RECURRENCE PROPERTIES AND DISJOINTNESS ON THE INDUCED SPACES

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ABSTRACT. A topological dynamical system induces two natural systems, one is on the hyperspace and the other one is on the space of probability measures. The connection among some dynamical properties on the original space and on the induced spaces are investigated. Particularly, a minimal weakly mixing system which induces a P-system on the probability measures space is constructed and some disjointness result is obtained.

1. INTRODUCTION

Throughout this paper, by a topological dynamical system (t.d.s. for short) we mean a pair $(X, T)$, where $X$ is a compact metric space with a metric $\rho$ and $T$ is a continuous surjective map from $X$ to itself. A non-empty closed invariant subset $Y \subset X$ (i.e., $TY \subset Y$) defines naturally a subsystem $(Y, T)$ of $(X, T)$.

A t.d.s. $(X, T)$ induces two natural systems, one is $(K(X), T_K)$ on the hyperspace $K(X)$ consisting of all non-empty closed subsets of $X$ with the Hausdorff metric, and the other one is $(M(X), T_M)$ on the probability measures space $M(X)$ consisting of all Borel probability measures with the weak*-topology. Bauer and Sigmund [3] first gave a systematic investigation on the connection of dynamical properties among $(X, T)$, $(K(X), T_K)$ and $(M(X), T_M)$. It was proved that $(X, T)$ is weakly mixing (resp. mildly mixing, strongly mixing) if and only if $(K(X), T_K)$ (or $(M(X), T_M)$) has the same property. We remark that later it was shown that the transitivity of $(K(X), T_K)$ (resp. $(M(X), T_M)$) is equivalent to the weak mixing property of $(K(X), T_K)$ (resp. $(M(X), T_M)$), see [2] and [24].

Since then the connection of dynamical properties among $(X, T)$, $(K(X), T_K)$ and $(M(X), T_M)$ has been studied by many authors, see, e.g., [2] [8] [9] [13] [14] [15] [18] [23]. A remarkable result by Glasner and Weiss [8] stated that the topological entropy of $(X, T)$ is zero if and only if so is $(M(X), T_M)$ (similar results related to nullness and tameness can be found in [13] [14]). Recently, Li [18] observed that $(K(X), T_K)$ is a $P$-system if and only if $(X, T)$ is a weakly mixing system with dense small periodic sets (first defined by Huang and Ye in [12] and called an HY-system in [17]).

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In this paper we further exploit the connection, and focus on periodic systems, $P$-systems, $M$-systems, $E$-systems and disjointness. On the way to do this we define an almost HY-system and show that $(M(X), T_M)$ is a $P$-system if and only if $(X, T)$ is a weakly mixing almost-HY-system (Theorem 4.10). A minimal weakly mixing system which induces a $P$-system on $(M(X), T_M)$ is constructed showing that HY-systems and almost HY-systems are different property (Theorem 4.11). We conjecture that there is a weakly mixing proximal $E$-system inducing a $P$-system on $(M(X), T_M)$ (this will be answered affirmatively in a forthcoming paper [19]). See the following two tables for the further connection (see Section 3 and Section 4 for details).

**Table 1. The connection with hyperspace**

| $(X, T)$ | periodic | HY-system | w.m. $M$-system | w.m. $E$-system |
|----------|----------|-----------|-----------------|----------------|
| $(K(X), T_K)$ | pointwise periodic | $P$-system | $M$-system | $E$-system |

**Table 2. The connection with probability measures space**

| $(X, T)$ | periodic | almost HY-sys. | not nece. $M$-sys. | w.m. $E$-sys. |
|----------|----------|----------------|------------------|--------------|
| $(M(X), T_M)$ | p.w. periodic | $P$-system | $M$-system | $E$-system |

The notion of disjointness of two t.d.s. was introduced by Furstenberg in his seminar paper [6]. It is known if two t.d.s. are disjoint then one of them is minimal. It is an open question which system is disjoint from all minimal system. It was shown that if a transitive t.d.s. is disjoint from all minimal system then it is weakly mixing with dense minimal points [12], and a weakly mixing system with dense distal points is disjoint from all minimal systems [5, 21]. In this paper we show that if $(K(X), T_K)$ is disjoint from all minimal system, then so is $(X, T)$ (Theorem 5.2). It seems that there are examples $(X, T)$ which do not have dense distal points and at the same time $(K(X), T_K)$ do. Unfortunately we could not provide one at this moment.

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2. Preliminary

2.1. Basic definitions and notations. In the article, the sets of integers, nonnegative integers and natural numbers are denoted by $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$, respectively.

A t.d.s. $(X, T)$ is transitive if for each pair non-empty open subsets $U$ and $V$, $N(U, V) = \{ n \in \mathbb{Z}_+ : T^{-n}V \cap U \neq \emptyset \}$ is infinite; it is totally transitive if $(X, T^n)$ is transitive for each $n \in \mathbb{N}$; and it is weakly mixing if $(X \times X, T \times T)$ is transitive. We say that $x \in X$ is a transitive point if its orbit $\text{orb}(x, T) = \{ x, Tx, T^2x, \ldots \}$ is dense.
in $X$. The set of transitive points is denoted by $\text{Tran}_T$. It is well known that if $(X, T)$ is transitive, then $\text{Tran}_T$ is a dense $G_δ$-set.

A t.d.s. $(X, T)$ is minimal if $\text{Tran}_T = X$, i.e., it contains no proper subsystems. A point $x \in X$ is called a minimal point or almost periodic point if $(\text{orb}(x, T), T)$ is a minimal subsystem of $(X, T)$. We say that $x \in X$ is a periodic point if $T^n x = x$ for some $n \in \mathbb{N}$. The set of all periodic points (resp. minimal points) of $(X, T)$ is denoted by $P(T)$ (resp. $\text{AP}(T)$). A t.d.s. $(X, T)$ is called

- a $P$-system if it is transitive and the set of periodic points is dense;
- an $M$-system if it is transitive and the set of minimal points is dense;
- an $E$-system if it is transitive and there is an invariant Borel probability measure $\mu$ with full support, i.e., $\text{supp}(\mu) = X$.

Let $S$ be a subset of $\mathbb{Z}_+$. The upper Banach density and upper density of $S$ are defined by

$$BD^*(S) = \limsup_{|I| \to \infty} \frac{|S \cap I|}{|I|} \quad \text{and} \quad D^*(S) = \limsup_{n \to \infty} \frac{|S \cap [0, n-1]|}{n},$$

where $I$ is taken over all non-empty finite intervals of $\mathbb{Z}_+$ and $| \cdot |$ denote the cardinality of the set.

A subset $S$ of $\mathbb{Z}_+$ is syndetic if it has a bounded gap, i.e., there is $N \in \mathbb{N}$ such that $\{i, i+1, \ldots, i+N\} \cap S \neq \emptyset$ for every $i \in \mathbb{Z}_+$; $S$ is thick if it contains arbitrarily long runs of positive integers, i.e., for every $n \in \mathbb{N}$ there exists some $a_n \in \mathbb{Z}_+$ such that $\{a_n, a_n+1, \ldots, a_n+n\} \subseteq S$.

