MEAN ERGODIC COMPOSITION OPERATORS ON SPACES OF SMOOTH FUNCTIONS AND DISTRIBUTIONS

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ABSTRACT. We investigate (uniform) mean ergodicity of weighted composition operators on the space of smooth functions and the space of distributions, both over an open subset of the real line. Among other things, we prove that a composition operator with a real analytic diffeomorphic symbol is mean ergodic on the space of distributions if and only if it is periodic with period 2. Our results are based on a characterization of mean ergodicity in terms of Cesàro boundedness and a growth property of the orbits for operators on Montel spaces which is of independent interest.

Keywords: Mean ergodic operator; Uniformly mean ergodic operator; Weighted composition operator; Spaces of smooth functions; Spaces of distributions

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1. INTRODUCTION

In this note we contribute to the investigation of the dynamical behaviour of weighted composition operators $C_{w,\phi}(u) = w \cdot (u \circ \phi)$ on the space of smooth (i.e. infinitely many times differentiable) functions $E(X)$ on an open subset $X \subseteq \mathbb{R}$ as well as on the space of distributions $\mathcal{D}'(X)$, where the symbol $\phi$ of $C_{w,\phi}$ is a smooth self map of $X$, respectively a diffeomorphism of $X$ when dealing with $\mathcal{D}'(X)$, and the weight $w$ is a complex valued smooth function on $X$. The space of smooth functions $E(X)$ is endowed with its natural topology of uniform convergence on compact subsets of $X$ of all derivatives up to an arbitrary finite order while $\mathcal{D}'(X)$ is equipped with its strong dual topology, being the topological dual space of $\mathcal{D}(X)$, the space of test functions on $X$. Thus, both spaces are Montel spaces. We are interested in when such weighted composition operators are mean ergodic (definitions will be given in Section 2 below).

In recent years there have been several articles studying mean ergodicity and related properties of (weighted) composition operators on various spaces of functions, such as spaces of holomorphic functions in finite dimensions [7], [8], [3], [11], [3], [15], [22], spaces of holomorphic functions on infinite dimensional Banach spaces [17], spaces of homogeneous polynomials on infinite dimensional Banach spaces [16], spaces of real analytic functions [8], the Schwartz space of rapidly decreasing functions on $\mathbb{R}$ [9], spaces of meromorphic functions [10], and within the general framework of function spaces defined by local properties [18].

This note is organized as follows. In Section 2 we show that for a continuous linear operator on a Montel space the properties of mean ergodicity and uniform mean ergodicity...
coincide, and we give a characterization of these properties in terms of Cesàro boundedness of the operator together with a growth property of its orbits (Theorem 2.5 (b)). In Section 3, based on the aforementioned result, we derive necessary and sufficient conditions for $C_{w, \rho}$ to be mean ergodic on $\mathcal{E}(X)$ (Theorem 3.2). Under the additional assumption that $\phi$ is a diffeomorphism and $(x \in X; \omega(x) \neq 0)$ is dense in $X$, in Section 4 we show that mean ergodicity of $C_{w, \rho}$ on $\mathcal{D}(X)$ forces a rather restrictive behaviour of the symbol $\phi$, namely $\phi$ as well as $\phi^{-1}$ have stable orbits (Theorem 4.3). This restrictive property is then used to show that for a real analytic diffeomorphism $\phi$ the corresponding unweighted composition operator $C_{\phi} := C_{1, \phi}$ is mean ergodic on $\mathcal{D}(X)$ if and only $C_{\phi}$ is periodic of period 2 (Theorem 4.9).

2. General abstract results

Let $E$ be a locally convex Hausdorff space (briefly, lCHs) and $T \in \mathcal{L}(E)$, where as usual we denote by $\mathcal{L}(E)$ the space of continuous linear operators on $E$. Moreover, by $c(s)(E)$ we denote the set of continuous seminorms on $E$. $T$ is said to be topologizable if for every $p \in c(s)(E)$ there is $q \in c(s)(E)$ such that for every $m \in \mathbb{N}$ there is $\gamma_m > 0$ with

$$p(T^n x) \leq \gamma_m q(x)$$

for all $x \in E$.

For the special case that in the above inequality one can take $\gamma_m = 1$ for all $m \in \mathbb{N}$ we say that $T$ is power bounded. In this case the family $\{T^m : m \in \mathbb{N}\}$ is an equicontinuous subset of $\mathcal{L}(E)$. Moreover, $T$ is Cesàro bounded if the family $\{T^n : n \in \mathbb{N}\}$ is an equicontinuous subset of $\mathcal{L}(E)$, where $T^n$ denotes the $n$-th Cesàro mean given by

$$\frac{1}{n} \sum_{m=1}^{n} T^m$$

An operator $T \in \mathcal{L}(E)$ is called mean ergodic if there is $P \in \mathcal{L}(E)$ such that for each $x \in E$ it holds $\lim_{n \to \infty} T^n x = Px$. In case that the convergence is uniform on bounded subsets of $E$ then $T$ is called uniformly mean ergodic. For $T \in \mathcal{L}(E)$ and $n \in \mathbb{N}$ we have the following identities (where $T^{[0]} = I$)

$$(1) \quad \frac{1}{n} T^n = T^{[n]} - \frac{n-1}{n} T^{[n-1]},$$

$$(2) \quad (I - T) T^n = T^{[n]} (I - T) = \frac{1}{n} (T - T^{n+1})$$

so that $\lim_{n \to \infty} \frac{1}{n} T^n x = 0$ for every $x \in E$ whenever $T$ is mean ergodic.

The following theorem is a special case of Eberlein’s mean ergodic theorem which is proved by a straightforward modification of the proof in [21, Chapter 2, § 2.1, Theorem 1.5, p. 76]. In our context, one has to set the semigroup of operators $\mathcal{S} = \{T^n : n \in \mathbb{N}_0\}$ and the ergodic net $\{T^{[n]} : n \in \mathbb{N}\}$, where one has to take into account the fact that due to

$$T^k T^{[n]} x = T^{[n]} T^k x - T^{[n]} x = \frac{k}{n} \sum_{m=1}^{n} T^{n+m} x - \frac{k}{n} \sum_{m=1}^{n} T^m x$$

it follows $\lim_{n \to \infty} T^{[n]} T^k x = T^{[n]} x = \lim_{n \to \infty} T^k T^{[n]} x - T^{[n]} x = 0$ in $E$ whenever $x \in E$ satisfies

$$(3) \quad \lim_{n \to \infty} \frac{1}{n} T^n x = 0.$$

**Theorem 2.1.** Let $E$ be a lCHS and let $T \in \mathcal{L}(E)$ be Cesàro bounded and let $x \in E$ be such that $\lim_{n \to \infty} \frac{1}{n} T^n x = 0$. The following conditions are equivalent for $y \in E$:

(a) $Ty = y$ and $y$ belongs to the closed convex hull of the orbit $O(x, T) := \{T^m x : m \in \mathbb{N}_0\}$ of $x$. 
We recall that a locally convex Hausdorff space is barrelled and every bounded subset of $\mathcal{L}(E)$ As in the proof of Corollary 2.2, for mean ergodic principle $T$ is Cesàro bounded and $\lim_{n \to \infty} T^n x = 0$ for every $x \in E$. Consequendy, given $x \in E$, the set $\{ T^n x : n \in \mathbb{N} \}$ is relatively $\sigma(E, E')$-compact, and therefore $\{ T^n x : n \in \mathbb{N} \}$ has a $\sigma(E, E')$-cluster point $y \in E$. By Theorem 2.1 necessarily $y = \lim_{n \to \infty} T^n x$. We define

$$P_x := \lim_{n \to \infty} T^n x$$

for each $x \in E$. Since $\{ T^n x \}_{n \in \mathbb{N}}$ is equicontinuous in $\mathcal{L}(E)$ we obtain that $P \in \mathcal{L}(E)$ and $T$ is mean ergodic.

The following result is a version of [1, Theorem 2.4] for reflexive locally convex spaces.

**Theorem 2.3.** Let $E$ be a reflexive lcHs and $T \in \mathcal{L}(E)$. Then $T$ is mean ergodic if and only if $T$ is Cesàro bounded and $\lim_{n \to \infty} \frac{1}{n} T^n x = 0$ for every $x \in E$.

**Proof.** We recall that a locally convex Hausdorff space $E$ is reflexive if and only if it is barrelled and every bounded subset of $E$ is relatively $\sigma(E, E')$-compact.

As in the proof of Corollary 2.2 for mean ergodic $T$ we have that $T$ is uniformly mean ergodic and $\lim_{n \to \infty} \frac{1}{n} T^n x = 0$ for every $x \in E$. Consequently, $\{ T^n x : n \in \mathbb{N} \}$ is a pointwise bounded set and by the Uniform Boundedness Principle $T$ is Cesàro bounded.

Conversely, if $\{ T^n x : n \in \mathbb{N} \}$ is equicontinuous, for each $x \in E$ the set $\{ T^n x : n \in \mathbb{N} \}$ is bounded. Thus, as $E$ is reflexive, this set is relatively $\sigma(E, E')$-compact. An application of Corollary 2.2 concludes the proof.

Since in semi-reflexive spaces bounded subsets are relatively $\sigma(E, E')$-compact and since for power bounded $T$ condition (3) trivially holds, Corollary 2.2 yields the next result from [6, Proposition 3.3].

**Corollary 2.4.** Every power bounded operator on a semi-reflexive locally convex Hausdorff space is mean ergodic.

We are now ready to prove the following result which contains a characterization of (uniform) mean ergodicity for an operator on Montel spaces in terms of Cesàro boundedness and a growth property of its orbits.

**Theorem 2.5.** Let $E$ be a Montel space and let $T \in \mathcal{L}(E)$.

(a) $T$ is mean ergodic if and only if $T$ is uniformly mean ergodic.
(b) The following are equivalent.
   (i) $T$ is Cesàro bounded and $\lim_{n \to \infty} \frac{T^n x}{n} = 0$, pointwise in $E$.
   (ii) $T$ is mean ergodic on $E$.
   (iii) $T$ is uniformly mean ergodic on $E$. 

As a consequence of Theorem 2.1 we obtain the next result.

**Corollary 2.2.** Let $T \in \mathcal{L}(E)$ be Cesàro bounded. Then $T$ is mean ergodic if and only if $T$ is satisfied for all $x \in E$ and $\{ T^n x : n \in \mathbb{N} \}$ is relatively $\sigma(E, E')$-compact for each $x \in E$.

**Proof.** If $T$ is mean ergodic, (3) holds by identity (1). Clearly $\{ T^n x : n \in \mathbb{N} \}$ is relatively $\sigma(E, E')$-compact for each $x \in E$ because $\{ T^n x : n \in \mathbb{N} \}$ converges in $E$.

Conversely, given $x \in E$, the set $\{ T^n x : n \in \mathbb{N} \}$ is relatively $\sigma(E, E')$-compact, and therefore $\{ T^n x : n \in \mathbb{N} \}$ has a $\sigma(E, E')$-cluster point $y \in E$. By Theorem 2.1 necessarily $y = \lim_{n \to \infty} T^n x$. We define

$$P_x := \lim_{n \to \infty} T^n x$$

for each $x \in E$. Since $\{ T^n x \}_{n \in \mathbb{N}}$ is equicontinuous in $\mathcal{L}(E)$ we obtain that $P \in \mathcal{L}(E)$ and $T$ is mean ergodic. \qed
(iv) \( T^t \) is mean ergodic on \((E', \beta(E', E))\).
(v) \( T^t \) is uniformly mean ergodic on \((E', \beta(E', E))\).
(vi) \( T^t \) is Cesàro bounded on \((E', \beta(E', E))\) and \( \lim_{n \to \infty} \frac{(T^t)^n}{n} = 0 \), pointwise in \((E', \beta(E', E))\).

Proof. Trivially, every uniformly mean ergodic operator is mean ergodic. Let \( T \) be mean ergodic. Then, \( \{T^n : n \in \mathbb{N}\} \) is equicontinuous and since \( E \) is a Montel space, every bounded subset \( B \) of \( E \) is relatively compact. Since on equicontinuous subsets of \( \mathcal{L}(E) \) pointwise convergence on \( E \) and uniform convergence of relatively compact subsets of \( E \) coincide, it follows that \( \{T^n\}_{n \in \mathbb{N}} \) converges uniformly on bounded subsets of \( E \).

Thus, (a) is proved.

