MODULI AND PERIODS OF SIMPLY CONNECTED ENRIQUES SURFACES

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It is well known that the presence of automorphisms and vector fields on a projective variety complicates the nature of the corresponding moduli spaces. The aim of this paper is to prove some general results that quantify this and to use them to describe various moduli spaces, both local and global, of algebraic surfaces. Our approach is naïve: suppose given a Keel–Mori stack, that is, an algebraic stack \( F \) over a noetherian base \( B \) such that the representable ([LMB00], 4.1) diagonal morphism \( F \to F \times_B F \) is finite, so that, by the fundamental theorem of Keel and Mori [KM97], there is a geometric quotient \( F \), which is the coarse moduli space. Then we try to describe \( F \) in elementary quasi-projective terms. We can sometimes take a slightly more refined point of view, which is to construct a Deligne–Mumford stack \( F_{DM} \) that lies between \( F \) and \( F \), so that, in particular, \( F \) is also the geometric quotient of \( F_{DM} \), and then describe \( F_{DM} \) so as to make transparent the passage to \( F \). We cannot, however, prove what ought obviously to be true, namely, that any Keel–Mori stack has a universal Deligne–Mumford quotient, although we can do this for stacks of Enriques surfaces. (It is easy to find examples where there is a universal Deligne–Mumford quotient whose construction is not local, unlike the construction of the geometric quotient.) We are interested mainly in Enriques surfaces (for whose moduli there do exist universal Deligne–Mumford quotients); recall that in the classification of surfaces by Mumford and Bombieri over an algebraically closed field \( k \) of arbitrary characteristic \( p \) these form one of the simplest classes beyond the ruled surfaces. They are defined by the properties that \( 2K \sim 0 \) and \( b_2 = 10 \) and do sometimes, if \( p = 2 \), possess vector fields.

If \( Y \) is a \( G \)-Enriques surface (that is, if \( G = \text{Pic}^+(Y) \)), then \( G \) has order two, so that there is a \( G^\vee \)-torsor \( X \to Y \) such that \( X \) is reduced and irreducible and the dualizing sheaf \( \omega_X \) is trivial. If \( p \neq 2 \), then \( G^\vee \cong \mathbb{Z}/2 \), \( X \) is a smooth K3 surface, \( Y \) is the quotient of \( X \) by a fixed–point–free involution and, in consequence of the Rudakov–Shafarevich theorem on the non-existence of vector fields on a K3 surface, \( Y \) has no vector fields. If, however, \( p = 2 \) (this we regard as the interesting case), then \( G \) is isomorphic to one of \( \mathbb{Z}/2, \mu_2, \) or \( \alpha_2 \), and \( Y \) is the quotient of a K3 surface by a fixed-point-free involution if and only if \( G = \mu_2 \). In the other cases \( G \) is unipotent (i.e., not linearly reductive) and the Cartier dual \( G^\vee \) is not reduced, so that the canonical \( G^\vee \)-torsor \( X \to Y \) is purely inseparable. In this case \( Y \) is simply connected; we shall also say that \( Y \) is unipotent.

**Definition 0.1** An RDP–K3 surface is a surface with rational double points (RDPs) whose minimal resolution is K3. An Enriques surface \( Y \) is a K3–Enriques surface if its canonical double cover is RDP–K3.
The chief global result is, in over-simplified terms, that there is a period morphism for simply connected K3–Enriques surfaces that describes the moduli space as essentially an open piece of a $\mathbb{P}^1$-bundle over the period space. (To be a little more precise, if the canonical double cover $X$ has only RDPs, then its minimal resolution $\tilde{X}$ is a supersingular K3 surface, for which Ogus, Rudakov and Shafarevich have constructed a period map and proved it to be an isomorphism. We can construct a period map for such Enriques surfaces, by taking the periods of $\tilde{X}$.) At the level of geometric points the fact that the period map is a fibration by curves of genus zero arises from the fact that $X$ has free tangent sheaf and every vector field on it is 2-closed; the genus zero curve is the set of lines of vector fields. (This idea is that of Moret-Bailly’s construction of complete $\mathbb{P}^1$’s of abelian surfaces by taking quotients of products of supersingular elliptic curves, although our construction does not give complete rational curves of Enriques surfaces; they degenerate when the line of vector fields on $X$ specializes to vanish at a singular point of $X$.) However, to make this precise, first at the level of stacks and then at the level of coarse moduli, and to show that this fibration really is a $\mathbb{P}^1$-bundle requires more. We prove some general results on stacks in order to deal easily with these issues (these results first appeared in a Mittag-Leffler preprint, [EHS07]). In particular, we can describe the difference between the number of local moduli and the number of global moduli of a variety in terms of the extent to which its automorphism group scheme $G$ fails to be reduced.

**Theorem 0.2** (= 2.8) Suppose that $f : \mathcal{X} \to S$ is proper and flat and is minimally versal at $s \in S$. Then the set of points $t \in S$ such that $\mathcal{X}_t$ is geometrically isomorphic to $\mathcal{X}_s$ forms a smooth subscheme of $S$ of dimension equal to $\dim \text{Lie}(G) - \dim G$.

Recently Liedtke has proved the following four basic results about moduli of Enriques surfaces, especially in characteristic 2. His key idea is to allow the surfaces $X$ to have RDPs and to put a Cossec–Verra polarization on them.

**Theorem 0.3** The stack $\mathcal{E}$ of appropriately polarized Enriques surfaces is algebraic over $\text{Spec } \mathbb{Z}$. All of its geometric components are 10-dimensional.

**Theorem 0.4** If $p \neq 2$, then $\mathcal{E} \otimes \mathbb{F}_p$ is absolutely irreducible.

The remaining results concern the case where $p = 2$.

**Theorem 0.5** An RDP-Enriques surface with a Cossec–Verra polarization can be lifted to $W(k)[\sqrt{2}]$.

**Theorem 0.6** $\mathcal{E} \otimes \mathbb{F}_2$ has just two components, $\mathcal{E}_{\text{uni}}$, the locus where $\text{Pic}^r$ is unipotent, and $\mathcal{E}_{\text{inf}}$, where $\text{Pic}^r$ is infinitesimal. Both are absolutely irreducible of dimension 10. $\mathcal{E}_{\text{uni}}$ is the locus of simply connected surfaces and $\mathcal{E}_{\text{inf}}$ is the closure of the locus of $\mu_2$–surfaces. They intersect in the locus $\mathcal{E}_\alpha$ of $\alpha_2$–surfaces.

As mentioned, these are due to Liedtke. Our contribution is as follows.

**Theorem 0.7** (= 4.6) Every $\alpha_2$-Enriques surface has a miniversal deformation space that is a regular scheme of the form $\text{Spec } W(k)[[x_1, \ldots, x_{12}]]/(FG - 2)$.
Theorem 0.8 (= 5.7) Every Enriques surface has a lifting to a characteristic zero DVR whose absolute ramification index divides $2^9N$, where $N$ is one of the numbers 192, 128, 60, 56, 40, 9.

There is a period map defined on the open substack $\mathcal{E}_{K3,\text{uni}}$ of $\mathcal{E}_{\text{uni}}$ corresponding to K3–Enriques surfaces. (The complement of this open substack has codimension at least 3 everywhere. This can be proved by counting constants for surfaces $Y$ whose canonical double cover $X$ is normal but not RDP-K3; when $X$ is not normal it is possible to make an estimate by finding a certain negative definite configuration $C$ of (-2)-curves on $Y$ and showing that $H^1(Y, T_Y(\log C))$ has high codimension in $H^1(Y, T_Y)$. However, the details involve a messy and unenlightening consideration of different cases, so are omitted.)

The image of the period map lies in a quotient $\mathcal{M}_N/\mathcal{S}_{12}$, where $\mathcal{M}_N$ is the period space for appropriately marked K3 surfaces [Og79].

For a morphism $X \to S$ of schemes or algebraic stacks in characteristic $p$, we denote by $X \to X^{(n)}$ the $n$'th Frobenius relative to $S$.

Theorem 0.9 (= 8.11, 8.12)(1) There is a canonical Deligne–Mumford quotient of the stack $\mathcal{E}_{K3,\text{uni}}$, denoted $(\mathcal{E}_{K3,\text{uni}})_{DM}$.

(2) There is an open piece $M^0_N$, the complement of an explicit divisor $D$ in a period space $M_N$, and a period morphism

$$\psi : (\mathcal{E}_{K3,\text{uni}})_{DM} \to [M^0_N/\mathcal{S}_{12}]^{(1)},$$

where $\mathcal{S}_{12}$ is the symmetric group on 12 letters, that identifies $(\mathcal{E}_{K3,\text{uni}})_{DM}$ with an open piece of a Zariski $\mathbb{P}^1$-bundle over the quotient Deligne–Mumford stack $[M^0_N/\mathcal{S}_{12}]^{(1)}$.

(3) In particular, the geometric quotient $(\mathcal{E}_{K3,\text{uni}})_{\text{geom}}$ is the quotient of an open piece of a $\mathbb{P}^1$-bundle over $M^0_N$ by an action of $\mathcal{S}_{12}$.

It follows from this that, since $\mathcal{S}_{12}$ acts generically freely on $M_N$, the geometric quotient of $\mathcal{E}_{\text{uni}}$ is birationally ruled over $M^0_N/\mathcal{S}_{12}$.

Theorem 0.10 (= 8.13, 8.14) There is a closed substack $\mathcal{E}_\alpha$ of $\mathcal{E}_{\text{uni}}$ whose geometric points correspond to Enriques surfaces with $\text{Pic}^\tau \cong \alpha_2$, that is an irreducible divisor. The intersection $\mathcal{E}_{K3,\alpha} = \mathcal{E}_{K3,\text{uni}} \cap \mathcal{E}_\alpha$ has a canonical Deligne–Mumford quotient $(\mathcal{E}_{K3,\alpha})_{DM}$ that is generically a section of the period map $\psi$ of Theorem 0.9.

Notation: Recall [CD89] that an $\alpha_2$-surface $Y$ has a global vector field $D$, unique modulo scalars. We say that $Y$ is additive (resp., multiplicative) if $D^2 = 0$ (resp., $D^2 \neq 0$).

Theorem 0.11 (= 7.7, 8.16) For every multiplicative $\alpha_2$–surface the canonical double cover is RDP-K3. The period map is defined for them and the period map is an isomorphism on the appropriate stack.

The paper is arranged as follows. In §1 there are various constructions and theorems of a general nature concerning algebraic stacks and particularly
quotients. §2 contains theorems relating miniversal deformation spaces to global moduli, particularly when there are non-integrable vector fields (i.e., when automorphism groupschemes are not reduced). §3 contains straightforward extensions of various well known results about foliations on smooth varieties to the singular case, mostly in order to deal with the singular $K3$ surfaces that arise.

Then we specialize to consider Enriques surfaces. §4 describes the local moduli space of an Enriques surface in a way that refines some of Liedtke’s results. §5 describes how to mark and polarize Enriques surfaces, and bounds the ramification involved in a lift to characteristic zero. §6 recalls the results of Ogus and others on periods of supersingular $K3$ surfaces and adapts them to our context. §7 contains lemmas on automorphism groupschemes. §8 gives the main results on the existence and structure of the period map; they are derived as easy consequences of the general results on stacks and the geometrical results of the other sections. In §9 we examine a particular open substack of $E_{uni}$, consisting of those surfaces where the singular locus of the $K3$ cover is $12 \times A_1$.

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1 Results about stacks

This section is devoted to some general results about algebraic stacks. They are of the kind found in the early chapters of [LMB00]; the main definition that is not found there is that of a relative groupscheme. This construction introduces problems of coherence, which we solve in the context of representable morphisms by using the notion of a “local construction” ([LMB00], ch. 14). These results will be used in the description given subsequently of the various stacks of Enriques surfaces and their double covers. There will be a fixed noetherian base scheme $B$; this will be excellent and have perfect residue fields in the geometric applications. The algebraic stacks and spaces under consideration will be over $B$; that is, they will have morphisms to $(Aff/B)$.

We start with some remarks about fibre products. Suppose that $F' : \mathcal{Y}' \to \mathcal{X}$ and $G' : \mathcal{Z}' \to \mathcal{X}$ are morphisms of stacks. Then the fibre product $\mathcal{Y}' \times_{\mathcal{X}} \mathcal{Z}'$ has no natural morphism to $\mathcal{X}$, even if $F'$ and $G'$ are representable. Rather, there are two obvious such morphisms, namely $F' \circ pr_1$ and $G' \circ pr_2$, and a 2-isomorphism between them. This leads to difficulties in the definition of group objects. However, a representable morphism $F' : \mathcal{Y}' \to \mathcal{X}$ is equivalent to the datum of a “local construction” of a sheaf of algebraic spaces on $\mathcal{X}$, $\mathcal{Y} : \mathcal{X} \to Sp_B$, where $Sp_B$ is the category of algebraic spaces over $B$ ([LMB00], ch. 14). The basic properties of this are summarized in the next two lemmas.

**Lemma 1.1** (1) The representable morphism $F' : \mathcal{Y}' \to \mathcal{X}$ defines a local construction $\mathcal{Y} : \mathcal{X} \to Sp_B$. 

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(2) A local construction $\mathcal{Y} : \mathcal{X} \to Sp_B$ defines a representable morphism $F : \mathcal{Y} \to \mathcal{X}$.

(3) If $\mathcal{Y}$ is obtained from $\mathcal{Y}'$ by composing these constructions, then $\mathcal{Y}$ is 1-isomorphic to $\mathcal{Y}'$.

(4) The category $LC/\mathcal{X}$ of local constructions on $\mathcal{X}$ is isomorphic (not just equivalent) to a full subcategory $Rep_{lc}/\mathcal{X}$ of the category $Rep/\mathcal{X}$ of representable morphisms $\mathcal{Y} \to \mathcal{X}$, and is equivalent to $Rep/\mathcal{X}$.

(5) $LC/\mathcal{X}$ has fibre products.

(6) $Rep_{lc}/\mathcal{X}$ has fibre products, whereas $Rep/\mathcal{X}$ does not, as already remarked.

(7) If $\mathcal{Y} : \mathcal{X} \to Sp_B$ and $\mathcal{Z} : \mathcal{X} \to Sp_B$ are local constructions on $\mathcal{X}$ and $\mathcal{W} = \mathcal{Y} \times \mathcal{Z}$, leading variously to representable morphisms $\mathcal{Y} \to \mathcal{X}$, $\mathcal{Z} \to \mathcal{X}$ and $\mathcal{W} \to \mathcal{X}$ in $Rep_{lc}/\mathcal{X}$, then $\mathcal{W}$ is isomorphic, and not just 1-isomorphic, to $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$.

(8) A morphism $\mathcal{Y} \to \mathcal{Z}$ of local constructions, induces a morphism $\mathcal{Y} \to \mathcal{Z}$ in $Rep_{lc}/\mathcal{X}$ that is strict; that is, the composite $\mathcal{Y} \to \mathcal{Z} \to \mathcal{X}$ is equal to, and not just 2-isomorphic to, the morphism $\mathcal{Y} \to \mathcal{X}$ coming from the local construction.

Lemma 1.2

1) If $\mathcal{G}$ is a local construction on $\mathcal{X}$, then there is an obvious notion of a group structure (namely, a multiplication morphism $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, an inverse morphism $\iota : \mathcal{G} \to \mathcal{G}$ and an identity morphism $e : \mathcal{X} \to \mathcal{G}$, where $\mathcal{X}$ is the trivial local construction on $\mathcal{X}$) on $\mathcal{G}$. We say that $\mathcal{G}$ is a local construction of a relative group scheme on $\mathcal{X}$.

2) If $\mu : \mathcal{G} \times_{\mathcal{X}} \mathcal{G} \to \mathcal{G}$ etc. are the morphisms obtained from $\mu$ etc., then they define a group structure on the representable morphism $\mathcal{G} \to \mathcal{X}$. In particular, the various diagrams defining associativity etc. are commutative, not merely 2-commutative. We say that $\mathcal{G} \to \mathcal{X}$ is a relative groupscheme.

3) If $\mathcal{G}$ is a local construction of a relative group scheme on $\mathcal{X}$, and $\mathcal{Z}$ a local construction on $\mathcal{X}$, then there is an obvious notion of an action of $\mathcal{G}$ on $\mathcal{Z}$.

4) An action of $\mathcal{G}$ on $\mathcal{Z}$ leads to an action of $\mathcal{G}$ on $\mathcal{Z}$ in which the various diagrams are commutative, not merely 2-commutative.

In short, if we restrict to the category $Rep_{lc}/\mathcal{X}$, then difficulties concerning fibre products and coherence in the definitions of group objects and their actions evaporate. So we shall assume, usually tacitly, that representable morphisms to $\mathcal{X}$ are in $Rep_{lc}/\mathcal{X}$. When we construct a representable morphism to $\mathcal{X}$ it will usually be clear that the construction lies in $Rep_{lc}/\mathcal{X}$.

Definition 1.3

Suppose that $\mathcal{G}$ acts on $\mathcal{Z}$ in $Rep_{lc}/\mathcal{X}$. Then $\mathcal{Z} \to \mathcal{X}$ is a pseudo-torsor if the action induces an isomorphism $\mathcal{G} \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Z} \times_{\mathcal{X}} \mathcal{Z}$. It is a torsor if also $\mathcal{Z} \times_{\mathcal{X}} X$ is a torsor under $\mathcal{G} \times_{\mathcal{X}} X$ for all spaces $X \to \mathcal{X}$.

Proposition 1.4

Suppose that $F : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic $B$-stacks. Then there is a relative groupscheme $\phi : \mathcal{G} = \mathcal{G}_F \to \mathcal{X}$, the stabilizer of $F$, where, for $U \in Ob(Aff/B)$, the objects of $(\mathcal{G}_F)_U$ are pairs $(x, x^a \to x)$ with $F(a) = 1_{F(x)}$.
and the morphisms \((x, a) \to (y, b)\) are those morphisms \(x \to y\) with \(bg = ga\) and \(\phi(x, a) = x\).

**Proof:** We exhibit the local construction \(G\). This is defined by taking \(G(x)\) to be the algebraic space that represents the sheaf \(I_{\text{som}}(x, x) = \text{Aut}(x)\) on \((\text{Aff}/U)\), where \(x \in \text{Ob}(\mathcal{U})\). \(\square\)

**Remark:** Without making a local construction, things are a little less tidy. Consider, for example, this approach.

The diagonal \(\Delta_F = \Delta_{X/Y} : \mathcal{X} \to \mathcal{X} \times_Y \mathcal{X}\) is representable. So the morphisms \(pr_i : G' := \mathcal{X} \times_{\Delta_{X/Y}, \mathcal{X} \times_Y \mathcal{X}, \Delta_{X/Y}} \mathcal{X} \rightarrow \mathcal{X}\), for \(i = 1, 2\), are representable. It is easy to prove that \(G'\) is 1-isomorphic to the stabilizer groupscheme \(G_F\) constructed above. Putting this aside, there is a groupoid structure on \(G'\) over \(\mathcal{X}\), and one can turn this into a group structure, in some sense, using the existence of a 2-isomorphism \(pr_1 \Rightarrow pr_2\). However, it is precisely the fact that we have a 2-isomorphism rather than an equality of morphisms that leads to problems of coherence.

Note that, if \(\mathcal{X} = [X/R]\) for an algebraic groupoid \(R = X \times_X X \Rightarrow X\), then \(G \to \mathcal{X}\) pulls back over \(X\) to the stabilizer groupscheme that is the restriction of \(R \to X \times_B X\) to the diagonal in \(X \times_B X\).

**Lemma 1.5** Given \(\mathcal{X} \xrightarrow{F} \mathcal{Y} \xrightarrow{G} \mathcal{Z}\), there is an exact sequence

\[1 \to G_{X/Y} \xrightarrow{A} G_{X/Z} \xrightarrow{B} G_{Y/Z} \times_Y \mathcal{X}\]

of relative groupschemes over \(\mathcal{X}\). The morphisms \(A, B\) are strict.

**Proof:** This is an immediate consequence of the definitions in terms of local constructions. \(\square\)

**Lemma 1.6** Suppose that \(F : \mathcal{X} \to \mathcal{Y}\) is a morphism of stacks, that \(\mathcal{Y}\) is algebraic and that for all spaces \(Y\) and morphisms \(Y \to \mathcal{Y}\), the fibre product \(\mathcal{X} \times_Y Y\) is 1-isomorphic to an algebraic stack. Then \(\mathcal{X}\) is algebraic.

**Proof:** For relative representability of \(\mathcal{X}\), suppose that \(V, W\) are algebraic spaces and \(v : V \rightarrow \mathcal{X}\), \(w : W \rightarrow \mathcal{X}\) are morphisms. There is a lift \(\tilde{w} = (w, 1) : W \to \mathcal{X} \times_Y W\) of \(w\). Then there is a 1-isomorphism

\[V \times_X W \to (V \times_X (\mathcal{X} \times_Y W)) \times_{\mathcal{X} \times_Y W, \tilde{w}} W.\]

Since \(V \times_Y W\) is representable and \(\mathcal{X} \times_Y W\) is an algebraic stack, it follows that the right hand term is representable, which is enough.

To find a smooth presentation of \(\mathcal{X}\), take a smooth presentation \(Y \to \mathcal{Y}\) of \(\mathcal{Y}\). Then \(\mathcal{X} \times_Y Y \to \mathcal{X}\) is smooth and surjective, so that a smooth presentation \(X \to \mathcal{X} \times_Y Y\) will suffice. \(\square\)

The next result generalizes the fact that the classifying stack of a flat groupscheme is algebraic ([LMB00], Proposition (10.13.1)).
Lemma 1.7 Suppose that $\mathcal{G} \to \mathcal{X}$ is a flat relative groupscheme. Denote the corresponding local construction by $\mathcal{G}$. Then there is an algebraic stack $BG$, the classifying stack, with a structural morphism $\pi : BG \to \mathcal{X}$ and a tautological section $s : \mathcal{X} \to BG$, defined by the property that for $U \in Ob(Aff/B)$, the objects of $(BG)_U$ are pairs $(x \in Ob(X_U), P \to U)$, where $P \to U$ is a torsor under the groupscheme $\mathcal{G}(x) \to U$, $\pi(x, P) = x$ and $s(x) = (x, \mathcal{G}(x) \to U)$. Moreover, $\pi : BG \to \mathcal{X}$ is smooth.

**PROOF:** It is clear that $BG$ is a $B$-stack and that for all morphisms $x : X \to \mathcal{X}$ with $X$ an algebraic space, $BG \times_X X \to X$ is 1-isomorphic to $B(\mathcal{G} \times_X X) \to X$. So $BG$ is algebraic, by 1.6.

The smoothness is local on $\mathcal{X}$, so we can take $X$ to be a space. In this case the smoothness is well known, although not to be found explicitly (by us) in [LMB00]. It follows from the statement that if $\mathcal{X} \xrightarrow{F} \mathcal{Y} \xrightarrow{G} \mathcal{Z}$ are 1-morphisms of algebraic stacks such that $GF$ is smooth and surjective and $F$ is flat, then $G$ is smooth. After pulling back by various smooth presentations this reduces to the analogous statement for algebraic spaces, where it is even more well known: to get the smoothness of $BG \to X$, take $X = \mathcal{X} = \mathcal{Z}$ and $Y = BG$, so that $\mathcal{X} \xrightarrow{F} \mathcal{Y}$ is a $G$-torsor, so flat, and $GF$ is the identity.

Various useful constructions involving groupschemes carry over to relative groupschemes in an unsurprising way. For example, it is well known that for a group $G$ over a space $X$, representable morphisms $W \to BG$ correspond to $G$-spaces $Z \to X$; given $Z$, $W$ is $\{Z/G\}$ and, given $W$, $Z$ is $X \times s_{BG} W$.

Lemma 1.8 Assume that $\mathcal{G} \to \mathcal{X}$ is a flat relative groupscheme. Then for any representable $a : Z \to \mathcal{X}$ with a $G$-action, there is a representable morphism $b : W \to BG$ such that for all $X \to \mathcal{X}$, $\mathcal{W} \times_{\pi_{\text{ob}, \mathcal{X}}} X$ is 1-isomorphic to the quotient stack $[Z \times_X X/G \times_X X]$. Conversely, every representable $a : Z \to \mathcal{X}$ with a $G$-action arises in this way, up to 1-isomorphism.

**PROOF:** To construct $W$ it is enough to produce the corresponding local construction $W$ on $BG$. So assume that $Z \in \text{Rep}_{lc}/\mathcal{X}$. Now an object $\tilde{x}$ of $(BG)_U$ is a pair $\tilde{x} = (x \in Ob(X_U), P \to U)$, where $P \to U$ is a torsor under $\mathcal{G}(x) \to U$. We have a $U$-space $\mathcal{Z}(x)$, and we define $W(\tilde{x})$ to be the quotient $U$-space $(\mathcal{Z}(x) \times_U P)/\mathcal{G}(x)$, where $\mathcal{G}(x)$ acts diagonally.

There is a diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{a} & W \\
\downarrow & & \downarrow b \\
\mathcal{X} \xrightarrow{s} BG & \xrightarrow{\pi} & \mathcal{X}
\end{array}
$$

in which the square is clearly commutative, not just 2-commutative. To show that it is 2-Cartesian, it is enough to show that it is Cartesian after base change by every morphism $X \to \mathcal{X}$, where $X$ is a space. But this is just the statement.
of the lemma for classifying stacks over a space, which is, as already stated, well known to be true.

The converse is immediate.

We write \( W = [Z/G] \). This agrees with the usual notation, in the sense that if we take the Cartesian product of the diagram above by a morphism \( X \to \mathcal{X} \) when \( X \) is a space, then \( [Z/G] \times_\mathcal{X} X \) is 1-isomorphic to \([Z/G]\), where \( Z \) is a space 1-isomorphic to \( Z \times_\mathcal{X} X \) and \( G \) is a groupscheme 1-isomorphic to \( G \times_\mathcal{X} X \).

**Proposition 1.9** Suppose that \( G \to \mathcal{X} \) is a flat relative groupscheme and that \( F : \mathcal{W} \to \mathcal{X} \) is a morphism of algebraic stacks. Suppose that \( f : \mathcal{P} \to \mathcal{W} \) is a torsor under \( G \) with \( \mathcal{P} = [\mathcal{P}/G] \). Then there is a morphism \( H : \mathcal{W} \to BG \) under which \( \mathcal{P} \) is 1-isomorphic to \( \mathcal{W} \times_H BG \).

**Proof:** It is clear that there is a 2-factorization of \( f \) through \([\mathcal{P}/G_W]\), and then, from the fact that \( \mathcal{P} \to \mathcal{W} \) is a torsor, that the resulting morphism \([\mathcal{P}/G_W] \to \mathcal{W} \) is a 1-isomorphism. So, by 1.8, there is a morphism \( t : \mathcal{W} \to B(G_W) \) and a 2-Cartesian square

\[
\begin{array}{ccc}
\mathcal{P} & \longrightarrow & \mathcal{W} \\
\downarrow & & \downarrow t \\
\mathcal{W} & \rightarrow & B(G_W)
\end{array}
\]

where \( s_W \) is the tautological section. Now \( B(G_W) \) is 1-isomorphic to \( BG \times_\mathcal{X} \mathcal{W} \), so there is another 2-Cartesian square

\[
\begin{array}{ccc}
\mathcal{W} & \rightarrow & B(G_W) \\
\downarrow s_W & & \downarrow s_X \\
\mathcal{X} & \rightarrow & BG
\end{array}
\]

putting these squares together gives the result.

It is convenient to “stackify” the construction of geometric quotients, at first in the context of group actions.

**Proposition 1.10** Suppose that \( G \to \mathcal{X} \) is a flat relative groupscheme acting properly on the representable morphism \( Z \to \mathcal{X} \). Then there is a 2-commutative diagram

\[
\begin{array}{ccc}
Z & \longrightarrow & [Z/G] \\
\downarrow a & & \downarrow b \\
\mathcal{X} & \rightarrow & BG \quad \pi \\
\end{array}
\]

whose left hand square is 2-Cartesian and whose vertical arrows are representable.

The morphism \((Z/G) \to \mathcal{X}\) has the property that for all spaces \( X \) and flat morphisms \( X \to \mathcal{X} \), the fibre product \((Z/G) \times_\mathcal{X} X\) is 1-isomorphic to the geometric quotient \((Z \times_\mathcal{X} X)/(G \times_\mathcal{X} X)\).
**PROOF:** The fundamental result of [KM97] is that proper actions of flat group-schemes have geometric quotients, and the formation of these quotients commutes with all flat base change. Then for any \( Y \to \mathcal{X} \), \((\mathcal{Z}/\mathcal{G}) \times_{\mathcal{X}} Y\) is constructed by first computing \((\mathcal{Z} \times_{\mathcal{X}} X)/(\mathcal{G} \times_{\mathcal{X}} X)\) for flat \( X \to \mathcal{X} \), pulling this back to \( Y \times_{\mathcal{X}} X\) and then making a flat descent to get an algebraic space over \( Y \). This gives a local construction \( \mathcal{Z}/\mathcal{G} \) on \( \mathcal{X} \), which leads in turn to the existence of \( \mathcal{Z}/\mathcal{G} \).

Of course, if \( \mathcal{G} \) acts freely on \( \mathcal{Z} \), then \([\mathcal{Z}/\mathcal{G}] \to \mathcal{Z}/\mathcal{G}\) is a 1-isomorphism, but not otherwise.

It is natural and useful in this context to introduce gerbes.

**Definition 1.11** An algebraic gerbe is an epimorphism \( F : \mathcal{X} \to \mathcal{Y} \) of algebraic stacks such that the diagonal morphism \( \Delta_{\mathcal{X}/\mathcal{Y}} : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \) is also an epimorphism. A gerbe is neutral if it has a 2-section.

(Cf. [LMB00], Déf. 3.15, p. 22, where “gerbe” is defined when \( \mathcal{Y} \) is a space.)

The meaning of this definition is that for all schemes \( X \) and morphisms \( \mathcal{X} \to \mathcal{Y} \), there is a faithfully flat \( Y' \to Y \) such that \( Y' \) lifts to \( \mathcal{X} \) and for all schemes \( X \) and morphisms \( v : X \to \mathcal{X} \), \( w : \mathcal{X} \to Y \), the morphism \( X \times_{\mathcal{X},w} Y \to X \times_{\mathcal{Y},fv,Y,fw} Y \) is surjective as a map of sheaves on \( X \), where both sides are regarded as \( X \)-schemes via the first projection.

**Lemma 1.12** Suppose that \( Z \to \mathcal{Y} \) is any morphism, and \( \mathcal{X} \to \mathcal{Y} \) is an algebraic gerbe. Then \( \mathcal{X} \times_{\mathcal{Y}} Z \to Z \) is an algebraic gerbe.

**Lemma 1.13** Suppose that \( F : \mathcal{X} \to \mathcal{Y} \) is an algebraic gerbe and that \( \mathcal{G}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X} \) is flat. Then \( F \) is smooth.

**PROOF:** Note first that the hypotheses on \( F \) and \( \mathcal{G}_{\mathcal{X}/\mathcal{Y}} \) are preserved under any base change \( Z \to \mathcal{Y} \).

