A sharp Trudinger-Moser type inequality involving $L^n$ norm in the entire space $\mathbb{R}^n$

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Abstract

Let $W^{1,n}(\mathbb{R}^n)$ be the standard Sobolev space and $\|\cdot\|_n$ be the $L^n$ norm on $\mathbb{R}^n$. We establish a sharp form of the following Trudinger-Moser inequality involving the $L^n$ norm

$$
\sup_{\|u\|_{W^{1,n}(\mathbb{R}^n)}=1} \int_{\mathbb{R}^n} \Phi \left( \alpha_n \|u\|_n^{\frac{n}{n-1}} \left(1 + \alpha \|u\|_n^{\frac{n}{n-1}}\right) \right) dx < +\infty
$$

in the entire space $\mathbb{R}^n$ for any $0 \leq \alpha < 1$, where $\Phi(t) = e^t - \sum_{j=0}^{n-2} \frac{t^j}{j!}$, $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$ and $\omega_{n-1}$ is the $n-1$ dimensional surface measure of the unit ball in $\mathbb{R}^n$. We also show that the above supremum is infinity for all $\alpha \geq 1$. Moreover, we prove the supremum is attained, namely, there exists a maximizer for the above supremum when $\alpha > 0$.

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is sufficiently small. The proof is based on the method of blow-up analysis of the nonlinear Euler-Lagrange equations of the Trudinger-Moser functionals.

Our result sharpens the recent work [12] in which they show that the above inequality holds in a weaker form when \( \Phi(t) \) is replaced by a strictly smaller \( \Phi^*(t) = e^t - \sum_{j=0}^{n-1} \frac{t^j}{j!} \).

(Note that \( \Phi(t) = \Phi^*(t) + \sum_{j=0}^{n-1} \frac{t^j}{j!} \)).

1 Introduction

Let \( \Omega \subseteq \mathbb{R}^n \) be an open set and \( W^{1,q}_0(\Omega) \) be the usual Sobolev space, that is, the completion of \( C_0^\infty(\Omega) \) under the norm

\[
\|u\|_{W^{1,q}(\Omega)} = \left( \int_\Omega (|u|^q + |\nabla u|^q) \, dx \right)^{\frac{1}{q}}.
\]

If \( 1 \leq q < n \), the classical Sobolev embedding says that \( W^{1,q}_0(\Omega) \hookrightarrow L^s(\Omega) \) for \( 1 \leq s \leq q^* \), where \( q^* := \frac{nq}{n-q} \). When \( q = n \), it is known that

\[
W^{1,n}_0(\Omega) \hookrightarrow L^s(\Omega) \quad \text{for any} \quad 1 \leq s < +\infty,
\]

but \( W^{1,n}_0(\Omega) \not\subset L^\infty(\Omega) \). The analogue of the optimal Sobolev embedding is the well-known Trudinger-Moser inequality ([23], [28]) which states as follows

\[
(1.1) \quad \sup_{u \in W^{1,n}_0(\Omega)} \int_\Omega e^{\alpha |u|^n} \frac{|u|}{\|\nabla u\|_{L^p(\Omega)}} \, dx < \infty \quad \text{iff} \quad \alpha \leq \alpha_n = n \omega_{n-1}^{-\frac{1}{n-1}},
\]

where \( \omega_{n-1} \) is the \( n-1 \) dimensional surface measure of the unit ball in \( \mathbb{R}^n \) and \( \Omega \) is a domain of finite measure in \( \mathbb{R}^n \).

Due to a wide range of applications in geometric analysis and partial differential equations (see [7], [8], [14] and references therein), numerous generalizations, extensions and applications of the Trudinger-Moser inequality have been given. We recall in particular the result obtained by P.-L. Lions [19], which says that if \( \{u_k\} \) is a sequence of functions in \( W^{1,n}_0(\Omega) \) with \( \|\nabla u_k\|_{L^n(\Omega)} = 1 \) such that \( u_k \rightharpoonup u \) weakly in \( W^{1,n}_0(\Omega) \), then for any \( 0 < p < \left( 1 - \|\nabla u\|_{L^n(\Omega)} \right)^{-1/(n-1)} \), one has

\[
\sup_k \int_\Omega e^{\alpha_n p |u_k|^{n-1}} \, dx < \infty.
\]

This conclusion gives more precise information than \((1.1)\) when \( u_k \rightharpoonup u \neq 0 \) weakly in \( W^{1,n}_0(\Omega) \). Based on the result of Lions and the blowing up analysis method, Adimurthi
and O. Druet [4] obtained an improved Trudinger-Moser type inequality in \( \mathbb{R}^2 \) on bounded domains \( \Omega \), which can be described as follows

\[
\sup_{u \in W^{1,2}(\Omega), \|\nabla u\|_2 \leq 1} \int_{\mathbb{R}^2} e^{4\pi|u|^2(1+\alpha\|u\|^2)} \, dx < \infty, \text{ iff } \alpha < \inf_{u \in W^{1,2}(\Omega), u \neq 0} \frac{\|\nabla u\|^2}{\|u\|^2}.
\]

Subsequently, this result was extended to \( L^p \) norm in two dimension and high dimension as well in Yang [29], Lu and Yang [20, 21] and Zhu [30].

Another interesting extension of (1.1) is to construct Trudinger-Moser inequalities for unbounded domains. In fact, we note that, even in the case \( \alpha < \alpha_n \), the supremum in (1.1) becomes infinite for domains \( \Omega \subseteq \mathbb{R}^n \) with \( |\Omega| = +\infty \). Related inequalities for unbounded domains have been first considered by D.M. Cao [5] in the case \( N = 2 \) and for any dimension by J.M. do ´O [10] and Adachi-Tanaka [11] in the subcritical case, that is \( \alpha < \alpha_n \). In [24], B. Ruf showed that in the case \( N = 2 \), the exponent \( \alpha_2 = 4\pi \) becomes admissible if the Dirichlet norm \( \int_\Omega |\nabla u|^2 \, dx \) is replaced by Sobolev norm \( \int_\Omega (|u|^2 + |\nabla u|^2) \, dx \), more precisely, he proved that

\[
(1.2) \sup_{u \in W^{1,2}(\mathbb{R}^2)} \int_{\mathbb{R}^2} \Phi(\alpha |u|^2) \, dx < +\infty, \text{ iff } \alpha \leq 4\pi,
\]

where \( \Phi(t) = e^t - 1 \). Later, Y. X. Li and B. Ruf [17] extended Ruf’s result to arbitrary dimension.

Recently, M. de Souza and J. M. do ´O [9] obtained an Adimurthi-Druet type result in \( \mathbb{R}^2 \) for some weighted Sobolev space

\[
E = \left\{ u \in W^{1,2}(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x) u^2 \, dx < \infty \right\},
\]

where the potential \( V \) is radially symmetric, increasing and coercive.

In this paper, we will try to remove the potential \( V \) in [9], and we obtain an Adimurthi-Druet type result for \( W^{1,n}(\mathbb{R}^n) \). Our main results read as follows

**Theorem 1.1.** For any \( 0 \leq \alpha < 1 \), the following holds:

\[
(1.3) \sup_{\|u\|_{W^{1,n}(\mathbb{R}^n)} = 1} \int_{\mathbb{R}^n} \Phi\left(\alpha_n |u|^n (1 + \alpha \|u\|^n) \frac{1}{n-1} \right) \, dx < \infty,
\]

where \( \Phi(t) = e^t - \sum_{j=0}^{n-2} \frac{t^j}{j!} \). Moreover, for any \( \alpha \geq 1 \), the supremum is infinite.
At this point, we call attention to the recent work of M. de Souza and J. M. do Ó in [12], where the authors establish an analogue of (1.3) under the additional assumption that $\Phi(t)$ is substituted by a smaller function $\Psi(t) = e^t - \sum_{j=0}^{n-1} \frac{\mu_j}{j!}$. But, they did not address whether the supremum is finite when $\alpha = 1$. Here, we remark that by using the test function sequence constructed in Section 2, we can show that the supreme in (1.3) is infinity when $\alpha = 1$. Therefore, our results indeed improve substantially the result in [12].

We set

$$S = \sup_{\|u\|_{W^{1,n}(\mathbb{R}^n)} = 1} \int_{\mathbb{R}^n} \Phi \left( \alpha_n \|u\|_n \left( 1 + \alpha \|u\|_n^{n-1} \right) \right) dx.$$ 

The existence of an extremal function for the above supremum is only known when $\alpha = 0$ as shown in [17]. However, whether an extremal function for the above supremum exists or not is not known for $\alpha > 0$. Our next aim is to show that the supremum above is attained when $\alpha$ is chosen small enough, that is

**Theorem 1.2.** There exists $u_\alpha \in W^{1,n}(\mathbb{R}^n)$ with $\|u_\alpha\|_{W^{1,n}(\mathbb{R}^n)} = 1$ such that

$$S = \int_{\mathbb{R}^n} \Phi \left( \alpha_n \|u_\alpha\|_n \left( 1 + \alpha \|u_\alpha\|_n^{n-1} \right) \right) dx$$

for sufficiently small $\alpha$.

The first result about existence of the extremal function for Trudinger-Moser inequality was given by L. Carleson and S.Y.A. Chang in [6], where it is proved that the supremum in (1.1) indeed has extremals by using symmetrization argument, when $\Omega$ is a ball in $\mathbb{R}^n$. This actually brings a surprise, since it is well-known that the Sobolev-inequality has no extremals on any finite domain $\Omega \neq \mathbb{R}^n$. Later, M. Flucher [13] showed that this result continues to hold for any smooth domain in $\mathbb{R}^2$ and Lin in [18] generalized the result to any dimension. More existence results can be found in several papers, see e.g. Y.X. Li [15] and [16] for Trudinger-Moser inequalities on compact Riemannian manifold, [24] and [17] for on unbounded domains in $\mathbb{R}^n$, and Lu and Yang [20, 21] and Zhu [30] for Trudinger-Moser inequalities involving a remainder term. For the existence of critical points for the supercritical regime, i.e. the Trudinger-Moser energy functionals constrained to manifold $M = \left\{ u \in W^{1,n}_0(\Omega), \|\nabla u\|_{L^n(\Omega)} > 1 \right\}$, see del Pino, Musso and Ruf [8] and Malchiodi and Martinazzi [22], and references therein.

We now sketch the idea of proving Theorem 1.1 and Theorem 1.2.

1. The proof of the second part of Theorem 1.1 is based on a test function argument. Unlike in the case for bounded domains [29], we cannot construct the test function by the eigenfunction of the first eigenvalue problem:

$$\inf_{u \in W^{1,n}_0(\Omega), u \neq 0} \frac{\|\nabla u\|^n_n}{\|u\|^n_n}.$$
since the above infimum is actually not attained when $\Omega = \mathbb{R}^n$. To overcome this difficulty, we will construct a new test function sequence (see Section 2 for more details).

2. For the proof of the first part of Theorem 1.1, we will carry out the standard blowing up analysis procedure. This method is based on a blowing up analysis of sequences of solutions to $n$-Laplacian in $\mathbb{R}^n$ with exponential growth, and it has been successfully applied in the proof of the Trudinger-Moser inequalities and related existence results in bounded domains (see [3], [14], [20] and [21]). In the unbounded case, one will encounter many new difficulties. For instance, when the blowing up phenomenon arises, a crucial step is to show the strong convergence of $u_k$ in $L^n$ norm ($u_k$ are the maximizers for a sequence of subcritical Trudinger-Moser energy functionals). We recall that in [12], the authors proved the strong convergence under the additional assumption that $\Phi(t)$ is substituted by a smaller function $\Psi(t) = e^t - \sum_{j=0}^{n-1} \frac{t^j}{j!}$. In our case, we remove that unnatural assumption, and in order to prove the strong convergence, we will need more careful analysis and different technique (see §4.1 for more details).

3. To prove Theorem 1.2, we will adapt ideas in the spirit of proofs given in, e.g., Y. X. Li [15, 16] and Y. X. Li and B. Ruf in [17]. We first derive the upper bound for the Trudinger-Moser inequality from a result of L. Carleson and S.Y.A. Chang [6] when the blowing up arises, and then construct a function sequence to show that the upper bound can actually be surpassed.

This paper is organized as follows. In Section 2 we prove the sharpness of the inequality in Theorem 1.1 by constructing a appropriate test function sequence; Section 3 is devoted to proving the existence of radially symmetric maximizing sequence for the critical functional; in Section 4, we apply the blowing up analysis to analyze the asymptotic behavior of the maximizing sequence near and far away from the origin, and give the proof for the first part of Theorem 1.1; in Section 5, we prove the existence result—Theorem 1.2 by constructing a test function sequence.

