Transversally strictly hyperbolic systems

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Abstract

We consider the Cauchy problem for first order systems. Assuming that the set Σ of the singular points of the characteristic variety is a smooth manifold and the characteristic values are real and semi-simple we introduce a new class which is strictly hyperbolic in the transverse direction to Σ. We prove that if Σ is either involutive or symplectic then these transversally strictly hyperbolic systems are strongly hyperbolic. On the other hand if Σ is neither involutive nor symplectic transversally strictly hyperbolic systems are much more involved which is discussed taking an interesting example.

Keywords: Transversally strictly hyperbolic, strongly hyperbolic, uniformly diagonalizable, involutive characteristics, symplectic characteristics, propagation cone

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1 Introduction

In this note, we continue to study a new class of first order systems

\[ L(t, x, D_t, D_x) = D_t - \sum_{j=1}^{d} A_j(t, x)D_{x_j} = D_t - A(t, x, D_x) \]

which we call transversally strictly hyperbolic systems in [4] and we discuss whether transversally strictly hyperbolic systems are strongly hyperbolic, which means by definition that for all lower order term B the Cauchy problem for \( L + B \) with initial data on the surface \( \{ t = 0 \} \) is well-posed in \( C^\infty \). Here we use the notation \( D = -i \partial \) for partial derivatives.

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Assumption 1.1. The coefficient matrices $A_j(t,x)$ are $C^\infty$ and constant outside a compact set and they act on $\mathbb{C}^N$. Moreover, for all $(t,x,\xi)$ the eigenvalues of $A(t,x,\xi) = \sum \xi_j A_j(t,x)$ are real and semi-simple.

We denote by $\Sigma$ the characteristic variety of $L$, that is the set of $(t,x,\tau,\xi) \in T^*\mathbb{R}^{1+d}\setminus\{0\}$ such that $\det L(t,x,\tau,\xi) = 0$.

Assumption 1.2. $\Sigma$ is a smooth $C^\infty$ manifold in $T^*\mathbb{R}^{1+d}\setminus\{0\}$ and on each component of $\Sigma$ the dimension of $\ker L(\rho)$ is constant.

Note that Assumption 1.1 implies that at characteristic points, (1.2) $\dim \ker L(t,x,\tau,\xi) =$ multiplicity of the eigenvalue $\tau$.

Definition 1.3. Recall first the invariant definition of the localized system (or localization) $L_\rho$ at $\rho \in \Sigma$:

(1.3) $L_\rho(\dot{\rho}) = \varpi_\rho(L'(\rho) \cdot \dot{\rho}) \iota_\rho$

where $\iota_\rho$ is the injection of $\ker L(\rho)$ into $\mathbb{C}^N$, $\varpi_\rho$ is the projection from $\mathbb{C}^N$ onto $\mathbb{C}^N/\text{range} L(\rho)$ and $L'$ is the derivative of $L$. Because $\ker L(\rho) \cap \text{range} L_\rho = \{0\}$ by Assumption 1.1, $L_\rho$ can also be seen as a matrix with values in $\text{Hom}(\ker L(\rho))$. Recall that $L_\rho$ is hyperbolic in the time direction, that is $\det L_\rho$ is a hyperbolic polynomial in the time direction.

We now introduce transversally strictly hyperbolic systems:

Definition 1.4. If for all $\rho \in \Sigma$, $L_\rho(\dot{\rho})$ is strictly hyperbolic in the time direction, on $T^*\mathbb{R}^{d+1}/T_\rho \Sigma$ then we call $L$ a transversally strictly hyperbolic system.

Here we note [4, Proposition 2.2]

Lemma 1.5. If $L$ is transversally strictly hyperbolic then $L$ is uniformly symmetrizable (equivalently uniformly diagonalizable), that is there is a family of hermitian symmetric matrices $S(t,x,\xi)$, such that $S$ and $S^{-1}$ are uniformly bounded and $SA$ is symmetric.

