A complete basis of nonlocal invariants in quantum gravity theory is built to third order in spacetime curvature and matter-field strengths. The nonlocal identities are obtained which reduce this basis for manifolds with dimensionality $2\omega < 6$. The present results are used in heat-kernel theory, theory of gauge fields and serve as a basis for the model-independent approach to quantum gravity and, in particular, for the study of nonlocal vacuum effects in the gravitational collapse problem.
I. INTRODUCTION

The concept of nonlocal invariant of a given order in the curvature (or, more generally, in the field strength) comes from quantum theory of gauge fields. The effective action of quantum operator fields is a nonlocal functional of the respective c-number fields which, for given quantum states, represent the matrix elements or expectation values.

For certain quantum states this functional can be expanded in powers of the deviation of the field argument from its value in a flat and empty spacetime, and the coefficients of this expansion are exact vertex functions of the theory. In the case of gauge fields, the effective action is an invariant functional, and the calculations can be done so that each term of the expansion is invariant. The expansion is then in powers of field strengths (curvatures) but its coefficients (the nonlocal form factors) are also field dependent. The meaning of such an expansion is that the effective action is obtained with accuracy $O[\mathcal{R}^N]$ i.e. up to $N$-th power in $\mathcal{R}$ where $\mathcal{R}$ is the collective notation for the full set of the field strengths of the theory. Each term of the expansion contains $N$ curvatures $\mathcal{R}$ explicitly and is defined up to $O[\mathcal{R}^{N+1}]$. By definition, such term is a nonlocal invariant of order $N$ in the curvature.

All invariants of order $N$ form a linear space. One still needs a specification of the class of nonlocal form factors which serve as coefficients of linear combinations in this space. Quantum field theory picks up a quite definite class which we consider below. With these specifications, we describe the general procedure of building the basis of nonlocal invariants of $N$-th order and build it explicitly to third order in the curvature inclusive.

The results obtained in this way are used in the model-independent approach to quantum gravity. The central object of this approach is the effective action for the in-vacuum state which is built by adopting the kinematic rules of quantum field theory. It is then a nonlocal functional of the class considered here. Owing to the properties of the state, it can be expanded in a basis of $N$-th order curvature invariants but the nonlocal form factors in this expansion are left unspecified. These form factors stand for the dynamical information contained in an unknown model of the vacuum. The purpose of the approach is to relate...
the properties of the spacetime solving the expectation-value equations to the properties of
the vacuum form factors. Thus, it has been shown that the requirement of the asymptotic
flatness of the solution leads to quite definite asymptotic behaviours of the form factors.

The physical problem which is considered within this approach and for which the nonlocal
vacuum effects play an important role is the gravitational collapse problem. The Hawking
radiation and its backreaction on the metric is one such effect. It has been shown that in four dimensions this effect starts with cubic in the curvature terms of the vacuum action. For this reason the explicit construction of the basis in the present paper extends to
the third order.

Another application that the present technique finds is in heat kernel theory. One needs
the heat kernel as an intermediate stage in calculating the effective action in quantum field
theory, but heat kernel is, of course, of self value. The heat kernel is a nonlocal functional
of the background fields contained in its Hamiltonian operator, and it can be calculated as
an expansion in nonlocal invariants of $N$-th order.

In the case of the in-vacuum state, the effective equations both in expectation-value
theory and transition-amplitude theory are obtained by certain rules from the effective
action of euclidean theory (see also Ref. 8). Therefore, in the present paper we confine
ourselves to the consideration of asymptotically flat euclidean spaces.

The basis of nonlocal invariants is built for an arbitrary dimension $2\omega$ but, for low-
dimensional manifolds, it is redundant because there exist hidden identities between the
nonlocal invariants of a given order. Most of the present paper is in fact devoted to a
systematic analysis of these identities. In the case of cubic invariants, the redundancy takes
place for $2\omega < 6$. For $2\omega = 4$, there exists a single nontrivial identity reducing the basis of
purely gravitational cubic invariants. As a by-product, we discover a mechanism by which
the Gauss-Bonnet invariant becomes topological in four dimensions.

The paper is organized as follows. In Sect.II we use the Bianchi identity to eliminate the
Riemann tensor in terms of the Ricci curvature. In Sects.III and IV we formulate the notion
of nonlocal curvature invariants with the Ricci tensor and build the basis of such invariants
to third order. Sect.V contains the construction of matter invariants and presents the full table of nonlocal invariants to third order in curvatures – gravitational and matter-field strengths. Finally, in Sect.VI we derive the hidden identities between local and nonlocal cubic invariants in low-dimensional manifolds.

II. ELIMINATION OF THE RIEMANN TENSOR

An important feature of nonlocal curvature invariants in asymptotically flat spacetime is that their purely gravitational strength boils down to the Ricci tensor, because the Riemann tensor can always be eliminated via the known corollary of the Bianchi identity

\[ \Box R^{\alpha \beta \mu \nu} \equiv \frac{1}{2} \left( \nabla^\mu \nabla^\alpha R^{\nu \beta} + \nabla^\alpha \nabla^\mu R^{\nu \beta} - \nabla^\nu \nabla^\alpha R^{\mu \beta} - \nabla^\alpha \nabla^\nu R^{\mu \beta} - \nabla^\mu \nabla^\beta R^{\nu \alpha} - \nabla^\beta \nabla^\mu R^{\nu \alpha} + \nabla^\nu \nabla^\beta R^{\mu \alpha} + \nabla^\beta \nabla^\nu R^{\mu \alpha} \right) 

+ R^{[\mu \lambda}] R^{\gamma \beta \lambda} - R^{[\mu \lambda]} R^{\gamma \beta \lambda} - 4 R^{[\lambda \mu \sigma]} R^{\alpha \beta \gamma \lambda} - R^{[\alpha \beta \sigma]} R^{\mu \nu \sigma \lambda}, \tag{2.1} \]

which can be iteratively solved for \( R^{\alpha \beta \mu \nu} \) in terms of \( R^{\mu \nu} \). In this equation the Ricci tensor plays the role of a source which determines the Riemann tensor up to initial or boundary conditions for the operator \( \Box \). In the case of positive-signature asymptotically flat spaces, which guarantee that a Ricci-flat space is flat, the iterational solution is uniquely determined by the Green function \( 1/\Box \) with zero boundary conditions at infinity. Up to the second order in \( R^{\mu \nu} \), necessary for the construction of the third order nonlocal invariants below, this solution is as follows

\[
R^{\alpha \beta \mu \nu} = \frac{1}{2} \left( \nabla^\mu \nabla^\alpha R^{\nu \beta} + \nabla^\alpha \nabla^\mu R^{\nu \beta} - \nabla^\nu \nabla^\alpha R^{\mu \beta} - \nabla^\alpha \nabla^\nu R^{\mu \beta} - \nabla^\mu \nabla^\beta R^{\nu \alpha} - \nabla^\beta \nabla^\mu R^{\nu \alpha} + \nabla^\nu \nabla^\beta R^{\mu \alpha} + \nabla^\beta \nabla^\nu R^{\mu \alpha} \right) 

+ \frac{1}{\Box} \left( 2 R^{\mu \lambda} (\nabla^\lambda \nabla^\alpha \frac{1}{\Box} R^{\nu \beta}) - 2 R^{\mu \lambda} (\nabla^\nu \nabla^\beta \frac{1}{\Box} R^{\lambda \alpha}) - 4 (\nabla^\lambda \nabla^\alpha \frac{1}{\Box} R^{\nu \beta}) (\nabla^\lambda \nabla^\alpha \frac{1}{\Box} R^{\nu \beta}) - 8 (\nabla^\lambda \nabla^\alpha \frac{1}{\Box} R^{\mu \lambda}) (\nabla^\lambda \nabla^\alpha \frac{1}{\Box} R^{\nu \beta}) 

+ 4 (\nabla^\lambda \nabla^\alpha \frac{1}{\Box} R^{\mu \lambda}) (\nabla^\sigma \nabla^{[\mu} \frac{1}{\Box} R^{\nu \beta]) - 4 (\nabla^\sigma \nabla^{[\mu} \frac{1}{\Box} R^{\lambda \alpha}) (\nabla^\lambda \nabla^\alpha \frac{1}{\Box} R^{\nu \beta}) 

+ 8 (\nabla^\lambda \nabla^\alpha \frac{1}{\Box} R^{[\mu \alpha})(\nabla^\lambda \nabla^\alpha \frac{1}{\Box} R^{\nu \beta}) + 8 (\nabla^\lambda \nabla^\alpha \frac{1}{\Box} R^{[\mu \alpha})(\nabla^\nu \nabla^\beta \frac{1}{\Box} R^{\lambda \sigma}) \right). 
\]
\[-4(\nabla_\lambda \nabla_\sigma \frac{1}{\Box} R^{\mu[\alpha]} (\nabla^\nu \nabla^\beta \frac{1}{\Box} R^{\lambda\sigma}) - 4(\nabla_\lambda \nabla_\sigma \frac{1}{\Box} R^{[\mu[\alpha]} (\nabla^\lambda \nabla^\sigma \frac{1}{\Box} R^{\nu]\beta]) \]
\[-4(\nabla^\nu \nabla^{[\alpha} \frac{1}{\Box} R_{\lambda\sigma]} (\nabla^\lambda \nabla^\beta \frac{1}{\Box} R^{\nu]}\beta) + \left[ (\alpha\beta) \leftrightarrow (\mu\nu) \right] \right\} + O[R^3], \quad (2.2)\]

where the antisymmetrizations on the right-hand side are with respect to \( \mu\nu \) and \( \alpha\beta \) and \( (\alpha\beta) \leftrightarrow (\mu\nu) \) denote the terms with the obvious permutation of pairs of indices which reproduces the symmetries of the Riemann tensor. Here we use a simplified notation for the Green functions introduced in Ref. For example, the quantity

\[ X^{\mu\nu} = \frac{1}{\Box} R^{\mu\nu} \quad (2.3) \]

denotes the solution of the tensor equation \( \Box X^{\mu\nu} = R^{\mu\nu} \) subject to zero boundary conditions at the infinity of the Euclidean asymptotically flat spacetime. The same convention applies also to other tensor quantities like \( \nabla^\mu \nabla^\alpha R_{\nu\beta} \), so that the tensor properties of the inverse box operator being determined by the tensor properties of the quantity which it acts on.

