Decoupling of Tensor factors in Cross Product and Braided Tensor Product Algebras

Gaetano Fiore,

Dip. di Matematica e Applicazioni, Fac. di Ingegneria
Università di Napoli, V. Claudio 21, 80125 Napoli
I.N.F.N., Sezione di Napoli,
Complesso MSA, V. Cintia, 80126 Napoli

October 7, 2018

Abstract

We briefly review and illustrate our procedure to ‘decouple’ by transformation of generators: either a Hopf algebra $H$ from a $H$-module algebra $A_1$ in their cross-product $A_1 \bowtie H$; or two (or more) $H$-module algebras $A_1, A_2$. These transformations are based on the existence of an algebra map $A_1 \bowtie H \to A_1$.
1 Decoupling of tensor factors in cross product algebras

Let $H$ be a Hopf algebra over $\mathbb{C}$, say, $A$ a unital (right, say) $H$-module algebra. We denote by $\triangleleft$ the right action, namely the bilinear map such that, for any $a, a' \in A$ and $g, g' \in H$

$$\triangleleft : (a, g) \in A \times H \to a \triangleleft g \in A,$$

$$a \triangleleft (gg') = (a \triangleleft g) \triangleleft g', \quad (aa') \triangleleft g = (a \triangleleft g) (a' \triangleleft g).$$

We have used the Sweedler-type notation $\Delta(g) = g_1 \otimes g_2$ for the coproduct $\Delta$. The cross-product algebra $A \rtimes H$ is $H \otimes A$ as a vector space, and so we denote as usual $g \otimes a$ simply by $ga$; $H_1A, 1_HA$ are subalgebras isomorphic to $H, A$, and so we omit to write either unit $1_A, 1_H$ whenever multiplied by non-unit elements; for any $a \in A, g \in H$ the product fulfills

$$ag = g_1(a \triangleleft g_2).$$

$A \rtimes H$ is a $H$-module algebra itself, if we extend $\triangleleft$ on $H$ as the adjoint action: $h \triangleleft g = Sg_1 h g_2$. If $H$ is a Hopf $*$-algebra, $A$ a $H$-module $*$-algebra, then, as known, these two $*$-structures can be glued in a unique one to make $A \rtimes H$ a $*$-algebra itself.

**Theorem 1** [6] Let $H$ be a Hopf algebra, $A$ a right $H$-module algebra. If there exists a “realization” $\tilde{\varphi}$ of $A \rtimes H$ within $A$ acting as the identity on $A$, i.e. an algebra map

$$\tilde{\varphi} : A \rtimes H \to A, \quad \tilde{\varphi}(a) = a,$$

then $\tilde{\zeta}(g) := g_1 \tilde{\varphi}(Sg_2)$ defines an injective algebra map $\tilde{\zeta} : H \to A \rtimes H$ such that

$$[\tilde{\zeta}(g), A] = 0$$

for any $g \in H$. Moreover $A \rtimes H = \tilde{\zeta}(H)A$. Consequently, the center of the cross-product algebra $A \rtimes H$ is given by $Z(A \rtimes H) = Z(A) \tilde{\zeta}(Z(H))$, and if $H_c, A_c$ are Cartan subalgebras of $H$ and $A$ respectively, then $A_c \tilde{\zeta}(H_c)$ is a Cartan subalgebra of $A \rtimes H$. Finally, if $\tilde{\varphi} : A \rtimes H \to A$ is a $*$-algebra map, then also $\tilde{\zeta} : H \to \tilde{C}$ is.
The equality $\mathcal{A} \rtimes H = \tilde{\zeta}(H) \mathcal{A}$ means that $\mathcal{A} \rtimes H$ is equal to a product $\mathcal{A} H'$, where $H' \equiv \tilde{\zeta}(H) \subset \mathcal{A} \rtimes H$ is a subalgebra isomorphic to $H$ and commuting with $\mathcal{A}$, i.e. $\mathcal{A} \rtimes H$ is isomorphic to the ordinary tensor product algebra $H \otimes \mathcal{A}$. In other words, if $\{a_I\}, \{g_J\}$ are resp. sets of generators of $\mathcal{A}, H$, then $\{a_I\} \cup \{\tilde{\zeta}(g_J)\}$ is a more manageable set of generators of $\mathcal{A} \rtimes H$ than $\{a_I\} \cup \{g_J\}$. The other statements allow to determine Casimirs and complete sets of commuting observables, key ingredients to develop representation theory.

