Smallness of Faltings heights of CM abelian varieties

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Abstract

We prove that assuming the Colmez conjecture and the “no Siegel zeros” conjecture, the stable Faltings height of a CM abelian variety over a number field is less than or equal to the logarithm of the root discriminant of the field of definition of the abelian variety times an effective constant depending only on the dimension of the abelian variety. In view of the fact that the Colmez conjecture for abelian CM fields, the averaged Colmez conjecture, and the “no Siegel zeros” conjecture for CM fields with no complex quadratic subfields are already proved, we prove unconditional analogues of the result above. In addition, we also prove that the logarithm of the root discriminant of the field of everywhere good reduction of CM abelian varieties can be “small”.

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1 Introduction

Let $E$ be a CM-field, and let $\Phi$ be a CM-type of $E$. Let $A$ be an abelian variety over a number field $K$ such that we have an embedding $i: \mathcal{O}_E \hookrightarrow \text{End}_K(A)$ such that $(A, i)$ has CM-type $\Phi$. It is proved by Colmez in [Col93] that the stable Faltings height $h^\text{st}(A)$ of the abelian variety $A$ depends only on the CM-field $E$ and the CM-type $\Phi$ and not on the abelian variety $A$. We denote it as $h^\text{st}(E, \Phi)$. In [Col93] Colmez proposed a conjecture relating $h^\text{st}(E, \Phi)$ to the logarithmic derivatives at $s = 0$ of certain Artin $L$-functions defined by $(E, \Phi)$. We will refer to this conjecture as the Colmez conjecture. The precise statement is as follows:

We define a function $A_0^0(E, \Phi)$ from $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $\mathbb{C}$ by

$$A_0^0(E, \Phi)(\sigma) = \frac{1}{[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \text{Stab}(\Phi)]} \sum_{\nu \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/\text{Stab}(\Phi)} |\nu \Phi \cap \sigma \nu \Phi|, \forall \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),$$

where $\text{Stab}(\Phi)$ is the subgroup of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ consisting of the stabilizers of $\Phi$.

This function is locally constant and constant on conjugacy classes. Therefore, there is a unique decomposition of $A_0^0(E, \Phi)$ into $\mathbb{C}$-linear combinations of irreducible Artin characters $\chi$ (i.e. characters $\chi$ of irreducible continuous representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on finite-dimensional $\mathbb{C}$-vector spaces)

$$A_0^0(E, \Phi) = \sum_{\chi} m(E, \Phi)(\chi) \chi, \quad m(E, \Phi)(\chi) \in \mathbb{C}.$$

It can be shown that for any irreducible Artin character $\chi$ such that $m(E, \Phi)(\chi) \neq 0$, the Artin $L$-function $L(s, \chi, \mathbb{Q})$ is defined and nonzero at $s = 0$. We define

$$Z(E, \Phi) := -\frac{1}{2} g \log(2\pi) + \sum_{\chi} m(E, \Phi)(\chi) \frac{L'(0, \chi, \mathbb{Q})}{L(0, \chi, \mathbb{Q})},$$

and

$$\mu(E, \Phi) := \sum_{\chi} m(E, \Phi)(\chi) \log(f(\chi, \mathbb{Q})),$$

where $f(\chi, \mathbb{Q})$ is the Artin conductor of the Artin character $\chi$ (a positive integer).
The Colmez conjecture says that we have
\[ h_{\text{Falt}}(E, \Phi) = -Z(E, \Phi) - \frac{1}{2} \mu(E, \Phi). \]

When \( E \) is a complex quadratic field, the Colmez conjecture is the same as the classical Chowla-Selberg formula (see for example Page 91 and 92 of [Wei76]), and so it is in fact a theorem.

Colmez [Col93] and Obus [Obu13] proved that the Colmez conjecture is true if the extension \( E/\mathbb{Q} \) is Galois with abelian Galois group (Theorem 4.8). Yuan–Zhang [YZ18] and Andreatta–Goren–Howard–Madapusi-Pera [AGHMP18] independently proved that the Colmez conjecture is true when one averages over all CM-types of a given CM-field (Theorem 5.2).

Let \(-d \in \mathbb{Z}_{\leq -2}\) be a fundamental discriminant, so \( \text{disc}(\mathbb{Q}(\sqrt{-d})) = d \). Let \( \chi_d \) be the quadratic character associated to the quadratic field extension \( \mathbb{Q}(\sqrt{-d})/\mathbb{Q} \), so \( \chi_d(p) = (-d)^{1/2} \) for any prime \( p \). Let \( L(s, \chi_d) \) be the Dirichlet \( L \)-function of the character \( \chi_d \), so \( L(s, \chi_d) = \zeta_{\mathbb{Q}(\sqrt{-d})}(s) \zeta_{\mathbb{Q}}(s) \).

It is known that there is at most one zero of \( L(s, \chi_d) \) in the region
\[ 1 - \frac{1}{4 \log(d)} \leq \Re(s) < 1, \quad |\Im(s)| \leq \frac{1}{4 \log(d)}, \]
and if such a zero exists it is real and simple.

For any \( 0 < c \leq \frac{1}{4} \), we define the \( c \)-Siegel zero of \( L(s, \chi_d) \) to be the zero of \( L(s, \chi_d) \) in the region
\[ 1 - \frac{c}{\log(d)} \leq \Re(s) < 1, \quad |\Im(s)| \leq \frac{1}{4 \log(d)}, \]
(if it exists). We define the Siegel zero of \( L(s, \chi_d) \) to be the \( \frac{1}{4} \)-Siegel zero of \( L(s, \chi_d) \).

The conjecture \( \text{No} \frac{1}{\Omega(1)} \)-Siegel zero of \( L(s, \chi_d) \) is as follows:

**Conjecture 1.1** (\( \text{No} \frac{1}{\Omega(1)} \)-Siegel zero of \( L(s, \chi_d) \)). There exists some effectively computable absolute constant \( C_{\text{zero}} \in \mathbb{R}_{\geq 4} \) such that for any fundamental discriminant \( -d \in \mathbb{Z}_{\leq -2} \), the Dirichlet \( L \)-function \( L(s, \chi_d) \) has no zeros in the region \( 1 - \frac{1}{C_{\text{zero}} \log(d)} \leq \Re(s) < 1, \quad |\Im(s)| \leq \frac{1}{4 \log(d)} \).

Now let \( E \) be an arbitrary CM-field. Let \( F \) be the maximal totally real subfield of \( E \), and so \( E \) is a totally complex quadratic field extension of \( F \). Let \( \chi_{E/F} \) be the quadratic character associated to the quadratic field extension \( E/F \), so for any prime ideal \( \mathfrak{p} \) of \( \mathcal{O}_F \),
\[ \chi_{E/F}(\mathfrak{p}) = \begin{cases} 1 & \text{if } \mathfrak{p} \text{ splits completely in } \mathcal{O}_E, \\ -1 & \text{if } \mathfrak{p} \text{ is unramified but does not split completely in } \mathcal{O}_E, \\ 0 & \text{if } \mathfrak{p} \text{ is ramified in } \mathcal{O}_E. \end{cases} \]
Let \( L(s, \chi_{E/F}) \) be the \( L \)-function of the character \( \chi_{E/F} \), and so \( L(s, \chi_{E/F}) = \frac{\zeta_E(s)}{\zeta_F(s)} \).

Similarly to the case where \( E \) is a complex quadratic field, by Lemma 3 of \([Sta74]\), for any CM-field \( E \) with maximal totally real subfield \( F \), \( L(s, \chi_{E/F}) \) has at most one zero in the region

\[
1 - \frac{1}{4 \log |\text{disc}(E)|} \leq \text{Re}(s) < 1, |\text{Im}(s)| \leq \frac{1}{4 \log |\text{disc}(E)|}
\]

If such a zero exists, it is real and simple.

For any \( 0 < c \leq \frac{1}{4} \), we define the generalized \( c \)-Siegel zero of \( L(s, \chi_{E/F}) \) to be the zero of \( L(s, \chi_{E/F}) \) in the region

\[
1 - \frac{c}{\log |\text{disc}(E)|} \leq \text{Re}(s) < 1, |\text{Im}(s)| \leq \frac{1}{4 \log |\text{disc}(E)|}
\]

(if it exists). We define the generalized Siegel zero of \( L(s, \chi_{E/F}) \) to be the generalized \( \frac{1}{4} \)-Siegel zero of \( L(s, \chi_{E/F}) \).

The conjecture No generalized \( \frac{1}{O(1)} \)-Siegel zero of \( L(s, \chi_{E/F}) \) is as follows:

**Conjecture 1.2** (No generalized \( \frac{1}{O(1)} \)-Siegel zero of \( L(s, \chi_{E/F}) \)). For any \( g \in \mathbb{Z}_{\geq 1} \), there exists some effectively computable constant \( C_{\text{zero}}(g) \in \mathbb{R}_{\geq 4} \) depending only on \( g \) such that for any CM-field \( E \) with maximal totally real subfield \( F \) such that \([F : \mathbb{Q}] = g\), the function \( L(s, \chi_{E/F}) \) has no zeros in the region

\[
1 - \frac{1}{C_{\text{zero}}(g) \log |\text{disc}(E)|} \leq \text{Re}(s) < 1, |\text{Im}(s)| \leq \frac{1}{4 \log |\text{disc}(E)|}.
\]

It is proved by Stark (Lemma 9 of \([Sta74]\)) that Conjecture 1.1 implies Conjecture 1.2. He also proved that Conjecture 1.2 is true whenever the CM-field \( E \) contains no complex quadratic subfields.

We show that assuming the Colmez conjecture, the nonexistence of the generalized Siegel zero of \( L \)-functions of quadratic characters associated to CM extensions is closely related to the stable Faltings height of CM abelian varieties being bounded by the logarithm of the root discriminant of the field of definition. More precisely, we prove the following theorem:

**Theorem 1.3.** Suppose that the Colmez conjecture holds. Suppose further that No \( \frac{1}{O(1)} \)-Siegel zero of \( L(s, \chi_d) \) holds. Then for any \( g \in \mathbb{Z}_{\geq 1} \), there exist effectively computable constants \( C_1(g) > 0, C_2(g) \in \mathbb{R} \) depending only on \( g \) such that

\[
h_{\text{Falt}}^+(A) \leq \frac{C_1(g)}{[K : \mathbb{Q}]} \log |\text{disc}(K)| + C_2(g),
\]

for any dimension-\( g \) abelian variety \( A \) defined over a number field \( K \) with complex multiplication by \( \mathcal{O}_E \) for some CM-field \( E \).
Since the Colmez conjecture for abelian CM-fields is already proved, and since Conjecture 1.2 is true when the CM-field \( E \) contains no complex quadratic subfields, we can also prove an unconditional version of the theorem above:

**Theorem 1.4.** For any \( g \in \mathbb{Z}_{>1} \), there exists effectively computable constants \( C_3(g) > 0, C_4(g) \in \mathbb{R} \) depending only on \( g \) such that

\[
h^+_\text{Fal}(A) \leq C_3(g) \frac{1}{[K : \mathbb{Q}]} \log |\text{disc}(K)| + C_4(g),
\]

for any dimension-\( g \) abelian variety \( A \) over a number field \( K \) with complex multiplication by \( \mathcal{O}_E \) for some CM-field \( E \) such that the extension \( E/\mathbb{Q} \) is Galois with abelian Galois group and \( E \) does not contain any complex quadratic subfields.

**Remark 1.5.** To show that the condition “\( E \) does not contain any complex quadratic subfields” in the hypotheses in Theorem 1.4 is possible, we give examples of CM fields \( E \) containing no complex quadratic subfields such that the extension \( E/\mathbb{Q} \) is Galois with abelian Galois group.

Let \( n \) be an integer greater than or equal to 3 such that the group \( (\mathbb{Z}/n\mathbb{Z})^\times \) is a cyclic group and such that \( #(\mathbb{Z}/n\mathbb{Z})^\times \) divides 4. (Equivalently, \( n = p^k \) or \( n = 2p^k \) for some odd prime \( p \) such that \( p \equiv 1 \mod 4 \).)

