INTERACTION AND MODULAR INVARIANCE OF STRINGS ON CURVED MANIFOLDS

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Abstract

We review and present new results for a string moving on an $SU(1,1)$ group manifold. We discuss two classes of theories which use discrete representations. For these theories the representations forbidden by unitarity decouple and, in addition, one can construct modular invariant partition functions. The partition functions do, however, contain divergencies due to the time-like direction of the $SU(1,1)$ manifold. The two classes of theories have the corresponding central charges $c = 9, 6, 5, 9/2, \ldots$ and $c = 9, 15, 21, 27, \ldots$. Subtracting two from the latter series of central charges we get the Gervais-Neveu series $c - 2 = 7, 13, 19, 25$. This suggests a relationship between the $SU(1,1)$ string and the Liouville theory, similar to the one found in the $c = 1$ string. Modular invariance is also demonstrated for the principal continuous representations. Furthermore, we present new results for the Euclidean coset $SU(1,1)/U(1)$. The same two classes of theories will be possible here and will have central charges $c = 8, 5, 4, \ldots$ and $c = 8, 14, 20, 26, \ldots$, where the latter class includes the critical 2d black hole. The partition functions for the coset theory are convergent.

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1 Introduction

In string theory one usually considers string backgrounds of the following type $M^{(d)} \otimes I^{d'}$, where $M^{(d)}$ is a flat $d$-dimensional Minkowski space and $I^{d'}$ an internal $d'$-dimensional compact space. This latter space is represented by some unitary conformal field theory. Much work has been devoted to a classification of these and although an enormous class of possibilities have been found, a complete classification is still lacking. The type of spaces described above, however, is not the most general background one may think of. A more general class of theories is obtained if we also allow ourselves to replace the flat Minkowski space by a more general non-compact space. We consider, therefore, a class of backgrounds of the following general form: $\mathcal{M}^{(d)} \otimes I^{d'}$. Here $\mathcal{M}^{(d)}$ is a non-compact $d$ dimensional space. A natural restriction on $\mathcal{M}^{(d)}$ is that it should have at most one "time" direction. Such a restriction may not be neccessary, but it is clearly the most obvious generalization of the flat Minkowski case. A simple counting argument indicates that for theories with only conformal symmetry, unitarity will allow at most one time direction.

The motivations for studying these more general backgrounds are several. I will only mention a few. Firstly, we do not really know if flat space is a stable background. It may be that dynamically it will flow to a fixed point, which is some non-trivial curved space. Such a possibility can not be ruled out, especially since we aim to describe $d$-dimensional quantum gravity. A further motivation is that these new theories will, as it seems, be truly low-dimensional strings, i.e. for which the total dimension of space-time is less than 26 or 10. This also means that we would be able to construct new unitary conformal field theories if we could factor out all non-unitary states in a consistent fashion. Such theories may be "non-rational" and contain an infinite number of primary fields with respect to some extended symmetry algebra. Finally, on quite general grounds, it will lead to a deeper understanding of string theory. Since our past experience of string theory is to a large extent based on having a flat Minkowski space, going beyond this case will sharpen our intuition as to why string theory seems to be consistent. Let me give one example. In formulating the string perturbation theory, we are used to thinking of this in terms of Euclidean world-sheet surfaces, where the order of perturbation is given by the genus of the surface. This was most elegantly exploited by Polyakov in his path-integral approach [1], but was already known in the first era of string theory. We know, however, that the string, as analyzed in the first quantized form, is really defined on a Minkowski world-
sheet. The transition from the Minkowskian to the Euclidean signature is seldomly, if ever, commented upon and the validity of the transition seems never to be questioned. An issue which is, at first sight, not related to this transition, is the corresponding rotation in space-time. In computing loops, we need to regularize the amplitudes due to the Minkowski time. The standard procedure is to make a Wick rotation. The transitions from Minkowskian to Euclidian signature for the world-sheet and space-time geometries are usually treated independently, but in fact, they are related. The direction of the Wick rotation in space-time is connected, by the requirement of convergent loop amplitudes, with the direction of the corresponding rotation on the world-sheet. The question is now, for more general space-time geometries including cases with compact time, if we can still do Wick rotations on the world-sheet as well as in the space-time and if so, what is the connection between the two. This question is of importance in dealing with the divergencies encountered in the work I will present here.

