Mathematical modeling and analysis of conditions for the robot's uniformly accelerated motion along a straight rail track, taking into account the delay intervals

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Abstract. The problem of the straight-line motion of a robot along a rectilinear rail track with the constant acceleration with delay intervals is considered. The aim of the paper is to determine and analyse the conditions describing the straight-line uniformly accelerated motion of the robot, taking into account the delay intervals. The law of speed change of the robot is simulated, using linear piecewise continuous functions of the time of motion. The necessary conditions for the uniformly accelerated motion of the robot in the acceleration and deceleration sections are formulated. Additionally, the criteria for uniformly variable acceleration and deceleration in the straight-line motion of the robot without delay intervals are recorded. It should be noted that the conditions and patterns presented in this paper have many practical applications. For example, they can be used in the development of traffic routes or in the operation of railway trains.

1. Introduction
Fundamental physics knows the phenomenon of a decrease in the friction coefficient with an increase in the sliding speed during the beginning of the wheel rolling on the rolling surface [1-5]. Undoubtedly, this phenomenon has important practical significance [6-7]. Actually, it significantly affects the dynamics of the motion of various rail vehicles, such as traditional railway trains, monorail trains, funiculars on a cable suspension, etc. Moreover, this phenomenon in the mathematical modeling of the motion of rail vehicles should be taken into account when predicting wheel wear [8] and when dealing with the wheel squeaking on curved sections of the track [9].

This paper considers the problem of mathematical modeling of straight-line accelerated motion of a robot along a rail guide of a finite specified length. The guide is an elongated rectangular parallelepiped made of aluminum sheet. Two pairs of small plastic wheels mounted on the robot roll along the surface of the guide and ensure that the entire robot is fixed within this guide. The laws of motion of the robot are described by various piecewise continuous dependencies of the speed on the time of motion. Common methods of mathematical analysis of piecewise continuous functions and methods of the kinematics of translational straight-line motion of a solid body are used for the purpose of this paper. Differential and integral calculus allows analyzing the change in acceleration and determining the position of the robot at any time of its motion along the guide. The theory of the kinematics of the motion of a solid body provides recording the laws of change in the speed of the robot. Moreover, mathematical conditions, which ensure the motion of the robot with a given constant acceleration in the areas of acceleration and deceleration, are formulated. The formulated conditions
are planned to be used in the motion control programs of the robot under consideration. Generally, it should be noted that, owing to the increase in the speed and load capacity of modern trains [10-12], the process of acceleration and deceleration with the constant acceleration contributes to the growth of durability of elements of the truck and reduces the wear of the rails, improves the process of current collection by pantographs and helps to prevent interruption of air lines.

2. **Robot design**

It is assumed that the electronic circuit of the robot design includes an Arduino type microcontroller that establishes the motion direction, a driver allowing to control the voltage on the electric motor terminals, the electric motor itself, and the power source. Figure 1 shows the robot positioned on a rectangular rail.

![Image of a robot on a straight-line rail track.](image)

**Figure 1.** The image of a robot on a straight-line rail track.

In this case, the program of the controlled motion of the robot is recorded in the permanent memory of the onboard microcontroller. A camera, as well as various sensors allowing measuring the speed of the robot's motion, its coordinates, and other physical parameters, can be placed onboard the robot as target equipment.

3. **Mathematical analysis of robot's acceleration with and without the initial delay interval**

3.1. **Mathematical model of robot motion during acceleration**

Let the law of speed change during the straight-line motion of the tangible robot contain the initial delay interval, \( t \in [0,1] \). Practically, this interval may describe a situation when a real solid body under the action of an increasing force stays in a position of equilibrium due to the counteraction of the static friction force. Additionally, let us assume that the robot's motion occurs with constant positive acceleration. Consider the definition.

**Definition 1.** The robot accelerates if the absolute value of its variable speed increases.

In this case, it is assumed that the law of speed change during acceleration of the robot is being simulated by a linear function that is continuous on the interval \( t \in (1, t_d] \), where \( t_d \) is the end time of acceleration of the robot. This pattern is described as follows:

\[
V = (t - 1) \cdot 1(t - 1),
\]

where \( 1(t - 1) \) is the Heaviside function.
By calculating the time derivative, $t$, from the speed (1), the expression for the robot's acceleration is found:

$$ W = t - 1 \quad \text{(2)} $$

Expression (2) shows that the acceleration value is constant on the robot acceleration interval, $t \in (1, t_d]$. It is obvious that the law of the change in the coordinate of the robot during acceleration on the interval $t \in (1, t_d]$ is a quadratic power function of the time of motion with the initial delay interval, $t \in [0, 1]$. It should be noted that it does not fit into the context of this study. For this reason, this law has no further consideration.

