THE IVP FOR A CERTAIN DISPERSION GENERALIZED
ZK EQUATION IN BI-PERIODIC SPACES

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Abstract. We establish well-posedness conclusions for the Cauchy problem
associated to the dispersion generalized Zakharov-Kutnetsov equation in bi-
periodic Sobolev spaces $H^s(T^2)$, $s > (\frac{3}{2} - \frac{1}{1 + \alpha} + \beta)$.

1. Introduction

In the present paper, we deal with the well-posedness of the initial value problem
(IVP):

$$\begin{cases}
\partial_t u - \partial_x \left( D_x^{1+\alpha} \pm D_y^{1+\beta} \right) u + uu_x = 0 & (x, y) \in T^2, \ t \in \mathbb{R}, \\
u (0) = \phi & \phi \in H^s(T^2),
\end{cases}$$

(1.1)

with $\alpha = 1, 2$ or $3$ and $0 < \beta \leq 1$, where $T^2 = \mathbb{R}^2/(2\pi \mathbb{Z})^2$ and $D_y^\beta = (-\partial_y^2)^{\beta}$ is a homogeneous fractional derivative in the variable $y$, and it is defined via the
Fourier transform by $(\widehat{D_y^\beta f})(m, n) := |n|^\beta \hat{f}(m, n)$, analogously is defined $D_x^\alpha$ for the variable $x$. When, $\alpha = \beta = 1$ and the $+$ sign, this equation corresponds to the well-known Zakharov-Kuznetsov (ZK) equation, that is a natural bi-dimenmsional extension of the well-known Korteweg de Vries (KdV) equation. The ZK equation describes the propagation of nonlinear ion-acoustic waves in magnetized plasma
and it's well-posedness has been extensively studied in \([24, 25, 10, 2, 5]\). Recently, models that generalize the ZK equation have emerged, see \([22]\). An important case in the study of this type of equations is the well-posedness in the periodic Sobolev space. In this regard, Linares et al. in \([23]\) obtained well-posedness for the ZK equation in the periodic Sobolev spaces $H^s(T^2)$, for $s > \frac{3}{2}$. Improved by Schippa in \([29]\) to $s > \frac{3}{4}$ via short-time bilinear Strichartz estimates adapting the bilinear arguments in the periodic Sobolev space $H^s(T^2)$. Recently, Kinosita and Schippa proved local well-posedness in $H^s(T^2)$ for $s > 1$ in \([20]\), the ingredient to improve on previous results is a nonlinear Loomis-Whitney-type inequality.

For $\alpha = 0$, $\beta = 1$ and the $+$ sign, the equation (1.1) coincides with the
Benjamin-Ono-Zakharov-Kuznetsov (BOZK) equation that is a model for thin
nano-conductors on a dielectric substrate in \([22]\). For result concerning well-posedness
(see \([8, 9, 6]\)). When $\alpha = 1, 3, 5 \cdots$ the equation (1.1) in one dimension is,

$$\partial_t u - \partial_x^\alpha u + uu_x = 0,$$

(1.2)

that is known as the KdV equation with higher dispersion(see \([11]\) and the references contained therein). Results regarding local well-posedness for (1.2) in $H^s(\mathbb{T})$,

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\[ s \geq -\frac{1}{2} \] were obtained in [11], using bilinear estimates in Bourgain spaces. Unlike our work, \( s \) does not depend on the order of the dispersion term. Another similar equation studied in \( \mathbb{R}^2 \) is the two-dimensional dispersive generalized g-BOZK equation (see [28]), in this case the general dispersion is in \( x \) with \( 0 \leq \alpha \leq 1 \) and \( \beta = 1 \), where local well-posedness was established in \( H^s(\mathbb{R}^2) \) for \( s > \frac{2}{1+\alpha} - \frac{3}{4} \).

We observe that the IVP (1.1) satisfies at least following conserved quantities:

\[ M(u) = \int_{\mathbb{T}^2} u^2 dx dy \quad (1.3) \]

and

\[ E(u) = \frac{1}{2} \int_{\mathbb{T}^2} \left( D_x^{1+\alpha} u \right)^2 \pm \left( D_y^{1+\beta} u \right)^2 - \frac{u^3}{3} \right) dx dy, \quad (1.4) \]

so that (1.3) and (1.4) are useful to extend the local solution to global one, to attain the global well-posedness in anisotropic Sobolev spaces \( H^{\frac{4}{1+\alpha} - \frac{3}{4}}(\mathbb{R}^2) \). Unfortunately, we are not dealing with the well-posedness in this spaces here. Our goal in this work is to improve the local well-posedness in periodic Sobolev spaces \( H^s(\mathbb{T}^2) \), \( s > 2 \) for (1.1), which is obtained from a parabolic regularization argument, following Iorio’s ideas in [16] Chapter 6. One way to improve this result is to use again energy estimates in smooth solutions, what leads us to obtain control of the norms \( \| \nabla u \|_{L_x^1 L_y^\infty} \). If the Sobolev embedding is used, we achieve this estimate, but we cannot improve regularity. Therefore, we use the short-time Strichartz linear approach introduced by Koch and Tzvetkov in [21], to get local well-posedness of Benjamin-Ono (BO) equation in \( \mathbb{R} \), which has been proved to be useful for these types of equations (see [14, 26, 17, 18]). But to perform this task in two dimensions with periodic context, we adapt the method used to prove local well-posedness for the Cauchy problem associated to the third-order KP-I and fifth-order KP-I equations on \( \mathbb{R} \times \mathbb{T} \) and \( \mathbb{T}^2 \) proposed by Ionescu and Kenig in [15] (for other applications, see [25][4] and the references therein). First, a localized Strichartz-type estimate for the linear part of the equation is obtained, where the main difficulty lies in obtaining bounds for exponential sums in the periodic case (see [13], such sums have been treated in different contexts in number theory. Then fixing \( \alpha \), we can apply a lemma due to H. Weyl (Lemma 2.3 below) that combined with some Strichartz estimates give a control to \( \| u \|_{L_x^1 L_y^\infty} \), \( \| u_x \|_{L_x^1 L_y^\infty} \) and \( \| u_y \|_{L_x^1 L_y^\infty} \). Finally by standard compactness methods, we get the result. Although the method used by Kinoshita and Shippa [20] produces better results for the ZK equation in the bi-periodic setting, it is not clear how to extended this technique to equations involving fractional operators in both spatial variables \( x \) and \( y \).