For a t.d.s. $(X, T), x \in X$ and $U \subset X$ let

$$N(x, U) = \{n \in \mathbb{Z}_+: T^n x \in U\}.$$  

It is well know that $x \in X$ is a minimal point if and only if $N(x, U)$ is syndetic for any neighborhood $U$ of $x$; a t.d.s. $(X, T)$ is weakly mixing if and only if $N(U, V)$ is thick for any non-empty open subsets $U, V$ of $X$ (see, for example, [6, 7]); and a t.d.s. $(X, T)$ is an $E$-system if and only if there is a transitive point $x \in X$ such that $N(x, U)$ has a positive upper Banach density for any neighborhood $U$ of $x$ (see, for example, [11, Lemma 3.6]).

Let $(X, T)$ be a t.d.s. and $(x, y) \in X^2$. It is a proximal pair if

$$\liminf_{n \to +\infty} d(T^n x, T^n y) = 0;$$

and it is a distal pair if it is not proximal. Denote by $P(X, T)$ or $P_X$ the set of all proximal pairs of $(X, T)$. A point $x$ is said to be distal if whenever $y$ is in the orbit closure of $x$ and $(x, y)$ is proximal, then $x = y$. A t.d.s. $(X, T)$ is called distal if $(x, x')$ is distal whenever $x, x' \in X$ are distinct.

Let $\{p_i\}_{i=1}^\infty$ be an infinite sequence in $\mathbb{N}$. One defines

$$FS(\{p_i\}_{i=1}^\infty) := \{p_{i_1} + p_{i_2} + \cdots + p_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k, k \in \mathbb{N}\}.$$  

A subset $F \subset \mathbb{N}$ is called an IP-set if it contains some $FS(\{p_i\}_{i=1}^\infty)$. A subset of $\mathbb{N}$ is called an IP*-set if it has non-empty intersection with any IP-sets. Denote by $\mathcal{F}_{ip}^*$
the family of all IP\(^*-1\)-sets. It is known that \( F_1, F_2 \in \mathcal{F}_{ip} \) implies \( F_1 \cap F_2 \in \mathcal{F}_{ip} \) (see [\ref{21} p. 179]); and \( x \) is distal if and only if \( x \) is IP\(^*-1\)-recurrent, i.e., \( N(x, U) \in \mathcal{F}_{ip} \) for any neighborhood \( U \) of \( x \) (see, for example, [\ref{21} Theorem 9.11] or [\ref{22} Proposition 2.7]).

2.2. Hyperspace. Let \( X \) be a compact metric space with a metric \( \rho \). Let \( K(X) \) be the hyperspace on \( X \), that is, the space of non-empty closed subsets of \( X \) equipped with the Hausdorff metric \( d_H \) defined by

\[
d_H(A, B) = \max \left\{ \min_{x \in A} \max_{y \in B} \rho(x, y), \max_{y \in B} \min_{x \in A} \rho(x, y) \right\}
\]

for \( A, B \in K(X) \).

This metric turns \( K(X) \) into a compact space. It is easy to see that the finite subsets of \( X \) are dense in \( K(X) \).

For any non-empty open subsets \( U_1, \ldots, U_n \) of \( X \), let

\[
\langle U_1, \ldots, U_n \rangle = \left\{ K \in K(X) : K \subset \bigcup_{i=1}^{n} U_i \text{ and } K \cap U_i \neq \emptyset \text{ for each } i = 1, \ldots, n \right\}.
\]

The following family

\[
\{ \langle U_1, \ldots, U_n \rangle : U_1, \ldots, U_n \text{ are non-empty open subsets of } X, n \in \mathbb{N} \}
\]

forms a basis for the topology obtained from the Hausdorff metric \( d_H \), which is called the Vietoris topology (see [\ref{23} Theorem 4.5]).

Now let \( (X, T) \) be a t.d.s. The transformation \( T \) induces a continuous map \( T_K : K(X) \to K(X) \) defined by

\[
T_K(C) = TC \text{ for } C \in K(X).
\]

It is easy to check that \( (K(X), T_K) \) is also a t.d.s.

2.3. Probability measures spaces. Let \( M(X) \) denote the space of Borel probability measures on \( X \) equipped with the Prohorov metric \( D \) defined by

\[
D(\mu, \nu) = \inf \left\{ \varepsilon : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \text{ for all Borel subsets } A \subset X \right\}
\]

for \( \mu, \nu \in M(X) \), where \( A^\varepsilon = \{ x \in X : \rho(x, A) < \varepsilon \} \). The induced topology is just the weak\(^*\)-topology for measures. It turns \( M(X) \) into a compact metric space. A basis is given by the collection of all sets of the form

\[
V_{\mu}(f_1, \ldots, f_k; \varepsilon) = \left\{ \nu \in M(X) : \left| \int_X f_i d\mu - \int_X f_i d\nu \right| < \varepsilon, 1 \leq i \leq k \right\},
\]

where \( \mu \in M(X), k \geq 1, f_i \in C(X, \mathbb{R}) \) (here \( C(X, \mathbb{R}) \) denote the Banach space of continuous real-valued functions on \( X \) with the supremum norm \( \| \cdot \| \) and \( \varepsilon > 0 \). If \( \{f_n\}_{n=1}^{\infty} \) is a dense subset of \( C(X, \mathbb{R}) \), then

\[
d(\mu, \nu) = \sum_{n=1}^{\infty} \frac{|\int f_n d\mu - \int f_n d\nu|}{2^n (\|f_n\| + 1)}
\]

is also a metric on \( M(X) \) giving the weak\(^*\)-topology.
Lemma 2.1. ([23, pp. 149]) The following statements are equivalent:

1. \( \mu_n \to \mu \) in the weak*-topology;
2. For each closed subset \( F \) of \( X \), \( \limsup_{n \to \infty} \mu_n(F) \leq \mu(F) \);
3. For each open subset \( U \) of \( X \), \( \liminf_{n \to \infty} \mu_n(U) \geq \mu(U) \).

For \( x \in X \), let \( \delta_x \in M(X) \) denote the Dirac point measure of \( x \) defined by
\[
\delta_x(A) = \begin{cases} 
1, & x \in A \\
0, & x \notin A.
\end{cases}
\]
It is easy to see that the map \( x \mapsto \delta_x \) imbeds \( X \) inside \( M(X) \). Note that \( M(X) \) is convex and that the point measures are just the extremal points of \( M(X) \). It follows that the convex combinations of point measures are dense in \( M(X) \).

For \( \mu \in M(X) \) and a Borel subset \( A \) of \( X \) with \( \mu(A) > 0 \), the conditional measure of \( A \) is defined by
\[
\mu_A(B) = \frac{\mu(A \cap B)}{\mu(A)}
\]
for all Borel subsets \( B \subset X \).

