Abstract. This paper solves a new class of optimization problems under uncertainty which optimizes an objective function of decision variables and subjects to a set of probability contour constraints (PCC). The proposed PCC logically means that an optimal solution should satisfy a set of algebraic constraints for all possible high-probability realizations of the uncertain parameters. The PCC is an alternative to the conventional chance constraint while the latter cannot guarantee the solution’s feasibility to high-probability realizations of uncertainty. Given that the existing solution methods of the conventional chance-constrained optimization are not suitable for solving the proposed probability contour constrained optimization (PCCO), we develop a novel data-based solution paradigm that uses historical measurements of the uncertain parameters as input samples. This solution paradigm is conceptually simple and allows us to develop effective data-reduction schemes which reduces computational burden while reserves high accuracy.

Key words. Chance-constrained optimization, data-driven optimization, optimization under uncertainty

MSC codes. 90C15, 62C05, 65C20

1. Introduction. Uncertainties exist in the decision-making processes of many engineering systems. Generally, the procedure of solving decision-making problems under uncertainty mainly consists of two steps. First, a logic model, such as stochastic [7], robust [5], chance-constrained [23] optimization models, and their variants [35], is chosen to mathematically formulate this decision-making problem under uncertainty. Second, the logic model is replaced by a deterministic approximation and, then solved by mature optimization algorithms [31] /solvers [3]. This paper proposes the following novel logic model for mathematically formulating decision-making problems under uncertainty:

\[ \begin{align*}
\text{PCCO:} & \quad \min_{x \in \mathbb{R}^n} f(x) \\
\text{s.t.} & \quad g(x, \xi) \leq 0, \ \forall y \in \{y \in \Xi | \mathbb{P}[^{\xi}] \geq \alpha\}
\end{align*} \]

where \( f : \mathbb{R}^n \to \mathbb{R} \), and \( g : \mathbb{R}^n \times \Xi \to \mathbb{R}^m \) are general functions, and \( \mathbb{P}[^{\cdot}] \) represents the probability of an event. Without loss of generality, only inequality constraints are considered in (1.1b) since there are explicit and implicit methods of equivalent reformulating equations as inequations. The vector of uncertain parameters \( \xi \in \Xi \subset (\mathbb{Z}^{r_1}, \mathbb{R}^{r_2}) \) \( (r_1 + r_2 = r) \) follows certain probability distributions which may be unknown. The proposed constraint (1.1b) is called probability contour constraint (PCC), since \( \mathbb{P}[\xi = y] = \alpha \) specifies a contour in \( \Xi \) (see two examples in Figure 1), and consequently, problem (1.1) is called probability contour constrained optimization (PCCO) in this paper.

The PCC (1.1b) logically means that “a solution of problem (1.1) should satisfy \( g(x, \xi) \leq 0 \) for all high-probability realizations of the uncertain parameters \( \xi \in \Xi \), i.e., the realizations whose probabilities are not less than \( \alpha \). It also guarantees that “the probability of that a solution violates \( g(x, \xi) \leq 0 \) is not bigger than a certain value.”

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1 Denoted as logical meaning (LM)#1
2 Denoted as LM#2

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Fig. 1. Two illustrative examples of PCC, where \( P[\xi = y] = P[\xi_1 = y_1]P[\xi_2 = y_2] \). In the first example, \( P[\xi_1 = y_1] \sim N(0,1) \) and \( P[\xi_2 = y_2] \sim N(0,1.8) \). In the second example, \( \xi_1 \) and \( \xi_2 \) follow bimodal distributions, where \( P[\xi_1 = y_1] \sim [0.5N(-2,1) + 0.5N(3,1.8)] \) and \( P[\xi_2 = y_2] \sim [0.5N(0,1.2) + 0.5N(5,1.6)] \).

The proposed PCC (1.1b) is considered an alternative to the following conventional chance-constraint (CC):

\[
P[g_i(x, \xi) > 0, \ i = 1, \ldots, m] \leq \beta,
\]

where \( \beta \) is the allowed probability of constraint violation. While the CC (1.2) logically means that “the probability of that a solution violates \( g(x, \xi) \leq 0 \) is not bigger than \( \beta \),” it cannot guarantee the feasibility of a solution to all high-probability realizations of \( \xi \) which is desirable for many engineering systems.

The novelty and contributions of this paper reside in that:

1. The proposed PCC (1.1b) is considered an interesting alternative to the conventional CC (1.2), since the it can guarantee both LM#1 and LM#2 while CC guarantees only LM#2 (see the related analysis and proofs in Section 2);
2. While the existing solution methods for the conventional chance-constrained optimization (CCO) are not applicable to PCCO problems, we developed a novel, conceptually simple solution paradigm which uses real-world historical data as input samples (see Section 3);
3. Algorithms of strategic data selection are developed for improving the computational efficiency of the proposed data-based solution paradigm by effectively eliminating inactive data points (see Section 4).

Note that, while the PCCO is a newly defined problem, a literature review on the existing methods of solving its conventional counterpart—the CCO—is provided in Subsection 2.2.

2. Probability Contour Constraint. When the PCC (1.1b) is considered an alternative to the CC (1.2), the PCCO (1.1) is an alternative to the following conventional chance-constrained optimization:

\[
\text{CCO: } \min_x (1.1a) \quad \text{s.t. } (1.2).
\]

Let \( x^*_P \) and \( x^*_C \) be the optimal solutions of PCCO (1.1) and CCO (2.1) respectively, and \( \xi \) be an arbitrary realization of \( \xi \in \Xi \), the logical means of the PCC and CC are compared in what follows for clarity:
• **PCC**: \( g(x^n_r, \xi) \leq 0 \) holds if the joint probability of \( \xi \) is not less than \( \alpha \);

• **CC**: the probability of event \( g(x^n_r, \xi) \leq 0 \) should not be less than \( 1 - \beta \) (or, the probability of event \( g(x^n_r, \xi) > 0 \) should not be bigger than \( \beta \)).

This section: 1) shows the relations between PCCO and CCO, 2) reviews the existing solution methods of CCO, and 3) applies these methods to solve the PCCO and shows that these existing methods are not adequate for solving PCCO.

### 2.1. Relations between PCC and CC.

Since the PCCO (1.1) and CCO (2.1) share the same objective function, the difference between them resides in the constraints. Let \( P: \Xi \to [0, 1] \) denote the probability distribution function of \( \mathbb{P}[\xi = y] \), i.e., \( \mathbb{P}[^{\star}y = y] = P(y) \), whose detailed expression may be unknown. Recall the logical meaning of PCC explained at the beginning of this section and let \( \Xi_{\alpha} = \{ \xi \in \Xi | P(\xi) \geq \alpha \} \), the particle-based deterministic formulation of the feasible space of \( x \) that is specified by the PCC (1.1b) is given as

\[
\mathcal{X}_P = \{ x \in \mathbb{R}^n | g(x, \xi^{(k)}) \leq 0, k = 1, \ldots, |\Xi_{\alpha}| \},
\]

where \( |\Xi_{\alpha}| \) is the number of all possible outcomes of \( \xi \) in \( \Xi_{\alpha} \). For \( \xi \) which is continuous or mixed-integer, \( |\Xi_{\alpha}| \) is generally infinite. We have the following proposition.

**Proposition 2.1.** \( \mathcal{X}_P(\alpha_1) \supseteq \mathcal{X}_P(\alpha_2) \) if \( \alpha_1 \geq \alpha_2 \).

The proof of this proposition is given in Appendix A.1. Recall the logical meaning of CC (1.2) presented at the beginning of this section, if the probability function \( P(\xi) \) is known, a deterministic formulation of the feasible space of \( x \) that is specified by the CC is given as

\[
\mathcal{X}_C = \left\{ x \in \mathbb{R}^n \left| \int \cdots \int_{M(x)} P(\xi)d\xi_\cdot \cdots d\xi_1 \geq 1 - \beta \right. \right\},
\]

where \( M(x) := \{ \xi \in \Xi | g(x, \xi) \leq 0, x \in \mathbb{R}^n \} \) is the set of \( \xi \) that \( g(x, \xi) \leq 0 \) holds for a specific \( x \). We have the following propositions.

**Proposition 2.2** (on the relations between \( \mathcal{X}_P \) and \( \mathcal{X}_C \)). If

\[
\alpha = \arg \left\{ \int \cdots \int_{\{\xi \in \Xi | P(\xi) \geq \nu\}} P(\xi)d\xi_\cdot \cdots d\xi_1 = 1 - \beta \right\},
\]

where \( \arg \) means the argument of a function, we have the following relations:

1. \( \mathcal{X}_P \subseteq \mathcal{X}_C \);
2. \( \mathcal{X}_P = \mathcal{X}_C \) if, when \( \xi^{(a)} \) and \( \xi^{(b)} \) are arbitrarily realizations in \( M(x) \) and \( \Xi \setminus M(x) \) respectively, \( P(\xi^{(a)}) \geq P(\xi^{(b)}) \).

While the proof of Proposition 2.2 is provided in Appendix A.2, an illustrative example is given in Figure 2, where \( r_1 = 0 \) and \( r_2 = 1 \) (i.e., \( \xi \) is a continuous scalar). Let \( x^{(a)} \) and \( x^{(b)} \) be two solutions, \( M(x^{(a)}) := \{ \xi \in \Xi | g(x^{(a)}, \xi) \leq 0 \} = \{ \xi | 1 \leq \xi \leq \xi_2 \}, \)

\[
M(x^{(b)}) := \{ \xi \in \Xi | g(x^{(b)}, \xi) \leq 0 \} = \{ \xi | \xi_3 \leq \xi \leq \xi_1 \} \cup \{ \xi | \xi_5 \leq \xi \leq \xi_6 \},
\]

Figure 2 shows that

\[
\int_{\xi \in M(x^{(a)})} P(\xi)d\xi = \int_{\xi_1}^{\xi_2} P(\xi)d\xi = 1 - \beta
\]

\[
\int_{\xi \in M(x^{(b)})} P(\xi)d\xi = \int_{\xi_3}^{\xi_4} P(\xi)d\xi + \int_{\xi_5}^{\xi_6} P(\xi)d\xi + \int_{\xi_7}^{\xi_8} P(\xi)d\xi = 1 - \beta,
\]

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3In this paper, particle means sample, realization, or data.

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which implies that both \( x(a) \) and \( x(b) \) are feasible to CC, i.e., \( x(a), x(b) \in X_C \). It suffices to know that only \( x(a) \) is feasible to PCC, i.e., \( x(a) \in X_P \) while \( x(b) \notin X_P \), since \( g(x(b), \xi) \leq 0 \) does not hold for \( \xi_4 \leq \xi \leq \xi_5 \) or \( \xi_6 \leq \xi \leq \xi_2 \) where \( P(\xi) \geq \alpha \). Although a normal distribution is considered in this illustrative example, the assertions of Proposition 2.2 are not sensitive to the type of distributions.

**Remark on the advantages of PCC.** First, although CC restricts that its feasible solutions can ensure a satisfactory probability of constraint violation, it cannot guarantee that its optimal solution is feasible to a possible high-probability realization, which is not desirable in engineering. In contrast, PCC can guarantee both a satisfactory probability of constraint violation and the feasibility to possible high-probability realizations. Second, (2.2)–the deterministic formulation of the feasible set \( X_P \) in the \( x \)-space–has less mathematical complexity than (2.3)–the one of \( X_C \)–does. In (2.3), the set \( M(x) \) is hard to quantify under the optimization framework as it varies following \( x \). In contrast, (2.2) is easier to be quantified, which lays a solid foundation for the development of an effective data-based solution paradigm as presented in Section 3.

### 2.2. Existing solution methods for CCO (2.1).

There exist a number of methods for solving the conventional CCO problems. This paper classifies these that are most commonly used into three categories:

1) **Direct sample-based methods** which construct the deterministic approximation of a CCO by directly using samples of uncertain parameters. An approach of this kind, called scenario method [13, 12, 18, 13, 12], starts with drawing \( N \) independent and identically distributed (iid) samples of \( \xi \) for its probability distribution functions and then replace (1.2) with \( g(x, \xi(k)) \leq 0 \) (\( k = 1, \ldots, N \)). While this approach has some interesting advantages, its limitations are also evident. On one hand, it is conceptually simple and does not require pre-known probability distribution function \( P(\xi) \) if a sufficient amount of realizations of \( \xi \) are given. On the other hand, such a naïve implementation of the realizations is known to be excessively conservative. The sample average approximation (SAA) [21] has also been extended to approximate (1.2) into

\[
\frac{1}{N} \sum_{k=1}^{N} \mathbf{1}_{(-\infty, 0]}[g(x, \xi(k))] \geq 1 - \beta, \quad \mathbf{1}_{(-\infty, 0]}[g(x, \xi(k))] = \begin{cases} 
1 & \text{if } g(x, \xi(k)) \leq 0 \\
0 & \text{if } g(x, \xi(k)) > 0
\end{cases}
\]
where \( 1_{(-\infty,0]} \) is an indicator function of \((-\infty,0] \) [2, 32, 15, 33, 14, 39]. The SAA (2.5) may be less conservative than the scenario method in approximating the CC. However, the use of the indicator function results in a mixed integer optimization where the number of binary variables would be \( s^r \), where \( s \) is the number of samples of each uncertain parameter and \( r \) is the number of uncertain parameters. For instance, suppose that \( \xi \) has 5 elements, even if we have 20 samples of each of them, the SAA of CC would involve \( 20^5 \) binary variables which is computationally intractable to solve.

