Study of the $B_s \rightarrow \phi f_0(980) \rightarrow \phi \pi^+ \pi^-$ decay with perturbative QCD approach

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Abstract

The rare cascade $B_s \rightarrow \phi f_0(980) \rightarrow \phi \pi^+ \pi^-$ decay is studied with the perturbative QCD approach based on the formula for the quasi two-body decay, where the two-pion pair originates from the $S$-wave resonant $f_0(980)$ state. It is found that with the introduction of the nonperturbative two-pion distribution amplitudes and the Flatté parameterization of the scalar form factor for the $f_0(980)$ resonance, the branching ratio in the mass range $400 \text{ MeV} < m(\pi^+ \pi^-) < 1600 \text{ MeV}$ is $B_{\text{theo}}(B_s^0 \rightarrow \phi f_0(980) \rightarrow \phi \pi^+ \pi^-) = [1.31^{+0.40}_{-0.31}(a_{\pi\pi})^{+0.19}_{-0.16}(m_b)^{+0.10}_{-0.09}(\text{CKM})] \times 10^{-6}$, where the uncertainties come from the parameter $a_{\pi\pi}$ of the two-pion distribution amplitudes, the $b$ quark mass $m_b$, the Cabibbo-Kobayashi-Maskawa (CKM) factors, respectively. This result agrees with the recent LHCb measurement within uncertainties, $B_{\text{exp}}(B_s^0 \rightarrow \phi f_0(980) \rightarrow \phi \pi^+ \pi^-) = (1.12^{+0.16+0.09}_{-0.08-0.11}) \times 10^{-6}$, where the errors are statistical, systematic and from the normalization, respectively.

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I. INTRODUCTION

With the great progress and good performance of the Belle, BaBar and LHCb experiments, many three-body nonleptonic $B$ meson weak decay channels are accessible and have been measured [1]. Recently, based on the $3\, fb^{-1}$ $pp$ collision data recorded by the LHCb detector, the three-body nonleptonic decay $B_s^0 \to \phi \pi^+ \pi^-$ was investigated with the requirements on the $\pi^+ \pi^-$ invariant mass in the range $400\, \text{MeV} < m(\pi\pi) < 1600\, \text{MeV}$, then an analysis of the $m(\pi\pi)$ spectrum including the $S$-, $P$-, and $D$-wave amplitudes was further performed to study the possible resonant contributions [2]. Some prominent maxima in the $m(\pi\pi)$ spectrum are observed around the $\rho(770)$, $f_0(980)$, $f_2(1270)$ and $f_0(1500)$ resonant regions. One of the formal public announcement of the LHCb Collaboration is that [2] the three-body sequential rare decay $B_s^0 \to \phi f_0(980) \to \phi \pi^+ \pi^-$ was first observed with a statistical significance of $8\, \sigma$ and the branching fraction of

$$B(B_s^0 \to \phi f_0(980) \to \phi \pi^+ \pi^-) = (1.12 \pm 0.16^{+0.09}_{-0.08} \pm 0.11) \times 10^{-6},$$

where the errors are statistical, systematic and from the normalization, respectively.

Although it is still a controversial issue whether the isospin-singlet particle $f_0(980)$ should be regarded as the conventional $q\bar{q}$ meson, or the exotic tetraquark $q\bar{q}q\bar{q}$ state, or the meson-meson $K\bar{K}$ molecule, it is usually suggested that the unflavored scalar $f_0(980)$ meson has a substantial $s\bar{s}$ component, and decays dominantly into the $\pi\pi$ final states [1]. Therefore, on the one hand, the $B_s^0 \to \phi f_0(980) \to \phi \pi^+ \pi^-$ decay is interesting and helpful to explore the compositive structure of the $f_0(980)$; on the other hand, the importance of the $B_s^0 \to \phi f_0(980) \to \phi \pi^+ \pi^-$ decay is obvious, i.e., this decay is induced by the flavor-changing-neutral-current (FCNC) $\bar{b} \to \bar{s}s\bar{s}$ process at the elementary particle level within the Standard model (SM), which is absolutely forbidden at the tree level by the Cabibbo-Kobayashi-Maskawa (CKM) quark-mixing mechanism within SM but sensitive to the new physics effects beyond SM.

Along with the experimental advances, the theoretical research on the three-body nonleptonic $B$ weak decay is really necessary. Although there exist some attractive QCD-inspired phenomenological methods to deal with the two-body nonleptonic $B$ decays, such as the perturbative QCD (PQCD) approach [3–11], the QCD factorization (QCDF) approach [12–19], and so on, the theoretical description of the three-body nonleptonic $B$ decays is still in the early stage of modeling. This is not surprising because that the more hadrons participated, the more intricate the interferences among different contributions (such as the
possible resonances and final state interactions) will certainly become. Moreover, for the three-body hadronic decays, the kinematical configurations will vary from region to region in the Dalitz plot, and in principle correspond to different dynamical components and theoretical treatments with special scales. The resonant contributions are entirely engulfed by the blurry background clouds, so any phenomenological parametrization and interpretations of the resonant structures are process- and model-dependent. The effective separations between the perturbative and nonperturbative contributions to the three-body nonleptonic $B$ decays will be much more complicated, which is by no means trivial. However at the same time, the phase space distributions make the theoretical calculation of three-body nonleptonic $B$ meson decays to be very meaningful for exploring some fresh and potentially important information, such as the natures and effects of possible resonances, the energy dependence of observables, the local $CP$ asymmetry distributions in the Dalitz plot, and so on. In the past years, there were plenty of theoretical studies of the three-body nonleptonic $B$ meson decays, such as Refs.[20–39] based on SU(3) relations, Refs.[40–55] based on both heavy quark effective theory and chiral perturbation theory, Refs.[56–71] with factorization approach, Refs.[72–87] with the QCDF approach, Refs.[88–103] with the PQCD approach.

Both the PQCD and QCDF approaches have been widely employed in the two-body nonleptonic $B$ meson decays in recent years. In Ref.[88], Chen and Li attempted to generalize the PQCD approach to the three-body nonleptonic $B^+ \to K^+\pi^+\pi^-$ decay for the particular configuration topologies where the kinematics is very similar to a two-body decay. In this paper, we shall follow the method of Ref.[88] to investigate the $B_s^0 \to \phi f_0(980) \to \phi \pi^+\pi^-$ decay with the PQCD approach. The overall layout of this paper is as follows. The theoretical framework and the amplitudes for the $B_s^0 \to \phi f_0(980) \to \phi \pi^+\pi^-$ decay are elaborated in section II. The numerical results and discussion are presented in Section III. The last section is a short summary.

II. THEORETICAL FRAMEWORK

A. The effective Hamiltonian

The nonleptonic weak decays of the $B$ mesons involve three fundamental scales, including the weak interaction scale $M_W$, the $b$ quark mass scale $m_b$, and the QCD characteristic
scale $\Lambda_{\text{QCD}}$, which are strongly ordered: $M_W \gg m_b \gg \Lambda_{\text{QCD}}$. To deal with the multi-scale problems, one usually has to resort to the effective theory approximation. Using the operator product expansion and the renormalization group (RG) equation, the low energy effective Hamiltonian for the FCNC process expansion and the renormalization group (RG) equation, the low energy effective problems, one usually has to resort to the effective theory approximation. Using the operator

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \left( V_{ub}^* V_{us} + V_{cb}^* V_{cs} \right) \sum_{i=3}^{10} C_i(\mu) Q_i(\mu) + \text{h.c.},$$

where the Fermi coupling constant $G_F \simeq 1.166 \times 10^{-5} \text{GeV}^{-2}$ [1]. $V_{ub}^* V_{us}$ and $V_{cb}^* V_{cs}$ are the CKM factors. The scale $\mu$ separates the effective Hamiltonian into two distinct parts: the Wilson coefficients $C_i$ and the local four-quark operators $Q_i$.

The expressions of the operators $Q_i$ are written as:

$$Q_3 = \sum_q (\bar{b}_\alpha s_\alpha)_{V-A} (\bar{q}_\beta q_\beta)_{V-A}, \quad Q_4 = \sum_q (\bar{b}_\alpha s_\alpha)_{V-A} (\bar{q}_\beta q_\alpha)_{V-A},$$

$$Q_5 = \sum_q (\bar{b}_\alpha s_\alpha)_{V-A} (\bar{q}_\beta q_\beta)_{V+A}, \quad Q_6 = \sum_q (\bar{b}_\alpha s_\alpha)_{V-A} (\bar{q}_\beta q_\alpha)_{V+A},$$

$$Q_7 = \frac{3}{2} \sum_q Q_q (\bar{b}_\alpha s_\alpha)_{V-A} (\bar{q}_\beta q_\beta)_{V+A}, \quad Q_8 = \frac{3}{2} \sum_q Q_q (\bar{b}_\alpha s_\alpha)_{V-A} (\bar{q}_\beta q_\alpha)_{V+A},$$

$$Q_9 = \frac{3}{2} \sum_q Q_q (\bar{b}_\alpha s_\alpha)_{V-A} (\bar{q}_\beta q_\beta)_{V-A}, \quad Q_{10} = \frac{3}{2} \sum_q Q_q (\bar{b}_\alpha s_\alpha)_{V-A} (\bar{q}_\beta q_\alpha)_{V-A},$$

where $Q_3, \ldots, 6$ and $Q_7, \ldots, 10$ are called as the QCD and electroweak penguin operators, respectively. $\alpha$ and $\beta$ are color indices. $q$ denotes all the active quark at the scale of $\mathcal{O}(m_b)$, i.e., $q = u, d, s, c, b$. $Q_q$ is the electric charge of quark $q$ in the unit of $|e|$. The operators $Q_i$ govern the dynamics of the $B$ meson weak decay. The coupling strength of the effective interactions among four quarks of the operators $Q_i$ is proportionate to the Wilson coefficients $C_i$. The physical contributions from the scale higher than $\mu$ are summarized in the Wilson coefficients $C_i$, while the physical contributions from the scale lower than $\mu$ are incorporated into the hadronic matrix elements (HMEs) where the operators $Q_i$ are sandwiched between the initial and final hadron states. The Wilson coefficients $C_i$ are process independent and computable order by order with the RG improved perturbative theory as long as the scale $\mu$ is not too small. The expressions of the Wilson coefficients $C_i$ including the next-to-leading order corrections can be found in Ref.[104]. The HMEs describe the transition from the quarks of the operators $Q_i$ to the participating hadrons. The operators $Q_i$ comprise of four quarks at the local interaction point, the initial and final
states are hadronic states. The transition between the quarks and hadrons necessarily involves the hadronization and other rescattering effects. Due to the low-energy long-distance QCD effects and the entanglement of nonperturbative and perturbative contributions, the main obstacles of the calculation of the nonleptonic $B$ decays is how to properly evaluate the HMEs of the local four-quark operators.

