GRMHD simulations of accreting neutron stars I: nonrotating dipoles

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ABSTRACT
We study the general-relativistic dynamics of matter being accreted onto and ejected by a magnetised and nonrotating neutron star. The dynamics is followed in the framework of fully general relativistic magnetohydrodynamics (GRMHD) within the ideal-MHD limit and in two spatial dimensions. More specifically, making use of the numerical code BHAC, we follow the evolution of a geometrically thick matter torus driven into accretion by the development of a magnetorotational instability. By making use of a number of simulations in which we vary the strength of the stellar dipolar magnetic field, we can determine self-consistently the location of the magnetospheric (or Alfvén) radius $r_{\text{msph}}$ and study how it depends on the magnetic moment $\mu$ and on the accretion rate. Overall, we recover the analytic Newtonian scaling relation, i.e., $r_{\text{msph}} \propto B^{4/7}$, but also find that the dependence on the accretion rate is very weak. Furthermore, we find that the material torque correlates linearly with the mass-accretion rate, although both of them exhibit rapid fluctuations. Interestingly, the total torque fluctuates drastically in strong magnetic field simulations and these unsteady torques observed in the simulations could be associated with the spin fluctuations observed in X-ray pulsars.

Key words: accretion, accretion discs – stars: neutron – MHD – methods: numerical

1 INTRODUCTION
Since the discovery of the first pulsating X-ray sources Centaurus X-3 (Giacconi et al. 1971; Schreier et al. 1972) and Hercules X-1 (Tananbaum et al. 1972), accretion of matter transferred from a binary companion onto a neutron star (Pringle & Rees 1972; Davidson & Ostriker 1973; Lamb et al. 1973) is known to power many classes of X-ray sources from the nuclear- and accretion-powered millisecond X-ray pulsars (Wijnands & van der Klis 1998) in low-mass X-ray binaries to pulsating ultra-luminous X-ray sources (Bachetti et al. 2014) in high-mass X-ray binaries. In the former case, mass transfer occurs by Roche-lobe overflow and proceeds by accretion from a viscously evolving disc around the compact object.

Furthermore, if the dipolar magnetic field of the neutron star is sufficiently strong, the inner portions of the disc are disrupted and the flow is funneled onto the star along the magnetic field lines. The magnetospheric (or Alfvén) radius of the disc is located at the distance where the material stresses are balanced by the magnetic stresses (Ghosh et al. 1977; Ghosh & Lamb 1978, 1979a,b; Wang 1987; Spruit & Taam 1990; Campbell 1992; Wang 1996) (see the detailed discussion in Sect. 3.2).

Some of the early models assumed that the disc, due to the high conductivity of its ionised plasma, would exhibit a diamagnetic behaviour i.e., expel the stellar magnetic field lines except for the innermost boundary where matter is coupled to the field lines (Ichimaru 1978; Scharlemann 1978; Aly 1980). This mode of interaction would then lead only to a spin-up of the neutron star via accretion. However, the discovery of X-ray pulsars that spin-down in the accretion stage motivated the “magnetically threaded disc” model (Ghosh & Lamb 1979a,b; Kaburaki 1986), where the stellar field lines are allowed to penetrate the disc over a broad region via several instabilities. This model assumes that the poloidal stellar field couples to the toroidal field generated from the angular-velocity difference between the star $\Omega_*$ and that of the matter in the disc $\Omega(r)$. After defining the magnetic pitch factor as the ratio between the toroidal ($B_{\text{tor}}$) and poloidal ($B_{\text{pol}}$) components, i.e., $\gamma_\phi := B_{\text{tor}} / B_{\text{pol}}$, the magnetically threaded disc model implies that $\gamma_\phi \propto (\Omega_* - \Omega(r))/\Omega_\kappa(r)$, where $\Omega_\kappa(r)$ is the Keplerian angular velocity – so that a negative magnetic torque proportional to $\gamma_\phi$ is exerted onto the star by the magnetic-field lines penetrating the disc beyond the corotation radius, which is where the disc rotates at the same angular velocity of the star.

The magnetic pitch factor is an important ingredient of the magnetically threaded model and the dependence of the pitch factor on the velocity difference – as well as on the processes limiting the growth of the toroidal magnetic field in regions where the velocity difference is large – has been explored in the literature (Wang 1987; Campbell 1992; Wang 1995; Uzdensky et al. 2002). In particular, it

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has been pointed out that the model of Ghosh & Lamb (1979a) tends to produce excessively large toroidal magnetic fields at large distances from the corotation axis, so large that the resulting magnetic pressure would disrupt the disc beyond the corotation radius (Wang 1987), unless some magnetic-field diffusive mechanism is invoked in the disc (Campbell 1992; Wang 1995).

More recently, steady-state configurations of the toroidal magnetic field was studied by Naoz & Miller (2011), who solved the full induction equation and observed that the profile of the toroidal magnetic field is rather different from that proposed in previous analytical models of Wang (1987) and Campbell (1992). We note that the spin-up timescale of X-ray pulsars, \( \tau := \Omega / |\mathbf{b}| \leq 10^4 \) yr, is much less than their lifetime as accretors (\( r_\chi \gtrsim 10^6 \) yr). This implies that the long-term spin-up trends observed from the first discovered X-ray pulsars do not represent a steady-state behaviour (Elsner et al. 1980). Indeed, it was soon observed that some accreting pulsars, such as GX 1+4 (Makishima et al. 1988), 4U 1626-67 (Chakrabarty et al. 1997; Camero-Arranz et al. 2010) and 4U 1907+09 (Inam et al. 2009), show torque reversals with secular spin-up or spin-down on sufficiently long timescales. Furthermore, observations of accreting pulsars with Burst and Transient Source Experiment (BATSE) on the Compton Gamma-Ray Observatory showed that all accreting pulsars, independently of whether they are disc-fed or wind-fed, exhibit stochastic variations in their spin frequencies and luminosities (Bildsten et al. 1997). BATSE observations also showed that sources exhibiting secular spin-up and spin-down trends also have stochastic variations in their spin frequencies.

In view of these considerations, over the last decade a number of numerical simulations covering the full set of the magnetohydrodynamics (MHD) equations, either in two or three spatial dimensions, have explored the complex behaviour of the interaction between a magnetised star and an accretion disc. While these simulations are also subject to simplifications and approximations, they are expected to provide a more accurate description of the complex non-stationary behaviour that is expected in accreting plasmas and that is suggested by the observations. Indeed, such simulations have revealed that the accretion flow onto a compact object from a magnetised plasma is intrinsically accompanied by the development in the disc of a magnetorotational instability (MRI; Velikhov 1959; Chandrasekhar 1960) that leads to a turbulent flow with a steady behaviour accompanied by fluctuations in all the physical quantities (Balbus & Hawley 1991).

Simulations providing useful insight for our understanding of the disc-magnetosphere interaction have been carried out initially within a Newtonian description of gravity (Hayashi et al. 1996; Miller & Stone 1997; Romanova et al. 2002; Long et al. 2005; Bessolaz et al. 2008; Romanova et al. 2008; Kulkarni & Romanova 2008; Romanova et al. 2011, 2012; Kulkarni & Romanova 2013; Romanova & Owocki 2015; Ireland et al. 2022). However, general-relativistic effects might play an important role in accretion onto neutron stars, especially in the case of weakly magnetised neutron stars in low-mass X-ray binaries, where the disc can extend closer to the star (Psaltis & Chakrabarty 1999). Hence, general-relativistic MHD (GRMHD) simulations have been performed recently to explore ultra-luminous X-ray sources (Takahashi & Ohsuga 2017; Takahashi et al. 2018; Abarca et al. 2018, 2021), and accreting millisecond X-ray pulsars for dipole magnetic field (Parfrey & Tchekhovskoy 2017) and multipole magnetic field geometries (Das et al. 2022).

