Principle of Pattern Selection for Nonequilibrium Phenomena

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Abstract

A general principle is advanced allowing the classification of nonunique solutions to nonlinear evolution equations, corresponding to different spatio-temporal patterns. This is done by defining the probability distribution of patterns, which characterizes multiple solutions as more or less probable with respect to each other. The most probable pattern is naturally defined by the maximum of the pattern distribution. This maximum is shown to be equivalent to the minimum of local contraction. The formulated principle plays for nonequilibrium dynamical systems the same ordering role as the condition of minimal free energy for equilibrium statistical systems. The generality of the principle is illustrated by several examples of different nature.

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1 Introduction

The problem of pattern selection is an old problem that has not yet found an appropriate solution. This problem appears when a system of nonlinear evolution equations possesses, under the same initial and boundary conditions, several solutions describing different spatio-temporal structures. Suppose, for convenience, that these solutions can be parametrized by a multiparameter $\beta$ from a manifold $\mathcal{B}$. It often happens that all solutions labelled by different $\beta \in \mathcal{B}$ are stable, then manifold $\mathcal{B}$ is called the stability balloon [1]. Since all solutions from the ensemble attached to the stability balloon are stable, there is no general way of distinguishing between such solutions and therefore between the related patterns. But the necessity of distinguishing them arises because the real life, that is assumed to be described by the corresponding evolution equations, does distinguish between different patterns: some of the latter appear in a given experimental protocol more often than other. Experiments demonstrate that Nature does prefer some patterns as more probable.

A similar problem exists in equilibrium statistical mechanics where it often happens that a nonlinear equilibrium system possesses several solutions for an observable quantity, say for an order parameter. The way of treating such a nonuniqueness of solutions for equilibrium systems is well known, being given by the condition of the minimal free energy: More stable is that solution and, respectively, that thermodynamic state which corresponds to the lower free energy, the lowest free energy defining an absolutely stable state. But for nonequilibrium systems, there is no such a general ordering principle permitting one to distinguish between more and less probable solutions. This problem of pattern selection has been thoroughly described in the detailed review [1] where one can find numerous references.

The aim of the present paper is threefold: (i) To formulate a general principle of pattern selection for nonequilibrium systems; (ii) To derive its equivalent representations that could be convenient for different cases; (iii) To emphasize the generality of the suggested principle by analysing different special applications.

2 Pattern Distribution

Consider a system of nonlinear evolution equations, which displays the multiplicity of solutions corresponding to different patterns. Let these solutions be parametrized by a multiparameter $\beta$ from a manifold $\mathcal{B}$. For the simplicity of notation, let us examine, first, the case when the considered problem can be reduced to a $d$-dimensional system of ordinary differential equations written in the normal form

$$\frac{d}{dt} y(t) = v(y, t),$$

(1)

where the state $y(t) = \{y_i(t) = y_i(\beta, t) | i = 1, 2, \ldots, d\}$ and velocity field $v(y, t) = \{v_i(y, t) | i = 1, 2, \ldots, d\}$ are the vectors, in which $\beta \in \mathcal{B}$, and $d$ pertains to a countable set. Evolution equations of many rather complicated systems can often be presented in the form (1), including many continuous systems after the reduction of their dynamics to the center manifold [2]. After examining the form (1), it will be straightforward to generalize the consideration for the system of partial differential equations.
As far as Nature does prefer some of the dynamical states, labelled by \( \beta \), as more probable, there should exist a probability measure on the manifold \( B \). If we were able to define a probability distribution \( p(\beta, t) \) for a dynamical system having at the moment \( t \) a dynamical state \( y(t) = y(\beta, t) \), this would be actually the solution of the problem of pattern selection: Then a pattern labelled by \( \beta_1 \) would be preferred over another one labelled by \( \beta_2 \) if \( p(\beta_1, t) > p(\beta_2, t) \). The most probable pattern would be given by the maximum of pattern distribution.

