Multiple model approach for robust state estimation in presence of model uncertainty and bounded disturbances*

Nestor N. Deniz\textsuperscript{a}, Marina H. Murillo\textsuperscript{a}, Guido Sanchez\textsuperscript{a}, Lucas M. Genzelis\textsuperscript{a}, Leonardo L. Giovanini\textsuperscript{a}

\textsuperscript{a}Instituto de Investigacion en Senales, Sistemas e Inteligencia Computacional, sinc(i), UNL, CONICET, Ciudad Universitaria UNL, 4to piso FICH, (S3000) Santa Fe, Argentina

Abstract

In the present work, an optimization-based algorithm for state estimation under model uncertainty and bounded disturbances is presented. In order to avoid to solve a non-convex optimization problem, model and state estimation problems are divided into two convex formulations which are solved within a fixed-point iteration scheme with standard available solvers. Guaranty of robust global stability are given for the case of bounded disturbances and uncertainty, and convergence to the true system and vector state are given for the case of vanishing disturbances.

Key words: Multiple model adaptive control, Moving horizon estimation, Robust stability, Nonlinear systems.

1 Introduction

When model uncertainties are small enough to neglect them, Moving Horizon Estimation (MHE) is the framework most widely used at the present time to estimate the system states. In Addition, as MHE is an optimization-based algorithm, it allows incorporating systems constraints in a natural fashion (see

* The material in this paper was not presented at any conference.

Email addresses: ndeniz@sinc.unl.edu.ar (Nestor N. Deniz), mmurillo@sinc.unl.edu.ar (Marina H. Murillo), gsanchez@sinc.unl.edu.ar (Guido Sanchez), lgenzelis@sinc.unl.edu.ar (Lucas M. Genzelis), lgiovanini@sinc.unl.edu.ar (Leonardo L. Giovanini).
Moreover, state estimation accuracy will depend on process and measurement noises, neglecting the effect of the initial state as time increases (see Müller (2017)).

However, in practice, not always is available an accurate model of the system, and state estimation must be carried out with model uncertainty, which implies a deteriorated accuracy. In order to improve performance, several mechanisms to estimate the plant model were developed. In Landau et al. (1998), parameter adaptation algorithms based on a least-squares approach for deterministic and stochastic environments are studied. Moreover, recursive plant model identification in open and closed loop are analyzed. Narendra & Balakrishnan (1997) introduces a scheme of multiple models where the plant is estimated as a combination of the models and the fast variations in parameters are handled by means of switching amongst the models. Whereas in Narendra & Xiang (2000) the problem of adaptive control for linear time-invariant discrete-time systems using multiple models is covered. The approach consists in an adaptive scheme based on the prediction errors of a finite number of fixed models. Proof of global stability for the overall system is given for the deterministic case. In Narendra et al. (2015) a detailed examination of theoretical and practical advantages is carried out for the mechanism which combines the models. However, only cases of systems unaffected by process and measurement noises are considered.

Multiple model adaptive control is threatened also in Anderson et al. (2000) and Hespanha et al. (2001). In the former, not only the problem of model uncertainty is addresses, but also the problem of structure uncertainty. In the later, design and analysis of hysteresis-based supervisory control algorithms for uncertain linear systems is discussed. The Vinnicombe metric is used to guaranty the commutation to a model whose feedback connection is stable as well as the problem of determining the appropriate model set.

In Hassani et al. (2009) a multiple model adaptive estimation and model identification using a minimum energy criterion for discrete-time linear time-invariant with multiple inputs and outputs systems with parameter uncertainty and unmodeled dynamics is addressed. The algorithm relies on a finite number of local observers and the one with smallest output prediction error energy is selected.

Kuipers & Ioannou (2010) present a Multiple Model Adaptive Control architecture, with adaptive mixing control, where interpolation of the controllers is performed in order to accomplish with required specification of performance.

In the work of Han & Narendra (2012), a multiple model adaptive control scheme is presented where only \( n + 1 \) models are required to satisfactorily control the uncertain system. Whereas in Narendra et al. (2014) the notion of
second level adaptation is reviewed, taking in account the case of the availability of the vector state and the case of output feedback.

Nguyen (2014) threat the subject of constrained control of uncertain, time-varying systems, where the control law is designed to drive a region which contain all possible models determined by uncertainties.

At the best of our knowledge, there are not many works which deals explicitlty with the problem of state estimation under model uncertainty. Research efforts have been focused separately in model estimation, control under model uncertainty and in a lesser fashion state estimation. Even though the problem of state estimation and control with model uncertainty can be addressed from the set theory, and design an optimal control lay to drive a region of (possibly infinite) plants with stability guaranties, the performance of the overall system can be sub-optimal. In order to improve state estimation, we propose an optimization-based scheme to estimate both states and parameters of the system. The behavior of the plant is assumed to be known (i.e., the dimension of the system in the state space is known), but an accurate model is not available. In order to avoid the formulation of a non-convex problem, the problems of state and parameters estimation are split into two separated problems within a fixed-point iteration scheme.

Since we regard the estimation problem, we assume the past inputs satisfies the statement of the persistence of excitation condition (PEC).

2 Preliminaries and setup

2.1 Notation

Let $\mathbb{Z}_{[a,b]}$ denotes the set of integers in the interval $[a, b] \subseteq \mathbb{R}$, and $\mathbb{Z}_{\geq a}$ denotes the set of integers greater or equal to $a$. Boldface symbols denote sequences of finite or infinite length, i.e., $w := \{w_{k_1}, \ldots, w_{k_2}\}$ for some $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ and $k_1 < k_2$, respectively. We denote $x_{jk}$ as the element of the sequence $x$ given at time $k \in \mathbb{Z}_{\geq 0}$ and $j \in [k_1, k_2]$. By $|x|$ we denote the Euclidean norm of a vector $x \in \mathbb{R}^n$. Let $\|x\| := \sup_{k \in \mathbb{Z}_{\geq 0}} |x_k|$ denote the supreme norm of the sequence $x$ and $\|x\|_{[a,b]} := \sup_{k \in \mathbb{Z}_{[a,b]}} |x_k|$. The symbol $\oplus$ denotes the Minkowski sum of two sets $A$ and $B$, which is defined as: $A \oplus B := \{a+b : a \in A, b \in B\}$.

A function $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if $\gamma$ is continuous, strictly increasing and $\gamma(0) = 0$. If $\gamma$ is also unbounded, it is of class $\mathcal{K}_\infty$. A function $\zeta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class $\mathcal{L}$ if $\zeta(k)$ is non increasing and $\lim_{k \to \infty} \zeta(k) = 0$.

A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(\cdot, k)$ is of class $\mathcal{K}$ for each fixed $k \in \mathbb{Z}_{\geq 0}$, and $\beta(r, \cdot)$ of class $\mathcal{L}$ for each fixed $r \in \mathbb{R}_{\geq 0}$. 

3
The following inequalities hold for all $\beta \in \mathcal{K} \mathcal{L}$, $\gamma \in \mathcal{K}$ and $a_j \in \mathbb{R}_{\geq 0}$ with $j \in \mathbb{Z}_{[1,n]}$

$$
\gamma \left( \sum_{j=1}^{n} a_i \right) \leq \sum_{j=1}^{n} \gamma (n a_i), \quad \beta \left( \sum_{j=1}^{n} a_i, k \right) \leq \sum_{j=1}^{n} \beta (n a_i, k). \quad (1)
$$

The preceding inequalities hold since $\max\{a_j\}$ is included in the sequence \{a_1, a_2, \ldots, a_n\} and $\mathcal{K}$ functions are non-negative strictly increasing functions.

A sequence $w$ is bounded if $\|w\|$ is finite. The set of bounded sequences $w$ is denoted as $\mathcal{W}(w_{\text{max}}) := \{w : w \leq w_{\text{max}}\}$ for some $w_{\text{max}} \in \mathbb{R}_{\geq 0}$. A bounded infinite sequence $w$ is convergent if $|w_k| \to 0$ as $k \to \infty$. Let us denote the set of convergent sequences $\mathcal{C}_w := \{w \in \mathcal{W}(w_{\text{max}}) \mid w \text{ is convergent}\}$. Analogously, $\mathcal{C}_v$ is defined for the sequence $v$.

