Mathematical representation: the roles, challenges and implication on instruction

A F Samsuddin¹, H Retnawati¹

¹ Mathematics Education Department of Graduate School, Yogyakarta State University Jl. Colombo No.1, Sleman, D.I. Yogyakarta, Indonesia 55281

Corresponding author: auliaulfitrah.2017@student.uny.ac.id

Abstract. Mathematics is often seen as completely separated from the daily life. This is why people need some connecting bridge. Many said that mathematical representation serves the bridge role for people to understand and express mathematical ideas. Representation consists of internal and external representation. However, representation term that researchers used mostly refers to only external part. Real-world problems can be represented using formula, visual, concrete, etc. This paper aimed to review results of researches related to mathematical representation. This paper will emphasize the role of using multiple representations in mathematics learning, the challenges that students and teachers face related to representation and teachers’ role in promoting students’ mathematical representation.

1. Introduction

If the real world is assumed as a land and mathematics is another land separated by a river, different and strange to each other. A bridge is needed to link those lands. Representation served the role of such bridge connecting the abstract mathematics concept with daily life context. In line with that, Matteson [1] analogued learning mathematics as learning foreign language and representation is the key elements for those who wants to understand and express mathematical ideas conveniently. In line with that Kilpatrick [2] stated that representation is a complex system and is like a foreign language for students since students learn to use representation while simultaneously using it to learn othe things. Duval [3] added mathematical objects (ideas, concepts and relations) can only be accessed through representation and its activities are affected by how the representation is used.

This paper aims to present the readers a deeper insight on mathematical representation from literature and empirical study. As the beginning, representation will be elaborated as well as the role representation has in mathematics learning. The discussion will be continued with definition and role of using multiple representation in mathematics. The paper will be completed with several challenges on using multiple representation and the possible strategies which teachers can implement to minimize or overcame the challenges faced by students.

2. Representations: What and Why?

Mathematical entities can be represented in multiple ways. For example, the number six can be represented by collection of real objects, such as six beads, six apples or six straws, by iconic pictures, such as images of six circles, or by abstract symbols like 6 or VI. These representations are observable. Generally, mathematical representation consists of two inseparable parts, namely: (1) external
representation, one that physically exist and observable, such as graphic, pictures, equations and table; (2) internal representation, model, scheme or concepts which is mental or cognitive and cannot be directly observable [4]. Hereinafter, the term representation refers merely to external representation.

Miller & Hudson [5] classified representation into three types, namely: (1) concrete, (2) representational, and (3) abstract. This classification is similar to representation mode by Bruner in Hebert & Powell [6] which are (1) enactive, (2) iconic, (3) symbolic. Suppose a teacher is using plastic cube to represent problems on basic operations to primary students. Another teacher is asking his or her students to walk around the edge of the classroom to introduce the concept of perimeter. Both teachers are actually utilizing concrete representation. In the representational stage teacher can use picture representing the plastic cube, instead of the real one. Gradually, students can use only number or abstract representation. Figure 1 showed simple subtraction problem represented concretely, pictorial/representationally and abstractly.

![Concrete, pictorial and abstract representations on simple subtraction problems](image)

Figure 1 Concrete, pictorial and abstract representations on simple subtraction problems

(Miller & Hudson, 2006 : 29).

Drawn from numbers of literature, the roles of representation in mathematics learning are : (1) representation helps students in making sense mathematics tasks and concepts [7], (2) representation facilitates students’ learning process [8], (3) representation helps students in managing and expressing their thinking as well as making mental model of their mathematical ideas [9], (4) representation helps students understanding abstract mathematical concepts [10], as well as (5) being used to mathematics problems with multiple representation helps students in analyzing problems [10].

Representation, whether it is physical, spoken or written, is necessary to communicate about mathematical numbers and operations [1]. Plenty sources suggested the need for employing multiple representation in mathematics instruction. This is because mathematical ideas are enhanced through multiple representation. The multiple representation possesses roles not only as the instructional tricks but also as a source of mathematical reasoning [2]. Havin experiences with multiple representation (graphic, table and algebraic) helps students construct a broader synergy related to Mathematics and enables them to minimize their difficulties in a certain topics, Function, for instance [11]. Another evidence was provided by Cai’s findings [12] about Chinese students showing a better result than students in America on process-restricted problems, but on non routine questions that does not have a standard algorithm, American students showed a better performance than those in China. Cai & Cifarelli [13] tried to dig deeper about this issue and found that there is a correlation between multiple representation in problem solving and students’ performance in a various assesment task.

