Elementary methods for evaluating Jordan’s sums

\[ \sum_{n \geq 1} \left( 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} \right) \frac{1}{n^{2a}} \quad \text{and} \quad \sum_{n \geq 1} \left( 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} \right) \frac{1}{(2n-1)^{2a}} \]

and analogous Euler’s type sums and for setting a \( \sigma \)-sum theorem

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Abstract

Our aim is first to calculate the following sums:

\[ \sum_{n \geq 1} \left( 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} \right) \frac{1}{n^{2a}} \quad \text{and} \quad \sum_{n \geq 1} \left( 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} \right) \frac{1}{(2n-1)^{2a}} \] (J)

when \( a \) is an integer \( \geq 1 \), by mean of double sum methods or integral representation. The values of sum in (J) are due P.F. Jordan, in Infinite sums of psi functions, Bulletin of American Mathematical Society (79) 4, 1973. We think that we give here the first simple an elementary proof of (J) and of other formulas which are deduced in the litterature from Jordan ones. As a consequence of our calculations, we find an expression for the sums

\[ \sum_{n \geq 1} \left( \frac{1}{2} + \cdots + \frac{1}{n} \right) \frac{1}{(2n-1)^{2a}}. \]

Moreover, in the last section, we find some relations between the sums \( \sigma(s, t) := \sum_{n \geq 1} \left( 1 + \frac{1}{3} + \cdots + \frac{1}{(2n-1)} \right) \frac{1}{n^s} \) (with \( s \) and \( t \) integers, \( s \geq 2 \)), which give the sums

\[ \sum_{n \geq 1} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) \frac{1}{(2n-1)^{2a+1}}. \]

Finally, we prove a new additive relation on the \( \sigma(s, t) \) of same weight \( s + t \) which is a “\( \sigma \)-sum theorem” analogous to this one involving the classical Euler’s sums \( \zeta(s, t) \).

Key words and phrases: multiple zeta values, Jordan’s sums, Euler’s Sums, harmonic and semi harmonic numbers, sum theorem.

Introduction and notations

We define harmonic numbers \( H_n \) and semi-harmonic numbers \( S_n \) by

\[ H_0 = 0, \quad H_n = H_{n-1} + \frac{1}{n} \quad (n \geq 1), \quad S_0 = 0, \quad S_n = S_{n-1} + \frac{1}{2n-1} \quad (n \geq 1) \] (1.1)

and we put

\[ J(b) := \sum_{n \geq 1} \frac{S_n}{n^b}, \quad \bar{J}(b) := \sum_{n \geq 1} \frac{S_n}{(2n-1)^b}. \] (1.2)
These series converge when $b > 1$. and we denote them by “Jordan’s sums”

We will give expression of $J(2a)$ and $\bar{J}(2a)$ when $a$ is an integer $\geq 1$. In the present paper, we propose an elementary and autonomous proof of their value. In [5], R.Sitaramachandrarao uses the values of $J(2a)$ and $\bar{J}(2a)$ given by Jordan in [4] to evaluate some series analogous to the so-called Euler’s sums $\sum_{n \geq 1} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \frac{1}{n^b}$, as, for example, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{n^{2a}}$. In this paper, we inverse the order of the proof, deducing for instance $J(2a)$ from the previous sum that we prove directly.

We use $\lambda(s) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} = \left(1 + \frac{1}{2^s}\right) \zeta(s)$, where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $s > 1$. We use also the classical Euler star sum

$$\zeta^*(b, 1) = \sum_{n \geq 1} \frac{H_n}{n^b} = \left(1 + \frac{b}{2}\right) \zeta(b + 1) - \frac{1}{2} \sum_{j=2}^{b-1} \zeta(j) \zeta(b + 1 - j)$$

(1.3)

when the integer $b$ is greater than 1. This expression has a simpler form for even $b$ (say $2a$)

$$\zeta^*(2a, 1) = (1 + a) \zeta(2a + 1) - \sum_{j=1}^{a-1} \zeta(2j) \zeta(2a + 1 - 2j).$$

(1.4)

We denote by $\tilde{\zeta}^*(b, 1)$ the alternating corresponding sum, that is:

$$\tilde{\zeta}^*(b, 1) = \sum_{n \geq 1} (-1)^n \frac{H_n}{n^b}.$$

Besides the formulas giving $J(2a)$ and $\bar{J}(2a)$, we obtain close formulas for the sums

$$\sum_{n \geq 1} \frac{H_n}{(2n + 1)^q}$$

when $q$ is an integer $\geq 2$. The result when $q$ is odd is obtained by searching linear relations between the sums $\sigma(s, t) := \sum_{n \geq 1} \left(1 + \frac{1}{3^s} + \cdots + \frac{1}{(2n - 1)^s}\right) \frac{1}{n^t}$ (with $s$ and $t$ integers, $s \geq 2$). We give also explicit values of some of these $\sigma(s, t)$, which we call $\sigma$-Euler sums in term of Riemann’s series or analogous ones and establish a sum theorem for all $\sigma$-sums of same weight. Finally, we sketch out a calculation of them when $s + t$ is odd.

I. The results:

Principal formulas for Jordan’s and analogous sums and $\sigma$-Euler sums

A. Jordan’s sums

$$J(2a) = \sum_{n \geq 1} \frac{S_n}{n^{2a}} = \frac{2^{2a+1} - 1}{4} \zeta(2a + 1) - \frac{1}{2} \sum_{j=1}^{a-1} (2^{2j+1} - 1) \zeta(2j + 1) \zeta(2a - 2j)$$

(a)
This formula becomes, by using \( \lambda \)'s series:

\[
J(2a) = \sum_{n \geq 1} \frac{S_n}{n^{2a}} = 2^{2a-1} \lambda(2a + 1) - \sum_{j=1}^{a-1} 2^{2j} \lambda(2j + 1) \zeta(2a - 2j).
\]  

\( (a') \)

\[
\bar{J}(2a) = \sum_{n \geq 1} \frac{S_n}{(2n - 1)^{2a}} = \lambda(2a) \ln 2 + \frac{1}{2} \lambda(2a + 1) - \sum_{j=1}^{a-1} \frac{1}{2^{2j+1}} \lambda(2a - 2j) \zeta(2j + 1).
\]

\( (b) \)

These are, with slightly different notations, the formulas given by Jordan in [4].

B. Sums involving \( H_n \)'s

We get two classes of closed formulas:

\[
\sum_{n \geq 1} \frac{H_n}{(2n + 1)^{2a}} = -2\lambda(2a) \ln 2 + 2a \lambda(2a + 1) - 2 \sum_{j=1}^{a-1} \lambda(2j) \lambda(2a + 1 - 2j).
\]

\( (c) \)

\[
\sum_{n \geq 1} \frac{H_n}{(2n + 1)^{2a-1}} = -2\lambda(2a - 1) \ln 2 + \left( a - \frac{1}{2} \right) \lambda(2a) - \sum_{q=1}^{a-2} \lambda(2q + 1) \lambda(2a - 2q - 1).
\]

\( (d) \)

When \( a = 2b \) it becomes:

\[
\sum_{n \geq 1} \frac{H_n}{(2n + 1)^{4b-1}} = -2\lambda(4b - 1) \ln 2 + \left( 2b - \frac{1}{2} \right) \lambda(4b) - 2 \sum_{q=1}^{b-1} \lambda(2q + 1) \lambda(4b - 2q - 1).
\]

\( (e) \)

and when \( a = 2b + 1 \):

\[
\sum_{n \geq 1} \frac{H_n}{(2n + 1)^{4b+1}} = -2\lambda(4b + 1) \ln 2 + \left( 2b + \frac{1}{2} \right) \lambda(4b + 2) - \lambda^2(2b + 1) - 2 \sum_{q=1}^{b-1} \lambda(2q + 1) \lambda(4b - 2q + 1).
\]

\( (e') \)

C. Some close formulas for \( \sigma \)-Euler sums and sum theorem

Clearly, by definition, \( \sigma(2a, 1) = J(2a) \). We prove the two relations:

\[
\sigma(2, 2a - 1) = 2a(2a - 1) \lambda(2a + 1) - 8 \sum_{j=1}^{a-1} j \lambda(2a - 2j) \lambda(2j + 1).
\]

\( (f) \)

\[
\sigma(2a - 1, 2) = -a 2^{2a-1} \lambda(2a + 1) + \frac{2^{2a-1}(2a + 1)}{3} \lambda(2) \lambda(2a - 1) + \sum_{j=1}^{a-2} j 2^{2j} \lambda(2j + 1) \zeta(2a - 2j).
\]

\( (g) \)
Theorem. [of the σ-sum] The sum of all the σ’s of same weight is calculable: for \( w \geq 3 \), we have:

\[
\sum_{i=1}^{w-2} \sigma(w - i, i) = (w - 1)\lambda(w).
\]

(h)

II. The first result: evaluation of sums involving the \( H_{2n} \)’s

The principal objective of this section is to evaluate the new sum \( Z(2a) := \sum_{n \geq 1} \frac{H_{2n}}{n^{2a}} \).

We begin by the sum \( \tilde{\zeta}^*(2a, 1) \).

