Localization of energy for a Kerr black hole

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Abstract

In the teleparallel equivalent of general relativity the energy density of asymptotically flat gravitational fields can be naturally defined as a scalar density restricted to a three dimensional spacelike hypersurface Σ. The scalar density has a simple expression in terms of the trace of the torsion tensor. Integration over the whole Σ yields the standard ADM energy. Here we obtain the formal expression of the localized energy for a Kerr black hole. The expression of the energy inside a surface of constant radius can be explicitly calculated in the limit of small $a$, the specific angular momentum. Such expression turns out to be exactly the same as the one obtained by means of the method proposed recently by Brown and York.

PACS numbers: 04.20.Cv, 04.20.Fy

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I. Introduction

Although there is a strong belief that Einstein’s equations describe the dynamics of the gravitational field, it has not been possible so far to arrive at a definite expression for the gravitational energy in the context of Einstein’s general relativity. On the one hand there are prejudices based on the principle of equivalence, which is often invoked to assure that the gravitational energy cannot be localized\textsuperscript{1, 2}. This principle, however, is not widely accepted as an essential element of the mathematical structure of vacuum general relativity\textsuperscript{3}. On the other hand, attempts based on the Hilbert-Einstein action integral fail to yield an unambiguous expression for the gravitational energy. The latter is normally associated with surface terms in the action or in the Hamiltonian, and these surface terms do not exhibit the appropriate transformation properties under coordinate transformations (see the clear discussion in \textsuperscript{4} in the case of asymptotically flat gravitational fields), a fact which prevents the construction of localized energy density.

Recently an expression for quasi-local energy has been proposed by Brown and York\textsuperscript{5}. Such expression is derived directly from the action functional $A_{cl}$. The latter is identified as Hamilton’s principal function and, in similarity with the classical Hamilton-Jacobi equation, which expresses the energy of a classical solution as minus the time rate of the change of the action, the quasilocal gravitational energy is identified as minus the proper time rate of change of the Hilbert-Einstein action (with surface terms included). Expressions for the quasilocal energy have been obtained for the Schwarzschild solution\textsuperscript{5} and for the Kerr solution\textsuperscript{6}.

Einstein’s equations can also be obtained from the teleparallel equivalent of general relativity (TEGR). The latter ammounts to a formulation which is geometrically different from the standard one based on the Hilbert-Einstein action integral, but whose dynamical
content is the same. In the TEGR the metrical quantity is the tetrad field and the action principle does not require the addition of non-covariant surface terms in the case of asymptotically flat gravitational fields. Because of this property, a localized energy density $\varepsilon(x)$ can be naturally defined from the action principle of the TEGR as minus the variation of the action with respect to the proper time $N(x)$. Integration of $\varepsilon(x)$ over the whole three-dimensional space yields the ADM energy. Moreover, $\varepsilon(x)$ also appears in the expression of the Hamiltonian constraint $C$ of the TEGR, a fact which allows the integral form of $C = 0$ to be written as $H - E_{ADM} = 0$.

We have calculated the energy inside a sphere of radius $r_o$ in a Schwarzschild spacetime by means of $\varepsilon(x)$\cite{7}. The expression turns out to be exactly the same as the one obtained through the procedure of ref.\cite{5} (expression (6.14) of \cite{5}). In this paper we consider the Kerr black-hole. We obtain the formal expression for the total energy in terms of non-trivial integrals in the angular variable $\theta$. In the limit of slow rotation (small specific angular momentum) the energy contained within a surface of constant radius $r_o$ can be calculated. Again the result obtained here is exactly the same as that obtained by Martinez\cite{6} who adopted Brown and York’s procedure. The advantage of our procedure rests on the fact that the localized energy associated with a Kerr spacetime can be calculated in the general case, without recourse to particular limits, at least by means of numerical integration, whereas in Brown and York’s procedure one has to calculate the subtraction term $\varepsilon^0$ and for this purpose it is necessary to embed an arbitrary two dimensional boundary surface of the Kerr space $\Sigma$ in the appropriate reference space ($E^3$, say), which is not always possible\cite{6}.

In section II we present the mathematical preliminaries of the TEGR, its Hamiltonian formulation and the expression of the energy for an arbitrary asymptotically flat spacetime. In section III we carry out the construction of triads for a three dimensional spacelike
hypersurface of the Kerr type, obtain the general expression of the energy contained in
a volume $V$ of space and provide the exact expression of the latter in the limit of slow
rotation. Comments and conclusions are presented on section IV.

