MINIMAL SURFACES IN THE THREE-DIMENSIONAL SPHERE AND
MINIMAL HYPERSURFACES OF TYPE NUMBER TWO

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ABSTRACT. We introduce canonical principal parameters on any strongly regular minimal
surface in the three dimensional sphere and prove that any such a surface is determined
up to a motion by its normal curvature function satisfying the Sinh-Poisson equation. We
obtain a classification theorem for bi-umbilical hypersurfaces of type number two. We prove
that any minimal hypersurface of type number two with involutive distribution is generated
by a minimal surface in the three-dimensional Euclidean space, or in the three dimensional
sphere. Thus we prove that the theory of minimal hypersurfaces of type number two with
involutive distribution is locally equivalent to the theory of minimal surfaces in the three
dimensional Euclidean space or in the three-dimensional sphere.

1. Introduction

The aim of this paper is to show the deep relation between the theory of minimal hyper-
surfaces of type number two in arbitrary dimension and the theory of minimal surfaces in
the three-dimensional Euclidean space \( \mathbb{R}^3 \) or in the three-dimensional Euclidean sphere \( S^3 \).

Section 1 is devoted to the invariant theory of minimal surfaces in \( S^3 \).

Let \( \mathcal{M} : z = l(u,v), \ (u,v) \in \mathcal{D} \) be a regular surface in \( S^3 \) parameterized by principal
parameters \((u,v)\) and consider the following four invariant functions: the principle normal
curvatures \( \nu_1, \nu_2 \); the principal geodesic curvatures (the geodesic curvatures of the principal
lines) \( \gamma_1, \gamma_2 \). We introduce the class of strongly regular surfaces by the condition
\[(\nu_1 - \nu_2) \gamma_1 \gamma_2 \neq 0.\]

In Subsection 2.2 we prove the Bonnet type Theorem 2.2 for strongly regular surfaces in
terms of the four invariants \( \nu_1, \nu_2, \gamma_1, \gamma_2 \).

The main obstacle to formulate the Bonnet type fundamental theorem for surfaces similar-
ly to the fundamental theorem for the curves is the lack of natural parameters in the
theory of surfaces. In [6] we have shown that the class of Weingarten surfaces in \( \mathbb{R}^3 \) admits
geometrically determined parameters, which we called canonical parameters. Here we show
that any minimal strongly regular surface in \( S^3 \) admits geometrically determined canonical
parameters.

Our main result for minimal strongly regular surfaces in \( S^3 \) is Theorem 2.3:

Any solution \( \nu > 0 \) to the Sinh-Poisson equation
\[
\Delta \ln \nu = 2 \frac{1 - \nu^2}{\nu}
\]

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satisfying the condition $\nu_1 \nu_2 \neq 0$ determines uniquely (up to a motion in $S^3$) a minimal strongly regular surface with invariants $\nu_1 = \nu, \nu_2 = -\nu, \gamma_1 = (\sqrt{\nu})_v, \gamma_2 = -(\sqrt{\nu})_u.$

Furthermore $(u, v)$ are canonical parameters.

This result allows us to introduce a family $\{M_t\}$ of minimal surfaces associated with a given minimal strongly regular surface $M$. The minimal surfaces of the family $\{M_t\}$ are isometric to the surface $M$.

Section 2 is devoted to minimal hypersurfaces of type number two. Let $(M^n, g), n \geq 3$ be a regular hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. If the shape operator of the hypersurface $M^n$ has two eigenvalues $\nu_1, \nu_2$ different from zero and the other $(n-2)$ eigenvalues are zero at each point, then $M^n$ is said to be of type number two. Thus, the hypersurfaces of type number two are characterized in terms of their fundamental form $h$ as follows:

$$h = \nu_1 \eta_1 \otimes \eta_1 + \nu_2 \eta_2 \otimes \eta_2, \quad \nu_1 \nu_2 \neq 0,$$

where $\eta_1, \eta_2$ are unit one-forms.

These hypersurfaces considered as Riemannian manifolds are semi-symmetric spaces foliated by Euclidean leaves of codimension two, i.e. they are foliated semi-symmetric spaces [10]. Conversely, any foliated semi-symmetric hypersurface in $\mathbb{R}^{n+1}$ is a hypersurface of type number two. Foliated semi-symmetric spaces have been studied in [4] with respect to their metrics under the name Riemannian manifolds of conullity two. Thus, the hypersurfaces of type number two can also be considered as hypersurfaces of conullity two.

In [8] we considered the hypersurfaces of type number two as the envelope of a two parameter system of hyperplanes. Then, a hypersurface $M^n$ of type number two can be considered as a two-parameter system of planes of codimension three $\{E^{n-2}(u, v)\}, (u, v) \in D$ with the property that the tangent hyperplane to $M^n$ at any point of an arbitrary generator $E^{n-2}$ is one and and the same. Briefly, the hypersurfaces of type number two are exactly the "developable" two parameter systems of planes of codimension three.

The unit eigenvector fields $X$ and $Y$ corresponding to $\nu_1$ and $\nu_2$ determine a two-dimensional distribution $\Delta = \text{span}\{X, Y\}$, which plays an essential role in the geometry of hypersurfaces of type number two. We denote by $\mathcal{K}_0$ the class of hypersurfaces of type number two, whose distribution $\Delta$ is involutive.

In [7] we proved that a hypersurface $M^n$ of type number two is in the class $\mathcal{K}_0$, then any two-dimensional integral surface $M^2$ of its distribution $\Delta$ has flat normal connection. Conversely, for any surface $M^2$ with flat normal connection, using the set of its parallel surfaces, we gave a natural geometric construction of a family of hypersurfaces of type number two belonging to the class $\mathcal{K}_0$. Thus the local differential geometry of hypersurfaces in the class $\mathcal{K}_0$ is equivalent to the local differential geometry of the surfaces with flat normal connection. Generally speaking, the theory of the surfaces with flat normal connection can be considered as a model of the theory of the hypersurfaces in the class $\mathcal{K}_0$. The aim of this paper is to realize this correspondence for the the minimal hypersurfaces from the class $\mathcal{K}_0$.

