Equilibrium states which are not Gibbs measure on hereditary subshifts*

Zijie Lin and Ercai Chen§

Abstract
In this paper, we consider which kind of invariant measure on hereditary subshifts is not Gibbs measure. For the hereditary closure of a subshift \((X, S)\), we prove that in some situation, the invariant measure \(\nu * B_{p,1-p}\) can not be a Gibbs measure where \(\nu\) is an invariant measure on \((X, S)\). As an application, we show that for some \(\mathcal{B}\)-free subshifts, the unique equilibrium state \(\nu_{\eta} * B_{p,1-p}\) is not Gibbs measure.

1 Introduction
Recall that a subshift \((X, S)\) is a subsystem of full shift \((\{0, 1\}^\mathbb{Z}, \sigma)\) where \(\{0, 1\}^\mathbb{Z} = \{(x_i)_{i \in \mathbb{Z}} : x_i \in \{0, 1\}\}\) and \(\sigma : \{0, 1\}^\mathbb{Z} \to \{0, 1\}^\mathbb{Z}\) with \(\sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}\). It means that \(X\) is a closed \(\sigma\)-invariant subset of \(\{0, 1\}^\mathbb{Z}\) and \(S = \sigma|_{X}\).

Denote by \(\mathcal{M}(X, S)\) (resp. \(\mathcal{M}^\ast(X, S)\)) the set of all the Borel \(S\)-invariant (resp. ergodic \(S\)-invariant) probability measure on \((X, S)\).

For a subshift \((X, S)\), recall that the set of all the \(n\)-length word is the set \(\mathcal{L}_n(X) = \{W = [w_0w_1 \cdots w_{n-1}] : \text{there exists } x \in X, x_i = w_i \text{ for } i = 0, 1, \ldots, n-1\}\) and the language is the set \(\mathcal{L}(X) = \bigcup_{n \in \mathbb{N}} \mathcal{L}_n(X)\). For each word \(W \in \mathcal{L}(X)\), denote by \(|W|\) the length of the word \(W\), that is, \(|W| = n\) if and only if \(W \in \mathcal{L}_n(X)\). For a word \(W \in \mathcal{L}(X)\) or a point \(x \in X\), let \(W[i, j] = [w_i \cdots w_j]\) and \(x[i, j] = [x_i \cdots x_j]\) for any suitable \(i \leq j\). Each word \(W\) also stands for the corresponding cylinder set \(W = \{x \in X : x[0,|W| - 1] = W\}\) with the same denotation. For any word \(W\), define \(\#_1 W = \#\{1 \leq i < |W| : w_i = 1\}\).

For two words \(W = [w_0 \cdots w_{n-1}], W' = [w'_0 \cdots w'_{n-1}] \in \mathcal{L}_n(X)\), we call \(W \leq W'\) if \(w_i \leq w'_i\) for each \(i = 0, 1, \ldots, n-1\). Also, for two points \(x = (x_i)_{i \in \mathbb{Z}}, y = (y_i)_{i \in \mathbb{Z}} \in X\), we call \(y \leq x\) if \(y_i \leq x_i\) for each \(i \in \mathbb{Z}\). The subshift \((X, S)\) is hereditary if for any \(W \in \mathcal{L}(X)\) and any \(W' \leq W\), the word \(W' \in \mathcal{L}(X)\). Define the hereditary closure of \((X, S)\) by

\[\bar{X} = \{y \in \{0, 1\}^\mathbb{Z} : \text{there exists } x \in X \text{ such that } y \leq x\}\].

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*§ School of Mathematical Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210093, P.R.China. Email: zjlin137@126.com; School of Mathematical Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210093, P.R.China, and Center of Nonlinear Science, Nanjing University, Nanjing 210093, P.R.China. Email: echen@njnu.edu.cn.
It follows that \((X, S)\) is hereditary if and only if \(\tilde{X} = X\). Examples of hereditary subshift include many \(B\)-free systems, introduced in Section 5. The basic properties of hereditary shifts are showed in [12, 13]. In [18], J.K.-Przymus, M. Lemańczyk and B. Weiss studied the invariant measure on \(B\)-free subshifts. In [10], A. Dydek, S. Kasjan, J.K.-Przymus and M. Lemańczyk studied entropy and intrinsic ergodicity of \(B\)-free subshifts.

Equilibrium states play an important role on complicated physical systems. Bowen[2] and Ruelle[19] have studied the existence of equilibrium states for continuous functions on shifts of finite type. In [8], the authors show that an equilibrium state exists if and only if the function is positive recurrent, and in this case the equilibrium state is unique. In [21], the author shows the existence of equilibrium states for Hölder continuous positive recurrent functions for which the Ruelle–Perron–Frobenius operator maps the constant function 1 to a bounded function.

Gibbs measures have strong relationship with the equilibrium states. The idea of Gibbs measures comes from statistical physics([14, 20]). The basic properties of Gibbs measures were introduced in [3, 25]. In [219], the authors proved the existence of Gibbs measures on topological Markov shifts. In [15], Mauldin and Urbański found sufficient topological conditions for the existence of Gibbs measures. In [23], the author showed that Mauldin and Urbański’s sufficiency result can be derived from the generalized Ruelle’s Perron–Frobenius theorem of [22], and gave a new proof of their result.

In [16], J.K.-Przymus and M. Lemańczyk proved that for some of hereditary subshifts, the maximal entropy measure does not have the Gibbs property(See details in [16]). This work motivates us to consider that for hereditary subshifts, when the equilibrium state is not Gibbs measure.

**Theorem 1.1.** For the hereditary closure \((\tilde{X}, S)\) of a subshift \((X, S)\), nonatomic measure \(\nu \in \mathcal{M}(X, S)\) with \(D_\nu = D\), \(\kappa = \nu * B_{q,1-q} \in \mathcal{M}(\tilde{X}, S)\) with some \(0 < q < 1\), and a continuous function \(\tilde{\phi} : \tilde{X} \to \mathbb{R}\) with \(D_{\tilde{\phi}} = D_{\tilde{\phi}}\). If

\[
\tilde{P} \leq (\text{Var}\tilde{\phi}(\{0\}) - \log(1-q) - \text{Var}\tilde{\phi}(\{1\})) + d\tilde{\phi} - \text{Var}\tilde{\phi}(\{0\}),
\]

\[
\sup \tilde{\phi}(\{1\}) \geq \sup \tilde{\phi}(\{0\}),
\]

and

\[
\text{Var}\tilde{\phi}(\{1\}) \leq \text{Var}\tilde{\phi}(\{0\}) - \log(1-q),
\]

then \(\kappa\) is not the Gibbs measure for \(\tilde{\phi}\).

As an application, we consider some \(B\)-free systems which are shown that its unique equilibrium state is not Gibbs measure. As a generalization of square-free numbers, \(B\)-free numbers and \(B\)-free systems were studied for several years (See details for [1, 6, 18]). Fix an infinite set \(B = \{b_1, b_2, \ldots\} \subset \{2, 3, \ldots\}\). The set \(B\) is said to be pairwise coprime if \(\gcd(b_i, b_j) = 1\) for any \(i \neq j\). We consider \(B\) satisfies the following conditions:

\[
B \text{ is infinite and pairwise coprime, and satisfies } \sum_{b \in B} \frac{1}{b} < \infty. \quad (1)
\]
For example, $\mathcal{B} = \{ p^2 : p \text{ is prime number} \}$ satisfies the above condition. When $\mathcal{B}$ satisfies condition (1), ergodic and topological properties of the corresponding $\mathcal{B}$-free systems were studied in [1, 6, 18].

In the present paper, we prove the following theorem.

**Theorem 1.2.** Suppose that $\mathcal{B} = \{ b_1, b_2, \cdots \}$ satisfies (1) and $b_1 = 2$. For $\phi = a_{00} \mathbb{1}_{[00]} + a_{01} \mathbb{1}_{[01]} + a_1 \mathbb{1}_{[1]}$, the unique equilibrium state $\nu_\eta * B_{p,1-p}$ for $\phi$ is not Gibbs measure, where $p = 2^{2a_{00}}$.  

This paper is organized as follows. In Section 2, we recall some basic notions and their properties. In Section 3, we introduce the densities for a continuous map $\phi$ and prove an inequality for them. Section 4 is the proof of Theorem 1.1. In Section 5, we prove Theorem 1.2 which gives some $\mathcal{B}$-free subshifts whose unique equilibrium state is not Gibbs measure, as an application of Theorem 1.1.

## 2 Preliminaries

For hereditary subshifts, we consider the invariant measure given by the following ways. Let $Q : X \times \{0, 1\}^\mathbb{Z} \to \bar{X}$ be the coordinatewise multiplication:

$$Q(x, y) = (\ldots, x_{-1}y_{-1}, x_0y_0, x_1y_1, \ldots)$$

for $x = (x_i)_{i \in \mathbb{Z}} \in X$ and $y = (y_i)_{i \in \mathbb{Z}} \in \{0, 1\}^\mathbb{Z}$. For any $\nu \in \mathcal{M}(X, S)$ and $\mu \in \mathcal{M}(\{0, 1\}^\mathbb{Z}, S)$ the multiplicative convolution of $\nu$ and $\mu$ is the measure $\nu * \mu \in \mathcal{M}(\bar{X}, S)$ given by:

$$\nu * \mu = (\nu \otimes \mu) \circ Q^{-1}.$$

For a subshift $(X, S)$, the topological entropy $h = h(X, S)$ is defined as follows:

$$h(X, S) = \lim_{n \to \infty} \frac{\log \# \mathcal{L}_n(X)}{n}.$$ 

And for each $\mu \in \mathcal{M}(X, S)$, the measure entropy of $\mu$ is defined as follows:

$$h_\mu(X, S) = \lim_{n \to \infty} \frac{h_\mu(\mathcal{L}_n(X))}{n},$$

where $h_\mu(\mathcal{L}_n(X)) = -\sum_{W \in \mathcal{L}_n(X)} \mu(W) \log \mu(W)$. By the variational principle, $h(X, S) = \sup_{\mu \in \mathcal{M}(X, S)} h_\mu(X, S)$.

For a subshift $(X, S)$ and a continuous function $\phi : X \to \mathbb{R}$, the topological pressure $P = P(X, \phi)$ is defined as follows:

$$P(X, \phi) = \lim_{n \to \infty} \frac{\log Z_n(X, S, \phi)}{n}.$$
where $Z_n(X,S,\phi) = \sum_{W \in L_n(X)} 2^{\sup_{z \in W} \sum_{i=1}^{n-1} \phi(S^i z)}$. By the variational principle,

$$P(X,\phi) = \sup_{\mu \in \mathcal{M}(X,S)} \left( h_\mu(X,S) + \int \phi d\mu \right).$$

The measure $\mu$ is called an equilibrium state if it satisfies $P(X,\phi) = h_\mu(X,S) + \int \phi d\mu$.