2) Stochastic-reformulation methods which approximate the CCO into an stochastic optimization (SO) problem then use SO solution methods, such as the sample-based methods, to solve the resulting SO problem. The reformulation is generally based on replacing the CC (1.2) with its expectation-based bounds, of which an example is given as

\[
E[g(x, \xi)] + \epsilon \sqrt{\mathbb{V}[g(x, \xi)]} \leq 0
\]

where \( \epsilon > 0 \), and \( E[\cdot] \) and \( \mathbb{V}[\cdot] \) represent expectation and variance respectively [30, 36, 29, 17, 27, 8]. According to Cantelli’s inequality [10], a solution that satisfies (2.6) also satisfies the CC (1.2) if \( \beta \leq 1 - (\epsilon^2/(1 + \epsilon^2)) \) which, nevertheless, can be rather loose. An advantage of (2.6) lies in that it is applicable to a wide range of CCs. However, (2.6) may not be a suitable deterministic approximation for CC if \( E[\cdot] \) and \( \mathbb{V}[\cdot] \) are hard to compute, such as when \( g(\cdot) \) is nonlinear or/and \( P(\xi) \) is unknown. Other expectation-based bounds include but are not limited to the Markov bound, conditional value-at-risk, Chebyshev bound [16] and Chernoff bound.

3) Robust-reformulation methods which approximate the CCO into an robust optimization (RO) problem and then use RO solution methods, such as the Wald’s maximin method [37], to solve this resulting problem. To be specific, the CC (1.2) is approximated by the following robust constraint:

\[
g(x, \xi) \leq 0, \forall \xi \in \Lambda \subset \Xi
\]

where \( \int \cdots \int_\Lambda P(\xi)d\xi \cdots d\xi \geq 1 - \beta \). Namely, \( \Lambda \) is a subset of the uncertainty set \( \Xi \) with a probability of at least \( 1 - \beta \). The RO-reformulation (2.7) of CC has some shortcomings. First, there may be multiple \( \Lambda \)'s that satisfy \( \int \cdots \int_\Lambda P(\xi)d\xi \cdots d\xi \geq 1 - \beta \) as observed in the illustrative example in Figure 2. That means the feasible set in the \( x \)-space that is specified by (2.7) is only a subset of \( \Lambda \). Second, it’s not necessarily easy to find a \( \Lambda \). The existing research generally constructs simplified \( \Lambda \) under some assumptions [4, 28, 25, 38, 20]. For example, [28] approximates \( \Lambda \) with a set of element-wise bounds, i.e., \( \xi_i \leq \xi_i \leq \bar{\xi}_i \) \( (i = 1, \ldots, r) \), which is actually a “box” in the \( \mathbb{R}^r \)-space. The accuracy and tractability of this box approximation rely on some strict assumptions, such as that \( g(x, \xi) \) achieves its maximum with respect to \( \xi \) at a vertex of the box. Last but not least, an RO problem, e.g., the problem that optimizes (1.1a) subjecting to (2.7), can be very hard to solve if it’s nonlinear and nonconvex.

2.3. Applying the existing solution methods to solve PCCO. Given that the PCC (1.1b) is a different constraint under uncertainty from the CC (1.2), the solution methods of CCO may not be applicable for solving PCCO. If there exists a pre-known closed-form expression of set \( \Xi_\alpha \), the PCCO can be reformulated as a classic RO [6] that can be solved by the Wald’s maximin method under some assumptions, e.g., \( \Xi_\alpha \) is convex and \( g(\cdot) \) should be linear or convex in both \( x \) and \( \xi \). It’s worth noting that these assumptions may not be practical for real-world engineering systems. The
illustrative example in Figure 1 shows that, even if uncertainty set $\Xi$ is convex, $\Xi_{\alpha}$ can be nonconvex or even discontinuous due to the probability distribution properties of $\xi$.

If the scenario-based method is directly applied, the PCCO is approximated by

\begin{align}
\min \quad & f(x) \\
\text{s.t.} \quad & g(x, \xi^{(k)}_{\text{iid}}) \leq 0, \quad (k = 1, \ldots, N)
\end{align}

where $\xi^{(k)}_{\text{iid}}$ is an iid sample. Recall that $\mathcal{X}_P$ is the feasible set of PCCO and further let $\mathcal{X}_S = \{x \in \mathbb{R}^n \mid (2.8b) \text{ holds}\}$ denote the feasible set of the scenario-based deterministic approximation (2.8) of PCCO, we have the following proposition:

**Proposition 2.3.** $\mathcal{X}_P \subset \mathcal{X}_S$ and $\mathcal{X}_P \neq \mathcal{X}_S$ are two high-probability events if $N \leq \frac{1}{\alpha}$ and $N > \frac{1}{\alpha}$ respectively.

Readers can find the proof of this proposition in Appendix A.3. When, in CCO (2.1), $f(\cdot)$ is linear and $g(\cdot)$ is convex on $x$, let $x_{\mathcal{C}}^*$ and $x_{\mathcal{S}}^*$ denote the optimal solutions of CCO (2.1) and its scenario-based deterministic approximation (2.8) respectively, and $\varepsilon = P[x_{\mathcal{S}}^* = x_{\mathcal{C}}^*]$, Theorem 1 in [13] asserts that $\varepsilon \geq \epsilon$ if

$$N \geq \frac{e(n - \ln \epsilon)}{\beta(e - 1)},$$

where $e$ is Euler’s number. Nevertheless, the $N$ that satisfies (2.9) is significantly bigger than $1/\alpha$ under condition (2.4). With the $N$ determined by (2.9), it cannot guarantee that $\mathcal{X}_S$ is an accurate approximation of $\mathcal{X}_P$ according to Proposition 2.3.

To sum up, the existing solution methods of CCO are either inapplicable or ineffective for solving PCCO. The purpose of next section is developing a novel solution method that is 1) probability distribution-free (it’s based on historical data rather than pre-known probability distribution functions), 2) general (it does not rely on linearity or convexity assumptions on system constraints), and 3) computationally effective (it facilitates the development of effective data reduction methods).

### 3. The Proposed Solution Paradigm for PCCO: Data-based Deterministication

Assuming that the probability distribution function $P(\xi)$ is not perfectly known, and instead, a set of historical measurements of $\xi$, i.e., $D = \{\xi^{(k)}_{\text{iid}} \mid k = 1, \ldots, D\}$ where $D$ is a finite but big number, is available, this section aims at developing a novel concept of data-based deterministication\(^4\). There are basically two factors that determines the quality of deterministication: accuracy and computational tractability.

#### 3.1. Preliminaries

This subsection presents the prerequisites needed to build our main results.

**Definition 3.1** (on “realizations,” “data points,” and “scenarios” of the uncertain parameter $\xi$).

\(^4\)In most of the solution processes of optimization problems under uncertainty, the first and foremost step is converting the logic models into their deterministic approximations, i.e. linear, nonlinear, or integer programs. Inspired by the widely adopted term “convexification” in the optimization field, which refers to the process of converting or approximating non-convex problems into convex ones, we define a verb “determinisfy” to refer to convert a logic model into its deterministic approximations, and the term “deterministication” to refer to the process of determinisfying.
• Realization $\xi$: the value of $\xi$ that is produced by random methods such as the Monte Carlo method if probability distribution functions $P(\xi)$ are known, namely an iid sample.

• Data point $\xi_d$: the value of $\xi$ that is actually measured in history, which can be considered an iid sample

• Scenario $\xi$: a possible outcome of $\xi$ to which a probability is assigned, which can be a set of realizations or data points.

An example is given in Figure 3 to illustrate the difference and relation between a scenario and a data point of $\xi$, where the uncertain vector $\xi = [\xi_1, \xi_2]^T$ ($\xi_1, \xi_2 \in \{1, 2, 3\}$) and there are 100 data points that are measured in history, namely the data set $D = \{\xi_d^{(k)}, k = 1, \ldots, 100\}$. As shown in Fig. 3, the area enclosed by the solid curve is the continuous sample space of $\xi$. As a result, the valid scenario space is $\Xi = \{\xi_1^{(k)}, k = 1, \ldots, 8\}$ since point $\xi_1 = (3, 3)$ is outside the sample space $\Xi$. For this particular example, the realized scenario set that corresponds to $D$ is $\mathcal{S} = \{\xi_d^{(k)}; k = 2, \ldots, 8\}$ since scenario $\xi_1^{(1)} = (1, 1)$ was not observed in history although it’s feasible, where a realized scenario means an observed scenario in history. If we order the realized scenarios as $\xi_d^{(2)} = (1, 2)$, $\xi_d^{(3)} = (1, 3)$, $\xi_d^{(4)} = (2, 1)$, $\xi_d^{(5)} = (2, 2)$, $\xi_d^{(6)} = (2, 3)$, $\xi_d^{(7)} = (3, 1)$, and $\xi_d^{(8)} = (3, 2)$, we have, for instance, $\xi_d^{(k)} = \xi_d^{(2)} (k = 1, \ldots, 9)$. Since scenario $\xi_1^{(2)}$ appears 9 times in the data set $D$, one can consider its joint probability as $P[\xi = \xi_d^{(2)}] = 9/100 = 0.09$. The probabilities of other scenarios can be obtained in the same way. Also note that, although scenario $\xi_1^{(1)} = (1, 1)$ is a feasible point in $\Xi$, its probability is considered zero since it is never observed in history.

We consider that data set $D$ has the following properties:

i) it is a finite multiset with a large $D$;

ii) the elements therein are historical measurements which are considered iid-samples of $\xi$.

A more general definition of the joint probability of historical data points of uncertain parameter $\xi$ is given as

DEFINITION 3.2 (on the joint probability of historical data point). Let $D(\xi_d^{(j)}, \zeta)$ be the set of data points in the $\zeta$-vicinity of data point $\xi_d^{(j)}$ in data set $D$, i.e., $D(\xi_d^{(j)}, \zeta) = \{\forall \xi_d^{(k)} \in D: \|\xi_d^{(k)} - \xi_d^{(j)}\| \leq \zeta\}$, and $D_\zeta = |D(\xi_d^{(j)}, \zeta)|$, the joint probability

$$P[\xi = \xi_d^{(j)}] = D_\zeta / D,$$

where $\zeta$ is a small positive scalar.

A pictorial interpretation of the above definition is given in Figure 4. In Definition 3.2, the selection of the bandwidth $\zeta$ influences the accuracy of the estimated joint probability $P[\xi = \xi_d^{(j)}]$. The optimal value of $\zeta$ varies from case to case. Fortunately, there exist various methods, such as the plug-in [9] and cross validation [19] selectors, that one can use for determining the optimal $\zeta$ for a specific case. A numerical example of selecting the best $\zeta$ is provided in Subsection 5.2.
3.2. A data-based deterministic approximation of PCCO. This subsection defines the following data-based deterministic programming problem:

\[
\text{D-DA: } \min_x f(x) \\
\text{s.t. } g(x, \xi_d^{(k)}) \leq 0, (\forall \xi_d^{(k)} \in \square)
\]

where \(\square\) represents a specific data set. Let \(\mathcal{D}_\alpha = \{\xi_d^{(k)}, k = 1, \ldots, D_\alpha\} \subset \mathcal{D}\) be a set that contains all data points in \(\mathcal{D}\) whose joint probabilities are not less than \(\alpha\), this paper calls the process of producing set \(\mathcal{D}_\alpha\) from \(\mathcal{D}\) as \(\alpha\)-process. We developed an algorithm of \(\alpha\)-process as given in the following table. Further let \(\mathcal{D}_\alpha^z \subset \mathcal{D}_\alpha\) be a set of \(z\) data points that are randomly selected from \(\mathcal{D}_\alpha\), next subsection will evaluate the accuracy of D-DA(\(\mathcal{D}_\alpha^z\)), which is the D-DA in (3.1) with \(\square = \mathcal{D}_\alpha^z\), as the deterministic approximation of the PCCO (1.1). Note that, otherwise stated, D-DA(\(\square\)) represents the D-DA in (3.1) with \(\square\) as input data set in the rest of the paper.

