Shelling and triangulating
the (extra)ordinary polytope

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Abstract

Ordinary polytopes were introduced by Bisztriczky as a (nonsimpli-
cial) generalization of cyclic polytopes. We show that the colex order
of facets of the ordinary polytope is a shelling order. This shelling
shares many nice properties with the shellings of simplicial polytopes.
We also give a shallow triangulation of the ordinary polytope, and
show how the shelling and the triangulation are used to compute the
toric $h$-vector of the ordinary polytope. As one consequence, we get
that the contribution from each shelling component to the $h$-vector is
nonnegative. Another consequence is a combinatorial proof that the
entries of the $h$-vector of any ordinary polytope are simple sums of
binomial coefficients.

1 Introduction

1.1 Motivation

This paper has a couple of main motivations. The first comes from the
study of toric $h$-vectors of convex polytopes. The $h$-vector played a crucial
role in the characterization of face vectors of simplicial polytopes [5 12 13].

*This research was supported by the sabbatical leave program of the University of
Kansas, and was conducted while the author was at the Mathematical Sciences Research
Institute, supported in part by NSF grant DMS-9810361, and at Technische Universität
Berlin, supported in part by Deutsche Forschungs-Gemeinschaft, through the DFG Re-
search Center “Mathematics for Key Technologies” (FZT86) and the Research Group
“Algorithms, Structure, Randomness” (FOR 13/1-1).
In the simplicial case, the $h$-vector is linearly equivalent to the face vector, and has a combinatorial interpretation in a shelling of the polytope. The $h$-vector of a simplicial polytope is also the sequence of Betti numbers of the associated toric variety. In this context it generalizes to nonsimplicial polytopes. However, for nonsimplicial polytopes, we do not have a good combinatorial understanding of the entries of the $h$-vector.

The definition of the (toric) $h$-vector for general polytopes (and even more generally, for Eulerian posets) first appeared in [14]. Already there Stanley raised the issue of computing the $h$-vector from a shelling of the polytope. Associated with any shelling, $F_1, F_2, \ldots, F_n$, of a polytope $P$ is a partition of the faces of $P$ into the sets $G_j$ of faces of $F_j$ not in $\cup_{i<j} F_i$. The $h$-vector can be decomposed into contributions from each set $G_j$. When $P$ is simplicial, the set $G_j$ is a single interval $[G_j, F_j]$ in the face lattice of $P$, and the contribution to the $h$-vector is a single 1 in position $|G_j|$. For nonsimplicial polytopes, the set $G_j$ is not so simple. It is not clear whether the contribution to the $h$-vector from $G_j$ must be nonnegative, and, if it is, whether it counts something natural. (Tom Braden [8] has announced a positive answer to this question, based on [1, 11].) Another issue is the relation of the $h$-vector of a polytope $P$ to the $h$-vector of a triangulation of $P$. This is addressed in [2, 15].

A problem in studying nonsimplicial polytopes is the difficulty of generating examples with a broad range of combinatorial types. Bisztriczky [7] discovered the fascinating “ordinary” polytopes, a class of generally nonsimplicial polytopes, which includes as its simplicial members the cyclic polytopes. These polytopes have been studied further in [3, 11, 12]. In particular, in [3], it is shown that ordinary polytopes have surprisingly nice $h$-vectors, namely, the $h$-vector is the sum of the $h$-vector of a cyclic polytope and the shifted $h$-vector of a lower-dimensional cyclic polytope. These $h$-vectors were calculated from the flag vectors, and the calculation did not give a combinatorial explanation for the nice form that came out. So we were motivated to find a combinatorial interpretation for these $h$-vectors, most likely through shellings or triangulations of the polytopes.

This paper is organized as follows. In the second part of this introduction we give the main definitions. The brief Section 2 warms up with the natural triangulation of the multiplex. Section 3 is devoted to showing that the colex order of facets is a shelling of the ordinary polytope. The proof, while laborious, is constructive, explicitly describing the minimal new faces of the polytope as each facet is shelled on. We then turn in Section 4 to $h$-vectors of multiplicity polytopes in general, and of the ordinary polytope in particular. Here a “fake simplicial $h$-vector” arises in the shelling of the
ordinary polytope. In Section 5, the triangulation of the multiplex is used to triangulate the boundary of the ordinary polytope. This triangulation is shown to have a shelling compatible with the shelling of Section 3. The shelling and triangulation together explain combinatorially the $h$-vector of the ordinary polytope.

Finally, a comment about the title of this paper. Bisztriczky named these polytopes “ordinary polytopes” to invoke the idea of ordinary curves. The name is, of course, a bit misleading, as it is applied to a truly extraordinary class of polytopes. We feel that these polytopes are extraordinary because of their special structure, but we hope that they will also turn out to be extraordinary for their usefulness in understanding general convex polytopes.

1.2 Definitions

For common polytope terminology, refer to [16].

The toric $h$-vector was defined by Stanley for Eulerian posets, including the face lattices of convex polytopes.

**Definition 1 ([14])** Let $P$ be a $(d-1)$-dimensional polytopal sphere. The $h$-vector and $g$-vector of $P$ are encoded as polynomials: $h(P, x) = \sum_{i=0}^{d} h_i x^{d-i}$ and $g(P, x) = \sum_{i=0}^{\lfloor d/2 \rfloor} g_i x^i$, with the relations $g_0 = h_0$ and $g_i = h_i - h_{i-1}$ for $1 \leq i \leq d/2$. Then the $h$-polynomial and $g$-polynomial are defined by the recursion

1. $g(\emptyset, x) = h(\emptyset, x) = 1$, and
2. $h(P, x) = \sum_{G \text{ face of } P \atop G \neq P} g(G, x)(x - 1)^{d - 1 - \dim G}$.

It is easy to see that the $h$-vector depends linearly on the flag vector. In the case of simplicial polytopes, the formulas reduce to the well-known transformation between $f$-vector and $h$-vector.

**Definition 2 ([16])** Let $C$ be a pure $d$-dimensional polytopal complex. If $d = 0$, then a shelling of $C$ is any ordering of the points of $C$. If $d > 0$, then a shelling of $C$ is a linear ordering $F_1, F_2, \ldots, F_s$ of the facets of $C$ such that for $2 \leq j \leq s$, $F_j \cap (\cup_{i<j} F_i)$ is nonempty and is the union of ridges of $C$ that form the initial segment of a shelling of $F_j$.

**Definition 3 ([2])** A triangulation $\Delta$ of a polytopal complex $C$ is shallow if and only if every face $\sigma$ of $\Delta$ is contained in a face of $C$ of dimension at most $2 \dim \sigma$. 

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Theorem 1.1 ([2]) If $\Delta$ is a simplicial sphere forming a shallow triangulation of the boundary of the convex $d$-polytope $P$, then $h(\Delta, x) = h(P, x)$.

Note: in [2] Theorem 4 gives $h(P, x) = h(\Delta, x)$ for a shallow subdivision $\Delta$ of the solid polytope $P$. The proof goes through for shallow subdivisions of the boundary, because it is based on the uniqueness of low-degree acceptable functions [14], which holds for lower Eulerian posets.

Definition 4 ([6]) A $d$-dimensional multiplex is a polytope with an ordered list of vertices, $x_0, x_1, \ldots, x_n$, with facets $F_0, F_1, \ldots, F_n$ given by

$$F_i = \text{conv}\{x_{i-d+1}, x_{i-d+2}, \ldots, x_{i-1}, x_{i+1}, x_{i+2}, \ldots, x_{i+d-1}\},$$

with the conventions that $x_i = x_0$ if $i < 0$, and $x_i = x_n$ if $i > n$.

Given an ordered set $V = \{x_0, x_1, \ldots, x_n\}$, a subset $Y \subseteq V$ is called a Gale subset if between any two elements of $V \setminus Y$ there is an even number of elements of $Y$. A polytope $P$ with ordered vertex set $V$ is a Gale polytope if the set of vertices of each facet is a Gale subset.

Definition 5 ([7]) An ordinary polytope is a Gale polytope such that each facet is a multiplex with the induced order on the vertices.

Cyclic polytopes can be characterized as the simplicial Gale polytopes. Thus the only simplicial ordinary polytopes are cyclics. In fact, these are the only ordinary polytopes in even dimensions. However, the odd-dimensional, nonsimplicial ordinary polytopes are quite interesting.

We use the following notational conventions. Vertices are generally denoted by integers $i$ rather than by $x_i$. Where it does not cause confusion, a face of a polytope or a triangulation is identified with its vertex set. Interval notation is used to denote sets of consecutive integers, $[a,b] = \{a, a+1, \ldots, b-1, b\}$.

2 Triangulating the multiplex

Multiplexes have minimal triangulations that are particularly easy to describe.

Theorem 2.1 Let $M_{d,n}$ be a multiplex with ordered vertices $0, 1, \ldots, n$. For $0 \leq i \leq n - d$, let $T_i$ be the convex hull of $[i,i+d]$. Then $M_{d,n}$ has a shallow triangulation as the union of the $n - d + 1$ $d$-simplices $T_i$. 


Proof: The proof is by induction on \( n \). For \( n = d \), the multiplex \( M^{d,d} \) is the simplex \( T_0 \) itself. Assume \( M^{d,n} \) has triangulation into simplices \( T_i \), \( 0 \leq i \leq n - d \). Consider the multiplex \( M^{d,n+1} \) with ordered vertices 0, 1, \( \ldots, n \). Then \( M^{d,n+1} = \text{conv}(M^{d,n} \cup \{n+1\}) \), where \( n+1 \) is a point beyond facet \( F_n \) of \( M^{d,n} \), beneath the facets \( F_i \) for \( 0 \leq i \leq n-d+1 \), and in the affine hulls of the facets \( F_i \) for \( n-d+2 \leq i \leq n-1 \). (See [3].) Thus, \( M^{d,n+1} \) is the union of \( M^{d,n} \) and \( \text{conv}(F_n \cup \{n+1\}) = T_{n+1-d} \), and \( M^{d,n} \cap T_{n+1-d} = F_n \). By the induction assumption, the simplices \( T_i \), with \( 0 \leq i \leq n+1-d \), form a triangulation of \( M^{d,n+1} \).

The dual graph of the triangulation is simply a path. (The dual graph is the graph having a vertex for each \( d \)-simplex, and an edge between two vertices if the corresponding \( d \)-simplices share a \((d-1)\)-face.) The ordering \( T_0, T_1, T_2, \ldots, T_{n-d} \) is a shelling of the simplicial complex that triangulates \( M^{d,n} \). So the \( h \)-vector of the triangulation is \((1, n-d, 0, 0, \ldots)\). This is the same as the \( g \)-vector of the boundary of the multiplex, which is the \( h \)-vector of the solid multiplex. So by [2], the triangulation is shallow. \( \square \)

Note, however, that for \( n \geq d+2 \), \( M^{d,n} \) is not weakly neighborly (as observed in [3]). This means that it has nonshallow triangulations. This is easy to see because the vertices 0 and \( n \) are not contained in a common proper face of \( M^{d,n} \).

