ON THE 8-RANK OF CLASS GROUPS OF $\mathbb{Q}(\sqrt{-4pq})$ AND $\mathbb{Q}(\sqrt{8pq})$

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Abstract. Let $d \in \{-4, 8\}$. We show that in the set of fundamental discriminants of the form $dpq$, where $p \equiv q \equiv 1 \mod 4$ are prime numbers and for which the class group $\text{Cl}(dpq)$ of the quadratic number field $\mathbb{Q}(\sqrt{dpq})$ has 4-rank equal to 2, the subset of those discriminants for which $\text{Cl}(dpq)$ has an element of order 8 has lower density at least $1/|d|$.

1. Introduction

Let $\text{Cl}(D)$ denote the narrow class group of the quadratic number field $\mathbb{Q}(\sqrt{D})$ of discriminant $D$. The isomorphism class of the finite abelian group $\text{Cl}(D)$ is determined by its $\ell^k$-ranks (where $\ell$ is a prime number and $k \geq 1$ is an integer), i.e., by the numbers

$$\text{rk}_\ell \text{Cl}(D) = \dim_{\mathbb{F}_\ell} \left( \text{Cl}(D)^{\ell^{k-1}} / \text{Cl}(D)^{\ell^k} \right).$$

In other words, the $\ell$-rank $\text{rk}_\ell \text{Cl}(D)$ is the number of cyclic $\ell$-groups appearing in the decomposition of $\text{Cl}(D)$ as a direct sum of cyclic subgroups of prime-power order, and $\text{rk}_\ell \text{Cl}(D)$ is the number of these cyclic $\ell$-groups that have an element of order $\ell^k$. Hence the $\ell$-rank measures the “width” of the $\ell$-part, while the $\ell^k$-rank as $k$ increases measures the “depth” of the $\ell$-part.

Thus to study the average behavior of $\text{Cl}(D)$ as $D$ ranges over some natural family of discriminants, we can study the distribution of $\text{rk}_\ell \text{Cl}(D)$ for various prime powers $\ell^k$. Density results about the $\ell^k$-rank of class groups in natural families of quadratic number fields exist only for $\ell^k$ equal to 2, 4, 8, 16, and 3 (see [22], [6], [23], [8], [19], [18], [5], [1]).

We will focus on the case $\ell = 2$, for which the most is known. The “width” of the 2-part of $\text{Cl}(D)$ is given by Gauss’s genus theory. More precisely, we have

$$\text{rk}_2 \text{Cl}(D) = \omega(|D|) - 1,$$

where $\omega(|D|)$ denotes the number of distinct primes dividing $|D|$. Rédei [22] gave formulas for the 4-rank in terms of the individual primes dividing the discriminant. Fouvry and Klüners [6] reinterpreted these formulas in a way that allowed them to attack the problem with an array of analytic tools. They obtained density results about the 4-rank in several settings (see also [7] and [8]), and their methods likely extend to almost any natural family of discriminants.

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Much less is known about the 8-rank. In [8], Fouvry and Klüners proved certain distribution results about the 8-rank in a special family of positive discriminants with an arbitrary 2-rank, but under the constraint that the 4-rank is 1. On the other hand, Stevenhagen [23] proved that if $d \neq 0$ and $k \geq 0$ are integers, then the set of primes $p$ such that $\text{rk}_8 \text{Cl}(dp) = k$ and such that $dp$ is a fundamental discriminant has a density in the set of all primes which is a rational number. Note that the families studied by Stevenhagen have a fixed 2-rank, have a 4-rank that is entirely determined by the congruence class of $p$ modulo $d$, and are parametrized by a single prime.

We prove a result about the 8-rank in a very special two-parameter family. For this family, our result goes a step beyond what could be deduced from either [8] or [23]. Our main theorem is as follows.

**Main Theorem.** Let $p$ and $q$ denote distinct prime numbers congruent to 1 modulo 4. Then for $d \in \{-4, 8\}$, we have

$$\liminf_{X \to \infty} \frac{\# \{pq \leq X : \text{rk}_4 \text{Cl}(dpq) = 2, \text{rk}_8 \text{Cl}(dpq) \geq 1\}}{\# \{pq \leq X : \text{rk}_4 \text{Cl}(dpq) = 2\}} \geq \frac{1}{|d|}.$$  

The asymptotic formula for the denominator in the ratio above is

$$\# \{pq \leq X : p \equiv q \equiv 1 \text{ mod } 4, \text{rk}_4 \text{Cl}(dpq) = 2\} \sim \frac{1}{32} \frac{X \log \log X}{\log X}$$

as $X \to \infty$ (for both $d = -4$ and $d = 8$). This formula is a slight variation of [10, Equation (2.12), p. 493], whose proof for our particular case can be found in [12].

Cohen and Lenstra [2] have developed a heuristic model for the average behavior of class groups of quadratic number fields. This heuristic model predicts that the limit in the Main Theorem exists and is equal to $5/8$ in the case $d = -4$ and $11/32$ in the case $d = 8$. See Section 3.2 for more details.

The proof of the Main Theorem exploits a new type of lower bound for the 8-rank. In [8], Fouvry and Klüners define a quantity $\lambda_D$ conducive to analytic techniques which gives a good upper bound for the 8-rank of the narrow class group $\text{Cl}(D)$ for a special class of positive discriminants $D$. This upper bound $\lambda_D$ actually coincides with $\text{rk}_8 \text{Cl}(D)$ when $\text{rk}_4 \text{Cl}(D) = 1$. However, when $\text{rk}_4 \text{Cl}(D) \geq 2$, the quantity $\lambda_D$ is only an upper bound for $\text{rk}_8 \text{Cl}(D)$ and hence cannot be used to deduce that $\text{rk}_8 \text{Cl}(D) \geq 1$. Therefore, the Main Theorem cannot be deduced from the techniques in [8].

We also note that the Main Theorem cannot be deduced from [23]. It follows from [23] that for every fixed prime $p \equiv 1 \text{ mod } 8$, the set $S_p$ of primes $q \equiv 1 \text{ mod } 8$ such that $\text{rk}_8 \text{Cl}(dpq) \geq 1$ is a Čebotarev set with some density $\delta_p$, so that

$$N_p(X) = \# \{q \in S_p : q \leq X\} = \delta_p \frac{X}{\log X} + E_p(X),$$

where $E_p(X) = o(X/\log X)$ as $X \to \infty$. However, the uniformity with respect to $p$ in the best known bounds for the error term $E_p(X)$ is too poor to give directly
the asymptotics for the sum
\[ \sum_{p \leq X} N_p(X/p) \]
as \( X \to \infty \).

The main novelty in the proof of the Main Theorem compared to previous work on the 8-rank is the analysis of certain kinds of Hecke characters for \( \mathbb{Z}[i] \) and \( \mathbb{Z}[\sqrt{2}] \) that detect the \( a \) of a particular element of order 8 in the class groups \( \text{Cl}(-4pq) \) and \( \text{Cl}(8pq) \), respectively. In the case \( d = 8 \), these Hecke characters already appear in [8, Theorem 6, p. 356], but they are not interpreted as such and hence not fully exploited in the subsequent analysis.

We also prove double oscillation results in the rings \( \mathbb{Z}[i] \) and \( \mathbb{Z}[\sqrt{2}] \) that are reminiscent of [9, Proposition 21.3, p. 1027] and [18, Proposition 7, p. 29], respectively. In our case, however, we need somewhat more precise estimates – the term \((MN)^{\epsilon}\) must be replaced by an arbitrary power of \( \log(MN) \).

A general approach to proving these types of double oscillation results was already developed in [15], so, after making appropriate adjustments to work inside more general number rings instead of the rational integers, the heart of the proof of both [9, Proposition 21.3, p. 1027] and [18, Proposition 7, p. 29] lies in achieving cancellation in characters sums as in [9, Lemma 21.1, p. 1025] and [18, Proposition 9, p. 35]. In Proposition 7 of this paper, we give a shorter and more natural proof of both of these results in one go.

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2. Algebraic Criteria for the 8-rank

2.1. Preliminaries. We will make extensive use of the following facts.

Let \( K \) be a quadratic number field of discriminant \( D \), \( \mathcal{O}_K \) its maximal order, and \( \text{Cl} \) the narrow class group of \( \mathcal{O}_K \). The narrow Hilbert class field \( H \) of \( K \) is the maximal unramified at finite places abelian extension of \( K \). Hereafter, we will use the word “unramified” to mean “unramified at finite places,” and we will discuss infinite places separately when necessary. If \( K \) is imaginary, then the two complex conjugate embeddings are always unramified in \( H \), and so \( \text{Cl} \) coincides with the ordinary class group of \( K \). If \( K \) is real, then the real embeddings might ramify in \( H \), and so the ordinary class group might be a quotient of \( \text{Cl} \) by a subgroup of order 2. The Artin map induces a canonical isomorphism of groups

\[ (3) \quad \left( \frac{H}{K} \right) : \text{Cl} \to \text{Gal}(H/K). \]

The above isomorphism allows us to deduce information about the class group \( \text{Cl} \) by constructing and studying unramified abelian extensions of \( K \).
The 2-torsion subgroup of the class group Cl is generated by the classes of the ramified primes in $K/Q$, i.e.,

$$\text{Cl}[2] = \langle [p] : p \text{ prime ideal of } \mathcal{O}_K \text{ such that } p|D \rangle.$$ 

We will use the two facts above in tandem as follows. Hereafter, $C_n$ will denote the cyclic group of order $n$.

**Lemma 1.** Let $K$ be a quadratic number field. Suppose that $L/K$ is an unramified $C_{2n}$-extension for some $n \geq 1$. Then every prime ideal $p$ of $\mathcal{O}_K$ that is ramified in $K/Q$ splits completely in $L/K$ if and only if there exists an unramified $C_{2n+1}$-extension $L'$ of $K$ containing $L$.

**Proof.** As $L/K$ is unramified and abelian, $L$ must be contained in the Hilbert class field $H$. A prime ideal $p$ of $\mathcal{O}_K$ splits completely in $L/K$ if and only if $(p|L/K) = 1 \in \text{Gal}(L/K) \cong \text{Cl}/\text{Gal}(H/L)$.

Hence, by (4), every prime ideal $p$ of $\mathcal{O}_K$ that is ramified in $K/Q$ splits completely in $L/K$ if and only if $\text{Cl}[2] \subseteq \text{Gal}(H/L)$. Hence, to prove the lemma, it remains to show that $\text{Cl}[2] \subseteq \text{Gal}(H/L)$ if and only if there exists a quadratic extension $L'/K$ contained in $H$ such that $\text{Gal}(L'/K) \cong C_{2n+1}$.

We proceed via a counting argument. Given a field extension $E/F$, define $S(E/F)$ to be the set of quadratic extensions of $F$ contained in $E$, i.e.,

$$S(E/F) = \{ F' \subset F \subset E \}.$$ 

If $E/F$ is a finite abelian extension, then $\#S(E/F) = \#\text{Gal}(E/F)[2] - 1$.

If $\text{Cl}[2] \subseteq \text{Gal}(H/L)$, then $\text{Cl}[2] = \text{Gal}(H/L)[2]$, and so $\#(H/K) = \#S(H/L)$.

As $L/K$ is cyclic, there is a unique quadratic extension $K \subset K_2 \subset L$. Hence there is a map

$$S(H/K) \setminus \{ K_2 \} \to S(H/L)$$

sending $K'$ to $L \cdot K'$. But as

$$\#(S(H/K) \setminus \{ K_2 \}) = \#S(H/L) - 1,$$

there exists a quadratic extension $L \subset L' \subset H$ such that $K_2$ is the unique quadratic extension of $K$ contained in $L'$. Thus $L'/K$ is cyclic of degree $2^{n+1}$.

Conversely, suppose that $L'$ is a $C_{2n+1}$-extension of $K$ containing $L$ and contained in $H$. Let $c \in \text{Cl}$ such that $c^2 = 1_H \in \text{Cl} = \text{Gal}(H/K)$. Then $(c|_{L'})^2 = 1_L$, so $L'c(L')$ is a subfield of $L'$ containing $K$, and, as $L'/K$ is cyclic, this subfield must be either $L'$ or $L$. In either case, we deduce that $c|_L = 1_L$, and so $c \in \text{Gal}(H/L)$. □

We will also make use of the following lemma from Galois theory (see [17], Chapter VI, Exercise 4, p.321)).
Lemma 2. Let $F$ be a field of characteristic different from 2, let $E = F(\sqrt{d})$, where $d \in F^\times \setminus (F^\times)^2$, and let $L = E(\sqrt{N})$, where $x \in E^\times \setminus (E^\times)^2$. Let $N = \text{Norm}_{E/F}(x)$. Then we have three cases:

1. If $N \notin (E^\times)^2 \cup d \cdot (F^\times)^2$, then $L/F$ has normal closure $L(\sqrt{N})$ and $\text{Gal}(L(\sqrt{N})/F) \cong D_8$, the dihedral group of order 8.

2. If $N \in (F^\times)^2$, then $L/F$ is normal and $\text{Gal}(L/F) \cong V_4$, the Klein four-group.

3. If $N \in d \cdot (F^\times)^2$, then $L/F$ is normal and $\text{Gal}(L/F) \cong C_4$, the cyclic group of order 4.

2.2. Special two-parameter families. Let $d \in \{-4, 8\}$, and let $p$ and $q$ be odd primes congruent to 1 modulo 4. Let

$$K = \mathbb{Q}(\sqrt{dpq}),$$

and let $H$ denote its Hilbert class field. Let

$$d_0 = \frac{d}{4},$$

so that the maximal order of $K$ is

$$\mathcal{O}_K = \mathbb{Z}[\sqrt{d_0pq}].$$

We are ultimately interested in the average value of the 8-rank of the class group $\text{Cl}(dpq)$ of $\mathcal{O}_K$ as $p$ and $q$ range among prime numbers satisfying

$$pq \leq X,$$

for a real parameter $X$ going to infinity.

