p-TORAL APPROXIMATIONS COMPUTE BREDON HOMOLOGY

GREGORY ARONE, W. G. DWYER, AND KATHRYN LESH

Abstract. We study Bredon homology approximations for spaces with an action of a compact Lie group \( G \). We show that if \( M \) is a coMackey functor satisfying mild \( p \)-locality conditions, then Bredon homology of a \( G \)-space \( X \) with coefficients in \( M \) is determined by fixed points of \( p \)-toral subgroups of \( G \) acting on \( X \). As an application we prove a vanishing result for the Bredon homology of the complex \( \mathcal{L}_n \) of direct-sum decompositions of \( \mathbb{C}^n \).

1. Introduction

Let \( X \) be a space with an action of a compact Lie group \( G \), and let \( p \) be a fixed prime. The \( G \)-equivariant homotopy type of \( X \), and hence its Bredon homology\(^1\), is determined by the fixed point spaces \( X^H \) of all closed subgroups \( H \subseteq G \). In this paper, we establish that the Bredon homology of \( X \) can sometimes be computed from knowledge of \( X^H \) for only a subset of the subgroups of \( G \). Our main results in this direction say that if a coefficient system \( M \) comes from a coMackey functor and satisfies certain \( p \)-locality conditions, then Bredon homology of \( X \) with coefficients in \( M \) is determined by the mod \( p \) homology of fixed point spaces of the \( p \)-toral subgroups of \( G \).

Our main application concerns the Bredon homology of \( \mathcal{L}_n \), the complex of proper direct-sum decompositions of \( \mathbb{C}^n \), which is a finite complex with an action of \( U(n) \). The space \( \mathcal{L}_n \) was introduced in [Aro02] and studied in detail, largely from first principles, in [BJL⁺¹⁵] and [BJL⁺]. We leverage the results of [BJL⁺], together with Smith theory to obtain Theorem 1.1 below. Let \( T \text{Sp}_k(\mathbb{F}_p) \) denote the symplectic Tits building, and let \( X^0 \) denote the unreduced suspension of \( X \). Let \( \Gamma_k \subset U(p^k) \) denote the unique subgroup (up to conjugacy) that acts irreducibly on \( \mathbb{C}^{p^k} \) and is extension of the central \( S^1 \subset U(p^k) \) by an elementary abelian \( p \)-group. The following theorem is our main result for \( \mathcal{L}_n \). (See Definition 4.1 for the transfer condition.)

**Theorem 1.1.** Suppose that \( M \) is a (co)Mackey functor for \( U(n) \) that takes values in \( p \)-local abelian groups and satisfies the transfer condition for the prime \( p \).

\( 1 \) If \( n \) is not a power of \( p \), then the map \( \mathcal{L}_n \to * \) induces an isomorphism on Bredon (co)homology with coefficients in \( M \).

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\(^1\)In general, we are interested in both homology and cohomology, so we write “(co)homology” throughout the paper to indicate both together. To avoid cluttering the exposition, we will focus on homology in the introduction.
If $n = p^k$, there is a map inducing an isomorphism on Bredon (co)homology with coefficients in $M$:

$$U(p^k) \wedge_{N(\Gamma_k)} T \text{Sp}_k(\mathbb{F}_p)^\circ \to \mathcal{L}_n^\circ.$$ 

The equivariant homotopy type of $\mathcal{L}_n^\circ$ came up in the first author’s work on orthogonal calculus [Aro02] and in the first and third author’s joint work on the rank filtration [AL07, AL10]. It has also played a role in the study of the Balmer spectrum of the equivariant stable homotopy category [BHN+]. The results of our current work will be used in a forthcoming paper on the K-theoretic analogue of the Whitehead conjecture.

Our results and methods are analogous to those used in [ADL16], where we proved a result similar to Theorem 1.1 for the $\Sigma_n$-space $P_n$, the poset of nontrivial, proper partitions of the set $\{1, \ldots, n\}$. One expects this analogy because of the dictionary provided by [Aro02], which gives a correspondence between objects associated to the Goodwillie tower for the identity (the “discrete case”) and the Weiss tower for the functor $V \mapsto B\text{Aut}(V)$ (the “orthogonal” or “unitary” or “compact Lie” case). As is standard for translation between discrete groups and compact Lie groups, the role of finite $p$-groups in the discrete case is taken by $p$-toral groups (extensions of a torus by a finite $p$-group) in the compact Lie case. In the remainder of the introduction, we outline our methods and other results in more detail. As is often the case, some things are harder in the compact Lie case than in the discrete case, and some are easier.

The first step of [ADL16], when $G$ was a finite group, relied on approximating a $G$-space $X$ by a space that uses only $p$-subgroups of $G$ as isotropy groups. Similar construction of an approximation in the compact Lie case is not difficult once we set up the appropriate language (Section 2). Let $\mathcal{C}$ be a collection of (closed) subgroups of $G$, that is, a set of subgroups of $G$ that is closed under conjugation. In Section 2 we construct a $G$-CW-complex $X_\mathcal{C}$, equipped with a natural map $\alpha_X : X_\mathcal{C} \to X$, which is an approximation of $X$ by a space whose isotropy is contained in $\mathcal{C}$.

**Proposition 2.10.** Let $X$ be a $G$-CW-complex and let $\mathcal{C}$ be a collection of subgroups of $G$. There exists a functorial $\mathcal{C}$-approximation $\alpha_X : X_\mathcal{C} \to X$ by a $G$-CW-complex $X_\mathcal{C}$ that has the following properties.

1. If $\text{Iso}(X) \subseteq \mathcal{C}$, then $\alpha_X$ is a weak $G$-equivalence.
2. $\text{Iso}(X_\mathcal{C}) \subseteq \mathcal{C}$.
3. If $H \in \mathcal{C}$, then $\alpha_X$ induces a weak equivalence $(X_\mathcal{C})^H \to X^H$.

The map $\alpha_X : X_\mathcal{C} \to X$ is characterized up to a weak $G$-equivalence by properties (2) and (3).

Turning our attention to the prime $p$, let $\mathcal{A}_p$ be the collection of all $p$-toral subgroups of $G$ (including the trivial subgroup). We would like to know when $X_{\mathcal{A}_p} \to X$ is an isomorphism on Bredon homology, so that we can reduce computing Bredon homology of $X$ to computing Bredon homology of $X_{\mathcal{A}_p}$. When $G$ is discrete, establishing such an isomorphism involves having a suitable coMackey functor for the coefficient system, and a standard chain-level transfer argument. In the compact Lie case, chain methods have to be replaced with stable homotopy theoretic ones. Because of this, we need to revisit the precise definitions of Mackey functors and Bredon homology. Moreover, it is another feature of the compact Lie
This is something we need for forthcoming applications. In the introduction we focus on the case of coMackey functors and homology, but each statement has a corresponding statement involving Mackey functors and cohomology. All this is discussed systematically in Section 4.

An appropriate $p$-locality condition is required on our coMackey coefficient system $M$ in order for us to hope that $X_{A_p}$ can compute Bredon homology of $X$. Bredon homology with coefficients in $M$ can be extended to an $RO(G)$-graded homology theory. This is the phenomenon that equips Bredon homology with coefficients in $M$ with transfers. Let $P$ be a maximal $p$-toral subgroup of $G$. We say that $M$ satisfies the transfer condition for $p$ if for every $G$-space $X$, the composition

$$H_*(X; M) \xrightarrow{tr} H_*(G/P \times X; M) \rightarrow H_*(X; M)$$

is an isomorphism. Here the first homomorphism is induced by the Becker-Gottlieb transfer associated with the projection $G/P \times X \rightarrow X$.

**Theorem 4.2.** Let $G$ be a compact Lie group, and let $M$ be a (co)Mackey functor for $G$ that satisfies the transfer condition for a prime $p$. Then for all $G$-CW-complexes $X$, the map $\alpha_X : X_{A_G} \rightarrow X$ is an $M$-(co)homology isomorphism.

One difference between finite groups and topological groups is that for finite groups the product of two orbits is a disjoint union of orbits, while for topological groups this is not the case. Therefore, for finite groups, it is clear what it means to evaluate a Mackey functor at a product of orbits, while for more general topological groups it is less clear. Indeed we take the “value” of a Mackey functor at any space to be the Bredon homology of the space with coefficients in the Mackey functor. As a consequence, conditions such as the transfer condition for $p$ are considerably harder to formulate, let alone verify, for compact Lie groups than for finite groups.

For future applications we are especially interested in coMackey functors that come from Borel homology. In Section 5 we develop a criterion for a Mackey functor to satisfy the transfer condition for $p$, and we verify it for Mackey functors that arise from Borel homology. More precisely, let $E$ be a spectrum with an action of $G$, and let $\tilde{h}$ denote based homotopy orbits. Let $\pi^* E$ be the coMackey functor defined by the formula $\pi^*_h E(\Sigma^\infty O_+) = \pi_*(E \wedge O_+)^{\tilde{h}G}$. Dually, let $\Delta^* E$ be the Mackey functor defined by the formula $\Delta^*_h E(\Sigma^\infty O_+) = \pi_*(E \wedge D(O_+))^{\tilde{h}G}$, where $D(-)$ denotes the Spanier–Whitehead dual. We show that if the non-equivariant homotopy groups of $E$ are $p$-local, then this (co)Mackey functor satisfies the transfer condition for $p$.

This is something we need for forthcoming applications.

Section 6 deals with the applications to $L_n$. In this section we prove Theorem 1.1 by first using Theorem 4.2 to approximate $L_n$ using $p$-toral subgroups of $U(n)$. The approach is analogous to that of [ADL16], where we began by approximating the partition complex $P_n$ using $p$-subgroups of $\Sigma_n$. In both [ADL16] and in the present work, it is necessary to find an argument to “discard” a subgroup from the approximating collection if that subgroup’s fixed point space is not contractible. In the argument of [ADL16], the technology used is a “pruning” argument, based on homological properties of the poset of nontrivial $p$-subgroups of the Weyl group of the problematic subgroup. The argument needs to be applied to a large number of $p$-subgroups of $\Sigma_n$ that might have non-contractible fixed point spaces on $P_n$.

In the case of $U(n)$ acting on $L_n$, there can likewise be $p$-toral subgroups of $U(n)$ whose fixed point spaces are not contractible and need to be handled. However, it...
is a pleasant feature of $\mathcal{L}_n$ that we can use Smith theory to show that unwanted subgroups can be disregarded. When $n$ is not a power of $p$, arguments using Smith theory are available because, unlike the partition complex $\mathcal{P}_n$, the space $\mathcal{L}_n$ is mod $p$ acyclic. When $n$ is a power of $p$, Smith theory can be used because, again unlike the situation for $\mathcal{P}_{p^k}$, the approximating space for $\mathcal{L}_{p^k}$ involving a Tits building already has the correct mod $p$ homology. It seems plausible that the pruning method could also be adapted to the present situation, but we decided not to go down that road.

By combining the results of Section 5 with Theorem 1.1 we obtain the following. Let $\tilde{H}_*$ denote reduced Bredon homology.

**Corollary 6.6.** Let $p$ be a prime. Let $E$ be a spectrum with an action of $U(n)$ whose non-equivariant homology groups are $p$-local. If $n$ is not a power of $p$, then $\tilde{H}_i(\mathcal{L}_n^\circ; \pi_b^*(E)) = 0$ for all $i \geq 0$. If $n = p^k$, this holds for $i \neq k$. A similar statement holds for cohomology with coefficients in $\pi_b^*(E)$.