**Algorithm 3.1 of \(\alpha\)-process**

Given a data set \(\mathcal{D}\) (\(\mathcal{D} = |\mathcal{D}|\)) and set \(i = 1:\)

1. count the number of data points in the \(\zeta\)-vicinity of \(\xi_d^{(i)}\) and save it to set \(\mathcal{D}_\alpha\) if \(D_\alpha^\zeta \geq \alpha D\),

where \(\zeta\) is the optimal bandwidth that was discussed in Definition 3.2;

2. stop if \(i = D\), otherwise set \(i = i + 1\) and repeat step 1.

3.3. Relations between the feasible spaces of D-DA(\(\mathcal{D}_\alpha^z\)) and PCCO. Denoting

\[
\mathcal{X}_\text{D} (\square) = \{x \in \mathbb{R}^n \mid g(x, \xi_d^{(k)}) \leq 0 (\forall \xi_d^{(k)} \in \square)\}
\]

as the feasible space of D-DA (3.1), we have the following definitions.

**Definition 3.3 (on a deterministic equivalent of PCCO).** Let \(\mathcal{S}^\alpha \) be the particle representation of the uncertainty set \(\Xi_\alpha\), i.e., \(\mathcal{S}^\alpha\) contains all possible scenarios of \(\Xi_\alpha\), D-DA(\(\mathcal{S}^\alpha\)) is considered a deterministic equivalent to PCCO, i.e., \(\mathcal{X}_\text{D}(\mathcal{S}^\alpha) = \mathcal{X}_P\).
Note that $S^r_\alpha$ is not a multiset and $\chi_D(S^r_\alpha)$ means the feasible space of D-DA($S^r_\alpha$) following (3.2).

DEFINITION 3.4 (on the boundary-forming data points of the feasible space of D-DA). If a data set $B^{FS}_\Box = \{s_d^{(k)}, k = 1, \ldots, B^{FS}_\Box\}$ is the SMALLEST subset of $\Box$ that satisfies:

\[(3.3) \quad \chi_D(B^{FS}_\Box) = \{x \in \mathbb{R}^n \mid g(x, \xi_d^{(k)}) \leq 0 (\forall \xi_d^{(k)} \in B^{FS}_\Box)\} = \chi_D(\Box),\]

the elements in $B^{FS}_\Box$ are the boundary-forming data points of feasible space $\chi_D(\Box)$.

Note that $|S^r_\alpha| = \infty$ when $\xi$ is continuous or mixed-integer. Following Definition 3.4, the boundary-forming data set of $\chi_D(S^r_\alpha)$ is denoted as $B_{\Box,\alpha}^{FS}$. For the general case of $\xi$ (i.e., $\xi$ can be integer, continuous, or mixed-integer), we have the following proposition.

PROPOSITION 3.5 (on the relations between the feasible spaces of D-DA($D^z_\alpha$) and PCCO). $\chi_D(D^z_\alpha) \supseteq \chi_P$ and, if $B_{\Box,\alpha}^{FS} \subseteq D^z_\alpha$, $\chi_D(D^z_\alpha) = \chi_P$.

Fig. 5. Pictorial interpretation of boundary-forming, active, and inactive data points in a 2-D $x$-space, where constraints 1-4 are the boundary-forming constraints of the feasible space, 3 and 4 are the boundary-forming constraints of the optimal solution, 3-5 are active constraints, and the rest are inactive constraints. A boundary-forming/active data point is a data point that contributes at least one boundary-forming/active constraint.

Readers can find the proof of this proposition in Appendix B.1. A pictorial explanation of definition 3.4 and proposition 3.5 is provided in Figure 5, which shows that the feasible space of a D-DA for a PCCO problem is determined by a limited number of boundary-forming data points. The rest data points are inactive which can be removed without impacting the feasible space. Proposition 3.5 indicates that, if one can guarantee that the finite data set $D^z_\alpha$ contains all the boundary-forming data points of the deterministic equivalent of PCCO, i.e., the D-DA($S^r_\alpha$), the finite deterministic optimization problem D-DA($D^z_\alpha$) is equivalent to PCCO. Nevertheless, it’s extreme hard to figure out the exact boundary-forming data points for a complex D-DA. Next subsection will investigate the relation between $z$ (i.e., the size of data set $D^z_\alpha$) and the probability that the D-DA($D^z_\alpha$) is equivalent to PCCO without knowing any information about the boundary-forming data points constraints in advance.
3.4. The Accuracy of Approximating PCCO with the finite D-DA($D^*_\alpha$).

Let $D_{\alpha}^{z-1} = D^*_\alpha \setminus \xi_{\alpha}^{(b)}$ and consider D-DA($D_{\alpha}^{z-1}$), i.e., the D-DA($D^*_\alpha$) with the constraints that correspond to data point $\xi_{\alpha}^{(b)}$ removed. Further let $x_D^*(D^*_\alpha)$ and $x_D(D_{\alpha}^{z-1})$ be the optimal solutions of D-DA($D^*_\alpha$) and D-DA($D_{\alpha}^{z-1}$) respectively, we have the following definition on the boundary-forming data point of an optimal solution while Definition 3.4 is on that of a feasible set.

**Definition 3.6 (on the boundary-forming data points of an optimal solution).**

Data point $\xi_{\alpha}^{(k)}$ is a boundary-forming data point for $x_D^*(D^*_\alpha)$ if $x_D^*(D_{\alpha}^{z-1}) < x_D^*(D^*_\alpha)$.

Recall that $x_D^*$ is the optimal solution of the original logic model PCCO (1.1) and $S^\gamma_\alpha$ is the particle representation of the uncertainty set $\Xi_\alpha$, and let $B_{z,\alpha}^{OS}$ denote the boundary-forming data set of $x_D^*(S^\gamma_\alpha)$, we have the following proposition.

**Proposition 3.7.** $x_D^*(D_{\alpha}^{z}) = x_D^*$ if $B_{z,\alpha}^{OS} \subseteq D_{\alpha}^{z}$.

The proofs of proposition 3.7 can be found in Appendix B.2. Let $\rho$ denote the probability that $x_D^*(D_{\alpha}^{z})$ is optimal to PCCO, i.e., $\rho = P[x_D^*(D_{\alpha}^{z}) = x_D^*]$, and recall that $D$, $D_\alpha$, $z$, and $B_{z,\alpha}^{OS}$ denote the numbers of data points in sets $D$, $D_\alpha$, $D_{\alpha}^{z}$, and $B_{z,\alpha}^{OS}$ respectively, we have the following theorem on the lower bound of $\rho$ for a given $z$ under Assumption 1.

**Assumption 1.** $D_\alpha$ contains all the boundary-forming data points of $x_D^*(S^\gamma_\alpha)$, i.e., $B_{z,\alpha}^{OS} \subseteq D_\alpha$.

**Theorem 3.8.** Let $\bar{B}_{z,\alpha}^{OS}$ be an upper bound of $B_{z,\alpha}^{OS}$, a lower bound of $\rho$ is given as

\[
\rho(z) = 1 + \sum_{k=1}^{n} (-1)^k \frac{\bar{B}_{z,\alpha}^{OS}}{k} \binom{D_\alpha - kaD}{D_z}
\]

under Assumption 1, where $\binom{\cdot}{\cdot}$ is the binomial coefficient.

While the proof can be found in Appendix B.3, the estimation of $\bar{B}_{z,\alpha}^{OS}$ and the rationality of Assumption 1 are discussed in next subsection. The assertion in this theorem implies that a sufficient condition for $\rho \geq \rho$ is $z = \bar{\rho}^{-1}(\rho)$.

3.5. Discussions and Summary. To sum up, in this section, we consider D-DA($D^*_\alpha$) as the deterministic approximation for the newly defined optimization problem under uncertainty—the PCCO (1.1)—and revealed some of its important properties. This subsection provides some further discussions on the proposed data-based deterministication.

3.5.1. A proper estimation of $\bar{B}_{z,\alpha}^{OS}$. It’s generally very hard to obtain the exact number of boundary-forming data points for the optimal solution of a D-DA.

Therefore, it’s necessary to obtain a proper estimation of $\bar{B}_{z,\alpha}^{OS}$ before the assertion of Theorem 3.8 can be used to determine the needed $z$ for a required $\rho$. We start discussing how to obtain a proper estimation of $\bar{B}_{z,\alpha}^{OS}$ from the following proposition.

**Proposition 3.9.** For a D-DA in (3.1), if $f(x)$ is linear and $g(x, \xi)$ is convex on $x$, the number of boundary-forming data points of its optimal solution is not more than $n$.

While the proof is provided in Appendix B.4, this proposition asserts that $\bar{B}_{z,\alpha}^{OS} = n$, where $n$ is the size of $x$, for D-DAs which are convexly-constrained linear programs. Although this assertion cannot be directly applied to a more general case, it can
provide a proper estimation. From the proof, we know that \( n \) is in fact a very loose upper bound of \( B_{\Omega}^{\text{OS}} \) for convexly constrained linear programs. To be specific, \( n \) is an upper bound for the number of support constraints (see Definition 4 in [11]) and the number of boundary-forming data points is generally less than that of the support constraints. Therefore, \( n \) is a valid upper bound of \( B_{\Omega}^{\text{OS}} \) (including \( B_{\eta,\alpha}^{\text{OS}} \)) for most of the general cases although it cannot guarantee for all.

3.5.2. Rationality of the data-based determinization. We first discuss the rationality of the direct use of historical data as input for determinisfying a PCCO problem. For the uncertainty in engineering problems, the construction of the probability triple, i.e., \((\Xi, \mathcal{F}, P)\), is generally based on engineers’ experience which mainly comes from the analysis on historical data and/or observations together with some assumptions. While mistakes and/or inaccuracy can occur in the process of constructing the probability triple, this issue does not exist in the data-based methods since they don’t need this process. Second, we discuss the rationality of Assumption 1. Normally, we care about the outcomes of \( \xi \) since they don’t need this process. While mistakes and/or inaccuracys can occur in the process of collecting the probability triple, this issue does not exist in the data-based methods.

4. The Proposed Solution Paradigm for PCCO: Strategic Data Selection. If D-DA(\( D_{\alpha} \)) investigated in Section 3 is computational intractable, the proposed solution paradigm will use D-DA(\( D_{\alpha} \)) as the DA of PCCO instead, where \( D_{\alpha} \subseteq D_{\alpha}^{\xi} \). In other words, a subset \( D_{\alpha} \) of \( D_{\alpha}^{\xi} \) is used as the input to D-DA (3.1). Since \( D_{\alpha} \subseteq D_{\alpha}^{\xi} \subseteq D_{\alpha} \), It’s straightforward to know that, the feasible sets

\[
\mathcal{X}_{D}(D_{\alpha}) \supseteq \mathcal{X}_{D}(D_{\alpha}^{\xi}) \supseteq \mathcal{X}_{D}(D_{\alpha}) \quad \text{and, hence,} \quad x_{D}(D_{\alpha}) \leq x_{D}(D_{\alpha}^{\xi}) \leq x_{D}(D_{\alpha}).
\]

Proposition 3.7 indicates that \( x_{D}(D_{\alpha}) = x_{D}^{\ast} \) under Assumption 1, and hence \( x_{D}(D_{\alpha}^{\xi}) \leq x_{D}^{\ast}(D_{\alpha}) \leq x_{D}^{\ast}(D_{\alpha}) \). While the optimality gap between \( x_{D}^{\ast}(D_{\alpha}) \) and \( x_{D}^{\ast} \) was discussed in the previous section, this section aims at developing a methodology of strategic data selection (SDS) for obtaining the data set \( D_{\alpha}^{\xi} \) which possesses two features: 1) the number of elements \( z_{\eta} = |D_{\alpha}^{\xi}| \ll z_{\eta} \), and 2) the gap between \( x_{D}^{\ast}(D_{\alpha}) \) and \( x_{D}^{\ast}(D_{\alpha}) \) is sufficiently small such that \( x_{D}^{\ast}(D_{\alpha}) \) is also a good estimation to \( x_{D}^{\ast} \). The SDS algorithms for continuous and integer/mixed-integer cases of \( \xi \) are presented in Subsections 4.1 and 4.3, respectively.

4.1. The SDS algorithm for continuous \( \xi \). When the uncertain parameters are continuous, i.e., \( \xi \in \Xi \subseteq \mathbb{R}^{r} \), we developed an SDS algorithm as detailed in Algorithm 4.1, where \( \eta \) is a key parameter. Then, we have the following theorem on the relation between \( z_{\eta} \) and \( \eta \).