According to the ideas of statistical mechanics [3], a probability \( p \) is related to entropy \( S \) as \( p \sim e^{-S} \). Or, since it is not entropy itself but rather its variation \( \Delta S \) that is measurable, it is more appropriate to write \( p \sim e^{-\Delta S} \). Hence the probability distribution can be presented as

\[
p(\beta, t) = \frac{1}{Z(t)} e^{-\Delta S(\beta, t)} , \quad Z(t) = \int_B e^{-\Delta S(\beta, t)} d\beta .
\] (2)

Thus, the most probable pattern corresponds to the minimum of entropy variation.

In order that the principle of pattern selection would be not just a declaration but a working tool, it is necessary to express the entropy variation through the dynamical states \( y(\beta, t) \). For a nonequilibrium system, the entropy variation can be naturally defined as the difference \( \Delta S(t) = S(t) - S(0) \) (3) with respect to the initial time, which is a kind of relative entropy [4]. Here and in the following intermediate expressions, we shall omit, for brevity, the labelling multiparameter \( \beta \), always keeping in mind its existence and restoring it in final formulas. The entropy may be defined as the logarithm of an elementary phase volume, \( S(t) = \ln |\delta \Gamma(t)| \); the latter, for a dynamical system, being \( \delta \Gamma(t) = \prod_i \delta y_i(t) \). Therefore, the entropy variation (3) writes

\[
\Delta S(t) = \ln |\delta \Gamma(t)|/|\delta \Gamma(0)| .
\] (4)

Introducing [5] the multiplier matrix \( \hat{M}(t) \), with the elements \( M_{ij}(t) \equiv \delta y_i(t)/\delta y_j(0) \), and initial condition \( M_{ij}(0) = \delta_{ij} \), the elementary phase volume can be presented as \( \delta \Gamma(t) = \prod_i \sum_j M_{ij}(t) \delta y_j(0) \), from where \( \Delta S(t) = \ln |\prod_i \sum_j M_{ij}(t) M_{ji}(0)| \). Taking into account the initial condition for the multiplier matrix, the entropy variation (4) becomes

\[
\Delta S(t) = \sum_i \ln |M_{ii}(t)| .
\] (5)

Consequently, the pattern distribution (2) acquires the form

\[
p(\beta, t) = \frac{1}{Z(t)} \exp \left\{ -\sum_i \ln |M_{ii}(\beta, t)| \right\} = \frac{1}{Z(t)} \prod_i |M_{ii}(\beta, t)|^{-1} .
\] (6)

In this way, for an ensemble of dynamical states \( y(\beta, t) \), one may define the multiplier matrix and calculate the pattern distribution (6).

In addition to expression (6), it is useful to give one more representation for the pattern distribution. For this purpose, let us introduce the matrix \( \hat{L}(t) \) with the elements \( L_{ij}(t) \equiv \ln |M_{ij}(t)| \). Then the entropy variation (5) reads

\[
\Delta S(t) = \text{Tr} \hat{L}(t) \equiv \sum_i L_{ii}(t) .
\] (7)
Varying the evolution equation (1), one gets the equation
\[ \frac{d}{dt} \dot{M}(t) = \dot{J}(y, t) \dot{M}(t), \quad J_{ij}(y, t) \equiv \frac{\delta v_i(y, t)}{\delta y_j(t)}, \] (8)
for the multiplier matrix, where \( \dot{J} \) is the Jacobian matrix.