We here report GRMHD simulations of accretion onto a magnetised nonrotating neutron star from a MRI driven accretion torus. In particular, we examine general relativistic effects on the disc-magnetosphere interaction, specifically the modifications on the magnetospheric radius. To this end, we perform simulations for ten different magnetisations of the neutron star and study how the magnetospheric radius depends on the stellar magnetic-field strength and on the properties of the accretion flow. We also investigate the properties of the magnetic pitch factor in the presence of MRI-induced turbulent fields.

The structure of the paper is as follows. The numerical setup and the initial conditions of our simulations are introduced in Sec. 2, while the results of the simulations are presented in Sec. 3. Finally, Sec. 4 collects the discussion of the results and our conclusions. Hereafter, and unless indicated otherwise, we adopt geometrised units where \( G = c = 1 \), with \( G \) and \( c \) being the gravitational constant and the speed of light, respectively.

## 2 NUMERICAL SETUP

In our simulations we employ BHAC (Porth et al. 2017; Olivares et al. 2019) to solve numerically the full set of the GRMHD equations over the fixed spacetime of a nonrotating star

\[
\nabla_\mu J^\mu = 0, \quad T^{\mu \nu} = 0, \quad \nabla_\mu F^{\mu \nu} = 0, \quad \nabla_\mu \rho \mu = 0, \quad T^{\mu \nu} = \left( \rho h + b^2 \right) u^\mu u^\nu + \left( p + \frac{b^2}{2} \right) g^{\mu \nu} - b^\mu b^\nu, \quad \text{where} \quad J^\mu = \text{the rest-mass current}, \quad T^{\mu \nu} = \text{the total energy-momentum tensor}, \quad F^{\mu \nu} = \text{the Faraday tensor.}
\]

Specifically, the explicit expression for rest-mass current and the energy-momentum tensor of a magnetised perfect fluid are given by

\[
J^\mu = \rho u^\mu, \quad T^{\mu \nu} = \left( \rho h + b^2 \right) u^\mu u^\nu + \left( p + \frac{b^2}{2} \right) g^{\mu \nu} - b^\mu b^\nu,
\]

where \( h, \rho, p, u^\mu, g_{\mu \nu}, b^\mu \) are, respectively, the specific enthalpy, the rest-mass density, the pressure, the fluid four-velocity, the metric tensor, and the magnetic field measured in the fluid frame, so that \( b^2 := b^\mu b_\mu \). In the ideal-MHD limit considered here, the electric field in the comoving frame is zero, i.e., \( F^{\mu \nu} u_\mu = 0 \), and the dual of the Faraday tensor is given by

\[
F^{\mu \nu} = u^{\mu} b^\nu - u^{\nu} b^\mu.\]

Additionally quantities used hereafter are: the Eulerian three-velocity,

\[
\nu^i := u^i / \Gamma + b^i / \alpha, \quad \text{where} \quad \Gamma := \left( 1 - \nu^i v^i \right)^{-1/2} \text{is the Lorentz factor} \quad \alpha \quad \text{and} \quad b^i \quad \text{are, respectively, the lapse function and the components of the shift vector in a 3+1 decomposition of the four-metric} - \text{and the Eulerian magnetic field}, \quad B^i := \nabla^i n^\nu, \quad \text{where} \quad n_\mu = \left(-1 / \sqrt{-g^{TT}}, 0, 0, 0\right) \quad \text{represents the one-form associated to an Eulerian observer} \quad \text{(Rezzolla & Zanotti 2013)}.
\]

We initialise the simulations with a Fishbone & Moncrief (FM) torus (Fishbone & Moncrief 1976), where its inner edge and the rest-mass density maximum are located at \( r_{\text{in}} = 200 M_\odot \) and \( r_{\text{max}} = 260 M_\odot \), respectively. At such a distance, the dipole magnetic field of the star is too weak to spoil the equilibrium of the torus. The system of GRMHD equations is closed with an equation of state with of an ideal fluid \( p = \rho e (\gamma - 1) \), where \( e \) is the specific internal energy and we assume an adiabatic index \( \gamma = 5/3 \) as for a completely degenerate non-relativistic electron fluid (Rezzolla & Zanotti 2013).

On the other hand, the mass and the radius of the neutron star are taken to be \( M = 1.4 M_\odot \) and \( R = 14 \) km, respectively. Furthermore, because the mass of the accreting fluid is much smaller than that of the nonrotating neutron star, the spacetime outside the star is described by...
by the Schwarzschild metric

\[ ds^2 = -\left(1 - \frac{2M}{r}\right) dr^2 + \frac{4M}{r} \frac{dr}{r} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \] (7)

The interior spacetime, on the other hand, does not need to be specified in our implementation since suitable boundary conditions will be imposed at the stellar surface (see discussion below).

Since the star is magnetised, a proper general-relativistic solution of the Maxwell equations for the external dipolar magnetic field needs to be specified. We here employ the analytic expressions derived by Wasserman & Shapiro (1983) and Rezzolla et al. (2001), and consider ten different strengths of the stellar magnetic field. In particular, we specify the dipolar magnetic moment \( \mu \) such that the maximum magnetisation parameter is

\[ \sigma_{\text{max}} := \sigma_{\text{max}}^\text{pole} \times 0.08433 \times \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}. \] (8)

where \( b_{\text{pole}}^2 \) is the strength of the magnetic field at the pole of the star and \( \rho_{\text{max}} \) is the maximum rest-mass density of the torus. In addition to the dipolar field of the star, a weak poloidal magnetic field is seeded in the torus via the toroidal component of a vector potential \( A_B \). The maximum of the rest-mass density is also used to introduce the following dimensionless dynamical quantities

\[ \dot{\rho} := \frac{\rho}{\rho_{\text{max}}}, \quad \ddot{\rho} := \frac{\rho}{\rho_{\text{max}}}, \quad \ddot{B} := \frac{\vec{B}}{\sqrt{\rho_{\text{max}}}^2}. \] (9)

so that all of our results can be scaled simply in terms of the choice for \( \rho_{\text{max}} \) (to keep the notation compact, we will omit the tildes on \( \rho \) on the relevant quantities).

Furthermore, since we do not include radiation hydrodynamics in our simulations [see instead Takahashi & Ohsuga (2017); Abarca et al. (2018)] where this is considered for accretion onto a non-magnetised star or Takahashi et al. (2018) for a magnetised star], our results are viable only in the sub-Eddington regime. Hence, in order to keep the mass-accretion rate lower than the Eddington limit and, at the same time, have the stellar magnetic field sufficiently strong so that the magnetospheric radius is larger than the stellar radius, we have set \( \rho_{\text{max}} = 0.001 \text{ g cm}^{-3} \). Hereafter, we present our results in units of

\[ B_0 := \left( \frac{B}{10^9 G} \right) \left( \frac{10^{-3} \text{ g cm}^{-3}}{\rho_{\text{max}}} \right)^{1/2}, \] (10)

\[ \mu_\text{Edd} := \left( \frac{\mu}{10^{27} \text{ G cm}^3} \right) \left( \frac{10^{-3} \text{ g cm}^{-3}}{\rho_{\text{max}}} \right)^{1/2}, \] (11)

\[ \dot{\nu} := \left( \frac{\dot{M}}{\eta \dot{M}_{\text{Edd}}} \right) \left( \frac{10^{-3} \text{ g cm}^{-3}}{\rho_{\text{max}}} \right), \] (12)

\[ j := \left( \frac{J}{10^{34} \text{ g cm}^2 s^{-2}} \right) \left( \frac{10^{-3} \text{ g cm}^{-3}}{\rho_{\text{max}}} \right), \] (13)

where \( \dot{M}_{\text{Edd}} := 2.2 \times 10^{-9} (M/M_\odot) \text{ M}_\odot \text{ yr}^{-1} \) is the Eddington mass-accretion rate and \( \eta := \left( GM/R_c^2 \right) \approx 0.15 \) is the efficiency coefficient. We note that magnetic-field values of the order of \( \lesssim 10^{10} \text{ G} \) are necessary to avoid regions of excessive magnetizations that cannot be handled within a GRMHD solution; similar values of the magnetic field have been used in the literature (Romanova et al. 2011, 2012; Romanova & Owocki 2015), but see also Parfrey & Tchekhovskoy (2017) and Das et al. (2022).