The superindice in $\Psi_x^i (\cdot)$ denotes the $i$-th times that $\Psi_x (\cdot)$ is computed in the same sampling time.

### 3 Problem statement

Let us consider a varying nonlinear discrete-time system with the following behaviour

$$
x_{k+1} = f (k; x_k, u_k, w_k, d_k), \quad y_k = h (x_k) + v_k, \quad \forall k \in \mathbb{Z}_{\geq 0} \quad (2)
$$

where $x_k \in \mathcal{X} \subset \mathbb{R}^n$ is the system state, $u_k \in \mathcal{U} \subset \mathbb{R}^m$ is the system input, $w_k \in \mathcal{W} \subset \mathbb{R}^n$ is the additive process disturbance, $y_k \in \mathcal{Y} \subset \mathbb{R}^p$ is the system measurements and $v_k \in \mathcal{V} \subset \mathbb{R}^p$ is the measurement noise. The additive uncertainty signal $d_k$ is modelled as follows,

$$
d_k = \delta_k \eta (x_k, u_k), \quad \forall k \in \mathbb{Z}_{\geq 0} \quad (3)
$$

where $\eta : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ is a known function such that it is continuous and $\eta (0, 0) = 0$, $\delta \in \mathbb{R}^n$ is an exogenous signal contained in a known compact set such that $d_k \in D \subset \mathbb{R}^n$. The sets $\mathcal{X}, \mathcal{U}, \mathcal{W}, \mathcal{D}, \mathcal{Y}$ and $\mathcal{V}$ are known compact sets containing the origin in its interior.

As the only information available is the output’s noisy measurements and the inputs, it becomes necessary to estimate the actual state of the system. However, there is not available an accurate model of the system. Therefore, the model needs to be estimated together with the states. In this work, we
propose a moving horizon estimation algorithm to simultaneously estimate the state of the system and the model. The sequence of states is estimated based upon the measurements and the system model available at the moment. The model is estimated as a convex combination of \( q \) fixed linear models \((A^1, B^1, C^1), \ldots, (A^q, B^q, C^q)\). It is assumed that the true model belong to the convex hull of the fixed models. When the system generating the measurements is a nonlinear one, it is assumed that its linear approximation belong to the convex hull. Once a new model is available, the old is replaced by the more accurate one. The problem to be solved at each sampling time is the following,

\[
\min_{\hat{x}_{k-N|k}, \hat{w}, \hat{d}, \alpha_k} \Psi_{x, \alpha} := \Gamma_{k-N}(\hat{w}_{k-N-1|k}) + \Lambda_k(\hat{w}_\alpha) + \sum_{j=k-N}^{k} \ell(\hat{w}_{j|k}, \hat{v}_{j|k}, \hat{d}_{j|k})
\]

\[
\begin{align*}
\hat{x}_{k-N|k} &= \bar{x}_{k-N} + \hat{w}_{k-N-1|k} \\
\alpha_{k|k} &= \alpha_{k-1|k} + \hat{w}_\alpha \\
\hat{x}_{j+1|k} &= (A^1 \hat{x}_{j|k} + B^1 u_{j|k}) \alpha_{1|k} + \ldots + (A^q \hat{x}_{j|k} + B^q u_{j|k}) \alpha_{q|k} + \\
&\quad \hat{w}_{j|k} + \hat{d}_{j|k}, \quad j \in \mathbb{Z}_{[k-N,k-1]}.
\end{align*}
\]

\[
y_j = C^1 \hat{x}_{j|k} \alpha_{1|k} + \ldots + C^q \hat{x}_{j|k} \alpha_{q|k} + \hat{v}_{j|k}, \quad j \in \mathbb{Z}_{[k-N,k]}.
\]

\[
\sum_{i=1}^{q} \alpha_{i|k} = 1 \\
\alpha_{i|k} \geq 0 \\
\hat{x}_{j|k} \in \mathcal{X}, \quad \hat{w}_{j|k} \in \mathcal{W}, \quad \hat{v}_{j|k} \in \mathcal{V}, \quad \hat{d}_{j|k} \in \mathcal{D}.
\]

As is common in moving horizon estimation approach, the term \( \Gamma_{k-N}(\cdot) \) is the so-called *arrival-cost*. The term \( \Lambda_k(\hat{w}_\alpha) \) just attempts to smooth the evolution of the vector \( \alpha \), which is defined as \( \alpha := [\alpha^1, \ldots, \alpha^q]^T \). The problem estimates simultaneously the optimal sequence of states and the model of the system. However, this problem looses the convexity property due to the simultaneous estimation and model identification.

In order to avoid to solve a non-convex problem, the optimization problem (4) is reformulated into a fixed-point iteration problem.

3.1 *Fixed-point iteration*

The proposed fixed-point iteration problem overcome the difficulty to solve a non-convex optimization problem solving separately the state estimation and the model identification. For the same sampling-time, the state estimation
and the model identification problems are solved several times until a stop condition is reached. At time $k$, the optimal state trajectory is estimated using the system model available. Once the state trajectory was estimated, it remains constant, and the model is re-identified. This procedure is repeated several times for the same sampling-time. The problems to be solved iteratively are the followings,

\[
\begin{align*}
\min_{\hat{x}_{k-N|k}, \hat{\alpha}} \Psi_x := \Gamma_{k-N} \left( \hat{w}_{k-N-1|k} \right) + \sum_{j=k-N}^{k} \ell \left( \hat{w}_{j|k}, \hat{v}_{j|k} \right) \\
\text{s.t.} \quad \begin{cases}
\hat{x}_{k-N|k} = \bar{x}_{k-N} + \hat{w}_{k-N-1|k} \\
\hat{x}_{j+1|k} = \hat{A}_k \hat{x}_{j|k} + \hat{B}_k u_{j|k} + \hat{d}_{j|k} + \hat{w}_{j|k}, j \in \mathbb{Z}_{[k-N,k-1]} \\
y_j = \hat{C}_k \hat{x}_{j|k} + \hat{v}_{j|k}, j \in \mathbb{Z}_{[k-N,k]} \\
\hat{x}_{j|k} \in \mathcal{X}, \hat{w}_{j|k} \in \mathcal{W}, \hat{v}_{j|k} \in \mathcal{V}.
\end{cases}
\end{align*}
\]

where the system matrices are given by

\[
\begin{align*}
\hat{A}_k &= \sum_{i=1}^{q} \alpha_k^i A^i, \quad \hat{B}_k = \sum_{i=1}^{q} \alpha_k^i B^i, \quad \hat{C}_k = \sum_{i=1}^{q} \alpha_k^i C^i \\
\sum_{i=1}^{q} \alpha_k^i &= 1, \quad \alpha_k^i \geq 0
\end{align*}
\]

Note that matrices $\hat{A}$, $\hat{B}$ and $\hat{C}$ are combined with the same coefficients $\alpha$. Regard that it can be always done by mean of a matrix transformation. For proof, the reader can see Nguyen (2014).

Once problem (5) is solved, the state trajectory remains constant, and the
model is estimated,

$$\min_{\alpha, \hat{w}} \Psi_{\alpha} := \Lambda_k(\hat{w}_{\alpha}) + \sum_{j=k-N}^{k} \ell(\hat{d}_{j|k}, \hat{v}_{j|k})$$

$$\begin{align*}
\alpha_{k|k} &= \hat{\alpha}_{k-1|k} + \hat{w}_{\alpha} \\
x_{j+1|k} &= (A^1 x_{j|k} + B^1 u_{j|k}) \alpha_{1|k} + \ldots + (A^q x_{j|k} + B^q u_{j|k}) \alpha_{q|k} + \hat{w}_{j|k} + \hat{d}_{j|k}, \\
y_j &= (C^1 x_{j|k}) \alpha_{1|k} + \ldots + (C^q x_{j|k}) \alpha_{q|k} + \hat{v}_{j|k}, \\
\sum_{i=1}^{q} \alpha_{i|k} &= 1 \\
\alpha_{i|k} &\geq 0 \\
\hat{d}_{j|k} &\in \mathcal{D}, \hat{w}_{\alpha} \in \mathcal{W}_{\alpha}, \hat{v}_{j|k} \in \mathcal{V}.
\end{align*}$$

(7)

Problems (5) and (7) are solved iteratively several times for the same sampling-time.