Let \( E \) be the \( n \)-th cyclotomic field \( \mathbb{Q}(\mu_n) \), where \( \mu_n \) denotes a primitive \( n \)-th root of unity. Then \( E \) is a CM-field with maximal totally real subfield \( F = \mathbb{Q}(\mu_n + \mu_n^{-1}) \). The extension \( E/\mathbb{Q} \) is Galois and \( \text{Gal}(E/\mathbb{Q}) \) is isomorphic to \( (\mathbb{Z}/n\mathbb{Z})^\times \). Since \( \text{Gal}(E/\mathbb{Q}) \) is cyclic and of even order, there is a unique subgroup \( H \) of \( \text{Gal}(E/\mathbb{Q}) \) of index 2, and so there is a unique quadratic subfield \( K \) of \( E \). Let \( \iota \) be the nontrivial element of \( \text{Gal}(E/F) \subset \text{Gal}(E/\mathbb{Q}) \). Then \( \iota \) is the unique element in \( \text{Gal}(E/\mathbb{Q}) \) of order 2. Since \( \#\text{Gal}(E/\mathbb{Q}) = #(\mathbb{Z}/n\mathbb{Z})^\times \) divides 4, we have \( \iota \in H \). Thus, \( K \) is fixed by the element \( \iota \). Therefore, \( K \) is a real quadratic field and so \( E \) contains no complex quadratic subfields.

More generally, let \( E \) be any totally imaginary number field such that the extension \( E/\mathbb{Q} \) is Galois with abelian Galois group. Then \( E \) is a CM-field (any abelian extension over \( \mathbb{Q} \) is either totally real or a CM field). We know that

\[
\text{Gal}(E/\mathbb{Q}) \cong \mathbb{Z}/q_1\mathbb{Z} \times \mathbb{Z}/q_2\mathbb{Z} \times \cdots \times \mathbb{Z}/q_m\mathbb{Z},
\]

where \( q_1, q_2, \cdots, q_m \) are powers of prime numbers \( (q_1, q_2, \cdots, q_m \text{ are not necessarily distinct}) \). Suppose further that the number “2" does not appear in \( q_1, q_2, \cdots, q_m \), i.e., each \( q_i \) is either \( 2k_i \) for \( k_i \geq 2 \) or a power of an odd prime. Then each component \( \mathbb{Z}/q_i\mathbb{Z} \) such that \( q_i = 2k_i \) (\( k_i \geq 2 \)) contains a unique subgroup \( H_i \) of index 2 and a unique element \( \sigma_i \) of order 2, and \( \sigma_i \in H_i \). Let \( \iota \) be the nontrivial element of \( \text{Gal}(E/F) \subset \text{Gal}(E/\mathbb{Q}) \), where \( F \) is the maximal real subfield of \( E \). Then \( \iota \in H \) for any subgroup \( H \) of \( \text{Gal}(E/\mathbb{Q}) \) of index 2. Therefore, \( E \) contains no complex quadratic subfields.

Since the averaged Colmez conjecture is already proved, we can also prove averaged analogues of the theorems above.
Theorem 1.6. For any $g \in \mathbb{Z}_{\geq 1}$, there exists effectively computable constants $C_5(g) > 0$, $C_6(g) \in \mathbb{R}$ depending only on $g$ such that

$$\frac{1}{2} \left( h_{\text{Fal}}^g(A_1) + h_{\text{Fal}}^g(A_2) \right)$$

$$\leq C_5(g) \cdot \frac{1}{2} \left( \frac{1}{[K_1 : \mathbb{Q}]} \log |\text{disc}(K_1)| + \frac{1}{[K_1 : \mathbb{Q}]} \log |\text{disc}(K_2)| \right) + C_6(g),$$

for any pair $A_1, A_2$ of dimension-$g$ abelian varieties defined over number fields $K_1, K_2$ respectively, such that the following holds:

- There exists a CM-field $E$ of degree $[E : \mathbb{Q}] = 2g$ and embeddings $i_1 : \mathcal{O}_E \hookrightarrow \text{End}_{K_1}(A_1)$, $i_2 : \mathcal{O}_E \hookrightarrow \text{End}_{K_2}(A_2)$ such that $E$ does not contain any complex quadratic subfields and the CM-type $\Phi_1$ of $(A_1, i_1)$ and the CM-type $\Phi_2$ of $(A_2, i_2)$ satisfy:

  $$|\Phi_1 \cap \Phi_2| = g - 1.$$

Theorem 1.7. Let $g$ be a positive integer. Suppose that there exists some effectively computable constant $C_{\text{zero}}(g) \in \mathbb{R}_{\geq 4}$ depending only on $g$ such that for any CM-field $E$ with maximal totally real subfield $F$ such that $[F : \mathbb{Q}] = g$, the function $L(s, \chi_{E/F})$ has no zeros in the region

$$1 - \frac{1}{C_{\text{zero}}(g) \log |\text{disc}(E)|} \leq \text{Re}(s) < 1, |\text{Im}(s)| \leq \frac{1}{4 \log |\text{disc}(E)|},$$

i.e. the conjecture No generalized $\frac{1}{2}(g+1)$-Siegel zero of $L(s, \chi_{E/F})$ holds for $g$.

Then there exist effectively computable constants $C_7(g) > 0$, $C_8(g) \in \mathbb{R}$ depending only on $g$ such that

$$\frac{1}{2} \left( h_{\text{Fal}}^g(A_1) + h_{\text{Fal}}^g(A_2) \right)$$

$$\leq C_7(g) \cdot \frac{1}{2} \left( \frac{1}{[K_1 : \mathbb{Q}]} \log |\text{disc}(K_1)| + \frac{1}{[K_1 : \mathbb{Q}]} \log |\text{disc}(K_2)| \right) + C_8(g),$$

for any pair $A_1, A_2$ of dimension-$g$ abelian varieties defined over number fields $K_1, K_2$ respectively, such that the following holds:

- There exists a CM-field $E$ of degree $[E : \mathbb{Q}] = 2g$ and embeddings $i_1 : \mathcal{O}_E \hookrightarrow \text{End}_{K_1}(A_1)$, $i_2 : \mathcal{O}_E \hookrightarrow \text{End}_{K_2}(A_2)$ such that the CM-type $\Phi_1$ of $(A_1, i_1)$ and the CM-type $\Phi_2$ of $(A_2, i_2)$ satisfy:

  $$|\Phi_1 \cap \Phi_2| = g - 1.$$

It might be interesting to know that if we only make use of the (proved) averaged Colmez conjecture, then we cannot obtain results stronger than Theorem 1.6 and Theorem 1.7 (i.e. the “average” condition in these theorems cannot be dropped), even if we further assume that the abelian variety over the number field has everywhere good reduction. In particular, we prove the following theorems, which show that the logarithm of the root discriminant of the field of everywhere good reduction of CM abelian varieties can be “small”.
Theorem 1.8. Assume the Generalized Riemann Hypothesis. For any \( g \in \mathbb{Z}_{\geq 1} \), there exist effectively computable constants \( C_{13}(g) > 0, C_{14}(g) \in \mathbb{R} \), such that for any CM-field \( E \) with \( [E : \mathbb{Q}] = 2g \), for any CM-type \( \Phi \) of \( E \), there exists a number field \( K' \) and a CM abelian variety \( (A, i: \mathcal{O}_E \hookrightarrow \mathrm{End}_{K'}(A)) \) over \( K' \) of CM-type \( \Phi \) such that the abelian variety \( A \) over \( K' \) has everywhere good reduction and

\[
\frac{1}{[K' : \mathbb{Q}]} \log |\text{disc}(K')| \leq C_{13}(g) \log \log |\text{disc}(E)| + C_{14}(g). \quad (1)
\]

Theorem 1.9. For any \( g \in \mathbb{Z}_{\geq 1} \), there exist effectively computable constants \( C_{15}(g) > 0, C_{16}(g) \in \mathbb{R} \), such that for any CM-field \( E \) with \( [E : \mathbb{Q}] = 2g \), for any CM-type \( \Phi \) of \( E \), there exists a number field \( K' \) and a CM abelian variety \( (A, i: \mathcal{O}_E \hookrightarrow \mathrm{End}_{K'}(A)) \) over \( K' \) of CM-type \( \Phi \) such that the abelian variety \( A \) over \( K' \) has everywhere good reduction and

\[
\frac{1}{[K' : \mathbb{Q}]} \log |\text{disc}(K')| \leq C_{15}(g) \log |\text{disc}(E)| + C_{16}(g). \quad (2)
\]

This will be discussed in detail in section 6.

In Theorem 6(ii) of [Col98], Colmez has proved that there exist effectively computable absolute constants \( C_{\text{Col}, 1} > 0, C_{\text{Col}, 2} \in \mathbb{R} \) such that for any CM-field \( E \) of degree \( [E : \mathbb{Q}] = 2g \) and any CM-type \( \Phi \) of \( E \) such that the following hold:

1. \((E, \Phi)\) satisfies the Colmez conjecture,

2. For any irreducible Artin character \( \chi \) such that \( m_{(E, \Phi)}(\chi) \neq 0 \), the Artin conjecture for \( \chi \) holds (i.e. the Artin \( L \)-function \( L(s, \chi, \mathbb{Q}) \) is holomorphic everywhere except possibly for a simple pole at \( s = 1 \)),

3. For any irreducible Artin character \( \chi \) such that \( m_{(E, \Phi)}(\chi) \neq 0 \), the Artin \( L \)-function \( L(s, \chi, \mathbb{Q}) \) has no zeros on the ball of radius \( \frac{1}{4} \) centered at 0,

we have

\[
h_{\text{Falh}}^{(E, \Phi)} \leq C_{\text{Col}, 1} \cdot \mu_{(E, \Phi)} + gC_{\text{Col}, 2}.
\]

The proofs of Theorem 1.3 and Theorem 1.4 show that that we can actually remove the second hypothesis that the Artin \( L \)-functions involved satisfy the Artin conjecture. Moreover, in the remark after Theorem 6 of [Col98], Colmez asked whether it is possible to remove the third hypothesis that the Artin \( L \)-functions involved have no zeros on the ball of radius \( \frac{1}{4} \) centered at 0 and making use of “no Siegel zeros” instead. The proofs of Theorem 1.3 and Theorem 1.4 is more or less a positive answer to this question.
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2 The Faltings height

Let $A$ be a dimension-$g$ abelian variety defined over a number field $K$. Let $\pi: A \rightarrow \text{Spec}(O_K)$ be the Néron model of $A$, and take $\omega$ to be any global section of $L := \pi^*\Omega^1_{A/\text{Spec}O_K}$. We define the unstable Faltings height of $A$ as follows:

$$h_{\text{unst}}(A/K) := \frac{1}{[K:Q]} \left( \log \# \left( H^0(\text{Spec}O_K,L)/(O_K \cdot \omega) \right) \right) - \frac{1}{2} \sum_{\sigma: K \rightarrow \mathbb{C}} \log \left( \frac{1}{(2\pi)^g} \left| \int_{A(\mathbb{C})} \sigma(\omega \wedge \overline{\omega}) \right| \right).$$

This definition is independent of the choice of $\omega \in H^0(\text{Spec}O_K,L)$.

We define the stable Faltings height of $A$ to be

$$h_{\text{st}}(A) := h_{\text{unst}}(A_K'/K'),$$

where $K'$ is a finite extension of $K$ such that $A_K'/K'$ has everywhere semistable reduction. This definition does not depend on the choice of the finite extension $K'/K$. Unlike the unstable Faltings height, the stable Faltings height does not depend on the field of definition of the abelian variety.

The following is a theorem of Bost ([Bos96]).

**Theorem 2.1.** There exists an effectively computable absolute constant $C_{\text{lower}} > 0$ such that for any dimension-$g$ abelian variety $A$ over a number field, we have

$$h_{\text{st}}(A) \geq -gC_{\text{lower}}.$$

As we have mentioned in section 1, it is proved by Colmez in [Col93] that for any CM-field $E$ and any CM-type $\Phi$ of $E$, if $(A_1, i_1: O_E \hookrightarrow \text{End}_{K_1}(A_1))$ and $(A_2, i_2: O_E \hookrightarrow \text{End}_{K_2}(A_2))$ are CM abelian varieties over number fields $K_1$ and $K_2$, both with CM-type $\Phi$, then

$$h_{\text{st}}(A_1) = h_{\text{st}}(A_2).$$

We denote this stable Faltings height as $h_{\text{st}}(E,\Phi)$. 

8
3 The Colmez conjecture revisited

Throughout this section \( g \) is an arbitrary positive integer, \( E \) is an arbitrary CM-field of degree \( [E : \mathbb{Q}] = 2g \) and \( \Phi \) is an arbitrary CM-type of \( E \). We denote as \( E^\ast_\Phi \) the reflex field of \((E, \Phi)\).