**Organization and notation:**

In Section 2 we review some basic facts regarding actions of compact Lie groups on spaces, such as Elmendorf’s theorem and approximation relative to a collection of subgroups. In Section 3 we review coefficient systems, and Bredon homology and cohomology of spaces with an action of a compact Lie groups. In Section 4 we review (co)Mackey functors for compact Lie groups, and equivariant homology and cohomology with coefficients in an (co)Mackey functors. We then prove our main approximation result, Theorem 4.2. In Section 5 we verify that the required hypothesis for Theorem 4.2 is satisfied by Borel homology associated with a $G$-spectrum that is non-equivariantly $p$-local. In Section 6 we apply our results to prove some facts about the Bredon homology of the complex of direct-sum decompositions. This will be used in a future work on the K-theoretic analogue of the Whitehead conjecture.

Throughout the paper, $p$ is a fixed prime, “group” means compact Lie group, and “subgroup” means closed subgroup. We write $\text{Iso}(X)$ to indicate the collection of isotropy subgroups of a $G$-space $X$. Unless otherwise indicated, the word “spectrum” means a $G$-equivariant spectrum indexed on a complete $G$-universe. In particular, phrases like “suspension spectrum,” “Eilenberg-Mac Lane spectrum,” etc., refer to the $G$-equivariant versions of such concepts. A “collection of subgroups” of a group $G$ means a set of subgroups that is closed under conjugation.

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2. **$G$-spaces as diagram categories**

Let $G$ be a compact Lie group. In this section we collect some background about the homotopy theory of $G$-spaces, including the construction of an equivariant approximation relative to a collection of subgroups.

Let $\mathcal{O}_G$ denote the orbit category of $G$, whose objects are $G$-orbits and whose morphisms are $G$-equivariant maps. The category $\mathcal{O}_G$ is a topologically enriched category: the morphism sets are naturally topologized as subspaces of orbits of $G$. Elmendorf’s seminal work [Elm83] established that the homotopy category of $G$-spaces is equivalent to the homotopy category of continuous functors $\mathcal{O}_G^{\text{op}} \to \text{Top}$. 
In this section we will review Elmendorf’s construction with appropriate language for our needs, and extend it to spaces whose isotropy groups are restricted to a collection of subgroups. We use this generalization to establish the criterion, standard for finite groups, for a map to be a weak G-equivalence (Corollary 2.11).

Elmendorf’s equivalence of categories is induced by the following functor.

**Definition 2.1.** Given a G-space X, let \( \Phi_X \) be the functor \( \mathcal{O}_G^{op} \to \text{Top} \) given by the formula \( \Phi_X(O) = \text{map}_G(O, X) \).

We may view \( \Phi \) as a functor from G-spaces (i.e. X) to the category of continuous functors \( \mathcal{O}_G^{op} \to \text{Top} \) (i.e. \( \Phi_X \)). By Elmendorf’s theorem, \( \Phi \) induces an equivalence of homotopy categories. Further, \( \Phi \) is actually the right adjoint of a Quillen equivalence [Pia91]. The (derived) left adjoint of \( \Phi \) can be constructed as a (derived) enriched CoEnd, as we discuss below.

**Enriched categories and functors.** Let \( (\mathcal{V}, \boxtimes, 1) \) be a closed symmetric monoidal category with small limits and colimits. Our prime examples of categories \( \mathcal{V} \) are the category Top, topological spaces, and the category Ch\(_{\mathbb{Z}}\), chain complexes of abelian groups. Suppose \( \mathcal{C} \) is a category enriched over \( \mathcal{V} \). Thus \( \mathcal{C} \) consists of a set of objects, and for every two objects \( x, y \), the morphisms from \( x \) to \( y \) are given by an object of \( \mathcal{V} \), denoted \( \mathcal{C}(x, y) \). There are unit morphisms \( 1 \to \mathcal{C}(x, x) \) for every object \( x \), and composition morphisms \( \mathcal{C}(x, y) \boxtimes \mathcal{C}(y, z) \to \mathcal{C}(x, z) \), and these structure maps are associative and unital. An enriched functor \( F: \mathcal{C} \to \mathcal{V} \) associates to every object \( x \) of \( \mathcal{C} \) an object \( F(x) \) of \( \mathcal{V} \); to any two objects \( x, y \) of \( \mathcal{C} \), the functor \( F \) associates a \( \mathcal{V} \)-morphism \( F(x) \boxtimes \mathcal{C}(x, y) \to F(y) \). Once again, these structure maps are required to be associative, and unital. There is a similar definition for a contravariant enriched functor \( G: \mathcal{C} \to \mathcal{V} \), with the structure morphisms having the form \( \mathcal{C}(x, y) \boxtimes G(y) \to G(x) \).

**Remark 2.2.** When \( \mathcal{V} \) is a nonspecific symmetric monoidal category, we will use the symbol \( \boxtimes \) to denote the monoidal product. When \( \mathcal{V} \) is a specific category, however, we will use the standard notation for the symmetric monoidal product for that category. Thus when \( \mathcal{V} = \text{Top} \), we use \( \times \) to denote the monoidal product, and when \( \mathcal{V} = \text{Ch}_{\mathbb{Z}} \) we will use \( \otimes \).

**Enriched constructions.** Next we look at enriched CoEnd and End (natural transformations). A covariant functor \( F \) and a contravariant functor \( G \) from \( \mathcal{C} \) to \( \mathcal{V} \) are analogous to a right and a left module, respectively, over a ring. The enriched CoEnd of \( F \) and \( G \) is analogous to the tensor product of modules. Dually, the enriched End is analogous to hom of modules.

The enriched CoEnd of \( F \) and \( G \), denoted \( F \boxtimes_{\mathcal{C}} G \) is defined by the usual coequalizer diagram in \( \mathcal{V} \):

\[
F \boxtimes_{\mathcal{C}} G \leftarrow \bigoplus_{x_0} F(x_0) \boxtimes G(x_0) \leftarrow \bigoplus_{x_0, x_1} F(x_0) \boxtimes \mathcal{C}(x_0, x_1) \boxtimes G(x_1)
\]

Dually, if \( F \) and \( G \) are both covariant enriched functors, then the object of enriched natural transformations from \( F \) to \( G \) (sometimes called the enriched End of \( F \) and \( G \)) is defined by the following equalizer diagram, where \( \text{hom}_\mathcal{V} \) denotes the internal hom object in \( \mathcal{V} \):

\[
\text{nat}_\mathcal{C}(F, G) \rightarrow \prod_{x_0} \text{hom}_\mathcal{V}(F(x_0), G(x_0)) \rightrightarrows \prod_{x_0, x_1} \text{hom}_\mathcal{V}(F(x_0) \boxtimes \mathcal{C}(x_0, x_1), G(x_1))
\]
Derived constructions. Adding the next piece of structure, we suppose that $\mathcal{V}$ is actually a closed symmetric monoidal Quillen model category. Limits and colimits do not always preserve weak equivalences. Therefore in the homotopy setting one often needs to work with derived (“homotopy”) limits and colimits. These constructions constitute universal approximations to the strict limits and colimits by homotopy invariant constructions. We will focus on derived CoEnd and End in particular.

Under mild assumptions on $\mathcal{V}$, the category of $\mathcal{V}$-enriched functors is equipped with the projective Quillen model structure—the structure where fibrations and weak equivalences are determined level-wise. One way to define the derived CoEnd of $F$ and $G$ is by taking the strict CoEnd of cofibrant replacements of $F$ and $G$. Dually, derived natural transformations from $F$ to $G$ are the strict natural transformations from a cofibrant replacement of $F$ to a fibrant replacement of $G$. This is the viewpoint taken, for example, in \cite{ADLINS}. In the current work, however, we chose to define the derived CoEnd/End via the bar/cobar constructions. This definition is sometimes convenient for calculations, and is also well-suited for proving specialized invariance results such as Lemma \ref{lem:1} later on. For detail beyond what is given in Definition \ref{def:1} and Remark \ref{rem:1} we refer the reader to \cite{Shu09, Rie14}.

Definition 2.3.

(1) Suppose $F$ is an enriched covariant functor $C \to \mathcal{V}$, and that $G$ is an enriched contravariant functor $C \to \mathcal{V}$. We assume that all the objects $C(x, y)$ are cofibrant in $\mathcal{V}$, and that $F$ and $G$ are objectwise cofibrant. The derived CoEnd of $F$ and $G$ will be denoted by $F \boxtimes_{C}^{h} G$, or sometimes by $F(x) \boxtimes_{x \in C}^{h} G(x)$, or by $F(x) \boxtimes_{x \in C}^{h} G(x)$, depending on what needs to be emphasized. It is defined to be the realization of the following simplicial object in $\mathcal{V}$:

$$\bigoplus_{x_{0}} F(x_{0}) \boxtimes G(x_{0}) \equiv \bigoplus_{x_{0}, x_{1}} F(x_{0}) \boxtimes C(x_{0}, x_{1}) \boxtimes G(x_{1})$$

$$\equiv \bigoplus_{x_{0}, x_{1}, x_{2}} F(x_{0}) \boxtimes C(x_{0}, x_{1}) \boxtimes C(x_{1}, x_{2}) \boxtimes G(x_{2}) \ldots$$

(2) Suppose $F$ and $G$ are both covariant functors from $C$ to $\mathcal{V}$. Assume that $F$ is objectwise cofibrant and $G$ is objectwise fibrant. The derived natural transformations from $F$ to $G$, denoted $\text{hnat}_{\mathcal{V}}(F, G)$, is the totalization of the following cosimplicial object in $\mathcal{V}$:

$$\prod_{x_{0}} \text{hom}_{\mathcal{V}}(F(x_{0}), G(x_{0})) \equiv \prod_{x_{0}, x_{1}} \text{hom}_{\mathcal{V}}(F(x_{0}) \boxtimes C(x_{0}, x_{1}), G(x_{1}))$$

$$\equiv \bigoplus_{x_{0}, x_{1}, x_{2}} \text{hom}_{\mathcal{V}}(F(x_{0}) \boxtimes C(x_{0}, x_{1}) \boxtimes C(x_{1}, x_{2}), G(x_{2})) \ldots$$

Remark 2.4. By geometric realization of a simplicial object (resp. the totalization of a cosimplicial object) in $\mathcal{V}$, we mean the homotopy colimit (resp. homotopy limit) of the underlying diagram, taken in $\mathcal{V}$. Homotopy limits and colimits can be defined in any model category. They are only defined up to a natural weak equivalence, but it does not really matter which model we use.

In some cases there is a particularly nice model for the geometric realization. For example, suppose $\mathcal{V} = \text{Top}$, and that for every object $x$ of $C$, the mapping
space \( C(x, x) \) is well-pointed by the identity map; then the simplicial bar resolution of \( F \times_C^h G \) is a Reedy cofibrant simplicial space, and one may use the classic construction of geometric realization. For another example, if \( V = \mathbb{Z}_2 \), then a good model for the geometric realization/totalization is obtained by first applying the normalized chains functor to obtain a bicomplex, and then taking the total complex. See [Dug14] Proposition 16.9 for the simplicial case, and [Bou87] Lemma 2.2 for the cosimplicial case.