Throughout this paper, the letter $c$ denotes a constant which may vary from line to line.

2 The test functions argument

In this section, we prove the sharpness of the inequality in Theorem 1.1. Namely, we will show that if $\alpha \geq 1$, then the supremum is infinity.

Proof of the Second Part of Theorem 1.1 Setting

$$u_k = \frac{1}{\omega_{n-1}^\frac{1}{n}} \begin{cases} \left(\log k\right)^{-\frac{1}{n}} \log \frac{R_k}{|x|} & \frac{R_k}{k} < |x| \leq R_k \\ \left(\log k\right)^{\frac{n-1}{n}} & 0 < |x| \leq \frac{R_k}{k} \end{cases}$$
where $R_k := \frac{(\log k)^{1/n}}{\log \log k} \to \infty$, as $k \to \infty$. We can easily verify that

$$\int_{\mathbb{R}^n} |\nabla u_k|^n \, dx = 1$$

and

$$\|u_k\|_{W^{1,n}(\mathbb{R}^n)}^n = \left(\int_{B_{R_k/k}} + \int_{B_{R_k/k} \setminus B_{R_k/k}}\right) |u_k|^n \, dx$$

$$= \left(\frac{\log k}n \right)^{n-1} \left(\frac{R_k}k\right)^n + \frac{R_k^n}{\log k} \int_k^1 \left(\log r\right)^n r^{n-1} \, dr$$

$$= C_n \frac{R_k^n}{\log k} (1 + o(1)) \to 0 \text{ as } k \to \infty,$$

where $C_n = \int_1^2 \left(\log r\right)^n r^{n-1} \, dr$. Therefore we have

$$\|u_k\|_{W^{1,n}(\mathbb{R}^n)}^n = 1 + \frac{C_n R_k^n}{\log k} (1 + o(1))$$

Since

$$1 + \frac{\|u_k\|_{W^{1,n}}^n}{\|u_k\|_{W^{1,n}}^n} = \frac{1 + 2 \|u_k\|_{W^{1,n}}^n}{1 + \|u_k\|_{W^{1,n}}^n},$$

then on the ball $B_{R_k/k}$, we have

$$\alpha_n \frac{|u_k|_{W^{1,n}(\mathbb{R}^n)}^{n-1}}{\|u_k\|_{W^{1,n}(\mathbb{R}^n)}^{n-1}} \left(1 + \frac{\|u_k\|_{W^{1,n}(\mathbb{R}^n)}^n}{\|u_k\|_{W^{1,n}(\mathbb{R}^n)}^n}\right) \frac{1}{n-1}$$

$$= n \omega_{n-1} \frac{\|u_k\|_{W^{1,n}(\mathbb{R}^n)}^{n-1}}{n-1} \left(1 + \frac{\|u_k\|_{W^{1,n}(\mathbb{R}^n)}^n}{\|u_k\|_{W^{1,n}(\mathbb{R}^n)}^n}\right) \frac{1}{n-1}$$

$$= n \log k \left(1 - \frac{1}{n-1} \|u_k\|_{W^{1,n}(\mathbb{R}^n)}^{2n} + \frac{2}{n-1} \|u_k\|_{W^{1,n}(\mathbb{R}^n)}^{2n} (1 + o(1))\right)$$

and $|B_{R_k/k}| = \frac{\omega_{n-1}}{n} \exp (n \log R_k - n \log k)$.

Thus

$$\sup_{\|u\|_{W^{1,n}(\mathbb{R}^n)}} \int_{\mathbb{R}^n} \Phi \left(\alpha_n \frac{|u|_{W^{1,n}(\mathbb{R}^n)}^{n-1}}{\|u\|_{W^{1,n}(\mathbb{R}^n)}^{n-1}} \left(1 + \frac{\|u\|_{W^{1,n}(\mathbb{R}^n)}^n}{\|u\|_{W^{1,n}(\mathbb{R}^n)}^n}\right)\right) \, dx$$

$$\ge c \int_{B_{R_k/k}} \exp \left(\alpha_n \frac{|u_k|_{W^{1,n}(\mathbb{R}^n)}^{n-1}}{\|u_k\|_{W^{1,n}(\mathbb{R}^n)}^{n-1}} \left(1 + \frac{\|u_k\|_{W^{1,n}(\mathbb{R}^n)}^n}{\|u_k\|_{W^{1,n}(\mathbb{R}^n)}^n}\right)\right) \, dx$$
\[ \geq c \exp \left( n \log k \left( 1 - \frac{1}{n-1} \|u_k\|_{2n}^2 + \frac{2}{n-1} \|u_k\|_{3n}^3 (1 + o(1)) \right) + n \log R_k - n \log k \right) \]
\[ = c \exp \left( n \log k \left( -\frac{1}{n-1} \|u_k\|_{2n}^2 + \frac{2}{n-1} \|u_k\|_{3n}^3 (1 + o(1)) \right) + n \log R_k \right) \]

Because
\[ n \log R_k = n \log \left( \frac{(\log k)^{1/2n}}{\log \log k} \right) = \frac{1}{2} \log \log k - n \log \log \log k \]

and
\[ n \log k \left( -\frac{1}{n-1} \|u_k\|_{2n}^2 + \frac{2}{n-1} \|u_k\|_{3n}^3 (1 + o(1)) \right) \]
\[ = -\frac{n}{n-1} C_n^2 \frac{R_k^{2n}}{\log k} (1 + o(1)) \]
\[ = -\frac{n}{n-1} C_n^2 \frac{1}{(\log \log k)^{2n}} (1 + o(1)) , \]

we can get
\[ \int_{\mathbb{R}^n} \Phi \left( \alpha_n |u_k|^\frac{n}{n-1} (1 + \|u_k\|_{n}^n) \right) \, dx \]
\[ \geq c \exp \left( n \log k \left( -\|u_k\|_{2n}^2 (1 + o(1)) \right) + n \log R_k \right) \]
\[ = c \exp \left( \frac{1}{2} \log \log k - n \log \log \log k - \frac{nC_n^2}{n-1} \frac{1}{(\log \log k)^{2n}} (1 + o(1)) \right) \]
\[ \to \infty . \]

The proof is finished. \[ \square \]

3 The maximizing sequence for critical functional

We first present a technical lemma contributed by João Marco do Ó, et al [11].

Lemma 3.1. Let \( \{u_k\} \) be a sequence in \( W^{1,n}(\mathbb{R}^n) \) such that \( \|u_k\|_{W^{1,n}} = 1 \) and \( u_k \to u \neq 0 \), weakly in \( W^{1,n}(\mathbb{R}^n) \). If
\[ 0 < q < q_n(u) := \frac{1}{(1 - \|u\|_{W^{1,n}}^{n/(n-1)})^{1/(n-1)}} , \]
then
\[ \sup_k \int_{\mathbb{R}^n} \Phi \left( \alpha_n q |u_k|^{\frac{n}{n-1}} \right) \, dx < \infty . \]
Let \( \{R_k\} \) be an increasing sequence which diverges to infinity, and \( \{\beta_k\} \) an increasing sequence which converges to \( \alpha_n \). Setting
\[
I_{\alpha \beta_k}^\alpha(u) = \int_{B_{R_k}} \Phi(\beta_k |u|^\frac{n}{n-1} (1 + \alpha \|u\|_n) \frac{1}{n-1}) \, dx
\]
and
\[
H = \{ u \in W^{1,n}_0(B_{R_k}) \mid \|u\|_{W^{1,n}} = 1 \}.
\]
We have

**Lemma 3.2.** For any \( 0 \leq \alpha \leq 1 \), there exists an extremal function \( u_k \in H \) such that
\[
I_{\alpha \beta_k}^\alpha(u_k) = \sup_{u \in H} I_{\alpha \beta_k}^\alpha(u).
\]

**Proof.** There exists a sequence of \( \{v_i\} \in H \) such that
\[
\lim_{i \to \infty} I_{\alpha \beta_k}^\alpha(v_i) = \sup_{u \in H} I_{\alpha \beta_k}^\alpha(u).
\]
Since \( v_i \) is bounded in \( W^{1,n}(\mathbb{R}^n) \), there exists a subsequence which will still be denoted by \( v_i \), such that
\[
v_i \to u_k \text{ weakly in } W^{1,n}(\mathbb{R}^n),
\]
\[
v_i \to u_k \text{ strongly in } L^s(B_{R_k}),
\]
for any \( 1 < s < \infty \) as \( i \to \infty \). Hence \( v_i \to u_k \) a.e. in \( \mathbb{R}^n \), and
\[
g_i = \Phi \left( \beta_k |v_i|^\frac{n}{n-1} (1 + \alpha \|v_i\|_n) \frac{1}{n-1} \right)
\]
\[
\to g_k = \Phi \left( \beta_k |u_k|^\frac{n}{n-1} (1 + \alpha \|u_k\|_n) \frac{1}{n-1} \right)
\]
a.e. in \( \mathbb{R}^n \). We claim that \( u_k \neq 0 \). If not, \( 1 + \alpha \|v_i\|_n \to 1 \), and then \( g_i \) is bounded in \( L^r(B_{R_k}) \) for some \( r > 1 \), thus \( g_i \to 0 \). Therefore, \( \sup_{u \in H} I_{\alpha \beta_k}^\alpha(u) = 0 \), which is impossible. By Lemma 3.1 we have for any \( q < q_n(u_k) := \frac{1}{(1 - \|u_k\|_{W^{1,n}})^{1/(n-1)}} \),
\[
\limsup_{i \to \infty} \int_{\mathbb{R}^n} \Phi(\alpha_n g_i |v_i|^\frac{n}{n-1}) \, dx < \infty.
\]
Since \( \alpha \leq 1 \), we have
\[
1 + \alpha \|u_k\|_n < 1 + \|u_k\|_{W^{1,n}} \frac{n}{1 - \|u_k\|_{W^{1,n}}} = q_n(u_k),
\]
then \( g_i \) is bounded in \( L^s \) for some \( s > 1 \), and \( g_i \to g_k \) strongly in \( L^1(B_{R_k}) \), as \( i \to \infty \). Therefore, the extremal function is attained for the case \( \beta_k < \alpha_n \) and \( \|u_k\|_{W^{1,n}} = 1 \). \qed
Similar as in [17], we give the following

**Lemma 3.3.** Let $u_k$ be as above. Then

(i) $u_k$ is a maximizing sequence for $S$;

(ii) $u_k$ may be chosen to be radially symmetric and decreasing.

**Proof.** (i) Let $\eta$ be a cut-off function which is 1 on $B_1$ and 0 on $\mathbb{R}^n \setminus B_2$. Then given any $\varphi \in W^{1,n}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} (|\varphi|^n + |\nabla \varphi|^n) \, dx = 1$, we have

$$
\tau^n(L) := \int_{\mathbb{R}^n} \left( |\nabla \eta \left( \frac{x}{L} \right) \varphi|^n + |\eta \left( \frac{x}{L} \right) \varphi|^n \right) \, dx \to 1, \text{ as } L \to +\infty.
$$

Hence for a fixed $L$ and $R_k > 2L$,

$$
\int_{B_L} \Phi \left( \beta_k \left( \left| \frac{\varphi}{\tau(L)} \right|^n \right)^{\frac{n}{n-1}} \left( 1 + \alpha \left| \frac{\eta \left( \frac{x}{L} \right) \varphi}{\tau(L)} \right|^n \right)^{\frac{1}{n-1}} \right) \, dx
\leq \int_{B_{2L}} \Phi \left( \beta_k \left( \left| \frac{\eta \left( \frac{x}{L} \right) \varphi}{\tau(L)} \right|^n \right)^{\frac{n}{n-1}} \left( 1 + \alpha \left| \frac{\eta \left( \frac{x}{L} \right) \varphi}{\tau(L)} \right|^n \right)^{\frac{1}{n-1}} \right) \, dx
\leq \int_{B_{R_k}} \Phi \left( \beta_k \left| u_k \right|^n \left( 1 + \alpha \|u_k\|_n \right)^{\frac{1}{n-1}} \right) \, dx.
$$

By the Levi Lemma, we have

$$
\int_{B_L} \Phi \left( \alpha_n \left| \frac{\varphi}{\tau(L)} \right|^n \left( 1 + \alpha \left| \frac{\eta \left( \frac{x}{L} \right) \varphi}{\tau(L)} \right|^n \right)^{\frac{1}{n-1}} \right) \, dx
\leq \lim_{k \to \infty} \int_{\mathbb{R}^n} \Phi \left( \beta_k \left| u_k \right|^n \left( 1 + \alpha \|u_k\|_n \right)^{\frac{1}{n-1}} \right) \, dx.
$$

