Odd Invariant Semidensity 
and 
Divergence–like Operators on an Odd Symplectic Superspace*

O.M. Khudaverdian

Laboratory of Computing Technique and Automation. Joint Institute for Nuclear Research 
Dubna, Moscow Region 141980, Russia **

e-mails: "khudian@main1.jinr.dubna.su", "khudian@sc2a.unige.ch"

The divergence–like operator on an odd symplectic superspace which acts invariantly on a 
specially chosen odd vector field is considered. This operator is used to construct an odd 
invariant semidensity in a geometrically clear way. The formula for this semidensity is 
similar to the formula of the mean curvature of hypersurfaces in Euclidean space.

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** on leave of absence from Department of Theoretical Physics of Yerevan State University 
375049 Yerevan, Armenia
In this paper we construct a differential first order divergence-like operator on a superspace endowed with an odd symplectic structure. This operator is applied for constructing invariant differential objects in this superspace. In particular an odd semidensity invariant under the transformations preserving the odd symplectic structure of the superspace and the volume form is constructed in a geometrically clear way.

The odd symplectic structure plays essential role in Lagrangian formulation of the BRST formalism (Batalin-Vilkovisky formalism) [1]. In supermathematics it is a natural counterpart of even symplectic structure [2,3]. On the other hand it has some ”odd” features which have no analogues in usual mathematics. Canonical transformations preserving odd symplectic structure (non-degenerate odd Poisson bracket) do not preserve any volume form. This fact, indeed, is the reason why in the case of odd symplectic structure some invariant geometrical objects have no natural homologues as it is for the even symplectic structure, for which the superconstructions are rather straightforward generalizations of those for the symplectic structures in usual spaces. We consider some examples.

In order to construct geometrical integration objects one needs to consider a pair, the volume form and the odd canonical structure. This pair is in fact the geometrical background for the formulation of Batalin-Vilkovisky formalism (See [4,5,6]). The so called $\Delta$ operator which plays essential role in this formalism can be defined in the following way: its action on the function $f$ is equal to the divergence with respect to the given volume form (defined by the action of the theory) of the Hamiltonian vector field, corresponding to the function $f$. It is a second order differential operator. In the case of even symplectic structure there exists the volume form which naturally corresponds to the canonical structure, and $\Delta$ operator is evidently equal to zero (Liouville Theorem). Even if the volume form is arbitrary, one arrives at a first order differential operator [4].

The second example is the invariant volume form (density) which can be defined on the Lagrangian supersurfaces embedded in the superspace endowed with an odd symplectic structure and a volume form. It is nothing but the integrand for the partition function in the space of fields and antifields in the BV formalism [5,6].

We focus on the third example, on the problem of finding an analogue of Poincare-Cartan integral invariants for the odd symplectic structure. In usual mathematics to the Poincare-Cartan integral invariant (invariant volume form on the embedded surface) there corresponds the wedged power of a differential two-form which defines the symplectic structure. In case of even symplectic structure in spite of the fact that differential form has nothing in common with invariant integration objects, the superdeterminant of the two-form induced on the embedded supersurface from the two-form which defines the symplectic structure leads us to the Poincare–Cartan invariant [7]. For the odd symplectic structure the situation is essentially different. In [8] there was considered the problem of constructing of the invariant densities for the superspace endowed with an odd symplectic structure.

In the case of symplectic structure in usual mathematics as well as in the case of even symplectic structure invariant densities are exhausted by densities $^* \, \, \!$ of the rank $k = 1$ which correspond to Poincare–Cartan integral invariants. This is not the case for

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$^*$ Density is the object which defines a volume form on embedded (super)surfaces.—

If $z = z(\zeta)$ is the local parametrization of a (super)surface then a density $A$ is a function
the odd symplectic structure. One can show that on \((p,p)\)-dimensional non-degenerate supersurfaces embedded in a superspace \(E^{n,n}\) which is endowed with a volume form and an odd symplectic structure there are no invariant densities of the rank \(k = 1\) (except of the volume form itself), and in the class of densities of the rank \(k = 2\) and of the weight \(\sigma\) (i.e. which are multiplied by the \(\sigma\)-th power of the superdeterminant of the reparametrization) there exists a unique (up to multiplication by a constant) semidensity \((\sigma = \frac{1}{2})\) in the case of \(p = n - 1\) [8]. In fact, in [8] this semidensity was constructed in a non-explicit way in terms of dual densities: If \((n - 1,n - 1)\) supersurface \(M^{n-1,n-1}\) is given by the equations: \(f = 0, \varphi = 0\) where \(f\) is even function and \(\varphi\) an odd function then to this semidensity there corresponds the function

\[
\tilde{A} \Big|_{f=\varphi=0} = \frac{1}{\sqrt{\{f,\varphi\}}} \left( \Delta f - \frac{1}{2\{f,\varphi\}} \Delta \varphi - \frac{\{f,\{f,\varphi\}\}}{\{f,\varphi\}} - \frac{\{f,\varphi\}}{2\{f,\varphi\}}^2 \{\varphi,\{f,\varphi\}\} \right). \quad (1.1)
\]

which depends on the second derivatives \((k = 2)\), is invariant under the transformations preserving the odd symplectic structure and the volume form, and is multiplied by the square root of the corresponding Berezinian (superdeterminant) under the transformation \(f \rightarrow af + \alpha \varphi, \varphi \rightarrow \beta f + b \varphi\) (which does not change the supersurface \(M\)). The semidensity (1.1) takes odd values. It is an exotic analogue of Poincare–Cartan invariant: \(\tilde{A}^2 = 0\), so it cannot be integrated over supersurfaces.

To clarify the geometrical meaning of density (1.1), in this paper a special geometrical object for odd symplectic superspace is considered.

As it was mentioned above, to a symplectic structure in usual space there corresponds a volume form (Liouville form):

\[
dv = \rho(x)dx^1...dx^{2n} = \sqrt{\text{det}(\Omega_{ik})}dx^1...dx^{2n} \quad (1.2)
\]

where \(\Omega = \Omega_{ik}dx^i \wedge dx^k\) is the closed non-degenerated two-form which defines the symplectic structure.

