WEIGHTED MORREY-HERZ SPACE ESTIMATES FOR
ROUGH HAUSDORFF OPERATOR AND ITS
COMMUTATORS

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Abstract. In this paper, we give necessary and sufficient conditions for
the boundedness of rough Hausdorff operators on Herz, Morrey and Morrey-
Herz spaces with absolutely homogeneous weights. Especially, the estimates
for operator norms in each case are worked out. Moreover, we also establish
the boundedness of the commutators of rough Hausdorff operators on the
two weighted Morrey-Herz type spaces with their symbols belonging to
Lipschitz space.

1. Introduction

Let $\Phi(t)$ be a locally integrable function in $(0, \infty)$. The one dimensional
Hausdorff operator is defined in terms of the integral form as follows
\[
H_{\Phi} f(x) = \int_0^\infty \frac{\Phi(t)}{t} f\left(\frac{x}{t}\right) dt.
\]
(1.1)

It is well known that the Hausdorff operator is one of important operators
in harmonic analysis, and it closely related to the summability of the clas-
sical Fourier series. It is worth pointing out that if the kernel function $\Phi$
is taken appropriately, then the Hausdorff operator reduces to many class-
cial operators in analysis such as the Hardy operator, the Cesàro operator,
the Riemann-Liouville fractional integral operator and the Hardy-Littlewood
average operator (see, e.g., [2], [8], [12], [17] and references therein).

In 2002, Brown and Móricz [3] extended the study of Hausdorff operator to
the high dimensional space which is defined as follows
\[
H_{\Phi,A}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} f(A(t)x) dt, \, x \in \mathbb{R}^n,
\]
(1.2)

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where \( \Phi \) is a locally integrable function on \( \mathbb{R}^n \), and \( A(t) \) is an \( n \times n \) invertible matrix for almost everywhere \( t \) in the support of \( \Phi \). It should be pointed out that if the kernel function \( \Phi \) and \( A(t) \) are chosen suitably, then \( H_{\Phi,A} \) reduces to the weighted Hardy-Littlewood average operator, the weighted Hardy-Cesàro operator (see [11, 9]). More generally, Chuong, Duong and Dung [5] recently have introduced a more general multilinear operator of Hausdorff type

\[
H_{\Phi,A}(f)(x) = \int_{\mathbb{R}^n} \Phi(t) \prod_{i=1}^{m} f_i(A_i(t)x)dt, \quad x \in \mathbb{R}^n,
\]

where \( \Phi : \mathbb{R}^n \rightarrow [0, \infty) \) and \( A_i(t) \), for \( i = 1, ..., m \) are \( n \times n \) invertible matrices for almost everywhere \( t \) in the support of \( \Phi \), and \( f_1, f_2, ..., f_m : \mathbb{R}^n \rightarrow \mathbb{C} \) are measurable functions.

It is interesting to see that the theory of weighted Hardy-Littlewood average operators, Hardy-Cesàro operators and Hausdorff operators has been significantly developed into different contexts (for more details see [9], [3], [7], [5], [10], [28] and references therein). In 2016, Chuong, Duong and Hung [6] studied the boundedness of the Hardy-Cesàro operators and their commutators on weighted Herz, Morrey and Morrey-Herz spaces with absolutely homogeneous weights. In 2012, Chen, Fan and Li [7] introduced another version of Hausdorff operators, so-called the rough Hausdorff operator, as follows

\[
H_{\Phi,\Omega}(f)(x) = \int_{\mathbb{R}^n} \Phi(x|y|^{-1}) \Omega(y|y|^{-1}) f(y)dy, \quad x \in \mathbb{R}^n,
\]  

(1.3)

where \( \Phi : \mathbb{R}^n \rightarrow \mathbb{C} \) and \( \Omega : S^{n-1} \rightarrow \mathbb{C} \) are Lebesgue measurable functions. Note that if \( \Phi \) is a radial function, then by using the change of variable in polar coordinates, the operator \( H_{\Phi,\Omega} \) is rewritten in terms of the following form

\[
H_{\Phi,\Omega}(f)(x) = \int_{0}^{\infty} \int_{S^{n-1}} \Phi(t) \frac{\Omega(y')f(t^{-1}|x|y')d\sigma(y')}{t} dt.
\]  

(1.4)

It is useful to remark that if we choose \( \Phi(t) = t^{-n} \chi_{(1,\infty)}(t) \) and \( \Omega \equiv 1 \), the rough Hausdorff operator \( H_{\Phi,\Omega} \) reduces to the famous Hardy operator

\[
\mathcal{H}(f)(x) = \frac{1}{|x|^n} \int_{|y| \leq |x|} f(y)dy.
\]  

(1.5)

Also, if \( \Omega \equiv 1 \) and \( \Phi(t) = \chi_{(0,1)}(t) \), the \( H_{\Phi,\Omega} \) reduces to the adjoint Hardy operator

\[
\mathcal{H}^*(f)(x) = \int_{|y| > |x|} \frac{f(y)}{|y|^n} dy.
\]  

(1.6)
Moreover, Chen, Fan and Li [7] revealed that the rough Hausdorff operators have better performance on the Herz type Hardy spaces \( H^{\alpha, p}_q (\mathbb{R}^n) \) than their performance on the Hardy spaces \( H^p (\mathbb{R}^n) \) when \( 0 < p < 1 \). Meanwhile, the authors obtained some new results and generalized some known results for the high dimensional Hardy operator as well as the adjoint Hardy operator.

Let \( b \) be a measurable function. Let \( M_b \) be the multiplication operator defined by \( M_b f(x) = b(x)f(x) \) for any measurable function \( f \). If \( \mathcal{H} \) is a linear operator on some measurable function space, the commutator of Coifman-Rochberg-Weiss type formed by \( M_b \) and \( \mathcal{H} \) is defined by \( [M_b, \mathcal{H}] f(x) = (M_b \mathcal{H} - \mathcal{H} M_b) f(x) \). In particular, if \( \mathcal{H} = \mathcal{H}_{\Phi, \Omega} \), then we have the commutators of Coifman-Rochberg-Weiss type of the rough Hausdorff operator given as follows

\[
\mathcal{H}^b_{\Phi, \Omega} f(x) = b(x)\mathcal{H}_{\Phi, \Omega} f(x) - \mathcal{H}_{\Phi, \Omega}(bf)(x)
= \int_0^\infty \int_{S^{n-1}} \Phi(t) \Omega(t'y') f(|x|t^{-1}y') \left[ b(x) - b(|x|t^{-1}y') \right] d\sigma(y')dt.
\]

Inspired by above mentioned results, the goal of this paper is to extend and develop the known results in [6] to rough Hausdorff operators setting. More precisely, we establish the necessary and sufficient conditions for the boundedness of rough Hausdorff operators on weighted Herz, central Morrey, and Morrey-Herz spaces with absolutely homogeneous weights. In each case, the estimates for operator norms are worked out. Also, the sufficient conditions for the boundedness of the commutators of rough Hausdorff operators on the two weighted Morrey-Herz type spaces with their symbols belonging to Lipschitz space is given.

Our paper is organized as follows. In Section 2, we give necessary preliminaries for Herz spaces, central Morrey spaces and Morrey-Herz spaces as well as the class of absolutely homogeneous weights. Our main theorems are given and proved in Section 3.

2. Preliminaries

Before stating our results in the next section, let us give some basic facts and notations which will be used throughout this paper. By \( \|T\|_{X \rightarrow Y} \), we denote the norm of \( T \) between two normed vector spaces \( X, Y \). The letter \( C \) denotes a positive constant which is independent of the main parameters, but may be different from line to line. For any \( a \in \mathbb{R}^n \) and \( r > 0 \), we shall denote by \( B(a, r) \) the ball centered at \( a \) with radius \( r \). We also denote \( S^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \} \) and \( |S^{n-1}| = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \). For any real number \( p > 0 \), denote by \( p' \) conjugate real number of \( p \), i.e. \( \frac{1}{p} + \frac{1}{p'} = 1 \).
Next, we write \( a \lesssim b \) to mean that there is a positive constant \( C \), independent of the main parameters, such that \( a \leq Cb \). The symbol \( f \approx g \) means that \( f \) is equivalent to \( g \) (i.e. \( C^{-1}f \leq g \leq Cf \)). Throughout the paper, the weighted function \( \omega(x) \) will be denoted a nonnegative measurable function on \( \mathbb{R}^n \), and let \( L^q_{\omega}(\mathbb{R}^n) \) \((0 < q < \infty)\) be the space of all Lebesgue measurable functions \( f \) on \( \mathbb{R}^n \) such that

\[
\|f\|_{q,\omega} = \left( \int_{\mathbb{R}^n} |f(x)|^q \omega(x) dx \right)^{\frac{1}{q}} < \infty.
\]

The space \( L^q_{\omega,\text{loc}}(\mathbb{R}^n) \) is defined as the set of all measurable functions \( f \) on \( \mathbb{R}^n \) satisfying \( \int_K |f(x)|^q \omega(x) dx < \infty \) for any compact subset \( K \) of \( \mathbb{R}^n \). The space \( L^q_{\omega,\text{loc}}(\mathbb{R}^n) \) is also defined in a similar way to the space \( L^q_{\omega,\text{loc}}(\mathbb{R}^n) \).

In the following definitions \( \chi_k = \chi_{C_k}, \ C_k = B_k \setminus B_{k-1} \) and \( B_k = \{ x \in \mathbb{R}^n : |x| \leq 2^k \} \), for all \( k \in \mathbb{Z} \). Now, we are in a position to give some definitions of the Lipschitz, Herz, Morrey and Morrey-Herz spaces. For further information on these spaces as well as their deep applications in analysis, the interested readers may refer to the work \([1]\) and to the monograph \([23]\).

**Definition 2.1.** Let \( 0 < \beta \leq 1 \). The Lipschitz space \( \text{Lip}^\beta(\mathbb{R}^n) \) is defined as the set of all functions \( f : \mathbb{R}^n \to \mathbb{C} \) such that

\[
\|f\|_{\text{Lip}^\beta(\mathbb{R}^n)} := \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.
\]

**Definition 2.2.** Let \( \lambda \in \mathbb{R} \) and \( 1 \leq p < \infty \). The weighted central Morrey space \( B^\lambda_p(\omega, \mathbb{R}^n) \) is defined as the set of all locally \( p \)-integrable functions \( f \) satisfying

\[
\|f\|_{B^\lambda_p(\omega, \mathbb{R}^n)} = \sup_{R > 0} \left( \frac{1}{\omega(B(0, R))^{1 + \lambda/p}} \int_{B(0, R)} |f(x)|^p \omega(x) dx \right)^{1/p} < \infty.
\]

**Definition 2.3.** Let \( \alpha \in \mathbb{R}, 0 < q < \infty \), and \( 0 < p < \infty \). The weighted homogeneous Herz-type space \( K^\alpha_{p,q}(\omega) \) is defined by

\[
K^\alpha_{p,q}(\omega) = \{ f \in L^q_{\omega,\text{loc}}(\mathbb{R}^n) \setminus \{0\}, \omega) : \|f\|_{K^\alpha_{p,q}(\omega)} < \infty \},
\]

where \( \|f\|_{K^\alpha_{p,q}(\omega)} = \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f \chi_k\|_{p,q,\omega}^p \right)^{\frac{1}{p}} \).