The stellar dipolar magnetic field and poloidal field of the torus are set to be anti-parallel, although this configuration has been shown to lead to a smoother accretion of matter (Romanova et al. 2011), essentially because the larger amount of reconnection taking place at the magnetospheric boundary releases larger amounts of magnetic energy and favours a more copious accretion; simulations in which the magnetic fields are parallel will be presented in a subsequent work. The magnitude of the poloidal magnetic field in the torus is set such that the plasma parameter \( \beta_g \) in the torus as a maximum at \( \beta_{g,\text{max}} := 2p_{\text{max}}/b_{\text{max}}^2 = 100 \), where \( p_{\text{max}} \) and \( b_{\text{max}} \) are the maximum pressure and magnetic fields, respectively. Note that the maximum of the total pressure and of the magnetic field do not coincide and that, in order to excite the development of the MRI, the thermal pressure of the torus is perturbed with a random noise with maximum relative amplitude of 2%.

As customary in codes solving the equations of general-relativistic hydrodynamics or of relativistic MHD (Rezzolla & Zanotti 2013), the whole computational domain needs to be filled with a fluid, including the regions outside the compact objects, which acts as a tenuous atmosphere. In our simulations, the rest-mass density in the atmosphere is initialised following two power laws, namely,

\[ \rho_{\text{atm}} = 3 \times 10^{-4} (R/r)^{3/2} p_{\text{max}}, \] (14)

\[ p_{\text{atm}} = 3 \times 10^{-6} (R/r)^{5/2} p_{\text{max}}, \] (15)

\[ v_{\text{atm}} = 0. \] (16)

Furthermore, to prevent small fluctuations from developing in the atmosphere, the rest-mass density, the pressure, and the velocity are reset to their floor values whenever \( p < 1.001 \rho_{\text{atm}} \) or \( p < 1.001 p_{\text{atm}} \).

In order to increase the resolution in the radial direction, we make use of a logarithmic radial coordinate, \( s(r) := \ln r \), which is uniformly spaced in the range \( s \in [2, 26, 69] \), thus corresponding to a radial coordinate \( r \in [R, 1000 M] \). The angular dimension with \( \theta \in [0, \pi] \), on the other hand, is covered with a uniform grid, so that the two-dimensional domain is covered at the coarsest level with an array of \( 320 \times 128 \) grid cells. Furthermore, to further increase the resolution efficiently, we employ a three-level adaptive mesh refinement (AMR) based on a Löhner scheme (Löhner 1987) to estimate the errors. Additionally, after defining the MRI quality factor as the ratio of the fastest growing MRI mode to the resolution in the locally non-rotating reference frame (LNRF)

\[ \mathcal{Q}^{(\theta)} := \frac{2\pi b^{\theta} \dot{e}_{\mu}^{(\theta)}}{\sqrt{(\rho h + b^2) (u^\theta |u|^2)}} \frac{1}{\Delta \theta \dot{e}_{\mu}^{(\theta)}}, \quad \Delta \theta := (0, 0, \Delta \theta, 0), \] (17)

where \( \Delta \theta \) is the distance between two adjacent grids along the \( \theta \)-direction and \( \dot{e}_{\mu}^{(\theta)} \) are the orthonormal bases of the LNRF, we refine the grid wherever \( \mathcal{Q}^{(\theta)} < 5 \) (Sano et al. 2004; Noble et al. 2010; McKinney et al. 2012).

Unlike black holes, whose event horizon acts as an absorbing null surface, neutron stars have a physical timelike surface that needs to be modeled suitably. A first possibility is to describe the neutron star as a self-gravitating fluid and to describe the interaction of the accretion flow onto the star as the interaction between two fluids. This approach has a long history in numerical relativity (see, e.g., Font et al. 2002; Baiotti et al. 2005) but is not particularly convenient here for at least two reasons. Firstly, it leads to very small timesteps in the central cells of the star, which have a very small volume and a rather simple dynamics. Secondly, the interaction of the accretion fluid with the stellar surface and the large differences in their rest-mass densities
can lead to strong shocks in energy and velocity triggering numerical artefacts.

A second possibility, and the one employed here, is to treat the stellar surface as a surface across which suitable boundary conditions are imposed. In particular, following Abarca et al. (2018) and Takahashi et al. (2018), at the inner boundary of the radial domain, we impose a “reflective” boundary condition on the radial (and hence normal) component of the fluid three-velocity, while no change is made to the tangential components for conservation of linear momentum. More specifically, we set

\[ v^r_{\text{in}} = -v^r_{\text{out}}, \quad v^\theta_{\text{in}} = v^\theta_{\text{out}}, \quad v^\phi_{\text{in}} = v^\phi_{\text{out}}, \quad (18) \]

where the subscripts “in” and “out” refer to the innermost cell in the domain and to the ghost cells, respectively. On the other hand, “continuous” boundary conditions – where the value of the innermost cell is copied to all the ghost cells – are applied to all components of the magnetic field

\[ B^i_{\text{out}} = B^i_{\text{in}}(s_{\text{min}}), \quad (19) \]

and “symmetric” boundary conditions are used for all the primitive variables

\[ \psi_{\text{out}} = \psi_{\text{in}}, \quad (20) \]

where \( \psi \) represents a generic scalar primitive variable. Furthermore, reflective boundary conditions are also applied on the conservative variables at the poles \( \theta = 0 \) and \( \theta = \pi \), i.e.,

\[ B^r_{\text{out}} = B^r_{\text{in}}, \quad S^r_{\text{out}} = S^r_{\text{in}}, \quad (21) \]

\[ B^\theta_{\text{out}} = -S^\theta_{\text{in}}, \quad S^\theta_{\text{out}} = -S^\theta_{\text{in}}, \quad (22) \]

\[ B^\phi_{\text{out}} = B^\phi_{\text{in}}, \quad S^\phi_{\text{out}} = S^\phi_{\text{in}}, \quad (23) \]

\[ \chi_{\text{out}} = \chi_{\text{in}}, \quad (24) \]

where \( S^i \) is the Eulerian momentum density and \( \chi \) represents a generic scalar conserved variable (Porth et al. 2017; Olivares et al. 2019). Finally, “outgoing” boundary conditions are applied to the primitive variables at the outer radial boundary i.e.,

\[ v^r_{\text{out}} = \max\left(0, v^r_{\text{in}}(s_{\text{max}})\right), \quad v^\theta_{\text{out}} = v^\theta_{\text{in}}(s_{\text{max}}), \quad v^\phi_{\text{out}} = v^\phi_{\text{in}}(s_{\text{max}}), \quad (25) \]

\[ B^r_{\text{out}} = B^r_{\text{in}}(s_{\text{max}}), \quad B^\theta_{\text{out}} = B^\theta_{\text{in}}(s_{\text{max}}), \quad B^\phi_{\text{out}} = B^\phi_{\text{in}}(s_{\text{max}}) \quad (26) \]

\[ \psi_{\text{out}} = \psi_{\text{in}}(s_{\text{max}}), \quad (27) \]

As a concluding remark, we note that all of the simulations performed have been evolved up to a time \( t = 75000 M_{\odot} \approx 370 \text{ ms} \approx 3.39 \tau_0 \) where \( \tau_0 \) is the initial orbital period at \( r_{\text{max}} \).

3 RESULTS

In what follows we discuss in detail the various aspects of our simulations, concentrating first on the plasma dynamics, so as to move, later on, on the properties of the angular velocity in the accreting flow and of the pitch factor across the torus.

3.1 General plasma dynamics

Driven by MRI, the matter in the torus flows towards the star advecting with it the magnetic field lines. From time to time, and responding to the turbulent nature of the accretion process, the inflow is halted by the concentration of magnetic field lines that accumulate around the magnetospheric radius. However, as additional matter continues to inflow and accumulates, the magnetic-pressure barrier is overcome, restoring the accretion onto the star. This final part of the accretion process – namely the one made by the plasma before reaching the stellar surface – does not proceed along the equatorial plane, but along the magnetic tubes produced by the most external magnetic field lines of the stellar magnetosphere (see Figs. 1 and 2), which become considerably stronger near the star and thus channel the matter from the accretion torus onto the stellar poles.

