Quantum Criticality out of Equilibrium: Steady State in a Magnetic Single-Electron Transistor

Stefan Kirchner\textsuperscript{1,2} and Qimiao Si\textsuperscript{1}

\textsuperscript{1}Department of Physics \& Astronomy, Rice University, Houston, Texas, 77005, USA
\textsuperscript{2}Max Planck Institute for the Physics of Complex Systems, 01187 Dresden, Germany

Quantum critical systems out of equilibrium are of extensive interest, but are difficult to study theoretically. We consider here the steady state limit of a single-electron transistor with ferromagnetic leads. In equilibrium (i.e., bias voltage $V = 0$), this system features a continuous quantum phase transition with a critical destruction of the Kondo effect. We construct an exact quantum Boltzmann treatment in a dynamical large-N limit, and determine the universal scaling functions of both the non-linear conductance and fluctuation-dissipation ratios. We also elucidate the decoherence properties as encoded in the local spin response.

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Quantum phase transitions are of extensive current interest in a variety of strongly correlated electronic and atomic systems\cite{1}. A quantum critical point occurs when such transitions are second order\cite{2}. In a quantum critical state, there is no intrinsic energy scale in the excitation spectrum. External probes will readily drive the system out of equilibrium, such that the standard notion of linear response breaks down. The response of such a system to an external drive cannot be directly related to its intrinsic fluctuations in equilibrium.

Compared to their classical counterparts, quantum critical systems are more difficult to study already at the equilibrium level. A quantum critical point involves the mixing of statics and dynamics. This complicates the determination of the equilibrium fluctuation spectrum, especially in the long-time “quantum relaxational” regime ($\hbar \omega \ll k_B T$) at finite temperatures\cite{2}. In addition, the classification of the universality classes is delicate, since the critical modes may incorporate inherently quantum ones that go beyond the order-parameter fluctuations. A typical example is for Kondo systems, in which quantum criticality can involve a critical destruction of the Kondo entanglement\cite{3}.

To make progress theoretically, we need approaches that can, on the one hand, capture the scaling properties of a quantum critical point in equilibrium, and, on the other hand, be extended to settings out of equilibrium. This is impeded by the lack of a general understanding of how to generalize the free energy functional to stationary nonequilibrium states from which scaling relations could be obtained. Some progress has been made in recent years. In the case of the two-dimensional superconductor-insulator transition, the steady-state conductance in the non-linear regime [$V(\gg k_B T/e) \to 0$] is found to differ from its counterpart in the linear-response regime [$T(\gg eV/k_B) \to 0$]\cite{4,5}, in analogy to the non-commutativity of the $\omega \to 0$ and $T \to 0$ limits of the equilibrium current-current correlation function\cite{6}. A complementary issue, \textit{viz.}, quantum criticality induced by a steady-state voltage drive, has also been addressed in the contexts of itinerant magnetism\cite{7,8}. Still, there is a considerable need for insights into many open issues regarding quantum critical systems out of equilibrium. How do we connect the scaling behavior of the equilibrium fluctuation spectrum with that of the out-of-equilibrium properties? How to calculate the universal out-of-equilibrium scaling functions? Is the fluctuation-dissipation theorem violated in a universal way in the quantum critical regime?

In this letter, we study these issues in the steady state of a quantum critical system out of equilibrium. We have chosen a ferromagnetic single-electron transistor (SET)\cite{9} as a case study. It was shown earlier that this system undergoes a gate-voltage-induced quantum phase transition, which is characterized by a continuous suppression of the Kondo screening\cite{12,13}. A dynamical large-N limit was found to capture the universal properties of the quantum criticality in equilibrium\cite{12,14}.