Ainsworth [14] stated functions of using multiple representation in instructions, such as : 1) Conveying different information. Sometimes, single representation is not enough to carry all the informations to be represented or would be too complicated for students, 2) minimizing possible misinterpretations of a representation or domain, and 3) promoting deeper understanding of ideas being represented.
3. Challenges in using Representations in Mathematics Instruction

Since there is an urge to use multiple representation in mathematics learning, one should be able to choose and translate among representations. The ability to choose the best representation for mathematical ideas is necessary since algorithms depend upon representation [2]. This is in line with what Retnawati et al. [15] found when investigating the difficulties students faced in solving geometry problems in National Examination in Indonesia. One of factors found is students’ lack of mathematical representation. Unfortunately, behind the broad advantages representation has, researches showed that students face difficulties related to mathematical representation. These hardships due to the lack of diagrammatic knowledge needed in representation, ability to interpret representation by connecting it to the real world, or being unable to translate among representations in the same domain [16].

Another distress happened when the use of representation is not accompanied by students’ mathematical comprehension. Students are forced to follow teachers’ preferred procedures without opportunity to reflect the activities and assistance to link representation with the underlaid mathematical ideas [17].

Difficulties can also be emerged because of the double yet not compatible conditions in Mathematics. Students have to use representation for abstract mathematical objects, and at the same time they have to understand those objects [3]. Based on Leikin’s findings [18] it can be implied that working in mathematics with representation can foster students’ speed in solving the problem, but unfortunately it is not the case with their accuracy. This is explained by the split-attention effect representation has.

It is a challenge as well when students regard representation and the concept it represents as two separated things. This is validated by Adu-Gyamfi & Bosse [19] who conducted research on 8 high school students majoring pre-calculus in south-eastern United States. It is found that most students answer correctly questions measuring representation skill generally. However, when students are asked to identify function displayed in graph, table and equation among relations (not function), also represented in multiple representation, the results is not in harmony with the first result.

Stylianou [20] found that representation is also affected by teachers’ perception about representation. Teachers who perceive representation as another learning topic or concept in mathematics, not as a tool to understand the concept itself, usually does not consider representation has a central role in mathematics learning. A teacher said “you still have to teach them all the other things”. Another teacher expressed similar idea which is “in the State exam practices representation is not required.” Stylianou also stated that teachers who have such opinion believe that representation is more suitable to high-performed students and will only make the other students confused.

According to Ainsworth [14] students face several learning demands of multiple representation, which are: a) students need to learn the format and operators of a representation. In the case of graph, the format would be lines, labels and axes, while the operators are how to find gradient of lines, intercepts, etc, b) students have to learn the relation between the representation and the concept represented and c) students have to learn the relation between the representations. Students who fail to satisfy one of the preceding demands are predicted not to be able to benefit fully from multiple representation advantage.

Adu-gyamfi [21] classified three types of error that students made in translation process (convert from one representation mode to another): a) Implementation error, b) Interpretation error, and c) Preservation error. Implementation error is related to computational or algorithmic mistakes. Interpretation error occurs when students fail to understand the characteristics or properties of either the source or the target of representation. Preservation error occurs when students are able to maintain the semantic congruence of some, not all the attributes of representation.

Teachers’ content knowledge is also a central issue related to the success of mathematics learning with representation. How can a teacher implement a successful instruction, in top of that with multiple representation, if the teacher does not have sufficient knowledge about the topics. An example is provided by Retnawati et al [22]) that showed teachers in the research have difficulties with problems related to function, sub-topic with multiple possible representation.
4. Implication in Instruction
As stated before, one of the challenges in mathematics using representation has to do with the teachers. Teachers should have a clear concept about representation so that they can employ it in the classroom. Based on NCTM [23], learning standards expected to be satisfied for level pre-kindergarten to K-12 are:
1. Create and use representations to organize, record, and communicate mathematical ideas
2. Select, apply, and translate among mathematical representations to solve problems
3. Use representations to model and interpret physical, social, and mathematical phenomena.