This process of mass transfer from the torus to the stellar surface starts well before the bulk of the torus has reached the magnetospheric radius and a quasi-stationary configuration is reached. Indeed, with the exception of the case with a weak stellar magnetic field, i.e., for \( B_0 = 1 \), the matter of the torus reach the stellar surface only along the flux tubes in the polar regions and not with a direct accretion along the equatorial plane. As a representative example, snapshots from the simulation with \( B_0 = 5 \) are presented in Figs. 1 and 2, whose different columns refer to snapshots at different times during the evolution, while the different rows report the rest-mass density (first row from the top), the magnetisation (second row), the Bernoulli parameter (third row) and the radial component of the three velocity (fourth row).

Together with the inflow, the dynamics of the torus is accompanied by a net outflow of matter, some of which is actually gravitationally unbound and launched at large distances. The ejection of matter can take place for at least three different reasons. First, as mentioned above, a significant amount of magnetic reconnection takes place at the outer parts of the stellar magnetosphere and around the equatorial plane. The consequent conversion of magnetic energy into internal energy can be so efficient (see third rows in Figs. 1 and 2) such that the matter in the torus becomes gravitationally unbound. Second, matter becomes heated as it is channelled in the tight flux tubes on the stellar poles, again reaching internal energies that can, episodically, lead to matter ejection. Finally, some of the accreting matter is simply reflected from the stellar surface, especially when the density in the polar flux tubes decreases because of a smaller accretion rate and the matter can freely fall onto the star. Under these conditions, strong shocks can develop at the stellar surface triggering powerful outflows with velocity \( > 0.1 \). Clearly, given the nature of these processes, which are basically triggered by stochastic magnetic reconnection, the ejection of matter is rather episodic and it is the most copious in the time window \( \sim 200 - 300 \text{ ms} \), that is, in the transition between the initial inflow and the reaching of a stationary solution. Overall, we do not find a periodic or quasi-periodic behaviour in the mass-ejection process in our simulations.

A couple of remarks are useful at this point. First, we observe that some of the matter escapes from the flux tubes as a result of magnetic reconnection and forms magnetic islands (plasmoids). The presence of these plasmoids can be appreciated, in particular, in the third-column, third-row panels of Figs. 1 and 2, where these magnetic islands are particularly visible. Although the magnetisation of these magnetic islands is larger than that of the rest of plasma, it is nevertheless modest and of order unity. Nevertheless, these structures share many of the properties of the plasmoids found in accreting supermassive black holes (Nathanail et al. 2020, 2021). Second, because the stellar magnetic field around the torus is weaker than the one initially seeded in the torus, the latter expands due to magnetic buoyancy as the simulation proceeds, giving rise to a low-density magnetised plasma, i.e., a “corona” (see top rows of Figs. 1 and 2). The plasma in this corona can then be energised by the matter.
reflected off the stellar surface and thus become unbound, leaving the system (see bottom rows of Figs. 1 and 2).

To determine whether a fluid element is unbound we use the so-called “Bernoulli parameter”, which we choose to set to $B_e := -(1 + hu_t)$, so that fluid elements with a positive Bernoulli parameter can escape to infinity$^2$. Given the duality in the inflowing and outflowing

$^2$ We note that several definitions are possible for the Bernoulli parameter (see Bovard & Rezzolla 2017, for a discussion of different measurements)
material, we measure the “inflowing” mass-accretion rate in terms of the outflowing material in a numerical simulation) and that numerical simulations of accretion flows either neglect inertial terms using a purely kinematic criterion (see, e.g., Porth et al. 2019a), or add contributions from the magnetic field (see, e.g., Narayan et al. 2012). Because both approaches tend to normally increase the amount of ejected matter, we here take a rather conservative view and thus employ a definition that takes into account inertial terms but only at the hydrodynamical level

\[ M_{\text{in}} := -2\pi \int \rho u' \sqrt{\frac{\rho}{\rho g}} d\theta , \]  

(28)

where the integrand refers to matter with \( u' < 0 \). Using similar filtering criteria, the rate of inflowing angular momentum is defined
Figure 3. Mass-accretion rate (left panel) and angular-momentum transport rate (right panel) for all simulations calculated at $r = 1.1 R$. They are presented in dimensionless units as given in Eq. 13 and their running averages are taken in 50 ms window.

Figure 4. Left panel: Evolution of the magnetospheric radius of the torus in time. The horizontal dashed lines denote the inner radius of the torus in steady state. Right panel: The magnetospheric radius vs. the best-fitting estimate of the magnetic dipole moment over the mass accretion rate. The top axis corresponds the magnetic field strength of the each simulation. The red points are the average of the last 20 ms of the inner radius and error bars denote the maximum and minimum values in last 20 ms. The black line is the best-fitting of the magnetic radius. The result for the $B_9 = 1$ simulation is excluded from the fitting since the torus extends to the star in this case.

As

$$J_{\text{in}} := -2\pi \int T_{\phi} \sqrt{-g} \, d\theta$$

$$= -2\pi \int \left( (\rho u^\tau + b^\tau) u_\phi - b^\tau b_\phi \right) \sqrt{-g} \, d\theta,$$  \hspace{1cm} (29)

$$J_{\text{in,matt}} := -2\pi \int \rho u^\tau u_\phi \sqrt{-g} \, d\theta,$$  \hspace{1cm} (30)

where $T_{\phi}$ is the only relevant component of the energy-momentum tensor and $J_{\text{in,matt}}$ refers to the portion of the angular-momentum transfer rate related to the matter only.

When inspecting the evolution of the mass-accretion rate at a high cadence we observe that it exhibits large fluctuations around an average, but with secularly varying value. These fluctuations reflect the turbulent and chaotic nature of the accretion flow, which is triggered both by the development of the MRI and by the reconnection processes taking place at the edge of the stellar magnetosphere.

Figure 3 reports the evolution of the inflowing mass-accretion rate.
larger for more copious mass-accretion rates – and the accretion of nonrotating. However, we measure both the mass-accretion rate – and we measure the spinup of the star, which is always treated as luminosity. In our simulations we do not compute the X-ray luminosity. In such sources, the accretion rate of angular momentum. The existence of a tight correlation between the mass-accretion rate and the stellar magnetic field, but also on whether a quasi-stationary equilibrium between the accretion flow and the matter ejection is reached. Hence, we measure the magnetospheric radius, which is always treated as nonrotating. However, we do measure both the mass-accretion rate – which is naturally associated to the X-ray luminosity, the latter being larger for more copious mass-accretion rates – and the accretion of angular momentum, which can naturally be associated with changes in the spin of the star. Indeed, using the angular-momentum transfer rate measured from the simulations, we estimate that the changes induced in the stellar spin would be of the order of \( \dot{\Omega} \approx 10^{-12} \text{ Hz s}^{-1} \) for \( B_0 = 10 \), which is in very good agreement with the spin-up rates measured in observations, i.e., \( \dot{\Omega} \approx 10^{-12} - 10^{-11} \text{ Hz s}^{-1} \) (Sugizaki et al. 2017). Hence, it is reasonable to assume a correlation of the type

\[
J_{\text{in, acc}} \propto m^{\lambda}_{\text{in}},
\]

where the exponent \( \lambda \) has been estimated in various analytical models and, for instance, is set to \( \lambda = 0.86 \) in the Ghosh & Lamb model (Ghosh & Lamb 1979b), \( \lambda = 0.9 \) in Kluźniak & Rappaport (2007) and \( \lambda = 0.64 \) in Shakura et al. (2012). We note that here we consider only the matter part of the angular-momentum flux, thus neglecting the magnetic contributions to the total torque. The latter, in fact, are produced at the edge of the magnetosphere and, being mostly stochastic in nature, are not correlated with the mass-accretion rate and thus cannot be modelled with the simple ansatz given in Eq. (31).

