FUNDAMENTAL SOLUTIONS OF NONLOCAL HÖRMANDER’S OPERATORS

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Dedicated to the memory of Professor Paul Malliavin

ABSTRACT. Consider the following nonlocal integro-differential operator: for \( \alpha \in (0, 2) \),
\[
\mathcal{L}_{\sigma,b}^{(\alpha)} f(x) := \text{p.v.} \int_{|z|<\delta} \frac{f(x + \sigma(x)z) - f(x)}{|z|^{d+\alpha}} dz + b(x) \cdot \nabla f(x) + \mathcal{L} f(x),
\]
where \( \sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d \) and \( b : \mathbb{R}^d \to \mathbb{R}^d \) are two \( C^\infty \)-functions, \( \delta \) is a small positive number, \( \text{p.v.} \) stands for the Cauchy principal value, \( \mathcal{L} \) is a bounded linear operator in Sobolev spaces. Define \( B_1(x) := \sigma(x) \) and \( B_{j+1}(x) := b(x) \cdot \nabla B_j(x) - \nabla b(x) \cdot B_j(x) \) for \( j \in \mathbb{N} \). Under the following uniform Hörmander’s type condition: for some \( j_0 \in \mathbb{N} \),
\[
\inf_{x \in \mathbb{R}^d} \inf_{|u|=1} \sum_{j=1}^{j_0} |u B_j(x)|^2 > 0,
\]
by using Bismut’s approach to the Malliavin calculus with jumps, we prove the existence of fundamental solutions to operator \( \mathcal{L}_{\sigma,b}^{(\alpha)} \). In particular, we answer a question proposed by Nualart [13] and Varadhan [20].

1. INTRODUCTION

Consider the following nonlocal integro-differential operator: for \( \alpha \in (0, 2) \),
\[
\mathcal{L}_{\sigma,b}^{(\alpha)} f(x) := \text{p.v.} \int_{\mathbb{R}_0^d} \frac{f(x + \sigma(x)z) - f(x)}{|z|^{d+\alpha}} dz + b(x) \cdot \nabla f(x),
\]
(1.1)
where \( \mathbb{R}_0^d := \mathbb{R}^d - \{0\} \) and \( \sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d \) and \( b : \mathbb{R}^d \to \mathbb{R}^d \) are two smooth functions and have bounded derivatives of all orders. Define \( B_1(x) := \sigma(x) \) and \( B_{j+1}(x) := b(x) \cdot \nabla B_j(x) - \nabla b(x) \cdot B_j(x) \) for \( j \in \mathbb{N} \). Recently, in a previous work [22], we have proved that if for each \( x \in \mathbb{R}^d \), there is an \( n(x) \in \mathbb{N} \) such that
\[
\text{Rank}[B_1(x), \cdots, B_{n(x)}(x)] = d,
\]
then the heat kernel \( \rho_t(x,y) \) of operator \( \mathcal{L}_{\sigma,b}^{(\alpha)} \) exists, and as a function of \( y \), it is continuous in \( L^1(\mathbb{R}^d) \) with respect to \( t, x \). Moreover, when \( \sigma(x) = \sigma \) is constant, under the following uniform Hörmander’s type condition: for some \( j_0 \in \mathbb{N} \),
\[
\inf_{x \in \mathbb{R}^d} \inf_{|u|=1} \sum_{j=1}^{j_0} |u B_j(x)|^2 > 0,
\]
the smoothness of \( (t, x, y) \mapsto \rho_t(x,y) \) is also obtained. The proofs in [22] are based on the Malliavin calculus to the subordinated Brownian motion (cf. [11]), i.e., to consider the following stochastic differential equation (abbreviated as SDE):
\[
dX_t(x) = b(X_t(x))dt + \sigma(X_{t-}(x))dW_t, \quad X_0(x) = x,
\]
(1.2)
where \( W_t \) is an \( \alpha \)-subordinated Brownian motion. It is well-known that the generator of Markov process \( X_t(x) \) is given by \( \mathcal{L}_{\sigma,b}^{(\alpha)} \). Thus, the main purpose is to study the existence and smoothness of the distribution density \( \rho_t(x,y) \) of \( X_t(x) \). If \( \sigma(x) \) depends on \( x \), since the solution \( \{X_t(x), x \in \mathbb{R}^d\} \) does not form a stochastic diffeomorphism flow in general (cf. [15]), it seems
impossible to prove the smoothness of $\rho_t(x, y)$ in the framework of [22]. In this work, we shall study the smoothness of $\rho_t(x, y)$ for non-constant coefficient $\sigma(x)$ in a different framework.

As far as we know, Bismut [3] first used Girsanov’s transformation to study the smoothness of distribution densities to SDEs with jumps. Later, in the monograph [2], Bichteler, Gravereaux and Jacod systematically developed the Malliavin calculus with jumps and studied the smooth density for SDEs driven by nondegenerate jump noises. In [14], Picard used difference operator to give another criterion for the smoothness of the distribution density of Poisson functionals, and also applied it to SDEs driven by pure jump Lévy processes. By combining the classical Malliavin calculus and Picard’s difference operator argument, Ishikawa and Kunita in [6] obtained a new criterion for the smooth density of Wiener-Poisson functionals (see also [9]). On the other hand, Cass in [4] established a Hörmander’s type theorem for SDEs with jumps by proving a Norris’ type lemma for discontinuous semimartingales, but the Brownian diffusion term can not disappear. In the pure jump degenerate case, by using a Komatsu-Takeuchi’s estimate proven in [7] for discontinuous semimartingales, Takeuchi [18] and Kunita [8, 10] also obtained similar Hörmander’s theorems. However, their results do not cover operator (1.1). More discussions about their results can be found in [22].

Let us now consider the following nonlocal integro-differential operator

$$
\mathcal{L}_0 f(x) := \text{p.v.} \int_{\Gamma^0_0} (f(x + \sigma(x, z)) - f(x)) \nu(dz) + b(x) \cdot \nabla f(x),
$$

where $\Gamma^0_0 := \{0 < |z| < \delta\}$, $\sigma(x, z) : \mathbb{R}^d \times \Gamma^0_0 \to \mathbb{R}^d$, $b : \mathbb{R}^d \to \mathbb{R}^d$, $\nu(dz)$ is a Lévy measure on $\Gamma^0_0$ and satisfies

$$
\int_{\varepsilon < |z| < \delta} \sigma(x, z) \nu(dz) = 0, \quad \forall \varepsilon \in (0, \delta), \quad x \in \mathbb{R}^d,
$$

and p.v. stands for the Cauchy principal value. Notice that (1.4) is a common assumption in the study of non-local operators.

We make the following assumptions:

(H') $b$ and $\sigma$ are smooth and for any $m \in \{0\} \cup \mathbb{N}$, $j \in \mathbb{N}$ and some $C_m, C_{mj} \geq 1$,

$$
|\nabla^m b(x)| \leq C_m, \quad |\nabla^m \nabla^j \sigma(x, z)| \leq C_{mj} |z|^{(1-j)\nu/v_0}.
$$

(H") $\nu(dz)|_{\Gamma^0_0} = \nu_0(z)dz|_{\Gamma^0_0}$, $\nu_0 \in C^\infty(\Gamma^0_0; (0, \infty))$ satisfies the following order condition:

$$
\lim_{\varepsilon \downarrow 0} \varepsilon^{a-2} \int_{|z| \leq \varepsilon} |z|^2 \kappa(z) dz =: c_1 > 0,
$$

and bounded condition: for any $m \in \mathbb{N}$ and some $C_m \geq 1$,

$$
|\nabla^m \log \kappa(z)| \leq C_m |z|^{-m}, \quad z \in \Gamma^0_0.
$$

(UH) Let $B_1(x) := \nabla_x \sigma(x, 0)$ and define $B_{j+1}(x) := b(x) \cdot \nabla B_j(x) - B_j(x) \cdot \nabla b(x)$ for $j \in \mathbb{N}$. Assume that for some $j_0 \in \mathbb{N}$,

$$
\inf_{x \in \mathbb{R}^d} \inf_{|a| = 1} \sum_{j=1}^{j_0} |uB_j(x)|^2 =: c_0 > 0.
$$

The aim of this paper is to prove the following Hörmander’s type theorem.

**Theorem 1.1.** Let $\mathcal{L}$ be a bounded linear operator in Sobolev spaces $\mathbb{W}^{k,p}(\mathbb{R}^d)$ for any $p > 1$ and $k \in \{0\} \cup \mathbb{N}$. Under (H'), (H") and (UH), if $\delta \leq \frac{1}{2C_{10}}$, where $C_{10}$ is the same as in (1.5), then there exists a measurable function $\rho_t(x, y)$ on $(0, 1) \times \mathbb{R}^d \times \mathbb{R}^d$ called fundamental solution of operator $\mathcal{L}_0 + \mathcal{L}$ with the properties that
i) For each $t \in (0,1)$ and almost all $y \in \mathbb{R}^d$, $x \mapsto \rho_t(x,y)$ is smooth, and there is a $\gamma = \gamma(\alpha, j_0, d)$ such that for any $p \in (1, \infty)$ and $k \in \{0\} \cup \mathbb{N}$,
\[ \|\nabla^k \rho_t(x, \cdot)\|_p \leq C t^{-(k+\gamma)d}, \quad t \in (0,1). \] (1.9)

(ii) For any $p \in (1, \infty)$ and $\varphi \in L^p(\mathbb{R}^d)$, $\mathcal{T}_t \varphi(x) := \int_{\mathbb{R}^d} \varphi(y) \rho_t(x,y) dy \in \cap_{k=0} \mathbb{W}^{k,p}(\mathbb{R}^d)$ satisfies
\[ \partial_t \mathcal{T}_t \varphi(x) = (\mathcal{L}_0 + \mathcal{L}) \mathcal{T}_t \varphi(x), \quad \forall (t,x) \in (0,1) \times \mathbb{R}^d. \] (1.10)

**Remark 1.2.** The role of operator $\mathcal{L}$ is usually referred to the large jump part as shown in Corollary 1.3 below.

Let us briefly introduce the strategy of proving this theorem. Let $N(dr, dz)$ be a Poisson random measure with intensity $d\nu(dz)$, and $\tilde{N}(dr, dz) := N(dr, dz) - d\nu(dz)$ the compensated Poisson random measure. Consider the following SDE:
\[ X_t(x) = x + \int_0^t b(X_s(x)) ds + \int_0^t \int_{\mathbb{R}^d} \sigma(X_s, z) \tilde{N}(ds, dz). \]

Under (H1), it is well-known that the above SDE has a unique solution $X_t(x)$, which defines a Markov process with generator $\mathcal{L}_0$. Let $\mathcal{T}_t^0 f(x) := \mathbb{E} f(X_t(x))$. Our first aim is to show that under (1.5)-(1.8), $X_t(x)$ admits a smooth density, which is a consequence of the following gradient type estimate: for any $m, k \in \{0\} \cup \mathbb{N}$, $p \in (1, \infty)$ and $f \in L^p(\mathbb{R}^d)$,
\[ \|\nabla^m \mathcal{T}_t^0 \nabla^k f\|_p \leq C t^{-\gamma_{mk}} \|f\|_p, \] (1.11)
where $\nabla$ stands for the gradient operator and $\gamma_{mk} > 0$. This will be realized by using Bismut’s approach to the Malliavin calculus with jumps. Of course, the core task is to prove the $L^p$-integrability of the inverse of the reduced Malliavin matrix (see Section 3). In order to treat operator $\mathcal{L}_0 + \mathcal{L}$, letting $\mathcal{T}_t$ be the corresponding semigroup, by Duhamel’s formula, we have
\[ \mathcal{T}_t \varphi = \mathcal{T}_0^0 \varphi + \int_0^t \mathcal{T}_{t-s}^0 \mathcal{L} \mathcal{T}_s \varphi ds. \]

Using the short-time estimate (1.11) and suitable interpolation techniques, we can prove similar gradient estimates for $\mathcal{T}_t \varphi$, which shall produce the desired results by Sobolev’s embedding theorem.

As an application of Theorem 1.1 we consider operator $\mathcal{L}_{\sigma,b}^{(a)}$ in (1.1) with $\sigma$ taking the following form:
\[ \sigma(x) = \begin{pmatrix} 0_{d_1 \times d_1} & 0_{d_1 \times d_2} \\ 0_{d_2 \times d_1} & \sigma_0(x) \end{pmatrix}, \] (1.12)
where $d_1 + d_2 = d$ and $\sigma_0(x)$ is a $d_2 \times d_2$-matrix-valued invertible function.

**Corollary 1.3.** Assume that $\sigma_0$, $b$ are smooth and have bounded partial derivatives of all orders, and (UH) holds. If $\sigma_0$ satisfies
\[ \|\sigma_0^{-1}\|_\infty < \infty, \] (1.13)
then the conclusions in Theorem 1.1 hold for operator $\mathcal{L}_{\sigma,b}^{(a)}$.

**Proof.** Let $\chi_\delta : [0, \infty) \to [0, 1]$ be a smooth function with
\[ \chi_\delta(x) = 1, \quad x \in [0, \frac{\delta}{2}], \quad \chi_\delta(x) = 0, \quad x \in [\delta, \infty). \]

We can write
\[ \mathcal{L}_{\sigma,b}^{(a)} f(x) = \mathcal{L}_0 f(x) + \mathcal{L} f(x), \]
If can use Theorem 1.1 to conclude the proof.

The proof of claim is complete.

By the chain rule and cumbersome calculations, one sees that (1.2) has a density, which is smooth in the first variable. Even in this case, this result seems to the boundedness of $L$.

Claim: $\mathcal{L}$ is a bounded linear operator in Sobolev spaces $\mathbb{W}^{k,p}(\mathbb{R}^d)$.

Proof of Claim: Let $z = (z_1, z_2)$ with $z_1 \in \mathbb{R}^{d_1}$ and $z_2 \in \mathbb{R}^{d_2}$. Define

$$\xi(x, z) := (z_1, \sigma_0^{-1}(x)z_2) \in \mathbb{R}^d.$$ 

Notice that by (1.13), there is a positive constant $c_0 > 0$ such that for all $x, z$,

$$c_0|z| \leq |\xi(x, z)| \leq c_0^{-1}|z|.$$

By the change of variables, we have

$$\mathcal{L} f(x) = \int_{\mathbb{R}^d} (f(x + (0, z_2)) - f(x)) \frac{1 - \chi_\delta(|\xi(x, z)|)}{|\xi(x, z)|^{d+\alpha}} \det(\sigma_0^{-1}(x)) dz$$

$$= \int_{|z_2| \leq \frac{1}{c_0}} (f(x + (0, z_2)) - f(x)) \frac{1 - \chi_\delta(|\xi(x, z)|)}{|\xi(x, z)|^{d+\alpha}} \det(\sigma_0^{-1}(x)) dz.$$ 

Thus, by Minkovskii’s inequality, we have

$$\|\mathcal{L} f\|_p \leq \int_{|z_2| \leq \frac{1}{c_0}} \|f(\cdot + (0, z_2)) - f(\cdot)\|_p \frac{\|\det(\sigma_0^{-1})\|_{\infty}}{(c_0|z|)^{d+\alpha}} dz \leq C\|f\|_p.$$ 

By the chain rule and cumbersome calculations, one sees that

$$\|\nabla^k \mathcal{L} f\|_p \leq C \sum_{j=0}^k \|\nabla^j f\|_p.$$ 

The proof of claim is complete.