The volume form (1.2) is preserved under canonical transformations (i.e., the transformations preserving the two-form \(\Omega\)). If \(X = X^i \frac{\partial}{\partial x^i}\) is an arbitrary vector field one can consider its divergence:

\[
\text{div}X = \frac{dL}{dv} = \frac{\partial X^i}{\partial x^i} + X^i \frac{\partial \log \rho}{\partial x^i}. \quad (1.3)
\]

In a symplectic space the canonical transformations preserve not only the volume form (1.2) but also its projection on an arbitrary symplectic plane. Moreover, if \(L(z)\) is a of \(z(\zeta), \frac{\partial z}{\partial \zeta}, \frac{\partial^2 z}{\partial \zeta \partial \zeta}, \ldots, A = A \left( z(\zeta), \frac{\partial z}{\partial \zeta}, \frac{\partial^2 z}{\partial \zeta \partial \zeta}, \ldots, \frac{\partial^k z}{\partial \zeta^k} \right)\) subject to the condition that under reparametrization \(\zeta \rightarrow \zeta(\tilde{\zeta})\), \(A \rightarrow A \cdot \text{sdet} \left( \frac{\partial \zeta}{\partial \tilde{\zeta}} \right)\). \(A\) is a density of the rank \(k\) if it depends on the tangent vectors of the order \(\leq k\) (i.e. on the derivatives of the order \(\leq k\)). In usual mathematics densities are the natural generalization of differential forms (if \(k = 1\)). In supermathematics even in the case of \(k = 1\) this object is of more importance, since differential forms in supermathematics are no more integration objects [9,10,11].
projector-valued function such that $\text{Im} \hat{L}(z)$ is a symplectic subspace (plane) in $T_zE$ then it is easy to see that to $\hat{L}(z)$ one can associate divergence-like invariant operator whose action on an arbitrary vector field $X$ is given by the expression

$$\partial(\hat{L}, X) = \left( \frac{\partial X^k}{\partial x^i} + \frac{1}{2} X^m \frac{\partial \Omega_{ip}}{\partial x^m} \Omega^{pk} \right) L_i^k. \quad (1.4)$$

(Compare with (1.3) in case of $\hat{L} = \text{id}$).

The formulae (1.2)–(1.4) have the straightforward generalization to the case of even symplectic structure in superspace (by changing determinant to superdeterminant and adding the powers of $(-1)$ wherever necessary), but it is not the case for the odd symplectic structure.

Nevertheless, it turns out that the analogue of the formula (1.4) can be considered for odd symplectic structure in a case where $\hat{L}$ is a projector on $(1.1)$-dimensional symplectic subspace and $X$ is the odd vector field which belongs to this subspace and is symplectoorthogonal to itself. In the next two sections we perform the corresponding constructions which are essentially founded on the following remark. Let $E^{1,1}$ be $(1.1)$-dimensional odd symplectic superspace and $(x, \theta)$ be Darboux coordinates on it: $\{x, \theta\} = 1$, $\{x, x\} = 0$ where $\{, \}$ is the odd Poisson bracket (Buttin bracket) corresponding to this symplectic structure. Let $\Psi$ be an odd vector field in this superspace which is equal to $\Psi(x, \theta) \partial_\theta$ in these Darboux coordinates. Then this vector field has the same form and its divergence $(\partial \Psi(x, \theta)/\partial \theta)$ remains the same in arbitrary Darboux coordinates. (See Example 2).

In the 4-th section we consider a $(n - 1, n - 1)$-dimensional supersurface embedded in an odd symplectic superspace $E^{n,n}$ endowed with a volume form. The differential operator described above can be naturally applied to the odd vector field which is defined only in the points of this supersurface and is symplectoorthogonal to itself and to this supersurface. In spite of the fact that this field is not defined on the whole superspace one can define the invariant ”truncated divergence” of this vector field. The analogue of this construction for usual symplectic structure is trivial. On the other hand this construction can be considered as an analogue of the corresponding operator acting on the vector field which is defined in a Euclidean (Riemannian) space on the points of embedded hypersurface and which takes values in the normal bundle to this surface. But the essential difference is that the group of transformations which preserve metrics in Euclidean (Riemannian) space is exhausted by linear transformations (the linear part of transformation defines uniquely all higher terms) and this is not true for symplectic case where the group of canonical transformations is infinite-dimensional.

In the 5-th Section we apply our geometrical construction to obtain the formula for odd semidensity (1.1) in a geometrically clear way: it turns out that on a $(n - 1, n - 1)$-dimensional supersurface embedded in a $(n,n)$-dimensional odd symplectic superspace endowed with a volume form one can define in a natural way the odd semidensity whose values are odd vectors, symplectoorthogonal to this supersurface, and the ”truncated divergence” of this vector-valued semidensity is invariant semidensity (1.1). It has to be noted that our formula for this semidensity is very similar to the formula for the density corresponding to the mean curvature of the 1-codimensional surface in the Euclidean space.
Section 2. Odd symplectic superspace.

Let $E^{n,n}$ be a superspace with coordinates $(z^A = x^1, \ldots, x^n, \theta^1, \ldots, \theta^n)$. We say that this superspace is odd symplectic superspace if it is endowed with odd symplectic structure, i.e., if an odd closed non-degenerated 2-form

$$\Omega = \Omega_{AB}(z)dz^Adz^B \quad (p(\Omega) = 1, \ d\Omega = 0)$$

(2.1)

is defined on it [2,3] ($p$ is a parity: $p(x^i) = 0, p(\theta^i) = 1$). To the differential form (2.1) on the superspace $E^{n,n}$ one can relate a function which for every point $^*$ defines the following skewsymmetric (in a supersense) odd bilinear form on tangent vectors:

$$p(\Omega(X,Y)) = 1 + p(X) + p(Y),$$

$$\Omega(X,Y) = -\Omega(Y,X)(-1)^{p(X)p(Y)};$$

$$\Omega(\lambda Y + \mu Z, X) = \lambda \Omega(Y, X) + \mu \Omega(Z, X).$$

(2.2)

In the coordinates:

$$\Omega_{AB} = -\Omega_{BA}(-1)^{p(A)p(B)} = \Omega \left( \frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B} \right), (p(\Omega_{AB}) = 1 + p(A) + p(B)).$$

(2.3)

Here $X, Y, Z$ are the vector fields $X^A(z)\frac{\partial}{\partial z^A}, Y^A(z)\frac{\partial}{\partial z^A}, Z^A(z)\frac{\partial}{\partial z^A}$ (the left derivations of functions on $E^{n,n}$).

(Differential form is usually considered as an element of an algebra generated by $z^A$ and $dz^A$, where the parity of $dz^A$ is opposite to the parity of $z^A$. In this case $\Omega_{AB} = \Omega_{BA}(-1)^{(p(A)+1)(p(B)+1)}$ instead of (2.3). The slight difference is eliminated by the transformation $\Omega_{AB} \rightarrow \Omega_{AB}(-1)^B$.