Figure 5 reports the behaviour of the mass-accretion and of the angular-momentum transfer rates for the representative case with stellar magnetic field \( B_0 = 5 \) and relative to the quasi-stationary part of the evolution, i.e., for \( t > 300 \text{ ms} \); very similar correlations are found in all cases considered. As suggested by Eq. (31), Fig. 5 shows a clear nonlinear correlation between these two quantities, highlighting that not only the steady-state accretion of matter should lead to a steady variation of the stellar spin, but also that fluctuations in \( J_{\text{in,acc}} \) are directly related to fluctuations in \( m_{\text{in}} \). This behaviour provides strong support to the idea that the episodic reconnection processes taking place at the edge of the magnetosphere should lead both to an increased luminosity and to a stellar spin-up.

Using the various simulations performed, we have estimated the values of the correlation exponent \( \lambda \), after removing from our data sets those variations\(^4\) in the mass-accretion rate or in the angular-momentum transfer rate that are larger by a factor of seven with respect to the corresponding average values. In this way we can filter out the most extreme fluctuations and find that the exponent varies in the range \( 0.93 \leq \lambda \leq 1.43 \), with an average value of \( \langle \lambda \rangle = 1.18 \) and no clear correlation with the value of the stellar magnetic field. Interestingly, this result is in very good agreement with the analysis of 12 X-ray sources reported by (Sugizaki et al. 2017), where \( \lambda \) is estimated to be 1.03.

\[\text{3.2 Inner edge of the torus and magnetospheric radius}\]

A very important quantity in our analysis is the determination of the magnetospheric radius \( r_{\text{msph}} \); namely of the radius at which the disc accretion on the equatorial plane, is diverted into the polar flux tubes reaching the stellar surface. Clearly, the location of the inner radius \( r_{\text{in}} \) where this process takes place will depend not only on the strength of the stellar magnetic field, but also on whether a quasi-stationary equilibrium between the accretion flow and the matter ejection is reached. Hence, we measure the magnetospheric radius as the asymptotic value attained by the inner radius when a quasi-stationary flow has been established. Stated differently, we define the magnetospheric radius as \( r_{\text{msph}} = \lim_{t \to r \to r_{\text{in}}} r_{\text{in}} \) and monitor the evolution of the inner radius till reaches a stationary value.

There are several different ways in which the inner radius can be

\[\text{\(3\)}\]

We recall we have considered ten different magnetisations, i.e., \( B_0 = 1 - 10 \), but report only five in Fig. 3.

\[\text{\(4\)}\]

We note that there is no excluded data for our representative example \( B_0 = 5 \) shown in Fig. 5.
extracted from the numerical simulations and this is because the edge of the magnetosphere is always very straightforward to determine when looking either at the rest-mass density (see Figs. 1 and 2) or at the velocity field (see Fig. 6). Out of these possibilities — all of which yield very similar results — we adopt the simplest one, namely, we set the inner radius to be the location where, moving along the equatorial plane out from the stellar surface, the rest-mass density reaches the threshold value $\rho_{\text{th}} = 10^{-2} \rho_{\text{max}}$. This radius obviously changes with time and we report in the left panel of Fig. 4 its evolution for the same representative values of the magnetic field shown in Fig. 3. Using the values of $r_{\text{in}}$, we therefore calculate the magnetospheric radius by taking the corresponding value when averaged over the last 20 ms of the simulation (see left panel of Fig. 4).

The importance of the magnetospheric radius is that its value can be estimated already in Newtonian gravity and is given by

$$r_{\text{msph}} = \xi \left( \frac{M^4}{8 \dot{m} \dot{m}_{\text{in}}} \right)^{1/7},$$

where $\mu$ is the magnetic moment of a star mass with mass $M$ and subject to a mass-accretion rate $\dot{m}$. In expression (32), $\xi$ is a dimensionless coefficient of order unity whose value depends on the assumptions about the details of the disc-magnetosphere interaction, i.e., the width of the zone where the magnetic field of the star can penetrate the disc and the physical processes limiting the growth of the toroidal field generated by the stellar field lines penetrating the disc and the physical processes limiting the growth of the zone where the magnetic field of the star can penetrate the disc and the physical processes limiting the growth of the zone where the magnetic field of the star can penetrate the disc and the physical processes limiting the growth of the zone where the magnetic field of the star can penetrate the disc. For instance, $\xi = 0.5$ in the model of Ghosh & Lamb (Ghosh & Lamb 1979a), while it is estimated to be $0.3 \sim 1.2$ in other studies (Wang 1987, 1996; Psaltis & Chakrabarty 1999; Erkut & Alpar 2004; Dall’Osso et al. 2016). Moreover, $\xi$ might depend on the ratio $r_{\text{msph}}/r_{\text{co}}$, where $r_{\text{co}} := (M/\Omega_z^2)^{1/3}$ is the corotation radius, with $\Omega_z$, the angular velocity of the star, and on the inclination angle between rotation and magnetic axis (Bozzo et al. 2018).

In order to verify whether expression (32) is valid also in a non-trivial and general-relativistic context, we have a more generic expression of the magnetospheric radius, which includes information on the mass-accretion rate, via a power-law of the type

$$r_{\text{msph}} = a_1 \left( \mu_{27} \right)^{a_2} \left( \dot{m}_{\text{in}} \right)^{a_3} \left( \frac{M}{M_\odot} \right),$$

where the inflow mass-accretion rate $\dot{m}_{\text{in}}$ has been measured at $r = 1.1 \: R$ and time-averaged over the last 110 ms. We have reported in the right panel of Fig. 4 the measured values of the magnetospheric radius and estimated the fitting coefficients via the ansatz (33). In this way, we have found that

$$a_1 = 7.66 \pm 1.6, \quad a_2 = 0.59 \pm 0.06, \quad a_3 = -0.08 \pm 0.06.$$ (34)

Interestingly, while the magnetospheric radius depends on the strength of the stellar magnetic field with the almost same power-law as in the Newtonian expression (the latter predicts that $a_2 \approx 4/7 \approx 0.571$), we find that the correlation between the mass-accretion rate and the location of the magnetospheric radius is a weaker than in the Newtonian estimate (the latter predicts that $a_3 \approx -2/7 \approx -0.286$). More importantly, however, the relevance of expressions (33) and (34) is that they do not depend on the idealised, stationary, axisymmetric assumptions behind the Newtonian expression (32). Rather, they reflect the quasi-stationary accretion flow that may be present near an accreting pulsar, where all quantities have a fully turbulent nature and the magnetic field does not have a simple power-law scaling.

In Appendix A we will provide an analytic expression for the magnetospheric radius for a spherically symmetric accretion flow onto a magnetised star up to the second order in the relativistic corrections $\mathcal{O}(M/r)$. Also in this case we find that $r_{\text{msph}} \approx \mu^{4/7}$, but also that the magnetospheric radius is smaller than the Newtonian counterpart for the same set of parameters [see Eq. (A8)]. This is indeed rather natural as the general-relativistic gravity will be more intense and require a comparatively larger magnetic field to push out the magnetospheric radius.

Figure 5. The matter part of the angular momentum transport rate $j_{\text{mat.,in}}$ [see Eq. (31)] shown as function of the mass-accretion rate for the $B_9 = 5$ simulation.

