VIRTUAL CLASSES VIA MATRIX FACTORIZATIONS

MARK SHOEMAKER

ABSTRACT. These expository notes are based on a series of lectures given at the May 2018 Snowbird workshop, Crossing the Walls in Enumerative Geometry. We give an introductory treatment of the notion of a virtual fundamental class in algebraic geometry, and describe a new construction of the virtual fundamental class for Gromov–Witten theory of a hypersurface via the derived category of factorizations. The results presented here are based on joint work with I. Ciocan-Fontanine, D. Favero, J. Guéré, and B. Kim.

CONTENTS

0. Introduction 1
1. Stable maps to projective space 5
2. Stable maps to a hypersurface \((n = 0)\) 13
3. Stable maps to a hypersurface \((n > 0)\) 19
4. Extracting invariants 26
5. Technical assumptions and the “two-step procedure” 33
6. Conclusions and further directions 40
References 40

0. INTRODUCTION

0.1. Motivation. Let \(\overline{\mathcal{M}}_{g,n}\) denote the moduli space of stable (complex) curves. This space is a compactification of the moduli space \(\mathcal{M}_{g,n}\) itself parametrizing smooth genus \(g\) curves with \(n\) distinct marked points. The compactification \(\overline{\mathcal{M}}_{g,n}\) introduces nodal curves of genus \(g\), with the requirement that the automorphisms of the marked curve are finite. With the introduction of marked points and nodal curves, we can define maps between these spaces for different choices of \(g\) and \(n\). For instance one can forget a marked point, or one can glue two marked points together to obtain a nodal curve. We obtain maps

\[(0.1)\]

\[
\begin{align*}
\text{for: } & \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}, \\
g_1^1: & \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}, \\
g_1^2: & \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}. 
\end{align*}
\]
These maps are the first evidence of connections between the spaces $\mathcal{M}_{g,n}$ for varying $g$ and $n$. In 1991, Witten proposed a remarkable conjecture [Wit91], which can be briefly summarized by stating that a large class of integrals over these spaces have a beautiful and surprising recursive structure. This was proven by Kontsevich in [Kon92] and now goes by the name of the Witten–Kontsevich theorem.

Since the Witten–Kontsevich theorem an effort has been made to understand how far and in what directions this recursive structure can be generalized. A natural approach to generalization is to enhance the moduli space of curves, defining new moduli spaces which parametrize curves together with some “extra structure.” This extra structure could be for instance a vector bundle with a section, a map from the curve to a target variety $X$, or something more exotic. Let us consider the second approach.

Fix $X$ a smooth projective variety. One can define a moduli space $\overline{\mathcal{M}}_{g,n}(X, d)$ parametrizing \textit{stable maps}

$$f : C \to X,$$

where $C$ is a smooth or nodal curve of genus $g$ with $n$ marked points, $f$ is a regular map of degree $d$, and the automorphisms of $C$ which preserve $f$ are required to be finite. One then considers integrals over these moduli spaces, with the hopes of obtaining useful invariants of the variety $X$. These are known as \textit{Gromov–Witten invariants} of $X$. In particularly simple cases these invariants give counts of curves in $X$, although in general such a direct enumerative interpretation is not possible.

Gromov–Witten theory, the study of these integrals, has proven to be a deep and exciting field. For instance when $X$ is a single point the Gromov–Witten invariants are simply integrals over $\overline{\mathcal{M}}_{g,n}$, the moduli space of stable curves. These invariants already have a rich structure, and form the basis of the Witten–Kontsevich theorem mentioned above.

In attempting to rigorously define Gromov–Witten invariants, a serious technical challenge arises. In contrast to $\overline{\mathcal{M}}_{g,n}$, the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X, d)$ is generally not smooth and often contains many irreducible components of different dimension. The fundamental class of $\overline{\mathcal{M}}_{g,n}(X, d)$ is not well-defined; it is therefore unclear how one should “integrate” over $\overline{\mathcal{M}}_{g,n}(X, d)$.

0.2. \textbf{The “virtual” fundamental class.} The solution is the notion of a \textit{virtual fundamental class}, a choice of element in the Chow group of $\overline{\mathcal{M}}_{g,n}(X, d)$ which behaves in many ways like a fundamental class. To be more precise, one hopes to construct an element of $A_*(\overline{\mathcal{M}}_{g,n}(X, d))$ which satisfies a number of properties analogous to those of a fundamental class, together with some additional properties reflecting the goals of the previous section. In particular one requires:
The virtual fundamental class is supported entirely in a single degree of \( A^* (\overline{\mathcal{M}}_{g,n}(X,d)) \), the so-called expected dimension (see Section 1.4);

- the virtual fundamental class is compatible with the various gluing and forgetful maps of (0.1) (see [Beh99, Section 2] for precise statements);

- Gromov–Witten invariants of \( X \), defined by integrating certain classes against the virtual fundamental class, are invariant under deformations of \( X \).

It is not clear a-priori that such a class should exist, and in general there is no guarantee that a choice of such a class is unique. The construction of the virtual fundamental class has been a fundamental challenge in Gromov–Witten theory (as well as other curve counting theories). Unfortunately any construction of the virtual fundamental class is necessarily technical, especially in its full generality. As such, the virtual fundamental class is one of the more formidable hurdles for graduate students learning the theory.

Constructing the virtual fundamental class has now been successfully carried out in a number of different ways. In Gromov–Witten theory this has been done by Behrend–Fantechi in [BF97], Fukaya–Ono in [FO99], Li–Tian in [LT98], Ruan in [Rua99], and others.

### 0.3. Summary

These expository notes have two complementary goals. The first is to give a friendly introduction to the concept of a virtual fundamental class. This is done in Section 1, where we explain the connection to more classical intersection theory. There we detail the case of \( \overline{\mathcal{M}}_{g,n}(P^r,d) \) where the construction can be simplified significantly.

The second goal of these notes is to present a new method, developed by Ciocan-Fontanine, Favero, Guéré, Kim, and the author in [CFG18], of constructing a virtual class for the Gromov–Witten theory of a hypersurface using the derived category of factorizations.

The construction of the virtual class using the category of factorizations has a number of advantages. First, it applies in a general setting which goes beyond Gromov–Witten theory. Second, the construction yields an object in a certain derived category (See Section 4) and so is in fact more akin to a virtual structure sheaf. It is therefore useful for defining more refined invariants, such as \( K \)-theoretic counterparts to Gromov–Witten invariants (see for instance [Lee04, GT14]). Finally, we hope this perspective will lead to new computational advances in Gromov–Witten and related theories. As evidence of this possibility, see [Gué16].

### 0.4. A note on generality

The method described in these notes is a particular case of a much more general construction developed in [CFG18]. The setting in which these techniques apply is that of a gauged linear sigma model (GLSM), which simultaneously generalizes Gromov–Witten theory of hypersurfaces as well as FJRW theory [FJR13] of homogeneous singularities.
GLSMs were first described in physics by Witten in [Wit97], while the first mathematical definition was given by Fan–Jarvis–Ruan in [FJR17].

In the particular case of FJRW theory, Polishchuk–Vaintrob used the derived category of factorizations to give an algebraic construction of the virtual fundamental class in [PV16]. The paper [CFG+18] was inspired by the work in [PV16] and [FJR17]. Our goal was to use factorizations in a manner analogous to that of [PV16] to define Gromov–Witten-like invariants for a large class of GLSMs. For a more detailed discussion of connections to other results, see Section 6.

Although these notes do not make further mention of the GLSM, it is in the background of everything that follows. We hope that by focusing on a special case, these notes will provide a simplified roadmap for readers interested in [CFG+18].

0.5. Plan of paper. In each of the first three sections we construct a virtual fundamental class (or some analogous object) in successively greater generality.

In the Section 1 we provide a motivating example of a virtual class and give a general procedure to define a virtual class on a space of sections. As a special case, we then construct a virtual class for the moduli space \( \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \) of stable maps to projective space.

In Section 2 we consider a hypersurface \( X_k \subset \mathbb{P}^r \), and define, with the help of one simplifying assumption, a virtual class for the moduli space \( \overline{\mathcal{M}}_{g,0}(X_k, d) \) of stable maps to \( X_k \) with no marked points. It is in this section that we first encounter a Koszul factorization, an object which plays a crucial role in the rest of the paper.

In Section 3 we extend the considerations of Section 2 to the case of curves with marked points. We construct, again under a simplifying assumption, a Koszul factorization which can be thought of as playing the role of the virtual fundamental class (or, more accurately, the virtual structure sheaf) for \( \overline{\mathcal{M}}_{g,n}(X_k, d) \). We call this the fundamental factorization.

In Section 4 we introduce the derived category of factorizations. We show how the fundamental factorization can be used as the kernel of an integral transform to define curve counting invariants for \( X_k \).

In Section 5 we remove all simplifying assumptions from the constructions of Sections 2 and 3 via a “two-step procedure,” so-called because it involves twice resolving the pushforward of a certain tautological vector bundle. We conclude by defining curve-counting invariants for \( X_k \) via the fundamental factorization and state a comparison result with Gromov–Witten theory.

We conclude in Section 6 with connections to related works, and give suggestions for further reading.
0.6. **Acknowledgments.** These notes are based on a series of lectures given at the 2018 Crossing the Walls in Enumerative Geometry Summer Workshop, and those lectures are based upon the paper [CFG+18].

I am grateful first and foremost to my collaborators on [CFG+18], I. Ciocan-Fontanine, D. Favero, J. Guéré, and B. Kim. I hope these expository notes have succeeded in faithfully and succinctly summarizing some of the main ideas behind that paper. I am grateful to the University of Minnesota’s Department of Mathematics for the use of their facilities during several extended collaborative meetings during the writing of [CFG+18].

I would like to extend special thanks to the organizers of the 2018 Crossing the Walls in Enumerative Geometry Summer Workshop, T. Jarvis, N. Prid- dis, and Y. Ruan, as well as D. Coffman, the project manager who took care of much of the work required to make the workshop such a success.

I would like to thank T. Jarvis and Y. Ruan a second time for many discussions explaining their mathematical construction of the GLSM.

This work was partially supported by NSF grant DMS-1708104.

1. **Stable maps to projective space**

In this section we construct a virtual fundamental class for the moduli space of stable maps to projective space, $\mathcal{M}_{g,n}(\mathbb{P}^r, d)$. We begin with an extended motivating example, the zero locus of a section of a vector bundle. In this context, virtual fundamental classes arise naturally and are relatively simple to define. The hope is that this example will motivate the construction of virtual classes as a natural outgrowth of more classical intersection theory.

In the remainder of the section we will see that the virtual class for $\mathcal{M}_{g,n}(\mathbb{P}^r, d)$ is in fact a special case of this example.

1.1. **The zero locus of a section.** The material of this section is described in detail in [Ful13].

Let $X$ be a smooth variety of dimension $n$ and let $E \to X$ be a vector bundle of rank $r$. Let $s \in \Gamma(X, E)$ be a section, and define $Z := Z(s)$ to be the scheme-theoretic zero locus of the section $s$. Recall that the total space of the vector bundle $E$ is given by

$$
tot(E) := \text{Spec} \left( \text{Sym} E^\vee \right).
$$

If $s$ is a regular section, then $Z$ will be of dimension

$$n - r = (n + r) - r = \dim(tot(E)) - \text{codim}(X) - \text{codim}(s(X)).$$

The class

$$[Z] \in A_{n-r}(Z)$$

can be viewed as the intersection of $[X]$ and $[s(X)]$ in $\text{tot}(E)$, where $X$ is embedded in $\text{tot}(E)$ via the zero section.
We write $[X] \cdot [s(X)] = [Z]$. The quantity $n - r$ is called the expected dimension of the intersection, it is the dimension in the particularly nice case that $s$ is regular.

If $s \in \Gamma(X, E)$ is no longer a regular section, then $\dim(Z)$ will be strictly greater than the expected dimension $n - r$. In this case we would still like to construct a class in the Chow group of $Z$ which represents the “correct” intersection of $[X]$ and $[s(X)]$. In particular the class should be of the expected dimension $n - r$.

**Example 1.1.** Let us consider what is in some sense the worst possible case, that is, $s \in \Gamma(X, E)$ is the zero section $0 \in \Gamma(X, E)$. In this case $s(X) = X$ and so we are looking for the intersection of $X$ with itself in $\text{tot}(E)$. In this case we define

$[X] \cdot [X] = e(E) \in A_{n-r}(X),$

where $e(E)$ denotes the Euler class (equal to the top Chern class) of $E$ [Ful13]. Under the cycle map $A_*(X) \to H_*(X) \cong H^*(X)$, $e(E)$ maps to the topological Euler class of, e.g. [BT82].

If there exists a regular section $s' \in \Gamma(X, E)$ and $Z' = Z(s')$ is the zero locus, then the class $[Z'] \in A_{n-r}(X)$ is equal to $e(E)$. In this sense $e(E)$ is the correct intersection of $[X]$ and $[X] = [s(X)]$, as it coincides with the class obtained by deforming $s$ into $s'$ and then taking the intersection of $[X]$ with $[s'(X)]$.

For a more general section $s \in \Gamma(X, E)$, not necessarily regular or zero, we would like to be able to define a class

$[X] \cdot [s(X)] \in A_{n-r}(Z).$

In this case we use the so-called refined Euler class $e(E, s)$ which is still of degree $n - r$, but is supported on the zero locus $Z$. It agrees with the usual Euler class in the sense that under the pushforward map

$i_s : A_*(Z) \to A_*(X),$

e(E, s)$ maps to $e(E)$. To define it we require some machinery from intersection theory:

---

1The problem of defining an intersection product has a long history. It has been definitively answered in a very general context. See for instance [Ful13] and the references therein. The case at hand of $[X]$ intersected with $[s(X)]$ becomes a special case of this general theory.
Definition 1.2. If \( i : Z \hookrightarrow X \) is defined by a sheaf of ideals \( \mathcal{I} \), the normal cone of \( Z \) in \( X \) is

\[
C_Z X := \text{Spec} \left( \bigoplus_{k=0}^{\infty} \mathcal{I}^k / \mathcal{I}^{k+1} \right).
\]

In the case that \( Z \hookrightarrow X \) is a regular embedding, this is actually equal to (the total space of) the normal bundle of \( Z \) in \( X \): \( C_Z X = N_Z X \).

It is a fact [Ful13, Appendix B.6.6] that \( C_Z X \) is of pure dimension \( n \).