Consider the induced triangulation of the boundary of \( M^{d,n} \). For notational purposes we consider \( T_0 \) and \( T_n \) separately. All facets of \( T_0 \) except \([1, d]\) are boundary facets of \( M^{d,n} \). Write \( T_{0,0} = [0, d-1] = F_0 \), and \( T_{0,j} = [0, d \setminus \{j\}] \) for \( 1 \leq j \leq d-1 \). Write \( T_{n-d,n} = [n-d+1, n] = F_n \), and \( T_{n,j} = [n-d, n \setminus \{j\}] \) for \( n-d+1 \leq j \leq n-1 \). For \( 1 \leq i \leq n-d-1 \), the facets of \( T_i \) are \( T_{i,j} = [i, i+d] \setminus \{j\} \). Two of these facets \((j = i \text{ and } j = i+d)\) intersect the interior of \( M^{d,n} \). For \( 1 \leq j \leq n-1 \), the facet \( F_j \) is triangulated by \( T_{i,j} \) for \( j-d+1 \leq i \leq j-1 \) (and \( 0 \leq i \leq n-d \)). The facet order \( F_0, F_1, \ldots, F_n \), is a shelling of the multiplex \( M^{d,n} \). The \((d-1)\)-simplices \( T_{i,j} \) in the order \( T_{0,0}, T_{0,1}, T_{0,2}, T_{1,2}, \ldots, T_{n-d-1,n-2}, T_{n-1,n-2}, T_{n-2,n-2}, T_{n-d+1,n} \) (increasing order of \( j \) and, for each \( j \), increasing order of \( i \)), form a shelling of the triangulated boundary of \( M^{d,n} \).

3 Shelling the ordinary polytope

Shelling is used to calculate the \( h \)-vector, and hence the \( f \)-vector of simplicial complexes (in particular, the boundaries of simplicial polytopes). This is possible because (1) the \( h \)-vector has a simple expression in terms of the
f-vector and vice versa; (2) in a shelling of a simplicial complex, among the faces added to the subcomplex as a new facet is shelled on, there is a unique minimal face; (3) the interval from this minimal new face to the facet is a Boolean algebra; and (4) the numbers of new faces given by (3) match the coefficients in the f-vector/h-vector formula. These conditions all fail for shellings of arbitrary polytopes. However, some hold for certain shellings of ordinary polytopes.

As mentioned earlier, noncyclic ordinary polytopes exist only in odd dimensions. Furthermore, three-dimensional ordinary polytopes are quite different combinatorially from those in higher dimensions. We thus restrict our attention to ordinary polytopes of odd dimension at least five. It turns out that these are classified by the vertex figure of the first vertex.

**Theorem 3.1** ([7, 9]) For each choice of integers \( n \geq k \geq d = 2m + 1 \geq 5 \), there is a unique combinatorial type of ordinary polytope \( P = P_{d,k,n} \) such that the dimension of \( P \) is \( d \), \( P \) has \( n + 1 \) vertices, and the first vertex of \( P \) is on exactly \( k \) edges. The vertex figure of the first vertex of \( P_{d,k,n} \) is the cyclic \((d-1)\)-polytope with \( k \) vertices.

We use the following description of the facets of \( P_{d,k,n} \) by Dinh [9]. For any subset \( X \subseteq \mathbb{Z} \), let \( \text{ret}_n(X) \) (the “retraction” of \( X \)) be the set obtained from \( X \) by replacing every negative element by 0 and replacing every element greater than \( n \) by \( n \). Let \( X_n \) be the collection of sets

\[
X = [i, i + 2r - 1] \cup Y \cup [i + k, i + k + 2r - 1],
\]

where \( i \in \mathbb{Z} \), \( 1 \leq r \leq m \), \( Y \) is a paired \((d - 2r - 1)\)-element subset of \([i + 2r + 1, i + k - 2]\), and \( |\text{ret}_n(X)| \geq d \). Dinh’s theorem says that \( \text{ret}_n(X_n) = \mathcal{F}(P_{d,k,n}) \), the set of facets of \( P_{d,k,n} \). It is easy to check that when \( n = k \), \( |\text{ret}_n(X)| = d \) for all \( X \in X_n \), and that \( \text{ret}_n(X_n) \) is the set of \( d \)-element Gale subsets of \([0, k]\), that is, the facets of the cyclic polytope \( P_{d,k,k} \).

Note that \( X_{n-1} \subseteq X_n \). We wish to describe \( \mathcal{F}(P_{d,k,n}) \) in terms of \( \mathcal{F}(P_{d,k,n-1}) \); for this we need the following shift operations. If \( F = \text{ret}_{n-1}(X) \in \mathcal{F}(P_{d,k,n-1}) \), let the right-shift of \( F \) be \( \text{rsh}(F) = \text{ret}_n(X + 1) \). Typically, \( \text{rsh}(F) \) contains \( F + 1 \). If it does not, then \( 0 \in F \setminus X \) and \( \text{rsh}(F) = ((F + 1) \setminus \{1\}) \cup \{0\} \). In either case, \( |\text{rsh}(F)| \geq |F| \geq d \). If \( F = \text{ret}_n(X) \in \mathcal{F}(P_{d,k,n}) \), let the left-shift of \( F \) be \( \text{lsh}(F) = \text{ret}_{n-1}(X - 1) \). Note that \( \text{lsh}(F) \setminus \{0\} = (F - 1) \cap [1, n] \); \( \text{lsh}(F) \) contains 0 if either 0 or 1 is in \( F \).

**Lemma 3.2** If \( n \geq k + 1 \) and \( F \in \mathcal{F}(P_{d,k,n}) \) with \( \max F \geq k \), then \( \text{lsh}(F) \in \mathcal{F}(P_{d,k,n-1}) \).
**Proof:** Let \( F = \text{ret}_n(X) \), with \( X = [i, i + 2r - 1] \cup Y \cup [i + k, i + k + 2r - 1] \). Then \( X - 1 \) also has the form of equation (11) (for \( i - 1 \)). The set \( \text{lsh}(F) \) is the vertex set of a facet of \( P^{d,k,n-1} \) as long as \( |\text{lsh}(F)| \geq d \). We check this in three cases.

**Case 1.** If \( k \leq i + k + 2r - 1 \leq n \), then \( i + 2r - 1 \geq 0 \), so \( Y \subseteq [i + 2r + 1, i + k - 2] \subseteq [2, i + k - 2] \). Then

\[
\text{lsh}(F) \supseteq \max\{i + 2r - 2, 0\} \cup (Y - 1) \cup [i + k - 1, i + k + 2r - 2],
\]

so \( |\text{lsh}(F)| \geq 1 + (d - 2r - 1) + 2r = d \).

**Case 2.** If \( i + k \geq n \), then \( i \geq n - k \geq 1 \). Also, \( |F| \geq d \) implies \( \max Y \leq n - 1 \). So

\[
\text{lsh}(F) = [i - 1, i + 2r - 2] \cup (Y - 1) \cup \{n - 1\},
\]

so \( |\text{lsh}(F)| = 2r + (d - 2r - 1) + 1 = d \).

**Case 3.** If \( i + k < n < i + k + 2r - 1 \), then \( i + 2r - 1 \geq n - k \geq 1 \), and

\[
F = [\max\{0, i\}, i + 2r - 1] \cup Y \cup [i + k, n],
\]

so

\[
|F| = (i + 2r - \max\{0, i\}) + (d - 2r - 1) + (n - i - k + 1)
\]

\[
= d + n - k - \max\{i, 0\} \geq d + 1.
\]

Then \( |\text{lsh}(F)| \geq |F| - 1 \geq d \).

Thus, \( \text{lsh}(F) \) is a facet of \( P^{d,k,n-1} \). \( \square \)

Identify each facet of the ordinary polytope \( P^{d,k,n} \) with its ordered list of vertices. Then order the facets of \( P^{d,k,n} \) in colex order. This means, if \( F = i_1 i_2 \ldots i_p \) and \( G = j_1 j_2 \ldots j_q \), then \( F \prec_c G \) if and only if for some \( t \geq 0 \), \( i_{p-t} < j_{q-t} \) while for \( 0 \leq s < t \), \( i_{p-s} = j_{q-s} \).

**Lemma 3.3** If \( n \geq k + 1 \) and \( F_1 \) and \( F_2 \) are facets of \( P^{d,k,n} \) with \( \max F_i \geq k \), then \( F_1 \prec_c F_2 \) implies \( \text{lsh}(F_1) \prec_c \text{lsh}(F_2) \).

**Proof:** Suppose \( F_1 \prec_c F_2 \), and let \( q \) be the maximum vertex in \( F_2 \) not in \( F_1 \). Then \( \text{lsh}(F_1) \prec_c \text{lsh}(F_2) \) as long as \( q \geq 2 \), for in that case \( q - 1 \in \text{lsh}(F_2) \setminus \text{lsh}(F_1) \), while \( [q, n - 1] \setminus \text{lsh}(F_1) = [q, n - 1] \cap \text{lsh}(F_2) \). (If \( q = 1 \), then \( q \) shifts to 0 in \( \text{lsh}(F_2) \), but 0 may be in \( \text{lsh}(F_1) \) as a shift of a smaller element.) So we prove \( q \geq 2 \). Write

\[
F_2 = \text{ret}_n([i, i + 2r - 1] \cup Y \cup [i + k, i + k + 2r - 1])
\]
and
\[ F_1 = \text{ret}_n ([i', i' + 2r' - 1] \cup Y' \cup [i' + k, i' + k + 2r' - 1]). \]

Since \( \max F_2 \geq k, i + 2r - 1 \geq 0 \), so \( Y \cup [i + k, i + k + 2r - 1] \subseteq [2, n] \). Thus, if \( q \in Y \cup [i + k, i + k + 2r - 1] \), then \( q \geq 2 \). Otherwise \( Y \cup [i + k, i + k + 2r - 1] = Y' \cup [i' + k, i' + k + 2r' - 1] \), but \( Y \neq Y' \). This can only happen when \( Y \cup [i + k, i + k + 2r - 1] \) is an interval; in this case \( i + k + 2r - 1 \geq n + 1 \). Then \( q = i + 2r - 1 = (i + k + 2r - 1) - k \geq n + 1 - k \geq 2 \).

\[ \Box \]

**Proposition 3.4** Let \( n \geq k + 1 \). The facets of \( P_{d,k,n} \) are
\[
\{ F : F \in \mathcal{F}(P_{d,k,n-1}) \text{ and max } F \leq n - 2 \}
\]
\[ \cup \{\text{rsh}(F) : F \in \mathcal{F}(P_{d,k,n-1}) \text{ and max } F \geq n - 2 \}. \]

**Proof:** If \( \max X \leq n - 2 \), then \( \text{ret}_n(X) = \text{ret}_{n-1}(X) \); in this case, letting \( F = \text{ret}_n(X) \), \( F \in \mathcal{F}(P_{d,k,n-1}) \) if and only if \( F \in \mathcal{F}(P_{d,k,n}) \). If \( F \in \mathcal{F}(P_{d,k,n-1}) \) with \( \max F \geq n - 2 \), then \( \text{rsh}(F) \in \mathcal{F}(P_{d,k,n}) \) with \( \max \text{rsh}(F) \geq n - 1 \). Now suppose that \( G = \text{ret}_n(X) \in \mathcal{F}(P_{d,k,n}) \) with \( \max G \geq n - 1 \). Let \( F = \text{lsh}(G) = \text{ret}_{n-1}(X - 1) \in \mathcal{F}(P_{d,k,n-1}) \); then \( \max F \geq n - 2 \). By definition, \( \text{rsh}(F) = \text{ret}_n((X - 1) + 1) = \text{ret}_n(X) = G \).

\[ \Box \]

**Theorem 3.5** Let \( F_1, F_2, \ldots, F_v \) be the facets of \( P_{d,k,n} \) in colex order. Then

1. \( F_1, F_2, \ldots, F_v \) is a shelling of \( P_{d,k,n} \).
2. For each \( j \) there is a unique minimal face \( G_j \) of \( F_j \) not contained in \( \cup_{i=1}^{j-1} F_i \).
3. For each \( j, 2 \leq j \leq v - 1 \), \( G_j \) contains the vertex of \( F_j \) of maximum index, and is contained in the \( d - 1 \) highest vertices of \( F_j \).
4. For each \( j \), the face \( G_j \) is a simplex and the interval \([G_j, F_j]\) is a Boolean lattice.