Let $\text{Cl} = \text{Cl}(dpq)$. Gauss’s genus theory implies that $\text{rk}_2 \text{Cl} = 2$, i.e., that

$$\text{Cl}/\text{Cl}^2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

and that the genus field, the maximal abelian extension of $\mathbb{Q}$ contained in $H$, is

$$G = H^{\text{Cl}^2} = \mathbb{Q}(\sqrt{d}, \sqrt{p}, \sqrt{q}).$$

The three quadratic subfields $G_1 = K(\sqrt{d})$, $G_2 = K(\sqrt{p})$, and $G_3 = K(\sqrt{q})$ of $G$ correspond to the three proper subgroups of $\text{Cl}/\text{Cl}^2$. The three ramified primes $t$, $p$, and $q$ of $\mathcal{O}_K$ that lie above 2, $p$, and $q$, respectively, generate the 2-torsion subgroup $\text{Cl}[2]$ and will play a prominent role in the subsequent discussions.

We see from \eqref{5} that $\text{rk}_4 \text{Cl} \leq 2$, and in fact the 4-rank of $\text{Cl}$ is the largest it could be exactly when $p$ and $q$ satisfy

$$p \equiv q \equiv 1 \mod 8,$$

and

$$\left(\frac{p}{q}\right) = 1.$$
Proposition 1. Let \( d \in \{-4, 8, 4\} \), and let \( p \) and \( q \) be odd prime numbers congruent to 1 modulo 4. Let \( \text{Cl} = \text{Cl}(dpq) \) denote the class group of the quadratic number field \( \mathbb{Q}(\sqrt{dpq}) \). Then

\[
\text{rk}_4 \text{Cl} = 2
\]

if and only if \( p \) and \( q \) satisfy (6) and (7).

Proof. The extension \( G_i/\mathbb{Q} \) is a \( V_4 \)-extension for \( i = 1, 2, 3 \), so the splitting behavior of \( t, p, \) and \( q \) in \( G_i/K \) is determined by the splitting behavior of 2, \( p, \) and \( q \), respectively, in quadratic subfields of \( G_i \). Conditions (6) and (7) imply that \( t, p, \) and \( q \) all split in \( G_i/K \) for \( i = 1, 2, 3 \). For instance, by (6), the prime \( p \) splits in \( \mathbb{Q}(\sqrt{d})/\mathbb{Q} \), and so \( p \) splits in \( G_1 \). Now Lemma 1 implies that \( p \) and \( q \) satisfy (6) and (7) if and only if there exists an unramified \( C_4 \)-extension \( L_i/K \) containing \( G_i \) for \( i = 1, 2, 3 \). By Galois theory, (3), and (5), this happens if and only if \( \text{Cl}/\text{Cl}^4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), and the result is proved. \( \square \)

From now on, suppose \( p \) and \( q \) satisfy (6) and (7). Although Proposition 1 demonstrates the existence of three distinct unramified \( C_4 \)-extensions of \( K \), it may be difficult to construct these extensions explicitly from \( d, p, \) and \( q \). In one case, however, we can do exactly this.

By (6), both \( p \) and \( q \) split in \( \mathbb{Z}[\sqrt{d}] \), so there exist primes \( w, z \in \mathbb{Z}[\sqrt{d}] \) such that \( \text{Norm}(w) = p \) and \( \text{Norm}(z) = q \). If \( d = -4 \), then, again by (6), we have

\[
w, z \equiv \pm 1 \equiv \square \mod 4\mathbb{Z}[i].
\]

If \( d = 8 \), then we can replace \( w \) by \(-w\) and/or \( z \) by \(-z\) if necessary to ensure that

\[
w, z \equiv 1 \text{ or } 3 + 2\sqrt{2} \equiv \square \mod 4\mathbb{Z}[\sqrt{2}].
\]

In any case, we can choose primes \( w \) and \( z \) in \( \mathbb{Z}[\sqrt{d}] \) such that

\[
\text{Norm}(w) = p, \quad \text{Norm}(z) = q, \quad \text{and} \quad w, z \equiv \square \mod 4\mathbb{Z}[\sqrt{d}].
\]

Define \( \alpha \in \mathbb{Z}[\sqrt{d}] \) and \( x, y \in \mathbb{Z} \) by the equation

\[
\alpha = wz = x + y\sqrt{d} \in \mathbb{Q}(\sqrt{d}) \subset G_1.
\]

Then \( \alpha \) satisfies the condition

\[
\alpha \equiv \square \mod 4\mathbb{Z}[\sqrt{d}],
\]

and \( p, q, x, \) and \( y \) satisfy the relation

\[
pq = x^2 - doy^2.
\]

For an element \( a \) in \( \mathbb{Q}(\sqrt{d}) \), we will denote the conjugate of \( a \) in \( \mathbb{Q}(\sqrt{d})/\mathbb{Q} \) (or \( G_1/K \)) by \( \overline{a} \), so that

\[
\overline{a} = wz = x - y\sqrt{d}.
\]

Let \( L_1 \) be a field extension of \( K \) defined by

\[
L_1 = G_1(\sqrt{\alpha}) = K(\sqrt{d}, \sqrt{\alpha}).
\]

The extension \( L_1/K \) is exactly the unramified \( C_4 \)-extension containing \( G_1 \).

Proposition 2. Let \( \alpha \in \mathbb{Q}(\sqrt{d}) \) be given by (9), and let \( L_1 \) be defined as in (13). Then \( L_1/K \) is an unramified \( C_4 \)-extension.
Proof. We have
\[
\text{Norm}_{G_1/K}(\alpha) = pq = d \cdot \left(\frac{1}{7} \sqrt{dpq}\right)^2 \in d \cdot (K^\times)^2,
\]
so Lemma 2 implies that \(L_1/K\) is a \(C_4\)-extension.

The only primes that can ramify in \(L_1/K\) are \(t\), \(p\), and \(q\). We will show that \(p\) is unramified in \(L_1/K\), and by symmetry this will imply that \(q\) is also unramified in \(L_1/K\). Recall from (9) that the prime \(w\) divides \(\alpha\). As \(p\) and \(q\) are distinct primes, \(\alpha\) and \(\overline{\alpha}\) are coprime in \(\mathbb{Z}[[\sqrt{d}]\) and hence \(w\) does not ramify in \(A = \mathbb{Q}(\sqrt{d}, \sqrt{\alpha})\). Thus the ramification index of \(p\) in \(L_1/K\) is at most 2. But \(p\) already ramifies in \(K/Q\), and hence \(p\) must be unramified in \(L_1/K\).

It remains to prove that \(L_1/K\) is unramified over \(t\). This can be done locally, so we may pass to the completion with respect to \(t\) and show that \(\mathbb{Q}_2(\sqrt{d}, \sqrt{\alpha})/\mathbb{Q}_2(\sqrt{d})\) is unramified. This is the case if and only if \(\alpha\) is a square in \(\mathbb{Q}_2(\sqrt{d})\), which happens if and only if \(\alpha\) is a square modulo \(t^5\), where by abuse of notation \(t\) is now the maximal ideal in the discrete valuation ring \(\mathbb{Z}_2[\sqrt{d}_0]\), and this is ensured by condition (10).

Now that we constructed \(L_1/K\) explicitly, we can apply Lemma 1 to determine when \(L_1\) is contained in an unramified \(C_8\)-extension \(M_1/K\). We must determine when \(t\), \(p\), and \(q\) all split completely in \(L_1\).

For the prime \(t\), this can once again be determined locally. Indeed, \(t\) splits completely in \(L_1/K\) if and only if the extension of local fields \(\mathbb{Q}_2(\sqrt{d}, \sqrt{\alpha})/\mathbb{Q}_2(\sqrt{d})\) is trivial. This occurs if and only if \(\alpha\) is a square in \(\mathbb{Q}_2(\sqrt{d})\), which happens if and only if \(\alpha\) is a square modulo \(t^5\), where by abuse of notation \(t\) is now the maximal ideal in the discrete valuation ring \(\mathbb{Z}_2[\sqrt{d}_0]\). Explicitly, this means that

\[
\alpha \equiv \square \mod t^5 \equiv \begin{cases} 
\pm 1 \mod 4(1 + i)\mathbb{Z}_2[\sqrt{-1}] & \text{if } d = -4, \\
1 \text{ or } 3 + 2\sqrt{2} \mod 4\sqrt{2}\mathbb{Z}_2[\sqrt{2}] & \text{if } d = 8.
\end{cases}
\]

For primes \(p\) and \(q\), the splitting criterion is somewhat different. We may again use the auxiliary extension \(A = \mathbb{Q}(\sqrt{d}, \sqrt{\alpha})\) from proof of Proposition 2. Let \(w\) and \(z\) be as in (9). We have \(p = w\overline{\alpha}\) with \(w\) dividing \(\alpha\), so \(p\) splits completely in \(L_1/K\) if and only if \(w\) splits in \(A/Q(\sqrt{d})\). We use a quadratic residue symbol in \(\mathbb{Q}(\sqrt{d})\) to
detect this, i.e., \( \mathfrak{p} \) splits completely in \( \mathcal{O} / \mathbb{Q}(\sqrt{d}) \) if and only if

\[
\left( \frac{\mathfrak{p}}{\omega} \right) = 1.
\]

Similarly, the prime \( q \) splits completely in \( L_1 / K \) if and only if

\[
\left( \frac{\mathfrak{p}}{\varpi} \right) = 1.
\]

We will now explore the link between the quadratic residue symbols \( \left( \frac{\mathfrak{p}}{\omega} \right) \) and \( \left( \frac{\mathfrak{p}}{\varpi} \right) \). Using (12) and the fact that the primes \( \omega \) and \( \varpi \) are of degree one over \( \mathfrak{p} \) and \( q \), respectively, we find that

\[
\left( \frac{\mathfrak{p}}{\omega} \right) \left( \frac{\mathfrak{p}}{\varpi} \right) = \left( \frac{x - y\sqrt{d_0}}{x + y\sqrt{d_0}} \right) = \left( \frac{2x}{pq} \right),
\]

where the last symbol is simply a Jacobi symbol. Using (11) and the fact that \( pq \equiv 1 \pmod{8} \), we find that

\[
\left( \frac{2x}{pq} \right) = \left( \frac{|x|}{pq} \right) = \left( \frac{x^2 - d_0y^2}{|x|} \right) = \left( \frac{-d_0}{|x|} \right).
\]

We now make a distinction between the cases \( d = -4 \) and \( d = 8 \). First suppose \( d = -4 \). Then

\[
\left( \frac{-d_0}{|x|} \right) = \left( \frac{1}{|x|} \right) = 1,
\]

and so

\[
\left( \frac{\mathfrak{p}}{\omega} \right) = \left( \frac{\mathfrak{p}}{\varpi} \right).
\]

In other words, if \( d = -4 \), then \( \mathfrak{p} \) splits completely in \( L_1 / K \) if and only if \( q \) does. Therefore, if \( d = -4 \), to ensure that \( L_1 \) is contained in an unramified \( C_8 \)-extension \( M_1 / K \), we only need to verify that (14) and (15) are satisfied.

Now suppose \( d = 8 \). Then

\[
\left( \frac{-d_0}{|x|} \right) = \left( \frac{-2}{|x|} \right),
\]

and so

\[
\left( \frac{\mathfrak{p}}{\omega} \right) = \left( \frac{\mathfrak{p}}{\varpi} \right) \iff |x| \equiv 1, 3 \pmod{8}.
\]

Thus if \( |x| \equiv 5, 7 \pmod{8} \), there is no chance that both (15) and (16) are satisfied and so \( L_1 \) is not contained in an unramified \( C_8 \)-extension \( M_1 / K \). Looking back at (14), we see that \( t \) splits in \( L_1 / K \) if and only if \( x \) satisfies

\[
x \equiv 1, 3 \pmod{8}.
\]

Hence, assuming that (14) holds, we find that \( \left( \frac{\mathfrak{p}}{\omega} \right) = \left( \frac{\mathfrak{p}}{\varpi} \right) \) if and only if

\[
x > 0.
\]

Note that (11) implies that \( |x| > |y\sqrt{2}| \), so that

\[
x > 0 \iff \alpha, \bar{\alpha} > 0.
\]
In other words, the field $L_1$ cannot be contained in an unramified $C_8$-extension $M_1/K$ unless $L_1$ is totally real, i.e., unless $L_1/K$ is unramified also at the infinite places.

We summarize the results of this section in the following proposition.