Moreover, if we let $\kappa(z) := \chi_\delta(|z|)|z|^{-d-\alpha}$, then it is easy to check that (1.7) is true. Thus, we can use Theorem 1.1 to conclude the proof.

Remark 1.4. If $\sigma(x)$ is non-degenerate and satisfies (1.13), then the law of solutions to SDE (1.2) has a density, which is smooth in the first variable. Even in this case, this result seems to be new as all of the well-known results require that $x \mapsto x + \sigma(x)z$ is invertible (cf. [14, 1]).

In Corollary 1.3, $\sigma$ is required to take a special form (1.12), which has been used to show the boundedness of $\mathcal{L}$ defined by (1.14) in $L^p$-space. Without assuming (1.12), due to the non-invertibility of $x \mapsto x + \sigma(x)z$, it is not any more true that operator $\mathcal{L}$ in (1.14) is bounded in $L^p$-space. Consider the following operator:

$$\mathcal{L}_{\sigma, b}^\nu f(x) := \text{p.v.} \int_{\mathbb{R}^d} (f(x + \sigma(x, z)) - f(x)) \nu(dz) + b(x) \cdot \nabla f(x),$$ 

where $\sigma, b$ and $\nu$ are as above. Let $T_t$ be the corresponding semigroup associated to $\mathcal{L}_{\sigma, b}^\nu$. Instead of working in the Sobolev space, if we consider the Hölder space, then we have

Theorem 1.5. Under (H$\nu$), (H$\nu'$) and (UH), if $\int_{|z|^q} |\nu|^q(dz) < \infty$ for some $q > 0$, then there exists a probability density function $\rho_t(x, y)$ such that for any $\varphi \in L^\infty(\mathbb{R}^d)$, $T_t \varphi(x) = \int_{\mathbb{R}^d} \varphi(y) \rho_t(x, y) dy$ belongs to Hölder space $C^{\alpha + \varepsilon}$, where $\varepsilon \in (0, 1)$ only depends on $\alpha$, $j_0$, $d$ with $\alpha$ from (1.6) and $j_0$ from (1.8). Moreover, if $\alpha < q + \varepsilon$, then $\partial_t T_t \varphi(x) = \mathcal{L}_{\sigma, b}^\nu T_t \varphi(x)$ for all $(t, x) \in (0, 1) \times \mathbb{R}^d$. 




Remark 1.6. Consider operator $L_{\sigma,b}^{(\alpha)}$ in \((1.7)\). Assume that $\sigma$ and $b$ have bounded derivatives of all orders and satisfy (UH). Let $X_t(x)$ be the unique solution of SDE \((1.2)\). For any $\phi \in L^\infty(\mathbb{R}^d)$, by the above theorem, $T_\tau\phi(x) = \mathbb{E}\phi(X_t(x)) \in C^{r+\varepsilon}$ is a classical solution of equation
\[
\partial_t u(t, x) = L_{\sigma,b}^{(\alpha)}u(t, x).
\]
Here we do not assume that $\sigma$ has the form \((1.12)\). The price we have to pay is that the regularity of $T_\tau\phi$ depends on the moment of the Lévy measure.

This paper is organised as follows: In Section 2, we first recall the Bismut’s approach to the Malliavin calculus with jumps, and an inequality for discontinuous semimartingales proven in \([22]\) which is originally due to Komatsu-Takeuchi \([7]\). Moreover, we also prove an estimate for exponential Poisson random integrals. In Section 3, we prove a quantity estimate for the Laplace transform of a reduced Malliavin matrix, which is the key step in our proofs and can be read independently. We believe that it can be used to other frameworks such as Picard \([14]\) or Ishikawa and Kunita \([6]\). In Section 4, we prove the existence of smooth densities for SDEs without big jumps, which corresponds to the operator $L_0$ in \((1.3)\). In Section 5, we treat big jump part and prove our main Theorems \([1.1]\) and \([1.5]\) by interpolation and bootstrap arguments. Finally, in Appendix, two technical lemmas are proven.

Before concluding this introduction, we collect some notations and make some conventions.

- Write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}^d_0 := \mathbb{R}^d - \{0\}$.
- $\nabla := (\partial_1, \cdots, \partial_d)$ denotes the gradient operator.
- For a càdlàg function $f : \mathbb{R}_+ \to \mathbb{R}^d$, $\Delta f := f_s - f_{s-}$.
- The inner product in Euclidean space is denoted by $\langle x, y \rangle$ or $x \cdot y$.
- For $p \in [1, \infty)$, $(L^p(\mathbb{R}^d), \| \cdot \|_p)$ is the $L^p$-space with respect to the Lebesgue measure.
- $\mathcal{W}^{k,p}$: Sobolev space; $\mathcal{H}^{\beta,p}$: Bessel potential space; $\mathcal{H}^{\beta,\infty} = \mathcal{O}^\beta$: Hölder space.
- For a smooth function $f : \mathbb{R}^d \to \mathbb{R}^d$, $(\nabla f)_{i,j} := (\partial_i f^j)$ denotes the Jacobian matrix of $f$.
- $C_0(\mathbb{R}^d)$: The space of all continuous functions with values vanishing at infinity.
- $C_0^k(\mathbb{R}^d)$: The space of all bounded continuous functions with bounded partial derivatives up to $k$-order. Here $k$ can be infinity.
- The letters $c$ and $C$ with or without indices will denote unimportant constants, whose values may change in different places.

2. Preliminaries

2.1. Bismut’s approach to the Malliavin calculus with jumps. In this section, we first recall some basic facts about Bismut’s approach to the Malliavin calculus with jumps (cf. \([3, 2]\) and \([16, \text{Section } 2]\)). Let $\Gamma \subset \mathbb{R}^d$ be an open set containing the original point. Let us define
\[
\Gamma_0 := \Gamma \setminus \{0\}, \quad \varrho(z) := 1 \wedge d(z, \Gamma_0^{-1}),
\]
where $d(z, \Gamma_0)$ is the distance of $z$ to the complement of $\Gamma_0$. Notice that $\varrho(z) = \frac{1}{d(z, \Gamma_0)}$ near 0.

Let $\Omega$ be the canonical space of all integer-valued measure $\omega$ on $[0, 1] \times \Gamma_0$ with $\mu(A) < +\infty$ for any compact set $A \subset [0, 1] \times \Gamma_0$. Define the canonical process on $\Omega$ as follows:
\[
N(\omega; dt, dz) := \omega(dt, dz).
\]

Let $\mathcal{F}_{t \in [0,1]}$ be the smallest right-continuous filtration on $\Omega$ such that $N$ is optional. In the following, we write $\mathcal{F} := \mathcal{F}_1$, and endow $(\Omega, \mathcal{F})$ with the unique probability measure $\mathbb{P}$ such that $N$ is a Poisson random measure with intensity $dv(dz)$, where $\nu(dz) = \kappa(z)dz$ with
\[
k \in C^1(\Gamma_0; (0, \infty)), \quad \int_{\Gamma_0} (1 \wedge |z|^2)\kappa(z)dz < +\infty, \quad |\nabla \log \kappa(z)| \leq C_0(\varrho(z)),
\]
where $g(z)$ is defined by (2.1). In the following we write
\[ \tilde{N}(dt, dz) := N(dt, dz) - d\nu(dz). \]
Let $p \geq 1$ and $k \in \mathbb{N}$. We introduce the following spaces for later use.

- $\mathbb{L}^p$: The space of all predictable processes: $\xi : \Omega \times [0, 1] \times \Gamma_0 \to \mathbb{R}^k$ with finite norm:
  \[ \|\xi\|_{\mathbb{L}^p} := \left[ \mathbb{E} \left( \int_0^1 \int_\Gamma |\xi(s, z)|^p \nu(dz) ds \right)^{\frac{1}{p}} \right]^{\frac{1}{p}} < \infty. \]

- $\mathbb{L}^p_{\infty}$: The space of all predictable processes: $\xi : \Omega \times [0, 1] \times \Gamma_0 \to \mathbb{R}^k$ with finite norm:
  \[ \|\xi\|_{\mathbb{L}^p_{\infty}} := \left[ \mathbb{E} \left( \int_0^1 \int_\Gamma |\xi(s, z)|^p \nu(dz) ds \right)^{\frac{1}{p}} \right]^{\frac{1}{p}} < \infty. \]

- $\mathbb{V}_p$: The space of all predictable processes $v : \Omega \times [0, 1] \times \Gamma_0 \to \mathbb{R}^d$ with finite norm:
  \[ \|v\|_{\mathbb{V}_p} := \|\nabla v\|_{\mathbb{L}^p} + \|v g\|_{\mathbb{L}^p} < \infty, \]
  where $g(z)$ is defined by (2.1). Below we shall write
  \[ \mathbb{V}_{\infty-} := \bigcap_{p \geq 1} \mathbb{V}_p. \]

Moreover, $\mathbb{V}_0$ is dense in $\mathbb{V}_p$ for all $p \geq 1$ (cf. [16, Lemma 2.1]).

Let $C^\infty_p(\mathbb{R}^m)$ be the class of all smooth functions on $\mathbb{R}^m$ which together with all the derivatives have at most polynomial growth. Let $\mathcal{FC}^\infty_p$ be the class of all Poisson functionals on $\Omega$ with the following form:
\[ F(\omega) = f(\omega(g_1), \ldots, \omega(g_m)), \]
where $f \in C^\infty_p(\mathbb{R}^m)$ and $g_1, \ldots, g_m \in \mathbb{V}_0$ are non-random, and
\[ \omega(g_j) := \int_0^1 \int_\Gamma g_j(s, z) \omega(ds, dz). \]

Notice that
\[ \mathcal{FC}^\infty_p \subset \cap_{p \geq 1} L^p(\Omega, \mathcal{F}, \mathbb{P}). \]

For $v \in \mathbb{V}_{\infty-}$ and $F \in \mathcal{FC}^\infty_p$, let us define
\[ D_v F := \sum_{j=1}^m (\partial_j f)(\cdot) \int_0^1 \int_\Gamma v(s, z) \cdot \nabla g_j(s, z) \omega(ds, dz), \]
where “($\cdot$)” stands for $(\omega(g_1), \ldots, \omega(g_m))$.

We have the following integration by parts formula (cf. [16, Theorem 2.9]).

**Theorem 2.1.** Let $v \in \mathbb{V}_{\infty-}$ and $p > 1$. The linear operator $(D_v, \mathcal{FC}^\infty_p)$ is closable in $L^p$. The closure is denoted by $\mathbb{W}^1_p(\Omega)$, which is a Banach space with respect to the norm:
\[ \|F\|_{\mathbb{W}^1_p} := \|F\|_{L^p} + \|D_v F\|_{L^p}. \]
Moreover, for any $F \in \mathbb{W}^1_p(\Omega)$, we have
\[ \mathbb{E}(D_v F) = -\mathbb{E}(F \text{ div}(v)), \] (2.3)
where $\text{div}(v)$ is defined by

$$
\text{div}(v) := \int_0^1 \int_{\Gamma_0} \frac{\text{div}(\kappa v)(s, z)}{\kappa(z)} \tilde{N}(ds, dz).
$$

Below, we shall write

$$
\mathbb{W}^{1,\infty}_V(\Omega) := \bigcap_{p > 1} \mathbb{W}^{1,p}_V(\Omega).
$$

The following Kusuoka and Stroock’s formula is proven in [16, Proposition 2.11].

**Proposition 2.2.** Fix $v \in \mathbb{W}^{1,\infty}_V$. Let $\eta(\omega, s, z) : \Omega \times [0, 1] \times \Gamma_0 \to \mathbb{R}$ be a measurable map and satisfy that for each $(\omega, s, z) \in \Omega \times [0, 1] \times \Gamma_0$, $\eta(\cdot, s, z) \in \mathbb{W}^{1,\infty}_V(\Omega)$, $\eta(\omega, \cdot, \cdot) \in C^1(\Gamma_0)$, and $s \mapsto \eta(s, z)$, $D_v \eta(s, z)$, $\nabla_z \eta(s, z)$ are left-continuous and $\mathcal{F}_s$-adapted,

and for any $p \geq 1$,

$$
\mathbb{E}\left[ \sup_{s \in [0,1]} \sup_{z \in \Gamma_0} \left( \frac{|\eta(s, z)|^p + |D_v \eta(s, z)|^p}{1 + |z|^p} + |\nabla_z \eta(s, z)|^p \right) \right] < +\infty.
$$

Then $\mathcal{I}(\eta) := \int_0^1 \int_{\Gamma_0} \eta(s, z)\tilde{N}(ds, dz) \in \mathbb{W}^{1,\infty}_V(\Omega)$ and

$$
D_v\mathcal{I}(\eta) = \int_0^1 \int_{\Gamma_0} D_v \eta(s, z)\tilde{N}(ds, dz) + \int_0^1 \int_{\Gamma_0} \langle \nabla_z \eta(s, z), v(s, z) \rangle N(ds, dz).
$$

We also need the following Burkholder’s inequalities (cf. [16, Lemma 2.3]).

**Lemma 2.3.** (i) For any $p > 1$, there is a constant $C_p > 0$ such that for any $\xi \in L^1_p$,

$$
\mathbb{E}\left[ \sup_{s \in [0,1]} \left| \int_0^{\infty} \xi(s, z)N(ds, dz) \right|^p \right] \leq C_p \|\xi\|_{L^1_p}^p.
$$

(ii) For any $p \geq 2$, there is a constant $C_p > 0$ such that for any $\xi \in L^2_p$,

$$
\mathbb{E}\left[ \sup_{s \in [0,1]} \left| \int_0^{\infty} \xi(s, z)N(ds, dz) \right|^p \right] \leq C_p \|\xi\|_{L^2_p}^p.
$$

2.2. Two Lemmas. We first recall the following important Komatsu-Takeuchi’s type estimate proven in [22, Theorem 4.2], which will be used in Section 3.

**Lemma 2.4.** Let $(f_i)_{i \geq 0}$ and $(f^0_i)_{i \geq 0}$ be two $m$-dimensional semimartingales given by

$$
f_i = f_0 + \int_0^{\infty} f_s^0 ds + \int_0^{\infty} \int_{|z| \leq \delta} g_s(z)\tilde{N}(ds, dz),
$$

$$
f^0_i = f^0_0 + \int_0^{\tau} f^0_s ds + \int_0^{\tau} \int_{|z| \leq \delta} g^0_s(z)\tilde{N}(ds, dz),
$$

where $\delta \in (0, 1]$, $\tau$ is a stopping time and $f_i, f^0_i, f_s^0$ and $g_i(z), g^0_i(z)$ are càdlàg $\mathcal{F}_t$-adapted processes. Assume that for some $\kappa \geq 1$,

$$
|f_i|^2 \vee |f^0_i|^2 \vee \sup_z \left| \frac{g_i(z)}{1 + |z|^2} \right|^2 \leq \kappa, \ a.s.
$$

Then for any $\varepsilon, T \in (0, 1]$, there exists a positive random variable $\zeta$ with $\mathbb{E}\zeta \leq 1$ such that

$$
c_0 \int_0^T |f^0_i|^2 dt \leq (\delta^{-\frac{1}{4}} + \varepsilon^{-\frac{1}{2}}) \int_0^T |f_i|^2 dt + \kappa \delta^{\frac{1}{4}} \log \zeta + \kappa (\varepsilon \delta^{-\frac{1}{4}} + \varepsilon^{\frac{1}{2}} + T \delta^{\frac{1}{4}}),
$$

where $c_0 \in (0, 1)$ only depends on $\int_{|z| \leq 1} |z|^2 v(dz)$.