Consider the following diagram:

\[
\begin{array}{ccc}
Z & \xrightarrow{i} & X \\
\downarrow & & \downarrow \\
X & \xleftarrow{0} & \text{tot}(E).
\end{array}
\]

Let \( \mathcal{I} \) denote the ideal sheaf associated to \( i : Z \hookrightarrow X \), and let \( \mathcal{J} \) denote the ideal sheaf of \( X \) in \( \text{tot}(E) \) (embedded as the zero section). There is a natural surjection

\[
s^*(\mathcal{J}) \twoheadrightarrow \mathcal{I}
\]

which in turn induces a surjection

\[
\bigoplus_{k=0}^{\infty} s^* \left( \mathcal{J}^k / \mathcal{J}^{k+1} \right) \twoheadrightarrow \bigoplus_{k=0}^{\infty} \mathcal{I}^k / \mathcal{I}^{k+1}.
\]

We obtain a closed embedding

\[
C_Z X \hookrightarrow i^*(C_X \text{tot}(E)) = i^*(N_X \text{tot}(E)) = i^*(\text{tot}(E)) = \text{tot}(E)|_Z,
\]

where \( N_X \text{tot}(E) \) is the total space of the normal bundle of \( X \) in \( \text{tot}(E) \). Note that the inclusion of \( X \) in \( \text{tot}(E) \) via the zero section is regular, so the normal cone of \( X \) in \( \text{tot}(E) \) is the normal bundle, which is simply \( \text{tot}(E) \). In particular, the normal cone can be viewed as lying inside the restriction of \( \text{tot}(E) \) to \( Z \). The projection \( \pi|_Z : \text{tot}(E)|_Z \to Z \) is flat, and so defines a map

\[
\pi|_Z^* : A_k(Z) \to A_{k+r}(\text{tot}(E)|_Z)
\]

\[
[V] \mapsto [\pi|_Z^* 1(V)].
\]

It is a (nontrivial) fact that this map is an isomorphism. We denote its inverse by \( 0|_Z^* : A_{k+r}(\text{tot}(E)|_Z) \to A_k(Z) \), as we may view it as pullback by the zero section \( 0|_Z : Z \to \text{tot}(E)|_Z \). With this setup, one can define the following:

Definition 1.3. The refined Euler class of \( E \to X \) with respect to \( s \in \Gamma(X, E) \) is the pullback of the normal cone of \( Z \) in \( X \) via the zero section:

\[
e(E, s) := 0|_Z^* ([C_Z|X])
\]

The pullback preserves codimension, so this class lies in \( A_{n-r}(Z) \).
Definition 1.4. Define the intersection

$$[X] \cdot [s(X)] := e(E, s).$$

One checks that this agrees with the definition in the previous two extreme cases where $s$ is regular or $s$ is the zero section. This class lies in the scheme-theoretic intersection of $X$ with $s(X)$, however it always has the dimension one would expect from taking the zero locus of a section of a rank $r$ vector bundle. This can be viewed as the first example of a virtual fundamental class.

1.2. Spaces of sections. In this section we demonstrate a special case of the example given above.

Let $\pi : C \to S$ be a flat (and proper) family of pre-stable curves lying over a smooth variety (or stack) $S$. Let $V \to C$ be a vector bundle on $C$. Then for each geometric point $s \in S$, the fiber of $\pi$ is a (at worst) nodal curve $C_s$, equipped with a vector bundle $V_s \to C_s$.

![Diagram of a family of vector bundles](image)

We would like to define a space, which we denote by $\text{tot}(\pi_* V)$, lying over $S$, whose fiber over a closed point $s \in S$ is the vector space of sections $\Gamma(C_s, V_s)$. We refer to this as the space of sections of $V$. Note that $\pi_* V$ is not usually a vector bundle, in particular the rank of $\Gamma(C_s, V_s)$ will vary as we vary $s$. Consequently the space $\text{tot}(\pi_* V)$ will not be smooth.

It is tempting to simply define this space as the “total space” of the pushforward sheaf $\pi_* V$, in analogy with the total space of a vector bundle (1.1). This cannot work, however, because the pushforward does not commute with base change. In particular, it often happens that

$$\pi_*(V)|_s \neq \pi_*(V_s).$$

For a simple example of this consider a trivial family of elliptic curves $C = E \times S$ with $V$ a nontrivial family of degree zero line bundles on $E$. Then $\pi_* V$ will be zero, but $\Gamma(C_s, V_s) \cong C$ whenever $V_s \cong O_{C_s}$.

On the other hand, because $\pi : C \to S$ is flat of relative dimension one, $\mathbb{R}^1 \pi_*(-)$ does commute with base change. Thus we can employ Serre duality to construct the desired space.

Definition 1.5. Let $\pi : C \to S$ and $V \to C$ be defined as in the introduction to this section. Define the space of sections of $V$ to be

$$\text{tot}(\pi_* V) := \text{Spec} \left( \text{Sym} \left( \mathbb{R}^1 \pi_*(\omega_\pi \otimes V^\vee) \right) \right).$$
where $\omega_\pi$ is the relative dualizing sheaf for $\pi$.

Given a sheaf $\mathcal{A}$ over $S$, recall that the $s$-points of $\text{Spec}(\text{Sym}(\mathcal{A}))$ are the elements of $\text{hom}(\mathcal{A}, \mathcal{O}_s)$. With this we observe that

$$\text{tot}(\pi_* V)|_s = \text{hom}\left(\mathbb{R}^1\pi_* (\omega_\pi \otimes V^\vee)_s, \mathcal{O}_s\right) = \text{hom}\left(\mathbb{R}^1\pi_* \omega_\pi, \mathcal{O}_s\right) = \Gamma(C_s, V_s)$$

as desired. Here the second equality is by base change (see [GD63], or [Oss, Theorem 1.2] for a summary of the case at hand). The third equality is Serre duality.

As mentioned above, the space $\text{tot}(\pi_* V)$ is not usually smooth and in fact is not even of pure dimension. Nevertheless we can exploit the ideas of Section 1.1 to construct a virtual class which is supported on $\text{tot}(\pi_* V)$ and of pure degree. The idea is to embed $V$ into a larger vector bundle $\mathcal{A} \to \mathcal{C}$ which is $\pi$-acyclic.

**Lemma 1.6.** If $\pi : \mathcal{C} \to S$ is projective, there exists an embedding of $V$ into a vector bundle $\mathcal{A}$ satisfying $\mathbb{R}^1\pi_* (\mathcal{A}) = 0$.

**Proof.** Let $\mathcal{O}(1) \to \mathcal{C}$ be a $\pi$-relatively ample line bundle. Then for sufficiently large $n$,

$$\mathbb{R}^1\pi_* (V^\vee(n)) = \mathbb{R}^1\pi_* (\mathcal{O}(n)) = 0$$

and the map

$$\pi^* (\pi_* (V^\vee(n))) \to V^\vee(n)$$

is surjective. Twisting by $\mathcal{O}(-n)$, we see

$$\pi^* (\pi_* (V^\vee(n)))(-n) \to V^\vee$$

is surjective. Dualizing this map, we see that $V$ embeds into

$$\mathcal{A} := \pi^* (\pi_* (V^\vee(n))^\vee(n)).$$

Given such an embedding $V \hookrightarrow \mathcal{A}$, let $\mathcal{B}$ denote the cokernel. Define

$$A := \pi_* (\mathcal{A}), \quad B := \pi_* (\mathcal{B}).$$

The short exact sequence

$$0 \to V \to \mathcal{A} \to \mathcal{B} \to 0$$

induces a long exact sequence

$$0 \longrightarrow \pi_* V \longrightarrow A \longrightarrow B \longrightarrow \mathbb{R}^1\pi_* V \longrightarrow \mathbb{R}^1\pi_* A \longrightarrow \mathbb{R}^1\pi_* B \longrightarrow 0.$$
Note, however, that $R^1 \pi_* A = 0$ by construction and so $R^1 \pi_* B = 0$ as well. Because the Euler characteristic is constant in flat families, the fibers of $A$ and $B$ have constant rank. By Grauert’s criterion [Har77, 12.9], they are vector bundles. We conclude that the two-term complex $[A \to B]$ is a resolution of $R\pi_* V$ by vector bundles.

Consider the total space of $A$,

$$X := \text{tot}(A),$$

and the forgetful map $\tau : X \to S$. Define the vector bundle $E := \tau^*(B)$ over $X$. The map $A \to B$ induces a natural section $s \in \Gamma(X, E)$. One can check using (1.2) that $\text{tot}(\pi_* V)$ is exactly the zero section $Z(s) \hookrightarrow X$. We are now exactly in the situation of Section 1.1. We proceed as before.

**Definition 1.7.** Define the virtual fundamental class of the space of sections $\text{tot}(\pi_* V)$ to be

$$[\text{tot}(\pi_* V)]^{\text{vir}} := e(E, s) \in A_k(\text{tot}(\pi_* V)).$$

Note in the above that $[\text{tot}(\pi_* V)]^{\text{vir}} \in A_k(\text{tot}(\pi_* V))$, where

$$k = \dim(\text{tot}(A)) - \text{rank}(E) = \dim(S) + \text{rank}(A) - \text{rank}(B) = \dim(S) + \chi(C_s, V_s),$$

for $s$ a geometric point of $S$.

One must check the following:

**Proposition 1.8.** The class $[\text{tot}(\pi_* V)]^{\text{vir}}$ is independent of the choice of embedding $V \to A$.

**Proof.** This follows from more general constructions such as in [BF97]. However in this simple case one can prove it more easily. The sketch of the proof is the following:

1. Reduce to the case of two embeddings $V \to A$ and $V \to A'$, where the second map factors through an embedding $A \to A'$. This can be accomplished by embedding $A$ and $A'$ in a common larger $\pi$-acyclic vector bundle $\tilde{A}$.
2. By a deformation argument, reduce to the case that $A' = A \oplus A''$ and the embedding $A \to A'$ is given by $(\text{id}_A, 0)$ (see [Ful13, Proposition 18.1] for such an argument).
3. In this simple case, one observes from the definition that $e(E', s')$ and $e(E, s)$ coincide.

$\square$
1.3. **Stable maps as a space of sections.** Recall that the moduli space \( \mathcal{M}_{g,n}(\mathbb{P}^r, d) \) is the stack representing families

\[
\begin{array}{c}
C \\
\downarrow \pi \\
S
\end{array}
\xrightarrow{f}
\mathbb{P}^r
\]

where

1. \( \pi \) is a flat family of prestable curves of genus \( g \). Included in this is the condition that the sections \( \{\sigma_i\}_{1 \leq i \leq n} \) are disjoint from each other and from the nodes of \( C_s \);
2. \( f \) restricts on each fiber to a degree \( d \) map \( f_s : C_s \to \mathbb{P}^r \);
3. the line bundle \( \omega_{\pi,\log} \otimes f^*(\mathcal{O}_{\mathbb{P}^r}(3)) \) is ample.

Denote by \( p_i \subset C \) the \( i \)th marked point divisor defined by \( \sigma_i \). Let \( \Sigma := \sum_{i=1}^n p_i \). Recall that the log canonical bundle \( \omega_{\pi,\log} \) is defined as

\[
\omega_{\pi,\log} := \omega_{\pi} \otimes \mathcal{O}(\Sigma).
\]

There is a natural notion of equivalence of such families. See [FP96] for a construction of \( \mathcal{M}_{g,n}(\mathbb{P}^r, d) \), and a proof that the coarse moduli space is projective. We argue below that \( \mathcal{M}_{g,n}(\mathbb{P}^r, d) \) can be realized as (a substack of) a space of sections, this time over a (smooth!) Artin stack.

First, we rephrase the functor. A degree \( d \) map \( f : C \to \mathbb{P}^r \) determines (up to a scaling) a degree \( d \) line bundle \( L \to C \) together with a nowhere-vanishing section \( s \in \Gamma(C, L^{\oplus r+1}) \). Thus we can view \( \mathcal{M}_{g,n}(\mathbb{P}^r, d) \) as representing families

\[
\begin{array}{c}
\mathcal{L}^{\oplus r+1} \\
\downarrow \pi \\
S
\end{array}
\xrightarrow{s}
C
\]

where \( L \) is degree \( d \) on each fiber \( C_s \), and the family satisfies:

1. \( \pi \) is a flat family of prestable curves of genus \( g \). The sections \( \{\sigma_i\}_{1 \leq i \leq n} \) are disjoint from each other and from the nodes of \( C_s \);
2. The section \( s \in \Gamma(C, L^{\oplus r+1}) \) is nowhere vanishing;
3. The line bundle \( \omega_{\pi,\log} \otimes L^{\otimes 3} \) is ample.

Here we must specify that two families are equivalent if the sections of \( L^{\oplus r+1} \) differ only by a scaling. This can all be made precise as in, e.g. [CFK10, Man14].

Forgetting the section \( s \), we obtain a map to the following stack.
Definition 1.9. Define the stack $\mathcal{Bun}_{g,n,d}$ to be the stack representing families

$$
\begin{array}{c}
\mathcal{L} \\
\downarrow \\
\mathcal{C} \\
\downarrow \pi \\
S
\end{array}
$$

where $\mathcal{L}$ is degree $d$ on each fiber and condition (1) is satisfied.

Define $\mathcal{Bun}_{g,n,d}^\circ \subset \mathcal{Bun}_{g,n,d}$ to be the open substack consisting of families such that Condition (3) above is also satisfied.

Note that all families in $\mathcal{Bun}_{g,n,d}^\circ$ have (in addition to automorphisms of the underlying curve $C$) a $\mathbb{C}^*$-family of automorphisms obtained by the scaling automorphisms of $\mathcal{L}$.

Proposition 1.10. [CKM14, Proposition 2.1.1] The stack $\mathcal{Bun}_{g,n,d}$ is a smooth Artin stack of dimension $4g - 4 + n$. Over the open locus $\mathcal{Bun}_{g,n,d}^\circ$, the universal curve $\pi : \mathcal{C} \to \mathcal{Bun}_{g,n,d}^\circ$ is projective.

Note that with regards to the second point of the Proposition, the line bundle $\omega_{\pi,\text{log}} \otimes \mathcal{L}^{\otimes 3}$ is relatively ample by construction (see Condition (3) above).

Consider the universal curve $\pi : \mathcal{C} \to \mathcal{Bun}_{g,n,d}^\circ$, and the universal line bundle $\mathcal{L} \to \mathcal{C}$. Define the vector bundle $\mathcal{V} := \mathcal{L}^{\oplus r+1}$ over $\mathcal{C}$. We see from the discussion above that $\underline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ is a substack of $\text{tot}(\pi_* \mathcal{V})$ (Definition 1.5). More precisely, $\underline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ is described in $\text{tot}(\pi_* \mathcal{V})$ by the condition that $s \in \Gamma(C_b, V_b)$ is nowhere vanishing. The complement of $\underline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ in $\text{tot}(\pi_* \mathcal{V})$ is seen to be a closed set, and

$$
\underline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \hookrightarrow \text{tot}(\pi_* \mathcal{V})
$$

is an open immersion.

1.4. The virtual class. Finally, we construct a virtual class for $\underline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ via a very minor modification of the procedure in Section 1.2. Let $\mathcal{V}$ be as in the previous paragraph. Note that $\pi : \mathcal{C} \to \mathcal{Bun}_{g,n,d}^\circ$ is projective. As described in Section 1.2, embed $\mathcal{V}$ into a $\pi$-acyclic vector bundle $\mathcal{A} \to \mathcal{C}$. Define $\mathcal{B}$ as the cokernel of $\mathcal{V} \to \mathcal{A}$, and let

$$
\begin{align*}
A &:= \pi_* (\mathcal{A}), \\
B &:= \pi_* (\mathcal{B}).
\end{align*}
$$

Then $[A \to B]$ is a two-term resolution of $R\pi_* \mathcal{V}$ by vector bundles. Define $\tau : \text{tot}(A) \to \mathcal{Bun}_{g,n,d}^\circ$ and $E := \tau^*(B)$ as before. Because $\mathcal{Bun}_{g,n,d}^\circ$ is a smooth Artin stack, $\text{tot}(A)$ will be as well.
For $b \in \text{Bun}_{g,n,d}$, the fiber of $\text{tot}(A)$ over $b$ is given by

$$\text{tot}(A)_b = \{ s \in \Gamma(C_b, A|_{C_b}) \}.$$ 

Define the open substack

$$U \subset \text{tot}(A)$$

by the condition that the section $s$ is nowhere vanishing on each fiber. Again this is an open condition. One can check that $U$ is a Deligne–Mumford stack. In particular the automorphism group of each point of $U$ is finite. As described in Section 1.2, the map $A \to B$ of vector bundles induces a natural section which we denote $\beta$ in $\Gamma(U, E)$. The zero locus of $\beta$ cuts out exactly those sections of $A$ which are sections of $V = L^{\oplus r+1}$. As we have restricted to the locus of nowhere vanishing sections, we see that $M_{g,n}(\mathbb{P}^r, d) = Z(\beta) \subset U \subset \text{tot}(A)$.

