Lattice equations arising from discrete Painlevé systems: II. $A_{3}^{(1)}$ case

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Lattice equations arising from discrete Painlevé systems: II. $A_4^{(1)}$ case

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Abstract
In this paper, we construct two lattices from the $\tau$ functions of $A_4^{(1)}$-surface $q$-Painlevé equations, on which quad-equations of ABS type appear. Moreover, using the reduced hypercube structure, we obtain the Lax pairs of the $A_4^{(1)}$-surface $q$-Painlevé equations.

Keywords: discrete Painlevé equation, ABS equation, Lax pair, $\tau$ function, affine Weyl group

(Some figures may appear in colour only in the online journal)

1. Introduction

Two longstanding classifications of integrable discrete systems in different dimensions, one by Adler–Bobenko–Suris (ABS) [1, 2, 6–8] and the other by Sakai [55], have been widely studied, but the mathematical connection between them remains incomplete. How to reduce the ABS partial difference equations to Sakai’s discrete Painlevé equations is a natural question that has inspired many authors [12, 14, 16, 17, 39, 48, 49]. However, these earlier approaches focused on taking periodic constraints in two-dimensions that lead to equations with a restricted set of parameters, manually extending these by adding gauge transformations in order to introduce more parameters. Another rich vein of inquiry reduces the Lax pairs of ABS equations to provide these elusive linear problems for discrete Painlevé equations. We provide a different approach grounded in higher-dimensional geometry associated naturally with full-parameter discrete Painlevé equations [23–25]. In this paper, we review our approach and illustrate it for $A_4^{(1)}$-surface type $q$-discrete Painlevé equations, providing new Lax pairs for these equations.

The geometric setting of reflection groups is essential to our approach. Within this framework, we construct higher dimensional lattices, called $\omega$-lattices, from discrete Painlevé equations. These lattices also arise from integer lattices associated with ABS classification...
and thereby provide a bridge between the two classifications. In an earlier series of papers [23–25], we constructed \( \omega \)-lattices for \( A_5^{(1)} \) and \( A_6^{(1)} \) surface \( q \)-Painlevé equations. The \( A_4^{(1)} \)-case is a simpler (less degenerate) surface than these earlier cases, but it is well known that when the surface is simpler, the corresponding symmetry groups and discrete Painlevé equations become more complex [55].

Despite the increasing complexity, our approach connects discrete Painlevé equations to partial difference equations through reductions of hypercubes and polytopes. We construct two lattices in two ways, one through reduction of polytopes and the other by reduction of hypercubes. Both lattices arise from the \( \tau \) functions of \( A_4^{(1)} \)-surface type \( q \)-discrete Painlevé equation. They share fundamental variables (called \( \omega \)-variables) and both give rise to ABS equations and to \( q \)-discrete Painlevé equations. The polytope case will be investigated further in future work. The hypercube lattice is referred to below as \( \omega_{A_{4+1}} \). (More details are given in section 1.2 and section 3.)

A fundamental property of the ABS equations is their consistency around each cube of the integer lattice. The reduced hypercube structure of the lattice \( \omega_{A_{4+1}} \) then provides us with reductions of the Lax pairs of ABS equations, which turn out to be new Lax pairs for \( q \)-Painlevé equations (1.1). Our results show that these equations share one monodromy problem. Moreover, the coefficient matrices in each case are factorized into product of matrices that are linear in the monodromy variable \( x \). We remark that in each case, we also obtain Lax pairs for the scalar form of the equations.

In this paper, we construct two important lattices, where quad-equations are observed, from the \( \tau \) functions of \( A_4^{(1)} \)-surface type \( q \)-discrete Painlevé equation. One is the \( \omega \)-lattice of type \( A_4^{(1)} \) investigated in section 2. An \( \omega \)-lattice provides informations about how a system of partial difference equations can be reduced to discrete Painlevé equations. It provides not only the type of equation, but also the combinatorial structure of the lattice before reduction (see [24, 25] for details). The other lattice is the \( \omega_{A_{4+1}} \) investigated in section 3. The lattice \( \omega_{A_{4+1}} \) can be obtained from an integer lattice, given by the space-filling of the hypercube on whose faces quad-equations of ABS type are assigned, by the geometric reduction. By using this reduced hypercube structure, we obtain the Lax pairs of the \( q \)-Painlevé equations (1.1). These Lax pairs differ from the ones in the literature [35]. Moreover, our result show that four equations of \( A_4^{(1)} \)-surface type share the same \( q \)-discrete monodromy problem (1.4) with differing deformation equations given by (1.7). Other distinctive properties of our Lax pairs are that their coefficient matrices occur as products of matrices of degree one in the spectral parameter \( x \) and elements of the coefficient matrices given by the rational functions of Painlevé variables.

### 1.1. \( A_4^{(1)} \)-surface \( q \)-Painlevé equations

In this paper, we collectively call the following \( q \)-difference equations as \( A_4^{(1)} \)-surface \( q \)-Painlevé equations since they are of \( A_4^{(1)} \)-surface type in Sakai’s classification [55]:

\[
\begin{align*}
FF &= \frac{c_1 + tG}{c_3 + G} \left( c_1 + tG \right) \left( c_2 + tG \right), \\
GG &= \frac{c_2}{t^2} \left( c_3^{-1} + tF \right) \left( qc_1c_2c_3^{-2} + tF \right) \frac{1}{c_3^{-1} + F}.
\end{align*}
\]
where \( \mathcal{F} \), \( \mathcal{G} \), and \( \mathcal{H} \) are known as a \( q \)-discrete analog of the Painlevé V equation [55], that of the Painlevé V equation [59], that of the Painlevé III equation of \( D_7 \) surface type [38] and that of the Painlevé IV equation [54], respectively.

Remark 1.1. It is known that \( q\)-Painlevé (1.1a) and \( q\)-Painlevé * (1.1b) can be reduced to \( q\)-Painlevé III (1.1c) [38] and \( q\)-Painlevé IV (1.1d) [37] by the projective reductions:

\[
\begin{align*}
    c_2 &= p^{-1}, \quad c_3 = 1, \quad q = p^2, \quad F = G, \\
    c_2 c_3 &= p^{-1}, \quad q = p^2, \quad F = G,
\end{align*}
\]

respectively. In this sense, \( q\)-Painlevé III (1.1c) and \( q\)-Painlevé IV are sometimes called as the scalar forms of \( q\)-Painlevé and \( q\)-Painlevé *, respectively.

1.2. Main results

In this section, we outline two main results of this paper.

Firstly, in section 4.1, we prove the following theorem.

**Theorem 1.2.** The lattice \( \omega_{A_4} \) has a reduced hypercube structure.

The lattice \( \omega_{A_4} \) is a three-dimensional integer lattice on which ABS equations (2.39) and (2.40) and \( q\)-Painlevé equations (1.1) appear. This lattice is constructed from the \( \tau \) functions of \( A_4 \)-surface \( q\)-Painlevé equations (see section 3). Theorem 1.2 means that the lattice \( \omega_{A_4} \) can be also obtained from the four-dimensional hypercube lattice on whose faces ABS equations are assigned. This reduced hypercube structure turn out to be essential in the construction of Lax pairs for discrete Painlevé equations [23].

Our second main result, theorem 1.3, concerns the Lax pairs of the \( q\)-Painlevé equations (1.1). Equation (1.1) share one spectral linear problem, which takes the factorized form

\[
\Phi(px) = \begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & 0 \end{pmatrix} \cdot \begin{pmatrix} * & x & * \\ * & * & x \\ * & * & x \end{pmatrix} \cdot \begin{pmatrix} * & x & * \\ * & * & x \\ * & * & x \end{pmatrix} \cdot \begin{pmatrix} * & x & * \\ * & * & x \\ * & * & x \end{pmatrix} \cdot \Phi(x) = A\Phi(x).
\]
Here, the $2 \times 2$ matrix $A = A(x)$ is given by (4.28) whose elements are expressed by the non-zero complex parameters $b_i$, $i = 0, \ldots, 3$, and $p$ and unknown functions $f_i^{(1)}$, $i = 1, 2, 3$. Note that the functions $f_i^{(1)}$ satisfy the following relation:
\begin{align*}
b_1 b_2 b_3 + p b_1^3 b_2 f_1^{(1)} - p^2 b_0^3 b_3^{3/2} f_1^{(3)} + p^2 b_0 b_1 b_3^{3/2} f_1^{(1)} f_1^{(2)} f_1^{(3)} &= 0. \quad (1.5)
\end{align*}
We introduce the deformation operators $T_0$, $T_{13}$, $R_0$ and $R_{13}$ whose actions on the parameters $b_i$, $i = 0, \ldots, 3$, and $p$ are given by
\begin{align*}
T_0 &: (b_0, b_1, b_2, b_3, p) \mapsto (p b_0, p b_1, b_2, b_3, p), \quad (1.6a) \\
T_{13} &: (b_0, b_1, b_2, b_3, p) \mapsto (b_0, b_1, p^2 b_2, b_3, p), \quad (1.6b) \\
R_0 &: (b_0, b_1, b_2, b_3, p) \mapsto (b_1, p b_0, b_2, b_3^{-1}, p), \quad (1.6c) \\
R_{13} &: (b_0, b_1, b_2, b_3, p) \mapsto (b_0, b_1, p b_2, b_3^{-1}, p), \quad (1.6d)
\end{align*}
while those on the spectral parameter $x$ and the wave function $\Phi = \Phi(x)$ are given by
\begin{align*}
T_0(x) &= T_{13}(x) = R_0(x) = R_{13}(x) = x, \quad (1.7a) \\
T_0(\Phi) &= \left( * x * x \right) \left( * x * x \right) \Phi(x) = B_{T0}\Phi, \quad (1.7b) \\
T_{13}(\Phi) &= \left( * x * 0 \right) \left( * x * 0 \right) \Phi(x) = B_{T13}\Phi, \quad (1.7c) \\
R_0(\Phi) &= \left( * x * x \right) \Phi(x) = B_{R0}\Phi, \quad (1.7d) \\
R_{13}(\Phi) &= \left( * x * 0 \right) \Phi(x) = B_{R13}\Phi, \quad (1.7e)
\end{align*}
where the $2 \times 2$ matrices $B_{T0} = B_{T0}(x)$, $B_{T13} = B_{T13}(x)$, $B_{R0} = B_{R0}(x)$ and $B_{R13} = B_{R13}(x)$ are given by (4.29). Equations (1.6) and (1.7) provide us with the deformation of the spectral problem.

**Theorem 1.3.** The compatibility conditions of the linear equation (1.4) with the operators $T_0$, $T_{13}$, $R_0$ and $R_{13}$:
\begin{align*}
T_0(A)B_{T0} &= B_{T0}(px)A, \quad T_{13}(A)B_{T13} = B_{T13}(px)A, \quad (1.8a) \\
R_0(A)B_{R0} &= B_{R0}(px)A, \quad R_{13}(A)B_{R13} = B_{R13}(px)A, \quad (1.8b)
\end{align*}
are equivalent to
\begin{align*}
\begin{cases}
T_0(f_1^{(3)})f_1^{(3)} &= \frac{b_1 (b_3 + p b_1 f_1^{(1)})(-b_0 b_2 b_1^{1/2} + p b_1 f_1^{(1)})}{p^2 b_0 b_3^{3/2} b_1 f_1^{(1)}}, \\
T_0^{-1}(f_1^{(1)})f_1^{(1)} &= \frac{p b_0 (b_3^{-1} + p b_0 f_1^{(1)})(-p b_1 b_2 b_3^{-1/2} + p b_0 f_1^{(3)})}{b_1^2 b_3^{-2} (b_1 b_3^{-1} + p b_0 f_1^{(3)})}.
\end{cases} \quad (1.9a)
\end{align*}
\[
\begin{align*}
\begin{cases}
T_{13}(f_1^{(1)})f_1^{(2)} - \frac{b_0^2}{b_1^4}f_1^{(1)}f_1^{(2)} - \frac{b_3^2}{b_1^4} = \frac{b_0^2b_2^2(pb_0^2 + b_1^2b_3f_1^{(2)})(1 + pb_1^2b_3f_1^{(2)})}{pb_1^2b_3^{3/2}(-b_0b_2 + b_1^3b_3^{1/2}f_1^{(2)})}, \\
f_1^{(1)}f_1^{(2)} - \frac{b_0^2}{b_1^4}f_1^{(1)}T_{13}^{-1}(f_1^{(2)}) - \frac{b_3^2}{b_1^4} = \frac{b_0^2b_2^2(pb_0^2b_3 + b_1^2f_1^{(1)})(b_3 + pb_1^2f_1^{(1)})}{pb_1^2b_3^{1/2}b_0(-b_0b_2b_3^{1/2} + pb_1^2f_1^{(1)})},
\end{cases}
\tag{1.9b}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
R_0(f_1^{(3)}) = f_1^{(1)}, \\
R_0(f_1^{(1)})f_1^{(3)} = \frac{b_1(b_3 + pb_1^2f_1^{(1)})(-b_0b_2b_3^{1/2} + pb_1^2f_1^{(1)})}{p^4b_0^4b_3^2(pb_0^2b_3 + b_1^2f_1^{(1)})},
\end{cases}
\tag{1.9c}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
R_3(f_1^{(1)}) = f_1^{(2)} , \\
R_3(f_1^{(2)})f_1^{(3)} - \frac{b_0^2}{b_1^4}f_1^{(1)}f_1^{(2)} - \frac{b_3^2}{b_1^4} = \frac{b_0^2b_2^2(pb_0^2 + b_1^2b_3f_1^{(2)})(1 + pb_1^2b_3f_1^{(2)})}{pb_1^2b_3^{1/2}(-b_0b_2 + b_1^3b_3^{1/2}f_1^{(2)})},
\end{cases}
\tag{1.9d}
\end{align*}
\]

respectively.

This theorem is proven in section 4.2. The actions (1.6) and (1.9) correspond to the q-Painlevé equations (1.1).

\textbf{Remark 1.4.} Equations (1.9a) and (1.9b) are equivalent to q-P\textsubscript{V} (1.1a) and q-P\textsubscript{V\textsuperscript{\textast}} (1.1b) by the following correspondences:

\[
\begin{align*}
\begin{cases}
- = T_0, & t = b_1^2, & c_1 = -\frac{b_0b_2b_3^{1/2}}{pb_1}, & c_2 = \frac{b_3}{p}, & c_3 = \frac{pb_0^2b_3}{b_1^2}, & q = p^2, \\
F = f_1^{(3)}, & G = f_1^{(1)},
\end{cases}
\tag{1.10a}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
- = T_{13}, & t = p^{1/2}b_2, & c_1 = -\frac{b_0}{b_1^{3/2}}, & c_2 = \frac{b_1}{p^{1/2}b_3^{1/2}}, & c_3 = \frac{1}{p^{1/2}b_1b_3^{1/2}}, \\
q = p^2, & F = -\frac{b_1^2}{b_0}f_1^{(1)}, & G = -\frac{b_2}{b_0}f_1^{(2)},
\end{cases}
\tag{1.10b}
\end{align*}
\]

respectively. Moreover, letting

\[
\begin{align*}
b_1 = p^{1/2}b_0, & \quad b_3 = 1,
\end{align*}
\tag{1.11}
\]

and setting

\[
\begin{align*}
- = R_0, & \quad t = b_1^2, & c_1 = -\frac{b_2}{p^{3/2}}, & G = f_1^{(1)},
\end{align*}
\tag{1.12}
\]

we obtain q-P\textsubscript{III}(D\textsubscript{1}^{(1)}) (1.1c) from the action (1.9c). Similarly, by letting

\[
\begin{align*}
b_3 = 1,
\end{align*}
\tag{1.13}
\]
and setting
\[ \sim = R_{13}, \quad t = p^{1/2}b_2, \quad c_1 = -\frac{b_0}{b_1^2}, \quad c_2 = \frac{b_1}{p^{1/2}}, \quad G = -\frac{b_1^2}{b_0}f_1^{(2)}, \]

the action (1.9d) gives \( q\)-P\(_{1V}\) (1.1d).