To define the entropy variation (7), one, actually, does not need to know the whole multiplier matrix but only the trace of the matrix \( \dot{L}(t) \). As far as the trace of a matrix does not depend on its representation, one can accomplish intermediate transformations in a chosen particular representation, returning at the end to the form valid for arbitrary representations. Here, at intermediate steps, we may consider the representation where the multiplier matrix is diagonal. If so, equation (8) yields for the diagonal elements of the multiplier matrix
\[ M_{ii}(t) = \exp \left\{ \int_0^t J_{ii}(y(t'), t') dt' \right\}. \]

Then, the diagonal elements of the matrix \( \dot{L}(t) \) are
\[ L_{ii}(t) = \int_0^t \text{Re} J_{ii}(y(t'), t') dt'. \]
Let us introduce the quantity
\[ K(t) \equiv \sum_i \text{Re} J_{ii}(y, t). \]
Without the loss of generality, one may assume that the evolution equation (1) is written for real functions, since an equation for a complex function can always be presented as a pair of equations for real functions. Hence the eigenvalues of the Jacobian matrix are either real or, if complex, come in complex conjugate pairs. This implies that \( \sum_i \text{Re} J_{ii}(y, t) = \text{Tr} \dot{J}(y, t) \). Therefore the notation for \( K \) can be written as
\[ K(t) = \text{Tr} \dot{J}(y, t). \] (9)

The latter, in dynamical theory [6], is termed the contraction rate. Using the equality \( \text{Tr} \dot{L}(t) = \int_0^t K(t') dt' \), we find for the entropy variation (7)
\[ \Delta S(t) = \int_0^t K(t') dt'. \] (10)

With Eq. (10), we get the pattern distribution
\[ p(\beta, t) = \frac{1}{Z(t)} \exp \left\{ - \int_0^t K(t') dt' \right\}. \] (11)

Defining the local contraction
\[ \Lambda(t) \equiv \frac{1}{t} \int_0^t K(t') dt', \] (12)
one may write \( \Delta S(t) = \Lambda(t) t \). Then the pattern distribution (11) takes the form
\[ p(\beta, t) = \frac{1}{Z(t)} \exp \left\{ -\Lambda(\beta, t) t \right\}. \] (13)

The latter shows that the most probable pattern is given by the minimum of local contraction.

The generalization to the case when evolution equations (1) represent a set of partial differential equations is straightforward. Then the dynamical state \( y(t) = \{ y_i(x, t) \} \) consists of functions of time as well as of a set \( x \) of continuous, say spatial, variables. Then \( y(t) \) can be treated as a vector with respect to the discrete index \( i \) and to the
continuous multi-index $x$. The multiplier and Jacobian matrices are to be considered as matrices in $i$ as well as in $x$, having the elements

$$M_{ij}(x, x', t) \equiv \frac{\delta y_i(x, t)}{\delta y_j(x', 0)}, \quad J_{ij}(x, x', y, t) \equiv \frac{\delta v_i(x, y, t)}{\delta y_j(x', t)},$$

the initial conditions for the multiplier matrix being $M_{ij}(x, x', 0) = \delta_{ij} \delta(x - x')$. Thus, employing the matrix notation [5], we may literally repeat the same steps as above, keeping in mind that, instead of one index $i$, we have a pair of $i$ and $x$. Then the sums $\sum_i$ are to be accompanied by the integrals $\int dx$. The product over a continuous variable can be naturally defined [7] as $\prod_x f(x) \equiv \exp \int \ln f(x) dx$. As is clear, for the contraction rate (9), we have

$$K(t) = \sum_i \int J_{ii}(x, x, y, t) \, dx . \quad (14)$$

The pattern distribution retains the same form (13), with the same local contraction (12), where the contraction rate is given by expression (14).

In this way, we have shown that each dynamical state $y(\beta, t)$, corresponding to a pattern labelled by a multi-index $\beta$, can always be equipped with a weight defining the probability distribution of patterns. The latter can be presented in several equivalent forms as (2), (6), (11), or (13). Defining the pattern weights makes it possible to organize a hierarchy among different dynamical states of an ensemble $\{y(\beta, t) | \beta \in \mathcal{B}\}$ of admissible solutions. That state is more preferable, which has a higher weight. The largest weight describes the most probable pattern. It is also possible to define the average pattern as ascribed to the average $\overline{\beta}(t) \equiv \int \beta p(\beta, t) d\beta$. As for any statistical ensemble, one may define the pattern dispersion $\sigma^2(t) \equiv \int \beta^2 p(\beta, t) d\beta - [\overline{\beta}(t)]^2$, the standard deviation, variance coefficient, and so on.

