Integrability of Particle Motion and Scalar Field Propagation in Kerr-(Anti) de Sitter Black Hole Spacetimes in All Dimensions

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ABSTRACT

We study the Hamilton-Jacobi and massive Klein-Gordon equations in the general Kerr-(Anti) de Sitter black hole background in all dimensions. Complete separation of both equations is carried out in cases when there are two sets of equal black hole rotation parameters. We analyze explicitly the symmetry properties of these backgrounds that allow for this Liouville integrability and construct a nontrivial irreducible Killing tensor associated with the enlarged symmetry group which permits separation. We also derive first-order equations of motion for particles in these backgrounds and examine some of their properties. This work greatly generalizes previously known results for both the Myers-Perry metrics, and the Kerr-(Anti) de Sitter metrics in higher dimensions.
1 Introduction

A number of recent developments in high energy physics have generated great interest in vacuum solutions of Einstein equations describing higher dimensional black holes, and the properties of these spacetimes. Models of spacetimes with large extra dimensions have been proposed to deal with several questions arising in modern particle phenomenology (e.g. the hierarchy problem) [1–3]. Higher dimensional black hole solutions arise naturally in such models. These models are also of interest in the context of mini-black hole production in high energy particle colliders, which would provide a window into non-perturbative gravitational physics [4, 5].

Superstring and M-theory also naturally give rise to higher dimensional black holes in their 10 or 11 dimensional ambient spacetimes. P-branes present in these theories can also support black holes, thereby making black hole solutions in an intermediate number of dimensions physically interesting as well. Solitonic objects in superstring theory frequently find a natural description in terms of higher dimensional black holes. They provide important keys to understanding strongly coupled non-perturbative phenomena which cannot be ignored at the Planck/string scale [6, 7].

The Kerr metric describes astrophysically relevant black hole spacetimes, to a very good approximation [8]. One generalization of the Kerr metric to higher dimensions is given by the Myers-Perry construction [9]. With interest now in a nonzero cosmological constant, it is worth studying spacetimes describing rotating black holes with a cosmological constant. Another motivation for including a cosmological constant is driven by the AdS/CFT correspondence. The study of black holes in an Anti-de Sitter background could give rise to interesting descriptions in terms of the conformal field theory on the boundary leading to better understanding of the correspondence [10, 11]. The general Kerr-de Sitter metrics describing rotating black holes in the presence of a cosmological constant have been constructed explicitly in [12, 13].

There is also a very strong need to understand the structure of geodesics in the background of black holes in Anti-de Sitter backgrounds in the context of string theory and the AdS/CFT correspondence. This is due to the recent work in exploring black hole singularity structure using geodesics and correlators on the dual CFT on the boundary [14–19].

In this paper we study the separability of the Hamilton-Jacobi equation in these spacetimes, which can be used to describe the motion of classical massive and massless particles (including photons). We also investigate the separability of the Klein-Gordon equation describing a massive scalar field propagating in this background. We explicitly perform the
separation in the case where there are only two sets of equal rotation parameters describing
the black hole. We use this explicit separation to obtain first-order equations of motion for
both massive and massless particles in these backgrounds. The equations are obtained in a
form that could be used for numerical study, and also in the study of black hole singularity
structure using geodesic probes and the AdS/CFT correspondence.

We also study the Klein-Gordon equation describing the propagation of a massive scalar
field in this spacetime. Separation is again explicitly shown for the case of two sets of equal
black hole rotation parameters. We construct the separation of both equations explicitly in
these cases.

This paper greatly generalizes the results of [20, 21] for the Myers-Perry metric in five
dimensions, [22] which separates the equations in the case of equal rotation parameters in the
odd dimensional Kerr-(A)dS spacetimes, and [23] which separates the equations in the case
of two independent sets of rotation parameters in the Myers-Perry metrics in all dimensions,
as well as some related results in five dimensional black hole spacetimes in [24, 25].

Separation is possible for both equations in this case due to the existence of a second-
order non-trivial irreducible Killing tensor. This is a generalization of the Killing tensor in
the Kerr black hole spacetime in four dimensions constructed in [26] which was subsequently
described by Chandrasekhar as the “miraculous property of the Kerr metric”. A similar
construction for the Myers-Perry metrics in higher dimensions has also been done [20,
23]. The Killing tensor provides an additional integral of motion necessary for complete
integrability. We also construct Killing vectors, which exist due to the additional symmetry,
and which permit the separation of these equations.

2 Construction and Overview of the Kerr-de Sitter Metrics

One of the most useful properties of the Kerr metric is that it can be written in the Kerr-
Schild [27] form, where the metric $g_{\mu\nu}$ is given exactly by its linear approximation around
the flat metric $\eta_{\mu\nu}$ as follows:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu\nu}dx^\mu dx^\nu + \frac{2M}{U}(k_\mu dx^\mu)^2,$$

where $k_\mu$ is null and geodesic with respect to both the full metric $g_{\mu\nu}$ and the flat metric
$\eta_{\mu\nu}$.

The Kerr-de Sitter metrics in all dimensions were obtained in [12] by using the de Sitter
metric instead of the flat background $\eta_{\mu\nu}$, with coordinates chosen appropriately to allow
for the incorporation of the Kerr metric via the null geodesic vectors \( k_\mu \). We quickly review the construction here.

In \( D \)-dimensional spacetime, we introduce \( n = [D/2] \) coordinates \( \mu_i \), where \([i]\) denotes the integer part of \( i \), subject to the constraint

\[
\sum_{i=1}^{n} \mu_i^2 = 1, \tag{2.2}
\]

together with \( N = [(D-1)/2] \) azimuthal angular coordinates \( \phi_i \), the radial coordinate \( r \), and the time coordinate \( t \). When the total spacetime dimension \( D \) is odd, \( D = 2n + 1 = 2N + 1 \), there are \( n \) azimuthal coordinates \( \phi_i \), each with period \( 2\pi \). If \( D \) is even, \( D = 2n = 2N + 2 \), there are only \( N = n - 1 \) azimuthal coordinates \( \phi_i \). Define \( \epsilon \) to be 1 for even \( D \), and 0 for odd \( D \).

The Kerr-de Sitter metric \( ds^2 \) in \( D \) dimensional spacetime satisfies the Einstein equation with cosmological constant \( \lambda \):

\[
R_{\mu\nu} = (D - 1) \lambda g_{\mu\nu}. \tag{2.3}
\]

Define functions \( W \) and \( F \) as follows:

\[
W = \sum_{i=1}^{n} \frac{\mu_i^2}{1 + \lambda a_i^2}, \quad F = \frac{r^2}{1 - \lambda r^2} \sum_{i=1}^{n} \frac{\mu_i^2}{r^2 + a_i^2}. \tag{2.4}
\]

In \( D \) dimensions, the Kerr-de Sitter metrics are given by

\[
ds^2 = ds^2 + \frac{2M}{U} (k_\mu dx^\mu)^2, \tag{2.5}
\]

where the de Sitter metric \( ds^2 \), the null vector \( k_\mu \), and the function \( U \) are now given by

\[
ds^2 &= -W (1 - \lambda r^2) dt^2 + F dr^2 + \sum_{i=1}^{n} \frac{r^2 + a_i^2}{1 + \lambda a_i^2} d\mu_i^2 + \sum_{i=1}^{n-\epsilon} \frac{r^2 + a_i^2}{1 + \lambda a_i^2} \mu_i^2 d\phi_i^2 \\
&\quad + \frac{\lambda}{W (1 - \lambda r^2)} \left( \sum_{i=1}^{n} \frac{(r^2 + a_i^2) \mu_i d\mu_i}{1 + \lambda a_i^2} \right)^2, \tag{2.6}
\]

\[
k_\mu dx^\mu = W dt + F dr - \sum_{i=1}^{n-\epsilon} \frac{a_i \mu_i^2}{1 + \lambda a_i^2} d\phi_i, \tag{2.7}
\]

\[
U = r^\epsilon \sum_{i=1}^{n} \frac{\mu_i^2}{r^2 + a_i^2} \prod_{j=1}^{n-\epsilon} (r^2 + a_j^2). \tag{2.8}
\]

In the even-dimensional case, where there is no azimuthal coordinate \( \phi_n \), there is also no associated rotation parameter; i.e., \( a_n = 0 \). Note that the null vector corresponding to the null one-form is

\[
k^\mu \partial_\mu = -\frac{1}{1 - \lambda r^2} \frac{\partial}{\partial t} + \frac{\partial}{\partial r} - \sum_{i=1}^{n-\epsilon} \frac{a_i}{r^2 + a_i^2} \frac{\partial}{\partial \phi_i}. \tag{2.9}
\]
This is easily obtained by using the background metric to raise and lower indices rather than the full metric, since \( k \) is null with respect to both metrics.