The main novelty of the proposed algorithm is that an improvement in the state estimation and model identification can be guaranteed for a certain number of iterations when some assumptions are fulfilled. Moreover, the number of iterations can be computed offline.

4 Stability of the fixed-point iteration

As stated formerly, problems (5) and (7) are solved sequentially within the fixed-point iteration for the same sampling time. In order to be clear, let us define an iteration as the process of solving (5) first and then (7), both one time. In this work, we will use as an indicator of improvement in states estimation and model identification within the fixed-point iteration the reduction in the costs as the iterations are performed. In the following, we will state the conditions required to achieve effectively a decreasing behaviour of the costs inside the fixed-point iteration. Before to start, we will state some necessary assumptions.

**Assumption 1** The prior weighting \( \Gamma_{k-N} (\cdot) \) is a continuous function \( \Gamma_{k-N} (\cdot) : \mathbb{R}^n \to \mathbb{R} \) lower bounded by \( \gamma_p (\cdot) \in \mathcal{K}_\infty \) and upper bounded by \( \bar{\gamma}_p (\cdot) \in \mathcal{K}_\infty \) such
that
\[ \gamma_p \left( |\hat{x}_{k-N}| - |\bar{x}_{k-N}| \right) \leq \Gamma_{k-N} \left( |\hat{x}_{k-N}| \right) \leq \bar{\gamma}_p \left( |\hat{x}_{k-N}| - |\bar{x}_{k-N}| \right) \] (8)
for all \( \hat{x} \in X \) and
\[ \gamma_p (r) \geq \underline{\gamma}_p r^a, \quad \bar{\gamma}_p (r) \leq \bar{\gamma}_p r^a \] (9)
where \( 0 \leq \underline{\gamma}_p \leq \bar{\gamma}_p \) and \( a \in R_{\geq 1} \). Moreover, if the arrival cost is updated using the method developed in [Sánchez et al. (2017)],
\[ \gamma_p (r) \geq |P_0^{-1}| r^a, \quad \bar{\gamma}_p (r) \leq |P_\infty^{-1}| r^a \] (10)

**Assumption 2** The stage cost \( \ell (\cdot, \cdot) : R^n \times R^m \rightarrow R \) is a continuous function bounded by \( \underline{\gamma}_w, \bar{\gamma}_v \in X_\infty \) such that the following inequalities are satisfied \( \forall w \in \mathcal{W} \) and \( v \in \mathcal{V} \)
\[ \underline{\gamma}_w (\hat{w}) + \bar{\gamma}_v (\hat{v}) \leq \ell (\hat{w}, \hat{v}) \leq \bar{\gamma}_w (\hat{w}) + \underline{\gamma}_v (\hat{v}) \] (11)

**Assumption 3** The function \( \beta (r, s) \in \mathcal{KL} \) and satisfies the following inequality:
\[ \beta (r, s) \leq c_\beta r^p s^{-q} \] (12)
for some \( c_\beta, p, q \in R_{\geq 0} \) and \( q \geq p \).

We have now all the necessary ingredients to enunciate the first theorem,

**Theorem 1** The sequences of costs \( \{ \Psi^1_x, \Psi^2_x, \ldots, \Psi^l_x \} \) and \( \{ \Psi^1_a, \Psi^2_a, \ldots, \Psi^l_a \} \) generated in the fixed-point iteration are decreasing if \( l \) is selected as:
\[ \log_2 \left( \frac{c \Psi^1_x - \Gamma_{k-N} \left( \hat{w}_{k-N-1|k}^l \right)}{\Gamma_{k-N} \left( \hat{w}_{k-N-1|k}^m \right)} + 1 \right) + 1 < l \] (13)
with \( c \in R_{\geq 0} \) and \( \Gamma_{k-N} \left( \hat{w}_{k-N-1|k}^m \right) = \min \{ \Gamma_{k-N} \left( \hat{w}_{k-N-1|k}^i \right) \}, \ i \in Z_{[1,l]} \).

**Proof.** In order to proof Theorem 1 let us consider the sequence of costs \( \Psi^i_x \) and \( \Psi^i_a \) generated in the fixed-point iteration problem, with \( i \in Z_{[1,l]} \). Note that due to optimality the following is satisfied,
\[ \Psi^1_x \geq \Psi^2_x \geq \ldots, \geq \Psi^l_x, \quad \Psi^1_a \geq \Psi^2_a \geq \ldots, \geq \Psi^l_a \] (14)
As an iteration takes in account both \( \Psi^i_x \) and \( \Psi^i_a \), and due to the sequences are non increasing, we just need to prove the decreasing behaviour for only one of the sequences, let’s say, \( \Psi^i_x \).

Let us define the sequence \( g(k, i) : Z_{\geq 0} \times Z_{[1,l]} \rightarrow R \) as the normalized version of \( \Psi^i_x \),
\[ g(k, i) := \{ 1, \frac{\Psi^2_x}{\Psi^1_x}, \ldots, \frac{\Psi^l_x}{\Psi^1_x} \} \] (15)
for $i \in \mathbb{Z}_{[1,l]}$.

We want to find the necessary and sufficient conditions under which sequence (15) is decreasing. In order to do that, let us consider first three non-negative sequences: the sequence of costs $\Psi^i_x$, the sequence of arrival-costs $\Gamma_{k-N} \left( \hat{w}_{k-N-1|k}^i \right)$ and the sequence defined in (15), i.e., $g(k,i)$. We will make use of the discrete version of the Gronwall inequality (see Ames & Pachpatte (1997) and Holte (2009)), which states that, for any three non-negative sequences $y_n, f_n$ and $g_n$,

$$y_n \leq f_n + \sum_{k=0}^{n} g_k y_k$$

(16)

then

$$y_n \leq f_n + \sum_{k=0}^{n-1} f_k g_k \prod_{j=k+1}^{n-1} (1 + g_j)$$

(17)

for $n \geq 0$. Taking $y_i = \Psi^i_x$, $f_i = \Gamma_{k-N} \left( \hat{w}_{k-N-1|k}^i \right)$, $g_i = g(k,i)$ and $n = l$, the condition (16) becomes

$$\Psi^l_x \leq \Gamma_{k-N} \left( \hat{w}_{k-N-1|k}^l \right) + \sum_{i=1}^{l-1} g(k,i) \Psi^i_x$$

(18)

The reader can verify that inequality (18) is always verified for the sequences selected. Applying the Gronwall inequality, (18) can be rewritten as

$$\Psi^l_x \leq \Gamma_{k-N} \left( \hat{w}_{k-N-1|k}^l \right) + \sum_{i=1}^{l-1} g(k,i) \prod_{j=i+1}^{l-1} (1 + g(k,j))$$

$$\Psi^l_x = g(k,l) \leq \frac{\Gamma_{k-N} \left( \hat{w}_{k-N-1|k}^l \right) + \sum_{i=1}^{l-1} \Gamma_{k-N} \left( \hat{w}_{k-N-1|k}^i \right) g(k,i) \prod_{j=i+1}^{l-1} (1 + g(k,j))}{\Psi^1_x}$$

(19)

$$\Gamma_{k-N} \left( \hat{w}_{k-N-1|k}^l \right) + \sum_{i=1}^{l-1} \Gamma_{k-N} \left( \hat{w}_{k-N-1|k}^i \right) g(k,i) \prod_{j=i+1}^{l-1} (1 + g(k,j))$$

$$\Psi^1_x = \epsilon$$
then,

\[
\sum_{i=1}^{l-1} \Gamma_{k-N} \left( \hat{w}^i_{k-N-1|k} \right) g(k,i) \prod_{j=i+1}^{l-1} (1 + g(k,j)) = e\Psi_x^1 - \Gamma_{k-N} \left( \hat{w}^l_{k-N-1|k} \right)
\]