Note that this theorem is not saying that the faces of \( P_{d,k,n} \) in the interval \([G_j, F_j]\) are all simplices.

**Proof:** We construct explicitly the faces \( G_j \) in terms of \( F_j \). The reader may wish to refer to the example that follows the proof.
Cyclic polytopes. We start with the cyclic polytopes. (For the cyclics, the theorem is generally known, or at least a shorter proof based on [5] is possible, but we will need the description of the faces $G_j$ later.)

Let $F_1$, $F_2$, ..., $F_v$ be the facets, in colex order, of $P^{d,k,n}$, the cyclic $d$-polytope with vertex set $[0,k]$. Each facet $F_j$ can be written as $F_j = I^0_j \cup I^1_j \cup I^2_j \cup \cdots \cup I^p_j \cup I^k_j$, where $I^0_j$ is the interval of $F_j$ containing $0$, if $0 \in F_j$, and $I^0_j = \emptyset$ otherwise; $I^k_j$ is the interval of $F_j$ containing $k$, if $k \in F_j$, and $I^k_j = \emptyset$ otherwise; and the $I^\ell_j$ are the other (even) intervals of $F_j$ with the elements of $I^\ell_j$ preceding the elements of $I^{\ell+1}_j$. (For example, in $P^{7,9,9}$, $F_0 = \{0,1,2,4,5,7,8\}$, $I^0_0 = \{0,1,2\}$, $I^1_0 = \{4,5\}$, $I^2_0 = \{7,8\}$, and $I^3_0 = \emptyset$.) For the interval $[a,b]$, write $E([a,b])$ for the integers in the even positions in the interval, that is, $E([a,b]) = [a,b] \cap \{a+2i+1 : i \in \mathbb{N}\}$. Let $G_j = \bigcup_{\ell=0}^{p-1} E(I^\ell_j) \cup I^k_j$. Since $I^0_j = F_j$ if and only if $j = 1$, $G_1 = \emptyset$, and for all $j > 1$, $G_j$ contains the maximum vertex of $F_j$. Since $F_j$ is a simplex, $[G_j,F_j]$ is a Boolean lattice.

To show that $F_1$, $F_2$, ..., $F_v$ is a shelling of $P^{d,k,n}$ we show that $G_j$ is not in a facet before $F_j$ and that every ridge of $P^{d,k,n}$ in $F_j$ that does not contain $G_j$ is contained in a previous facet. For $j > 0$ the face $G_j$ consists of the right end-set $I^k_j$ (if nonempty) and the set $\bigcup_{\ell=1}^p E(I^\ell_j)$ of singletons. Note that $G_j$ is contained in the $d-1$ highest vertices of $F_j$ (equivalently, $G_j$ does not contain the minimum element of $F_j$) unless $j = v$, in which case $G_v = F_v$. Any facet $F$ of $P^{d,k,n}$ containing $G_j$ must satisfy Gale’s evenness condition and therefore must contain an integer adjacent to each element of $\bigcup_{\ell=1}^p E(I^\ell_j)$. If any element of the form $I^\ell_j + 1$ is in $F$, then $F$ occurs after $F_j$ in colex order. This implies that any $F_i$ previous to $F_j$ and containing $G_j$ also contains $\bigcup_{\ell=1}^p I^\ell_j \cup I^k_j$. But $F_j$ is the first facet in colex order that contains $\bigcup_{\ell=1}^p I^\ell_j \cup I^k_j$. So $G_j$ is not in a facet before $F_j$.

Now let $g \in G_j$; we wish to show that $F_j \setminus \{g\}$ is in a previous facet. If $g \in E(I^\ell_j)$ for $\ell > 0$, let $F = F_j \setminus \{g\} \cup \{\min I^\ell_j - 1\}$. Then $F$ satisfies Gale’s evenness condition and is a facet before $F_j$. Otherwise $g \in I^k_j \setminus E(I^\ell_j)$; in this case let $F = F_j \setminus \{g\} \cup \{\max I^0_j + 1\}$ (where we let $\max I^0_j + 1 = 0$ if $I^0_j = \emptyset$). Again $F$ satisfies Gale’s evenness condition and is a facet before $F_j$.

Thus the colex order of facets is a shelling order for the cyclic polytope $P^{d,k,n}$, and we have an explicit description for the minimal new face $G_j$ as $F_j$ is shelled on.

General ordinary. Now we prove the theorem for general $P^{d,k,n}$ by induction on $n \geq k$, for fixed $k$. Among the facets of $P^{d,k,n}$, first in colex order are those with maximum vertex at most $n-2$. These are also the
first facets in colex order of $P^{d,k,n-1}$. Thus the induction hypothesis gives us that this initial segment is a partial shelling of $P^{d,k,n}$, and that assertions 2–4 hold for these facets.

**Later facets.** It remains to consider the facets of $P^{d,k,n}$ ending in $n-1$ or $n$. These facets come from shifting facets of $P^{d,k,n-1}$ ending in $n-2$ or $n-1$. Our strategy here will be to prove statement 2 of the theorem for these facets. The intersection of $F_j$ with $\cup_{i=1}^{j-1} F_i$ is then the antistar of $G_j$ in $F_j$, and so it is the union of $(d-2)$-faces that form an initial segment of a shelling of $F_j$. This will prove that the colex order $F_1, F_2, \ldots, F_v$ is a shelling of $P^{d,k,n}$.

Note that there is nothing to show for the last facet of $P^{d,k,n}$ in colex order. It is $F_v = [n-d+1,n]$, and is the only facet (other than the first) whose vertex set forms a single interval. Assume from now on that $j$ is fixed, with $j \leq v-1$. Later we will describe recursively the minimal new face $G_j$ as $F_j$ is shelled on. It will always be the case that $\max F_j \in G_j$. We will prove that $G_j$ is truly a new face (is not contained in a previous facet), and that every ridge not containing all of $G_j$ is contained in a previous facet.

**Ridges not containing the last vertex.** It is convenient to start by showing that every ridge of $P^{d,k,n}$ contained in $F_j$ and not containing $\max F_j$ is contained in an earlier facet. This case does not use the recursion needed for the other parts of the proof. Write

$$X = [i, i+2r-1] \cup Y \cup [i+k, i+k+2r-1]$$

and $F_j = \mathrm{ret}_n(X) = \{z_1, z_2, \ldots, z_p\}$ with $0 \leq z_1 < z_2 < \cdots < z_p \leq n$. The facet $F_j$ is a $(d-1)$-multiplex, so its facets are of the form

$$F_j(\hat{z}_t) = \{z_\ell : 1 \leq \ell \leq p, 0 < |\ell-t| \leq d-2\}$$

for $2 \leq t < p-1$, $F_j(\hat{z}_1) = \{z_1, z_2, \ldots, z_d-1\}$, and $F_j(\hat{z}_p) = \{z_{p-d+2}, \ldots, z_{p-1}, z_p\}$. If $F_j(\hat{z}_t)$ does not contain $\max F_j = z_p$, then $t \leq p-d+1$ and this implies $i \leq z_t \leq i+2r-1$. Consider such a $z_t$.

**The first ridge.** For $t = 1$, there are three cases to consider.

**Case 1.** Suppose $z_1 \geq 1$. Then $F_j(\hat{z}_1) = [i, i+2r-1] \cup Y$. Let $I$ be the right-most interval of $F_j(\hat{z}_1)$. Let $Z = (I - k) \cup F_j(\hat{z}_1)$, and $F = \mathrm{ret}_n(Z)$. Since $i \geq 1$ and $\max F_j(\hat{z}_1) \leq i+k-2$, the interval $I - k$ contributes at least one new element to $F$, so $|F| \geq d$.

**Case 2.** Suppose $z_1 = 0$ and the right-most interval of $F_j(\hat{z}_1)$ is odd. In this case the left-most interval of $F_j$ must also be odd, so $i < 0$, and $F_j(\hat{z}_1)$ contains $i+k$ but not $i+k-1$. Let $F = F_j(\hat{z}_1) \cup \{i+k-1\}$.
Case 3. Suppose $z_t = 0$ and the right-most interval of $F_j(\hat{z}_t)$ is even (and then so is the left-most interval). Then $F_j(\hat{z}_t) = [0, i + 2r - 1] \cup Y \cup [i + k, k - 1]$ (where the last interval is empty if $i = 0$). Let

$$F = F_j(\hat{z}_t) \cup \{i + 2r\} = \{0\} \cup [1, i + 2r] \cup Y \cup [i + k, k - 1].$$

(When $i = 0$ and $r = (d - 1)/2$, this gives $F = [0, d - 1]$.)

In all cases $F$ is a facet of $P^{d,k,n}$ containing $F_j(\hat{z}_t)$. It does not contain max $F_j$, so $F \prec_c F_j$.

Deleting a later vertex. Now assume $2 \leq t \leq p - d + 1$; then $z_t \geq \max\{i + 1, 1\}$. Here

$$F_j(\hat{z}_t) = \max\{i, 0\}, z_t - 1\} \cup [z_{t+1}, i + 2r - 1] \cup Y \cup [i + k, z_t - 1 + k],$$

and $|F_j(\hat{z}_t)| = z_t - \max\{i, 0\} + d - 2 \geq d - 1$. Also note that $z_t - 1 + k$ is the $(d - 2)$nd element of $\{z_1, z_2, \ldots, z_p\}$ after $z_t$, so $z_t - 1 + k = z_{t+d-2} < z_p = \max F_j$.

Case 1. If $z_t - i$ is even, let $F = F_j(\hat{z}_t) \cup \{i + 2r\}$. Then $F = \ret_n(Z)$, where

$$Z = [i, z_t - 1] \cup [z_t + 1, i + 2r] \cup Y \cup [i + k, z_t - 1 + k],$$

and $|F| \geq d$.

Case 2. If $z_t - i$ is odd and $\max([i, i + 2r - 1] \cup Y) < i + k - 2$, let $F = \ret_n(Z)$, where

$$Z = [i - 1, z_t - 1] \cup [z_t + 1, i + 2r - 1] \cup Y \cup [i + k - 1, z_t - 1 + k].$$

Then $F \supseteq F_j(\hat{z}_t) \cup \{i + k - 1\}$, so $|F| \geq d$.

