E^m_{z_0}(r) \subset (\Delta^m \times \Delta^j) \cap H^n(r) \) the map \( f \) by the induction hypothesis extends onto \( \Delta^{n-j-1} \times \Delta \times \Delta^{j-1}_r(z'_0) \times \Delta \). But

\[
\bigcup_{z'_0 \in \Delta^{j-1}} \Delta^{n-j-1}_r \times \Delta \times \Delta^{j-1}_r(z'_0) \times \Delta \subset \Delta^{n-j-1} \times \Delta^{j+1}_r(z'_0) \times \Delta
\]

and the Lemma is proved.

q.e.d.

Now left to prove the following

**Lemma 2.2.2.** Let \( f : E^{n+1}(r) \to X \) be a holomorphic mapping into a complex space \( X \). Suppose that for every \( z_0^1 \in \Delta_r \) the restriction \( f_{z_0^1} \) of \( f \) onto \( \{z^1 = z_0^1\} \cap E^{n+1}(r) := E^{m}_{z_0^1}(r) \)
extends holomorphically onto \( \Delta^{n+1}_{z_0^1}(r) := \{z^1 = z_0^1\} \times \Delta^{n+1}_r \times \Delta^2 \). Then \( f \) holomorphically extends onto \( \Delta^{n+1}_r \times \Delta^2 \).

**Proof.** Shrinking disks and annuli involved in the definitions of \( E^{n+1}(r) \) and \( E^{m}_{z_0^1}(r) \) we can suppose that \( f \) is defined in the neighborhood of \( \overline{E^{n+1}(r)} \) and for every \( z_0^1 \in \Delta_r \) extends
holomorphically to the neighborhood of \( \{z^1 = z_0^1\} \times \Delta^{n+2}_r \times \Delta^2 \) on the hyperplane \( \{z^1 = z_0^1\} \),
which doesn’t depend on \( z_0^1 \). Making homoethy by \( \frac{r}{r} \) on first \( (n-1) \) coordinates we can suppose that \( f \) is holomorphic in the neighborhood of the closure of domain \( \Delta^{n-1} \times \Delta \times \Delta^{n+1}_r \times A_{1-r,1} \) and for every \( z_0^1 \in \Delta \) the restriction \( f_{z_0^1} \) extends holomorphically to the neighborhood of the closure of \( \{z^1 = z_0^1\} \times \Delta^{n+2}_r \times \Delta^2 \) on the hyperplane \( \{z^1 = z_0^1\} \) not depending on \( z_0^1 \).

It is enough to prove that for every \( z_0^1 \in \Delta \) the map \( f \) extends holomorphically onto the \( (n+1) \)-dimensional neighborhood of \( \{z^1 = z_0^1\} \times \Delta^n := \Delta^n_{z_0^1} \). Because \( f_{z_0^1} \) is holomorphic on \( \Delta^n_{z_0^1} \) the graph \( \Gamma_{f_{z_0^1}} \) is an imbedded \( n \)-disk in the space \( \Delta^{n+1} \times X \). Let \( W \) be a Stein neighborhood of \( \Gamma_{f_{z_0^1}} \), see [Si-3]. From the holomorphicity of \( f \) on \( \Delta^{n-1} \times \Delta \times \Delta^{n+1}_r \times A_{1-r,1} \)
and \( \Delta^{n-1} \times \Delta \times \Delta^{n+1}_r \times A_{1-r,1} \) follows that for \( z^1 \) in some neighborhood \( U \ni z_0^1 \), \( f_{z^1}((\Delta^{n-1} \times \Delta \times \Delta^{n+1}_r \times A_{1-r,1})) \subset W \). But \( W \) is Stein, so \( f_{z^1}(\Delta^{n+2}_r \times \Delta^2) \subset W \) for all \( z_0^1 \in U \).

I.e. the map \( f \) maps \( U \times \Delta^{n+2}_r \times \Delta^2 \) to \( W \). So the statement of the Lemma is reduced to the analogous statement for mappings into Stein manifolds, and for them our Lemma is obvious.

q.e.d.
As we already had mentioned in the Introduction, our example shows that the analogous statement for meromorphic mappings is not valid. Here we only want to point out that the proof of Lemma 2.2.2 for the meromorphic case fails, because if $f_{z_0}$ is essentially meromorphic (i.e. not holomorphic) then $\Gamma_{f_{z_0}}$ doesn’t have Stein neighborhoods.

2.3. Sequences of analytic sets of bounded volume.

A Hermitian metric form on the complex space $X$ we define in the following way. Let an open covering $U_\alpha$ of $X$ is given together with proper holomorphic injections $\phi_\alpha : U_\alpha \to V_\alpha$ into a domains $V_\alpha \subset \mathbb{C}^n(\alpha)$. Let $U'_\alpha$ be the images of $U_\alpha$. Let $\{w_\alpha\}$ are positive $(1,1)$-forms on $V_\alpha$. $\{w_\alpha\}$ defines a Hermitian metric form on $X$ if $(\phi_\alpha \circ \phi^{-1}_\beta) w_\beta = w_\alpha$ for all $\alpha, \beta$. Note that $\phi_\alpha \circ \phi^{-1}_\beta$ is defined in some neighborhood of $\phi_\beta(U_\alpha \cap U_\beta)$ in $\mathbb{C}^n(\beta)$. We say that the metric $w$ is Kähler if $dw_\alpha = 0$ on $V_\alpha$ for all $\alpha$.

Fix a complex space $X$, equipped with some Hermitian metric $h$. By $w_h$, or simply by $w$ denote the $(1,1)$-form canonically associated with $h$. Let $\Delta^g$ be a polydisk in $\mathbb{C}^g$ with standard Euclidean metric $e$. The associated form will be denoted by $w_e = dd^c \|z\|^2 = i/2 \sum_{j=1}^g dz_j \wedge d\bar{z}_j$. By $p_1 : \Delta^g \times X \to \Delta^g$ and $p_2 : \Delta^g \times X \to X$ we denote the projections onto the first and second factors. On the product $\Delta \times X$ we consider the metric form $w = p_1^* w_e + p_2^* w_h$.

Definition 2.3.1. By a meromorphic $q$-disk in the complex space $X$ we shall understand a meromorphic mapping $\phi : \Delta^q \to X$, which is defined in some neighbourhood of the closure $\bar{\Delta}^q$.

It will be convenient for us to consider instead of mappings $\phi : \Delta^q \to X$ their graphs $\Gamma_{\phi}$. By $\hat{\phi} = (z, \phi(z))$ we shall denote the mapping into the graph $\Gamma_\phi \subset \Delta^q \times X$. The volume of the graph $\Gamma_{\phi}$ of the mapping $\phi$ is given by

$$\text{vol}(\Gamma_{\phi}) = \int_{\Gamma_{\phi}} w^q = \int_{\Delta^q} (\phi^* w_h + dd^c \|z\|^2)^q$$ (2.3.1)

Here by $\phi^* w_h$ we denote the preimage of $w_h$ under $\phi$, i.e. $\phi^* w_h = (p_1)_* p_2^* w_h$.

Recall that the Hausdorff distance between two subsets $A$ and $B$ of the metric space $(Y, \rho)$ is a number $\rho(A,B) = \inf \{ \varepsilon : A^\varepsilon \supset B, B^\varepsilon \supset A \}$. Here by $A^\varepsilon$ we denote the $\varepsilon$-neighborhood of the set $A$, i.e. $A^\varepsilon = \{ y \in Y : \rho(y,A) < \varepsilon \}$.

Further, let $\{\phi_r\}$ be the sequence of meromorphic mappings of the complex space $D$ into the complex space $X$.

Definition 2.3.2. We shall say that $\{\phi_r\}$ converge on the compacts in $D$ to the meromorphic mapping $\phi : D \to X$, if for every relatively compact open $D_1 \subset D$ the graphs $\Gamma_{\phi_r} \cap (D_1 \times X)$ converge in the Hausdorff metric on $D_1 \times X$ to the graph $\Gamma_{\phi} \cap (D_1 \times X)$.

First we shall prove the following

Lemma 2.3.1. Let $\{\phi_r\}$ be a sequence of meromorphic $q$-disks in complex space $X$. Suppose that there exists a compact $K \subset X$ and a constant $C < \infty$ such that:

a) $\phi_r(\Delta^q) \subset K$ for all $r$;

b) $\text{vol}(\Gamma_{\phi_r}) \leq C$ for all $r$.

Then there exists a subsequence $\{\phi_{r_j}\}$ and a proper analytic set $A \subset \Delta^q$ such that:
1) the sequence \( \{ \Gamma_{\phi, r} \} \) converges in the Hausdorff metric to the analytic subset \( \Gamma \) of \( \Delta^q \times X \) of pure dimension \( q \);

2) \( \Gamma = \Gamma_{\phi} \cup \hat{\Gamma} \), where \( \Gamma_{\phi} \) is the graph of some meromorphic mapping \( \phi : \Delta^q \to X \), and \( \hat{\Gamma} \) is a pure \( q \)-dimensional analytic subset of \( \Delta^q \times X \), mapped by the projection \( p_1 \) onto \( A \);

3) \( \phi_{r_j} \to \phi \) on compacts in \( \Delta^q \setminus A \);

4) one has

\[
\lim_{j \to \infty} \text{vol}(\Gamma_{\phi, r_j}) \geq \text{vol}(\Gamma_{\phi}) + \text{vol}(\hat{\Gamma}).
\]  

(2.3.2)

5) For every \( 1 \leq p \leq \dim X - 1 \) there exists a positive constant \( \nu_p = \nu_p(K, h) \) such that the volume of every pure \( p \)-dimensional compact analytic subset of \( X \) which is contained in \( K \) is not less then \( \nu_p \).

6) Put \( \hat{\Gamma} = \bigcup_{p=0}^{q-1} \Gamma_p \), where \( \Gamma_p \) is a union of all irreducible components of \( \hat{\Gamma} \) such that \( \dim[p_1(\Gamma_p)] = p \). Then

\[
\text{vol}_{2q}(\hat{\Gamma}) \geq \sum_{p=0}^{q-1} \text{vol}_{2p}(A_p) \cdot \nu_{q-p}
\]  

(2.3.3)

where \( A_p = p_1(\Gamma_p) \).

**Proof.** 1) The first statement of this lemma is exactly the theorem of Bishop about sequences of analytic sets of bounded volume, see [St]. Because \( \Gamma_{\phi, r_j} \subset \Delta^q \times K \), so also a limit \( \Gamma \subset \Delta^q \times K \). Consequently \( p_1 |_\Gamma : \Gamma \to \Delta^q \) is proper. Further, because \( p_1(\Gamma_{\phi, r_j}) = \Delta^q \) for all so \( p_1(\Gamma) = \Delta^q \). That is our map \( p_1 |_\Gamma : \Gamma \to \Delta^q \) is surjective.

Let \( \Gamma \subset \Delta^q \times X \) be an analytic subset and let \( p_1 |_\Gamma : \Gamma \to \Delta^q \) be the restriction of the natural projection onto \( \Gamma \). We shall say that \( p_1 |_\Gamma \) is one to one over a generic point, if there exists an open \( D \subset \Delta^q \) such that \( p_1 |_{\Gamma \cap p_1^{-1}(D)} : \Gamma \cap p_1^{-1}(D) \to D \) is one to one. In this case, using the Remmert proper mapping theorem one can show that \( D \) can be taken to be \( \Delta^q \) minus proper analytic subset.