For the purposes of analyzing the equations of motion and the Klein-Gordon equation, it is very convenient to work with the metric expressed in Boyer-Lindquist coordinates. In these coordinates there are no cross terms involving the differential \( dr \). In both even and odd dimensions, the Boyer-Lindquist form is obtained by means of the following coordinate transformation:

\[
\begin{align*}
\dot{t} &= \dot{\tau} + \frac{2M \, d\tau}{(1 - \lambda \tau^2)(V - 2M)}, \\
\phi_i &= \phi_i - \lambda a_i \, d\tau + \frac{2M \, a_i \, d\tau}{(r^2 + a_i^2)(V - 2M)}.
\end{align*}
\] (2.10)

In Boyer-Lindquist coordinates in \( D \) dimensions, the Kerr-de Sitter metrics are given by

\[
\begin{align*}
ds^2 &= -W(1 - \lambda r^2) \, d\tau^2 + \frac{U \, dr^2}{V - 2M} + \frac{2M}{U} \left( \dot{\tau} - \sum_{i=1}^{n-\epsilon} \frac{a_i \mu_i^2 \, d\varphi_i}{1 + \lambda a_i^2} \right)^2 \\
&\quad + \sum_{i=1}^n \frac{r^2 + a_i^2}{1 + \lambda a_i^2} \, d\mu_i^2 + \sum_{i=1}^{n-\epsilon} \frac{r^2 + a_i^2}{1 + \lambda a_i^2} \mu_i^2 \left( d\varphi_i - \lambda a_i \, d\tau \right)^2 \\
&\quad + \frac{\lambda}{W(1 - \lambda r^2)} \left( \sum_{i=1}^n \frac{(r^2 + a_i^2)\mu_i \, d\mu_i}{1 + \lambda a_i^2} \right)^2,
\end{align*}
\] (2.11)

where \( V \) is defined here by

\[
V \equiv r^{\epsilon-2}(1 - \lambda r^2) \prod_{i=1}^{n-\epsilon}(r^2 + a_i^2) = \frac{U}{F}.
\] (2.12)

Note that obviously \( a_n = 0 \) in the even dimensional case, as there is no rotation associated with the last direction.

### 3 Inverting the Kerr-(A)dS metric in all dimensions

We briefly review the process of inversion of the metric using the Kerr-Schild formalism. More extensive details of this type of procedure can be found in [22, 23]. This section will also help establish some useful notation and conventions for the rest of the paper. Note that the metric is block diagonal in the \((\mu_i)\) and the \((r, \tau, \varphi_i)\) sectors and so can be inverted separately.

To deal with the \((r, \tau, \varphi_i)\) sector, the most efficient method is to use the Kerr-Schild construction of the metric. From (2.1) and using the fact that \( k \) is null, we can write

\[
g^{\mu\nu} = \eta^{\mu\nu} - \frac{2M}{U} k^\mu k^\nu,
\] (3.1)
where \( \eta \) here is the de Sitter metric rather than the flat metric, and we raise and lower indices with \( \eta \). Since the null vector \( k \) has no components in the \( \mu_i \) sector, we can regard the above equation as holding true in the \((r, \tau, \varphi_i)\) sector with \( k \) null here as well. Then we can explicitly perform the coordinate transformation (2.10) (or rather its inverse) on the raised metric to obtain the components of \( g^{\mu \nu} \) in Boyer-Lindquist coordinates in the \((r, \tau, \varphi_i)\) sector.

We get the following components for the \((r, \tau, \varphi_i)\) sector of \( g^{\mu \nu} \):

\[
\begin{align*}
g^{rr} &= g^{\varphi_i r} = 0, \\
g^{\tau \tau} &= \frac{V - 2M}{U}, \\
g^{\tau \varphi_i} &= Q - \frac{4M^2}{U(1 - \lambda r^2)^2(V - 2M)}, \\
g^{\varphi_i \varphi_j} &= \lambda a_i Q - \frac{4M^2 a_i(1 + \lambda a_i^2)}{U(1 - \lambda r^2)^2(V - 2M)(r^2 + a_i^2)} - 2M \frac{a_i}{U} (1 - \lambda r^2)(r^2 + a_i^2), \\
g^{\varphi_i \varphi_j} &= \frac{(1 + \lambda a_i^2)}{(r^2 + a_i^2) \mu_i^2} Q^{ij} + \lambda^2 a_i a_j Q + \frac{Q^{ij} U}{U}
\end{align*}
\]

where \( Q \) and \( Q^{ij} \) are defined to be

\[
Q = -\frac{1}{W(1 - \lambda r^2)} - \frac{2M}{U} \frac{1}{(1 - \lambda r^2)^2},
\]

\[
Q^{ij} = \frac{-4M^2 \lambda a_i a_j (1 + \lambda a_i^2)(r^2 + a_i^2) + (1 + \lambda a_i^2)(r^2 + a_j^2)}{(1 - \lambda r^2)^2(V - 2M)(r^2 + a_i^2)(r^2 + a_j^2)} - 2M \frac{a_i a_j}{(r^2 + a_i^2)(r^2 + a_j^2)}
\]

\[
- \frac{2M \lambda a_i a_j}{(1 - \lambda r^2)} \left[ \frac{1}{(r^2 + a_i^2)} + \frac{1}{(r^2 + a_j^2)} \right] - \frac{4M^2 a_i a_j [(1 + \lambda a_i^2) + (1 + \lambda a_j^2)]}{(1 - \lambda r^2)^2(V - 2M)(r^2 + a_i^2)(r^2 + a_j^2)}. \tag{3.4}
\]

These results were compared to previously known ones in the case of \( \lambda = 0 \) and showed agreement [20]. Also, we used the GRTensor package for Maple explicitly to check that this is the correct inverse metric [28].

Note that the functions \( W \) and \( U \) both depend explicitly on the \( \mu_i \)'s. Unless the \((r, \tau, \varphi_i)\) sector can be decoupled from the \( \mu \) sector, complete separation is unlikely. If however, all the \( a_i = a \) for some non-zero value \( a \), then \( W \) and \( U \) are no longer \( \mu \) dependent (taking the constraint into account) and separation seems likely. Note, however, that in this case we cannot deal with even dimensional spacetimes, since \( a_n = 0 \) is different from the other \( a_i = a \). The analysis in this case has been done in detail in [22].
We will actually work with a much more general case, in which separation works in both even and odd dimensional spacetimes. We consider the situation in which the set of rotation parameters $a_i$ take on at most only two distinct values $a$ and $b$ ($a = b$ can be obtained as a special case). In even dimensions at least one of these values must be zero, since $a_n = 0$. As such, in even dimensions we take $b = 0$ and $a$ to be some (possibly different) value. In the odd dimensional case, there are no restrictions on the values of $a$ and $b$. We adopt the convention

$$a_i = a \quad \text{for} \quad i = 1, ..., m \quad , \quad a_{j+m} = b \quad \text{for} \quad j = 1, ..., p,$$

(3.5)

where $m + p = N + \epsilon = n$.

Since the $\mu_i$’s are constrained by (2.2), we need to use suitable independent coordinates instead. We use the following decomposition of the $\mu_i$:

$$\mu_i = \lambda_i \sin \theta \quad \text{for} \quad i = 1, ..., m \quad , \quad \mu_{j+m} = \nu_j \cos \theta \quad \text{for} \quad j = 1, ..., p,$$

(3.6)

where the $\lambda_i$ and $\nu_j$ have to satisfy the constraints

$$\sum_{i=1}^{m} \lambda_i^2 = 1 \quad , \quad \sum_{j=1}^{p} \nu_j^2 = 1.$$

(3.7)

Since these constraints describe unit $(m - 1)$ and $(p - 1)$ dimensional spheres in the $\lambda$ and $\nu$ spaces respectively, the natural choice is to use two sets of spherical polar coordinates. We write

$$\lambda_i = \left( \prod_{k=1}^{m-i} \sin \alpha_k \right) \cos \alpha_{m-i+1},$$

$$\nu_j = \left( \prod_{k=1}^{p-j} \sin \beta_k \right) \cos \beta_{p-j+1},$$

(3.8)

with the understanding that the products are one when $i = m$ or $j = p$ respectively, and that $\alpha_m = 0$ and $\beta_p = 0$.