Pick a flat presentation \( Y \to \mathcal{Y} \). By assumption on \( F \), there is a flat surjective map \( Y' \to Y \) such that \( Y' \to \mathcal{Y} \) lifts to \( \mathcal{X} \). Put \( \mathcal{X}' = \mathcal{X} \times_{\mathcal{Y}} Y' \). Then \( \mathcal{X}' \to Y' \) has a section, say \( s \), so that, by [LMB00], Lemme 3.21, \( \mathcal{X}' \) is naturally identified with \( B(G'/Y') \), where \( G' = \mathcal{G}_{\mathcal{X}/\mathcal{Y}} \times_{\mathcal{X},s} Y' \). Since, by assumption, \( G' \to Y' \) is flat, it follows that \( B(G'/Y') \to Y' \), and so \( \mathcal{X}' \to Y' \), is smooth, by 1.7. Now \( Y' \to \mathcal{Y} \) is a flat presentation, and so \( F \) is smooth.

It is not clear whether a smooth gerbe necessarily has a flat stabilizer. This amounts to whether a groupscheme \( K \to Y \) is necessarily flat if \( BK \) is algebraic and smooth over \( Y \).

The next result extends Lemme 3.21 of [LMB00].

**Proposition 1.14** Suppose that \( F : \mathcal{X} \to \mathcal{Y} \) is a neutral gerbe, with section \( s \). Define \( \mathcal{K} \to \mathcal{Y} \) by \( \mathcal{K} = \mathcal{G}_{F} \times_{\mathcal{X},s} \mathcal{Y} \). Then \( \mathcal{X} \) is 1-isomorphic to \( BK \) over \( \mathcal{Y} \).

**PROOF:** We want to show that \( s : \mathcal{Y} \to \mathcal{X} \) is a torsor over \( \mathcal{X} \) under \( \mathcal{K} \times_{\mathcal{Y},F} \mathcal{X} \to \mathcal{X} \). For this, we need an action morphism \( \sigma : (\mathcal{K} \times_{\mathcal{Y}} \mathcal{X}) \times_{\mathcal{X},s} \mathcal{Y} \to \mathcal{Y} \) that gives an isomorphism \( \Sigma : (\mathcal{K} \times_{\mathcal{Y}} \mathcal{X}) \times_{\mathcal{X},s} \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \). Note that \((\mathcal{K} \times_{\mathcal{Y}} \mathcal{X}) \times_{\mathcal{X},s} \mathcal{Y} \) is 1-isomorphic to \( \mathcal{K} \), and then the only possible choice for \( \sigma \) is, up to 2-isomorphism,
the structure morphism. Now it is shown in [LMB00] that, when \( Y \) is a space, there is a 1-isomorphism \( X \to \mathcal{B}K \) under which \( s : \mathcal{Y} \to X \) is 2-isomorphic to the natural morphism \( \mathcal{Y} \to \mathcal{B}K \), so that \( \Sigma \) is an isomorphism everywhere locally on \( \mathcal{Y} \). Hence \( \Sigma \) is a 1-isomorphism and \( s : \mathcal{Y} \to X \) is a torsor under \( K \times Y \mathcal{X} \). This gives a morphism \( X \to \mathcal{B}K \) over \( \Sigma \). It is shown in [LMB00] that this is an isomorphism everywhere locally on \( \mathcal{Y} \), and so is a 1-isomorphism.

\[ \tag{\text{-}Isom} \]

**Lemma 1.15** Suppose that \( R \to R \) is a smooth (resp. flat, resp. Cohen–Macaulay, resp. ...) algebraic groupoid and that there is a closed \( S \)-flat normal subgroupscheme \( H \to S \) of the stabilizer groupscheme \( R|_{\Delta_S} \to S \). Then there is a quotient \( H' \) that is also a smooth (resp. flat, resp. ...) algebraic groupoid over \( S \).

**Proof:** Assume that \( R \to R \) is smooth; the proof is the same for any of the other local properties. Put \( G = R|_{\Delta_S} \). There is a left action of \( G \to S \) on \( j : R \to S \times S \). Note that this left action makes \( R \) into a pseudo–torsor under \( pr_1^*G \) over \( S \times S \). In particular, these actions are free and, since \( H/S \) is flat, there is a geometric quotient \( \pi : R \to R_1 = H \backslash R \) and a factorization \( j = j_1 \circ \pi \) [Ar74a], Cor. 6.3. Via the identification of the two actions just mentioned, we regard \( R_1 \) indifferently as \( H \backslash R \) or \( pr_1^*H \backslash R \).

For smoothness, regarding \( R_1 \) as \( H \backslash R \) shows that the morphism \( p_1 : R_1 \to S \) induced by \( p \) is smooth. Next, let \( i : R \to R \) and \( c : R \times_{p,S,q} R \to R \) denote the inverse, resp. composition, morphisms. To show that \( i \) descends to \( i_1 : R_1 \to R_1 \), note that for \( g \in H(s) \) and \( f \in R(s,t) \), we have \( g(f) = a(g,f) = f \circ g^{-1} \), so that \( i(g(f)) = g'(i(f)) \), where \( g' = c(f^{-1},g(f)) \). So \( i \) preserves \( H \)-orbits, and so descends to \( i_1 \). As for \( c \), we identify \( c \) with a morphism \( c' : R \times_{p,s,p} R \to R \) via the isomorphism \( (1,i) : R \times_{p,s,q} R \to R \times_{p,s,p} R \). Then \( H \times S \) acts on \( R \times_{p,s,p} R \), and we must check that \( c' \) takes \( H \times S \)-orbits to \( H \)-orbits. For this, note that

\[
\tag{\text{orch}}
\]

where \( g' = h g_2^{-1} g_1 h^{-1} \), and so there is a composition \( c_1 : R_1 \times R_1 \to R_1 \). Finally, define \( q_1 : R_1 \to S \) by \( q_1 = p_1 \circ i_1 \) and \( j_1 = (p_1,q_1) : R \to S \times B \). So \( R_1 \) is a smooth groupoid over \( S \).

**Remark:** One crucial step in the construction given in [KM97] of geometric quotients is the formation of the quotient groupoid, in certain circumstances, of a groupoid by a normal subgroupoid. However, it is not clear that 1.15 is a special case of this, because it is not clear that the subgroupscheme \( H \) arises as the stabilizer of a subgroupoid of \( R \).

The next result restates this lemma in terms of algebraic stacks.
Definition 1.16 Suppose that $F : \mathcal{X} \to \mathcal{Y}$ is an algebraic gerbe and that $G = G_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is flat. Then $F : \mathcal{X} \to \mathcal{Y}$ is a quotient by $G$ or an extension by $G$. 

Remark: For example, if $\mathcal{H} \to \mathcal{Y}$ is a flat relative groupscheme, then $\pi : \mathcal{B}\mathcal{H} \to \mathcal{Y}$ is an extension by $\pi^*\mathcal{H}$. In view of 1.14 and the idea that the classifying stack $\mathcal{B}\mathcal{H} \to \mathcal{Y}$ of a groupscheme $G \to \mathcal{Y}$ is “the quotient of $\mathcal{Y}$ by $\mathcal{H}$”, the definition of “quotient” given above might seem to be made the wrong way round. However, it will turn out to be the right way round for our analysis of moduli.

Proposition 1.17 Suppose that $\mathcal{X}$ is an algebraic stack and that $H \to \mathcal{X}$ is a normal flat closed relative subgroupscheme of the stabilizer groupscheme $G_{\mathcal{X}/\mathcal{B}} \to \mathcal{X}$. Then there is an essentially unique smooth algebraic gerbe $F : \mathcal{X} \to \mathcal{Y}$ that is an extension (or quotient) by $H$.

Proof: Pick a flat presentation $S \to \mathcal{X}$ and put $R = R_S := S \times_{\mathcal{X}} S \rightarrow S$, so that $\mathcal{X} \cong [S/R]$. Then $\mathcal{H} \times_{\mathcal{X}} S$ is a normal flat subschemes of the $S$-group scheme $(R \to S \times_{\mathcal{B}} S)|_{\Delta_S \to B}$. By 1.15 $\mathcal{H}_S \to R$ is a flat groupoid over $S$. Then define $\mathcal{Y} = [S/\mathcal{H}_S \to R]$. It is clear that the obvious morphism $F : \mathcal{X} \to \mathcal{Y}$ is a smooth algebraic gerbe and is a quotient by $H$.

If $T \to \mathcal{X}$ is another flat presentation, then, after replacing $T$ by $T \times_{\mathcal{X}} S$ if necessary, we can suppose that $T \to \mathcal{X}$ factors through a flat morphism $T \to R$. Then $R_T \cong R_S \times_{S \times_{\mathcal{B}} S} T \times_{S} T$, the restriction of $R_S$ to $T$, and $\mathcal{H}_T = \mathcal{H}_S \times_{S} T$. The uniqueness of $\mathcal{X} \to \mathcal{Y}$ is now clear. \hfill $\Box$

This justifies the next definition.

Definition 1.18 If $F : \mathcal{X} \to \mathcal{Y}$ is an algebraic gerbe whose stabilizer $G_F$ is flat over $\mathcal{X}$, then $F : \mathcal{X} \to \mathcal{Y}$ is the quotient of $\mathcal{X}$ by $\mathcal{H}$ or the extension by $G_F$, and we write $\mathcal{Y} = \mathcal{X}/\mathcal{H}$.

Lemma 1.19 If also $\mathcal{X} \to \mathcal{Y}$ is a quotient by a flat representable algebraic relative groupscheme $G \to \mathcal{X}$, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ is a quotient by the pullback of $G$ to $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.

Proposition 1.20 Suppose that $F : \mathcal{X} \to \mathcal{Y}$ is the extension by $G_F$ and that $H : \mathcal{X} \to \mathcal{Z}$ is a morphism such that $G_F$ is 1-isomorphic to a normal closed subgroupscheme of $G_H$. Then there is a morphism $G : \mathcal{Y} \to \mathcal{Z}$ and a 2-isomorphism $G \circ F \Rightarrow H$.

Proof: Note first that since $G_F \to \mathcal{X}$ and $G_H \to \mathcal{X}$ are representable, we can suppose that they are objects of $\text{Rep}_{lc}/\mathcal{X}$, and then that $G_F$ is isomorphic to a normal closed subgroupscheme of $G_H$.

Pick a smooth presentation $Z_0 \to \mathcal{Z}$ and a smooth presentation $X_0 \to \mathcal{X} \times_{\mathcal{Z}} Z_0$. Then $X_0 \to \mathcal{X}$ is a smooth presentation, so that $\mathcal{X}$ is 1-isomorphic to $[X_0/X_1]$ and $\mathcal{Z}$ to $[Z_0/Z_1]$, where $X_1 = X_0 \times_{\mathcal{X}} X_0$ and $Z_1 = Z_0 \times_{\mathcal{Z}} Z_0$. So there
is a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f} & Z_1 \times_{Z_0 \times B Z_0} (X_0 \times_B X_0) \\
& & \downarrow h \\
X_0 \times_B X_0 & \xrightarrow{g} & Z_0 \times_B Z_0
\end{array}
\]

and \( \mathcal{X} \to \mathcal{Z} \) factors through \( \mathcal{X} \to [X_0/Z_1 \times_{Z_0 \times B Z_0} (X_0 \times_B X_0)] \). Also, \( f \) is a pseudo-torsor under \( pr_1^*(G_{\mathcal{X}/B} \times_{\mathcal{X}} X_0) \) and \( g \) is a pseudo-torsor under \( pr_1^*(G_{\mathcal{Z}/B} \times_{\mathcal{Z}} Z_0) \). So \( h \) is a pseudo-torsor under \( pr_1^*((G_{\mathcal{Z}/B} \times_{\mathcal{Z}} \mathcal{X}) \times_{\mathcal{X}} X_0) \).

Now \( G_H \) is the kernel of \( G_{\mathcal{X}/B} \to G_{\mathcal{Z}/B} \times_{\mathcal{X}} \mathcal{X} \), so that \( X_1 \to Z_1 \times_{Z_0 \times B Z_0} (X_0 \times_B X_0) \) factors through the quotient morphism \( X_1 \to X_1/pr_1^*(G_H \times_{\mathcal{X}} X_0) \).

Since \( \mathcal{Y} \) is 1-isomorphic to \( [X_0/(X_1/pr_1^*(G_H \times_{\mathcal{X}} X_0))] \), it follows that \( \mathcal{X} \to [X_0/Z_1 \times_{Z_0 \times B Z_0} (X_0 \times_B X_0)] \) factors through \( \mathcal{X} \to [X_0/(X_1/pr_1^*(G_H \times_{\mathcal{X}} X_0))] \).

Since \( G_F \) is a closed subgroupscheme of \( G_H \), \( \mathcal{X} \to [X_0/(X_1/pr_1^*(G_F \times_{\mathcal{X}} X_0))] \) factors through \( \mathcal{X} \to [X_0/(X_1/pr_1^*(G_F \times_{\mathcal{X}} X_0))] \). However, this last morphism is exactly the extension by \( G_F \), as constructed in terms of algebraic groupoids in 1.17.

\[\square\]

**Proposition 1.21** Suppose that \( F : \mathcal{X} \to \mathcal{Y} \) is an algebraic gerbe. Then one of \( \mathcal{X} \) and \( \mathcal{Y} \) has a geometric quotient if and only if the other does, and when the quotients exist they are naturally isomorphic.

**PROOF:** This follows from the fact that \( \mathcal{X} \) and \( \mathcal{Y} \) have the same associated sheaves of connected components. (Cf. [LMB00], Lemme 3.18.) \[\square\]

**Proposition 1.22** Suppose that \( F : \mathcal{X} \to \mathcal{Y} \) is the quotient of algebraic stacks by a flat relative groupscheme \( G \to \mathcal{X} \). Then every closed substack \( \mathcal{Z} \) of \( \mathcal{X} \) is of the form \( \mathcal{W} \times_\mathcal{X} \mathcal{X} \) for a unique closed substack \( \mathcal{W} \) of \( \mathcal{Y} \) that is identified with the quotient of \( \mathcal{Z} \) by \( G \times_\mathcal{X} \mathcal{Z} \).

**PROOF:** For any \( S \to \mathcal{X} \), put \( G_S = G \times_\mathcal{X} S \). Pick a flat presentation \( f : X_0 \to \mathcal{X} \) and put \( X_1 = X_0 \times_\mathcal{X} X_0 \Rightarrow X_0 \). Then \( \mathcal{X} \) is isomorphic to \( [X_0/X_1] \) and \( \mathcal{Y} \) is isomorphic to \( [X_0/(X_1/p^*G_{X_0})] \).

Now suppose that \( i : \mathcal{Z} \to \mathcal{X} \) is a closed embedding. Then \( Z_0 = X_0 \times_\mathcal{X} \mathcal{Z} \) is a closed algebraic subspace of \( X_0 \). Put \( Z_1 = Z_0 \times_\mathcal{Z} Z_0 \Rightarrow Z_0 \); then \( Z_1 \) is isomorphic to \( X_1 \times_\mathcal{X} \mathcal{Z} \), since \( i \) is representable. Also \( \mathcal{Z} \) is isomorphic to \( [Z_0/Z_1] \).

Put \( \mathcal{W} = [Z_0/(Z_1/s^*G_{Z_0})] \), so that \( \mathcal{W} = \mathcal{Z}/G \times_\mathcal{X} \mathcal{Z} \) and there is a 2-commutative square

\[
\begin{array}{ccc}
\mathcal{Z} & \xrightarrow{h} & \mathcal{W} \\
\downarrow i & & \downarrow j \\
\mathcal{X} & \xrightarrow{F} & \mathcal{Y}
\end{array}
\]

where \( h \) is a quotient by \( G \times_\mathcal{X} \mathcal{Z} \). The questions of whether this square is 2-Cartesian and \( j \) is a closed embedding are local on \( \mathcal{Y} \). Since \( F \) is a flat epimorphism, the composite \( g = F \circ f : X_0 \to \mathcal{Y} \) is a flat presentation. After pulling back
by $g$, we can suppose that $\mathcal{Y}$ is identified with $X_0$ and that $F$ has a section. Then ([LMB00], Lemme 3.21) $F : \mathcal{X} \to X_0$ is identified with $B(G_{X_0}/X_0) \to X_0$. Then $X_1 = G_{X_0}$ and $p,q$ are both the structure morphism, and the groupoid $X_1/p^*G_{X_0}$ is trivial because $(p, q) : X_1 \to X_0 \times X_0$ is a pseudo-torsor under $p^*G_{X_0} \to X_0 \times X_0$. Pulling back under $Z_0 \times Z_0 \to X_0 \times X_0$ shows that $(s, t) : Z_1 \to Z_0 \times Z_0$ is a pseudo-torsor under $s^*G_{Z_0} \to Z_0 \times Z_0$. Then $\mathcal{W}$ is identified with $Z_0$ and $j$ with the inclusion. Finally, the groupoid $Z_1 \to Z_0 \times Z_0$ is identified with the groupoid associated to the groupscheme $G_{Z_0} \to Z_0$.

**Proposition 1.23** With the previous notation and assumptions, suppose also that $\mathcal{H} \to [\mathcal{Z}/\mathcal{G}]$ is a closed normal flat relative subgroupscheme of the relative stabilizer groupscheme $\mathcal{G}|_{\mathcal{Z}/\mathcal{G}} \to \mathcal{X}$. Suppose that $[\mathcal{Z}/\mathcal{G}] \to \mathcal{E}$ is the extension by $\mathcal{H}$. Then $[\mathcal{Z}/\mathcal{G}] \to \mathcal{Z}/\mathcal{G}$ factors uniquely, up to 2-isomorphism, through $\mathcal{E}$.

**PROOF:** This is a special case of 1.20.

We expand upon this in a special case.

**Lemma 1.24** Suppose that $F : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, that $\mathcal{Y}$ is Deligne-Mumford and that whenever $H : \mathcal{X} \to \mathcal{Z}$ is a morphism to a Deligne-Mumford stack there is a morphism $G : \mathcal{Y} \to \mathcal{Z}$ and a 2-isomorphism $GF \Rightarrow H$. Then for any $F' : \mathcal{X} \to \mathcal{Y}'$ with the same property there is a 1-isomorphism $\phi : \mathcal{Y} \to \mathcal{Y}'$ and a 2-isomorphism $\phi \circ F \Rightarrow F'$.

**Definition 1.25** (1) In this case we say that $\mathcal{Y}$ is a Deligne-Mumford quotient of $\mathcal{X}$ and denote $\mathcal{Y}$ by $\mathcal{X}_{DM}$.

(2) A Keel-Mori stack is an algebraic stack whose stabilizer groupscheme is finite.

Recall that Keel-Mori stacks possess geometric quotients [KM97]. It seems plausible that any Keel-Mori stack should possess a Deligne-Mumford quotient, although our attempts to prove this have failed.

**Proposition 1.26** Suppose that $\mathcal{X}$ is an algebraic stack and there is a flat closed subrelative groupscheme $\mathcal{H} \to \mathcal{X}$ of $\mathcal{G}_\mathcal{X}/\mathcal{B} \to \mathcal{X}$ such that $\mathcal{H}$ has connected geometric fibres and $\mathcal{G}_\mathcal{X}/\mathcal{B}/\mathcal{H} \to \mathcal{X}$ is unramified. Then $\mathcal{H}$ is normal in $\mathcal{G}_\mathcal{X}$ and $\mathcal{X}/\mathcal{H}$ is a Deligne–Mumford quotient of $\mathcal{X}$.

**PROOF:** It is well known that if $G \to S$ is a groupscheme and $H$ is a closed flat subgroupscheme of $G$ with connected geometric fibres and unramified quotient $G/H \to S$, then $H$ is normal in $G$. The normality of $\mathcal{H}$ is now (even more) immediate. It is clear from the construction of $\mathcal{X}/\mathcal{H}$ in terms of smooth groupoids that it has unramified stabilizer. Then it is a Deligne–Mumford stack, by Th. 8.1 of [LMB00].

For the universal property, suppose that $\mathcal{X} \to Z$ is a morphism to a Deligne–Mumford stack. Pick an étale presentation $Z \to Z$ and put $T = Z \times_Z Z \rightrightarrows Z$. Pick a smooth presentation $X \to \mathcal{X} \times_Z Z$; in particular, $X \to \mathcal{X}$ is a smooth presentation. Put $S = X \times_X X \rightrightarrows X$ and $T|_X = T \times_{Z \times_Z Z} (X \times_B X)$;
then there is a factorization $S \to T \mid_X \cong X$. Since $T \mid_X \to X \times_B X$ is unramified, $S \to T \mid_X$ factors through $S / H_X$, and we are done.

**Proposition 1.27** With the assumptions and notation of 1.23 suppose also that $X$ is a Deligne–Mumford stack and that $G \to X$ is a finite flat relative groupscheme all of whose geometric fibres are connected. Then the sequence of morphisms $[\mathcal{Z} / \mathcal{G}] \to \mathcal{E} \to \mathcal{Z} / \mathcal{G}$ identifies $\mathcal{Z} / \mathcal{G}$ with the Deligne-Mumford quotient of each of the stacks $[\mathcal{Z} / \mathcal{G}]$ and $\mathcal{E}$.

**PROOF:** This is certainly true after pulling back by a flat presentation of $X$. Since the Deligne-Mumford quotient is unique up to $1$-isomorphism, when it exists, the result follows.

1.10 has a straightforward generalization; the proof is similar, so omitted.

**Proposition 1.28** Suppose that $F : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of algebraic stacks and that the relative stabilizer groupscheme $G_F \to \mathcal{X}$ is finite. Then $F$ factors as $\mathcal{X} \to \mathcal{Y} \to \mathcal{X}$, where $\mathcal{Y}$ is representable and for all spaces $Y$ and flat morphism $Y \to \mathcal{Y}$, the morphism $\pi_Y : \mathcal{X} \times_Y Y \to \mathcal{X} \times Y$ is a geometric quotient.

**Lemma 1.29** Suppose that $a : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks and that $\mathcal{X}$ is Deligne–Mumford. Assume that for all algebraically closed fields $k$ and points $x, x' : \text{Spec} \, k \to \mathcal{X}$, the morphism $a_{x,x'} : \text{Spec} \, k \times_{\mathcal{X},x'} \text{Spec} \, k \to \text{Spec} \, k \times_{\mathcal{Y},ax',Y} \text{Spec} \, k$ is an injection. Then $a$ is representable.

**PROOF:** We can suppose that $\mathcal{Y}$ is representable, and the nit is enough too show that for all $U \in \text{Ob}(A_{/f/B})$ and for all $\xi \in \text{Ob}(\mathcal{X}_U)$, the sheaf $\mathcal{A} = \text{Isom}(\xi, \xi)$ on $U$ is trivial. Note that this sheaf is represented by a groupscheme over $U$, which, since $\mathcal{X}$ is Deligne-Mumford, is finite and unramified.

Pick an arbitrary geometric point $z : \text{Spec} \, k \to U$. Now $z$ is $\xi : U \to \mathcal{X}$; put $\xi \circ z = x$. Then $\text{Isom}(\xi, \xi) \times_{U,x} \text{Spec} \, k$ is identified with $\text{Spec} \, k \times_{\mathcal{X},x} \text{Spec} \, k$, which, by the assumption of injectivity, has only one $k$-point. Hence $\text{Isom}(\xi, \xi)$ is trivial.

**Proposition 1.30** Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are separated algebraic stacks and that $a : \mathcal{X} \to \mathcal{Y}$ is a proper morphism with finite fibres. Suppose that $\mathcal{X}$ is Deligne–Mumford, that $\mathcal{Y}$ is normal and that both are irreducible. Assume also that $a$ is a 1-isomorphism over a non–empty open substack of $\mathcal{Y}$ and that for all geometric points $x, x' : \text{Spec} \, k \to \mathcal{X}$, the morphism $a_{x,x'} : \text{Spec} \, k \times_{\mathcal{X},x'} \text{Spec} \, k \to \text{Spec} \, k \times_{\mathcal{Y},ax'} \text{Spec} \, k$ is an injection. Then $a$ is a 1-isomorphism.

**PROOF:** By 1.29 $a$ is representable. The statement that $a$ is a 1-isomorphism is local on $\mathcal{Y}$, so we are reduced to proving that a proper finite morphism of
separated normal irreducible algebraic spaces that is generically an isomorphism is an isomorphism. This is well known, and trivial.

2 Local and global moduli: general results.

It is well known that the presence of vector fields complicates questions about the existence and structure of moduli spaces, and even their definitions. Our aim in this section is to clarify this; more precisely, we prove results concerning the local nature of the morphism from a local moduli space to a global coarse moduli space.

First, recall some basic results of Artin [Ar74a] and Keel and Mori [KM97]. Fix a Noetherian base scheme $B$ with perfect residue fields. We shall consider proper families $f : \mathcal{X} \to S = \text{Spec} R$, where $R = \mathcal{O}_S$ is an $\mathcal{O}_B$-algebra of finite type, possibly with additional structures, such as a polarization, or cone of polarizations, or the datum of a line in a space of derivations. We shall not be explicit about these additional structures, although there will always be enough to make all Isom and Aut functors representable by schemes of finite type. In other words, the stack $\mathcal{F}$ under consideration will be algebraic, in Artin’s sense [Ar74a]. Fix a point $s \in S$, mapping to $b \in B$, say. We shall abuse notation by choosing a coefficient ring for $k(s)$ and denoting it by $W(k(s))$. Of course, if $k(s)$ is perfect (for example, if $k(s)/k(b)$ is algebraic), then this is the ring of Witt vectors and the choices of rings and homomorphisms evaporate. We shall abuse language by using the phrase “$s$ is a closed point of $S$” to mean “there is a morphism $\text{Spec} k(s) \to \text{Spec} k \to S$”, where $\text{Spec} k \to S$ is a closed embedding and $k(s)/k$ is algebraic, and we shall make the running assumption that $k(s)/k(b)$ is algebraic whenever $s$ is closed.

According to Artin’s definition, $f$ is versal, resp. formally versal, at $s$ if, for any Henselian, resp. Artin, local $\mathcal{O}_B$-algebra $A$ with residue field $k(s)$ and any deformation $Y \to \text{Spec} A$ of $\mathcal{X}_s$, there is an $\mathcal{O}_B$-homomorphism $g : R \to A$ and an $A$-isomorphism $Y \to \mathcal{X} \otimes_R A$. Artin proved that in these circumstances, the formal versality of $f$ at $s$ implies its versality at $s$ and that versality is a Zariski open condition on $S$.

Let $\hat{\mathcal{O}}$, resp. $\hat{R}$, denote the $b$-adic completion of $\mathcal{O}_{B,b}$, resp. the $s$-adic completion of $R$. There are compatible homomorphisms $W(k(b)) \to \hat{\mathcal{O}}$ and $W(k(s)) \to \hat{R}$, which are natural if $k(s)$ is perfect. Put $\Lambda = \hat{\mathcal{O}} \otimes_{W(k(b))} W(k(s))$, so that $W(k(s)) \to \hat{R}$ factors through $\Lambda$. Note that $\Lambda$ is a complete local ring with residue field $k(s)$. Let $\mathcal{C}_\Lambda$ denote the category of Artin local $\Lambda$-algebras with residue field $k(s)$ and $F$ the set-valued deformation functor of $\mathcal{X}_s$ defined on $\mathcal{C}_\Lambda$. There is a hull for $F$, say $D_{\mathcal{X}_s}$, and a classifying $\Lambda$-algebra homomorphism $\phi : D_{\mathcal{X}_s} \to \hat{R}$ that is unique on tangent spaces.

**Lemma 2.1** Suppose that $k(s)$ is algebraic over $k(b)$. Then $f$ is formally versal.
at \( s \) if and only if \( \phi \) is formally smooth.

**PROOF:** Assume that \( f \) is formally versal at \( s \). Suppose that \( A \in \mathcal{C}_\Lambda \) and that \( \xi \in F(A) \). Then there is an \( \tilde{O} \)-homomorphism \( \pi : \tilde{R} \to A \) such that \( \mathcal{X} \) induces \( \xi \). Let \( i : \Lambda \to \tilde{R} \) and \( j : \Lambda \to A \) be the structural maps. It is enough to know that \( \pi \circ i = j \), that is, that \( \pi \) is a \( A \)-homomorphism. For this, argue by induction: pick a small principal surjection \( \sigma : A \to A_1 \) with kernel \( I \), put \( \sigma \circ \pi = \pi_1 \) and assume that \( \pi_1 \circ i = \sigma \circ j \). Then \( \pi \circ i - j \in \text{Der}_O(\Lambda, I) \), which vanishes since \( k(s)/k(b) \) is algebraic and separable.

The converse is immediate. \( \Box \)

**Definition 2.2** With the notation as above, a family is miniversal at \( s \) if \( k(s) \) is algebraic over \( k(b) \) and a classifying map \( \phi : D_{\mathcal{X}} \to \tilde{R} \) is an isomorphism. By a small deformation of a proper variety \( X \) we mean a proper family \( f : \mathcal{X} \to (S, s) \) such that \( S \) is local and \( \mathcal{X}_s \cong X \).

The main result of [KM97] is that the geometric quotient \( F \) of the stack \( \mathcal{F} \) exists, as an algebraic space of finite type over \( B \), if all \( \text{Isom} \) schemes are finite. Their idea is first to construct the quotient locally in the étale topology, and then glue. The local part of the argument goes as follows. Start with \( f : \mathcal{X} \to S \) that is everywhere versal and everywhere surjective and consider the groupoid (that is, the groupoid object in the category of schemes or algebraic spaces) \( R = \text{Isom}_{S \times_B S}(pr_1^*\mathcal{X}, pr_2^*\mathcal{X}) \), with source and target maps \( p, q : R \to S \), composition \( c : R \times_{pr_1, pr_2} R \to R \) and identity map \( e : S \to R \). In particular, \( e \) is a section of both \( p \) and \( q \). (We fix this notation.) The key tautologies (due to Artin [Ar74a]) are that \( R \) represents the set–valued functor \( S \times_{f, f, f} S \) and that the versality of \( f \) implies that both projections \( S \times_{f, f, f} S \to S \), and so the maps \( p, q : R \to S \), are smooth. Assume that \( j = (p, q) : R \to S \times_B S \) is finite. This is automatic if, for example, \( \mathcal{F} \) is a stack of polarized non-ruled surfaces with only RDPs, by the theorem of Matsusaka and Mumford. For \( t \in S \), put \( P_t = p^{-1}(t) \), with projection \( q_t : P_t \to S \). Define the orbit \( O(t) \) to be the image of \( q_t \). The essential local steps of [KM97] are to slice \( S \) by a subscheme \( W \) (so that, in particular, \( W \) is regularly embedded in \( S \)) transverse to a given orbit in such a way that the induced groupoid \( R|_W \) is quasi-finite and flat (in fact, locally complete intersection) over \( W \) and then to show that there is a connected component \( P \) of \( R|_W \) such that \( P \) is a subgroupoid that contains the inverse image of the diagonal and is finite over \( W \). They then construct the quotient \( W/P \) directly and check that it is, locally, the quotient \( S/R \). (They work in an apparently more general context, where \( p, q \) are only assumed to be flat, and then slice through a Cohen-Macaulay point, for example, a generic point of each orbit. When \( p, q \) are Cohen-Macaulay morphisms, the slice can be taken through any point. However, Artin showed [Ar74a] that any flat groupoid is equivalent to a smooth one.)

In terms of an arbitrary algebraic stack \( \mathcal{F}/B \) with a smooth presentation \( S \to \mathcal{F} \), the finiteness of \( R \to S \times_B S \) is equivalent to the finiteness of the
representable diagonal morphism \( F \to F \times_B F \). However, we have been unable to state all of our results in this generality, as explained below.