Well-known examples of maps (3) are the “vector field” or the “Jordan-Schwinger” realizations of the UEA $U\mathfrak{g}$’s ($\mathfrak{g}$ being a Lie algebra), where $\mathcal{A}$ is resp. the Heisenberg algebra on a $\mathfrak{g}$-covariant space or a $\mathfrak{g}$-covariant Clifford algebra. In the case of e.g. the Heisenberg algebra on the Euclidean space $\mathbb{R}^3$, the well-known realization of the three generators $J^{ij}, (i \neq j)$ of $\mathfrak{g} = \mathfrak{so}(3)$ as “vector fields” (namely homogeneous first order differential operators)

$$\tilde{\phi}(J^{jk}) := x^j p^k - x^k p^j$$

gives nothing but the orbital angular momentum operator in 1-particle quantum mechanics ($x_i, p^i \equiv -i\partial^i$ denote the position and momentum components respectively). Then

$$\tilde{\zeta}(J^{jk}) = J^{jk} - \tilde{\phi}(J^{jk}) = J^{jk} - (x^j p^k - x^k p^j),$$

gives the difference between the total and the orbital angular momentum, i.e. the “intrinsic” angular momentum (or “spin”), which indeed commutes with the $x^i, p^i$’s. Maps $\tilde{\phi}$ have been determined [2, 5] also for a number of $U_q \mathfrak{g}$-covariant, i.e. quantum group covariant, deformed Heisenberg (or Clifford) algebras. So the theorem is immediately applicable to them.

No map $\tilde{\phi}$ can exist if we take as $\mathcal{A}$ just the space on which the Heisenberg algebra is built (in the previous example $\mathbb{R}^3$), since one cannot realize the non abelian algebra $\mathcal{A} \rtimes H$ in terms of the abelian $\mathcal{A}$. Surprisingly, a map $\tilde{\phi}$ may exist if we deform the algebras. For instance, in Ref. [8] a map $\tilde{\phi}$ realizing $U_q \mathfrak{so}(3)$ has been determined for $\mathcal{A}$ the $q$-deformed fuzzy sphere $\hat{S}^2_q$. To treat other examples we generalize the previous results by weakening our assumptions. Namely, we require at least that $H$ admits a Gauss decomposition

$$H = H^+ H^- = H^- H^+$$

into two Hopf subalgebras $H^+, H^-$ for each of which analogous maps $\tilde{\phi}^+, \tilde{\phi}^-$ (3) coinciding on $H^+ \cap H^-$ exist. Then Theorem 1 will apply separately to
\(\mathcal{A} \rtimes H^+\) and \(\mathcal{A} \rtimes H^-\). What about the whole \(H \ltimes \mathcal{A}\)?

**Theorem 2** [6] Under the above assumptions setting \(\tilde{\zeta}^\pm(\delta) := g^{1\pm}_1(Sg^{2\pm}_2)\) (where \(g^{\pm} \in H^\pm\) respectively) defines injective algebra maps \(\tilde{\zeta}^\pm : H^\pm \to \mathcal{A} \rtimes H^\pm\) such that \([\tilde{\zeta}^\pm(\delta), \mathcal{A}] = 0\) for any \(g^\pm \in H^\pm\). Moreover

\[
\mathcal{A} \rtimes H = \tilde{\zeta}^+(H^+) \tilde{\zeta}^-(H^-) \mathcal{A} = \tilde{\zeta}^-(H^-) \tilde{\zeta}^+(H^+) \mathcal{A}.
\]

(5)

