Abstract We develop a general theory of ‘quantum’ diffeomorphism groups based on the universal comeasuring quantum group $M(A)$ associated to an algebra $A$ and its various quotients. Explicit formulae are introduced for this construction, as well as dual quasitriangular and braided $R$-matrix versions. Among the examples, we construct the $q$-diffeomorphisms of the quantum plane $yx = qxy$, and recover the quantum matrices $M_q(2)$ as those respecting its braided group addition law.

1 Introduction

In [1] one finds a standard construction for a measuring coalgebra $M(A, B)$ as a universal object for coalgebras ‘acting’ from algebras $A$ to $B$. The diagonal case $M(A, A)$ is a bialgebra. The construction is the analogue of the classical automorphism group except now as the universal or ‘maximal’ object in the category of bialgebras rather than of groups. Until now, however, this measuring bialgebra construction has been little studied in the modern quantum groups literature, although see [2][3]. Probably the main reason for this is the lack of explicit formulae: being defined as a universal object it is generally hard to compute.

In the present paper we develop a much more explicit and computable version of this construction, namely a bialgebra $M(A)$ defined directly by the structure constants of an associative algebra $A$, which we call the ‘comeasuring bialgebra’. It is the arrows-reversed version of the standard construction but, by contrast, is defined by generators and relations in a familiar way. It has many applications and, in fact, a quasi-quantum group version is used in our forthcoming paper [4] with H. Albuquerque as a definition of the automorphism object of quasi-associative algebras such as the octonions. By contrast, here we study in much more detail the strictly
associative setting and its ‘geometrical’ applications. We also generalise the construction to the general braided, where $A$ is an algebra in a braided category such as that associated to a Yang-Baxter matrix. Such algebras abound in the theory of $q$-deformations as the ‘geometrical objects’ on which quantum groups act as symmetries.

The automorphism object $M(A)$ clearly plays the role of ‘diffeomorphisms’ in a noncommutative geometry setting (where an algebra $A$ is viewed as like the ‘functions’ on some space). This is our point of view, and we will use the corresponding terminology throughout the paper. We also consider briefly quotients of $M(A)$ that preserve a given differential structure on $A$, but for the most part we work with the universal differential calculus canonically associated to an algebra. Also, we use the terms ‘quantum group’ and ‘braided group’ a little loosely, without requiring the existence of an antipode or ‘group inversion’. In examples, the restricted comeasuring bialgebras $M_0(A)$ will often have an antipode or will admit one by adjoining inverse determinants etc. Within these limitations, we provide a general approach to quantum diffeomorphisms which includes $q$-diffeomorphisms of the line (i.e. some version of a $q$-Virasoro quantum group) and of the quantum plane, as well as of finite-dimensional algebras. Our approach is further justified by showing that elements of ‘quantum geometry’ may be built around these objects, along the lines of [5][6]. The notion of quantum diffeomorphism group should also be viewed as a step towards the notion of ‘quantum manifold’, which is a long-term motivation for the present work.

An outline is the following. In the preliminary Section 2 we formulate the arrows-reversed measuring bialgebra construction and obtain explicit formulae for it and its natural quotients. Apart from the forthcoming paper [4] in the quasiassociative setting, these formulae appear to be new. In Section 3 we compute several examples of comeasuring algebras, exploring their role as ‘quantum diffeomorphisms’ of polynomial and discrete spaces. Section 3.7 contains the maximal comeasuring bialgebra of the quantum plane, while Section 3.8 obtains the $2 \times 2$ quantum matrices $M_q(2)$ as the $q$-diffeomorphisms respecting addition. Note that this result is somewhat different from the characterisation of quantum matrices in [5], which is based on a certain construction of ‘endomorphisms’ for quadratic algebras and their duals. In Section 4 we turn to general $R$-matrix constructions. We give a dual quasitriangular version $M(R,A)$ of the comeasuring construction, including the applications to the line and the quantum plane. Also
using R-matrices in this section are braided group versions of our constructions.

2 General constructions

The abstract definition of the comeasuring bialgebra is obtained from [1] by reversing arrows giving a universal comeasuring algebra \( M(A, B) \) ‘coacting’ from algebras \( A \) to \( B \). Here the arrows of the coacted-upon objects \( A, B \) are not reversed, i.e. we leave these as algebras and do not make them into coalgebras as a full dualisation would do. Also, we concentrate on the diagonal case \( M(A) = M(A, A) \) since we will not have much to say about the general non-diagonal case. Their formulation is, however, strictly analogous. We work over a general field \( k \).

Thus, by definition, a comeasuring of a unital algebra \( A \) is a pair \((B, \beta)\) where \( B \) is a unital algebra and \( \beta : A \to A \otimes B \) is an algebra map to the tensor product algebra. We define \((M(A), \beta_U)\), when it exists, to be the initial object in the category of comeasurings of \( A \), i.e. a comeasuring such that for any \((B, \beta)\) there exists a unique algebra map \( \pi : M(A) \to B \) such that \( \beta = (\text{id} \otimes \pi) \beta_U \).

**Proposition 2.1** cf[1] \( M(A) \), when it exists, is a bialgebra and \( \beta_U \) is a coaction of it on \( A \) as an algebra. Any other coaction of a bialgebra on \( A \) as an algebra is a quotient of this one.

**Proof** This is elementary. We note that \( M(A) \otimes M(A) \), \((\beta_U \otimes \text{id}) \circ \beta_U \) is also a comeasuring. Hence there is an algebra map \( \Delta : M(A) \to M(A) \otimes M(A) \) and \((\beta_U \otimes \text{id}) \circ \beta_U = (\text{id} \otimes \Delta) \circ \beta_U \). It remains to show that \( \Delta \) is coassociative. For this, consider \( M(A)^{\otimes 3} \), \((\beta_U \otimes \text{id} \otimes \text{id}) \circ (\beta_U \otimes \text{id}) \circ \beta_U \) as another comeasuring. The map \( \pi : M(A) \to M(A)^{\otimes 3} \) in this case is such that \((\text{id} \otimes \pi) \circ \beta_U \) is the comeasuring map associated to \( M(A)^{\otimes 3} \). Both \((\Delta \otimes \text{id}) \circ \Delta \) and \((\text{id} \otimes \Delta) \circ \Delta \) clearly fulfill the role of \( \pi \), and since \( \pi \) is unique, these maps coincide. Finally, \( k, \beta(a) = a \otimes 1 \) is a comeasuring and \( \epsilon : M(A) \to k \) is the induced map. It is easy to see that it provides a counit. Given any other coaction of a bialgebra \( B \) on \( A \) as an algebra (i.e. \( A \) a \( B \)-comodule algebra), the fact that \( B \) comeasures gives the required map \( \pi : M(A) \to B \).

When \( A \) is nonunital, we can follow the same definitions while omitting the conditions that \( B \) is unital and that \( \beta, \pi \) respect the unit. In this case it is clear that the universal object \( M(A) \) is a not-necessarily unital bialgebra. We call it the ‘nonunital version’ of the comeasuring
bialgebra. We can, however, always formally adjoin a unit to it, extending $\Delta, \epsilon$ by $\Delta(1) = 1 \otimes 1$ and $\epsilon(1) = 1$. We denote this extension by $M_1(A)$.

Suppose now that $A$ is finite-dimensional and let $\{e_i\}$ be a basis. We let $e_ie_j = c_{ij}^k e_k$ define its structure constants.

**Proposition 2.2** $M_1(A)$ is generated by $1$ and a matrix $t = (t^i_j)$ of generators, with relations and coproduct

$$c_{ij}^k t^k_i = c_{ab}^k t^a_i t^b_j, \quad \Delta t^i_j = t^i_a \otimes t^a_j, \quad \epsilon(t^i_j) = \delta^i_j.$$

The map $\beta_U(e_i) = e_a \otimes t^a_i$ is the coaction.

**Proof** It is easy to verify that $\Delta$ extends as an algebra map and that $\beta_U$ is a coaction. For the former, the proof is

$$\Delta(c_{ab}^k t^a_i t^b_j) = c_{ab}^k t^a_{im} t^a_{jn} \otimes t^m_i t^n_j = c_{mn}^a t^a_{km} \otimes t^m_i t^n_j = c_{ij}^b t^b_i \otimes t^a_j = \Delta(c_{ij}^a t^a_k).$$

(This is also a special case of the quasi-Hopf algebra construction in [4] or of the braided case in Section 4). Now, let $(B, \beta)$ be a comeasuring and define $\pi(t^i_j) \in B$ by $\beta(e_i) = e_a \otimes \pi(t^a_i)$. Then $\beta$ is an algebra map is the assertion that $c_{ij}^a \pi(t^k_a) = c_{ab}^k \pi(t^a_i) \pi(t^b_j)$, i.e. $\pi : M(A) \rightarrow B$ defined in this way extends as an algebra map. Thus $M(A)$ as generated by the $t^i_j$ has the required universal property. When we adjoin 1, we obtain $M_1(A)$ as stated. •

Note that the explicit proof that $\Delta$ provides a bialgebra works similarly for any tensor $c_{i_1 \cdots i_m j_1 \cdots j_n}$ of rank $(m, n)$, i.e. one has an associated bialgebra with the matrix coalgebra and the relations

$$c_{a_1 \cdots a_m} t^{a_1}_{i_1} \cdots t^{a_m}_{i_m} = t^{i_1}_{a_1} \cdots t^{i_m}_{a_m} e_{j_1 \cdots j_m}.$$  \hspace{1cm} (1)

The associated bialgebra may, however, be trivial. In our case the associativity of $A$, which is the equation $c_{ij}^l c_{ab}^k = c_{ia}^l c_{jk}^a$, is needed for the interpretation as universal comeasuring object. It also tends to ensure that the ideal generated by the stated relations is not too big. Specifically,

$$c_{ab} c_{ij}^k c_{ca}^l t^{i_1}_{i} t^{j_1}_{j} t^{k_1}_{k} = c_{ij}^k c_{ab}^l c_{ca}^k = c_{jk}^l c_{ca}^k = c_{ab}^l c_{ca}^k c_{jk}^l = c_{ab}^l c_{ca}^k c_{jk}^l = c_{abc} c_{ij}^k c_{ca}^l t^{i_1}_{i} t^{j_1}_{j} t^{k_1}_{k}$$

holds automatically in $M_1(A)$.

Now suppose that $A$ is unital and $e_0 = 1$ is a basis element. We let $e_i, i = 1, \cdots, \dim(A) - 1$ be the remaining basis elements.
Proposition 2.3  The comeasuring bialgebra $M(A)$ is the quotient of $M_1(A)$ by $t^0_i = 1, t^i_0 = 0$.

Explicitly, it has the relations

$$c_{ij} a^k t^a_a = c_{ab}^k t^a_i t^b_j + b_i t^k_j + t^k_j b_j, \quad c_{ij}^0 + c_{ij} a b_a = c_{ab}^0 t^a_i t^b_j + b_i b_j$$

and coproduct, coaction

$$\Delta t^i_j = t^i_a \otimes t^a_j, \quad \Delta b_i = b_a \otimes t^a_i, \quad \epsilon(t^i_j) = \delta^i_j, \quad \epsilon(b_i) = 0$$

$$\beta_U(e_i) = 1 \otimes b_i + e_a \otimes t^a_i,$$

where $b_i = t^0_i$.

**Proof** Here $t^0_0 - 1, t^i_0$ generates a biideal of $M_1(A)$ and hence provides a quotient bialgebra. For a direct proof that $\Delta$ as stated extends to products as an algebra map, we have

$$\Delta c_{ij} a^k t^b_b = c_{ij} a^k t^b_b \otimes t^a_a = c_{ab} c^k \otimes t^a_i t^b_j + t^k_c \otimes b_i t^c_j + t^c_j b_j$$

$$= c_{cd} c^k c^d \otimes t^a_i t^b_j + b_a t^k_b \otimes t^a_i t^b_j + t^k_a b_a \otimes t^a_i t^b_j + t^k_c \otimes b_i t^c_j + t^c_j b_j$$

$$= \Delta(c_{ab} c^k a^b_i t^a_j + b_i t^k_j + t^k_j b_j)$$

using the relations and coproduct stated for $M(A)$. Similarly,

$$\Delta(c_{ab}^0 t^a_i t^b_j + b_i b_j) = c_{ab}^0 t^a_i t^b_j \otimes t^a_i t^b_j + b_a b_b \otimes t^a_i t^b_j + 1 \otimes b_i b_j + b_a \otimes t^a_i t^b_j + b_a \otimes b_i t^a_j$$

$$= 1 \otimes c_{ab}^0 t^a_i t^b_j + 1 \otimes b_i b_j + b_a \otimes c_{ij} b_a = c_{ij}^0 1 \otimes 1 + c_{ij} a^0 \otimes b_a + c_{ij} b_a \otimes t^a_b = \Delta(c_{ij}^0 + c_{ij} a b_a)$$

for the second set of relations. The coproduct has a standard form which is clearly coassociative, hence we obtain a bialgebra. It inherits the coaction as shown. Conversely, suppose that $B, \beta$ is a comeasuring and let $\pi(b_i), \pi(t^i_j) \in B$ be defined by $\beta(e_i) = 1 \otimes \pi(b_i) + e_a \otimes \pi(t^a_i)$. Similarly to the nonunital case, it follows from $\beta$ a unital algebra map that $\pi$ extends as a unital algebra map and $\beta = (\text{id} \otimes \pi) \circ \beta_U$ by construction. ●

Note that these constructions are independent of the choice of basis (beyond the choice $e_0 = 1$ in the unital case) because they are abstractly defined. In a new basis $e_i' = e_a \Lambda^a_i$, the generators of $M_1(A)$ are

$$t^i_j = \Lambda^{i-a} \Lambda^b_j.$$

In the unital case we require that the transformation is of the form $e_i' = e_0$ and $e_i' = e_0 \lambda_i + e_a \Lambda^a_i$ where the indices on $\lambda_i, \Lambda$ run from 1, \cdots, dim$(A) - 1$. Then the transformed generators of $M(A)$

\[\text{dim}(A) = 5\]
are
\[ b'_{i} = b_{a}A'^{a}_{i} + \lambda_{i} - \lambda_{c}A^{-1}_{b}A'^{c}_{a}A'_{d}, \quad t'^{i}_{j} = \Lambda^{-1}_{a}A'^{a}_{b}A'_{j}. \] (3)

If we are given slightly more structure, namely a linear splitting \( A = 1 \oplus A' \) then we can define a restricted comeasuring bialgebra \( M_{0}(A) \) as the universal object for comeasurings that respect the splitting, i.e. such that \( \beta(A') \subset A' \otimes B \) in addition to \( \beta(1) = 1 \) as before.