(20)

Defining \( \Gamma_{k-N} \left( \hat{w}^m_{k-N-1|k} \right) := \min \left( \Gamma_{k-N} \left( \hat{w}^i_{k-N-1|k} \right) \right) \), for \( i \in \mathbb{Z}_{[1,l]} \) and recalling that \( g(k,i) \) is a non-increasing sequence, i.e., \( g(k,l) \leq g(k,i) \), we can write,

\[
\Gamma_{k-N} \left( \hat{w}^m_{k-N-1|k} \right) g(k,l) \sum_{i=1}^{l-1} \prod_{j=i+1}^{l-1} (1 + g(k,j)) < c
\]

(21)

Before to continue, let us note that,

\[
\sum_{i=1}^{l-1} \prod_{j=i+1}^{l-1} (1 + g(k,l)) = \sum_{i=0}^{l-2} (1 + g(k,l))^i = \frac{(1 + g(k,l))^{l-1} - 1}{g(k,l)}
\]

(22)

Therefore, Inequality (21) can be rewritten as,

\[
\Gamma_{k-N} \left( \hat{w}^m_{k-N-1|k} \right) g(k,l) \left( \frac{(1+g(k,l))^{l-1} - 1}{g(k,l)} \right) < e\Psi_x^1 - \Gamma_{k-N} \left( \hat{w}^l_{k-N-1|k} \right)
\]

\[ (1 + g(k,l))^{l-1} < \frac{e\Psi_x^1 - \Gamma_{k-N} \left( \hat{w}^l_{k-N-1|k} \right)}{\Gamma_{k-N} \left( \hat{w}^m_{k-N-1|k} \right) + 1} \]

\[ (l - 1) \log (1 + g(k,l)) < \log \left( \frac{e\Psi_x^1 - \Gamma_{k-N} \left( \hat{w}^l_{k-N-1|k} \right)}{\Gamma_{k-N} \left( \hat{w}^m_{k-N-1|k} \right) + 1} \right) \]

\[ \log (1 + g(k,l)) < \frac{\log \left( \frac{e\Psi_x^1 - \Gamma_{k-N} \left( \hat{w}^l_{k-N-1|k} \right)}{\Gamma_{k-N} \left( \hat{w}^m_{k-N-1|k} \right) + 1} \right)}{(l - 1)} \]

(23)

Taking \( \log(\cdot)^{-1} \) and solving for \( g(k,l) \),

\[
g(k,l) < \left( \frac{e\Psi_x^1 - \Gamma_{k-N} \left( \hat{w}^l_{k-N-1|k} \right)}{\Gamma_{k-N} \left( \hat{w}^m_{k-N-1|k} \right) + 1} \right)^{l-1} \frac{1}{l-1} - 1 < 1
\]

(24)

Selecting a value of \( l \) large enough, we can guarantee that the sequence \( g(k,i) \) is decreasing, i.e., the sequence of costs within the fixed-point iteration is de-
creasing. Moreover, solving for \( l \) in the second Inequality of (24),

\[
\frac{1}{l-1} \log \left( \frac{e\Psi_x^1 - \Gamma_{k-N} (\hat{w}_{k-N-1|k}^l) + 1}{\Gamma_{k-N} (\hat{w}_{k-N-1|k}^m)} \right) < \log (2)
\]

\[
\log_2 \left( \frac{e\Psi_x^1 - \Gamma_{k-N} (\hat{w}_{k-N-1|k}^l) + 1}{\Gamma_{k-N} (\hat{w}_{k-N-1|k}^m)} + 1 \right) + 1 < l
\]

(25)

An approximate (and conservative) value for Inequality (25) can be computed for \( l \) taking into account the worst scenario case.

\[
\lceil \log_2 (\mathcal{E} N (\bar{\gamma}_w (\|w\|) + \bar{\gamma}_v (\|v\|)) + 1) \rceil + 1 \leq l
\]

(26)

with \( \mathcal{E} := \epsilon / \Gamma_{k-N} (\hat{w}_{k-N-1|k}^m) \). Inequalities (25)-(26) allow to compute the required value of \( l \) to guarantee the costs decreasing within the fixed-point iteration.

**Remark 1** Note that for the noiseless case, only one iteration is needed after the transient due to the uncertainty in the initial condition has vanished.

5 Robust stability of moving horizon estimation under bounded disturbances and model uncertainty

In the previous section was shown that the sequence of cost decreases within the fixed-point iteration. At each sampling time, the model used by the estimator is replaced with the newly available. In this section, we will prove robust stability for the estimator under bounded disturbances and model uncertainty assuming that the system is \( i \)-IOSS. Moreover, if the length \( N \) of the horizon of the estimator is larger than a certain value \( N' \) that can be computed offline, the number of iterations \( l \) is chosen according to Equations (25)-(26), then, the effects of uncertainty in the initial condition vanish, as well as the disturbances due to model uncertainty. In the absence of process and measurement noises, states and model converges to the true ones.

**Definition 1** A system \( x_{k+1} = f(x_k, u_k, w_k), y_k = h(x_k) + v_k \) is incrementally input/output-to-state stable if there exist some functions \( \beta \in \mathcal{K}\mathcal{L} \) and \( \gamma_1, \gamma_2 \in \mathcal{K} \) such that for every two initial states \( z_1, z_2 \in \mathbb{R}^n \), and any two
disturbances sequences $\mathbf{w}_1, \mathbf{w}_2$ the following holds for all $k \in \mathbb{Z}_{\geq 0}$:

$$
|x(k, z_1, \mathbf{w}_1) - x(k, z_2, \mathbf{w}_2)| \leq \max \{ \beta (|z_1 - z_2|, k), \gamma_1 \left( \|\mathbf{w}_1 - \mathbf{w}_2\|_{[0,k-1]} \right), \\
\gamma_2 \left( \|\mathbf{v}_1 - \mathbf{v}_2\|_{[0,k-1]} \right) \}
$$

(27)

This definition is a notion of detectability (see Sontag & Wang (1997)) for nonlinear systems. It means that the vector state can be reconstructed from the input-output information. Furthermore, under the assumption that the plant belongs to a convex set, it can be approximated by a convex combination of a finite number of fixed models.

**Definition 2** Consider the system described by Equation (7) subject to disturbances $\mathbf{w} \in \mathcal{W}(w_{\text{max}})$ and $\mathbf{v} \in \mathcal{V}(v_{\text{max}})$ for $w_{\text{max}} \in \mathbb{R}_{\geq 0}$, $v_{\text{max}} \in \mathbb{R}_{\geq 0}$ with prior estimate $\bar{x}_0 \in \mathcal{X}(e_{\text{max}})$ for $e_{\text{max}} \in \mathbb{R}_{\geq 0}$. The moving horizon state estimator given by Equation (7) is robustly globally asymptotically stable (RGAS) if there exists functions $\Phi \in \mathcal{KL}$ and $\pi_w, \pi_v \in \mathcal{K}$ such that for all $x_0 \in \mathcal{X}$, all $\bar{x}_0 \in \mathcal{X}_0(e_{\text{max}})$, the following is satisfied for all $k \in \mathbb{Z}_{\geq 0}$

$$
|x_k - \hat{x}_k| \leq \Phi (|x_0 - \bar{x}_0|, k) + \pi_w \left( \|\mathbf{w}\|_{[0,k-1]} \right) + \pi_v \left( \|\mathbf{v}\|_{[0,k-1]} \right)
$$

(28)

We want to show that if system (5) is i-IOSS, Assumptions (1), (2) and (3) are fulfilled, then the proposed estimator with adaptive arrival cost is RGAS. Furthermore, if the process disturbance and measurement noise sequences are convergent (i.e., $\mathbf{w}, \mathbf{v} \in \mathcal{C}$), the estimation converges to the true state.

