Rational Convolution Roots of Isobaric Polynomials

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Abstract

In 2006, MacHenry and Tudose exhibited a formula for the rational convolution roots of the generalized Fibonacci polynomials (GFPs), as well as a more general formula for all weighted isobaric polynomials (WIPs). These formulas make use of two types of operators, which are derived from the generating functions for Stirling numbers of the first and second kind. Hence we have called these operators Stirling operators. For the roots of GFPs we produce matrix representations which use the Stirling operators of the first kind. Stirling operators of the second kind appear in the formulas for the rational convolution roots of the WIPs. We have given explicit examples to show how the Stirling operators appear in the low dimensional cases for the WIP-roots. As a consequence of these embeddings we have explicit embeddings of both the group of multiplicative arithmetic functions, and the group of additive arithmetic functions with respect to the Dirichlet product into their divisible closures. As a bi-product of this construction, we have embeddings of both the complete symmetric polynomials and the power sum symmetric polynomials into their injective hulls.

Keywords: isobaric polynomial, multiplicative function, additive function, generalized Fibonacci polynomial, generalized Lucas polynomial, matrix representation.

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1. Introduction

In 1975, Carroll and Gioia \cite{Gio91} gave a construction within the ring of arithmetic functions for the divisible closure of the group of multiplicative arithmetic functions (MF). In 2000, MacHenry \cite{Mac00} gave a somewhat more general proof of this same theorem. In 2005, MacHenry and Tudose \cite{Tud05} constructed the injective hull of the generalized Fibonacci polynomials (GFP) and extended this construction to the injective hull of the WIP-module, that is, the module of weighted isobaric polynomials (WIP) with respect to the convolution product. The isobaric polynomials are the symmetric polynomials over the elementary symmetric polynomial.
(ESP) basis; the isobaric ring is isomorphic to the ring of symmetric polynomials. In 2012, MacHenry and Wong showed that GFPs together with the convolution inverse give a faithful representation of the group of multiplicative arithmetic functions under the Dirichlet product, which in turn induces the embedding of the group $MF$ into its injective hull, that is, adjoins a $q$-th root to each multiplicative arithmetic functions for all non-zero rational numbers $q$ in $\mathbb{Q}$.

In 2013, Li and MacHenry gave two representations of the WIP-module in terms of Hessenberg matrices; they showed that the determinants of one of the sets of matrices were isomorphic to the permanents of the other set of matrices, and that the values of these determinants and permanents were just the elements of the WIP-module.

In this paper, we shall also use Hessenberg matrices to give a matrix representation of the convolution roots of generalized Fibonacci polynomials. As with the isobaric functions themselves, it is a representation by determinants and permanents.

2. Isobaric Polynomials

An isobaric polynomial in $k$ variables $\{t_1, \ldots, t_k\}$ of degree $n$ is of this form

$$P_{k,n} = \sum_{\alpha \vdash n} C_{\alpha} t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k},$$

where $C_{\alpha} \in \mathbb{Z}$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \vdash n$ means that $(1^{\alpha_1}, 2^{\alpha_2}, \ldots, k^{\alpha_k})$ is a partition of $n$ with $\sum_{j=1}^{k} j \alpha_j = n$. You can think that an isobaric polynomial is a symmetric polynomial written on the elementary symmetric polynomial (ESP) basis.

A special case of isobaric polynomials is the set of weighted isobaric polynomials defined by

$$P_{\omega,k,n} = \sum_{\alpha \vdash n} \frac{|\alpha|}{\alpha_1, \ldots, \alpha_k} \sum_{\alpha_i \omega_i} t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k},$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_j, \ldots)$ with $\omega_j \in \mathbb{Z}$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$.

The indexing set $\{n\}$ for these polynomials is the set of integers, positive, negative and zero, i.e., $n \in \mathbb{Z}$. In particular, $P_{\omega,k,0} = \omega_k, k \geq 1$, and $P_{\omega,k,0} = 1, k = 0$.