**Definition 2.4.** Let \( \alpha \in \mathbb{R}, 0 < p < \infty, 0 < q < \infty, \lambda \geq 0 \) and \( \omega \) be non-negative weighted function. The homogeneous weighted Morrey-Herz-type space \( M K^\alpha_{p,q,\lambda}(\omega) \) is defined by

\[
M K^\alpha_{p,q,\lambda}(\omega) = \{ f \in L^q_{\omega,\text{loc}}(\mathbb{R}^n) \setminus \{0\}, \omega) : \|f\|_{M K^\alpha_{p,q,\lambda}(\omega)} < \infty \},
\]
where \( \| f \|_{MK_{p,q}^{\alpha,\lambda}(\omega)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k = -\infty}^{k_0} 2^{k \alpha p} \| f \chi_k \|_{L^q(\mathbb{R}^n)}^{p} \right)^{\frac{1}{p}} \).

Note that \( K_{p}^{\alpha,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \) for \( 0 < p < \infty \), and \( \dot{K}_{q}^{\alpha/p,p}(\mathbb{R}^n) = L^p(|x|^\alpha dx) \) for all \( 0 < p < \infty \) and \( \alpha \in \mathbb{R} \). Since \( MK_{p,q}^{\alpha,0}(\mathbb{R}^n) = \dot{K}_{q}^{\alpha/p,p}(\mathbb{R}^n) \), it follows that the Herz spaces are the special cases of Morrey-Herz spaces. Therefore, it is said that the Herz spaces and Morrey-Herz spaces are natural generalizations of the Lebesgue spaces associated with power weights.

Next, let us give some definitions of the two weighted Herz, Morrey, and Morrey-Herz spaces.

**Definition 2.5.** Let \( 0 < p < \infty \) and \( \lambda > 0 \). Suppose \( \omega_1, \omega_2 \) are two weighted functions. Then, the two weighted Morrey space is defined by

\[ \dot{B}^p,\lambda(\omega_1, \omega_2) = \{ f \in L^p_{\text{loc}}(\omega_1) : \| f \|_{\dot{B}^p,\lambda(\omega_1, \omega_2)} < \infty \} \]

where

\[ \| f \|_{\dot{B}^p,\lambda(\omega_1, \omega_2)} = \sup_{R > 0} \left( \frac{1}{\omega_2(B(0, R))^\lambda} \int_{B(0, R)} |f(x)|^{p} \omega_1(x) dx \right)^{\frac{1}{p}}. \]

**Definition 2.6.** Let \( 0 < p < \infty \), \( 0 < q < \infty \), and \( \alpha \in \mathbb{R} \). Let \( \omega_1 \) and \( \omega_2 \) be nonnegative weighted functions. The homogeneous two weighted Herz space \( \dot{K}_{q}^{\alpha,p}(\omega_1, \omega_2) \) is defined to be the set of all \( f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}; \omega_2) \) such that

\[ \| f \|_{\dot{K}_{q}^{\alpha,p}(\omega_1, \omega_2)} = \left( \sum_{k \in \mathbb{Z}} \omega_1(B_k)^{\frac{p}{q} p} \| f \chi_k \|_{L^q(\mathbb{R}^n)}^{p} \right)^{\frac{1}{p}} < \infty. \]

**Definition 2.7.** Let \( \alpha \in \mathbb{R}, 0 < p < \infty, 0 < q < \infty, \lambda \geq 0 \) and \( \omega_1, \omega_2 \) be weighted functions. The two weighted Morrey-Herz space \( MK_{p,q}^{\alpha,\lambda}(\omega_1, \omega_2) \) is defined as the space of all functions \( f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}; \omega_2) \) such that \( \| f \|_{MK_{p,q}^{\alpha,\lambda}(\omega_1, \omega_2)} < \infty \), where

\[ \| f \|_{MK_{p,q}^{\alpha,\lambda}(\omega_1, \omega_2)} = \sup_{k_0 \in \mathbb{Z}} \left( \omega_1(B_{k_0})^{-\frac{\lambda}{p}} \left( \sum_{k = -\infty}^{k_0} \omega_1(B_k)^{\frac{p}{q} p} \| f \chi_k \|_{L^q(\mathbb{R}^n)}^{p} \right)^{\frac{1}{p}} \right). \]

It is obvious that for \( \lambda = 0 \), we have \( MK_{p,q}^{\alpha,0}(\omega_1, \omega_2) = \dot{K}_{q}^{\alpha,p}(\omega_1, \omega_2) \). Also, note that if we take \( \omega_1(x) = |B_0|^{-1} \), then \( MK_{p,q}^{\alpha,\lambda}(\omega_1, \omega_2) \) reduces to the usual one weighted Morrey-Herz space \( MK_{p,q}^{\alpha,\lambda}(\omega) \). For further the applications of these spaces in analysis, the readers can refer to the monograph [23].

**Definition 2.8.** Let \( \gamma \) be a real number. Let \( \mathcal{W}_\gamma \) be the set of all Lebesgue measurable functions \( \omega \) on \( \mathbb{R}^n \) such that \( \omega(x) > 0 \) for almost every where
x ∈ ℝ^n, 0 < \int_{S^{n-1}} \omega(x)\sigma(x) < ∞, and \omega is absolutely homogeneous of degree γ, that is, \omega(tx) = |t|^{\gamma}\omega(x) for all t ∈ ℝ\{0\}, x ∈ ℝ^n.

Let us denote \mathcal{W} = \bigcup_{\gamma} \mathcal{W}_\gamma. It is easy to see that \mathcal{W} contains strictly the class of power weights of the form |x|^\gamma. For further discussions, the readers can refer to [9] and [6]. Throughout the whole paper, we will denote by \omega a weight in \mathcal{W}_\gamma. Next, we recall the following result related to the class of weight functions \mathcal{W}_\gamma, which is used in the sequel.

**Lemma 2.9** ([6]). Let \omega ∈ \mathcal{W}_\gamma for γ > -n. Then, there exists a constant C = C(\omega, n) > 0 such that

\[ \omega(B_m) = C|B_m|^{\frac{\gamma + n}{n}} \quad \text{and} \quad \omega(C_m) = (1 - 2^{-\gamma - n})\omega(B_m), \]

for any m ∈ ℤ.

### 3. Main results and their proofs

Before stating our main results, we introduce some notations which will be used throughout this section. Assume that \Phi : ℝ^n → ℂ is a radial measurable function, that is, \Phi(x) = \Phi(|x|) for all x ∈ ℝ^n, and Ω : S^{n-1} → ℂ is a measurable function such that Ω(x) ≠ 0 for almost everywhere x in S^{n-1}. Let us recall that the rough Hausdorff operator is defined by

\[ \mathcal{H}_{\Phi, \Omega}(f)(x) = \int_{ℝ^n} \frac{\Phi(x|y|^{-1})}{|y|^n} \Omega(y|y|^{-1}) f(y)dy, \quad x ∈ ℝ^n. \quad (3.1) \]

Using polar coordinates and changing variables, it is easy to see that

\[ \mathcal{H}_{\Phi, \Omega}f(x) = \int_{0}^{+\infty} \left[ \int_{S^{n-1}} \frac{\Phi(t)}{t} \Omega(y')f(|x|t^{-1}y')d\sigma(y') \right] dt. \quad (3.2) \]

For \( b \in Lip^\beta(0 < \beta \leq 1) \), the commutator of Coifman-Rochberg-Weiss type of rough Hausdorff operator with the Lipschitz functions is defined as follows

\[ \mathcal{H}_{\Phi, \Omega}^b f(x) = b(x)\mathcal{H}_{\Phi, \Omega}f(x) - \mathcal{H}_{\Phi, \Omega}(bf)(x) \]

\[ = \int_{0}^{+\infty} \int_{S^{n-1}} \frac{\Phi(t)}{t} \Omega(y')f(|x|t^{-1}y') \left[ b(x) - b(|x|t^{-1}y') \right] d\sigma(y')dt, \quad (3.3) \]

where f : ℝ^n → ℂ are measurable functions.

Now, we are in a position to give the first our main results concerning the boundedness of the rough Hausdorff operator on the weighted Morrey spaces.
Theorem 3.1. Let $\gamma > -n, 1 \leq p < \infty, 1 + \lambda p > 0, \lambda \in \mathbb{R}$ and $\Omega \in L^{p'}(S^{n-1})$.

(i) If $\omega(x') \geq c > 0$ for all $x' \in S^{n-1}$, and

$$C_1 = \int_0^\infty \frac{|\Phi(t)|}{t^{1+(n+\gamma)\lambda}} dt < \infty,$$

we have $\mathcal{H}_{\Phi,\Omega}$ is a bounded operator on $\dot{B}^{p,\lambda}_\omega(\mathbb{R}^n)$. Moreover,

$$\|\mathcal{H}_{\Phi,\Omega}\|_{\dot{B}^{p,\lambda}_\omega(\mathbb{R}^n) \rightarrow B^{p,\lambda}_\omega(\mathbb{R}^n)} \lesssim C_1 \|\Omega\|_{L^{p'}(S^{n-1})}.$$

(ii) Conversely, suppose $\Omega \in L^{p'}(S^{n-1}, \omega(x') d\sigma(x'))$ and $\Phi$ is a real function with a constant sign in $\mathbb{R}^n$. Then, if $\mathcal{H}_{\Phi,\Omega}$ is bounded on $\dot{B}^{p,\lambda}_\omega(\mathbb{R}^n)$, we have $C_1 < \infty$. Furthermore,

$$\|\mathcal{H}_{\Phi,\Omega}\|_{\dot{B}^{p,\lambda}_\omega(\mathbb{R}^n) \rightarrow B^{p,\lambda}_\omega(\mathbb{R}^n)} \geq C_1 \frac{\|\Omega\|_{L^{p'}(S^{n-1})}}{\|\Omega\|_{L^{p'}(S^{n-1}, \omega(x') d\sigma(x'))}}.$$

Proof. (i) From (3.3) and by the Minkowski inequality, we have

$$\|\mathcal{H}_{\Phi,\Omega} f\|_{\dot{B}^{p,\lambda}_\omega(\mathbb{R}^n)}$$

$$= \sup_{R > 0} \left( \frac{1}{\omega(B(0, R))^{1+\lambda p}} \int_{B(0, R)} \left| \frac{\Phi(t)}{t} \Omega(y') f(|x| t^{-1/\lambda} y') d\sigma(y') \right|^p \omega(x) dx \right)^{1/p}$$

$$\leq \sup_{R > 0} \int_0^\infty \left( \int_{B(0, R)} \frac{1}{\omega(B(0, R))^{1+\lambda p}} \left| \frac{\Phi(t)}{t^p} \Omega(y') f(|x| t^{-1/\lambda} y') d\sigma(y') \right| \omega(x) dx \right)^{1/p} dt.$$

Using change of variable $u = xt^{-1}$, it is easy to show that

$$\|\mathcal{H}_{\Phi,\Omega} f\|_{\dot{B}^{p,\lambda}_\omega(\mathbb{R}^n)}$$

$$\leq \sup_{R > 0} \int_0^\infty \left( \int_{B(0, t^{-1} R)} \frac{|\Phi(t)|}{t^{1-p} \omega(B(0, R))^{1+\lambda p}} \int_{S^{n-1}} \Omega(y') f(|u| y') d\sigma(y') \right)^{1/p} \omega(u) du dt.$$

Note that, by the H"older inequality, we have

$$\int_{S^{n-1}} \Omega(y') f(|u| y') d\sigma(y') \leq \left( \int_{S^{n-1}} |f(|u| y')|^p d\sigma(y') \right)^{1/p} \left( \int_{S^{n-1}} |\Omega(y')|^{p'} d\sigma(y') \right)^{1/p}$$

$$= \left( \int_{S^{n-1}} |f(|u| y')|^p d\sigma(y') \right)^{1/p} \|\Omega\|_{L^{p'}(S^{n-1})}. \quad (3.4)$$
Thus, we obtain
\[
\| \mathcal{H}_{\Phi \cdot \Omega} f \|_{\dot{B}^0_{p, \lambda}(\mathbb{R}^n)} \lesssim \| \Omega \|_{L^p(S^{n-1})} \sup_{R > 0} \int_0^\infty \frac{|\Phi(t)|}{t^{1+\frac{n}{p}}} \Psi(t, R) dt, \tag{3.5}
\]
where \( \Psi(t, R) := \left( \int_{B(0, t^{-1}R)} \frac{1}{\omega(B(0, R))^{1+\lambda p}} \left( \int_{S^{n-1}} |f(|ru|^p)\omega(y') \right) \omega(u) du \right)^{\frac{1}{p}} \).

