Operator Analysis of Neutrinoless Double Beta Decay

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(December 17, 2021)

Abstract

We study the effective operators of the standard model fields which would yield an observable rate of neutrinoless double beta decay. We particularly focus on the possibility that neutrinoless double beta decay is dominantly induced by lepton-number-violating higher dimensional operators other than the Majorana neutrino mass. Our analysis can be applied to models in which neutrinoless double beta decay is induced either by a strong dynamics or by quantum gravity effects at a fundamental scale near the TeV scale as well as the conventional models in which neutrinoless double beta decay is induced by perturbative renormalizable interactions.

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I. INTRODUCTION

The neutrinoless double beta decay ($\beta\beta_0\nu$) provides a very sensitive probe of lepton-number ($L$) violating interactions. The most commonly quoted origin of $\beta\beta_0\nu$ is the $ee$-component of Majorana neutrino mass matrix in the charged lepton mass eigenbasis, which is given by

$$(m'^\nu)_{ee} = \sum_i U^2_{ei} m_i = c^2_{12} c^2_{13} e^{i a_1} m_1 + s^2_{12} c^2_{13} e^{i a_2} m_2 + s^2_{13} e^{i a_3} m_3,$$

(1)

where $m_i$ ($i = 1, 2, 3$) are the neutrino mass eigenvalues, $\theta_{ij}$ ($i \neq j$) and $\alpha_i$ denote the mixing angles and CP phases in the $3 \times 3$ neutrino mixing matrix $U$, and $c_{ij} = \cos \theta_{ij}$, $s_{ij} = \sin \theta_{ij}$. Recently there has been a report to claim $\beta\beta_0\nu$ with half-life time $\tau_{1/2} \approx 10^{25}$ yrs [1]. Though the claim is still debatable [2], some implications of this observation have been discussed already in many papers [3]. If the origin of $\beta\beta_0\nu$ were due to $(m'^\nu)_{ee}$, the data suggest

$$|(m'^\nu)_{ee}| = 0.1 - 0.6 \text{ eV}. \quad (2)$$

When combined with the information from atmospheric and solar neutrino data, this value of $(m'^\nu)_{ee}$ severely constrains the possible form of neutrino mass matrix. In particular, it does not allow the neutrino mass eigenvalues in normal hierarchy. As is well known, the atmospheric neutrino data imply [4]

$$\Delta m^2_{\text{atm}} \equiv |m^2_3 - m^2_2| \approx 3 \times 10^{-3} \text{ eV}^2, \quad (3)$$

As for the solar neutrino anomaly, the following four-solutions are possible [5]:

$$\Delta m^2_{\text{sol}} = |m^2_3 - m^2_1| \quad \sin^2 2\theta_{12}$$

$$\begin{align*}
LMA : & \quad 3.2 \times 10^{-5} \text{ eV}^2 \quad 0.75 \\
SMA : & \quad 5.0 \times 10^{-6} \text{ eV}^2 \quad 2.4 \times 10^{-3} \\
LOW : & \quad 1.0 \times 10^{-7} \text{ eV}^2 \quad 0.96 \\
VAC : & \quad 8.6 \times 10^{-10} \text{ eV}^2 \quad 0.96 \quad (4)
\end{align*}$$

where LMA, SMA, LOW and VAC mean the large mixing angle MSW, small mixing angle MSW, low mass, and vacuum oscillation solution, respectively, and the numbers in each solution represent the best-fit values. There exists also a constraint on $\theta_{13}$ from the CHOOZ reactor experiment [6]:

$$\sin \theta_{13} \lesssim 0.2. \quad (5)$$

If the neutrino mass eigenvalues are in normal hierarchy, i.e. $m_3 \gg m_2 \gg m_1$, which is one of the plausible scenarios, the above information on $\theta_{ij}$ and $m_i$ imply

$$|(m'^\nu)_{ee}| = \begin{cases} 
(2 - 5) \times 10^{-3} \text{ eV} & (LMA) \\
10^{-6} \text{ eV} - \text{Max}(m_1, s^2_{13} m_3) & (SMA) \\
10^{-4} \text{ eV} - s^2_{13} m_3 & (LOW) \\
10^{-5} \text{ eV} - s^2_{13} m_3 & (VAC)
\end{cases} \quad (6)$$
where

\[ s_{13}^2 m_3 = 2 \times 10^{-3} \left( \frac{s_{13}^2}{4 \times 10^{-2}} \right) \text{eV} \lesssim 2 \times 10^{-3} \text{eV}. \]  \hspace{1cm} (7)

Obviously, these values of \((m'^e)_{ee}\) are too small to induce \(\beta\beta_{0\nu}\) with \(\tau_{1/2} \approx 10^{25}\) yrs. So if the claimed \(\beta\beta_{0\nu}\) turns out to be correct, we would have either (approximately) degenerate neutrino masses or the observed \(\beta\beta_{0\nu}\) is not due to \((m'^e)_{ee}\), but due to some other \(L\)-violating interactions. This would be true as long as \(\tau_{1/2} \ll 10^{29}\) yrs which can be tested in future experiments \cite{4}.