We note that the theory of hypersurfaces of type number two carries some features of the usual theory of surfaces in the Euclidean space $\mathbb{R}^3$.

A point of a hypersurface $M^n$ of type number two is said to be bi-umbilical if $\nu_1 = \nu_2 \neq 0$ at this point. This notion corresponds to the notion of an umbilical point in Euclidean differential geometry. In Section 2 we obtain the classification Theorem 3.1 for bi-umbilical
hypersurfaces of type number two (corresponding to the classical result for umbilical surfaces). Our scheme is the following:

We prove that the integral surfaces of the distribution $\Delta$ of any bi-umbilical hypersurface of type number two lie on two-dimensional spheres. Conversely, any two-dimensional sphere generates a family of bi-umbilical hypersurfaces of type number two.

Any bi-umbilical hypersurface of type number two is determined by a unit vector function $l(u,v) \in \mathbb{R}^{n+1}$ and a scalar function $r(u,v)$ of two variables satisfying the system of partial differential equations [8]:

\[
\begin{align*}
l_{uu} - l_{vv} &= \frac{E_u}{E} l_u - \frac{E_v}{E} l_v, \\
2l_{uv} &= \frac{E_v}{E} l_u + \frac{E_u}{E} l_v, \\
r_{uu} - r_{vv} &= \frac{E_v}{E} r_u - \frac{E_u}{E} r_v, \\
2r_{uv} &= \frac{E_v}{E} r_u + \frac{E_u}{E} r_v,
\end{align*}
\]

Theorem 3.1 implies that the solutions of the above system can be found explicitly.

The classification of minimal ruled hypersurfaces (helicoids) in $\mathbb{R}^{n+1}$ was treated in a series of papers (e.g. [5], [2]) and was completed in 1981 [1]. It occurred that essential helicoids exist only in $\mathbb{R}^3$ (the usual helicoids) and in $\mathbb{R}^4$ (second type helicoids). While the usual helicoid $\mathcal{M}^2$ in $\mathbb{R}^3$ is a complete surface, the three-dimensional second type helicoid $\mathcal{M}^3$ in $\mathbb{R}^4$ has one singular point. All minimal ruled hypersurfaces in an arbitrary dimension are generated by these helicoids.

We note that the ruled hypersurfaces are in the class $\mathcal{K}_0$. In the present paper we obtain a geometric description of the minimal hypersurfaces in the class $\mathcal{K}_0$. This is obtained in Theorem 3.2. Our scheme is the following:

We prove that any integral surface of the distribution $\Delta$ of a minimal hypersurface of the class $\mathcal{K}_0$ is a minimal surface in the three-dimensional Euclidean space $\mathbb{R}^3$, or a minimal surface in the three-dimensional Euclidean sphere $S^3$. Conversely, any minimal surface in $\mathbb{R}^3$ or in $S^3$ generates a minimal hypersurface of the class $\mathcal{K}_0$.

Briefly, Theorem 3.2 means that the model theory of minimal hypersurfaces in the class $\mathcal{K}_0$ is the theory of minimal surfaces in $\mathbb{R}^3$ and in $S^3$.

Any minimal hypersurface of the class $\mathcal{K}_0$ is determined by a unit vector function $l(u,v) \in \mathbb{R}^{n+1}$ and a scalar function $r(u,v)$ of two variables satisfying the system of partial differential equations [8]:

\[
\begin{align*}
l_{uu} + l_{vv} + 2E l &= 0, \\
r_{uu} + r_{vv} + 2E r &= 0, \\
l_u^2 = l_v^2 = E, \\
l_u l_v &= 0.
\end{align*}
\]

Theorem 3.2 implies that any example of a minimal surface in $\mathbb{R}^3$ or in $S^3$ generates a family of solutions to the above system.

2. AN INVARIANT THEORY OF MINIMAL SURFACES IN THE THREE-DIMENSIONAL SPHERE

2.1. Strongly regular surfaces in $S^3$. We consider the three dimensional sphere $S^3$ as a unit hyper-sphere centered at the origin in the Euclidean space $\mathbb{R}^4$. The unit normal vector field to $S^3$ is denoted by $l$ and the flat Levi-Civita connection of the standard metric in $\mathbb{R}^4$ is denoted by $\nabla$. 


Let \( M : z = l(u, v), \quad (u, v) \in D \) be a surface in \( S^3 \). We denote by \( N \) the unit normal vector field to \( M \) and by \( E, F, G; \ e, f, g \) - the coefficients of the first and the second fundamental forms, respectively.

We suppose that the surface has no umbilical points and the principal lines on \( M \) form a parametric net, i.e.
\[
F(u, v) = f(u, v) = 0, \quad (u, v) \in D.
\]
Then the principal curvatures \( \nu_1, \nu_2 \) and the principal geodesic curvatures (geodesic curvatures of the principal lines) \( \gamma_1, \gamma_2 \) are given by
\[
(2.1) \quad \nu_1 = \frac{e}{E}, \quad \nu_2 = \frac{g}{G}, \quad \gamma_1 = -\frac{E_v}{2E\sqrt{G}}, \quad \gamma_2 = \frac{G_u}{2G\sqrt{E}}.
\]

We consider the canonical tangential frame field \( \{X, Y\} \) determined by
\[
X := \frac{z_u}{\sqrt{E}}, \quad Y := \frac{z_v}{\sqrt{G}}.
\]

The following Frenet type formulas for the orthonormal frame field \( X, Y, N, l \) associated with every point of \( M \) are valid
\[
(2.2) \quad \nabla_X X = \gamma_1 Y + \nu_1 N + l, \\
\nabla_X Y = -\gamma_1 X, \\
\nabla_X N = -\nu_1 X, \\
\nabla_X l = -X, \\
\n\nabla_Y X = \gamma_2 Y, \\
\nabla_Y Y = -\gamma_2 X + \nu_2 N + l, \\
\nabla_Y N = -\nu_2 Y, \\
\n\nabla_Y l = -Y.
\]