Let \( A^0_{(E, \Phi)} \) be the function from \( \text{Gal}^g(\overline{\mathbb{Q}}/\mathbb{Q}) \) to \( \mathbb{C} \) defined as in section 1. Then since \( \text{Stab}(\Phi) \subset \text{Gal}(\overline{\mathbb{Q}}/E^\ast_\Phi) \), we have: \( A^0_{(E, \Phi)} \) factors as

\[
\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \text{Gal}(\overline{E^\ast_\Phi}/\mathbb{Q}) \rightarrow \mathbb{C},
\]

where we denote as \( \overline{E^\ast_\Phi} \) the Galois closure of the extension \( E^\ast_\Phi/\mathbb{Q} \). For the following we view the function \( A^0_{(E, \Phi)} \) as a (class) function from \( \text{Gal}(\overline{E^\ast_\Phi}/\mathbb{Q}) \) to \( \mathbb{C} \). For any irreducible Artin character \( \chi \) such that \( m_{(E, \Phi)}(\chi) \neq 0 \), \( \chi \) also factors as

\[
\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \text{Gal}(\overline{E^\ast_\Phi}/\mathbb{Q}) \rightarrow \mathbb{C},
\]

and so we also view \( \chi \) as a character of \( \text{Gal}(\overline{E^\ast_\Phi}/\mathbb{Q}) \).

We fix an embedding \( \overline{\mathbb{Q}} \hookrightarrow \mathbb{C} \) and let \( \iota \) be the element of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) induced by complex conjugation. Let \( \chi \) be an irreducible Artin character. We say that \( \chi \) is odd if \( \chi(\iota) = -\chi(1) \). Some computations show that for the trivial character \( \chi = 1 \), we have \( m_{(E, \Phi)}(1) = \frac{1}{2}g \); and for any nontrivial irreducible Artin character \( \chi \), \( m_{(E, \Phi)}(\chi) = 0 \) unless \( \chi \) is odd.

Therefore, we have

\[
A^0_{(E, \Phi)} = \frac{1}{2}g \cdot 1 + \sum_{\chi \in \text{Irr}(\text{Gal}(\overline{E^\ast_\Phi}/\mathbb{Q})) \atop \chi \neq 1} m_{(E, \Phi)}(\chi) \chi. \tag{3}
\]

This implies that for any irreducible Artin character \( \chi \) such that \( m_{(E, \Phi)}(\chi) \neq 0 \), the Artin \( L \)-function \( L(s, \chi, \mathbb{Q}) \) is defined and nonzero at \( s = 0 \).

Let \( Z_{(E, \Phi)} \) and \( \mu_{(E, \Phi)} \) be as in section 1. Since

\[
\zeta_{\mathbb{Q}}(0) = \log(2\pi) \quad \text{and} \quad f(1, \mathbb{Q}) = 0,
\]

we can deduce that

\[
Z_{(E, \Phi)} = \sum_{\chi \in \text{Irr}(\text{Gal}(\overline{E^\ast_\Phi}/\mathbb{Q})) \atop \chi \neq 1} m_{(E, \Phi)}(\chi) \frac{L'(0, \chi, \mathbb{Q})}{L(0, \chi, \mathbb{Q})}, \tag{4}
\]
and  
\[ \mu(E, \Phi) = \sum_{\chi \in \text{Irr}(\text{Gal}(\tilde{E}_\Phi/Q))} m(E, \Phi)(\chi) \log f(\chi, Q). \]  \hspace{1cm} (5)

4 The zero of the Artin L-function near 1

4.1 Relation between the zero near 1 and the logarithmic derivative at 0 of the Artin L-function

Throughout this subsection \( g \) is an arbitrary positive integer, \( E \) is an arbitrary CM-field of degree \([E:Q] = 2g\) and \( \Phi \) is an arbitrary CM-type of \( E \). We denote as \( E_\Phi^\ast \) the reflex field of \((E, \Phi)\). We denote as \( \tilde{E}_\Phi^\ast \) the Galois closure of the extension \( E_\Phi^\ast/Q \).

By Chapter 2, Section 5 of [MM97], for any nontrivial irreducible character \( \chi \) of \( \text{Gal}(\tilde{E}_\Phi^\ast/Q) \), the functions \( \zeta_{\tilde{E}_\Phi^\ast}(s) \) and \( L(s, \chi, Q) \) are both holomorphic except for a simple pole at \( s = 1 \). By Lemma 3 of [Sta74], for any number field \( K \) such that \( K \neq Q \), the function \( \zeta_K(s) \) has at most one zero in the region

\[ 1 - \frac{1}{4 \log |\text{disc}(K)|} \leq \text{Re}(s) < 1, |\text{Im}(s)| \leq \frac{1}{4 \log |\text{disc}(K)|}. \]

If such a zero exists, it is real and simple.

Therefore, the function \( L(s, \chi, Q) \) has at most one zero in the region

\[ 1 - \frac{1}{4 \log |\text{disc}(E_\Phi^\ast)|} \leq \text{Re}(s) < 1, |\text{Im}(s)| \leq \frac{1}{4 \log |\text{disc}(E_\Phi^\ast)|}. \]

If such a zero exists, it is real and simple.

**Proposition 4.1.** Let \( \chi \) be a nontrivial odd irreducible character of \( \text{Gal}(\tilde{E}_\Phi^\ast/Q) \). Denote as \( \beta_0 \) the (necessarily real and simple) zero of \( L(s, \chi, Q) \) in the region

\[ 1 - \frac{1}{4 \log |\text{disc}(E_\Phi^\ast)|} \leq \text{Re}(s) < 1, |\text{Im}(s)| \leq \frac{1}{4 \log |\text{disc}(E_\Phi^\ast)|}. \]

(if it exists).

Let \( \delta_\chi \) be 1 if \( \beta_0 \) exists, and let \( \delta_\chi \) be 0 otherwise. We have

\[ - \left( \frac{L'(0, \chi, Q)}{L(0, \chi, Q)} + \frac{L'(0, \chi, Q)}{L(0, \chi, Q)} + \frac{2\delta_\chi}{1 - \beta_0} \right) \]

\[ > -75 \log |\text{disc}(E_\Phi^\ast)| + \left( \log \left( \frac{f(\chi, Q)}{\pi^{\chi(1)}} \right) + \chi(1) \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} \right), \]
\[
-L'(0, \chi, Q) + L'(0, \overline{\chi}, Q) + \frac{2\delta}{1 - \beta_0} \\
< 75 \log |\text{disc}(\overline{E}_q)| + \left( \log \left( \frac{f(\chi, Q)}{\pi^{(1)}} \right) + \chi(1) \Gamma' \left( \frac{1}{2} \right) \right). \tag{6}
\]

**Proof.** We define the function \( \Lambda(s, \chi, Q) \) to be

\[
\Lambda(s, \chi, Q) := \left( \frac{f(\chi, Q)}{\pi^{(1)}} \right)^s \pi^{-(s+1)/2} \Gamma \left( \frac{s+1}{2} \right) L(s, \chi, Q).
\]

We have the functional equation

\[
\Lambda(s, \chi, Q) = W(\chi) \Lambda(1-s, \chi, Q)
\]

for some \( W(\chi) \in \mathbb{C} \) with absolute value 1.

We define the function \( \xi_{\overline{E}_q} \) to be

\[
\xi_{\overline{E}_q}(s) := s(s-1)|\text{disc}(\overline{E}_q)|^{s/2} \left( 2(2\pi)^{-s} \Gamma(s) \right)^{\overline{E}_q(1)/2} \zeta_{\overline{E}_q}(s).
\]

We have the functional equation

\[
\xi_{\overline{E}_q}(s) = \xi_{\overline{E}_q}(1-s).
\]

First consider the function \( f_1(s) := (\xi_{\overline{E}_q}(s))^2 \Lambda(s, \chi, Q) \Lambda(s, \overline{\chi}, Q) \). It is entire and satisfies the functional equation

\[
f_1(s) = f_1(1-s).
\]

Since \( f_1(s) \) is real for \( s \) real, for any \( \rho \in \mathbb{C} \) the order of the zero of \( f_1(s) \) at \( s = \rho \) is equal to that at \( s = \overline{\rho} \). Moreover, all zeros of \( f_1(s) \) lie in the critical strip \( 0 < \Re(s) < 1 \). Therefore, by logarithmically differentiating the Hadamard product formula for \( f_1(s) \) at \( s = 1 \) we get

\[
\sum_{\rho: f_1(\rho) = 0} \frac{1}{2} \left( \frac{1}{1 - \rho} + \frac{1}{1 - \overline{\rho}} \right) = \frac{f_1'(1)}{f_1(1)},
\]

i.e.

\[
\sum_{\rho: f_1(\rho) = 0} \frac{1}{2} \left( \frac{1}{1 - \rho} + \frac{1}{1 - \overline{\rho}} \right)
= \frac{\xi_{\overline{E}_q}'(1)}{\xi_{\overline{E}_q}(1)} + \frac{\Lambda'(1, \chi, Q)}{\Lambda(1, \chi, Q)} + \frac{\Lambda'(1, \overline{\chi}, Q)}{\Lambda(1, \overline{\chi}, Q)}. \tag{7}
\]
Let $\delta_{\tilde{E}_6}$ be 1 if $\beta_0$ is a zero of $\zeta_{\tilde{E}_6}(s)$, and let $\delta_{\tilde{E}_6}$ be 0 otherwise. Then $\delta_{\tilde{E}_6} - \delta_\chi$ is equal to 0 or 1 and the order of the zero of $f_1(s)$ at $s = \beta_0$ is equal to $2\delta_{\tilde{E}_6} + 2\delta_\chi$. Thus, we have

$$
\sum_{\rho: \ f_1(\rho) = 0, \ \rho \neq \beta_0} \frac{1}{2} \left( \frac{1}{1 - \rho} + \frac{1}{1 - \overline{\rho}} \right)
= \left( \frac{\xi_{\tilde{E}_6}(1)}{2 \xi_{\tilde{E}_6}(1)} - \frac{2\delta_{\tilde{E}_6}}{1 - \beta_0} \right) + \left( \frac{\Lambda'(1, \chi, Q)}{\Lambda(1, \chi, Q)} + \frac{\Lambda'(1, \overline{\chi}, \overline{Q})}{\Lambda(1, \overline{\chi}, \overline{Q})} - \frac{2\delta_\chi}{1 - \beta_0} \right). \tag{8}
$$

Since the function $\frac{(\xi_{\tilde{E}_6}(s))^2}{\Lambda(s, \chi, Q)\Lambda(s, \overline{\chi}, \overline{Q})}$ is holomorphic on $0 < \text{Re}(s) < 1$, for any $\rho \in \mathbb{C}$ such that $0 < \text{Re}(\rho) < 1$, the order of the zero at $s = \rho$ of the function $f_1(s)$ is less than or equal to 4 times the order of the zero at $s = \rho$ of the function $\zeta_{\tilde{E}_6}(s)$. In view of the fact that all zeros of $f_1(s)$ lie in the critical strip $0 < \text{Re}(s) < 1$, we have

$$
0 \leq \sum_{\rho: \ f_1(\rho) = 0, \ \rho \neq \beta_0} \frac{1}{2} \left( \frac{1}{1 - \rho} + \frac{1}{1 - \overline{\rho}} \right) \leq 4 \sum_{\rho: \ \xi_{\tilde{E}_6}(\rho) = 0, \ \text{Re}(\rho) < 1, \ \rho \neq \beta_0} \frac{1}{2} \left( \frac{1}{1 - \rho} + \frac{1}{1 - \overline{\rho}} \right). \tag{9}
$$

Then consider the function $f_2(s) := \frac{(\xi_{\tilde{E}_6}(s))^2}{\Lambda(s, \chi, Q)\Lambda(s, \overline{\chi}, \overline{Q})}$. Similar to the case of $f_1$, we have

$$
\sum_{\rho: \ f_2(\rho) = 0, \ \rho \neq \beta_0} \frac{1}{2} \left( \frac{1}{1 - \rho} + \frac{1}{1 - \overline{\rho}} \right)
= \left( \frac{\xi_{\tilde{E}_6}(1)}{2 \xi_{\tilde{E}_6}(1)} - \frac{2\delta_{\tilde{E}_6}}{1 - \beta_0} \right) - \left( \frac{\Lambda'(1, \chi, Q)}{\Lambda(1, \chi, Q)} + \frac{\Lambda'(1, \overline{\chi}, \overline{Q})}{\Lambda(1, \overline{\chi}, \overline{Q})} - \frac{2\delta_\chi}{1 - \beta_0} \right), \tag{10}
$$

and

$$
0 \leq \sum_{\rho: \ f_2(\rho) = 0, \ \rho \neq \beta_0} \frac{1}{2} \left( \frac{1}{1 - \rho} + \frac{1}{1 - \overline{\rho}} \right) \leq 4 \sum_{\rho: \ \xi_{\tilde{E}_6}(\rho) = 0, \ \text{Re}(\rho) < 1, \ \rho \neq \beta_0} \frac{1}{2} \left( \frac{1}{1 - \rho} + \frac{1}{1 - \overline{\rho}} \right). \tag{11}
$$

By logarithmically differentiating the functional equation of $\Lambda(s, \chi, Q)\Lambda(s, \overline{\chi}, \overline{Q})$ at $s = 1$, we have

$$
\frac{\Lambda'(0, \chi, Q)}{\Lambda(0, \chi, Q)} + \frac{\Lambda'(0, \overline{\chi}, \overline{Q})}{\Lambda(0, \overline{\chi}, \overline{Q})} = - \left( \frac{\Lambda'(1, \chi, Q)}{\Lambda(1, \chi, Q)} + \frac{\Lambda'(1, \overline{\chi}, \overline{Q})}{\Lambda(1, \overline{\chi}, \overline{Q})} \right).$$
The result then follows from subtracting Equation (8) by Equation (10) and the following Lemma 4.2.