**Proposition 3.** Let $d \in \{-4, 8\}$, and let $p$ and $q$ be prime numbers satisfying (6) and (7). Let $w$ and $z$ be primes in $\mathbb{Z}[\sqrt{d}]$ satisfying (8). Let $\alpha$ and $x$ be defined as in (9). Suppose $\alpha$ satisfies (14). Furthermore, if $d = 8$, also suppose $x$ satisfies (18). Then there is an unramified $C_8$-extension of $\mathbb{Q}(\sqrt{dpq})$ containing $\mathbb{Q}(\sqrt{d}, \sqrt{pq})$ if and only if

$$\left( \frac{\alpha}{w} \right) = 1.$$

Consequently, under the assumptions above, if the equality above holds, then

$$\text{rk}_8 \text{Cl}(dpq) \geq 1.$$

2.3. **The splitting condition for $t$.** We now delve a bit deeper into the meaning of condition (14). Let $w$ and $z$ be primes in $\mathbb{Z}[\sqrt{d}]$ satisfying (8), and let $t$ be the prime ideal of $\mathbb{Z}[\sqrt{d}]$ lying above $2$. Let $t$ be a generator of $t$ defined by

$$(19) \quad t = \begin{cases} 1 + i & \text{if } d = -4 \\ \sqrt{2} & \text{if } d = 8. \end{cases}$$

In [24, proof of Theorem 1, p. 5], Stevenhagen proved that

$$(20) \quad \left( \frac{t}{w} \right) = \begin{cases} 1 & \text{if } w \equiv \Box \mod t^3 \\ -1 & \text{otherwise,} \end{cases}$$

and likewise for $z$. If we define

$$(21) \quad \chi_4(a) = \left( \frac{t}{a} \right)$$

for odd prime ideals $a$ in $\mathbb{Z}[\sqrt{d}]$ and extend multiplicatively to the group $\mathcal{I}(t)$ of fractional ideals of $\mathbb{Z}[\sqrt{d}]$ coprime to $t$, then $\chi_4$ is a quadratic Hecke character on $\mathbb{Z}[\sqrt{d}]$. The significance of (20) is twofold: first, the variables $p$ and $q$, which are inextricably linked in the definition of $\alpha$, are now separated; and second, condition (14) can now be written in terms of the quadratic Hecke character $\chi_4$ on $\mathbb{Z}[\sqrt{d}]$, i.e.,

$$(22) \quad \alpha \equiv \Box \mod t^3 \iff \chi_4((w))\chi_4((z)) = 1.$$ 

2.4. **Positivity condition on $x$.** The variables $p$ and $q$ are also inextricably linked in the definition of variable $x$ appearing in (9). However, the positivity condition (18) on $x$ can be unfolded via a theorem of Fouvry and Klüners [7, Proposition 6, p.2063]. We have

$$x > 0 \iff [2, pq]_4 = [pq, 2]_4,$$

where $[\cdot, \cdot]_4$ is the symbol defined in [17, p. 2061], i.e.,

$$[2, pq]_4 = [2, p]_4[2, q]_4,$$
where

\[
[2, p]_4 = \begin{cases} 
1 & \text{if 2 is a fourth power modulo } p \\
-1 & \text{if 2 is a square, but not a fourth power modulo } p \\
0 & \text{otherwise}
\end{cases}
\]

and similarly for \([2, q]_4\), and

\[
[pq, 2]_4 = \begin{cases} 
1 & \text{if } pq \equiv 1 \mod 16 \\
-1 & \text{if } pq \equiv 9 \mod 16 \\
0 & \text{otherwise.}
\end{cases}
\]

When \(p \equiv q \equiv 1 \mod 8\), then

\[
[2, pq]_4 = [2, p]_4[2, q]_4 = \chi_4((w))\chi_4((z)) = \chi_4((wz)).
\]

where \(w, z,\) and \(\chi_4\) are as in Section 2.3. Provided (22) is satisfied, we deduce from the equations and definitions above that

\[
(23) \quad x > 0 \iff pq \equiv 1 \mod 16.
\]

3. Strategy for the Proof of the Main Theorem

As before, let \(d \in \{-4, 8\}\). The ultimate goal is to prove, in the set of fundamental discriminants \(D = dpq\) satisfying \(\text{rk}_4 \text{Cl}(D) = 2\), a lower bound for the density of those \(D\) that also satisfy \(\text{rk}_8 \text{Cl}(D) \geq 1\).

Suppose \(p\) and \(q\) are prime numbers satisfying (6). Let \(w\) and \(z\) be primes in \(\mathbb{Z}[\sqrt{d}]\) satisfying (8), and define \(\alpha\) and \(x\) as in (9). Set \(p = (w)\) and \(q = (z)\). If \(d = 8\), suppose that \(x > 0\). We define the symbol \(\varepsilon(p, q)\) by

\[
(24) \quad \varepsilon(p, q) = \left( \frac{\alpha}{(w)} \right) = \left( \frac{\alpha}{(z)} \right).
\]

Recall from (23) that the positivity condition on \(x\) can be detected via congruence conditions on \(p\) and \(q\) modulo 16. Given that \(p \equiv q \equiv 1 \mod 8\), there are four choices for \((p, q)\) mod 16. When \(d = -4\), the positivity condition on \(x\) is irrelevant, so all four of the choices are valid; however, when \(d = 8\), exactly two of the choices correspond to the condition (23).

The splitting condition at the prime \(t\) lying above 2 can be detected via the Hecke character \(\chi_4\) as in (22). If \(\chi_4(p) = s_1\) and \(\chi_4(q) = s_2\) with \(s_1, s_2 \in \{\pm 1\}\), then \(\alpha \equiv \square \mod t^5\) if and only if \(s_1s_2 = 1\).

In light of Proposition 3, the asymptotic formula (2), and the remarks above, the Main Theorem is a consequence of the following theorem.

**Theorem 1.** Let \(d\) be \(-4\) or \(8\). Given primes \(p\) and \(q\) satisfying (6), let \(x\) and \(\alpha\) be defined as in (9). Let \(r_1, r_2 \in \{1, 9\}\) and, in case \(d = 8\), suppose that \(r_1r_2 \equiv
ON THE 8-RANK OF CLASS GROUPS OF $\mathbb{Q}(\sqrt{-4pq})$ AND $\mathbb{Q}(\sqrt{8pq})$ 11

1 mod 16. Let $s_1, s_2 \in \{\pm 1\}$ and suppose that $s_1s_2 = 1$. Then, as $X \to \infty$, we have

$$\sum_{pq \leq X} 1 \sim \frac{1}{1024} \frac{X \log \log X}{\log X}.$$ 

Theorem 1 can be interpreted as follows. Classical theory of the distribution of prime numbers (see for instance [21, Section 7.4, p.228]) gives the count of positive integers that are a product of two primes in fixed congruence classes modulo 16, i.e., we have the asymptotic formula

$$(25) \quad \sum_{pq \leq X} 1 \sim \frac{1}{64} \frac{X \log \log X}{\log X}$$

as $X \to \infty$. The conditions $\chi_1(p) = s_1$ and $\chi_1(q) = s_2$ can likewise be inserted without any trouble because $\chi_1$ is a multiplicative character of a fixed conductor not depending on $p$ or $q$. Hence, we have

$$(26) \quad \sum_{pq \leq X} 1 \sim \frac{1}{256} \frac{X \log \log X}{\log X}$$

as $X \to \infty$. Each of the remaining two conditions under the summation in Theorem 1 can then be viewed as an event that occurs with probability one-half. Moreover, these two events are independent. To make this argument rigorous, we make use of the following formulas. Given a mathematical statement $P$, we define the indicator function of $P$ to be

$$1(P) = \begin{cases} 1 & \text{if } P \text{ is true} \\ 0 & \text{if } P \text{ is false}. \end{cases}$$

For distinct odd primes $p$ and $q$, set

$$\chi_p(q) = \left(\frac{p}{q}\right).$$

Then we have

$$(27) \quad 1(p \equiv \square \mod q) = \frac{1}{2} (1 + \chi_p(q))$$

Now we wish to generalize the character $\chi_1$ to a function $\chi_2$ defined on all rational primes in a way that $\chi_2(p) = \chi_1(p)$ for a prime $p \equiv 1 \mod 8$. We set

$$\chi_2(p) = \frac{1}{\#\{p|p\}} \sum_{p|p} \chi_1(p),$$
where the sum is over all prime ideals $p$ in $\mathbb{Z}[\sqrt{d_0}]$ lying above $p$. With $t$ defined as in (19), $p$ a prime congruent to 1 modulo 8, and $p_1$ and $p_2$ the two primes in $\mathbb{Z}[\sqrt{d_0}]$ lying above $p$, we have

$$\chi_1(p_1)\chi_1(p_2) = \left( \frac{t}{p_1} \right) \left( \frac{t}{p_2} \right) = \frac{\text{Norm}(t)}{p_1} = \frac{\text{Norm}(t)}{p} = 1,$$

so that $\chi_1(p_1) = \chi_1(p_2)$. Thus indeed $\chi_2(p) = \chi_2(p')$ whenever $p \equiv 1 \mod 8$.

Given primes $p$ and $q$, an ordered pair of integers $r = (r_1, r_2) \in \{1, 9\} \times \{1, 9\}$, and an ordered pair of integers $s = (s_1, s_2) \in \{\pm 1\} \times \{\pm 1\}$, set

$$c(p, q; r, s) = 1 \quad \text{if} \quad (p, q) \equiv r \mod 16 \quad \text{and} \quad (\chi_2(p), \chi_2(q)) = s.$$

Now let $p$ and $q$ be distinct primes, let $r$ and $s$ be as above, and suppose that $s_1 s_2 = 1$, and if $d = 8$, also that $r_1 r_2 \equiv 1 \mod 16$. Then we have

$$1((p, q) \equiv r \mod 16, \quad (\chi_2(p), \chi_2(q)) = s, \quad \text{and} \quad \varepsilon(p, q) = 1) = c(p, q; r, s) \cdot \frac{1}{2} (1 + \varepsilon(p, q)).$$

Finally, given a vector $e = (e_1, e_2) \in \mathbb{F}_2^2$ and $p, q, r, s$ as above, define

$$f(p, q) = f(p, q; r, s, e) = c(p, q; r, s)\chi_p(q)^{e_1} \varepsilon(p, q)^{e_2}.$$

Then, putting together the formulas above, we deduce that

$$\sum_{p \leq X \atop p \equiv r \mod 16} \sum_{q \leq X \atop q \equiv s \mod q \atop \varepsilon(p, q) = 1} 1 = \frac{1}{4} \sum_{e \in \mathbb{F}_2^2} \sum_{p \leq X \atop p \equiv r \mod 16} \sum_{q \leq X \atop q \equiv s \mod q} f(p, q; r, s, e)$$

whenever $s$ satisfies $s_1 s_2 = 1$ and, if $d = 8$, $r$ satisfies $r_1 r_2 \equiv 1 \mod 16$. If $e = (0, 0)$, then, as we noted above in (26), we have

$$\sum_{p \leq X \atop p < q} f(p, q; r, s, e) \sim \frac{1}{256} \times \frac{X \log \log X}{\log X},$$

as $X \to \infty$. Hence Theorem 1 follows from the following oscillation statement.

**Theorem 2.** Let $r = (r_1, r_2) \in \{1, 9\} \times \{1, 9\}$ be such that $r_1 r_2 \equiv 1 \mod 16$ if $d = 8$, let $s = (s_1, s_2) \in \{\pm 1\} \times \{\pm 1\}$ be such that $s_1 s_2 = 1$, let $e \in \mathbb{F}_2^2$, and let $f(p, q; r, s, e)$ be defined as in (28). If $e \neq (0, 0)$, then

$$\sum_{p \leq X \atop p < q} f(p, q; r, s, e) = o \left( \frac{X \log \log X}{\log X} \right)$$

as $X \to \infty$.

The rest of the paper is devoted to proving Theorem 2.
3.1. Summing under a hyperbola. We now describe how to handle sums of the form

\[ S(X; f) = \sum_{pq \leq X; p < q} f(p, q), \]

where \( f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C} \) is supported on pairs of prime numbers. The goal is to give good upper bounds for \( S(X; f) \) when \( f \) oscillates. Let \( Y \) be a positive real number. Then

\[ S(X; f) = A(X, Y; f) + B(X, Y; f), \]

where

\[ A(X, Y; f) = \sum_{pq \leq X; p < q; p \leq Y} f(p, q), \]

and

\[ B(X, Y; f) = \sum_{pq \leq X; q > p > Y} f(p, q). \]

Usually \( Y \) is chosen small enough compared to \( X \) so that the sum \( A(X, Y; f) \) can be handled using the Siegel-Walffisz theorem and variations thereof. Bounding the sum \( B(X, Y; f) \) then usually proceeds by proving a double oscillation theorem for \( f \), and this type of theorem is generally useful only when \( Y \) is not too small. We make these techniques precise in the following proposition.
Proposition 4. Let $X > 1$ be a real number, let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ be a function satisfying $\|f\|_\infty \leq 1$, and let $S(X; f)$ be defined as in (29). Let $Y$ be a real number satisfying $1 < Y < X^{\frac{1}{4}}$. Suppose that there exist positive real numbers $\delta_1, \delta_2, \delta_3$ satisfying $\delta_3 < 2\delta_2$ such that

\begin{equation}
A_p(X; f) = \sum_{q \leq X} f(p, q) \ll XY^{-\delta_1}
\end{equation}

for all $p \leq Y$, where the implied constant is absolute, and such that

\begin{equation}
B(M, N; f, \Delta) = \sum_{M < p \leq M + \Delta M \atop N < q \leq N + \Delta N} f(p, q) \ll \Delta^{-\delta_2} (M^{-\delta_3} + N^{-\delta_3}) \Delta^2 MN,
\end{equation}

for all $M, N > 1$ and $\Delta \in (0, 1)$ satisfying $\Delta M, \Delta N > 1$, where the implied constant is absolute. Then there exists a positive real number $\delta$ in $(0, 1)$ such that

\[ S(X; f) \ll Y^{-\delta}X \log X, \]

where the implied constant is absolute. Moreover, we can take

\[ \delta = \min \left( \frac{\delta_1}{2}, \frac{\delta_3}{2\delta_2}, \frac{\delta_3}{2} \right). \]

Proof. With $A(X, Y; f)$ defined as in (31), using hypothesis (A), we deduce that

\begin{align}
A(X, Y; f) &= \sum_{p \leq Y} (A_p(X/p; f) - A_p(p; f)) \\
&\ll \sum_{p \leq Y} \left( Xp^{-1}Y^{-\delta_1} + pY^{-\delta_1} \right) \\
&\ll Y^{-\delta_1}X \log \log Y + Y^{2-\delta_1}.
\end{align}

Let

\[ \Delta = Y^{-\frac{\delta_1}{2\delta_2}}. \]

For each $k \geq 0$, we define real numbers $M_k$ and $N_k$ by

\[ M_k = N_k = Y(1 + \Delta)^k. \]

Let $\mathcal{R}(X)$ be the region in $\mathbb{R}^2$ defined by

\[ \mathcal{R}(X) = \{(x, y) \in \mathbb{R}^2 : x \geq Y, xy \leq X(1 + \Delta)^{-2}, x(1 + \Delta) \leq y \}, \]

and let $\Sigma(X)$ be the subset of $\mathbb{Z}_{>0}^2$ defined by

\[ \Sigma(X) = \{(j, k) \in \mathbb{Z}_{>0}^2 : (M_j, N_k) \in \mathcal{R}(X) \}. \]