The following result will be used in Section 4.
Lemma 2.5. Let \( g_s(z), \eta_s \) be two left continuous \( \mathcal{F}_s \)-adapted processes satisfying
\[
0 \leq g_s(z) \leq \eta_s, \quad |g_s(z) - g_s(0)| \leq \eta_s|z|, \quad \forall |z| \leq 1,
\]
and for any \( p \geq 2 \),
\[
\mathbb{E} \left( \sup_{x \in [0,1]} |\eta_t|^p \right) < +\infty.
\]
If for some \( \alpha \in (0,2) \),
\[
\lim_{\varepsilon \to 0} e^{\alpha - 2} \int_{|z| \leq \varepsilon} |z|^2 \nu(dz) =: c_1 > 0,
\]
then for any \( \delta \in (0,1) \), there exist constants \( c_2, \theta \in (0,1), C_2 \geq 1 \) such that for all \( \lambda, p \geq 1 \) and \( t \in (0,1) \),
\[
\mathbb{E} \exp \left\{-\lambda \int_0^t \int_{\mathbb{R}^d} g_s(z) \zeta(z) N(ds, dz) \right\} \leq C_2 \left( \mathbb{E} \exp \left\{-c_2 \lambda^p \int_0^t g_s(0)ds \right\} \right)^{\frac{1}{p}} + C_2 p \lambda^{-p},
\]
where \( \zeta(z) = \zeta_\delta(z) \) is a nonnegative smooth function with
\[
\zeta_\delta(z) = |z|^3, \quad |z| \leq \delta / 4, \quad \zeta_\delta(z) = 0, \quad |z| > \delta / 2.
\]

Proof. For \( \lambda \geq 1 \) and \( \beta > 0 \), define a stopping time
\[
\tau := \inf\{s > 0 : \eta_s \geq \lambda^\beta \} \wedge 1.
\]
Set
\[
h_t^1 := \int_{\mathbb{R}^d} (1 - e^{-\lambda g_s(\zeta(z))}) \nu(dz)
\]
and
\[
M_t^4 := -\lambda \int_0^{\tau \wedge T} \int_{\mathbb{R}^d} g_s(z) \zeta(z) N(ds, dz) + \int_0^{\tau \wedge T} h_t^4 ds.
\]
By Itô’s formula, we have
\[
e^{M_t^4} = 1 + \int_0^{\tau \wedge T} \int_{\mathbb{R}^d} e^{M_t^4 - (e^{-\lambda g_s(\zeta(z))} - 1)} \tilde{N}(ds, dz).
\]
Since for any \( x \geq 0 \),
\[
1 - e^{-x} \leq 1 \wedge x,
\]
by (2.11) and definition of \( \tau \), we have
\[
M_t^4 \leq \int_0^{\tau \wedge T} h_t^4 ds \leq \int_0^{\tau \wedge T} \int_{\mathbb{R}^d} (1 \wedge (\lambda g_s(\zeta(z))) \nu(dz) ds
\]
\[
\leq \int_{\mathbb{R}^d} (1 \wedge (\lambda^{1+\beta} \zeta(z))) \nu(dz) < \infty.
\]
Hence, \( \mathbb{E} e^{M_t^4} = 1 \) and by (2.14) and Hölder’s inequality,
\[
\mathbb{E} \exp \left\{-\lambda \int_0^{\tau \wedge T} \int_{\mathbb{R}^d} g_s(z) \zeta(z) N(ds, dz) \right\} \leq \left( \mathbb{E} \exp \left\{-\int_0^{\tau \wedge T} h_t^4 ds \right\} \right)^{\frac{1}{2}}.
\]
Since \( 1_{\tau \wedge T} g_s(z) \leq \lambda^\beta \) by (2.11) and definition of \( \tau \), and for any \( x \leq 1 \),
\[
1 - e^{-x} \geq \frac{1}{e},
\]
for any \( q \geq 1 + \beta \), there exists \( c \in (0, 1) \) small enough such that for all \( \lambda \geq 1 \) and \( s < \tau \),

\[
h^1_s \geq \int_{|z| < \xi} (1 - e^{-\lambda g_s(z)|z|^\beta}) v(dz) \geq \frac{\lambda}{e} \int_{|z| < \xi} g_s(z)|z|^{1+\alpha} v(dz) = \frac{\lambda g_s(0)}{e} \int_{|z| < \xi} |z|^{1+\beta} v(dz) \geq \frac{\lambda}{e} \int_{|z| < \xi} (g_s(z) - g_s(0))|z|^{1+\alpha} v(dz). \tag{2.16}
\]

Notice that by (2.12), for any \( p \geq 2 \), there exist constants \( c_0, C_0 > 0 \) such that for all \( \varepsilon \in (0, 1) \) (cf. [16, Lemma 5.2]),

\[
c_0 e^{\beta - \alpha} \leq \int_{|z| \leq \varepsilon} |z|^p \nu(dz) \leq C_0 e^{\beta - \alpha}. \tag{2.17}
\]

If we choose \( \beta \in (0, \frac{q+1}{3-\alpha}) \), \( q = \left\{ \begin{array}{ll} 1 + \beta, & \alpha \in (0, 1], \\ \frac{4(1+\beta)}{4-\alpha}, & \alpha \in (1, 2) \end{array} \right. \),

then by (2.16), (2.11) and (2.17), we further have for all \( \lambda \geq 1 \) and \( s < \tau \),

\[
h^1_s \geq c_2 g_s(0)\lambda^{1-\frac{(q+1)}{4-\alpha}} - C_1 \lambda^{1+\beta - \frac{(q+1)}{2}} \geq c_2 g_s(0)\lambda^{1-\frac{(q+1)}{4-\alpha}} - C_1. \tag{2.18}
\]

On the other hand, by Chebyshev’s inequality, we have for any \( p \geq 2 \),

\[\mathbb{P}(\tau \leq t) = \mathbb{P}\left( \sup_{s \in [0,t]} \eta_s > \lambda^2 \right) \leq \lambda^{-2p} \mathbb{E}\left( \sup_{s \in [0,t]} |\eta_s|^p \right), \]

which together with (2.15) and (2.18) yields the desired estimate (2.13). \( \Box \)

3. Estimate of Laplace transform of reduced Malliavin matrix

This section is devised to be independent of the settings in Subsection 2.1 so that it can be used to other framework such as Picard [14]. Let \( L_t \) be a \( d \)-dimensional Lévy process with Lévy measure \( \nu \). We assume that the Lévy measure \( \nu \) satisfies the following conditions: for some \( \alpha \in (0, 2) \),

\[
\int_{|z| < \delta} |z|^\gamma \nu(dz) \leq C\delta^{2-\alpha}, \quad \forall \delta \in (0, 1), \quad \int_{|z| > 1} |z|^m \nu(dz) < \infty, \quad \forall m \in \mathbb{N}. \tag{3.1}
\]

Let \( N(dr, dz) \) be the Poisson random measure associated with \( L_t \), i.e.,

\[N((0, t] \times E) = \sum_{s \in E} 1_E(\Delta L_s), \quad E \in \mathcal{B}(\mathbb{R}_0^d).\]

Let \( \tilde{N}(dr, dz) := N(dr, dz) - dr \nu(dz) \) be the compensated Poisson random measure. Consider the following SDE:

\[
X_t(x) = x + \int_0^t b(X_s(x))ds + \int_0^t \int_{|z| < R} \sigma(X_{s-}(x), z)\tilde{N}(ds, dz), \tag{3.2}
\]

where \( b : \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \) are two functions satisfying that for any \( m \in \mathbb{N}_0 \) and \( j = 0, 1 \),

\[
|\nabla^m b(x)| \leq C, \quad |\nabla^m \nabla_z^j \sigma(x, z)| \leq C|z|^{1-j} \tag{3.3}
\]

and

\[
\int_{r < |z| < R} |\sigma(x, z)| \nu(dz) = 0, \quad 0 < r < R < \infty. \tag{3.4}
\]
Under (3.3), it is well-known that SDE (3.2) has a unique solution denoted by \( X_t := X_t(x) \), which defines a \( C^\infty \)-stochastic flow (cf. [5] and [15]). Let \( J_t := J_t(x) := \nabla X_t(x) \) be the Jacobian matrix of \( X_t(x) \), which solves the following linear matrix-valued SDE:

\[
J_t = I + \int_0^t \nabla b(X_s) J_s ds + \int_0^t \nabla \sigma(X_{s-}, z) J_s \tilde{N}(ds, dz).
\] (3.5)

If we further assume

\[
\inf_{x \in \mathbb{R}^d} \inf_{z \in \mathbb{R}^d} \det(\mathbb{I} + \nabla \sigma(x, z)) > 0,
\] (3.6)

then the matrix \( J_t(x) \) is invertible (cf. [5]). Let \( K_t = K_t(x) \) be the inverse matrix of \( J_t(x) \). By Itô’s formula, it is easy to see that \( K_t \) solves the following linear matrix-valued SDE (cf. [22 Lemma 3.2]):

\[
K_t = I - \int_0^t K_s \nabla b(X_s) ds + \int_0^t \int_{\mathbb{R}^d} K_s - Q(X_{s-}, z) \tilde{N}(ds, dz)
\]

\[- \int_0^t \int_{\mathbb{R}^d} K_s - Q(X_{s-}, z) \nabla \sigma(X_{s-}, z) \nu(dz) ds,
\] (3.7)

where

\[
Q(x, z) := (\mathbb{I} + \nabla \sigma(x, z))^{-1} - I.
\] (3.8)

First of all, we have the following easy estimate. Since the proof is standard by Gronwall’s inequality and Burkholder’s inequality, we omit the details.

**Lemma 3.1.** Under (3.3) and (3.6), we have for any \( p \geq 1 \),

\[
\sup_{x \in \mathbb{R}^d} \mathbb{E} \left( \sup_{t \in [0, 1]} |J_t(x)|^p + |K_t(x)|^p \right) < +\infty.
\] (3.9)

We now prove the following crucial lemma.

**Lemma 3.2.** Let \( V : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d \) be a bounded smooth function with bounded derivatives of all orders. Under (3.3), (3.4) and (3.6), there exist \( \beta_1, \beta_2 \in (0, 1), \beta_3 \geq 1 \) only depending on \( \alpha \) and constants \( C_1 \geq 1 \) and \( c_1 \in (0, 1) \) only depending on \( b, V \) and \( \alpha, \beta_1, \nu \) such that for all \( \delta, t \in (0, 1) \) and \( p \geq 1 \),

\[
\sup_{|\xi| = 1} \mathbb{E} \left( \int_0^t |uK_s[b, V](X_s)|^2 ds \right) \geq \tau_2^{\beta_1}, \quad \int_0^t |uK_s V(X_s)|^2 ds \leq \tau_2^{\beta_2} \leq C_1 e^{-c_1 \delta^{\beta_3}} + c_2 \delta^p,
\] (3.10)

where \( [b, V] := b \cdot \nabla V - V \cdot \nabla b \).

**Proof.** We divide the proof into four steps.

(1) Fixing \( \delta \in (0, 1) \), we decompose the Lévy process as the small and large jump parts, i.e., \( L_t = L_t^\delta + \tilde{L}_t^\delta \), where

\[
L_t^\delta := \int_{|z| \leq \delta} \tilde{N}((0, t], dz), \quad \tilde{L}_t^\delta := \int_{|z| > \delta} \tilde{N}((0, t], dz).
\]

Clearly, \( L_t^\delta \) and \( \tilde{L}_t^\delta \) are independent.

Let us fix a path \( h \) with finitely many jumps on any finite time interval. Let \( X_t^h(x; h) \) solve the following SDE:

\[
X_t^h(x; h) = x + \int_0^t b(X_s^h(x; h)) ds + \sum_{s \in h} \sigma(X_s^h(x; h); \Delta h_s).
\]
Let \( K^\delta_t(x; h) := [\nabla X^\delta_t(x; h)]^{-1} \). Clearly, by (3.4) we have
\[
X_t(x) = X^\delta_t(x; h)_{|\delta=\delta}, \quad K_t(x) = K^\delta_t(x; h)_{|\delta=\delta}. \tag{3.12}
\]
Moreover, \( K^\delta_t := K^\delta_t(x; 0) \) solves the following equation
\[
K^\delta_t = \mathbb{I} - \int_0^t K^\delta_s \nabla b(X^\delta_s) ds + \int_0^t \int_{|z|\leq \delta} K^\delta_s Q(X^\delta_s, z) \tilde{N}(ds, dz) + \int_0^t \int_{|z|\leq \delta} K^\delta_s \nabla \sigma(X^\delta_s, z) \nu(dz) ds. \tag{3.13}
\]
(2) Define functions:
\[
H_V(x, z) := V(x + \sigma(x, z)) - V(x) + Q(x, z)V(x + \sigma(x, z)), \quad G_V(x, z) := H_V(x, z) + \nabla_x \sigma(x, z) \cdot V(x) - \sigma(x, z) \cdot \nabla V(x) \tag{3.14}
\]
and
\[
V_0(x) := [b, V](x) + \int_{|z|\leq \delta} G_V(x, z) \nu(dz), \quad V_1(x) := [b, V_0](x) + \int_{|z|\leq \delta} G_{V_0}(x, z) \nu(dz). \tag{3.15}
\]
It is easy to see by (3.3) that
\[
|H_V(x, z)| \leq C(1 + |z|), \quad |G_V(x, z)| \leq C(1 \wedge |z|^2). \tag{3.16}
\]
For a row vector \( u \in \mathbb{R}^d \), we introduce the processes:
\[
f_t := uK_t^\delta V(X^\delta_t), \quad f_t^0 := uK_t^\delta V_0(X^\delta_t), \quad f_t^{00} := uK_t^\delta V_1(X^\delta_t),
\]
\[
g_t(z) := uK_t^\delta H_V(X^\delta_t, z), \quad g_t^0(z) := uK_t^\delta H_{V_0}(X^\delta_t, z),
\]
where
\[
X^\delta_t := X^\delta_t(x; 0), \quad K_t^\delta := K_t^\delta(x; 0).
\]
By equations (3.13), (3.11) with \( \delta = 0 \) and using Itô’s formula, we have
\[
f_t = uV(x) + \int_0^t uK^\delta_s [b, V](X^\delta_s) ds + \int_0^t \int_{|z|\leq \delta} g_s(z) \tilde{N}(ds, dz) + \int_0^t \int_{|z|\leq \delta} uK^\delta_s G_V(X^\delta_s, z) \nu(dz) ds
\]
\[
= uV(x) + \int_0^t f_s^0 ds + \int_0^t \int_{|z|\leq \delta} g_s(z) \tilde{N}(ds, dz)
\]
and
\[
f_t^{00} = uV_0(x) + \int_0^t f_s^{00} ds + \int_0^t \int_{|z|\leq \delta} g_s^0(z) \tilde{N}(ds, dz).
\]
For \( \gamma \in (0, \frac{1}{4}) \), define a stopping time
\[
\tau := \tau_u(x) := \inf \{ s \geq 0 : |uK^\delta_s(x; 0)|^2 > \delta^{-\gamma} \}.
\]
Since \( V \) has bounded derivatives of all orders, there exists a constant \( \kappa_0 \geq 1 \) only depending on \( b, \sigma \) and \( V \) such that for all \( t \in [0, \tau) \) and \( z \in \mathbb{R}^d \),
\[
|f_t|^2, |f_t^0|^2, |f_t^{00}|^2 \leq \kappa_0 \delta^{-\gamma}, \quad |g_t(z)|^2, |g_t^0(z)|^2 \leq \kappa_0 \delta^{-\gamma}(1 \wedge |z|^2).
\]
If we make the following replacement in Theorem 2.4:

\[ f_t, g_t(z), f_t^0, g_t^0(z) \Rightarrow f_{t,t'}^0, 1_{t < t'} g_t(z), f_{t,t'}^0, 1_{t < t'} g_t^0(z), \]

then by (2.10) with \( \varepsilon = \delta^\gamma \) and \( \kappa = \kappa_0 \delta^\gamma \), we obtain

\[
c_0 \int_0^t |f_t|^2 \, ds \leq (\delta^{-\frac{1}{2}} + \delta^{-\frac{1}{4}}) \int_0^t |f_{t,t'}|^2 \, ds + \kappa_0 \delta^{\frac{1}{2} - \gamma} \log \zeta + \kappa_0 \delta^{-\gamma} (\delta^{\frac{1}{2} - \frac{1}{2}} + \delta^\gamma + t \delta^\frac{1}{2})
\]

\[
\leq 2\kappa_0 \delta^{-\frac{15}{4}} \int_0^t |f_{t,t'}|^2 \, ds + \kappa_0 \delta^{\frac{1}{2} - \gamma} \log \zeta + 2\kappa_0 (\delta^{\frac{1}{2} - \gamma} + t \delta^\frac{1}{2} - \gamma) \text{ a.s.,}
\]

where \( c_0 \in (0, 1) \) only depends on \( \int_{|z| \leq 1} |z|^2 \nu(dz) \). From this, dividing both sides by \( 2\kappa_0 \delta^{\frac{1}{2} - \gamma} \) and taking exponential, then multiplying \( 1_{t > \tau} \) and taking expectations, we derive that

\[
\mathbb{E}\left( \exp\left( c_0 \frac{\delta^{\gamma - \frac{1}{2}}}{2\kappa_0} \int_0^t |f_t|^2 \, ds - \delta^{\gamma - 8} \int_0^t |f_t|^2 \, ds \right) 1_{\tau > t} \right) \leq \mathbb{E}(1_{\tau > \tau} \xi) \exp(\delta^2 + t) \leq \exp(\delta^2 + t). \tag{3.17}
\]

Recalling the definition of \( f_t^0 \) and by \( |x + y|^2 \geq \frac{|x|^2}{2} - |y|^2 \), we have for \( t < \tau \),

\[
|f_t^0|^2 \geq \frac{|u K_t^0(b, \nabla \phi(X_t^0))|^2}{2} - C |u K_t^0|^2 \left( \int_{|z| \leq \delta} |z|^2 \nu(dz) \right)^2 \geq \frac{|u K_t^0|^2}{2} - C_2 \delta^{4 - 2a - \gamma} . \tag{3.16}
\]

Thus, by (3.17) there exist \( c_1 \in (0, 1) \) and \( C_3 \geq 1 \) independent of the starting point \( x \) such that for all \( \delta, t \in (0, 1) \),

\[
\sup_{x \in \mathbb{R}^d} \mathbb{E} \left( \exp \left( c_1 \delta^{\gamma - \frac{1}{2}} \int_0^t |u K_t^0(b, \nabla \phi(X_t^0))|^2 \, ds - \delta^{\gamma - 8} \int_0^t |u K_t^0 V(X_t^0)|^2 \, ds \right) 1_{\tau > t} \right) \leq \exp \left( \delta^2 + t(C_3 \delta^{\frac{1}{2} - 2a} + 1) \right). \tag{3.18}
\]

(3) For \( t \in (0, 1) \) and \( u \in \mathbb{R}^d \), define a random set

\[
\Omega_t^u(x; \delta) := \left\{ \sup_{s \in [0,t]} |u K_s^0(x; \delta)|^2 \leq \delta^{-\gamma} \right\} ,
\]

and let

\[
\mathcal{F}_t^u(x; \delta) := \exp \left( c_1 \delta^{\gamma - \frac{1}{2}} \int_0^t |u K_s^0(x; \delta)|^2 \, ds - \delta^{\gamma - 8} \int_0^t |u K_s^0 V(X_s^0)|^2 \, ds \right) 1_{\Omega_t^u(x; \delta)} \tag{3.19}
\]

Since \( \Omega_t^u(x; 0) \subseteq \{ \tau_u(x) \geq t \} \), by (3.18) we have

\[
\sup_{x \in \mathbb{R}^d} \mathbb{E} \sup_{u \in \mathbb{R}^d} \mathcal{F}_t^u(x; 0) \leq \exp \left( \delta^2 + t(C_3 \delta^{\frac{1}{2} - 2a} + 1) \right). \tag{3.20}
\]

Let \( 0 = t_0 < t_1 < \cdots < t_n \leq t_{n+1} = t \) be the jump times of \( \delta \). If we set

\[
\phi_t(x; \delta) := X_{t_{j-1}}^\delta(x; \delta) + \sigma(X_{t_{j-1}}^\delta(x; \delta), \Delta H_j),
\]

then for \( s \in [0, t_{j+1} - t_j) \),

\[
X^\delta_{s+t_j}(x; \delta) = X^\delta_{t_j}(\phi_t(x; \delta); 0) \Rightarrow K^\delta_{s+t_j}(x; \delta) = [\nabla \phi_t(x; \delta)]^{-1} K^\delta_s(\phi_t(x; \delta); 0)
\]
and
\[ \Omega_{\phi_t}^u(x; \hat{h}) = \Omega_{\phi_t}^u(x; \hat{h}) \cap \left\{ \sup_{s \in [0, \phi_t - 1]} |uK_{\phi_t}^\delta(x; \hat{h})|^2 \leq \delta^{-y} \right\}. \]

Thus, by the Markovian property, we have for all \( u \in \mathbb{R}^d \),
\[
\mathbb{E} \mathcal{F}_{n+1}^u(x; \hat{h}) = \mathbb{E} \left( \mathcal{F}_{n+1}^u(x; \hat{h}) \cdot (\mathbb{E} \mathcal{F}_{n+1}^u(y; \hat{h}) \bigg| u' = u[y \phi_{t+1}]^{-1}, y = \phi_{t+1}(x; \hat{h}) \right) 
\leq \mathbb{E} \mathcal{F}_{n+1}^u(x; \hat{h}) \exp \left\{ \delta^2 + (t_{n+1} - t_n)(C_3 \delta^{z-2\alpha} + 1) \right\}
\leq \cdots \cdots \cdots \cdots
\leq \Pi_{j=0}^n \exp \left\{ \delta^2 + (t_{j+1} - t_j)(C_3 \delta^{z-2\alpha} + 1) \right\}
= \exp \left\{ \delta^2 (n + 1) + t_{n+1}(C_3 \delta^{z-2\alpha} + 1) \right\}. \tag{3.21}
\]

Let \( N_t^\delta \) be the jump number of \( \hat{L}_t^\delta \) before time \( t \), i.e.,
\[ N_t^\delta = \sum_{s \in (0, t]} 1_{|\Delta_s^\delta| > 0} = \int_{|z| > \delta} N((0, t], dz) = \sum_{s \in (0, t]} 1_{|\Delta_s^\delta| > 0}, \]
which is a Poisson process with intensity \( \int_{|z| > \delta} \nu(dz) =: \lambda_\delta \). If we let \( m = [\log \delta^{-1}/\log 2] \), then by (3.1), we have
\[
\lambda_\delta \leq \int_{|z| > 1} \nu(dz) + \sum_{k=0}^m \int_{2^k \delta < |z| < 2^{k+1} \delta} \nu(dz)
\leq C + \sum_{k=0}^m (2^k \delta)^{-2} \int_{2^k \delta < |z| < 2^{k+1} \delta} |z|^2 \nu(dz)
\leq C + C \sum_{k=0}^m (2^k \delta)^{-2} (2^{k+1} \delta)^{2-\alpha}
= C + C 2^{2-\alpha} \sum_{k=0}^m (2^k \delta)^{-\alpha} \leq C \delta^{-\alpha}. \tag{3.22}
\]

Recalling (3.12), (3.19) and the independence of \( L_t^\delta \) and \( \hat{L}_t^\delta \), we have for any \( u \in \mathbb{R}^d \),
\[
\mathbb{E} \left( \exp \left\{ c_1 \delta^{y-\frac{1}{2}} \int_0^t |uK_s[b, V](X_s)|^2 ds - \delta^{y-8} \int_0^t |uK_s V(X_s)|^2 ds \right\} 1_{\Omega_{\phi_t}^u(x; \hat{h})} \right)
= \mathbb{E} \left( \mathbb{E} \mathcal{F}_{n+1}^u(x; \hat{h})_{n=L_s} \bigg| N_s^\delta = n \right)
\leq \sum_{n=0}^{\infty} \exp \left\{ \delta^2 (n + 1) + t(C_3 \delta^{z-2\alpha} + 1) \right\} \mathbb{E} (N_s^\delta = n)
= \exp \left\{ \delta^2 + t(C_3 \delta^{z-2\alpha} + 1) \right\} \sum_{n=0}^{\infty} \frac{e^{\delta^2 n}(tL_\delta)^n}{n!} e^{-tL_\delta}
= \exp \left\{ \delta^2 + t(C_3 \delta^{z-2\alpha} + 1) + (e^{\delta^2} - 1)tL_\delta \right\}
\leq \exp \left\{ 2 + C_4 t \delta^{z-2\alpha} + C_5 t \delta^{-\alpha} \right\}, \ \forall t, \delta \in (0, 1), \tag{3.23}
\]
where in the last step we have used that $e^x - 1 \leq 3x$ for $x \in (0, 1)$.

(4) By (3.23) and Chebyshev’s inequality, we have for any $\beta \in (0, 1)$,
\[
P\left\{ c_1 \delta^{\beta - 1} \int_0^t |uK_s[b, V](X_s)|^2 \, ds - \delta^{\beta - 8} \int_0^t |uK_s V(X_s)|^2 \, ds \geq t \delta^{\beta - 2}, \Omega^\beta_s(x; \hat{L}) \right\} \\
\leq \exp \left\{ 2 + C_4 t \delta^{2 - 2\alpha} + C_5 t \delta^{2 - \alpha} - t \delta^{\beta - 2} \right\}
\]
and by (3.9),
\[
P(\Omega^\beta_s(x; \hat{L})) = \mathbb{P}\left( \sup_{s \in [0, t]} |uK_s(x)|^2 > \delta^{-\gamma} \right) \\
\leq \delta^{\gamma p} \mathbb{E}\left( \sup_{s \in [0, t]} |uK_s(x)|^{2p} \right) \leq C_p |u|^{2p} \delta^{\gamma p}, \quad \forall p \geq 1.
\]
In particular, if $\beta \in (0 \lor (4\alpha - 7), 1)$, then there exists a constant $\delta_0$ such that for all $\delta \in (0, \delta_0)$, $t \in (0, 1)$ and $p \geq 1$,
\[
\sup_{|u| = 1} \mathbb{P}\left\{ \int_0^t |uK_s[b, V](X_s)|^2 \, ds \geq \frac{2t \delta^{1 - \lambda - \gamma}}{c_1}, \int_0^t |uK_s V(X_s)|^2 \, ds \leq t \delta^{8 - \frac{5}{2} - \gamma} \right\} \\
\leq \exp\left( 3 - t \delta^{\frac{5}{2}} \right) + C_\rho \delta^p,
\]
which then gives the desired estimate by adjusting the constants and rescaling $\delta$.

The reduced Malliavin matrix is defined by
\[
\hat{\Sigma}_s(x) := \int_0^t K_s(x)[(\nabla_x \sigma)(\nabla_x \sigma)^*](X_s(x), 0)K^*_s(x) \, ds. \tag{3.24}
\]
We are now in a position to prove the following main result of this section.

**Theorem 3.3.** Let $B_1(x) := \nabla_x \sigma(x, 0)$ and define $B_{j+1}(x) := b(x) \cdot \nabla B_j(x) - B_j(x) \cdot \nabla b(x)$ for $j \in \mathbb{N}$. Assume that for some $j_0 \in \mathbb{N}$,
\[
\inf_{x \in \mathbb{R}^d} \inf_{|u| = 1} \sum_{j=1}^{j_0} |uB_j(x)|^2 =: c_0 > 0. \tag{3.25}
\]
Under (3.3), (3.4) and (3.6), there exist $\gamma = \gamma(\alpha, j_0) \in (0, 1)$ and constants $C_2 \geq 1, c_2 \in (0, 1)$ such that for all $t \in (0, 1), \lambda \geq 1$ and $p \geq 1$,
\[
\sup_{|u| = 1} \mathbb{E} \exp \left\{ -\lambda u \hat{\Sigma}_s(x) u^* \right\} \leq C_2 \exp\left\{ -c_2 t \lambda^\gamma \right\} + C_\rho (\lambda t)^{-p}. \tag{3.26}
\]

**Proof.** Let $\beta_1, \beta_2, \beta_3$ be as in (3.10). Set $a := \frac{\beta_1}{\beta_2} \leq 1$ and define for $j = 1, \ldots, j_0$,
\[
E_j := \left\{ \int_0^t |uK_s B_j(X_s)|^2 \, ds \leq t \delta^{\alpha_j \beta_j} \right\}.
\]
Since $a^{j_1} \beta_2 = a^j \beta_1$ and $B_{j+1} = [b, B_j]$, by (3.10) with $\delta$ replaced by $\delta^{a^j}$, we have for any $p \geq 1$,
\[
P(E_j \cap E_{j+1}^c) \leq C_1 \exp\left\{ -c_1 t \delta^{a^j \beta_1} \right\} + C_\rho \delta^{a^j \beta_1}. \tag{3.27}
\]
Noticing that
\[
E_1 \subset \left( \bigcap_{j=1}^{j_0} E_j \right) \cup \left( \bigcup_{j=1}^{j_0-1} (E_j \cap E_{j+1}^c) \right),
\]
we have
\[ \mathbb{P}(E_1) \leq \mathbb{P}\left( \bigcap_{j=1}^{j_0} E_j \bigg) + \sum_{j=1}^{j_0-1} \mathbb{P}(E_j \cap E_{j+1}^c). \] (3.28)

On the other hand, if we define
\[ \tau := \inf\{ t \geq 0 : |J_t| \geq \delta^{-a/b} \}, \]
then for any \( s \leq \tau \) and \(|u| = 1\),
\[ |uK_s|^2 \geq |J_s^{-2} \geq \delta^{2a/b} \].

