On the Problem of Maximal $L^q$-regularity for Viscous Hamilton–Jacobi Equations

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Abstract

In this paper we prove a conjecture by P.-L. Lions on maximal regularity of $L^q$-type for periodic solutions to $-\Delta u + |Du|^{\gamma} = f$ in $\mathbb{R}^d$, under the (sharp) assumption that $q > d^{\frac{\gamma-1}{\gamma}}$.

1. Introduction

We address here the so-called problem of maximal $L^q$-regularity for equations of the form

$$-\Delta u(x) + |Du(x)|^{\gamma} = f(x) \quad \text{in } \mathbb{R}^d,$$

where $\gamma > 1$, $f : \mathbb{R}^d \to \mathbb{R}$ is 1-periodic (i.e. $f(x + z) = f(x)$ for all $x \in \mathbb{R}^d$, $z \in \mathbb{Z}^d$), $d \geq 1$; that is,

for all $M > 0$, there exists $K > 0$ (possibly depending on $M, \gamma, q, d$) such that

$$-\Delta u + |Du|^{\gamma} = f \quad \text{in } \mathbb{R}^d, \quad \|f\|_{L^q(Q)} \leq M$$

$$\implies \quad \|\Delta u\|_{L^q(Q)} + \||Du|^{\gamma}\|_{L^q(Q)} \leq K,$$

(M)

$Q$ being the $d$-dimensional unit cube $(-1/2, 1/2)^d$. This regularity problem has been proposed by P.-L. Lions in a series of seminars and lectures (e.g. [31,32]), where he conjectured its general validity under the assumption that

$$q > d^{\frac{\gamma-1}{\gamma}} \quad \text{(and } q > 1).$$

(A)

Some special cases have been addressed in these seminars, but the general problem has remained so far unsolved, to the best of our knowledge. We present here a proof...
of (M) + (A), under the sole restriction $q > 2$ (which is always realized when $\gamma > d/(d - 2)$).

Equations of the form (1) are prototypes of semilinear uniformly elliptic equations with superlinear growth in the gradient, and arise for example in ergodic stochastic control [4] and in the theory of growth of surfaces [25]. The study of regularity of their solutions has recently received a renewed interest in the theory of Mean Field Games [11,28]. There is a vast literature on such equations and more general quasilinear problems. While the existence of classical (or strong) solutions was investigated first (see for example [3,26,29,38]), the attention has since been largely focused on the existence (and uniqueness) of solutions $u \in W^{1,\gamma}(Q)$ satisfying (1) in the weak or generalized sense (typically with Dirichlet boundary conditions; see, for example, [1,6,8–10,14,18,22,35] and more recent works [2,7,16,17]. It has been observed that due to the superlinear nature of the problem, its (weak) solvability requires $f \in L^{q}$, where

$$q \geq d \frac{\gamma - 1}{\gamma}.$$  

Such a condition has been improved in the finer scale of Lorentz-Morrey spaces, and end-point situations typically require additional smallness assumptions [19,23]. It is worth observing that many results in the literature cover the case $1 < \gamma \leq 2$; that is, when the gradient term has at most natural growth. General results in the full range $\gamma > 1$, based on methods from nonlinear potential theory, appeared quite recently in [34,36,37].

Roughly speaking, properties (M)+(A) say that if $f$ belongs to a sufficiently small Lebesgue space, then solutions should enjoy much better regularity than $W^{1,\gamma}$, namely, be in $W^{1,q\gamma}(Q)$ (and even in $W^{2,q}(Q)$, by standard Calderón-Zygmund theory). Still, additional gradient regularity is typically achieved via methods that require much stronger hypotheses on the summability of $f$, being based on the classical or weak maximum principle: viscosity theory indeed requires $f$ to be bounded [24], while the Aleksandrov-Bakel’man-Pucci estimate needs $f \in L^{d}$, as in [33]. The situation is even worse when $\gamma > 2$, as one observes that general weak solutions are just Hölder continuous [15], so one has to select $u$ in a suitable class.

Here, we look at solutions to (1) that can be approximated by classical ones. Therefore, we will prove (M) in the form of an a priori estimate. It is known that in such a form, (M) cannot be expected in general if

$$1 < q \leq d \frac{\gamma - 1}{\gamma},$$

as described in Remark 1. On the other hand, P.-L. Lions indicated that (M)+(A) can be obtained in some particular cases. First, when $\gamma = 2$, the so-called Hopf-Cole transformation $v = e^{-u}$ reduces (1) to a linear elliptic equation, and one has the result employing (maximal) elliptic regularity and the Harnack inequality. Special cases $d = 1$ and $\gamma < d/(d - 1)$ can be also treated. As a final suggestion, an integral version of the Bernstein method [30] could be implemented to prove (M) when $q$ is
close enough to $d$ (see also [27], and [5] for further refinements of this technique), but the full regime (A) seems to be out of range using these sole arguments.