Lemma 2.2. Let \( X, Y \) be two compact metric spaces, \( \mu \in M(X) \) and \( \nu \in M(Y) \).

1. If \( A = \bigcup_{i=1}^n A_i \), where \( A_1, \ldots, A_n \) are Borel subsets of \( X \) with \( \mu(A_i) > 0 \) and \( \mu(A_i \cap A_j) = 0 \) for all \( 1 \leq i < j \leq n \), then \( \mu_A = \sum_{i=1}^n \frac{\mu(A_i)}{\mu(A)} \mu_{A_i} \).
2. Let \( \epsilon > 0 \) and \( A \) be a Borel subset of \( X \) with \( \mu(A) > 0 \). If \( B \) is a Borel subset of \( X \) such that \( \mu(B) > 0 \) and \( \mu(A \triangle B) < \mu(A) \cdot \epsilon \), then \( d(\mu_A, \nu_B) \leq 2\epsilon \).
3. If \( \pi : (X, \mu) \to (Y, \nu) \) is measurable and \( \pi \mu = \nu \), then \( \pi \mu \pi^{-1}A = \nu_A \) for each Borel subset \( A \) of \( Y \).

Proof. (1) Note that for any Borel subsets \( B \) of \( X \), we have
\[
\mu_A(B) = \frac{\mu(A \cap B)}{\mu(A)} = \sum_{i=1}^n \frac{\mu(A_i \cap B)}{\mu(A)} \mu_{A_i}(B).
\]

(2) Assume that \( A, B \) are two Borel subsets of \( X \) such that \( \mu(A) \mu(B) > 0 \) and \( \mu(A \triangle B) < \mu(A) \cdot \epsilon \). Then for each \( f \in C(X, \mathbb{R}) \) we have
\[
\left| \int_A f \, d\mu_A - \int_B f \, d\mu_B \right| = \left| \frac{1}{\mu(A)} \int_A f \, d\mu - \frac{1}{\mu(B)} \int_B f \, d\mu \right| \
\leq \frac{1}{\mu(A)} \left| \int_A f \, d\mu - \int_B f \, d\mu \right| + \frac{|\mu(A) - \mu(B)|}{\mu(A) \cdot \mu(B)} \cdot \left| \int_B f \, d\mu \right| 
\leq \frac{2 \cdot (\mu(A \triangle B) \cdot ||f||)}{\mu(A)} < 2 ||f|| \cdot \epsilon.
\]
Hence,
\[
d(\mu_A, \mu_B) = \sum_{n=1}^\infty \frac{1}{2^n} \left( \int f_n \, d\mu_A - \int f_n \, d\nu_B \right) \leq \sum_{n=1}^\infty \frac{\epsilon}{2^{n-1}} = 2\epsilon.
\]
(3) is an obvious fact. \( \square \)
Let \((X, T)\) be a t.d.s. The transformation \(T\) induces a continuous map \(T_M : M(X) \to M(X)\) defined by
\[
(T_M\mu)(A) = \mu(T^{-1}A), \mu \in M(X), A \subset X \text{ Borel.}
\]
It is easy to check that \((M(X), T_M)\) is also a t.d.s. For \(n \in \mathbb{N}\), define
\[
M_n(X) = \left\{ \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} : x_i \in X \text{ (not necessarily distinct)} \right\}.
\]

**Lemma 2.3.** \(M_n(X)\) is closed in \(M(X)\) and invariant under \(T_M\). \(\bigcup_{n=1}^{\infty} M_n(X)\) is dense in \(M(X)\).

Let \(M(X, T) = \{\mu \in M(X) : T_M\mu = \mu\}\) be the set of all \(T\)-invariant measures and \(\mathcal{E}(X, T)\) be the set of all ergodic measures in \(M(X, T)\). It is well known that \(M(X, T)\) is nonempty, compact and convex. If \(M(X, T)\) consists of a single point, then \((X, T)\) is said to be uniquely ergodic.

### 2.4. Product system, factor and extension.

For two t.d.s. \((X, T)\) and \((Y, S)\), their product system \((X \times Y, T \times S)\) is defined by
\[
T \times S(x, y) = (Tx, Sy) \text{ for } x \in X \text{ and } y \in Y.
\]

Higher order product systems are defined analogously and we write \((X^n, T^{(n)})\) for the \(n\)-fold product system \((X \times \cdots \times X, T \times \cdots \times T)\).

Let \((X, T)\) and \((Y, S)\) be two t.d.s. A continuous map \(\pi : X \to Y\) is called a homomorphism or factor map between \((X, T)\) and \((Y, S)\) if it is onto and \(\pi \circ T = S \circ \pi\). In this case we say \((X, T)\) is an extension of \((Y, S)\) or \((Y, S)\) is a factor of \((X, T)\). It is easy to see that \(\pi\) induces in an obvious way a homomorphism from \((M(X), T_M)\) onto \((M(Y), S_M)\) and from \((K(X), T_K)\) onto \((K(Y), S_K)\).

### 3. Dynamic properties on hyperspace

In this section, we will study some dynamic properties of \((K(X), T_K)\). Firstly, we recall the following useful lemma.

**Lemma 3.1.** (see \[2\]) Let \((X, T)\) be a t.d.s. Then the following statements are equivalent:

1. \((K(X), T_K)\) is weakly mixing;
2. \((K(X), T_K)\) is transitive;
3. \((X, T)\) is weakly mixing.

Let \((X, T)\) be a t.d.s. We say that \((X, T)\) has dense small periodic sets \([12]\) if for any non-empty open subset \(U\) of \(X\) there exists a closed subset \(Y\) of \(U\) and \(k \in \mathbb{N}\) such that \(T^kY \subset Y\). Clearly, every \(P\)-system has dense small periodic sets. If \((X, T)\) is transitive and has dense small periodic sets, then it is an \(M\)-system.

The system \((X, T)\) is called an HY-system if it is totally transitive and has dense small periodic sets. In \([12]\), Huang and Ye showed that an HY-system is also weakly mixing and disjoint from any minimal systems. There exists an HY-system without periodic points \([12, \text{Example 3.7}]\). In \([17]\), the author characterized HY-systems by
transitive points via the family of weakly thick sets. Recently, Li \[18\] studied the Devaney’s chaos on the hyperspace. That is, he obtained

**Theorem 3.2.** (see \[18\]) \((X, T)\) is an HY-system if and only if \((K(X), T_K)\) is a P-system.

We recall that a t.d.s. \((X, T)\) is said to be pointwise periodic if all points in \(X\) are periodic; and it is said to be periodic if there exists \(m \in \mathbb{N}\) such that \(T^m\) is the identity map of \(X\). It is clear that if \((K(X), T_K)\) is pointwise periodic, then \((X, T)\) is also pointwise periodic. However, the following example shows that the converse is not true.