1.3. Background

In the 1900s, in order to find new class of special functions, Painlevé and Gambier classified all differential equations in the form of \( y'' = F(y', y, t) \), where \( y = y(t) \), \( t' = d/dt \) and \( F \) is a rational function, by imposing the condition that the solutions should admit only poles as movable singular points. As a result, they showed that the resulting equations can be reduced to one of the six equations, which are now called the Painlevé I through VI equations, unless it can be integrated algebraically, or transformed into a simpler equations such as a linear equation or the differential equation of elliptic functions. Moreover, it is known that Painlevé equations can be classified into eight types by the geometrical classification of space of initial conditions [42, 43, 55]. From the view point of this classification, \( P_{1II} \) can be divided into \( P_{1II}(D_6^{(1)}), P_{1II}(D_7^{(1)}) \) and \( P_{1II}(D_8^{(1)}) \) by the values of parameters.

Discrete Painlevé equations are nonlinear ordinary difference equations of second order, which include discrete analogs of the Painlevé equations. The geometric classification of discrete Painlevé equations, based on types of rational surfaces connected to affine Weyl groups, is well known [55]. Together with the Painlevé equations, they are now regarded as one of the most important classes of equations in the theory of integrable systems (see, e.g., [13, 30]).

It is well known that the \( \tau \) functions, which gives rise to various bilinear equations, play a crucial role in the theory of integrable systems [34]. The same is true in the theory of continuous and discrete Painlevé equations [19–21, 40, 44–47]. A representation of the affine Weyl groups can be lifted to the level of the \( \tau \) functions [25–27, 32, 33, 60, 62].

Discrete Painlevé equations are called integrable because they arise as compatibility conditions of associated linear problems called Lax pairs. The search for and construction of Lax pairs of discrete Painlevé equations has been a very active research area. Noteworthy approaches include extensions of Birkhoff’s study of linear \( q \)-difference equations [22, 56, 57], periodic-type reductions from ABS equations or the discrete KP/UC hierarchy [15, 16, 23, 31, 48, 50, 51, 53, 61], extensions of Schlesinger transformations [4, 10, 11], search for linearizable curves in the space of initial values [65, 66], Padé approximation or interpolation [18, 36, 41] and the theory of orthogonal polynomials [3, 5, 9, 52, 63, 64].

1.4. Plan of the paper

This paper is organized as follows: in section 2, we introduce the \( \tau \) functions of \( A_4^{(1)} \)-surface \( q \)-Painlevé equations, which have the extended affine Weyl group symmetry \( \tilde{W}(A_4^{(1)}) \), and show that the \( q \)-Painlevé equations (1.1) can be derived from a birational representation of \( \tilde{W}(A_4^{(1)}) \). Moreover, we construct the \( \omega \)-lattice of type \( A_4^{(1)} \) and then derive various quadratures of \( A_4^{(1)} \) type, as relations on the \( \omega \)-lattice. In section 3, we construct the lattice \( \omega_{A_1 + A_2} \) and show its properties. In section 4, we give the proofs of theorems 1.2 and 1.3 by using the geometric reduction from the integer lattice \( \mathbb{Z}^2 \) with the integrable \( P\Delta E \)s to the lattice \( \omega_{A_1 + A_2} \). Some concluding remarks are given in section 5.
2. Construction of the $\omega$-lattice of type $A_4^{(1)}$

In this section, we define $\tau$ functions by using the transformation group $\tilde{W}(A_4^{(1)})$. Then, we derive the $q$-Painlevé equations (1.1) and construct the $\omega$-lattice of type $A_4^{(1)}$ from the $\tau$ functions.

For convenience, throughout this paper we use the following notation for compositions of arbitrary mappings $w_1, \ldots, w_n$:

$$ w_1 \cdots w_n \equiv w_1 \circ \cdots \circ w_n. \quad (2.1) $$

2.1. $\tau$ functions

In this section, we define the $\tau$ functions by using the transformation group $\tilde{W}(A_4^{(1)}) = \langle s_0, s_1, s_2, s_3, s_4, \sigma, \iota \rangle$, which forms the extended affine Weyl group of type $A_4^{(1)}$ (see appendix A).

Below, we describe the actions of $\tilde{W}(A_4^{(1)})$ on the five parameters $\mathbf{\tau} = (\tau_1^{(1)}, \ldots, \tau_5^{(1)})$ and on the ten variables $\mathbf{t} = (t_{ij})$, $i = 1, 2$, $j = 1, \ldots, 5$, which satisfy the following three relations:

$$
\tau_i = a_0 a_i \tau_1^{(1)} + a_0 a_i \tau_2^{(1)}, \quad \tau_i = a_1 a_i \tau_1^{(1)} + a_1 a_i \tau_2^{(2)}, \quad \tau_i = a_2 a_i \tau_1^{(2)} + a_2 a_i \tau_2^{(2)}, \quad (2.2a)
$$

$$
\tau_i = a_3 a_i \tau_1^{(3)} + a_3 a_i \tau_2^{(3)}, \quad \tau_i = a_4 a_i \tau_1^{(4)} + a_4 a_i \tau_2^{(4)}, \quad \tau_i = a_5 a_i \tau_1^{(5)} + a_5 a_i \tau_2^{(5)}. \quad (2.2b)
$$

Remark 2.1. Below we use the index $j$ to denote an element of $\mathbb{Z}/5\mathbb{Z}$ with a slightly different enumeration for transformations $s_0, \ldots, s_4$, parameters $a_0, \ldots, a_4$ and variables $\tau_1^{(1)}, \ldots, \tau_5^{(1)}$ ($i = 1, 2$). To avoid confusion, we point out, for example, that $j = 0$ for $s_j$ and $a_j$ would imply $j = 5$ for $\tau_j^{(1)}$.

Lemma 2.2. The action of $\tilde{W}(A_4^{(1)})$ on the parameters are given by

$$
\tau_i^{(1)} = a_0 a_i \tau_1^{(1)} + a_0 a_i \tau_2^{(1)}, \quad \tau_i^{(2)} = a_1 a_i \tau_1^{(1)} + a_1 a_i \tau_2^{(2)}, \quad \tau_i^{(3)} = a_2 a_i \tau_1^{(2)} + a_2 a_i \tau_2^{(2)}, \quad (2.2a)
$$

where $i, j \in \mathbb{Z}/5\mathbb{Z}$ and

$$
(a_{ij})^T_{j=0} = \begin{pmatrix}
2 & -1 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
-1 & 0 & 0 & -1 & 2
\end{pmatrix}. \quad (2.4)
$$

is the Cartan matrix of type $A_4^{(1)}$, while their actions on the variables are given by

$$
s_j(\tau_1^{(j)}) = \tau_2^{(j+4)}, \quad s_j(\tau_2^{(j)}) = \frac{a_i a_{j+4} (a_i a_{j+1} \tau_1^{(j+1)} \tau_2^{(j+3)} + a_i a_{j+3} \tau_2^{(j+2)} \tau_1^{(j+1)})}{a_i a_{j+1} \tau_1^{(j+1)}}. \quad (2.5a)
$$
\( s_j(\tau_{2}^{(j+4)}) = \tau_{1}^{(j)} \), \( s_j(\tau_{2}^{(j)}) = \frac{a_j a_{j+1} \tau_{1}^{(j+2)} + a_j a_{j+1} \tau_{2}^{(j+2)} \tau_{2}^{(j+4)}}{a_j a_{j+1} \tau_{2}^{(j+2)}} \) \hspace{1cm} (2.5b)

\( \sigma(\tau_{1}^{(j)}) = \tau_{1}^{(j+1)} \), \( \sigma(\tau_{2}^{(j)}) = \tau_{2}^{(j+1)} \), \( \iota(\tau_{1}^{(j)}) = \tau_{1}^{(j-1)} \), \( \iota(\tau_{2}^{(j)}) = \tau_{2}^{(j-1)} \) \hspace{1cm} (2.5c)

where \( j \in \mathbb{Z}/5\mathbb{Z} \). In general, for a function \( F = F(a_i, \tau_{i}^{(k)}) \), we let an element \( w' \in \widehat{W}(A_4^{(1)}) \) act as \( w' F = F(w w a_i, w^{k}_{w}) \), that is, \( w \) acts on the arguments from the left.

The proof of lemma 2.2 is given in appendix A.

Remark 2.3. The action of \( \widehat{W}(A_4^{(1)}) \) in lemma 2.2 was first obtained by Tsuda in [60] without the details of the proof. The notations in this paper are related to those in [60] by the following correspondence:

\[
(\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8, \pi_9, \pi_{10}) \rightarrow (\pi_2, \pi_1, \pi_0, \pi_9, \pi_8, \pi_7, \pi_6, \pi_5, \pi_4, \pi_3, \pi_2),
\]

We also note that in [60] each element \( w' \in \widehat{W}(A_4^{(1)}) \) acts on the arguments from the right, whereas in the present paper it acts from the left.

To iterate each variable \( \tau_{i}^{(j)} \), we need the following transformations:

\[
T_0 = \sigma \sigma s_4 s_3 s_2 s_1, \quad T_1 = \sigma s_0 s_4 s_3 s_2, \quad T_2 = \sigma s_1 s_0 s_4 s_3,
\]

\[
T_3 = \sigma s_2 s_0 s_4 s_3, \quad T_4 = \sigma s_3 s_2 s_0 s_4,
\]

which are translations on the root lattice \( \hat{Q}(A_4^{(1)}) \) (A.8) (see appendix A). Note that \( T_i \), \( i = 0, \ldots, 4 \), commute with each other and

\[
T_0 T_1 T_2 T_3 T_4 = 1.
\]

Their actions on the parameters are given by

\[
T_i(a_i) = qa_i, \quad T_i(a_{i+1}) = q^{-1} a_{i+1}, \quad i \in \mathbb{Z}/5\mathbb{Z},
\]

where \( q = a_0 a_1 a_2 a_3 a_4 \), \( i \in \mathbb{Z}/5\mathbb{Z} \).

where \( a_i \) are invariant under the actions of \( T_i \). We define \( \tau \) functions by

\[
\tau_{i_0 i_1 i_2 i_3 i_4} = T_0^{i_0} T_1^{i_1} T_2^{i_2} T_3^{i_3} T_4^{i_4} (\tau_2^{(3)}),
\]

where \( i_i \in \mathbb{Z} \). We note that

\[
\tau_{1}^{(1)} = \tau_{1,0,0,0,0}, \quad \tau_{1}^{(2)} = \tau_{1,0,0,0,0}, \quad \tau_{1}^{(3)} = \tau_{1,0,0,0,0}, \quad \tau_{1}^{(4)} = \tau_{1,0,0,0,0},
\]

\[
\tau_{2}^{(1)} = \tau_{2,0,0,0,0}, \quad \tau_{2}^{(2)} = \tau_{2,0,0,0,0}, \quad \tau_{2}^{(3)} = \tau_{2,0,0,0,0}, \quad \tau_{2}^{(4)} = \tau_{2,0,0,0,0}.
\]
2.2. Discrete Painlevé equations

In this section, we define the $f$-variables by rational functions of the $\tau$-variables. Then, we demonstrate that elements of infinite order of $\hat{W}(A_4^{(1)})$ give various $q$-Painlevé equations.

Let us define the ten $f$-variables by

$$f_1^{(j)} = \frac{\tau_1^{(j+1)} \tau_2^{(j)}}{\tau_1^{(j)} \tau_2^{(j+3)}}, \quad f_2^{(j)} = s_j(f_1^{(j+1)}) = \frac{a_j a_{j+1}(a_{j+2} a_{j+3} + a_j f_1^{(j+3)})}{a_{j+3} f_1^{(j+1)}}, \quad (2.12)$$

where $j \in \mathbb{Z}/5\mathbb{Z}$. From the definition above and the relations (2.2), the following relations hold:

$$a_{j+2} a_{j+3} f_1^{(j)} f_1^{(j+1)} = a_j a_{j+1} (a_{j+3} + a_j f_1^{(j+3)}), \quad (2.13)$$

where $j \in \mathbb{Z}/5\mathbb{Z}$. The relations above look like five equations, but the relations represent only three. Therefore, there are only two essential $f$-variables. The action of $\hat{W}(A_4^{(1)})$ on these variables $f_i^{(j)}$ is given by the lemma below, which follows from the actions (2.5).

**Lemma 2.4.** The action of $\hat{W}(A_4^{(1)})$ on variables $f_i^{(j)}$ is given by

$$s_j(f_1^{(j+3)}) = f_2^{(j+3)}, \quad s_j(f_1^{(j)}) = \frac{a_{j+4} (a_{j+2} + a_j f_1^{(j+2)})}{a_{j+1} f_1^{(j+1)} f_2^{(j+4)}}, \quad s_j(f_2^{(j+3)}) = f_1^{(j+3)}, \quad (2.14a)$$

$$s_j(f_2^{(j+2)}) = \frac{a_j a_{j+3} a_{j+4} (a_{j+2} + a_j f_1^{(j+2)}) + a_{j+2} f_1^{(j)} f_2^{(j+4)}}{a_{j+1} f_1^{(j+1)} f_2^{(j+3)}}, \quad (2.14b)$$

$$s_j(f_2^{(j+4)}) = \frac{a_j a_{j+1} (a_{j+2} f_1^{(j+4)}) f_1^{(j+4)}}{a_{j+4} (a_{j+2} + a_j f_1^{(j+2)})}, \quad (2.14c)$$

$$s_j(f_2^{(j)}) = \frac{a_j a_{j+1} a_{j+4} a_j f_1^{(j+1)} + a_{j+2} a_{j+3} f_1^{(j+4)}}{a_{j+1} a_{j+3} f_1^{(j+1)} f_1^{(j+4)}}, \quad (2.14d)$$

$$\sigma(f_1^{(j)}) = f_1^{(j+1)}, \quad \sigma(f_2^{(j)}) = f_2^{(j+1)}, \quad \iota(f_1^{(j)}) = f_1^{(3-j)}, \quad \iota(f_2^{(j)}) = \frac{a_{2-j} (a_{5-j} + a_{2-j} a_{3-j} f_1^{(5-j)})}{a_{5-j} a_{4-j} a_{5-j} f_1^{(2-j)}}, \quad (2.14e)$$

where $j \in \mathbb{Z}/5\mathbb{Z}$.