The Bernstein method is the starting point of our work. It consists in shifting the attention from the equation (1) for $u$ to the equation for a suitable function of $|Du|^2$, i.e. $w = g(|Du|^2)$; if $g$ is properly chosen, the equation for $w$ enjoys a strong degree of coercivity with respect to $w$ itself, which stems from uniform ellipticity and the coercivity of the gradient term in (1). By a delicate combination of these two regularising effects, it is possible to produce a crucial estimate on superlevel sets of $|Du|$, i.e.

$$
\left[ \int_{|Du| \geq k} \left( |Du| - k \right)^{\gamma q} \right]^{\frac{d-2}{d}} \leq \omega \left( \left| \left\{ |Du| \geq k \right\} \right| \right) + \int_{|Du| \geq k} \left( |Du| - k \right)^{\gamma q}
$$

(2)

for any $k \geq 0$, where $\omega(t) \to 0$ as $t \to 0$. This inequality again reflects the superlinear nature of the problem, being the exponents in the two sides unbalanced. Nevertheless, it is possible to control on $\| \nabla u \|_{L^q}$ as follows: (2) guarantees that $\int_{|Du| \geq k} \left( |Du| - k \right)^{\gamma q}$ is either belonging to a neighborhood of zero, or to an unbounded interval (for $k$ large enough, but independent of $\| \nabla u \|_{L^q}$). By the fact that $k \mapsto \int_{|Du| \geq k} \left( |Du| - k \right)^{\gamma q}$ is continuous and vanishes as $k \to \infty$, the second case can be ruled out, and boundedness of $\int_{|Du| \geq k} \left( |Du| - k \right)^{\gamma q}$ can be then recovered up to $k = 0$. This second key step has been inspired by an interesting argument that appeared in [20] (see also [21]), where $W^{1,2}$ estimates of (powers of) $u$ are obtained arguing similarly on superlevel sets of $|u|$.

Our result reads as follows:

**Theorem 1.1.** Let $f \in C^1(Q)$, $d \geq 3$, $\gamma > 1$ and

$$
q > d^{\gamma - 1} \gamma, \quad q > 2.
$$

For all $M > 0$, there exists $K = K(M, \gamma, q, d) > 0$ such that if $u \in C^3(Q)$ is a classical solution to (1) and

$$
\| f \|_{L^q(Q)} + \| Du \|_{L^1(Q)} \leq M,
$$

then

$$
\| \Delta u \|_{L^q(Q)} + \| \nabla u \|_{L^q(Q)} \leq K.
$$

We stress that our approach is not perturbative, in the sense that the gradient term is not treated as a perturbation of a uniformly elliptic operator (which would be natural under the growth condition $\gamma < 2$), nor vice-versa. It applies also to equations that have the gradient term with reversed sign (since there are no sign constraints on $f$, just reverse $u \mapsto -u$), and to solutions in a strong sense (Remark 3). As far as periodicity is concerned, it is common in applications to ergodic control
and Mean Field Games. The study of (M) in cases where \( u \) satisfies boundary conditions, or a local version of the estimate, will be matter of future work. We also conjecture that (M) holds in the limiting case \( q = d^\gamma \gamma - 1 \) under an additional smallness assumption on \( M \), which controls the norm of \( \| f \|_q \). This would be coherent with known results on the existence of weak solutions. Nevertheless, it does not seem evident how to adapt our proof to cover this end-point case.

Finally, our technique does not apply to the parabolic counterpart of (M). In this direction, some results based on rather different duality methods developed in [12] to get Lipschitz regularity, have been obtained in [13].

2. Proof of the Main Theorem

\( \partial_t, D, D^2 \) will denote the partial derivative in the \( i \)-th direction, the gradient, and the Hessian operator respectively. For the sake of brevity, we will often drop the \( x \)-dependence of \( u, Du, \ldots \), and the \( d \)-dimensional Lebesgue measure \( dx \) under the integral sign. \((x)^+ = \max\{x, 0\}\) will denote the positive part of \( x \), and for any \( p > 1 \), \( p' = p/(p - 1) \). For any measurable and 1-periodic set \( \Omega \subseteq \mathbb{R}^d \), \( |\Omega| \) will be the Lebesgue measure of its representative set, i.e. \( |\Omega| = \int_{\Omega \cap Q} dx \).

This section is devoted to the proof of Theorem 1.1, which will be based on the following lemma:

**Lemma 2.1.** There exists \( \delta \in (0, 1) \) (depending on \( \gamma, q, d \)) and \( \omega : [0, +\infty) \to [0, +\infty) \) (depending on \( M, \gamma, q, d \)) such that

\[
\lim_{t \to 0^+} \omega(t) = 0,
\]

and for all \( k \geq 1 \),

\[
\left( \int_{\Omega} \left( \left( 1 + |Du|^2 \right)^{\frac{1+\delta}{2}} - k \right)^{\frac{q \gamma}{1+\delta}} \right)^{d-2 \over d} \leq \omega \left( |\{1 + |Du|^2 > k^2 \gamma \}| \right) + \int_{\Omega} \left( \left( 1 + |Du|^2 \right)^{\frac{1+\delta}{2}} - k \right)^{\frac{q \gamma}{1+\delta}}. \quad (3)
\]

We postpone the proof of the lemma, and show first how (3) yields the conclusion of Theorem 1.1. Setting \( Y_k := \int_{\Omega} \left( \left( 1 + |Du|^2 \right)^{\frac{1+\delta}{2}} - k \right)^{\frac{q \gamma}{1+\delta}} \), then (3) reads as

\[
Y_k^{d-2 \over d} \leq Y_k + \omega \left( |\{1 + |Du|^2 > k^2 \gamma \}| \right) \quad \text{for all } k \geq 1. \quad (4)
\]