The Codazzi equations have the following form
\[
(2.3) \quad \gamma_1 = \frac{Y(\nu_1)}{\nu_1 - \nu_2} = \frac{(\nu_1)_v}{\sqrt{G}(\nu_1 - \nu_2)}, \quad \gamma_2 = \frac{X(\nu_2)}{\nu_1 - \nu_2} = \frac{(\nu_2)_u}{\sqrt{E}(\nu_1 - \nu_2)}
\]
and the Gauss equation can be written as follows:
\[
Y(\gamma_1) - X(\gamma_2) - (\gamma_1^2 + \gamma_2^2) = 1 + \nu_1 \nu_2,
\]
or
\[
(2.4) \quad \frac{(\gamma_1)_u}{\sqrt{G}} - \frac{(\gamma_2)_u}{\sqrt{E}} - (\gamma_1^2 + \gamma_2^2) = 1 + \nu_1 \nu_2.
\]

Remark 2.1. The mean curvature of \( M \) in \( S^3 \) is the function \( (\nu_1 + \nu_2)/2 \), while the Gauss curvature (the sectional curvature) of \( M \) is \( K = 1 + \nu_1 \nu_2 \).

A surface \( M : z = l(u, v), \quad (u, v) \in D \) parameterized with principal parameters is said to be strongly regular if (cf \[\text{[6]}\])
\[
\gamma_1(u, v)\gamma_2(u, v) \neq 0, \quad (u, v) \in D.
\]

Since
\[
\gamma_1 \gamma_2 \neq 0 \iff (\nu_1)_v(\nu_2)_u \neq 0,
\]
then the following formulas

\[ (2.5) \quad \sqrt{E} = \frac{(\nu_2)_u}{\gamma_2(\nu_1 - \nu_2)} > 0, \quad \sqrt{G} = \frac{(\nu_1)_v}{\gamma_1(\nu_1 - \nu_2)} > 0. \]

are valid for any strongly regular surface. Because of (2.5) formulas (2.2) become

\[ X_u = \frac{\gamma_1(\nu_2)_u}{\gamma_2(\nu_1 - \nu_2)} Y + \frac{\nu_1(\nu_2)_u}{\gamma_2(\nu_1 - \nu_2)} N + \frac{(\nu_2)_u}{\gamma_2(\nu_1 - \nu_2)} l, \]

\[ Y_u = -\frac{\gamma_1(\nu_2)_u}{\gamma_2(\nu_1 - \nu_2)} X, \]

\[ N_u = -\frac{\nu_1(\nu_2)_u}{\gamma_2(\nu_1 - \nu_2)} X, \]

\[ l_u = -\frac{(\nu_2)_u}{\gamma_2(\nu_1 - \nu_2)} X; \]

\[ (2.6) \quad X_v = \frac{\gamma_2(\nu_1)_v}{\gamma_1(\nu_1 - \nu_2)} Y, \]

\[ Y_v = -\frac{\gamma_2(\nu_1)_v}{\gamma_1(\nu_1 - \nu_2)} X + \frac{\nu_2(\nu_1)_v}{\gamma_1(\nu_1 - \nu_2)} N + \frac{(\nu_1)_v}{\gamma_1(\nu_1 - \nu_2)} l, \]

\[ N_v = -\frac{\nu_2(\nu_1)_v}{\gamma_1(\nu_1 - \nu_2)} Y, \]

\[ l_v = -\frac{(\nu_1)_v}{\gamma_1(\nu_1 - \nu_2)} Y. \]

Then the fundamental theorem for strongly regular surfaces states in terms of the four invariants \( \nu_1, \nu_2, \gamma_1, \gamma_2 \) as follows:

**Theorem 2.2. (Bonnet type fundamental theorem)** Given four functions \( \nu_1(u,v), \nu_2(u,v), \gamma_1(u,v), \gamma_2(u,v), \) \((u,v) \in D\) satisfying the following conditions:

1) \( \nu_1 - \nu_2 > 0, \quad \gamma_1(\nu_1)_v > 0, \quad \gamma_2(\nu_2)_u > 0; \)

2.1) \( \left( \ln \frac{(\nu_1)_u}{\gamma_1} \right)_u = \frac{(\nu_1)_u}{\nu_1 - \nu_2}, \quad \left( \ln \frac{(\nu_2)_u}{\gamma_2} \right)_v = -\frac{(\nu_2)_v}{\nu_1 - \nu_2}; \)

2.2) \( \nu_1 - \nu_2 \quad \left( \frac{(\gamma_1^2)_v}{(\nu_1)_v} - \frac{(\gamma_2^2)_u}{(\nu_2)_u} \right) - (\gamma_1^2 + \gamma_2^2) = 1 + \nu_1 \nu_2, \)

and an initial right oriented orthonormal frame \( l_0X_0Y_0N_0. \)

Then there exists a unique strongly regular surface \( \mathcal{M} : z = l(u,v), \) \((u,v) \in D_0 \) \( \left( (u_0,v_0) \in D_0 \subset D \right) \) in \( S^3 \), such that

(i) \( (u,v) \) are principal parameters;

(ii) \( l(u_0,v_0) = l_0, \quad X(u_0,v_0) = X_0, \quad Y(u_0,v_0) = Y_0, \quad N(u_0,v_0) = N_0; \)

(iii) the invariants of \( \mathcal{M} \) are the given functions \( \nu_1, \nu_2, \gamma_1, \gamma_2. \)
Proof: Let \( X(u, v), Y(u, v), N(u, v), l(u, v); (u, v) \in \mathcal{D} \) be four unknown vector valued functions in \( \mathbb{R}^4 \), satisfying the system (2.6). We write the system (2.6) in the form

\[
\begin{pmatrix}
X_u \\ Y_u \\ N_u \\ l_u
\end{pmatrix} = A \begin{pmatrix} X \\ Y \\ N \\ l \end{pmatrix},
\begin{pmatrix} X_v \\ Y_v \\ N_v \\ l_v \end{pmatrix} = B \begin{pmatrix} X \\ Y \\ N \\ l \end{pmatrix},
\]

where \( A \) and \( B \) are the skew symmetric \( 4 \times 4 \) matrices determined by (2.6).