From (2.2, 2.3) it follows that

$$\Omega \left( X^A \frac{\partial}{\partial z^A}, Y^B \frac{\partial}{\partial z^B} \right) = X^A \Omega_{AB} Y^B (-1)^{p(Y)p(B)+p(Y)}.$$ 

(2.4)

In the same way as in the standard symplectic calculus one can relate to the odd symplectic structure (2.1) the odd Poisson bracket (Buttin bracket) [3]

$$\{f, g\} = \frac{\partial f}{\partial z^A}(-1)^{p(f)p(A)+p(A)}\Omega^{AB} \frac{\partial g}{\partial z^B}$$

(2.5)

where

$$\Omega^{AB} = -\Omega^{BA}(-1)^{(p(A)+1)(p(B)+1)} = \{z^A, z^B\}$$

* More precisely, a point of superspace $E^{n,n}$ is $\Lambda$-point— $2n$-plet $(a^1, \ldots, a^n, \alpha^1, \ldots, \alpha^n)$ where $(a^1, \ldots, a^n)$ are arbitrary even and $(\alpha^1, \ldots, \alpha^n)$ are arbitrary odd elements of an arbitrary Grassman algebra $\Lambda$. (We use the most general definition of superspace suggested by A.S. Schwarz as the functor on the category of Grassman algebras.)
is the inverse matrix to $\Omega_{AB}$:
\[ \Omega^{AC}\Omega_{CB} = \delta^A_B. \] (2.6)

To a function $f$ via (2.5) there corresponds the Hamiltonian vector field
\[ D_f = \{ f, z^A \} \frac{\partial}{\partial z^A} \quad \text{and} \quad D_f g = \{ f, g \}, \quad \Omega(D_f, D_g) = \{ f, g \}. \] (2.7)

The condition of the closedness of the form (2.1) leads to the Jacoby identities:
\[ \{ f, \{ g, h \} \} ((-1)^{(p(f)+1)(p(h)+1)} + \{ g, \{ h, f \} \} ((-1)^{(p(g)+1)(p(f)+1)} + \{ h, \{ f, g \} \} ((-1)^{(p(h)+1)(p(g)+1)} = 0. \] (2.8)

Using the analog of Darboux Theorem [13] one can consider the coordinates in which the symplectic structure (2.1) and the corresponding Buttin bracket have the canonical expressions. We call the coordinates $w^A = (x^1, \ldots, x^n, \theta^1, ..., \theta^n)$ Darboux coordinates if in these coordinates holds
\[ \Omega = I_{AB} dw^A dw^B : \Omega \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = 0, \Omega \left( \frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j} \right) = 0, \Omega \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial \theta^j} \right) = -\delta^i_j, \] (2.9)
respectively
\[ \{ f, g \} = \sum_{i=1}^n \left( \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial \theta^i} + (-1)^{p(f)} \frac{\partial f}{\partial \theta^i} \frac{\partial g}{\partial x^i} \right). \] (2.10)

Now on the odd symplectic superspace $E^{n,n}$ endowed with an odd symplectic structure (2.1) we consider the following geometrical constructions: Let $\Psi$ be an odd nondegenerated vector field symplectoorthogonal to itself:
\[ \Omega(\Psi, \Psi) = 0 \] (2.11)
where
\[ \Psi = \sum_{i=1}^n \left( \Psi^i \frac{\partial}{\partial \theta^i} + \Phi^i \frac{\partial}{\partial x^i} \right) \quad (p(\Psi^i) = 0, p(\Phi^i) = 1) \] (2.12)
and at least for some index $i_0$ the coefficient $\Psi^{i_0}$ is not nilpotent. (It is easy to see that this condition is invariant under the coordinate transformations.).

(For example, to the even function $f$ such that $\{ f, f \} = 0$ there correspond the Hamiltonian vector field $D_f$ defined by (2.7), subject to condition (2.11) and to condition (2.12) if the gradient of $f$ is not nilpotent.)

Let $\Pi(z)$ be a field of (1.1)-dimensional subspaces (planes) ($\Pi(z) \in TE^{n,n}_z$) which contain the vector field $\Psi(z)$ and the symplectic structure induced on these planes is not degenerate. It means that there exists an even vector field $H(z)$ such that
\[ \Psi(z), H(z) \in \Pi(z) \quad \text{and} \quad \Omega(H(z), \Psi(z)) = 1. \] (2.13)
To this field of planes $\Pi(z)$ there corresponds the symplectoorthogonal projector $\hat{\Pi}(z)$ of the vectors in the tangent space to these planes:

$$
\hat{\Pi} : T_z E \to \Pi(z), \quad \hat{\Pi} \big|_{\Pi}(z) = \text{id}, \quad \hat{\Pi}X = 0 \quad \text{if} \quad \Omega(X, \Pi) = 0.
$$

In the coordinates $\{z^A\}$ to the projector $\hat{\Pi}$ there corresponds the matrix-valued function $\Pi^B_A(z)$:

$$
\hat{\Pi} \left( \frac{\partial}{\partial z^A} \right) = \Pi^B_A \frac{\partial}{\partial z^B}, \quad \text{so} \quad \hat{\Pi}(X^A \frac{\partial}{\partial z^A}) = X^B \Pi^A_B \frac{\partial}{\partial z^A} \quad \text{and} \quad \Omega(X, \hat{\Pi}Y) = \Omega(\Pi X, Y).
$$

Later on we call $(\Pi(z), \Psi(z))$ or equivalently $(\hat{\Pi}(z), \Psi(z))$ an odd normal pair if $\Psi(z)$ and $\Pi(z)$ are defined by (2.11—2.15).

**Section 3. Special geometrical construction.**

In this section for odd normal pair $(\Pi(z), \Psi(z))$ in an odd symplectic superspace we construct a first-order divergence-like differential operator which transforms it to a function on this superspace.

Let in a superspace $E$, $X(z)$ and $\hat{L}(z)$ be a vector field and a linear operators field defined on $T_z E$ respectively. If $\{z^A\}$ are arbitrary coordinates then for the pair $(\hat{L}, X)$ one can consider the function which depends on the coordinate system $\{z^A\}$:

$$
\partial(\hat{L}, X)^{\{z\}} = \frac{\partial X^A(z)}{\partial z^B} L^B_A(z)(-1)^{p(X)p(B)+p(B)}.
$$

Expression (3.1) is invariant under linear transformations of the coordinates $\{z^A\}$. In the general case if $\{w^A\}$ and $\{z^A\}$ are two different coordinate systems on $E$ then for the pair $(\hat{L}, X)$ we consider

$$
\Gamma(\hat{L}, X)^{\{w\}}_{\{z\}} = \partial(\hat{L}, X)^{\{w\}} - \partial(\hat{L}, X)^{\{z\}}.
$$

From (3.1) and (3.2) it follows that

$$
\Gamma(\hat{L}, X)^{\{w\}}_{\{z\}} = X^Q(z)\Gamma^A_{QB}(z \{w\}, \{z\})L^B_A(z)(-1)^{p(B)}
$$

where

$$
\Gamma^A_{BC}(z \{w\}, \{z\}) = \frac{\partial^2 w^K(z) \partial z^A(w)}{\partial z^B \partial z^C \partial w^K}.
$$

(In (3.3) the components of $\hat{L}$ and $X$ are in the coordinates $\{z^A\}$.)

