LAPLACIANS ON PERIODIC GRAPHS WITH GUIDES

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Abstract. We consider Laplace operators on periodic discrete graphs perturbed by guides, i.e., graphs which are periodic in some directions and finite in other ones. The spectrum of the Laplacian on the unperturbed graph is a union of a finite number of non-degenerate bands and eigenvalues of infinite multiplicity. We show that the spectrum of the perturbed Laplacian consists of the unperturbed one plus the additional so-called guided spectrum which is a union of a finite number of bands. We estimate the position of the guided bands and their length in terms of geometric parameters of the graph. We also determine the asymptotics of the guided bands for guides with large multiplicity of edges. Moreover, we show that the possible number of guided bands, their length and position can be rather arbitrary for some specific periodic graphs with guides.

1. Introduction

Laplacians on periodic discrete graphs have attracted a lot of attention due to their applications to the study of electronic properties of real crystalline structures, see, e.g., [H02], [NG04] and the survey [CGPNG09]. However, the arrangement of atoms or molecules in most crystalline materials is not perfect. The regular patterns are interrupted by crystalline defects. These defects are the most important features of the engineering material and are manipulated to control its behavior.

We consider Laplace operators on periodic discrete graphs perturbed by guides (i.e., graphs which are periodic in some directions and finite in others). For example, a guide is a periodic graph embedded into a strip in the case of planar graphs. It is well known that the spectrum of discrete Laplacians on periodic graphs has a band structure with a finite number of flat bands (eigenvalues of infinite multiplicity) [HN09], [HS04], [KS14], [RR07]. The spectrum of the Laplacian on the perturbed graph consists of the spectrum of the Laplacian on the unperturbed periodic graph plus the so-called guided spectrum. The additional guided spectrum is a union of a finite number of bands and the corresponding wave-functions are mainly located along the guides. In our paper we study guided spectra of Laplacians. We describe our main goals:

• to estimate the position of guided bands and their lengths in terms of geometric parameters of graphs;
• to determine asymptotics of the guided spectrum for guides with large multiplicity of edges;
• to show that a possible number of guided bands (including flat bands), their length and positions can be rather arbitrary for some specific periodic graphs with guides.

1.1. Discrete Laplacians on periodic graphs. Let $\Gamma = (V, E)$ be a connected infinite graph, possibly having loops and multiple edges, where $V$ is the set of its vertices and $E$ is the set of its unoriented edges. From the set $E$ we construct the set $\mathcal{A}$ of oriented edges by considering each edge in $E$ to have two orientations. An edge starting at a vertex $u$ and ending...
at a vertex \( v \) from \( V \) will be denoted as the ordered pair \((u, v) \in \mathcal{A}\). Vertices \( u, v \in V \) will be called \textit{adjacent} and denoted by \( u \sim v \), if \((u, v) \in \mathcal{A}\). We define the degree \( \kappa_v \) of the vertex \( v \in V \) as the number of all edges from \( \mathcal{A} \) starting at \( v \). We consider graphs with uniformly bounded degrees.

Let \( \ell^2(V) \) be the Hilbert space of all functions \( f : V \to \mathbb{C} \) equipped with the norm
\[
\|f\|_{\ell^2(V)}^2 = \sum_{v \in V} |f(v)|^2 < \infty.
\]

We define the discrete Laplacian (i.e., the combinatorial Laplace operator) \( \Delta \) on \( \ell^2(V) \) by
\[
(\Delta f)(v) = \sum_{e=(v,u) \in \mathcal{A}} (f(v) - f(u)), \quad f = (f(v))_{v \in V} \in \ell^2(V),
\]
where the sum is taken over all oriented edges starting at the vertex \( v \in V \). It is well known, see, e.g., \cite{M91}, that \( \Delta \) is self-adjoint and its spectrum satisfies: the point 0 belongs to the spectrum \( \sigma(\Delta) \) containing in \([0, 2\kappa_+]\), i.e.,
\[
0 \in \sigma(\Delta) \subset [0, 2\kappa_+], \quad \text{where} \quad \kappa_+ = \sup_{v \in V} \kappa_v < \infty. \tag{1.2}
\]

We consider a \( \mathbb{Z}^d \)-periodic graph \( \Gamma_0 = (V_0, \mathcal{E}_0) \), i.e., a graph satisfying the following conditions:

1) \( \Gamma_0 \) is equipped with an action of the free abelian group \( \mathbb{Z}^d \);

2) the quotient graph \( \Gamma_\ast = (V_\ast, \mathcal{E}_\ast) = \Gamma_0/\mathbb{Z}^d \) is finite.

We assume that the graphs are embedded into Euclidean space, since in many applications such a natural embedding exists. For example, in the tight-binding approximation real crystalline structures are modeled as discrete graphs embedded into \( \mathbb{R}^d \) \((d=2,3)\) and consisting of vertices (points representing positions of atoms) and edges (representing chemical bonding of atoms), by ignoring the physical characters of atoms and bonds that may be different from one another. But all results of the paper stay valid in the case of abstract periodic graphs (without the assumption of graph embedding into Euclidean space).

For a periodic graph \( \Gamma_0 \) embedded into the space \( \mathbb{R}^d \), the quotient graph \( \Gamma_0/\mathbb{Z}^d \) is a graph on the \( \tilde{d} \)-dimensional torus \( \mathbb{R}^d/\mathbb{Z}^d \). Due to the definition, the graph \( \Gamma_0 \) is invariant under translations through vectors \( a_1, \ldots, a_{\tilde{d}} \) which generate the group \( \mathbb{Z}^d \):
\[
\Gamma_0 + a_s = \Gamma_0, \quad \forall s \in \mathbb{N}_{\tilde{d}}.
\]

Here and below for each integer \( m \) the set \( \mathbb{N}_m \) is given by
\[
\mathbb{N}_m = \{1, \ldots, m\}. \tag{1.3}
\]

We will call the vectors \( a_1, \ldots, a_{\tilde{d}} \) the periods of the graph \( \Gamma_0 \). In the space \( \mathbb{R}^d \) we consider a coordinate system with the origin at some point \( O \) and with the basis \( a_1, \ldots, a_{\tilde{d}} \). Below the coordinates of all graph vertices will be expressed in this coordinate system.

Let \( A \) be a self-adjoint operator, we denote by \( \sigma(A) \), \( \sigma_{ac}(A) \), \( \sigma_p(A) \), and \( \sigma_{fb}(A) \) its spectrum, absolutely continuous spectrum, point spectrum (eigenvalues of finite multiplicity), and the flat band spectrum (eigenvalues of infinite multiplicity), respectively.

We consider the Laplacian defined by (1.1) on the periodic graph \( \Gamma_0 \) as an \textit{unperturbed operator} and denote it by \( \Delta_0 \). It is well known that the spectrum \( \sigma(\Delta_0) \) of the Laplacian on
periodic graphs is a union of $\nu$ spectral bands $\sigma_n(\Delta_0)$:

$$\sigma(\Delta_0) = \bigcup_{n=1}^{\nu} \sigma_n(\Delta_0) = \sigma_{ac}(\Delta_0) \cup \sigma_{fb}(\Delta_0),$$

(1.4)

where $\nu = \#V_*$ is the number of vertices of the quotient graph $\Gamma_*$, the absolutely continuous spectrum $\sigma_{ac}(\Delta_0)$ consists of non-degenerate bands $\sigma_n(\Delta_0)$. Note that each flat band is a degenerate band. The spectrum $\sigma(\Delta_0)$ is a subset of the interval $[0, \varrho]$:

$$\sigma(\Delta_0) \subset [0, \varrho], \quad \inf \sigma(\Delta_0) = 0, \quad \varrho = \sup \sigma(\Delta_0).$$

(1.5)

1.2. Results overview. There are results about spectral properties of the Schrödinger operator $H_0 = \Delta_0 + W$ with a periodic potential $W$. The decomposition of the operator $H_0$ into a constant fiber direct integral was obtained in [HN09], [HS04], [RR07] without an exact form of fiber operators and in [KS14], [KS17] with an exact form of fiber operators. In particular, this yields the band-gap structure of the spectrum of the operator $H_0$. In [GKT93] the authors described different properties of Schrödinger operators with periodic potentials on the lattice $\mathbb{Z}^2$, the simplest $\mathbb{Z}^2$-periodic graph. In [LP08], [KS15] the positions of the spectral bands of the Laplacians were estimated in terms of eigenvalues of the operator on finite graphs (the so-called eigenvalue bracketing). The estimate of the total length of all bands $\sigma_n(H_0)$ given by

$$\sum_{n=1}^{\nu} |\sigma_n(H_0)| \leq 2\beta,$$

(1.6)

was obtained in [KS14], where $\beta = \#E_* + 1 - \nu$ is the so-called Betti number, $\#E_*$ is the number of edges of the quotient graph $\Gamma_*$. Moreover, a global variation of the Lebesgue measure of the spectrum and a global variation of the gap-length in terms of potentials and geometric parameters of the graph were determined. Note that the estimate (1.6) also holds true for magnetic Schrödinger operators with periodic magnetic and electric potentials (see [KS17]). Estimates of the Lebesgue measure of the spectrum of $H_0$ in terms of eigenvalues of Dirichlet and Neumann operators on a fundamental domain of the periodic graph were described in [KS15]. Estimates of effective masses, associated with the ends of each spectral band, in terms of geometric parameters of the graphs were obtained in [KS16]. Moreover, in the case of the bottom of the spectrum two-sided estimates on the effective mass in terms of geometric parameters of the graphs were determined. The proof of all these results in [KS14]-[KS17] is based on Floquet theory and the exact form of fiber Schrödinger operators from [KS14], [KS17]. The spectra of the discrete Schrödinger operators on graphene nanotubes and nano-ribbons in external fields were discussed in [KK10], [KK10a]. The spectrum of discrete magnetic Laplacians on some planar graphs (the hexagonal lattice, the kagome lattice and so on) was described in [HKR16] and see the references therein.

Discrete Laplacians for some class of periodic graphs with compact perturbations including the square, triangular, diamond, kagome lattices were discussed in [AIM14]. Laplacians on periodic graphs with non-compact perturbations and the stability of their essential spectrum were considered in [SS15]. The spectrum of Laplacians on the lattice $\mathbb{Z}^d$ with pendant edges was studied in [S13]. In the paper [KS16a], the authors considered Schrödinger operators with periodic potentials on periodic discrete graphs perturbed by so-called guided potentials, which are periodic in some directions and finitely supported in others. They described some properties of the additional guided spectrum. We remark that the case of guided potentials...
is simpler than the case of periodic graphs with guides and helps us to understand better the properties of the guided spectrum in the case of periodic graphs with guides. It is important that in the case of guided potentials all operators act in the same space. But in the case of periodic graphs with guides this is not true. Note that line defects on the lattice were considered in [C12], [Ku14], [Ku16], [OA12].

Scattering theory for self-adjoint Schrödinger operators with decreasing potentials was investigated in [BS99], [IK12] (for the lattice) and in [PR16] (for periodic graphs). Inverse scattering theory with finitely supported potentials was considered in [IK12] for the case of the lattice $\mathbb{Z}^d$ and in [A12] for the case of the hexagonal lattice. The absence of eigenvalues embedded in the essential spectrum of the operators was discussed in [IM14], [V14]. Trace formulae and global eigenvalues estimates for Schrödinger operators with complex decaying potentials on the lattice were obtained in [KL16]. The Cwikel-Lieb-Rosenblum type bound for the discrete Schrödinger operator on $\mathbb{Z}^d$ was computed in [Ka08], [RS09]. Finally, we note that different properties of Schrödinger operators on graphs were considered in [G15], [Sh98].