The $\mu$ sector metric can then be written as

$$ds^2_\mu = \frac{\rho^2}{\Delta \theta^2} + \frac{\sigma^2}{\Sigma a} \sin^2 \theta \sum_{i=1}^{m-1} \left( \prod_{k=1}^{i-1} \sin^2 \alpha_k \right) d\alpha_i^2 + \frac{\rho^2}{\Sigma \sigma} \cos^2 \theta \sum_{j=1}^{p-1} \left( \prod_{k=1}^{j-1} \sin^2 \beta_k \right) d\beta_j^2,$$

(3.9)
again with the understanding that the products are one when \(i = 1\) or \(j = 1\), and we use the definitions

\[
\rho^2 = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta ,
\]
\[
\Delta_\theta = 1 + \lambda a^2 \cos^2 \theta + \lambda b^2 \sin^2 \theta ,
\]
\[
\Sigma_a = 1 + \lambda a^2 ,
\]
\[
\Sigma_b = 1 + \lambda b^2 ,
\]
\[
Z = r^\epsilon (r^2 + a^2)^{m-1}(r^2 + b^2)^{p-1-\epsilon} .
\]

(3.10)

This diagonal metric can be easily inverted to give

\[
g^{\theta \theta} = \frac{\Delta_\theta}{\rho^2} ,
\]
\[
g^{\alpha_i \alpha_j} = \frac{\Sigma_a}{(r^2 + a^2) \sin^2 \theta} \frac{1}{\prod_{k=1}^{i-1} \sin^2 \alpha_k} \delta_{ij} , \quad i, j = 1, ..., m ,
\]
\[
g^{\beta_i \beta_j} = \frac{\Sigma_b}{(r^2 + b^2) \cos^2 \theta} \frac{1}{\prod_{k=1}^{i-1} \sin^2 \beta_k} \delta_{ij} , \quad i, j = 1, ..., p .
\]

(3.11)

For the case of two sets of rotation parameters that we consider here, the following expressions will be extremely useful:

\[
U = \rho^2 Z ,
\]
\[
W = \frac{\Delta_\theta}{\Sigma_a \Sigma_b} .
\]

(3.12)

We note that both \(V\) and \(Z\) are functions of \(r\) only.

The following identity, which can be easily verified, will be crucial in the following:

\[
Q = \frac{\Sigma_a \Sigma_b}{\rho^2 \lambda \Delta_\theta} - \frac{\Sigma_a \Sigma_b}{\rho^2 \lambda (1 - \lambda r^2)} - \frac{2M}{\rho^2 Z (1 - \lambda r^2)^2 (V - 2M)} .
\]

(3.13)

4 The Hamilton-Jacobi Equation and Separation

The Hamilton-Jacobi equation in a curved background is given by

\[- \frac{\partial S}{\partial l} = H = \frac{1}{2} g^{\mu \nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} ,
\]

(4.1)

where \(S\) is the action associated with the particle and \(l\) is some affine parameter along the worldline of the particle. Note that this treatment also accommodates the case of massless particles, where the trajectory cannot be parameterized by proper time.
We can attempt a separation of coordinates as follows. Let
\[
S = \frac{1}{2} m^2 l - E\tau + \sum_{i=1}^{m} \Phi_i \varphi_i + \sum_{i=1}^{p} \Psi_i \varphi_{m+i} + S_r(r) + S_\theta(\theta) + \sum_{i=1}^{m-1} S_{\alpha_i}(\alpha_i) + \sum_{i=1}^{p-1} S_{\beta_i}(\beta_i).\tag{4.2}
\]
\(\tau\) and \(\varphi_i\) are cyclic coordinates, so their conjugate momenta are conserved. The conserved quantity associated with time translation is the energy \(E\), and the conserved quantity associated with rotation in each \(\varphi_i\) is the corresponding angular momentum \(\Phi_i\) or \(\Psi_j\). We also adopt the convention that \(\Psi_p = 0\) in an even number of spacetime dimensions.

Using (4.2), (4.10), (4.11), and (4.12) we write the Hamilton-Jacobi equation (4.1) as
\[
- m^2 = \left[ \frac{\Sigma_a \Sigma_b}{\lambda^2 \rho^2 \Delta \theta} - \frac{\Sigma_a \Sigma_b}{\rho^2 \lambda (1 - \lambda r^2)} - \frac{2M}{\rho^2 Z (1 - \lambda r^2)} - \frac{4M^2}{\rho^2 Z (1 - \lambda r^2)^2} \right] E^2
+ 2 \left[ \frac{a \Sigma_a \Sigma_b}{\rho^2 \Delta \theta} - \frac{a \Sigma_a \Sigma_b}{\rho^2 (1 - \lambda r^2)} - \frac{2M \lambda a}{\rho^2 Z (1 - \lambda r^2)^2} - \frac{4M^2 a \Sigma_a}{\rho^2 Z (1 - \lambda r^2)^2 (V - 2M) (r^2 + a^2)} \right]
+ 2 \left[ \frac{b \Sigma_a \Sigma_b}{\rho^2 \Delta \theta} - \frac{b \Sigma_a \Sigma_b}{\rho^2 (1 - \lambda r^2)} - \frac{2M \lambda b}{\rho^2 Z (1 - \lambda r^2)^2} - \frac{4M^2 b \Sigma_b}{\rho^2 Z (1 - \lambda r^2)^2 (V - 2M) (r^2 + b^2)} \right]
+ \frac{2Ma}{\rho^2 Z (1 - \lambda r^2)(r^2 + a^2)} \right] \sum_{i=1}^{m} (-E) \Phi_i
+ \frac{b \Sigma_a \Sigma_b}{\rho^2 \Delta \theta} - \frac{b \Sigma_a \Sigma_b}{\rho^2 (1 - \lambda r^2)} - \frac{2M \lambda b}{\rho^2 Z (1 - \lambda r^2)^2} - \frac{4M^2 b \Sigma_b}{\rho^2 Z (1 - \lambda r^2)^2 (V - 2M) (r^2 + b^2)}
+ \frac{2Mb}{\rho^2 Z (1 - \lambda r^2)(r^2 + b^2)} \right] \sum_{j=1}^{p} (-E) \Psi_j
+ \frac{\Sigma_a}{(r^2 + a^2) \sin^2 \theta} \sum_{i=1}^{m} \frac{\Phi_i^2}{\lambda_i^2} + \frac{\Sigma_b}{(r^2 + b^2) \cos^2 \theta} \sum_{i=1}^{p} \frac{\Psi_i^2}{\nu_i^2} + \frac{\Delta_\theta}{\rho^2} \left[ \frac{dS_\theta(\theta)}{d \theta} \right]^2
+ \frac{V - 2M}{\rho^2 Z} \left[ \frac{dS_r(r)}{dr} \right]^2 + \sum_{i=1}^{m-1} \frac{\Sigma_a}{(r^2 + a^2) \sin^2 \theta} \prod_{k=1}^{i-1} \sin^2 \alpha_k \left( \frac{dS_{\alpha_k}}{dx} \right)^2
+ \sum_{i=1}^{p-1} \frac{\Sigma_b}{(r^2 + b^2) \cos^2 \theta} \prod_{k=1}^{i-1} \sin^2 \beta_k \left( \frac{dS_{\beta_k}}{dx} \right)^2
+ \sum_{i=1}^{m} \sum_{j=1}^{m} \left[ \lambda^2 a^2 \left( \frac{\Sigma_a \Sigma_b}{\rho^2 \Delta \theta} - \frac{\Sigma_a \Sigma_b}{\rho^2 \lambda (1 - \lambda r^2)} - \frac{2M}{\rho^2 Z (1 - \lambda r^2)} \right)
+ \frac{4M^2 a \Sigma_a^2}{\rho^2 Z (V - 2M) (r^2 + a^2)^2} + \frac{Q^{ij}}{\rho^2 Z} \right] \Phi_i \Phi_j
+ \sum_{i=1}^{p} \sum_{j=1}^{p} \left[ \lambda^2 b^2 \left( \frac{\Sigma_a \Sigma_b}{\rho^2 \Delta \theta} - \frac{\Sigma_a \Sigma_b}{\rho^2 \lambda (1 - \lambda r^2)} - \frac{2M}{\rho^2 Z (1 - \lambda r^2)} \right)
+ \frac{4M^2 b \Sigma_b^2}{\rho^2 Z (V - 2M) (r^2 + b^2)^2} + \frac{Q^{(i+m)(j+m)}}{\rho^2 Z} \right] \Psi_i \Psi_j
+ 2 \sum_{i=1}^{m} \sum_{j=1}^{m} \left[ \lambda^2 ab \left( \frac{\Sigma_a \Sigma_b}{\rho^2 \Delta \theta} - \frac{\Sigma_a \Sigma_b}{\rho^2 \lambda (1 - \lambda r^2)} - \frac{2M}{\rho^2 Z (1 - \lambda r^2)} \right)
+ \frac{4M^2 ab \Sigma_a \Sigma_b}{\rho^2 Z (V - 2M) (r^2 + a^2)(r^2 + b^2)} + \frac{Q^{ij+m}}{\rho^2 Z} \right] \Phi_i \Psi_j.\tag{4.3}
Note that here the $\lambda_i$ and $\nu_j$ are not coordinates, but simply quantities defined by $\Phi$. We continue to use the convention defined for products of $\sin^2 \alpha_i$ and $\sin^2 \beta_j$ defined earlier. Separate the $\alpha_i$ and $\beta_j$ coordinates from the Hamilton-Jacobi equation via