Thus, by (3.25) we have
\[
\begin{align*}
&\int_{j=1}^{j_0} E_j \cap \{ \tau \geq t \} \subset \left\{ \sum_{j=1}^{j_0} \int_0^{\tau} |uK_\lambda B_j(X_{\lambda})|^2 ds \leq t \sum_{j=1}^{j_0} \delta^{a/b}, \tau \geq t \right\} \\
&\subset \left\{ c_0 \sum_{j=1}^{j_0} |uK_\lambda|^2 ds \leq t \sum_{j=1}^{j_0} \delta^{a/b}, \tau \geq t \right\} \\
&\subset \left\{ \{ c_0 \delta^{a/b} \leq t \} \right\} = 0,
\end{align*}
\] (3.29)

provided \( \delta < \delta_1 = (c_0/j_0)^{3/(a/b)} \). On the other hand, by (3.29), we have for any \( p \geq 2 \),
\[ \mathbb{P}(\tau < t) \leq \mathbb{P}\left( \sup_{\in [0,t]} |J_s| \geq \delta^{-a/b} \right) \leq \mathbb{E} \left( \sup_{\in [0,t]} |J_s|^p \right) \delta^{a/b} \leq C_p \delta^{a/b} \].

Therefore, combining (3.27)-(3.29) and resetting \( \varepsilon = \delta^{\beta_1} \) and \( \theta = a/b \beta_3 / \beta_1 \), we obtain that for all \( \varepsilon \in (0,1) \), \( t \in (0,1) \) and \( p \geq 1 \),
\[ \sup_{\in [0,t]} \mathbb{P}\left\{ \int_0^t |uK_\lambda B_1(X_{\lambda})|^2 ds \leq t \varepsilon \right\} \leq C_2 \exp \left\{ -c_1 t \varepsilon^{-\theta} \right\} + C_p \varepsilon^p. \]

For \( \lambda \geq t \), setting \( r := (\lambda/t)^{\frac{\beta_3}{\beta_1}} \) and \( \xi := \frac{1}{t} \int_0^t |uK_\lambda B_1(X_{\lambda})|^2 ds \), we have
\[
\begin{align*}
\mathbb{E} e^{-\xi} &= \int_0^\infty \lambda e^{-\lambda \xi} \mathbb{P}(\xi \leq \varepsilon) d\varepsilon \\
&\leq \int_0^\infty \lambda e^{-\lambda \xi} \mathbb{E} e^{-c_1 t \varepsilon^{-\beta_1}} + C e^{-\lambda \xi} d\varepsilon \\
&= e^{-\lambda r} + C \int_0^{\lambda r} e^{-c_1 t \varepsilon^{-\beta_1}} ds + C \lambda^{-p} \int_0^{\lambda r} e^{-s} s^{-\beta} ds \\
&\leq e^{-\lambda r} + Ce^{-c_1 t \varepsilon^{-\beta_1}} \int_0^{\lambda r} e^{-s} ds + C \lambda^{-p} \\
&\leq e^{-t(\lambda/t)^{\frac{\beta_3}{\beta_1}} - \varepsilon^p} + Ce^{-c_1 t \varepsilon^{-\beta_1}} + C \lambda^{-p}.
\end{align*}
\]

By replacing \( \lambda \) with \( \lambda t \) and recalling (3.24), we obtain the desired estimate (3.26). \( \square \)

4. Smooth densities for SDEs without big jumps

In the remainder of this paper, we assume (H) and (H') and choose \( \delta \) being small enough so that
\[ |\nabla \chi (x, z)| \leq \frac{1}{2}, \ |z| \leq \delta, \] (4.1)
and set
\[ \Gamma_0^\delta := \{ z \in \mathbb{R}^d : 0 < |z| < \delta \}. \]
Let \( X_t(x) = X_t \) solve the following SDE:
\[
X_t = x + \int_0^t b(X_s)ds + \int_0^t \int_{\Gamma_0} \sigma(X_{s-}, z)\tilde{N}(ds, dz).
\] (4.2)

It is well known that the generator of \( X_t(x) \) is given by \( \mathcal{L}_0 \) in (1.3).

This section is based on Subsection 2.1, Lemma 2.5 and Theorem 3.3. We first prove the following Malliavin differentiability of \( X_t \) with respect to \( \omega \) in the sense of Theorem 2.1.

**Lemma 4.1.** Fix \( \nu \in \mathbb{V}_{\infty-} \). For any \( t \in [0, 1] \), we have \( X_t \in \mathbb{W}^{1, \infty-}(\Omega) \) and
\[
D_\nu X_t = \int_0^t \nabla b(X_s)D_\nu X_s ds + \int_0^t \int_{\Gamma_0} \nabla_\nu \sigma(X_{s-}, z)D_\nu X_{s-} \tilde{N}(ds, dz)
+ \int_0^t \int_{\Gamma_0} \langle \nabla_\nu \sigma(X_{s-}, z), \nu(s, z) \rangle N(ds, dz).
\] (4.3)

Moreover, for any \( p \geq 2 \), we have
\[
\sup_{x \in \mathbb{R}^d} \mathbb{E} \left( \sup_{t \in [0, 1]} |D_\nu X_t(x)|^p \right) < \infty.
\] (4.4)

**Proof.** (1) Consider the following Picard’s iteration: \( X^0_t \equiv x \) and for \( n \in \mathbb{N} \),
\[
X^n_t := x + \int_0^t b(X^n_{s-})ds + \int_0^t \int_{\Gamma_0} \sigma(X^n_{s-}, z)\tilde{N}(ds, dz).
\]

Since \( b \) and \( \sigma \) are Lipschitz continuous, it is by now standard to prove that for any \( p \geq 2 \),
\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left( \sup_{t \in [0, 1]} |X^n_t|^p \right) < \infty \text{ and } \lim_{n \to \infty} \mathbb{E} \left( \sup_{t \in [0, 1]} |X^n_t - X^0_t|^p \right) = 0.
\] (4.5)

(2) Now we use the induction to prove that for each \( n \in \mathbb{N} \),
\[
X^n_t \in \mathbb{W}^{1, \infty-}(\Omega) \quad \text{and} \quad \mathbb{E} \left( \sup_{t \in [0, 1]} |D_\nu X^n_t|^p \right) < +\infty, \quad \forall \ p \geq 2.
\] (4.6)

First of all, it is clear that (4.6) holds for \( n = 0 \). Suppose now that (4.6) holds for some \( n \in \mathbb{N} \).

By (4.5) and the induction hypothesis, it is easy to check that the assumptions of Proposition 2.2 are satisfied. Thus, \( X^{n+1}_t \in \mathbb{W}^{1, \infty-}(\Omega) \) and
\[
D_\nu X^{n+1}_t = \int_0^t \nabla b(X^n_s)D_\nu X^n_s ds + \int_0^t \int_{\Gamma_0} \nabla_\nu \sigma(X^n_{s-}, z)D_\nu X^n_{s-} \tilde{N}(ds, dz)
+ \int_0^t \int_{\Gamma_0} \langle \nabla_\nu \sigma(X^n_{s-}, z), \nu(s, z) \rangle N(ds, dz).
\]

By Lemma 2.3, we have for any \( p \geq 2 \),
\[
\mathbb{E} \left( \sup_{t \in [0, 1]} |D_\nu X^{n+1}_t|^p \right) \leq C \int_0^t \mathbb{E} |D_\nu X^n_t|^p ds + C \mathbb{E} \left( \int_0^t \int_{\Gamma_0} |\langle \nabla_\nu \sigma(X^n_{s-}, z), \nu(s, z) \rangle| \nu(dz) ds \right)^p
+ C \mathbb{E} \left( \int_0^t \int_{\Gamma_0} |\langle \nabla_\nu \sigma(X^n_{s-}, z), \nu(s, z) \rangle|^p \nu(dz) ds \right).
\]

Since \( \nu \in \mathbb{V}_{\infty-} \), by (H_\nu^p) we further have
\[
\mathbb{E} \left( \sup_{t \in [0, 1]} |D_\nu X^{n+1}_t|^p \right) \leq C \int_0^t \mathbb{E} |D_\nu X^n_t|^p ds + C \int_0^t \mathbb{E} \left( \sup_{t \in [0, s]} |D_\nu X^n_t|^p \right) ds + C,
\]
where $C$ is independent of $n$ and the starting point $x$. Thus, we have proved (4.6) by the induction hypothesis. Moreover, by Gronwall’s inequality, we also have

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left( \sup_{s \in [0,1]} |D_s X^n_s|^p \right) < +\infty. \quad (4.7)$$

(3) Let $Y_t$ solve the following linear matrix-valued SDE:

$$Y_t = \int_0^t \nabla b(X_s) Y_s ds + \int_0^t \int_{\Gamma_0^\delta} \nabla_x \sigma(X_{s-}, z) Y_s \tilde{N}(ds, dz)$$

$$+ \int_0^t \int_{\Gamma_0^\delta} \langle \nabla_z \sigma(X_{s-}, z), \nu(s, z) \rangle N(ds, dz).$$

By Fatou’s lemma and (4.5), (4.7), for any $p \geq 2$, we have

$$\lim_{n \to \infty} \mathbb{E}|D_s X^n_t - Y_t|^p \leq C \int_0^t \lim_{n \to \infty} \mathbb{E}|D_s X^{n-1}_s - Y_s|^p ds,$$

which then gives

$$\lim_{n \to \infty} \mathbb{E}|D_s X^n_t - Y_t|^p = 0.$$

Thus, $X_t \in \mathbb{W}^{1,p}_V(\Omega)$ and $D_s X_t = Y_t$. Moreover, the estimate (4.4) follows by (4.5) and (4.7). \qed

Let $J_t = J_t(x)$ be the Jacobian matrix of $x \mapsto X_t(x)$, and $K_t(x)$ the inverse of $J_t(x)$. Recalling equations (3.5) and (4.3), by the formula of constant variation, we have for any $v \in \mathbb{V}_{\infty-}$,

$$D_s X_t = J_t \int_0^s \int_{\Gamma_0^\delta} K_s \nabla_x \sigma(X_{s-}, z) \nu(s, z) N(ds, dz). \quad (4.8)$$

Here the integral is the Lebesgue-Stieltjes integral. Let

$$U(x, z) := (\mathbb{I} + \nabla_x \sigma(x, z))^{-1} \nabla_x \sigma(x, z), \quad x \in \mathbb{R}^d, \quad z \in \Gamma_0^\delta,$$

and define

$$v_j(x; s, z) := [K_{s-}(x) U(X_{s-}(x), z)]^j \zeta(z),$$

where $\zeta(z) = \zeta_0(z)$ is a nonnegative smooth function with

$$\zeta_0(z) = |z|^3, \quad |z| \leq \delta/4, \quad \zeta_0(z) = 0, \quad |z| > \delta/2.$$

The following lemma is easy to be verified by definitions and (3.9).

**Lemma 4.2.** For any $m \in \mathbb{N}_0$, there is a constant $C > 0$ such that for all $x \in \mathbb{R}^d$ and $z \in \Gamma_0^\delta$,

$$|\nabla_x^m U(x, z)|, \quad |\nabla_z^m U(x, z)| \leq C, \quad |U(x, z) - U(x, 0)| \leq C|z|. \quad (4.9)$$

Moreover, for each $j = 1, \cdots, d$ and $x \in \mathbb{R}^d$, $v_j(x) \in \mathbb{V}_{\infty-}$.

Write

$$\Theta(s, z) := \Theta(x; s, z) := (v_1(x; s, z), \cdots, v_d(x; s, z))$$

and

$$(D_\Theta X_t)_{ij} := D_{v_j} X_t^i.$$

Since by equation (3.7),

$$K_s = K_{s-}(\mathbb{I} + \nabla_x \sigma(X_{s-}, \Delta L_s))^{-1},$$

by (4.8) we have

$$D_\Theta X_t(x) = J_t(x) \Sigma_t(x), \quad (4.10)$$
where
\[ \Sigma(x) := \int_0^t \int_{\Gamma_0^s} K_{r_1}(x)(UU^*)(X_{r_1}(x), z)K_{r_2}(x)(x)\zeta(z)N(ds, dz). \] (4.11)

**Lemma 4.3.** For any \( p \geq 2 \) and \( m, k \in \mathbb{N}_0 \) with \( m + k \geq 1 \), we have
\[ \sup_{x \in \mathbb{R}^d} \mathbb{E} \left( \sup_{r \in [0,1]} |D_{v_{j_1}} \cdots D_{v_{j_m}} \nabla^k X_r(x)|^p \right) < \infty, \] (4.12)
\[ \sup_{x \in \mathbb{R}^d} \mathbb{E} \left( \sup_{r \in [0,1]} |D_{v_{j_1}} \cdots D_{v_{j_m}} \text{div}(v_r(x))|^p \right) < \infty, \] (4.13)
where \( j_1, \cdots, j_m \) and \( i \) runs in \( \{1, 2, \cdots, d\} \).

**Proof.** For \( m + k = 1 \), (4.12) has been proven in (3.9) and (4.4). For general \( k \) and \( m \), it follows by induction. Let us look at (4.13) with \( m = 1 \). Notice that by (2.4),
\[ \text{div}(v_i) = \int_0^1 \int_{\Gamma_0^s} \left[ \langle \nabla \log \kappa(z), v_i(s, z) \rangle + \text{div}_z(v_i)(s, z) \right] \tilde{N}(ds, dz). \]

By Proposition 2.2, we have
\[ D_{v_i} \text{div}(v_i) = \int_0^1 \int_{\Gamma_0^s} \left[ \langle \nabla \log \kappa(z), D_{v_i} v_i(s, z) \rangle + D_{v_i} \text{div}_z(v_i)(s, z) \right] \tilde{N}(ds, dz) \]
\[ + \int_0^1 \int_{\Gamma_0^s} \langle v_i(s, z), \nabla \langle \nabla \log \kappa(z), v_i(s, z) \rangle \rangle + \nabla \text{div}_z(v_i)(s, z) \rangle \rangle N(ds, dz). \]

In view of \( \text{supp}v_i(s, \cdot) \subset \Gamma_0^s \), by Lemma 2.3 and (1.7), (4.9), (4.12), one obtains (4.13) with \( m = 1 \). For general \( m \), it follows by similar calculations. \( \square \)

Below we define
\[ \mathcal{T}_t^0 f(x) := \mathbb{E} f(X_r(x)). \] (4.14)

The following lemma is proven in appendix.

**Lemma 4.4.** Under \((\mathbb{H}_b^n)\), there exists a constant \( C > 0 \) such that for any \( f \in L^1(\mathbb{R}^d) \),
\[ \sup_{r \in [0,1]} \int_{\mathbb{R}^d} |\mathcal{T}_t^0 f(x)|dx \leq C \int_{\mathbb{R}^d} |f(x)|dx. \] (4.15)

Now we can prove the following main result of this section.