B. Hadronic matrix element

As for the two-body nonleptonic $B$ decays with both the PQCD and QCDF approaches, the HMEs are usually written as the convolution of the universal wave functions (WFs) or distribution amplitudes (DAs) reflecting the nonperturbative contributions with the scattering amplitudes containing perturbative contributions, based on the factorization theorem for exclusive processes [105–109]. Similarly, the HMEs for the three-body nonleptonic $B$ decays could generally be written as:

$$\langle h_1 h_2 h_3 | Q_i | B \rangle \sim \int dk_1 dk_2 dk_3 dk \Phi_{h_1}(k_1) \Phi_{h_2}(k_2) \Phi_{h_3}(k_3) \Phi_B(k) \mathcal{T}(k_1, k_2, k_3, k),$$

or

$$\sim \int dx_1 dx_2 dx_3 dx \phi_{h_1}(x_1) \phi_{h_2}(x_2) \phi_{h_3}(x_3) \phi_B(x) \tilde{\mathcal{T}}(x_1, x_2, x_3, x),$$

where $\Phi_{h_i}(k_i)$ and $\phi_{h_i}(x_i)$ are the WFs and DAs for the $h_i$ hadron, respectively; $k_i (x_i)$ is the momentum (the longitudinal momentum fraction) of the valence quark; $\mathcal{T}$ and $\tilde{\mathcal{T}}$ are the scattering kernels. It is assumed that the nonperturbative contributions are contained within the WFs and DAs. The DAs are universal. The DAs either extracted from experimental data or obtained from nonperturbative means could be employed for other processes involving the same hadron. The scattering kernels, $\mathcal{T}$ and $\tilde{\mathcal{T}}$, could be computed systematically in an expansion in the strong coupling $\alpha_s$ and the power $1/m_b$ with the perturbation theory.

As analyzed in Refs.[77], the Dalitz plot for the three-body nonleptonic $B$ decays could be divided into different regions with distinct kinematic and dynamic properties. In the center region of the Dalitz plot, all three final hadrons have a large energy and none of them moves collinearly to the others. This kinematical configurations have two hard gluons and the perturbative calculation of the scattering kernels seems to be applicable. Unfortunately, as analyzed in Ref.[88], the scattering kernels of this region contain two virtual gluons at the lowest order with the PQCD approach, which is not practical due to a huge number of Feynman diagrams. In addition, the amplitudes for this region are power suppressed with
respect to the amplitude at the edges. At the edges of the Dalitz plot, two hadrons move collinearly or back-to-back, so the three-body decay could be approximately regarded as the quasi-two-body decay [77]. The $B_s \to \phi \pi^+\pi^-$ decay observed by the LHCb Collaboration [2] with the $\pi^+\pi^-$ invariant mass less than 1.6 GeV is the case, where the possible $\pi^+\pi^-$ resonant states show up. The three-body $B_s \to \phi \pi^+\pi^-$ decay could be approximated as the quasi two-body $B_s \to \phi (\pi^+\pi^-)$ decay. It seems reasonable to assume that the two-pion pair originates from a quark-antiquark state and postulate the validity of factorization for this quasi two-body $B_s$ decay. In this paper, we will follow Ref.[88] as a hypothesis, and write the HMEs for the three-body nonleptonic $B_s \to \phi \pi^+\pi^-$ decay as follow.

$$\langle \phi \pi^+\pi^-|Q_i|B_s\rangle \sim \int dx \, dy \, dz \, \phi_B(x) \phi_\phi(y) \phi_{\pi\pi}(z) \mathcal{T}(x,y,z),$$

(9)

where one new input, the $\pi^+\pi^-$ pair DA $\phi_{\pi\pi}$ parameterizing both the resonant and nonresonant contributions, is introduced in order to factorize the HMEs for the three-body decay. It is possible to combine the $\pi^+\pi^-$ pair with the $S$, $P$, $D$ waves. The $S$-, $P$- and $D$-wave transition matrix elements between the two-pion pair and the vacuum are proportional to the time-like scale, vector, and tensor form factors, respectively. It is clear that for the sequential $B^0_s \to \phi f_0(980) \to \phi \pi^+\pi^-$ decay in question, only the $S$ wave contribution from the scalar $f_0(980)$ meson needs to be considered.

C. Kinematic variable

It is convenient to describe the kinematical variables in terms of the light cone coordinates. The relations between the four-dimensional space-time coordinates $(x^0, x^1, x^2, x^3) = (t, x, y, z)$ and the light-cone coordinates $(x^+, x^-, x_\perp)$ are defined as $x^\pm = (x^0 \pm x^3)/\sqrt{2}$ and $x_\perp = (x^1, x^2)$. The scalar product of two vectors is given by $a \cdot b = a_\mu b^\mu = a^+b^- + a^-b^+ - a_\perp \cdot b_\perp$. $n_+^\mu = (1, 0, 0)$ and $n_-^\mu = (0, 1, 0)$ are the plus and minus null vectors, respectively [110]. The momenta of the participating mesons in the rest frame of the $B_s$ meson are defined as follow.

$$p_{B_s} = \frac{m_{B_s}}{\sqrt{2}} (1, 1, 0),$$

(10)

$$p_\phi = (p_\phi^+, p_\phi^-, 0),$$

(11)

$$\epsilon_\phi^\parallel = \frac{1}{m_\phi} (p_\phi^+, -p_\phi^-, 0),$$

(12)

$$p_{2\pi} = q = (q^-, q^+, 0),$$

(13)
\[ p_{\pi^+} = (\zeta q^-, \zeta q^+, +\sqrt{\zeta} w), \]  
(14)

\[ p_{\pi^-} = (\zeta q^-, \zeta q^+, -\sqrt{\zeta} w), \]  
(15)

\[ p_{\phi}^\pm = \left( E_\phi \pm p_{cm} \right) / \sqrt{2}, \]  
(16)

\[ q^\pm = \left( E_w \pm p_{cm} \right) / \sqrt{2}, \]  
(17)

\[ E_\phi = (m_{B_s}^2 + m_\phi^2 - w^2) / (2 m_{B_s}), \]  
(18)

\[ E_w = (m_{B_s}^2 - m_\phi^2 + w^2) / (2 m_{B_s}), \]  
(19)

\[ p_{cm} = \sqrt{\frac{[m_{B_s}^2 - (m_\phi + w)^2][m_{B_s}^2 - (m_\phi - w)^2]}{2 m_{B_s}}}, \]  
(20)

\[ q^2 = (p_{B_s} - p_\phi)^2 = (p_{\pi^+} + p_{\pi^-})^2 = w^2, \]  
(21)

where \( \epsilon^\parallel \) is the longitudinal polarization vector of the \( \phi \) meson. \( \bar{\zeta} = 1 - \zeta \). The variable \( \zeta (\bar{\zeta}) \) is the \( \pi^+ (\pi^-) \) meson momentum fraction of the \( \pi^+ \pi^- \) meson pair with the invariant mass \( w = m(\pi\pi) \). The momenta of the spectator quark of the \( B_s \) meson and the valence quarks of the final states are defined as \( p, k \) and \( l \) (see Fig.1 for detail) with the longitudinal momentum fraction of \( x, y, z \) and the transverse momentum of \( p_T, k_T, l_T \), respectively,

\[ p = (x p_{B_s}^+, x \bar{p}_{B_s}, p_T), \]  
(22)

\[ k = (y p_{\phi}^+, y \bar{p}_{\phi}, k_T), \]  
(23)

\[ l = (z q^-, z q^+, l_T). \]  
(24)

**D. The distribution amplitudes**

Within the pQCD framework, the WFs and/or DAs are the essential input parameters. Following the notations in Refs.[111–115], the WFs of the \( B_s \) meson and the longitudinally polarized \( \phi \) meson are defined as:

\[ \langle 0 | \bar{b}_i(0) s_j(z) | B_s(p) \rangle = - \frac{i f_{B_s}}{4} \int d^4k e^{-ik\cdot z} \left\{ \left[ \bar{\phi} \Psi_{B_s}^a(k) + m_{B_s} \Phi_{B_s}^a(k) \right] \gamma_5 \right\}_{ji}, \]  
(25)

\[ \langle \phi(p, \epsilon^\parallel) | s_i(z) \bar{s}_j(0) | 0 \rangle = \frac{1}{4} \int_0^1 dk e^{ik\cdot z} \left\{ \epsilon^\parallel m_\phi \Phi_{\phi}^a(k) + \epsilon^\parallel \bar{\phi} \Phi_{\phi}^a(k) - m_\phi \Phi_{\phi}^a(k) \right\}_{ji}, \]  
(26)
where \( f_{B_s} = 227.2 \pm 3.4 \) MeV [1] is the decay constant of the \( B_s \) meson. The WFs of \( \Phi^P_B \) and \( \Phi^v \) are twist-2, while the WFs of \( \Phi^P_B \) and \( \Phi^s \) are twist-3. By integrating out the transverse momentum from the wave functions, one can obtain the corresponding DAs.

