SPDEs with non-Lipschitz coefficients and nonhomogeous boundary conditions

Jie Xiong, Xu Yang

Abstract. In this paper we establish the strong existence, pathwise uniqueness and a comparison theorem to a stochastic partial differential equation driven by Gaussian colored noise with non-Lipschitz drift, Hölder continuous diffusion coefficients and the spatial domain in finite interval, [0, 1], and with Dirichlet, Neumann or mixed nonhomogeneous random conditions imposed on the endpoints. The Hölder continuity of the solution both in time and in space variables is also studied.

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1 Introduction and main results

It is first shown in Konno and Shiga [18] (also in Reimers [28]) that the one-dimensional super-Brownian motion with binary branching has a jointly continuous density \( \{X_t(x) : t > 0, x \in \mathbb{R}\} \) which satisfies the following stochastic partial differential equation (SPDE):

\[
\frac{\partial}{\partial t}X_t(x) = \frac{1}{2}\Delta X_t(x) + X_t(x)^2\dot{W}_t(x), \quad t > 0, \ x \in \mathbb{R},
\]

where \( \Delta \) denotes the one-dimensional Laplacian operator and \( \dot{W} \) is a space-time Gaussian white noise. The weak uniqueness of the solution to (1.1) follows from the uniqueness of solution to the martingale problem for the associated super-Brownian motion. We refer to [4, 9, 10, 20, 27] for introduction to super-Brownian motion. The pathwise uniqueness of the nonnegative solution to (1.1) remains open even though it is studied by many authors; see [5, 6, 23, 34] and the references therein. For an one-dimensional super-Brownian motion in random environment, its density process satisfies more general SPDE

1Department of Mathematics and SUSTech International center for Mathematics, Southern University of Science & Technology, Shenzhen, China. Southern University of Science and Technology Start up fund Y01286120 and NSFC (Nos. 61873325 and 11831010). Email: xiongj@sustech.edu.cn

2School of Mathematics and Information Science, North Minzu University, Yinchuan, China. Supported by NSFC (No. 11771018) and Major research project for North Minzu University (No. ZDZX201902). Email: xuyang@mail.bnu.edu.cn. Corresponding author.
than equation (1.1) (see Dawson et al. [7]), and the joint Hölder continuity is studied in Li et al. [21] and the improved result is obtained in Hu et al. [15]. If the noise $\dot{W}$ is colored in space and white in time, the existence and pathwise uniquenes of nonnegative solution to the SPDE were established by Sturm [31] and Mytnik et al. [24], respectively. Further work have been studied in Rippl and Sturm [29] and Neuman [25]. For certain diffusion coefficient depending on the spatial derivative of the solution, Gomez et al. [11] and Xiong and Yang [35] studied the pathwise uniqueness and strong existence of the solution, respectively. We also refer the reader to Xiong [33] for some other related SPDEs connected to superprocesses.

There are many authors studying the following SPDE where the spatial domain is a finite interval, $[0, 1]$, with Dirichlet or Neumann conditions imposed on the endpoints:

$$\frac{\partial Y_t(x)}{\partial t} = \frac{1}{2} \Delta Y_t(x) + G(Y_t(x)) + H(Y_t(x))\dot{W}_t(x),$$  \hspace{1cm} (1.2)

where $x \in (0, 1)$, $t \geq 0$, $G$ and $H$ are continuous functions on $\mathbb{R}$, and $\dot{W}$ denote the space-time Gaussian white noise on $[0, \infty) \times [0, 1]$. It is shown in [32, Chapter 3] that SPDE (1.2) has a unique solution under Lipschitz continuity on $H$ and $G$. If $H$ is a constant and $G$ is locally bounded and satisfies one sided linear growth condition, then SPDE (1.2) has a unique strong solution and the comparison theorem holds, which is given in [13] and extended to nonnecessarily locally bounded for $G$ in [14]. If the assumptions concerning $G$ are those of [13], and $H$ has locally Lipschitz derivative and satisfies a linear growth condition, then the existence and uniqueness and comparison theorem are studied in [1] and the results are extended to Lipschitz continuous $G$ in [12]. There are also many authors concerning on white noise driven SPDEs with reflection, which adds a certain random measure on the right hand of (1.2) and the existence and uniqueness of solution and comparison theorem are studied in [8, 26, 37, 40] with $G$ and $H$ satisfying Lipschitz and linear growth conditions. Various properties of the solution were studied in [2, 3, 16, 38, 39].

In this paper we study SPDE (1.2) with Dirichlet, Neumann or mixed *nonhomogeneous random boundary conditions*, where $\dot{W}$ is a Gaussian noise that is white in time and colored in space. Namely, it is a Gaussian martingale measure on $[0, \infty) \times [0, 1]$ in the sense of [32, Chapter 2] and can be characterized by its covariance functional

$$E[W(\phi)W(\psi)] := \int_0^\infty \int_0^1 \int_0^1 \phi(s, x)\psi(s, y)\kappa(x, y)dsdxdy$$

for each continuous functions $\phi, \psi$ on $[0, \infty) \times [0, 1]$, where $\kappa$ is a nonnegative bounded function on $[0, 1] \times [0, 1]$ and $\kappa(x, y) = \kappa(y, x)$ for all $x, y \in [0, 1]$.

To continue with the introduction we state some notation. Let $\mathcal{B}[0, 1]$ denote the set of Borel functions on $[0, 1]$. For $f, g \in \mathcal{B}[0, 1]$ let $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$ whenever it exists. Let $B[0, 1]$ be the Banach space of bounded measurable functions on $[0, 1]$ furnished with the supremum norm $\| \cdot \|_0$. We use $C[0, 1]$ to denote the subset of continuous functions on $[0, 1]$. Define $C(\mathbb{R})$, $C([0, \infty) \times [0, 1])$ and $C([0, T] \times [0, 1])$ similarly. For any integer $n \geq 1$ let $C^n[0, 1]$ be the subset of $C[0, 1]$ of functions with bounded continuous derivatives up
to the \( n \)th order. Let \( C^n_c[0,1] \) be the subset of \( C^n[0,1] \) of functions with the supports in \( (0,1) \).

Suppose that \( (\mu_0(t))_{t \geq 0} \) and \( (\mu_1(t))_{t \geq 0} \) are two continuous processes and that \( Y_0 \in C[0,1] \). In this paper we aim to prove the strong existence, pathwise uniqueness and to establish a comparison theorem for the solution to SPDE (1.2) with \( G \) satisfying certain non-Lipschitz condition and \( H \) satisfying Hölder condition and with one of the following boundary conditions:

(C1) \( Y_t(0) = \mu_0(t) \) and \( Y_t(1) = \mu_1(t) \) almost surely for all \( t \geq 0 \);
(C2) \( \nabla Y_t(0) = \mu_0(t) \) and \( \nabla Y_t(1) = \mu_1(t) \) almost surely for all \( t \geq 0 \);
(C3) \( Y_t(0) = \mu_0(t) \) and \( \nabla Y_t(1) = \mu_1(t) \) almost surely for all \( t \geq 0 \);
(C4) \( \nabla Y_t(0) = \mu_0(t) \) and \( Y_t(1) = \mu_1(t) \) almost surely for all \( t \geq 0 \).

To present the definition of the solution to SPDE (1.2) with various boundary conditions precisely, we introduce the corresponding boundary conditions on the test functions \( f \in C^2[0,1] \):

(D1) \( f(0) = f(1) = 0 \);
(D2) \( f'(0) = f'(1) = 0 \);
(D3) \( f(0) = f'(1) = 0 \);
(D4) \( f''(0) = f(1) = 0 \).

**Definition 1.1** Let \( i \in \{1,2,3,4\} \). We say that \( (Y_t)_{t \geq 0} \) on a filtered probability space is a weak solution to (1.2) with boundary condition (Ci) if the mapping \( (t,x) \mapsto Y_t(x) \) is in \( C([0,\infty) \times [0,1]) \) almost surely and there exist a Gaussian colored noise \( W \), an initial \( Y_0 \) and boundary conditions \( \mu_0(t), \mu_1(t) \), so that for any \( f \in C^2[0,1] \) satisfying (Di), we have

\[
\langle Y_t, f \rangle = \langle Y_0, f \rangle + \frac{1}{2} \int_0^t \left[ \langle Y_s, f'' \rangle + F_s^{(i)}(f) \right] ds
+ \int_0^t \langle G(Y_s), f \rangle ds + \int_0^t \int_0^1 H(Y_s(x)) f(x) W(ds, dx), \quad t \geq 0 \tag{1.3}
\]

almost surely, where

\[
F_s^{(1)}(f) := f'(0)\mu_0(s) - f'(1)\mu_1(s), \quad F_s^{(2)}(f) := -f(0)\mu_0(s) + f(1)\mu_1(s) \tag{1.4}
\]

and

\[
F_s^{(3)}(f) := f'(0)\mu_0(s) + f(1)\mu_1(s), \quad F_s^{(4)}(f) := -f(0)\mu_0(s) - f'(1)\mu_1(s). \tag{1.5}
\]

SPDE (1.2) with boundary condition (Ci) has a strong solution if for any Gaussian colored noise \( W \) on filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\), for given initial \( Y_0 \) and the continuous processes \( (\mu_0(t))_{t \geq 0} \) and \( (\mu_1(t))_{t \geq 0} \), there exists a process \( (Y_t)_{t \geq 0} \) so that \( (t,x) \mapsto Y_t(x) \) is in \( C([0,\infty) \times [0,1]) \) and (1.3) holds for all \( f \in C^2[0,1] \) satisfying (Di).
We also study the joint Hölder continuity of the solution. We aim to prove that the Hölder exponent in spatial variable is arbitrarily close to 1/2 and in time variable is arbitrarily close to 1/4 for the case of boundary condition (C2); the Hölder exponents of the solution in spatial and in time variables depend on the Hölder exponent of the non-derivative boundary condition for the case of boundary conditions (C1), (C3) and (C4). Throughout this paper we always assume that all random variables defined on the same filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\). Let \(E\) denote the corresponding expectation.

The following conditions will be imposed in the rest of the paper.

**Condition 1.2** There are constants \(\gamma \in [2^{-1}, 1]\) and \(C > 0\) so that
\[
|G(x)| \leq C|x| + 1, \quad x \in \mathbb{R}
\]
and
\[
|H(x) - H(y)| \leq C|x - y|^{\gamma}, \quad x, y \in \mathbb{R}.
\]

**Condition 1.3** Let \(r_0 > 0\) be a constant. For each \(n \geq 1\) there is a non-decreasing and concave function \(r_n\) on \([0, \infty)\) so that \(\int_0^{r_n(x)} 1 \, dx = \infty\) and
\[
(x - y)r_0 + G(x) - G(y) \leq r_n(x - y), \quad -n \leq y \leq x \leq n.
\]

**Theorem 1.4** (Comparison theorem) Suppose that \(G_1\) and \(G_2\) are two continuous functions satisfying \(G_1(y) \leq G_2(y)\) for all \(y \in \mathbb{R}\), and that Condition 1.3 holds for \(G_2\) and (1.7) holds for \(H\). For each \(i = 1, 2\) let \((Y_t^{(i)})_{t \geq 0}\) be a weak solution to (1.2) with \(G\) replaced by \(G_i\). We also assume that \((Y_t^{(1)})_{t \geq 0}\) and \((Y_t^{(2)})_{t \geq 0}\) satisfy the same boundary condition. If \(Y_0^{(1)}(x) \leq Y_0^{(2)}(x)\) for all \(x \in [0, 1]\), then \(P\{Y_t^{(1)}(x) \leq Y_t^{(2)}(x)\} = 1\) for all \(t \geq 0\) and \(x \in [0, 1]\).

Applying this theorem we obtain the pathwise uniqueness to (1.2) immediately.

**Corollary 1.5** (Pathwise uniqueness) Suppose that (1.7) and Condition 1.3 hold. Let \((Y_t^{(1)})_{t \geq 0}\) and \((Y_t^{(2)})_{t \geq 0}\) be two weak solutions to (1.2) with the same boundary condition. If \(Y_0^{(1)}(x) = Y_0^{(2)}(x)\) for all \(x \in [0, 1]\), then \(P\{Y_t^{(1)}(x) = Y_t^{(2)}(x)\} = 1\) for all \(t \geq 0\) and \(x \in [0, 1]\).