**Lemma 4.2.** Let $K$ be a number field such that $K \neq \mathbb{Q}$. Denote as $\beta_0$ the (necessarily real and simple) zero of $\zeta_K(s)$ in the region

$$1 - \frac{1}{4\log|\text{disc}(K)|} \leq \text{Re}(s) < 1, |\text{Im}(s)| \leq \frac{1}{4\log|\text{disc}(K)|}$$

(if it exists). Then we have

$$0 \leq \sum_{\rho: \zeta_K(\rho)=0 \atop 0<\text{Re}(\rho)<1 \atop \rho \neq \beta_0} \frac{1}{2} \left( \frac{1}{1-\rho} + \frac{1}{1-\bar{\rho}} \right) < \frac{75}{2} \log|\text{disc}(K)|.$$

**Proof.** For any $\rho \in \mathbb{C}$ such that $\text{Re}(\rho) < 1 - \frac{1}{4\log|\text{disc}(K)|}$, we have

$$0 < \frac{1}{1-\rho} + \frac{1}{1-\bar{\rho}} < 25 \cdot \left( \frac{1}{1 + \frac{1}{\log|\text{disc}(K)|} - \rho} + \frac{1}{1 + \frac{1}{\log|\text{disc}(K)|} - \bar{\rho}} \right).$$

For any $\rho \in \mathbb{C}$ such that $1 - \frac{1}{4\log|\text{disc}(K)|} \leq \text{Re}(\rho) < 1$ and $|\text{Im}(\rho)| > \frac{1}{4\log|\text{disc}(K)|}$, we have

$$0 < \frac{1}{1-\rho} + \frac{1}{1-\bar{\rho}} < 5 \cdot \left( \frac{1}{1 + \frac{1}{\log|\text{disc}(K)|} - \rho} + \frac{1}{1 + \frac{1}{\log|\text{disc}(K)|} - \bar{\rho}} \right).$$

Therefore, we have

$$0 \leq \sum_{\rho: \zeta_K(\rho)=0 \atop 0<\text{Re}(\rho)<1 \atop \rho \neq \beta_0} \frac{1}{2} \left( \frac{1}{1-\rho} + \frac{1}{1-\bar{\rho}} \right) < 25 \sum_{\rho: \zeta_K(\rho)=0 \atop 0<\text{Re}(\rho)<1 \atop \rho \neq \beta_0} \frac{1}{2} \left( \frac{1}{1 + \frac{1}{\log|\text{disc}(K)|} - \rho} + \frac{1}{1 + \frac{1}{\log|\text{disc}(K)|} - \bar{\rho}} \right) \leq 25 \sum_{\rho: \zeta_K(\rho)=0 \atop 0<\text{Re}(\rho)<1} \frac{1}{2} \left( \frac{1}{1 + \frac{1}{\log|\text{disc}(K)|} - \rho} + \frac{1}{1 + \frac{1}{\log|\text{disc}(K)|} - \bar{\rho}} \right).$$

By the proof of Lemma 3 of [Sta74], we have

$$0 \leq \sum_{\rho: \zeta_K(\rho)=0 \atop 0<\text{Re}(\rho)<1} \frac{1}{2} \left( \frac{1}{s-\rho} + \frac{1}{s-\bar{\rho}} \right) < \frac{1}{s-1} + \frac{1}{2} \log|\text{disc}(K)|, \quad (12)$$

for any $s$ real with $1 < s < 2.$
Taking \( s = 1 + \frac{1}{\log |\text{disc}(K)|} \) in Equation (12), we get:
\[
0 \leq \sum_{\rho: \zeta_K(\rho)=0 \atop 0<\Re(\rho)<1} \frac{1}{2} \left( \frac{1}{1 + \frac{1}{\log |\text{disc}(K)|}} - \rho \right) < \frac{3}{2} \log |\text{disc}(K)|.
\]

Therefore, we have
\[
0 \leq \sum_{\rho: \zeta_K(\rho)=0 \atop 0<\Re(\rho)<1} \frac{1}{2} \left( \frac{1}{1 - \rho} + \frac{1}{1 - \rho} \right) < \frac{75}{2} \log |\text{disc}(K)|.
\]

**Corollary 4.3.** Let \( c \) be a real number such that \( 0 < c \leq \frac{1}{4} \). Suppose that for any nontrivial odd irreducible character \( \chi \) of \( \text{Gal}(\widetilde{E}_0^*/\mathbb{Q}) \) such that \( m_{(E,\Phi)}(\chi) \neq 0 \), there is no zero of \( L(s, \chi, \mathbb{Q}) \) in the region
\[
1 - \frac{c}{2} \log |\text{disc}(E_0^*)| \leq \Re(s) < 1, \quad |\text{Im}(s)| \leq \frac{1}{4} \log |\text{disc}(E_0^*)|,
\]
then we have
\[
-Z_{(E,\Phi)} - \frac{1}{2} \mu_{(E,\Phi)} < \frac{1}{4} g \cdot (75 + 2c')(2g)! \log |\text{disc}(E_0^*)| + \frac{1}{4} \left( \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} - \log(\pi) \right),
\]
where \( c' \) is defined to be \( \frac{1}{c} \) if \( c < \frac{1}{4} \), and 0 if \( c = \frac{1}{4} \).

**Proof.** By Lemma 2 and Section 2 of [Col98], for any \( \chi \in \text{Irr}(\text{Gal}(\widetilde{E}_0^*/\mathbb{Q})) \), \( m_{(E,\Phi)}(\chi) \) is a non-negative real number and \( m_{(E,\Phi)}(\chi) = m_{(E,\Phi)}(\overline{\chi}) \). Hence, we have
\[
-Z_{(E,\Phi)} - \frac{1}{2} \mu_{(E,\Phi)}
= \sum_{\chi \in \text{Irr}(\text{Gal}(\widetilde{E}_0^*/\mathbb{Q}))} m_{(E,\Phi)}(\chi) \left( \frac{L'(0,\chi,\mathbb{Q})}{L(0,\chi,\mathbb{Q})} - \frac{1}{2} \log(f(\chi, \mathbb{Q})) \right)
\leq \frac{1}{2} \sum_{\chi \in \text{Irr}(\text{Gal}(\widetilde{E}_0^*/\mathbb{Q}))} m_{(E,\Phi)}(\chi) \left( \frac{L'(0,\chi,\mathbb{Q})}{L(0,\chi,\mathbb{Q})} + \frac{L'(0,\overline{\chi},\mathbb{Q})}{L(0,\overline{\chi},\mathbb{Q})} - \log(f(\chi, \mathbb{Q})) \right)
\leq \frac{1}{2} \sum_{\chi \in \text{Irr}(\text{Gal}(\widetilde{E}_0^*/\mathbb{Q}))} m_{(E,\Phi)}(\chi) \left( (75 + 2c') \log |\text{disc}(E_0^*)| + \chi(1) \left( \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} - \log(\pi) \right) \right),
\]
(14)
By Equation (6).

By the definition of $A_{(E,\Phi)}^0$ we have
\[
\sum_{\chi \in \text{Irr}(\text{Gal}(\tilde{E}_\Phi^*/\mathbb{Q}))) \atop \chi \neq 1} m((E,\Phi)\chi) \chi(1) = A_{(E,\Phi)}^0(1) - \frac{1}{2}g = \frac{1}{2}g.
\]

Since for any $\chi \in \text{Irr}(\text{Gal}(\tilde{E}_\Phi^*/\mathbb{Q})))$, $m((E,\Phi)\chi)$ is a non-negative real number and $\chi(1) \geq 1$, we have
\[
\sum_{\chi \in \text{Irr}(\text{Gal}(\tilde{E}_\Phi^*/\mathbb{Q}))) \atop \chi \neq 1} m((E,\Phi)\chi) \leq \frac{1}{2}g.
\]

By the following Equation (17) we have
\[
\frac{1}{[E_\Phi^* : \mathbb{Q}]} \log |\text{disc}(\tilde{E}_\Phi)| \leq \log |\text{disc}(E_\Phi)|.
\]

The reflex field $E_\Phi^*$ is contained in the Galois closure $\tilde{E}$ of the extension $E/\mathbb{Q}$, and so $E_\Phi^*$ is also contained in $\tilde{E}$. Thus, we have $[E_\Phi^* : \mathbb{Q}] \leq (2g)!$.

Hence, we get our claim.

\[\square\]

**Lemma 4.4.** Let $K_1$ and $K_2$ be number fields. Let $K_1K_2$ be the compositum of $K_1$ and $K_2$. Then we have
\[
|\text{disc}(K_1K_2)|^{1/[K_1K_2:\mathbb{Q}]} \leq |\text{disc}(K_1)|^{1/[K_1:\mathbb{Q}]}|\text{disc}(K_2)|^{1/[K_2:\mathbb{Q}]}, \quad (15)
\]
and
\[
|\text{disc}(K_1K_2)| \leq |\text{disc}(K_1)|^{[K_2:\mathbb{Q}]}|\text{disc}(K_2)|^{[K_1:\mathbb{Q}]}. \quad (16)
\]

In particular, let $K$ be a number field and let $\tilde{K}$ be the Galois closure of the extension $K/\mathbb{Q}$. Then
\[
|\text{disc}(\tilde{K})|^{1/[\tilde{K} : \mathbb{Q}]} \leq |\text{disc}(K)|, \quad (17)
\]
and
\[
|\text{disc}(\tilde{K})| \leq |\text{disc}(K)|^{[K : \mathbb{Q}]!}. \quad (18)
\]

*Proof.* This is Lemma 7 of [Sta74]. \[\square\]
4.2 Sufficient conditions for the nonexistence of the zero near 1 of the Artin L-function

By Theorem 3 of [Sta74], we have the following theorem.

**Theorem 4.5.** Let \( L/K \) be a finite Galois extension of number fields. Let \( s_0 \in \mathbb{C} \) be a simple zero of \( \zeta_L(s) \).

1. For any irreducible character \( \chi \) of \( \text{Gal}(L/K) \), \( L(s, \chi, K) \) is defined at \( s = s_0 \). There is a (unique) irreducible character \( \chi_{s_0, L/K} \) of \( \text{Gal}(L/K) \) such that for any irreducible character \( \chi \) of \( \text{Gal}(L/K) \), \( L(s_0, \chi, K) = 0 \) if and only if \( \chi = \chi_{s_0, L/K} \). \( \chi_{s_0, L/K} \) is a linear character of \( \text{Gal}(L/K) \) (so \( \chi_{s_0, L/K} \) is a group homomorphism from \( \text{Gal}(L/K) \) to \( \mathbb{C} \times \)).

2. There is a (unique) subfield \( K_{s_0, L/K} \) of \( L \) containing \( K \) such that for any field \( K' \) containing \( K \) and contained in \( L \), \( \zeta_{K'}(s_0) = 0 \) if and only if \( K' \) contains \( K_{s_0, L/K} \). The extension \( K_{s_0, L/K}/K \) is cyclic.

3. \( K_{s_0, L/K} \) is the fixed field of the kernel of \( \chi_{s_0, L/K} \).