In [16], a measure $\mu \in \mathcal{M}(X,S)$ is said to have the Gibbs property if there exists $a > 0$ such that for any $\mu$-positive measure block $C$,

$$\mu(C) \geq a \cdot 2^{-|C|h(X,S)}.$$

In [3], for a continuous function $\phi : X \to \mathbb{R}$, a measure $\mu \in \mathcal{M}(X,S)$ is called a Gibbs measure for $\phi$ if there exist $P = P(X,\phi) \geq 0$ and $c = c(X,\phi) > 0$ such that for any $n \in \mathbb{N}$, $\mu$-positive measure block $C$ of length $n$ and $x \in C$,

$$c^{-1} \leq \frac{\mu([x[0,n-1])}{2^{\sum_{i=0}^{n-1} \phi(S^i x) - nP(X,\phi)}} \leq c.$$

The constant $P = P(X,\phi)$ above is the topological pressure of $\phi$ on $(X,S)$.

### 3 Densities for a continuous function

In [16], it defines four notions of density. For a subshift $(X,S)$, let

$$d = \sup_{\mu \in \mathcal{M}(X,S)} \mu([1]),$$

$$D = \lim_{n \to \infty} \frac{1}{n} \max_{W \in L_n(X)} \#_1 W.$$

For $\mu \in \mathcal{M}(X,S)$, let

$$d_\mu = \mu([1]),$$

$$D_\mu = \lim_{n \to \infty} \frac{1}{n} \max_{W \in L_n(X), \mu(W) > 0} \#_1 W.$$

Similar with the above definitions, we define four notions of density for a continuous function. For a continuous function $\phi : X \to \mathbb{R}$, let

$$D^\phi = \lim_{n \to \infty} \frac{1}{n} \sup_{x \in X} \sum_{i=0}^{n-1} \phi(S^i x),$$

$$d^\phi = \sup_{\mu \in \mathcal{M}(X,S)} \int_X \phi d\mu.$$

For $\mu \in \mathcal{M}(X,S)$, define

$$D^\phi_\mu = \lim_{n \to \infty} \frac{1}{n} \max_{W \in L_n(X), \mu(W) > 0} \sup_{x \in W} \sum_{i=0}^{n-1} \phi(S^i x),$$

$$d^\phi_\mu = \sup_{\mu \in \mathcal{M}(X,S)} \int_X \phi d\mu.$$
\[ d_\mu^\phi = \int_X \phi d\mu. \]

Both \( D^\phi \) and \( D_\mu^\phi \) are exist, because for any \( n, m \in \mathbb{N}, \)

\[
\sup_{x \in X} \sum_{i=0}^{n+m-1} \phi(S^i x) = \sup_{x \in X} \left( \sum_{i=0}^{n-1} \phi(S^i x) + \sum_{i=0}^{m-1} \phi(S^{n+i} x) \right) \leq \sup_{x \in X} \sum_{i=0}^{n-1} \phi(S^i x) + \sup_{x \in X} \sum_{i=0}^{m-1} \phi(S^i x)
\]

and

\[
\max_{W \in \mathcal{L}_{n+m}(X), \mu(W) > 0} \sup_{x \in W} \sum_{i=0}^{n+m-1} \phi(S^i x) = \max_{W \in \mathcal{L}_{n+m}(X), \mu(W) > 0} \sup_{x \in W} \left( \sum_{i=0}^{n-1} \phi(S^i x) + \sum_{i=0}^{m-1} \phi(S^{n+i} x) \right) \leq \max_{W \in \mathcal{L}_{n+m}(X), \mu(W) > 0} \left( \sup_{x \in W[0,n-1]} \sum_{i=0}^{n-1} \phi(S^i x) + \sup_{x \in W[n,n+m-1]} \sum_{i=0}^{m-1} \phi(S^{n+i} x) \right) \leq \max_{W \in \mathcal{L}_{n+m}(X), \mu(W) > 0} \sup_{x \in W} \sum_{i=0}^{n-1} \phi(S^i x) + \max_{W \in \mathcal{L}_{n+m}(X), \mu(W) > 0} \sup_{x \in W} \sum_{i=0}^{m-1} \phi(S^i x).
\]

The last inequality is hold because if \( W \in \mathcal{L}_{n+m}(X) \) and \( \mu(W) > 0 \), then \( \mu(W[0,n-1]) \geq \mu(W) > 0 \) and \( \mu(S^n W[n,n+m-1]) = \mu(W[n,n+m-1]) \geq \mu(W) > 0 \). Therefore, by subadditivity, \( D^\phi \) and \( D_\mu^\phi \) are exist.

It is obvious that when \( \phi = 1_{[1]} \), we have \( d = d^\phi \), \( D = D^\phi \), \( d_\mu = d_\mu^\phi \) and \( D_\mu = D_\mu^\phi \) for any \( \mu \in \mathcal{M}(X, S) \).

It is proved in (10) that \( d_\mu \leq D_\mu \leq D = d \). Similarly, we also prove the corresponding theorem for the four notions of density for \( \phi \).

**Theorem 3.1.** For any \( \mu \in \mathcal{M}^*(X, S) \) and any continuous function \( \phi : X \to \mathbb{R} \), we have \( d_\mu^\phi \leq D_\mu^\phi \leq D^\phi = d^\phi \).

**Proof.**

(1) \( D^\phi = d^\phi \): For any \( n \in \mathbb{N} \), let \( x^{(n)} \) satisfy

\[
\sum_{i=0}^{n-1} \phi(S^i x^{(n)}) = \sup_{x \in X} \sum_{i=0}^{n-1} \phi(S^i x).
\]

Let

\[
\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{S^i x^{(n)}}.
\]
Without loss of generality, we can assume \( \mu_n \to \mu \), so
\[
\frac{1}{n} \sum_{i=0}^{n-1} \phi(S^i x^{(n)}) = \int_X \phi d\mu_n \to \int_X \phi d\mu \leq d^\phi.
\]

Therefore, \( D^\phi \leq d^\phi \). Let \( \nu \) satisfy \( \int_X \phi d\nu = \sup_{\mu \in \mathcal{M}(X,S)} \int_X \phi d\mu \) and \( x \) is a generic point of \( \nu \), then

\[
d^\phi = \int_X \phi d\nu = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(S^i x) \leq D^\phi.
\]

(2) \( D^\phi_\mu \leq D^\phi \): By the definition, it is obvious.

(3) \( d^\phi_\mu \leq D^\phi_\mu \): For any \( \epsilon > 0 \), fix large enough \( n \) such that
\[
\max_{W \in \mathcal{L}_n(X), \mu(W) > 0} \sup_{x \in W} \sum_{i=0}^{n-1} \phi(S^i x) < n(D^\phi + \epsilon).
\]

Fix \( x \) is a generic point of \( \mu \). For \( i \in \mathbb{N} \), define
(a) \( i \) is good: \( \mu(x[i, i + n - 1]) > 0 \);
(b) \( i \) is bad: \( \mu(x[i, i + n - 1]) = 0 \).

Set \( i_0 = -n \). For \( j = 1, 2, ... \), define inductively that
\[
i_j = \min\{i \geq i_{j-1} + n : i \text{ is good}\}.
\]

So for any \( i \in \bigcup_{j=1}^{\infty} [i_{j-1} + n, i_j - 1] \), \( i \) is bad. For any \( k \in \mathbb{N} \), because \( \{[i_j, i_j + n - 1] : j = 1, 2, ...\} \) is pairwise disjoint, we have
\[
\# \{i \in [0, k-1] : j = 1, 2, ...\} \leq \frac{k}{n} + 1.
\]

In addition,
\[
\frac{1}{k} \# \{i \in [0, k-1] : i \text{ is bad}\} \leq \frac{1}{k} \sum_{i=0}^{k-1} \sum_{W \in \mathcal{L}_n(X), \mu(W) = 0} 1_W(S^i x) \to 0.
\]

So let \( K \in \mathbb{N} \) such that if \( k > K \), then
\[
\# \{i \in [0, k-1] : i \text{ is bad}\} \leq \epsilon k.
\]

Therefore,
\[
\frac{1}{k} \sum_{i=0}^{k-1} \phi(S^i x) \leq \frac{1}{k} \left( \sum_{j \in [k]} \sum_{i=0}^{n-1} \phi(S^{ij} x) + \sum_{i \in [0, k-1], i \text{ is bad}} \phi(S^i x) \right) \leq \frac{1}{k} \left( \frac{k}{n} + 1 \right) (nD^\phi_\mu + \epsilon) + \epsilon k|\phi| \leq D^\phi_\mu + \epsilon (1 + |\phi|) + \frac{nD^\phi_\mu + \epsilon}{k}.
\]
where $|\phi| = \sup_{x \in X} |\phi(x)|$. Let $k \to \infty$, we have $d_{\mu}^k \leq D_{\mu}^0 + \epsilon (1 + |\phi|)$. By the arbitrariness of $\epsilon$, it shows that $d_{\mu}^k \leq D_{\mu}^0$.

By all of above, it ends the proof. \qed

4 Proof of Theorem 1.1

For convenience, we prove the case of $q = 1/2$. Let $\kappa = \nu * B_{1/2,1/2}$, where $\nu \in \mathcal{M}^c(X,S)$ and $B_{q,1-q}$ stands the Bernoulli measure on $\{0,1\}^Z$ with $B_{q,1-q}([0]) = q$ and $B_{q,1-q}([1]) = 1 - q$.

For the hereditary closure $(\tilde{X}, S)$ of a subshift $(X, S)$ and a continuous map $\tilde{\phi} : \tilde{X} \to \mathbb{R}$, denote by $\tilde{P} = P(\tilde{X}, \tilde{\phi})$ the topological pressure for $\tilde{\phi}$ on $(\tilde{X}, S)$.

Here, we need some lemmas in \cite{10}.

Lemma 4.1 (\cite{10}). Let $\nu \in \mathcal{M}(X, S)$. Then for $\kappa = \nu * B_{1/2,1/2}$, we have

$$\kappa(C) = \sum_{C \leq C' \in \mathcal{L}(X)} \nu(C') \cdot 2^{-\#C'}$$

for each $C \in \mathcal{L}(\tilde{X})$.

Lemma 4.2 (\cite{10}). Let $\nu \in \mathcal{M}(X, S)$ and $a > 0$. Suppose that there is a sequence of block $C_n$ such that $|C_n| \to \infty$ and $\nu(C_n) \geq a$. Then there exists $(n_k)$ such that $\bigcap_{k \geq 1} C_{n_k} \neq \emptyset$. Moreover, we have $\nu(\{x\}) \geq a$ for $\{x\} = \bigcap_{k \geq 1} C_{n_k}$.

For a continuous map $\tilde{\phi} : \tilde{X} \to \mathbb{R}$ and $A \subset \tilde{X}$, let $\operatorname{Var}(A) = \sup \tilde{\phi}(A) - \inf \tilde{\phi}(A)$.