Notation: spacetime indices $\mu, \nu, ...$ and local Lorentz indices $a, b, ...$ run from 0 to 3.
In the 3+1 decomposition latin indices from the middle of the alphabet indicate space
indices according to $\mu = 0, i, \quad a = (0), (i)$. The tetrad field $e^a_\mu$ and the spin con-
nection $\omega_{\mu ab}$ yield the usual definitions of the torsion and curvature tensors: $R^a_{\ b\mu\nu} =
\partial_\mu \omega^a_{\nu \ b} + \omega^a_{\mu \ c} \omega^c_{\nu \ b} - ...$, $T^a_{\ \mu\nu} = \partial_\mu e^a_\nu + \omega^a_{\mu \ b} e^b_\nu - ...$. The flat spacetime metric is
fixed by $\eta_{(0)(0)} = -1$.

II. The TEGR in Hamiltonian form

In the TEGR the tetrad field $e^a_\mu$ and the spin connection $\omega_{\mu ab}$ are completely inde-
pendent field variables. The latter is enforced to satisfy the condition of zero curvature.
The Lagrangian density in empty spacetime is given by

$$L(e, \omega, \lambda) = -ke \left( \frac{1}{4} T^{abc}T_{abc} + \frac{1}{2} T^{abc}T_{bac} - T^aT_a \right) + e\lambda^{ab\mu\nu}R_{ab\mu\nu}(\omega). \quad (1)$$

where $k = \frac{1}{16\pi G}$, $G$ is the gravitational constant; $e = det(e^a_\mu)$, $\lambda^{ab\mu\nu}$ are Lagrange
multipliers and $T_a$ is the trace of the torsion tensor defined by $T_a = T^b_{\ ba}$.

The equivalence of the TEGR with Einstein’s general relativity is based on the identity

$$eR(e, \omega) = eR(e) + e\left( \frac{1}{4} T^{abc}T_{abc} + T^{abc}T_{acb} - T^aT_a \right) - 2\partial_\mu (eT^\mu) \quad , \quad (2)$$

which is obtained by just substituting the arbitrary spin connection $\omega_{\mu ab} = \omega_{\mu ab}(e) +
K_{\mu ab}$ in the scalar curvature tensor $R(e, \omega)$ in the left hand side; $\omega_{\mu ab}(e)$ is the Levi-
Civita connection and $K_{\mu ab} = \frac{1}{2} \epsilon_a{}^\lambda \epsilon_b{}^\nu (T_{\lambda \mu \nu} + T_{\nu \lambda \mu} - T_{\mu \nu \lambda})$ is the contorsion tensor. The vanishing of $R^a{}_{b \mu \nu} (\omega)$, which is one of the field equations derived from (1), implies the equivalence of the scalar curvature $R(e)$, constructed out of $e^a{}_{\mu}$ only, and the quadratic combination of the torsion tensor. It also ensures that the field equation arising from the variation of $L$ with respect to $e^a{}_{\mu}$ is strictly equivalent to Einstein’s equations in tetrad form (we refer the reader to refs.[4, 8] for additional details).

It is important to note that for asymptotically flat spacetimes the total divergence in (2) does not contribute to the action integral. Therefore the latter does not require additional surface terms, as it is invariant under coordinate transformations that preserve the asymptotic structure of the field quantities[4]. In what follows we will be interested in asymptotically flat spacetimes.

The Hamiltonian formulation of the TEGR can be successfully implemented if we fix the gauge $\omega_{0ab} = 0$ from the outset, since in this case the constraints (to be shown below) constitute a first class set[3]. The condition $\omega_{0ab} = 0$ is achieved by breaking the local Lorentz symmetry of (1). We still make use of the residual time independent gauge symmetry to fix the usual time gauge condition $e_{(k)}{}^0 = e_{(0)i} = 0$. Because of $\omega_{0ab} = 0$, $H$ does not depend on $P^{kab}$, the momentum canonically conjugated to $\omega^{kab}$. Therefore arbitrary variations of $L = p\dot{q} - H$ with respect to $P^{kab}$ yields $\dot{\omega}^{kab} = 0$. Thus in view of $\omega_{0ab} = 0$, $\omega^{kab}$ drops out from our considerations. The above gauge fixing can be understood as the fixation of a global reference frame.