Note that the function \( F : Z \mapsto Z^{{d-2 \over d}} - Z \) has a unique maximizer \( Z^* = (d^{{d-2 \over d}})^d \) whose corresponding value is \( F(Z^*) = F^* > 0 \) (which depends on \( d \) only). For any \( 0 \leq \omega < F^* \) the equation

\[
F(Z) = \omega
\]
has two roots $0 < Z^- (\omega) < Z^* < Z^+ (\omega)$. Since $\lim_{t \to 0} \omega(t) = 0$, pick $t^* = t^*(M, \gamma, p, d)$ such that $\omega(t) < F^t$ for all $t < t^*$. By Chebyshev’s inequality,

$$\sqrt{k^{\frac{2}{1+s}}} - 1 > \frac{\|Du\|_{L^1(Q)}}{t^*} \implies |\{1 + |Du|^2 > k^{\frac{2}{1+s}}\}| < t^*, \quad \forall k > \left( \frac{\|Du\|_{L^1(Q)}}{t^*} + 1 \right)^{\frac{1+\delta}{2}} =: k^*, \quad Y_k < Z^* \text{ or } Y_k > Z^*.$$

hence (4) yields the alternative

$$\forall k > k^*, \quad Y_k < Z^*,$$

and finally,

$$\| |Du|^\gamma \|_{L^b(Q)}^{1+\delta} \leq \| |Du|^{1+\delta} \|_{L^k(Q)}^{\frac{\gamma}{1+\delta}} \leq \left( (1 + |Du|^2) \right)^{\frac{\gamma}{1+\delta} - k^*} + k^* \| |Du|^{\frac{\gamma}{1+\delta}} \|_{L^k(Q)} \leq \left( Z^* \right)^{\frac{\gamma}{1+\delta} + k^*}.$$

The estimate on $\| \Delta u \|_{L^p(Q)}$ is then straightforward.

Having proven Theorem 1.1, we now come back to the main estimate (3).

**Proof of Lemma 2.1.** Let $w(x) := g(|Du(x)|^2)$, where $g(s) = g_\delta(s) = \frac{2}{1+\delta} (1 + s)^{\frac{\delta}{2}}$, $\delta \in (0, 1)$ to be chosen later. Note that, for any $\delta \in (0, 1)$, $g$ enjoys the following properties: for all $s \geq 0$,

$$g'(s)s^{\frac{\delta}{2}} \leq (1 + s)^{\frac{\delta}{2}}, \quad (5)$$

$$g'(s) + 2s g''(s) \geq \delta g'(s). \quad (6)$$

Note also that

$$g'(|Du(x)|^2) = (1 + |Du(x)|^2)^{\frac{\delta-1}{2}} = \left( \delta + \frac{1}{2} - w \right)^{\frac{\delta-1}{2+\delta}}.$$

$(g, g', g''$ below will be always evaluated at $|Du(x)|^2$).

Define $w_k = (w - k)^+ \in W^{1, \infty}(Q)$ and set $\Omega_k := \{w > k\}$. We now use $\varphi = \varphi^{(j)} = -2\partial_j (g' \partial_j u w_k^\beta)$, $j = 1, \ldots, d$ and $\beta > 1$ to be chosen later as test functions in the Hamilton–Jacobi equation. First, integrating by parts and substi-tuting $\partial_i w = 2g' Du \cdot D\partial_i u$,

$$\sum_j \int_Q Du \cdot D\varphi = -2 \sum_{i,j} \int_Q \partial_i u \cdot \partial_j (\partial_i (g' \partial_j u w_k^\beta)) = 2 \sum_{i,j} \int_Q \partial_i u \partial_i (g' \partial_j u w_k^\beta)$$

$$= 4 \int_Q g'' \sum_j (Du \cdot D\partial_j u)^2 w_k^\beta + 2 \int_Q |D^2 u|^2 g^' w_k^\beta + \beta \int_Q w_k^{p-1} Dw_k \cdot Dw.$$
Moreover, again integrating by parts,
\[ -2 \sum_j \int_Q |Du|^{\gamma} \partial_j (g' \partial_j u \, w_k^{\beta}) = \gamma \sum_j \int_Q w_k^{\beta} |Du|^{\gamma-2} Du \cdot D\partial_j u \; 2g' \partial_j u \, w_k^{\beta} \]
\[ = \gamma \int_Q |Du|^{\gamma-2} Du \cdot Dw \, w_k^{\beta}. \]

Noting that \( w_k^{\beta-1} Dw = w_k^{\beta-1} Dw_k \) on \( Q \), we end up with
\[ \beta \int_Q w_k^{\beta-1} |Dw_k|^2 + \int_Q \left( 4g'' \sum_j (Du \cdot D\partial_j u)^2 + 2g'|D^2u|^2 \right) w_k^{\beta} \]
\[ + \gamma \int_Q |Du|^{\gamma-2} Du \cdot Dw_k \, w_k^{\beta} \]
\[ = -2 \int_Q f \text{div}(g' Du \, w_k^{\beta}). \] (7)

Note also that in (7) integrating on \( Q \) and on \( \Omega_k \) is the same, by the fact that \( w_k \) vanishes on \( Q \setminus \Omega_k \). We use first Cauchy–Schwarz inequality, the equation (1) and the inequality \((a - b)^2 \geq \frac{a^2}{2} - 2b^2\) for every \( a, b \in \mathbb{R} \) to get
\[ |D^2u|^2 \geq \frac{1}{d} (\Delta u)^2 \geq \frac{1}{2d} |Du|^{2\gamma} - \frac{2}{d} f^2. \]

Moreover, again by Cauchy–Schwarz inequality (be careful about \( g'' < 0 \)) and (6),
\[ g'|D^2u|^2 + 2g'' \sum_j (Du \cdot D\partial_j u)^2 \geq (g' + 2|Du|^2 g'') |D^2u|^2 \geq \delta g'|D^2u|^2. \]

The above inequalities then yield
\[ 2g'|D^2u|^2 + 4g'' \sum_j (Du \cdot D\partial_j u)^2 \geq \delta g'|D^2u|^2 + \frac{\delta}{2d} |Du|^{2\gamma} g' - \frac{2\delta}{d} f^2 g'. \]