Letting $L \to \infty$, we get

$$
\int_{\mathbb{R}^n} \Phi \left( \alpha_n \left| \varphi \right|^n \left( 1 + \alpha \|\varphi\|_n \right)^{\frac{1}{n-1}} \right) \, dx \leq \lim_{k \to \infty} \int_{\mathbb{R}^n} \Phi \left( \beta_k \left| u_k \right|^n \left( 1 + \alpha \|u_k\|_n \right)^{\frac{1}{n-1}} \right) \, dx.
$$

Hence,

$$
\lim_{k \to \infty} \int_{B_{R_k}} \Phi \left( \beta_k \left| u_k \right|^n \left( 1 + \alpha \|u_k\|_n \right)^{\frac{1}{n-1}} \right) \, dx = \sup_{\|u\|_{W^{1,n}(\mathbb{R}^n)} = 1} \int_{\mathbb{R}^n} \Phi \left( \alpha_n \left| u \right|^n \left( 1 + \alpha \|u\|_n \right)^{\frac{1}{n-1}} \right) \, dx.
$$
(ii) Let \( u_k^* \) be the radial rearrangement of \( u_k \). Then
\[
\tau_k^n := \int_{\mathbb{R}^n} (|\nabla u_k^n|^n + |u_k^*|^n) \, dx \leq \int_{\mathbb{R}^n} (|\nabla u_k^n|^n + |u_k|^n) \, dx = 1,
\]
thus
\[
\int_{B_{\tau_k}} \Phi \left( \beta_k \frac{|u_k^n|}{\tau_k} \left( \frac{n}{n-1} \right) \right) \, dx \geq \int_{B_{\tau_k}} \Phi \left( \beta_k |u_k^n|^{\frac{n}{n-1}} \left( 1 + \alpha \|u_k^n\|_{n}^{\frac{1}{n-1}} \right) \right) \, dx.
\]
Since
\[
\int_{B_{\tau_k}} \Phi \left( \beta_k |u_k^n|^{\frac{n}{n-1}} \left( 1 + \alpha \|u_k^n\|_{n}^{\frac{1}{n-1}} \right) \right) \, dx = \int_{B_{\tau_k}} \Phi \left( \beta_k |u_k^n|^{\frac{n}{n-1}} \left( 1 + \alpha \|u_k^n\|_{n}^{\frac{1}{n-1}} \right) \right) \, dx,
\]
we have \( \tau_k = 1 \). It is well-known that \( \tau_k = 1 \) iff \( u_k \) is radial. Therefore
\[
\int_{\mathbb{R}^n} \Phi \left( \beta_k |u_k^n|^{\frac{n}{n-1}} \left( 1 + \alpha \|u_k^n\|_{n}^{\frac{1}{n-1}} \right) \right) \, dx = \sup_{\|u\|_{W^{1,n}(B_{\tau_k})}} \frac{1}{\|u\|_{W^{1,n}(B_{\tau_k})}} \int_{B_{\tau_k}} \exp \left\{ \beta_k |u|^{\frac{n}{n-1}} \left( 1 + \alpha \|u\|_{n}^{\frac{1}{n-1}} \right) \right\} \, dx.
\]

So, we can assume \( u_k = u_k (|x|) \), and \( u_k (r) \) is decreasing. \( \square \)

## 4 Blow up analysis

In this section, the method of blow-up analysis will be used to analyze the asymptotic behavior of the maximizing sequence \( \{u_k\} \), and the first part of Theorem 11 will be finished.

After a direct computation, the Euler-Lagrange equation for the extremal function \( u_k \in W^{1,n}_0(B_{\tau_k}) \) of \( I_{\beta_k}^{V} (u) \) can be written as

\[
- \Delta_n u_k + u_k^{n-1} = \mu_k \lambda_k^{-1} \frac{1}{u_k^n} \Phi' \left\{ \alpha_k u_k^{\frac{n}{n-1}} \right\} + \gamma_k u_k^{n-1}
\]

where

\[
\left\{ \begin{array}{l}
u_k \in W^{1,n}_0(B_{\tau_k}), \|u_k\|_{W^{1,n}} = 1, \\
\alpha_k = \beta_k \left( 1 + \alpha \|u_k^n\|_{n}^{\frac{1}{n-1}} \right), \\
\mu_k = (1 + \alpha \|u_k^n\|_{n}) / (1 + 2 \alpha \|u_k^n\|_{n}), \\
\gamma_k = \alpha / (1 + 2 \alpha \|u_k^n\|_{n}), \\
\lambda_k = \int_{B_{\tau_k}} u_k^{\frac{n}{n-1}} \Phi' \left( \alpha_k u_k^{\frac{n}{n-1}} \right) .
\end{array} \right.
\]

In the following, we denote \( c_k = \max u_k = u_k (0) \). First, we give the following important observation.
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Lemma 4.1. \( \inf_k \lambda_k > 0 \).

Proof. Assume \( \lambda_k \to 0 \). Then

\[
\lambda_k = \int_{\mathbb{R}^n} u_k^{n-1} \Phi' \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx = \int_{\mathbb{R}^n} u_k^{n-1} \sum_{j=n-2}^{\infty} \frac{\left( \alpha_k u_k^{\frac{n}{n-1}} \right)^j}{j!} \, dx
\]

(4.2)

\[
= \int_{\mathbb{R}^n} \left( \alpha_k^j u_k^{n} + \ldots \right) \, dx \geq \frac{\alpha_k^j}{(n-2)!} \int_{\mathbb{R}^n} u_k^{n} \, dx.
\]

Since \( u_k (|x|) \) is decreasing, we have \( u_k^n (L) |B_L| \leq \int_{B_L} u_k^n \, dx \leq 1 \), and then

(4.3) \( u_k^n (L) \leq \frac{n}{\omega_{n-1} L^n} \).

Set \( \varepsilon = \frac{n}{\omega_{n-1} L^n} \). Then for any \( x \notin B_L \), we have \( u_k \leq \varepsilon \), and

\[
\int_{\mathbb{R}^n \setminus B_L} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx \leq c \int_{\mathbb{R}^n \setminus B_L} u_k^n \, dx \leq c \lambda_k \to 0.
\]

Since

\[
\Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) = \sum_{j=n-1}^{\infty} \frac{\left( \alpha_k u_k^{\frac{n}{n-1}} \right)^j}{j!} \leq \sum_{j=n-2}^{\infty} \frac{\alpha_k u_k^{\frac{n}{n-1}}}{{(j+1)} \, j!} \leq \alpha_k u_k^{\frac{n}{n-1}} \Phi' \left( \alpha_k u_k^{\frac{n}{n-1}} \right),
\]

we have

\[
\lim_{k \to \infty} \int_{B_L} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx = \lim_{k \to \infty} \left( \int_{B_L \cap \{u_k \geq 1\}} + \int_{B_L \cap \{u_k < 1\}} \right) \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx
\]

\[
\leq \lim_{k \to \infty} \left( c \int_{B_L} u_k^{n} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx + \int_{B_L \cap \{u_k < 1\}} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx \right)
\]

\[
\leq \lim_{k \to \infty} \left( c \lambda_k + c \int_{B_L} u_k^n \, dx \right).
\]

By (4.2), we see that \( \int_{B_L} u_k^q \, dx \to 0 \), for any \( q > 1 \), and then we have

\[
\lim_{k \to \infty} \int_{B_L} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx = 0.
\]

This is impossible. \( \square \)

Now, we introduce the concept of Sobolev-normalized concentrating sequence and concentration-compactness principle as in [24].
**Definition 4.1.** A sequence \( \{u_k\} \in W^{1,n}(\mathbb{R}^n) \) is a Sobolev-normalized concentrating sequence, if

i) \( \|u_k\|_{W^{1,n}(\mathbb{R}^n)} = 1 \);

ii) \( u_k \to 0 \) weakly in \( W^{1,n}(\mathbb{R}^n) \);

iii) there exists a point \( x_0 \) such that for any \( \delta > 0 \), \( \int_{\mathbb{R}^n \setminus B_{\delta}(x_0)} (|\nabla u_k|^n + |u_k|^n) \, dx \to 0 \).

From Lemma 4.2, we can derive the following

**Lemma 4.2.** Let \( \{u_k\} \) be a sequence satisfying \( \|u_k\|_{W^{1,n}(\mathbb{R}^n)} = 1 \), and \( u_k \to u \) weakly in \( W^{1,n}(\mathbb{R}^n) \). Then either \( \{u_k\} \) is a Sobolev-normalized concentrating sequence, or there exists \( \gamma > 0 \) such that \( \Phi((\alpha_n + \gamma)|u_k|^\frac{n}{n-1}) \) is bounded in \( L^1(\mathbb{R}^n) \).

**Lemma 4.3.** If \( \sup c_k < \infty \), then Theorem 1.1 and Theorem 1.2 hold.

**Proof.** For any \( \varepsilon > 0 \), by using (4.3) we can find some \( L \) such that \( u_k(x) \leq \varepsilon \) when \( x \notin B_L \). We rewrite

\[
\int_{\mathbb{R}^n} \left( \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) - \frac{\alpha_k^{n-1}u_k^n}{(n-1)!} \right) \, dx = \int_{B_L} + \int_{\mathbb{R}^n \setminus B_L} \left( \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) - \frac{\alpha_k^{n-1}u_k^n}{(n-1)!} \right) \, dx.
\]

Since

\[
\int_{\mathbb{R}^n \setminus B_L} \left( \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) - \frac{\alpha_k^{n-1}u_k^n}{(n-1)!} \right) \, dx = c \int_{\mathbb{R}^n \setminus B_L} u_k^{\frac{n^2}{n-1}} \, dx \leq c \varepsilon \frac{n^2}{n-1} \int_{\mathbb{R}^n} u_k^n \, dx = c \varepsilon \frac{n^2}{n-1},
\]

we have

\[
\int_{\mathbb{R}^n} \left( \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) - \frac{\alpha_k^{n-1}u_k^n}{(n-1)!} \right) \, dx = \int_{B_L} \left( \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) - \frac{\alpha_k^{n-1}u_k^n}{(n-1)!} \right) \, dx + O \left( \varepsilon \frac{n^2}{n-1} \right).
\]

It follows from \( \sup c_k < \infty \) that

\[
\int_{\mathbb{R}^n} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx \leq \int_{B_L} \left( \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) - \frac{\alpha_k^{n-1}u_k^n}{(n-1)!} \right) \, dx + \int_{\mathbb{R}^n} \frac{\alpha_k^{n-1}u_k^n}{(n-1)!} \, dx + O \left( \varepsilon \frac{n^2}{n-1} \right)
\]

thus, Theorem 1.1 holds. By Lemma 4.1 and applying the elliptic estimate in [27] to equation (4.1), we have \( u_k \to u \) in \( C^1_{\text{loc}}(\mathbb{R}^n) \).

When \( u = 0 \), we claim that \( \{u_k\} \) is not a Sobolev-normalized concentrating sequence. If not, by iii) of Definition 4.1 and the fact that \( |u_k| \) is bounded, we have for any \( \delta > 0 \),

\[
\int_{\mathbb{R}^n} u_k^n \, dx \leq \int_{B_{\delta}} u_k^n \, dx + \int_{\mathbb{R}^n \setminus B_{\delta}} u_k^n \, dx.
\]
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\[
\leq c\delta^n + o_k(1).
\]

Letting \( \delta \to 0 \), we have \( \int_{\mathbb{R}^n} u_k^n dx \to 0 \), as \( k \to \infty \). For any \( \varepsilon > 0 \), when \( L \) is large enough, we have by (4.4)

\[
S + o_k(1) = \int_{\mathbb{R}^n} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) dx
\]

\[
= \int_{\mathbb{R}^n} \frac{\alpha_k^{n-1} u_k^n}{(n-1)!} dx + \int_{B_L} \left( \Phi \left( \alpha_k \cdot \frac{u}{u^{n-1}} \right) - \int_{B_L} \frac{\alpha_k^{n-1} \cdot u^n}{(n-1)!} \right) dx + O\left( \varepsilon^{\frac{n^2}{n-1}} \right),
\]

then

\[
S \leq \int_{\mathbb{R}^n} \frac{\alpha_k^{n-1} u_k^n}{(n-1)!} dx \to 0,
\]

which is impossible, and thus the claim is proved.

By Lemma 4.2, we have

\[
\int_{\mathbb{R}^n} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) dx \to \int_{\mathbb{R}^n} \Phi \left( \alpha u_k^{\frac{n}{n-1}} \right) dx = 0,
\]

which is still impossible. Therefore, \( u \neq 0 \).