Theorem 1.6 ([3]). If there is a symmetrizer which is Lipschitz continuous in $(t,x,\xi) \in \mathbb{R}^{d+1} \times S^{d-1}$ then the Cauchy problem for $L$ is $L^2$ well-posed and hence $L$ is a strongly hyperbolic system.
The symmetrizer can not be chosen to be Lipschitz continuous, not even continuous in general ([3]). We discuss about this later. We say that $f(x) \in \gamma^s(\mathbb{R}^d)$, the Gevrey class of order $s$, if for any compact set $K \subset \mathbb{R}^d$ there exist $C > 0, A > 0$ such that we have

$$|D^\alpha f(x)| \leq CA^{(|\alpha|)}x^s, \ x \in K, \ \forall \alpha \in \mathbb{N}^d.$$  

We denote $\gamma^s_0(\mathbb{R}^d) = \gamma^s(\mathbb{R}^d) \cap C^\infty_0(\mathbb{R}^d)$.

**Theorem 1.7** ([6]). Assume that $L(t,x,\tau,\xi)$ is uniformly symmetrizable then the Cauchy problem for $L + B$ is well-posed in the Gevrey class of order $1 < s \leq 2$ for any $B$.

A simple proof of the result is found in [1].

**Theorem 1.8** ([4]). Assume that $\Sigma$ is an involutive submanifold of $T^*\mathbb{R}^{1+d}\{0\}$ and $L$ is transversally strictly hyperbolic. Then the Cauchy problem for $L$ with initial data on $\{t = 0\}$ is well-posed in $L^2$. In particular $L$ is a strongly hyperbolic system.

**Theorem 1.9.** Assume that $\Sigma$ is a symplectic submanifold of $T^*\mathbb{R}^{1+d}\{0\}$ and $L$ is transversally strictly hyperbolic. Then the Cauchy problem for $L + B$ with initial data on $\{t = 0\}$ is $C^\infty$ well-posed for any $B(t,x)$, that is $L$ is a strongly hyperbolic system.

**Sketch of the proof:** Writing $L(t,x,\tau,\xi) + B(t,x) = \tau I_N - A(t,x,\xi) + B(t,x)$ one can assume that $A(t,x,\xi) - B(t,x)$ is block diagonal. Thus we can assume that $A$ is a $r \times r$ matrix and $h(t,x,\tau,\xi) = \det L(t,x,\tau,\xi)$ vanishes of order $r$ on a symplectic manifold $\Sigma$. Denote by $M(t,x,\tau,\xi)$ the cofactor matrix of the symbol $L(t,x,\tau,\xi)$. We look for a solution to $(L + B)U = 0$ in the form $U = M(t,x,D_t,D_x)u$ so that we are led to consider the Cauchy problem

$$\begin{cases}
(L(t,x,D_t,D_x) + B(t,x))M(t,x,D_t,D_x)u = 0, \\
D_t^j u(0,x) = u_j(x), \ j = 0,\ldots,r-1.
\end{cases}$$

With $P(t,x,D_t,D_x) = (L(t,x,D_t,D_x) + B(t,x))M(t,x,D_t,D_x)$ one has

**Lemma 1.10.** Denoting $h(t,x,\tau,\xi) = \det L(t,x,\tau,\xi)$ we have

$$\mathcal{P}(t,x,\tau,\xi) = h(t,x,\tau,\xi)I_r + P_{r-1} + \cdots + P_0$$

where $P_j(t,x,\tau,\xi)$ are homogeneous of degree $j$ in $(\tau,\xi)$ and vanishes of order $r - 2j$ on $\Sigma$.  

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Proof. We may assume that Σ is given by (microlocally) φ₀ = τ = 0, φ₁(t, x, ξ) = 0, ..., φₖ(t, x, ξ) = 0. Then from the assumption one can write

\[ L(t, x, τ, ξ) = φ₀I_r + \sum_{j=1}^{k} A_j(t, x, ξ)φ_j(t, x, ξ) \]

and hence \( M(t, x, τ, ξ) \) is a homogeneous polynomial in \((φ₀, φ₁, ..., φₖ)\) of degree \( r - 1 \). Since

\[ P_{r-j}(t, x, τ, ξ) = \sum_{l+|α|=j} (-i)^j \partial_x^α \partial_y^l (L(t, x, τ, ξ) + B(t, x)) \partial_x^l \partial_y^α M(t, x, τ, ξ) \]

and \( \partial_x^l \partial_y^α M(t, x, τ, ξ) \) vanishes on Σ of order \( r - l - |α| \) then the assertion is clear because \( r - 2j \leq r - 1 - j \) for \( j \geq 1 \).

Then the solvability of the Cauchy problem follows from [7, Theorem 1.3].