**Theorem 1.6** (Existence) Suppose that Conditions 1.2 and 1.3 hold. Then (1.2) has a unique strong solution \((Y_t)_{t \geq 0}\) with one of the boundary conditions (C1) to (C4). Moreover, if \(Y_0 \geq 0\), \(G(0) = H(0) = 0\) and \(\mu_0(t) = \mu_1(t) = 0\) for all \(t \geq 0\), then \(P\{Y_t \geq 0\} = 1\) for all \(t \geq 0\).

The next result is concerned with the Hölder continuity of the solution both in time and in space variables. We first state the following assumption.
Condition 1.7 (i) There is a constant \( C > 0 \) so that
\[ |G(x)| \leq C(|x| + 1), \quad |H(x)| \leq C(|x| + 1), \quad x \in \mathbb{R}. \]
(ii) For each \( T > 0 \) and \( p > 1 \),
\[ E \left[ \sup_{0 < t \leq T} \left( |\mu_0(t)|^{2p} + |\mu_1(t)|^{2p} \right) \right] < \infty. \]
(iii) There are constants \( 0 < \gamma_0 < 1/2 \) and \( C > 0 \) so that
- For boundary condition (C1),
  \[ E \left[ |\mu_i(t) - \mu_i(s)|^p \right] \leq C|t - s|^{p\gamma_0}, \quad 0 < t, s \leq T \]
  holds for \( i = 1, 2 \);
- For boundary condition (C3), (1.8) holds for \( i = 0 \);
- For boundary condition (C4), (1.8) holds for \( i = 1 \).

Theorem 1.8 (Joint Hölder continuity) Suppose that \((Y_t)_{t \geq 0}\) is a weak solution to (1.2) with one of the boundary conditions (C1) to (C4) and that Condition 1.7 holds and \( 0 < T_1 < T \). Then \([T_1, T] \times [0, 1] \ni (t, x) \mapsto Y_t(x)\) is Hölder continuous with exponent \( \eta_1/2 \) in the time variable and with exponent \( \eta_2 \) in space variables, where \( \eta_1, \eta_2 \in (0, 1/2) \) in the case of boundary condition (C2) and \( \eta_1, \eta_2 \in (0, \gamma_0) \) in the case of boundary conditions (C1), (C3) and (C4). Namely, there exists a random variable \( K \geq 0 \) depending on \( \eta_1 \) and \( \eta_2 \) so that
\[ |Y_{t_1}(x_1) - Y_{t_2}(x_2)| \leq K(|t_1 - t_2|^\eta_1/2 + |x_1 - x_2|^\eta_2), \quad t_1, t_2 \in [T_1, T], \quad x_1, x_2 \in [0, 1]. \]

In the following we consider SPDE (1.2) on real semi-axis.

Theorem 1.9 Suppose that Conditions 1.2 and 1.3 hold. Then there is a unique strong solution \((Y_t)_{t \geq 0}\) to (1.2) with \( x \in (-\infty, 0) \) (or \( x \in (0, \infty) \)) and one of the following boundary conditions:
(C1’) \( Y_t(0) = \mu_0(t) \) almost surely for all \( t \geq 0 \);
(C2’) \( \nabla Y_t(0) = \mu_0(t) \) almost surely for all \( t \geq 0 \).

Moreover, if \( Y_0 \geq 0, G(0) = H(0) = 0 \) and \( \mu_0(t) = \mu_1(t) = 0 \) for all \( t \geq 0 \), then \( P\{Y_t \geq 0 \text{ for all } t \geq 0\} = 1 \).

The rest of the paper is organized as follows. In Section 2, we state some properties of the solution to heat equations with boundary conditions, which are used in the proofs of theorems. Theorem 1.4 is proved in Section 3. In Section 4, we establish the proof of Theorems 1.6 and 1.9. We derive the proofs of Theorem 1.8 in Section 5.

Notation: Let \( \nabla \) and \( \Delta \) be the first and the second order spatial differential operators, respectively. Similarly, \( \nabla_x \) and \( \Delta_x \) denote the first and the second order spatial differential operators with respect to the variable \( x \). Let \( \partial_x \) denote the first partial derivative with respect to the variable \( x \) and \( \kappa_0 := \sup_{y_1, y_2 \in [0, 1]} \kappa(y_1, y_2) \). We use \( C \) to denote a positive constant whose value might change from line to line.
2 Heat equations with boundary conditions

In this section we establish some properties of the solutions to the following heat equations with different boundary conditions:

\[
\begin{cases}
\partial_t p_t(x, y) = \frac{1}{2} \Delta_x p_t(x, y), & t > 0, \ x, y \in (0, 1), \\
\lim_{t \downarrow 0} p_t(x, y) = \delta_y(x), & x, y \in [0, 1].
\end{cases}
\] (2.1)

For \( t > 0 \) and \( x, y \in \mathbb{R} \) set \( q_t(x) := \frac{1}{\sqrt{2\pi t}} \exp\{-x^2/(2t)\} \) and \( q_t(x, y) := q_t(x - y) \). For each \( t > 0 \) and \( x, y \in \mathbb{R} \) let

\[
p_t^{(1)}(x, y) = \sum_{k=-\infty}^{\infty} [q_t(2k + x, y) - q_t(2k - x, y)],
\] (2.2)

\[
p_t^{(2)}(x, y) = \sum_{k=-\infty}^{\infty} [q_t(2k + x, y) + q_t(2k - x, y)]
\] (2.3)

and

\[
p_t^{(3)}(x, y) = 2 \sum_{k=-\infty}^{\infty} [q_t(4k + x, y) - q_t(4k - x, y)] - \sum_{k=-\infty}^{\infty} [q_t(2k + x, y) - q_t(2k - x, y)],
\] (2.4)

\[
p_t^{(4)}(x, y) = 2 \sum_{k=-\infty}^{\infty} [q_t(4k + x, y) + q_t(4k - x, y)] - \sum_{k=-\infty}^{\infty} [q_t(2k + x, y) + q_t(2k - x, y)].
\] (2.5)

It is elementary to check that for each \( i = 1, 2, 3, 4 \), \( p_t^{(i)}(x, y) \) is the unique solution to (2.1) with \( p_t^{(i)}(x, \cdot) \) satisfying the boundary condition (Di) for all \( t > 0 \) and \( x \in \mathbb{R} \).

For simplicity we write \( p_t(x, y) \) for \( p_t^{(i)}(x, y) \) when the conclusion is true for all \( i \). We write \( P_t f(y) = \int_0^1 p_t(x, y) f(x) \, dx \) for \( x \in [0, 1] \) and \( f \in \mathcal{B}[0, 1] \). It is elementary to check the following two lemmas.

**Lemma 2.1** For each \( t \geq 0 \) and \( f \in C[0, 1] \) we have

\[
p_t(x, y) = p_t(y, x), \quad \int_0^1 |p_t(x, y)| \, dy + \int_0^1 |p_t(x, y)| \, dx \leq 12, \quad x, y \in \mathbb{R}.
\]

and

\[
\lim_{t \to 0} \int_0^1 f(x) p_t(x, y) \, dx = f(y), \quad \lim_{t \to 0} \int_0^1 f(y) p_t(x, y) \, dy = f(x), \quad x, y \in \mathbb{R}.
\]

**Lemma 2.2** For each \( t, s \geq 0 \) and \( f \in C[0, 1] \) we have

\[
P_t P_s f(x) = P_{t+s} f(x), \quad t, s > 0.
\]
Lemma 2.3 Let $T > 0$ be fixed. Then
\[|p_t(x_1, y) - p_t(x_2, y)| \leq Ct^{-1}|x_1 - x_2|, \quad t \in (0, T], \; x_1, x_2, y \in [0, 1]\]
and
\[|p_t(x, y)| \leq Ct^{-1/2}, \; |p_{t+\epsilon}(x, y) - p_t(x, y)| \leq Ct^{-1}\epsilon^{1/2}, \quad t \in (0, T], \; \epsilon > 0, \; x, y \in [0, 1].\]

Proof. It is obvious that
\[\sum_{k=-\infty}^{\infty} q_t(k + x - y) \leq Ct^{-1/2}, \quad x, y \in [0, 1], \; t \in (0, T]. \quad (2.6)\]
It then follows from (2.2)-(2.5) that
\[|p_t(x, y)| \leq Ct^{-1/2}, \quad x, y \in [0, 1], \; t \in (0, T].\]
Due to (30) (2.4e) we obtain
\[|q_t(x_1, y) - q_t(x_2, y)| \leq Ct^{-1/2}|x_1 - x_2|\left[q_{4t}(x_1 - y) + q_{4t}(x_2 - y)\right] \quad (2.7)\]
for $x_1, x_2, y \in \mathbb{R}$ and $t > 0$. From (2.2)-(2.5) it follows that for $x_1, x_2, y \in [0, 1]$ and $t \in (0, T],$
\[
|p_t(x_1, y) - p_t(x_2, y)| \\
\leq Ct^{-1/2}|x_1 - x_2| \sum_{k=-\infty}^{\infty} [q_{4t}(4k + x_1, y) + q_{4t}(4k - x_2, y) \\
+ q_{4t}(2k + x_1, y) + q_{4t}(2k - x_2, y)]
\leq Ct^{-1}|x_1 - x_2|,
\]
which gives the first assertion (2.6). Observe that $q_t(x) = t^{-1/2}q_1(t^{-1/2}x)$ and $xq_t(x) \leq C$ for all $t, x > 0$. From (2.7) it follows that for $0 < t \leq T$ and $x > 0$,
\[
|q_{t+\epsilon}(x) - q_t(x)| \\
\leq |((t + \epsilon)^{-1/2} - t^{-1/2})q_1((t + \epsilon)^{-1/2}x) + t^{-1/2}|q_1((t + \epsilon)^{-1/2}x) - q_1(t^{-1/2}x)| \\
\leq \epsilon^{1/2}t^{-1/2}(t + \epsilon)^{-1/2}q_1((t + \epsilon)^{-1/2}x) \\
+ Ct^{-1/2}|(t + \epsilon)^{-1/2} - t^{-1/2}| \cdot [q_1((t + \epsilon)^{-1/2}x) + q_t(t^{-1/2}x)] \\
\leq \epsilon^{1/2}t^{-1}q_1((T + \epsilon)^{-1/2}x) + \epsilon^{1/2}t^{-1}C[q_8((T + \epsilon)^{-1/2}x) + q_8(T^{-1/2}x)] \\
\leq C\epsilon^{1/2}t^{-1}[q_8(T + \epsilon^{-1/2}x) + T^{-1/2}q_8(x)].
\]
Combining this with (2.2)-(2.5) we obtain the second assertion. \qed

Lemma 2.4 We have
\[
\sup_{0 < t \leq T, x \in [0, 1]} \sum_{n=-\infty}^{\infty} \int_0^t |\nabla q_s(n - x)| ds < \infty.
\]
Proof. Note that for \( v > 0, \)
\[
|\nabla q_v(1)| \leq C(1_{\{v \leq 1\}} + v^{-3/2}1_{\{v > 1\}}).
\] (2.8)

It then follows from a change of variable that for each \( y > 0, \)
\[
\int_0^t |\nabla q_s(y)|ds = \int_0^{ty^{-2}} |\nabla q_v(1)|dv \leq \int_0^{\infty} |\nabla q_v(1)|dv < \infty.
\] (2.9)

For all \( y > 1, \)
\[
|\nabla q_s(y)| \leq y^2s^{-1}q_s(y) \leq Cq_2s(y),
\]
which leads to
\[
\sum |n| \geq 2 \int_0^t |\nabla q_s(n - x)|ds \leq C \int_0^t \sum |q_2s(n - x)|ds \leq Ct, \quad x \in [0, 1].
\]

Combining the above inequality with (2.9) one ends the proof. \( \square \)

Lemma 2.5 Let \( p \geq 1 \) be fixed and \((h(t))_{t \geq 0}\) be a stochastic process satisfying
\[
E \left[ \sup_{0 < t \leq T} |h(t)|^p \right] < \infty.
\]