Now, we show that \( \int_{\mathbb{R}^n} u_k^n \to \int_{\mathbb{R}^n} u^n \). By (4.4), we have

\[
S = \lim_{k \to \infty} \int_{\mathbb{R}^n} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) dx
\]

\[
\leq \int_{\mathbb{R}^n} \frac{\alpha_k^{n-1} u_k^n}{(n-1)!} dx + \lim_{k \to \infty} \int_{\mathbb{R}^n} \frac{\alpha_k^{n-1} (u_k^n - u^n)}{(n-1)!} dx.
\]

Set

\[
\tau_k^n = \frac{\int_{\mathbb{R}^n} u_k^n}{\int_{\mathbb{R}^n} u^n}.
\]

By the Levi Lemma, we have \( \tau_k \geq 1 \). Let \( \bar{u} = u \left( \frac{x}{\tau_k} \right) \). Then, we have

\[
\int_{\mathbb{R}^n} |\nabla \bar{u}|^n dx = \int_{\mathbb{R}^n} |\nabla u|^n dx \leq \int_{\mathbb{R}^n} |\nabla u_k|^n dx
\]

and

\[
\int_{\mathbb{R}^n} |\bar{u}|^n dx = \tau_k^n \int_{\mathbb{R}^n} |u|^n dx \leq \int_{\mathbb{R}^n} |u_k|^n dx.
\]

Therefore

\[
\int_{\mathbb{R}^n} (|\nabla \bar{u}|^n + |\bar{u}|^n) dx \leq 1.
\]

Hence, we have by (4.5) that

\[
S \geq \int_{\mathbb{R}^n} \Phi \left( \alpha_n (1 + \alpha \|\bar{u}\|^n)^{\frac{1}{n-1}} \bar{u}^{\frac{n}{n-1}} \right) dx
\]
\[ = \tau^n \int_{\mathbb{R}^n} \Phi \left( \alpha_n (1 + \alpha \tau^n \|u\|^n) \frac{n}{n-1} u \frac{n}{n-1} \right) \, dx \]
\[ \geq \tau^n \int_{\mathbb{R}^n} \Phi \left( \lim_{k \to \infty} \alpha_k u \frac{n}{n-1} \right) \, dx + o(1) \]
\[ = \int_{\mathbb{R}^n} \left( \Phi \left( \lim_{k \to \infty} \alpha_k u \frac{n}{n-1} \right) + (\tau^n - 1) \int_{\mathbb{R}^n} \lim_{k \to \infty} \alpha_k^{n-1} u \frac{n}{n-1} \, dx \right) \]
\[ + (\tau^n - 1) \int_{\mathbb{R}^n} \left( \Phi \left( \lim_{k \to \infty} \alpha_k u \frac{n}{n-1} \right) - \int_{\mathbb{R}^n} \lim_{k \to \infty} \alpha_k^{n-1} u \frac{n}{n-1} \, dx \right) + o(1) \]
\[ = (\tau^n - 1) \left( \int_{\mathbb{R}^n} \Phi \left( \lim_{k \to \infty} \alpha_k u \frac{n}{n-1} \right) - \int_{\mathbb{R}^n} \lim_{k \to \infty} \alpha_k^{n-1} u \frac{n}{n-1} \, dx \right) \]
\[ + \lim_{k \to \infty} \int_{\mathbb{R}^n} \Phi \left( \alpha_k u \frac{n}{n-1} \right) \, dx + o(1) \]
\[ = S + (\tau^n - 1) \int_{\mathbb{R}^n} \left( \Phi \left( \lim_{k \to \infty} \alpha_k u \frac{n}{n-1} \right) - \int_{\mathbb{R}^n} \lim_{k \to \infty} \alpha_k^{n-1} u \frac{n}{n-1} \, dx \right) + o(1) \]

Since \( \Phi \left( \lim_{k \to \infty} \alpha_k u \frac{n}{n-1} \right) - \int_{\mathbb{R}^n} \lim_{k \to \infty} \alpha_k^{n-1} u \frac{n}{n-1} \, dx > 0 \), we have \( \tau = 1 \), then
\[ \lim_{k \to \infty} \int_{\mathbb{R}^n} \Phi \left( \alpha_k u \frac{n}{n-1} \right) \, dx = \int_{\mathbb{R}^n} \Phi \left( \alpha_n (1 + \alpha \|u\|^n) \frac{n}{n-1} u \frac{n}{n-1} \right) \, dx \]

Thus, \( u \) is an extremal function. \( \square \)

In the following, we assume \( c_k \to +\infty \) and perform a blow-up procedure.

### 4.1 The asymptotic behavior of \( u_k \)

In this subsection, we investigate the asymptotic behavior of \( u_k \). First, we introduce the following important quality

\[ r_k^n = \frac{\lambda_k}{\mu_k c_k^{n-1} e^\alpha c_k^{n-1}}. \]

By (4.3), we can find a sufficiently large \( L \) such that \( u_k \leq 1 \) on \( \mathbb{R}^n \setminus B_L \). Then \( (u_k - u_k(L))^+ \in W^{1,n}_0(B_L) \) and
\[
\int_{B_L} |\nabla (u_k - u_k (L))^+|^n \, dx \leq 1,
\]
hence by [29, Theorem 1.1], we have
\[
\int_{B_L} e^{\alpha_n (1 + \beta \|u_k - u_k (L)\|_n^{\frac{n}{n-1}})} \, dx \leq c (L),
\]
provided
\[
\beta < \inf_{u \in W_0^{1,n} (B_L)} \frac{\|\nabla u\|_n^n}{\|u\|_n^n}.
\]
For any \( q < \alpha_n \left(1 + \beta \|u_k - u_k (L)\|_n^{\frac{n}{n-1}}\right) \), we can find a constant \( c (q) \) such that
\[
qu_k^{\frac{n}{n-1}} \leq \alpha_n (1 + \beta \|u_k - u_k (L)\|_n^{\frac{n}{n-1}}) \left((u_k - u_k (L))^+\right)^{\frac{n}{n-1}} + c (q),
\]
and then we have
\[
(4.6) \quad \int_{B_L} e^{qu_k^{\frac{n}{n-1}}} \, dx \leq c (L, q).
\]
Now we take some \( 0 < A < 1 \) such that
\[
(1 - A) \beta_k (1 + \alpha \|u_k\|_n^{\frac{n}{n-1}}) < \alpha_n (1 + \beta \|u_k - u_k (L)\|_n^{\frac{n}{n-1}}).
\]
Then
\[
\lambda_k e^{-A \beta_k \left(1 + \alpha \|u_k\|_n^{\frac{n}{n-1}}\right)\frac{n}{n-1}} e^{\frac{n}{n-1} c_k^{\frac{n}{n-1}}}
= e^{-A \beta_k \left(1 + \alpha \|u_k\|_n^{\frac{n}{n-1}}\right)\frac{n}{n-1}} e^{\frac{n}{n-1} c_k^{\frac{n}{n-1}}} \left[ \left( \int_{\mathbb{R}^n \setminus B_L} + \int_{B_L} \right) u_k^{\frac{n}{n-1}} \Phi' \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx \right]
\leq ce^{-A \beta_k \left(1 + \alpha \|u_k\|_n^{\frac{n}{n-1}}\right)\frac{n}{n-1}} e^{\beta_k \left(1 + \alpha \|u_k\|_n^{\frac{n}{n-1}}\right)\frac{n}{n-1}} u_k^{\frac{n}{n-1}} \, dx
\leq c \int_{B_L} u_k^{\frac{n}{n-1}} e^{\left(1-A\right) \beta_k \left(1 + \alpha \|u_k\|_n^{\frac{n}{n-1}}\right)\frac{n}{n-1}} u_k^{\frac{n}{n-1}} \, dx + o (1).
\]
Since \( u_k \) converges strongly in \( L^s (B_L) \) for any \( s > 1 \), by using Hölder’s inequality and (4.6), we have
\[
\lambda_k \leq ce^{A \alpha_k c_k^{\frac{n}{n-1}}},
\]
hence
\[
(4.7) \quad r_k^{\frac{n}{n-1}} \leq Ce^{(A-1) \alpha_k c_k^{\frac{n}{n-1}}} = o \left(c_k^{-q}\right),
\]
for any $q > 0$.

Now, we set

\[
\begin{aligned}
m_k(x) &= u_k(r_k x), \\
\phi_k(x) &= \frac{m_k(x)}{c_k}, \\
\psi_k(x) &= \frac{n}{n-1} \alpha_k c_k^{\frac{1}{n-2}} (m_k - c_k),
\end{aligned}
\]

where $m_k, \phi_k$ and $\psi_k$ are defined on $\Omega_k := \{x \in \mathbb{R}^n : r_k x \in B_1\}$. From (4.7) and (4.1), we know $\phi_k(x), \psi_k(x)$ satisfy

\[
-\Delta_n \phi_k(x) = \frac{r^n_k}{c_k^{n-1}} \left( \mu_k \gamma_k^{-1/n} m_k^{1/n} \Phi' \left\{ \alpha_k m_k^{1/n} \right\} + (\gamma_k - 1) m_k^{n-1} \right)
\]

\[
= \left( \frac{n\alpha_k}{n-1} \right)^{n-1} \left( c_k r_k^n \left( \mu_k \gamma_k^{-1/n} m_k^{1/n} \Phi' \left\{ \alpha_k m_k^{1/n} \right\} + (\gamma_k - 1) m_k^{n-1} \right) \right)
\]

\[
-\Delta_n \psi_k(x) = \left( \frac{n\alpha_k}{n-1} \right)^{n-1} \left( c_k r_k^n \left( m_k^{1/n} \Phi' \left\{ \alpha_k \left( m_k^{n/n} - c_k^{n/n} \right) \right\} + o(1) \right) \right).
\]

We analyze the limit function of $\phi_k$ and $\psi_k(x)$. Since $u_k$ is bounded in $W^{1,n}(\mathbb{R}^n)$, there exists a subsequence such that $u_k \to u$ weakly in $W^{1,n}(\mathbb{R}^n)$. Because the right side of (4.8) vanishes as $k \to \infty$, then we have $\phi_k \to \phi$ in $C^1_{loc}(\mathbb{R}^n)$, as $k \to \infty$, by applying the classical estimates [27]. Therefore,

\[-\Delta_n \phi = 0 \text{ in } \mathbb{R}^n.
\]

Since $\phi_k(0) = 1$, by the Lionville-type theorem, we have $\phi \equiv 1$ in $\mathbb{R}^n$.

Now, we investigate the asymptotic behavior of $\psi_k$. By (4.7) and the fact that $\phi_k(x) \leq 1$, we can rewrite (4.9) as

\[-\Delta_n \psi_k(x) = O(1).
\]

By [25] Theorem 7, we know that $osc_{B_L} \psi_k \leq c(L)$ for any $L > 0$. Then from the result of [27], we have $\|\psi_k\|_{C^1(B_L)} \leq c(L)$ for some $\delta > 0$. Hence $\psi_k$ converges in $C^1_{loc}(B_L)$ and $m_k - c_k \to 0$ in $C^1_{loc}(B_L)$.

Since

\[
m_k^{\frac{n}{n-1}} = c_k^{\frac{n}{n-1}} \left( 1 + \frac{m_k - c_k}{c_k} \right)^{\frac{n}{n-1}} = c_k^{\frac{n}{n-1}} \left( 1 + \frac{n}{n-1} \frac{m_k - c_k}{c_k} + O \left( \frac{1}{c_k^2} \right) \right)
\]
we have
\begin{equation}
\alpha_k \left( m_k \frac{n}{n-1} - c_k \frac{n}{n-1} \right) = \alpha_k c_k \left( \frac{n}{n-1} \frac{m_k - c_k}{c_k} + O \left( \frac{1}{c_k^2} \right) \right) = \psi_k(x) + o(1) \to \psi(x) \text{ in } C_0^{\infty},
\end{equation}
and then
\begin{equation}
- \Delta_n \psi = \left( \frac{n c_n}{n-1} \right)^{n-1} \exp \{ \psi(x) \},
\end{equation}
where \( c_n = \lim_{k \to \infty} \alpha_k = \alpha_n \left( 1 + \alpha \lim_{k \to \infty} \| u_k \|_n^n \right)^{\frac{1}{n-1}}. \)

Since \( \psi \) is radially symmetric and decreasing, it is easy to see that (4.11) has only one solution. We can check that
\[
\psi(x) = -n \log \left( 1 + \frac{c_n}{n-1} |x|^{\frac{n}{n-1}} \right)
\]
and
\[
\int_{\mathbb{R}^n} e^{\psi(x)} dx = \omega_{n-1} \frac{n-1}{n} \left( \frac{n^{\frac{n}{n-1}}}{c_n} \right)^{n-1} \int_0^\infty (1 + t)^{-n} t^{n-2} dt = \omega_{n-1} \frac{n-1}{n} \left( \frac{n^{\frac{n}{n-1}}}{c_n} \right)^{n-1} \cdot \frac{1}{n-1} = \frac{1}{1 + \alpha \lim_{k \to \infty} \| u_k \|_n^n},
\]
For any \( A > 1 \), let \( u_k^A = \min \{ u_k, \frac{c_k}{A} \} \).