NCTM elaborated in detail teachers’ role in promoting students’ mathematical representation ability based on their school level. In kindergarten to K-2, teachers’ role is mainly to create learning situation in which students can use multiple representations. Beside that, teachers need to motivate students to communicate their preferred representation since written work usually cannot reveal students’ whole thinking process. In level K-2 to K-5, teachers may begin to invite students discussing why there is representation more effective than others in a certain context. For students in K-6 and K-8, teachers can help students developing their self-belief and competence in making their own representation in a certain context or problems. Teachers can help students linking the use of representation with daily life situation. Table 1 documented Mitchell’s role of teachers related to the use of representation generally (regardless school level). Representation brings positive influence towards problem solving. This applies conversely that by conducting mathematics instruction based on problem will enhance mathematical representation of students [24]

| Representing and solving problem/carrying out mathematical operations | Mitchell’s Role of Teachers |
|------------------------------------------------------------------------|-----------------------------|
| Representing and solving problem/carrying out mathematical operations | Recognizing and abiding by the representations’ conventions |
| Creating a context for connecting multiple representations             | Using representations as a means to illuminate certain mathematical ideas involved in a procedure |
| Creating a context for generalizing procedures                         | Employing appropriate language and notation when using representations |
| Scaffolding student work on representations and the mathematics        | Decomposing and unpacking mathematical rules and operations through careful use of representations |
|                                                                        | Selecting representations that lend themselves to explaining a mathematical procedure |
|                                                                        | Using one representation to help students make sense of another |
|                                                                        | Using representations to build generalizations and help students move to a more abstract level |
|                                                                        | Selecting and sequencing examples to support student ability to generalize |
|                                                                        | Using multiple representations to help students make sense of the underlying meaning of a mathematical procedure |
|                                                                        | Using representations to surface student misconceptions and emphasize important mathematical ideas |
Using representations to trigger and remediate student misconceptions
Flexibly moving between representations to support student understanding

Providing a balance between explaining the representation conventions and allowing students the space and time to make meaning of the representations and the mathematical ideas they are intended to illuminate

Examining whether students correctly follow the representations’ conventions and ascribe meaning to the representations’ manipulations

Pressing students to articulate the mathematical meaning they are making out of using representations

Listening to students and unpacking their (promising) productions around using representations

Differentiating the scaffolding provided to students depending on
(a) the anticipated level of transparency of a given representation and
(b) students’ differential needs and their progress toward abstracting the underlying mathematical ideas the representation are intended to illuminate.

Bosse [25] outlined recommendation for teachers to facilitate students’ mathematics learning in respect to translation among multiple representation, which are:
1. Teachers should realize that their beliefs regarding which representation mode students can do, cannot do, and hould be able to do may affect instructional plans
2. Teachers should recognize which translations are more difficult than others. For instance, students may find translation from symbolic to tabular representation easier than translation from graphical to symbolic translation.
3. Teachers have to assure that students learn all the representations with the translation, particularly those which are more difficult.
4. Teachers need to take everything that support students’ translation to consider in the learning process, for instance, teachers’ questioning techniques.
5. Teachers should use the assessment type where students are asked to apply multiple translations and not only translation that teacher believe students can perform.
6. Teacher can use real world contexts which are familiar to students.
7. Teacher can use a rich-tasks to engage students.

5. Conclusion
Representation has a central role in mathematics instruction. Representation facilitates students in understanding abstract mathematical concepts or ideas. Generally representation can be classified into three modes, concrete (real manipulative object or activity), pictorial/representational (pictures, graph, etc) and abstract representation (mathematics equation). Literatures suggested the need to use multiple representation in mathematics instruction. However, in the reality there are challenges in conducting
mathematics learning using representation. One of the case is when students perceive representation and the mathematics concept being represented are two separated things. It is followed by students know how to use multiple representation (graph, table and equation) with function, for example, but they cannot understand the function itself. Another challenge is when teachers as a learning facilitator see representation as a product only, not as a process in understanding mathematics. If it happens, teachers will face difficulties in implementing mathematics learning with representation. Teachers’ roles in mathematics classroom with representation are Representing and solving problem/carrying out mathematical operations, Creating a context for connecting multiple representations and for generalizing procedures, and Scaffold students work on representations and the mathematics.

