A FREQUENCY CRITERION FOR THE EXISTENCE OF AN OPTIMAL CONTROL FOR ITÔ EQUATIONS∗

Nikolai Dokuchaev

Abstract

The following optimization problem is considered. For the vector Itô equation

\[ dx(t) = [Ax(t) + bu(t)]dt + Cx(t)dw(t) \]

with initial conditions \( x(0) = a \) it is required to find an optimal deterministic control vector \( u(t) \in L^2((0, +\infty), \mathbb{R}^m) \) which minimizes the functional

\[ \Phi[u(\cdot)] = \int_0^\infty [E x(t)^\top Gx(t) + u(t)^\top \Gamma u(t)]dt. \]

A necessary and sufficient condition for the existence of an optimal control are formulated in the form of frequency inequalities for functions depending on the matrices \( A, b, C, G \) and \( \Gamma \).

It is shown that an optimal control \( u^0(t) \) can be found by solving a certain linear-quadratic deterministic optimization problem.

Key words: optimal control, frequency theorem, Itô equations

We consider the following optimization problem on a standard probability space \((\Omega, \mathcal{F}, P)\):

\[ dx(t) = (Ax(t) + bu(t))dt + Cx(t)dw(t). \]

\[ x_0 = a. \]

\[ \Phi[u(\cdot)] = \int_0^\infty [E x(t)^\top Gx(t) + u(t)^\top \Gamma u(t)]dt = \min. \]

Here \( t \geq 0 \), \( dw(t) \) is a random walk adapted to a nondecreasing flow of \( \sigma \)-algebras \( \mathcal{F}(t) \subset \mathcal{F} \), \( x_t \) is a random \( n \)-vector of states, \( u(t) \) is a non-random \( m \)-vector of controls and \( A, C, G = G^\top, \Gamma = \Gamma^\top \) and \( b \) are are constant matrices of respective order \( n \times n, n \times n, n \times n, m \times m \) and \( n \times m \). All the vectors and matrices in (1) to (3) are real and \( E \) denotes expectation.

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The norm of a complex or real vector (matrix) $z$ is understood to be the square root of the sum of the squares of the moduli of its elements, and is denoted by $|z|$. Also, let $|\xi|_k = (E |\xi|^k)^{1/k}$.

The random vector $a$ is measurable with respect to the $\sigma$-algebra $\mathcal{F}_0$, is independent of $dw(t)$ and satisfies $E|a|^2 < +\infty$. Equation (1) is the Itô equation.

It is assumed that $A$ is a Hurwitz matrix and that for $u(t) = 0$ the system in (1), (2) is exponentially stable in the mean square, i.e., there exist numbers $c, \varepsilon > 0$ such that $|x(t)|_2 < ce^{-\varepsilon t}|a|_2 (\forall t > 0)$; this holds, for example under the mildly restrictive conditions given in Levit and Yakubovich (1972).

We establish a criterion for the existence of an optimal solution in the class $U = L^2((0, +\infty), \mathbb{R}^m)$ of deterministic measurable $m$-vector valued functions $u(t)$ such that $|u(t)| \in L^2(0, +\infty)$.

If $u^0(t)$ is an optimal control, then

$$\Phi[u^0(\cdot)] \leq \Phi[u(\cdot)] \quad (\forall u(\cdot) \in U).$$

The proof of the proposed criterion is based on results obtained in Yakubovich (1975), and makes essential the use of the idea of a proof given in Yakubovich (1975) for ordinary differential equations.

Consider the matrix-valued function $g(\lambda) = (i\lambda I - A)^{-1}$. Here and below, $i$ is the imaginary unit, $\lambda$ is in $\mathbb{R}^1$, and $I$ is the identity matrix. Suppose that the matrix $\Theta$ satisfies the equation

$$\Theta = G + \frac{1}{2\pi} \int_{-\infty}^{\infty} C^\top g(-\lambda)^\top \Theta g(\lambda) Cd\lambda. \quad (4)$$

We consider on $\mathbb{C}^n \times \mathbb{C}^m$ the Hermitian form

$$F(x, u) = x^* \Theta x + u^* \Gamma u. \quad (5)$$

1. A preliminary result is the following theorem, which actually establishes a criterion for the existence of $u^0(t)$.

**THEOREM 1** If there exists an optimal control $u^0(t)$, then

$$F(g(\lambda)bu, u) \geq 0 \quad (\forall \lambda \in \mathbb{R}^1, \forall u \in \mathbb{C}^m) \quad (6)$$

