Abstract

The high-temperature susceptibility of the \( q \)-state Potts model behaves as \( \Gamma|T - T_c|^{-\gamma} \) as \( T \to T_c^+ \), while for \( T \to T_c^- \) one may define both longitudinal and transverse susceptibilities, with the same power law but different amplitudes \( \Gamma_L \) and \( \Gamma_T \). We extend a previous analytic calculation of the universal ratio \( \Gamma / \Gamma_L \) in two dimensions to the low-temperature ratio \( \Gamma_T / \Gamma_L \), and test both predictions with Monte Carlo simulations for \( q = 3 \) and 4. The data for \( q = 4 \) are inconclusive owing to large corrections to scaling, while for \( q = 3 \) they appear consistent with the prediction for \( \Gamma_T / \Gamma_L \), but not with that for \( \Gamma / \Gamma_L \). A simple extrapolation of our analytic results to \( q \to 1 \) indicates a similar discrepancy with the corresponding measured quantities in percolation. We point out that stronger assumptions were made in the derivation of the ratio \( \Gamma / \Gamma_L \), and our work suggests that these may be unjustified.
1 Introduction

Two statistical systems can be characterised by the same internal symmetries and still differ in their microscopic realisation. This difference will be observable as far as the correlation length is not much larger than the microscopic length scale. Nearby a second order phase transition point, however, the microscopic details become irrelevant and the two systems appear as representatives of the same universality class.

In the characterisation of universal behaviour, which is one of the basic tasks of statistical mechanics, one can distinguish different steps. The first one is the determination of the universal features of the critical point (first of all the critical exponents). In two dimensions this goal was achieved with the solution of conformal field theories \[1, 2\]. The second natural step is the study of the scaling region surrounding the fixed point. The leading behaviour of a physical quantity in this region is assigned in terms of a critical amplitude multiplying the suitable power of the temperature. The critical amplitudes depend on metric factors, but they can be used to construct universal combinations which characterise the scaling region \[3\].

It is clear that the computation of the universal amplitude combinations requires a solution of the theory away from criticality. This has become possible over the last years for a large class of two-dimensional quantum field theories characterised by the presence of an infinite number of integrals of motion (integrable field theories) \[4\]. They describe the scaling limit of the isotropic statistical models which are solved on the lattice, but also of many others whose lattice solution is not available. In particular, the universal amplitude ratios for such a basic model as the Ising model in a magnetic field have been computed exactly in this framework \[5\]. More generally, integrable field theory provides accurate approximations for the amplitude ratios \[6, 7\].

For the purpose of comparison with the results provided by integrable field theory, it is clearly desirable to obtain independent estimates for the universal quantities. In view of the difficulties of other traditional approaches (accurate series expansions are available only for few models, and \(d = 2\) is normally too far from the upper critical dimension to obtain reliable estimates through the \(\epsilon\)-expansion), numerical simulations appear as a most valuable source of data. The field theoretical predictions concerning the scaling limit of an infinite system can in principle be tested numerically working in a range of temperature for which the correlation length is much larger than the lattice spacing and much smaller than the lattice size. In practice, however, both the location of this temperature window and the required lattice sizes are model dependent and not obvious to identify.

This paper deals with universal amplitude ratios for the two-dimensional \(q\)-state Potts model and the related problem of isotropic percolation. These have been the subject of
a general study in the framework of integrable field theory in Ref. [7]. Here we mainly focalise on the susceptibility amplitude ratios presenting new theoretical predictions for the ratio (not considered in [7]) of the transverse and longitudinal susceptibilities below $T_c$, and a Monte Carlo study for $q = 3$ and 4.

The $q$-state Potts model [8, 9] is defined by the lattice Hamiltonian

$$H = -J \sum_{(x,y)} \delta_{s(x),s(y)},$$  \hspace{1cm} (1.1)$$

where the sum is over nearest neighbours and the site variable $s(x)$ can assume $q$ possible values (colours). The model is clearly invariant under the group of permutations of the colours. In the ferromagnetic case $J > 0$ we are interested in, the states in which all the sites have the same colour minimise the energy and the system exhibits spontaneous magnetisation at sufficiently low temperatures. There exists a critical temperature $T_c$ above which the thermal fluctuations become dominant and the system is in a disordered phase. We will consider the Potts model in two dimensions in the range of the parameter $q$ for which the phase transition at $T = T_c$ is continuous, namely $q \leq 4$ [10].