Under the above gauge fixing the canonical action integral obtained from (1) becomes

$$A_{TL} = \int d^4x \{\Pi^{i(\bar{j})k} \dot{e}_{(\bar{j})k} - H\}, \quad (3)$$
\[ H = NC + N^i C_i + \Sigma_{mn} \Pi^{mn} + \frac{1}{8\pi G} \partial_k (N e T^k) + \partial_k (\Pi^{jk} N_j). \] (4)

\( N \) and \( N^i \) are the lapse and shift functions, and \( \Sigma_{mn} = -\Sigma_{nm} \) are Lagrange multipliers. The constraints are defined by

\[ C = \partial_j (2keT^j) - ke\Sigma^{kij} T_{kij} - \frac{1}{4ke} (\Pi^{ij} \Pi_{ji} - \frac{1}{2} \Pi^2), \] (5)

\[ C_k = -e_{(j)k} \partial_i \Pi^{(j)i} - \Pi^{(j)i} T_{(j)ik}, \] (6)

with \( e = \text{det}(e_{(j)k}) \) and \( T^i = g^{ik} e^{(j)l} T_{(j)lk} \). We remark that (3) and (4) are invariant under global \( \text{SO}(3) \) and general coordinate transformations.

We assume the asymptotic behaviour \( e_{(j)k} \approx \eta_{jk} + \frac{1}{2} h_{jk}(\frac{1}{r}) \) for \( r \to \infty \). In view of the relation

\[ \frac{1}{8\pi G} \int d^3x \partial_j (eT^j) = \frac{1}{16\pi G} \int_S dS_k (\partial_i h_{ik} - \partial_k h_{ii}) \equiv E_{\text{ADM}} \] (7)

where the surface integral is evaluated for \( r \to \infty \), we note that the integral form of the Hamiltonian constraint \( C = 0 \) may be rewritten as

\[ \int d^3x \left\{ ke\Sigma^{kij} T_{kij} + \frac{1}{4ke} (\Pi^{ij} \Pi_{ji} - \frac{1}{2} \Pi^2) \right\} = E_{\text{ADM}}. \] (8)

The integration is over the whole three dimensional space. Given that \( \partial_j (eT^j) \) is a scalar density, from (7) and (8) we define the gravitational energy density enclosed by a volume \( V \) of the space as

\[ E_g = \frac{1}{8\pi G} \int_V d^3x \partial_j (eT^j). \] (9)
In similarity with Brown and York’s procedure, we can also define the energy density as minus the variation of the action $A_{TL}$ with respect to the proper time $N(x)$. Thus for a given set of solutions of the classical equations of motion the energy density $\varepsilon(x)$ can be defined as

$$
\varepsilon(x) = -\frac{\delta A_{TL}}{\delta N(x)} = \frac{1}{8 \pi G} \partial_j (\varepsilon T^j),
$$

(10)
in agreement with (9).

### III. Energy of the Kerr geometry

The Kerr solution\cite{9} describes the field of a rotating black hole. In terms of Boyer and Lindquist coordinates\cite{10} $(t, r, \theta, \phi)$ it is described by the metric

$$
ds^2 = -\frac{\Delta}{\rho^2} [dt - a \sin^2 \theta d\phi]^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2)d\phi - a dt]^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2,
$$

(11)

$$
\Delta \equiv r^2 - 2mr + a^2,
$$

$$
\rho^2 \equiv r^2 + a^2 \cos^2 \theta;
$$

$a$ is the specific angular momentum defined by $a = \frac{J}{m}$. The components of the metric restricted to the three dimensional spacelike hypersurface are given by $g_{11} = \frac{\mu^2}{\Sigma}$, $g_{22} = \rho^2$ and $g_{33} = \frac{\Sigma^2}{\rho^2} \sin^2 \theta$, where $\Sigma$ is defined by

$$
\Sigma^2 = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta.
$$

6
We define the triads \( e_{(k)i} \) as

\[
e_{(k)i} = \begin{pmatrix}
\rho \sqrt{\Delta} \sin \theta \cos \phi & \rho \cos \theta \cos \phi & -\frac{\Sigma}{\rho} \sin \theta \sin \phi \\
\rho \sqrt{\Delta} \sin \theta \sin \phi & \rho \cos \theta \sin \phi & \frac{\Sigma}{\rho} \sin \theta \cos \phi \\
\rho \sqrt{\Delta} \cos \phi & -\rho \sin \theta & 0
\end{pmatrix}
\]  
(12)