**Proof.**

We have stated that if $l$ is chosen according Equations (25)-(26), the following Inequality is verified,

$$
J^*_{x} \left( \hat{x}^*_{k-N|k} \{ \hat{\mathbf{w}}^* \} \right) < J^*_{x} \left( \hat{x}^*_{k-N|k} \{ \mathbf{w}^* \} \right)
$$

and, by definition of the function $g(k, l)$,

$$
J^*_{x} \left( \hat{x}^*_{k-N|k} \{ \hat{\mathbf{w}}^* \} \right) = \frac{J^*_{x} \left( \hat{x}^*_{k-N|k} \{ \mathbf{w}^* \} \right)}{g(k, l)}
$$

and due optimality,

$$
\frac{J^*_{x} \left( \hat{x}^*_{k-N|k} \{ \hat{\mathbf{w}}^* \} \right)}{g(k, l)} \leq J^1_{x} \left( x_{k-N} \{ \hat{\bar{\mathbf{w}}} \} \right)
$$

We are considering here disturbances due to model uncertainty as a noise
which is added to the process disturbances. The true process is affected only by process disturbances since the noise coming from model uncertainty appear only in the estimation procedure. However, differentiating process and noise due model uncertainty disturbances can be favorable in order to gain clarity in our proof.

\[
J_x^{(0)}(x_{k-N}, \{w\}) = J_x^{(0)}(x_{k-N}, \{w + dw\})
\]

\[
\therefore \frac{J_x^{(l)}(x_{k-N|k}, \{w^*(l)\})}{g_l(l)} \leq J_x^{(0)}(x_{k-N}, \{w + dw\})
\]

By means of Assumptions [1] and [2]

\[
\gamma_p \left( |\hat{x}_{k-N|k}^{(l)} - \bar{x}_{k-N}^{(l)}| \right) + N\gamma_w \left( |\hat{w}_{j|k}^{(l)}| \right) + N\gamma_v \left( \left| \hat{v}_{j|k}^{(l)} \right| \right) \leq g_l(l) \left( \gamma_p \left( |x_{k-N} - \bar{x}_{k-N}^{(0)}| \right) \right. \\
N\gamma_w \left( \left\| \hat{w}^{(0)} \right\| \right) + N\gamma_v \left( \left\| \hat{v}^{(0)} \right\| \right) \\
\leq g_l(l) \left( \gamma_p \left( |x_{k-N} - \bar{x}_{k-N}^{(0)}| \right) \right. \\
N \left( \gamma_w \left( 2 \left\| w \right\| \right) + \gamma_v \left( 2 \left\| v \right\| \right) \right) + \\
\left. \gamma_w \left( 2 \left\| dw \right\| \right) + \gamma_v \left( 2 \left\| dv \right\| \right) \right)
\]

Since the improvement in model estimation achieved at time \( k \) remains at time \( k+1 \), we can define a function \( G(g_0, k) : \mathbb{R} \times \mathbb{Z} \to \mathbb{R} \) which exhibits this improvement over time,

\[
G(g_0, k) := g_0 \prod_{j=0}^{k} g_l(j)
\]  

(29)

Note that \( G(g_0, k) \in \mathcal{H}\mathcal{L} \), and since the improvement in the model estimation can be stated as a convergence of sequences \( dw \) and \( dv \), one is able to write,

\[
\gamma_p \left( |\hat{x}_{k-N|k}^{(l)} - \bar{x}_{k-N}^{(l)}| \right) + N\gamma_w \left( |\hat{w}_{j|k}^{(l)}| \right) + N\gamma_v \left( \left| \hat{v}_{j|k}^{(l)} \right| \right) \leq \\
\gamma_p \left( |x_{k-N} - \bar{x}_{k-N}^{(0)}| \right) + N\gamma_w \left( \left\| w \right\| \right) + N\gamma_v \left( \left\| v \right\| \right) + \\
G(g_0, k) N \gamma_w \left( \left\| dw \right\| \right) + G(g_0, k) N \gamma_v \left( \left\| dv \right\| \right) 
\]

(30)

From Inequalities [1] and [30] and Assumption [1] one can find the followings bounds,
\[ |\hat{x}_{k-N}[k] - \bar{x}_{k-N}| \leq \gamma_p^{-1} \left( \tau_p \left( |x_{k-N} - \hat{x}^{(0)}_{k-N}| \right) + N\tau_w \left( 2 \|w\| \right) + N\gamma_v \left( 2 \|v\| \right) + NG \left( g_0, N \right) \tau_w \left( \|d^w\| \right) \right) \]

\[ |\hat{x}_{k-N}[k] - \bar{x}_{k-N}| \leq \frac{5}{|P_0|} \left( |P_{\infty}^{-1}| |x_{k-N} - \hat{x}^{(0)}_{k-N}| + N \frac{1}{2} \left( \tau_w \left( 2 \|w\| \right) \right)^{\frac{1}{2}} + \tau_v \left( 2 \|v\| \right)^{\frac{1}{2}} + G \left( g_0, N \right) \tau_w \left( 2 \|d^w\| \right)^{\frac{1}{2}} \right) \]

(31)

\[ |\hat{w}^{(l)}_{j}[k]| \leq \gamma_w^{-1} \left( \frac{5}{N} \tau_p \left( |x_{k-N} - \hat{x}^{(0)}_{k-N}| \right) \right) + \gamma^{-1} (5\tau_w \left( 2 \|w\| \right)) + \gamma_w^{-1} (5\tau_v \left( 2 \|v\| \right)) + \gamma_w^{-1} (5G \left( g_0, N \right) \tau_w \left( 2 \|d^w\| \right)) \]

(32)

\[ |\hat{v}^{(l)}_{j}[k]| \leq \gamma_v^{-1} \left( \frac{5}{N} \tau_p \left( |x_{k-N} - \hat{x}^{(0)}_{k-N}| \right) \right) + \gamma^{-1} (5\tau_w \left( 2 \|w\| \right)) + \gamma_v^{-1} (5\tau_v \left( 2 \|v\| \right)) + \gamma_v^{-1} (5G \left( g_0, N \right) \tau_v \left( 2 \|d^v\| \right)) \]

(33)

In order to proceed with a proof by construction, we have to assume that the system is \( i - IOSS \) (Definition 1). Under this assumption, we can state that the difference between the true trajectory of the system and the estimated one, always will be upper bounded by,

\[ |x_{k} - \hat{x}_{k}[k]| \leq \beta \left( |x_{0} - \hat{x}_{0}[k]|, k \right) + \gamma_1 \left( \|w - \bar{w}\|_{[0,k]} \right) + \gamma_2 \left( \|v - \bar{v}\|_{[0,k]} \right) \]

(34)

We need to find bounds for the three terms of Inequality 34 relating the variables and equations of our estimator. First let us find a bound for the first term of Inequality 34. Let us assume \( k = N \).

\[ \beta \left( |x_{k-N} - \hat{x}^{(l)}_{k-N}[k]|, N \right) \leq \beta \left( |x_{k-N} - \hat{x}_{k-N} + \bar{\hat{x}}_{k-N} - \hat{x}^{(l)}_{k-N}[k]|, N \right) \]

\[ \leq \beta \left( 2|x_{k-N} - \bar{\hat{x}}_{k-N}|, N \right) + \beta \left( 2|\hat{x}^{(l)}_{k-N}[k] - \bar{\hat{x}}_{k-N}|, N \right) \]

Taking in account Assumptions 1, 2 and 3 Inequality 1 and with some algebraic work, one can write,
\[
\beta \left( |x_{k-N} - \hat{x}_{k-N[k]}|, N \right) \leq \frac{|x_{k-N} - \bar{x}_{k-N}|^p}{N^q} \left( c_\beta 2^p + \frac{|P_\infty^{-1}|}{|P_0^{-1}|^p} c_\beta 10^p 5^p \right) + \frac{c_\beta 50^p}{|P_0^{-1}|^p} \left( \gamma_w (2 \| w \|)^p + \gamma_v (2 \| v \|)^p \right) \]