The general principle for pattern selection is, briefly speaking, the maximum of pattern distribution. This can be reformulated in several forms according to a representation employed. For instance, equation (13) shows that this principle can be formulated as the minimum of local contraction. The latter can be expressed either through the diagonal elements of the multiplier matrix or through those of the Jacobian matrix as

$$\Lambda(\beta, t) = \frac{1}{t} \sum_i \int \ln |M_{ii}(x, x, \beta, t)| \, dx = \frac{1}{t} \int_0^t \text{Tr} \, J(y(\beta(t'), \beta, t'), dt' , \quad (15)$$

where trace implies summation over discrete indices and integration over continuous variables. The conditions of minimum, $\partial \Lambda(\beta, t)/\partial \beta = 0$, $\partial^2 \Lambda(\beta, t)/\partial \beta^2 > 0$, define $\beta(t)$ corresponding to a pattern preferable at time $t$. For nonequilibrium dynamical systems, the local contraction plays the same ordering role as free energy for equilibrium statistical systems.

### 3 Particular Cases

In order to emphasize the generality of the advanced principle, we shall consider several particular cases of nonequilibrium systems having rather different properties.
A. One-Dimensional Systems. In this simplest case, the local contraction (12) is what is called the local Lyapunov exponent [9-13], which can also be written as \( \Lambda(t) = \frac{1}{t} \ln |M(t)| \). The limit \( \Lambda = \lim_{t \to \infty} \Lambda(t) \) is the global Lyapunov exponent. Thus, the local contraction is closely connected with the stability properties of dynamical systems. And it becomes clear why a smaller local contraction defines a more probable pattern. This is because a smaller local contraction corresponds to a more stable dynamical system.

B. Hamiltonian Systems. The dynamical state \( y = \{q, p\} \) consists of a pair of sets, \( q = \{q_i(x, t)\} \) and \( p = \{p_i(x, t)\} \), satisfying the system of Hamiltonian equations \( \frac{\partial q}{\partial t} = \delta H/\delta p \) and \( \frac{\partial p}{\partial t} = -\delta H/\delta q \). For the trace of the Jacobian matrix, one has \( \text{Tr} \dot{J} = \text{Tr} (\delta^2 H/\delta p \delta q - \delta^2 H/\delta q \delta p) = 0 \), where the trace includes the summation over \( i \) and integration over \( x \), as in Eq. (14). Consequently, \( \Lambda = K = 0 \), which means that there are no multiple patterns but each pattern is to be uniquely defined by initial and boundary conditions.

C. Chaotic Systems. In the spectrum of Lyapunov exponents, there are positive exponents. The sum of the latter is called the Kolmogorov-Sinai entropy [8]. The difference with the latter in our case is that the limit \( \Lambda = \lim_{t \to \infty} \Lambda(t) \) is the sum of all Lyapunov exponents, but not of only positive ones. Hence, the principle of the local-contraction minimum makes it possible to classify even chaotic systems onto more or less stable. It is worth emphasizing that the local contraction characterizes the local stability. This is important since the asymptotic, as \( t \to \infty \), stability is known to be a too rough notion for typical dynamical systems describing realistic physical situations, as far as the phase spaces of these systems are usually incredibly complicated, being composed of a mixture of stability islands and chaotic domains [14,15].