The derived constructions above come equipped with a derived (co)Yoneda lemma. Suppose \( z \) is a fixed object of \( C \). Let \( C(\cdot, z) \) be the contravariant functor represented by \( z \), i.e., the functor \( x \mapsto C(x, z) \). The following is a well-known version of the derived enriched (co)Yoneda lemma. As we will see, it is a basic tool for proving facts about the derived CoEnd and End (for example, Theorem 2.7, Proposition 2.10).

**Lemma 2.5.** [Rie14, Example 4.5.7, Theorem 5.1.1] Let \( F \) be a covariant functor enriched over \( C \). The evaluation maps \( F(x) \boxtimes C(x, z) \to F(z) \) induce a natural weak equivalence in \( V \)

\[
\text{(coYoneda lemma)} \\
F(x) \boxtimes_{x \in C} C(x, z) \xrightarrow{\sim} F(z).
\]

Dually, there is a natural weak equivalence

\[
\text{(Yoneda lemma)} \\
F(z) \xrightarrow{\sim} \text{hnat}_C \left( C(z, \cdot), F(\cdot) \right).
\]

**Elmendorf’s construction.** With enriched and derived CoEnd in hand, we describe Elmendorf’s construction in this language. Let \( X \) be a \( G \)-space. Recall that the functor \( \Phi_X : \mathcal{O}_G^{op} \to \text{Top} \) is defined by the formula \( \Phi_X(O) = \text{map}_G(O, X) \). One can consider \( \Phi \) as a functor of two variables: \( X \) and \( O \). A key property of \( \Phi \) is that it preserves the Bousfield-Kan model for a homotopy colimits in the variable \( X \).

**Lemma 2.6.** Suppose that \( \{ X_{\alpha} \}_{\alpha \in I} \) is a diagram of \( G \)-spaces. Let \( \text{hocolim} \) denote the Bousfield-Kan homotopy colimit, and set \( X = \text{hocolim}_I X_{\alpha} \). There is a natural isomorphism of functors

\[
\text{hocolim}_{\alpha \in I} \Phi_{X_{\alpha}} \to \Phi_X.
\]

**Proof.** We need to show that for every \( O \in \mathcal{O}_G \), the following map is a homeomorphism:

\[
\text{hocolim}_{\alpha \in I} \left( \text{map}_G(O, X_{\alpha}) \right) \to \text{map}_G \left( O, \text{hocolim}_{\alpha \in I} X_{\alpha} \right).
\]

Representing \( O \) as \( G/H \) for some subgroup \( H \) of \( G \), we can identify \( \text{map}_G(O, X) \) with \( X^H \). The lemma follows because fixed points of an action of a compact Lie group commute with Bousfield-Kan homotopy colimits [Mal14, Proposition 1.2].

The functor \( \Phi_X \) is a contravariant enriched functor \( \mathcal{O}_G \to \text{Top} \), and can be paired with a covariant enriched functor in a derived CoEnd. Let \( Q \) be a particular object of \( \mathcal{O}_G \), i.e., a \( G \)-orbit, and consider the representable covariant functor \( F = \text{map}_G(Q, \cdot) \). By Lemma 2.5, there is a natural equivalence \( F \times_Q^h \Phi_X \to F(X) \), or

\[
\left[ \text{map}_G(Q, O) \right] \times^h_{O \in \mathcal{O}_G} \left[ \text{map}_G(O, X) \right] \xrightarrow{\sim} \text{map}_G(Q, X).
\]
In particular, taking $Q$ to be the free $G$-orbit, $Q = G$, we have a natural isomorphism $\Phi_X(G) \cong X$, giving an assembly map

$$O \times^h_{O \in O_G} \Phi_X(O) \longrightarrow X.$$ 

Note that every orbit $O$ has a left action of $G$, and all the face and degeneracy maps in the simplicial object defining $O \times^h_{O \in O_G} \Phi_X(O)$ are $G$-equivariant. It follows that $O \times^h_{O \in O_G} \Phi_X(O)$ has a natural action of $G$ via the action on $O$. The assembly map is $G$-equivariant by inspection. Elmendorf’s theorem says that it is actually a weak $G$-equivalence.

**Theorem 2.7.** [Elm83, Theorem 1] The assembly map $O \times^h_{O \in O_G} \Phi_X(O) \longrightarrow X$ is a weak $G$-equivalence.

**Proof.** We need to prove that for every subgroup $H \subseteq G$, the map induces an equivalence of fixed points

$$\left( O \times^h_{O \in O_G} \Phi_X(O) \right)^H \longrightarrow X^H.$$ 

Since $H$ acts $O \times^h_{O \in O_G} \Phi_X(O)$ via its action on $O$, we see that for every $O$ there is a homeomorphism $\left( O \times \Phi_X(O) \right)^H \cong O^H \times \Phi_X(O)$. Furthermore, $H$-fixed points commute with geometric realization [Mal14]. Therefore, what we need to show is that the map $O^H \times^h_{O \in O_G} \Phi_X(O) \longrightarrow X^H$ is a weak equivalence. For any $G$-space $Y$, there is a natural homeomorphism $Y^H \cong \text{map}_G(G/H,Y)$, so it is enough to show that, for the orbit $Q = G/H$ of $G$, the following map is a weak equivalence of spaces:

$$\text{map}_G(Q, O) \times^h_{Q \in O} \Phi_X(O) \longrightarrow \text{map}_G(Q, X).$$

But $\text{map}_G(Q, Q) = \Phi_X(Q)$. The functor $\text{map}_G(Q, -)$ is a contravariant representable functor, so the result follows by the coYoneda lemma (Lemma 2.5). $\square$

**Approximation relative to a collection of subgroups.** A metastatement of Elmendorf’s theorem is that $G$-spaces are essentially the same thing as contravariant continuous functors from $O_G$ to Top. Let $C$ be a collection of subgroups of $G$, i.e., a set of closed subgroups, closed under conjugation (but not necessarily closed under passage to subgroups). Let $O_C$ be the corresponding orbit category, the full subcategory of $O_G$ consisting of orbits whose isotropy groups are in $C$. The metastatement of Elmendorf’s theorem can be generalized as follows: $G$-spaces whose isotropy groups are contained in $O_C$ are essentially the same thing as contravariant continuous functors from $O_C$ to Top. We will now make this more precise (see in particular Corollary 2.11 below). First, a definition. Recall that if $Z$ is a $G$-space, then $\text{Iso}(Z)$ denotes the collection of isotropy subgroups of $Z$.

**Definition 2.8.** Let $X$ be a $G$-space, and $C$ a collection of subgroups of $G$. A $G$-map $Z \rightarrow X$ is called a $C$-approximation of $X$ if it satisfies the following properties:

1. $\text{Iso}(Z) \subseteq C$
2. If $H \in C$, then the map $Z^H \rightarrow X^H$ is a weak equivalence.
Remark 2.9. When $G$ is a finite group, approximation with respect to a collection of subgroups of $G$ was discussed in detail in [AD01]. The construction and its properties extend to compact Lie groups without any difficulty, but we do not know if it is documented in the literature in the generality we need.

Approximations have the usual existence and uniqueness statement.

Proposition 2.10. Let $X$ be a $G$-CW-complex and let $C$ be a collection of subgroups of $G$. There exists a functorial $C$-approximation $\alpha_X : X_C \to X$ by a $G$-CW-complex $X_C$ that has the following properties.

1. If $\text{Iso}(X) \subseteq C$, then $\alpha_X$ is a weak $G$-equivalence.
2. $\text{Iso}(X_C) \subseteq C$.
3. If $H \in C$, then $\alpha_X$ induces a weak equivalence $(X_C)^H \to X^H$.

The map $\alpha_X : X_C \to X$ is characterized up to a weak $G$-equivalence by properties (2) and (3).

Proof. We define $X_C$ by the homotopy CoEnd

$$X_C := \text{Hom}_{\mathcal{O}_C}(O, \Phi_X(O)).$$

The inclusion of orbit categories $\mathcal{O}_C \hookrightarrow \mathcal{O}_G$ induces a map

$$\text{Hom}_{\mathcal{O}_C}(O, \Phi_X(O)) \to \text{Hom}_{\mathcal{O}_G}(O, \Phi_X(O)),$$

and therefore a map of $G$-spaces $\alpha_X : X_C \to X$. It is clear from this definition that $X_C$ depends functorially on $X$, and the map $X_C \to X$ is a natural transformation.

The verification of the required isotropy and fixed-point properties is very similar to the proof of Theorem 2.7. If $X \in \mathcal{O}_C$, then the functor $\Phi_X(O) = \text{map}_G(O, X)$ is (contravariant) representable when it is restricted to a functor $\mathcal{O}_C^{\text{op}} \to \text{Top}$. Using the coYoneda lemma (Lemma 2.5), it follows that the map

$$X_C = \text{Hom}_{\mathcal{O}_C}(O, \Phi_X(O)) \to X$$

is a weak $G$-equivalence. On the other hand, it follows from Lemma 2.6 that the functor $X \mapsto X_C$ preserves homotopy colimits. Therefore, the class of spaces $X$ for which the map $\alpha_X : X_C \to X$ is a weak $G$-equivalence is closed under homotopy colimits. It follows that this class contains all $G$-CW-complexes $X$ for which $\text{Iso}(X) \subseteq C$, thus establishing (1). Further, it follows immediately from the construction of $X_C$ that $\text{Iso}(X_C) \subseteq C$, establishing (2).

To establish (3), the fixed-point property, suppose that $H \in C$. The map $(X_C)^H \to X^H$ induced by $\alpha_X$ can be identified with the map

$$\text{map}_G(G/H, O) \times_{\mathcal{O}_C} \Phi_X(O) \to \Phi_X(G/H) = \text{map}_G(G/H, X).$$

Again by the coYoneda lemma, this map is an equivalence because the functor $\text{map}_G(G/H, -)$ is representable.

Finally, suppose that $f : Y \to X$ is another map satisfying properties (2) and (3), i.e. $\text{Iso}(Y) \subseteq C$ and $Y^H \to X^H$ is a weak equivalence for $H \in C$. Consider the diagram

$$
\begin{array}{ccc}
Y_C & \xrightarrow{=} & X_C \\
\downarrow^{\alpha_Y} & & \downarrow^{\alpha_X} \\
Y & \xrightarrow{f} & X.
\end{array}
$$
Corollary 2.11. Let $X$ and $Y$ be $G$-CW-complexes, and suppose that $\text{Iso}(X) \cup \text{Iso}(Y) \subseteq C$. Suppose that $f : X \to Y$ is a $G$ map, and that $f^H : X^H \to Y^H$ is a weak equivalence for all $H \in C$. Then $f$ is a weak $G$-equivalence.

Proof. Consider the commutative square

$$
\begin{array}{ccc}
X_C & \longrightarrow & Y_C \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

The vertical maps are weak $G$-equivalences by Proposition 2.10. The top horizontal map is a map between homotopy CoEnds

$$
O \times_{O \in \mathcal{O}_C} \Phi_X(O) \longrightarrow O \times_{O \in \mathcal{O}_C} \Phi_Y(O).
$$

By assumption, the map $\Phi_X(O) \to \Phi_Y(O)$ is a weak equivalence for every $O \in \mathcal{O}_C$. It follows that the top horizontal map (of homotopy CoEnds) is a weak $G$-equivalence, and hence the lower horizontal map is as well. □

3. Bredon (co)homology of spaces

In this section we review some standard material about Bredon (co)homology for a compact Lie group, with coefficients in a local system. The most important result is a criterion for a $G$-equivariant map to induce an isomorphism in Bredon homology (Lemma 3.7). The criterion is certainly known, but we are not sure if it is recorded in the literature in the generality that we need.

