MASS CONCENTRATION PHENOMENON TO THE
TWO-DIMENSIONAL CAUCHY PROBLEM OF THE
COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS

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(Communicated by Stefano Bianchini)

Abstract. This concerns the global strong solutions to the Cauchy problem of the compressible Magnetohydrodynamic (MHD) equations in two spatial dimensions with vacuum as far field density. We establish a blow-up criterion in terms of the integrability of the density for strong solutions to the compressible MHD equations. Furthermore, our results indicate that if the strong solutions of the two-dimensional (2D) viscous compressible MHD equations blowup, then the mass of the MHD equations will concentrate on some points in finite time, and it is independent of the velocity and magnetic field. In particular, this extends the corresponding Du’s et al. results (Nonlinearity, 28, 2959-2976, 2015, [4]) to bounded domain in $\mathbb{R}^2$ when the initial density and the initial magnetic field are decay not too show at infinity, and Ji’s et al. results (Discrete Contin. Dyn. Syst., 39, 1117-1133, 2019, [10]) to the 2D Cauchy problem of the compressible Navier-Stokes equations without magnetic field.

1. Introduction. This paper concerns the cauchy problem of the 2D MHD equations:

$$\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda)\nabla \text{div}u + \nabla P = (\nabla \times H) \times H, \\
H_t - \nabla \times (u \times H) = \nu \Delta H, \\
\text{div}H = 0,
\end{cases}$$

(1.1)

where $x = (x_1, x_2) \in \mathbb{R}^2$ is the spatial coordinate and $t \geq 0$ is the time. The unknown functions $\rho$, $u = (u^1, u^2)$, $H = (H^1, H^2)$ and $P = P(\rho)$ denote the density, velocity field, magnetic field and pressure, respectively. The equation of state is given by

$$P = A\rho^\gamma,$$

(1.2)

with $A$ being a positive constant and $\gamma > 1$. The real parameters $\mu$ and $\lambda$ are the shear viscosity and the bulk viscosity coefficients, respectively, which satisfy the following physical restrictions:

$$\mu > 0, \quad \mu + \lambda \geq 0,$$

(1.3)

and $\nu$ is the magnetic diffusive coefficient of the magnetic field.

2020 Mathematics Subject Classification. Primary: 35Q35, 76W05; Secondary: 76N10.

Key words and phrases. Compressible MHD equations, strong solution, Cauchy problem, blow-up criterion, vacuum.

The author is partially supported by NSFC grant 11801460.
System (1.1) will be investigated with initial conditions
\[ \rho(x,0) = \rho_0(x), \quad \rho u(x,0) = \rho_0 u_0(x), \quad H(x,0) = H_0(x), \quad x \in \mathbb{R}^2, \tag{1.4} \]
for given the initial data \( \rho_0, u_0 \) and \( H_0 \), and the far-field conditions
\[ (\rho, u, H)(x,t) \rightarrow (0,0,0) \text{ as } |x| \rightarrow \infty. \tag{1.5} \]

There is a considerable body of literature on the global regularity and large time behavior of the multi-dimensional compressible MHD flow. If there is no electromagnetic effect, namely, \( H \equiv 0 \), the MHD systems becomes the Navier-Stokes equations, there are huge literatures on the global regularity results, please see \([1, 3, 8, 9, 11, 12, 13, 18, 22, 23, 24, 25, 26, 28, 29]\) and references therein. There are some important progresses on mathematical analysis on MHD flows in recent decades and we would like to recall some results on compressible MHD flows in multi-dimension briefly. Kawashima \([14]\) obtained the global existence of smooth solutions to the 2D compressible MHD system, provided that the initial density was strictly positive. Hu and Wang \([6]\) showed the global existence and the large time behavior of the renormalized solutions to the 3D compressible MHD system for general large initial data. The local strong solutions to the compressible MHD with large initial data were obtained by Volpert-Hudjaev \([26]\) as the initial density is strictly positive and by Fan-Yu \([5]\) as the initial density may contain vacuum in \( \mathbb{R}^3 \), respectively. Recently, Li, Xu and Zhang \([15]\) established the global existence of the classical solution to three-dimensional compressible MHD flows with small total energy. Starting with the pioneering works by Beale-Kato-Majda \([1]\) and Serrin \([23]\), many articles have been study the blowup phenomenon to MHD equations in \( \mathbb{R}^3 \) or bounded domain of \( \mathbb{R}^2 \) (see \([4, 5, 7, 27, 30]\) and the references therein). However, many fundamental and interesting problems are still open even for one-dimension case due to the lack of smoothing mechanism and the strong nonlinearity.

On another side, Lü and Huang \([19]\) have established the local strong solutions to Cauchy problem (1.1)-(1.5). Lü, Shi and Xu \([20]\) established the global well-posedness to 2D MHD equations, as long as the initial total energy is small. Later, Lü, Xu and Zhong \([21]\) obtained the global existence and uniqueness of strong solutions to the 2D Cauchy problem of nonhomogeneous incompressible magneto-hydrodynamic equations provided that the initial density and the initial magnetic field decay not too slow at infinity. Furthermore, there are some progress to the Cauchy problem of 2D compressible Navier-Stokes equations, i.e., Li, Liang and Xin \([16, 17]\). In particular, the regularity and uniqueness of the weak solution to the compressible MHD flows for general initial data are still open and challenge problems even in two dimensions case. Therefore, it is important to study the blowup mechanism and structure of possible singularities of strong (or classical) solutions to the compressible MHD equations in \( \mathbb{R}^2 \).

There are a series of blowup criteria for the 2D compressible MHD equations, especially, Du-Wang \([4]\) gave a blowup criterion for strong solutions in a bounded domain of \( \mathbb{R}^2 \). More precisely, they proved that if \( T^* \) is the life span of the strong solution to system (1.1), then
\[ \lim_{T \to T^*} \| \rho \|_{L^\infty(0,T;L^\infty)} = \infty. \tag{1.6} \]

The goal of this paper is to extend the blowup criterion (1.6) in bounded domain of the 2D compressible MHD equations to Cauchy problem.
Before stating the main result, first, we introduce the following simplified notations
\[ \int f dx = \int_{\mathbb{R}^2} f dx, \]
and
\[ L^r = L^r(\mathbb{R}^2), \quad W^{k,r} = W^{k,r}(\mathbb{R}^2), \quad H^k = W^{k,2}(\mathbb{R}^2), \]
for \( 1 \leq r \leq \infty, \ k \geq 0. \)
Without loss of generality, we assume that the initial density \( \rho_0 \) satisfies
\[ \int \rho_0 dx = 1. \quad (1.7) \]
The local strong solution of Cauchy problem to the 2D MHD equations with vacuum was obtained in [19]. We state those results as follows.

**Proposition 1.** For given positive constants \( 0 < \eta_0 \leq 1, \ q > 2 \) and \( a > 1, \) we define
\[ \bar{x} \triangleq (e + |x|^2) \frac{\hat{a}}{2} \log^{1+\eta_0} (e + |x|^2), \quad (1.8) \]
and the initial data satisfy
\[ \rho_0 \geq 0, \quad \bar{x} \bar{\rho}_0 \in L^1 \cap H^1 \cap W^{1,q}, \quad \nabla u_0 \in L^2, \quad \sqrt{\rho}_0 u_0 \in L^2, \quad (1.9) \]
Then there exists a positive constant \( T_0 \) such that the Cauchy problem \( (1.1)-(1.5) \) has a unique strong solution \( (\rho, u, H) \) in \( \mathbb{R}^2 \times (0, T_0) \) satisfying that
\[ \rho \in C([0, T_0]; L^1 \cap H^1 \cap W^{1,q}), \quad \bar{x} \bar{\rho} \in L^\infty(0, T_0; L^1 \cap H^1 \cap W^{1,q}), \]
\[ \sqrt{\rho} u, \ \nabla u, \ \bar{x}^{-1} u, \ \sqrt{\bar{x}} \sqrt{\rho} u_t \in L^\infty(0, T_0; L^2), \]
\[ \bar{x} \bar{H}, \ \nabla^2 \bar{H}, \ \nabla \bar{H}, \ \bar{H}_t, \ \sqrt{\bar{H}} \bar{H}_t \in L^\infty(0, T_0; L^2), \quad (1.10) \]
\[ \nabla u \in L^\infty(0, T_0; H^1) \cap L^{2^*} \left( (0, T_0; W^{1,q}) \right), \ \sqrt{\nabla u} \in L^2(0, T_0; W^{1,q}), \]
\[ \sqrt{\rho} u_t, \ \bar{x} \nabla H, \ \sqrt{\nabla} u_t, \ \sqrt{\nabla} H_t, \ \sqrt{\bar{x}}^{-1} u_{tt} \in L^2(\mathbb{R}^2 \times (0, T_0)), \]
and
\[ \inf_{0 \leq t \leq T_0} \int_{B_N} \rho(x, t) dx \geq \frac{1}{4}, \quad (1.11) \]
for some constant \( N > 0 \) and \( B_N \triangleq \{x \in \mathbb{R}^2 \mid |x| < N\}. \)

Our main result of this paper can be stated as follows.

