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Research Article

Higher-Order Accurate and Conservative Hybrid Numerical Scheme for Relativistic Time-Fractional Vlasov-Maxwell System

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The historical analysis demonstrates that plasma scientists produced a variety of numerical methods for solving “kinetic” models, i.e., the Vlasov-Maxwell (VM) system. Still, on the other hand, a significant fact or drawback of most algorithms is that they do not preserve conservation philosophies. This is a crucial fact that cannot be disregarded since the Vlasov Maxwell system is associated with conservation rules and is capable of assessing after the accomplishment of certain helpful mathematical actions. To examine the fractional-order routine of charged particles, we constructed a fractional-order plasma model and proposed a higher-order conservative numerical approach based on operational matrices theory and Shifted Gegenbauer estimations. Numerical convergence is investigated to confirm its competence and compatibility. This concept may be used in problems involving variable order and multidimensionality, such as those involving Vlasov and Boltzmann systems.

1. Introduction

The VM system is an important instrument due to its vast variety of applications [1–3]. Modern numerical plasma research is separated into two independent disciplines based on kinetic theory, notably the “particle” in cell technique (PICT) and the Vlasov equation (VE). The primary portion couple’s plasma “particle” motion equations to Maxwell equations and numerically cracks them using the particle in approach [1, 4], and [5]. The second approach employs FEM [6] and FDM [3] to discretize the VE. These are grid-constructed procedures. The main downside of debated numerical techniques is that they violate conservation commitments. Noteworthy key concerns with PICT are that in some conditions, complete energies rise dramatically in the lack of any valid source of energy. In 2010, a significant numerical study on this subject was conducted based on conservation rules. The terms “Crank” Nicolson and temporal integration are used to characterize the “conservative” arrangements of PICT [7–10].

G. Lapenta [11] published research in 2017, in which he analyzed motion equations (ME) and “Maxwell” equations (ME) using “Crank” Nicolson and leap-frog discretization methods. The technique maintains “energy” conservation, but on the other hand, it contradicts Gauss’s rule. As a result, we can conclude that formulating the PIC method, which adheres to all conservation rules, is difficult because two distinct types of the scheme are utilized, namely PICT and FDM. The forms of the “particles” are strongly retained by the integration approach used in PIC techniques, and they
are also dispersion in the numerical sense, although one cannot state that FDM is dispersion-free. As a result of these mathematical conflicts, we may accomplish that it is dreadful to eloquent the PIC approach, which perfectly follows the conservation rules.

Spectral algorithms are a powerful and groundbreaking tool for tackling several kinds of mathematical models. Different modules of "polynomials" in the orthogonal form are available in the prevailing literature for spectral estimates [12–14]. Two types of structures, i.e., VM [15] and Vlasov-Poisson (VP) [15, 16], are handled (numerically) using spectral methods which further linked Crank-Nicolson. These approaches assist us in overcoming difficulties, mainly numerical dispersion [17], that the PIC method encountered.

The TFVMS is built and explored in this study utilizing a specific geometry and an improved structure of numerical scheme based on FD and SGP approximations. Caputo sense's conservative time-fractional order finite difference approximations [18–24] has been utilized. Assuming that the "function" is constructed on multiple variables, we appropriately estimate stated "polynomials" and continue with the construction of the operational "matrices". Recent advancement in this subject is detailed in ref [19, 20]. The project tightly enforces conservation laws, which will be discussed in further detail in the next segment. The numerical framework exhibits virtuous convergence, which is also discussed in the paper. Finally, we conclude our research.