It is well known that the translation part of $\hat{W}(A_4^{(1)})$ give discrete Painlevé equations [55]. For examples, from the translations $T_i$, $i = 1, \ldots, 4$, we obtain $q$-$P_{\text{V}}$ (1.1a) and from the translations $T_i T_j$, $0 \leq i < j \leq 4$, we obtain $q$-$P_{\text{V}}^*$ (1.1b). Indeed, the action of $T_0$:

$$T_0 : (a_0, a_1, a_2, a_3, a_4) \mapsto (a_0, q^{-1} a_1, a_2, a_3, a_4), \quad (2.15a)$$
\begin{align}
T_0(f^{(3)}_1 f^{(3)}_1) &= \frac{a_3}{a_0 a_1^2 a_4} \left( a_1 + a_3 a_4 f^{(1)}_1 \right) \left( a_1 + a_3 f^{(1)}_1 \right), \\
T_0^{-1}(f^{(1)}_1 f^{(1)}_1) &= \frac{a_0 a_1^3}{a_3^2} \left( a_2 a_3 + a_0 f^{(3)}_1 \right) \left( a_3 + a_0 a f^{(3)}_1 \right),
\end{align}
leads to \( q\text{-P}_V \) (1.1a) by the correspondences (1.10a) and
\begin{align}
b_0 &= \frac{a_0^{1/2}}{a_1^{1/2}}, \quad b_1 = \frac{1}{a_1^{1/2} a_2^{1/2}}, \quad b_2 = -\frac{q^{1/4}}{a_2^{1/2} a_4^{1/2}}, \\
b_3 &= \frac{a_0 a_1 a_4}{q^{1/2}}, \quad p = q^{1/2},
\end{align}
or, equivalently,
\begin{align}
a_0 &= p^2 b_0^2, \quad a_1 = -\frac{b_2 b_3^{1/2}}{p b_0 b_1}, \quad a_2 = -\frac{p b_0}{b_1 b_2 b_3^{1/2}}, \quad a_3 = -\frac{b_1 b_2}{b_0 b_3^{1/2}},
\end{align}
Moreover, the action of \( T_{13} = T_1 T_3 \):
\begin{align}
T_{13} : (a_0, a_1, a_2, a_3, a_4) \mapsto (a_0, qa_1, q^{-1}a_2, qa_3, q^{-1}a_4),
\end{align}
\begin{align}
\left( T_{13}(f^{(1)}_1)^{(2)} - \frac{a_1 a_2}{a_3 a_4} f^{(1)}_1 \right) \left( f^{(1)}_1 f^{(2)}_1 - \frac{a_1 a_2}{a_3 a_4} \right) &= \frac{a_1^3 a_2^3}{a_3^2} \left( a_2 + a_0 a_4 f^{(2)}_1 \right) \left( a_2 + a_4 f^{(2)}_1 \right),
\end{align}
gives \( q\text{-P}_V^* \) (1.1b) by the correspondences (1.10b) and (2.16).

It is also known that discrete Painlevé equations can be obtained from elements of infinite order of \( \~W(A_4^{(1)}) \) which are not necessarily translations of \( \~W(A_4^{(1)}) \) [29, 58]. We here show that how \( q\text{-P}_{III}(D_7^{(1)}) \) (1.1c) and \( q\text{-P}_V \) (1.1d) can be derived from the actions of \( \~W(A_4^{(1)}) \). Let
\begin{align}
R_0 = \sigma_3^3 \sigma_{21}, \quad R_{13} = \sigma_{62} \sigma_{54},
\end{align}
where \( R_0^2 = T_0 \) and \( R_{13}^2 = T_{13} \). Actions of these transformations in the parameter space are not translational motion:
\begin{align}
R_0 : (a_0, a_1, a_2, a_3, a_4) \mapsto (a_0 a_3 a_4, q^{-1}a_1 a_2 a_3, a_4, a_0 a_1, a_2), \\
R_{13} : (a_0, a_1, a_2, a_3, a_4) \mapsto (a_0, a_1 a_2 a_3, a_3^{-1}, qa_2^{-1}, q^{-1}a_2 a_3 a_4),
\end{align}
but under the special values of the parameters these actions become translational motion. Indeed, by imposing
which implies
\[ q^{1/2} = a_2a_3 = a_0a_1a_2 = a_3a_4 = a_0a_1a_4, \]

the action of \( R_0 \) becomes
\[ R_0 : (a_0, a_1, a_2, a_3, a_4) \mapsto (q^{1/2}a_0, q^{-1/2}a_1, a_2, a_3, a_4). \]

Similarly, under the condition of the parameters
\[ q^{1/2} = a_2a_3 = a_0a_1a_4, \]

the action of \( R_{13} \) becomes
\[ R_{13} : (a_0, a_1, a_2, a_3, a_4) \mapsto (a_0, q^{1/2}a_1, q^{-1/2}a_2, q^{1/2}a_3, q^{-1/2}a_4). \]

Therefore, the action of \( R_0 \):
\[ R_0(f^{(3)}_1) = f^{(1)}_1, \quad R_0(f^{(2)}_1)f^{(3)}_1 = \frac{a_3}{a_0^2a_4} \frac{(a_1 + a_3a_4f^{(1)}_1)(a_1 + a_3f^{(1)}_1)}{a_0a_1 + a_3f^{(1)}_1}, \]

with the condition (2.20), gives \( q\text{-P}_{3}(D^{(1)}_4) \) (1.1c) by the correspondences (1.12) and (2.16).
Moreover, the action of \( R_{13} \):
\[ R_{13}(f^{(1)}_1) = f^{(2)}_1, \]
\[ (R_{13}(f^{(2)}_1)f^{(2)}_1) = \frac{a_1a_2}{a_3a_4} (a_1f^{(2)}_1 + a_2(af^{(2)}_1 + a_2), \]

with the condition (2.23), gives \( q\text{-P}_{4}(1.1d) \) by the correspondences (1.14) and (2.16).

2.3. \( \omega \)-lattice

In this section, we define the \( \omega \)-variables by the ratios of the \( \tau \)-variables and then construct the \( \omega \)-lattice of type \( A^{(1)}_4 \).

Let us define the fifteen \( \omega \)-variables by
\[ \omega^{(j)}_1 = \frac{\tau^{(j)}_1}{\tau^{(j+1)}_1}, \quad \omega^{(j)}_2 = \frac{\tau^{(j)}_2}{\tau^{(j+1)}_2}, \quad \omega^{(j)}_3 = \frac{\tau^{(j-1)}_1}{\tau^{(j-1)}_2}, \quad j \in \mathbb{Z}/5\mathbb{Z}, \]

which satisfy
\[ f^{(j)}_1 = \frac{a_1a_2}{a_3a_4} \frac{\omega^{(j+1)}_1(a_1 + a_3a_4\omega^{(j+3)}_1 + a_2\omega^{(j+3)}_2)}{\omega^{(j+3)}_1\omega^{(j+1)}_2}, \quad j \in \mathbb{Z}/5\mathbb{Z}. \]

From the definition above and the relations (2.2), they satisfy the following nine relations:
\[ \omega^{(5)}_2 = \frac{\omega^{(1)}_1\omega^{(3)}_4}{\omega^{(3)}_1}, \quad \omega^{(5)}_3 = \frac{\omega^{(4)}_1\omega^{(3)}_1}{\omega^{(3)}_4}, \quad \omega^{(5)}_2 = \frac{\omega^{(2)}_1\omega^{(3)}_3}{\omega^{(3)}_2}, \quad \omega^{(5)}_2 = \frac{\omega^{(3)}_1\omega^{(4)}_4}{\omega^{(4)}_1}. \]
\[ \omega^{(5)}_1 = \frac{1}{\omega^{(1)}_1\omega^{(2)}_4\omega^{(3)}_4\omega^{(4)}_1}, \quad \omega^{(5)}_1 = \frac{\omega^{(1)}_1\omega^{(2)}_4}{\omega^{(1)}_1}, \quad \omega^{(5)}_1 = \frac{\omega^{(1)}_1\omega^{(2)}_4}{\omega^{(1)}_1}. \]
By inspection, we see that there are six essential \( \omega \)-variables. The action of \( \tilde{W}(A_4^{(1)}) \) on the \( \omega \)-variables is given by the lemma below, which follows from the action (2.5) and the definition (2.27).

**Lemma 2.5.** The action of \( \tilde{W}(A_4^{(1)}) \) on the fifteen \( \omega \)-variables is given by

\[
\begin{align*}
\sigma_j(\omega_1^{(j+4)}) &= \omega_j^{(j+1)}, \\
\sigma_j(\omega_2^{(j+1)}) &= \omega_j^{(j+1)}, \\
\sigma_j(\omega_3^{(j+4)}) &= \omega_j^{(j+1)},
\end{align*}
\]

(2.30a)

(2.30b)

(2.30c)

(2.30d)

(2.30e)

(2.30f)

(2.30g)

where \( j \in \mathbb{Z}/5\mathbb{Z} \).

We define \( \omega \)-functions by

\[
\omega^{(j)}_{l_0,l_1,l_2,l_3} = T_0^{l_0} T_1^{l_1} T_2^{l_2} T_3^{l_3} \omega^{(j)}(\omega^{(j)}),
\]

(2.31)

where \( j = 1, \ldots, 5 \) and \( l_0, \ldots, l_3 \in \mathbb{Z} \). We note that

\[
\begin{align*}
\omega_1^{(1)} &= \omega_1^{(0,0,1)}, & \omega_2^{(1)} &= \omega_2^{(0,1,0,0)}, & \omega_3^{(1)} &= \omega_3^{(0)}, \\
\omega_1^{(2)} &= \omega_1^{(1,0,0,0)}, & \omega_2^{(2)} &= \omega_2^{(1,0,0,0)}, & \omega_3^{(2)} &= \omega_3^{(1)}, \\
\omega_1^{(3)} &= \omega_1^{(0,0,1,0)} & \omega_2^{(3)} &= \omega_2^{(0,0,1,0)}, & \omega_3^{(3)} &= \omega_3^{(0,0,1,0)}.
\end{align*}
\]

(2.32a)

(2.32b)

(2.32c)
Now we are in a position to construct the \( \omega \)-lattice of type \( A_4^{(1)} \). Let us consider the following lattice (see figure 1):

\[
\sum_{i=0}^{4} l_i v_i \subset \mathbb{Z}^5,
\]

(2.33)

whose vertices \( v_i, i = 0, \ldots, 4 \), are defined by

\[
v_0 = (-1, -1, -1, -1, 4), \quad v_1 = (4, -1, -1, -1, -1), \quad v_2 = (-1, 4, -1, -1, -1),
\]

(2.34a)

\[
v_3 = (-1, -1, 4, -1, -1), \quad v_4 = (-1, -1, -1, 4, -1),
\]

(2.34b)

and satisfy \( v_0 + v_1 + v_2 + v_3 + v_4 = 0 \). For simplicity, we here use the following notation:

\[
v_{k_1 \ldots k_5} = \sum_{i=1}^{n} v_{k_i}, \quad k_i \in \{0, \ldots, 4\}.
\]

(2.35)

Let us assign the \( \tau \) functions \( \tau_{li,li,li,li,li} \) and the \( \omega \)-functions \( \omega_{li,li,li,li,li}^{(j)} \) to the vertices and the edges of the lattice (2.33) by the following correspondence:

\[
\tau_{li,li,li,li,li}^{(1)} \mapsto I + v_1243,
\]

(2.36a)

\[
\omega_{li,li,li,li,li}^{(1)} \mapsto \text{edge}(I + v_1234; 1),
\]

(2.36b)

\[
\omega_{li,li,li,li,li}^{(2)} \mapsto \text{edge}(I; 2),
\]

(2.36c)

\[
\omega_{li,li,li,li,li}^{(3)} \mapsto \text{edge}(I + v_1; 3),
\]

(2.36d)

\[
\omega_{li,li,li,li,li}^{(4)} \mapsto \text{edge}(I + v_12; 4),
\]

(2.36e)

\[
\omega_{li,li,li,li,li}^{(5)} \mapsto \text{edge}(I + v_123; 0),
\]

(2.36f)

where \( I = \sum_{i=0}^{4} l_i v_i \). Here, \( \text{edge}(A; i) \) is a edge connecting a vertex \( A \) to a vertex \( (A + v_i) \).

We refer to the lattice (2.33) with the \( \omega \)-functions \( \omega_{li,li,li,li,li}^{(j)} \) as \( \omega \)-lattice of type \( A_4^{(1)} \). We note that the configurations of the \( \tau \)-variables on the \( \omega \)-lattice are given by

\[
(\tau_1^{(1)}, \tau_1^{(2)}, \tau_1^{(3)}, \tau_1^{(4)}, \tau_1^{(5)}) \leftrightarrow (0, v_1, v_123, v_1234),
\]

(2.37a)

\[
(\tau_2^{(1)}, \tau_2^{(2)}, \tau_2^{(3)}, \tau_2^{(4)}, \tau_2^{(5)}) \leftrightarrow (v_2, v_123, v_1234, v_1234),
\]

(2.37b)

while those of the \( \omega \)-variables are given by

\[
(\omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)}) \leftrightarrow (\text{edge}(0; 1), \text{edge}(v_2; 1), \text{edge}(v_1234; 1)),
\]

(2.38a)

\[
(\omega_1^{(2)}, \omega_2^{(2)}, \omega_3^{(2)}) \leftrightarrow (\text{edge}(v_1; 2), \text{edge}(v_123; 2), \text{edge}(0; 2)),
\]

(2.38b)

\[
(\omega_1^{(3)}, \omega_2^{(3)}, \omega_3^{(3)}) \leftrightarrow (\text{edge}(v_123; 3), \text{edge}(v_1234; 3), \text{edge}(v_1; 3)),
\]

(2.38c)

\[
(\omega_1^{(4)}, \omega_2^{(4)}, \omega_3^{(4)}) \leftrightarrow (\text{edge}(v_1233; 4), \text{edge}(v_1023; 4), \text{edge}(v_12; 4)),
\]

(2.38d)

\[
(\omega_1^{(5)}, \omega_2^{(5)}, \omega_3^{(5)}) \leftrightarrow (\text{edge}(v_1234; 0), \text{edge}(v_11234; 0), \text{edge}(v_123; 0)).
\]

(2.38e)
On the $\omega$-lattice various quad-equations of ABS-type can be derived, e.g.

$$T_0 T_2 (\omega_{3}^{(1)}) \frac{1}{\omega_{3}^{(1)}} = a_0 a_2 (1 - a_2 a_3 T_0 (\omega_{3}^{(1)}) + (1 - a_1) T_2 (\omega_{3}^{(1)})), \quad (2.39a)$$

$$T_0 T_3 (\omega_{3}^{(1)}) \frac{1}{\omega_{3}^{(1)}} = a_0 a_2 a_3 (1 - a_2 a_3 T_0 (\omega_{3}^{(1)}) + (1 - a_1) T_3 (\omega_{3}^{(1)})), \quad (2.39b)$$

$$T_0 T_4 (\omega_{3}^{(1)}) \frac{1}{\omega_{3}^{(1)}} = a_0 a_2 a_3 a_4 (1 - a_2 a_3 a_4 T_0 (\omega_{3}^{(1)}) + (1 - a_1) T_4 (\omega_{3}^{(1)})), \quad (2.39c)$$

$$T_2 T_3 (\omega_{3}^{(1)}) \frac{1}{\omega_{3}^{(1)}} = a_0 a_1^2 (1 - a_2 a_3) T_2 (\omega_{3}^{(1)}) + (1 - a_2) T_3 (\omega_{3}^{(1)})), \quad (2.39d)$$

$$T_2 T_4 (\omega_{3}^{(1)}) \frac{1}{\omega_{3}^{(1)}} = a_0 a_1 a_2 a_3 a_4 (1 - a_2 a_3 a_4) T_2 (\omega_{3}^{(1)}) + (1 - a_1) T_4 (\omega_{3}^{(1)})), \quad (2.39e)$$

$$T_3 T_4 (\omega_{3}^{(1)}) \frac{1}{\omega_{3}^{(1)}} = a_0 a_1^2 a_2 a_3 a_4 (1 - a_2 a_3 a_4) T_3 (\omega_{3}^{(1)}) + (1 - a_2 a_3) T_4 (\omega_{3}^{(1)})), \quad (2.39f)$$

Figure 1. The lattice (2.33) around the origin, which is a two-dimensional projection of the Voronoi cell of type $A_4$. Refer to (2.34) and (2.35) for $v$. The directions from $0$ to $v_i, i = 0, \ldots, 4$, correspond to the $T_i$-directions, $i = 0, \ldots, 4$, respectively.
Note that equations (2.39) are relations between the $\omega$-function $\omega_{h,h,l,l,t,4}^{(1)}$, but equations (2.40) are the relations between $\omega_{h,h,l,l,t,4}^{(1)}$ and $\omega_{h,h,l,l,t}^{(1)}$, $\omega_{h,h,l,l,t}^{(2)}$ and $\omega_{h,h,l,l,t}^{(4)}$, respectively. Each equation of equations (2.39) and that of equations (2.40) are of $\mathcal{H}_3$- and $\mathcal{D}_4$-types in the ABS classification \[1, 2, 6 – 8\], respectively. Details of the $\omega$-lattice of type $A_4^{(1)}$ will be discussed in a forthcoming paper (Joshi, Nakazono and Shi, in preparation).