Let us introduce the spin variables

$$\sigma_i(x) = \delta_{s(x),i} - \frac{1}{q}, \quad i = 1, 2, \ldots, q$$  \hspace{1cm} (1.2)$$

satisfying the condition

$$\sum_{i=1}^{q} \sigma_i(x) = 0.$$  \hspace{1cm} (1.3)$$

When $T > T_c$ all values of the site variable occur with equal probability $1/q$. At low-temperature, however, one of the $q$ degenerate ground states is selected out by spontaneous symmetry breaking. This might be done either by imposing a symmetry-breaking field which is allowed to tend to zero after taking the thermodynamic limit, or by imposing symmetry-breaking boundary conditions before taking the limit. Without loss of generality we choose the colour of the selected ground state at $T < T_c$ to correspond to $i = 1$. Then, for any temperature, we can write

$$\langle \sigma_i \rangle = \frac{q\delta_{i1} - 1}{q - 1} M,$$  \hspace{1cm} (1.4)$$

where $M$ denotes the ‘longitudinal’ spontaneous magnetisation $\langle \sigma_1 \rangle$ and vanishes at $T > T_c$.

The connected spin–spin correlation functions are given by

$$G_{ij}(x) = \langle \sigma_i(x)\sigma_j(0) \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle.$$  \hspace{1cm} (1.5)$$
If $\nu_i$ denotes the fraction of sites with colour $i$, the magnetic susceptibilities per site can be written as

$$\chi_i = \sum_x G_{ii}(x) = \langle \nu_i^2 \rangle - \langle \nu_i \rangle^2 .$$

(1.6)

Of course, $\chi_i = \chi$ at $T > T_c$, while in the low-temperature phase we have to distinguish between the longitudinal susceptibility $\chi_L = \chi_1$ and the transverse susceptibility $\chi_T = \chi_i \neq 1$. In the vicinity of the critical point, for $q < 4$, the susceptibilities behave as

$$\chi_i \simeq \Gamma_i t^{-\gamma} ,$$

(1.7)

where $t = |T - T_c|/T_c$. Denoting $\Gamma$, $\Gamma_L$ and $\Gamma_T$ the critical amplitudes associated respectively to $\chi$, $\chi_L$ and $\chi_T$, we have the two universal amplitude ratios

$$\Gamma / \Gamma_L , \quad \Gamma_T / \Gamma_L .$$

(1.8)

For $q = 4$ it is well known that quantities like the susceptibility asymptotically exhibit multiplicative logarithmic correction factors of the form $|\ln|t||^{\bar{\gamma}}$ \cite{11, 12}. These are due to a marginally irrelevant operator. The analytic calculation of Ref. \cite{7} was performed in the continuum massive field theory corresponding to a point on the outflowing renormalisation group trajectory. From this point of view, the logarithmic factors arise only when the parameters of the continuum theory are expressed in terms of those of the bare theory. We therefore expect that predictions for such universal quantities as the amplitude ratios above still to be valid when applied to ratios in which the leading logarithmic factors cancel.

In the next section we recall the link between the isotropic percolation problem and the $q \to 1$ limit of the Potts model, and show how in this limit the ratios (1.8) provide some universal information about percolation clusters. In section 3 we recall the origin of the theoretical predictions and present the new analytic results for the ratio $\Gamma_T / \Gamma_L$ and its percolation analogue. Section 4 is devoted to a Monte Carlo study of the ratios (1.8) in the Potts model for $q = 3$ and 4 before discussing the theoretical and numerical results in the final section.