II.1 Calculation of \( \tilde{\zeta}^*(2a, 1) := \sum_{n \geq 1} (-1)^{n-1} \frac{H_n}{n^{2a}} \)

By rational decomposition, we get \( \frac{H_n}{n} = \sum_{q \geq 1} \frac{1}{q(q + n)} \). So for \( a \geq 1 \), by absolute convergence

\[
\tilde{\zeta}^*(2a, 1) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^{2a-1}} \sum_{q \geq 1} \frac{1}{q(q + n)} = \sum_{q \geq 1} \frac{1}{q} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^{2a-1}(q + n)}.
\]

(3)

We have the following decomposition in the variable \( n \):

\[
\frac{1}{n^{2a-1}(q + n)} = \sum_{j=1}^{2a-2} \frac{(-1)^j}{n^{j+1}q^{2a-j-1}} + \frac{1}{q^{2a-1}} \left( \frac{1}{n} - \frac{1}{n + q} \right),
\]

(4)

which gives:

\[
\tilde{\zeta}^*(2a, 1) = \sum_{q \geq 1} \sum_{n \geq 1} (-1)^{n-1} \sum_{j=1}^{2a-2} \frac{(-1)^j}{n^{j+1}q^{2a-j}} + \sum_{q \geq 1, n \geq 1} \frac{(-1)^{n-1}}{q^{2a}} \left( \frac{1}{n} - \frac{1}{n + q} \right)
\]

\[
= \sum_{j=1}^{2a-2} (-1)^j \sum_{q \geq 1, n \geq 1} \frac{(-1)^{n-1}}{n^{j+1}q^{2a-j}} + \sum_{q \geq 1, n \geq 1} \frac{(-1)^{n-1}}{q^{2a}} \left( \frac{1}{n} - \frac{1}{n + q} \right)
\]

(5)

by defining, for \( s > 1 \), \( \tilde{\zeta}(s) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^s} \) and \( u := \sum_{q \geq 1, n \geq 1} \frac{(-1)^{n-1}}{q^{2a}} \left( \frac{1}{n} - \frac{1}{n + q} \right) \). Remark that if \( a = 1 \), the first sum in the last line of (5) does not exist, as it was already the case in (4). In the sequel, all sums indexed by a void set are identically 0. Now:

\[
u = \sum_{n \geq 1} \sum_{q \geq 1} \frac{(-1)^{n-1}}{n^{2a-1}(n + q)} = \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} \sum_{q=1}^{m-1} \frac{(-1)^{q-1}}{q^{2a-1}(m - q)}
\]

(6)
by defining \( n + q = m \) in the first summation. By new partial fractions decomposition, one has:

\[
\frac{1}{(m-q)q^{2a-1}} = \sum_{j=1}^{2a-2} \frac{1}{q^{j+1}m^{2a-j-1}} + \frac{1}{m^{2a-1}} \left( \frac{1}{q} + \frac{1}{m-q} \right),
\]

which gives:

\[
u = \sum_{j=1}^{2a-2} \sum_{m>q \geq 1} \frac{(-1)^{m+q-1}}{q^{j+1}m^{2a-j}} + \sum_{m>q \geq 1} \frac{(-1)^{m+q-1}}{m^{2a}} \left( \frac{1}{q} + \frac{1}{m-q} \right).
\]

Let \( u' \) be the first sum of (8) and \( u'' \) the second one. The sum \( u' \) splits in two parts: we sum first from 1 to \( a-1 \), and from \( a \) to \( 2a-2 \). In the second part we put \( j = 2a-1-j' \). Then \( j+1 = 2a - j' \) and \( 2a-j = j' + 1 \): so this second part becomes (we replace index \( j' \) by \( j \))

\[
\frac{\sum_{j=1}^{a-1} \sum_{m>q \geq 1} \frac{(-1)^{m+q-1}}{m^{j+1}q^{2a-j}}}{\sum_{j=1}^{a-1} \sum_{q>m \geq 1} \frac{(-1)^{m+q-1}}{q^{j+1}m^{2a-j}}}.
\]

So

\[
u' = \sum_{j=1}^{a-1} \sum_{q \neq m} \frac{(-1)^{m+q-1}}{q^{j+1}m^{2a-j}}.
\]

Now we observe that

\[
\sum_{q \neq m} \frac{(-1)^{m+q}}{q^{j+1}m^{2a-j}} = \tilde{\zeta}(j+1)\tilde{\zeta}(2a-j) - \zeta(2a+1).
\]

So equality (9) becomes

\[
u' = -\sum_{j=1}^{a-1} \tilde{\zeta}(j+1)\tilde{\zeta}(2a-j) + (a-1)\zeta(2a+1).
\]

In fact, we may also write:

\[
u' = -\sum_{j=1}^{a-1} \tilde{\zeta}(2j)\tilde{\zeta}(2a+1-2j) + (a-1)\zeta(2a+1).
\]

Now look at the sum \( u'' \). We have:

\[
u'' = \sum_{m \geq 1} \frac{(-1)^m}{m^{2a}} \mu(m)
\]

where

\[
\mu(m) := \sum_{q=1}^{m-1} (-1)^{q-1} \left( \frac{1}{q} + \frac{1}{m-q} \right).
\]
We will give a reduction of $B$

Clearly,

$$\mu(2m) = 2 \sum_{q=1}^{2m-1} \frac{(-1)^{q-1}}{q} = 2(H_{2m-1} - H_{m-1}) = 2(H_{2m} - H_m) + \frac{1}{m}.$$ 

As a result:

$$u'' = \sum_{m \geq 1} \frac{1}{(2m)^2 a} \left(2(H_{2m} - H_m) + \frac{1}{m}\right)$$

$$= \frac{1}{2^{2a-1}} Z(2a) - \frac{1}{2^{2a-1}} \zeta(2a, 1) + \frac{1}{2^{2a}} \zeta(2a + 1),$$

Remark that

$$\zeta^*(2a, 1) - \bar{\zeta}^*(2a, 1) = \sum_{m \geq 1} \frac{1 - (-1)^{m-1}}{m^{2a}} H_m = \frac{1}{2^{2a-1}} Z(2a).$$

So collecting previous results we obtain:

$$\bar{\zeta}^*(2a, 1) = \sum_{j=1}^{2a-2} (-1)^j \bar{\zeta}(j + 1)\zeta(2a - j) - \sum_{j=1}^{a-1} \bar{\zeta}(j + 1)\zeta(2a - j) + (a - 1)\zeta(2a + 1)$$

$$+ (\zeta^*(2a, 1) - \bar{\zeta}^*(2a, 1)) - \frac{1}{2^{2a-1}} \zeta^*(2a, 1) + \frac{1}{2^{2a}} \zeta(2a + 1).$$

This gives

$$2\bar{\zeta}^*(2a, 1) = 2a - \frac{2a + 1}{2^{2a}} \zeta(2a + 1) = (2a + 1)\zeta(2a + 1) - \zeta(2a + 1)$$

putting

$$A = \left(2a - \frac{2a + 1}{2^{2a}}\right)\zeta(2a + 1)$$

and

$$B = \sum_{j=1}^{2a-2} (-1)^j \bar{\zeta}(j + 1)\zeta(2a - j) - \sum_{j=1}^{a-1} \bar{\zeta}(j + 1)\zeta(2a - j) + \left(1 - \frac{1}{2^{2a-1}}\right) \sum_{j=1}^{a-1} \zeta(j + 1)\zeta(2a - j)$$

We will give a reduction of $B$. We write first (by consideration of parity):

\begin{align*}
B &= \sum_{j=1}^{a-1} \bar{\zeta}(2j + 1)\zeta(2a - 2j) - \sum_{j=1}^{a-1} \bar{\zeta}(2j)\zeta(2a + 1 - 2j) \\
&\quad - \sum_{j=1}^{a-1} \bar{\zeta}(2j + 1)\zeta(2a - 2j) - \left(1 - \frac{1}{2^{2a-1}}\right) \sum_{j=1}^{a-1} \zeta(2j + 1)\zeta(2a - 2j) \\
&= \sum_{j=1}^{a-1} \frac{1}{2^{2a-2j-1}} \zeta(2j + 1)\zeta(2a - 2j) - \sum_{j=1}^{a-1} \zeta(2j)\zeta(2a + 1 - 2j) \\
&\quad - \left(1 - \frac{1}{2^{2a-1}}\right) \sum_{j=1}^{a-1} \zeta(2j + 1)\zeta(2a - 2j).
\end{align*}
By replacing \( \tilde{\zeta}(2j + 1) \) by \((1 - 2^{-2j})\zeta(2j + 1)\) in the first sum we obtain, by grouping with the third one:

\[
B = \sum_{j=1}^{a-1} \left( \frac{1}{2^{2a-2j-1}} - 1 \right) \zeta(2j + 1)\zeta(2a - 2j) - \sum_{j=1}^{a-1} \tilde{\zeta}(2j)\zeta(2a + 1 - 2j)
\]

\[
= - \sum_{j=1}^{a-1} \tilde{\zeta}(2j + 1)\zeta(2a - 2j) - \sum_{j=1}^{a-1} \tilde{\zeta}(2j)\zeta(2a + 1 - 2j)
\]

\[
= - 2 \sum_{j=1}^{a-1} \tilde{\zeta}(2j)\zeta(2a + 1 - 2j),
\]

the last equality resulting for the exchange \( j \mapsto a - j \). Using (14) and (15) we get finally:

\[
\tilde{\zeta}^*(2a, 1) = \left( a + \frac{1}{2} \right)\tilde{\zeta}(2a + 1) - \frac{1}{2}\zeta(2a + 1) - \sum_{j=1}^{a-1} \tilde{\zeta}(2j)\zeta(2a + 1 - 2j)
\]  \quad (19)

Examples

\[
a = 1: \sum_{n \geq 1} (-1)^{n-1} \frac{H_n}{n^2} = 3 \zeta(3) - \frac{1}{2} \zeta(3) = \left( \frac{3}{2} \times \frac{3}{4} - \frac{1}{2} \right) \zeta(3) = \frac{5}{8} \zeta(3);
\]

\[
a = 2: \sum_{n \geq 1} (-1)^{n-1} \frac{H_n}{n^4} = \frac{5}{2} \zeta(5) - \frac{1}{2} \zeta(5) - \tilde{\zeta}(2)\zeta(3) = \frac{59}{32} \zeta(5) - \frac{1}{2} \zeta(2)\zeta(3).
\]

In fact formula (19) was obtained in [2] by Flajolet and Salvy, but they used there residue’s theorem. And it was established by Sitaramachandrarao in [5] by using the result for \( J(2a) \) given by Jordan in [4], but we will inverse the processes of Sitaramachandrarao, because the proof in [4] is hard to understand.

II.2 The sum \( Z(2a) \)

Subtracting (19) from (1.4) and using (12) we get

\[
2^{-2a+1} Z(2a) =
\]

\[
(a+1)\zeta(2a+1) - \sum_{j=1}^{a-1} \zeta(2j)\zeta(2a+1-2j) - \left( a + \frac{1}{2} \right)\tilde{\zeta}(2a+1) + \frac{1}{2} \zeta(2a+1) + \sum_{j=1}^{a-1} \tilde{\zeta}(2j)\zeta(2a+1-2j)
\]

which gives

\[
2^{-2a+1} Z(2a) = \left( a + \frac{3}{2} \right)\zeta(2a + 1) - \left( a + \frac{1}{2} \right)\tilde{\zeta}(2a + 1) - \sum_{j=1}^{a-1} \frac{1}{2^{2j-1}}\zeta(2j)\zeta(2a + 1 - 2j).
\]

Replacing \( \tilde{\zeta}(2a + 1) \) by its expression in terms of \( \zeta(2a + 1) \), one obtains now:

\[
\sum_{n \geq 1} \frac{H_{2n}}{n^{2a}} = \frac{1}{4} (2a + 1 + 2^{2a+1})\zeta(2a + 1) - \sum_{j=1}^{a-1} 2^{2a-2j}\zeta(2j)\zeta(2a + 1 - 2j)
\]  \quad (20)
This last expression is given in [5], Theorem 1, but using Bernoulli polynomials and integral transformations. This relation (20) will be the cornerstone for our proof of first Jordan’s formula.