We will now give the precise statement of our result in spaces of lower regularity,

**Theorem 1.1.** Let \( \alpha = 1, 2 \) or 3, \( 0 < \beta \leq 1 \), \( \phi \in H^s(\mathbb{T}^2) \) and \( s > (\frac{3}{2} - \frac{1}{2+\beta})(\frac{3}{2} - \frac{1}{\alpha}) \) such that \( \int \phi(x,y) dx = 0 \) a.e. \( y \in \mathbb{T} \), there exist \( T = T(||\phi||_{H^s}) \) and a unique solution of IVP (1.1), such that \( u \in C \left([0,T];H^s(\mathbb{T}^2)\right) \) and \( u, \partial_x u, \partial_y u \in L_x^1 L_y^{\infty} \). Moreover, the map data-solution \( \phi \in H^s(\mathbb{T}^2) \mapsto u \in C \left([0,T];H^s(\mathbb{T}^2)\right) \) is continuous.

Where it is observed that when the dispersion in the \( x \) variable increases the regularity required increases slightly.
The paper is organized as follows. In the following sections we prove Localized Strichartz Estimate. Section 3, deals with Preliminary and Key estimates. Lastly, the main result is proved. Before setting our results, some notation is necessary:

**Notation.**
- $a \lesssim b$ (resp. $a \gtrsim b$) means that there exists a positive constant $c$, such that, $a \leq cb$ (resp. $a \geq cb$).
- $a \sim b$, when $a \lesssim b$ and $a \gtrsim b$.
- $C_p(X)$ for the function space $C_p$ class in $X$.

$$
\|u\|_{L^p_tX} := \left( \int_{\mathbb{R}} \|u(t)\|_X^p \, dt \right)^{\frac{1}{p}}
$$

$$
\|u\|_{L^{\infty}_tX} := \text{ess sup}_{t \in \mathbb{R}} \|u(t)\|_X,
$$

where $X$ is Banach space, $u : \mathbb{R} \to X$ is a measurable function and $p \in [1, \infty]$.

$$
\|u\|_{L^p_tX} := \|\chi_I(|t|)u\|_{L^p_tX},
$$

where $I \subseteq \mathbb{R}$ is interval, $\chi_I$ is the characteristic function of $I$ and $u : I \to X$ is a measurable function. In particular, for $T > 0$

$$
\|u\|_{L^p_{[-T,T]}X} := \|u\|_{L^p_{[-T,T]}X}.
$$

$$
\hat{f}(m,n) = C \int_{\mathbb{T}^2} f(x,y) e^{-ixm} e^{-iny} \, dx \, dy,
$$

where $f \in L^1(\mathbb{T}^2)$, $(m,n) \in \mathbb{Z}^2$ and $C$ as a universal constant. It is the Fourier transform of $f$.

- $\mathcal{S} = C^\infty(\mathbb{T}^2)$.
- $\mathcal{S}'$ is the biperiodic distribution space.
- $\hat{\mathcal{S}}(m,n) = \langle f; e^{-i(mx+ny)} \rangle$ is the Fourier transform of $f \in \mathcal{S}'$, where $\epsilon_{mn}(x,y) = e^{i(mx + ny)}$, $(m,n) \in \mathbb{Z}^2$, $(x,y) \in \mathbb{T}^2$, $C$ as a universal constant and $\langle ; \rangle$ is the duality bracket of $\mathcal{S}'$.

$$
H^s(\mathbb{T}^2) = \{ f \in \mathcal{S}' : \sum_{(m,n) \in \mathbb{Z}^2} (1 + m^2 + n^2)^s |\hat{f}(m,n)|^2 < \infty \},
$$

where $H^s(\mathbb{T}^2)$ denote the standard Sobolev spaces in $L^2(\mathbb{T}^2)$ for $s \in \mathbb{R}$, and

$$
H^\infty(\mathbb{T}^2) = \bigcap_{s \geq 0} H^s(\mathbb{T}^2).
$$

$$
\|f\|_{H^s} \sim \|J^s f\|_{L^2(\mathbb{T}^2)} + \|J^s f\|_{L^2(\mathbb{T}^2)},
$$

For integers $k = 0, 1, \cdots$ we define the operators $Q^k_x$ and $Q^k_y$ on $H^\infty(\mathbb{T}^2)$ by

\[ Q^k_x g(m,n) = \chi_{[0,1)}(|m|) \hat{g}(m,n) \]
\[ Q^k_y g(m,n) = \chi_{[2^k-1,2^k]}(|n|) \hat{g}(m,n) \quad \text{si } k \geq 1 \] \hfill (1.5)

and

\[ Q^k_x g(m,n) = \chi_{[0,1)}(|n|) \hat{g}(m,n) \]
\[ Q^k_y g(m,n) = \chi_{[2^k-1,2^k]}(|m|) \hat{g}(m,n) \quad \text{si } k \geq 1, \] \hfill (1.6)
Lemma 2.3. If

\[
\|g\|_{L^2(T^2)}^2 \sim \sum_{k,j \geq 0} \|Q_x^k Q_y^j g\|_{L^2(T^2)}^2
\]

(1.7)

and

\[
\|J_x^k g\|_{L^2(T^2)}^2 + \|J_y^k g\|_{L^2(T^2)}^2 \sim \sum_{k \geq 0, j \geq 1} (2^j)^2 \|Q_x^k Q_y^j g\|_{L^2(T^2)}^2
\]

(1.8)

\[
+ \sum_{k \geq 1, j \geq 0} (2^k)^2 \|Q_x^k Q_y^j g\|_{L^2(T^2)}^2
\]

\[
+ \sum_{k,j \geq 0} \|Q_x^k Q_y^j g\|_{L^2(T^2)}^2
\]

2. Localized Strichartz Estimate

In this section, we prove Strichartz estimate localized in frequency and time. First we recall the following lemmas.

Lemma 2.1 (Poisson Summation Formula). Let \( f, \hat{f} \) are in \( L^1(\mathbb{R}^n) \) and satisfy the condition

\[
|f(x)| + |\hat{f}(x)| \leq C (1 + |x|)^{-n-\delta}
\]

for some constant \( C, \delta > 0 \). Then \( f \) and \( \hat{f} \) are continuous an for all \( x \in \mathbb{R}^n \) we have

\[
\sum_{m \in \mathbb{Z}^n} \hat{f}(2\pi m) = \sum_{k \in \mathbb{Z}^n} f(k).
\]

Proof. See [12] Theorem 3.1.17.