Note that the monomials are indexed by the partitions $(1^{\alpha_1}, 2^{\alpha_2}, \ldots, k^{\alpha_k})$ with parts no larger than $k$. Moreover, they are known to occur in sequences which are linear recursions with respect to the recursion parameters $\{t_1, \ldots, t_k\}$, thus

$$P_{\omega,k,n} = t_1 P_{\omega,k,n-1} + t_2 P_{\omega,k,n-2} + \cdots + t_j P_{\omega,k,n-j} + \cdots + t_k P_{\omega,k,n-k}.$$

Two cases of special importance are those of the generalized Fibonacci polynomials (GFP)

$$F_{k,n} = \sum_{\alpha \vdash n} \frac{|\alpha|}{\alpha_1, \ldots, \alpha_k} t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k},$$
where the weight vector is \( \omega = (1, 1, \ldots, 1) \) with \( F_{k,0} = 1 \), and the generalized Lucas polynomials (GLP)

\[
G_{k,n} = \sum_{\alpha \vdash n} \binom{|\alpha|}{\alpha_1, \ldots, \alpha_k} n! \frac{t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k}}{|\alpha|!}
\]

where the weight vector is \( \omega = (1, 2, \ldots, j, \ldots) \) and \( G_{k,0} = k \).

**Remark 1.** The GFPs are the complete symmetric polynomials written on the ESP basis, and the GLPs are the power sum symmetric polynomials written on the ESP basis; each of these sequences of polynomials is a basis for the ring of symmetric polynomials.

**Remark 2.** The WIPs in general have special significance in the symmetric ring. In order to see how this comes about, it is convenient to consider the notation \( [t_1, \ldots, t_k] \) used above to indicate recursion parameters. More generally, we shall use this notation to indicate the monic polynomial \( C(X) = X^k - t_1 X^{k-1} - \cdots - t_k \), that is

\[
[t_1, \ldots, t_k] = X^k - t_1 X^{k-1} - \cdots - t_k.
\]

When we are regarding the \( t_j \) as variables, we shall often refer to \( C(X) \) as the *generic core*, and when we evaluate the \( t_j \) over the ring of integers, the term *numerical core* will sometimes be used.

**Remark 3.** It is trivial to verify that when \( k = 2 \), the GFPs are generalizations of the classical “generalized Fibonacci polynomials” and the GLPs are generalizations of the classical “generalized Lucas polynomials”; when \( t_1 = t_2 = 1 \), the GFPs and GLPs become the classical Fibonacci and Lucas sequences. It is also surprising that these older terms persist in the current literature in competition with the true generalizations.

Next, we consider the companion matrix of \( [t_1, \ldots, t_k] = X^k - t_1 X^{k-1} - \cdots - t_k \), namely, the \( k \times k \)-matrix

\[
A_k = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
t_k & t_{k-1} & \cdots & t_2 & t_1
\end{pmatrix}
\]

We use \( A_k \) to construct the following infinite matrix by appending the orbit of the row vectors generated by letting \( A_k \) act on the right of the last row vector in \( A_k \), and repeating the process on the successive last row vectors. Noting that \( A_k \) is non-singular exactly when \( t_k \neq 0 \), and adding this as an assumption, we can perform the analogous operation on the first row vector of \( A_k \), extending the rows northward, yielding a doubly infinite matrix with \( k \) columns. We shall call this the *infinite companion matrix*, and denote it by \( A_k^\infty \), or simply as \( A^\infty \) when the \( k \) is clear. As pointed out in a number of previous papers, *e.g.* [2], this matrix has some remarkable properties. Since it is completely determined by the polynomial \( C(X) = [t_1, \ldots, t_k] \) we shall call \( C(X) \) the *core polynomial*.

We enumerate some of these properties here:
• The $k \times k$ contiguous blocks of $A^\infty$ are the successive powers in the free abelian group generated by the companion matrix $A_k$.

• The rows of $A^\infty$ give a vector representation of the successive powers of the zeros of the core polynomial. This is essentially a consequence of the Hamilton-Cayley theorem.

• The right hand column of $A^\infty$ is just the GFPs.

• The traces of the $k \times k$ contiguous blocks give in succession the GLPs.

• The columns of $A^\infty$ are linearly recursive with respect to the coefficients of the core polynomial as recursion parameters.

• The columns of $A^\infty$ are weighted isobaric polynomials with weight vectors $\pm(0, \ldots, 0, 1, 1, \ldots, 1, \ldots)$.