**Example 3.3.** Let \(X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}\) with the subspace topology of the real line \(\mathbb{R}\). Define \(T : X \to X\) as
- \(T(0) = 0\) and \(T(1) = 1\);
- \(T\left(\frac{1}{2^n}\right) = \frac{1}{2^n+1}, \ldots, T\left(\frac{1}{2^n-1}\right) = \frac{1}{2^n}\) for each \(n \in \mathbb{N}\).

Note that \((X, T)\) is pointwise periodic. Let \(K = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}_+\}\). Then \(K\) is not a periodic point of \((K(X), T_K)\).

What we have is the following theorem and we omit the simple proof.

**Theorem 3.4.** The following statements are equivalent:
- (1) \((X, T)\) is periodic;
- (2) \((K(X), T_K)\) is periodic;
- (3) \((K(X), T_K)\) is pointwise periodic.

Next we study the characterization of \(M\)-systems and \(E\)-systems on the hyperspace.

**Theorem 3.5.** \((X, T)\) is a weakly mixing \(M\)-system if and only if \((K(X), T_K)\) is an \(M\)-system.

**Proof.** Let \((K(X), T_K)\) be an \(M\)-system. By Lemma\[3.1\] \((X, T)\) is weakly mixing. Now we show that the set of minimal points of \((X, T)\) is dense. Let \(U, V \subset X\) be two non-empty open subsets of \(X\) with \(V \subset U\). Then \(\langle V \rangle = \{A \in K(X) : A \subset V\}\) is a non-empty open subset of \(K(X)\). Since \((K(X), T_K)\) is an \(M\)-system, there exists a minimal point \(C \in \langle V \rangle\) of \(T_K\). It follows that there exists a syndetic subset \(F\) of \(\mathbb{Z}_+\) such that \(T_K^n(C) \in \langle V \rangle\), which implies that \(T_K^n(C) \subset V\) for all \(n \in F\). Let \(D = \bigcup_{n \in F} T_K^n(C)\), then \(D \subset V \subset U\). By \[4\] Theorem 7, \(D \cap AP(T) \neq \emptyset\), so \(U \cap AP(T) \neq \emptyset\). That is, \((X, T)\) is an \(M\)-system.

Now assume \((X, T)\) is a weakly mixing \(M\)-system. Then the product system \((X^n, T^{(n)})\) is an \(M\)-system for each \(n \in \mathbb{N}\). This implies that the restriction of \(T_K\) to \(K_n(X) = \{C \in K(X) : |K| \leq n\}\), as a factor of \(T^{(n)}\), is also an \(M\)-system. Notice that \(\bigcup_{n=1}^\infty K_n(X)\) is dense in \(K(X)\). Hence the set of minimal points of \((K(X), T_K)\) is dense in \(K(X)\). That is, \((K(X), T_K)\) is an \(M\)-system. \(\square\)

**Theorem 3.6.** \((X, T)\) is a weakly mixing \(E\)-system if and only if \((K(X), T_K)\) is an \(E\)-system.
Proof. By [3, Proposition 5] and Lemma 3.1, it remains to show that \((K(X), T_K)\) is an \(E\)-system implies that \((X, T)\) is an \(E\)-system. Assume that \((X, T)\) is not an \(E\)-system, then there is a non-trivial factor \((Y, S)\) of \((X, T)\) such that \((Y, S)\) is uniquely ergodic with a fixed point \(p\).

Let \(\nu\) be an invariant probability measure on \(K(Y)\) with full support. Assume \(U \subset Y\) is non-empty open and \(p \not\in U\). Since \(U\) is a non-empty open subset of \(K(Y)\), there is an ergodic measure (using ergodic decomposition) \(\nu\) on \(K(Y)\) with \(\nu(U) > 0\). Let \(C \in \langle U \rangle\) be a generic point for \(\nu\) (i.e., \(\frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^nC} \to \nu\)), then the return time set \(N(C, \langle U \rangle) = \{n \in \mathbb{Z}_+ : T^nC \subset U\}\) has positive upper density. This implies that each point \(x \in C\) returns to \(U\) with positive upper density. Fix \(x \in C\). Since \((Y, S)\) is uniquely ergodic, \(\frac{1}{n} \sum_{i=0}^{n-1} \delta_{F^nix} \to \delta_p\) and by Lemma 2.1, we have

\[
0 = \delta_p(U) \geq \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{F^nix}(U) = D^*(N(x, U)) > 0,
\]

a contradiction. This shows that \((X, T)\) is an \(E\)-system. \(\square\)

4. Dynamic properties on the space of probability measures

In this section, we will study some dynamic properties of \((M(X), T_M)\).

4.1. Distal points and minimal points. In [3], the authors showed the distality (resp. minimality) of \((X, T)\) does not necessarily imply the distality (resp. minimality) of \((M(X), T_M)\). But we have the following result.

**Theorem 4.1.** If \((X, T)\) is distal, then the set of distal points of \((M(X), T_M)\) is dense in \(M(X)\). If \((X, T)\) is an \(M\)-system, then the set of minimal points of \((M(X), T_M)\) is dense.

**Proof.** Assume that \((X, T)\) is a distal system. Then the product system \((X^n, T^{(n)})\) is distal for every \(n \in \mathbb{N}\). This implies that the restriction of \(T_M\) to \(M_n(X)\), as a factor of \(T^{(n)}\), is also distal. Notice that each \(\mu \in M_n(X)\) is a distal point of \((M(X), T_M)\). Hence the set of distal points of \((M(X), T_M)\) is dense in \(M(X)\).

Now assume \((X, T)\) is an \(M\)-system. Then the set of minimal points of the product system \((X^n, T^{(n)})\) is dense for each \(n \in \mathbb{N}\). This implies that the restriction of \(T_M\) to \(M_n(X)\), as a factor of \(T^{(n)}\), is also has dense minimal points. Notice that \(\bigcup_{n=1}^{\infty} M_n(X)\) is dense in \(M(X)\). Hence the set of minimal points of \((M(X), T_M)\) is dense in \(M(X)\). \(\square\)

4.2. Weakly mixing. In this subsection, we study the weakly mixing property of \((M(X), T_M)\). Firstly, we recall the following useful lemma.

**Lemma 4.2.** [22] \((X, T)\) is weakly mixing if and only if \(N(U, U) \cap N(U, V) \neq \emptyset\) for any non-empty open subsets \(U, V \subset X\).

The following fact is known, see for example [24]. We give a proof for completeness.

**Theorem 4.3.** Let \((X, T)\) be a t.d.s. Then the following statements are equivalent:

1. \((X, T)\) is weakly mixing;
(2) \((M(X), T_M)\) is weakly mixing;
(3) \((M(X), T_M)\) is transitive.