In constructing conformally invariant string theories corresponding to propagation on curved backgrounds $\mathcal{M}^{(d)}$, a large class of them may be constructed using Wess-Zumino-Novikov-Witten (WZNW) models. These should then be based on non-compact Lie-groups. The simplest of all these is the group $SU(1, 1)$, which has a Minkowski signature. This model was first discussed in ref.\[2\]. Generalizations to higher dimensions using WZNW theories have been given in \[3\]-\[5\]. $SU(1, 1)$ is a three dimensional group and, therefore, not a completely realistic example. It may instead be regarded as a toy model in which important properties of string theories with "curved time" could be studied. I am strongly convinced that in solving this simple model we will solve most of the problems encountered for more realistic theories. The versatility of this theory as a toy model is further emphasized by the interpretation due to Witten \[6\] as a two-dimensional black hole. The connection with the Liouville theory discussed in this work and further analyzed by many others e.g. in refs.\[7\]-\[9\], is a further motivation to study the $SU(1, 1)$ string. In fact, this was my original incentive to study this particular model in \[10\]. The connection with the Liouville theory was suggested by the free field formulation of $SU(1, 1)$, since it used a " Liouville like" free field \[11\]-\[18\], in which the background charge corresponded to the "strong-coupling" regime of $2d$-gravity.

Generally, the construction of consistent string theories requires at least the following properties:

i. Unitarity of physical states
ii. Modular invariance
iii. Decoupling of non-unitary states in string amplitudes
iv. Renormalizability and anomaly freedom
I will discuss the first three of these properties in this talk. The fourth issue is yet unsolved, although we have studied it to some extent. I will only give some brief comments later in this connection. Let me start by summarizing the results below for the case of the SU(1,1) string.

i. Unitarity

The unitarity of the SU(1,1) string was studied in [10] for the bosonic case and in [19] for the N = 1 fermionic case. By unitarity we here mean the question of whether the physical states, i.e. the states which satisfy the usual Virasoro conditions have non-negative norm. The results of our investigations were that for conformal anomalies \(c > 3\), the theory was indeed unitary for the principal continuous unitary representations of SU(1,1) and for the discrete unitary representations under the restriction on the spin, \(j > k/2\) (in our conventions both \(j\) and \(k\) are negative). An analogous result holds for the fermionic case.

ii. Modular invariance

In an earlier work [20], we proposed a partition function, including only the allowed discrete representations, which was modular invariant for integer values of \(k\), the Kac-Moody anomaly. This partition function included an infinite number of new sectors of states realizing momentum and winding states on a non-Abelian group manifold which were necessary due to the special topology of SU(1,1). The sectors of states were named "non-Abelian" winding sectors, although strictly speaking, a better terminology is Weyl translation sectors. We have extended these results and we now have modular invariance in the following cases:

1. \(k + 2\) negative integer, discrete representations
2. \(1/(k + 2)\) negative integer, discrete representations
3. Principal continuous representations
4. SU(1,1)/U(1) for the cases 1, 2 and 3 above.

It should be remarked here that the first three cases all involve partition functions that contain divergencies. Consequently, the statement of modular invariance is formal and may be invalidated by a proper regularization. In the last case of the coset SU(1,1)/U(1) the winding sectors are absent and these divergencies are, consequently, removed. It is modular invariant in the same way as any unitary CFT. The conformal anomaly for the case \(k\) integer is \(c = 9, 6, 5, 9/2, \ldots\) and for \(1/(k + 2)\) integer, \(c = 9, 15, 21, 27, \ldots\). The corresponding numbers for the coset theory are given by subtracting one from above. In particular we have for \(1/(k + 2) = -4\) that \(c = 26\) for the coset. This case is of particular interest, since it corresponds to the critical Euclidean black hole. By subtracting two from above we get another interesting series of central charges; \(c = 7, 13, 19, 25, \ldots\). This series
has been suggested \[21\], as the possible central charges for a consistent quantum Liouville theory. This connection is probably not a coincidence, since the coset theory may be described by a Liouville-like field coupled to a \(c=1\) matter field. Hence, the subtraction of one from the coset central charge is due to the matter field and we arrive at the central charge of the Liouville-like theory.

\[\text{iii. Decoupling in string amplitudes}\]

The question of whether the truncation of the discrete series of representations is consistent in scattering amplitudes, \(i.e.\) that no non-unitary states will propagate in amplitudes, is essential for a consistent interacting string. We have been able to show, by extending an argument due to Gepner and Witten for the compact case \[22\], that for the cases when we have modular invariance \(i.e.\) if \(k\) or \(1/(k + 2)\) are integers, then we either have that the forbidden representations decouple or there is no propagation at all. Thus, assuming propagation among the allowed representations, then we have our decoupling theorem. We have also studied the string amplitudes using a free field representation. In this representation the fields in the different sectors will be able to interact with each other through an "interpolating" field in much the same way as Ramond and Neveu-Schwarz fields can interact in the the ordinary fermionic string. The possible interactions among the different representations is at present under study. It appears that there will be some differences compared to the Clebsch-Gordon couplings of the horizontal \(SU(1,1)\). For instance, we will not be able to couple two discrete primaries to a principal continuous one.