### Table 1: Important symbols used in the theory.

| Symbols | Name of symbols |
|---------|----------------|
| $F$     | Distribution function |
| $t$     | Time |
| $x, y, z$ | Velocity components |
| $p_x, p_y, p_z$ | Momentum components |
| $\gamma$ | Relativistic parameter |
| $C_\mu^\nu$ | Gegenbauer polynomials |
| $\varphi_f(F)$ | Shifted gegenbauer |
| $\varphi_{E_x}(E_x)$ | Vector norms of $E_x$, Electric field |
| $E$ | Caputo time fractional |
| $B$ | Current density |
| $\zeta D_t^\alpha$ | Charge density |
| $f$ | Normalizing factor |
| $\rho$ | Weight function |
| $\psi_{B_x}(B_x)$ | Vector norms of $B_x$, |
| $\psi_{E_y}(E_y)$ | Vector norms of $E_y$ |

### 2. Mathematical Modelling and Conservative Numerical Scheme

2.1. Generalized Form. The most recent and generalized form of VMS with important laws [19–24] are explained as:

\[
\begin{align*}
C_0 D_t^\alpha F + \frac{\partial}{\partial r} \left( \frac{p}{ym} F \right) + \frac{\partial}{\partial p} \left( q \frac{p \times B}{ymc} \right) F, y = \left( 1 + \frac{|p|}{mc} \right)^{1/2},
\end{align*}
\]

\[
\begin{align*}
\frac{1}{c_0} D_t^\alpha E - \nabla \times B + \frac{4\pi}{c} J = 0, \nabla \cdot E = 4\pi \rho, \\
\frac{1}{c_0} D_t^\alpha B - \nabla \times E = 0, \nabla \cdot B = 0, \frac{1}{c_0} D_t^\alpha \rho + \nabla \cdot J = 0.
\end{align*}
\]

Table 1 contains an explanation of all the symbols used in the preceding system.

2.2. Mathematical Assumptions and Conversion. The significant assumptions for the problems are:
\[ C_0^2 D_t^2 F + \frac{1}{ym} \left( p_x \frac{\partial F}{\partial x} + p_y \frac{\partial F}{\partial y} \right) + q \left( E_x \frac{\partial F}{\partial p_x} + E_y \frac{\partial F}{\partial p_y} \right) \]
\[ + \frac{q}{mc} \left( \frac{p_y B_z}{\gamma} \frac{\partial F}{\partial p_x} - \frac{p_z B_x}{\gamma} \frac{\partial F}{\partial p_y} \right) = 0, \]
\[ \frac{1}{c^2} D_t^2 E_x + \frac{4\pi}{c} j_x = \frac{\partial B_z}{\partial y} \frac{1}{c} D_t^0 E_y + \frac{4\pi}{c} j_y = -\frac{\partial B_z}{\partial x}, \]
\[ \frac{1}{c^2} D_t^2 B_z = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \frac{\partial E_z}{\partial y} + \frac{\partial E_z}{\partial y} = 4\pi p, \rho = q \int_{\Omega \in (-\infty,\infty)} (F) d\Omega, \]
\[ \frac{C}{D_t^2 \rho} + \frac{\partial}{\partial x} j_x + \frac{\partial}{\partial y} j_y = 0, j_x = \frac{q}{m} \int_{\Omega \in (-\infty,\infty)} \left( \frac{p_x}{\gamma} F \right) d\Omega, j_y = \frac{q}{m} \int_{\Omega \in (-\infty,\infty)} \left( \frac{p_y}{\gamma} F \right) d\Omega. \]

2.3. Scheme Discretization. According to the defined process [19, 20], we presented our system into discretized such matrix form as follow:

\[ s_i^{\infty} E^{\infty+1/2} G + \frac{p_x}{ym} C_1^{\infty+1} D_x^1 G + \frac{p_y}{ym} C_1^{\infty+1} D_y^1 G + q \xi(G^{\infty+1} C_1^{\infty+1} D_y^1) G \]
\[ + q \xi(G^{\infty+1} C_1^{\infty+1} D_y^1) G = \frac{q}{mc} \left( \frac{p_y}{\gamma} C_1^{\infty+1} D_y^1 G \right) \]
\[ \frac{1}{c^2} E_x^{\infty+1/2} G' + \frac{4\pi}{c} (j_x^{\infty+1}) = C_4^{\infty+1} D_y^1 G', \]
\[ \frac{1}{c^2} B_z^{\infty+1/2} G' = C_2^{\infty+1} D_y^1 G' - C_3^{\infty+1} D_x^1 G', \]
\[ \frac{1}{c^2} E_y^{\infty+1/2} G' + \frac{4\pi}{c} (j_y^{\infty+1}) = -C_4^{\infty+1} D_x^1 G', \]
\[ C_2^{\infty+1} D_x^1 G' + C_3^{\infty+1} D_y^1 G' = 4\pi p^{\infty+1}, \]
\[ s_i^* \left\{ \int_{\Omega \in (-\infty,\infty)} \left( F^{\infty+1/2} \right) d\Omega \right\} + \]
\[ \frac{q}{m} \int_{\Omega \in (-\infty,\infty)} \left( \frac{p_x}{\gamma} C_1^{\infty+1} D_y^1 G \right) d\Omega + \frac{q}{m} \int_{\Omega \in (-\infty,\infty)} \left( \frac{p_y}{\gamma} C_1^{\infty+1} D_y^1 G \right) d\Omega = 0, \]
\[ \rho^{\infty+1} = q \int_{\Omega \in (-\infty,\infty)} (C_1^{\infty+1} G) d\Omega, \]
\[ j_x^{\infty+1} = \frac{q}{m} \int_{\Omega \in (-\infty,\infty)} \left( \frac{p_x}{\gamma} C_1^{\infty+1} G \right) d\Omega, \]
\[ j_y^{\infty+1} = \frac{q}{m} \int_{\Omega \in (-\infty,\infty)} \left( \frac{p_y}{\gamma} C_1^{\infty+1} G \right) d\Omega. \]
The definition of operator is [18].

\[ \frac{\xi^\alpha F_{\eta+1/2}}{\xi_t} = \xi_1 \left( C_{\eta+1/2} + \sum_{s=1}^{n} (C_{\eta+1/2-s}) \right) = \frac{\Delta t^{-a}}{\Gamma (2 - a)} \left\{ (C_{\eta+1}^\alpha - C^\alpha) + \frac{\Delta t^{-a}}{\Gamma (2 - a)} \sum_{s=1}^{n} (C_{\eta+1-s}^\alpha - C_{\eta-s}^\alpha) b_s^\alpha \right\}. \]  

(5)

We also have \( b_s^\alpha = (s + 1)^{\alpha - s} - s^{\alpha - s} \). We use \( G_{M_1,M_2,M_3,M_4} = G \), and \( G_{M_1,M_2} = G' \) for simplicity in discretization form and also \( \xi_t^\alpha \) is the operator and is specified by

\[ \xi_t^\alpha F_{\eta+1/2} = \left( \delta_1 (C_{\eta+1/2}) - C_{\eta+1/2-s} \right) - \left( \sum_{s=1}^{n} \delta_1 (C_{\eta+1/2-s}) \right) \left( C_{\eta+1}^\alpha - C^\alpha \right) + \left( \sum_{s=1}^{n} (C_{\eta+1-s}^\alpha - C_{\eta-s}^\alpha) b_s^\alpha \right). \]  

(6)

2.4. Conservativeness of the Scheme

2.4.1. Charge Conservation. Taking \( q \int_{\Omega (-\infty, \infty)} d\Omega \) of VE (5) and once obtained:

\[ \int_{\Omega (-\infty, \infty)} q \left( \xi_t^\alpha F_{\eta+1/2} G + \frac{p_x}{ym} C_{\eta+1}^\alpha D_x^1 G + \frac{p_y}{ym} C_{\eta+1}^\alpha D_y^1 G \right) d\Omega = 0, \]

\[ \xi_t^\alpha \left\{ q \int_{\Omega (-\infty, \infty)} F_{\eta+1/2} G d\Omega \right\} + \left\{ \frac{q}{m} \int_{\Omega (-\infty, \infty)} \left( \frac{p_x}{y} C_{\eta+1}^\alpha D_x^1 G \right) d\Omega \right\} + \left\{ \frac{q}{m} \int_{\Omega (-\infty, \infty)} \left( \frac{p_y}{y} C_{\eta+1}^\alpha D_y^1 G \right) d\Omega \right\} = 0. \]  

(7)

Hence proved that charge conservation accordingly.