**Lemma 2.3** Let \( S \) be a scheme, \( G \) a flat \( S \)-group scheme and \( P \to S \) a pseudo-torsor under \( G \). Then a geometric quotient \( P/G \) exists as an algebraic space and the morphism \( P \to S \) factors through a monomorphism \( P/G \to S \).

**PROOF:** The existence of the quotient follows from [Ar74a] Cor. 6.3, or from the fundamental theorem of Keel and Mori [KM97]. It is then enough to show that for each geometric point \( \bar{s} \to S \) the fibre of \( P/G \) over \( \bar{s} \) is either empty or isomorphic to \( \bar{s} \). So we are reduced to the case where \( S = \bar{s} \), where the result is obvious. \( \square \)

**Lemma 2.4** Suppose that \( I, S \) are algebraic spaces of finite type over a Noetherian base, that \( f : I \to S \) is a monomorphism and that \( I \) is not empty. Then there is an open subspace \( U \) of \( S \) such that \( f^{-1}(U) \) is not empty and \( f^{-1}(U) \to U \) is a closed embedding.

**PROOF:** Since \( f \) is quasi-finite, there is, by Zariski’s Main Theorem, an open embedding \( j : I \to J \) and a proper morphism \( g : J \to S \) with \( f = g \circ j \). After replacing \( J \) by the closure of \( j(I) \), it is necessary, we can assume that there is a closed subspace \( Z \) of \( J \), not containing any irreducible component of \( J \), such that \( j \) induces an isomorphism \( I \to J - Z \). Since \( \dim I = \dim J \) and \( \dim Z < \dim J \), if we define \( S_0 = S - g(Z) \), \( J_0 = J - g^{-1}(g(Z)) \) and \( I_0 = I - f^{-1}(Z) \), we see that \( I_0 \) is not empty, that \( j \) induces an isomorphism \( I_0 \to J_0 \) and that \( J_0 \to S_0 \) is proper. So \( U = S_0 \) will do. \( \square \)

We shall use 2.3 when \( \mathcal{X} \to S \) is proper and everywhere versal, \( t \) is a closed point in \( S \), \( P = P_t = \text{Isom}_S(\mathcal{X}_t \times S, \mathcal{X}) \) and \( G = G_t \times S \), where \( G_t = \text{Aut}_\mathcal{X}_t \). The identity of any group scheme will be denoted by 1.

We shall use 2.4 to show that the induced morphism \( \pi_t : I_t \to S \) is an isomorphism to a subscheme, so that \( I_t \) is identified with the largest subscheme of \( S \) over which all geometric fibres are geometrically isomorphic to \( \mathcal{X}_t \), and this subscheme is identified with \( O(t) \).

Note that, via \( q_t \), \( P_t \) is a pseudo-torsor under \( G_t \times S \). By 2.3 the quotient \( I_t = P_t/G_t \times S \) exists and the natural morphism \( \pi_t : I_t \to S \) is a monomorphism.

Now fix a closed point \( s \) of \( S \) and assume that \( f : \mathcal{X} \to S \) is everywhere versal, but not necessarily surjective.

**Proposition 2.5** \( \pi_s \) is an isomorphism to a subscheme of \( S \).

**PROOF:** By 2.4 there is a closed point \( u \) of \( I_s \) such that \( \pi_s(u) = v \), say, has a neighbourhood over which \( \pi_s \) is an isomorphism to a closed subscheme. Then \( X_v \cong X_s \), so that \( P_v \) is identified with \( P_s \) and \( I_v \) with \( I_s \). Moreover, comparison of the germs \( S_v = (s, \bar{s}) \) and \( S_v = (s, \bar{v}) \) with hulls for \( X_s \) and \( X_v \) shows that the induced families over \( S_v \) and \( S_v \) are smoothly equivalent. Since the question
of whether $\pi_s$ is an isomorphism to a subscheme is unaffected by smooth base change and is local on $S$, we are done. \hfill \qed

Remark: We do not know whether $I_t \to S$ is an isomorphism to a subscheme for every smooth groupoid. This is one reason why we do not try to state all our results in terms of general algebraic stacks. Another reason, which is related, is that we do not have a natural definition of what it means for a smooth chart of an algebraic stack to be miniversal at a point.

Henceforth we identify $I_s$ with its image $O(s) = q_s(I_s)$.

Assume, as above, that $f : \mathcal{X} \to S$ is a proper and flat family, maybe with further structure, and assume also that $f$ is versal everywhere.

Definition 2.6 The excess of $f$ at $s$, denoted $\text{exc}_s(f)$, is the difference between $\dim_s S$ and the dimension of a hull for $\mathcal{X}_s$.

For any group scheme, $e$ will denote the identity.

Theorem 2.7 Assume that $S$ is connected.

(1) The function $t \mapsto \dim T_e(G_t) + \text{exc}_t(f)$ is constant on the closed points of $S$.

(2) $P_t$ is everywhere smooth of dimension equal to $\dim T_e(G_t) + \text{exc}_t(f)$.

(3) $I_t$ is everywhere smooth of dimension equal to $\dim T_e(G_t) + \text{exc}_t(f) - \dim G_t$.

(4) If $f$ is miniversal at $s$, then $P_t$ is everywhere smooth of dimension equal to $\dim T_e(G_s)$ and $I_t$ is everywhere smooth of dimension equal to $\dim T_e(G_s) - \dim G_t$.

(5) Suppose that $f$ is miniversal at $s$. Then $f$ is miniversal everywhere if and only if the function $t \mapsto \dim T_e(G_t)$ is constant on $S$.

Proof: As already remarked, the versality of $f$ implies the smoothness of $p,q$. So $P_t$ is smooth, and then $I_t$ is too.

Pick a closed point $v \in P_t$ and put $q_t(v) = w$. Then $\mathcal{X}_t \cong \mathcal{X}_w$.

Assume first that $f$ is miniversal at $w$. Then the derivative of $q_t$ vanishes at $v$ and the derivative of $\pi_t$ is injective. Hence the tangent space $T_v(P_t)$ is isomorphic to $T_e(G_t)$, so that $P_t$ is smooth and $\dim_v(P_t) = \dim T_e(G_t)$.

In general, the morphisms $P_t \to I_t \to S$ are obtained from the corresponding morphisms where $S$ is replaced by a miniversal deformation space $S_1$ of $\mathcal{X}_w$ by a smooth base change $S \to S_1$ of fibre dimension $\text{exc}_w(f)$. Hence $\dim_v P_t = \dim T_e(G_t) + \text{exc}_w(f)$.

Now take $v = e(t)$. Then $w = t$, so that $\dim_{e(t)} P_t = \dim T_e(G_t) + \text{exc}_t(f)$.

Since $S$ is connected, the fibre dimension of the smooth morphism $p$ is constant along any section. So $t \mapsto \dim T_e(G_t) + \text{exc}_t(f)$ is constant. Also $\dim_v P_t = \dim T_e(G_w) + \text{exc}_w(f)$, since $\mathcal{X}_t \cong \mathcal{X}_w$, and so $P_t$ is of constant dimension.
For (4) note that \( \dim P_t = \dim T^e(G_t) + \text{exc}_t(f) = \dim T^e(G_s) + \text{exc}_s(f) = \dim T^e(G_s) \), and similarly for \( \dim I_t \). Finally, (5) is an immediate consequence of (1).

**Remark:** 2.7 (5) is reminiscent of Schlessinger’s result [Sch68], that the hull represents the infinitesimal deformation functor if and only if \( \text{Aut}^0_{X/S} \) is smooth over \( S \). However, the comparison is not exact; it is true that, for example, the flatness of \( \text{Aut}^0_{X/S} \) over \( S \) implies the miniversality of \( f \) everywhere, but it is not clear that the converse is true.

**Corollary 2.8** For every closed point \( s \in S \) mapping to \( b \in B \), the set of closed points \( t \) such that \( \mathcal{X}_t \) is geometrically isomorphic to \( \mathcal{X}_s \) is the support of a smooth subscheme of \( S \otimes k(b) \). If also \( f \) is miniversal at \( s \), then the dimension of this subscheme is \( \dim \text{Lie}(G) - \dim G \), where \( G \) is the automorphism group scheme of \( \mathcal{X}_s \).

**PROOF:** Immediate from 2.7.

**Corollary 2.9** Suppose that \( \mathcal{F} \) is as above and for every \( T \)-point \( Y \to T \) of \( \mathcal{F} \) the automorphism group scheme \( \text{Aut}_{Y/T} \) is finite. Then the fibres of the classifying morphism \( S \to \mathcal{F} \), the geometric quotient of \( \mathcal{F} \), are set-theoretically smooth. If \( f \) is miniversal at \( s \) and \( S \) is connected, then the fibres of \( S \to \mathcal{F} \) are all of constant dimension \( \dim T^e(\text{Aut}_{X_s}) \).

**PROOF:** The fibres of \( S \to \mathcal{F} \) are, as sets, the incidence subschemes \( I_t \), and we are done by 2.7.

We close this section with a discussion of the local structure of a Keel-Mori stack and the morphism to its geometric quotient in the context of the moduli of projective varieties. There is a well known theorem to the effect that if \( X \) is a projective variety with aa finite reduced groupscheme \( \text{Aut}_X \) of automorphisms, then \( \text{Aut}_X \) acts on the miniversal deformation space \( \text{Def}_X \) of \( X \) and the quotient \( \text{Def}_X / \text{Aut}_X \) is the formal germ of the coarse moduli space at the point corresponding to \( X \). In this section we give a proof of this in the algebraic (i.e., henselian) context; we could not locate one in the literature. (In particular, any apparent similarity between this and Théoréme 6.1 of [LMB00] is only apparent; the group \( G \) of loc. cit. is not intrinsically related to \( X \).) We also discuss how to extend this to cover what happens when \( \text{Aut}_X \) is more general.

Assume that we have a stack \( \mathcal{F} \) of projective varieties such that for all spaces \( S \) and \( T \) both projections of the fibre product \( S \times_{\mathcal{F}} \mathcal{T} \) to \( S \) and \( T \) are finite. That is, \( \text{Isom} \) schemes are finite. We want a description, local in the étale topology on the geometric quotient \( F \) of \( \mathcal{F} \), of \( S \to F \), where \( f : X \to S \) is everywhere versal, which is more informative than the tautological statement that it is the quotient by a smooth groupoid. However, we can only do this under the following hypothesis:
(*) there is a closed flat normal subgroup scheme $H/S$ of $\text{Aut}_{X/S}$ such that $\text{Aut}_{X/S}/H$ is unramified over $S$.

Equivalently, there is a closed flat normal subrelative groupscheme $\mathcal{H}$ of the stabilizer relative groupscheme $\mathcal{G} = \mathcal{G}_{X/B}$ such that $\mathcal{G}/\mathcal{H} \to \mathcal{F}$ is unramified.

The next definition helps in the comparison of different groupoids.

Suppose that $R \overset{p,q}{\implies} S$ and $\tilde{R} \overset{\tilde{p},\tilde{q}}{\implies} \tilde{S}$ are groupoids, with compositions $c, \tilde{c}$ respectively, and that $a : \tilde{S} \to S$ is a morphism. Then a morphism $b : \tilde{R} \to R$ is an equivariant morphism of groupoids over $a : \tilde{S} \to S$ if the squares

\[
\begin{array}{ccc}
R & \xrightarrow{p} & S \\
\downarrow{b} & & \downarrow{a} \\
\tilde{R} & \xrightarrow{\tilde{p}} & \tilde{S}
\end{array}
\]

and

\[
\begin{array}{ccc}
R & \xrightarrow{q} & S \\
\downarrow{b} & & \downarrow{a} \\
\tilde{R} & \xrightarrow{\tilde{q}} & \tilde{S}
\end{array}
\]

are Cartesian and there is a commutative diagram

\[
\begin{array}{c}
\tilde{R} \times_{\tilde{p},\tilde{S},\tilde{q}} \tilde{R} \longrightarrow (R \times_{p,S,q} R) \times_{p\circ pr_S, \tilde{S}} \tilde{S} \\
\downarrow{m} & & \downarrow{m \times 1_{\tilde{S}}} \\
\tilde{R} & \longrightarrow R \times_{p,S} \tilde{S}
\end{array}
\]

where the horizontal maps are the isomorphisms given by the preceding Cartesian squares.

**Lemma 2.10** Suppose that $b : \tilde{R} \to R$ is an equivariant morphism of groupoids over $a : \tilde{S} \to S$.

1. If $j = (p,q) : R \to S \times S$ is finite, then so is $\tilde{j} : \tilde{R} \to \tilde{S} \times \tilde{S}$.
2. If $P$ is any of the properties flat, smooth, finite, étale, locally complete intersection (l.c.i.) and $R$ has property $P$, then so does $\tilde{R}$.
3. Suppose that $W \to S$ is a morphism and that $\tilde{W} = W \times_S \tilde{S}$. Then the induced groupoid

**Proof:** The first two parts are trivial. For the third, the groupoid $R|_W$ is
defined by the diagram

\[
\begin{array}{ccc}
R|_W & \xrightarrow{pw} & R \times_{p,S} W \\
qw \downarrow & & \downarrow \\
W \times_{S,q} R & \xrightarrow{p} & S
\end{array}
\]

where all squares are Cartesian. Then the analogous diagram defining \( \tilde{R}|_{\tilde{W}} \) lies over this in an obvious way, so that the resulting array (consisting of three cubes) is commutative and every square is Cartesian. \( \square \)

Fix a closed point \( s \in S \) and assume that \( f \) is miniversal at \( s \).

**Theorem 2.11** Assume hypothesis \((*)\). Then \( S \to F \) is, locally on \( F \), isomorphic to the quotient of a smooth morphism \( \tilde{W} \to W \), where \( W \hookrightarrow S \) is a slice, by an equivariant action of the unique finite étale group scheme \( \tilde{\Gamma} \) over \( B \) that extends \( G := (\text{Aut}_X/S/H)_s \).

**PROOF:** First, localize \( B \) so that it is a local scheme and \( k(s) = k(b) \). Put \( \text{Aut}_X/S = G \).

Now take an étale local slice \( W \) in \( S \) through \( s \), as in [KM97]. Put \( q_1^{-1}(W) = \tilde{W} \subset \tilde{S} \); this is a slice in \( \tilde{S} \) through any point lying over \( s \). According to [KM97], the natural map \( W/R_1|_W \to S/R_1 \) is étale, so the same is true of \( \tilde{W}/\tilde{R}_1|_W \to \tilde{S}/\tilde{R}_1 \). So it remains to describe the morphism \( \tilde{W}/\tilde{R}_1|_W \to W/R_1 \).

Since \( R_1 \to S \times S \) is unramified, the morphisms \( p_{1,W}, q_{1,W} \) are étale. So \( p_{1,W}^{-1}(s) = \coprod \text{Spec } k_i \), with \( k_i/k(s) \) finite and separable. Since \( W \) is local, it follows that \( p_{1,W}^{-1}(s) = j_{1,W}^{-1}(s,s) \subset j_{1,W}^{-1}(\Delta_W) \). There is [KM97] a decomposition \( R_1|_W = P \coprod Q \), where \( P \) is a subgroupoid finite over \( W \) and \( P \supset j_{1,W}^{-1}(\Delta_W) \). Then \( p_{1,W}^{-1}(s) \subset P \), and so, since \( W \) is local, \( R_1|_W = P \). That is, \( R_1|_W \) is a finite étale groupoid over \( W \). Note also that \( j_{1,W}^{-1}(s,s) \cong G \).

The existence and uniqueness of \( \tilde{\Gamma} \) follows from the fact that the étale covers of \( \text{Spec } k(s) \) and of \( B \) form equivalent categories. Moreover, the equivalence of categories of finite étale covers give isomorphisms \( \psi : R_1|_W \to W \times_B \tilde{\Gamma} \) and \( \tilde{\psi} : \tilde{R}_1|_{\tilde{W}} \to \tilde{W} \times_B \tilde{\Gamma} \) such that \( pr_1 \circ \psi = p_{1,W} \) and \( pr_1 \circ \tilde{\psi} = p_{1,\tilde{W}} \). Then \( q_{1,W} \circ (\psi^{-1}) \), respectively \( q_{1,\tilde{W}} \circ (\tilde{\psi}^{-1}) \), defines an action of \( \tilde{\Gamma} \) on \( W \), respectively \( \tilde{W} \) (the groupoid axioms carry over to the axioms describing the action of a group scheme) and these actions are equivariant with respect to \( \tilde{W} \to W \). Finally, \( W/(R_1|_W) \cong \text{Spec } O_{F,x}^h \), and we are done. \( \square \)

There are two special cases of this that we make explicit. One, when \( \text{Aut}_{X_s} \) is reduced, is the following well known folk theorem (see, for example, [FC])
which lacked, apparently, a published proof except when $k(s)$ is separably closed ([LMB00] 6.2.1). Note that, despite its apparent similarity, this result is not equivalent to Th. 6.2 of [LMB00]. For one thing, the group $G$ appearing there is constructed in the course of the proof as a symmetric group $\mathfrak{S}_d$, where $d$ is not intrinsic to the context.

**Theorem 2.12** If $B$ is henselian, $k(b) = k(s)$, $G = \text{Aut}_X$ is reduced and $f : \mathcal{X} \to S$ is miniversal at $s$, then $F$ is locally isomorphic at $\pi(s)$ to the quotient of $S$ by an action of $\tilde{\Gamma}$.

**PROOF:** Take $H = 1$ above. □

**Remark:** This combines with Th. 6.2 of [LMB00] to permit the definition and construction of the henselization of a Deligne–Mumford stack $\mathcal{F}$ at a point $P : T = \text{Spec} K \to \mathcal{F}$. Consider the inverse system of 2-Cartesian diagrams

$$
\begin{array}{ccc}
T & \longrightarrow & G \\
\downarrow & \equiv & \downarrow \\
T & \longrightarrow & \mathcal{F}
\end{array}
$$

where $\phi$ is étale and representable, and we demand also that the natural morphism $\mathcal{G} \to \mathcal{G} \times_{\mathcal{F}} \mathcal{G}$ be an isomorphism. The inverse limit of this system is the stack $[S/\tilde{\Gamma}]$ of 2.12 and is the henselization of $\mathcal{F}$ at $P$.

**Theorem 2.13** If $f : \mathcal{X} \to S$ is everywhere versal and $\text{Aut}_{\mathcal{X}/S}$ is flat over $S$, then $\pi : S \to F$ is smooth.

**PROOF:** Take $H = \text{Aut}_{\mathcal{X}/S}$ above. □

### 3 Tangent sheaves and foliations on singular varieties

We will need to generalise the notion of a smooth (height 1) foliation from the case of a smooth variety discussed in [Ek86] to the case of singular varieties.

**Definition 3.1** Suppose that $f : X \to S$ is of finite type. A smooth 1-foliation on $X/S$ is a subsheaf $\mathcal{F}$ of the tangent sheaf $T_{X/S}$ such that for every closed point $x$ of $X$ the induced map $\mathcal{F} \otimes_{O_X} k(x) \to \text{Hom}_{k(x)}(m_x/m_{f(x)} + m_x^2, k(x))$ is injective and such that $\mathcal{F}$ is closed under commutators (and, when $S$ has positive characteristic $p$, under $p$'th powers).

**Remark:** When $X/S$ is smooth, the injectivity condition simply says that $\mathcal{F}$ is a subbundle of $T_X$ so this corresponds to the already established notion.

We shall see that, just as in the smooth case, a smooth 1-foliation gives rise to a flat infinitesimal equivalence relation on $X$. We shall need the following lemma.
Lemma 3.2 Let \((A, m, k)\) be a local ring, \(M\) a finitely generated \(A\)-module and \(N\) a finitely generated \(A\)-module provided with an \(A\)-homomorphism \(N \to M^* := \text{Hom}_A(M, A)\). Assume that the map \(N/mN \to \text{Hom}_k(M/mM, k)\) induced by this is injective. Then \(N\) is free and a direct factor of \(M^*\). Furthermore, \(M\) can be written as a direct sum \(M_1 \oplus M_2\) such that \(M_1\) is free and \(M_1^* \hookrightarrow M^*\) can be identified with \(N \hookrightarrow M^*\).

**PROOF:** Let \(n := \dim_k N/mN\) and pick \(m_1, \ldots, m_n \in M\) such that the composite of \(N/mN \to \text{Hom}_k(M/mM, k)\) and the map \(\text{Hom}_k(M/mM, k) \to k^n\) given by evaluation at the residues \(\overline{m_i} \in M/mM\) is an isomorphism. Evaluation at the \(m_i\) gives a map \(M^* \to A^n\) and the composite \(N \to M^* \to A^n\) induces an isomorphism upon reduction modulo \(m\). By Nakayama’s lemma it is then surjective and by the projectivity of \(A^n\) and Nakayama’s lemma again it is an isomorphism. This shows that \(N\) is a direct factor of \(M^*\) as well as isomorphic to \(A^n\). By construction the basis of \(N\) thus constructed is dual to \(\{m_i\}\) and hence the \(m_i\) generate a direct summand of \(M\).

**Proposition 3.3** Suppose that \(X \to S\) is a finite type morphism, that \(S\) has positive characteristic \(p\) and \(\mathcal{F} \subseteq T_{X/S}\) a smooth 1-foliation. Then \(\mathcal{F}\) is locally free and locally a direct summand of \(T_{X/S}\). Furthermore, if \(f: X \to Y\) is the “scheme of constants” of \(\mathcal{F}\), i.e., \(O_Y = O^\mathcal{F}_X\), then \(f\) is a flat map of degree \(p^n\), where \(n := \text{rank} \mathcal{F}, \mathcal{F} = T_{X/Y}\) and \(\mathcal{F}^* = \Omega^1_{X/Y}\). Finally, there are locally sections \(f_1, f_2, \ldots, f_n \in O_X\) such that they generate \(O_X\) as \(O_Y\)-algebra. Put \(f_i^p = g_i\); then \(X\) is recovered from \(Y\) from the formula \(O_X = O_Y[z_1, \ldots, z_n]/(z_1^p - g_1, \ldots, z_n^p - g_n)\).

**PROOF:** If we apply the lemma with \(M = \Omega^1_{X/S}\) we immediately get that \(\mathcal{F}\) is locally free and locally a direct summand of \(T_{X/S}\). Let now \(\mathcal{A}\) be the subalgebra of the ring of differential operators relative to \(S\) of order \(< p\) generated by \(O_X\) and \(\mathcal{F}\). If we filter \(\mathcal{A}\) by the order filtration then there is a map \(\text{Sym}^{<p} \mathcal{F} \to \text{gr} \mathcal{A}\). We also have an obvious map \(\text{gr} \mathcal{A} \to \text{gr} \text{Diff}^{<p}\) and an evaluation map \(\text{Diff}^{<p} \to \text{Hom}_O(\text{Sym}^{<p} \Omega^1, O)\) and finally a restriction map \(\text{Hom}_O(\text{Sym}^{<p} \Omega^1, O) \to \text{Hom}_O(\text{Sym}^{<p} \mathcal{F}^*, O)\). The composite of these maps give a map \(\text{Sym}^{<p} \mathcal{F} \to (\text{Sym}^{<p} \mathcal{F}^*)^*\). This map is easily seen to be the standard map which is an isomorphism as we are only considering degrees \(< p\). This shows that \(\text{gr} \mathcal{A} = \text{Sym}^{<p} \mathcal{F}\), that \(\mathcal{A}\) is locally free and that its dual defines a closed subscheme of \(X \times_S X\) which is a finite flat equivalence relation. Its quotient is exactly \(Y\) and therefore the map \(X \to Y\) is flat of degree \(p^n\). If \(x \in X\) and \(y := f(x)\) we now have an exact sequence

\[
m_y/m_y^2 \to m_x/m_x^2 \to \Omega^1_{X/Y} \otimes k(x) \to 0
\]

which shows that if we let \(f_1, \ldots, f_n \in O_{X,x}\) be such that their residues form a basis for \(\Omega^1_{X/Y} \otimes k(x)\) then they generate \(O_{X,x}\) as \(O_{Y,y}\)-module and further the monomials where each \(f_i\) occurs to power \(< p\) form a set of module generators which have cardinality \(< p^n\) and hence form a basis. □
Corollary 3.4 If $X$ is a normal $k$-variety and $T_X$ is a smooth 1-foliation, then $X$ is smooth.

PROOF: Since $X$ is normal, the quotient morphism $X \to X/T_X/S$ is the relative Frobenius. This is flat, by the previous result, and now Kunz’ criterion shows that $X$ is a regular scheme, so smooth over $S$. □

Corollary 3.5 Suppose that $X$ is a normal $k$-variety and that $V \subset H^0(X, T_X)$ is a finite-dimensional vector space that generates $T_X$. Then for every singular point $P$ of $X$ there is a non-zero element $v$ of $V$ that vanishes at $P$. Moreover, if $\text{char } k = p > 0$ and $V$ is a $p$-Lie algebra, then we can take $v$ to be $p$-closed.

PROOF: We can assume that $\dim X \geq 2$.

If $v(P)$ is never zero, then $T_X$ is a smooth 1-foliation near $P$. If $\text{char } k = p > 0$, then the set $W = \{v \in V | v(P) = 0\}$ is a non-zero sub-$p$-Lie algebra of $V$, and any such contains a $p$-closed line. □

Corollary 3.6 Given $X \to S$ and $\mathcal{F}$ as above, the formation of the quotient $X/\mathcal{F}$ commutes with base change.

Corollary 3.7 (1) Let $X \to Y$ be the quotient by a smooth 1-foliation, where $X$ is assumed to be normal. Then $\Omega^1_{X/Y}$ is locally free. Thus $T_X/Y$ is locally a direct factor of $T_X$ and the quotient is reflexive. In particular, if it is of rank 1 it is a line bundle.

(2) Suppose $X$ is a normal surface and $D$ a global vector field on it which generates a smooth 1-foliation (i.e., $\mathcal{O}_XD \to T_X$ is a smooth 1-foliation). Then $T_X/\mathcal{O}_XD$ is a line bundle and in particular $T_X$ is locally free.

(3) Let $X$ be a $\mathbb{Z}/2$- or $\alpha_2$-Enriques surface such that the canonical double cover $Y$ has isolated singularities. Then the tangent bundle $T_Y$ is trivial.

PROOF: The first two parts follows directly from the proposition. For the third we get from the second and the fact that $T_X/\mathcal{O}_XD$ is the inverse image of the dual of $\Omega^1/\mathcal{O}_X \omega$ where $\omega$ is a non-zero form combined with the fact that by assumption the zeros of $\omega$ are isolated that $T_Y$ is an extension of $\mathcal{O}_Y$ by itself. As $H^1(Y, \mathcal{O}_Y) = 0$ this extension is trivial. □

A smooth 1-foliation on a smooth variety has a smooth quotient simply because the quotient map is flat. This leads to the impression that in general the singularities of a quotient should be simpler than that of the base variety. We do not have a general statement to that effect but the following lemma may be seen as further evidence of its existence.

Proposition 3.8 Let $R$ be a complete 2-dimensional local $k$-algebra of characteristic $p$ with an RDP, $\mathcal{F}$ a smooth 1-foliation on $R$ and $S$ its ring of constants. Then the list of possibilities is as follows.

(1) $S$ is regular and $R$ has a Zariski RDP.

(2) $p = 2$, $S$ has an $A_n$-singularity and $R$ is either of type $A_{2n+1}$ or, when $n = 1$, $D_{2m+1}^0$ or, when $n = 2$, $E_6^0$. 

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(3) $p \geq 3$, $R$ is of type $A_{pr-1}$ and $S$ is of type $A_{r-1}$.

In particular, $R$ has a Zariski RDP if and only if $S$ is regular.

PROOF: $R$ has embedding dimension 3 and by 3.3 $R = S[[v]]/(v^p - g)$ where $g \in m_S$. So $S$ has either embedding dimension two, in which case $S$ is regular and $R$ has a Zariski RDP, or it has embedding dimension 3 and then $0 \neq \bar{g} \in m_S/m^2_S$. We may assume the latter. Thus we can assume that $S = k[[x, y, z]]/(f)$ and $g = z$. Then $R = k[[x, y, v]]/(F)$, where $F = f(x, y, v^p)$. Then we can write

$$f = a_2(x, y) + (x, y)^3 + z^m u + z^n h(x, y, z) + z(x, y)^2,$$

where $u \in k[[z]]^*, h = \sum h_i(x, y)z^i$, $h_i$ is homogeneous and linear in $x, y$ and $h_0 \neq 0$.

Now suppose that $p \geq 3$. RDPs are characterized as the double points that are absolutely isolated, or as the double points whose blow-up at the origin has only RDPs. Then the quadratic term of the strict transform of $F$ under this blow-up is $a_2(x_1, y_1)$, where $x_1 = x/v$ and $y_1 = y/v$, and its cubic term is zero. Hence $a_2$ has rank 2, so that $R, S$ are of type $A_2, A_N$, say. Moreover, by analyzing what happens under a blow-up, the form of $f$ reveals that $N = \min\{m - 1, 2n - 1\}$. Applying this analysis to $F$ shows that $M = \min\{pm - 1, 2pn - 1\}$, which proves 3.8 3.

Now suppose that $p = 2$. Since $R$ has multiplicity 2 $f \in m^2 \setminus m^3$, where $m := (x, y, z)$. We now divide the discussion into three cases according to the rank of $f \in m^2/m^3$.

Assume first that $\bar{f}$ has rank 3. Up to linear transformations of $m/m^2$ there are two possible configurations for $\bar{g}$ and $\bar{f}$, seen as curves in $\mathbb{P}(m/m^2)$ they intersect in either one or two points. In the first case, we may, after a formal change of coordinates, assume that $f = z^2 + zj_2(x, y) + h_2(x, y)$, where the index indicates that $h_2, j_2 \in m^2$, and $g = x$. By assumption $h_2 \in m^2/m^3$ is the product of two distinct linear factors and we may assume that the second is $y$. After further formal change of coordinates we may assume that $h_2 = xy + k_3(y)$ and $j_2 = j_2(y)$. Then $R$ has the form $k[[t, y, z]]/(z^2 + zj_2 + x^2y + k_3)$. This is a $D_{2m+1}^0$-singularity (when it has an isolated singularity). In the second case may assume that $f = z^2 + xy$ and $g = z$ which leads to an $A_3$-singularity.

Assume now that $\bar{f}$ has rank 2 so that it is a product of two distinct linear factors. When $\bar{g}$ does not lie in the space spanned by them we may assume that $g = z$ and that $f = z^{n+1} + xy$ and then $S$ has an $A_n$-singularity and $R$ an $A_{2n+1}$-singularity. If it does lie in the space spanned by these two factors then it cannot divide $\bar{f}$ because if it did $R$ would have multiplicity three which it doesn’t. This leads to the form $f = z^2 + yz + h_3(x, z)$ and $g = y$. This gives that $R$ has equation $z^2 + y^2z + h_3(x, z)$. In order for this to be an RDP we need that $x^3$ occur in $h_3$. This leads to a singularity of type $E_6^0$ for $R$ while $S$ has type $A_2$.

Assume finally that $\bar{f}$ has rank 1 so that it is a square of a linear form. If $\bar{g}$ divides $\bar{f}$ $\bar{R}$ would have multiplicity 4 which is not possible. We may therefore
assume that \( f = z^2 + h_3(x, y, z) \) and \( g = x \) so that \( R \) is defined by \( z^2 + h_3(x^2, y, z) \).