From definition (3.2) of the $\Gamma(\hat{L}, X)^{\{w\}}_{\{z\}}$ it follows that for three different coordinate systems $\{w^A\}$, $\{z^A\}$ and $\{u^A\}$

$$
\Gamma(\hat{L}, X)^{\{w\}}_{\{z\}} + \Gamma(\hat{L}, X)^{\{z\}}_{\{u\}} + \Gamma(\hat{L}, X)^{\{u\}}_{\{w\}} = 0.
$$
Let $\mathcal{F}$ be a class of some coordinate systems such that for a given pair $(\hat{L}, X)$
\[ \forall\{z\}, \{w\} \in \mathcal{F} \quad \Gamma(\hat{L}, X)_{\{w\}}^{\{z\}} = 0. \quad (3.5) \]

Then to the class $\mathcal{F}$ one can relate the first-order differential operator $D$:
\[ D(\hat{L}, X) = \partial(\hat{L}, X)_{\{z\}}^{\{w\}} + \circ[6 \Gamma(\hat{L}, X)_{\{w\}}^{\{z\}}] \quad (3.6) \]
where $\{z\}$ are arbitrary coordinates on $E$ and $\{w\}$ are arbitrary coordinates from the class $\mathcal{F}$. From (3.2, 3.5) it follows that the r.h.s. of (3.6) does not depend on the choice of these coordinates.

Before going to the considerations for an odd symplectic superspace we consider an example where we come to the standard definition of the divergence in superspace using (3.1—3.6).

**Example 1.** Let $E$ be a superspace with a volume form $dv$ which in coordinates $\{w^A_0\} = \{x^1, \ldots, x^n, \theta^1, \ldots, \theta^m\}$ on $E$ is equal to
\[ dv = dx^1 \ldots dx^ndx^1 \ldots d\theta^m. \quad (3.7) \]

We define $\mathcal{F}$ as a class of coordinate systems in which the volume form $dv$ is given by (3.7):
\[ \mathcal{F} = \{ \{w\}: Ber\left(\frac{\partial w}{\partial w_0}\right) = 1 \}. \quad (3.8) \]

($BerA$ is the superdeterminant of $A$.) It is easy to see that in this case for arbitrary coordinates $\{z^A\}$ if $\hat{L} = \text{id}$ is identity operator and $X$ is an arbitrary vector field then
\[ \Gamma(\hat{L}, X)_{\{w\}}^{\{z\}} = X^Q(z)\frac{\partial^2 w^K(z) \partial z^A(w)}{\partial z^Q \partial z^A}(-1)^p(A) = \]
\[ X^Q(z)\frac{\partial}{\partial z^Q} \log \left(Ber\left(\frac{\partial w}{\partial z}\right)\right) = X^Q(z)\frac{\partial \log \rho(z)}{\partial z} \quad (3.9) \]
where $\{w^A\}$ are arbitrary coordinates from the class (3.8) and $\rho(z)dz^1 \ldots dz^{m+n}$ is the volume form (3.7) in the coordinates $\{z^A\}$. The condition (3.5) is fulfilled and we come to the standard definition of the divergence in a superspace endowed with volume form. For the pair $(\hat{L}, X)$ where $\hat{L} = \text{id}$ and for class (3.8) the operator $D(\hat{L}, X)$ is the divergence of the vector field $X$ corresponding to the volume form $dv$:
\[ D(\hat{L}, X) = \partial(\hat{L}, X)_{\{z\}}^{\{w\}} + \Gamma(\hat{L}, X)_{\{w\}}^{\{z\}} = \]
\[ \frac{\partial X^A(z)}{\partial z^A}(-1)^p(X)p(A) + X^A \frac{\partial \log \rho(z)}{\partial z^A} = div dv X. \quad (3.10) \]

Now we return to the considerations of Section 2.

For the superspace $E^{n,n}$ endowed with the odd symplectic structure we consider a field $(\hat{\Pi}(z), \Psi(z))$, where $(\hat{\Pi}(z), \Psi(z))$ is an odd normal pair in a vicinity of some point.
(See the end of the previous Section). We denote by $F_D$ the class of Darboux coordinates (2.9, 2.10) on $E_{n,n}$ and apply constructions (3.1—3.6) in this case.

**Lemma.** If $(\tilde{\Pi}, \Psi)$ is an odd normal pair in $E_{n,n}$ then

$$\forall \{w\}, \{\tilde{w}\} \in F_D, \quad \Gamma(\tilde{\Pi}, \Psi)_{\{\tilde{w}\}}^{\{w\}} = 0.$$  \hspace{1cm} (3.11)

Using the statement of the Lemma we consider the action of the operator $D_{\text{can}}$ corresponding to the class $F_D$ of Darboux coordinates by (3.6), on the odd normal pair $(\tilde{\Pi}(z), \Psi(z))$:

$$D_{\text{can}}(\tilde{\Pi}, \Psi) = \left( \frac{\partial \Psi^A(z)}{\partial z^B} + \Psi^Q(z) \frac{\partial^2 w^K(z)}{\partial z^Q \partial z^B} \frac{\partial z^A(w)}{\partial w^K} (-1)^{p(B)} \right) \Pi^B_A(z) \hspace{1cm} (3.12)$$

where $\{w\}$ are arbitrary Darboux coordinates. From (3.6) and the Lemma follows

**Theorem.** For the odd normal pair $(\tilde{\Pi}(z), \Psi(z)), D_{\text{can}}(\tilde{\Pi}, \Psi)$ is an invariant geometrical object.

In particular if $\{w\}$ are arbitrary Darboux coordinates then

$$D_{\text{can}}(\tilde{\Pi}, \Psi) = \frac{\partial \Psi^A(w)}{\partial w^B} \Pi^B_A(w) \hspace{1cm} (3.13)$$

does not depend on the choice of the Darboux coordinates $\{w\}$.

Before proving the Lemma we will consider

**Example 2.** Let $E_{1.1}$ be a (1.1)-dimensional superspace endowed with odd symplectic structure (2.1). Let $w = (x, \theta)$ be some Darboux coordinates on it:

$$\{x, \theta\} = 1, \{x, x\} = 0. \hspace{1cm} (3.14)$$

It is easy to see that in this case the odd vector field obeying to (2.11, 2.12) is of the form

$$\Psi = \Psi(x, \theta) \frac{\partial}{\partial \theta} \hspace{1cm} (3.15)$$

where $\Psi(x, \theta)$ is non-nilpotent even function. The projector operator (2.14) is evidently identity operator. So a normal pair is of the form $(\text{id}, \Psi(x, \theta) \partial_\theta)$ and

$$D_{\text{can}}(\text{id}, \Psi) = \frac{\partial \Psi(x, \theta)}{\partial \theta}. \hspace{1cm} (3.16)$$

It is easy to see that if $x', \theta'$ are some other Darboux coordinates then they are related with coordinates $x, \theta$ by canonical transformation

$$x' = f(x) \hspace{1cm} (3.17)$$

$$\theta' = \frac{\theta}{df(x)/dx} + \beta(x)$$

where $f(x)$ and $\beta(x)$ are given functions.
where \( f(x) \) and \( \beta(x) \) are even and odd functions on \( E^{1,1} \) respectively. (To obtain (3.17) from (3.14) one has to note that in \( E^{1,1} \) \( \{x',x'\} = 0 \to x'_\theta = 0 \).) It is easy to see from (3.17) that

\[
\Psi = \Psi(x, \theta) \frac{\partial}{\partial \theta} = \frac{\Psi(x, \theta)}{df(x)/dx \partial \theta'}
\]

and (3.16) does not change under transformation (3.17), so in this case the statements of the Lemma and of the Theorem hold.