2.4.2. Gauss Law and Solenoidal Constrains. (1) Differential Arrangement. Using the concepts of divergence and equation (2), we obtained:

\[ \frac{1}{c} \text{div} (C D_t^\alpha E) - \text{div} (\nabla \times B) + \frac{4\pi}{c} \text{div} (I) = 0 \Rightarrow \frac{1}{c} C D_t^\alpha (\text{div} (E) - 4\pi) = 0. \]  

(8)

Hence, we can say that the above relation will satisfy if we have

\[ \text{div} (E) = 4\pi. \]  

(9)
(2) Discretization Arrangement. Apply the divergence based on the assumptions, and we contract:

\[
\frac{1}{c^2} \partial_t \tilde{E}_x^{\eta+1/2} G' + \frac{4\pi}{c^2} \left( \frac{q}{m} \int_{\Omega \in (-\infty, \infty)} \frac{P_x C_1^{\eta+1} D_x G}{\gamma} \right) d\Omega = C_4^{\eta+1} D_x^1 D_y^1 G',
\]

\[
\frac{1}{c^2} \partial_t \tilde{E}_y^{\eta+1/2} G' + \frac{4\pi}{c^2} \left( \frac{q}{m} \int_{\Omega \in (-\infty, \infty)} \frac{P_y C_1^{\eta+1} D_y G}{\gamma} \right) d\Omega = -C_4^{\eta+1} D_x^1 D_y^1 G',
\]

\[
\frac{1}{c^2} \partial_t \tilde{E}_x^{\eta+1/2} G' + \partial_t \tilde{E}_y^{\eta+1/2} G' + \frac{4\pi}{c^2} \left( \frac{q}{m} \int_{\Omega \in (-\infty, \infty)} \frac{P_x C_1^{\eta+1} D_x G}{\gamma} \right) d\Omega + \frac{q}{m} \int_{\Omega \in (-\infty, \infty)} \left( \frac{P_y C_1^{\eta+1} D_y G}{\gamma} \right) d\Omega = 0,
\]

\[
\frac{1}{c^2} \partial_t \tilde{E}_x^{\eta+1/2} G' + \frac{4\pi}{c^2} \left( \frac{q}{m} \int_{\Omega \in (-\infty, \infty)} \frac{P_x C_1^{\eta+1} D_x G}{\gamma} \right) d\Omega + \frac{q}{m} \int_{\Omega \in (-\infty, \infty)} \left( \frac{P_y C_1^{\eta+1} D_y G}{\gamma} \right) d\Omega = 0.
\]

As a result, we may claim that the Gauss rule is true if and only if the following conditions are met:

\[
D_x^1 E_x^0 G' + D_y^1 E_y^0 G' = -4\pi \rho^0. \tag{11}
\]

It is not necessary to establish solenoidal limitations.

2.4.3. Momentum Conservation

(1) Differential Arrangement.

\[
D_t^\alpha \int_{\Omega \in (-\infty, \infty)} (p F) d\Omega + \frac{\partial}{\partial r} \int_{\Omega \in (-\infty, \infty)} \left( p \frac{P}{\gamma m} F \right) d\Omega = -\int_{\Omega \in (-\infty, \infty)} \left( \frac{\partial}{\partial p} \cdot \left( q \left( \frac{E + B}{\gamma mc} \right) F \right) \right) d\Omega,
\]

\[
D_t^\alpha \int_{\Omega \in (-\infty, \infty)} (q F) d\Omega + \frac{\partial}{\partial r} \int_{\Omega \in (-\infty, \infty)} \left( p \frac{P}{\gamma m} F \right) d\Omega = \int_{\Omega \in (-\infty, \infty)} \left( \left\{ q EF + qF \frac{P \times B}{\gamma mc} \right\} \right) d\Omega. \tag{12}
\]