• The elements of $A^\infty$ are Schur-hook polynomials $S_{(n, 1^r)}$, of arm-length $n - 1$ and leg length $r$. In particular, $F_{k,n} = S_{(n)}$.

• The WIPs form a free $\mathbb{Z}$-module whose basis consists exactly of the columns of $A^\infty$.

Moreover, there is a second matrix that is induced by the core polynomial, which also has some remarkable properties.

We consider the derivative of the core polynomial

$$C'(X) = kX^{k-1} - t_1X^{k-2} - \cdots - t_{k-1},$$

out of which, we manufacture the vector $(-t_{k-1}, \ldots, -t_1, k)$. Again letting the companion matrix $A_k$ act on this vector on the right and appending the resulting orbit as additional row vectors, we get a $k \times k$-matrix, which we call the different matrix, with label $D$, from which we can construct an infinite matrix, $D^\infty$ as we did above, and doubly infinite, in the case, that the derivative of the core is irreducible. Call this matrix the infinite different matrix. It too has some useful and remarkable properties.

• The determinant $\det D = \Delta$ is the discriminant of the core polynomial,

• The right hand column of $D^\infty$ is the sequence GLP.

• There is a bijection $\mathcal{L}$ from $A^\infty$ to $D^\infty$ which takes the element $a_{i,j}$ in $A^\infty$ to $d_{i,j}$ in $D^\infty$, which has the properties of a logarithm on elements, and which implies that $\mathcal{L}(F_{k,n}) = G_{k,n}$.

• The columns of $D^\infty$ are linear recursions with recursion parameters $\{t_1, \ldots, t_k\}$.

Next, we would like to point out in what way the sequences discussed here are important. In a serious of papers [5, 6, 7, 8, 2, 3] it has been shown that subgroups of the ring of arithmetic functions, namely, the Dirichlet group of multiplicative arithmetic functions, and the additive group of additive arithmetic
functions have faithful representations using the GFP sequence and the GLP sequence; they also show up in the character theory of the symmetric groups and Pólya’s Theory of Counting. In the following section, we shall produce some representations of the GFP, the GLP and in general, the WIP sequences, which will prove useful for calculation.

But first it is convenient to introduce the convolution product of weighted isobaric polynomials.

**Definition 4.** Let $P_{\omega,k,n}$ and $P_{\upsilon,k,n}$ be weighted isobaric polynomials of isobaric degree $n$. Define the convolution product of $P_{\omega,k,n}$ and $P_{\upsilon,k,n}$ by

$$P_{\omega,k,n} \ast P_{\upsilon,k,n} = \sum_{j=0}^{n} P_{\omega,k,j} P_{\upsilon,k,n-j}.$$  

Note that the product is also a weighted isobaric polynomial of isobaric degree $n$. In the case where we have two integer evaluations of $P_{\omega,k,n}$ and $P_{\upsilon,k,n}$, we denote them as, respectively, $P'_{\omega,k,n}$ and $P''_{\upsilon,k,n}$, and their numerical convolution product is

$$P'_{\omega,k,n} \ast P''_{\upsilon,k,n} = \sum_{j=0}^{n} P'_{\omega,k,j} P''_{\upsilon,k,n-j}.$$  

It is with respect to this product and the ordinary addition of polynomials that the logarithm operator $\mathcal{L}$ is defined.

### 3. Permanent and Determinant Representations

In a formula was given for the elements of the divisible closure of the WIP-module, i.e., each element in this module was given a $q$-th root for all $q \in \mathbb{Q}$, where these roots are unique up to sign. In [2, Theorems 33, 34, 35], an interesting representation of the elements of WIP were given in terms of determinants and in terms of permanents of the following Hessenberg matrices.

$$H_{+ (\omega,k,n)} = \begin{pmatrix} t_1 & 1 & 0 & \cdots & 0 \\ t_2 & t_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & 1 \\ \omega_n t_n & \omega_{n-1} t_{n-1} & \omega_{n-2} t_{n-2} & \cdots & \omega_1 t_1 \end{pmatrix},$$

and

$$H_{- (\omega,k,n)} = \begin{pmatrix} t_1 & -1 & 0 & \cdots & 0 \\ t_2 & t_1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & -1 \\ \omega_n t_n & \omega_{n-1} t_{n-1} & \omega_{n-2} t_{n-2} & \cdots & \omega_1 t_1 \end{pmatrix}.$$
The principal results are that
\[ \text{perm } H_{+}(\omega, k, n) = P_{\omega, k, n} = \det H_{-}(\omega, k, n). \]