**Theorem 4.1** (on \( z_{\eta} \) obtained by Algorithm 4.1). When \( \xi \in \Xi \subseteq \mathbb{R}^{r} \), we have

\[
\begin{align*}
  z_{\eta} = z & \quad \eta = 0 \\
  z_{\eta} \leq \bar{z}_{\eta} & \quad 0 < \eta < \bar{\eta} \\
  z_{\eta} = 1 & \quad \eta \geq \bar{\eta}
\end{align*}
\]

If \( 0 < \eta < \bar{\eta} \), \( z_{\eta} \) is a parameterized random variable whose expectation and upper
Algorithm 4.1 of strategic data selection for continuous $\xi$

Given a data set $D_\alpha^z$:
1. randomly select a data point $\xi_d^{(0)}$ and set $i = 1$;
2. randomly select another data point $\xi_d^{(i)}$ in $D_\alpha^z$ which satisfies

\begin{align*}
\|\xi_d^{(i)} - \xi_d^{(j)}\| & \geq 2\eta, \forall j = 0, \ldots, i - 1
\end{align*}

where $0 < \eta \leq \bar{\eta}$;
3. save $\xi_d^{(i)}$ to a new set $D_\alpha^n$ and discard all data points in the $\eta$-vicinity of $\xi_d^{(i)}$ (including $\xi_d^{(i)}$) from $D_\alpha^z$;
4. stop and report set $D_\alpha^n$ if there is no data point $\xi_d^{(i)}$ in $D_\alpha^z$ that satisfies (4.2), otherwise go to step 5;
5. set $i = i + 1$ and repeat steps 2-4.

Note: $\bar{\eta}$ is the minimum $\eta$ which results in that there do not exist two data points in $D_\alpha^z$ which satisfy condition (4.2) (see an illustrative example in Figure 4).

The proof of theorem 4.1 is provided in Appendix C.1. The second expression in (4.3) indicates that, when $\eta$ is small, $z_\eta$ drops significantly as $\eta$ increases.

Given that a data point $\xi_{d}^{(k)}$ contributes $m$ constraints to D-DA (3.1) since $g : \mathbb{R}^{n+r} \rightarrow \mathbb{R}^m$ (note that $\xi \in \Xi \subset \mathbb{R}^r$) in the PCC (1.1b), there are $z \times m$ inequality constraints in D-DA($D_\alpha^z$). We call the constraint $\tilde{g}_i(x, \xi_{d}^{(k)}) \leq 0$ ($i = 1, \ldots, B_c$) a boundary-forming constraint of the optimal solution $x_{D_\alpha}^*$ if removing it from D-DA($D_\alpha^z$) results in a change in $x_{D_\alpha}^*$, where the superscript $k$ means that it is contributed by the $k$th boundary-forming data point $\xi_{d,B}^{(k)}$ (as defined in Definition 3.6). It’s straightforward to know that a boundary-forming constraint is also an active constraint. In other words, the equal sign holds when $x_{D_\alpha}^*$ is substituted into this constraint, i.e., $\tilde{g}_i(x_{D_\alpha}^*, \xi_{d,B}^{(k)}) = 0$. Then, suppose there are $B_c$ boundary-forming constraints for $x_{D_\alpha}^*$, we have:

\[
\tilde{g}(x_{D_\alpha}^*, \xi_{d,B}) = \begin{cases} 
\tilde{g}_1(x_{D_\alpha}^*, \xi_{d,B}^{(1)}) = 0 \\
\vdots \\
\tilde{g}_{B_c}(x_{D_\alpha}^*, \xi_{d,B}^{(B_c)}) = 0
\end{cases}, \text{ where } \tilde{g}(x, \xi) = \begin{cases} 
\tilde{g}_1(x, \xi_1) \\
\vdots \\
\tilde{g}_{B_c}(x, \xi_{B_c})
\end{cases},
\]

$\tilde{g} : \mathbb{R}^{n+(\times B_c)} \rightarrow \mathbb{R}^{B_c}$, $\xi_i \in \mathbb{R}^r$ ($i = 1, \ldots, B_c$), $\tilde{\xi} = [\xi_1^T, \ldots, \xi_{B_c}^T]^T \in \mathbb{R}^{(\times B_c) \times 1}$ and $\xi_{d,B}^{(1)} = [\xi_{d,B}^{(1)}]^T, \ldots, [\xi_{d,B}^{(B_c)}]^T \in \mathbb{R}^{(\times B_c) \times 1}$ which implies that the $B_c$ boundary-forming constraints are contributed by $B(\tilde{\xi})$ boundary-forming data points (note that $B_c \geq B(\tilde{\xi})$). In other words, it’s possible that two or more elements in $\xi_{d,B}$ are identical, which means this data point (say $\xi_{d,B}^{(i)}$) contributes more than one boundary-forming constraints. Functions $\tilde{g}$ are the $B_c$ constraints out of $z \times m$ whose equal sign holds for the optimal solution $x_{D_\alpha}^*$. According to the discussion in Subsubsection 3.5.1, it’s reasonable to assume that $B_c \leq n$. Then, we have the following proposition.
Proposition 4.2 (Implicit function theorem). When \( B_z = n \), there exists a vector-valued function \( x = h(\xi) \) (\( h : \mathbb{R}^{(r \times n)} \to \mathbb{R}^n \)) which is equivalent to \( \tilde{g}(x, \tilde{\xi}) = 0 \) (\( \tilde{g} : \mathbb{R}^{n + (r \times n)} \to \mathbb{R}^n \)) in the vicinity of \((x_D^\alpha(D^\alpha), \xi_{d,B})\) if \( \tilde{g} \) is continuously differentiable and the Jacobian matrix of \( \tilde{g} \) with respect to \( x \) evaluated at \( x_D^\alpha(D^\alpha) \) is invertible.

Proposition 4.2 is actually the Implicit Function Theorem whose proof can be found in [22]. Algorithm 4.1 guarantees that, for any boundary-forming data point \( \xi_{d,B}^{(i)}(i = 1, \ldots, B(z)) \) of \( x_D(D^\alpha_z), D^\alpha_z \) contains a data point \( \tilde{\xi}_{d,B}^{(i)} \) which satisfies \( \| \tilde{\xi}_{d,B}^{(i)} - \xi_{d,B}^{(i)} \| \leq 2\eta \). Recall that we used \( x_D(D^\alpha_z) \) to denote the optimal solution of D-DA\((D^\alpha_z)\) and let \( \varphi = 1 - \frac{\| x_D(D^\alpha_z) - x_D(\tilde{D}^\alpha_z) \|}{\| x_D(D^\alpha_z) \|} \) denote the accuracy of approximating D-DA\((D^\alpha_z)\) with D-DA\((\tilde{D}^\alpha_z)\). We have \( \varphi = 1 \) when D-DA\((D^\alpha_z)\) is equivalent to D-DA\((\tilde{D}^\alpha_z)\). It suffices to know that \( \varphi \) is partially determined by \( \eta \) for a given \( D^\alpha_z \) and we have the following theorem.

Theorem 4.3 (on a lower bound of \( \varphi \) under Algorithm 4.1). If the boundary-forming constraints are the only active constraints at \( x_D(D^\alpha_z) \) and \( \eta \) is small, following the conditions in Proposition 4.2, a lower bound of \( \varphi \) is given as

\[
\varphi(\eta) = 1 - \frac{2\sqrt{n}\| H(\xi_{d,B}) \| \| \eta + \frac{\sqrt{n}}{3!} \| H'(\xi_{d,B}) \| \| \eta \|^2 + \frac{2\sqrt{n}}{3!} \| H''(\xi_{d,B}) \| \| \eta \|^3 + \cdots}{\| x_D(D^\alpha_z) \|}
\]

where \( H(\xi_{d,B}) \) and \( H'(\xi_{d,B}) \) are the Jacobian matrix and the Hessian tensor, respectively, of \( h(\xi) \) evaluated at \( \xi_{d,B} \), and \( H''(\xi_{d,B}) \) is a tensor which is a higher-dimensional generalization of a matrix and contains the third mixed partial derivatives.

The proof of Theorem 4.3 is provided in Appendix C.2 while a discussion on the benefits of SDS is provided in next subsection.

4.2. Discussion on the benefits of SDS. First, it’s worth noting that the computational complexity of the D-DA (3.1) is related to their numbers of input data points, i.e., the size of \( \square \). For simple cases, such as linear cases, the computational complexity is approximately proportional to the numbers of input data points. However, for nonconvex cases, the computational burden generally grows exponentially as numbers of input data points increase. Second, from theorems 4.1 and 4.3, we know that both \( z_n \), which is related to computational complexity, and \( \varphi \), which is related to accuracy, decrease as \( \eta \) increases. To be specific, Theorem 4.3 implies that

\[
\varphi(\eta) \approx \begin{cases} 
1 - \frac{2\sqrt{n}\| H(\xi_{d,B}) \| \| \eta \|}{\| x_D(D^\alpha_z) \|} & \eta \leq \eta_1 \\
1 - \frac{2\sqrt{n}\| H(\xi_{d,B}) \| \| \eta + \frac{\sqrt{n}}{3!} \| H'(\xi_{d,B}) \| \| \eta \|^2}{\| x_D(D^\alpha_z) \|} & \eta_1 < \eta \leq \eta_2 \\
1 - \frac{2\sqrt{n}\| H(\xi_{d,B}) \| \| \eta + \frac{\sqrt{n}}{3!} \| H'(\xi_{d,B}) \| \| \eta \|^2 + \frac{2\sqrt{n}}{3!} \| H''(\xi_{d,B}) \| \| \eta \|^3}{\| x_D(D^\alpha_z) \|} & \eta_2 < \eta \leq \eta_3 , \\
\vdots & \\
1 - \frac{2\sqrt{n}\| H(\xi_{d,B}) \| \| \eta + \frac{\sqrt{n}}{3!} \| H'(\xi_{d,B}) \| \| \eta \|^2 + \frac{2\sqrt{n}}{3!} \| H''(\xi_{d,B}) \| \| \eta \|^3 + \cdots}{\| x_D(D^\alpha_z) \|} & \eta_j < \eta \leq \tilde{\eta}
\end{cases}
\]

which means, when \( \eta \) is small, \( \varphi(\eta) \) is close to be linear to \( \eta \). By comparing \( \tilde{\eta} \) and \( \varphi \), we realize that \( \tilde{\eta} \) drops significantly while \( \varphi \) decreases in a much slower rate. Therefore, the main benefit of SDS is that it significantly reduces the computational burden of the proposed data-based DA, i.e., D-DA\((D^\alpha_z)\), with a tradeoff of slightly decrease in accuracy.

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4.3. The SDS algorithm for integer/mixed-integer $\xi$. When the uncertain parameters are mixed-integer, i.e., $\xi \in \mathcal{Z} \subset (\mathbb{Z}^{r_1}, \mathbb{R}^{r_2})$ ($r_1 + r_2 = r$), let $\mathcal{D}_\alpha^z|_\mathcal{Z} = \{c_{d,\mathcal{Z}}^{(k)}; k = 1, \ldots, z\}$ be the projection of $\mathcal{D}_\alpha^z$ in the $\mathbb{Z}^{r_1}$-space, and $\mathcal{I}$ be the index set of the elements in $\mathcal{U}[\mathcal{D}_\alpha^z|_\mathcal{Z}]$ (where $\mathcal{U}[\cdot]$ denotes the underlying set$^5$ of a multiset), we divide $\mathcal{D}_\alpha^z$ into $I = |\mathcal{I}|$ subsets, i.e., $\mathcal{D}_{\alpha,i}^z$ ($i \in \mathcal{I}$), making the elements in each of these subsets have the same integer part. Namely, if we denote all data points in $\mathcal{D}_{\alpha,i}^z$ as $\xi_{d,i}^{(i,j)} = (\xi_{d,\mathcal{Z}}^{(i,j)}, \xi_{d,\mathcal{R}}^{(i,j)}$ ($j = 1, \ldots, z_i$), they share the same integer part, i.e., $\xi_{d,\mathcal{Z}}$, where $z_i$ is the number of data points in $\mathcal{D}_{\alpha,i}^z$. Based on these notations, Algorithm 4.2 of SDS is developed for the cases where $\xi$ is integer/mixed-integer. Then, we have the following corollary of Theorem 4.1.