The possibility that \(\beta\beta_{0\nu}\) is dominantly induced by \(L\)-violating interactions other than \((m'^e)_{ee}\) has been discussed before in the context of specific models \cite{8–14}, and also a brief operator analysis of \(\beta\beta_{0\nu}\) has been made in \cite{15}. In this paper, we wish to provide a more detailed operator analysis of \(\beta\beta_{0\nu}\) by studying generic \(L\)-violating but baryon-number \((B)\) conserving higher-dimensional operators of the standard model (SM) fields which may induce \(\beta\beta_{0\nu}\). The main focus will be given to the possibility that \(\tau_{1/2}(\beta\beta_{0\nu}) \approx 10^{25}\) yrs though \((m'^e)_{ee}\) is in the range of \((6)\). Our result can be easily matched to the previous studies on specific models in which \(\beta\beta_{0\nu}\) is induced by perturbative renormalizable interactions. It can be applied also to models in which \(\beta\beta_{0\nu}\) is induced by either a strong dynamics or quantum gravity effects at energy scales near the TeV scale.

The organization of this paper is as follows. In section II, we classify the \(L\)-violating operators of the SM fields which can trigger \(\beta\beta_{0\nu}\). In section III, we tabulate the constraint on those \(\Delta L = 2\) operators from the condition \(\tau_{1/2}(\beta\beta_{0\nu}) \gtrsim 10^{25}\) yrs, and also estimate \((m'^e)_{ee}\) which would be radiatively-induced by the operators triggering \(\beta\beta_{0\nu}\). In section IV, we consider two specific models, i.e. a left-right symmetric model \cite{16,11} and a model with scalar diquark and dilepton \cite{17,13}, which can give \(\tau_{1/2}(\beta\beta_{0\nu}) \approx 10^{25}\) yrs, while having \((m'^e)_{ee}\) in the range of \((6)\). We match our results to the previous analysis on these models. Section V is the conclusion.

II. \(L\)-VIOlATING OPERATORS

In this section, we classify the \(L\)-violating but \(B\)-conserving operators of the SM fields which would trigger \(\beta\beta_{0\nu}\). A complete analysis of \(\Delta L = 2\) operators which would induce a nonzero Majorana neutrino mass can be found in \cite{15}. The \(\Delta L = 2\) \(\beta\beta_{0\nu}\) process may be induced by a double insertion of \(\Delta L = 1\) interactions or a single insertion of \(\Delta L = 2\) interaction. However with the SM fields alone, there is no way to construct a \(B\)-conserving operator with \(\Delta L = 1\). Since \(\beta\beta_{0\nu}\) occurs at energy scales far below the weak scale, the effects of quark-flavor-changing SM interactions on \(\beta\beta_{0\nu}\) are suppressed by the small Fermi constant and also the small quark-mixing angles. Also there is no renormalizable lepton-flavor-changing interaction in the SM. With these observations, we can ignore the effects of fermion-flavor-violation, so limit the analysis to the \(\Delta L = 2\) operators containing only the first generation of quarks and leptons. We also limit our study only to the operators without spacetime derivative or gauge field.

We use a notation in which all fermions are two-component Weyl spinors, i.e. \(\psi\) is a left-handed spinor and \(\bar{\psi}\) is its right-handed hermitian conjugate. Generic fermion bilinear
can be a Lorentz scalar, vector, or tensor:

\[(\psi \chi)_S = (\psi \chi), \quad (\psi \bar{\chi})_V = (\psi \sigma^\mu \bar{\chi}), \quad (\psi \chi)_T = (\psi \sigma^{\mu \nu} \chi),\]

where \(\sigma_{\mu \nu} = (\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu) / 4\). Left-handed fermions relevant for \(\beta \beta\) are

\[
\ell = (1, 2)_{-1/2}, \quad e^c = (1, 1)_1, \quad q = (3, 2)_{1/6}, \quad u^c = (3, 1)_{-2/3}, \quad d^c = (3, 1)_{1/3}, \quad (8)
\]

where \(SU(3)_C \times SU(2)_L \times U(1)_Y\) quantum numbers are indicated in parentheses. The SM Higgs doublet is denoted by \(H = (h^+, h^0) = (1, 2)_{1/2}\) and \(\bar{H}\) is its hermitian conjugate. We then have the following \(\Delta L = 2\) lepton bilinears

\[
(\ell^i \ell^j)_S = (\ell^i \ell^j)_S, \quad (\ell^i \ell^j)_T = - (\ell^i \ell^j)_T, \quad (\ell^i e^c)_V, \quad (e^c e^c)_S, \quad (9)
\]

where \(i, j\) are \(SU(2)\)-doublet indices.