Note that for \( \gamma > 1 \) it holds that
\[ (1 + |Du|^2)^\gamma \leq 2^{\gamma-1}(1 + |Du|^{2\gamma}), \] so \( |Du|^{2\gamma} \geq \frac{(1 + |Du|^2)^\gamma}{2^{\gamma-1}} - 1, \)
and hence, we are allowed to conclude
\[ \frac{\delta}{2d} |Du|^{2\gamma} g' \geq \frac{\delta}{2^{\gamma-1}d} (1 + |Du|^2)^{\gamma} g' - \frac{\delta}{2d} g' = \frac{\delta}{2d} (1 + |Du|^2)^{\gamma} + \frac{\delta - 1}{2d} g'. \]

This gives, going back to (7) and substituting \( (1 + |Du|^2)^{\frac{1}{2}} = \left( \frac{\delta + 1}{2} w \right)^{\frac{1}{\gamma+1}} \),
\[ \beta \int_{\Omega_k} w_k^{\beta-1} |Dw_k|^2 + \delta \int_{\Omega_k} g' w_k^{\beta} |D^2u|^2 + c_1 \int_{\Omega_k} w_k^{2\gamma + \delta - 1} w_k^{\beta} \]
\[ \leq \frac{\delta}{2d} \int_{\Omega_k} (1 + 4f^2) g' w_k^{\beta} - 2 \int_{\Omega_k} f \Delta u g' w_k^{\beta} - 4 \int_{\Omega_k} f g'' Du \cdot (D^2u Du) w_k^{\beta} \]
The first three terms are somehow similar: using Cauchy–Schwarz inequality and where

\[ c_1 = c_1(\delta, d, \gamma) > 0. \]

We now estimate the five terms on the right hand side of the previous inequality. The first three terms are somehow similar: using Cauchy–Schwarz inequality and that \( 2s g'' \leq g' \), we have for some \( c_2 = c_2(\delta, d) > 0 \) that

\[
\frac{\delta}{2d} \int_{\Omega_k} (1 + 4f^2)g'w_k^\beta - 2 \int_{\Omega_k} f\Delta u g'w_k^\beta - 4 \int_{\Omega_k} fg''Du \cdot (D^2u Du)w_k^\beta
\]

\[
\leq \frac{\delta}{2d} \int_{\Omega_k} (1 + 4f^2)g'w_k^\beta - (2d + 2) \int_{\Omega_k} |f||D^2u g'w_k^\beta
\]

\[
\leq \delta \int_{\Omega_k} |D^2u|^2 g'w_k^\beta + c_2 \int_{\Omega_k} (1 + f^2)w^{\frac{\beta-1}{\beta}}w_k^\beta. \tag{9}
\]

At this stage, we make some choices for the coefficients. Recalling that \( \frac{d}{\gamma'} < q \), we take

\[
p = \frac{2d}{d\gamma'} + \frac{d-2}{d}q, \quad \text{and} \quad \beta = \frac{1}{1+\delta}[\gamma(p-2) + 1 - \delta]. \tag{10}
\]

Note that \( \frac{d}{\gamma'} < p < q \). Assuming that \( p > 2 \) (which is always true when \( \gamma > \frac{d}{d-2} \)), otherwise see the remark at the end of the proof), we have \( \beta > 1 \) whenever \( \delta \) is close enough to zero. Moreover,

\[
\frac{2\gamma + \delta - 1}{1+\delta} = \frac{\delta - 1}{1+\delta} \frac{p}{p-2} + \beta \frac{2}{p-2}, \tag{11}
\]

\[
(\beta + 1) \frac{d}{d-2} = \gamma q \frac{1}{1+\delta}. \tag{12}
\]

Therefore, we apply Hölder’s inequality (with conjugate exponents \( p/2 \) and \( p/(p-2) \)) and Young’s inequality, and then \( w_k \leq w \) together with (11) to obtain

\[
c_2 \int_{\Omega_k} (1 + f^2)w^{\frac{\beta-1}{\beta}}w_k^\beta \leq c_2 \left( \int_{\Omega_k} (1 + f^2)^{\frac{\beta}{2}} \right)^\frac{2}{\beta} \left( \int_{\Omega_k} w^{\frac{\beta-1}{2} \frac{p}{p-2}}w_k^\beta \right) \left( \int_{\Omega_k} w^{\frac{\beta-1}{2} \frac{p}{p-2}}w_k^\beta \right)^{1-\frac{2}{p}}
\]

\[
\leq c_3 \int_{\Omega_k} (1 + |f|)^p + c_3 \int_{\Omega_k} w^{\frac{\beta-1}{2} \frac{p}{p-2} + \beta}w_k^\beta
\]

\[
\leq c_3 \int_{\Omega_k} (1 + |f|)^p + c_3 \int_{\Omega_k} w^{\frac{\beta-1}{2} \frac{p}{p-2} + \beta}w_k^\beta
\]

\[
\leq c_3 \int_{\Omega_k} (1 + |f|)^p + c_3 \int_{\Omega_k} w^{\frac{2\gamma + \delta - 1}{1+\delta}}w_k^\beta,
\]

where \( c_3 = c_3(\delta, d, \gamma, p) > 0 \). Plugging the previous inequality into (9) yields

\[
\frac{\delta}{2d} \int_{\Omega_k} (1 + f^2)g'w_k^\beta - 2 \int_{\Omega_k} f\Delta u g'w_k^\beta - 4 \int_{\Omega_k} fg''Du \cdot (D^2u Du)w_k^\beta
\]

\[
\leq \delta \int_{\Omega_k} |D^2u|^2 g'w_k^\beta + c_3 \int_{\Omega_k} (1 + |f|)^p + c_3 \int_{\Omega_k} w^{\frac{2\gamma + \delta - 1}{1+\delta}}w_k^\beta. \tag{13}
\]
The fourth term in (8) is a bit more delicate. We first use that \( s^\frac{1}{2} g'(s) \leq (1+s)^{\frac{\delta}{2}} \), in conjunction with Hölder’s and Young’s inequality, to get