Algorithm 4.2 of strategic data selection for cases of integer/mixed-integer $\xi$

Given a data set $\mathcal{D}_\alpha^z = \bigcup_{i \in \mathcal{I}} \mathcal{D}_{\alpha,i}^z$:

1. set $i = 1$
2. randomly select a data point $\xi_{d,i}^{(i,0)}$ in $\mathcal{D}_{\alpha,i}^z$ and set $j = 1$;
3. randomly select another data point $\xi_{d,i}^{(i,j)}$ in $\mathcal{D}_{\alpha,i}^z$ which satisfies
   \[(4.6) \quad \|\xi_{d,\mathcal{R}}^{(i,j)} - \xi_{d,\mathcal{R}}^{(i,k)}\| \geq 2\eta_i, \forall k = 0, \ldots, j - 1;\]
4. save $\xi_{d,i}^{(i,j)}$ to a new set $\mathcal{D}_\alpha^y$ and discard all data points in the $\eta_i$-vicinity of $\xi_{d,i}^{(i,j)}$ (including $\xi_{d,i}^{(i,j)}$) from $\mathcal{D}_{\alpha,i}^z$;
5. go to step 7 if there is no data points in $\mathcal{D}_{\alpha,i}^z$ that satisfies (4.6), otherwise go to step 6;
6. set $j = j + 1$ and repeat steps 3-5.
7. stop and report set $\mathcal{D}_\alpha^y$ if $i = |\mathcal{I}|$, otherwise go to step 8;
8. set $i = i + 1$ and repeat steps 2-7.

Note: $\bar{\eta}_i$ is the minimum $\eta_i$ which results in that there do not exist two data points in $\mathcal{D}_{\alpha,i}^z$ which satisfy condition (4.6).

Corollary 4.4 (on $\eta_{\bar{\eta}}$ obtained by Algorithm 4.2). When $\xi \in \mathcal{Z} \subset (\mathbb{Z}^{r_1}, \mathbb{R}^{r_2})$ ($r_1 + r_2 = r$), we have

\[(4.7) \quad \begin{cases} 
\frac{\partial (\mathbb{E}[z_{\eta}])}{\partial \eta_i} \leq 0, & \bar{\eta}_i = \sum_i \tilde{\eta}_{i,i} \text{ and } \tilde{\eta}_{i,i} \propto \frac{1}{\eta_i^2} \ (i \in \mathcal{I}).
\end{cases}\]

The proof of Corollary 4.4 is in Appendix C.3.

For the cases where $\xi$ is mixed-integer, we also denote the boundary-forming constraints of $x_0^*(\mathcal{D}_\alpha^z)$ in D-DA($\mathcal{D}_\alpha^z$) as $\tilde{g}(x, \xi_{d,B}) \leq 0$ ($\tilde{g}: \mathbb{R} \times \mathcal{Z}^{B_L} \to \mathbb{R}^{B_L}$), which are contributed by $B(z)$ boundary-forming data points. Let $\varepsilon_{d,B}^{(k)} (k = 1, \ldots, B(z))$ denote

\footnote{A underlying set is the set of distinct elements of a multiset}
a boundary-forming data point of \( x^*_D(D^*_\alpha) \) and assume that it belongs to subset \( D^*_\alpha,i \) (\( i \in I \)). When \( \eta_i \) (\( \forall i \in I \)) are small, algorithm 4.2 guarantees that \( D^*_\alpha \) contains a data point \( \xi^{(k)} \) which satisfies \( \xi^{(k)} = \xi^{(k)}_{d,\mathcal{Z}} \) and \( \|\xi^{(k)}_{d,\mathcal{R}} - \xi^{(k)}_{d,\mathcal{B},\mathcal{R}}\| \leq 2\eta_i \). Given that, for mixed-integer \( \xi \), the data points in a small vicinity of \( \xi^{(k)} \) have the same integer part as \( \xi^{(k)} \), one can consider \( \xi^{(k)} \in \mathbb{Z}^n \) fixed and reformulate \( \tilde{g}(x, \xi) \) as \( \tilde{g}(x, \xi_{\mathcal{R}}) \) (\( \tilde{g} : \mathbb{R}^{n+(r_2 \times B_\mathcal{R})} \rightarrow \mathbb{R}^{B_\mathcal{R}} \)). According to Proposition 4.2 and under the conditions therein, we know that there exists a vector-valued function \( x = h(\xi_{\mathcal{R}}) \) (\( h : \mathbb{R}^{(r_2 \times n)} \rightarrow \mathbb{R}^n \)) which is equivalent to \( \tilde{g}(x, \xi_{\mathcal{R}}) = 0 \) in the vicinity of \((x^*_D(D^*_\alpha), \xi_{d,\mathcal{B}}) \). (Note that \( B_\mathcal{R} = n \under the condition in Proposition 4.2.). Then, we have the following corollary of Theorem 4.3.

**Corollary 4.5** (on \( \varphi \) under Algorithm 4.2). If the boundary-forming constraints are the only active constraints at \( x^*_D(D^*_\alpha) \) and \( \eta_i \) (\( \forall i \in I \)) are small enough, following the conditions in Proposition 4.2, a lower bound of \( \varphi \) is given as (4.9)

\[
\varphi(\eta_1, \ldots, \eta_\mathcal{B}) = 1 - \frac{\|H(\xi_{d,\mathcal{B},\mathcal{R}})\|\dot{\eta} + \frac{1}{2}\|H'(\xi_{d,\mathcal{B},\mathcal{R}})\|\dot{\eta}^2 + \frac{1}{3!}\|H''(\xi_{d,\mathcal{B},\mathcal{R}})\|\dot{\eta}^3 + \cdots}{\|x^*_D(D^*_\alpha)\|}
\]

where \( \dot{\eta} = 2\sqrt{\eta_1^2 + \cdots + \eta_\mathcal{B}^2} \).

The proof of Corollary 4.5 can be found in Appendix C.4.

When the uncertain parameters are pure integers, i.e., \( \xi \in \Xi \subset \mathbb{Z}^r \), step 3 in Algorithm 4.2 is not necessary since \( r_2 = 0 \) and, consequently, \( \xi_{d,\mathcal{R}} = 0 \). Moreover, the data points in each subset \( D^*_\alpha,i \) (\( i \in I \)) are identical. Thus, Algorithm 4.2 picks one data point from each of these subsets, which results in that \( D^*_\alpha \) is the underlying set of \( D^*_\alpha,i \), i.e., \( D^*_\alpha = \Xi[D^*_\alpha] \) and \( z_{\eta} = I \). It’s worth noting that, generally, \( I << |D^*_\alpha| \) (see the illustrative example in Subsection 3.1). Then, we have the following lemma on Algorithm 4.2 for the cases of pure integer \( \xi \).

**Lemma 4.6** (on SDS algorithm for discrete \( \xi \)). When \( \xi \in \Xi \subset \mathbb{Z}^r \), \( x^*_D(D^*_\alpha) = x^*_D(z) \). Further, if \( D^*_\alpha = D_\alpha \) (i.e., Algorithm 4.2 uses \( D_\alpha \) as input data set), \( x^*_D(D^*_\alpha) = x^*_p \) under Assumption 1.

The proof of this lemma is provided in Appendix C.5. Let \( \omega \) denote the probability that \( x^*_D(D^*_\alpha) \) is optimal to PCCO, i.e., \( \omega = P[x^*_D(D^*_\alpha) = x^*_p] \). Lemma 4.6 implies that \( \omega = 1 \) when Algorithm 4.2 uses \( D_\alpha \) as input data set and Assumption 1 holds. The following proposition presents a trivial property of \( \omega \) for the PCCO problems with continuous/mixed-integer \( \xi \), of which the proof is given in Appendix C.6.

**Proposition 4.7** (on \( \omega \) for continuous/mixed-integer \( \xi \)). If \( \eta_1 < \eta_2 \), we have \( P[\omega(\eta_1) > \omega(\eta_2)] \geq P[\omega(\eta_1) < \omega(\eta_2)] \).

5. Application and Numerical Experiments. This section applies the proposed methods to a fundamental decision-making problem under uncertainty in electric power systems, i.e., the optimal power flow (OPF) with uncertain renewable energy, e.g., solar and wind power. The OPF determines the best operating levels of power generators/plants in order to meet demands given throughout a transmission/distribution network, usually with the objective of minimizing generation cost [24]. The original formulation of OPF, i.e., the formulation without uncertainty, is
given as

\[
\text{OPF}: \quad \min_{p^G, \theta} \quad c(p^G) = \sum_{i \in G} (c_{i,2}(p_i^G)^2 + c_{i,1}p_i^G + c_{i,0})
\]

\[
\text{s.t.} \quad B\theta = Ap^G + Cp_R + d,
\]

\[
|B_{ij}(\theta_i - \theta_j)| \leq P_{ij}^{\text{max}}, \forall \{i, j\} \in \mathcal{E}
\]

\[
p_i^{\text{min}} \leq p_i^G \leq p_i^{\text{max}}, \forall i \in \mathcal{G}
\]

where (5.1b) is the DC power flow (DCPF) equation, and A, B, and C are \(n \times g\), \(n \times n\), and \(n \times r\) matrices whose elements are given as follows respectively:

\[
A_{ij} = \begin{cases} 
1, & \text{if the } j\text{th power generator is connected to the } i\text{th node} \\
0, & \text{otherwise}
\end{cases}
\]

\[
B_{ij} = \begin{cases} 
-\frac{b_{ij}}{\sum_{k:(k,j) \in \mathcal{E}} b_{kj}}, & \{i, j\} \in \mathcal{E} \\
0, & \text{otherwise}
\end{cases}
\]

\[
C_{ij} = \begin{cases} 
1, & \text{if the } j\text{th renewable generator is connected to the } i\text{th node} \\
0, & \text{otherwise}
\end{cases}
\]

A nomenclature is given in the Table 1.

| A. Sets and Indices | B. Parameters | C. Decision Variables | D. Uncertain Variable |
|---------------------|---------------|-----------------------|-----------------------|
| \(\mathcal{E}\) Set of transmission/distribution lines | \(b_{ij}\) Susceptance of the transmission line \(\{i, j\} \in \mathcal{E}\) | \(p^G\) \(g \times 1\) vector of baseline generations for meeting the demand \(d\) with \(p^R\) | \(\xi_d\) Difference between the forecasted and real net demands of the \(d\)th load |
| \(\mathcal{G}\) Set of \(g\) power generators/plants | \(c_i\) Unit fuel cost of the \(i\)th power plant in $/\text{MWh}, \text{where } i \in \mathcal{G}\) | \(p^R\) \(r \times 1\) vector of the forecast outputs of renewable generators | \(\xi\) Difference between the forecasted and real net demands of the \(d\)th load |
| \(\mathcal{N}\) Set of \(n\) nodes/buses | \(d\) \(n \times 1\) vector of electricity demands | \(\lambda_i\) Participation factor of the \(i\)th power generator/plant on meeting the uncertain net load, where \(0 \leq \lambda_i \leq 1\) and \(\sum_{i \in \mathcal{R}} \lambda_i = 1\) | \(\theta\) Phase angle of the \(d\)th node/bus, where \(\theta_1 = 0\) |
| \(\mathcal{R}\) Set of \(r\) renewable generators | \(p_i^{\text{max}}\) Power limit on the transmission line \(\{i, j\} \in \mathcal{E}\) | \(p_i^{\text{max}}\) Lower bound on power generation of \(i\)th generator/plant, where \(i \in \mathcal{G}\) | \(\theta_d\) Phase angle of the \(d\)th node/bus, where \(\theta_1 = 0\) |
| \(p_i^{\text{max}}_g\) Upper bound on power generation of \(i\)th generator/plant, where \(i \in \mathcal{G}\) | \(p_i^{\text{max}}_R\) Upper bound on power generation of \(g\)th generator/plant | \(\theta_d\) Phase angle of the \(d\)th node/bus, where \(\theta_1 = 0\) | \(\theta_d\) Phase angle of the \(d\)th node/bus, where \(\theta_1 = 0\) |

### 5.1. PCCO formulation for OPF under uncertainty

When the uncertainty is considered, the real-time renewable generation \(\hat{p}^R = p^R + \xi\), where \(\xi \in \Xi \subset \mathbb{R}^r\) is the uncertain component. The power generation \(\hat{p}^G\) needs to be adjusted in real-time such that the DCPF equation holds in real-time, i.e., \(B\hat{\theta} = Ap^G + C\hat{p}^R + d\) where the real-time phase angle \(\hat{\theta} = \theta - \Delta\theta\). With a so-called affine control, i.e.,
\( \hat{p}_i^G = p_i^G + \lambda_t \sum_{j \in G} \xi_j \) (\( \forall i \in G \)), the real-time DCPF equation is given as:

\[ B(\theta - \Delta \theta) = A(p^G + e^T \xi \lambda) + C(p^R + \xi) + d \]

where \( e \) is a \( r \times 1 \) identity vector. By comparing (5.2) to (5.1b), we know that

\[ \Delta \theta = -\hat{B}(Ae^T \xi \lambda + C \xi) \]

where \( \hat{B} = \begin{bmatrix} 0 & 0 \\ 0 & \hat{B}^{-1} \end{bmatrix} \) and \( \hat{B} \) is a \((n - 1) \times (n - 1)\) matrix obtained by removing the first row and column from \( B \). Denote \( \Delta \theta_i = -[\hat{B}(Ae^T \xi \lambda + C \xi)]_{ii} \), the PCCO formulation of OPF under uncertainty of renewable energy is given as

\[ \text{(5.3a)} \quad \text{p-OPF:} \quad \min_{p^G, \theta, \lambda} \mathbb{E}[c(p^G + e^T y \lambda)] \]

s.t. (5.1b)

\[ \left| B_{ij} \left( \theta_i - [\hat{B}(Ae^T y \lambda + C y)]_{ij} - \theta_j + [\hat{B}(Ae^T y \lambda + C y)]_{ij} \right) \right| \leq p_{ij}^{\text{max}}, \forall \{i, j\} \in \mathcal{E} \]

\[ \sum_{i \in \mathcal{G}} \lambda_i = 1 \]

\[ p_{i, \min} \leq p_i^G + \lambda_i e^T y \leq p_{i, \max}, \forall i \in \mathcal{G} \]

\[ \forall y \in \{ y \in \mathbb{E} \mid \mathbb{P}[\xi = y] \geq \alpha \} \],

where \( \mathbb{E}[c(p^G + e^T y \lambda)] = \sum_{i \in \mathcal{G}} (c_{i, 2}(p_i^G)^2 + \mathbb{V}[e^T y \lambda_i^2] + c_{i, 1}p_i^G + c_{i, 0}) = c(p^G) + \mathbb{V}[e^T y \sum_{i \in \mathcal{G}} (c_{i, 2} \lambda_i^2)], \) given that \( \mathbb{E}[e^T y] = 0 \) and \( \mathbb{V}[\cdot] \) denotes variance.