There is a unique \(\Delta L = 2\), dimension \((d) = 5\) operator

\[
\mathcal{L}_{d=5} = - \frac{\xi v^2}{\Lambda} (\ell^i \ell^j)_S H^k H^l \epsilon_{ik} \epsilon_{jl} + h.c., \quad (10)
\]

where \(\Lambda\) denotes the mass scale of \(L\)-violating new physics which is assumed to exceed the weak scale, and the dimensionless \(\xi\) represents the strength of the couplings and/or the possible loop suppression factor involved in the mechanism to generate the above \(d = 5\) operator. After the electroweak symmetry breaking by \(\langle H \rangle = (0, v/\sqrt{2})\) with \(v = 246\) GeV, it gives the \(ee\)-component of the \(3 \times 3\) Majorana neutrino mass matrix:

\[
(m^\nu)_{ee} = \frac{\xi v^2}{\Lambda}. \quad (11)
\]

This neutrino mass is bounded to be less than 1 eV by \(\beta \beta\). Such a small neutrino mass can be a consequence of very large value of \(\Lambda\) [18,19], e.g. \(\xi \sim 1\) and \(\Lambda \sim 10^{14}\) GeV. Alternatively, \(\Lambda\) can be of order TeV, but \(m^\nu\) is small because \(\xi\) is small due to small couplings in the underlying dynamics which may be a consequence of some flavor symmetries [20], e.g. \(\xi \sim 10^{-11}\) and \(\Lambda \sim 1\) TeV. At any rate, when combined with the double insertion of the charged-current weak interaction, this Majorana neutrino mass leads to \(\beta \beta\) as in Fig. 1.a.

It is rather easy to see that there is no \(B\)-conserving \(d = 6\), \(\Delta L = 2\) operator. As for \(d = 7, \Delta L = 2\) operators which would trigger \(\beta \beta\), we have

\[
\mathcal{L}_{d=7} = \frac{1}{\Lambda^3} \left[ \lambda_1^{ST} \epsilon_{ik} \epsilon_{lj} (\ell^i \ell^j)_S (q^k d^c)_S T H^l + \lambda_2^{ST} \epsilon_{ij} \epsilon_{kl} (\ell^i \ell^j)_T (q^k d^c)_T H^l \right.
\]

\[
+ \lambda_3^{ST} \epsilon_{jl} (\ell^i \ell^j)_S (q^k \bar{u} c^c) S T H^l + \lambda_4^{ST} \epsilon_{ij} (\ell^i \ell^j)_T (q^k \bar{u} c^c)_T H^k \right.
\]

\[
+ \lambda_5 (\ell^i e^c)_V (d^c \bar{u} c^c)_V H^j \epsilon_{ij} \bigg] + h.c., \quad (12)
\]

where again \(\Lambda\) is the mass scale at which the above operators are generated. After the electroweak symmetry breaking, these \(d = 7\) operators yield the following 4-fermion operators:

\[
\frac{v}{\sqrt{2} \Lambda^3} \left[ \lambda_1 (\nu_e c)_S (u d^c) S + \lambda_2 (\nu_e c)_T (u d^c)_T + \lambda_3 (\nu_e c)_S (d \bar{u} c) S \right.
\]

\[
+ \lambda_4 (\nu_e c)_T (d \bar{u} c)_T + \lambda_5 (\nu_e \bar{e} c)_V (d \bar{u} c)_V \bigg] + h.c., \quad (13)
\]
where
\[ \lambda_1 = \lambda_1^5, \quad \lambda_2 = \lambda_2^7 - \lambda_1^7, \quad \lambda_3 = \lambda_3^5, \quad \lambda_4 = \lambda_4^7 - \lambda_3^7. \]

When combined with a single insertion of the standard charged current weak interaction, the above \( \Delta L = 2 \) four-fermi interactions lead to \( \beta\beta_{0\nu} \) as in Fig. 1.b.

As for the operators with \( d \geq 8 \), we are interested in the operators which can induce \( \beta\beta_{0\nu} \) without involving the SM weak interaction. Such operators should contain two electrons, so can be written as
\[
\mathcal{L}_{d=8} = (\ell \bar{\ell})_S \mathcal{O}_{ij}^S + (e \bar{e})_V \mathcal{O}_i^V + (\bar{e} e')_S \mathcal{O}^S
\]
(14)

where \( \mathcal{O}_I = \{ \mathcal{O}_{ij}^S, \mathcal{O}_i^V, \mathcal{O}^S \} \) are the operators made of the quarks and Higgs fields. In order to be \( \Delta Q_{em} = 2 \), \( \mathcal{O}_I \) must contain at least 4-quarks, so we need \( d \geq 9 \). Any \( \Delta Q_{em} = 2 \) four-quark operator can be written as a product of two \( \Delta Q_{em} = 1 \) quark-antiquark bilinears: \( \mathcal{O}_I = J_I J'_I \) where the quark-antiquark bilinears \( J_I, J'_I \) can be either color-singlet or color-octet. The hadronic matrix element for \( \beta\beta_{0\nu} \) can be approximated as \( \langle Z + 2| \mathcal{O}_I |Z \rangle \propto \langle p|J_I|n \rangle \langle p|J'_I|n \rangle \) for the neutron state \( |n \rangle \) and the proton state \( |p \rangle \), and then \( \mathcal{O}_I \) with color-octet \( J_I \) can be ignored in the operator analysis of \( \beta\beta_{0\nu} \). The most general \( d = 9 \) operators containing two color-singlet quark-antiquark bilinears together with two electrons are given by
\[
\mathcal{L}_{d=9} = \frac{1}{\Lambda^5} \left[ \kappa_1^{ST} \epsilon_{ik} \epsilon_{jl}(\ell \bar{\ell})_S(q^k d^e)_S(q^j d^e)_S + \kappa_2^{ST} (\ell \bar{\ell})_S(q^k \bar{d}^e)_S(q^j \bar{d}^e)_S + \kappa_3^{ST} \epsilon_{ik} (\ell \bar{\ell})_S(q^k \bar{u}^e)_S(q^j \bar{u}^e)_S + \kappa_4^{ST} \epsilon_{ik} \epsilon_{jl} (\ell \bar{\ell})_S(q^k \bar{u}^e)_S(q^j \bar{u}^e)_S \right] + h.c.
\]
(15)