D. Dissipative Systems. These, by definition, have negative contraction rates, \( K = \text{Tr} \dot{J} < 0 \). The evolution equations for dynamical states \( y = y(\beta, t) \) are often written in the form \( \frac{\partial y}{\partial t} = -\delta F/\delta y \), where \( \text{Tr} \delta^2 F/\delta y^2 > 0 \), with \( F = F[y] \) being a functional of \( y \). Such a form of the evolution equation is typical, e.g., for the evolution of order parameters [16]. The local contraction (12) becomes

\[
\Lambda(\beta, t) = -\frac{1}{t} \int_0^t \frac{\text{Tr} \delta^2 F[y]}{\delta y^2(\beta, t')} dt'.
\]

In the process of evolution, the state \( y \) tends to a solution minimizing the functional \( F[y] \). If the latter possesses two or several minima, one encounters the so-called bistability or, respectively, multistability effects. The minima of \( F[y] \) are attractors of the dynamical system. Each attractor is surrounded by its basin of attraction. For initial conditions inside a basin of attraction, the solution always tends to the corresponding attractor. The problem of pattern selection arises when initial conditions are on the Julia set separating different basins of attraction. Then the solution may tend to different attractors. The probability of ending at the corresponding attractor, labelled by the index \( \beta \), is given by the pattern distribution (13). The highest probability is defined by the minimum of the local contraction \( \Lambda(\beta, t) \).

E. Isolated Systems. Such systems are characterized by nonnegative entropy production [17]. The latter, as follows from Eqs. (3) and (10), is directly expressed
through the contraction rate $K(\beta, t) = dS(\beta, t)/dt \geq 0$. The second law of thermodynamics for an isolated system tells that the entropy does not decrease, that is $K \geq 0$. The system, as time increases, tends to an equilibrium state, so that $K(\beta, t) \to K(\beta)$ as $t \to \infty$. Then the local contraction (12) also tends to a constant, $\Lambda(\beta, t) \to K(\beta)$.

The thermodynamic system at equilibrium acquires a structure providing the maximum of entropy and the minimum of entropy production [17]. Labelling this structure by $\beta_0$, one has $K(\beta_0) = 0$ and $K'(\beta_0) = 0$, with $K''(\beta_0) > 0$, where primes imply the derivatives over $\beta_0$. Assume that in the process of evolution, a nonequilibrium system allows the existence of several structures classified by the pattern distribution (13). With time, one has

$$p(\beta, t) \approx \frac{1}{Z(\beta, t)} \exp\{-K(\beta) t\}, \quad Z(\beta, t) \approx \int \exp\{-K(\beta) t\} d\beta,$$

as $t \to \infty$. When the system tends to equilibrium, where the contraction rate is minimal, and $K(\beta_0) = 0$, then, employing the Laplace method, we find $Z(\beta, t) \approx \sqrt{2\pi \gamma(t)}$, $\gamma(t) \equiv [K''(\beta_0)t]^{-1/2}$. Expanding $K(\beta)$ in the vicinity of its minimum, we have

$$p(\beta, t) \approx \frac{1}{\sqrt{2\pi \gamma(t)}} \exp\left\{-\frac{(\beta - \beta_0)^2}{2\gamma^2(t)}\right\}.$$

From here, we see that $p(\beta, t) \to \delta(\beta - \beta_0)$ as $t \to \infty$. This implies that an isolated system tending to equilibrium, even if it possessed the possibility of having several structures in the process of evolution, finally acquires the sole structure defined by the minimum of local contraction or by the maximum of entropy. The latter two conditions, for an equilibrium system, coincide.

**F. Filamentary Structures.** It would be instructive to mention an example for which theoretical results could be directly compared with experimental observations. A good case for this purpose is the problem of the turbulent photon filamentation in resonant media with high Fresnel numbers (see reviews [18,19] and references therein). The problem of pattern formation in nonlinear optics for high Fresnel numbers $F > 10$ is principally different from that for low Fresnel numbers $F < 10$. In the latter case, all arising patterns are uniquely described by the empty-cavity Gauss-Laguerre modes. While at high Fresnel numbers the filament patterns have nothing to do with these modes displaying a nonunique variety of filaments with different radii. First, it has been suggested [20–22] that the problem of pattern selection in nonlinear optics at high Fresnel numbers can be treated being based on the condition of minimal average energy, by analogy with equilibrium systems. However, such a condition of minimal energy, in general, has no grounds for nonequilibrium systems. The latter are to be treated by the approach suggested in this paper. For this purpose, let us consider a system of resonant two-level atoms. The variables describing interlevel transitions and the population difference are given by the statistical averages

