Two problems of movement of multi-link wheeled vehicles

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Abstract. The paper presents the results of a study of the dynamics of multi-link wheeled vehicles with nonholonomic connections superimposed on this mechanical system. The article is of an overview nature. The mathematical formulation is a system of ordinary differential equations of motion in the form of Lagrange with undefined multipliers, solved together with derivatives of nonholonomic constraints. The results are presented in the case of controlled motion, when the law of motion of the first link is known, as well as in the case of uncontrolled motion, when the law of motion of the leading link is also the desired function of time. General equations are obtained for a mechanical system consisting of an arbitrary number of links. Numerical results are presented for the case of three coupled trolleys. The software package Maple was used to perform numerical calculations and plotting of the desired mechanical parameters.

1. Introduction
It is difficult to imagine the modern world without the use of various vehicles. It includes the operation of cars, the use of industrial transport for the movement of goods, and the use of mobile robotic devices in various spheres of life. The operation of multi-link wheeled vehicles pursues several goals:

- optimize transportation costs;
- to make the transportation process itself more maneuverable and safe, even in the absence of a person.

This paper presents the results of a study of a simplified mathematical model of the movement of a multi-link wheeled carriage, which can be the basis for creating more complex devices. The system consists from \( n \) links, each of which has one wheel pair, and the center of mass of each link coincides with the geometric center of the wheel pair and is located on the axis of symmetry of the link. The trolleys are interconnected in their geometric centers by a metal frame of fixed length. Each subsequent link is rigidly connected to the frame, which can freely rotate around the attachment point with the previous link.

To date, a number of results have been obtained on the movement of vehicles. The first works were made by Rocard [1], Stückler [2,3]. A little later, Martynyuk and Lobas were interested in this issue [4-5]. Recently, scientists of the Keldysh Institute of Applied Mathematics [6-7], scientists of the Izhevsk Institute of Computer Research [8-11] have been studying the dynamics of vehicles of various modifications (Chaplygin sleigh, snakeboard, roller-racer, robotrain). Their research concerned models consisting of one, two or three links.
The author of this work has carried out a number of studies on the movement of multi-link wheeled vehicles in the works [12, 13]. In this paper, the studies already done are supplemented, systematized and illustrated by the example of a crew consisting of three trolleys. Cases are considered when the leading link is under control, that is, the law of its movement is known, and without control, when the law of movement of the leading link is unknown. The difference in mathematical statements of problems will be shown depending on the nature of the movement of the first trolley: controlled or uncontrolled.

2. Mathematical formulation of the problem

A system consisting of \( n \) trolleys moves along a plane. One wheel pair is fixed on the trolleys so that the center of mass of the link coincides with the geometric center of the wheel pair and lies on the axis of symmetry of the trolley. Each trolley is rigidly fixed with a metal frame of length \( l \). The frame attaches the trolley to the previous trolley in its geometric center and rotates freely around the attachment point.

![Construction of a multi-link wheeled vehicle.](image)

We introduce two coordinate systems: fixed coordinate system \( Oxy \) and moving coordinate system \( O_1x_1y_1 \), connected to the first trolley and with the center \( O_1 \) in the center of mass of the leading (first) trolley.

Let’s introduce the notation:

- \( \mathbf{v}_i, i = 1, n \) is the linear velocity of the wheelset of the \( i \)-th trolley,
- \( \mathbf{v}_1 = (u, v) \) is the linear velocity of the wheelset of the first trolley,
- \( \omega_i, i = 1, n \) is the angular velocity of the \( i \)-th trolley,
- \( M \) is a weight of each trolley,
- the angle \( \varphi \) is the angle between the first trolley axis (the positive direction of the \( O_1x_1 \) axis of the moving coordinate system) and the positive direction of the \( Ox \) axis of the fixed coordinate system.
- \( \theta_i, i = 2, n \) is the angle between the \( i \)-th link axis and the positive direction of the \( O_1x_1 \) axis of the moving coordinate system.

All further calculations will be performed in a movable coordinate system. Since the velocity of each cart will be found by the formula

\[
\mathbf{v}_i = \mathbf{v}_1 + \sum_{k=2}^{l} \omega_k \xi_k
\]

and the angular velocity of each cart will be found the formulas
where \( \zeta_k = (\sin \theta_k, -\cos \theta_k) \), the kinetic energy can be written as

\[
T = \frac{1}{2} \sum_{i=1}^{n} I_i \omega_i^2 + \frac{1}{2} M \sum_{i=1}^{n} v_i^2,
\]

where \( I_i \) is the constant moment of inertia of the \( i \)-th trolley.