Relations (20) et (1.4) give easily the formula:

\[
\sum_{n \geq 1} \frac{H_{2n-1}}{(2n-1)^{2a}} = \frac{2a + 1}{2} \lambda(2a + 1) - \sum_{j=1}^{a-1} \zeta(2a + 1 - 2j) \lambda(2j),
\]

(20′)

III. Proof of the first Jordan formula

In this section, we give a close formula for the sum \( J(2a) \) in terms of \( \zeta \) and \( \lambda \) series, first by using previous results and second by an integral representation of it.

III.1 First proof by sums of series

Clearly, we have

\[ H_{2n} = S_n + \frac{1}{2} H_n, \]

which gives

\[
\sum_{n \geq 1} \frac{H_{2n}}{n^{2a}} = J(2a) + \frac{1}{2} \zeta^*(2a, 1).
\]

(21)

By (20), (21), and (1.4) we obtain:

\[
J(2a) = \frac{1}{4} (2a + 1 + 2^{2a+1}) \zeta(2a + 1) - \frac{1}{2} \sum_{j=1}^{a-1} 2^{2a-2j} \zeta(2j) \zeta(2a + 1 - 2j)
\]

\[ - \frac{1}{2} (\zeta(2a + 1) + \zeta(2a + 1)) + \frac{1}{2} \sum_{j=1}^{a-1} \zeta(2j) \zeta(2a + 1 - 2j) \]

\[ = \frac{2^{2a+1} - 1}{4} \zeta(2a + 1) - \frac{1}{2} \sum_{j=1}^{a-1} (2^{2a+1-2j} - 1) \zeta(2j) \zeta(2a + 1 - 2j). \]

The transformation \( j \mapsto a - j \) furnishes the final expression:

\[
J(2a) = \sum_{n \geq 1} \frac{S_n}{n^{2a}} = \frac{2^{2a+1} - 1}{4} \zeta(2a + 1) - \frac{1}{2} \sum_{j=1}^{a-1} (2^{2j+1} - 1) \zeta(2j + 1) \zeta(2a - 2j).
\]

(22)

This formula becomes, by using \( \lambda \)'s series:

\[
J(2a) = \sum_{n \geq 1} \frac{S_n}{n^{2a}} = 2^{2a-1} \lambda(2a + 1) - \sum_{j=1}^{a-1} 2^{2j} \lambda(2j + 1) \zeta(2a - 2j).
\]

(23)

Examples

\[
\sum_{n \geq 1} \frac{S_n}{n^2} = \frac{7}{4} \zeta(3) \quad ; \quad \sum_{n \geq 1} \frac{S_n}{n^4} = \frac{31}{4} \zeta(5) - \frac{7}{2} \zeta(3) \zeta(2).
\]

III.2 Second method for calculating \( J(2a) \), by an integral representation.

We base this new method on the following claim:
Claim. For \( n \geq 0 \), we have the relation
\[
2S_n = \int_0^{\pi} \frac{\sin^2 nx}{\sin x} \, dx.
\]

Proof. Denote by \( I_n \) the integral in the claim. For \( n \geq 1 \), by the classical relation
\[
\sin^2 nx - \sin^2 (n-1)x = \sin x \sin(2n-1)x,
\]
we have:
\[
I_n - I_{n-1} = \int_0^{\pi} \sin(2n-1)x \, dx = \frac{2}{2n-1},
\]
and this prove our claim, since \( I_0 = 0 \).

It follows from the claim the crucial integral representation for \( J(2a) \):
\[
2J(2a) = \int_0^{\pi} \frac{\varphi_a(x)}{\sin x} \, dx,
\]
where
\[
\varphi_a(x) := \sum_{p \geq 1} \frac{\sin^2 px}{p^{2a}}. \tag{24}
\]
The function \( \varphi_a \) is expressible by Bernoulli polynomials, but we did not use them explicitly. In fact, we use successive partial integrations in (24), which correspond to induction relations between this polynomials. Put
\[
\alpha_n(a) = \int_0^{\pi} \frac{\varphi_a(x)}{\sin x} \cos 2nx \, dx.
\]
We have \( 2J(2a) = \alpha_0(a) \). For all \( a \geq 1 \), the functions \( \psi_a : x \mapsto \frac{\varphi_a(x)}{\sin x} \) are integrable on the interval \([0, \pi] \). For \( a \geq 2 \), this results from the inequality \(|\sin px| \leq p |\sin x| \), which gives,
\[
|\psi_a(x)| \leq \zeta(2a-1) < +\infty.
\]
And for \( a = 1 \), the integrability results from the classical expansion (for \( 0 \leq x \leq \pi \)):
\[
\frac{1}{2} x(\pi - x) = \sum_{p \geq 1} \frac{\sin^2 px}{p^2} = \varphi_2(x)
\]
and from the fact that \( \psi_2(x) \) tends to \( \frac{\pi}{2} \) when \( x \) tends to 0 or to \( \pi \). Hence, by Riemann-Lebesgue theorem, the sequence \( \alpha_n(a) \) tends to 0 if \( n \) goes to infinity.

For \( n \geq 1 \), we have :
\[
\alpha_{n-1}(a) - \alpha_n(a) = \int_0^{\pi} \frac{\varphi_a(x)}{\sin x} (\cos(2n-2)x - \cos 2nx) \, dx = 2 \int_0^{\pi} \varphi_a(x) \sin(2n-1)x \, dx := 2v_n.
\]

So, by summation, \( \alpha_0(a) - \alpha_n(a) = 2 \sum_{k=1}^{n} v_k \). Then, \( \alpha_0(a) = 2 \sum_{n=1}^{\infty} v_n \), and finally
\[
J(a) = \sum_{n=1}^{\infty} v_n.
\]
In the following, we drop the subscript \( a \) in \( \varphi_a \).

**Calculation of** \( v_n = \int_0^\pi \varphi(x) \sin(2n-1)x \, dx \).

We proceed by successive integrations by part, involving the derivatives of \( \varphi \):

\[
\varphi^{(2j)}(x) = (-1)^{j-1} 2^{2j-1} \sum_{k=1}^\infty \frac{\cos 2kx}{k^{2a-2j}}, \quad \varphi^{(2j-1)}(x) = (-1)^{j-1} 2^{2j-2} \sum_{k=1}^\infty \frac{\sin 2kx}{k^{2a-2j+1}}
\]

when \( 1 \leq j \leq a - 1 \) for even derivatives and \( 1 \leq j \leq a \) for odd ones. We observe that if \( j = a \), the second relation makes sense only if \( x \neq 0 \) (mod \( \pi \)). In this case we have

\[
\varphi^{(2a-1)}(x) = (-1)^{a-1} 2^{2a-2} \sum_{k=1}^\infty \frac{\sin 2kx}{k}.
\]

Using Fourier’s series or complex logarithm, this is (when \( 0 < x < \pi \)):

\[
\varphi^{(2a-1)}(x) = (-1)^{a-1} 2^{2a-3}(\pi - 2x).
\]

In the following calculation, \( n \) appears only by mean of \( 2n-1 \), so we set \( 2n-1 = m \). Since \( \varphi(0) = \varphi(\pi) = 0 \), we obtain first:

\[
v_n = \left[ - \frac{\varphi(x) \cos mx}{m} \right]_0^\pi + \frac{1}{m} \int_0^\pi \varphi'(x) \cos mx \, dx = \frac{1}{m} \int_0^\pi \varphi'(x) \cos mx \, dx.
\]

When \( a = 1 \), this process stops \( (\varphi'(x) = \frac{1}{2}(\pi - 2x)) \) and we obtain:

\[
v_n = \frac{1}{m} \int_0^\pi \left( \frac{\pi}{2} - x \right) \cos mx \, dx = \frac{1}{m^2} \left[ \left( \frac{\pi}{2} - x \right) \sin mx \right]_0^\pi + \frac{1}{m^2} \int_0^\pi \sin mx \, dx = \frac{2}{m^3} \quad (24')
\]

For \( a \geq 2 \), we use two integrations by part:

\[
v_n = \frac{1}{m^2} \left[ \varphi'(x) \sin mx \right]_0^\pi - \frac{1}{m^2} \int_0^\pi \varphi''(x) \sin mx \, dx = -\frac{1}{m^2} \int_0^\pi \varphi''(x) \sin mx \, dx
\]

\[
= \frac{1}{m^3} \left[ \varphi''(x) \cos mx \right]_0^\pi - \frac{1}{m^3} \int_0^\pi \varphi'''(x) \cos mx \, dx
\]

\[
= -\frac{4}{m^3} \zeta(2a-2) - \frac{1}{m^3} \int_0^\pi \varphi'''(x) \cos mx \, dx,
\]

the last equality coming from the expression of \( \varphi'' \) (remember that \( m \) is odd). When \( a = 2 \), we stop. Eventually, we stop when the last integral is \( \int_0^\pi \varphi^{(2a-1)}(x) \cos mx \, dx \) and obtain the relation:

\[
v_n = -\sum_{j=1}^{a-1} \frac{2^2 j}{m^{2j+1}} \zeta(2a - 2j) + \frac{(-1)^{a-1}}{m^{2a-1}} \int_0^\pi \varphi^{(2a-1)}(x) \cos mx \, dx.
\]
which gives, by taking the value of $\varphi^{(2a-1)}(x)$ into account and the previous relation $(24')$:

$$v_n = -\sum_{j=1}^{a-1} \frac{2^{2j}}{m^{2j+1}} \zeta(2a-2j) + \frac{2^{2a-1}}{m^{2a+1}}.$$ 

Finally, we obtain:

$$J(2a) = 2^{2a-1} \lambda(2a + 1) - \sum_{j=1}^{a-1} 2^{2j} \lambda(2j + 1) \zeta(2a-2j),$$

which is the already seen formula (23).