If $(j, k) \in \Sigma(X)$, then the box $[M_j, M_{j+1}] \times [N_k, N_{k+1}]$ is completely contained in the region

\[ T(X) = \{(x, y) \in \mathbb{R}^2 : x \geq Y, xy \leq X, x \leq y \}. \]

Let $B(X, Y; f)$ be the sum defined in (32). Then we can partition $B(X, Y; f)$ as

\begin{equation}
B(X, Y; f) = \sum_{(j, k) \in \Sigma(X)} B(M_j, N_k; f, \Delta) + R(X, Y; f, \Delta).
\end{equation}

As $\|f\|_\infty \leq 1$, we give a trivial upper bound for $R(X, Y; f, \Delta)$ by counting lattice points in the region $T(X) \setminus \mathcal{R}(X)$, i.e.,

\[ |R(X, Y; f, \Delta)| \leq \# (\mathbb{Z}^2 \cap (T(X) \setminus \mathcal{R}(X))). \]
The right-hand side above can be approximated by the area of the region $\mathcal{T}(X) \setminus \mathcal{R}(X)$, with an error term bounded by the sum of the lengths of the projections of $\mathcal{T}(X) \setminus \mathcal{R}(X)$ to the axes (this is known as the Lipschitz principle; see [3] and [4]). Thus we have

$$R(X, Y; f, \Delta) \ll \int_0^{X^{\frac{1}{2}}} \Delta x dx + \int_Y^{X^{\frac{1}{2}}} \frac{(X - x/(1 + \Delta)^2)}{x} dx + X^{\frac{1}{2}} + XY^{-1} + 1$$

$$\ll \Delta X + X \frac{2\Delta + \Delta^2}{(1 + \Delta)^2} \log \left(\frac{X^{\frac{1}{2}}}{Y}\right) + X^{\frac{1}{2}} + XY^{-1} + 1$$

$$\ll \Delta X + \Delta X \log X + XY^{-1}$$

$$\ll Y^{-\frac{\delta_2}{2}} X \log X + Y^{-1} X.$$ 

As $\delta_3 < 2\delta_2$ and $M_j, N_k \geq Y$, we have $\Delta M_j, \Delta N_k > 1$. Thus we can use hypothesis (B) to give the bound

$$\sum\sum_{(j,k) \in \Sigma(X)} B(M_j, N_k; f, \Delta) \ll \Delta^{-\delta_2} Y^{-\delta_3} \sum\sum_{(j,k) \in \Sigma(X)} \Delta^2 M_j N_k$$

$$\ll Y^{-\frac{\delta_3}{4}} \cdot \text{Area} \mathcal{T}(X)$$

$$\ll Y^{-\frac{\delta_2}{2}} \cdot X \log X.$$

Combining (33), (35), and (36), we deduce the proposition. \hfill \square

To apply Proposition 4, we will prove the following two propositions. In the following, define $f(p, q; r, s, e)$ as in Theorem 2, and suppose $e \neq (0, 0)$.

**Proposition 5.** Let $f(p, q) = f(p, q; r, s, e)$. Then there is a constant $c > 0$ such that for all $p \leq (\log X)^{100}$, we have

$$A_p(X; f) = \sum_{q \leq X} f(p, q) \ll X \exp \left( -c \sqrt{\log X} \right),$$

where the implied constant is absolute (but ineffective).

**Proposition 6.** Let $f(p, q) = f(p, q; r, s, e)$. Then, for all $M, N > 1$ and $\Delta \in (0, 1)$ satisfying $\Delta M, \Delta N > 1$, we have

$$B(M, N; f, \Delta) = \sum\sum_{M < p \leq M + \Delta M \atop N < q \leq N + \Delta N} f(p, q) \ll \Delta^{-\frac{11}{12}} \left( M^{-\frac{1}{12}} + N^{-\frac{1}{12}} \right) \Delta^2 MN,$$

where the implied constant is absolute.

Hence, assuming Propositions 5 and 6, we can apply Proposition 4 with $Y = (\log X)^{100}$, $\delta_1 = 1$, $\delta_2 = \frac{11}{12}$, and $\delta_3 = \frac{1}{12}$ to obtain Theorem 2. Our goal is now to prove Propositions 5 and 6.

### 3.2. Heuristics.

We now briefly discuss the conjectural limit of the ratio in the Main Theorem and the limitations of our methods towards a proof of such a conjecture.

Let $G$ be a finite abelian group, and let $\#\text{Aut}(G)$ be the number of automorphisms
of $G$. Cohen and Lenstra [2] developed a heuristic model for the average structure of class groups of quadratic number fields. Their model is based on the assumption that $G$ occurs as the class group of an imaginary (resp. a real) quadratic field with probability proportional to the inverse of $\#\text{Aut}(G)$ (resp. $G \cdot \#\text{Aut}(G)$).

Although they stated their model only for the prime-to-2 part of the class group, Gerth [11] extended the model to the 2-primary part of the class group by stating that it is $\text{Cl}(D^2)$ instead of $\text{Cl}(D)$ that behaves like a random group in the sense of Cohen and Lenstra.

Under these assumptions, we can compute a conjectural density for the ratio

$$\frac{\#\{pq \leq X : \text{rk}_4\text{Cl}(dpq) = 2, \text{rk}_8\text{Cl}(dpq) \geq 1\}}{\#\{pq \leq X : \text{rk}_4\text{Cl}(dpq) = 2\}}$$

from the Main Theorem. Given that $\text{rk}_4\text{Cl}(D) = 2$, the 2-part of $\text{Cl}(D)$ must be of the form $\mathbb{Z}/2^m\mathbb{Z} \times \mathbb{Z}/2^n\mathbb{Z}$ for some $n \geq m \geq 1$. In this notation $\text{rk}_8\text{Cl}(D) \geq 1$ precisely when $n \geq 2$. An elementary computation yields

$$\#\text{Aut}(\mathbb{Z}/2^m\mathbb{Z} \times \mathbb{Z}/2^n\mathbb{Z}) = \begin{cases} 3 \cdot 2^{4m-3} & \text{if } m = n \\ 2^{3m+n-2} & \text{if } m < n. \end{cases}$$

Suppose now that $d = -4$, so that we’re in the imaginary case. The total weight of all groups of the form $\mathbb{Z}/2^m\mathbb{Z} \times \mathbb{Z}/2^n\mathbb{Z}$ is

$$\sum_{m \geq 1} \frac{1}{3 \cdot 2^{4m-3}} + \sum_{m \geq 1} \sum_{n \geq m+1} \frac{1}{2^{3m+n-2}} = \frac{4}{9}.$$ 

The case when $\text{rk}_8\text{Cl}(D) = 0$, i.e. $m = n = 1$, has weight $1/6$. The probability of the complement is thus

$$\frac{\frac{4}{9} - \frac{1}{6}}{\frac{4}{9}} = \frac{5}{8}.$$ 

Thus we are led to conjecture

**Conjecture 1.** Let $d = -4$. Then

$$\lim_{X \to \infty} \frac{\#\{pq \leq X : \text{rk}_4\text{Cl}(dpq) = 2, \text{rk}_8\text{Cl}(dpq) \geq 1\}}{\#\{pq \leq X : \text{rk}_4\text{Cl}(dpq) = 2\}} = \frac{5}{8}$$

as $X \to \infty$.

Similarly, in the real case, when $d = 8$, the total weight is

$$\sum_{m \geq 1} \frac{1}{3 \cdot 2^{6m-3}} + \sum_{m \geq 1} \sum_{n \geq m+1} \frac{1}{2^{4m+2n-2}} = \frac{4}{63},$$

while the weight of the case $m = n = 1$ is $1/24$. The probability of the complement is then

$$\frac{\frac{4}{63} - \frac{1}{24}}{\frac{4}{63}} = \frac{11}{32},$$

so that we conjecture

**Conjecture 2.** Let $d = 8$. Then

$$\lim_{X \to \infty} \frac{\#\{pq \leq X : \text{rk}_4\text{Cl}(dpq) = 2, \text{rk}_8\text{Cl}(dpq) \geq 1\}}{\#\{pq \leq X : \text{rk}_4\text{Cl}(dpq) = 2\}} = \frac{11}{32}$$

as $X \to \infty$. 

Both Conjectures 1 and 2 closely match numerical data generated in Sage.

There is another way to obtain the same conjectures that more closely matches our strategy of proof of the Main Theorem. For the sake of simplicity, we focus on the case \( d = -4 \). As we saw in Proposition 3, the existence of an unramified \( C_8 \)-extension of \( \mathbb{Q}(\sqrt{-4pq}) \) containing \( \mathbb{Q}(\sqrt{-4}, \sqrt{pq}) \) is contingent upon two events. The first is

Event A: condition holds, the splitting condition at 2,

and the second is

Event B: condition holds, the splitting condition at \( p \).

We already saw in (17) that the splitting condition at \( q \) is automatically satisfied if it is satisfied at \( p \). Both Events A and B are determined by the values of certain quadratic residue symbols depending on \( p \) and \( q \). Assuming these symbols take values +1 and −1 equally often and independently of each other, the probability that both Events A and B occur is \( \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \). This is exactly how we prove Theorem 1.

When \( \text{rk}_4 \text{Cl}(-4pq) = 2 \), there also exists an unramified \( C_4 \)-extension \( L' \) (resp. \( L'' \)) of \( \mathbb{Q}(\sqrt{-4pq}) \) that contains \( \mathbb{Q}(\sqrt{-4p}, \sqrt{q}) \) (resp. \( \mathbb{Q}(\sqrt{-4p}, \sqrt{-4q}) \)). For \( L' \) (resp. \( L'' \)) to be contained in an unramified \( C_8 \)-extension of \( \mathbb{Q}(\sqrt{-4pq}) \), there are again two events that must occur. One of them once again concerns the splitting condition at 2, say Event \( A' \) (resp. Event \( A'' \)). The other event, say Event \( B' \) (resp. \( B'' \)), concerns the splitting condition at \( q \) (resp. \( p \)).

We can once again expect Events \( A', A'', B', \) and \( B'' \) to be determined by values of certain quadratic residue symbols, except this time in \( \mathbb{Z}[\sqrt{-4p}] \) or \( \mathbb{Z}[\sqrt{-4q}] \). And we can again conjecture that each of these symbols takes the values +1 and −1 equally often. However, these events are not independent. If both \( \mathbb{Q}(\sqrt{-4}, \sqrt{pq}) \) and \( \mathbb{Q}(\sqrt{-4p}, \sqrt{q}) \) or \( \mathbb{Q}(\sqrt{-4p}, \sqrt{-4q}) \) are contained in (distinct) unramified \( C_8 \)-extensions of \( \mathbb{Q}(\sqrt{-4pq}) \), then so is \( \mathbb{Q}(\sqrt{p}, \sqrt{-4q}) \). One can check that out of the events \( A, A', A'' \), either exactly one or all three events occur, and similarly for \( B, B', \) and \( B'' \). Hence, using the principle of inclusion-exclusion, we may conjecture that \( \text{rk}_8 \text{Cl}(D) \) is at least 1 with probability

\[
\mathbb{P}(A \& B) + \mathbb{P}(A' \& B') + \mathbb{P}(A'' \& B'') - 2\mathbb{P}(A \& A' & B \& B') - \frac{3}{2} = \frac{5}{8}.
\]

Thus the discrepancy between our lower bound of 1/4 from the Main Theorem and the conjectural limit 5/8 comes from not taking into account unramified \( C_8 \)-extensions of \( \mathbb{Q}(\sqrt{-4pq}) \) containing \( \mathbb{Q}(\sqrt{-4p}, \sqrt{q}) \) or \( \mathbb{Q}(\sqrt{p}, \sqrt{-4q}) \).

The main obstacle in extending the ideas of this paper to handle Events \( B' \) and \( B'' \) is that \( \mathbb{Z}[\sqrt{-4p}] \) and \( \mathbb{Z}[\sqrt{-4q}] \) are no longer principal ideal domains, and in fact \( \text{Cl}(-4p) \) or \( \text{Cl}(-4q) \) (or both) may have non-trivial odd torsion. Thus it is difficult to control (in a uniform way as \( p \) and \( q \) vary) the size of the analogues of \( \alpha \) from (9), and a genuinely new idea would be required to apply similar analytic techniques. Theorem 1 already achieves a new lower bound for the 8-rank, so we leave the task of sharpening this lower bound for a future project.
4. QUASI-BILINEAR SYMBOLS AND HECKE CHARACTERS

In this section, we give an interpretation of \( \varepsilon(p, q) \) first as a value of a quasi-bilinear symbol on \( \mathbb{Q}(\sqrt{d}) \), and then, with \( p \) fixed, as a value of a certain Hecke character for \( \mathbb{Q}(\sqrt{d}) \). Recall that

\[
d_0 = \begin{cases} 
-1 & \text{if } d = -4, \\
2 & \text{if } d = 8,
\end{cases}
\]

so that \( \mathbb{Z}[\sqrt{d_0}] \) is the ring of integers of the quadratic number field \( \mathbb{Q}(\sqrt{d}) \). It will be convenient for us that, for \( d_0 \) as above, the ring \( \mathbb{Z}[\sqrt{d_0}] \) is a principal ideal domain.

