On the equilibriums stability in an approximate problem of the dynamics of a rigid body with a suspension point vibrating along an inclined straight line

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Abstract. We consider the heavy rigid body dynamics under the assumption that one of the body points (the suspension point) performs the specified high-frequency vibrations of small amplitude along an inclined straight line, and the body mass centre lies on the principal axis of inertia for the suspension point. In the framework of an approximate autonomous system of canonical equations of motion, the question of existence, number, and stability of the body relative equilibrium positions is solved. It is shown that for all such equilibria, the mass centre radius-vector lies in the vertical plane containing the vibration axis. The number of equilibria is four, six or eight depending on the vibration intensity. Sufficient and necessary conditions for their stability are found. The existence of high-frequency periodic motions of the initial non-autonomous system, which are generated by the investigated equilibria, is justified using Poincaré's method. Conclusions about stability (in linear approximation) of these periodic motions are drawn.

1. Introduction
The influence of high-frequency vibrations on the rigid body or rigid bodies system dynamics is the subject of many studies. Mainly pendulum systems were studied until recently. In the paper [1], approximate autonomous equations of motion of a body with arbitrary mass geometry, whose suspension point performs arbitrary periodic or conditionally periodic fast vibrations of small amplitude in three-dimensional space, are obtained for the first time. This work initiated a number of researches of the dynamics of a body with different mass geometries in the presence of suspension point vibrations [2, 3, 4, 5, 6].

This paper investigates the effect of a suspension point vibrations performed along an inclined straight line on the dynamics of a rigid body with a three-axis ellipsoid of inertia and a mass centre on the principal inertia axis (for a suspension point). Within the framework of approximate autonomous problem the question of existence, number and stability of the body relative equilibrium positions is solved. The existence of high-frequency periodic motions of the initial non-autonomous system generated by the above equilibria is justified using Poincaré’s method, and conclusions about their stability (in linear approximation) are drawn.

Previously, the relative equilibria of a heavy rigid body with arbitrary mass geometry were studied for the case of vertical [2] and horizontal [3] high-frequency harmonic suspension point
vibrations. The dynamics of a mathematical pendulum with a suspension point vibrating along an inclined straight line was studied in [5], where it is shown that the case of the pendulum suspension point arbitrary vibrations in the its vertical motions plane can be reduced to this problem. The dynamics of a Lagrange’s top with a suspension point vibrating along an inclined straight line was studied in [5].

2. Statement of the problem

We consider the motion of a rigid body of mass \(m\) in a uniform gravity field. Let one of the body points \(O\) (suspension point) performs a specified motion relative to a fixed space point \(O_s\) along a straight line inclined at an angle \(\alpha\) to the horizontal \((\alpha \in (0; \pi/2))\), according to the law \(O_sO = \xi(t)\). We assume that the function \(\xi(t)\) is periodic with frequency \(\Omega\), its time average value \(\langle \xi(t) \rangle\) per period is zero.

We introduce a translationally moving coordinate system \(OXYZ\) with the axis \(OZ\) directed vertically upwards and the axis \(OY\) lying in the vertical plane containing the vibration axis, as well as the rigidly attached with the body coordinate system \(Oxyz\) with axes directed along the principal inertia axes of the body for the point \(O\). The orientation of the coordinate system \(Oxyz\) with respect to \(OXYZ\) is given by the Euler angles \(\psi, \theta, \varphi\). We denote the principal moments of inertia by \(A, B\) and \(C\). Assume that the body mass centre \(G\) is on the axis \(Oz\), \(OG = z_G\). Without generality restriction, we assume that \(A > B\).

Let the maximum deviation \(h_s\) of the suspension point from the point \(O_s\) be small compared to the reduced body length \(l = B/(mz_G)\) and the vibration frequency \(\Omega\) of the suspension point is large relative to the characteristic frequency \(\omega_s = \sqrt{g/l}\). We introduce a small parameter \(\varepsilon^2 = h_s/l\) and assume that \(\varepsilon \sim \sqrt{\varepsilon^2 \Omega} (h_s \sim \varepsilon \omega_s)\).