where all quark bilinears in the parentheses are color-singlet. These \( d = 9 \) operators give the following 6-fermion operators which would trigger \( \beta\beta_{0\nu} \) as in Fig. 1.c:
\[
\frac{1}{\Lambda^5} \left[ \kappa_1^{ST} (ee)_S (ud\bar{e})_S (ud\bar{e})_S + \kappa_2^{ST} (ee)_S (d\bar{u}e)_S (d\bar{u}e)_S + \kappa_3^{ST} (ee)_S (ud\bar{d})_S (u\bar{d}d)_S + \kappa_4^{ST} (ee)_S (d\bar{u}\bar{e})_S (u\bar{d}d)_S + \kappa_5^{ST} (ee)_S (d\bar{d}\bar{e})_S (u\bar{d}d)_S \right] + h.c.
\]
(16)

As for the next higher dimensional \( d = 11 \) operators, we are interested in the operators which can not be obtained by multiplying the gauge-invariant Higgs bilinear \( H^I H_I \) to \( d = 9 \) operators in (15). Among such operators, the following ones are relevant for \( \beta\beta_{0\nu} \):
\[
\mathcal{L}_{d=11} = \frac{1}{\Lambda^7} \left[ \eta_1^{I} \epsilon_{km} \epsilon_{ln}(\ell \bar{\ell})_S(q^k \bar{q}^l)_V (q^m \bar{q}^n)_V H^m H^n + \eta_2^{I} \epsilon_{ik} \epsilon_{jm}(\ell \bar{\ell})_S(q^k \bar{q}^l)_V (q^m \bar{q}^n)_V H^m H^n \right]
\]
(17)
After the EWSB, these \( d = 11 \) operators give the following 6-fermion operators leading to \( \beta \beta_{0\nu} \) as in Fig. 1.c:

\[
\frac{v^2}{2\Lambda^7} \left[ \eta_1 (ee)_S (ud)_V (ud)_V + \eta_2 (ee)_{S'} (d\bar{u}c)_V (d\bar{u}c)_V \\
+ \eta_3^{ST} (e\bar{e}c)_V (udc)_S (udc)_T + \eta_4^{ST} (e\bar{e}c)_V (udc)_S (udc)_T \\
+ \eta_5^{ST} (e\bar{e}c)_S (udc)_S (udc)_T + \eta_6^{ST} (e\bar{e}c)_S (udc)_S (udc)_T \\
+ \eta_7^{ST} (e\bar{e}c)_S (udc)_S (udc)_T + \eta_8 (e\bar{e}c)_S (udc)_V (d\bar{u}c)_V \right] + h.c.,
\]

where

\[
\eta_1 = \eta_1' + \eta_1'' + \eta_1''', \\
\eta_3^{ST} = \eta_3^{ST} + \eta_3^{S'T} + \eta_3^{ST'}, \\
\eta_4^{ST} = \eta_4^{S'T} + \eta_4^{ST'} + \eta_4^{S'T}.
\]

III. CONSTRAINTS FROM \( \beta \beta_{0\nu} \)

To determine the \( \beta \beta_{0\nu} \) rate induced by the operators presented in section II, one needs to compute the nuclear matrix elements of the involved multi-quark operators. In this paper, we will use the results of [21] for the necessary nuclear matrix elements. We will also assume that \( \beta \beta_{0\nu} \) is dominated by one of the operators in (10), (12), (15) and (17), so ignore possible interference effects, however still it can be used to constrain \( L \)-violating interactions with a reasonable accuracy.

If \( \beta \beta_{0\nu} \) is induced dominantly by \( (m^\nu)_{ee} \), one finds [1]

\[
\tau_{1/2}^{-1} = 1.1 \times 10^{-13} \left( \frac{v^2}{\Lambda m_e} \right)^2 |\xi|^2 \text{ yr}^{-1} = 7.4 \times 10^{-30} \left( \frac{(m^\nu)_{ee}}{4 \times 10^{-3} \text{ eV}} \right)^2 \text{ yr}^{-1}
\]

In case that \( \beta \beta_{0\nu} \) is dominated by one of \( d \geq 7 \) operators in section II, the resulting \( \tau_{1/2}^{-1} \) can be written as \( \tau_{1/2}^{-1} = |\epsilon|^2 \Phi_e \) where \( \epsilon \) contains the operator coefficient, while \( \Phi_e \) contains the phase space factors and nuclear matrix elements depending on the Lorentz structure of the corresponding operator. Using the results of [21], the numerical values of \( \Phi_e \) can be obtained as summarized in Tables I and II. For \( d = 7 \) operators of (12) giving the 4-fermion operators (13), \( \beta \beta_{0\nu} \) occurs as in Fig. I(b). We then find the corresponding half-life time

\footnote{In fact, since the \( \Delta L = 2 \) operators are assumed to be generated at scale \( \Lambda \), one also needs to compute the renormalization group evolution of the operators over the scales from \( \Lambda \) to \( \Lambda_{QCD} \sim 1 \) GeV. Taking into account the effects of such renormalization group evolution is beyond the scope of this paper, so will be ignored.}
\[ \tau_{1/2} = \frac{1}{128} \left( \frac{v^3}{\Lambda^3} \right)^2 \left[ 16|\lambda_{1,3,5}|^2, |\lambda_{2,4}|^2 \right] \Phi_{\lambda_i} \text{ yr}^{-1}, \]  

(21)

where the numerical values of \( \Phi_{\lambda_i} \) are listed in Table I. The upperbound on \( \lambda \)'s resulting from the condition \( \tau_{1/2} \gtrsim 10^{25} \) yrs are summarized in Table III.