**IV. Calculation of $\bar{J}(2a) = \sum_{n \geq 1} S_n (2n-1)^{2a}$ and applications**

**IV.1 Calculation of $\bar{J}(2a)$**

First we write a relation between $\bar{J}(b)$ and $J(b)$ for all integer $b \geq 2$. We start by the following easy identity:

$$S_n = \sum_{p \geq 1} \frac{2n}{(2p-1)(2n + 2p - 1)} = \sum_{p \geq 1} \left( \frac{1}{2p-1} - \frac{1}{2n + 2p - 1} \right). \quad (25)$$

So, we have

$$\bar{J}(b) = \sum_{n \geq 1, p \geq 1} \frac{2n}{(2n-1)^b(2p-1)(2n + 2p - 1)} = \sum_{p \geq 1} \frac{1}{2p-1} \sum_{n \geq 1} \frac{2n}{(2n-1)^b(2n + 2p - 1)}.$$ 

(26)

We denote by $\rho(p)$ the inner summation in (26) and put $Q(x, \mu) = \frac{x + 1}{x^b(x - \mu)}$. So the generic term in $\rho(p)$ is $Q(2n - 1, -2p)$. We decompose the rational fraction $Q(x, \mu)$ into partial fractions in the variable $x$:

$$Q(x, \mu) = -\frac{1}{\mu x^b} - \sum_{j=1}^{b-2} \frac{\mu + 1}{\mu^{j+1} x^{b-j}} - \frac{\mu + 1}{\mu^b} \left( \frac{1}{x} - \frac{1}{x - \mu} \right),$$

so that

$$\rho(p) = \sum_{n \geq 1} \left( \frac{1}{2p(2n-1)^b} + \sum_{j=1}^{b-2} \frac{(-1)^{j+1}(2p - 1)}{(2p)^{j+1}(2n-1)^{b-j}} + (-1)^b \frac{2p-1}{(2p)^b} \left( \frac{1}{2n-1} - \frac{1}{2n + 2p - 1} \right) \right)$$

or using (25)

$$\rho(p) = \frac{1}{2p} \lambda(b) + (2p-1) \sum_{j=1}^{b-2} \frac{(-1)^{j+1}}{(2p)^{j+1}} \lambda(b - j) + (-1)^b \frac{2p-1}{(2p)^b} S_p.$$
By carrying this value in (26), we get:

\[
J(b) = \lambda(b) \sum_{p \geq 1} \frac{1}{2p(2p - 1)} + \sum_{j=1}^{b-2} (-1)^{j+1} \lambda(b - j) \sum_{p \geq 1} \frac{1}{(2p)^{j+1}} + (-1)^b \frac{S_p}{(2p)^b}.
\]

The first sum in the previous line is ln 2. One gets the general relation, for all integer \(b\):

\[
\bar{J}(b) + \frac{(-1)^{b-1}}{2^b} J(b) = \lambda(b) \ln 2 + \sum_{j=1}^{b-2} \frac{(-1)^{j+1}}{2^{j+1}} \lambda(b - j) \zeta(j + 1).
\]  

(28)

So we can calculate \(\bar{J}(b)\) if \(J(b)\) is known, in particular when \(b\) is even or when \(b = 3\) (see remark below). Look at \(b = 2a\). We use then (23) by replacing \(j\) by \(a - j\) in the sum:

\[
J(2a) = \frac{1}{2} \left(1 - \frac{1}{2^{2a+1}}\right) \zeta(2a + 1) + \lambda(2a) \ln 2 - \frac{1}{2^{2a+1}} \sum_{j=1}^{a-1} 2^{2a-2j+1} \lambda(2a - 2j + 1) \zeta(2j) \\
+ \sum_{j=1}^{2a-2} \frac{(-1)^{j+1}}{2^{j+1}} \lambda(2a - j) \zeta(j + 1).
\]

(29)

We split the second sum in (29) accordingly with the parity of the index \(j\). This gives:

\[
\sum_{n \geq 1} \frac{S_n}{(2n - 1)^{2a}} = \lambda(2a) \ln 2 + \frac{1}{2} \left(1 - \frac{1}{2^{2a+1}}\right) \zeta(2a + 1) - \sum_{j=1}^{a-1} \frac{1}{2^{2j+1}} \lambda(2a - 2j) \zeta(2j + 1),
\]

and finally:

\[
\bar{J}(2a) = \sum_{n \geq 1} \frac{S_n}{(2n - 1)^{2a}} = \lambda(2a) \ln 2 + \frac{1}{2} \lambda(2a + 1) - \sum_{j=1}^{a-1} \frac{1}{2^{2j+1}} \lambda(2a - 2j) \zeta(2j + 1).
\]

(30) = (b)

This formula is given by Jordan in [4] and is used in [5].

Examples:

\[
\bar{J}(2) = \frac{3}{4} \zeta(2) \ln 2 + \frac{7}{16} \zeta(3), \quad \bar{J}(4) = \frac{15}{16} \zeta(4) \ln 2 + \frac{31}{64} \zeta(5) - \frac{3}{32} \zeta(2) \zeta(3)
\]

The case \(b = 3\)

By using the classical series \(\lim_{n \to \infty} \frac{1}{2^n n^3} = \sum_{n \geq 1} \frac{1}{2^n n^3}\) the sums \(\bar{\zeta}^*(3, 1)\) (and then \(J(3)\)) can be evaluated (see for instance [2]):

\[
\bar{\zeta}^*(3, 1) = -2 \text{li}_4 \frac{1}{2} + \frac{11}{4} \zeta(4) + \frac{1}{2} \zeta(2) \ln^2 2 - \frac{1}{12} \ln^4 2 - \frac{7}{4} \zeta(3) \ln 2.
\]

This gives first:

\[
J(3) = \sum_{n \geq 1} \frac{H_{2n}}{n^3} - \frac{1}{2} \zeta^*(3, 1) = 4(\zeta^*(3, 1) - \bar{\zeta}^*(3, 1)) - \frac{1}{2} \zeta^*(3, 1) = \frac{7}{2} \zeta^*(3, 1) - 4 \bar{\zeta}^*(3, 1).
\]
But we know $\zeta^*(3, 1) = \frac{5}{4} \zeta(4)$ (formula (1.3) for $b = 3$). So we obtain

$$J(3) = 8 \text{li}_4 \left( \frac{1}{2} \right) - \frac{53}{8} \zeta(4) - 2 \zeta(2) \ln^2 2 + \frac{1}{3} \ln^4 2 + 7 \zeta(3) \ln 2.$$

Then, formula (28) gives, when $b = 3$:

$$J(3) + \frac{1}{8} J(3) = \lambda(3) \ln 2 + \frac{1}{4} \lambda(2) \zeta(2) = \frac{7}{8} \zeta(3) \ln 2 + \frac{15}{32} \zeta(4),$$

this gives:

$$J(3) = -\text{li}_4 \left( \frac{1}{2} \right) + \frac{83}{64} \zeta(4) + \frac{1}{4} \zeta(2) \ln^2 2 - \frac{1}{24} \ln^4 2.$$

Infortunatly, it is conjectured that $\zeta^*(k, 1)$ when $k \geq 5$ is odd are not expressible by usual constants and so probably the $J(b)$ are not simple when $b$ is odd $\geq 5$.

### III.2 Application to the calculation of new sums involving the $H_n$’s

We establish a pretty formula: by using (20’) (in which we change the $j$ index by $a - j$), previous relation (30) and the simple fact that

$$H_{2n-1} = \frac{1}{2} H_{n-1} + S_n$$

we obtain

$$\frac{1}{2} \sum_{n \geq 1} \frac{H_{n-1}}{(2n - 1)^{2a}} = -\lambda(2a) \ln 2 + a \lambda(2a + 1) + \sum_{j=1}^{a-1} \lambda(2j) \zeta(2a + 1 - 2j) \left( \frac{1}{2^{2a+1-2j}} - 1 \right),$$

or

$$\sum_{n \geq 1} \frac{H_n}{(2n + 1)^{2a}} = -2 \lambda(2a) \ln 2 + 2a \lambda(2a + 1) - 2 \sum_{j=1}^{a-1} \lambda(2j) \lambda(2a + 1 - 2j). \quad (31) = (c)$$

Thus, the closed form of the previous sum involves only $\lambda$ series.

**Examples:**

$$\sum_{n \geq 1} \frac{H_n}{(2n + 1)^2} = -\frac{\pi^2}{4} \ln 2 + 2 \lambda(3), \quad \sum_{n \geq 1} \frac{H_n}{(2n + 1)^4} = -\frac{\pi^4}{48} \ln 2 + 4 \lambda(5) - \frac{\pi^2}{4} \lambda(3).$$

### IV. Series including the sums $\sum_{1}^{n} \frac{(-1)^{k-1}}{k^{2a}}$

Now we define, for $b \geq 1$ $\tilde{H}^{(b)}_0 = 0$ and, if $n \geq 1$

$$\tilde{H}^{(b)}_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^b}.$$
We wish to get a formula for the sum

\[
\sum_{n=1}^{\infty} \frac{(-1)^n \tilde{H}^{(2a)}_{n-1}}{n}
\]  

(32)

when \( a \) is a positive integer. This sum exists because we can write \( \tilde{H}^{(2a)}_{n-1} = \tilde{\zeta}(2a) + r_n \), where the “remainder” \( r_n = O\left(\frac{1}{n^{2a}}\right) \) and the series (32) is the sum of two convergent series.

Return to the second line of (5) and its second sum. We have, by absolute convergence

\[
\sum_{n \geq 1, q \geq 1} \frac{(-1)^{n-1}}{q^{2a}} \left( \frac{1}{n} - \frac{1}{n+q} \right) = \lim_{r \to \infty} \left( \sum_{k=1}^{r} \sum_{q+n=k} \frac{(-1)^{n-1}}{q^{2a}n} - \sum_{k=1}^{r} \sum_{q+n=k} \frac{(-1)^{n-1}}{q^{2a}(n+q)} \right). 
\]  

(33)

But the first term in (33) is the Cauchy product of two convergent series, one of them being absolutely convergent and so the limit for \( r \) infinite is \( \ln 2\zeta(2a) \). Thus we have:

\[
\tilde{\zeta}^*(2a, 1) = \ln 2\zeta(2a) + \sum_{j=1}^{2a-2} (-1)^j \tilde{\zeta}(j+1)\zeta(2a-j) - \sum_{k=1}^{\infty} \sum_{q+n=k} \frac{(-1)^{n-1}}{q^{2a}(n+q)}.
\]

But

\[
\sum_{q+n=k} \frac{(-1)^{n-1}}{q^{2a}(n+q)} = -\frac{(-1)^{k-1}}{k} \sum_{q=1}^{k-1} \frac{(-1)^{q-1}}{q^{2a}} = -\frac{(-1)^{k-1} \tilde{H}^{(2a)}_{k-1}}{k}, \text{ which gives}
\]

\[
\tilde{\zeta}^*(2a, 1) = \ln 2\zeta(2a) + \sum_{j=1}^{2a-2} (-1)^j \tilde{\zeta}(j+1)\zeta(2a-j) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \tilde{H}^{(2a)}_{k-1}}{k}.
\]