For example, we can look at the following matrix when \( n = 4 \)
\[
\begin{pmatrix}
  t_1 & 1 & 0 & 0 \\
  t_2 & t_1 & 1 & 0 \\
  t_3 & t_2 & t_1 & 1 \\
  \omega_4 t_4 & \omega_3 t_3 & \omega_2 t_2 & \omega_1 t_1 \\
\end{pmatrix}
\]
whose permanent is easily seen to be
\[
\omega_1 t_4^4 + (2\omega_1 + \omega_2) t_1^2 t_2 + \omega_2 t_2^2 + (\omega_1 + \omega_3) t_1 t_3 + \omega_4 = P_{\omega, 4, 4}.
\]

Moreover, it is easy to see that there is a nesting of the Hessenberg matrices from lower right hand corner to upper left. We shall call these representations \textit{Hessenberg representations}. It turns out that we can use these to go further and give a useful representation of the \( q \)-th convolution roots of generalized Fibonacci polynomials in terms of matrices of this type.

4. Convolution Roots

MacHenry and Tudose [5, Theorems 5.1 and 5.7] gave a general expression for the \( q \)-th, \( q \in \mathbb{Q} \), convolution roots of the GFPs, and more generally a general expression for the \( q \)-th convolution roots of all of the WIPs.

The formula for \( q \)-th roots of polynomials in GFP is given by
\[
F_{q, k, n}^q = \sum_{\alpha \vdash n} \frac{1}{|\alpha|!} B_{|\alpha|-1} \left( \begin{array}{c}
|\alpha| \\
\alpha_1, \ldots, \alpha_k
\end{array} \right) t_{\alpha_1}^{\alpha_2} \cdots t_{\alpha_k}^{\alpha_k}.
\]

For \( n = 3 \) and \( n = 4 \), we have the following determinantal representations

\[ F_{q, 3}^q = \det \begin{pmatrix}
  qt_1 & -1 & 0 \\
  qt_2 & \frac{1}{2}(q+1)t_1 & -1 \\
  qt_3 & \frac{1}{3}(2q+1)t_2 & \frac{1}{3}(q+2)t_1
\end{pmatrix} \]

and

\[ F_{q, 4}^q = \det \begin{pmatrix}
  qt_1 & -1 & 0 & 0 \\
  qt_2 & \frac{1}{2}(q+1)t_1 & -1 & 0 \\
  qt_3 & \frac{1}{3}(2q+1)t_2 & \frac{1}{3}(q+2)t_1 & -1 \\
  qt_4 & \frac{1}{4}(3q+1)t_3 & \frac{1}{4}(2q+2)t_2 & \frac{1}{4}(q+3)t_1
\end{pmatrix}, \]

where \( B_j \) is the polynomial generating function for the Stirling numbers of the 1st kind evaluated at \( q \). \( (B_{-j} \text{ the analogue, determined by the polynomial generating function for Stirling numbers of the 2nd kind}); \)

namely, \( B_j = q(q+1) \cdots (q+j) \) and \( B_{-j} = q(q-1) \cdots (q-j) \). We call \( B_j \) and \( B_{-j} \) \textit{Stirling operators} of 1st kind and 2nd kind, respectively.

The main theorem of this paper is a generalization to arbitrary \( n \) of the two matrices which appear above.
The first five such roots, starting with \( F_{k,0}^q \) for an arbitrary \( q \) are

\[
\begin{align*}
F_{k,0}^q &= 1 \\
F_{k,1}^q &= qt_1 \\
F_{k,2}^q &= \frac{1}{2}q(q+1)t_1^2 + qt_2 \\
F_{k,3}^q &= \frac{1}{3!}q(q+1)(q+2)t_1^3 + q(q+1)t_1 t_2 + qt_3 \\
F_{k,4}^q &= \frac{1}{4!}q(q+1)(q+2)(q+3)t_1^4 + \frac{1}{2!}q(q+1)(q+2)t_1^2 t_2 + \frac{1}{2!}q(q+1)t_1 t_3 + qt_4 \\
F_{k,5}^q &= \frac{1}{5!}q(q+1)(q+2)(q+3)(q+4)t_1^5 + \frac{1}{3!}q(q+1)(q+2)(q+3)t_1^3 t_2 + \frac{1}{2!}q(q+1)(q+2)t_1^2 t_3 + \frac{1}{2!}q(q+1)(q+2)t_1 t_4 + qt_5
\end{align*}
\]