## 2 Connection with percolation

Percolation is the geometrical problem in which bonds are randomly distributed on a lattice with occupation probability $p$ \cite{13}. A set of bonds forming a connected path on the lattice is called a cluster. There exists a critical value $p_c$ of the occupation probability above which an infinite cluster appears in the system; $p_c$ is called the percolation threshold. If $N$ is the total number of bonds in the lattice, the probability of a configuration with
occupied bonds is \( p^{N_b} (1 - p)^{N - N_b} \). Hence, the average of a quantity \( X \) over all configurations \( G \) is
\[
\langle X \rangle = \sum_G X \ p^{N_b} (1 - p)^{N - N_b} .
\] (2.1)

It is well known that the percolation problem can be mapped onto the limit \( q \to 1 \) of the \( q \)-state Potts model [14]. In fact, if we define \( z = e^{J/T} - 1 \), the partition function of the Potts model can be written in the form
\[
Z = \text{Tr}_s \ \prod_{\langle x, y \rangle} (1 + z\delta_{s(x), s(y)}) .
\] (2.2)

A graph \( G \) on the lattice can be associated to each Potts configuration by drawing a bond between two sites with the same colour. In the above expression, a power of \( z \) is associated to each bond in the graph. Taking into account the summation over colours one arrives to the expansion [15]
\[
Z = \sum_G q^{N_c} z^{N_b} ,
\] (2.3)

where \( N_b \) is the total number of bonds in the graph \( G \) and \( N_c \) is the number of clusters in \( G \) (each isolated site is also counted as a cluster). In terms of the partition function (2.3) the \( q \)-state Potts model is well defined even for noninteger values of \( q \). The average of a quantity \( X \) can be written as
\[
\langle X \rangle_q = Z^{-1} \sum_G X \ q^{N_c} z^{N_b} .
\] (2.4)

Hence, it is sufficient to make the formal identification \( z = p/(1 - p) \) to see that \( \langle X \rangle_1 \) coincides with the percolation average (2.1). For \( q \neq 1 \) the Potts model describes a generalised percolation problem in which each cluster can assume \( q \) different colours. The presence of a spontaneous magnetisation \( M \) at \( T < T_c \) reflects the appearance of an infinite cluster at \( p > p_c \).

Let \( P \) denote the probability that a site belongs to the infinite cluster (\( P = 0 \) for \( p < p_c \)). Then, for any value of \( p \), the probability that the site \( x \) has colour \( k \) is
\[
\langle \delta_{s(x), k} \rangle = P\delta_{k1} + \frac{1}{q} (1 - P) .
\] (2.5)

Recalling Eqs. (1.2) and (1.4), we obtain
\[
P = \frac{q}{q - 1} M .
\] (2.6)

Consider now two sites located at \( x \) and \( y \), and call \( P_i \) the probability that they are both in the infinite cluster, \( P_f \) the probability that they are in the same finite cluster, \( P_{ff} \) the
probability that they are in different finite clusters, and \( P_{if} \) the probability that \( x \) is in the infinite cluster while \( y \) is in a finite one. The probability that \( x \) has colour \( k \) and \( y \) has colour \( j \) can be expressed as

\[
\langle \delta_{s(x),k}\delta_{s(y),j} \rangle = P_i \delta_{k1}\delta_{j1} + P_f \frac{1}{q} \delta_{kj} + P_{ff} \frac{1}{q^2} + P_{if} \frac{1}{q} (\delta_{k1} + \delta_{j1}).
\]  

(2.7)

Since \( P_i + P_f + P_{ff} + 2P_{if} = 1 \) and \( P_i + P_{if} = P \), the two-point correlations only depend on two independent functions of \( x - y \), say \( P_i \) and \( P_f \). For the connected spin–spin correlators (1.5) one finds

\[
G_{kj}(x) = \left( \delta_{k1}\delta_{j1} - \frac{1}{q} (\delta_{k1} + \delta_{j1}) + \frac{1}{q^2} \right) P_i(x) + \left( \frac{1}{q} \delta_{kj} - \frac{1}{q^2} \right) P_f(x) - \left( \delta_{k1} - \frac{1}{q} \right) \left( \delta_{j1} - \frac{1}{q} \right) P^2.
\]  

(2.8)

Restrict from now on our attention to the case of ordinary percolation, so that the limit \( q \to 1 \) is understood in all the subsequent equations. From the previous equation we obtain

\[
G_{11}(x) = (q - 1) P_f(x),
\]

(2.9)

\[
G_{kk}(x) = P_i(x) - P^2, \quad k \neq 1.
\]

(2.10)

The average size of finite clusters is given by

\[
S = \sum_x P_f(x) = \frac{1}{q - 1} \sum_x G_{11}(x).
\]

(2.11)

Nearby the percolation threshold this quantity behaves as

\[
S \simeq \sigma_\pm |p_c - p|^{-\gamma},
\]

