Thin convex shells in Micromagnetics

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Abstract. Micromagnetic distributions of the vortex and onion type have been widely studied in the context of planar structures. Recently a significant interest to nanomagnets with curved shape has appeared. In particular, spherical shells are currently of great interest due to their capability to support skyrmion solutions which can be stabilized by curvature effects only, in contrast to the planar case where the intrinsic Dzyaloshinsky-Moriya interaction is required. It is well established that the effects of the demagnetizing field operator can be reduced to an effective easy-surface anisotropy for planar thin shells whose thickness is much less than the size of the system. The result has later been extended to surfaces whose closure is diffeomorphic to the closed unit disk of $\mathbb{R}^2$. The aim of the paper is to perform a rigorous $\Gamma$-development analysis of the micromagnetic energy functional, when the shell is generated, like in the case of a sphere, by a bounded and convex smooth surface.

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1. Introduction and Physical Motivations

According to Landau-Lifshitz-Brown micromagnetic theory of ferromagnetic materials (cf. [LL35, Bro62, Bro63, Ber98, HS08, BMS09]), a ferromagnet occupying a region $\Omega \subseteq \mathbb{R}^3$ and subject to a given external magnetic field $h_a$ is a mesoscopic medium whose possible equilibrium states are described by a vector field, the magnetization $M$, verifying the so-called fundamental constraint of micromagnetic theory: A ferromagnetic body is always locally saturated, i.e. there exists a material dependent positive constant $M_s$ such that $|M| = M_s(T)$ a.e. in $\Omega$. The spontaneous magnetization $M_s$ depends on the temperature $T$, and vanishes above a temperature (characteristic of each crystal type) known as the Curie point. Since we will assume that the specimen is at a fixed temperature well below the Curie point of the material, the value $M_s$ will be constant in $\Omega$. Due to the fundamental constraint, it is possible to express the magnetization $M$ under the form $M := M_s n$ where $n: \Omega \to S^2$ is a vector field taking values on the unit sphere $S^2$ of $\mathbb{R}^3$. Without loss of generality, in the sequel, we shall assume $M_s = 1$.

Even though the magnitude of the magnetization vector is constant in space, in general, it is not the case for its direction, and the observable states can be mathematically characterized as the local minimizers (among vector fields with values on $S^2$) of the Gibbes-Landau free energy functional associated to the ferromagnetic particle [Bro63, Ber98, HS08, BMS09]:

$$
\mathcal{G}_L(m, \Omega) := \int_{\Omega} a_{ex} |\nabla m|^2 \, dx + \int_{\Omega} \varphi_{an}(m) \, dx - \frac{\mu_0}{2} \int_{\Omega} h_a[m] \cdot m \, dx - \mu_0 \int_{\Omega} h_a \cdot m \, dx.
$$

The first term, $\mathcal{E}(m)$, is called exchange energy and penalizes nonuniformities in the magnetization orientation. The constant $a_{ex} \in \mathbb{R}^+$ is the so-called exchange stiffness constant, a phenomenological positive constant which summarizes the effect of short-range exchange interactions.

The second term, $\mathcal{A}(m)$, the anisotropy energy, describes crystal anisotropy effects. It models the existence of preferred directions of the magnetization (the so-called easy axes). The anisotropy energy density $\varphi_{an}: S^2 \to \mathbb{R}^+$ is assumed to be a non-negative Lipschitz continuous function, that vanishes only on a finite set of directions, the easy directions, which depend on the crystallographic symmetry of the sample.
The third term, $W(m)$, is called the magnetostatic self-energy and is the energy due to the (dipolar) magnetic field $h_d[m]$, also known in the literature as the demagnetizing field (or the stray field) due to its tendency to act on the magnetization $m$ so as to reduce the total magnetic moment. The constant factor $\mu_0$ is the value of magnetic permeability in the vacuum. From the mathematical point of view, any magnetization $m \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ generates the demagnetizing field $h_d[m] := \nabla u_m$, where the potential $u_m$ is the unique solution in the space $S'(\mathbb{R}^3)$ of the tempered distributions of the linear elliptic equation $\Delta u_m = -\div m$ (see section 2.2).

Eventually, the fourth term $Z(m)$, is called the interaction energy and models the tendency of a specimen to have its magnetization aligned with the (externally) applied field $h_a$, assumed to be unaffected by variations of $m$.

The four terms in the expression of the functional (1) include effects originating from different spatial scales such as short-range exchange forces and long-range magnetostatic effects. Together they model a multiscale system involving both intrinsic parameters, which are due to the internal physical structure of the ferromagnetic media, and extrinsic parameters arising from the geometry of the sample. Depending on the order of magnitude relation among the material and geometrical dependent parameters, different asymptotic regimes appear that can be efficiently investigated by the use of the dimension reduction techniques of calculus of variations [DeS95, GJ97, DKMO02, KS05, DKMO06, AL07, Ig09, SS12].

Recently a significant interest to nanomagnets with curved shape has appeared [LAE+07, YAK+11, KSS+12, GRS14, GKS14]. In particular, spherical shells are currently worth of interests due to their ability to support skyrmion solutions which can be stabilized by curvature effects only, in contrast to the planar case where the intrinsic Dzyaloshinsky-Moriya interaction is required [Dzy58, Mor60]. Nevertheless, and this is the main motivation for our paper, as remarked in [KRV+16]: «It is well established in numerous studies on rigorous micromagnetism that the effects of nonlocal dipole-dipole interaction can be reduced to an effective easy-surface anisotropy for thin shells when thickness is much less than the size of the system [...]». Being aware that these results were obtained for plane films, we assume that the same arguments are valid for smoothly curved shells».

Indeed, in [GJ97] it is shown that the effects of the demagnetizing field operator can be reduced to an effective easy-surface anisotropy for planar thin shells. A generalization of this result can be found in [Car01] where, for thin shells generated by extruding surfaces whose closure
isdieomorphictotheclosedunitdiskof$\mathbb{R}^2$ (see Figure 1), the asymptotic behavior of the minimizers of (1) is investigated. In [Sla05] a $\Gamma$-convergence analysis is performed on pillow-like thin shells, i.e. on shells of the type $\Omega := \{(x, z) \in \omega \times \mathbb{R} : \epsilon f_2(x) \leq z \leq \epsilon f_1(x)\}$ where $\omega \subset \mathbb{R}^2$ is a planar surface and $f_1, f_2$ functions vanishing on the boundary of $\omega$ (see Figure 1). In both cases the investigation leaves out very interesting scenarios like the spherical one (cf. Figure 2) which cannot be easily recovered by simply gluing local patches (for example via a partition of unity): Indeed the presence of the demagnetizing field, which is a non-local operator, makes necessary to look at the «global picture» of the surface from the very beginning, and indeed our analysis focuses on thin shells generated by bounded convex smooth surfaces.

In this paper we perform, in the spirit of the theoretical framework presented in [AB93] (see also Remark 1), a $\Gamma$-development analysis of the Gibbs-Landau energy functional (1), when the shell is generated, like in the case of a sphere, by a bounded and convex smooth surface (cf. Figure 2 and Definition 2).

Let us briefly explain why one needs a first order expansion. For any $\epsilon \in I_\delta := (0, \delta)$, with $0 < \delta < 1$, we consider the spherical shell $\Omega_\epsilon := B_{1+\epsilon} \setminus B_{1-\epsilon}$ of thickness $2\epsilon$, and denote by $G_\epsilon$ the Gibbs-Landau energy functional defined for every $m \in H^1(\Omega_\epsilon, S^2)$ by (cf. (1))

$$G_\epsilon(m) := \mathcal{E}_\epsilon(m) + W_\epsilon(m) + A_\epsilon(m) + Z_\epsilon(m)$$

$$= \int_{\Omega_\epsilon} |\nabla m|^2 \, dx - \frac{1}{2} \int_{\Omega_\epsilon} |h_d[m \chi_{\Omega_\epsilon}] \cdot m \, dx + \int_{\Omega_\epsilon} \varphi_{\text{ext}}(m) \, dx - \int_{\Omega_\epsilon} h_d \cdot m \, dx,$$

where $m \chi_{\Omega_\epsilon}$ denotes the extension of $m$ to all $\mathbb{R}^3$ which is equal to zero in $\mathbb{R}^3 \setminus \Omega_\epsilon$. Note that the expression of $G_\epsilon$ differs from the one given in (1) by the absence of the physical constants $\alpha_{\text{ext}}$ and $\mu_0$ which play no role in the forthcoming arguments. The existence for any $\epsilon \in I_\delta$ of at least a minimizer for $G_\epsilon$ in $H^1(\Omega_\epsilon, S^2)$ is easily obtained by the direct method of the calculus of variations (cf. [Vis85]) and here we are interested in the asymptotic behavior of the family of minimizers of $(G_\epsilon)_{\epsilon \in I_\delta}$ in the limit $\epsilon \to 0$.