In order to prove (b), we observe that (i) and (ii) as well as (iv) and (vi) are equivalent by Theorem 2.3, while (ii) and (iii) are equivalent by part (a). Since with \( E \) also \((E', \beta(E', E))\) is a Montel space, the equivalence of (iv) and (v) follows from part (a) as well. Finally, by [2] Corollary 2.7 (iii) and the fact that Montel spaces are reflexive, (iii) and (v) are equivalent. \( \square \)

3. Weighted composition operators on \( \mathcal{E}(X) \)

In this section we study the mean ergodicity of weighted composition operators \( C_{w, \phi} \) on the space of smooth functions \( \mathcal{E}(X) \), where \( X \subseteq \mathbb{R} \) is an open set. Here, \( w : X \to \mathbb{C} \) and \( \phi : X \to X \) are smooth functions and \( \mathcal{E}(X) \) is equipped with its standard topology, i.e. with the Fréchet space topology generated by the seminorms

\[
\forall K \subset X \text{ compact}, s \in \mathbb{N}_0, f \in \mathcal{E}(X): \|f\|_{s,K} := \sup_{x \in K, \beta \leq s} |f^{(r)}(x)|.
\]

As usual, \( C_{w, \phi} : \mathcal{E}(X) \to \mathcal{E}(X) \) is defined as

\[ C_{w, \phi} f(x) := w(x) f(\phi(x)), \quad x \in X, \]

for all \( f \in \mathcal{E}(X) \). Then, \( \mathcal{E}(X) \) is a nuclear Fréchet space and thus, in particular a Montel space and clearly \( C_{w, \phi} \in \mathcal{L}(\mathcal{E}(X)) \). Thus, by Theorem 2.3 (b), \( C_{w, \phi} \) is mean ergodic if and only if it is uniformly mean ergodic.

For \( s \in \mathbb{N} \) we have, using Leibniz’ rule and Faà di Bruno’s formula [14],

\[
(C_{w, \phi}^{n})^{(s)} = \left( \prod_{i=0}^{n-1} w(\phi^i) \right)^{(s)} f(\phi^n) \]

\[
+ \sum_{r=1}^{s} \binom{s}{r} \left( \prod_{i=0}^{n-1} w(\phi^i) \right)^{(s-r)} \sum_{j=1}^{r} f^{(j)}(\phi^n) B_{r,j}(1, \ldots, \phi^n)^{r-j+1},
\]

where \( B_{r,j} \) denote the corresponding Bell polynomials. Please note that by \( \phi^j \) we denote the \( j \)-fold composition of \( \phi \) with itself etc. In order to simplify our notation, we abbreviate

\[
\forall r \in \mathbb{N}, j \in \{1, \ldots, r\}, n \in \mathbb{N}_0: B_{r,j,n} := B_{r,j}(1, \ldots, (\phi^n)^{r-j+1})
\]

as well as

\[
\forall r \in \mathbb{N}, n \in \mathbb{N}_0, x \in \mathbb{R}: B_{0,0,n}^\infty(x) := 1, B_{r,0,n}^\infty(x) := 0.
\]

With this notation, we have

\[
(C_{w, \phi}^{n})^{(s)} = \sum_{0 \leq j \leq r \leq s} \binom{s}{r} \left( \prod_{i=0}^{n-1} w(\phi^i) \right)^{(s-r)} f^{(j)}(\phi^n) B_{r,j,n}^\infty
\]
for every $f \in \mathcal{E}(X)$ and $s \in \mathbb{N}_0$. Evaluating this equality for the special case of $f_s(y) := \exp(\lambda y)$, $\lambda \in \mathbb{C}$, $y \in \mathbb{R}$, yields

\begin{align}
\forall \lambda \in \mathbb{C}, s \in \mathbb{N}_0 : \left(C^n_{\phi, \lambda} f_s\right)^{(s)} &= f_s \circ \phi^n \sum_{0 \leq j, r \leq s} \left(\prod_{i=0}^{n-1} w(\phi^i)\right)^{(s-r)} \lambda_j^j B_{r,j,n}^\phi.
\end{align}

(5)

Now we discuss necessary and sufficient conditions involving mean ergodicity. The following result is a characterization of the property $[3]$ for $C_{\phi, \lambda}$. Recall, that $\phi : X \to X$ is said to have stable orbits if for each compact $K \subset \mathbb{R}$ there is another compact subset $L \subset X$ with $\phi^n(K) \subseteq L$ for every $n \in \mathbb{N}_0$.

**Proposition 3.1.** Let $\phi : X \to X$ and $w : X \to \mathbb{C}$ be smooth functions such that $\langle x \in X ; w(\phi^n(x)) \neq 0 \rangle$ is dense in $X$ for every $m \in \mathbb{N}_0$. Then, the following are equivalent:

(i) $\lim_{n \to \infty} \frac{1}{n} C^n_{\phi, \lambda} f = 0$ in $\mathcal{E}(X)$.

(ii) $\phi$ has stable orbits and for every compact $K \subset X$, $s \in \mathbb{N}_0$, and $h \in \{0, \ldots, s\}$ it holds

\begin{align}
\lim_{n \to \infty} \frac{1}{n} \left\| \sum_{r=h}^s \left(\prod_{i=0}^{n-1} w(\phi^i)\right)^{(s-r)} B_{r,h,n}^\phi \right\|_{0, K} = 0.
\end{align}

Proof. In order to show that (i) implies (ii), for $s \in \mathbb{N}$ we set $\lambda_s := \exp\left(\frac{2\pi i}{s}\right)$ and for $h \in \{1, \ldots, s\}$ we define

\begin{align}
Q_{s,h}(x) := \prod_{1 \leq j \neq h, j \neq s} (\lambda_s^j - x), x \in \mathbb{R}.
\end{align}

Thus, $Q_{s,h}$ is a polynomial of degree $s - 1$ with $Q_{s,h}(\lambda_s^h) \neq 0$. Then,

\begin{align}
P_{s,h}(x) := \frac{1}{Q_{s,h}(\lambda_s^h)} Q_{s,h}(x), x \in \mathbb{R},
\end{align}

is a polynomial of degree $s - 1$ satisfying $P_{s,h}(\lambda_s^j) = \delta_{j,h}$ for $j \in \{1, \ldots, s\}$ where $\delta_{j,h}$ denotes Kronecker’s delta. Let $\alpha_{0,h}^{(s,h)}, \ldots, \alpha_{s-1,h}^{(s,h)} \in \mathbb{C}$ be such that $P_{s,h}(x) = \sum_{k=0}^{s-1} \alpha_k^{(s,h)} x^k$.