As before, let $G$ be a compact Lie group, and let $\mathcal{O}_G$ denote its orbit category, whose morphism sets are naturally topologized as subspaces of orbits of $G$. Let $\pi_0 \mathcal{O}_G$ be the homotopy category of $\mathcal{O}_G$, that is, the category obtained by taking path components of the morphism spaces.

Definition 3.1. Let $\mathcal{A}$ be an abelian category. A coefficient system for $G$ with values in $\mathcal{A}$ is a functor $M : \mathcal{O}_G \to \mathcal{A}$ that factors through the projection functor $\mathcal{O}_G \to \pi_0 \mathcal{O}_G$. We say $M$ is a homological coefficient system if $M$ is covariant, and a cohomological coefficient system if $M$ is contravariant.

By including $\mathcal{A}$ in the category $\text{Ch}_\mathcal{A}$ of chain complexes (over $\mathcal{A}$) concentrated in dimension 0, we may consider a coefficient system $M$ as a homotopy functor from $\mathcal{O}_G$ to $\text{Ch}_\mathcal{A}$. Let $\text{Top}_G$ be the category of $G$-spaces. The derived Kan extension of $M$ along the inclusion $\mathcal{O}_G \hookrightarrow \text{Top}_G$ is a homotopy functor from $\text{Top}_G$ to $\text{Ch}_\mathcal{A}$. The homology of this functor defines Bredon (co)homology with coefficients in $M$, according to the variance of $M$. The rest of this section gives details.

---

2We do not actually need the full generality of using an abelian category as the target of a coefficient system, but it is sometimes convenient to have $M$ at least take values in graded abelian groups rather than just abelian groups.
Definition of Bredon (co)homology. Let \( \text{Ch}_\mathbb{Z} \) denote the category of chain complexes of abelian groups, and let \( C_* : \text{Top} \rightarrow \text{Ch}_\mathbb{Z} \) be the normalized singular chains functor, which is a lax symmetric monoidal functor. Suppose \( O \) (shortly to be the orbit category \( O_G \)) is a category enriched over \( \text{Top} \). We define \( C_*^O \) as the category obtained by applying the normalized singular chains functor \( C_*(\quad; \mathbb{Z}) \) to the hom objects of \( O \). The category \( C_*^O \) has the same objects as \( O \), and it is enriched over \( \text{Ch}_\mathbb{Z} \). Furthermore, suppose \( F : O \rightarrow \text{Top} \) is a topologically enriched functor. Then the functor \( C_*F : C_*^O \rightarrow \text{Ch}_\mathbb{Z} \) is a functor enriched over \( \text{Ch}_\mathbb{Z} \).

After composing \( C_* \) with \( \Phi_X \), we obtain the functor \( C_* \Phi_X : O^\text{op}_G \rightarrow \text{Ch}_\mathbb{Z} \) by the formula \( O \mapsto (C_* \Phi_X)(O) \). Let \( M \) be a (co)homological coefficient system, which one may think of as a (possibly contravariant) enriched functor \( M : C_*^O \rightarrow \text{Ch}_\mathbb{Z} \).

Definition 3.2.

1. If \( M \) is a homological coefficient system, the Bredon homology of \( X \) with coefficients in \( M \), denoted \( H^G_0(X; M) \), is the homology of the chain complex

\[
M \otimes^b_{C_*^O} C_* \Phi_X.
\]

2. If \( M \) is a cohomological coefficient system, then the Bredon cohomology of \( X \) with coefficients in \( M \), denoted \( H^G_*(X; M) \), is the homology of the cochain complex

\[
\text{hnat}_{C_*^O}(C_* \Phi_X, M).
\]

The Bredon (co)homology groups of Definition 3.2 satisfy an equivariant version of the Eilenberg-Steenrod axioms. In particular, Lemma 3.3 below says that Bredon (co)homology satisfies an equivariant version of the dimension axiom. All the other Eilenberg-Steenrod axioms are exactly the same for equivariant (co)homology as their non-equivariant counterparts [Wil75].

Lemma 3.3. If \( X \in O_G \) then there is an isomorphism \( H^G_0(X; M) \cong M(X) \), natural in \( X \). Moreover, \( H^G_*(X; M) = 0 \) for \( i > 0 \). The dual statement holds for cohomology.

Proof. If \( X \in O_G \), then \( C_* \Phi(X) \) is a representable contravariant functor, so the claim follows by the enriched (co)Yoneda lemma (Lemma 2.5). \qed

Remark 3.4. An alternative, commonly used construction of Bredon homology uses cellular chains. For example, see [Wil75] or [May96, Chapter I.4]. Any two theories that satisfy the Eilenberg-Steenrod axioms and agree on \( O_G \) are naturally isomorphic on the category of \( G \)-CW-complexes [Wil75, Corollary 3.2]. Therefore the cellular definition agrees with Definition 3.2. We chose to use homotopy \( \text{CoEnd} \) and \( \text{End} \) in our definition, because some invariance results seem most apparent from this perspective. See in particular, Lemma 3.7 below.

Spaces with isotropy in a collection. As in Section 2 suppose \( C \) is a collection of subgroups of \( G \) and \( O_C \) is the corresponding subcategory of the orbit category \( O_G \). We will now see another manifestation of the metastatement that if \( C \) is a collection of subgroups, then spaces whose isotropy is contained in \( C \) are essentially the same as contravariant functors from \( O_C \) to \( \text{Top} \). The following lemma is closely analogous to Proposition 2.10 (1).
Lemma 3.5. Suppose that $X$ is a $G$-CW-complex all of whose isotropy groups are in $C$. Then the inclusion of subcategories $\mathcal{O}_C \to \mathcal{O}_G$ induces a quasi-isomorphism of chain complexes

$$ M \otimes_{\mathcal{O}_C} C_* \Phi_X \to M \otimes_{\mathcal{O}_G} C_* \Phi_X. $$

In particular, either chain complex can be used to calculate $H^G_*(X; M)$. A dual statement holds for cohomology.

Proof. If $X$ is an object of $\mathcal{O}_C$, the result follows by applying the (co)Yoneda lemma to both sides of (3.6). However, the functor $\Phi_X$ preserves homotopy colimits in the variable $X$ (Lemma 2.6). Therefore the class of spaces $X$ for which the lemma holds is closed under arbitrary homotopy colimits, and in particular it contains the class of spaces whose isotropy is contained in $C$. □

Criterion for isomorphic Bredon (co)homology. For our applications we need determine circumstances under which homological properties of the fixed point diagram $\Phi_X$ determine Bredon (co)homology. The following lemma is an algebraic analogue of Corollary 2.11. We will use the lemma later to prove that if a $G$-map induces a mod $p$ homology isomorphism on certain fixed point sets, then it actually induces an isomorphism on Bredon homology for coefficients that are $p$-local in a suitable sense (Proposition 4.12).

Lemma 3.7. Let $f : X \to Y$ be a map of $G$-spaces, and let $M$ be a coefficient system for Bredon (co)homology. Assume that for all isotropy groups $H_1, H_2 \in \text{Iso}(X) \cup \text{Iso}(Y)$, the map $f : X^{H_1} \to Y^{H_2}$ induces an isomorphism of ordinary (co)homology groups with coefficients in $M(G/H_2)$. Then $f$ induces an isomorphism in Bredon (co)homology with coefficients in $M$.

Proof. Consider the case of homology. Let $C = \text{Iso}(X) \cup \text{Iso}(Y)$. The inclusion $\mathcal{O}_C \to \mathcal{O}_G$ gives us a commutative square

$$ M \otimes_{\mathcal{O}_C} C_* \Phi_X \to M \otimes_{\mathcal{O}_G} C_* \Phi_X $$

Our goal is to show that the right vertical map is a quasi-isomorphism. The horizontal maps are quasi-isomorphisms by Lemma 3.5, so it is enough to prove that the left vertical map is a quasi-isomorphism. That map is induced by a homomorphism between simplicial chain complexes so it is enough to prove that it is a quasi-isomorphism in simplicial degree $n$ for all $n$.

Suppose that $H_0, ..., H_n$ are subgroups of $G$, and let $Z(H_0, ..., H_n)$ denote the chain complex $C_* ((G/H_1)^{H_0}) \otimes \cdots \otimes C_* ((G/H_n)^{H_{n-1}})$. The chain complex $Z(H_0, ..., H_n)$ could be zero, if one of the subgroups is not subconjugate to the next, but it is certainly a complex of free abelian groups, by definition of $C_*$. In simplicial degree $n$ of the left vertical map of (3.8), we have
the homomorphism
\[
\bigoplus_{H_0, \ldots, H_n \in C} \left[ M(G/H_0) \otimes Z(H_0, \ldots, H_n) \otimes C_*(X^{H_n}) \right] \]
(3.9)
\[
\bigoplus_{H_0, \ldots, H_n \in C} \left[ M(G/H_0) \otimes Z(H_0, \ldots, H_n) \otimes C_*(Y^{H_n}) \right].
\]

Our assumption implies that the homomorphism
\[
M(G/H_0) \otimes C_*(X^{H_n}) \rightarrow M(G/H_0) \otimes C_*(Y^{H_n})
\]
is a quasi-isomorphism. It follows that (3.9) is also a quasi-isomorphism, because \(Z(H_0, \ldots, H_n)\) is free. Therefore the left vertical map in (3.8), an induced map of homotopy CoEnds, is a quasi-isomorphism as well.

The proof in the cohomology case is similar. □

**Remark.** The main application of Lemma 3.7 will be to a situation where \(M\) takes values in \(p\)-local abelian groups and \(f\) induces an isomorphism in homology with \(p\)-local coefficients on fixed point spaces \(X^H \rightarrow Y^H\) for every \(H \in \text{Iso}(X) \cup \text{Iso}(Y)\). See Proposition 4.12 below.

4. \(p\)-Toral Approximations

Given a compact Lie group \(G\), let \(\mathcal{A}_G\) denote the collection of all \(p\)-toral subgroups of \(G\). We want to show that approximations of \(G\)-spaces by this collection can be used to compute Bredon (co)homology for suitable choices of the coefficient system \(M\), that is, those satisfying Definition 4.1 below. After the statement of Theorem 4.2, our main result along these lines, we spend the first part of the section discussing (co)Mackey functors, to make sense of Definition 4.1. We then discuss restriction and induction of (co)Mackey functors, and finally we prove Theorem 4.2 in several stages.

**Definition 4.1.** We say that a (co)Mackey functor \(M\) for \(G\) satisfies the transfer condition for the prime \(p\) if, for all \(G\)-spaces \(X\) and a maximal \(p\)-toral subgroup \(P\) of \(G\), the appropriate resulting composite map on Bredon (co)homology is an isomorphism:
\[
H^G_*(X; M) \xrightarrow{\text{tr}} H^G_*(G/P \times X; M) \rightarrow H^G_*(X; M)
\]
\[
H^G_*(X; M) \rightarrow H^G_*(G/P \times X; M) \xrightarrow{\text{tr}} H^G_*(X; M).
\]

In this section, our goal is to establish the following theorem.

**Theorem 4.2.** Let \(G\) be a compact Lie group, and let \(M\) be a (co)Mackey functor for \(G\) that satisfies the transfer condition for a prime \(p\). Then for all \(G\)-CW-complexes \(X\), the map \(\alpha_X : X_{\mathcal{A}_G} \rightarrow X\) is an \(M\)-(co)homology isomorphism.