If there exists a number $\delta > 0$ such that

$$F(g(\lambda)bu, u) \geq \delta|u|^2 \quad (\forall \lambda \in \mathbb{R}^1, \forall u \in \mathbb{C}^m) \quad (7)$$

then there exists an optimal control $u^0(t)$, and it is unique to within equivalence.
PROOF. Let us consider the real Hilbert space $Y = \{y\} = L^2(\Omega, \mathcal{F}, \mathbf{P})$ and the complex Hilbert space $Y_c = \{y\} = L^2(\Omega, \mathcal{F}, \mathbf{P}, \mathbb{C}^n)$ of random $n$-vectors $y$ $\mathbf{P}$-equivalent vectors being identified) with respective inner products $\mathbf{E} y_1^\top y_2$ and $\mathbf{E} y_1^* y_2$.

For an arbitrary Hilbert space $H = \{h\}$ with inner product $(h_1, h_2)$, and for an interval $T \subset \mathbb{R}$ we denote by $Z = L^2(T, H)$ the Hilbert space of strongly measurable $H$-valued functions $h(t)$ such that $||h(t)|| \in L^2(T)$. If $H$ is a real (complex) space, then we regard $Z$ as a real (complex) space.

As usual, the inner product and the norm in $Z$ are

$$(h_1(\cdot), h_2(\cdot))_Z = \sum_{t \in T} (h_1(t), h_2(t)) \quad \text{and} \quad ||h(\cdot)||_Z = (h(\cdot), h(\cdot))_Z^\frac{1}{2}$$

Consider the Hilbert spaces

$$X = L^2(\mathbb{R}), \quad \bar{U} = L^2(\mathbb{R}), \quad \mathbb{C}^m.$$

For all $x(t) \in X$, $u(t) \in \bar{U}$, $\alpha > 0$, $\lambda \in \mathbb{R}$, let

$$\dot{x}_\alpha(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^\alpha e^{-i\lambda t} x(t) dt \quad \text{and} \quad \dot{u}_\alpha(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^\alpha e^{-i\lambda t} u(t) dt.$$

For $x(t)$ and $u(t)$ we define the Fourier transforms $\dot{x}_\alpha(\lambda)$ and $\dot{u}_\alpha(\lambda)$ in a way similar to that in Yakubovich (1975) (Part I, §I) so that $|\dot{x}(\cdot) - \dot{x}_\alpha(\cdot)|_X \rightarrow 0$ and $|\dot{u}(\cdot) - \dot{u}_\alpha(\cdot)|_U \rightarrow 0$ as $\alpha \rightarrow +\infty$.

We now consider the functions $u(t) \in \bar{U}$. Let $x(t)$ satisfy (1) and (2) for $t \geq 0$. Then $x(t)$ is a real random function. For $m(t) = \mathbf{E} x(t)$ and $M(t) = \mathbf{E} x(t) x(t)^\top$ we have the relations

$$m(t) = A m(t) + b u(t)$$

$$M(t) = AM(t) + M(t) A^\top + bu(t)m(t)^\top + m(t)u(t)^\top b^\top + CM(t)C^\top.$$

Let

$$|z(t)|_{L_k} = \left( \int_0^{+\infty} |z(t)|^k dt \right)^{\frac{1}{k}}$$

Then we can see that there are numbers $c_j > 0$ such that

$$|M(t)|_{L_k} \leq c_1 |m(t)u(t)|_{L_1} + c_2 |a|_2^2 \leq c_3 |m(t)|_{L_2} |u(t)|_{L_2} + c_2^2 |a|_2^2 \leq c_4 |u(t)|^2_{L_2} + c_2^2 |a|_2^2$$

Equation (8) means that

$$\int_0^\alpha e^{-i\lambda t} x(t) dt = \int_0^\alpha e^{-i\lambda t} [Ax(t) dt + bu(t) dt + Cx(t) dw(t)]$$

$$= i\lambda \int_0^\alpha e^{-i\lambda t} x(t) dt + e^{-i\lambda x(\alpha)} - x(0).$$
It follows from (8) that if \( u(t) \in U \), then \(|x(t)|_2 \in L^2(0, +\infty)\). We regard \( x(t) \) and \( u(t) \) as elements of \( X \) and \( U \) by setting \( x(t) = 0 \) and \( u(t) = 0 \) for \( t < 0 \). Let

\[
f_\alpha(\lambda) = \frac{1}{\sqrt{2\pi}} C \int_0^\alpha e^{-i\lambda t} x(t) dw(t), \quad h_\alpha(\lambda) = g(\lambda) f_\alpha(\lambda).
\]