\[
2\beta \int_{\Omega_k} f g' Du \cdot Dw_k w_k^{\beta - 1} \leq 2\beta \int_{\Omega_k} |f|(1 + |Du|^2)^{\frac{\delta}{2}} |Dw_k| w_k^{\beta - 1}
\]

\[
\leq 2\beta \left( \int_{\Omega_k} w_k^{\beta - 1} |Dw_k|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega_k} |f|^q \right)^{\frac{1}{q}}
\]

\[
\left( \int_{\Omega_k} (1 + |Du|^2)^{\frac{4}{2} \frac{pq}{q-p}} \int_{\Omega_k} w_k^{(\beta - 1) \frac{p}{p-2}} \right)^{\frac{p-2}{p}}.
\]

\[
\leq \frac{\beta}{3} \int_{\Omega_k} w_k^{\beta - 1} |Dw_k|^2 + \frac{c_1}{3} \int_{\Omega_k} w_k^{(\beta - 1) \frac{p}{p-2}} + c_4 \left( \int_{\Omega_k} |f|^q \right)^{\frac{p}{q}}
\]

\[
\left( \int_{\Omega_k} (1 + |Du|^2)^{\frac{4}{2} \frac{pq}{q-p}} \right)^{\frac{p-2}{p}}.
\]

where \( c_4 = c_4(\delta, d, \gamma, \beta) > 0 \). Since \( k \geq 1 \), we have \( w \geq 1 \) on \( \Omega_k \). Hence, recalling also (11),

\[
\int_{\Omega_k} w_k^{(\beta - 1) \frac{p}{p-2}} = \int_{\Omega_k} w_k^{\frac{2}{p-2} - \frac{p}{p-2}} w_k^\beta \leq \int_{\Omega_k} w_k^{\frac{2}{p-2} - \frac{p}{p-2}} w_k^\beta \leq \int_{\Omega_k} w_k^{\beta} = \int_{\Omega_k} w_k^{\frac{2\gamma + \delta - 1}{1 + \delta}} w_k^\beta.
\]

so

\[
2\beta \int_{\Omega_k} f g' Du \cdot Dw_k w_k^{\beta - 1}
\]

\[
\leq \frac{\beta}{3} \int_{\Omega_k} w_k^{\beta - 1} |Dw_k|^2 + \frac{c_1}{3} \int_{\Omega_k} w_k^{\frac{2\gamma + \delta - 1}{1 + \delta}} w_k^\beta
\]

\[
+ c_4 \| f \|_{L^q(Q)} \left( \int_{\Omega_k} (1 + |Du|^2)^{\frac{\delta}{2} \frac{pq}{q-p}} \right)^{\frac{q-p}{q}}.
\]

We now focus on the fifth term in (8). By Young’s inequality,

\[
- \gamma \int_{\Omega_k} |Du|^{\gamma - 2} Du \cdot Dw_k w_k^\beta \leq \frac{3\gamma^2}{4\beta} \int_{\Omega_k} |Du|^{2\gamma - 2} w_k^{\beta + 1} + \frac{\beta}{3} \int_{\Omega_k} |Dw_k|^2 w_k^{\beta - 1}.
\]

Furthermore, letting

\[
\eta = \frac{2\gamma + \delta - 1}{1 + \delta}, \quad \text{so that} \beta + \eta = \frac{p\gamma}{1 + \delta},
\]

we get (it holds that \( s^\frac{1}{2} \leq g^\frac{1}{1+\delta} \))
\[
\int_{\Omega_k} |Du|^{2\gamma-2} w_k^{\beta+1} \leq \int_{\Omega_k} w_k^{\frac{2\gamma-2}{1+\varepsilon}} w_k^{\beta+1} = \int_{\Omega_k} w^{\eta-1} w_k^{\beta/\eta} w_k^{\beta/\eta+1}
\leq \left( \int_{\Omega_k} w^{\eta} w_k^{\beta} \right)^{\frac{1}{\eta}} \left( \int_{\Omega_k} w_k^{\beta+\eta} \right)^{\frac{1}{\eta}}.
\]

Plugging the previous inequality into (15) and using Young’s inequality again leads to
\[
-\gamma \int_{\Omega_k} |Du|^{\gamma-2} Du \cdot Dw_k w_k^{\beta} \leq c_1 \int_{\Omega_k} w_k^{\frac{2\gamma+\delta-1}{1+\varepsilon}} w_k^{\beta} + c_5 \int_{\Omega_k} w_k^{\frac{p\gamma}{p+\varepsilon}}
+ \frac{\beta}{3} \int_{\Omega_k} |Dw_k|^2 w_k^{\beta-1}
\]
for some \( c_5 = c_5(\delta, d, \gamma, p) > 0. \)

Plug now (13), (14) and (16) into (8) to obtain
\[
\frac{\beta}{3} \int_{\Omega_k} w_k^{\beta-1} |Dw_k|^2 \leq c_3 \int_{\Omega_k} (1 + |f|)^p + c_4 \| f \|_{L^p(\Omega)}^p \left( \int_{\Omega_k} (1 + |Du|^2)^{\frac{\delta}{2} \frac{p\gamma}{\gamma-p}} \right)^{\frac{q-p}{q}}
+ c_5 \int_{\Omega_k} w_k^{\frac{p\gamma}{p+\varepsilon}}.
\]