Case 3. Finally, suppose $z_t - i$ is odd and $\max Y = i + k - 2$. Let $[q, i + k - 2]$ be the right-most interval of $Y$, and let $F = \ret_n(Z)$, where

$$Z = [q - k, z_t - 1] \cup [z_t + 1, i + 2r - 1] \cup (Y \setminus [q, i + k - 2]) \cup [q, z_t - 1 + k].$$

Then $F \supseteq F_j(\hat{z}_t) \cup \{i + k - 1\}$, so $|F| \geq d$.

In all cases, $F$ is a facet of $P^{d,k,n}$ containing $F_j(\hat{z}_t)$ and max $F_j \not\subseteq F$, so $F$ occurs before $F_j$ in colex order.

Determining the minimal new face. We now describe the faces $G_j$ recursively. (We are still assuming that $\max F_j \geq n - 1$.) Let $G$ be the face of $\lsh(F_j)$ that is the minimal new face when $\lsh(F_j)$ is shelled on, in the colex shelling of the polytope $P^{d,k,n-1}$. Let $G_j = G + 1$; then $G_j$ contains $\max F_j$. Also, $G_j$ is a subset of the last $d - 1$ vertices of $F_j$, which form a facet of $F_j$; hence $G_j$ is a face of $F_j$. For any facet $F_i$ of $P^{d,k,n}$, $G_j \subseteq F_i$ if
and only if \( G \subseteq \operatorname{lsh}(F) \). So by the induction hypothesis, \( G_j \) is not contained in a facet occurring before \( F_j \) in colex order.

**Ridges in previous facets.** It remains to show that any ridge of \( P_{d,k,n} \) contained in \( F_j \) but not containing all of \( G_j \) is contained in a facet prior to \( F_j \). Note that we have already dealt with those ridges not containing \( \max(F_j) \).

Now let \( g \in G \), \( g_j = g + 1 \in G_j \), and assume \( g_j \neq \max(F_j) \). The only ridge of \( P_{d,k,n} \) contained in \( F_j \), containing \( \max(F_j) \), and not containing \( g_j \) is \( F_j(\hat{g}_j) \).

Let \( H \) be the unique ridge of \( P_{d,k,n-1} \) in \( \operatorname{lsh}(F_j) \) containing \( \max(\operatorname{lsh}(F_j)) \), but not containing \( g \). By the induction hypothesis, \( H \) is contained in a facet \( F \) of \( P_{d,k,n-1} \) occurring before \( \operatorname{lsh}(F_j) \) in colex order. Suppose \( F_j(\hat{g}_j) \) is contained in a facet \( F_\ell \) of \( P_{d,k,n} \) occurring after \( F_j \) in colex order. Then \( H \) is contained in \( \operatorname{lsh}(F_\ell) \). Thus the ridge \( H \) of \( P_{d,k,n-1} \) is contained in three different facets: \( F \) (occurring before \( \operatorname{lsh}(F_j) \) in colex order), \( \operatorname{lsh}(F_j) \), and \( \operatorname{lsh}(F_\ell) \) (occurring after \( \operatorname{lsh}(F_j) \) in colex order). This contradiction shows that the ridge \( F_j(\hat{g}_j) \) can only be contained in a facet of \( P_{d,k,n} \) occurring before \( F_j \) in colex order.

**Boolean intervals.** Finally to verify assertion 4 of the theorem, observe that every facet \( F_j \) is a \((d-1)\)-dimensional multiplex. The vertices of \( G_j \) are among the last \( d \) vertices of \( F_j \) and so are affinely independent \[6\]; thus \( G_j \) is a simplex. The maximum vertex \( u \) of \( F_j \) is contained in \( G_j \). The vertex figure of the maximum vertex in any multiplex is a simplex \[6\]. So the interval \([G_j, F_j]\), which is an interval in \([u, F_j]\), is a Boolean lattice. \( \blacksquare \)

A nonrecursive description of the faces \( G_j \), generalizing that for the cyclic case in the proof, is as follows. Write the facet \( F_j \) as a disjoint union,

\[
F_j = A_j^0 \cup I_j^1 \cup I_j^2 \cup \cdots \cup I_j^p \cup I_j^n,
\]

where \( I_j^n \) is the interval of \( F_j \) containing \( n \) if \( n \in F_j \), and \( I_j^n = \emptyset \) otherwise; the \( I_j^\ell \) \((1 \leq \ell \leq p)\) are even intervals of \( F_j \) written in increasing order; and \( A_j^0 \) is

- the interval containing 0, if \( \max(F_j) \leq k - 1 \);
- the union of the interval containing \( \max(F_j - k) \) and the interval containing \( \max(F_j - k + 2) \) (if the latter exists), if \( k \leq \max(F_j) \leq n - 1 \);
- the interval containing \( n - k \), if \( \max(F_j) = n \) and \( n - k \in F_j \);
- \( \emptyset \), if \( \max(F_j) = n \) and \( n - k \not\in F_j \).

Then \( G_j = \bigcup_{\ell=1}^p E(I_j^\ell) \cup I_j^n \).

**Example.** Table\[7\] gives the faces \( F_j \) and \( G_j \) for the colex shelling of the ordinary polytope \( P_{5,6,8} \).
Let us look at what happens when facet $F_{13}$ is shelled on. The ridges of $P^{5,6,8}$ contained in $F_{13}$ are 0123, 0236, 01367, 012678, 12378, 2368, and 3678. The first ridge, 0123, is contained in $F_1 = 01234$. The ridge 0236 is $F_{13}(\hat{z}_2) = F_{13}(\hat{1})$, and $\text{max}(\{i, i + 2r - 1\} \cup Y) = 3 < 4 = i + k - 2$, so we find that 0236 is contained in $F_4 = 02356$. The ridge 01367 is $F_{13}(\hat{z}_3) = F_{13}(\hat{2})$, so we find that 01367 is contained in $F_6 = 013467$. This facet $F_{13} = 0123678$ is shifted from the facet 012567 of $P^{5,6,7}$, which in turn is shifted from the facet 01456 of the cyclic polytope $P^{5,6,6}$. When 01456 occurs in the shelling of the cyclic polytope, its minimal new face is its right interval, 456. In $P^{5,6,8}$, then, the minimal new face when $F_{13}$ is shelled on is 678. The other ridges of $F_{13}$ not containing 678 are 12378 and 2368. The interval $[G_{13}, F_{13}]$ contains the triangle 678, the 3-simplex 3678, the 3-multiplex 012678, and $F_{13}$ itself (which is a pyramid over 012678).

Note that for the multiplex, $M^{d,n} = P^{d,d,n}$, this theorem gives a shelling different from the one mentioned in Section 2. In the standard notation for the facets of the multiplex (see Definition 2), the colex shelling order is $F_0$, $F_1$, $\ldots$, $F_{n-d}$, $F_{n-1}$, $F_{d-2}$, $\ldots$, $F_{n-d+1}$, $F_n$. The statements of this section hold also for even-dimensional multiplexes.

### 4 The $h$-vector from the shelling

The $h$-vector of a simplicial polytope can be obtained easily from any shelling of the polytope. For $P$ a simplicial polytope, and $\cup [G_j, F_j]$ the partition of a face lattice of $P$ arising from a shelling, $h(P, x) = \sum_j x^{d-\lvert G_j \rvert}$. For general polytopes, the (toric) $h$-vector can also be decomposed according to the
shelling partition. For a shelling, $F_1$, $F_2$, \ldots, $F_n$, of a polytope $P$, write $G_j$ for the set of faces of $F_j$ not in $\cup_{i<j} F_i$. Then $h(P, x) = \sum_{j=1}^n h(G_j, x)$, where $h(G_j, x) = \sum_{G \subseteq G_j} g(G, x)(x-1)^{d-\dim G}$. However, in general we do not know that the coefficients of $h(G_j, x)$ count anything natural, nor even that they are nonnegative. Stanley raised this issue in [14, Section 6]. It has apparently been settled by Tom Braden [8].

We turn now to $h$-vectors of ordinary polytopes. In [3] we used the flag vector of the ordinary polytope to compute its toric $h$-vector.

**Theorem 4.1** ([3]) For $n \geq k \geq d = 2m + 1 \geq 5$ and $1 \leq i \leq m$,

$$h_i(P^{d,k,n}) = \binom{k-d+i}{i} + \binom{k-d+i-1}{i-1}.$$  

We did not understand why the $h$-vector turned out to have such a nice form. Here we show how the $h$-vector can be computed from the colex shelling. Properties 2 and 4 of Theorem 3.5 are critical.

In [3] we showed that the flag vector of a multiplicial polytope depends only on the $f$-vector. However, for our purposes here it is more useful to write the $h$-vector in terms of the $f$-vector and the flag vector entries of the form $f_0i$ (the sum of the number of vertices on the $i$-faces) We introduce a modified $f$-vector. Let $\bar{f}_1 = f_1 = 1$, $\bar{f}_0 = f_0$, and $\bar{f}_{d-1} = f_{d-1} + (f_{0,d-1} - df_{d-1})$; and for $1 \leq j \leq d - 2$, let

$$\bar{f}_j = f_j + (f_{0,j+1} - (j + 2)f_{j+1}) + (f_{0,j} - (j + 1)f_j).$$

**Theorem 4.2** If $P$ is a multiplicial $d$-polytope, then

$$h(P, x) = \sum_{i=0}^{d} h_i(P)x^{d-i} = \sum_{i=0}^{d} \bar{f}_{i-1}(P)(x-1)^{d-i}.$$  

**Proof:** As observed in the proof of Theorem 2.1, the $g$-polynomial of an $e$-dimensional multiplex $M$ with $n+1$ vertices is $g(M, x) = 1 + (n-e)x$. So for a multiplicial $d$-polytope $P$,

$$h(P, x) = \sum_{\substack{G \text{ face of } P \\ G \neq P}} g(G, x)(x-1)^{d-\dim G}$$

$$= \sum_{\substack{G \text{ face of } P \\ G \neq P}} (1 + (f_0(G) - 1 - \dim G)x)(x-1)^{d-\dim G}$$
the colex shelling of \( P \cup P \) has exactly \( P \) partition of the face lattice of \( i \). All faces of \( i \).