3. Construction of the lattice $\omega_{A_2+A_1}$

In this section, we consider the extended affine Weyl group $\widetilde{W}((A_2 \times A_1)^{(1)})$ given by the following six generators:

\[ w_0 = s_0, \quad w_1 = s_1s_2s_1, \quad w_2 = s_3s_4s_3, \quad r_0 = t, \quad r_1 = \sigma s_2s_4, \quad \pi = \sigma^3s_4. \quad (3.1) \]

The details of $\widetilde{W}((A_2 \times A_1)^{(1)})$ is discussed in appendix B. Using this group, we construct another important lattice $\omega_{A_2+A_1}$. Moreover, we show that the $q$-Painlevé equations (1.1) can be derived also as the relations on the lattice $\omega_{A_2+A_1}$.

3.1. Affine Weyl group $\widetilde{W}((A_2 \times A_1)^{(1)})$

In this section, we consider the birational action of $\widetilde{W}((A_2 \times A_1)^{(1)})$ on the parameters $b_i$, $i = 0, 1, 2, 3$, and $p$ defined by (2.16) and on the particular $\omega$-variables $\omega_i^{(j)}$, $i = 1, 2, 3$ and $j = 1, 3$, given by (2.27). We note that from the relations (2.9), these six $\omega$-variables satisfy the following two relations:

\[
\begin{align*}
\frac{\omega_i^{(1)}}{\omega_i^{(3)}} &= \frac{a_0^2a_4}{a_2^2a_3} \cdot \frac{\omega_i^{(3)}}{\omega_i^{(2)}} = \frac{a_0}{a_2}, \quad (2.40a) \\
\frac{\omega_i^{(1)}}{\omega_i^{(3)}} &= \frac{a_0^2a_4}{a_2^2a_3} \cdot \frac{T_3(\omega_i^{(2)})}{\omega_i^{(2)}} = \frac{a_0}{a_2a_3}, \quad (2.40b) \\
\frac{\omega_i^{(1)}}{\omega_i^{(3)}} &= \frac{a_0^2}{a_2^2a_3} \cdot \frac{\omega_i^{(4)}}{T_3^{-1}(\omega_i^{(4)})} = \frac{a_3a_4}{a_2}. \quad (2.40c)
\end{align*}
\]

Therefore, essential $\omega$-variables used here are four. The action of $\widetilde{W}((A_2 \times A_1)^{(1)})$ on the parameters is given by

\[
\begin{align*}
w_0 &: (b_0, b_1, b_2, b_3, p) \mapsto (b_0^{-1}p^{-2}, p^{-1}b_0^{-1}b_1, p^{-1}b_0^{-1}b_2, b_3, p), \quad (3.3a) \\
w_1 &: (b_0, b_1, b_2, b_3, p) \mapsto (b_0b_1^{-1}, b_1^{-1}, b_1^{-1}b_2, b_3, p), \quad (3.3b) \\
w_2 &: (b_0, b_1, b_2, b_3, p) \mapsto (b_1, b_0, b_2, b_3, p), \quad (3.3c) \\
r_0 &: (b_0, b_1, b_2, b_3, p) \mapsto (b_0^{-1}, b_0^{-1}b_1, p^{-1}b_0^{-1}b_2, b_3^{-1}, p^{-1}), \quad (3.3d) \\
r_1 &: (b_0, b_1, b_2, b_3, p) \mapsto (pb_1, pb_0, p^{-1}b_2, b_3^{-1}, p^{-1}), \quad (3.3e)
\end{align*}
\]
while that on the six $\omega$-variables is given by

\[
\begin{align*}
\omega^{(1)}(3) & = \frac{p^3 b_0 \omega^{(3)}(3)(p b_0 b_3 \omega^{(1)} + b_1 \omega^{(1)})}{b_1^2 (p b_0 b_3 \omega^{(2)} + b_1 \omega^{(1)})}, \\
\omega^{(2)}(3) & = \frac{p^3 b_0 \omega^{(3)}(3)(p b_0 b_3 \omega^{(1)} + b_1 \omega^{(1)})}{b_1^2 (p b_0 b_3 \omega^{(2)} + b_1 \omega^{(1)})}, \\
\omega^{(3)}(3) & = \frac{p b_1 \omega^{(1)}(3)(p b_0 b_3 \omega^{(2)} + b_1 \omega^{(1)})}{b_1^2 (p b_0 b_3 \omega^{(2)} + b_1 \omega^{(1)})}, \\
\omega^{(4)}(3) & = \frac{b_1 \omega^{(1)}(3)(p b_0 b_3 \omega^{(2)} + b_1 \omega^{(1)})}{b_1^2 (p b_0 b_3 \omega^{(2)} + b_1 \omega^{(1)})}, \\
\omega^{(5)}(3) & = \frac{b_1 \omega^{(1)}(3)(p b_0 b_3 \omega^{(2)} + b_1 \omega^{(1)})}{b_1^2 (p b_0 b_3 \omega^{(2)} + b_1 \omega^{(1)})}, \\
\omega^{(6)}(3) & = \frac{b_1 \omega^{(1)}(3)(p b_0 b_3 \omega^{(2)} + b_1 \omega^{(1)})}{b_1^2 (p b_0 b_3 \omega^{(2)} + b_1 \omega^{(1)})},
\end{align*}
\]

where $i \in \mathbb{Z}/3\mathbb{Z}$, which follow from (2.3), (2.16), (2.30) and (3.1).

Let

\[
\rho_1 = \pi r_0 w_1 w_2, \quad \rho_2 = \pi r_0 w_0 w_1, \quad \rho_3 = \pi r_0 w_2 w_0, \quad \rho_4 = \pi r_0 r_1 r_2.
\]

Note here that the transformations $\rho_i, i = 1, \ldots, 4$, are translations on the root system $Q((\Lambda_2 + \Lambda_3)^{(1)})$ (B.1) (see appendix B for details). The translations $\rho_i, i = 1, \ldots, 4$, commute with each other and

\[
\rho_1 \rho_2 \rho_3 \rho_4 = 1.
\]

Their actions on the parameters are given by

\[
\rho_1 : (b_0, b_1, b_2, b_3) \mapsto (p b_0 b_1^{-1}, b_1, p^{-1} b_1^{-1} b_2, b_3, p^{-1}),
\]

(3.7a)
\[
\rho_2 : (b_0, b_1, b_2, b_3) \mapsto (b_0, pb_1, b_2, b_3^{-1}),
\]
(3.7b)

\[
\rho_3 : (b_0, b_1, b_2, b_3) \mapsto (p^{-1}b_0, p^{-1}b_1, p^{-1}b_2, b_3^{-1}),
\]
(3.7c)

\[
\rho_4 : (b_0, b_1, b_2, b_3) \mapsto (b_0, b_1, pb_2, b_3^{-1}),
\]
(3.7d)

where \( p \) is invariant under their actions.

### 3.2. Lattice \( \omega_{A_2 + A_1} \)

In this section, we define the \( \omega \)-functions associated with the translations on the root system \( Q(A_2 + A_1) \) and then construct the lattice \( \omega_{A_2 + A_1} \).

We define \( \omega \)-functions by using the translations \( \rho_i, i = 1, \ldots, 4 \), as follows:

\[
\omega_{h_i, t_i, l_i} = \rho_i^1 \rho_i^2 \rho_i^3 \rho_i^4 (\omega_i^{(1)}),
\]
(3.8)

where \( l_i \in \mathbb{Z} \). We note that

\[
\omega_i^{(1)} = \omega_{1,1,0,0}, \quad \omega_i^{(1)} = \omega_{2,1,1,0}, \quad \omega_i^{(1)} = \omega_{0,0,0,0},
\]
(3.9a)

\[
\omega_i^{(3)} = \omega_{1,1,1,0}, \quad \omega_i^{(3)} = \omega_{1,1,0,1}, \quad \omega_i^{(3)} = \omega_{0,1,1,1}.
\]
(3.9b)

Let us assign the \( \omega \)-functions \( \omega_{h_i, t_i, l_i} \) to the vertices of the lattice

\[
\left\{ \sum_{i=1}^{4} l_i v_i \mid l_1, \ldots, l_4 \in \mathbb{Z} \right\}
\]
(3.10)

by the following correspondence:

\[
\omega_{h_i, t_i, l_i} \leftrightarrow l_1 v_1 + l_2 v_2 + l_3 v_3 + l_4 v_4.
\]
(3.11)

Here, \( v_i, i = 1, \ldots, 4 \), are defined by

\[
v_1 = (1, 1, 1), \quad v_2 = (-1, -1, 1), \quad v_3 = (1, -1, -1), \quad v_4 = (-1, 1, -1),
\]
(3.12)

and satisfy \( v_1 + v_2 + v_3 + v_4 = 0 \). We here refer to the lattice (3.10) with the \( \omega \)-functions \( \omega_{h_i, t_i, l_i} \) as lattice \( \omega_{A_2 + A_1} \). We note that the configurations of the \( \omega \)-variables on the lattice \( \omega_{A_2 + A_1} \) are given by

\[
(\omega_i^{(1)}, \omega_i^{(1)}, \omega_i^{(3)}, \omega_i^{(3)}, \omega_i^{(3)}) \leftrightarrow (v_1 + v_2, v_1 - v_4, 0, v_3, v_1, -v_4, -v_3).
\]
(3.13)

See the example given in figure 2 to see the quadrilateral associated with \( \omega_i^{(1)}, \omega_i^{(3)}, \omega_i^{(3)} \) and \( \omega_i^{(3)} \).

The 14 vertices around \( l \in \omega_{A_2 + A_1} \):

\[
\{ l \pm v_i, l + v_i + v_j \mid i, j = 1, \ldots, 4, \ i \neq j \},
\]
(3.14)

collectively forms the rhombic dodecahedron (see figure 3). Letting \( \mathcal{V}(l) \) be the rhombic dodecahedron with the center \( l \in \omega_{A_2 + A_1} \):

\[
\mathcal{V}(l) = \{ l \} \cup V(l),
\]
(3.15)

then the following holds:

\[
\omega_{A_2 + A_1} = \bigcup_{l \in \omega_{A_2 + A_1}} \mathcal{V}(l).
\]
(3.16)

Henceforth, let us consider the quad-equations appearing on the lattice \( \omega_{A_2 + A_1} \).
Lemma 3.1. The following quad-equations hold on the lattice $\omega_{\lambda_1+A}$:

\[
\frac{\omega_{l_1+l_2+l_3+l_4}}{\omega_{l_1+l_2+l_3+l_4}} = p^{2(i-i_1)+1}b_{l_1}^2\lambda_{l_1+l_2+l_3+l_4} - \frac{p^{4(i_1-i_2)}b_{l_1}b_{l_2}b_{l_3}b_{l_4}}{b_{l_1}b_{l_2}b_{l_3}b_{l_4}} \omega_{l_1+l_2+l_3+l_4},
\]

(3.17a)

\[
\frac{\omega_{l_1+l_2+l_3+l_4}}{\omega_{l_1+l_2+l_3+l_4}} = p^{2(i_i)+1}b_{l_1}^2\lambda_{l_1+l_2+l_3+l_4} - \frac{p^{4(i_1-i_2)}b_{l_1}b_{l_2}b_{l_3}b_{l_4}}{b_{l_1}b_{l_2}b_{l_3}b_{l_4}} \omega_{l_1+l_2+l_3+l_4},
\]

(3.17b)

\[
\frac{\omega_{l_1+l_2+l_3+l_4}}{\omega_{l_1+l_2+l_3+l_4}} = p^{2(i_i)+1}b_{l_1}^2\lambda_{l_1+l_2+l_3+l_4} - \frac{p^{4(i_1-i_2)}b_{l_1}b_{l_2}b_{l_3}b_{l_4}}{b_{l_1}b_{l_2}b_{l_3}b_{l_4}} \omega_{l_1+l_2+l_3+l_4},
\]

(3.17c)
\[ \frac{\omega_{l_0,l_0,l_0}}{\omega_{l_0+1,l_0+1,l_0+1}} = \left( \frac{p^{2l_0+2l_1+3l_2+3l_3+4l_4}b_0^3}{b_0^3} \right)^2 \frac{\omega_{l_0+1,l_0+1,l_0+1}}{\omega_{l_0+1,l_0+1,l_0+1}} + \frac{p^{3l_0+3l_1+3l_2+3l_3+4l_4}}{b_0^3}, \]

(3.17d)

\[ \frac{\omega_{l_0,l_0,l_0}}{\omega_{l_0+1,l_0+1,l_0+1}} = \left( \frac{p^{4l_0+2l_1+4l_2+4l_3+4l_4}b_1^2}{b_1^2} \right)^2 \frac{\omega_{l_0+1,l_0+1,l_0+1}}{\omega_{l_0+1,l_0+1,l_0+1}} + \frac{p^{4l_0+4l_1+4l_2+4l_3+4l_4}}{b_1^2}, \]

(3.17e)

\[ \frac{\omega_{l_0,l_0,l_0}}{\omega_{l_0+1,l_0+1,l_0+1}} = \left( \frac{p^{5l_0+3l_1+5l_2+5l_3+4l_4}b_2}{b_2} \right)^2 \frac{\omega_{l_0+1,l_0+1,l_0+1}}{\omega_{l_0+1,l_0+1,l_0+1}} + \frac{p^{5l_0+5l_1+5l_2+5l_3+4l_4}}{b_2}, \]

(3.17f)

where

\[ \lambda_i = b_j^{(-i)j}. \]

(3.18)

Note that each equation of equations (3.17a)–(3.17c) and that of equations (3.17d)–(3.17f) are of H3- and D4-types in the ABS classification [1, 2, 6–8], respectively.