\[\text{iv. Divergencies}\]

In computing the partition functions we will encounter two types of divergencies. The first one is due to the infinite dimensional representations of \(SU(1,1)\). It is always possible to introduce a regulator to make the infinite sums well-defined. For the discrete representations, the divergencies will cancel in our modular invariant partition functions and we can remove the regulator. The finite expression found in this way is the same as one would find by a \(\zeta\)-function regularization. For the continuous representations, the removal of the regulator is a more delicate problem than for the discrete case. The divergencies of this first type exist for the full \(SU(1,1)\) theory, but are absent in the Euclidean coset.

The second type of divergence is due to the time-direction of the \(SU(1,1)\) theory and hence, is also not present for the Euclidean coset. It is analogous to the divergence found for the flat case prior to Wick rotation. This type of divergence has proven to be very difficult to deal with. It appears not to be a problem of a curved time, but rather of the \textit{compactness} of the time direction which leads to divergent sums instead of integrals. Although we have studied the problem to some depth, we have not found any
satisfactory resolution. Our approach relies on starting on a Minkowskian world-sheet and then deriving a generalized Wick rotation. In doing this, one encounters a modified modular transformation on a Minkowski world-sheet. The Wick rotation derived by such an approach does not seem to give a Euclidean modular invariance for our theory. This is, however, a preliminary result and we will not discuss it further in this talk. One may hope that these divergencies may be regularized on the infinite covering space of $SU(1,1)$. We have at present no understanding of the string theory on this space. The main unsolved problems are the decoupling of non-unitary representations and construction of modular invariant partition functions.

Let me now present and discuss some of our results a little more extensively.

2 The $SU(1,1)$ theory

The $\hat{su}(1,1)$ currents $J^\pm(z)$ and $J^3(z)$ and their counterparts in the other chiral sector satisfy the $\hat{su}(1,1)$ current algebra

\[
    J^+(z)J^-(w) = \frac{k}{(z-w)^2} + \frac{2}{z-w}J^3(w)
\]

\[
    J^3(z)J^3(w) = \frac{k/2}{(z-w)^2}
\]

\[
    J^3(z)J^\pm(w) = \pm \frac{1}{z-w}J^\pm(w)
\]

In addition to these currents we have the primary fields $V_{j,m}(z)$, where $j$ is the spin corresponding to a Casimir $j(j+1)$ and $m$ is the eigenvalue of $J^3_0$. For the discrete unitary representations of $SU(1,1)$, $j$ belongs to the set of negative integers or half-integers. Other values are possible for multi-valued representations. For the principal continuous representations $j = -1/2 + i\rho$, with $\rho$ being real. When acting on the $SL_2$ invariant vacuum these primary fields will define primary states, $| 0; j, m > = V_{j,m}(0) | 0 >$. The energy-momentum tensor is constructed by the standard procedure

\[
    T(z) = \frac{1}{k+2} \left[ (J^3(z))^2 + \frac{1}{2}J^+(z)J^-(z) + \frac{1}{2}J^-(z)J^+(z) \right]
\]

and has a conformal anomaly $c = 3k/(k+2)$. Note that if $k < -2$, which is the case assumed here, then $c$ is greater than three, which is the physical dimension of $SU(1,1)$.

The state space, constructed by acting with the negative modes of the currents on the primary states, will in general contain negative norm states. This is normal for a string

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2This idea is due to Bo Sundborg
theory and to project out a physical and unitary state space, we require states to satisfy
the Virasoro conditions

\[(L_n - \delta_n) \mid \text{phys} \geq 0, \ n \geq 0. \quad (2.3)\]

The unitarity of this physical subspace, for \( k < -2 \), is achieved for the principal continuous
representations and for the discrete representations for \( k/2 < j < 0 \) \(^{[10]}\). An important
property of the state space is that for certain values of \( j \) and \( k \) for the discrete representa-
tions there exists null-states, \( i.e. \) states which are both primary and descendent. Such
states are orthogonal to all other states and are, consequently, formally zero. For integer
values of \( k \) we have a particularly simple class of such null-primaries, \( e.g. \) of highest
weight

\[(J^+(z))^NV_{j,j}(z) = 0. \quad (2.4)\]