**Proof:** Direct evaluation gives \( f(P) = f(P) \) when \( h \) entries in the \( \{i\} \) case of ordinary polytopes something special happens. Write \( h'(P,x) = \sum_{i=0}^{d} h_i(P)x^{d-i} \) for the \( h \)-polynomial that \( P \) would have if it were simplicial, by letting \( h'_i(P) \) be the coefficient of \( x^{d-i} \) in \( \sum_{i=0}^{d} f_i(P)(x-1)^{d-i} \).

\[
\begin{align*}
&= \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} + \sum_{i=1}^{d-1} (f_{0i} - (i + 1)f_i)x(x-1)^{d-1-i} \\
&= \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} + \sum_{i=1}^{d-1} (f_{0i} - (i + 1)f_i)(x-1)^{d-i} + (x-1)^{d-1-i} \\
&= (x-1)^d + f_0(x-1)^{d-1} \\
&+ \sum_{i=2}^{d-1} (f_{i-1} + (f_{0i} - (i + 1)f_i) + (f_{0,i-1} - if_{i-1}))((x-1)^{d-i} \\
&+ (f_{d-1} + (f_{0,d-1} - df_{d-1})) \\
&= \sum_{i=0}^{d} \hat{f}_{i-1}(P)(x-1)^{d-i}.
\end{align*}
\]

\[\square\]

Simplicial polytopes are a special case of multiplicial polytopes. Clearly, when \( P \) is simplicial, \( f(P) = f(P) \), and we recover the definition of the simplicial \( h \)-vector in terms of the \( f \)-vector. The multiplicial \( h \)-vector formula can be thought of as breaking into two parts: one involving the \( f \)-vector, and matching the simplicial \( h \)-vector formula; the other involving the “excess vertex counts,” \( f_{0,j} - (j + 1)f_j \). In the simplicial case the sum of the entries in the \( h \)-vector is the number of facets. For multiplicial polytopes \( \sum_{i=0}^{d} h_i(P) = \hat{f}_{d-1}(P) = f_{d-1} + (f_{0,d-1} - df_{d-1}) \).

In general, applying the simplicial \( h \)-formula to a nonsimplicial \( f \)-vector produces a vector with no (known) combinatorial interpretation. This vector is neither symmetric nor nonnegative in general. We will see that in the case of ordinary polytopes something special happens. Write \( h'(P,x) = \sum_{i=0}^{d} h'_i(P)x^{d-i} \) for the \( h \)-polynomial that \( P \) would have if it were simplicial, by letting \( h'_i(P) \) be the coefficient of \( x^{d-i} \) in \( \sum_{i=0}^{d} f_i(P)(x-1)^{d-i} \).

**Theorem 4.3** Let \( P^{d,k,n} \) be an ordinary polytope. Let \( \cup_{j=1}^{n} [G_j, F_j] \) be the partition of the face lattice of \( P^{d,k,n} \) associated with the colex shelling of \( P^{d,k,n} \). Then for all \( i, 0 \leq i \leq d \), \( h'(P^{d,k,n}, x) = \sum_{j=1}^{n} x^{d-|G_j|} \).

Furthermore, if \( C^{d,k} \) is the cyclic \( d \)-polytope with \( k + 1 \) vertices, then for all \( i, 0 \leq i \leq d \), \( h'_i(P^{d,k,n}) \geq h_i(C^{d,k}) \), with equality for \( i > d/2 \).

**Proof:** Direct evaluation gives \( h_0(P) = h'_0(P) = 1 \). Let \( F_1, F_2, \ldots, F_v \) be the colex shelling of \( P^{d,k,n} \). By Theorem 3.5 part 2, the set of faces of \( P^{d,k,n} \) has a partition as \( \cup_{j=1}^{n} [G_j, F_j] \). By Theorem 3.5 part 4, the interval \([G_j, F_j]\) has exactly \( \binom{d-1-\dim G_j}{\dim G_j} \) faces of dimension \( \ell \) for \( \dim G_j \leq \ell \leq d - 1 \). Let

\[\binom{d-1-\dim G_j}{\dim G_j} \]

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So $k_i = |\{j : \dim G_j = i - 1\}| = |\{j : |G_j| = i\}|$ (since all the $G_j$ are simplices).

Then $f_\ell = \sum_{i=0}^{d-\ell} \binom{d-i}{i} k_i$. These are the (invertible) equations that give $f_\ell$
in terms of $h'_i$, so for all $i$, $h'_i = k_i = |\{j : |G_j| = i\}|$.

The second part we prove by induction on $n \geq k$. Recall from the proof of Theorem 3.5 that for each facet $F_j$ of $P^{d,k,n}$, $G_j$ is the same size as the minimum new face $G$ of the corresponding facet of $P^{d,k,n-1}$; that facet is the same (as vertex set) as $F_j$, if max $F_j \leq n - 2$, and is $\text{ls}h(F_j)$, if max $F_j \geq n - 1$. From Proposition 3.3 we see that each facet of $P^{d,k,n-1}$
with maximum vertex $n - 2$ gives rise to two facets of $P^{d,k,n}$, while all others give rise to exactly one facet each. So for all $i$,

$$h'_i(P^{d,k,n}) = h'_i(P^{d,k,n-1})$$

+ |\{j : F_j is a facet of $P^{d,k,n}$ with max $F_j = n - 1 and |G_j| = i\}|.

Thus, $h'_i(P^{d,k,n}) \geq h'_i(P^{d,k,n-1})$, so by induction, $h'_i(P^{d,k,n}) \geq h'_i(C^{d,k})$ for all $i$.

Now consider $F_j$ with max $F_j = n - 1$. In the description of the faces $G_j$ following the proof of Theorem 3.5, $F_j = \emptyset$ and $F_j \cap A^0_j \subseteq Y \cup [i + k, i + k + 2r - 1]$. Thus $|G_j| \leq |E(Y \cup [i + k, i + k + 2r - 1])| = (d - 1)/2$. So for $i \geq d/2$, $h'_i(P^{d,k,n}) = h'_i(P^{d,k,n-1}) = h'_i(C^{d,k}).$

Note that for the multiplex $M^{d,n}$ (d odd or even), $h'(M^{d,n}) = (1, n - d + 1, 1, 1, \ldots, 1, 1)$, while $h(M^{d,n}) = (1, n - d + 1, n - d + 1, \ldots, n - d + 1, 1)$.

Now for multiplicial polytopes, we consider the remaining part of the h-vector, coming from the parameters $f_{0,j} - (j+1)f_j$. This is

$$h(P, x) - h'(P, x) = (f_{0,d-1} - df_{d-1}) + \sum_{i=2}^{d-1} ((f_{0,i} - (i+1)f_i) + (f_{0,i-1} - if_{i-1}))(x-1)^{d-i}.$$ 

So

$$h(P, x + 1) - h'(P, x + 1) = (f_{0,d-1} - df_{d-1}) + \sum_{i=2}^{d-1} ((f_{0,i} - (i+1)f_i) + (f_{0,i-1} - if_{i-1}))(x+1)^{d-i}$$

$$= \sum_{i=2}^{d-1} (f_{0,i} - (i+1)f_i)(x+1)x^{d-1-i}.$$ 

So

$$\sum_{i=2}^{d-1} (h_i(P) - h'_i(P))(x + 1)^{d-1-i} = \sum_{i=2}^{d-1} (f_{0,i} - (i+1)f_i)x^{d-1-i}.$$
For the ordinary polytope, this equation can be applied locally to give
the contribution to \( h(P^{d,k,n}, x) - h'(P^{d,k,n}, x) \) from each interval \([G_j, F_j]\) of
the shelling partition. For each \( j \), and each \( i \geq \dim G_j \), let \( b_{j,i} = \sum (f_0(H) - (i + 1)) \), where the sum is over all \( i \)-faces \( H \) in \([G_j, F_j]\). Let
\( b_j(x) = \sum_{i=\dim G_j}^{d-1} b_{j,i} x^{d-1-i} \). Write \( b_j(x) \) in the basis of powers of \((x+1)\):
\( b_j(x) = \sum a_{j,i}(x+1)^{d-1-i} \). Then \( a_{j,i} = h_i(G_j) - h_i'(G_j) \), the contribution to
\( h_i(P^{d,k,n}) - h_i'(P^{d,k,n}) \) from faces in the interval \([G_j, F_j]\). Note that for
fixed \( j \), \( \sum_i a_{j,i} = b_j(0) = f_0(F_j) - d \). We will return to the coefficients \( a_{j,i} \) after
triangulating the ordinary polytope.

**Example.** The \( h \)-vector of \( P^{5,6,8} \) is \( h(P^{5,6,8}) = (1, 4, 7, 7, 4, 1) \). The
sum of the \( h_i \) is 24, which counts the 16 facets plus one for each of the
four 6-vertex facets, plus two for each of the two 7-vertex facets. Referring
to Table 1, we see that \( h'(P^{5,6,8}) = (1, 4, 5, 3, 2, 1) \); from this we compute
\( f(P^{5,6,8}) = (9, 31, 52, 44, 16) \). The nonzero \( a_{j,i} \) here are \( a_{6,2} = a_{7,3} = a_{11,2} = a_{12,3} = 1 \) and \( a_{13,3} = a_{15,4} = 2 \). In this case each interval \([G_j, F_j]\) contributes
to \( h_i(P^{d,k,n}) - h_i'(P^{d,k,n}) \) for at most one \( i \), but this is not true in general.

## 5 Triangulating the ordinary polytope

Triangulations of polytopes or of their boundaries can be used to calculate
the \( h \)-vector of the polytope if the triangulation is shallow [2]. The solid
ordinary polytope need not have a shallow triangulation, but its boundary
does have a shallow triangulation. The triangulation is obtained simply by
triangulating each multiplex as in Section 2. This triangulation is obtained
by pushing the vertices in the order 0, 1, \ldots, \( n \).

**Theorem 5.1** The boundary of the ordinary polytope \( P^{d,k,n} \) has a shallow
triangulation. The facets of one such triangulation are the Gale subsets of
\([i, i + k] \) (where \( 0 \leq i \leq n - k \)) of size \( d \) containing either 0 or \( n \) or the set
\( \{i, i + k\} \).

**Proof:** First we show that each such set is a consecutive subset of some
facet of \( P^{d,k,n} \). Suppose \( Z \) is a Gale subset of \([i, i + k] \) of size \( d \) containing
\( \{i, i + k\} \). Write \( Z = [i, i + a - 1] \cup Y \cup [i + k - b + 1, i + k] \), where \( a \geq 1 \),
\( b \geq 1 \), and \( Y \cap \{i + a, i + k - b\} = \emptyset \). Since \( Z \) is a Gale subset, \(|Y| \) is even;
let \( r = (d - 1 - |Y|)/2 \). Since \(|Z| = d, a + b = 2r + 1 \), so \( a \) and \( b \) are each at
most \( 2r \). Define \( X = [i + a - 2r, i + a - 1] \cup Y \cup [i + k - b + 1, i + k - b + 2r] \).
Note that \( i + k - b + 1 = (i + a - 2r) + k \). Then \( \text{ret}_n(X) \) is the vertex set of
a facet of \( P^{d,k,n} \), and \( Z \) is a consecutive subset of \( \text{ret}_n(X) \).
Now suppose that $Z$ is a Gale subset of $[0,k]$ of size $d$ containing 0, but not $k$. Write $Z = \{0\} \cup Y \cup [j - 2r + 1, j]$, where $j < k$, $r \geq 1$, and $j - 2r \not\in Y$. Then $|Y| = d - 2r - 1$, and $Z = \text{ret}_n(X)$, where $X = [j - 2r + 1 - k, j - k] \cup Y \cup [j - 2r + 1, j]$. So $Z$ itself is the vertex set of a facet of $P^{d,k,n}$. The case of sets containing $n$ but not $n - k$ works the same way.