2. Main results

2.1. The unperturbed case: periodic graphs. We define the infinite fundamental graph $C_0$ of the $\mathbb{Z}^d$-periodic graph $\Gamma_0$ by

$$C_0 = (V_0^c, E_0^c) = \Gamma_0/\mathbb{Z}^d,$$

where $V_0^c$ is its vertex set and $E_0^c$ is its set of unoriented edges. Remark that the graph $C_0$ is a graph on the cylinder $\mathbb{R}^d/\mathbb{Z}^d$ and is $\mathbb{Z}^{d-d}$-periodic. We also call the fundamental graph $C_0$ a discrete cylinder or just a cylinder. We identify the vertices of the cylinder $C_0$ with the vertices of the periodic graph $\Gamma_0$ from the strip $S = [0, 1)^d \times \mathbb{R}^{d-d}$. We will call this infinite vertex set a fundamental vertex set of $\Gamma_0$ and denote it by the same symbol $V_0^c$:

$$V_0^c = V_0 \cap S,$$

where $V_0 = [0, 1)^d \times \mathbb{R}^{d-d}$.

Edges of the periodic graph $\Gamma_0$ connecting the vertices from the fundamental vertex set $V_0^c$ with the vertices from $V_0 \setminus V_0^c$ will be called bridges. Bridges always exist and provide the connectivity of the periodic graph. The set of all bridges of the graph $\Gamma_0$ we denote by $\mathcal{B}$.

2.2. The perturbed case: periodic graphs with guides. We define the union of two graphs $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ is a graph $G$ given by

$$G = G_0 \cup G_1 = (\bar{V}, \bar{E}), \quad \bar{V} = V_0 \cup V_1, \quad \bar{E} = E_0 \cup E_1.$$

Now we define a periodic graph with guides. Let $\Gamma_1 = (V_1, E_1)$ be a finite graph, possibly not connected, such that all vertices of $\Gamma_1$ are contained in the strip $S$ and the graph $\Gamma_0 \cup \Gamma_1$ is connected. A $\mathbb{Z}^d$-periodic graph

$$\Gamma_1^g = \bigcup_{m \in \mathbb{Z}^d} (\Gamma_1 + m)$$

will be called a guide with the fundamental graph $\Gamma_1$. We define a perturbed graph $\Gamma$ as a union of the unperturbed periodic graph $\Gamma_0$ and the perturbation $\Gamma_1^g$:

$$\Gamma = \Gamma_0 \cup \Gamma_1^g.$$

We will call the graph $\Gamma$ a periodic graph with a guide $\Gamma_1^g$ or a perturbed graph.
Due to the definition (2.3) of the perturbed graph \( \Gamma \), the \textit{perturbed} cylinder \( \mathcal{C} = \Gamma / \mathbb{Z}^d = (V^c, \mathcal{E}^c) \) for \( \Gamma \) is a union of the unperturbed cylinder \( \mathcal{C}_0 = (V_0^c, \mathcal{E}_0^c) \) and the finite graph \( \Gamma_1 = (V_1, \mathcal{E}_1) \):

\[
\mathcal{C} = \mathcal{C}_0 \cup \Gamma_1,
\]

i.e.,

\[
V^c = V_0^c \cup V_1, \quad \mathcal{E}^c = \mathcal{E}_0^c \cup \mathcal{E}_1, \quad \mathcal{A}^c = \mathcal{A}_0^c \cup \mathcal{A}_1,
\]

where \( \mathcal{A}^c, \mathcal{A}_0^c \) and \( \mathcal{A}_1 \) are the sets of all doubled oriented edges of \( \mathcal{C}, \mathcal{C}_0 \) and \( \Gamma_1 \), respectively.

**Figure 1.** a) The square lattice \( \mathbb{L}^2 \); the vertices from the set \( V_0^c \) are big black points; the strip \( S \) is shaded; b) the unperturbed cylinder \( \mathcal{C}_0 = \mathbb{L}^2 / \mathbb{Z} \) (the edges of the strip are identified); c) a perturbation \( \Gamma_1 \) with two connected components; d) the perturbed square lattice \( \Gamma = \mathbb{L}^2 / \Gamma_1^q \); e) the perturbed cylinder \( \mathcal{C} = \Gamma / \mathbb{Z} \) (the edges of the strip are identified).

**Example.** For the square lattice \( \mathbb{L}^2 \) with the periods \( a_1, a_2 \), see Fig.1a, the unperturbed cylinder \( \mathcal{C}_0 = \mathbb{L}^2 / \mathbb{Z} = (V_0^c, \mathcal{E}_0^c) \) is shown in Fig.1b. The vertices from the set \( V_0^c \) are big black points in Fig.1a. A perturbation \( \Gamma_1 \), the perturbed square lattice \( \Gamma = \mathbb{L}^2 / \Gamma_1^q \) and the perturbed cylinder \( \mathcal{C} = \Gamma / \mathbb{Z} \) are shown in Fig.1c,d,e.

### 2.3. Floquet decomposition and the spectrum of perturbed Laplacians.

We describe the basic spectral properties of Laplacians on periodic graphs with guides.

**Proposition 2.1.** i) The Laplacian \( \Delta \) on a perturbed graph \( \Gamma \) has the following decomposition into a constant fiber direct integral for some unitary operator \( U : \ell^2(V) \to \mathcal{H} : \)

\[
\mathcal{H} = \int_{\mathbb{T}^d} \ell^2(V^c) \frac{d\vartheta}{(2\pi)^d}, \quad U \Delta U^{-1} = \int_{\mathbb{T}^d} \Delta(\vartheta) \frac{d\vartheta}{(2\pi)^d},
\]

where \( \mathbb{T}^d = \mathbb{R}^d / (2\pi \mathbb{Z})^d \) and the fiber Laplacian \( \Delta(\vartheta) \) on the fiber space \( \ell^2(V^c) \) is given by

\[
(\Delta(\vartheta)f)(v) = \sum_{e=(v, u) \in \mathcal{A}^c} \left( f(v) - e^{i(\tau(e), \vartheta)} f(u) \right), \quad v \in V^c, \quad f \in \ell^2(V^c).
\]

Here \( \tau(e) \in \mathbb{Z}^d \) is the index of the edge \( e \in \mathcal{A}^c \) defined by \( (3.3), (3.4) \). \( V^c \) and \( \mathcal{A}^c \) are the vertex set and the set of oriented edges of the cylinder \( \mathcal{C} \), respectively; \( \langle \cdot, \cdot \rangle \) is the inner product in \( \mathbb{R}^d \).

ii) For each \( \vartheta \in \mathbb{T}^d \) the spectrum of the fiber operator \( \Delta(\vartheta) \) has the form

\[
\sigma(\Delta(\vartheta)) = \sigma_{ac}(\Delta(\vartheta)) \cup \sigma_{fb}(\Delta(\vartheta)) \cup \sigma_p(\Delta(\vartheta)),
\]

\[
\sigma_{ac}(\Delta(\vartheta)) = \sigma_{ac}(\Delta_0(\vartheta)), \quad \sigma_{fb}(\Delta(\vartheta)) = \sigma_{fb}(\Delta_0(\vartheta)),
\]

where \( \sigma_{ac} \) and \( \sigma_{fb} \) are the spectra of the constant and finite fiber Laplacians, respectively.
Proposition 2.2. Let $\vartheta \in \mathbb{T}^d$ be a periodic graph. Assume that $\zeta$ is an eigenvalue of the Laplacian $\Delta_1$ on a finite graph $\Gamma_1$ with an eigenfunction $f \in L^2(V_1)$ equal to zero on $V_0^c \cap V_1$, i.e.,

$$f(v) = 0, \quad \forall v \in V_{01} = V_0^c \cap V_1.$$  

Then $\{\zeta\}$ is a guided flat band of the Laplacian $\Delta_1$ on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_1$. 

**Example.** We consider the perturbed square lattice $\Gamma = \mathbb{Z}^2 \cup \Gamma_1^g$ shown in Fig.2a. For each $\vartheta \in \mathbb{T} = (-\pi, \pi]$ the spectrum of the fiber Laplacian $\Delta(\vartheta)$ has the form

$$\sigma(\Delta(\vartheta)) = \sigma_{ac}(\Delta(\vartheta)) \cup \sigma_p(\Delta(\vartheta)),$$

where $\sigma_{ac}(\Delta(\vartheta))$ consists of two eigenvalues $\lambda_1(\vartheta)$ and $\lambda_2(\vartheta) = 3$, see Fig.2d.

The spectrum of the Laplacian $\Delta$ on $\Gamma$ has the form

$$\sigma(\Delta) = \sigma(\Delta_0) \cup \sigma(\Delta), \quad \sigma(\Delta_0) = \{0, 8\},$$

where the guided spectrum $\sigma(\Delta)$ is given by (see Fig.2d and details in Proposition 5.3)

$$\sigma(\Delta) = \sigma_{ac}(\Delta) \cup \sigma_{fb}(\Delta), \quad \sigma_{ac}(\Delta) = \sigma_1(\Delta) = \lambda_1(\mathbb{T}) \approx [10.6; 13.9], \quad \sigma_{fb}(\Delta) = \sigma_2(\Delta) = \lambda_2(\mathbb{T}) = \{3\}.$$
The Laplacian $\Delta_1$ on the finite graph $\Gamma_1$ has the eigenvalue $\lambda = 3$ with an eigenfunction $f$ such that $f(0) = 0$. Then, due to Proposition 2.2, \{3\} is a guided flat band of $\Delta$ on the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma^q_1$. Here we remark that we do not know an example of a Schrödinger operator with a guided potential on a periodic graph having a guided flat band.

2.4. ** Estimates of guided bands.** We consider the guided bands from (2.11) (or their parts) above the spectrum of the unperturbed Laplacian $\Delta_0$:

$$s_j'(\Delta) = s_j(\Delta) \cap [\varrho, +\infty) \neq \emptyset, \quad j = 1, \ldots, N_g, \quad N_g \leq N;$$

(2.14)

recall that $\varrho = \sup \sigma(\Delta_0)$. The Laplacian $\Delta_1$ on the finite graph $\Gamma_1 = (V_1, \mathcal{E}_1)$ has the eigenvalue 0 of multiplicity $c_{\Gamma_1}$ and $p$ positive eigenvalues $\zeta_j$ labeled by

$$0 < \zeta_p \leq \ldots \leq \zeta_2 \leq \zeta_1, \quad p = \nu_1 - c_{\Gamma_1}, \quad \nu_1 = \#V_1,$$

(2.15)

counting multiplicity, where $c_{\Gamma_1}$ is the number of connected components of the graph $\Gamma_1$.