**Proposition 2.4** \( M_{0}(A) \) is the quotient of \( M(A) \) by \( b_{i} = 0 \), i.e. it is the associative algebra generated by 1, \( t'_{ij} \) where \( i, j = 1, \ldots \dim(A) - 1 \), and the relations
\[ c_{ij}^{a}t^{k}_{a} = c_{ab}^{k}t^{a}_{i}t^{b}_{j}, \quad c_{ij}^{0} = c_{ab}^{0}t^{a}_{i}t^{b}_{j} \]
and matrix coalgebra. Its coaction is \( \beta_{U}(e_{i}) = e_{a} \otimes t^{a}_{i} \).

**Proof** We suppose that a splitting \( A = 1 \oplus A' \) is given and \( \{e_{i}\} \) are a basis of \( A' \). We clearly have a biideal of \( M(A) \) generated by the \( b_{i} \) and hence a quotient bialgebra. It inherits the relations and coaction shown. Conversely, if \( (B, \beta) \) is a comeasuring preserving \( A' \), we define \( \pi(t'_{ij}) \in B \) by \( \beta(e_{i}) = e_{a} \otimes \pi(t^{a}_{i}) \) and \( \pi(1) = 1 \). •

This recovers the associative case of the formulae used in [4], but now as a universal comeasuring object. It depends on the basis only through the splitting of 1 defined by it, i.e. is independent of the basis of \( A' \) (one may make transformations as above with \( \lambda = 0 \)). In summary, associated to a split unital algebra \( A \), we have a sequence of bialgebras
\[ M_{1}(A) \rightarrow M(A) \rightarrow M_{0}(A) \]
where \( M(A) \) is the ‘usual’ definition dual to the construction in [4], respecting the unit but not its splitting, \( M_{1}(A) \) is the unital extension of the nonunital version, and \( M_{0}(A) \) is the version respecting the unit and in addition its splitting.

We have similar formulae for \( M_{1}(A, B), M(A, B), M_{0}(A, B) \) as algebras of ‘maps’ between algebras. For example, if \( B \) has structure constants \( d_{\alpha\beta\gamma} \) in basis \( \{f_{\alpha}\} \), then \( M_{1}(A, B) \) is generated by 1 and a rectangular matrix \( \{t'^{a}_{\alpha}\} \) of generators with relations
\[ c_{ij}^{k}t'^{a}_{\alpha}t'^{j}_{\beta} = d_{\alpha\beta\gamma}t'^{k}_{\gamma}. \]
Similarly for its quotients. Also, although we have assumed finite bases for our explicit formulae, similar formulae hold more generally with countable basis. In this case one may have to work
formally or, more precisely, with a suitable completion of tensor products. In this respect, the original measuring (rather than comeasuring) construction in [1] is better behaved. On the other hand, we have the advantage in the dual formulation of explicit formulae as algebras with generators and relations.

3 Quantum diffeomorphisms of polynomial algebras

We now compute the comeasuring bialgebras for polynomials \( \mathbb{C}[x] \) and their quotients, and justify their role as ‘diffeomorphisms’. Further justification is in Section 5 where we consider the preservation of nonuniversal differential calculi; for the moment the underlying calculus is the universal one canonically associated to the algebra. In the case of \( \mathbb{C}[x] \), the full comeasuring bialgebras \( M_1(\mathbb{C}[x]) \) and \( M(\mathbb{C}[x]) \) require formal power series and are included for motivation only, but \( M_0(\mathbb{C}[x]) \) is a completely algebraic object.

Clearly, the algebraic version of a diffeomorphism \( \mathbb{C}[x] \rightarrow \mathbb{C}[x] \) is a polynomial map \( x \mapsto a(x) = a_0 + a_1 x + a_2 x^2 + \cdots \), i.e. a polynomial. If we compose two such diffeomorphisms \( a, b \), then

\[
a \circ b(x) = \sum_j a(x^j) b_j = \sum_j \sum_{n_1, \ldots, n_j} x^{n_1 + \cdots + n_j} a_{n_1} \cdots a_{n_j} b_j = \sum_i x^i \sum_j \sum_{n_1 + \cdots + n_j = i} a_{n_1} \cdots a_{n_j} b_j
\]

where \( i, j, n_1 \in \mathbb{Z}_+ \), etc. Here \( \mathbb{Z}_+ \) denotes the natural numbers including 0. On the other hand, there is not necessarily a polynomial inverse. Therefore the ‘coordinate ring’ of the semigroup of such diffeomorphisms is the commutative bialgebra \( \text{Diff}(\mathbb{C}[x]) = \mathbb{C}[t_i | i \in \mathbb{Z}_+] \) with countable generators \( t_i \) and the coaction

\[
x \mapsto \sum_i x^i \otimes t_i
\]

as a formal power series. The corresponding coproduct and counit is

\[
\Delta t_i = \sum_j \sum_{n_1 + \cdots + n_j = i} t_{n_1} \cdots t_{n_j} \otimes t_j, \quad \epsilon(t_i) = \delta_{i,1}
\]

and also involves power series. On the other hand, if we restrict attention to diffeomorphisms which preserve the point zero in the line, we consider only polynomials \( a(x) = a_1 x + a_2 x^2 + \cdots \), i.e. without constant term. This restricted diffeomorphism bialgebra is the commutative bialgebra \( \text{Diff}_0(\mathbb{C}[x]) = \mathbb{C}[t_i | i \in \mathbb{N}] \) with the same formulae as above but with \( i, j, n_1 \in \mathbb{N} \) etc. Here \( \mathbb{N} \) denotes the natural numbers not including 0. In this case (3) has only a finite number of terms.
Thus the algebraic diffeomorphisms fixing 0 form a bialgebra with algebraic coalgebra structure and countable generators.

3.1 Comeasurings of the line

We now apply the constructions in Section 2, and recover (noncommutative) versions of the above diffeomorphism bialgebras as \( M(\mathbb{C}[x]) \) and \( M_0(\mathbb{C}[x]) \) respectively. Thus, we take \( A = \mathbb{C}[x] \) and basis \( e_0 = 1 \) and \( e_i = x^i \) for \( i \in \mathbb{N} \). Then the structure constants are

\[
  c_{ij}^k = \delta_{i+j}^k, \quad c_{ij}^0 = 0
\]

and the comeasuring bialgebra \( M(\mathbb{C}[x]) \) from Proposition 2.3 has generators 1, \( b_i \) and \( t^i_j \) with relations

\[
  b_{i+j} = b_i b_j, \quad t^k_{i+j} = \sum_{m+n=k} t^m_i t^n_j + b_i t^k_j + t^k_i b_j
\]

for all indices in \( \mathbb{N} \). These relations allow us to consider \( t_0 \equiv b_1 \) and \( t_i \equiv t^1_i \) as the generators, obtaining the other \( b_i, t^i_j \) inductively as

\[
  t^i_j = \sum_{n_1+\cdots+n_j=i} t_{n_1} \cdots t_{n_j}, \quad b_i = (t_0)^i
\]

for \( i, j \in \mathbb{N} \) and \( n_1 \cdots n_j \in \mathbb{Z}_+ \). In this case the relations \( (7) \) become empty, so \( M(\mathbb{C}[x]) = \mathbb{C}(t_i \mid i \in \mathbb{Z}_+) \), the free algebra on countable generators. Its coproduct and coaction from Proposition 2.3 therefore takes the form \( (6) \) and \( (4) \) with all indices from \( \mathbb{Z}_+ \), i.e. \( M(\mathbb{C}[x]) \) has the same form as \( \text{Diff}(\mathbb{C}[x]) \) except that the generators \( t_i \) are totally noncommuting.

The restricted comeasuring bialgebra \( M_0(\mathbb{C}[x]) \) is the quotient of this where we set \( t_0 = 0 \). Equivalently, working from Proposition 2.4, it has generators \( t^i_j \) for \( i, j \in \mathbb{N} \) with relations

\[
  \sum_{m+n=k} t^m_i t^n_j = t^k_{i+j}
\]

where all indices are in \( \mathbb{N} \). As before, this implies

\[
  t^i_j = \sum_{n_1+\cdots+n_j=i} t_{n_1} \cdots t_{n_j}
\]

where, \( t_i \equiv t^1_i \) and now the \( n_1, \cdots, n_j \in \mathbb{N} \). This implies in particular that \( t^i_j = 0 \) for \( j > i \), which ensures that the matrix coproduct is now a finite sum of terms. In this way, \( M_0(\mathbb{C}[x]) = \mathbb{C} < t_i \mid i \in \mathbb{N} > \), the free algebra, with coproduct \( (8) \) and coaction \( (4) \) where all indices are...
in \( \mathbb{N} \). Thus, \( M_0(\mathbb{C}[x]) \) has the same form as \( \text{Diff}_0(\mathbb{C}[x]) \) except that its generators are totally noncommuting.

Finally, the extended comeasuring bialgebra \( M_1(\mathbb{C}[x]) \) has no classical analogue (since usual diffeomorphisms preserve the constant function \( 1 \in \mathbb{C}[x] \)). From Proposition 2.2 it is generated by \( 1 \) and \( t^i_j \), where \( i, j \in \mathbb{Z}_+ \) and relations (3) (but now with all indices in \( \mathbb{Z}_+ \)). This time we can reduce the matrix generators to two sequences \( E_i \equiv t^i_0 \) and \( t^i_i \) for \( i \in \mathbb{Z}_+ \). The others are recovered by (8) where now \( i, n_1, \ldots, n_j \in \mathbb{Z}_+ \) etc., and \( j \in \mathbb{N} \). In this way we find that \( M_1(\mathbb{C}[x]) \) is generated by \( 1, E_i, t^i_i \) for \( i \in \mathbb{Z}_+ \), with the residual relations and coalgebra

\[
\Delta E_j = \sum_i E_i \otimes \prod_{n_1+\cdots+n_j=i} t_{n_1} \cdots t_{n_j}, \quad \Delta t^i_i = \sum_j \prod_{n_1+\cdots+n_j=i} t_{n_1} \cdots t_{n_j} \otimes t_j
\]

and \( \epsilon(E_i) = \delta^i_0, \epsilon(t^i_i) = \delta^i_1 \), where all indices are from \( \mathbb{Z}_+ \). We note that the constrained sums in these expressions are the usual convolution product \(*\) on sequences, and in that notation the relations and coproduct of \( M_1(\mathbb{C}[x]) \) are

\[
E * E = E, \quad E * t = t = t * E, \quad \Delta E_j = \sum_i E_i \otimes (t * \cdots * t)_i, \quad \Delta t^i_i = \sum_j (t * \cdots * t)_i \otimes t_j
\]

where the \(*\)-products are \( j\)-fold. Finally, the coaction is

\[
1 \mapsto \sum_i x^i \otimes E_i, \quad x \mapsto \sum_i x^i \otimes t^i_i.
\]

The \( M(\mathbb{C}[x]) \) above is quotients of this non-unit-preserving diffeomorphism group by setting \( E_i = \delta^i_0 \).

### 3.2 Comeasurings of Grassmann and anyonic variables

Here we consider ‘diffeomorphisms’ of the finite-dimensional quotients \( A = \mathbb{C}[x]/x^N = 0 \). The case \( N = 2 \) is where \( x \) is a fermionic or ‘Grassmann’ variable. The case \( N > 2 \) is an anyonic variable. The addition law and geometry of this ‘anyonic line’ can be found in [8].

We take basis \( e_i = x^i \) where \( i = 1, \cdots, N-1 \) and \( e_0 = 1 \). Then \( c_{ij}^k \) have the same form (3) but restricted in range to \( 0 < i, j, k < N \). In this case \( M = \mathbb{C} < t_i | i = 0, \cdots, N-1 > \) is the free algebra and has coproduct (3) with indices including 0. The restricted comeasuring bialgebra is the free algebra \( M_0 = \mathbb{C} < t_i | i = 1, \cdots, N-1 > \) with coproduct likewise from (3) but with all
indices now excluding 0. On the other hand the comeasuring bialgebra is now finitely generated and does not need any formal powerseries. (One can then obtain $M(\mathbb{C}[x])$ in the projective limit $N \to \infty$.)