In our calculation, the expressions of the \( B_s \) DAs are [113–115]:

\[
\phi^a_B(x) = N_a x \bar{x} \exp\left\{ -\frac{1}{8} \omega_B \left( \frac{m_s^2}{x} + \frac{m_b^2}{\bar{x}} \right) \right\},
\]

\[
\phi^p_B(x) = N_p \exp\left\{ -\frac{1}{8} \omega_B \left( \frac{m_s^2}{x} + \frac{m_b^2}{\bar{x}} \right) \right\},
\]

where \( x \) and \( \bar{x} = 1 - x \) are the longitudinal momentum fractions of light and heavy quarks, respectively; \( m_b = 4.78 \pm 0.06 \) GeV [1] and \( m_s = \simeq 0.51 \) GeV [116] are the mass of the \( b \) and \( s \) quarks. The parameter \( \omega_B \) determines the average transverse momentum of partons, and \( \omega_B \simeq m_i \alpha_s \). The parameters \( N_a \) and \( N_p \) are the normalization coefficients,

\[
\int_0^1 dx \phi^{a,p}_B(x) = 1.
\]

One distinguish feature of the above DAs is the exponential functions, which strongly suppress the contribution from the end point of \( x, \bar{x} \rightarrow 0 \) and naturally provide the effective truncation for the end point and soft contributions. In addition, the exponential factors are proportional to the ratio of the parton mass squared \( m_i^2 \) to the momentum fraction \( x_i \). Hence, the above DAs are generally consistent with the ansatz that the momentum fractions are shared among the valence quarks according to the quark mass, i.e., the light \( s \) quark carries relatively less momentum fraction in the heavy-light \( B_s \) meson.

The expressions of the two-particle DAs of the \( \phi \) meson are [111, 112]:

\[
\phi^v_\phi(x) = 6 f_\phi x \bar{x} \left\{ 1 + a^\phi_2 C^{3/2}_2(\xi) + \cdots \right\},
\]

\[
\phi^t_\phi(x) = 3 f^T_\phi \xi^2,
\]

\[
\phi^s_\phi(x) = 3 f^T_\phi \xi,
\]

where \( \xi = x - \bar{x} \); \( f_\phi = (215 \pm 5) \) MeV and \( f^T_\phi = (186 \pm 9) \) MeV [111] are the longitudinal and transverse decay constants for the \( \phi \) meson. \( C^{3/2}_2(\xi) \) is the Gegenbauer polynomial. The nonperturbative parameter \( a^\phi_2 = 0.18 \pm 0.08 \) [111] is the Gegenbauer moment.

The \( S \)-wave two-pion WFs have been defined in Ref.[117, 118]

\[
\Phi_{\pi^+\pi^-} = \frac{1}{4} \left\{ \delta \phi_-(z, \zeta, w^2) + \omega \phi_+(z, \zeta, w^2) - \omega \left( \delta \phi_+ \delta \phi_- - 1 \right) \phi_+(z, \zeta, w^2) \right\},
\]
where the variable $z$ gives the momentum fraction of the quark. The variables $\zeta$ and $w^2$ concern the hadronic system but not the partons. The asymptotic expressions of the two-pion DAs are the variable $\zeta$ independent [92, 93, 117, 118],

$$\phi_-(z, \zeta, w^2) = 18 F_s(w^2) a_{\pi\pi} z \bar{z} (\bar{z} - z) = \phi_-, \quad (34)$$

$$\phi_s(z, \zeta, w^2) = F_s(w^2) = \phi_s, \quad (35)$$

$$\phi_+(z, \zeta, w^2) = F_s(w^2) (\bar{z} - z) = \phi_+, \quad (36)$$

where $F_s(w^2)$ is the time-like scalar form factor, and the parameter $a_{\pi\pi} = 0.2 \pm 0.2$ [92]. Clearly, the DAs of $\phi_-$ and $\phi_+$ are antisymmetric under the interchange $z \leftrightarrow \bar{z}$. The $F_s(w^2)$ involves the strong interaction between the $S$-wave resonance and two-pion, as well as elastic rescattering of pion pair. Because the mass of the $f_0(980)$ meson is near the $KK$ threshold, the form factor $F_s(w^2)$ for the $S$-wave $f_0(980)$ resonance is usually parameterized with the Flatté model [119–122].

$$F_s(w^2) = \frac{m^2_{f_0(980)}}{m^2_{f_0(980)} - w^2 - im_{f_0(980)}(g_{\pi\pi}\rho_{\pi\pi} + g_{KK}\rho_{KK})}, \quad (37)$$

where in our calculation, the mass of the $f_0(980)$ meson is fixed to the value used by the LHCb Collaboration in the amplitude analysis for the $B_s \rightarrow \phi \pi^+\pi^-$ decay, i.e., $m_{f_0(980)} = 0.98$ GeV [2]. The parameters of $g_{\pi\pi}$ and $g_{KK}$ are the $f_0(980)$ couplings to the $\pi\pi$ and $KK$ states, respectively. Their values are fitted by the LHCb Collaboration through the $B_s \rightarrow J/\psi \pi^+\pi^-$ decay, and $g_{\pi\pi} = 167 \pm 7$ MeV and $g_{KK} = (3.47 \pm 0.12)g_{\pi\pi}$ [122]. The expressions of the phase space factors are written as [120–122]:

$$\rho_{\pi\pi} = \frac{2}{3} \sqrt{1 - \frac{4m^2_{\pi\pi}}{w^2}} + \frac{1}{3} \sqrt{1 - \frac{4m^2_{\pi0}}{w^2}}, \quad (38)$$

$$\rho_{KK} = \frac{1}{2} \sqrt{1 - \frac{4m^2_{KK}}{w^2}} + \frac{1}{2} \sqrt{1 - \frac{4m^2_{K^0}}{w^2}}. \quad (39)$$

### E. Decay amplitude

The Feynman diagrams for the $B_s \rightarrow \phi f_0(980) \rightarrow \phi \pi^+\pi^-$ decay within the pQCD framework are shown in Fig.1, including (1) the $\phi$ meson emission while the $B_s$ meson transition into the two-pion pair through the $f_0(980)$ resonance in Fig.1(a-d), (2) the two-pion pair...
emission while the $B_s$ meson transition into the $\phi$ meson in Fig.1(e-h), (3) the $B_s$ annihilation in Fig.1(i-p). In addition, the diagrams in the first (last) two columns are called the (non)factorizable topologies. In general, the amplitudes of the factorizable topologies have the relatively simple structures. For the factorizable topologies of Fig.1(a,b), the $\phi$ meson can be isolated from the $B_s\pi\pi$ system, so the amplitudes can be written as the product of the decay constant $f_\phi$ and the $B_s \rightarrow \pi\pi$ transition form factors. Similarly, for the factorizable topologies of Fig.1(e,f), the two-pion pair can be isolated from the $B_s\phi$ system, and the transition matrix elements between the vacuum and the two-pion pair can be expressed as
the time-like scalar form factor $F_s(w^2)$. So the HMEs of the local operators can be written as the $B_s \rightarrow \phi$ transition form factors multiplied by the form factor $F_s(w^2)$. Likewise, for the factorizable topologies of Fig.1(i,j) and (m,n), the $B_s$ meson can be isolated from the final states, so the amplitudes can be written as the product of the decay constant $f_{B_s}$ and the time-like form factors for the transition between the $\phi$ meson and the two-pion pair. The amplitudes of the nonfactorizable topologies involve the DAs of all participating mesons.

Using the PQCD formula in Eq.(9) for the quasi two-body decay, the amplitude for the $B_s \rightarrow \phi f_0(980) \rightarrow \phi \pi^+\pi^-$ decay is written as follow.

$$
\mathcal{A} = (V_{us} V_{us} + V_{cb} V_{cs}) \left\{ \mathcal{A}_{ef}^{LL}[a_3 - \frac{1}{2}a_9] + \mathcal{A}_{ef}^{LR}[a_5 - \frac{1}{2}a_7] + \mathcal{A}_{ef}^{SP}[a_0 - \frac{1}{2}a_8] + \mathcal{A}_{nf}^{LL}[C_4 - \frac{1}{2}C_{10}] + \mathcal{A}_{nf}^{LR}[C_6 - \frac{1}{2}C_{8}] + \mathcal{A}_{nf}^{SP}[C_5 - \frac{1}{2}C_{7}] \right\},
$$

$$
\mathcal{A}_{ef}^\rho = \mathcal{A}_c^\rho + \mathcal{A}_d^\rho + \mathcal{A}_e^\rho + \mathcal{A}_f^\rho + \mathcal{A}_g^\rho + \mathcal{A}_h^\rho + \mathcal{A}_i^\rho + \mathcal{A}_l^\rho, \quad \text{for } \rho = LL, LP, SP \quad (40)
$$

$$
\mathcal{A}_{nf}^\rho = \mathcal{A}_c^\rho + \mathcal{A}_d^\rho + \mathcal{A}_g^\rho + \mathcal{A}_h^\rho + \mathcal{A}_i^\rho + \mathcal{A}_l^\rho + \mathcal{A}_m^\rho + \mathcal{A}_n^\rho, \quad \text{for } \rho = LL, LP, SP \quad (41)
$$

$$
a_i = \begin{cases} 
C_i + C_{i+1}/N_c & \text{for odd } i; \\
C_i + C_{i-1}/N_c & \text{for even } i,
\end{cases} \quad (43)
$$

where the Wilson coefficients $C_i$ are looked as the function variables of the amplitudes of $\mathcal{A}_{ef}^\rho$, $\mathcal{A}_{nf}^\rho$ and $\mathcal{A}_\rho^\sigma$, and $N_c = 3$ is the color number. $\mathcal{A}_{ef}^\rho$ ($\mathcal{A}_{ff}^\rho$) is the sum of the amplitudes for the (non)factorizable topologies. The superscript $\rho$ of the amplitude building block $\mathcal{A}_\rho^\sigma$ refers to the three possible Dirac structures $\Gamma_1 \otimes \Gamma_2$ of the operators $(\bar{q}_1 q_2)\Gamma_1 (\bar{q}_3 q_4)\Gamma_2$, namely $\rho = LL$ for $(V - A) \otimes (V - A)$, $\rho = LP$ for $(V - A) \otimes (V + A)$ and $\rho = SP$ for $-2(S - P) \otimes (S + P)$. The subscript $\sigma$ of $\mathcal{A}_\rho^\sigma$ ($\sigma = a b, \cdots, p$) corresponds to the sub-diagram indices of Fig.1. $\mathcal{A}_{ef}^\rho$, $\mathcal{A}_{ff}^\rho$ and $\mathcal{A}_\rho^\sigma$ are the functions of the Wilson coefficient $C_i$. The analytical expressions of the amplitude building blocks $\mathcal{A}_\rho^\sigma$ are listed in Appendix A in detail.