Any \(c \in \mathcal{Z}(\mathcal{A} \rtimes H)\) can be expressed in the form

\[
c = \tilde{\zeta}^+ (c^{(1)}) \tilde{\zeta}^- (c^{(2)}) c^{(3)},\]

where \(c^{(1)} \otimes c^{(2)} \otimes c^{(3)} \in H^+ \otimes H^- \otimes \mathcal{Z}(\mathcal{A})\) and \(c^{(1)}c^{(2)}c^{(3)} \in \mathcal{Z}(H) \otimes \mathcal{Z}(\mathcal{A})\); conversely any such \(c \in \mathcal{Z}(\mathcal{A} \rtimes H)\). If \(H_c \subset H^+ \cap H^-\) and \(\mathcal{A}_c\) are Cartan subalgebras resp. of \(H\) and \(\mathcal{A}\), then \(\mathcal{A}_c \tilde{\zeta}^+(H_c) \equiv [\mathcal{A}_c \tilde{\zeta}^-(H_c)]\) is a Cartan subalgebra of \(\mathcal{A} \rtimes H\).

If \(\tilde{\varphi}^\pm\) are \(*\)-algebra map, then also \(\tilde{\zeta}^\pm : H^\pm \to \tilde{\mathcal{C}}\) are. If \(\tilde{\varphi}^\pm((\alpha^\pm)^*) = [\tilde{\varphi}^\pm(\alpha^\pm)]^* \forall \alpha^\pm \in \mathcal{A} \rtimes H^\pm\), then \(\tilde{\zeta}^\pm(\delta^\pm)^* = [\tilde{\zeta}^\pm(\delta^\pm)]^*\), with \(\delta^\pm \in H^\pm\).

As a consequence of this theorem, \(\forall g^+ \in H^+, g^- \in H^- \exists c^{(1)} \otimes c^{(2)} \otimes c^{(3)} \in \mathcal{Z}(\mathcal{A}) \otimes H^- \otimes H^+\) (depending on \(g^+, g^-\)) such that

\[
\tilde{\zeta}^+(\delta^+)\tilde{\zeta}^-(\delta^-) = c^{(1)}\tilde{\zeta}^-(c^{(2)})\tilde{\zeta}^+(c^{(3)}).
\]

(7)

These will be the ”commutation relations” between elements of \(\tilde{\zeta}^+(H^+\) and \(\tilde{\zeta}^-(H^-)\). Their form will depend on the specific algebras considered.

As an application we consider now the pair \((H, \mathcal{A})\) with \(H = U_q so(3)\) and \(\mathcal{A} = \mathbb{R}^3_q\), the (algebra of functions on) the 3-dim quantum Euclidean space, whose generators we denote resp. by \(E^+, E^-, K, K^{-1}\) and \(p^+, p^-, p^0\). In our present conventions

\[
p^0 p^\pm = q^\pm p^\pm p^0 \quad [p^+, p^-] = (1 - q^{-1}) p^0 p^0 \quad (8)
\]

\[
K E^\pm = q^\pm E^\pm K \quad [E^+, E^-]_{q^{-1}} = \frac{K^2 - 1}{q^2 - 1} \quad (9)
\]

\[
K p^0 = p^0 K, \quad K p^\pm = q^\pm p^\pm K \quad [p^0, E^\pm] = \mp p^\pm \quad (10)
\]

\[
\Delta(K) = K \otimes K \quad \Delta(E^\pm) = E^\pm \otimes K + 1 \otimes E^\pm \quad (11)
\]

\[
(p^0)^* = p^0, \quad (p^-)^* = p^+, \quad K^* = K, \quad (E^+)^* = E^-, \quad (12)
\]
where \([a, b]_w := ab - wba\). The first three relations give the algebra structure of \(R^3_q \bowtie U_q so(3)\) (this underlies the quantum group of inhomogenous transformations of \(R^3_q\)), (11) together with \(\varepsilon(E^\pm) = 0, \varepsilon(K) = 1\) the coalgebra of \(H\), (12) the \(*\)-structure corresponding to compact \(H\) and “real” \(R^3_q\) (this requires \(q \in \mathbb{R}\)). The element \(P^2 := qp^+p^- + p^0p^0 + p^-p^+\) is central, positive definite under this \(*\)-structure and real under the other one (that requires \(|q| = 1\)). We enlarge the algebra by introducing also the square root and the inverse \(P, P^{-1}\). Setting \(P = 1\) we obtain the quantum Euclidean sphere \(S^2_q\), and \(S^2_q \bowtie U_q so(3)\) so (3) can be interpreted as the algebra of observables of a quantum particle on \(S^2_q\). The maps \(\tilde{\varphi}^+, \tilde{\varphi}^-, \tilde{\zeta}^+, \tilde{\zeta}^-, \) the algebra relations among the new generators \(e^+ := \tilde{\zeta}^+(E^+), e^- := \tilde{\zeta}^-(E^-), k := \tilde{\zeta}^+(K) = \tilde{\zeta}^-(K)\), and the additional central element \(c\) of the form (6) are given by