\begin{align*}
J_1^2 &= \sum_{i=1}^{m} \left[ \frac{\Phi_i^2}{\lambda_i^2} + \frac{1}{\prod_{k=1}^{i-1} \sin^2 \alpha_k} \left( \frac{dS_\alpha}{d\alpha_i} \right)^2 \right], \\
L_1^2 &= \sum_{i=1}^{p} \left[ \frac{\Psi_i^2}{\nu_i^2} + \frac{1}{\prod_{k=1}^{i-1} \sin^2 \beta_k} \left( \frac{dS_\beta}{d\beta_i} \right)^2 \right], \quad (4.4)
\end{align*}

where $J_1^2$ and $L_1^2$ are separation constants. Then the remaining terms in the Hamilton-Jacobi equations can be explicitly separated to give ordinary differential equations for $r$ and $\theta$:

\begin{align*}
K &= m^2 r^2 - \left[ \frac{\Sigma_a \Sigma_b}{\lambda(1 - \lambda r^2)} + \frac{2M}{Z(1 - \lambda r^2)} + \frac{4M^2}{Z(1 - \lambda r^2)^2} \right] E^2 + \frac{V - 2M}{Z} \left[ \frac{dS_r(r)}{dr} \right]^2 \\
&+ 2 \left[ \frac{a \Sigma_a \Sigma_b}{(1 - \lambda r^2)} + \frac{2M \lambda a}{Z(1 - \lambda r^2)^2} + \frac{4M^2 a \Sigma_a}{4M^2 a \Sigma_a} \right] + \frac{2M a}{Z(1 - \lambda r^2)(r^2 + a^2)} \sum_{i=1}^{m} (-E) \Phi_i \\
&+ 2 \left[ \frac{b \Sigma_a \Sigma_b}{(1 - \lambda r^2)} + \frac{2M \lambda b}{Z(1 - \lambda r^2)^2} + \frac{4M^2 b \Sigma_b}{4M^2 b \Sigma_b} \right] + \frac{2Mb}{Z(1 - \lambda r^2)(r^2 + b^2)} \sum_{j=1}^{p} (-E) \Psi_j + \sum_{i=1}^{m} \sum_{j=1}^{p} \left[ \lambda^2 a^2 \left( \frac{\Sigma_a \Sigma_b}{\lambda(1 - \lambda r^2)} + \frac{2M}{Z(1 - \lambda r^2)} \right) \right] \\
&- \frac{4M^2 a^2 \Sigma^2_a}{Z(2M)(r^2 + a^2)^2} - \frac{Q_{ij}^2}{Z} \Phi_i \Phi_j \\
&+ \sum_{i=1}^{m} \sum_{j=1}^{p} \left[ \lambda^2 b^2 \left( \frac{\Sigma_a \Sigma_b}{\lambda(1 - \lambda r^2)} + \frac{2M}{Z(1 - \lambda r^2)} \right) - \frac{4M^2 b^2 \Sigma^2_a}{Z(2M)(r^2 + b^2)^2} \right] \Psi_i \Psi_j + \sum_{i=1}^{m} \sum_{j=1}^{p} \left[ \lambda^2 ab \left( \frac{\Sigma_a \Sigma_b}{\lambda(1 - \lambda r^2)} + \frac{2M}{Z(1 - \lambda r^2)} \right) \right] \\
&- \frac{4M^2 ab \Sigma_a \Sigma_b}{Z(2M)(r^2 + b^2)(r^2 + a^2)} - \frac{Q_{ij}^2}{Z} \Phi_i \Psi_j + \frac{\Sigma_a(r^2 + b^2)}{r^2 + a^2} J_1^2 + \frac{\Sigma_b(r^2 + a^2)}{r^2 + b^2} L_1^2, \quad (4.5)
\end{align*}

and

\begin{align*}
- K &= m^2 a^2 \cos^2 \theta + m^2 b^2 \sin^2 \theta + \Delta_\theta \left( \frac{dS_\theta}{d\theta} \right)^2 + \Sigma_a \cot^2 \theta J_1^2 + \Sigma_b \tan^2 \theta L_1^2 \\
&+ \frac{\Sigma_a \Sigma_b}{\lambda \Delta_\theta} E^2 - 2 \sum_{i=1}^{m} \frac{a \Sigma_a \Sigma_b}{\Delta_\theta} E \Phi_i - 2 \sum_{i=1}^{p} \frac{b \Sigma_a \Sigma_b}{\Delta_\theta} E \Psi_i + \sum_{i=1}^{m} \sum_{j=1}^{p} \lambda^2 a^2 \Sigma_a \Sigma_b \Phi_i \Phi_j \\
&+ \sum_{i=1}^{m} \sum_{j=1}^{p} \lambda^2 b^2 \Sigma_a \Sigma_b \Psi_i \Psi_j + 2 \sum_{i=1}^{m} \sum_{j=1}^{p} \lambda^2 ab \Sigma_a \Sigma_b \Phi_i \Psi_j, \quad (4.6)
\end{align*}
where $K$ is a separation constant.

In order to show complete separation of the Hamilton-Jacobi equation, we analyze the $\alpha$ and $\beta$ sectors in (4.4) and demonstrate separation of the individual $\alpha_i$ and $\beta_j$. The pattern here is that of a Hamiltonian of non-relativistic classical particles on the unit $(m-1)$-\(\alpha\) and the unit $(p-1)$-\(\beta\) spheres, with some potential dependent on the squares of the $\mu_i$. This can easily be additively separated following the usual procedure, one angle at a time, and the pattern continues for all integers $m, p \geq 2$.

The separation has the following inductive form for $k = 1, \ldots, m-2$, and $l = 1, \ldots, p - 2$:

\[
\left( \frac{dS_{\alpha_k}}{d\alpha_k} \right)^2 = J_k^2 - \frac{J_{k+1}^2}{\sin^2 \alpha_k} - \frac{\Phi_{m-k+1}^2}{\cos^2 \alpha_k},
\]
\[
\left( \frac{dS_{\alpha_{m-1}}}{d\alpha_{m-1}} \right)^2 = J_{m-1}^2 - \frac{\Phi_1^2}{\sin^2 \alpha_{m-1}} - \frac{\Phi_2^2}{\cos^2 \alpha_{m-1}},
\]
\[
\left( \frac{dS_{\beta_l}}{d\beta_l} \right)^2 = L_l^2 - \frac{L_{l+1}^2}{\sin^2 \beta_l} - \frac{\Psi_{p-l+1}^2}{\cos^2 \beta_l},
\]
\[
\left( \frac{dS_{\beta_{p-1}}}{d\beta_{p-1}} \right)^2 = L_{p-1}^2 - \frac{\Psi_1^2}{\sin^2 \beta_{p-1}} - \frac{\Psi_2^2}{\cos^2 \beta_{p-1}}.
\] (4.7)

Thus, the Hamilton-Jacobi equation in the Kerr-(Anti) de Sitter rotating black hole background in all dimensions with two sets of possibly unequal rotation parameters has the general separation

\[
S = \frac{1}{2} m^2 l - E \tau + \sum_{i=1}^{m} \Phi_i \varphi_i + \sum_{i=1}^{p} \Psi_i \varphi_{m+i} + S_r(r) + S_\theta(\theta) + \sum_{i=1}^{m-1} S_{\alpha_i}(\alpha_i) + \sum_{i=1}^{p-1} S_{\beta_i}(\beta_i),
\] (4.8)

where the $\alpha_i$ and $\beta_j$ are the spherical polar coordinates on the unit $(m-1)$ and unit $(p-1)$ spheres respectively. $S_r(r)$ can be obtained by quadratures from (4.5), $S_\theta(\theta)$ by quadratures from (4.6), and the $S_{\alpha_i}(\alpha_i)$ and the $S_{\beta_j}(\beta_j)$ again by quadratures from (4.7).