### III. NUMERICAL RESULTS AND DISCUSSION

The differential branching ratio for the sequential $B_s \rightarrow \phi f_0(980) \rightarrow \phi \pi^+\pi^-$ decay is [1]:

$$
\frac{d\mathcal{B}}{dw} = \frac{\tau_{B_s} p^*_{\pi} p_{cm} \cosh}{4(2\pi)^3 m_{B_s}^2} |\mathcal{A}|^2, \quad (44)
$$

where $\tau_{B_s} = (1.510 \pm 0.005)$ ps is the lifetime of the $B_s$ meson [1]. The kinematic variable $p^*_{\pi}$ is the pion momentum in the rest frame of the two-pion pair,

$$
p^*_{\pi} = \frac{1}{2} \sqrt{w^2 - 4m_{\pi \pm}^2}. \quad (45)
$$
In our calculation, besides the aforementioned parameters, other related parameters, such as the mass of the mesons and quarks, will take their values given in Ref.[1]. And if it is not specified explicitly, their central values will be fixed as the default inputs. Our numerical result of the branching ratio is

\[
B(B_s \to \phi f_0(980) \to \phi \pi^+ \pi^-) = \left[1.31^{+0.40}_{-0.31}(a_{\pi\pi})^{+0.19}_{-0.16}(m_b)^{+0.10}_{-0.09}(\text{CKM})\right] \times 10^{-6},
\]

where the uncertainties come from the parameter \(a_{\pi\pi}\) of DA in Eq.(34), the \(b\) quark mass \(m_b\), the CKM factors \(V_{ub}^* V_{us}\) and \(V_{cb}^* V_{cs}\), respectively. It is clear that the result in Eq.(46) agrees with the LHCb measurement in Eq.(1) within uncertainties.

![Graph showing the distributions from some resonances versus the invariant mass of the \(\pi^+\pi^-\) pair for the \(B_s \to \phi f_0(980) \to \phi \pi^+ \pi^-\) decay.](Image)

**FIG. 2:** The distributions from some resonances versus the invariant mass of the \(\pi^+\pi^-\) pair for the \(B_s \to \phi f_0(980) \to \phi \pi^+ \pi^-\) decay, where the solid (green) line is our result with the PQCD approach, the dot-dashed (blue) line is the \(f_0(980)\) meson contribution given by the LHCb Collaboration in Fig.7(d) of Ref.[2], and a full explanation of other lines can be found in Ref.[2].

To illustrate the \(S\)-wave \(f_0(980)\) contribution to the decay in question, and to compare our result with the experimental measurement, the dependence of the calibrated differential branching ratio \(dB/dw\) on the pion-pair invariant mass \(w = m(\pi^+ \pi^-)\) is shown in Fig.2. It
is seen that the result with the PQCD approach is generally consistent with the shape line of the $f_0(980)$ meson fitted by the LHCb Collaboration [2].

In addition, the contributions from different topologies to the $B_s \rightarrow \phi f_0(980) \rightarrow \phi \pi^+ \pi^-$ decay are investigated. It shows that (1) the main contributions come from the factorizable $\phi$ emission topologies of Fig.1(a,b). (2) Due to the renormalization conditions of the two-pion DAs, only the amplitudes corresponding to the Dirac current structure of $\Gamma_1 \otimes \Gamma_2 = -2(S-P) \otimes (S+P)$ have nonzero contributions [see Eq.(A14)-Eq.(A17)] for the factorizable two-pion emission topologies of Fig.1(e,f). (3) Because of the opposite sign of the quark propagators between the factorizable annihilation topologies of Fig.1(i) [Fig.1(j)] and Fig.1(n) [Fig.1(m)], the interference cancelation mechanism results in the relatively small total contributions from the factorizable annihilation topologies. (4) For each type diagrams, such as the $\phi$ emission diagrams in Fig.1(a-d) or the two-pion emission diagrams in Fig.1(e-h), the nonfactorizable contributions are small relative to the factorizable contributions because of the $1/N_c$ suppression. (5) The relative magnitudes of decay amplitudes basically correspond with the power estimations in Ref.[88], i.e.,

$$\sum_{\alpha=LL,LR,SP} A_{i}^\alpha : \sum_{\beta=LL,LR,SP} A_{j}^\beta : \sum_{\rho=LL,LR,SP} A_{k}^\rho = 1 : \frac{w}{m_{B_s}} : \frac{\Lambda_{QCD}}{m_{B_s}}.$$  \hspace{1cm} (47)

It should be pointed out that the $B_s \rightarrow \phi \pi^+ \pi^-$ decay could be approximately handled as the quasi two-body sequential $B_s \rightarrow \phi f_0(980) \rightarrow \phi \pi^+ \pi^-$ decay at the edges of the Dalitz plot, and there are still many factors that can affect the theoretical result. For example, the contributions from the center regions of the Dalitz plot and the nonresonant contributions to the $B_s \rightarrow \phi \pi^+ \pi^-$ decay are not considered in this paper. It has shown in Refs.[52–54] that the nonresonant contributions are important and deserve much attention, which is beyond the scope of this paper.

IV. SUMMARY

The rare cascade $B_s \rightarrow \phi f_0(980) \rightarrow \phi \pi^+ \pi^-$ decay is induced by the FCNC $b \rightarrow \bar{s}s\bar{s}$ process within SM, where the isoscalar $f_0(980)$ meson has a substantial $s\bar{s}$ component. Given the two-pion pair with small invariant mass GeV comes from the $S$-wave resonant $f_0(980)$ state, the three-body $B_s \rightarrow \phi f_0(980) \rightarrow \phi \pi^+ \pi^-$ decay can be approximated as the quasi two-body decay. By introducing the nonperturbative two-pion DAs to describe the two-pion system,
and parameterizing the scalar form factor for the $f_0(980)$ resonance with the Flatté model, the $B_s \rightarrow \phi f_0(980) \rightarrow \phi \pi^+\pi^-$ decay is studied with the PQCD approach. It is found that with appropriate parameters, the theoretical result of the branching ratio in the mass range $400\text{ MeV} < m(\pi^+\pi^-) < 1600\text{ MeV}$ is in agreement with the recent LHCb data [2] within uncertainties.

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Appendix A: The amplitude building blocks for the $B_s \rightarrow \phi f_0(980) \rightarrow \phi \pi^+\pi^-$ decay

\[ \mathcal{F} = -i \pi C_F f_{B_s}, \]  \hspace{1cm} (A1)

\[ \mathcal{A}_{a}^{LL}[C_i] = 2 \mathcal{F} f_\phi \int_0^1 dx \int_0^\infty db \int_0^1 dz \int_{0}^{\infty} db_b f_b \alpha_s(t_a) H_{ef}(\alpha_a, \beta_a, b, b_f) E_\phi(t_a) C_i(t_a) \]
\[ \times m_{B_s} \left[ \phi_\phi^2(x) m_{B_s} \left[ \phi_- m_{B_s} p_{cm} \left( 1 - z + z r_\phi^2 + \phi_\phi r_\phi w p_{cm} \right) + \phi_+ r_\phi w E_\phi \right] - 2 \phi_\phi^2(x) \left[ \phi_+ w (m_{B_s} E_\phi + z w^2 - z E_w m_{B_s}) + \phi_- r_\phi m_{B_s} p_{cm} \phi_\phi m_{B_s} E_\phi \right] \right], \]  \hspace{1cm} (A2)

\[ \mathcal{A}_{a}^{LR}[C_i] = \mathcal{A}_{a}^{LL}[C_i], \]  \hspace{1cm} (A3)

\[ \mathcal{A}_{a}^{SP}[C_i] = 0, \]  \hspace{1cm} (A4)

\[ \mathcal{A}_{b}^{LL}[C_i] = 2 \mathcal{F} f_\phi \int_0^1 dx \int_0^\infty db \int_0^1 dz \int_{0}^{\infty} db_b f_b \alpha_s(t_b) H_{ef}(\alpha_a, \beta_b, b, b_f) E_\phi(t_b) C_i(t_b) \]
\[ \times m_{B_s} p_{cm} \left[ \phi_\phi^2(x) \phi_- \left( \bar{x} w^2 + x m_\phi^2 \right) - \phi_\phi^2(x) \phi_\phi 2 m_{B_s} w \bar{x} \right], \]  \hspace{1cm} (A5)

\[ \mathcal{A}_{b}^{LR}[C_i] = \mathcal{A}_{b}^{LL}[C_i], \]  \hspace{1cm} (A6)

\[ \mathcal{A}_{b}^{SP}[C_i] = 0, \]  \hspace{1cm} (A7)