Here, we show that this approach can be reliably generalized to the case of steady states under a finite bias, and use it to determine the full scaling functions of nonequilibrium properties. Our results establish the system as a prototype model setting in which the general issues raised earlier can be systematically studied. In addition to elucidating the nonequilibrium quantum criticality, our work also contributes to the general understanding of the Kondo effect out of equilibrium\cite{15,16,17,18,19,20}. The system, the model, and its large-N generalization: The system we consider is illustrated in Fig. 1a, showing a quantum dot coupled to two leads of ferromagnetic metals whose magnetizations are antiparallel. It was demonstrated experimentally\cite{9} and theoretically\cite{21,22} that a Kondo effect can still occur in the Coulomb-blockade regime, in spite of a large Stoner splitting of the conduction electron bands. It was shown\cite{12,13} that the system should be modeled in terms of a Bose-Fermi Kondo Hamiltonian. A localized moment on the dot is coupled to a fermionic bath, corresponding to the Stoner continuum of the ferromagnetic leads, with a coupling constant $J$. Simultaneously, it is coupled to a...
bosonic bath, associated with the magnons of the leads, with a coupling constant \( g \approx J^2/\Gamma \) (where \( \Gamma \) is the bare resonance width). Tuning the gate voltage varies the ratio \( (g/J) \) of the two couplings, giving rise to a continuous transition between the Kondo phase and a critical local moment (LM) phase.

Of interest to us here is the universal behavior of the quantum critical point, referred to henceforth as C, and the quantum critical LM phase. The latter is exactly marginal\([10, 11, 12]\), and would only be a coupling constant \( g \) separates the Kondo phase from a critical local moment (LM) phase. A critical point at coupling \( g \) separates a N° case.

Anticipating the large-N limit to be taken, we write the effective low-energy model in the following form:

\[
H = \sum_{i,\sigma,\sigma'} \left( \frac{J}{N} \right) S^\alpha \cdot c_{i\sigma}^\dagger c_{i\sigma'} S^\alpha + H_0(c) + \left( g / \sqrt{N} \right) S \cdot \Phi + \sum_q \omega_q \Phi_q^\dagger \Phi_q. \tag{1}
\]

Here, \( c_{i\sigma}^\dagger = (t_L c_{i\sigma,L}^\dagger + t_R c_{i\sigma,R}^\dagger) / \sqrt{|t_L|^2 + |t_R|^2} \) creates an electron in a linear combination of the local states in the leads for spin \( \sigma \) and channel index \( i = 1, \ldots, N \), \( i = 1, \ldots, M \), and \( t_L \) and \( t_R \) are the hybridization matrix elements with the left and right leads, respectively. \( H_0(c) = \sum_{\alpha=L,R} H_{0\alpha}(c_{\alpha}) \), with \( H_{0\alpha}(c_{\alpha}) = \sum_{\kappa_{\alpha'}} \epsilon_{\kappa_{\alpha'}} - \mu_{\alpha} c_{\kappa_{\alpha'}} \Phi_{\kappa_{\alpha'}} \). The energy dispersion, \( \epsilon_{\kappa_{\alpha}} \), contains the appropriate Zeeman shifting, but is otherwise taken to be featureless and has in particular a finite density of states in between the two chemical potentials \( \mu_L \) and \( \mu_R \). \( \Phi = \sum_q (\Phi_q^\dagger + \Phi_q) \) is also a linear combination of the spin waves on the left and right leads; it has been written to contain \( N^2 - 1 \) components. The matrices \( \tau^\alpha, \alpha = 1, \ldots, N^2 - 1 \) span the generators of SU(N). For simplicity, we consider \( t_L = t_R \) (otherwise, a local magnetic field will arise\([13, 21]\)). It follows from \( \omega_q \propto q^2 / \sqrt{N} \) that the spectrum of the bosonic modes is

\[
\sum_q [\delta(\omega - \omega_q) - \delta(\omega + \omega_q)] \sim |\omega|^{1/2} sgn(\omega), \tag{2}
\]

with \( \epsilon = 1/2 \). (Our results can be easily generalized to the generic sub-Ohmic case, \( 0 < \epsilon < 1 \).) The power-law form given in Eq. (2) is valid for \( \omega \) below some cutoff frequency, \( \Lambda \). For simplicity, we consider the particle-hole symmetric case without any potential-scattering term. The latter is exactly marginal\([10, 11, 12]\), and would hence only add a constant offset without modifying the \( T \)- and \( V \)-dependent scaling terms. The explicit Keldysh calculation in the presence of the potential scattering term is more difficult, because solving the pseudofermion constraint in the absence of particle-hole symmetry is more involved; we have nonetheless used such a calculation to confirm the above\[23\].