Pattern (1) simulates the behavior of the speed against the time of motion at two intervals simultaneously, i.e. on the delay interval, $t \in [0, 1]$, and on the interval of acceleration, $t \in (1, t_d]$. This pattern is continuous individually on each interval. However, function (1) is, generally, a piecewise linear function of the time of motion on two intervals. A similar conclusion can be drawn with respect to function (2). Actually, function (2) on these two intervals is a piecewise constant function of the robot's time of motion.

Figure 1 shows the change in the speed of the robot during acceleration with the initial interval of rest. Figure 2 shows the acceleration of the robot during the acceleration with a constant acceleration of $1 \text{m/s}^2$.

Gathering speed by the robot with the constant acceleration without taking into account the delay interval is simulated as follows:

$$ V = (t - 1) \quad \text{(3)} $$

By calculating the time derivative, $t$, from the speed (3), the expression for the robot's acceleration is obtained:

$$ W = 1 \quad \text{(4)} $$

Problem statement is to formulate a new algorithm that allows finding all binary palindromes of an arbitrary given finite bit $n$.

![Figure 2. The dependence of the robot's speed on the time of its acceleration.](image)
Let us note that the law of change in the robot’s coordinate during acceleration on the interval $t \in (1,t_d]$ is a quadratic power function of the time of motion with zero initial speed without taking into account the delay interval, $t \in [0,1]$. Pattern (1) is continuous on the considered interval $t \in (1,t_d]$. 

![Graph of W vs t]

**Figure 3.** The dependence of the robot’s acceleration on the time of its acceleration.

### 3.2 The necessary condition for accelerating the robot, taking into account the delay interval

The following theorem holds.

**Theorem 1.** To accelerate the robot, as described by pattern (1) on the interval $t \in (1,t_d]$ with the initial interval of the robot's delay, $t \in [0,1]$, the following necessary condition should be met:

$$V \cdot \dot{W} \geq 0.$$  

(5)

**Proof.** There is no motion on the delay interval, $t \in [0,1]$. Hence, the equality $V \cdot \dot{W} = 0$ is satisfied on this interval. On the acceleration interval, $t \in (1,t_d]$, $V \cdot \dot{W} = (t-1) \cdot [t(t-1)]^2 > 0$ is found, taking into account expressions (1) and (2). Thus, for the acceleration of the robot described by the regularity (1) on the interval $t \in (1,t_d]$ with the initial interval of the motionlessness of the robot $t \in [0,1]$, the fulfillment of condition (5) is required. Actually, the condition $t > 1$ holds on the acceleration interval. The theorem is proved.

Thus, condition (5) is necessary for the robot to accelerate on the interval $t \in (1,t_d]$, as described by pattern (1), taking into account the delay interval, $t \in [0,1]$. Therefore, condition (5) is a property that characterizes the acceleration of the robot on the interval $t \in (1,t_d]$, as described by pattern (1), taking into account the delay interval, $t \in [0,1]$.

Let us note that the converse statement of this theorem in the general case is not true. Actually, for example, when condition (5) is met, particularly, when $V \cdot \dot{W} = 0$, there is no motion on both intervals under consideration.

### 3.3 The criterion for accelerating the robot without taking into account the delay interval

The following theorem holds.

**Theorem 2.** In order for a robot, which has the law of the straight-line motion (3), to accelerate on the interval $t \in (1,t_d]$ from zero (without taking into account the zero value of the initial speed), the following condition is necessary and sufficient to be met:

$$V \cdot \dot{W} > 0.$$  

(6)

**Proof of necessity.** On the acceleration interval, $t \in (1,t_d]$, from the zero initial speed of the robot
(without taking into account this zero value of the initial speed), the equalities (3) and (4) are observed. Since the condition \( t > 1 \) is satisfied, when multiplying the right parts of expressions (3) and (4), \( V \cdot W = (t - 1) > 0 \) is found.

Proof of sufficiency. The fulfillment of condition (6) assumes that the marks of the speed and acceleration of the robot on the considered interval of continuous change of these values, \( t \in (1, t_d) \), coincide. Therefore, when the robot moves at a speed, as described by a linear function, we invariably obtain the constant acceleration with a mark that coincides with the mark of the speed. Thus, the robot moving at a positive speed will accelerate on the considered interval \( t \in (1, t_d) \). The theorem is proved.

4. Mathematical analysis of robot’s deceleration with and without the finite delay interval

4.1. Mathematical model of robot motion during deceleration

Consider the definition.

Definition 2. The robot will decelerate if the absolute value of its variable speed decreases to zero.