4. Suppose further that \( s_0 \) is real. Then exactly one of the following holds:

   1. \( K_{s_0, L/K} \) is equal to \( K \) and \( \chi_{s_0, L/K} \) is the trivial character.
   2. \( K_{s_0, L/K} \) is quadratic over \( K \) and \( \chi_{s_0, L/K} \) is the group homomorphism from \( \text{Gal}(L/K) \) to \( \mathbb{C} \times \) with kernel \( \text{Gal}(L/K_{s_0, L/K}) \) and image \( \{\pm 1\} \). In particular, \( \chi_{s_0, L/K} \) is a nontrivial real linear character.

For the rest of this subsection \( E \) is an arbitrary CM-field and \( \Phi \) is an arbitrary CM-type of \( E \). We denote as \( E^*_\Phi \) the reflex field of \((E, \Phi)\). We denote as \( \tilde{E}_\Phi^* \) the Galois closure of the extension \( E^*_\Phi/Q \).

**Corollary 4.6.** Suppose that one (or two, or all) of the following conditions hold:

1. The Galois closure \( \tilde{E} \) of the extension \( E/Q \) does not contain any complex quadratic subfields.
2. \( \tilde{E}_\Phi^* \) does not contain any complex quadratic subfields.
3. There does not exist a nontrivial irreducible real linear character \( \chi \) of \( \text{Gal}(\tilde{E}_\Phi^*/Q) \) such that \( m_{(E, \Phi)}(\chi) \neq 0 \) and the homomorphism \( \chi \) from \( \text{Gal}(\tilde{E}_\Phi^*/Q) \) to \( \mathbb{C} \times \) has image \( \{\pm 1\} \) and kernel \( \text{Gal}(\tilde{E}_\Phi^*/K) \) for some complex quadratic subfield \( K \) of \( \tilde{E}_\Phi^* \).

(Note that Condition 1 implies Condition 2 since \( E^*_\Phi \subset \tilde{E} \), and Condition 2 implies Condition 3.)
Then for any nontrivial odd irreducible character \( \chi \) of \( \text{Gal}(\bar{E}_\Phi^*/\mathbb{Q}) \), there is no zero of \( L(s, \chi, \mathbb{Q}) \) in the region

\[
1 - \frac{1}{4\log|\text{disc}(E_\Phi^*)|} \leq \text{Re}(s) < 1, \left| \text{Im}(s) \right| \leq \frac{1}{4\log|\text{disc}(E_\Phi^*)|}.
\]

**Proof.** Let \( \chi \) be a nontrivial odd irreducible character of \( \text{Gal}(\bar{E}_\Phi^*/\mathbb{Q}) \) such that such a zero exists. Denote this zero as \( \beta_0 \). Then \( \beta_0 \) must be real and \( \beta_0 \) is also a simple zero of \( \zeta_{E_\Phi^*}(s) \).

Therefore, by Theorem 4.5, \( \chi \) is a real linear character of \( \text{Gal}(\bar{E}_\Phi^*/\mathbb{Q}) \), and the homomorphism \( \chi \) from \( \text{Gal}(\bar{E}_\Phi^*/\mathbb{Q}) \) to \( \mathbb{C}^\times \) has image \{±1\} and kernel \( \text{Gal}(E_\Phi^*/K) \) for some quadratic subfield \( K \) of \( E_\Phi^* \).

Since \( \chi \) is an odd character, we have \( \chi(i) = -\chi(1) \), where \( i \) is the element in \( \text{Gal}(E_\Phi^*/\mathbb{Q}) \) induced by complex conjugation, and so \( K/\mathbb{Q} \) must be a complex quadratic extension.

Therefore, our claim follows. \( \square \)

Since the compositum of two CM-fields is also a CM-field, the Galois closure of a CM-field (viewed as an extension over \( \mathbb{Q} \)) is also a CM-field. We know that the reflex field \( E_\Phi^* \) of \((E, \Phi)\) is a CM-field. Therefore, \( E_\Phi^* \) is also a CM-field. We denote as \((E_\Phi^*)_+\) the maximal totally real subfield of \( E_\Phi^* \).

**Proposition 4.7.** Let \( c \) be a real number such that \( 0 < c \leq \frac{1}{4} \). Suppose that the function \( L(s, \chi, \bar{E}_\Phi^*/(E_\Phi^*)_+) = \frac{\zeta_{(E_\Phi^*)_+}}{\zeta(E_\Phi^*)} \) has no zero in the region

\[
1 - \frac{c}{\log|\text{disc}(E_\Phi^*)|} \leq \text{Re}(s) < 1, \left| \text{Im}(s) \right| \leq \frac{1}{4\log|\text{disc}(E_\Phi^*)|}
\]

Then for any nontrivial odd irreducible character \( \chi \) of \( \text{Gal}(E_\Phi^*/\mathbb{Q}) \), there is no zero of \( L(s, \chi, \mathbb{Q}) \) in the above region either.

**Proof.** Let \( \chi \) be a nontrivial odd irreducible character of \( \text{Gal}(\bar{E}_\Phi^*/\mathbb{Q}) \) such that such a zero exists. Denote this zero as \( \beta_0 \). Then \( \beta_0 \) must be real and \( \beta_0 \) is also a simple zero of \( \zeta_{E_\Phi^*}(s) \). By our assumption on \( L(s, \chi, \bar{E}_\Phi^*/(E_\Phi^*)_+) \), \( \beta_0 \) cannot be a zero of \( L(s, \chi, \bar{E}_\Phi^*/(E_\Phi^*)_+) \). Therefore, \( \beta_0 \) is a zero of \( \zeta_{(E_\Phi^*)_+}(s) \). Therefore, the field \( K_{\beta_0,E_\Phi^*/\mathbb{Q}} \) in Theorem 4.5 must be contained in the field \( (E_\Phi^*)_+ \), and so \( K_{\beta_0,E_\Phi^*/\mathbb{Q}} \) is a real quadratic field. By Theorem 4.5, since \( L(\beta_0, \chi, \mathbb{Q}) = 0 \), \( \chi \) is a group homomorphism from \( \text{Gal}(\bar{E}_\Phi^*/\mathbb{Q}) \) to \( \mathbb{C}^\times \) with kernel \( \text{Gal}(E_\Phi^*/K_{\beta_0,E_\Phi^*/\mathbb{Q}}) \), and so \( \chi(i) = \chi(1) = 1 \), where \( i \) is the element in \( \text{Gal}(\bar{E}_\Phi^*/\mathbb{Q}) \) induced by complex conjugation. This is a contradiction since the character \( \chi \) is assumed to be odd. \( \square \)
4.3 Proofs of Theorem 1.3 and Theorem 1.4

Proof of Theorem 1.3. Let g be a positive integer. Let E be a CM-field with maximal totally real subfield F of degree \([F : \mathbb{Q}] = g\). Let \((A, i : \mathcal{O}_E \to \text{End}_K(A))\) be a CM abelian variety over a number field \(K\) and let \(\Phi\) be the CM-type of \((A, i)\). Then the field \(K\) contains the reflex field \(E_\Phi^*\). Thus, we have

\[
\frac{1}{[K : \mathbb{Q}]} \log |\text{disc}(K)| \geq \frac{1}{[E_\Phi^* : \mathbb{Q}]} \log |\text{disc}(E_\Phi^*)| \geq \frac{1}{(2g)!} \log |\text{disc}(E_\Phi^*)|,
\]

where the last inequality follows from the fact that the reflex field \(E_\Phi^*\) is contained in the Galois closure \(\tilde{E}\) of the extension \(E/\mathbb{Q}\).

By Lemma 8 and Lemma 9 of [Sta74], suppose that there is a (necessarily real and simple) zero \(\beta_0\) of \(L(s, \chi_{E_\Phi^*/\mathbb{Q}})\) in the range

\[
1 - \frac{1}{4 \log |\text{disc}(E_\Phi^*)|} \leq \text{Re}(s) < 1, |\text{Im}(s)| \leq \frac{1}{4 \log |\text{disc}(E_\Phi^*)|},
\]

then there exists a complex quadratic subfield \(K\) of \(\tilde{E_\Phi^*}\) such that \(\zeta_K(\beta_0) = 0\) also. Since the Riemann zeta function \(\zeta_{\mathbb{Q}}(s)\) has no real zeros in the range \(0 < s < 1\), this means that \(\beta_0\) is a zero of the function \(L(s, \chi_K/\mathbb{Q}) = \frac{\zeta_K(s)}{\zeta_{\mathbb{Q}}(s)}\).

Since \(K\) is contained in \(\tilde{E_\Phi^*}\), we have \(|\text{disc}(E_\Phi^*)| \geq |\text{disc}(K)|\). Therefore, \(\beta_0\) is a Siegel zero of \(L(s, \chi_K/\mathbb{Q})\).

The result then follows from Proposition 4.7 and Corollary 4.3.

It is proved by Colmez ([Col93]) and Obus ([Obu13]) that the Colmez conjecture is true when the CM-field is abelian:

**Theorem 4.8.** Let \(E\) be a CM-field such that the extension \(E/\mathbb{Q}\) is Galois with abelian Galois group. Then we have

\[
h_{\text{Falt}}^{(E, \Phi)} = -Z_{(E, \Phi)} - \frac{1}{2} \mu_{(E, \Phi)}
\]

for any CM-type \(\Phi\) of \(E\).

As a corollary, we can prove an unconditional analogue of Theorem 1.3.

Proof of Theorem 1.4. Similar to the above proof of Theorem 1.3, the statement follows from the above-mentioned Lemma 8 and Lemma 9 of [Sta74], Corollary 4.6, Corollary 4.3, and Theorem 4.8.
5 The (proved) averaged Colmez conjecture

Although the formula $-Z_{(E, \Phi)} - \frac{1}{2} \mu_{(E, \Phi)}$ in the Colmez conjecture appears very complicated, the average over all CM-types \( \Phi \) of a CM-field \( E \) is much simpler: As is conjectured in Page 634 of [Col93] and proved in [YZ18] and [AGHMP18], we have the following proposition.

**Proposition 5.1.** Let \( E \) be a CM-field with maximal totally real subfield \( F \). Then we have

$$\frac{1}{2[E:Q]} \sum_{\Phi} \left( -Z_{(E, \Phi)} - \frac{1}{2} \mu_{(E, \Phi)} \right) = -\frac{1}{2} \frac{L'(0, \chi_{E/F})}{L(0, \chi_{E/F})} - \frac{1}{4} \log(|\text{disc}(E)|/|\text{disc}(F)|),$$

where the sum on the left-hand-side is over all CM-types \( \Phi \) of \( E \).

In other words, the Colmez conjecture implies the (Proved) averaged Colmez conjecture stated below.

**Theorem 5.2** ((Proved) averaged Colmez conjecture). Let \( E \) be a CM-field with maximal totally real subfield \( F \). Then we have

$$\frac{1}{2[F:Q]} \sum_{\Phi} \beta_{\text{Falt}}(E, \Phi) = -\frac{1}{2} \frac{L'(0, \chi_{E/F})}{L(0, \chi_{E/F})} - \frac{1}{4} \log(|\text{disc}(E)|/|\text{disc}(F)|),$$

where the sum on the left-hand-side is over all CM-types \( \Phi \) of \( E \).

This is proved independently by Yuan–Zhang [YZ18] and Andreatta–Goren–Howard–Madapusi-Pera [AGHMP18].

In the following, we use the proved averaged Colmez conjecture to prove averaged analogues of **Theorem 1.3** and **Theorem 1.4**.

**Proposition 5.3.** Let \( g \) be a positive integer. Suppose that there exists some effectively computable constant \( C_{\text{zero}}(g) \in \mathbb{R}_{\geq 4} \) depending only on \( g \) such that for any CM-field \( E \) with maximal totally real subfield \( F \) such that \([F:Q] = g\), the function \( L(s, \chi_{E/F}) \) has no zeros in the region

$$1 - \frac{1}{C_{\text{zero}}(g) \log |\text{disc}(E)|} \leq \text{Re}(s) < 1, |\text{Im}(s)| \leq \frac{1}{4 \log |\text{disc}(E)|},$$

then there exist effectively computable constants \( C_9(g) > 0, C_{10}(g) \in \mathbb{R} \) depending only on \( g \) such that

$$h_{\text{Falt}}^*(A) \leq C_9(g) \log |\text{disc}(E)| + C_{10}(g)$$

for any CM-field \( E \) of degree \([E:Q] = 2g\) and for any abelian variety \( A \) over a number field with complex multiplication by \( \mathcal{O}_E \).

