INSTABILITY OF A WITNESS BUNCH IN A PLASMA BUBBLE

A. Burov, V. Lebedev, and S. Nagaitsev
Fermilab, Batavia, IL 60510

Abstract
The stability of a trailing witness bunch, accelerated by a plasma wake accelerator (PWA) in a blow-out regime, is discussed. The instability growth rate as well as the energy spread, required for BNS damping, are obtained. A relationship between the PWA power efficiency and the BNS energy spread is derived.

THE LU EQUATION

When the drive bunch is intense enough, it produces an electron-free plasma bubble, with an average radius, \( R \), exceeding the characteristic plasma length, \( R \gg k_p^{-1} \). In this case, the shape of the bubble \( r_b(z) \) can be described by the Lu equation \([@Lu]\):

\[
r''_b r'_b + r'_b + 1 = \frac{4 \lambda(z)}{r_b^3};
\]

\[
\lambda(z) = \frac{N_d}{(2\pi)^{3/2} \sigma n_e} \exp \left( -\frac{z^2}{2\sigma^2} \right);
\]

\[
\int \lambda(z) dz = \frac{N_d}{2\pi n_e}.
\]

Here \( \lambda(z) \) is the beam line density normalized in agreement with Ref \([@Lu]\), \( N_d \) is the number of electrons in the drive bunch, \( n_e \) is the plasma density and \( z \) is the distance from the drive bunch; zero boundary conditions sufficiently ahead of the bunch are assumed for \( r_b \) and \( r'_b \). After the substitution \( r_b \rightarrow \sigma r'_b; \ z \rightarrow \sigma z \), the equation is made to depend on a single parameter \( q \):

\[
r''_b + 2r'_b + 1 = \frac{4 \lambda(z)}{r_b^3};
\]

\[
\lambda(z) = \frac{N_d}{(2\pi)^{3/2} \sigma} \exp \left( -\frac{z^2}{2} \right);\]

\[
q = \frac{N_d}{n_e \sigma^3}.
\]

Figures 1 and 2 show the bubble boundary for \( q = 100 \) and \( q = 800 \) respectively. As one can see, the bubble is approximately spherical, except for its very frontal part, where the drive bunch is located. The bubble radius is seen to be well approximated by \( R = \sigma q^{1/3} = (N_d / n_e)^{1/3} \), so it does not depend on the bunch length \( \sigma \). Note that the related range of parameters corresponds to \( R \gg \sigma \). The longitudinal electric field on the axis is given by \([@Lu]\)

\[
E_z = 2\pi n_e e r'_b \approx 2\pi n_e e R \zeta = E_0 \zeta
\]

(3)

where \( \zeta = (z - z_0) / R \) is a dimensionless distance to the bubble centre, and the approximation assumes reasonable closeness to the centre. The transverse focusing field, under the same conditions, is given by \([@Lu]\)

\[
F_r = 2\pi n_e r / R.
\]

(4)

Fig. 1: Solution of the Lu equation for its parameter \( q=100 \). The line density \( \lambda \) is in arbitrary units, the bubble radius is in the units of the rms bunch length \( \sigma \) and the longitudinal field \( E_z \) is in units of \( 4\pi n_e e \sigma \).
It is worth mentioning that, according to Eq. (2), the number of plasma ions in the bubble is proportional to the number of electrons in the drive bunch:

\[ N_i = \frac{4}{3} \pi R^3 n_e \approx 4 N_d \]  

Let us define the transformer ratio as

\[ r = \frac{E_0}{\langle E_z \rangle_d} \approx \frac{11}{q^{1/4}}; \quad (q \geq 100) \]  

\[ \langle E_z \rangle_d = \int dz \lambda E_z \]  

With that, the power transfer ratio can be expressed as

\[ \frac{P_w}{P_d} = r^2 \frac{N_w}{N_d} \]  

where \( N_w \) is a number of electrons in the witness bunch.