2 Neither involutive nor symplectic case

If Σ is neither involutive nor symplectic, we will see that transversally strictly hyperbolic systems are more involved. We make detailed study on the following 3 × 3 system

\[ L_a = \frac{∂}{∂t} + \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \frac{∂}{∂x} + x \left( \begin{array}{ccc} 0 & a & 1 \\ -a & 0 & 0 \\ 1 + a² & 0 & 0 \end{array} \right) \frac{∂}{∂y} = \frac{∂}{∂t} + A \frac{∂}{∂x} + xB \frac{∂}{∂y} = \frac{∂}{∂t} + G_a \]

where \( a \in \mathbb{C} \) which is given in [3]. Note that

\[ \det L_a(x, τ, ξ, η) = τ(τ^2 - ξ^2 - x^2η^2) \]

and the characteristic manifold is \( Σ = \{ τ = 0, ξ = 0, x = 0 \} \) which is neither symplectic nor involutive. Let \( ρ = (t, 0, y, 0, 0, η) \in Σ \) where \( η \neq 0 \). Since

\[ (L_a)_ρ(\dot{x}, \dot{t}, \dot{ξ}) = \dot{τ}I + \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \dot{ξ} + \dot{η} \left( \begin{array}{ccc} 0 & a & 1 \\ -a & 0 & 0 \\ 1 + a² & 0 & 0 \end{array} \right) \dot{x} \]

then \( \det(L_a)_ρ(\dot{x}, \dot{t}, \dot{ξ}) = \dot{τ}(\dot{τ}^2 - ξ^2 - η^2x^2) \) is a strictly hyperbolic polynomial in \((\dot{x}, \dot{τ}, ξ, η)\). Therefore \( L_a \) is transversally strictly hyperbolic system for any \( a \in \mathbb{C} \) by definition with characteristic manifold \( Σ \) which is neither symplectic nor involutive.
2.1 Ill-posedness

Note

\[ L^* = -\frac{\partial}{\partial t} - A^* \frac{\partial}{\partial x} - xB^* \frac{\partial}{\partial y} = -\frac{\partial}{\partial t} + G^* \]

and consider the eigenvalue problem \( G^* a V(x, y) = i\beta V(x, y) \). We look for \( V(x, y) \) in the form \( V(x, y) = e^{\pm iy}\bar{V}(x) \) so that the problem is reduced to \( (A^* \partial_x \pm ixB^*)\bar{V}(x) = -i\beta \bar{V}(x) \). Denote

\[ \bar{V}(x) = \begin{pmatrix} u(x) \\ v(x) \\ w(x) \end{pmatrix}. \]

If \( u(x) \) satisfies

\[(2.1) \quad (\partial_x^2 - x^2 + \beta^2 \pm i\bar{a})u(x) = 0 \]

then with

\[ v(x) = \frac{i}{\beta}(\partial_x \pm i\bar{a}x)u(x), \]
\[ w(x) = \mp \frac{x}{\beta}u(x) \]

we conclude that \( G^* a V^\pm = i\beta V^\pm \). Since \( G^* a V^\pm(\eta x, \eta^2 y) = i\eta\beta V^\pm(\eta x, \eta^2 y) \)

one has

\[ L^* \left( e^{i\beta \eta t} V^\pm(\eta x, \eta^2 y) \right) = 0. \]

The following lemma is easily checked.

**Lemma 2.1.** Assume that \( ia \not\in [-1, 1] \). Then either \( \beta^2 + i\bar{a} = 1 \) or \( \beta^2 - i\bar{a} = 1 \) has a root \( \beta \in \mathbb{C} \) with \( \text{Im} \beta > 0 \).

Assume that \( \beta \in \mathbb{C} \) is chosen such that \( \beta^2 \pm i\bar{a} = 1 \) so that \( (2.1) \) is verified by \( u^\pm(x) = e^{-x^2/2} \). Therefore

\[ W^\pm(t, x, y) = \exp \left( i\beta \eta t \pm iy\eta^2 - \frac{1}{2} \eta^2 x^2 \right) (W_0 + \eta x W_1) \]

solves \( L^* a W^\pm = 0 \) where

\[ W_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad W_1 = \begin{pmatrix} 0 \\ -i(1 \mp i\bar{a})/\beta \end{pmatrix}. \]
We now consider the following Cauchy problem