Since $\mathcal{E}(X)$ is a Fréchet space, by the Uniform Boundedness Principle, (i) implies the equicontinuity of $\left\{ \frac{1}{n} C^n_{\phi, \lambda} ; n \in \mathbb{N}_0 \right\}$. In particular, $C_{\phi, \lambda}$ is topologizable, so that by [10] Corollary 3.12 and proof of Corollary 3.15 $\phi$ has stable orbits. Now we fix $s \in \mathbb{N}$ and a compact $K \subset X$. For arbitrary $k \in \mathbb{N}_0$, applying (i) and (5) to $f_{s,k}^\phi$ and $\lambda_s^k$, respectively, yields

\begin{align}
0 = \lim_{n \to \infty} \frac{1}{n} \left\| f_{s,k}^\phi \left(\sum_{0 \leq j \leq r \leq s} \left(\prod_{i=0}^{n-1} w(\phi^i)\right)^{(s-r)} \lambda_j^j B_{r,j,n}^\phi \right) \right\|_{0, K}.
\end{align}

(6)

Because $\phi$ has stable orbits, there is a compact set $L \subset X$ such that $\phi^n(K) \subseteq L$ for every $n \in \mathbb{N}_0$. By compactness of $L \times \{ \lambda \in \mathbb{C} ; |\lambda| = 1 \}$, there is $C > 0$ such that

\begin{align}
\forall n, k \in \mathbb{N}_0, x \in K : |f_{s,k}^\phi(\phi^n(x))| \leq C.
\end{align}
Moreover, evaluating (i) for $f$ implies
\[0 = \lim_{n \to \infty} \frac{1}{n} \left| \sum_{k=0}^{n-1} \alpha_k \sum_{0 \leq j < r \leq s} \left( \prod_{i=0}^{n-r} w(\phi^i) \right) \lambda_{s}^{\lambda k} B_{r,j,n}^{k} \right|_{0,K}.\]

Thus, combining the last two equalities gives
\[\forall h \in [0, s]: 0 = \lim_{n \to \infty} \frac{1}{n} \left| \sum_{r=h}^{s} \left( \prod_{i=0}^{n-r} w(\phi^i) \right) \right|_{0,K}.\]

Finally, evaluating (i) again for $f = 1$ implies for $s = 0$ that
\[0 = \lim_{n \to \infty} \frac{1}{n} \left| \prod_{r=0}^{n-r} w(\phi^i) \right|_{0,K},\]

so that (ii) follows.

Conversely, in order to show that (ii) implies (i), we note that, since $\phi$ has stable orbits, for every $j \in \mathbb{N}_0$, the values of $f^{(j)}(\phi^n)$ on $K$ are contained in a compact set which is independent of $n$ and thus can be estimated by a constant. Hence, for arbitrary $f \in \mathcal{E}(X)$, by (4) and the limits appearing in (ii),
\[\forall K \subset X \text{ compact, } s \in \mathbb{N}_0 : \lim_{n \to \infty} \frac{1}{n} \left| C_{\phi,\phi}^{n} f \right|_{1,K} = 0\]

which shows (i).

\[\square\]

**Theorem 3.2.** Let $\phi: X \to X$ and $w: X \to \mathbb{C}$ be smooth functions with $\langle x \in X; w(\phi^n(x)) \neq 0 \rangle$ being dense in $X$ for every $m \in \mathbb{N}_0$. Consider the following conditions.

(i) $\phi$ has stable orbits and for every compact set $K \subset X$, $s \in \mathbb{N}_0$, and $h \in [0, s]$ there holds
\[\lim_{n \to \infty} \frac{1}{n} \left| \sum_{r=h}^{s} \left( \prod_{i=0}^{n-r} w(\phi^i) \right) B_{r,h,n}^{\phi} \right|_{0,K} = 0\]
as well as
\[\sup_{m \in \mathbb{N} \setminus 0} \frac{1}{m} \sum_{n=1}^{m} \left| \sum_{r=h}^{s} \left( \prod_{i=0}^{n-r} w(\phi^i) \right) B_{r,h,n}^{\phi} \right|_{0,K} < \infty.\]

(ii) $C_{\phi,\phi}$ is (uniformly) mean ergodic on $\mathcal{E}(X)$.

(iii) $\phi$ has stable orbits and for every compact set $K \subset X$, $s \in \mathbb{N}_0$, and $h \in [0, s]$ condition (8) holds.

Then (i) implies (ii) and (ii) implies (iii).
Proof. If (i) holds, it follows from Proposition 3.4 that \( \frac{1}{n} C_{w,\phi} f \) converges to 0 in \( \mathcal{E}(X) \) for every \( f \in \mathcal{E}(X) \). Moreover, since \( \phi \) has stable orbits, for a fixed compact set \( K \subset X \) there is a compact set \( L \subset X \) with \( \phi^n(K) \subset L \) for every \( n \in \mathbb{N} \). For arbitrary \( f \in \mathcal{E}(X) \) and \( s \in \mathbb{N}_0 \) it follows from (9) for \( x \in K \) and \( m \in \mathbb{N} \) that

\[
\left| \frac{1}{m} \sum_{n=1}^{m} \left( C_{w,\phi} f \right)^{(s)}(x) \right| \leq \sum_{h=0}^{s} \frac{1}{m} \sum_{n=1}^{m} \left( \frac{1}{n} \prod_{r=0}^{s} \|w(\phi^r)\| \right)^{(s-r)}(x) B_{r,h,n}(x) \times f^{(h)}(\phi^n(x)) \leq \|f\|_{s,L} \sum_{h=0}^{s} \frac{1}{m} \sum_{n=1}^{m} \sum_{r=h}^{s} \left( \frac{1}{n} \prod_{r=0}^{s} \|w(\phi^r)\| \right)^{(s-r)} B_{r,h,n}^{(\phi)} \bigg|_{0,K}
\]

which by (9) implies the Cesàro boundedness of \( C_{w,\phi} \) on \( \mathcal{E}(X) \). Since \( \mathcal{E}(X) \) is a Montel space, by Theorem 2.5(b) we conclude that \( C_{w,\phi} \) is uniformly mean ergodic.

Next, if \( C_{w,\phi} \) is mean ergodic, by Proposition 3.1(iii) follows.

\[ \square \]

Remark 3.3. 
(i) In case \( \phi : X \rightarrow X \) is a diffeomorphism it follows from Proposition 3.9 together with Brouwer’s Invariance of Domain Theorem, that \( \{x \in X; w(\phi^n(x)) \neq 0\} \) is dense in \( X \) for every \( m \in \mathbb{N}_0 \) if (and only if) \( \{x \in X; w(x) \neq 0\} \) is dense in \( X \).