It is straightforward to check that with respect to the weak topology of $H^1(\Omega_\delta)$, the family $(G_\epsilon)_{\epsilon \in I_\delta}$ is equi-mildly coercive and hence the fundamental theorem of $\Gamma$-convergence applies [BD98, Dal93]. On the other hand, a simple computation shows that the $\Gamma$-limit of $(G_\epsilon)_{\epsilon \in I_\delta}$ coincides with the identically null functional, so that every element of $H^1(\Omega_\delta, S^2)$ is a minimum point for $\tilde{G}_0 := \Gamma\lim_{\epsilon \to 0} G_\epsilon$ and higher order terms are necessary to gain information on the asymptotic behavior of the minimizing sequences for $(G_\epsilon)_{\epsilon \in I_\delta}.$
Remark 1. Let us recall the following important result, the simple proof of which can be found in [AB93, Bra02, BT08]. We state it in a form that better fits our setting: Let \( (\mathcal{G}_\epsilon) \) be a family of functionals all defined in the same Hilbert space \( H \). Let us suppose that \( \Gamma\lim_{\epsilon\to 0} \mathcal{G}_\epsilon = 0 \) and that \( \mathcal{G}'_\epsilon := \Gamma\lim_{\epsilon\to 0} \epsilon^{-1} \mathcal{G}_\epsilon \) with respect to the weak topology of \( H \). If the family \( (\epsilon^{-1} \mathcal{G}_\epsilon) \) is equi-mildly coercive and \( \min \mathcal{G}'_\epsilon \neq +\infty \) then
\[
\min \mathcal{G}_\epsilon = \epsilon \min \mathcal{G}'_\epsilon + o(\epsilon).
\]

Moreover, if \( (u_\epsilon) \) is a minimizing family for \( (\mathcal{G}_\epsilon) \), there exists a subsequence of \( (u_\epsilon) \) which weakly converges to a minimum point of \( \mathcal{G}_\epsilon \).

The expansion (4) is referred to as the first order \( \Gamma \)-development of \( (\mathcal{G}_\epsilon)_{\epsilon \in I}\), and the aim of the paper is to perform a first order \( \Gamma \)-development of the family of Gibbs-Landau energy functionals (3). More precisely, the paper is organized as follows: In section 2 we describe the functional and geometric framework in which we carry out our investigations and then state the main result of the paper (cf. Theorem 4). The proof of the main result is conceptually divided into four steps: the first two, concerning the reformulation of the minimization problem for the family \( (\mathcal{G}_\epsilon)_{\epsilon \in I}\), and the proof of the equi-mildly coercivity condition required in Remark 1, are the object of section 3. Section 4, devoted to the identification of the first order \( \Gamma \)-expansion of the family \( (\mathcal{G}_\epsilon)_{\epsilon \in I}\), completes the proof of the main result.

2. The functional setting and the statement of the main result

2.1. Notation and Setup. Before computing the first order \( \Gamma \)-development of the family of Gibbs-Landau energy functionals, we need to precise some differential geometric notion that will be used throughout the paper.

Let \( S \) be an orientable regular surface in \( \mathbb{R}^3 \), and let us denote by \( \nu : S \to S^2 \) the normal field associated to the choice of an orientation for \( S \). For every \( \sigma \in S \) and every \( \delta \in \mathbb{R}^+ \) we denote by \( \ell_\delta(\sigma) := \{ \sigma + t\nu(\sigma) \}_{|t|<\delta} \) the normal segment to \( S \) of radius \( \delta \) centered at \( \sigma \). The surface \( S \) admits a tubular neighbourhood, if for some \( \delta \in \mathbb{R}^+ \) the following properties hold:

- **TN1.** For every \( \sigma_1, \sigma_2 \in S \) one has \( \ell_\delta(\sigma_1) \cap \ell_\delta(\sigma_2) = \emptyset \) whenever \( \sigma_1 \neq \sigma_2 \).
- **TN2.** The union \( \Omega_\delta := \bigcup_{\sigma \in S} \ell_\delta(\sigma) \) is an open set of \( \mathbb{R}^3 \) containing \( S \). We say that \( \Omega_\delta \) is a tubular neighbourhood of \( S \) of radius \( \delta \).
- **TN3.** The map \( \psi : (\sigma, s) \in S \times I \mapsto \sigma + s\delta \nu(\sigma) \in \Omega_\delta \), where \( I = (-1,1) \), is a diffeomorphism of the product manifold \( S \times I \) onto \( \Omega_\delta \). In particular, the projection \( \pi : \Omega_\delta \to S \) which assigns to every \( x \in \Omega_\delta \) the unique \( \sigma \in S \) such that \( x \in \ell_\delta(\sigma) \), is a smooth map usually referred to as the nearest point projection of \( \Omega_\delta \) on \( S \).

It is well known (cf. [Do 76]) that any compact regular surface is orientable and admits a tubular neighbourhood. It turns out that the existence of a tubular neighbourhood of \( S \) is sufficient to investigate the \( \Gamma \)-development of the family of exchange energy functionals \( \mathcal{E}_\epsilon \) on the other hand, due to its non-locality, something more we require for the identification of the first order \( \Gamma \)-development of the family of magnetostatic self-energies \( W_\epsilon \). In this respect, we give the following definitions.

Definition 2. We say that an orientable smooth surface \( S \) is convex if it admits an orientation \( \nu \) such that the conditions **TN1, TN2 and TN3**, still hold when one replace the normal segment \( \ell_\delta(\sigma) \) with the normal half-line \( \ell^+_\delta(\sigma) := \{ \sigma + t\nu(\sigma) \}_{t \in (-\delta, +\infty)} \). We shall then denote by \( \Omega_\delta^+ \) the unbounded open set \( \bigcup_{\sigma \in S} \ell^+_\delta(\sigma) \) and refer to it as a tubular strip of \( S \).

Remark 3. Simple examples of convex surfaces are the sphere \( S^2 \) (as well as the triaxial ellipsoid \( \mathbb{E}^2 \)), the unit cylinder \( S^1 \times I \), the infinite cylinder \( S^1 \times \mathbb{R} \) and the plane \( \mathbb{R}^2 \) (cf. Figure 2). The name «convex» given to this class of surfaces is motivated by the fact that in the compact case such kind of surfaces are intimately related to the convexity of the domain they bound, but we do not dwell on that here and refer to [Do 76, Fed14, Kli13].
If $S$ is a smooth surface in $\mathbb{R}^3$, for every $\sigma \in S$ we denote by $\tau_1(\sigma), \tau_2(\sigma)$ the orthonormal basis of $T_\sigma S$ made by its principal directions, i.e. the orthonormal basis induced by the eigenvectors of the shape operator of $S$ [Do 76]. We then denote by $\kappa_1(\sigma), \kappa_2(\sigma)$ the associated principal curvatures. Note that if $S$ is convex then for any $x \in \Omega_\epsilon^+$ the trihedron
\[
(\tau_1(\sigma), \tau_2(\sigma), \nu(\sigma)) \quad \text{with} \quad \sigma := \pi(x)
\]
constitutes a basis of $\mathbb{R}^3$ which depends on $S$ only. Next, for every $\epsilon \in I_\delta$ we denote by $\psi_\epsilon$ the diffeomorphism of $\mathcal{M}$ onto $\Omega_\epsilon$ given by
\[
\psi_\epsilon : (\sigma, s) \in \mathcal{M} \mapsto \sigma + \epsilon s \nu(\sigma) \in \Omega_\epsilon.
\]
We then denote by $g_\epsilon$ the metric factor which relates the volume form on $\Omega_\epsilon$ to the volume form on $\mathcal{M}$, and by $h_{1,\epsilon}, h_{2,\epsilon}$ the metric coefficients which relate the gradient on $\Omega_\epsilon$ to the gradient on $\mathcal{M}$. A direct computation shows that their expression is given by
\[
g_\epsilon(\sigma, s) := |1 + 2\epsilon s H(\sigma) + (\epsilon s)^2 G(\sigma)|, \quad h_{i,\epsilon}(\sigma, s) := \frac{g_\epsilon(\sigma, s)}{(1 + \epsilon s \kappa_i(\sigma))^2} \quad (i \in \mathbb{N}_2),
\]
where $\kappa_1(\sigma), \kappa_2(\sigma)$ are the principal curvature at $\sigma \in S$, i.e. the eigenvalues associated to the eigenvectors $\tau_1(\sigma), \tau_2(\sigma)$ of the shape operator at $\sigma \in S$, while $H(\sigma), G(\sigma)$ are, respectively, the mean and gaussian curvature at $\sigma \in S$.

Let $S$ be a smooth and bounded convex surface and $\Omega_\delta$ the tubular neighbourhood of $S$ of radius $\delta \in \mathbb{R}^+$. For any $\epsilon \in I_\delta := (0, \delta)$ we denote by $G_\delta$ the Gibbs-Landau energy functional defined on $\Omega_\delta$ by (2). We then set $I := (-1, 1)$, denote by $\mathcal{M}$ the product manifold $S \times I$, and denote by $H^1(\mathcal{M}, \mathbb{R}^3)$ the Sobolev space of vector-valued functions defined on the manifold $\mathcal{M}$ endowed with the norm $\|u\|_{H^1(\mathcal{M})} := \|\nabla u\|_{L^2(S)} + \|\partial_\nu u\|_{L^2(I)}$ where $\nabla$ stands for the tangential gradient of $u$ on $S$. Eventually $H^1(\mathcal{M}, \mathbb{S}^2)$ is the subset of $H^1(\mathcal{M}, \mathbb{R}^3)$ made by vector valued functions taking values in $\mathbb{S}^2$.