5 The Equations of Motion

5.1 Derivation of the Equations of Motion

To derive the equations of motion, we will write the separated action $S$ from the Hamilton-Jacobi equation in the following form:

\[
S = \frac{1}{2} m^2 l - E \tau + \sum_{i=1}^{m} \Phi_i \varphi_i + \sum_{i=1}^{p} \Psi_i \varphi_{m+i} + \int_r^{r'} \sqrt{R(r')} dr' + \int_{\theta}^{\Theta(\theta')} d\theta' \\
+ \sum_{i=1}^{m-1} \int_{\alpha_i}^{\alpha_i'} \sqrt{A_i(\alpha_i')} d\alpha_i' + \sum_{i=1}^{p-1} \int_{\beta_i}^{\beta_i'} \sqrt{B_i(\beta_i')} d\beta_i',
\] (5.1)
where

\[
A_k = \frac{J_k^2}{\sin^2 \alpha_k} - \frac{J_k^2}{\cos^2 \alpha_k}, \quad k = 1, \ldots, m - 2,
\]

\[
A_{m-1} = \frac{J_{m-1}^2}{\sin^2 \alpha_{m-1}} - \frac{\Phi_1\Phi_2}{\cos^2 \alpha_{m-1}},
\]

\[
B_k = \frac{L_k^2}{\sin^2 \beta_k} - \frac{\Psi_1\Psi_2}{\cos^2 \beta_k}, \quad k = 1, \ldots, p - 2,
\]

\[
B_{p-1} = \frac{L_{p-1}^2}{\sin^2 \beta_{p-1}} - \frac{\Psi_1\Psi_2}{\cos^2 \beta_{p-1}},
\]

(5.2)

\[\Theta\] is obtained from (4.6) as

\[
\Delta \Theta = - m^2 a^2 \cos^2 \theta - m^2 b^2 \sin^2 \theta - \Sigma_a \cot^2 \theta J_1^2 - \Sigma_b \tan^2 \theta L_1^2 - \frac{\Sigma_a \Sigma_b}{\Delta \theta} E^2
\]

\[+ 2 \sum_{i=1}^{m} \frac{a \Sigma_a \Sigma_b}{\Delta \theta} E \Phi_i + 2 \sum_{i=1}^{m} \frac{b \Sigma_a \Sigma_b}{\Delta \theta} E \Psi_i - \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\lambda^2 a^2 \Sigma_a \Sigma_b}{\Delta \theta} \Phi_i \Phi_j
\]

\[- \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\lambda^2 b^2 \Sigma_a \Sigma_b}{\Delta \theta} \Psi_i \Psi_j - 2 \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\lambda^2 a b \Sigma_a \Sigma_b}{\Delta \theta} \Phi_i \Phi_j - K
\]

(5.3)

and \(R\) is obtained from (4.5) as

\[
\frac{V - 2M}{Z} R = - m^2 r^2 + \left[ \frac{\Sigma_a \Sigma_b}{\lambda(1 - \lambda r^2)} + \frac{2M}{Z(1 - \lambda r^2)} + \frac{4M^2}{Z(1 - \lambda r^2)^2} \right] E^2
\]

\[+ 2 \left[ \frac{a \Sigma_a \Sigma_b}{(1 - \lambda r^2)} + \frac{2M \lambda a}{Z(1 - \lambda r^2)^2} + \frac{4M^2 a \Sigma_a}{Z(1 - \lambda r^2)^2} \right] \sum_{i=1}^{m} (-E) \Phi_i
\]

\[+ 2 \left[ \frac{b \Sigma_a \Sigma_b}{(1 - \lambda r^2)} + \frac{2M \lambda b}{Z(1 - \lambda r^2)^2} + \frac{4M^2 b \Sigma_b}{Z(1 - \lambda r^2)^2} \right] \sum_{i=1}^{m} (-E) \Psi_i
\]

\[+ 2 \left[ \frac{2M b}{Z(1 - \lambda r^2)} \right] \sum_{j=1}^{p} (-E) \Psi_j - \sum_{i=1}^{m} \sum_{j=1}^{m} \left[ \frac{\lambda^2 a^2 \Sigma_a \Sigma_b}{\lambda(1 - \lambda r^2)} \right] \Phi_i \Phi_j
\]

\[+ \sum_{i=1}^{p} \sum_{j=1}^{p} \left[ \frac{\lambda^2 b^2 \left( \frac{\Sigma_a \Sigma_b}{Z(1 - \lambda r^2)} + \frac{2M}{Z(1 - \lambda r^2)} \right)}{Z} \right] \Phi_i \Psi_j
\]

\[- \sum_{i=1}^{p} \sum_{j=1}^{p} \left[ \frac{\lambda^2 a b \left( \frac{\Sigma_a \Sigma_b}{Z(1 - \lambda r^2)} + \frac{2M}{Z(1 - \lambda r^2)} \right)}{Z} \right] \Phi_i \Phi_j
\]

\[-\frac{4M^2 b \Sigma_a \Sigma_b}{Z(1 - \lambda r^2)^2} \left( \frac{V - 2M}{r^2} \right) \sum_{i=1}^{p} \sum_{j=1}^{p} \left[ \frac{\lambda^2 a b \left( \frac{\Sigma_a \Sigma_b}{Z(1 - \lambda r^2)} + \frac{2M}{Z(1 - \lambda r^2)} \right)}{Z} \right] \Phi_i \Psi_j
\]

\[+ \frac{4M^2 a b \Sigma_a \Sigma_b}{Z(1 - \lambda r^2)^2} \left( \frac{V - 2M}{r^2} \right) \sum_{i=1}^{p} \sum_{j=1}^{p} \left[ \frac{\lambda^2 a b \left( \frac{\Sigma_a \Sigma_b}{Z(1 - \lambda r^2)} + \frac{2M}{Z(1 - \lambda r^2)} \right)}{Z} \right] \Phi_i \Phi_j
\]

\[- \frac{\Sigma_a (r^2 + b^2)}{r^2 + a^2} J_1^2 - \frac{\Sigma_b (r^2 + a^2)}{r^2 + b^2} L_1^2 + K.
\]

(5.4)
To obtain the equations of motion, we differentiate $S$ with respect to the parameters $m^2, K, E, J^2_i, L^2_j, \Phi_i, \Psi_j$ and set these derivatives equal to other constants of motion. However, we can set all these new constants of motion to zero (following from freedom in choice of origin for the corresponding coordinates, or alternatively by changing the constants of integration). Following this procedure, we get the following equations of motion:

\[
\frac{\partial S}{\partial m^2} = 0 \Rightarrow l = \int \frac{Z r^2}{V - 2M \sqrt{R}} \, dr + \int \frac{(a^2 \cos^2 \theta + b^2 \sin^2 \theta) d\theta}{\Delta \sqrt{\Theta}},
\]

\[
\frac{\partial S}{\partial K} = 0 \Rightarrow \int \frac{d\theta}{\Delta \sqrt{\Theta}} = \int \frac{Z}{V - 2M \sqrt{R}} \, dr,
\]

\[
\frac{\partial S}{\partial J^2_i} = 0 \Rightarrow \int \frac{d\alpha_k}{\sqrt{A_k}} = \int \frac{Z}{V - 2M} \frac{\Sigma_a (r^2 + b^2)}{r^2 + a^2} \, dr + \int \frac{\Sigma \cot^2 \theta d\theta}{\Delta \sqrt{\Theta}},
\]

\[
\frac{\partial S}{\partial J^2_k} = 0 \Rightarrow \int \frac{d\alpha_k}{\sqrt{A_k}} = \int \frac{1}{\sin^2 \alpha_{k-1} \sqrt{A_{k-1}}} \, d\alpha_{k-1}, \quad k = 2, ..., m - 2,
\]

\[
\frac{\partial S}{\partial L^2_i} = 0 \Rightarrow \int \frac{d\beta_l}{\sqrt{B_l}} = \int \frac{1}{\sin^2 \beta_{l-1} \sqrt{B_{l-1}}} \, d\beta_{l-1}, \quad l = 2, ..., p - 2.
\] (5.5)

We can obtain $N$ more equations of motion for the variables $\varphi_i$ by differentiating $S$ with respect to the angular momenta $\Phi_i$ and $\Psi_j$. Another equation can also be obtained by differentiating $S$ with respect to $E$ involving the time coordinate $\tau$. However, these equations are not particularly illuminating, but can be written out explicitly if necessary by following this procedure. It is often more convenient to rewrite these in the form of first-order differential equations obtained from (5.5) by direct differentiation with respect to the affine parameter. We only list the most relevant ones here:

\[
\rho^2 \frac{dr}{dl} = \frac{V - 2M \sqrt{R}}{Z},
\]

\[
\rho^2 \frac{d\theta}{dl} = \Delta \sqrt{\Theta},
\]

\[
\frac{\Sigma_a}{\sqrt{A_k}} \frac{d\alpha_k}{dl} = \frac{\sqrt{A_k}}{\sin^2 \theta \prod_{i=1}^{k-1} \sin^2 \alpha_i}, \quad k = 1, ..., m - 1,
\]

\[
\frac{\Sigma_b}{\sqrt{B_l}} \frac{d\beta_k}{dl} = \frac{\cos^2 \theta \prod_{i=1}^{l-1} \sin^2 \beta_i}{\prod_{i=1}^{l-1} \sin^2 \beta_i}, \quad l = 1, ..., p - 1.
\] (5.6)

5.2 Analysis of the Radial Equation

Worldlines of particles in these backgrounds are completely specified by the values of the conserved quantities $E, K, L^2_i, J^2_j$, and by the initial values of the coordinates. We will consider particle motion in the black hole exterior. Allowed regions of particle motion
necessarily need to have positive value for the quantity \( R \), owing to equation (5.6). We determine some of the possibilities of the allowed motion.