(2.12)

where the subscripts + and − refer to \( p < p_c \) and \( p > p_c \), respectively, and \( \gamma \) is the Potts critical exponent evaluated at \( q = 1 \). Equation (2.11) implies

\[
\frac{\sigma_+}{\sigma_-} = \frac{\Gamma}{\Gamma_L}.
\]

(2.13)

The quantity

\[
S' = \sum_x (P_i(x) - P^2) = \sum_x G_{kk}(x), \quad k \neq 1
\]

(2.14)

is a measure of the short range correlations inside the infinite cluster and behaves near criticality as

\[
S' \simeq \sigma'|p_c - p|^{-\gamma}.
\]

(2.15)

One can then introduce a second universal ratio \( \sigma'/\sigma_- \) whose relation with the Potts susceptibility amplitudes is

\[
\frac{\sigma'}{\sigma_-} = (q - 1) \frac{\Gamma_T}{\Gamma_L}.
\]

(2.16)
3 Analytic results

The scaling limit of the $q$-state Potts model is an integrable field theory \cite{16, 4}, and this fact allows the evaluation of the correlation functions through the form factor approach, which is of general applicability within integrable field theory. This programme was carried out for the $q$-state Potts model in Ref. \cite{7}. We just recall here the basic steps of the procedure, referring the reader to that paper for all the details.

The starting point is the exact scattering description of the low-temperature phase of the model determined by Chim and Zamolodchikov \cite{17}. Since at $T < T_c$ the model exhibits $q$ degenerate vacua, the elementary excitations entering this scattering description are kinks interpolating among the different vacua. The knowledge of the $S$-matrix allows the computation of the matrix elements (form factors) $\langle 0|\Phi(0)|n \rangle$ entering the spectral decomposition of the correlation functions:

$$
\langle \Phi_1(x)\Phi_2(0) \rangle = \sum_{n=0}^{\infty} \langle 0|\Phi_1(0)|n \rangle \langle n|\Phi_2(0)|0 \rangle e^{-|x|E_n},
$$

(3.1)

where $E_n$ denotes the total energy of the $n$-particle state $|n\rangle$. It is known that in integrable models the spectral series (3.1) exhibit remarkable convergence properties, and that, in particular, very accurate estimates of integrated correlators can be obtained retaining only the terms of the series containing no more than two particles (two-particle approximation). In Ref. \cite{7}, the one- and two-particle form factors of the energy, spin and disorder operators were computed in both phases of the model, the information about the high-temperature phase being obtained by duality. The two-particle approximation for the correlators was then used to evaluate a series of universal amplitude ratios, including $\Gamma_L/\Gamma$. Ref. \cite{4} also contains all the necessary information for the computation (within the same approximation) of the ratio $\Gamma_T/\Gamma_L$, which however was not discussed in that paper. We give in Table 1 the results corresponding to $q = 2, 3, 4$.

Due to technical difficulties, the form factor equations for the spin operator could be solved only for $q = 2, 3, 4$ in Ref. \cite{7}, a limitation which prevents the analytic continuation to the percolation point $q = 1$ for those amplitude ratios which are related to correlation functions of the spin operator. An estimate of these percolation ratios ($\sigma_+ / \sigma_-$, in particular) was however proposed in terms of a simple (quadratic) extrapolation to $q = 1$ of the results obtained for $q = 2, 3, 4$. We do here the same thing for the low-temperature ratio $\sigma'/\sigma_-$. Using the values of $\Gamma_T/\Gamma_L$ given in Table 1 for the extrapolation of Eq. (2.10),
Table 1: Analytic results for susceptibility and correlation length ratios. Those for the susceptibility are in the two-particle approximation, while those for the correlation length are exact.

| \( q \) | 2 | 3 | 4 |
|-------|---|---|---|
| \( \Gamma_T/\Gamma_L \) | 1 | 0.327 | 0.129 |
| \( \Gamma/\Gamma_L \) | 37.699 | 13.848 | 4.013 |
| \( \xi_+/\xi_- \) | 2 | 2 | \( \sqrt{3} \) |

and explicitly incorporating the factor of \((q - 1)\), we find

\[
\frac{\sigma'}{\sigma_-} \approx 1.49 .
\]

According to the considerations developed in [7], we estimate an accuracy of order 1% for our predictions of \( \Gamma_T/\Gamma_L \), while we allow for a 10% error on the value of \( \sigma'/\sigma_- \) to take into account the uncertainty coming from the extrapolation procedure.