2) Define \( \Gamma_1 = \{(z, x) \in \Gamma : \text{dim}_{p_1}(z, x)(p_1 |_\Gamma)^{-1}(z) \geq 1\} \). \( \Gamma_1 \) is an analytic subset of \( \Gamma \). By the properness of \( p_1 |_\Gamma \) \( A_1 = p_1(\Gamma_1) \) is analytic subset of \( \Delta^q \). By \( \hat{\Gamma} \) denote the union of irreducible components of \( \Gamma_1 \) of dimension \( q \) ant put \( A = p_1(\Gamma) \). Then \( \Gamma = \Gamma_{\phi} \cup \hat{\Gamma} \), where \( \Gamma_{\phi} \) is the union of those irreducible components of \( \Gamma \) which are not in \( \hat{\Gamma} \). Remark that because \( \text{dim} A \leq q - 1 \) the restriction \( p_1 |_{\Gamma_{\phi}} \) is surjective and one to one over the general point. To prove that \( \Gamma_{\phi} \) is a graph of some meromorphic mapping it is sufficient to show that \( \Gamma_{\phi} \) is irreducible. Suppose that \( \Gamma_2 \) is some nontrivial irreducible component of \( \Gamma_{\phi} \). Because \( p_1 |_{\Gamma_{\phi}} \) is one to one over a general point then \( p_1(\Gamma_2) = A_2 \) is a proper analytic subset of \( \Delta^q \). But then \( \Gamma_2 \subset \hat{\Gamma} \) by the definition of \( \hat{\Gamma} \). So \( \Gamma_{\phi} \) is irreducible and thus is a graph of a meromorphic mapping which we denote by \( \phi \). Let \( \Gamma' \) be the union of all irreducible components of \( \Gamma_1 \) of dimension less then \( q \). Then \( p_1(\Gamma') = F \) is exactly the set of points of indeterminacy of the map \( \phi \).

3) \( \Gamma_{\phi, r_j} \to \Gamma = \Gamma_{\phi} \cup \hat{\Gamma} \) on \( \Delta^q \times X \). So, because \( \hat{\Gamma} = p_1^{-1}(A) \) we have that \( \Gamma_{\phi, r_j} \to \Gamma_{\phi} \) on compacts in \( (\Delta^q \setminus A) \times X \). By definition this means that \( \phi_{r_j} \to \phi \) on compacts in \( \Delta^q \setminus A \).

4) This statement follows from the generalisation of the theorem of Bishop made by Harvey-Shiffman, see [H-S]. The theorem of Harvey-Shiffman states that the sequence of
analytic sets \( \{ \Gamma_{\varphi r_j} \} \), which in our case converges to the analytic set \( \Gamma \) in the Hausdorff metric, converges to \( \Gamma \) also in locally flat topology, see Theorem 3.9 from [H-S]. The latter means, in particular, that for any \((k,k)\)-form \( \Omega \) on \( \Delta^q \times X \) with compact support \( \int_{\Gamma_{\varphi r_j}} \Omega \to \int_{\Gamma} \Omega \) when \( j \to \infty \).

If one takes as \( \Omega \) the \( q \)-th exterior power of \( w \) times nonnegative function \( \psi \) with compact support, i.e. \( \Omega = \psi \cdot w^q \), and takes into account that \( \text{vol}(\Gamma) = \int_{\Gamma} w^q \), then one gets that for every compact \( K \subset \Delta^q \times X \):

\[
\text{vol}(\Gamma_{\varphi r_j} \cap K) = \int_{\Gamma_{\varphi r_j}} w^q \to \int_{\Gamma \cap K} w^q = \text{vol}(\Gamma_{\varphi} \cap K) + \text{vol}(\hat{\Gamma} \cap K)
\]

(2.3.4)

Now take into account that \( K \) is arbitrary and that the limit in (2.3.4) one should understand with multiplicities. Withouth multiplicities one gets from (2.3.4) the stated inequality (2.3.2).

5) Proof of this item we shall derive from contradiction. Suppose that there exists a sequence of pure \( p \)-dimensional compact analytic subsets \( \{ C_j \} \) in \( X \) such that \( \text{vol}(C_j) \to 0 \) while \( j \to \infty \). Going to a subsequence one obtains that \( C_{j_k} \to C \) in the Hausdorff metrick to the compact \( p \)-dimensional analytic set \( C \). From (2.3.2) we see that \( \text{vol}(C) \leq \lim_{k \to \infty} \inf \text{vol}(C_{j_k}) = 0 \) which is impossible.

6) The relation (2.3.3) reflects the fact that our metric \( w \) on the product is a sum of the metrics on the factors. Recall that \( A = p_1(\hat{\Gamma}) \), where \( \hat{\Gamma} \) is a union of the irreducible components of the limit \( \Gamma \) other then \( \Gamma_{\varphi} \). As above denote by \( A_p = (p_1|_{\Gamma})(\Gamma_p) \), where \( \Gamma_p \) as above is a union of irreducible components of \( \hat{\Gamma} \) such that \( \text{dim} A_p = p \). For the proof of (2.3.3) it is sufficient to show that

\[
\text{vol}(\Gamma_p) \geq \text{vol}(A_p) \cdot \nu_{q-p}
\]

(2.3.5)

Let us underline that \( \Gamma_p \) is \( q \)-dimensional as a union of some \( q \)-dimensional components of \( \hat{\Gamma} \), and that \( A_p \) is \( p \)-dimensional. We have

\[
\text{vol}(\Gamma_p) = \int_{\Gamma_p} w^q = \int_{\Gamma_p} (p_1^* w_e + p_2^* w_h)^q \geq \int_{\Gamma_p} (p_1^* w_e)^p \wedge (p_2^* w_h)^{q-p} = \int_{A_p} \left( \int_{(p_1|_{\Gamma})^{-1}(z)} (w_e^{q-p}) w_e^p \right) = \nu_{q-p} \cdot \text{vol}(A_p).
\]

In the first inequality we used the fact that all terms of decomposition

\[
(p_1^* w_e + p_2^* w_h)^q = \sum_{j=0}^{k} C_j^q (p_1^* w_e)^j \wedge (p_2^* w_h)^{q-j}
\]
are positive. In the second one we used that $\nu_{q-p}$ is a minima of volumes of $(q-p)$-dimensional compact analytic subsets of $X$ contained in $K$.

q.e.d.

2.4. Meromorphic families of analytic subsets.

Let $S$ be a set, and $W \subset \subset \mathbb{C}^q$ an open subset. $W$ is equipped with the usual Euklidean metric from $\mathbb{C}^q$.

**Definition 2.4.1.** By a family of $q$-dimensional analytic subsets in complex space $X$ we shall understand an subset $F \subset S \times W \times X$ such that, for every $s \in S$ the set $F_s = F \cap \{s\} \times W \times X$ is a graph of a meromorphic mapping of $W$ into $X$.

Suppose further that the set $S$ is equipped with topology and let our space $X$ be equipped with some Hermitian metric $h$.

**Definition 2.4.2.** We shall say that the family $F$ is continuous at point $s_0 \in S$ if

$$\mathcal{H}\lim_{s \to s_0} F_s = F_{s_0}.$$ 

Here by $\mathcal{H}\lim_{s \to s_0} F_s$ we denote the limit of closed subsets of $F_s$ in the Hausdorff metric on $W \times X$. $F$ is continuous if it is continuous at each point of $S$. If $W_0$ is open in $W$ then the restriction $F_{W_0}$ is naturally defined as $F \cap (S \times W_0 \times X)$.

When $S$ is a complex space itself, we give the following

**Definition 2.4.3.** Call the family $F$ meromorphic if the closure $\hat{F}$ of the set $F$ is an analytic subset of $S \times W \times X$.

We shall be interested with an interaction of notions of continuity and meromorphicity of families of meromorphic polydisks.

Let us prove our main statement about meromorphic families. Consider a meromorphic mapping $f : V \times W_0 \to X$ into a complex space $X$, where $V$ is a domain in $\mathbb{C}^p$. Let $S$ be some closed subset of $V$ and $s_0 \in S$ some accumulation point of $S$. Suppose that for each $s \in S$ the restriction $f_s = f|_{\{s\} \times W_0}$ meromorphically extends onto $W_0 \supset W_0$. We suppose additionally that there is a compact $K \subset \subset X$ such that for all $s \in S f_s(W) \subset K$.

Let, as in Lemma 2.3.1 $\nu_j$ denotes the minima of volumes of $j$-dimensional compact analytic subsets contained in our compact $K \subset X$. Fix some $W_0 \subset \subset W_1 \subset \subset W$ and put

$$\nu = \min\{ \text{vol}(A_{q-j}) \cdot \nu_j : j = 1, \ldots, q \},$$

(2.4.1)

where $A_{q-j}$ are running over all $(q-j)$-dimensional analytic subsets of $W$, intersecting $W_1$. Clearly $\nu > 0$. In the following Lemma the volumes of graphs over $W$ are taken. More precisely, having an Euklidean metric form $w_e = dd^c \|z\|^2$ on $W \subset \mathbb{C}^q$ and Hermitian metric form $w_h$ on $X$, we consider $\Gamma_{f_s}$ for $s \in S$ as an analytic subsets of $W \times X$ and their volumes are

$$\text{vol}(\Gamma_{f_s}) = \int_{\Gamma_{f_s}} (p_1^*w_e + p_2^*w_h)^q = \int_W (w_e + (p_1)_*p_2^*w_h)^q,$$

where $p_1 : W \times X \to W$ and $p_2 : W \times X \to X$ are natural projections.

**Lemma 2.4.1.** Suppose that there exists a neighbourhood $U \ni s_0$ in $V$ such that, for all $s_1, s_2 \in S \cap U$
If $s_0$ is a locally regular point of $S$ then there exists a neighbourhood $V_1 \ni s_0$ in $V$ such, that $f$ meromorphically extends onto $V_1 \times W_1$.

**Proof.** Step 1. Let $s_0$ be a locally regular point of $S$. First of all we remark that the family of analytic subsets $\{\Gamma_{f_n}\}_{n \in S}$ is continuous at $s_0$ in $W_1 \times X$. Indeed, let $s_n \in S, s_n \rightarrow s_0$ as $n \rightarrow \infty$. Then from (2.4.2) we see that $\lim \nu(f_{s_n})$ are uniformly bounded and thus by Lemma 2.3.1 $\Gamma_{f_{s_n}} \subset (W_0 \times X)$ extends to a graph of meromorphic mapping over $W$. This will be also denoted as $\Gamma_{f_{s_0}}$. Now if one could find a sequence $s_n \in S, s_n \rightarrow s_0$ as $n \rightarrow \infty$ such that $\Gamma_{f_{s_n}} \rightarrow \Gamma_{f_{s_0}}$ in Hausdorff metric, then by Lemma 2.3.1, using the boundedness of volumes of $\Gamma_{f_n}$, one finds a subsequence, still denoted as $s_n$ such that $\Gamma_{f_{s_n}} \rightarrow \Gamma \supset \Gamma_{f_{s_0}}$, but not equal $\Gamma_{f_{s_0}}$.

But then, by the relations (2.3.2) and (2.3.3) of Lemma 2.3.1 one has

$$\lim_{n \rightarrow \infty} \nu(f_{s_n}) \geq \nu + \nu(f_{s_0}),$$

which contradicts (2.4.2).

Let us prove now that the family $F = \bigcup_{t \in S} \Gamma_{f_{s}} \subset S \times W_0 \times X$ extends to a meromorphic family on $V_1 \times W_1 \times X$ for some neighbourhood $V_1 \ni s_0$.