**Theorem 4.5.** Assume \((\mathbb{H}_b^n)\), \((\mathbb{H})\) and \((\mathbb{UH})\) and let \( \delta \) be as in (4.1). For any \( k, n, m \in \mathbb{N}_0 \) and \( p \in (1, \infty) \), there exist \( \gamma_{kmn} \geq 0 \) only depending on \( k, m, n, \alpha, j_0, d \) and a constant \( C \geq 1 \) such that for all \( f \in \mathcal{H}^{n,p}(\mathbb{R}^d) \) and \( t \in (0, 1) \),
\[ \|\nabla^k \mathcal{T}_t^0 \nabla^m f\|_p \leq C t^{-\gamma_{kmn}} \|f\|_{n,p}, \] (4.16)
where \( \gamma_{kmn} \) is increasing with respect to \( k, m \) and decreasing in \( n \), and \( \gamma_{kmn} = 0 \) for \( n \geq k + m \). In particular, \( X_r(x) \) admits a smooth density \( \rho_r(x) \) with \( \rho_r \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^d) \) such that
\[ \partial_t \rho_r(x, y) = \mathcal{L}_0 \rho_r(\cdot, y)(x), \quad \forall (t, x, y) \in (0, 1) \times \mathbb{R}^d \times \mathbb{R}^d. \]

**Proof.** Below we only prove (4.16) for \( p \in (1, \infty) \). For \( p = \infty \), it is similar and simpler. We assume \( f \in C_0^\infty(\mathbb{R}^d) \) and divide the proof into four steps.

1. Let \( \Sigma_t(x) \) be defined by (4.11). In view of \( U(x, 0) = \nabla \sigma(x, 0) \), by (2.13) and (3.26), there
are constants $C_3 \geq 1$, $c_3 \in (0, 1)$ and $\gamma = \gamma(\alpha, j_0) \in (0, 1)$ such that for all $t \in (0, 1)$, $\lambda \geq 1$ and $p \geq 1$,

$$\sup_{x \in \mathbb{R}^d} \sup_{|\ell| = 1} \mathbb{E} \exp \{ -\lambda t \Sigma_0(x) u^* \} \leq C_3 \exp \{ -c_3 t \lambda^p \} + C_\rho (\lambda t)^{-p}. \quad (4.17)$$

where $\Sigma_0(x)$ is defined by (4.11). As in [22, Lemma 5.3], for any $p \geq 1$, there exist constant $C \geq 1$ an $\gamma' = \gamma'(\alpha, j_0, d)$ such that for all $t \in (0, 1)$,

$$\sup_{x \in \mathbb{R}^d} \mathbb{E} \left( (\det \Sigma_0(x))^{-p} \right) \leq C t^{-\gamma' p},$$

which in turn gives that for all $p \geq 1$,

$$\sup_{x \in \mathbb{R}^d} \| \Sigma_0^{-1}(x) \|_{L^p(\Omega)} \leq C t^{-\gamma'}. \quad (4.18)$$

(2) For $t \in (0, 1)$ and $x \in \mathbb{R}^d$, let $\mathcal{C}_i(x)$ be the class of all polynomial functionals of

$$\text{div} \Theta, \Sigma_i^{-1}, K_i, (\nabla^k X_i)_{k=1}^{\ell_1}, (D_{v, j_1} \cdots D_{v, j_m}(X_i, \cdots, \nabla^\ell_i X_i, K_i, \text{div} \Theta, \Sigma_i))_{j=1}^{\ell_3},$$

where $\ell_1, \ell_2, \ell_3 \in \mathbb{N}$, $j_i \in \{1, \cdots, d\}$ and the starting point $x$ is dropped in the above random variables. By (4.18) and Lemma 4.3, for any $H_i(x) \in \mathcal{C}_i(x)$, there exists a $\gamma(H) \geq 0$ only depending on the degree of $\Sigma_i^{-1}$ and $\alpha, j_0, d$ such that for all $t \in (0, 1)$ and $p \geq 1$,

$$\sup_{x \in \mathbb{R}^d} \| H_i(x) \|_{L^p(\Omega)} \leq C \rho t^{-\gamma(H)}. \quad (4.19)$$

Notice that if $H_i$ does not contain $\Sigma_i^{-1}$, then $\gamma(H) = 0$.

(3) Let $\xi \in \mathcal{C}_i(x)$. Recalling that $D_\Theta X$ is an invertible matrix, by (4.10) and the integration by parts formula (2.3), we have

$$\mathbb{E} (\nabla f)(X_i) \xi = \mathbb{E} (\nabla f)(X_i) D_\Theta X_i \cdot (D_\Theta X_i)^{-1} K_i \xi = \mathbb{E} (D_\Theta f(X_i) \Sigma_i^{-1} K_i \xi) = \mathbb{E} (f(X_i) \xi'),$$

where $\xi' \in \mathcal{C}_i(x)$. Starting from this formula, by the chain rule and induction, we have

$$\nabla^k \mathbb{E} (\nabla^m f)(X_i) = \sum_{j=0}^{k} \mathbb{E} (\nabla^{m+j} f)(X_i) G_j(\nabla X_i, \cdots, \nabla^k X_i) = \sum_{j=0}^{n} \mathbb{E} (\nabla^j f)(X_i) H_j,$$

where $\{G_j, j = 1, \cdots, k\}$ are real polynomial functions and $H_j \in \mathcal{C}_i(x)$. Notice that if $n = k + m$, then $H_j$ will not contain $\Sigma_i^{-1}$.

(4) Now, for any $p \in (1, \infty)$, by Hölder’s inequality, we have

$$\| \nabla^k t^{-\gamma(H)} \|_{L^p(\mathbb{R}^d)} \leq C \sum_{j=0}^{n} t^{-\gamma(H)} \left( \int_{\mathbb{R}^d} \mathbb{E} (|\nabla^j f|^p)(X_i(x)) \right)^{\frac{1}{p}} \leq C \sum_{j=0}^{n} t^{-\gamma(H)} \left( \int_{\mathbb{R}^d} \mathbb{E} (|\nabla^j f|^p)(X_i(x)) \right)^{\frac{1}{p}} \leq C t^{-\gamma(\alpha, j_0)} \| f \|_{p, \infty}, \quad t \in (0, 1).$$
As for the second conclusion, it follows by (4.16) and Sobolev’s embedding theorem (cf. [12]). The proof is thus complete.

5. Proofs of Theorems 1.1 and 1.5

We first recall some definitions about the Sobolev and Hölder spaces. For \( k \in \mathbb{N}_0 \) and \( p \in [1, \infty] \), let \( \mathbb{W}^{k,p} = \mathbb{W}^{k,p}(\mathbb{R}^d) \) be the usual Sobolev spaces with the norm:

\[
||\varphi||_{k,p} := \sum_{j=0}^{k} ||\nabla^j \varphi||_p.
\]

For \( \beta \geq 0 \) and \( p \in [1, \infty) \), let \( \mathbb{H}^\beta,p := (I - \Delta)^{-\beta/2} (L^p(\mathbb{R}^d)) \) be the usual Bessel potential space. For \( p = \infty \), let \( \mathbb{H}^\beta,\infty \) be the usual Hölder space, i.e., if \( \beta = k + \theta \) with \( \theta \in [0, 1) \), then

\[
||\varphi||_{\beta,\infty} := ||\varphi||_{k,\infty} + ||\nabla^k \varphi||_\theta < \infty,
\]

where \([\nabla^k \varphi]_\theta := 0 by convention and for \( \theta \in (0, 1) \),

\[
[\nabla^k \varphi]_\theta := \sup_{x \neq y} \frac{|\nabla^k \varphi(x) - \nabla^k \varphi(y)|}{|x - y|^{\theta}}.
\]

Notice that \( \mathbb{H}^{k,\infty} = C^k(\mathbb{R}^d) \) for \( k \in \mathbb{N}_0 \). It is well known that for any \( k \in \mathbb{N}_0 \) and \( p \in (1, \infty) \) (cf. [17]),

\[
\mathbb{H}^{k,p} = \mathbb{W}^{k,p},
\]

and for any \( \beta_1, \beta_2 \geq 0 \), \( p \in (1, \infty) \) and \( \theta \in [0, 1) \),

\[
[\mathbb{H}^{\beta_1,p}, \mathbb{H}^{\beta_2,p}]_\theta = \mathbb{H}^{\beta_1 + \theta(\beta_2 - \beta_1),p},
\]

and if \( \beta_1 + \theta(\beta_2 - \beta_1) \) is not an integer, then

\[
(\mathbb{H}^{\beta_1,\infty}, \mathbb{H}^{\beta_2,\infty})_{\theta,\infty} = \mathbb{H}^{\beta_1 + \theta(\beta_2 - \beta_1),\infty},
\]

where \([\cdot, \cdot]_\theta \) (resp. \((\cdot, \cdot)_{\theta,\infty}\)) stands for the complex (resp. real) interpolation space.

We recall the following interpolation theorem (cf. [19], p.59, Theorem (a)).

**Theorem 5.1.** Let \( A_i \subset B_i, i = 0, 1 \) be Banach spaces. Let \( \mathcal{T} : A_i \to B_i, i = 0, 1 \) be bounded linear operators. For \( \theta \in [0, 1] \), we have

\[
||\mathcal{T}||_{A_\theta \to B_\theta} \leq ||\mathcal{T}||_{A_\theta \to B_\theta}^{1-\theta}{||\mathcal{T}||_{A_\theta \to B_\theta}^\theta},
\]

where \( A_\theta := [A_0, A_1]_\theta, B_\theta := [B_0, B_1]_\theta \), and \( ||\mathcal{T}||_{A_\theta \to B_\theta} \) denotes the operator norm of \( \mathcal{T} \) mapping \( A_\theta \) to \( B_\theta \). The same is true for real interpolation spaces.

Let \( T_1^0 \) be the semigroup defined by (4.14), whose generator is given by \( L_0 \). We have

**Lemma 5.2.** Let \( \gamma_{100} \) be the same as in Theorem 4.5. For any \( p \in (1, \infty) \), \( \theta \in [0, 1) \) and \( \beta \geq 0 \), there exit constants \( C_1, C_2 > 0 \) such that for all \( t \in (0, 1) \),

\[
||T_1^0 \varphi||_{\beta+\theta,p} \leq C_1 t^{-\theta \gamma_{100}} ||\varphi||_{\beta,p},
\]

and if \( \beta \) and \( \theta + \beta \) are not integers, then

\[
||T_1^0 \varphi||_{\beta+\theta,\infty} \leq C_2 t^{-\theta \gamma_{100}} ||\varphi||_{\beta,\infty}.
\]

**Proof.** Let \( \theta \in [0, 1) \) and \( \beta \geq 0 \). For any \( p \in (1, \infty) \), by Theorem 4.5 and interpolation Theorem 5.1 there exists a constant \( C > 0 \) such that for all \( t \in (0, 1) \),

\[
||T_1^0 \varphi||_{\beta,p} \leq C ||\varphi||_{\beta,p}
\]

and

\[
||T_1^0 \varphi||_{1,p} \leq C t^{-\gamma_{100}} ||\varphi||_p.
\]
On the other hand, noticing that by (5.1),

\[ [H^{\beta, p}, H^{1, p}]_0 = H^{\beta + \theta, p}, \quad [H^{\beta, p}, H^{0, p}]_0 = H^{\beta, p}, \]

and if \( \beta \) and \( \theta + \beta \) are not integers, then by (5.2),

\[ (H^{\beta, \infty}, H^{1, \infty})_{\theta, \infty} = H^{\beta + \theta, \infty}, \quad (H^{\beta, \infty}, H^{0, \infty})_{\theta, \infty} = H^{\beta, \infty}, \]

by interpolation Theorem 5.1 again, we obtain the desired estimate. \( \square \)

5.1. **Proof of Theorem 1.1.** Let \( \mathcal{L} \) be a bounded linear operator in \( C_0(\mathbb{R}^d) \) and Sobolev spaces \( W^{k,p}(\mathbb{R}^d) \) for any \( p > 1 \) and \( k \in \mathbb{N}_0 \). Let \( \mathcal{T}_t \) be the semigroup in \( L^p(\mathbb{R}^d) \) associated with \( \mathcal{L}_0 + \mathcal{L} \), i.e., for any \( \varphi \in L^p(\mathbb{R}^d) \),

\[ \partial_t \mathcal{T}_t \varphi = \mathcal{L}_0 \mathcal{T}_t \varphi + \mathcal{L} \mathcal{T}_t \varphi. \]

By Duhamel’s formula, we have

\[ \mathcal{T}_t \varphi = \mathcal{T}_t^0 \varphi + \int_0^t \mathcal{T}^{-1}_{t-s} \mathcal{L} \mathcal{T}_s \varphi \, ds. \quad (5.5) \]

**Lemma 5.3.** Let \( \gamma_{100} \) be as in Theorem 4.5. Fix \( \theta \in (0, \frac{1}{\gamma_{100}} \land 1) \). For any \( m \in \mathbb{N} \) and \( p \in (1, \infty) \), there exists a constant \( C > 0 \) such that for all \( t \in (0, 1) \) and \( \varphi \in L^p(\mathbb{R}^d) \),

\[ \| \mathcal{T}_t \varphi \|_{m^p, p} \leq C t^{-\theta \gamma_{100}} \| \varphi \|_p. \quad (5.6) \]

**Proof.** First of all, since \( \mathcal{L} \) is a bounded linear operator in \( W^{k,p} \), by interpolation Theorem 5.1 we have for all \( \beta \geq 0 \) and \( p \in (1, \infty) \),

\[ \| \mathcal{L} \varphi \|_{\beta, p} \leq C \| \varphi \|_{\beta, p}. \]

Let \( \theta \in (0, \frac{1}{\gamma_{100}} \land 1) \) and \( m \in \mathbb{N} \). By (5.5) and Lemma 5.2, we have

\[ \| \mathcal{T}_t \varphi \|_{m^p, p} \leq \| \mathcal{T}^0_t \varphi \|_{m^p, p} + \int_0^t \| \mathcal{T}^{-1}_{t-s} \mathcal{L} \mathcal{T}_s \varphi \|_{m^p, p} \, ds \]

\[ \leq C t^{-\theta \gamma_{100}} \| \varphi \|_{(m-1)^\theta, p} + C \int_0^t \| \mathcal{T}_s \varphi \|_{m^p, p} \, ds, \]

which, by Gronwall’s inequality, yields that for all \( t \in (0, 1) \),

\[ \| \mathcal{T}_t \varphi \|_{m^p, p} \leq C t^{-\theta \gamma_{100}} \| \varphi \|_{(m-1)^\theta, p}. \]

Thus, by iteration we obtain

\[ \| \mathcal{T}_t \varphi \|_{m^p, p} \leq C t^{-\theta \gamma_{100}} \| \mathcal{T}_{(m-1)^\theta} \varphi \|_{(m-1)^\theta, p} \leq \cdots \leq C t^{-\theta \gamma_{100}} \| \varphi \|_p, \]

which gives the estimate (5.6) by resetting \( mt \) with \( t \). \( \square \)

Now we can give

**Proof of Theorem 1.1.** For any \( p \in (1, \infty) \) and \( \varphi \in L^p(\mathbb{R}^d) \), by Lemma 5.3 and Sobolev’s embedding theorem, we have \( \mathcal{T}_t \varphi \in C_0^\infty(\mathbb{R}^d) \) and for any \( k \in \mathbb{N}_0 \) and \( t \in (0, 1) \),

\[ \| \mathcal{T}_t \varphi \|_{k, \infty} \leq C \| \mathcal{T}_t \varphi \|_{k+d, \infty} \leq C t^{-\gamma_{100}} \| \varphi \|_p. \quad (5.7) \]

In particular, there is a function \( \rho_t(x, \cdot) \in L^1(\mathbb{R}^d) \) such that for any \( \varphi \in L^p(\mathbb{R}^d) \),

\[ \mathcal{T}_t \varphi(x) = \int_{\mathbb{R}^d} \varphi(y) \rho_t(x, y) \, dy. \]

By (5.7), we obtain

\[ \| \nabla^k \rho_t(x, \cdot) \|_{\frac{p}{k+1}} \leq C t^{-\gamma_{100}}, \]

for all \( k \in \mathbb{N}_0 \).
where \( \nabla^k \) stands for the distributional derivative, \( C \) is independent of \( x \). By Fubini’s theorem, we have for any \( R > 0 \),
\[
\int_{\mathbb{R}^d} \int_{B_R} |\nabla^k \rho_t(x,y)|^{\frac{1}{p'}} \, dx \, dy < \infty,
\]
which, by Sobolev’s embedding theorem again, produces that for almost all \( y \in \mathbb{R}^d \),
\[
x \mapsto \rho_t(x,y)
\]
is smooth.