**Theorem 1.1.** Suppose that the initial data \( (\rho_0, u_0, H_0) \) satisfy \( (1.9) \), and \( (\rho, u, H) \) be a strong solution to the Cauchy problem \( (1.1)-(1.5) \) satisfying \( (1.10)-(1.11) \) in \( \mathbb{R}^2 \times (0, T^*) \). If \( T^* < \infty \) is the maximal existence time of the strong solution, then there exists some positive constant \( s_0 \) such that
\[ \lim_{T \to T^*} \| \rho \|_{L^\infty(0, T; L^r)} = \infty, \quad (1.12) \]
for all \( s \geq s_0. \)

**Remark 1.** Similar arguments as Remark 1.2 in [7], we distinguish three possible cases as follows. It follows from continuity equation that
\[ \rho_t + u \cdot \nabla \rho + \rho d u = 0, \]
which implies that if the velocity field is regular, the characteristic line and the
density can be defined as follows:

\[
\frac{dy}{ds}(s; x, t) = u(y(s; x, t), s), \quad y(t; x, t) = x,
\]

and

\[
\rho(x, t) = \rho_0(y(0; x, t)) \exp \left( - \int_0^t \text{div}(y(s; x, t), s) ds \right),
\]

which implies that if the singularity of the solution to the compressible MHD equa-
tions (1.1) formulates in finite time \( T^* \), we distinguish three possible cases in the
following.

1. The density may concentrate, that is to say,

\[
\lim_{T \to T^*} \| \rho \|_{L^\infty(0, T; L^\infty)} = \infty. \quad (1.13)
\]

2. Vacuum states may appear in the non-vacuum region: there exist some \( \tilde{x} \in \mathbb{R}^2 \)
and \( \tilde{x}(t) \) satisfying

\[
\rho_0(\tilde{x}) > 0 \quad \text{and} \quad y(0; \tilde{x}(t), t) = \tilde{x},
\]
such that

\[
\lim_{t \to T^*} \rho(\tilde{x}(t), t) = 0. \quad (1.14)
\]

3. Vacuum states may vanish: there exist some \( x_0 \in \mathbb{R}^2 \) and \( x_0(t) \) satisfying

\[
\rho_0(x_0) = 0 \quad \text{and} \quad y(0; x_0(t), t) = x_0,
\]
such that

\[
\lim_{t \to T^*} \rho(x_0(t), t) \geq c_0 > 0. \quad (1.15)
\]

A natural and interesting question may be asked: Which one or some of (1.13)-(1.15) will happen when the singularity formulates? Theorem 1.1 gives an answer to this question and show that the mass of the fluid will concentrate on some points before other cases (2) and (3) happen.

**Remark 2.** The approach can also be applied to deal with the bounded domain in \( \mathbb{R}^2 \). Roughly speaking, we generalize the results in [4] to the Cauchy problem of 2D compressible MHD equations. This criterion is analogous to the 2D barotropic compressible Navier-Stokes equations without magnetic field, in particular, it is independent of the magnetic field and is just the same as that of the barotropic compressible Navier-Stokes equations [10]. On the other hand, it would be interesting to study wether (1.12) is a necessary condition.

**Remark 3.** It follows from (1.1) and (1.7) that \( \| \rho \|_{L^\infty(0, T; L^1)} \leq C \). In view of the standard interpolation inequality and \( \| \rho \|_{L^\infty(0, T; L^{q_2})} \leq C \), we obtain the bound of \( \| \rho \|_{L^\infty(0, T; L^{q_2})} \leq C \) for any \( q_2 \in (1, q_1) \). Hence, it suffices to prove the main Theorem 1.1 holds for \( s = s_0 \).

**Remark 4.** Indeed, the positive constant \( s_0 \) can be chosen in Section 3, which depends on \( \lambda, \mu \) and \( \gamma \).

We now comment on the analysis of this paper.

Compared with [4, 10] for 2D compressible barotropic MHD equations in bounded domain and 2D Cauchy problem of compressible barotropic Navier-Stokes equations, some new difficulties arise in the Cauchy problem of compressible MHD system. The first difficult lies in the fact that the Brezis-Waigner’s inequality [2] fails for the 2D Cauchy problem, and it seems difficult to estimate \( \| u \|_{L^q(\mathbb{R}^2)} \) for any \( q > 1 \) just
in terms of \( \| \sqrt{\rho} u \|_{L^2(\mathbb{R}^2)} \) and \( \| \nabla u \|_{L^2(\mathbb{R}^2)} \). One way to overcome this difficulty is to estimate the momentum \( \rho u \) instead of the velocity \( u \), since \( \rho \) decays for large \( x \), the momentum \( \rho u \) decays faster than \( u \) itself. Moreover, we use the variant of Gagliardo-Nirenberg inequality and more finely estimate for the nonlinear coupling term \( \rho u \cdot \nabla u \). In particular, the high order estimates on \( \rho, u \) and \( H \) will not be improved as in the bounded domain case in [4]. Therefore, the Hardy-type inequality is introduced to control the \( L^p \)-norm of \( \rho u \). On the other hand, compared with the 2D compressible Navier-Stokes system [10], there are some new nonlinear coupling terms, such as \( u \cdot \nabla H, H \cdot \nabla H \) and \( H \cdot \nabla u \). Indeed, in order to control the \( \| u \| \nabla H \|_{L^2} \), our new observation is to obtain some weighted estimates on both \( H \) and \( \nabla H \), namely, \( \| H x^2 \|_{L^2(\mathbb{R}^2)} \) and \( \| \nabla H x^2 \|_{L^2(\mathbb{R}^2)} \). Furthermore, the initial density vacuum is allowed in this paper.

The rest of this paper is organized as follows. Some important inequalities and auxiliary lemmas will be given in Section 2. We will prove Theorem 1.1 in Section 3.

2. Preliminaries. In this section, some elementary lemmas will be used later. One is the variant of Gagliardo-Nirenberg inequality. For its proof, refer to [22].

**Lemma 2.1.** Assume \( f \in W^{1,m}(\mathbb{R}^2) \cap L^r(\mathbb{R}^2) \), it holds that

\[
\| f \|_{L^q} \leq C \| \nabla f \|_{L^m}^{\theta} \| f \|_{L^r}^{1-\theta},
\]

where \( \theta = (\frac{1}{r} - \frac{1}{q})/(\frac{1}{r} - \frac{1}{m} + \frac{1}{2}) \), and if \( m < 2 \), then \( \theta \) is between \( r \) and \( \frac{2m}{2-m} \), if \( m = 2 \), then \( \theta \) is in \( [2, \infty) \), if \( m > 2 \), then \( \theta \) is in \( [2, \infty] \) and constant \( C \) depending on \( q, m \) and \( r \).

Next, the material derivative \( \dot{f} \), the effective viscous flux \( G \), and the vorticity \( \omega \) are defined as follows.

\[
\dot{f} = f_t + u \cdot \nabla f, \quad G = (2\mu + \lambda) \text{div} u - P - \frac{1}{2} |H|^2, \quad \omega = \nabla \times u = \partial_{x_1} u_2 - \partial_{x_2} u_1.
\]

We obtain two key elliptic system of \( G \) and \( \omega \).

\[
\Delta G = \text{div} (\rho \dot{u} - H \cdot \nabla H), \quad \mu \Delta \omega = \nabla \times (\rho \dot{u} - H \cdot \nabla H),
\]

where we have used (1.1)$_2$ and (2.2).

From the standard \( L^p \)-estimate of elliptic system (2.3), we have the following estimate.

**Lemma 2.2.** Let \( (G, \omega) \) be a strong solution of (2.3). Then there exists a generic positive constant \( C \) depending only on \( \mu, \lambda \) and \( p \), such that

\[
\| \nabla G \|_{L^p} + \| \nabla \omega \|_{L^p} \leq C \| \rho \dot{u} - H \cdot \nabla H \|_{L^p},
\]

where \( 1 < p < \infty \).

We introduce the Hardy-type inequality as follows, which plays a crucial role in the estimate. The proof can be found in [16].

**Lemma 2.3.** Let \( \bar{x} \) and \( \eta_0 \) be as in (1.8) and \( B_{N_1} = \{ x \mid |x| < N_1 \} \) with \( N_1 \geq 1 \). Assume that \( \rho \in L^1 \cap L^{\infty} \) is a non-negative function such that

\[
\int_{B_{N_1}} \rho dx \geq M_1, \quad \int_{\mathbb{R}^2} \rho^q dx \leq M_2,
\]
for positive constants $M_1$ and $M_2$. Then there exists a positive constant $C$ depending on $M_1$, $M_2$, $\gamma$, $N_1$ and $\eta_0$ such that

\[
\|\nabla \bar{v}^{-1}\|_{L^2} \leq C\|\sqrt{\rho v}\|_{L^2} + C\|\nabla v\|_{L^2},
\]  
\[\tag{2.5}
\]

for any $v \in \dot{D}^{1,2} \triangleq \{v \in H^1_{\text{loc}} \mid \nabla v \in L^2\}$. Furthermore, for $\varepsilon > 0$ and $\eta > 0$, there exists a positive constant $C$ depending on $\varepsilon$, $\eta$, $M_1$, $M_2$, $\gamma$, $N_1$ and $\eta_0$ such that every function $v \in \dot{D}^{1,2}$ satisfies

\[
\|v \bar{v}^{-\eta}\|_{L^{2p_0}} \leq C\|\sqrt{\rho v}\|_{L^2} + C\|\nabla v\|_{L^2},
\]  
\[\tag{2.6}
\]

with $\bar{\eta} = \min\{1, \eta\}$.