Fix a point $z_0 \in \Reg\Gamma_{f_{s_0}} \cap (W_0 \times X)$. Here we consider $\Gamma_{f_{s_0}}$ as analytic space itself. So $\Reg\Gamma_{f_{s_0}}$ is connected dense subset in $\Gamma_{f_{s_0}}$ and $\Sing\Gamma_{f_{s_0}} := \Gamma_{f_{s_0}} \setminus \Reg\Gamma_{f_{s_0}}$ is an analytic subset of $\Gamma_{f_{s_0}}$.

Now take a point $z_1 \in \Reg\Gamma_{f_{s_0}} \cap (W_0 \times X)$. Take a path $\gamma : [0,1] \rightarrow \Reg\Gamma_{f_{s_0}}$ from $z_0$ to $z_1$. We shall prove that there is a neighborhood $\Omega$ of $\gamma([0,1])$ in $W \times X$ and a neighborhood $V \ni s_0$ such that $F \cap (V \times \Omega)$ extends to an analytic set in $V \times \Omega$.

By $T$ denote the set of those $t \in [0,1]$ that there exists a neighborhoods $\Omega_t \supset \gamma([0,t])$ and $V_t \ni s_0$ such that $F \cap V_t \times \Omega_t$ extends to an analytic set in $V_t \times \Omega_t$. Note that $T$ is open and contains the origin. Now let $t_0$ be the cluster point of $T$. Find the chart $\Sigma \cong \Delta^q \times \Delta^n$ for the space $W \times X$ in the neighborhood of $\gamma(t_0)$ with coordinates $u_1, \ldots, u_q, v_1, \ldots, v_n$ in such a way that $\gamma(t_0) = 0$ and $\Gamma_{f_{s_0}} \cap \Sigma = \{(u,v) : v = F_0(u)\}$ for some holomorphic map $F_0 : \Delta^q \rightarrow \Delta^n$. By the Hausdorff continuity of our family $\{\Gamma_{f_s}\}$ in $s_0 \Gamma_{f_s} \cap \Omega = \{(u,v) : v = F_s(u)\}$ for $s$ close to $s_0$, $F_s$ holomorphic and continuously depending on $s$.

Take $t_1 \in T$ close to $t_0$, such that $\gamma([0,t_1]) \subset \Sigma$. We have some neighborhoods $V_{t_1} \ni s_0, \Omega_{t_1} \ni \gamma(t_1)$ such that $F$ extends analytically to $V_{t_1} \times \Omega_{t_1}$. Let $u_1 \in \Delta^q$ be such that $\gamma(t_1) = (u_1, F_0(u_1))$. Then there is a neighborhood, say $\Delta^q_1(u_1)$, such that $\Gamma_{f_{s_1}} \cap (V_{t_1} \times \Delta^q_1(u_1) \times \Delta^n)$ is defined by the equation $v = F(u,s)$, where $F(u,s) = F_s(u) : V_{t_1} \times \Delta^q_1(u_1) \rightarrow \Delta^n$ as above. From the condition of the Lemma we see that for $s \in S$ close to $s_0$ $F(u,s)$ extends onto $\Delta^q$. So by Lemma 2.1.1 $F(u,s)$ extends holomorphically to $V_{t_0,\epsilon} \times \Delta^q_{1-\epsilon}$, where $\epsilon$ is arbitrarily small ($V_{t_0,\epsilon}$ depending on $\epsilon$). But this means that $F$ extends analytically onto $V_{t_0} \times \Delta^q_{1-\epsilon} \times \Delta^n$. Thus $T$ is closed and coincides with $[0,1]$.

We proved in fact that for any compact subset $R \subset \Reg\Gamma_{f_{s_0}} \cap (W_1 \times X)$ there are neighborhoods $V_R \ni s_0, \Omega_R \ni R$ such that $\Gamma_f$ analytically extend to $V_R \times \Omega_R$. 16
Step 2. Cover the set $\text{Sing}\, \Gamma_{f_{s_0}} \cap \{(s_0) \times \bar{W}_1 \times X\}$ with a finite number of open charts of the form $1/2V_\alpha \times \Omega_\alpha$, where $V_\alpha \cong \Delta^q$ and $\Omega_\alpha \cong \Delta^n$, and such that $\Gamma_{f_{s_0}} \cap (V_\alpha \times \Omega_\alpha)$ is analytic cover of $V_\alpha$. By Step 1 we can find an open neighborhoods $V_R \ni s_0$ and $\Omega_R \ni \Gamma_{f_{s_0}} \backslash [\bigcup_\alpha 1/2V_\alpha \times \Omega_\alpha]$ such that $\mathcal{F}$ analytically extends to $V_R \times \Omega_R$.

Fix now some $\alpha$. All that remained to prove is that $\Gamma_\alpha$ analytically extends to $V_\alpha' \times V_\alpha \times \Omega_\alpha$ for some neighborhood of $V_\alpha' \ni s_0$. But this again follows from Lemma 2.1.1 applied to the coefficients of polynomials which define the cover $\Gamma_{f_\alpha} \cap (V_\alpha \times \Omega_\alpha) \rightarrow V_\alpha$.

q.e.d.

Divide the variables in $\mathbb{C}^n = \mathbb{C}^{p+q}$ into two groups: $(z_1, \ldots, z_p)$ and $(u_1, \ldots, u_q)$. Let $\Gamma$ an analytic set in $\Delta^q \times X$. Fix some $0 < r < 1$ and put $\Gamma_z = \Gamma \cap \{(z) \times \Delta^q \times X\}$. Consider a function

$$v_\Gamma(z) = \text{vol}_{2q}(\Gamma_z) = \int_{\Gamma_z} (dd^c \|u\|^2 + w_h)^q.$$  \hfill (2.4.3)

Here, as usually $w_h$ denotes the $(1,1)$-form canonically associated to $h$. This is well defined for $z \in \Delta^p$, for which $\dim(\Gamma_z) = q$. Denote the set of such $z$ as $u$. $T := \Delta^p \backslash U$ is contained in at most countable union of locally closed proper analytic subvarieties of $\Delta^p$, see [F]. We shall make use of the following result of D. Barlet:

the function $v_\Gamma$ is locally bounded on $\Delta^p$.

see [B] Théorème 3.

Corollary 2.4.2. If $f : \Delta^q \rightarrow X$ is a meromorphic map then the Lelong numbers $\Theta((f^*w)^q,0)$ are finite for all $q = 1, \ldots, n$.

Proof. This is the same as to bound the Lelong numbers of the currents $(dd^c \|z\|^2 + f^*w_h)^q$ Put $z' = (z_{i_1}, \ldots, z_{i_q})$ and $z'' = (z_{j_1}, \ldots, z_{j_{n-q}})$. We have

$$\frac{1}{r^{2p}} \int_{\Delta^p_\alpha \times \Delta^q_\alpha} (f^*w + dd^c \|z'\|^2)^q \wedge (dd^c \|z''\|^2)^p = \frac{1}{r^{2p}} \int_{\Delta^p_\alpha} (dd^c \|z''\|^2)^p \int_{\Gamma_z} (dd^c \|z'\|^2 + w)^q =$$

$$= \frac{1}{r^{2p}} \int_{\Delta^p_\alpha} (dd^c \|z''\|^2)^p v_\Gamma(z'') \leq C \cdot \frac{\text{vol}(\Delta^p_\alpha)}{r^{2p}} \leq C_1$$

by the Barlet Theorem.

q.e.d.

2.5. The Rothstein-type extension theorem and separate meromorphicity.

Lemmas 2.3.1 and 2.4.1 together with estimate in 3.2 will play a key role in the proof of the main result of this paper. However we shall break here the main line of exposition to show how they can be already applied to the classical question entitled above.

We are going to treat the question of separate meromorphicity in the form proposed by Kazaryan and moreover, we are going to show the condition on the image space $X$ to possess the mer.ext.property is "almost" not needed. It was F. Hartogs who proved his
"Hartogs-type" extension theorem for holomorphic functions in order to derive from this his separate holomophicity theorem. Rothstein did this for meromorphic functions, starting from E. Levi extension theorem. B. Shiffman fulfilled this program for meromorphic mappings. Namely he proved in [Sh-2] that if the complex space $X$ possess a meromorphic extension property in dimension $n$ then separately meromorphic mappings from $\Delta^n$ to $X$ are meromorphic.

We shall prove here the Rothstein-type extension and separate meromorphicity in the form of Siciak and Kazaryan. If the space $X$ posses a hol.ext.prop. the separate holomorphy in the form of Siciak was obtained, using the approach of Shiffman, by O. Alehyane in [A-1].

**Corollary 2.5.1** Let $V \subset \mathbb{C}^p$ and $W_0 \subset \subset W \subset \mathbb{C}^q$ be a domains and let $E$ a nonpluripolar subset of $V$. Let further a meromorphic mapping $f : V \times W_0 \to X$ into a complex space $X$ is given. Suppose that for $z \in E$ the restriction $f_z := f|_{\{z\} \times W_0}$ is well defined and extends meromorphically onto $\{z\} \times W$. Then there is a pluripolar subset $E' \subset E$ such that $f$ meromorphically extends onto a neighborhood of $V \times W_0 \cup (E \setminus E') \times W$.

**Proof.** Take an exhaustion $W_0 \subset \subset W_1 \subset \subset ...$ of $W$ by relatively compact domains. For every $n$ we shall find a pluripolar subset $E'_n \subset E$ such that $f$ extends meromorphically to the neighborhood $P_n$ of $V \times W_0 \cup (E \setminus E'_n) \times W$. Then $P := \bigcup_{n\geq 1} W_n$ will be a neighborhood of $V \times W_0 \cup (E \setminus \bigcup_{n \geq 1} E'_n) \times W = V \times W_0 \cup (E \setminus E') \times W$, where $E' := \bigcup_{n \geq 1} E'_n$ is pluripolar in $V$ by the Josefson theorem. Finally $f$ is extended to this neighborhood.

So, we can suppose additionally that there is a domain $\tilde{W} \supset \supset W$ such that for any $z \in E$ $f_z$ is well defined and extends meromorphically onto $cl(\tilde{W})$-closure of $\tilde{W}$.

Next, fix some compact exhaustion $K_1 \subset \subset ... \subset \subset K_n \subset \subset ...$ of $X$. Denote now by $E_n$ the set of those $z \in E$ that $f_z(\tilde{W}) \subset K_n$. While $\bigcup E_n = E$, starting from some $n_0$ all $E_n$ are not pluripolar. If we shall prove that there exists a pluripolar $E_n$ and extension of $f$ into the neighborhood of $V \times W_0 \cup (E_n \setminus E'_n) \times W$, then again using Josefson theorem we can finish the proof.

Thus we may additionally suppose that there is a compact $K \subset \subset X$ such that $f_z(\tilde{W}) \subset K$ for all $z \in E$.

Take $\nu$ as in Lemma 2.4.1 for $W_0 \subset \subset W \subset \subset \tilde{W}$ and $K$. Let $R$ be the maximal open subset of $E$ such that $f$ extends to the neighborhood of $V \times W_0 \cup R \times W$. Put $S = E \setminus R$. Put further

$$S_k = \{z \in S : vol(\Gamma_{f_z}) \leq k\frac{\nu}{2}\},$$

where graphs are taken over $\tilde{W}$. Note that by Lemma 2.3.1 $S_k$ are closed and they are increasing. Also $\bigcup_{k \geq 1} S_k = S$. By Lemma 2.4.1 and maximality of $R = E \setminus S$ the sets $S_{k+1} \setminus S_k$ are not locally regular at any of their points. Thus the sets $S_1, S_2 \setminus S_1, ...$ are pluripolar and so is $S$.

q.e.d.