By the expression (19) of \( \tilde{\zeta}^*(2a, 1) \) we get:

\[
\sum_{n=1}^{\infty} \frac{(-1)^n \tilde{H}^{(2a)}_{n-1}}{n} = \ln 2\zeta(2a) + \sum_{j=1}^{2a-2} (-1)^j \tilde{\zeta}(j+1)\zeta(2a-j) - \left( a + \frac{1}{2} \right) \tilde{\zeta}(2a+1) + \frac{1}{2} \tilde{\zeta}(2a+1) + \sum_{j=1}^{a-1} \tilde{\zeta}(2j)\zeta(2a+1 - 2j).
\]

Separating odd indices \( j \) from even one’s in the first sum of right hand member, we obtain:

\[
\sum_{n=1}^{\infty} \frac{(-1)^n \tilde{H}^{(2a)}_{n-1}}{n} = \frac{1}{2} \zeta(2a+1) - \left( a + \frac{1}{2} \right) \tilde{\zeta}(2a+1) + \ln 2\zeta(2a) + \sum_{j=1}^{a-1} \tilde{\zeta}(2j+1)\zeta(2a-2j).
\]  

(34)

**Examples**

\[
\sum_{n=1}^{\infty} \frac{(-1)^n \tilde{H}^{(2)}_{n-1}}{n} = \ln 2\zeta(2) + \frac{1}{2} \zeta(3) - \frac{3}{2} \tilde{\zeta}(3) = \ln 2\zeta(2) - \frac{5}{8} \zeta(3).
\]
\[
\sum_{n=1}^{\infty} \frac{(-1)^n \tilde{H}_{n-1}^{(4)}}{n} = \ln 2 \zeta(4) - \frac{59}{32} \zeta(5) - \frac{3}{4} \zeta(3) \zeta(2).
\]

V. Other new sums

We define for all integers \( t \geq 1 \) and \( s \geq 2 \), \( S_n^{(t)} = \sum_{k=1}^{n} \frac{1}{(2k-1)^t} \) and

\[
\sigma(s, t) = \sum_{n \geq 1} \frac{S_n^{(t)}}{n^s}
\]

(whose “weight” is \( s + t \)) and we investigate some relations between these new sums, by imitation of well known one’s regarding the \( \zeta(s, t) = \sum_{n \geq 1} \frac{H_n^{(t)}}{n^s} \). For these last sums, results was first given by Euler for odd weights \( \leq 13 \) and more recently established for all odd weights by D. Borwein, J.M. Borwein and R. Girgensohn in [1] and Flajolet, Salvy in [2].

V.1 Linear relations between \( \sigma(s, t) \) of same weight \( s + t \) and \( \sum_{n \geq 1} \frac{H_n}{(2n+1)^{s+t-1}} \)

We write

\[
\sigma(s, t) = \sum_{n \geq m \geq 1} \frac{1}{n^s(2m-1)^t} = \sum_{p \geq 0} \sum_{m \geq 1} \frac{1}{(m+p)^s(2m-1)^t}.
\]

Recall the decomposition in \( \mathbb{C}[[u]] \):

\[
\frac{1}{(r-u)^tu^s} = \sum_{i=0}^{s-1} \binom{t+i-1}{i} \frac{1}{r^{t+i}u^{s-i}} + \sum_{j=0}^{t-1} \binom{s+j-1}{j} \frac{1}{r^{s+j}(r-u)^{t-j}}, \tag{35}
\]

where \( r \) is some parameter. We set \( u = m+p, r = \frac{1}{2}+p \) and we get, by (35):

\[
\frac{1}{(m+p)^s(2m-1)^t} = (-1)^t \sum_{i=0}^{s-1} \binom{t+i-1}{i} 2^i (2p+1)^{t+i}(m+p)^{s-i} + 2^s \sum_{j=0}^{t-1} \binom{s+j-1}{j} (-1)^j (2p+1)^{s+j}(2m-1)^{t-j}.
\]

We cut the right hand of previous equality in three parts:

\[
A(m, p) = (-1)^t \sum_{i=0}^{s-2} \binom{t+i-1}{i} 2^i (2p+1)^{t+i}(m+p)^{s-i},
\]

\[
B(m, p) = 2^s \sum_{j=0}^{t-2} \binom{s+j-1}{j} (-1)^j (2p+1)^{s+j}(2m-1)^{t-j}.
\]
and the part corresponding to \( i = s - 1 \) and \( j = t - 1 \) (then the corresponding binomial coefficients are equal):

\[
C(m, p) = (-1)^t 2^{s-1} \binom{s + t - 2}{s - 1} \times \frac{1}{(2p + 1)^{s+t-1}} \left( \frac{1}{m + p} - \frac{2}{2m - 1} \right)
\]

(note that the sum of residue must be 0). Now we have

\[
\sigma(s, t) = \sum_{p \geq 0, m \geq 1} A(m, p) + B(m, p) + C(m, p),
\]

\[
\sum_{p \geq 0, m \geq 1} A(m, p) = (-1)^t \sum_{i=0}^{s-2} 2^i \binom{t + i - 1}{i} \sigma(s - i, t + i),
\]

\[
\sum_{p \geq 0, m \geq 1} B(m, p) = 2^s \sum_{j=0}^{t-2} (-1)^j \binom{s + j - 1}{j} \lambda(s + j) \lambda(t - j).
\]

In order to evaluate the last term, we write

\[
\frac{1}{m + p} - \frac{2}{2m - 1} = \frac{1}{m + p} - \frac{1}{m} + 2 \left( \frac{1}{2m} - \frac{1}{2m - 1} \right).
\]

By summing (in \( m \)) we get \(-H_p - 2 \ln 2\), for any \( p \geq 0 \) and so

\[
\sum_{p \geq 0, m \geq 1} C(m, p) = (-1)^t 2^{s-1} \binom{s + t - 2}{s - 1} \sum_{p \geq 1} \frac{H_p}{(2p + 1)^{s+t-1}}
\]

\[
+ (-1)^t 2^s \binom{s + t - 2}{s - 1} \lambda(s + t - 1) \ln 2.
\]

It follows the general relation

\[
(-1)^t \sigma(s, t) = \sum_{i=0}^{s-2} 2^i \binom{t + i - 1}{i} \sigma(s - i, t + i)
\]

\[
+ (-1)^t 2^s \sum_{j=0}^{t-2} (-1)^j \binom{s + j - 1}{j} \lambda(s + j) \lambda(t - j)
\]

\[
- 2^{s-1} \binom{s + t - 2}{s - 1} \sum_{p \geq 1} \frac{H_p}{(2p + 1)^{s+t-1}} - 2^s \binom{s + t - 2}{s - 1} \lambda(s + t - 1) \ln 2
\]

(36)

If \( t = 1 \) and \( s \geq 2 \), the relation (36) reduces to

\[
\sigma(s, 1) + \sum_{i=1}^{s-2} 2^{i-1} \sigma(s - i, 1 + i) = 2^{s-2} \sum_{p \geq 0} \frac{H_p}{(2p + 1)^s} + 2^{s-1} \lambda(s) \ln 2.
\]

(36.1)
When \( s = 2 \), it gives \( \sigma(2, 1) = \sum_{p \geq 1} \frac{H_p}{(2p + 1)^2} + 2\lambda(2) \ln 2 \), which results also from (23) and (32.1).

When \( t \) is even, the left hand member and the first term of right hand member cancel out and we can write, for all \( t = 2r, r \geq 1 : \)

\[
\begin{align*}
\sum_{i=1}^{s-2} 2^{i-1} \binom{2r + i - 1}{i} \sigma(s - i, 2r + i) + 2^{s-1} \sum_{j=0}^{2r-2} (-1)^j \binom{s + j - 1}{j} \lambda(s + j) \lambda(2r - j) \\
-2^{s-2} \binom{s + 2r - 2}{s - 1} \sum_{p \geq 0} \frac{H_p}{(2p + 1)^{s+2r-1}} - 2^{s-1} \binom{s + 2r - 2}{s - 1} \lambda(s + 2r - 1) \ln 2 = 0
\end{align*}
\]

(37)

V.2 Application to sums \( \sum_{p \geq 1} \frac{H_p}{(2p + 1)^{2r+1}} \)

For example, when \( s = 2 \) and \( r = 1 \) we get the relation

\[
0 = 4\lambda(2)^2 - 4 \sum_{p \geq 0} \frac{H_p}{(2p + 1)^3} - 8 \ln 2 \lambda(3).
\]

But \( \lambda(2)^2 = \frac{\pi^4}{64} \) et \( \lambda(3) = \frac{7}{8} \zeta(3) \). So, we obtain:

\[
\sum_{p \geq 1} \frac{H_p}{(2p + 1)^3} = \lambda(2)^2 - 2\lambda(3) \ln 2 = \frac{\pi^4}{64} - \frac{7}{4} \zeta(3) \ln 2.
\]

(38)

More generally, \( s = 2 \) gives following formula (compare with formula (32.1))

\[
\sum_{p \geq 1} \frac{H_p}{(2p + 1)^{2r+1}} = -2\lambda(2r + 1) \ln 2 + \frac{1}{r} \sum_{j=0}^{2r-2} (-1)^j (j + 1) \lambda(2 + j) \lambda(2r - j).
\]

In order to transform the previous sum for \( r \geq 2 \), denote it by \( D \) and set \( k = 2r - 2 - j \). We get

\[
D = \sum_{k=0}^{2r-2} (-1)^k (2r - 1 - k) \lambda(2r - k) \lambda(2 + k) = \sum_{j=0}^{2r-2} (-1)^j (2r - 1 - j) \lambda(2r - j) \lambda(2 + j)
\]

Writing \( D = \frac{1}{2} (D + D) \), we get

\[
\frac{D}{r} = \sum_{j=0}^{2r-2} (-1)^j \lambda(2 + j) \lambda(2r - j) = \sum_{q=1}^{r} \lambda(2q) \lambda(2r - 2q + 2) - \sum_{q=1}^{r-1} \lambda(2q + 1) \lambda(2r - 2q + 1).
\]

