Odd-parity superconductivity near an inversion breaking quantum critical point
in one dimension

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We study how an inversion-breaking quantum critical point affects the ground state of a one-dimensional electronic liquid with repulsive interaction and spin-orbit coupling. We find that regardless of the interaction strength, the critical fluctuations always lead to a gap in the electronic spin-sector. The origin of the gap is a two-particle backscattering process, which becomes relevant due to renormalization of the Luttinger parameter near the critical point. The resulting spin-gapped state is topological and can be considered as a one-dimensional version of spin-triplet superconductor. Interestingly, in the case of a ferromagnetic critical point the Luttinger parameter is renormalized in the opposite manner, such that the system remains non-superconducting.

Evidence for odd-parity superconductivity were found in doped Bi$_2$Se$_3$ [25–28], via NMR [29] and specific heat [30] measurements, which agree with theoretical proposals [22, 31, 32]. The superconducting state was shown to be correlated with structural transitions induced by external pressure [33], which may also be induced internally by the doping process [34]. Thus, it is interesting to understand the interplay between structural transitions and superconductivity in topological materials.

In this paper we study the impact of an inversion breaking structural transition on metallic states in one-dimension. The advantage of one dimension is that we have a good description of the interacting electronic state in terms of a Luttinger liquid (LL). It is important to note that one-dimensional superconductivity differs from higher dimensional superconductivity. Fluctuations prevent the establishment of true long-range order; instead, the order parameter exhibits only power law correlations, while spin excitations become gapped. This state is known as a Luther-Emery liquid (LEL) [35].

For concreteness we study a specific model. The model consists of two electronic wires coupled to a soft transverse optical phonon that undergoes a transition into a polarized state, where it breaks inversion between the wires. We note, however, that the low-energy theory we obtain is generic and describes any spin-orbit coupled system, which is coupled to an inversion breaking transition in one dimension. We tune the phonon through an inversion-breaking QCP and find that despite the repulsive electron-electron interactions there is always a region, close enough to the critical point, where a LEL is formed. The electron-electron interactions are crucial for the formation of the gap; they provide a backscattering term, which becomes relevant near the critical point. We also show that the most divergent superconducting order parameter is odd under inversion, and therefore we identify this state as the one-dimensional version of an odd-parity superconductor. Interestingly, this type of LEL was shown to be a gapless topological state protected by time-reversal symmetry [36], and similar to gapped states [37–40]. Note that an analogous construction with a ferromagnetic QCP, where time-reversal symmetry is broken instead of inversion, does not lead to superconductivity.

Model – We consider a minimal model for a one-dimensional metal with inversion symmetry and spin-orbit coupling. The model consists of two wires, which are interchanged under inversion and thus have Rashba spin-orbit coupling of opposite signs, as shown in Fig. 1(a). The structural inversion-breaking transition is considered to be due to an ionic distortion between these wires, represented by the grey sites in Fig. 1(a). The electrons are coupled to the ions through the local deformation potential generated by the distortion.

We now elaborate on each ingredient of the model starting with the two fermionic wires, which are described
polarization where \( \psi \) near the QCP. This mode can be described by a scalar can focus on the lower energy one, which becomes soft lift the degeneracy between these modes, such that one
described by a scalar

doing by their helicity
in the spin space.
local repulsive interaction, and respond to
distortions of the ions between the wires.

\[ \mathbf{t}_\perp = \sum_j \psi_j^\dagger \left( \psi_\perp \psi_2 + \psi_\perp \psi_1 \right) + V_{\text{int}}, \] (1)

where \( \psi_1 = (\psi_{11}, \psi_{12})^T \) and \( \psi_2 = (\psi_{21}, \psi_{22})^T \) are the fermionic fields of the two wires, \( m \) is the mass band, \( \alpha \) is the strength of Rashba spin-orbit coupling and \( \pm \) correspond to \( j = 1, 2 \), respectively. \( \mu \) is the chemical potential. The term \( V_{\text{int}} \) describes a generic \( S_z \) conserving local repulsive interaction, and \( \sigma^z \) are the Pauli matrices in the spin space.

Tunneling between the wires, \( t_\perp \), opens a gap at the \( \Gamma \)-point, as shown in Fig. 1(b). We consider the case \( |\mu| < t_\perp \), where only two points cross the Fermi energy. These two modes are denoted by their helicity \( \nu = \pm \), which is interchanged under inversion.