Suppose that there is a constant \( 0 < \bar{\gamma}_0 < 1/2 \) so that
\[
E \left[ |h(s + t) - h(t)|^p \right] \leq Cs^{\bar{\gamma}_0}, \quad s, t > 0.
\]
Then
\[
E \left[ \left| \int_0^t h(s) \sum_{n = -\infty}^{\infty} |\nabla q_s(2n - x_1) - \nabla q_s(2n - x_2)|ds \right|^p \right] \leq C[t^{-p/2} + 1]|x_1 - x_2|^{p\bar{\gamma}_0}
\]
for all \( t \in (0, T] \) and \( x_1, x_2 \in [0, 1]. \)

Proof. Let
\[
I_t(x, y) := \int_0^t h(s)\nabla q_s(x + y)ds - \int_0^t h(s)\nabla q_s(y)ds, \quad 0 \leq x, y \leq 1.
\]

We first show that for \( 0 < x, y \leq 1, \)
\[
E \left[ |I_t(x, y)|^p \right] \leq C[t^{-p/2} + 1]x^{p\bar{\gamma}_0}.
\] (2.10)

By a change of variable and (2.8),
\[
|I_t(x, y)| = \left| \int_0^{t(x+y)^{-2}} h(y^2x^2)\nabla q_v(1)dv - \int_0^{ty^{-2}} h(vy^2)\nabla q_v(1)dv \right|
\]
\[ \int_0^{ty^{-2}} |h(vy^2)\nabla q_v(1)|dv + I_{t,1}(x, y) \leq C \sup_{0<v\leq T} |h(v)|^{1/2}x + I_{t,1}(x, y), \tag{2.11} \]

where

\[ I_{t,1}(x, y) = \int_0^{t(x+y)^{-2}} |[h(v(x + y)^2) - h(vy^2)]\nabla q_v(1)|dv. \]

Taking \( \delta > 0 \) satisfying \( \tilde{\gamma}_0 + 1/p < 3\delta/2 < 1/2 + 1/p \), then

\[ 3(1 - \delta)p > 1, \quad \frac{3p\delta}{2} - p\tilde{\gamma}_0 > 1. \]

By Hölder’s inequality and (2.8) we find

\[ \mathbb{E}\left[ |I_{t,1}(x, y)|^p \right] \leq \int_0^\infty |\nabla q_v(1)|^{(1-\delta)p} |v|^{p-1} \mathbb{E}\left[ |h(v(x + y)^2) - h(vy^2)|^p \right] \cdot |\nabla q_v(1)|^{p\delta} dv \]
\[ \leq Cx^{p\tilde{\gamma}_0} \int_0^\infty \left[ 1_{\{v\leq 1\}} + v^{-3(1-\delta)p/2} 1_{\{v>1\}} \right] dv \int_0^\infty \left[ 1_{\{v\leq 1\}} + v^{p\tilde{\gamma}_0 - 3\delta/2} 1_{\{v>1\}} \right] dv \]
\[ \leq Cx^{p\tilde{\gamma}_0} \]

for all \( 0 < x, y \leq 1 \) and \( t > 0 \), which gives (2.10) by (2.11).

Observe that

\[ |\Delta q_t(x)| \leq Cq_t(x), \quad t > 0, \quad x \geq 1. \]

Then, for all \( x, t > 0 \) and \( y \geq 1 \), we have

\[ |\nabla q_t(x + y) - \nabla q_t(y)| \leq x \int_0^1 |\Delta q_t(y + x\theta)|d\theta \leq Cx \int_0^1 q_{2t}(y + x\theta)d\theta \leq Cxq_{2t}(y), \]

which implies that for all \( x, y, t > 0 \) and \( x + y \leq 1 \),

\[ \sum_{|n| \geq 1} \int_0^t |\nabla q_n(2n - x - y) - \nabla q_n(2n - y)|ds \]
\[ \leq Cx \int_0^t \left[ \sum_{n \leq -1} q_{2n}(2n - y) + \sum_{n \geq 1} q_{2n}(2n - 1 - y) \right] ds \leq Cxt. \]

Combining this with (2.10) one ends the proof. \( \square \)

### 3 Proof of Theorem 1.4

In this section we prove Theorem 1.4 by a modification of the Yamada-Watanabe argument for ordinary stochastic differential equations.
Proof of Theorem 1.4. For \( k \geq 1 \) put \( a_k = \exp \{-k(k + 1)/2\} \). Let \( \psi_k \in C^2_c[0,1] \) satisfy \( \text{supp}(\psi_k) \subset (a_k, a_{k-1}) \), \( \int_{a_k}^{a_{k-1}} \psi_k(x)\,dx = 1 \), and \( 0 \leq \psi_k(x) \leq 2/(kx) \) for all \( x > 0 \) and \( k \geq 1 \). Let \( x^+ := x \vee 0 \) and \( \phi_k(x) = \int_0^{x^+} dy \int_0^y \psi_k(z)\,dz \) for \( x \in \mathbb{R} \) and \( k \geq 1 \). Then for all \( x \in \mathbb{R} \), \( 0 \leq \phi_k(x) \leq 1 \), \( \phi_k'(x) \to 1_{(x > 0)} \) and \( \phi_k(x) \to x^+ \) as \( k \to \infty \).

For \( x \in \mathbb{R} \) define \( J(x) = \int_{\mathbb{R}} e^{-|y|} \rho_0(x-y)\,dy \) with the mollifier \( \rho_0 \) given by

\[
\rho_0(x) = c_0 \exp \left(-1/(1-x^2)\right) 1_{\{|x| < 1\}},
\]

where \( c_0 > 0 \) is a constant so that \( \int_{\mathbb{R}} \rho_0(x)\,dx = 1 \). Moreover, due to (2.1) of [22], for each \( n \geq 0 \) there exist constants \( \tilde{C}_n, \tilde{C}'_n > 0 \) so that

\[
\tilde{C}_n e^{-|x|} \leq |J^{(n)}(x)| \leq \tilde{C}'_n e^{-|x|}, \quad x \in \mathbb{R}.
\]

For \( \zeta > 0 \) and \( x \in \mathbb{R} \) let \( J_\zeta(x) = (J(x))^\zeta \). Then by using (3.1) one can see that there is a constant \( C_0 > 0 \) independent of \( \zeta \) so that

\[
|J_\zeta'(x)| \leq C_0 \zeta (\zeta + 1) J_\zeta(x), \quad \zeta > 0, \quad x \in \mathbb{R}.
\]

For the solutions \((Y^{(1)}_t)_{t \geq 0}\) and \((Y^{(2)}_t)_{t \geq 0}\) to (1.2) with boundary condition (Ci), let \( p_t(x,y) \) denote the solution to (2.1) with the boundary condition (Di). It then follows that for \( k = 1, 2 \),

\[
\langle Y^{(k)}_t, p_0^x \rangle = \langle Y^{(k)}_0, p_0^x \rangle + \frac{1}{2} \int_0^t [\Delta_x \langle Y^{(k)}_s, p_0^x \rangle + E_s^{(i)}(p_0^x)]\,ds + \int_0^t \langle G(Y^{(k)}_s), p_0^x \rangle\,ds
\]

\[
+ \int_0^t \int_0^1 H(Y^{(k)}(y)) p_0^x(y) W(ds, dy),
\]

(3.3)

where \( p_0^x = p_0(x, \cdot) \), \( E_s^{(i)}(p_0^x) \) is defined in (1.4) and (1.5). For each \( t \geq 0 \) and \( k = 1, 2 \) we extend the definition of \( Y^{(k)}_t \) on \( \mathbb{R} \) by \( Y^{(k)}_t(x) = Y^{(k)}_t(0) \) for \( x \leq 0 \) and \( Y^{(k)}_t(x) = Y^{(k)}_t(1) \) for \( x \geq 1 \).

For \( n \geq 1 \), we define stopping time

\[
\tau_n := \inf \{ t \geq 0 : \|Y^{(1)}_t\|_0 + \|Y^{(2)}_t\|_0 \geq n \}.
\]

Set \( v_t(x) = Y^{(1)}_t(x) - Y^{(2)}_t(x) \) and \( v_0^x = \langle v_t, p_0^x \rangle \). From (3.3) it follows that

\[
v_{t, \tau_n}^x = v_0^x + \frac{1}{2} \int_0^{t, \tau_n} \Delta_x v_s^x(x)\,ds + \int_0^{t, \tau_n} [G_s^{(1)}(x) + G_s^{(2)}(x)]\,ds
\]

\[
+ \int_0^{t, \tau_n} \int_0^1 M_s^x(x, y) W(ds, dy),
\]

(3.4)

where

\[
G_s^{(1)}(x) := \langle G_1(Y^{(1)}_s) - G_2(Y^{(1)}_s), p_0^x \rangle, \quad G_s^{(2)}(x) := \langle G_2(Y^{(1)}_s) - G_2(Y^{(2)}_s), p_0^x \rangle
\]

and

\[
M_s^x(x, y) := [H(Y^{(1)}_s(y)) - H(Y^{(2)}_s(y))] p_0^x(y).
\]
By (1.7) and Hölder’s inequality we obtain
\[
M^\delta_s(x) := \left| \int_0^1 dy_1 \int_0^1 M^\delta_s(x, y_1) M^\delta_s(x, y_2) \kappa(y_1, y_2) dy_2 \right|
\leq \kappa_0 \left[ \int_0^1 \left| [H(Y^1_s(y)) - H(Y^2_s(y))] p^\delta_s(y) \right| dy \right]^2
\leq C \left[ \int_0^1 \left| v_s(y) \right|^\gamma \cdot \left| p^\delta_s(y) \right| dy \right]^2 \leq C \left( \left| v_s \right|, \left| p^\delta_s \right| \right)^{2\gamma}.
\tag{3.5}
\]

It then follows from (3.4) and Itô’s formula that
\[
\phi_k(v^\delta_{t \wedge \tau_n}(x)) = \phi_k(v^\delta_0(x)) + \frac{1}{2} \int_0^{t \wedge \tau_n} \left[ \phi_k'(v^\delta_s(x)) \Delta_x v^\delta_s(x) + \phi_k''(v^\delta_s(x)) M^\delta_s(x) \right] ds
+ \int_0^{t \wedge \tau_n} \phi_k'(v^\delta_s(x)) \left[ G^{\delta,1}(x) + G^{\delta,2}(x) \right] ds
+ \int_0^{t \wedge \tau_n} \phi_k'(v^\delta_s(x)) M^\delta_s(x, y) W(ds, dy),
\]
which leads to
\[
\int_\mathbb{R} \mathbb{E} \left[ \phi_k(v^\delta_{t \wedge \tau_n}(x)) \right] J_\zeta(x) dx = \int_\mathbb{R} \phi_k(v^\delta_0(x)) J_\zeta(x) dx
= \mathbb{E} \left[ \int_0^{t \wedge \tau_n} \left[ 2^{-1} \int_\mathbb{R} \phi_k'(v^\delta_s(x)) \Delta_x v^\delta_s(x) J_\zeta(x) dx + 2^{-1} \int_\mathbb{R} \phi_k''(v^\delta_s(x)) M^\delta_s(x) J_\zeta(x) dx \right.ight.
\left. + \int_\mathbb{R} \phi_k'(v^\delta_s(x)) G^{\delta,1}(x) J_\zeta(x) dx + \int_\mathbb{R} \phi_k'(v^\delta_s(x)) G^{\delta,2}(x) J_\zeta(x) dx \right] ds
=: \mathbb{E} \left[ \int_0^{t \wedge \tau_n} \left[ 2^{-1} I^\delta_{1,k}(s) + 2^{-1} I^\delta_{2,k}(s) + I^\delta_{3,k}(s) + I^\delta_{4,k}(s) \right] ds \right].
\tag{3.6}
\]