Next we show that all consecutive $d$-subsets of facets $F$ of $P^{d,k,n}$ are of one of these types. Let $F = \text{ret}_n(X)$, where $X = [i, i + 2r - 1] \cup Y \cup [i + k, i + k + 2r - 1]$, with $Y$ a paired subset of size $d - 2r - 1$ of $[i + 2r + 1, i + k - 2]$. Suppose first that $i + 2r - 1 \geq 0$ and $i + k \leq n$. Let $Z$ be a consecutive $d$-subset of $F$. Since $|Y| = d - 2r - 1$, $|[i, i + 2r - 1] \cap F| \leq 2r$, and $|[i + k, i + k + 2r - 1] \cap F| \leq 2r$, it follows that $i + 2r - 1$ and $i + k$ must both be in $Z$. Thus we can write $Z = [i + 2r - a, i + 2r - 1] \cup Y \cup [i + k, i + k + b - 1]$, with $a + b = 2r + 1$, $i + 2r - a \geq 0$, and $i + k + b - 1 \leq n$. Let $\ell = i + 2r - a$. Then $i + k + b - 1 = \ell + k$, so $0 \leq \ell \leq n - k$, and $Z$ is a Gale subset of $[\ell, \ell + k]$ containing the vertex set of $F$.

If $i + 2r - 1 < 0$, then $i + k + 2r - 1 < k \leq n$, and $F = \{0\} \cup Y \cup [i + k, i + k + b - 1]$. Then $|F| = d$ and $F$ itself is a Gale subset of $[0,k]$ of size $d$ containing 0. Similarly for the case $i + k > n$.

The sets described are exactly the $(d - 1)$-simplices obtained by triangulating each facet of $P^{d,k,n}$ according to Theorem 2.1. The fact that this triangulation is shallow follows from the corresponding fact for the triangulations of the multiplexes. □

Let $T = T(P^{d,k,n})$ be this triangulation of $\partial P^{d,k,n}$. Since $T$ is shallow, $h(P^{d,k,n}, x) = h(T, x)$. We calculate $h(T, x)$ by shelling $T$.

**Theorem 5.2** Let $F_1, F_2, \ldots, F_c$ be the colex order of the facets of $P^{d,k,n}$. For each $j$, if $F_j = \{z_1, z_2, \ldots, z_{p_j}\}$ ($z_1 < z_2 < \cdots < z_{p_j}$), and $1 \leq p_j \leq p_j - d + 1$, let $T_{j,\ell} = \{z_{\ell}, \ldots, z_{\ell+d-1}\}$. Then $T_{1,1}, T_{1,2}, \ldots, T_{1,p_1-d+1}, T_{2,1}, \ldots, T_{p_1, p_1-d+1}, \ldots, T_{c,1}, \ldots, T_{p_{p_d-d+1}}$ is a shelling of $T(P^{d,k,n})$.

Let $U_{j,\ell}$ be the minimal new face when $T_{j,\ell}$ is shelled on. As vertex sets, $U_{j,\ell-p_{j-d+1}} = G_j$.

**Proof:** Throughout the proof, write $F_j = \{z_1, z_2, \ldots, z_{p_j}\}$ ($z_1 < z_2 < \cdots < z_{p_j}$). We first show that $G_j$ is the unique minimal face of $T_{j,\ell-p_{j-d+1}}$ not contained in $(\cup_{l=1}^{j-1} U_{l,\ell-p_{l-d+1}} T_{i,\ell}) \cup (\cup_{l=1}^{p_{j-d+1}} T_{j,\ell})$. The set $G_j$ is not contained in a facet of $P^{d,k,n}$ earlier than $F_j$. So $G_j$ does not occur in a facet of $T$ of the form $T_{i,\ell}$ for $i < j$. Also, $\max F_j \in G_j$, so $G_j$ does not occur in a facet.
of $\mathcal{T}$ of the form $T_{j,\ell}$ for $\ell \leq p_j - d$. Thus $G_j$ does not occur in a facet of $\mathcal{T}$ before $T_{j,p_j-d+1}$.

We show that for $z_q \in G_j$, $T_{j,p_j-d+1} \setminus \{z_q\}$ is contained in a facet of $\mathcal{T}$ occurring before $T_{j,p_j-d+1}$. There is nothing to check for $j = v$, because $T_{v,p_v-d+1} = F_v$ is the last simplex in the purported shelling order. So we may assume that $j < v$ and thus $G_j$ is contained in the last $d - 1$ vertices of $F_j$.

**Case 1.** If $p_j > d$ and $q = p_j$ ($z_q$ is the maximal element of $F_j$), then $T_{j,p_j-d+1} \setminus \{z_{p_j}\} \subseteq T_{j,p_j-d}$. 

**Case 2.** Suppose $p_j - d + 2 \leq q \leq p_j - 1$. Then $T_{j,p_j-d+1} \setminus \{z_q\} \subseteq \{z_{q-d+2}, \ldots, z_{q-1}, z_q, \ldots, z_{p_j}\}$. This is a ridge of $P^{d,k,n}$ in $F_j$ not containing $G_j$, and hence $H$ is contained in a previous facet $F_\ell$ of $P^{d,k,n}$. Since $H$ is a ridge in both $F_j$ and $F_\ell$, $H$ is obtained from each facet by deleting a single element from a consecutive string of vertices in the facet. So $|H| \leq |F_\ell \cap [z_{q-d+2}, z_{p_j}]| \leq |H| + 1$, and so $d - 1 \leq |F_\ell \cap [z_{p_j-d+1}, z_q]| \leq d$. So $T_{j,p_j-d+1} \setminus \{z_q\}$ is contained in a consecutive set of $d$ elements of $F_\ell$, and hence a $(d - 1)$-simplex of $\mathcal{T}(P^{d,k,n})$ belonging to $F_\ell$. This simplex occurs before $T_{j,p_j-d+1}$ in the specified shelling order.

**Case 3.** Otherwise $p_j = d$ (so $p_j - d + 1 = 1$) and $q = d$. Then $T_{j,1} = F_j$, and $H = T_{j,1} \setminus \{z_d\}$ is a ridge of $P^{d,k,n}$ in $F_j$ not containing max $F_j$, so $H$ is contained in a previous facet $F_\ell$ of $P^{d,k,n}$. As in Case 2, $d - 1 \leq |F_\ell \cap [z, z_{d-1}]| \leq d$. So $T_{j,1} \setminus \{z_d\}$ is contained in a consecutive set of $d$ elements of $F_\ell$, and hence a $(d - 1)$-simplex of $\mathcal{T}(P^{d,k,n})$ belonging to $F_\ell$. This simplex occurs before $T_{j,p_j-d+1}$ in the specified shelling order.

So in the potential shelling of $\mathcal{T}$, $G_j$ is the unique minimal new face as $T_{j,p_j-d+1}$ is shelled on. Write $U_{j,p_j-d+1} = G_j$. At this point we need a clearer view of the simplex $T_{j,\ell}$. Recall that $F_j$ is of the form ret$_n(X)$, where $X = [i, i + 2r - 1] \cup Y \cup [i + k, i + k + 2r - 1]$, with $Y$ a subset of size $d - 2r - 1$. If $i + 2r - 1 < 0$ or $i + k > n$, then $p_j = |F_j| = d$, and $T_{j,1} = T_{j,p_j-d+1} = F_j$; we have already completed this case. So assume $i + 2r - 1 \geq 0$ and $i + k \leq n$. A consecutive string of length $d$ in ret$_n(X)$ must then be of the form $[i + s, i + 2r - 1] \cup Y \cup [i + k, i + k + s]$ for some $s$, $0 \leq s \leq 2r - 1$. (All such strings—with appropriate $Y$—having $i + s \geq 0$ and $i + k + s \leq n$ occur as $T_{j,\ell}$.) In particular, for $\ell < p_j - d + 1$, $T_{j,\ell} = T_{j,\ell+1} \setminus \{\max T_{j,\ell+1}\} \cup \{\min T_{j,\ell+1} - 1\}$ and $\max T_{j,\ell} = \min T_{j,\ell+1} + k$.

Now define $U_{j,\ell}$ for $\ell \leq p_j - d$ recursively by $U_{j,\ell} = U_{j,\ell+1} \setminus \{z\} \cup \{z - k, z - 1\}$, where $z = \max T_{j,\ell+1}$. By the observations above, $U_{j,\ell} \subseteq T_{j,\ell}$.

We prove by downward induction that $U_{j,\ell}$ is not contained in a facet of $F_\ell$ of $P^{d,k,n}$ before $F_j$, that $U_{j,\ell}$ is not contained in a facet of $\mathcal{T}$ occurring before
$T_{j,\ell}$, and that any ridge of $\mathcal{T}$ in $T_{j,\ell}$ not containing all of $U_{j,\ell}$ is in an earlier facet of $\mathcal{T}$. The base case of the induction is $\ell = p_j - d + 1$, and this case has been handled above.

Note that \{z - k, z - 1\} is a diagonal of the 2-face \{z - k - 1, z - k, z - 1, z\} of $P_{d,k,n}$. So if $F_i$ is a facet of $P_{d,k,n}$ containing $U_{j,\ell}$, then $F_i$ contains \{z - k - 1, z - k, z - 1, z\}. Thus $F_i$ contains $U_{j,\ell+1}$, so, by the induction assumption, $i \geq j$. Therefore, for $i < j$, and any $r$, $T_{i,r}$ does not contain $U_{j,\ell}$. For $r < \ell$, $T_{j,r}$ does not contain $z - 1 = max T_{j,\ell}$, so $T_{j,r}$ does not contain $U_{j,\ell}$.

Now we wish to show that for any $g \in U_{j,\ell}$, $T_{j,\ell} \setminus \{g\}$ is in a previous facet of $\mathcal{T}$.

**Case 1.** If $g = z - 1 = max T_{j,\ell}$ and $\ell \geq 2$, then $T_{j,\ell} \setminus \{g\} \subset T_{j,\ell-1}$.

**Case 2.** If $g = z - 1 = max T_{j,\ell}$ and $\ell = 1$, then $T_{j,\ell} \setminus \{g\}$ is the leftmost ridge of $P_{d,k,n}$ in $F_j$ and, in particular, does not contain max $F_j$. So $H = T_{j,\ell} \setminus \{g\}$ is contained in a previous facet $F_e$ of $P_{d,k,n}$. As in the $\ell = p_j - d + 1$ case, $F_e \cap [min T_{j,\ell}, max T_{j,\ell}]$ is contained in a consecutive set of $d$ elements of $F_e$, and hence in a $(d - 1)$-simplex of $\mathcal{T}(P_{d,k,n})$ belonging to $F_e$. So $T_{j,\ell} \setminus \{g\}$ is contained in a previous facet of $\mathcal{T}$.