**Proof.** Let \( E \) be any CM-field with maximal totally real subfield \( F \) such that \([E:Q] = g\). Denote as \( \beta_0 \) the (necessarily real and simple) zero of \( L(s, \chi_{E/F}) \) in the region

$$1 - \frac{1}{4 \log |\text{disc}(E)|} \leq \text{Re}(s) < 1, |\text{Im}(s)| \leq \frac{1}{4 \log |\text{disc}(E)|}$$

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We define \( \delta_{\chi_{E/F}} \) to be 1 if \( \beta_0 \) exists, and we define \( \delta_{\chi_{E/F}} \) to be 0 otherwise. By an argument similar to the proof of Proposition 4.1, we have

\[
- L'(0, \chi_{E/F}) \frac{\delta_{\chi_{E/F}}}{1 - \beta_0} \geq \frac{1}{2} \log(|\text{disc}(E)|/|\text{disc}(F)|) + \frac{g}{2} \left( \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} - \log(\pi) \right),
\]

and

\[
- \frac{L'(0, \chi_{E/F})}{L(0, \chi_{E/F})} \frac{\delta_{\chi_{E/F}}}{1 - \beta_0} < \frac{75}{2} \log |\text{disc}(E)| + \frac{1}{2} \log(|\text{disc}(E)|/|\text{disc}(F)|) + \frac{g}{2} \left( \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} - \log(\pi) \right).
\]

By our assumption, we then have

\[
- \frac{1}{2} \frac{L'(0, \chi_{E/F})}{L(0, \chi_{E/F})} \left( \frac{1}{4} \log(|\text{disc}(E)|/|\text{disc}(F)|) \right)
\]

\[
< \frac{1}{2} \left( \frac{75}{2} \log |\text{disc}(E)| + C_{\text{zero}}(g) \log |\text{disc}(E)| + \frac{g}{2} \left( \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} - \log(\pi) \right) \right).
\]

Let \((A, i: \mathcal{O}_E \hookrightarrow \text{End}_K(A))\) be any CM abelian variety over a number field \(K\). Let \(\Phi_0\) be the CM-type of \((A, i)\). By Theorem 5.2, we have

\[
h_{\text{Falt}}(\Phi_0) = - \sum_{\Phi \neq \Phi_0} h_{\text{Falt}}(\Phi) + 2^g \left( - \frac{1}{2} \frac{L'(0, \chi_{E/F})}{L(0, \chi_{E/F})} - \frac{1}{4} \log(|\text{disc}(E)|/|\text{disc}(F)|) \right).
\]

Let \(C_{\text{lower}} > 0\) be as in Theorem 2.1. Then by Theorem 2.1 we have

\[
h_{\text{Falt}}(A) = - \sum_{\Phi \neq \Phi_0} h_{\text{Falt}}(\Phi) + 2^g \left( - \frac{1}{2} \frac{L'(0, \chi_{E/F})}{L(0, \chi_{E/F})} - \frac{1}{4} \log(|\text{disc}(E)|/|\text{disc}(F)|) \right)
\]

\[
\leq (2^g - 1)gC_{\text{lower}} + 2^g \left( \frac{75}{2} \log |\text{disc}(E)| + C_{\text{zero}}(g) \log |\text{disc}(E)| + \frac{g}{2} \left( \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} - \log(\pi) \right) \right).
\]

\[
\text{Proposition 5.4.} \quad \text{For any } g \in \mathbb{Z}_{\geq 1}, \text{ there exist constants } C_{11}(g) > 0, C_{12}(g) \in \mathbb{R} \text{ depending only on } g \text{ such that }
\]

\[
h_{\text{Falt}}^*(A) \leq C_{11}(g) \log |\text{disc}(E)| + C_{12}(g)
\]

for any CM-field \(E\) of degree \([E: \mathbb{Q}] = 2g\) such that \(E\) has no complex quadratic subfields and for any abelian variety \(A\) over a number field with complex multiplication by \(\mathcal{O}_E\).
Proof. Let \( g \) be a positive integer. Let \( E \) be a CM-field with maximal totally real subfield \( F \) with degree \([F : \mathbb{Q}] = g\). By Lemma 9 of \cite{Sta74}, suppose that there exists a (necessarily real and simple) zero \( \beta_0 \) of \( L(s, \chi_{E/F}) \) in the range

\[
1 - \frac{1}{16g \log |\text{disc}(E)|} \leq \text{Re}(s) < 1, \quad |\text{Im}(s)| \leq \frac{1}{4 \log |\text{disc}(E)|},
\]

then there exists a complex quadratic subfield \( K \) of \( E \) such that \( \zeta_K(\beta_0) = 0 \) as well. So if \( E \) does not contain any complex quadratic fields, then there is no such zero.

The rest of the proof is similar to that of Proposition 5.3. \( \Box \)

**Lemma 5.5.** Let \( E \) be a CM-field with maximal totally real subfield \( F \) of degree \([F : \mathbb{Q}] = g\). Let \( \Phi_1, \Phi_2 \) be CM-types of \( E \) such that \(|\Phi_1 \cap \Phi_2| = g - 1\). Let \( \varphi_0 \) be the unique element in \( \text{Hom}_{\mathbb{Q}}(F, \mathbb{R}) \) such that the element \( \varphi_1 \) in \( \Phi_1 \) lying above \( \varphi_0 \) is not equal to the element \( \varphi_2 \) in \( \Phi_2 \) lying above \( \varphi_0 \). We have \( \varphi_1 = \varphi_2 \circ i \), where \( i \) is the nontrivial element of \( \text{Gal}(E/F) \). It is easy to see that the subfield \( \varphi_1(E) \) of \( C \) is equal to the subfield \( \varphi_2(E) \) of \( C \). Let \( E_{\Phi_1}^*, E_{\Phi_2}^* \) be the reflex fields of \((E, \Phi_1), (E, \Phi_2)\), respectively. Then the compositum of fields \( E_{\Phi_1}^*, E_{\Phi_2}^* \) contains the field \( \varphi_1(E) = \varphi_2(E) \).

**Proof.** Since \( E \) is a totally complex quadratic extension of the totally real field \( F \), we can write \( E = F[\sqrt{-\alpha_E}] \) for some totally positive \( \alpha_E \in F \), where \( \sqrt{-\alpha_E} \) is any square root of \(-\alpha_E \) in \( \mathbb{Q} \). Thus, \( \sum_{\varphi \in \Phi_1} \varphi(\sqrt{-\alpha_E}) \in E_{\Phi_1}^* \) and \( \sum_{\varphi \in \Phi_2} \varphi(\sqrt{-\alpha_E}) \in E_{\Phi_2}^* \). By our assumption on \( \Phi_1, \Phi_2 \) and \( \varphi_0 \), we have

\[
\sum_{\varphi \in \Phi_1} \varphi(\sqrt{-\alpha_E}) - \sum_{\varphi \in \Phi_2} \varphi(\sqrt{-\alpha_E}) = \varphi_1(\sqrt{-\alpha_E}) - \varphi_2(\sqrt{-\alpha_E}) = 2\varphi_1(\sqrt{-\alpha_E}) - 2\varphi_2(\sqrt{-\alpha_E}).
\]

Therefore, the compositum of fields \( E_{\Phi_1}^*, E_{\Phi_2}^* \) contains the element \( \varphi_1(\sqrt{-\alpha_E}) = -\varphi_2(\sqrt{-\alpha_E}) \).

Let \( \alpha_F \) be an element of \( F \) such that \( F = \mathbb{Q}[\alpha_F] \). Then similar to above, since

\[
\sum_{\varphi \in \Phi_1} \varphi(\alpha_E \sqrt{-\alpha_E}) - \sum_{\varphi \in \Phi_2} \varphi(\alpha_E \sqrt{-\alpha_E}) = \varphi_1(\alpha_E \sqrt{-\alpha_E}) - \varphi_2(\alpha_E \sqrt{-\alpha_E}) = 2\varphi_1(\alpha_F \sqrt{-\alpha_E}) - 2\varphi_2(\alpha_F \sqrt{-\alpha_E}) = 2\varphi_0(\alpha_F) \varphi_1(\sqrt{-\alpha_E}) = -2\varphi_0(\alpha_F) \varphi_2(\sqrt{-\alpha_E}),
\]

the compositum of fields \( E_{\Phi_1}^*, E_{\Phi_2}^* \) contains the element

\( \varphi_0(\alpha_F) \varphi_1(\sqrt{-\alpha_E}) = -\varphi_0(\alpha_F) \varphi_2(\sqrt{-\alpha_E}) \).

Combined with above, we have: the compositum of fields \( E_{\Phi_1}^*, E_{\Phi_2}^* \) contains the element \( \varphi_0(\alpha_F) \) and the element \( \varphi_1(\sqrt{-\alpha_E}) = -\varphi_2(\sqrt{-\alpha_E}) \), and so it contains the field \( \varphi_1(E) = \varphi_2(E) \). \( \Box \)
Remark 5.6. The CM-types $\Phi_1, \Phi_2$ in Lemma 5.5 is a pair of “nearby” CM-types considered in [YZ18].

Corollary 5.7. Let $E$ be a CM-field with maximal totally real subfield $F$. Let $\Phi_1, \Phi_2$ be CM-types of $E$ such that $|\Phi_1 \cap \Phi_2| = g - 1$. Then we have
$$|\text{disc}(E^{\Phi_1})|^{1/|\Phi_1 : \mathbb{Q}|}|\text{disc}(E^{\Phi_2})|^{1/|\Phi_2 : \mathbb{Q}|} \geq |\text{disc}(E)|^{1/|E : \mathbb{Q}|},$$
where $E^{\Phi_1}, E^{\Phi_2}$ are the reflex fields of $(E, \Phi_1), (E, \Phi_2)$, respectively.

Proof. This follows from Lemma 5.5 and Equation (15).

Proof of Theorem 1.6. This follows from Corollary 5.7, Proposition 5.4, and the fact that the field of definition of any CM abelian variety contains the reflex field.

Proof of Theorem 1.7. This follows from Corollary 5.7, Proposition 5.3, and the fact that the field of definition of any CM abelian variety contains the reflex field.

6 Field of everywhere good reduction of CM abelian varieties

We know that any abelian variety over a number field with complex multiplication by a CM-field has potential good reduction everywhere. In this section, we show that the logarithm of the root discriminant of the field of everywhere good reduction can be small compared with the logarithm of the discriminant of the CM-field.

Lemma 6.1. Let $A$ be an abelian variety over a number field $K$. Let $L_1, L_2$ be number fields containing $K$. If the abelian variety $A_{L_1}/L_1$ and the abelian variety $A_{L_2}/L_2$ both have everywhere good reduction, then the abelian variety $A_{L_1 \cap L_2}/L_1 \cap L_2$ has everywhere good reduction.

Proof. This follows from the Neron-Ogg-Shafarevich criterion.

By part (b) of Corollary 2 to Theorem 2 of [ST68], we have the following theorem:

Theorem 6.2. Let $A$ be an abelian variety over a number field $K$. Let $p$ be a prime ideal of $\mathcal{O}_K$. Let $p$ be the characteristic of the residue field $\mathcal{O}_K/p$. Suppose that $A/K$ has potential good reduction at $p$. Let $m$ be any integer $\geq 3$ and prime to $p$. Let $K(A[m])$ be the minimal field of definition of the set of $m$-torsion points $A[m]$ of $A$. The following are equivalent:
(a) The extension $K[A[m]]/K$ is unramified at $\mathfrak{p}$.

(b) The abelian variety $A/K$ has good reduction at $\mathfrak{p}$.

**Corollary 6.3.** Let $K$ be a number field. Let $A$ be an abelian variety over $K$ with potential good reduction everywhere. Let $S_{A/K}$ be the set of all prime ideals of $\mathcal{O}_K$ where the abelian variety $A$ over $K$ does not have good reduction. There exists a finite Galois extension $L/K$, $L/K$ unramified at all primes $\mathfrak{p}$ of $\mathcal{O}_K$ with $\mathfrak{p} \notin S_{A/K}$, such that the abelian variety $A_L/L$ has good reduction everywhere.