Proposition 2.1 and the standard perturbation theory give the estimates of the position of the bands $s_j'(\Delta)$ and their number $N_g$ by (for more details see Corollary 4.1)

$$s_j'(\Delta) \subset [\zeta_j, \zeta_j + \varrho], \quad N_g \geq \#\{j \in \mathbb{N}_p : \zeta_j > \varrho\},$$

(2.16)

where $\#A$ is the number of elements of the set $A$. In particular, this yields that if the eigenvalues of $\Delta_1$ satisfy $\zeta_p > \varrho$ and $\zeta_j - \zeta_{j+1} > \varrho$ for all $j \in \mathbb{N}_{p-1}$, then the guided spectrum of the Laplacian $\Delta$ consists of exactly $p$ guided bands separated by gaps.

In order to formulate our main result we define the set $\mathcal{B}^c = \mathcal{B}/\mathbb{Z}^d$ of all bridges of the cylinder $\mathcal{C} = (V^c, \mathcal{E}^c)$ and the modified cylinder $\mathcal{C}^m = (V^c, \mathcal{E}^c \setminus \mathcal{B}^c)$, which is obtained from $\mathcal{C}$ by deleting all its bridges. We consider the Laplacian $\Delta^m$ defined by (1.11) on the modified cylinder $\mathcal{C}^m$. This Laplacian $\Delta^m$ has at most $p = \operatorname{rank} \Delta_1$ eigenvalues $\tilde{\mu}_1 \geq \tilde{\mu}_2 \geq \ldots$. Define $\mu_j$ by

$$\mu_j = \max\{\tilde{\mu}_j, \sup \sigma_{ess}(\Delta^m)\}, \quad j = 1, 2, \ldots, p.$$  

(2.17)

We estimate the position of the guided bands $s_j'(\Delta)$ defined by (2.14) in terms of the eigenvalues of the operator $\Delta^m$ and the number of bridges on the cylinder $\mathcal{C}$.
Theorem 2.3. i) Let $\Delta$ be the Laplacian on a perturbed graph $\Gamma$ and let $\mu_j$ be defined by (2.18). Then each guided band $s_j^o(\Delta)$, $j = 1, \ldots, N_g$, defined by (2.14) satisfies

$$s_j^o(\Delta) \subset [\mu_j, \mu_j + 2\beta_j], \quad \beta_j = \max_{v \in V_c} \beta_v,$$

(2.18)

where $\beta_v$ is the number of bridges on $C$ starting at the vertex $v \in V_c$.

ii) Moreover, for any $\varepsilon > 0$ there exists a perturbed graph $\Gamma$ such that each non-degenerate guided band length $|s_j^o(\Delta)| > 2\beta_j - \varepsilon$, $j = 1, \ldots, N_g$.

Remark. For most of graphs the number $\beta_+ = 1$, then the guided band length $|s_j^o(\Delta)| \leq 2$ for all $j = 1, \ldots, N_g$, but for specific graphs $\beta_+$ may be any given integer number.

Let $\Gamma_t = (V_1, E_t)$ be a finite graph obtained from the graph $\Gamma_1 = (V_1, E_1)$ considering each edge of $\Gamma_1$ to have the multiplicity $t \in \mathbb{N}$. We consider the Laplacian $\Delta_t$ acting on a perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_t^0$ and discuss the guided spectrum of $\Delta_t$ for large $t$.

Theorem 2.4. Let $\Delta_t$ be the Laplacian on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_t^0$, where $\Gamma_0$ is any periodic graph and $t \in \mathbb{N}$ is large enough. Then the guided spectrum of the Laplacian $\Delta_t$ consists of exactly $p$ guided bands separated by gaps, where $p$ is defined in (2.13), and the following statements hold true:

i) Let $\zeta_j$ for some $j \in \mathbb{N}_p$ be a simple positive eigenvalue of the Laplacian $\Delta_1$ on the graph $\Gamma_1$ with a normalized eigenfunction $f_j \in \ell^2(V_t)$.

- If $f_j = 0$ on $V_0 = V_0^c \cap V_1$, then $\{t\zeta_j\}$ is a guided flat band of the Laplacian $\Delta_t$.

- If $f_j \neq 0$ on $V_0$, then the guided band $s_j(\Delta_t) = [\lambda_j^-(t), \lambda_j^+(t)]$ satisfies

$$\lambda_j^+(t) = t\zeta_j + W_j^+ + O(1/t), \quad |s_j(\Delta_t)| = W_j^* + O(1/t),$$

(2.19)

as $t \to \infty$, where

$$W_j^- = \min_{\vartheta \in \ell^d} W_j(\vartheta), \quad W_j^+ = \max_{\vartheta \in \ell^d} W_j(\vartheta),$$

$$W_j^* = W_j^+ - W_j^-, \quad W_j^* \leq 2\beta_0,$$

(2.20)

for some function $W_j$ defined by the formula (4.13). Here $\beta_0$ is the number of all oriented bridges connecting the vertices from $V_0$ on the cylinder $C$.

ii) In particular, if the set $V_0$ consists of one vertex $v$, then

$$f_j^2(v)\beta_0 \leq W_j^* \leq 2f_j^2(v)\beta_0.$$

(2.21)

Moreover, $W_j^* = 0$ iff $\beta_0 = 0$.

iii) Let all positive eigenvalues $\zeta_j$, $j \in \mathbb{N}_p$, of $\Delta_1$ be distinct. Then the Lebesgue measure $|s(\Delta_t)|$ of the guided spectrum of the Laplacian $\Delta_t$ satisfies

$$|s(\Delta_t)| = \sum_{j=1}^p W_j^* + O(1/t).$$

(2.22)

iv) In particular, if there is no bridge connecting the vertices from the set $V_0$ on the cylinder $C$, then $W_j^* = 0$ for each $j \in \mathbb{N}_p$ and the second identity in (2.19) and the formula (2.22) take the form $|s_j(\Delta_t)| = O(1/t)$, $j \in \mathbb{N}_p$, and $|s(\Delta_t)| = O(1/t)$, respectively.

Now we describe geometric properties of the guided spectrum for periodic graphs with specific guides.
Corollary 2.5. Let $\Gamma_0$ be a periodic graph with an unperturbed cylinder $C_0 = (V_0^c, E_0^c)$. Then the following statements hold true.

i) For any constant $C > 0$ there exists a finite graph $\Gamma_t$, $t \in \mathbb{N}$, such that the Lebesgue measure of the guided spectrum $s(\Delta)$ of the perturbed Laplacian $\Delta$ on $\Gamma = \Gamma_0 \cup \Gamma_t^g$ satisfies $|s(\Delta)| > C$ and all guided bands are non-degenerate.

ii) Let, in addition, there exist a vertex $v \in V_0^c$ such that there is no bridge on $C_0$ starting at $v$. Then for any small $\varepsilon > 0$ there exists a finite graph $\Gamma_t$ such that the Lebesgue measure of the guided spectrum $s(\Delta)$ of the perturbed Laplacian $\Delta$ on $\Gamma = \Gamma_0 \cup \Gamma_t^g$ satisfies $|s(\Delta)| < \varepsilon$.

iii) For any constant $\lambda_0 > 0$ there exists a finite graph $\Gamma_t$ such that the guided spectrum $s(\Delta)$ of the perturbed Laplacian $\Delta$ on $\Gamma = \Gamma_0 \cup \Gamma_t^g$ satisfies $s(\Delta) \cap (\lambda_0, +\infty) \neq \emptyset$.

iv) There exists a finite graph $\Gamma_1$ such that the guided spectrum $s(\Delta)$ of the perturbed Laplacian $\Delta$ on $\Gamma = \Gamma_0 \cup \Gamma_1^g$ has a degenerate guided band.

Thus, roughly speaking, the guided spectrum can be any set above the unperturbed spectrum. Its Lebesgue measure can be arbitrarily large or arbitrarily small.

We present the plan of our paper. In Section 3 we introduce the notion of edge indices and prove Proposition 2.1 about the decomposition of the Laplacian on periodic graphs with guides into a constant fiber direct integral. In Section 4 we prove Theorems 2.3, 2.4 and Corollary 2.5. Section 5 is devoted to properties of the guided spectrum for the square lattice with specific guides.

3. Direct integral for Laplacians on periodic graphs with guides

3.1. Edge indices. In order to give a decomposition of Laplacians on periodic graphs with guides into a constant fiber direct integral with a precise representation of fiber operators we need to define an edge index. Recall that an edge index was introduced in [KS14] and it was important to study the spectrum of Laplacians and Schrödinger operators on periodic graphs, since fiber operators are expressed in terms of edge indices (see (2.7)).

For any $v \in V$ the following unique representation holds true:

$$v = v_0 + [v], \quad v_0 \in V^c, \quad [v] \in \mathbb{Z}^d,$$

(3.1)

where $V^c$ is the fundamental vertex set of the graph $\Gamma = (V, E)$ defined by

$$V^c = V \cap S, \quad S = [0, 1)^d \times \mathbb{R}^{d-d}. $$

(3.2)

In other words, each vertex $v$ can be obtained from a vertex $v_0 \in V^c$ by the shift by a vector $[v] \in \mathbb{Z}^d$. For any oriented edge $e = (u, v) \in A$ we define the edge "index" $\tau(e)$ as the integer vector given by

$$\tau(e) = [v] - [u] \in \mathbb{Z}^d,$$

(3.3)

where, due to (3.1), we have

$$u = u_0 + [u], \quad v = v_0 + [v], \quad u_0, v_0 \in V^c, \quad [u], [v] \in \mathbb{Z}^d.$$

We note that edges connecting vertices from the fundamental vertex set $V^c$ have zero indices.

We define a surjection $f_A : A \rightarrow A^c = A/\mathbb{Z}^d$, which map each edge to its equivalence class. If $e$ is an oriented edge of the graph $\Gamma$, then there is an oriented edge $e_* = f_A(e)$ on the cylinder $C = \Gamma/\mathbb{Z}^d$. For the edge $e_* \in A^c$ we define the edge index $\tau(e_*)$ by

$$\tau(e_*) = \tau(e).$$

(3.4)
In other words, edge indices of the cylinder $C$ are induced by edge indices of the graph $\Gamma$. Edges with nonzero indices are called bridges. Edge indices, generally speaking, depend on the choice of the coordinate origin $O$ and the periods $a_1, \ldots, a_d$ of the graph $\Gamma$. But in a fixed coordinate system indices of the cylinder edges are uniquely determined by (3.4), since

$$\tau(e + m) = \tau(e), \quad \forall (e, m) \in \mathcal{A} \times \mathbb{Z}^d.$$  

We note that, due to the definition of periodic graphs with guides, 

$$\tau(e) = 0, \quad \forall e \in \mathcal{A}_1. \quad (3.5)$$

### 3.2. Direct integrals

We prove Proposition 2.1 about the decomposition of Laplacians on periodic graphs with guides into a constant fiber direct integral.

**Proof of Proposition [2.1i)** Repeating the arguments from the proof of Theorem 1.1 in [KSL14] we obtain (2.6), (2.7), where the unitary operator $U : \ell^2(\Gamma) \to \mathcal{H}$ has the form

$$(U f)(\vartheta, v) = \sum_{m \in \mathbb{Z}^d} e^{-i(m, \vartheta)} f(v + m), \quad (\vartheta, v) \in \mathbb{T}^d \times \mathbb{V}^c, \quad f \in \ell^2(\Gamma). \quad (3.6)$$

The Hilbert space $\mathcal{H}$ defined in (2.6) is equipped with the norm $\|g\|_{\mathcal{H}}^2 = \int_{\mathbb{T}^d} \|g(\vartheta, \cdot)\|_{\ell^2(\mathbb{V}^c)}^2 \frac{d\vartheta}{(2\pi)^d}$, where the function $g(\vartheta, \cdot) \in \ell^2(\mathbb{V}^c)$ for almost all $\vartheta \in \mathbb{T}^d$. 