**Remarks.**
1. If $E = V$ and $X$ posses a mer.ext. property then we obtain the result of B. Shiffman, see [Sh-2].
2. If $X$ is compact and Kähler then it satisfies this assumption, see [Iv-2].
3. Let $X$ posses a meromorphic extension property in dimension $p+q$. Then $f$ extends onto the envelope of holomorphy of the neighborhood of $V \times W_0 \cup (E \setminus E') \times W$, which is
obviously a neighborhood of $V \times W_0 \cup E^* \times W$. Here we denote by $E^*$ the set of locally regular points of $E$.

4. Let us give the following

**Definition 2.5.1.** We shall say that meromorphic mappings into complex space $X$ satisfy:

(h) a Hartogs-type extension theorem in bidimension $(p,q)$ if every meromorphic map from

$$H^p_q (r) = \{(z_1, \ldots, z_{p+q}) \in \mathbb{C}^{p+q} : \|z'\| < r, \|z''\| < 1 \text{ or } \|z'\| < 1, 1-r < \|z''\| < 1\},$$

(2.5.2)

to $X$ extend meromorphically onto $\Delta^{p+q}$. Here $z' = (z_1, \ldots, z_p)$, $z'' = (z_{p+1}, \ldots, z_{p+q})$;

(t) a strong Thullen-type extension theorem in bidimension $(p,q)$ if for any closed pluripolar subset $S$ of $\Delta^q$ every meromorphic map from

$$T^p_q (S, a, b) := (\Delta^q \times \Delta^p (a)) \cup ((\Delta^q \setminus S) \times \Delta^p (b))$$

(2.5.3)
to $X$ extends meromorphically onto $\Delta^{p+q}$, here $a < b$.

Note that Hartogs $(p,q)$-extendibility obviously implies the strong Thullen-type one. Vice versa is not true. Example is given by a 3-fold constructed by M. Kato in [Ka-2].

Namely Kato had constructed a compact three-fold $X$, which is a quotient of $D = \{(z_0; \ldots; z_3) \in \mathbb{C}^3 : |z_0|^2 + |z_1|^2 < |z_2|^2 + |z_3|^2\}$ by a co-compact properly discontinuous subgroup $\mathbb{G} \subset \text{Aut}(D)$. Denote by $\pi : D \rightarrow X$ the natural projection. Consider a hyperplane $P = \{z \in \mathbb{C}^3 : z_0 = 0\} \cong \mathbb{C}^2$. Then $D \cap P = \mathbb{C}^2 \setminus \overline{B}^2$, here $\overline{B}^2$ is a ball $\{|z|^2 > |z_2|^2 + |z_3|^2\}$ in $\mathbb{C}^3$. Thus $\pi_{|D\cap P} : \mathbb{C}^2 \setminus \overline{B}^2 \rightarrow X$ cannot be extended onto the neighborhood of any point on $\partial \overline{B}^2$. So meromorphic mappings into $X$ are not Hartogs $(1,1)$-extendable.

Let now $f : T^1_1 (S,1/2,1) \rightarrow X$ be some meromorphic map. While $f : T^1_1 (S,1/2,1)$ is simply-connected we can consider a lifting $F = \pi^{-1} \circ f : T^1_1 (S,1/2,1) \rightarrow D \subset \mathbb{C}^3$. By a Thullen-type extension theorem for meromorphic functions $F$ extends onto $\Delta^2$. But $F(\Delta^2) \cap \partial D = \emptyset$ because one cannot touch $\partial D$ by bidisk. Thus $\pi \circ F$ gives an extension of $f$ onto $\Delta^2$.

We have the following obvious corollary.

**Corollary 2.5.2.** If in the conditions of the Corollary 2.5.1 a complex space $X$ possess a strong Thullen-type extension property in bidimension $(p,q)$ and $E = V$ then $f$ extends meromorphically onto $V \times W$.

Let us turn now to the separate meromorphicity.

**Corollary 2.5.3.** Let $E$ and $G$ be a nonpluripolar subsets in domains $V \subset \mathbb{C}^p$ and $W \subset \mathbb{C}^q$ respectively. Let $F$ be some pluripolar subset of in $V \times W$. Let further some mapping $f : E \times G \setminus F \rightarrow X$ into a complex space $X$ is given. Suppose that:

(i) for every $z \in E$, such that $\{z\} \times G \not\subset F$ the restricton $f_z := f|_{\{z\} \times G}$ is well defined and meromorphically extends meromorphically onto $\{z\} \times G$, and

(ii) for every $w \in G$, such that $V \times \{w\} \not\subset F$ the restricton $f^w_z := f|_{V \times \{w\}}$ is well defined and meromorphically extends meromorphically onto $V \times \{w\}$.

Then there are pluripolar subsets $E' \subset E$, $G' \subset G$ and a meromorphic mapping $\tilde{f}$ of some neighborhood of $(E \setminus E') \times W \cup V \times (G \setminus G')$ into $X$ which extends $f$. 

19
Proof. Without loss of generality as in the proof of Rothstein-type theorem, we suppose that \( f_z \) extends onto \( cl(\tilde{W}) \) for some \( \tilde{W} \supset W \) and for all \( z \in E \), and that there is a compact \( K \subset X \) such that \( f_z(\tilde{W}) \subset K \) for all \( z \in E \). The same for \( f^{w-s} \).

**Step 1.** There is a point \( Z_0 = (z_0, w_0) \in E \times G \) such that \( f \) holomorphically extend to the neighborhood of \( Z_0 \).

Define
\[
E_k = \{ z \in E : vol(\Gamma_{f_z}) \leq k^{\nu}\}, \tag{2.5.4}
\]

While \( E \) is not plurioliar, there exists \( k \) and \( z_1 \in E_{k+1} \setminus E_k \) such that \( E_{k+1} \setminus E_k \) is locally regular at \( z_1 \). The same reasoning as at the beginning of the proof of Lemma 2.4.1 shows that the family \( \{ \Gamma_{f_z} : z \in E_{k+1} \setminus E_k \} \) is continuous in the neighborhood of \( z_1 \). Take a point \( w_0 \in W \) such that \( f_{z_1} \) is holomorphic in the neighborhood of \( w_0 \). Remark that this \( w_0 \) can be taken to be a locally regular point of \( G_{l+1} \setminus G_l \) for some \( l \). From Hausdorff continuity of our family in the neighborhood of \( z_1 \) we get immediately that all \( f_z \) are holomorphic in the neighborhood of \( w_0 \) for \( z \in E_{k+1} \setminus E_k \) close to \( z_1 \). Find a point \( z_0 \) close to \( z_0 \) where \( f^{w_0} \) is holomorphic. Now the separate analyticity theorem for functions tells us that the point \( Z_0 = (z_0, w_0) \) is as needed.

**Step 2. End of the proof.**

Applying two times coordinatevise the Rothstein theorem we get the statement.

q.e.d.

**Remark.**
1. Again, as above, if \( X \) posseds a mer.ext.prop. then we can take \( E \setminus E' = E^* \) and \( G \setminus G' = G^* \). This case was studied recently in [A-2], using approach developped in [Sh-2] and [Sh-3].
2. In the case \( X = \mathbb{CP}^1 \) we obtain the theorem of Kazaryan, see [Kz].

### 3. Estimates of Lelong numbers from below.

#### 3.1. Generalities on blowings-up.

First we recall the Hironaka Resolution Singularities Theorem. We shall use the so called embedded resolution of singularities, see [H], [B-M]. Let us recall the notion of the sequence of local blowings up over a complex manifold \( D \). Take a point \( s_0 \in D_0 := D \). Let \( V_0 \) be some neighbourhood of \( s_0 \) and \( l_0 \) smooth, closed submanifold of \( V_0 \) of codimension at least two, passing through \( s_0 \). Denote by \( \pi_1 : D_1 \to V_0 \) the blowing up of \( V_0 \) along \( l_0 \). Call this a local blowing up of \( D_0 \) along the locally closed center \( l_0 \). The exceptional divisor \( \pi_1^{-1}(l_0) \) of this blowing up we denote by \( E_1 \).

We can repeat this procedure, taking a point \( s_1 \in D_1 \), a neighborhood \( V_1 \) of that point in \( D_1 \) and smooth closed submanifold \( l_1 \) in \( V_1 \) of codimension at least two.

**Definition 3.1.1.** A finite sequence \( \{ \pi_j \}_{j=1}^N \) of such local blowings up we call a sequence of local blowings up over \( s_0 \in D \), or a local regular modification.

By \( \{ l_j \}_{j=0}^{N-1} \) we denote the corresponding centers in the neighborhoods \( \{ V_j \}_{j=0}^{N-1} \) of points \( \{ s_j \}_{j=0}^{N-1} \), and by \( \{ E_j \}_{j=1}^N \) the exceptional divisors, \( s_j \in D_j \).
If $V_j = D_j$ for all $j = 0, \ldots, N-1$, and points $s_j$ are not specified we call this sequence a sequence of (global) blowings up or a regular modification. In this case we put $\pi = \pi_1 \circ \cdots \circ \pi_N$, $\hat{D} = D_N$, $E$ denotes the exceptional divisor of $\pi$, i.e. $E = \pi_N^{-1}(l_{N-1} \cup \cdots \cup (\pi_1 \circ \cdots \circ \pi_N)^{-1}(l_0)$.

**Proof.** Let $L$ be a subvariety of $D$. Then there exists a regular (not local!) modification $\pi : \hat{D} \to D$ such that:

1) the strict transform $\hat{L}$ of $L$ is smooth;

2) $l_i \subset \text{Sing} L_i$, where $L_i$ is a strict transform of $L$ by $\pi_1 \circ \cdots \circ \pi_j$.

See [H].

For the proof we refer to [B-M].

**Remark.** We shall use this Theorem only for the case $\dim D = 3$ and $\dim L = 1$. In this case (while $\dim \text{Sing} L = 0$), we need only blowings up of points to resolve the singularities of $L$. We shall need also the following three Lemmas about the behavior of meromorphic mappings under the modifications. First let us introduce some more notations. Let $f : D \setminus S \to X$ be a meromorphic map into a complex space $X$. Here $D$ is a manifold and $S$ supposed to be closed and zero dimensional.

**Definition 3.1.2.** Recall that a closed subset $S$ of a metric space is called zero dimensional if for any $s_0 \in S$ and for almost all $r > 0$ the sphere centered at $s_0$ of radii $r$ do not intersect $S$.

By $I(f)$ we shall denote the set of points of indeterminacy of $f$. By $I_p(f)$ those components of $I(f)$ which have dimension at least $p$. By $I_{p,s}(f)$ the set of components from $I_p(f)$ which pass through the point $s$; by $I_{p,R}(f)$ the set of those components from $I_p(f)$ which intersect the set $R$.

Remark also that if $l$ is an (irreducible) analytic set in $D \setminus S$ of pure dimension $p \geq 1$, then its closure is an (irreducible) analytic subset of $D$, provided $S$ is zero-dimensional.

**Lemma 3.1.1.** Let $f : B^n \setminus S \to X$ be a holomorphic map of punctured $n$-ball, $n \geq 2$, into a complex space $X$ which meromorphically extends onto $B^n$. Suppose one can find a sequence $\{\pi_j\}_{j=1}^\infty$ of blowings-up in such a way that:

(i) $\pi_1$ is a blow-up of $B^n_0 := B^n$ at $s_{0,1} = 0$; $B^n_1 := \pi_1^{-1}(B^n_0)$.

(ii) $\pi_{j+1}$ is a blow-up of $B^n_j$ at points $\{s_{j,1}, \ldots, s_{j,N_j}\} \subset \bigcup_{i=1}^{N_j} \pi_j^{-1}((s_{j-1,i})$; here $N_0 = 1$.