**Proof.** We first fix a prime $p_1$ such that the abelian variety $A/K$ has good reduction at every prime ideal $\mathfrak{p}_1$ of $\mathcal{O}_K$ above $p_1$. Let $L_1 := K(A[p_2])$ be the minimal field of definition of the set of $p_1$-torsion points $A[p_2]$ of $A$. By Theorem 6.2, we can show that $L_1/K$ is a finite Galois extension unramified at any prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ such that $\mathfrak{p} \notin S_{A/K}$ and the characteristic of the residue field $\mathcal{O}_K/\mathfrak{p}$ is not equal to $p_1$, and the abelian variety $A_{L_1}/L_1$ has everywhere good reduction.

Next, we fix a prime $p_2$ not equal to $p_1$ such that the abelian variety $A/K$ has good reduction at every prime ideal $\mathfrak{p}_2$ of $\mathcal{O}_K$ above $p_2$. Let $L_2 := K(A[p_2])$ be the minimal field of definition of the set of $p_2$-torsion points $A[p_2]$ of $A$. Again by Theorem 6.2, we can show that $L_2/K$ is a finite Galois extension unramified at any prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ such that $\mathfrak{p} \notin S_{A/K}$ and the characteristic of the residue field $\mathcal{O}_K/\mathfrak{p}$ is not equal to $p_2$, and the abelian variety $A_{L_2}/L_2$ has everywhere good reduction.

Now consider the extension $L_1 \cap L_2$ of $K$. It is a finite Galois extension unramified at any prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ such that $\mathfrak{p} \notin S_{A/K}$ (since $p_1 \neq p_2$). Since the abelian varieties $A_{L_1}/L_1$ and $A_{L_2}/L_2$ both have everywhere good reduction, by Lemma 6.1, the abelian variety $A_{L_1 \cap L_2}/L_1 \cap L_2$ also has everywhere good reduction. Taking $L = L_1 \cap L_2$, we get our claim.

The following lemma shows that in terms of unramifiedness, the extension $L/K$ in Corollary 6.3 is the “best possible”.

**Lemma 6.4.** Let $K$ be a number field. Let $K'/K$ be a finite extension. Let $\mathfrak{p}'$ be a prime ideal of $\mathcal{O}_{K'}$, lying above a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$. Let $A$ be an abelian variety over $K$. Suppose that the extension $K'/K$ is unramified at $\mathfrak{p}'$, and the abelian variety $A_{K'}/K'$ has good reduction at $\mathfrak{p}'$, then the abelian variety $A/K$ has good reduction at $\mathfrak{p}$.

**Proof.** This follows from the Neron-Ogg-Shafarevich criterion.

By Theorem 7 and the remarks before Theorem 7 of [ST68], we have the following theorem:

**Theorem 6.5.** Let $K$ be a number field. Let $E$ be a CM-field. Let $A$ be an abelian variety over $K$ with complex multiplication by $E$. Let $\mu(E)$ be the group
of all roots of unity in $E$. There exists a cyclic extension $C$ of $K$ of degree $[C : K] \leq 2 \cdot \#\mu(E)$, such that the abelian variety $A_C$ over $C$ has everywhere good reduction.

**Corollary 6.6.** Let $K$ be a number field. Let $E$ be a CM-field. Let $A$ be an abelian variety over $K$ with complex multiplication by $E$. Let $\mu(E)$ be the group of all roots of unity in $E$. Let $S_{A/K}$ be the set of all prime ideals of $O_K$ where the abelian variety $A$ over $K$ does not have good reduction. There exists a cyclic extension $K'$ of $K$ of degree $[K' : K] \leq 2 \cdot \#\mu(E)$, such that the abelian variety $A_{K'}$ over $K'$ has everywhere good reduction.

**Proof.** Let $C/K$ be the finite cyclic extension in Theorem 6.5 and let $L/K$ be the finite Galois extension in Corollary 6.3. Let $K' = C \cap L$. Then $K'/K$ is a cyclic extension of degree $[K' : K] \leq 2 \cdot \#\mu(E)$ and $K'/K$ is unramified at any prime ideal $p$ of $O_K$ such that $p \notin S_{A/K}$. By Lemma 6.1, the abelian variety $A_{K'/K'}$ has everywhere good reduction. Hence we get our claim.

In order to prove Theorem 1.8 and Theorem 1.9, we will also need the following theorem, which is a combination of Corollary A.4.6.5, Theorem A.4.5.1 and Remark A.4.5.2 of [CCO14].

**Theorem 6.7.** Let $E$ be a CM-field and let $\Phi$ be a CM-type of $E$. Let $E^*_\Phi$ be the reflex field of $(E, \Phi)$ and let $M$ be the field of moduli for the reflex norm of $(E, \Phi)$ ($M$ is an everywhere unramified finite abelian extension of $E^*_\Phi$).

There exists a prime $p$ and a CM abelian variety $(A, i: O_E \hookrightarrow \text{End}_M(A))$ over $M$ of CM-type $\Phi$ such that $A$ has good reduction at every prime ideal of $O_M$ outside $p$.

Moreover, we can choose $p$ such that

$$p \leq 2 \cdot |\text{disc}(E(\mu_{mp_1p_2\cdots p_s}))|^{C_{\text{prime}}},$$

where $C_{\text{prime}}$ is an effectively computable absolute constant in $\mathbb{R}_{>0}$, for any positive integer $n$, $\mu_n$ denotes a primitive $n$-th root of unity, $m$ is the order of the group $\mu(E)$ of all roots of unity in $E$, and $p_1, p_2, \ldots, p_s$ are the distinct prime divisors of $m$.

Assuming the Generalized Riemann Hypothesis, the above bound on $p$ can be improved to

$$p \leq 70 \cdot \left(\log |\text{disc}(E(\mu_{mp_1p_2\cdots p_s}))|\right)^2.$$

**Proof of Theorem 1.8.** Assume the Generalized Riemann Hypothesis.

Let $g$ be a positive integer. Let $E$ be a CM-field such that $[E : Q] = 2g$. Let $\Phi$ be a CM-type of $E$. Let $E^*_\Phi$ be the reflex field of $(E, \Phi)$ and let the field $M$,
the abelian variety $A$ over $M$ and the prime $p$ be as in Theorem 6.7, such that the upper bound on $p$ is given by Equation (21).

Denote $K := M$. As in Corollary 6.6, let $S_{A/K}$ be the set of all prime ideals of $O_K$ where the abelian variety $A$ over $K$ does not have good reduction. By our choice of $A$, for any $p \in S_{A/K}$, $p$ lies above the prime $p$. Let $K'$ be the cyclic extension of $K$ in Corollary 6.6 of degree $[K' : K] \leq 2 \cdot \# \mu(E)$, $K'/K$ unramified at any prime ideal $p$ of $O_K$ such that $p \notin S_{A/K}$, such that the abelian variety $A_{K'}$ over $K'$ has everywhere good reduction.

Therefore, the extension $K'/K$ is ramified only at the prime ideals $q$ of $O_{K'}$ such that $q$ lies above the prime $p$. Let $D_{K'/K}$ be the different of the extension $K'/K$. By Chapter 3, Section 6 of [Ser79], we have

$$e_{q/p} - 1 \leq \text{val}_q(D_{K'/K}) \leq e_{q/p} - 1 + \text{val}_q(e_{q/p}),$$

for any prime ideal $q$ of $O_{K'}$, lying above a prime ideal $p$ of $O_K$, where $e_{q/p}$ is the ramification index of $q|p$.

This means that we have

$$\text{val}_q(D_{K'/K}) \leq 2e_{q/p}e_{q/p} \leq 2e_{q/p}[K' : K],$$

where $e_{q/p}$ is the ramification index of the prime ideal $q$ of $O_{K'}$, lying above the prime ideal $(p)$ of $\mathbb{Z}$.

Therefore, we have

$$D_{K'/K} \bigg| \prod_{q \in O_{K'}} q^{e_{q/p} \cdot 2 \cdot \# \mu(E)},$$

where the product is over the prime ideals $q$ of $O_{K'}$ above $p$.

Thus, we have

$$\log \left( \text{Norm}_{K'/\mathbb{Q}}(D_{K'/K}) \right) \leq \log \left( \text{Norm}_{K'/\mathbb{Q}} \left( \prod_{q \in O_{K'}} q^{e_{q/p} \cdot 2 \cdot \# \mu(E)} \right) \right)$$

$$\leq 2 \cdot \# \mu(E) \sum_{q \in O_{K'}} \frac{e_{q/p} \cdot \log \left( \text{Norm}_{K'/\mathbb{Q}}(q) \right)}{q|p}$$

$$= 2 \cdot \# \mu(E) \sum_{q \in O_{K'}} e_{q/p}f_{q/p} \log(p)$$

$$= 2 \cdot \# \mu(E)[K' : \mathbb{Q}] \log(p),$$

where $f_{q/p}$ is the residue degree of the prime ideal $q$ of $O_{K'}$, lying above the prime ideal $(p)$ of $\mathbb{Z}$.

Since the extension $K/E_{p^*}$ is unramified, the different $D_{K/E_{p^*}}$ of the extension
$K/E^*_\Phi$ is equal to the unit ideal of $\mathcal{O}_K$. Thus, we have

$$\frac{1}{[K' : \mathbb{Q}]} \log |\text{disc}(K)| = \frac{1}{[K' : \mathbb{Q}]} \log (\text{Norm}_{K'/\mathbb{Q}}(\mathcal{D}_{K'/\mathbb{Q}}))$$

$$= \frac{1}{[K' : \mathbb{Q}]} \log (\text{Norm}_{K'/\mathbb{Q}}(\mathcal{D}_{K'/K} \mathcal{O}_K \mathcal{D}_{E^*_\Phi} \mathcal{D}_{E^*_\Phi}/\mathbb{Q}))$$

$$= \frac{1}{[K' : \mathbb{Q}]} \log (\text{Norm}_{K'/\mathbb{Q}}(\mathcal{D}_{K'/K} \mathcal{O}_K \mathcal{D}_{E^*_\Phi}/\mathbb{Q}))$$

$$= \frac{1}{[K' : \mathbb{Q}]} \log (\text{Norm}_{K'/\mathbb{Q}}(\mathcal{D}_{K'/K} \mathcal{O}_K \mathcal{D}_{E^*_\Phi}/\mathbb{Q})) + \frac{1}{[K' : \mathbb{Q}]} \log (\text{Norm}_{K'/\mathbb{Q}}(\mathcal{D}_{E^*_\Phi}/\mathbb{Q}))$$

$$\leq 2 \cdot \# \mu(E) \log(p) + \frac{1}{[E^*_\Phi : \mathbb{Q}]} \log |\text{disc}(E^*_\Phi)|,$$

where the last inequality follows from Equation (22).

By our assumption on $p$, we have

$$p \leq 70 \cdot \left( \log |\text{disc}(E(\mu_{mp_1p_2 \cdots p_s}))| \right)^2.$$

Thus, we have

$$\log(p) \leq \log(70) + 2 \log |\text{disc}(E(\mu_{mp_1p_2 \cdots p_s}))|$$

$$\leq \log(70) + 2 \log \left( (4g)^4 \left( (4g)^{4g} \cdot 2g(\log(\text{disc}(E)))^{1+g} \right) \right),$$

where the last inequality is by the following Lemma 6.9.

By the following Lemma 6.8, we have

$$\mu(E) \leq (4g)^2.$$  \hspace{1cm} (25)

Plugging Equation (24) and Equation (25) into Equation (23), we get our claim.

\begin{lemma}
Let $K$ be a number field of degree $[K : \mathbb{Q}] = n$. Let $\mu(K)$ be the group of all roots of unity in $K$. Then $\mu(K)$ is a finite cyclic group of order less than or equal to $(2n)^2$.
\end{lemma}

\begin{proof}
Since $[K : \mathbb{Q}]$ is finite, $\mu(K)$ is a finite group. It is easy to see that $\mu(K)$ is a cyclic group. Let $m$ be the order of the group $\mu(K)$, and let $p_1, p_2, \cdots, p_s$ be the distinct prime divisors of $m$. Denote as $\mu_m$ the primitive $m$-th root of
unity. Then $K$ contains the $m$-th cyclotomic field $\mathbb{Q}(\mu_m)$. Since $[\mathbb{Q}(\mu_m) : \mathbb{Q}] = \#(\mathbb{Z}/m\mathbb{Z})^\times = \frac{m(p_1 - 1)(p_2 - 1) \cdots (p_s - 1)}{p_1 p_2 \cdots p_s}$ and $[K : \mathbb{Q}] = n$, this means that 

$$m(p_1 - 1)(p_2 - 1) \cdots (p_s - 1) \leq n.$$ 

For any prime $p \neq 2$, we have $\sqrt{p} \leq p - 1$. Thus, we have 

$$\frac{m(p_1 - 1)(p_2 - 1) \cdots (p_s - 1)}{p_1 p_2 \cdots p_s} \geq \frac{m\sqrt{p_1}\sqrt{p_2} \cdots \sqrt{p_s}}{2p_1 p_2 \cdots p_s} = \frac{2\sqrt{p_1}\sqrt{p_2} \cdots \sqrt{p_s}}{m} \geq \frac{m}{2\sqrt{m}} = \sqrt{m}/2.$$ 

Therefore, we have 

$$m \leq (2n)^2.$$ 

\[\square\]

**Lemma 6.9.** Let $K$ be a number field of degree $[K : \mathbb{Q}] = n$. Then we have 

$$|\text{disc}(K(\mu_{mp_1 p_2 \cdots p_s}))| \leq ((2n)^4)^{(2n)^4} n \cdot |\text{disc}(K)|^{(2n)^4},$$ 

where for any positive integer $k$, $\mu_k$ denotes a primitive $k$-th root of unity, $m$ is the order of the group $\mu(K)$ of all roots of unity in $K$, and $p_1, p_2, \cdots, p_s$ are the distinct prime divisors of $m$.