Note, that in case of the Lorentzian asymptotically flat spacetime Ricci-flat geometry is also flat, provided there is no incoming gravitational wave\[3\], and the equation of the above type also holds with \( 1/\Box \) replaced by the retarded Green’s function \( 1/\Box^{\text{ret}} \).

### III. NONLOCAL INVARIANTS OF N-TH ORDER WITH THE RICCI TENSOR

We shall start with the construction of purely gravitational invariants. In this case, since the Riemann tensor is eliminated, we are left with the Ricci tensor. A regular procedure of building the basis of nonlocal invariants referred to in the Introduction looks as follows.

To begin with, consider the diagrams of effective action in quantum field theory and assume (for the moment) that all propagators in these diagrams are massive. We can, then, expand them in inverse masses, in which case all nonlocal invariants forming the effective action will formally be expanded in an infinite series of local invariants

\[ \int dx \ g^{1/2} \sum_{n=0}^{\infty} c_n (\nabla_n \cdots \nabla_2) \underbrace{R \cdots R}_{N} + O[R^{N+1}]. \quad (3.1) \]
Here the covariant derivatives somehow acting on the product of $N$ Ricci tensors can be commuted freely because the contribution of their commutator is already $O[R^{N+1}]$. Among $(\nabla...\nabla)$ only a limited number ($\leq 2N$) of derivatives can have indices contracted with the indices of $R_{\bullet\bullet}R_{\bullet\bullet}...R_{\bullet\bullet}$, while the rest of the derivatives contract with each other to form boxes – covariant D’Alambertians acting on separate Ricci tensors or their pairs. This follows from a trivial identity

$$2\nabla_i\nabla_k = (\nabla_i + \nabla_k)^2 - \nabla_i^2 - \nabla_k^2 \equiv \Box_{i+k} - \Box_i - \Box_k$$  \hspace{1cm} (3.2)$$

where $\nabla_i$ denotes the covariant derivative acting on the $i$-th Ricci tensor in the product $R_{\bullet\bullet}...R_{\bullet\bullet} = R_{1\bullet\bullet}...R_{i\bullet\bullet}...R_{N\bullet\bullet}$, the same notation being assumed for covariant boxes $\Box_i$ and $\Box_{i+k}$ acting on the separate $i$-th Ricci tensor and the separate $ik$-th pair of those respectively.

The rearrangement of derivatives in (3.1) according to (3.2) obviously leads to the following general structure of $N$-th order nonlocal invariant

$$\int dx \, g^{1/2} \, F(\Box_1, ..., \Box_N, \Box_{1+2}, \Box_{1+3}, ...) (\nabla...\nabla) R_{1\bullet\bullet}R_{2\bullet\bullet}...R_{N\bullet\bullet},$$  \hspace{1cm} (3.3)$$

where $F(\Box_1, ..., \Box_N, \Box_{1+2}, \Box_{1+3}, ...)$ is an operatorial function of boxes accumulating the result of the infinite summation of derivatives in (3.1). As above the D’Alambertians $\Box_i$ and $\Box_{i+k}$ here are acting on the corresponding Ricci tensors or their pairs and, similarly to eq.(2.3), have the tensor properties determined by the tensor nature of the quantity which they act on. This operatorial function is a nonlocal formfactor which serves as a coefficient of the invariant $(\nabla...\nabla) R_{1\bullet\bullet}R_{2\bullet\bullet}...R_{N\bullet\bullet}$ – the member of the basis in the linear space of nonlocal invariants of order $N$ referred to in the Introduction.

In the first three orders these form factors take a simpler form due to the integration by parts – the property which, under the asymptotically flat boundary conditions providing the absence of surface terms, can be written as

$$\nabla_1 + \nabla_2 + ... \nabla_N = 0.$$  \hspace{1cm} (3.4)$$

For $N = 1$ this relation implies that the first-order formfactor is always a local constant, $F(\Box_1) = F(0) = \text{const.}$ The second-order form factor is a function of one box $\Box_1 = \Box_2 =$
\[ F(\Box_1, \Box_2) = f(\Box) \], and the third-order formfactor is a function of three boxes acting on three separate Ricci curvatures:

\[
F(\Box_1, \Box_2, \Box_3)(\nabla \cdots \nabla) R^1_{\Box\cdots\Box} R^2_{\Box\cdots\Box} R^3_{\Box\cdots\Box}.
\] (3.5)

The mixed-box arguments \( \Box_{1+2} \), \( \Box_{1+3} \), and \( \Box_{1+4} \) exactly correspond to the Mandelstam variables \( s = (p_1 + p_2)^2 \), \( t = (p_1 + p_3)^2 \) and \( u = (p_1 + p_4)^2 \) of the formfactor rewritten in the momentum representation.

Thus, the effective action analytic in spacetime curvature can be represented as a series in nonlocal invariants \( \text{(3.3)} \). It should be mentioned that certain caution is needed while working with this expansion. In particular, the operatorial arguments \( \Box_1, \Box_2, \ldots \) commute with each other because they act on different functions, but the mixed-box arguments \( \Box_{1+2}, \Box_{1+3}, \ldots \) do not commute with each other and with \( \Box_1, \Box_2, \ldots \). Therefore, beginning with the fourth order one must take trouble in ordering these arguments in a definite way.

Of course, we don’t assume that the form factors are expandable in any local series. We took this only for the purpose of building the basis of invariants. What we shall really assume about the form factors is their analytic properties. In quantum field theory we get the form factors in the Euclidean spacetime, i.e. for real negative \( \Box \)'s. We next assume that they can be analytically continued to the whole of the complex plane in each of the arguments, and that their singularities are only at real non-negative \( \Box \). This allows us to write and use the spectral representation which in the simplest case has the form

\[
F(\Box_1, \Box_2, \ldots) = \int_0^\infty dm_1^2 dm_2^2 \cdots \frac{\rho(m_1^2, m_2^2, \ldots)}{(m_1^2 - \Box_1)(m_2^2 - \Box_2) \cdots},
\] (3.6)

where all the information about non-local form factors is now contained in the spectral densities \( \rho(m_1^2, m_2^2, \ldots) \) and the only non-local structure that remains is contained in massive
Green’s functions. Generally this representation should be modified by subtraction terms accounting for a possible growth of the form factors at large boxes, while their growth at small arguments is severely limited by the requirement of the asymptotic flatness of the theory and provides the consistency of the representation at small energies.

The spectral forms constitute the basis of the model-independent approach to quantum gravity theory. They encode all the unknown information about the fundamental model in the set of spectral densities and have a number of advantages. In particular, they allow one to order the noncommuting arguments of the form factors, provide simple variational rules which, even for a nonlocal operator function of one self-commuting argument, would generally be a problem and make the equations of the theory manageable, since the only nonlocal ingredient that enters the formalism is the Green function. Most important, however, is that the spectral forms provide a technical means of imposing correct boundary conditions in nonlocal effective equations. As shown in these equations for expectation values of fields in the standard in-vacuum can be obtained by varying the Euclidean effective action with a subsequent replacement of nonlocal Euclidean form factors by their retarded Lorentzian analogues (see also Ref. for a similar recipe in the first order of the perturbation theory). This procedure boils down to using the retarded massive Green functions in the spectral representation of the form factors in effective equations for expectation values.

IV. THE BASIS OF GRAVITATIONAL INVARIANTS TO THIRD ORDER

Now we shall implement the procedure of the above type to build explicitly the basis of Ricci-tensor invariants to third order. To begin with, consider the first order $N = 1$ in which case the the invariant $\left( \nabla \ldots \nabla \right) R$, with derivatives not boiling down to a box, obviously reduces to the Ricci scalar. Due to vanishing of total derivative terms in asymptotically flat spacetime mentioned above (cf. eq.) its coefficient is always a pure numerical constant,
so that the first order invariant is
\[ \text{const} \times R, \quad R \equiv g^{\mu\nu} R_{\mu\nu} \]  
(4.1)

Here "const" is a simplest example of an unknown form factor. From experiment we know its value – it is the inverse Newton constant. Generally, similar considerations apply to all vacuum form factors, the experiment allowing us to extract a part of information about them and encode in the spectral densities discussed above. It should be emphasized that not all details of the form factors are equally important. In the example above it is only important to know that the Newton constant is positive. The above first-order invariant is what Einstein incidentally confined himself with. However, this is not the full effective action. To learn more about it we must proceed further.

In the second order the set of invariants \( (\nabla \ldots \nabla) R_{1\bullet\bullet} R_{2\bullet\bullet} \) with covariant derivatives not forming boxes simplifies to two structures
\[ R_{1\mu\nu} R_{2\mu\nu}^{\bullet\bullet}, \quad R_{1\bullet\bullet} R_{2\bullet\bullet}, \]  
(4.2)
on account of the contracted form of Bianchi identity \( \nabla^\mu R_{\mu\nu} = \frac{1}{2} \nabla_\nu R \) and the integration by parts. As was mentioned above, the coefficients of these structures in the effective action are already the nonlocal formfactors of one argument \( f(\Box_1) = f(\Box_2) \).