\[ \mathcal{A}_{c}^{LL}[C_i] = \frac{\mathcal{F}}{N_c} \int_0^1 dx dy dz \int_{0}^{\infty} db db_b \phi_\phi \alpha_s(t_c) H_{en}(\alpha_a, \beta_c, b, \phi, b_f) E_n(t_c) C_i(t_c) \]
\[ \times \phi_\phi^2(y) \left[ 4 m_{B_s}^2 p_{cm} \phi_B^2(x) \phi_- \left( y E_\phi + z E_w - x m_{B_s} \right) + \phi_+ m_{B_s} p_{cm} \left( x - z \right) \right] \]
\[ + \phi_+ \left( x m_{B_s} E_\phi - y m_\phi^2 - z m_{B_s} E_w + z w^2 \right) 2 m_{B_s} w \phi_B^2(x) \], \]  \hspace{1cm} (A8)
\[ A^{LR}_{c}[C_i] = \frac{F}{N_c} \int_0^1 dx dy dz \int_0^\infty db db_\phi db_f b_\phi b_f \alpha_s(t_c) H_{en}(\alpha_a, \beta_c, b, b_f) E_n(t_c) C_i(t_c) \]
\[ \times 2 m_{B_s} \phi_\phi^\nu(y) \left\{ 2 p_{cm} \phi_B^\nu(x) \phi_-(y p_{cm}^2 + y E_w E_\phi + z w^2 - x E_w m_{B_s}) \right. \]
\[ + w \phi_B^p(x) \left[ \phi_s m_{B_s} p_{cm} (x - z) - \phi_+ (x m_{B_s} E_\phi - y m_\phi^2 - z m_{B_s} E_w + z w^2) \right] \} \}, \quad (A9) \]
\[ A^{SP}_{c}[C_i] = -\frac{2F}{N_c} \int_0^1 dx dy dz \int_0^\infty db db_\phi db_f b_\phi b_f \alpha_s(t_c) H_{en}(\alpha_a, \beta_c, b, b_f) E_n(t_c) C_i(t_c) \]
\[ \times m_{B_s} \phi_\phi \left\{ \phi_B^p(x) \phi_2 p_{cm} \left[ x m_{B_s} E_w - \bar{y} (m_{B_s} E_\phi - m_\phi^2) - z w^2 \right] \right. \]
\[ + \phi_B^p(x) \left[ \phi_s m_{B_s} p_{cm} (z - x) + \phi_+ (x m_{B_s} E_\phi - \bar{y} m_\phi^2 - z m_{B_s} E_w + z w^2) \right] \}, \quad (A10) \]
\[ A^{LL}_{d}[C_i] = \frac{F}{N_c} \int_0^1 dx dy dz \int_0^\infty db db_\phi db_f b_\phi b_f \alpha_s(t_d) H_{en}(\alpha_a, \beta_d, b, b_f) E_n(t_d) C_i(t_d) \]
\[ \times 2 m_{B_s} \phi_\phi^\nu(y) \left\{ \phi_B^p(x) \phi_2 p_{cm} \left[ x m_{B_s} - \bar{y} (m_{B_s} E_\phi - m_\phi^2) - z w^2 \right] \right. \]
\[ + \phi_B^p(x) \left[ \phi_s m_{B_s} p_{cm} (z - x) - \phi_+ (x m_{B_s} E_\phi - \bar{y} m_\phi^2 - z m_{B_s} E_w + z w^2) \right] \}, \quad (A11) \]
\[ A^{LR}_{d}[C_i] = \frac{F}{N_c} \int_0^1 dx dy dz \int_0^\infty db db_\phi db_f b_\phi b_f \alpha_s(t_d) H_{en}(\alpha_a, \beta_d, b, b_f) E_n(t_d) C_i(t_d) \]
\[ \times 2 m_{B_s} \phi_\phi^\nu(y) \left\{ \phi_B^p(x) \phi_2 m_{B_s} p_{cm} \left[ (x m_{B_s} E_\phi - \bar{y} m_\phi^2 - z m_{B_s} E_w + z w^2) \right] \right. \]
\[ + \phi_B^p(x) \left[ \phi_s m_{B_s} p_{cm} (z - x) - \phi_+ (x m_{B_s} E_\phi - \bar{y} m_\phi^2 - z m_{B_s} E_w + z w^2) \right] \}, \quad (A12) \]
\[ A^{SP}_{d}[C_i] = -\frac{2F}{N_c} \int_0^1 dx dy dz \int_0^\infty db db_\phi db_f b_\phi b_f \alpha_s(t_d) H_{en}(\alpha_a, \beta_d, b, b_f) E_n(t_d) C_i(t_d) \]
\[ \times m_{B_s} \phi_\phi \left\{ \phi_B^p(x) \phi_2 p_{cm} \left[ x m_{B_s} - \bar{y} (m_{B_s} E_\phi - m_\phi^2) - z w^2 \right] \right. \]
\[ + \phi_B^p(x) \left[ \phi_s m_{B_s} p_{cm} (z - x) - \phi_+ (x m_{B_s} E_\phi - \bar{y} m_\phi^2 - z m_{B_s} E_w + z w^2) \right] \}, \quad (A13) \]
\[ \mathcal{A}_{c}^{LL}[C_i] = \mathcal{A}_{c}^{LR}[C_i] = 0, \quad (A14) \]
\[ \mathcal{A}_{c}^{SP}[C_i] = -4 m_{B_s} F \int_0^1 dx dy \int_0^\infty db db_\phi b_\phi \alpha_s(t_e) H_{ef}(\alpha_c, \beta_e, b, b_\phi) E_f(t_e) C_i(t_e) \]
\[ \times \phi_s \left\{ \phi_B^p(x) \phi_s(y) m_\phi (m_{B_s} - y E_\phi) - \phi_s(y) m_b p_{cm} - \phi_\phi^\nu(y) m_\phi p_{cm} y \right. \]
\[ + 2 \phi_B^p(x) \left[ \phi_\phi^\nu(y) m_{B_s} p_{cm} - \phi_\phi^\nu(y) m_\phi m_b \right] \}, \quad (A15) \]
\[ A_{f}^{LL}[C_{i}] = A_{f}^{LR}[C_{i}] = 0, \quad (A16) \]

\[ A_{f}^{SP}[C_{i}] = -4 m_{B_{s}} F \int_{0}^{1} x dx \int_{0}^{\infty} dy db b_{\phi} b_{\rho}(t_{f}) H_{\gamma}(\alpha_{e}, \beta_{f}, b_{\phi}, b) E_{f}(t_{f}) C_{i}(t_{f}) \times \phi_{s} \{ \phi_{B}^{a}(x) \phi_{B}^{s}(y) 2 m_{\phi} (E_{\phi} - x m_{B_{s}}) - \phi_{B}^{p}(x) \phi_{B}^{u}(y) m_{B_{s}} p_{cm} x \}, \quad (A17) \]

\[ A_{g}^{LL}[C_{i}] = \frac{F}{N_{C}} \int_{0}^{1} x dx \int_{0}^{\infty} dy db b_{\phi} b_{\rho}(t_{g}) H_{\gamma}(\alpha_{e}, \beta_{g}, b_{f}, b_{\phi}, b) E_{g}(t_{g}) C_{i}(t_{g}) \times 2 m_{B_{s}} \phi_{s} - \{ \phi_{B}^{a}(x) \phi_{B}^{u}(y) 2 p_{cm} (y E_{\phi} + z E_{w} - x m_{B_{s}}) + \phi_{B}^{p}(x) r \phi_{B}^{u}(y) [E_{w} (y E_{\phi} - x m_{B_{s}}) + y p_{cm}^{2} + z w^{2}] - \phi_{B}^{l}(y) m_{B_{s}} p_{cm} (y - x) \}, \quad (A18) \]

\[ A_{g}^{LR}[C_{i}] = \frac{F}{N_{C}} \int_{0}^{1} x dx \int_{0}^{\infty} dy db b_{\phi} b_{\rho}(t_{g}) H_{\gamma}(\alpha_{e}, \beta_{g}, b_{f}, b_{\phi}, b) E_{g}(t_{g}) C_{i}(t_{g}) \times 2 m_{B_{s}} \phi_{s} - \{ \phi_{B}^{a}(x) \phi_{B}^{u}(y) 2 p_{cm} (y E_{\phi} + z E_{w} - x m_{B_{s}}) + \phi_{B}^{p}(x) r \phi_{B}^{u}(y) [E_{w} (y E_{\phi} - x m_{B_{s}}) + y p_{cm}^{2} + z w^{2}] + \phi_{B}^{l}(y) m_{B_{s}} p_{cm} (y - x) \}, \quad (A19) \]

\[ A_{g}^{SP}[C_{i}] = \frac{(-2) F}{N_{C}} \int_{0}^{1} x dx \int_{0}^{\infty} dy db b_{\phi} b_{\rho}(t_{g}) H_{\gamma}(\alpha_{e}, \beta_{g}, b_{f}, b_{\phi}, b) E_{g}(t_{g}) C_{i}(t_{g}) \times m_{B_{s}} \{ r m_{B_{s}} \phi_{B}^{a}(x) [ (\phi_{B}^{l}(y) \phi_{s} - \phi_{B}^{s}(y) \phi_{B}^{l}(y) m_{B_{s}} p_{cm} (y - z) + (\phi_{B}^{l}(y) \phi_{s} - \phi_{B}^{s}(y) \phi_{B}^{l}(y) m_{B_{s}} p_{cm} (y - z) + \phi_{B}^{p}(x) \phi_{B}^{u}(y) [\phi_{s} m_{B_{s}} p_{cm} (y - x) + \phi_{s} (x m_{B_{s}} E_{\phi} - y m_{\phi}^{2} + z w^{2} - z E_{w} m_{B_{s}}) \}, \quad (A20) \]

\[ A_{h}^{LL}[C_{i}] = \frac{F}{N_{C}} \int_{0}^{1} x dx \int_{0}^{\infty} dy db b_{\phi} b_{\rho}(t_{h}) H_{\gamma}(\alpha_{e}, \beta_{h}, b_{f}, b_{\phi}, b) E_{h}(t_{h}) C_{i}(t_{h}) \times 2 m_{B_{s}} \phi_{s} - \{ \phi_{B}^{a}(x) \phi_{B}^{u}(y) 2 E_{w} p_{cm} (x + z - 1) + \phi_{B}^{p}(x) r \phi_{B}^{u}(y) [E_{w} (x m_{B_{s}} - y E_{\phi}) - y p_{cm}^{2} + (z - 1) w^{2}] - \phi_{B}^{l}(y) m_{B_{s}} p_{cm} (y - x) \}, \quad (A21) \]
\[ A_{h}^{LR}[C_{i}] = \frac{\mathcal{F}}{N_{C}} \int_{0}^{1} dx \, dy \, dz \int_{0}^{\infty} bdb \, b_{db} \, b_{f} \, db \, \alpha_{s}(t_{h}) \, H_{cm}(\alpha_{c}, \beta_{h}, b_{f}, b_{\phi}, b_{e}) \, E_{n}(t_{h}) \, C_{i}(t_{h}) \]
\times 2 \, m_{B_{s}}^{2} \, \phi_{-} \left\{ \phi_{B_{x}}^{a}(x) \, \phi_{B_{y}}^{\alpha}(y) \, 2 \, p_{cm} \left[ x \, m_{B_{s}} - y \, E_{\phi} + (z - 1) \, E_{w} \right] \right. \\
+ \phi_{B_{x}}^{p}(x) \, r \left[ \phi_{B_{x}}^{s}(y) \, m_{B_{s}} \, p_{cm} (y - x) \right. \\
+ \phi_{B_{y}}^{s}(y) \left[ E_{w} (x \, m_{B_{s}} - y \, E_{\phi}) - y \, p_{cm}^2 + (z - 1) \, w^2 \right] \right\}, \quad (A22) \\