and in the Stirling operator notation, these translate into:

\[
\begin{align*}
F_{k,0}^q &= 1 \\
F_{k,1}^q &= B_0 t_1 \\
F_{k,2}^q &= \frac{1}{2!}B_1 t_1^2 + B_0 t_2 \\
F_{k,3}^q &= \frac{1}{3!}B_2 t_1^3 + \frac{1}{2!}2B_1 t_1 t_2 + B_0 t_3 \\
F_{k,4}^q &= \frac{1}{4!}B_3 t_1^4 + \frac{1}{3!}3B_2 t_1^2 t_2 + \frac{1}{2!}2B_1 t_1 t_3 + B_0 t_4 \\
F_{k,5}^q &= \frac{1}{5!}B_4 t_1^5 + \frac{1}{4!}4B_3 t_1^3 t_2 + \frac{1}{3!}3B_2 t_1 t_2 t_3 + \frac{1}{2!}2B_1 t_1 t_4 + B_0 t_5
\end{align*}
\]

A rule of thumb for writing the \( q \)-th convolution root is as follows: First write the polynomial \( F_n \) as a function of \( t_j, j = 1, \ldots, k \), then, observing the exponent sum \( |\alpha| \), monomial by monomial, enter the fraction \( \frac{1}{|\alpha|} \), and the Stirling function values \( B_{|\alpha|-1} \). There will usually be some cancellations among the fractions for the most economical expression.

For example,

\[
F_{k,3} = t_1^3 + 2t_1 t_2 + t_3
\]

and

\[
F_{k,3}^q = \frac{1}{3!}B_2 t_1^3 + \frac{1}{2!}2B_1 t_1 t_2 + B_0 t_3.
\]

**Theorem 5.**

\[
F_{k,n}^q = \det \left( \begin{array}{cccc}
qt_1 & -1 & 0 & \cdots & 0 \\
qt_2 & \frac{1}{2}q(q+1)t_1 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
qt_{n-1} & \frac{1}{n-1}(n-2)q+1)t_{n-2} & \frac{1}{n-1}(n-3)q+2)t_{n-3} & \cdots & -1 \\
qt_n & \frac{1}{n}(n-1)q+1)t_{n-1} & \frac{1}{n}(n-2)q+2)t_{n-2} & \cdots & \frac{1}{n}(q+(n-1))t_1
\end{array} \right)
\]

\[
= \perm \left( \begin{array}{cccc}
qt_1 & 1 & 0 & \cdots & 0 \\
qt_2 & \frac{1}{2}q(q+1)t_1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
qt_{n-1} & \frac{1}{n-1}(n-2)q+1)t_{n-2} & \frac{1}{n-1}(n-3)q+2)t_{n-3} & \cdots & 1 \\
qt_n & \frac{1}{n}(n-1)q+1)t_{n-1} & \frac{1}{n}(n-2)q+2)t_{n-2} & \cdots & \frac{1}{n}(q+(n-1))t_1
\end{array} \right)
\]
Proof. Note that the determinants and permanents are nested, that is, \( F_{k,j}^{q} \) is the \( j \times j \) principal minor in the upper left hand corner of the matrices. This allows us to use induction in our proof. We shall carry out the computations for the determinant case. The proof for permanent case is similar.

**Lemma 6.** \( F_{k,n}^{q} \) satisfies the recursive formula

\[
F_{k,n}^{q} = s_1 F_{k,n-1}^{q} + s_2 F_{n-2}^{q} + s_3 F_{k,n-3}^{q} + \cdots + s_{n-1} F_{k,1}^{q} + s_n F_{k,0}^{q},
\]

where the recursion parameters \( s_j = \frac{1}{n}(jq + n - j)t_j \).