With reference to the family $(W_\epsilon)_{\epsilon \in I_\delta}$ of magnetostatic self-energies, let us recall that from the mathematical point of view, for any tempered distribution $m \in S'(\mathbb{R}^3)$ the corresponding stray field $h_d[m]$ is the gradient field whose potential $u_m$ is the unique solution of the elliptic equation:
\[
\Delta u_m = -\text{div} m \quad \text{in} \quad S'(\mathbb{R}^3).
\]
It is straightforward to check, via Lax-Milgram theorem, that for every $m \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ the elliptic equation (8) possesses a unique solution in the Beppo-Levi space (cf. [DL00])
\[
BL^1(\mathbb{R}^3) = \{ u \in S'(\mathbb{R}^3) : u \omega \in L^2(\mathbb{R}^3), \nabla u \in L^2(\mathbb{R}^3, \mathbb{R}^3) \}, \quad \omega(x) := \frac{1}{\sqrt{1 + |x|^2}},
\]
which is a Hilbert space when endowed with the norm $\|u\|_{BL^1(\mathbb{R}^3)}^2 := \|\nabla u\|_{L^2(\mathbb{R}^3, \mathbb{R}^3)}^2 + \|\partial_\nu u\|_{L^2(I)}^2$. Moreover, the map $m \mapsto h_d[m] := \nabla u_m$ defines a bounded, linear, definite negative and non-expansive map from $L^2(\mathbb{R}^3, \mathbb{R}^3)$ into itself:
\[
\|h_d[m]\|_{L^2(\mathbb{R}^3, \mathbb{R}^3)}^2 = -\langle h_d[m], m \rangle_{L^2(\mathbb{R}^3, \mathbb{R}^3)} \leq \|m\|_{L^2(\mathbb{R}^3, \mathbb{R}^3)}^2 \quad \forall m \in L^2(\mathbb{R}^3, \mathbb{R}^3).
\]
In particular, for any $\epsilon \in I_\delta$ and any $m \in H^1(\Omega_\epsilon, \mathbb{S}^2)$, we have $m \chi_{\Omega_\epsilon} \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ and therefore $h_d[m \chi_{\Omega_\epsilon}] \in L^2(\mathbb{R}^3, \mathbb{R}^3)$. Furthermore, since $h_d[m]$ is a gradient field, for any $\epsilon \in I_\delta$ the following system of variational equations is satisfied
\[
\int_{\mathbb{R}^3} \langle h_d[m \chi_{\Omega_\epsilon}], m \chi_{\Omega_\epsilon} \odot \nabla \phi \rangle \, dx = 0 \quad \forall \phi \in BL^1(\mathbb{R}^3),
\]
\[
\int_{\mathbb{R}^3} h_d[m \chi_{\Omega_\epsilon}] \cdot \text{curl} \phi \, dx = 0 \quad \forall \phi \in BL^1(\mathbb{R}^3, \mathbb{R}^3).
\]
For what concerns the family \((A_\epsilon)_{\epsilon \in I_\delta}\) of anisotropy energies, the energy density \(\varphi_{an}: S^2 \to \mathbb{R}^+\), which does not depend on \(\epsilon \in I_\delta\), is assumed to be a non-negative Lipschitz continuous function that vanishes only on a finite set of directions, the so-called easy directions. The hypotheses assumed on \(\varphi_{an}\) are sufficiently general to treat the most common classes of crystal anisotropy energy densities arising in applications. Eventually, with reference to the family of interaction energies \((Z_\epsilon)_{\epsilon \in I_\delta}\), the vector field \(h_\epsilon\), which does not depend on \(\epsilon \in I_\delta\), is assumed to be Lipschitz continuous in \(\Omega_\delta\).

### 2.2. Statement of the main result.

Let us define the following functionals on \(H^1(\mathcal{M}, S^2)\), which can be read as the reading of \(\mathcal{E}_s, \mathcal{W}_e, \mathcal{A}_s\) and \(\mathcal{Z}_s\) on \(\mathcal{M}\):

- The exchange energy on \(\mathcal{M}\), defined for every \(u \in H^1(\mathcal{M}, S^2)\) by \(\mathcal{E}_s(u) : = \mathcal{E}_s^e(u) + \mathcal{E}_s^i(u)\), where the tangential and normal component of the exchange energy are respectively given by

\[
\mathcal{E}_s^e(u) := \frac{1}{\epsilon^2} \int_{\mathcal{M}} |\partial_u u(\sigma, s)|^2 \mathbf{g}_e(\sigma, s) \, d\sigma \, ds. \tag{14}
\]

- The magnetostatic self-energy on \(\mathcal{M}\), defined for every \(u \in H^1(\mathcal{M}, S^2)\) by \(\mathcal{W}_e(u) : = \mathcal{W}_e^s(u) + \mathcal{W}_e^i(u)\), where the tangential and normal component of the energy are respectively given by

\[
\mathcal{W}_e^s(u) := -\frac{1}{2} \int_{\mathcal{M}} \sum_{i \in \mathbb{N}_d} (h_i[u](\sigma, s) \cdot \tau_i(\sigma)) (u(\sigma, s) \cdot \tau_i(\sigma)) \mathbf{g}_e(\sigma, s) \, d\sigma \, ds, \tag{15}
\]

\[
\mathcal{W}_e^i(u) := -\frac{1}{2} \int_{\mathcal{M}} (h_i[u](\sigma, s) \cdot \nu(\sigma)) (u(\sigma, s) \cdot \nu(\sigma)) \mathbf{g}_e(\sigma, s) \, d\sigma \, ds. \tag{16}
\]

Here we have denoted by \(h_i[u] \in L^2(\mathcal{M}, \mathbb{R}^d)\) the demagnetizing filed read on \(\mathcal{M}\) and defined by:

\[
h_e[u](\sigma, s) := h_0((u \chi_{\mathcal{T}}) \circ \psi^{-1}) \circ \psi. \tag{17}
\]

The family of diffeomorphisms \((\psi_\epsilon)_{\epsilon \in I_\delta}\) is the one given by (6).

- The anisotropy and interaction energies on \(\mathcal{M}\), respectively, given for every \(u \in H^1(\mathcal{M}, S^2)\) by

\[
\mathcal{A}^s(u) := \int_{\mathcal{M}} \varphi_{an}(u(\sigma, s)) \mathbf{g}_e(\sigma, s) \, d\sigma \, ds, \tag{18}
\]

\[
\mathcal{Z}^s(u) := -\int_{\mathcal{M}} h_0(\sigma, s) \cdot u(\sigma, s) \mathbf{g}_e(\sigma, s) \, d\sigma \, ds. \tag{19}
\]

Here, for every \(\epsilon \in I_\delta\), we have denoted by \(h_0^s\) the reading of \(h_0\) on \(\mathcal{M}\) defined for every \((\sigma, s) \in \mathcal{M}\) by the position \(h_0^s(\sigma, s) : = h_0(\psi_\epsilon(\sigma, s))\). Note that, in the new coordinate system, the applied field depends on \(\epsilon \in I_\delta\).

We can now state the main result of the paper.

**Theorem 4.** When \(\epsilon \in I_\delta\) is sufficiently small, any minimum point, as well as any minimum value, of the Gibbs-Landau energy functional \(\mathcal{G}_\epsilon\) (see (3)) can be described by a suitable minimum point of the energy functional \(\mathcal{F}_\epsilon\), defined for every \(u \in H^1(S, S^2)\) by

\[
\mathcal{F}_\epsilon(u) := \mathcal{E}_\epsilon(u) + \mathcal{W}_\epsilon(u) + \mathcal{A}_\epsilon(u) + \mathcal{Z}_\epsilon(u) \tag{20}
\]

\[
= 2 \int_S |\nabla u|^2 \, d\sigma + \int_S (w_0 \cdot \nu)^2 \, d\sigma + 2 \int_S \varphi_{an}(u) \, d\sigma - 2 \int_S h_\epsilon \cdot u \, d\sigma. \tag{21}
\]
Indeed, if \((m_\epsilon)_\epsilon \in I_\epsilon\) is a minimizing family for \((G_\epsilon)_\epsilon \in I_\epsilon\), then \((m_\epsilon \circ \psi_\epsilon)_\epsilon \in I_\epsilon\) converges, weakly in \(H^1(M)\) and up to the extraction of a subsequence, to a minimum point of \(F_0\).

More precisely, for any \(\epsilon \in I_\epsilon\), the minimization problem for \(G_\epsilon \in H^1(\Omega_\epsilon, S^2)\) is equivalent to the minimization in \(H^1(M, S^2)\) of the functional \(F_\epsilon\) defined by (cf. (13),(14),(15),(16),(18) and (19))

\[
F_\epsilon(u) := \epsilon E_M^\epsilon(u) + W_M^\epsilon(u) + A_M^\epsilon(u) + Z_M^\epsilon(u).
\]

in the sense that the state \(m_\epsilon \in H^1(\Omega_\epsilon, S^2)\) minimizes \(G_\epsilon\) if and only if the state \(u_\epsilon(\sigma, s) := m(\psi_\epsilon(\sigma, s)) \in H^1(M, S^2)\) minimizes \(F_\epsilon\), and one has \(G_\epsilon(m_\epsilon) = F_\epsilon(u_\epsilon)\).

The families \((F_\epsilon)_\epsilon \in I_\epsilon\), and \((F'_\epsilon)_\epsilon \in I_\epsilon\) := \((\epsilon^{-1} F_\epsilon)_\epsilon \in I_\epsilon\), are both equi-mildly coercive with respect to the weak topology of \(H^1(M, S^2)\). Moreover \(F_0 := \Gamma\text{-}\lim_{\epsilon \to 0} F_\epsilon = 0\) is identically equal to zero, while \(F'_0 := \Gamma\text{-}\lim_{\epsilon \to 0} F'_\epsilon\) is given by

\[
F'_0(u) \in H^1(M, S^2) \mapsto \left\{ \begin{array}{ll} E_0^\epsilon(u) + W_0^\epsilon(u) + A_0^\epsilon(u) + Z_0^\epsilon(u) & \text{if } \partial \epsilon u = 0, \\ +\infty & \text{otherwise}, \end{array} \right.
\]

with \(E_0^\epsilon, W_0^\epsilon, A_0^\epsilon, Z_0^\epsilon\) given by (4). Furthermore the following first order \(\Gamma\)-development holds

\[
\min_{H^1(\Omega_\epsilon, S^2)} G_\epsilon = \min_{H^1(M, S^2)} F_\epsilon = \epsilon \left( \min_{H^1(M, S^2)} F'_\epsilon \right) + o(\epsilon),
\]

and if \((u_\epsilon)_\epsilon \in I_\epsilon\) is a minimizing family for \((F_\epsilon)_\epsilon \in I_\epsilon\), there exists a subsequence of \((u_\epsilon)_\epsilon \in I_\epsilon\) which weakly converges in \(H^1(M, S^2)\) to a minimum point of \(F'_0\).