**Definition 1.11.** Define the virtual class

$$[\mathcal{M}_{g,n}(\mathbb{P}^r, d)]_{\text{vir}} := e(E, \beta) \in A_*(\mathcal{M}_{g,n}(\mathbb{P}^r, d)).$$

As a quick check that this definition is reasonable, we do a dimension count. The refined Euler class $e(E, \beta)$ lies in $A_k(\mathcal{M}_{g,n}(\mathbb{P}^r, d))$, where

$$k = \dim(\text{Bun}_{g,n,d}) + \rank(A) - \rank(B)
= \dim(\text{Bun}_{g,n,d}) + \chi(V_s)
= 4g - 4 + n + (r + 1)(1 - g + d)
= (3g - 3)r + n + (r + 1)d
= \text{virdim}(\mathcal{M}_{g,n}(\mathbb{P}^r, d)).$$

This is the expected dimension of $\mathcal{M}_{g,n}(\mathbb{P}^r, d)$.

**Proposition 1.12.** The virtual class of Definition 1.11 agrees with the Gromov–Witten theory virtual fundamental class for $\mathcal{M}_{g,n}(\mathbb{P}^r, d)$ as defined by Behrend–Fantechi in [BF97].

**Proof.** Definition 1.11 is essentially a special case of the construction given in [BF97].

**Remark 1.13.** The construction of this section can be generalized to define the virtual class $\mathcal{M}_{g,n}(X, d)$ whenever $X = [V // G]$ is a GIT quotient. By changing the stability condition one can also define the virtual class for quasi-maps in this way. See e.g. [CFK10] for the toric case, where they use a similar method to this.

## 2. Stable maps to a hypersurface ($n = 0$)

Let $w = w(x_0, \ldots, x_r)$ be a non-degenerate homogeneous polynomial of degree $k$. Let $X_k \subset \mathbb{P}^r$ be the smooth hypersurface defined as the vanishing locus of $w$. In this section we expand the ideas of the previous section to...
construct a virtual class on $\overline{\mathcal{M}}_{g,0}(X_k, d)$. Here we restrict to the special case of no marked points. We will add marked points in Section 3.

Related to the moduli space of stable maps to $X_k$ is the space of maps to $\mathbb{P}^r$ with $p$-fields, which has the advantage of being a space of sections of a vector bundle as in Section 1.2. This moduli space and various generalizations have appeared in e.g. [CL11, Cla17, FJR17].

Definition 2.1. [CL11] Given $k \in \mathbb{Z}_{>0}$, define the space of stable maps to $\mathbb{P}^r$ with a $p$-field of degree $k$ to be the moduli space lying over $\overline{\mathcal{M}}_{g,0}((\mathbb{P}^r, d))$ parametrizing families of stable maps $f : C \to \mathbb{P}^r$ of degree $d$ from genus $g$ curves $C$, together with a section

$$p \in \Gamma(C, f^*(\mathcal{O}_{\mathbb{P}^r}(-k) \otimes \omega_C)).$$

This moduli stack is denoted by $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^r, d)^p$.

The stack $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^r, d)^p$ is easily seen to be a Deligne–Mumford stack because $\overline{\mathcal{M}}_{g,0}((\mathbb{P}^r, d))$ is. We can also view the stack as representing families of the form

$$\begin{array}{c}
\mathcal{L} \\
\mathcal{L}^{r+1} \\
\mathcal{L}^{\otimes k} \otimes \omega_\pi
\end{array}$$

$$\begin{array}{c}
C \\
\pi \\
S
\end{array}$$

where $\mathcal{L}$ is degree $d$ on each fiber $C_s$, and the family satisfies:

1. $\pi$ is a flat family of prestable curves of genus $g$.
2. The section $s \in \Gamma(C, \mathcal{L}^{\otimes r+1})$ is nowhere vanishing;
3. The line bundle $\omega_\pi \otimes \mathcal{L}^{\otimes 3}$ is ample.

Again we see that $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^r, d)^p$ is (an open subset of) a space of sections over $\mathcal{Bun}^0_{g,0,d}$.

We can repeat the argument of the previous section to realize $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^r, d)^p$ as the zero locus of a section of a natural vector bundle defined over a smooth Deligne–Mumford stack.

2.1. Smooth embedding via resolutions. Let

$$\begin{array}{c}
\mathcal{L} \\
\mathcal{L} \\
\pi \\
S
\end{array}$$

be a family in $\mathcal{Bun}^0_{g,0,d}$, viewed as pulled back from a map $S \to \mathcal{Bun}^0_{g,0,d}$.
Definition 2.2. Define the fiber products

\[
\overline{\mathcal{M}}_{g,0}(\mathbb{P}^r, d)_S := S \times_{\mathfrak{Bun}^\circ_{g,0,d}} \overline{\mathcal{M}}_{g,0}(\mathbb{P}^r, d)
\]
\[
\overline{\mathcal{M}}_{g,0}(\mathbb{P}^r, d)_S^p := S \times_{\mathfrak{Bun}^\circ_{g,0,d}} \overline{\mathcal{M}}_{g,0}(\mathbb{P}^r, d)^p
\]
\[
\overline{\mathcal{M}}_{g,0}(X_k, d)_S := S \times_{\mathfrak{Bun}^\circ_{g,0,d}} \overline{\mathcal{M}}_{g,0}(X_k, d)
\]

Let

\[\mathcal{V}_1 := L^{r+1},\]
\[\mathcal{V}_2 := L^{-k} \otimes \omega,\]
\[\mathcal{V} := \mathcal{V}_1 \oplus \mathcal{V}_2.\]

Choose $\pi$-acyclic vector bundles $A_1$ and $A_2$ together with embeddings of $\mathcal{V}_1$ and $\mathcal{V}_2$ to obtain short exact sequences

\[0 \to \mathcal{V}_{1/2} \to A_{1/2} \to B_{1/2} \to 0\]

where $B_1$ and $B_2$ are defined as the cokernels. Define

\[A := A_1 \oplus A_2 := \pi_*(A_1) \oplus \pi_*(A_2)\]
\[B := B_1 \oplus B_2 := \pi_*(B_1) \oplus \pi_*(B_2).\]

Then the two-term complex of vector bundles $A \to B$ gives a resolution of $\mathbb{R}\pi_* (\mathcal{V})$ over $\mathfrak{Bun}^\circ_{g,0,d}$.

Definition 2.3. Define

\[U \subset \text{tot}(A) = \text{tot}(A_1 \oplus A_2)\]

to be the open locus where the section $s \in \Gamma(C, A_1)$ is nowhere vanishing. Let $\tau : U \to \mathfrak{Bun}^\circ_{g,0,d}$ be the forgetful map, and define

\[E := \tau^*(B).\]

The map $A \to B$ induces a section $\beta \in \Gamma(U, E)$. We conclude that

\[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^r, d)_S^p = Z(\beta) \subset U\]

as desired.

Remark 2.4. One is tempted at this point to construct a virtual class in $A_*(\overline{\mathcal{M}}_{g,0}(\mathbb{P}^r, d)_S^p)$ using the refined Euler class $e(E, \beta)$ as in the previous section. However, because $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^r, d)_S^p$ is non-compact, one cannot integrate over this class. In a sense, we must refine the Euler class even further. The data which has not yet been used is the polynomial $w = w(x_0, \ldots, x_r)$ defining $X_k$. 
2.2. Incorporating $\mathring{w}$. Recall that $\mathring{w}(x_0, \ldots, x_r)$ is a homogeneous polynomial of degree $k$ defining a hypersurface $X_k \subset \mathbb{P}^r$. Define

$$\mathring{w} = \mathring{w}(x_0, \ldots, x_r, y) := y \cdot w(x_0, \ldots, x_r).$$

If $y$ is given degree $-k$, then $\mathring{w}$ is homogeneous of degree zero. In particular observe that $\mathring{w}$ defines a function on $\text{tot}(\mathcal{O}_{\mathbb{P}^r}(-k))$. Note further that the degeneracy locus of $\mathring{w}$ is exactly $X_k$:

$$Z(d\mathring{w}) := \left\{ \frac{\partial}{\partial y} \mathring{w} = 0, \frac{\partial}{\partial x_i} \mathring{w} = 0 \right\} = X_k.$$

Let $\pi : \mathcal{C} \to S$ be the universal curve. If $s_0, \ldots, s_r \in \Gamma(\mathcal{C}, \mathcal{L})$ and $p \in \Gamma(\mathcal{C}, \mathcal{L} \otimes \omega_\pi)$ are sections, we see that $\mathring{w}(s_0, \ldots, s_r, p)$ defines a section of $\omega_\pi$. Thus $\mathring{w}$ defines a map of vector bundles

$$\text{Sym}^{k+1} \mathcal{V} \to \mathcal{O}_S.$$

Pushing forward, we obtain a morphism $\mathbb{R}\pi_* (\text{Sym}^{k+1} \mathcal{V}) \to \mathbb{R}\pi_* (\omega_\pi)$ in the derived category of $S$. Consider the following composition in the derived category:

\[
\begin{array}{c}
\text{Sym}^{k+1} ([A \to B]) \xrightarrow{\sim} \text{Sym}^{k+1} ([\mathbb{R}\pi_* \mathcal{V}]) \xrightarrow{\text{nat}} \mathbb{R}\pi_* (\text{Sym}^{k+1} \mathcal{V}) \xrightarrow{\downarrow} \mathbb{R}\pi_* (\omega_\pi) \\
\xrightarrow{[\tilde{\alpha}]} \mathbb{R}^1\pi_* (\omega_\pi)[-1] \xrightarrow{\text{trace}} \mathcal{O}_S[-1].
\end{array}
\]

For simplicity of exposition, we will make the following assumption for the remainder of this section.

**Assumption 2.5.** Assume that the bundles $A$ and $B$ are such that the map $[\tilde{\alpha}]$ exists as a map of complexes.

We warn the reader that this assumption may not in fact hold in all cases. We will explain how to work around it in Section 5.

Given Assumption 2.5, $[\tilde{\alpha}]$ can be represented by a map of chain complexes

\[
\begin{array}{c}
\text{Sym}^{k+1} (A) \xrightarrow{d_{k+1}} \text{Sym}^k (A) \otimes B \xrightarrow{d_k} \text{Sym}^{k-1} \otimes \wedge^2 (B) \xrightarrow{d_{k-1}} \cdots \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \quad \to \mathcal{O}_S \quad \to \mathbb{R} \to \cdots
\end{array}
\]

where we denote by $\tilde{\alpha} : \text{Sym}^k (A) \otimes B \to \mathcal{O}_S$ the only non-zero vertical map. A map of vector bundles $\text{Sym}^k (A) \otimes B \to \mathcal{O}_S$ is equivalent to a section of $E^\vee = \tau^!(B^\vee)$ on $\text{tot}(A)$ or, alternatively, a cosection

$$\alpha : E \to \mathcal{O}_{\text{tot}(A)}.$$
We now have

\[
\begin{array}{ccc}
O_U & \xrightarrow{\beta} & E \\
\downarrow & & \downarrow \\
\mathcal{M}_{g,0}(\mathbb{P}', d)^p_S & = Z(\beta) & \hookrightarrow U
\end{array}
\]

The differential \(d\hat{w}\) defines a map of sheaves over \(S\):

\[
(2.4)
\]

\[\tilde{d}\hat{w} : \text{Sym}^k(\pi_* \mathcal{V}) \otimes \mathbb{R}^1 \pi_*(\mathcal{V}) \to O_S \]

\[
(s, p) \otimes (\dot{s}, \dot{p}) \mapsto \frac{1}{k+1} \left( k \sum_{i=0}^r \frac{\partial \hat{w}}{\partial x_i} |_{(s_0, \ldots, s_r, p)} \dot{s}_i + \dot{p} w(s_0, \ldots, s_r) \right).
\]

This is described carefully in [CFG+18 Equation (3.29)]. In the above equation, we view \(\dot{s}_i \in \mathbb{R}^1 \pi_*(\mathcal{L})\) as an element of \(\pi_* (\mathcal{L}^\vee \otimes \omega)^\vee\), noting that \(\frac{\partial \hat{w}}{\partial x_i} |_{(s_0, \ldots, s_r, p)}\) lies in \(\pi_* (\mathcal{L}^\vee \otimes \omega)\).

The map \(\tilde{d}\hat{w}\) corresponds to a cosection over \(\mathcal{M}_{g,0}(\mathbb{P}', d)^p_S\) (which by abuse of notation we also denote by \(d\hat{w}\)):

\[d\hat{w} : \tau^* \left( \mathbb{R}^1 \pi_* \mathcal{V} \right) \to O_{\mathcal{M}_{g,0}(\mathbb{P}', d)^p_S} \]

Since \(w\) is assumed to be non-degenerate, the cosection \(d\hat{w}\) is identically zero when \(p \equiv 0\) and \(w(s_0, \ldots, s_r) = f^*(w) \equiv 0\). But \(\mathcal{M}_{g,0}(X_k, d)_S \subset \mathcal{M}_{g,0}(\mathbb{P}', d)_S\) is exactly the locus where \(f^*(w) \equiv 0\), so

\[\{d\hat{w} \equiv 0\} = \mathcal{M}_{g,0}(X_k, d)_S. \]

**Proposition 2.6.** [CFG+18 Lemma 3.6.3] The following diagram commutes

\[
\begin{array}{ccc}
\text{Sym}^k(\pi_* \mathcal{V}) \otimes B & \xrightarrow{\kappa} & O_S \\
\downarrow \text{id} \otimes H^1(-) & & \downarrow \tilde{d}\hat{w} \\
\text{Sym}^k(\pi_* \mathcal{V}) \otimes \mathbb{R}^1 \pi_*(\mathcal{V}) & &
\end{array}
\]

where \(H^1(-)\) is the map \(B \to B/A = \mathbb{R}^1 \pi_* \mathcal{V}\).

The above proposition should be interpreted as follows. We have a cosection \(\alpha : E \to O_U\) of \(E\) on \(U\). After restricting to \(Z(\beta)\) and identifying this locus with \(\mathcal{M}_{g,0}(\mathbb{P}', d)^p_S\), the proposition implies that \(\alpha\) factors as a surjection followed by \(d\hat{w}\). Thus the locus \(\{\alpha \equiv 0\}\) must be equal to \(\{d\hat{w} \equiv 0\} = \mathcal{M}_{g,0}(X_k, d)_S\). This proves the following.

**Corollary 2.7.** The closed substack \(Z(\beta) \cap \{\alpha \equiv 0\}\) is equal to \(\mathcal{M}_{g,0}(X_k, d)_S \subset U\).
2.3. \textbf{$Z_2$-localized Chern character}. In Section 1.1 we saw that given a vector bundle with a section, one could construct a refined Euler class supported on the vanishing locus of the section. In this section we have constructed a vector bundle $E \to U$ together with a section and cosection of $E$, which simultaneously vanish on a closed substack of $U$. We would like to “refine further” to incorporate the data of the cosection. This was done in [PV01]. The key is a modification of MacPherson’s graph construction [Ful13, Mac74].