**Proof.** (1) ⇒ (2) can see [3, Theorem 1]; and (2) ⇒ (3) is obvious.

(3) ⇒ (1) Let \(U, V\) be two non-empty open subsets of \(X\) and let \(W_1 = \{\mu \in M(X) : \mu(U) > \frac{1}{2}\}\) and \(W_2 = \{\mu \in M(X) : \mu(U) > \frac{1}{2} \text{ and } \mu(V) > \frac{2}{3}\}\). It is clear that \(W_i\) is non-empty open in \(M(X)\) for \(i = 1, 2\).

By the transitivity of \(T_M\), there is \(k \in \mathbb{N}\) such that \(W_1 \cap T^{-k}_MW_2 \neq \emptyset\). Let \(\mu \in W_1 \cap T^{-k}_MW_2\). Then we have \(\mu \in W_1\) and \(T^k \mu \in W_2\), which implies that \(U \cap T^{-k}U \neq \emptyset\) and \(U \cap T^{-k}V \neq \emptyset\). By Lemma 4.2 \((X, T)\) is weakly mixing. \(\square\)

### 4.3. E-system

In this subsection, we will study the characterization of \(E\)-systems on the space of probability measures. That is, we have the following result.

**Theorem 4.4.** \((X, T)\) is a weakly mixing \(E\)-system if and only if \((M(X), T_M)\) is an \(E\)-system.

**Proof.** Assume that \((X, T)\) is a weakly mixing \(E\)-system. It remains to show that there is a \(T_M\)-invariant measure \(\mu \in \hat{M}(M(X))\) with full support. It is easy to see that \(T^{(n)}\) admits an invariant measure with full support. This implies that the restriction of \(T_M\) to \(M_n(X)\), as a factor of \(T^{(n)}\), admits an invariant measure \(\mu_n\) with full support on \(M_n(X)\). Since \(\bigcup_{n=1}^{\infty} M_n(X)\) is dense in \(M(X)\), the \(T_M\)-invariant measure \(\mu = \sum_{n=1}^{\infty} \frac{1}{2^n}\mu_n \in M(M(X))\) with \(\text{supp}(\mu) = M(X)\). That is, \((M(X), T_M)\) is an \(E\)-system.

Now assume that \((M(X), T_M)\) is an \(E\)-system. By Theorem 4.3 \((X, T)\) is weakly mixing. Let \(\nu\) be a \(T_M\)-invariant measure on \(M(X)\) with full support. Then the barycenter \(\mu = \int_{M(X)} \theta d\nu(\theta)\) is a \(T\)-invariant measure on \(X\). Let \(U\) be a non-empty subset of \(X\) and let \(V = \{m \in M(X) : m(U) > \frac{1}{2}\}\). Then we have

\[
\mu(U) \geq \int_V \theta(U) d\nu(\theta) \geq \frac{1}{2} \nu(V) > 0.
\]

This implies \(\text{supp}(\mu) = X\). \(\square\)

### 4.4. P-system

Note that for each \(x \in X\), \(\delta_x\) is a periodic point of \((M(X), T_M)\) which implies that \(x\) is a periodic point of \((X, T)\). Hence, if \((M(X), T_M)\) is pointwise periodic, then \((X, T)\) is also pointwise periodic. However, the following example shows that the converse is not true.

**Example 4.5.** Let \(X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}\) with the subspace topology of the real line \(\mathbb{R}\). Define \(T : X \rightarrow X\) as

- \(T(0) = 0\) and \(T(1) = 1\);
- \(T\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+1}}, \ldots, T\left(\frac{1}{2^n-1}\right) = \frac{1}{2^n}\) for each \(n \in \mathbb{N}\).

Note that \((X, T)\) is pointwise periodic. Let \(\mu = \sum_{n=1}^{\infty} \frac{1}{2^n}\delta_{\frac{1}{2^n-1}}\). Then \(\mu \in M(X)\) is not periodic.

Again we omit the simple proof of the following fact.

**Theorem 4.6.** The following statements are equivalent:
(1) $(X, T)$ is periodic;
(2) $(M(X), T_M)$ is periodic;
(3) $(M(X), T_M)$ is pointwise periodic.

In order to characterize $P$-systems on the space of probability measures, we need a notion of an almost HY-system.

**Definition 4.7.** Let $(X, T)$ be a t.d.s. We say that $(X, T)$ has almost dense periodic sets if for each non-empty open subset $U \subset X$ and $\varepsilon > 0$, there are $k \in \mathbb{N}$ and $\mu \in M(X)$ with $T_M^k \mu = \mu$ such that $\mu(U^c) < \varepsilon$, where $U^c = \{x \in X : x \notin U\}$. We say that $(X, T)$ is an almost HY-system if it is totally transitive and has almost dense periodic sets.

We note that $(X, T)$ has almost dense periodic sets if and only if for each non-empty open subset $U \subset X$, there are periodic points $\mu_k \in M(X)$ such that $\mu_k(U) \rightarrow 1$. It is easy to see that if $(X, T)$ has dense small periodic sets, then it has almost dense periodic sets, and hence an HY-system is a weakly mixing almost HY-system. If $(X, T)$ has almost dense periodic sets, then it has an invariant measure with full support. Moreover, we also have the following result.

**Proposition 4.8.** Let $(X, T)$ and $(Y, S)$ be two t.d.s.

1. If $(X, T)$ has almost dense periodic sets and $\pi : (X, T) \rightarrow (Y, S)$ is a factor map, then $(Y, S)$ has almost dense periodic sets.
2. If $(X, T)$ and $(Y, S)$ have almost dense periodic sets, then $(X \times Y, T \times S)$ has almost dense periodic sets.

**Proof.** (1) Let $V \subset Y$ be a non-empty open subset and $\varepsilon > 0$. Then $U = \pi^{-1}(V) \subset X$ is open and non-empty, and there is $T^k$-invariant measure $\mu$ with $\mu(U^c) < \varepsilon$. This implies that $\pi \mu(V^c) < \varepsilon$. It is clear that $\nu = \pi \mu$ is $S^k$-invariant, and hence $(Y, S)$ has almost dense periodic sets.