4.1. Primitivity. We say that an ideal \( a \) in \( \mathbb{Z}[\sqrt{d_0}] \) is primitive if it is a product of unramified prime ideals of residue degree one. The main property of primitive ideals that we will use is that the inclusion

\[
\mathbb{Z} \rightarrow \mathbb{Z}[\sqrt{d_0}]
\]

induces an isomorphism

\[
\mathbb{Z}/(\text{Norm}(a)) \cong \mathbb{Z}[\sqrt{d_0}]/a.
\] (37)

We call an ideal \( a \) (resp. element \( u \)) in \( \mathbb{Z}[\sqrt{d_0}] \) odd if \( \text{Norm}(a) \) (resp. \( \text{Norm}(u) \)) is an odd integer. An ideal in \( \mathbb{Z}[\sqrt{d_0}] \) is odd if and only if every prime ideal that divides \( a \) is unramified. Hence, an ideal \( a \) is primitive if and only if \( a \) is odd and there is no rational prime \( p \) dividing \( a \) (i.e., no rational prime \( p \) such that \( (p) \) divides \( a \)).

Remark. For instance, in \( \mathbb{Z}[i] \), \( \text{Norm}(5) = 25 \), but \( \mathbb{Z}[i]/(5) \ncong\mathbb{Z}[i]/(2+i) \times \mathbb{Z}[i]/(2-i) \ncong\mathbb{Z}/(5) \times \mathbb{Z}/(5) \ncong\mathbb{Z}/(25) \).

For every integer \( n \) we have the equality of quadratic residue symbols

\[
\left( \frac{n}{\text{Norm}(a)} \right) = \left( \frac{n}{a} \right) \text{,}
\] (38)

where the symbol on the left is the usual Jacobi symbol while the symbol on the right is the two-power-residue symbol in \( \mathbb{Z}[\sqrt{d_0}] \), i.e.,

\[
\left( \frac{n}{a} \right) = \prod_{p | a} \left( \frac{n}{p} \right),
\]

where

\[
\left( \frac{n}{p} \right) = \begin{cases} 
1 & \text{if } (n, p) = 1 \text{ and } n \equiv \square \mod p \\
-1 & \text{if } (n, p) = 1 \text{ and } n \not\equiv \square \mod p \\
0 & \text{otherwise.}
\end{cases}
\]

Now it follows immediately from (37) and (38) that

\[
\sum_{z \in \mathbb{Z}[\sqrt{d_0}]/a} \left( \frac{z}{a} \right) = \sum_{n \in \mathbb{Z}/(\text{Norm}(a))} \left( \frac{n}{\text{Norm}(a)} \right) \text{.}
\] (39)

The following is yet another characterization of primitive ideals.

Lemma 3. Suppose \( a \subset \mathbb{Z}[\sqrt{d_0}] \) is an odd ideal. Then \( a \) is primitive if and only if \( \gcd(a, \overline{a}) = (1) \).
Proof. If \(a\) is not primitive, then there is a rational prime \(p\) dividing \(a\). As \(p\) is rational, it also divides \(\overline{a}\), and so \(\gcd(a, \overline{a}) \neq (1)\). Conversely, if \(\gcd(a, \overline{a}) \neq (1)\), then there is a prime ideal \(p\) in \(\mathbb{Z}[\sqrt{d_0}]\) such that both \(p\) and \(\overline{p}\) divide \(a\). If \(p\) is a prime of degree 2, then \(p = (p)\) for some rational prime \(p\) and automatically \(a\) is not primitive. Otherwise, as \(a\) is odd and the only prime that ramifies in \(\mathbb{Q}(\sqrt{d})/\mathbb{Q}\) is 2, we conclude that \(p\) and \(\overline{p}\) are coprime, and hence that \(p\overline{p}\) divides \(a\). Once again, as \(p\overline{p} = (p)\) for a rational prime \(p\), \(a\) is not primitive. \(\Box\)

Suppose \(a\) and \(b\) are ideals in \(\mathbb{Z}[\sqrt{d_0}]\). If one of \(a\) and \(b\) is not primitive, then clearly their product \(ab\) is not primitive. Even if both \(a\) and \(b\) are primitive, the product \(ab\) need not be primitive. Nonetheless, we have the following lemma.

Lemma 4. Suppose \(a\) and \(b\) are primitive. Let \(r = \gcd(a, b)\) and \(r = \text{Norm}(r)\). Then \(ab/r\) is primitive. In particular, \(ab\) is primitive if and only if \(\gcd(a, b) = (1)\).

Proof. Suppose \(p\) is a rational prime such that \(p\) divides \(ab\). Then \((p)\) cannot be a prime in \(\mathbb{Z}[\sqrt{d_0}]\), because otherwise either \(a\) or \(b\) is not primitive. Hence there exists a prime ideal \(p \in \mathbb{Z}[\sqrt{d_0}]\) such that \(p = p\overline{p}\) and \(p|a\). If \(p^k\) is the exact power of \(p\) dividing \(ab\), then the assumption that \(a\) and \(b\) are primitive implies that \(p^k|a\) and \(p^k|b\), which is true if and only if \(p^k|r\). \(\Box\)

There is another way to obtain a primitive ideal from a product of two odd primitive ideals \(a\) and \(b\). We can write

\[
a = \prod_{p \text{ split}} p^{a_p} \overline{p}^{\bar{a}_p},
\]

and

\[
b = \prod_{p \text{ split}} p^{b_p} \overline{p}^{\bar{b}_p},
\]

where \(a_p \overline{a}_p = b_p \overline{b}_p = 0\) for every \(p\). Let \(r = \gcd(a, b)\) and let \(r = \text{Norm}(r)\). If a prime \(p\) divides \(r\), after possibly interchanging the roles of \(p\) and \(\overline{p}\) in the products above, we can assume that \(p\) divides \(r\). For every such prime \(p\), define

\[
c_{a, p} = \begin{cases} p^{a_p} & \text{if } a_p \leq \overline{b}_p, \\ 1 & \text{otherwise}, \end{cases}
\]

\[
c_{b, p} = \begin{cases} 1 & \text{if } a_p \leq \overline{b}_p, \\ \overline{p}^{\bar{b}_p} & \text{otherwise}, \end{cases}
\]

and set

\[
c_a = \prod_p c_{a, p},
\]

\[
c_b = \prod_p c_{b, p},
\]

and

\[
c = c_a c_b.
\]

Then clearly

\[
\text{Norm}(c) = \text{Norm}(r) = r.
\]
Moreover, by construction
\[ \gcd \left( \frac{a}{\epsilon_a}, \frac{b}{\epsilon_b} \right) = (1), \]
so by Lemma 3 we conclude \( \mathfrak{a} \mathfrak{b} / \mathfrak{c} \) is primitive. By construction, \( \mathfrak{c} \) is also primitive and coprime to \( \mathfrak{a} \mathfrak{b} / \mathfrak{c} \). Therefore, using the Chinese Remainder Theorem and applying (37) twice, we conclude that
\[
\mathbb{Z}[\sqrt{d_0}]/\mathfrak{a} \mathfrak{b} \cong \mathbb{Z}[\sqrt{d_0}]/(\mathfrak{a} \mathfrak{b} / \mathfrak{c}) \times \mathbb{Z}[\sqrt{d_0}] / \mathfrak{c} \cong \mathbb{Z} / (W/r) \times \mathbb{Z} / (r),
\]
where \( W = \text{Norm}(\mathfrak{a} \mathfrak{b}) \).

Finally, we say that an element \( w \in \mathbb{Z}[\sqrt{d_0}] \) is primitive if and only if the principal ideal generated by \( w \) is primitive. An equivalent definition is that \( w = a + b\sqrt{d} \) is odd and \( \gcd(a, b) = 1 \).

4.2. A quasi-bilinear symbol with a reciprocity law. The rough idea behind proving that the symbol \( \varepsilon(p, q) \) defined in (24) oscillates as \( p \) and \( q \) vary in a box where neither \( p \) nor \( q \) is too small is to give meaning to \( \varepsilon(m, n) \) for all integers \( m \) and \( n \), then to prove that the bilinear sum
\[
\sum_{m,n} a_m b_n \varepsilon(m, n)
\]
oscillates for any bounded sequences \( \{a_m\}_m \) and \( \{b_n\}_n \), and finally to apply this result to sequences \( \{a_m\}_m \) and \( \{a_n\}_n \) supported on the primes. The following definition generalizes the symbol \( \varepsilon(p, q) \) in a way that will allow us to apply this method.

Suppose \( w \) and \( z \) are any elements in \( \mathbb{Z}[\sqrt{d_0}] \), not necessarily prime. If \( w \) is odd, we define the \textit{generalized Dirichlet symbol} \( \gamma(w, z) \) to be
\[
\gamma(w, z) = \left( \frac{wz}{(w)} \right),
\]
where \( (\cdot) \) is the quadratic residue symbol in \( \mathbb{Q}(\sqrt{d}) \). Our choice of terminology is inspired by the Dirichlet symbol defined in a slightly different context in [9, Section 19, p.1018-1021].

When \( w \) and \( z \) are primes in \( \mathbb{Z}[\sqrt{d_0}] \) lying above rational primes \( p \) and \( q \), respectively, satisfying \([6], [7], [8], [14]\), and, if \( d = 8 \), also \([18]\), then \( \gamma(w, z) \) coincides with the symbol \( \varepsilon(p, q) \) defined in \([24]\). Indeed, we have
\[
\varepsilon(p, q) = \left( \frac{w}{(w)} \right) = \gamma(w, z).
\]
The symbol \( \gamma(w, z) \) factors as
\[
\gamma(w, z) = m(w) \chi_w(z),
\]
where, for odd \( w \in \mathbb{Z}[\sqrt{d_0}] \), we define
\[
m(w) = \gamma(w, 1) = \left( \frac{w}{(w)} \right)
\]
and
\[
\chi_w(z) = \left( \frac{z}{(w)} \right).
\]
By Lemma 38, if $w \in \mathbb{Z}[\sqrt{d_0}]$ is odd, then
\[ m(w) \neq 0 \iff \gcd((w), (\overline{w})) = (1) \iff w \text{ is primitive}. \]

Hence the factor $m(w)$ restricts the support of $\gamma(w, z)$ to $w$ that are primitive. Furthermore, if $w$ and primitive, then $\gcd((w), (\overline{w})) = \gcd((w), (\overline{x}))$, and so $\gamma(w, z) = 0$ if and only if $\gcd((w), (\overline{x})) \neq (1)$.

The factor $\chi_w(z)$ is completely multiplicative in $z$, so it follows from (43) that
\[ \gamma(w, z_1)\gamma(w, z_2) = \gamma(w, z_1z_2)m(w), \]
for any $w$, $z_1$, and $z_2$ in $\mathbb{Z}[\sqrt{d_0}]$ such that $w$ is odd. Hence the symbol $\gamma(w, z)$ is multiplicative in $z$ except for a twist by $m(w)$.

The symbol $\gamma(w, z)$ also satisfies a reciprocity law, which is a key ingredient in the proof of Proposition 6.

**Lemma 5.** Let $w, z \in \mathbb{Z}[\sqrt{d_0}]$ such that both $w$ and $z$ are odd. Then
\[ \gamma(w, z)\gamma(z, w) = m(wz). \]

**Proof.** We have
\[ \gamma(w, z)\gamma(z, w) = \left( \frac{\overline{w}z}{w} \right) \left( \frac{\overline{w}z}{z} \right) = \left( \frac{\overline{w}z}{wz} \right) = m(wz). \]

Finally, we note that $\gamma(w, z)$ is periodic in the second argument. In fact, $\gamma(w, z_1) = \gamma(w, z_2)$ whenever $z_1 \equiv z_2 \mod (\overline{w})$. In other words, $\gamma(w, \cdot)$ is a function on $\mathbb{Z}[\sqrt{d_0}]/(\overline{w})$. This allows us to prove the following key proposition, which will provide all of the cancellation that we need for Proposition 6.

**Proposition 7.** Let $w_1, w_2 \in \mathbb{Z}[\sqrt{d_0}]$ be primitive. Let $r = \gcd((w_1), (\overline{w_2}))$, $r = \text{Norm}(r)$, $W = \text{Norm}(w_1w_2)$. Then
\[ \sum_{z \in \mathbb{Z}[\sqrt{d_0}]/(w)} \gamma(w_1, z)\gamma(w_2, z) = \begin{cases} W \varphi(r)\varphi(W/r) & \text{if } W \text{ and } r \text{ are squares} \\ 0 & \text{otherwise.} \end{cases} \]

**Proof.** By (43), we have
\[ \gamma(w_1, z)\gamma(w_2, z) = m(w_1)m(w_2) \left( \frac{\overline{w}}{(w_1w_2)} \right), \]
and, as $w_1$ and $w_2$ are odd and primitive, $m(w_1)m(w_2) \neq 0$. Hence
\[ \sum_{z \in \mathbb{Z}[\sqrt{d_0}]/(w)} \gamma(w_1, z)\gamma(w_2, z) = \sum_{z \in \mathbb{Z}[\sqrt{d_0}]/(w)} \left( \frac{\overline{z}}{(w_1w_2)} \right). \]

Now, as $W$ is rational, the map $z \mapsto \overline{z}$ is an automorphism of the group $\mathbb{Z}[\sqrt{d_0}]/(W)$. Thus, we obtain
\[ \sum_{z \in \mathbb{Z}[\sqrt{d_0}]/(w)} \left( \frac{\overline{z}}{(w_1w_2)} \right) = \sum_{z \in \mathbb{Z}[\sqrt{d_0}]/(w)} \left( \frac{z}{(w_1w_2)} \right). \]
As \( \left( \frac{\cdot}{(w_1, w_2)} \right) \) is already a function on \( \mathbb{Z}[\sqrt{d_0}] / (w_1 w_2) \), and

\[
\# \left( \mathbb{Z}[\sqrt{d_0}] / (W) \right) = W \cdot \# \left( \mathbb{Z}[\sqrt{d_0}] / (w_1 w_2) \right),
\]

we have

\[
\sum_{z \in \mathbb{Z}[\sqrt{d_0}] / (w_1 w_2)} \left( \frac{z}{w_1 w_2} \right) = W \sum_{z \in \mathbb{Z}[\sqrt{d_0}] / (w_1 w_2)} \left( \frac{z}{w_1 w_2} \right).
\]