As for equation (1.10), it follows by (5.5).

\[ \square \]

### 5.2. Proof of Theorem 1.5

Let \( \delta \) be as in (4.1). We decompose the operator \( \mathcal{L}^\sigma_{\alpha,b} \) as
\[
\mathcal{L}^\sigma_{\alpha,b} f(x) = \mathcal{L}_0 f(x) + \mathcal{L}_1 f(x),
\]
where
\[
\mathcal{L}_0 f(x) := \text{p.v.} \int_{|z| < \delta} f(x + \sigma(x,z)) - f(x) \nu(dz) + b(x) \cdot \nabla f(x)
\]
and
\[
\mathcal{L}_1 f(x) := \int_{|z| \geq \delta} f(x + \sigma(x,z)) - f(x) \nu(dz).
\]

**Lemma 5.4.** If \( \int_{|z| > 1} |z|^q \nu(dz) < \infty \) for some \( q > 0 \), then for any \( \beta \in [0, q] \), there exists a constant \( \beta > 0 \) such that for all \( f \in \mathbb{H}^{\beta, \infty} \),
\[
\|\mathcal{L}_1 f\|_{\beta, \infty} \leq C \|f\|_{\beta, \infty}. \tag{5.8}
\]

**Proof.** First of all, (5.8) is clearly true for \( \beta = 0 \). By interpolation Theorem 5.1, it suffices to prove (5.8) for \( \beta \in [0, q] \cap \mathbb{N} \) and \( \beta = q \). Setting \( \phi_\sigma(x) := x + \sigma(x,z) \), by (1.5), we have
\[
\|\nabla^m \phi_\sigma\|_{\infty} \leq C(1 + |z|), \quad \forall m \in \mathbb{N}. \tag{5.9}
\]

If \( q \in (0, 1) \), then
\[
[f \circ \phi_\sigma]_q = \sup_{x \neq y} \frac{|f(\phi_\sigma(x)) - f(\phi_\sigma(y))|}{|x - y|^q} \leq [f]_q \sup_{x \neq y} \frac{|\phi_\sigma(x) - \phi_\sigma(y)|^q}{|x - y|^q} \tag{5.9}
\]
\[
\leq [f]_q \|\nabla \phi_\sigma\|_{\infty} \leq C [f]_q (1 + |z|^q).
\]
Hence,
\[
[\mathcal{L}_1 f]_q \leq C [f]_q \int_{|z| > 1} (1 + |z|^q) \nu(dz).
\]

For \( q = 1 \), it is easy to see that (5.8) is true by the chain rule. Now assume \( q \in (1, 2) \). By the chain rule we have
\[
[\nabla (f \circ \phi_\sigma)]_{q-1} = \sup_{x \neq y} \frac{|(\nabla f) \circ \phi_\sigma(x) \cdot \nabla \phi_\sigma(x) - (\nabla f) \circ \phi_\sigma(y) \cdot \nabla \phi_\sigma(y)|}{|x - y|^{q-1}} \leq \sup_{x \neq y} \frac{|(\nabla f) \circ \phi_\sigma(x) - (\nabla f) \circ \phi_\sigma(y)| \cdot \|\nabla \phi_\sigma\|_{\infty}}{|x - y|^{q-1}} + \sup_{x \neq y} \frac{\|\nabla f\|_{\infty} |\nabla \phi_\sigma(x) - \nabla \phi_\sigma(y)|}{|x - y|^{q-1}} \tag{5.9}
\]
\[
\leq C \|\nabla f\|_{q-1} (1 + |z|) \sup_{x \neq y} \frac{|\phi_\sigma(x) - \phi_\sigma(y)|^{q-1}}{|x - y|^{q-1}} + C \|\nabla f\|_{\infty} (1 + |z|)^{2-q} \sup_{x \neq y} \frac{|\nabla \phi_\sigma(x) - \nabla \phi_\sigma(y)|^{q-1}}{|x - y|^{q-1}}
\]

This completes the proof of Lemma 5.4.
Thus,

\[ [L_1f]_{q-1} \leq C\|f\|_{q-1,\infty} \int_{|z|>1} (1 + |z|^q)\nu(dz). \]

For \( q \geq 2 \), it follows by similar calculations.

Let \( T_t \) be the semigroup associated with \( L'_{\sigma,b} \). For any \( \varphi \in C^\infty_b(\mathbb{R}^d) \), as above by Duhamel’s formula, we have

\[ T_t\varphi(x) = T_t^0\varphi(x) + \int_0^t T_{t-s}^0 L_1 T_s \varphi(x)ds. \]  

(5.10)

**Lemma 5.5.** Let \( \gamma_{100} \) be as in Theorem 4.5. If \( \int_{|z|>1} |z|^q\nu(dz) < \infty \) for some \( q > 0 \), then for any \( \beta \in (0, \frac{1}{\gamma_{100}} \wedge 1) \) with \( q + \beta \) being not an integer, there exists a constant \( C > 0 \) such that for all \( t \in (0, 1) \) and \( \varphi \in L^\infty(\mathbb{R}^d) \),

\[ \|T_t\varphi\|_{q+\beta,\infty} \leq C t^{-q+\beta} \gamma_{100} \|\varphi\|_\infty. \]

**Proof.** Fix an irrational number \( q_0 \in (0, q) \) and choose \( m \in \mathbb{N} \) being large so that

\[ \theta := \frac{qm}{m} < \frac{1}{\gamma_{100}} \wedge 1. \]

By (5.10), (5.8) and Lemma 5.2, we have

\[ \|T_t\varphi\|_{m\theta,\infty} \leq \|T_t^0\varphi\|_{m\theta,\infty} + \int_0^t \|T_{t-s}^0 L_1 T_s \varphi\|_{m\theta,\infty}ds \]

\[ \leq C t^{-m\theta_{100}} \|\varphi\|_{(m-1)\theta,\infty} + C \int_0^t \|T_s \varphi\|_{m\theta,\infty}ds, \]

which, by Gronwall’s inequality, yields that for all \( t \in (0, 1) \),

\[ \|T_t\varphi\|_{m\theta,\infty} \leq C t^{-m\theta_{100}} \|\varphi\|_{(m-1)\theta,\infty}. \]

Since \( j\theta \) is not an integer for any \( j \in \mathbb{N} \), by iteration we obtain

\[ \|T_t\varphi\|_{q_0,\infty} = \|T_t\varphi\|_{m\theta,\infty} \leq C t^{-m\theta_{100}} \|\varphi\|_{q_0,\infty} = C t^{-q_0\gamma_{100}} \|\varphi\|_{\infty}. \]

Next we choose \( \theta_0 \in (0, \frac{1}{\gamma_{100}} \wedge 1) \) and an irrational number \( q_0 \leq q \) so that \( q_0 + \theta_0 = q + \beta \). As above, we have

\[ \|T_t\varphi\|_{q_0+\theta_0,\infty} \leq C t^{-\theta_0\gamma_{100}} \|\varphi\|_{q_0,\infty} + C \int_0^t (t-s)^{-\theta_0\gamma_{100}} \|L_1 T_s \varphi\|_{q_0,\infty}ds \]

\[ \leq C t^{-\theta_0\gamma_{100}} \|\varphi\|_{q_0,\infty} + C \|\varphi\|_{q_0,\infty} \int_0^t (t-s)^{-\theta_0\gamma_{100}}ds \]

\[ \leq C (t^{-\theta_0\gamma_{100}} + t^{1-\theta_0\gamma_{100}}) \|\varphi\|_{q_0,\infty}. \]

Thus,

\[ \|T_t\varphi\|_{q_0+\theta_0,\infty} \leq C t^{-\theta_0\gamma_{100}} \|T_t\varphi\|_{q_0,\infty} \leq C t^{-q_0\gamma_{100}} \|\varphi\|_{\infty}. \]

The proof is complete.

**Lemma 5.6.** Let \( \gamma_{010} \) be as in Theorem 4.5. For any \( \theta \in (0, 1/\gamma_{010} \wedge 1) \), there exists a constant \( C > 0 \) such that for all \( t \in (0, 1) \) and \( \varphi \in C^\infty_b(\mathbb{R}^d) \),

\[ \|T_t \Delta \varphi\|_{\infty} \leq C t^{-\theta \gamma_{010}} \|\varphi\|_{\infty}. \]
Proof. First of all, we show that
\[ \| T_t^0 \Delta^\varphi \|_\infty \leq C t^{-\theta y_{010}} \| \varphi \|_\infty. \]  
(5.11)

Notice that
\[ T_t^0 \Delta^\varphi(x) = \mathbb{E} \int_{|z|^\theta > 0} \frac{\varphi(X_t(x) + z) - \varphi(X_t(x))}{|z|^{d+\theta}} dz = I_1(x) + I_2(x), \]  
(5.12)

where
\[ I_1(x) := \mathbb{E} \int_{|z|^\theta > 0} \frac{\varphi(X_t(x) + z) - \varphi(X_t(x))}{|z|^{d+\theta}} dz, \]
\[ I_2(x) := \mathbb{E} \int_{|z|^\theta > 0} \frac{\varphi(X_t(x) + z) - \varphi(X_t(x))}{|z|^{d+\theta}} dz. \]

For \( I_1(x) \), setting \( \varphi_{xz}(x) := \varphi(x + sz) \), we have
\[
I_1(x) = \mathbb{E} \int_{|z|^\theta > 0} \left( \int_0^1 z \cdot \nabla \varphi(X_t(x) + sz) ds \right) \frac{dz}{|z|^{d+\theta}} \\
= \int_{|z|^\theta > 0} \left( \int_0^1 z \cdot T_t^0 \nabla \varphi_{xz}(x) ds \right) \frac{dz}{|z|^{d+\theta}}.
\]

Hence,
\[
\| I_1 \|_\infty \leq \int_{|z|^\theta > 0} \left( \int_0^1 \| T_t^0 \nabla \varphi_{xz} \|_{\infty} ds \right) \frac{|z| dz}{|z|^{d+\theta}} \\
\overset{(4.16)}{\leq} C t^{-\gamma_{010}} \| \varphi \|_\infty \int_{|z|^\theta > 0} \frac{dz}{|z|^{d+\theta-1}} \leq C t^{-\theta y_{010}} \| \varphi \|_\infty. \]  
(5.13)

For \( I_2(x) \), we have
\[
\| I_2 \|_\infty \overset{(4.15)}{\leq} C \| \varphi \|_\infty \int_{|z|^\theta > 0} \frac{1}{|z|^{d+\theta}} dz \leq C t^{-\gamma_{010}} \| \varphi \|_\infty. \]  
(5.14)

Combining (5.12)-(5.14), we obtain (5.11). Now, by (5.10) and (5.11), we have
\[
\| T_t^0 \Delta^\varphi \|_\infty \leq \| T_t^0 \Delta^\varphi \|_\infty + \int_0^t \| T_{t-s}^0 L_1 T_s^0 \Delta^\varphi \|_\infty ds \\
\leq C t^{-\theta y_{010}} \| \varphi \|_\infty + \int_0^t \| T_s^0 \Delta^\varphi \|_\infty ds,
\]
which in turn gives the desired estimate by Gronwall’s inequality. \( \Box \)

Proof of Theorem 1.5 Let \( X_t(x) \) solve SDE (3.2). By Lemma 5.6 and Lemma 6.4 in Appendix, there exists a function \( \rho_t(x, y) \in (L^1 \cap L^p)(\mathbb{R}^d) \) for some \( p > 1 \) such that for all \( \varphi \in C_0(\mathbb{R}^d), \)
\[ T_t \varphi(x) = \mathbb{E} \varphi(X_t(x)) = \int_{\mathbb{R}^d} \varphi(y) \rho_t(x, y) dy. \]

By a further approximation, the above equality also holds for any \( \varphi \in L^\infty(\mathbb{R}^d) \). The \( q + \varepsilon \)-order Hölder continuity of \( x \mapsto T_t \varphi(x) \) follows by Lemma 5.5. \( \Box \)
6. Appendix

6.1. Proof of Lemma 6.4. Let δ be as in (4.1). For \(0 < \varepsilon < \delta\), let \(X_t^\varepsilon(x) = X_t\) solve the following SDE:

\[
X_t^\varepsilon = x + \int_0^t b(X_s^\varepsilon) ds + \int_0^t \int_{\Gamma_e^\varepsilon} \sigma(X_s^\varepsilon, z) \tilde{N}(ds, dz),
\]

(6.1)

where \(\Gamma_e^\varepsilon := \{z \in \mathbb{R}^d : \varepsilon \leq |z| < \delta\}\). We first prove the following limit theorem.