Theorem 4.3. For the hereditary closure $(\tilde{X}, S)$ of a subshift $(X, S)$, nonatomic measure $\nu \in \mathcal{M}^c(X, S)$ with $D_{\nu} = D_{\nu}$, $\kappa = \nu * B_{1/2,1/2} \in \mathcal{M}^c(\tilde{X}, S)$ and a continuous map $\tilde{\phi} : \tilde{X} \to \mathbb{R}$ with $D_{\kappa}^0 = D_{\tilde{\phi}}^0$. If

$$\tilde{P} \leq (1 + \operatorname{Var}(\tilde{\phi}([0])) - \operatorname{Var}(\tilde{\phi}([1])))d + d_{\tilde{\phi}} - \operatorname{Var}(\tilde{\phi}([0])),$$

$$\sup \tilde{\phi}([1]) \geq \sup \tilde{\phi}([0]),$$

and

$$\operatorname{Var}(\tilde{\phi}([1])) \leq \operatorname{Var}(\tilde{\phi}([0])) + 1,$$

then $\kappa$ is not the Gibbs measure for $\tilde{\phi}$.

Proof. If $\kappa = \nu * B_{1/2,1/2}$ is Gibbs measure for $\tilde{\phi}$. Because $D_{\kappa}^0 = D_{\tilde{\phi}}^0$, for any $n \in \mathbb{N}$, there exists $x^{(n)} \in \tilde{X}$ such that

$$\sum_{i=0}^{n-1} \tilde{\phi}(S^i x^{(n)}) = \max_{W \in \mathcal{L}_n(\tilde{X})} \sup_{y \in W} \sum_{i=0}^{n-1} \tilde{\phi}(S^i y) \geq nD_{\tilde{\phi}}^0$$

and $\kappa(x^{(n)}[0, n - 1]) > 0$. 

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Since $D_\nu = D$, for any $n \in \mathbb{N}$, there exists $C_n \in \mathcal{L}_n(X)$ such that $\#_1 C_n = \max_{W \in \mathcal{L}_n(X), \nu(W) > 0} \#_1 W \geq nD = nd$.

Let $a_n = \#_1 C_n - \#_1 x^{(n)}[0, n-1]$. By Lemma 4.1,

$$\kappa(x^{(n)}[0, n-1]) = \sum_{W \in \mathcal{L}_n(X), W \geq x^{(n)}[0, n-1]} \nu(W) \cdot 2^{\#_1 W} > 0.$$ 

So there exists $W$ with $\nu(W) > 0$ such that $\#_1 W \geq \#_1 x^{(n)}[0, n-1]$, which implies that $a_n \geq 0$. Now fix $y \in C_n$,

$$0 \leq \sum_{i=0}^{n-1} \bar{\phi}(S^i x^{(n)}) - \sum_{i=0}^{n-1} \bar{\phi}(S^i y)$$

$$\leq \#_1 x^{(n)}[0, n-1] \sup \bar{\phi}(1) + (n - \#_1 x^{(n)}[0, n-1]) \sup \bar{\phi}(0) - \#_1 C_n \inf \bar{\phi}(1) - (n - \#_1 C_n) \inf \bar{\phi}(0)$$

$$= \#_1 C_n \inf \bar{\phi}(1) - (n - \#_1 C_n) \inf \bar{\phi}(0)$$

$$\leq \#_1 C_n (\text{Var} \bar{\phi}(1) - \text{Var} \bar{\phi}(0)) + n \text{Var} \bar{\phi}(0) + n \text{Var} \bar{\phi}(0).$$

Therefore,

$$c^{-1} \leq \kappa(C_n) \cdot 2^n \bar{\phi}(1) - \sum_{i=0}^{n-1} \bar{\phi}(S^i y)$$

$$\leq \nu(C_n) \cdot 2^n \bar{\phi}(1) - \sum_{i=0}^{n-1} \bar{\phi}(S^i x^{(n)})$$

$$\leq \nu(C_n) \cdot 2^n \bar{\phi}(1) - \nu(C_n) \cdot 2^n \bar{\phi}(0)$$

$$\leq \nu(C_n) \cdot 2^n \bar{\phi}(1) - \nu(C_n) \cdot 2^n \bar{\phi}(0)$$

$$\leq \nu(C_n) \cdot 2^n \bar{\phi}(1) - \nu(C_n) \cdot 2^n \bar{\phi}(0)$$

$$\leq \nu(C_n).$$

By Lemma 4.2, $\nu$ is atomic, which is a contradiction. 

If $\bar{\phi} = a_0 \mathbb{1}_{[0]} + a_1 \mathbb{1}_{[1]}$ for some $a_0 \leq a_1$, then $\text{Var} \bar{\phi}(0) = \text{Var} \bar{\phi}(1) = 0$. So we have the following corollary.

**Corollary 4.4.** For the hereditary closure $(\bar{X}, S)$ of a subshift $(X, S)$, nonatomic measure $\nu \in \mathcal{M}(\bar{X}, S)$ with $D_\nu = D$ and $\kappa = \nu * B_{1/2, 1/2} \in \mathcal{M}(\bar{X}, S)$, suppose that $a_0 = 0$ or $a_1 = 1$ with $a_0 \leq a_1$ and $D_0 = D$. If $\bar{P} \leq d + d$, then $\kappa$ is not the Gibbs measure for $\bar{\phi}$.

More than Lemma 4.1, we prove that:
Lemma 4.5. Let $\nu \in \mathcal{M}(X, S)$ and $0 < q < 1$. Then for $\kappa = \nu \ast B_{q, 1-q}$, we have

$$\kappa(C) = \sum_{C \leq C' \in \mathcal{L}(X)} \nu(C') \cdot q^{\#_{1-C'} C}(1-q)^{\#_{1-C}}$$

for each $C \in \mathcal{L}(\tilde{X})$.

Proof. For any $n \in \mathbb{N}$ and any $C \in \mathcal{L}_n(\tilde{X})$, we have

$$Q^{-1}(C) = \bigcup_{C \leq C' \in \mathcal{L}(X)} \bigcup_{C', D \in \mathcal{L}_n((0,1)^2)} C' \times D.$$ 

For each $C' \geq C$, if $C'[i] = 1$, $D[i] = C[i]$. So $\#\{i : D[i] = 1\} = \#_1 C$. On the other hand, if $C'[i] = 0$, $D[i]$ is arbitrary. So $\#\{i : D[i] = 1\} = \#_1 C$. Then

$$\kappa(C) = \sum_{C \leq C' \in \mathcal{L}(X)} \nu(C') \cdot q^{n-\#_1 D}(1-q)^{\#_1 D}$$

$$= \sum_{C \leq C' \in \mathcal{L}(X)} \nu(C') \sum_{i=0}^{n-\#_1 C'} \binom{n}{i} q^{n-\#_1 C'-i}(1-q)^{\#_1 C+i}$$

$$= \sum_{C \leq C' \in \mathcal{L}(X)} \nu(C') q^{\#_{1-C'}-\#_1 C}(1-q)^{\#_1 C}.$$ 

Proof of Theorem 1.1. By Lemma 4.5, if $C \in \mathcal{L}_n(X)$ attaches the maximum of the number of ones, that is, $\#_1 C = \max_{W \in \mathcal{L}_n(X)} \#_1 W$, then

$$\nu \ast B_{q, 1-q}(C) = \nu(C) \cdot (1-q)^{\#_1 C}.$$ 

Therefore, Theorem 1.1 can be proved by a similar proof of Theorem 4.3. □

5 $\mathcal{B}$-free systems

In this section, we consider some $\mathcal{B}$-free systems as an application of Theorem 1.1 and Theorem 4.3. Firstly, we show some basic notions about $\mathcal{B}$-free systems.

Let $\mathcal{B} = \{b_1, b_2, \cdots\}$ be an infinite subset of $\{2, 3, \cdots\}$. In the rest of this section, we always assume that $\mathcal{B}$ satisfies condition (1).

For $A \subseteq \mathbb{Z}$, define the densities of the positive part of $A$:

- lower density: $d(A) = \liminf_{N \to \infty} \frac{\#(A \cap [1, N])}{N}$,
- upper density: $\bar{d}(A) = \limsup_{N \to \infty} \frac{\#(A \cap [1, N])}{N}$.
If \( d(A) = \bar{d}(A) \), we set \( d(A) := \bar{d}(A) = \tilde{d}(A) \), called the density of \( A \). Also, the lower logarithmic density \( \delta(A) \) and the upper logarithmic density \( \bar{\delta}(A) \) of \( A \) is defined as follows:

\[
\delta(A) = \liminf_{N \to \infty} \frac{1}{\log N} \sum_{1 \leq a \leq N, a \in A} \frac{1}{a},
\]

\[
\bar{\delta}(A) = \limsup_{N \to \infty} \frac{1}{\log N} \sum_{1 \leq a \leq N, a \in A} \frac{1}{a}.
\]

If \( \delta(A) = \bar{\delta}(A) \), we set \( \delta(A) := \delta(A) = \bar{\delta}(A) \), called the logarithmic density of \( A \).

Let \( M_B = \bigcup_{b \in B} b\mathbb{Z} \), and \( F_B = \mathbb{Z} \setminus M_B \). By our assumptions of \( B \) and [7, 9], the density of \( M_B \) exists, which means that \( B \) is Besicovitch. In Section 2 of [6], since \( \sum_{b \in B} \frac{1}{b} < \infty \) (called thin in [6]), \( B \) has light tails (See the definition in [10]).

Let \( \eta = \mathbb{1}_{F_B} \in \{0, 1\}^\mathbb{Z} \), that is,

\[
\eta[n] = 1 \text{ if and only if } n \in F_B.
\]

Let

\[
X_\eta = \{ y \in \{0, 1\}^\mathbb{Z} : \text{ for any } i, j \in \mathbb{N}, \ y[i, i + j] = \eta[k, k + j] \text{ for some } k \}.
\]

Recall that a point \( y \in \{0, 1\}^\mathbb{Z} \) is \( B \)-admissible if \( \#(\text{supp}(y) \mod b) < b \) for each \( b \in B \), where \( \text{supp}(y) = \{ n \in \mathbb{Z} : y[n] = 1 \} \).

**Lemma 5.1** ([7, 24]). The space \( X_\eta = X_B := \{ y \in \{0, 1\}^\mathbb{Z} : y \text{ is } B \text{-admissible} \} \).

In particular, \( X_\eta \) is hereditary, that is, \( \tilde{X}_\eta = X_\eta \).

The following theorems and proposition are proved in [1, 4, 5, 18].