**Lemma 4.4.** For any \( A > 1 \), there holds
\[
\limsup_{k \to \infty} \int_{\mathbb{R}^n} \left( |u_k^A|^n + |\nabla u_k^A|^n \right) dx \leq 1 - \frac{A - 1}{A} \frac{1}{\alpha \lim_{k \to \infty} \| u_k \|_n^n}.
\]

**Proof.** Since \( \left\{ x : u_k \geq \frac{c_k}{A} \right\} \left| \frac{c_k}{A} \right|^n \leq \int_{\left\{ u_k \geq \frac{c_k}{A} \right\}} |u_k|^n dx \leq 1 \), we can find a sequence \( \rho_k \to 0 \) such that
\[
\left\{ x : u_k \geq \frac{c_k}{A} \right\} \subset B_{\rho_k}.
\]
Since \( u_k \) converges in \( L^s(B_1) \) for any \( s > 1 \), we have
\[
\lim_{k \to \infty} \int_{\left\{ u_k \geq \frac{c_k}{A} \right\}} |u_k^A|^s dx \leq \lim_{k \to \infty} \int_{\left\{ u_k \geq \frac{c_k}{A} \right\}} |u_k|^s dx = 0,
\]
and then for any $s > 0$,
\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \left( u_k - \frac{c_k}{A} \right)^+ |u_k|^s \, dx = 0.
\]

Testing (4.1) with $\left( u_k - \frac{c_k}{A} \right)^+$ we have
\[
\int_{\mathbb{R}^n} \left( \left| \nabla \left( u_k - \frac{c_k}{A} \right) \right|^n + \left( u_k - \frac{c_k}{A} \right)^+ |u_k|^{n-1} \right) \, dx
= \int_{\mathbb{R}^n} \left( u_k - \frac{c_k}{A} \right)^+ \mu_k \lambda_k^{-1} \frac{1}{u_k^{n-1}} \Phi^\prime \left\{ \alpha_k u_k^{\frac{n}{n-1}} \right\} \, dx + o(1)
\geq \int_{B_{R_k}} \left( u_k - \frac{c_k}{A} \right)^+ \mu_k \lambda_k^{-1} \frac{1}{u_k^{n-1}} \exp \left\{ \alpha_k u_k^{\frac{n}{n-1}} \right\} \, dx + o(1)
= \int_{B_R} \left( \frac{m_k - \frac{c_k}{A}}{c_k} \right) \frac{1}{c_k} \left( \frac{m_k - c_k}{c_k} + 1 \right)^{1/n-1} \exp \left\{ \psi_k(x) + o(1) \right\} \, dx + o(1)
\geq \frac{A-1}{A} \int_{B_R} e^{\psi(x)} \, dx.
\]

Letting $R \to \infty$, $k \to \infty$, by (4.12), we have
\[
\liminf_{k \to \infty} \int_{\mathbb{R}^n} \left( \left| \nabla \left( u_k - \frac{c_k}{A} \right) \right|^n + \left( u_k - \frac{c_k}{A} \right)^+ |u_k|^{n-1} \right) \, dx \geq \frac{A-1}{A} \frac{1}{\left( 1 + \alpha \lim_{k \to \infty} \|u_k\|_n \right)}.
\]

Now, observe that
\[
\int_{\mathbb{R}^n} \left( |\nabla u_k|^n + |u_k|^n \right) \, dx
= 1 - \int_{\mathbb{R}^n} \left( \left| \nabla \left( u_k - \frac{c_k}{A} \right) \right|^n + \left( u_k - \frac{c_k}{A} \right)^+ |u_k|^{n-1} \right) \, dx
+ \int_{\mathbb{R}^n} \left( u_k - \frac{c_k}{A} \right)^+ |u_k|^{n-1} \, dx - \int_{\left\{ u_k > \frac{c_k}{A} \right\}} |u_k|^n \, dx + \int_{\left\{ u_k > \frac{c_k}{A} \right\}} |u_k|^n \, dx
\leq 1 - \frac{A-1}{A} \frac{1}{1 + \alpha \lim_{k \to \infty} \|u_k\|_n} + o(1),
\]
the proof is finished. \hfill \Box

Lemma 4.5. $\lim_{k \to \infty} \|u_k\|_n = 0$.

Proof. If $\{u_k\}$ is a Sobolev-normalized concentrating sequence, then $\lim_{k \to \infty} \|u_k\|_n = 0$. If $\{u_k\}$ is not a Sobolev-normalized concentrating sequence, and $\lim_{k \to \infty} \|u_k\|_n \neq 0$. For $A$ large enough, there exist some constant $\varepsilon_0 > 0$ such that
\[
\int_{\mathbb{R}^n} \left( |\nabla u_k|^n + |u_k|^n \right) \, dx = 1 - \frac{1}{1 + (\alpha + \varepsilon_0) \lim_{k \to \infty} \|u_k\|_n} < 1.
\]

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By [17, Theorem 1.1], we have \( \int_{\mathbb{R}^n} \Phi \left( q \alpha_n \left| u_k^A \right|^\frac{n}{n-1} \right) \, dx \leq \infty \), for any

\[
q < \left( \frac{1 + (\alpha + \varepsilon_0) \lim_k \| u_k \|_n^n}{(\alpha + \varepsilon_0) \lim_k \| u_k \|_n^n} \right)^{\frac{1}{n-1}}.
\]

Since \( \alpha < 1 \), \( \| u_k \|_{W^{1,n}} = 1 \) and \( \lim_k \| u_k \|_n^n \neq 0 \), we can take some \( \varepsilon_0 \) such that \( (\alpha + \varepsilon_0) \lim_k \| u_k \|_n^n < 1 \), and then

\[
1 + \alpha \lim_k \| u_k \|_n^n \rightarrow < \left( \frac{1 + (\alpha + \varepsilon_0) \lim_k \| u_k \|_n^n}{(\alpha + \varepsilon_0) \lim_k \| u_k \|_n^n} \right)^{\frac{1}{n-1}}.
\]

Therefore

\[
(4.13) \quad \int_{\mathbb{R}^n} \Phi \left( p' \alpha_k \left| u_k^A \right|^\frac{n}{n-1} \right) \, dx < \infty,
\]

for some \( p' > 1 \).

Now, we claim that \( \Delta_n u_k \in L^r \) for some \( r > 1 \).

When \( \int_{\{ u_k > \frac{c}{k} \}} |\nabla u_k|^n \, dx \rightarrow 0 \). In this case, we can easily derive the above claim by the Trudinger-Moser inequalities on bounded domains and (4.13). When \( \int_{\{ u_k > \frac{c}{k} \}} |\nabla u_k|^n \, dx \geq c \), for some \( c > 0 \). We split \( u_k \) as \( u_k^1 + u_k^2 \), with \( u_k^1 \rightarrow c\delta_0 \) and \( \int_{\{ u_k > \frac{c}{k} \}} |\nabla u_k^2|^n \, dx \rightarrow 0 \). Since \( \alpha < 1 \), we have

\[
1 + \alpha \| u_k \|_n^n = 1 + \alpha \| u_k^2 \|_n^n + o_k(1) < \frac{1}{1 - \| u_k^2 \|_{W^{1,n}}^n} + o_k(1)
\]

\[
\leq \frac{1}{\| \nabla u_k^2 \|_n^n} + o_k(1) \leq \frac{1}{\| \nabla u_k \|_n^n(\{ u_k > \frac{c}{k} \})} + o_k(1),
\]

and then there exists some constant \( s > 1 \) such that \( (1 + \alpha \| u_k \|_n^n) s \leq \frac{1}{\| \nabla u_k \|_n^n(\{ u_k > \frac{c}{k} \})} \), therefore by the classic Trudinger-Moser inequality on the bounded domain and (4.13), the claim is proved.

Based on the the claim above and the classic elliptic estimate, we know that \( u_k \) is bounded near 0, and which contradicts the assumption that \( c_k \rightarrow \infty \). Therefore \( \lim_k \| u_k \|_n^n = 0 \), and the lemma is proved.

\[ \square \]

**Remark 4.1.** From the above lemma, we have

\[
\lim_k \alpha_k = \alpha_n, \lim_k \mu_k = 1,
\]
\[ \limsup_{k \to \infty} \int_{\mathbb{R}^n} \left( |\nabla u_k^A|^n + |u_k^A|^n \right) dx = \frac{1}{A}, \]

\[ \psi(x) = -n \log \left( 1 + \left( \frac{\omega_{n-1}}{n} \right)^{\frac{1}{n-1}} |x|^{\frac{n}{n-1}} \right), \]

and

\[ \lim \lim_{R \to \infty} \lim_{k \to \infty} \frac{1}{\lambda_k} \int_{B_{Rk}} u_k^{\frac{n}{n-1}} \exp \left( \alpha_k u_k^{\frac{n}{n-1}} \right) dx = \lim \lim_{R \to \infty} \lim_{k \to \infty} \frac{1}{\mu_k} \int_{B_R} e^{\psi(x)} dx \]

\[(4.14) = \lim_{k \to \infty} \frac{1}{1 + \alpha \lim \|u_k\|^n} \mu_k = 1. \]

**Corollary 4.1.** We have \( \lim_{k \to \infty} \int_{\mathbb{R}^n \setminus B_\delta} (|\nabla u_k|^n + |u_k|^n) dx = 0, \) for any \( \delta > 0, \) and then \( \lim_{k \to \infty} u_k : \equiv 0. \)

**Lemma 4.6.** We have

\[ \lim_{k \to \infty} \int_{\mathbb{R}^n} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) dx \leq \lim \lim_{R \to \infty} \lim_{k \to \infty} \int_{B_{Rk}} \left( \exp \left( \alpha_k u_k^{\frac{n}{n-1}} \right) - 1 \right) dx = \lim_{k \to \infty} \frac{\lambda_k}{c_k^{\frac{n}{n-1}}}, \]

moreover,

\[ \frac{\lambda_k}{c_k} \to \infty \text{ and } \sup_k \frac{c_k^\frac{n}{n-1}}{\lambda_k} \leq \infty. \]

**Proof.** For any \( A > 1, \) from the expression of \( \lambda_k, \) we have

\[ \int_{\mathbb{R}^n} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) dx \leq \int_{u_k < \frac{A}{c_k}} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) dx + \int_{u_k \geq \frac{A}{c_k}} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) dx \]

\[ \leq \int_{\mathbb{R}^n} \Phi \left( \alpha_k u_k^A \right) dx + \int_{u_k \geq \frac{A}{c_k}} \Phi' \left( \alpha_k u_k^{\frac{n}{n-1}} \right) dx \]

\[ \leq \int_{\mathbb{R}^n} \Phi \left( \alpha_k u_k^A \right) dx + \left( \frac{A}{c_k} \right)^{\frac{n}{n-1}} \lambda_k \int_{u_k \geq \frac{A}{c_k}} \frac{u_k^{\frac{n}{n-1}}}{\lambda_k} \Phi' \left( \alpha_k u_k^{\frac{n}{n-1}} \right) dx. \]

Thanks to Remark 4.1 and [17, Theorem 1.1], \( \Phi \left( \alpha_k u_k^A \right) \) is bounded in \( L^r \) for some \( r > 1. \) Since \( u_k^A \to 0 \) a.e. in \( \mathbb{R}^n \) as \( k \to \infty, \) we have

\[ \int_{\mathbb{R}^n} \Phi \left( \alpha_k u_k^A \right) dx \to \int_{\mathbb{R}^n} \Phi (0) dx = 0, \text{ as } k \to \infty. \]
Hence, we have by (4.14) that
\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx \leq \left( \frac{A}{c_k} \right)^{\frac{n}{n-1}} \lambda_k \int_{u_k \geq \frac{c_k}{A}} \frac{u_k^{\frac{n}{n-1}}}{\lambda_k} \Phi' \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx + o(1)
\]
\[
= \lim_{k \to \infty} A^{\frac{n}{n-1}} \frac{\lambda_k}{c_k^{\frac{n}{n-1}}} + o(1)
\]
Letting \( A \to 1 \) and \( k \to \infty \) we obtain (4.15).