Now, by putting \( u = rx' \) and using the condition \( \omega(x') \geq c > 0 \) for all \( x' \in S^{n-1} \), we have
\[
\Psi(t) = \left( \frac{1}{\omega(B(0, R))^{1+\lambda p}} \int_{S^{n-1}} \int_0^{t^{-1}R} \left( \int_{S^{n-1}} |f(rx)|^p \omega(y') \right) \omega(rx) d\sigma(x) r^{n-1} dr \right)^{\frac{1}{p}}
\]
\[
\lesssim \omega(S^{n-1})^{\frac{1}{p}} \left( \frac{1}{\omega(B(0, R))^{1+\lambda p}} \int_0^{t^{-1}R} r^{\gamma+n-1} \left( \int_{S^{n-1}} |f(y')|^p \omega(y') \right) dr \right)^{\frac{1}{p}}
\]
\[
\lesssim \omega(S^{n-1})^{\frac{1}{p}} \left( \frac{1}{\omega(B(0, R))^{1+\lambda p}} \int_{B(0, t^{-1}R)} |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}}. \tag{3.6}
\]

We have
\[
\omega(B(0, t^{-1}R)) = \int_{B(0, t^{-1}R)} \omega(z) dz
\]
\[
= \int_{B(0, R)} t^{-(\gamma+n)} \omega(y) dy = t^{-(\gamma+n)} \omega(B(0, R)),
\]
so
\[
\frac{1}{\omega(B(0, R))^{1+\lambda p}} = \frac{1}{t^{(\gamma+n)(1+\lambda p)} \omega(B(0, t^{-1}R))^{1+\lambda p}}. \tag{3.7}
\]

Hence, from (3.5), (3.6) and (3.7), we obtain
\[
\| \mathcal{H}_{\Phi \cdot \Omega} f \|_{\dot{B}^0_{p, \lambda}(\mathbb{R}^n)} \lesssim \| \Omega \|_{L^p(S^{n-1})} \| f \|_{\dot{B}^0_{p, \lambda}(\mathbb{R}^n)} \int_0^\infty \frac{|\Phi(t)|}{t^{1+(\gamma+n)\lambda}} dt
\]
\[
\lesssim C \| \Omega \|_{L^p(S^{n-1})} \| f \|_{\dot{B}^0_{p, \lambda}(\mathbb{R}^n)}.
\]
which completes the proof of the part (i).

(ii) Conversely, suppose $\mathcal{H}\Phi,\Omega$ is bounded on the space $\dot{B}^{p,\lambda}_{p',\omega}(\mathbb{R}^n)$. As it is known, a standard approach to proving for the part (ii) of the theorem is to take a appropriately radial function. Here, let us also choose the function as follows

$$f(x) = |x|^{(n+\gamma)\lambda}|\Omega(x')|^{p'-2}\Omega(x'), \text{ for } x' = \frac{x}{|x|}.$$ 

We then have

$$\|f\|_{\dot{B}^{p,\lambda}_{p',\omega}(\mathbb{R}^n)} = \sup_{R > 0} \left( \frac{1}{\omega(B(0, R))^{1+\lambda p}} \int_{B(0, R)} |x|^{(n+\gamma)\lambda p}|\Omega(x')|(p'-2)^{p'}|\Omega(x')|^p\omega(x)dx \right)^{\frac{1}{p'}}$$

$$= \sup_{R > 0} \left( \frac{1}{\omega(B(0, R))^{1+\lambda p}} \int_{B(0, R)} |x|^{(n+\gamma)\lambda p}|\Omega(x')|^{p'}\omega(x)dx \right)^{\frac{1}{p'}}.$$ 

Since $\gamma > -n$, a simple computation shows that

$$\omega(B(0, R)) = \frac{R^{n+\gamma}}{n+\gamma}\omega(S^{n-1}),$$

and we get

$$\int_{B(0, R)} |x|^{(n+\gamma)\lambda p}|\Omega(x')|^{p'}\omega(x)dx = \int_0^R \int_{S^{n-1}} |r, x'|^{(n+\gamma)\lambda p}|\Omega(x')|^{p'}\omega(rx')r^{n-1}d\sigma(x')dr$$

$$= \left( \int_0^R r^{(n+\gamma)(1+\lambda p)-1}dr \right) \left( \int_{S^{n-1}} |\Omega(x')|^{p'}\omega(x')d\sigma(x') \right)$$

$$= \frac{R^{(n+\gamma)(1+\lambda p)}}{(n+\gamma)(1+\lambda p)} \|\Omega\|_{L^{p'}(S^{n-1}, \omega(x')d\sigma(x'))}.$$ 

Consequently,

$$\|f\|_{\dot{B}^{p,\lambda}_{p',\omega}(\mathbb{R}^n)} = \left( \frac{n+\gamma}{\omega(S^{n-1})} \right)^{\lambda} \frac{1}{(1+\lambda p)^{\frac{1}{p'}}} \frac{1}{\omega(S^{n-1})^{\frac{1}{p}} \|\Omega\|_{L^{p'}(S^{n-1}, \omega(x')d\sigma(x'))}^{\frac{1}{p'}}} < \infty.$$ 

On the other hand, by choosing $f$ as above, we get

$$\mathcal{H}\Phi,\Omega f(x) = \int_0^\infty \left( \int_{S^{n-1}} \frac{\Phi(t)}{t} \Omega(y') f(|x|t^{-1}y')d\sigma(y') \right) dt$$
Then, Corollary 3.2. Let \( \gamma > -n, 1 \leq p < \infty, 1 + \lambda p > 0, \lambda \in \mathbb{R} \). Suppose \( \Omega \in L^{p'}(S^{n-1}), \omega(x) = |x|^{\gamma} \) for \( \gamma > -n \), and \( \Phi \) is a nonnegative radial function. Then, \( \mathcal{H}_{\Phi, \Omega} \) is a bounded operator on \( B^{p, \lambda}_{\gamma}(\mathbb{R}^n) \) if and only if

\[
C_{1.1} = \int_0^\infty \frac{\Phi(t)}{t^{1+(n+\gamma)\lambda}} dt < \infty.
\]

Moreover,

\[
\|\mathcal{H}_{\Phi, \Omega}\|_{B^{p, \lambda}_{\gamma}(\mathbb{R}^n) \to B^{p, \lambda}_{\gamma}(\mathbb{R}^n)} \simeq C_{1.1} \cdot \|\Omega\|_{L^{p'}(S^{n-1})}.
\]
Next, we also give the boundedness and bound of the rough Hausdorff operator on the weighted Herz spaces.

**Theorem 3.3.** Let \(1 \leq p, q < \infty\) and \(\Omega \in L^{q'}(S^{n-1})\).

(i) If \(\omega(x') \geq c > 0\) for all \(x' \in S^{n-1}\) and
\[
C_2 = \int_0^\infty |\Phi(t^{-1})| t^{1-2n-\frac{2q}{q'}} dt < \infty,
\]
we have \(\mathcal{H}_{\Phi,\Omega}\) is a bounded operator on \(\dot{K}^{\alpha,p}_q(\omega)\). Moreover,
\[
\|\mathcal{H}_{\Phi,\Omega}\|_{\dot{K}^{\alpha,p}_q(\omega) \to \dot{K}^{\alpha,p}_q(\omega)} \leq C_2 \|\Omega\|_{L^{q'}(S^{n-1})}.
\]

(ii) Conversely, suppose \(\Omega \in L^{q'}(S^{n-1}, \omega(x')d\sigma(x'))\) and \(\Phi\) is a real function with a constant sign in \(\mathbb{R}^n\). Then, if \(\mathcal{H}_{\Phi,\Omega}\) is bounded on the space \(\dot{K}^{\alpha,p}_q(\omega)\), we have \(C_2 < \infty\). Furthermore,
\[
\|\mathcal{H}_{\Phi,\Omega}\|_{\dot{K}^{\alpha,p}_q(\omega) \to \dot{K}^{\alpha,p}_q(\omega)} \geq C_2 \frac{\|\Omega\|_{L^{q'}(S^{n-1}, \omega(x')d\sigma(x'))}}{\|\Omega\|_{L^{q'}(S^{n-1})}}.
\]

**Proof.** (i) For every \(k \in \mathbb{Z}\), by changing of variable \(u = t^{-1}\), we have
\[
\|\mathcal{H}_{\Phi,\Omega}f \chi_k\|_{q,\omega} = \left(\int_{\mathbb{R}^n} |\mathcal{H}_{\Phi,\Omega}f(x) \chi_k(x)|^q \omega(x) dx\right)^{\frac{1}{q}}
\]
\[
= \left(\int_{\mathbb{R}^n} \int_0^\infty \left(\int_{S^{n-1}} |\Phi(u^{-1})| u^{1-2n} |\Omega(y') f(|u|y')| d\sigma(y')\right) du \right)^{\frac{1}{q}} \omega(x) dx.
\]

By Minkowski’s inequality and changing of variable \(v = ux\), we obtain
\[
\|\mathcal{H}_{\Phi,\Omega}f \chi_k\|_{q,\omega} \leq \int_0^\infty \int_{\mathbb{R}^n} |\Phi(u^{-1})| u^{1-2n-\frac{2q}{q'}} \left(\int_{S^{n-1}} \Omega(y') f(|u|y') d\sigma(y')\right)^q \omega(v) dv du.
\]

On the other hand, by the Hölder inequality, we have the following estimate
\[
\int_{S^{n-1}} \Omega(y') f(|u|y') d\sigma(y') \leq \left(\int_{S^{n-1}} |f(|u|y')|^p d\sigma(y')\right)^{\frac{1}{p}} \left(\int_{S^{n-1}} |\Omega(y')|^q d\sigma(y')\right)^{\frac{1}{q}}
\]
\[
= \left(\int_{S^{n-1}} |f(|u|y')|^p d\sigma(y')\right)^{\frac{1}{p}} \|\Omega\|_{L^{q'}(S^{n-1})}.
\]
Therefore, by combining (3.8) and (3.9), one has

\[
\|H_{\Phi, \Omega} f \chi_k\|_{q, \omega} \\
\leq \int_0^\infty |\Phi(u^{-1})| u^{1-2n-\frac{2}{q} - \frac{2}{\gamma}} \left( \int_{uC_k} \left( \int_{S^{n-1}} |f(v)|^q d\sigma(y') \right) \right)^{\frac{1}{q}} \left( \frac{\|\Omega\|_{L^{q}(S^{n-1})}}{\Omega(v) dv} \right)^{\frac{1}{q}} du \\
= \|\Omega\|_{L^{q}(S^{n-1})} \int_0^\infty |\Phi(u^{-1})| u^{1-2n-\frac{2}{q} - \frac{2}{\gamma}} \mathcal{J}(u)^{\frac{1}{q}} du,
\]

where \( \mathcal{J}(u) := \int_{uC_k} \left( \int_{S^{n-1}} |f(v)|^q d\sigma(y') \right) \omega(v) dv \). By changing of variable \( v = rx' \) and using \( \omega(x') \geq c > 0 \) for all \( x' \in S^{n-1} \), we get

\[
\mathcal{J}(u) = \int_{uC_k} \int_{S^{n-1}} \left( \int_{S^{n-1}} |f(|rx'|)|^q d\sigma(y') \right) d\sigma(x') \omega(rx') r^{n-1} dr \\
= \omega(S^{n-1}) \int_{uC_k} r^{n-1+\gamma} \left( \int_{S^{n-1}} |f(ry')|^q d\sigma(y') \right) dr \\
\leq \int_{uC_k} r^{n-1+\gamma} \left( \int_{S^{n-1}} |f(ry')|^q \omega(y') d\sigma(y') \right) dr = \|f \chi_{uC_k}\|_{q, \omega}^q.
\]