**Theorem 5.2** (Theorem 5.3 in [1]). The topological entropy of the subshift \( X_B \) is given by

\[
h_{\text{top}}(X_B) = \prod_{i \in \mathbb{N}} \left( 1 - \frac{1}{b_i} \right).
\]

**Theorem 5.3** ([4, 5]). For any \( B \subset \mathbb{N} \), the logarithmic density \( \delta(M_B) \) of \( M_B \) exist. Moreover,

\[
\delta(M_B) = d(M_B) = \lim_{K \to \infty} d(M_{\{b \in B : b \leq K\}}).
\]

**Proposition 5.4** (Proposition K in [6]). For any \( B \subset \mathbb{N} \), we have \( h_{\text{top}}(X_\eta) = h_{\text{top}}(X_B) = \delta(F_B) \).
By our assumption of $B$, for subshift $(X_B, S)$, we have
\[ d = d(F_B) = h_{top}(X_B) = \prod_{i \in \mathbb{N}} \left( 1 - \frac{1}{b_i} \right). \]

Recall some known facts about the dynamical systems associated to $B$-free numbers. Let
\[ \Omega := \prod_{i \geq 1} \mathbb{Z}/b_i\mathbb{Z} = \{ \omega = (\omega(1), \omega(2), \ldots) : \omega(i) \in \mathbb{Z}/b_i\mathbb{Z}, i = 1, 2, \ldots \}. \]

With the product topology and the coordinatewise addition, $\Omega$ becomes a compact metrizable Abelian group. Let $P$ be the normalized Haar measure of $\Omega$ (which is the product of uniform measures on $\mathbb{Z}/b_k\mathbb{Z}$). Denote $T : \Omega \rightarrow \Omega$ the homeomorphism given by
\[ T \omega = (\omega(1) + 1, \omega(2) + 1, \ldots) \]
where $\omega = (\omega(1), \omega(2), \ldots)$. It is known that $(\Omega, T)$ has zero entropy. Define $\phi : \Omega \rightarrow \{0, 1\}$ by
\[ \phi(\omega)(n) = \begin{cases} 1, & \text{if for any } i \geq 1, \omega(i) + n \not\equiv 0 \mod b_i, \\ 0, & \text{otherwise}. \end{cases} \]

It is not hard to see that $\phi$ is Borel, equivariant (that is, $\phi \circ T = S \circ \phi$) and $\eta = \phi(0, 0, \ldots)$. Let $\nu_\eta = \phi^*(P) := P \circ \phi^{-1}$ be the image of $P$ via $\phi$, which is called the Mirsky measure of $(X_\eta, S)$. By [6], $\eta$ is generic point of the Mirsky measure $\nu_\eta$. So for any $A \subset X_\eta$,
\[ \nu_\eta(A) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_A(S^i \eta). \]

Since $B$ is infinite, by the tautness of $B$ and Proposition 3.5 in [18], $\nu_\eta$ is non-atomic. Also, in [11], it is proved that the tautness of $B$ implies that $\nu_\eta$ is full support on $X_\eta$.

Now, we turn to focus on some $B$-free systems, and show that for some $\phi$, its unique equilibrium state is not Gibbs measure.

Theorem [12] will be proved by several steps. First, to calculate the topological pressure on $(X_\eta, S)$, we need the following lemma, proved in [9].

Lemma 5.5 ([9], p.242). For any $b_k$, $k \geq 1$, any $r_k \in \mathbb{Z}/b_k\mathbb{Z}$, and $K \geq 1$, we have
\[ d \left( \bigcup_{k=1}^{K} (b_k \mathbb{Z} + r_k) \right) \geq d(\mathcal{M}_{\{b_1, \ldots, b_K\}}). \]

Proposition 5.6. Suppose that $B = \{b_1, b_2, \ldots\}$ satisfies (7) and $b_1 = 2$. For $\phi = a_{00} \mathbb{1}_{[00]} + a_{01} \mathbb{1}_{[01]} + a_1 \mathbb{1}_{[1]}$, the topological pressure
\[ P(X_\eta, \phi) = a_{00}(1 - 2d) + d \log(2^{a_2} + a_{01} + 2^{2a_{00}}). \]
Proof. For \( n \in \mathbb{N} \), since \( X_n = X_\mathcal{B} \),

\[
\mathcal{L}_n := \mathcal{L}_n(X_n) = \{W \in \{0,1\}^n : W \text{ is } \mathcal{B}-\text{admissible}\}.
\]

For \( K \in \mathbb{N} \), let

\[
\mathcal{L}_{n,K} = \{W \in \{0,1\}^n : W \text{ is } \{b_1, b_2, \ldots, b_K\}-\text{admissible}\}.
\]

So \( \mathcal{L}_n \subseteq \mathcal{L}_{n,K} \) for any \( K \geq 1 \). Let \( N_{n,K} := nb_1b_2\cdots b_K \). Similar with the proof of Proposition K in [6], we can obtain \( W \in \mathcal{L}_{n,K} \) by the following ways:

(a) choose any \((r_1, \ldots, r_K) \in \prod_{k=1}^K \mathbb{Z}/b_k\mathbb{Z} \). Then for \( j \in \bigcup_{k=1}^K (r_k + b_k\mathbb{Z}) \), set \( W[j] = 0 \) when \( j \in [0, N_{n,K} - 1] \);

(b) for \( j \in [0, N_{n,K} - 1] \setminus \bigcup_{k=1}^K (r_k + b_k\mathbb{Z}) \), complete the word \( W \) by choosing arbitrarily \( W[i] \in \{0, 1\} \).

So \( \#_1 W \) is ranged over 0 to \( N_{n,K}(1 - d(\bigcup_{k=1}^K (r_k + b_k\mathbb{Z})) \). By Lemma 5.3

\[
N_{n,K}(1 - d(\bigcup_{k=1}^K (r_k + b_k\mathbb{Z})) \leq N_{n,K}(1 - d_K) \text{ where } d_K := d(M_{\{b_1, \ldots, b_K\}}).\]

Fixed any \( \epsilon > 0 \), by Theorem 5.3 choose large enough \( K \), such that \( 1 - d - \epsilon < d_K \leq 1 - d + \epsilon \) (notice that \( d = d(F_\mathcal{B}) = 1 - d(M_\mathcal{B}) \)). Because \( 2 \in \mathcal{B} \), the word 11 does not appear in any \( x \in X_n \).

We claim that, for any \( W \in \mathcal{L}_{n,K} \),

\[
\sup_{x \in W} \sum_{i=0}^{N_{n,K}-1} \phi(S^i x) \leq a_1 \#_1 W + a_{01} \#_1 W + a_{00}(N_{n,K} - 2 \#_1 W) + \phi(0).
\]

For \( W \in \mathcal{L}_{n,K} \), there are three cases:

(1) \( W[N_{n,K} - 1] = 1 \): Because \( b_1 = 2 \), \( N_{n,K} \) is even, which implies that \( W[0] = 0 \).

So \( \# \{i \in [0, N_{n,K} - 1] : W[i, i + 1] = [01] \} = \#_1 W \) and

\[
\sup_{x \in W} \sum_{i=0}^{N_{n,K}-1} \phi(S^i x) \\
=a_1 \#_1 W + a_{01} \#_1 W + a_{00}(N_{n,K} - 2 \#_1 W) \\
\leq a_1 \#_1 W + a_{01} \#_1 W + a_{00}(N_{n,K} - 2 \#_1 W) + \phi(0);
\]

(2) \( W[0] = W[N_{n,K} - 1] = 0 \): Because \( W[0] = 0 \), we also have \( \# \{i \in [0, N_{n,K} - 1] : W[i, i + 1] = [01] \} = \#_1 W \). So

\[
\sup_{x \in W} \sum_{i=0}^{N_{n,K}-1} \phi(S^i x) \\
\leq a_1 \#_1 W + a_{01} \#_1 W + a_{00}(N_{n,K} - 1 - 2 \#_1 W) + \sup \phi(0) \\
\leq a_1 \#_1 W + a_{01} \#_1 W + a_{00}(N_{n,K} - 2 \#_1 W) + \phi(0);
\]

(3) \( W[0] = 1 \) and \( W[N_{n,K} - 1] = 0 \): The quantity \( \# \{i \in [0, N_{n,K} - 1] : W[i, i + 1] = [01] \} = \#_1 W \) and
1] = [01] = \#_{1}W - 1. So

$$
\sup_{x \in W} \sum_{i=0}^{N_{n,K}-1} \phi(S^{i}x) 
\leq a_{1} \#_{1}W + a_{01}(\#_{1}W - 1) + a_{00}(N_{n,K} - 2 \#_{1}W) + \sup \phi([0]) 
\leq a_{1} \#_{1}W + a_{01} \#_{1}W + a_{00}(N_{n,K} - 2 \#_{1}W) + \Var \phi([0]).
$$

It ends the proof of the claim.

So

$$
Z_{N_{n,K}}(X_{\eta}, \phi) 
\leq \sum_{W \in \mathcal{L}_{n,K}} 2^{\sup_{x \in W} \sum_{i=0}^{N_{n,K}-1} \phi(S^{i}x)} 
\leq \sum_{W \in \mathcal{L}_{n,K}} 2^{a_{1} \#_{1}W + a_{01} \#_{1}W + a_{00}(N_{n,K} - 2 \#_{1}W) + \Var \phi([0])} 
\leq K \prod_{k=1}^{N_{n,K}(1-d_{n,K})} \left( N_{n,K}(1-d_{n,K}) \right)^{2^{a_{1}i+a_{01}i+a_{00}(N_{n,K}-2i)+\Var \phi([0])}} 
= \prod_{k=1}^{K} b_{k} \cdot 2^{a_{00}(N_{n,K}(1-d_{n,K}))} \left( 2^{a_{1}+a_{01}} + 2^{2a_{00}} \right)^{N_{n,K}(1-d_{n,K})}. 
$$

Thus,

$$
P(X_{\eta}, \phi) 
\leq a_{00}(2d_{K} - 1) + (1 - d_{K}) \log(2^{a_{1}+a_{01}} + 2^{2a_{00}}) 
\leq a_{00}(1 - 2d + 2\epsilon) + (d + \epsilon) \log(2^{a_{1}+a_{01}} + 2^{2a_{00}}) ,
$$

which shows that $P(X_{\eta}, \phi) \leq a_{00}(1 - 2d) + d \log(2^{a_{1}+a_{01}} + 2^{2a_{00}})$ by the arbitrariness of $\epsilon$.

To complete the proof, it remains to show that the inverse inequality. For any $n \in \mathbb{N}$, let

$$
p(n) := \#([1,n] \cap \mathcal{F}_{\#:}.)
$$

The set

$$
\{W \in \{0,1\}^{n} : W \leq \eta[1,n]\} = \prod_{i \in \mathbb{Z} \cap [1,n] \setminus \mathcal{F}_{\#}} \{0\} \times \prod_{i \in [1,n] \setminus \mathcal{F}_{\#}} \{0,1\} \subset \mathcal{L}_{n},
$$

Thus

$$
Z_{n}(X_{\eta}, \phi) \geq \sum_{i=0}^{p(n)} \binom{p(n)}{i} 2^{a_{1}i+a_{01}i+a_{00}(n-2i)-2|\phi|} 
= 2^{2a_{00}p(n)-2|\phi|}(2^{a_{1}+a_{01}} + 2^{2a_{00}})^{p(n)},
$$

13
where $|\phi| = \sup_{x \in X_\eta} |\phi(x)|$. So

$$P(X_\eta, \phi) \geq \lim_{n \to \infty} a_{00} \left( 1 - \frac{2p(n)}{n} \right) - \frac{2|\phi|}{n} + \frac{p(n)}{n} \log(2^{a_1 + a_{01}} + 2^{2a_{00}})$$

$$= a_{00}(1 - 2d) + d \log(2^{a_1 + a_{01}} + 2^{2a_{00}}),$$

which ends the proof.