\[ A_{h}^{SP}[C_{i}] = \frac{(-2)\mathcal{F}}{N_{C}} \int_{0}^{1} dx \, dy \, dz \int_{0}^{\infty} bdb \, b_{db} \, b_{f} \, db \, \alpha_{s}(t_{h}) \, H_{cm}(\alpha_{c}, \beta_{h}, b_{f}, b_{\phi}, b_{e}) \, E_{n}(t_{h}) \, C_{i}(t_{h}) \]
\times m_{B_{s}} \, w \left\{ r \, m_{B_{s}} \, \phi_{B_{x}}^{a}(x) \left[ (\phi_{s}^{t}(y) \, \phi_{s}^{t}(y) + \phi_{s}^{t}(y) \, \phi_{s}^{t}(y)) \, p_{cm} (1 - y - z) \right. \\
+ (\phi_{s}^{t}(y) \, \phi_{s}^{t}(y)) \, (y \, E_{\phi} - x \, m_{B_{s}} - z \, E_{w} + E_{w}) \right. \\
+ \phi_{B_{x}}^{p}(x) \phi_{B_{x}}^{s}(y) \left[ \phi_{s} \, m_{B_{s}} \, p_{cm} (x + z - 1) \right. \\
+ \phi_{s} \left[ x \, m_{B_{s}} \, E_{\phi} - y \, m_{\phi}^2 + (w^2 - m_{B_{s}} \, E_{w}) (1 - z) \right] \right\}, \quad (A23) \\

\[ A_{i}^{LL}[C_{i}] = \mathcal{F} \int_{0}^{1} dy \, dz \int_{0}^{\infty} b_{db} \, b_{db} \, b_{f} \, db \, \alpha_{s}(t_{i}) \, H_{af}(\alpha_{i}, \beta_{i}, b_{f}, b_{\phi}, b_{e}) \, E_{B}(t_{i}) \, C_{i}(t_{i}) \]
\times 2 \, m_{B_{s}}^{2} \, \left[ \phi_{-} \phi_{d}^{s}(y) \, m_{B_{s}} \, p_{cm} (z \, r^2 - r^2 - z) \right. \\
- 2 \, r \, w \, \phi_{s}^{s}(y) \left[ \phi_{s} \left( z \, E_{w} + E_{\phi} \right) - \phi_{+} \, p_{cm} (1 - z) \right] \right\}, \quad (A24) \\
\[ A_{i}^{LR} = A_{i}^{LL}, \quad (A25) \]

\[ A_{i}^{SP}[C_{i}] = \frac{(-2)\mathcal{F}}{N_{C}} \int_{0}^{1} dy \, dz \int_{0}^{\infty} b_{db} \, b_{db} \, b_{f} \, db \, \alpha_{s}(t_{i}) \, H_{af}(\alpha_{i}, \beta_{i}, b_{f}, b_{\phi}, b_{e}) \, E_{B}(t_{i}) \, C_{i}(t_{i}) \]
\times 2 \, m_{B_{s}} \left[ w \, \phi_{d}^{s}(y) \left[ \phi_{s} \, m_{B_{s}} \, p_{cm} + \phi_{+} \left( z \, w^2 - m_{\phi}^2 - z \, m_{B_{s}} \, E_{w} \right) \right. \\
+ \phi_{s}^{s}(y) \left. \phi_{-} \right] \, 2 \, r \, m_{B_{s}} \left( p_{cm}^2 + z \, w^2 + E_{\phi} \, E_{w} \right) \right\], \quad (A26) \\

\[ A_{j}^{LL}[C_{i}] = \mathcal{F} \int_{0}^{1} dy \, dz \int_{0}^{\infty} b_{db} \, b_{db} \, b_{f} \, db \, \alpha_{s}(t_{j}) \, H_{af}(\alpha_{j}, \beta_{j}, b_{f}, b_{\phi}, b_{e}) \, E_{B}(t_{j}) \, C_{i}(t_{j}) \]
\times 2 \, m_{B_{s}} \left\{ \phi_{-} \phi_{d}^{s}(y) \, p_{cm} \left[ (1 - y) \, (m_{B_{s}} \, E_{\phi} + p_{cm}^2 + E_{\phi} \, E_{w}) + w^2 \right] \right. \\
- 2 \, r \, w \, m_{B_{s}} \, \phi_{s} \left[ \phi_{d}^{s}(y) \, y \, p_{cm} - \phi_{d}^{s}(y) \, \left[ E_{w} - E_{\phi} (y - 1) \right] \right] \right\}, \quad (A27) \\
\[ A_{j}^{LR}[C_{i}] = A_{j}^{LL}[C_{i}], \quad (A28) \]

\[ A_{j}^{SP}[C_{i}] = \frac{(-2)\mathcal{F}}{N_{C}} \int_{0}^{1} dy \, dz \int_{0}^{\infty} b_{db} \, b_{db} \, b_{f} \, db \, \alpha_{s}(t_{j}) \, H_{af}(\alpha_{j}, \beta_{j}, b_{f}, b_{\phi}, b_{e}) \, E_{B}(t_{j}) \, C_{i}(t_{j}) \]
\times 2 \, m_{B_{s}} \left\{ r \, \phi_{-} \left[ \phi_{d}^{s}(y) (y - 1) \, m_{B_{s}} \, p_{cm} - \phi_{d}^{s}(y) \, [(y - 1) \, p_{cm}^2 + (y - 1) \, E_{\phi} \, E_{w} - w^2] \right. \\
+ \phi_{s} \phi_{d}^{s}(y) 2 \, w \, p_{cm} \right\}, \quad (A29) \]
$$A^{LL}_{k}[C_i] = \frac{F}{NC} \int_0^1 dx \, dy \int_0^\infty db db db db db, \quad f \alpha_s(t_k) H_{an}(\alpha_1, \beta, b, b, f) E_n(t_k) C_i(t_k)$$

$$\times 2 m_{B_s}^2 \left\{ \phi_B^p(x) \left[ \phi_+ \phi^+(y) \right] 2 E_w p_{cm} + (x + z - 1) \right\}$$

$$++ (\phi_+ \phi^+(y)) r w p_{cm} (y + z - 1)$$

$$++ (\phi_+ \phi^+(y)) r w [(x - 1) m_{B_s} - (y - 1) E_\phi + z E_w]$$

$$+ r_B m_{B_s} \phi_B^p(x) \left[ \phi_+ \phi^+(y) p_{cm} + \phi_+ \phi^+(y) 2 r w \right], \quad (A30)$$

$$A^{LR}_{k}[C_i] = \frac{F}{NC} \int_0^1 dx \, dy \int_0^\infty db db db db db, \quad f \alpha_s(t_k) H_{an}(\alpha_1, \beta, b, b, f) E_n(t_k) C_i(t_k)$$

$$\times 2 m_{B_s}^2 \left\{ \phi_B^p(x) \left[ \phi_+ \phi^+(y) \right] 2 p_{cm} (x m_{B_s} E_w - y E_\phi - y p_{cm} - m_{B_s}^2 + z w^2) \right\}$$

$$++ (\phi_+ \phi^+(y)) r w p_{cm} (1 - y - z)$$

$$++ (\phi_+ \phi^+(y)) r w [(x - 1) m_{B_s} - (y - 1) E_\phi + z E_w]$$

$$+ r_B m_{B_s} \phi_B^p(x) \left[ \phi_+ \phi^+(y) p_{cm} + \phi_+ \phi^+(y) 2 r w \right], \quad (A31)$$

$$A^{SP}_{k}[C_i] = \frac{(2)F}{NC} \int_0^1 dx \, dy \int_0^\infty db db db db db, \quad f \alpha_s(t_k) H_{an}(\alpha_1, \beta, b, b, f) E_n(t_k) C_i(t_k)$$

$$\times m_{B_s} \left\{ \phi_B^p(x) \left[ - r m_{B_s} \phi_+ \phi^+(y) \right] r_B p_{cm} + \phi_+ \phi^+(y) E_w \right\}$$

$$++ r_B w m_{B_s} \phi_B^p(x) \left[ \phi_+ \phi^+(y) p_{cm} + \phi_+ \phi^+(y) \right]$$

$$++ \phi_B^p(x) \left[ r_B \phi_+ \phi^+(y) m_{B_s} p_{cm} (x - y) \right. \right.$$

$$- \phi_+ \phi^+(y) \left[ (1 - x) m_{B_s} E_w - (1 - y) E_\phi E_w - (1 - y) p_{cm}^2 - z w^2 \right] \right. \right.$$

$$- w \phi_+ \phi^+(y) \left[ \phi_+ m_{B_s} p_{cm} (x + z - 1) \right. \right.$$

$$- \phi_+ \phi^+(y) \left[ (y - 1) m_{B_s}^2 + (1 - x) m_{B_s} E_\phi + z (w^2 - m_{B_s} E_w) \right], \quad (A32)$$

$$A^{LL}_{i}[C_i] = \frac{F}{NC} \int_0^1 dx \, dy \int_0^\infty db db db db db, \quad f \alpha_s(t_i) H_{an}(\alpha_1, \beta, b, b, f) E_n(t_i) C_i(t_i)$$