**Mackey functors and compact Lie groups.** In Section 3 we recalled the Bredon (co)homology of \(G\)-spaces with coefficients in a functor from \(O_G\) to an abelian category (a “coefficient system”). We now focus on coefficient systems that have an additional structure, that of a (co)Mackey functor (Definition 4.3). The new tool that becomes available as a consequence of this assumption is the transfer homomorphism.
Let \( S(O_G) \) be the stable homotopy category of \( G \)-orbits. Thus objects of \( S(O_G) \) are equivariant suspension spectra \( \Sigma^\infty O_+ \), where \( O \) is a \( G \)-orbit, and morphisms are stable \( G \)-equivariant homotopy classes of maps between such spectra.

**Definition 4.3.** Let \( A \) be an abelian category. A coMackey functor for \( G \) with values in \( A \) is an additive functor \( M: S(O_G) \to A \). A Mackey functor for \( G \) with values in \( A \) is an additive functor \( M: S(O_G)^{\text{op}} \to A \).

(The convention that Mackey functors are contravariant and coMackey functors are covariant is established in the literature, and we are not going to try to subvert it.)

**Remark.** When \( G \) is a finite group, Spanier-Whitehead duality induces an isomorphism between the category \( S(O_G) \) and \( S(O_G)^{\text{op}} \) that is the identity on objects. This in turn induces an isomorphism between the categories of Mackey and coMackey functors that is the identity on objects. But for compact Lie groups that are not finite groups, this duality does not exist: the Spanier-Whitehead dual of \( G/H \) is not equivalent to \( G/H \), or to any other object of \( S(O_G) \). This is why we must deal explicitly with both Mackey and coMackey functors in this work, where in [ADL16] we were able to suppress the distinction.

Note that the assignment \( O \mapsto \Sigma^\infty O_+ \) defines a functor \( O_G \to S(O_G) \). Thus a coMackey functor \( M \) (in the sense of Definition 4.3) gives rise to an “ordinary” coefficient system (in the sense of Definition 3.1). We will generally denote the coefficient system defined by a coMackey functor \( M \) with the same letter \( M \). Dually, a Mackey functor gives rise to a cohomological coefficient system.

The key point is that when a coefficient system comes from a (co)Mackey functor in this way, one can extend the associated (co)homology theory from the category of \( G \)-spaces to the category of \( G \)-spectra. The constructions of (co)homology groups of \( G \)-spectra with coefficients in a (co)Mackey functor are very similar to the constructions of Bredon homology and cohomology alluded to in Remark 3.4. Namely, when \( X \) is a \( G \)-CW spectrum, the cellular chain complex of \( X \) is a chain complex of coMackey functors (and dually, cellular cochains are Mackey functors). One then defines the homology groups as the homology of a homotopy (Co)End, just as in Definition 3.2. Standard cellular techniques guarantee that the construction is homotopy invariant. For more details, see [May96, Chapter XIII, Section 4] or [LMM81]. If \( X \) is a \( G \)-space and \( M \) is a coMackey functor, then the Bredon homology groups \( H^G_*(X; M) \) are canonically isomorphic to the new homology groups \( H^G_*(\Sigma^\infty X_+; M) \) [May96 page 138].

(Co)homology theories on the category of \( G \)-spectra are sometimes referred to as \( RO(G) \)-graded theories. However, we will only consider \( \mathbb{Z} \)-graded (co)homology groups associated with such theories. In any case, \( RO(G) \)-graded cohomology theories on spaces are the same thing as \( \mathbb{Z} \)-graded theories defined on the category of \( G \)-spectra [May96, Chapter XIII Corollary 3.3].

The most important consequence of a (co)homology theory being \( RO(G) \)-graded is that it has transfers. More specifically, suppose \( \eta: E \to B \) is a \( G \)-equivariant fibration, whose fiber is a finite complex \( F \). The equivariant Becker-Gottlieb transfer of \( \eta \) is a \( G \)-map \( \text{tr}: \Sigma^\infty B_+ \to \Sigma^\infty E_+ \). Therefore, for any coMackey functor \( M \), a fiber bundle \( \eta \) as above gives rise to a transfer homomorphism \( H^G_*(B; M) \to H^G_*(E; M) \). Dually, if \( M \) is a Mackey functor, then a fiber bundle \( \eta \) gives rise to a transfer homomorphism \( H^G_*(E; M) \to H^G_*(B; M) \).
Finally, suppose $M$ is a coMackey functor. The groups $H^G_\ast(E; M)$ satisfy a dimension axiom analogous to the one stated in Lemma 3.3. Namely, there is an isomorphism $H^G_\ast(-; M) \cong M(-)$ of functors from $S(O_G)$ to $Ab$, and for $n \neq 0$, $H^G_n(-; M) \cong \{0\}$. (Similarly for cohomology with coefficients in a Mackey functor.) Homology and cohomology with coefficients in $M$ also satisfy all the other Eilenberg-Steenrod axioms for (co)homology theory on the category of $G$-spectra.

**Restriction and induction.** Suppose $G$ is a group and $H$ a subgroup of $G$. There is an induction functor $i: O_H \rightarrow O_G$ given by $O \mapsto G \times H O$. If $M$ is a coefficient system for $G$, the composition $Mi$ is a coefficient system for $H$. We call $Mi$ the restriction of $M$ to $H$, and denote it by $M|_H$. The following is a standard result, a consequence of the fact that (derived) left Kan extension is (derived) left adjoint to restriction.

**Lemma 4.4.** With $G$ and $H$ as above, let $X$ be a space with an action of $H$ and let $M$ be a coefficient system for $G$. There is a natural isomorphism

$$H^H_\ast(X; M|_H) \cong H^G_\ast(G \times_H X; M),$$

where if $M$ is cohomological then homology is replaced with cohomology.

Note also that if $X$ is a space with an action of $G$, there is a canonical homeomorphism of $G$-spaces $G/H \times X \cong G \times_H X$. Therefore in this case we have an isomorphism $H^G_\ast(X; M|_H) \cong H^G_\ast(G/H \times X; M)$

**Proving an approximation theorem.** Our approach to proving Theorem 4.2 is to formulate it as a property of $G$ and $M$, and then use induction over the size of $p$-toral subgroups to establish the theorem.

**Definition 4.5.** Let $G$ be a compact Lie group and let $M$ be a Bredon coefficient system for $G$. We say that the group $G$ has the $p$-toral approximation property for $M$ if, for all $G$-CW-complexes $X$, the map $X_{AG} \rightarrow X$ is an $M$-(co)homology isomorphism.

We first recall two standard lemmas, with the goal of reducing the problem to approximating a point.

**Lemma 4.6.** If $(G/K)_{AG} \rightarrow G/K$ induces an $M$-(co)homology isomorphism for all subgroups $K \subseteq G$, then $G$ has the $p$-toral approximation property for $M$.

**Proof.** We focus on the homology case. The approximation functor $X \mapsto X_{AG}$ preserves homotopy colimits. The “Bredon chains” functor $X \mapsto M \otimes^{hc}_{C_*O_G} C_*\Phi_X$ also preserves homotopy colimits. It follows that the class of spaces $X$ for which the lemma holds is closed under homotopy colimits. If that class includes all orbits of $G$, then it includes all $G$-CW-complexes. 

**Lemma 4.7.** Suppose $K \subseteq G$. If the map $*_{AK} \rightarrow *$ is an $M|_K$-(co)homology isomorphism, then $(G/K)_{AG} \rightarrow G/K$ is an $M$-(co)homology isomorphism.

**Proof.** There is a canonical $G$-equivalence

$$G \times_K (*_{AK}) \rightarrow (G \times_K *)_{AG}. $$

(4.8)
But since these spaces are induced up from $K$-spaces, (4.8) is an $M$-(co)homology equivalence (of $G$-spaces) iff $\ast_{A_{K}} \to \ast$ is an $M|_{K}$-(co)homology equivalence (of $K$-spaces).

Corollary 4.9. If for all $K \subseteq G$, the map $\ast_{A_{K}} \to \ast$ is an $M|_{K}$-(co)homology equivalence (of $K$-spaces), $\blacksquare$

Corollary 4.9 reduces Theorem 4.2 from a problem of approximating a general $G$-space to a problem of approximating a point. The next step is to reduce from a general compact Lie group $G$ to its maximal $p$-toral subgroup.

Proposition 4.10. Let $G$ be a compact Lie group with maximal $p$-toral subgroup $P$, and let $M$ be a (co)Mackey functor for $G$ that satisfies the transfer condition for all $G$-spaces. If $P$ has the $p$-toral approximation property for $M|_{P}$, then $G$ has the $p$-toral approximation property for $M$.

Proof. We will consider the case when $M$ is a coMackey functor and focus on homology. The proof of the other case is essentially the same.

Consider the diagram

$$
\begin{array}{c}
G \times_{P} X_{A_{G}} \hspace{1cm} \longrightarrow \hspace{1cm} G \times_{P} X \\
\downarrow \hspace{1cm} \downarrow \\
X_{A_{G}} \hspace{1cm} \longrightarrow \hspace{1cm} X
\end{array}
$$

Because $X_{A_{G}}$ and $X$ are $G$-spaces, and not just $P$-spaces, we know that the top row is homeomorphic to $G/P \times X_{A_{G}} \longrightarrow G/P \times X$. The coefficients $M$ are assumed to satisfy the transfer condition for $X$, so we know that on $H^{G}_{*}(\_ ; M)$ the lower map is a retract of the upper map. Hence it is sufficient to show that the top row,

$$
H^{G}_{*}(G \times_{P} X_{A_{G}} ; M) \longrightarrow H^{G}_{*}(G \times_{P} X ; M)
$$

is an isomorphism. Since these spaces are induced up, this means we want to show that

$$
H^{P}_{*}(X_{A_{G}} ; M|_{P}) \longrightarrow H^{P}_{*}(X ; M|_{P})
$$

is an isomorphism.

Consider the following diagram of $P$-spaces:

$$
\begin{array}{c}
(X_{A_{G}})_{A_{P}} \hspace{1cm} \longrightarrow \hspace{1cm} X_{A_{P}} \\
\downarrow \hspace{1cm} \downarrow \\
X_{A_{G}} \hspace{1cm} \longrightarrow \hspace{1cm} X
\end{array}
$$

The top row is actually a $P$-equivalence, as we can tell by checking the fixed point sets, and is therefore an $M|_{P}$-homology isomorphism. The two vertical maps are $M|_{P}$-homology isomorphisms because we assumed that $P$ has the $p$-toral approximation property for $M|_{P}$. Hence the bottom row is an $M|_{P}$-isomorphism, and the proposition follows. $\blacksquare$

The proof of Theorem 4.2 goes by induction over the $p$-toral subgroups of $G$, and for this, we need a notion of the size of a $p$-toral group.

Definition 4.11. Assume $P$ is a $p$-toral group. The size of $P$ is defined as $\text{size}(P) = (r, c)$, where $r$ is the rank of the identity component of $P$ (which is a torus) and $c$ is the number of components of $P$. 
We order $p$-toral subgroups by size lexicographically. Note that if $Q \subset P$ is a strict containment of $p$-toral subgroups, then size $(Q) < size (P)$.

Proof of Theorem 4.2. By Proposition 4.10, it is sufficient to establish the theorem for the maximal $p$-toral subgroup of $G$. We use induction to prove that all $p$-toral subgroups $P$ of $G$ have the $p$-toral approximation property for $M|_{p}$.