By integration by parts and (3.2),
\[
I^\delta_{1,k}(s) = \int_\mathbb{R} \Delta_x (\phi_k(v^\delta_s(x))) J_\zeta(x) dx - \int_\mathbb{R} \phi_k''(v^\delta_s(x)) |\nabla_x v^\delta_s(x)|^2 J_\zeta(x) dx
\leq \int_\mathbb{R} \phi_k(v^\delta_s(x)) J_\zeta(x) dx \leq C_1 \zeta + 1 \int_\mathbb{R} \phi_k(v^\delta_s(x)) J_\zeta(x) dx
\]
for some constant \( C_1 > 0 \). As \( G_1(y) \leq G_2(y) \), we have \( I^\delta_{3,k}(s) \leq 0 \). Combining the above inequalities with (3.5) and (3.6) we obtain
\[
\int_\mathbb{R} \mathbb{E} \left[ \phi_k(v^\delta_{t \wedge \tau_n}(x)) \right] J_\zeta(x) dx - \int_\mathbb{R} \phi_k(v^\delta_0(x)) J_\zeta(x) dx
\leq \mathbb{E} \left[ \int_0^{t \wedge \tau_n} \left[ \int_\mathbb{R} \left[ 2^{-1} C_1 \zeta + 1 \phi_k(v^\delta_s(x)) \right.ight.ight.
\left. + C \phi_k''(v^\delta_s(x)) (|v_s|, p^\delta_s)^{2\gamma} \right] J_\zeta(x) dx + I^\delta_{4,k}(s) \right] ds.
\]

Letting \( \delta \to 0 \) and using Lemma 2.1 and dominated convergence we get
\[
\int_\mathbb{R} \mathbb{E} \left[ \phi_k(v_{t \wedge \tau_n}(x)) \right] J_\zeta(x) dx - \int_\mathbb{R} \phi_k(v_0(x)) J_\zeta(x) dx
\]
Recalling the constant $r_0$ in Condition 1.3 and taking suitable $\zeta > 0$ so that $r_0 = 2^{-1}C_1\zeta(\zeta + 1)$. Since $|v_0^+(x)| \equiv 0$ and $0 \leq |y|\phi_k^+(y) \leq 2/k$ for all $y \in \mathbb{R}$, letting $k \to \infty$ in the above inequality we get

$$
\langle \mathbb{E}[v_{r \wedge \tau_n}^+], C_2J_\zeta \rangle = C_2 \int_\mathbb{R} \mathbb{E}[v_{r \wedge \tau_n}^+(x)]J_\zeta(x)dx
$$

\[ \leq C_2\mathbb{E}\left[ \int_0^{r \wedge \tau_n} ds \int_\mathbb{R} \left[r_0v_+^+(x) + G_2(Y_s^{(1)}(x)) - G_2(Y_s^{(2)}(x))\right]1_{\{v_s(x) > 0\}}J_\zeta(x)dx \right] \]

\[ \leq \int_0^t ds \int_\mathbb{R} \mathbb{E}[r_n(v_{r \wedge \tau_n}^+(x))]C_2J_\zeta(x)dx \leq \int_0^t (\mathbb{E}[v_{r \wedge \tau_n}^+], C_2J_\zeta)ds,
\]

where $C_2 := (J_\zeta, 1)^{-1}$, $v_+^+(x) := v_+(x) \vee 0$ and we used Condition 1.3 concaveness of $y \mapsto r_n(y)$ and Jensen’s inequality in the last inequality. It is then easy to show that

$$
\int_\mathbb{R} \mathbb{E}[v_{r \wedge \tau_n}^+(x)]J_\zeta(x)dx = 0,
$$

and hence, $\int_0^1 v_+^+(x)dx = 0$ for all $t \leq \tau_n$ and $n \geq 1$. Letting $n \to \infty$ we obtain $\int_0^1 v_+^+(x)dx = 0$, which ends the proof by the continuities of $(x, t) \mapsto Y_t^{(1)}(x)$ and $(x, t) \mapsto Y_t^{(2)}(x)$.

\[ \square \]

### 4 Proof of Theorems 1.6 and 1.9

This section is devoted to the proofs of Theorems 1.6 and 1.9. Recall that $\kappa_0 := \sup_{x,y \in [0,1]} \kappa(x, y)$. Let $T > 0$ and $p > 1$ be fixed. For simplicity we write $F_s^t-p_s(x) := F_s(p_t-p_s(x.), \cdot))$ for $F_s^t= p_t^i(x, \cdot)$ in the rest of this paper.

First we establish Theorem 1.6 under Lipschitz condition.

**Proposition 4.1** Suppose that Condition 1.7(i)–(ii) hold, and $G$ and $H$ satisfy the following Lipschitz condition:

$$
|G(x) - G(y)| + |H(x) - H(y)| \leq C|x - y|, \quad x, y \in \mathbb{R}.
$$

(4.1)

Then for each boundary conditions (C1) to (C4), 1.2 has a strong solution $(Y_t)_{t \geq 0}$.

We now proceed to proving Proposition 4.1. Let $(\hat{Y}_t)_{t \geq 0}$ be the solution to

$$
\begin{cases}
\partial_t \hat{Y}_t(x) = \frac{1}{2} \Delta \hat{Y}_t(x), & t > 0, \ x \in (0, 1), \\
\hat{Y}_0(x) = Y_0(x), & x \in [0, 1]
\end{cases}
$$

(4.2)

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with boundary condition (Ci), \( i = 1, 2, 3, 4 \). For \( n \geq 0, t > 0 \) and \( x \in [0, 1] \) define \( Y^{(n)}_t \) by the following:

\[
Y^{(n)}_t(x) = \bar{Y}^{(n)}_t(x) + \tilde{Y}_t(x),
\]

(4.3)

where \( \bar{Y}^{(n)}_t \) are defined recursively by \( \bar{Y}^{(0)}_t(x) = 0 \) and

\[
\bar{Y}^{(n+1)}_t(x) = \int_0^t ds \int_0^1 G(\bar{Y}^{(n)}_s(y))p_{t-s}(x, y)dy
+ \int_0^t \int_0^1 H(\bar{Y}^{(n)}_s(y))p_{t-s}(x, y)W(ds, dy).
\]

(4.4)

Recall that \( P_t f(y) = \int_0^1 p_t(x, y) f(x)dx \) for \( t > 0, y \in [0, 1] \) and \( f \in \mathscr{B}[0, 1] \). For \( n, t \geq 0 \) and \( \delta > 0 \) define \( \bar{Y}^{(n, \delta)}_t(x) = P_{\delta} \bar{Y}^{(n)}_t \). It follows from (4.4) that

\[
\bar{Y}^{(n+1, \delta)}_t(x) = \int_0^t ds \int_0^1 G(\bar{Y}^{(n)}_s(y))p_{t-s+\delta}(x, y)dy
+ \int_0^t \int_0^1 H(\bar{Y}^{(n)}_s(y))p_{t-s+\delta}(x, y)W(ds, dy).
\]

(4.5)

Then for all \( t, \delta > 0 \) and \( n \geq 1, \bar{Y}^{(n, \delta)}_t \in C^2[0, 1] \) and satisfies boundary condition (Di).

**Lemma 4.2** There is a constant \( C_1 > 0 \) so that

\[
\sup_{t \in [0, T]} \| \bar{Y}^t \|_0 \leq C_1 \left[ \| Y_0 \|_0 + \sup_{t \in [0, T]} \| | \mu_0(t) | + | \mu_1(t) | \right].
\]

**Proof.** By Definition 1.1, for each \( f \in C^2[0, 1] \) satisfying Condition (Di),

\[
\langle \bar{Y}_t, f \rangle = \langle Y_0, f \rangle + \frac{1}{2} \int_0^t \langle [\bar{Y}_s, f''] + F_s(f) \rangle ds,
\]

(4.6)

which can be written into the following mild form

\[
\langle \bar{Y}_t, f \rangle = \langle Y_0, P_t f \rangle + \frac{1}{2} \int_0^t F_s(P_{t-s} f) ds.
\]

It follows that

\[
\bar{Y}_t(x) = \langle Y_0, p^x_t \rangle + \frac{1}{2} \int_0^t F_s^{t-s}(x) ds.
\]

Using Lemmas 2.3 and 2.4 we obtain the assertion. \( \square \)

For \( f \in \mathscr{B}[0, 1] \) define \( \| f \|^2 := \int_0^1 | f(x) |^2 dx \).

**Lemma 4.3** Suppose that Condition 1.7(i)-(ii) holds. Then

\[
\sup_{0 \leq t \leq T, n \geq 1} \mathbb{E}[\| \bar{Y}^{(n)}_t \|^2 + \| Y^{(n)}_t \|^2] < \infty.
\]

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Proof. Applying Hölder's inequality, Lemma 2.1 and Condition 1.7(i), we get

\[
\left| \int_0^t ds \int_0^1 G(Y_s^{(n)}(y))p_{t-s+\delta}(x,y)dy \right|^2 \\
\leq Ct \int_0^t ds \int_0^1 |G(Y_s^{(n)}(y))|^2 \cdot |p_{t-s+\delta}(x,y)|dy \\
\leq Ct \int_0^t ds \int_0^1 \|Y_s^{(n)}(y)\|^2 + 1 \cdot |p_{t-s+\delta}(x,y)|dy
\]

(4.7)

and

\[
M^{(n)}(t, s, x) := \left| \int_0^1 dy \int_0^1 H(Y_s^{(n)}(y))p_{t-s+\delta}(x,y)H(Y_s^{(n)}(z))p_t(x,z)\kappa(y,z)dz \right| \\
\leq \kappa_0 \left[ \int_0^1 |H(Y_s^{(n)}(y))p_{t-s+\delta}(x,y)|dy \right]^2 \leq C \int_0^1 |H(Y_s^{(n)}(y))|^2 \cdot |p_t(x,y)|dy \\
\leq C \int_0^1 \|Y_s^{(n)}(y)\|^2 + 1 \cdot |p_t(x,y)|dy.
\]

It then follows from Doob's inequality that

\[
\mathbb{E}\left[ \int_0^t \int_0^1 H(Y_s^{(n)}(y))p_{t-s+\delta}(x,y)W(ds, dy) \right]^2 \leq 4\mathbb{E}\left[ \int_0^t M^{(n)}(t-s+\delta, s, x)ds \right] \\
\leq C\mathbb{E}\left[ \int_0^t ds \int_0^1 \|Y_s^{(n)}(y)\|^2 + 1 \cdot |p_{t-s+\delta}(x,y)|dy \right].
\]

Now by using (3.4), (4.7) and Lemma 1.2,

\[
\mathbb{E}\left[ \|Y_t^{(n+1,\delta)}\|^2 \right] \leq 2\mathbb{E}\left[ \|\bar{Y}_t\|^2 \right] + 2\mathbb{E}\left[ \|\bar{Y}_t^{(n+1)}\|^2 \right] \leq C + C \int_0^t \mathbb{E}\left[ \|Y_s^{(n)}\|^2 + 1 \right]ds.
\]

It follows from Lemma 2.1 and Fatou’s lemma that

\[
\mathbb{E}\left[ \|Y_t^{(n+1)}\|^2 \right] = \mathbb{E}\left[ \lim_{\delta \to 0} \|Y_t^{(n+1,\delta)}\|^2 \right] \leq \lim_{\delta \to 0} \mathbb{E}\left[ \|Y_t^{(n+1,\delta)}\|^2 \right] \\
\leq C + C \int_0^t \mathbb{E}\left[ \|Y_s^{(n)}\|^2 + 1 \right]ds.
\]

Then by an induction argument one ends the proof. \(\square\)

Lemma 4.4 Under Condition 1.7(i)-(ii), for each \(f \in B[0, 1], \delta > 0\) and \(n \geq 1\), we have

\[
\langle Y_t^{(n+1,\delta)}, f \rangle = \frac{1}{2} \int_0^t \langle \Delta Y_t^{(n+1,\delta)}, f \rangle ds + \int_0^t \langle G(Y_s^{(n)}), P_{t-s+\delta}f \rangle ds \\
+ \int_0^t \int_0^1 H(Y_s^{(n)}(y))P_{t-s+\delta}f(y)W(ds, dy), \quad t \geq 0.
\]