We describe the body motion using the canonical Hamilton equations. By perturbation theory methods it is possible to reduce the Hamiltonian function to a form whose main part is autonomous (see, e.g., [1]). Leaving behind the variables the former notation, we write the Hamiltonian function of the system in the form

\[
H = H_0 + O(\varepsilon),
\]

\[
H_0 = \frac{(A\cos^2 \varphi + B\sin^2 \varphi)(P_\varphi - P_\varphi \cos \theta)^2}{2AB\sin^2 \theta} + \frac{A\sin^2 \varphi + B\cos^2 \varphi}{2AB} P_\theta^2 + \frac{P_\varphi^2}{2C} + \frac{(B - A)\sin \varphi \cos \varphi(P_\psi - P_\psi \cos \theta)}{AB \sin \theta} P_\theta + \hat{\Pi},
\]

\[
\hat{\Pi} = mgz_G \cos \theta + \hat{\Pi}_v,
\]

\[
\hat{\Pi}_v = \frac{m^2 z_G^2 \sigma}{2AB} \left( A\nu^2 + Bu^2 \right),
\]

Here \(P_\psi, P_\theta, P_\varphi\) — impulses conjugate to the Euler angles, \(\hat{\Pi}\) — potential energy of the system, consisting of potential energy of gravity, and vibrational potential \(\hat{\Pi}_v\) [1]. The parameter \(\sigma = \langle \xi^2(t) \rangle\) characterizes the intensity of the suspension point motion.

The accuracy of the solutions of the approximated system with Hamiltonian (2) with respect to the solutions of the full initial system at the time interval \(t \sim \varepsilon^{-1/2}\) is determined by the relations [1]

\[
\tilde{\psi} = \psi + O(\varepsilon^{3/2}), \quad \tilde{\theta} = \theta + O(\varepsilon^{3/2}), \quad \tilde{\varphi} = \varphi + O(\varepsilon^{3/2}),
\]

\[
\tilde{P}_\psi = P_\psi - mz_G \xi \cos \psi \sin \theta + O(\varepsilon^{1/2}),
\]

\[
\tilde{P}_\theta = P_\theta - mz_G \xi \sin \psi \cos \theta + O(\varepsilon^{1/2}), \quad \tilde{P}_\varphi = P_\varphi + O(\varepsilon^{1/2}).
\]
Here the tilde marks the solutions of the full initial system.

The purpose of the paper is to solve the question of existence, number and stability of relative equilibrium positions of the system with approximate Hamiltonian (2), as well as to solve the question of existence and stability (in linear approximation) of high-frequency periodic motions of the full initial system corresponding to these positions.

3. Equilibrium positions
In the first step, we will consider the body motion within an approximate system (2). The system with Hamiltonian \( H_0 \) is conservative, so the equilibrium positions will be searched as stationary points of potential energy \( \tilde{\Pi} \). The equilibrium values of the Euler angles are the solutions of equations

\[
\tilde{\Pi}_\psi = \frac{m^2 z G \sigma}{AB} (Av \cos \psi \cos \varphi - \sin \psi \sin \varphi \cos \theta) - Bu (\cos \psi \sin \varphi + \sin \psi \cos \varphi \cos \theta) = 0, \tag{5}
\]

\[
\tilde{\Pi}_\theta = -mgzG \sin \theta - \frac{m^2 z G \sigma}{AB} (\cos \psi \sin \theta \cos \alpha - \sin \alpha \cos \theta) (Av \sin \varphi + Bu \cos \varphi) = 0, \tag{6}
\]

\[
\tilde{\Pi}_\varphi = \frac{m^2 z G \sigma}{AB} (A - B)uv = 0. \tag{7}
\]

It follows from (7) that \( uv = 0 \). It can be shown that at \( \sin \psi \neq 0 \) this condition is incompatible with equations (5) and (6).

In the case of \( \sin \psi = 0 \), the condition \( uv = 0 \) take a form

\[
\sin \varphi \cos \varphi (\sin \theta \sin \alpha + \cos \theta \cos \alpha \cos \psi)^2 = 0.
\]

From this we obtain two solutions \( \sin \varphi = 0 \) and \( \cos \varphi = 0 \). Note that the equality to zero of the multiplier in brackets is inconsistent with equation (6). Next, we will consider two pairs of values \( \varphi_1 = 0, \psi_1 = 0 \) and \( \varphi_2 = \pi/2, \psi_2 = 0 \), corresponding to two qualitatively different body positions. For both pairs of values, the \( Oz \) axis containing the body mass centre is in the vertical plane \( OYZ \) containing the vibration axis, and one of the two body principal axes of inertia \( Ox \) (for the pair \( (\varphi_1, \psi_1) \)) or \( Oy \) (for the pair \( (\varphi_2, \psi_2) \)) containing no mass centre, is perpendicular to this plane.