Thus, we obtain

\[
\|H_{\Phi, \Omega} f \chi_k\|_{q, \omega} \leq \|\Omega\|_{L^{q}(S^{n-1})} \int_0^\infty |\Phi(u^{-1})| u^{1-2n-\frac{2}{q} - \frac{2}{\gamma}} \|f \chi_{uC_k}\|_{q, \omega} du.
\]

Noting that for each \( u \in (0, \infty) \), one can find an integer number \( \ell = \ell(u) \) such that \( 2^{\ell-1} < u \leq 2^\ell \). This implies that \( uC_k \) is a subset of \( C_{k+\ell-1} \cup C_{k+\ell} \). Thus, we obtain

\[
\|f \chi_{uC_k}\|_{q, \omega} \leq \|f \chi_{k+\ell-1}\|_{q, \omega} + \|f \chi_{k+\ell}\|_{q, \omega}.
\]

So, one has

\[
\|H_{\Phi, \Omega} f \chi_k\|_{q, \omega} \leq \|\Omega\|_{L^{q}(S^{n-1})} \int_0^\infty |\Phi(u^{-1})| u^{1-2n-\frac{2}{q} - \frac{2}{\gamma}} \left( \|f \chi_{k+\ell-1}\|_{q, \omega} + \|f \chi_{k+\ell}\|_{q, \omega} \right) dt.
\]

(3.10)
On the other hand, by $1 \leq p < \infty$, we have
\[
\|\mathcal{H}_\Phi \Omega f\|_{K^{\alpha,p}_q(\omega)} = \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|\mathcal{H}_\Phi \Omega f(x)\chi_k(x)\|_{q,\omega}^p \right)^{1/p} 
\]
\[
\lesssim \|\Omega\|_{L^q(S^{n-1})} \int_0^\infty \Phi(u) \left( u^{1-2n-\frac{2}{q}-\frac{n}{q}} \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha p} (\|f(x)\chi_{k+1}\|_{q,\omega} + \|f(x)\chi_{k+\ell}\|_{q,\omega})^p \right)^{1/p} \right) \, du.
\]
Since $2^{\ell-1} < u \leq 2^\ell$, it follows that
\[
\left( \sum_{k=-\infty}^{\infty} 2^{k\alpha p} (\|f(x)\chi_{k+1}\|_{q,\omega} + \|f(x)\chi_{k+\ell}\|_{q,\omega})^p \right)^{1/p} \leq \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f(x)\chi_{k+1}\|_{q,\omega}^p \right)^{1/p} + \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f(x)\chi_{k+\ell}\|_{q,\omega}^p \right)^{1/p} \leq (2^{(-\ell+1)\alpha} + 2^{-\ell\alpha}) \|f\|_{K^{\alpha,p}_q(\omega)} \lesssim u^{-\alpha} \|f\|_{K^{\alpha,p}_q(\omega)}.
\]
Consequently,
\[
\|\mathcal{H}_\Phi \Omega f\|_{K^{\alpha,p}_q(\omega)} \lesssim \|\Omega\|_{L^q(S^{n-1})} \cdot \|f\|_{K^{\alpha,p}_q(\omega)} \cdot \int_0^\infty \Phi(u) \left( u^{1-2n-\frac{2}{q}-\frac{n}{q}-\alpha} \right) \, du.
\]
This shows that the operator $\mathcal{H}_\Phi \Omega$ is boundedness on the space $K^{\alpha,p}_q(\omega)$ and $\|\mathcal{H}_\Phi \Omega\|_{K^{\alpha,p}_q(\omega) \to K^{\alpha,p}_q(\omega)} \lesssim C_2 \|\Omega\|_{L^q(S^{n-1})}$.

(ii) Now, we will give the proof for part (ii) of the theorem. For $m \in \mathbb{Z}$, we choose $m$ sufficiently large such that $\alpha + \frac{1}{2m} \neq 0$. Let us choose the functions
\[
f_m(x) = \begin{cases} 0, & \text{if } |x| < 1, \\ |x|^{-\frac{\alpha}{\frac{2}{q} - \frac{n}{q}}} \Omega(x'), & \text{if } |x| \geq 1. \end{cases}
\]
By similar argument as in [8], one can show that $f_m \in K^{\alpha,p}_q(\omega)$. But, for convenience to the reader, we provide details for the proof here. First, we remark that for $k \in \mathbb{Z}$, $k \geq 0$, we have
\[
\|f_m \chi_k\|_{q,\omega} \leq \left( \int_{C_k} \left| |x|^{-\frac{\alpha}{\frac{2}{q} - \frac{n}{q}} \Omega(x')|q-2\Omega(x')\chi_k(x)| \omega(x) \right|^q \, dx \right)^{1/q} \leq \left( \int_{C_k} \int_{S^{n-1}} \left| r x' \right|^{-\alpha q - n - \frac{2}{q} \alpha} \Omega(x') |q-2\Omega(x') \omega(r x') r^{n-1} \, d\sigma(x') \, dr \right)^{1/q}
\]
where $C_k \subset \mathbb{R}^n$. \qed
Because, for each $k$

$$\mathcal{H}_{\Phi,\Omega}f_m$$

$$= \begin{cases}
0, & \text{if } |x| < 1,

|x|^{-\alpha - \frac{q}{q'} - \frac{n}{2} - \frac{1}{4}} \int_{S(x)} \left( \int_{S(x)} \Phi(u^{-1})u^{1-2n-\frac{q}{q'} - \frac{1}{2}} \Omega(y')^\|d\sigma(y') \right) du, & \text{if } |x| \geq 1,
\end{cases}$$

where $S(x) = \{ u \in (0, \infty) : |x|u^{q'} \geq 1 \}$. For $k \in \mathbb{Z}$ such that $k \geq 1$, let $S_k = \left\{ u \in (0, \infty) : |u| \geq \frac{1}{2_k-1} \right\}$.

It is clear that the sequence $\{S_k\}_{k \geq 0}$ is increasing and tends to $(0, \infty)$. Let $1 \leq m \leq k$. Then, for all $x \in C_k$, there exits a measurable subset $A$ of $(0, \infty)$ with $|A| = 0$ such that $S(x) \supset S_m \setminus A$.

Because, for each $k \leq 0$, $\mathcal{H}_{\Phi,\Omega}f_m \chi_k = 0$, we have

$$\| \mathcal{H}_{\Phi,\Omega}f_m \chi_k \|_{q,\omega}$$
It is clear that $\| f'_m \chi_k \|_{L^q(S^{n-1})} = 0$ for all $k \leq 0$. Therefore,

$$\| \mathcal{H}_{\Phi, \Omega} f_m \|_{K^\alpha,p_q(\omega)} \geq \left( \sum_{k=-\infty}^{\infty} 2^{kpq} \int_{S^{n-1}} |\Phi(u^{-1})| u^{1-2n-\frac{7}{4}-\frac{3}{4p} - \frac{3}{2m}} du |f'_m \chi_k|_{q,\omega} \|\Omega\|^q_{L^q(S^{n-1})} \right)^\frac{1}{p} \geq \|\Omega\|^q_{L^q(S^{n-1})} \left( \sum_{k=-\infty}^{\infty} 2^{kpq} \|f'_m \chi_k\|_{q,\omega} \right)^\frac{1}{p} \left( \int_{S^{n-1}} |\Phi(u^{-1})| u^{1-2n-\frac{7}{4}-\frac{3}{4p} - \frac{3}{2m}} du \right)^\frac{1}{p} \geq \|\Omega\|^q_{L^q(S^{n-1})} \sum_{k=-\infty}^{\infty} 2^{kpq} \left( \sum_{k=m}^{\infty} 2^{kpq} |f'_m \chi_k|_{q,\omega} \right)^\frac{1}{p} \left( \int_{S^{n-1}} |\Phi(u^{-1})| u^{1-2n-\frac{7}{4}-\frac{3}{4p} - \frac{3}{2m}} du \right)^\frac{1}{p} \geq \|\Omega\|^q_{L^q(S^{n-1})} 2^{-\frac{7}{4} - \frac{3}{4p} - \frac{3}{2m}} \left( \sum_{k=0}^{\infty} 2^{-\frac{kpq}{2}} \right)^\frac{1}{p} \left( \int_{S^{n-1}} |\Phi(u^{-1})| u^{1-2n-\frac{7}{4}-\frac{3}{4p} - \frac{3}{2m}} du \right)^\frac{1}{p} \geq \|\Omega\|^q_{L^q(S^{n-1})} 2^{-\frac{7}{4} - \frac{3}{4p} - \frac{3}{2m}} \sum_{k=0}^{\infty} 2^{-\frac{kpq}{2}} \left( \frac{2^{q(pq+\alpha)} - 1}{(\frac{1}{2m} + \alpha)q} \right)^\frac{1}{p} C_2(m),$$

where $C_2(m) := \int_{S^{n-1}} |\Phi(u^{-1})| u^{1-2n-\frac{7}{4}-\frac{3}{4p} - \frac{3}{2m}} du$. Now, since the operator $\mathcal{H}_{\Phi, \Omega}$ is bounded on the space $K^\alpha,p_q(\omega)$, we yield

$$\| \mathcal{H}_{\Phi, \Omega} f_m \|_{K^\alpha,p_q(\omega)} \geq \frac{\| \mathcal{H}_{\Phi, \Omega} f_m \|_{K^\alpha,p_q(\omega)}}{\| f_m \|_{K^\alpha,p_q(\omega)}}.$$
Theorem 3.5. Let \( \Omega \) be a nonnegative radial function. Then, \( \mathcal{H}_{\Phi,\Omega} \) is a bounded operator on \( \dot{K}^{\alpha,p}_q(\omega) \) if and only if
\[
C_{2,1} = \int_0^\infty |\Phi(t^{-1})| t^{1-2n-\gamma - \frac{\alpha}{\gamma}} \, dt < \infty.
\]
Moreover,
\[
\|\mathcal{H}_{\Phi,\Omega}\|_{\dot{K}^{\alpha,p}_q(\omega)} \simeq C_{2,1} \cdot \|\Omega\|_{L^{q'}(S^{n-1})}.
\]

Next, we also give the boundedness and bound of the rough Hausdorff operator on the weighted Morrey-Herz spaces.

Theorem 3.6. Let \( 1 \leq q < \infty, 0 < p < \infty, \gamma \in \mathbb{R}, \lambda > 0, \) and \( \Omega \in L^{q'}(S^{n-1}) \).
(i) If \( \omega(x') \geq c > 0 \) for all \( x' \in S^{n-1} \) and
\[
C_3 = \int_0^\infty \frac{|\Phi(t)|}{t^{1-\frac{\gamma}{q} - \frac{\alpha}{p} + \lambda - \alpha}} \, dt < \infty,
\]
then \( \mathcal{H}_{\Phi,\Omega} \) is a bounded operator on \( MK^{\alpha,\lambda}_{p,q}(\omega) \). Moreover,
\[
\|\mathcal{H}_{\Phi,\Omega}\|_{MK^{\alpha,\lambda}_{p,q}(\omega)} \lesssim C_3 \|\Omega\|_{L^{q'}(S^{n-1})}.
\]
(ii) Conversely, suppose \( \Omega \in L^{q'}(S^{n-1}, \omega(x')d\sigma(x')) \) and \( \Phi \) is a real function with a constant sign in \( \mathbb{R}^n \). Then, if \( \mathcal{H}_{\Phi,\Omega} \) is bounded on the \( MK^{\alpha,\lambda}_{p,q}(\omega) \), then

By Theorem 3.3, we also have the following useful corollary.