Lemma 2.2 (Van der Corput). Let \( p \geq 2, I = [a,b] \varphi \in C^p(I) \) be a real value function such that \( |\varphi^{(p)}(x)| \geq \lambda > 0, \psi \in L^\infty(I) \) and \( \psi' \in L^1(I) \). Then,

\[
\left| \int_I e^{i\varphi(x)} \psi(x) \, dx \right| \leq C_p \lambda^\frac{1}{p} (\|\psi\|_{L^\infty} + \|\psi'\|_{L^1})
\]

Proof. See [30] Chapter 8.

Lemma 2.3. If \( h(x) = \omega_0 x^d + ... + \omega_1 x + \omega_0 \) is a polynomial with real coefficients and \( |\omega_d - \frac{a}{q}| \leq \frac{1}{q} \) for some \( a \in \mathbb{Z} \) and \( q \in \mathbb{Z}^+ \) with \( (a,q) = 1 \) (and and b are relatively prime) then for any \( \delta > 0 \),

\[
\left| \sum_{m=1}^{N} e^{2\pi i h(m)} \right| \leq C_{h,d} N^{1+\delta} \left[ q^{-1} + N^{-1} + qN^{-d} \right]^{\frac{1}{m+1}}
\]

where the constant \( C_{h,d} \) only depends on \( \delta \) and \( d \).

Proof. See [27] Theorem 4.3.

Lemma 2.4. For any integer \( \Lambda \geq 1 \) and any \( r \in \mathbb{R} \), there are integers \( q \in \{1,2,...,\Lambda\} \) and \( a \in \mathbb{Z} \), \( (a,q) = 1 \), such that

\[
|r - \frac{a}{q}| \leq \frac{1}{\Lambda q}
\]
Theorem 2.5. Let $l$ for any measurable functions $\phi$ and $g$. By duality, for any $\psi \in L^2 \cap \mathcal{D}'$, we get,

$$W_0^\alpha (t) \phi = \sum_{m \in \mathbb{Z}^n} \sum_{n \in \mathbb{Z}} \hat{\phi} (m) e^{i \alpha (m \cdot x + m (|x|^{\alpha} + |x|^\beta)) t} + \sum_{n \in \mathbb{Z}} \hat{\phi} (0, n) e^{i n y}$$

where $m = (m, n), x = (x, y)$ and $m \cdot x = mx + ny$.

Proof. Let $\psi_1 : \mathbb{R} \to [0, 1]$, denote a smooth even function supported in $\{ r : |r| \leq \frac{1}{4} \}$ and $\psi_1 \equiv 1$ in $\{ r : |r| \leq \frac{1}{2} \}$. Let $a (m) = (Q^k \psi_1) \hat{(m)}, \psi_1 \left( \frac{m}{2^j} \right) = 1$ in $[-2^{k-j}, -2^{k-j+1}] \times [-2^{k-j}, -2^{k-j+1}]$ and $\supp Q^k \psi_1 \subset \supp \psi_1, j, k \geq 1$. Then,

$$W_0^\alpha (t) Q^k \psi_1 = \sum_{(m, n) \in \mathbb{Z}^n \times \mathbb{Z}} \frac{a (m)}{n} \psi_1 (\frac{m}{2^j}) \psi_1 (\frac{n}{2^k}) e^{i (m \cdot x + m (|x|^{\alpha} + |x|^\beta)) t}$$

It suffice to prove that,

$$\left\| \chi_{[0, 2^{-(k+j)}]} (t) \sum_{(m, n) \in \mathbb{Z}^n \times \mathbb{Z}} a (m) \psi_1 (\frac{m}{2^j}) \psi_1 (\frac{n}{2^k}) e^{i (m \cdot x + m (|x|^{\alpha} + |x|^\beta)) t} \right\|_{L^2} \leq C e^{2 \left( \frac{| \alpha |}{\alpha + \epsilon} t \right) j + \left( \frac{2}{\epsilon} + \epsilon \right) k} \left\| a \right\|_{L^2}.$$
with \( t \neq t' \) and \( t, t' \in [\frac{-2}{k+j}, \frac{2}{k+j}] \). Thus, it is enough to prove that,

\[
\left| \sum_{m \in \mathbb{Z}^+} \sum_{n \in \mathbb{Z}} \psi^2_1 \left( \frac{m}{2} \right) e^{i\left(mx+tm\lfloor m \right]} \cdot \psi^2_1 \left( \frac{n}{2 \nu} \right) e^{i\left[ny+tm\lfloor n \right]} \right| \lesssim 2 \nu \left( -\frac{1}{2 \nu^2} - 2 \right) j + \left( -\frac{1}{2 \nu^2} + 2 \right),
\]

(2.1)

for any \( x, y \in [0, 2 \pi) \) and \( \nu \in [\frac{1}{2}, \frac{1}{2}] \). The cases \( k, j = 0 \) is immediate, if \( j, k > 0 \) using the Poisson summation formula (Lemma 2.1) in the suma in \( n \), we get

\[
\sum_{m \in \mathbb{Z}^+} \psi^2_1 \left( \frac{m}{2} \right) e^{i\left(mx+tm\lfloor m \right]} \left( \sum_{n \in \mathbb{Z}} \psi^2_1 \left( \frac{n}{2 \nu} \right) e^{i\left[ny+tm\lfloor n \right]} d\eta \right).
\]

We will use integration by parts to solve the integral term in the above expression. We define \( A := \text{supp} \psi_1 \left( \frac{m}{2} \right) = \{ \eta : 2^{k-1} \leq |\eta| \leq 2^{k+1} \} \),

\[
\int_{A} \psi^2_1 \left( \frac{n}{2 \nu} \right) e^{i\left((y-2 \pi \nu \eta \pm tm|\eta|^{1+\alpha}) \right] d\eta
\]

\[
= \int_{A} \psi^2_1 \left( \frac{n}{2 \nu} \right) e^{i\left((y-2 \pi \nu \eta \pm (1 + \beta) \text{sgn}(\eta) |\eta|^\beta mt \right] d\eta
\]

\[
= \frac{-1}{i} \int_{A} \frac{2 \cdot 2^{-j} \psi_1 \psi'_1}{\left[ (y-2 \pi \nu \pm (1 + \beta) \text{sgn}(\eta) |\eta|^\beta mt \right] ^2 \left[ (y-2 \pi \nu \pm (1 + \beta) \text{sgn}(\eta) |\eta|^\beta mt \right] ^2 \left[ e^{i\left((y-2 \pi \nu \eta \pm tm|\eta|^{1+\beta}) \right] d\eta
\]