The integrability condition of the system (2.7) is given by the equality

\[
B_u - A_v = [A, B].
\]

Taking into account (2.6), we find that (2.8) is equivalent to the conditions 2.1 and 2.2 of the theorem.

Now, let \( l_0X_0Y_0N_0 \) be an initial orthonormal right oriented coordinate system at the point \( l_0 \in S^3 \). Applying the theorem of existence and uniqueness of a solution to (2.6) with initial conditions

\[
l(u_0, v_0) = l_0, \quad X(u_0, v_0) = X_0, \quad Y(u_0, v_0) = Y_0, \quad N(u_0, v_0) = N_0,
\]

we obtain a uniquely determined solution \( l(u, v), X(u, v), Y(u, v), N(u, v); (u, v) \in \mathcal{D}', (u_0, v_0) \in \mathcal{D}' \subset \mathcal{D} \).

Further, we have to prove that \( lXYN \) form an orthonormal right oriented frame field in \( \mathcal{D}' \).

Let \( l(l^1, l^2, l^3, l^4), X(X^1, X^2, X^3, X^4), Y(Y^1, Y^2, Y^3, Y^4), N(N^1, N^2, N^3, N^4) \), and set

\[
f^{ij}(u, v) := l^i l^j + X^i X^j + Y^i Y^j + N^i N^j, \quad i, j = 1, 2, 3, 4.
\]

Differentiating (2.9) with respect to \( u \) and \( v \) and taking into account (2.6), we obtain that \( f^{ij}(u, v) = \text{const} = \delta_{ij}, \delta_{ij} \) being the Kronecker’s deltas. This proves that \( lXYN \) form an orthonormal right oriented frame field at any point \( (u, v) \in \mathcal{D}' \).

Now let us consider the surface \( \mathcal{M} : z = l(u, v), (u, v) \in \mathcal{D}' \).

Taking into account that the vector valued functions \( X, Y, N, l \) satisfy (2.6), we find that the functions \( E = l_u^2, G = l_v^2 \) satisfy (2.5), which implies that the invariants of \( \mathcal{M} \) are the given functions \( \nu_1, \nu_2, \gamma_1, \gamma_2 \).

2.2. Minimal strongly regular surfaces in \( S^3 \). Let \( \mathcal{M} : z = l(u, v), (u, v) \in \mathcal{D} \) be a minimal surface in \( S^3 \), i.e. \( \nu_1 + \nu_2 = 0 \). We set \( \nu = \nu_1 \) and further assume that \( \nu > 0 \). Then we have

\[
\nu_1 = \nu, \quad \nu_2 = -\nu.
\]

Further we assume that \( \mathcal{M} \) is a strongly regular surface, i.e.

\[
\nu_u \nu_v \neq 0.
\]

Taking into account the equalities (2.1) and (2.3) we find

\[
\gamma_1 = \frac{(\ln \sqrt{E})_v}{\sqrt{G}} = \frac{(\ln \sqrt{\nu})_v}{\sqrt{G}}, \quad \gamma_2 = \frac{(\ln \sqrt{G})_u}{\sqrt{E}} = \frac{(\ln \sqrt{\nu})_u}{\sqrt{E}},
\]

which imply that \( \sqrt{\nu E} \) does not depend on \( v \), while \( \sqrt{\nu G} \) does not depend on \( u \).
Let \((u_0, v_0)\) be a fixed point in \(\mathcal{D}\). We introduce new parameters in a neighborhood of \((u_0, v_0)\) by the formulas

\[
\begin{align*}
\bar{u} &= \int_{u_0}^{u} \sqrt{\nu} E \, du, \\
\bar{v} &= \int_{v_0}^{v} \sqrt{\nu} G \, dv.
\end{align*}
\]

and call them *canonical parameters* (cf [6]). It is easy to check that

\[
\bar{E} = \bar{G} = \frac{1}{\nu}
\]

with respect to the canonical parameters \((\bar{u}, \bar{v})\).

Further we assume, that the minimal strongly regular surface \(M : z = l(u, v), \ (u, v) \in \mathcal{D}\) is parameterized with canonical principal parameters. Then we have

\[
(2.10) \quad E = \frac{1}{\nu}, \ G = \frac{1}{\nu}, \ \gamma_1 = (\sqrt{\nu})_v, \ \gamma_2 = -(\sqrt{\nu})_u.
\]

Theorem 2.2, applied to minimal strongly regular surfaces parameterized with canonical principal parameters states as follows:

**Theorem 2.3.** Given a function \(\nu(u, v) > 0\) in a neighborhood \(\mathcal{D}\) of \((u_0, v_0)\) with \(\nu_u \nu_v \neq 0\), satisfying the partial differential equation

\[
(2.11) \quad \Delta \ln \nu = 2 \frac{1 - \nu^2}{\nu} \quad (\Delta \text{ – Laplace operator})
\]

and an initial right oriented orthonormal frame \(l_0X_0Y_0N_0\).

Then there exists a unique minimal strongly regular surface \(M : z = l(u, v), \ (u, v) \in \mathcal{D}_0 (\{(u_0, v_0) \in \mathcal{D}_0 \subset \mathcal{D}\})\) in \(S^3\), such that

(i) \((u, v)\) are canonical principal parameters;

(ii) \(l(u_0, v_0) = l_0, \ X(u_0, v_0) = X_0, \ Y(u_0, v_0) = Y_0, \ N(u_0, v_0) = N_0;\)

(iii) the invariants of \(M\) are the following functions

\[
\nu_1 = \nu, \ \nu_2 = -\nu, \ \gamma_1 = (\sqrt{\nu})_v, \ \gamma_2 = -(\sqrt{\nu})_u.
\]

**Remark 2.4.** Under the equalities (2.10) the integrability conditions 2.1 and 2.2 in Theorem 2.2 reduce to (2.11). Putting \(f = \ln \nu\), the partial differential equation (2.11) gets the form

\[
\Delta f + 4 \sinh f = 0 \quad (\sinh \text{ – Poisson equation}).
\]

Thus the sinh-Poisson equation is the natural partial differential equation of minimal (strongly regular) surfaces in \(S^3\).