It is also straightforward to compute \( \tau_{1/2} \) for \( \beta\beta_{0\nu} \) induced by \( d = 9 \) and \( d = 11 \) operators of (13) and (17). For \( d = 9 \) operators, we find

\[ \tau_{1/2}^{-1} = \left( \frac{m_p v^4}{8 \Lambda^3} \right)^2 \left[ 16|\kappa_{1,2,3,4,5}^S|^2, |\kappa_{3}^V|^2, 4|\kappa_{4,5}^T|^2, |\kappa_{1,2,3}^T|^2, 16|\kappa_6|^2 \right] \Phi_{\kappa_i} \text{ yr}^{-1}, \]  

(22)

and for \( d = 11 \) operators,

\[ \tau_{1/2}^{-1} = \left( \frac{m_p v^4}{16 \Lambda^3} \right)^2 \left[ 16|\eta_{1,2,7}|^2, 16|\eta_{3,4,5,6,7}^S|^2, 4|\eta_{3,4,5}^T|^2, |\eta_{5,6,7}^T|^2 \right] \Phi_{\eta_i} \text{ yr}^{-1}, \]  

(23)

where \( m_p \) is the proton mass and the numerical values of \( \Phi_{\kappa_i} \) and \( \Phi_{\eta_i} \) are listed in Table I. The resulting constraints on \( \kappa \)'s and \( \eta \)'s for \( \tau_{1/2} \gtrsim 10^{25} \) yrs are summarized in Table IV and V.

The above equations (21), (22) and (23) summarizing the \( \beta\beta_{0\nu} \) rate of \( d = 7, 9, 11 \) operators and the resulting constraints on the operator coefficients \( \lambda \)'s, \( \kappa \)'s and \( \eta \)'s listed in Tables III, IV, V are the main results of this paper. Still one of our major concern is the possibility that \( \beta\beta_{0\nu} \) is induced dominantly by one of \( d \geq 7 \) operators, not by the \( d = 5 \) operator for \( (m^\nu)_{ee} \). This would occur for instance if some of \( \lambda \)'s or \( \kappa \)'s saturate their bounds from \( \beta\beta_{0\nu} \), while \( \xi \) is small enough to give \( (m^\nu)_{ee} \ll 1 \) eV. In fact, the condition \( (m^\nu)_{ee} \ll 1 \) eV constrains the coefficients of \( d \geq 7 \) operators also since those operators can generate \( (m^\nu)_{ee} \) through loops. For instance, the \( d = 7 \) operator with coefficients \( \lambda_{1,3}^S \) in (12) generate the \( d = 5 \) operator for \( (m^\nu)_{ee} \) through the one-loop diagram of Fig. 2.a, yielding

\[ \delta_\lambda \xi \sim \frac{y_{d,u}}{16\pi^2} \lambda_{1,3}^S, \]  

(24)

where we have taken the cutoff of the loop momenta to be \( \Lambda \) and \( y_{d,u} \) is the down(up)-quark Yukawa couplings. Other \( d = 7 \) operators can generate \( \xi \) also, however it involves more loops and/or more insertions of small Yukawa couplings. For instance, the operator with coefficient \( \lambda_5 \) in (12) generates \( \xi \) through the 2-loop diagram of Fig. 2.b, yielding

\[ \delta_\lambda \xi \sim \frac{y_u y_d y_e}{(16\pi^2)^2} \lambda_5, \]  

(25)

where \( y_e \) is the electron Yukawa coupling. As a result, \( \delta_\lambda \xi \) from other \( d = 7 \) operators are negligibly small compared to \( \delta_\lambda \xi \) from \( \lambda_{1,3}^S \).

Similarly, the \( d = 9 \) operators with coefficients \( \kappa_{1,2,3}^S \) in (13) generate the \( d = 5 \) operator for \( (m^\nu)_{ee} \) at two-loop order (Fig. 2.c):

\[ \delta_\kappa \xi \sim \frac{(y_u^2, y_d^2, y_d y_u)}{(16\pi^2)^2} \kappa_{1,2,3}^S, \]  

(26)

where again the cutoff of the loop momenta is chosen to be \( \Lambda \). Other \( d = 9 \) operators and also the \( d = 11 \) operators can generate \( \xi \), however again the corresponding diagrams involve...
more loops and/or more insertions of small Yukawa couplings. For instance, the \( d = 9 \) operator with coefficient \( \kappa_6 \) generates \( \xi \) through the 4-loop diagram of Fig. 2d, yielding

\[
\delta \kappa \xi \sim \frac{y_u^2 y_d^2 y_e^2}{(16\pi^2)^4} \kappa_6,
\]

which is absolutely negligible even when \( \kappa_6 \) saturates the bound from \( \beta \beta_0 \nu \).