The kinetic energy will take the form

\[
T = \frac{1}{2} I_i \omega_i^2 + \frac{1}{2} \sum_{i=2}^{n} J_i \left( \omega_i + \dot{\theta}_i \right)^2 + \frac{Mn}{2} \left( u^2 + v^2 \right) + M \sum_{i=2}^{n} (n+1-i) \left( \omega_i + \dot{\theta}_i \right) \left( u \sin \theta_i - v \cos \theta_i \right) + M^2 \sum_{m=2}^{n-1} \sum_{i=m+1}^{n} (n+1-i) \left( \omega_i + \dot{\theta}_m \right) \left( \omega_i + \dot{\theta}_i \right) \cos \left( \theta_i - \theta_m \right),
\]

where \( J_i = I_i + M^2(n+1-i) \).

Nonholonomic constraints are imposed on the wheels at each of the links; these constraints mean that the velocity vector of the center of mass of the trolley is always co-directional with the axis of the trolley [12, 13]:

\[
f_i = v = 0,
\]

\[
f_i = u \sin \theta_i + l \sum_{k=2}^{i-1} \left( \omega_k + \dot{\theta}_k \right) \cos \left( \theta_i - \theta_k \right) + I \left( \omega_i + \dot{\theta}_i \right) = 0.
\]

We believe that \( \sum_{k=2}^{i-1} g_k = 0 \).

Equations of motion with indefinite multipliers will take the form

\[
d \frac{\partial T}{\partial u} - \frac{\partial T}{\partial v} = \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial u},
\]

\[
d \frac{\partial T}{\partial v} + \frac{\partial T}{\partial u} = \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial v},
\]

\[
d \frac{\partial T}{\partial \omega_i} + u \frac{\partial T}{\partial \omega_i} - v \frac{\partial T}{\partial \omega_i} = \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial \omega_i},
\]

\[
d \frac{\partial T}{\partial \dot{\theta}_i} - \frac{\partial T}{\partial \theta_i} = \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial \dot{\theta}_i}.
\]

To define the constants \( \lambda_i \) one needs jointly solve the system of equations (3) and the time derivatives of the constraints (2).

The equations of dynamics (3) define the phase flow on the \((2n + 1)\)-dimensional space \( M^{2n+1} = \left\{ (u,v,\omega_i,\theta_i,\dot{\theta}_i) \right\} \) and the equations of constraints (2) are integrals of the flow.

Next, we can consider two cases: the first link is under control, that is, the functions \( u \) and \( \omega_1 \) are known (\( v = 0 \)) or the coordinates of point \( O_1 (x(t), y(t)) \) in a fixed coordinate system \( Oxy \) is known; the
first link is unmanaged, that is, the functions $u$ and $\omega_i$ are unknown or the coordinates of point $O_i \left( x(t), y(t) \right)$ in a fixed coordinate system $Oxy$ is unknown.

3. A controlled wheeled vehicle

Let the first cart be controllable. If the functions $u, \varphi, \omega_i$ are known, then to find the angles $\theta_i$ it is necessary to solve equations (2) alternately. If the coordinates of point $O_i \left( x(t), y(t) \right)$ in a fixed coordinate system $Oxy$ is known, then from the following relations

$$\dot{x}(t) = u \cos \varphi, \quad \dot{y}(t) = u \sin \varphi,$$

the functions $u, \varphi, \omega_i$ can be determined:

$$u = \dot{x}(t) \cdot \cos \varphi + \dot{y}(t) \cdot \sin \varphi,$$
$$\varphi = ATAN2(\dot{y}(t), \dot{x}(t)),$$
$$\omega_i = \frac{\dot{y}(t) \dot{x}(t) - \dot{x}(t) \dot{y}(t)}{\dot{x}(t)^2 + \dot{y}(t)^2}.$$

The functions $\theta_i$ can be found from the sequential solution of equations (2). Thus, in the case of a controlled wheeled carriage, there is no need to solve the system of equations of motion.

**Example.** Let the linear velocity and angular velocity of the center of mass of the first link of a three-link wheeled vehicle be given:

$$u = u_0, \quad \omega_i = \Omega, \quad \varphi = \Omega t.$$

Then the angles $\theta_2, \theta_3$ are determined from the differential equations:

$$\dot{\theta}_2 = -\frac{u_0 \sin \theta_2}{l} - \Omega,$$
$$\dot{\theta}_3 = -\frac{u_0 \cos \theta_2 \sin \left( \theta_3 - \theta_2 \right)}{l} - \Omega$$

Equations (4) can be solved both numerically and analytically. Figure 2 and figure 3 shows graphs of the desired functions $\theta_2, \theta_3$ for different various input parameters.

The nature of the behavior of the system (4) can be classified as follows.

- In the case $\left| \frac{\Omega u}{u_0} \right| < 1$ the system has stable equilibrium states and for $t \to +\infty$ the both trolleys tend to a certain limit position.
- In the case $\left| \frac{\Omega u}{u_0} \right| > 1$ the system has no stable equilibrium states and the functions $\theta_2, \theta_3$ are periodic or quasi-periodic in nature.
- In the case $\left| \frac{\Omega u}{u_0} \right| = 1$ the system has no stable equilibrium states and the second trolley will tend to its limit position, and the third trolley will make periodic or quasi-periodic oscillations.
A more complete study of the phase space \( (\theta_2, \theta_3) \) showed the absence of chaotic oscillations in the system.