\[
(L_{a}U = 0, \\
U(0, x, y) = \phi(x)\psi(y)W_0)
\]

where \(\phi, \psi \in C^\infty_0(\mathbb{R})\) are real valued. We remark that we can assume that solutions \(U\) to (2.2) have compact supports with respect to \((x, y)\). To examine this we recall the Holmgren uniqueness theorem (see for example [5, Theorem 4.2]). For \(\delta > 0\) we denote

\[
D_\delta = \{(t, x, y) \in \mathbb{R}^3 \mid x^2 + y^2 + |t| < \delta\}
\]

then we have

**Proposition 2.2** (Holmgren). There exists \(\delta_0 > 0\) such that if \(U(t, x, y) \in C^1(D_\delta)\) with \(0 < \delta \leq \delta_0\) verifies

\[
\begin{align*}
L_{a}U = 0 & \quad \text{in } D_\delta, \\
U(0, x, y) = 0 & \quad \text{on } (x, y) \in D_\delta \cap \{t = 0\}
\end{align*}
\]

then \(U(t, x, y)\) vanishes identically in \(D_\delta\).

**Proposition 2.3.** Assume that \(\psi \in C^\infty_0(\mathbb{R})\) is an even function such that \(\psi \notin \gamma^{(2)}_0(\mathbb{R})\). We assume also \(\phi(0) \neq 0\). Let \(\Omega\) be a neighborhood of the origin of \(\mathbb{R}^3\) such that \(\text{supp } \phi \psi \subset \Omega \cap \{t = 0\}\). Then the Cauchy problem (2.2) has no \(C^1(\Omega)\) solution.

**Proof.** Suppose that there is a neighborhood \(\Omega\) of the origin verifying \(\text{supp } \phi \psi \subset \Omega \cap \{t = 0\}\) such that the Cauchy problem would have \(C^1(\Omega)\) solution \(U\). From Proposition 2.2 one can assume \(U(t, x, y) = 0\) for \(x^2 + y^2 \geq r^2\) if \(|t| \leq T\) with some small \(T > 0\) and \(r > 0\). Denote

\[
W_\eta^\pm(t, x, y) = e^{\pm i\eta^2 y - i\beta \eta(t - T)} e^{-\eta^2 x^2/2}
\]

then we have \(L^*_a W_\eta^\pm = 0\). From

\[
0 = \int_0^T (L^*_a W_\eta^\pm, U) dt = \int_0^T (W_\eta^\pm, L_a U) dt + (W_\eta^\pm(T), U(T)) - (W_\eta^\pm(0), U(0))
\]

it follows that

\[
(W_\eta^\pm(T), U(T)) = (W_\eta^\pm(0), U(0)).
\]
Note that the left-hand side on (2.3) is $O(\eta)$ as $\eta \to \infty$ while the right-hand side is
\begin{equation}
\eta^{-1}e^{-i\beta \eta T} \hat{\psi}(\eta^2) \int e^{-x^2/2} \phi(\eta^{-1}x) dx
\end{equation}
where $\hat{\psi}$ denotes the Fourier transform of $\psi$. Then from (2.4) we conclude that there is $C > 0$ such that for large positive $\eta$ one has
\[ |\hat{\psi}(\eta^2)| \leq C\eta^2 e^{(-\text{Im}\beta)\eta T}. \]
Since $\psi$ is even this implies that $|\hat{\psi}(\eta)| \leq C'e^{-c|\eta|^{1/2}}$ with some $c > 0$ and hence $\psi \in \gamma_0^2(\mathbb{R})$ which is a contradiction. \hfill \qed

**Theorem 2.4.** Assume that $ia \notin [-1, 1]$. Then the Cauchy problem for $L_a$ is not locally solvable in the Gevrey class of order $s > 2$.

**Corollary 2.5.** Assume that $ia \notin [-1, 1]$. Then the Cauchy problem for $L_a$ is $C^\infty$ ill-posed.

### 2.2 Well-posedness

Recall
\begin{equation}
L_a = \partial_t + \begin{pmatrix} 0 & \partial_x + ax\partial_y & x\partial_y \\ \partial_x - ax\partial_y & 0 & 0 \\ (1 + a^2)x\partial_y & 0 & 0 \end{pmatrix} = \partial_t + G_a.
\end{equation}

**Lemma 2.6 (3).** If $a \neq 0$ is a constant, there are bounded symmetrizers $S(x, \xi, \eta)$ for $L_a$, but no continuous symmetrizers at $(x, \xi) = (0, 0)$ when $\eta = 1$.