**Proof.** Recalling the definitions of \( \rho_i \) given in (3.5) and the relations (3.2), we have the actions shown below:

\[ \rho_2(\omega_1^{(1)}) = \frac{b_1^2 \omega_1^{(3)}(pb_0^2 b_3 \omega_3^{(1)} + \omega_1^{(1)})}{b_0^2(pb_0^2 b_3 \omega_3^{(1)} + \omega_1^{(1)})}, \]

(3.19a)

\[ \rho_3(\omega_1^{(3)}) = \frac{\omega_1^{(1)}(pb_0^2 b_3 \omega_3^{(1)} + \omega_1^{(1)})}{b_0^2(pb_0^2 b_3 \omega_3^{(1)} + \omega_1^{(1)})}, \]

(3.19b)

\[ \rho_3^{-1}(\omega_1^{(3)}) = \frac{p^3 b_0^2 b_3 \omega_3^{(1)} + b_3 b_2^2 \omega_1^{(1)}}{p b_0^2 b_3 \omega_3^{(1)} + \omega_3^{(3)}}, \]

(3.19c)

\[ \rho_3^{-1}(\omega_1^{(3)}) = \omega_2^{(1)} = \frac{b_0 b_2^3 \omega_3^{(1)}(p^2 b_0^3 b_4^3 \omega_3^{(1)} - b_1 b_2 \omega_1^{(3)})}{b_3^2 \omega_3^{(1)}}, \]

(3.19d)

\[ \rho_3(\omega_1^{(3)}) = \frac{b_0^2 \omega_1^{(1)} \omega_3^{(3)}}{b_1 b_3^2 \omega_3^{(1)} - b_0 b_2 \omega_3^{(3)}}. \]

(3.19e)

This leads to

\[ \frac{\omega_1^{(1)}}{\omega_3^{(1)}} = \frac{pb_1^2 b_2 \omega_1^{(3)} - b_0^4 b_1^{-4} \rho_2(\omega_3^{(1)})}{b_0^2 b_1^{-2} \rho_2(\omega_3^{(1)}) - \omega_1^{(1)}}, \]

(3.20a)

\[ \frac{\omega_3^{(3)}}{\omega_3^{(3)}} = \frac{b_3 \omega_1^{(1)} - b_4 \rho_3(\omega_3^{(1)})}{b_0^2 \rho_3(\omega_3^{(1)}) - \omega_1^{(1)}}, \]

(3.20b)
\[
\frac{\omega^{(3)}}{\omega^{(3)}} = \frac{b_{3} \rho_{3}^{-1}(\omega^{(3)}) - \rho_{3} b_{0}^{1/2}}{p b_{3}^{1/2} p^{2} b_{0}^{3} \omega^{(3)}} \tag{3.20c}
\]

\[
\frac{\omega^{(3)}}{\omega^{(3)}} = \left( \frac{b_{1} b_{3}^{-1}}{p b_{0}^{2}} \right)^{2} \frac{\rho_{1}^{-1}(\omega^{(3)})}{\omega^{(3)}} + \frac{b_{1} b_{2} b_{3}^{-1/2}}{b_{1}^{2}} \tag{3.20d}
\]

\[
\frac{\omega^{(3)}}{\omega^{(3)}} = \left( \frac{b_{2} b_{3}^{-1}}{b_{1}^{2}} \right)^{2} \frac{\rho_{1}^{-1}(\omega^{(3)})}{\omega^{(3)}} + \frac{b_{0} b_{2} b_{3}^{-1/2}}{b_{1}^{2}} \tag{3.20e}
\]

which in turn lead immediately to equations (3.17a)–(3.17e). Moreover, we get equation (3.17f) from the relation (3.2b) or, equivalently,

\[
\frac{\omega^{(1)}}{\omega^{(1)}} = \frac{(p b_{0} b_{3})^{2} \omega^{(2)}}{\omega^{(3)}} + p b_{0} b_{1} b_{3}^{1/2}. \tag{3.21}
\]

Therefore we have completed the proof. □

**Lemma 3.2.** The quad-equations (3.17) are fundamental relations on the lattice \( \omega_{\lambda_{2} + \lambda_{3}} \).

**Proof.** In this proof we will show that any \( \omega \)-function \( \omega_{b_{1}, b_{2}, b_{3}, b_{4}} \) can be calculated by the quad-equations (3.17) with four initial values: \( \omega_{1}^{(1)}, \omega_{1}^{(1)}, \omega_{1}^{(3)} \) and \( \omega_{1}^{(3)} \) (or, \( \omega_{1,1,0,0}, \omega_{0,0,0,0}, \omega_{1,0,0,0} \) and \( \omega_{1,1,0,0} \)).

First, we obtain the values of all \( \omega \)-functions on \( \bar{V}(\emptyset) \) from the initial values by the following steps.

**Step 1:** By using equations (3.17a)\(_{0,0,0,0}\), (3.17b)\(_{1,0,0,0}\) and (3.17f)\(_{1,1,0,0}\), the functions on \( v_{2}, v_{3} \) and \( -v_{3} \) can be calculated, respectively.

**Step 2:** By using equations (3.17c)\(_{0,0,0,0}\), (3.17c)\(_{0,1,0,0}\), (3.17d)\(_{0,1,0,0}\), (3.17e)\(_{1,0,0,0}\) and (3.17e)\(_{1,0,1,0}\), the functions on \( v_{3}, v_{2} + v_{3}, v_{2} + v_{4}, v_{1} + v_{4} \) and \( -v_{2} \) can be calculated, respectively.

**Step 3:** By using equations (3.17a)\(_{0,0,0,1}\), (3.17d)\(_{0,0,1,0}\) and (3.17f)\(_{0,1,0,0}\), the functions on \( v_{4}, v_{3} \) and \( -v_{4} \) can be calculated, respectively.

Note that the subscripts of the equation numbers \( (l_{1}, l_{2}, l_{3}, l_{4}) \) denote the values of the parameters \( l_{i}, i = 1, \ldots, 4 \), in the equations.

Next, we consider \( \bar{V}(v_{i}) \). From the determined \( \omega \)-functions on

\[
\bar{V}(\emptyset) \cap \bar{V}(v_{i}) = \{\emptyset, v_{i}, v_{1} + v_{i}, -v_{i} \mid i = 2, 3, 4\}, \tag{3.22}
\]

we can obtain the values of the \( \omega \)-functions on

\[
\bar{V}(v_{i}) - \bar{V}(\emptyset) = \{v_{1} - v_{i}, 2v_{1} - v_{i}, 2v_{1} + v_{i} \mid i = 2, 3, 4\}, \tag{3.23}
\]

by the following steps.

**Step 1:** By using equations (3.17a)\(_{0,0,1,0,0}\), (3.17c)\(_{0,0,0,1,0}\) and (3.17d)\(_{0,0,0,1,0}\), the functions on \( v_{1} - v_{2}, v_{1} - v_{3} \) and \( v_{1} - v_{4} \) can be calculated, respectively.

**Step 2:** By using equations (3.17c)\(_{1,1,0,0}\), (3.17a)\(_{1,1,0,1,0}\) and (3.17a)\(_{1,0,1,0,1}\), the functions on \( 2v_{1} + v_{2}, 2v_{1} + v_{3} \) and \( 2v_{1} + v_{4} \) can be calculated, respectively.

**Step 3:** By using equation (3.17b)\(_{0,0,0,0}\), the function on \( 2v_{1} \) can be calculated.

In a similar manner, we can calculate all \( \omega \)-functions on \( \bar{V}(l + v_{i}) \), \( i = 1, \ldots, 4 \), from those on \( \bar{V}(l) \) for any \( l \in \omega_{\lambda_{2} + \lambda_{3}} \). Therefore we have completed the proof. □
For later convenience, we here make the mention of $R_0$ briefly. Its action on the parameters $b_i$ and $p$ is given by

$$R_0 : (b_0, b_1, b_2, b_3, p) \mapsto (b_1 b_0, b_2, b_3^{-1}, p),$$

while that on the restricted $\omega$-functions, which are on the following sublattice:

$$\left\{ \sum_{i=1}^{4} l_i \bigg| l_i = l_2, \ l_i \in \mathbb{Z} \right\} \bigcup \left\{ \sum_{i=1}^{4} l_i \bigg| l_i = l_2 + 1, \ l_i \in \mathbb{Z} \right\} \subset \omega_{A_1 \times A_1},$$

is given by

$$R_0 : \omega_{l_0, l_2, l_2, l_2} \mapsto \begin{cases} \omega_{l_1 + 1, l_2, l_2, l_2} & \text{if } l_i = l_2, \\ \omega_{l_0, l_2 + 1, l_2, l_2} & \text{if } l_i = l_2 + 1. \end{cases}$$

### 3.3. Discrete Painlevé equations

In this section we consider the particular $f$-variables $f_i^{(j)}$, $j = 1, 2, 3$, given by (2.12), which can be expressed by the ratios of the $\omega$-functions $\omega_{l_0, l_2, l_2, l_2}$ as follows:

$$f_i^{(1)} = \frac{\omega_i^{(1)}}{\omega_i^{(3)}}, \quad f_i^{(2)} = \frac{\omega_i^{(2)}}{\omega_i^{(3)}}, \quad f_i^{(3)} = \frac{\omega_i^{(3)}}{\omega_i^{(3)}} = \frac{\omega_{1,1,1,0}}{\omega_{1,0,0,0}}.$$

These $f$-variables satisfy the relation (1.5), which follows from the relations (2.13). The action of $W((A_1 \times A_1)\wedge 1)$ on the three $f$-variables is given by

$$\begin{align*}
w_0(f_i^{(3)}) &= \frac{p^3 b_0^2 f_i^{(3)}(pb_0 b_3 + b_i f_i^{(1)}))}{b_3 + pb_i f_i^{(1)}}, \\
w_1(f_i^{(1)}) &= \frac{b_i^2 f_i^{(1)}(b_1^2 + pb_0^2 b_3 f_i^{(1)}))}{1 + pb_0^2 b_3 f_i^{(1)}}, \\
w_2(f_i^{(2)}) &= \frac{b_i^2 f_i^{(2)}(pb_i^2 b_3 + r_0 f_i^{(3)}))}{b_0^2 (pb_i^2 b_3 + r_0 f_i^{(3)})}, \\
r_0(f_i^{(3)}) &= \frac{b_0^2 f_i^{(3)}(pb_0^2 b_3 + r_0 f_i^{(3)}))}{b_1^2 f_i^{(3)}}, \quad r_0(f_i^{(2)}) = f_i^{(2)}, \\
r_1(f_i^{(1)}) &= \frac{b_0^2 f_i^{(1)}(pb_0^2 b_3 + r_0 f_i^{(3)}))}{b_1^2 f_i^{(1)}}, \quad r_1(f_i^{(3)}) = f_i^{(3)}, \\
r_1(f_i^{(2)}) &= -\frac{b_0^3 b_1^{1/2}}{p^3 b_0^2 b_0 f_i^{(1)} f_i^{(3)}} - \frac{b_0^3 b_2 b_3^{1/2}}{p^2 b_0^2 b_0 f_i^{(1)}} + \frac{b_0^2 b_2 b_3^{3/2}}{p^2 b_0^2 b_0 f_i^{(1)}}.
\end{align*}$$
\[ \pi(f_1^{(1)}) = r_0(f_1^{(3)}), \quad \pi(f_1^{(2)}) = \frac{b_1(-b_0b_2b_3^{1/2} + pb_1^3f_1^{(1)})}{pb_0^2b_3^2f_1^{(3)}}. \] (3.28g)

Note that
\[ r_0(f_1^{(3)}) = \frac{\omega_1^{(1)}}{\omega_3^{(1)}}. \] (3.29)

Moreover, the time evolutions of the \(q\)-Painlevé equations shown in section 2.2 can be expressed by the elements of \(\tilde{W}(A_2 \times A_1)^{(1)}\) as follows:
\[ T_0 = \rho_1\rho_2, \quad T_{13} = \rho_2^2, \quad R_0 = \pi r_0 w_1, \quad R_{13} = \rho_1. \] (3.30)
where \(\rho_i\) are defined by (3.5). Therefore, the birational actions of \(T_0, T_{13}, R_0\) and \(R_{13}\) are given by (1.6) and (1.9). As mentioned in remark 1.4, these actions give \(q\)-Painlevé equations (1.1).

4. Proofs of theorems 1.2 and 1.3

In this section, we consider the following system of the partial difference equations:
\[ \frac{u(l + e_1 + e_2)}{u(l)} = -\frac{\alpha_i u(l + e_1) - \beta_i u(l + e_2)}{\alpha_i u(l + e_2) - \beta_i u(l + e_1)}, \quad (4.1a) \]
\[ \frac{u(l + e_2 + e_3)}{u(l)} = -\frac{\beta_i u(l + e_2) - \gamma_i u(l + e_3)}{\beta_i u(l + e_3) - \gamma_i u(l + e_2)}, \quad (4.1b) \]
\[ \frac{u(l + e_3 + e_1)}{u(l)} = -\frac{\gamma_i u(l + e_3) - \alpha_i u(l + e_1)}{\gamma_i u(l + e_1) - \alpha_i u(l + e_3)}, \quad (4.1c) \]
\[ \frac{u(l + e_1 + e_2)}{u(l)} + \frac{u(l + e_3)}{u(l + e_1)} = -\frac{\alpha_i K_i}{u(l + e_1)}, \quad (4.1d) \]
\[ \frac{u(l + e_2 + e_3)}{u(l)} + \frac{u(l + e_3)}{u(l + e_2)} = -\frac{\beta_i K_i}{u(l + e_2)}, \quad (4.1e) \]
\[ \frac{u(l + e_3 + e_2)}{u(l)} + \frac{u(l + e_3)}{u(l + e_3)} = -\frac{\gamma_i K_i}{u(l + e_3)}, \quad (4.1f) \]

where \(l = \sum_{i=1}^{4} l_i e_i \in \mathbb{Z}^4\) and \(\{e_1, \ldots, e_4\}\) is a standard basis for \(\mathbb{R}^4\). Here, \(u(l)\) is a function from \(\mathbb{Z}^4\) to \(\mathbb{C}\) and \(\{\alpha_i\}_{e_i}, \{\beta_i\}_{e_i}, \{\gamma_i\}_{e_i}\) and \(\{K_i\}_{e_i}\) are complex parameters. This system is obtained by assigning the quad-equations of ABS type to the faces of each four-dimensional hypercube (4-cube) (see [23] and references therein). The Lax equations for system (4.1) are given by the following [23]:
\[ \Psi_{l+1,l_1,l_2,l_3} = \delta^{(1)} \left( \begin{array}{cc} \frac{\mu}{\alpha_i} & -\frac{u(l + e_1)}{u(l)} \\ \frac{1}{\alpha_i} & \frac{\mu}{u(l)} \frac{u(l + e_1)}{u(l)} \end{array} \right) \psi_{l_1,l_2,l_3,l_4}. \] (4.2a)
\[ \Psi_{i,j,z+1,j,l+1} = \delta^{(2)} \left( \frac{\mu}{\beta_{l_2}} \frac{1}{u(l)} - \frac{\mu}{\beta_{l_2}} \frac{u(l+e_2)}{u(l)} \right) \Psi_{i,j,z,l+1}, \quad (4.2b) \]

\[ \Psi_{i,j,z+1,j+1,l} = \delta^{(3)} \left( \frac{\mu}{\gamma_{l_1}} \frac{1}{u(l)} - \frac{\mu}{\gamma_{l_1}} \frac{u(l+e_3)}{u(l)} \right) \Psi_{i,j,z+1,j+1,l}, \quad (4.2c) \]

\[ \Psi_{i,j,z,j+1,l+1} = \delta^{(4)} \left( \frac{-\mu K_{i_2}}{u(l)} - \frac{-u(l+e_4)}{u(l)} \right) \Psi_{i,j,z,j+1,l+1}, \quad (4.2d) \]

where \( \delta^{(i)}, i = 1, \ldots, 4 \), are arbitrary constants and \( \mu \) is a spectral parameter. The pairs of equations (4.2) give the Lax pairs of \( \rho \Delta \mathcal{E}s \) (4.1) (see table 1).