**Remark 5.** For the identification of the \(\Gamma\)-limit of the family \((E_M^\epsilon)_\epsilon \in I_\epsilon\), no convexity assumption is needed on \(S\). Indeed this requirement is used in the identification of the \(\Gamma\)-limit of the family of magnetostatic self-energies \((W_M^\epsilon)_\epsilon \in I_\epsilon\), where a clever idea suggested in [Ca01] is used. Although the results in [Ca01] do not cover shells domains like the spherical one, they do not depend on the sign of the principal curvatures of \(S\), so that it is still not clear if something different from (23) appears for shells generated by non-convex compact surfaces. Nevertheless we think that the techniques proposed here can be used to treat also non-convex scenarios by considering them as the result of a non-uniform extrusion of the convex surface \(S\) along its normal direction \(\nu\), i.e. by considering domains of the type

\[
\Omega_t := \{ x \in \mathbb{R}^3 : x = \sigma + \epsilon t(\sigma) \nu(\sigma), \sigma \in S \},
\]

where \(t\) is now a suitable real-valued function defined on \(S\). This approach is currently under investigation.

**Remark 6.** In order to simplify the proof, it is important to note that then families \((A_M^\epsilon)_\epsilon \in I_\epsilon\) and \((Z_M^\epsilon)_\epsilon \in I_\epsilon\) can be interpreted as the two-variables functions \((\epsilon, u) \mapsto A_M^\epsilon(u)\) and \((\epsilon, u) \mapsto Z_M^\epsilon(u)\), both defined on the product topological space \(I_\epsilon \times H^1(M, S^2)\), for which one immediately gets

\[
\lim_{(\epsilon, u) \to (0, u_0)} A_M^\epsilon(u) = \int_M \varphi_{\text{min}}(u(\sigma)) \, d\sigma, \quad \lim_{(\epsilon, u) \to (0, u_0)} Z_M^\epsilon(u) = -\int_M h_\nu(\sigma) \cdot u(\sigma, s) \, d\sigma \, ds.
\]

Hence the families \((A_M^\epsilon)_\epsilon \in I_\epsilon\) and \((Z_M^\epsilon)_\epsilon \in I_\epsilon\) constitute a so-called continuous perturbation for the family \((E_M^\epsilon + W_M^\epsilon)_\epsilon \in I_\epsilon\), so that the theorem on the sum of \(\Gamma\)-limits holds, namely:

\[
\Gamma\text{-}\lim_{\epsilon \to 0} F'_\epsilon = \Gamma\text{-}\lim_{\epsilon \to 0} (E_M^\epsilon + W_M^\epsilon) + \Gamma\text{-}\lim_{\epsilon \to 0} A_M^\epsilon + \Gamma\text{-}\lim_{\epsilon \to 0} Z_M^\epsilon
\]

\[
= \Gamma\text{-}\lim_{\epsilon \to 0} (E_M^\epsilon + W_M^\epsilon) + A'_0 + Z'_0.
\]

Therefore, in the sequel, in identifying the \(\Gamma\)-limit we will focus on the family \((E_M^\epsilon + W_M^\epsilon)_\epsilon \in I_\epsilon\).
The proof of Theorem 4 is divided into several steps and given in section 3. Precisely, in subsection 3.1 we prove that for any \( \epsilon \in I_5 \) and any \( m \in H^1(\Omega_\epsilon, S^2) \) one has \( \mathcal{G}_\epsilon(m) = F_\epsilon(m \circ \psi_\epsilon) \) where \( \psi_\epsilon \) stands for the diffeomorphism of \( \mathcal{M} \) onto \( \Omega_\epsilon \) given by \( \psi_\epsilon(\sigma, s) := \sigma + \epsilon s \nu(\sigma) \). In particular one has \( \inf_{m \in H^1(\Omega_\epsilon, S^2)} \mathcal{G}_\epsilon(m) = \inf_{u \in H^1(M, S^2)} F_\epsilon(u) \). In subsection 3.2 we prove that the family \((F'_\epsilon)_{\epsilon \in I_5}\) is equi-mildly coercive with respect to the weak topology of \( H^1(M, S^2) \). In subsection 3.3 we prove that \( \Gamma_{\text{lim} \epsilon \to 0} F_\epsilon = 0 \) and introduce the development (24). The complete characterization of the \( \Gamma \)-limit of \( (F'_\epsilon)_{\epsilon \in I_5} \) will be given in section 4.

3. ASYMPOTIC \( \Gamma \)-DEVELOPMENT OF \( F_\epsilon \)

3.1. The reading of \( \mathcal{G}_\epsilon \) on \( \mathcal{M} \) and the equivalence of \( \mathcal{G}_\epsilon \) and \( F_\epsilon \). In this section we prove the first part of Theorem 4, namely that once introduced, for any \( \epsilon \in I_5 \), the diffeomorphism of \( \mathcal{M} \) onto \( \Omega_\epsilon \) given by \( \psi_\epsilon(\sigma, s) \in \mathcal{M} \mapsto \sigma + \epsilon s \nu(\sigma) \in \Omega_\epsilon \), one has \( \mathcal{G}_\epsilon(m) = F_\epsilon(m \circ \psi_\epsilon) \), and therefore \( u_\epsilon \) minimizes \( \mathcal{G}_\epsilon \), if and only if \( u_\epsilon(\sigma, s) := m(\psi_\epsilon(\sigma, s)) \) minimizes \( F_\epsilon \). The important point here, as usual in dimension reduction investigations, is that we rewrite the functional \( \mathcal{G}_\epsilon \) defined in \( H^1(\Omega_\epsilon, S^2) \), whose domain of definition varies with \( \epsilon \in I_5 \), as an equivalent functional \( F_\epsilon \) on \( H^1(M, S^2) \) whose domain of definition does not depend on \( \epsilon \in I_5 \), and for which the \( \Gamma \)-asymptotic analysis results more natural.

Here we only show that for any \( m \in H^1(\Omega_\epsilon, S^2) \) we have \( E_\epsilon(m) = E_\epsilon^M(m \circ \psi_\epsilon) \), the proof of the inequalities \( W_\epsilon(m) = W_\epsilon^M(m \circ \psi_\epsilon) , A_\epsilon(m) = A_\epsilon^M(m \circ \psi_\epsilon) \) and \( Z_\epsilon(m) = Z_\epsilon^M(m \circ \psi_\epsilon) \) being trivial. Let us consider the family of exchange energy functionals \((E_\epsilon)_{\epsilon \in I_5}\). For any \( m \in H^1(\Omega_\epsilon, S^2) \), by coarea formula we get

\[
E_\epsilon(m) := \int_{\Omega_\epsilon} |\nabla m(x)|^2 dx \quad = \quad \epsilon \int_{\psi_\epsilon(S) \times I} |\nabla m \circ \psi_\epsilon(\sigma, s)|^2 d\sigma ds \quad (29)
\]

\[
= \epsilon \int_{\mathcal{M}} |\nabla m \circ \psi_\epsilon(\sigma, s)|^2 d\mathcal{g}_\epsilon(\sigma, s) d\sigma ds. \quad (30)
\]

In deriving the last equality we have taken into account that for any \( (\epsilon, s) \in I_5 \times I \) the volume form on \( \psi_\epsilon(S) \) is related to the volume form on \( S \) by the metric factor \( \mathcal{g}_\epsilon(\sigma, s) := |1 + 2(1 + \epsilon s)H(\sigma) + (\epsilon s)^2 G(\sigma)| \). Next, we project the gradient onto the orthonormal (moving) frame \((\tau_1(\sigma), \tau_2(\sigma), \nu(\sigma))\) induced by \( S \) on \( \mathbb{R}^3 \) (cf. (5)). For any \( x \in \Omega_\epsilon \), we get \( |\nabla m(x)|^2 = \sum_{i \in \mathbb{N}_2} |\partial_{\tau_i}(\sigma)m(x)|^2 + |\partial_{\nu}(\sigma)m(x)|^2 \) with \( \sigma = \pi(x) \). On the other hand we have the relations

\[
|\partial_{\tau_i}(\sigma)m(\psi_\epsilon(\sigma, s))|^2 \quad = \quad \frac{1}{(1 + \epsilon s K_i(\sigma))^2} |\partial_{\tau_i}(\sigma)m(\sigma, s)|^2 ; 
\]

\[
|\partial_{\nu}(\sigma)m(\psi_\epsilon(\sigma, s))|^2 \quad = \quad \frac{1}{\epsilon^2} |\partial_{\nu}(\sigma)m(\sigma, s)|^2 . 
\]

from which the equality of \( E_\epsilon(m) \) and \( E_\epsilon^M(m \circ \psi_\epsilon) \) follows. Note that the previous computation also shows that \( m \in H^1(\Omega_\epsilon, S^2) \) if and only if \( m \circ \psi_\epsilon \in H^1(M, S^2) \). Since for every \( \epsilon \in I_5 \) the superposition operator \( m \in H^1(\Omega_\epsilon, S^2) \mapsto (m \circ \psi_\epsilon) \in H^1(M, S^2) \) is surjective we finish with:

\[
\inf_{m \in H^1(\Omega_\epsilon, S^2)} \epsilon^{-1} \mathcal{G}_\epsilon(m) = \inf_{u \in H^1(M, S^2)} E_\epsilon^M(u) + W_\epsilon^M(u) + A_\epsilon^M(u) + Z(\epsilon(u). \quad (33)
\]

This concludes the proof of the first part of Theorem 4.