We conclude this section with a reminder of the theoretical predictions for the correlation length amplitude ratio which will also be considered in the next section. The “true” correlation length \( \xi \) is defined through the large distance decay of the spin-spin correlation function

\[
\langle \sigma_i(x)\sigma_i(0) \rangle \sim e^{-|x|/\xi} ,
\]

and is determined as the inverse of the total mass of the lightest state entering the spectral series (3.1). For the spin operator at \( q \leq 3 \), this lightest state is the one-kink state for \( T > T_c \), and the two-kink state for \( T < T_c \). For \( q > 3 \) one has to take into account that the two-kink state gives rise to a bound state whose mass equals \( \sqrt{3}m \) at \( q = 4 \), \( m \) being the mass of the kink. Hence, denoting by \( \xi_{\pm} \) the critical amplitudes of the “true” correlation length in the two different phases, the exact results for the ratio for integer \( q \) are those given in Table 1. We also include our results for \( \Gamma/\Gamma_L \) taken from Ref. [7].

4 Computer simulations

Computer simulations are performed on the three- and four-states Potts model on an \( L \times L \) square lattice with periodic (helical) boundary conditions. The Wolff single-spin cluster algorithm Wolff [18] was used, implemented as described in Ref. [19]. A random

\footnote{This is the result of the extrapolation performed in the variable \( \lambda \) (related to \( q \) by \( \sqrt{q} = 2\sin(\pi\lambda/3) \)) in which all the results originating from the scattering theory are analytic. Extrapolating in \( q \) gives \( \sigma'/\sigma_- \approx 1.43 \).}
configuration is generated, and thermalised by applying cluster moves until each spin is statistically flipped 500 times for the three-states Potts model, 1000 times for the four-state Potts model with \( L \leq 400 \), or 2000 times for the four-state Potts model with larger lattices. After the thermalisation, a sequence of 1000 (or for the smallest lattice size 2000) configurations is generated, in which consecutive configurations are separated by a sequence of cluster moves in which each spin is statistically visited 10 times. In each configuration, the magnetisation \( \nu_i \), the fraction of spins in each state \( i \), is determined. From these numbers, the magnetic susceptibility above the critical temperature is calculated using Eq. (1.6) and averaging over \( i \). Below the critical temperature, we expect to find spontaneous symmetry breaking, but of course this cannot occur in a finite system. One way to implement this would be to choose open boundaries and to fix the spins on the boundary to a preferred state. However, this brings in boundary effects and would necessitate using much larger systems. Instead, we simply observe that in any given configuration one colour of spin dominates, and in the thermodynamic limit this will be the preferred orientation of the ground state. Thus we estimate the longitudinal susceptibility from the fluctuations in the fraction of spins which are in the majority state, averaged over all configurations, and the transverse susceptibility from the fluctuations of the fraction of spins in each minority state, averaged over the \((q - 1)\) minority states in each configuration, and then over all configurations. We expect that the difference between the result of this method of estimation and that using fixed boundary conditions to be of order \( e^{-L/\xi} \), and thus exponentially suppressed in the region we study.

We are interested in the ratios \( \Gamma_T/\Gamma_L \) and \( \Gamma/\Gamma_L \), as discussed in the previous sections. The measurements should be performed at temperatures where the correlation length \( \xi \) is large compared to the lattice spacing, but small compared to the system size. To determine approximately the middle of the appropriate temperature regime, we first measured the spin-spin correlation function above and below the critical temperature, in simulations of the \( 200 \times 200 \) three- and four-states Potts models. In figure 4 we plot the correlation length as a function of reduced temperature \( t = (T - T_c)/T_c \) above the critical temperature, and on top of that the correlation length as a function of the scaled reduced temperature \( t' = -c(T - T_c)/T_c \) below the critical temperature. For the three-state Potts model, the curves collapse for \( c_{Q=3} = 2.7 \pm 0.5 \), for the four-state Potts model for \( c_{Q=4} = 2.2 \pm 0.3 \); both values are in agreement with the theoretical expectations \( c_{Q=3} = 2^{1/\nu} = 2.297 \) and \( c_{Q=4} = (\sqrt{3})^{1/\nu} = 2.280 \).