Figure 6. The ratio of the square of the poloidal velocity $v_2^\ast := v^\prime v_\parallel + v^\prime v_\perp$ to the poloidal Alfvén velocity [Eq. (35)] (top row), and the Eulerian velocity $v^a = v^\prime v_\parallel$ to the free-fall velocity [Eq. (36)] (bottom row) at the final time of three representative simulations. The dashed vertical lines mark the magnetospheric radii in each simulation.
3.3 Angular velocities in the torus

Although the flow inside the torus is nonrelativistic, matter in the outer parts of the corona can reach quite high speeds, i.e., $\gtrsim 0.1 c$ (see Figs. 1 and 2). A convenient way to present the velocity field is to compare it with two other characteristic velocity scales, namely, the poloidal Alfvén velocity

$$v_{\text{Alf},p} := \sqrt{\frac{B^\prime B_x + B^0 B_0}{\rho}},$$

and the free-fall velocity in a spherically symmetric metric of mass $M$ (Rezzolla & Zanotti 2013)

$$v_{\text{ff}} := \sqrt{\frac{2M}{r \left(1 - \frac{2M}{r}\right)}},$$

Figure 6 provides a view of the velocity field in terms of the square of the poloidal velocity, $v_p^2 := v^r v_r + v^\theta v_\theta$, normalised to the poloidal Alfvén velocity (top row) and the Eulerian velocity, $v^2 := v^i v_i$, normalised to the free-fall velocity (bottom row). Note that the flow is super-Alfvénic (and supersonic) up to the interaction between the accretion flow and the magnetosphere, and it becomes sub-Alfvénic (and subsonic) inside the funnel, where however the matter content is very small. Similarly, matter in the torus and near the equatorial plane moves towards the stellar surface with speeds larger than the free-fall velocity. On the other hand, a significant acceleration is experienced by the plasma as it reaches the stellar surface along the funnel walls.

Also interesting is to consider the radial profiles of the angular velocity in the torus. While the motion there is very turbulent, it is nevertheless possible to perform a polar average defined as

$$\langle \Omega \rangle_p(r) := \frac{\int \Omega(r,\theta) \sqrt{g} d\theta}{\int \sqrt{g} d\theta},$$

where, in order to exclude any contribution from the corona, we have restricted the integrals to a wedge of $\sim 20^\circ$ around the equatorial plane and to densities in the range $\rho/\rho_{\text{max}} > 10^{-2}$. The corresponding angular velocities at the final moments of the simulation are presented in Fig. 7 for some of the representative stellar magnetic-field strengths. Note that, in all cases, the averaged angular velocity profiles have a global maximum near to the inner edge of the torus.

Following Ghosh et al. (1977), we define the transition region as the region between the magnetospheric radius and the location where the stellar dipolar magnetic field modifies the angular velocity away from a Keplerian profile. Using this definition across the various cases of stellar magnetic fields, we find that the width of the transition region varies between 2 and 9 km. More importantly, we find that the width of the transition region is not correlated with the strength of the stellar magnetic field. Not surprisingly, given the turbulent nature of the accretion flow and the sharp transition between the torus and the corona, the angular velocity can become locally negative (see Fig. 7). However, this behaviour is not stationary and over a time average the $\theta$-averaged angular velocity in Eq. 37 is always positive at all latitudes considered.

Although no reflection symmetry is imposed in the simulations and the accretion flow is highly turbulent, matter is channelled onto almost symmetrical locations at the north and south hemispheres of the star, i.e., at latitudes of $15^\circ \sim 31^\circ$ and $163^\circ \sim 152^\circ$, respectively. The exact position of these “hot spots” obviously depends on the strength of the stellar dipolar magnetic field, so that the elevation increases with the stellar magnetic field; this is to be expected when considering the Newtonian estimate of the boundary of the polar cap $\theta_c$, i.e., $\sin \theta_c = \sqrt{R/r_{\text{msph}}}$; note that the Newtonian estimate for $\theta_c$ is $25 - 45\%$ larger than what we measure in our general-relativistic calculations. At the same time, the fraction of the stellar surface where matter can accrete becomes progressively smaller as the stellar magnetic field increases. More specifically, the fraction of the stellar surface area interested by the accreting plasma can be large as 28% for the $B_9 = 2$ models, while it reduces to only 5% when considering the $B_9 = 10$ model.

3.4 Secular toroidal magnetic field and pitch factor

We have already discussed how the development of the MRI is responsible for the transport of angular momentum and the accretion of matter towards the stellar magnetosphere. Also, while the initial magnetic field is purely poloidal, the turbulent motion produced by the MRI generates, as a result of the large conductivity of the plasma, a toroidal magnetic field on those length scales over which the instability develops. This is a well-known process (Balbus & Hawley 1998), which has been studied in great detail in the accretion onto supermassive black holes (see, e.g., Porth et al. 2019b). However, accretion onto a magnetised star introduces an important difference with respect to the accretion onto a black hole, and this is in the generation of a globally coherent, secular toroidal magnetic field in the disc. The source of this difference is indeed the presence of a stellar magnetic field, which is clearly absent in the case of a neutrally charged hole, and that add to the locally turbulent toroidal magnetic field and is stretched in the azimuthal direction by the global rotation of the accretion disc (see Rezzolla et al. 2000, for an analogous process in the case of a $r$-mode unstable neutrons star).

The growth of this magnetic field can be easily deduced from
the Newtonian expression of the induction equation for the toroidal magnetic field that, near the equatorial plane, where the radial component of the magnetic field is much smaller than the polar one, i.e., \( B^r \ll B^\theta \), reads
\[
\langle \partial_t B^\theta \rangle_t = \frac{1}{r} \langle \partial_\theta (\nu B^\theta) \rangle_t ,
\]
where the \( \langle \cdot \rangle_t \) brackets indicate a time average. Hence, toroidal magnetic field is produced by the shearing of the poloidal magnetic field, grows linearly in time, at least in the ideal-MHD approximation of infinite conductivity, and changes polarity across the equator.

In the magnetically threaded disc model (Ghosh & Lamb 1979a; Wang 1987; Campbell 1992), the toroidal magnetic field is generated as a result of the angular velocity difference between the star, \( \Omega_s \), and the rotating plasma \(^5\), hence, the field is generated at a rate,
\[
\left\langle \partial_t B^\theta \right\rangle_t = \frac{1}{r} \left\langle \frac{\partial \theta}{\partial t} (\nu B^\theta) \right\rangle_t \propto \pm (\Omega - \Omega_s) B^\theta ,
\]
where “\( \pm \)” arises due to the vertical gradient switching sign above and below the disc midplane.

The generated toroidal magnetic field is illustrated in Fig. 8, which reports a snapshot of the toroidal magnetic field at time \( t = 369 \) ms (left panel) normalised to the initial stellar value, but also its time integration over the last 10 ms (right panel). Clearly, the instantaneous toroidal magnetic field changes polarity also on small scales and does not show a globally coherent structure. This behaviour is very similar to that produced in simulations of accretion discs onto black holes (see, e.g., Nathanael et al. 2020) and onto magnetised neutron stars (Naso & Miller 2011). However, when averaged over sufficiently long timescales, it is possible to appreciate also the appearance of a clear polarity change across the equatorial plane.

\(^5\) The mentioned models give the steady-state toroidal magnetic field by the balancing of the advection and diffusivity terms in the induction equation. Yet, we solve the GRMHD equations within the ideal-MHD approximation, therefore, there is no mechanism to limit the growth of the toroidal magnetic field in our numerical setup.

Another useful diagnostic of the properties of the accreting plasma is the so-called “pitch factor”, namely, the polar-averaged ratio of the toroidal magnetic field to the poloidal magnetic field. Here we compute this quantity at \( r = 81 \text{ km} \)
\[
\langle \gamma_{\phi} \rangle_{\theta} (t) := \frac{\int B^\phi (B^r B_\theta + B^\theta B_\phi)^{-1/2} \sqrt{\rho} \, d\theta}{\int \sqrt{\rho} \, d\theta} ,
\]
where for symmetry reasons the average is restricted to the upper hemisphere, i.e., \( 0 < \theta < \pi/2 \), and to densities in the range \( \rho/\rho_{\text{max}} > 10^{-2} \).

Figure 9 reports in its left panel the evolution of the pitch factor over 50 ms, together with the time-averaged value marked with a red dashed line. Note that the pitch factor is essentially zero before a steady-state accretion is established and that it fluctuates significantly in value, sometimes changing sign as a result of the turbulent nature of the accretion flow; furthermore, the fluctuations in time are significantly larger than its average value. Overall, when averaged over the time window shown in the left panel of Figure 9, the pitch factor is positive with a value \( \langle \gamma_{\phi} \rangle_{\theta} \approx 0.06 \).