In the third order the full set of invariants \( (\nabla \ldots \nabla) R_{1\bullet\bullet\bullet} R_{2\bullet\bullet\bullet} R_{3\bullet\bullet\bullet} \) includes the structures with six, four, two and zero number of derivatives whose indices are contracted with the indices of Ricci tensors. Let us consequitively build all of them. For this purpose note the following general rule: in any structure with derivatives saturating the contraction of Ricci curvature indices the covariant derivative acting on one of Ricci curvatures can be contracted only with the index of another curvature, for otherwise the Bianchi identity generates the invariant with two derivatives contracted with one another, which in view of (3.2) should be absorbed into nonlocal formfactor and don’t enter the tensor invariant itself. The same rule together with integration by parts says that the following identification can be assumed in such invariants
\[ R_{1\mu\bullet} \nabla^\mu R_{2\bullet\bullet\bullet} R_{3\bullet\bullet\bullet} = - R_{1\mu\bullet} R_{2\bullet\bullet\bullet} \nabla^\mu R_{3\bullet\bullet\bullet} + (\ldots) \]  
(4.3)
modulo the box formfactor terms and higher orders of the curvature denoted by (...). From this it immediately follows that the only independent invariant with six derivatives can be cast in the form

\[\nabla_\alpha \nabla_\beta R^\gamma_1 \nabla_\gamma \nabla_\delta R^\mu_2 \nabla_\mu \nabla_\nu R^{\alpha\beta}_3. \quad (4.4)\]

In the invariant with four derivatives one pair of indices belonging to three Ricci curvatures must be contracted, the rest of them being contracted with indices of derivatives. There are two such contractions: \(R^\mu_1 R^\alpha_2 R^\beta_3\) and \(R^\alpha_1 R^\beta_2 R^{\mu\nu}_3\) which generate in view of (4.3) the following independent invariants with four derivatives

\[\nabla_\alpha \nabla_\beta R^\mu_1 \nabla_\mu R^{\alpha\beta}_2 \nabla_\nu R^{\mu\nu}_3, \quad \nabla_\mu R^\alpha_1 \nabla_\nu R^\beta_2 \nabla_\alpha \nabla_\beta R^{\mu\nu}_3 \quad (4.5)\]

(the independent invariants with \(\nabla_\alpha\) or \(\nabla_\beta\) acting respectively on \(R^\alpha_1\) or \(R^\beta_2\) above are ruled out by integration by parts which reduces them to the first of the structures (4.5)).

There are four tensors with two uncontracted indices \(R^\alpha_1 R^\beta_2 R^\mu_3, R^\nu_1 R^\alpha_2 R^\beta_3, R^\mu_1 R^\nu_2 R^\alpha_3, R^\mu_1 R^\nu_2 R^\alpha_3\) that give rise to the following invariants with two derivatives

\[R^\alpha_1 \nabla_\alpha R^\beta_2 \nabla_\beta R^\mu_3, \quad \nabla_\mu R^\alpha_1 \nabla_\nu R^\beta_2 \nabla_\alpha R^\nu_3, \quad R^\mu_1 \nabla_\mu R^\alpha_2 \nabla_\nu R^\beta_3, \quad R^\mu_1 \nabla_\nu R^\alpha_2 \nabla_\beta R^\nu_3. \quad (4.6)\]

Finally, there are three obvious invariants that contain no derivatives

\[R^\mu_1 R^\nu_2 R^\alpha_3, \quad R^\mu_1 R^\nu_2 R^\beta_3, \quad R^\mu_1 R^\nu_2 R^\nu_3. \quad (4.7)\]

It should be emphasized that the full set of cubic invariants follows from the above structures (4.4) - (4.7) by all possible permutations of three curvature labels 1, 2 and 3. But according to our notations in eqs. (3.3) and (3.5) such permutations in \((\nabla...\nabla) R_{1...2...3...}\) can always be replaced by the corresponding permutation of arguments in the form factor \(F(\Box_1, \Box_2, \Box_3)\), so that the independent set of invariants can be fixed once and for all with the enumeration of curvatures chosen above. In what follows we shall assume this rule which allows one to reduce the set of independent nonlocal invariants with the proper account for the permutation of boxes in their form factors.
V. THE BASIS OF MATTER INVARIANTS TO THIRD ORDER

The consideration of the above type applies also to matter field invariants provided that we know the full set of the field strengths of the theory. We shall consider a very general set of field strengths which appears in a class of field theories where the Hessian of the action, or the inverse propagator for the full collection of fields, can be written as

\[ g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \delta_{B}^{A} + \left( P_{B}^{A} - \frac{1}{6} R \delta_{B}^{A} \right). \]  

This operator is built of a covariant derivative \( \nabla_{\mu} \) and a potential term and is applied to an arbitrary set of fields \( \varphi^{B}(x) \), so that \( A \) and \( B \) stand for any set of discrete indices. The covariant derivative (with an arbitrary connection) is characterized by its commutator curvature

\[ (\nabla_{\mu} \nabla_{\nu} - \nabla_{\nu} \nabla_{\mu}) \varphi^{A} = R_{B \mu \nu} \varphi^{B}. \]  

The matrix \( P_{B}^{A} \) in the potential term of (5.1) is also arbitrary (the term \( \frac{1}{6} R \delta_{B}^{A} \) with the Ricci scalar is added for convenience), and terms of first order in derivatives are omitted since they can always be removed by a redefinition of the covariant derivative. In what follows we shall use a hat to indicate the matrix quantities acting in the vector space of \( \varphi^{A} \) and also denote the matrix trace operation in this space by tr:

\[ \delta_{B}^{A} = \hat{1}, \quad P_{B}^{A} = \hat{P}, \quad R_{B \mu \nu} = \hat{R}_{\mu \nu}, \text{ etc.} \]

\[ \text{tr} \hat{1} = \delta_{A}^{A}, \quad \text{tr} \hat{P} = P_{A}^{A}, \quad \text{tr} \hat{R}_{\mu \nu} = P_{B}^{A} R_{A \mu \nu}, \text{ etc.} \]  

Thus there are three independent inputs in the operator (5.1): \( g^{\mu\nu}, \nabla_{\mu} \) and \( \hat{P} \) - the metric contracting the second derivatives, the connection which defines the covariant derivative and the potential matrix. They may be regarded as background fields to which correspond their field strengths or curvatures. There is the Riemann curvature associated with \( g_{\mu\nu} \), the commutator curvature (5.2) associated with \( \nabla_{\mu} \) and the potential \( \hat{P} \) which is its own "curvature". Since the Riemann tensor was eliminated above in terms of the Ricci curvature,
the full set of field strengths characterizing the operator (5.1), for which we shall use the collective notation \( \mathcal{R} \), includes

\[
\mathcal{R} = (R_{\mu\nu}, \hat{R}_{\mu\nu}, \hat{P}).
\] (5.4)

Purely gravitational invariants to third order in the Ricci tensor have been constructed above, so let us consider now the full set of invariants involving the matter field strengths \( \hat{R}_{\mu\nu} \) and \( \hat{P} \) to the cubic order in \( \mathcal{R} \).

In the first order the matter field strengths can contribute only one invariant which, for the same reasons as in the gravitational case, can enter the effective action of the theory only with a local purely numerical coefficient:

\[
\text{const} \times \text{tr} \hat{P}
\] (5.5)

In the second order, modulo total derivative, there are only two invariants of the form \( (\nabla \cdots \nabla)\hat{R}_{1\bullet \bullet} \hat{R}_{2\bullet \bullet} \) in which derivatives do not form boxes or commutators leading to extra power of \( \mathcal{R} \): \( \text{tr} \hat{R}_{1\nu} \hat{R}_{2\mu} \) and \( \text{tr} \nabla^\nu \nabla_\alpha \hat{R}_{1\mu} \hat{R}_{2\nu}^{\alpha \nu} \). However, the cyclic Jacobi identity for the commutator curvature

\[
\nabla_\alpha \hat{R}_{\mu\nu} + \nabla_\mu \hat{R}_{\nu\alpha} + \nabla_\nu \hat{R}_{\alpha\mu} = 0
\] (5.6)

leads to the relation

\[
\nabla_\alpha \hat{R}_{\mu\nu} \hat{R}_{\alpha\nu} = \frac{1}{2} \nabla_\mu \hat{R}_{\nu\alpha} \hat{R}_{\nu\alpha},
\] (5.7)

which reduces the second invariant up to a box coefficient to the first one: \( \text{tr} \hat{R}_{1\mu} \hat{R}_{2\nu}^{\mu\nu} \). The rest of the second-order invariants include \( \text{tr} \hat{P}_1 \hat{P}_2 \), \( \text{tr} \hat{P}_1 R_2 \), because in view of integration by parts and the contracted Bianchi identity they also exhausts the other possible mixed gravity-matter structure \( \text{tr} R_1^{\mu\nu} \nabla_\mu \nabla_\nu \hat{P}_2 \). Thus, together with (4.2), the basis of structures forming all quadratic invariants consists of

\[
\mathcal{R}_1 \mathcal{R}_2(1) = R_{1\mu\nu} R_{2\mu\nu}^{\nu\mu},
\]
\[
\mathcal{R}_1 \mathcal{R}_2(2) = R_1 R_2^{\nu\mu} \hat{1},
\]

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\[ \mathcal{R}_1 \mathcal{R}_2(3) = \hat{P}_1 R_2, \]
\[ \mathcal{R}_1 \mathcal{R}_2(4) = \hat{P}_1 \hat{P}_2, \]
\[ \mathcal{R}_1 \mathcal{R}_2(5) = \hat{\mathcal{R}}_{1\mu\nu} \hat{\mathcal{R}}_{2}^{\mu\nu}. \]  

(5.8)