(iii) $f_j := f \circ (\pi_1 \circ \cdots \circ \pi_j)$ is holomorphic on $B^n_{j} \setminus \{s_{j,1}, \ldots, s_{j,N_j}\}$.

Then there exists $j_0$ such that $f_{j_0}$ is holomorphic on $B^n_{j_0}$.

**Proof.** By $\Gamma_f \subset B^n \times X$ denote the graph of $f$. Put $\Gamma = \Gamma_f \cap \{(0)\} \times X$. Write $\Gamma = \bigcup_{i=1}^{N} \Gamma_i$-decomposition into irreducible components. Put $E_{j+1,i} = \pi_j^{-1}(s_{j,i})$, $i = 1, \ldots, N$, $E_{j+1} = \bigcup_{i=1}^{N_j} E_{j+1,i}$ - the exceptional divisor of $\pi_{j+1}$. Usually we denote by $E_{j+1,i}$ also all strict transforms of it by subsequent blowings-up.

**Step 1.** For every $i = 1, \ldots, N$ there is a $j$ and $1 \leq k \leq N_j$ such that $f_{j+1}(E_{j+1,k}) = \Gamma_i$.

We shall prove this for $i = 1$. Take a point $a \in \text{Reg} \Gamma_1 \setminus (\bigcup_{i=2}^{N} \Gamma_i)$. There is a holomorphic map $\phi : \Delta^2 \to \Gamma_f$, such that:

(i) $\phi(\Delta \times \{(0)\}) = a$,

(ii) $\phi(\Delta^2 \setminus (\Delta \times \{(0)\})) \subset \Gamma_f \cap (B^n \times X)$,
(iii) for any \( z' \in \Delta \) the map \( \phi_{z'} := \phi |_{\{z'\} \times \Delta} \) is proper and primitive (i.e. not multiple covered).

Such map can be constructed first as imbedding into the smooth manifold \( \tilde{\Gamma}_h \), which is a modification of \( \Gamma_f \). One should only chose \( \phi \) to be transversal to the exceptional divisor of smoothing modification \( \pi : \tilde{\Gamma}_f \to \Gamma_f \) on each disk \( \{z'\} \times \Delta \), and then take \( \pi \circ \phi \) as \( \phi \).

Put \( \psi = p_1 \circ \phi \), where \( p_1 : B^n \times X \to B^n \) is a natural projection, and \( \psi_{z'} := \psi |_{\{z'\} \times \Delta} \) for \( z' \in \Delta \). By \( \psi^j : \Delta^2 \to B^n_j \) denote the lift \( (\pi_1 \circ \ldots \pi_j)^{-1} \circ \psi \) by \( \pi_1 \circ \ldots \circ \pi_j \).

There is a \( j \) such that \( \psi_{z'}^j \) is an imbedding of \( \Delta \) (after shrinking \( \Delta \) if necessary). So (after shrinking of \( \Delta \)), \( \psi_{z'}^j \) is also imbedding for all \( z' \in \Delta \). Remark that for all \( z' \in \Delta \) by construction one has \( \lim_{z_k \to 0}(f_j \circ \psi_{z'}^j)(z_k) = a \).

**Case 1.** \( \psi_{0'}^j(0) \notin \{s_{j,1}, \ldots, s_{j,N_j}\} \) for some \( j \).

In this case clearly \( f_j(E_{j,i}) = \Gamma_1 \), where \( i \in \{1, \ldots, N_j\} \) is such that \( E_{j,i} \ni \psi_{0'}^j(0) \).

**Case 2.** \( \psi_{0'}^j(0) = s_{j,i} \) for all \( j \).

Then there exists \( j_1 \) such that \( \psi_{z_0}^{j_1}(0) \neq \psi_{0'}^j(0) \) for \( z' \neq 0 \), but \( \psi_{z_0}^{j_1-1}(0) = \psi_{0'}^j(0) \). Now one can take some \( z'_0 \) instead of \( 0' \) for which \( \psi_{z_0}^{j_1} \notin \{s_{j_1,1}, \ldots, s_{j_1,N_{j_1}}\} \) and repeat the Case 1.

**Step 2.** Let \( \nu \) be from (2.4.1) for some fixed Hermitian metric \( h \) on \( X \) and some compact \( K \supset f(B^n_f) \). Then by Lemma 3.2.1 the sum of Lelong numbers of currents \( f^*w + dd^c\|z\|^{2}, \ldots, (f^*w + dd^c\|z\|^{2})^n \) is \( \geq N \cdot \nu \).

If \( f_{j_1} \) is not holomorphic say in \( s_{j_1,1} \) we can take this point instead of zero and repeat the Step 1. If this procedure doesn’t stop then the sum of Lelong numbers must be infinite. This contradics the meromorphy of \( f \) at zero, see Corollary 2.4.2.

q.e.d.

Let \( f_0 : D_0 \setminus S_0 \to X \) be a meromorphy mapping into a complex space \( X \), \( S_0 \)-being zero-dimensional, \( s_0 \in S_0 \), \( \dim D_0 = 3 \). Put \( \{l_1^{(0)}, \ldots, l_R^{(0)}\} = I_{1,s_0}(f_0) \).

Suppose that \( l_1^{(0)} \) is smooth. Consider the following sequence of blowings-up.

1) \( p_1 : D_1 \to D_0 \) is a blowing up of \( D_0 \) along \( l_1^{(0)} \). By \( E_1 \) denote the exceptional divisor of \( p_1 \).

Suppose that \( f_1 := f_0 \circ p_1 \) meromorphically extends onto \( D_1 \setminus S_1 \) with \( S_1 \) being zero-dimensional. Take some \( s_1 \in p_1^{-1}(s_0) \cap S_1 \) (if such exists). Let \( p_1' : D_1' \to D_1 \) be a regular modification resolving the singularities of all noncompact components of \( I_{1,s_1}(f_1) \) which are contained in \( E_1 \). Denote them (and their strict transforms) by \( l_{1,1}'^{(0)}, \ldots, l_{1,R_1}'^{(0)} \).

2) \( p_2 : D_2 \to D_1 \) is the composition with \( p_1' \) of the regular modification \( p_1'' : D_2 \to D_1' \) which is successive blown-up of \( l_{1,1}'^{(1)} \), then \( l_{1,1}'^{(1)} \), and so on till \( l_{P_1}^{(1)} \). By \( E_2 \) we denote the exceptional divisor of \( p_1 \circ p_2 \).

3) \( p_n : D_n \to D_{n-1} \) is constructed.

By \( E_n \) denote the exceptional divisor of \( p_1 \circ \ldots \circ p_n \). Suppose that \( f_n := f_{n-1} \circ p_n \) meromorphically extends onto \( D_n \setminus S_n \) with \( S_n \) being zero-dimensional. Take some \( s_n \in p_n^{-1}(s_{n-1}) \cap S_n \) (if such exists). Let \( p_n : D_n' \to D_n \) be a regular all noncompact components
of $I_1,s_n(f_n)$ which are contained in $E_n$. Denote them (and their strict transforms) by $l_1^{(n)}, \ldots, l_p^{(n)}$.

$n+1$ $p_{n+1} : D_{n+1} \to D_n$ is the composition with $p_n'$ of the regular modification $p_n'' : D_{n+1} \to D_n$ which is successive blown-up of $l_1^{(n)}$, then $l_2^{(n)}$, and so on till $l_p^{(n)}$. By $E_{n+1}$ we denote the exceptional divisor of $p_1 \circ \ldots \circ p_{n+1}$.

**Lemma 3.1.2.** There exists a $n_0$ and $s_{n_0} \in S_{n_0} \cap (p_1 \circ \ldots \circ p_{n_0})^{-1}(s_0)$ such that no noncompact component of $I_1,s_{n_0}(f_{n_0})$ is contained in $E_{n_0}$ - the exceptional divisor of $p_1 \circ \ldots \circ p_{n_0}$.

**Remark.** This means that after applying $R_0$-times this Lemma we get a regular modification $p_{n_1} : (D_n, S_n, f_{n_1}) \to (D_0, S_0, f_0)$ and a point $s_{n_1} \in S_{n_1} \cap p_n^{-1}(s_0)$ such that all components of $I_1,s_{n_1}(f_{n_1})$ are compact and contained in $p_n^{-1}(s_0)$.

**Proof.** The proof follows from the previous Lemma 3.1.1 by sections. Take a point $a_0' \in l_1^{(0)} \setminus S_0$. Find a coordinates $z_1, z_2, z_3$ in the polydisk neighborhood $\Delta^3$ of $a_0'$ such that

1. $a_0' = 0$,
2. $\Delta^3 \cap S_0 = \emptyset$,
3. $\Delta^3 \cap l_1^{(0)} = \{ z : z_1 = z_2 = 0 \}$.

For $z' \in \Delta_2$, we put $\Delta_z = \{ z' \} \times \Delta$ and $f_z := f | \Delta_z$. Let $\nu_1$ be from Lemma 2.3.1. Put $\Sigma_0 = \emptyset$ and

$$
\Sigma_j = \{ z' \in \Delta^2 : \mathrm{vol}(\Gamma_{f_z'}) \leq \frac{\nu_1}{2} \cdot j \}, \text{ for } j \geq 1.
$$

From (2.3.2) we see that $\Sigma_j$ are closed, $\Sigma_j \subset \Sigma_{j+1}$, and $\bigcup_{j=1}^{\infty} \Sigma_j = \Delta^2$. Find $j_1 \geq 1$ such that $\Sigma_{j_1} \setminus \Sigma_{j_1-1}$ contains a closed disk $\Delta(a_0') \subset l_1^{(0)}$ centered at $a_0' \in l_1^{(0)}$. Note that by Lemma 2.3.1 and the fact the $f$ is not holomorphic in the neighborhood of $a_0''$ it follows that $\Sigma_{j_1}$ is contained in a proper analytic set in the neighborhood of $a_0''$ in $\Delta^2$. So we can find a disk $\Delta(a_0) \subset \Delta(a_0'')$ such that for some $r_0 > 0$ $(\Delta_{r_0} \times \Delta(a_0')) \setminus \{ 0 \} \times \Delta(a_0'') \subset \bigcup_{j > j_1} \Sigma_j$.

Remark that if $z_1' \in \{ 0 \} \times \Delta(a_0)$ and $z_2' \in (\Delta_{r_0} \times \Delta(a_0') \setminus \{ 0 \} \times \Delta(a_0'))$ one has a jump of volume: $|\mathrm{vol}(\Gamma_{f_{z_1'}}) - \mathrm{vol}(\Gamma_{f_{z_2''}})| \geq \nu_1/2$.

Suppose that for all $n$ the set $E_n$ contains a noncompact component of $I(f_n)$, which projects by $p_1 \circ \ldots \circ p_n$ onto $l_1^{(0)}$. Repeating the arguments above we can construct a next sequence:

1. $a_n$ a point on $l_1^{(n)}$;
2. a disk $\Delta(a_n)$ and disk $\Delta_{r_n}$ such that if $z_1' \in \{ 0 \} \times \Delta(a_n)$ and $z_2' \in \Delta_{r_n} \times \Delta(a_n)$ one has a jump of volume: $|\mathrm{vol}(\Gamma_{f_{z_1'}}) - \mathrm{vol}(\Gamma_{f_{z_2''}})| \geq \nu_1/2$.
3. $p_n(\Delta(a_n)) \subset \subset \Delta(a_{n-1})$.