To start the induction, suppose that $P$ is a $p$-toral subgroup of $G$ with size $(P) = (0, c)$ for some $c$, i.e., $P$ is a finite $p$-group. For any $P$-space $X$, the isotropy groups of $X$ are then necessarily also finite $p$-groups and therefore in $AP$. Because the isotropy groups of $X$ are already in $AP$, we can actually take $X_{AP} = X$, so $X_{AP} \rightarrow X$ is certainly an $M|_{p}$-(co)homology isomorphism.

Now suppose that size $(P) = (r, c)$ with $r > 0$, and that any $p$-toral subgroup $Q \subseteq G$ of smaller size than $P$ has the approximation property for $M|_{Q}$. By Lemmas 4.6 and 4.7, if we want to know that $P$ has the $p$-toral approximation property for $M|_{P}$, it is sufficient to show that for all $K \subseteq P$, the map $\ast_{AK} \rightarrow \ast$ is an $M|_{K}$-(co)homology isomorphism.

Certainly for $K = P$ we know $\ast_{AK} \rightarrow \ast$ is an $M|_{K}$-(co)homology isomorphism, since $\ast$ considered as a $P$-space has $p$-toral isotropy and we can take $\ast_{AP} = \ast$. Suppose that $K$ is a proper subgroup of $P$, and consider $\ast_{AK} \rightarrow \ast$. Let $Q$ be a maximal $p$-toral subgroup of $K$. Since $Q$ is properly contained in $P$, we know that size $(Q) < size (P)$, so $Q$ has the $p$-toral approximation property for $M|_{Q}$. Then by Proposition 4.10, the map $\ast_{AK} \rightarrow \ast$ is an $M|_{K}$-(co)homology isomorphism. It follows from Corollary 4.9 that $P$ has the $p$-toral approximation property for $M|_{P}$.

In particular, we have now proved that the maximal $p$-toral subgroup $P$ of $G$ has the $p$-toral approximation property for $M|_{P}$, and the theorem follows from Proposition 4.10.

As a consequence, we can prove the following proposition, which develops a criterion for a map to induce an isomorphism in Bredon (co)homology with coefficients systems that take values in $p$-local abelian groups. We say that a space is **homologically of finite type** if all its (ordinary, integral) homology groups are finitely generated. For example, a finite $G$-CW-complex is homologically of finite type.

**Proposition 4.12.** Suppose that $f : X \rightarrow Y$ is a map of $G$-CW-complexes. Assume that for all $p$-toral subgroups $H \subseteq G$, the spaces $X^{H}$ and $Y^{H}$ are homologically of finite type, and the map $X^{H} \rightarrow Y^{H}$ is an isomorphism in mod $p$ homology. Let $M$ be a (co)Mackey functor satisfying the transfer condition for the prime $p$ and taking values in $p$-local abelian groups. Then $f$ induces an isomorphism in Bredon (co)homology with coefficients in $M$.

**Proof.** Consider the diagram of $G$-spaces

$$
\begin{array}{ccc}
X_{AP} & \longrightarrow & Y_{AP} \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y.
\end{array}
$$

(4.13)

Since $M$ satisfies the transfer condition for the prime $p$, the vertical maps induce isomorphisms in Bredon (co)homology by Theorem 4.2.

On the other hand, we are set up to apply Lemma 3.7 to show that the top horizontal map of (4.13) is a Bredon (co)homology isomorphism. We need to consider...
subgroups $H \in \text{Iso}(X_{A_p}) \cup \text{Iso}(Y_{A_p})$, and such $H$ are necessarily $p$-toral. Therefore

\[
\begin{array}{ccc}
(X_{A_p})^H & \longrightarrow & (Y_{A_p})^H \\
\cong & & \cong \\
X^H & \longrightarrow & Y^H.
\end{array}
\]

is a mod $p$ homology isomorphism because the bottom horizontal map is a mod $p$ homology isomorphism by assumption. Because of the finite-type hypothesis, it follows that the top horizontal map is an isomorphism on homology with coefficients in any $p$-local group. Thus the top horizontal map is actually an isomorphism of ordinary homology with coefficients in $M(G/K)$ for any subgroup $K \subseteq G$. We conclude that the top horizontal map of (4.13) induces an isomorphism in Bredon (co)homology by Lemma 3.7. It follows that the bottom horizontal map of (4.13) induces an isomorphism in Bredon (co)homology also. □

5. Satisfying the Transfer Condition for $p$

Our main goal in this section is to show that the transfer condition for the prime $p$, which was a key assumption in Theorem 4.2, is satisfied by a certain class of (co)Mackey functors. More precisely, suppose that $E^G_\ast$ is a generalized equivariant homology theory for which it is true that the composed homomorphism

\[
E^G_\ast(X) \xrightarrow{\text{tr}} E^G_\ast(G/H \times X) \longrightarrow E^G_\ast(X)
\]

is an isomorphism for all $X$. We then show that the same holds for homology with coefficients in the coMackey functor $\pi_\ast(E)$ (see below) and cohomology with coefficients in the Mackey functor $\underline{\pi}_\ast(E)$. This is the content of Proposition 5.3. In particular, if $H$ is a maximal $p$-toral subgroup of $G$, then the transfer condition for $p$ is satisfied.

Recall from the first part of Section 4 that to a coMackey functor one can associate a homology theory on $G$-spectra, and to a Mackey functor one can associate a cohomology theory on $G$-spectra. These theories satisfy an equivariant version of the dimension axiom. In other words, (co)Mackey functors represent “ordinary” (co)homology theories on $G$-spectra. More general homology and cohomology theories are represented by (genuine) $G$-spectra. Indeed, let $E$ be a $G$-spectrum. Then $E$ represents a homology theory on $G$-spectra by the formula

\[
E^G_\ast(X) := \pi_\ast((E \wedge X)^G).
\]

Dually, $E$ represents a cohomology theory via the formula

\[
E^\ast_G(X) := \pi_\ast(\text{map}(X, E)^G).
\]

In fact, every (co)homology theory on $G$-spectra is represented by a $G$-spectrum in this way [May96, Chapter XIII Corollaries 3.3 and 3.5].

Any homology (resp. cohomology) theory defined on $G$-spectra gives rise to a coMackey (resp. Mackey) functor by restriction to the category $S(O_G)$. In particular, a $G$-spectrum $E$ gives rise to a coMackey and a Mackey functor, which we denote $\pi_\ast E$ and $\underline{\pi}_\ast E$. Explicitly, they are given by the formulas

\[
\pi_\ast E(\Sigma^\infty G/H_+) = E^G_\ast(G/H_+) \\
\underline{\pi}_\ast E(\Sigma^\infty G/H_+) = E^\ast_G(G/H_+).
\]
One may either think of $*$ as a fixed integer, or consider all values of $*$ simultaneously and view $\pi$ and $\varpi$ as (co)Mackey functors with values in graded abelian groups. We will adopt the latter point of view.

The (bigraded) (co)homology theories $H^G_\ast(X, \varpi_\ast(E))$ and $H^G_\ast(X, \varpi_\ast(E))$ are the ordinary (co)homology theories that agree with $E^G_\ast$ and $E^G_\ast$, respectively, on the category $S(O_G)$. Hence for an object $X$ of $S(O_G)$, there are natural isomorphisms

$$H^G_\ast(X, \varpi_\ast(E)) \cong E^G_\ast(X)$$
$$H^G_\ast(X, \varpi_\ast(E)) \cong E^G_\ast(X),$$

and if $i > 0$, then $H^G_i(X, \pi_\ast(E)) \cong H^G_i(X, \varpi_\ast(E)) = 0$. The theories $H^G_\ast(X, \varpi_\ast(E))$ and $H^G_\ast(X, \varpi_\ast(E))$ are represented by a product of equivariant Eilenberg-Mac Lane spectra that have the same equivariant homotopy groups as $E$.

**Example 5.1.** We describe a family of generalized equivariant homology theories and their associated (co)Mackey functors. This example is needed for the intended applications. For a $G$-spectrum $E$, we define the **Borel homology associated with $E$** by the formula

$$E^G_\ast(X) := \pi_\ast((E \wedge X)_{hG}).$$

Borel homology is an $RO(G)$-graded homology theory. Indeed, it is represented in the usual way by the free $G$-spectrum $\Sigma^{-\text{ad}G}E \wedge EG_+$. This is because of the Adams isomorphism, which is a homotopy equivalence $(\Sigma^{-\text{ad}G}E \wedge EG_+ \wedge X)^G \cong (E \wedge X)_{hG}$ (see [May96, Ch. XVI Theorem 5.4]).

It is worth noting that Borel homology is well-defined for naive $G$-spectra. More precisely, this means that the Borel homology functor factors through the forgetful functor from the homotopy category of genuine $G$-spectra to the homotopy category of naive $G$-spectra. In other words, Borel homology has the following homotopy invariance property: if $f: X \to Y$ is a $G$ map that is a non-equivariant equivalence, then $f$ induces an isomorphism in Borel homology. Another pleasant invariance property of Borel homology is that if $f$ induces an isomorphism $E_\ast(X) \to E_\ast(Y)$ of non-equivariant $E$-homology, then $f$ induces an isomorphism on Borel homology.

Dually, the **Borel cohomology associated with $E$** is defined by the formula

$$E^G_\ast(X) = E^G_\ast(DX) = \pi_\ast \text{map}(X, E)_{hG}.$$

Borel cohomology satisfies all the obvious analogues of the properties of Borel homology.

The (co)Mackey functor associated with Borel (co)homology will be denoted by $\varpi^G_\ast(E)$ and $\varpi^G_\ast(E)$. They are defined by the formulas

$$\varpi^G_\ast(E)(\Sigma^\infty G/H_+) = \pi_\ast((E \wedge G/H_+)_{hG})$$
$$\cong \pi_\ast(E_{hH})$$

and

$$\varpi^G_\ast(E)(\Sigma^\infty G/H_+) = \pi_\ast((E \wedge D(G/H_+))_{hH})$$
$$\cong \pi_\ast(\Sigma^{\text{ad}H} - \text{ad}G E_{hH}).$$

It is worth noting that if $H \subset G$, then the restriction of the coMackey functor $\varpi^G_\ast(E)$ to $H$ is given by the formula

$$\varpi^H_\ast(E)(\Sigma^\infty H/K_+) \cong \varpi^G_\ast(E)(\Sigma^\infty H/K_+),$$
where \(E|_H\) is the spectrum \(E\) with action restricted from \(G\) to \(H\). By contrast, the restriction of the Mackey functor \(\pi^b(E)\) to \(H\) is given by the formula

\[
\pi^b(E)|_H(\Sigma^\infty H/K_+) \cong \pi^b(\Sigma^{ad_H} - \Delta_{ad_G}E|_H)(\Sigma^\infty H/K_+)
\]

As we mentioned already, the important property provided by a (co)homology theory being \(RO(G)\)-graded is the existence of transfers. More specifically, suppose \(\eta: E \to B\) is a \(G\)-equivariant fibration, whose fiber is a finite complex \(F\). The equivariant Becker-Gottlieb transfer of \(\eta\) is a \(G\)-map \(tr: \Sigma^\infty B_+ \to \Sigma^\infty E_+\). Therefore, for any \(RO(G)\)-graded homology \(h_*\), a fibration \(\eta\) as above gives rise to a transfer homomorphism \(h_*(B) \to h_*(E)\). The following is a key property of the transfer map.