From [Li69] p. 268 it follows that this does not lead to an RDP.

**Definition-Lemma 3.9** Let \( X \) be a surface over a perfect field of characteristic \( p, x \in X, f \in \mathcal{O}_x \) and \( Y := \text{Spec} \mathcal{O}_x[z]/(z^p - f) \) an isolated Zariski singularity. Then there is a unique subsheaf \( G \subseteq T_Y \) of colength 1 such that a rank 1 subsheaf \( \mathcal{F} \subseteq T_Y \) closed under \( p \)'th powers is a smooth 1-foliation if and only if \( \mathcal{F} \nsubseteq G \).

We will call this subsheaf the sheaf of non-free vector fields.

**Proof:** This is just the statement that the kernel of the natural homomorphism \( T_Y \otimes k(y) \rightarrow \text{Hom}(m_y/m^2_y, k(y)) \) has a 1-dimensional kernel, where \( y \in Y \) is the point above \( x \). Using the fact that a basis for \( T_Y \) is given by \( \partial/\partial z \) and \( f_y \partial/\partial x - f_x \partial/\partial y \) and \( f_x, f_y \in m_x \) this follows immediately.

An isolated surface singularity in characteristic \( p > 0 \) will be called a Zariski singularity if it has the local form \( \mathcal{R}[[z]]/(z^p - f) \), where \( R \) is a regular local ring.

In characteristic 2, a Zariski RDP is an RDP which is also a Zariski singularity. In Artin’s notation [Ar77] these are the ones of type \( A_1, D^0_{2n}, E^0_7 \) and \( E^0_8 \). The underlying Dynkin diagrams are the simply laced diagrams whose root lattice is 2-elementary. Note that contrary to the case of characteristic 0 there are in general several RDPs with the same Dynkin diagram, though only one Zariski RDP.

**Remark:** The terminology is motivated by the Zariski surfaces, surfaces in characteristic \( p \) birational to surfaces with equation \( z^p - f(x, y) \).

**Definition-Lemma 3.10** Let \( f \in k[[x, y]] \), where \( k \) is a field of characteristic 2, be such that \( (f_x', f_y') \) is of finite codimension. Suppose \( X \rightarrow \text{Spec} R \) is a deformation of the singularity \( X_0 = \text{Spec} k[[x, y, z]]/(z^2 - f) \) and that \( X/R \) admits a 2-closed smooth 1-foliation \( \mathcal{F} \). Then this deformation is isomorphic to one of the type \( \mathcal{R}[[x, y, z]]/(z^2 - F) \). When \( X \rightarrow \text{Spec} T \) is a miniversal deformation we will call the locus of such singularities the Zariski locus. It is smooth of dimension equal to \( \dim k[[x, y]]/(f_x', f_y') \). A deformation of a Zariski singularity of this type will be called a Zariski deformation.

**Proof:** By 3.6 \( X/\mathcal{F} = Y \), say, is formally smooth over \( \text{Spec} R \). Let \( Z \in \mathcal{O}_X \) be a lifting of \( z \). Then \( Z^2 := F \in \mathcal{O}_Y \) and we get a map \( \mathcal{O}_Y[Z]/(Z^2 - F) \rightarrow \mathcal{O}_X \).

As this is an isomorphism over the closed point of \( \text{Spec} R \) and \( \mathcal{O}_X \) is projective of rank 2 over \( \mathcal{O}_Y \) this is an isomorphism.

**Lemma 3.11** Suppose that the henselian germ \( (X, 0) \) is an RDP of index \( n \) and characteristic 2 admitting an equicharacteristic deformation \( \mathfrak{x} \rightarrow B \) such that \( B \) is reduced and for every geometric generic point \( \bar{\eta} \) of \( B \) the fibre \( \mathfrak{x}_{\bar{\eta}} \) has only Zariski RDPs of total index \( n \). Then \( (X, 0) \) is a Zariski RDP and \( \mathfrak{x} \rightarrow B \) is a Zariski deformation.

**Proof:** The existence of \( \mathfrak{x} \rightarrow B \) shows that in an algebraic representative \( \mathfrak{x}_1 \rightarrow V \) of a miniversal deformation of \( (X, 0) \), there is a closed point \( v \in V \)
such that $\mathcal{X}_{1,v}$ has $n$ nodes. In characteristic 2 a Zariski RDP of index $r$ has $2r$ local moduli, so that, by the openness of versality and the formal smoothness of the natural map from the miniversal deformation space of a normal affine surface to the product of the miniversal deformation spaces of neighbourhoods of its singularities, $\dim V \geq 2n$. An inspection of Artin’s tables [Ar77] shows that $(X, 0)$ is Zariski, and $\dim V = 2n$.

In particular, for every geometric $b \in B$, the module $T_{\mathcal{X}_b}^1$ has length $2n$. So $T_{\mathcal{X}/B}^1$ is a locally free $\mathcal{O}_B$-module, and then 3.16 $T_{\mathcal{X}/B}$ is $B$–flat and its formation commutes with base change. So $T_{\mathcal{X}/B}$ is $\mathcal{O}_X$–free. We conclude by 3.10.

Suppose $X$ is a local complete intersection surface with isolated singularities and $\pi: \tilde{X} \to X$ a minimal resolution of singularities. The local Chern classes $c_i^\pi(X) \in A_*(Z)$, where $Z := \pi^{-1}(X^{\text{sing}})$, are defined as $c_i(T_{\tilde{X}/X}^\pi) \cap [\tilde{X}]$, where $\pi$ is the relative tangent complex (the derived dual of the cotangent complex) $T_{\tilde{X}/X}$, considered as a perfect complex on $\tilde{X}$ with support in $Z$ and the $c_i$ are the localized Chern classes in $A^*(Z \to X)$ of [Fu84], Example 18.1.3. If $x \in X^{\text{sing}}$ we define $c_i(X)_x \in A_*(\pi^{-1}(x))$ to be the component in $A_*(\pi^{-1}(x))$ of $c_i^\pi(X)$. Also we define the local Chern numbers $c_1^\pi(X)_x$ and $c_2(X)_x$ as $\deg(c_1(T_{\tilde{X}/X}^\pi) \cap c_1(X)_x)$ and $\deg(c_2(X)_x)$ respectively.

**Proposition 3.12** Let $X$ be a local complete intersection surface with isolated singularities, $\pi: \tilde{X} \to X$ its minimal resolution and $i: Z := \pi^{-1}(X^{\text{sing}}) \to X$.

1. For every irreducible component $E$ of $Z$ we have that $c_1^\pi(X) \cap i^*\mathcal{O}(E) = i^*c_i(T_{\tilde{X}/X}^\pi)$. In particular, $2g(E) - 2 = E^2 + \deg(c_1^\pi(X) \cap i^*\mathcal{O}(E))$ and this determines completely $c_1^\pi(X) \in A_0(Z)$.

2. (Local Noether’s formula) For each $x \in X^{\text{sing}}$ we have that

$$-\text{length}_x(R^1\pi_*\mathcal{O}_{\tilde{X}}) = \frac{c_1^2(X)_x + c_2(X)_x}{12}.$$ 

3. We have that $c_i(\tilde{X}) = \pi^*c_i(X) + i_*c_i^\pi(X)$ and, when $X$ is proper, $c_1^\pi(\tilde{X}) = c_1^\pi(X) + \sum_{x \in X^{\text{sing}}} c_1^\pi(X)_x$ as well as $c_2(\tilde{X}) = c_2(X) + \sum_{x \in X^{\text{sing}}} c_2(X)_x$.

**PROOF:** We have that $i^*\mathcal{O}(E) \cap c_1^\pi(X) = i^*\mathcal{O}(E) \cap c_i(T_{\tilde{X}/X}^\pi) \cap [\tilde{X}]$ and this equals $c_i(T_{\tilde{X}/X}^\pi) \cap \mathcal{O}(E) \cap [\tilde{X}] = c_i(T_{\tilde{X}/X}) \cap i_*[E]$ by [Fu84], Prop. 17.3.2, and this in turn equals $c_i(Li^*T_{\tilde{X}/X}) \cap [E]$ by the definition of the localized Chern classes. However, localised Chern classes with support equal to the whole space are the ordinary Chern classes by the analogue for Chern classes of [Fu84], Prop. 18.1, which gives the first part of (1). The second part follows from the adjunction formula and the negative definiteness of the intersection matrix for $Z$.

Continuing with (3) we have that $c(T_{\tilde{X}}^\pi) = \pi^*c(T_X)c(T_{\tilde{X}/X}^\pi)$ because of the usual distinguished triangle. Furthermore, $c_i(T_{\tilde{X}/X}^\pi)$ for $i > 0$ has support on $Z$ so have zero intersection with $\pi^*c_j(T_X)$ for $j > 0$. This shows that $c_i(T_{\tilde{X}}^\pi) = \pi^*c(T_X) = \pi^*c(T_X)c(T_{\tilde{X}/X}^\pi)$ because of the usual distinguished triangle. Furthermore, $c_i(T_{\tilde{X}/X}^\pi)$ for $i > 0$ has support on $Z$ so have zero intersection with $\pi^*c_j(T_X)$ for $j > 0$. This shows that $c_i(T_{\tilde{X}}^\pi) =$
\[ \pi^*c_i(T_X) + c_i(T_{X/X}) \] and [Fu84], Prop. 17.3.2 shows that \( c_i(T_{X/X}) = i_*c_i^!(X) \). The rest of (3) follows from this and the orthogonality of \( \pi^*c_i(T_X) \) and \( i_*c_i^!(X) \).

Finally, for (2) we may assume that \( X \) is proper and that \( x \) is its only singular point. Then (2) follows from (3) and the Riemann-Roch formula for \( X \) and \( \tilde{X} \).

We will use this result for rational double points and for it it becomes necessary to compute the local Chern classes for them.

**Corollary 3.13**

1. A surface RDP singularity has trivial local Chern classes.
2. Suppose that \( X \) is a surface with only isolated singularities and whose singularities have the local form \( R[z]/(z^2 - f) \) where \( R \) is smooth. Then \( c_1(T_X) = c_1(T_{X}) \) and

\[
\begin{align*}
    c_2(T_X) \cap [X] &= c_2(T_{X}) \cap [X] - \sum_{x \in X^{\text{sing}}} r_x[x],
\end{align*}
\]

where \( r_x := \text{length}_x(T^1(X)) \).

**Proof:** (1): That the first local Chern class is trivial follows from the fact that the relative \( c_1 \) is trivial and that the second one is follows from the local Noether’s formula and the fact the degree gives an isomorphism \( A_1(Z) \cong \mathbb{Z} \) (using the notations of the proposition).

As for (2) we have, by definition, that \( T^1_X = H^1(T^*(X)) \) and \( H^0(T_X) = T_X \) and the other cohomology sheaves are zero.

**Lemma 3.14** Let \( X \) be a smooth surface in characteristic 2 and \( F \hookrightarrow T_X \) a 1-foliation. Then \( F \otimes \omega_X \) is the square of a line bundle.

**Proof:** By restricting to the part of \( X \) where \( F \) is smooth (and using that the singular locus of \( F \) is of codimension \( \geq 2 \)) we may assume that \( F \) is smooth. If \( X \to Y \) is the quotient by \( F \) we have a double inseparable cover \( Y^{(-1)} \to X \). It corresponds to an \( \mathcal{L} \)-torsor for some line bundle \( \mathcal{L} \) (cf. [Ek88], Prop. 1.11). It therefore gives a vector bundle embedding \( \mathcal{L}^2 \hookrightarrow \Omega^1_X \) which is orthogonal to \( F \). This gives the lemma.

**Proposition 3.15** Let \( \pi: X \to S \) be a proper cohomologically flat morphism of noetherian schemes and \( E \) a coherent \( \mathcal{O}_X \)-module which is flat over \( S \) such that its fibres are locally free.

1. The set \( N \) of points \( s \in S \) where \( E_s \) is free is locally closed.
2. If \( S \) is regular and \( N \) is dense then its complement has everywhere codimension 1.
3. If \( N = S \) then \( E = \pi^*\pi_*E \) and \( \pi_*E \) is a locally free \( \pi_*\mathcal{O}_X \)-module that commutes with base change.

**Proof:** It is clear that \( E \) is a locally free \( \mathcal{O}_X \)-module and hence its rank is locally constant on \( X \). To prove (1) by the semi-continuity theorem we may assume that \( h^0(E_s) \) equals \( h^0(\mathcal{O}_{X_s}^{\text{rank}E}) \) and that \( S \) is reduced. Then \( \pi_*E \) is locally
free and commutes with base change so that the locus where $\mathcal{E}$ is not free is the image of the support of $\pi^* \pi_* \mathcal{E} \to \mathcal{E}$ which is closed as $\pi$ is proper.

(3) is now standard and for (2) if it is not true we may localize at a generic point of the complement of $N$ and then take a suitable quotient so as to reduce to the case when $S$ is 2-dimensional and $N$ contains the complement of one closed point $s$ and we need to show that it contains $s$ as well. Being flat over $S$ $\mathcal{E}$ has depth 2 at $s$ as $\mathcal{O}_S$-module. This implies that $\pi_* \mathcal{E}$ is reflexive and hence locally free. The map $\pi^* \pi_* \mathcal{E} \to \mathcal{E}$ is an isomorphism outside of the complement of $s$. As both have depth 2 at $s$ it is an isomorphism.

Proposition 3.16  For a flat family $\pi: X \to S$ of local complete intersection surfaces with isolated singularities we set $T^1_{X/S} := H^1(T_{X/S})$, where $T_{X/S}$ is the tangent complex.

(1) $T^1_{X/S}$ is a coherent $\mathcal{O}_S$-module that commutes with base change.

(2) $T^1_{X/S}$ is a locally free $\mathcal{O}_S$-module if and only if $T_{X/S}$ is a flat $\mathcal{O}_S$-module that commutes with base change.

Proof: This follows immediately from the fact that $T_{X/S}$ commutes with base change as $\pi$ is flat and has universally amplitude $[0,1]$ with $H^0(T_{X/S}) = T_{X/S}$.

Lemma 3.17  Let $\pi: X \to S$ be a proper morphism of noetherian schemes and $\mathcal{F}$ a coherent $\mathcal{O}_X$-sheaf that is flat as $\mathcal{O}_S$-module. Then there is a subscheme stratification, $\{S_i\}$, of $S$ such that for a morphism $f: T \to S$ $f^* \mathcal{F}$ is cohomologically flat with $\pi_* f^* \mathcal{F}$ locally free of rank $e$ if and only if $f$ factors through $S_e \hookrightarrow S$. If $\mathcal{F}$ is cohomologically flat in degree 0 the same result holds true for cohomological flatness in degree 1.

Proof: Recall (cf. [EGAIII:2], §7) that there is a coherent $\mathcal{O}_S$-module $\mathcal{M}$ that represents $B \mapsto \pi_* (\mathcal{F} \otimes \pi^* B)$, that it commutes with base change and is locally free if and only if $\mathcal{F}$ is cohomologically flat. Furthermore, when locally free its rank equals that of $\pi_* \mathcal{F}$. The existence of the stratification then follows immediately from [Mu66], p. 56. The case of cohomological flatness in degree 1 is completely similar.

Lemma 3.18  Suppose $f: X \to S$ is a morphism of schemes and $G$ is an $S$-group scheme acting on $X$. Let $x \in X$ be a fixed point of $G$ such that $f$ is formally smooth at $x$ and suppose that $H^1(G_s, (\Omega_{X/S})_x) = 0$, where $s := \pi(x)$. Then the fixed point locus of $G$ on $X$ is formally smooth at $x$.

Remark: The only consequence of this that we will use, that if $G$ is linearly reductive then the fixed point locus is formally smooth, is of course well known.

Proof: To prove this we may assume we have a small infinitesimal $S$-extension $T \hookrightarrow T'$ and an $S$-map of $T$ to the fixed point locus of $G$ and we want to show that it is liftable to $T'$. Now, such a lifting is nothing but a lifting to $X$ that is invariant under $G$. A lifting to $X$ always exist because of formal smoothness and
the set a liftings is a torsor under \((\Omega_{X/S})_x\). This torsor is a \(G_s\)-equivariant torsor and hence the obstruction to a \(G\)-fixed lifting is an element of \(H^1(G_s, (\Omega_{X/S})_x)\) and by assumption that obstruction is trivial.

Suppose now that \(X\) is a surface with Zariski RDPs of total index \(r\) and minimal resolution \(\pi: \tilde{X} \to X\).

**Theorem 3.19** The sheaf \(T_X\) is locally free and \(c_2(T_X) = c_2(\tilde{X}) - 2r\).

**PROOF:** The local freeness of \(T_X\) is 3.7 2 and the Chern class formula is 3.13 and 3.12 3 combined with the fact that the length of \(T^1\) is twice the index.

**Proposition 3.20** Suppose that \(X\) is Zariski RDP–K3 and that \(r = 12\). Then \(T_X\) has trivial Chern classes and either \(T_X\) is free or there is a decomposition

\[
0 \to \mathcal{O}(A) \to \pi^* T_X \to I_Z \mathcal{O}(-A) \to 0, 
\]

where \(A\) is effective and non–zero. Moreover, in the second case \(T_X\) is \(H\)-unstable for all ample \(H \in \text{Pic} X\).

**PROOF:** The triviality of the Chern classes follows directly from Theorem 3.19. The Riemann-Roch theorem then shows that \(h^0(X, T_X) + h^2(X, T_X) \geq 4\) but as \(T_X\) is of rank 2 with trivial determinant it is self-dual and as \(\omega_X\) is trivial this gives that \(h^0(X, T_X) = h^2(X, T_X)\) and thus \(h^0(X, T_X) \geq 2\). This gives a map \(\mathcal{O}^2_X \to T_X\) and if its image has rank 2 it must be an isomorphism by taking its determinant. The case when the image has rank 1 leads to the second part of the proposition.

The next result gives the key relationship between Enriques and K3 surfaces over an algebraically closed field.

**Theorem 3.21** (1) i Suppose that \(X\) is Zariski RDP–K3, that \(r = 12\) and that \(T_X\) is free. Then for any 2-closed vector field \(\xi\) on \(X\) that does not vanish at a singular point, the quotient \(Y = X/\xi\) is smooth and Enriques.

(2) ii Suppose that \(Y\) is a unipotent Enriques surface whose canonical double cover \(X \to Y\) is RDP-K3. Then \(X\) is Zariski RDP of total index 12 and \(T_X\) is free.

**PROOF:** For (1) the smoothness of \(Y\) is an immediate consequence of the non-vanishing of \(\xi\). Then the adjunction formula applied to the quotient map \(\rho: X \to Y\) shows that \(K_Y\) is numerically trivial. Since \(\rho\) is an étale homeomorphism, it follows that \(b_1(Y) = 0\) and \(e(Y) = 12\). Then \(Y\) is Enriques, by the Bombieri–Mumford classification of surfaces.

For (2) the only thing to prove is that \(T_X\) has no subsheaf \(\mathcal{O}(A)\) with \(A > 0\). Regard \(\rho: X \to Y\) as the quotient of \(X\) by a rank 1 foliation \(F \hookrightarrow T_X\); the adjunction formula for \(\rho\) shows that \(c_1(F)\) is numerically trivial, and comparison of the saturated subsheaf \(F \hookrightarrow T_X\) with the putative subsheaf \(\mathcal{O}(A) \hookrightarrow T_X\) gives a contradiction.
4 Further deformation theory: Enriques surfaces

A smooth, proper and polarized surface $X$ is correctly obstructed if $h^0(T_X) = 0$, there is an effective formal deformation over its hull $D_X$ and $\dim D_X = h^1(T_X) - h^2(T_X)$.

**Theorem 4.1** Suppose that $X$ is a smooth projective surface over $k$, that $f: \mathcal{X} \to S$ is versal and is miniversal at the closed point $s$. Assume that $\text{Aut}_X$ is finite and that $S$ is connected.

1. Every irreducible component of $S$ whose geometric generic fibre is correctly obstructed has dimension $h^1(X, T_X) - h^2(X, T_X)$.

2. If the set of geometric points in $S$ with correctly obstructed fibres is dense, then $(S, s)$ a local complete intersection over $\mathbf{W}$ of dimension $h^1(X, T_X) - h^2(X, T_X)$.

3. Suppose that $h^2(T_X) = 1$ and $f: \mathcal{X} \to S$ contains a correctly obstructed geometric fibre. Then $\dim_s S = h^1(X, T_X) - 1$.

**Proof:** For any geometric point $t \in S$, there is a smooth morphism $\alpha: (S, t) \to D_{\mathcal{X}_t}$. If $h^0(T_{\mathcal{X}_t}) = 0$, then $D_{\mathcal{X}_t}$ is universal, so that the classifying map $D_{\mathcal{X}_t} \to \hat{M}$, where $\hat{M}$ is the formal completion of the coarse moduli space $M$ at the appropriate point, is identified with a quotient by $\text{Aut}_{\mathcal{X}_t}$. Then, by 2.9, the fibredimension of $\alpha$ is $h^0(T_{\mathcal{X}_t})$. Hence $\dim_{\mathcal{X}_t} S = \dim D_{\mathcal{X}_t} + h^0(T_X)$. If $\mathcal{X}_t$ is correctly obstructed, then $\dim D_{\mathcal{X}_t} = h^1(T_{\mathcal{X}_t}) - h^2(T_{\mathcal{X}_t}) = -\chi(T_{\mathcal{X}_t})$ and $\dim_{\mathcal{X}_t} S = h^1(T_X) - h^2(T_X)$.

This proves (1), and now (2) is clear.

Finally, (3) follows from (2) since, if the dimension is $h^1(T_X)$, then a small miniversal deformation is irreducible at $X$. \hfill \square

The Picard scheme plays an important role in the study of Enriques surfaces, so it is not surprising that for the deformation theory we need to know how the Picard scheme varies in a family of Enriques surfaces. It would seem that the fact that we may have $h^2(O) \neq 0$ could cause problems but the following result shows that this is not so.

**Proposition 4.2** Suppose that $f: X \to S$ is smooth and proper and that for every geometric point $s \to S$ we have $h^2(O_{X_s}) = h^1(O_{X_s}) - b_1(X_s)/2$. Then $\text{Pic}(X/S)$ is flat over $S$ and so if $f$ is projective, $\text{Pic}(X/S) / \text{Pic}^\tau(X/S)$ is locally constant.

**Proof:** We may assume that $S = \text{Spec } R$, where $R$ is local and artinian, the closed point is $s$ and that $k = k(s)$ is algebraically closed. The condition $h^2(O_{X_s}) = h^1(O_{X_s}) - b_1(X_s)/2$ is equivalent to $H^2(X_s, WO_{X_s})$ being of finite length and $H^0(X_s, WO_{X_s})$ being $V$-torsion free if char $k > 0$ and $H^2(X_s, O) = 0$ if char $k = 0$. Thus the case of characteristic zero is clear and we may assume that $k$ is of positive characteristic.

Let us first show that $\text{Pic}(X/S)$ is flat along the zero section. Indeed, the completion $T$ of the local ring at $0$ of $\text{Pic}(X/S)$ is the quotient of some power
series ring $P$ over $R$ by an ideal $\mathcal{I}$ generated by $h^2(\mathcal{O}_{X_s})$ elements. However, as a finite type group scheme over a field is a local complete intersection the conditions of the proposition implies that the minimal number of generators of $\ker(P/mP \rightarrow T/mT)$ is $h^1(\mathcal{O}_{X_s}) - b_1(X_s)/2$ and the hypotheses then imply that $\mathcal{I}$ is a complete intersection ideal, so that $T$ is $R$-flat. When $\text{Pic}(X/S)$ is flat along the zero section $R^2f_\ast\mathcal{G}_m$ is prorepresentable ([Ra79], 2.7.5.3) and so is zero as its restriction to $k$ is zero, again by the assumptions and [Ek85], Prop. 8.1. To prove flatness it is sufficient to show that any $k$-point of $\text{Pic}(X/S)$ lifts to a $Y$-point where $Y \rightarrow S$ is flat as then translation gives an isomorphism of local rings. However, the obstruction for lifting this $k$-point to an $S$-point is an element of $H^2(X, \hat{\mathcal{G}}_m)$ and as $R^2f_\ast\mathcal{G}_m = 0$ this element is killed by some flat extension. Finally, if $f$ is projective then $\text{Pic}^\tau(X/S)$ is an open sub-algebraic space and so is flat. This means that $\text{Pic}(X/S)/\text{Pic}^\tau(X/S)$ is a flat group-algebraic space and as it is always unramified, it is étale. As each component is proper it is therefore locally constant.

As was mentioned our intended application of this result is to a family of Enriques surfaces.

**Corollary 4.3** (1) Let $X \rightarrow S$ be a family of Enriques surfaces. Then $\text{Pic}^\tau(X/S)$ is a flat group scheme of order two and $\text{Pic}(X/S)/\text{Pic}^\tau(X/S)$ is a locally constant sheaf of torsion free finitely generated abelian groups.

(2) A formal deformation of Enriques surfaces is effective.

**PROOF:** Indeed, an Enriques surface fulfils the conditions of the proposition and over an algebraically closed field $\text{Pic}^\tau$ is of order 2.

As for the second part, the first part shows that the square of any line bundle lifts over any formal deformation and thus it is projective and hence effective. □

**Proposition 4.4** If $Y$ is an Enriques surface over an algebraically closed field, then $h^2(Y, T_Y) = h^0(Y, T_Y) \leq 1$.

**PROOF:** $h^0 = h^2$ is proved in [CD89] and both vanish if $p \neq 2$. In [CD89] it is shown that $h^0(Y, T_Y) = 0$ if $Y$ is a $\mu_2$-surface and $h^0(Y, T_Y) = 1$ if $Y$ is an $\alpha_2$-surface. In [SB96] it is claimed that $h^0(Y, T_Y) = 0$ for $\mathbb{Z}/2$-surfaces, but this is false; the true statement is that $h^0(Y, T_Y) \leq 1$ [ESB99].

There is not much to say about miniversal deformations of $\mathbb{Z}/2$- and $\mu_2$-surfaces, except that we would like to be able to say more about the case where there are non-trivial vector fields.

**Theorem 4.5** Let $\mathcal{X} \rightarrow S \ni s$ be a family of $\mu_2$- or $\mathbb{Z}/2$-Enriques surfaces miniversal (for all deformations not just modulo 2) at $s$ and let $X := \mathcal{X}_s$.

(1) If $h^0(X, T_X) = 0$ then $S$ is formally smooth over $\mathbb{Z}$ of relative dimension 10 at $s$.

(2) If $h^0(X, T_X) = 1$ then $S$ is flat over $\mathbb{Z}$ of relative dimension 11 at $s$ and it has a hypersurface singularity there. The incidence scheme of $\mathcal{X} \times_\mathbb{Z} k(s) \rightarrow S \times_\mathbb{Z} k(s)$ and $S \times_\mathbb{Z} k(s) \times X$ is formally smooth of dimension 1 over $k(s)$.  

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PROOF: The first part is clear. For the second, note that $S \otimes \mathbf{F}_2$ is formally smooth over $k$ outside a subset $S_{bad}$ of codimension at least 3 and every geometric point of $S - S_{bad}$ corresponds to a surface with $h^0(T) = h^2(T) = 0$. The result, except the flatness, then follows from 4.1 3. Flatness also follows from this, since $S - S_{bad}$ is $\mathbf{Z}$-flat.

We now direct our attention towards the deformation theory of $\alpha_2$-Enriques surfaces which, as we will see, is much richer. First some preliminaries. A Pic-rigidification of a family $\pi: X \to S$ of Enriques surfaces is a trivialisation of the line bundle $\mathcal{O}_{\text{Pic}}/\mathcal{O}_S$. Note that for a family of $\mu_2$- or $\mathbf{Z}/2$-surfaces there is a natural such identification coming from the unique isomorphism of $\text{Pic}$ with $\mu_2$ resp. $\mathbf{Z}/2$ but for $\alpha_2$-surfaces this is not the case.

If $G \to S$ is a finite flat group scheme of order 2 and $t$ a generator of $\mathcal{L} := \mathcal{O}_G/\mathcal{O}_S$ then there is a unique element, also denoted $t$, of the augmentation ideal of $\mathcal{O}_G$ which maps to $t$ and $\{1, t\}$ is a basis for $\mathcal{O}_G$. There then are unique elements $f$ and $g$ of $\mathcal{O}_S$ such that $t^2 = ft$ and the coproduct takes $t$ to $1 \otimes t - g(t \otimes t) + t \otimes 1$; one sees immediately that this defines a group scheme if and only if $fg = 2$. Clearly if $t$ is replaced by $\lambda t$ then $f$ is replaced by $\lambda f$ and $g$ by $\lambda^{-1} g$. This means that we have maps $\mathcal{L} \to \mathcal{O}_S$ and $\mathcal{L}^{-1} \to \mathcal{O}_S$ taking $t$ to $f$ and $t^{-1}$ to $g$ respectively such that the map induced by multiplication $\mathcal{O}_S = \mathcal{L} \otimes \mathcal{L}^{-1} \to \mathcal{O}_S$ takes 1 to 2. The subschemes defined by the images of $\mathcal{L}$ resp. $\mathcal{L}^{-1}$ will be called the locus of infinitesimality and the locus of unipotence respectively.

**Proposition 4.6** Let $X$ be an $\alpha_2$-Enriques surface over $k$. Its miniversal deformation space $S = \text{Spec } R$ is isomorphic to $\text{Spec } \mathbb{W}(k)[[x_1, \ldots, x_{12}]]/(FG - 2)$ where $F,G$ lie in the ideal $(2, x_1, \ldots, x_{12})$. In particular, $S$ is a regular scheme. The isomorphism may be chosen such that there is a Pic-rigidification of a miniversal deformation over $S$ such that $F$ and $G$ map to the $f$ and $g$ just defined.

PROOF: Since $h^1(X, T_X) = 12$, we can write a hull $R$ as a quotient of the power series ring $\mathbb{W}(k)[[x_1, \ldots, x_{12}]]$. If we choose a Pic-rigidification we get elements $f$ and $g$ in $R$ such that $fg = 2$. This means that we can write $R$ as a quotient of $\mathbb{W}(k)[[x_1, \ldots, x_{12}]]/(FG - 2)$. However, as $f$ and $g$ lie in the maximal ideal of $R$, $\mathbb{W}(k)[[x_1, \ldots, x_{12}]]/(FG - 2)$ is regular and as $h^2(X, T_X) = 1$, $R$ has dimension $\geq 11$. This shows that the quotient map must be an isomorphism.

In order to get more information we will now determine the image of $f$ and $g$ in the cotangent space of the hull. Note that as $\omega_X$ for an $\alpha_2$-surface is trivial we can identify $H^1(X, T_X)$ with $H^1(X, \Omega_X^1)$ and by Serre duality we can thus identify the cotangent space of a miniversal deformation with $H^1(X, \Omega_X^1)$. The images sought will therefore be cohomology classes in $H^1(X, \Omega_X^1)$. They are only defined as elements when a Pic-rigidification has been chosen and we will start by describing certain elements in the cohomology of $X$ that can be constructed from such a rigidification.

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Definition 4.7 Let \((X, t)\) be a Pic-rigidified \(\alpha_2\)-Enriques surface over a field \(k\). Associated to it are the following elements of Hodge cohomology.