$$u(r, t) \equiv 2 < S^{-}(r, t) >, \quad s(r, t) \equiv 2 < S^{z}(r, t) >$$

of the quasispin operators [23]. The evolution equations for these quantities can be derived by invoking the method of eliminating field variables [19,23], which yields

$$\frac{\partial u}{\partial t} = -(i\omega_0 + \gamma_2)u + fs, \quad \frac{\partial |u|^2}{\partial t} = -2\gamma_2|u|^2 + (u^*f + f^*u)s,$$
\[ \frac{\partial s}{\partial t} = - \frac{1}{2} (u^* f + f^* u) - \gamma_1 (s - \zeta) \, . \] (17)

Here \( \omega_0 \) is the atomic transition frequency, \( \gamma_1 \) and \( \gamma_2 \) are the longitudinal and transverse attenuation parameters, \( \zeta > 0 \) is a pumping parameter, and

\[ f(r, t) = -2i d \cdot E_0(r, t) + f_{rad}(r, t) \, , \quad \varphi(r) \equiv \frac{\exp(i k_0 |r|)}{k_0 |r|} \, , \]

\[ f_{rad}(r, t) = - \frac{3}{4} i \gamma \rho \int \left[ \varphi(r - r') u(r', t) - e^{2\gamma} \varphi^*(r - r') u^*(r', t) \right] \, dr' \, , \]

with \( \gamma \equiv (4/3) k_0^2 d_0^2 \), \( k_0 \equiv \omega_0 / c \), \( \rho \) being the density of atoms, and \( d \equiv d_0 e_d \) being the transition dipole. The seed field \( E_0 = \frac{1}{2} (E_1 e^{i(kz - \omega t)} + \frac{1}{2} E_1^* e^{-i(kz - \omega t)} \) selects the longitudinal mode with \( \omega = kc \) and with a small detuning from the resonance, \( |\Delta| \ll \omega_0, \Delta \equiv \omega - \omega_0 \). The evolution equations (17) are nonlinear integro-differential equations which may possess several solutions. Despite that the pumping is uniform, there may appear self-organized transverse modes visible as radiating filaments. The solutions describing a filamentary structure can be presented as the sums over \( N_f \) filaments,

\[ u(r, t) = \sum_{n=1}^{N_f} u_n(r_, t)e^{ikz} \, , \quad s(r, t) = \sum_{n=1}^{N_f} s_n(r_, t) \, , \]

where \( r_\perp \equiv \sqrt{x^2 + y^2} \) and the functions \( u_n \) and \( s_n \) are assumed to be essentially nonzero around the axis of an \( n \)-th filament but fastly decreasing outside the filament, so that \( u_n u_m \sim \delta_{mn} \), \( s_m s_n \sim \delta_{mn} \), and \( u_n s_m \sim \delta_{mn} \). Substituting expansions (18) into Eqs. (17), we obtain the evolution equations for the filament functions \( u_n \) and \( s_n \). These equations compose an infinite-dimensional dynamical system. To simplify the problem, we may pass from the infinite-dimensional system to its center manifold that would be of finite dimensionality. To this end, let us introduce the averaged functions

\[ u(t) \equiv \frac{1}{V_n} \int_{V_n} u_n(r_, t) \, dr \, , \quad s(t) \equiv \frac{1}{V_n} \int_{V_n} s_n(r_, t) \, dr \, , \]