For the case of $x$-component, we obtained

\[
D_t^\alpha \int_{\Omega \in (-\infty, \infty)} (F p_x) d\Omega + E_x \int_{\Omega \in (-\infty, \infty)} (q F) d\Omega + \frac{\partial}{\partial r} \int_{\Omega \in (-\infty, \infty)} \left( p_x \frac{P}{\gamma m} F \right) d\Omega = \frac{q}{mc} \int_{\Omega \in (-\infty, \infty)} \left( \frac{1}{\gamma} F \left( B_z p_y \right) \right) d\Omega,
\]

\[
D_t^\alpha \int_{\Omega \in (-\infty, \infty)} (p_x F) d\Omega + \frac{\partial}{\partial r} \int_{\Omega \in (-\infty, \infty)} \left( p_x \frac{P}{\gamma m} F \right) d\Omega = \rho E_x + \frac{1}{c} \left( B_z J_y \right). \tag{13}
\]

Additionally, we shall express the "Maxwell" equation in its momentum form as:
\[
\begin{align*}
B_z \left( \frac{1}{4\pi} \left( \frac{\partial B_x}{\partial z} - \frac{\partial B_y}{\partial x} \right) + \frac{E_y}{4\pi} \left( \frac{1}{c} D_t B_z + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \right) &= -\frac{B_z}{4\pi} 4\pi \frac{j_y}{c}, \\
B_x \left( \frac{1}{4\pi} \left( \frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} \right) - \frac{E_z}{4\pi} \left( \frac{1}{c} D_t B_y + \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial y} \right) \right) &= -\frac{B_x}{4\pi} 4\pi \frac{j_y}{c}.
\end{align*}
\]

(14)

Simplified form is

\[
\begin{align*}
\frac{B_z}{4\pi} \left( \frac{1}{c} D_t E_y + \frac{\partial B_x}{\partial y} - \frac{\partial B_z}{\partial x} \right) + \frac{E_y}{4\pi} \left( \frac{1}{c} D_t B_z + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) &= -\frac{B_z}{4\pi} 4\pi \frac{j_y}{c}, \\
\frac{B_x}{4\pi} \left( \frac{1}{c} D_t E_z + \frac{\partial B_y}{\partial x} - \frac{\partial B_z}{\partial y} \right) - \frac{E_z}{4\pi} \left( \frac{1}{c} D_t B_y + \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial y} \right) &= -\frac{B_x}{4\pi} 4\pi \frac{j_y}{c}.
\end{align*}
\]

(15)

Additionally, we get

\[
\frac{1}{4\pi c} D^2 \left\{ B_z E_y \right\} + \frac{1}{8\pi} \left( \frac{\partial}{\partial x} (E^2 + B^2) \right) - \frac{1}{4\pi} \left( \text{div}(E_x E + B_x B) \right) + \frac{1}{4\pi} \left[ B_s \text{div}(B) + E_x \text{div}(E) \right] = -\frac{j_y B_z}{c}.
\]

(16)

From (2), we get

\[
\frac{1}{4\pi c} D^2 \left\{ B_z E_y \right\} + \frac{1}{8\pi} \left( \frac{\partial}{\partial x} (E^2 + B^2) \right) - \frac{1}{4\pi} \left( \text{div}(E_x E + B_x B) \right) = -\frac{E_x \rho - j_y B_z}{c}.
\]

(17)

Further from equations (14) and (17), we get

\[
D^2 \int_{\Omega \in (-\infty, \infty)} \left( p_x F + \frac{B_z E_y}{4\pi c} \right) d\Omega + \begin{cases}
\frac{\partial}{\partial x} \int_{\Omega \in (-\infty, \infty)} \left( \frac{p_x}{\gamma m} F - \frac{E_x^2 + B_x^2}{4\pi} + \frac{E_x E_y + B_x B_y}{8\pi} \right) d\Omega + \frac{\partial}{\partial y} \int_{\Omega \in (-\infty, \infty)} \left( \frac{E_y}{\gamma m} + \frac{E_x E_y}{4\pi} \right) d\Omega \\
\frac{\partial}{\partial x} \int_{\Omega \in (-\infty, \infty)} \left( \frac{p_x}{\gamma m} F - \frac{E_x^2 + B_x^2}{4\pi} + \frac{E_x E_y + B_x B_y}{8\pi} \right) d\Omega + \frac{\partial}{\partial y} \int_{\Omega \in (-\infty, \infty)} \left( \frac{E_y}{\gamma m} + \frac{E_x E_y}{4\pi} \right) d\Omega
\end{cases} = 0.
\]