At large radius \( r \), the dominant contribution to \( R \), in the case of \( \lambda = 0 \), is \( E^2 - m^2 \). Thus we can say that for \( E^2 < m^2 \), we cannot have unbounded orbits, whereas for \( E^2 > m^2 \), such orbits are possible. For the case of nonzero \( \lambda \), the dominant term at large \( r \) in \( R \) (or rather the slowest decaying term) is \( \frac{m^2}{r^2} \). Thus in the case of the Kerr-Anti-de Sitter background, only bound orbits are possible, whereas in the Kerr-de Sitter backgrounds, both unbounded and bound orbits may be possible.

In order to study the radial motion of particles in these metrics, it is useful to cast the radial equation of motion into a different form. Decompose \( R \) as a quadratic in \( E \) as follows:

\[
R = \alpha E^2 - 2\beta E + \gamma ,
\]

where

\[
\alpha = \frac{Z}{V - 2M} \left[ \frac{\Sigma_a \Sigma_b}{\lambda (1 - \lambda r^2)} + \frac{2M}{Z(1 - \lambda r^2)} + \frac{4M^2}{Z(1 - \lambda r^2)^2} \right],
\]

\[
\beta = \frac{-Z}{V - 2M} \left[ \frac{a\Sigma_a \Sigma_b}{(1 - \lambda r^2)} + \frac{2M\lambda a}{Z(1 - \lambda r^2)^2} + \frac{4M^2 a\Sigma_a}{Z(1 - \lambda r^2)^2(V - 2M)(r^2 + a^2)} \right]
\]

\[
+ \frac{2Ma}{Z(1 - \lambda r^2)(r^2 + a^2)} \sum_{i=1}^m \Phi_i ,
\]

\[
- \frac{Z}{V - 2M} \left[ \frac{b\Sigma_a \Sigma_b}{(1 - \lambda r^2)} + \frac{2M\lambda b}{Z(1 - \lambda r^2)^2} + \frac{4M^2 b\Sigma_b}{Z(1 - \lambda r^2)^2(V - 2M)(r^2 + b^2)} \right]
\]

\[
+ \frac{2Mb}{Z(1 - \lambda r^2)(r^2 + b^2)} \sum_{j=1}^p \Psi_j ,
\]

\[
\gamma = \left\{ -\sum_{i=1}^m \sum_{j=1}^m \left[ \lambda^2 a^2 \left( \frac{\Sigma_a \Sigma_b}{\lambda (1 - \lambda r^2)} + \frac{2M}{Z(1 - \lambda r^2)} \right) - \frac{4M^2 a^2 \Sigma_a^2}{Z(V - 2M)(r^2 + a^2)^2} - \frac{Q^{ij}}{Z} \right] \Phi_i \Phi_j 
\]

\[- \sum_{i=1}^m \sum_{j=1}^m \left[ \lambda^2 b^2 \left( \frac{\Sigma_a \Sigma_b}{\lambda (1 - \lambda r^2)} + \frac{2M}{Z(1 - \lambda r^2)} \right) - \frac{4M^2 b^2 \Sigma_a^2}{Z(V - 2M)(r^2 + b^2)^2} \right] \Phi_i \Phi_j 
\]

\[- \frac{Q^{(i+m)(j+m)}}{Z} \Psi_i \Psi_j - 2 \sum_{i=1}^m \sum_{j=1}^m \left[ \lambda^2 ab \left( \frac{\Sigma_a \Sigma_b}{\lambda (1 - \lambda r^2)} + \frac{2M}{Z(1 - \lambda r^2)} \right) \right] \Phi_i \Psi_j 
\]

\[- \frac{4M^2 ab \Sigma_a \Sigma_b}{Z(V - 2M)(r^2 + a^2)(r^2 + b^2)} - \frac{Q^{(i+m)}}{Z} \Phi_i \Psi_j 
\]

\[- \frac{\Sigma_a (r^2 + b^2) J_1}{r^2 + a^2} - \frac{\Sigma_b (r^2 + a^2) L_1^2 + K - m^2 r^2}{r^2 + a^2} \right\} \frac{Z}{V - 2M} .
\]

The turning points for trajectories in the radial motion (defined by the condition \( R = 0 \)) are given by \( E = V_\pm \) where

\[
V_\pm = \frac{\beta \pm \sqrt{\beta^2 - \alpha \gamma}}{\alpha} .
\]
These functions, called the effective potentials [20], determine allowed regions of motion. In this form, the radial equation is much more suitable for detailed numerical analysis for specific values of parameters.

### 5.3 Analysis of the Angular Equations

Another class of interesting motions possible describes motion at a constant value of $\alpha_i$ or $\beta_j$. These are analogous to the same class of motions analyzed in [23]. We briefly summarize them here. These motions are described by the simultaneous equations

$$A_i(\alpha_i = \alpha_{i0}) = \frac{dA_i}{d\alpha_i}(\alpha_i = \alpha_{i0}) = 0, \quad i = 1, \ldots, m - 1, \quad (5.10)$$

in the case of constant $\alpha_i$ motion, where $\alpha_{i0}$ is the constant value of $\alpha_i$ along this trajectory, or by the simultaneous equations

$$B_i(\beta_i = \beta_{i0}) = \frac{dB_i}{d\beta_i}(\beta_i = \beta_{i0}) = 0, \quad i = 1, \ldots, p - 1, \quad (5.11)$$

in the case of constant $\beta_i$ motion, where $\beta_{i0}$ is the constant value of $\beta_i$ along this trajectory.

These equations can be explicitly solved. In the case of constant $\alpha_i$ motion, we get the relations

$$\frac{J_{i+1}^2}{\sin^4 \alpha_i} = \frac{\Phi_{m-i-1}^2}{\cos^4 \alpha_i},$$

$$J_i^2 = \frac{J_{i+1}^2}{\sin^2 \alpha_i} + \frac{\Phi_{m-i+1}^2}{\cos^2 \alpha_i}, \quad i = 1, \ldots, m - 1. \quad (5.12)$$

Note that if $\alpha_{i0} = 0$, then $J_{i+1}^2 = 0$, and if $\alpha_{i0} = \pi/2$, then $\Phi_{m-i+1}^2 = 0$. Similarly, in the case of constant $\beta_i$ motion, we get the relations

$$\frac{L_{i+1}^2}{\sin^4 \beta_i} = \frac{\Psi_{p-i-1}^2}{\cos^4 \beta_i},$$

$$L_i^2 = \frac{L_{i+1}^2}{\sin^2 \beta_i} + \frac{\Psi_{p-i+1}^2}{\cos^2 \beta_i}, \quad i = 1, \ldots, p - 1. \quad (5.13)$$

Again if $\beta_{i0} = 0$, then $L_{i+1}^2 = 0$, and if $\beta_{i0} = \pi/2$, then $\Phi_{p-i+1}^2 = 0$.

Examining $A_k$ in the general case, $\alpha_k = 0$ can only be reached if $J_{k+1} = 0$, and $\alpha_k = \pi/2$ can be only be reached if $\Phi_{m-k+1} = 0$. The orbit will completely be in the subspace $\alpha_k = 0$ only if $J_k^2 = \Phi_{m-k+1}^2$ and will completely be in the subspace $\alpha_k = \pi/2$ only if $J_k^2 = J_{k+1}^2$. Analogous results hold for constant $\beta_i$ motion.