In order to prove the next item of Proposition 2.1 we need the following lemma.

**Lemma 3.1.** Let $P$ and $P_1$ be the orthogonal projections of $\ell^2(\mathbb{V}^c)$ onto the subspaces $\ell^2(V_0^c)$ and $\ell^2(V_1^c)$, respectively. Then each fiber Laplacian $\Delta(\vartheta), \vartheta \in \mathbb{T}^d$, defined by (2.7) has the following decomposition:

$$\Delta(\vartheta) = P\Delta_0(\vartheta)P + P_1\Delta_1 P_1, \quad (3.7)$$

where $\Delta_0(\vartheta)$ is the fiber operator for the unperturbed Laplacian $\Delta_0$ on the periodic graph $\Gamma_0$, $\Delta_1$ is the Laplacian on the finite graph $\Gamma_1 = (V_1, E_1)$.

**Proof.** The Laplacian $\Delta_0$ on the unperturbed periodic graph $\Gamma_0 = (V_0, E_0)$ has a decomposition into a constant fiber direct integral for some unitary operator $U_0 : \ell^2(V_0) \to \mathcal{H}_0$:

$$\mathcal{H}_0 = \int_{\mathbb{T}^d} \ell^2(V_0^c) \frac{d\vartheta}{(2\pi)^d}, \quad U_0 \Delta_0 U_0^{-1} = \int_{\mathbb{T}^d} \Delta_0(\vartheta) \frac{d\vartheta}{(2\pi)^d}, \quad (3.8)$$

where the fiber Laplacian $\Delta_0(\vartheta)$ acts on the fiber space $\ell^2(V_0^c)$ and is given by

$$(\Delta_0(\vartheta)f_0)(v) = \sum_{e=(u, v) \in \mathcal{A}_0^c} (f_0(u) - e^{i(\vartheta(e), \vartheta)}) f_0(u), \quad v \in V_0^c, \quad f_0 \in \ell^2(V_0^c). \quad (3.9)$$

For each $f \in \ell^2(\mathbb{V}^c)$ we have

$$\langle (P\Delta_0(\vartheta)P + P_1\Delta_1 P_1)f, f \rangle_{\mathbb{V}^c} = \langle \Delta_0(\vartheta)Pf, Pf \rangle_{V_0^c} + \langle \Delta_1 P_1 f, P_1 f \rangle_{V_1}$$

$$= \sum_{v \in V_0^c} \langle \Delta_0(\vartheta)Pf \rangle(v) \overline{f}(v) + \sum_{v \in V_1} \langle \Delta_1 P_1 f \rangle(v) \overline{f}(v) = \sum_{v \in V_0^c \setminus V_0^c} \langle \Delta_0(\vartheta)Pf \rangle(v) \overline{f}(v)$$

$$+ \sum_{v \in V_1 \setminus V_0^c} \langle \Delta_1 P_1 f \rangle(v) \overline{f}(v) + \sum_{v \in V_0^c} \langle (\Delta_0(\vartheta)P + \Delta_1 P_1)f \rangle(v) \overline{f}(v),$$
where \( \langle \cdot, \cdot \rangle_V \) denotes the inner product in \( \ell^2(V) \). Substituting the definitions \( (1.1) \) and \( (3.9) \) of the spectrum of the unperturbed fiber Laplacian \( \Delta_0 \) and the fiber Laplacian \( \Delta_0(\vartheta) \) into this formula and using the identities \( (2.5), (3.5) \) and \( (2.7) \), we obtain

\[
\langle (P\Delta_0(\vartheta)P + P_1\Delta_1P_1)f, f \rangle_{V^c} = \sum_{v \in V_0^c} \sum_{e = (v, u) \in \mathcal{A}_0^c} (f(v) - e^{i(\tau(e), \vartheta)} f(u)) \overline{f(v)} + \sum_{v \in V_0^c} \sum_{(v, u) \in \mathcal{A}_1} (f(v) - f(u)) \overline{f(v)}
\]

which implies \( (3.7) \).

**Proof of Proposition 2.1.ii)** For each \( \vartheta \in \mathbb{T}^d \) the unperturbed fiber Laplacian \( \Delta_0(\vartheta) \) is \( \mathbb{Z}^{d-d} \)-periodic. Then, using standard arguments (see Theorem XIII.85 in [RS78]), we obtain that the spectrum of \( \Delta_0(\vartheta) \) has the form

\[
\sigma(\Delta_0(\vartheta)) = \sigma_{ac}(\Delta_0(\vartheta)) \cup \sigma_{fb}(\Delta_0(\vartheta)).
\]

Since the graph \( \Gamma_1 \) is finite, for each \( \lambda \in \sigma_{fb}(\Delta_0(\vartheta)) \) there exists a corresponding eigenfunction \( f \in \ell^2(V_0^c) \) with a finite support (see, e.g., Theorem 4.5.2 in [BK13]) not intersecting with \( V_1 \). Due to \( (3.7) \), \( (f, 0) \in \ell^2(V^c) \) is an eigenfunction of \( \Delta(\vartheta) \) with the same finite support and the same eigenvalue \( \lambda \). Thus, \( \lambda \in \sigma_{fb}(\Delta(\vartheta)) \) and vice versa. Since the operator \( \Delta_1 \) has finite rank \( p \), where \( p \) is defined in \( (2.10) \), for each \( \vartheta \in \mathbb{T}^d \) the spectrum \( \sigma(\Delta(\vartheta)) \) of the fiber Laplacian \( \Delta(\vartheta) \) is given by \( (2.8) \), where \( \sigma_{ac}(\Delta(\vartheta)), \sigma_{fb}(\Delta(\vartheta)) \) satisfy \( (2.9) \) and \( \sigma_p(\Delta(\vartheta)) \) consists of \( N_\vartheta \leq p \) eigenvalues \( (2.10) \).

4. **Proof of the main results**

In this section we prove Theorem 2.3 about the position of guided bands and Theorem 2.4 about the asymptotics of the guided bands for guides with large multiplicity of their edges. We prove Corollary 2.5 about geometric properties of the guided spectrum for periodic graphs with specific guides.

4.1. **Estimates for the guided spectrum.** Denote by \( m_\pm(\vartheta) \) the upper and lower endpoints of the spectrum of the unperturbed fiber Laplacian \( \Delta_0(\vartheta) \):

\[
m_-(\vartheta) = \inf \sigma(\Delta_0(\vartheta)), \quad m_+(\vartheta) = \sup \sigma(\Delta_0(\vartheta)).
\]

Then \( (1.5) \) yields

\[
\min_{\vartheta \in \mathbb{T}^d} m_-(\vartheta) = 0, \quad \max_{\vartheta \in \mathbb{T}^d} m_+(\vartheta) = \varrho.
\]

We need a simple estimate for eigenvalues of bounded self-adjoint operators [RS78]: Let \( A, B \) be bounded self-adjoint operators in a Hilbert space \( \mathcal{H} \) and let \( \lambda_j(A) = \max \{ \lambda_j(A), \sup \sigma_{ess}(A) \} \), \( j = 1, 2, \ldots \), where \( \tilde{\lambda}_1(A) \geq \tilde{\lambda}_2(A) \geq \ldots \) are the eigenvalues of \( A \). Then

\[
\lambda_j(A) + \inf \sigma(B) \leq \lambda_j(A + B) \leq \lambda_j(A) + \sup \sigma(B), \quad j = 1, 2, 3, \ldots
\]
The following simple corollary about the position of the guided bands $s_j^\vartheta(\Delta)$ defined by (2.14) is a direct consequence of Proposition 2.1.

**Corollary 4.1.** Let $\Delta$ be the Laplacian on a perturbed graph $\Gamma$ and let $\varrho = \sup \sigma(\Delta_0)$. Then each guided band $s_j^\vartheta(\Delta)$, $j = 1, \ldots, N_\varrho$, and their number $N_\varrho$ satisfy

$$s_j^\vartheta(\Delta) \subset [\zeta_j, \zeta_j + \varrho],$$

$$N_\varrho \geq \# \{ j \in \mathbb{N}_p : \zeta_j > \varrho \},$$

where $\zeta_1 \geq \ldots \geq \zeta_p$ are the positive eigenvalues of the Laplacian $\Delta_1$ and $p = \text{rank} \Delta_1$.

**Proof.** Each fiber Laplacian $\Delta(\vartheta)$, $\vartheta \in T^d$, is given by (3.7). Then, due to (4.1)–(4.3), each eigenvalue $\lambda_j(\vartheta)$ of $\Delta(\vartheta)$ above its essential spectrum satisfy

$$\zeta_j \leq \lambda_j(\vartheta) \leq \zeta_j + m_+(\vartheta) \leq \zeta_j + \varrho,$$

which yields (4.4). Let $\zeta_j > \varrho$ for some $j = 1, \ldots, p$. Then, due to (4.6), $\lambda_j(\vartheta) > \varrho$ for all $\vartheta \in T^d$. Thus, $\lambda_j$ creates the guided band $s_j^\vartheta(\Delta) = \lambda_j(T^d)$. This yields (4.5).

**Remark.** It is well known, see, e.g., [F73], that the positive eigenvalues $\zeta_1 \geq \ldots \geq \zeta_p$ of the Laplacian $\Delta_1$ on a finite graph $\Gamma_1 = (V_1, E_1)$ satisfy

$$\frac{\nu_1}{\nu_1 - 1} \max_{v \in V_1} \zeta_v \leq \zeta_1 \leq \max_{u,v \in V_1} (\zeta_u + \zeta_v),$$

$$\min_{u,v \in V_1} (\zeta_u + \zeta_v) - (\nu_1 - 2) \leq \zeta_p \leq \frac{\nu_1}{\nu_1 - 1} \min_{v \in V_1} \zeta_v,$$

where $\nu_1 = \# V_1$, $\zeta_v$ is the degree of the vertex $v \in V_1$ on $\Gamma_1$. From these estimates and (4.4) it follows that increasing the degree of at least one vertex of the graph $\Gamma_1$ removes the first guided band $s_1^\vartheta(\Delta)$ arbitrarily far to the right. Increasing the degrees of all vertices of $\Gamma_1$ removes all guided bands arbitrarily far to the right.