Let us consider the case of the straight-line motion of the robot during its deceleration on the interval \( t \in [0,1] \) with a finite delay interval, \( t \in (1, t_d) \), where \( t_d \) is the end time of the considered delay interval of the robot. In practical applications, the situation under consideration simulates changes in speed when a real solid body stops and remains stationary on the given delay interval, \( t \in (1, t_d) \), as a result of a decrease in speed. Let the motion of the robot on the deceleration interval, \( t \in [0,1] \), occur with constant negative acceleration. In this case, the law of speed change during deceleration of the robot is simulated by a linear function, which is continuous on the interval \( t \in [0,1] \). The chosen pattern is described as follows:

\[
V = (1 - t) \cdot 1(1-t).
\]

(7)

here, \( 1(1-t) \) is the Heaviside function.

By calculating the time derivative, \( t \), from the speed (7), the expression for the robot’s acceleration is determined:

\[
W = -1(1-t).
\]

(8)

It follows from expression (8) that the acceleration value on the robot's deceleration interval, \( t \in [0,1] \), is constant and negative.

In the case under consideration, the law of change in the robot's coordinate on the deceleration interval, \( t \in [0,1] \), is a quadratic power function of the time of motion turning into the robot's interval of rest, \( t \in (1, t_d) \). Let us note that this law is not considered.

Pattern (7) models the behavior of the speed against the time of motion at two intervals simultaneously, i.e. on the deceleration interval, \( t \in [0,1] \), and on the delay interval, \( t \in (1, t_d) \). This pattern is continuous individually on each of two intervals. However, function (7) is, generally, a piecewise linear function of the time of motion on two intervals, similar to function (1). Moreover, function (8) on these two intervals is a piecewise constant function of the robot's time of motion.

Figure 4 shows the change in the robot’s speed during deceleration with a finite delay interval. Figure 5 shows the acceleration of the robot during the deceleration with a constant acceleration of -1 m/s².

4.2. Necessary condition for the deceleration of the robot

The following theorem holds.

Theorem 3. To decelerate the robot, as described by pattern (4) on the interval \( t \in [0,1] \) with the finite interval of the robot’s delay, \( t \in (1, t_d) \), the following condition should be met:

\[
V \cdot W \leq 0.
\]

(9)
Proof. Let us find \( V \cdot W = -(1-t) \cdot [(1-t)^2] < 0 \) on the deceleration interval, \( t \in [0,1] \), taking into account expressions (4) and (5). On the delay interval, \( t \in (1,t_d) \), there is no motion of the robot. Hence, the equality \( V \cdot W = 0 \) is satisfied on this interval. Thus, to decelerate the robot, as described by pattern (4) on the interval \( t \in [0,1] \) with the finite interval of the robot’s delay, \( t \in (1,t_d) \), the condition (9) should be met. The theorem is proved.

Thus, condition (9) is necessary for the robot to brake on the interval \( t \in [0,1] \), as described by pattern (7), taking into account the delay interval, \( t \in (1,t_d) \). Therefore, condition (9) is a property that characterizes the acceleration of the robot on the interval \( t \in [0,1] \), as described by pattern (7), taking into account the delay interval, \( t \in (1,t_d) \).

Let us note that the converse statement of this theorem in the general case is not true. Actually, for example, when condition (9) is met, particularly, when \( V \cdot W = 0 \), there is no motion on both intervals under consideration.

![Figure 4](image1.png)  
**Figure 4.** The dependence of the speed on the time when decelerating the robot.

![Figure 5](image2.png)  
**Figure 5.** The dependence of the acceleration on the time when decelerating the robot.

4.3 The criterion for decelerating the robot without taking into account the delay interval

The following theorem holds.

**Theorem 4.** In order for a robot with the law of the straight-line motion (7) to decelerate on the interval \( t \in [0,1] \) (without taking into account the zero finite value of the speed), the following condition is necessary and sufficient to be met:

\[
V \cdot W < 0.
\]

(10)
Proof of the necessity. On the robot's deceleration interval, \( t \in (0,1) \), from a single initial value of its speed (without taking into account the finite zero value of the speed), the equalities (7) and (8) are observed. Since the condition \( t < 1 \) is satisfied, when multiplying the right parts of expressions (7) and (8), \( V \cdot W = -(1-t) < 0 \) is found.

Proof of sufficiency. The fulfillment of condition (10) assumes that the marks of the speed and acceleration of the robot on the considered interval of continuous change of these values, \( t \in (0,1) \), are contrary. Therefore, when the robot moves at a speed, as described by a linear function, we invariably obtain the constant acceleration with a mark that is contrary to the one of the speed. Thus, the robot moving at a positive linear speed will have constant negative acceleration. Hence, the robot will decelerate on the considered interval \( t \in (0,1) \). The theorem is proved.