3.2. The equi-mildly coercivity of the families \((F_\epsilon)_{\epsilon \in I_5}\) and \((F'_\epsilon)_{\epsilon \in I_5} \). Here we prove that the families \((F_\epsilon)_{\epsilon \in I_5}\) and \((F'_\epsilon)_{\epsilon \in I_5} \) are both equi-mildly coercive with respect to the weak topology of \( H^1(M, S^2) \). This means, by definition (see [BD98]) and with reference to \( F_\epsilon \), that there exists a nonempty weakly compact set \( K \subseteq H^1(M, S^2) \) such that \( \inf_{H^1(M, S^2)} F_\epsilon = \inf_K F_\epsilon \) for every \( \epsilon \in I_5 \). The importance of this step in a \( \Gamma \)-convergence result is in that it assures the validity of the fundamental theorem of \( \Gamma \)-convergence concerning the convergence of minimum problems ([BD98, Da93]).
Clearly, it is sufficient to prove the equi-mildly coerciveness of $(\mathcal{F}_0^\epsilon)_{\epsilon \in I\delta}$. Moreover, since $\mathcal{A}_M$ and $\mathcal{Z}_M$ are uniformly (with respect to $\epsilon \in I\delta$) bounded terms, it is sufficient to show the equi-mildly coerciveness of the family $(\mathcal{E}_M^\epsilon + \mathcal{W}_M^\epsilon)_{\epsilon \in I\delta}$, that here we still denote by $(\mathcal{F}_0^\epsilon)_{\epsilon \in I\delta}$. To this end, we observe that for any uniform vector filed $v \in H^1(\mathcal{M}, S^2)$ we have
\[
\min_{u \in H^1(\mathcal{M}, S^2)} \mathcal{F}_0^\epsilon(u) \leq \mathcal{E}_M^\epsilon(v) + \mathcal{W}_M^\epsilon(v) = \mathcal{W}_M^\epsilon(v).
\] (34)

Next, taking into account (10) and the fact that $g_\epsilon$ is uniformly (with respect to $\epsilon \in I\delta$) bounded on $\mathcal{M}$, we finish with
\[
\min_{u \in H^1(\mathcal{M}, S^2)} \mathcal{F}_0^\epsilon(u) \leq \int_\mathcal{M} g_\epsilon(\sigma, s) \, d\sigma \, ds \leq \kappa_M|\mathcal{M}|,
\] (35)

for a suitable positive constant $\kappa_M$ depending on $\mathcal{M}$ only. Therefore, for ever $\epsilon \in I\delta$ the minimum of $(\mathcal{F}_0^\epsilon)_{\epsilon \in I\delta}$ is achieved in the set $K(\mathcal{M}, S^2) := \cup_{\epsilon \in I\delta} \{ u \in H^1(\mathcal{M}, S^2) : \mathcal{F}_0^\epsilon(u) \leq \kappa_M|\mathcal{M}| \}$. On the other hand, since the principal curvatures $\kappa_1, \kappa_2$ are bounded in $S$, whenever the radius $\delta \in \mathbb{R}^+$ of the tubular neighbourhood $\mathcal{D}_{\delta}$ is sufficiently small, there exists a positive constant $c_M$, independent from $\epsilon \in I\delta$, such that for any $i \in \mathbb{N}$ one has $\inf_{(\sigma, s) \in \mathcal{M}, i}(\sigma, s) \geq c_M$ for every $\epsilon \in I\delta$.

Therefore, since $\mathcal{W}_M^\epsilon$ is always non-negative due to (10), we get
\[
\|u\|^2_{H^1(\mathcal{M}, S^2)} = |\mathcal{M}| + \sum_{i \in \mathbb{N}^2} \int_\mathcal{M} |\partial_{(\sigma, s)} u(\sigma, s)|^2 \, d\sigma \, ds + \int_\mathcal{M} |\partial_{u} u(\sigma, s)|^2 \, d\sigma \, ds \leq |\mathcal{M}| + \frac{1}{c_M} \mathcal{F}_0^\epsilon(u)
\] (36)

and hence if $u \in K(\mathcal{M}, S^2)$ then $\|u\|^2_{H^1(\mathcal{M}, S^2)} \leq (1 + \kappa_M/c_M)|\mathcal{M}|$. In other terms, the set $K(\mathcal{M}, S^2)$ is contained in the bounded subset $H^1_b(\mathcal{M}, S^2) := \{ u \in H^1(\mathcal{M}, S^2) : \mathcal{F}_0^\epsilon(u) \leq \kappa_M|\mathcal{M}| \}$ of $H^1(\mathcal{M}, S^2)$, this by the Rellich-Kondrachov theorem, one has $u_n \to u_0$ strongly in $L^2(\mathcal{M}, S^2)$, and therefore, up to the extraction of a subsequence, $1 \equiv |u_n| \to |u_0|$ a.e. in $\mathcal{M}$. Thus $u_0(\sigma, s) \in S^2$ for a.e. $(\sigma, s) \in \mathcal{M}$ and this concludes the proof.

3.3. The necessity for a higher order development of $(\mathcal{F}_\cdot)$. Let us start by noting that, with respect to the weak topology on $H^1(\mathcal{M}, S^2)$, one has
\[
\mathcal{F}_0 := \Gamma\text{-lim}_{\epsilon \to 0} \mathcal{F}_\epsilon = 0.
\] (38)

Since the family $(\mathcal{E}_M^\epsilon + \mathcal{W}_M^\epsilon)_{\epsilon \in I\delta}$ is non-negative, one has $0 \leq \liminf_{\epsilon \to 0} (\mathcal{E}_M^\epsilon(u_n) + \mathcal{W}_M^\epsilon(u_n))$ for every $(u_n)_{n \in \mathbb{N}}$ weakly converging in $H^1(\mathcal{M}, S^2)$. On the other hand, for every $u \in H^1(\mathcal{M}, S^2)$ the constant family defined by $\epsilon \in I\delta \mapsto u \in H^1(\mathcal{M}, S^2)$ is a recovery sequence for the $\Gamma$-limit because, up to a multiplicative constant, one has $\mathcal{E}_M^\epsilon(u) + \mathcal{W}_M^\epsilon(u) \leq \epsilon \|u\|^2_{H^1(\mathcal{M}, S^2)}$. This concludes the proof of (38).

Until now we have proved that $(\mathcal{F}_\epsilon)_{\epsilon \in I\delta}$ is equi-mildly coercive and $\Gamma$-converges, and hence the fundamental theorem of $\Gamma$-convergence applies [BD98, Dal93]. But $\mathcal{F}_0$ coincides with the null functional, so that every element of $H^1(\mathcal{M}, S^2)$ is a minimum point for $\mathcal{F}_0$, and higher order terms are necessary to gain information on the asymptotic behavior of minimizing sequence of $(\mathcal{F}_\epsilon)_{\epsilon \in I\delta}$. Indeed, due to Remark 1, if $\mathcal{F}_0^\epsilon$ as in our case (cf. (23)) is not the null functional, then its minimizers reveal information on the minimizers of $\mathcal{F}_\epsilon$, at least when $\epsilon$ is sufficiently small.

Since we have already proved the equi-mildly coercivity of $(\mathcal{F}_\epsilon)_{\epsilon \in I\delta}$ in section 3.2, to complete the proof of Theorem 1 we have to show that $\mathcal{F}_0^\epsilon$ is given by (23). This is the object of the next section.
4. The $\Gamma$-limit of the first order family $\mathcal{F}^\epsilon_0$

In this section we focus on the identification of $\mathcal{F}^\epsilon_0 := \Gamma\lim_{\epsilon \to 0} (\epsilon^{-1} \mathcal{F}_\epsilon)$. As pointed out at the end of subsection 2.2, it is sufficient to focus on the $\Gamma$-convergence of the family $(\mathcal{E}_M^s + \mathcal{W}_M^s)_{\epsilon \in I_s}$ which hereinafter we denote by $\mathcal{V}^\epsilon_0$:

$$\mathcal{V}^\epsilon_0; u \in H^1(M, S^2) \mapsto \mathcal{E}_M^s(u) + \mathcal{W}_M^s(u). \quad (39)$$

We then set $\mathcal{V}^\epsilon_0 := \mathcal{E}^\epsilon_0 + \mathcal{W}^\epsilon_0$ with $\mathcal{E}^\epsilon_0$ and $\mathcal{W}^\epsilon_0$ given by (4). Note that due to (10) one has $\mathcal{V}^\epsilon_0(u) \geq 0$ for any $u \in H^1(M, S^2)$.