Let $a \in \bigcap_{n=1}^{\infty} (p_1 \circ \ldots \circ p_n)(\Delta(a_n))$. Take a two-ball $B_2^2$ centered at $a$ and transversal to $l_1^{(0)}$. Note that if this ball was chosen sufficiently small that $f |_{B_2^2}$ is meromorphic and by properties (i)-(iii) above all $f \circ (p_n \circ \ldots \circ p_1)$ are essentially meromorphic i.e. not holomorphic. This contradics Lemma 3.1.1.

Q.E.D.
In the sequel we shall repeatedly use the following statement.

**Lemma 3.1.3.** Let $f : H^{n+1}(r) \to X$ be a meromorphic map into a complex space $X$, which possess a meromorphic extension property in dimension $n$. Then $f$ meromorphically extends onto $\Delta^{n+1} \setminus S$, where $S = S_1 \times \ldots \times S_{n+1}$ and all $S_j \subset \Delta$ are of harmonic measure zero.

**Proof.** We shall prove that $f$ extends meromorphically onto $E^{n+1}(r) \setminus S$ with $S$ as described. Here

$$E^{n+1}(r) = (\Delta_r^{n-1} \times \Delta_r \times \Delta) \cup (\Delta_r^{n-1} \times \Delta \times A_{1-r,1}) = \bigcup_{z_1 \in \Delta_r} E_r^{n}(z_1)$$

in notations of 2.2. This will clearly imply the statement of Lemma.

According to the assumption of $n$-dimensional extension property of $X$, for every $z_1 \in \Delta_r$ the restriction $f_{z_1} = f|_{E_r^{n}(r)}(r)$ extends meromorphically onto $D_{z_1} := \{z_1\} \times \Delta_r^{n-2} \times \Delta^2$.

After shrinking, we can suppose that all $f_{z_1}$ are meromorphic in the neighborhood of $\bar{D}_{z_1}$ in $\{z_1\} \times \mathbb{C}^n$. Put $D := \Delta_r^{n-2} \times \Delta^2$. Take $\varepsilon > 0$ and consider $D_\varepsilon = \{z \in D : d(z, \partial D) \geq \varepsilon\}$. Denote by $\Omega_\varepsilon$ the maximal open subset of $\Delta_r$ such that $f$ meromorphically extends onto $\Omega_\varepsilon \times D_\varepsilon$. Let $S_\varepsilon = \Delta_r \setminus \Omega_\varepsilon$. Let $\nu_p$ be as in (5) of Lemma 2.1.1. Put $\nu = \inf \{\nu_p : \varepsilon^{2(n-p)} : p = 0, \ldots, n-1\}$.

Consider a following closed subsets of $S_\varepsilon$: $S_j^\varepsilon = \{z_1 \in S_\varepsilon : \text{vol}(\Gamma_{f_{z_1}}) \leq \frac{\nu}{2} \cdot j\}$. Note that $S_{j+1}^\varepsilon \supset S_j^\varepsilon$ and $S_\varepsilon = \bigcup (S_j^\varepsilon).$ $S_{j+1}^\varepsilon \setminus S_j^\varepsilon$ is of harmonic measure zero in $\Delta_r \setminus S_j^\varepsilon$. Really, would $s_0 \in S_{j+1}^\varepsilon \setminus S_j^\varepsilon$ be some regular point of $S_{j+1}^\varepsilon$, then by Lemma 2.4.1 $f$ would meromorphically extend onto $V \times D$ for some neighborhood $V \ni s_0$. This contradicts the maximality of $\Omega_\varepsilon$. By the Josefson $S_\varepsilon$ is polar.

Further $S_1 = \bigcup S_\varepsilon$ is polar, so $f$ extends onto $(\Delta_r \setminus S_1) \times D$. Repeating the same arguments for other coordinates we obtain the statement of Lemma.

q.e.d.

Consider a meromorphic map $f_0 : D_0 \setminus S_0 \to X$ into a complex space which possesses a meromorphic extension property in dimension $n$, $S_0$ is zerodimensional and $\text{dim}D_0 = n+1$. Let $\pi_1 : D_1 \to D_0$ be some regular modification.

**Lemma 3.1.4.** $f_0 \circ \pi_1$ extends to a meromorphic map $f_1 : D_1 \setminus S_1 \to X$, where $S_1$ is zero dimensional, closed and pluripolar subset of $D_1$.

The proof, which is similar to that of previous Lemma, will be omitted.

We shall also make use of the fact that the set of harmonic measure zero on the plain has zero dimension, see [Gl].

Note that Lemma 3.1.3 gives part (i) of Theorem 2.

3.2. *Estimates.*

Let $X$ be a complex space, equipped with some Hermitian metric $h$. By $\omega$ we denote, as usually, $(1,1)$-form canonically associated with $h$. Let $S_0$ be some zero dimensional closed subset of a complex manifold $D_0$ and let $f : D_0 \setminus S_0 \to X$ be some meromorphic map such that $\text{cl}(f(D_0 \setminus S_0)) \subset K$-some compact in $X$. Suppose that some sequence $\{\pi_j\}_{j=1}^N$ of local blowings-up $\pi_j : D_j \to D_{j-1}$ over the point $s_0 \in S_0$ is given. Denote by
for the Lelong numbers of the currents $T$ and compact neighborhood of $T$. To simplify the notations in what follows we drop the subindex $i$ from here we put $\tilde{f}$ the lifting of $f_{j-1}$ onto $D_j$, i.e. $f_j = f_{j-1} \circ \pi_j$, where $f_0 = f$. Suppose now that all $f_j$ extend meromorphically onto $D_j \setminus S_j$, where $S_j$ are closed zero dimensional subsets of $D_j$. As before, by $l_{j-1}$ we denote the center of $\pi_j$ and by $E_j = \pi_j^{-1}(l_{j-1})$ the corresponding exceptional divisor (and all its strict transforms under $\pi_{j+1}, \ldots, \pi_N$). Denote by $\hat{S} = S_N$ and by $\hat{D} = D_N$. Put also $\pi = \pi_1 \circ \ldots \circ \pi_N : \hat{D} \to D_0$. We suppose moreover that $f_j|_{E_j \setminus S_j}$ meromorphically extends onto $E_j$. For every $j = 1, \ldots, N$ by $r_j$ denote the rank of $f_j|_{E_j}$.

**Remark.** In what follows the local blowings up will appear in the following context. Take a point $s_j \in E_j$ and let $l_j$ be the smooth compact component of $I_{1,s_j}(f_j)$ (note that codim $l_j \geq 2$). In this case we shall take $V_j = D_j$ and blow up $D_j$ along $l_j$ to obtain an exceptional divisor $E_{j+1}$. If $l_j$ was compact then $E_{j+1}$ is also compact. Such configurations will contribute to our estimates of Lelong numbers.

Consider now currents $T_p = (f^* \omega + dd^c \|z\|^2)^p$ on $D_0 \setminus (S_0 \cup I(f_0))$. More accurately the currents $T_p$ one can define as $T_p = ((p_1 \mid_{\Gamma_j}), p_2^* w + dd^c \|z\|^2)^p$. For each $T_p$ we consider the Lelong number of $T_p$ in $s_0$:

$$\Theta(T_p, s_0) = \lim_{\varepsilon \searrow 0} \sup \int_{B_{s_0}(\varepsilon) \setminus (S_0 \cup I(f_0))} T_p \wedge (dd^c \|z\|^2)^{2(n-p)},$$

which can of course take infinite value, $n = \dim D_0$. By $\nu_p = \nu_p(K)$ we denote, as before the minima of volumes of $p$-dimensional compact subvarieties in $X$, which are contained in $K$. By $\sigma_p$ denote the number of those $j$ that $l_j$ is compact and $r_{j+1} = p$.

**Lemma 3.2.1.** There is a constant $C = C(h, K)$, depending only on Hermitian metric $h$ and compact $K$, such that for any meromorphic map $f$ as above one has the next estimate for the Lelong numbers of the currents $T_p$ in the point $s_0 \in S$:

$$\Theta(T_p, s_0) \geq C \cdot \sigma_p \cdot \nu_p$$

Proof. Take only those $E_{j_1}$ which are blowings-up of compact $l_{j_1-1}$ and $\operatorname{rk} f_{j_1} |_{E_{j_1}} = p$. To simplify the notations in what follows we drop the subindex $1$ in $j$-s.

Take an $\varepsilon$-neighborhood $V^\varepsilon$ of $\bigcup_{i=1}^{p} E_j$ with respect to some metric on $\hat{D}$. Remove from $V^\varepsilon$ the $\varepsilon$-neighborhood of intersections $E_i \cap E_j, i \neq j$, $\varepsilon$-neighborhood of $\hat{S}$ and $\varepsilon$-neighborhood of $I(f_N)$. For $\varepsilon > 0$ small enough we obtain the union $\bigcup_{j=1}^{p} V^\varepsilon_j$ of pairwise disjoint open sets: $\bigcup_{j=1}^{p} V^\varepsilon_j = V^\varepsilon \setminus (\bigcup_{i \neq j} E_i \cap E_j) \varepsilon \cup \hat{S} \cup I(f_N)$. By $\tilde{V}^\varepsilon_j$ denote the closure of $V^\varepsilon_j$ in $\hat{D}$.

Denote by $W^\varepsilon_j$ the image of $\tilde{V}^\varepsilon_j$ under the blown-down mapping $\pi : \hat{D} \to D_0$. Starting from here we put $s_0 = 0$. Note that for $\varepsilon > 0$ sufficiently small $W^\varepsilon_j \cap W^\varepsilon_i = \{0\}$ for $i \neq j$. Note further that $\pi^{-1} \mid_{W^\varepsilon_j \setminus \{0\}} : W^\varepsilon_j \setminus \{0\} \to \hat{D}$ is correctly defined and, in fact, $\pi^{-1}(W^\varepsilon_j \setminus \{0\}) = \tilde{V}^\varepsilon_j \setminus E_j$. Denote by $F^\varepsilon_j$ the closure in $D_0 \times X$ of the graph of $f \mid_{W^\varepsilon_j \setminus \{0\}}$.

So $F^\varepsilon_j \cap (\{0\} \times X)$ is compact in $\{0\} \times X$ of finite Hausdorff $p$-measure. In fact $F^\varepsilon_j \cap (\{0\} \times X)$ is the closure of the image of $f_N(E_j \setminus (S_j \cup \bigcup_{i \neq j} E_i) \cup I(f_N))$, so $F^\varepsilon_j \cap (\{0\} \times X)$ is contained in a compact subvariety $A_j = f(E_j)$ of $\{0\} \times X$ of complex dimension $p$. 

25
Let $K$ be a compact in $X$ containing $cl(f(D_0 \setminus S_0))$. Cover $K$ by a finite number of open sets $\{U_\alpha\}$, which are biholomorphic to an analytic subsets of $\Delta^{m_1}_\alpha$ unit polydisk in $\mathbb{C}^{m_1}, m_1 \geq m = \text{dim}X$. Appropriate coordinate functions on $U_\alpha$ are denoted by $u_1^\alpha, ..., u_{m_1}^\alpha$. In each $\Delta^{m_1}_\alpha$ find an orthonormal basis $e_1^\alpha, ..., e_{m_1}^\alpha$ of $(1,0)$-forms with respect to the Hermitian metric $h_\alpha$ on $\Delta^{m_1}_\alpha$, so that the form $w_\alpha$ can be written as $w_\alpha = \sum_{j=1}^{m_1} \frac{i}{2} e_j^\alpha \wedge \bar{e}_j^\alpha$.