To obtain the equilibrium values of the angle \( \theta \) determining the position of the axis \( Oz \) in the plane \( OYZ \), we substitute pairs of solutions \( (\varphi_1, \psi_1) \) and \( (\varphi_2, \psi_2) \) in equation (6). We get the equations

\[
f_1(\theta, \alpha, b_1) = \sin \theta + b_1 \sin 2(\theta - \alpha) = 0, \quad b_1 = \frac{mzG\sigma}{2Ag}, \tag{8}
\]

and

\[
f_2(\theta, \alpha, b_2) = \sin \theta + b_2 \sin 2(\theta - \alpha) = 0, \quad b_2 = \frac{mzG\sigma}{2Bg}, \tag{9}
\]

respectively. These equations are similar to those obtained in [7, 5], when studying the dynamics of a mathematical pendulum and a Lagrange’s top with a suspension point vibrating along an inclined line.

Consider the equation (8). The qualitatively different behaviour of the function \( f_1 \) depending on the parameters \( \alpha, b_1 \) is shown in Fig. 1. In the sub-area 1 of the parameter plane \( \alpha, b_1 \) in Fig. 2 a, the equation (8) has two roots (see Fig. 1 a)

\[
\theta_{11} \in (0; \alpha), \quad \theta_{12} \in (\frac{\pi}{2} + \alpha; \pi).
\]
Figure 1. Behaviour of function $f_1$. 

Figure 2. Bifurcation diagrams.

In sub-area 2 in Fig. 2 a the equation has four roots (see Figure 1 b)

$$\theta_{11} \in (0; \alpha), \quad \theta_{12} \in \left(\frac{\pi}{2} + \alpha; \pi\right), \quad \theta_{13,14} \in (\pi + \alpha; \frac{3\pi}{2} + \alpha).$$

At the boundary of regions 1 and 2, both the function $f_1$ and its derivative on $\theta$ are simultaneously zero

$$f_1' = \cos \theta + 2b_1 \cos(\theta - \alpha) = 0.$$  \hspace{1cm} (10)

Equations (8) and (10), where the value of $\theta$ should be treated as a parameter, set the bifurcation curve $b_1 = b_{1*}(\alpha)$ in the parameter plane $\alpha, b_1$ in Fig. 2 a. This curve passes through the points $\alpha = 0, b_1 = 0.5; \alpha = \pi/4, b_1 = 1$ and $\alpha = \pi/2, b_1 = 0.5$.

Similar reasoning holds for the equation (9), which can have two or four roots ($\theta_{21, \theta_{22}, \theta_{23}, \theta_{24}$) in regions separated by a similar bifurcation curve $b_{2*}(\alpha)$ in the parameter plane $\alpha, b_2$ (see Fig. 2 a).

Let us turn from the parameter planes $\alpha, b_1$ and $\alpha, b_2$ to the plane $\alpha, \sigma$ (Fig. 2 b). The bifurcation curves $b_1 = b_{1*}(\alpha)$ and $b_2 = b_{2*}(\alpha)$ correspond to curves $\sigma = \sigma_{1*}(\alpha)$ and $\sigma = \sigma_{2*}(\alpha)$, constructed by taking into account formulas (8) and (9). The ending points of these curves at $\alpha = 0$ and $\alpha = \pi/2$ correspond to the values

$$\sigma_{10} = \frac{Ag}{mzG}, \quad \sigma_{20} = \frac{Bg}{mzG} \quad (\sigma_{10} > \sigma_{20}).$$
The curves $\sigma = \sigma_{1*}(\alpha)$ and $\sigma = \sigma_{2*}(\alpha)$ divide the parameter plane into three regions 1, 2 and 3, in which the system can have four, six or eight qualitatively different equilibrium positions depending on the parameter $\sigma$. For regions 1, 2 and 3 in Fig. 3 a, b and c dots (in different colours) show the location of the centre of mass of the body in the $OYZ$ plane at the equilibrium positions. Here, the angle $\theta$ is counted from the vertical axis $OZ$ to the mass centre radius-vector (lying on the $OZ$ axis).