Analogous lowest weight fields also exist. Here \( j = k/2 - (N - 1)/2 \) for \( N = 1, 2, \ldots \).
We should here observe that these null-primaries are absent in the range of spins, \( k/2 < j < 0 \),
which is the same range required by unitarity of the physical space. In fact, by studying the Kac-Kazhdan determinant \(^{[23]}\), one can conclude that for the range of spins \( k/2 < j < -k/2 - 1 \), there are no null-primaries at all. This is in contrast to the \( \hat{su}(2) \) case, where the unitary representations occur for spins which have null-states. In
this case we know by the analysis of Gepner and Witten \(^{[22]}\) that these null-primaries
play an essential rôle in consistency of the truncation to the allowed representations. For
\( \hat{su}(1, 1) \) we may modify this argument and prove a similar decoupling theorem for \( k \)
being an integer. We study a three-point function with one primary field having a spin \( j_1 \)
in the forbidden region. This is sufficient for establishing the decoupling. We use eq.(2.4)
inserted into the three-point function

\[0 = \langle (J^+(z_1))^NV_{j_1,j_1}(z_1)V_{j_2,m_2}(z_2)V_{j_3,m_3}(z_3) \rangle \quad (2.5)\]

We proceed by using the OPE of currents with primaries to eliminate \( J^+(z_1) \). Then \( (2.5) \)
will imply

\[\langle V_{j_1,j_1}(z_1)V_{j_2,m_2}(z_2)V_{j_3,m_3}(z_3) \rangle = 0. \quad (2.6)\]

The decoupling theorem is then concluded from this equation by observing that for \( \hat{su}(1, 1) \)
the identity representation contains no null-primaries. Consequently, if there exists any
propagation of states in the unitary sector, then the non-unitary sector must decouple.
For \( \hat{su}(2) \) the situation is in a sense reverse, the identity representation contains null-
primaries, so that only representations which do not contain nulls of the form \( (2.4) \) will
decouple. Another important consequence of this difference between $\hat{su}(1, 1)$ and $\hat{su}(2)$ is that eq. (2.4) will not, in the former case, yield additional selection rules in correlation functions. Our argument this far has been for integer $k$. One may generalize this result for the case of $1/(k+2)$ being an integer. This case is more complicated. For $2j = k, k-1, \ldots$, we can still use the null-states above. These $j$-values are rational, and hence, we must consider representations on covering spaces of $SU(1, 1)$. There are, however, always more $j$-values on a given covering than those above. The trick is to generate the rest of them by Weyl translations (which will be discussed in the next section). One can show that precisely in the cases of $1/(k+2)$ being integers will we generate all spins on the cover of order $-1/(k+2)$. For other values of $k$ we have at present no similar construction and, consequently, we do not know of the consistency of the unitarity truncation.

Let us end this section by presenting the characters of $\hat{su}(1, 1)$. We require the representations to belong to the unitary sector. Since this sector does not contain any null states, it is straightforward to compute the characters. The main difficulty is the fact that we are dealing with infinite dimensional representations and, therefore, we need to regularize the traces. We define the characters by

$$\chi_{j,0}(\tau, \theta) = Tr \{ e^{2\pi i (L_0-c/24)\tau + J_0^2 \theta} \}$$

(2.7)

The trace is here taken over the states in the respective representations. The sum over $J_0^2$ eigenvalues is regularized by letting $\theta$ have an imaginary part of the appropriate sign. The explicit form for the discrete representations is

$$\chi_{j,0}^\pm(\tau, \theta) = e^{2\pi i (2j+1)^2/(k+2)} \pm 2\pi ij \theta} R^{-1}(\tau, \pm \theta)$$

(2.8)

with

$$R(\tau, \theta) = (1 - e^{-2\pi i \theta}) e^{\pi i/4} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) e^{2\pi i n \theta} (1 - e^{-2\pi i n \tau}) e^{-2\pi i n \theta}.$$  

(2.9)

For the principal continuous representations it is

$$\chi_{j,0}^a(\tau, \theta) = \eta(\tau)^{-3} e^{-2\pi i ((a^2)/(k+2)) \tau} \sum_{n=-\infty}^{\infty} e^{2\pi i (n+a) \theta},$$

(2.10)

where $a = 0$ or $1/2$ for single-valued representations. We notice that the characters in eq. (2.8) and (2.10) diverge for $\theta \to 0$, as expected from the infinite dimensionality of the representations. The sum $\chi_{j,0}^- + \chi_{j,0}^+$ is, however, finite in this limit. It corresponds to regularizing $\sum_{-\infty}^j 1$ to the value $2j + 1$. The same finite result is found by using a $\zeta$-function regularization and it equals the sum of the Plancherel measures of the two representations.\footnote{We thank Brian Greene for this remark.} Using the characters above we have not found any way to construct

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$\hat{su}(1, 1)$
modular invariant partition functions. Thus, one may suspect that we lack some essential ingredient for completing the $SU(1,1)$ string theory. This is what we will discuss in the next section.