The equation $L_a U = F$ reads
\begin{align*}
\partial_t u + (\partial_x + ax\partial_y)v + x\partial_y w &= f \\
\partial_t v + (\partial_x - ax\partial_y)u &= g \\
\partial_t w + (1 + a^2)x\partial_y u &= h
\end{align*}
where $U = (u, v, w)$ and $F = (f, g, h)$. Thus
\begin{equation}
P u = \partial_t f - (\partial_x + ax\partial_y)g - x\partial_y h
\end{equation}
where
\begin{equation}
P = \partial_t^2 - (\partial_x + ax\partial_y)(\partial_x - ax\partial_y) - (1 + a^2)x^2 \partial_y^2 \\
= \partial_t^2 - \partial_x^2 - x^2 \partial_y^2 + a\partial_y.
\end{equation}
We choose $a = i\mu$ with $\mu \in \mathbb{R}$ and denote

$$A = -\partial_x^2 - x^2 \partial_y^2 + i\mu \partial_y$$

so that $P = \partial_t^2 + A$ where $A$ is self adjoint and positive if $|\mu| < 1$:

**Lemma 2.7.** We have $(Au, u) = (u, Au)$ for $u \in C_0^\infty(\mathbb{R}^2)$. For $|\mu| < 1$ we have

$$(Au, u) \geq (1 - |\mu|)(\|\partial_x u\|^2 + \|x\partial_y u\|^2), \quad u \in C_0^\infty(\mathbb{R}^2).$$

**Proof.** The first assertion is clear. Note that

$$|(\partial_y u, u)| = |([\partial_x, x\partial_y]u, u)| \leq 2\|\partial_x u\|\|x\partial_y u\| \leq \|\partial_x u\|^2 + \|x\partial_y u\|^2.$$  \hfill (2.9)

Since $(Au, u) = \|\partial_x u\|^2 + \|x\partial_y u\|^2 + (i\mu x\partial_y u, u)$ and

$$|(i\mu \partial_y u, u)| \leq |\mu|(|\partial_y u, u|) \leq |\mu|(\|\partial_x u\|^2 + \|x\partial_y u\|^2)$$

by (2.9) the proof is clear. \hfill \Box

Assume that $f, g, h$ in (2.6) are linear combinations of $u, v, w$ with coefficients which are $C^\infty$ and constant outside a compact set. Then in

$$\partial_t f - (\partial_x + ax\partial_y)g - x\partial_y h$$

replacing $\partial_t v$ and $\partial_t w$ by the second and third equations of (2.6) one can write

$$\begin{align*}
\partial_t^2 u + Au &= c\partial_t u + b_1 u + b_2 v + b_3 w + \tilde{f} \\
\partial_t v + (\partial_x - a\partial_y)u &= \tilde{g} \\
\partial_t w + ax\partial_y u &= \tilde{h}
\end{align*}$$  \hfill (2.10)

where $b_j = b_{j1}(t, x, y)\partial_x + b_{j2}(t, x, y)x\partial_y$ and $c(t, x, y)$, $b_{ji}(t, x, y)$ are $C^\infty$ and constant outside a compact set and

$$|\tilde{f}| + |\tilde{g}| + |\tilde{h}| \lesssim |u| + |v| + |w|. \hfill (2.11)$$

Taking this into account we consider

$$\begin{align*}
\partial_t^2 u + Au &= B_0 u + B_1 \partial_t u + B v \\
\partial_t v &= Cu
\end{align*}$$  \hfill (2.12)
where we assume
\[ (2.13) \quad \|Cu\|_2 \lesssim (Au, u), \quad \|B^*u\|_2 \lesssim (Au, u), \quad \|(\partial_t B)^*u\|_2 \lesssim (Au, u), \quad \|B_0u\|_2 \lesssim (Au, u), \quad \|B_1u\| \lesssim \|u\|. \]

Introduce the energy
\[ E(t) = \frac{1}{2} (\|\partial_t u(t)\|^2 + (Au, u) + \lambda^2 \|u(t)\|^2 + \lambda^2 \|v(t)\|^2) \]
where \( \lambda > 0 \) is a positive constant. Then
\[ \partial_t E(t) = \text{Re} (Bv, \partial_t u) + \text{Re} (B_0u + B_1 \partial_t u, \partial_t u) + \lambda^2 \text{Re} (\partial_t u, u) + \lambda^2 \text{Re} (Cu, v). \]