**Proof of Theorem 2.3.** i) We rewrite the fiber Laplacian $\Delta(\vartheta)$, $\vartheta \in T^d$, defined by (2.7) in the form:

$$\Delta(\vartheta) = \Delta^m + \Delta_\vartheta(\vartheta),$$

$$\left( \Delta_\vartheta(\vartheta)f \right)(v) = \sum_{e=(u,v) \in A_\vartheta} (f(v) - e^{i(\tau(e),\vartheta)} f(u)), \quad v \in V^c,$$

where $\tau(e) \in \mathbb{Z}^d$ is the index of the edge $e \in A^c$ defined by (3.3), (3.4). Each operator $\Delta_\vartheta(\vartheta)$, $\vartheta \in T^d$, is the magnetic Laplacian on the graph $\mathcal{C}_\vartheta = (V^c, B^\vartheta)$ and the degree of each vertex $v \in V^c$ on $\mathcal{C}_\vartheta$ is equal to the number $\beta_v$ of all bridges starting at $v$. Then the spectrum $\sigma(\Delta_\vartheta(\vartheta)) \subset [0, 2\beta_+]$ for each $\vartheta \in T^d$ (see, e.g., [HS99]) and, due to (4.3), each eigenvalue $\lambda_j(\vartheta)$ of $\Delta(\vartheta)$ above its essential spectrum satisfy $\mu_j \leq \lambda_j(\vartheta) \leq \mu_j + 2\beta_+$, which yields (2.18).

ii) The existence of such graphs is proved in Propositions 4.2 and 4.3. 

4.2. Proof of Theorem 2.4. Let $\Gamma = (V_1, E_1)$ be a finite graph obtained from the graph $\Gamma_1 = (V_1, E_1)$ considering each edge of $\Gamma_1$ to have the multiplicity $t \in \mathbb{N}$. Then the Laplacian on the graph $\Gamma$ has the form $t\Delta_1$. If $t$ is large enough, then all positive eigenvalues $t\zeta_0 \leq \ldots \leq t\zeta_1$ of the Laplacian $t\Delta_1$ on the graph $\Gamma$, satisfy $t\zeta_p > \varrho$ and $t(\zeta_j - \zeta_{j+1}) > \varrho$ for all $j \in \mathbb{N}_{p-1}$, where $\varrho$ is defined in (1.5). Then, due to Corollary 4.1, the guided spectrum of the Laplacian $\Delta_t$ consists of exactly $p$ guided bands separated by gaps.

Let $f_0 = 0$ on $V_{01}$, then, due to Proposition 2.2 $\{t\zeta_j\}$ is a guided flat band of the Laplacian $\Delta_t$ on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_0^\varrho$.

Let $f_j = (f_{j,01}, f_{j,11}) \in \ell^2(V_1)$, where $0 \neq f_{j,01} \in \ell^2(V_{01})$ and $f_{j,11} \in \ell^2(V_1 \setminus V_{01})$. Using (3.7), we rewrite the fiber Laplacian $\Delta_t(\vartheta)$, $\vartheta \in \mathbb{T}^d$, for the Laplacian $\Delta_t$ acting on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_0^\varrho$ in the form

$$\Delta_t(\vartheta) = P\Delta_0(\vartheta)P + tP_1\Delta_1P_1 = tK_t(\vartheta), \quad K_t(\vartheta) = P_1\Delta_1P_1 + \varepsilon P\Delta_0(\vartheta)P, \quad \varepsilon = \frac{1}{t}.$$ 

We denote the eigenvalues of the operator $K_t(\vartheta)$ above its essential spectrum by

$$E_0(\vartheta, t) \leq \ldots \leq E_1(\vartheta, t) \leq E_1(\vartheta, t), \quad \vartheta \in \mathbb{T}^d.$$ 

Then the eigenvalue $E_j(\vartheta, t)$ of the operator $K_t(\vartheta)$ has the following asymptotics:

$$E_j(\vartheta, t) = \zeta_j + \varepsilon W_j(\vartheta) + O(\varepsilon^2) \quad (4.10)$$

(see pp. 7–8 in [RS78]) uniformly in $\vartheta \in \mathbb{T}^d$ as $t \to \infty$, where

$$W_j(\vartheta) = \langle (0, f_{j,01}), \Delta_0(\vartheta)(0, f_{j,01}) \rangle_{V_0^c}, \quad (4.11)$$

$\langle \cdot, \cdot \rangle_{V_0^c}$ denotes the inner product in $\ell^2(V_0^c)$. This yields the asymptotics of the eigenvalue $\lambda_j(\vartheta, t)$ of the operator $\Delta_t(\vartheta)$:

$$\lambda_j(\vartheta, t) = t E_j(\vartheta, t) = t\zeta_j + W_j(\vartheta) + O(1/t). \quad (4.12)$$

Using this asymptotics for $\lambda_j^-(t) = \min_{\vartheta \in \mathbb{T}^d} \lambda_j(\vartheta, t)$ and $\lambda_j^+(t) = \max_{\vartheta \in \mathbb{T}^d} \lambda_j(\vartheta, t)$, we obtain

$$\lambda_j^\pm(t) = t\zeta_j + W_j^\pm + O(1/t),$$

where $W_j^\pm$ are defined in (2.20). Since $s_j(\Delta_t) = [\lambda_j^-(t), \lambda_j^+(t)]$, the asymptotics (4.12) also gives the second formula in (2.19). Using the formula (3.9) for the fiber Laplacian $\Delta_0(\vartheta)$, we obtain

$$W_j(\vartheta) = \sum_{v \in V_{01}} (\Delta_0(\vartheta)(0, f_{j,01}))(v) \overline{f}_j(v) = \sum_{v \in V_{01}} \sum_{e = (v, u) \in A_0^c} (f_j(v) - e^{i\tau(e, \vartheta)} f_j(u)) \overline{f}_j(v)$$

$$= \sum_{v \in V_{01}} \varkappa_0^v |f_j(v)|^2 - \sum_{e = (v, u) \in A_0^c} e^{i\tau(e, \vartheta)} f_j(u) \overline{f}_j(v) \overline{f}_j(v)$$

$$= \sum_{v \in V_{01}} \varkappa_0^v |f_j(v)|^2 - \sum_{e = (v, u) \in A_0^c} \cos(\tau(e, \vartheta)) f_j(u) \overline{f}_j(v), \quad (4.13)$$

where $\varkappa_0^v$ is the degree of the vertex $v$ on the unperturbed cylinder $C_0$ and $\tau(e) \in \mathbb{Z}^d$ is the edge index defined by (3.3), (3.4). Using this formula we rewrite the constant $W_j^\pm$ defined in
in the form
\[ W_j^* = \max_{\vartheta \in \mathbb{T}^d} \Omega_j(\vartheta) - \min_{\vartheta \in \mathbb{T}^d} \Omega_j(\vartheta), \quad \Omega_j(\vartheta) = \sum_{e = (v, u) \in A_0^c \atop v \in V_01, \vartheta(e) \neq 0} \cos(\tau(e), \vartheta)f_j(u)f_j(v). \]  

We have
\[ |\Omega_j(\vartheta)| \leq \sum_{e = (v, u) \in A_0^c \atop v \in V_01, \tau(e) \neq 0} |\cos(\tau(e), \vartheta)| \leq \beta_{01}, \]
which yields \( W_j^* \leq 2\beta_{01} \).

ii) Let \( V_{01} = \{v\} \). Then for \( \Omega_j \) defined in (4.14) we have
\[ \Omega_j(\vartheta) = f_j^2(v) \sum_{e = (v, u) \in A_0^c \atop \tau(e) \neq 0} \cos(\tau(e), \vartheta), \quad \max_{\vartheta \in \mathbb{T}^d} \Omega_j(\vartheta) = f_j^2(v)\beta_{01}. \]  

Using the identity \( \int_{\mathbb{T}^d} \cos(\tau(e), \vartheta) \, d\vartheta = 0 \) for each \( \tau(e) \neq 0 \), we obtain
\[ -f_j^2(v)\beta_{01} \leq \min_{\vartheta \in \mathbb{T}^d} \Omega_j(\vartheta) \leq 0. \]  

Then (4.14) – (4.16) yield \( f_j^2(v)\beta_{01} \leq W_j^* \leq 2f_j^2(v)\beta_{01} \). From (4.15) and the condition \( f_j(v) \neq 0 \) it follows that \( \Omega_j(\cdot) = \text{const} \) if \( \beta_{01}=0 \). This yields the last statement of the item.

iii) If all positive eigenvalues \( \zeta_1 > \ldots > \zeta_p \) of the Laplacian \( \Delta_1 \) are distinct, then summing the second asymptotics in (2.19) over \( j = 1, \ldots, p \) we obtain (2.22).

iv) If there is no bridge connecting the vertices from the set \( V_{01} \) on the cylinder \( C \), then for each \( j \in \mathbb{N}_p \) the function \( W_j \) defined by (4.13) is constant, i.e., \( W_j^* = 0 \), and, the second asymptotics in (2.19) and the asymptotics (2.22), take the form \( |\delta_j(\Delta_1)| = O(1/t) \) and \( |s(\Delta_1)| = O(1/t) \), respectively. ■

**Remark.** The set \( A_0^c \) of all oriented edges of the cylinder \( C_0 \) is infinite, but the sum in (4.13) is taken over a finite (maybe empty) set of edges \( (v, u) \in A_0^c \) for which \( v, u \in V_{01} \).

### 4.3. Geometric properties of the guided spectrum.

Now we prove Proposition 2.2 and Corollary 2.5 about geometric properties of the guided spectrum for specific graphs.

**Proof of Proposition 2.2.** Let \( \nu_1 \times \nu_1 \) matrix \( \Delta_1 = \{\Delta_{uv}\}_{u,v \in V_1} \) associated to the Laplacian \( \Delta_1 \) on a finite graph \( \Gamma_1 = (V_1, \mathcal{E}_1) \) in the standard orthonormal basis is given by
\[ \Delta_{1uv} = \delta_{uv} \kappa_v^1 - \kappa_{uv}, \quad \nu_v = \#V_1, \]  

where \( \delta_{uv} \) is the Kronecker delta, \( \kappa_v^1 \) is the degree of the vertex \( v \in V_1 \) on the graph \( \Gamma_1 \), \( \kappa_{uv} \) is the number of edges \( (u, v) \in A_1 \).

2) The sufficient conditions from Proposition 2.2 are equivalent to \( \zeta \) being an eigenvalue of two operators: the Laplacian \( \Delta_1 \) and the Laplacian \( \Delta_D \) on \( \Gamma_1 \) with Dirichlet boundary condition
\[ f(v) = 0, \quad \forall v \in V_{01}, \]  
i.e., \( \zeta \) is an eigenvalue of two matrices: \( \Delta_1 = \{\Delta_{1uv}\}_{u,v \in V_1} \) and its submatrix \( \Delta_D = \{\Delta_{1uv}\}_{u,v \in V_1 \setminus V_{01}} \).
Proof of Corollary 2.5. i) Due to the connectivity of the periodic graph $\Gamma_0$, there exists a bridge on the cylinder $C_0$. First we consider the case when there exists a bridge-loop at some vertex $u_1 \in V_0^c$. Due to the periodicity of the cylinder $C_0$, for each $j \in \mathbb{Z}$ we have $u_j = u_1 + ja_{\tilde{d}} \in V_0^c$, where $a_{\tilde{d}}$ is one of the periods of $C_0$.

**Figure 3.** A finite graph $\Gamma_1$ with $p$ connected components $\Gamma_{11}, \ldots, \Gamma_{1p}$.

Let $\Gamma_1 = (V_1, E_1)$ be a finite graph consisting of $p \in \mathbb{N}$ connected components $\Gamma_{11}, \ldots, \Gamma_{1p}$, where $\Gamma_{1j}$ is a graph consisting of two vertices $u_j \in V_0^c, v_j \not\in V_0^c$ and $j$ edges connecting these vertices (see Fig.3). The Laplacian $\Delta_1$ on the finite graph $\Gamma_1$ has exactly $p$ simple positive eigenvalues $\zeta_1 = 2p, \zeta_2 = 2(p - 1), \ldots, \zeta_p = 2$.