3 New sectors of states

The topology of the $SU(1,1)$ manifold is equivalent to the space $\mathbb{R}^2 \times S^1$. This means that we have the possibility for the string to wind around the compact direction yielding winding states. This is a well-known situation for a free field compactified on $S^1$. In our case there is a complication, since the circle is embedded in a non-Abelian manifold. In [20] we proposed a construction of the new sectors of states, realizing winding states for our case. Let me here explain this construction by making comparisons with the simple Abelian case. The discussion here will apply to the case of integer $k$.

In the $U(1)$ theory, when we compactify a boson on a circle of radius $r$, where $2r^2$ is an integer (which corresponds to $k$ being integer), we may write the spectrum of momenta as

\begin{align}
< \alpha_0 > &= p_L = \frac{a}{2r} + 2sr \\
< \overline{\alpha}_0 > &= p_R = \frac{a}{2r} + 2\pi r.
\end{align}

Here $a, s, \pi$ are integers with $1 \leq a \leq 4r^2$. With a set of values in the form eq.(3.1) we can introduce a unitary transformation

\begin{align}
\alpha_n &\rightarrow \alpha_n^s = \alpha_n + 2sr\delta_n \\
\overline{\alpha}_n &\rightarrow \overline{\alpha}_n^s = \overline{\alpha}_n + 2\pi r\delta_n.
\end{align}

On the Virasoro modes this transformation yields

\begin{align}
L_n \rightarrow L_n^{(s)} = L_n - 2rs\alpha_n - 2r^2s^2\delta_n,
\end{align}

and we have the analogous expression for the opposite chirality. It is clear that this transformation will transform the full set of momentum states into a new and equivalent set of momentum states and, therefore, it is a symmetry of the theory. This symmetry is present due to the compactness of space, yielding a discrete spectrum of momentum and winding states. We could have arrived at this symmetry in a different way. Let us restrict the set of values in eq.(3.1) to a much smaller set, namely those in eq.(3.1) for which $s$ and $\pi$ are zero. We could then recover the symmetry by simply applying the transformation (3.2) and adding all the new sectors of states corresponding to different values of $s$ and $\pi$. 
The unitary transformations will generate an orbit of $p_L$-values for each value of $a$. It is worth noticing that in computing the partition function, the different orbits, labelled by $a$, will in a natural way define characters

$$X_a = \sum_{s \in \mathbb{Z}} Tr\{e^{2\pi i r_L^{(s)} e^{2\pi i \theta_0^{(s)}}}\} \quad (3.4)$$

The transformations defined above may be interpreted as Weyl translations for an Abelian group.

In the case of $\hat{su}(1,1)$ we now proceed in a similar fashion. For integer $k$, taking into consideration the different normalization as compared to the Abelian case, we have in place of eq.(3.1)

$$< J_0^3 > = \frac{a}{2} + 2sr_1^2$$

$$< J_0^3 > = \frac{a}{2} + 2sr_1^2, \quad (3.5)$$

with $2r_1^2 = -k \in \mathbb{Z}$, $1 \leq a \leq -2k$. In analogy with eq.(3.2), we define the transformations

$$J_3 \rightarrow J_3^{(s)} = J_3 + 2sr_1^2 \delta_n$$

$$\mathcal{J}_3 \rightarrow \mathcal{J}_3^{(s)} = \mathcal{J}_3 + 2sr_1^2 \delta_n. \quad (3.6)$$

In order to proceed, we must solve a problem, which we did not encounter in the Abelian case. The transformation (3.6) must be compatible with the $\hat{su}(1,1)$ symmetry. Indeed, if it is unitary, it will preserve the $\hat{su}(1,1)$ algebra. In addition, it will transform primary fields into primary fields with respect to the transformed currents. These properties follow directly from the $\hat{su}(1,1)$ algebra and the definition of primary fields and they are enough to determine the transformation. One finds [20]