The last two terms are \( O(E(t)) \), thus
\[ E(t) \lesssim E(0) + \text{Re} \int_0^t (Bv, \partial_t u) dt' + \int_0^t E(t') dt'. \]

Integrating by parts in time, then from (2.13) the first integral is
\[ -\text{Re} \int_0^t ((\partial_t v, B^*u) + (v, (\partial_t B)^*u)) dt' + \text{Re} (v, B^*u)]_0^t. \]

Thanks to (2.13) one has
\[ -\text{Re} \int_0^t ((Cu, B^*u) + (v, (\partial_t B)^*u)) dt' \lesssim \int_0^t E(t') dt'. \]

and the boundary term at \( t \) is
\[ \text{Re} (v(t), B^*u(t)) \lesssim (Au(t), u(t))^{1/2} \|v(t)\| \lesssim E(t)^{1/2} \left( \|v(0)\| + \int_0^t E(t')^{1/2} dt' \right) \lesssim \varepsilon E(t) + C_\varepsilon \left( \|v(0)\|^2 + t \int_0^t E(t') dt' \right). \]

Since \( \text{Re} (v(0), (B^*u)(0)) \lesssim E(0) \) by (2.13) again choosing \( \varepsilon \) small enough, we get

**Lemma 2.8.** Assume \( |\mu| < 1 \). Then we have
\[ (2.14) \quad E(t) \lesssim E(0) + (1 + t) \int_0^t E(t') dt'. \]
For any given $B = B(t, x, y)$ which is $C^\infty$ and constant outside a compact set we consider the Cauchy problem

\begin{equation}
L_a U = BU, \quad U(0, x, y) = U_0(x, y). \tag{2.15}
\end{equation}

**Lemma 2.9.** Assume $|\mu| < 1$. For any $T > 0$, $p \in \mathbb{N}$ and $R > 0$ there exists $C > 0$ such that for any $U \in C^1([0, T]; H^{p+1}(\mathbb{R}^2))$ supported in $x^2 + y^2 \leq R^2$ and satisfying (2.15) we have

\begin{equation}
\|U(t)\|_p \leq C\|U_0\|_{p+1}, \quad 0 \leq t \leq T. \tag{2.16}
\end{equation}

**Proof.** We denote

$$E(t) = \frac{1}{2}(\|\partial_t u(t)\|^2 + (Au, u) + \lambda^2\|u(t)\|^2 + \lambda^2\|v(t)\|^2 + \lambda^2\|w(t)\|^2).$$

If $U = (u, v, w)$ satisfies (2.15) then $u, v, w$ verifies (2.10) which is a bounded perturbation of a system of the form (2.12) with $(v, w)$ in place of $v$ for the second equation of (2.12). Therefore from Lemma 2.8 we have

$$E(t) \lesssim E(0) + (1 + t) \int_0^t E(t')dt'.$$

From Gronwall’s lemma for any $T > 0$ there is $C > 0$ such that $E(t) \leq CE(0)$ for $0 \leq t \leq T$. This shows that

\begin{equation}
\|U(t)\| \leq C\left(\|\partial_t U(0)\| + \|\partial_x U(0)\| + \|x\partial_y U(0)\| + \|U(0)\|\right). \tag{2.17}
\end{equation}

From the equation $\partial_t U(0)$ can be estimated by $C(\|\partial_x U_0\| + \|x\partial_y U_0\| + \|U_0\|)$ then (2.17) proves (2.16) when $p = 0$. Then applying $\partial_x^k \partial_y^\ell$ to (2.15) and repeating the same arguments we obtain (2.16) for general $p \in \mathbb{N}$.