4. An uncontrolled wheeled vehicle

We will assume that the law of motion of the first cart is unknown. In this case, after the joint solution of equations (3) and derivatives of the time coupling equations (2) and reduction to the level of the first integrals, we obtain the reduced system for three trolleys.

\[
\dot{u} = \frac{1}{R(\theta_2, \theta_3)} \left[ \sin \theta_2 \cos \theta_2 \left( (J_2 - 4) - (J_3 - 2) \sin^2 \alpha \right) u \omega_1 + \left( (J_2 - 4) \sin^2 \theta_2 \cos \theta_2 + (J_3 - 2) \cos \theta_2 \sin \alpha \left( -\sin \theta_2 \cos \theta_2 \cos \alpha - \sin^2 \theta_2 \sin \alpha + \cos^2 \theta_2 \cos \alpha \sin \alpha \right) \right) u^2 \right],
\]

\[
\dot{\omega}_1 = 0,
\]
\[
\dot{\theta}_2 = -u \sin \theta_2 - \omega_1,
\]
\[
\dot{\theta}_3 = -u \cos \theta_2 \sin \alpha - \omega_1,
\]

where \( M = l = 1, \alpha = \theta_3 - \theta_2, R(\theta_2, \theta_3) = (J_2 - 4) \sin^2 \theta_2 + (J_3 - 2) \cos^2 \theta_2 \sin^2 \alpha + 3. \)
In this case, the movement of the system occurs with a constant angular velocity $\omega_0$. The system (5) has the first integral

$$I_1\omega_0^2 + R(\theta_2, \theta_3)u^2 = F.$$  \hfill (6)

From where

$$u = \pm \sqrt{\frac{F - I_1\omega_0^2}{R(\theta_2, \theta_3)}}.$$

Because $J_2 > 2$, $J_3 > 1$, the function $R(\theta_2, \theta_3) > 0$. The system (5) has a solution for $F \geq I_1\omega_0^2$.

We will further assume that $u \geq 0$. Reducing the system to the level of the first integral (6) and considering that the angular velocity is constant, we obtain the reduced system of equations

$$\dot{\theta}_2 = -\sqrt{\frac{F - I_1\omega_0^2}{R(\theta_2, \theta_3)}} \sin \theta_2 - \omega_0,$$

$$\dot{\theta}_3 = -\sqrt{\frac{F - I_1\omega_0^2}{R(\theta_2, \theta_3)}} \cos \theta_2 \sin \alpha - \omega_0.$$ \hfill (7)

To determine the trajectory of the center of mass of the first trolley, it is necessary to additionally solve differential equations.

$$\dot{x} = u \cos(\omega_0 t), \quad \dot{y} = u \sin(\omega_0 t).$$

In this case, the dynamics of the system (6) is somewhat similar to the dynamics discussed in the previous section. Based on the results of numerical experiments and analytical analysis, the following classification of the possible behavior of the mechanical system under consideration can be proposed.

Depending on the moments of inertia and initial conditions:

- the system can have a stable state of equilibrium, in which the center of mass of the leading link will eventually move into a uniform movement around the circle, and the second and third trolleys will occupy the limiting position relative to the first link (figure 4);
- the system does not have stable equilibrium positions, the solutions of system (6) are quasi-periodic, the center of mass of the leading link will move along a curved trajectory, the second and third trolleys will be constantly in motion relative to the first link (figure 5);
- the first trolley will rotate in place at a constant angular velocity for $F = I_1\omega_0^2$. 

Figure 4. Graph of function $u$ and trajectory of the center of mass of the first cart for $I_1 = 4, J_2 = 5, J_3 = 6, F = 600, \omega_0 = 5, \theta_2(0) = \frac{\pi}{4}, \theta_3(0) = \frac{\pi}{2}$.

Figure 5. Graph of function $u$ and trajectory of the center of mass of the first cart for $I_1 = 4, J_2 = 5, J_3 = 6, F = 220, \omega_0 = 5, \theta_2(0) = \frac{\pi}{4}, \theta_3(0) = \frac{\pi}{2}$.

In the case of an uncontrolled wheeled multi-link crew, there are also no chaotic fluctuations in the system.

5. Conclusion
This article is of an overview nature and contains some results of a study of the dynamics of multi-link wheeled vehicles. Cases were considered when the first trolley is under control and when the first trolley is unmanageable, that is, the trajectory of its movement is also unknown. Using the example of three trolleys, the nature of the movement of this mechanical system was demonstrated. There were no chaotic fluctuations in the system in any case. The desired mechanical parameters had either a quasi-periodic (periodic) character, or asymptotic behavior.

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