If \( \frac{\lambda_k}{c_k} \) is bounded or \( \sup_k \frac{\lambda_k}{c_k} = \infty \), from (4.15), we have
\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx = 0,
\]
which is impossible. \( \square \)

**Lemma 4.7.** For any \( \varphi \in C_0^\infty (\mathbb{R}^n) \), we have
\[
\int_{\mathbb{R}^n} \varphi \mu_k \lambda_k^{-1} c_k u_k^{\frac{1}{n-1}} \Phi' \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx = \varphi(0).
\]

**Proof.** As [17, Lemma 3.6], we split the integral as follows
\[
\int_{\mathbb{R}^n} \varphi \mu_k \lambda_k^{-1} c_k u_k^{\frac{1}{n-1}} \Phi' \left( \alpha_k (u_k)^{\frac{n}{n-1}} \right) \, dx \leq \left( \int_{\{u_k \geq \frac{c_k}{A}\} \setminus B_{Rk}} + \int_{B_{Rk}} + \int_{\{u_k < \frac{c_k}{A}\}} \right) \ldots dx
\]
\[
= I_1 + I_2 + I_3.
\]
Now, we have
\[
I_1 \leq A \Vert \varphi \Vert_{L^\infty} \int_{\{u_k \geq \frac{c_k}{A}\} \setminus B_{Rk}} \mu_k \lambda_k^{-1} c_k u_k^{\frac{1}{n-1}} \Phi' \left( \alpha_k (u_k)^{\frac{n}{n-1}} \right) \, dx
\]
\[
\leq A \Vert \varphi \Vert_{L^\infty} \left( \int_{\mathbb{R}^n} - \int_{B_{Rk}} \right) \mu_k \lambda_k^{-1} u_k^{\frac{n}{n-1}} \Phi' \left( \alpha_k (u_k)^{\frac{n}{n-1}} \right) \, dx
\]
\[
\leq A \Vert \varphi \Vert_{L^\infty} \left( 1 - \int_{B_R} \exp \left( \alpha_k m_k^{\frac{n}{n-1}} - \alpha_k c_k^{\frac{n}{n-1}} \right) \right)
\]
\[
= A \Vert \varphi \Vert_{L^\infty} \left( 1 - \int_{B_R} \exp \left( \psi_k (x) + o(1) \right) \right)
\]
and
\[
I_2 = \int_{B_{R_k}} \varphi \mu_k \lambda_k^{-1} c_k u_k^{\frac{n}{n-1}} \Phi' \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx
\]
\[
= \int_{B_R} \varphi (r_k x) \left( \frac{m_k}{c_k} \right)^{\frac{1}{n-1}} \exp \left( \alpha_k m_k^{\frac{n}{n-1}} - \alpha_k c_k^{\frac{n}{n-1}} \right) \, dx + o(1)
\]
\[
= \varphi (0) \int_{B_R} \exp (\psi_k (x) + o(1)) \, dx + o(1) = \varphi (0) + o(1) \to \varphi (0), \text{ as } k \to \infty.
\]

By (4.16) and Hölder’s inequality, we obtain
\[
I_3 = \int_{\{u_k < \frac{c_k}{4}\}} \varphi \mu_k \lambda_k^{-1} c_k u_k^{\frac{n}{n-1}} \Phi' \left( \alpha_k \left( u_k \right)^{\frac{n}{n-1}} \right) \, dx
\]
\[
= \int_{\mathbb{R}^n} \varphi \mu_k \lambda_k^{-1} c_k \left( u_k^A \right)^{\frac{n}{n-1}} \Phi' \left( \alpha_k \left( u_k^A \right)^{\frac{n}{n-1}} \right) \, dx
\]
\[
\leq c_k \| \varphi \|_{L^\infty} \lambda_k^{-1} \left( \int_{\mathbb{R}^n} \left( u_k^A \right)^{\frac{\alpha}{n-1}} \, dx \right)^{\frac{1}{\alpha}} \left( \int_{\mathbb{R}^n} \Phi' \left( \alpha_k \left( u_k^A \right)^{\frac{n}{n-1}} \right) \, dx \right)^{\frac{1}{\gamma}} \to 0, \text{ as } k \to \infty,
\]
for any \( q' < A^{\frac{1}{n-1}} \), such that \( q = \frac{q'}{\gamma - 1} \) large enough. Letting \( R \to \infty \), by Remark 4.1, the lemma is proved. \( \square \)

**Lemma 4.8.** On any \( \Omega \subset \mathbb{R}^n \setminus \{0\} \), we have \( c_k^{\frac{1}{n-1}} u_k \to G_\alpha \in C^{1,\alpha} (\Omega) \) weakly in \( W^{1,q} (\Omega) \) for any \( 1 < q < n \), where \( G_\alpha \) is a Green function satisfying

\[
(4.17) \quad - \Delta_n G_\alpha = \delta_0 + (\alpha - 1) G_\alpha^{n-1}.
\]

**Proof.** Setting \( U_k = c_k^{\frac{1}{n-1}} u_k \). By (4.11), \( U_k \) satisfy:

\[
(4.18) \quad - \Delta_n U_k = \mu_k c_k \lambda_k^{-1} u_k^{\frac{1}{n-1}} \Phi' \left\{ \alpha_k u_k^{\frac{n}{n-1}} \right\} + (\gamma_k - 1) U_k^{n-1}.
\]

For \( t \geq 1 \), denote \( U_k^t = \min \{ U_k, t \} \) and \( \Omega_k^t = \{ 0 \leq U_k \leq t \} \). Testing (4.18) with \( U_k^t \), we have

\[
\int_{\mathbb{R}^n} - U_k^t \Delta_n U_k \, dx + (1 - \gamma_k) \int_{\mathbb{R}^n} U_k^t U_k^{n-1} \, dx \leq \int_{\mathbb{R}^n} U_k^t \mu_k c_k \lambda_k^{-1} u_k^{\frac{1}{n-1}} \Phi' \left\{ \alpha_k u_k^{\frac{n}{n-1}} \right\} \, dx.
\]

Since \( \gamma_k \to \alpha < 1 \), as \( k \to \infty \), we have

\[
\int_{\Omega_k^t} |\nabla U_k^t|^n \, dx + \int_{\Omega_k^t} |U_k^t|^n \, dx \leq \int_{\mathbb{R}^n} \left( - U_k^t \Delta_n U_k \, dx + U_k^t U_k^{n-1} \right) \, dx
\]
A sharp Trudinger-Moser type inequality in $\mathbb{R}^n$

\[
\leq c \int_{\mathbb{R}^n} U_k^t \mu_k c_k \lambda_k^{-1} u_k^{\frac{n}{n-1}} \Phi' \left\{ \alpha_k u_k^{\frac{n}{n-1}} \right\} \, dx \leq ct.
\]

Let $\eta$ be a radially symmetric cut-off function which is 1 on $B_R$ and 0 on $B_{2R}^c$, and satisfy $|\nabla \eta| \leq 1$ (when $R$ large enough). Then

\[
\int_{B_{2R}} |\nabla \eta U_k^t|^n \, dx \leq \int_{B_{2R}} |\nabla \eta|^n |U_k^t|^n \, dx + \int_{B_{2R}} \eta |\nabla U_k^t|^n \, dx \leq c_1(R) t + c_2(R),
\]

taking $t$ large enough, we have

\[
\int_{B_{2R}} |\nabla \eta U_k^t|^n \, dx \leq 2c(R) t.
\]

Then by an adaptation of an argument due to Struwe [26] (also see [17]), we can obtain that $\|\nabla U_k\|_{L^q(B_R)} \leq c(q, n, \alpha, R)$ for any $1 < q < n$, and thus $\|U_k\|_{L^p(B_R)} \leq \infty$, for any $0 < p < \infty$. By Corollary 4.1, we know $\exp \left\{ \alpha_k u_k^{\frac{n}{n-1}} \right\}$ is bounded in $L^r(\Omega \setminus \{B_\delta\})$ for any $r > 0$ and $\delta > 0$. Then applying [25, Theorem 2.8] and the result of [27], we have $\|U_k\|_{C^1(\Omega \setminus \{B_\delta\})} \leq c$, then $c_k^{\frac{1}{n-1}} u_k \to G_\alpha$ weakly in $W^{1,q}(B_R)$. So we are done.

Next, as [17, Lemma 3.8], we can obtain the following asymptotic representation of $G_\alpha$, which will be used to prove the existence of Trudinger-Moser functions.

**Lemma 4.9.** $G_\alpha \in C^{1,\beta}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ for some $\beta > 0$, and near 0, we have

\[
G_\alpha = -\frac{n}{\alpha_n} \log r + A (n \log n) r.
\]

Moreover, for any $\delta > 0$, we have

\[
\lim_{k \to \infty} \left( \int_{\mathbb{R}^n \setminus B_\delta} |\nabla U_k|^n \, dx + (1 - \alpha) \int_{\mathbb{R}^n \setminus B_\delta} U_k^n \, dx \right) = \omega_{n-1} |G'_\alpha(\delta)|^{n-1} \delta^{n-1}.
\]

**Proof.** The proof of (4.19) is similar as [17, Lemma 3.8], here we only give the proof for (4.20).

By Corollary 4.1, we have

\[
\int_{\mathbb{R}^n \setminus B_\delta} u_k^{\frac{n}{n-1}} \Phi' \left\{ \alpha_k u_k^{\frac{n}{n-1}} \right\} \, dx \leq c \int_{\mathbb{R}^n \setminus B_\delta} u_k^n \, dx \to 0.
\]

Testing (4.18) with $U_k$, we get

\[
\int_{\mathbb{R}^n \setminus B_\delta} |\nabla U_k|^n \, dx + \int_{\partial B_\delta} |\nabla U_k|^{n-2} U_k \frac{\partial U_k}{\partial n} \, dx = -\int_{\mathbb{R}^n \setminus B_\delta} \text{div} \left( |\nabla U_k|^{n-2} \nabla U_k \right) U_k \, dx
\]
by Lemma 4.8, we know $c_k u_k \rightarrow u_k$ in $C^{n-1}(\mathbb{R}^n \setminus \bar{B}_L)$. Now, we consider the case on $B_L$. Since $(u_k - u_k(L))^+ \in W^{1,n}_0(B_L)$ and for some $c > 0$, $u_k \rightarrow u_k(L)$ in $C^{n-1}(\mathbb{R}^n \setminus \bar{B}_L)$, we have

\begin{align*}
\int_{\mathbb{R}^n \setminus \bar{B}_L} \exp \left\{ \beta_k |u_k|^{\frac{n}{n-1}} \left( 1 + \alpha \|u_k\|_n^{\frac{1}{n-1}} \right) \right\} dx &\leq c \int_{\mathbb{R}^n \setminus \bar{B}_L} |u_k|^n dx \\
&\leq c.
\end{align*}

By (4.21), (4.16), we have

\begin{align*}
\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n \setminus \bar{B}_L} |\nabla U_k|^n dx &= - \lim_{k \rightarrow \infty} \int_{\partial B_L} |\nabla U_k|^{n-2} U_k \frac{\partial U_k}{\partial n} dx + (\alpha - 1) \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n \setminus \bar{B}_L} U_k^{n-1} U_k dx \\
&= - G_\alpha(\delta) \int_{\partial B_L} |\nabla G_\alpha|^{n-2} \frac{\partial G_\alpha}{\partial n} dx + (\alpha - 1) \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n \setminus \bar{B}_L} U_k^{n-1} U_k dx \\
&= \omega_{n-1} |G_\alpha'(\delta)|^{n-1} \delta^{n-1} + (\alpha - 1) \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n \setminus \bar{B}_L} U_k^{n-1} U_k dx.
\end{align*}

Thus

\begin{align*}
\lim_{k \rightarrow \infty} \left( \int_{\mathbb{R}^n \setminus \bar{B}_L} |\nabla U_k|^n dx + (1 - \alpha) \int_{\mathbb{R}^n \setminus \bar{B}_L} U_k^{n} dx \right) &= \omega_{n-1} |G_\alpha'(\delta)|^{n-1} \delta^{n-1}.
\end{align*}

The proof is finished.