with the averaging accomplished over the cylinder enveloping the \( n \)-th filament. The volume of the enveloping cylinder is \( V_n = \pi b_n^2 L \), where \( b_n \) is the cylinder radius and \( L \) is the length of the sample. The enveloping cylinder radius \( b_n \) is related to the filament radius \( r_n \) by the conservation-energy relation \( \int |u_n(r_, t)|^2 dr = V_n |u_n(r_n, t)|^2 \). This relation involves the function \( |u_n|^2 \) since a filament as such is defined by its radiation intensity which is proportional to \( |u_n|^2 \). Assuming that the profile of the function \( |u_n(r_, t)|^2 \) is well approximated by the normal law \( \exp(- r_\perp^2 / 2 r_n^2) \), we obtain

\[ b_n = (4e)^{1/4} r_n = 1.82 r_n \, . \] (20)

The whole sample is supposed to have the cylindrical shape of radius \( R \) and length \( L \), with the relation \( R \ll L \) typical of lasers. The radiation wavelength is \( \lambda \ll R \). The evolution equations for the averages (19) take the form

\[ \frac{du}{dt} = -(i \Omega + \Gamma) u + f_1 s \, , \quad \frac{d|u|^2}{dt} = -2i \Gamma |u|^2 + (u^* f_1 + f_1^* u) s \, , \]
\[
\frac{ds}{dt} = -g\gamma_2 |u|^2 - \frac{1}{2}(u^* f_1 + f_1^* u) - \gamma_1 (s - \zeta),
\]

(21)

where \( \Omega \equiv \omega_0 + \gamma_2 g \)'s is the collective frequency, \( \Gamma \equiv \gamma_2 (1 - gs) \) is the collective width, \( f_1 \equiv -id \cdot E_1 e^{-i\omega t} \), and the effective coupling parameters are

\[
g \equiv \frac{3\gamma \rho}{4\gamma_2} \int_{V_n} \frac{\sin(k_0 r - k z)}{k_0 r} \, dr, \quad g' \equiv \frac{3\gamma \rho}{4\gamma_2} \int_{V_n} \frac{\cos(k_0 r - k z)}{k_0 r} \, dr.
\]

(22)

Equations (21) can be solved by invoking the scale separation approach [24]. Taking into account the standard inequalities \( \gamma \ll \omega_0, \gamma_1 \ll \omega_0 \), and \( \gamma_2 \ll \omega_0 \), the solutions to Eqs. (21) can be classified onto fast, \( u \), and slow, \( |u|^2 \) and \( s \). Treating the slow functions as quasi-invariants, for the fast solutions one has

\[
u(t) = \left( u_0 - \frac{sd \cdot E_1}{\omega - \Omega + i\Gamma} \right) e^{-(\Omega + \Gamma)t} + \frac{sd \cdot E_1}{\omega - \Omega + i\Gamma} e^{-i\omega t}.
\]

Substituting this into the equations for the slow functions, we average the right-hand sides of these equations over time and introduce the function

\[
w \equiv |u|^2 - \alpha s^2, \quad \alpha \equiv \frac{\text{Re}}{s\gamma} \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau u^*(t)f_1(t) \, dt = \frac{|d \cdot E_1|^2}{(\omega - \Omega)^2 + \Gamma^2}.
\]

(23)

The role of the seed field is to select a longitudinal mode, the amplitude \( E_1 \) being small, so that \( \alpha \ll 1 \). Finally, for the slow functions, we obtain the equations

\[
\frac{dw}{dt} = -2\gamma_2 (1 - gs)w, \quad \frac{ds}{dt} = -2\gamma_2 w - \gamma_1 (s - \zeta),
\]