The third order is much richer for it includes all possible cubic combinations of \( R_{\mu\nu}, \hat{R}_{\mu\nu} \) and \( \hat{P} \). Let us first consider \( \hat{P}^3 \) and \( \hat{R}^{3\bullet} \) invariants. The single \( \hat{P}^3 \) invariant is trivial \( \text{tr} \hat{P}_1 \hat{P}_2 \hat{P}_3 \), while the whole hierarchy of structures

\[ \sum_{\leq 6} (\nabla \ldots \nabla) \hat{\mathcal{R}}_{1\bullet} \hat{\mathcal{R}}_{2\bullet} \hat{\mathcal{R}}_{3\bullet} \]

includes apriori the invariants with 6, 4, 2 and zero number of derivatives. However, the Jacobi identity \((5.6)\) essentially reduces their number. Consider first the invariant with six derivatives. Up to total derivative terms and commutators of \( \nabla \)'s it can be represented as

\[ \text{tr} \nabla_{\alpha} \hat{J}_{1\nu} \nabla_{\beta} \hat{J}_{2\nu} \nabla_{\gamma} \hat{J}_{3\mu} \]

in terms of the transverse vector

\[ \hat{J}^\alpha = \nabla_\lambda \hat{R}^{\alpha\lambda}, \quad \nabla_\alpha \hat{J}^\alpha = 0. \]

(5.9)

In view of \((5.8)\) this vector satisfies the relation

\[ \nabla_\mu \hat{J}^\alpha = \nabla^\alpha \hat{J}_\mu - \Box \hat{R}^\alpha_{\cdot \cdot \mu} + O[\mathcal{R}^2], \]

(5.10)

which means that the above cubic invariant with six derivatives not forming boxes is absent. The set of invariants with four derivatives again reduces to the only one structure \( \text{tr} \nabla_{\alpha} \hat{J}_{1\nu} \nabla_{\beta} \hat{J}_{2\nu} \nabla_{\gamma} \hat{J}_{3\mu} \) which does not straightforwardly disappear (up to irrelevant terms) due to eqs.\((5.6)\) and \((5.10)\). However, in view of \((5.10)\) one can rewrite this structure as \( \text{tr} \nabla_{\mu} \hat{J}_{1\nu} \nabla_{\nu} (\hat{J}_{2\mu} \hat{J}_{3\nu}) \) which disappears up to the total derivative term in view of the transversality of \((5.9)\). Thus, there is no invariant with four derivatives either. The invariants with two derivatives originate from all possible differentiations of the following two structures \( \text{tr} \hat{R}^{\mu\nu}_{1\bullet} \hat{R}^{\alpha\beta}_{2\bullet} \hat{R}^{3\alpha\beta}_{3\bullet} \) and \( \text{tr} \hat{R}^{\alpha\beta}_{1\bullet} \hat{R}^{\mu\nu}_{2\bullet} \hat{R}^{3\mu\nu}_{3\bullet} \) and, by integration by parts and use of the Jacobi identity, boil down to \( \text{tr} \hat{R}^{\mu\nu}_{1\bullet} \nabla_{\mu} \hat{R}^{\alpha\beta}_{2\bullet} \nabla_{\nu} \hat{R}^{3\alpha\beta}_{3\bullet} \) and \( \text{tr} \hat{R}^{\alpha\beta}_{1\bullet} \hat{J}_{2\alpha} \hat{J}_{3\beta} \). But the first of these invariants reduces to the second one by the following sequence of transformations. First integrate it by parts and use \((5.7)\) to convert it to the form \( 2 \text{tr} \hat{J}^\mu_1 \hat{R}^{\alpha\beta}_{2\bullet} \nabla_{\alpha} \hat{R}^{3\alpha\beta}_{3\bullet} \). Then
integrate by parts again and use \(5.10\) to interchange the indices in \(\nabla_\alpha \hat{J}_i^\nu\). The sequence of a new integration by parts and the use of \(5.7\) eventually leads to the equation

\[
\text{tr} \hat{R}_1^{\mu
u} \nabla_\mu \hat{R}_2^{\alpha\beta} \nabla_\nu \hat{R}_3^{\alpha\beta} - 2 \text{tr} \hat{R}_3^{\alpha\beta} \hat{J}_1 \hat{J}_2^\beta + 2 \text{tr} \hat{R}_2^{\alpha\beta} \hat{J}_3 \hat{J}_1^\beta + 2 \text{tr} \hat{R}_1^{\alpha
u} \hat{R}_2^{\beta\nu} \hat{R}_3^{\alpha\beta} - \text{tr} \hat{R}_1^{\mu
u} \nabla_\mu \hat{R}_2^{\alpha\beta} \nabla_\nu \hat{R}_3^{\alpha\beta} + O[\Re^4] + \text{a total derivative},
\]

which can be easily solved for \(\text{tr} \hat{R}_1^{\mu
u} \nabla_\mu \hat{R}_2^{\alpha\beta} \nabla_\nu \hat{R}_3^{\alpha\beta}\) in terms of the invariant \(\text{tr} \hat{R}_1^{\alpha\beta} \nabla_\mu \hat{R}_2^{\alpha\beta} \nabla_\nu \hat{R}_3^{\alpha\beta}\) and the only possible \(\Re^3\)-invariant without derivatives \(\text{tr} \hat{R}_1^{\alpha
u} \hat{R}_2^{\beta\nu} \hat{R}_3^{\alpha\beta}\) (under a proper permutation of curvature labels, cf. discussion at the end of Sect.IV).

A similar technique can be applied for a construction of all the rest gravity-matter invariants. Again their number is essentially reduced by using the identities of the above type. Here are some examples of such a reduction\(^4\) which we present without derivation starting with a corollary of \(5.11\)

\[
\text{tr} \hat{R}_1^{\alpha\beta} \nabla_\alpha \hat{R}_2^{\mu\nu} \nabla_\beta \hat{R}_3^{\mu\nu} = \text{tr} \left( \Box_1 \hat{R}_1^{\mu\nu} \hat{R}_2^{\alpha\beta} \hat{R}_3^{\mu\nu} \right) + O[\Re^4] + \text{a total derivative},
\]

\[
\text{tr} \nabla_\alpha \hat{R}_1^{\beta\nu} \nabla_\beta \hat{R}_2^{\alpha\mu} \nabla_\gamma \hat{R}_3^{\mu\nu} = -\frac{1}{2} \text{tr} \left( \Box_1 \hat{R}_1^{\beta\nu} \nabla_\mu \hat{R}_2^{\alpha\mu} \nabla_\nu \hat{R}_3^{\gamma\beta} \right) + O[\Re^4] + \text{a total derivative},
\]

\[
\text{tr} \nabla_\alpha \hat{R}_1^{\beta\nu} \nabla_\beta \hat{R}_2^{\alpha\mu} \nabla_\gamma \hat{R}_3^{\mu\nu} = -\frac{1}{2} \text{tr} \left( \Box_1 \hat{R}_1^{\beta\nu} \nabla_\mu \hat{R}_2^{\alpha\mu} \nabla_\nu \hat{R}_3^{\gamma\beta} \right) + O[\Re^4] + \text{a total derivative},
\]

\[
\text{tr} \nabla_\alpha \hat{R}_1^{\beta\nu} \nabla_\beta \hat{R}_2^{\alpha\mu} \nabla_\gamma \hat{R}_3^{\mu\nu} = -\frac{1}{2} \text{tr} \left( \Box_1 \hat{R}_1^{\beta\nu} \nabla_\mu \hat{R}_2^{\alpha\mu} \nabla_\nu \hat{R}_3^{\gamma\beta} \right) + O[\Re^4] + \text{a total derivative},
\]

\[
\text{tr} \nabla_\alpha \hat{R}_1^{\beta\nu} \nabla_\beta \hat{R}_2^{\alpha\mu} \nabla_\gamma \hat{R}_3^{\mu\nu} = -\frac{1}{2} \text{tr} \left( \Box_1 \hat{R}_1^{\beta\nu} \nabla_\mu \hat{R}_2^{\alpha\mu} \nabla_\nu \hat{R}_3^{\gamma\beta} \right) + O[\Re^4] + \text{a total derivative},
\]

\[
\text{tr} \nabla_\alpha \hat{R}_1^{\beta\nu} \nabla_\beta \hat{R}_2^{\alpha\mu} \nabla_\gamma \hat{R}_3^{\mu\nu} = -\frac{1}{2} \text{tr} \left( \Box_1 \hat{R}_1^{\beta\nu} \nabla_\mu \hat{R}_2^{\alpha\mu} \nabla_\nu \hat{R}_3^{\gamma\beta} \right) + O[\Re^4] + \text{a total derivative},
\]

\[
\text{tr} \nabla_\alpha \hat{R}_1^{\beta\nu} \nabla_\beta \hat{R}_2^{\alpha\mu} \nabla_\gamma \hat{R}_3^{\mu\nu} = -\frac{1}{2} \text{tr} \left( \Box_1 \hat{R}_1^{\beta\nu} \nabla_\mu \hat{R}_2^{\alpha\mu} \nabla_\nu \hat{R}_3^{\gamma\beta} \right) + O[\Re^4] + \text{a total derivative},
\]

\[
\text{tr} \nabla_\alpha \hat{R}_1^{\beta\nu} \nabla_\beta \hat{R}_2^{\alpha\mu} \nabla_\gamma \hat{R}_3^{\mu\nu} = -\frac{1}{2} \text{tr} \left( \Box_1 \hat{R}_1^{\beta\nu} \nabla_\mu \hat{R}_2^{\alpha\mu} \nabla_\nu \hat{R}_3^{\gamma\beta} \right) + O[\Re^4] + \text{a total derivative},
\]

\[
\text{tr} \nabla_\alpha \hat{R}_1^{\beta\nu} \nabla_\beta \hat{R}_2^{\alpha\mu} \nabla_\gamma \hat{R}_3^{\mu\nu} = -\frac{1}{2} \text{tr} \left( \Box_1 \hat{R}_1^{\beta\nu} \nabla_\mu \hat{R}_2^{\alpha\mu} \nabla_\nu \hat{R}_3^{\gamma\beta} \right) + O[\Re^4] + \text{a total derivative},
\]