Proof. The nesting of the matrices, and hence of the determinants and permanents, implies recursion:

Let \( M_n = \det \begin{pmatrix}
q_1 & -1 & 0 & \cdots & 0 \\
q_2 & \frac{1}{2}(q+1)t_1 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
q_{n-1} & \frac{1}{n-1}((n-2)q + 1)t_{n-2} & \frac{1}{n-1}((n-3)q + 2)t_{n-3} & \cdots & -1 \\
q_n & \frac{1}{n}((n-1)q + 1)t_{n-1} & \frac{1}{n}((n-2)q + 2)t_{n-2} & \cdots & \frac{1}{n}(q + (n-1))t_1
\end{pmatrix} \)

1 and \( m_{i,j} \) be the \((i,j)\)th entry in the matrix. To prove the recursive formula, it is equivalent to prove

\[
M_n = m_{n,n}M_{n-1} + m_{n,n-1}M_{n-2} + \cdots + m_{n,2}M_1 + m_{n,1}M_0.
\]

We do the cofactor expansion along \( n \)th column from bottom to top and we get:

\[
M_n = m_{n,n}M_{n-1} + \det \begin{pmatrix}
m_{1,1} & -1 & 0 & \cdots & 0 \\
m_{2,1} & m_{2,2} & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
m_{n-2,1} & m_{n-2,2} & m_{n-2,3} & \cdots & -1 \\
m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \cdots & m_{n,n-1}
\end{pmatrix}
\]

Then do the cofactor expansion along the last column from bottom to top.

\[
= m_{n,n}M_{n-1} + m_{n,n-1}M_{n-2} + \det \begin{pmatrix}
m_{1,1} & -1 & 0 & \cdots & 0 \\
m_{2,1} & m_{2,2} & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
m_{n-3,1} & m_{n-3,2} & m_{n-3,3} & \cdots & -1 \\
m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \cdots & m_{n,n-2}
\end{pmatrix}
\]

Continue to do cofactor along the last column.

\[
= m_{n,n}M_{n-1} + m_{n,n-1}M_{n-2} + \cdots + m_{n,2}M_1 + m_{n,1}M_0.
\]

Let \( s_j = m_{n,n-j+1} \). We then have

\[
M_n = s_1 M_{n-1} + s_2 M_{n-2} + \cdots + s_{n-1} M_1 + s_n M_0.
\]
Putting $F^q_{k,n-j} = M_{n-j}$, we assume inductively that $M_{n-j} = F^q_{k,n-j}$, $j = 0, 1, \ldots, n - 1$, we have

$$M_n = s_1 F^q_{k, n-1} + s_2 F^q_{k, n-2} + \cdots + s_n F^q_{k, 0}.$$  

Now we only need to show that $M_n = F^q_{k, n}$.

Recall that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \vdash n$ means $\alpha_1 + 2\alpha_2 + \cdots + k\alpha_k = n$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$.

To prove $F^q_{k, n} = \sum_{\alpha \vdash n} \frac{1}{|\alpha|!} B_{|\alpha|-1} \left( \binom{\alpha_1}{\alpha}, \ldots, \binom{\alpha_k}{\alpha} \right) t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k} = \sum_{\alpha \vdash n} \frac{B_{|\alpha|-1}}{\alpha_1! \alpha_2! \cdots \alpha_k!} t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_k^{\alpha_k}$ is the determinant or permanent of the matrices in Theorem 5, we only need to show that $F^q_{k, n}$ satisfies the recursive formula in Lemma 6 which is equivalent to show that

$$\frac{B_{|\alpha|-1}}{\alpha_1! \alpha_2! \cdots \alpha_k!} = \frac{k}{n} \frac{B_{|\alpha|-2}}{\alpha_1! \alpha_2! \cdots (\alpha_i - 1)! \cdots \alpha_k!} \left( (q + n - 1) B_{|\alpha|-2} + \frac{(2q + n - 2) B_{|\alpha|-2}}{n \alpha_1! (\alpha_2 - 1)! \cdots \alpha_k!} + \cdots + \frac{(kq + n - k) B_{|\alpha|-2}}{n \alpha_1! \cdots (\alpha_k - 1)!} \right)$$

$$= \frac{B_{|\alpha|-2}}{n \alpha_1! \alpha_2! \cdots \alpha_k!} \left( \alpha_1 (q + n - 1) + \alpha_2 (2q + n - 2) + \cdots + \alpha_k (kq + n - k) \right)$$