Proof for the first part of Theorem 1.1. By (4.3), we can choose some $L > 0$ such that $u_k(L) < 1$, and then

\begin{align*}
\int_{\mathbb{R}^n \setminus \bar{B}_L} \exp \left\{ \frac{1}{c_k} \left( (u_k - u_k(L))^+ + u_k(L) \right)^{\frac{n}{n-1}} \right\} dx &\leq c \int_{\mathbb{R}^n \setminus \bar{B}_L} |u_k|^n dx \\
&\leq c.
\end{align*}

Now, we consider the case on $B_L$. Since $(u_k - u_k(L))^+ \in W^{1,n}_0(B_L)$ and for some $c > 0$, $u_k \rightarrow u_k(L)$ in $C^{n-1}(\mathbb{R}^n \setminus \bar{B}_L)$, we have

\begin{align*}
\int_{B_L} \exp \left\{ \beta_k |u_k|^{\frac{n}{n-1}} \left( 1 + \alpha \|u_k\|_n^{\frac{1}{n-1}} \right) \right\} dx &\leq c_k \int_{B_L} |u_k|^n dx \\
&\leq c_k.
\end{align*}

By Lemma 4.8, we know $c_k u_k \rightarrow G_\alpha$, then $u_k(L) = \frac{G_\alpha(L)}{c_k}$. Therefore, we have

\begin{align*}
\int_{B_L} \exp \left\{ \beta_k |u_k|^{\frac{n}{n-1}} \left( 1 + \alpha \|u_k\|_n^{\frac{1}{n-1}} \right) \right\} dx &\leq \left( (u_k - u_k(L))^+ \right)^{\frac{n}{n-1}} + \frac{1}{c_k} \int_{B_L} |u_k|^n dx \\
&\leq \left( (u_k - u_k(L))^+ \right)^{\frac{n}{n-1}} + \frac{1}{c_k} \int_{B_L} |u_k|^n dx.
\end{align*}

Thus

\begin{align*}
\int_{B_L} \exp \left\{ \beta_k |u_k|^{\frac{n}{n-1}} \left( 1 + \alpha \|u_k\|_n^{\frac{1}{n-1}} \right) \right\} dx &\leq \left( (u_k - u_k(L))^+ \right)^{\frac{n}{n-1}} + \frac{1}{c_k} \int_{B_L} |u_k|^n dx.
\end{align*}
\[ \leq c \int_{B_L} \exp \left\{ \beta_k \left( (u_k - u_k(L))^+ \right)^{\frac{n}{n-1}} \left( 1 + \alpha \|u_k\|_n^{\frac{n}{n-1}} \right) \right\} dx \]

\[ \leq c \int_{B_L} \exp \left\{ \beta_k \left( (u_k - u_k(L))^+ \right)^{\frac{n}{n-1}} \left( 1 + \alpha \|u_k\|_n^{\frac{n}{n-1}} - 1 \right) \right\} \exp \left( \beta_k \left( (u_k - u_k(L))^+ \right)^{\frac{n}{n-1}} \right) dx \]

\[ \leq c \exp \left\{ \beta_k c_k^{\frac{n}{n-1}} \left( 1 + \alpha \|u_k\|_n^{\frac{n}{n-1}} - 1 \right) \right\} \int_{B_L} \exp \left( \beta_k \left( (u_k - u_k(L))^+ \right)^{\frac{n}{n-1}} \right) dx \]

From Lemma 4.8 and Lemma 4.9, we know \( \|u_k\|_n \) is bounded. Recalling the fact that \( \|u_k\|_n \to 0 \), and applying the classic Trudinger-Moser inequality, we have

\[ \int_{B_L} \exp \left\{ \beta_k |u_k|^{\frac{n}{n-1}} \left( 1 + \alpha \|u_k\|_n^{\frac{n}{n-1}} \right) \right\} dx \]

\[ \leq c \exp \left\{ \frac{\alpha \beta_k c_k^{\frac{n}{n-1}}}{n-1} \|u_k\|_n \right\} \int_{B_L} \exp \left( \beta_k \left( (u_k - u_k(L))^+ \right)^{\frac{n}{n-1}} \right) dx \]

\[ = c \exp \left\{ \frac{\beta_k \alpha}{n-1} \left( c_k \right) \|u_k\|_n \right\} \int_{B_L} \exp \left( \beta_k \left( (u_k - u_k(L))^+ \right)^{\frac{n}{n-1}} \right) dx \]

\[ \leq c. \]

\[ \square \]

5 Existence of the extremal function

In this section, we will show that the existence of the extremal functions of the Trudinger-Moser inequality involving \( L^n \) norm in \( \mathbb{R}^n \). For this, we first establish the upper bound for critical functional when \( c_k \to \infty \), and then construct an explicit test function, which provides a lower bound for the supremum of our Trudinger-Moser inequality, meanwhile, this lower bound equals to the upper bound.

In order to prove the existence of the extremal functions, we need the following famous result due to L. Carleson and S.Y.A. Chang [6], which often plays a key role in proof of existence result (see [17], [20], [21] and [30]).

**Theorem 5.1** (Carleson and Chang). Let \( B \) be a unit ball in \( \mathbb{R}^n \). Given a function sequence \( \{u_k\} \subset W^{1,n}_0(B) \) with \( \|\nabla u_k\|_n = 1 \). If \( u_k \to 0 \) weakly in \( W^{1,n}_0(B) \), then

\[ \limsup_{k \to \infty} \int_B e^{\alpha_n |u_k|^{\frac{n}{n-1}}} dx \leq B \left( 1 + e^{1 + \frac{1}{2} + \ldots + \frac{1}{n-1}} \right). \]
Proposition 5.1. If $S$ can not be attained, then

$$S \leq \frac{\omega_{n-1}}{n} \exp \left\{ \alpha_n A + 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} \right\},$$

where $A$ is the constant in (4.19).

Proof. By the Lemma 4.9 we get

$$\int_{\mathbb{R}^n \setminus B_\delta} (|\nabla u_k|^n + |u_k|^n) \, dx$$

$$= c_k \left( \alpha \int_{\mathbb{R}^n \setminus B_\delta} U_n^\alpha \, dx + G_\alpha (\delta) \omega_{n-1} |G'| (\delta) \right)$$

$$= c_k \left( \alpha \lim_{k \to \infty} \|U_k\|^n_n - \frac{n}{\alpha_n} \log \delta + A + o_k (1) + \alpha (1) \right).$$

Setting $\bar{u}_k (x) = (u_k (x) - u_k (\delta))^+$. Then we have

$$\int_{B_\delta} |\nabla \bar{u}_k|^n dx \leq \int_{B_\delta} |\nabla u_k|^n dx = \tau_k := 1 - \int_{\mathbb{R}^n \setminus B_\delta} (|\nabla u_k|^n + |u_k|^n) \, dx - \int_{B_\delta} |u_k|^n \, dx$$

(5.1)

$$= 1 - c_k^{\frac{1}{n-1}} \left( \alpha \lim_{k \to \infty} \|U_k\|^n_n - \frac{n}{\alpha_n} \log \delta + A + o_k (1) + \alpha (1) \right).$$

When $x \in B_{Lr_k}$, by (5.1) and Lemma 4.8, we have

$$\alpha_k \bar{u}_k = \alpha_n (1 + \alpha \|u_k\|^n_n) \left( \bar{u}_k + u_k (\delta) \right)$$

$$\leq \alpha_n \|\bar{u}_k\|_{n-1}^{\frac{1}{n-1}} + \frac{n \alpha_n}{n-1} \|\bar{u}_k\|_{n-1}^{\frac{1}{n-1}} \alpha |u_k (\delta)| + \frac{\alpha_n \alpha}{n-1} \left\| \frac{1}{\alpha_k} u_k \right\|_n^{n-1} + o_k (1)$$

$$\leq \alpha_n \|\bar{u}_k\|_{n-1}^{\frac{1}{n-1}} + \frac{n \alpha_n}{n-1} \|\bar{u}_k\|_{n-1}^{\frac{1}{n-1}} \alpha |u_k (\delta)| + \frac{\alpha_n \alpha}{n-1} \lim_{k \to \infty} \|U_k\|^n_n + o_k (1)$$

$$\leq \alpha_n \|\bar{u}_k\|_{n-1}^{\frac{1}{n-1}} - \frac{n^2}{n-1} \log \delta + \frac{n \alpha_n}{n-1} A + \frac{\alpha_n \alpha}{n-1} \lim_{k \to \infty} \|U_k\|^n_n + o_k (1)$$

$$\leq \frac{\alpha_n \|\bar{u}_k\|_{n-1}^{\frac{1}{n-1}}}{\tau_k^{n-1}} + \alpha_n A - \log \delta^n + o_k (1) + o_k (1)$$

Integrating the above estimates on $B_{Lr_k}$, we have

$$\int_{B_{Lr_k}} \left( \exp \left\{ \alpha_k \bar{u}_k^{\frac{1}{n-1}} \right\} - 1 \right) \, dx \leq \delta^{-n} \exp \{ \alpha_n A + o_k (1) \}.$$
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\[ \cdot \int_{B_{lr_k}} \left( \exp \left( \alpha_k u_k^n \frac{n}{\tau_k} \right) \right) dx + o_k(1). \]

By the Lemma 5.1, we have

\[ \int_{B_{lr_k}} \left( \exp \left( \alpha_k u_k^n \right) - 1 \right) dx \leq \frac{\omega_{n-1}}{n} \exp \left\{ \alpha_n A + 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} \right\}, \]

thanks to Lemma 4.6, we get

\[ \lim_{k \to \infty} \int_{B_{lr_k}} \Phi \left( \alpha_k u_k^n \right) dx \leq \lim_{L \to \infty} \lim_{k \to \infty} \int_{B_{lr_k}} \left( \exp \left( \alpha_k u_k^n \right) - 1 \right) dx \]

\[ \leq \frac{\omega_{n-1}}{n} \exp \left\{ \alpha_n A + 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} \right\}. \]

(5.2)

Combining (5.2) and Lemma 4.3, the proposition is proved.

In this subsection, we will construct a function sequence \( \{u_\varepsilon\} \subset W^{1,n}(\mathbb{R}^n) \) with \( \|u_\varepsilon\|_{W^{1,n}} = 1 \) such that

\[ \int_{\mathbb{R}^n} \Phi \left( \alpha_n u_\varepsilon^n \right) dx > \frac{\omega_{n-1}}{n} \exp \left\{ \alpha_n A + 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} \right\}. \]

Proof of Theorem 1.2. Let

\[ u_\varepsilon = \begin{cases} 
\frac{C - C^n \frac{1}{n} \log \left( 1 + c_n |x|^\frac{n}{\alpha} \right) B_\varepsilon}{\alpha_n n^{-1} \log \left( 1 + c_n |x|^\frac{n}{\alpha} \right) - B_\varepsilon} & |x| \leq R\varepsilon, \\
\frac{\left( 1 + a C^n \frac{1}{n} \|G_\alpha\|^n \right) \frac{n}{G_\alpha(|x|)} + B_\varepsilon \left( C^n \frac{1}{n} + a \|G_\alpha\|^n \right) ^{-1}}{C^n \frac{1}{n} + a \|G_\alpha\|^n} & R\varepsilon < |x|,
\end{cases} \]

where \( c_n = \left( \frac{\omega_{n-1}}{n} \right)^{n-1} \), \( B_\varepsilon \), \( R \) and \( c \) depending on \( \varepsilon \) will also be determined later, such that

i) \( R\varepsilon \to 0, R \to \infty \) and \( C \to \infty \), as \( \varepsilon \to 0 \);

ii) \( B_\varepsilon \left( 1 + a C^n \frac{1}{n} \|G_\alpha\|^n \right) \frac{n}{G_\alpha(|x|)} + B_\varepsilon \left( C^n \frac{1}{n} + a \|G_\alpha\|^n \right) ^{-1} \)