For the second term of the previous integral, we have that, \( (1 + \beta) |\eta|^{\beta-1} mt \leq 4(2 \cdot 2^k)^{\beta-1} 2^{j-2(k+j)} \lesssim 2^{(\beta-2)k} \), \( (1 + \beta) |\eta|^\beta mt \leq 2 \cdot 2^k \nu \lesssim 2 \) and \( y \in [0, 2 \pi) \), \( \nu > 0 \), then \( |(y-2 \pi \nu \pm (1 + \beta) \text{sgn}(\eta) |\eta|^\beta mt \right| ^2 \sim \nu^2 \right. \). We get,

\[
\left| \int_{A} \frac{\psi_1^2((1 + \beta) |\eta|^{\beta-1} mt)}{(y-2 \pi \nu \pm (1 + \beta) \text{sgn}(\eta) |\eta|^\beta mt \right] ^2 e^{i\left((y-2 \pi \nu \eta \pm tm|\eta|^{1+\beta}) \right] d\eta}
\]

On the other hand, for the first term of the right-hand side of (2.1), we again use integration by parts,

\[
\int_{A} \frac{2 \cdot 2^{-j} \psi_1 \psi'_1}{\left[ (y-2 \pi \nu \pm (1 + \beta) \text{sgn}(\eta) |\eta|^\beta mt \right] ^2 \left[ e^{i\left((y-2 \pi \nu \eta \pm tm|\eta|^{1+\beta}) \right] d\eta
\]

\[
= \frac{1}{i} \int_{A} \frac{2 \cdot 2^{-j} \psi_1 \psi'_1}{\left[ (y-2 \pi \nu \pm (1 + \beta) \text{sgn}(\eta) |\eta|^\beta mt \right] ^2 \left[ e^{i\left((y-2 \pi \nu \eta \pm tm|\eta|^{1+\beta}) \right] d\eta
\]

\[
= \frac{-1}{i} \int_{A} \frac{2 \cdot 2^{-2j} (\psi_1)^2 + 2 \cdot 2^{-2j} \psi_1 \psi'_1}{\left[ (y-2 \pi \nu \pm (1 + \beta) \text{sgn}(\eta) |\eta|^\beta mt \right] ^2 \left[ e^{i\left((y-2 \pi \nu \eta \pm tm|\eta|^{1+\beta}) \right] d\eta
\]
Now, we use the summation by parts formula, \( \psi \) of Lemma 2.2 with that, if \(|\nu| > 100\), \( \int_{\mathbb{R}} \psi_1^2 \left( \frac{\eta}{2^k} \right) e^{i[(y-2\pi \nu)\eta \pm t \eta |\eta|^{1+\beta}]} d\eta \leq \frac{C}{|\nu|^2} + \frac{C}{|\nu|^3} \). Then,

\[
\sum_{m} \int_{\mathbb{R}} \psi_1^2 \left( \frac{\eta}{2^k} \right) e^{i[(y-2\pi \nu)\eta \pm t \eta |\eta|^{1+\beta}]} d\eta = \sum_{|\nu|\leq 100} \int_{\mathbb{R}} \psi_1^2 \left( \frac{\eta}{2^k} \right) e^{i[(y-2\pi \nu)\eta \pm t \eta |\eta|^{1+\beta}]} d\eta + O(1).
\]

So it is enough to estimate,

\[
\left| \sum_{m=1}^{\infty} \psi_1^2 \left( \frac{m}{2^k} \right) e^{i(mx + tm^{2+\alpha})} \int_{\mathbb{R}} \psi_1^2 \left( \frac{\eta}{2^k} \right) e^{i[\nu \eta \pm t \eta |\eta|^{1+\beta}]} d\eta \right| \leq 2^{4+2k} (2^{2-k} + 2^{2+2k}),
\]

with \(|t| \in [2^{-l}, 2^{-l+1}]\), \(m \sim 2^{k}\) and \(0 \leq \beta \leq 1\), being the estimate of the other sum similar. The estimation of the following oscillatory integral, is a consequence of Lemma 2.2 with \( \varphi(\eta) = y' \eta \pm t \eta |\eta|^{1+\beta} \) and \(|\eta| \sim 2^k\),

\[
\left| \int_{\mathbb{R}} \psi_1^2 \left( \frac{\eta}{2^k} \right) e^{i[y' \eta \pm t \eta |\eta|^{1+\beta}]} d\eta \right| \leq \frac{2^{4+2k}}{|mt|^{\frac{1}{2}}}, \quad (2.3)
\]

Now, we use the summation by parts formula,

\[
\sum_{m=1}^{\infty} a_m b_m = \sum_{N=1}^{\infty} \left( \sum_{m=1}^{N} a_m \right) (b_N - b_{N+1})
\]

for any compactly supported sequences \(a_m\) and \(b_m\). To get,

\[
\left| \sum_{m=1}^{\infty} \psi_1^2 \left( \frac{m}{2^k} \right) e^{i(mx + tm^{2+\alpha})} \int_{\mathbb{R}} \psi_1^2 \left( \frac{\eta}{2^k} \right) e^{i[y' \eta \pm t \eta |\eta|^{1+\beta}]} d\eta \right| \leq 2^{4+2k} \| \beta^\frac{1}{2} \psi_1 \|_0^2
\]

where \(a_m = e^{i(mx + tm^{2+\alpha})}, \quad b_m = \psi_1^2 \left( \frac{m}{2^k} \right) \int_{\mathbb{R}} \psi_1^2 \left( \frac{\eta}{2^k} \right) e^{i[y' \eta \pm t \eta |\eta|^{1+\beta}]} d\eta \) and \(N \in [2^{-l}, 2^{-l+1}]\).