Next, we consider the phonon mode that becomes soft at the QCP. This mode describes the motion of the ions located in between the two fermionic wires, as shown in Fig. 1. These ions are localized, but may fluctuate around their equilibrium positions. We consider tuning these ions to a critical point where they become soft and condense in a different configuration, which breaks the inversion symmetry between the wires [16].

Among the three (one longitudinal and two transverse) phonon modes, only the transverse modes become soft at the transition [41]. The transverse motion is decomposed into two inversion-breaking polarizations, \( \hat{y} \) and \( \hat{z} \) [see Fig. 1(c)]. The electronic wires, lying in the \( xz \)-plane, lift the degeneracy between these modes, such that one can focus on the lower energy one, which becomes soft near the QCP. This mode can be described by a scalar

\[ \mathcal{L}_f = -\lambda \varphi \left( \psi_1^\dagger \psi_1 - \psi_2^\dagger \psi_2 \right). \] (3)

In what follows we consider only this polarization. We note that in the case of the \( \hat{y} \)-polarization, shown on the right panel of Fig. 1(c), the lattice distortion generates a Rashba-like effect for tunneling between the wires, leading to a coupling of the form \( \mathcal{L}_{fb} = -i \lambda \varphi \left( \psi_1^\dagger \sigma^z \psi_1 - \psi_2^\dagger \sigma^z \psi_2 \right) \). This expression can be reduced to Eq. (3) by a unitary transformation, so the case of \( \hat{y} \)-polarization leads to the same results.

**Analysis of the model near the QCP** – To analyze the model given by Eqs. (1)-(3) we bosonize the two fermionic modes crossing the Fermi energy \( \psi_{\nu \nu'} \approx (F_{\nu} / \sqrt{2\pi a}) \exp(i \nu k_0^\perp x) \exp[-i (\nu \theta_{\nu} - \phi_{\nu'})] \). Here \( F_{\nu} \) are Klein factors, \( a \) is the short distance cutoff, \( \varepsilon = R, L \) and \( \nu = \pm \) denote the chirality and helicity of the modes, respectively (\( R, L \) correspond to \( \varepsilon = +, - \), respectively). Bosonic fields \( \theta_{\nu} \) and \( \phi_{\nu'} \) obey commutation relations \( [\theta_{\nu}(x), \phi_{\nu'}(x')] = i \varepsilon \theta_{\nu} \gamma_{R,L}(x - x') \), where \( \gamma_{R,L} \) is the Heaviside step function. In these notations, the charge densities \( \rho_{\nu} = -(1/\pi) \partial_x \theta_{\nu} \) and \( J_{\nu} = (1/\pi) \partial_x \phi_{\nu} \) are the charge and current densities of the helical band \( \nu \), respectively (the uniform part of density is omitted). Because of the helical structure of the bands, spin density \( \rho_{\nu}^s \) and spin current density \( J_{\nu}^s \) can be expressed through charge components as \( \rho_{\nu}^s = -\nu J_{\nu} \) and \( J_{\nu}^s = \nu \rho_{\nu} \).

After the bosonization, the fermionic Lagrangian (1) becomes a sum of two decoupled LLs describing charge and spin degrees of freedom:

\[ \mathcal{L}_f = \mathcal{L}_\rho + \mathcal{L}_\sigma = \frac{1}{2} \sum_{\eta = \rho, \sigma} \Phi_\eta^T G^{-1}_\eta \Phi_\eta + \frac{g}{2(\pi a)} \cos \sqrt{8}\rho_{\sigma}, \] (4)

where \( G^{-1}_\eta = \frac{1}{\pi} \left( -u_\eta K_\eta^\perp \partial_x^2 \right) \) is the bare
Green’s function and \( \Phi_\eta = (\phi_{\eta}, \theta_{\eta})^T \). Here we defined ‘charge’ and ‘spin’ variables \( \theta_{\eta} = (1/\sqrt{2}) (\theta_+ + \theta_-), \)
we assume that in the vicinity of \( r_c \) the system is already deep in the spin-gapped state, so that the cosine term in Eq. (6) can be expanded near one of its minima, leading
to an effective gap $\Delta$. We will justify this assumption later, by explicitly calculating $\Delta$ using the variational principle. In this case we can integrate out the field $\theta_\sigma$ in Eq. (6) and express the resulting Lagrangian in terms of the spin current, $J_\sigma = (\sqrt{2}/\pi) \partial_x \phi_\sigma$. In the long wavelength limit (which implies $r \gg q, \omega$ and $\Delta \gg u_\sigma q$), it becomes.