1. \(\beta \in H^1(X, \mathcal{O}_X)\) which corresponds to \(t\) under the identifications between \(H^1(X, \mathcal{O}_X)\) and the augmentation ideal of \(\mathcal{O}_{\text{Pic}^\tau(X/k)}\).
2. \(\beta \in H^1(X, \mathcal{O}_X^\tau)\) which is dual to \(\beta\) (i.e., \(\text{Tr}(\beta \delta) = 1\)).
3. \(\eta \in H^0(X, \Omega^1_{X/k})\) which is the image of the inclusion \(H^1(X, B_1) \rightarrow H^0(X, \Omega^1_{X/k})\) of the (uniquely determined) element of \(H^0(X, B_1)\) which maps to \(\eta\) under the boundary map of the long exact sequence of cohomology associated to \(0 \to \mathcal{O}_X \to \mathcal{O}_X \to B_1 \to 0\).
4. \(\beta^2 \in H^2(X, \mathcal{O}_X)\) which is the cup square of \(\beta\).
5. \(\delta^2 \in H^0(X, \Omega^2_{X/k})\) which is the dual of \(\beta^2\).
6. \(D \in H^0(X, T_X)\) is the global vector field defined by \(D(f)\beta^2 = \eta \wedge df\).

Some of the statements we will make about these elements will be independent on the rigidification chosen. In such situations we will use these elements without always explicitly choosing a rigidification. Let us also note that with respect to changing a rigidification \(\beta, \beta^2, \eta, \beta, \beta^2, D\) will be homogeneous of degrees 1, 2, 2, \(-1, -2, -4\) respectively.

Using the computation of the cohomology of Enriques surfaces it is clear that all these elements are non-zero. From them we can construct two elements of \(H^1(X, \Omega^1_X)\) namely \(d\beta\) and \(\eta\beta\).

Proposition 4.8 Let \(\pi: \mathcal{X} \to S\) be a family of Enriques surfaces.

1. The infinitesimal locus of \(\text{Pic}^\tau(\mathcal{X}/S)\) equals the locus where \(\mathcal{O}_X\) is cohomologically flat in degree 1 and \(R^1\pi_*\mathcal{O}_X\) has rank 1 (cf., 3.17).
2. The unipotent locus of \(\text{Pic}^\tau(\mathcal{X}/S)\) equals the locus where \(\Omega^1_X\) is cohomologically flat (in degree 0) and \(\pi_*\Omega^1_X\) has rank 1 (cf., 3.17).
3. Let \(X\) be an \(\alpha_2\)-Enriques surface. Then the tangent space of the infinitesimal locus of \(\text{Pic}^\tau(X)\) in a miniversal deformation of \(X\) is the orthogonal complement of \(d\beta\), where \(\beta\) corresponds to a Pic-rigidification as above.
4. Let \(X\) be an \(\alpha_2\)-Enriques surface. Then the tangent space of the unipotent locus of \(\text{Pic}^\tau(X)\) in a miniversal deformation of \(X\) is the orthogonal complement of \(\eta\beta\), where \(\beta\) and \(\eta\) correspond to a Pic-rigidification as above.

Proof: We start with (2): We are immediately reduced to assuming that \(S = \text{Spec} R\) is local artinian with the fibre \(X\) of \(\pi\) over the closed point of \(S\) an \(\alpha_2\)-surface and we want to show that \(\Omega^1_X\) is cohomologically flat if and only if \(\text{Pic}^\tau(\mathcal{X}/S)\) is unipotent.

We start by showing the equality modulo 2. In general, for a family \(\pi: \mathcal{X} \to S\) of smooth and proper varieties in characteristic \(p\) we have a morphism \(\text{Pic}(\mathcal{X}/S) \to \pi_*\Omega^1_{\mathcal{X}/S}\) defined as follows (following the lines of [Od69]). If we represent a line bundle on \(\mathcal{X}\) by a cocycle \(f_{ij}\), then if the \(p\)'th power of it is trivial there are functions \(f_i\) such that \(f_{ij}^p = f_{i}f_{j}^{-1}\). Applying the logarithmic derivative gives us a global 1-form \(d\log(f_i)\). This gives a map upon sheafification.
and it is natural for all base changes in the sense that for any map \( f: T \to S \) the following diagram commutes

\[
\begin{array}{ccc}
\Pic(X \times_S T/T) & \xrightarrow{f_*} & \Pic(X \times_S T/T) \\
\downarrow & & \downarrow \\
p \Pic(X/S) & \xrightarrow{\pi_*} & \pi_* \Omega^1_{X/S}.
\end{array}
\]

Returning to the case at hand we note that this map is injective for \( X \to k \). Note that for any rigidified order 2 finite flat group scheme \( G \) over \( S \) with functions \( f \) and \( g \) as above, \( \text{Hom}(G, G_a) \) equals the annihilator of \( g \) in \( R \). Assume first that \( \Omega^1 \) is cohomologically flat and choose a Pic-rigidification of \( X \) (which gives us \( f \) and \( g \)) as well as a generator of \( 0(\mathcal{X}, \Omega^1_{X/S}) \). In particular this means that if \( f: k \to S \) is the inclusion of the closed point then \( H^0(\mathcal{X}, \Omega^1_{X/S}) \to H^0(X, \Omega^1_X) \) is surjective but also that the map \( 2 \Pic(\mathcal{X}/S) \to \pi_* \Omega^1_{X/S} \) can be seen as a map \( 2 \Pic(\mathcal{X}/S) \to G_a \). By the commutativity of the diagram this means that the reduction map \( \text{Ann}_R g \to k \) takes an element of \( \text{Ann}_R g \) to a non-zero element of \( k \) which means that by Nakayama’s lemma that \( \text{Ann}_R g = R \) and so \( g = 0 \).

Conversely, assume, with the same notations as previously, that \( g = 0 \) so that \( 2 \Pic(\mathcal{X}/S) \) is isomorphic to \( \alpha_2 \). If \( \Omega^1_{X/S} \) is not cohomologically flat then the map \( H^0(\mathcal{X}, \Omega^1_{X/S}) \to H^0(X, \Omega^1_X) \) is not surjective and thus zero. Restricting to the flat site of \( S \) it then remains zero. Looking at the commutative diagram and using the injectivity of \( 2 \Pic(X/k) \to H^0(X, \Omega_X) \) we see that this means that \( 2 \Pic(\mathcal{X}/S) \to f_*(2 \Pic(\mathcal{X}/S)) \) is zero on the flat site of \( S \). However, this is false as \( 2 \Pic(\mathcal{X}/S) \) is isomorphic to \( \alpha_2 \) and \( \alpha_2 \to f_* \alpha_2 \) is non-zero on the flat site of \( S \) as can be seen by considering it for \( S[a] \).

This proves the statement when restricted to characteristic 2. For the general case we may assume that the base is a deformation hull and note that the ideal generated by \( g \) (using notation as before) contains \( 2 \) as \( fg - 2 = 0 \). As the ideal are equal modulo 2 this means that the ideal \( I \) defining the locus where \( \Omega^1 \) is cohomologically flat with \( \pi_* \Omega \) of rank 1 is contained in the ideal generated by \( g \). Now by what we have just proved there is an element \( g' \) in \( I \) which modulo 2 is \( g \). As \( g' = hg \) for some element we see immediately that \( h \) is a unit so the two ideals are the same.

The proof of (1) is similar but simpler.

Turning to (3) we need by (1) to show that for a deformation \( \mathcal{X} \) of \( X \) over \( k[\delta] \) the boundary map \( H^1(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X) \) is zero if and only if the deformation class in \( H^1(X, T_X) = H^1(X, \Omega^1_X) \) is orthogonal to \( d\beta \). In general if \( \mathcal{E} \) is a sheaf of tensors on \( X \), we have an action \( L_D \) by a vector field \( D \). The boundary map \( H^i(X, \mathcal{E}) \to H^{i+1}(X, \mathcal{E}) \) given by a deformation over the dual numbers is given by cupping with \( L_\nu \), where \( \nu \in H^1(X, T_X) \) is the class of the deformation. When \( \mathcal{E} = \mathcal{O}_X \), \( L_D \) is simply \( D \) itself. Now, we have to look into the identification between \( T_X \) and \( \Omega^1_X \). If \( \omega \) is a non-zero global 2-form then the
derivation $D$ corresponds to the form $\nu$ when $D(f)\omega = \nu \wedge df$ for all 1-forms $\nu$. This implies that $(D, \psi)\omega = \nu \wedge \psi$ for all 1-forms $\nu$ and $\psi$. Now choose a Pic-rigidification and let $\omega = \tilde{\beta}$. Then if we let $\nu \in H^1(X, \Omega^1_X)$ correspond to $D \in H^1(X, T_X)$ then if $\alpha \in H^1(X, \mathcal{O}_X)$ we have that $L_D(\alpha)\tilde{\beta} = \nu \wedge d\alpha$. As $H^1(X, \mathcal{O})$ is spanned by $\beta$ and $H^2(X, \mathcal{O}_X)$ by $\beta^2$ which is dual to $\tilde{\beta}$ this shows that the boundary map is zero if and only if $\nu$ is orthogonal to $d\beta$.

To prove (4) similarly requires an elucidation of $L_D(\eta)$. Here we use Cartan’s formula $L_D\omega = d(D, \omega) + D, \omega$. As $\eta$ is closed we get that $L_D\eta = d(D, \eta)$. As $d: H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \Omega^1_X)$ is injective this shows that $L_D\eta$ is zero if and only if $D, \eta$ is. Now multiplication by $\beta$ on $H^1(X, \mathcal{O}_X)$ is injective so this is if and only if $\beta(D, \eta) = 0$ and this in turn is true if and only if $\beta(D, \eta)\tilde{\beta} = 0$ and this equals $\beta\eta\nu$.

This proposition has several interesting consequences.

**Corollary 4.9** Let $X$ be an $\alpha_2$-Enriques surface over $k$ and let $S = \text{Spec} R$ be a deformation hull of $X$. Choose a Pic-rigidification of the miniversal deformation of $X$ over $S$ so that the elements $f, g, \beta, \eta$ and so on are defined.

1. The formal subscheme of $S$ where $\text{Pic}^\tau$ is infinitesimal is formally smooth of dimension 11 over $k$. In particular $R \cong W(k)[[f, x_2, \ldots, x_{12}]]/(fg - 2)$.

2. The formal subscheme of $S$ where $\text{Pic}^\tau$ is unipotent is formally smooth over $k$ precisely when $\eta\beta \neq 0$. It always has dimension 11 and embedding dimension $\leq 12$.

3. The formal subscheme of $S$ where $\text{Pic}^\tau$ is isomorphic to $\alpha_2$ is formally smooth over $k$ precisely when $d\beta$ and $\eta\beta$ are linearly independent. It is reduced and 10-dimensional and has embedding dimension $\leq 11$. In particular, $g$ is divisible by neither $f$ nor 2 and $R$ is flat over $W(k)$.

4. $R \cong W(k)[[f, g, x_3, \ldots, x_{12}]]/(fg - 2)$ when $d\beta$ and $\eta\beta$ are linearly independent. In particular the map from the deformation problem of $X$ to that of deforming $\text{Pic}^\tau(X)$ as a flat group scheme is formally smooth in this case.

5. $X$ can be lifted to characteristic 0. It cannot be lifted to $W(k)$.

6. If $d\beta$ and $\eta\beta$ are non-proportional then $X$ can be lifted to any DVR of characteristic zero that is ramified over $W(k)$.

7. If $d\beta$ and $\eta\beta$ are proportional but $\eta\beta \neq 0$ then $X$ has a lifting over some characteristic zero DVR with absolute ramification index 2.

8. If $\eta\beta = 0$ then $X$ cannot be lifted to any DVR with absolute ramification index $\leq 2$.

**PROOF:** Recall that $f, g \in m_R$, the maximal ideal of $R$, and that always $d\beta \neq 0$. This means that $f$ is non-zero modulo $m^2_R$, where $m_R$ is the maximal ideal of $R$. Hence we can chose $f$ as one of the parameters when writing $R$ as a quotient of a power series ring over $W(k)$, so that $R$ is isomorphic to the quotient ring $W(k)[[f, x_2, \ldots, x_{12}]]/(fg - 2)$. This implies that $R/Rf$ is isomorphic to $k[[x_2, \ldots, x_{12}]]$ which proves (1).
To continue with (2) we get similarly that $g$ modulo $m_R^2$ is non-zero if and only if $\eta \beta$ is. This proves the first part. To prove the second we need to show that $g \neq 0$. If $g = 0$ then $R$ has characteristic 2 but this is not possible as that would force any versal deformation of $X$ to have the base killed by 2 and by (1), there are $\mu_2$-surfaces in any neighbourhood of any versal deformation. By openness of versality a versal deformation of such $\mu_2$-surface would also have base killed by 2 which is absurd.

As for (3) $d\beta$ and $\eta \beta$ are linearly independent if and only if the images of $f$ and $g$ modulo $m_R^2$ are, and the $\alpha_2$-locus is smooth if and only if these images are linearly independent, since it is defined by the vanishing of $f$ and $g$. In general, $g$ cannot be divisible by $f$, because if so $R/2R$ is non-reduced. However, a versal deformation is generically reduced and $R/2R$ is a local complete intersection. Similarly, if $g$ is divisible by 2 then $g$ would vanish in $R/2R$ and then any mod 2-deformation of $X$ would have unipotent $\text{Pic}^*$, in contradiction to the fact that, as we have just seen, there are $\mu_2$-deformations. Flatness over $W(k)$ now follows as $fg - 2$ is not divisible by 2. Further, if $f$ and $g$ modulo $m_R^2$ are linearly independent, then they may be chosen as parameters when writing $R$ as a quotient of $W(k)[[x_1, \ldots, x_{12}]]$, which proves (4).

Next, we consider the question of liftability. Liftability follows from the flatness over $W(k)$ of a versal deformation. For any lifting over a complete DVR $V$ we get elements $\overline{f}$ and $\overline{g}$ in $m_V$, such that $\overline{f} \overline{g} = 2$ which forces $V$ to be ramified. If $f$ and $g$ can be chosen to be parameters in $R$ then it is clear that we can find a lifting over any ramified $V$. If $d\beta$ and $\eta \beta \neq 0$ are proportional then we can write $g$ as $\lambda f + O(2)$ where $\lambda$ is a unit in $W(k)$, so that $R/(x_2, \ldots, x_{12})$ is of the form $W(k)[[f]]/(h(f))$, where $h(f)$ is of the form $\overline{f}^2 + O(2)$ modulo 2. By the Weierstrass preparation theorem $W(k)[[f]]/(h(f))$ is then free of rank 2 over $W(k)$ and there is a lifting over it. Pull back to the normalization of this ring to get the result.

Finally, if $\eta \beta = 0$, then $g \in m_R^2$, so that $2 \in m_R^3$, and we are done. \hfill \Box

We will now spend some time studying multiplicative $\alpha_2$-surfaces.

**Proposition 4.10** Let $X$ be a multiplicative $\alpha_2$-Enriques surface.

(1) $D$ acts non-trivially on $H^1(X, \mathcal{O}_X)$.

(2) $\eta \beta d \beta \neq 0$ and in particular $d(\eta \beta) \neq 0$.

(3) $d\beta$ and $\eta \beta$ are linearly independent.

**PROOF:** Rescale $D$ so that $D^2 = D$. This means that $\mathcal{O}_X$ is the direct sum $\mathcal{E} \oplus \mathcal{F}$ where $D$ acts by zero on $\mathcal{E}$ and by 1 on $\mathcal{F}$. If $D$ acts trivially on $H^1(X, \mathcal{O}_X)$ then $h^1(X, \mathcal{E}) = 1$, but $\mathcal{E}$ is the structure sheaf of the quotient of $X$ by $D$ and this quotient has only rational double points with minimal resolution rational or K3 so $h^1(X, \mathcal{E}) = 0$. This means that $D \beta \neq 0$ and as $D^2 = D$ we get $D \beta = \beta$. Now there is a $0 \neq \lambda \in k$ s.t. for $f \in \mathcal{O}_X$, $D(f) \beta_2 = \lambda \eta \wedge df$, as this is true for any non-zero global vector field. Hence $0 \neq \beta \beta_2 = D(\beta) \beta_2 = \lambda \eta \wedge d\beta$; i.e., $\eta \wedge d\beta$ and so $d\beta \eta \beta$ is different from zero. By duality we have that multiplication by $\beta$
on $H^1(X, \Omega^2_X)$ is injective so that $\eta d\beta = d(\eta \beta) \neq 0$. Finally $d\beta d\beta = 0$ so that $d\beta$ and $\eta \beta$ are linearly independent.

Corollary 4.11

1. A versal deformation of $\alpha_2$-Enriques surfaces is smooth at a surface of multiplicative type.

2. The moduli problem of Pic-rigidified Enriques surfaces is prorepresentable and smooth of dimension 11 at a multiplicative $\alpha_2$-Enriques surface.

Proof: The first part follows immediately from the proposition and 4.9. For the second we note that as the global non-trivial vector fields act non-trivially on $H^1(X, \mathcal{O}_X)$ we see that two different Pic-rigidifications of the same deformation of $X$ over a local artinian base $S$ that are the same when restricted to a subscheme $T \hookrightarrow S$ which is defined by an ideal of length 1 are isomorphic. Hence, by decomposing a general map into such substeps we see that over a local artinian base isomorphism classes of deformations of $X$ are in bijection with isomorphism classes of Pic-rigidified deformations of $X$. Similarly, using small steps as above one shows that the automorphism group of a Pic-rigidified deformation of $X$ is trivial, hence that deformation problem is prorepresentable.

Remark: This gives, in the multiplicative type case, a clearer picture of the rather strange non-separation of the moduli stack of Enriques surfaces at $\alpha_2$-surfaces: The moduli stack of Pic-rigidified $\mu_2$, $\mathbb{Z}/2$-, or multiplicative $\alpha_2$-Enriques surfaces has an unramified diagonal and the stack of un-rigidified surfaces is the quotient of it by the natural $G_m$-action.

Even though we are not going to use it we finish with a description of an invariant of a multiplicative $\alpha_2$-Enriques surface which gives an étale map from the moduli stack.

Proposition 4.12 Let $\pi: X \to S$ be a Pic-rigidified family of multiplicative $\alpha_2$-Enriques surfaces. Then there is a canonical map $\overline{\text{Pic}}(X/S) \to G_a$ such that $\overline{\text{Pic}}^\tau(X/S)$ maps isomorphically to $\alpha_2$. Dividing by $\text{Pic}^\tau(X/S)$ gives an additive map $NS \to G_a$. Suppose that $S$ is of finite type over a field $k$ and that we have a marking $\mathbb{Z}/10 \cong NS$ so that the map $NS \to G_a$ can be interpreted as a map $\rho: S \to \text{Hom}(\mathbb{Z}/10, G_a)$. At any point at which $\pi$ is miniversal, $\rho$ is étale.

Proof: We can define $\beta$ and $\eta$ as before even though we work in a family. Define $\overline{\text{Pic}}(X/S) \to G_a$ by $L \mapsto \text{Tr}(\eta \beta c_1(L))$ where $c_1(L)$ is the de Rham Chern class. Certainly $\overline{\text{Pic}}^\tau(X/S)$ maps into $\alpha_2$ and to show that it maps onto it suffices to show that the tangent map is surjective. Hence we may suppose that $S = k[\delta]$ for some field $k$ and we want to show that $\eta \beta c_1(L) \neq 0$ for $L$ given by $1 + \delta \beta \in H^1(X, 1 + \delta \mathcal{O}_X)$. However, $c_1(L) = d \log(1 + \delta \beta) = \delta d\beta$ and we have just seen that $\eta \beta d\beta \neq 0$. Furthermore, as a miniversal deformation is smooth, to show that $\rho$ is étale it is sufficient to show that it is an isomorphism on tangent vectors. As the tangent spaces have the same dimension it will be enough to show that the tangent map is injective. Suppose therefore that we have a deformation $Y \to \text{Spec} k[\delta]$, 

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the pullback of \( X \) by a map \( k[\delta] \to S \), for which the map \( NS \to G_\alpha \) is constant. As we have seen, for any deformation, the extension

\[ 0 \to \text{Pic}^r \to \text{Pic} \to NS \to 0 \]

is isomorphic to the pull back of the extension

\[ 0 \to \alpha_p \to G_\alpha \to G_\alpha \to 0 \]

by the map \( NS \to G_\alpha \). Hence we have seen that \( Y \) is an \( \alpha_2 \)-surface means that \( \alpha \) has cup product zero with \( \eta \beta \) and \( d\beta \). Now, as \( d(\eta \beta) \neq 0 \) it is easy to see that \( \eta \beta, d\beta \) and \( c_1(\text{Pic}(Y_k)) \) span \( H^1(Y_k, \Omega^1_{Y_k/k}) \) as a \( k \)-vector space and so by duality and the isomorphism \( \omega_{Y_k} \leftarrow \mathcal{O}_{Y_k}; \alpha = 0 \).

\[ \square \]

5 Marking and polarizations of Enriques surfaces

We start this section with a definition that provides a solution to the problem of how to mark an Enriques surface and simultaneously polarize it. Then we discuss the morphism from the corresponding stack to the stack considered by Liedtke [L10] of Enriques surfaces with a Cossec–Verra polarization and derive an upper bound on the ramification involved in lifting an Enriques surface to characteristic zero. After that we go on to consider those (singular) K3 surfaces that arise as canonical double covers of an Enriques surface over an algebraically closed field of characteristic 2.

Denote by \( E \) the lattice \( E_{10}(-1), O(E) \) its orthogonal group and \( O^+(E) \) the index 2 subgroup of \( O(E) \) consisting of elements that preserve the two cones of positive vectors in \( E_\mathbb{R} \). It is known [CD] that \( O^+(E) \) is the Weyl group \( W(E) \), the group generated by reflections in the roots of \( E \). For any Enriques surface \( Y \) over an algebraically closed field, \( \text{Num}(Y) \cong E \). Fix, once and for all, a chamber \( \mathcal{D}_0 \) defined by the roots (that is, the \((-2)\)-vectors) in the positive cone of \( E \otimes \mathbb{R} \). (We shall not always be scrupulous in distinguishing between \( \mathcal{D} \) and its closure.) This defines a root basis \( \alpha_1, \ldots, \alpha_{10} \) of \( E \) that in turn defines a Dynkin diagram of type \( E_{10} = T_{2,3,7} \). We recover \( \mathcal{D}_0 \) as \( \mathcal{D}_0 = \sum \mathbb{R}_{\geq 0} \omega_i \), where \( \omega_1, \ldots, \omega_{10} \) are the fundamental dominant weights defined by the root basis. That is, \( \omega_i.\alpha_j = \delta_{ij} \).

We label the simple roots \( \alpha_1, \ldots, \alpha_{10} \) according to the diagram

\[ \begin{array}{cccccccccccc}
\alpha_1 & \alpha_2 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 & \alpha_9 & \alpha_{10} \\
\alpha_3 & & & & & & & & \\
\end{array} \]

If \( f : Y \to S \) is a family of smooth Enriques surfaces, then the chambers in the Néron-Severi groups of the geometric fibres define a local system of sets on \( S \); that is, an étale covering \( \tilde{S}_f \to S \).
Definition 5.1 An E-marking of a family $Y \to S$ of Enriques surfaces is an isometry $\phi: E_S \to \text{Num}(Y/S)$. A D-polarization of $Y \to S$ is a choice of nef classes $[L_1], \ldots, [L_{10}]$ in $\text{Num}(Y/S)$ for which there is a marking $\phi$ with $\phi(\omega_i) = [L_i]$. Equivalently, a D-polarization of $Y \to S$ is a choice of section $\sigma$ of the covering $\tilde{S}_f \to S$ such that $\sigma(s)$ lies in the nef cone of every geometric fibre $f^{-1}(s)$.

Lemma 5.2 The datum of a D-polarization is equivalent to the datum of an E-marking $\phi$ such that $\phi(D_0)$ lies in the ample cone of $Y$. In particular, a D-polarization determines an E-marking.

Proof: The Dynkin diagram $E_{10}$ has no symmetries, so the same is true of the chamber $D_0$. That is, the stack $E_D$ of D-polarized Enriques surfaces is isomorphic to the stack of smooth Enriques surfaces $Y$ with an E-marking $\phi$ such that $\phi(D_0)$ lies in the nef cone. (To be precise, the morphisms in the latter stack are defined as follows: if $(Y, \phi: E \to \text{NS}(Y))$ and $(Z, \psi: E \to \text{NS}(Z))$ are objects, so that $\phi(E_0)$ and $\psi(E_0)$ lie in the relevant nef cones, then a morphism from $(Y, \phi)$ to $(Z, \psi)$ is an isomorphism $f: Y \to Z$ such that $f^* \circ \psi = \phi$.)

Liedtke [L10] considers the stack $E_{CV, l.b.}$ of RDP-Enriques surfaces $X$ that are equipped with a line bundle $L$, not just a polarization, such that

(i) $c_1(L)^2 = 4$,
(ii) $L$ is ample and
(iii) for every elliptic half-fibre $\Phi$ on the minimal resolution $\tilde{X}$ of $X$, we have $c_1(L).\Phi \geq 2$.

One of his main results is that, via the morphism sending an RDP-Enriques surface to its $\text{Pic}^r$, the stack $E_{CV, l.b.}$ is smooth over the stack of group schemes of order 2, so smooth over $\text{Spec} \mathbb{Z}[x, y]/(xy - 2)$. For us it is more convenient to consider the stack $E_{CV}$ of RDP-Enriques surfaces with an ample Cossec-Verra polarization rather than a line bundle. Since the forgetful morphism $E_{CV, l.b.} \to E_{CV}$ is a torsor under $\text{Pic}^r$, it follows that $E_{CV}$ is also smooth over $\text{Spec} \mathbb{Z}[x, y]/(xy - 2)$. Liedtke goes on to point out that this reduces the question of controlling the ramification in a characteristic zero lift of a smooth Enriques surface to controlling the ramification in a simultaneous resolution of RDPs (the existence of such a resolution, even in bad characteristics, was proved by Artin [Ar74a]).

Lemma 5.3 Suppose that $X$ is a smooth D-polarized Enriques surface. Then the weight $\omega_1$ is a Cossec-Verra polarization.

Proof: By assumption, each $\omega_i$ is nef. The calculation of each $\omega_1$ is simplified by first calculating the discriminant of $\alpha^+_1$ in the unimodular lattice $E$. In particular, it is easy to verify that

$$\omega_1 = 4\alpha_1 + 9\alpha_2 + 14\alpha_4 + 7\alpha_3 + 12\alpha_5 + 10\alpha_6 + 8\alpha_7 + 6\alpha_8 + 4\alpha_9 + 2\alpha_{10}$$
and that
\[ \varpi_{10} = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 + \alpha_9. \]

So \( \varpi_1^2 = 4, \varpi_{10}^2 = 0 \) and \( \varpi_1 \cdot \varpi_{10} = 2. \)

Now suppose that \( \Phi \) is an elliptic half-fibre. Then \( \varpi_1 \cdot \Phi \geq 1; \) suppose that \( \varpi_1 \cdot \Phi = 1. \) Note that also \( \Phi \cdot \varpi_{10} \geq 1. \) Then \( \varpi_1 \cdot (2\Phi + \varpi_{10}) = 4 \) and \( (2\Phi + \varpi_{10})^2 \geq 4; \) it follows from the index theorem that \( \varpi_1 = 2\Phi + \varpi_{10}, \) so that \( \varpi_1 - \varpi_{10} \) is even. But this contradicts the expressions above.

**Lemma 5.4** Every Cossec–Verra polarization \( H \) is \( O^+(E) \)-equivalent to \( \varpi_1. \)

**Proof:** \( H \) is equivalent to a C-V polarization in \( D, \) so we can assume that \( H \in D. \) Then \( H = \sum_1^{10} n_i \varpi_i \) with \( n_i \in \mathbb{N}, \) and \( H^2 = \sum n_i^2 \varpi_i^2. \) Since \( \varpi_{10}^2 = 0, \varpi_3^2 = 2 \) and \( \varpi_j^2 > 4 \) otherwise, the result follows at once.

There is an obvious forgetful morphism \( \mathcal{E}_D \to \mathcal{E}_{CV}: \) given a family \( X \to S \) of \( D \)-polarized Enriques surfaces, take the corresponding family \( X' \to S \) of RDP-Enriques surfaces that arises from the polarization \( \varpi_1. \) The singularities in the fibres of \( X' \to S \) are configurations whose root lattices embed in the root lattice \( D_9, \) since \( D_9 \) is what arises by deleting the vertex \( \alpha_1 \) from the diagram \( E_10. \) These are classified by Oshima \([Os]\).

Artin’s extension \([Ar74a]\) of Brieskorn’s construction \([Br68]\) of simultaneous resolution for deformations of RDPs in the complex case to all contexts (bad characteristic, mixed characteristic) came at the cost of losing some of the explicit information available in good characteristic, when the corresponding simple algebraic group tells us everything \([Br70], [Sl80], [SB01]\). Now 2 is not a good characteristic for RDPs of type \( D \) or \( E, \) so the next result is not an immediate consequence of the group theory. It depends much more upon Brieskorn’s construction.

Suppose that \( \mathcal{O}_S \to \mathcal{O}_T \) is a finite extension of mixed characteristic DVRs, \( S = \text{Spec} \mathcal{O}_S, T = \text{Spec} \mathcal{O}_T \) and \( s,t \) are the generic points of \( S,T. \)

**Proposition 5.5** Assume that \( X \to S \) is a 1-parameter deformation of an RDP \( (X_0, x) \) and that the absolute Galois group of the fraction field \( k(t) \) acts trivially on the Picard group of the geometric generic fibre \( X_t \) of \( X_T = X \times_S T \to T. \) Then \( X_T \to T \) has a simultaneous resolution, with no further base change.

**Proof:** By Artin’s result, there is a base change \( V \to T \) and a resolution \( \pi: \tilde{X}_V \to X_V. \) According to \([Br68], p. 257, \) there is a divisorial ideal \( J \) of \( X_V \) such that \( \pi \) is the blow-up of \( J. \) Now the restriction map from the class group \( \text{Cl}(X_V) \) of \( X_V \) to the Picard group \( \text{Pic}(X_v) \) of the generic fibre is an isomorphism (this holds for any 1-parameter deformation of a normal variety) and, by the assumption on the Galois action, the natural homomorphism \( \text{Pic}(X_t) \to \text{Pic}(X_v) \) is an isomorphism. Hence the natural homomorphism \( \text{Cl}(X_T) \to \text{Cl}(X_V) \) is an isomorphism.
According to Theorem 6.2 of [Sa64], this homomorphism is defined by \( I \mapsto I.\mathcal{O}_{X_V} \). Therefore there is a divisorial ideal \( I \) of \( \mathcal{O}_{X_T} \) such that \( I.\mathcal{O}_{X_V} \) is the product of \( J \) with a principal fractional ideal. Since \( X_V \to X_T \) is flat, we have \( I^n.\mathcal{O}_{X_V} = I^n \otimes \mathcal{O}_{X_V} \) for all \( n \), so that

\[
\widetilde{X}_V = \text{Bl}_J X_V = (\text{Bl}_I X_T) \times_T V.
\]

Since \( \widetilde{X}_V \to V \) is smooth, so is \( \text{Bl}_I X_T \to T \), and we are done. \( \square \)

**Corollary 5.6** (1) Every smooth Enriques surface \( X \) over an algebraically closed field \( k \) of characteristic 2 has a lifting over a DVR \( A \) that is a finite Galois extension of \( W(k)[\sqrt{2}] \) whose Galois group \( G \) is a subgroup of the Weyl group \( W(D_9) \).