**Lemma 5.2** (Becker-Gottlieb \([BG76]\)). The composition \(\Sigma^\infty B_+ \xrightarrow{tr} \Sigma^\infty E_+ \xrightarrow{\eta} \Sigma^\infty X_+\) induces multiplication by the Euler characteristic of the fiber \(F\) on non-equivariant homology.

Now we can state and prove the main result of this section.

**Proposition 5.3.** Let \(G\) be a compact Lie group and \(H \subseteq G\) a subgroup. Let \(E\) be a \(G\)-spectrum. Suppose that for every \(G\)-space \(X\), the composed map

\[
\Sigma^\infty X_+ \xrightarrow{tr} \Sigma^\infty (G/H \times X)_+ \to \Sigma^\infty X_+
\]

induces an isomorphism on \(E_*(X_+)\). Then the induced homomorphism on homology with coefficients in \(\pi_*(E)\) is an isomorphism as well. That is, for all \(i, j\) the following composed homomorphism is an isomorphism:

\[
H^i_0(X, \pi_j(E)) \xrightarrow{tr} H^i_0(G/H \times X, \pi_j(E)) \to H^i_0(X, \pi_j(E)).
\]

The analogous result for cohomology with coefficients in \(\pi_*(E)\) also holds.

**Remark 5.4.** Before we get to the proof of Proposition 5.3 we point out that the proposition is essentially obvious when \(G\) is a finite group. Meyer-Vietoris tells us that it is enough to prove its conclusion when \(X = G/K\) for some subgroup \(K \subseteq G\). For finite \(G\), a product \(G/H \times G/K\) is a disjoint union of \(G\)-orbits, and therefore there is a natural isomorphism \(E_j^G(G/H \times G/K) \cong H_0^j(G/H \times G/K; \pi_*(E))\). It follows that there is a commutative diagram

\[
H^0_0(G/K; \pi_*(E)) \to H^0_0(G/H \times G/K; \pi_*(E)) \to H^0_0(G/K; \pi_*(E)).
\]

But if \(G\) is a compact Lie group that is not finite, then in most cases \(G/H \times G/K\) is not a disjoint union of \(G\)-orbits. So we have to check that the following composed homomorphisms are the same, even though the intermediate groups may not be:

\[
E^0_j(G/K) \to E^0_j(G/H \times G/K) \to E^0_j(G/K) \quad (5.5)
\]

\[
H^0_j(G/K; \pi_*(E)) \to H^0_j(G/H \times G/K; \pi_*(E)) \to H^0_j(G/K; \pi_*(E)).
\]

**Proof of Proposition 5.3.** By a Meyer-Vietoris argument, it is enough to establish the result when \(X = \Sigma^\infty (G/K_+)\) is an object of \(\mathcal{S}(O_G)\). In this case, as in the finite case, there is an isomorphism \(H^0_j(G/K; \pi_*(E)) \cong E^0_j(G/K)\) on the left side of diagram (5.5). Moreover the morphism \(H^0_j(G/K, \pi_j(E)) \to H^0_j(G/K, \pi_j(E))\) across
the bottom of \((5.5)\) can be identified with the morphism \(E^G_j(G/K) \rightarrow E^G_j(G/K)\) across the top row, because it is induced by a self-map of \(G/K\) in \(S(O_G)\). By our assumption, the composed top row of \((5.5)\) is an isomorphism, and therefore so is the bottom row. This proves the claim for \(i = 0\). For \(i > 0\) there is nothing to prove, because \(H^*_G(X, \pi_j(E)) = 0\), and the proof is complete for the case of homology. The proof of the cohomology case is the same. \(\square\)

**Example 5.6.** Fix a prime \(p\). In the examples of Proposition\(5.3\) that we have in mind, \(E\) is a spectrum with an action of \(G\), and the non-equivariant homotopy groups of \(E\) are \(p\)-local. Let us take \(H = P\), a maximal \(p\)-toral subgroup of \(G\). Then the Borel homology \(E^*_bG\) satisfies the hypothesis of Proposition\(5.3\). To check this, it is enough to show that an isomorphism in homotopy is induced by the composed map

\[
\left( E \land X_+ \right)_{hG} \xrightarrow{\text{tr}} \left( E \land (U(n) / P \times X)_+ \right)_{hG} \xrightarrow{} \left( E \land X_+ \right)_{hG}.
\]

An isomorphism on homotopy follows from showing that, before taking homotopy orbits, we have an isomorphism on non-equivariant homology:

\[
(5.7) \quad \left( E \land X_+ \right) \xrightarrow{\text{tr}} \left( E \land U(n) / P \times X_+ \right) \rightarrow \left( E \land X_+ \right).
\]

But by Lemma\(5.2\) on homology \((5.7)\) induces multiplication by the Euler characteristic of \(U(n)/P\), which is invertible mod \(p\) because \(P\) is a maximal \(p\)-toral subgroup of \(G\). Since the homotopy groups of \(E\) are \(p\)-local, multiplication by the Euler characteristic is an isomorphism.

Dually, the Borel cohomology theory \(E^*_{bG}\) also satisfies the hypothesis of Proposition\(5.3\).

6. The Complex of Direct-Sum Decompositions

In this section we apply our general results, Propositions\(4.12\) and \(5.3\) and Example\(5.6\) to the space \(\mathcal{L}_n\), the complex of direct-sum decompositions of \(\mathbb{C}^n\). The space \(\mathcal{L}_n\) was introduced in \([\text{Aro02}]\) and studied in detail in \([\text{BJL}^+15, \text{BJL}^+]\). Throughout the section, let \(\mathbb{C}^n\) be equipped with the standard inner product.

**Definition 6.1.** A proper direct-sum decomposition of \(\mathbb{C}^n\) is an unordered set \(\lambda\) of proper, non-zero, pairwise orthogonal subspaces of \(\mathbb{C}^n\), whose sum is \(\mathbb{C}^n\). The subspaces are called the components of \(\lambda\). Given two direct-sum decompositions \(\lambda\) and \(\theta\), we say that \(\lambda\) is a refinement of \(\theta\), and write \(\lambda \leq \theta\), if every component of \(\lambda\) is a subspace of some component of \(\theta\).

The set of direct-sum decompositions has a natural action of \(U(n)\), and the partial ordering by refinement is respected by that action. Moreover, the set of direct-sum decompositions is equipped with a natural topology, defined by the disjoint union of \(U(n)\)-orbits. The morphisms (unique between any two comparable objects) likewise have an inherited topology. The set of proper direct-sum decompositions therefore forms a topological poset, internal to the category of topological spaces.

**Definition 6.2.** The space \(\mathcal{L}_n\) is the topological realization of the category of proper direct-sum decompositions of \(\mathbb{C}^n\).
As usual, we denote the unreduced suspension of $L_n$ by $L_n^\circ$. The space $L_n^\circ$ arose as a building block in the study of the stable rank filtration of complex $K$-theory \cite{Al07, Al10}. The Spanier-Whitehead dual of $L_n^\circ$ plays a similar role in the Taylor tower (in the sense of orthogonal calculus) of the functor $V \mapsto BU(V)$. This paper is a part of a long-term program to understand these two filtrations and the relationship between them.

In order to use results of previous sections, we need to know about the mod $p$ homology of $L_n$. We need a little more notation to recall those results. Let $\Gamma_k \subset U(p^k)$ be the (unique up to conjugacy) $p$-radical subgroup of the unitary group that acts irreducibly on $\mathbb{C}^{p^k}$. It fits into a short exact sequence

$$1 \to S^1 \to \Gamma_k \to \mathbb{F}_p^{2k} \to 1$$

where $S^1$ is the center of $U(p^k)$. Let $\text{Sp}_k(\mathbb{F}_p)$ denote the group of automorphisms of $\mathbb{F}_p^{2k}$ equipped with the standard symplectic form. The normalizer of $\Gamma_k$ in $U(p^k)$, denoted $N(\Gamma_k)$, fits into a short exact sequence

$$1 \to \Gamma_k \to N(\Gamma_k) \to \text{Sp}_k(\mathbb{F}_p) \to 1.$$

In particular, the Weyl group of $\Gamma_k$ in $U(p^k)$ is the symplectic group $\text{Sp}_k(\mathbb{F}_p)$.

Lastly, let $T\text{Sp}_k(\mathbb{F}_p)$ denote the Tits building for $\text{Sp}_k(\mathbb{F}_p)$: the geometric realization of the poset of proper coisotropic subspaces of $(\mathbb{F}_p^{2k})$. For details and further references, see \cite{BJL+}.

The mod $p$ homology of $L_n$ was described in \cite{Aro02}.

Theorem 6.4.

1. [Aro02, Theorem 4] If $n$ is not a power of $p$, then $L_n$ is mod $p$ acyclic.
2. [Aro02, Theorem 1] If $n = p^k$, there is a mod $p$ homology equivalence

$$U(p^k) \wedge_{N(\Gamma_k)} T\text{Sp}_k(\mathbb{F}_p) \to L_n^\circ.$$ 

The principal result of this paper, Theorem 1.1, is a strengthening of Theorem 6.4. We then use Theorem 1.1 to deduce Corollary 6.6 which will provide a key computational input in future work on the K-theoretic analogue of the Whitehead conjecture. It is worth mentioning that Corollary 6.6 is closely analogous to one of the main results of \cite{ADL16}. Theorem 1.1 and Corollary 6.6 do for the complex of direct-sum decompositions $L_n$ what Theorem 1.1 and Corollary 1.2 of \cite{ADL16} did for the complex of partitions $P_n$.

Theorem 1.1. Suppose that $M$ is a (co)Mackey functor for $U(n)$ that takes values in $p$-local abelian groups and satisfies the transfer condition for the prime $p$.

1. If $n$ is not a power of $p$, then the map $L_n \to \ast$ induces an isomorphism on Bredon (co)homology with coefficients in $M$.
2. If $n = p^k$, there is a map inducing an isomorphism on Bredon (co)homology with coefficients in $M$:

$$U(p^k) \wedge_{N(\Gamma_k)} T\text{Sp}_k(\mathbb{F}_p) \to L_n^\circ.$$ 

Proof of Theorem 1.1. First, consider the case when $n$ is not a power of $p$. Let $P$ be a $p$-toral subgroup of $U(n)$. We would like to apply Proposition 4.12 for which we need to know that the fixed point space $(L_n)^P$ is mod $p$ acyclic. In fact, [BJL+] Theorem 1.2] tells us that for most $p$-toral subgroups $P \subset U(n)$, the space $(L_n)^P$
is actually contractible. If $P$ has the property that $(\mathcal{L}_n)^P$ is not contractible, then $P$ must fit in a short exact sequence

$$1 \longrightarrow S^1 \longrightarrow P \longrightarrow P/S^1 \longrightarrow 1,$$

where $S^1$ is the center of $U(n)$, and $P/S^1$ is a finite $p$-group (in fact, an elementary abelian $p$-group). It follows that $(\mathcal{L}_n)^P = (\mathcal{L}_n^{S^1})^{P/S^1}$. However, $S^1$ acts trivially on $\mathcal{L}_n$, since $S^1$ acts on any subspace of $\mathbb{C}^n$ by multiplication by scalars. Thus $(\mathcal{L}_n)^P = (\mathcal{L}_n)^{P/S^1}$.