**Remark 5.7.** For $n \geq 1$, let $\mathcal{E}_n = \{ \sum_{W \in \mathcal{L}_n(X_\eta)} a_W \mathbb{I}_W : a_W \in \mathbb{R}, W \in \mathcal{L}_n(X_\eta) \}$. We also consider the function $\phi \in \mathcal{E}_n \setminus \mathcal{E}_2$ for $n \geq 3$. But in the calculation of topological pressure on $X_\eta$, it is not easy to estimate the frequency of the $n$-length word $A$ with $#_1A \geq 2$ appearing in $W \in \mathcal{L}(X_\eta)$. This difficulty arises for $X_\eta$ with $2 \in \mathcal{B}$. Also, for $\phi \in \mathcal{E}_2$ and $X_\eta$ with $2 \notin \mathcal{B}$, this difficulty will arise because the word 11 will appear in some $W \in \mathcal{L}(X_\eta)$. So in such cases, it is not easy to calculate or estimate the topological pressure for $\phi$.

The measure entropy $h_{\nu_\eta * B_{p,1-p}}(X_\eta, S)$ is given in [18].

**Proposition 5.8** (Proposition 2.1.9 [18]). If $\kappa = B_{p,1-p}$, then

$$h_{\nu_\eta * \kappa}(X_\eta, S) = (-p \log p - (1-p) \log(1-p)) \prod_{i=1}^{\infty} \left( 1 - \frac{1}{b_i} \right).$$

The next proposition shows that for some $p$, $\nu_\eta * B_{p,1-p}$ is an equilibrium state for $\phi = a_{00} \mathbb{I}_{[00]} + a_{01} \mathbb{I}_{[01]} + a_1 \mathbb{I}_{[1]}$.

**Proposition 5.9.** Suppose that $\mathcal{B} = \{ b_1, b_2, \ldots \}$ satisfies (7) and $b_1 = 2$. For $\phi = a_{00} \mathbb{I}_{[00]} + a_{01} \mathbb{I}_{[01]} + a_1 \mathbb{I}_{[1]}$, $\nu_\eta * B_{p,1-p}$ is an equilibrium state for $\phi$ where

$$p = \frac{2^{2a_{00}}}{2^{a_1 + a_{01}} + 2^{2a_{00}}}.$$ 

**Proof.** Since $2 \in \mathcal{B}$, the word 11 does not appear in $\eta$, which implies that $[11] \cap X_\eta = \emptyset$. For $[00]$, $[01]$ and $[10]$, their measures are given as follows:

$$\nu_\eta([00]) = \lim_{n \to \infty} \frac{\# \{ i \in [0, n-1] : S^i \eta \in [0] \text{ and } S^{i+1} \eta \in [0] \}}{n}$$

$$= \lim_{n \to \infty} \frac{n - 2 \#([0, n-1] \cap \mathcal{F}_\mathcal{B})}{n}$$

$$= 1 - 2d; \quad (2)$$

$$\nu_\eta([01]) = \lim_{n \to \infty} \frac{\# \{ i \in [0, n-1] : S^{i+1} \eta \in [1] \}}{n}$$

$$= \lim_{n \to \infty} \frac{\#([1, n] \cap \mathcal{F}_\mathcal{B})}{n}$$

$$= d; \quad (3)$$

The measure entropy $h_{\nu_\eta * B_{p,1-p}}(X, S)$ is given in [18].
\[
\nu_\eta([10]) = \lim_{n \to \infty} \frac{\# \{ i \in [0, n-1] : S^i \eta \in [1] \}}{n} = \lim_{n \to \infty} \frac{\#([0, n-1] \cap \mathcal{F}_{\eta})}{n} = d.
\]

Those three equations follow from the fact that 11 does not appear in \( \eta \). Thus, by the Lemma 4.5 and equations (2), (3) and (4),
\[
\nu_\eta * B_{p,1-p}(\{0\}) = \nu_\eta(\{0\}) + \nu_\eta(\{1\})p + \nu_\eta([10])p = 1 - 2d + 2dp,
\]
\[
\nu_\eta * B_{p,1-p}(\{1\}) = \nu_\eta(\{1\})(1-p) = d(1-p),
\]
\[
\nu_\eta * B_{p,1-p}(\{10\}) = \nu_\eta(\{10\})(1-p) = d(1-p).
\]

So
\[
\int \phi d\nu_\eta * B_{p,1-p} = a_{00}(1 - 2d) + d(2a_{00}p + (a_{01} + a_1)(1 - p)).
\]

Notice that
\[
\log(2^{a_1 + a_{01}} + 2^{2a_{00}}) = (-p \log p - (1 - p) \log(1 - p)) + 2a_{00}p + (a_{01} + a_1)(1 - p)
\]
if and only if
\[
p = \frac{2^{2a_{00}}}{2^{a_1 + a_{01}} + 2^{2a_{00}}},
\]
It is showed before that \( d = \prod_{i=1}^{\infty} (1 - 1/b_i) \). So when
\[
p = \frac{2^{2a_{00}}}{2^{a_1 + a_{01}} + 2^{2a_{00}}},
\]
we have
\[
P(X_\eta, \phi) = h_{\nu_\eta * B_{p,1-p}}(X_\eta, S) + \int \phi d\nu_\eta * B_{p,1-p},
\]
which implies that \( \nu_\eta * B_{p,1-p} \) is the equilibrium state for \( \phi \).

Next, we will prove the uniqueness of equilibrium state. In [17] and [18], they prove intrinsic ergodicity of the squarefree flow and \( \mathcal{B} \)-free system. We mainly use their methods to prove the uniqueness of the equilibrium state for \( \phi = a_{00} \mathbb{1}_{[00]} + a_{01} \mathbb{1}_{[01]} + a_1 \mathbb{1}_{[1]} \).

Let \( I = (i_1, i_2, \ldots) \) where \( i_k \in \{1, 2, \ldots, b_k - 1\} \) for each \( k \geq 1 \). Define
\[
X_I = \{ x \in X_\eta : \text{ for any } k \geq 1, |\text{supp}(x) \mod b_k| = b_k - i_k \},
\]
and for any \( K \geq 1 \),
\[
d_{I,K} = \prod_{k=1}^{K} \left( 1 - \frac{i_k}{b_k} \right), \quad d_I = \prod_{k=1}^{\infty} \left( 1 - \frac{i_k}{b_k} \right).
\]
For convenience, we set \( X_1 := X_{(1,1,\ldots)} \). Notice that \( X_1 \) is Borel and \( SX_1 = X_1 \).
Lemma 5.10. Suppose that $\mathcal{B} = \{b_1, b_2, \ldots\}$ satisfies $[1]$ and $b_1 = 2$. Let $I = (i_1, i_2, \ldots)$ where $i_k \in \{1, 2, \ldots, b_k\}$ for each $k \geq 1$. For $\phi = a_{00} 1_{[00]} + a_{01} 1_{[01]} + a_1 1_{[1]}$, we have

$$P(\overline{X}_I, S) \leq a_{00}(1 - 2d_I) + d_I \log(2^{2a_{00}} + 2^{a_{01} + a_1}),$$

where $\overline{X}_I$ is the closure of $X_I$.

Proof. For $n \in \mathbb{N}$, notice that

$$\mathcal{L}_n(\overline{X}_I) = \{W \in \{0, 1\}^n : |\text{supp}(W) \mod b_k| \leq b_k - i_k \text{ for each } k \geq 1\}.$$

For $K \in \mathbb{N}$, let

$$\mathcal{L}_{n,I,K} = \{W \in \{0, 1\}^n : |\text{supp}(W) \mod b_k| \leq b_k - i_k \text{ for each } k = 1, 2, \ldots, K\}.$$

So $\mathcal{L}_n(\overline{X}_I) \subset \mathcal{L}_{n,I,K}$ for any $K \geq 1$. Let $N_{n,K} := nb_1 b_2 \cdots b_K$.

We can obtain $W \in \mathcal{L}_{N_{n,K},I,K}$ by the following ways:

(a) choose any $(Z_1, \ldots, Z_K)$ with $Z_k \subset \mathbb{Z}/b_k \mathbb{Z}$ and $\#Z_k = i_k$ for each $k = 1, \ldots, K$. Then let

$$Z(Z_1, \ldots, Z_K) = \{0 \leq j < N_{n,K} : \text{for any } k = 1, 2, \ldots, K, j \mod b_k \in Z_k\},$$

and for $j \in Z(Z_1, \ldots, Z_K)$, set $W[j] = 0$;

(b) for $j \in \{0, N_{n,K} - 1\} \setminus Z(Z_1, \ldots, Z_K)$, complete the word $W$ by choosing arbitrarily $W[j] \in \{0, 1\}$.

Since $\#Z(Z_1, \ldots, Z_K) = N_{n,K}(1 - d_{I,K})$, we have $\#W$ is ranged over $0$ to $N_{n,K}d_{I,K}$. And there are at most $\binom{b_1}{i_1} \cdots \binom{b_K}{i_K}$ choices of $(Z_1, \ldots, Z_K)$ in (a). Similar with the claim in the proof of Proposition 5.9 we have

$$\sup_{x \in W} \sum_{i=0}^{N_{n,K}-1} \phi(S^i x) \leq a_1 \#_1 W + a_{01} \#_1 W + a_{00}(N_{n,K} - 2\#_1 W) + \text{Var}\phi([0])$$

for each $W \in \mathcal{L}_{N_{n,K},I,K}$. So

$$Z_{N_{n,K}}(\overline{X}_I, \phi) \leq \sum_{W \in \mathcal{L}_{N_{n,K},I,K}} 2^{\sup_{x \in W} \sum_{i=0}^{N_{n,K}-1} \phi(S^i x)} \leq \sum_{W \in \mathcal{L}_{N_{n,K},I,K}} 2^{a_1 \#_1 W + a_{01} \#_1 W + a_{00}(N_{n,K} - 2\#_1 W) + \text{Var}\phi([0])} \leq \prod_{k=1}^{K} \left(\frac{b_k}{i_k}\right)^{N_{n,K}d_{I,K}} \sum_{i=0}^{N_{n,K}d_{I,K}} \left(\binom{N_{n,K}d_{I,K}}{i}\right) 2^{a_1 i + a_{01} i + a_{00}(N_{n,K} - 2i) + \text{Var}\phi([0])} \leq \prod_{k=1}^{K} \left(\frac{b_k}{i_k}\right)^{N_{n,K}d_{I,K}} 2^{a_{00} N_{n,K} + \text{Var}\phi([0]) - 2a_{00} N_{n,K}d_{I,K} (2^{a_1 + a_{01}} + 2^{2a_{00}}) N_{n,K}d_{I,K} + \text{Var}\phi([0])} = \prod_{k=1}^{K} \left(\frac{b_k}{i_k}\right)^{N_{n,K}d_{I,K}} 2^{a_{00} N_{n,K} (1 - 2d_{I,K}) + \text{Var}\phi([0]) (2^{a_1 + a_{01}} + 2^{2a_{00}}) N_{n,K}d_{I,K}}.$$
Thus
\[ P(X_I, \phi) \leq a_{00}(1 - 2d_I) + d_I \log(2^{a_1+a_{01}} + 2^{2a_{00}}), \]
and let \( K \to \infty \), which ends the proof. \( \Box \)

The next lemma shows that we can use the methods in the proof of intrinsic ergodicity in [17] and [18].