**Theorem 2.10.** If $ia \in (-1, 1)$ then $L_a$ is strongly hyperbolic.

**Proof.** Approximate $B(t, x, y)$ and $U_0(x, y) \in C^\infty_0(\mathbb{R}^2)$ by Gevrey functions $B^\varepsilon$ and $U_0^\varepsilon$ for which the Cauchy problem

$$L_a U = B^\varepsilon U, \quad U(0, x, y) = U_0^\varepsilon(x, y).$$

admits a unique solution $U^\varepsilon \in C^1([0, T]; C^\infty_0(\mathbb{R}^2))$. From Lemma 2.9 we see that $\sup_{0 \leq t \leq T} \|U^\varepsilon(t)\|_p$ is uniformly bounded in $\varepsilon$. From Ascoli-Arzela’s theorem one can pick a uniformly convergent subsequence of $\{U^\varepsilon\}$ such that their derivatives also uniformly convergent. Then the limit function $U$ solves the Cauchy problem (2.15).
From Lemma 2.6 there is no symmetrizer of $G_a$ which is continuous at $(x, \xi, \eta) = (0, 0, 1)$ if $a \neq 0$ while one can construct a smooth symmetrizer of second order if $ia \in (-1, 1)$. Set $a = i\mu$ with $\mu \in \mathbb{R}$. We denote

$$S = \begin{pmatrix} A + \lambda & 0 & 0 \\ 0 & B + \lambda & D \\ 0 & D^* & C + \lambda \end{pmatrix}$$

with 

$$X = \partial_x - i\mu x \partial_y, \quad Y = x \partial_y, \quad A = X^* X + (1 - \mu^2)Y^* Y, \quad B = XX^*, \quad C = YY^* = Y^* Y, \quad D = XY^*$$

where $\lambda > 0$ is a positive constant. Note that

$$\langle SU, U \rangle = \|Xu\|^2 + (1 - \mu^2)\|Yu\|^2 + \|X^* v + Yw\|^2 + \lambda(\|u\|^2 + \|v\|^2 + \|w\|^2)$$

with $U = t(u, v, w)$.

**Lemma 2.11.** For $|\mu| < 1$, one has

$$\|Xu\|^2 + \|Yu\|^2 + \lambda\|U\|^2 \leq \langle SU, U \rangle, \quad U = t(u, v, w).$$

Moreover

$$\text{Re} \langle SG_a U, U \rangle \leq \langle SU, U \rangle.$$

**Proof.** The first estimate follows immediately from (2.19). Next we compute

$$SG_a = \begin{pmatrix} A + \lambda & 0 & 0 \\ 0 & B + \lambda & D \\ 0 & D^* & C + \lambda \end{pmatrix} \begin{pmatrix} 0 & -X^* & -Y^* \\ X & 0 & 0 \\ (1 - \mu^2)Y & 0 & 0 \end{pmatrix}$$

where we note that

$$BX + (1 - \mu^2)DY = XX^* X + (1 - \mu^2)XY^* Y = KA = KA^*,$$

$$D^* X + (1 - \mu^2)CY = YX^* X + (1 - \mu^2)YY^* Y = YA = YA^*.$$
Therefore one has

\[
\text{Re } SG_a = \begin{pmatrix}
0 & 0 & -\mu^2 Y^* \\
0 & 0 & 0 \\
-\mu^2 Y & 0 & 0
\end{pmatrix}
\]

and (2.21) follows from (2.20) and \( |\mu| < 1 \).

Thanks to Lemma 2.11 we have

\[
\frac{1}{2} \partial_t (SU, U) = -\text{Re } (SG_a U, U) + \text{Re } (SL_a U, U) \\
\leq (SU, U) + |(SL_a U, U)|.
\]

(2.22)

Since \( |(SL_a U, U)| \leq (SL_a U, L_a U)^{1/2} (SU, U)^{1/2} \) and hence

\[
(SU(t), U(t))^{1/2} \leq e^t (SU(0), U(0))^{1/2} \\
+ \int_0^t e^{t-s} (SL_a U(s), L_a U(s))^{1/2} ds.
\]

Then denoting

\[
[[U]]_1^2 = \|U\|^2 + \|\partial_x U\|^2 + \|x \partial_y U\|^2
\]

there is \( C > 0 \) such that

\[
\|U(t)\| \leq C \left( e^t [[U(0)]_1] + \int_0^t e^{t-s} [[L_a U(s)]_1(s)] ds \right).
\]