### WAKE FUNCTIONS

The Lu equation (1) allows for obtaining the longitudinal wake function for witness beam particles. Indeed, an addition of the line density perturbation

\[ \delta \lambda(z) = \frac{\delta(z - z_1)}{2\pi \epsilon n_e} \]  

to Eq. (1) yields a step in the longitudinal field

\[ \delta E_z = W(0) = \frac{4}{r_b^2} \]  

which is, by definition, the longitudinal wake function at distances \( |\Delta z| \ll R, k_p^{-1} \). Note that this wake value is identical to the one for a hollow cylindrical plasma channel of the radius \( r_b \), as found in Ref. [Schroeder99]. The same value of the short-range wake is valid for dielectric channels and resistive walls as well [@BurNov, @Chao]. For any channel, the Panofsky-Wenzel relation between the longitudinal and transverse wakes is given by [@Bane]:

\[ W_\perp = \frac{2}{r_b^3} \int W_\parallel dz = \frac{8 \Delta z}{r_b^4}. \]  

Below, we are assuming that this transverse wake is valid for the particles of a witness bunch, which length is small enough, \( \Delta z \ll R, k_p^{-1} \).

### INSTABILITY

The transverse instability of an accelerated short homogeneous bunch a zero momentum spread in a plasma channel was described in Ref. [@Schroeder]; its result for a slow instability is reproduced below. Assuming that focusing scales with the energy as \( k_p \propto \gamma^{-1/2} \), which is valid for the plasma density being kept constant, the equation of motion has been solved with the following result for the local transverse offsets along the bunch:

\[ \frac{X(I,s)}{X_0} = \frac{3^{1/4}}{2^{3/2}} \pi^{1/2} \left( \frac{\gamma_0'}{\gamma_0} \right)^{1/4} \exp(A) \cos \psi; \]  

\[ \psi = \theta - A/3^{1/2} + \pi/12. \]  

Asymptotically, \( A \to (s/L)^{1/6} \), with the instability growth length

\[ L = \frac{2}{3} \left( \frac{I_0}{I} \right)^2 \left( k_0 \gamma_0^{1/2} r_b^4 \right)^2 \]  

\[ k_0 \gamma_0^{1/2} = k_p \gamma^{1/2} = k_p / 2^{1/2}; \]  

\[ I_0 = mc^3 / e; \quad I = N_w e c / l. \]  

For the plasma bubble, where

\[ k_p^2 \gamma_0 = k_p^2 / 2; \quad \gamma' = k_p^2 \zeta / 2; \]  

\[ r_b^2 = R^2 (1 - \zeta^2); \quad R = (N_d / n_e)^{1/3}; \]  

the instability length is simplified to

\[ L \approx 0.03 \left( \frac{N_d}{N_w} \right)^2 R^3 \zeta (1 - \zeta^2)^4. \]  

The growth length \( L \) reaches its maximum at \( \zeta = 1/3 \):

\[ L_{\max} \approx 0.006 \left( \frac{N_d}{N_w} \right)^2 R^3 \zeta (1 - \zeta^2)^4. \]  

Assuming, for example, \( N_d / N_w = 10 \), and \( R / l = 10 \), we get \( L_{\max} \approx 60 R \). Due to the extremely slow growth of the exponent power \( A \), the available acceleration length \( s_{\max} \) can exceed the instability length.
Let's assume that the offset growth shown in Eq. (9) is limited by a factor of 10: $X / X_0 = 10(\gamma_0 / \gamma)^{1/4}$. This limitation yields a maximum for the exponent power $A \leq A_{\text{max}} \approx 4.0$, and correspondingly,

$$s_{\text{max}} = A_{\text{max}}^6 L_{\text{max}} \approx 4 \cdot 10^3 L_{\text{max}} \approx 2 \cdot 10^5 R$$

for this numerical example, leading to the maximal energy

$$\gamma_{\text{max}} = A_{\text{max}}^6 L_{\text{max}} \gamma' \approx 2 \cdot 10^5 \Delta \gamma R;$$

$$\Delta \gamma R = \gamma' R = k_p^2 R^2 / 6.$$  

Note that

$$\gamma_{\text{max}} \approx N_{d}^{10/3} N_{w}^{-2} l^2 n_e^{-1/3}$$

For $n_e = 10^{17} \text{cm}^{-3}$ and $N_d = 6 \cdot 10^{10}$, this results in $R \approx 100 \mu m$, $k_p R \approx 5$ and $\gamma_{\text{max}} \approx 10^6$. This limit is very sensitive to the maximally tolerable exponent power $A$, which, in its turn, depends on the accuracy of beam transfer from one plasma section into another. For example, if this exponent factor was a bit lower, $A_{\text{max}} = 3$, the maximal energy would be about 5 times lower.