Indeed, for this case one can say more about (3.16). Let \((y, \eta)\) be an arbitrary coordinates on the \( E^{1,1} \) and a volume form \( dv \) on \( E^{1,1} \) is defined by the equation

\[
dv = dy d\eta \{y, \eta\}.
\]

Then one can check using (3.10) that \( D_{can}(\text{id}, \Psi) \) in (3.16) is the divergence of the vector field \( \Psi \) by the volume form (3.19) and does not depend on the choice of coordinates \((y, \eta)\).

Now we prove the Lemma. Let

\[
z^A = w^B L^A_B + w^B w^C T^A_{BC} + o(w^2)
\]

be arbitrary canonical transformations from Darboux coordinates \( \{z\} \) to Darboux coordinates \( \{w\} \) in a vicinity of the point \( z = 0 \). For proving the Lemma we have to show that for transformation (3.20)

\[
\Gamma(\hat{\Pi}, \Psi)_{\{w\}}_{\{z\}} \big|_{z=0} = \Psi^{Q}(z) \frac{\partial^2 w^K(z)}{\partial z^Q \partial z^B} \frac{\partial z^A(w)}{\partial w^K} \Pi^B_A(-1)^B \big|_{z=0} = 0.
\]

(We can consider (3.20) without loss of generality, since in Darboux coordinates the translation is obviously the canonical transformation.) We include the Darboux transformation (3.20) in the chain of Darboux transformations:

\[
\{z\}^{linear} \rightarrow \{\tilde{z}\} \rightarrow \{w\}^{linear} \rightarrow \{\tilde{w}\}
\]

which obey to the following conditions:

a) The transformation

\[
\{z\} \rightarrow \{\tilde{z}\}
\]

is the linear canonical transformation such that in the Darboux coordinates \( \tilde{z} = (\tilde{x}^i, \tilde{\theta}^k) \)

\[
\hat{\Pi} \frac{\partial}{\partial \tilde{x}^i} = \frac{\partial}{\partial \tilde{x}^i}, \quad \hat{\Pi} \frac{\partial}{\partial \tilde{\theta}^1} = \frac{\partial}{\partial \tilde{\theta}^1}.
\]

b) The transformation

\[
\{w\} \rightarrow \{\tilde{w}\}
\]

is the linear canonical transformation such that

\[
\tilde{w}^A = \tilde{z}^A + o(\tilde{z}).
\]
From (3.2, 3.4) it follows that

$$\Gamma(\hat{\Pi}, \Psi)_{\{\tilde{w}\}} = \Gamma(\hat{\Pi}, \Psi)_{\{\tilde{w}\}} + \Gamma(\hat{\Pi}, \Psi)_{\{\tilde{z}\}}.$$  (3.27)

But $\Gamma(\hat{\Pi}, \Psi)_{\{\tilde{w}\}}$ and $\Gamma(\hat{\Pi}, \Psi)_{\{\tilde{z}\}}$ in (3.27) are zero because the corresponding transformations are linear. The transformation $\{\tilde{w}\} \rightarrow \{\tilde{z}\}$ is not linear but from (3.24, 3.26, 2.11, 2.12) it follows that $\Psi_{\{\tilde{z}\}}|_{z=0} = \Psi_{\partial_{\theta_1}^1}$ and

$$\Gamma(\hat{\Pi}, \Psi)_{\{\tilde{w}\}} |_{z=0} = \Psi \frac{\partial^2 x'^1}{\partial \theta_1 \partial x^1} = 0$$  (3.28)

because $\{x'^1, x'^1\} = 0$. ($\{\tilde{w}\} = (x'^1, \cdots)$). (Compare with Example 2). So (3.21) is obeyed. Lemma is proved.

Considering a pair $(\hat{L}(z), X(z))$ we constructed in this section the divergence-like operator (3.12, 3.13) in an odd symplectic superspace in the case when $\hat{L}$ is a projector on a (1,1)-dimensional symplectic subspace and $X$ is an odd vector in it which is symplectoorthogonal to itself. In the case of even symplectic structure this construction can be carried out in a more general case and it is trivial because in this case there exists a volume form corresponding to the symplectic structure. Indeed if in a superspace $E^{2m.n}$ endowed with even symplectic structure there is given a pair $(\hat{L}(z), X(z))$ where $\hat{L}(z)$ is a symplectoorthogonal projector on $(2p,q)$-dimensional symplectic subspaces in $T_z E^{2m.n}$ and $X(z)$ is an arbitrary vector field then to the class $\mathcal{F}$ of the Darboux coordinates on this superspace there corresponds $\tilde{D}(\hat{L}, X)$ defined by (3.6) which is a straightforward generalization of (1.4).

Section 4. Truncated divergence.

We consider in this section the odd vector field which is defined on the points of $(n-1, n-1)$-dimensional nondegenerate supersurfaces embedded in the $(n,n)$-dimensional odd symplectic superspace with a volume form. In case of this vector field being symplectoorthogonal to this supersurface and to itself using the geometrical object $\mathcal{D}_{\text{can}}(\hat{\Pi}, \Psi)$ for an odd normal pair we define the first order differential operator on it (truncated divergence) whose action on this field gives the function on this supersurface.

Let $M^{n-1,n-1}$ be an arbitrary nondegenerate $(n-1, n-1)$-dimensional supersurface embedded in a superspace $E^{n,n}$ which is endowed with odd symplectic structure (2.1) and the volume form $d\mathbf{v}$

$$d\mathbf{v} = \rho(z)dz^1 \cdots dz^{2n}.$$  (4.1)

(The supersurface $M$ is nondegenerate if the symplectic structure of $E^{n,n}$ induces nondegenerate symplectic structure on $M$.) Let $z^A = z^A(\zeta^\alpha)$ be a local parametrization of the supersurface $M$. (For us it will be convenient to denote by the letter ”$\alpha$” the coordinates on $M$.) In the coordinates the induced two-form $\Omega_{\alpha\beta} d\zeta^\alpha d\zeta^\beta$ on $M$ is given by the following equation (See Section 2):

$$\Omega_{\alpha\beta}(z(\zeta)) = \Omega \left( \partial_{\alpha} z^A \frac{\partial}{\partial z^A}, \partial_{\beta} z^B \frac{\partial}{\partial z^B} \right) = \partial_{\alpha} z^A \Omega_{AB} \partial_{\beta} z^B (-1)^{s(B,\beta)}.$$  (4.2)
Hereafter we use notations \( \partial_\alpha z^A = \frac{\partial z^A}{\partial \xi^\alpha}, \partial_\alpha f = \frac{\partial f}{\partial \xi^\alpha}, \ldots \) for derivatives along surface and

\[
(-1)^{s(B, \beta)} = (-1)^{p(B)}p(\beta) + p(\beta)
\] (4.2a)

for sign factor.