Let us start by focusing on the $\Gamma$-limit inequality for $(\mathcal{V}^\epsilon_0)_{\epsilon \in I_s}$, i.e. on proving that for any family $(u_\epsilon)_{\epsilon \in I_s}$ weakly converging to some $u_0 \in H^1(M, S^2)$ one has $\mathcal{V}^\epsilon_0(u_0) \leq \liminf_{\epsilon \to 0} \mathcal{V}^\epsilon_0(u_\epsilon)$. Without loss of generality, we can restrict ourselves to the subset of sequences in $H^1(M, S^2)$ such that $\liminf_{\epsilon \to 0} \mathcal{V}^\epsilon_0(u_\epsilon) < +\infty$. We then necessarily have (see (14))

$$+\infty > \liminf_{\epsilon \to 0} \mathcal{F}^\epsilon_0(u_\epsilon) \geq \liminf_{\epsilon \to 0} \mathcal{E}^\epsilon_0(u_\epsilon) = \liminf_{\epsilon \to 0} \frac{1}{\epsilon} \int_M |\partial_s u_\epsilon(\sigma, s)|^2 g_\epsilon(\sigma, s) \, d\sigma ds. \quad (40)$$

Moreover, for any $i \in \mathbb{N}_2$, since $\sup_{\sigma \in S} |\kappa_\epsilon(\sigma)| < \infty$, there exists a strictly positive real-valued function $\gamma: I_s \to \mathbb{R}^+$ such that in a neighbourhood of $0 \in \mathbb{R}$ the following estimate holds:

$$\inf_{(\sigma, s) \in M} h_i(\sigma, s) = \inf_{(\sigma, s) \in M} \frac{g_\epsilon(\sigma, s)}{(1 + \kappa_\epsilon(\sigma))^{2i}} \geq \gamma(\epsilon) \quad \text{with} \quad \gamma(\epsilon) = 1 + o(1). \quad (41)$$

From (40) and (41) one has that $\lim_{\epsilon \to 0} \|\partial_s u_\epsilon\|_{L^2(M)} = 0$, and since $\partial_s u_\epsilon \rightharpoonup \partial_s u_0$ in $D'(M)$ we get

$$\partial_s u_\epsilon \rightharpoonup \partial_s u_0(\sigma, s) \quad \text{strongly in} \quad L^2(M), \quad \partial_s u_0(\sigma, s) = 0 \quad \text{a.e. in} \quad M. \quad (42)$$

Therefore, for the identification of the $\Gamma$-limit of $(\mathcal{V}^\epsilon_0)_{\epsilon \in I_s}$ is sufficient to restrict ourselves on the families made by $H^1(M, S^2)$ functions which weakly converge to an element of $H^1(M, S^2)$ and have the tensor product form

$$u_0(\sigma, s) = \chi_{t}(s)u_0(\sigma), \quad (43)$$

i.e. not depending on the $s$ variable.

In computing $\mathcal{V}^\epsilon_0$, we first show that the $\Gamma$-limit of the families $(\mathcal{E}_M^s)_{\epsilon \in I_s}$, $(\mathcal{W}_M^s)_{\epsilon \in I_s}$, is respectively equal to $\mathcal{E}^0_0$ and $\mathcal{W}^0_0$ (cf. (4)), then we prove that $\mathcal{V}^\epsilon_0 := \Gamma\lim_{\epsilon \to 0} \mathcal{V}^\epsilon_0 = \mathcal{E}^0_0 + \mathcal{W}^0_0$.

4.1. The $\Gamma$-limit of the family $(\mathcal{E}^s_M)_{\epsilon \in I_s}$. This section is devoted to the identification of the $\Gamma$-limit of the family $(\mathcal{E}^s_M)_{\epsilon \in I_s}$ of exchange energies on $M$. Note that in what follows it is not necessary to suppose that $S$ is convex since we just need to work with a tubular neighbourhood of $S$ having a sufficiently small radius. We start by addressing the $\Gamma$-limit inequality for $(\mathcal{E}^s_M)_{\epsilon \in I_s}$.

Taking into account the lower semicontinuity of the norm, for any $u_\epsilon \rightharpoonup u_0$ in $H^1(M, S^2)$, with $u_0$ of the type (43), we get

$$\|u_0\|_{H^1(M, S^2)}^2 = \int_M |u_\epsilon(\sigma, s)|^2 \, d\sigma ds + \sum_{i \in \mathbb{N}_2} \int_M |\partial_{\tau_i(\sigma)} u_\epsilon(\sigma, s)|^2 \, d\sigma ds \quad (44)$$

$$\leq |M| + \liminf_{\epsilon \to 0} \left( \sum_{i \in \mathbb{N}_2} \int_M |\partial_{\tau_i(\sigma)} u_\epsilon(\sigma, s)|^2 \, d\sigma ds + \int_M |\partial_s u_\epsilon(\sigma, s)|^2 \, d\sigma ds \right) \quad (45)$$

$$= |M| + \liminf_{\epsilon \to 0} \int_M |\nabla_{\tau(\sigma)} u_\epsilon(\sigma, s)|^2 \, d\sigma ds. \quad (46)$$
In deriving the last equality we have made use of (42) and have denoted by $\nabla_{\tau(\sigma)} u_e(\sigma, s)$ the tangential gradient of $u_e$ on $S$ which, with respect to an orthonormal basis $(\tau_1(\sigma), \tau_2(\sigma))$ of $T_\sigma S$, can be expressed in the form $|\nabla_{\tau(\sigma)} u_e(\sigma, s)|^2 := \sum_{i \in \mathbb{N}_2} |\partial_{\tau_i(\sigma)} u_e(\sigma, s)|^2$. Thus

$$
\|\nabla_{\tau} u_0\|_{H^1(S, \mathbb{S}^2)}^2 = 2 \sum_{i \in \mathbb{N}_2} \int_S |\partial_{\tau_i(\sigma)} u_0(\sigma)|^2 \, d\sigma \, ds \leq \liminf_{\epsilon \to 0} \int_M |\nabla_{\tau} u_\epsilon|^2. \tag{47}
$$

Next, making use of well-known properties of the $\liminf$ operator and taking into account relation (41), we compute:

$$
\liminf_{\epsilon \to 0} \int_M |\nabla_{\tau} u_\epsilon|^2 = \left( \liminf_{\epsilon \to 0} \gamma(\epsilon) \right) \left( \liminf_{\epsilon \to 0} \int_M |\nabla_{\tau(\sigma)} u_\epsilon(\sigma, s)|^2 \, d\sigma \, ds \right) \tag{48}
$$

$$
\leq \liminf_{\epsilon \to 0} \gamma(\epsilon) \int_M |\nabla_{\tau(\sigma)} u_\epsilon(\sigma, s)|^2 \, d\sigma \, ds \tag{49}
$$

$$
= \liminf_{\epsilon \to 0} \left( \sum_{i \in \mathbb{N}_2} \gamma(\epsilon) \int_M |\partial_{\tau_i(\sigma)} u_\epsilon(\sigma, s)|^2 \, d\sigma \, ds \right) \tag{50}
$$

$$
\leq \liminf_{\epsilon \to 0} \left( \sum_{i \in \mathbb{N}_2} \int_M |\partial_{\tau_i(\sigma)} u_\epsilon(\sigma, s)|^2 \frac{g_\epsilon(\sigma, s)}{(1 + \epsilon g_\epsilon(\sigma))^2} \, d\sigma \, ds \right) \tag{51}
$$

$$
= \liminf_{\epsilon \to 0} E^*_\epsilon(u_\epsilon). \tag{52}
$$

Substituting (51) into (47) we finish the proof of the following

**Lemma 7.** If $u_\epsilon \to u_0$ weakly in $H^1(M, \mathbb{S}^2)$ and $\liminf_{\epsilon \to 0} \mathcal{V}_1(u_\epsilon) < +\infty$, the following estimate holds

$$
2 \|\nabla_{\tau} u_0\|_{H^1(S, \mathbb{S}^2)}^2 \leq \liminf_{\epsilon \to 0} E^*_\epsilon(u_\epsilon) \leq \liminf_{\epsilon \to 0} (E^*_\epsilon(u_\epsilon) + E^*_\epsilon(u_\epsilon)). \tag{53}
$$

We now address the existence of a recovery sequence for the $\Gamma$-liminf. To this end, we note that for every $u_\epsilon \in H^1(M, \mathbb{S}^2) $ having the tensor product form $u_\epsilon(\sigma, s) = \chi(\sigma) u_0(s)$ we get (from (41))

$$
\limsup_{\epsilon \to 0} \left( \sum_{i \in \mathbb{N}_2} \int_M |\partial_{\tau_i(\sigma)} u_0(\sigma)|^2 \frac{g_\epsilon(\sigma, s)}{(1 + \epsilon g_\epsilon(\sigma))^2} \, d\sigma \, ds \right) \tag{54}
$$

$$
= \sum_{i \in \mathbb{N}_2} \int_M |\partial_{\tau_i(\sigma)} u_0(\sigma)|^2 \, d\sigma \, ds \tag{55}
$$

$$
= 2 \|\nabla_{\tau} u_0\|_{L^2(S, \mathbb{S}^2)}^2. \tag{56}
$$

Therefore

**Proposition 8.** Let $S$ be a smooth compact surface (convex or not) and $M := S \times I$. The family $(\mathcal{E}_\lambda)_{\lambda \in \mathcal{I}_s}$ of exchange energy on $M$, $\Gamma$-converges with respect to the weak topology of $H^1(M, \mathbb{S}^2)$ to the functional

$$
\mathcal{E}_0: u \in H^1(M, \mathbb{S}^2) \mapsto \begin{cases} 2 \|\nabla_{\tau} u\|_{L^2(S, \mathbb{S}^2)}^2 & \text{if } \partial_{\tau} u = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{57}
$$

4.2. The $\Gamma$-limit of the family $(\mathcal{W}_\lambda)_{\lambda \in \mathcal{I}_s}$. This section is devoted to the identification of the $\Gamma$-limit of the family $(\mathcal{W}_\lambda)_{\lambda \in \mathcal{I}_s}$ of magnetostatic self-energies on $M$. In contrast to what has been done for $(\mathcal{E}_\lambda)_{\lambda \in \mathcal{I}_s}$, it is now essential to make use of the convexity assumption on $S$. 