4.1. Proof of theorem 1.2

In this section, we show that the lattice \( \omega_{A_k} + A_k \) can be obtained from the integer lattice \( \mathbb{Z}^4 \) with the \( \rho \Delta \mathcal{E}s \) (4.1) by a geometric reduction.

Let

\[ u(l) = \frac{\lambda_{l+1,l+1+4} t(l+1,l+1+4-2l)/2}{U(l)}, \quad (4.3) \]

where \( l = \sum_{i=1}^{4} l_i e_i \in \mathbb{Z}^4 \). Here, \( \lambda_0 \) is a non-zero complex parameter and

\[ \lambda_i = \begin{cases} \lambda_0 & \text{if } l = 2n, \\ \frac{1}{\lambda_0} & \text{if } l = 2n + 1. \end{cases} \quad (4.4) \]

Then, system (4.1) can be rewritten as the following:

\[ \frac{U(l + e_1 + e_2)}{U(l)} = -\lambda_{l+1,l+1+4} \frac{\alpha_{l_2} U(l + e_1) - \beta_{l_2} U(l + e_2)}{\alpha_{l_2} U(l + e_1) - \beta_{l_2} U(l + e_2)}, \quad (4.5a) \]

\[ \frac{U(l + e_2 + e_3)}{U(l)} = -\lambda_{l+1,l+1+4} \frac{\beta_{l_2} U(l + e_2) - \gamma_{l_1} U(l + e_3)}{\beta_{l_2} U(l + e_2) - \gamma_{l_1} U(l + e_3)}, \quad (4.5b) \]

\[ \frac{U(l + e_3 + e_1)}{U(l)} = -\lambda_{l+1,l+1+4} \frac{\gamma_{l_1} U(l + e_3) - \alpha_{l_1} U(l + e_1)}{\gamma_{l_1} U(l + e_3) - \alpha_{l_1} U(l + e_1)}, \quad (4.5c) \]

\[ \frac{U(l)}{U(l + e_1 + e_4)} + \lambda_{l+1,l+1+4} \frac{U(l + e_4)}{U(l + e_4)} = -\alpha_{l_1} K_{i_2} \lambda_{l+1,l+1+4}/2, \quad (4.5d) \]

\[ \frac{U(l)}{U(l + e_2 + e_4)} + \lambda_{l+1,l+1+4} \frac{U(l + e_2)}{U(l + e_2)} = -\beta_{l_2} K_{i_2} \lambda_{l+1,l+1+4}/2, \quad (4.5e) \]
Moreover, by imposing the following \(1, 1, 1, 1\)-periodic condition:

\[
\int_{\mathbb{R}} \ldots + \lambda_{l_{1}+l_{2}+l_{3}+l_{4}} \int_{\mathbb{R}} \ldots = -\gamma_{l_{1}}K_{l_{1}} \lambda_{l_{1}+l_{2}+l_{3}+l_{4}}^{1/2}. \tag{4.5f}
\]

for \(l \in \mathbb{Z}^{4}\), with the following condition of the parameters:

\[
\alpha_{l} = p^{-1}\alpha_{0}, \quad \beta_{l} = p^{-1}\beta_{0}, \quad \gamma_{l} = p^{-1}\gamma_{0}, \quad K_{l} = p^{l}K_{0}, \tag{4.7}
\]

where \(p\) is a non-zero complex parameter, system (4.5) becomes the system of \(q\)-difference equations (in this case the shift parameter is given by \(p\)).

We define the transformations \(\hat{\rho}_{i}, i = 1, \ldots, 4\), by the following actions:

\[
\hat{\rho}_{1} : (U(l), \alpha_{0}, \beta_{0}, \gamma_{0}, K_{0}, \lambda_{0}, p) \mapsto (U(l + \epsilon_{1}), p^{-1}\alpha_{0}, \beta_{0}, \gamma_{0}, K_{0}, \lambda_{0}^{-1}, p), \tag{4.8a}
\]

\[
\hat{\rho}_{2} : (U(l), \alpha_{0}, \beta_{0}, \gamma_{0}, K_{0}, \lambda_{0}, p) \mapsto (U(l + \epsilon_{2}), \alpha_{0}, p^{-1}\beta_{0}, \gamma_{0}, K_{0}, \lambda_{0}^{-1}, p), \tag{4.8b}
\]

\[
\hat{\rho}_{3} : (U(l), \alpha_{0}, \beta_{0}, \gamma_{0}, K_{0}, \lambda_{0}, p) \mapsto (U(l + \epsilon_{3}), \alpha_{0}, \beta_{0}, \gamma_{0}, p^{-1}K_{0}, \lambda_{0}^{-1}, p), \tag{4.8c}
\]

\[
\hat{\rho}_{4} : (U(l), \alpha_{0}, \beta_{0}, \gamma_{0}, K_{0}, \lambda_{0}, p) \mapsto (U(l + \epsilon_{4}), \alpha_{0}, \beta_{0}, \gamma_{0}, pK_{0}, \lambda_{0}^{-1}, p), \tag{4.8d}
\]

which imply that \(\hat{\rho}_{i}\) is a shift operator of \(\epsilon_{i}\)-direction on \(\mathbb{Z}^{4}\). In addition, we also introduce a transformation \(\hat{R}_{0}\) as follows. Its action on the parameters is defined by

\[
\hat{R}_{0} : (\alpha_{0}, \beta_{0}, \gamma_{0}, K_{0}, \lambda_{0}, p) \mapsto (\beta_{0}, p^{-1}\alpha_{0}, \gamma_{0}, K_{0}, \lambda_{0}^{-1}, p), \tag{4.9}
\]
while that on the function \( U(L) \) is defined by

\[
\hat{R}_0(U(L)) = \begin{cases} 
U(L + e_1) & \text{if } L \in \nu^{(1)}, \\
U(L + e_2) & \text{if } L \in \nu^{(2)}, 
\end{cases}
\]

where

\[
\nu^{(1)} = \bigg\{ \sum_{i=1}^4 l_i e_i \bigg| l_i \in \mathbb{Z}, \ l_1 = l_2 \bigg\}, \quad \nu^{(2)} = \bigg\{ \sum_{i=1}^4 l_i e_i \bigg| l_i \in \mathbb{Z}, \ l_1 = l_2 + 1 \bigg\}.
\]

These imply that \( \hat{R}_0 \) is a zigzag-shift operator on the sublattice

\[
\mathcal{R} = \nu^{(1)} \cup \nu^{(2)} \subset \mathbb{Z}^4,
\]

that is,

\[
\hat{R}_0(L) = \begin{cases} 
\hat{\rho}_1(L) & \text{if } L \in \nu^{(1)}, \\
\hat{\rho}_2(L) & \text{if } L \in \nu^{(2)}, 
\end{cases} \quad \hat{R}_0^{-1}(L) = \begin{cases} 
\hat{\rho}_1^{-1}(L) & \text{if } L \in \nu^{(1)}, \\
\hat{\rho}_2^{-1}(L) & \text{if } L \in \nu^{(2)}. 
\end{cases}
\]

In general, for a function \( F = F(U(L), \alpha_0, \beta_0, \gamma_0, K_0, \lambda_0, p) \), we let a transformation \( w \in \langle \hat{\rho}_1, \ldots, \hat{\rho}_4, \hat{R}_0 \rangle \) act as

\[
w(F) = F(w(U(L)), w(\alpha_0), w(\beta_0), w(\gamma_0), w(K_0), w(\lambda_0), w(p)),
\]

that is, the transformation \( w \) act on the arguments from the left.

Finally, letting

\[
\omega_{l_1,l_2,l_3,l_4} = H_{l_1,l_2,l_3,l_4} U(L), \quad b_0 = \frac{\gamma_0}{\alpha_0}, \quad b_1 = \frac{\gamma_0}{\beta_0}, \quad b_2 = \gamma_0 K_0,
\]

where \( L = \sum_{i=1}^4 l_i e_i \), we obtain the fundamental relations on the lattice \( \omega_{A_1 + A_4} \) (3.17) from system (4.5). Here, the gauge factor \( H_{l_1,l_2,l_3,l_4} \) is defined by

\[
H_{l_1,l_2,l_3,l_4} = \left( \frac{\gamma_l}{\alpha_l^{1/2}\beta_l^{1/2}} \right)^{3/2} \times \frac{\log \alpha_0 \beta_0 \gamma_0 K_0}{\log \rho_0 (\log \rho_0^4 \alpha_0^2 \beta_0 \gamma_0 K_0 + 12 (\log \alpha_0 \beta_0^{-1})^3)} / 16 \log \rho.
\]

Figure 4. A (1, 1, 1, 1)-reduction from a 4-cube to a rhombic dodecahedron with a center.
where \( i = \sqrt{-1} \). Furthermore, the actions of transformations \( \hat{\rho}_i, i = 1, \ldots, 4 \), and \( \hat{R}_0 \) correspond to those of \( \rho_i, i = 1, \ldots, 4 \), and \( R_0 \) which are elements of \( \hat{W}((A_2 \times A_1)^{(1)}) \), respectively. We note here that the reduction from system (4.1) to (3.17) causes the reduction of the underlying lattice (see figure 4):

\[
\mathbb{Z}^4 \rightarrow \mathbb{Z}^4/\mathbb{Z}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4) \cong \omega_{A_2 + A_1}.
\]

The reduction from \( \mathbb{Z}^4 \) with system (4.1) to the lattice \( \omega_{A_2 + A_1} \) is referred to as geometric reduction [24] and then the lattice \( \omega_{A_2 + A_1} \) is said to have the reduced hypercube structure. Therefore, we have completed the proof of theorem 1.2.

4.2. Proof of theorem 1.3

In this section, we construct the Lax pairs of the \( q \)-Painlevé equations (1.1) from the Lax equations (4.2) by using the reduction given in section 4.1.

By the gauge transformations (4.3) and

\[
\Psi_{l_1l_2l_3l_4} = i^{-l_1-l_2-l_3-l_4+\lambda_4} \begin{pmatrix} U(l)^{-1} & 0 \\ i \lambda_{l_1+l_2+l_3+l_4} & (l_1+l_2+l_3-2l_4)/2 \end{pmatrix} \phi_{l_1l_2l_3l_4},
\]

the Lax equations (4.2) can be rewritten as the following:

\[
\phi_{l_1+1,l_2+1,l_3+1,l_4} = \delta^{(1)} \begin{pmatrix} i \mu U(l + \epsilon_1) \alpha_{l_1} & 1 \\ \lambda_{l_1+l_2+l_3+l_4}^{1/2} & i \mu U(l) \alpha_{l_1} \end{pmatrix} \phi_{l_1l_2l_3l_4},
\]

(4.18a)

\[
\phi_{l_1,l_2+1,l_3+1,l_4} = \delta^{(2)} \begin{pmatrix} i \mu U(l + \epsilon_2) \beta_{l_2} & 1 \\ \lambda_{l_1+l_2+l_3+l_4}^{1/2} & i \mu U(l) \beta_{l_2} \end{pmatrix} \phi_{l_1l_2l_3l_4},
\]

(4.18b)

\[
\phi_{l_1,l_2,l_3+1,l_4} = \delta^{(3)} \begin{pmatrix} i \mu U(l + \epsilon_3) \gamma_{l_3} & 1 \\ \lambda_{l_1+l_2+l_3+l_4}^{1/2} & i \mu U(l) \gamma_{l_3} \end{pmatrix} \phi_{l_1l_2l_3l_4},
\]

(4.18c)

\[
\phi_{l_1,l_2,l_3,l_4+1} = \delta^{(4)} \begin{pmatrix} -i \mu K_{l_4} U(l + \epsilon_4) \lambda_{l_1+l_2+l_3+l_4} & 1 \\ \lambda_{l_1+l_2+l_3+l_4} & 0 \end{pmatrix} \phi_{l_1l_2l_3l_4}.
\]

(4.18d)

These give the Lax pairs of \( P\Delta E \)s (4.5) (see table 1). Moreover, by the reduction (4.6) with (4.7) and the replacement (4.15), the Lax equations (4.18) can be rewritten as the
following:

\[
\begin{aligned}
\phi_{h+1,l_2,l_3,l_4} &= \delta^{(1)} \left( -p^{-l_0+l_2+l_3-1} \frac{b_1}{b_0} \frac{\omega_{l_0,l_2,l_3,l_4}}{\omega_{h,l_2,l_3,l_4}} x \right) \left( 1 - \frac{1}{\lambda_{h+1,l_2+l_3+l_4}^{1/2}} - p^{3l_0-l_2-l_3-1} \frac{b_0^3}{b_1} \frac{\omega_{l_0,l_2,l_3,l_4}}{\omega_{h+1,l_2,l_3,l_4}} x \right) \phi_{h,l_2}, \\
\phi_{h,l_2+l_3,l_3,l_4} &= \delta^{(2)} \left( -p^{l_0-l_2+l_3-1} \frac{b_0}{b_1} \frac{\omega_{l_0,l_2+l_3,l_4}}{\omega_{h,l_2+l_3,l_4}} x \right) \left( 1 - \frac{1}{\lambda_{h+l_2+l_3+l_4}^{1/2}} - p^{3l_0-l_2-l_3+1} \frac{b_0^3}{b_1} \frac{\omega_{l_0,l_2+l_3,l_4}}{\omega_{h+l_2+l_3+l_4}} x \right) \phi_{h,l_2}, \\
\phi_{h,l_2,l_3+l_4} &= \delta^{(3)} \left( -p^{l_0+l_2+l_3+l_4-1} b_0 b_1 \frac{\omega_{l_0,l_2,l_3+l_4}}{\omega_{h,l_2,l_3+l_4}} x \right) \left( 1 - \frac{1}{\lambda_{h+l_2+l_3+l_4}^{1/2}} - p^{3l_0-l_2-l_3+1} \frac{b_0^3}{b_1} \frac{\omega_{l_0,l_2,l_3+l_4}}{\omega_{h+l_2+l_3+l_4}} x \right) \phi_{h,l_2+l_3,l_4}, \\
\phi_{h,l_2,l_3,l_4+l_1} &= \delta^{(4)} \left( p^{l_0} b_2 \frac{\omega_{l_0,l_2,l_3+l_4}}{\omega_{h,l_2,l_3+l_4}} x \frac{\lambda_{h+l_2+l_3+l_4}}{\lambda_{h+l_2+l_3+l_4}} \right) \left( 1 - \frac{1}{\lambda_{h+l_2+l_3+l_4}^{1/2}} - p^{3l_0-l_2-l_3+1} \frac{b_0^3}{b_1} \frac{\omega_{l_0,l_2,l_3+l_4}}{\omega_{h+l_2+l_3+l_4}} x \right) \phi_{h,l_2,l_3,l_4}. \\
\end{aligned}
\]

(4.19a)

\[x = \frac{\mu}{\gamma_0}. \]

(4.20)

These give the Lax pairs of PΔEs (3.17) (see table 1).