The normalized eigenfunction $f_j \in \ell^2(V_1)$ of the Laplacian $\Delta_1$ corresponding to the eigenvalue $\zeta_j$ has the form

$$f_j(v) = \begin{cases} \frac{1}{\sqrt{2}}, & \text{if } v = u_j \\ \frac{-1}{\sqrt{2}}, & \text{if } v = v_j \\ 0, & \text{otherwise} \end{cases}, \quad j = 1, \ldots, p. \quad (4.19)$$

Let $\Gamma_t$, $t \in \mathbb{N}$, be a finite graph obtained from the graph $\Gamma_1$ considering each edge of $\Gamma_1$ to have the multiplicity $t$. Let $\Delta_t$ be the Laplacian on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_t$. Due to Theorem 2.4, for $t$ large enough the guided spectrum of the Laplacian $\Delta_t$ consists of exactly $p$ guided bands $s_j(\Delta_t) = s_j(\Delta_t)$ separated by gaps and these bands satisfy

$$|s_j(\Delta_t)| = W_j^\bullet + O(1/t), \quad j \in \mathbb{N}_p, \quad (4.20)$$

where $W_j^\bullet$ is defined in (2.20). Substituting the identities (4.19) into (4.13), we obtain the following expression for the function $W_j$:

$$W_j(\vartheta) = \frac{\pi u_1}{2} - \sum_{e=(u_j, v_j) \in A_0^c} \cos(\tau(e), \vartheta). \quad (4.21)$$

Due to the periodicity of the cylinder $C_0$, for each $\vartheta \in \mathbb{T}^d$ we have

$$\sum_{e=(u_1, u_1) \in A_0^c} \cos(\tau(e), \vartheta) = \sum_{e=(u_2, u_2) \in A_0^c} \cos(\tau(e), \vartheta) = \ldots. \quad (4.22)$$

This and (4.21) yield that $W_1^\bullet = \ldots = W_p^\bullet \neq 0$. Then, due to (4.20), all guided bands are non-degenerate and

$$|s(\Delta_t)| = \sum_{j=1}^p |s_j(\Delta_t)| = p W_1^\bullet + O(1/t). \quad (4.23)$$

Choosing $p > \frac{C}{W_1^\bullet}$, we obtain $|s(\Delta_t)| > C$ for $t$ large enough.
Now let there be no bridge-loop on the cylinder $C_0$. Then the existing bridge connects some distinct vertices $u_1, v_1 \in V_0^c$. Repeating the above arguments but with $u_j = u_1 + ja_g \in V_0^c$ and $v_j = v_1 + ja_g \in V_0^c$, $j \in \mathbb{N}_p$, we also obtain the required statement. Note that in this case the function $W_j$ has the form

$$W_j(\vartheta) = \frac{1}{2}(x_{u_j}^0 + x_{v_j}^0) + \sum_{e=(u_j, v_j) \in A_0^c} \cos(\tau(e), \vartheta), \quad j \in \mathbb{N}_p. \quad (4.24)$$

ii) Let there exist a vertex $v \in V_0^c$ such that there is no bridge on $C_0$ starting at $v$. We consider a finite graph $\Gamma_t = (V_t, E_t)$ consisting of two vertices $u \notin V_0^c$ and $v$ and the edge $(u, v)$ of multiplicity $t \in \mathbb{N}$. The Laplacian on the finite graph $\Gamma_t$ has one simple positive eigenvalue $\zeta_1 = 2t$. Let $\Delta_t$ be the Laplacian on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_t$. Due to Theorem 2.4, for $t$ large enough the guided spectrum of $\Delta_t$ consists of exactly one guided band $\mathbf{s}_1(\Delta_t)$ and the length of this guided band satisfies $|\mathbf{s}_1(\Delta_t)| = O(1/t)$. Thus, for any small $\varepsilon > 0$ there exists $t \in \mathbb{N}$ such that $|\mathbf{s}(\Delta_t)| < \varepsilon$.

iii) This follows from (2.16) and the fact that the eigenvalues of the Laplacian on the finite graph $\Gamma_t$ can be arbitrary large as $t \to \infty$.

iv) The sufficient conditions for the existence of guided flat bands are proved in Proposition 2.2. For examples of finite graphs $\Gamma_1$ satisfying these conditions see Propositions 5.2, 5.3. 

4.4. Reduction to operators on unperturbed cylinder. We reduce the eigenvalue problem for the fiber Laplacian on the perturbed cylinder $C$ to that on the unperturbed cylinder $C_0$. In order to do this we use the following well-known theorem [BFS98].

**Theorem 4.2** (Feshbach projection method). Let $P$ be an orthogonal projection on a separable Hilbert space $\mathcal{H}$, and let $P^\perp = \mathbb{I} - P$ be its complement. Let $T$ be a bounded self-adjoint operator. Assume that $P^\perp TP^\perp$ is invertible on $P^\perp \mathcal{H}$. Then

i) $T$ is invertible on $\mathcal{H}$ if and only if its Feshbach map

$$\mathcal{F} = PTP - PTP^\perp (P^\perp TP^\perp)^{-1} P^\perp TP$$

is invertible on $P\mathcal{H}$; in this case $\mathcal{F}^{-1} = PT^{-1}P$;

ii) if $T\psi = 0$ for some vector $0 \neq \psi \in \mathcal{H}$, then $\mathcal{F}P\psi = 0$, where $P\psi \neq 0$;

iii) if $\mathcal{F}\varphi = 0$ for some vector $0 \neq \varphi = P\varphi$, then $T\psi = 0$, where

$$0 \neq \psi = [P - P^\perp (P^\perp TP^\perp)^{-1} P^\perp TP] \varphi;$$

iv) the kernels of $T$ and $\mathcal{F}$ have equal dimensions.

**Remark.** The operator $T$ in our consideration is $\Delta(\vartheta) - \lambda$, where $\Delta(\vartheta)$ is the fiber Laplacian on the perturbed cylinder $C = (V^c, E^c)$.

**Proposition 4.3.** Let $P$ be the orthogonal projection of $\ell^2(V^c)$ onto the subspace $\ell^2(\tilde{V}_0^c)$. Then the following statements hold true.

i) If $P^\perp (\Delta_1 - \lambda)P^\perp$ is invertible on $P^\perp \ell^2(V^c)$, then for each $\vartheta \in \mathbb{T}^d$

$$\lambda \in \sigma_p(\Delta(\vartheta)) \iff 0 \in \sigma_p(\mathcal{F}(\vartheta, \lambda)), \quad (4.26)$$

and the kernels of $\Delta(\vartheta) - \lambda$ and $\mathcal{F}(\vartheta, \lambda)$ have equal dimensions. Here $\mathcal{F}(\vartheta, \lambda)$ is the Feshbach map (4.25) for the operator $\Delta(\vartheta) - \lambda$, defined by (2.7), and $\mathcal{F}(\vartheta, \lambda)$ has the form

$$\mathcal{F}(\vartheta, \lambda) = P(\Delta_0(\vartheta) - \lambda)P + P_{01}(\Delta_1 - \Delta_1 P^\perp (P^\perp (\Delta_1 - \lambda)P^\perp)^{-1} P^\perp \Delta_1)P_{01}, \quad (4.27)$$
$P_{01}$ is the orthogonal projection of $\ell^2(V^c)$ onto $\ell^2(V_{01})$.

ii) Let, in addition, a finite graph $\Gamma_1$ consist of $c := c_{\Gamma_1}$ connected components

\[ \Gamma_{11} = (V_{11}, E_{11}), \quad \ldots, \quad \Gamma_{1c} = (V_{1c}, E_{1c}) \]

each of which has exactly one common vertex $v_1, \ldots, v_c$, respectively, with the unperturbed cylinder $C_0 = (V_0^c, E_0^c)$. Then for each $\vartheta \in \mathbb{T}^d$ the Feshbach map \((4.27)\) for the operator $\Delta(\vartheta) - \lambda$ is just a fiber Schrödinger operator given by

\[ \mathcal{F}(\vartheta, \lambda) = P(\Delta_0(\vartheta) - \lambda)P + Q(\lambda), \quad (4.28) \]

where $Q(\lambda) = Q(\lambda, \cdot)$ is a potential with the compact support $V_{01} = \{v_1, \ldots, v_c\}$:

\[ Q(\lambda, v_j) = \left( \mathcal{P}_j(\Delta_{1j} - \Delta_{1j}^{\perp} P_{1j}^{\perp} (\Delta_{1j} - \lambda P_{1j}^{\perp})^{-1} P_{1j}^{\perp} \Delta_{1j}) \mathcal{P}_j \right) |_{\ell^2(v_j)}, \quad j = 1, \ldots, c, \quad (4.29) \]

$\Delta_{1j}$ is the Laplacian on the finite graph $\Gamma_{1j}$, $\mathcal{P}_j$ is the orthogonal projection of $\ell^2(V_{1j})$ onto the one-dimensional subspace $\ell^2(v_j)$. In particular, if the operator $\Delta_{1j} - \lambda P_{1j}^{\perp}$ is invertible, then the expression \((4.29)\) can be written in the form

\[ Q^{-1}(\lambda, v_j) = \left( \mathcal{P}_j(\Delta_{1j} - \lambda P_{1j}^{\perp})^{-1} P_{1j}^{\perp} \right) |_{\ell^2(v_j)}. \quad (4.30) \]

iii) If $P^{\perp}(\Delta_{1j} - \lambda P_{j}^{\perp})$ is not invertible on $P^{\perp} \ell^2(V^c)$, then $\lambda$ is a guided flat band of the Laplacian $\Delta$ on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_9$.

**Proof.** i) Let $P_1$ be the orthogonal projection of $\ell^2(V^c)$ onto $\ell^2(V_1)$. Then, using \((4.25), (3.7)\) and the identities

\[ PP_{1} = P_{1}P = P_{01}, \quad PP_{1}^{\perp} = P^{\perp} = P = P_{1}P_{1}^{\perp} = P_{1}, \quad (4.31) \]

we have

\[
\begin{align*}
\mathcal{F}(\vartheta, \lambda) &= P(\Delta(\vartheta) - \lambda)P - P\Delta(\vartheta)P^{\perp}(P^{\perp}(\Delta(\vartheta) - \lambda)P^{\perp})^{-1}P^{\perp} \Delta(\vartheta)P \\
&= P(P_{\Delta_0}(\vartheta)P + P_1 \Delta_1 P_1 - \lambda)P \\
&
- P(P_{\Delta_0}(\vartheta)P + P_1 \Delta_1 P_1)P^{\perp}(P^{\perp}(P_{\Delta_0}(\vartheta)P + P_1 \Delta_1 P_1 - \lambda)P^{\perp})^{-1}P^{\perp}(P_{\Delta_0}(\vartheta)P + P_1 \Delta_1 P_1)P \\
&= P(\Delta_{0}(\vartheta) - \lambda)P + P_{01}\Delta_{1}\lambda P_{01} - P_{01}\Delta_{1}\lambda P_{1}^{\perp}(P^{\perp}(\Delta_{1} - \lambda)P^{\perp})^{-1}P^{\perp} \Delta_{1} P_{01}.
\end{align*}
\]

This and Theorem \((4.2)\) yield the required statement.

ii) In this case $\Delta_1 = \Delta_{11} \oplus \ldots \oplus \Delta_{1c}$. Then \((4.27)\) has the form \((4.28), (4.29)\). Let the operator $\Delta_{1j} - \lambda P_{j}^{\perp}$ be invertible. Then, using the partitioned presentation of the inverse matrix (see p.18 in \([11, 85]\)), we obtain \((4.30)\).

iii) Let $P^{\perp}(\Delta_{1j} - \lambda P_{j}^{\perp})$ not be invertible on $P^{\perp} \ell^2(V^c)$, then $\lambda$ is an eigenvalue of the operator $P_{1}^{\perp} \Delta_{1} P_{1}^{\perp}$ and, due to Proposition \((2.22)\) \{ $\lambda$ \} is a guided flat band of the Laplacian $\Delta$ on the perturbed graph $\Gamma$.

\[ \blacksquare \]

5. Square lattice with guides.

In this section we consider the square lattice with specific guides. We obtain some properties of the guided spectrum on such graphs and give examples of the guided spectrum.