$$L_{\text{Ising}} = \frac{1}{2} J_\sigma \left( -\frac{\partial^2}{\zeta} - \rho_\sigma \partial_z^2 + \alpha \right) J_\sigma + V J_\sigma^4, \quad (8)$$

where $\zeta = 2 \Delta^2 r^2 / \pi u_\sigma K_\sigma (r^2 + \Delta^2 r_c)$, $\rho_\sigma = \pi u_\sigma K_\sigma r_c / 2r^2$ and $\alpha = \pi u_\sigma K_\sigma (1 - r_c/r) / 2$. The Lagrangian (8) describes the Ising transition in 1+1 dimensions. Here $\alpha$ plays the role of the tuning parameter. The term $V$ is responsible for quantum fluctuations, which shift the transition point. We note that in the absence of particle-hole symmetry the Ising theory, Eq. (8), is also coupled to the charge sector by $L_{\rho\sigma} = \frac{1}{2} \partial_x \theta J_\rho^2$, which modifies the nature of the transition [48, 49]. However, it couples the gapless-charge to gapped-Ising modes and thus does not shift the transition point or change the nature of the Ising phases.

We now contrast the results we have obtained for inversion-breaking transition with the ferromagnetic case considered in Ref. [50]. In the latter, the ferromagnetic fluctuations couple to the spin density as opposed to spin current, implying that $\partial_x \theta_\sigma$ substitutes $\partial_x \phi_\sigma$ in Eq. (5). The renormalized Luttinger parameter then equals $K_\sigma (r) = K_\sigma / \sqrt{1 - (r_c K_\sigma^2 / r)}$, which is only enhanced in the vicinity of the QCP, making the cosine term even more irrelevant. Therefore, in the case of a ferromagnetic transition, the fermionic spin sector remains gapless all the way to the QCP, which is characterized by the dynamical exponent $z = 2$, as opposed to the Ising transition in our case. In this case the superconducting correlations are not divergent.

**Calculation of the spin gap** – We now turn to explicitly calculate the magnitude of the spin gap, $\Delta$, as a function of the tuning parameter $r$. For this purpose we employ the variational principle, which is known to capture the qualitative behavior of the gapped LEL [42]. We introduce the variational action $L_{\text{var}} = \frac{1}{2} \Phi_\sigma^T G_\sigma^{-1} \Phi_\sigma - \frac{r}{\pi} \partial_x \phi_\sigma G_\sigma \partial_x \phi_\sigma + \Delta^2 \theta^2 / u_\sigma K_\sigma$, where $\Delta$ is the variational parameter, which represents the gap of spin excitations [the same $\Delta$ appears in Eq. (8)]. Therefore, we minimize the free energy corresponding to Eq. (6) with respect to the $\Delta$ [42, 44]. On the ordered side of the transition ($r < r_c$) the spin stiffness is negative. In this case we expand around the broken symmetry state, taking into account the higher order term presented in Eq. (8). Consequently, another BKT transition occurs on the ordered side, at $r = r_{**} \equiv r_c / \left[ 1 + (2K_\sigma^2)^{-1} \right]$ [44].

We note that the variational approach breaks down at two points: near $r = r_s$ and near $r = r_c$. In the former, the gap vanishes like $\Delta \approx \sqrt{r_s} \exp \left( -\frac{\pi}{A \sqrt{1 - r/r_c}} \right)$, where $A$ is defined in the SI [44]. In the latter, the quartic term $V$ in Eq. (8), which we have neglected in this calculation, becomes important in the regime $|r - r_c| \sim \rho_\sigma V / \zeta$ [51].