(2) Let $U$ be a non-empty open subset of $X \times Y$ and $\varepsilon > 0$. Then there are non-empty open subsets $U_1 \subset X$ and $U_2 \subset Y$ such that $U_1 \times U_2 \subset U$. Since $(X, T)$ and $(Y, S)$ have almost dense periodic sets, there are $T^{k_1}$-invariant measure $\mu \in M(X)$ and $S^{k_2}$-invariant measure $\nu \in M(Y)$ with $\mu(U_1^c) < \frac{\varepsilon}{4}$ and $\nu(U_2^c) < \frac{\varepsilon}{4}$. Set $k = k_1 \times k_2$, then $\mu \times \nu \in M(X \times Y)$ is $T^k \times S^k$-invariant and

$$
(\mu \times \nu)(U^c) \leq (\mu \times \nu)((U_1 \times U_2)^c) \leq \mu(U_1^c) + \nu(U_2^c) < \varepsilon.
$$

That is, $(X \times Y, T \times S)$ has almost dense periodic sets. \hfill \Box

It is well known that a totally transitive $P$-system is weakly mixing [1]. In [12], Huang and Ye showed that a totally transitive system with dense small periodic sets is weakly mixing. Now we improve these results by showing that each almost HY-system is weakly mixing. That is,

**Proposition 4.9.** If $(X, T)$ is a totally transitive system with almost dense periodic sets, then $(X, T)$ is weakly mixing.
Proof. Let $U, V$ be two non-empty open subsets of $X$. Since $(X, T)$ has almost dense periodic sets, there exist $k \in \mathbb{N}$ and a $T^k$-invariant measure $\mu \in M(X)$ such that $\mu(U) > \frac{4}{5}$. Hence, $\{ki : i \in \mathbb{Z}_+\} \subset N(U, U)$. Note that $(X, T^k)$ is topological transitive, and thus $N(U, U) \cap N(U, V) \neq \emptyset$. By Lemma 4.2, $(X, T)$ is weakly mixing. 

Theorem 4.10. The following statements are equivalent:

1. $(X, T)$ is an almost HY-system.
2. $(X, T)$ is weakly mixing, and for each open non-empty $U \subset X$ and $\varepsilon > 0$
   there are $k \in \mathbb{N}$ and $\mu \in \mathcal{E}(X, T^k)$ such that $\mu(U^c) < \varepsilon$.
3. $(M(X), T_M)$ is a $P$-system.
4. $(M(X), T_M)$ is an almost HY-system.

Proof. (1) $\Rightarrow$ (2) Let $(X, T)$ be an almost HY-system. By Proposition 4.9, $(X, T)$ is weakly mixing. For each non-empty open subset $U \subset X$ and $\varepsilon > 0$, there are $k \in \mathbb{N}$ and $\nu \in \mathcal{E}(X, T^k)$ such that $\nu(U^c) < \varepsilon$. Using the ergodic decomposition we have $\nu = \int_{\mathcal{E}(X, T^k)} m \, d\tau(m)$, where $\tau \in M(M(X))$ with $\tau(\mathcal{E}(X, T^k)) = 1$. It follows that

$$\int_{\mathcal{E}(X, T^k)} m(U^c) \, d\tau(m) = \nu(U^c) < \varepsilon.$$ 

Thus there is $\mu \in \mathcal{E}(X, T^k)$ such that $\mu(U^c) < \varepsilon$.

(2) $\Rightarrow$ (3) Let $x \in X$. For any $f_1, \ldots, f_s \in C(X, \mathbb{R})$ and $\varepsilon > 0$, there is $r > 0$ such that if $d(y, x) < r < \varepsilon/2$ then $|f_j(y) - f_j(x)| < \varepsilon/2$ for all $j = 1, \ldots, s$. Let $U = B(x, r)$. Then there is $k \in \mathbb{N}$ and $\mu \in \mathcal{E}(X, T^k)$ such that $\mu(U^c) < (\max\{||f_j|| : 1 \leq j \leq s\})^{-1} \varepsilon/4$. Thus

$$\int_X |f_j(y) - f_j(x)| \, d\mu(y) \leq \int_{U^c} |f_j(y) - f_j(x)| \, d\mu(y) + \int_U |f_j(y) - f_j(x)| \, d\mu(y) < \varepsilon,$$

for each $1 \leq j \leq s$. That is, $\mu \in V_{\delta_x}(f_1, \ldots, f_s; \varepsilon)$. So $\delta_x$ is a limit point of $P(T_M)$.

For $x_1, \ldots, x_n \in X$, let $\nu = \frac{1}{n} \sum_{j=1}^n \delta_{x_j}$. For give $f_1, \ldots, f_s \in C(X, \mathbb{R})$ and $\varepsilon > 0$, let

$$\mu_j \in P(T_M) \cap V_{\delta_{x_j}}(f_1, \ldots, f_s; \varepsilon)$$

and $\mu = \nu \delta_x$. It is clear that $\mu$ is a periodic point of $T_M$ and $\mu \in V_\nu(f_1, \ldots, f_s; \varepsilon)$. This implies that $M_n(X) \subset P(T_M)$. Therefore, the set of periodic points of $(M(X), T_M)$ is dense in $M(X)$.

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1) Let $U$ be a non-empty open subset of $X$ and $\varepsilon > 0$. Let $V = \{m \in M(X) : m(U) \geq 1 - \frac{\varepsilon}{2}\}$. Then there are $k \in \mathbb{N}$ and $T^k_M$-invariant measure $\nu$ on $M(X)$ with $\nu(V^c) < \frac{\varepsilon}{2}$. Let $\mu = \int_M(\nu) \, d\theta(\nu)$. It is clear that $T^k_M \mu = \mu$ and

$$\mu(U^c) = \int_V \theta(U^c) \, d\nu(\theta) + \int_{V^c} \theta(U^c) \, d\nu(\theta) \leq \frac{\varepsilon}{2} \nu(V) + \nu(V^c) < \varepsilon.$$

That is, $(X, T)$ is an almost HY-system. 

$\square$
Let $\Sigma_2 = \prod_{n=1}^{\infty} \{0,1\}$, where $\{0,1\}$ and $\Sigma_2$ are equipped with the discrete and the product topology respectively. For $n \in \mathbb{N}$ and $(x_1, \cdots, x_n) \in \{0,1\}^n$, define

$$[x_1, \cdots, x_n] := \{y \in \Sigma_2 : y_i = x_i, i = 1, \ldots, n\},$$

which is called an $n$-cylinder. It is known that the set of all cylinders form a semi-algebra which generates the Borel $\sigma$-algebra of $\Sigma_2$. If $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$ are two elements of $\Sigma_2$, then their sum $x \oplus y = (z_1, z_2, \cdots)$ is defined as follows. If $x_1 + y_1 < 2$, then $z_1 = x_1 + y_1$; if $x_1 + y_1 \geq 2$, then $z_1 = x_1 + y_1 - 2$ and we carry 1 to the next position. The other terms $z_2, \cdots$ are successively determined in the same fashion. Let $T : \Sigma_2 \to \Sigma_2$ be defined by $T(z) = z \oplus 1$ for each $z \in \Sigma_2$, where $1 = (1, 0, 0, \cdots)$. It is known that $T$ is minimal and unique ergodic, which is called a dyadic adding machine. Now we have

Theorem 4.11. There is a minimal weakly mixing almost-HY-system.

Proof. Let $(\Sigma_2, T)$ be the dyadic adding machine with a unique ergodic measure $\mu$. A remarkable result due to Lehrer [16] which generalized the famous Jewett-Krieger Theorem says that $(\Sigma_2, T, \mu)$ has a minimal weakly mixing uniquely ergodic model, i.e., there exists a system $(Y, S, \nu)$ isomorphic to $(\Sigma_2, T, \mu)$, where $(Y, S)$ is a minimal, unique ergodic and weak mixing t.d.s. We shall show that $(Y, S)$ is an almost-HY-system. By Theorem 4.10 it remains to show that the set of periodic points of $(M(Y), S_M)$ is dense.