$$\times 2 m_{B_s} \phi_B^p(x) \left\{ \phi_+ \phi^+(y) 2 p_{cm} [x m_{B_s} E_w - z w^2 + (y - 1) (m_{B_s} E_\phi - m_{B_s}^2)] \right\}$$

$$++ r w m_{B_s} \phi_B^p(x) \left[ \phi_+ \phi^+(y) p_{cm} (y + z - 1) \right.$$

$$+ r w [(x - 1) m_{B_s} + (y - 1) E_\phi - z E_w] \right\}, \quad (A33)$$

$$A^{LR}_{i}[C_i] = \frac{F}{NC} \int_0^1 dx \, dy \int_0^\infty db db db db db, \quad f \alpha_s(t_i) H_{an}(\alpha_1, \beta, b, b, f) E_n(t_i) C_i(t_i)$$
\[\times 2 m_B^2 \phi_B^0(x) \left\{ \phi_- \phi_\phi^v(y) 2 E_w p_{cm} (x - z) + r w p_{cm} (\phi_s \phi_\phi^0(y) + \phi_+ \phi_\phi^0(y)) (1 - y - z) + r w (\phi_s \phi_\phi^0(y) + \phi_+ \phi_\phi^0(y)) [x m_B + (y - 1) E_{\phi} - z E_w] \right\}, \quad (A34)\]

\[\mathcal{A}_i^{SP}[C_i] = \frac{(-2)F}{N_C} \int_0^1 dx \, dy \, dz \int_0^\infty \int_0^{\infty} bdb db \rho b_f db_f \alpha_s(t_i) H_{an}(\alpha_i, \beta_i, b, b_\phi, b_f) E_n(t_i) C_i(t_i) \times \phi_B^0(x) \left\{ r m_B^2 \phi_- \phi_\phi^0(y) m_B, p_{cm} (y + x - 1) - \phi_\phi^0(y) [(1 - y) E_{\phi} E_w - x m_B E_w - (y - 1) p_{cm}^2 + z w^2] - w m_B \phi_\phi^0(y) \left[ \phi_s m_B, p_{cm} (x - z) + \phi_+ [(y - 1) m_B^2 + x m_B E_{\phi} + z (w^2 - E_w m_B)] \right] \right\}; \quad (A35)\]

\[\mathcal{A}_m^{LL}[C_i] = -2 F dy \, dz \int_0^\infty bdb db \rho b_f db_f \alpha_s(t_m) H_{af}(\alpha_m, \beta_m, b_f, b_\phi) E_B(t_m) C_i(t_m) \times \left\{ m_B, p_{cm} \phi_- \phi_\phi^0(y) \left[ y (E_{\phi} m_B + p_{cm}^2 + E_{\phi} E_w) + w \right] + 2 r w m_B^2 \phi_s \phi_\phi^0(y) \left[ 1 - y + \phi_\phi^0(y) (y E_{\phi} + E_w) \right] \right\}, \quad (A36)\]

\[\mathcal{A}_m^{LR}[C_i] = \mathcal{A}_m^{LL}[C_i], \quad (A37)\]

\[\mathcal{A}_m^{SP}[C_i] = 4 F \int_0^1 dy \, dz \int_0^\infty bdb db \rho b_f db_f \alpha_s(t_m) H_{af}(\alpha_m, \beta_m, b_f, b_\phi) E_B(t_m) C_i(t_m) \times m_B^2 \left\{ r \phi_- \phi_\phi^0(y) m_B, p_{cm} + \phi_\phi^0(y) (y p_{cm}^2 + y E_{\phi} E_w + w^2) \right\} + \phi_s \phi_\phi^0(y) 2 w p_{cm}, \quad (A38)\]

\[\mathcal{A}_n^{LL}[C_i] = \int_0^1 dy \, dz \int_0^\infty bdb db \rho b_f db_f \alpha_s(t_n) H_{af}(\alpha_m, \beta_n, b_f, b_\phi) E_B(t_n) C_i(t_n) \times 2 m_B^2 \left\{ \phi_- \phi_\phi^0(y) m_B, p_{cm} (z r^2 - z + 1) + 2 r w \phi_\phi^0(y) \left[ \phi_s [E_{\phi} - (z - 1) E_w] + \phi_+ z p_{cm} \right] \right\}, \quad (A39)\]

\[\mathcal{A}_n^{LR}[C_i] = \mathcal{A}_n^{LL}[C_i], \quad (A40)\]

\[\mathcal{A}_n^{SP}[C_i] = (-2)F \int_0^1 dy \, dz \int_0^\infty bdb db \rho b_f db_f \alpha_s(t_n) H_{af}(\alpha_m, \beta_n, b_f, b_\phi) E_B(t_n) C_i(t_n) \times 2 m_B \left\{ \phi_- \phi_\phi^0(y) 2 r m_B [z - 1] w^2 - p_{cm}^2 - E_{\phi} E_w \right\} \left\{ - w \phi_\phi^0(y) \left[ \phi_s (1 - z) m_B, p_{cm} - \phi_+ [(z - 1) (w^2 - E_w m_B) - m_B^2] \right] \right\}; \quad (A41)\]
\[ A^{LL}_o[C_i] = \frac{F}{N_C} \int_0^1 dx \, dy \int_{-\infty}^{\infty} db db_f db_b \, \alpha_s(t_o) \, H_{am}(\alpha_m, \beta_o, b, b_\phi, b_f) \, E_n(t_o) \, C_i(t_o) \times 2 m_{B_s} \left\{ \phi_B^o(x) \left[ \phi_+ \phi_\phi^o(y) 2 p_{cm} [E_w((x-1) m_{B_s} + y E_\phi) + y p_{cm}^2 - (z - 1) w^2] + r w m_{B_s} p_{cm} (\phi_s \phi_\phi^i(y) + \phi_+ \phi_\phi^i(y)) (1 - y - z) + r w m_{B_s} (\phi_s \phi_\phi^o(y) + \phi_+ \phi_\phi^o(y)) [(x - 1) m_{B_s} + y E_\phi - (z - 1) E_w] + r_b m_{B_s}^2 \phi_B^o(x) \left[ \phi_+ \phi_\phi^o(y) p_{cm} + \phi_s \phi_\phi^i(y) 2 r w \right] \right\}, \] (A42)

\[ A^{LR}_o[C_i] = \frac{F}{N_C} \int_0^1 dx \, dy \int_{-\infty}^{\infty} db db_f db_b \, \alpha_s(t_o) \, H_{am}(\alpha_m, \beta_o, b, b_\phi, b_f) \, E_n(t_o) \, C_i(t_o) \times 2 m_{B_s}^2 \left\{ \phi_B^o(x) \left[ \phi_+ \phi_\phi^o(y) 2 E_w p_{cm} (x - z) + r w p_{cm} (\phi_s \phi_\phi^i(y) + \phi_+ \phi_\phi^i(y)) (y + z - 1) + r w (\phi_s \phi_\phi^o(y) + \phi_+ \phi_\phi^o(y)) [(x - 1) m_{B_s} + y E_\phi - (z - 1) E_w] \right\} + r_b m_{B_s} \phi_B^o(x) \left[ \phi_+ \phi_\phi^o(y) p_{cm} + \phi_s \phi_\phi^i(y) 2 r w \right], \] (A43)

\[ A^{SP}_o[C_i] = \frac{-2F}{N_C} \int_0^1 dx \, dy \int_{-\infty}^{\infty} db db_f db_b \, \alpha_s(t_o) \, H_{am}(\alpha_m, \beta_o, b, b_\phi, b_f) \, E_n(t_o) \, C_i(t_o) \times m_{B_s} \left\{ \phi_B^o(x) \left[ r_r_b m_{B_s}^2 \phi_+ (\phi_\phi^i(y) p_{cm} - \phi_\phi^o(y) E_w) + r_b w m_{B_s} \phi_\phi^o(y) (\phi_s p_{cm} - \phi_+ E_\phi) \right] + \phi_B^o(x) \left[ r m_{B_s} \phi_+ [\phi_\phi^i(y) m_{B_s} p_{cm}(1 - y - x) - \phi_\phi^o(y)((z - 1) w^2 - y p_{cm}^2 + (1 - x) m_{B_s} E_\phi E_w)] - w \phi_\phi^o(y) [\phi_s m_{B_s} p_{cm} (x - z) + \phi_+ ((1 - y) m_{B_s}^2 - x m_{B_s} E_\phi + z (m_{B_s} E_\phi - w^2))] \right\}, \] (A44)

\[ A^{LL}_p[C_i] = \frac{F}{N_C} \int_0^1 dx \, dy \int_{-\infty}^{\infty} db db_f db_b \, \alpha_s(t_p) \, H_{am}(\alpha_m, \beta_p, b, b_\phi, b_f) \, E_n(t_p) \, C_i(t_p) \times 2 m_{B_s}^2 \phi_B^o(x) \left\{ 2 E_w p_{cm} \phi_+ \phi_\phi^o(y) (x + z - 1) + r w p_{cm} (\phi_s \phi_\phi^i(y) + \phi_+ \phi_\phi^i(y)) (1 - y - z) + r w (\phi_s \phi_\phi^o(y) + \phi_+ \phi_\phi^o(y)) [x m_{B_s} - y E_\phi + (z - 1) E_w] \right\}, \] (A45)

\[ A^{LR}_p[C_i] = \frac{F}{N_C} \int_0^1 dx \, dy \int_{-\infty}^{\infty} db db_f db_b \, \alpha_s(t_p) \, H_{am}(\alpha_m, \beta_p, b, b_\phi, b_f) \, E_n(t_p) \, C_i(t_p) \]
\[ \times 2 m_{B_s} \phi^p_\phi(x) \left\{ 2 p_{cm} \phi_- \phi_\phi(y) [E_w(x m_{B_s} - E_\phi y) - y p^2_{cm} + (z - 1) w^2] 
+ r w p_{cm} m_{B_s} (\phi_\phi \phi_\phi(y) + \phi_+ \phi_\phi(y)) (y + z - 1) 
+ r w m_{B_s} (\phi_\phi \phi_\phi(y) + \phi_+ \phi_\phi(y)) [x m_{B_s} - y E_\phi + (z - 1) E_w] \right\}, \quad (A46) \]