Theorem 2.3 gives locally a one-to-one correspondence between minimal strongly regular surfaces (considered up to a motion in \(S^3\)) and the solutions of the natural partial differential equation (2.11), satisfying the conditions

\[
(2.12) \quad \nu > 0, \ \nu_u \nu_y \neq 0.
\]

Associated minimal surfaces to a given minimal strongly regular surface

A beautiful fact in the theory of minimal surfaces in the three-dimensional Euclidean space is that any minimal surface generates a family of associated minimal surfaces which are isometric to the given one.

Using Theorem 2.3 we introduce a family of associated minimal surfaces in the following way.

Let \(M : z = z(u, v), \ (u, v) \in \mathcal{D}\) be a minimal strongly regular surface in \(S^3\) parameterized by canonical parameters. Assume that the domain \(\mathcal{D}\) is a disc centered at the origin of the
parametric plane. If $\nu(u,v)$ is the normal curvature function of $\mathcal{M}$, then it is a solution to (2.11) satisfying the conditions (2.12). Putting

$$\nu_t(u,v) := \nu(\cos t u - \sin t v, \sin t u + \cos t v), \quad t \in [0, 2\pi), \quad (u,v) \in \mathcal{D},$$

we obtain a family of solutions $\{\nu_t, t \in [0, 2\pi)\}$ satisfying (2.12). Applying Theorem 2.3 to any solution $\nu_t$, we obtain a family of minimal strongly regular surfaces $\{\mathcal{M}_t\}$. It follows immediately that

$$E_t(u,v) = E(\cos t u - \sin t v, \sin t u + \cos t v)$$

$$G_t(u,v) = G(\cos t u - \sin t v, \sin t u + \cos t v).$$

i.e. any surface $\mathcal{M}_t$ is isometric to the the given surface $\mathcal{M}$. It is natural to call $\{\mathcal{M}_t, t \in [0, 2\pi)\}$ a family of minimal surfaces associated with a given strongly regular surface $\mathcal{M}$.

**Remark 2.5.** The above approach for introducing a family of associated isometric surfaces can be applied to many other cases. For example, applying the results in [6], [9], it gives a family of:

- surfaces of constant mean curvature isometric with a given CMC-surface in $\mathbb{R}^3$;
- surfaces of constant Gauss curvature ($K = -1$) isometric with a given surface of constant Gauss curvature ($K = -1$) in $\mathbb{R}^3$;
- minimal surfaces isometric with a given minimal surface in $\mathbb{R}^4$.

### 3. Minimal hypersurfaces of conullity two

#### 3.1. Hypersurfaces of conullity two.

Let $(\mathcal{M}^n, g)$ be a regular hypersurface in the Euclidean space $(\mathbb{R}^{n+1}, g)$ with the induced metric $g$ and shape operator $A$. The standard flat Levi-Civita connection of $(\mathbb{R}^{n+1}, g)$ is denoted by $\nabla$.

Let the hypersurface $\mathcal{M}^n$ be of *type number two*, i.e. its shape operator $A$ has two different from zero eigenvalues $\nu_1$ and $\nu_2$ and the other $n - 2$ eigenvalues are zero at each point. We recall that the case $\nu_1 = \nu_2 = 0$ characterizes locally a hyperplane and the case $\nu_1 \neq 0$, $\nu_2 = 0$ (hypersurfaces with type number one) characterizes locally a developable hypersurface. Here we study the case

$$\nu_1(p) \nu_2(p) \neq 0, \quad p \in \mathcal{M}^n.$$

Since our considerations are local, we can choose two unit tangent vector fields $X$ and $Y$, such that

$$AX = -\nu_1 X, \quad AY = -\nu_2 Y.$$ 

Then for any tangent vector $x_0 \perp X, Y$ we have

$$Ax_0 = 0.$$ 

We denote by $\Delta$ the distribution span${\{X, Y\}}$, and by $\Delta^\perp$ the distribution, orthogonal to $\Delta$.

Let us introduce the 1-form $\sigma$ on the distribution $\Delta^\perp$ by the formula

$$\sigma(x_0) = g(\nabla_{x_0} X, Y), \quad x_0 \in \Delta^\perp.$$
Then the standard Codazzi equations for the shape operator $A$ imply the following formulas

\begin{align}
  g(\nabla_X X, x_0) &= d \ln \nu_1(x_0), \\
  g(\nabla_Y Y, x_0) &= d \ln \nu_2(x_0), \\
  g(\nabla_X Y, x_0) &= \frac{\nu_1 - \nu_2}{\nu_2} \sigma(x_0), \\
  g(\nabla_Y X, x_0) &= \frac{\nu_1 - \nu_2}{\nu_1} \sigma(x_0), \\
  \nabla_{x_0} X &= \sigma(x_0) Y, \\
  \nabla_{x_0} Y &= -\sigma(x_0) X, \\
  (\nu_1 - \nu_2)^2 \gamma_1 &= (\nu_1 - \nu_2) Y(\nu_1), \\
  (\nu_1 - \nu_2)^2 \gamma_2 &= (\nu_1 - \nu_2) X(\nu_2),
\end{align}

(3.1)