**Proof.** For any $k \in \mathbb{Z}_{\geq 3}$, the $k$-th cyclotomic field $\mathbb{Q}(\mu_k)$ has degree $[\mathbb{Q}(\mu_k) : \mathbb{Q}] = \#(\mathbb{Z}/k\mathbb{Z})^\times$. By [MP05], we have:

$$\text{disc}(\mathbb{Q}(\mu_k)) = (-1)^{\varphi(k)/2} \frac{k^{\varphi(k)}}{\prod_{p|k} p^{\varphi(k)/(p-1)}},$$

where $\varphi(k) = \#(\mathbb{Z}/k\mathbb{Z})^\times$ is Euler’s totient function, and the product in the denominator on the right-hand-side is over primes $p$ dividing $k$.

Thus, we have 

$$|\text{disc}(\mathbb{Q}(\mu_k))| \leq k^k.$$ 

By **Lemma 6.8**, we have 

$$m \leq (2n)^2.$$ 

Thus, the $mp_1 p_2 \cdots p_s$-th cyclotomic field $\mathbb{Q}(\mu_{mp_1 p_2 \cdots p_s})$ has degree 

$$[\mathbb{Q}(\mu_{mp_1 p_2 \cdots p_s}) : \mathbb{Q}] = \#(\mathbb{Z}/mp_1 p_2 \cdots p_s\mathbb{Z})^\times$$

$$= m(p_1 - 1)(p_2 - 1) \cdots (p_s - 1)$$

$$\leq m^2$$

$$\leq (2n)^4.$$
Moreover, \( mp_1p_2 \cdots p_s \leq m^2 \leq (2n)^4 \). Thus, we have
\[
|\text{disc}(\mathbb{Q}(\mu_{mp_1p_2 \cdots p_s}))| \leq ((2n)^4)^{(2n)^4}.
\]

We know that \( K(\mu_{mp_1p_2 \cdots p_s}) \) is equal to the compositum of \( K \) and \( \mathbb{Q}(\mu_{mp_1p_2 \cdots p_s}) \). Thus, by Equation (16), we have
\[
|\text{disc}(K(\mu_{mp_1p_2 \cdots p_s}))| \leq |\text{disc}(K)[\mathbb{Q}(\mu_{mp_1p_2 \cdots p_s}):\mathbb{Q}]| \text{disc}(\mathbb{Q}(\mu_{mp_1p_2 \cdots p_s}))[K:\mathbb{Q}]
\leq ((2n)^4)^{(2n)^4} n(\text{disc}(K))^{(2n)^4}.
\]

\[ \square \]

**Proof of Theorem 1.9.** The proof is similar to that of Theorem 1.8 (using Theorem 6.7 and Corollary 6.6). The difference is that since we do not assume the Generalized Riemann Hypothesis, we use Equation (20) in Theorem 6.7 to bound the prime \( p \) instead of Equation (21). The term \( \frac{1}{[E_\Phi:Q]} \log |\text{disc}(E_\Phi)| \) of Equation (1) is submerged into the term \( C_{15}(g)\log |\text{disc}(E)| \) of Equation (2) by the fact that the reflex field \( E_\Phi \) is contained in the Galois closure \( \tilde{E} \) of the extension of \( E/Q \) (and so \( \frac{1}{[E_\Phi:Q]} \log |\text{disc}(E_\Phi)| \) is less than or equal to \( \frac{1}{[E:Q]} \log (|\text{disc}(\tilde{E})|) \)) and the fact that \( \frac{1}{[E:Q]} \log (|\text{disc}(\tilde{E})|) \leq \log |\text{disc}(E)| \) by Equation (17).

**Remark 6.10.** In comparison with Theorem 1.8, one might ask how the right-hand-side of the formula in (Proved) averaged Colmez conjecture behaves under the Generalized Riemann Hypothesis.

The right-hand-side of Equation (19) is equal to
\[
\frac{1}{2} \frac{L'(0, \chi_{E/F})}{L(0, \chi_{E/F})} - \frac{1}{4} \log (|\text{disc}(E)|/|\text{disc}(F)|)
= \frac{1}{2} \frac{L'(1, \chi_{E/F})}{L(1, \chi_{E/F})} + \frac{1}{4} \log (|\text{disc}(E)|/|\text{disc}(F)|)
+ \frac{1}{2} \frac{g}{2} \left( \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})} + \frac{\Gamma(1)}{\Gamma(1)} - 2 \log (\pi) \right),
\]
where \( g := [F:Q] \). The equality follows from logarithmically differentiating the functional equation of \( L(s, \chi_{E/F}) \) at \( s = 0 \).

Assume the Generalized Riemann Hypothesis. Then for any \( g \in \mathbb{Z}_{\geq 1} \), there exist constants \( C_{\text{GRH},1}(g) > 0, C_{\text{GRH},2}(g) \in \mathbb{R} \) depending only on \( g \) such that
\[
\left| \frac{L'(1, \chi_{E/F})}{L(1, \chi_{E/F})} \right| \leq C_{\text{GRH},1}(g) \log \log |\text{disc} E| + C_{\text{GRH},2}(g)
\]
for any CM-field \( E \) with maximal totally real subfield \( F \) such that \( [F:Q] = g \). Then it is easy to see that for any \( g \in \mathbb{Z}_{\geq 1} \), for any \( \epsilon > 0 \), there exists a constant
\[ c(g, \epsilon) > 0 \text{ depending only on } g \text{ and } \epsilon \text{ such that} \]
\[ \left| \frac{-1L'(0, \chi_{E/F})}{2L(0, \chi_{E/F})} - \frac{1}{4} \log(|\text{disc}(E)|/|\text{disc}(F)|) \right| < \epsilon \log |\text{disc}(E)|, \]
for any CM-field \( E \) with maximal totally real subfield \( F \) such that \( [E : Q] = 2g \) and \( |\text{disc}(E)| \geq c(g, \epsilon) \).

Since \( |\text{disc}(E)|/|\text{disc}(F)| \leq |\text{disc}(E)| \leq (|\text{disc}(E)|/|\text{disc}(F)|)^2 \), this means that assuming the Generalized Riemann Hypothesis, the right-hand-side of Equation (19) is “approximately some constant times \( \log |\text{disc}(E)| \)”.

**Remark 6.11.** One might wonder whether there is a lower bound for \( |\text{disc}(E^*_{\Phi})| \) in terms of \( |\text{disc}(E)| \) and \( [E : Q] \). The following example shows that the answer is no:

Let \( F \) be any totally real number field. Let \( -d \in \mathbb{Z}_{\leq -2} \) be any fundamental discriminant (so \( \text{disc}(Q(\sqrt{-d})) = d \)) such that \( -d \) is prime to \( \text{disc}(F) \). (For any totally real number field \( F \), there are infinitely many such \( -d \).) Let \( E \) be the compositum of the fields \( F \) and \( Q(\sqrt{-d}) \).

Then \( E \) is a CM-field with maximal totally real subfield \( F \). Let \( \Phi \) be the CM-type defined as follows:

For any \( \varphi_0 \in \text{Hom}_Q(F, \mathbb{R}) \), the element \( \phi : E \to \mathbb{C} \) in \( \Phi \) lying above \( \varphi_0 \) always sends \( \sqrt{-d} \) to \( \sqrt{-d} \).

Then it is easy to see that \( E^*_{\Phi} = Q(\sqrt{-d}) \). Thus, \( \text{disc}(E^*_{\Phi}) = \text{disc}(Q(\sqrt{-d})) = d \).

Since \( \text{disc}(Q(\sqrt{-d})) = d \) is coprime to \( \text{disc}(F) \), by Theorem 4.26 of [Nar90], for example, we have

\[ |\text{disc}(E)| = d^{[F : Q]}|\text{disc}(F)|^2. \]

Therefore, for any fixed \( g \in \mathbb{Z}_{\geq 2} \), the quotient

\[ \frac{\log |\text{disc}(E^*_{\Phi})|}{\log |\text{disc}(E)|}, \]

where \( E \) is a CM-field of degree \( [E : Q] = 2g \) and \( \Phi \) is a CM-type of \( E \), can be arbitrarily small.

Combining Remark 6.11 with Theorem 1.8, we have shown the following:

**Proposition 6.12.** Assume the Generalized Riemann Hypothesis. For any \( g \in \mathbb{Z} \) such that \( g \geq 2 \), for any \( \epsilon > 0 \), there exists a CM-field \( E \) with \( [E : Q] = 2g \), a CM-type \( \Phi \) of \( E \), a number field \( K' \) and a CM abelian variety \((A, i : O_E \hookrightarrow \)
End_{K'}(A) over $K'$ of CM-type $\Phi$ such that the abelian variety $A$ over $K'$ has everywhere good reduction and

$$\frac{1}{[K' : \mathbb{Q}]} \log |\text{disc}(K')| \leq \epsilon \log |\text{disc}(E)|.$$ 

**Remark 6.13.** In view of Remark 6.10 and Proposition 6.12, we cannot remove the “average” condition in Theorem 1.6 and Theorem 1.7—Using only the (Proved) averaged Colmez conjecture, we can only prove averaged analogues of Theorem 1.3 and Theorem 1.4.

**Remark 6.14.** In Theorem 6(i) of [Col98], Colmez has proved that there exist effectively computable absolute constants $C_{\text{Col},3} > 0$, $C_{\text{Col},4} \in \mathbb{R}$ such that for any CM-field $E$ of degree $[E : \mathbb{Q}] = 2g$ and any CM-type $\Phi$ of $E$ such that the following hold:

1. $(E, \Phi)$ satisfies the Colmez conjecture,
2. For any irreducible Artin character $\chi$ such that $m_{(E, \Phi)}(\chi) \neq 0$, the Artin conjecture for $\chi$ holds (i.e. the Artin $L$-function $L(s, \chi, \mathbb{Q})$ is holomorphic everywhere except possibly for a simple pole at $s = 1$),

we have

$$h_{(E, \Phi)}^\text{Falt} \geq C_{\text{Col},3} \cdot \mu_{(E, \Phi)} + gC_{\text{Col},4}.$$ 

Let $E$ be a CM-field of degree $[E : \mathbb{Q}] = 2g$ and let $\Phi$ be a CM-type of $E$. It is easy to see that for the function $A_{(E, \Phi)}^0$ from $\text{Gal}(\bar{E}_{\Phi}/\mathbb{Q})$ to $\mathbb{C}$, for any $\sigma \in \text{Gal}(\bar{E}_{\Phi}/\mathbb{Q})$, $A_{(E, \Phi)}^0(\sigma) = g$ if and only if $\sigma = 1$. Therefore, some calculations using the definition of the Artin conductor of Artin characters show that for any $g \in \mathbb{Z}_{\geq 1}$, there exist effectively computable constants $C_{\mu,1}(g) > 0$, $C_{\mu,2}(g) \in \mathbb{R}$ such that

$$\mu_{(E, \Phi)} \geq C_{\mu,1}(g) \frac{1}{[E_{\Phi}^* : \mathbb{Q}]} \log |\text{disc}(E_{\Phi}^*)| + C_{\mu,2}(g)$$

for any CM-field $E$ of degree $[E : \mathbb{Q}] = 2g$ and any CM-type $\Phi$ of $E$. We can compare this to Theorem 1.8.

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