The results of the variational calculation [44] are summarized in Fig. 2, where we plot the gap, $\Delta$, as a function of the tuning parameter $r$. Starting from the disordered side and reducing the tuning parameter $r$ towards the critical value $r_c$, the fermionic sector first undergoes a BKT transition into the LEL at $r = r_s$. Then, at $r = r_c$ the Ising transition occurs and inversion becomes spontaneously broken, where the gap reaches its maximal value. Finally, at $r = r_{**}$, there is another BKT transition to the gapless LL state.

**2D wire construction** – We now extend our results to two dimensions by discussing the phase diagram of an array of wires, which are individually described by Eqs. (6) and the gapless charge sector. Close to the QCP and in the weak coupling limit, $\theta_\sigma$ is pinned on each wire and there is a finite gap $\Delta$. In this case, for weakly coupled wires we are left with an array of LL in the charge sector, $L_j = \frac{1}{2} \Phi_\rho^T T \Phi_\rho$, interacting by

$$L_{j,j+1} = \frac{1}{2\pi} \left( W_\theta \partial_x \phi_\rho^j \partial_x \phi_\rho^{j+1} + W_\theta \partial_x \phi_\rho^j \partial_x \phi_\rho^{j+1} \right) + g_\phi \cos \sqrt{2} \left( \phi_\rho^j - \phi_\rho^{j+1} \right) + g_\theta \cos \sqrt{2} \left( \theta_\rho^j - \theta_\rho^{j+1} \right).$$

$W_\phi$ and $W_\theta$ describe forward scattering (for repulsive interaction $W_\phi < W_\theta$). The cosine term $g_\phi$ is proportional to the strength of pair hopping between the wires and $g_\theta$ results form wire-number conserving processes.

This model was studied in Refs. [52–54]. It was shown that a superconducting phase, where $g_\phi$ is the most relevant perturbation, exists over a wide range of parameters in the case of spin-gapped wires [54]. In Fig. 3 we plot...
the phase diagram based on the scaling equations for $g_\phi$ and $g_\theta$ [54]. However, in our case the spin variable $\phi_s$ is locked to $\pm \pi/\sqrt{8}$ rather than $0, \pi$. As a result the two gapped phases are a spin-density wave and odd-parity superconductor rather than a charge density wave and even-parity superconductor. The novel aspect of this result is that inversion breaking clearly enhances the odd-parity channel over the even-parity one.

Conclusions – We have shown that a one-dimensional electronic liquid with strong spin-orbit coupling and repulsive interactions coupled to an inversion breaking QCP develops a spin gap close enough to the transition due to the establishment of a paired state. We have argued that this state indicates the emergence of odd-parity superconductivity near inversion breaking QCP. We have also shown that inversion breaking is distinct from the ferromagnetic transition, where the spin sector remains gapless.

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FIG. 3. Phase diagram obtained from the RG equations in Refs. [52, 53]. The blue region denotes odd-parity superconductivity, white slither denotes gapless sliding LL state and pink denotes a spin ordered state.

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Supplementary material for ”Odd-parity superconductivity near an inversion breaking quantum critical point in one dimension”

This supplementary material consists of two sections. In Section I, we elaborate on the variational method used to calculate the gap $\Delta$. This method breaks down near the BKT transition where $\Delta$ vanishes. Therefore in Section II we use the RG flow equations for the BKT transition to calculate $\Delta$ near the transition point $r = r_*$.

DETAILS OF THE VARIATIONAL CALCULATION OF THE SPIN GAP

In this section we elaborate on the variational method we used to calculate the gap in Fig. 2 of the main text. The calculation follows Ref. [42]. We start by introducing the variational Lagrangian $\mathcal{L}_V = (1/2)\Phi^2 \mathcal{G}^{-1}_V \Phi$, where

$$\mathcal{G}^{-1}_V = \frac{1}{\pi} \left( u_\sigma K_\sigma \left[ 1 - \frac{r_*}{\omega_q \sqrt{r_* \omega_q}} \right] q^2 \left( \frac{i \omega q}{q^2 + \Delta^2/\omega_q^2} \right) \right).$$

(9)

We then minimize the expectation value of the free energy corresponding to the Lagrangian given by Eq. (6) of the main text. The expectation value is taken with respect to the Gaussian Lagrangian $\mathcal{L}_V^G$, and $\Delta$ is the variational parameter [42]. As a result, the following self-consistent equation for $\Delta$ can be found:

$$\frac{\Delta^2}{\pi u_\sigma K_\sigma} = \frac{16 g}{(2\pi \alpha)^2} \exp \left[ -\frac{4}{\beta L} \sum_{\omega,q} \mathcal{G}^{\theta\theta}_V (i \omega, q) \right],$$