**Lemma 5.11.** Suppose that \( B = \{b_1, b_2, \ldots \} \) satisfies (1) and \( b_1 = 2 \). Let \( \mu \in \mathcal{M}(X, S) \) be an equilibrium state for \( \phi = a_{00} \mathbb{I}_{[00]} + a_{01} \mathbb{I}_{[01]} + a_1 \mathbb{I}_{[1]} \). Then \( \mu(X_1) = 1 \).

**Proof.** First, there is unique \( I = (i_1, i_2, \ldots) \in \prod_{k \geq 1} \{1, 2, \ldots, b_k - 1\} \) such that \( \mu(X_I) = 1 \), which can be proved by a similar method in [17, Lemma 3.3] although it is the case of \( B = \{p^2 : p \text{ is prime number}\} \). Thus
\[ P(X_{\eta}, \phi) = h_\mu(X_{\eta}, S) + \int_{X_\eta} \phi d\mu = P(X_I, \phi) \leq a_{00}(1 - 2d_I) + d_I \log(2^{a_1+a_{01}} + 2^{2a_{00}}). \]

Since \( P(X_{\eta}, \phi) = a_{00}(1 - 2d) + d \log(2^{a_1+a_{01}} + 2^{2a_{00}}) \), we have \( d_I \geq d \). When \( B \) satisfies the condition (1), it follows by [1] that \( d > 0 \), which implies that \( i_k = 1 \) for each \( k \geq 1 \) and \( X_I = X_1 \).

By this lemma, we can use the methods in [1] and [18] to prove the uniqueness of equilibrium state. Given \( k \geq 1 \) and \( z \in \mathbb{Z}/b_k \mathbb{Z} \), set
\[
\Omega_{k,z} = \{ \omega \in \Omega : \omega(k) = z \},
\]
\[
E_{k,z} = \{ \omega \in \Omega : \text{for any } s \geq 1, \varphi(\omega)(-z + sb_k) = 0 \}
\]
and
\[
\Omega_0' = \bigcap_{k \geq 1} \bigcap_{z \in \mathbb{Z}/b_k \mathbb{Z}} (E_{k,z}^c \cup \Omega_{k,z}) , \Omega_0 = \bigcap_{k \in \mathbb{Z}} T^k \Omega_0'.
\]

**Lemma 5.12** (Proposition 3.2 in [1]). We have \( \mathbb{P}(\Omega_0) = 1 \) and \( \varphi|_{\Omega_0} \) is 1-1.

Define a Borel map \( \theta : X_1 \to \Omega \) (cf. [17]) satisfy that
\[-\theta(y)(i) \notin \text{supp}(y) \text{ mod } b_i \text{ for all } i \geq 1.\]
Since \( |\text{supp}(y)\text{ mod } b_i| = b_i - 1 \) for each \( y \in X_1 \) and each \( i \geq 1 \), the map \( \theta \) is well defined.

**Lemma 5.13** (Lemma 2.5 in [18]). We have:

(i) \( T \circ \theta = \theta \circ S \);

(ii) for each \( y \in X_1 \), \( y \leq \varphi(\theta(y)) \);

(iii) \( \varphi(\Omega_0) \subset X_1 \) (In particular, \( \theta \circ \varphi|_{\Omega_0} = \text{id}_{\Omega_0} \)).
Fix a measure $\mu$ on $X_1$ is the equilibrium state for $\phi = a_{00}1_{[00]} + a_{01}1_{[01]} + a_11_{[1]}$.

**Lemma 5.14.** We have $\theta_*(\mu) = \mathbb{P}$.

**Proof.** By Lemma 5.13 and the fact that $(\Omega, T)$ is uniquely ergodic, it ends the proof. \hfill \Box

Let $Y := \theta^{-1}(\Omega_0)$. By Lemma 5.13 and Lemma 5.14 we have $\mu(Y) = \theta_*(\mu)(\Omega_0) = P(\Omega_0) = 1$.

Here, we recall some observations in [18, Section 2.2]. Let $Q = \{Q_0 = [0] \cap Y, Q_1 = [1] \cap Y\}$ be the generating partition of $Y$. Set

$$Q^- := \bigvee_{j \geq 1} S^{-j}Q,$$

where $\mathcal{B}(\cdot)$ stands for the Borel $\sigma$-algebra. Since $Q$ is a generating partition, the $\sigma$-algebra $\cap_{m \geq 0} S^{-m}Q^-$ is the Pinsker $\sigma$-algebra of $(Y, \mathcal{B}(Y), \mu, S)$. By [18, Lemma 2.11], $A \subset \cap_{m \geq 0} S^{-m}Q^-$ modulo $\mu$. It follows that almost every atom of the partition corresponding to the Pinsker $\sigma$-algebra of $(Y, \mathcal{B}(Y), \mu, S)$ is contained in an atom of the partition of $Y$ corresponding to $A$. Also, we have $A \subset S^{-m}Q^-$. Fix $m \geq 0$. Let $\pi_m$ be the quotient map from $Y$ to the quotient space $Y/S^{-m}Q^-$. Let $\bar{\mu}_m := (\pi_m)_*(\mu)$. So $S$ acts naturally on the quotient space $Y/S^{-m}Q^-$ as an endomorphism preserving $\bar{\mu}_m$ and $\pi_m \circ S = S \circ \pi_m$. Also, it can define the quotient map $\rho_m : Y/S^{-m}Q^- \rightarrow \Omega$ with $\rho_m \circ S = T \circ \rho_m$. Then $(\rho_m)_*(\bar{\mu}_m) = \mathbb{P}$. Thus it follows that $S^k \circ \varphi \circ \rho_m = \varphi \circ \rho_m \circ S^k$, that is,

$$\varphi \circ \rho_m(y)(m + k) = \varphi \circ \rho_m(S^k y)(m) \text{ for any } k \in \mathbb{Z}.$$ (6)

We identify points in $Y/S^{-m}Q^-$ by the following ways: for $y \in Y$, let $\bar{y}$ be the atom of the partition associated to $S^{-m}Q^-$ which contains $y$, that is, 

$$\bar{y} = \cdots \bar{y}_{i-1} \bar{i} \iff y \in S^{-m-1}Q_{i-1} \cap S^{-m-2}Q_{i-1} \cap \cdots.$$

**Lemma 5.15** (Lemma 2.13 in [18]). For each $m \geq 0$, $r = 0, 1, \ldots, 2m$ and $\bar{y} \in Y/S^{-m}Q^-$, we have

$$\bar{\mu}_m(S^{m-r}Q_{i_{m-1}} S^{m-r-1}Q_{i_{m-2}} \cdots \cap S^{-m-r}Q_{i_{m-2}} \cdots \cap S^{-m}Q_{i_m} \cap S^{-m}Q^-)(\bar{y}) = \bar{\mu}_m(S^{m-r}Q_{i_{m-1}} S^{-m-r}Q^-)(\bar{y} - m \cdots i_{m-r}).$$

for each choice of $i_k \in \{0, 1\}, -m \leq k \leq m$.

Now, we can prove the uniqueness of equilibrium state for $\phi = a_{00}1_{[00]} + a_{01}1_{[01]} + a_11_{[1]}$ in the $\mathcal{B}$-free system with $2 \in \mathcal{B}$.

**Theorem 5.16.** Suppose that $\mathcal{B} = \{b_1, b_2, \cdots\}$ satisfies (7) and $b_1 = 2$. Let $\phi = a_{00}1_{[00]} + a_{01}1_{[01]} + a_11_{[1]}$. Then $\phi$ has unique equilibrium state.
Proof. Let \( \mu \) be an ergodic equilibrium state for \( \phi \). We will show that the conditional measures \( \mu_\omega \) in the disintegration

\[
\mu = \int_\Omega \mu_\omega d\mathbb{P}
\]

of \( \mu \) over \( \mathbb{P} \) given by the mapping \( \theta : X_1 \to \Omega \) are unique \( \mathbb{P} \)-a.e. \( \omega \in \Omega_0 \). This will show the uniqueness of equilibrium state. We define another measure \( \mu^* \) in the following way, which will be showed that \( \mu = \mu^* \). For each \( \omega \in \Omega_0 \), we have \( \varphi(\omega) \in Y \). By Lemma 5.13, \( \varphi(\omega) \) is the largest element in \( \theta^{-1}(\omega) \). Thus for each \( u = u_{-k} \cdots u_k \in \{0,1\}^{2k+1} \leq \varphi(\omega)[-k,k] \), we set

\[
\mu^*_\omega([u]) := \prod_{-k \leq i \leq k} \lambda_{u_i}^{1},
\]

where \( \lambda_0^1 = \frac{2^{2n+1}}{2^{2n+1} + 2^{2n+1} + \cdots} \) and \( \lambda_1^1 = 1 - \lambda_0^1 \). Finally, we set

\[
\mu^* = \int_{\Omega_0} \mu^* d\mathbb{P}.
\]

We will show that \( \mu_\omega = \mu^*_\omega \) for \( \mathbb{P} \)-a.e. \( \omega \in \Omega_0 \), which implies that \( \mu = \mu^* \). We will prove that for any \( m \geq 0 \) and any \( A \in \bigcup_{j=-m}^{m} S^jQ \),

\[
\mu_\omega(A) = \mu^*_\omega(A), \ \mathbb{P} \text{-a.e.} \ \omega \in \Omega.
\] (7)

Recall that

\[
\mu_\omega(A) = \mathbb{E}^\mu(A|\Omega)(\omega).
\] (8)

To get the equation (7), we will step by step make use of the equality

\[
\mathbb{E}^\mu(A|\Omega)(\omega) = \mathbb{E}^\mu(\mathbb{E}^\mu(A|S^-mQ^-)(\bar{y}_m)|\Omega)(\omega)
\] (9)

where \( A \in \bigcup_{j=-m}^{m} S^jQ \), \( m \geq 0 \) and show that

\[
\mathbb{E}^\mu(A|S^-mQ^-)(\bar{y}_m) = \mu^*_\omega(A)
\] (10)

for all \( \bar{y}_m \) having the same \( \rho_m \)-projection \( \omega \).

First, we need some denotations. For \( m \geq 0 \), let

\[
\hat{C}^0_m := \varphi^{-1}(S^{-m}(Q_0 \cap S^{-1}Q_1)) = \{ \omega \in \Omega_0 : \varphi(\omega)(m) = 0, \varphi(\omega)(m+1) = 0 \},
\]

\[
\hat{C}^1_m := \varphi^{-1}(S^{-m}(Q_0 \cap S^{-1}Q_1)) = \{ \omega \in \Omega_0 : \varphi(\omega)(m) = 0, \varphi(\omega)(m+1) = 1 \},
\]

\[
\hat{C}^1_m := \varphi^{-1}(S^{-m}Q_1) = \{ \omega \in \Omega_0 : \varphi(\omega)(m) = 1 \}.
\]

Then \( \Omega_0 = \hat{C}^0_m \cup \hat{C}^1_m \cup \hat{C}^1_m \) and \( Y = \theta^{-1}(\Omega_0) = \theta^{-1}(\hat{C}^0_m) \cup \theta^{-1}(\hat{C}^1_m) \cup \theta^{-1}(\hat{C}^1_m) \). Let \( B^j_m := \rho_m^{-1}(C^j_m) \) for \( j \in \{00,01,1\} \). Then we have

\[
Y/S^{-m}Q^- = B^0_m \cup B^1_m \cup B^1_m.
\]
By the definition of $\rho_m$,

$$
\bar{\mu}_m(B_m^{00}) = \mu(\theta^{-1}(\hat{C}_m^{00})) = \nu(\hat{C}_m^{00}) = \nu_\eta([00]) = 1 - 2d,
$$

and $\bar{\mu}_m(B_m^{01}) = \bar{\mu}_m(B_m^{11}) = \nu_\eta([01]) = \nu_\eta([11]) = d$.