(2) \( G \) is a semi-direct product \( G = H \rtimes C \), where \( H \) is a 2-group and \( C \) is a cyclic group of odd order.

**PROOF:** Choose a \( D \)-polarization on \( X \). This defines a Cossec-Verra polarization on \( X \), and so an RDP model \( X' \) that can, by Liedtke’s result, be lifted over \( W(k)[\sqrt{2}] \). The previous proposition on simultaneous resolution, combined with the fact that the stabilizer of \( \varpi_1 \) in the orthogonal group \( Q(E) \) is isomorphic to \( W(D_9) \), then gives (1). Standard results about local fields then give (2). \( \square \)

Recall that \( W(D_9) \) is a semi-direct product

\[
W(D_9) = (\mathbb{Z}/2)^8 \rtimes S_9.
\]

Let \( G' \) be the image of \( G \) in the quotient group \( S_9 \) of \( W(D_9) \). Then \( G' = H' \rtimes C' \), with \( H' \) a 2-group and \( C' \) cyclic of odd order. MAGMA reveals that there are just 171 conjugacy classes in \( S_9 \) of such subgroups, and their orders divide an element of the set

\[
\Sigma = \{2^63, \ 2^7, \ 2.3.5, \ 2^37, \ 2^35, \ 3^2\}.
\]

**Corollary 5.7** Every smooth Enriques surface over an algebraically closed field \( k \) of characteristic 2 can be lifted to a DVR of characteristic zero whose absolute ramification index divides \( 2^9N \), where \( N \in \Sigma \).

### 6 Periods, automorphisms and moduli for singular K3 surfaces.

Now move to K3 surfaces. Let \( M \) denote the lattice \( \bigoplus_1^{12} \mathbb{Z} e_i + \mathbb{Z} \frac{1}{2} \sum e_i \), where \( e_i.e_j = -2\delta_{ij} \) and \( N = M \oplus E(2) \). Fix a chamber \( \mathcal{D} \) in \( E(2) \otimes \mathbb{R} \) defined by the roots in \( E \). Note that \( N \) is 2-elementary (that is, \( 2N^\vee \subset N \), where \( N^\vee \) is the dual lattice), \( \sigma(N) = 10 \) and \( x^2 \equiv 0 \pmod{4} \) for all \( x \in 2N^* \). Put \( N_0 = 2N^\vee / 2N \). Then \( \dim_{\mathbb{F}_2} N_0 = 2\sigma(N) \).
Fix, in the positive cone \( N_+ \) of \( N \otimes \mathbb{R} \), a chamber \( C \) of the decomposition of \( N_+ \) defined by the roots. Fix also a chamber \( D \) in the decomposition of \( E(2) \otimes \mathbb{R} \) defined by the roots in \( E \) that lies in the closure of \( C \).

**Definition 6.1** An \( N \)-marked K3 surface over \( B \) is a pair \( (S \to B, \phi) \), where \( S \to B \) is a K3 surface and \( \phi: N_B \to \text{Pic}_{S/B} \) is an embedding.

**Definition 6.2** \((S \to B, \phi)\) as above is polarized if \( \phi_b(C) \) contains the ample cone of \( S_b \) for every geometric point \( b \).

**Definition 6.3** The period space \( \mathcal{M}_N \) is the \( k \)-variety classifying maximal totally isotropic (that is, of dimension \( \sigma(N) \)) subspaces \( U \) of \( N_0 \otimes_{\mathbb{F}_2} k \) such that \( \dim(U \cap F(U)) = \sigma(N) - 1 \). (Note that when we are dealing, as we do here, with quadratic spaces, a subspace is totally isotropic when the restriction to it of the quadratic function – and not just the associated scalar product – is zero.)

\( \mathcal{M}_N \) is smooth and projective over \( \mathbb{F}_2 \). \( \mathcal{M}_N \otimes k \) has two irreducible components, both unirational and 9-dimensional.

**Definition 6.4** The period point of an \( N \)-marked K3 surface \((X, \phi)\) is \( \ker(\phi \otimes 1: N \otimes k \to H^2_{\text{dR}}(X/k)) \). (This space is a point of \( \mathcal{M}_N \).)

A point \( s \) in \( \mathcal{M}_N \) determines a lattice \( N(s) \) with \( N \subset N(s) \subset N^* \), so a root system \( \Delta(s) \) and a decomposition of \( N_+ \) into chambers [Ogus, pp. 373–374].

For the rest of this section, we shall assume that \( Y \) is a unipotent Enriques surface over an algebraically closed field of characteristic 2 whose canonical double cover \( \rho: X \to Y \) is RDP-K3. Denote by \( \pi: \tilde{X} \to X \) the minimal resolution; it is well known that \( \tilde{X} \) is supersingular. Put \( \rho \circ \pi = \alpha \) and let \( \Gamma \) denote the exceptional locus of \( \pi \).

**Lemma 6.5** Suppose that \((Z, P)\) is a Zariski RDP of rank \( r \) with minimal resolution \( f: \tilde{Z} \to Z \) and that \( \xi \) is a 2-closed vector field on \( Z \) with \( \xi(P) \neq 0 \). Denote by \( \mathcal{F} \) the 1-foliation on \( \tilde{Z} \) that is generically generated by \( \xi \). Then \( \mathcal{F} \) is smooth and \( c_1(\mathcal{F}) = -A \), where \( A \) is given by the following diagram. Moreover, if \( L \) is the root lattice corresponding to \((Z, P)\), then there is an embedding \( A_1^r \hookrightarrow L \) such that the sum of the positive roots is \( A \).

\[ \begin{array}{c}
A_1:
\bullet \\
1
\end{array} \]

\[ \begin{array}{c}
D_{2n}:
2 \quad 2 \quad 4 \quad 4 \quad 6 \quad \ldots \quad 2n-2 \quad 2n-2
\end{array} \]

\[ \begin{array}{c}
E_7:
\bullet \\
\end{array} \]
PROOF: Let \( \rho : Z \to (W,Q) = Z/\xi \) be the quotient, so that \( W \) is a smooth germ. Then there is a sequence \( g : \tilde{W} \to W \) of \( r \) blow-ups and a commutative diagram

\[
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{\tilde{\rho}} & \tilde{W} \\
\downarrow f & & \downarrow g \\
Z & \xrightarrow{\rho} & W
\end{array}
\]

where \( \tilde{\rho} \) is the quotient by \( F \). Then \( F \) is smooth and for every exceptional curve \( E \) on \( \tilde{Z} \), either \( F \) is tangent to \( E \), in which case there is an isomorphism \( F|_E \to T_E \) and \( \tilde{\rho}(E) \) is a \((-1)\)-curve, or \( F \) is normal to \( E \), there is an isomorphism \( F|_E \to N_E/\tilde{Z} \) and \( \tilde{\rho}(E) \) is a \((-4)\)-curve. Since the geometry of \( \tilde{W} \) is determined by \( Z \), the intersection numbers \( c_1(F).E \) are now determined. Define \( A \) to be the cycle in the diagram above. It is then trivial to check that \( -A.E = c_1(F).E \) for every \( E \), and so \( c_1(F) = -A \).

Finally, \(-\tilde{\rho}^* K_{\tilde{W}} = c_1(F)\). Since \( K_{\tilde{W}} \) is the sum of the \((-1)\)-classes in each of the blow-ups comprising \( g \), it follows that \(-c_1(F) \) is the sum of the positive roots in an embedding of \( A^1 \).

**Lemma 6.6** Suppose that \( \text{Pic}^r(Y) = G^r \). Then \( \pi^* \text{Pic}(Y) = \text{Pic}(X)^G \) and if \( G \) is infinitesimal it acts trivially on \( \text{Pic}(X) \) so that in that case \( \pi^* \text{Pic}(Y) = \text{Pic}(X) \).

**PROOF:** Since \( X \to Y \) is a \( G \)-torsor, there is a Hochschild-Serre spectral sequence

\[
E_2^{ij} = H^i(G, H^j(X, G_m)) \Rightarrow H^{i+j}(Y, G_m).
\]

As \( H^i(G, G_m) = 0 \) for \( i = 1, 2 \) (cf. \[Mu70\] Lemma 23.1 ii) this gives that \( \pi^* \text{Pic}(Y) = \text{Pic}(X)^G \). As \( H^1(X, \mathcal{O}_X) = 0 \), \( \text{Pic}(X) \) is étale and so \( G \) acts trivially on it when it is infinitesimal.

Each of the finitely many lattices \( L \) between \( N \) and \( N^\vee = M^\vee + E(2)^\vee \) determines a locally finite collection of walls in the positive cone \( C \) of \( N \otimes \mathbb{R} \). Say that \( L \) is *good* if \( L \cap E(2)^\vee \) contains no roots. Take the union, over the good lattices, of the corresponding sets of walls. This is still locally finite, and decomposes \( C \) and \( H \) into not necessarily congruent regions. Such a region is a *good subchamber* if it meets \( E \otimes \mathbb{R} \).
Lemma 6.7  The chamber $\mathcal{D}$ in $E \otimes \mathbb{R}$ lies in a unique good subchamber $\mathcal{R}$ of $N \otimes \mathbb{R}$.

PROOF: This is equivalent to the statement that, for every root $\delta$ in a good lattice, the wall $\delta^\perp$ does not meet the interior $\mathcal{D}^o$ of $\mathcal{D}$. So suppose $\delta^\perp$ meets $\mathcal{D}^o$. Choose $x$ in $\mathcal{D}^o$ with $x.\delta = 0$. Note that $x^2 > 0$. Write $\delta = \delta_1 + \delta_2$, with $\delta_1 \in M^\vee$ and $\delta_2 \in E(2)^\vee \cong E(\frac{1}{2})$. Then $\delta_2 \neq 0$ and $\delta_1.\delta_2 = x.\delta_1 = x.\delta_2 = 0$, so that $\delta_2 \neq 0$ and $\delta_2^2 \leq 0$. Also, $\delta_2^2 \in \mathbb{Z}$, from the nature of $E(\frac{1}{2})$, so that either

(1)  $\delta_1^2 = \delta_2^2 = -1$ or

(2)  $\delta_1 = 0$ and $\delta_2^2 = -2$.

In case 1), $2\delta_2 = \eta$, say, is a root in $E$, so that by construction $\mathcal{D}^o$ is disjoint from $\eta^\perp$, and so from $\delta^\perp$. In case 2) $\delta = \delta_2$, and $L$ is not good. □

From now on, we fix this good subchamber $\mathcal{R}$. An $\mathcal{R}$-polarization on a K3 surface $S \to B$ is an $N$-marking $\psi : N_B \to NS_{S/B}$ such that $\psi_b(\mathcal{R}^o)$ meets the ample cone in $NS_{S_b}$ for every geometric point $b$ of $B$.

Lemma 6.8  (1) If $NS_{S_b}$, regarded as a lattice between $N$ and $N^\vee$, is good, then $\phi_b(\mathcal{R}^o)$ meets the ample cone in $NS_{S_b}$ if and only if it is contained in it.

(2) $NS_{S_b}$ is good if and only if the period point of $S_b$ lies in the open subscheme $\mathcal{M}^o_{N_b}$ (see p. 46).

Proposition 6.9  Fix a $\mathcal{D}$-polarization $\phi$ of $Y$. Then $\phi$ extends to an $\mathcal{R}$-polarization $\psi$ of $\widetilde{X}$. Moreover, $\alpha^*NS(Y)$ is a saturated sublattice of $NS(\tilde{X})$ and the image of the sublattice $\bigoplus \mathbb{Z}e_i$ of $M$ is the sublattice of the lattice generated by the exceptional $(-2)$-curves given by 6.5.

PROOF: There is a commutative diagram

$\tilde{X} \xrightarrow{\pi} X$

$\tilde{Y} \xrightarrow{\tilde{\rho}} Y$

where $\tau$ is the composite of 12 blow-ups and $\tilde{\rho}$ is the quotient by a 1-foliation $\mathcal{F} \hookrightarrow T_{\tilde{X}}$. Suppose that $C_i$ is the class in Pic($\tilde{Y}$) of the exceptional curve of the $i$th blow-up and put $\tilde{\rho}^*C_i = D_i$. Define $\psi_1 : \bigoplus \mathbb{Z}e_i \oplus E(2) \to NS(\tilde{X})$ by $\psi_1(e_i) = D_i$ and $\psi_1(x) = \alpha^*\phi(x)$ for $x \in E$. Next, note that $-c_1(\mathcal{F}) = \sum D_i$, so is the sum of the positive roots in $12 \times A_1$ embedded in the sum of the root lattices generated by $\Gamma$ as in 6.5. Since $c_1(\mathcal{F})$ is even in $NS(\tilde{X})$, by 3.14, $\psi_1$ extends to $\psi : N \to NS(\tilde{X})$. That $\psi$ is an $\mathcal{R}$-polarization follows from the fact that $\phi(\mathcal{D})$ meets the ample cone of $Y$.

As $\pi$ is a resolution of RDPs we know that $\pi^*NS(X)$ is the orthogonal complement of the exceptional set and as such it is saturated. Hence to show
that $\alpha^*NS(Y)$ is saturated it is enough to show that $NS(X) = \rho^*NS(Y)$ but this follows directly from 6.6.

We have seen that the period point of a K3 surface $\tilde{X}$ as above has the particular property that for any of the $N$-markings obtained by choosing an embedding of $M$ into the saturation of the lattice generated by the exceptional $(-2)$-curves the $E(2)$-part is saturated. In the following we will only use that there are no $-2$-curves in the saturation. The final conclusion then implies that this implies that there is no even lattice in the saturation. As a motivation for the considerations to follow we prove this directly.

**Lemma 6.10** Any even lattice containing $E(2)$ as a proper sublattice of finite index is generated by $E_2$ and $(-2)$-vectors.

**PROOF:** It is enough to prove that such a lattice that contains $E(2)$ as a sublattice of index 2 contains a $(-2)$-vectors. However, the group of orthogonal transformations of $E$ maps onto the orthogonal group of $E/2E$ (cf. [Ni80], Thm. 1.14.2) so it is enough to find one such lattice containing a $(-2)$-vector but that is obvious.

**Lemma 6.11** The locus $D_1$ in $\mathcal{M}_N$ whose points correspond to $N$–marked K3 surfaces where there is a root in the saturation of $\phi(E(2))$ is a divisor.

**PROOF:** Suppose that $\phi: N \to NS_S$ is an $N$-marking of $S$. Then $S$ corresponds to a point of $D_1$ if and only if there is a root $\delta$ in the saturation $\tilde{E}$ of $\phi(E)$. Since $NS_S$ is 2-elementary, we have $\phi(E) \subset \tilde{E} \subset (\phi(E))^\vee = \frac{1}{2}\phi(E)$, so that $2\delta \in \phi(E)$.

Put $E_1 = E + \mathbb{Z}.\phi^{-1}(\delta)$; then $\text{disc}(E_1) = -2^8$ and $E_1$ is 2–elementary, so is unique up to isomorphism. Moreover, there are only finitely many copies of $E_1$ between $E$ and $E^\vee$, so only finitely many copies of $F$ inside $E_1$. Since $S$ is a point in $D_1$ if and only if $\phi$ extends to $\phi_1: M \oplus E_1 \to NS_S$, we have identified $D_1$ with the union of a finite number of copies of the period space of $(M \oplus E_1)$–marked K3 surfaces.

Put $\mathcal{M}_N^0 = \mathcal{M}_N - D_1$, $B_N^0 = \pi^{-1}(\mathcal{M}_N^0)$ and let $S^0 \to B_N^0$ be the induced universal family.

Recall that a morphism of finite type is *almost proper* if it satisfies the surjectivity part of the valuative criterion for properness.

Recall that $\varpi_1, \ldots, \varpi_{10}$ denote the fundamental dominant weights for the labelling of the nodes of the Dynkin diagram $E_{10}$ fixed earlier; they generate the cone $\mathcal{D}$.

The next result is Ogus’ global Torelli theorem for supersingular K3 surfaces, adapted very slightly for our purposes.

**Theorem 6.12** (1) There is an algebraic space $B_N$ and a universal $\mathcal{R}$–polarized $N$–marked K3 surface $S \to B_N$.

(2) The period map $\pi: B_N \to \mathcal{M}_N$ is étale, surjective, of degree one and almost proper.
(3) $B_N^0 \to M_N^0$ is an isomorphism.

(4) There is a contraction $S^0 := S \mid_{B_N^0} \to X \to B_N^0$ of the induced family $f: S^0 \to B_N^0$ such that for each geometric $b \in B_N^0$ the exceptional locus of $S^0 = S_b \to X_b$ is the unique rank 12 configuration $\Gamma_b$ of $(-2)$–curves lying in $\phi_b(E(2))$. In particular, the configuration of RDPs on $X_b$ has rank 12.

(5) There are line bundles $A_1, \ldots, A_{10}$ on $X$ such that $c_1(g^* A_i) = \omega_i$.

PROOF: (1) is a matter of observing that the stack of $R$–polarized $N$–marked K3 surfaces is algebraic, as follows from the criteria of [Ar74b], and that the automorphism group schemes of these objects are trivial.

(2) is a very slight variant of Ogus’ global Torelli theorem, with the additional datum of an $R$–polarization. His argument shows, in fact, that $\pi$ is an isomorphism over the locus where $\phi_b(R^0)$ lies in the ample cone, which is (3).

For (4) and (5), note that $\phi_b(R)$ lies in the nef cone of $S_b$ for every $b \in B_N^0$. So the same holds for $\phi_b(D)$. Then the line bundles $L_i = \phi(\omega_i)$ on $S^0$ define the contraction $g: S^0 \to X$; that is, there are line bundles $A_1, \ldots, A_{10}$ on $X$ with $g^* A_i = L_i$.

On $S_b$ the exceptional locus of $g$ is the union of the $(-2)$–curves orthogonal to every $L_i$. These are clearly the curves whose classes span the saturation in $NS_{S_b}$ of the sublattice $\phi_b(M)$.

We shall be careless in distinguishing between $B_N^0$ and $M_N^0$.

**Definition 6.13** Suppose that $Y \to S$ is a family of RDP–K3 surfaces whose geometric fibres are of RDP index 12. Then a $D$–polarization of $Y \to S$ is an embedding $\phi: E(2)_S \to \text{Pic}_{Y/S}$ such that $\phi_s(D)$ lies in the ample cone of $Y_s$ for all geometric $s \in S$.

So, for example, the line bundles $A_i$ provide a $D$–polarization of $q: X \to B_N^0$.

Fix a geometric point $b \in B_N^0$.

**Lemma 6.14** The connected components of $\Gamma_b$ are of type $A_1, D_{2n}, E_7$ and $E_8$.

PROOF: Let $\Lambda_b$ be the $\mathbb{Z}$–span of $\Gamma_b$. Then $\Lambda_b$ is 2–elementary and the result follows.

Define $\Gamma_b$ to be *even* if the divisor $\phi_b(\sum e_i)$ is divisible by 2 as a divisor, not just as a divisor class.

**Lemma 6.15** If there is an even set of $r$ disjoint $(-2)$–curves on $S$, then $r \equiv 0 \pmod{4}$ and $r \geq 8$.

PROOF: If $\sum E_i \sim 2M$, then $-2r = 4M^2 \equiv 0 \pmod{8}$. If $r = 4$, then $M^2 = -2$, so that $\pm M$ is effective. This is absurd.

**Corollary 6.16** If $\Gamma$ is odd, then it is one of $12 \times A_1$, $8 \times A_1 + D_4$, $6 \times A_1 + D_6$, $5 \times A_1 + E_7$. If $\Gamma$ is even, then it is one of $3 \times D_4$, $D_4 + D_8$, $D_4 + E_8$, $D_{12}$.

PROOF: Combine 6.14, 6.15 and 6.5.
Lemma 6.17 (1) The set \( \{e_1, \ldots, e_{12}\} \) can be split into disjoint subsets, one for each connected component of \( \Gamma_b \), such that the sum of the roots in the subset is the vector described in Lemma 6.5.

(2) \( \Gamma_b \) is even if and only if its connected components are of type \( D_{4n} \) or \( E_8 \).

Definition 6.18 \( X_b \) has even singularities if the configuration \( \Gamma_b \) on \( S_b \) is even.

Lemma 6.19 For a generic \( N \)-marked K3 surface \( S \), the configuration \( \Gamma \) given by Proposition 6.12 is \( 12 \times A_1 \).

PROOF: We have \( \text{disc}(NS_S) = -2^{20} = \text{disc}(N) \), so \( N = NS_S \). If \( \Gamma \neq 12 \times A_1 \), then \( |\text{disc}(Z, \Gamma)| \leq 2^{8-2} \times 2^2 \) (equality being achieved when \( \Gamma = 8 \times A_1 + D_4 \), in which case the \( 8 \times A_1 \) configuration is even), which gives \( |\text{disc}(NS_S)| \leq 2^{18} \).

Corollary 6.20 (1) Every geometric fibre of \( q: X \to \mathcal{M}_{0, N} \) has Zariski RDPs and its tangent bundle has trivial Chern classes.

(2) For each singular point in a fibre, the induced deformation is a Zariski deformation.

(3) \( T^1_{X/B^0_N} \) is \( \mathcal{O}_{B^0_N} \)-cohomologically flat and \( T^1_{X/B^0_N} \) is a locally free \( \mathcal{O}_{B^0_N} \)-module of rank 24.

PROOF: (1) and (2) follow from 3.20 and 3.11.

To verify (3) it is enough, by 3.16, to show that \( T^1_{X/B^0_N} \) is flat of rank 24. As \( B^0_N \) is reduced it is enough to show that the dimension of \( T^1 \) is 24 for each fibre. However, for a Zariski RDP this dimension is twice the index and the sum of all the indices is 12. \( \square \)

Proposition 6.21 The locus \( V \) in \( \mathcal{M}_{0, N} \) corresponding to surfaces with even singularities is irreducible and of codimension 2.

PROOF: There is a unique embedding \( M \hookrightarrow D_4^{\oplus 3} \), and the closure of \( V \) in \( \mathcal{M}_N \) is then identified with the period space \( \mathcal{M}_{(D_4^{\oplus 3} \oplus E)} \). This is 7-dimensional. \( \square \)

We shall refer to \( V \) as the \( 3 \times D_4 \) locus.

Theorem 6.22 \( T_X \) is free for every fibre \( X \) of \( q \).

PROOF: The idea is to show that the locus \( D_2 \) of surfaces \( X \) such that \( T_X \) is not free is a divisor in \( \mathcal{M}_{0, N} \) and is empty in codimension 1.

First, \( D_2 \) is not all of \( \mathcal{M}_{0, N} \), since, for example, there are canonical double covers of Enriques surfaces giving points in \( \mathcal{M}_{0, N} - D_2 \). The fact that \( D_2 \) is everywhere of codimension 1 is 3.15 1 and proposition 6.12.

Lemma 6.23 The locus in \( B^0_N \) (or, equivalently, \( \mathcal{M}_{0, N} \)) where \( \Gamma \) is of type neither \( 12 \times A_1 \) nor \( 8 \times A_1 + D_4 \) is everywhere of codimension \( \geq 2 \).

PROOF: If \( \Gamma \neq 12 \times A_1, 8 \times A_1 + D_4 \), then \( |\text{disc}(NS_S)| \leq 2^{16} \). \( \square \)
So it is enough to prove 6.22 in the two cases $\Gamma = 12 \times A_1, 8 \times A_1 + D_4$. We assume that $T_X$ is not free and that $\pi: \tilde{X} \rightarrow X$ is the minimal resolution, so that, by 3.20, $\pi^*T_X$ has a destabilizing subsheaf, and deduce that there is a $(-2)$-curve in the smooth locus of $X$. The existence of this curve means that the period point of $\tilde{X}$ lies in the forbidden divisor $D_1$.

**Case (1):** $\Gamma = 12 \times A_1$. Let $E_1, \ldots, E_{12}$ be the $(-2)$-curves in $\Gamma$, so that $\sum E_i$ is even. Then the existence of the corresponding square root fibration gives a 2-integrable 1-foliation $\mathcal{F} = \mathcal{O}(B - \sum E_i)$ in $T_{\tilde{X}}$, where $B \geq 0$ and $B$ is disjoint from $\sum E_i$. By comparing $\mathcal{F}$ with the destabilizing subsheaf of $\pi^*T_X$, we see that $B > 0$. We also know (Lemma 3.14) that $c_1(B)$ is even in $\text{NS}_{\tilde{X}}$; say $B \sim 2C$.

Let $\rho: \tilde{X} \rightarrow Y = \tilde{X}/\mathcal{F}$ be the quotient and $\alpha: Y \rightarrow Y$ the minimal resolution. Then, arguing as in §1, $\tilde{Y}$ is rational and $Y$ has only RDPs, of total index $r$, say. Computing $c_2(\tilde{X})$ gives $24 = -(B - \sum E_i)^2 + r$, while $\rho^*K_Y \sim -B + \sum E_i$. Then Noether’s formula applied to $\tilde{Y}$ gives $12 = \frac{3}{2}(B - \sum E_i)^2 + 24 + r$. Hence $r = 0$ (and $B^2 = 0$), so that $Y$ is smooth. Put $\rho(E_i) = F_i$, so that $F_i$ is a $(-1)$-curve on $Y$ and $\rho^*F_i = E_i$.

There is a commutative diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\rho} & Y \\
\pi \downarrow & & \downarrow r \\
X & \xrightarrow{\rho_1} & Y_1
\end{array}
$$

where $\tau: Y \rightarrow Y_1$ contracts the curves $F_i$ to points $P_i$. Note that any curve $C$ on $Y$ with $C^2 < 0$ has $C^2 = -1$ or $C^2 = -4$, according as $\rho^*C = D$ or $\rho^*C = 2D$.

Suppose that $l_1$ is a $(-1)$-curve on $Y_1$, with strict transform $l$ on $Y$. If $l^2 = -4$, then $\rho^*l = 2m$ and $\tau^*l_1 = l + \sum_{i}^3 F_i$, so that $l.\sum_{i}^{12} F_i = 3$. Hence $m.\sum E_i = 3$. But $\sum E_i$ is even, so that $l^2 = -1$. Then $\rho^*l$ is the $(-2)$-curve we sought.

**Case (2):** $\Gamma = 8 \times A_1 + D_4$. Then, if $E_1, \ldots, E_8$ form the $8 \times A_1$ configuration, $\sum E_i$ is even. Suppose that $\sum G_i$ is the $D_4$-configuration, with $G_1$ the central curve. As in (1), there is a 2-integrable 1-foliation $\mathcal{F} = \mathcal{O}(B - \sum E_i)$ with $B \geq 0$ and $B \cap \sum E_i = \emptyset$, and comparison with a destabilizing subsequence of $\pi^*T_X$ shows that $B.\pi^*H > 0$ for ample $H \in \text{Pic } X$, so that $B > 0$. A calculation similar to that in (1) shows that $B^2 = -8$ and $\tilde{X}/\mathcal{F} = Y$ is smooth. Moreover, $B$ is even; say $B \sim 2C$. Then $C^2 = -2$ and $C.\pi^*H > 0$, so that, by Riemann-Roch, $C$ is effective. Moreover, $\text{Supp } C$ is not contained in $\sum G_i$.

Consider the commutative diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\rho} & Y \xrightarrow{\sigma} \tilde{X}^{(1)} \\
\phi \downarrow & & \downarrow \phi^{(1)} \\
X_2 & \xrightarrow{\rho_2} & Y_2 \xrightarrow{\sigma_2} X_2^{(1)}
\end{array}
$$

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where $\phi$ contracts the $E_i$ and $\psi$ contracts their images $F_i = \rho(E_i)$ to points $P_i$. So $Y_2$ is a smooth rational surface and $\rho^*_2 c_1(Y_2) \sim B \sim 2C$. So $-K_{Y_2} \sim \sigma_2^* \mathcal{C}(1)$ and is thus effective. Let $H_i$ denote both $\rho(G_i)$ and its image in $Y_2$.

**Lemma 6.24** Suppose that $D$ is an effective anti-canonical divisor on a smooth rational surface.

1. $D$ is connected.
2. Either $D$ is reduced and $h^1(\mathcal{O}_D) = 1$ or $D_{\text{red}}$ is a tree of $\mathbb{P}^1$'s.
3. If $E \leq D$ with $h^1(\mathcal{O}_E) = 1$ and $E$ is reduced, then $E = D$.

**PROOF:** This is easy and well known.

As in Case (1), no $(-1)$-curve on $Y_2$ passes through any $P_i$.

**Lemma 6.25** $C.G_i = \pm 1$.

**PROOF:** $C.G_i = 1$ if and only if $\mathcal{F}$ is tangent to $G_i$ and $C.G_i \leq -1$ otherwise. If $C.G_i \leq -2$, then the non-zero map $\mathcal{F}|_{G_i} \to \mathcal{N}_{G_i/Y_2}$ is not an isomorphism, so that $H_i$ is singular. Also, $G_i \leq C$, so that there exists $H \in |-K_{Y_2}|$ with $H \geq H_i$. Then $H = H_i$, so that $C$ is supported on $\sum G_i$.

**Lemma 6.26** $C.(G_1 + G_i) = 0$ for $i \neq 1$.

**PROOF:** Suppose that $C.G_1 = C.G_2 = -1$. Then $G_1, G_2 \leq C$ and $H_1, H_2 = 2$. So there exists $H \in |-K_{Y_2}|$ with $H \geq H_1 + H_2$, so that $H = H_1 + H_2$ and again $C$ is supported on $\sum G_i$.

So there are two possibilities for the configuration of intersection numbers $C.G_i$.

**Subcase (2.1):** $C.G_1 = -1$ and $C.G_i = 1$ for $i \neq 1$. Then $\sum H_i$ contracts to a smooth point, say via $Y_2 \to Y_1$. Suppose that $l_1$ is a $(-1)$-curve on $Y_1$ and $l$ its strict transform on $Y$. If $l^2 = -1$, then $\rho^* l$ is the $(-2)$-curve we sought. So suppose that $l^2 = -4$. Then, as in Case (1), $l. \sum F_i$ is even. It is then easy to verify that $l$ meets just two of the $F_i$, say $F_1, F_2$, and that $l.H_1 = 1, l.H_i = 0$ for $i \neq 1$. Moreover, $\rho^* H_1 = 2G_1$ and $\rho^* l = 2m$, so that $4m.G_1 = 2$, which is absurd.

**Subcase (2.2):** $C.G_1 = 1$ and $C.G_i = -1$ for $i \neq 1$. Then $G_i \leq C$ for $i \neq 1$ and $H_i$ is a $(-1)$-curve. Let $Y_1 \to Y'_1$ be the contraction of $H_1$ and $H'_i$ the image of $H_i$ in $Y'_1$. Then there exists $H \in |-K_{Y_2}|$ with $H \geq \sum H'_i$, so that $H = \sum H'_i$ and $C$ is supported on $\sum G_i$. This contradiction completes the proof of 6.22.

**Corollary 6.27** The formation of $T_{X/\mathcal{M}_N^0}$ commutes with base change and $q_* T_{X/\mathcal{M}_N^0}$ is locally free of rank 2, its formation commute with base change and the map $q^* q_* T_{X/\mathcal{M}_N^0} \to T_{X/\mathcal{M}_N^0}$ is an isomorphism.