\[
\text{tr} \nabla_\alpha \hat{R}_1^{\beta\nu} \nabla_\beta \hat{R}_2^{\alpha\mu} \nabla_\gamma \hat{R}_3^{\mu\nu} = -\frac{1}{2} \text{tr} \left( \Box_1 \hat{R}_1^{\beta\nu} \nabla_\mu \hat{R}_2^{\alpha\mu} \nabla_\nu \hat{R}_3^{\gamma\beta} \right) + O[\Re^4] + \text{a total derivative},
\]
As a result of such a reduction we have for the full basis of gravitational and matter invariants eleven structures without derivatives

\[ R_1 R_2 R_3(1) = \hat{P}_1 \hat{P}_2 \hat{P}_3, \]
\[ R_1 R_2 R_3(2) = \hat{R}_1^{\mu} \hat{R}_2^{\alpha} \hat{R}_3^{\beta} \hat{R}, \]
\[ R_1 R_2 R_3(3) = \hat{R}_1^{\mu \nu} \hat{R}_2_{\mu \nu} \hat{P}_3, \]
\[ R_1 R_2 R_3(4) = R_1 R_2 \hat{P}_3, \]
\[ R_1 R_2 R_3(5) = R_1^{\mu \nu} R_2_{\mu \nu} \hat{P}_3, \]
\[ R_1 R_2 R_3(6) = \hat{P}_1 \hat{P}_2 R_3, \]
\[ R_1 R_2 R_3(7) = R_1 R_2^{\mu \nu} \hat{R}_3_{\mu \nu}, \]
\[ R_1 R_2 R_3(8) = R_1^{\alpha \beta} \hat{R}_2^{\alpha} \hat{R}_3^{\beta} \hat{R}, \]
\[ R_1 R_2 R_3(9) = R_1 R_2 R_3 \hat{1}, \]
\[ R_1 R_2 R_3(10) = R_1^{\mu} R_2^{\alpha} R_3^{\beta} \hat{1}, \]
\[ R_1 R_2 R_3(11) = R_1^{\mu \nu} R_2_{\mu \nu} R_3 \hat{1}, \]

(5.17)

fourteen structures with two derivatives

\[ R_1 R_2 R_3(12) = \hat{R}_1^{\alpha \beta} \nabla^{\mu} \hat{R}_2_{\mu \alpha} \nabla^{\nu} \hat{R}_3_{\nu \beta}, \]
\[ R_1 R_2 R_3(13) = \hat{R}_1^{\mu} \nabla_\mu \hat{P}_2 \nabla_\nu \hat{P}_3, \]
\[ R_1 R_2 R_3(14) = \nabla_\mu \hat{R}_1^{\mu \alpha} \nabla_\nu \hat{R}_2_{\nu \alpha} \hat{P}_3, \]
\[ R_1 R_2 R_3(15) = \hat{R}_1^{\mu \nu} \nabla_\mu R_2 \nabla_\nu \hat{P}_3, \]
\[ R_1 R_2 R_3(16) = \nabla^{\mu \nu} \hat{P}_1 \nabla_\nu R_2_{\mu \alpha} \hat{P}_3, \]
\[ R_1 R_2 R_3(17) = \hat{R}_1^{\mu} \nabla_\mu \nabla_\nu \hat{P}_2 \hat{P}_3, \]
\[ R_1 R_2 R_3(18) = R_1^{\alpha \beta} \nabla_\mu \hat{R}_2^{\mu \alpha} \nabla_\nu \hat{R}_3^{\nu \beta}, \]
\[ R_1 R_2 R_3(19) = R_1^{\alpha \beta} \nabla_\alpha \hat{R}_2^{\mu \nu} \nabla_\beta \hat{R}_3_{\mu \nu}, \]
\[ R_1 R_2 R_3(20) = R_1 \nabla_\alpha \hat{R}_2^{\alpha \mu} \nabla_\beta \hat{R}_3^{\beta \mu}, \]
\[ R_1 R_2 R_3(21) = R_1^{\mu \nu} \nabla_\mu \nabla_\nu \hat{R}_2^{\lambda \alpha} \hat{R}_3^{\lambda \alpha \nu}, \]
\begin{align*}
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (22) &= R_1^{\alpha \beta} \nabla_\alpha R_2 \nabla_\beta R_3, \\
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (23) &= \nabla^\mu R_1^{\alpha \mu} \nabla_\nu R_2 \nabla_\mu R_3, \\
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (24) &= R_1^{\mu \nu} \nabla_\mu R_2^{\alpha \beta} \nabla_\nu R_3 \nabla_\alpha \nabla_\beta, \\
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (25) &= R_1^{\mu \nu} \nabla_\alpha R_2 \nabla_\mu \nabla_\beta R_3, \\
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (26) &= \nabla_\alpha \nabla_\beta R_1^{\mu \nu} \nabla_\mu \nabla_\nu R_3, \\
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (27) &= \nabla_\alpha \nabla_\beta R_1^{\mu \nu} \nabla_\mu \nabla_\nu R_2 \nabla_\alpha \nabla_\beta R_3, \\
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (28) &= \nabla_\mu R_1^{\alpha \lambda} \nabla_\nu R_2 \nabla_\beta \nabla_\alpha \nabla_\beta R_3, \\
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (29) &= \nabla_\alpha \nabla_\beta R_1^{\mu \nu} \nabla_\mu \nabla_\nu R_2 \nabla_\alpha \nabla_\beta R_3,
\end{align*}

three structures with four derivatives

\begin{align*}
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (30) &= R_1^{\mu \alpha} R_2^{\alpha \beta} \nabla_\beta R_3, \\
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (31) &= \nabla_\mu \nabla_\nu R_1 \nabla_\alpha \nabla_\beta R_3, \\
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (32) &= \nabla_\beta \nabla_\nu R_1^{\alpha \beta} \nabla_\alpha R_3, \\
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (33) &= \nabla_\mu \nabla_\nu R_1 \nabla_\alpha \nabla_\beta R_3, \\
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (34) &= \nabla_\mu \nabla_\nu R_1 \nabla_\alpha \nabla_\beta R_3, \\
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (35) &= \nabla_\mu \nabla_\nu R_1 \nabla_\alpha \nabla_\beta R_3, \\
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (36) &= \nabla_\beta \nabla_\nu R_1^{\alpha \beta} \nabla_\alpha R_3, \\
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (37) &= \nabla_\mu \nabla_\nu R_1 \nabla_\alpha \nabla_\beta R_3, \\
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 (38) &= \nabla_\alpha \nabla_\beta R_1 \nabla_\mu \nabla_\nu R_2 \nabla_\alpha \nabla_\beta R_3.
\end{align*}

(5.18)

Thus, the structures (4.1), (5.5), (5.8) and (5.17) - (5.21) form a complete basis of non-local invariants to third order in the curvature. The background-field functionals belonging to the class of invariants discussed in Introduction have, in the notations of Sect.III, the
expansion of the form

$$\int dx \, g^{1/2} \tr \left\{ \text{const} \, \hat{R} \hat{1} + \text{const} \, \hat{P} + \sum_{i=1}^{5} f_i(\Box_2) \, \mathcal{R}_1 \mathcal{R}_2(i) \\
+ \sum_{i=1}^{38} F_i(\Box_1, \Box_2, \Box_3) \, \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i) + O[\mathcal{R}^4] \right\}. \quad (5.22)$$

In the third order, which is a matter of our prime interest here, there are thirty eight invariants admissible by the requirements of covariance and asymptotic flatness. (Ten of them are purely gravitational, and with gravity switched off there are six.) In the field-theoretic calculations, however, the last nine structures have a special status which makes us to present them in a separate list (5.21). As shown in 14, the trace of the heat kernel for the operator (5.1) and the corresponding one-loop effective action do not contain these structures because their form factors $F_i(\Box_1, \Box_2, \Box_3)$, $i = 30, \ldots, 38$, either identically vanish or have such symmetries under the permutation of the box arguments that make their contribution vanishing in view of the symmetries of the structures themselves. For example, the structure 36 is antisymmetric in labels 2 and 3,

$$\nabla_\beta \hat{R}_1^{\beta \alpha} \nabla_\alpha R_2 R_3 = -\nabla_\beta \hat{R}_1^{\beta \alpha} \nabla_\alpha R_3 R_2 + \text{a total derivative},$$

while its heat-kernel form factor is symmetric 14, $F_{36}(\Box_1, \Box_2, \Box_3) = F_{36}(\Box_1, \Box_3, \Box_2)$. The same symmetry of the heat-kernel form factor $F_{13}(\Box_1, \Box_2, \Box_3)$ does not, however, annihilate the contribution of the structure $\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3$ (13) linear in $\hat{R}_{\mu \nu}$, because all of its three curvatures are matrices, and its antisymmetrization in the labels 2 and 3 is proportional to the nonvanishing commutator $[\nabla_\mu \hat{P}, \nabla_\nu \hat{P}]$ (see Refs. 14,23).