(ii) For the special case of a constant weight \( w(x) = \alpha \in C \setminus \{0\} \), (9) in Theorem 3.2 simplifies to

\[
\lim_{n \rightarrow \infty} \frac{|\alpha|^n}{n} \|B_{s,h,n}^{(\phi)}\big|_{0,K} = 0.
\]

while (9) turns into

\[
\sup_{m \in \mathbb{N}} \frac{1}{m} \sum_{n=1}^{m} |\alpha|^n \|B_{s,h,n}^{(\phi)}\big|_{0,K} < \infty
\]

4. Mean ergodic composition operators on \( \mathcal{D}(X) \)

In this section, for an open subset \( X \subset \mathbb{R} \), we study mean ergodicity of weighted composition operators on \( \mathcal{D}(X) \) with diffeomorphic symbol, where as usual \( \mathcal{D}(X) \) is equipped with the strong dual topology \( \beta(\mathcal{D}(X), \mathcal{D}(X)) \), i.e. the topology of uniform convergence on bounded subsets of \( \mathcal{D}(X) \). Recall that for a diffeomorphism \( \phi : X \rightarrow X \) (or, more generally, a smooth function \( \phi : X \rightarrow X \) for which \( \phi' \) does not have zeros - so that \( \phi \) is injective, in particular) there is a unique continuous linear operator \( C_{\phi} \) on \( \mathcal{D}(X) \) which satisfies \( C_{\phi} f = f \circ \phi \) for every \( f \in C(X) \). It holds

\[
\forall u \in \mathcal{D}(X), \phi \in \mathcal{D}(X) : \langle C_{\phi} u, \phi \rangle = \left\langle u, \left( \phi \frac{1}{|\phi'|} \circ \phi^{-1} \right) \right\rangle,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the duality bracket between \( \mathcal{D}(X) \) and \( \mathcal{D}(X) \) (cf. [12] Section 6.1). If additionally \( w : X \rightarrow C \) is smooth we define the weighted composition operator \( C_{w,\phi} \) with weight \( w \) and symbol \( \phi \) as \( C_{w,\phi} := M_{w} \circ C_{\phi} \), where \( M_{w} \) denotes the multiplication operator by \( w \) on \( \mathcal{D}(X) \), i.e. \( \langle M_{w} u, \phi \rangle = \langle u, w \phi \rangle, u \in \mathcal{D}(X), \phi \in \mathcal{D}(X) \). Hence, for \( m \in \mathbb{N}_0 \) and \( u \in \mathcal{D}(X), \phi \in \mathcal{D}(X) \)

\[
\left\langle C_{w,\phi}^{m} u, \phi \right\rangle = \left\langle u, \left( \frac{\prod_{r=0}^{m-1} w(\phi^r)}{|(\phi^m)|} \right) \circ (\phi^{-1}) \right\rangle
\]

\[
= \left\langle u, \left( \prod_{r=0}^{m-1} \frac{w(\phi^r)}{|\phi^r \circ \phi|} \right) \circ (\phi^{-1}) \right\rangle.
\]

We begin this section with a trivial but important remark.
Remark 4.1. Let $\phi : X \to X$ be a diffeomorphism and let $w : X \to \mathbb{C}$ be smooth. Then
\[
\forall u \in \mathcal{D}(X) : \text{supp } (C_{w,\phi} u) \subseteq \phi \left( \text{supp } u \right).
\]
In particular, for every distribution with compact support $u$ it follows that $C_{w,\phi} u$ has again compact support, i.e. $C_{w,\phi} \left( \mathcal{E}'(X) \right) \subseteq \mathcal{E}'(X)$. Now, we fix $u \in \mathcal{D}(X)$ and let $\varphi \in \mathcal{D}(X)$ be such that $\varphi = 1$ in a neighborhood of $\phi(\text{supp } u)$. Denoting the duality bracket between $\mathcal{D}'(X)$ and $\mathcal{D}(X)$ with $(\cdot, \cdot)_{\mathcal{D}'(X), \mathcal{D}(X)}$ and between $\mathcal{E}'(X)$ and $\mathcal{E}(X)$ with $(\cdot, \cdot)_{\mathcal{E}'(X), \mathcal{E}(X)}$ for a moment, for $f \in \mathcal{E}'(X)$ we have
\[
(C_{w,\phi} u, f)_{\mathcal{E}'(X), \mathcal{E}(X)} = (C_{w,\varphi} u, \varphi f)_{\mathcal{D}'(X), \mathcal{D}(X)} = \left< u, \left( \frac{w}{|\varphi'|} f \circ \varphi^{-1} \right) \circ \varphi^{-1} \right>_{\mathcal{D}'(X), \mathcal{D}(X)},
\]
where we have used $\varphi \circ \varphi^{-1} = 1$ in a neighborhood of $\text{supp } u$ in the last equality. Thus, for the transpose of the restriction of $C_{w,\varphi}$ to $\mathcal{E}'(X)$ we have
\[
(C_{w,\varphi} f)_{\mathcal{E}'(X), \mathcal{E}(X)} = \left< u, \left( \frac{w}{|\varphi'|} f \right) \circ \varphi^{-1} \right>_{\mathcal{E}'(X), \mathcal{E}(X)}.
\]

Proposition 4.2. Let $\phi : X \to X$ be a diffeomorphism and $w : X \to \mathbb{C}$ be smooth such that $(x \in X; w(x) \neq 0)$ is dense in $X$ and such that $C_{w,\phi}$ is mean ergodic on $\mathcal{D}'(X)$. Then, $C_{w,\phi} \mathcal{E}'(X)$ is mean ergodic on $\mathcal{E}'(X)$, where the latter is equipped with the strong dual topology $\beta(\mathcal{E}'(X), \mathcal{E}(X))$.

Note that $\mathcal{D}'(X)$ as well as $\mathcal{E}'(X)$ are Montel spaces, so by Theorem 2.3, mean ergodicity and uniform mean ergodicity of operators on these spaces are equivalent.

Proof. Since $C_{w,\phi}$ is mean ergodic on $\mathcal{D}'(X)$, by Theorem 2.3(b), it is in particular topologizable. Thus, by [19 Corollary 2.10], $\phi$ has stable orbits. Therefore, for fixed $u \in \mathcal{E}'(X)$ and $K := \text{supp } u$ there is a compact $L \subseteq X$ such that $\phi^n(K) \subseteq L$ for each $n \in \mathbb{N}_0$. In particular, $\text{supp } C_{w,\phi} u \subseteq L$ for every $n \in \mathbb{N}_0$, i.e. $C_{w,\phi} u \in \mathcal{E}'(L)$ for all $n \in \mathbb{N}$. Since $\mathcal{E}'(L)$ is a closed subspace of $\mathcal{D}'(X)$ and $(C_{w,\phi} u)^n_{n\in\mathbb{N}}$ converges in $\mathcal{D}'(X)$ by hypothesis, we conclude that $(C_{w,\phi} u)^n_{n\in\mathbb{N}}$ converges in $\mathcal{E}'(L)$ with respect to the topology $\beta(\mathcal{D}'(X), \mathcal{D}(X))$. However, by [19 Theorem 4.2.1], $\beta(\mathcal{D}'(X), \mathcal{D}(X))$ and $\beta(\mathcal{E}'(X), \mathcal{E}(X))$ induce the same topology on $\mathcal{E}'(L)$ so that $(C_{w,\phi} u)^n_{n\in\mathbb{N}}$ converges in $\mathcal{E}'(X)$. Since $u \in \mathcal{E}'(X)$ was chosen arbitrarily, the claim follows.