Let $\mathbb{L}^2 = (V, E)$ be the square lattice, where the vertex set $V = \mathbb{Z}^2$ and the edge set $E = \{(m, m + (1, 0)), (m, m + (0, 1)) \mid \forall m \in \mathbb{Z}^2\}$, see Fig.4a. The Laplacian $\Delta_0$ on $\mathbb{L}^2$ has
Figure 4. (a) The square lattice $L^2$; the vertices from the fundamental vertex set $V_0^c$ are big black points; the strip $S$ is shaded; (b) the cylinder $C_0 = L^2/Z$ (the edges of the strip are identified).

the form

$$(\Delta_0 f)(m) = 4f(m) - \sum_{|m-k|=1} f(k), \quad f \in \ell^2(Z^2), \quad m \in Z^2. \quad (5.1)$$

We consider the Laplacian $\Delta$ on the perturbed square lattice $\Gamma = L^2 \cup \Gamma_1^g$ with a guide $\Gamma_1^g$. Due to Proposition 2.1, the Laplacian $\Delta$ on $\Gamma$ has the decomposition (2.6), (3.7) into a constant fiber direct integral, where the fiber unperturbed Laplacian $\Delta_0(\vartheta)$ acts on $f \in \ell^2(Z)$ and is given by

$$\Delta_0(\vartheta) = 2(1 - \cos \vartheta) + h,$$

$$(hf)(n) = 2f(n) - f(n + 1) - f(n - 1), \quad n \in Z, \quad (5.2)$$

for all $\vartheta \in \mathbb{T} = (-\pi, \pi]$. It is well known that the spectrum of the Laplacian $h$ on $Z$ is given by $\sigma(h) = \sigma_{ac}(h) = [0, 4]$. Then the spectrum of each fiber Laplacian $\Delta(\vartheta)$, $\vartheta \in \mathbb{T}$, has the form

$$\sigma(\Delta(\vartheta)) = \sigma_{ac}(\Delta(\vartheta)) \cup \sigma_p(\Delta(\vartheta)), \quad (5.3)$$

$$\sigma_{ac}(\Delta(\vartheta)) = \sigma_{ac}(\Delta_0(\vartheta)) = [2 - 2 \cos \vartheta, 6 - 2 \cos \vartheta],$$

where $\sigma_p(\Delta(\vartheta))$ is the set of all eigenvalues of $\Delta(\vartheta)$ of finite multiplicity given by (2.10).

The spectrum of the Laplacian $\Delta$ on the perturbed square lattice $\Gamma = L^2 \cup \Gamma_1^g$ has the form

$$\sigma(\Delta) = \sigma(\Delta_0) \cup s(\Delta), \quad \sigma(\Delta_0) = [0, 8], \quad s(\Delta) = \bigcup_{j=1}^N s_j(\Delta) = s_{ac}(\Delta) \cup s_{fb}(\Delta),$$

where $N$ is defined in (2.11).

Now we consider the perturbed square lattice $\Gamma = L^2 \cup \Gamma_1^g$ when a finite graph $\Gamma_1$ has exactly one common vertex $v_1 = 0$ with the unperturbed cylinder $C_0 = L^2/Z = (Z, E_0^c)$, see Fig. 5.

**Proposition 5.1.** Let $\Gamma_1 = (V_1, E_1)$ be a finite connected graph and let $V_{01} = Z \cap V_1 = \{0\}$. Then the guided spectrum $s_{ac}(\Delta)$ and $s_{fb}(\Delta)$ of the Laplacian $\Delta$ on the perturbed square lattice $\Gamma = L^2 \cup \Gamma_1^g$ satisfy

$$s_{ac}(\Delta) = \left\{ \lambda \in \mathbb{R} : 2 \leq \lambda - \sqrt{4 + Q^2(\lambda)} \leq 6, \quad Q(\lambda) > 0 \right\}, \quad (5.4)$$

$$s_{fb}(\Delta) = \left\{ \lambda \in \mathbb{R} : \lambda \text{ is an eigenvalue of the operator } \mathcal{P} \Delta_1 \mathcal{P} \right\}, \quad (5.5)$$
Combining this with (5.9) and using the second identity in (5.8) and the fact that any eigen-
s of the Laplacian $\Delta$ on $\Gamma$. Thus,

$$P = \text{the Laplacian on the finite graph } \Gamma.$$ 

In particular, if the operator $\Delta_1 - \lambda P^\perp$ is invertible, then the expression (5.6) can be written in the form

$$Q^{-1}(\lambda) = (P(\Delta_1 - \lambda P^\perp)^{-1})P \big|_{\ell^2(\{0\})}. \quad (5.7)$$

**Proof.** Let $\lambda \in \mathbb{R}$ not be an eigenvalue of the operator $P\Delta_1 P$. For each $\vartheta \in \mathbb{T}$, using (4.28), (4.29) and (5.10), we obtain the Feshbach map for the operator $\Delta(\vartheta) - \lambda$:

$$\mathcal{F}(\vartheta, \lambda) = P(\Delta_1(\vartheta) - \lambda)P + Q(\lambda) = P(h - \mu)P + Q(\lambda), \quad (5.8)$$

where $P$ is the orthogonal projection of $\ell^2(V^c)$ onto the subspace $\ell^2(\mathbb{Z})$, $h$ is the Laplacian on $\mathbb{Z}$, the potential $Q(\lambda)$ has the support $\{0\}$ and is given by (5.7) or, in the case when $\Delta_1 - \lambda P^\perp$ is invertible, by (5.7). Then, due to Proposition 3.3 and the identity (5.8), we have

$$\lambda \in \sigma_p(\Delta(\vartheta)) \iff 0 \in \sigma_p(\mathcal{F}(\vartheta, \lambda)) \iff \mu \in \sigma_p(h + Q(\lambda)). \quad (5.9)$$

It is well known that the Schrödinger operator $h + Q$ on $\mathbb{Z}$ with a potential $Q$ having a support at a single point has the unique eigenvalue

$$\mu = \begin{cases} 2 + \sqrt{4 + Q^2}, & Q > 0, \\ 2 - \sqrt{4 + Q^2}, & Q < 0. \end{cases} \quad (5.10)$$

Combining this with (5.9) and using the second identity in (5.8) and the fact that any eigen-
value $\lambda$ of $\Delta(\vartheta)$ satisfies $\lambda \geq 2 - 2 \cos \vartheta$, we obtain

$$\sigma_p(\Delta(\vartheta)) = \{\lambda \in \mathbb{R} : \lambda - \sqrt{4 + Q^2(\lambda)} = 4 - 2 \cos \vartheta, \quad Q(\lambda) > 0\}, \quad (5.11)$$

which, due to the definition (2.12) of the guided spectrum, yields (5.9) and $\lambda \notin \mathcal{S}_h(\Delta)$.

Let $\lambda \in \mathbb{R}$ be an eigenvalue of the operator $P\Delta_1 P$. Then, due to Proposition 2.2, $\{\lambda\}$ is a guided flat band of the Laplacian $\Delta$ on $\Gamma$. Thus, $\mathcal{S}_h(\Delta)$ has the form (5.3).
Remark. If $Q(\lambda) < 0$ for some $\lambda > 8$, then $\lambda$ lies in the gap of the spectrum of the Laplacian $\Delta$ on the perturbed square lattice.

5.1. **Lattice with $p$ pendant edges at each vertex of $\mathbb{Z} \times \{0\}$**. Let $\Gamma_1 = (V_1, E_1)$ be a star graph of order $p + 1$, i.e., a tree on $p + 1$ vertices $v_1, v_2, \ldots, v_{p+1}$ with one vertex $v_1 = 0$ having degree $p$ and the other $p$ having degree $1$. We assume that $V_{01} = \mathbb{Z} \cap V_1 = \{0\}$ (5.12) and consider the perturbed square lattice $\Gamma = L^2 \cup \Gamma_1^g$, see Fig. 6.

![Figure 6](image)

**(a)** The perturbed square lattice $\Gamma = L^2 \cup \Gamma_1^g$; **(b)** the perturbed cylinder $C = \Gamma / \mathbb{Z}$.

**Proposition 5.2.** The guided spectrum of the Laplacian $\Delta$ on the perturbed square lattice $\Gamma = L^2 \cup \Gamma_1^g$, where $\Gamma_1$ is a star graph of order $p + 1$ satisfying (5.12), has the form

$$s(\Delta) = s_{ac}(\Delta) \cup s_{fb}(\Delta), \quad s_{ac}(\Delta) = s_1(\Delta),$$

$$s_{fb}(\Delta) = s_2(\Delta) \cup \ldots \cup s_p(\Delta) = \{1\},$$

i.e., the guided flat band $\{1\}$ has the multiplicity $p - 1$ and

$$s_1(\Delta) \subset [p + 3, p + 8], \quad |s_1(\Delta)| < 4.$$

Moreover, $\lim_{p \to \infty} |s_1(\Delta)| = 2\beta_+ = 4$, where $\beta_+$ is defined in (2.18).

**Remark.** For example, for $p = 1$ and $p = 2$ the guided band $s_1(\Delta)$ has the form

$$s_1(\Delta) \approx [4,38; 8,30] \quad \text{and} \quad s_1(\Delta) \approx [5,18; 9,01],$$

respectively.

**Proof.** The $(p + 1) \times (p + 1)$ matrix $\Delta_1 = \{\Delta_{uv}\}_{u,v \in V_1}$ defined by (4.17) for the Laplacian $\Delta_1$ on the star-graph $\Gamma_1$ has the form

$$\Delta_1 = \begin{pmatrix} p & b \\ b^* & I_p \end{pmatrix}, \quad b = (-1, \ldots, -1) \in \mathbb{C}^p,$$

(5.13)
where $I_p$ is the identity $p \times p$ matrix. Since $\lambda = 1$ is an eigenvalue of the matrix $\Delta_1$ of multiplicity $p - 1$, due to Proposition 2.2, the Laplacian $\Delta$ on $\Gamma$ has a guided flat band $\{1\}$ of multiplicity $p - 1$.