Then for \( \alpha = 0 \)

\[i\lambda \tilde{x}_\alpha(\alpha) = A\tilde{x}_\alpha(\lambda) + b\tilde{u}_\alpha(\lambda) + f_\alpha(\lambda) - \frac{1}{\sqrt{2\pi}} e^{-i\alpha x(\alpha)} \]

Note that \( E h_\alpha(\lambda) = 0 \); we have

\[
E \tilde{x}_\alpha(\lambda)^* G\tilde{x}_\alpha(\lambda) = E \{ (g(\lambda) b\tilde{u}_\alpha(\lambda))^* G [g(\lambda) b\tilde{u}_\alpha(\lambda)] \}
- \frac{2}{\sqrt{2\pi}} Re \{ g(\lambda) b\tilde{u}_\alpha(\lambda))^* G g(\lambda) e^{-i\alpha x(\alpha)} \}
+ \frac{1}{\sqrt{2\pi}} [g(\lambda) e^{-i\alpha x(\alpha})]^* G [g(\lambda) e^{-i\alpha x(\alpha)}] + h_\alpha(\lambda)^* G h_\alpha(\lambda) \}. 
\]

Hence for \( \beta \geq 0 \)

\[
\int_{-\beta}^\beta E \tilde{x}_\alpha(\lambda)^* G\tilde{x}_\alpha(\lambda)d\lambda = \int_{-\beta}^\beta \tilde{u}_\alpha(\lambda)^* b^\top (-\lambda)^T G g(\lambda) b\tilde{u}_\alpha(\lambda)d\lambda + \int_{-\beta}^\beta E h_\alpha(\lambda)^* G h_\alpha(\lambda)d\lambda + \Psi(\beta, \alpha).
\]

The function \( \Psi(\beta, \alpha) \) goes to zero uniformly in \( \beta \) as \( \alpha \to +\infty \), because \( |x(\alpha)|_2 \to 0 \), the function \( |f_\alpha(\lambda)|_2 \) is bounded, and the functions \( |g(\lambda)| \) and \( |g(\lambda) b\tilde{u}_\alpha(\lambda)| \) are in \( L^2(-\infty, +\infty) \). Let

\[
T(\beta)(G) = \frac{1}{2\pi} \int_{-\beta}^\beta C^\top G(-\lambda)^T Gg(\lambda)Cd\lambda,
\]

\[
T(G) = \frac{1}{2\pi} \int_{-\infty}^\infty C^\top G(-\lambda)^T Gg(\lambda)Cd\lambda.
\]

Then by a property of Itô integral,

\[
\int_{-\beta}^\beta E h_\alpha(\lambda)^* G h_\alpha(\lambda)d\lambda = \int_{-\alpha}^\alpha E x(t)^\top T(\beta)(G)x(t)dt.
\]

Suppose now that \( \alpha \to +\infty \) and \( \beta \to +\infty \) in (9). Note that \( g(\lambda) b\tilde{u}_\alpha(\lambda) \to g(\lambda) b\tilde{u}(\lambda) \) in the \( L^2 \)-norm. Hence

\[
\int_{-\infty}^\infty E \tilde{x}(\lambda)^* G\tilde{x}(\lambda)d\lambda = \int_{-\infty}^\infty \tilde{u}(\lambda)^* b^\top (-\lambda)^T G g(\lambda) b\tilde{u}(\lambda)d\lambda + \int_{-\infty}^\infty E x(t)^\top Gx(t)dt.
\]

We observe that, by Parseval’s formula,

\[
\Phi[u(\cdot)] = (x(\cdot), Gx(\cdot))_X + (u(\cdot), G\tilde{u}(\cdot))_U
= (\tilde{x}(\cdot), G\tilde{x}(\cdot))_X + (\tilde{u}(\cdot), G\tilde{u}(\cdot))_U.
\]
Here $Gx(\cdot)$ and $\Gamma u(\cdot)$ denote functions with values $Gx(t)$ and $\Gamma u(t)$. If the matrix $\Theta$ satisfies (4), i.e. $G = \Theta - T(\Theta)$, then for $\Pi(\lambda) = b^T g(-\lambda)^T \Theta(\lambda) b + \Gamma$ we have

$$\Phi[u(\cdot)] = \int_{-\infty}^{\infty} \tilde{u}(\lambda)^* \Pi(\lambda) \tilde{u}(\lambda) d\lambda.$$ 