Applying the findings in Section 3, the D-DA(\( \mathcal{D}_n^a \)) and D-DA(\( \mathcal{D}_n^p \)) of (p-OPF) are given as the follows respectively.

\[ \text{(5.4a)} \quad \text{d-OPF(} \mathcal{D}_n^a \text{):} \quad \min_{p^G, \theta, \lambda} \quad c(p^G) + \left( \sum_{i \in \mathcal{G}} (c_{i, 2} \lambda_i^2) \right) \frac{1}{z} \sum_{k=1}^{z} \left( e^T \xi_{kd}^{(k)} \right) \]

s.t. (5.1b)

\[ \left| B_{ij} \left( \theta_i - [\hat{B}(Ae^T \xi_{kd}^{(k)} \lambda + C \xi_{kd}^{(k)})]_{ij} - \theta_j + [\hat{B}(Ae^T \xi_{kd}^{(k)} \lambda + C \xi_{kd}^{(k)})]_{ij} \right) \right| \leq p_{ij}^{\text{max}}, \forall \{i, j\} \in \mathcal{E} \]

\[ \sum_{i \in \mathcal{G}} \lambda_i = 1 \]

\[ p_{i, \min} \leq p_i^G + \lambda_i e^T \xi_{kd}^{(k)} \leq p_{i, \max}, \forall i \in \mathcal{G} \]

\[ \forall \xi_{kd}^{(k)} \in \mathcal{D}_n^a, \]

where \( D_n^a \) is the number of data points in the \( \eta \)-vicinity of \( \xi_{kd}^{(k)} \) as defined in Definition 3.2.
5.2. Test Systems and Scenario Sets. We used three representative test systems, i.e., IEEE 6, 39, and 118-bus systems [1]. The problem sizes of the p-OPFs for these test systems are provided in Table 2 (recall that \( n \) and \( r \) are the numbers of decision and uncertain variables respectively, and \( m \) is the number of constraints that are contributed by one data point). In this numerical experiment, \( \alpha = 1\% \) is considered. Table 2 also tabulates the sizes of sets \( D, D_{\alpha}, D_{\alpha}^{x}, \) and \( D_{\eta}^{\alpha} \) (recall that they are the original historical data set, the data set after the \( \alpha \)-process, the input data sets of \( \text{D-DA}(D_{\alpha}^{x}) \) and \( \text{D-DA}(D_{\eta}^{\alpha}) \), respectively) for each case.

| Case     | \( n \) | \( r \) | \( m \) | \( D \)  | \( \alpha \) | \( \alpha_{\text{opt}} \) | \( \rho \) | \( z \) | \( z_{\eta=\alpha} \) |
|----------|---------|---------|---------|----------|--------------|----------------|--------|------|----------------|
| IEEE-6   | 9       | 2       | 16      | 1000     | 0.09         | 0.05           | 685    | 0.80 | 20             |
| IEEE-39  | 58      | 2       | 96      | 5000     | 0.12         | 0.01           | 4459   | 0.99 | 678            |
| IEEE-118 | 155     | 10      | 315     | 10000    | 0.16         | 0.01           | 9762   | 0.99 | 773            |

In this engineering application, both \( x \) and \( \xi \) are continuous. Hence, the above sets are determined following the path below:

\[
\begin{align*}
D & \xrightarrow{\alpha-\text{Process}} D_{\alpha} \quad \text{with the optimal } \zeta \\
& \xrightarrow{\text{Randomly pick } z \text{ data points}} D_{\alpha}^{z} \quad \text{with } \eta = \zeta \\
& \xrightarrow{\text{Algorithm 4.1}} D_{\eta}^{\alpha}
\end{align*}
\]

The value of \( \zeta \) for each case listed in Table 2 are the values that make the resulting probability distribution smoothest. The case of IEEE 39-bus system is given in Figure 6 as an example for elaborating this idea. It can be observed that, with an \( \zeta \) at around 0.12, we obtain a smooth probability distribution. Both bigger and smaller \( \zeta \) will result in less smooth distribution.

![Fig. 6. Example of determining \( \zeta \).](image)

5.3. Results and Analysis. Recall that \( x_{D}^{*}(D_{\alpha}) = x_{\text{opt}}^{*} \), under Assumption 1, we use D-DA\((D_{\alpha})\) as the reference for evaluating the performance of D-DA\((D_{\alpha}^{x})\) and D-DA\((D_{\eta}^{\alpha})\), i.e., the proposed DAs of PCCO (1.1). For each IEEE test case, the computational times and optimality gaps of D-DA\((D_{\alpha}^{x})\) and D-DA\((D_{\eta}^{\alpha})\) are compared in Table 3. First, we can observed that the numerical results satisfy relation (4.1).
Second, which is more important, D-DA(Dα) is an accurate approximation to D-DA(DZα) with a lower computational burden.

| Test system | D-DA | Number of constraints | Computational time (s) | Objective value ($) | Optimality gap (%) |
|-------------|------|-----------------------|------------------------|---------------------|-------------------|
| IEEE-6 | D-DA(Dα) | 10,960 | 0.66 | 2260.21 | 0.0065% |
| IEEE-39 | D-DA(Dα) | 960 | 0.28 | 2258.73 | 0.1106% |
| IEEE-118 | D-DA(Dα) | 115,920 | 0.0026% | 428,064 | 0.0026% |

6. Conclusion. For solving a specific PCCO problem, we assume that a big data set D of historical measurements of the uncertain parameters ξ is available. First, the α-process, i.e., Algorithm 3.1, is used to eliminate the low-probability data points leaving set Dα, which only contains data points whose joint probabilities are not less than α. Second, z data points are randomly selected from Dα and stored in DZα, where z is determined by a desired probability ρ via z = g−1(ρ). Finally, zq data points are further selected from DZα and stored in DZα by a corresponding SDS algorithm if D-DA(DZα) is still too big to compute.

This research rethinks the entire procedure of solving optimization problems under uncertainty from logic modeling to solution algorithm design. The logic model (1.1), i.e., PCCO, defined in this paper is a novel alternative to the existing CCO. The PCC (1.1b) therein logically means that an optimal solution should be feasible to all high-probability realizations of the uncertain variables. Such a logical meaning grants the PCCO a very high application value since it reflects the need of many engineering systems in terms of optimization under uncertainty. Since the existing deterministication methods are either inapplicable or inefficient to PCCO, another key contribution of this paper lies in the novel solution paradigm which consists of data-based deterministication (as detailed in Section 3) and strategic data selection (as detailed in Section 4). With the proposed solution paradigm, PCCO problems can be solved accurately with relatively-low computational complexity.

Appendix A. Proofs in Section 2.

A.1. Proof of Proposition 2.1. When α1 ≥ α2, for an arbitrary realization ˇξ of ξ, P[ξ = ˇξ] ≥ α2 if P[ξ = ˇξ] ≥ α1. In other words, ˇξ ∈ Ξα2 if ˇξ ∈ Ξα1, which implies that Ξα1 ⊆ Ξα2. Then,

\[ X_\alpha(\alpha_2) = \left\{ x \in \mathbb{R}^n \mid g(x, \xi^{(k)}) \leq 0, \ k = 1, \ldots, |\Xi_{\alpha_1}| \right\} \]

\[ = \left\{ x \in X_\alpha(\alpha_1) \mid g(x, \xi^{(k)}) \leq 0, \ k = |\Xi_{\alpha_1}| + 1, \ldots, |\Xi_{\alpha_2}| \right\} \]

\[ \subseteq X_\alpha(\alpha_1). \]

A.2. Proof of Proposition 2.2. We consider the experiment that a realization of ξ is randomly extracted from the sample space Ξ and let ˇξ denote this arbitrary realization. According to the probability theory, the probability of event P[ξ = ˇξ] ≥ α,
i.e., \( \mathbb{P}[\mathbb{P}[\xi = \xi] \geq \alpha] \), is given as

\[
(\text{A.1}) \quad \mathbb{P}[\mathbb{P}[\xi = \xi] \geq \alpha] = \mathbb{P}[\mathbb{P}(\xi) \geq \alpha] = \int_{\{\xi \in \mathbb{P}(\xi) \geq \alpha\}} P(\xi) d\xi. 
\]

As a result, condition (2.4) indicates that \( \mathbb{P}[\mathbb{P}(\xi) \geq \alpha] = 1 - \beta \). Let \( x_p \in X_p \) denote an arbitrarily feasible solution of PCCO, the logical meaning of PCC, i.e., \( g(x_p, \xi) \leq 0 \) holds if \( \mathbb{P}[\xi = \xi] \geq \alpha \), implies that the probability of event \( g(x_p, \xi) \leq 0 \) is not less than the probability of event \( \mathbb{P}[\xi = \xi] \geq \alpha \), i.e., \( \mathbb{P}[g(x_p, \xi) \leq 0] \geq \mathbb{P}[\mathbb{P}(\xi) \geq \alpha] = 1 - \beta \). Therefore, \( x_p \in X_c \) according to the logical meaning of CC, which implies that \( X_p \subseteq X_c \).

The condition \( \mathbb{P}(\xi^{(a)}) \geq \mathbb{P}(\xi^{(b)}) \) when \( \xi^{(a)} \) and \( \xi^{(b)} \) are arbitrarily realizations in \( M(x) \) and \( \Xi \setminus M(x) \) respectively implies that there exists a probability \( v \) such that \( \mathbb{P}(\xi^{(a)}) \geq v \) and \( \mathbb{P}(\xi^{(b)}) \leq v \). Then, \( M(x) \) becomes

\[
(\text{A.2}) \quad M'(x) = \{ \xi \in \Xi | g(x, \xi) \leq 0 \text{ and } \mathbb{P}(\xi) \geq v \}. 
\]

Under condition (2.4), it’s not hard to know that \( v = \alpha \) in (A.2). Let \( \hat{\xi} \) be an arbitrary realization of \( \xi \) in \( M'(x) \), we have \( \mathbb{P}[\xi = \hat{\xi}] \geq \alpha \). Let \( x_C \) denote an arbitrarily feasible solution of CCO that satisfies the above condition, we have \( g(x_C, \xi) \leq 0 \), which implies that \( x_C \in X_p \) and, namely, \( X_p \supseteq X_c \). Therefore, we have \( X_p = X_c \).

### A.3. Proof of Proposition 2.3.

Considering the experiment of randomly generating \( N \) iid samples, it’s of high probability that these samples belong to scenarios whose probabilities are not less than \( 1/N \). When \( N \leq \frac{1}{\alpha} \), it’s of high probability that these samples belong to scenarios whose probabilities are not less than \( \alpha \). Note that \( X_S \) is a set in \( x \)-space specified by \( N \) samples whose probabilities are not less than \( \alpha \), we have \( X_p \subseteq X_S \) since \( X_p \) is specified by all possible scenarios whose probabilities are not less than \( \alpha \). Given that \( N \leq \frac{1}{\alpha} \) is a relatively small number, it’s of high probability that \( X_p \subseteq X_S \). When \( N > \frac{1}{\alpha} \), a scenario whose probability is less than \( \alpha \) may be included in the \( N \) samples, which results in that an element in \( X_p \) may not be feasible to \( X_S \). Hence, \( X_p \neq X_S \).