(18)
(2) Discretization Arrangement:

\[
\int_{\Omega \in (-\infty, \infty)} \left\{ \frac{p_x y}{ym} \right\} d\Omega + \int_{\Omega \in (-\infty, \infty)} \left\{ \frac{p_x y}{ym} \right\} d\Omega + \int_{\Omega \in (-\infty, \infty)} \left\{ \frac{p_x y}{ym} \right\} d\Omega + \int_{\Omega \in (-\infty, \infty)} \left\{ \frac{p_x y}{ym} \right\} d\Omega.
\]

Moreover, we have

\[
\frac{1}{4\pi} C_4^{\eta+1} G' \left( C_2 G' + C_4^{\eta+1} D_x^1 G' \right) = \frac{1}{4\pi} C_4^{\eta+1} \frac{\pi}{c} \left( j_y^{\eta+1} \right).
\]

From (15), we can write as:

\[
\frac{1}{4\pi} \left( C_4^{\eta+1} G' + C_2 G' \delta_s^{1/2} G' \right) + \frac{1}{4\pi} C_2^{\eta+1} G' \left( C_2 G' + C_4^{\eta+1} D_x^1 G' \right)
\]

\[
+ \frac{1}{4\pi} \left( 2C_2^{\eta+1} G' + C_4^{\eta+1} D_x^1 G' \right) = \frac{1}{4\pi} C_4^{\eta+1} G' \left( j_y^{\eta+1} + 4\pi p^{\eta+1} \right).
\]
As a result of (20) and (21), we may conclude that the scheme follows the momentum conservation rule.

### 2.4.4. Energy Conservation

We shall use

\[
\int_{\Omega \in (-\infty, \infty)} (\gamma m c^2) d\Omega \quad \text{and obtained further:}
\]

\[
D_t^\alpha \int_{\Omega \in (-\infty, \infty)} (\gamma m c^2) d\Omega + \frac{\partial}{\partial \tau} \int_{\Omega \in (-\infty, \infty)} (c^2 p F) d\Omega = \int_{\Omega \in (-\infty, \infty)} \left( p \left( E + \frac{p \times B}{\gamma m c} \right) F \right) d\Omega,
\]

\[
D_t^\alpha \int_{\Omega \in (-\infty, \infty)} (\gamma m c^2) d\Omega + \frac{\partial}{\partial \tau} \int_{\Omega \in (-\infty, \infty)} (c^2 p F) d\Omega = \int_{\Omega \in (-\infty, \infty)} \left( \frac{1}{\gamma m c} \left( q (p \cdot E) + \frac{1}{m c} \left( p \cdot \frac{p \times B}{\gamma} F \right) \right) \right) d\Omega
\]

(22)

\[
D_t^\alpha \int_{\Omega \in (-\infty, \infty)} (\gamma m c^2) d\Omega + \frac{\partial}{\partial \tau} \int_{\Omega \in (-\infty, \infty)} (c^2 p F) d\Omega = J \cdot E,
\]

In the next we have;

\[
\frac{c}{4\pi} E_x \left( \frac{1}{c} D_t^\alpha E_x - \frac{\partial B_z}{\partial y} \right) + \frac{c}{4\pi} E_y \left( \frac{1}{c} D_t^\alpha E_y + \frac{\partial B_z}{\partial x} \right) = \frac{4\pi}{c} c \cdot E_x j_x - \frac{4\pi}{c} c \cdot E_y j_y
\]

\[
\frac{c}{4\pi} B_z \left( \frac{1}{c} D_t^\alpha B_z + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = 0,
\]