The right panel of Fig. 9, on the other hand, reports the radial profile of the time-averaged pitch factor (right). As in previous figures, the shaded area marks the maximum and the minimum values of the pitch factor at a given radius over the last 50 ms. Alternative definitions of the pitch factor – e.g., computed in terms of the magnetic fields measured in the frame comoving with the fluid – do not change the qualitative behaviour of the pitch factor reported in Fig. 9, but do change its range, which can vary of two orders of magnitude when applied to our simulations (see Hawley et al. 2011, and references therein).

Simple models, as those proposed by Ghosh & Lamb (1979a); Campbell (1992); Wang (1995), have suggested a simple relation between the pitch factor and the angular velocity of the accreting plasma.
flow $\Omega$, namely, that $\gamma_\phi \propto (\Omega - \Omega_K)/\Omega_K$. As result, the pitch factor should undergo a sign change at the corotation radius, where the angular velocity of the plasma equals to the spin of the star. Because our simulations consider nonrotating stars, we are unable to verify this prediction, but we can nevertheless explore the relation between the pitch factor and the angular velocity of the accreting flow.

This is done in Fig. 10, which shows the relation between the polar-averaged pitch factor and the polar-averaged angular velocity, where the latter is normalised to the Keplerian angular velocity, i.e., $(\Omega)_r/r/\Omega_K$. Because the accretion flow is far from being laminar near the magnetospheric radius and the angular velocity of the accreting plasma becomes Keplerian only at sufficiently large distances (see Fig. 7), we report in Fig. 10 the pitch factor between an inner radius coinciding with the magnetospheric radius $r_{\text{msh}} = 60$ km (marked with a black filled circle) and up to an outer radius at 140 km.

Clearly, this relation should be a constant according to the analytic models (Ghosh & Lamb 1979a; Campbell 1992; Wang 1995), but this is not what the simulations actually reveal. This is expected since the ideal-MHD approximation in our setup does not allow the stellar magnetic field to penetrate the disc, in contrast with what assumed in the models mentioned above. In particular, for values of $(\Omega)_r/r/\Omega_K \lesssim 0.75$, which corresponds to the inner parts of the accretion flow and essentially from the local maximum of the angular velocity in Fig. 7 down to the magnetospheric radius, the pitch factor is smoothly correlated with $(\Omega)_r/r/\Omega_K$. It is zero at the magnetospheric radius and becomes increasingly negative when moving outwards, reaching a minimum of $(\gamma_\phi)_{r,t} \approx -0.1$ for $(\Omega)_r/r/\Omega_K \approx 0.75$. This is not surprising as in these regions of the flow the angular velocity is highly turbulent and rather different from the Keplerian one. However, for $(\Omega)_r/r/\Omega_K \gtrsim 0.75$, the pitch factor rapidly changes sign, becoming positive and reaches values of the order of $(\gamma)_r \approx 0.2$ in the bulk of the flow. Finally, we note that we find a systematic and long-lasting sign change in the radial profile of the pitch factor (see right panel of Fig. 9) even though this is not contemplated in the stationary analytic models; this clearly points out to the inability of such models to capture the highly dynamical flows encountered in our simulations.

We conclude this section by exploring the properties of the turbulence in our simulations that, as discussed earlier, is generated by the development of the MRI. In the classical Shakura-Sunyaev discs, on the other hand, the molecular turbulent viscosity is expressed in terms of the shear-viscosity coefficient $\nu := \tilde{c}_5 H$ (Shakura & Sunyaev 1973), where $c_s$ is the speed of sound, $H$ is the vertical scale-height of the disc, and $\tilde{\alpha}$ – also referred to as the “alphaviscosity” parameter – is an unknown dimensionless coefficient to be determined by the observations. Typical values for the dimensionless constant needed to reproduce to some extent the astronomical observations are $\tilde{\alpha} \approx 0.1 - 0.4$ (see, e.g., King et al. 2007; Martin et al. 2019).

In our simulations, we can associate and measure the alpaviscosity parameter in terms of the ratio of the Maxwell stresses and of the total pressure as measured in the frame comoving with the fluid, namely, we define the polar-averaged MRI-driven alpaviscosity parameter $\langle \tilde{\alpha} \rangle_\theta$ (Pessah et al. 2006; Shafee et al. 2008; Porth et al. 2019a) as

$$\langle \tilde{\alpha} \rangle_\theta := \frac{\int (b^2/b' \sqrt{\rho p_{\text{rad}}}) \sqrt{\frac{\rho}{ho_{\text{msh}}}} d\theta}{\int (p + b^2/2) \sqrt{\frac{\rho}{ho_{\text{msh}}}} d\theta},$$

where the integral is restricted to to densities in the range $\rho/\rho_{\text{msh}} > 10^{-2}$.

We report in Fig. 11 the evolution of $(\alpha)_\theta$-viscosity parameter at a fixed radius of 81 km as well as with its variance in space (left panel), and its radial profile averaged over the last 50 ms, together with its fluctuations (right panel). Both panels refer to the $B_0 = 5$ model and indicate that the time-averaged value of the MRI-driven alpaviscosity is $(\alpha)_\theta = 0.11$ at $r = 81$ km.

Clearly, our simulations reveal that $(\tilde{\alpha})_\theta$ fluctuates significantly both in space and time. While this was already suggested by Lyubarskiï (1997), it points out to important differences between an MRI-driven alpaviscosity – which is unsteady in space and

Figure 9. Evolution of the pitch factor (left panel) and radial profile of the time-averaged pitch factor (right panel). As in previous figures, the shaded region marks the maximum and the minimum values of the pitch factor measured on a radial shell at 81 km and over a timescale of 50 ms; the red-dashed line denotes the time-averaged value. For symmetry reasons, the polar average is restricted to the upper hemisphere, i.e., $0 < \theta < \pi/2$.

Figure 10. The pitch factor as a function of the angular velocity scaled by the Keplerian angular velocity. Both quantities are averaged over the time (last 50 ms) and $\theta$ (between 0 and $\pi/2$).
time – and the alpha-viscosity used in the simulations of Romanova et al. (2002) – which is instead constant in space and time. On the other hand, similar results have been presented by Romanova et al. (2011), who report an MRI-driven alpha viscosity in range of $\langle \alpha \rangle_\theta \approx 0.01 - 0.1$.

4 DISCUSSION AND CONCLUSION

We have reported 2D general-relativistic magnetohydrodynamics (GRMHD) simulations of matter being accreted onto and ejected by a magnetised and nonrotating neutron star. The dynamics is followed within the ideal-MHD limit and making use of the numerical code BHAC.

Employing a number of simulations and considering various strengths of the stellar dipolar magnetic field, we have determined self-consistently the location of the magnetospheric radius $r_{\text{msph}}$, and study how it depends on the magnetic moment $\mu$ and on the accretion rate. Overall, we recover the analytic Newtonian scaling relation, i.e., $r_{\text{msph}} \propto B^4/\lambda$, confirming the behaviour explored by Parfrey & Tchekhovskoy (2017). At the same time, we find that the correlation with the mass-accretion rate is different from the Newtonian expectation, i.e., $r_{\text{msph}} \propto m^{-2/7}_{\text{in}}$, and recently observed in the Newtonian simulations of Ireland et al. (2022). The weaker correlation with the accretion rate found here, i.e., $r_{\text{msph}} \propto m^{-1/12}_{\text{in}}$, is unlikely to be due to our general-relativistic framework, which we show in Appendix A to provide only second-order corrections to the magnetospheric radius. Rather, we believe that the turbulent nature of the accreting flow produced in our simulations weakens the dependence of the magnetospheric radius on the accretion rate, leaving the magnetic field as the main regulator of its location.