(10)

where $\mathcal{G}^{\theta\theta}_V$ is the $\theta\theta$ component of the Green’s function (9) given by

$$\mathcal{G}^{\theta\theta}_V (\omega, q) = \frac{\pi K_\sigma u_\sigma (\omega^2 + B)}{AB + (r_c + A + B) \omega^2 + \omega^4} ; \quad A \equiv u_\sigma^2 q^2 + \Delta^2 ; \quad B \equiv q^2 + r - r_c ,$$

$L$ is the system size and $\beta$ is the inverse temperature.

To obtain Fig. 2 of the main text we solve Eq. (10) numerically. In the limit of small $\Delta$, it can be resolved analytically (we consider zero-temperature limit):

$$\frac{\Delta^2}{\pi u_\sigma K_\sigma} = \frac{4g}{\pi^2 \alpha^2} \left( \frac{\Lambda}{\Lambda_0} \right)^{2K_\sigma} \left( \frac{\Delta}{u_\sigma \Lambda} \right)^{2K_\sigma \sqrt{(r - r_c)/r}},$$

(11)
where $\tilde{\Lambda} \sim \sqrt{\tau}$ is an effective energy scale below which the renormalized Luttinger parameters are described by Eq. (7) of the main text, and $\Lambda_0 \sim 1/\alpha$ is the ultraviolet cutoff. This result coincides with the RG calculation deep in the massive phase [42], and correctly predicts BKT transition point $r_\ast = r_c/(1 - K_{\sigma}^{-2})$, defined as a point where non-zero solution for $\Delta$ appears.

On the ordered side of the transition, $r < r_c$, the $q = 0, \omega = 0$ part of the $\phi \phi$ component of the matrix (9) becomes negative, signaling the instability to the broken symmetry state. As explained in the main text, in this case we must stabilize the theory with higher order terms of the form $V (\partial_x \phi)^4$, which are irrelevant near the Luttinger liquid fixed point, but are clearly important here. We make use of the fact that the mass term of a Ginzburg-Landau theory can be related to the mass on the disordered side $M_d$ by $M_o = -2 M_d$, and importantly is independent of $V$. Thus, to describe the system in the ordered phase we shift the prefactor of $(\partial_x \phi)^2$ by $3 u_{\sigma} K_{\sigma} (1 - r_c/r)$. Afterwards, the analysis above can be simply repeated for the regime $r < r_c$. Consequently, another BKT transition occurs on the ordered side, at $r = r_{\ast\ast} = r_c/\left[1 + (2K_{\sigma}^2)^{-1}\right]$. This second BKT transition must appear, because deep in the broken symmetry phase the $\phi$ degrees of freedom freeze. In this limit the only effect of the bosonic degrees of freedom is to break inversion, thus leading to a helical band structure at the Fermi energy equivalent to a single wire with Rashba spin-orbit coupling. Such a model does not have a spin gap when the interactions are strongly repulsive (see for example Ref. [?])

**TWO STEP RG CALCULATION OF THE SPIN GAP NEAR THE BKT TRANSITION**

In this section we analyze the BKT transition near $r = r_\ast$. We first obtain the value of $r_\ast$ for a finite value of $g$. Then we estimate the single particle gap $\Delta$ close to $r = r_\ast$. As explained above, the variational method breaks down at the transition because it does not take into account fluctuations which renormalize $K_{\sigma}$ [42]. For this purpose we consider the RG flow equations

$$\frac{dy(l)}{dl} = [2 - 2K(l)] y(l)$$

$$\frac{dK(l)}{dl} = -\frac{K^2(l) y^2(l)}{2}.$$  

Here $K(l)$ is the running Luttinger parameter of the spin sector, $y(l) = g(l)/\pi u_{\sigma}$ is the dimensional coupling constant and $l = \log (\Lambda_0/\Lambda)$ is the RG time. The initial values for these parameters are, therefore, $y(0) = g/\pi u_{\sigma}$ and $K(0) = K_{\sigma}$.

