Almost Sure Invariance Principle for Nonuniformly Hyperbolic Systems

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Abstract: We prove an almost sure invariance principle that is valid for general classes of nonuniformly expanding and nonuniformly hyperbolic dynamical systems. Discrete time systems and flows are covered by this result. In particular, the result applies to the planar periodic Lorentz flow with finite horizon.

Statistical limit laws such as the central limit theorem, the law of the iterated logarithm, and their functional versions, are immediate consequences.

1. Introduction

Statistical properties of uniformly expanding maps and uniformly hyperbolic (Axiom A) diffeomorphisms are by now classical. Hölder observations satisfy exponential decay of correlations and the central limit theorem (CLT), see for example Bowen [8], Ratner [34], Ruelle [35], Parry and Pollicott [31]. Furthermore, Denker and Philipp [17] proved an almost sure invariance principle (ASIP) for Hölder observations. Immediate consequences of the ASIP are the CLT, the law of the iterated logarithm (LIL), and their functional versions, see [32].

Many proofs of the CLT for dynamical systems use directly the martingale approximation method of Gordin [20], see [25, 26, 29]. The ASIP can often also be obtained in this way see [16, 19, 29, 39] and indeed this method yields a better error estimate in the ASIP than the usual one, see Field et al. [19]. However, it should be emphasised that the martingale approximation of Gordin [20] leads directly only to a reverse martingale increment sequence and so the ASIP is obtained in backwards time in the first instance. This is not an issue for distributional results such as the CLT, but the ASIP in [16, 19, 29] uses explicitly the fact that the class of systems being studied is closed under time-reversal. To obtain forward martingale approximations, it is necessary to use more sophisticated versions of Gordin’s approach [32].

Recently, there has been an explosion of interest in nonuniformly expanding maps and nonuniformly hyperbolic diffeomorphisms (possibly with singularities). We refer
to the articles of Young [40, 41] as well as Aaronson [1], Baladi [4, 5], Gouëzel [22], Viana [38] and references therein. In particular, decay of correlations and the CLT are studied extensively in these references. However, such classes of dynamical systems are intrinsically time-orientation specific, and largely for this reason the ASIP has not previously been proved. Similarly, the LIL was previously unproved for such systems.

In this paper, we establish the ASIP, and hence the (functional) LIL, for nonuniformly expanding/hyperbolic systems. Both discrete time systems and flows are covered by our results.

Remark 1.1. We note that [33] attempted to apply the approach in [19] to nonuniformly expanding systems. However, it appears that the time-orientation issue discussed above was overlooked in [33], and that this is a gap. Hence it seems necessary to find an alternative approach to the one in [19], and that is what is done in the current paper.

Precise formulations are given in the body of the paper, but here is an outline of our main result, and the strategy behind its proof, for a nonuniformly expanding map $T : M \to M$ where $(M, d)$ is a metric space. By standard methods, $T$ can be modelled by a discrete-time suspension over a Gibbs-Markov map $f : Y \to Y$ with return time function $R : Y \to \mathbb{Z}^+$. (Roughly speaking, a Gibbs-Markov map is like a uniformly expanding map with possibly countably many inverse branches.) There exists a unique ergodic $T$-invariant probability measure equivalent to Lebesgue, and the following result is formulated with this measure in mind.

Theorem 1.2. Let $T : M \to M$ be a nonuniformly expanding map. Assume moreover that $R \in L^{2+\delta}(Y)$. Let $\phi : M \to \mathbb{R}$ be a mean zero Hölder observation. Then $\phi$ satisfies the ASIP. That is, there exists $\epsilon > 0$, a sequence of random variables $\{S_N\}$ and a Brownian motion $W$ with variance $\sigma^2 \geq 0$ such that $\left( \sum_{j=0}^{N-1} \phi \circ T^j \right) \Rightarrow_d \{S_N\}$, and

$$S_N = W(N) + O(N^{1/2-\epsilon}) \quad \text{as } N \to \infty,$$

almost everywhere.

Using a method due to Hofbauer and Keller [24] which exploits a result of Philipp and Stout [32, Theorem 7.1], we obtain the ASIP (in the correct time direction but without the improved error term) for $f : Y \to Y$ and a class of “weighted Lipschitz” observations. Theorem 1.2 then follows directly by Melbourne and Torok [30]. (We note that the method in [30] has independently been used by Gouëzel [21] to obtain a simplified derivation of the CLT and stable laws.)

A precise version of Theorem 1.2 is stated and proved in Sect. 2(e). The ASIP for nonuniformly hyperbolic maps extends easily to a class of nonuniformly expanding semiflows, see Sect. 2(e).

Our results for nonuniformly hyperbolic diffeomorphisms and nonuniformly hyperbolic flows are completely analogous, but the set-up is more technical and we postpone further details until Sect. 3.

Planar periodic Lorentz gas. The planar periodic Lorentz gas is a class of examples introduced by Sinai [36]. See [15] for a survey of results about Lorentz gases. The Lorentz flow is a billiard flow on $\mathbb{T}^2 - \Omega$, where $\Omega$ is a disjoint union of convex regions with $C^3$ boundaries. (The phase-space of the flow is three-dimensional, planar position and direction.) The flow has a natural global cross-section $M = \partial \Omega \times [-\pi/2, \pi/2]$ corresponding to collisions and the Poincaré map $T : M \to M$ is called the billiard