For the fermionic case $x^2 = 0$, the comeasuring bialgebra is $M = \mathbb{C} < b, t >$ with

$$\Delta b = 1 \otimes b + b \otimes t, \quad \Delta t = t \otimes t, \quad \epsilon(b) = 0, \quad \epsilon(t) = 1,$$

i.e. the matrix of generators has the form $\begin{pmatrix} 1 & b \\ 0 & t \end{pmatrix}$. The restricted comeasuring bialgebra is just $M_0 = \mathbb{C}[t]$ with $\Delta t = t \otimes t$ and $\epsilon(t) = 1$. The extended comeasuring bialgebra $M_1$ is generated by 1 and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with the relations

$$a^2 = a, \quad ac + ca = c, \quad ab = ba, \quad ad + cb = d = da + bc$$

and the usual matrix coalgebra.

For an anyonic variable with $x^3 = 0$, the comeasuring bialgebra is $M = \mathbb{C} < b, t, s >$ with

$$\Delta b = 1 \otimes b + b \otimes t + b^2 \otimes s, \quad \Delta t = t \otimes t + (bt + tb) \otimes s$$

$$\Delta s = s \otimes t + (t^2 + sb + bs) \otimes s, \quad \epsilon(b) = \epsilon(s) = 0, \quad \epsilon(t) = 1,$$

i.e. the matrix of generators takes the form $\begin{pmatrix} 1 & b & b^2 \\ 0 & t & bt + tb \\ 0 & s & t^2 + sb + bs \end{pmatrix}$. The restricted comeasuring bialgebra is $M_0 = \mathbb{C} < t, s >$ with

$$\Delta t = t \otimes t, \quad \Delta s = s \otimes t + t^2 \otimes s, \quad \epsilon(t) = 1, \quad \epsilon(s) = 0,$$

i.e. the matrix of generators take the form $\begin{pmatrix} t & 0 \\ s & t^2 \end{pmatrix}$. 

### 3.3 Comeasurings of roots of unity

For another class of discrete spaces (which we consider as discrete models of a circle), we consider the algebra $A = \mathbb{C}Z_N = \mathbb{C}[x]/x^N = 1$. We take basis $e_0 = 1$ and $e_i = x^i$ as before, with

$$c_{ij}^k = \delta_{i+j}^k, \quad c_{ij}^0 = \delta_{i-j},$$

where $0 < i, j, k < N$ are added modulo $N$. One may put these into Propositions 2.2-2.4 to obtain the comeasuring bialgebras.

Actually, it is easier in this class of examples to compute $M_1$ first and then quotient it. This has generators $1, t^i j$ with relations $[\square]$, where $i, j \in \mathbb{Z}_N$ including zero. The convolution formula
now applies for all \( i, j \) since \( \mathbb{Z}_N \) is a group, so generators may all be written in terms of \( t_i \equiv t^i_1 \) for \( i \in \mathbb{Z}_n \). In particular,

\[
E_i \equiv t^i_0 = \sum_{n_1 + \cdots + n_N} t_{n_1} \cdots t_{n_N},
\]

where all indices are in \( \mathbb{Z}_N \). Thus, the extended comeasuring bialgebra \( M_1 \) is generated by \( 1, t_i \) for \( i \in \mathbb{Z}_N \) with the relations

\[
\sum_{n_1 + \cdots + n_N = i} t_{n_1} \cdots t_{n_N + 1} = t_i, \quad \forall i \in \mathbb{Z}_N
\]

and coproduct (3) with all indices in \( \mathbb{Z}_N \).

The comeasuring algebra \( M \) is given by setting \( E_i = \delta^i_0 \). Hence it is generated by \( 1, t_i \) for \( i \in \mathbb{Z}_N \) modulo the relations

\[
\sum_{n_1 + \cdots + n_N = i} t_{n_1} \cdots t_{n_N} = \delta^i_0, \quad \forall i \in \mathbb{Z}_N
\]

and the same form of coproduct. Here

\[
b_i \equiv t^0_i = \sum_{n_1 + \cdots + n_i = 0} t_{n_1} \cdots t_{n_i} \quad \text{for } i, j = 1, \cdots, N - 1 \text{ and } n_1 \in \mathbb{Z}_N, \text{ etc., provide the generators in the form of Proposition 2.3.}
\]

Finally, the restricted comeasuring bialgebra \( M_0 \) is given by setting \( b_i = 0 \). Hence \( M_0 \) is generated by \( 1, t_i \) where \( i = 1, \cdots, N - 1 \) modulo the relations

\[
\sum_{n_1 + \cdots + n_N = 0} t_{n_1} \cdots t_{n_N} = 1, \quad \sum_{n_1 + \cdots + n_N = i} t_{n_1} \cdots t_{n_N} = 0, \quad \sum_{n_1 + \cdots + n_i = 0} t_{n_1} \cdots t_{n_i} = 0
\]

for \( i, n_1, \text{ etc., are in the range } 1, \cdots, N - 1 \text{ and addition modulo } \mathbb{Z}_N. \) The coproduct is (3) with all indices similarly in this range.

For \( x^2 = 1 \), the extended comeasuring bialgebra \( M_1 \) is generated by \( 1, b, t \) modulo the relations and coalgebra

\[
(b \pm t)^3 = (b \pm t), \quad \Delta t = t \otimes t + (bt + tb) \otimes b, \quad \Delta b = (b^2 + t^2) \otimes b + b \otimes t, \quad \epsilon(t) = 1, \quad \epsilon(b) = 0.
\]

The matrix of generators has the form \( \begin{pmatrix} b^2 + t^2 & b \\ bt + tb & t \end{pmatrix} \). The comeasuring bialgebra \( M \) is obtained by setting \( b^2 + t^2 = 1, \ bt + tb = 0 \) and is therefore generated by \( 1, b, t \) modulo the stronger relations and coalgebra

\[
(b \pm t)^2 = 1, \quad \Delta t = t \otimes t, \quad \Delta b = 1 \otimes b + b \otimes t, \quad \epsilon(t) = 1, \quad \epsilon(b) = 0.
\]
Although not a Hopf algebra, there is a formal antipode
\[ St = \frac{t}{1 - b^2}, \quad Sb = -\frac{bt}{1 - b^2} \]
as a powerseries in \( b \). The restricted comeasuring bialgebra is just \( M_0 = \mathbb{C}[t]/t^2 = 1 \) and \( \Delta t = t \otimes t, \epsilon(t) = 1 \), and is a Hopf algebra with \( St = t \).

For \( x^3 = 1 \), the comeasuring bialgebra \( M \) is generated by \( 1, b, t, s \) modulo the relations
\[ ba + t\gamma + s\beta = 1, \quad ta + s\gamma + b\beta = 0, \quad s\alpha + b\gamma + t\beta = 0 \]
where
\[ \alpha \equiv b^2 + ts + st, \quad \beta \equiv s^2 + bt + tb, \quad \gamma \equiv t^2 + sb + bs. \]
The coalgebra is
\[ \Delta b = 1 \otimes b + b \otimes t + \alpha \otimes s, \quad \Delta t = t \otimes t + \beta \otimes s, \quad \Delta s = s \otimes t + \gamma \otimes s, \quad \epsilon(b) = \epsilon(s) = 0, \quad \epsilon(t) = 1, \]
i.e., the matrix of generators has the form
\[ \begin{pmatrix} 1 & b & \alpha \\ 0 & t & \beta \\ 0 & s & \gamma \end{pmatrix}. \]
The extended \( M_1 \) is similar with weaker relations. Finally, the restricted \( M_0 \) is generated by \( 1, t, s \) with the relations and coproduct
\[ ts + st = 0, \quad t^3 + s^3 = 1, \quad t^2 s = ts^2 = 0, \quad \Delta t = t \otimes t + s^2 \otimes s, \quad \Delta s = s \otimes t + t^2 \otimes s \]
and \( \epsilon(t) = 1, \epsilon(s) = 0 \), i.e. the matrix of generators has the form \( \begin{pmatrix} t & s^2 \\ s & t^2 \end{pmatrix} \). The second relation is equivalent to \( (t + s)^3 = 1 \) given the others.

Similarly, one may compute the measuring bialgebra for \( k_\lambda = k[\lambda]/m(\lambda) = 0 \) for general fields and general polynomials \( m \). When \( m \) is monic and irreducible, \( k_\lambda \) is a field extension of \( k \) and \( M(k_\lambda) \) should be viewed as the ‘quantum Galois group’ for the field extension. Also, we are not limited to commutative algebras and fields. The comeasuring bialgebras for the complex numbers and the quaternions, as algebras of dimension 2, 4 respectively over \( \mathbb{R} \) are computed in [4] as preludes to the octonion case. In the quaternion case one has a noncommutative and noncocommutative bialgebra projecting onto the coordinate ring of \( SO_3 \). There are of course plenty of other interesting examples according to ones favourite algebra.
3.4 Comeasurings of finite sets

For completeness, we conclude our collection of general classes of examples with the case $A = \mathbb{C}(\Sigma)$ where $\Sigma$ is a finite set. In this case we compute only $M_1(\mathbb{C}(\Sigma))$ and $M(\mathbb{C}(\Sigma))$ since there is no particularly natural splitting of the identity without more structure. We take basis 
\[ \{ e_i = \delta_i, \ i \in \Sigma \}, \] the delta-functions at element in the set. The structure constants are

\[ c_{ij}^k = \delta_{i,j} \delta_j^k \]  

(11)

and hence we find from Section 2 that $M_1(\mathbb{C}(\Sigma))$ is generated by 1 and $\tau^i_j$ (say) with relations

\[ \tau^k_i \tau^k_j = \delta_{i,j} \tau^k_i \]  

(12)

(no sum) and the matrix coalgebra structure. This means for each row $k$ the matrix of generators forms an orthogonal family of projectors, i.e. a copy of $\mathbb{C}(\Sigma)$, while there are no relations between the different rows. We note that such algebras have recently been considered in [3] in an interesting $C^*$-algebra setting, again as some kind of universal automorphism objects for finite sets. In our case we obtain them as an elementary example of the general (but algebraic) construction in Proposition 2.2 dual to [1]. We also have a unit $1 = \sum_i \delta_i$ and the corresponding $M(\mathbb{C}(\Sigma))$ can be computed as follows. We take a different basis $e_0 = 1$ and $\{ e_i | i \in \Sigma - * \}$ where * is a basepoint (a fixed element of $\Sigma$). Then $M(\mathbb{C}(\Sigma))$ is generated by $t^i_j, b_i$ with $i,j \in \Sigma - *$ and relations

\[ t^k_i t^k_j + b_i t^k_j + t^k_i b_j = \delta_{ij} t^k_j, \quad b_i b_j = \delta_{ij} b_j \]  

(13)

and the coalgebra in Proposition 2.3. We see that $M(\mathbb{C}(\Sigma)) \supset \mathbb{C}(\Sigma)$ embedded as $\{1, b_i\}$. Note that if we want to describe $M(\mathbb{C}(\Sigma))$ in our original basepoint-free delta-function basis, it consists of the quotient of (12) by the relations

\[ \sum_j \tau^i_j = 1 \]  

(14)

where $i,j \in \Sigma$. This is equivalent to (13) via

\[ \tau^i_j = t^i_j + b_j, \quad \tau^0_0 = 1 - \sum_i b_i, \quad \tau^0_i = b_i, \quad \tau^i_0 = 1 - \sum_j t^i_j - \sum_j b_j \]  

(15)

for $i,j \in \Sigma - *$, which is the transformation induced by the change between the two bases.
We note that the algebras $A = \mathbb{C}[x]/x^N = 1$ are isomorphic by Fourier transform to $\mathbb{C}(\mathbb{Z}_N)$ so their comeasuring bialgebras are isomorphic to those for a finite set with $N$ elements. Explicitly,
\[
\tau_{ij} = N^{-1} \sum_{m,n=0}^{N-1} e^{\frac{2\pi i}{N}(mn)} t^m n
\]
is the matrix of projectors (12) generating $M_1(\mathbb{C}(\Sigma))$ in terms of the matrix generators of $M_1$ for $\mathbb{C}[x]/x^N = 1$ in the preceding subsection. For example, for $N = 2$ the matrix of projectors is
\[
\tau = \frac{1}{2} \begin{pmatrix} (b + t)^2 + (b + t) & (b + t)^2 - (b + t) \\ (b - t)^2 + (b - t) & (b - t)^2 - (b - t) \end{pmatrix}.
\]
That the elements of each row are orthogonal projectors is equivalent to the relations $(b \pm t)^3 = (b \pm t)$. For the quotient $M(\mathbb{C}(\Sigma))$ we obtain
\[
\tau = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}, \quad \tau_{ij} = \frac{1}{2} (1 \pm (b \pm t))
\]
i.e. the algebra generated by two projectors $g_\pm$ and the coalgebra
\[
\Delta g_\pm = g_\pm \otimes g_\pm + (1 - g_\pm) \otimes (1 - g_\mp), \quad \epsilon(g_\pm) = 1
\]
as isomorphic to $M$ for $\mathbb{C}[x]/x^2 = 1$ in the preceding subsection. This is the basepoint-free description via (12) and (14). On the other hand, if we compute $M(\mathbb{C}(\Sigma))$ from Proposition 2.3 according to (13) (which is in a different basis) we have an equivalent matrix of generators
\[
\begin{pmatrix} \Sigma b \\ \Sigma t \end{pmatrix}, \quad \Sigma b = 1 - g_+ = \frac{1}{2} (1 - (b + t)), \quad \Sigma t = g_- - 1 + g_+ = t
\]
and obeying
\[
\Sigma b^2 = \Sigma b, \quad (\Sigma t + \Sigma b)^2 = \Sigma t + \Sigma b, \quad \Delta \Sigma b = \Sigma b \otimes \Sigma b + \Sigma b \otimes \Sigma t, \quad \Delta \Sigma t = \Sigma t \otimes \Sigma t
\]
and $\epsilon(\Sigma t) = 1$, $\epsilon(\Sigma b) = 0$.