The main ingredient of the proof is that we decompose the velocity field into two parts, namely $u = v + w$, where $v$ is the solution to Lamé system

\[
\begin{cases}
\mu \Delta v + (\mu + \lambda) \nabla \text{div} v = \nabla P, & x \in \mathbb{R}^2, \\
v(x) = 0, & |x| \to \infty,
\end{cases}
\]  
\[\tag{2.7}
\]

and $w$ satisfies the following elliptic boundary value problem

\[
\begin{cases}
\mu \Delta w + (\mu + \lambda) \nabla \text{div} w = \rho \dot{u} - H \cdot \nabla H + \frac{1}{2} \nabla |H|^2, & x \in \mathbb{R}^2, \\
w(x) = 0, & |x| \to \infty.
\end{cases}
\]  
\[\tag{2.8}
\]

Using $L^p$-estimate ($1 < p < \infty$), we obtain the following estimates.

\textbf{Lemma 2.4.} \textit{Let $v$ and $w$ be a solution of (2.7) and (2.8), respectively. Then there exists a generic positive constant $C$ depending only on $\mu$, $\lambda$, $p$ such that}

\[
\|\nabla v\|_{L^p} \leq C\|P\|_{L^p},
\]  
\[\tag{2.9}
\]

and

\[
\|\nabla^2 w\|_{L^p} \leq C\|\rho \dot{u} - H \cdot \nabla H + \frac{1}{2} \nabla |H|^2\|_{L^p},
\]  
\[\tag{2.10}
\]

for any $1 < p < \infty$.

\textit{Proof.} We can prove (2.9) in a similar way as in the proof of Lemma 2.3 in [7]. The standard $L^p$-estimate of elliptic system (2.8) yield to (2.10). \hfill \Box

\textbf{Remark 5.} Indeed, taking $p_0 \in (1, 2)$, using Hölder’s inequality yield to

\[
\|\nabla^2 w\|_{L^{p_0}} \leq C\left(\|\rho \dot{u}\|_{L^{p_0}} + \|H \cdot \nabla H\|_{L^{p_0}}\right)
\leq C\left(\|\sqrt{\rho} \dot{u}\|_{L^\frac{2p_0}{p_0}} + \|\nabla H\|_{L^2} \|H\|_{L^2} \frac{2p_0}{2p_0}\right). \tag{2.11}
\]

Finally, we introduce the following Beale-Kato-Majda type inequality to estimate the term $\|\nabla u\|_{L^\infty}$, which was first proved in [1] when $\text{div} u = 0$. The proof for a general situation can be found in [13].

\textbf{Lemma 2.5.} \textit{Suppose that $\nabla u \in L^2(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2)$ for any $q \in (2, \infty)$, there exists a constant $C$ depending only on $q$, such that}

\[
\|\nabla u\|_{L^\infty} \leq C(\|\text{div} u\|_{L^\infty} + \|\omega\|_{L^\infty}) \log(e + \|\nabla^2 u\|_{L^q}) + C(1 + \|\nabla u\|_{L^2}). \tag{2.12}
\]

\textit{Proof.} We can prove (2.12) in a similar way as in the proof of Lemma 2.3 in [7]. The standard $L^p$-estimate of elliptic system (2.8) yield to (2.10). \hfill \Box
3. Proof of The main results. Let \((\rho, u, H)\) be a strong solution of (1.1)-(1.3) on \(\mathbb{R}^2 \times [0, T^\ast]\). We will prove our main Theorem 1.1 by contradiction arguments. Suppose not, for some sufficiently large \(1 < s_0 < \infty\), there exists a positive constant \(M\) such that
\[
\lim_{T \to T^\ast} \|\rho\|_{L^\infty(0, T; L^{s_0})} \leq M < \infty.
\] (3.1)
The combination of (1.7) and the mass conservation equation (1.1)\(_1\) yields
\[
\int \rho \, dx = \int \rho_0 \, dx = 1.
\] (3.2)

First, we obtain the following standard energy estimate for \((\rho, u, H)\).

**Lemma 3.1.** There exists a positive constant \(N_1\) which holds that for \(0 \leq T < T^\ast\),
\[
\sup_{0 \leq t \leq T} \int (\rho|u|^2 + H^2) \, dx + \int_0^T \left( |\nabla u|^2 + |\nabla H|^2 \right) \, dx \, dt \leq C,
\] (3.3)
and
\[
\inf_{0 \leq t \leq T} \int_{B_{N_1}} \rho \, dx \geq \frac{1}{4}.
\] (3.4)
Here and after, \(C\) and \(C_i\) \((i = 0, 1, 2, \ldots)\) denote the generic positive constants depending on \(M, \mu, \lambda, \nu, T^\ast, N_1\) and the initial data.

**Proof.** Indeed, the proof of (3.3) is standard. Multiplying the momentum equation by \(u\), the magnetic field equation by \(H\) and using (1.1)\(_1\), we can prove the estimate of (3.3).

Next, for \(N_1 > 1\), a cutoff function \(\eta_{N_1}(x) \in C_0^\infty(\mathbb{R}^2)\) is defined by
\[
0 \leq \eta_{N_1}(x) \leq 1, \quad \eta_{N_1}(x) = \begin{cases} 1, & \text{if } |x| \leq N_1, \\ 0, & \text{if } |x| \geq 2N_1, \end{cases} \quad |\nabla \eta_{N_1}| \leq \frac{2}{N_1}.
\]
Multiplying (1.1) by \(\eta_{N_1}\), using Hölder’s inequality, (3.3) and integrating by parts give that
\[
\frac{d}{dt} \int \rho \eta_{N_1} \, dx = \int \rho u \cdot \nabla \eta_{N_1} \, dx \\
\geq -2N_1^{-1} \left( \int \rho \, dx \right)^{\frac{1}{2}} \left( \int \rho|u|^2 \, dx \right)^{\frac{1}{2}} \geq -C_1N_1^{-1},
\] (3.5)
this gives
\[
\inf_{0 \leq t \leq T} \int \rho \eta_{N_1} \, dx \geq \int \rho_0 \eta_{N_1} \, dx - C_1N_1^{-1}T,
\] (3.6)
which together with (3.2) imply that,
\[
\inf_{0 \leq t \leq T} \int_{B_{N_1}} \rho \, dx \geq \frac{1}{4},
\]
for \(N_1\) suitably large. Therefore, we complete the proof of Lemma 3.1. \(\square\)

Next, there is an important spatial weighted estimates about \(\rho\) and \(H\).