(23)

\[
\frac{1}{8\pi} D_t^\alpha \left( E_x^2 + E_y^2 + B_z^2 \right) + \frac{c}{4\pi} \left( -E_x \frac{\partial B_z}{\partial y} + E_y \frac{\partial B_z}{\partial x} + B_z \frac{\partial E_y}{\partial x} - B_z \frac{\partial E_x}{\partial y} \right) = -j \cdot E,
\]

Following a series of specific stages, we get to

\[
\frac{1}{8\pi} D_t^\alpha \left( E_x^2 + E_y^2 + B_z^2 \right) + \frac{c}{4\pi} \text{div}(E \times B) = -j \cdot E
\]

(24)

We have obtained from equations (23) and (24) as follow:

\[
D_t^\alpha \int_{\Omega \in (-\infty, \infty)} \left( \gamma m c^2 F + \frac{E_x^2 + B_z^2}{8\pi} \right) d\Omega + \text{div}\left( \int_{\Omega \in (-\infty, \infty)} \left( c^2 p f + c \frac{\text{div}(E \times B)}{4\pi} \right) d\Omega \right) = 0.
\]

(25)
(2) Discretization Arrangement:

\[
D_t \int_{\Omega(-\infty, \infty)} (\gamma mc^2 \mathbf{s}_t^p e^{\gamma + \frac{1}{2}G}) d\Omega + \int_{\Omega(-\infty, \infty)} (\mathbf{c}_x^p C_1^{\gamma + \frac{1}{2}D_x^1} G) d\Omega + \int_{\Omega(-\infty, \infty)} (\mathbf{c}_y^p C_1^{\gamma + \frac{1}{2}D_y^1} G) d\Omega
\]

\[
= \left( \frac{q_p C_1^{\gamma + \frac{1}{2}D_y^1} G}{m_y} \right) d\Omega + \int_{\Omega(-\infty, \infty)} \left( \frac{q_p C_1^{\gamma + \frac{1}{2}D_y^1} G}{m_y} \right) d\Omega D_t^\alpha,
\]

\[
\int_{\Omega(-\infty, \infty)} (\gamma mc^2 \mathbf{s}_t^p e^{\gamma + \frac{1}{2}G}) d\Omega + \int_{\Omega(-\infty, \infty)} (\mathbf{c}_x^p C_1^{\gamma + \frac{1}{2}D_x^1} G) d\Omega + \int_{\Omega(-\infty, \infty)} (\mathbf{c}_y^p C_1^{\gamma + \frac{1}{2}D_y^1} G) d\Omega = C_2^G j_x + C_3^G j_y,
\]

\[
D_t^\alpha \int_{\Omega(-\infty, \infty)} (\gamma mc^2 \mathbf{s}_t^p e^{\gamma + \frac{1}{2}G}) d\Omega + \int_{\Omega(-\infty, \infty)} (\mathbf{c}_x^p C_1^{\gamma + \frac{1}{2}D_x^1} G) d\Omega + \int_{\Omega(-\infty, \infty)} (\mathbf{c}_y^p C_1^{\gamma + \frac{1}{2}D_y^1} G) d\Omega = C_2^G j_x + C_3^G j_y.
\]

The discretization procedure is elucidated as:

\[
\frac{c}{4\pi} C_4^G \left\{ c_1^p E_x^{\gamma + \frac{1}{2}G} - C_1^D_y^1 G \right\} + \frac{c}{4\pi} C_3^G \left\{ c_1^p E_y^{\gamma + \frac{1}{2}G} + C_1^D_x^1 G \right\} = -\left( \frac{4\pi}{c} C_2^G j_x + \frac{4\pi}{c} C_3^G j_y \right).
\]

\[
\frac{c}{4\pi} C_4^G \left\{ c_1^p B_x^{\gamma + \frac{1}{2}G} + C_2^D_y^1 G - C_1^D_y^1 G \right\} = 0,
\]