### Appendix B. Proofs in Section 3.

#### B.1. Proof of Proposition 3.5.

Given that data set \( D^*_{\alpha} \) is generally a multiset, let \( \mathfrak{U}[D^*_{\alpha}] \) denote its underlying set, we have that D-DA(\( \mathfrak{U}[D^*_{\alpha}] \)) is equivalent to D-DA(D^*_{\alpha}). Because, for any data point in the complement of \( \mathfrak{U}[D^*_{\alpha}] \) in \( D^*_{\alpha} \), i.e., \( D^*_{\alpha} \setminus \mathfrak{U}[D^*_{\alpha}] \), there is a data point \( \mathfrak{U}[D^*_{\alpha}] \) that is the same as it and, therefore removing the data points in \( D^*_{\alpha} \setminus \mathfrak{U}[D^*_{\alpha}] \) does not change the feasible space of D-DA(D^*_{\alpha}). Since \( S^S_{\alpha} \) is defined as the set that contains all possible scenarios of \( \Xi_{\alpha} \), we have \( \mathfrak{U}[D^*_{\alpha}] \subseteq S^S_{\alpha} \) and, consequently \( X_D(D^*_{\alpha}) \supseteq X_D(S^S_{\alpha}) \) according to Proposition 2.1. According to Definition 3.3, we have \( X_D(S^S_{\alpha}) = X_p \) and, consequently, \( X_D(D^*_{\alpha}) \supseteq X_p \). Condition \( B^{FS}_{\hat{V},\alpha} \subseteq D^*_{\alpha} \) indicates that \( X_D(B^{FS}_{\hat{V},\alpha}) \subseteq X_D(D^*_{\alpha}) \). According to Definition 3.4, we have \( X_D(B^{FS}_{\hat{V},\alpha}) = X_D(S^S_{\alpha}) = X_p \), which means \( X_p \subseteq X_D(D^*_{\alpha}) \). Therefore, we have \( X_D(D^*_{\alpha}) = X_p \) if \( B^{FS}_{\hat{V},\alpha} \subseteq D^*_{\alpha} \).

#### B.2. Proof of Proposition 3.7.

Recall that \( x_D^*(S^S_{\alpha}) \) is the optimal solution of D-DA(S^S_{\alpha}) and \( B^{FS}_{\hat{V},\alpha} \) is the set of boundary-forming data points at \( x_D^*(S^S_{\alpha}) \), we know that \( x_D^*(B^{FS}_{\hat{V},\alpha}) = x_D^*(S^S_{\alpha}) \) according to Definition 3.6. Namely, removing any data point in the complement of \( B^{FS}_{\hat{V},\alpha} \) with respect to \( S^S_{\alpha} \), i.e., \( S^S_{\alpha} \setminus B^{FS}_{\hat{V},\alpha} \), will not affect the solution of D-DA(S^S_{\alpha}). If \( B^{FS}_{\hat{V},\alpha} \subseteq D^*_{\alpha} \), i.e., \( D^*_{\alpha} \) contains all boundary-forming data.
points of $x^*_D(S^*_D)$, we have $B^\text{OS}_{\forall,\alpha} \subseteq \mathcal{U}[D^*_\alpha]$. From the proof of Proposition 3.5, we know that $\mathcal{U}[D^*_\alpha] \subseteq S^*_\alpha$ and, hence $(\mathcal{U}[D^*_\alpha] \setminus B^\text{OS}_{\forall,\alpha}) \subseteq (S^*_\alpha \setminus B^\text{OS}_{\forall,\alpha})$. As a result, with all data points in $\mathcal{U}[D^*_\alpha] \setminus B^\text{OS}_{\forall,\alpha}$ removed, the optimal solution does not change while the optimization problem changes from D-DA$(\mathcal{U}[D^*_\alpha])$ to D-DA$(B^\text{OS}_{\forall,\alpha})$, which means $x^*_D(\mathcal{U}[D^*_\alpha]) = x^*_D(B^\text{OS}_{\forall,\alpha})$. Consequently, we have $x^*_D(D^*_\alpha) = x^*_D(S^*_\alpha) = x^*_D$.

**B.3. Proof of Theorem 3.8.** Let $\xi_{\text{d,B}}^{(i)} (i = 1, \ldots, B^\text{OS}_{\forall,\alpha})$ denote the boundary-forming data points for $x^*_D(S^*_D)$, we have $P[\xi = \xi_{\text{d,B}}^{(i)}] \geq \alpha (\forall i = 1, \ldots, B^\text{OS}_{\forall,\alpha})$. Therefore, the expectation of the times that $\xi_{\text{d,B}}^{(i)}$ appears in $D$ is not less than $\alpha D$. Then, we consider the following events:

- $E^o$: when a data point is randomly selected from $D_\alpha$, it is one of the $B^\text{OS}_{\forall,\alpha}$ boundary-forming data points;
- $E^*$: when a data point is randomly selected from $D_\alpha$, at least one of each of the $B^\text{OS}_{\forall,\alpha}$ active points is selected;
- $E_i (i = 1, \ldots, B^\text{OS}_{\forall,\alpha})$: when a data point is randomly selected from $D_\alpha$, no $\xi_{\text{d,B}}^{(i)}$ is selected.

It suffices to show that $P[E^*] \geq \alpha D/D_\alpha$ and $P[E^*] = \bigcup_{i=1}^{B^\text{OS}_{\forall,\alpha}} E_i$ which is the complement of $E^*$. Then, $P[E^*] = 1 - P[E^*] = 1 - P[\bigcup_{i=1}^{B^\text{OS}_{\forall,\alpha}} E_i]$. Moreover,

\begin{equation}
(1.1) \quad P\left[ \bigcup_{i=1}^{B^\text{OS}_{\forall,\alpha}} E_i \right] = \sum_{i=1}^{B^\text{OS}_{\forall,\alpha}} P[E_i] - \sum_{i=1}^{B^\text{OS}_{\forall,\alpha}} \sum_{j>i} P[E_i \cap E_j] + \sum_{i=1}^{B^\text{OS}_{\forall,\alpha}} \sum_{j>i} \sum_{k>j} P[E_i \cap E_j \cap E_k]
\end{equation}

\[ \vdots \quad + (-1)^{(B^\text{OS}_{\forall,\alpha} - 1)} P\left[ \bigcap_{i=1}^{B^\text{OS}_{\forall,\alpha}} E_i \right] \]

It's not hard to know that $P[E_i], P[E_i \cap E_j], P[E_i \cap E_j \cap E_k], \ldots,$ and $P[\bigcap_{i=1}^{B^\text{OS}_{\forall,\alpha}} E_i]$ follow the hypergeometric distribution [34]. Hence, we have

\begin{equation}
(2.2) \quad P[K] = P\left[ \bigcap_{i=1}^{K} E_i \right] = \frac{(D_\alpha - K P[E^*] D_\alpha)}{(D^*_\alpha K)}, \quad K = 1, \ldots, B^\text{OS}_{\forall,\alpha}
\end{equation}

and

\begin{equation}
(3.3) \quad P[E^*] = 1 - P\left[ \bigcup_{i=1}^{B^\text{OS}_{\forall,\alpha}} E_i \right] = 1 - \sum_{K=1}^{B^\text{OS}_{\forall,\alpha}} \left( -1 \right)^{K-1} \left( \frac{B^\text{OS}_{\forall,\alpha}}{K} \right) P[K] .
\end{equation}

Recall that $P[E^*] \geq \alpha D/D_\alpha$ and moreover, $B^\text{OS}_{\forall,\alpha} \leq B^\text{OS}_{\forall,\alpha}$, we have

\begin{equation}
(4.4) \quad P[E^*] \geq 1 - \sum_{K=1}^{B^\text{OS}_{\forall,\alpha}} \left( -1 \right)^{K-1} \left( \frac{B^\text{OS}_{\forall,\alpha}}{K} \right) \frac{(D_\alpha - K \alpha D)}{(D^*_\alpha K)} = \varrho .
\end{equation}

According to Proposition 3.7, event $E^*$ implies that $x^*_D(D^*_\alpha) = x^*_D(S^*_\alpha)$ and, consequently $x^*_D(D^*_\alpha) = x^*_D$. Therefore, $\varrho = P[x^*_D(D^*_\alpha) = x^*_D] = P[E^*]$ and the $\varrho$ in (4.4) is a lower bound of $\varrho$.  

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B.4. Proof of Proposition 3.9. To prove this proposition, we utilize the finding on the support constraint, as defined in [11], whose removal changes the solution of the optimization problem. According to Proposition 1 in [11], for a convexly constrained linear program, the number of support constraints is at most \( n \) (read [11] for a detailed proof). Let \( x_D^n(\square) \) denote the optimal solution of D-DA (3.1), by comparing Definition 3.6 of this paper and the definition of the support constraint, one can consider a boundary-forming data point of \( x_D^n(\square) \) a data point that contributes at least one support constraint to \( x_D^n(\square) \) (note that a data point contributes \( m \) constraints to the D-DA). As such, we have “\( n \geq m \) the number of support constraints \( \geq \) the number of boundary-forming data points.”

Appendix C. Proofs in Section 4.

C.1. Proof of Theorem 4.1. When \( \xi \in \Xi \subset \mathbb{R}^r \), it’s straightforward to know that \( D^a_0 = D^a_{\infty} \) if \( \eta = 0 \). Hence, \( z_\eta = z \) when \( \eta = 0 \). When \( \eta \geq \bar{\eta} \), only one data point (i.e., \( \xi_0^{(1)} \)) is selected from \( D^a_0 \) and stored in \( D^a_\alpha \). In the \( r \)-dimensional Euclidean space, the \( \eta \)-vicinity of a data point is actually an \( r \)-ball whose volume is \( V^n_{\eta} = 2(2\pi)^{r/2} \eta^r/\Gamma(r/2)! \) [26], where \( r/\Gamma(r/2)! \) is the double factorial (semifactorial) of \( r \). Let \( A \subset \Xi \subset \mathbb{R}^r \) be the smallest continuous set that contains all the data points in \( D^a_\alpha \) and \( V^A_r \) denote the \( r \)-dimensional Euclidean volume of \( A \), we divide \( V_r^A \) by \( V^n_{\eta} \) and denote it as \( \bar{z}_\eta \):

\[
\bar{z}_\eta = \frac{V_r^A}{V^n_{\eta}} = \frac{r^r V_r^A}{(2\pi)^{r/2} \eta^r / \Gamma(r/2)!},
\]

where \( 0 < \eta < \bar{\eta} \). One can consider Algorithm 4.1 as “packing \( z_\eta \) \( r \)-balls in the continuous space \( A \).” It’s straightforward to know that \( z_\eta < \bar{z}_\eta \) due to the existence of “gaps” among the \( z_\eta \) \( r \)-balls. In other words, we can consider \( \bar{z}_\eta \) an upper estimation of \( z_\eta \) for \( 0 < \eta < \bar{\eta} \), and relation (C.1) implies \( \bar{z}_\eta \propto 1/\eta^r \).

Since the positions of the \( r \)-balls are randomly chosen when Algorithm 4.1 packs these \( r \)-balls in \( A \), the needed number \( z_\eta \) of \( r \)-balls for filling out \( A \) would slightly vary even when the radius \( \eta \) of these \( r \)-balls is fixed. Therefore, \( z_\eta \) is a random variable which is parameterized by \( \eta \). It’s also straightforward to know that, if the radius \( \eta \) of these \( r \)-balls is bigger, the number \( z_\eta \) of \( r \)-balls that can be packed in \( A \) is most likely less. Now, let’s consider two sets of random experiments where Algorithm 4.1 fills \( A \) with \( r \)-balls of radius \( \eta_1 \) in the first set of experiments and with \( r \)-balls of radius \( \eta_2 \) in the second. Further note that \( \eta_1 < \eta_2 \) and, in each experiment, the positions of \( r \)-balls are randomly selected. It’s not difficult to know that the expectation of \( z_{\eta_1} \) should not be less than that of \( z_{\eta_2} \), i.e., \( E[z_{\eta_1}] \geq E[z_{\eta_2}] \). Then,

\[
\frac{E[z_{\eta_1}] - E[z_{\eta_2}]}{\eta_1 - \eta_2} \leq 0.
\]

The limit of the left-hand-side of the above inequality as \( |\eta_1 - \eta_2| \to 0 \) yields \( \partial(E[z_\eta]) / \partial \eta \leq 0 \).