On the other hand, it follows from [Aro12 Theorem 4] that if $n$ is not a power of $p$, then $\mathcal{L}_n$ is mod $p$ acyclic. Because $P/S^1$ is a finite $p$-group, by Smith theory $(\mathcal{L}_n)^{P/S^1}$ is mod $p$ acyclic. By Proposition 4.12 it then follows that the map $\mathcal{L}_n \rightarrow \ast$ induces an isomorphism on Bredon (co)homology with coefficients in $M$. This proves the first part of the theorem.

In order to tackle the case of $\mathcal{L}_p^k$. Once again we want to apply Proposition 4.12 so we want to prove that the map

\begin{equation}
U(p^k)^+_\wedge N(\Gamma_k) \text{ Top}_k (\mathbb{F}_p)^\circ \longrightarrow \mathcal{L}_p^k
\end{equation}

induces a mod $p$ homology isomorphism on fixed point spaces of a $p$-toral subgroup $P \subset U(p^k)$.

Suppose first that $P$ is not subconjugate to $N(\Gamma_k)$. In this case the fixed points on the left side of (6.5) are just a point. Because $P$ is not subconjugate to $\Gamma_k$, we also know by results of [BLLP Theorem 1.2] that $(\mathcal{L}_p^k)^P \simeq \ast$, so the map of $P$-fixed points of (6.5) is a homotopy equivalence.

Next, suppose that $P$ is subconjugate to $N(\Gamma_k)$. For $P = \Gamma_k$ itself, the map of $P$-fixed points is actually a homeomorphism, by the calculation of [AL15 Theorem 1.2]. The same is true for $p$-toral subgroups of $N(\Gamma_k)$ that strictly contain $\Gamma_k$. For other values of $P$, however, we can only get a homology calculation, which we obtain by a relative Smith theory calculation.

By (6.5), we know that if $P$ is $p$-toral and subconjugate to $N(\Gamma_k)$, then $P$ is an extension of a finite $p$-group by the central $S^1$. Let $C$ be the homotopy cofiber of (6.5), which is mod $p$ acyclic by Theorem 6.4. The action of $S^1$ on both sides of (6.5) is trivial, so as in the previous case, $C^P = C^{P/S^1}$. But $P/S^1$ is a finite $p$-group, and $C$ is a finite, mod $p$ acyclic complex. By Smith theory, $C^P$ is mod $p$ acyclic. Since fixed point spaces commute with homotopy cofibers [Mal14], it follows that the map (6.5) induces a mod $p$ homology isomorphism on the fixed point space of $P$.

As in the previous case, by Proposition 4.12 we conclude that (6.5) induces an isomorphism on Bredon (co)homology with coefficients in $M$. □

The following corollary will be needed in a forthcoming paper where the results of this paper will be applied. Recall from Section 5 that if $E$ is a spectrum with an action of $G$, then $\mathcal{E}_b^G(\ast)$ is the Borel homology associated with $E$, and $\mathcal{E}_b^G(E)$ is the coMackey functor associated with Borel homology

**Corollary 6.6.** Let $p$ be a prime. Let $E$ be a spectrum with an action of $U(n)$ whose non-equivariant homology groups are $p$-local. If $n$ is not a power of $p$, then $\tilde{H}_i(\mathcal{E}_b^G(E)) = 0$ for all $i \geq 0$. If $n = p^k$, this holds for $i \neq k$. A similar statement holds for cohomology with coefficients in $\mathcal{E}_b^G(E)$. 


Proof. By Example 5.6 the (co)Mackey functors $\pi_*^E(E)$ and $\pi_*^p(E)$ satisfy the transfer condition for the prime $p$. Obviously they take values in $p$-local groups. Therefore they satisfy the assumptions of Theorem 1.1. It follows that if $n$ is not a power of $p$ then the map $\mathcal{L}_n \to \ast$ induces an isomorphism on the Bredon homology groups $H^i_{U(n)}(\ast; \pi_*^E(E))$, and similarly for cohomology. This is equivalent to saying that
$$
\tilde{H}^i_{U(n)}(\mathcal{L}^o_n; \pi_*^E(E)) = 0, 
$$
and similarly for cohomology.

Now suppose that $n = p^k$. We will focus on the homology case. The proof of the cohomology case is essentially the same, with the tweak that one needs to keep track of the adjoint representation sphere when applying Spanier-Whitehead duality to a compact Lie group. This does not affect the logic of the proof.

We need to show that $\tilde{H}^i_{U(n)}(\mathcal{L}^o_n; \pi_*^E(E)) = 0$ for $i \neq k$. By second part of Theorem 1.1 followed by a reduced version of Lemma 4.4, there are isomorphisms
$$
\tilde{H}^{U(p^k)}_{i}(\mathcal{L}^o_n; \pi_*^E(E)) \cong \tilde{H}^{U(p^k)}_{i}(U(p^k)_+ \wedge N(\Gamma_k) \cdot \text{Sp}_k(\mathbb{F}_p)^o; \pi_*^E(E)) \\
\cong \tilde{H}^{N(\Gamma_k)}_{i}(\text{Sp}_k(\mathbb{F}_p)^o; \pi_*^E(N(\Gamma_k)))
$$
(6.7)

We focus on the last group, and our goal is to show that it is zero except (possibly) when $i = k$.

In fact, we can simplify (6.7) further, because the action of $N(\Gamma_k) \cdot \text{Sp}_k(\mathbb{F}_p)^o$ restricts to a trivial action of $\Gamma_k$ itself on $\text{Sp}_k(\mathbb{F}_p)^o$. The reason is that coisotropic subgroups are normal in $\Gamma_k$, so $\Gamma_k$ acts trivially on the poset of coisotropic subgroups, i.e. on $\text{Sp}_k(\mathbb{F}_p)$. Our next step is to leverage the triviality of the action.

Recall that the coMackey functor $\pi_*^E(E)|_{N(\Gamma_k)}$ is the restriction of the coMackey functor $\pi_*^E(E)$ from $U(p^k)$ to $N(\Gamma_k)$. If $O$ is an orbit of $N(\Gamma_k)$, then by definition
$$
\pi_*^E(E)|_{N(\Gamma_k)}(\Sigma^\infty O_+) = \pi_*^E(E \wedge \Sigma^\infty O_+)_{hN(\Gamma_k)}.
$$
Moreover, if the group $\Gamma_k \subset N(\Gamma_k)$ acts trivially on $O$, then the formula for $\pi_*^E(E)|_{N(\Gamma_k)}$ can be rewritten as follows
$$
\pi_*^{p^k}(E)|_{N(\Gamma_k)}(\Sigma^\infty O_+) \cong \pi_*^E\left(E_{h\Gamma_k} \wedge \Sigma^\infty O_+\right)_{hN(\Gamma_k)/\Gamma_k} \\
\cong \pi_*^E\left(E_{h\Gamma_k} \wedge \Sigma^\infty O_+\right)_{h\text{Sp}_k(\mathbb{F}_p)}.
$$
(6.8)

Let $F = E_{h\Gamma_k}$. Then $F$ is a spectrum with an action of $\text{Sp}_k(\mathbb{F}_p)$, and it is clear by inspection that there is an isomorphism
$$
\pi_*^E\left(E_{h\Gamma_k} \wedge \Sigma^\infty O_+\right)_{h\text{Sp}_k(\mathbb{F}_p)} \cong \pi_*^E(F)(\Sigma^\infty O_+)
$$
(6.9)
Combining (6.8) and (6.9), we find that if $\Gamma_k$ acts trivially on $O$, then there is an isomorphism
$$
\pi_*^E(E)|_{N(\Gamma_k)}(\Sigma^\infty O_+) \cong \pi_*^{p^k}(F)(\Sigma^\infty O_+)
$$
(6.10)

Suppose $X$ is a CW-complex with an action of $N(\Gamma_k)$ that restricts to a trivial action of $\Gamma_k$. Then the action of $N(\Gamma_k)$ induces an action of $\text{Sp}_k(\mathbb{F}_p)$ on $X$. It follows from (6.10) that there is an isomorphism
$$
\tilde{H}^{N(\Gamma_k)}_{i}(X; \pi_*^E(E)|_{N(\Gamma_k)}) \cong \tilde{H}^{\text{Sp}_k(\mathbb{F}_p)}_{i}(X; \pi_*^{p^k}(F)).
$$
As discussed above, the action of \( N(\Gamma_k) \) on \( TSp_k(\mathbb{F}_p)^\circ \) does restrict to the trivial action of \( \Gamma_k \), and we can conclude that there is an isomorphism
\[
\bar{H}^{N(\Gamma_k)}_i\left( TSp_k(\mathbb{F}_p)^\circ; \pi^*_s(E)|_{N(\Gamma_k)} \right) \cong \bar{H}^{Sp_k(\mathbb{F}_p)}_i\left( TSp_k(\mathbb{F}_p)^\circ; \pi^*_s(F) \right).
\]

Our remaining goal is to show that the groups \( \bar{H}^{Sp_k(\mathbb{F}_p)}_i\left( TSp_k(\mathbb{F}_p)^\circ; \pi^*_s(F) \right) \) are zero for \( i \neq k \). Note that \( Sp_k(\mathbb{F}_p) \) is a finite group. Let \( \text{Syl}_p \) be its \( p \)-Sylow subgroup. The non-equivariant homotopy groups \( \pi_*(F) \) are \( p \)-local. It follows that homology with coefficients in the Borel coMackey functor \( \pi^*_s(F) \) has the property that the following composed homomorphism is an isomorphism
\[
\bar{H}^{Sp_k(\mathbb{F}_p)}_i\left( --; \pi^*_s(F) \right) \overset{\text{tr}}{\longrightarrow} \bar{H}^{\text{Syl}_p}_i\left( --; \pi^*_s(F)|_{\text{Syl}_p} \right) \longrightarrow \bar{H}^{Sp_k(\mathbb{F}_p)}_i\left( --; \pi^*_s(F) \right)
\]
It follows that the Bredon homology groups \( \bar{H}^{Sp_k(\mathbb{F}_p)}_i\left( TSp_k(\mathbb{F}_p)^\circ; \pi^*_s(F) \right) \) are direct summands of the Bredon homology groups \( \bar{H}^{\text{Syl}_p}_i\left( TSp_k(\mathbb{F}_p)^\circ; \pi^*_s(F)|_{\text{Syl}_p} \right) \). But as a \( \text{Syl}_p \)-space, the Tits building \( TSp_k(\mathbb{F}_p) \) is equivalent to \( (\text{Syl}_p)^+ \wedge S^k \), i.e., a bouquet of \( k \)-spheres freely permuted by \( \text{Syl}_p \). It follows that there are isomorphisms
\[
\bar{H}^{\text{Syl}_p}_i\left( TSp_k(\mathbb{F}_p)^\circ; \pi^*_s(F)|_{\text{Syl}_p} \right) \cong \bar{H}^{\text{Syl}_p}_i\left( (\text{Syl}_p)^+ \wedge S^k; \pi^*_s(F)|_{\text{Syl}_p} \right) \cong \bar{H}_i(S^k; \pi^*_s(F)).
\]
Here the right-hand side means the ordinary, non-equivariant reduced homology groups of the sphere \( S^k \) with coefficients in the group \( \pi^*_s(F) \). Clearly the right hand side is zero for \( i \neq k \). Since the right-hand side contains the group
\[\bar{H}^{Sp_k(\mathbb{F}_p)}_i\left( TSp_k(\mathbb{F}_p)^\circ; \pi^*_s(F) \right)\]as a direct summand, it follows that the latter group is also zero for \( i \neq k \). □

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