Now, we prove the equality $\overline{7}$ in two cases: $m = 0$ and $m > 0$, which the first case is not necessary but it can be seen as a toy model for the second case.

(1) Toy model: the case of $m = 0$.

We first show that $\theta^{-1}(\hat{C}_0^{00}) \subset Q_0 \cap S^{-1}Q_0$. For any $y \in \theta^{-1}(\hat{C}_0^{00})$, we have $\varphi(\theta(y))(0) = \varphi(\theta(y))(1) = 0$. Since $y \leq \varphi(\theta(y))$, we have $y \in Q_0 \cap S^{-1}Q_0$. Thus for any $\tilde{y} \in B_0^{00}$, $\pi_0^{-1}(\tilde{y}) \subset \pi_0^{-1}(B_0^{00}) = \theta^{-1}(\hat{C}_0^{00}) \subset Q_0 \cap S^{-1}Q_0$, which implies that for $\tilde{y} \in B_0^{00}$,

$$
\bar{\mu}_0(Q_0 \cap S^{-1}Q_0|Q^-)(\tilde{y}) = 1,
$$

$$
\bar{\mu}_0(Q_0 \cap S^{-1}Q_1|Q^-)(\tilde{y}) = 0,
$$

$$
\bar{\mu}_0(Q_1|Q^-)(\tilde{y}) = 0.
$$

Therefore,

$$
\int_{B_0^{00}} a_{00}\bar{\mu}_0(Q_0 \cap S^{-1}Q_0|Q^-)(\tilde{y}) + a_{01}\bar{\mu}_0(Q_0 \cap S^{-1}Q_1|Q^-)(\tilde{y})
\quad + a_1\bar{\mu}_0(Q_1|Q^-)(\tilde{y}) d\bar{\mu}_0(\tilde{y}) = a_{00}(1 - 2d). \tag{11}
$$

Also, we have $\theta^{-1}(\hat{C}_0^{00} \cup \hat{C}_0^{01}) \subset Q_0$. Thus for any $\tilde{y} \in B_0^{00} \cup B_0^{01}$,

$$
(\bar{\mu}_0(Q_0|Q^-)(\tilde{y}), \bar{\mu}_0(Q_1|Q^-)(\tilde{y})) = (1, 0) =: (\lambda_0, \lambda_0), \tag{12}
$$

which implies that $H(\mu(Q|Q^-))(\tilde{y}) = 0$. In particular, for $\tilde{y} \in B_0^{01}$, we have

$$
\bar{\mu}_0(Q_0 \cap S^{-1}Q_j|Q^-)(\tilde{y}) = \bar{\mu}_0(S^{-1}Q_j|Q^-)(\tilde{y}) \quad \text{for} \ j = 0, 1.
$$

We claim that $SB_0^{01} = B_0^{11}$. Indeed, for any $\tilde{y} \in B_0^{01}$, $\varphi(\rho_0(S\tilde{y})) = S\varphi(\rho_0(\tilde{y})) \in S\varphi(\hat{C}_0^{01}) \subset [1]$. Thus $S\tilde{y} \in B_0^{11}$. Conversely, for any $\tilde{y} \in B_0^{01}$, let $y$ with $\pi_0(\tilde{y}) = \tilde{y}$ and $y' = \pi_0(S^{-1}y)$. So $S\tilde{y}' = \tilde{y}$. Then $S\varphi(\rho_0(\tilde{y}')) = \varphi(\theta(y)) \in [1]$. Since $b_1 = 2$, we have $\varphi(\rho_0(\tilde{y}'))(0) = 0$, $\varphi(\rho_0(\tilde{y}'))(1) = 1$. Thus $\tilde{y}' \in B_0^{01}$, which ends the proof of the claim. Therefore,

$$
\int_{B_0^{01}} a_{00}\bar{\mu}_0(Q_0 \cap S^{-1}Q_0|Q^-)(\tilde{y}) + a_{01}\bar{\mu}_0(Q_0 \cap S^{-1}Q_1|Q^-)(\tilde{y})
\quad + a_1\bar{\mu}_0(Q_1|Q^-)(\tilde{y}) d\bar{\mu}_0(\tilde{y}) = \int_{B_0^{00}} a_{00}\bar{\mu}_0(S^{-1}Q_0|Q^-)(\tilde{y}) + a_{01}\bar{\mu}_0(S^{-1}Q_1|Q^-)(\tilde{y}) d\bar{\mu}_0(\tilde{y})
\quad = \int_{B_0^{01}} a_{00}\bar{\mu}_0(Q_0|Q^-)(\tilde{y}) + a_{01}\bar{\mu}_0(Q_1|Q^-)(\tilde{y}) d\bar{\mu}_0(\tilde{y}). \tag{13}
$$
Since $2 \in \mathcal{B}$, one can prove that $\theta^{-1}(\tilde{C}_0^1) \subset S^{-1}Q_0$. Indeed, if $y \in \theta^{-1}(\tilde{C}_0^1)$, we have $\varphi(\theta(y))(0) = 1$. Thus $y(1) \leq \varphi(\theta(y))(1) = 0$ since $2 \in \mathcal{B}$ and $y \leq \varphi(\theta(y))$. It follows that for $\bar{y} \in B_0^1$, we have $\pi_0^{-1}(\bar{y}) \subset S^{-1}Q_0$, that is, $\bar{\mu}(Q_0 \cap S^{-1}Q_1|Q^-)(\bar{y}) = \bar{\mu}_0(\bar{y}) = \bar{\mu}_0(Q_0|Q^-)(\bar{y})$. Therefore,

$$
\int_{B_0^1} a_0 \bar{\mu}_0(Q_0 \cap S^{-1}Q_0|Q^-)(\bar{y}) + a_0 \bar{\mu}_0(Q_0 \cap S^{-1}Q_1|Q^-)(\bar{y})
+ a_1 \bar{\mu}_0(Q_1|Q^-)(\bar{y})d\bar{\mu}_0(\bar{y})
= \int_{B_0^1} a_0 \bar{\mu}_0(Q_0|Q^-)(\bar{y}) + a_1 \bar{\mu}_0(Q_1|Q^-)(\bar{y})d\bar{\mu}_0(\bar{y}).
$$

Sum up with (11), (13) and (14), we have

$$P(X_\eta, \phi) = h_\mu(X_\eta, S) + \int \phi d\mu$$

$$= \int_{Y/Q^-} H_\mu(Q|Q^-)(\bar{y}) + a_0 \bar{\mu}_0(Q_0 \cap S^{-1}Q_0|Q^-)(\bar{y})
+ a_0 \bar{\mu}_0(Q_0 \cap S^{-1}Q_1|Q^-)(\bar{y}) + a_1 \bar{\mu}_0(Q_1|Q^-)(\bar{y})d\bar{\mu}_0(\bar{y})
= a_0(1 - 2d) + \int_{B_0^1} H_\mu(Q|Q^-)(\bar{y}) + a_0 \bar{\mu}_0(Q_0|Q^-)(\bar{y})
+ (a_0 + a_1) \bar{\mu}_0(Q_1|Q^-)(\bar{y})d\bar{\mu}_0(\bar{y})
\leq a_0(1 - 2d) + \bar{\mu}_0(B_0^1) \log(2^{2a_0} + 2^{a_0 + a_1})
= a_0(1 - 2d) + d \log(2^{2a_0} + 2^{a_0 + a_1}).$$

The inequality comes from $\sum p_i(b_i - \log p_i) \leq \log(\sum 2^{b_i})$ for any $\sum p_i = 1$ and any $b_i$. So for $\bar{\mu}_0$-a.e. $\bar{y} \in B_0^1$, we have

$$\bar{\mu}_0(Q_0|Q^-)(\bar{y}), \bar{\mu}_0(Q_1|Q^-)(\bar{y}) = (\lambda_0, \lambda_1).$$

Notice that (12) and (15) do not depend on $\bar{y}$ itself but only on the values $\varphi(\rho_0(\bar{y}))(0)$ and $\varphi(\rho_0(\bar{y}))(1)$, which implies that (10) holds. Sum up with (8), (9) and (10), we conclude that in the disintegration of $\mu$ over $\mathbb{P}$ via $\theta$, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\mu_\omega(Q_j) = \mu^\omega_j(Q_j)$ for $j = 0, 1$.

(2) General case: the case of $m \geq 0$.

Fix $m \geq 0$. As in the case of $m = 0$, we obtain that $\theta^{-1}(\tilde{C}_m^0) \subset S^{-m}(Q_0 \cap S^{-1}Q_0)$, which implies that for $\bar{y} \in B_m^0$,

$$\bar{\mu}_m(S^{-m}(Q_0 \cap S^{-1}Q_0)|S^{-m}Q^-)(\bar{y}) = 1,$$
$$\bar{\mu}_m(S^{-m}(Q_0 \cap S^{-1}Q_1)|S^{-m}Q^-)(\bar{y}) = 0,$$
$$\bar{\mu}_m(S^{-m}Q_1|S^{-m}Q^-)(\bar{y}) = 0.$$

Similar to the case of $m = 0$, we have $\theta^{-1}(\tilde{C}_m^0 \cup \tilde{C}_m^0) \subset S^{-m}Q_0$, which implies that for any $\bar{y} \in B_m^0 \cup B_m^0$,

$$\bar{\mu}_m(S^{-m}Q_0|S^{-m}Q^-)(\bar{y}), \bar{\mu}_m(S^{-m}Q_1|S^{-m}Q^-)(\bar{y}) = (1, 0) = (\lambda_0, \lambda_0),$$