Let $\pi : (\Sigma_2, T, \mu) \to (Y, S, \nu)$ be an isomorphism, that is, there are invariant Borel subsets $X_1 \subset \Sigma_2$ and $X_2 \subset Y$ with $\nu(X_1) = \nu(X_2) = 1$ and an invertible measure-preserving transformation $\pi : X_1 \to X_2$ such that $\pi(T_x) = S \pi(x)$ for all $x \in X_1$.

Let $\epsilon > 0$ and let $U$ be a non-empty open subset of $Y$. Since $(Y, S)$ is minimal and unique ergodic, we have $\nu(U) > 0$. Thus, there are finitely many pairwise disjoint cylinders $A_1, \ldots, A_k$ of $\Sigma_2$ such that $\mu(\pi^{-1} U \triangle A) < \nu(U) \cdot \epsilon$ with $A = \bigcup_{i=1}^{k} A_i$, which implies $\nu(U \triangle \pi(A \cap X_1)) < \nu(U) \cdot \epsilon$. Using Lemma 2.2 (2), $d(\nu_U, \nu_{\pi(A \cap X_1)}) \leq 2 \epsilon$. Since $T^{2^k} \pi C = C$ for each cylinder $C$ of $X$, where $|C|$ stands for the length of $C$, we conclude that $\mu_C$ is periodic. In particular, each $\mu_{A_i}$ is periodic. By Lemma 2.2 (3), each $\nu_{\pi(A \cap X_1)}$ is also periodic. By Lemma 2.2 (1), $\nu_{\pi(A \cap X_1)} = \sum_{i=1}^{k} p_i \nu_{\pi(A_i \cap X_1)}$, where $p_i = \mu(A_i) / \mu(A)$. Thus, $\nu_{\pi(A \cap X_1)}$ is periodic. It follows that $\nu_U$ is approached by periodic points of $(M(Y), S_M)$.

Now take $y \in Y$ and let $\{U_n\}_{n=1}^{\infty}$ be a sequence of open neighborhoods of $y$ such that $\text{diam}(U_n) \to 0$. For any $f \in C(Y, \mathbb{R})$, we have

$$\left| \int_Y f(z) \, d\nu_{U_n} - f(y) \right| \leq \int_{U_n} \left| f(z) - f(y) \right| \, d\nu_{U_n} \to 0.$$ 

A simple calculation shows $\nu_{U_n} \to \delta_y$, and hence $\delta_y$ is a limit point of $P(S_M)$. This implies that each element of $M_n(Y)$ is approached by elements of $P(S_M)$. Since $\bigcup_{n=1}^{\infty} M_n(Y)$ is dense in $M(Y)$, it follows that $(M(Y), S_M)$ is a P-system. □

Remark 4.12. (1) For the dyadic adding machine $(\Sigma_2, T)$, we remark that the convex combinations of conditional measures $\mu_B$ ($B$ is a cylinder of $\Sigma_2$) is dense in $M(\Sigma_2)$. More precisely, for each $n \in \mathbb{N}$, define $C_n(\Sigma_2) = \{\mu = \frac{1}{n} \sum_{i=1}^{n} \mu_{A_i} :
$A_i$ is a cylinder of $\Sigma_2\}$. Then each $\mu \in C_n(\Sigma_2)$ is periodic and $\bigcup_{n=1}^{\infty} C_n(\Sigma_2)$ is dense in $M(\Sigma_2)$. This shows that the set of periodic points of $(M(\Sigma_2), T_M)$ is dense.

(2) In [10], Theorem 5.10, Huang, Li and Ye showed that a minimal t.d.s. has dense small periodic sets if and only if it is an almost one-to-one extension of some adding machine. We remark that a nontrivial adding machine is never totally transitive, and hence the systems described in Theorem 4.11 do not have dense small periodic sets. This shows that any minimal almost HY-system is not an HY-system.

The following question is natural.

**Question 4.13.** Is there a weakly mixing, proximal $E$-system $(X, T)$ such that $(M(X), T_M)$ is a $P$-system?

This question will be answered affirmatively in a forthcoming paper of Lian, Shao and Ye by showing that each ergodic system has a weakly mixing and proximal topological model [19]. And it follows that $(X, T)$ needs not to be a weakly mixing $M$-system whenever $(M(X), T_M)$ is an $M$-system.

## 5. Disjointness

In this section, we discuss disjointness. The notion of disjointness of two t.d.s. was introduced by Furstenberg in his seminar paper [6]. Let $(X, T)$ and $(Y, S)$ be two t.d.s. We say $J \subset X \times Y$ is a *joining* of $X$ and $Y$ if $J$ is a non-empty closed invariant set, and is projected onto $X$ and $Y$, respectively. If each joining is equal to $X \times Y$, then we say that $(X, T)$ and $(Y, S)$ are *disjoint*, denoted by $(X, T) \perp (Y, S)$ or $X \perp Y$. Note that if $(X, T) \perp (Y, S)$, then one of them is minimal [6], and if $(Y, S)$ is minimal, then $(X, T)$ has dense minimal points and it is weakly mixing if it is transitive [12].

**Remark 5.1.** For any nontrivial t.d.s. $(X, T)$ we have at least the two distinct self-joinings $X \times X$ and $\triangle = \{(x, x) : x \in X\}$, and thus the only system which is disjoint from itself is the trivial t.d.s. Using Theorem 4.11 we obtain that there is a minimal almost HY-system is not in $\mathcal{M}^\perp$, where $\mathcal{M}^\perp$ denote the collection of all t.d.s. disjoint from all minimal systems.

We say that $(X, T)$ is *strongly disjoint from all minimal systems* if $(X^n, T^{(n)})$ is disjoint from all minimal systems for any $n \in \mathbb{N}$. Then we have

**Theorem 5.2.** Let $(X, T)$ be a t.d.s. Then

1. If $(K(X), T_K)$ is weakly mixing and is disjoint from all minimal systems, then $(X, T)$ is weakly mixing and is disjoint from all minimal systems.
2. If $(X, T)$ is strongly disjoint from all minimal systems, then both $(K(X), T_K)$ and $(M(X), T_M)$ are disjoint from all minimal systems.