As is natural to expect, the fluctuations in the mass-accretion rates are accompanied by the fluctuations in the matter-part (i.e., independent of the magnetic-field strength) of the angular-momentum transport rate. Our simulations exhibit a clear correlation between the mass-accretion rate and the matter-part of the rate of transport of angular momentum. This correlation supports the idea that the episodic reconnection processes taking place at the edge of the magnetosphere should lead both to an increased luminosity and to a stellar spin-up. Interestingly, when expressing this correlation as $J_{\text{matt,in}} \propto m^{4/7}_{\text{in}}$, we can estimate the exponent of the correlation to be $\langle \lambda \rangle \approx 1.18$, which is in good agreement with the analysis of 12 X-ray sources reported by Sugizaki et al. (2017).

We note that in our simulations the total angular-momentum transport rate exhibits large fluctuations that become larger as the magnetic field of the star is increased. These fluctuations are due to the complex plasma dynamics taking place at the magnetospheric radius and can be so intense that can lead to a change of sign in the total angular-momentum transport rate and could therefore lead to a spin-down of the accreting star. Although these sign changes occur over timescales that are much shorter than those measured in the observations, the phenomenology produced in our simulations may have implications on the torque reversals or on the noise in the measured spin frequencies in the case of slowly rotating stars. In addition, we have investigated the behaviour of the pitch factor and found it to be quite different from what is expected in simplified models of magnetically-threaded discs. More specifically, we have observed that the pitch factor undergoes significant fluctuations in time and space – sometimes undergoing sign changes – as a result of the turbulent nature of the accreting flow. These differences are not surprising given the ideal-MHD assumption under which our simulations are carried out, which prevents the stellar magnetic field from penetrating the accreting plasma. Yet, the significant differences – both qualitative and quantitative – found in the properties of the pitch factor casts doubts on the effectiveness of this diagnostic quantity in characterising the properties of the accreting flow.

Finally, our results confirm the findings of Romanova et al. (2011) in that discs with MRI-driven turbulence have substantial differences when compared to discs with constant $\alpha$-viscosity parameter. However, when expressing this parameter in terms of the ratio of the Maxwell stresses and of the total pressure as measured in the frame comoving with the fluid, it exhibits large fluctuations both in space and in time, but these average to values of $\alpha \sim 0.1$.

Future work will improve on several aspects of these simulations. First, while the numerical resistivity of our ideal-MHD simulations is very small and unable to change the bulk properties of the accretion flow (e.g., the diffusion of the stellar magnetic field lines in the accretion disc), it is important to assess how resistive effects impact on the dynamics described here and on the scaling relations found. To this scope, we will introduce a physical model of resistivity along the lines of similar simulations carried out by Ripperda et al. (2020) and Nathanail et al. (2021). This will allow us to investigate under more realistic conditions the generation of the toroidal magnetic field and its effect on the torque exerted onto the star. Second, the 2D nature of our simulations has the consequence that turbulence and magnetic fields intrinsically decay unless a proper dynamo mechanism is implemented (Sadowski et al. 2015), and the non-axisymmetric motion of the fluid that can create differences in the dynamics of the accretion (Igumenshchev 2008; Romanova et al. 2012; McKinney et al. 2012) is ignored. Hence, we will reconsider the ideal-MHD scenarios explored here also in fully three-dimensional (3D) simulations.
to better investigate the interaction between the magnetosphere and the accretion disc, our investigations (either in 2D or in 3D) will be extended by considering the impact of stellar rotation.

While completing this work, we have become aware of a similar and independent study of Das et al. (2022), which investigates accretion onto rotating stars with a dipolar and quadrupolar magnetic-field topologies. While the numerical code employed is the same (Porth et al. 2017; Olivares et al. 2019), the scope of the paper is different and differently from us, Das et al. (2022) introduce a force-free prescription in the magnetosphere necessary to handle a rotating star. Despite these differences, the phenomenology observed under the same physical conditions (i.e., dipolar magnetic field, comparable magnetisation) is similar, as is the location and scaling of the magnetospheric radii.

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DATA AVAILABILITY

The data underlying this article will be shared on reasonable request to the corresponding author.

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and the relativistic velocity, the corresponding mass-accretion rate can be written as

\[ \dot{m} = 4\pi \rho v^2 \Gamma \theta, \]

and retain terms at the second-order in the expansion \( O(M'r) \). In this framework, the condition of pressure balance between the total and the magnetic pressures can be written as

\[ \rho v^2 + \frac{B^2}{2} = \frac{\mu^2}{r_0^2} \left( 1 + \frac{M^2}{5r_0^2} + O(M'r) \right) \]

up to the second-order correction in an expansion in terms of \( M/r \) (Wasserman & Shapiro 1983; Rezzolla et al. 2001). Finally, the strength of the magnetic field in the fluid frame at the equatorial plane is given by

\[ B^\theta = 0, \]

\[ B^\phi = \frac{\mu}{r^2} \left[ 1 - \frac{2M}{r} \left( \frac{3^3}{4M^3} \ln \left( 1 - \frac{2M}{r} \right) + \frac{3r^2}{4M^2} \ln \left( 1 - \frac{2M}{r} \right) \right) + O \left( \frac{M^3}{r^3} \right) \right], \]

Inserting (A6) and (A7) in Eq. (A1), we obtain that the magnetospheric radius at the second order in the \( O(M/r) \) expansion can be approximated as

\[ r_{msph}^{7/2} \left( 1 + \frac{3M^2}{5r_{msph}^2} + O(M^3) \right) \approx \frac{\mu^2}{\sqrt{8M}\dot{m}}. \]

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APPENDIX A: MAGNETOSPHERIC RADIUS

We here present an analytical estimate of the magnetospheric radius in a general-relativistic framework by using the balance of pressures. Because the magnetospheric radius is located at a few times the radius of the star, we treat relativistic effects as small corrections, \( M/r \ll 1 \), and retain terms at the second-order in the expansion \( O(M/r) \). In this framework, the condition of pressure balance between the total and the magnetic pressures can be written as

\[ \rho v^2 + p = \frac{B^2}{2}. \]

Adopting an ideal-fluid equation of state for the left-hand side of (A1) yields

\[ \rho v^2 + p = (\rho + p\varepsilon + p) \Gamma^2 v^2 + p = \left( \rho + p \frac{\varepsilon}{\varepsilon} \right) \Gamma^2 v^2 + p, \]

where the last term in the equation above will be neglected hereafter since in the innermost regions of torus the pressure is much smaller than the rest-mass density. Assuming now, and for simplicity, a spherically symmetric accretion flow in steady state with \( v^\phi = v \), the corresponding mass-accretion rate can be written as

\[ \dot{m} = 4\pi \rho v^2 \Gamma v, \]

and the relativistic velocity \( \Gamma v \) be expressed as

\[ \Gamma^2 v^2 = g_{rr} \Gamma^2 (v^\phi)^2 = \left( 1 - \frac{2M}{r} \right)^{-1} \frac{2M}{r}, \]

so that the rest-mass density can be eliminated in Eq. (A1). Moreover, it is reasonable to assume that magnetic field at the magnetospheric radius is essentially the stellar one, so that on the equatorial plane, \( \theta = \pi/2 \), this is given by

\[ B^\theta = 0, \]

\[ B^\phi = \frac{\mu}{r^2} \left[ 1 - \frac{2M}{r} \left( \frac{3^3}{4M^3} \ln \left( 1 - \frac{2M}{r} \right) + \frac{3r^2}{4M^2} \ln \left( 1 - \frac{2M}{r} \right) \right) + O \left( \frac{M^3}{r^3} \right) \right], \]

up to the second-order correction in an expansion in terms of \( M/r \) (Wasserman & Shapiro 1983; Rezzolla et al. 2001). Finally, the strength of the magnetic field in the fluid frame at the equatorial plane is given by

\[ b^\phi = \frac{B^\phi}{\Gamma^2} \left( B^\phi \right)^2 = r^2 \left( B^\phi \right)^2 \left( 1 - \frac{2M}{r} \right), \]

\[ \approx \frac{\mu^2}{r^6} \left[ 1 + \frac{2M}{r} + \frac{17M^2}{5r^2} + O \left( \frac{M^3}{r^3} \right) \right]. \]