By [40], we have

\[
\mathbb{Z}[\sqrt{d_0}] / (w_1 w_2) \cong \mathbb{Z}[\sqrt{d_0}] / (\alpha) \times \mathbb{Z}[\sqrt{d_0}] / (\beta),
\]

where \( (\alpha) \) and \( (\beta) \) are coprime primitive ideals of norm \( W/r \) and \( r \), respectively, satisfying \( (w_1 w_2) = (\alpha \beta) \). Hence

\[
\sum_{z \in \mathbb{Z}[\sqrt{d_0}] / (w_1 w_2)} \left( \frac{z}{w_1 w_2} \right) = \sum_{z_{01} \mod \alpha} \sum_{z_{02} \mod \beta} \left( \frac{z}{\alpha \beta} \right),
\]

where

\[
z = z_{01} \cdot \beta' + z_{02} \cdot \alpha' \]

and \( \alpha' \) and \( \beta' \) are some elements of \( \mathbb{Z}[\sqrt{d_0}] \) such that \( \alpha \alpha' \equiv 1 \mod \beta \) and \( \beta \beta' \equiv 1 \mod \alpha \). With these choices, we have

\[
\left( \frac{z}{\alpha \beta} \right) = \left( \frac{z}{\alpha} \right) \left( \frac{z}{\beta} \right) = \left( \frac{z_{01}}{\alpha} \right) \left( \frac{z_{02}}{\beta} \right).
\]

Then, by [39], we have

\[
\sum_{z_{01} \mod \alpha} \left( \frac{z_{01}}{\alpha} \right) \sum_{z_{02} \mod \beta} \left( \frac{z_{02}}{\beta} \right) = \sum_{a \in \mathbb{Z} / (W/r)} \left( \frac{a}{W/r} \right) \sum_{b \in \mathbb{Z} / (r)} \left( \frac{b}{r} \right),
\]

where the symbols on the right-hand side of the equality are the usual Jacobi symbols. For any positive integer \( n \), we have

\[
\sum_{a \in \mathbb{Z} / (n)} \left( \frac{a}{n} \right) = \begin{cases} \phi(n) & \text{if } n \text{ is a square}, \\ 0 & \text{otherwise}. \end{cases}
\]

Combining all of the equations above, we conclude the proof of the proposition. \( \square \)

**Remark.** If we replace \( \gamma(w_1, z) \gamma(w_2, z) \) by \( \gamma(w_1, z) \gamma(w_2, z) \), the conclusion of Proposition [7] remains the same.

### 4.3. The link between \( \gamma(w, z) \) and \( \left( \frac{\zeta}{\beta} \right) \)

Let \( p \equiv 1 \mod 8 \) be a prime and let \( w \in \mathbb{Z}[\sqrt{d_0}] \) be such that \( \text{Norm}(w) = p \). The following lemma gives the link between \( \gamma(w, z) \), \( \gamma(w_1, z) \), and the Jacobi symbol \( \left( \frac{\text{Norm}(z)}{p} \right) \). This link is of crucial importance in handling the condition \( q \equiv \square \mod p \) in the sums \( A_0(X; d) \) and \( B(M, N; \Delta) \) from Propositions [5] and [6] respectively.

**Lemma 6.** Let \( p \equiv 1 \mod 8 \) be a prime and let \( w \in \mathbb{Z}[\sqrt{d_0}] \) be such that \( \text{Norm}(w) = p \). Then

\[
\gamma(w, z) \gamma(w_1, z) = \left( \frac{\text{Norm}(z)}{p} \right),
\]

where the symbol on the right side of the equality is a Jacobi symbol.
Proof. We have
\[
\gamma(w, z)\gamma(\overline{w}, \overline{z}) = \left( \frac{wz}{w} \right) \left( \frac{wz}{\overline{w}} \right) = \left( \frac{\text{Norm}(z)}{w} \right) = \left( \frac{\text{Norm}(z)}{p} \right),
\]
where the last equality follows because \(w\) is a prime of residue degree one over \(p\). \(\Box\)

4.4. A family of Hecke characters for \(\mathbb{Z}[\sqrt{d}]\). The function \(\chi_w\) is a character on \((\mathbb{Z}[\sqrt{d}] / (w)) \times\). We now show that this character can be completed into a Hecke character \(\psi_w\) for \(\mathbb{Q}(\sqrt{d})\) in the case that \(w\) is a prime of degree 1 in \(\mathbb{Z}[\sqrt{d}]\) satisfying \(\text{Norm}(w) = p \equiv 1 \mod 8\).

We must define a homomorphism \(\psi_w\) on the group \(I(\overline{w})\) of fractional ideals of \(\mathbb{Z}[\sqrt{d}]\) coprime to \((w)\), i.e.,
\[
\psi_w : I(\overline{w}) \rightarrow S^1 = \{ s \in \mathbb{C} : |s| = 1 \},
\]
such that there exists a continuous function \(\chi_w, \infty : F^\times \rightarrow S^1\) satisfying \(\chi_w(u)\chi_w,\infty(u) = \psi_w((u)) = 1\) for all units \(u \in \mathbb{Z}[\sqrt{d}]^\times\); here
\[
F^\times = \begin{cases} 
\mathbb{C}^\times & \text{if } d = -4, \\
\mathbb{R}^x \times \mathbb{R}^x & \text{if } d = 8.
\end{cases}
\]

4.4.1. The case \(d = -4\). Suppose \(w = a + bi\) with \(a, b \in \mathbb{Z}\), \(a\) odd, and \(b = 2^k b'\) with \(b'\) odd. As \(p = \text{Norm}(w)\) is prime, \(a\) and \(b\) are coprime, so we can write
\[
i \equiv a/b \mod \overline{w}.
\]
Then, as \(p = a^2 + b^2 \equiv 1 \mod 8\), we have
\[
\chi_w(i) = \left( \frac{i}{\overline{w}} \right) = \left( \frac{ab}{\overline{w}} \right) = \left( \frac{|a|}{a^2 + b^2} \right) \left( \frac{|b'|}{a^2 + b^2} \right) = 1 \cdot 1 = 1.
\]
As \(\chi_w\) is trivial on the units \(\mathbb{Z}[i] \times\), we can extend \(\chi_w\) to a character on ideals in \(\mathbb{Z}[i]\) by setting \(\chi_w(a) = \chi_w(z)\), where \(z\) is any generator of \(a\). Now it suffices to take \(\chi_w,\infty\) to be identically 1 on all of \(\mathbb{C}^\times\). Then
\[
(47) \quad \psi_w : I(\overline{w}) \rightarrow S^1
\]
defined by \( \psi_w(a) = \chi_w(a) \) is a Hecke character for \( \mathbb{Z}[i] \). Moreover, by (12), if \( p \) and \( q \) are primes satisfying (6) and (7), and \( w \) and \( z \) are primes in \( \mathbb{Z}[i] \) satisfying (8) and (14), then we have

\[
(48) \quad \varepsilon(p, q) = \psi_w((z)).
\]

4.4.2. The case \( d = 8 \). Suppose as before that \( w = a + b\sqrt{2} \) with \( a, b \in \mathbb{Z} \). Then, as \( p = \text{Norm}(w) \equiv 1 \mod 8 \), \( b \) must be even. The unit group \( \mathbb{Z}[\sqrt{2}]^\times \) is generated by \( -1 \) and \( \varepsilon = 1 + \sqrt{2} \). We have

\[
\chi_w(-1) = \left( \frac{-1}{w} \right) = \left( \frac{-1}{p} \right) = 1,
\]

and, if we write \( b = 2^kb' \) with \( b' \) odd, we have

\[
\chi_w(\varepsilon) = \left( \frac{1 + \sqrt{2}}{w} \right) = \left( \frac{1 + a/b}{w} \right) = \left( \frac{1 + a/b}{a^2 - 2b^2} \right) = \left( \frac{b^2 + ab}{a^2 - 2b^2} \right) = \left( \frac{b}{a^2 - 2b^2} \right) = \left( \frac{|b|}{|a + b|} \right).
\]

Every other generator for the ideal \( (w) \) of norm \( p \) is of the form \( \pm \varepsilon^{2^k}w \), where \( k \) is an integer. As

\[
\varepsilon^2(a + b\sqrt{2}) = (3a + 4b) + (2a + 3b)\sqrt{2}
\]

and

\[
(3a + 4b) + (2a + 3b) = 5a + 7b \equiv a + b \mod 4,
\]

the last line of (49) implies that \( \chi_w(\varepsilon) = \chi_{\varepsilon^2w}(\varepsilon) \). Moreover, again by the last line of (49), we have

\[
\chi_{-w}(\varepsilon) = \left( \frac{-1}{-a - b} \right) = \left( \frac{-1}{|a + b|} \right) = \chi_w(\varepsilon).
\]

Thus we cannot always choose a generator \( w \) of a prime ideal lying above \( p \) satisfying both \( \text{Norm}(w) = p \) and \( \chi_w(\varepsilon) = 1 \). In fact, we have

\[
\chi_w(\varepsilon) = \begin{cases} 
1 & \text{if } |a + b| \equiv 1 \mod 4, \\
-1 & \text{otherwise}.
\end{cases}
\]

We will define a different Hecke character \( \psi_w \) modulo \( (w) ) \infty_1 \infty_2 \) in each of the cases above; here \( \infty_1 \) and \( \infty_2 \) are the two embeddings \( \mathbb{Q}(\sqrt{2}) \hookrightarrow \mathbb{R} \). If \( \chi_w(\varepsilon) = 1 \), then \( \chi_w \) is already a character on fractional ideals in \( \mathbb{Z}[\sqrt{2}] \) and we simply define

\[
(50) \quad \psi_w : \mathcal{I}(w) \to S^1
\]

by setting \( \psi_w(a) = \chi_w(z) \), where \( z \) is any generator of \( a \). In this case, we again take \( \chi_{w, \infty} \) to be identically \( 1 \) on all of \( \mathbb{R}^\times \times \mathbb{R}^\times \).

If \( \chi_w(\varepsilon) = -1 \), we take \( \chi_{w, \infty}(z) = \text{sign}(\text{Norm}(z)) \), and define \( \psi_w(a) = \chi_w(z)\chi_{w, \infty}(z) \), where \( z \) is any generator of \( a \). The homomorphism \( \psi_w \) is well-defined because

\[
\chi_w(\varepsilon)\chi_{w, \infty}(\varepsilon) = -1 \cdot -1 = 1
\]
and
\[ \chi_w(-1)\chi_w placements\). We note that in both cases, if \( z \equiv 1 \mod \times (w) \), so that \( z, \overline{z} > 0 \), then
\[ \psi_w((z)) = 1 = \text{sign}(z). \]  
Furthermore, similarly as in the case of \( d = -4 \), if \( p \) and \( q \) are primes satisfying (6) and (7) and \( w \) and \( z \) are primes in \( \mathbb{Z}[\sqrt{d}] \) satisfying (8), then we have
\[ \epsilon(p, q) = \psi_w((z)). \]

5. Proof of Proposition \[\text{5}\]
In this section, we exploit the arithmetic of \( \mathbb{Q}(\sqrt{d}) \) to prove that \( \epsilon(p, q) \) oscillates when \( q \) varies over a range much bigger than the size of \( p \). The main tool is the theory of Hecke \( L \)-functions.

Let us first recall the sum from Proposition \[\text{5}\]. We let
\[ A_p(X; f) = \sum_{q \leq X} f(p, q; r, s, e), \]
where \( r = (r_1, r_2) \in \{1, 9\} \times \{1, 9\}, r_1 r_2 \equiv 1 \mod 16 \) if \( d = 8 \), \( s = (s_1, s_2) \in \{\pm 1\} \times \{\pm 1\}, s_1 s_2 = 1, e \in \mathbb{F}_2^2, e \neq (0, 0), \) and
\[ f(p, q; r, s, e) = \begin{cases} \chi_p(q)^{r_1} \epsilon(p, q)^{r_2} & \text{if } (p, q) \equiv r \mod 16 \text{ and } (\chi_2(p), \chi_2(q)) = s \\ 0 & \text{otherwise.} \end{cases} \]
Hence \( A_p(X; f) \) vanishes unless \( p \equiv r_1 \mod 16 \) and \( \chi_2(p) = s_1 \). So let \( p \) be a prime number satisfying \( p \equiv r_1 \mod 16 \) and \( \chi_2(p) = s_1 \). Let \( w \in \mathbb{Z}[\sqrt{d}] \) be a prime satisfying (8), and let \( \psi_w \) be the Hecke character on \( \mathbb{Q}(\sqrt{d}) \) defined in (47) or (50).