VI. HIDDEN IDENTITIES BETWEEN NONLOCAL CUBIC INVARIANTS

For low-dimensional manifolds the basis built above may be redundant. Thus, in the two-dimensional case $2\omega = 2$, because of the identity $R_{\mu \nu} = \frac{1}{2} g_{\mu \nu} R$, there is just one purely gravitational nonlocal structure at each order

$$\int dx \, g^{1/2} \, F(\Box_1, \ldots, \Box_N, \Box_{1+2}, \Box_{1+3}, \ldots) \, R_1 R_2 \ldots R_N, \quad (6.1)$$

and, moreover, at first order there is none because the Ricci scalar is a total derivative. For $2\omega \geq 2$ the first- and second-order bases are already irreducible but the third-order basis is
not. To show this, we begin the analysis of cubic invariants with the local identities for a tensor possessing the symmetries of the Weyl tensor:

\[
C_{\alpha\beta\gamma\delta} = C_{[\alpha\beta][\gamma\delta]} = C_{\alpha\beta[\gamma\delta]} = C_{\gamma\delta\alpha\beta},
\]

(6.2)

\[
C_{\alpha\beta\gamma\delta} + C_{\alpha\delta\beta\gamma} + C_{\alpha\gamma\delta\beta} = 0, \quad C^\alpha_{\beta\alpha\delta} = 0.
\]

(6.3)

Here the Ricci identity (6.3) can be rewritten as

\[
C_{\alpha[\beta\gamma]\delta} = -\frac{1}{2}C_{\alpha\delta\beta\gamma}
\]

(6.4)

and, with the use of (6.2), as

\[
C_{\alpha[\beta\gamma\delta]} = 0
\]

(6.5)

where the complete antisymmetrization in three indices is meant. Eq. (6.4) is useful when forming contractions because it shows that the contractions of the form \(C^\cdot\alpha\beta C_{\cdot\cdot\alpha\beta}\) and \(C^\cdot\cdot\alpha\beta C_{\cdot\alpha\beta}\) express through one another.

In view of applications to the gravitational equations, we first list all possible cubic contractions having two free indices. The symmetries above allow only

\[
J^\nu_{1\mu} = C_{\mu\beta\gamma\delta}C^{\nu\beta\alpha\sigma}C^\cdot_{\alpha\sigma\gamma\delta},
\]

(6.6)

\[
J^\nu_{2\mu} = C_{\mu\beta\gamma\delta}C^{\nu\alpha\gamma\sigma}C^\cdot_{\alpha\sigma\beta\delta},
\]

(6.7)

\[
J^\nu_{3\mu} = C_{\mu\beta\gamma\delta}C^{\nu\alpha\gamma\sigma}C^\cdot_{\sigma\beta\alpha\delta},
\]

(6.8)

\[
J^\nu_{4\mu} = C_{\mu\gamma\beta\delta}C^{\nu\beta\alpha\sigma}C^\cdot_{\alpha\sigma\gamma\delta},
\]

(6.9)

\[
J^\nu_{5\mu} = C^{\nu}_{\sigma\mu\kappa}C^\sigma_{\alpha\beta\gamma}C^\cdot_{\alpha\beta\gamma},
\]

(6.10)

and, furthermore, by (6.4), one has

\[
J^\nu_{4\mu} = \frac{1}{2}J^\nu_{1\mu},
\]

(6.11)

and, by applying the Ricci identity to the last \(C\) in (6.7), one obtains

\[
J^\nu_{3\mu} = J^\nu_{2\mu} - \frac{1}{2}J^\nu_{4\mu} = J^\nu_{2\mu} - \frac{1}{4}J^\nu_{1\mu}.
\]

(6.12)
Thus, for an arbitrary space-time dimension \(2\omega\), there are three independent contractions:

\[ J_{1\mu}^\nu, J_{2\mu}^\nu, J_{3\mu}^\nu. \]

For a particular space-time dimension, the number of independent contractions can be smaller because of the existence of identities obtained by antisymmetrization of \((2\omega + 1)\) indices. Note that such an antisymmetrization must not involve more than two indices of each \(C\) tensor; otherwise the identity will be satisfied trivially by virtue of \((6.5)\). Hence, for three \(C\) tensors, the number of indices involved in the antisymmetrization should not exceed six, and, therefore, the space-time dimension \(2\omega\) for which nontrivial identities exist cannot exceed five. For \(2\omega \leq 5\) we have

\[
C_{[\alpha_\beta}^\gamma\delta C_{\gamma_\delta}^{\mu\nu}C_{\nu\mu]}^{\alpha_\beta} \equiv 0, \quad 2\omega \leq 5
\]

with the complete antisymmetrization of six lower indices. When written down explicitly, this identity takes the form

\[
J_{1\mu}^\nu - 4J_{3\mu}^\nu - 2J_{5\mu}^\nu = 0, \quad 2\omega \leq 5
\]

or, by \((6.12)\), the form

\[
J_{2\mu}^\nu = \frac{1}{2}J_{1\mu}^\nu - \frac{1}{2}J_{5\mu}^\nu, \quad 2\omega \leq 5
\]

and reduces the number of independent contractions down to two: \(J_{1\mu}^\nu\) and \(J_{5\mu}^\nu\).

Finally, for \(2\omega = 4\) (the lowest dimension in which a nonvanishing Weyl tensor exists), the identity \((5.13)\) becomes a linear combination of the identities

\[
C_{[\alpha_\beta}^{\gamma\delta}C_{\gamma_\delta}^{\mu\nu}C_{\mu\nu]}^{\alpha_\beta} \equiv 0, \quad 2\omega = 4
\]

with the antisymmetrization over only five indices, and there is one more identity, quadratic in \(C\):

\[
C_{[\alpha_\beta}^{\gamma\delta}C_{\gamma_\delta}^{\alpha_\beta}C_{\mu\nu]}^{\delta_\mu} \equiv 0, \quad 2\omega = 4.
\]

Its explicit form is

\[
C^{\alpha\beta\gamma\nu}C_{\alpha\beta\gamma\mu} = \frac{1}{4}\delta_{\mu}^{\nu}C_{\alpha\beta\gamma}^{\delta}C^{\alpha\beta\gamma\delta}, \quad 2\omega = 4.
\]
When this relation is used in (6.10), the result is

\[ J_{5\mu}^{\nu} = 0, \quad 2\omega = 4 \] (6.19)

by the second of equations (6.3). Thus, in four dimensions, there is only one independent contraction: \( J_{\mu}^{\nu} \).

Similar results hold for invariants except that the complete contraction of \( J_5 \) in (6.10) is zero for any space-time dimension. Therefore, initially one has four different \( C^3 \) invariants

\[ I_1 = C_{\mu\beta\gamma\delta} C_{\mu\alpha\sigma} C_{\alpha\sigma}^{\gamma\delta}, \] (6.20)

\[ I_2 = C_{\mu\beta\gamma\delta} C_{\mu\alpha\gamma\sigma} C_{\alpha\sigma}^{\beta\delta}, \] (6.21)

\[ I_3 = C_{\mu\beta\gamma\delta} C_{\mu\alpha\gamma\sigma} C_{\sigma\alpha}^{\beta\delta}, \] (6.22)

\[ I_4 = C_{\mu\gamma\beta\delta} C_{\mu\beta\alpha\sigma} C_{\alpha\sigma}^{\gamma\delta} \] (6.23)

with the relations

\[ I_3 = I_2 - \frac{1}{4} I_1, \quad I_4 = \frac{1}{2} I_1, \] (6.24)

and, for \( 2\omega \leq 5 \), the identity (6.13) (contracted in \( \mu, \nu \)) adds one more relation:

\[ I_2 = \frac{1}{2} I_1, \quad 2\omega \leq 5. \] (6.25)

When going over from \( 2\omega = 5 \) to \( 2\omega = 4 \), the identity (6.18) leads to no further reduction. Thus, the dimension of the basis of local \( C^3 \) invariants is 2 for \( 2\omega > 5 \), and 1 for both \( 2\omega = 5 \) and \( 2\omega = 4 \).

For invariants with the Riemann tensor, the counting is different because, in this case, the quadratic identity (6.18) begins working. For \( 2\omega \leq 5 \), the identity (6.25) with the Weyl tensor expressed through the Riemann tensor reduces the number of independent cubic invariants by one. For \( 2\omega = 4 \), the identity (6.18) contracted with the Ricci tensor:

\[ C_{\alpha\beta\gamma\nu} C_{\alpha\beta\gamma\mu} R_{\nu}^{\mu} = \frac{1}{4} R C_{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta}^{\mu}, \quad 2\omega = 4 \] (6.26)

reduces this number by one more. These results agree with the group-theoretic analysis carried out in (25) (for the application of this analysis in a geometric classification of conformal
anomalies see Ref. 26). According to 25, the dimension of the basis of local cubic invariants with the Riemann tensor (without derivatives) is 8 for $2\omega > 5$, 7 for $2\omega = 5$, and 6 for $2\omega = 4$.

We shall now concentrate on the space-time dimension $2\omega = 4$ and go over to the consideration of nonlocal invariants cubic in the curvature. Owing to their algebraic nature, the identities above admit easily a nonlocal generalization. Indeed, the two cubic identities obtained by antisymmetrizations in four dimensions: eq. (6.16) contracted in $\mu, \nu$, and eq. (6.17) contracted with $R^\mu_\nu$ can in fact be written down for three different tensors and, in particular, for the curvature tensors at three different points. One can then multiply them by arbitrary form factors and next make the points coincident. Since the basis of nonlocal invariants above was built in terms of Ricci curvatures, rather than the Weyl tensors, we shall start such a derivation with equations (6.16) - (6.17) containing instead of the Weyl tensors the Riemann ones (which will be later eliminated in terms of Ricci curvatures). The nonlocal identities obtained in this way are of the form

$$\tilde{\tilde{F}}(\Box_1, \Box_2, \Box_3) R_1[\alpha\beta\gamma\delta] R_2[\kappa\mu] R_3[\lambda\mu] = 0, \quad 2\omega = 4,$$

(6.27)

with arbitrary $\tilde{\tilde{F}}(\Box_1, \Box_2, \Box_3)$ and $\tilde{\tilde{F}}(\Box_1, \Box_2, \Box_3)$. Since nothing is involved here except inexistence of five different indices in four dimensions, these identities are obviously correct in the present nonlocal form as well.