Combining (20), (21) and (22) with (24) and (26), one easily finds

\[
\frac{\tau^{-1}_{1/2}(\lambda_{1,3}^S)}{\tau^{-1}_{1/2}(\delta \lambda \xi)} \sim 3 \times 10^3 \left( \frac{\text{TeV}}{\Lambda} \right)^4,
\]

\[
\frac{\tau^{-1}_{1/2}(\kappa_{1,2,3}^S)}{\tau^{-1}_{1/2}(\delta \kappa \xi)} \sim 3 \times 10^7 \left( \frac{\text{TeV}}{\Lambda} \right)^8,
\]

implying that if the scale \( \Lambda \) of \( L \)-violating interactions is about 1 TeV, it is quite possible that \( \beta \beta_0 \nu \) is dominantly induced by one of \( \Delta L = 2, d = 7 \) or \( d = 9 \) operators. In particular, one of \( d = 7 \) or \( d = 9 \) operators can give \( \tau^{-1}_{1/2}(\beta \beta_0 \nu) \approx 10^{25} \) yrs even when \( (m^\nu)_{ee} \) is in the range of (9) as suggested by the atmospheric and solar neutrino oscillation data in normal neutrino mass hierarchy.

**IV. APPLICATIONS TO SOME MODELS**

Our results in the previous section can be applied to various kind of models providing \( L \)-violating interactions for \( \beta \beta_0 \nu \) and/or neutrino mass. In this section, we consider some specific models of \( \beta \beta_0 \nu \) which have been discussed in the literatures [11,13] and use our results to rederive the constraints on \( L \)-violating couplings from the condition \( \tau^{-1}_{1/2}(\beta \beta_0 \nu) \gtrsim 10^{25} \) yrs.

Let us first consider a model in which \( d = 7 \) operators can be a dominant source of \( \beta \beta_0 \nu \). An example of such model is the left-right symmetric model [16,11] with gauge group \( SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \). The Higgs sector of the model contains a bi-doublet \( \phi \) and also triplets \( \Delta_L \) and \( \Delta_R \) whose \( SU(2)_L \times SU(2)_R \times U(1)_{B-L} \) quantum numbers are given by

\[
\Delta_L = (3, 1)_2 = \begin{pmatrix} \delta_L^+ / \sqrt{2} \\ \delta_L^0 \\ -\delta_L^- / \sqrt{2} \end{pmatrix},
\]

\[
\Delta_R = (1, 3)_2 = \begin{pmatrix} \delta_R^+ / \sqrt{2} \\ \delta_R^0 \\ -\delta_R^- / \sqrt{2} \end{pmatrix},
\]

\[
\phi = (2, 2)_0 = \begin{pmatrix} \phi_0^0 \\ \phi_1^+ \\ \phi_2^- \end{pmatrix},
\]

where the subscripts are the \( U(1)_{B-L} \) charges. The model contains also the left and right-handed lepton doublets:

\[
\ell_L = \begin{pmatrix} \nu \\ e \end{pmatrix} = (2, 1)_{-1}, \quad \bar{\ell}_R = \begin{pmatrix} \bar{N}_c^c \\ e^c \end{pmatrix} = (1, 2)_{-1},
\]

as well as the quark doublets
\[ Q_L = \begin{pmatrix} u \\ d \end{pmatrix} = (2,1)_{1/3}, \quad \bar{Q}_R = \begin{pmatrix} \bar{u}^c \\ \bar{d}^c \end{pmatrix} = (1,2)_{1/3}. \]

Yukawa interactions of the 1st generation are given by

\[ \mathcal{L}_Y = h_L \ell_L \ell_R + h_L \sigma_2 \phi^* \sigma_2 \ell_R + h^Q Q_L \phi Q_R + \bar{h}^Q Q_L \bar{\phi} Q_R + f \ell_L i \sigma_2 \Delta_L \ell_L + f \ell_R i \sigma_2 \Delta_R \ell_R + h.c. \]

(29)

Parameters in the Higgs potential can be chosen to yield the Higgs vacuum expectation values \( \langle \delta^0_{L,R} \rangle = v_{L,R} \) and \( \langle \phi \rangle = \text{diag}(\kappa, \kappa') \) with the scale hierarchy \( v_L \gg \kappa \gg \kappa' \gg v_R \).

Then the fermion masses of the model are given by

\[ m_e \approx \tilde{h} \kappa, \quad m_u \approx h^Q \kappa, \quad m_d \approx \tilde{h}^Q \kappa \]

for \( \kappa \sim 180 \text{ GeV} \), and the neutrino mass

\[ (m'_{\nu})_{ee} \approx \frac{(h\kappa)^2 - 4f^2 v_L v_R}{2fv_R}. \]

(30)

The model can generate also the \( d = 7 \) operators of (12) (with coefficients \( \lambda^S_{1,3} \)) through the diagram of Fig.3.a, yielding

\[ \frac{\lambda^S_1}{\Lambda^3} \approx \frac{\gamma v_R f}{m^2_{\Delta_L}} \left( \frac{h^Q}{m^2_{\phi_1}} - \frac{\tilde{h}^Q}{m^2_{\phi_2}} \right), \]

\[ \frac{\lambda^S_2}{\Lambda^3} \approx \frac{\gamma v_R f}{m^2_{\Delta_L}} \left( \frac{h^Q}{m^2_{\phi_1}} - \frac{\tilde{h}^Q}{m^2_{\phi_2}} \right), \]