Let us start by observing that for every $u \in L^2(\mathcal{M}, \mathbb{R}^3)$ the distribution $(u \chi_I) \circ \psi_\epsilon^{-1}$, with $\psi_\epsilon$ given by (6), is in $L^2(\Omega_\epsilon, \mathbb{R}^3)$ and is therefore possible to evaluate the demagnetizing field $h_d$ on its extension by zero outside $\Omega_\epsilon$. For notational convenience we still denote by $(u \chi_I) \circ \psi_\epsilon^{-1}$ such an extension. Since $S$ is a convex surface (cf. Definition 2), for every $\epsilon \in I_5$ it is possible to define the tubular strip $\Omega_{\epsilon}^{M_\epsilon}$ of $S$ by the position $\Omega_{\epsilon}^{M_\epsilon} := \{ \sigma + \epsilon s \nu(\sigma) \}_{(\sigma, s) \in \mathcal{M}_\epsilon}$ with $\mathcal{M}_\epsilon := S \times (-1, +\infty)$.  

By first expressing (11) and (12) on the domain $\Omega_{\epsilon}^{M_\epsilon}$ and then rewriting them by the means of the diffeomorphism $\psi_\epsilon; (\sigma, s) \in \mathcal{M}_\epsilon \rightarrow \sigma + \epsilon s \nu(\sigma) \in \Omega_{\epsilon}^{M_\epsilon}$ we get that for any $\varphi \in \mathcal{D}(\Omega_{\epsilon}^{M_\epsilon})$ the following relations hold:

\[
\epsilon \int_{\mathcal{M}_\epsilon} (h_d[u](\sigma, s) + u(\sigma, s) \chi_I(s)) \cdot (\nabla \varphi \circ \psi_\epsilon) \, g_\epsilon(\sigma, s) \, d\sigma \, ds = 0 \quad \forall \varphi \in \mathcal{D}(\Omega_{\epsilon}^{M_\epsilon}),
\]

\[
\epsilon \int_{\mathcal{M}_\epsilon} h_d[u](\sigma, s) \cdot (\text{curl} \varphi \circ \psi_\epsilon) \, g_\epsilon(\sigma, s) \, d\sigma \, ds = 0 \quad \forall \varphi \in \mathcal{D}(\Omega_{\epsilon}^{M_\epsilon}, \mathbb{R}^3),
\]

with $h_d[u](\sigma, s) := h_d[(u \chi_I) \circ \psi_\epsilon^{-1}] \circ \psi_\epsilon$. The previous two relations are the key for the computation of $\mathcal{W}_0^\epsilon$ and play a different role in the analysis: Equation (58) permits to compute the $\Gamma$-limit of the normal part of the energy $\mathcal{W}_0^\epsilon$, while (59) is essential in understanding its tangential part.

Let $(u_\epsilon)_{\epsilon \in I_6}$ be a family of $H^1(\mathcal{M}, \mathbb{S}^2)$ functions weakly converging to some $u_0 \in H^1(\mathcal{M}, \mathbb{S}^2)$ and such that $\liminf_{\epsilon \rightarrow 0} \mathcal{F}_\epsilon(u_\epsilon) < +\infty$. By relations (10) and (17) we know that for any $m_\epsilon \chi_{\Omega_\epsilon} = (u \chi_I) \circ \psi_\epsilon^{-1}$ we have

\[
\frac{1}{2} \int_{\mathcal{M}_\epsilon} |h_d[u_\epsilon]|(\sigma, s)|^2 g_\epsilon(\sigma, s) \, d\sigma \, ds = \frac{1}{2} \int_{\Omega_{\epsilon}^{M_\epsilon}} |h_d[m_\epsilon \chi_{\Omega_\epsilon}]|^2 \, d\mu 
\leq \frac{1}{2} \int_{\mathbb{R}^3} |h_d[m_\epsilon \chi_{\Omega_\epsilon}]|^2 \, d\mu 
= - \frac{1}{2\epsilon} \int_{\mathcal{M}_\epsilon} h_d[m_\epsilon \chi_{\Omega_\epsilon}] \cdot m_\epsilon \chi_{\Omega_\epsilon} \, d\mu 
= \mathcal{W}_\epsilon(u_\epsilon) + \mathcal{W}_\epsilon^\prime(u_\epsilon),
\]

with $\mathcal{W}_\epsilon$ and $\mathcal{W}_\epsilon^\prime$ respectively given by (15) and (16). Hence, there exists a subsequence extracted from $(h_d[u_\epsilon \chi_I])_{\epsilon \in I_6}$, still denoted by $(h_d[u_\epsilon \chi_I])_{\epsilon \in I_6}$, and an element $h_0 \in L^2(\mathcal{M}_+, \mathbb{R}^3)$ such that $h_d[u_\epsilon \chi_I] \rightharpoonup h_0$ weakly in $L^2(\mathcal{M}_+, \mathbb{R}^3)$.

Let us consider the energy term $\mathcal{W}_\epsilon$, i.e. the normal part of the family of magnetostatic self-energy functionals defined by (16). Decomposing (58) into its normal and tangential part, and evaluating it on the weakly convergent sequence $(u_\epsilon)_{\epsilon \in I_6}$ we get that for any $\varphi \in \mathcal{D}(\mathcal{M}_+)$

\[
\int_{\mathcal{M}_+} [(h_d[u_\epsilon](\sigma, s) + u_\epsilon(\sigma, s) \chi_I(s)) \cdot \nu(\sigma)] \, \partial_\nu \varphi(\sigma, s) \, g_\epsilon(\sigma, s) \, d\sigma \, ds 
= - \epsilon \sum_{i \in \mathbb{N}_2} \int_{\mathcal{M}_+} (h_d[u_\epsilon](\sigma, s) + u_\epsilon(\sigma, s) \chi_I(s)) \cdot \tau_i(\sigma) \, \partial_\tau_i \varphi(\sigma, s) \, h_\epsilon(\sigma, s) \, d\sigma \, ds.
\]

Taking into account (41) and passing to the limit for $\epsilon \rightarrow 0$ in (64), up to the extraction of a subsequence, we get

\[
\int_{\mathcal{M}_+} [(h_0(\sigma, s) + u_0(\sigma) \chi_I(s)) \cdot \nu(\sigma)] \, \partial_\nu \varphi(\sigma, s) \, d\sigma \, ds = 0 \quad \forall \varphi \in \mathcal{D}(\mathcal{M}_+).
\]

Thus the quantity $(h_0(\sigma, s) + u_0(\sigma) \chi_I(s)) \cdot \nu(\sigma)$ is constant with respect to the $s$-variable, and since it is an element of $L^2(\mathcal{M}_+)$ one necessarily has

\[
h_0(\sigma, s) \cdot \nu(\sigma) = - u_0(\sigma) \chi_I(s) \cdot \nu(\sigma) \quad \forall (\sigma, s) \in \mathcal{M}_+.
\]
In particular, the normal component of the weak limit \( h_0 \in L^2(M_+, \mathbb{R}^3) \) does not depend on the extracted subsequence so that the full subsequence \( h_{i, \chi_i}(\sigma, s) \cdot \nu(\sigma) \) weakly converges to \(-w_0(\sigma) \chi_1(s) \cdot \nu(\sigma)\) in \( L^2(M_+, \mathbb{R}^3) \). By Rellich–Kondrachov theorem, the weak convergence of \((u_\epsilon)_{\epsilon \in I_\epsilon}\) to \(w_0(\sigma) \in H^1(M, \mathbb{S}^3)\) implies that \(u_{\epsilon}(\sigma, s) \to w_0(\sigma)\) strongly in \( L^2(M, \mathbb{R}^3)\). Hence by taking the limit for \( \epsilon \to 0 \) of both members of (64), taking into account (66), we finish with the following relation:

\[
\lim_{\epsilon \to 0} W_\epsilon^*(u_\epsilon) = \frac{1}{2} \int_{\mathcal{M}} (u_0(\sigma) \cdot \nu(\sigma))^2 d\sigma ds. \quad (67)
\]

Note that the right-hand side of (67) coincide with \( W_0^*\), and this is due to the fact that, as we are going to prove next, the tangential component of the demagnetizing field \( h_0 \) is zero, and hence \( \lim_{\epsilon \to 0} (W_\epsilon^*(u_\epsilon) + W_\epsilon^*(u_\epsilon)) = W_0^*(0)\).

We now address the tangential energy term \( W_\epsilon^*\) defined by (15). Information on the tangential component of the weak limit \( h_0 \) can be retrieved from relation (59). In this respect, we start by decomposing the integrand along its tangent and normal directions. More precisely, we observe that for any \( \varphi \in \mathcal{D}(\Omega_{M_+}, \mathbb{R}^3)\) one has (let us temporarily set \( \tau_0 := \nu(\sigma) \) in order to reduce the length of the relations)

\[
h_\epsilon[u] \cdot (\text{curl} \varphi \circ \psi_\epsilon) = \frac{1}{2} \sum_{i \in \mathbb{N}_3} (h_\epsilon[u] \times \tau_i) \cdot (\text{curl} \varphi \circ \psi_\epsilon \times \tau_i).
\]

We then denote by \( \nabla_{\text{skw}} \) the skew symmetric part of the gradient defined by \( \nabla_{\text{skw}} \varphi := (\nabla^T \varphi - \nabla \varphi)/2 \), and recall that for any \( \omega \in \mathbb{R}^3 \) one has \( \frac{1}{2} \text{curl} \varphi(x) \times \omega = (\nabla_{\text{skw}} \varphi)(x, \omega)\). From (67) we get

\[
h_\epsilon[u] \cdot (\text{curl} \varphi \circ \psi_\epsilon) = \sum_{i \in \mathbb{N}_3} (h_\epsilon[u] \times \tau_i) \cdot (\nabla_{\text{skw}} \varphi \circ \psi_\epsilon \tau_i).
\]