We can obtain the information of \( B_\varepsilon, C \) and \( R \) by normalizing \( u_\varepsilon \). By Lemma 4.9 we have

\[ \int_{\mathbb{R}^n \setminus B_{R\varepsilon}} (|\nabla u_\varepsilon|^n + |u_\varepsilon|^n) dx \]
\[ \frac{1}{C_{n-1}^\alpha + \alpha \| G_\alpha \|_n^n} \int_{\mathbb{R}^n \setminus B_{R\varepsilon}} (|\nabla G_\alpha|^n + |G_\alpha|^n) \, dx \]
\[ = \frac{1}{C_{n-1}^\alpha + \alpha \| G_\alpha \|_n^n} \left( -G_\alpha (R\varepsilon) \int_{\partial B_{R\varepsilon}} \left( |\nabla G_\alpha|^{n-2} \frac{\partial G_\alpha}{\partial n} \right) \, dx + \alpha \int_{\mathbb{R}^n \setminus B_{R\varepsilon}} |G_\alpha|^n \, dx \right) \]
\[ = \frac{G_\alpha (R\varepsilon) \omega_{n-1} |G'(R\varepsilon)|^{n-1} (R\varepsilon)^{-1} + \alpha \int_{\mathbb{R}^n \setminus B_{R\varepsilon}} |G_\alpha|^n \, dx}{C_{n-1}^\alpha + \alpha \| G_\alpha \|_n^n}, \]

and
\[ \int_{B_{R\varepsilon}} (|\nabla u_\varepsilon|^n) \, dx = \frac{n-1}{\alpha_n \left( C_{n-1}^\alpha + \alpha \| G_\alpha \|_n^n \right)} \int_0^{c_n R_{n-1}^\alpha} \frac{u^{n-1}}{(1+u)^2} \, du \]
\[ = \frac{n-1}{\alpha_n \left( C_{n-1}^\alpha + \alpha \| G_\alpha \|_n^n \right)} \int_0^{c_n R_{n-1}^\alpha} \frac{(1+u)^{-1}}{(1+u)^n} \, du \]
\[ = \frac{n-1}{\alpha_n \left( C_{n-1}^\alpha + \alpha \| G_\alpha \|_n^n \right)} \left( \sum_{k=0}^{n-2} c_{n-1}^k (-1)^{n-1-k} \frac{1}{n-k-1} \right) \]
\[ + \log \left( 1 + c_n L_{n-1}^\alpha \right) + O \left( R_{n-1}^\alpha \right), \]

using the fact that
\[ E := \sum_{k=0}^{n-2} c_{n-1}^k (-1)^{n-1-k} \frac{1}{n-k-1} = - \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \right), \]

we have
\[ \int_{B_{R\varepsilon}} (|\nabla u_\varepsilon|^n) \, dx = - \frac{n-1}{\alpha_n \left( C_{n-1}^\alpha + \alpha \| G_\alpha \|_n^n \right)} \left( E - \log \left( 1 + c_n R_{n-1}^\alpha \right) + O \left( R_{n-1}^\alpha \right) \right) \]

It is easy to check that
\[ \int_{B_{R\varepsilon}} (|u_\varepsilon|^n) \, dx = O \left( (R\varepsilon)^n C^n \right), \]

thus we get
\[ \int_{\mathbb{R}^n} (|\nabla u_\varepsilon|^n + |u_\varepsilon|^n) \, dx = \frac{1}{\alpha_n \left( C_{n-1}^\alpha + \alpha \| G_\alpha \|_n^n \right)} \left( (n-1) E + (n-1) \log \left( 1 + c_n R_{n-1}^\alpha \right) \right) \]
\[ - \log (R\varepsilon)^n + \alpha_n A + \alpha \| G_\alpha \|_n^n + O \left( \phi \right) \]

where
\[ \phi = (R\varepsilon)^n C^n + (R\varepsilon)^n \log^n R\varepsilon + R_{n-1}^{\frac{n}{n-1}} + C_{n-1}^{\frac{2n}{n-1}} + C_{n-1}^{\frac{n^2}{n-1}} R^n \varepsilon^n. \]
Setting \( \int_{\mathbb{R}^n} (|\nabla u_\varepsilon|^n + |u_\varepsilon|^n) \, dx = 1 \), we have

\[
\alpha_n \left( C^{n-1}_{\alpha} + \alpha \|G_\alpha\|_n^n \right) = (n - 1) E + (n - 1) \log \left( 1 + c_n R^{n}_{\alpha} \right) \\
- \log (R \varepsilon)^n + \alpha_n A + \alpha \alpha_n \|G_\alpha\|_n^n + O(\phi),
\]

that is

\[
(5.3) \quad \alpha_n C^{n-1}_{\alpha} = (n - 1) E + \log \frac{\omega_{n-1}}{n} - \log \varepsilon^n + \alpha_n A + O(\phi).
\]

On the other hand, by ii) we have

\[
C - C^{n-1}_{\alpha} \left( \frac{n - 1}{\alpha_n} \log \left( 1 + c_n |R|^{n}_{\alpha} \right) - B_\varepsilon \right) = \frac{-\frac{n}{\alpha_n} \log R \varepsilon + A + O(\phi)}{C^{n-1}_{\alpha}}
\]

which implies that

\[
(5.4) \quad C^{n-1}_{\alpha} = \frac{\frac{n}{\alpha_n} \log \varepsilon + \log \frac{\omega_{n-1}}{n} - B_\varepsilon + A + O(\phi)}{C^{n-1}_{\alpha}}.
\]

Combining (5.3) and (5.4), we have

\[
(5.5) \quad B_\varepsilon = -\frac{n - 1}{\alpha_n} E + O(\phi)
\]

Setting \( R = -\log \varepsilon \), which satisfies \( R \varepsilon \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \). We can easily verify that

\[
(5.6) \quad \|u_\varepsilon\|_n^n = \frac{\|G_\alpha\|_n^n + O \left( C^{n^2}_{\alpha} R^n \varepsilon^n \right) + O(R^n \varepsilon^n (-\log (R \varepsilon)^n))}{C^{n-1}_{\alpha} + \alpha \|G_\alpha\|_n^n}.
\]

It is well known that when \(|t| < 1\),

\[
(1 - t)^{\frac{n-1}{n}} \geq 1 - \frac{n}{n-1} t
\]

and

\[
(1 + t)^{-\frac{1}{n-1}} \geq 1 - \frac{t}{n-1}.
\]

By using above inequalities and (5.6), we have for any \( x \in B_{R \varepsilon} \),
\begin{align*}
\alpha_n \left| u_\varepsilon \right|_n^{\frac{n}{n-1}} (1 + \alpha \left\| u_\varepsilon \right\|_n^{n})^{\frac{1}{n-1}} \\
= \alpha_n C^{\frac{n}{n-1}} \left( 1 - C^{\frac{n}{n-1}} \left( \frac{n-1}{\alpha_n} \log \left( 1 + c_n \left| x \right|_\varepsilon^{\frac{n}{n-1}} \right) - B_\varepsilon \right) \right)^{\frac{n}{n-1}} (1 + \alpha \left\| u_\varepsilon \right\|_n^{n})^{\frac{1}{n-1}} \\
\geq \alpha_n C^{\frac{n}{n-1}} \left( 1 - \frac{n}{n-1} C^{\frac{n}{n-1}} \left( \frac{n-1}{\alpha_n} \log \left( 1 + c_n \left| x \right|_\varepsilon^{\frac{n}{n-1}} \right) - B_\varepsilon \right) \right)^{\frac{1}{n-1}} \cdot \left( 1 - \alpha C^{\frac{n-2}{n-1}} \left\| G_\alpha \right\|_n^{\frac{n}{n-1}} \right) \left( 1 + \alpha \left\| u_\varepsilon \right\|_n^{n} \right) \left( 1 + \alpha \left\| u_\varepsilon \right\|_n^{n} \right)^{\frac{1}{n-1}} \\
\geq \alpha_n C^{\frac{n}{n-1}} \left( 1 - \frac{n}{n-1} C^{\frac{n}{n-1}} \left( \frac{n-1}{\alpha_n} \log \left( 1 + c_n \left| x \right|_\varepsilon^{\frac{n}{n-1}} \right) - B_\varepsilon \right) \right)^{\frac{1}{n-1}} \cdot \left( 1 - \alpha C^{\frac{n-2}{n-1}} \left\| G_\alpha \right\|_n^{\frac{n}{n-1}} \right) \left( 1 + \alpha \left\| u_\varepsilon \right\|_n^{n} \right) \left( 1 + \alpha \left\| u_\varepsilon \right\|_n^{n} \right)^{\frac{1}{n-1}} \\
\cdot \left( 1 - \frac{1}{n-1} \log (1 + c_n \left| x \right|_\varepsilon^{\frac{n}{n-1}}) + \frac{n \alpha w}{n - 1} B_\varepsilon - \frac{\alpha_n \alpha^2 \left\| G_\alpha \right\|_n^{2n}}{\left( n - 1 \right) C^{\frac{n}{n-1}}} + O \left( \phi \right) \right).
\end{align*}

By (5.3) and (5.5), we obtain

\begin{align*}
\alpha_n \left| u_\varepsilon \right|_n^{\frac{n}{n-1}} (1 + \alpha \left\| u_\varepsilon \right\|_n^{n})^{\frac{1}{n-1}} \\
\geq -E + \log \left( \omega_{n-1} \frac{n}{n} - \log \varepsilon^n - n \log \left( 1 + c_n \left| x \right|_\varepsilon^{\frac{n}{n-1}} \right) \right) \\
- \frac{\alpha_n \alpha^2 \left\| G_\alpha \right\|_n^{2n}}{\left( n - 1 \right) C^{\frac{n}{n-1}}} + \alpha_n A + O \left( \phi \right).
\end{align*}

Then we have

\begin{align*}
\int_{B_{R\varepsilon}} \Phi \left( \alpha_n \left| u_\varepsilon \right|_n^{\frac{n}{n-1}} (1 + \alpha \left\| u_\varepsilon \right\|_n^{n})^{\frac{1}{n-1}} \right) dx
\end{align*}
\[ \geq \exp \left\{ -E + \alpha_n A + \log \omega_{n-1} - \log \varepsilon^n - \frac{\alpha_n \alpha^2 \| G_{\alpha} \|_{p}^{2n}}{(n-1) C^{n-1}} + O(\phi) \right\}. \]

\[ \cdot \int_{B_{R\varepsilon}} \exp \left\{ -n \log \left( 1 + c_n \left| \frac{x}{\varepsilon} \right|^{n-1} \right) \right\} \]

\[ \geq c_n^{n-1} \varepsilon^{-n} \exp \left\{ -E + \alpha_n A - \frac{\alpha_n \alpha^2 \| G_{\alpha} \|_{p}^{2n}}{(n-1) C^{n-1}} + O(\phi) \right\} \int_{B_{R\varepsilon}} \left( 1 + c_n \left| \frac{x}{\varepsilon} \right|^{n-1} \right)^{-n} dx \]

\[ \geq \frac{(n-1) \omega_{n-1}}{n} \exp \left\{ -E + \alpha_n A - \frac{\alpha_n \alpha^2 \| G_{\alpha} \|_{p}^{2n}}{(n-1) C^{n-1}} + O(\phi) \right\} \int_{0}^{c_n R^{\frac{n}{n-1}}} \frac{u^{n-2}}{(1+u)^n} du \]

\[ \geq \frac{(n-1) \omega_{n-1}}{n} \exp \left\{ -E + \alpha_n A - \frac{\alpha_n \alpha^2 \| G_{\alpha} \|_{p}^{2n}}{(n-1) C^{n-1}} + O(\phi) \right\} \left( \frac{1}{n-1} + o\left( R^{\frac{n}{n-1}} \right) \right) \]

\[ \geq \frac{\omega_{n-1}}{n} \exp \left\{ -E + \alpha_n A \right\} \left( 1 - \frac{\alpha_n \alpha^2 \| G_{\alpha} \|_{p}^{2n}}{(n-1) C^{n-1}} + O(\phi) \right) \]

On the other hand, we have

\[ \int_{\mathbb{R}^{n} \setminus B_{R\varepsilon}} \Phi \left( \alpha_n u_{\varepsilon}^{n-1} \right) dx \geq \frac{\alpha_n^{n-1}}{(n-1)! C^{n-1}} \int_{\mathbb{R}^{n} \setminus B_{R\varepsilon}} |G_{\alpha}|^{n} dx \]

\[ = \frac{\alpha_n^{n-1} \| G_{\alpha} \|_{p}^{n} + O\left( R^{n} \varepsilon^{n} \log (R\varepsilon)^{n}) \right)}{(n-1)! C^{n-1}}, \]

thus

\[ \int_{\mathbb{R}^{n}} \Phi \left( \alpha_n |u_{\varepsilon}|^{n-1} \left( 1 + \alpha \| u_{\varepsilon} \|_{p}^{n} \right) \frac{1}{n-1} \right) dx \]

\[ \geq \frac{\omega_{n-1}}{n} \exp \left\{ -E + \alpha_n A \right\} \left( 1 - \frac{\alpha_n \alpha^2 \| G_{\alpha} \|_{p}^{2n}}{(n-1) C^{n-1}} + O(\phi) \right) + \frac{\alpha_n^{n-1} \| G_{\alpha} \|_{p}^{n}}{(n-1)! C^{n-1}}. \]

Since \( R = \log \frac{1}{\varepsilon} \), by [3.4] we have \( R \sim C^{\frac{n}{n-1}} \), then we can easily verify that \( \phi = o\left( C^{\frac{n}{n-1}} \right) \).

Hence when \( \alpha \) small enough, we have

\[ \int_{\mathbb{R}^{n}} \Phi \left( \alpha_n |u_{\varepsilon}|^{n-1} \left( 1 + \alpha \| u_{\varepsilon} \|_{p}^{n} \right) \frac{1}{n-1} \right) dx > \frac{\omega_{n-1}}{n} \exp \left\{ -E + \alpha_n A \right\}. \]

Therefore the proof of Theorem [1.2] is completely finished. \( \square \)

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