Now we are in a position to construct the Lax pairs of the \(q\)-Painlevé equations. We first lift the action of \(\hat{\rho}_1, \ldots, \hat{\rho}_4, \hat{R}_0\) up to the Lax equations (4.19) by

\[
\begin{aligned}
\hat{\rho}_1 : (\delta^{(1)}, \delta^{(2)}, \delta^{(3)}, \delta^{(4)}, \mu, \phi_{h,l_2,l_3,l_4}) \mapsto (\delta^{(1)}, \delta^{(2)}, \delta^{(3)}, \delta^{(4)}, \mu, \phi_{h+1,l_2,l_3,l_4}), \\
\hat{\rho}_2 : (\delta^{(1)}, \delta^{(2)}, \delta^{(3)}, \delta^{(4)}, \mu, \phi_{h,l_2+l_3,l_4}) \mapsto (\delta^{(1)}, \delta^{(2)}, \delta^{(3)}, \delta^{(4)}, \mu, \phi_{h,l_2+l_3,l_4}), \\
\hat{\rho}_3 : (\delta^{(1)}, \delta^{(2)}, \delta^{(3)}, \delta^{(4)}, \mu, \phi_{h,l_2,l_3+l_4}) \mapsto (\delta^{(1)}, \delta^{(2)}, \delta^{(3)}, \delta^{(4)}, \mu, \phi_{h,l_2,l_3+l_4}), \\
\hat{\rho}_4 : (\delta^{(1)}, \delta^{(2)}, \delta^{(3)}, \delta^{(4)}, \mu, \phi_{h,l_2,l_3}) \mapsto (\delta^{(1)}, \delta^{(2)}, \delta^{(3)}, \delta^{(4)}, \mu, \phi_{h,l_2,l_3,l_4}), \\
\hat{R}_0 : (\delta^{(1)}, \delta^{(2)}, \delta^{(3)}, \delta^{(4)}, \mu, \phi_{h,l_2,l_3}) \mapsto (\delta^{(1)}, \delta^{(2)}, \delta^{(3)}, \delta^{(4)}, \mu, \phi_{h,l_2,l_3,l_4}).
\end{aligned}
\]

(4.21a)

where

\[
\phi_{h,l_2,l_3,l_4}^R = \begin{cases} 
\phi_{h+1,l_2,l_3,l_4} & \text{if } h = l_2, \\
\phi_{h,l_2+1,l_3,l_4} & \text{if } h = l_2 + 1.
\end{cases}
\]

(4.22)
By letting
\[
\phi_{0,0,0,-1} = \begin{pmatrix} \omega_{0,0,0,-1} & 0 \\ 0 & \omega_{0,0,0} \end{pmatrix} \Phi,
\]
the action of \( \{ \hat{\rho}_1, ..., \hat{\rho}_4, \hat{R}_0 \} \) on \( \Phi \) is given by
\[
\hat{\rho}_1(\Phi) = \delta_1 \begin{pmatrix} -\frac{b_1}{b_0 p^2} x & b_1^2 \\ \frac{b_0 b_3 (p^2 b_0^3 b_3^{1/2} f_1^{(3)} - b_1 b_2)}{b_3^{1/2}} & -\frac{b_3}{b_3^{1/2} (p^2 b_0^3 b_3^{1/2} f_1^{(3)} - b_1 b_2)} \end{pmatrix} \Phi,
\]
\[
\hat{\rho}_2(\Phi) = \delta_2 \begin{pmatrix} -\frac{b_0}{p b_1} x & b_0^2 \\ \frac{b_1 b_3 (p^2 b_1^3 b_3^{1/2} \hat{\rho}_1^{-1} f_1^{(1)}(1) - b_0 b_2)}{b_3^{1/2}} & -\frac{b_3}{b_3^{1/2} (p^2 b_1^3 b_3^{1/2} \hat{\rho}_1^{-1} f_1^{(1)}(1) - b_0 b_2)} \end{pmatrix} \Phi,
\]
\[
\hat{\rho}_3(\Phi) = \delta_3 \begin{pmatrix} -\frac{b_0 b_1}{p} x & b_1^{1/2} \\ \frac{b_0 b_1 (b_0 b_1 \hat{\rho}_3 (f_1^{(2)} f_1^{(3)}(1))) + b_2 b_3^{3/2}}{p^2 b_3^{3/2}} & -\frac{b_0 b_1 \hat{\rho}_3 (f_1^{(2)} f_1^{(3)}(1))) + b_2 b_3^{3/2}}{p b_2^2 \hat{\rho}_3 (f_1^{(2)} f_1^{(3)}(1)))} \end{pmatrix} \Phi,
\]
\[
\hat{\rho}_4(\Phi) = \delta_4 \begin{pmatrix} b_2 x & \frac{1}{b_3} \\ \frac{p b_0 f_1^{(3)} (p b_0^3 \hat{\rho}_4 (f_1^{(3)}(1)) - b_1 b_2 b_3^{1/2})}{b_1^2 b_3} & 0 \end{pmatrix} \Phi.
\]
\[
\hat{R}_0(\Phi) = \hat{\rho}_4(\Phi),
\]
where \( f_1^{(j)}, j = 1, 2, 3, \) are given by (3.27) and satisfy the relation (1.5). Next, let us define
\[
\hat{T}_{SP} = \hat{\rho}_1 \hat{\rho}_2 \hat{\rho}_3 \hat{\rho}_4, \quad \hat{T}_0 = \hat{\rho}_1 \hat{\rho}_2, \quad \hat{T}_{13} = \hat{\rho}_3, \quad \hat{R}_{13} = \hat{\rho}_4.
\]

**Remark 4.1.** Under the actions on the \( f \)-variables \( f_1^{(j)}, j = 1, 2, 3, \) and the parameters \( b_i, i = 0, ..., 3, \) and \( p, \) the transformations \( \hat{T}_{0}, \hat{T}_{13}, \hat{R}_0 \) and \( \hat{R}_{13} \) are respectively equivalent to the transformations \( T_{0}, T_{13}, R_0 \) and \( R_{13} \), which are elements of \( \hat{W}(\mathbb{A}_2 \times \mathbb{A}_3)^{(1)} \), and the spectral operator \( T_{SP} \) can be regarded as an identity mapping.

The actions of \( \hat{T}_{SP}, \hat{T}_0, \hat{T}_{13}, \hat{R}_0 \) and \( \hat{R}_{13} \) on the spectral parameter \( x \) are given by
\[
\hat{T}_{SP}(x) = px, \quad \hat{T}_0(x) = \hat{T}_{13}(x) = \hat{R}_0(x) = \hat{R}_{13}(x) = x,
\]
while those on the wave function \( \Phi \) are given by the following:
\[
\hat{T}_{SP}(\Phi) = \delta_1 \delta_2 \delta_3 \delta_4 A \Phi, \quad \hat{T}_0(\Phi) = \delta_1 \delta_2 B_{T_0} \Phi, \quad \hat{T}_{13}(\Phi) = \delta_4^2 B_{T_{13}} \Phi.
\]
\[ \hat{R}_0(\Phi) = \delta_1 B_{R0}\Phi, \quad \hat{R}_3(\Phi) = \delta_4 B_{R13}\Phi, \]  

where

\[
A = \begin{pmatrix}
  b_2 \frac{p_x}{p} & b_3 \\
  -b_1^{1/2}(p^2b_0^3b_3^{1/2}f_1^{(3)}) - b_1b_2(b_0b_2b_3^{1/2} + pb_0^3b_3^{3/2}f_1^{(3)}) - pb_1^3f_1^{(1)} & 0
\end{pmatrix}
\]

\[
B_{T0} = \begin{pmatrix}
  b_2 \frac{p_x}{p} & b_3 \\
  -b_1^{1/2}(p^2b_0^3b_3^{1/2}f_1^{(3)}) - b_1b_2(b_0b_2b_3^{1/2} + pb_0^3b_3^{3/2}f_1^{(3)}) - pb_1^3f_1^{(1)} & b_1
\end{pmatrix}
\]

\[
B_{T13} = \begin{pmatrix}
  pb_1b_3^{1/2}f_1^{(1)}(b_1b_2^{1/2}f_1^{(2)}) - b_0b_2(p^2b_0^3b_3^{1/2}f_1^{(3)}) - pb_1^3b_1^{1/2} & 0 \\
  b_2x \frac{p_x}{p} & b_3
\end{pmatrix}
\]

\[
\left( \begin{array}{c}
  b_2x \\
  b_3
\end{array} \right)
\]
\[
B_{R0} = \begin{pmatrix}
-\frac{b_1}{pb_0} x & b_1^2 \\
\frac{f_1^{(3)}}{b_3^{3/2}} & \frac{b_0b_1(p^2b_0^3b_3^{1/2}f_1^{(3)} - b_1b_2)}{b_3^{3/2}(p^2b_0^3b_3^{1/2}f_1^{(3)} - b_1b_2)}
\end{pmatrix},
\]
\(4.29c\)

\[
B_{R13} = \begin{pmatrix}
\frac{b_2}{p} x & 1 \\
\frac{p b_0f_1^{(3)}(p b_0^3 R_{13} f_1^{(3)}) - b_1b_2b_3^{1/2}}{b_2^2 b_3} & 0
\end{pmatrix}.
\]
\(4.29d\)

Therefore, we finally obtain theorem 1.3 by the following correspondence:

\[
\hat{T}_0 = T_0, \quad \hat{T}_{13} = T_{13}, \quad \hat{R}_0 = R_0, \quad \hat{R}_{13} = R_{13},
\]
\(4.30a\)

\[
\hat{\delta}_1 = \hat{\delta}_2 = \hat{\delta}_3 = \hat{\delta}_4 = 1.
\]
\(4.30b\)

5. Concluding remarks

In this paper, we constructed the \(\omega\)-lattice of type \(A_4^{(1)}\). The \(\omega\)-lattice provides the informations about how a system of partial difference equations can be reduced to \(A_4^{(1)}\)-surface \(q\)-Painlevé equations. We will show how to use this information in forthcoming paper (N Joshi, N Nakazono and Y Shi, in preparation). We also constructed another important lattice \(\omega A_1 + A_1\) and showed that it has the reduced hypercube structure. Moreover, by using this structure, we constructed the Lax pairs of the \(q\)-Painlevé equations (1.1). The distinguishing feature of the Lax pairs given in this paper as compared with those in the other works, e.g. [22, 35], is that their coefficient matrices can be factorized into the products of matrices which are of degree one in the spectral parameter \(x\). This property enables us to construct the Lax pairs of symmetric discrete Painlevé equations, e.g. \(q\)-P(III) (1.1c) and \(q\)-P(IV) (1.1d), which can be obtained by projective reductions [28, 29].

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Appendix A. Proof of lemma 2.2

In this section, we define the transformation group \(\hat{\mathcal{W}}(A_4^{(1)})\) with its linear action and show it forms the extended affine Weyl group of type \(A_4^{(1)}\). Moreover, we lift its action to the birational action on the parameters and the \(\tau\)-variables.

First, we define the transformation group \(\hat{\mathcal{W}}(A_4^{(1)}) = \langle s_0, s_1, s_2, s_3, s_4, \sigma, \iota \rangle\). Let \((f, g)\) be inhomogeneous coordinate of \(\mathbb{P}^1 \times \mathbb{P}^1\). We consider the following eight base points of
Figure A1. Dynkin diagram of type $A_4^{(1)}$.

\[ \begin{align*}
\mathbb{P}^1 \times \mathbb{P}^1: \\
p_1 : (f, g) &= (-a_0^{-1}a_2, 0), \\
p_2 : (f, g) &= (-a_0^{-1}a_1^{-1}a_3, 0), \\
p_3 : (f, g) &= (-a_0^{-1}a_3a_2, \infty), \\
p_4 : (f, g) &= (0, -a_0a_2^{-1}), \\
p_5 : (f, g) &= (0, -a_0a_2^{-1}a_4), \\
p_6 : (f, g) &= (\infty, -a_0a_2^{-1}a_3^{-1}), \\
p_{\infty} : (f, g) &= (\infty, \infty), \\
p_7 : (f, g; f/g) &= (\infty, \infty; -a_0^{-1}a_2a_3),
\end{align*}\]

where $a_i, i = 0, \ldots, 4$, are non-zero complex parameters. Let $\epsilon : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ denote blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the points (A.1). The linear equivalence classes of the total transform of the coordinate lines $f = \text{constant}$ and $g = \text{constant}$ are denoted by $h_1$ and $h_2$, respectively. The Picard group of $X$, denoted by $\text{Pic}(X)$, is given by

\[ \text{Pic}(X) = \mathbb{Z} h_1 \oplus \mathbb{Z} h_2 \oplus \mathbb{Z} e_i, \]

where $e_i = e^{-1}(p_i), i = 1, \ldots, 8, (p_8 = p_{\infty})$ are exceptional divisors. The intersection form $(\cdot | \cdot)$ is defined by

\[ (h_i | h_j) = 1 - \delta_{ij}, \quad (h_i | e_j) = 0, \quad (e_i | e_j) = -\delta_{ij}. \]

The anti-canonical divisor of $X$, denoted by $-K_X$, is uniquely decomposed into the prime divisors:

\[ -K_X = 2h_1 + 2h_2 - \sum_{i=1}^{8} e_i = 4 d_i = \delta, \]

where

\[ \begin{align*}
d_0 &= h_1 - e_6 - e_8, \\
d_1 &= e_6 - e_7, \\
d_2 &= h_2 - e_3 - e_8, \\
d_3 &= h_1 - e_4 - e_5, \\
d_4 &= h_2 - e_1 - e_2.
\end{align*}\]
The corresponding Cartan matrix

\[
(d_{ij})_{i,j=0}^4 = \begin{pmatrix}
2 & -1 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
-1 & 0 & 0 & -1 & 2 \\
\end{pmatrix}, \quad d_{ij} = \frac{2(d_i|d_j)}{(d_i|d_i)}, \tag{A.6}
\]

and Dynkin diagram (see figure A1) are of type $A_4^{(1)}$. Thus, we can set the root lattice as

\[
\mathcal{Q}(A_4^{(1)}) = \bigoplus_{i=0}^4 \mathbb{Z}d_i, \tag{A.7}
\]

and identify the surface $X$ as being type $A_4^{(1)}$ in Sakai’s classification [55]. Moreover, we obtain the following root lattice orthogonal to $\mathcal{Q}(A_4^{(1)})$:

\[
\hat{\mathcal{Q}}(A_4^{(1)}) = \bigoplus_{i=0}^4 \mathbb{Z}\alpha_i, \tag{A.8}
\]

where

\[
\alpha_0 = h_1 + h_2 - e_1 - e_4 - e_7 - e_8, \quad \alpha_1 = e_1 - e_2, \quad \alpha_2 = h_1 - e_1 - e_3, \tag{A.9a}
\]

\[
\alpha_3 = h_2 - e_4 - e_5, \quad \alpha_4 = e_4 - e_5, \tag{A.9b}
\]

and

\[
\delta = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \tag{A.10}
\]

by searching for elements of Pic $(X)$ that are orthogonal to all divisors $d_i, i=0,\ldots,4$. The root lattice $\hat{\mathcal{Q}}(A_4^{(1)})$ is also of $A_4^{(1)}$-type.