\((k)\) is the line index and \(i\) is the column index. The one form \( e^{(k)} \) is defined by

\[
e^{(k)} = e^{(k)r}dr + e^{(k)\theta}d\theta + e^{(k)\phi}d\phi,
\]

from what follows

\[
e^{(k)}e_{(k)} = \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2}{\rho^2} \sin^2 \theta d\phi^2
\]

We also obtain \( e = det(e_{(k)i}) = \frac{\rho \Sigma}{\sqrt{\Delta}} \sin \theta \). Therefore the triads given by (12) describe the components of the Kerr solution restricted to the three dimensional spacelike hypersurface.

One readily notices that there is another set of triads that yields the Kerr solution, namely, the set which is diagonal and whose entries are given by the square roots of \( g_{ii} \). This set is not appropriate for our purposes, and the reason can be understood even in the simple class of flat spacetime. In the limit when both \( a \) and \( m \) go to zero (12) describes flat space: the curvature tensor and the torsion tensor vanish in this case. However, for the diagonal set of triads (again requiring \( a \to 0 \) and \( m \to 0 \)),

\[
e^{(r)} = dr, \ e^{(\theta)} = r \ d\theta, \ e^{(\phi)} = r \sin \theta \ d\phi,
\]

some components of the torsion tensor do not vanish, \( T_{(2)12} = 1 \), \( T_{(3)13} = \sin \theta \), and \( E_{g} \) calculated out of the diagonal set above diverges when integrated over the whole space. Therefore the use of (12) is mandatory in the present context.
The components of the torsion tensor can be calculated in a straightforward way from (12). Only $T(3)_{13}$ and $T(3)_{23}$ are vanishing. The others are given by:

$$
T(1)_{12} = \cos\theta\cos\phi \left( \frac{r}{\rho} + \frac{a^2}{\rho\sqrt{\Delta}} \sin^2\theta - \frac{\rho}{\sqrt{\Delta}} \right)
$$

$$
T(1)_{13} = \sin\theta\sin\phi \left( -\frac{1}{\rho\Sigma} [2r(r^2 + a^2) - a^2\sin^2\theta(r - m)] + \frac{r\Sigma}{\rho^3} + \frac{\rho}{\sqrt{\Delta}} \right)
$$

$$
T(1)_{23} = \cos\theta\sin\phi \left( \rho - \frac{\Sigma}{\rho} + a^2\sin^2\theta \left( \frac{\Delta}{\rho\Sigma} - \frac{\Sigma}{\rho^3} \right) \right)
$$

$$
T(2)_{12} = \cos\theta\sin\phi \left( \frac{r}{\rho} + \frac{a^2}{\rho\sqrt{\Delta}} \sin^2\theta - \frac{\rho}{\sqrt{\Delta}} \right)
$$

$$
T(2)_{13} = -\sin\theta\cos\phi \left( -\frac{1}{\rho\Sigma} [2r(r^2 + a^2) - a^2\sin^2\theta(r - m)] + \frac{r\Sigma}{\rho^3} + \frac{\rho}{\sqrt{\Delta}} \right)
$$

$$
T(2)_{23} = -\cos\theta\cos\phi \left( \rho - \frac{\Sigma}{\rho} + a^2\sin^2\theta \left( \frac{\Delta}{\rho\Sigma} - \frac{\Sigma}{\rho^3} \right) \right)
$$

$$
T(3)_{12} = \sin\theta \left[ -\frac{r}{\rho} + \frac{\rho}{\sqrt{\Delta}} + \frac{a^2}{\rho\sqrt{\Delta}} \cos^2\theta \right]
$$