In region 1 (Fig. 2 b), when the vibration intensity is low ($\sigma < \sigma_{2*}(\alpha)$), the pairs $(\varphi_1; \psi_1)$ and $(\varphi_2; \psi_2)$ each have two equilibrium positions given by angles $\theta_{11}, \theta_{12}$ and $\theta_{21}, \theta_{22}$ (Fig. 3 a), respectively. In region 2, at more intense vibrations ($\sigma_{2*}(\alpha) < \sigma < \sigma_{1*}(\alpha)$) the two equilibria $\theta_{11}, \theta_{12}$ correspond to the pair $(\varphi_1; \psi_1)$, and four equilibria $\theta_{21}, \theta_{22}, \theta_{23}, \theta_{24}$ — to the pair $(\varphi_2; \psi_2)$ (Fig. 3 b). In region 3, with further increase of vibration intensity ($\sigma > \sigma_{1*}(\alpha)$) both pairs $(\varphi_1; \psi_1)$ and $(\varphi_2; \psi_2)$ are corresponded by four equilibrium positions $\theta_{11}, \theta_{12}, \theta_{13}, \theta_{14}$ and $\theta_{21}, \theta_{22}, \theta_{23}, \theta_{24}$ each (Fig. 3 c).

![Figure 3](image)

**Figure 3.** Equilibrium positions of the body with mass centre on the principal inertia axis.

As the vibration intensity increases, the equilibrium positions of the $OZ$ axis tend towards one of the dashed lines in Fig. 3. The angle values $\theta_{11}, \theta_{21}$ tend to the value of $\alpha$, the values $\theta_{12}, \theta_{22}$ — to the value of $\pi/2 + \alpha$, the values of $\theta_{13}, \theta_{23}$ — to the value of $\pi + \alpha$ and the values of $\theta_{14}, \theta_{24}$ — to the value of $3\pi/2 + \alpha$. Note that the mutual arrangement of the axis $OZ$ at the equilibrium positions (for different values $\sigma$ and all $\theta_{1j}$ and $\theta_{2j}, j = 1, ..., 4$) remains unchanged.

4. The study of the equilibrium positions stability

Sufficient stability conditions of the obtained equilibrium positions will be searched in the form of potential energy minimum conditions (4), and necessary stability conditions — by investigating the roots of the characteristic equation of the perturbed motion equations linearized system.

4.1. Sufficient stability conditions

We introduce perturbations using the formulas

$$x_1 = \psi - \psi_*, x_2 = \theta - \theta_*, x_3 = \varphi - \varphi_*, \quad y_1 = P_\psi, y_2 = P_\theta, y_3 = P_\varphi,$$

where $\psi_*, \theta_*, \varphi_*$ — equilibrium values of Euler angles.

Consider the perturbation quadratic part of the potential energy decomposed in the vicinity of the equilibrium position. To investigate the minimum conditions we apply the Sylvester criterion, which is reduced to the analysis of two inequalities.
For the first equilibrium pair \((\varphi_1, \psi_1)\) these conditions can be represented in the form

\[
\cos \theta_* + \frac{mzG\sigma}{Ag} \cos 2(\theta_* - \alpha) < 0, \quad (11)
\]
\[
\sin(\theta_* - \alpha) \sin \theta_* > 0. \quad (12)
\]

The left-hand side of the first inequality is a function \(f'_1\) and condition (11) is satisfied for the root \(\theta_{12}\) and for the root \(\theta_{14}\) in its existence region and is not satisfied for roots \(\theta_{11}\) and \(\theta_{13}\) (see Fig. 1 a and b). In the second inequality, the value of \(\sin \theta_*\) is positive for the roots \(\theta_{11}\) and \(\theta_{12}\) and negative for \(\theta_{13}\) and \(\theta_{14}\). The value of \(\sin(\theta_* - \alpha)\) is positive for the root \(\theta_{12}\) and negative for the remaining three values. Thus, the second condition is satisfied for the roots \(\theta_{12}, \theta_{13}\) and \(\theta_{14}\), and both conditions (11) and (12) are satisfied for the roots \(\theta_{12}\) and \(\theta_{14}\).

4.2. Necessary stability conditions

The characteristic equation of the perturbed motion equations linearised system will be written in the form

\[
(\lambda^4 + a\lambda^2 + b)(\lambda^2 + c) = 0, \quad (13)
\]

where \(a, b, c\) — are some real coefficients. If conditions

\[
a > 0, \quad b > 0, \quad c > 0, \quad D = a^2 - 4b > 0, \quad (14)
\]

is satisfied, then the roots of the equation (13) are strictly imaginary and the equilibrium position under study is stable in the linear approximation. If the sign of one inequality is reversed, the equation (13) has roots with a positive real part, and the equilibrium position is unstable not only in the linear, but also in the nonlinear autonomous problem.

All of the obtained sufficient stability conditions are also necessary conditions, so we will consider cases where only the necessary stability conditions are satisfied.