$$J_n^\pm \rightarrow J_n^{(s)\pm} = J_n^{\pm + 2s}, \quad (3.7)$$

and for the primary fields

$$V_{j,m}(z) \rightarrow V_{j,m}^{(s)}(z) = z^{-2ms}V_{j,m}(z). \quad (3.8)$$

On the Virasoro generators these transformations induce

$$L_n \rightarrow L_n^{(s)} = L_n - 2sJ_n^3 - 2s^2 r_1^2 \delta_n. \quad (3.9)$$

Here, $L_n$ are normal ordered with respect to $J_n^a$, and $L_n^{(s)}$ with respect to the transformed $J_n^{a(s)}$. The similarity between this equation and eq.(3.3) for the Abelian case is striking. It should be noted from eq.(3.7), that the state spaces for different values of $s$ are distinct.
from each other. This is in contrast to the Abelian case, where only the momentum values and not the higher modes are affected by the transformation. Just as for the Abelian case, the transformations defined above are Weyl translations of the original operators. It is clear that these transformations are not symmetries of the original SU(1, 1) theory, which means that the "momentum" and "winding" states in eq.(3.5) are not compatible with the state space of this theory. It corresponds instead to the case of restricting eq.(3.1) to the values \( s = \bar{s} = 0 \) in the U(1) theory. It should be remarked that even though the Weyl translations here, and in the U(1) theory, are consequences of the presence of an \( S^1 \), the converse is not necessarily true, as is the case of \( e.g. \) SU(2). Another example is the Euclidean coset \( SU(1, 1)/U(1) \). The corresponding Virasoro modes will be invariant under Weyl translations. This is as expected, since the compact time direction is factored out and, consequently, there is no \( S^1 \) embedded. We have now completed the construction of the \( SU(1, 1) \) string and we can proceed to construct characters for the new sectors of states and combine these into modular invariant partition functions.

4 Modular invariance

We begin our study of modular invariance by remarking that, since our string model has a compact time component, we will run into divergent sums due to "momentum" and "winding" states in the real time direction. In this section we will consider the divergent functions to be formally defined in the sense that the modular properties may be extracted following formal manipulations. When we consider the Euclidean coset \( SU(1, 1)/U(1) \), in which the time component has been removed, the partition functions will be convergent.

The transformation defined in the previous section induces the following new characters

\[
\chi_{j,0}(\tau, \theta) \rightarrow \chi_{j,s}(\tau, \theta) = \text{Tr}\{e^{2\pi i[(L_0-2sJ_0+s^2k-c/24)\tau+(J_0-sk)\theta]}\}. \tag{4.1}
\]

These can be easily computed from the old characters by noting that

\[
\chi_{j,s}(\tau, \theta) = e^{2\pi i s^2k} e^{-2\pi i sk} \chi_{j,0}(\tau, \theta - 2s\tau). \tag{4.2}
\]

Then we may write for the discrete representations

\[
\chi_{j,s}^{\pm}(\tau, \theta) = (-1)^{2s} e^{\pm \pi i \theta} e^{2\pi i r(k^\pm)(s\pm \frac{c+1}{2k+1})^2} e^{-2\pi i \theta(s(k+\pm)(j+1/2))} R^{-1}(\tau, \pm \theta). \tag{4.3}
\]

The transformed continuous characters are found similarly;

\[
\chi_{j,s}^a(\tau, \theta) = \eta(\tau)^{-3} e^{-2\pi i (\frac{c^2}{k+1})\tau} \sum_{n=-\infty}^{\infty} e^{2\pi i r(x+a-sk)^2} e^{2\pi i (x+a-sk)\theta}. \tag{4.4}
\]
Using the characters above we can proceed and seek modular invariant combinations. We first consider the simplest extension of our algebra where we assume \( k \in \mathbb{Z} \), and \( k < -2 \). In [20] the following extended character was presented for the discrete representations

\[
\chi_j^{\pm}(\tau, \theta) = \sum_{s \in \mathbb{Z}} \chi_{j,s}^{\pm}(\tau, \theta)
\]

\[
= \pm \frac{\Theta_{\mp(2j+1),k+2}(\tau, \theta)}{\Theta_{1,2}(\tau, \theta) - \Theta_{-1,2}(\tau, \theta)}
\]

where the formal theta function is defined by

\[
\Theta_{n,k}(\tau, \theta) = \sum_{m \in \mathbb{Z} + \frac{n}{2k}} \exp[2\pi ik(m^2 \tau - m\theta)]
\]

If we wish to describe a theory that enjoys time reversal symmetry, we may write a modular invariant partition function as

\[
Z^D(\tau, \theta) = \sum_{k/2 < j < 0} M_{\vec{j}}(\chi_j^{\dagger} + \chi_{-j}^{\dagger})^*(\chi_j + \chi_{-j}^-)
\]

where the coefficients \( M_{\vec{j}} \) are non-negative integers connected to the corresponding ones for \( \hat{su}(2) \) by the substitution \((k < -4) \ j, \vec{j} \rightarrow -j - 1, -\vec{j} - 1 \). We see from eq.(4.7) that we only include the allowed representations. In the diagonal combination where \( M_{\vec{j}} = \delta_{j\vec{j}} \), the range of \( j \) is \( k/2 + 1 \leq j \leq -1 \), since this combination of characters vanishes for non-square-integrable representation \( j = -1/2 \) as well as for \( j = 1/2(k+1) \). Note, that the partition function is finite in the limit \( \theta \rightarrow 0 \). The divergence has been removed in forming the combination of the two discrete representations, as remarked in section 2. The \( k = -3 \) partition function vanishes identically, and one may find two extra unitary solutions which are not symmetric under time reversal:

\[
Z_{-3}^{\pm}(\tau, \theta) = |\chi_{j=-1/2}^{\pm}(\tau, \theta)|^2 + |\chi_{j=-1}^{\pm}(\tau, \theta)|^2
\]

We see that here the non-square-integrable state \( j = -1/2 \) appears, which is typical of partition functions of the discrete series lacking time reversal symmetry.