We now use Lemma 1 in Yakubovich (1975), Part I, §2 which asserts the following. Let $U = \{u\}$ be an arbitrary real Hilbert space with inner product $(u_1, u_2)$ and norm $|.|$, with arbitrary quadratic functional

$$\Phi(u) = (u, Ru) + 2(r, u) + \rho$$

(10)

defined on it, where $R$ is a self-adjoint bounded operator in $U$, $r \in U$ and $\rho \in \mathbb{R}^1$. Then: a) if there exists an optimal point $u^0 \in U$, i.e., a point such that $\Phi(u^0) \leq \Phi(u) (\forall u \in U)$; and b) if there exists a number $\delta > 0$ such that $(u, Ru) \geq \delta |u|^2 (\forall u \in U)$, then there exists an optimal point $u^0 \in U$, and it is unique.

Take $U$ to be the space $U = L^2([0, +\infty), \mathbb{R}^m]$. We consider also the real Hilbert space $\mathcal{X} = L^2([0, +\infty), \mathcal{Y})$. Obviously, for $u(t) \in U$, the solution of the system (11), (2) has the form $x(\cdot) = Qu(\cdot) + la$ where $Q$ and $l$ are bounded linear mappings carrying the Hilbert spaces $U$ and $L^2(\Omega, \mathcal{F}_0, \mathcal{P}, \mathbb{R}^n)$ into $\mathcal{X}$. It follows from (3) that $||Q|| \leq c_2$. It is not hard to show that, the functional (3) has the form (11), where $||R|| < +\infty$ and $|r|_U < +\infty$. As shown above, for $a = 0.$

$$\Phi[u(\cdot)] = (u(\cdot), Ru(\cdot)) = \int_{-\infty}^{\infty} F(g(\lambda) b\tilde{u}(\lambda), \tilde{u}(\lambda)) d\lambda = \int_{-\infty}^{\infty} \tilde{u}(\lambda)^* \Pi(\lambda) \tilde{u}(\lambda) d\lambda.$$ 

Assume that there exists $\lambda_0 \in \mathbb{R}^1$ and $v \in \mathbb{C}^m$ such that $v^* \Pi(\lambda_0) v < 0$. Let $\tilde{u}(t) \in U$ be such that the condition

$$\int_{0}^{\infty} F[y(t), \tilde{u}(t)] dt < 0$$

holds for the solution of the deterministic system $\frac{dy}{dt}(t) = Ay(t) + bu(t)$, $y(0) = 0$, and for the Hermitian form (5). (We find such a sequence $\tilde{u}(t)$ by repeating the arguments in the proof of Lemma 4 in Yakubovich (1975), Part I, §2). Then ($\tilde{u}(\cdot), R\tilde{u}(\cdot))_U < 0$. If $u^* \Pi(\lambda) u \geq \delta |u|^2$, then obviously $(\tilde{u}(\cdot), R\tilde{u}(\cdot))_U \geq \delta |\tilde{u}|_U^2$. Thus, Theorem 1 follows from Lemma 1 in Yakubovich (1975), Part I, §2.

EXAMPLE. Let $n = m = 1$, and write $\alpha = -A$. A necessary and sufficient condition for the exponential stability of (11), (2) in the mean square for $u(t) = 0$ is the condition $2\alpha > C^2 \geq 0$ (see Levit and Yakubovich (1976)). Now $g(\lambda) = (i\lambda + \alpha)^{-1}$, and (1) takes the form

$$\Theta = G + \frac{C^2}{2\alpha} \Theta$$

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Hence, for $\gamma = (1 - C^2/2\alpha)^{-1}$ we have that $\Theta = \gamma G$ and $\gamma \geq 1$. Thus,

$$F(g(\lambda)bu, u) = \left(\frac{\gamma Gb^2}{\alpha^2 + \lambda^2 + \Gamma}\right) u^2.$$  

For $G \geq 0$ conditions (6) and (7) mean that $\Gamma \geq 0$ and $\Gamma > 0$, respectively. For $G < 0$ conditions (6) and (7) mean that $\Gamma \geq -\gamma Gb^2/\alpha^2$ and $\Gamma > -\gamma Gb^2/\alpha^2$ respectively. We remark that in this case a stronger restriction is imposed on $\Gamma$ than for $C = 0$, when (1) is an ordinary difference equation.