Corollary 3.4. Let \( 1 \leq p, q < \infty \) and \( \Omega \in L^{q'}(S^{n-1}), \omega(x) = |x|^{\gamma} \). Let \( \Phi \) be a nonnegative radial function. Then, \( \mathcal{H}_{\Phi,\Omega} \) is a bounded operator on \( \dot{K}^{\alpha,p}_q(\omega) \) if and only if
\[
C_{2,1} = \int_0^\infty |\Phi(t^{-1})| t^{1-2n-\gamma - \frac{\alpha}{\gamma}} \, dt < \infty.
\]
Moreover,
\[
\|\mathcal{H}_{\Phi,\Omega}\|_{\dot{K}^{\alpha,p}_q(\omega)} \simeq C_{2,1} \cdot \|\Omega\|_{L^{q'}(S^{n-1})}.
\]
$C_3 < \infty$. Furthermore,

$$\|H_{\Phi, \Omega}\|_{MK^\alpha_{p,q}(\omega) \to MK^\alpha_{p,q}(\omega)} \geq C_3 \frac{\|\Omega\|_{L^q(S^{n-1})}^\gamma}{\|\Omega\|_{L^q(S^{n-1}, \omega(x')d\sigma(x'))}^{\gamma/3}}.$$  

**Proof.** (i) From the Minkowski inequality and changing variable $u = xt^{-1}$, we obtain

$$\|H_{\Phi, \Omega}f\chi_k\|_{q, \omega} \leq \int_0^\infty \frac{|\Phi(t)|}{t} \left( \int_{C_k} \left( \int_{S^n-1} \Omega(y') f(|x|^{-1}y') d\sigma(y') \right)^q \omega(x) dx \right)^{\frac{1}{q}} dt$$

$$= \int_0^\infty \frac{|\Phi(t)|}{t^{1 - \frac{n}{q} - \frac{\gamma}{q}}} \left( \int_{C_k} \left( \int_{S^n-1} \Omega(y') f(|u|y') d\sigma(y') \right)^q \omega(u) du \right)^{\frac{1}{q}} dt.$$  

(3.11)

By (3.9) and (3.11), it follows that

$$\|H_{\Phi, \Omega}f\chi_k\|_{q, \omega} \leq \int_0^\infty \frac{|\Phi(t)|}{t^{1 - \frac{n}{q} - \frac{\gamma}{q}}} \left( \int_{C_k} \left( \int_{S^n-1} |f(|u|y')|^q d\sigma(y') \right)^\frac{1}{q} \|\Omega\|_{L^q(S^{n-1})}^\gamma \omega(u) du \right)^{\frac{1}{q}} dt$$

$$\leq \|\Omega\|_{L^q(S^{n-1})} \int_0^\infty \frac{|\Phi(t)|}{t^{1 - \frac{n}{q} - \frac{\gamma}{q}}} \mathcal{J}'(t)^{\frac{1}{q}} dt,$$

where $\mathcal{J}'(t) := \int_{C_k} \left( \int_{S^n-1} |f(|u|y')|^q d\sigma(y') \right) \omega(u) du$. By the similar estimate as $\mathcal{J}(u)$, we also have

$$\mathcal{J}'(t) \lesssim \|f\chi_{\frac{1}{t}C_k}\|_{q, \omega}^q.$$

Note that for each $t \in (0, \infty)$, we can find an integer number $\ell = \ell(t)$ such that $2^{\ell-1} < \frac{1}{t} \leq 2^\ell$. This implies that $\frac{1}{t}C_k$ is a subset of $C_{k+\ell-1} \cup C_{k+\ell}$. Thus, we obtain

$$\|f\chi_{\frac{1}{t}C_k}\|_{q, \omega} \leq \|f\chi_{k+\ell-1}\|_{q, \omega} + \|f\chi_{k+\ell}\|_{q, \omega}.$$
Hence,

\[
\| \mathcal{H}_{\Phi,\Omega} f \chi_k \|_{q,\omega} \lesssim \| \Omega \|_{L^p(S^{n-1})} \int_0^\infty \left| \frac{\Phi(t)}{t^{1-\frac{n}{p}+\frac{\lambda}{2}}} \right|^p \left( \| f \chi_{k+\ell-1} \|_{q,\omega} + \| f \chi_{k+\ell} \|_{q,\omega} \right) dt. 
\]

(3.12)

We consider two case as follows.

Case 1: \( 1 \leq p < \infty \). Then, we get

\[
\| \mathcal{H}_{\Phi,\Omega} f \|_{MK^{p,\lambda}_{\varphi}\varrho}(\omega)
= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k_0 p} \| \mathcal{H}_{\Phi,\Omega} f(x) \chi_k(x) \|_{p,\omega}^p \right)^{\frac{1}{p}} 
\lesssim \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k_0 p} \left( \| \Omega \|_{L^p(S^{n-1})} \int_0^\infty \left| \frac{\Phi(t)}{t^{1-\frac{n}{p}+\frac{\lambda}{2}}} \right|^p \left( \| f \chi_{k+\ell-1} \|_{q,\omega} + \| f \chi_{k+\ell} \|_{q,\omega} \right) dt \right) \right)^{\frac{1}{p}} 
\lesssim \| \Omega \|_{L^p(S^{n-1})} \int_0^\infty \left| \frac{\Phi(t)}{t^{1-\frac{n}{p}+\frac{\lambda}{2}}} \right| \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k_0 p} \left( \| f \chi_{k+\ell-1} \|_{q,\omega} + \| f \chi_{k+\ell} \|_{q,\omega} \right) \right)^{\frac{1}{p}} dt.
\]

It is clear that

\[
\sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k_0 p} \left( \| f \chi_{k+\ell-1} \|_{q,\omega} + \| f \chi_{k+\ell} \|_{q,\omega} \right) \right)^{\frac{1}{p}} 
\leq \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k_0 p} \| f \chi_{k+\ell-1} \|_{p,\omega}^p \right)^{\frac{1}{p}} + \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k_0 p} \| f \chi_{k+\ell} \|_{q,\omega}^p \right)^{\frac{1}{p}} 
\lesssim 2^{\lambda(\lambda-\alpha)} \| f \|_{MK^{p,\lambda}_{\varphi}\varrho}(\omega) \lesssim \left( \frac{1}{t} \right)^{\lambda-\alpha} \| f \|_{MK^{p,\lambda}_{\varphi}\varrho}(\omega).
\]

Consequently,

\[
\| \mathcal{H}_{\Phi,\Omega} f \|_{MK^{p,\lambda}_{\varphi}\varrho}(\omega) \lesssim \| \Omega \|_{L^p(S^{n-1})} \int_0^\infty \left| \frac{\Phi(t)}{t^{1-\frac{n}{p}+\frac{\lambda}{2}}} \right| \left( \frac{1}{t} \right)^{\lambda-\alpha} \| f \|_{MK^{p,\lambda}_{\varphi}\varrho}(\omega) dt
\lesssim \| \Omega \|_{L^p(S^{n-1})} \| f \|_{MK^{p,\lambda}_{\varphi}\varrho}(\omega) \int_0^\infty \left| \frac{\Phi(t)}{t^{1-\frac{n}{p}+\frac{\lambda}{2}+\lambda-\alpha}} \right| dt.
\]

Case 2: \( 0 < p < 1 \). It follows from the definition of weighted Morrey-Herz space that

\[
\| f \chi_k \|_{q,\omega} \leq 2^{k(\lambda-\alpha)} \| f \|_{MK^{p,\lambda}_{\varphi}\varrho}(\omega).
\]
For all $f \in M\hat{K}_{p,q}^{\alpha,\lambda}(\omega)$, by (3.12), we obtain
\[
\|\mathcal{H}_\Phi f\|_{L^p(S^{n-1})} \lesssim \|\Omega\|_{L^{q'}(S^{n-1})} \int_0^\infty \frac{|\Phi(t)|}{t^\frac{1}{q} - \frac{n}{q}} \left( \sum_{i=-1,0} 2^{(k+i)(\lambda-\alpha)} \|f\|_{M\hat{K}_{p,q}^{\alpha,\lambda}(\omega)} \right) dt,
\]
for all $k \in \mathbb{Z}$. Thus,
\[
\|\mathcal{H}_\Phi f\|_{M\hat{K}_{p,q}^{\alpha,\lambda}(\omega)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left( \sum_{k_{-\infty}}^{k_0} 2^{k_0\lambda} \|\mathcal{H}_\Phi f\|_{L^p(S^{n-1})} \|\Omega\|_{L^{q'}(S^{n-1})} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \times \left( \sum_{k=-\infty}^{k_0} 2^{k_0\lambda} \left( \int_0^\infty \frac{|\Phi(t)|}{t^{1-\frac{n}{q} - \frac{\lambda}{q}}} \left( \sum_{i=-1,0} 2^{(k+i)(\lambda-\alpha)} \right) dt \right) \right)^\frac{1}{p} \right)^\frac{1}{p}.
\]
Since $2^{q-1} < \frac{1}{t} \leq 2^q$ and $\lambda > 0$, we estimate
\[
\sum_{i=-1,0} 2^{(k+i)(\lambda-\alpha)} \lesssim \left( \frac{1}{t} \right)^{\lambda-\alpha} 2^{k(\lambda-\alpha)} \sum_{i=-1,0} 2^{i(\lambda-\alpha)}.
\]
Therefore,
\[
\|\mathcal{H}_\Phi f\|_{M\hat{K}_{p,q}^{\alpha,\lambda}(\omega)} \lesssim \|\Omega\|_{L^{q'}(S^{n-1})} \|f\|_{M\hat{K}_{p,q}^{\alpha,\lambda}(\omega)} \sum_{k_0 \in \mathbb{Z}} \left( \sum_{k=-\infty}^{k_0} 2^{(k-k_0)\lambda p} \left( \int_0^\infty \frac{|\Phi(t)|}{t^{1-\frac{n}{q} - \frac{\lambda}{q} + \lambda-\alpha}} dt \right) \right)^\frac{1}{p} \sum_{i=-1,0} 2^{i(\lambda-\alpha)}
\]
\[
\lesssim \|\Omega\|_{L^{q'}(S^{n-1})} \|f\|_{M\hat{K}_{p,q}^{\alpha,\lambda}(\omega)} \sum_{k_0 \in \mathbb{Z}} \left( \sum_{k=-\infty}^{k_0} 2^{(k-k_0)\lambda p} \left( \int_0^\infty \frac{|\Phi(t)|}{t^{1-\frac{n}{q} - \frac{\lambda}{q} + \lambda-\alpha}} dt \right) \right)^\frac{1}{p} \sum_{i=-1,0} 2^{i(\lambda-\alpha)}
\]
\[
\lesssim \|\Omega\|_{L^{q'}(S^{n-1})} \|f\|_{M\hat{K}_{p,q}^{\alpha,\lambda}(\omega)} \int_0^\infty \frac{|\Phi(t)|}{t^{1-\frac{n}{q} - \frac{\lambda}{q} + \lambda-\alpha}} dt.
\]