Let $N$ be the unit normal vector field to $M^n$ and $\{e_1, ..., e_{n-2}\}$ be an orthonormal frame field such that $\Delta^\perp = \text{span}\{e_1, ..., e_{n-2}\}$. Taking into account (3.1), we write the following Frenet type formulas for the derivatives with respect to $X$ and $Y$:

\begin{align}
  \nabla_X X &= \gamma_1 Y + \sum_{i} \lambda_i e_i + \nu_1 N, \\
  \nabla_X Y &= -\gamma_1 X + \frac{\nu_1 - \nu_2}{\nu_2} \sum_{i} \sigma_i e_i, \\
  \nabla_X e_i &= -\lambda_i X - \frac{\nu_1 - \nu_2}{\nu_2} \sigma_i Y, \\
  \nabla_X N &= -\nu_1 X, \\
  \nabla_Y X &= \gamma_2 Y + \frac{\nu_1 - \nu_2}{\nu_1} \sum_{i} \sigma_i e_i, \\
  \nabla_Y Y &= -\gamma_2 X + \sum_{i} \mu_i e_i + \nu_2 N, \\
  \nabla_Y e_i &= -\frac{\nu_1 - \nu_2}{\nu_1} \sigma_i X - \mu_i Y, \\
  \nabla_Y N &= -\nu_2 Y,
\end{align}

(3.2)

where

\[ \gamma_1 = g(\nabla_X X, Y), \quad \gamma_2 = -g(\nabla_Y Y, X); \quad \lambda_i = \frac{e_i(\nu_1)}{\nu_1}, \quad \mu_i = \frac{e_i(\nu_2)}{\nu_2}, \quad \sigma_i = \sigma(e_i); \quad i = 1, ..., n - 2. \]

First we shall consider the special class of bi-umbilical hypersurfaces of type number two, which is the analogue to the umbilical surfaces in $\mathbb{R}^3$. In this case the tangent space at each point of $M^n$ consists of two orthogonal umbilical distributions $\Delta$ and $\Delta^\perp$.

3.2. Bi-umbilical hypersurfaces of type number two. Let us consider the class of hypersurfaces of type number two satisfying the equality

\[ \nu_1(p) = \nu_2(p) = \nu(p), \quad p \in M^n. \]
Since the shape operator $A$ has equal eigen values at each of the mutually orthogonal distributions $\Delta$ and $\Delta^\perp$, we called these hypersurfaces bi-umbilical [8].

The first property of these hypersurfaces follows immediately from (3.2):

$$[X, Y] = \nabla_X Y - \nabla_Y X = -\gamma_1 X - \gamma_2 Y,$$

i.e. the distribution $\Delta$ of any bi-umbilical hypersurface of type number two is involutive.

Bi-umbilical hypersurfaces of type number two are described by the following statement.

**Theorem 3.1.** The integral surfaces of the distribution $\Delta$ of any bi-umbilical hypersurface $M^n$ of type number two are two-dimensional spheres.

Conversely, any two-dimensional sphere generates a family of bi-umbilical hypersurfaces of type number two.

**Proof:** Let $M^2 : z = z(u, v), (u, v) \in \mathcal{D}$ be a fixed regular integral surface of the distribution $\Delta$. Further we assume that the parametric lines of $M^2$ are orthogonal and denote $z_u^2 = E$, $z_v^2 = G$. Then we can choose $X = \frac{z_u}{\sqrt{E}}$, $Y = \frac{z_v}{\sqrt{G}}$. Taking into account (3.2), we obtain the following Frenet type formulas for the surface $M^2$:

$$\nabla_X X = \gamma_1 Y + \sum_{i=1}^{n-2} \lambda_i e_i + \nu N,$$

$$\nabla_X Y = -\gamma_1 X,$$

$$\nabla_X e_i = -\lambda_i X,$$

$$\nabla_X N = -\nu X,$$

$$\nabla_Y X = \gamma_2 Y,$$

$$\nabla_Y Y = -\gamma_2 X + \sum_{i=1}^{n-2} \lambda_i e_i + \nu N,$$

$$\nabla_Y e_i = -\lambda_i Y,$$

$$\nabla_Y N = -\nu Y.$$

(3.3)

In view of the equalities

$$\nabla_X \nabla_Y N - \nabla_Y \nabla_X N = \nabla_{[X,Y]} N,$$

(3.4)

$$\nabla_X \nabla_Y e_i - \nabla_Y \nabla_X e_i = \nabla_{[X,Y]} e_i, \quad i = 1, \ldots, n-2$$

(3.5)

we find that

$$X(\nu) = Y(\nu) = 0, \quad X(\lambda_i) = Y(\lambda_i) = 0.$$

Hence

$$\nu = \text{const} \neq 0, \quad \lambda_i = c_i = \text{const}.$$

If all $c_i = 0$, then the space span\{ $z_u, z_v, N$ \} is a constant three-dimensional subspace $\mathbb{R}^3$ of $\mathbb{R}^{n+1}$ and $M^2$ is a surface in $\mathbb{R}^3$. It follows immediately from (3.3) that $M^2$ lies on a sphere with radius $\frac{1}{|\nu|}$.
If \( \sum_i c_i^2 = c^2 > 0 \), we consider the unit vector field \( b = \frac{1}{c} \sum_i c_i e_i \). Then the equations (3.3) become

\[
\nabla_X X = \gamma_1 Y + c b + \nu N \\
\nabla_X Y = -\gamma_1 X, \\
\nabla_X b = -c X, \\
\nabla_X N = -\nu X;
\]

(3.6)

\[
\nabla_Y X = \gamma_2 Y, \\
\nabla_Y Y = -\gamma_2 X + c b + \nu N, \\
\nabla_Y b = -c Y, \\
\nabla_Y N = -\nu Y.
\]

The equalities (3.6) imply that the space span\( \{z_u, z_v, b, N\} \) is a constant four-dimensional subspace \( \mathbb{R}^4 \) of \( \mathbb{R}^{n+1} \) and \( \mathcal{M}^2 \) lies in \( \mathbb{R}^4 \). Further it follows that the vectors

\[
\bar{b} = \cos \alpha b + \sin \alpha N, \quad \bar{N} = -\sin \alpha b + \cos \alpha N,
\]

where \( \tan \alpha = -\frac{c}{\nu} \) satisfy the conditions

\[
\nabla_X \bar{b} = \nabla_Y \bar{b} = 0, \quad \nabla_X \bar{N} = -\sqrt{c^2 + \nu^2} X, \quad \nabla_Y \bar{N} = -\sqrt{c^2 + \nu^2} Y.
\]

The last equalities again show that \( \mathcal{M}^2 \) lies on a sphere with radius \( \frac{1}{\sqrt{c^2 + \nu^2}} \) in a constant three-dimensional subspace \( \mathbb{R}^3 \) in \( \mathbb{R}^4 \), orthogonal to the constant vector \( b \).