**Lemma 3.2.** Under condition (3.1), and \(s_0 \geq 2\), it holds that for \(0 \leq T < T^\ast\),
\[
\sup_{0 \leq t \leq T} \int (\rho \tilde{x}^a + |H \tilde{x}^\frac{a}{2}|^2 + H^4) \, dx + \int_0^T \left( |\nabla H \tilde{x}^\frac{a}{2}|^2 + |\nabla H|^2 |H|^2 \right) \, dx \, dt \leq C.
\] (3.7)
Proof. Firstly, multiplying (1.1) by $\tilde{x}^a$ and integrating the resulting equation over $\mathbb{R}^2$, integrating by parts and (2.6), one has
\[
\frac{d}{dt} \int \rho \tilde{x}^a dx \\
\leq C \int \rho |u| \tilde{x}^{a-1} \log^{1+n_0}(e + |x|^2) dx \\
\leq C \int \rho \frac{\nabla \tilde{x}^a}{\tilde{x}^a} \cdot \tilde{x}^{-\frac{a}{2}}|u| \tilde{x}^{-\frac{4}{2+n_0}} \log^{1+n_0}(e + |x|^2) dx \\
\leq C \left\| \tilde{x}^{-\frac{a}{2}} \log^{1+n_0}(e + |x|^2) \right\|_{L_{\infty}} \left\| u \tilde{x}^{-\frac{4}{2+n_0}} \right\|_{L_{16+2n}} \left( \int \rho^2 \, dx \right)^{\frac{a}{2+n_0}} \left( \int \rho \tilde{x}^a \, dx \right)^{\frac{2+n_0}{2+n_0}} \\
\leq C \left( \| \sqrt{\rho u} \|_{L^2} + \| \nabla u \|_{L^2} \right) \left( \int \rho \tilde{x}^a \, dx \right)^{\frac{2+n_0}{2+n_0}} \\
\leq C \left( 1 + \| \nabla u \|_{L^2}^2 \right) \left( \int \rho \tilde{x}^a \, dx + 1 \right), \quad (3.8)
\]
which together with Gronwall’s inequality and (3.3) give that
\[
\sup_{0 \leq t \leq T} \int \rho \tilde{x}^a \, dx \leq C. \quad (3.9)
\]
Next, multiplying (1.1) by $H \tilde{x}^a$ and integrating by parts gives
\[
\frac{1}{2} \frac{d}{dt} \int H^2 \tilde{x}^a dx + \nu \int |\nabla H|^2 \tilde{x}^a \, dx \\
= \nu \int |H|^2 \Delta \tilde{x}^a \, dx - \frac{1}{2} \int |H|^2 \tilde{x}^a \, dx \\
+ \int H \cdot \nabla u \cdot H \tilde{x}^a \, dx + \frac{1}{2} \int |H|^2 u \cdot \nabla \tilde{x}^a \, dx \\
= I_1 + I_2 + I_3 + I_4. \quad (3.10)
\]
Furthermore, due to (2.1), (2.6) and Hölder inequality, we estimate $I_1$-$I_4$ as follows.
\[
I_1 \leq C \int |H|^2 \tilde{x}^a \tilde{x}^{-2} \log^{2(1+n_0)}(e + |x|^2) \, dx \leq C \int |H|^2 \tilde{x}^a \, dx, \quad (3.11)
\]
\[
I_2 + I_3 \leq C \int |\nabla u| |H|^2 \tilde{x}^a \, dx \leq C \| \nabla u \|_{L^2} \left\| |H|^2 \tilde{x}^a \right\|_{L^4} \\\n\leq C \| \nabla u \|_{L^2} \left\| |H|^2 \tilde{x}^a \right\|_{L^2} \left( \| \nabla H \tilde{x}^a \|_{L^2} + \| H \nabla \tilde{x}^a \|_{L^2} \right) \\\n\leq C \left( 1 + \| \nabla u \|_{L^2}^2 \right) \left\| |H|^2 \tilde{x}^a \right\|_{L^2}^2 + \nu \frac{1}{4} \| \nabla H \tilde{x}^a \|_{L^2}^2, \quad (3.12)
\]
and
\[
I_4 \leq C \int u \tilde{x}^{-\frac{a}{2}} |H|^2 \tilde{x}^a \tilde{x}^{-\frac{a}{2}} \log^{1+n_0}(e + |x|^2) \, dx \\
\leq C \left\| |H|^2 \tilde{x}^a \right\|_{L^2} \left\| |H|^2 \tilde{x}^a \right\|_{L^4} \left\| u \tilde{x}^{-\frac{a}{2}} \right\|_{L^4} \\\n\leq C \left\| |H|^2 \tilde{x}^a \right\|_{L^4}^2 + C \left\| |H|^2 \tilde{x}^a \right\|_{L^2} \left( \| \sqrt{\rho u} \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 \right) \\\n\leq C \left\| |H|^2 \tilde{x}^a \right\|_{L^2} \left\| \nabla (|H|^2 \tilde{x}^a) \right\|_{L^2} + C \left\| |H|^2 \tilde{x}^a \right\|_{L^2} \left( \| \sqrt{\rho u} \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 \right) \\\n\leq C \left( 1 + \| \nabla u \|_{L^2}^2 \right) \left\| |H|^2 \tilde{x}^a \right\|_{L^2}^2 + \nu \frac{1}{4} \| \nabla |H|^2 \tilde{x}^a \|_{L^2}^2. \quad (3.13)
\]
Together (3.10)-(3.13) with (3.3) and Gronwall’s inequality, one has
\[
\sup_{0 \leq t \leq T} \int |H|^2 \bar{x}_a \, dx + \int_0^T \int |\nabla H|^2 \bar{x}_a \, dx dt \leq C. \tag{3.14}
\]
Finally, multiplying (1.1)_3 by $|H|^2 H$ and integrating by parts lead to
\[
\frac{d}{dt} \int |H|^4 \, dx + 4\nu \int (|\nabla H|^2 |H|^2 + |\nabla |H|^2|^2) \, dx
\]
\[
= 4 \int (H \cdot \nabla u - u \cdot \nabla H - H \text{div } u) |H|^2 H \, dx, \tag{3.15}
\]
where we have used the following fact
\[
\nabla \times (u \times H) = (H \cdot \nabla)u - (u \cdot \nabla)H - (\text{div } H).u.
\]
Furthermore, a direct computation gives that
\[
\int u \cdot \nabla H |H|^2 H \, dx = -\frac{1}{4} \int |H|^4 \text{div } u \, dx.
\]
Substituting this into (3.15) yields that
\[
\frac{d}{dt} \int |H|^4 \, dx + 4\nu \int (|\nabla H|^2 |H|^2 + |\nabla |H|^2|^2) \, dx
\]
\[
\leq C |\nabla u|_{L^2}^2 |H|^2_{L^4} \leq C |\nabla u|_{L^2}^2 |H|^2_{L^2} |\nabla |H|^2|_{L^2}
\]
\[
\leq 2\nu |H|^2 |\nabla H|_{L^2}^2 + C |\nabla u|_{L^2}^2 |H|^2_{L^4}.
\]
Due to the Gronwall’s inequality and (3.3), we can conclude that
\[
\sup_{0 \leq t \leq T} \int |H|^4 \, dx + \int_0^T \int |H|^2 |\nabla H|^2 \, dx dt \leq C,
\]
which together with (3.3) and (3.14) yield to (3.7).

Next, there is an important estimate about $\rho u$, the similar arguments of the following estimate come from [25].

**Lemma 3.3.** Under condition (3.1), there exist some positive constants $q_0 \in (2, \min \left(\frac{3\mu + 2\lambda}{\mu + \lambda}, \frac{\mu}{2}\right))$ and $s_0 \geq s_1 \equiv \frac{4 + 2\gamma - 2\alpha}{4 - q_0}$, such that
\[
\sup_{0 \leq t \leq T} \int \rho |u|^{q_0} \, dx \leq C, \tag{3.16}
\]
for any $0 \leq T < T^*$. 

**Proof.** Multiplying the momentum equations (1.1)_2 by $q|u|^{q-2}u$ $(2 < q < 4)$ and integrating the resulting equation yield to
\[
\frac{d}{dt} \int \rho |u|^q \, dx + q \int |u|^{q-2} [\mu |\nabla u|^2 + (\mu + \lambda)(\text{div } u)^2 + \mu(q - 2) |\nabla |u|^2] \, dx
\]
\[
= - (\mu + \lambda)q(q - 2) \int |u|^{q-3} u \cdot \nabla |u| \text{div } u \, dx + q \int \text{div } (|u|^{q-2} u) P \, dx
\]
\[
+ \frac{q}{2} \int |u|^{q-2} u \cdot (2H \cdot \nabla H - |H|^2) \, dx
\]
\[
\leq (\mu + \lambda)q(q - 2) \int |u|^{q-2} |\nabla |u|| |\text{div } u| \, dx
\]
\[
+ C \int \rho |u|^{q-2} |\nabla u| \, dx + C \int |H|^2 |\nabla u| |u|^{q-2} \, dx
\]
\[
\begin{align*}
&\leq (\mu + \lambda) q(q-2) \int |u|^{q-2} |\nabla u| |\text{div} u| dx + \frac{q\mu}{2} \int |u|^{q-2} |\nabla u|^2 dx \\
&\quad + C \left( \int |H|^4 |u|^{q-2} dx + \left( \int \rho \frac{|u|^{q-2}}{|\nabla u|^{\frac{q-2}{q}}} dx \right) \left( \int \rho |u|^q dx \right)^{\frac{q-2}{q}} \|\nabla u\|_{L^2}^2 \right) \\
&\leq (\mu + \lambda) q(q-2) \int |u|^{q-2} |\nabla u| |\text{div} u| dx + \frac{q\mu}{2} \int |u|^{q-2} |\nabla u|^2 dx \\
&\quad + C \int |H|^2 |H|^2 |u|^{q-2} x^q dx + C \int \rho |u|^q dx + C \|\nabla u\|_{L^2}^2 + C \\
&\leq (\mu + \lambda) q(q-2) \int |u|^{q-2} |\nabla u| |\text{div} u| dx + \frac{q\mu}{2} \int |u|^{q-2} |\nabla u|^2 dx \\
&\quad + C \int \rho |u|^q dx + C \|H\|_{L^2}^2 \|H\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + C \\
&\leq (\mu + \lambda) q(q-2) \int |u|^{q-2} |\nabla u| |\text{div} u| dx + C(1 + \|\nabla u\|_{L^2}^2) \left( 1 + \|\nabla u\|_{L^2}^2 \right) \\
&\quad + \frac{q\mu}{2} \int |u|^{q-2} |\nabla u|^2 dx + C \int \rho |u|^q dx + C \|\nabla u\|_{L^2}^2 + C \\
&\leq (\mu + \lambda) q(q-2) \int |u|^{q-2} |\nabla u| |\text{div} u| dx + C \|\nabla u\|_{L^2}^2 + C \|u\|_L^{4(q-2)} \\
&\quad + \frac{q\mu}{2} \int |u|^{q-2} |\nabla u|^2 dx + C \int \rho |u|^q dx + C \|\nabla u\|_{L^2}^2 + C, \quad (3.17)
\end{align*}
\]

due to (2.1), (2.6), (3.7), Hölder’s inequality and the fact $|\nabla u| \geq |\nabla u|$.

In addition, it is easy to see that
\[
|u|^{q-2} \left[ \mu |\nabla u|^2 + (\mu + \lambda) (\text{div} u)^2 + \mu(q-2)|\nabla u|^2 - (\mu + \lambda)(q-2)|\nabla u| |\text{div} u| \right]
\geq |u|^{q-2} \left[ \frac{\mu + \lambda}{2} (4-q)(\text{div} u)^2 + \left( \mu - \frac{\mu + \lambda}{2} (q-2) \right) |\nabla u|^2 \right]. \quad (3.18)
\]

Combining (3.17) with (3.18), choosing $q_0 \in (2, \min(\frac{3\mu + 2\lambda}{\mu + \lambda}, \frac{5}{2}))$, we obtain
\[
\|\nabla u\|_{L^2}^{4(q_0-2)} \leq \|\nabla u\|_{L^2}^2 + C,
\]
and
\[
\left( \mu - \frac{\mu + \lambda}{2} (q-2) \right) |\nabla u|^2 \geq \frac{\mu}{2} |\nabla u|^2,
\]
then
\[
\sup_{0 \leq t \leq T} \int \rho |u|^{q_0} dx \leq C.
\]

Thus, we finish the proof of Lemma 3.3.