Proposition 4.3. Let $\phi : X \to X$ be a diffeomorphism and $w : X \to \mathbb{C}$ be smooth such that $(x \in X; w(x) \neq 0)$ is dense in $X$ and such that $C_{w,\phi}$ is (uniformly) mean ergodic on $\mathcal{D}'(X)$. Then, the weighted composition operator with weight $w(|\phi^{-1}|)$ and symbol $\phi^{-1}$ on $\mathcal{E}'(X)$, $C_{w(|\phi^{-1}|)}$, is (uniformly) mean ergodic.

Proof. By Proposition 4.2 $C_{w,\phi}$ is (uniformly) mean ergodic on $\mathcal{E}'(X)$. Since $\mathcal{E}'(X)$ is a Fréchet-Montel space, the (uniform) mean ergodicity of $C_{w(|\phi^{-1}|)}$ follows from Theorem 2.3 and Remark 4.1.

Theorem 4.4. Let $\phi : X \to X$ be a diffeomorphism and $w : X \to \mathbb{C}$ be smooth such that $(x \in X; w(x) \neq 0)$ is dense in $X$ and such that $C_{w,\phi}$ is (uniformly) mean ergodic on $\mathcal{D}'(X)$. Then, $\phi$ and $\phi^{-1}$ have stable orbits.

Proof. Since $C_{w,\phi}$ is mean ergodic, $C_{w,\phi}$ is topologizable so that $\phi$ has stable orbits by [19 Corollary 2.10]. Moreover, by Proposition 4.2 $C_{w(|\phi^{-1}|)}$ is mean ergodic on $\mathcal{E}'(X)$. Thus, $\phi^{-1}$ has stable orbits by Theorem 4.3 together with the fact that with $(x \in X; w(x) \neq 0)$ being dense in $X$ the same holds for $(x \in X; w(x)|(|\phi^{-1}|)(x)| \neq 0)$. 


Example 4.5. Let \( \phi : \mathbb{R} \to \mathbb{R} \) be the diffeomorphism defined by \( \phi(x) = x/2 \). Then the operator \( C_\phi \) is topologizable but it is neither mean ergodic nor power bounded on \( \mathcal{D}'(\mathbb{R}) \).

Indeed, clearly \( \phi \) has stable orbits so that by [19, Corollary 2.10] \( C_\phi \) is topologizable. On the other hand, the inverse \( \phi^{-1}(x) = 2x \) of \( \phi \) does not have stable orbits. By Theorem [4.4] the operator \( C_\phi \) is not mean ergodic. Since \( \mathcal{D}'(\mathbb{R}) \) is Montel, in particular semi-reflexive, an application of Corollary [2.4] shows that \( C_\phi \) is not power bounded.

The symbol of the operator from the previous example is a real analytic diffeomorphism that allows us to construct an operator \( C_\phi \) which is topologizable and not mean ergodic on \( \mathcal{D}'(\mathbb{R}) \). However, it is not clear whether there is a mean ergodic composition operator on \( \mathcal{D}'(\mathbb{R}) \) which is not power bounded. In fact, Theorem [4.9] below shows that if such a composition operator exists on \( \mathcal{D}'(\mathbb{R}) \) then - recalling that the derivative of the symbol may have no zeros in order to induce a composition operator on \( \mathcal{D}'(\mathbb{R}) \) - its symbol cannot be real analytic.

The rest of this section is devoted to prove Theorem [4.9] which characterizes the real analytic diffeomorphisms \( \phi \) on an open interval \( X \subseteq \mathbb{R} \) for which \( C_\phi \) is mean ergodic on \( \mathcal{D}'(X) \) and by which this holds precisely when \( C_\phi^2 = id_{\mathcal{D}'(X)} \).

We denote by \( F_\phi \) the set of fixed points of the mapping \( \phi : X \to X \).

Lemma 4.6. Let \( \phi : \mathbb{R} \to \mathbb{R} \) be a diffeomorphism such that \( \phi \) and \( \phi^{-1} \) have stable orbits. Assume \( \phi' > 0 \), then for each \( x \notin F_\phi \) there exist \( x_1, x_2 \in F_\phi \) such that \( x_1 < x < x_2 \) and

\[
[|x_1, x_2| \cap F_\phi = \emptyset].
\]

Proof. Firstly, if there was no fixed point we had \( \phi(y) < y \) or \( \phi(y) > y \) for all \( y \in \mathbb{R} \). In any case, since \( \phi' > 0 \), we obtain that \( \phi \) cannot have stable orbits. This is a contradiction and therefore \( F_\phi \neq \emptyset \).

Now, we argue by contradiction. Without loss of generality we assume that there is \( x_1 < x \) such that \( |x_1, +\infty| \cap F_\phi = \emptyset \). If \( \phi(y) > y \) for all \( y > x_1 \), then \( \phi \) cannot have stable orbits.

Finally, if \( \phi(y) < y \) for all \( y > x_1 \), we have that \( \phi^{-1}(y) > y \) for all \( y > x_1 \) thus \( \phi^{-1} \) cannot have stable orbits. In any case we obtain a contradiction.

This completes the proof because \( x \) cannot be an accumulation point of \( F_\phi \) since the set of fixed points is closed. \( \square \)

Remark 4.7. Let a smooth function \( \phi : X \to X \) be given. For the special weight \( w = |\phi'| \), a straightforward calculation gives

\[
\forall f \in \mathcal{E}(X), n \in \mathbb{N}_0 : C_{|\phi'|, \phi}^n f = |(\phi^n)'| f (\phi^n), \text{ sign } \left((\phi^n)'\right) = (\text{sign } (\phi'))^n.
\]

Denoting a primitive function of \( f \in \mathcal{E}(X) \) by \( F \), we have for \( r, n \in \mathbb{N}_0 \)

\[
\left(C_{|\phi'|, \phi}^n f\right)^{(r)} = \left((f \circ \phi^n) \text{ sign } (\phi')^n (\phi^n)'\right)^{(r)} = \left((\text{sign } (\phi'))^n (F \circ \phi^n)'\right)^{(r)} = \left(C_{\text{sign } (\phi'), \phi}^n F\right)^{(r+1)}.
\]