$$= \frac{B_{|\alpha|-2}}{n \alpha_1! \alpha_2! \cdots \alpha_k!} \left( \alpha_1 q + n \alpha_1 - \alpha_1 + 2\alpha_2 q + \alpha_2 - 2\alpha_2 + \cdots + k\alpha_k q + n\alpha_k - k\alpha_k \right)$$

$$= \frac{B_{|\alpha|-2}}{n \alpha_1! \alpha_2! \cdots \alpha_k!} \left( (\alpha_1 + 2\alpha_2 + \cdots + k\alpha_k) q + n(\alpha_1 + \alpha_2 + \cdots + \alpha_k) - (\alpha_1 + 2\alpha_2 + \cdots + k\alpha_k) \right)$$

$$= \frac{B_{|\alpha|-2}}{n \alpha_1! \alpha_2! \cdots \alpha_k!} \left( q + n |\alpha| - n \right)$$

$$= \frac{B_{|\alpha|-2}}{\alpha_1! \alpha_2! \cdots \alpha_k!} \left( q + n |\alpha| - 1 \right)$$

$$= \frac{B_{|\alpha|-1}}{\alpha_1! \alpha_2! \cdots \alpha_k!}$$

It is of interest to see the matrix which represents the convolution roots in a form which explicitly displays the Stirling operators $B_j$, which we now do in

**Corollary 7.** $F^q_{k, n}$ is the determinant of the following matrix:

$$
\begin{pmatrix}
B_0 t_1 & -1 & 0 & 0 & \cdots & 0 \\
B_0 t_2 & \frac{1}{2} B_1 t_1 & -1 & 0 & \cdots & 0 \\
B_0 t_3 & \frac{1}{3} (2 B_1 - 1) t_2 & \frac{1}{3} B_2 t_1 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
B_0 t_n & \frac{1}{n} ((n - 1) B_1 - (n - 2)) t_{n-1} & \frac{1}{n} (n - 2) B_2 t_{n-1} & \frac{1}{n} B_3 t_{n-1} & \cdots & 1 \frac{1}{n} \left( \frac{B_{n-1}}{B_{n-2}} \right) t_1
\end{pmatrix}.
$$
So the recursion coefficients are
\[ s_j = \frac{1}{n} \left( j \frac{B_{n-j}}{B_{n-j-1}} - (n-j)(j-1) \right) t_j, \text{ for } j = 1, \ldots, n-1, \text{ and } s_n = B_0 t_n. \]

\[ \Box \]

We shall call these representations, \textit{Hessenberg-Stirling representations}. The Stirling part is due to the role that the Stirling operators play in the construction of the roots of the GPs.

The root formula for the WIP polynomials is a generalization of the root formula for the GFP, and is a bit more complicated.

\textbf{Theorem 8 (\cite{citation}).}

\[ P_{\omega,k,n}^q = \sum_{\alpha+n} L_{k,n,\omega}(\alpha) t_1^{\alpha_1} \cdots t_k^{\alpha_k}, \]

where
\[ L_{\omega,k,n}(\alpha) = \sum_{j=0}^{|\alpha|-1} \frac{1}{(\Pi_{i=1}^{|\alpha|})} \left( \begin{array}{c} |\alpha| - 1 \\ j \end{array} \right) B_{-j} D_{|\alpha|-j-1}(\omega_1^{\alpha_1} \cdots \omega_k^{\alpha_k}) \]

and \( D_j(\omega) = D_j(\omega_1^{\alpha_1} \cdots \omega_k^{\alpha_k}) \) is the total derivative of the expression \( j \) times.

The \textit{total differential operator} \( D_j \) is defined inductively by \( D_j = D_1(D_{j-1}) \) with \( D_1(\omega_1^{\alpha_1} \cdots \omega_k^{\alpha_k}) = \sum_{i=1}^k \partial_i(\omega_1^{\alpha_1} \cdots \omega_k^{\alpha_k}) = \sum_{i=1}^k \alpha_i(\omega_1^{\alpha_1} \cdots \omega_i^{\alpha_i-1} \cdots \omega_k^{\alpha_k}) \). For example, \( D_2(\omega_1^3 \omega_2^2) = 6\omega_1^4 \omega_2 + 12\omega_1^3 \omega_2^2 + 2\omega_1^2 \).