3.5 Elements of quantum geometry

In the remaining two subsections we consider briefly some steps towards a ‘differential geometry’ based on these ‘quantum diffeomorphisms’. We essentially use the quantum group approach to noncommutative geometry in [5] and especially the recent paper [6]. We recall that in the classical situation, if $\Sigma$ is a manifold one has, roughly speaking, an identification $\text{Diff}(\Sigma)/\text{Diff}_*(\Sigma) \cong \Sigma$ providing a principal bundle over $\Sigma$ with structure group $\text{Diff}_*(\Sigma)$. The canonical projection
sends a diffeomorphism $\sigma$ to $\sigma(*) \in \Sigma$. Moreover, this principal bundle has, again formally, a canonical form $\theta$ making this a frame resolution of $\Sigma$ in the language of [6]. The usual affine frame bundle and linear frame bundle are subbundles. Here $\theta$ is the projection to $T^*_\Sigma = \mathbb{R}^n$ of the Maurer-Cartan form on $\text{Diff}(\Sigma)$, where the Lie algebra of vector fields on $\Sigma$ modulo those that vanish at $*$ is identified with the value at $*$. Using the canonical form $\theta$, we have, essentially a correspondence between gauge-fields on this principal bundle and covariant derivatives on the cotangent bundle of $\Sigma$. In this way, differential geometry on the manifold may be developed strictly as $\text{Diff}_*(\Sigma)$ gauge theory. Of course, because these are large infinite dimensional groups, one needs topological considerations to make these ideas fully precise in the manifold setting.

One the other hand, in our algebraic setting we can try to keep everything algebraic, e.g. if the role of our manifold is played by a finite-dimensional (possibly noncommutative) algebra. We have seen above that for polynomials, $M(\mathbb{C}[x])$ plays the role of ‘diffeomorphisms’ and $M_0(\mathbb{C}[x])$ plays the role of diffeomorphisms fixing the origin. Motivated by these ideas we make the parallel choice for our finite dimensional algebras.

To recall the set-up, note that arrows are reversed in our co-ordinate ring formulation. Thus, a homogeneous quantum principal bundle arises from a Hopf algebra map $\pi : M \to M_0$ between Hopf algebras $M, M_0$ (say) such that the induced coaction $\Delta_L = (\pi \otimes \text{id})\Delta : M \to M_0 \otimes M$ has fixed point subalgebra $A = M^{M_0} \equiv \{h \in M | \Delta_L h = 1 \otimes h\}$ (it plays the role of the coordinates of the base manifold), and obeying a certain nondegeneracy condition (the ‘Hopf-Galois’ condition) that the map

$$\chi : M \otimes M \to M_0 \otimes M, \quad h \otimes g \mapsto \Delta_L(h)g$$

is invertible. There is a canonical form $\theta : V \to \Omega^1 M$ given by

$$V = \ker \epsilon|_A, \quad \theta = (\text{id} \otimes S)\Delta$$

in terms of the Hopf algebra structure of $M$. Here $\Omega^1 M \subset M \otimes M$ is the kernel of the product map (the universal differential calculus). A connection on the quantum bundle is a map $\omega : M_0 \to \Omega^1 M$ which is left $\text{Ad}$-covariant and obeys $\chi \circ \omega = \text{id} \otimes 1$ on $\ker \epsilon \subset M_0$ and $\omega(1) = 0$ as in [5], and in the presence of $\theta$ it defines a covariant derivative $\nabla : \Omega^1 A \to \Omega^1 A \otimes A \Omega^1 A \cong \Omega^2 A \subset A \otimes A \otimes A$ by

$$\nabla da = da \otimes 1 - a_{(1)} \otimes S a_{(2)} \omega(a_{(3)}) a_{(4)} - a_{(1)} \otimes S a_{(2)} \otimes a_{(3)} + 1 \otimes 1 \otimes a$$
where $d : A \to \Omega^1 A$ given by $da = a \otimes 1 - 1 \otimes a$ is the exterior derivative and $\Delta a = a_{(1)} \otimes a_{(2)}$ etc., is a notation for the coproduct, see [8].

As a first step towards applying this formalism, we consider the case where $\Sigma$ is a 2-point set. We take $M(C(\Sigma))$ in its basepoint form (13), namely generated by projectors $p = \Sigma b + \Sigma t$ and $q = \Sigma b$ with no further relations and the coalgebra

$$\Delta p = p \otimes p + (1 - p) \otimes q, \quad \Delta q = (1 - q) \otimes q + q \otimes p, \quad \epsilon(p) = 1, \quad \epsilon(q) = 1.$$  

Similarly, $M_0(C(\Sigma)) = \mathbb{C}[\bar{p}] / \bar{p}^2 = \bar{p}$ with $\Delta \bar{p} = \bar{p} \otimes \bar{p}$ and $\epsilon \bar{p} = 1$. The projection is $\pi(p) = \bar{p}$ and $\pi(q) = 0$. The induced coaction is

$$\Delta_L q = 1 \otimes q, \quad \Delta_L p = \bar{p} \otimes p + (1 - \bar{p}) \otimes q.$$  

Since $M$ is spanned by 1 and alternating words $h = pqpq \cdots$ or $h = qpqp \cdots$, one can show that

$$\Delta_L 1 = 1 \otimes 1, \quad \Delta_L h = \bar{p} \otimes h + (1 - \bar{p}) \otimes q$$  

for either kind of non-trivial $h$. Hence the fixed point subalgebra is the subalgebra spanned by 1, $q$, i.e.

$$M(C(\Sigma))^{M_0(C(\Sigma))} = C(\Sigma).$$  

One can also reach a similar conclusion

$$M(C[x])^{M_0(C[x])} = C[x]$$  

where $M(C[x]) = \mathbb{C} < t_i | i \in \mathbb{Z}_+ >$ and $M_0(C[x]) = \mathbb{C} < \bar{t}_i | i \in \mathbb{N} >$ as explained in Section 3.2. The projection is $\pi(t_i) = \bar{t}_i$ for $i > 0$ and $\pi(t_0) = 0$. The induced coaction is therefore

$$\Delta_L t_0 = 1 \otimes t_0, \quad \Delta_L t_i = \sum_{j=1}^{n_i} \sum_{n_1 + \cdots + n_j = i} \bar{t}_{n_1} \cdots \bar{t}_{n_j} \otimes t_j$$  

where all indices shown as in $\mathbb{N}$. Note that although the coproduct of $M(C[x])$ involves formal powerseries, its product structure and the induced coaction are algebraic. Since $M(C[x])$ and $M_0(C[x])$ are free algebras, the fixed point subalgebra is $\mathbb{C}[t_0]$.

To go further, one needs to have versions of $M, M_0$ which are actually Hopf algebras and not merely bialgebras. This can typically be done by localizing, i.e. by adjoining suitable inverses or powerseries. For example, we may take the same $M$ as for the two-point set but in the form
of the comeasuring bialgebra for \( \mathbb{C}[x]/x^2 = 1 \) as in Section 3.4 (generated by \( 1, b, t \)). For the splitting associated with this basis we have \( M_0 = \mathbb{C}[\bar{t}] / \bar{t}^2 = 1 \) which is actually a Hopf algebra. The map \( \pi \) is \( \pi(b) = 0 \) and \( \pi(t) = \bar{t} \). The induced coaction is

\[
\Delta_L b = 1 \otimes b, \quad \Delta_L t = \bar{t} \otimes t
\]

Since \( t^2 = 1 - b^2 \) in \( M \), every element of that can be written in the form \( f(b) + tg(b) \), of which the fixed elements are those with \( g = 0 \). Therefore the fixed subalgebra \( A \), the base of the bundle, is

\[
M^{M_0} = \mathbb{C}[b]
\]

Note that the base is no longer the original algebra of which we took the ‘diffeomorphisms’. (This is attributable to the nontrivial \( c_{ij}^0 \) in Proposition 2.3 for this basis). On the other hand, if we allow formal powerseries in \( b^2 \) then \( M \) has an antipode and, moreover, the non-degeneracy (Hopf-Galois) condition holds so that we have, at least formally, a homogeneous quantum principal bundle. The canonical form \( \theta \) is also formal, for the same reason. Here

\[
\ker \epsilon = \{ f(b) + tg(b) | f(0) + g(0) = 0 \}
\]

and hence \( V \) consists of polynomials in \( b \) vanishing at 0. Then

\[
\theta(b) = b \otimes \frac{t}{1 - b^2} - 1 \otimes \frac{bt}{1 - b^2}.
\]

On the other hand, the map \( \Phi(\bar{t}) = t \) is a coalgebra map splitting of \( \pi \) and makes this localized bundle trivial. It should therefore be viewed only as a local ‘patch’ of a more non trivial bundle without such localization. (This requires, however, a suitable extension of the theory in [5][6].) At least in this patch, a connection \( \omega \) is induced as in [5] by a ‘gauge field’ \( \alpha : M_0 \to \Omega^1 \mathbb{C}[b] \) such that \( \alpha(1) = 0 \), which in our case means a single element \( \alpha = \alpha(\bar{t}) \in \Omega^1 \mathbb{C}[b] \). This induces the covariant derivative

\[
\nabla : \Omega^1 \mathbb{C}[b] \to \Omega^1 \mathbb{C}[b] \otimes_{\mathbb{C}[b]} \Omega^1 \mathbb{C}[b], \quad \nabla db = (b \otimes 1 - 1 \otimes b) \alpha.
\]

### 3.6 Preserving nonuniversal differential calculi

Here we discuss briefly the further restrictions imposed by non-universal differential calculi. We recall that for any unital algebra \( A \) the universal calculus is \( \Omega^1 A \) is the kernel of the product map \( A \otimes A \to A \) as an \( A \)-bimodule, while a general differential calculus is a quotient \( \Omega^1(A) = \Omega^1 A / N \)
for some subbimodule \( N \). The differential is the universal one \( da = a \otimes 1 - 1 \otimes a \) (as above) projected down to \( \Omega^1(A) \) in the nonuniversal case. It is natural to restrict the notion of unital comeasurings to ones that preserve \( N \) in the sense

\[ \beta(N) \subseteq N \otimes B, \] (20)

where \( \beta \) is extended to the tensor square \( A \otimes A \) in the usual way, restricted to \( \Omega^1 A \). The universal object in this case is a quotient of \( M(A) \). It is easy to see that it remains a bialgebra, which we denote \( M(A, \Omega^1(A)) \), the comeasuring bialgebra with nonuniversal calculus.

Thus, when \( A = \mathbb{C}(\Sigma) \), the subbimodules \( N \) are classified by subsets of \( \Sigma \times \Sigma - \text{diag} \). In (11) the complement of this subset is denoted \( E \) and we write \( i - j \) when \( (i, j) \in E \) and \( i \# j \) when \( (i, j) \) is in the complement in \( \Sigma \times \Sigma - \text{diag} \). Thus, \( N = \{ \delta_i \otimes \delta_j | i \# j \} \). To describe the comeasuring bialgebra \( M(\mathbb{C}(\Sigma), \Omega^1(\mathbb{C}(\Sigma))) \) in this case we use the base-point free version of \( M(\mathbb{C}(\Sigma)) \) defined by (12) and (14), and quotient further by the relations

\[ \tau^i_j \tau^k_l = 0 \quad \forall i - k, \quad j \# l \] (21)

as the requirement that \( N \) is preserved by \( \delta_j \mapsto \delta_i \otimes \tau^i_j \). Note that in \( M(\mathbb{C}(\Sigma)) \) we have, for \( i - k \) and \( j \# l \),

\[ \Delta(\tau^i_j \tau^k_l) = \sum_{a-b} \tau^i_a \tau^k_b \otimes \tau^a_j \tau^b_l + \sum_{a \# b} \tau^i_a \tau^k_b \otimes \tau^a_j \tau^b_l. \]

Hence it is clear that the quotient by the relations (21) remains a bialgebra. The case \( a = b \) does not contribute here in view of \( j \neq l \) and (12).