\[ A_{SP}^{C_i} = \frac{(2)F}{N_C} \int_0^1 dx dy dz \int_0^\infty bdb \phi dbf \alpha_s(t_p) H_{an}(\alpha_m, \beta_m, b, b_\phi, b_f) E_n(t_p) C_i(t_p) 
\times m_{B_s} \phi^p_B(x) \left\{ r m_{B_s} \phi_- \left[ m_{B_s} p_{cm} \phi_\phi(y) (y - x) 
- \phi_\phi(y) [y E_\phi E_w - x m_{B_s} E_w + y p^2_{cm} - (z - 1) w^2] 
- w \phi_\phi(y) \left[ m_{B_s} p_{cm} \phi_\phi (x + z - 1) 
+ \phi_+ [y m^2_\phi - x m_{B_s} E_\phi + (1 - z)(m_{B_s} E_w - w^2)] \right] \right\}, \quad (A47) \]

where the color number \( N_c = 3 \) and the color factor \( C_F = 4/3 \). The superscript \( \rho \) of the amplitude building block \( A_{\rho} \) refers to the three possible Dirac structures \( \Gamma_1 \otimes \Gamma_2 \) of the operators \( (q_1q_2)_{\Gamma_1} (q_3q_4)_{\Gamma_2} \), namely \( \rho = LL \) for \( (V - A) \otimes (V - A) \), \( \rho = LR \) for \( (V - A) \otimes (V + A) \) and \( \rho = SP \) for \( -2(S - P) \otimes (S + P) \). The subscript \( \sigma \) of \( A_{\rho}^\sigma \) (\( \sigma = a, b, \cdots, p \)) corresponds to the sub-diagram indices of Fig.1.

The variables of \( b, b_\phi, b_f \) are the conjugate variables of the transverse momentum \( p_T, k_T, l_T \), respectively.

The function \( H_i \) and Sudakov factor \( E_i \) are defined as

\[ H_{ef}(\alpha, \beta, b_i, b_j) = K_0(b_i \sqrt{-\alpha}) \theta(b_i - b_j) K_0(b_j \sqrt{-\beta}) I_0(b_j \sqrt{-\beta}) + (b_i \leftrightarrow b_j) \]  
\[ \times \theta(-\beta) K_0(b_i \sqrt{-\alpha}) I_0(b_j \sqrt{-\alpha}) + (b_i \leftrightarrow b_j) \]  
\[ \left\{ \theta(b_i - b_j) \left[ i J_0(b_i \sqrt{-\beta}) - Y_0(b_i \sqrt{-\beta}) \right] \right\} \]  
\[ \left\{ \theta(b_i - b_j) \left[ i J_0(b_i \sqrt{-\alpha}) - Y_0(b_i \sqrt{-\alpha}) \right] \right\} \]  
\[ \left\{ \theta(b_i - b_j) \left[ i J_0(b_i \sqrt{-\alpha}) - Y_0(b_i \sqrt{-\alpha}) \right] J_0(b_j \sqrt{-\beta}) + (b_i \leftrightarrow b_j) \right\} \]  
\[ \times \frac{\pi^2}{4} \left\{ i J_0(b_i \sqrt{-\alpha}) - Y_0(b_i \sqrt{-\alpha}) \right\}, \quad (A50) \]

\[ H_{an}(\alpha, \beta, b_i, b_j) = \frac{\pi}{2} \left\{ \theta(-\beta) K_0(b_i \sqrt{-\beta}) + \theta(-\beta) \left[ i J_0(b_i \sqrt{-\beta}) - Y_0(b_i \sqrt{-\beta}) \right] \right\} \]  
\[ \times \left\{ \theta(b_i - b_j) \left[ i J_0(b_i \sqrt{-\alpha}) - Y_0(b_i \sqrt{-\alpha}) \right] J_0(b_j \sqrt{-\beta}) + (b_i \leftrightarrow b_j) \right\} \]  
\[ \delta(b_j - b_k), \quad (A51) \]

\[ E_\phi(t) = \exp \left\{ - S_{B_s}(t) - S_{f_0}(t) \right\}, \quad (A52) \]

\[ E_f(t) = \exp \left\{ - S_{B_s}(t) - S_\phi(t) \right\}, \quad (A53) \]

\[ E_B(t) = \exp \left\{ - S_{f_0}(t) - S_\phi(t) \right\}, \quad (A54) \]

\[ E_n(t) = \exp \left\{ - S_{B_s}(t) - S_{f_0}(t) - S_\phi(t) \right\}, \quad (A55) \]
where the subscripts \( i = ef, en, af, an \) of the function \( H_i \) correspond to the factorizable emission topologies, the nonfactorizable emission topologies, the factorizable annihilation topologies, and the nonfactorizable annihilation topologies, respectively. \( I_0, J_0, K_0 \) and \( Y_0 \) are the Bessel functions. The expression of \( s(x, b, Q) \) can be found in of Ref.[4]. \( \gamma_q = -\alpha_s/\pi \) is the quark anomalous dimension.

The parameters of \( \alpha_i \) and \( \beta_i \) are the virtualities of gluons and quarks. The subscript \( i \) of \( \alpha_i, \beta_i, t_i \) corresponds to the indices of Fig.1. The explicit definitions of the virtualities and typical scale \( t_i \) are given as follows.

\[
\alpha_a = x^2 m_{Bs}^2 + z^2 w^2 - 2 x z m_{Bs} E_w, \tag{A59}
\]
\[
\alpha_e = x^2 m_{Bs}^2 + y^2 m_\phi^2 - 2 x y m_{Bs} E_\phi, \tag{A60}
\]
\[
\alpha_i = \bar{y}^2 m_\phi^2 + z^2 w^2 + \bar{y} z (m_{Bs}^2 - m_\phi^2 - w^2), \tag{A61}
\]
\[
\alpha_m = y^2 m_\phi^2 + \bar{z}^2 w^2 + y \bar{z} (m_{Bs}^2 - m_\phi^2 - w^2), \tag{A62}
\]
\[
\beta_a = (1 - r_b^2) m_{Bs}^2 + z^2 w^2 - 2 z m_{Bs} E_w, \tag{A63}
\]
\[
\beta_b = w^2 + x^2 m_{Bs}^2 - 2 x m_{Bs} E_w, \tag{A64}
\]
\[
\beta_c = (x - y) (x - z) m_{Bs}^2
\]
\[
+ (y - z) (y - x) m_\phi^2
\]
\[
+ (z - x) (z - y) w^2, \tag{A65}
\]
\[
\beta_d = \beta_c|_{y \rightarrow \bar{y}}, \tag{A66}
\]
\[
\beta_e = (1 - r_b^2) m_{Bs}^2 + y^2 m_\phi^2 - 2 y m_{Bs} E_\phi, \tag{A67}
\]
\[
\beta_f = m_\phi^2 + x^2 m_{Bs}^2 - 2 x m_{Bs} E_\phi, \tag{A68}
\]
\[
\beta_g = \beta_c, \tag{A69}
\]
\[
\beta_h = \beta_g|_{z \rightarrow \bar{z}}, \tag{A70}
\]
\[
\beta_i = m_\phi^2 + z^2 w^2 + z (m_{Bs}^2 - m_\phi^2 - w^2), \tag{A71}
\]
\[
\beta_j = \bar{y}^2 m_\phi^2 + w^2 + \bar{y} (m_{Bs}^2 - m_\phi^2 - w^2), \tag{A72}
\]
$$\beta_k = \beta_c \frac{(x \to \bar{x})}{y \to \bar{y}} - m_b^2,$$  \hspace{1cm} (A73)

$$\beta_l = \beta_c \frac{(y \to \bar{y})}{},$$  \hspace{1cm} (A74)

$$\beta_m = w^2 + y^2 m_\phi^2 + y (m_{B_s}^2 - m_\phi^2 - w^2),$$  \hspace{1cm} (A75)

$$\beta_n = \bar{z}^2 w^2 + m_\phi^2 + \bar{z} (m_{B_s}^2 - m_\phi^2 - w^2),$$  \hspace{1cm} (A76)

$$\beta_o = \beta_c \frac{(x \to \bar{x})}{z \to \bar{z}} - m_b^2,$$  \hspace{1cm} (A77)

$$\beta_p = \beta_c \frac{(z \to \bar{z})}{},$$  \hspace{1cm} (A78)

$$t_{a,b} = \max\{\sqrt{-\alpha_a} \sqrt{|\beta_{a,b}|}, 1/b, 1/b_f\},$$  \hspace{1cm} (A79)

$$t_{c,d} = \max\{\sqrt{-\alpha_a} \sqrt{|\beta_{c,d}|}, 1/b, 1/b_\phi\},$$  \hspace{1cm} (A80)

$$t_{e,f} = \max\{\sqrt{-\alpha_e} \sqrt{|\beta_{e,f}|}, 1/b, 1/b_\phi\},$$  \hspace{1cm} (A81)

$$t_{g,h} = \max\{\sqrt{-\alpha_e} \sqrt{|\beta_{g,h}|}, 1/b, 1/b_f\},$$  \hspace{1cm} (A82)

$$t_{i,j} = \max\{\sqrt{-\alpha_i} \sqrt{|\beta_{i,j}|}, 1/b_\phi, 1/b_f\},$$  \hspace{1cm} (A83)

$$t_{k,l} = \max\{\sqrt{-\alpha_i} \sqrt{|\beta_{k,l}|}, 1/b, 1/b_f\},$$  \hspace{1cm} (A84)

$$t_{m,n} = \max\{\sqrt{-\alpha_m} \sqrt{|\beta_{m,n}|}, 1/b_\phi, 1/b_f\},$$  \hspace{1cm} (A85)

$$t_{o,p} = \max\{\sqrt{-\alpha_m} \sqrt{|\beta_{o,p}|}, 1/b, 1/b_f\}.$$  \hspace{1cm} (A86)

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