Given a smooth variety $X$ and a rank $r$ vector bundle $E \to X$, recall the identity
\begin{equation}
\tag{2.5}
e(E) = \text{ch}(\wedge^* E^\vee) \text{Td}(E)
\end{equation}
where \text{ch} is the Chern character, \text{Td} is the Todd class (viewed as an element of $A_*(X)$), and $\wedge^* E^\vee$ is the class in $K$-theory given by the alternating sum $\oplus_{k=0}^r (-1)^k \wedge^k E^\vee$. This identity can be refined to include a section. Let $s \in \Gamma(X, E)$. Define the Koszul complex

$$K := \wedge^r E^\vee \to \cdots \to \wedge^2 E^\vee \to E^\vee \to 0,$$

where the differential is given by contraction with respect to $s, - \lrcorner s$. Since $s$ is non-vanishing outside of $Z = Z(s)$, by a standard linear algebra exercise this complex is exact outside of $Z(s)$, i.e. $K$ is supported on $Z$. MacPherson’s graph construction (see [Ful13, Section 18.1] and [Mac74] for details) defines a localized Chern character:

$$\text{ch}_{Z}^X(K) : A_*(X) \to A_*(Z).$$

The following identity holds:
\begin{equation}
\tag{2.6}
e(E, s) = \text{ch}_{Z}^X(K) (\text{Td}(E)).
\end{equation}

An insight of Polishchuk–Vaintrob in [PV01] was to adapt the definition of the localized Chern character to the case of 2-periodic complexes. Given an infinite 2-periodic complex $K$ of vector bundles, let $Z$ denote the support of $K$. In [PV01] a $Z_2$-localized Chern character is defined:

$$\text{ch}_{Z_2}^X(K) : A_*(X) \to A_*(Z).$$

The construction is a $Z_2$-graded version of the original construction.

Assume now we have $E \to X$ a vector bundle with a section $t \in \Gamma(X, E)$ and a cosection $s \in \text{hom}(E, \mathcal{O}_X)$.

\textit{Definition 2.8.} Define the vector bundles

$$F_0 := \oplus_k \wedge^{2k} E^\vee$$

$$F_{-1} := \oplus_k \wedge^{2k+1} E^\vee,$$

and define a differential between them $d := - \wedge s + - \lrcorner t$. Define the Koszul factorization $\{s, t\}$ to be the 2-periodic chain of vector bundles:

$$\{s, t\} := \cdots \to F_{-1} \to F_0 \to F_{-1} \to F_0 \to \cdots.$$
One can check that $d^2 = \text{id}_E \cdot (s \circ t)$.

Therefore in the case that $s \circ t = 0$, $\{s, t\}$ is a 2-periodic complex. One can check that the support of $\{s, t\}$ is $Z(t) \cap \{s \equiv 0\}$. In this case $\{s, t\}$ may be viewed as a 2-periodic generalization of the Koszul complex to include a cosection; we refer it as a Koszul factorization of 0.

2.4. The virtual class. Given Assumption 2.5, we have a vector bundle $E \to U$, together with a section $\beta \in \Gamma(U, E)$ and a cosection $\alpha \in \text{hom}(E, \mathcal{O}_U)$. The composition $\alpha \circ \beta$ is given by the linear map

$$\tilde{\alpha} \circ d_{k+1}: \text{Sym}^{k+1}(A) \to \mathcal{O}_S$$

which is zero by (2.3). Furthermore, the locus $Z(\beta) \cap \{\alpha \equiv 0\} = \overline{\mathcal{M}}_{g,0}(X_k, d)_S$. Construct the Koszul factorization $\{\alpha, \beta\}$ as above. Then the support of $\{\alpha, \beta\}$ is $\overline{\mathcal{M}}_{g,0}(X_k, d)_S$. In analogy with (2.6), we can make the following definition.

Definition 2.9. Let $\mathcal{M}_{X_k} := \overline{\mathcal{M}}_{g,0}(X_k, d)_S$. Define the virtual class

$$[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^r, d)]^\text{vir} = \mathbb{Z} \cdot \text{ch}^U_{\mathcal{M}_{X_k}}(\{\alpha, \beta\})(\text{Td}(E)) \in A_*(\overline{\mathcal{M}}_{g,0}(X_k, d)_S).$$

Remark 2.10. If Assumption 2.5 held for $S = \mathfrak{Bun}_{g,0, d}^p$, we would obtain a virtual class

$$[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^r, d)]^\text{vir} \in A_*(\overline{\mathcal{M}}_{g,0}(X_k, d))$$

as desired. Unfortunately Assumption 2.5 may not hold in this case. We explain in Section 5 how to overcome this difficulty.

3. Stable maps to a hypersurface ($n > 0$)

As before, let $X_k$ be a smooth hypersurface of degree $k$ in $\mathbb{P}^r$, defined by the vanishing of a polynomial $w$. In the previous section we defined a virtual fundamental class for $n = 0$. We would like to construct a virtual class on $\overline{\mathcal{M}}_{g,n}(X_k, d)$ for all $g, n, d$ such that $2g - 2 + n \geq 0$. In this section we incorporate marked points and evaluation maps into the previous construction. We will construct a Koszul factorization which will be used in Section 4 to define enumerative invariants. The use of a Koszul factorization to define enumerative invariants was first considered by Polishchuk–Vaintrob in [PV16].

3.1. Gluing. A crucial feature of Gromov–Witten theory, FJRW theory, or related enumerative theories is the existence of gluing maps between different moduli spaces obtained by gluing marked points. Consider for instance the case of stable maps to a smooth variety $X$. Let

$$\mathcal{D} \subset \overline{\mathcal{M}}_{g+1,0}(X, d)$$

denote the divisor of nodal curves. Let

$$\tilde{\mathcal{D}} \subset \overline{\mathcal{M}}_{g,2}(X, d)$$

denote the closed subvariety defined by the condition that \( ev_1 = ev_2 \). In other words, \( \tilde{D} \) parametrizes families of stable maps

\[
\{ \tilde{f} : \tilde{C} \to X; p_1, p_2 \in \tilde{C} \mid f(p_1) = f(p_2) \}.
\]

Then by gluing the points \( p_1 \) and \( p_2 \) together, one obtains a nodal curve \( C \) of genus \( g + 1 \), together with a map \( f : C \to X \). This defines a gluing morphism

\[
gl : \tilde{D} \to D \subset \mathcal{M}_{g+1,0}(X,d).
\]

These types of morphisms between moduli spaces, and corresponding compatibilities between the virtual classes (see \([\text{Beh99}]\)) give the Gromov–Witten invariants of \( X \) the structure of a cohomological field theory. When working with moduli of sections, one would like a similar structure.

Consider \( \mathcal{M}_{g+1,0}(\mathbb{P}^r,d)^p \) as in the previous section, parametrizing families \( C \to S \) of genus \( g + 1 \) curves together with a line bundle \( \mathcal{L} \) and a section

\[
(s, p) \in \Gamma(C, \mathcal{L}^\oplus r + L^{-k} \otimes \omega_C).
\]

Assume we have such a family, with exactly one node at each fiber. By normalizing the curve \( C \), one would expect to obtain a family parametrizing the same type of object, but with genus \( g \) and with 2 marked points. A subtlety arises, however, due to the well-known fact that under the normalization map

\[
v : \tilde{C} \to C
\]

the canonical bundle \( \omega_C \) does not pull back to the canonical bundle \( \omega_{\tilde{C}} \) of the source curve, but rather

\[
v^* (\omega_C) = \omega_{\tilde{C}}(p_1 + p_2)
\]

where \( p_1 \) and \( p_2 \) are the two points in the preimage of the node of \( C \). By normalizing we obtain a genus \( g \) curve \( \tilde{C} \), a line bundle \( \tilde{\mathcal{L}} = v^*(\mathcal{L}) \to \tilde{C} \), and a section

\[
(\tilde{s}, \tilde{p}) \in \Gamma(\tilde{C}, \tilde{\mathcal{L}}^\oplus r + \tilde{\mathcal{L}}^{-k} \otimes \omega_{\tilde{C}}(p_1 + p_2)).
\]

Alternatively, given a family of genus \( g \) marked curves with sections of \( \tilde{\mathcal{L}}^\oplus r + \tilde{\mathcal{L}}^{-k} \otimes \omega_{\tilde{C}}(p_1 + p_2) \), if \( (\tilde{s}, \tilde{p})|_{p_1} \) is equal to \( (\tilde{s}, \tilde{p})|_{p_2} \) up to a scaling of the fiber of \( \tilde{C} \), we can glue the sections at the marked points to obtain a map to \( \mathcal{M}_{g+1,0}(\mathbb{P}^r,d)^p \). It is this gluing property which we would like to preserve. We are lead to consider the relative log canonical bundle, \( \omega_{\pi, \log} = \omega_\pi(\Sigma) \), where recall that \( \Sigma \) is the divisor of marked points \( \sum_{i=1}^n p_i \), and \( p_i \) is shorthand for \( c_1(S) \).

**Definition 3.1.** \([\text{CL11}, \text{FJR17}]\) Fix \( g, n \in \mathbb{Z}_{\geq 0} \). Given \( k \in \mathbb{Z}_{\geq 0} \), define the space of stable maps to \( \mathbb{P}^r \) with a \( p \)-field of degree \( k \) to be the moduli space lying over \( \mathcal{M}_{g,n}(\mathbb{P}^r,d) \) parametrizing families \( \pi : C \to S \) of pre-stable
genus $g$, $n$-marked curves together with a line bundle $\mathcal{L} \to \mathcal{C}$ and a section

$$(s, p) \in \Gamma(\mathcal{C}, \mathcal{L}^r \oplus \mathcal{L}^{-k} \otimes \omega_{\pi, \log})$$

such that $s \in \Gamma(\mathcal{C}, \mathcal{L}^r)$ is nowhere vanishing and $\omega_{\pi, \log} \otimes \mathcal{L}^{-k}$ is ample. This moduli stack is denoted by $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. Given a map $S \to \text{Bun}_{g,n,d}$, define

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)_S := S \times_{\text{Bun}_{g,n,d}} \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d).$$

3.2. Evaluation maps. There is a second benefit to using $\omega_{\pi, \log}$ in place of $\omega_{\pi}$ in the presence of marked points. That is the existence of evaluation maps. Near a marked point $p_i \in \mathbb{C}$, sections of the log canonical bundle are given by differential forms with a pole at $p_i$. Taking the residue at the pole, we obtain a canonical trivialization $\omega_{\pi, \log}|_{p_i} \cong \mathcal{O}_{\mathbb{C}}|_{p_i}$. Therefore at the marked point we have a canonical isomorphism

$$\left(\mathcal{L}^r \oplus \mathcal{L}^{-k} \otimes \omega_{\pi, \log}\right)|_{p_i} \cong \left(\mathcal{L}^r \oplus \mathcal{L}^{-k}\right)|_{p_i}.$$

The section $(s, p)|_{p_i}$ gives a well-defined point in $\text{tot}(\mathcal{O}_{\mathbb{P}^r}(-k))$. We obtain evaluation maps:

$$\text{ev}_i : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)_S \to \left(\text{tot}(\mathcal{O}_{\mathbb{P}^r}(-k))\right).$$

3.3. Admissible resolutions. Given a family $S \to \text{Bun}_{g,n,d}$, as in the previous two sections we may view $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)_S$ as a space of sections over $\text{Bun}_{g,n,d}$. Let

$$\begin{array}{ccc}
\mathcal{L} & \to & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{C} & \to & \mathcal{S} \\
\sigma & \pi & \\
\mathcal{L} & \to & \mathcal{C} \\
\end{array}$$

be a family in $\text{Bun}_{g,n,d}$, viewed as pulled back from a map $S \to \text{Bun}_{g,n,d}$. Let

$$\mathcal{V}_2 := \mathcal{L}^{-k} \otimes \omega_{\pi, \log},$$

We repeat the construction of Section 2.1 but replacing $\mathcal{L}^{-k} \otimes \omega_{\pi}$ with $\mathcal{L}^{-k} \otimes \omega_{\pi, \log}$ in the definition of $\mathcal{V}_2$. We obtain a two term resolution

$$[A \to B] \sim \mathbb{R}_{\pi_*}(\mathcal{V}).$$

Let

$$\text{ev}_i : \mathbb{R}_{\pi_*}(\mathcal{V}) \to \mathbb{R}_{\pi_*}(\mathcal{V}|_{p_i}) \sim \mathcal{V}|_{p_i}.$$
denote the map in the derived category $D(S)$ corresponding to evaluation at the $i$th marked point.

Consider the function on $\mathcal{M}_{g,n}(\mathbb{P}^r,d)^p_S$ given by $ev_i^*(\hat{w})$. This is related to a map $\text{Sym}(\mathbb{R}\pi_*(\mathcal{V})) \to O_S$ in the derived category. Namely, we have the composition

$$(3.2) \quad [\tilde{Z}_i] : \text{Sym}^{k+1}(\mathbb{R}\pi_*(\mathcal{V})) \to \text{Sym}^{k+1}(\mathbb{R}\pi_*(\mathcal{V}|_p))$$

$$\quad \to \mathbb{R}\pi_* \text{Sym}^{k+1}(\mathcal{V}|_p) \xrightarrow{\hat{w}|_p} \mathbb{R}\pi_* \omega_{\pi,\log}|_p \to O_S$$

Exactly as in (2.2), the potential $\hat{w}$ defines a map

$$(3.3) \quad [\tilde{Z}_i] : \text{Sym}^{k+1}(\mathbb{R}\pi_*(\mathcal{V})) \to \mathbb{R}\pi_* \omega_{C,\log} \to \mathbb{R}\pi_* \omega_{C,\log}|_p \to O_S,$$

where the first arrow is the map defined above.

**Proposition 3.2.** [CFG+18, Corollary 3.4.2] The resolution $[A \to B]$ may be chosen such that the evaluation maps extend to $U$. In other words one can construct a resolution $[A \to B]$ such that there exist maps

$$ev_i : A \to \mathcal{V}|_p$$

so that the diagram

$$(3.4) \quad \begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow^{ev_i} & & \downarrow \\
\mathcal{V}|_p & \longrightarrow & 0.
\end{array}$$

realizes (3.1) at the level of complexes.

Furthermore, the induced map

$$(3.5) \quad \tilde{Z}_i : \text{Sym}^{k+1}([A \to B]) \xrightarrow{\text{Sym}^{k+1}(ev_i)} \text{Sym}^{k+1}(\mathcal{V}|_p) \to \omega_{C,\log}|_p \to O_S$$

gives a chain level realization of map $[\tilde{Z}_i]$ of (3.2) and (3.3).

**Remark 3.3.** The first part of the proposition corresponds to Condition 1 of Definition 3.2.1 of [CFG+18], the last part corresponds to Condition 3.

Note that $ev_i$ induces a map

$$ev_i : \text{tot}(A) \to [(C^{r+1} \oplus C)/C^*],$$

where the action of $C^*$ on $C^{r+1} \oplus C$ has weight $(1, \ldots, 1, -k)$. One can construct $ev_i$ such that the restriction of $ev_i$ to $U \subset \text{tot}(A)$ lands in the stable locus $\text{tot}(O_{2^r}(-k)) \subset [C^{r+1} \oplus C/C^*]$ (see Lemma 3.4.1 of [CFG+18] for details).
**Definition 3.4.** Define
\[ ev_i : U \to \text{tot} (\mathcal{O}_{\mathbb{P}^r}(-k)) \]
to be the map induced by \( \tilde{ev}_i \). Define
\[ \tilde{Z} : \text{Sym}([A \to B]) \to \mathcal{O}^n_S \]
to be the direct sum of \( \tilde{Z}_i \) for \( 1 \leq i \leq n \). Denote by
\[ Z_i : \text{tot}(A) \to C \]
the function on \( \text{tot}(A) \) corresponding to \( \tilde{Z}_i \) and let
\[ Z : \text{tot}(A) \to C^n \]
denote the direct sum.

By construction the following diagram commutes:
\[
\begin{array}{ccc}
\mathcal{M}_{g,n}(\mathbb{P}^r,d)_{S}^P & \xrightarrow{ev_i} & U \\
\downarrow{\text{tot}(\mathcal{O}_{\mathbb{P}^r}(-k))} & & \downarrow{\text{tot}(\mathcal{O}_{\mathbb{P}^r}(-k))} \\
C & \xrightarrow{Z_i} & C. \\
\end{array}
\]

**Remark 3.5.** The above diagram shows that \( Z_i \) may be understood as an extension of \( ev_i^* (\hat{\omega}) \) from \( \mathcal{M}_{g,n}(\mathbb{P}^r,d)_{S}^P \) to \( U \).