Clearly, from the above equality we derive that \( C_{|\phi'|, \phi} \) is (uniformly) mean ergodic in \( \mathcal{E}(X) \) whenever \( C_{\text{sign } (\phi'), \phi} \) is. On the other hand, suppose that \( C_{\text{sign } (\phi'), \phi} \) is (uniformly) mean ergodic in \( \mathcal{E}(X) \). Then, \( \phi \) has stable orbits (see Theorem [4.9]) so that the sequence \( \left(C_{|\phi'|, \phi}^n F\right)_{n \in \mathbb{N}_0} \) is bounded with respect to the compact open topology for every \( F \in \mathcal{E}(X) \). Additionally, by the fact that \( \phi \) has stable orbits, it also holds true that \( \left(C_{\text{sign } (\phi'), \phi}^n F\right)_{n \in \mathbb{N}_0} \) tends to 0 with respect to the compact open topology. Thus, by the
above equation $C_{\text{sign}(\phi^t), \phi}$ is Cesàro bounded and satisfies that $\lim_{n \to \infty} \frac{1}{n} C_{\text{sign}(\phi^t), \phi}^n = 0$
pointwise in $\mathcal{E}(X)$. Theorem 2.3 (b) yields that $C_{\text{sign}(\phi^t), \phi}$ is (uniformly) mean ergodic on $\mathcal{E}(X)$ if the same is true for $C_{\phi^t, \phi}$.

**Proposition 4.8.** Let $\phi : \mathbb{R} \to \mathbb{R}$ be a real analytic diffeomorphism. Assume $C_{\phi}$ is mean ergodic on $\mathcal{D}'(\mathbb{R})$. Then, $\phi(\phi(x)) = \phi^2(x) = x$, $x \in \mathbb{R}$, and consequently $C_{\phi, \phi}^2(u) = u$, $u \in \mathcal{D}'(\mathbb{R})$.

**Proof.** We proceed by contradiction, assuming there is $y \in \mathbb{R}$ such that $\phi^{-2}(y) \neq y$. By Theorem 4.4 for each $K \subset \mathbb{R}$ compact set there is $L \subset \mathbb{R}$ compact such that

$$\bigcup_{n=0}^{\infty} (\phi^2)^n(K) \subseteq \bigcup_{n=0}^{\infty} \phi^n(K) \subseteq L.$$ 

Thus we obtain that $\phi^2$ has stable orbits and we can apply the same argument to obtain that $\phi^{-2}$ has stable orbits. Since $\phi$ is a diffeomorphism we have $\phi'(x) \neq 0$ for every $x \in \mathbb{R}$ so that $\phi' > 0$ or $\phi' < 0$. Thus, by the chain rule $(\phi^2)' > 0$, so that by Lemma 3.5 there exist $x_1, x_2 \in F_{\phi^2}$ such that $x_1 < y < x_2$ and

$$y = \lim_{n \to \infty} \frac{1}{n} \phi^n(x) = \lim_{n \to \infty} \frac{1}{n} \phi^{2n}(x) = 0,$$

On the other hand, by Proposition 4.3 and Remark 4.7 we obtain that the weighted composition operator $C_{\text{sign}(\phi^{-1}), \phi^{-1}} : \mathcal{E}(\mathbb{R}) \to \mathcal{E}(\mathbb{R})$ is mean ergodic. Since $\mathcal{E}(\mathbb{R})$ is reflexive we can apply Theorem 4.3. Observe that Proposition 3.1 implies that

$$\lim_{n \to \infty} \frac{1}{n} \text{sign}(\phi^{-1})^n \cdot (\phi^{-1})^n(x) = \lim_{n \to \infty} \frac{1}{n} \phi^{-1}(x) = 0,$$

where we have taken $s \geq 1$, $h = 1$ and an arbitrary $K = \{x\} \subset \mathbb{R}$ and where we have used that $B_{s, h}(y_1, \ldots, y_s) = y_s$. Denoting $\psi = \phi^{-2}$ we obtain

$$\lim_{n \to \infty} \left| \frac{1}{2n} (\psi^{n}(x))' \right| = 0,$$

for an arbitrary $x \in \mathbb{R}$.

**Auxiliary Claim 1:** $\psi'(x_1) = 1$ or $\psi'(x_2) = 1$.

Indeed, by the chain rule we have that $(\psi^n)'(x) = (\psi'(x))^n$ for any fixed point $x$. Then by (11) we have

$$0 < |\psi(x)| = 1 \quad \text{and} \quad 0 < |\psi(x)| = 1.$$

Suppose that $0 < \psi(x_1) < 1$ and $0 < \psi(x_2) < 1$, then there are $y_1, y_2 \in [x_1, x_2]$ such that $\psi(y_1) < y_1$ and $\psi(y_2) > y_2$. Thus by Bolzano’s Theorem there is a fixed point between $y_1$ and $y_2$ which contradicts (10) and proves the Auxiliary Claim 1.

Without loss of generality we can assume $\psi'(x_1) = 1$.

By induction one proves $(\psi^n)'(x_1) = n \cdot \psi'(x_1)$ for all $n \in \mathbb{N}$. Then for $s = 2$ and $x_1$ we have by (11)

$$0 = \lim_{n \to \infty} \left| \frac{1}{2n} (\psi^n)(x_1) \right| = \lim_{n \to \infty} \left| \frac{n}{2n} \psi'(x_1) \right| = \frac{1}{2} \left| \psi'(x_1) \right|.$$ 

**Auxiliary Claim 2:** Fix $s \geq 3$. Assume that $\psi'(x_1) = 1$ and $(\psi^n)^{(j)}(x_1) = 0$ hold for all $n \in \mathbb{N}$ and all $2 \leq j \leq s - 1$. Then $(\psi^n)^{(s)}(x_1) = n \cdot \psi^{(s)}(x_1)$ for all $n \in \mathbb{N}$.
Indeed, fixing \( n \geq 2 \) and using the original version of Faà di Bruno’s Formula for \((ψ ◦ ψ^{n−1})^{(j)}\) (see (14)) we have

\[
(ψ ◦ ψ^{n−1})^{(j)}(x_1) = \sum_{b_1+b_2+⋯+s b = s} \frac{s!}{b_1! b_2!⋯b_s!} ψ^{(b_1+⋯+b_s)}(ψ^{−1}(x_1)) × \times \prod_{j=1}^{s} \left( \left( ψ^{−1}(j) (x_1) \right)^{b_j} \right).
\]

By the assumptions, given a summand if there is \( 2 \leq j \leq s − 1 \) with \( b_j ≠ 0 \) then, the summand is automatically 0. Therefore the non-zero summands must satisfy \( b_1 + s b_s = s \). By this together with the fact that \( x_1 \) is a fixed point we obtain

\[
(ψ^{−1})^{(j)}(x_1) = ψ^{(j)}(x_1) \left( (ψ^{−1})^{(j)} (x_1) \right)^\prime + ψ^{(j)}(x_1) (ψ^{−1})^{(j)}(x_1)
\]

because \( (ψ^{−1})^{(j)}(x_1) = (ψ^{(j)}(x_1))^{n−1} = 1 \). Auxiliary Claim 2 is now obtained by applying this argument recursively to \((ψ^{−1})^{(j)}(x_1)\).