But the first sum in the right hand side is \( \left( r + \frac{1}{2} \right) \lambda(2r + 2) \). This last fact follows from the identity (valid for all integer \( n \geq 1) :\)

\[
\sum_{j=1}^{n-1} \lambda(2j) \lambda(2n - 2j) = \left( n - \frac{1}{2} \right) \lambda(2n), \quad (E)
\]
This classical identity can be proved by using the expansion (for $|x| < 1$):

$$\frac{\pi x}{4} \tan \frac{\pi x}{2} = \sum_{n \geq 1} \lambda(2n)x^{2n}.$$ 

So, by setting $r = a - 1$, we obtain:

$$\sum_{p \geq 1} \frac{H_p}{(2p + 1)^{2a-1}} = -2\lambda(2a - 1) \ln 2 + \left(a - \frac{1}{2}\right)\lambda(2a) - \sum_{q=1}^{a-2} \lambda(2q + 1)\lambda(2a - 2q - 1).$$  \hspace{1cm} (39) = (d)

When $a = 2b$ the relation (39) becomes:

$$\sum_{p \geq 1} \frac{H_p}{(2p + 1)^{4b-1}} = -2\lambda(4b - 1) \ln 2 + \left(2b - \frac{1}{2}\right)\lambda(4b) - 2\sum_{q=1}^{b-1} \lambda(2q + 1)\lambda(4b - 2q - 1)$$ 

and when $a = 2b + 1$, it becomes

$$\sum_{p \geq 1} \frac{H_p}{(2p + 1)^{4b+1}} = -2\lambda(4b + 1) \ln 2 + \left(2b + \frac{1}{2}\right)\lambda(4b + 2)$$

$$- \lambda^2(2b + 1) - 2\sum_{q=1}^{b-1} \lambda(2q + 1)\lambda(4b - 2q + 1)$$ 

\hspace{1cm} (39.1) = (e')

**Examples:**

1. If $b = 1$ (39.1) gives

$$\sum_{p \geq 1} \frac{H_p}{(2p + 1)^3} = -2\lambda(3) \ln 2 + \frac{3}{2}\lambda(4),$$

which is the same thing as (38).

2. If $b = 2$ (39.1) gives

$$\sum_{p \geq 1} \frac{H_p}{(2p + 1)^7} = -2\lambda(7) \ln 2 + \frac{7}{2}\lambda(8) - 2\lambda(3)\lambda(5).$$

3. If $b = 1$ (39.2) gives

$$\sum_{p \geq 1} \frac{H_p}{(2p + 1)^5} = -2\lambda(5) \ln 2 + \frac{5}{2}\lambda(6) - \lambda^2(3)$$

and for $b = 2$:

$$\sum_{p \geq 1} \frac{H_p}{(2p + 1)^9} = -2\lambda(9) \ln 2 + \frac{9}{2}\lambda(10) - 2\lambda(3)\lambda(7).$$

V.3 Linear relations between the $\sigma$’s of same weight

(1) A general relation

(a) In (37) we put $s = 2v$ and replace the sum $\sum_{p \geq 1} \frac{H_p}{(2p + 1)^{2v+2r-1}}$ in it by the value
given by (39) when \( a = v + r \). So we obtain (remark that the term with \( \ln 2 \) cancels out)

\[
\begin{align*}
\sum_{i=1}^{2v-2} 2^{i-1} \binom{2r + i - 1}{i} \sigma(2v - i, 2r + i) \\
+ 2^{2v-1} \sum_{j=0}^{2r-2} (-1)^j \binom{2v + j - 1}{j} \lambda(2v + j)\lambda(2r - j) \\
- 2^{2v-3} \binom{2v + 2r - 2}{2v - 1} (2v + 2r - 1)\lambda(2v + 2r) \\
+ 2^{2v-2} \binom{2v + 2r - 1}{2v - 1} \sum_{j=1}^{r+v-2} \lambda(2j + 1)\lambda(2r + 2v - 2j - 1) = 0
\end{align*}
\]  

(40.1)

(b) Now we put \( s = 2v + 1 \) in (37), by use of (32.1) (in which we put \( a = v + r \) ); we obtain similarly:

\[
\begin{align*}
\sum_{i=1}^{2v-1} 2^{i-1} \binom{2r + i - 1}{i} \sigma(2v + 1 - i, 2r + i) \\
+ 2^{2v} \sum_{j=0}^{2r-2} (-1)^j \binom{2v + j}{j} \lambda(2v + 1 + j)\lambda(2r - j) \\
- 2^{2v} \binom{2v + 2r - 1}{2v} (v + r)\lambda(2v + 2r + 1) \\
+ 2^{2v} \binom{2v + 2r - 1}{2v} \sum_{j=1}^{r+v-1} \lambda(2j)\lambda(2r + 2v - 2j + 1) = 0
\end{align*}
\]  

(40.2)

(2) Application to a formula for \( \sigma(2, 2a - 1) \)

In (36) we set \( s = 2 \) and \( t = 2a - 1 \), where \( a \) is an integer \( \geq 2 \). This furnishes, by setting

\[
h_q := \sum_{p \geq 1} \frac{H_p}{(2p + 1)^q}
\]

\[
\sigma(2, 2a - 1) = 2 \sum_{j=0}^{2a-3} (-1)^j (j + 1)\lambda(j + 2)\lambda(2a - 1 - j) + (2a - 1)[h_{2a} + 2\lambda(2a) \ln 2].
\]

(41.1)

The sum in square brackets is given by (32.1) and we obtain:

\[
\frac{1}{2} \sigma(2, 2a - 1) = \sum_{j=0}^{2a-3} (-1)^j (j + 1)\lambda(j + 2)\lambda(2a - 1 - j) \\
+ a(2a - 1)\lambda(2a + 1) - (2a - 1) \sum_{j=1}^{a-1} \lambda(2j)\lambda(2a + 1 - 2j).
\]

(41.2)
By splitting the first sum in (41.2) in $\sum_{j=1}^{a-1} (2j - 1)\lambda(2j)\lambda(2a + 1 - 2j)$ corresponding to even indices and $-\sum_{j=1}^{a-1} (2a - 2j)\lambda(2a + 1 - 2j)\lambda(2j)$ corresponding to odd indices, (41.2) becomes (after changing $j$ into $a-j$)

$$
\sigma(2, 2a - 1) = 2a(2a - 1)\lambda(2a + 1) - 8 \sum_{j=1}^{a-1} j\lambda(2a - 2j)\lambda(2j + 1). \quad (42) = (f)
$$

(3) Examples and applications

$\begin{align*}
a = 1 : & \quad \sigma(2, 1) = 2\lambda(3) = \frac{7}{4}\zeta(3) \\
a = 2 : & \quad \sigma(2, 3) = 12\lambda(5) - 8\lambda(2)\lambda(3) = \frac{93}{8}\zeta(5) - \frac{21}{4}\zeta(2)\zeta(3) \\
a = 3 : & \quad \sigma(2, 5) = 30\lambda(7) - 8\lambda(4)\lambda(3) - 16\lambda(5)\lambda(2)
\end{align*}$

If we test $s = 3$ in (36.1), we get $\sigma(3, 1) + \sigma(2, 2) = 2 \sum_{p \geq 1} \frac{H_p}{(2p + 1)^3} + 4\lambda(3)\ln 2$, that is

$$
\sigma(3, 1) + \sigma(2, 2) = 3\lambda(4) = \frac{\pi^4}{32} = \frac{45}{16}\zeta(4). \quad (43)
$$

We have already seen that

$$
J(3) = \sigma(3, 1) = 8\text{li}_4 \frac{1}{2} - 2\zeta(2)\ln^2 2 + 7\zeta(3)\ln 2 + \frac{1}{3}\ln^4 2 - \frac{53}{8}\zeta(4).
$$

Hence (43) furnishes :

$$
\sigma(2, 2) = -8\text{li}_4 \frac{1}{2} + 2\zeta(2)\ln^2 2 - \frac{1}{3}\ln^4 2 - 7\zeta(3)\ln 2 + \frac{151}{16}\zeta(4).
$$

Setting $s = 4$ in (36.1) we get

$$
\sigma(3, 2) + 2\sigma(2, 3) = -\sigma(4, 1) + 4 \sum_{p \geq 1} \frac{H_p}{(2p + 1)^4} + 8\lambda(4)\ln 2.
$$

By the previous results, this gives

$$
\sigma(3, 2) = -\frac{31}{2}\zeta(5) + \frac{35}{4}\zeta(2)\zeta(3) = -16\lambda(5) + \frac{40}{3}\lambda(2)\lambda(3).
$$
(4) Application of a result of [6] to a formula for $\sigma(2a - 1, 2)$

In [6], the authors gives a formula which leads to the calculation of $\sigma(2a - 1, 2)$. They define, for $q \geq 2$,

$$E_{p, q} = \sum_{n \geq 1} \frac{1}{n^q} \sum_{1 \leq k \leq 2n} \frac{1}{k^p}. \tag{44}$$

With our notations, this turns to

$$E_{p, q} = \sigma(q, p) + \frac{1}{2^p} \zeta^*(q, p),$$

where

$$\zeta^*(q, p) = \sum_{1 \leq m \leq n} \frac{1}{n^q m^p} = \zeta(p + q) + \zeta(q, p).$$

In particular, we get

$$\sigma(2a - 1, 2) = E_{2, 2a - 1} - \frac{1}{4} \zeta^*(2a - 1, 2)$$

Theorem 2 in [6] give a formula for $E_{2, 2a - 1}$ which may be written as:

$$E_{2, 2a - 1} = \sum_{j=1}^{a-2} j 2^{2j} \zeta(2j + 1) \zeta(2a - 2j) + \left( (2a + 1)2^{2a-3} - \frac{1}{2} \right) \zeta(2) \zeta(2a - 1)$$

$$- \left( a2^{2a-1} + \frac{2a^2 - a - 1}{8} \right) \zeta(2a + 1). \tag{45}$$

But (see [1] or [2], th. (3.1))

$$\zeta^*(2a - 1, 2) = -\frac{1}{2} (2a^2 + a - 1) \zeta(2a + 1) + (2a - 1) \zeta(2) \zeta(2a - 1) + 2 \sum_{j=1}^{a-2} j \zeta(2j + 1) \zeta(2a - 2j),$$

so we obtain, by some manipulations:

$$\sigma(2a - 1, 2) = -a2^{2a-1} \lambda(2a + 1) + \frac{2^{2a-1}(2a + 1)}{3} \lambda(2) \lambda(2a - 1)$$

$$+ \sum_{j=1}^{a-2} j 2^{2j} \lambda(2j + 1) \zeta(2a - 2j) \tag{46} = (g)$$

When $a = 2$ this last formula gives again

$$\sigma(3, 2) = -16 \lambda(5) + \frac{40}{3} \lambda(2) \lambda(3).$$

For $a = 3$, (46) gives:

$$\sigma(5, 2) = -96 \lambda(7) + \frac{224}{3} \lambda(2) \lambda(5) + 4\lambda(3) \zeta(4) = -96 \lambda(7) + \frac{224}{3} \lambda(2) \lambda(5) + \frac{64}{15} \lambda(3) \lambda(4)$$
(5) A first weighted sum of \( \sigma \)'s.