Again these equations are in a form suitable for numerical analysis for specific values of the black hole and particle parameters.
6 Dynamical Symmetry

The spacetimes discussed here are stationary and “axisymmetric”; i.e., $\partial/\partial\tau$ and $\partial/\partial\varphi_i$ are Killing vectors and have associated conserved quantities, $-E$, $\Phi_i$, and $\Psi_i$. In general if $\eta$ is a Killing vector, then $\eta^\mu p_\mu$ is a conserved quantity, where $p$ is the momentum. Note that this quantity is first order in the momenta.

In the case of only two sets of possibly unequal rotation parameters, more Killing vectors exist since the spacetime acquires additional dynamical symmetry. We have complete symmetry between the various planes of rotation characterized by the same value of rotation parameter $a_i = a$, and we can “rotate” one into another. Similarly, we have symmetry between the planes of rotation characterized by the same value of the rotation parameter $a_i = b$, and we can “rotate” these into one another as well. The vectors that generate these transformations are the required Killing vectors. The explicit construction of such Killing vectors is done in [22]. In this case, we get two independent sets of such Killing vectors, associated with the constant $a$ and $b$ value rotations.

In an odd number of spacetime dimensions, if $a \neq b$ and both are nonvanishing, then the rotational symmetry group is $U(m) \times U(p)$. If one of them is zero, but the other is nonzero (we take the nonzero one to be $a$), then the rotational symmetry group is $U(m) \times O(2p)$.

In the case when $a = b \neq 0$, the rotational symmetry group is $U(m + p)$. In the case when $a = b = 0$, i.e. in the Schwarzschild metric, the rotational symmetry group is $O(2m + 2p)$.

In an even number of spacetime dimensions, $b = 0$ in the cases we have analyzed. If $a \neq 0$, then the rotational symmetry group is $U(m) \times O(2p - 1)$, and in the case when $a = b = 0$, i.e. in the Schwarzschild metric, the rotational symmetry group is $O(2m + 2p - 1)$. Note that since these metrics are stationary, the full dynamical symmetry group is the direct product of $\mathbb{R}$ and the rotational symmetry group, where $\mathbb{R}$ is the additive group of real numbers parameterizing $\tau$.

We also obtain a non-trivial irreducible second-order Killing tensor, whose existence is the principal reason that permits the separation of the $r - \theta$ equations. This Killing tensor is a generalization of the result obtained in the five dimensional case in [20]. This is obtained from the separation constant $K$ in (4.5) and (4.6). We choose to analyze the latter.

\[
K = -m^2 a^2 \cos^2 \theta - m^2 b^2 \sin^2 \theta - \Sigma_a \Sigma_b \frac{E^2}{\lambda \Delta_\theta} - \Sigma_a \cot^2 \theta J_1^2 - \Sigma_b \tan^2 \theta L_1^2 - \Delta_\theta \left( \frac{\partial S}{\partial \theta} \right)^2 + 2 \sum_{i=1}^m \frac{a \Sigma_a \Sigma_b}{\Delta_\theta} E \Phi_i + 2 \sum_{j=1}^p \frac{b \Sigma_a \Sigma_b}{\Delta_\theta} E \Psi_i - \sum_{i=1}^m \sum_{j=1}^p \frac{\lambda^2 a^2 \Sigma_a \Sigma_b}{\Delta_\theta} \Phi_i \Phi_j
\]
The Killing tensor $K^{\mu\nu}$ is obtained from this separation constant (which is quadratic in the canonical momenta) using the relation $K = K^{\mu\nu}p_\mu p_\nu$. It is then easy to see that

$$K^{\mu\nu} = -g^{\mu\nu}(a^2 \cos^2 \theta + b^2 \sin^2 \theta) - \sum a^2 \frac{\delta^{\mu}_{\varphi_i} \delta^{\nu}_{\varphi_j}}{\Delta_\theta} - \sum b^2 \frac{\delta^{\mu}_{\phi_{i+1}} \delta^{\nu}_{\phi_{j+1}}}{\Delta_\theta}$$

$$- \sum \lambda^2 a^2 \frac{\delta^{\mu}_{\phi_{i+1}} \delta^{\nu}_{\phi_{j+1}}}{\Delta_\theta} + \sum \lambda^2 b \frac{\delta^{\mu}_{\phi_{i+1}} \delta^{\nu}_{\phi_{j+1}}}{\Delta_\theta} + \sum \lambda^2 \frac{\delta^{\mu}_{\phi_{i+1}} \delta^{\nu}_{\phi_{j+1}}}{\Delta_\theta}$$

where $J^{\mu\nu}_1$ and $L^{\mu\nu}_1$ are the reducible Killing tensors associated with the $\alpha$ and $\beta$ separation.

The existence of these additional Killing vectors and the nontrivial irreducible Killing tensor, is the principal reason behind the complete separation of the Hamilton-Jacobi equation. The nontrivial Killing tensor, in particular, exists due to the detailed structure of the metrics under consideration and is a surprising result.

7 The Scalar Field Equation

Consider a scalar field $\Psi$ in a gravitational background with the action

$$S[\Psi] = -\frac{1}{2} \int d^Dx \sqrt{-g}((\nabla \Psi)^2 + \alpha R \Psi^2 + m^2 \Psi^2),$$

where we have included a curvature dependent coupling. However, in the Kerr-(Anti) de Sitter background, $R = \lambda$ is constant. As a result we can trade off the curvature coupling for a different mass term. So it is sufficient to study the massive Klein-Gordon equation in this background. We will simply set $\alpha = 0$ in the following. Variation of the action leads to the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Psi) = m^2 \Psi.$$ 

As discussed by Carter [26], the assumption of separability of the Klein-Gordon equation usually implies separability of the Hamilton-Jacobi equation. Conversely, if the Hamilton-Jacobi equation does not separate, the Klein-Gordon equation seems unlikely to separate. We can also see this explicitly (as in the case of the Hamilton-Jacobi equation), since the
the (r, τ, φ_i) sector has coefficients in the equations that explicitly depend on the μ_i except when of all a_i = a, in which case separation seems likely. We will again consider the much more general case of two sets of possibly unequal sets of rotation parameters a and b. We continue using the same numbering conventions for the variables.

Once again, we impose the constraint (2.2) and decompose the μ_i in two sets of spherical polar coordinates as in (3.6) and (3.8). We calculate the determinant of the metric to be

\[ g = \frac{-r^2 \rho^4 (r^2 + a^2)^2 (r^2 + b^2)^2 p^{-2}}{\Sigma_a \Sigma_b \Sigma_\rho} \sin^{4m-2} \theta \cos^{4p-2 \epsilon} \theta \]

\[ \times \left[ \prod_{i=1}^{m-1} \sin^{4m-4j-2} \alpha_j \cos^2 \alpha_j \right] \left[ \prod_{k=1}^{p-1} \sin^{4p-4k-2} \beta_k \cos^2 \beta_k \right] \cos^{-2 \epsilon} \beta_1. \] (7.3)

For convenience we write \( g = -\frac{RTAB \rho^4}{\Sigma_a \Sigma_b \Sigma_\rho} \), where

\[ R = \frac{r^2 (r^2 + a^2)^2 (r^2 + b^2)^2 p^{-2}}{\Sigma_a \Sigma_b \Sigma_\rho}, \]

\[ T = \sin^{4m-2} \theta \cos^{4p-2 \epsilon} \theta, \]

\[ A = \prod_{j=1}^{m-1} \sin^{4m-4j-2} \alpha_j \cos^2 \alpha_j, \]

\[ B = \prod_{k=1}^{p-1} \sin^{4p-4k-2} \beta_k \cos^2 \beta_k \cos^{-2 \epsilon} \beta_1. \] (7.4)

Note that R and T are functions of r and θ only, and A and B only depend on the set of variables α_i and β_j respectively. Then the Klein-Gordon equation in this background (2.2) becomes

\[ m^2 \Psi = \frac{1}{\rho^2 \sqrt{R}} \left( \sqrt{R} \frac{V - 2M}{Z} \partial_r \Psi \right) + \frac{\Sigma_a}{(r^2 + a^2) \sin^2 \theta} \sum_{i=1}^{m-1} \frac{1}{\lambda_i} \partial^2_{r \phi_i} \Psi \]

\[ + \frac{\Sigma_b}{(r^2 + b^2) \cos^2 \theta} \sum_{i=1}^{p} \frac{1}{\nu_i} \partial^2_{r \theta_i} \Psi + \frac{1}{\rho^2 \sqrt{T}} \partial_\theta \left( \sqrt{T} \Delta_\theta \partial_\theta \Psi \right) \]

\[ + \frac{\Sigma_a \Sigma_b}{\lambda \rho^2 \Delta_\theta} \left[ \frac{2M}{\rho^2 (1 - \lambda^2)} - \frac{4M^2}{\rho^2 Z (1 - \lambda^2)^2 (V - 2M)(r^2 + a^2)} \right] \partial^2_{r \phi_i} \Psi \]

\[ + \frac{2}{\rho^2 Z (1 - \lambda^2)^2 (V - 2M) (r^2 + a^2)} \sum_{i=1}^{m-1} \partial^2_{\phi_i} \Psi \]
We attempt the usual multiplicative separation for $\Psi$ in the following form:

$$
\Psi = \Phi_r(r)\Phi_\theta(\theta)e^{-iE_r r}e^{i\sum_i \Psi_i r + \sum_{i<j} \Psi_{i,j} r + \sum_{i<j} \Phi_{i,j}(\phi_{i,j})}.
$$

where we again adopt the convention that $\Psi_p = 0$ in the case of even dimensional spacetimes.