The dynamical large-N limit in equilibrium was considered in\[14\], and this limit has been shown to correctly capture the dynamical scaling properties (including, in particular, the finite-temperature relaxation properties) of the \( N = 2 \) case for both the quantum critical point \((g = g_c)\) and quantum critical LM phase \((g > g_c)\[12, 14]\). In the large-N limit\[21, 22\], \( S \) is expressed in terms of pseudofermions, \( S_{\sigma\sigma'} = f_{\sigma}^c f_{\sigma'} - \delta_{\sigma,\sigma'} q_{\delta_{\sigma,\sigma'}} \), where \( q_\delta = Q/N \) is related to the chosen irreducible representation of SU(N) and it also appears in the pseudo-fermion occupancy constraint \( \sum_{\sigma} f_{\sigma}^c f_{\sigma} = Q \). The quartic interaction is decoupled via a Hubbard-Stratonovich field \( B_{\delta}(i = 1, \ldots, M) \), which is conjugate to \( \sum_{\sigma} c_{i\sigma}^\dagger f_{\sigma} / \sqrt{N} \).

For a finite bias voltage, we work on the Keldysh contour, depicted in Fig. 1a. Since we are interested in the steady state limit, we specify the state of the system at the infinite past \( t_0 = -\infty \), so that at finite time all initial correlations will have washed out. A finite bias voltage, \( eV = \mu_L - \mu_R \), is incorporated in the following lesser function for the fermionic bath\[15, 26]\:

\[
G^<_{\Phi}(\omega) = \pi i [f_R(\omega) + f_L(\omega)] \rho(\omega), \tag{3}
\]

with \( f_{L/R}(x) = [\exp((x - \mu_{L/R})/T) + 1]^{-1} \) and \( \mu_L = eV/2 = -\mu_R \). Eq. (3) also defines the lead distribution function. Both leads are held at temperature \( T \) which completely characterizes the magnon distribution function. The magnons, therefore, retain thermal equilibrium, and the corresponding lesser function is

\[
G^<_{\Phi} = -2\pi ib(\omega) A_{\Phi}(\omega). \tag{4}
\]

Here, \( A_{\Phi}(\omega) = [1/\Gamma(1/2)] |\omega|^{1/2} sgn(\omega) \), where \( \Gamma \) is the Euler Gamma function, and \( b(\omega) = [\exp(\omega/T) - 1]^{-1} \).

**Large-N limit on the Keldysh contour:** In the equilibrium case, the large-N limit yields saddle-point equations for the pseudo-particle self-energies\[14\]. On the
can then be alternatively cast into:

\[
I(V, T) = \frac{e}{2h} \int d\omega \rho(\omega) \left[ f_L(\omega) - f_R(\omega) \right] \times \text{Im} \left[ T_{RL}^\alpha(\omega, T, V) + T_{RR}^\alpha(\omega, T, V) \right].
\]

Here, \( T_{\alpha\beta}, (\alpha, \beta = L/R) \), is the T-matrix of the Bose-Fermi Kondo model. In the large-N limit we consider, \( T(t) \equiv T_{LL} + T_{RR} = (i/N) G_B^{-1}(t) G_f(t) \).

**V/T scaling of the I-V characteristics:** We are now in a position to discuss the quantum critical behavior out of equilibrium. Fig. 2(a) shows the current-voltage characteristics at the quantum critical point, \( g = g_c \). In the linear-response regime, for \( T \to 0 \) after taking \( V \to 0 \), the conductance is a universal constant in the leading order [12]. In the non-linear regime, for \( V \to 0 \) at \( T = 0 \), we find that the conductance is also a universal constant, but differs from the linear-response value. This is reminiscent of what happens [4] at the two-dimensional superconductor-insulator transition. It is to be contrasted with the result in the two-channel Kondo problem, in which the two constants are identical: V/T scaling is then analyzed in an unorthodox fashion, in terms of \( \Delta G \equiv G(V, T) - G(V = 0, T = 0) \) [30, 31, 32].

Beyond these limits, we are able to determine the conductance for arbitrary \( V, T \). We find that \( I/T \) shows a scaling collapse of \( V/T \), provided \( V, T \approx T_K^0 \).

Consider next the quantum critical LM phase at \( g > g_c \). In the linear response regime, the conductance is proportional to \( T^{1/2} \) [12]. In the non-linear regime, \( V \to 0 \) at \( T = 0 \), we find that the conductance goes as \( V^{1/2} \). Fig. 2(b) now demonstrates a scaling collapse of \( I/T^{3/2} \) vs. \( V/T \). For both cases, Fig. 2 shows the full scaling function for the non-linear conductance.