For the three- and four-states Potts models, we measured the susceptibility \( \chi \) for

\(^2\)It should be noted that this is no longer the case as \( T \to T_c^- \) at fixed \( L \), and indeed we found that our susceptibility ratios as estimated this way did not converge to unity as the true ones must in this limit.
lattice sizes $L = 200$ to $1200$, at temperatures $t_+$ above $T_c$. We found surprising difficulty in identifying a window where both finite-size effects and corrections to scaling may simultaneously be ignored. The former are negligible in the region for sufficiently large $t_+$ when the data for different, sufficiently large, values of $L$ all collapse. The latter are negligible if we observe a plateau in the collapsed data when it is multiplied by $(t_+)^\gamma$. An example of such a plateau is shown in Fig. 2 for $q = 3$. We also found it more difficult to identify the scaling window for $T > T_c$ than below the critical temperature. This may be for two reasons: first the correlation length amplitudes are larger above $T_c$, so that one needs to go to larger values of $|t|$ to get rid of finite size effects; and second, periodic boundary conditions move the peak in the finite-size susceptibility to higher temperatures, pushing away the plateau in $\chi(t_+)^\gamma$.

After these initial difficulties we decided to repeat the exercise for the $q = 2$ Ising model and found precisely the same effect. Examination of all three cases led us to the following prescription: we measured the susceptibility above $T_c$ around the temperature where the correlation length is around $\xi = \sqrt{L/2}$; we next measured the longitudinal and transverse susceptibilities $\chi_l$ and $\chi_t$ at the corresponding temperatures $t_- = -t_+ / c$ below $T_c$ where the correlation length is the same; the exact temperatures used in the simulations are listed in table 2. From $\chi(t_+), \chi_L(t_-)$ and $\chi_T(t_-)$ we obtain the required ratios using

$$\frac{\Gamma_T}{\Gamma_L} = \frac{\chi_T(t_-)}{\chi_L(t_-)}$$

(4.1)

$$\frac{\Gamma}{\Gamma_L} = \frac{\chi(t_+)}{\chi_L(t_-)} \left( \frac{t_+}{t_-} \right)^\gamma$$

(4.2)

where $\gamma_{Q=3} = 13/9$ and $\gamma_{Q=4} = 7/6$. The results are presented in table 2. The statistical errors are two standard deviations wide; they are obtained by repeating the same procedure five or ten times, with different random number generator seeds.

5 Analysis and comparison with other work

We first discuss the comparison of our numerical results in Table 2 with the analytic predictions presented in Table 1. The most stable results are those for the ratio $\Gamma_T/\Gamma_L$ for $q = 3$. The analytic prediction of 0.327 is just outside the statistical error bars, but we note that for a fixed temperature there is a consistent trend towards this value with increasing $L$. We conclude that the data support the analytic prediction in this case, particularly given the small but unknown errors of the two-particle truncation, which might be expected to lie in the third decimal place. However, for the ratio $\Gamma/\Gamma_L$, while
Table 2: Numerical estimates for the ratios $\Gamma_T/\Gamma_L$ and $\Gamma/\Gamma_L$.