Now we find the absolutely continuous guided spectrum $s_{ac}(\Delta)$. Since the number of guided bands $N \leq p$ and the guided flat band $\{1\}$ has multiplicity $p - 1$, due to Proposition 2.2, the Laplacian $\Delta$ on $\Gamma$ has an absolutely continuous guided spectrum $s_{ac}(\Delta)$ consists of at most one band $s_1(\Delta) = [\lambda_1^-, \lambda_1^+]$ and, due to Proposition 5.1, has the form (5.4), where $Q(\lambda) = \frac{p \lambda}{\lambda^2 - 1}$. The inequality $Q(\lambda) > 0$ gives that $\lambda \neq 1$. The graphs of the function $f(\lambda) = \sqrt{4 + Q^2(\lambda)}$ for $\lambda > 1$ and of the functions $g_1(\lambda) = \lambda - 2$, $g_2(\lambda) = \lambda - 6$ are shown in Fig.7. Thus, the absolutely continuous guided spectrum $s_{ac}(\Delta)$ consists of exactly one guided band $s_1(\Delta) = [\lambda_1^-, \lambda_1^+]$ and $|s_1(\Delta)| = \lambda_1^+ - \lambda_1^- < 4$, see Fig.7.

From the identity $f(\lambda_1^-) = g_1(\lambda_1^-)$ we obtain that $\lambda_1^-$ is a root of the equation

$$h(\lambda) \equiv \lambda^3 - 6\lambda^2 + (9 - p^2)\lambda - 4 = 0. \quad (5.14)$$

Using that $h(p + 3) = -4 < 0$, $h(p + 4) = 2p^2 + 9p > 0$, we obtain that the equation (5.14) has a unique solution

$$\lambda^-_1 \in [p + 3, p + 4]. \quad (5.15)$$

This and $|s_1(\Delta)| < 4$ give that $s_1(\Delta) \subset [p + 3, p + 8]$.

We have $\lambda_1^+ \to \infty$ as $p \to \infty$ and $f(\lambda) = \sqrt{4 + p^2} + o(1)$ as $\lambda \to \infty$. Then

$$\lambda^-_1 \to 2 + \sqrt{4 + p^2}, \quad \lambda_1^+ \to 6 + \sqrt{4 + p^2},$$

which yields that $|s_1(\Delta)| \to 4$ as $p \to \infty$. On the other hand, the number $\beta_+ = 2$, since for each vertex of the cylinder $C$ there are two bridges starting at this vertex. Thus, we have $|s_1(\Delta)| = 2\beta_+ + o(1)$ as $p \to \infty$. $\blacksquare$

### 5.2. Lattice with mandarin guides

An $s$-mandarin graph is a graph consisting of two vertices and $s$ edges connecting these vertices. Let $\Gamma_1 = (V_1, E_1)$ be a union of two $s$-mandarin graphs with one common vertex $v_1 = 0$, see Fig.2b. We assume that

$$V_{01} = \mathbb{Z} \cap V_1 = \{0\} \quad (5.16)$$
and consider the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_1^g$, see Fig.2.

**Proposition 5.3.** The guided spectrum of the Laplacian $\Delta$ on the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_1^g$, where $\Gamma_1$ is a union of two $s$-mandarin graphs with one common vertex satisfying (5.16), has the form

$$s(\Delta) = s_{ac}(\Delta) \cup s_{fb}(\Delta), \quad s_{ac}(\Delta) = s_1(\Delta), \quad s_{fb}(\Delta) = s_2(\Delta) = \{s\},$$

where

$$s_1(\Delta) \subset [3s + 1, 3s + 6] \quad \text{for} \quad s \geq 2, \quad |s_1(\Delta)| < 4.$$

Moreover, $\lim_{s \to \infty} |s_1(\Delta)| = 4$.

**Remarks.** 1) For example, for $s = 1$ and $s = 2$ the guided band $s_1(\Delta)$ has the form

$$s_1(\Delta) \approx [5, 18; 9, 01] \quad \text{and} \quad s_1(\Delta) \approx [7, 75; 11, 26],$$

respectively.

2) For $s \leq 8$ the guided flat band $\{s\}$ is embedded into the absolutely continuous spectrum of the Laplacian $\Delta$. For $s > 8$ it lies in a gap.

**Proof.** The matrix $\Delta_1 = \{\Delta_{uv}\}_{u,v \in V_1}$ defined by (4.17) for the Laplacian $\Delta_1$ on the graph $\Gamma_1$ has the form

$$\Delta_1 = s \begin{pmatrix} 2 & b \\ b^* & I_2 \end{pmatrix}, \quad b = (-1, -1). \quad (5.17)$$

Since $\lambda = s$ is a simple eigenvalue of the matrix $\Delta_1$, due to Proposition 2.2, the Laplacian $\Delta$ on $\Gamma$ has a guided flat band $\{s\}$.

The absolutely continuous guided spectrum $s_{ac}(\Delta)$ consists of at most one band $s_1(\Delta) = [\lambda_1^-, \lambda_1^+]$ and, due to Proposition 5.1, has the form (5.4), where $Q(\lambda) = \frac{2\lambda}{1 + \lambda^2}$. The inequality $Q(\lambda) > 0$ gives that $\lambda > s$. The graphs of the function $f(\lambda) = \sqrt{4 + Q^2(\lambda)}$ for $\lambda > s$ and of the functions $g_1(\lambda) = \lambda - 2$, $g_2(\lambda) = \lambda - 6$ are shown in Fig.8. Thus, the absolutely continuous guided spectrum $s_{ac}(\Delta)$ consists of exactly one guided band $s_1(\Delta) = [\lambda_1^-, \lambda_1^+]$ and $|s_1(\Delta)| = \lambda_1^+ - \lambda_1^- < 4$, see Fig.8.
From the identity $f(\lambda^-) = g_1(\lambda^-)$ we obtain that $\lambda^-_{1}$ is a root of the equation
$$h(\lambda) \equiv (\lambda - 4)(\lambda - s)^2 - 4s^2\lambda = 0. \quad (5.18)$$
Using that
$$h(3s + 1) = -4s^2 - 9s - 3 < 0, \quad h(3s + 2) = 4(2s^2 - s - 2) > 0 \quad \text{for} \quad s \geq 2,$$
we obtain that for $s \geq 2$ the equation $h(\lambda) = 0$ has a unique solution
$$\lambda^-_{1} \in [3s + 1, 3s + 2]. \quad (5.19)$$
This and $|s_1(\Delta)| < 4$ give that $s_1(\Delta) \subset [3s + 1, 3s + 6]$.

We have $\lambda^+_1 \to \infty$ as $s \to \infty$ and $f(\lambda) = 2\sqrt{1 + s^2} + o(1)$ as $\lambda \to \infty$. Then
$$\lambda^-_{1} \to 2 + 2\sqrt{1 + s^2}, \quad \lambda^+_1 \to 6 + 2\sqrt{1 + s^2},$$
which yields that $|s_1(\Delta)| \to 4$ as $s \to \infty$. \[\blacksquare\]

5.3. Lattice with path guides. Let $\Gamma_1 = (V_1, E_1)$ be a path of length 2, i.e., a connected graph with two vertices $v_1$ and $v_3$ having degree 1 and a vertex $v_2$ having degree 2 and let $\Gamma_t = (V_1, E_t)$ be a finite graph obtained from the graph $\Gamma_1$ considering each edge of $\Gamma_1$ to have the multiplicity $t$. We assume that $V_{01} = \mathbb{Z} \cap V_1 = \{v_1 = 0\}$ and consider the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \mathbb{L}^2_t$, see Fig.\[9\].

**Figure 9.** a) The perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \mathbb{L}^2_t$; b) the perturbed cylinder $\mathcal{C} = \Gamma/\mathbb{Z}$.

**Proposition 5.4.** The guided spectrum of the Laplacian $\Delta$ on the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \mathbb{L}^2_t$ has the form
$$s(\Delta) = s_{ac}(\Delta) = \begin{cases} s_1(\Delta), & \text{if } t = 1, 2, \\ s_1(\Delta) \cup s_2(\Delta), & \text{if } t \geq 3 \end{cases},$$
where $s_2(\Delta) \subset [t, 2t]$ and $s_1(\Delta) = [\lambda^-_{1}, \lambda^+_1]$,
$$\lambda^-_{1} = 2 + \sqrt{4 + t^2} + o(1), \quad \lambda^+_1 = 6 + \sqrt{4 + t^2} + o(1), \quad \text{as} \quad t \to \infty. \quad (5.20)$$
Proof. The matrix $\Delta_1 = \{\Delta^1_{uv}\}_{u,v \in V_1}$ defined by (4.17) for the Laplacian $\Delta_1$ on the graph $\Gamma_t$ has the form

$$\Delta_1 = t \begin{pmatrix} 1 & b \\ b^* & \Delta_D \end{pmatrix}, \quad b = (-1, 0), \quad \Delta_D = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$ (5.21)

Since the matrices $\Delta_1$ and $\Delta_D$ have no equal eigenvalues, due to Proposition 5.1, the Laplacian $\Delta$ on $\Gamma$ has no guided flat bands. Then the absolutely continuous guided spectrum $s_{ac}(\Delta)$ consists of at most two bands and, due to Proposition 5.1, has the form (5.4), where

$$Q(\lambda) = \frac{t\lambda(\lambda - 2t)}{\lambda^2 - 3t\lambda + t^2}.$$ (5.22)

The inequality $Q(\lambda) > 0$ gives that

$$\lambda \in \left(\frac{3 - \sqrt{5}}{2} t, 2t\right) \cup \left(\frac{3 + \sqrt{5}}{2} t, +\infty\right).$$

For $t \geq 4$ the graphs of the function $f = \sqrt{4 + Q^2(\lambda)}$ for $\lambda > \frac{3 - \sqrt{5}}{2} t$ and of the functions $g_1(\lambda) = \lambda - 2$, $g_2(\lambda) = \lambda - 6$ are shown in Fig. 10. Thus, for $t \geq 4$ the absolutely continuous guided spectrum $s_{ac}(\Delta)$ consists of exactly two guided bands $s_s(\Delta) = [\lambda_s^-, \lambda_s^+]$, $s = 1, 2$. Since the Laplacian $\Delta_1$ on the graph $\Gamma_t$ has the positive eigenvalues $\zeta_1 = 3t$ and $\zeta_2 = t$, Corollary 4.1 and the formula (5.22) give that $s_s(\Delta) \subset [t, 2t]$.

We have $\lambda_s^+ \to \infty$ as $t \to \infty$ and $f(\lambda) = \sqrt{4 + t^2} + o(1)$ as $\lambda \to \infty$. Then we get (5.20).

From (5.21) by a direct calculation we obtain the absolutely continuous guided spectrum of the Laplacian $\Delta$ for $t \leq 3$:

$$s_{ac}(\Delta) = \begin{cases} s_1(\Delta) \approx [4, 47; 8, 31], & t = 1 \\ s_1(\Delta) \approx [6, 66; 9, 45], & t = 2 \\ s_1(\Delta) \cup s_2(\Delta) \approx [4, 62; 6] \cup [9, 51; 11, 44], & t = 3 \end{cases}.$$ (5.23)

Figure 10. The graphs of the functions $f, g_1, g_2$; $t \geq 4$.

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