By (48) and (52), we have
\[ f(p, q; r, s, e) = \begin{cases} \left( \frac{p}{q} \right)^{\epsilon_1} \psi_w(q)^{r_2} & \text{if } q \equiv r_2 \mod 16 \text{ and } \chi_2(q) = s_2 \\ 0 & \text{otherwise,} \end{cases} \]
and \( q \) is a prime ideal in \( \mathbb{Z}[\sqrt{d}] \) dividing \( q \). To use the theory of \( L \)-functions for the number field \( \mathbb{Q}(\sqrt{d}) \), we now define a function on all ideals \( q \) in \( \mathbb{Z}[\sqrt{d}] \). Let
\[ f_1(q; w, e) = \left( \frac{p}{q} \right)^{\epsilon_1} \psi_w(q)^{r_2}. \]
We can detect the congruence condition \( q \equiv r_2 \mod 16 \) via Dirichlet characters modulo 16 and the condition \( \chi_2(q) = s_2 \) via the formula
\[ \frac{1}{2} \left( 1 + s_2 \chi_2(q) \right) = \begin{cases} 1 & \text{if } \chi_2(q) = s_2 \\ 0 & \text{otherwise.} \end{cases} \]
We then have

\[ 2 \cdot A_p(X; f) = \frac{1}{16} \sum_{\chi_{16} \mod 16} \left( \sum_{e_3 \in \mathbb{F}_2} \sum_{q \text{ split}} \chi_{16}(r_2 \text{Norm}(q))(s_2 \chi_t(q))\chi^{e_3} f_1(q; w, e) \right) \]

\[ = \frac{1}{16} \sum_{\chi_{16} \mod 16} \left( \sum_{e_3 \in \mathbb{F}_2} \sum_{q \text{ split}} \chi_{16}(r_2 \text{Norm}(q))(s_2 \chi_t(q))\chi^{e_3} f_1(q; w, e) \right) \]

\[- \frac{1}{16} \sum_{\chi_{16} \mod 16} \left( \sum_{e_3 \in \mathbb{F}_2} \sum_{q \text{ inert}} \chi_{16}(r_2 \text{Norm}(q))(s_2 \chi_t(q))\chi^{e_3} f_1(q; w, e) \right), \]

where the outer sums are over Dirichlet characters \( \chi_{16} \mod 16 \) and elements \( e_3 \in \mathbb{F}_2 \). But if a prime ideal \( q = (q) \) in \( \mathbb{Z}[\sqrt{d_0}] \) is inert, then \( \text{Norm}(q) = q^2 \), so

\[ \sum_{q \text{ inert}} 1 \ll X^{\frac{2}{3}}. \]

Hence, to prove Proposition [5] it remains to show, for each Dirichlet character \( \chi_{16} \) and element \( e_3 \in \mathbb{F}_2 \), that there exists a constant \( c > 0 \) such that

\[ \sum_{\text{Norm}(q) \leq X} \chi_{16}(\text{Norm}(q))\chi_t(q)\chi^{e_3} f_1(q; w, e) \ll X \exp \left( c \sqrt{\log X} \right) \]

for all \( p = \text{Norm}(w) \leq (\log X)^{100} \). We now apply the theory of Hecke \( L \)-functions to obtain this bound. Define the Hecke character \( \psi \) for \( \mathbb{Z}[\sqrt{d_0}] \) by setting

\[ \psi(q) = \chi_{16}(\text{Norm}(q))\chi_t(q)\chi^{e_3} f_1(q; w, e). \]

We claim that the function \( q \mapsto \psi(q) \) is a non-trivial Hecke character for \( \mathbb{Z}[\sqrt{d_0}] \) of conductor \( f \) satisfying \( \text{Norm}(f) \ll p^2 \), where the implied constant is absolute. First, note that \( q \mapsto \chi_{16}(\text{Norm}(q))\chi_t(q)^e_3 \) is a Hecke character of conductor dividing a power of 2. If \( e_1 = 1 \) and \( e_2 = 0 \), then the claim follows because

\[ q \mapsto \left( \frac{p}{q} \right) \]

is a non-trivial Hecke character of conductor \( (p) \). If \( e_1 = 0 \) and \( e_2 = 1 \), then the claim follows because

\[ q \mapsto \psi_w(q) \]

is a non-trivial Hecke character of conductor \( (w) \), as shown in Section 4.4. Finally, if \( e_1 = e_2 = 1 \), then by Lemma 6, we have

\[ \left( \frac{p}{q} \right) \psi_w(q) = \psi_{\overline{w}}(q), \]

so that

\[ q \mapsto \left( \frac{p}{q} \right) \psi_w(q) \]

is a non-trivial Hecke character of conductor \( (w) \).
Now that we have established the claim, we use a version of the Siegel-Walfisz Theorem for Hecke $L$-functions. As usual, we define the Hecke $L$-function

$$L(s, \psi) = \sum a \psi(a) \text{Norm}(a)^{-s} \quad (\Re(s) > 1),$$

where the sum is over all non-zero ideals $a \subset \mathbb{Z}[\sqrt{d_0}]$. By [20, Theorem 3.3.1, p. 93], $L(s, \psi)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies a functional equation as well as other standard properties of $L$-functions. As $\psi$ is not the trivial character, the order of the pole at $s = 1$ of $L(s, \psi)$ is 0. Hence [13, Main Theorem, p.418] implies that there is a constant $c > 0$ such that for all $p \leq (\log X)^{100}$, we have

$$\sum_{\text{Norm}(q) \leq X} \psi(q) \ll X \exp\left(c\sqrt{\log X}\right).$$

This completes the proof of Proposition 5.

Remark. The range of $p$ for which the above bound holds could be extended to $\exp\left(c'\sqrt{\log X}\right)$ for some small $c' > 0$ instead of a power of $\log X$ if we were certain that $L(s, \psi)$ has no Siegel zeros. Although this is conjectured to be true in any case, we can only show it in the case $d = -4$ and $e_2 = 1$. In both cases $d = -4$ and $d = 8$, when $e_2 = 1$, the theta series

$$\Theta(z, \psi) = \sum a \psi(a) \exp(2\pi i \text{Norm}(a))$$

is a holomorphic modular form of weight 1 and level $4p$ (see [20, Theorem 4.8.2, p. 183]). Now a theorem of Hoffstein and Ramakrishnan [16, Theorem C, p.299] implies that the associated $L$-function $L(s, \psi)$ has no Siegel zero whenever $\Theta(z, \psi)$ is a cusp form. If $d = -4$, then this is indeed the case. Otherwise, if $d = 8$, unfortunately $\Theta(z, \psi)$ is not a cusp form.

6. Proof of Proposition 6

In this section we finish the proof of Proposition 6 and hence also the Main Theorem. We will use power-saving upper bounds for very general bilinear sums that were obtained in [9] for $d = -4$ and [18] for $d = 8$. These results are most naturally stated in the quadratic ring $\mathbb{Z}[\sqrt{d_0}]$, so we first develop a correspondence between rational primes and those in $\mathbb{Z}[\sqrt{d_0}]$.

6.1. From Primes in $\mathbb{Z}$ to Primes in $\mathbb{Z}[\sqrt{d_0}]$. To give bounds for the sum

$$B(M, N; f, \Delta) = \sum_{M < p \leq M + \Delta M} \sum_{N < q \leq N + \Delta N} f(p, q)$$

from Proposition 6, we will transform the summations over primes in $\mathbb{Z}$ to summations over primes in $\mathbb{Z}[\sqrt{d_0}]$. In $\mathbb{Z}$, there is a clear canonical choice for a generator of an ideal, namely the positive generator. In this way prime numbers correspond to prime ideals. In the rings $\mathbb{Z}[\sqrt{d_0}]$, where the unit groups are larger, this choice becomes more delicate. Given primes $p$ and $q$ satisfying (53), we have

$$\varepsilon(p, q) = \gamma(w, z) = \gamma(z, w)$$
for \( \text{any} \) choice of \( w, z \in \mathbb{Z}[^d] \) satisfying [8] and, if \( d = 8 \), also [18] and [22]. In this sense, it might appear as though making special choices for generators is superfluous. However, we must make these choices for the sake of subsequent analytic arguments. We often sum over connected regions in \( \mathbb{Z}[^d] \otimes \mathbb{Q} \mathbb{R} \), so it is important for the generators to be chosen in some structured way.

6.1.1. The case \( d = -4 \). In this case, given an odd ideal \( a \subset \mathbb{Z}[i] \), we choose a generator of \( a \) which is primary, i.e., an element \( z \in \mathbb{Z}[i] \) such that \((z) = a\) and
\[
z \equiv 1 \mod t^3,
\]
where now \( t = (1 + i) \) is the (ramified) prime of \( \mathbb{Z}[i] \) lying above 2. Each odd ideal \( a \) has exactly one primary generator. The benefit of choosing generators in this way is that they lie in a sub-lattice of \( \mathbb{Z}[i] \sim \mathbb{Z} \oplus \mathbb{Z} \). In fact, an odd element \( z = a + bi \in \mathbb{Z}[i] \) is primary if and only if \( a \equiv 1 \mod 2 \) and \( a + b \equiv 1 \mod 4 \).

To every prime \( p \equiv 1 \mod 8 \) we can now associate two elements of \( \mathbb{Z}[i] \), namely the two primary generators \( w \) and \( w \) of the two prime ideals of \( \mathbb{Z}[i] \) lying above \( p \). Although this correspondence is not \( 1 : 1 \) but \( 1 : 2 \), this is not a problem because multiplying the sums \( B(M, N; f, \Delta) \) by 2 does not change the outcome of Proposition 6.

6.1.2. The case \( d = 8 \). In \( \mathbb{Z}[^2] \), the unit group \( \mathbb{Z}[^2] \) if of rank 1 as a \( \mathbb{Z} \) module, so every non-zero ideal has an infinite number of generators. Since the norm of a fundamental unit
\[
e = 1 + \sqrt{2}.
\]
is \( \text{Norm}(\varepsilon) = -1 \), every ideal has a generator of positive norm, and we will want to choose among these generators because of the restrictions on \( w \) in the definition of \( \gamma(w, z) \) (see [11]). It is actually
\[
e^2 = 3 + 2\sqrt{2}
\]
that generates the units of norm 1 in \( \mathbb{Z}[^2] \). This means that if we have one representation of a prime \( p \equiv 1 \mod 8 \) as
\[
p = w\overline{w},
\]
when all such representations are given by
\[
p = \pm w\varepsilon^{2k} \cdot \pm \overline{w}\varepsilon^{2k}, \quad k \in \mathbb{Z}.
\]
Hence the following result from [13] is useful. Let
\[
\Omega := \left\{(u, v) \in \mathbb{R}^2 : u > 0, -u < \sqrt{2}v < u\right\}.
\]
Then the lattice points \( (u, v) \in \Omega \cap \mathbb{Z}^2 \) precisely enumerate the totally positive elements \( w = u + v^2 \) (i.e., elements such that both \( w \) and \( \overline{w} \) are positive). The ring \( \mathbb{Z}[^2] \) acts on itself by multiplication, and this induces an action
\[
\mathbb{Z}[^2] \times \Omega \to \Omega
\]
given by
\[
(a, b) \cdot (u, v) := (au + 2bv, bu + av).
\]
Since \( \text{Norm}(\varepsilon^2) = 1 \) and since the norm is multiplicative, it follows immediately that \( \varepsilon^2 \cdot \Omega \subset \Omega \).
Let $\mathcal{D}$ be the subset of $\Omega$ defined by
\begin{equation}
\mathcal{D} := \left\{ (u,v) \in \mathbb{R}^2 : u > 0, -2v \leq u \right\}
\end{equation}

In [18, Lemma 14, p.22], we proved that $\mathcal{D}$ is a fundamental domain for the action of $\varepsilon^2$ on $\Omega$ in the following sense.

**Lemma 7.** For each element $(u,v) \in \Omega \cap \mathbb{Z}^2$, there exists exactly one integer $k$ such that $\varepsilon^2 k \cdot (u,v) \in \mathcal{D}$.

From now on, we will say that $w = a + b\sqrt{2}$ is in $\mathcal{D}$ (or in $\Omega$) if $(a,b)$ is in $\mathcal{D}$ (or in $\Omega$).

### 6.2. General bilinear sum estimates.

We can now state a general bilinear sum estimate that we will use to prove Proposition 6. In the following, let $d = -4$ or $d = 8$.

Given two sequences of complex numbers $\alpha = \{\alpha_w\}$ and $\beta = \{\beta_z\}$, each indexed by non-zero elements in $\mathbb{Z}[\sqrt{d}]$, and positive real numbers $M$, $N$, and $\Delta$, we define the bilinear sum
\begin{equation}
B_0(M,N; \Delta, \alpha, \beta) := \sum_{w \in \mathcal{R}_{-4}(M)}^* \alpha_w \sum_{z \in \mathcal{R}_{-4}(N)}^* \beta_z \gamma(w,z),
\end{equation}

where

\begin{align*}
\mathcal{R}_{-4}(M) &= \left\{ w \in \mathbb{Z}[i] : w \text{ odd and primary, } M < \text{Norm}(w) \leq M(1 + \Delta) \right\}, \\
\mathcal{R}_8(N) &= \left\{ w \in \mathbb{Z}[\sqrt{2}] : w \text{ or } -w \in \mathcal{D}, \text{ w odd, } N < \text{Norm}(z) \leq N(1 + \Delta) \right\},
\end{align*}

and $\sum^*_w$ restricts the summation to primitive $w$. Our aim is to prove the following result.

**Proposition 8.** Let $\alpha = \{\alpha_w\}$ and $\beta = \{\beta_z\}$ be bounded sequences of complex numbers, supported on $w$ and $z$ each having at most $f$ prime factors in $\mathbb{Z}[\sqrt{d}]$ and satisfying $\text{Norm}(w) \equiv \text{Norm}(z) \equiv 1 \mod 8$. Then there is a constant $C > 0$ such that for all $M, N > 1$ and $\Delta \in (0,1)$ satisfying $\Delta M, \Delta N \geq 1$, we have
\begin{equation}
|B_0(M,N; \Delta, \alpha, \beta)| \leq C \cdot 2^f \Delta^{-\frac{1}{2}} \left( M^{-\frac{1}{2}} + N^{-\frac{1}{2}} \right) \Delta^2 MN.
\end{equation}

For $d = -4$, the proposition is essentially [9] Proposition 21.3, p. 1027 since $\gamma(w,z)$ coincides with the Dirichlet symbol $(\frac{z}{w})$ defined in [9] Section 19, p. 1018-1021. For $d = 8$, the proposition is essentially [18] Proposition 7, p.29. Here $\gamma(w,z)$ coincides with the symbol defined in [18] Equation (45), p.29. We now give a sketch of the proof and make note of the necessary changes.
We first define a closely related bilinear sum. Given two sequences of complex numbers \( \alpha = \{\alpha_w\} \) and \( \beta = \{\beta_z\} \) and real numbers \( M, N, \Delta > 0 \), we define

\[
Q(M, N; \Delta, \alpha, \beta) := \sum_{w \in \mathcal{R}_d(M)} \sum_{z \in \mathcal{R}_d(N)} \alpha_w \beta_z \gamma(w, z),
\]

so that the sum over \( z \) is no longer restricted to primitive \( z \). The standard first step is to apply the Cauchy-Schwarz inequality and Proposition 7.