The partition functions for the continuous representations with \( k \in \mathbb{Z} \) are quite simple to construct because the range of the Casimir eigenvalues are not restricted by unitarity. Summing over integral winding sectors \((j = -1/2 + i\rho)\), we have

\[
\chi_{\rho}^{\alpha}(\tau, \theta) = \sum_{s \in \mathbb{Z}} \chi_{\rho,s}^{\alpha}(\tau, \theta)
\]

\[
= \eta(\tau)^{-3} e^{-2\pi i \frac{1}{k+2} \sum_{n=0}^{n=k} \Theta_{2(n+\alpha),k}(\tau, 0)\Theta_{2(n+\alpha),-k}(\tau, \theta)}
\]
We see that the winding sectors correct for the measure of the moduli space by soaking up the contribution of the extra two \( \eta \)-functions. Taking the sum of both continuous characters \( \chi^0_\rho + \chi^{1/2}_\rho \), then the double theta function factor formally transforms under the modular group as a modular form which exactly cancels the contribution from two of the \( \eta \)-functions (disregarding a possible modular anomaly). Thus, we need only to integrate over \( \rho \) which acts like the momentum zero mode of an uncompactified boson:

\[
Z^C(\tau, \theta) = \int_0^\infty d\rho \left| \chi^0_\rho(\tau, \theta) + \chi^{1/2}_\rho(\tau, \theta) \right|^2.
\]  

(4.10)

This partition function is also finite in the limit \( \theta \to 0 \). The divergence has been absorbed into the divergent sum over \( s \).

The transformation in the previous section was defined for integer \( k \). One may extend this construction to the case \( k + 2 = p/q , p < 0, q > 1 \), where \( p \) and \( q \) are coprime integers. We must then allow spins on the \( q \)-th covering of \( SU(1,1) \). We have only succeeded in constructing modular invariant partition functions, using the representations allowed by unitarity, for the case \( p = -1 \). It is also consistent with the fact that only for this case we may generalize the decoupling theorem. The characters are of the following form

\[
\chi^{(r)\pm}_j(\tau, \theta) = \sum_{s' \in q\mathbb{Z}} \chi^{(r)\pm}_{j,s'}(\tau, \theta) = \Theta_{2r,p \pm (2j+1)q,pq}(\tau, \theta) e^{\pm \pi i \theta} R^{-1}(\tau, \pm \theta)
\]  

(4.11)

where the formal (divergent) theta function is defined as in eq.(4.6). In order to relate the partition functions to the covering groups of \( SU(1,1) \), we define

\[
j^{(r)} \equiv \frac{r}{q} + j,
\]  

(4.12)

so that we may consider \( j^{(r)} \) to be the spin of a representation on the \( q \)-th covering of \( SU(1,1) \). In terms of this notation we may write the modular invariant partition functions as

\[
Z^D(\tau, \theta) = \sum_{j^{(r)} \, \widetilde{\mathcal{F}}^{(r)}} M_{j^{(r)}, \widetilde{\mathcal{F}}^{(r)}} (\chi_{\widetilde{\mathcal{F}}^{(r)}}^+ + \chi_{\widetilde{\mathcal{F}}^{(r)}}^-)^* (\chi_{j^{(r)}}^+ + \chi_{j^{(r)}}^-).
\]  

(4.13)

The possible matrices, \( M_{j^{(r)}, \widetilde{\mathcal{F}}^{(r)}} \), may be found using the known classification of non-unitary \( SU(2) \) modular invariants [24] and the relations to \( SU(1,1) \) invariants given above.