When \( \Sigma \) is the two-point set, there is (up to equivalence) only one nonuniversal nontrivial differential calculus, namely \( E = \{(1, 2)\} \). Then \( M(\mathbb{C}(\Sigma), \Omega^1(\mathbb{C}(\Sigma))) \) is the quotient of the bialgebra (18), (13) by the additional relation

\[ (1 - g_+)(1 - g_-) = 0. \]

Equivalently, we quotient the comeasuring bialgebra \( M \) for \( \mathbb{C}[x]/x^2 = 1 \) in Section 3.4 by

\[ t^2 = t(1 + b). \]

When \( A = \mathbb{C}[x] \) the natural differential calculi are those bicovariant under the coaddition structure, and are labeled by a single parameter \( \lambda \in \mathbb{C} \) (and over a general field, the coirreducible
calculi correspond to field extensions). The standard commutative differential calculus is the one where $N$ is the submodule generated by $x dx - (dx)x$. This contains in particular all elements of the form $x^i dx^j - (dx^j)x^i$ for $i, j \in \mathbb{Z}_+$. Under the coaction, we have

$$x dx - (dx)x \mapsto \sum_{i,j} (x^i dx^j - (dx^j)x^i) \otimes t_i t_j + \sum (dx^j)x^i \otimes (t_i t_j - t_j t_i).$$

Since the $(dx^j)x^i$ are linearly independent elements of $\Omega^1 \mathbb{C}[x]$ for $j > 0$, the differentiable comeasurings $M(\mathbb{C}[x], \Omega^1(\mathbb{C}[x]))$ for the standard commutative differential calculi are precisely the quotient of $M(\mathbb{C}[x]) = \mathbb{C} < t_i | i \in \mathbb{Z}_+ >$ by the relations of commutativity of the generators i.e. precisely the usual $\text{Diff}(\mathbb{C}[x])$. Differential calculi in between the universal one and the commutative one therefore lead to quotients in between the free comeasuring bialgebra $M(\mathbb{C}[x])$ and the classical commutative one.

### 3.7 Diffeomorphisms of the quantum-braided plane

Until now, we have only considered diffeomorphism of the line and its quotients. We now move on to the plane, which is now nontrivial enough to have a noncommutative $q$-deformed version, the quantum-braided plane $\mathbb{C}_q^2$. This has generators $1, x, y$ with relations $yx = qxy$, where $q \neq 0$ is a parameter. It is infinite-dimensional but, as in Section 3.1, we find that $M_0(\mathbb{C}_q^2)$ is algebraic without the need for formal powerseries.

The computation of the comeasuring bialgebra $M(\mathbb{C}[x, y])$ for the classical plane follows just the same steps as in Section 3.1. With the basis $\{e_{m,n} = x^m y^n | m, n \in \mathbb{Z}_+\}$ one has the structure constants

$$c_{(i,j)(k,l)}(m,n) = \delta_{m+k}^{n+l},$$

and $M_1(\mathbb{C}[x, y])$ is generated by $1, t^{(i,j)}(k,l)$ with relations

$$\sum_{(a,b)+(c,d)=(m,n)} t^{(a,b)}(i,j) t^{(c,d)}(k,l) = t^{(m,n)}(i+j)(k+l) \tag{22}$$

and the matrix coalgebra. Similarly to Section 3.1, we find $M(\mathbb{C}[x, y])$ is generated by $1, s_{(i,j)}, t_{(i,j)}$ where $i, j \in \mathbb{Z}_+$ modulo the relations

$$\sum_{(a,b)+(c,d)=(i,j)} s_{(a,b)} t^{(c,d)} = \sum_{(a,b)+(c,d)=(i,j)} t_{(a,b)} s_{(c,d)}, \forall i, j \in \mathbb{Z}_+. \tag{23}$$

Here

$$s_{(i,j)} \equiv t^{(i,j)}(1,0), \quad t_{(i,j)} \equiv t^{(i,j)}(0,1) \tag{24}$$
generate the others by repeated ‘convolution’. Thus,

\[ t^{(i,j)}_{(k,0)} = \sum_{(a_1, b_1) + \cdots + (a_k, b_k) = (i,j)} s_{(a_1, b_1)} \cdots s_{(a_k, b_k)}, \quad t^{(i,j)}_{(0,k)} = \sum_{(a_1, b_1) + \cdots + (a_k, b_k) = (i,j)} t_{(a_1, b_1)} \cdots t_{(a_k, b_k)} \]

for \( k, l \geq 1 \). These follow from [22], while [23] is the residual content of [24] after making these substitutions. If we consider the \( s_{(i,j)}, t_{(i,j)} \) as sequences on \( \mathbb{Z}_+ \times \mathbb{Z}_+ \) with convolution product

\[ s \ast t^{(i,j)} = \sum_{(a,b) + (c,d) = (i,j)} s_{(a,b)} t^{(c,d)} \]

we can write the residual relations of \( M(\mathbb{C}[x,y]) \) compactly as \( s \ast t = t \ast s \). The coaction is as in (26). The higher generators and hence the coproduct (25) are given by

\[ \Delta t^{(i,j)}_{(k,l)} = \sum_{(a,b) + (c,d) = (i,j)} t^{(a,b)}_{(k,0)} t^{(c,d)}_{(0,l)} \]

The coaction on \( \mathbb{C}[x,y] \) is

\[ x \mapsto \sum_{(i,j)} x^i y^j \otimes s^{(i,j)}, \quad y \mapsto \sum_{(i,j)} x^i y^j \otimes t^{(i,j)}. \] (26)

The quotient \( M_0(\mathbb{C}[x,y]) \) has the same form but with the indices in (24), (25) and (26) excluding \((0,0)\). In this case \( t^{(i,j)}_{(k,l)} = 0 \) unless \( i + j \geq k + l \), so that the coproduct in this case is a finite sum. The geometric meaning of \( M_0(\mathbb{C}[x,y]) \) is the diffeomorphisms that fix the origin.

For the quantum-braided plane, we again have a basis \( \{ e_{m,n} = x^m y^n \mid m, n \in \mathbb{Z}_+ \} \), but now with the q-deformed structure constants and consequent relations

\[ c^{(m,n)}_{(i,j)(k,l)} = \delta^m_{i+k} \delta^n_{j+l} q^{j(k)}, \quad \sum_{(a,b) + (c,d) = (m,n)} q^{bc} t^{(i,j)}_{(a,b)} t^{(c,d)}_{(l,k)} = q^{ik} t^{(m,n)}_{(i,j)(k,l)}. \] (27)

**Proposition 3.1** The comeasuring bialgebra \( M(\mathbb{C}_q^2) \) of the quantum-braided plane is generated by 1, \( s^{(i,j)}, t^{(i,j)} \) for \( i, j \in \mathbb{Z}_+ \), modulo the relations

\[ q \sum_{(a,b) + (c,d) = (i,j)} q^{bc} s_{(a,b)} t^{(c,d)} = \sum_{(a,b) + (c,d) = (i,j)} q^{bc} t_{(a,b)} s^{(c,d)}, \quad \forall i, j \in \mathbb{Z}_+. \]

The coaction is as in [26]. The higher generators and hence the coproduct [22] are given by

\[ t^{(i,j)}_{(k,0)} = \sum_{(a_1, b_1) + \cdots + (a_k, b_k) = (i,j)} s_{(a_1, b_1)} \cdots s_{(a_k, b_k)} q^{s_{(b_1+\cdots+b_{s-1})a_s}}, \]

\[ t^{(i,j)}_{(0,k)} = \sum_{(a_1, b_1) + \cdots + (a_k, b_k) = (i,j)} t_{(a_1, b_1)} \cdots t_{(a_k, b_k)} q^{s_{(b_1+\cdots+b_{s-1})a_s}} \]
\[ t^{(i,j)}_{(k,l)} = \sum_{(a,b)+(c,d)=(i,j)} q^{bc} t^{(a,b)}_{(k,0)} t^{(c,d)}_{(0,l)} \]

for \( k \geq 1 \).

**Proof** The relations (27) allow one to define the general \( t^{(i,j)}_{(k,l)} \) as stated in terms of the \( s_{(i,j)}, t_{(i,j)} \). Putting these back into (27) leaves the residual relations between \( s, t \) as shown. These are obtained from \( t^{(i,j)}_{(1,0)+(0,1)} = t^{(i,j)}_{(1,1)} = t^{(i,j)}_{(0,1)+(1,0)} \) computed from (27), i.e. they reflect commutativity of the addition law on \( \mathbb{Z}_+ \times \mathbb{Z}_+ \). To see that these are all the relations, it is useful to define the \( q \)-deformed convolution product

\[ s_{(i,j)} \ast_q t_{(i,j)} = \sum_{(a,b)+(c,d)=(i,j)} q^{bc} s_{(a,b)} t_{(c,d)} \]

for sequences on \( \mathbb{Z}_+ \times \mathbb{Z}_+ \). One may check that \( \ast_q \) is associative. Then \( t^{(i,j)}_{(k,0)} = (s \ast_q \cdots \ast_q s)_{(i,j)} \) and \( t^{(i,j)}_{(0,k)} = (t \ast_q \cdots \ast_q t)_{(i,j)} \) (\( k \)-fold products) and \( t^{(i,j)}_{(k,l)} = (s \ast_q \cdots \ast_q s \ast_q t \ast_q \cdots \ast_q t)_{(i,j)} \) (\( k \)-fold and \( l \)-fold). The relations (27) then hold if \( t \ast_q s = qs \ast_q t \), which are the stated relations of \( M(\mathbb{C}_q^2) \).

The quotient \( M_0(\mathbb{C}_q^2) \) has the same form with the index \((0,0)\) excluded from the expressions in the proposition. In this case \( t^{(i,j)}_{(k,l)} = 0 \) unless \( i + j \leq k + l \), so that the coproduct is a finite sum. The lowest level generators of \( M_0(\mathbb{C}_q^2) \) are

\[ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \equiv \left( \begin{array}{cc} s_{(1,0)} & t_{(1,0)} \\ s_{(0,1)} & t_{(0,1)} \end{array} \right) = \left( \begin{array}{cc} t_{(1,0)}^{(1,0)} & t_{(1,0)}^{(0,1)} \\ t_{(0,1)}^{(1,0)} & t_{(0,1)}^{(0,1)} \end{array} \right) \]

and the relations among these are

\[ dc = qcd, \quad ba = qab, \quad ad - da = q^{-1}bc - qcb, \]

which are just half of the relations of the quantum matrices. Moreover, this is just the lowest level content of the relations \( t \ast_q s = qs \ast_q t \).

### 3.8 Preserving a coaddition

We are now ready to consider the general theory of restricting diffeomorphisms to those preserving a coalgebra structure. In our ‘coordinate ring’ setting it means a quotient of \( M_1(A) \), which we denote \( M_1(A, \Delta) \). Thus, a coalgebra structure on \( A \) corresponds in a basis \( \{e_i\} \) to structure constants defined by \( \Delta e_i = d^{jk} e_j \otimes e_k \). Along the same lines as in Section 2, the comeasurings that preserve this clearly means the quotient of \( M_1(A) \) by the additional relation

\[ d_{i}^{jk} t^{i}_{a} = d_{i}^{ab} t^{j}_{a} t^{k}_{b}. \]
As explained below Proposition 2.2, one has a bialgebra for any $d_{i}^{jk}$, i.e. one does not need $\Delta$ to be coassociative or to make $A$ into a bialgebra. The coassociativity is, however, natural to assume.

We also have quotients $M(A, \Delta), M_0(A, \Delta)$ etc., as before. And in principle, we have still further quotients where a counit $\epsilon$ is respected as well. However, the counit in the case of a bialgebra $A$ defines a natural splitting $A = 1 \oplus A'$, where $A' = \ker \epsilon$, i.e. it is natural to choose the basis so that $\epsilon(1) = 1$ and $\epsilon(e_i) = 0$ for $i > 0$. At least in this case, the bialgebra $M_0(A, \Delta)$ automatically preserves the counit without further quotients arising from that.

The quantum-braided plane $\mathbb{C}_q^2$ has such coalgebra structure, expressing the braided addition law. This is the braided ‘coaddition’ introduced in [10]. Explicitly, it is given by

$$\Delta_+(x^m y^n) = \sum_{r=0}^{m} \sum_{s=0}^{n} \left[ \begin{array}{c} m \\ r \end{array} \right] q^r x^r y^s \otimes x^{m-r} y^{n-s} q^{(m-r)s},$$

where we use the $q^2$-binomial coefficients defined as usual, but in terms of $q^2$-factorials

$$[n]_{q^2}! = [n]_{q^2} \cdots [1]_{q^2}, \quad [m]_{q^2} = \frac{1 - q^{2m}}{1 - q^2}.$$

Although this coproduct forms a coalgebra (it is coassociative), it does not make $\mathbb{C}_q^2$ into a usual bialgebra or Hopf algebra, but rather into a braided group[11]. We will say more about this in the next section; for the present purposes we need only to know its explicit form as stated here.

For our above basis, we have

$$d_{(m,n)}^{(i,j)(k,l)} = \delta_{m}^{i+k} \delta_{n}^{j+l} \left[ \begin{array}{c} m \\ i \end{array} \right] q^i \left[ \begin{array}{c} n \\ j \end{array} \right] q^j q^{jk}.$$

**Proposition 3.2** The quotient of the restricted comeasuring bialgebra $M_0(\mathbb{C}_q^2)$ respecting the braided coaddition on $\mathbb{C}_q^2$, for generic $q$, can be identified with the standard $2 \times 2$ quantum matrices $M_q(2)$.