**Case 3.** Suppose $g < z - 1$ and $g \in U_{j,\ell} \cap U_{j,\ell+1}$. Since \{z - 1, z\} $\subset T_{j,\ell+1}$, $T_{j,\ell+1}$ contains at most $d - 3$ elements less than $g$. The ridge $H$ of $P_{d,k,n}$ in $F_j$ containing $T_{j,\ell+1} \setminus \{g\}$ consists of the $d - 2$ elements of $F_j$ below $g$ and the (up to) $d - 2$ elements of $F_j$ above $g$. In particular, $H$ contains $min T_{j,\ell+1} = min T_{j,\ell}$. So $T_{j,\ell} \setminus \{g\} \subset H$. Since $dim T_{j,\ell+1} \setminus \{g\} = d - 2$, $H$ is the (unique) smallest face of $P_{d,k,n}$ containing $T_{j,\ell+1} \setminus \{g\}$. By the induction hypothesis $T_{j,\ell+1} \setminus \{g\}$ is contained in a previous facet $T_i$ of $\mathcal{T}$; here $i < j$ because $max T_{j,\ell+1} \in T_{j,\ell+1} \setminus \{g\}$. The $(d - 2)$-simplex $T_{j,\ell+1} \setminus \{g\}$ is then contained in a ridge of $P_{d,k,n}$ contained in $F_i$, but this ridge must be $H$, by the uniqueness of $H$. So $T_{j,\ell} \setminus \{g\} \subset H = F_i \cap F_j$. As in earlier cases, $F_i \cap [min T_{j,\ell}, max T_{j,\ell}]$ is contained in a consecutive set of $d$ elements of $F_i$, and hence in a $(d - 1)$-simplex of $\mathcal{T}(P_{d,k,n})$ belonging to $F_i$. So $T_{j,\ell} \setminus \{g\}$ is contained in a previous facet of $\mathcal{T}$.

**Case 4.** Finally, let $g = z - k$, which is $min T_{j,\ell} + 1$. Then $T_{j,\ell}$ contains $d - 2$ elements above $g$. Let $H$ be the ridge of $P_{d,k,n}$ in $F_j$ containing $T_{j,\ell} \setminus \{g\}$. Then $max H = max T_{j,\ell} < max F_j$, so $H$ does not contain $G_j$. So $H$ is in a previous facet $F_i$ of $P_{d,k,n}$. As in earlier cases, $F_i \cap [min T_{j,\ell}, max T_{j,\ell}]$ is contained in a consecutive set of $d$ elements of $F_i$, and hence in a $(d - 1)$-simplex of $\mathcal{T}(P_{d,k,n})$ belonging to $F_i$. So $T_{j,\ell} \setminus \{g\}$ is contained in a previous facet of $\mathcal{T}$.

Thus $T_{1,1}, T_{1,2}, \ldots, T_{1,p_1-d+1}, T_{2,1}, \ldots, T_{2,p_2-d+1}, \ldots, T_{v,1}, \ldots, T_{v,p_v-d+1}$ is a shelling of $\mathcal{T}(P_{d,k,n})$. □
Corollary 5.3 Let \( n \geq k \geq d = 2m + 1 \geq 5 \). Let \( \bigcup [G_j, F_j] \) be the partition of the face lattice of \( P^{d,k,n} \) from the colex shelling, and let \( \bigcup [U_{j,\ell}, T_{j,\ell}] \) be the partition of the face lattice of \( T(P^{d,k,n}) \) from the shelling of Theorem 5.2. Then

1. \( h(P^{d,k,n}, x) \geq h'(P^{d,k,n}, x) \).

2. The contribution to \( h_i(P^{d,k,n}) - h'_i(P^{d,k,n}) \) from the interval \([G_j, F_j]\) is

\[
 a_{j,i} = |\{ \ell : |U_{j,\ell}| = i, 1 \leq \ell \leq p_{\ell} - d \}| \geq 0.
\]

**Proof:** The \( h \)-vector of \( T = T(P^{d,k,n}) \) counts the sets \( U_{j,\ell} \) of each size. Among these are all the sets \( G_j \) counted by the \( h' \)-vector of \( P^{d,k,n} \). Thus

\[
 h_i(P^{d,k,n}) = h_i(T) = |\{(j, \ell) : |U_{j,\ell}| = i\}| \\
 \geq |\{(j, \ell) : |U_{j,\ell}| = i \text{ and } \ell = p_{\ell} - d + 1\}| = h'_i(P^{d,k,n}).
\]

Recall that we write \( G_j \) for the set of faces of \( F_j \) not in \( \cup_{i \neq j} F_i \); here \( G_j \) is the set of faces in \([G_j, F_j]\). Write also \( T G_j \) for the set of faces of \( T \) that are contained in \( F_j \) but not in \( \cup_{i \neq j} F_i \). By [2, Corollary 7], since \( T \) is a shallow triangulation of \( \partial P^{d,k,n} \),

\[
 g(G, x) = (x - 1)^{d - 1 - \dim \sigma},
\]

where the sum is over all faces \( \sigma \) of \( T \) that are contained in \( G \) but not in any proper subface of \( G \). Thus

\[
 h(G_j, x) = \sum_{G \in [G_j, F_j]} g(G, x)(x - 1)^{d - 1 - \dim G} \\
 = \sum_{\sigma \in T G_j} (x - 1)^{d - 1 - \dim \sigma} = \sum_{\ell=1}^{p_{\ell} - d + 1} x^{d - |U_{j,\ell}|}
\]

Since \( h'(G_j, x) = x^{d - |G_j|} = x^{d - |U_{j, p_{\ell} - d + 1}|} \),

\[
 \sum_i a_{j,i}x^i = h(G_j, x) - h'(G_j, x) = \sum_{\ell=1}^{p_{\ell} - d} x^{d - |U_{j,\ell}|},
\]

or

\[
 a_{j,i} = |\{ \ell : |U_{j,\ell}| = i, 1 \leq \ell \leq p_{\ell} - d \}| \geq 0.
\]

\( \square \)

**Example.** Table 2 gives the shelling of the triangulation of \( P^{5,6,8} \). (Refer back to Table 1 for the shelling of \( P^{5,6,8} \) itself.) Among the rows \((6, 1), (7, 1), \)
Table 2: Shelling of triangulation of $P_{5,6,8}$

| $(j, \ell)$ | $T_{j,\ell}$ | $U_{j,\ell}$ | $(j, \ell)$ | $T_{j,\ell}$ | $U_{j,\ell}$ |
|------------|-------------|-------------|------------|-------------|-------------|
| 1,1        | 01234       | $\emptyset$| 11,1       | 1234        | 7           |
| 2,1        | 012 45      | 5           | 11,2       | 234 78      | 78          |
| 3,1        | 02345       | 35          | 12,1       | 1245        | 7           |
| 4,1        | 023 56      | 6           | 12,2       | 245 78      | 578         |
| 5,1        | 03456       | 46          | 13,1       | 0123 6      | 126         |
| 6,1        | 0134 6      | 16          | 13,2       | 123 67      | 267         |
| 6,2        | 134 67      | 7           | 13,3       | 23678       | 678         |
| 7,1        | 01456       | 156         | 14,1       | 34 678      | 4678        |
| 7,2        | 14567       | 57          | 15,1       | 012 56      | 1256        |
| 8,1        | 2345 8      | 8           | 15,2       | 12 567      | 2567        |
| 9,1        | 23 56 8     | 68          | 15,3       | 2 5678      | 5678        |
| 10,1       | 3456 8      | 468         | 16,1       | 45678       | 45678       |

(11, 1), (12, 1), (13, 1), (13, 2), (15, 1), (15, 2) (rows $(j, \ell)$ that are not the last row for that $j$), count the $U_{j,\ell}$ of cardinality $i$ to get $h_i(P_{5,6,8}) - h'_i(P_{5,6,8})$. Note that $U_{13,3} = G_{13}$ (from Table 1), and that $U_{13,2} = U_{13,3} \setminus \{8\} \cup \{2, 7\}$. The ridges in $T_{13,2}$ are 1236, 1237, 1267, 1367, and 2367. The first ridge, 1236, falls under Case 1 of the (later part of the) proof of Theorem 5.2; it is contained in the previous facet, $T_{13,1}$. The next ridge, 1237, falls under Case 3; it is contained in the ridge 12378 of $P_{5,6,8}$ in $T_{0,13} = 0123678$, and 12378 also contains the ridge 2378 in $T_{13,3}$. The induction assumption says that 2378 is contained in an earlier facet, in this case $T_{11,2}$, and 12378 is contained in $F_{11}$. Finally, the ridge 1237 is contained in the simplex $T_{11,1}$, part of the triangulation of $F_{11}$. The last ridge of $T_{13,2}$ not containing 267 is 1367. It falls under Case 4. The set 1367 is contained in the ridge 01367 of $P_{5,6,8}$, contained in $T_{13,3}$. This ridge is also contained in the earlier facet $F_6$. The ridge 1367 of the triangulation is contained in the simplex $T_{0,2}$.

**Theorem 5.4** Let $n \geq d + k - 1$. For $1 \leq i \leq d - 1$, $h_i(P_{d,k,n}) - h_i(P_{d,k,n-1})$ is the number of facets $T_{j,\ell}$ of $T(P_{d,k,n})$ such that $\max F_j = n - 1$ and $|U_{j,\ell}| = i$. For $1 \leq i \leq (d - 1)/2$, this is $\binom{k - d + i - 1}{i - 1}$.

**Proof:** Refer to Proposition 3.4 for a description of the facets of $P_{d,k,n}$ in terms of those of $P_{d,k,n-1}$. For $n \geq d + k - 1$, for every facet $P_{d,k,n}$ with maximum element $n$, the translation $F - 1$ is a facet of $P_{d,k,n-1}$. (For smaller $n$, a facet of $P_{d,k,n}$ may contain both 0 and $n$, in which case lsh($F$) is a proper...
subset of $F - 1$.) The same holds for the simplices $T_{j,\ell}$ triangulating these facets, and for the sets $U_{j,\ell}$. The facets of $P^{d,k,n}$ with maximum element at most $n - 2$ are facets of $P^{d,k,n-1}$, and the same holds for the corresponding $T_{j,\ell}$ and $U_{j,\ell}$. The contributions to $h(P^{d,k,n})$ from facets ending in any element but $n - 1$ thus total $h(P^{d,k,n-1})$. So for $1 \leq i \leq d - 1$, $h_i(P^{d,k,n}) - h_i(P^{d,k,n-1})$ is the number of facets $T_{j,\ell}$ of $T(P^{d,k,n})$ such that $\max F_j = n - 1$ and $|U_{j,\ell}| = i$.

Now consider the set $S$ of facets $T_{j,\ell}$ of $T(P^{d,k,n})$ with $\max F_j = n - 1$. For each $T \in S$, $T$ is a set of $d$ elements occurring consecutively in some $F_j$ with maximum element $n - 1$. So $T$ can be written as

$$T = [b, n - k - 1] \cup [n - k + 1, c] \cup Y \cup [c, b + k],$$

(2)

where

1. $n - k - d + 1 \leq b \leq n - k - 1$;
2. $n - k \leq c \leq b + d - 1$ and $c - n + k$ is even (here $c = n - k$ means $[n - k + 1, c] = \emptyset$);
3. $Y$ is a paired subset of $[c + 2, e - 1]$;
4. $e = b + k - 1$ if $n - k - b$ is odd, and $e = b + k$ if $n - k - b$ is even; and
5. $|T| = d$.

In these terms, the minimum new face $U$ when $T$ is shelled on is $U = [b + 1, n - k - 1] \cup E(Y) \cup \{b + k\}$.