\[
\tilde{\varphi}^\pm(K) = \eta \frac{P}{p_0} e^\pm = \frac{\eta P}{E^+p^0 + \frac{qp^-}{1-q}} e^\pm = \frac{\eta}{P} \left( E^-p^0 + \frac{p^+}{1-q} \right) (13)
\]

\[
k = K \frac{\eta p_0}{P} e^\pm = q^\pm e^\pm k \quad [e^+, e^-]_{q^{-1}} = \frac{k^2 + \eta^2}{q^2 - 1} \quad c = e^+ e^- k^{-1} \frac{k - qk^{-1}}{1 + q} + \frac{k - qk^{-1}}{(1 - q^2)^2} (15)
\]

Here \(\eta \in \mathbb{C}, \eta \neq 0\). (15) translates (7), and differs from (9) by the presence at the rhs of the “central charge” \((\eta^2 + 1)/(q^2 - 1)\). Only for \(\eta^2 = -1\) can the maps \(\tilde{\varphi}^+, \tilde{\varphi}^-\) and \(\tilde{\zeta}^+, \tilde{\zeta}^-\) be glued into maps \(\varphi^+\) and \(\tilde{\zeta}\) respectively; then the latter will be \(*\)-maps under the non-compact \(*\)-structure where \(|q| = 1\), but not under (12). In order this to happen we need to take \(\eta \in \mathbb{R}\). The \(*\)-representations of the \(e^+, e^-, k\) subalgebra for real \(q\) differ from the ones of \(U_q so(2)\) in that they are lowest-weight but not highest-weight representations, or viceversa [4].

In Ref. [7, 6] we give maps \(\tilde{\varphi}^\pm\) for the cross products \(R^N_q \bowtie U_q so(N)\) for all \(N \geq 3\).

## 2 Decoupling of braided tensor products

As known, if \(H\) is a noncocommutative Hopf algebra (e.g. a quantum group \(U_q g\)) and \(A_1, A_2\) are two (unital) \(H\)-module algebras the tensor product algebra \(A_1 \otimes A_2\) will not be in general a \(H\)-module algebra. If \(H\) is quasitriangular a \(H\)-module algebra can be obtained as the braided tensor product
algebra $\mathcal{A}^+ := \mathcal{A}_1 \otimes \mathcal{A}_2$, which is defined as follows: the vector space underlying the latter is still the tensor product of the vector spaces underlying $\mathcal{A}_1, \mathcal{A}_2$, and so we shall denote as usual $a_1 \otimes a_2$ simply by $a_1 a_2$; $\mathcal{A}_1 \mathcal{A}_2, 1_{\mathcal{A}_1} \mathcal{A}_2$ are still subalgebras isomorphic to $\mathcal{A}_1, \mathcal{A}_2$, and so one can omit to write the units, whenever they are multiplied by non-unit elements; but the “commutation relation” between $a_1 \in \mathcal{A}_1, a_2 \in \mathcal{A}_2$ are modified:

\[ a_2 a_1 = (a_1 \triangleleft R^{(1)}) (a_2 \triangleleft R^{(2)}). \] 

(16)

Here $R \equiv R^{(1)} \otimes R^{(2)} \in H^+ \otimes H^-$ (again a summation symbol at the rhs has been suppressed) denotes the so-called universal $R$-matrix or quasitriangular structure of $H \equiv [3]$, and as before $H^\pm$ denote some positive and negative Borel Hopf subalgebras of $H$. If $H = U_q \mathfrak{g}$, then in the limit $q \to 1$ $H$ becomes the cocommutative Hopf algebra $U \mathfrak{g}$ and $R \to 1 \otimes 1$. As a consequence $a_2 a_1 \to a_1 a_2$ and thus $\mathcal{A}^+$ goes to the ordinary tensor product algebra. An alternative braided tensor product $\mathcal{A}^- = \mathcal{A}_1 \overleftarrow{\otimes} \mathcal{A}_2$ can be obtained by replacing in (16) $R$ by $R^{-1}$. This is equivalent to exchanging $\mathcal{A}_1$ with $\mathcal{A}_2$.