There exists a map from the cone \( C(\tilde{Z}) \) to \( \mathcal{O}_S \) obtained via the following commutative diagram (see [PV16]):
\[
\begin{array}{cccc}
\text{Sym}^{k+1}([A \to B]) & \xrightarrow{Z} & \mathcal{O}^n_S & \xrightarrow{=} & C(\tilde{Z}) \\
\downarrow & & \downarrow{\cong} & & \\
\mathbb{R} \pi_*(\omega_{\pi,\log}) & \xrightarrow{\text{sum}} & \mathbb{R} \pi_*(\omega_{\pi,\log} \mid \Sigma) & \xrightarrow{\text{trace}} & \mathbb{R} \pi_*(\omega_{\pi})[1]_{[a]} \\
& & \downarrow & & \\
& & \mathcal{O}_S.
\end{array}
\]

**Remark 3.6.** We warn the reader that the dashed vertical arrow is not canonical, even in the derived category. However in this situation there is a canonical way of choosing the map. The canonical choice was first described in [PV16]. See [CFG+18] for the construction in this context. Without further comment we will implicitly use this choice in the rest of the paper.

**Definition 3.7.** Define an admissible resolution to be a two-term resolution of \( \mathbb{R} \pi_*(\mathcal{V}) \) by vector bundles \([A \to B]\) such that

1. for \( 1 \leq i \leq n \), there exists a surjective evaluation map
\[ \tilde{ev}_i : A \to \mathcal{V}|_{p_i} \]
such that the diagram \((3.4)\) realizes \((3.1)\) at the level of complexes;
There exists a map $\tilde{\alpha}: \text{Sym}^k(A) \otimes B \to \mathcal{O}_S$ such that the vertical arrows in the diagram

$\begin{array}{cccc}
\text{Sym}^{k+1}(A) & \xrightarrow{(-d_{k+1}, \cdot)} & \text{Sym}^k(A) \otimes B \oplus \mathcal{O}_S^{\oplus n} & \xrightarrow{-d_k} & \text{Sym}^{k-1} \otimes \wedge^2(B) & \xrightarrow{-d_{k-1}} & \cdots \\
\downarrow & & \downarrow(\tilde{\alpha}, \text{sum}) & & \downarrow & & \\
0 & \rightarrow & \mathcal{O}_S & \rightarrow & 0 & \rightarrow & \cdots 
\end{array}$

give a map of complexes realizing $[\tilde{\alpha}]$ at the level of complexes. The map $[\tilde{\alpha}]$ is defined in (3.7) when $n > 0$ and in (2.2) when $n = 0$.

**Remark 3.8.** The conditions listed above to define an admissible resolution are described slightly differently than as originally presented in Definition 3.2.1 of [CFG+18]. This difference in presentation arises out of our choice to omit the construction of the dashed arrow of (3.7) from these notes.

We next give a condition for when an admissible resolution is known to exist.

**Definition 3.9.** We say that $S$ satisfies Condition $(\ast)$ if:

1. $S$ is a Deligne–Mumford stack of finite type over $\text{Spec}(\mathbb{C})$;
2. $S$ can be expressed as a global quotient stack by a linear algebraic group action; and
3. $S$ has projective coarse moduli space.

**Proposition 3.10.** [CFG+18 Proposition 3.5.2] Given a map $S \to \mathbb{B}\text{un}_{g,n,d}$, if $S$ satisfies Condition $(\ast)$, then there exists an admissible resolution $[A \to B]$ of $\mathbb{R}\pi_*(\mathcal{V})$.

**Remark 3.11.** Proposition 3.10 will be used in Section 5.2 to construct an admissible resolution. Note in the above that we do not require that $S$ map to the open locus $\mathbb{B}\text{un}_{g,n,d}^\circ \subset \mathbb{B}\text{un}_{g,n,d}$.

### 3.4. The Koszul factorization

For the remainder of this section we will assume the following.

**Assumption 3.12.** For a given family $S \to \mathbb{B}\text{un}_{g,n,d}^\circ$, assume that there exists over $S$ an admissible resolution $[A \to B]$ of $\mathbb{R}\pi_*(\mathcal{V})$.

As with Assumption 2.5 we warn the reader that this may not in fact hold in cases we care about (such as $S = \mathbb{B}\text{un}_{g,n,d}^\circ$). We explain how to work around it in Section 5.

Nevertheless, given Assumption 3.12 we may define the open subset $U \subset \text{tot}(A)$ and a vector bundle $E = \tau^r(B) \to U$ as in Definition 2.3.

As in Section 2 the map $A \to B$ defines a section $\beta \in \Gamma(U, E)$ such that

$\mathcal{M}_{g,n}(\mathbb{P}^r,d)^\flat_{\text{tot}(A)} \subset U$.

Furthermore, the map $\tilde{\alpha}: \text{Sym}^k(A) \otimes B \to \mathcal{O}_S$ defines a cosection of $E$,

$\alpha: E \to \mathcal{O}_{\text{tot}(A)}$.
on $\text{tot}(A)$ and on $U \subset \text{tot}(A)$.

Let $\{\alpha, \beta\}$ denote the Koszul factorization from Definition 2.8 associated to the section $\beta$ and the cosection $\alpha$. This is a 2-periodic chain of vector bundles over $U$. It is not, however, a complex. The composition $d^2$ is given by $\alpha \circ \beta$. This function on $\text{tot}(A)$ corresponds to the map of vector bundles defined by the composition

$$\text{Sym}^{k+1}(A) \xrightarrow{d_{k+1}} \text{Sym}^k(A) \otimes B \xrightarrow{\tilde{\alpha}} \mathcal{O}_S.$$  

By the commutativity of the first square of (3.8), this is equal to the composition

$$\text{Sym}^{k+1}(A) \xrightarrow{Z} \mathcal{O}_S^\oplus n \xrightarrow{\text{sum}} \mathcal{O}_S$$

which corresponds to the function $\text{sum} \circ Z : \text{tot}(A) \to \mathbb{C}$. When restricted to $U$, this is equal to

$$\text{sum} \circ Z|_U = \sum_{i=1}^n \text{ev}_i^* (\hat{w})$$

by (3.6). This proves the following:

**Proposition 3.13.** On $U$, the function $\alpha \circ \beta$ is equal to $\sum_{i=1}^n \text{ev}_i^* (\hat{w})$. Consequently, the composition of the differential of $\{\alpha, \beta\}$ is given by

$$d^2 = \text{id}_{F_{0/-1}} \cdot \left( \sum_{i=1}^n \text{ev}_i^* (\hat{w}) \right).$$

The 2-periodic chain of vector bundles $\{\alpha, \beta\}$ is called a Koszul factorization of $\sum_{i=1}^n \text{ev}_i^* (\hat{w})$. We will see in the next section that it gives an object in a triangulated category defined in terms of $U$ and the function $\sum_{i=1}^n \text{ev}_i^* (\hat{w})$.

Similarly to Proposition 2.6, we have the following.

**Proposition 3.14.** [CFG+18 Lemma 3.6.3] If $[A \to B]$ is an admissible resolution as in Definition 3.7, then there exists a vector bundle $A'$ together with a map $A' \to B$ such that

- $[A' \to B]$ is a two-term resolution of $\mathbb{R} \pi_*(\mathcal{V}(-\Sigma))$;
- the following diagram commutes

$$\begin{array}{ccc}
\text{Sym}^k(\pi_* \mathcal{V}) \otimes B & \longrightarrow & \text{Sym}^k(A) \otimes B \oplus \mathcal{O}_S^\oplus n \\
\downarrow & & \downarrow \text{(\hat{\alpha}, \text{sum})} \\
\text{Sym}^k(\pi_* \mathcal{V}) \otimes \mathbb{R}^1 \pi_* (\mathcal{V}(-\Sigma)) & \longrightarrow & \mathcal{O}_S \\
\end{array}$$

where the left vertical map is obtained from the quotient $B \to B/A'$ and the bottom arrow is the analogue of (2.4).

**Corollary 3.15.** Under the conditions of Proposition 3.14 we have

$$Z(\beta) \cap \{\alpha = 0\} \subset \mathcal{M}_{g,n}(\mathbb{P}^r,d)_S \cap \{d\hat{w} = 0\} = \mathcal{M}_{g,n}(X_k,d)_S.$$

**Remark 3.16.** Under Assumption 3.12 we have obtained the following:
• A smooth Deligne–Mumford stack $U$, together with a vector bundle $E \to U$ and a section $\beta \in \Gamma(U, E)$ such that $\mathcal{M}_{g, n}(\mathbb{P}^r, d)^P_S = Z(\beta);

• A cosection $\alpha \in \text{hom}(E, \mathcal{O}_U)$ such that
  – the 2-periodic chain of vector bundles $\{\alpha, \beta\}$ is a factorization of $\sum_{i=1}^n \text{ev}_i^*(\hat{w})$;
  – the locus $Z(\beta) \cap \{\alpha \equiv 0\}$ is contained in the substack $\mathcal{M}_{g, n}(X_k, d)_S$.

However because $\alpha \circ \beta$ is not zero, $\{\alpha, \beta\}$ is not a complex and we cannot apply the methods of the previous section to obtain a virtual class as a localized Chern character of $\{\alpha, \beta\}$. Nevertheless we would like to view the Koszul factorization $\{\alpha, \beta\}$ as the object taking the place of a virtual class. In the next section we will see how to use it to define enumerative invariants for $X_k$.

4. Extracting invariants

In this section we place the construction of the Koszul factorization $\{\alpha, \beta\}$ in its proper context, as an object in the derived category of factorizations. We then show how this perspective may be harnessed to construct numerical invariants.

4.1. The category of factorizations. We give a brief overview of the derived category of factorizations. For more details see [LP13, EP15, BDF+16].

Definition 4.1. [PV11, Section 3.1] An algebraic stack $Y$ is a nice quotient stack if $Y = [\tilde{Y}/H]$ where $\tilde{Y}$ is a noetherian scheme and $H$ is a reductive linear algebraic group such that $\tilde{Y}$ has an ample family of $H$-equivariant line bundles.

We will always assume that we are dealing with nice quotient stacks.

Definition 4.2. Let $Y$ be a nice quotient stack, let $L \to Y$ be a line bundle on $Y$, and let $w \in \Gamma(Y, L)$ be a section. We refer this structure as a Landau–Ginzburg space (or LG space) and sometimes denote it by $(Y, w)$. Given a pair of LG spaces $(Y_1, w_1 \in \Gamma(Y_1, L_1))$ and $(Y_2, w_2 \in \Gamma(Y_2, L_2))$, a function $f : Y_1 \to Y_2$ is a map of LG spaces, denoted

$$f : (Y_1, w_1) \to (Y_2, w_2)$$

if $f^*(L_2) = L_1$ and $f^*(w_2) = w_1$.

Definition 4.3. A factorization of $w$ on $Y$ is a pair of quasi-coherent sheaves $\mathcal{E}_{-1}, \mathcal{E}_0$ on $Y$ together with maps

$$\phi_0 : \mathcal{E}_{-1} \to \mathcal{E}_0$$

$$\phi_{-1} : \mathcal{E}_0 \to \mathcal{E}_{-1} \otimes_{\mathcal{O}_Y} L$$

such that we have the following identities on the compositions

$$\phi_{-1} \circ \phi_0 = \text{id}_{\mathcal{E}_{-1}} \otimes w : \mathcal{E}_{-1} \to \mathcal{E}_{-1} \otimes_{\mathcal{O}_Y} L$$

$$\phi_0 \circ \phi_{-1} \otimes \text{id}_L = \text{id}_{\mathcal{E}_0} \otimes w : \mathcal{E}_0 \to \mathcal{E}_0 \otimes_{\mathcal{O}_Y} L$$
We denote such a factorization by $E_\bullet = (E_{-1}, E_0, \phi_{-1}, \phi_0)$. We may think of the factorization as defining a sort of twisted complex

$$\cdots \to E_0 \otimes L^\vee \to E_{-1} \to E_0 \to E_{-1} \otimes L \to E_0 \otimes L \to \cdots$$

where the composition $d^2$ is equal to $w$ rather than 0.

Given two factorizations $E_\bullet = (E_{-1}, E_0, \phi_{-1}, \phi_0)$ and $F_\bullet = (F_{-1}, F_0, \psi_{-1}, \psi_0)$ of $w$ on $Y$, a morphism $g_\bullet : E_\bullet \to F_\bullet$ is a pair of maps $g_\bullet = (g_{-1} : E_{-1} \to F_{-1}, g_0 : E_0 \to F_0)$ such $g_0 \circ \phi_0 = \psi_0 \circ g_{-1}$ and $g_{-1} \otimes \text{id}_L \circ \phi_{-1} = \psi_{-1} \circ g_0$. In other words the following diagram commutes

$$\cdots \to E_0 \otimes L^\vee \to E_{-1} \to E_0 \to E_{-1} \otimes L \to E_0 \otimes L \to \cdots$$

$$\cdots \to F_0 \otimes L^\vee \to F_{-1} \to F_0 \to F_{-1} \otimes L \to F_0 \otimes L \to \cdots.$$

This defines an abelian category, denoted $\text{Qcoh}(Y, w)$. There is a natural notion of homotopy between two such morphisms. The homotopy category has the structure of a triangulated category. Taking the Verdier quotient by acyclic objects (see [EP15] for the definition), one obtains the derived category.

**Definition 4.4.** [EP15,BDF+16] Let $D(\text{Qcoh}(Y, w)) := K(\text{Qcoh}(Y, w))/\text{Acyc}(Y, w)$

denote the derived category of quasi-coherent factorizations of $(Y, w)$, defined as the Verdier quotient of the homotopy category by the subcategory of acyclic factorizations. Define the derived category of coherent factorizations of $(Y, w)$, $D(Y, w)$ to be the full subcategory of $D(\text{Qcoh}(Y, w))$ generated by factorizations with coherent components. This has the structure of a dg category by Corollary 2.23 of [BDF+16].

**Definition 4.5.** Let $M$ be a closed substack of $Y$ and let $E_\bullet$ be a factorization of $Y$. If for any morphism $S \to Y$ from a scheme $S$, the restriction of $E_\bullet$ to $S \setminus S \times Y M$ is acyclic, we say $E_\bullet$ is supported on $M$. Let $D(Y, w)_M$ denote the subcategory of $D(Y, w)$ consisting of factorizations supported on $M$.

We collect here the some important facts on functors between derived categories of factorizations.

**Proposition 4.6.** [EP15,BDF+16,CFG+18] One may construct the following functors.

1. Let $f : (X, v) \to (Y, w)$ be a map of LG spaces. There exists a derived pullback functor $f^* : D(Y, w) \to D(X, v)$. 

(2) Let \(u, w \in \Gamma(Y, L)\) be two sections of the same line bundle. Let \(F_\bullet \in D(Y, u)\). There exists a derived tensor product

\[- \otimes_{O_Y} F_\bullet : D(Y, w) \to D(Y, w + u).\]

Note that the potentials are added.