**Proof** The additional quotient of $M_0(\mathbb{C}_q^2)$ is by the relation

$$\sum_{(a,b)+(c,d)=(m,n)} \left[ \begin{array}{c} m \\ a \end{array} \right] q^a \left[ \begin{array}{c} n \\ b \end{array} \right] q^b q^{i+j} \left[ \begin{array}{c} i+k \\ i \end{array} \right] q^i \left[ \begin{array}{c} j+l \\ j \end{array} \right] q^j q^{jk} t^{(i,j)+(k,l)(m,n)}.$$
and then the additional relations become

$$\sum_{(a,b)+(c,d)=(m,n)} q^{bc} \tau^{(i,j)}_{(a,b)} \tau^{(k,l)}_{(c,d)} = q^{jk} \tau^{(i,j)+(k,l)}_{(m,n)}$$

which, by similar reasoning as for the comeasuring bialgebra, implies that the generators can be obtained by convolution from the generators $\sigma^{(i,j)} = \tau^{(i,j)}$ and $\tau^{(i,j)} = \tau^{(i,j)}$. This time

$$t^{(k,0)}_{(i,j)} = \sum_{(a,b_1)+\cdots+(a_k,b_k)=(i,j)} \sigma^{(a_1,b_1)} \cdots \sigma^{(a_k,b_k)} q^{\sum_{s=2}^k (b_1+\cdots+b_{s-1}) a_s}$$

$$t^{(0,k)}_{(i,j)} = \sum_{(a_1,b_1)+\cdots+(a_k,b_k)=(i,j)} \tau^{(a_1,b_1)} \cdots \tau^{(a_k,b_k)} q^{\sum_{s=2}^k (b_1+\cdots+b_{s-1}) a_s}$$

$$\tau^{(k,l)}_{(i,j)} = \sum_{(a,b)+(c,d)=(i,j)} q^{bc} \tau^{(k,0)}_{(a,b)} t^{(0,l)}_{(c,d)}$$

for $k, l \geq 1$. The residual relations are

$$q \sum_{(a,b)+(c,d)=(i,j)} q^{bc} \sigma^{(a,b)} \tau^{(c,d)} = \sum_{(a,b)+(c,d)=(m,n)} q^{bc} \tau^{(a,b)} \sigma^{(c,d)}, \quad \forall i, j \in \mathbb{Z}^+$$

of the convolution notation, $\tau * q \sigma = q \sigma * q \tau$. This is the bialgebra respecting only the coalgebra $\Delta_+$ (indeed, the quantum-braided plane is self-dual as a braided group and this is why this algebra has the same form as the comeasuring bialgebra).

These convolution formulae imply that $t^{(i,j)}_{(k,l)} = 0$ unless $i + j \leq k + l$. Combined with the reverse inequality for $M_0(\mathbb{C}_q^2)$, we see that $t^{(i,j)}_{(k,l)} = 0$ in $M(\mathbb{C}_q^2, \Delta_+)$ unless $i + j = k + l$. It follows that $M_0(\mathbb{C}_q^2, \Delta_+)$ is generated by 1 and the lowest level generators (28). Half their relations are given above, inherited from $M_0(\mathbb{C}_q^2)$. The other half come from the relations $\tau * q \sigma = q \sigma * q \tau$, which are computed similarly as

$$db = qbd, \quad ca = qac, \quad ad - da = q^{-1} cb - qbc.$$

Thus, $M_0(\mathbb{C}_q^2, \Delta_+) = M_q(2)$, the standard $2 \times 2$ quantum matrices in the conventions of [8]. The coaction reduces to the standard coaction on $\mathbb{C}_q^2$.

## 4 R-matrix constructions for comeasuring bialgebras

In this section we consider some general constructions possible when our algebra is braided, i.e. in the presence of a Yang-Baxter operator. As a first application, we note that until now we have studied the maximal comeasuring objects in the category of bialgebras. Since they
are maximal they tend to be free, with only the minimal relations compatible with coacting on
our algebra. However, when the algebra is itself braided, we can look for objects maximal in
some braided-commutative sense. We then give more functorial braided group versions of these
constructions, related by transmutation.

We recall that an algebra $A$ is braided if it comes with an operator $\Psi : A \otimes A \to A \otimes A$
obracket
obeying the braid relations and functorial with respect to the product of $A$ in the sense
\[
\Psi(ab \otimes c) = (\text{id} \otimes \cdot)(\Psi \otimes \text{id})(\text{id} \otimes \Psi)(a \otimes b \otimes c), \quad \Psi(a \otimes bc) = (\cdot \otimes \text{id})(\text{id} \otimes \Psi)(\Psi \otimes \text{id})(a \otimes b \otimes c),
\]
for all $a, b, c \in A$. More generally, a braided algebra means $A$ an object in a braided category
(such as that generated by a single braiding operator) with the product a morphism. We recall
that in a braided category there is a braiding between any two objects playing the role of
‘transposition’. Also, the (co)modules of any (dual) quasitriangular bialgebra or Hopf algebra
(i.e., of any ‘strict’ quantum group) form a braided category, so any algebra covariant under
a strict quantum group is a braided algebra. See [8] or papers such as [11][12] where braided
algebras and groups have been introduced. In this setting we introduce $M_1(R, A)$ as again a
dual quasitriangular bialgebra or ‘strict’ quantum group.

Thus, when $A$ has a basis $\{e_i\}$, a braided algebra translates into the existence of a matrix
$R \in M_n \otimes M_n$ (where $M_n$ denotes $n \times n$-matrices and $n = \text{dim}(A)$) obeying the Quantum
Yang-Baxter Equations (QYBE) $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ (a so-called R-matrix), such that
\[
c_{12}^3 R_{14} R_{24} = R_{34} c_{12}^3, \quad R_{12} c_{34}^2 c_{34}^2 R_{14} R_{13}
\]
We use the standard compact notation where $R_{12} = R \otimes \text{id}$ and we suppose that $R$ is invertible.
In explicit component terms, the requirement is
\[
c_{ij}^a R_{i m a}^k R_{m c a}^j = c_{ab}^k R_{i c b}^a R_{k i a}^c, \quad c_{jk}^a R_{i m a}^m R_{i c b}^j = c_{ab}^i R_{k m a}^b R_{c i a}^c,
\]
where
\[
\Psi(e_i \otimes e_j) = e_b \otimes e_a R_{i b}^a.
\]
4.1 Dual quasitriangular comeasuring bialgebras

We also recall that a bialgebra $M$ is dual quasitriangular or $R$-commutative if there exists $R : M \otimes M \to k$ such that

$$R(ab, c) = R(a, c_{(1)})R(b, c_{(2)}), \quad R(a, bc) = R(a_{(1)}, c)R(a_{(2)}, b), \quad b_{(1)}a_{(1)}R(a_{(2)}, b_{(2)}) = R(a_{(1)}, b_{(1)})a_{(2)}b_{(2)}$$

for all $a, b, c \in M$. The axioms are dual to the quasitriangular structures introduced for the quantum groups $U_q(g)$ by Drinfeld\[13\]. On the other hand, given any $R$-matrix there is a dual quasitriangular bialgebra $A(R)$ of ‘quantum matrices’ with generators $R_{i_1i_2}$ (in a compact notation where $t_1 = t \otimes \text{id}$ etc.). These are the FRT relations\[14\] while the dual quasitriangularity for general $R$ is due to the author\[15\] and takes the form $R(t_{ij}, t_{kl}) = R_{ij kl}$.

See [8].

**Proposition 4.1** If $A$ is a braided algebra with braiding defined by an $R$-matrix, then $M_1(R, A)$ defined by the matrix coalgebra and the relations

$$R_{a^i j} t^a_{k^i} t^b_{l^j} = t^j_{b^i} t^a_{k^i} R_{a^j b^i}, \quad c_{ij} a^k = c_{ab} k^a t^i_{l^j} t^b_{l^j}$$

is a dual quasitriangular bialgebra.

**Proof** We check that this quotient of $A(R)$, in which the relations of Proposition 2.2 are further imposed, inherits the linear functional $R$. If so then it will remain a dual quasitriangular structure for the quotient. Thus,

$$R(t_{ij}, c_{ab} k^a t^b_{mn}) = R_{c^i n} R_{a^j m} c_{ab} k = c_{mn} a^i k^a R_{a^j} = R(t_{ij}, c_{mn} a^i k^a)$$

$$R(c_{ab} k^a t^b_{ij}, t^m_{ln}) = c_{ij} a^k R_{a^i} m c_{ij} a^k R_{a^i} m = R(c_{ij} a^k a^i, t^m_{ln})$$

using the covariance conditions (32). The proof for $t_{ij}$ and $t^m_{ln}$ replaced by general strings of generators has just the same form, with repeated use of the covariance conditions; the general proof thereby proceeds by a straightforward induction. •

While this quotient will always be dual quasitriangular, it may also be trivial, i.e. the ideal generated by both sets of relations may be too large. Although not necessary, if $A$ is itself ‘braided commutative’ in some sense then one may expect that the above construction is
more natural. The appropriate form of commutativity is, as for braided matrices and braided planes, the existence of a matrix \( R' \in M_n \otimes M_n \) obeying

\[
R'_{12} R_{13} R_{23} = R_{23} R_{13} R'_{12}, \quad R_{12} R_{13} R'_{23} = R'_{23} R_{13} R_{12}.
\]

(33)

More precisely, these matrix conditions follow from and are essentially equivalent to the algebraic condition

\[
R_{a}^t - a_1^t R_{a}^t = a_2^t R_{a}^t - a_3^t R_{a}^t
\]

in the quantum matrix algebra. Typically, \( R' \) is built from \( R \) and the relations with \( R' \) are equivalent to the defining relations with \( R \). With respect to this, braided-commutativity is

\[
c_{ij}^k = c_{ba}^k R_{i,j}^{a,b}.
\]

(34)

Note that the more naive definition of braided commutativity would be \( \cdot \circ \Psi(a \otimes b) = ab \) for all \( a, b \), but this is only natural when \( \Psi^2 = \text{id} \). It corresponds to the choice \( R' = R \), which is too restrictive to apply in most examples of q-deformation. Then

\[
c_{ba}^k (t_{j}^b R_{i,j}^{a,b} m_{n}) = c_{ba}^k R_{i,j}^{a,b} t_{j}^b m_{n} = c_{ij}^k t_{m}^j t_{n}^i = c_{mn}^k t_{a}^k
\]

holds automatically in \( M_1(R, A) \).

As before, there are quotients \( M(R, A) \) and \( M_0(R, A) \) when \( \Psi \) respects \( e_0 = 1 \). The natural condition is \( \Psi(a \otimes 1) = 1 \otimes a \) and \( \Psi(1 \otimes a) = a \otimes 1 \) for all \( a \in A \).

As a finite-dimensional example, the \( q \)-epsilon tensor and \( q \)-metric associated to the \( SO_q(3) \)-covariant quantum plane may be used to define an associative \( q \)-quaternion algebra \( \mathbb{H}_q \). Its automorphism bialgebra then recovers \( M_0(R, \mathbb{H}_q) = SO_q(3) \) along the same lines as for the \( q = 1 \) case in [4].

For the simplest ‘geometrical’ example, namely \( A = \mathbb{C}[x] \), we have more than one way to consider it as a braided algebra. The simplest is as the braided line where

\[
\Psi(x^i \otimes x^j) = q^{ij} x^j \otimes x^i, \quad R_{j}^{i,k} = \delta_{j}^{i} \delta_{k}^{i} q^{ik}.
\]

(35)

Then \( M(R, \mathbb{C}[x]) = \text{Diff}(\mathbb{C}[x]) \) is the classical commutative diffeomorphism group. Here \( q \) cancels from both sides and the result is the same as for \( q = 1 \), i.e. the imposition of commutativity between all the generators.

A different braiding on \( A = \mathbb{C}[x] \) is the one introduced in [10],

\[
\Psi(x^i \otimes x^j) = \sum_{k=0}^{i} \left[ \begin{array}{c} i \\ k \end{array} \right]_q q^{i(k-i)} (1 - q)^{k} \frac{(j + k - 1)!}{(j - 1)!} x^{j+k} \otimes x^{i-k}, \quad \forall i, j \in \mathbb{N}
\]

(36)
and the trivial transposition when \( i \) or \( j = 0 \). This is the canonical ‘double’ braiding associated to any braided group, in this case the braided line. The algebra of the braided group is automatically a braided algebra under its canonical braiding. From another point of view, the double bosonisation of a braided group canonically acts on the braided group. Here the double bosonisation of the braided line is \( U_q(su_2) \) and acts as a \( q \)-deformation of \( so(2,1) \) by ‘conformal transformation’ on \( \mathbb{C}[x][\mathfrak{l}] \). The above braiding is the one induced from the quasitriangular structure of \( U_q(su_2) \) by this action. We therefore call this the ‘conformally braided’ line.

**Proposition 4.2** The restricted dual-quasitriangular comeasuring bialgebra \( M_0(R, \mathbb{C}[x]) \) is generated by \( 1, t_i \) for \( i \in \mathbb{N} \) with the relations

\[
qt_j t_i = \sum_{k=0}^{i-1} \frac{[i+k]! [j-1]_q!}{[j-k-1]_q! [i]_q! [k]_q!} q^{i(j-k)} (1-q)^k t_{i+k} t_{j-k}
\]

and the coalgebra \( \mathfrak{l} \) as in Section 3.1. The dual quasitriangular structure is \( \mathcal{R}(t_i, t_j) = q \delta_i^j \delta_1^1 \).