Sobolev’s inequality, related to the continuous embedding of \( W^{1,2}(\Omega) \) into \( L^{\frac{2d}{d-2}}(\Omega) \), gives (for \( c_6 = c_6(d, \delta, \gamma, p) \))
\[
\frac{\beta}{3} \int_{\Omega} w_k^{\beta-1} |Dw_k|^2 \geq c_6 \left( \int_{\Omega} w_k^{(\beta+1) \frac{d}{d-2}} \right)^{\frac{d-2}{d}} - \frac{\beta}{3} \int_{\Omega} w_k^{\beta+1},
\]
hence
\[
c_6 \left( \int_{\Omega_k} w_k^{(\beta+1) \frac{d}{d-2}} \right)^{\frac{d-2}{d}}
\leq c_3 \int_{\Omega_k} (1 + |f|)^p + c_4 \| f \|_{L^p(\Omega)}^p \left( \int_{\Omega_k} (1 + |Du|^2)^{\frac{\delta}{2} \frac{p\gamma}{\gamma-p}} \right)^{\frac{q-p}{q}}
+ c_5 \int_{\Omega_k} w_k^{\frac{p\gamma}{p+\varepsilon}} + \frac{\beta}{3} \int_{\Omega_k} w_k^{\beta+1}.
\]

We finally choose \( \delta > 0 \) small enough so that \( \delta \frac{p\gamma}{\gamma-p} < 1 \). Recall that \( p < q \), so, using Hölder’s and Young’s inequalities repeatedly, we obtain
\[
c_3 \int_{\Omega_k} (1 + |f|)^p \leq c_3 \| 1 + |f| \|_{L^p(\Omega)}^p \| \Omega_k \|^{\frac{q-p}{q}},
\]
\[
c_4 \| f \|_{L^p(\Omega)}^p \left( \int_{\Omega_k} (1 + |Du|^2)^{\frac{\delta}{2} \frac{p\gamma}{\gamma-p}} \right)^{\frac{q-p}{q}} \leq c_4 \| f \|_{L^p(\Omega)}^p \| 1 + |Du|^2 \|_{L^1(\Omega)}^\frac{\delta}{2} \| \Omega_k \|^{1-\frac{\delta}{2} \frac{p\gamma}{\gamma-p} \frac{q-p}{q}},
\]
\[
c_5 \int_{\Omega_k} w_k^{\frac{p\gamma}{p+\varepsilon}} \leq c_6 \int_{\Omega_k} w_k^{\frac{p\gamma}{p+\varepsilon}} + c_7 |\Omega_k|,
\]
\[
\frac{\beta}{3} \int_{\Omega_k} w_k^{\beta+1} \leq c_6 \int_{\Omega_k} w_k^{(\beta+1) \frac{d}{d-2}} + c_8 |\Omega_k|.
\]
Recalling that \((\beta + 1)\frac{d}{d-2} = \frac{d\gamma}{1+\delta}\) and \(\|f\|_{L^\gamma(Q)} + \|Du\|_{L^1(Q)} \leq M\), we obtain
\[
\left( \int_Q w_k^{\frac{d\gamma}{1+\delta}} \right)^{\frac{1+\delta}{d\gamma}} \leq \int_Q w_k^{\frac{d\gamma}{1+\delta}} + C_3 \left[ 1 + \|f\|_{L^\gamma(Q)} \right]^{\frac{d\gamma}{1+\delta}} \frac{d\gamma}{1+\delta} \nu_k \left( \frac{\nu_k}{\nu_k} \right) \frac{d\gamma}{1+\delta} + C_4 + C_8 \left[ \Omega_k \right] \leq \int_Q w_k^{\frac{d\gamma}{1+\delta}} + C_4 \left( 1 + M \right) \frac{d\gamma}{1+\delta} \frac{d\gamma}{1+\delta} \nu_k \left( \frac{\nu_k}{\nu_k} \right) \frac{d\gamma}{1+\delta} + C_4 + C_8 \left[ \Omega_k \right].
\]
Replacing \(w_k\) by its definition provides the assertion (up to an additional constant in front of \(\omega\)).

If the choice of \(p\) in (10) does not satisfy \(p > 2\), just pick \(\tilde{p} < q\) and \(\tilde{p} > 2\), and proceed in the same way. Then, (12) becomes
\[
(\beta + 1)\frac{d}{d-2} > \frac{\gamma q}{1+\delta},
\]
so it suffices once again to apply Hölder’s and Young’s inequalities to get the same assertion (with an additional term in \(\omega\)).

\[\square\]

3. Further Remarks

Remark 1. General failure of (M) when \(q \leq \frac{d-1}{\gamma}\). In the critical case \(q = \frac{d-1}{\gamma}\), one may consider the family of functions \(v_\varepsilon\) defined as follows for \(\varepsilon \in (0, 1]\): let \(\chi \in C_0^\infty((1, +\infty))\) be a non-negative cutoff function, \(\chi \equiv 1\) on \([2, +\infty)\), and \(v_\varepsilon(x) = v_\varepsilon(|x|)\), where
\[
v_\varepsilon(r) = c \int_r^{1/2} s^{-\frac{1}{\gamma-1}} \chi \left( \frac{s}{\varepsilon} \right) ds, \quad |c|^\gamma = -\left( d - 1 - \frac{1}{\gamma - 1} \right) c.
\]
Then, on \(B_{1/2} := \{|x| < 1/2\},\)
\[
-\Delta v_\varepsilon + |Dv_\varepsilon|^\gamma = \frac{c}{\varepsilon} |x|^{-\frac{1}{\gamma-1}} \chi'(\varepsilon^{-1}|x|) + |c|^\gamma \left( \chi^\gamma(\varepsilon^{-1}|x|) \right) - \chi(\varepsilon^{-1}|x|)|x|^{-\frac{\gamma}{\gamma-1}} =: f_\varepsilon(x),
\]
and \(v_\varepsilon = 0\) on \(\partial B_{1/2}\). Therefore, there exists \(\overline{M} > 0\), depending on \(c, d, \gamma, \chi\) only, such that
\[
\|f_\varepsilon\|_{L^{\frac{d-1}{\gamma-1}}(B_{1/2})} = \overline{M} \quad \text{for all} \ \varepsilon \in (0, 1/4],
\]
but \(\||Dv_\varepsilon|^\gamma\|_{L^{\frac{d-1}{\gamma-1}}(B_{1/2})} \to \infty \ \text{as} \ \varepsilon \to 0\).