**PROOF:** The first part is 6.20 and the rest 3.15.
7 Some automorphism group schemes.

Here we make some calculations concerning automorphism group schemes of certain unipotent Enriques surfaces and RDP-K3 surfaces. These will be useful in the analysis in §8 of the relationships between various stacks of surfaces.

We fix the following notation to be used in this section.

**Notation:** The base is an algebraically closed field $k$ of characteristic 2. For a group scheme $H$, denote by $\text{Exp}^{n}(H)$ the kernel of $\text{Frob}^{n}: H \rightarrow H^{(n)}$ (the height $n$ part of $H$). Denote by $\text{Aff}$ the group of affine transformations of the affine line. $(S, \psi)$ will be an $\mathcal{R}$-polarized $N$-marked K3 surface whose period point lies in $\mathcal{M}_{S}^{0}$ and $\pi: S \rightarrow S_{1}$ the contraction of the configuration $\Gamma$ given by Proposition 6.12. Then $T_{S_{1}}$ is free, so that $H^{0}(T_{S_{1}})$ is a 2-dimensional 2-Lie algebra $\mathfrak{g}$; it is the Lie algebra of $G = G_{S_{1}} := \text{1}(\text{Aut}_{S_{1}})$.

Recall that there is a bijection between connected height 1 groupschemes $H$ over $k$ and 2-Lie algebras $\mathfrak{h}$ over $k$; if $\mathfrak{h} = \text{Lie}(H)$, then we write $H = \text{Exp}(\mathfrak{h})$.

**Proposition 7.1** (1) Suppose that $\xi \in \mathfrak{g}$ is 2-closed and that $Y = S_{1}/\xi$ is smooth. Then $Y$ is an Enriques surface and $\rho: S_{1} \rightarrow Y$ is the canonical double cover.

(2) There is a unique $\mathcal{D}$-polarization $\phi: E \rightarrow \text{NS}(Y)$ such that $\rho^{*}\phi: E(2) \rightarrow \text{NS}(S)$ extends to $\psi$ and $\phi(\mathcal{D})$ lies in $\mathcal{R}$.

**Proof:** (1) $V$ generates a copy of $\mathcal{O}_{S_{1}}$ in $T_{S_{1}}$, so that $\rho^{*}c_{1}(Y) \sim 0$. Then $2K_{Y} \sim 0$ and $b_{2}(Y) = 10$, so that $Y$ is Enriques. Since $\text{Pic}_{S_{1}} = 0$, $\rho$ factors through the canonical double cover $Z$ of $Y$. So $S_{1}$ is the normalization of $Z$. Since $\omega_{Z}$ and $\omega_{S_{1}}$ are trivial the conductor is empty and $S_{1} \rightarrow Z$ is an isomorphism.

(2) Since $\psi(E(2))$ is orthogonal to the exceptional curves of $\pi$ it is contained in $\rho^{*}(\text{NS}(Y))$. Then $\psi|_{E(2)}$ is an isometry $\phi: E \rightarrow \text{NS}(Y)$. Moreover, $\phi(\mathcal{D})$ lies in $\psi(\mathcal{R})$, which meets the ample cone of $S$, so that $\phi(\mathcal{D})$ meets the ample cone of $Y$ and therefore lies in it.

**Proposition 7.2** Suppose that $(Y, \psi)$ is a unipotent $\mathcal{D}$-polarized Enriques surface and that its canonical double cover $X$ is RDP-K3 with minimal resolution $\pi: \tilde{X} \rightarrow X$. Then $X \cong S_{1}$ for some $S_{1}$ as above.

**Proof:** This is just 6.9

We know that $Y \cong X/\mathcal{F}$ for some smooth 1-foliation $\mathcal{F}$, that $X \cong S_{1}$ for some $S_{1}$ as described, and that $c_{1}(\mathcal{F})$ is numerically trivial in codimension one. It follows at once that $\mathcal{F}$ is globally generated by a 2-closed vector field.

**Corollary 7.3** Every vector in $\mathfrak{g}$ is 2-closed.

**Proof:** We know that an Enriques surface has at least 10 moduli (that is, every component of the coarse moduli space is at least 10-dimensional), while a supersingular K3 surface has 9 moduli. Hence there is at least one degree of freedom in the choice of 2-closed line in $H^{0}(T_{S_{1}})$. 

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Lemma 7.4  $G$ is either $\text{Aff}_1$ or $\alpha_2 \times \alpha_2$.

PROOF: We shall solve the equivalent problem of determining $g$.

Suppose first that $G$ is not commutative. Let $H$ denote the derived subgroup scheme $[G, G]$. Then $g$ has a basis $\{\xi, \eta\}$ such that $\xi$ generates Lie $H$ and $[\xi, \eta] = \xi$. Rescale $\xi$ so that $\xi^2 = \xi$ or 0. Say $\eta^2 = b\eta$. Then, since $(\lambda \xi + \eta)^2$ is proportional to $\lambda \xi + \eta$ for all $\lambda \in k$, it follows at once that $\xi^2 = 0$. Then rescale $\eta$ so that $\eta^2 = \eta$ or 0. In the first case $G \cong \text{Aff}_1$ and in the second the proportionality of $(\lambda \xi + \eta)^2$ and $\lambda \xi + \eta$ for all $\lambda \in k$ gives a contradiction.

Now suppose that $G$ is commutative. Then pick a basis $\{\xi, \eta\}$ and consider $(\lambda \xi + \eta)^2$, as before. It is immediate that $\xi^2 = \eta^2 = 0$, so that $G \cong \alpha_2 \times \alpha_2$.

Remark: The list of all connected height 1 group schemes of order 4 is rather longer. It includes a non-commutative family parametrized by $\mathbb{P}^1$, of which $\text{Aff}_1$ is one member, whose general member has Lie algebra generated by $\xi, \eta$, with $\xi^2 = 0$, $[\xi, \eta] = \xi$ and $\eta^2 = a\xi + \eta$.

Lemma 7.5  If $0 \neq \xi \in H^0(T_{S_1})$ and $\xi^2 = 0$, then $\xi$ does not vanish at any $A_1$ singularity of $S_1$.

PROOF: At an $A_1$ singularity $P$, $S_1$ is locally given by an equation $xy + z^2 = 0$. Then $\{D_1 = x\partial_x + y\partial_y, D_2 = \partial_z\}$ is a local basis of $T_{S_1}$. Since $\xi$ is an element of a global basis, we can write $\xi = fD_1 + gD_2$, where $f, g \in \mathcal{O}_{S_1,P}$ and at least one of them is a unit. Assume that $\xi(P) = 0$; then $g \in m_P$, so that $f$ is a unit. Since $D_1^2 = D_1$ and $D_2^2 = 0$, taking the coefficient of $D_1$ in the equation $\xi^2 = 0$ gives

$$fD_1(f) + f + gD_2(f) = 0.$$

However, $D_1(f), g \in m_P$, which is absurd. □

Theorem 7.6  If $\Gamma$ is even, then $G \cong \alpha_2 \times \alpha_2$. Otherwise, $G \cong \text{Aff}_1$.

PROOF: Since $g$ is 2-dimensional, there is, for any singular point $P$ on $S_1$, a vector field $\xi \in g$ with $\xi(P) = 0$. If $\Gamma$ is odd, then $S_1$ has an $A_1$ singularity, and the result follows from 7.5.

For the converse, suppose that $G \cong \text{Aff}_1$. Denote the radical of $g$ by $\mathfrak{a}$ and put $Y^{(1)}_1 = S_1/\mathfrak{a}$. Then $Y \rightarrow S$ is a principal $\mu_2$-bundle over $S_1 - \text{Sing}(S_1) = S_0$, say. The exact sequence

$$0 \rightarrow \bigoplus_{i \in I} \mathbb{Z}.E_i \rightarrow \text{Pic}(S) \rightarrow \text{Pic}(S_0) \rightarrow 0$$

shows that then there is a subset $J \subset I$ such that $\sum_{i \in J} E_i$ is even. Since $\#J = 8$ or 12, by 6.15, it is easy to check that this is impossible for each of the even configurations listed in Corollary 6.16. So $\Gamma$ is odd. □
Proposition 7.7 A multiplicative $\alpha_2$-Enriques surface $X$ is $12A_1$-Enriques.

PROOF: The multiplicative vector field corresponds to an action of $\mu_2$ on $X$. We know by 3.18 that the fixed point scheme of such an action is smooth. We first want to show that this fixed point scheme is non-reduced at all of its codimension 1 points. We choose a Pic-rigidification of $X$ and employ the notations of Definition 4.7. As we have $D(f)\tilde{\beta}_2 = \eta \wedge df$ we see that the zero-scheme of $D$ equals that of $\eta$. On the other hand, $\eta$ comes from $H^0(X, B_1)$ so is locally of the form $df$. However, in characteristic 2 the divisorial part of the zero scheme of such a form is twice a divisor which can be seen either by make a calculation in coordinates at a point where the zero set of $f$ is smooth or by using [Ek88], Prop. 1.11. Hence the multiplicity of the codimension 1-part of the zero set of $D$ is even and as it is also smooth it must be empty.

We now note that if $\eta$ locally can be written $df$ then, over the same open subset, the canonical double cover can be written $\text{Spec} \mathcal{O}_X[z]/(z^2 - f)$ so that to begin with the canonical double cover has only isolated singularities. Furthermore, under a singularity, $df$ has a smooth zero scheme which means that it must have the form $\lambda + xy$ in suitable local coordinates and hence the double cover has an $A_1$-singularity and $X$ is $K3$-Enriques.

Define $S_1$ to be of type $12 \times A_1$ if it has 12 singularities, each of type $A_1$.

Theorem 7.8 $G = \text{Aut}_{S_1}^0$ if $S_1$ is of type $12 \times A_1$.

Remark: In Corollary 8.9 this will be extended to all surfaces $S_1$ arising from a K3 surface $S$ whose period point lies in $\mathcal{M}_N^0$.

PROOF: Put $A = \text{Aut}_{S_1}^0$ and assume the result false. So $G = 1 A \neq A$. Then $\text{Frob} : A \to A^{(1)}$ kills $G$ but not all of $A$. Then there is an order 2 subgroup scheme in the image; define $B$ to be its inverse image. Then $B$ is an order 8 subgroup scheme of $A$ with $G = 1B$ and $G$ normal in $B$, since $G$ is normal in $A$.

By 7.6, $G = 1 \text{Aff}$. Then the unique copy $\alpha$ of $\alpha_2$ in $G$ is normal in $B$. Conjugation gives a homomorphism $\phi : B \to \text{Hom}_{\text{gp-sch}}(\alpha, \alpha) \cong G_m$. Say $\ker \phi = H$. Then $H$ contains $\alpha$ but not $G$. So either $H = \alpha$ or $H$ has order 4. Let $\rho : S_1 \to S_1/\alpha = X$ denote the quotient map; then $X$ is smooth, by 7.5, and so is $K3$-Enriques with $B/\alpha \subset \text{Aut}_X^0$.

Suppose first that $H$ has order 4. Then $G, H$ are distinct normal order 2 subgroupschemes of $B$ and $G \cap H = \alpha$. Choose generators $\xi$, resp. $\eta$, of $\text{Lie}(G/\alpha)$, resp. $\text{Lie}(H/\alpha)$. Then $\text{ad}(\xi)(\eta) \in k\eta$ and $\text{ad}(\eta)(\xi) \in k\xi$. So $[\xi, \eta] = 0$, and we deduce that $B/\alpha = G/\alpha \times H/\alpha$. Then $\xi, \eta$ give linearly independent elements of $H^0(X, T_X)$, which is impossible.

Hence $H = \alpha$ and there is an exact sequence

$$1 \to \alpha \to B \to \mu_4 \to 1.$$ 

In particular, $\mu_4 \leftrightarrow \text{Aut}_X^0$. Moreover, the sequence splits, as follows. The adjoint representation gives an embedding $B \leftrightarrow GL_2$, and $B$ preserves a line, namely
Then (Noether) $\chi$ double cover is $S\mu$

**Lemma 7.9** If $\mu_2 \rightarrow \mu$ acts freely at a point $s \in S_1$, then so does $\mu$.

**PROOF:** Since $\mu_2$ is the unique proper subgroup scheme of $\mu$, Stab($s$) is trivial if it does not contain $\mu_2$.

**Lemma 7.10** $B$ acts freely on $S_1 \setminus \text{Sing}(S_1)$.

**PROOF:** $G$ acts freely on $S_1 \setminus \text{Sing}(S_1)$ and meets every non-trivial subgroup scheme of $B$ non-trivially.

Now let $\rho : S_1 \rightarrow Y$ denote the quotient by $\mu_2$ and $\sigma : Y \rightarrow Z$ the quotient by $\mu/\mu_2$. Put $\rho(P_i) = Q_i$ and $\sigma(Q_i) = R_i$.

**Lemma 7.11** If $\mu_2$ acts freely at $P_i$, then $Y$ is smooth at $Q_i$ and $Z$ is smooth at $R_i$.

**PROOF:** Same arguments as before.

**Notation:** We let $[a_1, \ldots, a_n]$ denote a chain of $n$ transverse smooth rational curves with self-intersections successively $a_1, \ldots, a_n$.

**Lemma 7.12** If $\mu_2$ fixes $P_i$, then the minimal resolution of $(Y, Q_i)$ is either $[-2, -2, -2]$, in which case $(Y, Q_i)$ is an $A_3$ singularity, or $[-4]$.

**PROOF:** Since $\mu_2$ fixes $P_i$ it acts on its blow-up, which is the minimal resolution. Let $E$ denote the exceptional curve. Either $\mu_2$ acts trivially on $E$, giving the first case, or it acts with just two fixed points, which gives the second.

Hence $\text{Sing} Y$ consists of $a$ of the $[-4]$ singularities and $b$ of the $A_3$ singularities, where $a + b \leq 12$. Let $f : \tilde{Y} \rightarrow Y$ be the minimal resolution. Recall that $K_Y$ is numerically trivial and that the irregularity $q(\tilde{Y}) = 0$.

Suppose that $a = 0$. Then $\tilde{Y}$ is $K3$ or Enriques and $c_2(\tilde{Y}) = 12 + 3b$. So $b = 0$ or 4. If $b = 0$, then $Y$ is a smooth $\mathbb{Z}/2$-Enriques surface whose canonical double cover is $S_1$. So $H^0(T_Y) = 0$. However, $\mu_4/\mu_2$ acts on $Y$, and so $b = 4$. Then (Noether) $\chi(O_Y) = 2$, while $Y$ can be deformed, by varying a vector field on $S_1$, to a smooth Enriques surface with $\chi(O) = 1$. (Note that this relies upon the linear reductivity of $\mu_2$, which ensures that taking invariants commutes with specialization.)

Hence $a > 0$. Then $\tilde{Y}$ is rational, $K^2_{\tilde{Y}} = -a$ and $c_2(\tilde{Y}) = 12 + a + 3b$. So $b = 0$.

**Lemma 7.13** The action of $\mu_4/\mu_2$ on $Y$ lifts to $\tilde{Y}$.

**PROOF:** Pick a singular point $Q_i$ on $Y$ and put $f^{-1}(Q_i) = E$. The action of $\mu_4/\mu_2$ on $Y$ defines a 2-closed 1-foliation $\mathcal{F}$ on $\tilde{Y}$, with $\mathcal{F} \cong \mathcal{O}(aE)$ near $E$. 
Comparing the induced map $\mathcal{F}|_E \rightarrow T_{\tilde{Y}}|_E$ with the normal bundle sequence of the embedding $E \hookrightarrow \tilde{Y}$ shows that $a \geq 0$.

Recall that $\mu_4/\mu_2$ acts freely on $Y \setminus \text{Sing} Y$. The fixed locus of the action on $\tilde{Y}$ is smooth, so near an exceptional curve $E$ consists either of $E$ or of two points on $E$. It is then easy to see that if $\tilde{Z} \rightarrow Z = Y/(\mu_4/\mu_2)$ is the minimal resolution, the exceptional locus in $\tilde{Z}$ is $c \times [-2, -3, -2] + d \times [-8]$ with $c + d = a$. Moreover, $K_Z$ is numerically trivial, so that $\tilde{Z}$ is rational, $K_{\tilde{Z}}^2 = -c/2 - 9d/2$ and $c_2(\tilde{Z}) = 12 + 3c + d$. Then $5c = 7d$, so that $c = 7$ and $d = 5$. Then $r = a = 12$, so that $\text{Sing} S_1 = 12 \times A_1$ and the $\mu_2$-action fixes every singular point. Then the $\mu_2$-action lifts to the minimal resolution of $S_1$, which contradicts the Rudakov-Shafarevich theorem.

**Proposition 7.14** Suppose that $\Xi$ is a line in $\mathfrak{g}$. Put $Y = S_1/\Xi$. Then the kernel $A$ of the natural homomorphism $\text{Aut}_{(X, \Xi)} \rightarrow \text{Aut}_Y$ is $\exp(\Xi)$.

**Proof:** Put $\exp(\Xi) = H$. Certainly $H \subset A$.

Suppose first that $g \in A(k)$. Since $S_1$ and $Y$ have the same underlying topological space and $g$ maps to the identity in $\text{Aut}_Y(k)$, it follows that $g = 1$. So $A$ is connected.

Next, suppose that $\eta \in \text{Lie}(A) \setminus \Xi$. Then $\eta$ is a vector field on $S_1$ that commutes with $\Xi$, and so descends to a vector field $\bar{\eta}$ on $Y$. By assumption, $\bar{\eta} = 0$. Since $\Xi, \eta$ generate $T_{S_1}$, it follows that $\mathcal{O}_Y$ is invariant under $T_{S_1}$, so that $S_1 \rightarrow Y$ factors through the Frobenius $S_1 \rightarrow S_1^{(1)}$, which is absurd. Hence $\text{Lie}(A) = \Xi$, so that $A$ has height $\geq 2$.

Since $S_1 \rightarrow Y$ is a torsor under the $Y$-groupscheme $H \times Y \rightarrow Y$ and $A \times Y$ is a subgroupscheme of the $Y$-groupscheme $\text{Aut}_{X/Y}$, it follows that $A$ embeds into the semi-direct product $H \rtimes \text{Aut}_{\text{gpsch}}(H)$. Since $\text{Aut}_{\text{gpsch}}(\mu_2) = 1$, it follows that $H = \alpha_2$ and $A$ embeds into $\alpha_2 \rtimes \mathbb{G}_m = B$, say. However, $A$ contains $H$ and has 1-dimensional Lie algebra, so this is impossible.

**8 From surfaces to periods**

In this section we shall look more closely at the passage from Enriques surfaces to periods. This restricts us to K3-Enriques surfaces. It is clear from what we now know that the coarse moduli space of K3-Enriques surfaces whose $\text{Pic}^*$ is unipotent is, modulo finite (reduced) groups of automorphisms, an open piece of something that is fibred over something that is equivalent, under a radicial map, to an explicit open piece of the period space of K3 surfaces, and the reduced geometric fibres are curves of geometric genus zero. Obviously, this description is very crude; for example, the geometric generic fibre just referred to might, a priori, fail to be reduced. To get a more satisfactory description, we must make a more careful examination of the stacks involved and the relevant morphisms.
between them. It turns out to be easier to go in the other direction and examine the passage from periods to Enriques surfaces. We need a definition.

**Definition 8.1** A family of RDP-K3 surfaces is a Zariski family if the singularities in each geometric fibre are Zariski RDPs and the induced deformation of each of these singularities is a Zariski deformation.

Here is a list of most of the stacks that we shall investigate. They will be algebraic and of finite type over the base $B = \text{Spec} \mathbb{F}_2$.

1. The stack $\mathcal{E}_{uni}$ of unipotent $D$–polarized Enriques surfaces. The open substack corresponding to the K3-Enriques surfaces is $\mathcal{E}_{K3,uni}$.

2. The stack $K$ of pairs $(f: Y \to S, \psi)$ such that $f$ is a Zariski family of RDP–K3 surfaces of RDP rank 12, the sheaf $f_*T_{Y/S}$ of Lie algebras is locally free of rank 2, the natural homomorphism $f_*f^*T_{Y/S} \to T_Y/S$ is an isomorphism, every sub-bundle of $f_*T_{Y/S}$ is 2-closed and $\psi$ is an embedding $E(2)_{\mathcal{S}} \to \text{Pic}_{Y/S}$ such that $\psi_s(\mathcal{D})$ lies in the ample cone of $Y_s$ for all geometric points $s$ of $S$. (These we abbreviate to “$D$-polarized 12-nodal RDP K3 surfaces with trivial tangent bundle” or just “12-nodal RDP K3 surface”.)

3. The open substack $K_{odd}$ of $K$ where the geometric fibres have odd singularities.

4. The relative stabilizer groupscheme $G = G_{k/B} \to K$ and its height $1$ subrelative groupscheme $1_G \to K$. Since for every morphism $s: S = \text{Spec} \mathcal{O} \to K$ of a local scheme to $K$ the family $s: X \to S$ gives a free 2-closed $\mathcal{O}$-Lie algebra of rank 2, the Lie algebra of $G \to K$ or $1_G \to B$ is a rank 2 locally free sheaf of 2-Lie algebras $G \to K$.

5. The bundle $\pi: P = \mathbb{P}(G^\vee) \to K$ ([LMB00], p. 137). We identify $P$ with the stack of 12-nodal RDP K3 surfaces with a choice of a line of vector fields; recall that every such line is 2-closed. So an $S$-point of $P$ is a pair $(X \to S, \Xi)$, where $X \to S$ is an $S$-point of $K$ and $\Xi$ is a line of vector fields. Note that there is a tautological finite flat closed subrelative groupscheme $H \to P$ of $\pi^*(1_G) \to P$ that is of order 2; its Lie algebra is the kernel of the natural homomorphism $\pi^*\mathfrak{g} \to \mathcal{O}(1)$ and over an $S$-point $(X \to S, \Xi)$ of $P$ this Lie algebra also coincides with $\Xi$.

6. The open substack $U$ of $P$, the complement of the closed substack where $\Xi$ vanishes at a singularity. Of course, $\pi: U \to K$ is smooth and representable.

7. The closed substack $P_{nilpt}$ of $P$ where $\Xi$ is nilpotent, and its complement $P_0$. (That $P_{nilpt}$ is a closed substack follows from the formula $(a + b)^{[p]} = a^{[p]} + b^{[p]} + \sum s_i(a, b)$ that holds in a $p$-Lie algebra, where the $s_i$ are universal Lie polynomials.) Note that $P_{nilpt}$ is a quasi-section of $P \to K$, in the sense that it restricts to a section over $K_{odd}$.

**Notation:** Recall that every finite locally free bundle $\mathfrak{h} \to S$ of $p$-Lie algebras is the Lie algebra of a unique connected height 1 finite flat groupscheme $H \to S$; we shall write $H = \exp(\mathfrak{h})$. 

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Theorem 8.2 There is an algebraic gerbe \( \phi : \mathcal{U} \to \mathcal{E}_{K3,uni} \) that is an extension by \( \mathcal{H} \). In particular, \( \phi \) is smooth.

PROOF: According to 3.21, dividing a Zariski RDP-K3 surface with trivial tangent sheaf by a nowhere vanishing 2-closed vector field gives an Enriques surface. Then the morphism \( \phi \) is given at the level of \( S \)-points by associating to the point \( x = (X \to S, \Xi) \) in \( \mathcal{U}(S) \) the point \( X/\Xi \to S \) in \( \mathcal{E}_{K3,uni}(S) \). Note that because \( \xi \) is nowhere zero, the groupscheme \( H_S \to S \) acts freely on \( X \to S \). So taking the quotient by \( \Xi \) commutes with base change.

Next, recall [Ra70], 6.2.1, that for an object \( f : Y \to S \) of \( \mathcal{E}(S) \) and a finite flat commutative \( S \)-group scheme \( A \) there is a natural isomorphism \( R^1f_*A \cong \text{Hom}(A^\vee, \text{Pic}^r(Y/S)) \), where \( A^\vee \) is the Cartier dual of \( A \). To begin with, this means that locally (in the fppf topology) there is a \( \text{Pic}^r(Y/S) \)-torsor which corresponds to the identity element. Over each fibre this then is the double canonical cover.

If \( f : Y \to S \) is an object of \( \mathcal{E}_{K3,uni} \) this gives a point in \( \mathcal{U}(S) \), which shows local essential surjectivity.

Now note that a non-trivial torsor over an Enriques surface \( Y \) (over an algebraically closed field \( k \)) under a groupscheme \( A/k \) of order 2 is isomorphic to the canonical double cover of \( Y \). Indeed, the torsor corresponds to a homomorphism \( \psi : A^\vee \to \text{Pic}^r(Y) \), which is non-trivial as the torsor is, and so \( \psi \) induces an isomorphism \( \Psi : A^\vee \cong \text{Pic}^r(Y) \) as both are group schemes of order 2. Then the torsor is, again by Raynaud’s isomorphism, isomorphic to the one obtained from the canonical double cover by \( \Psi \).

Assume now that we have two objects \( (X, \Xi) \) and \( (X', \Xi') \) of \( \mathcal{U}(S) \) and an isomorphism between the associated objects of \( \mathcal{E}_{K3,uni}(S) \). Composing this isomorphism with the quotient map gives us two non-trivial torsors, over the same base \( f : Y \to S \), under \( A = \exp(\Xi) \) and \( A' = \exp(\Xi') \). As has just been observed they are both isomorphic, fibre by fibre, to the canonical double cover. This shows that \( A \) and \( A' \) are isomorphic to the dual of \( \text{Pic}^r(Y/S) \). Then, using these isomorphisms, we get isomorphisms between \( A \) and \( A' \) such that the two elements of \( R^1f_*A \) to which the two torsors give rise are the same, since they are the same under the identification of \( R^1f_*A \) with \( \text{Hom}(A^\vee, \text{Pic}^r(Y/S)) \). This shows that the two torsors become isomorphic after a flat base extension and hence \( \phi : \mathcal{U} \to \mathcal{E}_{K3,uni} \) induces surjective maps on Hom-sheaves and is therefore an algebraic gerbe.

It remains to show that \( G_\phi = \mathcal{H} \). Certainly, \( \mathcal{H} \subset G_\phi \). Since \( \mathcal{K} \) is a stack of non-ruled polarized surfaces with only RDPs, the relative groupscheme \( G_{K/B} \to \mathcal{K} \) is finite, by the Matsusaka–Mumford theorem, so the closed subrelative groupscheme \( G_\phi \to \mathcal{K} \) is finite. Since \( \mathcal{H} \to \mathcal{K} \) is finite and flat of order 2, it is enough to show that \( G_\phi \) has order 2 over every geometric point of \( \mathcal{K} \). But this is the content of 7.14. Finally, the smoothness of \( \phi \) follows from 1.13. □
We now focus on the diagram

\[
\begin{array}{c}
\mathcal{U} \xrightarrow{\phi} \mathcal{E}_{K^3,uni} \\
\downarrow \pi \\
\mathcal{K}
\end{array}
\]

The morphisms are smooth and \(\pi\) is representable. The next step is to analyze \(\mathcal{K}\) in terms of Ogus’ periods.

For this, return to the diagram

\[
\begin{array}{c}
S^0 \xrightarrow{g} \mathcal{X} \\
\downarrow \mathcal{K} \\
B^0_N \cong \mathcal{M}^0_N
\end{array}
\]

of 6.12. The symmetric group \(S_{12}\) acts on \(M\), and so on \(N\), by permuting the \(e_i\), and is the stabilizer in the orthogonal group \(O(M)\) of a suitable chamber \(\mathcal{C}\).

**Corollary 8.3** The family \(q: \mathcal{X} \to \mathcal{M}^0_N\) is Zariski.

**PROOF:** This is an infinitesimal problem so we may assume the base is local infinitesimal. Then by 6.27 every vector field on the closed fibre can be lifted and as there is one that acts freely at a given (in fact on all) singular point(s) we have a Zariski deformation. \(\square\)

**Lemma 8.4** The action of \(S_{12}\) interchanges the two geometric components of \(\mathcal{M}^0_N\).

**PROOF:** Trivial. \(\square\)

**Proposition 8.5** Suppose that \(X/k\), with \(k = \bar{k}\), is a \(k\)-point of \(\mathcal{K}\). Let \(D_X\) be a hull of \(X\) and \(T_{D_X}\) its tangent space.

1. The natural map \(\text{res} : t_{D_X} \to H^0(X, T^1_X)\) given by restricting deformations of \(X\) to deformations of its singularities is an inclusion.

2. \(D_X\) and \(\mathcal{K}\) are each locally a hypersurface.

**PROOF:** The computation of the hypercohomology of the tangent complex gives a short exact sequence

\[
H^1(X, T_X) \to t_{D_X} \to H^0(X, T^1_X),
\]

where \(t_{D_X}\) is the tangent space to a hull \(D_X\). Now \(T_X\) is free, so that \(H^1(X, T_X) = 0\) and \(\text{res}\) is an injection.

A Zariski RDP of rank \(r\) has a smooth hull of dimension \(2r\), and the subspace of Zariski deformations is smooth of dimension \(r\). So the subspace \(D^Z_X\) of \(D_X\) corresponding to Zariski deformations has embedding dimension at most 12.

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On the other hand, we know that $X$ has 9 moduli as a supersingular K3 surface, so that by 2.9 $\dim D_{X}^{\text{ar}} \geq 11$. 

Denote by $Q$ the Deligne–Mumford stack $[\mathcal{M}_{N}^{0}/\mathcal{S}_{12}]$. There is an extension of stacks $K \to K/G$ by $G$.

**Proposition 8.6** The morphism $b : \mathcal{M}_{N}^{0} \to K$ that is the family $q : X \to \mathcal{M}_{N}^{0}$ is proper and factors through $Q^{(1)}$. The induced morphism of geometric quotients induces a bijection on geometric points.

**PROOF:** The properness of $b$ is immediate from its construction in 6.12. The group $\mathcal{S}_{12}$ acts on the lattice $N$, and so on the functor represented by $S \to B_{N}$. So it acts equivariantly on $S \to B_{N}$. Then it acts on the restriction $S^{0} \to B_{N}^{0}$, so as to preserve the exceptional locus of $g$, so it acts on the contracted family $q : X \to \mathcal{M}_{N}^{0}$. Regard this action as a groupoid $\mathcal{M}_{N}^{0} \times \mathcal{S}_{12} \rightrightarrows \mathcal{M}_{N}^{0} \times \mathcal{M}_{N}^{0}$; this groupoid maps naturally to the Isom-scheme $R := \mathcal{M}_{N}^{0} \times_{q, K, q} \mathcal{M}_{N}^{0} \rightrightarrows \mathcal{M}_{N}^{0} \times \mathcal{M}_{N}^{0}$, so there are morphisms $\mathcal{M}_{N}^{0} \to Q \to [\mathcal{M}_{N}^{0}/R]$. But $q : \mathcal{M}_{N}^{0} \to K$ factors naturally through $[\mathcal{M}_{N}^{0}/R]$; this is a general statement about morphisms from schemes to algebraic stacks. So $b : \mathcal{M}_{N}^{0} \to K$ factors through $Q$. To show that $b$ factors through $Q^{(1)}$ it is enough to show that it factors through $\mathcal{M}_{N}^{0(1)}$. For this, it is enough to show that the derivative of $b$ vanishes at any point corresponding to a surface $X = \mathcal{X}_{i}$ with 12 $A_{1}$ points.