In order to evaluate (9) we need to obtain $T^i$. After a long calculation we arrive at

$$
T^1 = \frac{\sqrt{\Delta}}{\rho^2} + \frac{\sqrt{\Delta}}{\Sigma} - \frac{\Delta}{\rho^2\Sigma^2} [2r(r^2 + a^2) - a^2\sin^2\theta(r - m)] ,
$$

$$
T^2 = \sin\theta \cos\theta \frac{a^2}{\rho^4} + \frac{1}{\rho\Sigma} \frac{\cos\theta}{\sin\theta} \left[ \rho - \frac{\Sigma}{\rho} + a^2\sin^2\theta \left( \frac{\Delta}{\rho\Sigma} - \frac{\Sigma}{\rho^3} \right) \right] ,
$$

$$
T^3 = 0 .
$$
The gravitational energy density inside a volume $V$ of a three dimensional spacelike hypersurface of the Kerr solution can now be easily calculated. It is given by

$$E_g = \frac{1}{8\pi} \int_V dr \, d\theta \, d\phi \left\{ \frac{\partial}{\partial r} \left[ \sin \theta \left( \rho + \frac{\Sigma}{\rho} \right) - \sqrt{\Delta} \left( 2r(r^2 + a^2) - a^2 \sin^2 \theta (r - m) \right) \right] \right\}$$

$$+ \frac{\partial}{\partial \theta} \left[ \frac{\Sigma a^2}{\sqrt{\Delta} \rho^3} \sin^2 \theta \cos \theta + \frac{\cos \theta}{\sqrt{\Delta}} \left( \rho - \frac{\Sigma}{\rho} + a^2 \sin^2 \theta \left( \frac{\Delta}{\rho \Sigma} - \frac{\Sigma}{\rho^3} \right) \right) \right] \right\} \quad (13)$$

Next we specialize $E_g$ to the case when the volume $V$ is contained within a constant radius surface $r = r_o$, assuming $r_o$ to be greater than the outer horizon $r_+ = m + \sqrt{m^2 - a^2}$. The integrations in $\phi$ and $r$ are trivial. Also, because we integrate $\theta$ between 0 and $\pi$, the second line of the expression above vanishes. We then obtain

$$E_g = \frac{1}{4} \int_0^\pi d\theta \sin \theta \left\{ \rho + \frac{\Sigma}{\rho} - \frac{\sqrt{\Delta}}{\rho \Sigma} \left( 2r(r^2 + a^2) - a^2 \sin^2 \theta (r - m) \right) \right\}_{r=r_o}. \quad (14)$$

We have not managed to evaluate exactly the integral above. However, in the limit of slow rotation, namely, when $\frac{a}{r_o} << 1$ all integrals have a simple structure and we can obtain the approximate expression of $E_g$. It reads

$$E_g = r_o \left( 1 - \sqrt{1 - \frac{2m}{r_o} + \frac{a^2}{r_o^2}} \right) + \frac{a^2}{6r_o} \left[ 2 + \frac{2m}{r_o} + \left( 1 + \frac{2m}{r_o} \right) \sqrt{1 - \frac{2m}{r_o} + \frac{a^2}{r_o^2}} \right] \quad (15)$$

This is exactly the expression found by Martinez[6] for the energy inside the surface of constant radius $r_o$ in a spacelike hypersurface of a Kerr black hole, in the limit of small specific angular momentum. As in ref.[6], we have not expanded the square root which appears in (15) in powers in $\frac{a^2}{r_o^2}$.

Let us mention finally that the expansion of $\rho + \frac{\Sigma}{\rho}$ in the integrand of (14) yields $-\varepsilon_0$, whereas the remaining term corresponds exactly to $\varepsilon$, expressions (3.17) and (3.1)
respectively of \[6\]. It does not seem to be possible, however, to split \(\partial_i(eT^i)\) into two terms such that their integrals arise in the form \(\varepsilon - \varepsilon_0\).

**Comments**

The gravitational energy \(E_g\) defined by (14) can be evaluated for an arbitrary value of \(a\) by means of numerical integration. This is the major advantage of our procedure as compared to that of Brown and York\[5\]. By means of the latter one cannot construct expressions like (13) and (14), which may be useful in the study of astrophysical problems, since in a general situation Brown and York’s procedure requires the embedding of an arbitrary two dimensional boundary surface of the Kerr space in the reference space \(E^3\), a construction which is not possible in general\[6\] (the evaluation of \(\varepsilon_0\) in \[6\] is only possible in the limit \(a/r_o << 1\)). Therefore the present approach is more general than that of ref.\[6\]. Finally we remark that we expect expressions (9) and (10) to be useful in the study of the thermodynamics of self-gravitating systems, where the gravitational energy plays the role of the thermodynamical internal energy that is conjugate to the inverse temperature. We hope to come to this issue in the future.

**Acknowledgements**

This work was supported in part by CNPQ.
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