(ii) Conversely, suppose $\mathcal{H}_\Phi$ is bounded on the space $M\hat{K}_{p,q}^{\alpha,\lambda}(\omega)$. Then, let us choose the function
\[
f(x) = |x|^{-\alpha - \frac{n}{q} - \frac{\lambda}{q} + \lambda} |\Omega(x')|^{q'-2} |\Omega(x')^\alpha |^{q'.}
\]
We have
\[
\|f \chi_k\|_{\Omega,\omega} = \left( \int_{\mathbb{R}^n} |x|^{-\alpha - \frac{n}{q} - \frac{\lambda}{q} + \lambda} |\Omega(x')|^{q'-2} |\Omega(x')^\alpha |^{q'} |\omega(x) dx \right)^\frac{1}{q}.
\]
\[
\begin{align*}
&= \left( \int_{\mathbb{R}^n} \int_{S^{n-1}} r^{-\alpha q - n + \lambda q} |\Omega(x')| q r^\gamma \omega(x') r^{n-1} d\sigma(x') dr \right)^{1/q} \\
&= \left( \int_{\mathbb{R}^n} \int_{S^{n-1}} r^{-\alpha q + \lambda q - 1} dr \int_{S^{n-1}} |\Omega(x')| q r^\gamma \omega(x') d\sigma(x') \right)^{1/q} \\
&= \left( \int_{\mathbb{R}^n} r^{-\alpha q + \lambda q - 1} dr \right)^{1/q} \| \Omega \|_{\mathbb{L}^q(S^{n-1}, \omega(x') d\sigma(x'))} \\
&= \begin{cases} 
\ln 2 \| \Omega \|_{\mathbb{L}^q(S^{n-1}, \omega(x') d\sigma(x'))}, & \text{if } \alpha = \gamma, \\
2^{k(\lambda - \alpha)} \left| \frac{1 - 2^{-q(\lambda - \alpha)}}{q(\lambda - \alpha)} \right| \| \Omega \|_{\mathbb{L}^q(S^{n-1}, \omega(x') d\sigma(x'))}, & \text{if } \alpha \neq \gamma.
\end{cases}
\end{align*}
\]

Therefore, an easy computation shows that

\[
\|f\|_{MK_{p,q}^{\alpha,\lambda}(\omega)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k = -\infty}^{k_0} 2^{k \alpha p} \| f \chi_k \|_{q,\omega}^p \right)^{1/p} \\
\lesssim \| \Omega \|_{\mathbb{L}^q(S^{n-1}, \omega(x') d\sigma(x'))} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k = -\infty}^{k_0} 2^{k \alpha p} (2^{k(\lambda - \alpha)})^p \right)^{1/p} \\
\lesssim \| \Omega \|_{\mathbb{L}^q(S^{n-1}, \omega(x') d\sigma(x'))} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k = -\infty}^{k_0} 2^{k \lambda p} \right)^{1/p} < \infty.
\]

On the other hand, we also get

\[
\mathcal{H}_{\Phi,\Omega} f(x) = \int_0^\infty \left( \int_{S^{n-1}} \frac{\Phi(t)}{t} \Omega(y') f(|x| t^{-\gamma} y') d\sigma(y') \right) dt \\
= |x|^{-\alpha - \frac{n}{q} + \lambda} \int_0^\infty \frac{\Phi(t)}{t^{\gamma + \frac{n}{q} - \lambda}} \left( \int_{S^{n-1}} |\Omega(y')| q r^\gamma d\sigma(y') \right) dt \\
= |x|^{-\alpha - \frac{n}{q} + \lambda} \| \Omega \|_{\mathbb{L}^q(S^{n-1})} \int_0^\infty \frac{\Phi(t)}{t^{1 - \gamma + \lambda - \alpha}} dt.
\]
Hence, it immediately follows that
\[
\|\mathcal{H}_\Phi \Omega f\|_{MK_{p,q}^{\alpha,\lambda}(\omega)} \simeq \|x|^{-\alpha - \frac{n}{q} + \frac{2}{q} + \lambda}\|MK_{p,q}^{\alpha,\lambda}(\omega)\| \Omega\|_{L^q(S^{n-1})}^q \int_0^\infty \frac{\Phi(t)}{t^{1 - \frac{n}{q} + \frac{2}{q} + \lambda - \alpha}} dt.
\]

Therefore,
\[
\|\mathcal{H}_\Phi \Omega\|_{MK_{p,q}^{\alpha,\lambda}(\omega)\to MK_{p,q}^{\alpha,\lambda}(\omega)} \simeq \frac{\|\mathcal{H}_\Phi \Omega f\|_{MK_{p,q}^{\alpha,\lambda}(\omega)}}{\|f\|_{MK_{p,q}^{\alpha,\lambda}(\omega)}} \geq \frac{\int_0^\infty \frac{\Phi(t)}{t^{1 - \frac{n}{q} + \frac{2}{q} + \lambda - \alpha}} dt \|\Omega\|_{L^q(S^{n-1})}^q \|x|^{-\alpha - \frac{n}{q} + \frac{2}{q} + \lambda}\|MK_{p,q}^{\alpha,\lambda}(\omega)}{\|\Omega\|_{L^q(S^{n-1})}^q \|x|^{-\alpha - \frac{n}{q} + \frac{2}{q} + \lambda}\|MK_{p,q}^{\alpha,\lambda}(\omega)} \geq \frac{\int_0^\infty \frac{\Phi(t)}{t^{1 - \frac{n}{q} + \frac{2}{q} + \lambda - \alpha}} dt \|\Omega\|_{L^q(S^{n-1})}^q}{\|\Omega\|_{L^q(S^{n-1})}^q}.
\]

This ends the proof of theorem. \(\square\)

By Theorem 3.5, we have the following useful corollary when \(\omega\) is a power weight function and \(\Phi\) is a nonnegative function.

**Corollary 3.6.** Let \(1 \leq q < \infty, 0 \leq p < \infty, \gamma \in \mathbb{R}, \lambda > 0\). Suppose \(\Omega \in L^q(S^{n-1}), \omega(x) = |x|^\gamma, \) and \(\Phi\) is a nonnegative radial function. Then, \(\mathcal{H}_\Phi \Omega\) is a bounded operator on \(MK_{p,q}^{\alpha,\lambda}(\omega)\) if and only if
\[
\mathcal{C}_{3.1} = \int_0^\infty \frac{\Phi(t)}{t^{1 - \frac{n}{q} + \frac{2}{q} + \lambda - \alpha}} dt < \infty.
\]

Moreover,
\[
\|\mathcal{H}_\Phi \Omega\|_{MK_{p,q}^{\alpha,\lambda}(\omega)} \simeq \mathcal{C}_{3.1} \|\Omega\|_{L^q(S^{n-1})}.
\]

Next, we will give the boundedness of the commutator of rough Hausdorff operator on weighted spaces of Morrey-Herz type with their symbols \(b\) belonging to Lipschitz space \(\text{Lip}^\beta(\mathbb{R}^n)\) \((0 < \beta \leq 1)\). Before stating our next results, we want to give the following useful inequality
\[
|b(x) - b(|x|^{-1}y')| \leq \|b\|_{\text{Lip}^\beta} \|x - |x|^{-1}y'\|^\beta = \|b\|_{\text{Lip}^\beta} \|x\|^\beta (1 + t^{-1})^\beta, \forall t > 0, y' \in S^{n-1}.
\]  \(3.13\)

**Theorem 3.7.** Let \(1 \leq p < \infty, \omega_1, \omega_2 \in \mathcal{W}_\gamma\) for \(\gamma > -n\), and \(b \in \text{Lip}^\beta(\mathbb{R}^n)\) for \(0 < \beta \leq 1\). Let \(\Omega \in L^q(S^{n-1})\) and \(\omega_2(x') \geq c > 0\) for all \(x' \in S^{n-1}\).
Suppose that $\lambda_1 = \lambda - \frac{2p}{n+7} > 0$. Then, if

$$C_4 = \int_0^\infty \frac{\Phi(t)}{t^{1+(\gamma+n)\frac{\lambda_1-1}{p}}(1+t)^{-\beta}} dt < \infty,$$

the commutator $\mathcal{H}^{b}_{\Phi, \Omega}$ is a bounded operator from $\dot{B}^{p,\lambda_1}(\omega_1, \omega_2)$ to $\dot{B}^{p,\lambda}(\omega_1, \omega_2)$.

**Proof.** It is easy to see that for any $x \in B(0, R)$, then $|x|^{\beta} \leq |B(0, R)|^{\frac{\beta}{p}}$. It is also important to note that $\omega_2(B(0, R)) \simeq |B(0, R)|^{\frac{\gamma-n}{p}}$ for all $\gamma > -n$. From this and by (3.13) above, for all $f \in \dot{B}^{p,\lambda_1}(\omega_1, \omega_2)$ we have

$$\|\mathcal{H}^{b}_{\Phi, \Omega} f\|_{\dot{B}^{p,\lambda}(\omega_1, \omega_2)} = \sup_{R > 0} \left( \frac{1}{\omega_2(B(0, R))^{\lambda_1}} \int_{B(0, R)} \left| \mathcal{H}^{b}_{\Phi, \Omega} f \right|^p \omega_1(x) dx \right)^{\frac{1}{p}}$$

$$\leq \|b\|_{Lip^p} \sup_{R > 0} \left( \frac{1}{\omega_2(B(0, R))^{\lambda_1}} \int_{B(0, R)} \frac{\Phi(t)}{t(1+t)^{-\beta}} \Omega(y') f(|x|t^{-1}y') \right)^{\frac{1}{p}}$$

$$\times |x|^{\beta} d\sigma(y') dt \left| \omega_1(x) dx \right|^{\frac{1}{p}},$$

where $\lambda_1 = \lambda - \frac{2p}{n+7}$. Now, using the Minkowski inequality and changing variable $u = xt^{-1}$, we get

$$\|\mathcal{H}^{b}_{\Phi, \Omega} f\|_{\dot{B}^{p,\lambda}(\omega_1, \omega_2)}$$

$$\lesssim \|b\|_{Lip^p} \sup_{R > 0} \int_0^\infty \left| \frac{\Phi(t)}{t(1+t)^{-\beta}} \left( \frac{1}{\omega_2(B(0, R))^{\lambda_1}} \int_{B(0, R)} \left| \Omega(y') f(|x|t^{-1}y') \right| d\sigma(y') \right|^{p}$$

$$\times \omega_1(x) dx \right|^{\frac{1}{p}} dt$$

$$\lesssim \|b\|_{Lip^p} \sup_{R > 0} \int_0^\infty \left| \frac{\Phi(t)}{t^{1+\frac{\beta}{p}}(1+t)^{-\beta}} \left( \frac{1}{\omega_2(B(0, R))^{\lambda_1}} \int_{B(0, t^{-1}R)} \left| \int_{S^{n-1}} \Omega(y') f(|u|y') d\sigma(y') \right|^{p}$$

$$\times \omega_1(u) du \right|^{\frac{1}{p}} dt.$$
It follows from (3.4) that

\[
\| \mathcal{H}_\Phi f \|_{B^{p,\lambda}(\omega_1,\omega_2)} \lesssim \| b \|_{Lip} \| \Omega \|_{L^p(S^{n-1})} \sup_{R > 0} \int_0^\infty \frac{|\Phi(t)|}{t^{1+\frac{n+1}{p} - \frac{n}{p}(1 + t^{-1}) - \beta}} F(t) dt,
\]

(3.14)

where \( F(t) := \left( \frac{1}{\omega_2(B(0, R))^{\lambda_1}} \int_{B(0, R)} \left( \int_{S^{n-1}} |f(|x|^2 \Omega_y | y')|^p d\sigma(y') \right) \omega_1(u) du \right)^{\frac{1}{p}} \).