(31)

where \( \gamma \) is the coefficient of the term \( \text{tr}(\Delta^\dagger_L \phi \Delta_R \phi^\dagger) \) in the Higgs potential. Then there exists a parameter range of the model in which \( \beta \beta_{0\nu} \) is dominated by these \( d = 7 \) operators. For instance, if \( f v_R \sim m_{\Delta_L} \sim 10^5 \text{ GeV}, m_{\phi} \sim 2 \times 10^2 \text{ GeV}, \) and \( \gamma \sim 10^{-1} \), the resulting \( \lambda^S_{1,3}/\Lambda^3 \sim 10^{-6}/(\text{TeV})^3 \) saturates the bound listed in Table 11 so lead to \( \beta \beta_{0\nu} \) with \( \tau_{1/2} \sim 10^{25} \) yrs. Though not very natural, still the parameters of the model can be tuned to yield \( (m'_{\nu})_{ee} = O(10^{-3}) \text{ eV} \), while keeping \( \lambda^S_{1,3}/\Lambda^3 \sim 10^{-6}/(\text{TeV})^3 \). So the model can accommodate \( \tau_{1/2}(\beta \beta_{0\nu}) \sim 10^{25} \) yrs together with the atmospheric and solar neutrino oscillation data in hierarchical neutrino mass scenario.

As an example of model in which \( d = 9 \) operators of (15) can be a dominant source of \( \beta \beta_{0\nu} \), let us consider a model containing scalar-diquarks and scalar-dilepton with the following \( SU(3)_c \times SU(2)_L \times U(1)_Y \) quantum numbers:

\[ \Delta_u = (6,1)_{4/3}, \quad \Delta_d = (6,1)_{-2/3}, \quad \Delta_e = (1,1)_{-2}. \]

(32)

The Yukawa couplings of the model are assumed to include

\[ h_u \Delta_u u^c u^c + h_d \Delta_d d^c d^c + h_e \Delta_e e^c e^c \]

(33)

and also the Higgs potential contains

\[ \mu \Delta_u \Delta_d^\dagger \Delta_e \]

(34)

which breaks \( L \)-conservation.
With the above interactions, a \( d = 9 \) operator for \( \beta\beta_0\nu \) is generated at tree level as depicted in Fig. 3b. The resulting operator corresponds to the \( \kappa_6 \) term of (15):

\[
\frac{\kappa_6}{\Lambda^5}(e\bar{e})S(d\bar{u})V(d\bar{u})V, \tag{35}
\]

where

\[
\frac{\kappa_6}{\Lambda^5} = \frac{\mu h_u h_d h_e}{8m^2_{\Delta_u}m^2_{\Delta_d}m^2_{\Delta_e}}. \tag{36}
\]

If \( h_u, d, e \sim 1 \), \( m_{\Delta_u, d, e} \sim 1 \) TeV and \( \mu \sim 250 \) GeV, the resulting \( \kappa_6/\Lambda^5 \sim 3 \times 10^{-2}/(\text{TeV})^5 \) saturates the bound in Table IV, so the model leads to \( \beta\beta_0\nu \) with \( \tau_{1/2} \sim 10^{25} \) yr. On the other hand, \( (m^\nu)_{ee} \) induced by the \( L \)-violating interaction (34) is 4-loop suppressed and involves 6-powers of small Yukawa couplings as in Fig. 2.d:

\[
(m^\nu)_{ee} \sim \frac{y^2_u y^2_d y^2_e}{(16\pi^2)^4} \frac{h_u h_d h_e \mu \langle H \rangle^2}{m^2_{\Delta}}. \tag{37}
\]

This \( (m^\nu)_{ee} \) is absolutely negligible, \( (m^\nu)_{ee} \sim 10^{-30} \) eV, even when \( \kappa_6/\Lambda^5 \) saturates its bound. So one needs additional \( L \)-violating interactions to generate neutrino masses which would explain the atmospheric and solar neutrino oscillation data in hierarchical neutrino mass scenario.

V. CONCLUSION

Motivated by the recent claim of observing \( \beta\beta_0\nu \) with \( \tau_{1/2} \approx 10^{25} \) yrs, we studied the effective \( \Delta L = 2 \) operators of the SM fields which would generate \( \beta\beta_0\nu \). We classified such operators up to mass dimension \( d = 11 \), and find the upper bound on each operator coefficient resulting from the condition \( \tau_{1/2} \gtrsim 10^{25} \) yrs. Our results are summarized in Tables III, IV, V. We also examined the possibility that \( d = 7 \) or 9 operators are a dominant source of \( \beta\beta_0\nu \) in the context of generic operator analysis, particularly the possibility that \( \tau_{1/2}(\beta\beta_0\nu) \sim 10^{25} \) yrs, while \( (m^\nu)_{ee} = \mathcal{O}(10^{-3}) \) eV as suggested by the atmospheric and solar neutrino oscillation data in hierarchical neutrino mass scenario. As we have demonstrated in section IV, our results can be easily matched to the previous analysis on specific models in which \( \beta\beta_0\nu \) is induced by perturbative renormalizable interactions. They can be also applied to models in which \( \beta\beta_0\nu \) is induced by either a strong dynamics or quantum gravity effects at scales near the TeV scale.