2. Observe now that in (10) $(r, u) = (Qu(\cdot), Gla)^T X = \int_{\infty}^{\infty} E x_u(t)^\top G x_a(t) dt,$

where $x_u(t)$ and $x_a(t)$ are solutions to the system (1), (2) when $a = 0$ and $u(t) = 0$, respectively.

By arguing as in the proof in Theorem 1 it is not hard to get that

$$(r, u) = \int_{-\infty}^{\infty} E \tilde{x}_u(\lambda)^* G \tilde{x}_a(\lambda) d\lambda$$

(it is assumed that $\tilde{x}_u(\lambda)$ and $\tilde{x}_a(\lambda)$ are their Fourier transforms). If $G = \Theta - T(\Theta)$, then

$$(r, u) = \int_{-\infty}^{\infty} [g(\lambda)bu(\lambda)]^* \Theta g(\lambda) E d\lambda.$$  

For $u \in U$ we consider the optimization problem $(u = u(\cdot))$

$$y(t) = Ay(t) + bu(t), \quad y_0 = E a.$$  

$$\Phi_1[u(\cdot)] = \int_{0}^{\infty} [y(t)^\top \Theta y(t) + u(t)^\top \Gamma u(t)] dt = \min.$$  

Obviously $\Phi_1(u) = (u, R_1u) + 2(u, r_1) + \rho_1$ exists, with $R = R_1$ and $r = r_1$, for some $\rho_1 \in \mathbb{R}$. It is known (Yakubovich (1975), Part I, §1) that optimal controls in the problems (11)-(3) and (11)-(12) are determined from the respective equations $Ru^0 + r = 0$ and $R_1u^0 + r_1 = 0$. Note that $u^0(t)$ is uniquely determined from these equations when (7) holds, and $u^0(t) = h^\top y^0(t)$, where $h$ is a certain $n \times m$ matrix and $y^0(t) = e^{(A+bh^\top)^\top} y_0$. Thus we obtain the following results.

**THEOREM 2** An optimal control $u^0(t)$ exists in the stochastic optimization problem (1)-(3) if and only if an optimal control exists in the optimization problem (11)-(12). If condition (7) holds, then optimal controls in the optimization problem (1)-(3) and (11), (12) exist and are identical (and unique within equivalence).
REMARK 1 It can be shown that

\[ \mathbf{\rho} = \int_0^\infty \mathbf{E} x_a(t)\mathbf{G} x_a(t) dt = \int_{-\infty}^{\infty} \mathbf{E} a^\top g(-\lambda)^\top \Theta g(\lambda) d\lambda. \]

Hence if a us a deterministic vector then \( \mathbf{\rho} = \mathbf{\rho}_1 \).

REMARK 2 The proofs in Theorems 1 and 2 do not change for the following cases:

(a) \( Cx(t)dw(t) \) is replaced in (I) by \( \sum_{j=1}^d C_j x(t)dw^{(j)}(t) \), where the \( C_j \) are constant \( n \times n \) matrices, \( dw(t) = [dw^{(1)}(t), \ldots, dw^{(d)}(t)] \) is a standard \( d \)-dimensional Wiener process, and (4) is replaced by the equation

\[ \Theta = \mathbf{G} + \frac{1}{2\pi} \sum_{j=1}^d \int_{-\infty}^{+\infty} C_j^\top g(-\lambda) \Theta g(\lambda) C_j d\lambda. \]

(b) In the definition of the spaces \( \mathbf{U} \) and \( \bar{\mathbf{U}} \) we replace \( \mathbf{R}^m \) by \( L^2(\Omega, \mathcal{F}(0), \mathbf{P}, \mathbf{R}^m) \) and \( \mathbf{C}^m \) by \( L^2(\Omega, \mathcal{F}(0), \mathbf{P}, \mathbf{C}^m) \), in (11) we replace the condition \( y_0 = \mathbf{E} a \) by \( y_0 = a \), and in front of the right-hand sides of (5) and (12) we place the expectation sign, \( \mathbf{E} \). Then \( \mathbf{\rho} = \mathbf{\rho}_1 \) and \( \Phi_1[u(\cdot)] = \Phi[u(\cdot)] \).

REMARK 3 Equation (4) means that \( A\mathbf{D} + \mathbf{D} A^\top = -\Theta \) and \( \Theta = \mathbf{G} + \mathbf{C}^\top \mathbf{D} \mathbf{C} \) for some \( n \times n \) matrix \( \mathbf{D} \).

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