Here we give some low-dimensional examples for the \( q \)-th roots of weighted isobaric polynomials:

\[ P_{\omega,k,0}^q = 1 \]
\[ P_{\omega,k,1}^q = q \omega_1 t_1 \]
\[ P_{\omega,k,2}^q = (q \omega_1 + 2q(q-1)\omega_1^2)t_2^2 + q\omega_2 t_2 \]
\[ P_{\omega,k,3}^q = (q \omega_1 + q(q-1)\omega_1^2 + \frac{1}{3!}q(q-1)(q-2)\omega_1^3)t_1^3 + [q(\omega_1 + \omega_2) + q(q-1)\omega_1 \omega_2]t_1 t_2 + q\omega_3 t_3 \]

and in the Stirling operator notation, these translate into:

\[ P_{\omega,k,0}^q = 1 \]
\[ P_{\omega,k,1}^q = B_0 \omega_1 t_1 \]
\[ P_{\omega,k,2}^q = [B_0 \omega_1 + \frac{1}{2}B_{-1} \omega_1^2] t_1^2 + B_0 \omega_2 t_2 \]
\[ P_{\omega,k,3}^q = [B_0 \omega_1 + B_{-1} \omega_1^2 + \frac{1}{3!}B_{-2} \omega_1^3] t_1^3 + [B_0 (\omega_1 + \omega_2) + B_{-1} \omega_1 \omega_2] t_1 t_2 + B_0 \omega_3 t_3 \]

\textbf{Remark 9.} A more precise notation for the roots is \( P^{**} \), emphasizing that this root is to be taken with respect to the convolution product, that is, to retrieve the original function after have taken the \( q \)-th root, one must take the convolution product \( \frac{1}{q} \) times. We shall use the shorter form \( P^q \) with the meaning \( P^q = P^{**} \).

We would like to point out the rather unexpected usefulness of what we have called the Stirling operators, that is, the generating functions of the Stirling numbers of both first and second kind, in order to produce these convolution roots. So far, we know of no such previous applications using the Stirling generating functions. As for the usefulness of these results, we would point out that they give a complete answer to a concern that arose in arithmetic number theory, namely, that of the construction of the rational roots of the group of multiplicative arithmetic functions under the Dirichlet product \( \square \). In the next section, we shall
describe how isobaric polynomials (in particular generalized Fibonacci polynomials, and generalized Lucas polynomials) are used to produce isobaric isomorphic copies of the groups of multiplicative and of additive arithmetic function \[2\]. Here the key factor is that the convolution product discussed in this paper has as its arithmetic analogue, the Dirichlet product, thus our result is transmitted to the arithmetic function cases through this isomorphism. However, it is also evident that its use has wider applications, say to other groups and certain rings as well.

5. Multiplicative Arithmetic Functions

The ring (UFD) of arithmetic functions consists of the functions \(\alpha : \mathbb{Z} \to \mathbb{Q}\). The Dirichlet product of two arithmetic functions \(\alpha\) and \(\beta\) is given by

\[
\alpha \ast \beta(n) = \sum_{d|n} \alpha(d)\beta\left(\frac{n}{d}\right),
\]

where \(d|n\) \[9\].

The multiplicative arithmetic functions (MF) are those functions \(\alpha\) such that

\[
\alpha(mn) = \alpha(m)\alpha(n)
\]

whenever \((m, n) = 1\). This is equivalent to saying that a multiplicative function is completely determined by its values at primes. We shall say that such functions are determined locally, so that we are interested in the products

\[
\alpha \ast \beta(p^n) = \sum_{i=0}^{n} \alpha(p^i)\beta(p^{n-i}).
\]

If we consider the set GFP with the convolution product as multiplication, then we also get an abelian group. And if we consider all of the evaluations of the variables \(t_j\) over the integers, we produce a group that is locally isomorphic to the group MF \[7, 2\]. It was shown in \[5\] that this induces a mapping from the divisible closure of the group GFP to the divisible closure of MF, and this mapping is a local isomorphism.

Thus the matrix representation of \(F_{k,n}^q\) carries over to a matrix representation of the divisible closure of MF. (There are analogous results for the group GLP and the group AddF of additive arithmetic functions.)

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