(3) Let \(f : (X, v) \to (Y, w)\) be a map of LG spaces. Let \(M\) be a closed substack of \(X\) and assume that \(f|_M : M \to Y\) is proper. Assume that \(f : X \to Y\) has finite cohomological dimension and that \(X\) has a smooth atlas. Then there is a well-defined derived pushforward

\[f_* : D(X, v)_M \to D(Y, w).\]

4.2. Koszul factorizations revisited. Let \((Y, w)\) be an LG space where \(w \in \Gamma(Y, L)\). Let \(V\) be a locally free sheaf on \(Y\). Assume we have sections \(\alpha^\vee \in \Gamma(Y, V^\vee \otimes L)\) and \(\beta \in \Gamma(Y, V)\). Assume that the following commutes

\[\beta \quad \xrightarrow{\alpha^\vee} \quad \phi^\vee\]

where \(\alpha \in \text{hom}(V, L)\) is the \(O_Y\)-linear map of vector bundles corresponding to \(\alpha^\vee\). Define

\[E_0 := \bigoplus_{i \geq 0} \left( \wedge^{2i} V^\vee \right) \otimes L^i\]

and define maps

\[\phi_0, \phi_{-1} := - \beta + - \wedge \alpha^\vee\]

where \(- \beta\) and \(- \wedge \alpha^\vee\) denote contraction with respect to \(\beta\) and the wedge with respect to \(\alpha^\vee\).

**Definition 4.7.** Define the Koszul factorization \(\{\alpha, \beta\} \in D(Y, w)\) to be

\[\{\alpha, \beta\} := E_\bullet\]

where \(E_\bullet = (E_{-1}, E_0, \phi_{-1}, \phi_0)\) from above.

**Proposition 4.8.** [PV16/CFG+18] The support of \(\{\alpha, \beta\}\) is given by

\[\text{Supp} (\{\alpha, \beta\}) = \{\beta = 0\} \cap \{\alpha^\vee = 0\} = \{\beta = 0\} \cap \{\alpha \equiv 0\}.\]

**Example 4.9.** A special case of the above construction is particularly relevant to our setting. Let \(w = w(x_0, \ldots, x_r)\) be a homogeneous polynomial of degree \(k\), defining a section \(s^\vee \in \Gamma(P^r, O_{P^r}(k))\). Recall \(\hat{w} = y \cdot w\) defines a function on

\[T := \text{tot}(O_{P^r}(-k)).\]

Let \(V = O_T(-k)\) denote the pullback of \(O_{P^r}(-k)\) to \(T\). There exists a tautological section \(t = \text{taut} \in \Gamma(T, V)\). Let \(s^\vee \in \Gamma(T, V^\vee)\) denote the pullback of \(s^\vee\). It is easy to check that \(s \circ t = \langle s^\vee, t \rangle\) is equal to \(\hat{w}\). Thus the Koszul factorization \(\{s, t\}\) is a factorization in \(D(T, \hat{w})\).
Example 4.10. We can refine the above construction by incorporating a \(C^\ast\)-action, called the R-charge. Let \(C^\ast_R = C^\ast\) denote a torus which acts on \(T\) by scaling the fiber coordinate. Let \(\chi : C^\ast_R \to C^\ast\) denote the identity character, and let

\[ L := \mathcal{O}_{[T/C^\ast_R]}(\chi) \]

denote the \(C^\ast_R\)-equivariant line bundle on \(T\) which is topologically trivial, with a weight one action by \(C^\ast_R\). Note that \(\tilde{w}\) may be viewed as a section of \(L\). Furthermore, if we now define

\[ V := \mathcal{O}_{[T/C^\ast_R]}(-k + \chi) \]

to be the line bundle which is topologically the pullback of \(O_{\mathbb{P}^r}(-k)\) but with a torus action of weight one, we retain from the previous discussion the existence of a tautological section \(t = \text{taut} \in \Gamma([T/C^\ast_R], V)\). The section \(s^\vee\) is now viewed as a section of \(V^\vee \otimes L\). With these considerations, we may define the Koszul factorization \(\{s, t\}\) as an element of \(D([T/C^\ast_R], \tilde{w})\), where \(w \in \Gamma([T/C^\ast_R], L)\).

Let the setup be as in the previous example. We have the following useful comparison.

Proposition 4.11. \([\text{Isi12, Shi12}]\) Let \(X_k = \{w = 0\}\) denote the degree \(k\) hypersurface in \(\mathbb{P}^r\) defined by \(w\). Assume that \(X_k\) is smooth. There exists an equivalence of categories

\[ \hat{\phi}_+ : D(X_k) \to D([T/C^\ast_R], \tilde{w}) \]

sending \(O_{X_k}\) to the Koszul factorization \(\{s, t\}\). Here \(D(X_k)\) denotes the bounded derived category of \(X_k\).

Proof. This was proven independently in \([\text{Isi12}]\) and \([\text{Shi12}]\). The precise formulation stated above can be found in \([\text{CFG}^{+18}, \text{Remark 2.5.6}]\). The equivalence \(\hat{\phi}_+\) is given in \([\text{CFG}^{+18}, \text{Definition 2.5.4}]\) \(\square\)

There are various choices one can make for the equivalence \(D(X_k) \to D([T/C^\ast_R], \tilde{w})\). In what follows we use \(\hat{\phi}_+\) as defined in \([\text{CFG}^{+18}, \text{Definition 2.5.4}]\).

The above statement generalizes by replacing \(\mathbb{P}^r\) with a smooth Deligne–Mumford global quotient stack, see \([\text{Hir17}]\). An especially simple case of the proposition is the following.

Corollary 4.12. Let \(X\) be a smooth variety or Deligne–Mumford stack. There exists an equivalence of categories

\[ D(X) \to D([X/C^\ast_R], 0) \]

where \(C^\ast_R\) is defined to act trivially on \(X\), and \(0\) is viewed as a section of \(O_{[X/C^\ast_R]}(\chi)\).

Remark 4.13. Factorizations of a section of a trivial bundle are only \(\mathbb{Z}_2\)-graded. The addition of the trivial action of \(C^\ast_R\) in (4.1) has the effect of equipping the right hand side with a \(\mathbb{Z}\)-grading.
4.3. **Hochschild homology.** Let $Y$ be a nice quotient stack. Recall that associated to the (bounded) derived category $\mathcal{D}(Y)$ is the Hochschild homology

$$\text{HH}_*(Y) := \bigoplus_i \text{HH}_i(Y),$$

$$\text{HH}_i(Y) := H^i(\Delta^*(\Delta_*(\mathcal{O}_Y)))$$

where $\Delta : Y \to Y \times Y$ is the diagonal map. Given an LG space $(Y, w)$, one can define the Hochschild homology as well, via an appropriate generalization of the above. Consider the product LG space $(Y \times Y, w \boxplus -w)$, where $w \boxplus -w$ is a section of $\pi_1^*(L) \otimes \pi_2^*(L)$ on $Y \times Y$. There is a diagonal map of LG spaces $\Delta : (Y, 0) \to (Y \times Y, w \boxplus -w)$.

**Definition 4.14.**

$$\text{HH}_*(Y, w) := \bigoplus_i \text{HH}_i(Y, w),$$

$$\text{HH}_i(Y, w) := H^i(\Delta^*(\Delta_*(\mathcal{O}_Y)))$$

where here $\mathcal{O}_Y$ denotes the factorization of 0 given by $E_\bullet = (\mathcal{O}_Y, 0, 0, 0)$.

In analogy with cohomology, there exists a categorical Chern character

$$\text{ch} : \mathcal{D}(Y) \to \text{HH}_*(Y),$$

$$\text{ch} : \mathcal{D}(Y, w) \to \text{HH}_*(Y, w).$$

Given a functor between derived categories, there is always an induced map on Hochschild homology which commutes with the Chern character.

As further evidence of the connection with cohomology we have the following theorem.

**Theorem 4.15.** [HKR09, Swa96, C˘al05] If $Y$ is a smooth and projective variety, there exists an HKR isomorphism

$$\phi_{\text{HKR}} : \text{HH}_*(Y) \to H^*(Y)$$

such that $\phi_{\text{HKR}} \circ \text{ch}$ is equal to the usual Chern character.

**Proof.** The HKR isomorphism was proven in the affine case in [HKR09] and in general in [Swa96]. The compatibility with the Chern character was proven in [C˘al05].

Via resolution of singularities of the coarse space, if $Y$ is a smooth Deligne-Mumford stack with projective coarse moduli space one can still define an HKR morphism:

$$\overline{\phi}_{\text{HKR}} : \text{HH}_*(Y) \to H^*(Y)$$

which is compatible with the Chern character. This is no longer an isomorphism.
4.4. Enumerative invariants. Let the setup be as in Section 3. Choose \( g, n, d \geq 0 \) such that \( 2g - 2 + n \geq 0 \). We can phrase the results of Section 3 in the language of the current section. Given a family \( S \to \mathcal{B}un_{g,n,d} \) satisfying Assumption 3.12, the constructions in Section 2 yield the collection of objects described in Remark 3.16. In particular we obtain a Koszul factorization. We will assume that the stack \( \mathcal{B}un_{g,n,d} \) itself satisfies Assumption 3.12 in the remainder of this section. We will show how, under this assumption, one can define enumerative invariants using the Koszul factorization. We will remove the need for this assumption in the next section.

Let \( S = \mathcal{B}un_{g,n,d} \). Assume \( S \) satisfies Assumption 3.12. Applying the constructions of Section 3 we obtain a smooth Deligne–Mumford stack \( U \) containing \( \mathcal{M}_{g,n}(\mathbb{P}^r, d) \) as a closed substack, together with a Koszul factorization \( \{ \alpha, \beta \} \) of \( \sum ev^*_i(\hat{\omega}) \), supported on \( \mathcal{M}_{g,n}(X_t, d) \). If we define an action of \( C^*_R \) on \( U \) by scaling in the \( A_2 \) direction, the evaluation maps

\[ ev_i : U \to T \]

become \( C^*_R \)-equivariant. Define the fundamental factorization

\[ K_{g,n,d} := \{ -\alpha, \beta \} \in D \left( [U/C^*_R], -\sum_{i=1}^n ev^*_i(\hat{\omega}) \right). \]

We have the following diagram between derived categories

\[
\begin{array}{ccc}
D \left( [U/C^*_R], \sum_{i=1}^n ev^*_i(\hat{\omega}) \right) & \xrightarrow{\otimes K_{g,n,d}} & D \left( [U/C^*_R], 0 \right) \mathcal{M}_{g,n}(X_t, d) \\
\downarrow{ev^*_i} & & \downarrow{\text{proj}} \\
D \left( [T/C^*_R], \hat{\omega} \right) \otimes \mathbb{C} & \xrightarrow{\cong} & D \left( [\mathcal{M}_{g,n}/C^*_R], 0 \right) \xrightarrow{\cong} D \left( \mathcal{M}_{g,n} \right)
\end{array}
\]

where the \( C^*_R \)-action on \( \mathcal{M}_{g,n} \) is trivial. To define enumerative invariants, we would like to take the induced functor on Hochschild homology, combined with the HKR morphism \( \bar{\varphi}_{HKR} : HH_*(\mathcal{M}_{g,n}) \to H^* (\mathcal{M}_{g,n}) \). However, as in Definition 2.9 we must adjust by a Todd correction to get invariants which are homogeneous (and which agree with Gromov–Witten theory). This requires compactifying \( U \) (or more precisely a subspace of \( U \)).

Let \( U_1 \) denote \( U \cap \text{tot}(A_1) \) where we view \( \text{tot}(A_1) \) as lying inside \( \text{tot}(A) \) via the zero section of \( A_2 \). Assume there exists a smooth Deligne–Mumford stack \( \tilde{U}_1 \) lying over \( \mathcal{B}un_{g,n,d} \), with projective coarse moduli space and containing \( U_1 \) as an open substack. Notice that on \( U_1 \), the relative tangent bundle is equal to the pullback of \( A_1 \),

\[ T_{U_1}/\mathcal{B}un_{g,n,d} = \tau^*(A_1). \]

Therefore in K-theory we have the equality

\[ E \otimes \tau^*(A_2) = \tau^*(B) \otimes \tau^*(A) \otimes T_{U_1}/\mathcal{B}un_{g,n,d} = T_{U_1}/\mathcal{B}un_{g,n,d} \otimes \tau^* \mathbb{R} \pi_*(\mathcal{V}). \]
The class $\text{Td}(E) / \text{Td}(\tau^*(A_2))$ may therefore be viewed as the restriction of
$$\text{Td}(T\bar{U}_1 / \mathbb{B}_{\text{un}_{g,n,d}}) / \text{Td}(\tau^*\mathcal{R}_s(V))$$
to $U_1$. Then consider the modified diagram
(4.2)
\[
\begin{array}{c}
D\left([U/C_R], \sum_{i=1}^n \text{ev}_i^* (\bar{w})\right) \overset{\otimes \Phi_{K_{g,n,d}}}{\longrightarrow} D\left([U/C_R], 0\right) \overset{\otimes \iota_*}{\longrightarrow} D\left(T\bar{U}_1 / \mathbb{B}_{\text{un}_{g,n,d}}\right)
\end{array}
\]
where $\iota : U \to U_1 \hookrightarrow \bar{U}_1$ is the projection followed by the inclusion. Define
$$\Phi_{K_{g,n,d}} : \text{HH}_*(T/C_R, \bar{w}) \otimes n \to \text{HH}_*(\bar{U}_1)$$
to be the map on Hochschild homology induced by the composition of functors (4.2). Let
$$\overline{\Phi}_{\text{HKR}} : \text{HH}_*(\bar{U}_1) \to H^*(\bar{U}_1)$$
denote the HKR morphism, and let
$$\text{proj}_* : H^*(\bar{U}_1) \to H^*(\overline{\mathcal{M}}_{g,n})$$
denote the map forgetting sections and stabilizing the curve.

**Definition 4.16.** Define
$$\Lambda_{g,n,d} : \text{HH}_*([T/C_R], \bar{w}) \otimes n \to H^*(\overline{\mathcal{M}}_{g,n})$$
as follows. For $s_1, \ldots, s_n \in \text{HH}_*(T/C_R, \bar{w})$, let
$$\Lambda_{g,n,d}(s_1, \ldots, s_n) = \text{proj}_* \left( \frac{\text{Td}(T\bar{U}_1 / \mathbb{B}_{\text{un}_{g,n,d}})}{\text{Td}(\tau^*\mathcal{R}_s(V))} \cup \overline{\Phi}_{\text{HKR}} \circ \Phi_{K_{g,n,d}}(s_1, \ldots, s_n) \right).$$

We may then define enumerative invariants:
$$\langle s_1, \ldots, s_n \rangle_{g,n,d} = \int_{\overline{\mathcal{M}}_{g,n}} \Lambda_{g,n,d}(s_1, \ldots, s_n).$$

In conclusion, the fundamental factorization $K_{g,n,d}$ defines an integral transform between categories of factorizations. The induced map on Hochschild homology allows us, after correcting by a Todd class and applying the HKR morphism to cohomology, to define numerical invariants as if we had a

\[\text{The reader may wonder why we are concerned here with Td}(E) / \text{Td}(\tau^*(A_2)), \text{ when in Section 3 we simply used Td}(E). \text{ The reason is that (see Definition 5.5) we apply the HKR morphism only after projecting from U to U_1. A Grothendieck–Riemann–Roch calculation then implies that the desired Todd correction after mapping to U_1 is Td}(E) / \text{Td}(T_{U_1/U_1}) = \text{Td}(E) / \text{Td}(\tau^*(A_2)).\]
virtual fundamental class. It will turn out that the maps \( \{ \Lambda_{g,n,d} \} \) define a cohomological field theory. Furthermore, via the isomorphism
\[
H^*(X_k) \cong HH_*(X_k) \cong HH_*(\mathbb{T}/C^+_R, \hat{w}),
\]
these invariants agree with Gromov–Witten invariants up to a sign.

5. Technical assumptions and the “two-step procedure”

The construction of enumerative invariants in Section 4.4 rely on Assumption 3.12 holding for \( \mathcal{Bun}^0_{g,n,d} \). This is not known, and likely not true. In this section we describe a “two-step procedure” to circumvent the problem. For details and a general formulation see Section 4 of [CFG+18].