(16)
and $H_\mu (S^{-m}Q|S^{-m}Q^{-})(\bar{y}) = 0$. In particular, for $\bar{y} \in B_{m}^1$, we have
\[
\mu_m(S^{-m}(Q_0 \cap S^{-1}Q_1)|S^{-m}Q^{-})(\bar{y}) = \mu_m(S^{-m}(S^{-1}Q_1)|S^{-m}Q^{-})(\bar{y}) \quad \text{for } j = 0, 1.
\]
And we also obtain that $SB_{m}^0 = B_1^m$ and $\theta^{-1}(C_m^1) \subset S^{-m}(S^{-1}Q_0)$. Therefore, similar to the case of $m = 0$, the computation of
\[
h_\mu (X_{\bar{y}}, S) + \int \phi \circ S^m d\mu
\]
leads to
\[
(\mu_m(S^{-m}Q_0|S^{-m}Q^{-})(\bar{y}), \mu_m(S^{-m}Q_1|S^{-m}Q^{-})(\bar{y})) = (\lambda_1^0, \lambda_1^1). \quad (17)
\]
for $\mu_m$-a.e. $\bar{y} \in B_1^m$. In order to prove that $\mu_\omega = \mu_\omega^*$ for $A \in \bigvee_{r=0}^m S^rQ$, choose $(i_{-m}, \ldots, i_{0}, \ldots, i_{m}) \in \{0, 1\}^{2m+1}$. By the chain conditional probabilities and Lemma 5.15, we have
\[
\bar{\mu}_m(\bigcap_{r=0}^{2m} S^{m-r}Q_{i_{m-r}}|S^{-m}Q^{-})(\bar{y})
\]
\[
= \prod_{r=0}^{2m} \bar{\mu}_m(S^{m-r}Q_{i_{m-r}}|S^{m-r-1}Q_{i_{m-r-1}} \cap \cdots \cap S^{-m}Q_{i_{-m}} \cap S^{-m}Q^{-})(\bar{y})
\]
\[
= \prod_{r=0}^{2m} \bar{\mu}_m(S^{-m}Q_{i_{m-r}}|S^{-m}Q^{-})(\bar{yi_{-m}} \cdots i_{m-r-1}).
\]
By (16) and (17), for $\mu_m$-a.e. $\bar{y} \in Y/S^{-m}Q^{-}$,
\[
\bar{\mu}_m(S^{-m}Q_{i_{m-r}}|S^{-m}Q^{-})(\bar{yi_{-m}} \cdots i_{m-r-1}) = \lambda_{i_{m-r}}^{j_r},
\]
where $j_r = \varphi(\mu_m(\bar{y}_{i_{-m}} \cdots i_{m-r-1}))(m)$. And by equation (9), $j_r = \varphi(\mu_m(\bar{y}))((m+2m-r) \text{ Sum up with } (8), (9) \text{ and (10)}. (7)$ is proved.

It follows that $\mu_\omega = \mu_\omega^*$ for $P$-a.e. $\omega \in \Omega$ and $\mu = \mu^*$, which ends the proof. \hfill \Box

**Remark 5.17.** Similar to Remark 5.7, we do not know whether the uniqueness of equilibrium state holds for $\phi \in \mathcal{C}_n \setminus \mathcal{C}_2$ where $n \geq 3$.

By using Theorem 5.1 we will show that $\nu_\phi \ast B_{p,1-p}$ is not Gibbs measure for some $\phi$. We consider $\phi = a_{00}1_{[00]} + a_{01}1_{[01]} + a_{11}1_{[1]}$ with $a_1 > \max \{a_{00}, a_{01}\}$. It is necessary to sure that the condition
\[
P \leq (\text{Var}\phi([0]) - \log(1 - p))d + d^\phi - \text{Var}\phi([0]) \quad (18)
\]
can be satisfied, where $p = \frac{2^{2a_0}}{2^{a_1} + 2^{a_0}}$. Firstly, we estimate the quantity $d^\phi$. It is showed that for $0 < q < 1$, 

$$\int \phi d\nu \ast B_{q,1-q} = a_{00}(1-2d) + d(2a_{00}q + (a_{01} + a_1)(1-q)).$$

So

$$d^\phi \geq \sup_{0 < q < 1} \int \phi d\nu \ast B_{q,1-q} = a_{00}(1-2d) + d \max\{2a_{00}, a_1 + a_{01}\}. \quad (19)$$

Since

$$P = a_{00}(1-2d) + d \log(2^{a_1 + a_{01}} + 2^{2a_{00}}) = a_{00}(1-2d) + d(a_1 + a_{01}) - d \log(1-q),$$

we can replace the condition (18) by the condition

$$d(a_1 + a_{01}) \leq (d-1) \text{Var}^\phi[0] + d \max\{2a_{00}, a_1 + a_{01}\}. \quad (20)$$

Notice that $2 \in \mathcal{B}$ implies that $0 \leq d < 1/2$. If $2a_{00} \leq a_1 + a_{01}$, then the condition (20) is satisfied when $\text{Var}^\phi[0] = 0$, which means that $a_{00} = a_{01}$ and $2a_{00} \leq a_1 + a_{01}$ is natural.

If $2a_{00} > a_1 + a_{01}$, then we have $a_1 > a_{00} > a_{01}$. So the condition (20) becomes to be

$$(2d-1)(a_{00} - a_1) + d(a_{00} - a_1) \geq 0,$$

which cannot be satisfied when $a_1 > a_{00} > a_{01}.$

So, by the above consideration, we have

**Proposition 5.18.** Suppose that $\mathcal{B} = \{b_1, b_2, \ldots \}$ satisfies (1) and $b_1 = 2$. For $\phi = a_0 \mathbb{1}_{[0]} + a_1 \mathbb{1}_{[1]}$, if $a_1 \geq a_0$, then the equilibrium state $\kappa = \nu_\eta \ast B_{p,1-p}$ is not Gibbs measure for $\phi$, where

$$p = \frac{2^{a_0}}{2^{a_1} + 2^{a_0}}.$$

**Proof.** With the assumptions of $a_0$ and $a_1$, we have

$$\sup \phi([1]) \geq \sup \phi([0]), \quad \text{and} \quad \text{Var}^\phi([1]) = 0 \leq \text{Var}^\phi([0]) - \log(1-p).$$

Since $a_1 \geq a_0$, by inequality (19),

$$d^\phi \geq a_0(1-d) + a_1 d.$$

So we have

$$P(X_\eta, \phi) = a_0(1-d) + d \log(2^{a_1} + 2^{a_0}) \leq d^\phi - d \log(1-p).$$

Since $\nu_\eta$ is full support on $X_\eta$, $\kappa = \nu_\eta \ast B_{p,1-p}$ is full support on $X_\eta$. Therefore, $D_{\nu_\eta} = D$ and $D^\phi = D^\phi$. Then by Theorem 1.1, $\kappa = \nu_\eta \ast B_{p,1-p}$ is not Gibbs measure for $\phi$. \qed
Here, we give an example that the equilibrium state $\nu_\eta \ast B_{p,1-p}$ is not Gibbs measure for more general $\phi = a_{00}1_{[00]} + a_{01}1_{[01]} + a_11_{[1]}$ on $(X_\eta, S)$ with $2 \in \mathcal{B}$, but we cannot use Theorem 1.1 directly.

**Proposition 5.19.** Suppose that $\mathcal{B} = \{b_1, b_2, \cdots \}$ satisfies (1) and $b_1 = 2$. For $\phi = a_{00}1_{[00]} + a_{01}1_{[01]} + a_11_{[1]}$, if $a_1 \geq \max\{a_{00}, a_{01}\}$ and $2a_{00} \leq a_1 + a_{01}$, then the equilibrium state $\kappa = \nu_\eta \ast B_{p,1-p}$ is not Gibbs measure for $\phi$, where

$$p = \frac{2^{2a_{00}}}{2^{a_1 + a_{01}} + 2^{2a_{00}}}.$$

**Proof.** Since $\kappa$ is full support on $X_\eta$, $D_\kappa^\phi = D^\phi$. So for any $n \in \mathbb{N}$, there exists $x^{(n)} \in X_\eta$ such that

$$\sum_{i=0}^{n-1} \phi(S^i x^{(n)}) = \sup_{y \in X_\eta} \sum_{i=0}^{n-1} \phi(S^i y) \geq nD^\phi.$$

Since $\nu_\eta$ is full support on $X_\eta$, $D_{\nu_\eta} = D$. So for any $n \in \mathbb{N}$, there exists $C_n \in \mathcal{L}_n(X_\eta)$ such that $\#_1 C_n = \max_{W \in \mathcal{L}_n(X_\eta)} \#_1 W \geq nD = nd$.

Let $A_n = \#_1 C_n - \#_1 x^{(n)}[0, n-1] \geq 0$.

Let $|\phi| = \sup_{x \in X_\eta} |\phi(x)|$. For any $W \in \mathcal{L}_n(X_\eta)$ and $x \in W$, we have

$$\sum_{i=0}^{n-1} \phi(S^i x) \leq a_1 \#_1 W + a_{01} \#_1 W + a_{00}(n - 2\#_1 W) + 2|\phi|,$$

and

$$\sum_{i=0}^{n-1} \phi(S^i x) \geq a_1 \#_1 W + a_{01} \#_1 W + a_{00}(n - 2\#_1 W) - 2|\phi|.$$

Now fix $y \in C_n$.

$$0 \leq \sum_{i=0}^{n-1} \phi(S^i x^{(n)}) - \sum_{i=0}^{n-1} \phi(S^i y) \leq (a_1 + a_{01}) \#_1 x^{(n)}[0, n-1] + a_{00}(n - 2\#_1 x^{(n)}[0, n-1]) + 2|\phi|$$

$$- (a_1 + a_{01}) \#_1 C_n - a_{00}(n - 2\#_1 C_n) + 2|\phi|$$

$$= - A_n(a_1 + a_{01} - 2a_{00}) + 4|\phi|$$

$$\leq 4|\phi|.$$  

Therefore, if $\kappa$ is Gibbs measure for $\phi$, then there exists $c > 0$ such that

$$c^{-1} \leq \kappa(C_n) \cdot 2^{nP - \sum_{i=0}^{n-1} \phi(S^i y)} \leq \kappa(C_n) \cdot 2^{nP - \sum_{i=0}^{n-1} \phi(S^i x^{(n)})} \cdot 2^{4|\phi|}$$

$$\leq \nu_\eta(C_n) \cdot 2^{\#_1 C_n \log(1-p)} \cdot 2^{nP - nd^\beta + 4|\phi|}$$

$$\leq \nu_\eta(C_n) \cdot 2^{nd \log(1-p) + nP - nd^\beta + 4|\phi|},$$  (21)

$$24$$
noticed that $\kappa(C_n) > 0$. We claim that $P \leq -d \log(1 - p) + d^\phi$. Since $a_1 + a_{01} \geq 2a_{00}$, by inequality (10),

\[d^\phi \geq a_{00}(1 - 2d) + d(a_1 + a_{01}).\]

By Proposition 5.6
\[P = a_{00}(1 - 2d) + d \log(2^{a_1 + a_{01} + 2^2a_{00}})\]
\[= a_{00}(1 - 2d) + d(a_1 + a_{01}) - d \log(1 - p)\]
\[\leq d^\phi - d \log(1 - p).\]

Combined with the inequality (21), we have $\nu_\eta(C_n) \geq c^{-1} \cdot 2^{-4|\phi|}$. By Lemma 4.2 $\nu_\eta$ is atomic, which is a contradiction.

Proof of Theorem 1.2. It immediately follows from Proposition 5.9, Theorem 5.16 and Proposition 5.19.

Remark 5.20. For $\phi \in \mathcal{C}_n \setminus \mathcal{C}_2$ where $n \geq 3$, we do not know whether $\nu_\eta * B_{p, 1-p}$ for some $p$ can be the equilibrium state for such $\phi$.

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