Conversely, let \( S^2(r) : z = z(u, v), \ (u, v) \in \mathcal{D} \) be a sphere in \( \mathbb{R}^3 \subset \mathbb{R}^{n+1} \). Taking into account the above arguments, we shall give a construction of the bi-umbilical hypersurfaces generated by a given sphere \( S^2(r) \).

Let \( \bar{N} \) denote the normal vector field to \( S^2(r) \) in \( \mathbb{R}^3 \) and \( e \) be a constant unit vector in \( \mathbb{R}^{n-2} \) orthogonal to \( \mathbb{R}^3 \). Choose an orthonormal basis \( \{e, e_2, ..., e_{n-2}\} \) of \( \mathbb{R}^{n-2} \) and consider the vectors

\[
N = \cos \alpha \bar{N} + \sin \alpha e, \quad e_1 = -\sin \alpha \bar{N} + \cos \alpha e, \quad \alpha = \text{const}.
\]

Then the required hypersurfaces are constructed as follows:

\[
\mathcal{M}^n : X(u, v, w^1, ..., w^{n-2}) = z(u, v) + \sum_{\alpha=1}^{n-2} w^\alpha e_\alpha; \quad (u, v) \in \mathcal{D}, \ w^\alpha \in \mathbb{R}.
\]

Direct calculations show that \( \mathcal{M}^n \) is a hypersurface of type number two with normal vector field \( N \) and \( \nu_1 = \nu_2 \). Hence \( \mathcal{M}^n \) is a bi-umbilical hypersurface of type number two. \( \Box \)

Let \( \mathcal{M}^n \) be a hypersurface in \( \mathbb{R}^{n+1} \) of type number two, which is the envelope of a two-parameter family of hyperplanes \( \{\mathbb{R}^n(u, v)\}; \ (u, v) \in \mathcal{D} \), defined in a domain \( \mathcal{D} \subset \mathbb{R}^2 \). We denote by \( l = l(u, v) \) the unit normal vector field of the hyperplane \( \mathbb{R}^n(u, v) \) (\( l \) is determined up to a sign) and by \( r = r(u, v) - \) the oriented distance from the origin of \( \mathbb{R}^{n+1} \) to \( \mathbb{R}^n(u, v) \).
We assume that \( l, l_u \) and \( l_v \) are linearly independent (otherwise \( \mathcal{M}^n \) is a developable ruled hypersurface, which is a hypersurface with type number one). Then, locally there exist \( n-2 \) mutually orthogonal unit vectors \( b_1(u,v), \ldots, b_{n-2}(u,v), (u,v) \in \mathcal{D}, \) which are orthogonal to \( \operatorname{span}\{l, l_u, l_v\} \). Further we denote

\[
E(u,v) = g(l_u, l_u), \quad F(u,v) = g(l_u, l_v), \quad G(u,v) = g(l_v, l_v),
\]

satisfying the inequalities \( E > 0, \ G > 0, \ E \ G - F^2 > 0. \)

Then \( \mathcal{M}^n \) can be parameterized locally as follows:

\[
(3.7) \quad X(u,v,w^\alpha) = r l + \frac{G r_u - F r_v}{W^2} l_u + \frac{E r_v - F r_u}{W^2} l_v + \sum_{\alpha=1}^{n-2} w^\alpha b_{\alpha},
\]

where \( (u,v) \in \mathcal{D}, \ w^\alpha \in \mathbb{R}, \ \alpha = 1, \ldots, n-2. \)

Thus each pair of a unit vector-valued function \( l = l(u,v) \) and a scalar function \( r = r(u,v) \) determines a hypersurface \( \mathcal{M}^n \) of type number two by the equation (3.7).

In [8] we proved that a hypersurface of type number two given by (3.7) is bi-umbilical if and only if

\[
(3.8) \quad l_{uu} - l_{vv} = \frac{E_u}{E} l_u - \frac{E_v}{E} l_v, \quad 2l_{uv} = \frac{E}{E_u} l_u + \frac{E}{E_v} l_v,
\]

\[
r_{uu} - r_{vv} = \frac{E_u}{E} r_u - \frac{E_v}{E} r_v, \quad 2r_{uv} = \frac{E}{E_u} r_u + \frac{E}{E_v} r_v.
\]

Taking into account that the vector function \( l \) is the normal vector field to \( \mathcal{M}^n \), Theorem 3.1 implies that the solutions to the system (3.8) can be found explicitly.

In what follows we shall consider hypersurfaces of type number two satisfying the condition

\[
(3.9) \quad \nu_1(p) - \nu_2(p) \neq 0, \quad p \in \mathcal{M}^n.
\]

3.3. **Minimal hypersurfaces of type number two with involutive distribution.** Let \( \mathcal{M}^n \) be a regular hypersurface of type number two satisfying the condition (3.9). Formulas (3.1) imply that the distribution \( \Delta \) is involutive if and only if

\[
(3.10) \quad \sigma(x_0) = 0, \quad x_0 \in \Delta^\perp.
\]

Further we assume that \( \mathcal{M}^n \) is with involutive distribution \( \Delta \), i.e. the condition (3.10) is valid.

We denote by \( \mathcal{K}_0 \) the class of hypersurfaces of type number two with involutive distribution \( \Delta \).

A hypersurface \( \mathcal{M}^n \) is minimal if \( \nu_1 + \nu_2 = 0 \).

Minimal hypersurfaces of type number two with involutive distribution are described by the following statement.

**Theorem 3.2.** The integral surfaces of the distribution \( \Delta \) of any minimal hypersurface \( \mathcal{M}^n \) of the class \( \mathcal{K}_0 \) is a minimal surface in \( \mathbb{R}^3 \) or in \( S^3(r) \).