**Proof.** (1) Let $(Y, S)$ be a minimal system. Since $(K(X), T_K) \perp (Y, S)$, we have $\text{orb}((E, y), T_K \times S) = K(X) \times Y$ for any transitive points $E$ of $(K(X), T_K)$ and any $y \in Y$. Choose $x \in E$ and let $J \subset X \times Y$ be a joining. Then there is $y \in Y$ with $(x, y) \in J$. We will show that $\text{orb}((x, y), T \times S) = X \times Y$, which implies $J = X \times Y$, and hence $(X, T) \perp (Y, S)$.
In fact, for any pair \( u \in X \) and \( v \in Y \), there is a positive integers sequence \( \{n_i\}_{i=1}^{\infty} \) such that \( (T_K \times S)^{n_i}(E, y) \rightarrow (\{u\}, v) \). By the definition of the Hausdorff metric on \( K(X) \) we know that \( d(T^{n_i}x, u) \leq d_H(T^{n_i}_K E, \{u\}) \rightarrow 0 \).

This implies \( T^{n_i}x \rightarrow u \), and hence \( (T \times S)^{n_i}(x, y) \rightarrow (u, v) \).

(2) Assume that \((X, T)\) is strongly disjoint from all minimal systems. Then the restriction of \( T_K \) to \( K_0(X) \), as a factor of \( T^{(n)} \), is disjoint from all minimal systems for each \( n \in \mathbb{N} \). Since \( \bigcup_{n=1}^{\infty} K_0(X) \) is dense in \( K(X) \), \((K(X), T_K)\) is disjoint from all minimal systems.

The same argument shows that \((M(X), T_M)\) is disjoint from all minimal systems. \( \square \)

**Remark 5.3.** (1) Furstenberg [6] showed that each weakly mixing system with dense periodic points is disjoint from any minimal systems. It directly follows from Theorem 5.2 and Theorem 3.2 that each HY-system (i.e., weakly mixing system with dense small periodic sets) is disjoint from any minimal systems (first proved by Huang and Ye in [12, Theorem 3.4]).

(2) It follows from Theorem 4.11 that \((X, T)\) needs not to be disjoint from all minimal systems whenever \((M(X), T_M)\) is disjoint from all minimal systems.

We say that a t.d.s. \((X, T)\) has **dense distal sets** if for each non-empty open subset \( U \) of \( X \), there is a distal point \( C \) of \((K(X), T_K)\) such that \( C \subset U \).

**Proposition 5.4.** The following statements are equivalent:

1. \((X, T)\) is a weakly mixing system with dense distal sets;
2. \((K(X), T_K)\) is a weakly mixing system with dense distal points;
3. \((K(X), T_K)\) is a weakly mixing system with dense distal sets.

**Proof.** (1) \( \Rightarrow \) (2) By Lemma 5.1 \((K(X), T_K)\) is weakly mixing. Let \( n \in \mathbb{N} \) and \( U_1, \ldots, U_n \) be non-empty open subsets of \( X \). Since \((X, T)\) has dense distal sets, there exist distal points \( C_1, \ldots, C_n \) of \((K(X), T_K)\) such that \( C_i \subset U_i \). Let \( C = \bigcup_{i=1}^{n} C_i \). Clearly, \( C \subset \bigcup_{i=1}^{n} U_i \). We will show that \( C \) is a distal point of \((K(X), T_K)\), which implies \((K(X), T_K)\) has dense distal points.

Let \( V_1, \ldots, V_m \) be non-empty open sets of \( X \) with \( C \in \langle V_1, \ldots, V_m \rangle \). Then for each \( C_i \) there are \( s_1, \ldots, s_m \in \{1, \ldots, m\} \) such that \( C_i \in \langle V_{s_1}, \ldots, V_{s_m} \rangle \). Clearly, we have \( \bigcup_{i=1}^{m} \{s_1, \ldots, s_m\} = \{1, \ldots, m\} \). Notice that each \( C_i \) is distal and \( \mathcal{F}_{ip}^{+} \) is a filter, it is not hard to verify that \( N(C, \langle V_1, \ldots, V_m \rangle) \) contains an IP*-set \( \bigcap_{i=1}^{m} N(C_i, \langle V_{s_1}, \ldots, V_{s_m} \rangle) \), which implies that \( C \) is a distal point of \((K(X), T_K)\).

(2) \( \Rightarrow \) (3) is obvious.

(3) \( \Rightarrow \) (1) By Lemma 5.1 \((X, T)\) is weakly mixing. Now we show that \((X, T)\) has dense distal sets. Let \( U \) be a non-empty open subset of \( X \). Since \((K(X), T_K)\) has dense distal sets, there exists a distal point \( \mathcal{A} \) of \((K(K(X)), T_K)\) such that \( \mathcal{A} \subset \langle U \rangle \).

Let \( C = \bigcup \{A : \mathcal{A} \subset \langle U \rangle \} \). Clearly, \( C \subset U \). Next, we will show that \( C \in K(X) \). In fact, let \( \{x_n\}_{n=1}^{\infty} \) be a sequence of \( C \) and \( x_n \rightarrow x \). Then for each \( n \in \mathbb{N} \) there is \( A_n \in \mathcal{A} \)
such that $x_n \in A_n$. Since $\mathcal{A}$ is a non-empty closed subset of $K(X)$ and $K(X)$ is a compact metric space, without loss of generality, we may assume $A_n$ convergence to $A \in \mathcal{A}$. By the definition of Hausdorff metric, we have $x \in A \subset C$, which implies $C \in K(X)$. To complete the proof, it remains to show that $C$ is a distal point of $(K(X), T_K)$.

Let $V_1, \ldots, V_m$ be non-empty open sets of $X$ with $C \in \langle V_1, \ldots, V_m \rangle$. Then there exist non-empty open subsets $\mathcal{V}_1, \ldots, \mathcal{V}_s$ of $K(X)$ satisfies that

- $\mathcal{V}_i = \langle V_{k_1}, \ldots, V_{k_s(i)} \rangle$ for each $i = 1, \ldots, s$;
- $\mathcal{A} \in \langle \mathcal{V}_1, \ldots, \mathcal{V}_s \rangle$;
- $\bigcup_{i=1}^s \{V_{k_1}, \ldots, V_{k_s(i)}\} = \{V_1, \ldots, V_m\}$.

It follows that $N(\mathcal{A}, \langle \mathcal{V}_1, \ldots, \mathcal{V}_s \rangle) \subset N(C, \langle V_1, \ldots, V_m \rangle)$, and hence $C$ is IP-recurrent (see Section 2). That is, $C$ is a distal point of $(K(X), T_K)$. $\square$

In [12] it was showed that each weakly mixing system with dense regular minimal points is disjoint from any minimal systems; and in [5, 21], the authors showed that each weakly mixing system with dense distal points is disjoint from any minimal systems. Now we improve these results by showing that each weakly mixing system with a dense set of distal sets is disjoint from all minimal systems. That is, we have

**Theorem 5.5.** If $(X,T)$ is a weakly mixing t.d.s. and has dense distal sets, then $(X,T)$ is disjoint from all minimal systems.

**Proof.** It directly follows from Theorem 5.2, Proposition 5.4 and [5, Theorem 7.14]. $\square$

We strongly believe that there is a t.d.s. which has dense distal sets and does not have dense distal points.

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