3 Remarks

It seems to be natural to state Theorems 1.8 and 1.9 in terms of the propagation cone of \( L_\rho \). Consider \( L_\rho \) on \( T^*\mathbb{R}^{d+1} \) not on \( T^*\mathbb{R}^{d+1}/T_\rho \Sigma \) and hence \( L_\rho \) is independent of directions in \( T_\rho \Sigma \). Note that

\[
\det L_\rho(t, x, \tau, \xi) = h_\rho(t, x, \tau, \xi), \quad h = \det L
\]

where \( h_\rho \) is the first non-vanishing term of the Taylor expansion of \( h(t, x, \tau, \xi) \) around \( \rho \) which is a homogeneous polynomial in \( (t, x, \tau, \xi) \). Recall the hyperbolicity cone \( \Gamma(L_\rho) \) of \( L_\rho \) defined as the connected component of \( \theta = (0, \ldots, 0, 1, 0, \ldots, 0) = -H_i \) in

\[
\{(t, x, \tau, \xi) \in T^*\mathbb{R}^{d+1} \mid \det L_\rho(t, x, \tau, \xi) \neq 0\}.
\]
The propagation cone $C(L_ρ)$ of $L_ρ$ is given by
\[ C(L_ρ) = \{ X = (t, x, τ, ξ) \mid \sigma(X, Y) \leq 0, \forall Y \in \Gamma(L_ρ) \} \]
where $σ = \sum_{j=0}^d dξ_j ∧ dx_j$ with $x_0 = t$ and $ξ_0 = τ$ is the symplectic two form on $T^*R^{d+1}$. The propagation cone is the minimal cone including every bicharacteristic of $h$ which has $ρ$ as a limit point in the following sense:

**Lemma 3.1.** Let $ρ \in T^*R^{d+1}$ be a multiple characteristic of $h$. Assume that there are simple characteristics $ρ_j$ of $h$ and positive numbers $γ_j$ such that
\[ ρ_j → ρ \quad \text{and} \quad γ_j H_h(ρ_j) → X(0) \quad j → ∞. \]
Then $X \in C(L_ρ)$. Here $H_h$ denotes the Hamilton vector field of $h$.

Denote $\{ X \in R^{d+1} × R^{d+1} \mid (dξ ∧ dx)(X, Y) = 0, \forall Y \in T_ρΣ \}$ by $(T_ρΣ)^σ$, the $σ$ orthogonal space of $T_ρΣ$. Since $L_ρ$ is independent of directions in $T_ρΣ$ it is clear that $C(L_ρ) \subset (T_ρΣ)^σ$. Here we note

**Lemma 3.2** ([8]). The characteristic manifold $Σ$ is an involutive manifold if and only if $C(L_ρ) \subset T_ρΣ$ for all $ρ \in Σ$.

Thus Theorem [1.8] is stated as

**Theorem 3.3.** Assume that $C(L_ρ) \subset T_ρΣ$ for all $ρ \in Σ$ and $L$ is transversally strictly hyperbolic. Then the Cauchy problem for $L + B$ with initial data on $\{ t = 0 \}$ is $C^∞$ well-posed for any $B(t, x)$, that is $L$ is a strongly hyperbolic system.

If $Σ$ is a symplectic manifold, that is $T_ρΣ ∩ (T_ρΣ)^σ = \{ 0 \}$ then it is clear that $C(L_ρ) ∩ T_ρΣ = \{ 0 \}$ because $C(L_ρ) \subset (T_ρΣ)^σ$. Actually Theorem [1.9] could be generalized to

**Theorem 3.4.** Assume that $C(L_ρ) ∩ T_ρΣ = \{ 0 \}$ for all $ρ \in Σ$ and $L$ is transversally strictly hyperbolic. Then the Cauchy problem for $L + B$ with initial data on $\{ t = 0 \}$ is $C^∞$ well-posed for any $B(t, x)$, that is $L$ is a strongly hyperbolic system.

For $L_a$ which we have studied in Section 2 we see $C((L_a)ρ)$ is independent of $a \in C$ because $\det(L_a)ρ(\hat{x}, \hat{τ}, \hat{ξ}) = (\det L_a)ρ(\hat{x}, \hat{τ}, \hat{ξ}) = \hat{τ}(\hat{τ}^2 − \hat{ξ}^2 − i\hat{η}^2\hat{x}^2)$. It is also easy to check that
\[ \{ 0 \} ≠ C((L_a)ρ) ∩ T_ρΣ \subset C((L_a)ρ) \]
while $L_a$ is strongly hyperbolic system if $ia \in (-1, 1)$ and not if $ia ∉ [-1, 1]$. 

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