With the help of Lemma 2.2 and Lemma 2.4, we can prove the following key estimate of $\nabla u$ and $\nabla H$. \qed
Lemma 3.4. Under condition (3.1), and \( s_0 \geq \max\{s_2, s_3\} \) (\( s_2, s_3 \) can be chosen in the proof), it holds that for \( 0 \leq T < T^* \),

\[
\sup_{0 \leq t \leq T} (\|u\|^2_{L^2} + \|v\|^2_{L^2} + \|C\|^2_{L^2}) + \int_0^T \left( (\rho|\dot{u}|^2 + |\nabla^2 u|^2 + |H_1|^2 + |\nabla^2 H|^2) \right) dx dt \leq C. \tag{3.19}
\]

Proof. Multiplying the momentum equation (1.1) by \( u \), integrating the resulting equation over \( \mathbb{R}^2 \) and integrating by parts give that

\[
\frac{1}{2} \frac{d}{dt} \int (\mu|\nabla u|^2 + (\mu + \lambda)(\text{div}u)^2) \, dx + \int \rho|\dot{u}|^2 \, dx
\]

\[
= \int \rho \dot{u} (u \cdot \nabla)u \, dx + \int P \text{div}u \, dx + \frac{1}{2} \int (2H \cdot \nabla H - \nabla|H|^2) \cdot u \, dx
\]

\[
= \sum_{i=1}^{3} I_i. \tag{3.20}
\]

In view of (2.1), (2.9), (2.10), (2.11), (3.16), choosing \( p_2 > \frac{2q_0}{q_0 - 2} \) and \( p_1 \in \left( \frac{2p_2}{2p_2 - 1}, 2 \right) \), \( s_2 = \max\{s_0, 2p_2 \} \in \left( \frac{2p_2(q_0 - 2)}{q_0 - 2 - 2p_2}, 2 \right) \), Young’s inequality, Sobolev inequality and integrating by parts give that

\[
I_1 \leq \frac{1}{4} \int \rho|\dot{u}|^2 \, dx + C \int \rho u^2 |\nabla u|^2 \, dx
\]

\[
\leq \frac{1}{4} \int \rho|\dot{u}|^2 \, dx + C \|\nabla u\|_{L^{p_2}}^2 \left( \int \rho|u|^{q_0} \, dx \right)^{2/q_0} \left( \int \rho^{l_0} \, dx \right)^{(q_0 - 2)/q_0 l_0}
\]

\[
\leq \frac{1}{4} \int \rho|\dot{u}|^2 \, dx + C \|\nabla u\|_{L^{p_2}}^2
\]

\[
\leq \frac{1}{4} \int \rho|\dot{u}|^2 \, dx + C \varepsilon \left( \|\nabla u\|_{L^{2p_2}}^2 + \|\nabla^2 w\|_{L^{2p_2}}^2 + \|\nabla u\|_{L^2}^2 \right)
\]

\[
\leq \frac{1}{4} \int \rho|\dot{u}|^2 \, dx + C \|P\|_{L^{2p_2}}^2 + \varepsilon \|\rho \dot{u}\|_{L^{p_1}}^2 + \|\nabla H\|_{L^2}^2 \|H\|_{L^{2p_2}}^2 + \|\nabla u\|_{L^2}^2
\]

\[
\leq \frac{1}{4} \int \rho|\dot{u}|^2 \, dx + C \|P\|_{L^{2p_2}}^2 + C \varepsilon \|\rho \dot{u}\|_{L^{p_1}}^2 + \|\nabla H\|_{L^2}^2 \|\rho\|_{L^{2p_2}}^2 + \|\nabla u\|_{L^2}^2
\]

\[
\leq \frac{1}{3} \int \rho|\dot{u}|^2 \, dx + C \|\nabla H\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 + C. \tag{3.21}
\]

Indeed, we have the following fact.

\[
I_2 = \frac{d}{dt} \int P \text{div}u \, dx - \int \dot{P} \text{div}u \, dx. \tag{3.22}
\]

Consider the integral term

\[
\int \dot{P} \text{div}u \, dx = \int P \text{div}u + \int P \text{div}w \, dx = -\int \nabla P \text{div}u + \int P \text{div}w \, dx. \tag{3.23}
\]

where \( u \) is separated into \( v \) and \( w \), which satisfy (2.7) and (2.8), respectively. First, for the first term, thanks to the equation in (2.7), we have

\[
-\int \nabla P \text{div}u = -\int (\mu \Delta v + (\mu + \lambda) \nabla \text{div}v) \, dx
\]

\[
= \frac{1}{2} \frac{d}{dt} \int (\mu |\nabla v|^2 + (\mu + \lambda) (\text{div}v)^2) \, dx. \tag{3.24}
\]
For the second term, we obtain
\[
\int P_t \text{div} u dx = -\int \text{div}(Pu) \text{div} u dx - (\gamma - 1) \int P \text{div} u dx
\]
\[
= \int Pu \cdot \nabla v dx - (\gamma - 1) \int P \text{div} v dx,
\]
where have used the following fact
\[
P_t + \text{div}(Pu) + (\gamma - 1) P \text{div} u = 0.
\]
We choose \( p_1 \in \left( \frac{\eta_0}{\eta_0 - 1}, 2 \right). \) Let \( s_3 \equiv \max\{4\eta \left( 1 - p_1 \right), \frac{\eta_0 - p_1}{\eta_0 - 1}, \frac{p_1}{2 - p_1}, \right\}, \) in view of Young’s inequality, (2.10) and (3.1), one has
\[
\int P_t \text{div} u dx \leq C \left\| \rho^{-\frac{\eta_0}{\eta_0 - 1}} \right\|_{L^\infty([0,T]; L^{\eta_0 - 1}(\Omega))} \left\| \rho^{-\frac{\eta_0}{\eta_0 - 1}} u \right\|_{L^2(\Omega)} \left\| \nabla \text{div} u \right\|_{L^p(\Omega)}
\]
\[
+ C ||P||_{L^\infty} \left\| \nabla u \right\|_{L^2(\Omega)} \left\| \nabla w \right\|_{L^2(\Omega)}
\]
\[
\leq C_{s_3} \left( \left\| \nabla w \right\|_{L^p(\Omega)}^2 + C \left\| \nabla \text{div} u \right\|_{L^2(\Omega)} \left\| \nabla w \right\|_{L^2(\Omega)} \left\| \nabla \text{div} u \right\|_{L^p(\Omega)}^2 \right) + C
\]
\[
\leq C_{s_3} \left( \left\| \nabla w \right\|_{L^p(\Omega)}^2 \right) + C \left\| \nabla \text{div} u \right\|_{L^2(\Omega)} \left( \left\| \nabla w \right\|_{L^2(\Omega)} + \left\| \nabla v \right\|_{L^2(\Omega)} \right).
\]
(3.25)

On the other hand, integrating by parts yields to
\[
I_3 = \frac{1}{2} \int (|H|^2 \text{div} u - 2H \cdot \nabla u \cdot H) \ dx
\]
\[
= \frac{1}{2} \frac{d}{dt} \int (|H|^2 \text{div} u - H \cdot \nabla u \cdot H) \ dx
\]
\[
- \int (H \cdot H_t \text{div} u - H \cdot \nabla u \cdot H_t - H \cdot \nabla \text{div} u \cdot H) \ dx
\]
\[
\leq \frac{1}{2} \frac{d}{dt} \int (|H|^2 \text{div} u - 2H \cdot \nabla u \cdot H) \ dx + C \|\text{div} u\|_{H^1}^2 + C \left\| \nabla u \right\|_{L^2(\Omega)}^2
\]
\[
\leq \frac{1}{2} \frac{d}{dt} \int (|H|^2 \text{div} u - 2H \cdot \nabla u \cdot H) \ dx + C \|\text{div} u\|_{H^1}^2 + C \left\| \text{div} u \right\|_{L^2(\Omega)}^2
\]
\[
\leq \frac{1}{2} \frac{d}{dt} \int (|H|^2 \text{div} u - 2H \cdot \nabla u \cdot H) \ dx + C \|\text{div} u\|_{H^1}^2 + C \left\| \nabla u \right\|_{L^2(\Omega)}^2
\]
\[
+ C \left\| \nabla \text{div} u \right\|_{L^2(\Omega)} \left\| \nabla w \right\|_{L^2(\Omega)} \left\| \nabla \text{div} u \right\|_{L^p(\Omega)}^2
\]
\[
\leq \frac{1}{2} \frac{d}{dt} \int (|H|^2 \text{div} u - 2H \cdot \nabla u \cdot H) \ dx + C \|\text{div} u\|_{H^1}^2
\]
\[
+ C \left\| \nabla \text{div} u \right\|_{L^2(\Omega)} \left\| \nabla u \right\|_{L^2(\Omega)} + C \left\| \nabla v \right\|_{L^2(\Omega)} + C + \varepsilon_1 C \left\| \sqrt{\rho} \right\|_{L^2(\Omega)}^2 + C \left\| \nabla H \right\|_{L^2(\Omega)}^2 + 1,
\]
(3.26)
due to (2.1), (2.9), (2.10) and (3.7), where \( p_1 \) was defined as before.