\[
|b_N - b_{N+1}| \leq \psi_1^2 \left( \frac{N}{2^k} \right) - \psi_1^2 \left( \frac{N+1}{2^k} \right) \left| \int_{\mathbb{R}} \psi_1^2 \left( \frac{\eta}{2^k} \right) e^{i[y' \eta \pm t \eta |\eta|^{1+\beta}]} d\eta \right|
\]

\[
\leq 2^{-j} \frac{2^{1-\beta} k}{|N|^{\frac{1}{2}}} + |t| \int_{\mathbb{R}} \left| \psi_1^2 \left( \frac{\eta}{2^k} \right) \right|^2 d\eta
\]

\[
\leq 2^{-j} \frac{2^{1-\beta} k}{|N|^{\frac{1}{2}}} + |t| 2^k \| D^{\frac{1}{2}+\beta} \psi_1 \|_0^2
\]
\begin{align*}
\leq 2^{-j} \left( \frac{2^{2j \alpha}}{|N| \pi^2} + |t||2^k| \left\| \psi_1 \right\|^2_{\ell^2} \right) &
\leq 2^{2j - 2j \alpha - \frac{4}{2^k}} + 2^{2j - 2j \alpha - \frac{4}{2^k}}
\end{align*}

where \( |t| \in [2^{-l}, 2^{-l+1}] \), \( N \sim 2^j \), \( l \geq k + j \) and \( \left\| \psi_1 \right\|_{\ell^2} < \infty \). Then we must estimate,

\begin{align*}
\left| \sum_{N=2j-1}^{2j+1} \left( \sum_{m=1}^{N} e^{i(mx + tm^2 + na)} \right) (b_N - b_{N+1}) \right| &
\leq 2^{2j - 2j \alpha - \frac{4}{2^k}} \sum_{N=2j-1}^{2j+1} \left| \sum_{m=1}^{N} a_m \right|
\leq 2^{2j - 2j \alpha - \frac{4}{2^k}} \sum_{N=1}^{N} a_m 
\end{align*}

(2.5)

Following the proof of the Theorem 9.3.1 de [13], we use Lemma of H. Weyl in \( \left| \sum_{m=1}^{N} a_m \right| \). If \( \alpha = 1 \), \( h(m) = \frac{1}{2\pi} m^3 + \frac{1}{2\pi} m \), we fix \( \Lambda = 2^{2j+5} \) and apply Lemma 2.4 to \( r = \frac{4}{\pi} \) then, \( \left| \frac{4}{\pi} - \frac{a}{q} \right| \leq \frac{1}{2\pi q} \). Since \( j \) is large, \( N \sim 2^j \), \( l \in [2j, 2j+2] \) (the restriction \( l \leq 2j + 2 \) guarantees that \( \frac{a}{q} \neq 0 \) and \( 2^j \leq q \leq 2^{2j+5} \), then by Lemma 2.3 we get

\begin{align*}
\left| \sum_{m=1}^{N} e^{i(mx + tm^3)} \right| &
\leq C_\epsilon (2j)^{1+2\epsilon} \left( \frac{1}{2j} + \frac{1}{2j} + \frac{2^{2j+5}}{2^{3j}} \right)^{\frac{1}{4}} \leq 2^{\left( \frac{3}{4} + 2\epsilon \right) k}
\end{align*}

When \( \alpha = 2 \), \( h(m) = \frac{1}{2\pi} m^4 + \frac{1}{2\pi} m \), we fix \( \Lambda = 2^{3j} \) and apply Lemma 2.4 to \( r = \frac{4}{\pi} \) then, \( \left| \frac{4}{\pi} - \frac{a}{q} \right| \leq \frac{1}{2\pi q} \). Since \( j \) is large, \( N \sim 2^j \), \( l \in [2j, 3j] \) and \( 2^j \leq q \leq 2^{3j} \), then by Lemma 2.3 we get, \( \left| \sum_{m=1}^{N} e^{i(mx + tm^4)} \right| \leq C_2 \left( \frac{3}{4} + 2\epsilon \right) j \). And if \( \alpha = 3 \), \( h(m) = \frac{1}{2\pi} m^5 + \frac{1}{2\pi} m \), we fix \( \Lambda = 2^{4j} \) and apply Lemma 2.4 to \( r = \frac{4}{\pi} \) then, \( \left| \frac{4}{\pi} - \frac{a}{q} \right| \leq \frac{1}{2\pi q} \). Since \( j \) is large, \( N \sim 2^j \), \( l \in [2j, 3j] \) and \( 2^j \leq q \leq 2^{4j} \), then by Lemma 2.3 we get, \( \left| \sum_{m=1}^{N} e^{i(mx + tm^5)} \right| \leq C_2 \left( \frac{3}{4} + 2\epsilon \right) j \). In the case \( \alpha = 1 \). By replacing in (2.5), we get,

\begin{align*}
\left| \sum_{m=1}^{\infty} \psi_1^2 \left( \frac{m}{2\pi} \right) e^{i(mx + tm + na)} \right| &
\left| \int_{\mathbb{R}} \psi_1^2 \left( \frac{\eta}{2\pi} \right) e^{i(\eta' \eta + tm |\eta'|^{1+\alpha})} d\eta \right| \leq 2^{2j \left( -\frac{1}{2} + 2\epsilon \right)} (\frac{\alpha}{2} + 2\epsilon) k,
\end{align*}

Analogously in the other cases.

\( \square \)

**Remark 2.6.**

- As in the case \( \alpha = 2 \), we have,

\begin{align*}
\sum_{m \in \mathbb{Z}^*} \psi_1^2 \left( \frac{m}{2\pi} \right) e^{i(mx + ts\text{sgn}(m) m^4)} &
\int_{\mathbb{R}} \psi_1^2 \left( \frac{\eta}{2\pi} \right) e^{i(\eta' \eta + tm |\eta'|^{1+\alpha})} d\eta 
\end{align*}

\begin{align*}
= \sum_{m=1}^{\infty} \psi_1^2 \left( \frac{m}{2\pi} \right) e^{i(mx + tm^4)} &
\int_{\mathbb{R}} \psi_1^2 \left( \frac{\eta}{2\pi} \right) e^{i(\eta' \eta + tm |\eta'|^{1+\alpha})} d\eta 
+ \sum_{m=-\infty}^{\infty} \psi_1^2 \left( \frac{m}{2\pi} \right) e^{i(mx - tm^4)} \int_{\mathbb{R}} \psi_1^2 \left( \frac{\eta}{2\pi} \right) e^{i(\eta' \eta + tm |\eta'|^{1+\alpha})} d\eta.
\end{align*}
So it is enough to estimate,
\[
\sum_{m=1}^{\infty} \psi_\varepsilon^2 \left( \frac{m}{2} \right) e^{i \psi [mx + tm]} \frac{d}{d\eta} e^{i \psi [\eta^2 t m(\eta^{2} + \beta)]} \lesssim 2 \left( -\frac{1}{2} + 2\varepsilon \right) j + \left( -\frac{3}{4} + 2\varepsilon \right) k.
\]