(35)

For the second term of Inequality (34), we will make use of the triangular inequality and the bound found in Inequality (33),

\[
\gamma_1 \left( \| w_j - \bar{w}_j[k] \|_{j \in \mathbb{Z}_{[k-N,k]}} \right) \leq \gamma_1 \left( \| w_j \|_{j \in \mathbb{Z}_{[k-N,k]}} + \| \bar{w}_j[k] \|_{j \in \mathbb{Z}_{[k-N,k]}} \right)
\]

Again, by mean of Assumptions 1, 2 and 3, Inequality (11), the bound for the difference between the sequences of real process and the estimated one disturbances can be stated as,

\[
\gamma_1 \left( \| w_j - \bar{w}_j[k] \|_{j \in \mathbb{Z}_{[k-N,k]}} \right) \leq \frac{c_\gamma 50^p}{N^q} |P_\infty^{-1}| \| x_{k-N} - \bar{x}_{k-N} \|^{\alpha_1} + c_1 5^\alpha \gamma_w (2 \| w \|)^{\alpha_1} + 
\gamma_1 \left( 5 \left( \| w \| + \gamma_w^{-1} (5 \gamma_w (2 \| w \|)) \right) \right) + 
c_1 5^\alpha G (g_0, N) \gamma_w (2 \| d_w \|) + c_1 5^\alpha G (g_0, N) \gamma_v (2 \| d_v \|)
\]

(36)

Analogously, the third term of Inequality (34) can be upper bounded as,

\[
\gamma_2 \left( \| v_j - \bar{v}_j[k] \|_{j \in \mathbb{Z}_{[k-N,k]}} \right) \leq \frac{c_\gamma 50^p}{N^q} |P_\infty^{-1}| \| x_{k-N} - \bar{x}_{k-N} \|^{\alpha_1} + c_1 5^\alpha \gamma_v (2 \| v \|)^{\alpha_1} + 
\gamma_1 \left( 5 \left( \| w \| + \gamma_w^{-1} (5 \gamma_w (2 \| w \|)) \right) \right) + 
c_1 5^\alpha G (g_0, N) \gamma_v (2 \| d_v \|) + c_1 5^\alpha G (g_0, N) \gamma_v (2 \| d_v \|)
\]

(37)

Now we are able to establish an upper bound for $|x_k - \hat{x}_{k|k}|$ with Inequalities (35), (36) and (37).
\[ |x_k - \hat{x}_{k|k}| \leq \frac{|x_N - \hat{x}_N|}{N^q} \left( c_\beta 2^p + \frac{|P_\infty - P_1|^2}{P_0^{-1}|a\beta|} c_\beta 10^p 5^\frac{p}{2} \right) + \frac{c_\beta 50^p}{|P_0^{-1}|} (\gamma_v (2 \|u\|)^p + \gamma_{\nu} (2 \|v\|)^p + c_\nu 50^p |P_\infty - P_1| G (g_0, N) (\gamma_v (d^w)^p + \gamma_{\nu} (d^v)^p)) + c_\nu 50^p |P_\infty - P_1| \frac{|x_k - N - \hat{x}_N|^a_\alpha + c_1 50^a_\alpha \gamma_v (2 \|u\|)^a_\alpha + \gamma_1 (5 (\|w\| + \gamma_v^{-1} (5 \gamma_v (2 \|u\|))) + c_1 50^a_\alpha G (g_0, N)^{a_\alpha} \gamma_v (2 \|d^w\|) + c_1 50^a_\alpha G (g_0, N)^{a_\alpha} \gamma_v (2 \|d^v\|) + c_2 50^a_\alpha |P_\infty - P_1| \frac{|x_k - N - \hat{x}_N|^a_\alpha + c_1 50^a_\alpha \gamma_v (2 \|u\|)^a_\alpha + \gamma_1 (5 (\|w\| + \gamma_v^{-1} (5 \gamma_v (2 \|u\|))) + c_1 50^a_\alpha G (g_0, N)^{a_\alpha} \gamma_v (2 \|d^w\|) + c_1 50^a_\alpha G (g_0, N)^{a_\alpha} \gamma_v (2 \|d^v\|) (38) \]

For the sake of clarity, we will define the following functions \( \Phi_w (r) \in \mathcal{K}_\infty \) and \( \Phi_v (r) \in \mathcal{K}_\infty \) and constants \( \mathcal{C}_\beta, \phi \) and \( \eta \),

\[
\mathcal{C}_\beta := \left( c_\beta 2^p + \frac{|P_\infty - P_1|^2}{P_0^{-1}|a\beta|} c_\beta 10^p 5^\frac{p}{2} \right) + c_1 50^a_\alpha |P_\infty|^{a_\alpha} + c_2 50^a_\alpha |P_\infty|^{a_\alpha_2} \\
\Phi_w (r) := \frac{c_1 50^p}{|P_0^{-1}|} \gamma_v (2 \|r\|) + \gamma_1 (5 (\|r\| + \gamma_v^{-1} (5 \gamma_v (2 \|r\|))) + c_2 50^a_\alpha \gamma_v (2 \|r\|) \\
\Phi_v (r) := \frac{c_1 50^p}{|P_0^{-1}|} \gamma_v (2 \|r\|) + \gamma_2 (5 (\|r\| + \gamma_v^{-1} (5 \gamma_v (2 \|r\|))) + c_1 50^a_\alpha \gamma_v (2 \|r\|) (39) \\
\Psi_{d^w} (r) := \frac{c_1 50^p}{|P_0^{-1}|} \gamma_v (2 \|r\|) + c_1 50^a_\alpha \gamma_v (2 \|r\|) + c_2 50^a_\alpha \gamma_v (2 \|r\|) \\
\Psi_{d^v} (r) := \frac{c_1 50^p}{|P_0^{-1}|} \gamma_v (2 \|r\|) + c_1 50^a_\alpha \gamma_v (2 \|r\|) + c_2 50^a_\alpha \gamma_v (2 \|r\|) \\
\phi := \max\{a\alpha_1, a\alpha_2, p\} \\
\eta := \min\{q, \alpha_1, \alpha_2\} \]

Therefore, Inequality (38) can be rewritten as,

\[
|x_k - \hat{x}_{k|k}| \leq \frac{|x_k - N - \hat{x}_N|}{N^q} \mathcal{C}_\beta + \Phi_w (w) + \Phi_v (v) + G (g_0, N) (\Psi_{d^w} (\|d^w\|) + \Psi_{d^v} (\|d^v\|)) (40) \]

Now we can define a function \( \bar{\beta}(r, s) \in \mathcal{K} \mathcal{L} \) as \( \bar{\beta}(r, s) := \frac{s^\phi \epsilon_0}{\alpha} \). Inequality (40) can be rewritten as,
\[ |x_k - \hat{x}_{k|k}| \leq \bar{\beta} (|x_{k-N} - \bar{x}_{k-N}|, N) + \Phi_w (w) + \Phi_v (v) + G (g_0, N) (\Psi_{dw} (\|d^w\|) + \Psi_{dv} (\|d^v\|)) \]  

(41)

Note that functions \( \Phi_w (w) \) and \( \Psi_{dw} (d^w) \) could be put together into a more general function, since \( \bar{\beta} = w + d^w \). The same applies for \( \Phi_v (v) \) and \( \Psi_{dv} (d^v) \) (or avoiding split it as done in former Equations). However, we prefer to distinguish between disturbances due process and model uncertainty in order to clarify how model uncertainty is mitigated in the fixed-point iteration.