| $Q$ | $L$ | $t_+$ | $t_- = -t_+/c$ | $\Gamma_T/\Gamma_L$ | $\Gamma/\Gamma_L$ |
|-----|-----|-------|----------------|---------------------|-----------------|
| 3   | 200 | 0.006 | -0.002612 | 0.334 (0.004) | 19.6 (1.6) |
|     |     | 0.012 | -0.005224 | 0.336 (0.002) | 11.8 (0.6) |
|     |     | 0.024 | -0.01045  | 0.347 (0.006) | 9.5 (0.5)  |
|     | 400 | 0.0042| -0.001850 | 0.333 (0.005) | 14.7 (1.2) |
|     |     | 0.0085| -0.003700 | 0.333 (0.006) | 10.4 (0.8) |
|     |     | 0.017 | -0.007401 | 0.338 (0.004) | 9.3 (0.7)  |
|     | 800 | 0.003 | -0.001306 | 0.332 (0.004) | 11.4 (1.4) |
|     |     | 0.006 | -0.002612 | 0.333 (0.007) | 9.7 (0.4)  |
|     |     | 0.012 | -0.005224 | 0.336 (0.005) | 9.6 (0.4)  |
| 4   | 200 | 0.0025| -0.001096 | 0.153 (0.004) | 3.8 (0.4)  |
|     |     | 0.005 | -0.002193 | 0.160 (0.002) | 3.0 (0.3)  |
|     |     | 0.01  | -0.004386 | 0.166 (0.003) | 3.0 (0.2)  |
|     | 400 | 0.0015| -0.000658 | 0.151 (0.004) | 2.5 (0.6)  |
|     |     | 0.003 | -0.001316 | 0.157 (0.003) | 2.7 (0.5)  |
|     |     | 0.006 | -0.002632 | 0.163 (0.003) | 2.8 (0.2)  |
|     | 800 | 0.0009| -0.000395 | 0.146 (0.004) | 2.1 (0.4)  |
|     |     | 0.0018| -0.000789 | 0.152 (0.003) | 2.1 (0.3)  |
|     |     | 0.0036| -0.001579 | 0.159 (0.004) | 2.4 (0.2)  |
|     | 1200| 0.0005| -0.000219 | 0.153 (0.005) | 3.0 (0.7)  |
|     |     | 0.001 | -0.000439 | 0.146 (0.005) | 1.9 (0.6)  |
|     |     | 0.002 | -0.000877 | 0.154 (0.006) | 2.1 (0.4)  |
the data is less stable, there is clear trend towards a value below 10.0, with error bars which, although large, appear to exclude the analytic prediction of 13.8.

The situation for $q = 4$ is more complex. The data for $\Gamma_T/\Gamma_L$ appear fairly stable, yet even in the most favourable case lie at least three standard deviations above the analytic prediction. The situation for $\Gamma/\Gamma_L$ is even worse as the results do not appear to be stable. As remarked earlier, in the amplitude ratios the leading multiplicative logarithmic prefactors should cancel, and even some of the non-leading terms \cite{12}, but there is no reason to suppose this is true for the $O(1/\ln|t|)$ corrections and further. For $\Gamma_T/\Gamma_L$ it is conceivable that these corrections are responsible for the discrepancy with the analytic prediction, but we have not attempted to perform a fit including the $O(1/\ln|t|)$ corrections, since at the reduced temperatures we are working the neglect of the further corrections cannot be justified. The instability of the results for $\Gamma/\Gamma_L$ may be understood on plotting the scaled susceptibilities $\chi(t_+)^\gamma$ and $\chi_L(t_-)^\gamma$, which should, asymptotically, reveal the logarithmic prefactors $(\ln|t|)^{\tilde{\gamma}}$, with $\tilde{\gamma} = \frac{3}{4}$. In fact (see Fig. 3) these have opposite slopes, indicating that, in this region, the effective exponents $\tilde{\gamma}$ have different signs. Once again, this is probably explained by the importance of non-leading and non-universal $1/\ln|t|$ corrections. We deduce that our numerical results are inconclusive for $q = 4$.

We now compare our results with those of some other recent studies.

Salas and Sokal\cite{12} made a detailed study of logarithmic corrections in the $q = 4$ model, including the susceptibility for $T > T_c$. Our raw data appears to be consistent with theirs, in the ranges of $t_+$ and $L$ for which they overlap. However, their main goal was to extract the exponent $\tilde{\gamma}$ of the leading multiplicative logarithmic prefactor. They found that there was no region in which they could isolate such a prefactor and eliminate finite-size corrections. In fact, in order to determine $\tilde{\gamma}$ they had to take such corrections systematically into account using a modified form of finite-size scaling. It is therefore no surprise that it should be impossible to determine the asymptotic amplitude of such a term from data taken over similar ranges.

Caselle et al.\cite{20} have also performed Monte Carlo simulations of the $q = 4$ model with a view to extracting the susceptibility amplitude ratios and also that involving the magnetisation. Before taking into account any logarithmic corrections, these authors’ estimates for the ratio $\Gamma/\Gamma_L$ disagree with the predictions of Ref. \cite{7} by a factor of $\approx 2.5$. By performing a linear extrapolation versus $1/\ln|t|$ they then find a corrected result in reasonably good agreement. However, one might argue that it is difficult to justify such a linear extrapolation, ignoring $O(1/(\ln|t|)^2)$ terms, when the resultant correction is so large. In any case, the analysis of Salas and Sokal\cite{12} indicates that finite-size effects cannot be excluded in the region that the non-leading logarithmic corrections are small.
The agreement with the analytic predictions of Ref. [7] may therefore be fortuitous, particularly since, as we argue here, the latter may well be wrong.