**Proof** The R-matrix corresponding to (34) is

\[
R^{i,j}_{0,k} = \delta^i_0 \delta^j_k, \quad R^{i,k}_{j,0} = \delta^i_j \delta^k_0, \quad R^{j,k}_{j+1,i} = \delta^j_{j+1} \delta^k_i q^{i(j-1)} (1-q)^{j-i} \frac{[k-1]_q!}{[j-1]_q!}
\]

for \( j, k \in \mathbb{N} \) and \( i \leq j \) in the third expression (which is zero otherwise), which implies that \( R^{a,b}_{1,1} = q \delta^a_1 \delta^b_1 \) when \( a, b \in \mathbb{N} \). Hence the \( R t_i t_2 = t_2 t_1 R \) relations on the generators \( t_i \equiv t^i_1 \) have the form

\[
\sum_{a,b} R^{i,j}_{a,b} t_a t_b = qt_j t_i,
\]

which then computes as stated. The R-matrix also provides the dual-quasitriangular structure on the generators. •

In particular, the relations of the lowest order generators with the general generators are

\[
t_1 t_i = q^{i-1} t_i t_1, \quad t_2 t_i = q^{2i-1} t_i t_2 + (1-q^{i+1}) q^{i-1} t_{i+1} t_1 \\
t_3 t_i = q^{3i-1} t_i t_3 + (1-q^{i+1})(1+q) q^{2i-1} t_{i+1} t_2 + (1-q^{i+2})(1-q^{i+1}) q^{i-1} t_{i+2} t_1
\]

for all \( i \in \mathbb{N} \), etc. From this, for generic \( q \), one obtains explicitly

\[
t_1 t_2 = qt_2 t_1, \quad t_1 t_3 = q^2 t_3 t_1 = q(t_2)^2, \quad t_2 t_3 = qt_3 t_2 = q^2 t_1 t_1, \\
\Delta t_1 = t_1 \otimes t_1, \quad \Delta t_2 = t_2 \otimes t_1 + (t_1)^2 \otimes t_2, \quad \Delta t_3 = t_3 \otimes t_1 + (1+q) t_2 t_1 \otimes t_2 + (t_1)^3 \otimes t_3
\]
and so on, to all orders. The comeasuring bialgebra $M(R, \mathbb{C}[x])$ is similar, with the extra generator $t_0$. The coproduct in this case involves infinite sums, so has to be treated formally.

For anyonic variables $x^3 = 0$, we again have a braided group (the anyonic line), and hence a canonical 'double' braiding. Here $q^3 = 1$ and the resulting $9 \times 9$ $R$-matrix in this case is given in [10] Ex. 4.8. The restricted $M_0(R, \mathbb{C}[x]/x^3)$ is the quotient of $M_0(\mathbb{C}[x]/x^3)$ in Section 3.2 by the relation $ts = qst$, i.e. we have a dual-quasitriangular bialgebra

$$ts = qst, \quad \Delta t = t \otimes t, \quad \Delta s = s \otimes t + t^2 \otimes s, \quad \mathcal{R}(t^i, t^j) = q^{ij}$$

and $\mathcal{R}$ zero on any expression containing $s$.

Finally, we let $A = \mathbb{C}_q^2$ be the quantum-braided plane $\mathbb{C}_q^2$ with generators $x, y$ and relations $yx = qxy$, as in Section 3.7. This has a natural braiding $\Psi$ used to describe coaddition on the quantum plane [10], as studied in Section 3.8. The quantum-braided plane is also braided commutative with respect to a certain matrix $R'$. Explicitly,

$$\Psi(x \otimes x) = q^2x \otimes x, \quad \Psi(y \otimes y) = q^2y \otimes y, \quad \Psi(x \otimes y) = qy \otimes x, \quad \Psi(y \otimes x) = qx \otimes y + (q^2-1)y \otimes x$$

extended to products by functoriality. This is the braiding induced by the action of $U_q(\widetilde{su}_2)$ on the quantum-braided plane and is usually given by an array of $su_2$-type $R$-matrices, see [11] 8.

**Proposition 4.3** The restricted dual-quasitriangular comeasuring bialgebra $M(R, \mathbb{C}_q^2)$ is generated by $1, s_{(i,j)}, t_{(i,j)}, i, j \in \mathbb{Z}_+$ with the relations in Proposition 3.1 and the additional relations

$$R^{(i,j)}_{(a,b)(k,l)}(c,d)s_{(a,b)}s_{(c,d)} = q^2s_{(k,l)}s_{(i,j)}, \quad R^{(i,j)}_{(a,b)(k,l)}(c,d)t_{(a,b)}t_{(c,d)} = q^2t_{(k,l)}t_{(i,j)}$$

$$R^{(i,j)}_{(a,b)(k,l)}(c,d)t_{(a,b)}s_{(c,d)} = q^2t_{(k,l)}s_{(i,j)}, \quad R^{(i,j)}_{(a,b)(k,l)}(c,d)s_{(a,b)}s_{(c,d)} = q^2s_{(k,l)}t_{(i,j)} + (q^2-1)t_{(k,l)}s_{(i,j)}$$

where $R$ corresponds to the braiding $\Psi$ on $\mathbb{C}_q^2$. Summation over $a, b, c, d \in \mathbb{Z}_+$ should be understood.

**Proof** We already know $M(\mathbb{C}_q^2)$ from Section 3.7, and now quotient this further by the $Rt_1t_2 = t_2t_1R$ relations. On the other hand, the braiding on the generators (37) immediately gives $R^{(a,b)}_{(1,0)(c,d)}(1,0) = \delta^a_c \delta^b_d q$, corresponding to $\Psi(x \otimes x) = q^2 x \otimes x$, etc. Thus the additional relations on the generators $s_{(i,j)}, t_{(i,j)}$ reduce to the four sets as shown. They can be stated more
The third relation in Proposition 4.3 similarly reduces to the four relations of the quantum-plane in R-matrix form, i.e., compactly as the relations of a rectangular quantum matrix $A(R : R_{su_2})$, where $R_{su_2}$ is the standard $su_2$-type R-matrix. 

The relations of $M(R, C^2)$ may be further expressed in terms of $R_{su_2}$ or, alternatively, defined by induction. If $x_i = (x, y)$ is an $SU_q(2)$-covariant covector notation then

\[ x_{i_1} x_{i_2} \cdots x_{i_n} = q^{p(i_1, \ldots, i_n)} x^{\#(i_1, \ldots, i_n)} y^{n-\#(i_1, \ldots, i_n)} \]

defines $\mathbb{Z}_+$-valued functions $p, \#$ (here $\#$ is the number of indices with value 1, and $p$ is the number of indices with value 1 to the right of each index with value 2.) Then

\[ R^{(a,b)}_{(i,j)}(c,d)_{(k,l)} = \delta_{i+j}^{a+b} \delta_{c+d}^{c+d} \sum_{\#(b_i) = c} \sum_{\#(a_i) = a} q^{p(b_i)+p(a_i)} Z^{a_1 \cdots a_{i+j}} x^{b_1 \cdots b_{k+l-1} 2 \cdots} \]

where there are $i$ indices $1 \cdots$, $j$ indices $2 \cdots$, followed by $k$ indices $1 \cdots$, and $l$ indices $2 \cdots$, and where $Z$ is the ‘partition function’ array of copies of $R_{su_2}$ corresponding to $\Psi$ in the braided covector description of the quantum-braided plane (see \cite{8}, Thm. 10.2.1). Alternatively, the general $\Psi$ in our basis $e_{(i,j)} = x^i y^j$ may be obtained by induction, via the formulae

\[ \Psi(y^i \otimes x^j) = x \Psi(y^i \otimes x^{j-1}) q^i + (q^2 - 1)[i] q^2 q^{2(j-1)} y \Psi(y^{i-1} \otimes x^{j-1}) x \]

and

\[ \Psi(x^i y^j \otimes x^k y^l) = q^{i(2k+l-j)+l(2j-k)} y^l \Psi(y^j \otimes x^k) x^i. \]

These expressions follow from the functoriality (30) of the braiding.

**Corollary 4.4** There is a bialgebra surjection $M_0(R, C) \to M_q(2)$ and $M_q(2)$ appears as a subalgebra \cite{28} covered by this surjection. The dual-quasitriangular structure of $M_0(R, C)$ extends that of $M_q(2)$.

**Proof** Since $\Psi$ preserves the total degree, the only way to obtain a non-zero coefficient of $x \otimes x$, $x \otimes y$, $y \otimes x$ or $y \otimes y$ in $\Psi(x^i y^j \otimes x^k y^l)$ is with $i+j = 1 = k+l$. Hence to compute $R^{(1,0)}_{(i,j)}(1,0)_{(k,l)}$ etc., we need only to consider (37). Writing $1 \equiv (1,0)$ and $2 = (0,1)$, the only nonzero entries of this form are given by the standard $R_{su_2}$. Thus, the lowest level generators $s_{(1,0)}, s_{(0,1)}$ obey the relations of the quantum-plane in R-matrix form, i.e. $ca = qac$ in the notation (28). Similarly, $t_{(1,0)}, t_{(0,1)}$ form a quantum-braided plane, i.e. $db = qbd$. These relations already hold in $M_0(C^2_q)$. The third relation in Proposition 4.3 similarly reduces to the four relations $ba = gab$, $dc = qcd$, etc., we need only to consider (37). Writing 1 \equiv (1,0) and 2 = (0,1), the only nonzero entries of this form are given by the standard $R_{su_2}$. Thus, the lowest level generators $s_{(1,0)}, s_{(0,1)}$ obey the relations of the quantum-plane in R-matrix form, i.e. $ca = qac$ in the notation (28). Similarly, $t_{(1,0)}, t_{(0,1)}$ form a quantum-braided plane, i.e. $db = qbd$. These relations already hold in $M_0(C^2_q)$.
cb = bc and ad − da = (q^{-1} − q)bc, while the forth is then redundant. Hence the relations among these lowest level generators of $M_0(R, \mathbb{C}_q^2)$ are precisely the relations of the $2 \times 2$ quantum matrices $M_q(2)$.

The geometric meaning of this is as follows. The surjection corresponds classically to the inclusion of $2 \times 2$ linear transformations among the algebraic diffeomorphisms of the plane. The inclusion of $M_q(2)$ corresponds classically to the projection which associates to a diffeomorphism fixing zero its differential at zero, i.e., the linear transformation induced on the tangent space at zero.

Similarly, there is a surjection from $M(R, \mathbb{C}_q^2)$ to the $q$-deformed Weyl algebra $\mathbb{C}_q^2 \rtimes M_q(2)$ cf [8] and an inclusion of it. One replaces the dilaton-extended $SU_q(2)$, i.e., $GL_q(2)$, in the bosonisation construction $\mathbb{C}_q^2 \rtimes GL_q(2)$ from [10] [8] by the quantum matrices $M_q(2)$. Conversely, at least these quotients of $M(R, \mathbb{C}_q^2)$ become Hopf algebras by adjoining the inverse of the $q$-determinant. Moreover, our constructions are quite general and can be applied similarly to quantum-braided planes associated to other $R$-matrices than $R_{su_2}$.

### 4.2 Braided comeasuring bialgebras

Finally, we give braided versions of all our comeasuring constructions. From a categorical point of view we can fix any braided category in which our algebra lives and look for the universal comeasuring object in this braided category. The result will now be a braided group [12] or bialgebra in the braided category.

For the abstract aspect of these constructions, we use the diagrammatic notation for ‘braided algebra’ (due to the author) in which we write products etc. as nodes $\cdot = \Psi$ and ‘wire up’ our algebraic operation using $\Psi = \times$, $\Psi^{-1} = \times$ as necessary. Note that the use of ‘flow charts’ to express algebraic or other operations is nothing new – it is used routinely by physicists as Feynman diagrams and by engineers as wiring diagrams; the new feature in braided mathematics [11] [19] is to adapt the notation to algebraic constructions in braided categories, where under and over crossings are nontrivial and distinct operators. We use here the coherence theorem for braided categories [20].

In particular, using the braiding, one has a braided tensor product algebra $A \otimes B$ for $A, B$ algebras in the category. In a concrete setting it is $(a \otimes b)(c \otimes d) = a\Psi(b \otimes c)d$, but more
Figure 1: (a) Comeasuring property of \((B, \beta)\) (b) Proof that \(B \otimes B, (\beta \otimes \text{id}) \beta\) is another comeasuring (c) Proof that \(\Delta\) is coassociative

generally it is defined diagrammatically, see [8]. We define a comeasuring of \(A\) as a pair \((B, \beta)\) where \(\beta: A \to A \otimes B\) is a morphism and an algebra map.

**Proposition 4.5** If \(A\) is an algebra in a braided category, the universal comeasuring object \(M_1(A)\) is a braided group (a bialgebra in the braided category).

*Proof* This is shown in Figure 1. In part (a) we write the definition of comeasuring in diagrammatic form. In part (b) we check that comeasurings are closed under tensor product. The lower box is the braided tensor product algebra \(B \otimes B\). Hence if \((M, \beta_U)\) is the universal object, then \((M \otimes M, (\beta_U \otimes \text{id}) \beta_U)\) also comeasures, hence there is an induced algebra map \(\Delta : M \to M \otimes M\). Part (c) checks that it is coassociative.