Following the same procedure as in Muller (2016), function \( \bar{\beta} (r, s) \) can be extended to \( \bar{\beta} (r, 0) \) by mean of a redefinition of the function at \( s = 0 \). Therefore, we can define \( \bar{\beta} (r, 0) \geq k_\beta \bar{\beta} (r, 1) \), for any \( k \in \mathbb{R}_{\geq 1} \) satisfying the Inequality. Now, Inequality (41) holds for \( k = 0 \), and hence \( \forall k \in \mathbb{Z}_{[0,N]} \)

With a similar procedure as in Muller (2016), let us fix \( \mu > 0 \) and let \( r_{\text{max}} := \max \{ \frac{1}{2} \left( \bar{\beta} (e_{\text{max}}, 0) + \Phi (\|w\|) + \Phi (\|v\|) + \Psi_{dw} (\|d^w\|) + \Psi_{dv} (\|d^v\|) \right), (1 + \mu) (\Phi (\|w\|) + \Phi (\|v\|) + \Psi_{dw} (\|d^w\|) + \Psi_{dv} (\|d^v\|)) \} \). From function \( \bar{\beta} (r, s) \), we can calculate the minimum horizon size \( \mathcal{N} \) which guarantees a decreasing behavior of the disturbances due uncertainty in the initial condition. For each \( 0 \leq r \leq r_{\text{max}} \) and for \( k \geq N \), we have,

\[ \bar{\beta} (2r, N) \leq \frac{(2r)^{\phi_{\beta}}}{N_q} \]  

(42)

We desire that,

\[ \frac{(2r)^{\phi_{\beta}}}{N_q} \leq r \]  

(43)

Let denote as \( \mathcal{N} \) the minimum horizon size which satisfy Inequality (43),

\[ \left( \frac{2^{\phi_{\beta} - 1}}{N_q} \right)^{\frac{1}{\eta}} \leq \mathcal{N} \]  

(44)

From now to later \( N \geq \mathcal{N} \), we will assume that our estimator has a window length \( N \geq \mathcal{N} \). Let us consider the case when,

\[ |x_k - \hat{x}_{k|k}| \leq 2 (1 + \mu) (\Phi_w (\|w\|) + \Phi_v (\|v\|) + \Psi_{dw} (\|d^w\|) + \Psi_{dv} (\|d^v\|)) \]  

(45)

Recalling Inequalities (41) (for \( G (g_0, k) = 1 \)) and (43), we could write,
\[ |x_{k+N} - \hat{x}_{k+N}| \leq \bar{\beta} \left( |x_k - \hat{x}_{k+N}|, N \right) + \Phi_w(\|w\|) + \Phi_v(\|v\|) + \Psi_{d^w}(\|d^w\|) + \Psi_{d^v}(\|d^v\|) \]  

(46)

Using Inequality (45),

\[ |x_{k+N} - \hat{x}_{k+N}| \leq (2 + \mu) (\Phi_w(\|w\|) + \Phi_v(\|v\|) + \Psi_{d^w}(\|d^w\|) + \Psi_{d^v}(\|d^v\|)) \]  

(47)

On the other hand, when,

\[ 2 (1 + \mu) (\Phi_w(\|w\|) + \Phi_v(\|v\|) + \Psi_{d^w}(\|d^w\|) + \Psi_{d^v}(\|d^v\|)) < |x_k - \hat{x}_{k+N}| \leq 2r_{\max} \]  

(48)

Again, using Inequality (45),

\[ |x_{k+N} - \hat{x}_{k+N}| \leq \bar{\beta} \left( |x_k - \hat{x}_{k+N}|, N \right) + \Phi_w(\|w\|) + \Phi_v(\|v\|) + \Psi_{d^w}(\|d^w\|) + \Psi_{d^v}(\|d^v\|) \]  

\[ \leq \frac{|x_k - \hat{x}_{k+N}|}{2} + \frac{|x_k - \hat{x}_{k+N}|}{2(1 + \mu)} \]  

(49)

Note that \( \theta := \left( \frac{2 + \mu}{2(1 + \mu)} \right) < 1 \), exhibiting a contractive behavior when estimation error \( 2 (1 + \mu) (\Phi_w(\|w\|) + \Phi_v(\|v\|) + \Psi_{d^w}(\|d^w\|) + \Psi_{d^v}(\|d^v\|)) < |x_k - \hat{x}_{k+N}| \).

The cases studies up to this point were related with disturbances including both process and (additive) model uncertainty disturbances, assuming the worst scenario where disturbances due model uncertainty take their maxima value. However, taking advantage of the fixed-point iteration, we will see that choosing a proper value of \( l \) (number of iteration) which satisfies all Assump-
belonging to \( X \), i.e., we can define the Euclidean norm. The value of \( \delta_{\Omega} \) can be calculated by means of a Full Information Estimator, incorporating estimation that our estimator can achieve. In this sense, we could think about the computation of the CRLB for a particular case. As mentioned in (CRLB). The work in Taylor (1979) can be useful to give the reader an insight on how to compute this value.

\[
G(g_0, k) := g_0 \prod_{j=0}^{k} g_{\ell}(j) \\
\leq g_0 \prod_{j=0}^{k} \delta_{\Omega} \left( |\bar{z}_{j-N,\ell}^{(0)} - \bar{z}_{j-N,\ell}^{(i)}| \right) + N \left( \tau_{\omega} \left( \| \bar{w}^{(0)} \| \right) \right) + \delta_{\Omega} \left( \| \bar{z}^{(0)} \| \right)
\]

We can calculate the decreasing rate for this function every \( N \) samples. Let us define as before some variables denoting minimal and maximal values which will help us with calculations.

\[
\delta := \max \{|\bar{z}_{j-N,\ell}^{(0)} - \bar{z}_{j-N,\ell}^{(i)}|\}, \quad j \in Z_{[\ell, k]}
\]

\[
\Omega := \min \{|\bar{z}_{j-N,\ell}^{(i)} - \bar{z}_{j-N,\ell}^{(i)}|\}, \quad j \in Z_{[\ell, k]}
\]

The value of \( \Omega \) can be computed by mean of the Cramer-Rao Lower Bound (CRLB). The work in Taylor (1979) can be useful to give the reader an insight about the computation of the CRLB for a particular case. As mentioned in Taylor (1979), the CRLB define "the best that can be done", i.e., the best estimation that our estimator can achieve. In this sense, we could think that \( \Omega \) can be calculated by mean of a Full Information Estimator, incorporating the system constraints. However, until now, we do not have proof of it.

The value of \( \delta \) can be selected as the maximal distance between two points belonging to \( \mathcal{X} \) (Note that this can be done since \( \mathcal{X} \) is a topological space, i.e., we can define the Euclidean norm).

\[
\frac{G(g_0, k)}{G(g_0, k - \ell)} \leq g_0 \prod_{j=k - \ell}^{k} \left( \delta_{\Omega} \left( |\bar{z}_{j-N,\ell}^{(0)} - \bar{z}_{j-N,\ell}^{(i)}| \right) \right) + N \left( \tau_{\omega} \left( \| \bar{w}^{(0)} \| \right) \right) + \delta_{\Omega} \left( \| \bar{z}^{(0)} \| \right)
\]

\[
\frac{G(g_0, k)}{G(g_0, k - \ell)} \leq \left( \frac{G(g_0, k)}{G(g_0, k - \ell)} \right) \left( \tau_{\omega} \left( \| \bar{w}^{(0)} \| \right) \right) + \tau_{\omega} \left( \| \bar{z}^{(0)} \| \right)
\]

\[
\frac{G(g_0, k)}{G(g_0, k - \ell)} \leq g_0 \left( \delta_{\Omega} \left( |\bar{z}_{j-N,\ell}^{(0)} - \bar{z}_{j-N,\ell}^{(i)}| \right) \right) + N \left( \tau_{\omega} \left( \| \bar{w}^{(0)} \| \right) \right) + \delta_{\Omega} \left( \| \bar{z}^{(0)} \| \right)
\]

If we choose \( \epsilon > 1 \) we achieve a decreasing rate for he disturbances due model uncertainty. In order to accomplish that, we have to be able to increase \( l \) to the value,
\[
\frac{1}{g_0} \left( \frac{P_{x^{n+1}}}{P_{x^n}} \| \mathbf{z}(0) \| \right) e^{-\frac{1}{T_1}} \leq l \tag{53}
\]

**Remark 2** Note that at this point, we are able to state that the disturbances due initial condition and disturbances due model uncertainty are vanishing over time as long as the values of \( N \) and \( l \) are choose properly. Even though the bounds founds in this work can be conservative in the sense that they can be larger than expected, the steps which conduce us to overestimate some bounds are required to derive in a manageable fashion some mathematical expressions.