The induced Poisson bracket structure on \( M \{,\} \) according to (2.5, 2.6) is defined by the matrix \( \Omega^{\alpha\beta} \) which is inverse to the matrix \( \Omega_{\alpha\beta} \). Using this induced symplectic structure one can construct the symplectoorthogonal projector on \( TM \)

\[
T_{z(\zeta)}E^{n.n} \xrightarrow{\tilde{P}(z(\zeta))} T_{z(\zeta)}M^{n-1,n-1}
\]

which can be expressed in terms of \( \Omega_{AB} \) and \( \Omega^{\alpha\beta} \)

\[
\tilde{P} : P^B_A(z(\zeta)) = \Omega_{AK}(z(\zeta)) \cdot \{ z^K(\zeta), z^B(\zeta) \}_M = \Omega_{AK}(z(\zeta))\partial_\alpha z^K(-1)^{s(K, \alpha)}\Omega^{\alpha\beta}\partial_\beta z^B.
\] (4.3)

The operator

\[
\tilde{\Pi} = \text{id} - \tilde{P}
\] (4.4)

in every point \( z(\zeta) \) is a symplectoorthogonal projector on the \((1,1)\)-dimensional subspace \( \tilde{\Pi}(z(\zeta)) \) in \( T_{z(\zeta)}E \) which is symplectoorthogonal and transversal to \( T_{z(\zeta)}M \) (See 2.14, 2.15). It is easy to see that there exists the odd vector field \( \Psi(z(\zeta)) \) belonging to \( \tilde{\Pi}(z(\zeta)) \) which is defined on the points of the supersurface \( M \), is symplectoorthogonal to \( M \) and which is symplectoorthogonal to itself and non-degenerate (2.11, 2.12):

\[
\tilde{P}\Psi = 0, \Omega(\Psi, \Psi) = 0. \text{(In the components } \Psi^A P^B_A = 0, \partial_\alpha z^A P^B_A = \partial_\alpha z^B). \] (4.5)

(This vector field is fixed uniquely up to the multiplication by an even non-nilpotent function of \( \zeta \). \( \Psi \to f(z(\zeta))\Psi \).)

The field \( \Psi \) and the projector \( \tilde{\Pi} \) form the odd normal pair \( (\tilde{\Pi}, \Psi) \) in the points \( z(\zeta) \) of the supersurface \( M \).

For example if the supersurface \( M^{n-1,n-1} \) is defined by equations

\[
f(z) = 0, \varphi(z) = 0
\] (4.6)

then for arbitrary point \( z_0 \) on this supersurface, the vectors \( D\varphi \) and \( Df \) (see 2.7) are the basis vectors of the subspace \( \Pi(z_0) \) and the vector

\[
\Psi = \left( Df - \frac{\{f, f\}}{2\{f, \varphi\}} D\varphi \right) \big|_{z_0}
\] (4.7)

is subject to the conditions (4.5).

Now for this odd vector field which is defined only on the supersurface \( M \) we will construct the "truncated divergence". For this purpose we consider an odd normal pair \( (\tilde{\Pi}, \tilde{\Psi}) \) in \( E^{n.n} \) in the vicinity of the arbitrary point \( z(\zeta) \) which is a prolongation of the odd normal pair \( (\tilde{\Pi}, \Psi) \):

\[
\tilde{\Psi} \big|_{z=z(\zeta)} = \Psi(z(\zeta)), \quad \tilde{\Pi} \big|_{z=z(\zeta)} = \tilde{\Pi}(z(\zeta))
\] (4.8)
and define the truncated divergence in the following way:

\[
\text{Div}_{\text{trunc}} \Psi(z(\zeta)) = \left( \text{div}_{d\nu} \tilde{\Psi} - D_{\text{can}}(\tilde{\Pi}, \tilde{\Psi}) \right) \bigg|_{z=z(\zeta)} .
\] (4.9)

In (4.9) \(D_{\text{can}}(\tilde{\Pi}, \tilde{\Psi})\) is given by (3.12, 3.13) for the odd normal pair \((\tilde{\Pi}, \tilde{\Psi})\) and \(\text{div}_{d\nu} \Psi\) is divergence (3.10) of the vector field corresponding to the volume form \(d\nu\) defined by (4.1).

One can see that the r.h.s. of equation (4.9) indeed does not depend on the prolongation \((\tilde{\Pi}, \tilde{\Psi})\) of the odd normal pair \((\Pi, \Psi)\).

Using equations (4.3 – 4.5) for the projectors \(\hat{P}\) and \(\Pi\), formulae (3.9, 3.10) for the divergence, formula (3.12) for \(D_{\text{can}}(\tilde{\Pi}, \tilde{\Psi})\) and the fact that

\[
\partial_{\beta} z^B \frac{\partial \tilde{\Psi}(z)^A}{\partial z^B} \bigg|_{z=z(\zeta)} = \partial_{\beta} \Psi^A(z(\zeta))
\] (4.10)

depends only on \(\Psi(z(\zeta))\), we rewrite \(\text{Div}_{\text{trunc}} \Psi(z(\zeta))\) in arbitrary coordinates \(\{z^A\}\) in the following way

\[
\text{Div}_{\text{trunc}} \Psi(z(\zeta)) = \left( \partial_{\beta} \Psi^A + \partial_{\beta} z^B \Psi^Q \frac{\partial^2 w^K(z)}{\partial z^Q \partial z^B} (-1)^{p(B)} \right) \Omega_{AK} \partial_{\alpha} z^K \Omega^{\alpha\beta}(-1)^{s(K,\alpha)}
\] (4.11)

\[
+ \Psi^A \frac{\partial}{\partial z^A} \left( \log \rho(z) - \log \left( \text{Ber} \frac{\partial w}{\partial z} \right) \right) \bigg|_{z=z(\zeta)}
\]

where \(\{w^A\}\) are any Darboux coordinates.

In arbitrary Darboux coordinates \(\{w^A\}\) using formula (3.13) for \(D_{\text{can}}(\tilde{\Pi}, \tilde{\Psi})\) we arrive at the following expression for \(\text{Div}_{\text{trunc}} \Psi(w(\zeta))\):

\[
\text{Div}_{\text{trunc}} \Psi(w(\zeta)) = \partial_{\beta} \Psi^A I_{AK} \partial_{\alpha} w^K \Omega^{\alpha\beta}(w(\zeta))(-1)^{s(K,\alpha)} + \Psi^A \frac{\partial \log \rho(w)}{\partial w^A} \bigg|_{w=w(\zeta)} .
\] (4.12)

Indeed \(\text{Div}_{\text{trunc}} \Psi\) depends only on the values of \(\Psi\) and it does not depend on the derivatives \(\partial_{\alpha} \Psi^A\) because \(\Psi\) is symplectoorthogonal to \(M^{n-1.n-1}\) (\(\Psi^A \Omega_{AK} \partial_{\alpha} z^K (-1)^{s(K,\alpha)} = 0\)). One can rewrite for example (4.12) in the following way:

\[
\Psi^A \left( -I_{AK} \partial_{\beta} \partial_{\alpha} w^K \Omega^{\alpha\beta}(w(\zeta))(-1)^{s(K,\alpha+\beta)+p(\beta)} + \frac{\partial \log \rho(w)}{\partial w^A} \bigg|_{w=w(\zeta)} \right) .
\] (4.12a)

(For the definition of \(s(K,\alpha+\beta)\) see (4.2a).)