ACKNOWLEDGMENTS

This work is supported in part by BK21 Core program of MOE, KRF Grant No. 2000-015-DP0080, KOSEF Sundo-Grant, and KOSEF through CHEP at Kyungpook National University.
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FIG. 1. Feynman diagrams for $\beta\beta_{0\nu}$. (a) corresponds to the conventional $\beta\beta_{0\nu}$ induced by $(m'_{\nu})_{ee}$. (b) represents $\beta\beta_{0\nu}$ induced by the combined effects of a $d = 7$, $\Delta L = 2$ operator (dark blob) and the SM charged current weak interaction, while (c) represents $\beta\beta_{0\nu}$ induced by a $d = 9$ or 11, $\Delta L = 2$ operator.

FIG. 2. Feynman diagrams for the $d = 5$ operator for $(m'_{\nu})_{ee}$ radiatively induced by $d = 7$ or $d = 9$ operators
FIG. 3. (a) Feynman diagram for the $d = 7$ operator with coefficients $\lambda_{1,3}^S$ in the left–right symmetric model. (b) Feynman diagram for the $d = 9$ operator with coefficient $\kappa_6$ induced by the exchange of scalar diquarks and dilepton.
TABLE I. Numerical values of $\Phi_{\lambda_i}$ obtained using the results of Ref. [21].

| $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ |
|-------------|-------------|-------------|-------------|-------------|
| $6.9 \times 10^{-10}$ | $2.9 \times 10^{-8}$ | $6.9 \times 10^{-10}$ | $2.0 \times 10^{-7}$ | $1.7 \times 10^{-13}$ |

TABLE II. Numerical values of $\Phi_{\kappa_i}$ and $\Phi_{\eta_i}$ obtained using the results of Ref. [21].

| $\kappa_{1,2,3,\eta_{5,6,7}^S}$ | $\kappa_{1,2,3,\eta_{5,6,7}^T}$ | $\kappa_{6,\eta_{1,2}}$ | $\kappa_{3,\eta_{7}}$ | $\kappa_{4,5,\eta_{3,4}^S}$ | $\kappa_{4,5,\eta_{3,4}^T}$ |
|-------------------------------|-------------------------------|-------------------------|-------------------------|-------------------------------|-------------------------------|
| $6.2 \times 10^{-13}$ | $1.4 \times 10^{-8}$ | $3.5 \times 10^{-11}$ | $5.6 \times 10^{-10}$ | $1.4 \times 10^{-12}$ | $1.4 \times 10^{-10}$ |

TABLE III. Upperbounds on the coefficient of $d = 7$ operators from $\tau_{1/2} \geq 10^{25}$ yrs. Here $\Lambda_{TeV} = \Lambda/TeV$

| $\lambda_1/\Lambda_{TeV}^{3}$ | $\lambda_2/\Lambda_{TeV}^{3}$ | $\lambda_3/\Lambda_{TeV}^{3}$ | $\lambda_4/\Lambda_{TeV}^{3}$ | $\lambda_5/\Lambda_{TeV}^{3}$ |
|-------------------------------|-------------------------------|-------------------------|-------------------------|-------------------------------|
| $2.3 \times 10^{-6}$ | $1.4 \times 10^{-6}$ | $2.3 \times 10^{-6}$ | $5.3 \times 10^{-7}$ | $1.4 \times 10^{-4}$ |

TABLE IV. Upperbounds on the coefficients of $d = 9$ operators

| $\kappa_{1,2,3}/\Lambda_{TeV}^{5}$ | $\kappa_{1,2,3}^T/\Lambda_{TeV}^{5}$ | $\kappa_{3}/\Lambda_{TeV}^{5}$ | $\kappa_{4,5}/\Lambda_{TeV}^{5}$ | $\kappa_{4,5}^T/\Lambda_{TeV}^{5}$ | $\kappa_{6}/\Lambda_{TeV}^{5}$ |
|-------------------------------|-------------------------------|-------------------------|-------------------------|-------------------------------|-------------------------------|
| $2.3 \times 10^{-1}$ | $6.2 \times 10^{-3}$ | $7.7 \times 10^{-3}$ | $1.5 \times 10^{-1}$ | $3.1 \times 10^{-2}$ | $3.1 \times 10^{-2}$ |

TABLE V. Upper bounds on the coefficients of $d = 11$ operators

| $\eta_{1,2}/\Lambda_{TeV}^{7}$ | $\eta_{3,4}/\Lambda_{TeV}^{7}$ | $\eta_{3,4}^T/\Lambda_{TeV}^{7}$ | $\eta_{5,6,7}/\Lambda_{TeV}^{7}$ | $\eta_{5,6,7}^T/\Lambda_{TeV}^{7}$ | $\eta_{7}/\Lambda_{TeV}^{7}$ |
|-------------------------------|-------------------------------|-------------------------|-------------------------|-------------------------------|-------------------------------|
| $1.0$ | $5.1$ | $1.0$ | $7.6$ | $0.2$ | $0.3$ |