Generally, the relation (36.1) allows us to obtain the sum \( \sum_{i=1}^{s-2} 2^{i-1} \sigma(s - i, 1 + i) \) when \( s \) is even and \( \geq 4 \). Let be \( s = 2a, a \geq 2 \)

\[
\sum_{i=1}^{2a-2} 2^{i-1} \sigma(2a - i, 1 + i) = -J(2a) + 2^{2a-2}h_{2a} + 2^{2a-1}\lambda(2a) \ln 2. \quad (47)
\]

In (47) \( J(2a) \) is given by (23) and \( h_{2a} \) by (32.1):

\[
\sum_{i=1}^{2a-2} 2^{i-1} \sigma(2a - i, 1 + i) = -(2^{2a-1}\lambda(2a + 1) - \sum_{j=1}^{a-1} 2^{2j}\lambda(2j + 1)\zeta(2a - 2j)) + 2^{2a-2}(-2\lambda(2a) \ln 2 + 2a\lambda(2a + 1) - 2 \sum_{j=1}^{a-1} \lambda(2j)\lambda(2a + 1 - 2j)) + 2^{2a-1}\lambda(2a) \ln 2. \quad (48)
\]

In (47) the term in \( \lambda(2a) \ln 2 \) vanishes:

\[
\sum_{i=1}^{2a-2} 2^{i-1} \sigma(2a - i, 1 + i) = -(2^{2a-1}\lambda(2a + 1) - \sum_{j=1}^{a-1} 2^{2j}\lambda(2j + 1)\zeta(2a - 2j)) + 2^{2a-2}(2a\lambda(2a + 1) - 2 \sum_{j=1}^{a-1} \lambda(2j)\lambda(2a + 1 - 2j))
\]

\[
= (a - 1)2^{2a-1}\lambda(2a + 1) + \sum_{j=1}^{a-1} 2^{2a-2j}\zeta(2j)\lambda(2a + 1 - 2j)
\]

\[
- 2^{2a-1}\sum_{j=1}^{a-1} \lambda(2j)\lambda(2a + 1 - 2j).
\]

In terms of series \( \lambda \) this is:

\[
\sum_{i=1}^{2a-2} 2^{i-1} \sigma(2a - i, 1 + i) = 2^{2a-1}[(a - 1)\lambda(2a + 1) + \sum_{j=1}^{a-1} \frac{3 - 2^{2j}}{2^{2j} - 1}\lambda(2j)\lambda(2a + 1 - 2j)],
\]

but it is lightly simpler to write

\[
\sum_{i=1}^{2a-2} 2^{i-1} \sigma(2a - i, 1 + i) = 2^{2a-1}[(a - 1)\lambda(2a + 1) + \sum_{j=1}^{a-1} (3 \cdot 2^{-2j} - 1)\zeta(2j)\lambda(2a + 1 - 2j)].
\]

In the left hand member we pick the term corresponding to \( i = 2a - 2 \). So, in view of (42) we obtain

\[
\sum_{i=1}^{2a-3} 2^{i-1} \sigma(2a - i, 1 + i) = -2^{2a-2}[(2a^2 - 3a + 2)\lambda(2a + 1)
\]

\[
- 2 \sum_{j=1}^{a-1} \left( \frac{3 - 2^{2j}}{2^{2j} - 1} + 2a - 2j \right) \lambda(2j)\lambda(2a + 1 - 2j). \quad (49)
\]
We return to (37), in which we put now \( t = 2r = 2 \):

\[
\sum_{i=1}^{s-2} 2^{i-1}(i+1)\sigma(s-i, 2+i) = -2^{s-1}\lambda(s)\lambda(2) + 2^{s-2}sh_{s+1} + 2^{s-1}s\lambda(s+1)\ln 2.
\]

If \( s = 4 \), this gives

\[
2\sigma(3, 3) + 6\sigma(2, 4) = -8\lambda(4)\lambda(2) + 16h_5 + 32\ln 2\lambda(5).
\]

Taking account of the value of \( h_5 \) given after (39.2) and the relation \( \lambda(4)\lambda(2) = \frac{5}{4}\lambda(6) \) we obtain

\[
\sigma(3, 3) + 3\sigma(2, 4) = 15\lambda(6) - 8\lambda^2(3).
\]

(50)

One can verify that making \( s = t = 3 \) or \( s = 4, t = 2 \) in (36) gives the same result.

(6) Other linear links between the sum \( \sigma(k, \ell) \)

Now we give new linear relations similar to those linking the \( \zeta(k, \ell) \).

For integers \( k, \ell \geq 2 \) we can write

\[
\lambda(k)\lambda(\ell) = \sum_{1 \leq n \leq m} \frac{1}{(2n-1)^k(2m-2n+1)^\ell}.
\]

(51)

By using the decomposition in partial rational fractions given by (35), we get :

\[
\frac{1}{(2n-1)^k(2m-2n+1)^\ell} = \frac{1}{2^\ell} \sum_{i=0}^{k-1} \binom{\ell + i - 1}{i} \frac{2^{-i}}{m^{\ell+i}(2n-1)^{k-i}} + \frac{1}{2^k} \sum_{j=0}^{\ell-1} \binom{k + j - 1}{j} \frac{2^{-j}}{m^{k+j}(2m-2n+1)^{\ell-j}}.
\]

(52)

So, by summation, (52) gives:

\[
\lambda(k)\lambda(\ell) = \frac{1}{2^\ell} \sum_{i=0}^{k-1} 2^{-i} \binom{\ell + i - 1}{i} \sigma(\ell + i, k - i) + \frac{1}{2^k} \sum_{j=0}^{\ell-1} 2^{-j} \binom{k + j - 1}{j} \sigma(k + j, \ell - j).
\]

(53)

In order to verify, test with \( k = \ell = 2 : 4\lambda(2)^2 = 2(\sigma(2, 2) + \sigma(3, 1)) \), which gives the following already seen relation:

\[
\sigma(2, 2) + \sigma(3, 1) = 2\lambda(2)^2 = \frac{\pi^4}{32}.
\]
VI. Sum formula for the \( \sigma \)'s of same weight

We observe that \( J(2) = \sigma(2, 1) = 2\lambda(3) \), by (23), while the relation (46) gives us \( \sigma(2, 2) + \sigma(3, 1) = 3\lambda(4) \). By using the previous relations giving \( \sigma(4, 1), \sigma(3, 2) \) and \( \sigma(2, 3) \), it is easy to see that the sum of the \( \sigma \) series of weight 5 is \( 4\lambda(5) \).

So, this suggests to us that the following formula
\[
\sum_{i=1}^{w-2} \sigma(w-i, i) = (w-1)\lambda(w)
\]
holds for all weights \( w \geq 3 \). Particular checking of this relation for \( w \leq 10 \) have given to us the idea of the proof of the general result. This result is similar to those concerning the classical MZV: the so-call theorem of the sum asserts that the sum of all MZV of same depth (number of variable) and of same weight \( w \) is \( \zeta(w) \). For a proof, see Zagier [7] or Granville [3]. When the depth is 2, this gives
\[
\sum_{i=1}^{w-2} \zeta(w-i, i) = \zeta(w) \quad \text{(note that this last relation appears naturally in some proofs of Euler formula (2.1)).}
\]

Since \( \zeta^*(s, t) = \sum_{1 \leq m \leq n} \frac{1}{n^s m^t} = \zeta(s, t) + \zeta(s + t) \) we have:
\[
\sum_{i=1}^{w-2} \zeta^*(w-i, i) = (w-1)\zeta(w),
\]
which gives us a further motivation for the following theorem.

**Theorem.** [of the \( \sigma \)-sum] The sum of all the \( \sigma \)'s of same weight is calculable: for \( w \geq 3 \), we have:
\[
\sum_{i=1}^{w-2} \sigma(w-i, i) = (w-1)\lambda(w) \quad \text{(54) = (h)}
\]

Since \( \sigma(2, 1) = 2\lambda(3)(\text{see section V.3.3}) \), it suffices to prove (54) when \( w \geq 4 \). We start by transform the relation (53), for \( w = k + \ell \). First we set
\[
x_i = \sigma(w-i, i)
\]
for \( 1 \leq i \leq w-2 \). Put \( i' = k - i \) in the first summation of (53). So it becomes :
\[
\frac{1}{2^\ell} \sum_{i'=1}^{k} 2^{i'-k} \binom{w-i' - 1}{k-i'} \sigma(w-i', i') = \frac{1}{2^w} \sum_{i=1}^{k} 2^i \binom{w-i-1}{\ell-1} x_i = \frac{1}{2^w} \sum_{i=1}^{w-2} 2^i \binom{w-i-1}{\ell-1} x_i,
\]
because the binomial numbers in the last sum vanish if \( k+1 \leq i \leq w-2 \). By proceeding in same way for the second sum in (53) we obtain finally:
\[
\lambda(k)\lambda(\ell) = \frac{1}{2^w} \sum_{i=1}^{w-2} 2^i \left[ \binom{w-i-1}{\ell-1} + \binom{w-i-1}{k-1} \right] x_i.
\]

At this stage, the proof depends on wether the weight is odd or not.

**VI.1 The case \( w \) odd**

We set \( w = 2a+1 \). Return to the relation (42), which can be write:
\[
x_{2a-1} = 2a(2a-1)\lambda(2a+1) - 8A,
\]
(56.1)
by setting \( A = \sum_{j=1}^{a-1} j \lambda (2a - 2j) \lambda (2j + 1) \).

But we have, by (55) applied to \( k = 2j + 1, \ell = 2a - 2j, 1 \leq j \leq a - 1, \)

\[
2^{2a+1} A = \sum_{i=1}^{2a-1} 2^i x_i \sum_{j=1}^{a-1} j \left[ \binom{2a - i}{2a - 2j - 1} + \binom{2a - i}{2j} \right].
\]  

(56.2)

We denote by \( A_i \) the inner sum of relation (56.2): we prove that \( 2^i A_i \) does not depend of \( i \) when \( 1 \leq i \leq 2a - 2 \).