Now, we put \( u = r x' \), so

\[
F(t) = \left( \frac{1}{\omega_2(B(0, R))^{\lambda_1}} \int_{B(0, R) S^{n-1}} \left( \int_{S^{n-1}} |f(|r x'|^2 \Omega_y | y')|^p d\sigma(y') \right) d\sigma(x') \omega_1(r x') r^{\gamma+n-1} dr \right)^{\frac{1}{p}}
\]

\[= \omega(S^{n-1})^{\frac{1}{p}} \left( \frac{1}{\omega_2(B(0, R))^{\lambda_1}} \int_{B(0, R)} r^{\gamma+n-1} \left( \int_{S^{n-1}} |f(|r|^2 \Omega_y | y')|^p d\sigma(y') \right) dr \right)^{\frac{1}{p}}
\]

\[\lesssim \left( \frac{1}{\omega_2(B(0, R))^{\lambda_1}} \int_{B(0, R)} r^{\gamma+n-1} \left( \int_{S^{n-1}} |f(|r|^2 \Omega_y | y')|^p d\sigma(y') \right) dr \right)^{\frac{1}{p}}. \quad (3.15)
\]

Note that we have \( \frac{1}{\omega_2(B(0, R))^{\lambda_1}} = \frac{1}{t^{(\gamma+n)\lambda_1} \omega_2(B(0, t^{-1} R))^{\lambda_1}} \). Hence, by (3.14), (3.15) and the condition \( \omega_1(x') > c > 0 \) for all \( x' \in S^{n-1} \), we obtain

\[
\| \mathcal{H}_\Phi f \|_{B^{p,\lambda}(\omega_1,\omega_2)} \lesssim \| b \|_{Lip} \| \Omega \|_{L^p(S^{n-1})} \sup_{R > 0} \int_0^\infty \frac{|\Phi(t)|}{t^{1+\frac{n+1}{p} - \frac{n}{p}(1 + t^{-1}) - \beta}} \left( \frac{1}{t^{(\gamma+n)\lambda_1} \omega_2(B(0, t^{-1} R))^{\lambda_1}} \right) 
\]

\[\times \int_{B(0, t^{-1} R)} r^{\gamma+n-1} \left( \int_{S^{n-1}} |f(|r|^2 \Omega_y | y')|^p \omega_1(y') d\sigma(y') \right) dr \right)^{\frac{1}{p}} dt
\]

\[\lesssim \| b \|_{Lip} \| \Omega \|_{L^p(S^{n-1})} \int_0^\infty \frac{|\Phi(t)|}{t^{1+\frac{n+1}{p} - \frac{n}{p}(1 + t^{-1}) - \beta}} \sup_{R > 0} \left( \frac{1}{\omega_2(B(0, t^{-1} R))^{\lambda_1}} \right) dt.
\]
\begin{align*}
\times \int_{B(0,t^{-1}R)} r^{\gamma+n-1} \left( \int_{\mathbb{S}^{n-1}} |f(|r| y')|^p \omega_1(y') \, d\sigma(y') \right)^{\frac{1}{p}} \, dt \\
\lesssim \|b\|_{Lip^\beta} \|\Omega\|_{L^{p'}(\mathbb{S}^{n-1})} \|f\|_{\tilde{B}^{\alpha_1}(\omega_1,\omega_2)} \cdot \int_0^\infty \frac{|\Phi(t)|}{t^{1+\gamma+n-1} \omega_1(1+t^{-1})^{-\beta}} \, dt.
\end{align*}

This implies that the commutator $\mathcal{H}^b_{\Phi,\Omega}$ is determined as a bounded operator from $\tilde{B}^{\alpha_1}(\omega_1,\omega_2)$ to $\tilde{B}^{\alpha_2}(\omega_1,\omega_2)$. The proof of the theorem is completed. \(\square\)

Finally, it is also interesting to give the boundedness of the commutator $\mathcal{H}^b_{\Phi,\Omega}$ on the two weighted Herz type spaces and on the two weighted Morrey-Herz type spaces. More precisely, we have the results as follows.

**Theorem 3.8.** Let $1 \leq p < \infty, 1 \leq q < \infty, 0 < \beta \leq 1, \gamma > -n$ and $\alpha_1 = \alpha_2 + \frac{n\beta}{n+\gamma}$. Suppose $b \in Lip^\beta(\mathbb{R}^n), \omega_1, \omega_2 \in \mathcal{W}_\gamma$ with $\omega_2(x') \geq c > 0$ for all $x' \in S^{n-1}, \Omega \in L^q(S^{n-1})$, and

\[ C_3 = \int_0^\infty \frac{|\Phi(t)|}{t^{1-\frac{\gamma}{q} - \frac{n\beta}{n+\gamma} - \alpha_1(1+\beta)}(1+t^{-1})^{-\beta}} \, dt < \infty. \]

Then the commutator $\mathcal{H}^b_{\Phi,\Omega}$ is a bounded operator from $\dot{K}^{\alpha_1,p}_q(\omega_1,\omega_2)$ to $\dot{K}^{\alpha_2,p}_q(\omega_1,\omega_2)$.

**Proof.** Let $f \in \dot{K}^{\alpha_1,p}_q(\omega_1,\omega_2)$. For any $k \in \mathbb{Z}$, by (3.13) and the Minkowski inequality, we get

\[ \|\mathcal{H}^b_{\Phi,\Omega} f \chi_k\|_{q,\omega_2} \]

\[ = \left( \int_{\mathcal{C}_k} \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{\Phi(t)}{t} \Omega(y') f(|x| t^{-1} y') \left( b(x) - b(|x| t^{-1} y') \right) \, d\sigma(y') \, dt \right)^{\frac{1}{q}} \omega_2(x) \, dx \]

\[ \leq \left( \int_{\mathcal{C}_k} \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{|\Phi(t)|}{t} \Omega(y') f(|x| t^{-1} y') \left( \|b\|_{Lip^\beta} |x|^\beta(1+t^{-1})^\beta \right) \, d\sigma(y') \, dt \right)^{\frac{1}{q}} \omega_2(x) \, dx \]

\[ \lesssim \|b\|_{Lip^\beta} \left( \int_{\mathcal{C}_k} \int_0^\infty \frac{|\Phi(t)|}{t(1+t^{-1})^{-\beta}} \Omega(y') f(|x| t^{-1} y') |x|^\beta \, d\sigma(y') \, dt \right)^{\frac{1}{q}} \omega_2(x) \, dx \]

\[ \lesssim \|b\|_{Lip^\beta} \left( \int_0^\infty \frac{|\Phi(t)|}{t(1+t^{-1})^{-\beta}} \left( \int_{\mathbb{S}^{n-1}} \Omega(y') f(|x| t^{-1} y') |x|^\beta \, d\sigma(y') \right)^{\frac{1}{q}} \omega_2(x) \, dx \right) \, dt. \]
Using changing variable $u = xt^{-1}$ and by (3.9) again, we obtain

$$\|H^b_{\phi, \Omega} f \chi_k\|_{q, \omega_2}$$

$$\lesssim \|b\|_{Lip^\beta} |B_k|^{\frac{2}{n+1}} \int_0^\infty \frac{|\Phi(t)|}{t^{1 - \frac{2 \beta}{n} + \frac{n}{2} (1 + t^{-1})^{-\beta}}} \left( \int_{S^{n-1}} \Omega(y')f(|u|y')d\sigma(y') \right)^{\frac{q}{\gamma}} \omega_2(u)du \ dt$$

$$\lesssim \|b\|_{Lip^\beta} \|\Omega\|_{L^q(S^{n-1})} |B_k|^{\frac{2}{n+1}} \int_0^\infty \frac{|\Phi(t)|}{t^{1 - \frac{2 \beta}{n} + \frac{n}{2} (1 + t^{-1})^{-\beta}} \left( J(t, \omega_2) \right)^{\frac{q}{\gamma}} dt$$

$$\lesssim \|b\|_{Lip^\beta} \|\Omega\|_{L^q(S^{n-1})} |B_k|^{\frac{2}{n+1}} \int_0^\infty \frac{|\Phi(t)|}{t^{1 - \frac{2 \beta}{n} + \frac{n}{2} (1 + t^{-1})^{-\beta}} \left( \|f \chi_k + \ell - 1\|_{q, \omega_2} + \|f \chi_k + \ell\|_{q, \omega_2} \right) dt.$$

(3.16)

where $J(t, \omega_2) := \int_{\frac{t}{\lambda}C_k} \left( \int_{S^{n-1}} |f(|u|y')|^q d\sigma(y') \right) \omega_2(u)du$, and $\ell = \ell(t)$ is an integer number such that $2^\ell \simeq t^{-1}$. On the other hand, by the Minkowski inequality for $1 \leq p < \infty$, we have

$$\|H^b_{\phi, \Omega} f\|_{K^0_{\alpha_2} p, (\omega_1, \omega_2)} = \left( \sum_{k \in \mathbb{Z}} \omega_1(B_k)^{\alpha_2 \frac{n}{p}} \|H^b_{\phi, \Omega} f \chi_k\|_{L^q(\mathbb{R}^n, \omega_2)} \right)^{\frac{1}{p}}$$

$$\lesssim \|b\|_{Lip^\beta} \|\Omega\|_{L^q(S^{n-1})} \left( \sum_{k \in \mathbb{Z}} \omega_1(B_k)^{\alpha_2 \frac{n}{p}} \left( |B_k|^{\frac{2}{\gamma}} \int_0^\infty \frac{|\Phi(t)|}{t^{1 - \frac{2 \beta}{n} + \frac{n}{2} (1 + t^{-1})^{-\beta}} \right)^{\frac{q}{\gamma}} \right)^{\frac{1}{p}}$$

$$\times (\|f \chi_k + \ell - 1\|_{q, \omega_2} + \|f \chi_k + \ell\|_{q, \omega_2}) dt \right)^{\frac{1}{p}}$$