Proof. In view of (4.5), we deduce

\[
\langle \bar{Y}_t^{(n+1,\delta)}, f \rangle = \int_0^t \langle G(Y_s^{(n)}), P_{t-s+\delta}f \rangle ds + \int_0^t \int_0^1 H(Y_s^{(n)}(y))P_{t-s+\delta}f(y)W(ds, dy).
\]
Put \( k \geq 1 \) and \( t_i := it/k \). Then by virtue of Lemma 2.2 for \( i \geq 1 \),

\[
\langle \tilde{Y}^{(n+1,\delta)}_{t_i} , f \rangle - \langle \tilde{Y}^{(n+1,\delta)}_{t_{i-1}} , P_{t_i-t_{i-1}}f \rangle = \int_{t_{i-1}}^{t_i} \langle G(Y^{(n)}_s), P_{t_i-s+\delta}f(y) \rangle ds + \int_{t_{i-1}}^{t_i} \int_0^1 H(Y^{(n)}_s(y))P_{t_i-s+\delta}f(y)W(ds, dy). \tag{4.8}
\]

We deduce from (2.1) that

\[
P_{t_i-t_{i-1}}f(y) - f(y) = \int_0^{t_i-t_{i-1}} \partial_s(P_s f(y)) ds = \frac{1}{2} \int_0^{t_i-t_{i-1}} ds \int_0^1 \Delta_x(p_s(x, y))f(x) dx
\]

\[
= \frac{1}{2} \int_0^{t_i-t_{i-1}} ds \int_0^1 \Delta_y(p_s(x, y))f(x) dx = \frac{1}{2} \int_0^{t_i-t_{i-1}} \Delta_y(P_s f(y)) ds. \tag{4.9}
\]

By the boundary conditions on \( \tilde{Y}^{(n+1,\delta)}_{t_i} \) and \( P_{t_i-t_{i-1}} \),

\[
\tilde{G}_n(i, s) := \tilde{Y}^{(n+1,\delta)}_{t_{i-1}}(y) \nabla_y(P_{t_i-s}f(y)) \big|_0^1 - \nabla \tilde{Y}^{(n+1,\delta)}_{t_{i-1}}(y) P_{t_i-s}f(y) \big|_0^1 = 0.
\]

Applying integration by parts and (4.9) we get

\[
\langle \tilde{Y}^{(n+1,\delta)}_{t_{i-1}} , f - f \rangle = \frac{1}{2} \int_{t_{i-1}}^{t_i} \langle \tilde{Y}^{(n+1,\delta)}_{t_{i-1}} , \Delta(P_{t_i-s}f) \rangle ds
\]

\[
= \frac{1}{2} \int_{t_{i-1}}^{t_i} \left[ \langle \Delta \tilde{Y}^{(n+1,\delta)}_{t_{i-1}} , P_{t_i-s}f \rangle + \tilde{G}_n(i, s) \right] ds = \frac{1}{2} \int_{t_{i-1}}^{t_i} \langle \Delta \tilde{Y}^{(n+1,\delta)}_{t_{i-1}} , P_{t_i-s}f \rangle ds.
\]

It thus follows from (4.8) that

\[
\langle \tilde{Y}^{(n+1,\delta)}_t , f \rangle
\]

\[
= \sum_{i=1}^k \left[ \langle \tilde{Y}^{(n+1,\delta)}_{t_{i-1}} , P_{t_i-t_{i-1}}f \rangle - \langle \tilde{Y}^{(n+1,\delta)}_{t_{i-1}} , P_{t_i-t_{i-1}}f \rangle \right] = \frac{1}{2} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \langle \Delta \tilde{Y}^{(n+1,\delta)}_{t_{i-1}} , P_{t_i-s}f \rangle ds + \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \langle G(Y^{(n)}_s) , P_{t_i-s+\delta}f \rangle ds
\]

\[
+ \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \int_0^1 H(Y^{(n)}_s(y))P_{t_i-s+\delta}f(y)W(ds, dy)
\]

\[
= \frac{1}{2} \int_0^t \sum_{i=1}^k \int_{t_{i-1}}^{t_i} 1_{(t_{i-1},t_i]}(s) \langle \Delta \tilde{Y}^{(n+1,\delta)}_{t_{i-1}} , P_{t_i-s}f \rangle ds
\]

\[
+ \int_0^t \sum_{i=1}^k \int_0^1 1_{(t_{i-1},t_i]}(s) H(Y^{(n)}_s(y))P_{t_i-s+\delta}f(y)W(ds, dy).
\]

Letting \( k \to \infty \), and using Lemma 4.3 and dominated convergence one can conclude the assertion. \( \square \)
Proof of Proposition 4.1. For \( n, s \geq 0 \) and \( \delta > 0 \) let
\[
\begin{align*}
\varphi_n(s) &= Y_s(n+1) - Y_s(n) = Y_s(n+1) - Y_s(n), \\
\varphi_n(s) &= P_\delta \varphi_n(s)
\end{align*}
\]
and
\[
H_n(s)(y) := H(Y_s(n+1)(y)) - H(Y_s(n)(y)), \quad G_n(s)(y) := G(Y_s(n+1)(y)) - G(Y_s(n)(y)).
\]
Then \( \varphi_n(s) \in C^2[0,1] \) and satisfies boundary condition (Di) for each fixed \( s, n, \delta \). For \( \lambda, \delta > 0 \) let \( \varphi_n(s) = e^{-\lambda s} \| u_\delta(s) \|^2 \) and \( v_n(s, \delta) = e^{-\lambda s} \| u_\delta(s, \delta) \|^2 \). In virtue of Lemma 4.4, for each \( f \in B^2[0,1] \),
\[
\begin{align*}
\langle \varphi_n(s), f \rangle^2 e^{-\lambda t} &= \int_0^t \langle \varphi_n(s), f \rangle \langle \Delta \varphi_n(s), f \rangle e^{-\lambda s} ds + 2 \int_0^t \langle \varphi_n(s), f \rangle \langle G_n(s), P_\delta f \rangle e^{-\lambda s} ds \\
&+ \int_0^t e^{-\lambda s} ds \int_0^1 dy \int_0^1 H_n(n-1)(y) P_\delta f(y) H_n(n-1)(z) P_\delta f(z) \kappa(y, z) dz \\
&+ 2 \int_0^t \int_0^1 e^{-\lambda s} \langle \varphi_n(s), f \rangle H_n(n-1)(y) P_\delta f(y) W(\nu ds, dy) - \lambda \int_0^t \langle \varphi_n(s), f \rangle^2 e^{-\lambda s} ds.
\end{align*}
\]
Summing on \( f \) over a complete orthonormal system of \( L^2[0,1] := \{ f \in B[0,1] : \| f \| < \infty \} \) we obtain
\[
\begin{align*}
v_n(s, \delta) &= \int_0^t \langle \varphi_n(s), \Delta \varphi_n(s), f \rangle e^{-\lambda s} ds + 2 \int_0^t \langle \varphi_n(s), P_\delta G_n(s) \rangle e^{-\lambda s} ds \\
&+ \int_0^t e^{-\lambda s} ds \int_0^1 dy \int_0^1 H_n(n-1)(y) H_n(n-1)(z) p_{2\delta}(y, z) \kappa(y, z) dz \\
&+ 2 \int_0^t \int_0^1 e^{-\lambda s} H_n(n-1)(y) \varphi_n(n, \delta)(y) W(\nu ds, dy) - \lambda \int_0^t v_n(s, \delta) ds. \quad (4.10)
\end{align*}
\]
Observe that under boundary conditions (D1) to (D4),
\[
\begin{align*}
u_n(s, \delta)(0) \Delta \varphi_n(s)(0) &= \varphi_n(n, \delta)(1) \Delta \varphi_n(s)(1) = 0.
\end{align*}
\]
Then by integration by parts,
\[
\langle \varphi_n(s), \Delta \varphi_n(s) \rangle = -\| \nabla \varphi_n(s) \|^2,
\]
which deduces from (4.10) that
\[
\begin{align*}
v_n(s, \delta) &= -\int_0^t \| \nabla \varphi_n(s) \|^2 e^{-\lambda s} ds + 2 \int_0^t \langle \varphi_n(s), P_\delta G_n(s) \rangle e^{-\lambda s} ds \\
&+ \int_0^t e^{-\lambda s} ds \int_0^1 dy \int_0^1 H_n(n-1)(y) H_n(n-1)(z) p_{2\delta}(y, z) \kappa(y, z) dz \\
&+ 2 \int_0^t \int_0^1 e^{-\lambda s} H_n(n-1)(y) \varphi_n(n, \delta)(y) W(\nu ds, dy) - \lambda \int_0^t v_n(s, \delta) ds. \quad (4.11)
\end{align*}
\]
It follows that
\[
E[v_t^{(n,\delta)}] \leq 2 \int_0^t e^{-\lambda s} E[\langle u_s^{(n,\delta)}, P_s G_s^{(n-1)} \rangle] \, ds - \lambda \int_0^t E[v_s^{(n,\delta)}] \, ds \\
+ \int_0^t e^{-\lambda s} \, ds \int_0^1 dy \int_0^1 E[H_s^{(n-1)}(y) H_s^{(n-1)}(z)] p_{2\delta}(y, z) \kappa(y, z) \, dz.
\]

(4.12)

Under condition (4.1),
\[
||\langle u_s^{(n,\delta)}, P_s G_s^{(n-1)} \rangle|| \leq C ||u_s^{(n,\delta)}||^2 + C ||u_s^{(n-1)}||^2
\]

and
\[
\int_0^1 dy \int_0^1 E[H_s^{(n-1)}(y) H_s^{(n-1)}(z)] p_{2\delta}(y, z) \kappa(y, z) \, dz \\
\leq 2^{-1} \kappa_0 \int_0^1 dy \int_0^1 E[|H_s^{(n-1)}(y)|^2 + |H_s^{(n-1)}(z)|^2] p_{2\delta}(y, z) \, dz \\
\leq C \int_0^1 E[|H_s^{(n-1)}(y)|^2] \, dy \leq C E[||u_s^{(n-1)}||^2].
\]

Combining above inequalities and (4.12) we obtain
\[
E[v_t^{(n,\delta)}] \leq -\lambda \int_0^t E[v_s^{(n,\delta)}] \, ds + C \int_0^t E[v_s^{(n,\delta)} + v_s^{(n-1)}] \, ds \leq C \int_0^t E[v_s^{(n-1)}] \, ds
\]

for large \( \lambda > 0 \). From Fatou’s lemma it follows that
\[
E[v_t^{(n)}] \leq \liminf_{\delta \to 0} E[v_t^{(n,\delta)}] \leq C \int_0^t E[v_s^{(n-1)}] \, ds.
\]

Then it is elementary to see that \((Y_t^{(n)})_{t \geq 0}\) and \((\bar{Y}_t^{(n)})_{t \geq 0}\) are Cauchy sequences, and \((Y_t)_{t \geq 0}\) and \((\bar{Y}_t)_{t \geq 0}\) denote the limits. Moreover, \(Y_t(x) = \bar{Y}_t(x) + \bar{Y}_t(x)\). Letting \(n \to \infty\) in (4.4) we can obtain
\[
\bar{Y}_t(x) = \int_0^t ds \int_0^1 G(Y_s(y)) p_{t-s}(x, y) \, dy + \int_0^t \int_0^1 H(Y_s(y)) p_{t-s}(x, y) W(ds, dy).
\]

It follows that for any \(f \in C^2[0, 1]\) satisfying (Di), we have
\[
\langle \bar{Y}_t, f \rangle = \frac{1}{2} \int_0^t \langle \bar{Y}_s, f'' \rangle ds + \int_0^t \langle G(Y_s), f' \rangle ds + \int_0^t \int_0^1 H(Y_s(x)) f(x) W(ds, dx), \ t \geq 0.
\]

In view of (4.6), one sees that \((Y_t)_{t \geq 0}\) is a strong solution to (1.2), which ends the proof. \(\square\)

Next we establish the weak existence part of Theorem 1.6 under a weaker condition.