Note that the example is meaningful only if \(\gamma > \frac{d}{d-1}\), that is when \(\frac{d-1}{\gamma} > 1\). Note also that though \(v_\varepsilon\) is not periodic, being smooth on \(B_{1/2}\) and vanishing on \(\partial B_{1/2}\), it is straightforward to produce similar examples in the periodic setting. Finally,
different choices of the truncation \( \chi(|x|) = \chi_\varepsilon(|x|) \) lead to counterexamples in the regime \( q < d^{\frac{\gamma - 1}{\gamma}} \).

Note however that existence of weak solutions to the viscous Hamilton–Jacobi equation (1) can be obtained when \( f \in L^q(Q) \) and \( q = d^{\frac{\gamma - 1}{\gamma}} \) (at least for the Dirichlet problem), provided that \( \|f\|_{L^q} \) is small, see e.g. [18,20]. Therefore, we do not exclude that (M) holds even when \( q = d^{\frac{\gamma - 1}{\gamma}} \), under extra smallness assumptions on \( \|f\|_{L^q} \).

**Remark. 2.** \( d = 1, 2 \). Theorem 1.1 is stated in dimension \( d \geq 3 \), but the proof for \( d = 1, 2 \) follows identical lines. As it usually happens, the point is that in the latter case \( W^{1,2}(Q) \) is continuously embedded into \( L^p(Q) \) for all finite \( p \geq 1 \), and not only into \( L^{\frac{2d}{d-2}}(Q) \).

**Remark. 3.** Less regularity of \( u \). Theorem 1.1 holds more in general for (strong) solutions \( u \in W^{2,q} \cap W^{1,\gamma q}(Q) \) of the equation. Indeed, consider a sequence \( \psi_\varepsilon \) of standard compactly supported regularizing kernels, and observe that \( u_\varepsilon = u \star \psi_\varepsilon \) satisfies

\[
-\Delta u_\varepsilon + |Du_\varepsilon|^\gamma = f \star \psi_\varepsilon + |Du_\varepsilon|^\gamma - |Du|^\gamma \star \psi_\varepsilon.
\]

For \( 0 < \varepsilon \leq \varepsilon_0 \),

\[
\|f \star \psi_\varepsilon + |Du_\varepsilon|^\gamma - |Du|^\gamma \star \psi_\varepsilon\|_{L^q(Q)} + \|Du_\varepsilon\|_{L^1(Q)} \leq M + 1,
\]

so applying Theorem 1.1 to \( u_\varepsilon \) and passing to the limit \( \varepsilon \to 0 \) yields

\[
\|\Delta u\|_{L^q(Q)} + \||Du|^\gamma\|_{L^q(Q)} \leq K(M + 1, \gamma, q, d).
\]

More generally, Theorem 1.1 continues to hold for solutions that can be obtained as limits of smooth approximations.

**Remark. 4.** More general Hamiltonians. Theorem 1.1 can be easily generalized to more general equations of the form

\[
-\Delta u + H(Du) = f,
\]

where \( H : \mathbb{R}^d \to \mathbb{R} \) satisfies, e.g.,

\[
\left| H(r) - c_1|r|^{\gamma} \right| \leq c_2 \quad \text{for all } r \in \mathbb{R}^d
\]

for some \( c_1, c_2 \in \mathbb{R} \) and \( \gamma > 1 \). Indeed, any \( u \) solving (19) also solves

\[
-\Delta u + c_1|Du|^{\gamma} = f + f_H, \quad f_H = c_1|Du|^{\gamma} - H(Du).
\]

Since \( \|f + f_H\|_{L^q(Q)} \leq \|f\|_{L^q(Q)} + c_2 \), and \( c_2 \) does not depend on \( u \), it suffices to apply Theorem 1.1 (which is easily proven to hold for any \( c_1 \in \mathbb{R} \)) with \( f + f_H \).
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References