Next, we compute the relation between \( \nabla_{\text{skw}} \varphi \circ \psi_\epsilon \) and \( \nabla_{\mathcal{M}}(\varphi \circ \psi_\epsilon) \). To this end, let us first note that for any \( \varphi \in \mathcal{D}(\mathcal{M}_+), \) the function \( \varphi_\epsilon := \varphi \circ \psi_\epsilon^{-1} \) is in \( \mathcal{D}(\Omega_{\mathcal{M}_+}^+, \mathbb{R}^3) \), and moreover

\[
(\nabla_{\text{skw}} \varphi_\epsilon \circ \psi_\epsilon) \tau_1 \cdot \tau_2 = (d\varphi \circ \psi_\epsilon) \tau_1 \cdot \tau_2 - (d\varphi \circ \psi_\epsilon) \tau_2 \cdot \tau_1
\]

\[
= \frac{1}{1 + \epsilon s_{\mathcal{M}_+}} \partial_{\mathcal{M}_+}(\varphi \circ \psi_\epsilon) \tau_2 - \frac{1}{1 + \epsilon s_{\mathcal{M}_+}} \partial_{\mathcal{M}_+}(\varphi \circ \psi_\epsilon) \tau_1.
\]

Similarly, we compute the normal component of the tangential image of \( \nabla_{\text{skw}} \). For any \( i \in \mathbb{N}_2 \) we get

\[
(\nabla_{\text{skw}} \varphi_\epsilon \circ \psi_\epsilon) \tau_i \cdot \nu = (d\varphi \circ \psi_\epsilon) \tau_i \cdot \nu - (d\varphi \circ \psi_\epsilon) \nu \cdot \tau_i
\]

\[
= \frac{1}{1 + \epsilon s_{\mathcal{M}_+}} \partial_{\mathcal{M}_+}(\varphi \circ \psi_\epsilon) \nu - \frac{1}{\epsilon} \partial_{\mathcal{M}_+}(\varphi \circ \psi_\epsilon) \cdot \tau_i.
\]

By taking the limit for \( \epsilon \to 0 \), of both members of (72) and (73), we get

\[
\lim_{\epsilon \to 0} \epsilon (\nabla_{\text{skw}} \varphi_\epsilon \circ \psi_\epsilon) \tau_1 \cdot \tau_2 = 0,
\]

\[
\lim_{\epsilon \to 0} \epsilon (\nabla_{\text{skw}} \varphi_\epsilon \circ \psi_\epsilon) \tau_i \cdot \nu = - \partial_{\mathcal{M}_+}(\varphi \circ \psi_\epsilon) \tau_i \quad (i \in \mathbb{N}_2).
\]

Since \( h_\epsilon[u_\epsilon] \to h_0 \) weakly in \( L^2(\mathbb{R}^3, \mathbb{R}^3)\), and \( (h_\epsilon[u_\epsilon])_{\epsilon \in I_\epsilon} \) satisfies (59), for every \( \varphi \in \mathcal{D}(\mathcal{M}_+) \) we have (here we set as before \( \varphi_\epsilon := \varphi \circ \psi_\epsilon \))

\[
0 = \epsilon \int_{\mathcal{M}_+} h_\epsilon[u_\epsilon] \cdot (\text{curl} \varphi_\epsilon \circ \psi_\epsilon) \varphi \circ \psi_\epsilon \, d\sigma ds \quad (75)
\]

\[
= 2 \sum_{i < j \in \mathbb{N}_3} \int_{\mathcal{M}_+} [(h_\epsilon[u_\epsilon] \times \tau_i) \cdot \tau_j] \epsilon (\nabla_{\text{skw}} \varphi_\epsilon \circ \psi_\epsilon) \tau_i \cdot \tau_j \varphi \circ \psi_\epsilon \, d\sigma ds, \quad (76)
\]
and therefore, taking into account (41) and (74) and passing to the limit for $\epsilon \to 0$ in the previous expression, we finish with the relation
\begin{equation}
\int_{\mathcal{M}_+} (h_0(\sigma, s) \times \nu(\sigma)) \cdot \partial_s \varphi(\sigma, s) \, d\sigma \, ds = 0 \quad \forall \varphi \in \mathcal{D}(\mathcal{M}_+),
\end{equation}
from which we deduce that the quantity $h_0 \times \nu$ does not depend on the $s$-variable, and since $h_0 \times \nu \in L^2(\mathcal{M}_+)$ one necessarily has $h_0 \times \nu = 0$. Hence, the weak limit $h_0$ has no tangential component and that means (cf. (16)) that $\lim_{\epsilon \to 0} \mathcal{W}_\epsilon^\nu(u_\epsilon) = 0$. We have so proved the following

**Lemma 9.** If $u_\epsilon \rightharpoonup u_0$ weakly in $H^1(\mathcal{M}, \mathbb{S}^2)$ and $\liminf_{\epsilon \to 0} \mathcal{V}_\epsilon'(u_\epsilon) < +\infty$, then $\lim_{\epsilon \to 0} \mathcal{W}_\epsilon^\nu(u_\epsilon) = 0$ and
\begin{equation}
\mathcal{W}_0' = \lim_{\epsilon \to 0} \mathcal{W}_\epsilon^\nu(u_\epsilon) = \lim_{\epsilon \to 0} \mathcal{W}_\epsilon^\nu(u_0) = \int_S (u_0(\sigma) \cdot \nu(\sigma))^2 \, d\sigma.
\end{equation}

### 4.3. The $\Gamma$-limit of the family $(\mathcal{F}_\epsilon')_{\epsilon \in I_\epsilon}$
In this section, we complete the proof of theorem 4 by showing that $\mathcal{F}_\epsilon'$ is given by (23). As pointed out in Remark 6 it is sufficient to show that $\Gamma$-limit $\mathcal{V}_\epsilon' = \mathcal{E}_0' + \mathcal{W}_0'$. We note that if $u_\epsilon \rightharpoonup u$ in $H^1(\mathcal{M}, \mathbb{S}^2)$ and $\partial_s u \neq 0$ then $\liminf_{\epsilon \to 0} \mathcal{V}_\epsilon'(u_\epsilon) = +\infty$ and therefore the $\Gamma$-limit inequality is trivially satisfied. On the other hand, if $\partial_s u = 0$, then from Lemmas 7 and 9 we get
\begin{align}
\liminf_{\epsilon \to 0} \mathcal{V}_\epsilon'(u_\epsilon) &= \liminf_{\epsilon \to 0} (\mathcal{E}_\epsilon'(u_\epsilon) + \mathcal{E}_\epsilon'(u_0)) + \lim_{\epsilon \to 0} (\mathcal{W}_\epsilon^\nu(u_\epsilon) + \mathcal{W}_\epsilon^\nu(u_0)) \\
&\geq \mathcal{E}_0'(u_0) + \mathcal{W}_0'(u_0) \\
&= \mathcal{V}_0'(u_0).
\end{align}
Eventually, for any $u_0 \in H^1(\mathcal{M}, \mathbb{S}^2)$ such that $\partial_s u_0 = 0$, the constant (with respect to the index $\epsilon$) family $(u_{\epsilon})_{\epsilon \in I_\epsilon} = (u_0)_{\epsilon \in I_\epsilon}$ is a recovery sequence. Indeed we have
\begin{equation}
\limsup_{\epsilon \to 0} \mathcal{V}_\epsilon'(u_{\epsilon}) = \limsup_{\epsilon \to 0} (\mathcal{E}_\epsilon'(u_0) + \mathcal{E}_\epsilon'(u_0)) + \lim_{\epsilon \to 0} (\mathcal{W}_\epsilon^\nu(u_{\epsilon}) + \mathcal{W}_\epsilon^\nu(u_{\epsilon})) = \mathcal{V}_0'(u_0),
\end{equation}
and the proof of theorem 4 is complete.

5. Conclusion and Acknowledgment
We have given in this paper a first-order $\Gamma$-limit development analysis of the Gibbs-Landau energy functional when the shell is generated by a bounded and convex smooth surface. The result puts on a solid mathematical basis most of the micromagnetic studies on nanomagnets with curved shape, which are currently under investigation by the theoretical physicists community [LAE+07, YAK+11, KSS+12, SKSG13, GRS14, GKS14, SKG15, KRV+16].

For spherical shells, the concrete expression of the Gibbs-Landau first order $\Gamma$-limit, as well as its associated Euler-Lagrange equations, can be easily computed, while the mathematical characterization of their ground states is a challenging problem that could have many far-reaching consequences in modern magnetic storage technology. In that regard, let us observe that given a uniformly magnetized spherical region $B_\epsilon$ of radius $\epsilon$, the induced demagnetizing field is also uniform in $B_\epsilon$ (see [Osb45, Kel10, Di 16]). Moreover, when $\epsilon$ is under a critical size $\epsilon_0$, both the global and local minimizers of the Gibbs-Landau energy functional $\mathcal{G}_\epsilon(\cdot, B_\epsilon)$ (cf. (11)) are uniform in space [Bro69, Aha88, AB09, DSD12, ADM15]. Since for a spherical shell $B_\epsilon \setminus B_{\lambda \epsilon}$, with $\lambda < 1$, one has $\mathcal{G}_\epsilon(\cdot, B_\epsilon \setminus B_{\lambda \epsilon}) = \mathcal{G}_\epsilon(\cdot, B_\epsilon) - \mathcal{G}_\epsilon(\cdot, B_{\lambda \epsilon})$, with $\epsilon < \epsilon_0$, the ground states can be thought of as the result of a competition of uniform in space magnetizations, which tend to minimize $\mathcal{G}_\epsilon(\cdot, B_{\lambda \epsilon})$, with those which tend to maximize $\mathcal{G}_\epsilon(\cdot, B_{\lambda \epsilon})$. Depending on the geometrical parameters $\epsilon$ and $\lambda$, a competition prevails over the other, and this phenomenon is still present in the limiting behavior ($\epsilon \to 1$) stated in the paper (Theorem 4). This and many other aspects of the question will be the object of forthcoming works.

### 5.1. Acknowledgment
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