For a pair of equilibrium values \((\varphi_1, \psi_1)\) the coefficient \(c\) takes the form

\[
c = -\frac{mgzG}{A} \left( \cos \theta_* + \frac{mzG\sigma}{Ag} \cos 2(\theta_* - \alpha) \right). \quad (15)
\]

For the roots \(\theta_{12}\) and \(\theta_{14}\), the sufficient stability conditions are satisfied (and for them, taking into account (11), \(c > 0\)). For the roots \(\theta_{11}\) and \(\theta_{13}\) the coefficient \(c\) is negative and the corresponding equilibrium positions are unstable.

For the second pair of equilibrium values \((\varphi_2, \psi_2)\) the coefficients \(c\) and \(b\) take the form

\[
c = -\frac{mgzG}{B} \left( \cos \theta_* + \frac{mzG\sigma}{Bg} \cos 2(\theta_* - \alpha) \right), \quad (16)
\]
\[
b = \frac{(A - B)m^4zG^4\sigma^2 \sin \alpha \sin(\theta_* - \alpha)}{A^2B^4C} \cos^2(\theta_* - \alpha). \quad (17)
\]

The condition \(c > 0\) is satisfied for the roots \(\theta_{22}\) and \(\theta_{24}\), and is not satisfied for the roots \(\theta_{21}\) and \(\theta_{23}\).

Consider the condition \(b > 0\) for \(\theta_{22}\) and \(\theta_{24}\). The expressions \(\sin \theta_*\) and \(\sin(\theta_* - \alpha)\) for these roots have the same sign, so the condition \(A > B\) implies that \(b < 0\). Hence, the roots \(\theta_{22}\) and \(\theta_{24}\) are also unstable.

Thus, all the necessary stability conditions coincide with the sufficient conditions.
The stability investigation results are shown in Fig. 3. The stable equilibrium positions \( \theta_{12}, \theta_{14} \) corresponding to the pair \((\varphi_1, \psi_1)\) are shown in Fig. 3 with black colour. The unstable equilibria \( \theta_{11}, \theta_{13} \) (for the pair \((\varphi_1, \psi_1)\)) are shown with white colour, and all four unstable equilibria \( \theta_{2j} \) corresponding to the pair \((\varphi_2, \psi_2)\) are shown with grey colour.

Consider the boundary values of \( \alpha \). In the case \( \alpha = 0 \) of suspension point horizontal vibrations (see also [3]) the qualitative picture does not change, and for all \( \sigma \) equations (8) and (9) have solutions \( \theta = 0 \) and \( \theta = \pi \), corresponding to the upper and lower equilibrium positions when the radius vector \( OG \) is vertical and point \( G \) is above or below point \( O \). At values of \( \sigma > \sigma_{20} \) and \( \sigma > \sigma_{10} \) two pairs of lateral equilibria arise in which the radius-vector \( OG \) makes an obtuse angle with the \( OZ \) axis.

In the case of \( \alpha = \pi/2 \) vertical vibrations (see also [2]), the precession angle \( \psi \) becomes a cyclic coordinate and in the equilibrium positions the radius-vector \( OG \) can lie in an arbitrary vertical plane. The equilibrium positions form families, and at all values \( \sigma \) there are families of upper and lower vertical equilibrium positions. For values \( \sigma > \sigma_{20} \) and \( \sigma > \sigma_{10} \) there appear two families of lateral equilibria for which the radius-vector \( OG \) is inclined to the vertical, and makes a sharp angle with the axis \( OZ \).

5. On high-frequency periodic motions of a complete non-autonomous system
In the vicinity of the found equilibrium positions of the approximated autonomous system (2) (except for the bifurcation values \( \sigma_1 \), and \( \sigma_2 \)) the roots of the characteristic equation of the complete non-autonomous system (of order \( \omega_\ast \)) are incommensurate with the external disturbances frequency \( \Omega \) (of order \( \varepsilon^{-2} \)). Thus, there is a non-resonant case of the periodic motions Poincaré theory [8]. Then each found equilibrium position of the approximate system corresponds to a single analytic by \( \varepsilon \), \( 2\pi/\Omega \)-periodic by \( t \) solution of the complete non-autonomous system. Such solutions represent high-frequency body shaking in the vicinity of the approximate system equilibrium configurations.

The stable equilibrium positions \( \theta_{12}, \theta_{14} \) corresponding to the pair \((\varphi_1, \psi_1)\) will turn into stable in the linear approximation periodic motion of the complete system, while the other unstable equilibrium positions will turn into unstable periodic motion. This follows from the continuity by \( \varepsilon \) of the characteristic exponents of the corresponding perturbed motion linearized equations.

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