We give a bijection between the facets $T$ in $S$ with $|U| = i$ (where $1 \leq i \leq (d - 1)/2$) and the $(k - d)$-element subsets of $[1, k - d + i - 1]$. Let $T$ be as in Equation 2. Then $i = |U| = n - k - b + |Y|/2$. Write $[c + 1, e - 1] \setminus Y = \{x_1, x_2, \ldots, x_{k - d}\}$, with the $x_\ell$s increasing. (This set has $k - d$ elements because $d = (e - b) + |Y| + (b + k - e + 1)$, so $|[c + 1, e - 1] \setminus Y| = e - c - 1 - |Y| = k - d$.) For each $\ell$, let $y(x_\ell)$ be the number of pairs in $Y$ with both elements less than $x_\ell$. Let $a_1 = n - k - b = i - |Y|/2$. Set

$$A(T) = \{a_1 + y(x_\ell) + \ell - 1 : 1 \leq \ell \leq k - d\}.$$

To see that this is a subset of $[1, k - d + i - 1]$, note that the elements of $A(T)$ form an increasing sequence with minimum element $a_1$ and maximum element $a_1 + y(x_{k - d}) + (k - d - 1) \leq a_1 + |Y|/2 + (k - d - 1) = k - d + i - 1$.

For the inverse of this map, write a $(k - d)$-element subset of $[1, k - d + i - 1]$ as $A = \{a_1, a_2, \ldots, a_{k - d}\}$, with the $a_\ell$s increasing. Then $1 \leq a_1 \leq i$. Let

$$x_1 = n - k + d - 2i + a_1 - \chi(a_1 \text{ odd}).$$

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Set

\[ T(A) = [n - k - a_1, n - k - 1] \cup \{x_1, n - k + 1, x_1 - 1\} \]
\[ \cup Y \cup [n - a_1 - \chi(a_1 \text{ odd}), n - a_1], \]

where

\[ Y = [x_1, n - a_1 - 1 - \chi(a_1 \text{ odd})] \setminus \{x_1 + 2(a_\ell - a_1) - (\ell - 1) : 1 \leq \ell \leq k - d\}. \]

We check that this gives a set of the required form.

1. Since \(1 \leq a_1 \leq i \leq d - 1\), \(n - k - d + 1 \leq n - k - a_1 \leq n - k - 1\).
2. \(x_1 - 1 - n + k = d - 2i - 1 + (a_1 - \chi(a_1 \text{ odd}))\), which is nonnegative and even; \(x_1 - 1 = (n - k - a_1 + d - 1) - (2i - 2a_1 + \chi(a_1 \text{ odd})) \leq n - k - a_1 + d - 1\).
3. \(Y\) is clearly a subset of \([x_1 + 1, n - a_1 - \chi(a_1 \text{ odd}) - 1]\). To see that \(Y\) is paired, note that the difference between two consecutive elements in the removed set is \((x_1 + 2(a_\ell+1 - a_1) - \ell) - (x_1 + 2(a_\ell - a_1) - (\ell - 1)) = 2(a_{\ell+1} - a_\ell) - 1\).
4. This condition holds by definition.
5. Since \(A\) is a subset of \([1, k - d + i - 1]\),

\[
\begin{align*}
x_1 + 2(a_{k-d} - a_1) - (k - d - 1) \\
\leq x_1 + 2(k - d + i - 1) - 2a_1 - (k - d - 1) \\
= x_1 + k - d + 2i - 2a_1 - 1 = n - a_1 - \chi(a_1 \text{ odd}) - 1.
\end{align*}
\]

So

\[
\{x_1 + 2(a_\ell - a_1) - (\ell - 1) : 1 \leq \ell \leq k - d\} \subseteq [x_1, n - a_1 - 1 - \chi(a_1 \text{ odd})],
\]

and

\[
|Y| = (n - a_1 - \chi(a_1 \text{ odd}) - x_1) - (k - d) = 2i - 2a_1.
\]

So \(|T(A)| = x_1 - (n - k - a_1) + |Y| + \chi(a_1 \text{ odd}) = d\).

Also, in this case \(U = [n - k - a_1 + 1, n - k - 1] \cup E(Y) \cup \{n - a_1\}\), so \(|U| = i\).

It is straightforward to check that these maps are inverses. The main point is that, if \(a_\ell = a_1 + y(x_\ell) + \ell - 1\), then

\[
\begin{align*}
x_1 + 2(a_\ell - a_1) - (\ell - 1) &= x_1 + 2(y(x_\ell) + \ell - 1) - (\ell - 1) \\
&= x_1 + 2y(x_\ell) + \ell - 1 = x_\ell.
\end{align*}
\]

\(\square\)
Example. Consider the ordinary polytope $P^{7,9,15}$. There are six facets with maximum vertex 14; they are (with sets $G_j$ underlined) \{
4, 5, 7, 8, 9, 10, 13, 14\}, \{
4, 5, 7, 8, 10, 11, 13, 14\}, \{
4, 5, 8, 9, 10, 11, 12, 13, 14\}, \{
2, 3, 4, 5, 7, 8, 11, 12, 13, 14\}, \{
2, 3, 4, 5, 9, 11, 12, 13, 14\}, and \{
0, 1, 2, 3, 4, 5, 9, 10, 11, 12, 13, 14\}. Among the 6-simplices occurring in the triangulation of these facets, six have $|U_{j,\ell}| = 3$. Table 3 gives the bijection from this set of simplices to the 2-element subsets of $\{1, 2, 3, 4\}$.

| $T_{j,\ell}$ | $b$ | $c$ | $e$ | $Y$ | $a_1$ | $x_1, x_2$ | $y(x_i)$ | $A(T_{j,\ell})$ |
|-------------|-----|-----|-----|-----|------|----------|--------|--------------|
| $4, 5, 7, 8, 10, 11, 12$ | 4   | 8   | 13  | $10, 11$ | 2    | 9, 12    | 0, 1   | $\{2, 4\}$  |
| $5, 8, 9, 10, 11, 13, 14$ | 5   | 6   | 13  | $8, 9, 10, 11$ | 1    | 7, 12    | 0, 2   | $\{1, 4\}$  |
| $3, 4, 5, 7, 8, 11, 12$ | 3   | 8   | 11  | $\emptyset$ | 3    | 9, 10    | 0, 0   | $\{3, 4\}$  |
| $4, 5, 7, 8, 11, 12, 13$ | 4   | 8   | 13  | $11, 12$ | 2    | 9, 10    | 0, 0   | $\{2, 3\}$  |
| $5, 8, 9, 11, 12, 13, 14$ | 5   | 6   | 13  | $8, 9, 11, 12$ | 1    | 7, 10    | 0, 1   | $\{1, 3\}$  |
| $5, 9, 10, 11, 12, 13, 14$ | 5   | 6   | 13  | $9, 10, 11, 12$ | 1    | 7, 8     | 0, 0   | $\{1, 2\}$  |

Table 3: Bijection with 2-element subsets of $\{1, 2, 3, 4\}$

Again, the results of this section hold for even-dimensional multiplexes as well.

6 Afterword

The story of the combinatorics of simplicial polytopes is a beautiful one. There one finds an intricate interplay among the face lattice of the polytope, shellings, the Stanley-Reisner ring and the toric variety, tied together with the $h$-vector. (See, for example, \cite{16}, Lecture 8.) The cyclic polytopes play a special role, serving as the extreme examples, and providing the environment in which to build representative polytopes for each $h$-vector (the Billera-Lee construction \cite{25}). In the general case of arbitrary convex polytopes, the various puzzle pieces have not interlocked as well. In this paper we made progress on putting the puzzle together for the special class of ordinary polytopes. Since the ordinary polytopes generalize the cyclic polytopes, a natural next step would be to mimic the Billera-Lee construction, or Kalai’s extension of it \cite{10}, on the ordinary polytopes, as a way of generating multiplexial flag vectors. It would also be interesting to see if there is a ring associated with these polytopes, particularly one having a quotient with Hilbert function equal to the $h'$-polynomial. Another open problem is to determine the best even-dimensional analogues of the ordinary
polytopes. They may come from taking vertex figures of odd-dimensional ordinary polytopes, or from generalizing Dinh’s combinatorial description of the facets of ordinary polytopes. Looking beyond ordinary and multiplicial polytopes, we should ask what other classes of polytopes have shellings with special properties that relate to the $h$-vector?

**Acknowledgments**

My thanks go to the folks at University of Washington, the Discrete and Computational Geometry program at MSRI and the Diskrete Geometrie group at TU-Berlin, who listened to me when it was all speculation. Particular thanks go to Carl Lee for helpful discussions.

**References**

[1] Gottfried Barthel, Jean-Paul Brasselet, Karl-Heinz Fieseler, and Ludger Kaup, Combinatorial intersection cohomology for fans, *Tohoku Math. J.* (2), 54(1):1–41, 2002.

[2] Margaret M. Bayer, Equidecomposable and weakly neighborly polytopes, *Israel J. Math.*, 81 (1993), 301–320.

[3] Margaret M. Bayer, Flag vectors of multiplicial polytopes, Preprint 2003-016, Mathematical Sciences Research Institute, Berkeley, 2003.

[4] Margaret M. Bayer, Aaron M. Bruening, and Joshua D. Stewart, A combinatorial study of multiplexes and ordinary polytopes, *Discrete Comput. Geom.*, 27 (2002), no. 1, 49–63.

[5] Louis J. Billera and Carl W. Lee, A proof of the sufficiency of McMullen’s conditions for $f$-vectors of simplicial polytopes, *J. Combin. Theory Ser. A* 31 (1981), 237–255.

[6] T. Bisztriczky, On a class of generalized simplices, *Mathematika*, 43 (1996), 274–285.

[7] T. Bisztriczky, Ordinary $(2m + 1)$-polytopes, *Israel J. Math.*, 102 (1997), 101–123.

[8] Tom Braden, $g$- and $h$-polynomials of non-rational polytopes—recent progress, Abstract at Meeting on Topological and Geometric Combi-
natorics, Mathematisches Forschungsinstitut Oberwolfach, April 6–12, 2003.

[9] Thi Ngoc Dinh, *Ordinary Polytopes*, PhD thesis, The University of Calgary, 1999.

[10] Gil Kalai, Many triangulated spheres, *Discrete Comput. Geom.*, 3 (1988), 1–14.

[11] Kalle Karu, Hard Lefschetz Theorem for nonrational polytopes, [arXiv:math.AG/0112087](http://arxiv.org/abs/math.AG/0112087), 2001.

[12] P. McMullen and G. C. Shephard, *Convex polytopes and the upper bound conjecture*, Cambridge Univ. Press, London, 1971.

[13] Richard P. Stanley, The number of faces of simplicial convex polytopes, *Adv. Math.* 35 (1980), 236–238.

[14] Richard P. Stanley, Generalized $H$-vectors, intersection cohomology of toric varieties, and related results, In *Commutative algebra and combinatorics (Kyoto, 1985)*, volume 11 of *Adv. Stud. Pure Math.*, pages 187–213, Amsterdam-New York, 1987, North-Holland.

[15] Richard P. Stanley, Subdivisions and local $h$-vectors, *J. Amer. Math. Soc.*, 5 (1992), 805–851.

[16] Günter M. Ziegler, *Lectures on Polytopes*, Springer-Verlag, 1995.