The inclusion $t_{D_{X}} \hookrightarrow H^{0}(X, T_{X}^{1})$ described in 8.5 shows that it is enough to show that the deformation of each singular point induced by $q$ is trivial, locally on $\mathcal{X}$. Consider the blowing-down $g : S^{0} \to \mathcal{X}$; locally over $\mathcal{X}$, the dualizing sheaf $\omega_{\mathcal{X}_{i}}$ is trivial, so that $\mathcal{X}$ is singular along the critical locus $\text{Sing} q$ of $q$. In suitable local co–ordinates, $(\mathcal{X}, P)$ is given by an equation $xy + z^{2} = 0$. All geometric fibres of the deformation of this induced by $q$ are singular, so that the deformation is Zariski, so given by the equation $xy + z^{2} + F(t_{1}, \ldots, t_{9}) = 0$, where the $t_{i}$ are local co–ordinates on $\mathcal{M}_{N}^{0}$. Since $\text{Sing} \mathcal{X} = \text{Sing} q$, by construction of $q$, $F$ is a square, which proves the local triviality and the first part of the proposition.

The last part is an immediate consequence of Ogus’ global Torelli theorem.

Let $c : Q^{(1)} \to K$ be the morphism given by 8.6.

**Proposition 8.7** $K$ is normal.

**PROOF:** Consider the morphisms $\phi : U \to \mathcal{E}_{K3,uni}$ and $\pi : U \to K$. The stack $\mathcal{E}_{K3,uni}$ is smooth, since its $k$-points correspond to surfaces with $H^{2}(T) = 0$, and so $U_{0}$ is smooth. Let $K_{0}$ be the open substack of $K$ that is the image of $U_{0}$ (since $U \to K$ is flat and representable, this makes sense); then $K_{0}$ is smooth. Since the locus $V$ of points in $\mathcal{M}_{N}^{0}$ corresponding to K3 surfaces with even singularities is of codimension 2 everywhere, by 6.21, it follows that the complement of $K_{0}$ in $K$ has codimension 2. Since $K$ is everywhere locally a hypersurface, we are done.
Theorems 8.8 The composite morphism $f : Q^{(1)} \to \mathcal{K}/G$ induced by 8.6 is a 1-isomorphism and $\mathcal{K}/G$ is Deligne-Mumford.

PROOF: Let $\mathcal{K}_{12 \times A_1}$ denote the open substack of $\mathcal{K}$ defined by the property that all geometric fibres of its objects have 12 $A_1$ singularities and $\mathcal{G}_{12 \times A_1} \to \mathcal{K}_{12 \times A_1}$ the restriction of $\mathcal{G}$. There is a Galois étale cover $\tilde{K}_{12 \times A_1} \to \mathcal{K}_{12 \times A_1}$, with group $\mathcal{G}_{12}$ determined by ordering the singularities and an open subscheme $\mathcal{M}_{N,12 \times A_1}$, obtained by deleting the closure of the $E_7 + 5 \times A_1$ locus and the closure of the $D_4 + 8 \times A_1$ locus that fit into a Cartesian diagram

$$
\begin{array}{ccc}
\mathcal{M}_{N,12 \times A_1} & \longrightarrow & \tilde{K}_{12 \times A_1} \\
\downarrow & & \downarrow \\
Q_{12 \times A_1} & \longrightarrow & K_{12 \times A_1}
\end{array}
$$

where the vertical morphisms are stack quotients by $\mathcal{G}_{12}$. Since $\mathcal{M}_N$ has two geometric components that are exchanged by $\mathcal{G}_{12}$, it follows from Ogus’ global Torelli theorem that $\mathcal{K}$ and so $\mathcal{K}_{12 \times A_1}$, is irreducible.

Now pick a smooth presentation $v : V \to \mathcal{K}_{12 \times A_1}$. Put $\tilde{V} = V \times_{v, \mathcal{K}_{12 \times A_1}} \tilde{K}_{12 \times A_1}$. Then $v$ is a family $Y \to V$ of $K3$’s that have, amongst their other properties, 12 disjoint $A_1$ singularities. Consider $\tilde{Y} = Y \times_V \tilde{V} \to \tilde{V}$. We know, by a previous argument, that the fibre product $\tilde{Y} \times_{\tilde{V}} \tilde{V}^{(-1)} \to \tilde{V}^{(-1)}$ has a simultaneous resolution, so that there is a commutative diagram

$$
\begin{array}{ccc}
\tilde{V}^{(-1)} & \longrightarrow & \tilde{V} \\
\downarrow & & \downarrow \\
\mathcal{M}_{N,12 \times A_1} & \longrightarrow & \tilde{K}_{12 \times A_1}
\end{array}
$$

This shows that there is a factorization $\mathcal{M}_{N,12 \times A_1} \to \tilde{K}_{12 \times A_1} \to \mathcal{M}_{N,12 \times A_1}^{(1)}$ of the geometric Frobenius on $\mathcal{M}_{N,12 \times A_1}$, so a factorization $Q_{12 \times A_1} \to \mathcal{K}_{12 \times A_1} \to Q_{12 \times A_1}^{(1)}$. Since $Q_{12 \times A_1} \to \mathcal{K}_{12 \times A_1}$ factors through $Q_{12 \times A_1}^{(1)}$, it follows that the restriction $f_{12 \times A_1} : Q_{12 \times A_1}^{(1)} \to \mathcal{K}_{12 \times A_1}/\mathcal{G}_{12 \times A_1}$ of $f$ is split.

Now we know two things. One is Theorem 7.8, that for a surface $X/k$ corresponding to a geometric point Spec $k \to \mathcal{K}_{12 \times A_1}$ the connected component of the automorphism groupscheme of $X$ is $1$ Aff, so is the restriction of $\mathcal{G}$; this tells us that $\mathcal{K}_{12 \times A_1}/\mathcal{G}_{12 \times A_1}$ is a Deligne–Mumford stack. The other is the part of Ogus’ global Torelli theorem, saying that the Picard group of the minimal resolution $\tilde{X}$ of $X$ when $X/k$ corresponds to a geometric point Spec $k \to \mathcal{K}$ rigidifies $\tilde{X}$; since $\text{Aut}_X(k) = \text{Aut}_X(k)$ this tells us that, in the notation of 1.29, the morphisms $f_{x,x'}$ arising from $f : Q^{(1)} \to \mathcal{K}/G$ are embeddings. Then $f_{12 \times A_1}$ is a split morphism of irreducible normal algebraic stacks that is representable, by 1.29; it follows that $f_{12 \times A_1}$ is an isomorphism. Then $f$ is an isomorphism, by 1.30.
Corollary 8.9  (1) If $X$ is a singular K3 surface corresponding to a geometric point of $\mathcal{K}$, then $\text{Aut}_X^0$ is of height 1.

(2) $\mathcal{G} = 1\mathcal{G}$.

PROOF: This follows from the fact that $\mathcal{K}/\mathcal{G}$ is Deligne-Mumford. □

Now consider the diagram with 2-Cartesian squares

that extends our previous diagram. (The two upper vertical arrows are open embeddings.) Via 8.8, we identify $\mathcal{K}/\mathcal{G}$ with $\mathcal{Q}^{(1)}$ and $bc$ with $1_{\mathcal{Q}^{(1)}}$. Let $\pi_0 : \mathcal{U} \to \mathcal{K}$ be the composite.

We can recover a result of Liedtke [L10].

Corollary 8.10  $\mathcal{K}$, $\mathcal{U}$ and $\mathcal{E}_{K3,uni}$ are smooth and irreducible.

PROOF: Since $\mathcal{Q}^{(1)}$ is smooth and irreducible, the smoothness and irreducibility of $\mathcal{K}$ follows from 8.8 and 1.13. Then $\mathcal{U}$ is smooth and irreducible; since $\phi$ is smooth, so is $\mathcal{E}_{K3,uni}$. □

Consider the relative groupscheme $\mathcal{L} = \mathcal{G} \times_{\mathcal{K},c} \mathcal{Q}^{(1)} \to \mathcal{Q}^{(1)}$. This is finite, flat, of order 4 and of height 1. Put $\mathcal{L} = \text{Lie}(\mathcal{L})$; then $\mathcal{P}' = \mathbb{P}(\mathcal{L}')$. There is a tautological subgroupscheme $\mathcal{M} = j^*\mathcal{H}$ of order 2 in $\pi'^*\mathcal{L}$, where $\mathcal{H}$ is the tautological subgroupscheme of order 2 in $\pi^*\mathcal{G}$.

Theorem 8.11  (1) The morphisms $\mathcal{Q}^{(1)} \to \mathcal{K} \to \mathcal{Q}^{(1)}$ provide a 1-isomorphism from $\mathcal{K}$ to the classifying stack $\mathcal{B}\mathcal{L}$ with its canonical projection $\mathcal{B}\mathcal{L} \to \mathcal{Q}^{(1)}$ and its tautological section.

(2) $j : \mathcal{P}' \to \mathcal{P}$ and $i : \mathcal{U}' \to \mathcal{U}$ induce 1-isomorphisms $[\mathcal{P}'/\mathcal{L}] \to \mathcal{P}$ and $[\mathcal{U}'/\mathcal{L}] \to \mathcal{U}$.

(3) The stacks $\mathcal{U}$ and $\mathcal{E}_{K3,uni}$ have Deligne–Mumford quotients $U_{DM}$ and $\mathcal{E}_{K3,uni,DM}$, and the morphism $\mathcal{U} \to \mathcal{U}'/\mathcal{L}$ induced from the isomorphism of (2) factors through $\phi : \mathcal{U} \to \mathcal{E}_{K3,uni}$ and induces 1-isomorphisms $U_{DM} \to (\mathcal{E}_{K3,uni})_{DM}$ and $(\mathcal{E}_{K3,uni})_{DM} \to \mathcal{U}'/\mathcal{L}$.

(4) $\mathcal{P}' \to \mathcal{P}'/\mathcal{L}$ is the Frobenius relative to $\mathcal{Q}^{(1)}$ and $\mathcal{P}'/\mathcal{L} \to \mathcal{Q}^{(1)}$ is a $\mathbb{P}^1$-bundle.

PROOF: Except for (4), these all follow from the general results of §1. More precisely, (1) follows from 8.8 and 1.14. (2) is a consequence of 1.8. For (3), the first statement follows from 1.10 and 1.23 and the second from 1.27.

For (4) we make a calculation.
We know that outside the $3 \times D_4$ locus in $Q^{(1)}$, which is of pure codimension 2, the geometric fibres of $\mathcal{L}$ are $\mathbb{A}^1$-Aff, and over the $3 \times D_4$ locus they are $\alpha_2 \times \alpha_2$. Hence, locally on $Q^{(1)}$ around a point in the $3 \times D_4$ locus, there is a basis $\{e_1, e_2\}$ of $\text{Lie}(\mathcal{L})$ and a regular sequence $s_1, s_2$ in $\mathcal{O} = \mathcal{O}_{Q^{(1)}}$ such that $[e_1, e_2] = s_1 e_1 + s_2 e_2$. Let $\{e'_1, e'_2\}$ be the dual basis of $\text{Lie}(\mathcal{L})^*$ and $R = \mathcal{O}[e'_1, e'_2] = \text{Sym}^* \text{Lie}(\mathcal{L})^*$. We need to compute the $\mathcal{O}$-subalgebra of $\mathcal{L}$-invariants of $R$. Since $\mathcal{L}$ is of height 1, this is the algebra of $\text{Lie}(\mathcal{L})$-invariants, and is easily computed to be $\mathcal{O}[(e'_1)^2, (e'_2)^2]$. This completes the proof. 

**Corollary 8.12** There is a period morphism $\psi : (\mathcal{E}_{K3,\text{uni}})^{\text{DM}} \rightarrow Q^{(1)}$ that is an open piece of a $\mathbb{P}^1$-bundle.

**Theorem 8.13** There is a closed substack $\mathcal{E}_{K3,\alpha}$ of $\mathcal{E}_{K3,\text{uni}}$ whose geometric points are the geometric points of $\mathcal{E}_{K3,\text{uni}}$ whose $\text{Pic}^\tau$ is isomorphic to $\alpha_2$.

**Proof:** Define $\mathcal{E}_{K3,\alpha}$ to be the closed substack of $\mathcal{E}_{K3,\text{uni}}$ that corresponds under the bijection given by 1.22 to $\mathcal{U}_{\text{nilpt}}$. For any $h : S \rightarrow U$, the restriction $\mathcal{H}_S \rightarrow S$ of $\mathcal{H} \rightarrow U$ is a twisted form of $\alpha_2$ if and only if $h$ factors through $\mathcal{U}_{\text{nilpt}}$. Identify $h$ with $h : (X, \Xi) \rightarrow S$ and put $Y = X/\exp(\Xi) \rightarrow S$. Then $X \rightarrow Y$ is a torsor under $\mathcal{H}_S \rightarrow S$, so if $S$ is a geometric point, then $S \rightarrow \mathcal{E}_{K3,\text{uni}}$ defined by $Y$ factors through $\mathcal{E}_{K3,\alpha}$ if and only if $\text{Pic}^\tau(Y) \cong \alpha_2$.

**Remark:** It is not so clear how to give an *a priori* definition of $\mathcal{E}_{K3,\alpha}$ even as a stack, let alone a substack of $\mathcal{E}_{K3,\text{uni}}$; the difficulty stems from the fact that $\alpha_2$ has a non-trivial automorphism groupscheme, namely $G_m$.

To describe $\mathcal{E}_{K3,\alpha}$ further needs some more notation. We have the $3 \times D_4$-locus $Z \subset \mathcal{M}_{\overline{3}}$, which is smooth and of codimension 2. It maps to the closed substack $Z^{(1)} = [Z^{(1)}/\mathcal{G}_{12}]$ of $Q^{(1)}$. Then the inverse image $\mathcal{Y}$ of $Z^{(1)}$ in $\mathcal{K}$, via the isomorphism $f : Q^{(1)} \rightarrow \mathcal{K}/\mathcal{G}$, is smooth and $\mathcal{Y} \rightarrow Z^{(1)}$ is an extension by the restriction $\mathcal{G}_{\mathcal{Y}}$ of $\mathcal{G}$ to $\mathcal{Y}$. Note that the complement of $\mathcal{Y}$ in $\mathcal{K}$ is the open substack $\mathcal{K}^{\text{odd}}$. Let $\mathcal{W}$ be the inverse image of $\mathcal{Y}$ in $\mathcal{P}$ and $\mathcal{V} = W \cap U$.

**Proposition 8.14** The image $(\mathcal{E}_{K3,\alpha})^{\text{DM}}$ of $\mathcal{E}_{K3,\alpha}$ in $(\mathcal{E}_{K3,\text{uni}})^{\text{DM}}$ is a section of $\psi$ over the complement of the $3 \times D_4$-locus. That is, the period morphism is an isomorphism on the Deligne–Mumford quotient of the stack of multiplicative $\alpha_2$-Enriques surfaces.

**Theorem 8.15** $\mathcal{E}_{K3,\alpha}$ is irreducible and of codimension 1 in $\mathcal{E}_{K3,\text{uni}}$.

**Proof:** We know by 4.9 that $\mathcal{E}_{K3,\alpha}$ is everywhere of codimension 1 in $\mathcal{E}_{K3,\text{uni}}$. So it is enough to show that $\mathcal{P}_{\text{nilpt}}$ has a unique component of codimension 1. Now $\mathcal{P}_{\text{nilpt}}$ is the union of two pieces, namely $\pi^{-1}(\mathcal{Y})$ and a section of $\pi$ over $\mathcal{K}^{\text{odd}}$. The first is of codimension 2 and the second is irreducible, and we are done. 

**Theorem 8.16** There is an irreducible smooth open substack $\mathcal{E}_{K3,\alpha,\text{mult}}$ of $\mathcal{E}_{K3,\alpha}$ whose geometric points are multiplicative $\alpha_2$-Enriques surfaces. Its complement is
a smooth irreducible closed substack $\mathcal{E}_{K3,\alpha,\text{add}}$ whose geometric points are additive $\alpha_2$-Enriques surfaces.

**Proof:** Define $\mathcal{E}_{K3,\alpha,\text{add}}$ to be the closed substack of $\mathcal{E}_{K3,\text{uni}}$ that corresponds, according to 1.22, to $\mathcal{U} \cap \pi^{-1}(Y)$. Then take $\mathcal{E}_{K3,\alpha,\text{mult}}$ to be its complement in $\mathcal{E}_{K3,\alpha}$. $\square$

### 9 Periods of $12A_1$-Enriques surfaces

We will in this section make a special study of $12A_1$-Enriques surfaces and their periods. We begin by showing that all multiplicative $\alpha_2$-surfaces belong to this category. We will now make a closer study of the subset $\mathcal{P}_0$ of the $\mathcal{P}_1$-bundle $\mathcal{P} \to \mathcal{M}_0N$ or rather its complement in $\mathcal{P}$.

**Lemma 9.1** Let $\pi: \mathcal{X} \to S$ be a flat and proper family of RDP-K3 surfaces whose singularities are Zariski RDP’s of total index 12 and such that $T_{\mathcal{X}/S}$ is fibrewise free so that $\pi_*T_{\mathcal{X}/S}$ is locally free of rank 2 and commutes with base change. Let $Z$ be the singular locus of $\pi$ (i.e., locally defined by the partial derivatives of a defining equation for $\mathcal{X}$ over $S$). Then $Z$ is finite flat of degree 12 over $S$ and there is a natural map $Z \to \mathbb{P}(\pi_*T_{\mathcal{X}/S})$.

**Proof:** The fact that $\pi_*T_{\mathcal{X}/S}$ is locally free and commutes with base change as well as the fact that $Z$ is finite flat of degree 12 is clear. To continue we note that we have a natural evaluation map $\Omega^1_{\mathcal{X}/S} \to \text{Hom}_X(T_{\mathcal{X}/S}, \mathcal{O}_X)$ which is an injection and whose cokernel $\mathcal{L}_Z$ is a line bundle over $Z$. This can be verified locally and fibrewise and is then a direct computation. This gives us a map $\pi_*T_{\mathcal{X}/S} \to \mathcal{L}_Z$ and we claim that the induced map $\pi_*T_{\mathcal{X}/S} \otimes_{\mathcal{O}_S} \mathcal{O}_Z \to \mathcal{L}_Z$ is surjective. This however follows directly from the fact that $\pi^*\pi_*T_{\mathcal{X}/S} \to T_{\mathcal{X}/S}$ is surjective. The map $\pi_*T_{\mathcal{X}/S} \otimes \mathcal{O}_Z \to \mathcal{L}_Z$ now induces the desired map. $\square$

With the aid of this lemma and provided with a map $\pi: \mathcal{X} \to S$ as in it, we may now use [Mu65], §5.3 to define a Cartier divisor $\text{Div}(Z)$, the non-free locus, in $\mathbb{P}(\pi_*T_{\mathcal{X}/S})$. This divisor commutes with base change by [Mu65], §5.3 and is hence a relative Cartier divisor as it gives a divisor on each fibre. By construction it is of degree 12 over the base. It is not étale over a point of $S$ over which $Z$ is not étale and $Z \to S$ is étale precisely over the points at which $\mathcal{X}$ has $12A_1$-singularities. Hence, when we start with a family of K3-Enriques surfaces and let the RDP-K3-surface be the canonical double cover, the non-free locus has a chance of being étale only over points over which the Enriques surface an $12A_1$-surface. Our aim is not to show that in that case non-free locus is indeed étale or equivalently, $Z \to \mathbb{P}(\pi_*T_{\mathcal{X}/S})$ is an embedding. We will do this by reading off the position of the points of $Z$ in $\mathbb{P}(\pi_*T_{\mathcal{X}/S})$ from the period of the surface. To do this we start by characterising those periods that correspond to $12A_1$-surfaces.
Lemma 9.2 Any even superlattice of $M$ is generated by $M$ and the $-2$-elements it contains.

PROOF: It is enough to assume that such a superlattice $L$ contains $M$ as a subgroup of index 2 and is hence generated by $M$ and one other element. That extra element may be assumed to have the form $\frac{1}{2}(a_1, \ldots, a_{12})$ where $a_i \in \{0, 1\}$ and the fact that $L$ should be even forces the number $a_i$'s that are equal to 1 to be divisible by 4. As $\frac{1}{2}(1, 1, \ldots, 1)$ belongs to $M$ we see that $L$ must contain an element of the form $\frac{1}{2}(a_1, \ldots, a_{12})$ with exactly 4 $a_i$'s being equal to 1 but that is a $-2$-element.

We need to relate $N$-periods to the orthogonal decomposition $N = E(2) \perp M$.

Lemma 9.3 Let $V \subset N \otimes k$ be a period in $\mathcal{M}_N^0$ where $k$ is a perfect field of characteristic 2.

1. The projection of $V$ into $M \otimes k$ has as image an $\mathbb{F}_2$-rational subspace.
2. $V$ corresponds to an RDP-K3 surface with 12 $A_1$-singularities if and only if the projection of $V$ into $M \otimes k$ is surjective.

PROOF: If $\varphi$ is the Frobenius map on $N \otimes k$ with respect to the $\mathbb{F}_2$-rational structure given by $N$, then $V + \varphi(V)$ is 11-dimensional and so has a non-trivial intersection with $E(2) \otimes k$. We claim that $V \cap E(2) \otimes k \neq (V + \varphi(V)) \cap E(2) \otimes k$. In fact if they are equal, then $V \cap E(2) \otimes k$ is stable under $\varphi$ and hence is $\mathbb{F}_2$-rational. It is totally isotropic because it is a subspace of $V$. Hence, it corresponds to an even superlattice of $E(2)$ but this contradicts the definition of $\mathcal{M}_N^0$ and lemma 6.10. Hence, $V + \varphi(V)$ is the sum of $V$ and $(V + \varphi(V)) \cap E(2) \otimes k$. This means that the projection onto $M \otimes k$ is stable under $\varphi$ and is hence $\mathbb{F}_2$-rational. This proves (1).

As for (2), as $V$ is maximal isotropic, $V$ will contain the orthogonal complement in $M \otimes k$ to the image of $V$ in $M \otimes k$. As the image is $\mathbb{F}_2$-rational so is its orthogonal complement. That orthogonal complement thus equals $V \cap M$ so it is also $\mathbb{F}_2$-rational and it is, by 9.2 trivial if and only if $V$ corresponds to an RDP-K3 surface with 12 $A_1$-singularities. □

Definition 9.4 If $1 \leq i \neq j \leq 12$ we set $e_{ij} := e_i + e_j \in M/2M$. We let $\mathcal{M}_N^1$ be the open subvariety of $\mathcal{M}_N^0$ consisting of those periods $V \in N \otimes k$ for which $V \cap M/2M = \{0\}$ and which contains no element of the form $\alpha + e_{ij}$ with $\alpha \in E(2)/2E(2)$.

Remark: The complement of $\mathcal{M}_N^1$ in $\mathcal{M}_N^0$ contains points which correspond to RDP-K3 surfaces with 12 $A_1$-singularities. Indeed, a period in $\mathcal{M}_N^0$ corresponds to such a surface precisely when $V \cap M/2M = \{0\}$ and if we pick $\alpha \in E(2)/2E(2)$ with $\alpha^2 = 1$ then the set of such $V$ that contain $\alpha + e_{12}$, say, is non-empty and a generic point of it contains no other points of $N/2N$.
Our next step is to get a direct description of the global sections of the tangent bundle of an RDP-K3 surface with only $A_1$-singularities in terms of a resolution of singularities.

**Lemma 9.5** Let $X$ be a surface over a field $k$ whose singular locus consists of only $A_1$-singularities, let $\pi: \tilde{X} \to X$ be a minimal resolution of singularities, $Z$ the exceptional divisor for $\pi$, and $j: U \to X$ the inclusion of the non-singular locus.

1. We have $T_X = j_* T_U = \pi_* T_{\tilde{X}}(\log Z)(Z)$.
2. When the characteristic of $k$ equals 2, the sheaf of non-free vector fields equals the kernel of $\pi_*$ applied to the natural map $T_{\tilde{X}}(\log Z)(Z) \to T_Z(Z)$.
3. If $\omega_X$ is trivial then using a trivialisation of it $T_{\tilde{X}}(\log Z)(Z)$ is identified with $\Omega^1(\log Z)$, compatibly with the identification of $T_{\tilde{X}}$ and $\Omega^1_{\tilde{X}}$, and $T_Z(Z)$ with $\mathcal{O}_Z$. Under these identifications the map $T_{\tilde{X}}(\log Z)(Z) \to T_Z(Z)$ is identified with the Poincaré residue map $\Omega^1_{\tilde{X}}(\log Z) \to \mathcal{O}_Z$.

**Proof:** That $T_X = j_* T_U$ follows directly from $j_* \mathcal{O}_U = \mathcal{O}_X$. Furthermore, we have that $j_* T_U = \pi_* T_{\tilde{X}}(*Z)$ and what remains to show for (1) is that the inclusion $T_{\tilde{X}}(\log Z)(Z) \hookrightarrow T_{\tilde{X}}(*Z)$ induces an isomorphism upon applying $\pi$. This is local so we may assume that $X$ contains only one singular point and then $Z \cong \mathbb{P}^1$. Now, the short exact sequence

$$0 \to T_{\tilde{X}}(\log Z) \to T_{\tilde{X}} \to \mathcal{N}_{Z/\tilde{X}} \to 0$$

and the fact that $\mathcal{N}_{Z/\tilde{X}} \cong \mathcal{O}_Z(-2)$ shows that $\pi_* T_{\tilde{X}}(\log Z)(Z) = \pi_* T_{\tilde{X}}(Z)$ and the facts that $\mathcal{O}_Z(Z) = \mathcal{O}_Z(-2)$ and $T_{\tilde{X}}(\log Z)/T_{\tilde{X}}(-Z) = T_Z \cong \mathcal{O}_Z(2)$ shows that $\pi_* T_{\tilde{X}}(iZ) = \pi_* T_{\tilde{X}}((i + 1)Z)$ for all $i > 1$.

Continuing with (2), the kernel of $T_{\tilde{X}}(\log Z)(Z) \to T_Z(Z)$ is $T_{\tilde{X}}$ so that the kernel of the map induced by applying $\pi_*$ to it consists of those vector fields that lift to the resolution and it is a simple local calculation that these are exactly the non-free vector fields.

Finally, to prove (3) as the statement is independent of the choice of trivialisation of $\omega_X$ we may work locally and choose coordinates such that $Z$ is defined by the vanishing of one of them and the trivialisation is given by $dx \wedge dy$. Then the result is a simple calculation.

We are now prepared to state and prove the main result of this section.

**Theorem 9.6** Suppose that a period point $x$ in $\mathcal{M}^{0}_N$ corresponds to the RDP-K3 surface $X$. Then the non-free locus in $\mathbb{P}(H^0(X, T_X))$ is étale precisely when $x$ lies in $\mathcal{M}^{1}_N$.

**Proof:** As we have seen for the non-free locus to be étale it is necessary that $X$ has 12 $A_1$-singularities and if the period lies in $\mathcal{M}^{1}_N$ this is always the case. We may and will therefore assume that $X$ has 12 $A_1$-singularities.

If $W$ is the singular locus of $X$ then, as has already been noted, the non-free locus is étale precisely when the map $W \to \mathbb{P}(H^0(X, T_X))$ is injective. For each
x ∈ W let TX → ℓx be the quotient by the non-free subsheaf with respect to x. Then this means that for any two x ≠ y ∈ W, the map \( H^0(X, T_X) \to \ell_x \oplus \ell_y \) is surjective. Let \( \widetilde{X} \to X \) be a minimal resolution of singularities and Z its exceptional divisor. We have a short exact sequence

\[
0 \to \Omega^1_{\widetilde{X}} \longrightarrow \Omega^1_X(\log Z) \longrightarrow \mathcal{O}_Z \to 0,
\]

and using the fact that \( h^0(\tilde{X}, \Omega^1_{\tilde{X}}) = 0 \) we see that \( H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log Z)) \) can be identified with the kernel of the map \( H^0(Z, \mathcal{O}_Z) \to H^1(\tilde{X}, \Omega^1_{\tilde{X}}) \). Now, \( Z \) is the disjoint union of its irreducible components \( Z_x \) for \( x \in W \) and thus \( H^0(Z, \mathcal{O}_Z) = \bigoplus_{x \in W} k1_{Z_x} \). Furthermore, the image in \( H^1(\tilde{X}, \Omega^1_{\tilde{X}}) \) of \( 1_{Z_x} \) equals the de Rham cycle class of \( Z_x \). Hence, we may identify \( H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log Z)) \) with the kernel of the map \( \sum_{i=1}^{12} ke_i \to H^1(\tilde{X}, \Omega^1_{\tilde{X}}) \) which sends \( e_i \) to the first Chern class of \( \mathcal{O}(Z_x) \).

Using 9.5 we can identify \( H^0(X, T_X) \) and \( H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log Z)) \) and then the map \( H^0(X, T_X) \to \ell_x \) can be identified with the projection of \( H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log Z)) \) onto the \( ke_i \)-factor where \( x \in W \) is the \( i \)’th point (for some fixed ordering). Furthermore, if \( V \) is the period of \( \tilde{X} \) then the kernel of the map \( N \otimes k \to H^1(\tilde{X}, \Omega^1_{\tilde{X}}) \) is \( \alpha \varphi(V) \), where \( \varphi \) is the Frobenius map with respect to \( N/2N \) (cf. [Og79], 3.20.1). Hence, if we consider \( V \) as a subspace of \( E(2) \otimes k \), where \( M' = \sum_i Z_e_i \) then \( H^0(X, T_X) \) can be identified with \( (V + \varphi(V)) \cap M' \otimes k \).

We have thus reduced the theorem to proving that \( V \in M^1_{12} \) precisely when \( \lambda e_{ij} \) maps onto \( ke_i \otimes ke_j \) for all \( i ≠ j \). As \( \frac{1}{2} \sum_{i=1}^{12} e_i \in M \) we see that the diagonal \( \lambda e_{ij} \) will always lie in this image and hence the map fails to be surjective exactly when the image is this diagonal, i.e., when \( (V + \varphi(V)) \cap M' \otimes k \) is contained in \( M_{ij} \otimes k \), where \( M_{ij} := \{ \sum a_r e_r \in M | a_i = a_j \} \). If \( V \) contains an element of the form \( \alpha + e_{ij}, \alpha \in E(2)/2E(2) \), then \( V \) and therefore \( \varphi(V) \) as well as \( V + \varphi(V) \) is orthogonal to \( \alpha + e_{ij} \) and thus any element in \( (V + \varphi(V)) \cap M' \otimes k \) lies in \( M_{ij} \otimes k \). Conversely, let us assume that \( (V + \varphi(V)) \cap M' \otimes k \) lies in \( M_{ij} \otimes k \). Then \( V + M_{ij} \) is stable under \( \varphi \) and is hence \( \mathbb{F}_2 \)-rational. As \( V \) is the graph of an isomorphism between \( E(2) \otimes k \) and \( M' \otimes k \) the sum \( V + M_{ij} \) is direct and its dimension is therefore equal to 19. Its annihilator is a 1-dimensional subspace of \( N/2N \) and is contained in \( V \) as \( V \) is maximal totally isotropic. Furthermore, it does not lie in \( E(2)/2E(2) \) as \( V \in M^0_1 \) and as it is orthogonal to \( M_{ij} \) a non-zero element in it must therefore be of the form \( \alpha + e_{ij} \).

\[ \Box \]

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