5. Mathematical modelling of the robot's straight-line motion

5.1. Model of the robot's return motion without stopping at an intermediate robot

It is proposed to consider two models describing the change in the speed of the robot during the straight-line motion with the constant acceleration with delay interval. All the models under consideration describe the motion of the robot, which returns to its initial position at the finite moment of the time of motion. An example of such motions is the motion of the robot on a rectilinear section of the track from the initial station to the station closest to it, with subsequent returning to the initial station.

Let the robot be stationary on the first interval. Let it then make a non-stop straight-line motion with constant acceleration from robot A, through robot B, and to robot C. Then, without stopping at robot C, the robot performs a reverse motion with constant acceleration, passes through robot B without stopping, and returns to robot A. At the final robot A, the robot is again stationary. The law of speed change of the robot, in this case, is described as follows:

\[
\begin{align*}
V &= (t - 1) \cdot (1 - t), \quad \text{npu } t \in [0,2), \\
V &= 3 - t, \quad \text{npu } t \in (2,4), \\
V &= (5 - t) \cdot (5 - t), \quad \text{npu } t \in (4,6].
\end{align*}
\] (11)

Figure 6 shows the law of speed change as a function of the time of motion in the problem of the straight-line motion of the robot with the constant acceleration, as described by expression (11). Robots A, B, and C are shown in Figure 6 with the indication of the time the robot passed them and stops. From Figure 6, it follows that when moving on intervals A'B and CB, these intervals are those of the robot's acceleration. The acceleration of the robot on these intervals is 1 and \(-1 \, m/s^2\), respectively.

![Figure 6. The dependence of the speed on the time when the robot moves with one delay interval.](image)
On the contrary, when moving on intervals BC and BA', these intervals are those of the robot's deceleration. The acceleration of the robot on these intervals is \(-1\) and \(1 \text{ m/s}^2\), respectively.

The law of the robot's motion (11) envisage the following:
1. On interval AB (excluding robot B), condition (3) of Theorem 1 is satisfied;
2. On interval BC (excluding robots B and C), condition (10) of Theorem 1 holds;
3. On interval CB (excluding robots C and B), condition (6) of Theorem 2 holds; and
On interval BA (excluding robot B), condition (9) of Theorem 3 is satisfied.

5.2 Model of the robot's return motion with stopping at an intermediate robot
Let the robot perform the straight-line motion from robot A, through robot B, and to robot C. It repeats the same route on the way back. At the same time, the robot makes a stop at robot C. The robot's speed is zero at robots A and B but it has no intervals of rest at these robots. The motion outside the delay intervals and outside robots A and B occurs with the constant acceleration. The law of speed change of the robot, in this case, is as follows:

\[
\begin{align*}
V &= t, \text{ at } t \in [0,1), \\
V &= (2-t) \cdot 1(2-t), \text{ at } t \in (1,3), \\
V &= 3-t, \text{ at } t \in (3,4), \\
V &= t-5, \text{ at } t \in (4,5).
\end{align*}
\]

(12)

Figure 7 shows the law of speed change as a function of the time of motion in the problem of the straight-line motion of the robot with the constant acceleration, as described by expression (12).
Robots A, B, and C are shown in Figure 7 with the indication of the time the robot passed them or stops. From Figure 6, it follows that when the robot moves on intervals AB and C'B, it accelerates. The acceleration of the robot on these intervals is \(1\) and \(-1 \text{ m/s}^2\), respectively. On the contrary, the robot decelerates when moving on intervals BC and BA. The acceleration of the robot on these intervals is \(-1\) and \(1 \text{ m/s}^2\), respectively.

The law of the robot's motion (12) envisage the following:
1. On interval AB (excluding robots A and B), condition (6) of Theorem 2 is satisfied;
2. On interval BC' (excluding robots B and C), condition (9) of Theorem 1 holds;
3. On interval CB (excluding robots C and B), condition (6) of Theorem 2 holds; and
4. On interval BA (excluding robots A and B), condition (10) of Theorem 4 is satisfied.
6. Conclusion
This paper presents the results of a mathematical analysis applied to the problem of the straight-line motion of the robot with the constant acceleration with delay intervals. At the same time, the required conditions for the robot's uniformly accelerated motion in the acceleration and deceleration sections, which take into account delay intervals, were formulated. The main results of the paper also include the criteria for the uniformly variable acceleration and deceleration during the straight-line motion of the robot without delay intervals. It should be noted that the speed change of the robot is described by linear piecewise continuous functions of the time of its motion. The conditions and criteria obtained in the paper can be successfully applied in an entire class of problems of modern science and technology. Particularly, these theoretical results can be used in the development of traffic routes and in the operation of railway trains or other technical units that perform the straight-line uniformly accelerated motion on rail tracks. For example, metro trains, monorail trains, funiculars on a cable suspension, etc.

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