$\mathcal{A}^+$ (as well as $\mathcal{A}^-$) is a $*$-algebra if $H$ is a Hopf $*$-algebra, $\mathcal{A}_1, \mathcal{A}_2$ are $H$-module $*$-algebras (we use the same symbol $*$ for the $*$-structure on all algebras $H, \mathcal{A}_1, \text{etc.}$), and $R^* \equiv R^{(1)*} \otimes R^{(2)*} = R^{-1}$. In the quantum group case this requires $|q| = 1$. Under the same assumptions also $\mathcal{A}_1 \triangleleft H$ is a $*$-algebra.

If $\mathcal{A}_1, \mathcal{A}_2$ represent the algebras of observables of different two quantum systems, (16) will mean that in the composite system their degrees of freedom are "coupled" to each other. But again one can "decouple" them by a transformation of generators if there exists an algebra map $\tilde{\varphi}_1^+$, or an algebra map $\tilde{\varphi}_1^-$:

**Theorem 3** [7]. Let $\{H, R\}$ be a quasitriangular Hopf algebra and $H^+, H^-$ be Hopf subalgebras of $H$ such that $R \in H^+ \otimes H^-$. Let $\mathcal{A}_1, \mathcal{A}_2$ be respectively a $H^+$- and a $H^-$-module algebra, so that we can define $\mathcal{A}^+$ as in (16), and $\tilde{\varphi}_1^+$ be a map of the type (3), so that we can define the "unbraiding" map $\chi^+: \mathcal{A}_2 \to \mathcal{A}^+$ by

\[ \chi^+(a_2) := \tilde{\varphi}_1^+(R^{(1)}) (a_2 \triangleleft R^{(2)}). \]

(17)

Then $\chi^+$ is an injective algebra map such that

\[ [\chi^+(a_2), \mathcal{A}_1] = 0, \]

(18)
namely the subalgebra $\tilde{A}_2^+ := \chi^+(A_2) \approx A_2$ commutes with $A_1$. Moreover $A^+ = A_1 \tilde{A}_2^+$. Finally, if $R^* = R^{-1}$ and $\tilde{\phi}_1^+$ is a $*$-algebra map then $\chi^+$ is, and $A_1, \tilde{A}_2^+$ are closed under $*$.

By replacing everywhere $R$ by $R_{21}^{-1}$ we obtain an analogous statement valid for $A^-, \chi^-$. Of course, we can use the above theorem iteratively to completely decouple the braided tensor product algebra of an arbitrary number $M$ of copies of $A_1$. One can also combine the two methods illustrated here to decouple the tensor factors in ‘mixed’ tensor products such as

$$ (A_1 \otimes H) \otimes^\pm A_2, \quad A_1 \otimes^\pm (A_2 \otimes H), \quad (A_1 \otimes^\pm A_2) \otimes H. \quad (19) $$

References

[1] Cerchiai B.L., Madore J., Schraml S. and J. Wess 2000, Eur. Phys. J. C 16, 169.

[2] Chu C. S. and Zumino B. 1995, Proceedings of ICGTMP XX, Toyonaka (Japan) 1994, q-alg/9502005.

[3] Drinfeld V. 1986, Proceedings of the International Congress of Mathematicians, Berkeley (USA) 1986, Vol. 1, 798.

[4] Fiore G 1995, J. Math. Phys. 36, 4363-4405; 1996, Int. J. Mod. Phys. A11, 863-886.

[5] Fiore G. 1995, Commun. Math. Phys. 169, 475-500.

[6] Fiore G. 2002, J. Phys. A: Math. Gen. 35, 657-678.

[7] Fiore G., Steinacker H. and Wess J. “Unbraiding the braided tensor product”, math/0007174, to appear in J. Math. Phys.

[8] Grosse H., Madore J., Steinacker H. 2001, J.Geom.Phys. 38, 308-342