Let us now turn to the coset \( SU(1,1)/U(1) \), where \( U(1) \) is the compact time component. Although the question of modular invariance of these models has so far remained elusive, 4 we are now in the position to construct modular invariant partition functions

4In ref.[25] the 3d Euclidean space \( SL(2,\mathbb{C})/SU(2) \) is discussed and a modular invariant partition function is constructed. The coset \( SL(2,\mathbb{C})/SU(2) \ mod \mathbb{R} \) is also considered and some indications are given that this theory is identical to the coset \( SU(1,1)/U(1) \). We have been unable to verify this. The partition functions presented in [25] appear to be completely different from ours.
using the methods of [26]-[29]. The state space of $\hat{su}(1,1)$ may be decomposed as

$$\mathcal{H}_j^{SU(1,1)} = \bigoplus_m \mathcal{H}_{j,m}$$  \hspace{1cm} (4.14)

where $m$ is the eigenvalue of $J_0^3$ and $j$ the spin of the ground-state. We may further decompose the states into a direct product of the parafermionic and $U(1)$ state

$$\mathcal{H}_{j,m} = \mathcal{H}_{j,m}^{PF} \otimes \mathcal{H}_m$$  \hspace{1cm} (4.15)

As we have mentioned above, the coset Virasoro modes are invariant under the Weyl translations. On the other hand, the eigenvalues of $J_0^3$ are clearly not invariant. However, from the decomposition eq.(4.14) one observes that a translation $m \to m - sk$ can be absorbed into the infinite sum over different $m$-values (which is not restricted by the lowest or highest weights, since it refers to the total $J_0^3$-value). The characters are, therefore, invariant under the Weyl translations. From the explicit expression of the character $\chi_{j,0}$ eq.(2.8) and the decomposition eq.(4.15), one finds for the $SU(1,1)/U(1)$ parafermion theory

$$\chi_{j,m}^{PF \pm} = \eta(\tau)D_{j,m}^{(\pm)}(\tau).$$  \hspace{1cm} (4.16)

Here we have defined a string function

$$D_{j,m}^{(\pm)}(\tau) = \mp \eta(\tau)^{-3} \sum_{r=0}^{\infty} (-1)^r e^{2\pi i r[(k+2)(\frac{j+1}{2})^2 - k(\frac{j+1}{2})^2]}.$$  \hspace{1cm} (4.17)

It is convenient to introduce $N = -(k + 2) > 0$, and we write $m = m' - sk$. Summing over $s$ we may construct the following modular function,

$$c_{2j+1}^{m'-1}(\tau) = \sum_{s \in \mathbb{Z}} (D_{j,m'+s(N+2)}^{(+)}(\tau) + D_{j,m'+s(N+2)}^{(-)}(\tau)).$$  \hspace{1cm} (4.18)

It can be shown [28, 29, 30] that this function is equivalent to the absolutely convergent Hecke indefinite modular form

$$c_M^L(\tau) = \sum_{-|x| < y \leq |x|} \text{sign}(x)e^{2\pi i r(N+2)x^2 - Ny^2}$$  \hspace{1cm} (4.19)

where

$$(x, y) \text{ or } (\frac{1}{2} - x, \frac{1}{2} + y) \in (\frac{L+1}{2(N+2)}, \frac{M}{2N}) + \mathbb{Z}^2.$$  \hspace{1cm} (4.20)

The modular properties of these functions are well known [28, 30] and a modular invariant partition functions can be written as

$$Z_{D}^{PF}(\tau) = |\eta(\tau)|^2 \sum_{L,M,L,M} N_{L,M,L,M} c_M^L(\tau)c_M^L(\tau)^*.$$  \hspace{1cm} (4.21)
The coefficients $N_{L,M;L',M'}$ are related to the corresponding modular invariant partition function for the coset $SU(2)/U(1)$, which for rational values of $N$ are given in [24]. Note, however, that $j$ and $m$ have exchanged rôles with respect to the familiar string functions of $SU(2)/U(1)$ coset models. As remarked earlier, the above partition functions are absolutely convergent and contain only the allowed representations for $k$ or $1/(k + 2)$ being an integer. For $1/(k + 2) = -4$ the above partition functions represent the $c = 26$ Euclidean black hole.

We will finally consider the principal continous representations. Again, it is simple to repeat the steps above with the result

$$
\chi_{j,m}(\tau, \theta) = \eta(\tau)^{-2} e^{2\pi i r[\frac{m^2}{4(k+2)} - \frac{1}{k+2}]} , \quad j = -\frac{1}{2} + i\rho. \quad (4.22)
$$

Constructing modular invariant partition functions is straightforward with these string functions, since they are of the same form as two free bosons, one of which is compactified on a circle of radius $\sqrt{-k/2}$.

Let us end by a remark. The modular invariant partition functions for the coset did no require the Weyl translation sectors. Still, it is worth noticing that in deriving the partition functions, we made a decomposition, leading to eq.(4.18), which is clearly reminiscent of these sectors. In fact, if we were to reintroduce the $U(1)$ piece and maintain modular transformation properties, the sum over $s$ in eq.(4.18), will become the sum over winding sectors.

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