C.2. Proof of Theorem 4.3. Recall that we have assumed that there are \( B_c \) boundary-forming constraints at \( x^n_D(D^a_\alpha) \) and they are denoted as \( \bar{g}(x, \xi_{\text{B}}) \leq 0 \) (\( \bar{g} : \mathbb{R}^{n+(r \times B_c)} \to \mathbb{R}^{B_c} \)). In this proof, we first show that, in D-DA(\( D^a_\alpha \)), \( \bar{g}(x, \xi_{\text{d}}) \leq 0 \) are the \( B_c \) boundary-forming constraints at \( x^n_D(z_\eta) \) under the conditions that the \( B_c \) boundary-forming constraints are the only active constraints at \( x^n_D(D^a_\alpha) \) and \( \eta \) is small. When \( \eta \) is small enough, we have \( \xi_{\text{d}} \approx \xi_{\text{d,B}} \) and, consequently \( \bar{g}(x, \xi_{\text{d}}) \approx \bar{g}(x, \xi_{\text{d,B}}) \).
Given that \( \tilde{g}(x, \xi_{d,B}) = 0 \) has a solution \( x^*_{D} (D^*_{\alpha}) \), it’s reasonable to assume that \( \tilde{g}(x, \hat{\xi}_{d}) = 0 \) also has a solution, since \( \tilde{g} \) is continuously differentiable with respect to \( x \) at the vicinity of \( x^*_{D} (D^*_{\alpha}) \), and let \( x^* \) denote this solution. The condition of boundary-forming constraints being the only active constraints at \( x^*_{D} (D^*_{\alpha}) \) implies that \( \tilde{g}(x, \xi_{d,B}) = 0 \) are all the constraints that intersect a small enough vicinity of \( x^*_{D} (D^*_{\alpha}) \). Let \( \tilde{g}(x, \xi_{d(k)}) = 0 \) denote the closest inactive constraint to \( x^*_{D} (D^*_{\alpha}) \) in D-DA\((D^*_{\alpha})\), it’s not hard to know that, in D-DA\((D^*_{\alpha})\), there does not exist a constraint which is closer to \( x^*_{D} (D^*_{\alpha}) \) than \( \tilde{g}(x, \xi_{d(k)}) = 0 \) since \( D^*_{\alpha} \subset D^*_{\alpha} \). That means it’s possible that \( \|x^* - x^*_{D} (D^*_{\alpha})\| \) is less than or equals to the distance from \( x^*_{D} (D^*_{\alpha}) \) to \( \tilde{g}(x, \xi_{d(k)}) = 0 \).

In other words, when \( \xi_{d,B} \) is replaced by \( \hat{\xi}_{d} \), \( \tilde{g}(x, \xi_{d,B}) = 0 \) and \( x^*_{D} (D^*_{\alpha}) \) migrate to \( \tilde{g}(x, \hat{\xi}_{d}) = 0 \) and \( x^* \) respectively. In this process, no constraints that are originally inactive become boundary-forming. Hence, \( \tilde{g}(x, \xi_{d}) \leq 0 \) are the boundary-forming constraints of D-DA\((D^*_{\alpha}) \) and \( x^* = x^*_{D} (D^*_{\alpha}) \) under the above mentioned conditions.

Proposition 4.2 indicates the existence of a vector-valued function \( x = h(\hat{\xi}) \) which is equivalent to \( \tilde{g}(x, \hat{\xi}) = 0 \) in the vicinity of \((x^*_{D} (D^*_{\alpha}), \xi_{d,B}) \) under the conditions therein. Let \( \Delta x = x^*_{D} (D^*_{\alpha}) - x_{D}^* (D^*_{\alpha}) \) and \( \Delta \hat{\xi} = \hat{\xi}_{d} - \xi_{d,B} \), the Taylor series of \( x = h(\hat{\xi}) \) at \( \xi_{d,B} \) is

\[
\Delta x = H(\xi_{d,B}) \Delta \hat{\xi} + \frac{1}{2!} \Delta \hat{\xi}^{T} H'(\xi_{d,B}) \Delta \hat{\xi} + \frac{1}{3!} \Delta \hat{\xi}^{T} H''(\xi_{d,B})(\Delta \hat{\xi})^{2} + \cdots .
\]

Then, we have

\[
\|\Delta x\| \leq \|H(\xi_{d,B})\| \|\Delta \hat{\xi}\| + \frac{1}{2!} \|H'(\xi_{d,B})\| \|\Delta \hat{\xi}\|^{2} + \frac{1}{3!} \|H''(\xi_{d,B})\| \|\Delta \hat{\xi}\|^{3} + \cdots .
\]

Note that \( B_{c} = n \) under the condition in Proposition 4.2 and \( \Delta \hat{\xi} = [\Delta \hat{\xi}_{1}, \cdots, \Delta \hat{\xi}_{n}]^{T} \), where \( \|\Delta \hat{\xi}_{i}\| \leq 2\eta \) \( (i = 1, \cdots, n) \) according to Algorithm 4.1, we have \( \|\Delta \hat{\xi}\| \leq 2\sqrt{n}\eta \). Therefore, from (C.3), we have:

\[
\|\Delta x\| \leq 2\sqrt{n}\|H(\xi_{d,B})\| \eta + \frac{2^{2}n}{2!} \|H'(\xi_{d,B})\| \eta^{2} + \frac{2^{3}\sqrt{n^{3}}}{3!} \|H''(\xi_{d,B})\| \eta^{3} + \cdots .
\]

Then, we further have:

\[
\varphi = 1 - \frac{\|\Delta x\|}{x^*_{D} (D^*_{\alpha})} \geq 1 - \frac{2\sqrt{n}\|H(\xi_{d,B})\| \eta + \frac{2^{2}n}{2!} \|H'(\xi_{d,B})\| \eta^{2} + \frac{2^{3}\sqrt{n^{3}}}{3!} \|H''(\xi_{d,B})\| \eta^{3} + \cdots }{x^*_{D} (D^*_{\alpha})} = \varphi(\eta).
\]

**C.3. Proof of Corollary 4.4.** Recall that all elements in a subset \( D^*_{\alpha,i} \) \( (i \in \mathcal{I}) \) of \( D^*_{\alpha} \) have the same integer part. With the integer part being fixed, steps 2-7 in Algorithm 4.2 on \( D^*_{\alpha,i} \) is equivalent to Algorithm 4.1 on \( D^*_{\alpha} \). Therefore, one can apply the assertions in Theorem 4.1 to each \( i \in \mathcal{I} \). Then, we have

\[
z_{\eta,i} \begin{cases} = z_{i} & \eta_{i} = 0 \\ \leq \bar{z}_{\eta,i} \frac{0 < \eta_{i} < \bar{\eta}_{i} \cdot \bar{z}_{\eta,i} \propto \frac{1}{\eta_{i}^{2}}} & , & \partial \left\{ [E(z_{\eta,i})] \right\} \leq 0, \ i \in \mathcal{I}. \end{cases}
\]

Since \( z = \sum_{i \in \mathcal{I}} z_{i} \) and \( z_{\eta} = \sum_{i \in \mathcal{I}} z_{\eta,i} \), we have (4.7) and (4.8).
C.4. Proof of Corollary 4.5. We don’t need to consider the subsets \( D^i_{\alpha} \) \((i \in \mathcal{I})\) of \( D_{\alpha} \) which do not contain active data points of \( x^*_{D}(z) \) since removing any inactive data point will not impact the optimal solution. Under Algorithm 4.2, we have \( \|\Delta \xi_i\| \leq 2\eta_i \) \((i=1, \ldots, B_z)\), which implies that \( \|\Delta \xi\| \leq 2\sqrt{\eta^2_1 + \ldots + \eta^2_B_z} = \eta \).

Then, applying the conclusions in Theorem 4.3, we have (4.9).

C.5. Proof of Lemma 4.6. When \( \xi \in \Xi \subset \mathbb{Z}^r \), we already know that \( D^0_{\alpha} = \Omega[\mathcal{I}] \) and \( z_\eta = |\Omega[\mathcal{I}]| = I \). If we renumber the data points in \( D^i_{\alpha} = \{ \xi^{(k)}_d \} (k = 1, \ldots, z_\eta) \) as \( D^i_{\alpha} = \{ \xi^{(i,j)}_d \} (i = 1, \ldots, I; j = 1, \ldots, z_\eta) \), D-DA(\( D^i_{\alpha} \)) is equivalent to

\[
\begin{align*}
(C.6a) \quad \min_x & \quad f(x) \\
(C.6b) \quad \text{s.t.} & \quad g(x, \xi^{(i,1)}_d) \leq 0 \\
(C.6c) \quad & \quad g(x, \xi^{(i,j)}_d) \leq 0,
\end{align*}
\]

where \( i = 1, \ldots, I \) and \( j = 1, \ldots, z_\eta \). Since \( \xi^{(i,1)}_d, \ldots, \xi^{(i,z_\eta)}_d \), constraints in (C.6c) are redundant and can be remove without impacting the solution. Without constraints (C.6c), problem (C.6) is exactly the D-DA(\( D^0_{\alpha} \)). D-DA(\( D^0_{\alpha} \)) being equivalent to D-DA(\( D^i_{\alpha} \)) means \( x^*_D(D^0_{\alpha}) = x^*_D(D^i_{\alpha}) \).

Recall that \( x^*_D(D_{\alpha}) \) denotes the optimal solution of D-DA(\( D_{\alpha} \)), Assumption 1 implies that \( x^*_D(D_{\alpha}) = x^*_i \). If Algorithm 4.2 uses \( D_{\alpha} \) as input data set, it’s not hard to know that the resulting \( D^0_{\alpha} = \Omega[\mathcal{I}] \). Following the analysis in the previous paragraph, it suffices to have \( x^*_D(D^0_{\alpha}) = x^*_D(D_{\alpha}) \). Therefore, \( x^*_D(D^0_{\alpha}) = x^*_i \).

C.6. Proof of Proposition 4.7. Relation (4.3) in Theorem 4.1 indicates that \( \mathbb{E}[z_{\eta(1)}] \geq \mathbb{E}[z_{\eta(2)}] \) if \( \eta(1) < \eta(2) \), which implies that \( \mathbb{P}[z_{\eta(1)} > z_{\eta(2)}] \geq \mathbb{P}[z_{\eta(1)} < z_{\eta(2)}] \). Considering \( (z_{\eta(1)} - z_{\eta(2)})/(\eta(1) - \eta(2)) < 0 \) and \( (z_{\eta(1)} - z_{\eta(2)})/(\eta(1) - \eta(2)) > 0 \) as the first and second events respectively (denoted as E1 and E2 respectively), we have \( \mathbb{P}[E1] \geq \mathbb{P}[E2] \). Given that \( D^{(1)}_{\alpha} \) and \( D^{(2)}_{\alpha} \) are two random data sets, it suffices to show that, if \( z_{\eta(1)} > z_{\eta(2)} \), \( \mathbb{P}[\omega(\eta(1)) > \omega(\eta(2))] \geq \mathbb{P}[\omega(\eta(1)) < \omega(\eta(2))] \). Further let \( (\omega(\eta(1)) - \omega(\eta(2)))/(z_{\eta(1)} - z_{\eta(2)}) > 0 \) and \( (\omega(\eta(1)) - \omega(\eta(2)))/(z_{\eta(1)} - z_{\eta(2)}) < 0 \) be the third and fourth events respectively (denoted as E3 and E4 respectively), we have \( \mathbb{P}[E3] \geq \mathbb{P}[E4] \). Finally, considering \( (\omega(\eta(1)) - \omega(\eta(2)))/(\eta(1) - \eta(2)) < 0 \) and \( (\omega(\eta(1)) - \omega(\eta(2)))/(\eta(1) - \eta(2)) > 0 \) as the fifth and sixth events (denoted as E5 and E6 respectively), we have

\[
\begin{align*}
\mathbb{P}[E5] &= \mathbb{P}[E1] \cdot \mathbb{P}[E3] + \mathbb{P}[E2] \cdot \mathbb{P}[E4] \\
\mathbb{P}[E6] &= \mathbb{P}[E1] \cdot \mathbb{P}[E4] + \mathbb{P}[E2] \cdot \mathbb{P}[E3],
\end{align*}
\]

and \( \mathbb{P}[E5] - \mathbb{P}[E6] = (\mathbb{P}[E1] - \mathbb{P}[E2])(\mathbb{P}[E3] - \mathbb{P}[E4]) \geq 0 \).

Hence, we have we have \( \mathbb{P}[\omega(\eta(1)) > \omega(\eta(2))] \geq \mathbb{P}[\omega(\eta(1)) < \omega(\eta(2))] \) when \( \eta(1) < \eta(2) \).

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