Substituting (3.21)-(3.26) into (3.20), we obtain after choosing \( \varepsilon_1 \) suitably small that
\[
\frac{d}{dt} \int \Psi dx + \int \rho \dot{u}^2 dx \leq C \varepsilon_2 \|H_t\|_{L^2(\Omega)}^2 + C \left( \|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 + \|\nabla H\|_{L^2(\Omega)}^2 + 1 \right),
\]
(3.27)
where
\[
\Psi = \mu (|\nabla u|^2 + |\nabla v|^2) + (\mu + \lambda) ((\text{div} u)^2 + (\text{div} v)^2) - 2P \text{div} u - |H|^2 \text{div} u + 2H \cdot \nabla u \cdot H. \tag{3.28}
\]

Furthermore, due to (2.1) yields to
\[
\left| \int (-|H|^2 \text{div} u + 2H \cdot \nabla u \cdot H) dx \right| \leq C \|H\|_{L^2}^2 \|\nabla u\|_{L^2} \leq C \|H\|_{L^2} \|\nabla H\|_{L^2} \|\nabla u\|_{L^2} \leq \frac{\mu}{4} \|\nabla u\|_{L^2}^2 + C_2 \|\nabla H\|_{L^2}^2. \tag{3.29}
\]

Next, we will deal with the key term \(\|H_t\|_{L^2}^2\) on the right-hand side of (3.27). It follows from the fact
\[
\nu \frac{d}{dt} \|\nabla H\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \nu^2 \|\Delta H\|_{L^2}^2 = \int |H_t - \nu \Delta H|^2 dx,
\]
and the third equation in (1.1) that
\[
\nu \frac{d}{dt} \|\nabla H\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \nu^2 \|\Delta H\|_{L^2}^2 = \int |H_t - \nu \Delta H|^2 dx.
\]

due to (2.1), (2.6), (2.9), (2.10), (3.7), Hölder inequality and interpolation inequality. Furthermore, it follows the fact
\[
\|\nabla^2 H\|_{L^2} \leq C_3 \|\Delta H\|_{L^2},
\]
First, integrating by parts, we have that
\[
\nu \frac{d}{dt} \|\nabla H\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \frac{\nu^2}{2C_3^2} \|\nabla^2 H\|_{L^2}^2 \leq \varepsilon_3 \|\sqrt{\rho} u_t\|_{L^2}^2 + C \left( \|\nabla H\|_{L^2}^4 + \|\nabla u_t\|_{L^2}^4 + \|\nabla v\|_{L^2}^4 + \|\nabla H \bar{x}^2\|_{L^2}^2 \right) + 1.
\] (3.31)

Indeed, it is easy to check that
\[
\varepsilon_3 \|\sqrt{\rho} u_t\|_{L^2}^2 + C \left( \|\nabla H\|_{L^2}^4 + \|\nabla u_t\|_{L^2}^4 + \|\nabla v\|_{L^2}^4 + \|\nabla H \bar{x}^2\|_{L^2}^2 \right) + 1.
\] (3.30), choosing \(\varepsilon_4\) suitable small that
\[
\nu \frac{d}{dt} \|\nabla H\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \frac{\nu^2}{2C_3^2} \|\nabla^2 H\|_{L^2}^2 \leq \varepsilon_3 \|\sqrt{\rho} u_t\|_{L^2}^2 + C \left( \|\nabla H\|_{L^2}^4 + \|\nabla u_t\|_{L^2}^4 + \|\nabla v\|_{L^2}^4 + \|\nabla H \bar{x}^2\|_{L^2}^2 \right) + 1.
\] (3.32)

This together with Gronwall’s inequality, (3.3), (3.7), (3.32) and (3.29) yields (3.19).

Indeed, it is easy to check that
\[
\left| \int P \text{div} u dx \right| \leq \frac{\mu}{4} \|\nabla u\|_{L^2}^2 + C.
\]

This completes the proof of Lemma 3.4.

Next, we can improve the regularity estimates on \(u\) and \(H\).

**Lemma 3.5.** Under condition (3.1), and \(s_0 \geq \max \{4\gamma, s_4\}\) (\(s_4\) can be chosen in the proof), it holds that for \(0 \leq T < T^*\),
\[
\sup_{0 \leq t \leq T} \left( \|\sqrt{\rho} u\|_{L^2}^2 + |H_t|_2^2 + t \|\nabla u\|_{L^2}^2 \right) + \int_0^T \left( t|x|^{-1} \|\dot{u}\|_2^2 + |\nabla H|_2^2 + t |\nabla H_t|^2 \right) dx dt \leq C. \quad (3.33)
\]

**Proof.** Apply the operator \(\dot{u}^j[\partial_t + \text{div}(u)]\) to (1.1) \((j = 1, 2)\) and using the first equation of (1.1), integration by parts over \(\mathbb{R}^2\), one has
\[
\frac{1}{2} \frac{d}{dt} \int \rho |\dot{u}|^2 dx = - \int \dot{u}^j [\partial_j P_t + \text{div}(u \partial_j P)] dx + \mu \int \dot{u}^j \left[ \Delta u^j + \text{div}(u \Delta u^j) \right] dx
\]

\[
+ (\mu + \lambda) \int \dot{u}^j [\partial_j \text{div} u_t + \text{div}(u \partial_j \text{div} u)] dx
\]

\[
- \frac{1}{2} \int \dot{u}^j \left[ \partial_i \partial_j |H|^2 + \text{div} \left( u \partial_j |H|^2 \right) \right] dx
\]

\[
+ \int \dot{u}^j \left[ \partial_i \partial_j \left( H^i H^j \right) + \text{div} \left( u \partial_i \left( H^i H^j \right) \right) \right] dx
\]

\[
= \sum_{i=1}^5 N_i. \quad (3.34)
\]

First, integrating by parts, we have that
\[
N_1 = - \int \dot{u}^j \left[ \partial_j P_t + \text{div}(u \partial_j P) \right] dx
\]

\[
= \int \left( \partial_j \dot{u}^j P_t (\rho) \partial_t + \partial_k \dot{u}^j u^k \partial_j P \right) dx
\]

\[
= \int \left( -\rho P' (\rho) \partial_j \dot{u}^j \text{div} u - \partial_j \dot{u}^j u^k \partial_k P + \partial_k \dot{u}^j u^k \partial_j P \right) dx
\]

\[
= \int \left( -\rho P' (\rho) \partial_j \dot{u}^j \text{div} u + P \partial_k (\partial_j \dot{u}^j u^k) - P \partial_j (\partial_k \dot{u}^j u^k) \right) dx
\]
due to (3.1), (3.13) and Hölder’s inequality. Furthermore, integration by parts and together with (2.1), (2.6), (3.7) and (3.13) yield

due to Young’s inequality. Similarly, we obtain

Substituting (3.35)-(3.39) into (3.34), and choosing \( \varepsilon \) suitably small, we obtain

Moreover, integration by parts and together with (2.1), (2.6), (3.7) and (3.13) yield that

Similarly, we have

Substituting (3.35)-(3.39) into (3.34), and choosing \( \varepsilon \) suitably small, we obtain

Next, differentiating the third equation in (1.1) with respect to \( t \), and multiplying the resulting equation by \( H_i \) in \( L^2 \), after integration by parts, using (2.1), (2.6), (3.7) and (3.19) yield that

\[
\frac{1}{2} \frac{d}{dt} \int |H_i|^2 dx + \nu \int |\nabla H_i|^2 dx
\]
\[
= \int (H_t \cdot \nabla u + H \cdot \nabla u_t \cdot H_t - u_t \cdot \nabla H \cdot H_t) \, dx \\
- \int (u \cdot \nabla H_t \cdot H_t + |H_t|^2 \text{div} + H \cdot H_t \text{div} u) \, dx \\
= \int (H_t \cdot \nabla u \cdot H_t - H \cdot \nabla H_t \cdot H_t + (u \cdot \nabla u) \cdot \nabla H_t \cdot (u \cdot \nabla u)) \, dx \\
+ \int \left( \frac{1}{2} |H_t|^2 \text{div} + \dot{u} \cdot \nabla H_t \cdot H - (u \cdot \nabla u) \cdot \nabla H_t \cdot H - |H_t|^2 \text{div} \right) \, dx
\]

\[
\leq C \int |H_t|^2 |\nabla u| \, dx + C \int |u| H |\nabla u| |\nabla H_t| \, dx + C \int |H||\nabla H_t||\dot{u}| \, dx \\
\leq C \|\nabla u\|_{L^2} |H_t|_{H^1} \|u\|_{L^2} + C \|H\|_{L^\infty} \|H_t\|_{H^2} \|u\|_{H^1} \|\nabla u\|_{H^1} |\nabla H_t|_{L^2} \\
+ C \|\nabla H_t\|_{H^2} C \|H\|_{L^2} \|H_t\|_{H^2} \|u\|_{H^1} \|\nabla u\|_{H^1} |\nabla H_t|_{L^2} \\
C \|H_t\|_{L^2} \|\nabla H_t\|_{L^2} + C \|\nabla^2 H\|_{L^2} \|u\|_{H^1} \|\nabla H_t\|_{L^2} \\
+ C \|\nabla H_t\|_{L^2} (\|\sqrt{\rho} \dot{u}\|_{L^2} \|H_t\|_{L^2} + \|\nabla \dot{u}\|_{L^2}) \\
\leq \frac{\nu}{2} \|H_t\|_{L^2}^2 + C_\nu \|\nabla u\|_{L^2}^2 + C \|\nabla^2 H\|_{L^2} \|u\|_{H^1}^2 + C \|\dot{u}\|_{L^2}^2 + C \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + 1
\]

which implies that

\[
\frac{d}{dt} \int |H_t|^2 \, dx + \nu \int |\nabla H_t|^2 \, dx \\
\leq C \|H_t\|_{L^2}^2 + C \|\nabla^2 H\|_{L^2} \|u\|_{H^1}^2 + C \|\nabla u\|_{L^4}^4 + C \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C. \tag{3.41}
\]