Under the assumptions of Auxiliary Claim 2 and using (11) on \( x_1 \) we conclude that

\[
0 = \lim_{n→∞} \frac{n}{2n} |ψ^{(j)}(x_1)| = \frac{1}{2} |ψ^{(j)}(x_1)|,
\]

for each \( s ≥ 2 \). To summarize, \( x_1 \) satisfies \( (ψ^{(j)}(x_1)) = 1 \) and \( ψ^{(j)}(x_1) = 0, s ≥ 2 \). Because \( ψ \) is real analytic as the inverse of a real analytic function, it thus follows \( ψ(x) = x \) for every \( x ∈ \mathbb{R} \) which contradicts (10).

Our next result should be compared to [9, Theorem 3.8].

**Theorem 4.9.** Let \( φ : X → X \) be a real analytic diffeomorphism on the non-empty, open interval \( X ⊆ \mathbb{R} \). Then, the following are equivalent.

(i) \( C_φ : \mathcal{D}'(X) → \mathcal{D}'(X) \) is power bounded.

(ii) \( C_φ : \mathcal{D}'(X) → \mathcal{D}'(X) \) is mean ergodic

(iii) \( C_φ : \mathcal{D}'(X) → \mathcal{D}'(X) \) is uniformly mean ergodic.

(iv) \( φ^2(x) = x \) for each \( x ∈ X \).

(v) \( C_φ \) is periodic with period 2.

**Proof.** Let \( E \) and \( F \) be two lCHs and let \( T \) as well as \( S \) be continuous linear operators on \( E \) and \( F \), respectively. Moreover, let \( R : E → F \) be a continuous linear bijection such that \( R^{−1} \) is continuous, too, such that \( R ∘ T ∘ R^{−1} = S \). It is straightforward to show that \( T \) is power bounded or (uniformly) mean ergodic and so if only the same applies to \( S \). Moreover, there is a real analytic diffeomorphism \( χ : X → \mathbb{R} \) and \( C_χ : \mathcal{D}'(\mathbb{R}) → \mathcal{D}'(X) \), \( u → u ∘ χ \) is a continuous linear bijection with \( C_χ^{−1} = C_χ^{−1} \), where for \( u ∈ \mathcal{D}'(\mathbb{R}) \) and \( v ∈ \mathcal{D}(X) \) as usual \( \langle C_χ u, φ \rangle = \langle u, \left( \frac{dχ}{dx} \right)^{−1} \circ φ \rangle \). Clearly, \( C_χ^{−1} ∘ C_φ ∘ C_χ = C_χ ∘ χ^2 \), and the real analytic diffeomorphism \( χ ∘ φ ∘ χ^{−1} \) on \( \mathbb{R} \) satisfies \( (χ ∘ φ ∘ χ^{−1})^2 (x) = x \) for every \( x ∈ \mathbb{R} \) if and only if \( φ^2(x) = x \) for each \( x ∈ X \). Therefore, without loss of generality we can assume \( X = \mathbb{R} \).

Next, since \( \mathcal{D}'(\mathbb{R}) \) is a Montel space it is in particular a semi-reflexive space. Then by Corollary 2.14 every power bounded operator \( T \) on \( \mathcal{D}'(\mathbb{R}) \) is mean ergodic. An application of Theorem 2.25 (a) gives that \( T \) is also uniformly mean ergodic. Hence, (ii) follows from (i), and (ii) and (iii) are equivalent. By Proposition 1.13 (ii) implies (iv). Finally, (iv) trivially implies (v) which in turn immediately implies (i).

**Corollary 4.10.** Let \( φ : X → X \) be a real analytic diffeomorphism on an open interval \( X ⊆ \mathbb{R} \) such that there is \( p ∈ \mathbb{N} \) with \( φ^p(x) = x \) for all \( x ∈ X \). Then \( φ^p(x) = x \) for each \( x ∈ X \).
Contrary to the case of increasing real analytic diffeomorphisms, if we assume $\phi' > 0$ the unique possibility is the identity map. However, the case $\phi' < 0$ is richer. The following example should be compared to [9] Proposition 3.6 and Example 1.

**Example 4.11.** Let $\phi: X \to X$ be a real analytic diffeomorphism on an open interval $X \subseteq \mathbb{R}$ such that $\phi^2(x) = x$ for each $x \in X$.

Assuming that our real analytic diffeomorphism $\phi: X \to X$ is increasing, it follows that $\phi(x) = x$, for all $x \in X$. Indeed, if there is $y \neq \phi(y)$ we may assume without loss of generality that $y < \phi(y)$ (for otherwise, we consider $\phi^{-1}$). Now, since $\phi$ is increasing we obtain that $y < \phi(y) < \phi^2(y)$ which contradicts the assumption.

Contrary to the case of increasing real analytic diffeomorphisms, if we assume $\phi' < 0$, apart from the obvious example $\phi(x) = -(x + c)$ where $c \in \mathbb{R}$, we obtain a whole class of examples as in [9] Example 1. For this, let $f: \mathbb{R} \to \mathbb{R}$ be an even real analytic function such that $|f'(x)| \leq a < 1$ for all $x \in \mathbb{R}$. Then, except for referring to the Implicit Function Theorem in the real analytic class, see e.g. [20] Theorem 2.3.5, a verbatim repetition of the arguments presented in [9] Example 1, the equation

$$x + y = f(x - y)$$

defines a decreasing real analytic symbol $y = \phi(x)$, $x \in \mathbb{R}$, with $\phi^2(x) = x$. Thus, $\phi$ is bijective, hence a real analytic diffeomorphism.

For a concrete example, we consider the function $f(x) = \sqrt{\frac{x^2}{2}} + 1$ which yields the real analytic decreasing diffeomorphism

$$\phi: \mathbb{R} \to \mathbb{R}, x \mapsto -3x + \sqrt{8x^2 + 2}.$$

It is easy to see that $\phi = \phi^{-1}$.

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