When written down explicitly, the left-hand sides of eqs. (6.27), (6.28) take the form (for arbitrary dimension)

$$\tilde{\tilde{F}}(\Box_1, \Box_2, \Box_3) R_1[\alpha\beta\gamma\delta] R_2[\kappa\mu] R_3[\lambda\mu] = -\frac{2}{15} \tilde{\tilde{F}}(\Box_1, \Box_2, \Box_3) \left[ R_{1\alpha\beta\gamma\delta} R_2[\kappa\mu] R_3[\lambda\mu] \right]$$

(6.29)
\[
-\frac{1}{4} R_{1\alpha\beta\gamma\delta} R_{2}^{\alpha\beta\gamma\delta} R_{3} - 2 R_{1\alpha\beta} R_{2}^{\alpha\beta} R_{3}^{\alpha} - R_{1\alpha\beta\gamma\delta} R_{2}^{\alpha\beta\gamma\delta} R_{3}^{\alpha\beta} - R_{2\alpha\beta\gamma\delta} R_{1}^{\alpha\gamma} R_{3}^{\beta\delta} \\
+ R_{1\alpha\beta} R_{2}^{\alpha\beta} R_{3} + \frac{1}{2} R_{1} R_{2\alpha\beta} R_{3}^{\alpha} + \frac{1}{2} R_{2} R_{1\alpha\beta} R_{3}^{\alpha\beta} - \frac{1}{4} R_{1} R_{2} R_{3} \right].
\] (6.30)

Since these expressions contain arbitrary form factors, they suggest that in virtue of (5.27) - (5.28), in four dimensions, the basis of nonlocal gravitational invariants may be redundant. However, to convert relations (5.27), (5.28) into constraints between the basis structures one must make one more step: eliminate the Riemann tensor.

As discussed in Sect.II, the Riemann tensor expresses through the Ricci tensor in a non-local way once the boundary conditions for the gravitational field are specified. Elimination of the Riemann tensor from the expressions (6.29) and (6.30) with the use of (2.2) (to lowest order) brings them to the following form:

\[
\tilde{F}(\Box_{1}, \Box_{2}, \Box_{3}) R_{1[\alpha\beta} R_{2]\gamma\delta} R_{3}^{\alpha\beta\gamma\delta} R_{3}^{\alpha\beta} \equiv -\frac{1}{15} \left\{ \frac{1}{8} \tilde{F}' \left( \frac{\Box_{1}}{\Box_{2}} + \frac{\Box_{2}}{\Box_{1}} + \frac{\Box_{3}}{\Box_{1}} \right) R_{1} R_{2} R_{3} \\
+ \tilde{F}' \left( -\frac{1}{\Box_{1}} - \frac{1}{\Box_{2}} - \frac{1}{\Box_{3}} + \frac{1}{2} \Box_{1} \Box_{3} + \frac{1}{2} \Box_{2} \Box_{3} + \frac{1}{2} \Box_{1} \Box_{2} \right) R_{1}^{\alpha\mu} R_{2}^{\beta} R_{3}^{\mu} \\
+ \frac{1}{4} \left( \frac{1}{\Box_{1}} + \frac{1}{\Box_{2}} - \frac{3}{\Box_{1} \Box_{2}} \right) \left[ \tilde{F}' + \tilde{F}'_{|2\rightarrow3} + \tilde{F}'_{|1\rightarrow3} \right] R_{1}^{\mu\nu} R_{2} R_{3} \\
+ \frac{1}{4} \left( \frac{1}{\Box_{1}} + \frac{1}{\Box_{2}} + \frac{1}{\Box_{3}} \right) \left[ \tilde{F}' - \tilde{F}'_{|1\rightarrow2} - \tilde{F}'_{|1\rightarrow3} \right] R_{1}^{\alpha\beta} \nabla_{\alpha} R_{2} \nabla_{\beta} R_{3} \\
+ \frac{1}{\Box_{1}} \left[ \tilde{F}' + \tilde{F}'_{|2\rightarrow3} + \tilde{F}'_{|1\rightarrow3} \right] \nabla_{\alpha} R_{1}^{\alpha\beta} \nabla_{\beta} R_{3} \\
+ \frac{1}{\Box_{2}} \left[ \tilde{F}' + \tilde{F}'_{|1\rightarrow2} + \tilde{F}'_{|1\rightarrow3} \right] R_{1}^{\mu\nu} \nabla_{\mu} R_{2} \nabla_{\nu} R_{3} \\
+ \left( \frac{1}{\Box_{1}} + \frac{1}{\Box_{3}} - \frac{1}{\Box_{2} \Box_{3}} \right) \left[ \tilde{F}' + \tilde{F}'_{|1\rightarrow2} + \tilde{F}'_{|1\rightarrow3} \right] R_{1}^{\mu\nu} \nabla_{\alpha} R_{2} \nabla_{\beta} R_{3}^{\alpha} \\
+ \frac{1}{\Box_{1}} \left[ \tilde{F}' - \tilde{F}'_{|2\rightarrow3} - \tilde{F}'_{|1\rightarrow3} \right] \nabla_{\alpha} R_{1}^{\alpha\beta} \nabla_{\beta} R_{3}^{\alpha} \\
+ \frac{2}{\Box_{1} \Box_{2} \Box_{3}} \left[ \tilde{F}' - \tilde{F}'_{|2\rightarrow3} - \tilde{F}'_{|1\rightarrow3} \right] \nabla_{\mu} R_{1} \nabla_{\nu} R_{2}^{\beta} \nabla_{\alpha} R_{3}^{\alpha} \right\} \\
+ \text{a total derivative} + O[R_{4}^{i}], \quad \tilde{F}' \equiv (\Box_{1} - \Box_{2} - \Box_{3}) \tilde{F},
\] (6.31)
where

\[ \tilde{\mathcal{F}}'_{1+2} \equiv \tilde{\mathcal{F}}'(\Box_2, \Box_1, \Box_3), \text{ etc.} \]

It is now seen that, up to total derivatives and terms \( O[R^4] \), (6.27) is an equivalent form of (6.28) corresponding to

\[ \tilde{\mathcal{F}} = \frac{1}{2} \frac{1}{\Box_3} \mathcal{F}' = \frac{1}{2} \frac{1}{\Box_1 - \Box_2 - \Box_3} \mathcal{F}. \]

Thus, assuming integration over the space-time which will make irrelevant total derivatives, we conclude that, at third order in the curvature, of the two generally different identities for nonlocal invariants, existing in four dimensions, only one is independent: eq. (6.28).

This equation is brought to its final form by putting

\[ \tilde{\mathcal{F}}(\Box_1, \Box_2, \Box_3) = -\frac{1}{6} \Box_1 \Box_2 \mathcal{F}(\Box_1, \Box_2, \Box_3) \]

where \( \mathcal{F}(\Box_1, \Box_2, \Box_3) \) is a new arbitrary function, and taking into account the symmetries of the tensor structures entering (6.32). The result is the following constraint between the basis invariants listed in the table (5.17)-(5.20):

\[
\int dx \, g^{1/2} \text{tr} \mathcal{F}^\text{sym}(\Box_1, \Box_2, \Box_3) \{ -\frac{1}{48}(\Box_1^2 + \Box_2^2 + \Box_3^2) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(9) \\
- \frac{1}{12}(\Box_1^2 + \Box_2^2 + \Box_3^2 - 2\Box_1 \Box_2 - 2\Box_2 \Box_3 - 2\Box_1 \Box_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(10) \\
- \frac{1}{8}(\Box_1 + \Box_2 - \Box_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(11) + \frac{1}{8}(3\Box_1 + \Box_2 + \Box_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(22) \}
\]
where \( F^{\text{sym}}(\Box_1, \Box_2, \Box_3) \) is a completely symmetric but otherwise arbitrary function. This constraint, valid in four dimensions, reduces the basis of nonlocal gravitational invariants by one structure. With its aid one can exclude everywhere either the structure 9 or 10 or the completely symmetric (in the labels 1,2,3) part of anyone of the remaining purely gravitational structures except \( R_1 R_2 R_3(29) \). The latter structure which is the only one containing six derivatives is absent from the constraint (6.33) and is, therefore, inexcludable. This can be explained by the fact that its local version is the only independent contraction of three Weyl tensors: eq. (6.20) (cf. eq. (2.2)).

At least apparently, eqs. (6.27) and (6.28) are not the most general nonlocal identities that can be written down by antisymmetrizing five indices. More generally, one can apply this procedure to three tensors with arbitrary indices and arbitrary number of uncontracted derivatives. Therefore, to make sure that there are no more constraints between the basis invariants, an independent check is needed. Since, at third order in the curvature, the maximum number of derivatives that do not contract in the box operators is six, we begin this check with nonlocal structures having three Ricci tensors and six derivatives. There exist only two such, and, according to the discussion of Sect.IV (eq. (4.4)), only one of them is independent:

\[
\hat{\nabla}_\alpha \nabla_\beta R^\gamma_1 R^\delta_2 R_\gamma_1 R_\delta_2 R_\mu_1 R_\nu_1 R_\mu_2 R_\nu_2 = R_1 R_2 R_3(29), \tag{6.34}
\]

\[
\hat{\nabla}_\alpha \nabla_\beta R^\gamma_1 R^\delta_2 R_\gamma_1 R_\delta_2 R_\mu_1 R_\nu_1 R_\mu_2 R_\nu_2 = -R_1 R_2 R_3(29) + (\ldots) \tag{6.35}
\]

where the ellipses (\ldots) stand for total derivatives and terms with derivatives contracting in the box operators. Eq. (6.33) is obtained by three integrations by parts applied to \( \nabla_\alpha, \nabla_\mu \) and \( \nabla_\delta \). Since not more than one index of each Ricci tensor and not more than one derivative acting on each Ricci tensor may participate in the antisymmetrization (otherwise the result
will be either trivial or \( O[R^4] \), there are only two possible 5-antisymmetrizations of (6.34):

\[
\nabla_{\alpha} \nabla_{\beta} R_{\gamma\delta} R_{\mu\nu} \nabla_{\mu} \nabla_{\nu} R_{\gamma\delta} = 0, \quad \nabla_{\alpha} \nabla_{\beta} R_{\gamma\delta} R_{\mu\nu} \nabla_{\mu} \nabla_{\nu} R_{\gamma\delta} = 0.
\]