Let us consider the Cremona isometries for this setting. A Cremona isometry is defined by an automorphism of Pic $(X)$ which preserves

(i) the intersection form on Pic $(X)$;

(ii) the canonical divisor $K_X$;

(iii) effectiveness of each effective divisor of Pic $(X)$.

The reflections $s_i$ for simple roots $\alpha_i, i=0,\ldots,4$, defined by the following right actions:

\[
v.s_i = v - \frac{2(v|\alpha_i)}{(\alpha_i|\alpha_i)} \alpha_i, \tag{A.11}
\]

for all $v \in \text{Pic}(X)$ and the automorphisms of the Dynkin diagram:

\[
(d_0, d_1, d_2, d_3, d_4; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \cdot \sigma = (d_2, d_3, d_4, d_0, d_1; \alpha_4, \alpha_0, \alpha_1, \alpha_2, \alpha_3), \tag{A.12a}
\]

\[
(d_0, d_1, d_2, d_3, d_4; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \cdot \iota = (d_2, d_1, d_0, d_4, d_3; \alpha_0, \alpha_4, \alpha_3, \alpha_2, \alpha_1), \tag{A.12b}
\]
defined by the following right actions:

\[
(h_1, h_2, e_1, \ldots, e_8) \cdot \sigma = (h_1, h_2, e_1, \ldots, e_8) \cdot \sigma = (h_1, h_2, e_1, \ldots, e_8) .
\]

\[
\begin{pmatrix}
n & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
n & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
n & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
n & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
n & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 \\
n & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]  

\[\text{(A.13a)}\]

\[
\begin{pmatrix}
n & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
& 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
& 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]  

\[\text{(A.13b)}\]

are Cremona isometries and collectively form extended affine Weyl group of type $A_4^{(1)}$. Namely, we can easily verify that the following fundamental relations hold:

\[s^2_i = 1, \quad (s_is_{i+1})^3 = 1, \quad (s_is_{i\pm 1})^2 = 1, \quad j \neq i \pm 1, \quad \text{(A.14a)}\]

\[\sigma^5 = 1, \quad \sigma s_i = s_{i+1}\sigma, \quad i^2 = 1, \quad i\sigma = s_{i-1}\sigma, \quad \sigma i = i\sigma^{-1}, \quad \text{(A.14b)}\]

where $i, j \in \mathbb{Z}/5\mathbb{Z}$. Note here that the transformations $T_i$, $i = 0, \ldots, 4$, defined by (2.7) are translations on $\mathcal{Q}(A_4^{(1)})$:

\[\alpha_iT_i = \alpha_i - \delta, \quad \alpha_{i+1}T_i = \alpha_{i+1} + \delta, \quad \text{(A.15)}\]

where $i \in \mathbb{Z}/5\mathbb{Z}$.

Next, we lift the action of $\mathcal{W}(A_4^{(1)})$ to the birational action. We first define the variables $f_w, f_d, g_w$ and $g_d$ by

\[f = \frac{f_w}{f_d}, \quad g = \frac{g_w}{g_d}, \quad \text{(A.16)}\]

and their polynomial $F_\Lambda$ by

\[F_\Lambda = F_\Lambda(f_w, f_d, g_w, g_d), \quad \text{(A.17)}\]

where $\Lambda = mh_1 + nh_2 - \sum_{i=1}^8 \mu_i e_i$, which corresponds to a curve of bi-degree $(m, n)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ passing through base points $p_i$ with multiplicity $\mu_i$. For example,

\[F_{h_1h_2} = \gamma(a_0^2a_4f_wg_d + a_2a_3f_dg_w + a_0a_3a_4f_dg_d), \quad \text{(A.18)}\]
where $\gamma$ is an arbitrary non-zero complex parameter. We next define a mapping $\tau$ by the following definition.

**Definition A.1.** We define a mapping $\tau$ on the set

$$M = \{e, w\mid w \in \mathcal{W}(A^1_4), \ i = 1, \ldots, 8\}$$  \hspace{1cm} (A.19)

by the following:

(i) if under the blowing down map an exceptional line $e_i$ collapses to a base point $p_j$, put

$$\tau(e_i) = \tau(e_j);$$  \hspace{1cm} (A.20)

(ii) if $\Lambda = mh_1 + nh_2 = \sum_{i=1}^8 \mu_i e_i \in \{d_0, d_2, d_3, d_4\}$, then

$$\frac{F_\Lambda(f_\nu, f_d, g_\nu, g_d)}{F_\Lambda(1, 1, 1, 1)} = (e_i)^{\nu_1} \cdots (e_8)^{\nu_8},$$  \hspace{1cm} (A.21)

which give

$$f_\nu = \tau(e_1)\tau(e_2), \quad f_d = \tau(e_3)\tau(e_4), \quad g_\nu = \tau(e_5)\tau(e_6), \quad g_d = \tau(e_7)\tau(e_8);$$  \hspace{1cm} (A.22)

(iii) for $\Lambda = mh_1 + nh_2 = \sum_{i=1}^8 \mu_i e_i \in M$, $\tau(\Lambda)$ is defined by

$$\tau(\Lambda) = \frac{F_\Lambda(f_\nu, f_d, g_\nu, g_d)}{\tau(e_1)^{\nu_1} \cdots \tau(e_8)^{\nu_8}};$$  \hspace{1cm} (A.23)

(iv) $w \in \mathcal{W}(A^1_4)$ act on $\tau(\Lambda)$ as

$$w.\tau(\Lambda) = \tau(\Lambda.w^{-1}),$$  \hspace{1cm} (A.24)

where $\Lambda \in M$.

Finally, lemma 2.2 follows from the setting

$$\tau_1^{(1)} = \tau(e_2), \quad \tau_1^{(2)} = \tau(e_3), \quad \tau_1^{(3)} = \tau(e_4), \quad \tau_1^{(4)} = \tau(e_5), \quad \tau_1^{(5)} = \tau(e_6), \quad \tau_1^{(6)} = \tau(e_7) = \tau(e_8),$$  \hspace{1cm} (A.25a)

$$\tau_2^{(1)} = \tau(e_1)\sigma^4 = \frac{a_0a_1(a_3\tau_1^{(3)}\tau_1^{(2)}) + a_0\tau_1^{(4)}\tau_2^{(3)}}{a_2a_3\tau_2^{(2)}},$$  \hspace{1cm} (A.25b)

$$\tau_2^{(2)} = \tau(e_4)\sigma = \frac{a_1a_2(a_4\tau_1^{(4)}\tau_1^{(1)}) + a_1\tau_1^{(5)}\tau_2^{(4)}}{a_3a_4\tau_2^{(1)}}, \quad \tau_2^{(3)} = \tau(e_4),$$  \hspace{1cm} (A.25c)

$$\tau_2^{(4)} = \tau(e_1)\sigma = \frac{a_3a_4(a_1\tau_1^{(1)}\tau_1^{(4)}) + a_3\tau_1^{(2)}\tau_2^{(1)}}{a_0a_1\tau_2^{(5)}}, \quad \tau_2^{(5)} = \tau(e_1),$$  \hspace{1cm} (A.25d)

and the normalization of the polynomials $F_\Lambda$ to be designed to hold the fundamental relations (A.14). We note that the action of $\mathcal{W}(A^1_4)$ on the $\tau$-variables are directly obtained from the definition of the mapping $\tau$. For example,
where \( g \) is an arbitrary non-zero complex parameter. Moreover, figure A2 shows simple relations between the \( \tau \)-variables.

Appendix B. The linear action of \( \tilde{W} ((A_2 \times A_1)^{(1)}) \)

In this section, we give explanations of the transformation group \( \tilde{W} ((A_2 \times A_1)^{(1)}) \) and its translation part \( \{\rho_1, \rho_2, \rho_3, \rho_4\} \) with their linear actions on the root systems.

We here consider the following submodule of the root lattice \( Q(A_1^{(1)}) \) (A.8):

\[
Q(A_2 + A_1) = \mathbb{Z} \beta_0 \oplus \mathbb{Z} \beta_1 \oplus \mathbb{Z} \beta_2 \oplus \mathbb{Z} \gamma_0 \oplus \mathbb{Z} \gamma_1,
\]

where the simple roots \( \beta_i, i = 0, 1, 2, \) and \( \gamma_i, i = 0, 1, \) are defined by

\[
\begin{align*}
\beta_0 &= \alpha_0, \\
\beta_1 &= \alpha_1 + \alpha_2, \\
\beta_2 &= \alpha_3 + \alpha_4,
\end{align*}
\]

\[
\begin{align*}
\gamma_0 &= 2\alpha_1 - \alpha_2 + \alpha_3 - 2\alpha_4, \\
\gamma_1 &= \alpha_0 - \alpha_1 + 2\alpha_2 + 3\alpha_4,
\end{align*}
\]

and satisfy

\[
\delta = \beta_0 + \beta_1 + \beta_2 = \gamma_0 + \gamma_1.
\]

The root lattices \( Q(A_2^{(1)}) = \bigoplus_{i=0}^{2} \mathbb{Z} \beta_i \) and \( Q(A_1^{(1)}) = \bigoplus_{i=0}^{1} \mathbb{Z} \gamma_i \) are of \( A_2^{(1)} \) - and \( A_1^{(1)} \)-types, respectively:

\[
(b_{ij})_{i,j=0}^{2} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad (c_{ij})_{i,j=0}^{1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.
\]

where

\[
\begin{align*}
b_{ij} &= \frac{2(\beta_i|\beta_j)}{(\beta_j|\beta_j)}, \\
c_{ij} &= \frac{2(\gamma_i|\gamma_j)}{(\gamma_j|\gamma_j)}.
\end{align*}
\]

Let us discuss Cremona transformations for \( Q((A_2 + A_1)^{(1)}) \). The transformations \( w_i, i = 0, 1, 2, \) and \( \pi \), defined by (3.1), act on \( Q((A_2 + A_1)^{(1)}) \) as the following:

\[
(b_0, \beta_1, \beta_2, \gamma_0, \gamma_1)_w = (-\beta_0, \beta_1 + \beta_0, \beta_2 + \beta_0, \gamma_0, \gamma_1),
\]

\[
(\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1)_p = (\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1).
\]
The transformations \( w_i, i = 0, 1, 2, \) correspond to the reflections for the simple roots \( \beta_i, i = 0, 1, 2, \) respectively, that is, they satisfy 

\[
\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, w_1 = (\beta_0 + \beta_1, -\beta_1, \beta_2 + \beta_1, \gamma_0, \gamma_1),
\]

for all \( v \in \text{Pic}(X). \) Moreover, the transformation \( \pi \) corresponds to the automorphism of the Dynkin diagram:

\[
(d_0, d_1, d_2, d_3; \beta_0, \beta_1, \beta_2; \gamma_0, \gamma_1).\pi = (d_1, d_0, d_4, d_3, \beta_2, \beta_1, \beta_0; \gamma_1, \gamma_0).
\]

Note that there are no Cremona transformations correspond to the reflections for the simple roots \( \gamma_i, i = 0, 1, \) since

\[
2(h_i|\gamma_i) = -\frac{1}{15} \gamma_i \not\in \text{Pic}(X).
\]

From the fundamental relations (A.14), we can verified that the group of transformations \( \langle w_0, w_1, w_2, r_0, r_1, \pi \rangle \) satisfy the following relations:

\[
w_i^2 = (w_{i+1})^3 = 1, \quad r_0^2 = r_1^2 = (r_0r_1)^\infty = 1, \quad \pi^2 = 1, \quad \pi^2 = 1, \quad \pi = 1,
\]

where \( i \in \mathbb{Z}/3\mathbb{Z}. \) We note that the relation \((ww')^\infty = 1\) for transformations \( w \) and \( w' \) means that there is no positive integer \( N \) such that \((ww')^N = 1.\) Therefore, transformation group \( \langle w_0, w_1, w_2, r_0, r_1, \pi \rangle \) forms the extended affine Weyl group of type \( (A_2 \times A_1)^{(1)} \), denoted by \( W((A_2 \times A_1)^{(1)}). \) Here, \( W(A_2^{(1)}) = \langle w_0, w_1, w_2 \rangle \) and \( W(A_1^{(1)}) = \langle r_0, r_1 \rangle \) form affine Weyl groups of types \( A_2^{(1)} \) and \( A_1^{(1)} \), respectively. Moreover, \( W((A_2 \times A_1)^{(1)}) = \langle w_0, w_1, w_2, r_0, r_1 \rangle \) is the semi direct product of \( W(A_2^{(1)}) \) and \( W(A_1^{(1)}). \)

The transformations \( \rho_i, i = 1, \ldots, 4, \) defined by (3.5) are translations on \( Q((A_2 + A_1)^{(1)}) \) since they act on \( Q((A_2 + A_1)^{(1)}) \) as the following:

\[
(\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1), \rho_1 = (\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1) + (-1, 0, 1, 1, -1)\delta,
\]

\[
(\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1), \rho_2 = (\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1) + (0, 1, -1, 1, -1)\delta,
\]

\[
(\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1), \rho_3 = (\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1) + (1, -1, 0, 1, -1)\delta,
\]

\[
(\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1), \rho_4 = (\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1) + (0, 0, 0, -3, 3)\delta.
\]

Note that \( \rho_i, i = 1, \ldots, 4, \) are not translational motions on \( \hat{Q}(A_4^{(1)}) \) (A.8):

\[
(a_0, a_1, a_2, a_3, a_4)\rho_1 = (a_0 - \delta, a_1 + a_2 + a_3, -a_3, -a_2 + \delta, a_2 + a_3 + a_4).
\]

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\[(a_0, a_1, a_2, a_3, a_4), \rho_2 \]
\[= (a_0, a_1 + a_2 + a_3, -a_3 + \delta, -a_2, -a_0 - a_1), \quad (B.12b)\]
\[(a_0, a_1, a_2, a_3, a_4), \rho_3 \]
\[= (a_0 + \delta, -a_0 - a_4, -a_3, -a_2 + \delta, -a_0 - a_1), \quad (B.12c)\]
\[(a_0, a_1, a_2, a_3, a_4), \rho_4 \]
\[= (a_0, -a_0 - a_4, -a_3 + \delta, -a_2, a_2 + a_3 + a_4), \quad (B.12d)\]

but their squares are translations on \(\hat{Q}(A_4^{(1)})\):
\[\rho_1^2 = T_0 T_2 T_4^{-1}, \quad \rho_2^2 = T_0 T_2^{-1} T_4, \quad \rho_3^2 = T_0^{-1} T_2 T_3, \quad \rho_4^2 = T_1 T_3. \quad (B.13)\]

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