- In the case \(Q_y^0 Q_x^1 \phi, j > 0\), we used the summation by parts formula, with
  \(a_m = e^{i (mx + tm + \gamma)}\), \(b_m = \psi_\varepsilon^2 \left( \frac{m}{2} \right)\) and Lemma of H. Weyl,
  \[
  \|W_0(\cdot) Q_y^0 Q_x^1 \phi\|_{L^2_{x,y} L^\infty(T^2)} \lesssim 2 \left( -\frac{1}{2} + 2\varepsilon \right) j \|Q_y^0 Q_x^1 \phi\|_{L^2(T^2)}.
  \]

The case, \(Q_y^0 Q_x^0 \phi, k \geq 0\), is not contemplated, because, if \(\hat{\phi}(0, n) = 0\), for all \(n \in \mathbb{Z}\), then, \(u(0, n, t) = 0\), for all \(n \in \mathbb{Z}\).

### 3. Preliminary and Key estimates

As a consequence of the Strichartz inequality (Theorem 2.5), we obtain:

**Lemma 3.1.** Let \(a = 1, 2\) or \(3, 0 < \beta \leq 1\), \(u \in C \left( [0, T] ; H^\infty \left( \mathbb{T}^2 \right) \right) \cap C^1 \left( [0, T] ; H^\beta \left( \mathbb{T}^2 \right) \right)\)
and \(f \in C \left( [0, T] ; H^\infty \left( \mathbb{T}^2 \right) \right)\), \(T \in (0, 1]\) such that
\[
\partial_t u - \partial_x \left( D_x^{1 + \alpha} + D_y^{1 + \beta} \right) u = \partial_x f.
\]

Then,
\[
\|u\|_{L^1_x L^\infty_{x,y}(T^2)} \lesssim s_{1,2} \frac{T}{2} \left( \|J_x^{s_1} J_y^{s_2} u\|_{L^2_x L^\infty_{x,y}(T^2)} + \|J_x^{s_1} f\|_{L^1_x L^2_y(T^2)} \right),
\]
for any \(s_1 > \frac{1}{2} - \frac{1}{\alpha}, s_2 > \frac{1}{2} - \frac{1}{\beta}\).

**Proof.** We partition the interval \([0, T]\) into \(2^{j+k}\) equal intervals of length \(T 2^{-(j+k)}\),
denote by \([\bar{a}_k, m, a_k, (m+1)]\), \(m = 0, 1, 2 \cdots 2^{j+k}\). Then,
\[
\|Q_y^k Q_x^j u\|_{L^1_x L^\infty_{x,y}} \lesssim \sum_{m=1}^{2^{k+j}} \left\| \chi_{[\bar{a}_k, m, a_k, (m+1)]} (t) Q_y^k Q_x^j u \right\|_{L^1_x L^\infty_{x,y}}.
\]  

By Cauchy-Schwarz inequality in (3.2),
\[
\left\| \chi_{[\bar{a}_k, m, a_k, (m+1)]} (t) Q_y^k Q_x^j u \right\|_{L^1_x L^\infty_{x,y}} \lesssim \left( T \right)^{-\bar{k}} \left\| \chi_{[\bar{a}_k, m, a_k, (m+1)]} (t) Q_y^k Q_x^j u \right\|_{L^2_x L^\infty_{x,y}}.
\]  

By Duhamel’s formula, for \(t \in [\bar{a}_k, m, a_k, (m+1)]\),
\[
u(t) = W_e^n (t - a_k, m) \left( u (a_k, m) \right) + \int_{a_k, m}^t W_e^n (t - s) (\partial_x f (s)) ds
\]  

It follows from (3.1) and Theorem 2.5 that,
\[
\left\| \chi_{[\bar{a}_k, m, a_k, (m+1)]} (t) Q_y^k Q_x^j u \right\|_{L^2_x L^\infty_{x,y}} \lesssim 2 \left( -\frac{1}{2} + 2\varepsilon \right) j \left( -\frac{3}{4} + 2\varepsilon \right) k \|Q_y^k Q_x^j u (a_k)\|_{L^2_x L^\infty_{x,y}}.
\]
Theorem 3.4

Then, using Gronwall's inequality, we get the result.

Then, the left hand side of (3.3) is bounded by,

\[ 2^{-k} \int_{T^2} \sum_{m=1}^{2^{k+j}} \left( 2^{-\left(\frac{k}{2} + \epsilon \right)} j + \left( -\frac{k}{2} + \epsilon \right) k \right) \|Q_y^k Q_x^j u(a_{k,m})\|_{L^2_y}^2 \]

Then, the left hand side of (3.3) is bounded by,

\[ 2^{-k} \int_{T^2} \sum_{m=1}^{2^{k+j}} \left( 2^{-\left(\frac{k}{2} + \epsilon \right)} j + \left( -\frac{k}{2} + \epsilon \right) k \right) \|Q_y^k Q_x^j u(a_{k,m})\|_{L^2_y}^2 \]

where \( 2^{\left(\frac{k}{2} - \frac{4+\epsilon}{4} \right) k} \) < 1 for \( k \geq 1 \) and \( \epsilon < \frac{2+\beta}{6} \), the result is followed.

\[ \square \]

To obtain the energy estimate, we recall the periodic version of the Kato -Ponce commutator,

**Proposition 3.2.** Let \( s \geq 1 \) and \( f, g \in H^\infty (T^2) \). Then,

\[ \| J^s (fg) - f J^s g \|_0 \leq C_s \| J^s f \|_0 \| g \|_\infty + (\| f \|_\infty + \| \nabla f \|_\infty) \| J^{s-1} g \|_0. \]

**Proof.** See Lemma 9.A.1 in [15]. \( \square \)

**Lemma 3.3** (Energy estimate). Let \( \alpha = 1, 2 \) or 3, \( 0 < \beta \leq 1 \) and \( u \) solution of IVP (1.1) with \( \phi \in H^\infty (T^2) \), then, for any \( s \geq 1 \), we have, for any \( T \in [0,1] \), that

\[ \sup_{0 < t < T} \| u \|_{H^s(T^2)} \leq e^{C \left( \| u \|_{L^1_T L^2_y(T^2)} + \| \partial_x u \|_{L^1_T L^2_y(T^2)} + \| \partial_y u \|_{L^1_T L^2_y(T^2)} \right)} \| \phi \|_{H^\infty(T^2)}. \]