In the case if there exist Darboux coordinates \(\{w\}\) in which the volume form \(d\nu\) is trivial,

\[
d\nu = dz^1 \ldots dz^{2n} \quad (\rho = 1)
\] (4.13)
then
\[ \text{Div}_{\text{trunc}} \Psi(w(\zeta)) = -\Psi^A I_{AK} \partial_\beta \partial_\alpha w^K (-1)^{s(K,\alpha+\beta)+p(\beta)} \Omega^{\alpha\beta}(w(\zeta)). \] (4.14)

Equation (4.14) defines $\text{Div}_{\text{trunc}} \Psi(w(\zeta))$ in arbitrary Darboux coordinates (if they exist) which are subject to condition (4.13).

**Section 5. The odd invariant semidensity.**

Now we are well prepared for writing the formula for odd invariant semidensity using constructions (4.9—4.12) for truncated divergence. The constructions of this section are founded on the following remark. Let $M^{n-1.n-1}$ be an arbitrary nondegenerate $(n-1.n-1)$-dimensional supersurface embedded in the odd symplectic superspace $E^{n,n}$ and a field $\Psi$ on this supersurface obeys to conditions (4.5). Then the r.h.s. of (4.11, 4.12) by its definition is invariant under coordinate transformations of the superspace $E^{n,n}$ and does not depend on the parametrization $z^A = z^A(\zeta^\alpha)$ of the supersurface $M^{n-1.n-1}$. So the equations (4.11, 4.12) define a density of the weight $\sigma = 0$ which is defined on $(n-1.n-1)$-dimensional supersurfaces. Moreover, one can see from (4.12a) that if $\Psi$ is a density of an arbitrary weight $\sigma$ which is defined on supersurfaces $M^{n-1.n-1}$ and takes values in the odd vector fields obeying to (4.5) then the truncated divergence of this density is the density of the weight $\sigma$ which is defined on $(n-1.n-1)$-dimensional nondegenerate supersurfaces and takes numerical values.

Let, as in Section 4, $M^{n-1.n-1}$ be an arbitrary nondegenerate supersurface in an odd symplectic superspace $E^{n,n}$ which is endowed with symplectic structure (2.1) and volume form (4.1). Now we will construct the semidensity on the $M$ which takes values in the odd vectors $\Psi$ obeying to conditions (4.5). Let the vectors $(e_1, ..., e_{n-1}; f_1, ..., f_{n-1})$ constitute a basis of the tangent space $T_z(z(\zeta))$ in arbitrary point $z(\zeta)$ of the supersurface $M^{n-1.n-1}$ ( $e_i$ are even vectors and $f_i$ are odd ones). Let $\Psi(z(\zeta))$ and $H(z(\zeta))$ be respectively an odd and an even vector fields which belong to $\Pi(z(\zeta))$ (see (4.5)) such that $(\Pi(z(\zeta)), \Psi(z(\zeta)))$ form an odd normal pair

\[ \Omega(\Psi, \Psi) = 0 \quad \text{and} \quad \Omega(H, \Psi) = 1. \] (5.1)

These conditions fix the vector fields $H$ and $\Psi$ up to the transformation

\[ H \rightarrow \frac{1}{\lambda} H + \beta \Psi, \quad \Psi \rightarrow \lambda \Psi \] (5.2)

where $\lambda$ is an arbitrary even function (taking values in non-nilpotent numbers) and $\beta$ is an arbitrary odd function (compare with 3.17). Using (5.2) one can choose the vector field $\Psi$ (but not the vector field $H$) in the unique way by imposing the normalization condition via volume form (4.1):

\[ dv(e_1, ..., e_{n-1}, H; f_1, ..., f_{n-1}, \Psi) = 1 \] (5.3)

We arrive at the function

\[ \Psi = \Psi(z(\zeta), e_1, ..., e_{n-1}; f_1, ..., f_{n-1}). \] (5.4)
which depends on points \( z(\zeta) \) of the supersurface \( M_{n-1,n-1} \) and the bases \( (e_1, \ldots, e_{n-1}; f_1, \ldots, f_{n-1}) \) in the \( T_z(M_{n-1,n-1}) \) and which takes values in the odd vector fields obeying to condition (4.5). This function is defined uniquely by condition (5.1) and by normalization condition (5.3). It is easy to see that under the change of the basis the function (5.4) is multiplied by the square root of the corresponding Berezinian \(*\). For example if \( e_1 \to \lambda e_1 \) and \( f_1 \to \mu f_1 \) then \( \Psi \to \sqrt{\frac{\lambda}{\mu}}\Psi \).

If \( z^A = z^A(\zeta^\alpha) \) is any parametrization of the supersurface \( M_{n-1,n-1} \) where \( \{\zeta^\alpha\} = (\xi^1, \ldots, \xi^{n-1}; \nu^1, \ldots, \nu^{n-1}) \) are even and odd parameters of this supersurface then considering as the basis vectors

\[
e_1 = \frac{\partial z^A}{\partial \xi^1} \frac{\partial}{\partial z^A}, \ldots, e_{n-1} = \frac{\partial z^A}{\partial \xi^{n-1}} \frac{\partial}{\partial z^A}, f_1 = \frac{\partial z^A}{\partial \nu^1} \frac{\partial}{\partial z^A}, \ldots, f_{n-1} = \frac{\partial z^A}{\partial \nu^{n-1}} \frac{\partial}{\partial z^A},
\]

we see that (5.4) defines odd vectors valued semidensity \( \Psi(z(\zeta), \frac{\partial z}{\partial \zeta}) \) of the rank \( k = 1 \) on nondegenerate \((n-1,n-1)\)-dimensional supersurfaces. The truncated divergence of this semidensity is the odd semidensity of the rank \( k = 2 \). Using the formula (4.12a) we arrive at this odd invariant semidensity:

\[
A\left( w(\zeta), \frac{\partial w}{\partial \zeta}, \frac{\partial^2 w^A}{\partial \zeta \partial \zeta} \right) = \text{Div}_{\text{trunc}} \Psi \left( w(\zeta), \frac{\partial w}{\partial \zeta} \right) = \Psi^A \left( w(\zeta), \frac{\partial w}{\partial \zeta} \right) \left( -I_{AK} \partial_\beta \partial_\alpha w^K \Omega^{\alpha\beta}(w(\zeta))(-1)^{s(K,\alpha+\beta)+p(\beta)} + \frac{\partial \log \rho(w)}{\partial w^A} \right)_{w = w(\zeta)}
\]