**Proposition 4.5** Suppose that Condition 1.7(i)–(ii) holds. Then for each boundary conditions (C1) to (C4), (1.2) has a weak solution \((Y_t)_{t \geq 0}\).
We now proceed to proving Proposition 4.5. For \( n \geq 1 \) and \( x \in \mathbb{R} \) let

\[
G_n(x) = \int_{\mathbb{R}} q_{n-1}(x - y)((G(y) \wedge n) \vee (-n))dy
\]

and

\[
H_n(x) = \int_{\mathbb{R}} q_{n-1}(x - y)((H(y) \wedge n) \vee (-n))dy,
\]

where the function \( q_n(x) \) is defined in Section 2. Then \( \lim_{n \to \infty} G_n(x) = G(x) \) and \( \lim_{n \to \infty} H_n(x) = H(x) \) for all \( x \in \mathbb{R} \). Moreover, for each fixed \( n \geq 1 \), \( G_n \) and \( H_n \) satisfy Lipschitz condition, and hence by Proposition 4.1, equation (1.2) with \( G, H \) replaced by \( G_n, H_n \) has a strong solution \((U^{(n)}_t)_{t \geq 0}\). Let \((\tilde{Y}_t)_{t \geq 0}\) be the solution to (1.2) and

\[
\tilde{U}^{(n)}_t(x) := U^{(n)}_t(x) - \tilde{Y}_t(x), \quad t \geq 0, \quad n \geq 1, \quad x \in \mathbb{R}.
\] (4.13)

It follows that for \( f \in C^2[0, 1] \) and \( t \geq 0 \),

\[
\langle \tilde{U}^{(n)}_t, f \rangle = \frac{1}{2} \int_0^t \langle \tilde{U}^{(n)}_s, f'' \rangle ds + \int_0^t \langle G_n(U^{(n)}_s), f \rangle ds
+ \int_0^t \int_0^1 H_n(U^{(n)}_s(x))f(x)W(ds, dx),
\] (4.14)

which implies that

\[
\tilde{U}^{(n)}_t(x) = \int_0^t ds \int_0^1 G_n(U^{(n)}_s(y))p_{t-s}(x, y)dy
+ \int_0^t \int_0^1 H_n(U^{(n)}_s(y))H_n(U^{(n)}_s(z))p_{2\delta}(y, z)\kappa(y, z)dz.
\] (4.15)

for \( t \geq 0 \) and \( x \in \mathbb{R} \).

**Lemma 4.6** Suppose that Condition 1.7(i)-(ii) holds. Then

\[
\sup_{n \geq 1} \mathbb{E} \left[ \sup_{0 < t \leq T} \| U^{(n)}_t \|^2 \right] < \infty.
\]

**Proof.** Let \( \bar{U}^{(n, \delta)} = P_\delta \tilde{U}^{(n)} \) and \( U^{(n, \delta)} = P_\delta U^{(n)}_t \). Then \( U^{(n, \delta)} \) satisfies the boundary condition (Di). As the same argument in (4.14),

\[
\| U^{(n, \delta)} \|^2 = -\int_0^t \| \nabla \tilde{U}^{(n, \delta)} \|^2 ds + 2\int_0^t \langle \tilde{U}^{(n, \delta)}_s, P_\delta G_n(Y^{(n)}_s) \rangle ds
+ \int_0^t ds \int_0^1 dy \int_0^1 H_n(U^{(n)}_s(y))H_n(U^{(n)}_s(z))p_{2\delta}(y, z)\kappa(y, z)dz
+ 2\int_0^t \int_0^1 H_n(U^{(n)}_s(y))\tilde{U}^{(n, \delta)}_s(y)W(ds, dy).
\] (4.16)
We finish the proof by the following three steps.

**Step 1.** For \(\lambda, \delta > 0, t \geq 0\) and \(n \geq 1\) put

\[
f_n^\delta(t) = \|\nabla \bar{U}_t^{(n, \delta)}\|^2 \|\bar{U}_t^{(n, \delta)}\|^{2p-2}, \quad k_n^\delta(t) = \|\bar{U}_t^{(n, \delta)}\|^{2p-2} \bar{U}_t^{(n, \delta)}, P_3 G_n(U_t^{(n)})
\]

\[
g_n^\delta(t) = \|\bar{U}_t^{(n, \delta)}\|^{2p-2} \int_0^1 dy \int_0^1 H_n(U_t^{(n)}(y)) H_n(U_t^{(n)}(z)) p_{2\delta}(y, z) \kappa(y, z) dz,
\]

\[
h_n^\delta(t) = \|\bar{U}_t^{(n, \delta)}\|^{2p-4} \int_0^1 dy \int_0^1 M_n^\delta(t, y) M_n^\delta(t, z) \kappa(y, z) dz,
\]

\[
l_n^\delta(t) = \int_0^t \int_0^1 \|\bar{U}_s^{(n, \delta)}\|^{2p-2} M_n^\delta(s, y) W(ds, dy),
\]

where \(M_n^\delta(t, y) := H_n(U_t^{(n)}(y)) C_t^{(n, 2\delta)}(y)\). In this step we estimate \(g_n^\delta, h_n^\delta\) and \(k_n^\delta\).

Under Condition 1.7(i),

\[
|G_n(x)| \leq \int_\mathbb{R} q_1(y)|G(x - y/\sqrt{n})|dy \leq \int_\mathbb{R} q_1(y)(|x| + |y|/\sqrt{n}) + 1)dy \leq C(|x| + 1)
\]

and

\[
|H_n(x)| \leq C(|x| + 1). \tag{4.17}
\]

Then applying Lemmas 2.1 and 4.2, Condition 1.7(ii), and (4.13) one sees that

\[
2\|\bar{U}_t^{(n, \delta)}, P_3 G_n(U_t^{(n)})\| \leq \|\bar{U}_t^{(n, \delta)}\|^2 + \|G_n(U_t^{(n)})\|^2
\]

\[
\leq C[\|\bar{U}_t^{(n)}\|^2 + \|U_t^{(n)}\|^2 + 1] \leq C[\|\bar{U}_t^{(n)}\|^2 + 1]
\]

and

\[
2\int_0^1 dy \int_0^1 H_n(U_t^{(n)}(y)) H_n(U_t^{(n)}(z)) p_{2\delta}(y, z) \kappa(y, z) dz
\]

\[
\leq \kappa_0 \int_0^1 dy \int_0^1 [H_n(U_t^{(n)}(z))^2 + H_n(U_t^{(n)}(y))^2] p_{2\delta}(y, z) dz \leq C[\|\bar{U}_t^{(n)}\|^2 + 1].
\]

Similarly,

\[
\left| \int_0^1 dy \int_0^1 M_n^\delta(t, y) M_n^\delta(t, z) \kappa(y, z) dz \right|^{1/2} \leq \kappa_0^{1/2} \int_0^1 |M_n^\delta(t, y)| dy
\]

\[
\leq C \|H_n(U_t^{(n)})\|^2 + C \int_0^1 dy \int_0^1 \|\bar{U}_t^{(n)}(x)\| p_{2\delta}(x, y) dx
\]

\[
\leq C(\|H_n(U_t^{(n)})\|^2 + \|\bar{U}_t^{(n)}\|^2) \leq C[\|\bar{U}_t^{(n)}\|^2 + 1]. \tag{4.18}
\]

Therefore,

\[
g_n^\delta(t) \leq C(\|\bar{U}_t^{(n)}\|^2 + 1) \|\bar{U}_t^{(n)}\|^{2p-2}
\]

\[
\leq C[\|\bar{U}_t^{(n)}\|^{2p} + \|\bar{U}_t^{(n)}\|^{2p} + \|\bar{U}_t^{(n)}\|^{2p-2}] \leq C[\|\bar{U}_t^{(n)}\|^{2p} + 1] \tag{4.19}
\]

and

\[
h_n^\delta(t) \leq C(\|\bar{U}_t^{(n)}\|^2 + 1)^2 \|\bar{U}_t^{(n)}\|^{2p-4} \leq C[\|\bar{U}_t^{(n)}\|^{2p} + 1], \tag{4.20}
\]

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where we used the fact $x^r \leq x + 1$ and
\[ xy \leq p^{-1}x^p + (p - 1)p^{-1}y^{p/(p - 1)}, \quad (x + y)^{2p} \leq 2px^{2p} + 2py^{2p} \]
for all $x, y \geq 0$ and $0 \leq r \leq 1$. Similarly,
\[ k_n^\delta(t) \leq C\|\bar{U}_t^{(n)}\|^{2p-2} \cdot (\|\bar{U}_t^{(n)}\|^2 + 1) \leq C\|\bar{U}_t^{(n)}\|^{2p} + 1. \tag{4.21} \]

**Step 2.** We show that for all $p > 1$,
\[ \sup_{s \in [0,t], n \geq 1} E[\|U_s^{(n)}\|^{2p}] < \infty, \quad t > 0. \tag{4.22} \]

It follows from (4.16) and Itô’s formula that
\[ \|\bar{U}_t^{(n,\delta)}\|^{2p} = p \int_0^t \left[ g_n^\delta(s) - f_n^\delta(s) + 2(p - 1)h_n^\delta(s) + k_n^\delta(s) \right] ds + p\bar{f}_n^\delta(t). \tag{4.23} \]
Then by (4.19)-(4.21) and (4.23)
\[ E[\|\bar{U}_t^{(n,\delta)}\|^{2p}] \leq C \int_0^t E[\|\bar{U}_s^{(n)}\|^{2p} + 1] ds. \]

Using Lemma 2.1 and Fatou’s lemma,
\[ E[\|\bar{U}_t^{(n)}\|^{2p}] = E[\lim_{\delta \to 0} \|\bar{U}_t^{(n,\delta)}\|^{2p}] \leq \liminf_{\delta \to 0} E[\|\bar{U}_t^{(n,\delta)}\|^{2p}] \leq C \int_0^t E[\|\bar{U}_s^{(n)}\|^{2p} + 1] ds. \]

Now by Gronwall’s lemma we obtain (4.22).

**Step 3.** By Doob’s inequality and the arguments in (4.18) and (4.19), for all $t \geq 0$,
\[ E \left[ \sup_{s \in [0,t]} |U_s^{(n)}(s)|^2 \right] \leq 4E \left[ \int_0^t ds \int_0^1 dy \int_0^1 \|\bar{U}_s^{(n,\delta)}\|^{4p-4} M_n^\delta(s, y) M_n^\delta(s, z) |\kappa(y, z)| ds \right] \]
\[ \leq C E \left[ \int_0^t \|\bar{U}_s^{(n)}\|^{4p-4} \|\bar{U}_t^{(n)}\|^2 + 1)^2 ds \right] \leq C E \left[ \int_0^t (\|\bar{U}_s^{(n)}\|^{4p} + 1) ds \right]. \]

Combining the above inequality with (4.22) and (4.23) one can obtain
\[ \sup_{n \geq 1, \delta > 0} E \left[ \sup_{s \in [0,t]} \|U_s^{(n,\delta)}\|^{2p} \right] < \infty. \]

Now using Fatou’s lemma,
\[ \sup_{n \geq 1} E \left[ \sup_{s \in [0,t]} \|U_s^{(n)}\|^{2p} \right] \leq \sup_{n \geq 1} E \left[ \liminf_{\delta \to 0} \sup_{s \in [0,t]} \|U_s^{(n,\delta)}\|^{2p} \right] \]
\[ \leq \sup_{n \geq 1} \liminf_{\delta \to 0} E \left[ \sup_{s \in [0,t]} \|U_s^{(n,\delta)}\|^{2p} \right] < \infty, \]
which finishes the proof. □
Lemma 4.7 Suppose that Condition (i)-(ii) hold. Then for each $\alpha \in (0, 1)$,

$$E[|\bar{U}^{(n)}_t(x_1) - \bar{U}^{(n)}_t(x_2)|^{2p}] \leq C|x_1 - x_2|^\alpha, \quad t \in (0, T], \ x_1, x_2 \in [0, 1], \ n \geq 1$$

and

$$E[|\bar{U}^{(n)}_{t+\varepsilon}(x) - \bar{U}^{(n)}_t(x)|^{2p}] \leq C\varepsilon^{\alpha p/2}, \quad t \in (0, T], \ \varepsilon > 0, \ x \in [0, 1], \ n \geq 1.$$ 