Ziff and co-workers [21], following the appearance of Ref. [7], have reanalysed the percolation data which corresponds to the limit $q \to 1$. Previous quoted results for the ratio of mean cluster size below and above $p_c$ ($\sigma_+ / \sigma_-$, which is the $q \to 1$ limit of $\Gamma / \Gamma_L$, see Sec. 2) had ranged from 14 to 220. The prediction of Ref. [7], based on simple extrapolation of the results for $q = 2, 3, 4$ (which appears to work to 10% accuracy for other amplitude ratios) gives a value around 74. However, Ziff et al.’s value [21] is $163 \pm 2$, in complete disagreement. (Note that in percolation there are no finite-size effects and corrections to scaling are generally small.) Ziff et al. have also measured the ratio $\sigma' / \sigma_-$ of integrated correlations within the infinite cluster to those in the finite clusters and find a value $1.5 \pm 0.2$, in perfect agreement with our extrapolated value of 1.49 in Eq. 3.2.

What conclusion is to be drawn? There appears to be firm confirmation of our new analytic results for the low-temperature ratio $\Gamma_T / \Gamma_L$ both from $q = 3$ and from percolation, while there is strong evidence that the results of Ref. [7] for the ratio $\Gamma / \Gamma_L$ are incorrect. The most likely source of error lies in the computation of the correlation function for $T > T_c$, since the low-temperature calculations are verified by the other ratio. We recall that in fact all calculations in Ref. [7] were performed in the low-temperature phase, and that the order parameter form factors for $T > T_c$ were inferred from those of the disorder operator for $T < T_c$ by duality. In order to fix the ratio $\Gamma / \Gamma_L$ it is therefore crucial to be able to fix the relative normalisation of the order and disorder operator form factors. In Ref. [6] this was done assuming an extension (to the case of theories with internal symmetries) of the factorisation result of [22]. While this extension gives the correct result for $q = 2$, the analysis of this paper suggests that this may not be the case for $q = 3, 4$.

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\footnote{There appears to have been some confusion over this in the literature in the past. For example, Aharony and Stauffer\cite{13} on p. 60 (2nd. ed.) state that the mean cluster size for $p > p_c$ is found by summing the connectedness function $g(r)$ over $r$ and subtracting $P^2$, where $P$ is the probability of a given site belonging to the infinite cluster. However, this would give $\sigma_- + \sigma'$.

\footnote{This would affect the predictions of Ref. [6] for the ratios $\Gamma / \Gamma_L$ and $R_c$, which are the only ones to be sensitive to the relative normalisation of order and disorder operators.}
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Figure 1: Correlation length as a function of reduced temperature $t$ above the critical temperature (circles), and as a function of the scaled reduced temperature $t' = -ct$ (squares), for the three-states (top figure) and four-states Potts model (lower figure). The data points overlap for $c_{Q=3} = 2.7 \pm 0.5$ and $c_{Q=4} = 2.2 \pm 0.3$. The dashed lines show the expected behaviour $\xi \sim |t|^{-\nu}$. 
Figure 2: Susceptibilities for the 3-state Potts model, above and below $T_c$, rescaled by the expected asymptotic power-law dependence $|t|^{-\gamma}$. These plots exhibit the evidence for a plateau region in which both finite-size effects and corrections to scaling are small. The height of the plateau then indicates the relevant amplitude $\Gamma_i$. 
Figure 3: Susceptibilities for the 4-state Potts model, above and below $T_c$, rescaled by the expected asymptotic power-law dependence $|t|^{-\gamma}$. These plots show that although there are windows in which finite-size effects and power-law corrections to scaling may be ignored, the corresponding ‘plateaux’ are in fact sloping. This is presumably due to logarithmic corrections. Notice, however, that while the data for $T < T_c$ are consistent with a multiplicative correction $|\ln |t||^{\bar{\gamma}}$, with a positive $\bar{\gamma}$ as theoretically expected, those for $T > T_c$ exhibit the opposite slope.