5.1. Step 1: a projective embedding. Fix \( g, n, d \geq 0 \) such that \( 2g - 2 + n > 0 \). We first embed \( \mathcal{M}_{g,n} \) into (the smooth locus of) a larger space of stable maps.

Consider the stabilization map
\[
\text{st} : \mathcal{Bun}^0_{g,n,d} \to \mathcal{M}_{g,n},
\]
and the diagram
\[
\begin{array}{ccc}
\mathcal{L} & \to & O_{\mathbb{P}^N}(1) \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
where \( \mathcal{M} := \mathcal{L} \otimes \mathcal{N}, M = (r + 1)N \) and \( \mathcal{Q} = \mathcal{V}_1 \otimes \tilde{\text{st}}^* \circ \rho^* (\mathbb{P}^N). \) Pushing forward via \( \pi \) we obtain the long exact sequence on \( \text{Bun}^o_{g,n,d}. \)

\[
0 \to \pi_* (\mathcal{V}_1) \to \pi_* \left( \mathcal{M}^{\oplus M} \right) \to \pi_* (\mathcal{Q}) \\
\to \mathbb{R}^1 \pi_* (\mathcal{V}_1) \to 0
\]

where the last terms are zero because \( \mathcal{M} \) was constructed to be \( \pi \)-acyclic. Letting

\[
A'_1 := \pi_* \left( \mathcal{M}^{\oplus M} \right), \\
B'_1 := \pi_* (\mathcal{Q}),
\]

we see that \([A'_1 \to B'_1]\) is a two term resolution of \( \mathbb{R} \pi_* (\mathcal{V}_1) \) by vector bundles. We also obtain an embedding \( \text{tot}(\pi_* \mathcal{V}_1) \to \text{tot}(A'_1). \) Let

\[
\tau : \text{tot}(A'_1) \to \text{Bun}^o_{g,n,d}
\]

denote the map forgetting the sections. Then \( \text{tot}(\pi_* \mathcal{V}_1) \) is the zero locus of the section of \( \tau^* (B'_1) \) induced by the map \( A'_1 \to B'_1. \)

The section of \( \mathcal{N}^{\oplus N} \) obtained by pulling back the first terms of the Euler sequence is nowhere vanishing. Therefore nowhere vanishing sections of \( \mathcal{V}_1 \) are mapped to nowhere vanishing sections of \( \mathcal{M}^{\oplus M} \) which are nowhere vanishing. Note that these sections correspond to a map to \( \mathbb{P}^{M-1} \) of some degree \( e. \) This realizes \( U_1 \) as an open subset of \( \overline{\mathcal{M}}_{g,n} (\mathbb{P}^{M-1}, e). \) By construction \( U_1 \) is a smooth Deligne–Mumford stack, lying in the smooth locus of \( \overline{\mathcal{M}}_{g,n} (\mathbb{P}^{M-1}, e). \) Let \( \overline{U}_1 \) denote the closure of \( U_1 \) in \( \overline{\mathcal{M}}_{g,n} (\mathbb{P}^{M-1}, e). \) The zero locus of \( \tau^* (B'_1)|_{U_1} \) is equal to \( \overline{\mathcal{M}}_{g,n} (\mathbb{P}^r, d). \) We have

\[
\overline{\mathcal{M}}_{g,n} (\mathbb{P}^r, d) \hookrightarrow U_1 \subset \overline{U}_1 \subset \overline{\mathcal{M}}_{g,n} (\mathbb{P}^{M-1}, e).
\]

5.2. Step 2: an admissible resolution. The next step is to construct a second resolution, this time over \( U_1. \)

Let \( \pi_{U_1} : \mathcal{C}_{U_1} \to U_1 \) and \( \pi_{\overline{U}_1} : \mathcal{C}_{\overline{U}_1} \to \overline{U}_1 \) denote the universal curves over \( U_1 \) and \( \overline{U}_1 \) respectively. Let \( \mathcal{L}_{U_1}, \mathcal{N}_{U_1}, \) and \( \mathcal{M}_{U_1} \) denote the pullback of \( \mathcal{L}, \mathcal{N}, \) and \( \mathcal{M} \) from \( \mathfrak{c} \) to \( \mathcal{C}_{U_1}. \) These line bundles naturally extend to line bundles \( \mathcal{L}_{\overline{U}_1}, \mathcal{N}_{\overline{U}_1}, \) and \( \mathcal{M}_{\overline{U}_1} \) over \( \overline{U}_1 \) as follows. Since \( \overline{U}_1 \subset \overline{\mathcal{M}}_{g,n} (\mathbb{P}^{M-1}, e), \) there exists a universal map \( f : \mathcal{C}_{\overline{U}_1} \to \mathbb{P}^{M-1}. \) Define \( \mathcal{M}_{\overline{U}_1} \) to be the pullback of \( \mathcal{O}_{\mathbb{P}^{M-1}}(1). \) On the other hand by forgetting \( f \) and stabilizing the curve, we get a map \( \overline{U}_1 \subset \overline{\mathcal{M}}_{g,n} (\mathbb{P}^{M-1}, e) \to \overline{\mathcal{M}}_{g,n}, \) and consequently a map

\[
\mathcal{C}_{\overline{U}_1} \to \mathcal{C}_{\mathcal{st}} \to \mathcal{C}_{\mathcal{st}} \to \mathbb{P}^{N-1}.
\]

Define \( \mathcal{N}_{\overline{U}_1} \) to be the pullback of \( \mathcal{O}_{\mathbb{P}^{N-1}}(1). \) Finally, define

\[
\mathcal{L}_{\overline{U}_1} := \mathcal{M}_{\overline{U}_1} \otimes \mathcal{N}_{\overline{U}_1}.
\]
The line bundle $L \rightarrow C$ is a degree $d$ line bundle over the family $C \rightarrow U$ of pre-stable curves, therefore there is an induced map $U \rightarrow \text{Bun}_{g,n,d}$, which, when restricted to $U \subset U$, recovers the forgetful map $U \rightarrow \text{Bun}_{g,n,d}^\circ \subset \text{Bun}_{g,n,d}$.

Let $V^i := L^{r+1}$, $V^i := L^{-k} \otimes \omega_{\pi, \log}$, and define 

$$V := V^1 \oplus V^2.$$ 

Let $V_{U,1}, V_{U,2}$, and $V_U$ denote the corresponding restrictions to $C$.

Proposition 5.1. \cite[Proposition 3.5.2]{CFG} On $U$, there exist resolutions

$$[A_{U,1}'' \rightarrow B_{U,1}''] \sim \mathcal{R} \pi_* (V_{U,1}),$$

$$[A_{U,2}'' \rightarrow B_{U,2}''] \sim \mathcal{R} \pi_* (V_{U,2})$$

such that, if we let

$$[A_1'' \rightarrow B_1''] := [A_{U,1}'' \oplus A_{U,2}'' \rightarrow B_{U,1}'' \oplus B_{U,2}''],$$

there exists

(1) for $1 \leq i \leq n$, a surjective evaluation map

$$\tilde{ev}_i : A_{U,1}'' \rightarrow V_{U,1}|_i$$ 

which induce maps

$$\tilde{Z}_i : \text{Sym}^{k+1}(A_{U,1}'' \rightarrow B_{U,1}'') \rightarrow \text{Sym}^{k+1}(V_{U,1}|_i) \rightarrow \omega_{\pi, \log}|_i \rightarrow O_{U,1}$$

realizing the maps (3.2) and (3.3) at the chain level.

(2) a map

$$\tilde{\alpha} : \text{Sym}^{k}(A_{U,1}'') \otimes B_{U,1}'' \rightarrow O_{U,1}$$

fitting into the diagram (3.8) and realizing the map $[\tilde{\alpha}]$ of (3.7) at the level of complexes.

Proof. By \cite[Corollary 1.0.3]{AGOT}, the space $U$ satisfies Condition ($\star$). By Proposition 3.10 (\cite[Proposition 3.5.2]{CFG}), we can construct an admissible resolution over $U$. By \cite[Proposition 3.5.5]{CFG}, the admissible resolution can be assumed to split as $[A_{U,1}'' \oplus A_{U,2}'' \rightarrow B_{U,1}'' \oplus B_{U,2}'']$.

Let

$$A_{U,1,1}'' \rightarrow B_{U,1,1}'',$$

$$A_{U,1,2}'' \rightarrow B_{U,1,2}'',$
denote the restrictions of
\[ A''_{U_1,1} \rightarrow B''_{U_1,1}, \]
\[ A''_{U_1,2} \rightarrow B''_{U_1,2}, \]
to \( U_1 \). Let
\[ A'_{U_1,1} := \tau^*(A'_1), \]
\[ B'_{U_1,1} := \tau^*(B'_1). \]

In the derived category \( D(U_1) \) we have the equivalence
\[ [A''_{U_1,1} \rightarrow B''_{U_1,1}] \sim [A'_{U_1,1} \rightarrow B'_{U_1,1}]. \]

By standard arguments (see Lemma 3.6.5 of [CFG+18]) there exists another resolution \( [A_{U_1,1} \rightarrow B_{U_1,1}] \) of \( \mathbb{R}\pi_*(\mathcal{V}_{U_1}) \) and a roof diagram
\[
\begin{array}{ccc}
A_{U_1,1} & \xrightarrow{d_1} & B_{U_1,1} \\
\downarrow d'' & & \downarrow d'' \\
A''_{U_1,1} & \xrightarrow{d''} & B''_{U_1,1}
\end{array}
\]
(5.2)

where the diagonal maps of two-term complexes are quasi-isomorphisms. Consider the resolution of \( \mathbb{R}\pi_*(\mathcal{V}_{U_1}) \) given by
\[ [A_{U_1} \rightarrow B_{U_1}] := [A_{U_1,1} \oplus A''_{U_1,2} \rightarrow B_{U_1,1} \oplus B''_{U_1,2}]. \]

By composing the maps of Proposition 5.1 with the left diagonal map of (5.2), we have maps
\[
(5.3)
\begin{align*}
ev_i &: A_{U_i} \rightarrow \mathcal{V}_{U_i}|_{p_i} \\
\tilde{Z}_i &: \text{Sym}^{k+1}(A_{U_i}) \rightarrow \mathcal{O}_{U_i} \\
\tilde{k} &: \text{Sym}^{k}(A_{U_i}) \otimes B_{U_i} \rightarrow \mathcal{O}_{U_i},
\end{align*}
\]
which fit into diagrams (3.4), (3.5), and (3.8) realizing the maps (3.1), (3.2) (and (3.3)), and (3.7) at the level of complexes.

Define \( \text{tot}(A_{U_i})^0 \) to be the open subset of \( \text{tot}(A_{U_i}) \) such that \( \ev_i \) maps to the stable locus of \( \mathcal{V}_{U_i}|_{p_i} \) for all \( 1 \leq i \leq n \). The evaluation maps \( \ev_i \) induce maps
\[ \ev_i : \text{tot}(A_{U_i})^0 \rightarrow \text{tot}(\mathcal{O}_{\mathbb{P}^r}(-k)). \]

One may endow \( \text{tot}(A_{U_i})^0 \) with a \( C_R^* \) action by scaling in the \( A_{U_i,2} \) direction. With this the evaluation maps \( \ev_i \) are \( C_R^* \)-equivariant, where recall that \( C_R^* \) acts on \( \text{tot}(\mathcal{O}_{\mathbb{P}^r}(-k)) \) by scaling in the fiber direction. In other words we have a map
\[ \ev_i : [\text{tot}(A_{U_i})^0/C_R^*] \rightarrow [\text{tot}(\mathcal{O}_{\mathbb{P}^r}(-k))/C_R^*]. \]

Consider the morphism
\[ \tilde{\tau} : \text{tot}(A_{U_i})^0 \rightarrow U_1 \]
forgetting the section of $A_{U_1}$. Define $E := \tilde{\tau}^*(B_{U_1})$. The map $[A_{U_1} \to B_{U_1}]$ defines a section
\[ \beta \in \Gamma(\text{tot}(A_{U_1})^\circ, E) \]
and the map $\tilde{\alpha}$ of (5.3) defines a cosection
\[ \alpha \in \text{hom}(E, \mathcal{O}_{\text{tot}(A_{U_1})^\circ}). \]

The composition of the above maps is given by
\[ \alpha \circ \beta = \sum_{i=1}^{n} \text{ev}_i^*(\hat{\omega}). \]

The Koszul factorization of Definition 2.8 defines an element
\[ \{\alpha, \beta\} \in D \left( \left[ \text{tot}(A_{U_1})^\circ / \mathcal{C}_R^\ast \right], \sum_{i=1}^{n} \text{ev}_i^*(\hat{\omega}) \right). \]

5.3. The cut-down procedure. The space $\text{tot}(A_{U_1})^\circ$ lies over $U_1 \subset \text{tot}(A_1')$, so the relative dimension of $\text{tot}(A_{U_1})^\circ \to \mathcal{B}un_{g,n,d}^\circ$ is $\text{rank}(A_{U_1}) + \text{rank}(A_1')$.

If we view $E = \tilde{\tau}^*(B_{U_1})$ as an obstruction bundle, the relative virtual dimension over $\mathcal{B}un_{g,n,d}^\circ$ is
\[ \text{rank}(A_{U_1}) - \text{rank}(B_{U_1}) + \text{rank}(A_1') = \chi(\mathbb{R}\pi_\ast(\mathcal{V})) + \text{rank}(A_1') \]
which is too large by rank $(A_1')$. This overcounting is due to the fact that we resolved $\mathbb{R}\pi_\ast(\mathcal{V})$ twice. We must correct for this redundancy by choosing an appropriate closed subset of codimension equal to rank $(A_1')$.

We have tautological sections $\text{taut} \in \Gamma(\text{tot}(A_{U_1})^\circ, \tilde{\tau}^*(A_{U_1,1}))$ and $\text{taut}' \in \Gamma(U_1, A_{U_1,1})$. Let $f_1 : A_{U_1,1} \to A_{U_1,1}'$ denote the surjective map from (5.2).

Consider the section
\[ \tilde{\xi} := \tilde{\tau}^*(f_1) \circ \text{taut} - \tilde{\tau}^*(\text{taut}') \in \Gamma(\text{tot}(A_{U_1})^\circ, \tilde{\tau}^*(A_{U_1,1}')). \]

Definition 5.2. Define the substack of $\text{tot}(A_{U_1})^\circ$
\[ \square := \{\tilde{\xi} = 0\}. \]

By definition of $\tilde{\xi}$, $\square$ consists of triples $(a_1', a_1, a_2)$ where $a_1' \in U_1 \subset \text{tot}(A_1')$, $a_1 \in \text{tot}(A_{U_1,1})|_{a_1'}$, and $a_2 \in \text{tot}(A_{U_1,2})|_{a_1'}$ such that $f_1(a_1) = a_1'$.

Because $f_1$ is surjective, $\square$ will be smooth of relative dimension rank $(A_{U_1})$ over $\mathcal{B}un_{g,n,d}^\circ$. Note also that $\square$ is preserved by the action of $\mathcal{C}_R^\ast$.

Let $\iota : \square \hookrightarrow \text{tot}(A_{U_1})^\circ$
denote the inclusion.

Definition 5.3. Define the Fundamental factorization
\[ K_{g,n,d} := \iota^*(\{-\alpha, \beta\}) \in D \left( \left[ \square / \mathcal{C}_R^\ast \right], -\sum_{i=1}^{n} \text{ev}_i^*(\hat{\omega}) \right). \]

---

3 More precisely, $[A_1' \to B_1']$ resolved $\mathbb{R}\pi_\ast(\mathcal{V})$, and $[A_{U_1,1} \to B_{U_1,1}]$ resolved $\tau^*(\mathbb{R}\pi_\ast(\mathcal{V}))$. 
The terms of $K_{g,n,d}$ consist of the wedge powers of $i^*(E)$.