Proof. Since the proofs are similar, we only present the first one. It follows from (4.15) that

$$U^{(n)}_t(x_1) - U^{(n)}_t(x_2) = \int_0^t \int_0^1 M_n(x_1, x_2, t, s, y)W(ds, dy) + \int_0^t ds \int_0^1 G_n(U^{(n)}_s(y))k_{t-s}(x_1, x_2, y)dy$$

$$=: I^n_t(x_1, x_2) + J^n_t(x_1, x_2), \quad (4.24)$$

where

$$k_t(x_1, x_2, y) := p_t(x_1, y) - p_t(x_2, y), \ M_n(x_1, x_2, t, s, y) := H_n(U^{(n)}_s(y))k_{t-s}(x_1, x_2, y).$$

By Lemma 2.3

$$|k_t(x_1, x_2, y)| = |k_t(x_1, x_2, y)|^{1-\alpha}|k_t(x_1, x_2, y)|^\alpha \leq Ct^{-(\alpha+1)/2}|x_1 - x_2|^\alpha,$$

which deduces

$$\int_0^1 |k_t(x_1, x_2, y)|^2dy \leq Ct^{-(\alpha+1)/2}|x_1 - x_2|^\alpha \int_0^1 [|p_t(x_1, y)| + |p_t(x_2, y)|]dy$$

$$\leq Ct^{-(\alpha+1)/2}|x_1 - x_2|^\alpha, \quad (4.25)$$

where the last inequality follows from Lemma 2.1. Applying H"older's inequality, (4.17) and (4.25), for all $n \geq 1$,

$$\left| \int_0^1 dy \int_0^1 M_n(x_1, x_2, t, s, y)M_n(x_1, x_2, t, s, z)\kappa(y, z)dz \right|$$

$$\leq \kappa_0 \left[ \int_0^1 |M_n(x_1, x_2, t, s, y)|dy \right]^2$$

$$\leq \kappa_0 \|H_n(U^{(n)}_s)\|^2 \int_0^1 |k_{t-s}(x_1, x_2, y)|^2dy$$

$$\leq C(\|U^{(n)}_s\|^2 + 1)(t-s)^{-\alpha+1}|x_1 - x_2|^\alpha.$$ 

Using Burkholder-Davis-Gundy's inequality we then get

$$E[|I^n_t(x_1, x_2)|^{2p}] \leq C \mathbb{E}\left[ \left| \int_0^t ds \int_0^1 dy \int_0^1 M_n(x_1, x_2, t, s, y)M_n(x_1, x_2, t, s, z)\kappa(y, z)dz \right|^p \right]$$

$$\leq C|x_1 - x_2|^\alpha \mathbb{E}\left[ \sup_{0 \leq s \leq T} \|U^{(n)}_s\|^{2p} + 1 \right] \left| \int_0^t s^{-(\alpha+1)/2}ds \right|^p, \ t \in (0, T].$$
Similarly,
\[ E[|J^p_t(x_1, x_2)|^{2p}] \leq C|x_1 - x_2|^{2p} \sup_{0 \leq s \leq t} \left[ \frac{\|U^{(n)}_s\|^2 + 1}{s^{-(\alpha + 1)/2}} \right] \int_0^t s^{-(\alpha + 1)/2} ds, \quad t \in (0, T], \]
which yields the assertion by \((4.24)\) and Lemma \(4.6\).

Proof of Proposition 4.5. By Lemma \(4.7\) and Kolmogorov’s criteria (e.g. [71, Corollary 16.9]), for each \(T > 0\), the sequence \(\{\hat{U}^{(n)}_t(x) : (t, x) \in [0, T] \times [0, 1]\}\) is tight on \(C([0, T] \times [0, 1])\) and hence, has a convergent subsequence \((\hat{U}^{(nk)}_t)\) which converges in law to \((\hat{U}_t)\). It follows that \((U^{(nk)}_t)\) converges in law to \((U_t)\) as \(k \to \infty\). Thus
\[ M^{(nk)} := (U^{(nk)}_t, W_t, \mu_0(t), \mu_1(t))_{t \geq 0} \longrightarrow M := (U_t, W_t, \mu_0(t), \mu_1(t))_{t \geq 0} \]
in law as \(k \to \infty\). Applying Skorokhod’s representation, on another probability space, there are continuous processes
\[ \hat{M}^{(nk)} := (\hat{U}^{(nk)}_t, \hat{W}^{(nk)}_t, \hat{\mu}_0^{(nk)}(t), \hat{\mu}_1^{(nk)}(t))_{t \geq 0} \text{ and } \hat{M} := (\hat{U}_t, \hat{W}_t, \hat{\mu}_0(t), \hat{\mu}_1(t))_{t \geq 0} \]
with the same distribution as \(M^{(nk)}\) and \(M\), respectively. Moreover, \(\hat{M}^{(nk)}\) converges to \(\hat{M}\) almost surely. In the following let \(f \in C^2[0, 1]\) satisfy (Di). Since \((U^{(nk)}_t)\) is a strong solution to \((1.2)\) with \(G_n\) and \(H_n\) replaced by \(G\) and \(H\) by Proposition 4.1 then
\[
\langle U^{(nk)}_t, f \rangle = \langle Y_0, f \rangle + \frac{1}{2} \int_0^t [\langle U^{(nk)}_s, f'' \rangle + F_s(f)] ds + \int_0^t \langle G_n(U^{(nk)}_s), f \rangle ds + \int_0^t \int_0^1 H_n(U^{(nk)}_s(y)) f(y) W(ds, dy).
\]
Define \(\hat{F}_s^{(n)}(f)\) and \(\hat{F}_s(f)\) similar as \(F_s(f)\) with \(\mu_0, \mu_1\) replaced by \(\hat{\mu}_0^{(n)}, \hat{\mu}_1^{(n)}\) and \(\hat{\mu}_0, \hat{\mu}_1\), respectively. Therefore,
\[
\langle \hat{U}^{(nk)}_t, f \rangle = \langle Y_0, f \rangle + \frac{1}{2} \int_0^t [\langle \hat{U}^{(nk)}_s, f'' \rangle + \hat{F}_s^{(nk)}(f)] ds + \int_0^t \langle G_n(\hat{U}^{(nk)}_s), f \rangle ds + \int_0^t \int_0^1 H_n(\hat{U}^{(nk)}_s(y)) f(y) W_n(ds, dy).
\]
It thus follows from [36, Lemma 2.4] that
\[
\langle \hat{U}_t, f \rangle = \langle Y_0, f \rangle + \frac{1}{2} \int_0^t [\langle \hat{U}_s, f'' \rangle + \hat{F}_s(f)] ds + \int_0^t \langle G(\hat{U}_s), f \rangle ds + \int_0^t \int_0^1 H(\hat{U}_s(y)) f(y) W(ds, dy).
\]
This ends the proof. \(\square\)

Now we are ready to proving
Proof of Theorem 1.6. The second assertion follows from Theorem 1.4 immediately. Now let us prove the first one. For each $n \geq 1$ define stopping time  

$$\tau_n := \inf \{ t \geq 0 : \mu_0(t) + \mu_1(t) \geq n \}. $$

Then $\tau_n \to \infty$ almost surely as $n \to \infty$. By Corollary 1.5 and Proposition 4.5 for each $\mu^0_i(t) := \mu_i(t \wedge \tau_n)$ and $\mu^1_0(t) := \mu_0(t \wedge \tau_n)$, there is a unique strong solution $(Y^n_t)_{t \geq 0}$ to (1.2) satisfying the boundary condition by [19, Corollary 2.8]. Moreover, $Y^n_t = Y^n_{t+1}$ for all $t \leq \tau_n$. Let $Y_t = Y^n_t$ for each $t \leq \tau_n$. Then $(Y_t)_{t \geq 0}$ satisfies (1.2) with the boundary condition (C1) to (C4).

Proof of Theorem 1.9. The conclusions can be justified by using essentially the same argument as those in the proofs of Theorems 1.4 and 1.6 with $p^{(1)}_t(x, y)$ and $p^{(2)}_t(x, y)$ replaced by  

$$p^{(1)}_t(x, y) = q_t(x, y) - q_t(-x, y), \quad p^{(2)}_t(x, y) = q_t(x, y) + q_t(-x, y),$$

respectively. We omit the details. 

5 Proof of Theorem 1.8

In this section we present the proof of Theorem 1.8. Let $(Y_t)_{t \geq 0}$ be a weak solution to (1.2) with the boundary condition (Ci) and $(p_t)_{t \geq 0}$ be the solution to (2.1) with the boundary condition (Di) for $i \in \{1, 2, 3, 4\}$. Write $F_s^{t-s}(x)$ for $F_s^{(i)}(p_{t-s}^{(i)}(x, \cdot))$. Let arbitrary constants $T > 0$ and $p \geq 1$ be fixed and $Y_0 \in C[0, 1]$. We will prove our theorem by a sequence of lemmas.

Lemma 5.1 For each $t > 0$ and $f \in C[0, 1]$,  

$$\langle Y_t, f \rangle = \langle Y_0, P_t f \rangle + \int_0^t [2^{-1} \langle F_s^{t-s}, f \rangle + \langle G(Y_s), P_{t-s} f \rangle] ds$$

$$+ \int_0^t \int_0^1 H(Y_s(y)) P_{t-s} f(y) W(ds, dy).$$

Moreover, for $x \in (0, 1)$,  

$$Y_t(x) = \langle Y_0, p_t^x \rangle + \int_0^t [2^{-1} F_s^{t-s}(x) + \langle G(Y_s), p_{t-s}^x \rangle] ds$$

$$+ \int_0^t \int_0^1 H(Y_s(y)) p_{t-s}^x(y) W(ds, dy)$$

(5.2)

with $p_t^x(y) := p_t(x, y)$.

Proof. Let $t > 0$ be fixed. By Definition 1.11 for each $f \in C[0, 1]$,  

$$\langle Y_t, P_v f \rangle = \langle Y_0, P_v f \rangle + \frac{1}{2} \int_0^t ds \int_0^1 \langle Y_s, \Delta_v p_v(x, \cdot) \rangle f(x) dx + \frac{1}{2} \int_0^t \langle F_s^v, f \rangle ds$$

$$+ \int_0^t \int_0^1 H(Y_s(y)) p_{t-s}^v(y) W(ds, dy).$$

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Set \( t_i = it/n \) for \( 0 \leq i \leq n \) and \( n \geq 1 \). In view of (2.1) and (5.3), we obtain
\[
\langle Y_t, f \rangle - \langle Y_0, P_t f \rangle
= \sum_{i=1}^{n} \langle Y_{t_i}, P_{t-t_i} f - P_{t-t_{i-1}} f \rangle + \sum_{i=1}^{n} \langle Y_{t_i}, P_{t-t_{i-1}} f \rangle - \langle Y_{t_{i-1}}, P_{t-t_{i-1}} f \rangle
= \frac{1}{2} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \langle Y_t, \partial_s p_s(x, \cdot) \rangle f(x) dt + \frac{1}{2} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \langle Y_s, \Delta_x p_{t-t_{i-1}} (x, \cdot) \rangle f(x) ds + \frac{1}{2} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \int_{0}^{1} H(Y_s(y)) P_{t-t_{i-1}} f(y) W(ds, dy).
\]

Letting \( n \to \infty \) we obtain (5.1). Taking \( f(y) = p_0^2(y) \) in (5.1) and then letting \( \delta \to \infty \) and using dominated convergence and Lemma 5.1 we get (5.2). \( \square \)

**Lemma 5.2** Under Condition \( (i)-(ii) \), we have
\[ \sup_{0 < t \leq T, x \in [0,1]} E[|Y_t(x)|^{2p}] < \infty. \]

**Proof.** For each \( n \geq 1 \) define stopping time \( \tau_n \) by
\[ \tau_n := \inf\{t \geq 0 : \|Y_t\|_0 \geq n\}. \]
Then \( \tau_n \to \infty \) as \( n \to \infty \). Applying Lemma 5.1 we find
\[
Y_t(x)1_{\{t \leq \tau_n\}} = \langle Y_0, p_t^x \rangle 1_{\{t \leq \tau_n\}} + 2^{-1} 1_{\{t \leq \tau_n\}} \int_{0}^{t} F_{t-s}^y(x) ds + 1_{\{t \leq \tau_n\}} \int_{0}^{t} \langle G(Y_s), p_{t-s}^x \rangle ds
+ 1_{\{t \leq \tau_n\}} \int_{0}^{t} \int_{0}^{1} H(Y_s(y)) p_{t-s}^x(y) W(ds, dy)
=: I_t^{n}(x) + 2^{-1} I_t^{2n}(x) + I_t^{3n}(x) + I_t^{4n}(x). \]
By applying Lemma 2.1 we find
\[ E[|I_t^{1,n}(x)|^{2p}] \leq \|Y_0\|_{0}^{2p} \cdot \langle 1, |p_t^x| \rangle^{2p} \leq C\|Y_0\|_{0}^{2p}. \quad (5.5) \]