(5.5)

where \( \{w^A\} \) are arbitrary Darboux coordinates in the \( E^{n,n} \) and \( w(\zeta) \) is parametrization of supersurface \( M_{n-1,n-1} \). In a case if there exist Darboux coordinates \( \{w^A\} \) in which the volume form is trivial \((\rho = 1)\) the formula (5.5) according to (4.14) is reduced to

\[
A\left( w(\zeta), \frac{\partial w}{\partial \zeta}, \frac{\partial^2 w}{\partial \zeta \partial \zeta} \right) = -\Psi^A \left( w(\zeta), \frac{\partial w}{\partial \zeta} \right) I_{AK} \partial_\beta \partial_\alpha w^K \Omega^{\alpha\beta}(w(\zeta))(-1)^{s(K,\alpha+\beta)+p(\beta)}.
\]

(5.6)

The semidensity (5.5) is nothing but the semidensity obtained in [8]. (See (1.1).) To compare (5.5) with (1.1) we consider in the vicinity of arbitrary point \( z_0 \) the Darboux coordinates in which the supersurface is flat up to the second order derivatives: \( \{w^A\} = (x^1, \ldots, x^n, \theta^1, \ldots, \theta^n) \) and parameters \( \{\zeta^\alpha\} = (\xi^1, \ldots, \xi^{n-1}, \nu^1, \ldots, \nu^{n-1}) \) such that parametrization \( w^A = w^A(\zeta^\alpha) \) is of the form:

\[
\begin{align*}
x^n &= o(\zeta^2), \\
\theta^n &= o(\zeta^2), \\
x^i &= \xi^i + o(\zeta^2) \quad \text{if} \quad 1 \leq i \leq n - 1, \\
\theta^i &= \nu^i + o(\zeta^2) \quad \text{if} \quad 1 \leq i \leq n - 1.
\end{align*}
\]

(5.7)

* It is interesting to note that these considerations for obtaining the formula for invariant vector--valued semidensity are similar to the considerations for obtaining the formula for the invariant density on the lagrangian surfaces in \( E^{n,n} \) suggested by A.S. Schwarz [5].
(In (5.7) \( z_0 = 0 \).) It is evident that the vector valued semidensity (5.4) in the point \( z_0 \) in parametrization (5.7) is equal to \( \partial_{\theta^n} \) and \( u^{A}_{\alpha\beta}|_{z_0} = 0 \), so semidensity (5.5) in the point \( z_0 \) is equal to

\[
\frac{\partial \log \rho}{\partial \theta^n} \bigg|_{z=z_0} .
\]  

(5.8)

Returning to (1.1) we see that to (5.7) there correspond the functions \( f = x^n + o(z^2) \) and \( \phi = \theta^n + o(z^2) \), so the dual density \( \hat{A} \) in (1.1) in this point is equal to

\[
\Delta f \bigg|_{z=z_0} = \text{div}_x D_f \bigg|_{z=z_0} = \frac{\partial \log \rho}{\partial \theta^n} \bigg|_{z=z_0}
\]

and coincides (up to a constant) with (5.8).

Section 6. Discussions.

The constructions of truncated divergence and of odd semidensity have analogues in the standard differential geometry.

The constructions of Section 4 have evident analogues for the hypersurfaces in a Riemannian space. For example to (4.14) there corresponds the following construction.

Let \( C \) be an \((n - 1)\)-dimensional surface (hypersurface) embedded in an \( n \)-dimensional Euclidean space \( E^n \) and \( R \) be a vector field defined on the surface \( C \) which is orthogonal to \( C \). Then one can consider in analogy with (4.14) the function

\[
R^i G_{ik} \partial_{\beta} \partial_{\alpha} x^k g^{\alpha \beta}
\]  

(6.1)

which does not depend on the parametrization \( x^i = x^i(\xi^\alpha) \) of the surface \( C \) and on the choice of the Euclidean coordinates \( \{x^i\} \) in \( E^n \). In (6.1) \( G_{ik} = \delta_{ik} \) is the metric tensor in \( E^n \), \( g_{\alpha\beta} = \partial_{\alpha} x^i G_{ik} \partial_{\beta} x^k \) is the metrics induced on the surface \( C \), \( g^{\alpha\beta} = (g)^{-1}_{\alpha\beta} \) is inverse metric tensor.

More interesting is to compare the odd semidensity constructed in Section 5 with the mean curvature of the hypersurfaces in Euclidean space.

In analogy with considerations of Section 5 one can consider for the hypersurface \( C \) in \( E^n \) invariant density \( R^i(x(\xi), \frac{\partial x}{\partial \xi}) \) of the rank \( k = 1 \) which takes values in the vectors orthogonal to this surface. By these conditions it is fixed uniquely (up to multiplication by the constant):

\[
R^i(x(\xi), \frac{\partial x}{\partial \xi}) = n^i(x(\xi)) \sqrt{\text{det}(g_{\alpha\beta})}
\]  

(6.2)

where \( n^i(x(\xi)) \) is the unit vector field orthogonal to this surface and \( \sqrt{\text{det}(g_{\alpha\beta})} \) is the density of the volume form induced on the surface. Applying (6.1) to vector-valued density (6.2) we come in analogy with (5.6) to the following density of the second rank

\[
H(x(\xi), \frac{\partial x}{\partial \xi}, \frac{\partial^2 x}{\partial \xi \partial \xi}) = n^i(x(\xi)) \partial_{\alpha} \partial_{\beta} x^i g^{\alpha \beta} \sqrt{\text{det}(g_{\alpha\beta})} .
\]  

(6.3)

It is easy to see that \( n^i(x(\xi)) \partial_{\alpha} \partial_{\beta} x^i \) are components of the second quadratic form for the hypersurface \( C \) and the density (6.3) corresponds to the mean curvature [14].
Here we want to note that in spite of the fact that odd semidensity (5.5) can not be integrated over surfaces \( A^2 = 0 \) one can consider the equation

\[ A \equiv 0. \tag{6.4} \]

which extracts locally the class of \((n - 1, n - 1)\) supersurfaces (in an odd symplectic superspace endowed with a volume form) on which (6.4) is satisfied.

For example, from (5.6—5.8) it follows that to this class there belong supersurfaces which can be locally defined by equations \( x^n = \theta^n = 0 \) if there exist Darboux coordinates \((x^1, \ldots, x^n, \theta^1, \ldots, \theta^n)\) in which the volume form is trivial.

The analogous condition for mean curvature (6.3)

\[ H \equiv 0 \tag{6.5} \]

is the solution of the variational problem for minimal surfaces. Mean curvature (6.3) of the hypersurface \( C \) in \( E^n \) is identically equal to zero iff the surface \( C \) locally is extremal for the "surface " functional which is equal to the integral over the surface of the volume element (density) \( \sqrt{\text{det} g_{\alpha\beta}} \) corresponding to the metrics \( g_{\alpha\beta} \) induced on the surface [14].

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