The gap can be estimated from the RG equations in the standard way. Basically, we integrate over the RG equations between the initial point $l = 0$ and $l = l_\Delta$, where we define the scale $\Delta = \Lambda_0 e^{-l_{\Delta}}$ as the scale at which $y(l_\Delta) \sim 1$. However, in this case we must separate the integration in two regions:

(i) For $\Lambda < \Lambda < \Lambda_0$, where $\Lambda \sim \sqrt{\tau}$, we are integrating out high energy states which are not affected by the soft phonon mode $\varphi$, which has a gap of size $\tilde{\Lambda}$. Therefore $K(l)$ is not renormalized by the phonons. If we assume that $K_{\sigma} \sim 1 \gg y(0)/2$ then we are far from the separatrix and therefore the flow is approximately vertical. Therefore, $K(l) = K_{\sigma}$ and the flow is described solely by Eq. (12). Integrating over Eq. (12) in this region yields

$$y_r = y_0 \left( \frac{\Lambda_0}{\Lambda} \right)^{2-2K_{\sigma}}.$$  

(ii) In the second step we integrate in range $\Delta < \Lambda < \tilde{\Lambda}$. Here $K_{\sigma}$ and $u_{\sigma}$ become renormalized by the soft phonon mode $\varphi$, according to Eq. (7) of the main text

$$K(\tilde{\Lambda}) = \tilde{K}_{\sigma}(r) = K_{\sigma} \sqrt{1 - \frac{r_c}{r}}, \quad u_{\sigma} \to \tilde{u}_{\sigma}(r) = u_{\sigma} \sqrt{1 - \frac{r_c}{r}} \quad \text{and therefore} \quad y_r \to \frac{y_0}{\sqrt{1 - \frac{r_c}{r}}} \left( \frac{\Lambda_0}{\tilde{\Lambda}} \right)^{2-2K_{\sigma}}.$$  

Since we are interested in analyzing the transition point we seek the value of $r_\ast$ such that these renormalized parameters land on the separatrix. In this case the flow strongly modifies $K$ and we must use both equations. Demanding that $K_{\sigma}(r_\ast) = 1 + y_\ast/2$ (which defines the separatrix at the scale $\Lambda$), where $y_\ast = y_{r_\ast}$, we obtain from Eq. (15) the self-consistent equation for the transition point

$$r_\ast = \frac{r_c}{1 - K_{\sigma}^{-2} \left[1 + \frac{y_\ast}{2}\right]^2},$$  

where $y_\ast$ is the effective coupling constant at the transition point.
which reduces to $r_\ast = r_c / (1 - K_\sigma^{-2})$ in the limit of $y = 0$. In the main text (Fig. 2) we have used $g = 0.2$, $u_\sigma = 3/2$, $r_c = 0.1\Lambda^2$ and $K_\sigma = 1.2$. These parameters lead to $r_\ast \approx 0.361\ldots$

Defining the parameter $y_\parallel(l) \equiv 2K(l) - 2$ and combining Eqs. (12,13) one obtains

$$y = \sqrt{y_\parallel^2 + \delta^2},$$

where $\delta^2 = A^2 (1 - r/r_\ast)$ dictates the flow line, where

$$A = \sqrt{2(a_2 - a_1)y_\ast},$$

$$a_1 = K_\sigma r_c/r_\ast \sqrt{1 - r_c/r_\ast} \quad \text{and} \quad a_2 = \{ K_\sigma - 1 + \frac{1}{2} [r_c / (r_c - r_\ast)] \} y_\ast.$$

Plugging Eq. (17) in the RG equations Eqs. (12,13) we obtain

$$\frac{dy_\parallel(l)}{dl} = -\delta^2 - y_\parallel^2(l).$$

Integrating this equation between $\tilde{\Lambda}$ and $\Delta$ one obtains [42]

$$\Delta \approx \tilde{\Lambda} \exp \left( -\frac{\pi}{A\sqrt{1 - r/r_\ast}} \right)$$

Therefore close to the seperatrix the gap vanishes exponentially fast. This result indicates that all derivatives of the gap $\Delta(r)$ vanish at $r = r_\ast$, signaling that the free energy is analytic to all orders, a well known property of the BKT transition [55].