Using Lemmas 2.3 and 2.4 and Condition 1.7(ii) one can conclude that
\[ \sup_{n \geq 1, 0 < t \leq T, x \in [0,1]} E[|I_t^{2,n}(x)|]^{2p} \leq \sup_{0 < t \leq T, x \in [0,1]} E\left[ \int_0^t F_s^{t-s}(x)ds \right]^{2p} < \infty. \quad (5.6) \]

By applying Lemma 2.1 we find
\[ E[|I_t^{3,n}(x)|^{2p}] \leq C E\left[ \int_0^t |G(Y_s)|^{2p}, p_l^x(y) |1_{\{s \leq \tau_n\}} ds \right] \]
\[ \leq C E\left[ \int_0^t |Y_s|^{2p} 1_{\{s \leq \tau_n\}} + 1, p_l^x(y) | ds \right] \]
\[ \leq C \int_0^t \sup_{y \in [0,1]} E[|Y_s(y)|^{2p} 1_{\{s \leq \tau_n\}}] ds + Ct. \quad (5.7) \]

Applying Lemmas 2.1 and 2.3 Hölder’s inequality and Condition 1.7(i) we obtain
\[ M(t, s, x) := |\int_0^1 dy \int_0^1 H(Y_s(y)) p_l^x(y) H(Y_s(z)) p_{l-s}^z(z) \kappa(y, z) dz| \]
\[ \leq \kappa_0 \int_0^1 |H(Y_s(y)) p_l^x(y)| dy \leq C \int_0^1 |H(Y_s(y))|^2 |p_l^x(y)| dy \]
\[ \leq C \int_0^1 [|Y_s(y)|^2 + 1] \cdot |p_l^x(y)| dy. \]

It then follows from Hölder’s inequality again that
\[ \left| \int_0^t M(t - s, s, x) 1_{\{s \leq \tau_n\}} ds \right|^{2p} \leq C \int_0^t ds \int_0^1 [|Y_s(y)| 1_{\{s \leq \tau_n\}}|^{2p} + 1] \cdot |p_{l-s}^x(y)| dy \]
for $0 < t \leq T$ and $n \geq 1$. Then by virtue of Burkholder-Davis-Gundy’s inequality,
\[ E[|I_t^{1,n}(x)|^{2p}] \leq E\left[ \int_{t \leq \tau_n} \int_0^t \int_0^1 H(Y_s(y)) p_{l-s}^x(y) W(ds, dy) \right]^{2p} \]
\[ = E\left[ \int_{t \leq \tau_n} \int_0^t \int_0^1 H(Y_s(y)) p_{l-s}^x(y) W(ds, dy) \right]^{2p} \]
\[ \leq E\left[ \int_0^t \int_0^1 H(Y_s(y)) p_{l-s}^x(y) 1_{\{s \leq \tau_n\}} W(ds, dy) \right]^{2p} \]
\[ \leq CE\left[ \int_0^t M(t - s, s, x) 1_{\{s \leq \tau_n\}} ds \right]^{2p} \]
\[ \leq C \int_0^t \sup_{x \in [0,1]} E[|Y_s(y)| 1_{\{s \leq \tau_n\}}|^{2p} ds + C \]
for all $0 < t \leq T$ and $n \geq 1$. Combining this with (5.4) to (5.7) we have
\[ \sup_{x \in [0,1]} E[|Y_t(x)| 1_{\{t \leq \tau_n\}}|^{2p} \leq C + C \int_0^t \sup_{x \in [0,1]} E[|Y_s(x)| 1_{\{s \leq \tau_n\}}|^{2p} ds, \quad 0 < t \leq T, \ n \geq 1. \]
Now applying Gronwall’s lemma we know that
\[ \sup_{t \in (0,T], x \in [0,1], n \geq 1} E\left[|Y_t(x)1_{\{t \leq \tau_n\}}|^{2p}\right] < \infty. \]

From Fatou’s lemma it follows that
\[
\sup_{t \in (0,T], x \in [0,1]} E\left[|Y_t(x)|^{2p}\right] = \sup_{t \in (0,T], x \in [0,1]} E\left[\lim_{n \to \infty} |Y_t(x)1_{\{t \leq \tau_n\}}|^{2p}\right] \\
\leq \sup_{t \in (0,T], x \in [0,1], n \geq 1} E\left[|Y_t(x)1_{\{t \leq \tau_n\}}|^{2p}\right] < \infty,
\]
which completes the proof. \(\square\)

**Lemma 5.3** Suppose that Condition 1.1 holds. Then for boundary condition (C2),
\[
E\left[|Y_t(x_1) - Y_t(x_2)|^{2p}\right] \leq C[t^{-p} + 1]|x_1 - x_2|^p, \quad t \in (0, T], \ x_1, x_2 \in [0, 1]
\]
and for boundary conditions (C1), (C3) and (C4),
\[
E\left[|Y_t(x_1) - Y_t(x_2)|^{2p}\right] \leq C[t^{-p} + 1]|x_1 - x_2|^{2p\gamma_0}, \quad t \in (0, T], \ x_1, x_2 \in [0, 1].
\]

**Proof.** Since the proofs are similar, we only state that of boundary condition (C3). By applying Lemma 5.1
\[
Y_t(x_1) - Y_t(x_2) \\
= \langle Y_0, p_t^{x_1} - p_t^{x_2} \rangle + \frac{1}{2} \int_0^t \left[ F_{t-s}^{x_1}(s) - F_{t-s}^{x_2}(s) \right] ds + \int_0^t \left( \langle G(Y_s), p_{t-s}^{x_1} - p_{t-s}^{x_2} \rangle \right) ds \\
+ \int_0^t \int_0^1 H(Y_s(y))[p_{t-s}^{x_1}(y) - p_{t-s}^{x_2}(y)] W(ds, dy) \\
=: \sum_{i=1,2,3,4} I_i^x(x_1, x_2), \quad t \in (0, T], \ x_1, x_2 \in [0, 1]. \tag{5.8}
\]

In view of Lemmas 2.4 and 2.3
\[
\langle 1, |p_t^{x_1} - p_t^{x_2}| \rangle \leq C t^{-1/2}|x_1 - x_2|^{1/2} \langle 1, \sqrt{|p_t^{x_1} - p_t^{x_2}|} \rangle \\
\leq C t^{-1/2}|x_1 - x_2|^{1/2} \sqrt{\langle 1, |p_t^{x_1} - p_t^{x_2}| \rangle} \leq C t^{-1/2}|x_1 - x_2|^{1/2}
\]
for all \(t > 0\), which leads to
\[
|I_i^x(x_1, x_2)|^{2p} \leq \|Y_0\|^{2p}_0 \langle 1, |p_t^{x_1} - p_t^{x_2}| \rangle^{2p} \leq C t^{-p}|x_1 - x_2|^p, \quad t \in (0, T], \tag{5.9}
\]
and by using Condition 1.1(i) and Hölder’s inequality we deduce
\[
E\left[|I_i^x(x_1, x_2)|^{2p}\right] \leq C E\left[\int_0^t \langle |Y_s| + 1, |p_{t-s}^{x_1} - p_{t-s}^{x_2}| \rangle ds \right]^{2p} \\
\leq C E\left[\int_0^t \langle (|Y_s| + 1)^{2p}, |p_{t-s}^{x_1} - p_{t-s}^{x_2}| \rangle ds \right] \cdot \int_0^t \langle 1, |p_{t-s}^{x_1} - p_{t-s}^{x_2}| \rangle ds \right]^{2p-1}
\]
\[26\]
\[
\leq C_{c_0}|x_1 - x_2|^p, \quad t \in (0, T], \quad (5.10)
\]

where
\[
c_0 := \sup_{0 < t \leq T, y \in [0, 1]} \mathbb{E}[|Y_s(y)|^{2p} + 1] < \infty
\]

by Lemma 5.2. Moreover, by Condition 1.7 (i),

\[
M_{t,s}(x_1, x_2) := \left| \int_0^1 \int_0^1 H(Y_s(y))[p_{t}^{x_1}(y) - p_{s}^{x_2}(y)]H(Y_s(z))[p_{t}^{x_1}(z) - p_{s}^{x_2}(z)]\kappa(y, z)dz \right|
\]

\[
\leq C \left| \int_0^1 [|Y_s(y)| + 1]|p_{t}^{x_1}(y) - p_{s}^{x_2}(y)|dy \right|^2,
\]

which leads to

\[
\mathbb{E}\left[ \left( \int_0^t M_{t-s,s}(x_1, x_2)ds \right)^p \right] \leq C|x_1 - x_2|^p, \quad t \in (0, T].
\]

Then by Burkholder-Davis-Gundy’s inequality,

\[
\mathbb{E}\left[ |I_t^4(x_1, x_2)|^{2p} \right] \leq C \mathbb{E}\left[ \left( \int_0^t M_{t-s,s}(x_1, x_2)ds \right)^p \right] \leq C|x_1 - x_2|^p, \quad t \in (0, T]. \quad (5.11)
\]

Observe that

\[
F_{s-t}^t(x_1) - F_{s-t}^t(x_2) = \mu_0(s)[\nabla p_{t-s}^{x_1}(0) - \nabla p_{t-s}^{x_2}(0)] + \mu_1(s)J_{t-s}(x_1, x_2)
\]

with \( J_t(x_1, x_2) := p_{t}^{x_1}(1) - p_{s}^{x_2}(1) \). Then by using (2.4) and Lemma 2.5

\[
\mathbb{E}\left[ \left( \int_0^t \mu_0(s)[\nabla p_{t-s}^{x_1}(0) - \nabla p_{t-s}^{x_2}(0)]ds \right)^{2p} \right] \leq C[t^{-p} + 1]|x_1 - x_2|^{2p\gamma_0}. \quad (5.12)
\]

In view of Lemma 2.3,

\[
|J_t(x_1, x_2)| = \left| \frac{J_t(x_1, x_2)}{2} \right|^{1/2} \cdot \left| \frac{J_t(x_1, x_2)}{2} \right|^{1/2} \leq C^{-1/4} t^{-1/2} |x_1 - x_2|^{1/2} = C t^{-3/4} |x_1 - x_2|^{1/2},
\]

which yields

\[
\mathbb{E}\left[ \left( \int_0^t |\mu_1(s)J_{t-s}(x_1, x_2)|ds \right)^{2p} \right] \leq C t^{p/2} \mathbb{E}\left[ \sup_{s \in (0, T)} |\mu_1(s)|^{2p} \right] |x_1 - x_2|^p.
\]

Combining the above inequality with (5.12) we obtain

\[
\mathbb{E}\left[ |I_t^2(x_1, x_2)|^{2p} \right] \leq C[t^{-p} + 1]|x_1 - x_2|^{2p\gamma_0}, \quad t \in (0, T].
\]

Together this with (5.8)-(5.11) one ends the proof.

Similar argument as in the proof of Lemma 5.3, we can deduce the following lemma and omit the proof.

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Lemma 5.4 Suppose that Condition 1.7 holds and $0 < T_1 < T$. Then for boundary condition (C2),

$$E[|Y_{t_1}(x) - Y_{t_2}(x)|^{2p}] \leq C|t_1 - t_2|^{p/2}, \quad t_1, t_2 \in [0, T], \; x \in [0, 1]$$

and for boundary conditions (C1), (C3) and (C4),

$$E[|Y_{t_1}(x) - Y_{t_2}(x)|^{2p}] \leq C|t_1 - t_2|^{p\gamma_0}, \quad t_1, t_2 \in [T_1, T], \; x \in [0, 1].$$

Proof of Theorem 1.8 By Kolmogorov’s continuity criteria (see e.g. [32, Corollary 1.2(ii)]) and Lemmas 5.3 and 5.4 the assertion follows immediately. \qed

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