**Lemma 8.** Let \( \alpha = \{\alpha_w\} \) and \( \beta = \{\beta_z\} \) be bounded sequences of complex numbers, and suppose that \( \alpha \) is supported on \( w \) with at most \( f \) prime factors in \( \mathbb{Z}[\sqrt{d}] \). Then there is a constant \( C > 0 \) such that for all \( M, N > 1 \) and \( \Delta \in (0, 1) \) satisfying \( \Delta M \geq 1 \), we have

\[
|Q(M, N; \Delta, \alpha, \beta)| \leq C \cdot 2^f \Delta^{\frac{3}{2}} \left( M^\frac{7}{2} N + M^2 N^\frac{3}{2} + M^3 N^\frac{1}{2} \right),
\]

where \( Q(M, N; \Delta, \alpha, \beta) \) is defined as in (56).

**Proof.** Let \( Q(M, N) = Q(M, N; \Delta, \alpha, \beta) \). After applying the Cauchy-Schwarz inequality, using trivial estimates, and expanding and rearranging sums, we obtain

\[
|Q(M, N)|^2 \leq \Delta N \sum_{w_1 \in \mathcal{R}_d(M)} \sum_{w_2 \in \mathcal{R}_d(M)} \alpha_{w_1} \gamma(w_1, z) R(N; w_1, w_2),
\]

where

\[
R(N; w_1, w_2) = \sum_{z \in \mathcal{R}_d(N)} \gamma(w_1, z) \gamma(w_2, z).
\]

For each pair of primitive \( w_1 \) and \( w_2 \), set \( W = \text{Norm}(w_1 w_2) \) and \( r = \text{Norm}(\gcd((w_1), (w_2))) \). Using the Lipschitz principle and Proposition 7, we obtain the estimate

\[
R(N; w_1, w_2) \ll \begin{cases} \Delta N + W(\Delta N)^\frac{3}{2} + W^2 & \text{if } W \text{ and } r \text{ are squares} \\ W(\Delta N)^{\frac{3}{2}} + W^2 & \text{otherwise}. \end{cases}
\]

Using the bound \( W \leq ((1 + \Delta)M)^2 \ll M^2 \), and setting \( m_1 = \text{Norm}(w_1) \), \( m_2 = \text{Norm}(w_2) \), we get

\[
|Q(M, N)|^2 \ll \Delta N \left( \sum_{M < m_1, m_2 \leq ((1 + \Delta)M)^2} A^f \left( \Delta N + M^2 (\Delta N)^{\frac{3}{2}} + M^4 \right) + (\Delta M)^2 \left( M^2 (\Delta N)^{\frac{3}{2}} + M^4 \right) \right).
\]

Here we used the assumption that \( \alpha \) is supported on \( w \) with at most \( f \) prime factors; the number of such \( w \in \mathcal{R}_d(M) \) satisfying \( \text{Norm}(w) = m \) for a given \( m \) is at most \( 2^f \). We then deduce that

\[
Q(M, N) \ll 2^f \left( \Delta^\frac{3}{2} M^\frac{7}{2} N + \Delta^\frac{5}{4} M^2 N^{\frac{3}{2}} + \Delta^\frac{5}{4} M^2 N^{\frac{1}{2}} + \Delta^\frac{5}{4} M^2 N^{\frac{1}{2}} + \Delta^\frac{5}{4} M^3 N^{\frac{1}{4}} \right).
\]

The inequalities \( \Delta < 1 \) and \( \Delta M \geq 1 \) now imply the desired result.

The following method, which appears in [7], exploits the multiplicativity of \( \gamma(w, z) \) in \( z \) to improve the quality of the estimate when \( M \) and \( N \) are close to each other.
Lemma 9. Let \( \alpha = \{\alpha_w\} \) and \( \beta = \{\beta_z\} \) be bounded sequences of complex numbers, and suppose that \( \alpha \) is supported on \( w \) with at most \( f \) prime factors in \( \mathbb{Z}[\sqrt{a_0}] \). Then there is a constant \( C > 0 \) such that for all \( M, N > 1 \) and \( \Delta \in (0,1) \) satisfying \( \Delta M \geq 1 \), we have

\[
|Q(M, N; \Delta, \alpha, \beta)| \leq C \cdot 2^f \Delta^{\frac{11}{12}} \left( M^\frac{11}{12} N + M^\frac{5}{6} N^\frac{2}{3} + M^\frac{7}{12} N^\frac{1}{2} \right),
\]

where \( Q(M, N; \Delta, \alpha, \beta) \) is defined as in (56).

Proof. We apply Hölder’s inequality to get

\[
|Q(M, N)|^6 \leq \left( \sum_w |\alpha_w|^6 \right)^5 \sum_w |\beta_z \gamma(w, z)|^6.
\]

By (46), we can write the second factor above as

\[
\sum_w \left| \sum_z \beta_z \gamma(w, z) \right|^6 =: \sum_w \sum_z \alpha'_w \beta'_z \gamma(w, z),
\]

where \( \alpha'_w = m(w)^5 \) and

\[
\beta'_z = \prod_{z_1, \ldots, z_6 \in \mathcal{R}_d(N)} \beta_{z_1} \beta_{z_2} \cdots \beta_{z_5} \beta_{z_6}.
\]

Note that \( \beta_z \) is supported on \( z \) such that \( N^6 < \text{Norm}(w) \leq (1 + \Delta)^6 N^6 \leq N^6 + \Delta N^6 \cdot O(1) \). Now using Lemma 8 to estimate the sum (58), and substituting back into (57), we obtain the desired result. \( \square \)

The final step is to exploit the symmetry of the symbol \( \gamma(w, z) \) coming from its reciprocity law. Suppose \( x = a + b \sqrt{a_0} \in \mathbb{Z}[\sqrt{a_0}] \) is primitive and satisfies \( \text{Norm}(x) \equiv 1 \mod 8 \) and \( \text{Norm}(x) > 0 \). Then

\[
m(x) = \left( \frac{2}{\text{Norm}(x)} \right) \left( \frac{2}{\text{Norm}(x)} \right) \left( \frac{|a|}{\text{Norm}(x)} \right) \left( \frac{\text{sign}(a)}{\text{Norm}(x)} \right) = \left( \frac{|a|}{\text{Norm}(x)} \right) \left( \frac{\text{Norm}(x)}{|a|} \right) = \left( -d_0 \right) \left( \frac{|a|}{|a|} \right).
\]

If \( d = -4 \), then \( m(x) = 1 \), and so Lemma 5 implies that

\[
\gamma(w, z) = \gamma(z, w)
\]

for every pair \( w, z \) satisfying \( \text{Norm}(w) \equiv \text{Norm}(z) \equiv 1 \mod 8 \) (note that if \( \gcd((w), (z)) \neq (1) \), then both sides of the equation above are equal to 0). If \( d = 8 \), the situation is more complicated. In this case \( m(x) \) is determined by the congruence class of \( |a| \) modulo 8, which is in turn determined by the congruence class of \( a \) modulo 8 and the sign of \( a \).

Suppose \( d = 8 \), and let \( w = w_1 + w_2 \sqrt{2} \in \mathcal{R}_d(M) \) and \( z = z_1 + z_2 \sqrt{2} \in \mathcal{R}_d(N) \). Then either \( w \) or \( -w \) is in \( \Omega \) and similarly for \( z \) and, as \( \Omega \) is closed under multiplication, also for \( wz \). Thus \( \text{sign}(w_1)w_1, \text{sign}(z_1)z_1 \), and \( \text{sign}(w_1 z_1 + 2w_2 z_2)wz \) are all in \( \Omega \). Again, as \( \Omega \) is closed under multiplication, we find that

\[
\text{sign}(w_1 z_1 + 2w_2 z_2) = \text{sign}(z_1)\text{sign}(z_2).
\]
We recall the sum from Proposition 6. We defined Proposition 8 by making appropriate choices for the sequences \( \{ \}

Then, under the assumptions of Proposition 8, Lemma 9 gives the estimate

where \( \delta(\mod 8, \mod 8, \sign(w_1), \sign(z_1)) \) depends only on the congruence classes of \( w \) and \( z \) modulo \( 8\mathbb{Z}[\sqrt{d_0}] \) and the signs of \( w_1 \) and \( z_1 \).

We are thus led to decompose the sum \( B_0(M, N; \Delta, \alpha, \beta) \) as

where \( \alpha(w_0, \delta_1) \) and \( \beta(z_0, \delta_2) \) are sequences indexed by non-zero elements of \( 8\mathbb{Z}[\sqrt{d_0}] \) defined by

and

It now suffices to prove Proposition 8 for the sequences \( w_0 \) and \( z_0 \) modulo \( 8\mathbb{Z}[\sqrt{d_0}] \) and each pair of signs \( \delta_1, \delta_2 \in \{ \pm 1 \} \).

Fix congruence classes \( w_0 \) and \( z_0 \) modulo \( 8\mathbb{Z}[\sqrt{d_0}] \) and fix signs \( \delta_1, \delta_2 \in \{ \pm 1 \} \). Then, under the assumptions of Proposition 8 Lemma 9 gives the estimate

By Lemma 5 we have

Note that under the assumptions of Proposition 8 the sum \( B_0(N, M; \Delta, \beta(z_0, \delta_2), \alpha(w_0, \delta_1)) \) satisfies the assumptions of Lemma 9. Thus, applying Lemma 9 gives the estimate

Now taking the minimum of the terms in (59) and (60) in the appropriate ranges, we obtain

thus concluding the proof of Proposition 8.

6.3. From Proposition 8 Proposition 6. We will now prove Proposition 8 from Proposition 6 by making appropriate choices for the sequences \( \{ \alpha_w \} \) and \( \{ \beta_z \} \).

We recall the sum from Proposition 6. We defined

where

Hence \( m(wz) \) is determined by the congruence classes of \( w \) and \( z \) modulo \( 8\mathbb{Z}[\sqrt{d_0}] \) and the signs of \( w_1 \) and \( z_1 \). Lemma 8 then implies that for every \( w, z \in \mathbb{Z}[\sqrt{d_0}] \) satisfying \( \Norm(w) = \Norm(z) \equiv 1 \mod 8 \), we have

\[
\gamma(w, z) = \delta(\mod 8, \mod 8, \sign(w_1), \sign(z_1)) \cdot \gamma(z, w),
\]

where \( \delta(\mod 8, \mod 8, \sign(w_1), \sign(z_1)) \in \{ \pm 1 \} \) depends only on the congruence classes of \( w \) and \( z \) modulo \( 8\mathbb{Z}[\sqrt{d_0}] \) and the signs of \( w_1 \) and \( z_1 \).
where \( r = (r_1, r_2) \in \{1, 9\} \times \{1, 9\}, r_1r_2 \equiv 1 \mod 16 \) if \( d = 8 \), \( s = (s_1, s_2) \in \{\pm 1\} \times \{\pm 1\}, s_1s_2 = 1, e \in \mathbb{F}_{30}^\times, e \neq (0, 0) \), and

\[
f(p, q; r, s, e) = \begin{cases} \chi_p(r)\varepsilon(p, q)^{r_2} & \text{if } (p, q) \equiv r \mod 16 \text{ and } (\chi_2(p), \chi_2(q)) = s \\ 0 & \text{otherwise.} \end{cases}
\]

For \( w, z \in \mathbb{Z}[\sqrt{d_0}] \), we set

\[
\alpha_{f, w} = 1(w \text{ is a prime in } \mathbb{Z}[\sqrt{d_0}] \cdot 1(\text{Norm}(w) \equiv \delta_1 \mod 16) \\
\cdot 1(w \equiv \Box \mod 4\mathbb{Z}[\sqrt{d_0}]) \cdot 1(\chi_2(\text{Norm}(w)) = s_1)
\]

and

\[
\beta_{f, z} = 1(z \text{ is a prime in } \mathbb{Z}[\sqrt{d_0}] \cdot 1(\text{Norm}(z) \equiv \delta_2 \mod 16) \\
\cdot 1(z \equiv \Box \mod 4\mathbb{Z}[\sqrt{d_0}]) \cdot 1(\chi_2(\text{Norm}(z)) = s_2)
\]

Since there are exactly two primes in \( \mathbb{Z}[\sqrt{d_0}] \) lying above each rational prime congruent to 1 modulo 8, we have, by (53), that

\[
E(M, N; f, \Delta) = \frac{1}{2} \sum_{w \in \mathcal{R}_d(M)} \sum_{z \in \mathcal{R}_d(N)} \alpha_{f, w} \beta_{f, z} \left( \frac{\text{Norm}(w)}{\text{Norm}(z)} \right)^{e_1} \gamma(w, z)^{r_2}.
\]

If \( e_2 = 0 \), then Proposition 8 is a statement about double oscillation of the usual Jacobi symbol \( \left( \frac{\cdot}{q} \right) \), and the claim follows from very strong bounds due to Heath-Brown (see [14]). If \( e_2 = 1 \), then we apply Proposition 8. Indeed, if \( e_1 = 0 \) and \( e_2 = 1 \), we can apply Proposition 8 directly to obtain the desired result. If \( e_1 = 1 \) and \( e_2 = 1 \), then by Lemma 12 we have

\[
\left( \frac{\text{Norm}(w)}{\text{Norm}(z)} \right)^{e_1} \gamma(w, z) = \gamma(w, z)
\]

whenever \( (w) \) and \( (z) \) are degree-one primes in \( \mathbb{Z}[\sqrt{d_0}] \) satisfying \( \text{Norm}(w) \equiv \text{Norm}(z) \equiv 1 \mod 8 \). The conclusion of Proposition 8 does not change upon applying the non-trivial automorphism of \( \mathbb{Q}(\sqrt{d}) \), and so the desired result follows once again.

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