In each of these cases, upon calculation, the terms (6.34) and (6.35) appear in a sum with equal coefficients and, therefore, cancel. This proves that the structure with six derivatives remains unconstrained. Among the invariants with three Ricci tensors and four derivatives, only two are independent: the basis structures 27 and 28, and only the latter admits nontrivial 5-antisymmetrizations. There is, moreover, only one such:

\[
\nabla_{\mu} R_{\alpha\beta} \nabla_{\nu} R_{\gamma\delta} \nabla_{\alpha} \nabla_{\beta} R_{\mu\nu} = 0.
\]

Upon calculation and multiplication by an arbitrary form factor, the latter identity gives precisely the constraint (6.33). The invariants with the commutator curvature and four derivatives are all reducible except the structure \( R_1 R_2 R_3 (38) \) (see Sect.V) which only admits the nontrivial 5-antisymmetrization of five indices, but one can apply to this structure the same argumentation as in the case of \( R_1 R_2 R_3 (29) \) above, which proves that it also remains unconstrained. Finally, invariants with two derivatives do not admit a nontrivial 5-antisymmetrization since, for that, one needs at least ten indices: five uncontracted to be involved in the symmetrization, and five more to make a complete contraction.

Thus, in four dimensions, there is only one constraint between the basis structures, and the dimension of the basis of nonlocal cubic invariants which is generally 38 and in the case of the gravitational invariants 10 becomes respectively 37 and 9.

The nonlocal identity obtained above has a direct relation to the Gauss-Bonnet identity in four dimensions. Indeed, by calculating the square of the Riemann tensor with the aid of eq. (2.2), one finds for arbitrary dimension \( 2\omega \):

\[
\int dx \, \gamma^{1/2} \left( R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) = \int dx \, \gamma^{1/2} \left[ \frac{1}{2} \bigcirc_1 \bigcirc_3 - \bigcirc_1 \bigcirc_3 \right] R_1 R_2 R_3 \\
+ 2 \left( \frac{1}{2} \bigcirc_1 \bigcirc_3 - \frac{1}{2} \bigcirc_1 \bigcirc_3 \right) R_{\alpha} R_{\beta} R_{\gamma\delta} + \frac{1}{2} \bigcirc_1 \bigcirc_3 \bigcirc_3 \bigcirc_2 ) R_{\mu\nu} R_{\alpha\beta} R_{\gamma\delta} \\
+ \left( - \frac{1}{2} \bigcirc_1 \bigcirc_3 - \frac{1}{2} \bigcirc_1 \bigcirc_3 \right) R_{\alpha} R_{\beta} R_{\gamma\delta} + \frac{1}{2} \bigcirc_1 \bigcirc_3 \bigcirc_3 \bigcirc_2 ) R_{\mu\nu} R_{\alpha\beta} R_{\gamma\delta} \\
+ \left( - \frac{1}{2} \bigcirc_1 \bigcirc_3 - \frac{1}{2} \bigcirc_1 \bigcirc_3 \right) R_{\alpha} R_{\beta} R_{\gamma\delta} + \frac{1}{2} \bigcirc_1 \bigcirc_3 \bigcirc_3 \bigcirc_2 ) R_{\mu\nu} R_{\alpha\beta} R_{\gamma\delta}
\]
\begin{align*}
+4 \frac{1}{\Box_2 \Box_3} R_{1 \mu \nu}^{\alpha \beta} \nabla_\mu R_{2 \nu}^{\alpha \beta} R_{3 \alpha \beta} &+ 4 \left( \frac{1}{\Box_1 \Box_2} - \frac{1}{\Box_2 \Box_3} \right) R_{1 \mu \nu}^{\alpha \beta} \nabla_\alpha R_{2 \beta \mu}^{\alpha \beta} \nabla_\beta R_{3 \nu}^{\alpha \beta} \\
-4 \frac{1}{\Box_1 \Box_2 \Box_3} \nabla_\alpha \nabla_\beta R_{1 \mu \nu}^{\alpha \beta} \nabla_\mu \nabla_\nu R_{2 \alpha \beta}^{\alpha \beta} R_{3 \alpha \beta} - 8 \frac{1}{\Box_1 \Box_2 \Box_3} \nabla_\mu R_{1 \alpha \lambda}^{\alpha \lambda} \nabla_\nu R_{2 \beta \lambda}^{\beta \lambda} \nabla_\alpha \nabla_\beta R_{3 \mu \nu}^{\alpha \beta} \right) + O[R_{4 \mu \nu}^{\alpha \beta}]. \quad (6.36)
\end{align*}

In agreement with the result of Ref. 12, a contribution of second order in the curvature is absent from this expression for any space-time dimension. The third-order contribution (6.36) does not generally vanish but vanishes in four dimensions because it coincides with the left-hand side of the identity (6.33) if in the latter one puts

\[ F^{\text{sym}}(\Box_1, \Box_2, \Box_3) = -8(\text{tr} \hat{1})^{-1} \frac{1}{\Box_1 \Box_2 \Box_3}. \]

Comparison of eq. (6.36) and (6.32) gives

\[ \int dx \, g^{1/2} \left( R_{\alpha \beta \gamma \delta} R_{\alpha \beta \gamma \delta} - 4 R_{\mu \nu} R_{\mu \nu} + R^2 \right) = \int dx \, g^{1/2} \left( -10 \frac{1}{\Box_3} R_{1[\alpha \beta \gamma \delta} R_{2 \gamma \delta}^{\alpha \beta} R_{3 \mu \nu]} + O[R_{4 \mu \nu}^{\alpha \beta}] \right). \quad (6.37) \]

This relation \textit{valid for any number of space-time dimensions} elucidates the mechanism by which the Gauss-Bonnet identity arises in four dimensions.\[ \square \]

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\[ ^1 \text{B.S.DeWitt, in } \textit{Relativity, groups and topology}, 1963 \text{ Les Houches lectures, eds. C.DeWitt and B.S.DeWitt (Gordon and Breach, N.Y., 1964) p.587; Dynamical theory of groups and fields (Gordon and Breach, New York, 1965) } \]
2 G.Jona-Lasinio, Nuovo Cim. 34, 1790 (1964)

3 J.Honerkamp, Nucl.Phys. B 36, 130 (1972); G.t'Hooft, in Proc. XII Winter School in Carpackz, Acta Universitatis Wratislaviensis, No.368 (1975)

4 S.Deser, M.Duff and C.J.Isham, Nucl.Phys. B111, 45 (1976)

5 G.A.Vilkovisky, in Quantum theory of gravity, ed. S.M.Christensen (Hilger, Bristol, 1984) p.169

6 J.Schwinger, J.Math.Phys. 2, 407 (1961); L.V.Keldysh, Zh.Eksp.Teor.Fiz. 47, 1515 (1964); Yu.V.Gol'fand, Yad.Fiz. 8, 600 (1968); P.Hajicek, in Proc. Second Marcel Grossman meeting on general relativity, Trieste, 1979, ed. R.Ruffini (North Holland, 1982) p.483; E.S.Fradkin and D.M.Gitman, Fortschr. der Phys. 29, 381 (1981); J.L.Buchbinder, E.S.Fradkin and D.M.Gitman, Fortschr. der Phys. 29, 187 (1981); R.D.Jordan, Phys.Rev. D33, 44 (1986); E.Calzetta and B.L.Hu, Phys.Rev. D35, 495 (1987)

7 V.P.Frolov and G.A.Vilkovisky, Proc. Second Seminar on Quantum Gravity (Moscow, 1981), eds. M.A.Markov and P.C.West (Plenum, London, 1983) p.267; Phys.Lett. 106 B, 307 (1981)

8 J.B.Hartle and G.Horowitz, Phys.Rev. D24, 257 (1981)

9 A.O.Barvinsky and G.A.Vilkovisky, Phys. Reports 119, 1 (1985)

10 A.O.Barvinsky and G.A.Vilkovisky, in Quantum field theory and quantum statistics, vol. 1, eds. I.A.Batalin, C.J.Isham and G.A.Vilkovisky (Hilger, Bristol, 1987) p.245

11 A.O.Barvinsky and G.A.Vilkovisky, Nucl.Phys. B282, 163 (1987)

12 A.O.Barvinsky and G.A.Vilkovisky, Nucl. Phys. B333, 471 (1990)

13 A.O.Barvinsky and G.A.Vilkovisky, Nucl.Phys. B333, 512 (1990)

14 A.O.Barvinsky, Yu.V.Gusev, G.A.Vilkovisky and V.V.Zhytnikov, Covariant Perturbation Theory (IV). Third Order in the Curvature, Report of the University of Manitoba (University of Manitoba, Winnipeg, 1993)
A special status of structures (5.21) is not exceptional in field-theoretic calculations. For example, the first-order invariant $R$ is also absent in the expansion (5.22) for the heat kernel trace (although it usually emerges in such an expansion as a part of the invariant $\text{tr} \hat{P}$), while the second-order structure $\mathcal{R}_1 \mathcal{R}_2(3)$ has a peculiarity that its one-loop effective action form factor is a local numerical constant $\frac{1}{2}$.

In Ref. [4] this equation figures with the Weyl tensors instead of the Riemann ones – the fact brought to our attention by Hugo Osborn, and the authors are very grateful to him for pointing out this error.