Now put $\|u_\alpha\|^2 = \sum_{j=1}^{m_1} |u_j^\alpha|^2$ and express

$$\frac{i}{2} \partial \bar{\partial} \|u_\alpha\|^2 = \sum_{j,k=1}^{m_1} \frac{i}{2} c_{jk}^\alpha e_j^\alpha \wedge \bar{e}_k^\alpha,$$

where $[c_{jk}^\alpha]_{j,k=1}^{m_1}$ is strictly positive-definite matrix depending on $u_\alpha \in U_\alpha$. Find constants $C_\alpha^{1}, C_\alpha^{2}$ such that

$$C_\alpha^{1} \cdot w_\alpha \leq \frac{i}{2} \partial \bar{\partial} \|u_\alpha\|^2 = \sum_{j,k=1}^{m_1} \frac{i}{2} c_{jk}^\alpha e_j^\alpha \wedge \bar{e}_k^\alpha \leq C_\alpha^{2} \cdot \sum_{j=1}^{m_1} \frac{i}{2} e_j^\alpha \wedge \bar{e}_j^\alpha = C_\alpha^{2} w_\alpha.$$

Denote by $A_{j}^{-\varepsilon} := \text{Reg}(f_j(E_j \setminus S_j \cap \bigcup_{i \neq j} E_i^{\varepsilon})))$. Cover $A_{j}^{-\varepsilon}$ by a finite number of open sets $V_i$ satisfying the following conditions:

(i) $V_i \subset U_\alpha$ for some $\alpha$.

If $\phi_\alpha : U_\alpha \to \Delta^{m_1}_\alpha$ is coordinate imbedding, and $\phi_\alpha \mid V_i$ is a proper imbedding into some open subset $V_i'$ of $\Delta^{m_1}_\alpha$, then on $V_i'$ one can introduce coordinates $u_1^i, ..., u_{m_1}^i$ such that

(ii) $V_i'$ is polydisk in coordinates $u^i$ and $\phi_\alpha(V_i \cap A_{j}^{-\varepsilon}) = \{u^i : u^i_{p+1} = ... = u^i_{m_1} = 0\}$.

(iii) $\frac{1}{2} \cdot dd^c \|u_\alpha\|^2 \leq dd^c \|u^i\|^2 \leq 2 \cdot dd^c \|u_\alpha\|^2$.

(iv) Each point of $A_{j}^{-\varepsilon}$ belongs not more than to $2p+1$ of $V_i$.

If $p = m_1$, then the second condition is not needed. Put $U = V_i'$ to simplify the notations and denote by $B^p_\varepsilon$ the ball of radius $r$ centered at zero, $0 < r < \varepsilon$, where $\varepsilon$ is chosen sufficiently small to guarantee that the restriction onto $F_j^\varepsilon \cap (B^p_\varepsilon \times U)$ of the projection $pr : B^n \times U \to B^n \times U_{u_1}^{p}, ..., u_p^{p} \times U_{u_{p+1}}^{p}, ..., u_{m_1}^{p}$ is proper. Here $U = U_{u_1}^{p}, ..., u_p^{p} \times U_{u_{p+1}}^{p}, ..., u_{m_1}^{p}$. We had dropped the indice $i$ in $u^i$.

Now put $C(K) = \max_\alpha \{C_\alpha^{1}, C_\alpha^{2}\}$. Further put $Z_j = pr(F_j^\varepsilon \cap (B^n \times U))$. It is a closed subvariety of dimension $n$ in $B^n \times U_{u_1}^{p}, ..., u_p^{p}$.

Consider the following Hermitian metric on $B^n \times U_{u_1}^{p}, ..., u_p^{p}$: $e = \sum_{j=1}^{n} \frac{i}{2} dz_j \otimes d\bar{z}_j + \sum_{k=1}^{p} \frac{i}{2} du_k \otimes d\bar{u}_k$. Denote by $vol_e Z_j$ the volume of $Z_j$ with respect to $e$. Then putting $\|u\|^2 = \sum_{j=1}^{p} |u_j|^2$, we have

$$vol_e(Z_j) \leq \int_{(B^n_\varepsilon \times U) \cap F_j^\varepsilon} (dd^c (\|z\|^2 + \|u\|^2_1))^n \leq \frac{2^n}{n!} \int_{(B^n_\varepsilon \times U) \cap F_j^\varepsilon} (dd^c (\|z\|^2 + \|u\|^2_1))^p \wedge (dd^c (\|z\|^2))^n-p \leq$$
\[ \leq C^p(K) \frac{2^n}{n!} \int_{(B_r^p \times U) \cap F_j^c} (dd^c \| z \|^2 + w)^p \wedge (dd^c \| z \|^2)^{n-p} = \]

\[ = C(K)^p \frac{2^n}{n!} \int_{B_r^p \cap W_j^c} T_p \wedge (dd^c \| z \|^2)^{n-p}. \quad (3.2.5) \]

From the well known lower bound of volumes of analytic varieties, see [A-T-U], one has

\[ \text{vol}_{2n}(Z_j) \geq C \cdot r^{2n-2p} \cdot \text{vol}_{2p}(Z_j^0). \quad (3.2.6) \]

Here \( Z_j^0 = Z_j \cap \{0\} \times U = A_j^{-\varepsilon} \cap U \) and \( C \) is absolute constant. Now from (3.2.5) and (3.2.6) we get that

\[ C^p(K) \frac{2^n}{n!} \int_{B_r^p \cap W_j^c} T_p \wedge (dd^c \| z \|^2)^{n-p} \geq \text{vol}_e(Z_j) \geq \]

\[ \geq C \cdot r^{2n-2p} \cdot \text{vol}_e(Z_j^0) = r^{2n-2p} C \cdot \text{vol}_e(A_j \cap U) \geq r^{2n-2p} C \cdot \frac{1}{C^p(K)} \cdot \text{vol}_h(A_j \cap U). \]

So

\[ \frac{1}{r^{2n-2p}} \int_{B_r^p \cap W_j^c} T_p \wedge (dd^c \| z \|^2)^{2n-2p} \geq \frac{n! C^{2p}(K)}{2^n} \sum_i \text{vol}_h(A_j^{-\varepsilon} \cap V_i) \geq \]

\[ \geq \frac{n! C^{2p}(K)}{2^n (2p + 1)} \text{vol}_h(A_j^{-\varepsilon}). \]

Remained to take sum over \( j = 1, \ldots, \sigma_p \) and let \( \varepsilon \to 0 \).

q.e.d.

**Proof of Theorem 1.**

(a) is proved in Lemma 3.1.3.

(b) Take a ball \( D_0 \) centered at \( s_0 \) and such that all components of \( I_1(f_0) \) intersecting \( D_0 \) pass through \( s_0 \). Denote \( S_0 = D_0 \cap S \) and put \( f_0 = f \mid_{D_0 \setminus S_0} \). Denote by \( l_1^{(0)}, \ldots, l_{N_0}^{(0)} \) all components of codimension two in \( I_{s_0}(f_0) \). In the sequel we shall say that a regular modification \( \pi : (D_1, S_1, f_1) \to (D_0, S_0, f_0) \) is given, having in mind that \( f_0 \) is defined and meromorphic on \( D_0 \setminus S_0 \), where \( S_0 \) is zero dimensional, and that \( f_1 = f_0 \circ \pi \) extends meromorphically onto \( D_1 \setminus S_1 \), where \( S_1 \) again is zero dimensional by Lemma 3.1.4. We denote by \( I_R(f) \) the set of all irreducible components of \( I(f) \) which intersect the subset \( R \).

**Step 1.** There is a regular modification \( \pi_1 : (D_1, S_1, f_1) \to (D_0, S_0, f_0) \), such that the set \( I^{-1}_{\pi_1(s_0)}(f_1) \) doesn’t contain components of codimension two, which \( \pi_1 \) maps onto some of \( l_1^{(0)}, \ldots, l_{N_0}^{(0)} \).

**Proof of Step 1.** Let \( p_1 : (D_1, S_1, f_1) \to (D_0, S_0, f_0) \) be a regular modification which is a composition of resolution of singularities of \( l_1^{(0)} \) and blow-up of strict transform of \( l_1^{(0)} \). By \( l^{(1)} = \{l_1^{(1)}, \ldots, l_{N_1}^{(1)}\} \) denote the set of all components of \( I^{-1}_{p_1(s_0)}(f_1) \) whom \( p_1 \) projects onto \( l_1^{(0)} \).
Further, let $p_2 : (D_2, S_2, f_2) \to (D_1, S_1, f_1)$ be a regular modification, which is a composition of resolution of singularities of all $l^{(1)}_1, \ldots, l^{(1)}_{N_1}$ and successive blowings-up of their strict transforms, and $f_2 = f_1 \circ p_2$.

Suppose $p_n : (D_n, S_n, f_n) \to (D_{n-1}, S_{n-1}, f_{n-1})$ is constructed. Let us denote by $l^{(n)} = \{l^{(n)}_1, \ldots, l^{(n)}_{N_n}\}$ the set of all components of $I_{(p_1 \circ \cdots \circ p_n)^{-1}}(f_n)$ whom $p_n \circ \cdots \circ p_1$ projects onto $l^{(0)}_1$. Let $p_{n+1} : (D_{n+1}, S_{n+1}, f_{n+1}) \to (D_n, S_n, f_n)$ be a regular modification, which is a composition of resolution of singularities of all $l^{(n)}_1, \ldots, l^{(n)}_{N_n}$ and successive blowings-up of their strict transforms. Again $f_{n+1} := f_n \circ p_{n+1}$.

Lemma 3.1.2 tells us that for some $n_0$ we have $l^{(n_0)} = \emptyset$. Denote by $l^{(n_0)}_1, \ldots, l^{(n_0)}_{N_{n_0}}$ the set of all components of $I_{(p_1 \circ \cdots \circ p_{n_0})^{-1}}(f_{n_0})$ whom $p_1 \circ \cdots \circ p_{n_0}$ projects onto some of $l^{(0)}_1, \ldots, l^{(0)}_{N_0}$. We proved that $(p_1 \circ \cdots \circ p_{n_0})(\bigcup_{i=1}^{N_{n_0}} l^{(n_0)}_i) \subset \bigcup_{i=2}^{N_{n_0}} l^{(0)}_i$.

By $E_1$ denote a component of the exceptional divisor of $p_1 \circ \cdots \circ p_{n_0}$ such that $(p_1 \circ \cdots \circ p_{n_0})(E_1) = l^{(0)}_1$.

Replacing this procedure for the strict transform of $l^{(0)}_2$ instead of $l^{(0)}_1$ and so on till $l^{(0)}_{N_0}$ we get the regular modification $\pi_1 : (D_1, S_1, f_1) \to (D_0, S_0, f_0)$ such that no 1-dimensional component of $I(f_1)$ is mapped by $\pi_1$ onto some of $l^{(0)}_1, \ldots, l^{(0)}_{N_0}$. By $E_j$ denote the union of the components of the exceptional divisor of $\pi_1$ such that $\pi_1(E_j) = l^{(0)}_j$. Lemma 3.1.2 gives us now the statement of Step 1.

By $l^{(1)} = l^{(1)}_1, \ldots, l^{(1)}_{N_1}$ denote the set of all one-dimensional components of $I_{\pi_1^{-1}(s_0)}(f_1)$. Note that they all are contained in the fiber $\pi_1^{-1}(s_0)$. In particular they are compact.

**Step 2.** There is a regular modification $\pi_2 : (D_2, S_2, f_2) \to (D_1, S_1, f_1)$, such that the set $I_{(\pi_1 \circ \pi_2)^{-1}(s_0)}(f_2)$ doesn’t contain any component of codimension two. Moreover $\Theta_H(f^*w, s_0) \geq N_1 \cdot \nu$. Here the total Lelong number is defined as $\Theta_H(f^*w, s_0) := -\Theta(f^*w, s_0) + \Theta((f^*w)^2, s_0)$ and $N_1$ is a number of 1-dimensional components of $I_{\pi_1^{-1}(s_0)}(f)$.