\[
\frac{1}{4\pi} \left( C_4^G \left\{ c_1^p E_x^{\gamma + \frac{1}{2}G} \right\} + C_3^G \left\{ c_1^p E_y^{\gamma + \frac{1}{2}G} \right\} + C_4^G \left\{ c_1^p B_x^{\gamma + \frac{1}{2}G} \right\} \right)
\]

\[
+ \frac{c}{4\pi} C_4^G \left\{ C_4^D_x^1 G \right\} + \frac{c}{4\pi} C_3^G \left\{ C_3^D_y^1 G \right\} - \frac{c}{4\pi} C_2^G \left\{ C_1^D_y^1 G \right\}
\]

\[
- \frac{c}{4\pi} C_4^G \left\{ C_2^D_y^1 G \right\} = -\left( C_2^G j_y + C_3^G j_y \right).
\]

Hence proved.

3. Results and Discussion

Using MAPLE and Python, we generated generic code for assessing the numerical solution of the “model” using the specified numerical technique. We will explore the attitude of plasma particles for fractional concepts using the following initial perturbation.

\[
F(t, x, y, p_x, p_y) = \frac{1}{\sqrt{2\pi}} e^{\frac{-\left( p_x^2 + p_y^2 \right)}{2}} \left( 1 + e \cos(k_x x) \cos(k_y y) \right).
\]

3.1. Numerical Convergence. To demonstrate the numerical convergence, we use concepts of norm. Therefore, we have the following relation as:
Even under the most difficult conditions imposed by the challenge, the implemented system demonstrates efficient convergence and precision. In both cases, i.e., integer and fractional values of the fractional parameter, convergence rises steadily as the computing domain extends as shown in Figure 1(a)–1(d). Consequently, we can easily show that our technique is well-matched, competent, and appropriate for the models discussed before.

3.2. Behaviour of Charged Particles at $\alpha \in (0, 1]$. We have revealed one critical sort of graphical diagrams, namely "density". We picked two distinct time periods, $t = 1.33, 3.66$, and varied $0 < \alpha \leq 1$. While we were executing initial data, "plasma" particles were unexpectedly displaced from their initial positions. "Particles" also acquire energy as a result of this perturbation. As a result, plasma particles began moving abruptly in order to stabilize themselves.
Noted that excited “plasma” particles are accessible in cluster form in the range $400 < x, y < 800$. Formulation of clusters is due to its high rate of plasma particle. As we vary $t = 1.33, \alpha = 0.2$ to $t = 1.33, \alpha = 0.6$, the particles undergo modification, and the bunch magnitude and outline are repaired appropriately (see Figure 2(b)).

A thin layer is seen in Figure 2(b) around clusters of low “momentum” plasma particles. Additionally, the assortment of particles deviate position. These clusters are translated into the final location and “momentum” by selecting $t = 1.33, \alpha = 1.0$.

As a result, we can examine the particles’ locus as the fractional parameter is varied. We clearly noticed significant fluctuations in the location and velocity of plasma particles during the second time period, as seen in Figures 2(d)–2(f). Some of the particles are clustered, while others are uncontrolled, covering the computing area. This procedure will continue indefinitely until the equilibrium state is reached.

When the fractional parameter is modified, we may see statistically significant variations. Therefore, fractional concepts are utilized to indicate the density of plasma particles and the path taken as a consequence of this. It dives into the complex picture of plasma particle behaviour that has remained concealed.

4. Conclusion

The current work accomplished two critical objectives: it developed a multi-order (integer and fractional) “fully relativistic” model based on several concepts, and it proposed a conservative “hybrid” numerical technique for solving the anticipated “plasma” model. To deal with the time-“fractional” derivative, the Caputo sense definition is used. The reported findings unequivocally illustrate that plasma particles exhibit significant variances over a range of fractional parameter values. By assigning specific positions, momentum, and energy to plasma particles, we may now determine their eventual state. As a consequence, the proposed model has the potential to considerably advance our understanding of plasma particles in the field of computing. The technique is adequate, well-matched, and effectual for the suggested model based on numerical convergence. It has a high rate of “convergence” for both derivatives, which grows gradually as the computing domain is enlarged.

Data Availability

No data was required to perform this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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