In the first sum of \( A_i \), we exchange \( j \) into \( a - j \). So we get:

\[
A_i = a \sum_{j=1}^{a-1} \binom{2a - i}{2j - 1} + \sum_{j=1}^{a-1} j \binom{2a - i}{2j} - \sum_{j=1}^{a-1} j \binom{2a - i}{2j - 1}.
\]  

(57)

Then we evaluate each sum in (57). We use the binomial expansions

\[
(1 + x)^{2a-i} + (1 - x)^{2a-i} = 2 \sum_{0 \leq j \leq a-1} \binom{2a - i}{2j} x^{2j} = 2 \sum_{0 \leq j \leq a-1} \binom{2a - i}{2j} x^{2j},
\]  

(58)

\[
(1 + x)^{2a-i} - (1 - x)^{2a-i} = 2 \sum_{1 \leq j \leq a-i} \binom{2a - i}{2j - 1} x^{2j-1} = 2 \sum_{1 \leq j \leq a-1} \binom{2a - i}{2j - 1} x^{2j-1}
\]  

(59)

and their derivatives:

\[
(2a - i)((1 + x)^{2a-i-1} - (1 - x)^{2a-i-1}) = 4 \sum_{0 \leq j \leq a-1} j \binom{2a - i}{2j} x^{2j-1},
\]  

(60)

\[
(2a - i)((1 + x)^{2a-i-1} + (1 - x)^{2a-i-1}) = 2 \sum_{1 \leq j \leq a-1} (2j - 1) \binom{2a - i}{2j - 1} x^{2j-2}.
\]  

(61)

In the last formulas, we have introduce null binomial numbers in order to extend the summation to all integer lying between 1 and \( a - 1 \).

Now, suppose that \( 1 \leq i \leq 2a - 2 \). In (59), (60), (61) we set \( x = 1 \), so this gives

\[
2 \sum_{0 \leq j \leq a-1} \binom{2a - i}{2j - 1} = 2^{2a-i},
\]  

(62)

\[
4 \sum_{1 \leq j \leq a-1} j \binom{2a - i}{2j} = (2a - i)2^{2a-i-1},
\]  

(63)

\[
2 \sum_{1 \leq j \leq a-1} (2j - 1) \binom{2a - i}{2j - 1} = (2a - i)2^{2a-i-1}.
\]  

(64)

By relations (62) and (64) we obtain:

\[
4 \sum_{1 \leq j \leq a-1} j \binom{2a - i}{2j - 1} = (2a - i)2^{2a-i-1} + 2^{2a-i}.
\]  

(65)
Then, we put this results in relation (57) (when $1 \leq i \leq 2a - 2$):

$$A_i = a 2^{2a-i-1} - 2^{a-i-2} = 2^{2a-2-i}(2a - 1).$$

So we get finally, when $1 \leq i \leq 2a - 2$:

$$2^i A_i = 2^{2a-2}(2a - 1). \quad (66)$$

It remains the case when $i = 2a - 1$. Without difficulty, one obtains:

$$A_{2a-1} = a - 1. \quad (67)$$

Now we put this calculations in (56.2). This gives

$$2^{2a+1} A = 2^{2a-2}(2a - 1) + 2^{2a-1} x_{2a-1} (a - 1),$$

or

$$8A = (2a - 1) \sum_{i=1}^{2a-2} x_i + (2a - 2) x_{2a-1}.$$  

We put this last result in (56.1) and the theorem is proved for odd weight.

**VI.2 The case $w$ even**

The proof is similar, but easier. Let be $a \geq 2$. In the relation (55), we set $k = 2j$ and $\ell = 2a - 2j$, for $1 \leq j \leq a - 1$. So we have $w = 2a$ and

$$2^a \lambda(2j) \lambda(2a - 2j) = \sum_{i=1}^{2a-2} 2^i \left[ \binom{2a-i-1}{2a-2j-1} + \binom{2a-i-1}{2j-1} \right] x_i. \quad (68)$$

Summing the last equalities for $1 \leq j \leq a - 1$ and using the formula (E) of section V.2 we get:

$$2^{2a-1}(2a - 1) \lambda(2a) = \sum_{i=1}^{2a-2} 2^j x_j \sum_{i=1}^{a-1} \left[ \binom{2a-i-1}{2a-2j-1} + \binom{2a-i-1}{2j-1} \right]. \quad (69)$$

But, by changing $j$ into $a - j$, we see that

$$\sum_{j=1}^{a-1} \binom{2a-i-1}{2a-2j-1} = \sum_{j=1}^{a-1} \binom{2a-i-1}{2j-1}. \quad (70)$$

As in the odd case, we set $x = 1$ in the binomial expansion:

$$(1+x)^{2a-i-1} - (1-x)^{2a-i-1} = 2 \sum_{1 \leq j \leq a-\frac{i}{2}} \binom{2a-i-1}{2j-1} x^{2j-1} = 2 \sum_{1 \leq j \leq a-1} \binom{2a-i-1}{2j-1} x^{2j-1},$$
(if \( j > a - \frac{i}{2} \), the corresponding binomial coefficient vanishes). So we get:

\[
2 \sum_{1 \leq j \leq a-1} \binom{2a - i - 1}{2j - 1} = 2^{2a-i-1}.
\]

By transporting this value in (70), we obtain

\[
\sum_{i=1}^{2a-2} x_i = (2a - 1)\lambda(2a),
\]

which is the wanted result for \( w = 2a \).

**VII. The explicit evaluation of other \( \sigma \) series**

**(1) The case of odd weights**

Suppose that \( w \) is odd: the previous sections give explicit values of the series \( \sigma(2, w - 2) \), \( \sigma(w - 2, 2) \) and \( \sigma(w - 1, 1) \) in terms of \( \zeta \) or \( \lambda \) series. When \( w = 7 \), the unknown series are \( \sigma(4, 3), \sigma(3, 4) \) and the relation (55) furnishes easily the system (where \( x_i = \sigma(7 - i, i) \))

\[
\begin{align*}
6x_3 + 8x_4 &= 32\lambda(2)\lambda(5) - 5x_1 - 5x_2 - 8x_5 \\
4x_3 + 2x_4 &= 16\lambda(3)\lambda(4) - 5x_1 - 5x_2
\end{align*}
\]

which gives, taking account of the values of \( x_1, x_2 \) and \( x_5 \):

\[
\begin{align*}
\sigma(4, 3) &= 120\lambda(7) - 96\lambda(2)\lambda(5) \\
\sigma(3, 4) &= -80\lambda(7) + 8\lambda(3)\lambda(4) + \frac{176}{3}\lambda(2)\lambda(5).
\end{align*}
\]

When \( w = 2a + 1 \geq 9 \), the number of unknown \( \sigma \) series is \( w - 5 = 2a - 4 \), but the relations (55) give only \( a - 1 \) (which is \( < 2a - 4 \)) equations. We think that these equations are independent and that the other linear relations given by the formula (36) for various choices of \( s \) and \( t \) (with \( s + t = 2a + 1 \)) depend on them.

However, we have found an alternative but tedious method to explicit other \( \sigma \) series by integral representation of the finite sums \( S_n^{(t)} \), similar of these exposed in section **II.2**. As example, we sketch out the case of \( \sigma(2a - 1, 2) \), which will give an other proof of formula (46). This method uses trigonometric series which can be expressed by Bernoulli polynomials. It may be useful to compare our method with the computations given in [6].

First, we observe that, for all integer \( k \geq 1 \),

\[
\int_{0}^{\pi} x \cos(2k - 1)x \, dx = \frac{-2}{(2k - 1)^2},
\]

and

\[
-2S_n^{(2)} = \int_{0}^{\pi} x \left( \sum_{k=1}^{n} \cos(2k - 1)x \right) \, dx = \frac{1}{2} \int_{0}^{\pi} \frac{x \sin 2nx}{\sin x} \, dx.
\]

It follows:

\[
-4\sigma(2a - 1, 2) = \int_{0}^{\pi} \frac{\varphi(x)}{\sin x} \, dx,
\]
where 
\[ \varphi(x) = x \sum_{n \geq 1} \frac{\sin 2nx}{n^{2a-1}}. \]

As in in section II.2, we set 
\[ v_m = \int_0^\pi \varphi(x) \sin(2m-1)x \, dx \]
and prove that 
\[ \sigma(2a-1, 2) = -\frac{1}{2} \sum_{m=1}^\infty v_m. \]

Then, we calculate \( v_m \) by successive integrations by part, using at the last step the well known series (when \( 0 < x < \pi \)):
\[ \sum_{n=1}^\infty \frac{\cos 2nx}{n^2} = \zeta(2) - x(\pi - x) \quad \text{and} \quad \sum_{n=1}^\infty \frac{\sin 2nx}{n} = \frac{\pi}{2} - x \]

This process can be extended as follow. From (72), we obtain
\[ \frac{-2}{(2k-1)^2} = \left[ \frac{x^2}{2} \cos(2k-1)x \right]_0^\pi + \frac{1}{2}(2k-1) \int_0^\pi x^2 \sin(2k-1)x \, dx, \]
which gives
\[ \frac{4}{(2k-1)^3} = \frac{\pi^2}{2k-1} - \int_0^\pi x^2 \sin(2k-1)x \, dx \]
and
\[ 4S_n^{(3)} = \pi^2 S_n - \int_0^\pi x^2 \sum_{1 \leq k \leq n} \sin(2k-1)x \, dx = \pi^2 S_n - \int_0^\pi x^2 \frac{\sin^2 nx}{\sin x} \, dx. \]
So:
\[ 4\sigma(b, 3) = \pi^2 \sigma(b, 1) - \int_0^\pi \frac{x^2}{\sin x} \sum_{n=1}^\infty \frac{\sin^2 nx}{n^b} \, dx, \]
which gives, after similar calculations:
\[ \sigma(2a - 2, 3) = a(2a - 1)2^{2a-3} \lambda(2a + 1) \]
\[ - (a - 1)(2a + 3)2^{2a-4} \zeta(2) \lambda(2a - 1) \]
\[ - \sum_{j=2}^{a-2} j(2j - 1)2^{2j-2} \zeta(2a - 2j) \lambda(2j + 1) \]

Step by step, this process can give also \( \sigma(2a - 3, 4) \) etc.

(2) The case of even weights
When the weight is even, it seems that no \( \sigma \) series can be evaluate by classical functions, a part the case \( w = 4 \) which needs the special value \( \text{li}_4 \frac{1}{2} \). If we attempt to explicit this \( \sigma \)'s by using the previous integral method, we fall on the trigonometric Clausen series \( \sum_{n \geq 1} \frac{\cos 2nx}{n^{2a-1}}, \)
and their derivatives (when \( a > 1 \)). This series are not elementary, a part the case \( a = 1 \).
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