**Fluctuation-dissipation ratio and spin decoherence:** A number of features underlie the \( V/T \) scaling found here for the non-linear conductance. First, we observe that the kernel in the expression for the current in Eq. (6) is a function of \( \omega/T \) and \( V/T \) only: \( f_L(\omega) - f_R(\omega) \) = \( \frac{\sinh(\omega/V/2T)}{\cosh(\omega/V/2T) + \cosh(\omega/V/2T)} \). This is actually a manifestation of the particular form for the fluctuation-dissipation ratio of the T-matrix:

\[
FDR_T \equiv \frac{T^+ + T^-}{T^+ - T^-} = \frac{\sinh(\omega/T)}{\cosh(\omega/T) + \cosh(\omega/V/2T)}.
\]

This result for the fluctuation-dissipation ratio is valid for arbitrary bias voltage, regardless of whether the system is critical, and for any \( N \). It appears not to have been recognized before, but it can be traced back to the steady state condition [29]. In light of Eq. (7), the \( V/T \) scaling seen in the I-V characteristics would follow if the spectral density of the T-matrix exhibits simultaneous \( \omega/T \) and \( V/T \) scaling. The latter is indeed observed.

To explore the origin of such scaling behavior further, and especially to elucidate the spin decoherence, we have studied the local spin susceptibility. We find...
that it too displays a $V/T$ and $\omega/T$ scaling. Fig. 3 illustrates the scaling collapse of the spin spectral density, $\Im \chi$. It shows the $V/T$ scaling at a particular frequency, $\omega = V/4$; independent $\omega/T$ and $V/T$ scalings are obeyed when the $\omega$ and $V$ dependences are separately analyzed (not shown). Fig. 3 shows a similar scaling collapse for the fluctuation-dissipation ratio of the spin susceptibility. Note that, unlike for $T$, the steady state condition does not fix the fluctuation-dissipation ratio of $\chi$.

Two important points are in order. First, the violation of the fluctuation-dissipation theorem away from thermal equilibrium depends on the physical property, as had already been noted in other settings (e.g., Ref. [33]); relatedly, the notion of an effective temperature apparently fails here. Second, the fluctuation-dissipation ratio nonetheless still satisfies universal scaling.

The $\omega/T$ and $V/T$ scaling occurs only at the quantum critical point and in the scaling regime of the quantum critical LM phase. As in the equilibrium relaxational regime ($V = 0$, $\omega \ll k_B T/h$) where an $\omega/T$ scaling implies a linear-in-$T$ spin relaxation rate [2], in the non-equilibrium relaxational regime here ($T = 0$, $\omega \ll eV/h$), the $\omega/V$ scaling implies a linear-in-$V$ dependence of the decoherence rate, $I_V \equiv [-i \partial \ln \chi^a(\omega, T = 0, V)/\partial \omega]_{\omega = 0} = c(e/h)V$. It is worth stressing that, in contrast to its counterpart in the high-voltage ($V \gg k_B T_R/e$) perturbative regime [13, 14, 34], the decoherence rate here is universal: $c$ is a number characterizing the fixed point.

The Kondo effect involving ferromagnetic leads has already been observed [34], and it was argued that implementing the tuning to the quantum critical regime is feasible [12]. The scaling of the out-of-equilibrium steady-state current can accordingly be studied. Measuring the local spin response is more challenging experimentally. A promising route could be provided by the single-spin ESR technique [35].

In conclusion, we have identified a system in which quantum criticality out of equilibrium can be systematically studied. In this model system, we have shown that the universal scaling is obeyed by the steady state current, the pertinent spectral densities, and the associated fluctuation-dissipation ratios. We have been able to calculate the entire scaling function for each of these non-equilibrium quantities. Our theoretical approach can be extended to study the transient behavior and other non-equilibrium probes of quantum criticality.

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Note added- After this paper was posted (arXiv:0805.3717), another work (C.-H. Chung et al., arXiv:0811.1230, Phys. Rev. Lett. 102, 216803 (2009)) has appeared on a nonequilibrium dissipative problem with an Ohmic spectrum. Their case has a Kosterlitz-Thouless transition, while the sub-Ohmic case we study here contains a genuine quantum critical point. Their results are complementary to ours.

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