6. References
[1] Matteson S M 2006 Mathematical literacy and standardized mathematical assessments Read. Psychol. 27 no. 2–3 pp. 205–233
[2] Kilpatrick J, Swafford J, and Findell B 2001 Adding It Up
[3] Duval R 2006 A Cognitive Analysis of Problems of Comprehension in a Learning of Mathematics Author (s): Raymond Duval Source : Educational Studies in Mathematics , 61 , No. 1 / 2 Semiotic Perspectives in Mathematics Education : A PME Special Issue 2006 Educ. Stud. Math. vol. 61 no. 1 pp. 103–131
[4] Goldin G and Kaput J “A joint perspective on the idea of representation in learning and doing mathematics.” Theor. Math. Learn., no. September, pp. 397–430, 1996
[5] Miller S P and Hudson P J 2003 Techniques for Program Support Helping Students With Mathematics Means
[6] Hebert M A and Powell S R, 2016 Examining fourth-grade mathematics writing: features of organization, mathematics vocabulary, and mathematical representations Read. Writ., vol. 29 no. 7 pp. 1511–1537
[7] Mitchell R, Charalambous C Y, and Hill H C 2014 Examining the task and knowledge demands needed to teach with representations J. Math. Teach. Educ. 17 37
[8] Martin L C 2008 Folding back and the dynamical growth of mathematical understanding: Elaborating the Pirie-Kieren Theory J. Math. Behav. 27 64
[9] Schwarz B B, Kohn A S, and Resnick L B, 1994 Positives About Negatives: A Case Study of an Intermediate Model for Signed Numbers J. Learn. Sci. 3 37
[10] Kang R and Liu D 2018 The Importance of Multiple Representations of Mathematical Problems: Evidence from Chinese Preservice Elementary Teachers’ Analysis of a Learning Goal Int. J. Sci. Math. Educ. 16 125
[11] Even R 1998 Factors involved in linking representations of functions J. Math. Behav. 17 105
[12] Cai J 2000 “Mathematical Thinking Involved in U.S. and Chinese Students’ Solving of Process-Constrained and Process-Open Problems Math. Think. Learn. 2 309
[13] Cai J and Victor C 2004 Mathematical Thinking Involved in U.S. and Chinese Students’ Solving of Process-Constrained and Process-Open Problems in How Chinese Learn Mathematics: Perspectives from Insiders pp. 71–106.
[14] Ainsworth S E 1999 Designing Effective multirepresentational Learning Environments ESRC Cent. Res. Dev. Instr. Train. Univ. Nottingham vol. technical, p. PhD thesis and technical report number 47
[15] Retnama H, Arlinwibowo J, and Sulistyaningsih E, 2017 The Students’ Difficulties in Completing Geometry Items of National Examination Int. J. New Trends Educ. Their Implic. 8 28
[16] Bock D D, Dooren W V, and Verschaffel L, 2015 Students’ understanding of proportional inverse proportional, and affine functions: two studies on the role of external representations Int. J. Sci. Math. Educ. 13 47
[17] Stein M K and Bovalino J W 2001 Manipulatives: One piece of the puzzle *Math. Teach. Middle Sch.*

[18] Leikin R, Leikin M, Waisman I, and Shaul S 2013 Effect of the Presence of External Representations on Accuracy and Reaction Time in Solving Mathematical Double-Choice Problems By Students of Different Levels of Instruction *Int. J. Sci. Math. Educ.* vol. 11 no. 5 pp. 1049–1066

[19] Adu-Gyamfi K and Bossé M J 2014 Processes and reasoning in representations of linear functions *Int. J. Sci. Math. Educ.* vol. 12 no. 1 pp. 167–192

[20] Stylianou D A 2010 Teachers’ conceptions of representation in middle school mathematics *J. Math. Teach. Educ.* vol. 13 no. 4 pp. 325–343

[21] Adu-Gyamfi K, Bossé M J, and Stiff L V, 2012 Lost in Translation: Examining Translation Errors Associated With Mathematical Representations *Sch. Sci. Math.* vol. 112 no. 3 pp. 159–170

[22] N C of T. of Mathematics 2000 Principles and standards for school mathematics

[23] NCTM 2000 Principles and standards for school mathematics

[24] Farhan M and Retnawati H, 2014 Keefektifan PBL Dan IBL Ditinjau dari Prestasi Belajar, Kemampuan Representasi Matematis, dan Motivasi Belajar *J. Ris. Pendidik. Mat*

[25] Bossé M J, Adu-Gyamfi K, and Cheetham M R 2011 Assessing the difficulty of mathematical translations: Synthesizing the literature and novel findings *Int. Electron. J. Math. Educ.* vol. 6 no. 3 pp. 113–133