**Proposition 5.4.** The support of $K_{g,n,d}$ is contained in $\mathcal{M}_{g,n}(X_k,d)$.

**Proof.** By Proposition 4.8

$$\text{supp} \ (K_{g,n,d}) = Z(i^*(\beta)) \cap \{i^*(\alpha) \equiv 0\}.$$ 

The locus $Z(i^*(\beta))$ is equal to $\mathcal{M}_{g,n}(P', d)^p$ because we have cut down to $\square$. By Corollary 3.15, the intersection of this with $\{i^*(\alpha) \equiv 0\}$ is contained in $\mathcal{M}_{g,n}(P', d)^p \cap \{d\omega \equiv 0\} = \mathcal{M}_{g,n}(X_k,d)$. 

$\square$

5.4. **Enumerative invariants.** Let $\tilde{U}_1$ denote a smooth resolution of $U_1$. Let $i : \square \rightarrow \tilde{U}_1$ denote the composition $i : \square \hookrightarrow \text{tot}(A_{U_1})^\circ \rightarrow U_1 \hookrightarrow \tilde{U}_1$.

We mimic the procedure of Section 4.4. We have the following diagram between derived categories (5.4)

$$
\begin{array}{ccc}
D\left([\square/C^+_R]_*, \sum_{i=1}^n \text{ev}_i^*(\omega)\right) & \otimes_{K_{g,n,d}} & D\left([\square/C^+_R]_*, 0\right)_{\mathcal{M}_{g,n}(X_k,d)} \\
\uparrow \text{ev}^* & & \downarrow \text{id} \\
D\left([T/C^+_R], \omega\right)^{\otimes n} & \cong & D\left([\tilde{U}_1/C^+_R]_*, 0\right) \rightarrow D(\tilde{U}_1).
\end{array}
$$

Define

$$\Phi_{K_{g,n,d}} : \text{HH}^*_s([T/C^+_R], \omega)^{\otimes n} \rightarrow \text{HH}^*_s(\tilde{U}_1)$$

to be the map on Hochschild homology induced by the composition of functors (5.4). Let

$$\mathcal{F}_{\text{HKR}} : \text{HH}^*_s(\tilde{U}_1) \rightarrow H^*(\tilde{U}_1)$$

denote the HKR morphism, and let

$$\text{proj}^* : H^*(\tilde{U}_1) \rightarrow H^*(\mathcal{M}_{g,n})$$

denote the map forgetting sections and stabilizing the curve.

**Definition 5.5.** Define

$$\Lambda_{g,n,d} : \text{HH}^*_s([T/C^+_R], \omega)^{\otimes n} \rightarrow H^*(\mathcal{M}_{g,n})$$

as follows. For $s_1, \ldots, s_n \in \text{HH}^*_s([T/C^+_R], \omega)$, let

$$\Lambda_{g,n,d}(s_1, \ldots, s_n) = \text{proj}^* \left( \frac{\text{Td}(T_{U_1} \cup \text{Bun}_{g,n,d})}{\text{Td}(\tau^*\mathbb{R}\pi_*\mathcal{V})} \cup \mathcal{F}_{\text{HKR}} \circ \Phi_{K_{g,n,d}}(s_1, \ldots, s_n) \right).$$

We then define enumerative invariants:

$$\langle s_1, \ldots, s_n \rangle_{g,n,d}^{([T/C^+_R], \omega)} := \int_{\mathcal{M}_{g,n}} \Lambda_{g,n,d}(s_1, \ldots, s_n).$$
5.5. **Comparison.** These invariants have been proven to agree with the Gromov–Witten theory of $X_k$ in the following sense. Recall by Proposition 4.11, the existence of an equivalence $\tilde{\phi}_+ : D(X_k) \to D([T/C_R^k], \tilde{w})$. This induces a map on Hochschild homology. Composing with the HKR morphism gives an isomorphism

$$(\tilde{\phi}_+) \circ \phi_{HKR}^{-1} : H^*(X_k) \xrightarrow{\cong} \text{HH}_*(\pi^*([T/C_R^k], \tilde{w})).$$

In [CFG+18 Definition 2.5], this map is modified by precomposing with $\text{Td} (\mathcal{O}_{\mathbb{P}}(-k)) \cup - : H^*(X_k) \to H^*(X_k)$. Composing these two maps, we define

$$\phi_{\text{Td}} = (\tilde{\phi}_+) \circ \phi_{HKR}^{-1} \circ \text{Td} (\mathcal{O}_{\mathbb{P}}(-k)) \cup - .$$

See [CFG+18 Definition 2.5.7].

**Proposition 5.6.** Given $\gamma_1, \ldots, \gamma_n \in H^*(X_k)$, the quantity

$$\langle \phi_{\text{Td}}^* (\gamma_1), \ldots, \phi_{\text{Td}}^* (\gamma_n) \rangle_{g,n,d}(\mathbb{T}/C_R^k, \tilde{w})$$

agrees with the Gromov–Witten invariant

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{g,n,d}^{X_k}$$

up to a sign.

**Proof.** In [CL11], a virtual class similar in spirit to that of Definition 2.9 is constructed via cosection localization. In [CFG+18 Theorem 6.1.8] it is proven that the maps

$$H^*(X_k)^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n})$$

defined using the cosection-localized virtual class agree with $\Lambda_{g,n,d}$ from Definition 5.5 defined via the fundamental factorization. Next, the cosection-localized virtual class is proven to agree with the Gromov–Witten virtual class defined in [BP97] up to a sign of $(-1)^{\chi(\mathcal{L}^{\otimes-k})}$ where $\mathcal{L}$ is the universal line bundle over the universal curve $\mathcal{C} \to \text{Bun}_{g,n,d}$ and $\chi(\mathcal{L}^{\otimes-k})$ denotes the virtual rank of $\mathcal{R} \pi_* (\mathcal{L}^{\otimes-k})$. This comparison was shown in the case that $X_k$ is the quintic three-fold and $n = 0$ in [CL11]. It was proven in general in [KO18].

**Corollary 5.7.** The invariants $\Lambda_{g,n,d}$ form a cohomological field theory in the sense of [KM94].

**Proof.** Because Gromov–Witten theory of $X_k$ has the structure of a cohomological field theory, it suffices to observe that the axioms are preserved after modifying the maps

$$H^*(X_k)^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n})$$

of Gromov–Witten theory by the sign $(-1)^{\chi(\mathcal{L}^{\otimes-k})}$. 

6. CONCLUSIONS AND FURTHER DIRECTIONS

The introduction of factorizations to the setting of enumerative geometry yields a powerful new tool for defining and computing enumerative invariants. In the last four sections we have described how integral transforms between derived categories of factorizations can be used to give a new construction of the Gromov–Witten invariants of a projective hypersurface. The first instance in which factorizations were used to define enumerative invariants was by Polishchuk–Vaintrob in [PV16], where these ideas were developed and employed to give a new construction of the FJRW theory (see [FJR13]) of a homogeneous singularity.

This perspective has led to new methods of computation as well. In [Gue16], Guère used the definitions of [PV16] to compute FJRW invariants in the so-called non-concave setting in genus zero. The analogous invariants in Gromov–Witten theory have yet to be computed.

In fact the methods outlined above apply in much greater generality than has been treated here. The general context in which one might use such techniques is that of a gauged linear sigma model (GLSM). This includes both Gromov–Witten theory of a hypersurface as well as FJRW theory of a singularity as special cases. For a detailed description of the mathematical theory of the GLSM, as well as a construction of corresponding enumerative invariants in certain cases, see [FJR17].

Although less familiar to mathematicians than Gromov–Witten theory, one place in which GLSMs arise naturally is through mirror symmetry and wall crossing correspondences with Gromov–Witten theory. The most well-known example of this is the Landau–Ginzburg/Calabi–Yau correspondence of e.g. [CR10], relating Gromov–Witten theory and FJRW theory of the quintic. An analogous result involving more exotic GLSMs, known as hybrid models, appears in [Cla17].

For hybrid model GLSMs, which still includes Gromov–Witten theory and FJRW theory as special cases, factorizations have been employed to define enumerative invariants in [CFG+18]. These notes are based on that paper.

REFERENCES

[AGOT07] Dan Abramovich, Tom Graber, Martin Olsson, and Hsian-Hua Tseng. On the global quotient structure of the space of twisted stable maps to a quotient stack. J. Algebraic Geom., 16(4):731–751, 2007.

[BDF+16] Matthew Ballard, Dragos Deliu, David Favero, M Umut Isik, and Ludmil Katzarkov. Resolutions in factorization categories. Advances in Mathematics, 295:195–249, 2016.

[Beh99] K. Behrend. Algebraic Gromov-Witten invariants. In New trends in algebraic geometry (Warwick, 1996), volume 264 of London Math. Soc. Lecture Note Ser., pages 19–70. Cambridge Univ. Press, Cambridge, 1999.

[BF97] Kai Behrend and Barbara Fantechi. The intrinsic normal cone. Inventiones Mathematicae, 128(1):45–88, 1997.
[BT82] Raoul Bott and Loring W. Tu. Differential forms in algebraic topology, volume 82 of Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1982.

[Căl05] Andrei Căldăru. The Mukai pairing, II: the Hochschild–Kostant–Rosenberg isomorphism. Advances in Mathematics, 194(1):34–66, 2005.

[CFG+18] Ionuţ Ciocan-Fontanine, David Favero, Jérémie Guéré, Bumsig Kim, and Mark Shoemaker. Fundamental factorization of a GLSM, part I: Construction. preprint: arxiv.org/abs/1802.05247, 2018.

[CFK10] Ionuţ Ciocan-Fontanine and Bumsig Kim. Moduli stacks of stable toric quasimaps. Adv. Math., 225(6):3022–3051, 2010.

[CKM14] Ionuţ Ciocan-Fontanine, Bumsig Kim, and Davesh Maulik. Stable quasimaps to GIT quotients. Journal of Geometry and Physics, 75:17–47, 2014.

[CL11] Huai-Liang Chang and Jun Li. Gromov–Witten invariants of stable maps with fields. International mathematics research notices, 2012(18):4163–4217, 2011.

[Cla17] Emily Clader. Landau-Ginzburg/Calabi-Yau correspondence for the complete intersections $X_{3,3}$ and $X_{2,2,2,2}$. Adv. Math., 307:1–52, 2017.

[CR10] Alessandro Chiodo and Yongbin Ruan. Landau-Ginzburg/Calabi-Yau correspondence for quintic three-folds via symplectic transformations. Invent. Math., 182(1):117–165, 2010.

[EP15] Alexander I Efimov and Leonid Positselski. Coherent analogues of matrix factorizations and relative singularity categories. Algebra & Number Theory, 9(5):1159–1292, 2015.

[FJR13] Huijun Fan, Tyler Jarvis, and Yongbin Ruan. The Witten equation, mirror symmetry, and quantum singularity theory. Annals of Mathematics, 178(1):1–106, 2013.

[FJR17] Huijun Fan, Tyler Jarvis, and Yongbin Ruan. A mathematical theory of the gauged linear sigma model. Geometry & Topology, 22(1):235–303, 2017.

[FO99] Kenji Fukaya and Kaoru Ono. Arnold conjecture and Gromov-Witten invariant. Topology, 38(5):933–1048, 1999.

[FP96] William Fulton and Rahul Pandharipande. Notes on stable maps and quantum cohomology. preprint arXiv:9608011, 1996.

[Ful13] William Fulton. Intersection theory, volume 2. Springer Science & Business Media, 2013.

[GD63] A. Grothendieck and J. Dieudonné. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. II. Inst. Hautes Études Sci. Publ. Math., (17):91, 1963.

[GT14] Alexander Givental and Valentin Tonita. The Hirzebruch-Riemann-Roch theorem in true genus-0 quantum K-theory. In Symplectic, Poisson, and noncommutative geometry, volume 62 of Math. Sci. Res. Inst. Publ., pages 43–91. Cambridge Univ. Press, New York, 2014.

[Gué16] Jérémie Guéré. A Landau-Ginzburg mirror theorem without concavity. Duke Math. J., 165(13):2461–2527, 2016.

[Har77] Robin Hartshorne. Algebraic geometry. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Hir17] Yuki Hirano. Derived Knörrer periodicity and Orlov’s theorem for gauged Landau–Ginzburg models. Compositio Mathematica, 153(5):973–1007, 2017.

[HKR09] Gerhard Hochschild, Bertram Kostant, and Alex Rosenberg. Differential forms on regular affine algebras. In Collected Papers, pages 265–290. Springer, 2009.

[Isi12] Mehmet Umut Isik. Equivalence of the derived category of a variety with a singularity category. International Mathematics Research Notices, 2013(12):2787–2808, 2012.

[KM94] Maxim Kontsevich and Yu Manin. Gromov-Witten classes, quantum cohomology, and enumerative geometry. Communications in Mathematical Physics, 164(3):525–562, 1994.
[KO18] Bumsig Kim and Jeongseok Oh. Localized Chern characters for 2-periodic complexes. https://arxiv.org/pdf/1804.03774.pdf, 2018.

[Kon92] Maxim Kontsevich. Intersection theory on the moduli space of curves and the matrix airy function. Communications in Mathematical Physics, 147(1):1–23, Jun 1992.

[Lee04] Y.-P. Lee. Quantum $K$-theory. I. Foundations. Duke Math. J., 121(3):389–424, 2004.

[LP13] Kevin H. Lin and Daniel Pomerleano. Global matrix factorizations. Math. Res. Lett., 20(1):91–106, 2013.

[LT98] Jun Li and Gang Tian. Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties. J. Amer. Math. Soc., 11(1):119–174, 1998.

[Mac74] R. D. MacPherson. Chern classes for singular algebraic varieties. Ann. of Math. (2), 100:423–432, 1974.

[Man14] Cristina Manolache. Stable maps and stable quotients. Compos. Math., 150(9):1457–1481, 2014.

[Oss] Brian Osserman. Notes on cohomology and base change. https://www.math.ucdavis.edu/~osserman/math/cohom-base-change.pdf.

[PV01] Alexander Polishchuk and Arkady Vaintrob. Algebraic construction of Witten’s top Chern class, volume 276 of Contemp. Math. Amer. Math. Soc., Providence, RI, 2001.

[PV11] Alexander Polishchuk and Arkady Vaintrob. Matrix factorizations and singularity categories for stacks [factorisations matricielles et catégories des singularités pour les champs algébriques]. In Annales de l’Institut Fourier, volume 61, pages 2609–2642, 2011.

[PV16] Alexander Polishchuk and Arkady Vaintrob. Matrix factorizations and cohomological field theories. J. Reine Angew. Math., 714:1–122, 2016.

[Rua99] Yongbin Ruan. Virtual neighborhoods and pseudo-holomorphic curves. In Proceedings of 6th Gokova Geometry-Topology Conference, volume 23, pages 161–231, 1999.

[Shi12] Ian Shipman. A geometric approach to Orlov’s theorem. Compositio Mathematica, 148(5):1365–1389, 2012.

[Swa96] Richard G Swan. Hochschild cohomology of quasiprojective schemes. Journal of Pure and Applied Algebra, 110(1):57–80, 1996.

[Wit91] Edward Witten. Two-dimensional gravity and intersection theory on moduli space. Surveys Diff. Geom., 1:243–310, 1991.

[Wit97] Edward Witten. Phases of $N = 2$ theories in two dimensions. In Mirror symmetry, II, volume 1 of AMS/IP Stud. Adv. Math., pages 143–211. Amer. Math. Soc., Providence, RI, 1997.