Using (2.1), (2.9), (2.10), (3.1) and (3.19), we obtain

\[
\|\nabla u\|_{L^4}^4 \leq C \left( \|\nabla u\|_{L^2}^2 + \|\nabla H_t\|_{H^1} \|\nabla u\|_{L^2} \right) \\
\leq C \left( \|\nabla u\|_{L^2}^{\frac{3p_1}{2}} \|\nabla^2 w\|_{L^1} \right) \\
\leq C \left( \|\nabla u\|_{L^2}^{\frac{3p_1}{2}} + \|\nabla v\|_{L^2}^{\frac{3p_1}{2}} \right) \|\nabla^2 w\|_{L^1} \|\nabla v\|_{L^2} \|\nabla u\|_{L^4}^4 \\
\leq C \|\nabla^2 w\|_{L^1} \|\nabla v\|_{L^2} \|\nabla u\|_{L^4}^4 \\
\leq C \|\nabla \dot{u}\|_{L^2} \|\rho\|_{L^4} \|H_t\|_{L^4} + C \|\nabla H\|_{L^2} \|H\|_{L^4} \|H_t\|_{L^2} \|\dot{u}\|_{L^2} \|P\|_{L^4}^4 \\
\leq C \|\nabla \dot{u}\|_{L^2}^4 + C, \tag{3.42}
\]

where \( p_1 \in (\frac{5}{2}, 2) \) and \( s_4 = \frac{s_3}{2 - p_1} \). Multiply (3.40) by \( \frac{2C_0}{\mu} \) and adding it to (3.41), choosing \( \eta \) suitable small, and using (3.42), we obtain

\[
\frac{d}{dt} \int \left( \frac{2C_0}{\mu} \rho \|\dot{u}\|^2 + |H_t|^2 \right) \, dx + \int \left( C_0 \|\nabla \dot{u}\|^2 + \frac{\nu}{2} \int |\nabla H_t|^2 \right) \, dx \\
\leq C \|H_t\|_{L^2}^2 + C \|\nabla^2 H\|_{L^2}^2 + C \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C, \tag{3.43}
\]

which together with (3.19) give

\[
\sup_{0 \leq t \leq T} \left( \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + |H_t|^2 \right) + \int_0^T \left( \|\nabla \dot{u}\|^2 + |\nabla H_t|^2 \right) \, dx \, dt \leq C. \tag{3.44}
\]
By Gronwall’s inequality and multiplying (3.43) by \( t \) and integrating the resulting inequality over \((0, T)\) lead to
\[
\sup_{0 \leq t \leq T} t \left( \|\sqrt{\rho} \dot{u} \|_{L^2}^2 + \|H_t\|_{L^2}^2 \right) + \int_0^T t \left( \|\nabla \dot{u}\|^2 + |\nabla H_t|^2 \right) dt \leq C.
\]

Using the above estimate and (2.5), we obtain
\[
\int_0^T t \|\bar{x}^{-1} \dot{u}\|_{L^2}^2 dt \leq C \int_0^T (t\|\sqrt{\rho} \dot{u}\|_{L^2}^2 + t\|\nabla \dot{u}\|_{L^2}^2) dt \leq C.
\]

Therefore, the proof of Lemma 3.5 is finished. \( \square \)

Next, in view of Lemma 3.1–Lemma 3.5, we can prove the boundedness of \( \|\rho\|_{L^\infty(0, T; L^\infty)} \) as follows.

**Lemma 3.6.** Under the conditions of Theorem 1.1, let \( s_0 \geq s_5 \) (\( s_5 \) can be chosen in the proof), then
\[
\sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^\infty} \leq C,
\]
holds for any \( T \in [0, T^*) \).

**Proof.** On the one hand, multiplying (1.1) by \( pp^{p-1} \) \((p > 1)\) and integrating by parts over \( \mathbb{R}^2 \) yields to
\[
\frac{d}{dt} \int \rho^p dx = - \int (u \cdot \nabla \rho^p + pp^p \text{div} u) dx = (1 - p) \int \rho^p \text{div} u dx
\]
\[
= \frac{1 - p}{2\mu + \lambda} \int \rho^p G dx + \frac{1 - p}{2\mu + \lambda} \int \rho^p \left( P + \frac{|H|^2}{2} \right) dx
\]
\[
\leq \frac{p - 1}{2\mu + \lambda} \|G\|_{L^\infty} \int \rho^p dx.
\]

Set \( p_1 > 2, \ \varepsilon = \frac{2}{a - 1}, \) taking \( \eta = \frac{1}{p_1} \) and \( s_5 = \frac{2(a p_1 - 1)}{a - 1}, \) we obtain
\[
\|G\|_{L^\infty} \leq C \|G\|_{L^2} + C \|\nabla G\|_{L^{p_1}} \leq C \|\nabla G\|_{L^{p_1}} + C
\]
\[
\leq C \|p \dot{u}\|_{L^{p_1}} + C \|H\|_{L^1} \|\nabla \dot{u}\|_{L^{2\mu + \lambda}} + C \|\nabla^2 H\|_{L^2} + C
\]
\[
\leq C \left( \|\nabla \dot{u}\|_{L^2} + \|\nabla^2 H\|_{L^2} + 1 \right) \leq C \left( \|\nabla \dot{u}\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2 + 1 \right),
\]
which together with (2.1), (2.2), (2.6), (3.1), (3.7) with (3.1) and using Hölder’s inequality yield
\[
\frac{d}{dt} \|\rho\|_{L^p} \leq \frac{C(p - 1)}{p} \left( \|\nabla \dot{u}\|_{L^2} + \|\nabla^2 H\|_{L^2} + 1 \right) \|\rho\|_{L^p},
\]
where constant \( C \) independent of \( p \).

It follows form (3.47), (3.19), (3.33) and Gronwall’s inequality that
\[
\sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^p} \leq C.
\]
Let \( p \rightarrow \infty \), we finish the proof of Lemma 3.6. \( \square \)

Finally, the following Lemma gives the bounds of the first-order derivative of density \( \rho \), the second spatial derivative of the velocity \( u \) and \( H \).
Lemma 3.7. Under the condition (3.1), it holds that for some $q \in (2, \infty)$ and $0 \leq T < T^*$,

$$\sup_{0 \leq t \leq T} (\| \rho \|_{H^1 \cap W^{1,q}} + \| \nabla u \|_{H^1} + \| \nabla H \|_{H^1}) + \int_0^T \left( \| \nabla^2 u \|_{L^q}^\frac{q+1}{q} + t \| \nabla^2 u \|_{L^q}^2 \right) dt \leq C. \quad (3.48)$$

Proof. Firstly, $|\nabla \rho|^q$ satisfies

$$\frac{d}{dt} \| \nabla \rho \|_{L^q} \leq C (1 + \| \nabla u \|_{L^\infty}) \| \nabla \rho \|_{L^q} + C \| \nabla^2 u \|_{L^q}. \quad (3.49)$$

Next, we estimate terms $\| \nabla u \|_{L^\infty}$ and $\| \nabla^2 u \|_{L^q}$, respectively. In fact, due to (2.6) and the standard $L^p$-estimate $(p > 1)$ of elliptic system (1.1), we get

$$\sup_{0 \leq t \leq T} \| \nabla u \|_{L^\infty} \leq C (1 + \| \nabla u \|_{L^\infty}) \log (e + \| \nabla^2 u \|_{L^q}) + C. \quad (3.50)$$

Furthermore, we have the following estimate

for any $q \in (2, \infty)$, which implies

$$\frac{d}{dt} \| \nabla \rho \|_{L^q} \leq C \left( 1 + \| \nabla u \|_{L^\infty} + \| \omega \|_{L^\infty} \right) \| \nabla \rho \|_{L^q} + C \| \nabla^2 u \|_{L^q}. \quad (3.51)$$

Next, we estimate terms $\| \nabla u \|_{L^\infty}$ and $\| \nabla^2 u \|_{L^q}$, respectively. In fact, due to (2.6) and the standard $L^p$-estimate $(p > 1)$ of elliptic system (1.1), we get

$$\sup_{0 \leq t \leq T} \| \nabla \rho \|_{L^q} \leq C \left( 1 + \| \nabla u \|_{L^\infty} + \| \omega \|_{L^\infty} \right) \| \nabla \rho \|_{L^q} + C \| \nabla^2 u \|_{L^q}. \quad (3.52)$$

due to (2.1), (2.4), (2.6) and (3.19). Substituting (3.50)-(3.52) into (3.49), we get

$$\frac{d}{dt} \| \nabla \rho \|_{L^q} \leq C \left( 1 + \| \nabla u \|_{L^\infty}^2 + \| \nabla^2 H \|_{L^2}^2 \right) \log (e + \| \nabla \rho \|_{L^q}), \quad (3.53)$$

which implies

$$\frac{d}{dt} \| \nabla \rho \|_{L^q} \leq C \left( 1 + \| \nabla u \|_{L^\infty}^2 + \| \nabla^2 H \|_{L^2}^2 \right) \log (e + \| \nabla \rho \|_{L^q}), \quad (3.54)$$

Furthermore, we have

$$\sup_{0 \leq t \leq T} \| \nabla \rho \|_{L^q} \leq C. \quad (3.54)$$

due to (3.19), (3.33) and (3.54).

Indeed, $\| \nabla \rho \|_{L^2}$ satisfies

$$\frac{d}{dt} \| \nabla \rho \|_{L^2} \leq C \left( 1 + \| \nabla u \|_{L^\infty} \right) \| \nabla \rho \|_{L^2} + C \| \nabla^2 u \|_{L^2}$$

$$\leq C \left( 1 + \| \nabla u \|_{L^\infty} + \| \nabla^2 u \|_{L^q} \right) \| \nabla \rho \|_{L^q} + C \| \nabla^2 u \|_{L^2},$$
which together with (3.33), (3.50), (3.54), (3.55) and using Gronwall’s inequality imply that

$$\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^2} \leq C. \quad (3.56)$$

Moreover, the standard $L^2$-estimate of elliptic system (1.1)$_2$ gives that

$$\|\nabla^2 u\|_{L^2} \leq C (\|\sqrt{p}\rho\|_{L^2} + \|\nabla \rho\|_{L^2} + \|H \cdot \nabla H\|_{L^2}) \leq C. \quad (3.57)$$

Finally, we will prove the boundedness of $\|\nabla H\|_{H^1}$.