Now we are in condition to derive the last expression which probes Robust Global Asymptotic Stability (RGAS) for our estimator.

\[\square\]

6 **Examples**

The following examples will be used to illustrate the results presented in the previous sections and compare the performance of the estimators. In favor of simplicity, we will consider a system in companion form, and all states variables of the plant are inaccessible.

6.1 **Example 1: Linear System**

The first example considers the linear system with the following behavior

\[
A_p = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_p = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}
\]

\[
x_{k+1} = A_p x_k + B_p u_k + w_k
\]

\[
y_k = C_p x_k + v_k
\]

The fixed models are chosen as the vertex of a triangle with center at \((a_1, a_2)\) in the parameters space and a radius of \(6(S_w + S_v)\), where \(S_w\) and \(S_v\) are the process and measurement variance noises, with \(S_w = 0.1\), \(S_v = 0.05\) and \(\|u_k\| \leq 0.25\). The guess for the initial condition is \(\bar{x}_0 = [0, 0]^T\), whereas \(x_0 = \)
A nominal model is generated randomly as a convex combination of the fixed models. The nominal model is used to initialize the MHE estimator, and the model is updated at every sampling time. In order to compare performances, estimations are performed as well with a MHE which does not update the model and a Full Information Estimator which performs estimations with the true model $A_p$. The stage cost is chosen as $\ell(w, v) = w^T Q^{-1} w + v^T R^{-1} v$ with $Q^{-1} = \text{diag}(1e^3, 5e^3)$ and $R^{-1} = \text{diag}(5e^2, 5e^2)$. The prior weighting matrix is initialized as $P_0^{-1} = 1e^{-1} I_2$ and $P_k$ is updated using the algorithm developed in Sánchez et al. (2017) for $k \in \mathbb{Z}_{[1, \infty)}$, with $\sigma = 1$ and $c = 1e1$. Each point of Figure 1 represents an average of 100 trials for each value of $l$ and $N$. As expected, the performance of the FIE is better for this case.

![Graph](image)

**Fig. 1.** Performance of the proposed algorithm for different values of $l$ (Red: $N = 2$, Green: $N = 5$ and Blue, $N = 8$). The Black line represents the error for the FIE.

### 6.1.1 Example 3: Nonlinear system

As the second experiment, state estimation of a nonlinear time-varying system is carried out. The polyhedral is designed to guarantee that the nonlinear system always remains inside it. The system consists in a nonlinear part and a time-varying part, with the following behavior

$$x_{k+1} = \begin{bmatrix} x_k^2 \\ k_1^1 x_k^1 + k_2 \sin(x_k^2) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k + w_k$$

$$y_k = C_p x_k + v_k$$
For this case, $S_w = 0.1$, $S_v = 0.05$ and $\|u_k\| \leq 0.25$. The guess for the initial condition is $\bar{x}_0 = [0, 0]^T$, whereas $x_0 = [0.5, 0.3]^T$. Results of experiments are summarized in the Figure 2, where each point represent an average over 100 trials. Again, comparison between the proposed algorithm and linear FIE using as model the linearized version of the nonlinear time-varying system computed at time $k$ are performed. As can be seen from the Figure 2, the performance of the MHE with model updating is better even that of the FIE. It is due to the MHE estimates over a finite length windows, whereas the FIE attempt to estimates the trajectory from time $k = 0$ to the present with the model available at time $k$.

Fig. 2. Performance of the proposed algorithm for different values of $l$ (Red: $N = 2$, Green: $N = 5$ and Blue, $N = 8$). The Black line represent the error for the FIE.

7 Conclusions

In this work, we have proved that the fixed-point iteration improves significantly the estimation error, for either linear and nonlinear systems with bounded disturbances. At each sampling time, the adaption mechanism replaces the model of the system for a more accurate one, improving the state estimation. Equations derived in section 4 and 5 are supported by several simulations cases.
Acknowledgements

The authors wish to thank the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) from Argentina, for their support.
References

Ames, W. F. & Pachpatte, B. (1997), *Inequalities for differential and integral equations*, Vol. 197, Elsevier.

Anderson, B. D., Brinsmead, T. S., De Bruyne, F., Hespanha, J., Liberzon, D. & Morse, A. S. (2000), ‘Multiple model adaptive control. part 1: Finite controller coverings’, *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal* 10(11-12), 909–929.

Han, Z. & Narendra, K. S. (2012), ‘New concepts in adaptive control using multiple models’, *IEEE Transactions on Automatic Control* 57(1), 78–89.

Hassani, V., Aguiar, A. P., Athans, M. & Pascoal, A. M. (2009), Multiple model adaptive estimation and model identification using a minimum energy criterion, *in* ‘American Control Conference, 2009. ACC’09.’, IEEE, pp. 518–523.

Hespanha, J., Liberzon, D., Stephen Morse, A., Anderson, B. D., Brinsmead, T. S. & De Bruyne, F. (2001), ‘Multiple model adaptive control. part 2: switching’, *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal* 11(5), 479–496.

Holte, J. M. (2009), Discrete Gronwall lemma and applications, *in* ‘MAA-NCS meeting at the University of North Dakota’, Vol. 24, pp. 1–7.

Ji, L., Rawlings, J. B., Hu, W., Wynn, A. & Diehl, M. (2016), ‘Robust stability of moving horizon estimation under bounded disturbances’, *IEEE Transactions on Automatic Control* 61(11), 3509–3514.

Kuipers, M. & Ioannou, P. (2010), ‘Multiple model adaptive control with mixing’, *IEEE Transactions on Automatic Control* 55(8), 1822–1836.

Landau, I. D., Lozano, R., M’Saad, M. & Karimi, A. (1998), *Adaptive control*, Vol. 51, Springer London.

Muller, M. (2016), Nonlinear moving horizon estimation for systems with bounded disturbances, *in* ‘2016 American Control Conference (ACC), IEEE’, pp. 883–888.

Müller, M. A. (2017), ‘Nonlinear moving horizon estimation in the presence of bounded disturbances’, *Automatica* 79, 306–314.

Narendra, K. S. & Balakrishnan, J. (1997), ‘Adaptive control using multiple models’, *IEEE transactions on automatic control* 42(2), 171–187.

Narendra, K. S., Wang, Y. & Chen, W. (2014), Stability, robustness, and performance issues in second level adaptation, *in* ‘American Control Conference (ACC), 2014’, IEEE, pp. 2377–2382.

Narendra, K. S., Wang, Y. & Chen, W. (2015), ‘The rationale for second level adaptation’, *arXiv preprint arXiv:1510.04989*.

Narendra, K. S. & Xiang, C. (2000), ‘Adaptive control of discrete-time systems using multiple models’, *IEEE Transactions on Automatic Control* 45(9), 1669–1686.

Nguyen, H.-N. (2014), ‘Constrained control of uncertain, time-varying, discrete-time systems’, *An Interpolation-Based Approach (Cham: Springer)*.
Rao, C. V., Rawlings, J. B. & Mayne, D. Q. (2003), ‘Constrained state estimation for nonlinear discrete-time systems: Stability and moving horizon approximations’, *IEEE transactions on automatic control* 48(2), 246–258.

Sánchez, G., Murillo, M. & Giovanini, L. (2017), ‘Adaptive arrival cost update for improving moving horizon estimation performance’, *ISA transactions* 68, 54–62.

Sontag, E. D. & Wang, Y. (1997), ‘Output-to-state stability and detectability of nonlinear systems’, *Systems & Control Letters* 29(5), 279–290.

Taylor, J. (1979), ‘The cramer-rao estimation error lower bound computation for deterministic nonlinear systems’, *IEEE Transactions on Automatic Control* 24(2), 343–344.