$$\lesssim \|b\|_{Lip^\beta} \|\Omega\|_{L^q(S^{n-1})} \int_0^\infty \frac{|\Phi(t)|}{t^{1 - \frac{2 \beta}{n} + \frac{n}{2} (1 + t^{-1})^{-\beta}}} Bdt,$$
where \( B := \left( \sum_{k \in \mathbb{Z}} \omega_1(B_k)^{\alpha_2} |B_k|^{\frac{\beta}{p}} \left( \| f \chi_{k+\ell-1} \|_{q,\omega_2} + \| f \chi_{k+\ell} \|_{q,\omega_2} \right) \right)^{\frac{1}{\beta}}. \) It is not hard to see that
\[
B \leq \left( \sum_{k \in \mathbb{Z}} \omega_1(B_k)^{\alpha_2} |B_k|^{\frac{\beta}{p}} \| f \chi_{k+\ell-1} \|_{q,\omega_2} \right)^{\frac{1}{\beta}} + \left( \sum_{k \in \mathbb{Z}} \omega_1(B_k)^{\alpha_2} |B_k|^{\frac{\beta}{p}} \| f \chi_{k+\ell} \|_{q,\omega_2} \right)^{\frac{1}{\beta}}.
\]
Note that it follows from Lemma 2.9 that \( \frac{|B_k|}{\omega_1(B_k)^{\frac{\beta}{p}} \omega_1(B_{k+\ell-i})} \) is a constant and
\[
\frac{\omega_1(B_k)}{\omega_1(B_{k+\ell-i})} = 2^{-(\ell-i)(n+\gamma)}, \quad i = -1, 0.
\]
(3.17)
With \( \alpha_1 = \alpha_2 + \frac{n\beta}{n+\gamma} \), by \( 2^\ell \simeq t^{-1} \), we obtain
\[
B \leq \left( \sum_{k \in \mathbb{Z}} \omega_1(B_{k+\ell-1})^{\alpha_1} \| f \chi_{k+\ell-1} \|_{q,\omega_2} \right)^{\frac{1}{\beta}} \left( \frac{\omega_1(B_k)}{\omega_1(B_{k+\ell-i})} \right)^{\alpha_1} \left( \frac{|B_k|}{\omega_1(B_k)^{\frac{\beta}{p}}} \right)^{\frac{\beta}{p}}
\]
\[
+ \left( \sum_{k \in \mathbb{Z}} \omega_1(B_{k+\ell})^{\alpha_1} \| f \chi_{k+\ell} \|_{q,\omega_2} \right)^{\frac{1}{\beta}} \left( \frac{\omega_1(B_k)}{\omega_1(B_{k+\ell-i})} \right)^{\alpha_1} \left( \frac{|B_k|}{\omega_1(B_k)^{\frac{\beta}{p}}} \right)^{\frac{\beta}{p}}
\]
\[
\leq \left( \frac{\omega_1(B_k)}{\omega_1(B_{k+\ell-i})} \right)^{\frac{\alpha_1}{\beta}} + \left( \frac{\omega_1(B_k)}{\omega_1(B_{k+\ell-i})} \right)^{\frac{\alpha_1}{\beta}} \| f \|_{K_\alpha^{1,p}(\omega_1,\omega_2)}
\]
\[
\leq 2^{-(\ell-i)\alpha_1(1+\frac{\beta}{p})} + 2^{-(\ell-i)\alpha_1(1+\frac{\beta}{p})} \| f \|_{K_\alpha^{1,p}(\omega_1,\omega_2)}
\]
\[
\leq \left( \frac{1}{t} \right)^{-\alpha_1(1+\frac{\beta}{p})} \| f \|_{K_\alpha^{1,p}(\omega_1,\omega_2)}.
\]
Consequently,
\[
\| \hat{H}_{\Phi,\Omega} f \|_{K_\alpha^{2,p}(\omega_1,\omega_2)} \leq \left( \frac{1}{t} \right)^{-\alpha_1(1+\frac{\beta}{p})} \| f \|_{K_\alpha^{1,p}(\omega_1,\omega_2)}
\]
\[
\leq \int_0^\infty \Phi(t) \left( \frac{1}{t} \right)^{-\alpha_1(1+\frac{\beta}{p})} \| f \|_{K_\alpha^{1,p}(\omega_1,\omega_2)} dt
\]
\[
\leq \int_0^\infty \Phi(t) \left( \frac{1}{t} \right)^{-\alpha_1(1+\frac{\beta}{p})} \| f \|_{K_\alpha^{1,p}(\omega_1,\omega_2)} dt.
\]
Therefore, the theorem is completely proved. \( \Box \)

Similarly, we also have the following result for the two weighted Morrey-Herz spaces.
Theorem 3.9. Let $0 < p < \infty, 1 \leq q < \infty, 0 < \beta \leq 1, \gamma > -n$ and
\[\alpha_1 = \alpha_2 + \frac{\alpha_1}{\alpha_2 + \gamma}.\]
Suppose $b \in \text{Lip}^{\beta}(\mathbb{R}^n)$, $\omega_1, \omega_2 \in \mathcal{W}$, with $\omega_2(x') \geq c > 0$ for all $x' \in S^{n-1}$, $\Omega \in L^s(S^{n-1})$, and
\[
C_3 = \int_0^\infty \frac{|\Phi(t)|}{t^{1-\frac{\beta}{q} + \frac{\gamma}{(1+\gamma)\beta}}} (1 + t^{-1})^{-\beta} dt < \infty.
\]
Then the commutator $\mathcal{H}^{\gamma}_{\Phi, \Omega}$ is a bounded operator from $M\hat{K}_{p,q}^{\alpha_1, \lambda}(\omega_1, \omega_2)$ to $M\hat{K}_{p,q}^{\alpha_2, \lambda}(\omega_1, \omega_2)$.

Proof. The proof of the theorem is quite similar to one of Theorem 3.8, but to convenience to the readers, we also give the brief proof here. In order to estimate the right hand side of (3.16), we need to consider the following two cases.

Case 1: $1 \leq p < \infty$. By Minkowski’s inequality, it follows from Lemma 2.9 and (3.17) that

\[
\left\| \mathcal{H}^{\beta}_{\Phi, \Omega} f \right\|_{M\hat{K}_{p,q}^{\alpha_2, \lambda}(\omega_1, \omega_2)} \lesssim \|b\|_{Lip^{\beta}} \|\Omega\|_{L^q(S^{n-1})} \sup_{k_0 \in \mathbb{Z}} \left( \omega_1(B_{k_0})^{-\frac{\beta}{\alpha_2}} \left( \sum_{k=-\infty}^{k_0} \omega_1(B_k)^{\alpha_2} \left| B_k \right|^{\frac{\beta}{\alpha_2}} (\|f\chi_{k+\beta-1}\|_{q, \omega_2} + \|f\chi_{k+\beta}\|_{q, \omega_2}) dt \right)^{\frac{p}{\alpha_2}} \right)^{\frac{1}{p}}
\]

where

\[
\tilde{B} := \sup_{k_0 \in \mathbb{Z}} \left( \omega_1(B_{k_0})^{-\frac{\beta}{\alpha_2}} \left( \sum_{k=-\infty}^{k_0} \omega_1(B_k)^{\alpha_2} \left| B_k \right|^{\frac{\beta}{\alpha_2}} (\|f\chi_{k+\beta-1}\|_{q, \omega_2} + \|f\chi_{k+\beta}\|_{q, \omega_2}) dt \right)^{\frac{1}{p}} \right)^{\frac{1}{p}},
\]

and $\ell = \ell(t)$ is an integer number such that $2^\ell \simeq t^{-1}$. It is clear that

\[
\tilde{B} \leq \sup_{k_0 \in \mathbb{Z}} \omega_1(B_{k_0})^{-\frac{\beta}{\alpha_2}} \left( \sum_{k=-\infty}^{k_0} \omega_1(B_{k+\beta-1})^{\alpha_2} \|f\chi_{k+\beta-1}\|_{q, \omega_2} \right)^{\frac{1}{p}} \left( \frac{\omega_1(B_k)}{\omega_1(B_{k+\beta-1})} \right)^{\frac{\alpha_1}{\alpha_2}} \left( \frac{|B_k|}{\omega_1(B_{k+\beta-1})} \right)^{\frac{\beta}{\alpha_2}} + \sup_{k_0 \in \mathbb{Z}} \omega_1(B_{k_0})^{-\frac{\beta}{\alpha_2}} \left( \sum_{k=-\infty}^{k_0} \omega_1(B_{k+\beta})^{\alpha_2} \|f\chi_{k+\beta}\|_{q, \omega_2} \right)^{\frac{1}{p}} \left( \frac{\omega_1(B_k)}{\omega_1(B_{k+\beta})} \right)^{\frac{\alpha_1}{\alpha_2}} \left( \frac{|B_k|}{\omega_1(B_{k+\beta})} \right)^{\frac{\beta}{\alpha_2}}.
\]
Consequently, we have

\[
\| \mathcal{H}_b^b \Omega f \|_{M^\Lambda_{p,q}^\alpha (\omega_1, \omega_2)} \lesssim \| b \|_{\text{Lip}^p} \| \Omega \|_{L^q(S^{n-1})} \int_0^\infty \frac{|\Phi(t)|}{t^{1 - \frac{n}{q} - \frac{\alpha}{q} (1 + t^{-1}) - \beta}} \| f \|_{M^\Lambda_{p,q}^\alpha (\omega_1, \omega_2)} dt
\]

Case 2: \( 0 < p < 1 \). We first observe that

\[
\| f \chi_{k+\ell+i} \|_{q, \omega_2} \leq \omega_1(B_{k+\ell+i}) \frac{\lambda - \alpha_i}{n} \omega_1(B_{k+\ell+i})^{-\frac{\alpha}{n}} \left( \sum_{j=-\infty}^{k+\ell+i} \omega_1(B_j) \frac{\alpha}{n} \| f \chi_j \|_{p, \omega_2}^p \right)^{\frac{1}{p}}
\]

\[
\leq \omega_1(B_{k+\ell+i}) \frac{\lambda - \alpha_i}{n} \| f \|_{M^\Lambda_{p,q}^\alpha (\omega_1, \omega_2)}, \quad i = -1, 0.
\]

Combining this with (3.16), we obtain

\[
\| \mathcal{H}_b^b \Omega f \|_{M^\Lambda_{p,q}^\alpha (\omega_1, \omega_2)} \lesssim \sum_{i=-1,0} \| b \|_{\text{Lip}^p} \| \Omega \|_{L^q(S^{n-1})} \| f \|_{M^\Lambda_{p,q}^\alpha (\omega_1, \omega_2)} \sup_{k_0 \in \mathbb{Z}} \left( \omega_1(B_{k_0})^{-\frac{\alpha}{n}} \left( \sum_{k=-\infty}^{k_0} \omega_1(B_k) \frac{\alpha}{n} |B_k|^{\frac{\alpha}{n}} \right)^{\frac{1}{p}} \right)
\]

\[
\times \left( \int_0^\infty \frac{|\Phi(t)|}{t^{1 - \frac{n}{q} - \frac{\alpha}{q} (1 + t^{-1}) - \beta}} \omega_1(B_{k+\ell+i})^{\frac{\lambda - \alpha_i}{n}} dt \right)^{\frac{1}{p}}
\]

\[
\lesssim \sum_{i=-1,0} \| b \|_{\text{Lip}^p} \| \Omega \|_{L^q(S^{n-1})} \| f \|_{M^\Lambda_{p,q}^\alpha (\omega_1, \omega_2)} \sup_{k_0 \in \mathbb{Z}} \left( \sum_{k=-\infty}^{k_0} \left( \int_0^\infty \frac{|\Phi(t)|}{t^{1 - \frac{n}{q} - \frac{\alpha}{q} (1 + t^{-1}) - \beta}} T dt \right)^{\frac{1}{p}} \right).
\]
where $T := \omega_1(B_{k_0})^{\frac{\lambda}{n}} \omega_1(B_k)^{\frac{\lambda}{n}} |B_k|^{\frac{\lambda}{n}} \omega_1(B_{k+\ell+1})^{\frac{\lambda}{n}}$. Hence, by (3.17) and for any $k \leq k_0$, it follows that

$$T \lesssim 2^{(k-k_0)}(1+\frac{r}{n})^\lambda \left( \frac{1}{t} \right)^{(\lambda-\alpha_1)(1+\frac{r}{n})}.$$

Consequently, we obtain

$$\|H_b \Phi, \Omega f\|_{M^{\dot{K}_{\alpha_2,\lambda}}(\omega_1,\omega_2)} \lesssim \|f\|_{M^{\dot{K}_{\alpha_1,\lambda}}(\omega_1,\omega_2)} \sup_{k_0 \in \mathbb{Z}} \left( \sum_{k=-\infty}^{k_0} 2^{(k-k_0)}(1+\frac{r}{n})^\lambda \right)^{\frac{1}{p}} \int_0^\infty \frac{|\Phi(t)|}{t^{1-\frac{r}{q}-(\lambda-\alpha_1)(1+\frac{r}{n})} (1+t^{-1})^{-\beta}} dt \lesssim \|f\|_{M^{\dot{K}_{\alpha_1,\lambda}}(\omega_1,\omega_2)} \int_0^\infty \frac{|\Phi(t)|}{t^{1-\frac{r}{q}-(\lambda-\alpha_1)(1+\frac{r}{n})} (1+t^{-1})^{-\beta}} dt.$$

Therefore, the proof of the theorem is completed. \(\square\)

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