As before, we have quotients \(\underline{M}(A)\) and \(\underline{M}_0(A)\) (the latter when 1 is split). We have explicit formulae in the \(R\)-matrix case. For convenience we assume that \(R\) is bi-invertible in the sense that \(\tilde{R} = ((R^{2})^{-1})^{t_2}\) exists, where \(t_2\) denotes transposition in the second matrix factor.

**Proposition 4.6** If \(A\) is a finite-dimensional braided algebra with braiding determined by a bi-invertible \(R\)-matrix, then the universal comeasuring braided group has the explicit form \(\overline{M}_1(R, A)\) with generators 1 and \(u^{i,j}\) and the relations, coalgebra and braiding

\[ c_{ij} a^k_{\alpha} = c_{ab} b^k_{\alpha} R^{-1} a^{bc}_{\alpha} c^{de}_{\alpha} R^{ef}_{\alpha} d^{fg}_{\alpha} \]
\[ \Delta u^i_j = u^i_a \otimes u^a_j, \quad \epsilon(u^i_j) = \delta^i_j, \quad \Psi(u^i_j \otimes u^k_l) = u^m_n \otimes u^r_s R^d_i a_m R^{-1}a^{-1} R^b_j b^i \tilde{R}^c_k d. \]

The braided coaction and the braiding with A is

\[ \beta_U(e_j) = e_a \otimes u^a_j \quad \Psi(u^i_j \otimes e_k) = e_m \otimes u^a_b R^{-1}a^{-1} R^b_j n^k, \quad \Psi(e_k \otimes u^i_j) = u^a_b \otimes e_m \tilde{R}^n_k a R^m_b. \]

In compact form, the bialgebra structure is

\[ u_3 c_{12}^3 = c_{12}^3 R_{12}^{-1} u_1 R_1 u_2, \quad \Delta u = u \otimes u, \quad cu = id, \quad \Psi(R_{12}^{-1} u_1 \otimes R_{12} u_2) = u_2 R_{12}^{-1} \otimes u_1 R_{12}. \]

Proof: That we obtain here a bialgebra follows from the preceding proposition once we have established the universal property. Before doing this we first outline, for completeness, the direct algebraic proof. For this, we have to check that the coproduct extends as an algebra map to the braided tensor product with the stated braiding \( \Psi \). Thus,

\[
\begin{align*}
\Delta (c_{ab}^k R^{-1}_a b^i d e^f c_{ij}^j f u^j f j) &= c_{ab}^k R^{-1}_a b^i d e^f c_{ij}^j f (u^c_p \otimes u^e^p)(u^f q \otimes u^q a j) \\
&= c_{ab}^k R^{-1}_a b^i d e^f c_{ij}^j f u^c_p u^m_n \otimes u^r_s u^q_j R^p u^r u^z v R^w v q \tilde{R}^e f z \\
&= c_{ab}^k R^{-1}_a b^i d e^f c_{ij}^j f u^c_p R^p u^r u^m_n \otimes u^r_s u^q_j R^{-1}_a n^v v R^s v q \\
&= c_{ab} a^k u^a_a \otimes u^r_s R^{-1}_a n^v u^q_j R^s v q = c_{ij} a^k b \otimes u^a_a = \Delta c_{ij} a^k.
\end{align*}
\]

Or in the compact notation, this proof reads

\[
\begin{align*}
\Delta (c_{12}^3 R_{12}^{-1} u_1 R_{12} u_2) &= c_{12}^3 R_{12}^{-1} (u_1 \otimes u_1) R_{12} (u_2 \otimes u_2) = c_{12}^3 R_{12}^{-1} u_1 \Psi(u_1) R_{12} \otimes u_2) u_2 \\
&= c_{12}^3 R_{12}^{-1} u_1 R_{12} u_2 \otimes R_{12}^{-1} u_1 R_{12} u_2 = u_3 c_{12}^3 \otimes R_{12}^{-1} u_1 R_{12} u_2 = u_3 \otimes u_3 c_{12}^3 = \Delta u_3 c_{12}^3.
\end{align*}
\]

We also have to check that \( \Psi \) is itself well-defined when extended to products by functoriality (i.e. such that \( M_1(R, A) \) is a braided algebra). Here a direct proof is too complex to write out in explicit terms and we give it only with the compact notation. Thus,

\[
\begin{align*}
\Psi(u_1 R_{12} \otimes c_{34} R_{34}^{-1} u_3 R_{34} u_4) &= \Psi(u_1 \otimes c_{34} R_{14} R_{13} R_{34}^{-1} u_3 R_{34} u_4) \\
&= c_{34}^2 R_{34}^{-1} \Psi(u_1 \otimes R_{13} R_{14} u_3 R_{34} u_4) = c_{34}^2 R_{34}^{-1} R_{13} R_{34}^{-1} u_3 R_{13} \Psi(u_1 \otimes R_{14} R_{34} u_4) \\
&= c_{34}^2 R_{34}^{-1} R_{13} u_3 R_{13}^{-1} R_{34} \Psi(u_1 \otimes R_{14} u_4) R_{13} = c_{34}^2 R_{34}^{-1} R_{13} R_{34}^{-1} u_3 R_{13} R_{34}^{-1} u_4 R_{14}^{-1} R_{14} R_{13} \\
&= c_{34}^2 R_{34}^{-1} R_{13} R_{14} u_3 R_{34} R_{34}^{-1} R_{14}^{-1} \otimes u_1 R_{14} R_{13} = c_{34}^2 R_{14} R_{13} R_{34}^{-1} u_3 R_{34} u_4 R_{13} R_{14}^{-1} \otimes u_1 R_{14} R_{13} \\
&= R_{12} c_{34}^2 R_{34}^{-1} u_3 R_{34} u_4 R_{13}^{-1} R_{14}^{-1} \otimes u_1 R_{14} R_{13} = R_{12} u_2 c_{34}^2 R_{13}^{-1} R_{14}^{-1} \otimes u_1 R_{14} R_{13} \\
&= R_{12} u_2 \otimes R_{12}^{-1} c_{34}^2 u_1 R_{14} R_{13} = R_{12} u_2 \otimes R_{12}^{-1} u_1 R_{12} c_{34}^2 = \Psi(u_1 R_{12} \otimes u_2 c_{34}^2)
\end{align*}
\]
Thus we have
\[ \Psi(c_{12}^{-1}u_{112}R_{124}u_{2} \otimes R_{14}R_{24}u_{4}) = c_{12}^{-1}R_{12}^{-1}\Psi(u_{112}R_{12}R_{14}u_{2} \otimes R_{24}u_{4}) \]
\[ = c_{12}^{-1}R_{12}^{-1}\Psi(u_{1}R_{12}R_{14}R_{24} \otimes u_{4})R_{24}^{-1}u_{2}R_{24} = c_{12}^{-1}R_{12}^{-1}R_{24}^{1}\Psi(u_{1}R_{14} \otimes u_{4})R_{12}R_{24}^{-1}u_{2}R_{24} \]
\[ = c_{12}^{-1}R_{12}^{-1}R_{24}R_{14}u_{4}R_{14}^{-1} \otimes u_{1}R_{14}R_{12}R_{24}^{-1}u_{2}R_{24} = c_{12}^{-1}R_{14}R_{24}R_{12}^{-1}u_{4}R_{14}^{-1} \otimes R_{24}^{-1}u_{1}R_{12}R_{14}u_{2}R_{24} \]
\[ = R_{34}u_{4}c_{12}^{-1}R_{12}^{-1}R_{14}^{-1}u_{1}R_{12}R_{14}u_{2}R_{24} = R_{34}u_{4}R_{34}^{-1} \otimes u_{3}c_{12}^{-1}R_{12}R_{14} \]
\[ = R_{34}u_{4}R_{34}^{-1} \otimes u_{3}R_{34}c_{12}^{-1} = \Psi(u_{3}R_{34} \otimes u_{4}c_{12}^{-1}) = \Psi(u_{3}c_{12}^{-1} \otimes R_{14}R_{24}u_{4}) \]

using the QYBE and the other half of [14]. The proof for higher products is similar and the general case follows by induction.

Finally, we verify the universal property. Thus, if \( B \) is a braided comeasuring on \( A \), we define \( \pi(u^i_j) \in B \) as \( \beta(e_j) = e_a \otimes \pi(u^a_j) \). That this is a morphism implies that the braiding with \( A \) may be computed before or after applying \( \beta \). This is shown in Figure 2. Thus
\[ e_a \otimes \Psi(\pi(u^a_i) \otimes e_j) = \Psi^{-1}(e_b \otimes e_c) \otimes u^c_i R_{i,j}^{a,b} = e_m \otimes e_n R_{m,n}^{c,b} \otimes u^c_i R_{i,j}^{a,b} \]

which tells us that the braiding \( \Psi : B \otimes A \to A \otimes B \) is compatible via \( \pi \) with the braiding of \( M(R,A) \) with \( A \). The braiding with \( A \) on the other side similarly gives compatibility of \( \Psi : A \otimes B \to B \otimes A \). Once these are known then the braiding of \( A \) with \( \pi(u^i_k) \in B \) before and after \( \beta \) yields compatibility with \( \Psi : B \otimes B \to B \otimes B \). This is shown on the right in Figure 2. In this way, the braiding on \( \pi(u^i_j) \) has the same form as the braiding on \( u^i_j \). Next, the assumption that \( \beta \) is an algebra map to \( A \otimes B \) translates as
\[ \beta(e_i e_j) = c_{ij}e_k \otimes \pi(u^k_a) = (e_a \otimes \pi(u^a_i))(e_b \otimes \pi(u^b_j)) = e_a e_c \otimes \pi(u^d_e) R_{i,j}^{-1} \pi(u^e_f) R_{i,j}^{e,f} \pi(u^b_j). \]

Thus \( u^i_j \mapsto \pi(u^i_j) \) extends as an algebra map \( \pi : M(R,A) \to B \) and a morphism in the braided category. ●

Finally, we recall that the quantum matrices \( A(R) \) above have a braided group version \( B(R) \) defined by a matrix of generators \( u = (u^i_j) \) and relations \( R_{21}u_{1}R_{u_{2}} = u_{2}R_{21}u_{1}R \), forming a braided group with the above matrix coalgebra and braiding. The algebra relations here are known in other contexts too and sometimes called ‘reflection equations’, although
they have been introduced and studied as quadratic algebras by the author under the heading of the braided matrix relations. Key properties such as its covariance properties (as a braided algebra), the braided coproduct, results about the Poincaré series etc. were introduced in [12].

On the other hand, the quantum and braided matrices are closely tied by a theory of transmutation which relates their products, e.g.

$$t = u, \quad R_{12} t_1 t_2 = u_1 R u_2.$$ \hspace{1cm} (38)

See [8][19] for an introduction to this transmutation theory of braided groups.

**Corollary 4.7** The transmutation of $M_1(R, A)$ is the braided comeasuring bialgebra $\overline{M}_1(R, A)$ generated by 1 and $u^i_j$ with the braided matrix relations $R_{21} u_1 R u_2 = u_2 R_{21} u_1 R$ and the further relations $u_3 c_{12}^3 = c_{12}^1 R_{12}^{-1} u_1 R u_2$ and coalgebra of $\overline{M}_1(A)$ from the preceding proposition.

**Proof** This is immediate from (38) applied to the relations $t_3 c_{12}^3 = c_{12}^3 t_1 t_2$ of $M(R, A)$. •

One may then proceed to apply the extensive theory of braided groups to $\overline{M}_1(A)$, $\overline{M}_1(R, A)$ and their unital quotients. For example, associated to any braided group in the category of comodules of a quantum group $H$ (which is essentially the situation above, with $H$ obtained from $A(R)$), one has ordinary bialgebras $\overline{M}_1(A)\bowtie H$ etc., by the bosonisation construction.

As a simple example, we consider $\mathbb{C}[x]$ as a braided algebra with $R$-matrix (35). In this case the relations from Proposition 4.6 are

$$u^k_{i+j} = \sum_{a+b=k} u^a_i u^b_j q^{(i-a)b}.$$ 

Hence $\overline{M}(\mathbb{C}[x]) = \mathbb{C} < u_i | i \in \mathbb{Z}_+ >$ is the free algebra as in Section 3.1, but the other generators (and hence the matrix coalgebra) are given from these by

$$u^i_j = \sum_{a_1 + \cdots + a_j = i} u_{a_1} \cdots u_{a_j} q^{-(i-a_1) - \sum_{s=2}^{i}(a_1 + \cdots + a_{s-1})a_s}$$
, which is a q-deformation of $\mathbb{R}$ as a braided group. The braiding is

$$\Psi(u_i, u_j) = u_j \otimes u_i q^{(i-1)(j-1)}.$$

Similarly, the braided-commutativity in Corollary 4.7 in this simplest example reduces to the usual commutativity relations. For example, $M_0(R, \mathbb{C}[x]) = \mathbb{C}[u_i | i \in \mathbb{N}]$ is the same algebra as $\text{Diff}_0(\mathbb{C}[x])$ in the classical case, but the coalgebra is q-deformed and provides a nontrivial braided group structure on this algebra.

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