1. Alaa, N.E., Pierre, M.: Weak solutions of some quasilinear elliptic equations with data measures. SIAM J. Math. Anal. 24(1), 23–35, 1993
2. Alvino, A., Ferone, V., Mercaldo, A.: Sharp a priori estimates for a class of nonlinear elliptic equations with lower order terms. Ann. Mat. Pura Appl. (4) 194(4), 1169–1201, 2015
3. Amann, H., Crandall, M.G.: On some existence theorems for semi-linear elliptic equations. Indiana Univ. Math. J. 27(5), 779–790, 1978
4. Arisawa, M., Lions, P.-L.: On ergodic stochastic control. Commun. Partial Differ. Equ. 23(11–12), 2187–2217, 1998
5. Bardi, M., Perthame, B.: Uniform estimates for some degenerating quasilinear elliptic equations and a bound on the Harnack constant for linear equations. Asymptotic Anal. 4(1), 1–16, 1991
6. Barles, G., Porretta, A.: Uniqueness for unbounded solutions to stationary viscous Hamilton–Jacobi equations. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 5(1), 107–136, 2006
7. Betta, M.F., Di Nardo, R., Mercaldo, A., Perrotta, A.: Gradient estimates and comparison principle for some nonlinear elliptic equations. Commun. Pure Appl. Anal. 14(3), 897–922, 2015
8. Boccardo, L., Murat, F., Puel, J.-P.: Résultats d’existence pour certains problèmes elliptiques quasilinéaires. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 11(2), 213–235, 1984
9. Boccardo, L., Murat, F., Puel, J.-P.: $L^\infty$ estimate for some nonlinear elliptic partial differential equations and application to an existence result. SIAM J. Math. Anal. 23(2), 326–333, 1992
10. Cho, K., Choe, H.J.: Nonlinear degenerate elliptic partial differential equations with critical growth conditions on the gradient. Proc. Am. Math. Soc. 123(12), 3789–3796, 1995
11. Cirant, M.: Stationary focusing mean-field games. Commun. Partial Differ. Equ. 41(8), 1324–1346, 2016
12. Cirant, M., Goffi, A.: Lipschitz regularity for viscous Hamilton–Jacobi equations with $L^p$ terms. Ann. Inst. H. Poincaré Anal. Non Linéaire 37(4), 757–784, 2020
13. Cirant, M., Goffi, A.: Maximal $L^q$-regularity for parabolic Hamilton–Jacobi equations and applications to Mean Field Games (2020). arXiv:2007.14873
14. Dall’Aglio, A., Giachetti, D., Puel, J.-P.: Nonlinear elliptic equations with natural growth in general domains. *Ann. Mat. Pura Appl.* (4) **181**(4), 407–426, 2002
15. Dall’Aglio, A., Porretta, A.: Local and global regularity of weak solutions of elliptic equations with superquadratic Hamiltonian. *Trans. Am. Math. Soc.* **367**(5), 3017–3039, 2015
16. Del Piero, F.: Existence results for non-uniformly elliptic equations with general growth in the gradient. *Differ. Integral Equ.* **21**(9–10), 821–836, 2008
17. Del Piero, F., Gavitone, N.: Sharp estimates and existence for anisotropic elliptic problems with general growth in the gradient. *Z. Anal. Anwend.* **35**(1), 61–80, 2016
18. Ferone, V., Murat, F.: Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small. *Nonlinear Anal.* **42**(7, Ser. A: Theory Methods), 1309–1326 (2000)
19. Ferone, V., Murat, F.: Nonlinear elliptic equations with natural growth in the gradient and source terms in Lorentz spaces. *J. Differ. Equ.* **256**(2), 577–608, 2014
20. Ferone, V., Murat, F., Porretta, A.: Local and global regularity of weak solutions of elliptic equations with superquadratic Hamiltonian. *Trans. Am. Math. Soc.* **367**(5), 3017–3039, 2015
21. Grenon, N., Murat, F., Porretta, A.: Existence and a priori estimate for elliptic problems with subquadratic gradient dependent terms. *C. R. Math. Acad. Sci. Paris* **342**(1), 23–28, 2006
22. Grenon, N., Murat, F., Porretta, A.: A priori estimates and existence for elliptic equations with gradient dependent terms. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)** **13**(1), 137–205, 2014
23. Grenon, N., Trombetti, C.: Existence results for a class of nonlinear elliptic problems with $p$-growth in the gradient. *Nonlinear Anal.* **52**(3), 931–942, 2003
24. Hansson, K., Maz’ya, V.G., Verbitsky, I.E.: Criteria of solvability for multidimensional Riccati equations. *Ark. Mat.* **37**(1), 87–120, 1999
25. Ishii, H., Lions, P.-L.: Viscosity solutions of fully nonlinear second-order elliptic partial differential equations. *J. Differ. Equ.* **83**(1), 26–78, 1990
26. Krug, J., Spohn, H.: Universality classes for deterministic surface growth. *Phys. Rev. A* **38**, 4271–4283, 1988
27. Ladyzhenskaya, O.A., Ural’tseva, N.N.: *Linear and Quasilinear Elliptic Equations*. Academic Press, New York 1968
28. Lasry, J.-M., Lions, P.-L.: Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem. *Math. Ann.* **283**(4), 583–630, 1989
29. Lasry, J.-M., Lions, P.-L.: Mean field games. *Jpn. J. Math.* **2**(1), 229–260, 2007
30. Lions, P.-L.: Résolution de problèmes elliptiques quasilineaires. *Arch. Ration. Mech. Anal.* **74**(4), 335–353, 1980
31. Lions, P.-L.: Quelques remarques sur les problèmes elliptiques quasilineaires du second ordre. *J. Anal. Math.* **45**, 234–254, 1985
32. Lions, P.-L.: Recorded video of Séminaire de Mathématiques appliquées at Collège de France. https://www.college-de-france.fr/site/pierre-louis-lions/seminar-2014-11-14-11h15.htm, 14 November 2014
33. Lions, P.-L.: On mean field games. In: Seminar at the conference “Topics in Elliptic and Parabolic PDEs”, Napoli, 11–12 September 2014
34. Mauger, A., Palagachev, D.K., Softova, L.G.: *Elliptic and Parabolic Equations with Discontinuous Coefficients*. Mathematical Research, vol. 109. Wiley-VCH Verlag Berlin GmbH, Berlin 2000
35. Mengesha, T., Phuc, N.C.: Quasilinear Riccati type equations with distributional data in Morrey space framework. *J. Differ. Equ.* **260**(6), 5421–5449, 2016
36. Messano, B.: Symmetrization results for classes of nonlinear elliptic equations with $q$-growth in the gradient. *Nonlinear Anal.* **64**(12), 2688–2703, 2006
37. Phuc, N.C.: Morrey global bounds and quasilinear Riccati type equations below the natural exponent. *J. Math. Pures Appl.* (9) **102**(1), 99–123, 2014
37. Phuc, N.C.: Nonlinear Muckenhoupt-Wheeden type bounds on reifenberg flat domains, with applications to quasilinear Riccati type equations. Adv. Math. 250, 387–419, 2014
38. Serrin, J.: The problem of dirichlet for quasilinear elliptic differential equations with many independent variables. Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Sci. 264(1153), 413–496, 1969

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