**Proof.** We apply \( J^s \) to equation in (1.1) and multiply by \( J^s u \),

\[ \int_{T^2} J^s u J^s u dx dy - \int_{T^2} J^s \partial_x D_x^{1+\alpha} u J^s u dx dy \pm \int_{T^2} J^s \partial_x D_y^{1+\beta} u J^s u dx dy \]

\[ + \int_{T^2} J^s u u_x J^s u dx dy = 0. \]

By integrating by parts and applying Proposition 3.2 we get:

\[ \frac{1}{2} \frac{d}{dt} \| J^s u \|_0^2 \leq \| \partial_x u \|_\infty \| J^s u \|_0 + (\| u \|_\infty + \| \nabla u \|_\infty) \| J^{s-1} \partial_x u \|_0, \]

then, using Gronwall's inequality, we get the result.

\[ \square \]

Now, we obtain an estimate of \( u, u_x \) and \( u_y \) in \( L^1_T L^2_y(T^2) \).

**Theorem 3.4** (Product Lemma). Let \( s \geq 0 \) and \( f, g \in H^\infty (T^2) \). Then

\[ \| J^s (fg) \|_{L^p} \leq C \left( \| J^s f \|_{L^\infty} \| g \|_{L^p} + \| f \|_{L^\infty} \| J^s g \|_{L^p} \right), \]
**Proof.** See Lemma 4.2 in [4]. □

**Proposition 3.5.** Let $\alpha = 1, 2$ or $3$, $0 < \beta \leq 1$, $u$ be a solution del IVP (1.1) with $\phi \in H^\infty(T^2)$, then for any $s > (\frac{1}{2} - \frac{1}{1+\beta})\left(\frac{1}{2} - \frac{\beta}{4}\right)$, there exists $T = T(||\phi||, s)$ and a constant $C_T(||\phi||, s)$ such that,

\[
g(T) := \int_0^T \left( ||u||_{L^\infty(T^2)} + ||u_x||_{L^\infty(T^2)} + ||u_y||_{L^\infty(T^2)} \right) dt' \leq C_T
\]

**Proof.** We apply Lemma 3.1 with $s_1 > \frac{1}{2} - \frac{1}{1+\beta}$ and $s_2 > \frac{1}{2} - \frac{\beta}{4}$ in $u$, $\partial_xu$, $\partial_yu$ and respectively by $f = \frac{1}{2}u^2$, $\frac{1}{2}\partial_xu^2$, $\frac{1}{2}\partial_yu^2$. We get,

\[
\begin{align*}
||u||_{L^1_tL^\infty_x} + ||u_x||_{L^1_tL^\infty_x} + ||u_y||_{L^1_tL^\infty_x} \\
\lesssim s T^{\frac{1}{2}} \left( ||J_x J_y^s u||_{L^\infty_t L^2_x} + ||J_y J_x^s \partial_x u||_{L^\infty_t L^2_x} + ||J_x J_y^s \partial_y u||_{L^\infty_t L^2_x} + ||J_x^s u^2||_{L^1_t L^2_x} + ||J_y^s \partial_x (u^2)||_{L^1_t L^2_x} + ||J_y^s \partial_y (u^2)||_{L^1_t L^2_x} \right)
\end{align*}
\]

The first three terms above can be bounded by Young’s inequality with $p = \frac{s_2+1}{s_2}$ and $q = s_2 + 1$, $p = q = 2$ respectively,

\[
\begin{align*}
||J_x^s J_y^s \partial_x u||_{L^2} &\lesssim \sum_{(m,n) \in \mathbb{Z}^2} \left( 1 + m^2 \right)^{s_1 + 1} \left( 1 + n^2 \right)^{s_2} |\hat{u}|^2 \\
&\lesssim \left( 1 + m^2 \right)^{\frac{(s_1+1)(s_2+1)}{2}} \hat{u} \left( 1 + n^2 \right)^{\frac{(s_1+1)(s_2+1)}{2}} \hat{u} \\
&\lesssim ||J_x^{(s_2+1)(s_1+1)} u||_{L^2} + ||J_y^{s_2+1} u||_{L^2} \\
&\lesssim ||u||_s,
\end{align*}
\]

\[
\begin{align*}
||J_y J_x^s \partial_y u||_{L^2} &\lesssim \sum_{(m,n) \in \mathbb{Z}^2} \left( 1 + m^2 \right)^{s_1} \left( 1 + n^2 \right)^{s_2+1} |\hat{u}|^2 \\
&\lesssim \left( 1 + \xi^2 \right)^{\frac{s_1+1}{2}} \hat{u} \left( 1 + n^2 \right)^{\frac{(s_1+1)(s_2+1)}{2}} \hat{u} \\
&\lesssim ||J_x^{s_1+1} u||_{L^2} + ||J_y^{(s_2+1)(s_1+1)} u||_{L^2} \\
&\lesssim ||u||_s
\end{align*}
\]

and

\[
\begin{align*}
||J_x^s J_y^s u||_{L^2} &= \sum_{(m,n) \in \mathbb{Z}^2} \left( 1 + n^2 \right)^{s_1} \left( 1 + n^2 \right)^{s_2} |\hat{u}|^2 \\
&\lesssim \left( 1 + \xi^2 \right)^{s_1} \hat{u} + \left( 1 + n^2 \right)^{s_2} \hat{u} \\
&\lesssim ||J_x^{2s_1} u||_{L^2} + ||J_y^{2s_2} u||_{L^2} \\
&\lesssim ||u||_s
\end{align*}
\]

Applying Leibniz’s product rule periodic version and Young’s inequality,

\[
\begin{align*}
\int_0^T ||J_x^s u^2||_{L^2} dt' &\lesssim \int_0^T ||u||_{L^\infty} ||J_x^s u||_{L^2} dt' \\
&\leq ||u||_{L^1_t L^\infty_x} ||J_x^s u||_{L^2 L^2} \\
&\leq ||u||_{L^1_t L^\infty_x} ||u||_{L^\infty_{L^2}}^{s_2}
\end{align*}
\]

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To complete the proof, by an argument of continuity if \( g \) (Lemma 3.3) we obtain the inequality, proving Theorem.