**Proof of Step 2.** Construct inductively the following sequence of regular modifications:

1. $p_2 : (D_2, S_2, f_2) \to (D_1, S_1, f_1)$ is a composition of resolution of singularities of all $l^{(1)}_1, \ldots, l^{(1)}_{N_1}$ and successive blowings-up of their strict transforms.

Suppose $p_n : (D_n, S_n, f_n) \to (D_{n-1}, S_{n-1}, f_{n-1})$ is constructed. Let us denote by $l^{(n)} = \{l^{(n)}_1, \ldots, l^{(n)}_{N_n}\}$ the set of all one-dimensional components of $I_{(\pi_1 \circ p_2 \circ \cdots \circ p_n)^{-1}}(f_n)$ whom $p_1 \circ p_2 \circ \cdots \circ p_n$ projects onto some of $l^{(0)}_1, \ldots, l^{(0)}_{N_1}$.

$n+1$ $p_{n+1} : (D_{n+1}, S_{n+1}, f_{n+1}) \to (D_n, S_n, f_n)$ is a regular modification, which is a composition of resolution of singularities of all $l^{(n)}_1, \ldots, l^{(n)}_{N_n}$ and successive blowings-up of their strict transforms.

By Lemma 3.1.2 for some $n_1$ we have $l^{(n_1)} = \emptyset$. This means that there are compact divisors $E_1^{(1)}, \ldots, E_{N_1}^{(1)}$ such that $p^{n_1} := \pi_1 \circ p_2 \circ \cdots \circ p_{n_1}$ maps $E_i^{(1)}$ onto $l^{(1)}_i$ and $f_{n_1} |_{E_i^{(1)} \setminus S_{n_1}}$ meromorphically extends onto $E_i^{(1)}$. Lemma 3.2.1 gives us $\Theta_H(f^*w, s_0) \geq N_1 \cdot \nu$. Put $\pi_2 := p_2 \circ \cdots \circ p_{n_1}, D_2 := D_{n_1}, f_2 := f_{n_1}, S_2 := S_{n_1}$.
If \( I_{(\pi_1, \pi_2)^{-1}(s_0)}(f_{n_1}) \neq \emptyset \) we can repeat this procedure for \( f_{n_1} \) instead of \( f_1 \) and obtain the following sequence of regular modifications:

\[
\ldots \xrightarrow{\pi_{n+1}} D_n \rightarrow \ldots \rightarrow D_2 \xrightarrow{\pi_3} D_1 \xrightarrow{\pi_1} D_0
\]

Here, if we put \( \pi^n = \pi_1 \circ \ldots \circ \pi_n \) and \( l^{(n)} = \{ l_1^{(n)}(s_0), \ldots, l_{N(n)}^{(n)}(s_0) \} = I_{(\pi^n)^{-1}(s_0)}(f_{n_1}) \), then \( I_{(\pi^n)^{-1}(s_0)}(f_{n_1}) \) doesn’t contain components which \( \pi^n \) maps onto some of \( l_1^{(n)}, \ldots, l_{N(n)}^{(n)} \). So if \( I_{(\pi^n)^{-1}(s_0)}(f_{n_1}) \) is not empty for all \( n \), then \( \Theta_{hi}(f^*w, s_0) \geq \sum_{i=1}^n N_i \cdot \nu \) for all \( n \). This contradicts the condition of the Theorem. Thus \( I_{(\pi_1 \circ \pi_2)^{-1}(s_0)}(f_{n_1}) = \emptyset \) for some \( n \), and consequently \( l^{(n)} = \emptyset \) for some \( n \). Hence the Step 2 is proved.

**Step 3.** There is a regular modification \( \pi_3 : (D_3, S_3, f_3) \rightarrow (D_0, S_0, f_0) \), such that \( f_3 \) is holomorphic in the neighborhood of \( \pi_3^{-1}(s_0) \).

**Proof of Step 3.** Let \( \pi_1 \circ \pi_2 : (D_2, S_2, f_2) \rightarrow (D_0, S_0, f_0) \) be a regular modification constructed in Step 2. If \( S_2 \cap I_{(\pi_1 \circ \pi_2)^{-1}(s_0)}(f_2) \) is not empty then take a point \( s_2 \in S_2 \cap I_{(\pi_1 \circ \pi_2)^{-1}(s_0)}(f_2) \). If \( \pi_1 \circ \pi_2 \) is not biholomorphic, then there is a compact curve \( C \ni s_2 \), \( C \subset (\pi_1 \circ \pi_2)^{-1}(s_0) \), \( C \cong \mathbb{CP}^1 \). If \( \pi_1 \circ \pi_2 \) is biholomorphic take \( C = \{ z_2 = z_3 = 0 \} \) — line in \( D_0 = B^3 \) with \( s_0 \in C \).

By \( B^2 \) denote some two-ball transversal to \( C \) at \( s_2 \).

**Case 1.** \( f_2 \mid_{B^2 \setminus S_2} \) holomorphically extends onto \( B^2 \).

In this case \( f_2 \) is holomorphic in the neighborhood of \( s_2 \). To see this, take a Stein neighborhood \( W \) of \( \Gamma_{f(B^2)} \) in \( D_2 \times X \). Then the needed statement follows from the Hartogs extension theorem for holomorphic functions. But this contradicts the choice of \( s_2 \).

**Case 2.** \( f_2 \mid_{B^2} \) is not holomorphic (i.e. essentially meromorphic).

Denote by \( p_3 : \hat{B}^2 \rightarrow B^2 \) the finite sequence of blowings-up of points such that \( f_2 \circ p_3 : \hat{B}^2 \rightarrow X \) is holomorphic. Take a regular modification \( \pi_3 : D_3 \rightarrow D_2 \) as an extension of \( p_3 \) along \( C \). Let \( p'_3 \) (correspondingly \( \pi'_3 \)) be the last blowing-up in the sequence \( p_3 \) (corr. \( \pi_3 \)). Let \( l_3 \) (corr. \( E_3 \)) denote the exceptional divisor of \( p'_3 \) (corr. \( \pi'_3 \)). Then either \( (f_2 \circ \pi_3)(E_3) \) is nonconstant or \( l_3 \in I_{(\pi_1 \circ \pi_2)^{-1}(s_0)}(f_2 \circ \pi_3) \). In both cases we have a contribution to \( \Theta_{hi}(f^*w, s_0) \) of at least \( \nu \). If \( f_2 \circ \pi_3 \) is not holomorphic in the neighborhood of \( (\pi_2 \circ \pi_3)^{-1}(s_0) \) then take \( \pi_4 \) to be a composition of a regular modification described in Step 2 and just above. We get a sequence of regular modifications

\[
\ldots \xrightarrow{\pi_{n+1}} D_n \rightarrow \ldots \rightarrow D_2 \xrightarrow{\pi_3} D_1 \xrightarrow{\pi_1} D_0
\]

such that \( \Theta_{hi}(f^*w, s_0) \geq n \cdot \nu \). So it is finite by the assumption of the Theorem. That means that for \( f_n \circ \pi_n \) with \( n \) big enough only the first case can occur. Thus \( f_n \circ \pi_n \) is holomorphic in the neighborhood of \( (\pi_1 \circ \ldots \circ \pi_n)^{-1}(s_0) \).

The proof of Step 3 and thus of Theorem 1 is completed.

q.e.d.

References.
[A-1] Alehyane O.: Une extension du théorème de Hartogs pour les applications séparément holomorphes. C.R.Acad.Sci. Paris, t.323, Série I (1996).

[A-2] Alehyane O.: Applications séparément méromorphes dans les espaces analytique. Preprint.

[A-T-U] Alexander H., Taylor B., Ullman J.: Areas of Projections of Analytic Sets. Invent. math., 16, 335-341, (1972).

[Ba] Barlet D.: Majoration du volume des finres génériques et la forme géométrique du théorème d’aplatissement. Semiar P. Lelong - H. Skoda, Lect. Notes Math., 822, 1-17, (1978/79).

[B-T] Bedford E., Taylor B.: A new capacity for plurisubharmonic functions. Acta Math. (1982) 149, 1-40, (1982).

[B-M] Bierstone, E., Milman, P.D.: Local resolution of singularities. Proc. Symp. Pure. Math. 52, 42-64, (1991).

[Gl] Golysin G.: Geometric function theory. Nauka, Moscow (1966).

[H-S] Harvey R., Shiffman B.: A characterization of holomorphic chains. Ann. Math. 99, 553-587, (1974).

[Hi] Hironaka H.: Introduction to the theory of infinitely near singular points. Mem. Mat. Inst. Jorge Juan, 28, Consejo Superior de Investigaciones Cientificas, Madrid (1974).

[Hs] Hirschowitz A.: Les deux types de méromorphie différent. J. reine & ang. Math., 313 , 157 - 160, (1980).

[F] Frisch J. Points de platitude d’une morphisme d’espaces analytiques complexes. Invent. math. 4 , 118 - 138, (1967).

[Iv-1] Ivashkovich S.: The Hartogs phenomenon for holomorphically convex Kähler manifolds Math. USSR Izvestiya 29 N 1, 225-232 (1987).

[Iv-2] Ivashkovich S.: The Hartogs-type extension theorem for the meromorphic maps into compact Kähler manifolds. Invent. math. 109 , 47-54, (1992).

[Ka-1] Kato, M.: Compact complex manifolds containing ”global spherical shells”. Proc. Int. Symp. Alg. Geom. Kyoto. 45-84, (1977).

[Ka-2] Kato, M.: Examples on an Extension Problem of Holomorphic Maps and Holomorphic 1-Dimensional Foliations. Tokyo Journal Math. 13, n 1, 139-146, (1990).

[Kz] Kazaryan M.: Meromorphic extension with respect to groupos of variables. Math. USSR Sbornik 53, n 2, 385-398, (1986).

[Kl] Klimek M.: Pluripozential theory. London. Math. Soc. Monographs , New Series 6, (1991).
[Os] OSGOOD W.: *Lehrbuch der Funktionstheorie, Bd.II.1*. Leipzig. Teubner (1929).

[Re] REMMERT R.: *Holomorphe und meromorphe Abbildungen komplexer Räume*. Math. Ann. 133, 328-370, (1957).

[Sa] SADULLAEV A.: *Plurisubharmonic measures and capacities on complex manifolds*. Russian Math. Surv. 36, No 4, 61-119, (1981).

[Sh-1] SHIFFMAN B.: *Extension of Holomorphic Maps into Hermitian Manifolds*. Math. Ann. 194, 249-258, (1971).

[Sh-2] SHIFFMAN B.: *Separately meromorphic mappings into compact Kähler manifolds*. Contributions to Complex Analysis and Analytic Geometry (H. Skoda and J.-M. Trépreau, eds) Vieweg, Braunschweig, Germany, 243-250 (1994).

[Sh-3] SHIFFMAN B.: *Hartogs theorems for separately holomorphic mappings into complex spaces*. C.R.Acad.Sci. Paris, t.310, Série I, 89-84, (1990).

[Si-1] SIU Y.-T.: *Every stein subvariety admits a stein neighborhood*. Invent. Math. 38, N 1, 89-100, (1976).

[St] STOLZENBERG G.: *Volumes, Limits and Extension of Analytic Varieties*. Springer Verlag (1966).

Université de Lille-I
UFR Mathématiques
59655 Villeneuve d’Ascq Cedex
France
ivachkov@gat.univ-lille1.fr

IAPMM Acad Sci. of Ukraine
Naukova 3/b, 290053 Lviv
Ukraine