In principle, a proper momentum spread in the witness bunch could suppress the instability, by means of the BNS damping mechanism [@BNS]. This option is examined in the following section.

**BNS DAMPING**

In this section, the bunch is treated as consisting of two macroparticles with $N_w / 2$ electrons each. The longitudinal forces acting on them follow:

$$F_1 = eE_0 \zeta_1 - N_w e^2 W_{||} / 4;$$

$$F_2 = eE_0 \zeta_2 - N_w e^2 W_{||} / 4 - N_w e^2 W_{\perp} / 2.$$  

Due to the difference of these two forces, the particles are getting unequal momenta, which relative difference can be presented as

$$\frac{\delta p}{p} = \frac{2 F_1 - F_2}{F_1 + F_2} \zeta - \rho;$$

$$\zeta = (\zeta_1 + \zeta_2) / 2; \quad \delta \zeta = \zeta_2 - \zeta_1;$$

$$\rho = \frac{N_w e^2 W_{\perp}}{2 e E_0} = \frac{N_w}{\pi n R^3 (1 - \zeta^2)} \approx 0.3 \frac{N_w}{N_d (1 - \zeta^2)}.$$  

To provide BNS damping, the momentum spread has to be matched with the transverse wake [@Chao]:

$$\frac{\delta p}{p}_{\text{BNS}} = \frac{N_w e^2 W_{\perp}}{4 k_p^2 \gamma} = \rho \delta \zeta / 1 - \zeta^2.$$  

From here, a condition for the distance between the macroparticles $\delta \zeta$ follows:

$$\frac{\rho - \delta \zeta}{\zeta - \rho} = \frac{\rho \delta \zeta}{1 - \zeta^2}.$$  

The relative momentum spread required by Eq. (19) can be much smaller than 100% only if the intensity parameter

$$\rho \ll 1.$$  

With this condition, Eq. (20) yields:

$$\delta \zeta = \rho;$$

$$\frac{\delta p}{p}_{\text{BNS}} = \frac{\rho^2}{1 - \zeta^2}.$$  

Taking into account Eq. (18) and Eq. (5), the required momentum spread can be also presented as

$$\frac{\delta p}{p}_{\text{BNS}} \approx \frac{0.1 r^2}{l^2} \left( \frac{P_{w}}{P_d} \right)^2 \frac{1}{\zeta^2 (1 - \zeta^2)^3}.$$  

Minimal value of the energy spread is achieved when the witness bunch is positioned at half-radius, $\zeta = 1 / 2$, which leads to a limitation:

$$\frac{\delta p}{p}_{\text{BNS}} \geq \frac{1}{r^2 (P_{w}/P_d)^2}.$$  

Thus, high power efficiency entails large energy spread.

**CONCLUSIONS**

The transverse instability of the witness bunch sets a limit on the maximal energy up to which the bunch could be accelerated in the nonlinear regime of the plasma wake acceleration with a higher-intensity driving bunch. For the BNS damping, limitations on the acceptable momentum spread inevitably entail a corresponding reduction of the power efficiency of the acceleration.

FNAL is operated by Fermi Research Alliance, LLC under Contract No. De–AC02–07CH11359 with the United States Department of Energy.

**REFERENCES**

[@Lu] W. Lu et al., Phys. Plasmas, 13, 056709 (2006)

[@Schroeder] C. B. Schroeder, D. H. Whittum, J. S. Wurtele, PRL, 82, 6 (1999).

[@BurNov] A. Burov, A. Novokhatski, HEACC’92, p. 537

[@Bane] K. Bane, SLAC-PUB-9663

[@BNS] V. Balakin, A. Novokhatski, V. Smirnov, Proc. 12th Int. Conf. High Energy Accel, p. 112 (1983).

[@Chao] A. Chao, Physics of Collective Beam Instabilities, p. 142 (1993).