It follows from (2.1), (3.7), (3.19), (3.33), (3.42), (3.57), Young’s inequality and standard $L^2$-estimate of elliptic system (1.1)$_3$ that

$$\|\nabla^2 H\|_{L^2} \leq C \|H_t\|_{L^2} + C \|\nabla \rho\|_{H^1} + C \|H\|_{L^2} + C \|\nabla H\|_{L^2} + C \|\nabla u\|_{L^2} + C \|\nabla H\|_{L^2} + C \|\nabla u\|_{L^2} + C \|\nabla H\|_{L^2} + C \|\nabla H\|_{L^2} + C$$

which means

$$\|\nabla^2 H\|_{L^2} \leq C \|\nabla^2 H\|_{L^2} + C \|\nabla H\|_{L^2} + C \|\nabla u\|_{L^2} + C \|\nabla H\|_{L^2} + C. \quad (3.58)$$

Moreover, we estimate the boundedness of $\|\nabla H\|_{L^2}^2$. Multiplying (1.1)$_3$ by $\Delta H \bar{x}^a$, using (2.1), (2.12), (3.7), (3.19), (3.33), (3.55), integrating by parts yield

$$\frac{1}{2} \frac{d}{dt} \int |\nabla H|^2 \bar{x}^a dx + \nu \int |\Delta H|^2 \bar{x}^a dx$$

$$\leq C \int |H| \nabla \rho \|\nabla u\| \nabla \bar{x}^a dx + C \int |H| |\Delta H| \|\nabla u\| \bar{x}^a dx$$

$$+ C \int |\|\nabla H\|^2 |\nabla \bar{x}^a| dx + C \int |\nabla H| |\Delta H| \|\nabla u\|^2 \bar{x}^a dx$$

$$\leq C \int |H| \bar{x}^a |\nabla H| |\nabla u| \bar{x}^a dx + C \int |H| \bar{x}^a |\Delta H| \|\nabla u\| \bar{x}^a dx$$

$$+ C \int |\|\nabla H\|^2 |\nabla H| \|\bar{x}^a\| \|\nabla \bar{x}^a\| dx + C \int |\nabla H| |\Delta H| \|\bar{x}^a\| \|\nabla u\| \bar{x}^a dx$$

$$\leq C \|H \bar{x}^a\|_{L^1} \|\Delta H \bar{x}^a\|_{L^2} |\nabla u|_{L^2} + C \|H \bar{x}^a\|_{L^1} \|\Delta H \bar{x}^a\|_{L^2} |\nabla u|_{L^2} + C \|\nabla H\|_{L^2} \|\nabla \bar{x}^a\|_{L^2} + C \|\nabla H\|_{L^2} \|\Delta H \bar{x}^a\|_{L^2}$$

$$\leq C (1 + \|\nabla^2 u\|_{L^2} + \||\nabla H\|^2\|_{L^2}) (1 + \|\nabla H\|^2\|_{L^2} + \frac{\nu}{2} \|\Delta H\|^2\|_{L^2})$$

This gives that

$$\frac{d}{dt} \int |\nabla H|^2 \bar{x}^a dx + \nu \int |\Delta H|^2 \bar{x}^a dx$$

$$\leq C (1 + \|\nabla^2 u\|_{L^2} + \||\nabla H\|^2\|_{L^2}) (1 + \|\nabla H\|^2\|_{L^2}),$$

which together with Gronwall’s inequality yield to

$$\sup_{0 \leq t \leq T} \|\nabla H \bar{x}^a\|_{L^2} \leq C. \quad (3.59)$$
Lemma 3.8. With the assumption (3.1), it holds that for some $q \in (2, \infty)$,
\[
\sup_{0 \leq t \leq T} \|\rho \bar{x}^a\|_{H^{1+1/4}} \leq C. 
\]

Proof. It follows from (1.1) that $\rho \bar{x}^a$ satisfies
\[
(\rho \bar{x}^a)_t + u \cdot \nabla (\rho \bar{x}^a) - a p \rho \bar{x}^a u \cdot \nabla \log \bar{x} + \rho \bar{x}^a \text{div}u = 0. 
\]
Hence, for $p \in [2, q]$, $|\nabla (\rho \bar{x}^a)|^p$ satisfies
\[
\frac{d}{dt} \|\nabla (\rho \bar{x}^a)\|_{L^p} \leq C (1 + \|\nabla u\|_{L^\infty} + \|u \cdot \nabla \log \bar{x}\|_{L^\infty}) \|\nabla (\rho \bar{x}^a)\|_{L^p} 
+ C \|\rho \bar{x}^a\|_{L^\infty} \left( \|u \nabla \log \bar{x}\|_{L^p} + \|u^2 \log \bar{x}\|_{L^p} + \|\nabla^2 u\|_{L^p} \right) 
\leq C (1 + \|\nabla u\|_{W^{1,4}}) \|\nabla (\rho \bar{x}^a)\|_{L^p} 
+ C \|\rho \bar{x}^a\|_{L^\infty} \left( \|\nabla u\|_{L^p} + \|u \bar{x}^{-\frac{1}{2}}\|_{L^p} + \|\nabla^2 u\|_{L^p} \right) 
\leq C \left( 1 + \|\nabla^2 u\|_{L^p} + \|\nabla u\|_{W^{1,4}} \right) (1 + \|\nabla (\rho \bar{x}^a)\|_{L^p} + \|\nabla (\rho \bar{x}^a)\|_{L^q} ),
\]
due to (2.1), (2.6), (3.3), (3.19) and the following facts
\[
\|u \cdot \nabla \log \bar{x}\|_{L^\infty} \leq C \|u \bar{x}^{-\frac{1}{2}}\|_{L^\infty} \leq C \left( \|u \bar{x}^{-\frac{1}{2}}\|_{L^q} + \|\nabla \left( u \bar{x}^{-\frac{1}{2}} \right)\|_{L^q} \right) 
\leq C \left( \|u \bar{x}^{-\frac{1}{2}}\|_{L^q} + \|\nabla u\|_{L^q} \right) \leq C (1 + \|\nabla u\|_{H^1} ) \leq C,
\]
and in view of (2.6), (3.48), (2.1) and (3.7) yield to
\[
\|\rho \bar{x}^a\|_{L^\infty} \leq C \left( \|\rho \bar{x}^a\|_{L^2} + \|\nabla (\rho \bar{x}^a)\|_{L^q} \right) 
\leq C \left( \|\rho \bar{x}^a\|_{L^1} + \epsilon \|\rho \bar{x}^a\|_{L^\infty} + \|\nabla (\rho \bar{x}^a)\|_{L^q} \right) 
\leq C (1 + \epsilon \|\rho \bar{x}^a\|_{L^\infty} + \|\nabla (\rho \bar{x}^a)\|_{L^q} ),
\]
which together with $\epsilon$ suitable small implies
\[
\|\rho \bar{x}^a\|_{L^\infty} \leq C (1 + \|\nabla (\rho \bar{x}^a)\|_{L^q} ).
\]
Choosing $p = q$ in (3.61), using Gronwall’s inequality and (3.48) yields to
\[
\sup_{0 \leq t \leq T} \|\nabla (\rho \bar{x}^a)\|_{L^q} \leq C. 
\]
Furthermore, taking $p = 2$ in (3.61), in view of (3.48) and (3.62), one has
\[
\sup_{0 \leq t \leq T} \|\nabla (\rho \bar{x}^a)\|_{L^2} \leq C. 
\]
Proof of Theorem 1.1. In view of Proposition 1, there exists a $T^* > 0$ such that the boundary value problem (1.1)-(1.5) has a unique strong solution $(\rho, u, H)$ on $\mathbb{R}^2 \times (0, T^*)$. Next, we will extend the local strong solution to all times. Set

$$T^* = \sup \{ T \mid (\rho, u, H) \text{ is a strong solution on } \mathbb{R}^2 \times (0, T] \}.$$ 

It follows from the standard embedding

$$L^\infty(\tau, T; H^1) \cap H^1(\tau, T; L^2) \hookrightarrow C([\tau, T]; L^q), \text{ for any } q \in (2, \infty),$$

and Lemma 3.3-Lemma 3.5 and Lemma 3.8 that

$$\nabla u, \nabla H \in C([\tau, T]; L^2 \cap L^q), \quad \text{and} \quad \rho \in C ([\tau, T]; W^{1,q}).$$

Finally, we prove that

$$T^* = \infty.$$

Suppose not, $T^* < \infty$, in view of Lemma 3.3-Lemma 3.8, $$(\rho, u, H)(x, T^*) = \lim_{t \to T^*} (\rho, u, H)(x, t)$$ satisfy the conditions imposed on the initial data at the time $t = T^*$. Thus, with the help of Lemma 3.1-Lemma 3.8 and local existence in Proposition 1, we can extend the local strong solution of $(\rho, u, H)$ beyond $t > T^*$, which contradicts the maximality of $T^*$. The proof of Theorem 1.1 is completed. \qed

Acknowledgments. The author would like to thank the referees for their helpful suggestions and careful reading which has improved the presentation of this paper.

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Received February 2020; revised June 2020.

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