(24)

defining the guiding-center solutions. From here, we have the contraction rate, given by Eq. (9), as \( K = -\gamma_1 - 2\gamma_2 (1 - gs) \). The latter depends on the filament radius through relations (20) and (22). Therefore, the pattern distribution (11) gives the distribution of photon filaments with respect to their radii. The most probable filament radius corresponds to the maximum of the pattern distribution (11), that is, to the minimum of the local contraction (12). Filaments with different radii have different values of coupling parameters (22). The stability analysis of Eqs. (24) shows that all filaments, independently of their radii, are stable from the point of view of the asymptotic Lyapunov stability. The classification of filaments onto more or less probable happens at the initial stage of their formation, when the local contraction (12) is

\[
\Lambda(t) \simeq -\gamma_1 - 2\gamma_2 (1 - gs_0),
\]

(25)

where \( s_0 \equiv s(0) < 0 \). The minimum of \( \Lambda \) corresponds to the maximum of \( g \). Minimizing Eq. (25) with respect to the filament radius, we find the most probable radius \( r_f = 0.3\sqrt{\lambda L} \), which is in good agreement with experiments (reviewed in Ref. [19]) for various laser media.

G. Time Series. The general principle described in Sec. 2 can also be employed for analyzing time series by providing a probabilistic distribution of extrapolated scenarios. This is possible since a time series is a realization of a random process corresponding to a stochastic dynamical system. Different realizations of a random process can be
presented by different time series (which we may enumerate by the index $\beta = 1, 2, \ldots$) representing the same random process. This implies that there are several sets of data $f_k(\beta)$ associated with the moments of time $t_k(\beta)$, with $k = 0, 1, 2, \ldots$. Each set forms the data base $D_k(\beta) = \{f_k(\beta), f_{k-1}(\beta), \ldots, f_0 | t_k(\beta) < t_{k-1}(\beta) < \ldots < 0\}$, where the backward time ordering is accepted and the present time moments is set to be $t_0(\beta) = 0$. To extrapolate a time series means to construct a forecast, valid for $t > 0$, on the base $D_k(\beta)$ containing the data for $t \leq 0$. Assuming that the considered random process follows the self-similar dynamics [25], we obtain the self-similar forecast

$$f_k^*(\beta, t) = f_0 \exp (c_1 t \exp (c_2 t \ldots \exp (c_k t)) \ldots) ,$$

extrapolating the time series to $t > 0$. Here $c_k = c_k(\beta, t)$ are control functions, or controllers, defined by minimizing a cost functional. Each forecast $f_k^*(\beta, t)$ presents a possible scenario for the extrapolated behaviour of the considered time series. In the present case, scenario is a synonym for pattern. Hence, we may construct a scenario probability $p_k(\beta, t) = 1/Z_k(t) |M_k(\beta, t)|$ by using definition (6). Treating the family $\{f_k^*(\beta, t)| k = 1, 2, \ldots\}$ as the trajectory of a dynamical system with discrete time $k = 1, 2, \ldots$, we find the multipliers $M_k(\beta, t) \equiv \delta f_k^*(\beta, t)/\delta f_1^*(\beta, t)$. Introducing the average multiplier $\overline{M}_k(t)$ by the relation $Z_k(t) \equiv 1/\overline{M}_k(t) = \sum_\beta 1/|M_k(\beta, t)|$, we obtain the scenario distribution

$$p_k(\beta, t) = \frac{\overline{M}_k(t)}{|M_k(\beta, t)|} = \overline{M}_k(t) \exp \{-\Lambda_k(\beta, t) t\}$$

with the local contraction $\Lambda_k = \frac{1}{t} \ln |M_k|$, in agreement with Eq. (15). Thus, the most probable scenario is defined by the minimum of the local contraction $\Lambda_k(\beta, t)$ or, respectively, by the minimum of the multiplier modulus $|M_k(\beta, t)|$. Several examples of particular time series are considered in Ref. [25], where a variant of the scenario distribution was postulated.

In conclusion, a general principle for pattern selection is advanced, based on the definition of the probability distribution of patterns. It is demonstrated that the most probable pattern corresponds to the minimum of local contraction. The suggested approach is shown to be applicable to different dynamical systems describing various nonequilibrium phenomena.
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