**Lemma 6.1.** Under (H') \(\varepsilon\), there exist a subsequence \(\varepsilon_k \to 0\) and a null set \(\Omega_0\) such that for all \(\omega \notin \Omega_0\),

\[
\limsup_{k \to \infty} \sup_{|x| \leq R} |X_{t}^\varepsilon(x, \omega) - X_t(x, \omega)| = 0, \quad \forall R \in \mathbb{N}.
\]

**Proof.** Set \(Z_t^\varepsilon := X_t^\varepsilon - X_t\). By Burkholder’s inequality (2.9) and (H'), we have for any \(p \geq 2\),

\[
\mathbb{E} \left( \sup_{t \in [0, T]} |Z_t^\varepsilon|^p \right) \leq C \mathbb{E} \left( \int_0^T |b(X_s^\varepsilon) - b(X_s)| ds \right)^p + C \mathbb{E} \left( \sup_{t \in [0, T]} \left| \int_0^t \int_{\Gamma_e^\varepsilon} \sigma(X_s^\varepsilon, z) \tilde{N}(ds, dz) \right|^p \right)
\]

\[
+ C \mathbb{E} \left( \sup_{t \in [0, T]} \left| \int_0^t \int_{\Gamma_e^\varepsilon} (\sigma(X_s^\varepsilon, z) - \sigma(X_s, z)) \tilde{N}(ds, dz) \right|^p \right),
\]

\[
\leq C \int_0^T \mathbb{E} |Z_t^\varepsilon|^p ds + C \int_{\Gamma_e^\varepsilon} |z|^p \nu(dz) + C \int_{\Gamma_e^\varepsilon} |z|^2 \nu(dz)^{\frac{p}{2}},
\]

where \(C\) is independent of \(\varepsilon, t \in (0, 1)\) and \(x \in \mathbb{R}^d\). By Gronwall’s inequality, we obtain

\[
\lim_{\varepsilon \to 0} \sup_{x \in \mathbb{R}^d} \mathbb{E} \left( \sup_{t \in [0, 1]} |X_t^\varepsilon(x) - X_t(x)|^p \right) = 0.
\]

Similarly, we can prove that for any \(p \geq 2\),

\[
\lim_{\varepsilon \to 0} \sup_{x \in \mathbb{R}^d} \mathbb{E} \left( \sup_{t \in [0, 1]} |\nabla X_t^\varepsilon(x) - \nabla X_t(x)|^p \right) = 0.
\]

Thus, for any \(R > 0\), by Sobolev’s embedding theorem, we have

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{|x| < R} \sup_{t \in [0, 1]} |X_t^\varepsilon(x) - X_t(x)|^p \right) \leq C \lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{t \in [0, 1]} \|X_t^\varepsilon(\cdot) - X_t(\cdot)|\|_{W^{1,p}(B_R)}^p \right) = 0,
\]

where \(p > d\) and \(W^{1,p}(B_R)\) is the first order Sobolev space over \(B_R := \{x \in \mathbb{R}^d : |x| < R\}\). The desired limit follows by a suitable choice of subsequence \(\varepsilon_k\). \(\square\)

Define \(\phi(x, z) := x + \sigma(x, z)\). By (4.1), the mapping \(x \mapsto \phi(x, z)\) is invertible for each \(|z| \leq \delta\). Let \(\phi^{-1}(x, z)\) be the inverse of \(x \mapsto \phi(x, z)\). Write

\[
\hat{\sigma}(x, z) := \sigma(\phi^{-1}(x, z), z), \quad |z| \leq \delta
\]

(6.2)

and

\[
\hat{b}(x) := b(x) + \int_{\Gamma_e^\varepsilon} [\sigma(\phi^{-1}(x, z), z) - \sigma(x, z)] \nu(dz),
\]

(6.3)

\[
\hat{b}_e(x) := b(x) + \int_{\Gamma_e^\varepsilon} [\sigma(\phi^{-1}(x, z), z) - \sigma(x, z)] \nu(dz).
\]

(6.4)

By the chain rule, the following lemma is easy.
Lemma 6.2. Under \((H^\gamma_\delta)\), there exists a constant \(C > 0\) such that for all \(x \in \mathbb{R}^d\) and \(|z| \leq \delta\),
\[
|\hat{\sigma}(x, z)| \leq C|z|, \quad |\nabla_x \hat{\sigma}(x, z)| \leq C|z|.
\]
Moreover, \(\hat{b}, \hat{b}_e \in C_b^1(\mathbb{R}^d)\) and for some \(C > 0\),
\[
\|\hat{b}_e - \hat{b}\|_\infty + \|\nabla \hat{b}_e - \nabla \hat{b}\|_\infty \leq C \int_{\Gamma_0^\delta} |z|^2 \nu(dz).
\]

Fix \(T \in [0, 1]\). For \(t \in [0, T]\), define
\[
\hat{L}_t^T := L_{T-} - L_{T-T^+} \text{ with } L_{T-T^+} := \lim_{\delta \downarrow 0} L_{T-\delta}.
\]
In particular, \((\hat{L}_t^T)_{t \in [0, T]}\) is still a Lévy process with the same Lévy measure \(\nu\) and
\[
\Delta \hat{L}_t^T = \Delta L_{T-T^+}.
\] (6.5)

Let \(\hat{N}^T(ds, dz)\) be the Poisson random measure associated with \(\hat{L}_t^T\), i.e.,
\[
\hat{N}^T((0, t] \times E) := \sum_{0 < s \leq t} 1_E(\Delta \hat{L}_s^T), \quad E \in \mathcal{B}(\mathbb{R}^d),
\]
and \(\tilde{N}^T(ds, dz) := \hat{N}^T(ds, dz) - d\nu(dz)\) the compensated Poisson random measure. We have

Lemma 6.3. Let \(\hat{X}^T_t(x) = \hat{X}^T_t\) solve the following SDE:
\[
\hat{X}^T_t = x - \int_0^t \hat{b}(\hat{X}^T_s)ds - \int_0^t \int_{\Gamma^\delta_0} \hat{\sigma}(\hat{X}^T_s, z)\hat{N}^T(ds, dz),
\] (6.6)
where \(\hat{\sigma}\) and \(\hat{b}\) are defined by \((6.2)\) and \((6.3)\) respectively. Then
\[
\hat{X}^T_t(x) = X_t^{-1}(x), \quad \forall x \in \mathbb{R}^d, \text{ a.s.} \tag{6.7}
\]

Proof. For \(\varepsilon \in (0, \delta)\), let \(X^\varepsilon_t(x) = X^\varepsilon_t\) solve the following random ODE:
\[
X^\varepsilon_t = x + \int_0^t b(X^\varepsilon_s)ds + \int_0^t \int_{\Gamma^\delta_0} \sigma(X^\varepsilon_s, z)\hat{N}(ds, dz)
\]
\[
= x + \int_0^t \tilde{b}_e(X^\varepsilon_s)ds + \sum_{0 < s \leq t} \sigma(X^\varepsilon_{s-}, \Delta L_s)1_{\varepsilon}(\Delta L_s),
\]
where
\[
\tilde{b}_e(x) = b(x) - \int_{\Gamma^\delta_0} \sigma(x, z)\nu(dz).
\]

By the change of variables, we have
\[
X^\varepsilon_{T-t} = X^\varepsilon_t - \int_{T-t}^T \tilde{b}_e(X^\varepsilon_s)ds - \sum_{T-t < s < T} \sigma(X^\varepsilon_{s-}, \Delta L_s)1_{\varepsilon}(\Delta L_s)
\]
\[
= X^\varepsilon_T - \int_0^T \tilde{b}_e(X^\varepsilon_{T-s})ds - \sum_{0 \leq s < T} \sigma(X^\varepsilon_{T-s}, \Delta L_{T-s})1_{\varepsilon}(\Delta L_{T-s}).
\]

Noticing that if \(\Delta L_t \in \Gamma^\delta_0\), then
\[
X^\varepsilon_t - X^\varepsilon_{T-t} = \sigma(X^\varepsilon_{T-t}, \Delta L_t), \quad \Rightarrow X_{T-t} = \phi^{-1}(X^\varepsilon_{T-t}, \Delta L_t),
\]
and since \(\Delta L_T = 0\) almost surely, we further have
\[
X^\varepsilon_{T-t} = X^\varepsilon_T - \int_0^T \tilde{b}_e(X^\varepsilon_{T-s})ds - \sum_{0 \leq s < T} \hat{\sigma}(X^\varepsilon_{T-s}, \Delta L_{T-s})1_{\varepsilon}(\Delta L_{T-s}).
\]
Then there exists a measurable function \( \hat{\sigma} \) such that for any \( \epsilon > 0 \),
\[
\int_0^t \hat{b}_s(X_{t-s}^\epsilon)\,ds - \int_0^t \hat{\sigma}(X_{t-s}^\epsilon, \Delta L_s^T)1_{ \{ \epsilon \} }\,d\hat{\lambda}^s(\Delta L_s^T),
\]
where \( \hat{\sigma}(x, \cdot) \) is defined by (6.2). In particular,
\[
X_{T-t}^\epsilon = X_T^\epsilon - \int_0^t \hat{b}_s(X_{t-s}^\epsilon)\,ds - \int_0^t \hat{\sigma}(X_{t-s}^\epsilon, \cdot)\,d\hat{\lambda}^s,
\]
where \( \hat{b}_s(x) \) is defined by (6.4). On the other hand, let \( \hat{X}_t^T(x) = \hat{X}_t^T(x) \) solve the following SDE:
\[
\dot{\hat{X}}_t^T = x - \int_0^t \hat{b}_s(\hat{X}_s^T)\,ds - \int_0^t \hat{\sigma}(\hat{X}_s^T, \cdot)\,d\hat{\lambda}^s.
\]
By the uniqueness of solutions to random ODEs, we have
\[
X_{T-t}^\epsilon(x) = \hat{X}_t^T(X_T^\epsilon(x)), \quad \forall x \in \mathbb{R}^d, \text{ a.s.}
\]
In particular,
\[
x = \hat{X}_t^T(X_T^\epsilon(x)), \quad \forall x \in \mathbb{R}^d, \text{ a.s.}
\] (6.8)
By Lemmas 6.2, 6.1 and taking limits for (6.8), we obtain
\[
x = \hat{X}_t^T(X_T(x)), \quad \forall x \in \mathbb{R}^d, \text{ a.s.}
\]
The proof is complete. \(\square\)

Now we can give

**Proof of Lemma 4.4.** By equation (6.6) and a standard calculation, we have for any \( p \geq 2 \),
\[
\sup_{T \in [0,1]} \sup_{x \in \mathbb{R}^d} \mathbb{E}[|\nabla \hat{X}_T^T(x)|^p] < \infty,
\]
which, together with (6.7), implies that
\[
\sup_{T \in [0,1]} \sup_{x \in \mathbb{R}^d} \mathbb{E}(\det(\nabla X_T^{-1}(x))) < \infty.
\]
The desired estimate (4.15) then follows by the change of variables and the above estimate. \(\square\)

### 6.2. A criterion for the existence of density.

**Lemma 6.4.** Let \( T \) be a bounded linear operator in \( C_0(\mathbb{R}^d) \). Assume that for some \( \theta \in (0,1) \) and any \( \varphi \in C_0^\infty(\mathbb{R}^d) \),
\[
\|T\Delta^2 \varphi\|_\infty \lesssim C_0\|\varphi\|_\infty.
\] (6.9)
Then there exists a measurable function \( \rho(x,y) \) with \( \rho(x,\cdot) \in (L^1 \cap L^p)(\mathbb{R}^d) \) for some \( p > 1 \) and such that for any \( \varphi \in C_0(\mathbb{R}^d) \),
\[
T \varphi(x) = \int_{\mathbb{R}^d} \varphi(y)\rho(x,y)\,dy.
\] (6.10)

**Proof.** By Riesz’s representation theorem, there exists a family of finite signed measures \( \mu_x(dy) \) such that \( x \mapsto \mu_x(dy) \) is weakly continuous and for any \( \varphi \in C_0(\mathbb{R}^d) \),
\[
T \varphi(x) = \int_{\mathbb{R}^d} \varphi(y)\mu_x(dy).
\] (6.11)
Let \( \varrho \) be a nonnegative symmetric smooth function with compact support and \( \int_{\mathbb{R}^d} \varrho(y)dy = 1 \). Let \( \varrho_\varepsilon(y) := \varepsilon^{-d} \varrho(\varepsilon^{-1}y) \) be a family of mollifies. For \( R > 0 \), let \( \chi_R : \mathbb{R}^d \to [0, 1] \) be a smooth cutoff function with
\[
\chi_R(x) = 1, \quad |x| \leq R, \quad \chi_R(x) = 0, \quad |x| \geq 2R.
\]

For \( \varphi \in L^\infty(\mathbb{R}^d) \), set
\[
\varphi_\delta(x) := \varphi * \varrho_\delta(x), \quad \varphi^R_\delta(x) := (\varphi_\delta \chi_R) * \varrho_\varepsilon(x)
\]
and
\[
\mu^e_\delta(z) := \int_{\mathbb{R}^d} \varrho_\varepsilon(y-z) \mu(y)dy.
\]

It is easy to see that \( \mu^e_\delta \in \cap \mathcal{W}^{1,1}(\mathbb{R}^d) \) and \( \Delta^2 \varphi^R_\delta, \mu^e_\delta \in C_0(\mathbb{R}^d) \). Thus, by (6.11) we have
\[
\mathcal{T} \Delta^2 \varphi^R_\delta(x) = \int_{\mathbb{R}^d} (\Delta^2(\varphi_\delta \chi_R)) \cdot \varrho_\varepsilon(y) \mu(y)dy
\]
\[
= \int_{\mathbb{R}^d} \Delta^2(\varphi_\delta \chi_R)(z) \mu^e_\delta(z)dz = \int_{\mathbb{R}^d} \varphi_\delta \chi_R(z) \Delta^2 \mu^e_\delta(z)dz,
\]
which yields by (6.9) that
\[
\left| \int_{\mathbb{R}^d} \varphi_\delta \chi_R(z) \Delta^2 \mu^e_\delta(z)dz \right| \leq C_\delta \| \varphi^R_\delta \| \leq C_\delta \| \varphi \|_\infty.
\]

Letting \( R \to \infty \) and \( \delta \to 0 \), by the dominated convergence theorem, we obtain that for all \( \varphi \in L^\infty(\mathbb{R}^d) \),
\[
\left| \int_{\mathbb{R}^d} \varphi(z) \Delta^2 \mu^e_\delta(z)dz \right| \leq C_\delta \| \varphi \|_\infty,
\]
which gives
\[
\sup_{x \in \mathbb{R}^d} \sup_{\delta \in (0,1)} \| \Delta^2 \mu^e_\delta \|_1 \leq C_\delta.
\]

Moreover, we also have
\[
\sup_{x \in \mathbb{R}^d} \sup_{\delta \in (0,1)} \| \mu^e_\delta \|_1 \leq \sup_{\| \varphi \|_\infty < 1} \| \mathcal{T} \varphi \|_\infty.
\]

By Sobolev’s embedding theorem, there is a \( p > 1 \) such that
\[
\sup_{x \in \mathbb{R}^d} \sup_{\delta \in (0,1)} \| \mu^e_\delta \|_p < \infty.
\]

Since \( L^p(\mathbb{R}^d) \) is reflexive, for each fixed \( x \in \mathbb{R}^d \), there is a subsequence \( \varepsilon_k \to 0 \) and \( \rho(x, \cdot) \in (L^1 \cap L^p)(\mathbb{R}^d) \) such that for any \( \varphi \in C_0(\mathbb{R}^d) \subset L^{p-1}(\mathbb{R}^d) \),
\[
\mathcal{T} \varphi_{\varepsilon_k}(x) = \int_{\mathbb{R}^d} \mu^e_{\varepsilon_k}(z) \varphi(z)dz \xrightarrow{k \to \infty} \int_{\mathbb{R}^d} \rho(x, z) \varphi(z)dz.
\]

On the other hand, for any \( \varphi \in C_0(\mathbb{R}^d) \),
\[
\| \mathcal{T} \varphi_{\varepsilon_k} - \mathcal{T} \varphi \|_\infty \leq C \| \varphi_{\varepsilon_k} - \varphi \|_\infty \varepsilon_k \to 0,
\]
which together with (6.12) yields (6.10). \( \square \)

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