The Klein-Gordon equation then completely separates. The $r$ and $\theta$ equations are given as

$$
K = \frac{d}{dr} \left( \sqrt{r} \frac{V - 2M}{Z} \frac{d\Phi_r}{dr} \right) + \left[ \frac{\Sigma_a \Sigma_b}{\rho^2(1 - \lambda r^2)^2} - \frac{2M \lambda b}{\rho^2 Z(1 - \lambda r^2)^2} + \frac{2M \lambda b}{\rho^2 Z(1 - \lambda r^2)^2} \right] E^2
$$

We attempt the usual multiplicative separation for $\Psi$ in the following form:

$$
\Psi = \Phi_r(r)\Phi_\theta(\theta)e^{-iE_r r}e^{i\sum_i \Psi_i r + \sum_{i<j} \Psi_{i,j} r + \sum_{i<j} \Phi_{i,j}(\phi_{i,j})}.
$$

where we again adopt the convention that $\Psi_p = 0$ in the case of even dimensional spacetimes.
\[
\begin{align*}
&+ \frac{2Ma}{Z(1 - \lambda r^2)(r^2 + a^2)} \sum_{i=1}^{m} E\Phi_i \\
&- 2 \left[ \frac{b\Sigma_b b}{(1 - \lambda r^2)} + \frac{2M\lambda b}{Z(1 - \lambda r^2)^2} + \frac{4M^2 b\Sigma_b}{Z(1 - \lambda r^2)(r^2 + b^2)} \right] \\
&+ \frac{2Mb}{Z(1 - \lambda r^2)(r^2 + b^2)} \sum_{j=1}^{p} E\Psi_j - \Sigma_i \frac{r^2 + b^2}{r^2 + a^2} \sum_{i=1}^{m} K_1 \Phi_i^2 \\
&- \Sigma_i \frac{r^2 + a^2}{r^2 + b^2} \sum_{j=1}^{p} M_1 \Psi_j^2 - m^2 r^2, \quad \text{(7.7)}
\end{align*}
\]

and

\[
\begin{align*}
-K &= \frac{1}{\Phi_\theta \sqrt{T}} \frac{d}{d\theta} \left( \sqrt{T} \Delta_\theta d\Phi_\theta \right) - \frac{\Sigma_a \Sigma_b}{\lambda \Delta_\theta} E^2 - m^2 (a^2 \cos^2 \theta + b^2 \sin^2 \theta) \\
&+ K_1 \cot^2 \theta + M_1 \tan^2 \theta - 2\lambda a^2 \frac{\Sigma_a \Sigma_b}{\Delta_\theta} \sum_{i=1}^{m} \Phi_i \Phi_j - 2\lambda b^2 \frac{\Sigma_a \Sigma_b}{\Delta_\theta} \sum_{i=1}^{m} \mathrm{E} \Phi_i + 2 \frac{b\Sigma_a \Sigma_b}{\Delta_\theta} \sum_{j=1}^{p} E\Phi_j, \quad \text{(7.8)}
\end{align*}
\]

where \( K, K_1 \) and \( M_1 \) are separation constants. \( K_1 \) and \( M_1 \) encode all the \( \alpha \) and \( \beta \) dependence respectively and are defined explicitly as follows:

\[
K_1 = \sum_{i=1}^{k-1} A_i + \frac{K_k}{\prod_{i=1}^{k-1} \sin^2 \alpha_j}, \quad k = 1, \ldots, m - 1, \quad \text{(7.9)}
\]

where

\[
A_i = \frac{1}{\Phi_{\alpha_i} \cos \alpha_i \sin^{2m-2i-1} \alpha_i \prod_{k=1}^{i-1} \sin^2 \alpha_k} \frac{d}{d\alpha_i} \left( \cos \alpha_i \sin^{2m-2i-1} \alpha_i \frac{d\Phi_{\alpha_i}}{d\alpha_i} \right) \\
- \frac{\Phi_{m-i+1}^2}{\cos^2 \alpha_i \prod_{j=1}^{m-1} \sin^2 \alpha_j}, \quad \text{(7.10)}
\]

and

\[
M_1 = \sum_{i=1}^{k-1} B_i + \frac{M_k}{\prod_{i=1}^{k-1} \sin^2 \beta_j}, \quad k = 1, \ldots, p - 1, \quad \text{(7.11)}
\]

and where

\[
B_i = \frac{1}{\Psi_{\beta_i} \cos \beta_i \sin^{2p-2i-1} \beta_i \prod_{k=1}^{i-1} \sin^2 \beta_k} \frac{d}{d\beta_i} \left( \cos \beta_i \sin^{2p-2i-1} \beta_i \frac{d\Phi_{\beta_i}}{d\beta_i} \right) \\
- \frac{\Psi_{p-i+1}^2}{\cos^2 \beta_i \prod_{j=1}^{p-1} \sin^2 \beta_j}, \quad \text{(7.12)}
\]

Then we inductively have the complete separation of the \( \alpha_i \) dependence as

\[
K_k = \frac{K_{k+1}}{\sin^2 \alpha_k} - \frac{\Phi_{\alpha_k}^2}{\cos^2 \alpha_k} + \frac{1}{\Phi_{\alpha_k} \cos \alpha_k \sin^{2m-2k-1} \alpha_k} \frac{d}{d\alpha_k} \left( \cos \alpha_k \sin \alpha_k \frac{d\Phi_{\alpha_k}}{d\alpha_k} \right), \quad \text{(7.13)}
\]
where $k = 1, \ldots, m - 1$, and we use the convention $K_m = -\Phi_1^2$. Similarly, the complete separation of the $\beta_i$ dependence is given inductively by

$$M_k = \frac{M_{k+1}}{\sin^2 \beta_k} - \frac{\Psi_{p-k+1}^2}{\cos^2 \beta_k} + \frac{1}{\Phi_{\beta_k} \cos \beta_k \sin^{2p-2k-1} \beta_k} \frac{d}{d\beta_k} \left( \cos \beta_k \sin \beta_k \frac{d\Phi_{\beta_k}}{d\beta_k} \right), \quad (7.14)$$

where $k = 1, \ldots, p - 1$, and we use the convention $M_p = -\Psi_1^2$. These results agree with the previously known analysis in five dimensions [21].

At this point we have complete separation of the Klein-Gordon equation in the Kerr-(Anti) de Sitter black hole background in all dimensions with two sets of possibly unequal rotation parameters in the form given by (7.6) with the individual separation functions given by the ordinary differential equations above. Note that the separation of the Klein-Gordon equation in this geometry is again due to the existence of the non-trivial Killing tensor.

**Conclusions**

We studied the integrability properties of the Hamilton-Jacobi and the massive Klein-Gordon equations in the Kerr-(Anti) de Sitter black hole backgrounds in all dimensions. Complete separation of both equations in Boyer-Lindquist coordinates is possible for the case of two possibly unequal sets of rotation parameters. We discuss the Killing vectors and reducible Killing tensors that exist in the spacetime and also construct the nontrivial irreducible Killing tensor which explicitly permits complete separation. Thus we demonstrate the separability of the Hamilton-Jacobi and the Klein-Gordon equations as a direct consequence of the enhancement of symmetry. We also derive first-order equations of motion for classical particles in these backgrounds, and analyze the properties of some special trajectories.

Further work in this direction could include the study of higher-spin field equations in these backgrounds, which is of great interest, particularly in the context of string theory. Explicit numerical study of the equations of motion for specific values of the black hole parameters could lead to interesting results. The first order equations of motion presented here can also readily be used in the detailed study of black hole singularity structure in an AdS background geodesic probes and the AdS/CFT correspondence.

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