We deduce from energy estimate (Lemma 3.3) and previous inequality (4.1) that,

\[
\sup_{0 < t < T} \| u \|_{L^\infty} \leq C_T \| \phi \|_{L^\infty} \quad \text{for } T < T_0 
\]

Then adding the previous inequalities and applying the respective energy estimate (Lemma 3.3) we obtain the inequality,

\[
g(T) \lesssim \| \phi \|_s e^{g(T)} (1 + g(T)).
\]

To complete the proof, by an argument of continuity if \( T \leq T_0 C_T(\| \phi \|_s, s) \) is small enough, \( g(T) \leq C_T(\| \phi \|_s, s) \)

In this point, we can use standard compactness arguments as in Kenig [19] for proving Theorem.

### 4. Proof of Theorem 1.1

Let \( s > (\frac{1}{2} - \frac{1}{3m})(\frac{3}{2} - \frac{1}{4}) \), \( \phi \in H^s (T^2) \). We consider by density, \( \phi_\gamma \in H^s \cap H^s \) such that \( \lim_{\gamma \to \infty} \| \phi_\gamma - \phi \|_{H^s(T^2)} = 0 \) and \( \| \phi_\gamma \|_{H^s(T^2)} \leq C \| \phi \|_{H^s(T^2)} \). Let \( \{ u_\gamma \} \)

solutions associated to the initial data \( \{ \phi_\gamma \} \) such that \( u_\gamma \in C ([0, T']; H^s (T^2)) \),

\( T' > 0 \) guaranteed by the local well-posedness of (1.1) in \( H^s(T^2) \) for \( s > 2 \). We can extend \( u_\gamma \) on a time interval \( [0, T'] \), \( T = T(\| \phi \|_{H^s}, s) \) by Proposition 3.5 and also we show that there is a constant \( C_T \) such that,

\[
\int_0^T \left( \| u_\gamma \|_{L^\infty} + \| \partial_x u_\gamma \|_{L^\infty} + \| \partial_y u_\gamma \|_{L^\infty} \right) dt \leq C_T \quad (4.1)
\]

We deduce from energy estimate (Lemma 3.3) and previous inequality (4.1) that,

\[
\sup_{0 < t < T} \| u_\gamma \|_{H^s(T^2)} \leq C_T \quad (4.2)
\]

by using Gronwall’s inequality and inequality (4.1) we get,

\[
\lim_{\gamma \to \infty} \sup_{0 < t < T} \| u_\gamma - u_{\mu} \|_0 = 0 \quad (4.3)
\]

Now we consider the inequality (4.2) and (4.3), we can find \( u \in C ([0, T]; H^{s_1}(T^2)) \cap L^\infty ([0, T]; H^s (T^2)) \) with \( s_1 < s \), such that \( u_\gamma \to u \) in \( C ([0, T]; H^{s_1}(T^2)) \). By (4.3), \( u_\gamma \to u \) in \( C ([0, T]; L^2 (T^2)) \). Hence, by weak* compactness \( u \in L^\infty ([0, T]; H^s (T^2)) \). To establish the uniqueness of \( u \) we follow a similar argument as above and the continuous dependence one uses the Bona-Smith argument (see [3]).

**Remark 4.1.** Following the ideas of [7], we obtain a better result of local and global well-posedness to the regularized problem (4.4): \( u_\gamma \to u \) in \( C ([0, T]; H^{s_1}(T^2)) \cap L^\infty ([0, T]; H^s (T^2)) \) with \( s_1 < s \), such that \( u_\gamma \to u \) in \( C ([0, T]; H^{s_1}(T^2)) \). By (4.3), \( u_\gamma \to u \) in \( C ([0, T]; L^2 (T^2)) \). Hence, by weak* compactness \( u \in L^\infty ([0, T]; H^s (T^2)) \). To establish the uniqueness of \( u \) we follow a similar argument as above and the continuous dependence one uses the Bona-Smith argument (see [3]).
(1) For the local well-posedness with $\mu > 0$ and $-2 < s < 2$. We consider,

$$
\chi_T^s = \left\{ u_\mu \in C \left( [0, T]; H^s (\mathbb{T}^2) \right) : \| u \|_{\chi_T^s} < \infty \right\}
$$

with $\| u \|_{\chi_T^s} = \sup_{[0, T]} \left\{ \| u(t) \|_s + t^{\frac{s}{2}} \| u \|_0 \right\}$. We apply the fixed point theorem and respective linear and nonlinear estimates;

$$
\| \mathcal{W}_\mu (t) \phi \|_{\chi_T^s} \leq \| \mathcal{W}_\mu (t) \phi \|_s + t^{\frac{s}{2}} \| \mathcal{W}_\mu (t) \phi \|_0 \\
\leq \left( 1 + C_s \left( T^{\frac{|s|}{4}} + \mu^{-\frac{|s|}{4}} \right) \right) \| \phi \|_s
$$

and

$$
\left\| \int_0^t \mathcal{W}_\mu (t-t') \partial_x u^2 dt' \right\|_{\chi_T^s} \\
\leq \left\| \int_0^t \mathcal{W}_\mu (t-t') \partial_x u^2 dt' \right\|_s + t^{\frac{s}{2}} \left\| \int_0^t \mathcal{W}_\mu (t-t') \partial_x u^2 dt' \right\|_0 \\
\leq \left( \mu^{-\frac{(s+2)}{4}} + \mu^{-\frac{s}{4}} \right) T^{\frac{|s|}{2} - \frac{s}{4}} \| u \|_{\chi_T^s}^2.
$$

(2) The regularized problem \((4.4)\) satisfies,

$$
\| \phi \|_0^2 = \| u(t) \|_0^2 + \mu \int_0^t \| \Delta u \|_0^2 dt', \quad (4.5)
$$

and

$$
\| \partial_x u \|_0^2 \leq \| \partial_x \phi \|_0^2 e^{\epsilon_1 T} \\
(4.6)
$$

We get \((4.0)\) and \((4.4)\), applying Gagliardo-Nirenberg inequality (Theorem 3.70 in [11]) for compact surfaces with $u(0) = 0$. with these ingredients we establish global well-posedness for $s > -2$;

**Theorem 4.2.** Let $s \in (-2, 2)$ and $\mu > 0$. Then \((4.4)\) is global well-posedness.

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