Conversely, any minimal surface in \( \mathbb{R}^3 \) or in \( S^3(r) \) generates a minimal hypersurface of the class \( \mathcal{K}_0 \).
Proof: We put
\[(3.11) \quad \nu_1 = \nu > 0, \quad \nu_2 = -\nu.\]

Let \(\mathcal{M}^2 : z = z(u, v), (u, v) \in \mathcal{D}\) be a fixed regular integral surface of the distribution \(\Delta\). We assume that the parametric lines of \(\mathcal{M}^2\) are orthogonal and denote \(z_u = \frac{z_u}{\sqrt{E}}, \quad z_v = \frac{z_v}{\sqrt{G}}\). Taking into account (3.10) and (3.11), we obtain from (3.2) the following Frenet type formulas for the surface \(\mathcal{M}^2\):

\[(3.12)\]
\[
\begin{align*}
\nabla_X X &= \gamma_1 Y + \sum_i \lambda_i e_i + \nu N, \\
\nabla_X Y &= -\gamma_1 X, \\
\nabla_X e_i &= -\lambda_i X, \\
\nabla_X N &= -\nu X, \\
\nabla_Y X &= \gamma_2 Y, \\
\nabla_Y Y &= -\gamma_2 X + \sum_i \lambda_i e_i - \nu N, \\
\nabla_Y e_i &= -\lambda_i Y, \\
\nabla_Y N &= \nu Y.
\end{align*}
\]

Using (3.12) we find from (3.5)
\[(\lambda_i)_u = 0, \quad (\lambda_i)_v = 0.\]

The last equalities imply that
\[
\lambda_i = c_i = \text{const}, \quad i = 1, ..., n - 2.
\]

If all \(c_i = 0\), then the space span \(\{z_u, z_v, N\}\) is a constant \(\mathbb{R}^3\) and \(\mathcal{M}^2 : z = z(u, v)\) is a surface in \(\mathbb{R}^3\). It follows immediately from (3.12) that \(\mathcal{M}^2\) is a minimal surface in \(\mathbb{R}^3\).

If \(\sum_i c_i^2 = c^2 > 0\), we consider the unit vector field \(b = \frac{1}{c} \sum_i c_i e_i\). Then (3.12) become

\[
\begin{align*}
\nabla_X X &= \gamma_1 Y + c b + \nu N, \\
\nabla_X Y &= -\gamma_1 X, \\
\nabla_X b &= -c X, \\
\nabla_X N &= -\nu X; \\
\nabla_Y X &= \gamma_2 Y, \\
\nabla_Y Y &= -\gamma_2 X + c b - \nu N, \\
\nabla_Y b &= -c Y, \\
\nabla_Y N &= \nu Y.
\end{align*}
\]

These equalities show that \(\mathcal{M}^2\) is a minimal surface in \(\mathbb{S}^3 (r = 1/c)\).
Conversely, let \( \mathcal{M}^2 : z = z(u, v), (u, v) \in \mathcal{D} \) be a minimal surface in a fixed \( S^3(r) \subset \mathbb{R}^{n+1} \). Taking into account the above arguments, we shall give a construction of the minimal hypersurface of type number two generated by the given minimal surface \( \mathcal{M}^2 \).

Let \( N \) be the normal vector field to \( \mathcal{M}^2 \) in \( S^3 \). Denote by \( \mathbb{R}^{n-2}(u, v) \) the plane in \( \mathbb{R}^{n+1} \) orthogonal to \( \text{span}\{z_u, z_v, N\} \) at each point of \( \mathcal{M}^2 \) and choose a base \( \{b_\alpha(u, v)\}, \alpha = 1, \ldots, n-2 \) of \( \mathbb{R}^{n-2}(u, v) \). Then we consider the hypersurface in \( \mathbb{R}^{n+1} \) given by

\[
\mathcal{M}^n : X(u, v; w^1, \ldots, w^{n-2}) := z(u, v) + \sum w^\alpha b_\alpha, \quad (u, v) \in \mathcal{D}, \quad w^\alpha \in \mathbb{R}, \alpha = 1, \ldots, n-2.
\]

By straightforward computations it follows that \( \mathcal{M}^n \) is a minimal hypersurface of type number two. It is clear that the distribution \( \Delta \) of \( \mathcal{M}^n \) is involutive.

Further, let \( \mathcal{M}^2 : z = z(u, v), (u, v) \in \mathcal{D} \) be a minimal surface in a fixed \( \mathbb{R}^3 \subset \mathbb{R}^{n+1} \). Choose an orthonormal basis \( \{b_\alpha\}, \alpha = 1, \ldots, n-2 \) for the orthogonal complement \( \mathbb{R}^{n-2} \) of \( \mathbb{R}^3 \) in \( \mathbb{R}^{n+1} \) and consider the hypersurface in \( \mathbb{R}^{n+1} \) given by

\[
\mathcal{M}^n : X(u, v; w^1, \ldots, w^{n-2}) := z(u, v) + \sum w^\alpha b_\alpha, \quad (u, v) \in \mathcal{D}, \quad w^\alpha \in \mathbb{R}, \alpha = 1, \ldots, n-2.
\]

It is easy to check that \( \mathcal{M}^n \) is a minimal hypersurface of type number two with involutive distribution \( \Delta \).

Let \( \mathcal{M}^n \) be a hypersurface of type number two given by (3.5). In [8] we proved that \( \mathcal{M}^n \) is minimal if and only if

\[
\begin{align*}
    l_{uu} + l_{vv} + 2El = 0, \\
    r_{uu} + r_{vv} + 2Er = 0, \\
    l_u^2 = l_v^2 = E, \\
    l_u l_v = 0.
\end{align*}
\]

Theorem 3.2 means that the solutions to the above system are generated by the minimal surfaces in \( \mathbb{R}^3 \) and by the minimal surfaces in \( S^3 \).

In a next paper we will show in details that the minimal ruled hypersurfaces in \( \mathbb{R}^{n+1} \) are generated by the helicoids in \( \mathbb{R}^3 \) or by the minimal "ruled" surfaces [3] in \( S^3 \).

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