An integrable $BC(n)$ Sutherland model with two types of particles

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Abstract

A hyperbolic $BC(n)$ Sutherland model involving three independent coupling constants that characterize the interactions of two types of particles moving on the half-line is derived by Hamiltonian reduction of the free geodesic motion on the group $SU(n,n)$. The symmetry group underlying the reduction is provided by the direct product of the fixed point subgroups of two commuting involutions of $SU(n,n)$. The derivation implies the integrability of the model and yields a simple algorithm for constructing its solutions.
1 Introduction

The integrable many-body systems discovered by Calogero and Sutherland around 40 years ago still enjoy extraordinary popularity due to the wealth of their physical applications and connections to mathematics, which are described in the surveys [8, 12, 15, 17]. In correspondence to the many variants of these systems (associated with different interaction potentials, root systems, relativistic deformations, inclusion of two types of particles, and so on) there exist also several approaches to studying them. The systems based on trigonometric/hyperbolic interaction potentials are usually called Sutherland type and here we study a particular case of such systems in the classical Hamiltonian reduction framework, reviewed for example in [12].

A Sutherland type integrable system describing the interaction of \( m \) ‘positively charged’ and \((n - m)\) ‘negatively charged’ particles was first introduced by Calogero [2] by means of shifting the positions of \( m \) out of the \( n \) particles by \( i\pi/2 \). This trick converts the repulsive interaction potential \( \sinh^{-2}(q_j - q_k) \) into the attractive potential \( -\cosh^{-2}(q_j - q_k) \) between the particles of opposite charge (indexed say by \( 1 \leq j \leq m < k \leq n \)). Then Olshanetsky and Rogov [10] derived the Calogero-Sutherland model by Hamiltonian reduction of free motion on an affine symmetric space. The dynamics of the model and its relativistic deformation was analyzed in detail by Ruijsenaars [14], motivated mainly by the relation of this problem to the interaction of sine-Gordon solitons and anti-solitons.

In a little noticed paper Hashizume [7] generalized the Olshanetsky-Rogov derivation and thereby proved the integrability of a family of hyperbolic Sutherland models associated to the so-called root systems with signature. In the case of the \( BC(n) \) root system, his model involves two types of particles moving on the half-line with the interaction governed by two independent coupling parameters.

It is well-known that integrable \( BC(n) \) Sutherland models involve in general three arbitrary couplings corresponding to the three different root lengths. However, it has been explained only rather recently, by Pusztai and one of us [5], how the three couplings arise in the setting of Hamiltonian reduction. In the present paper, we generalize the result of [5] and derive the following \( BC(n) \) type Sutherland Hamiltonian,

\[
H = \frac{1}{2} \sum_{j=1}^{n} p_j^2 - \sum_{1 \leq j \leq m < k \leq n} \left( \frac{\kappa^2}{\cosh^2(q_j - q_k)} + \frac{\kappa^2}{\cosh^2(q_j + q_k)} \right) \\
+ \sum_{1 \leq j < k \leq m} \left( \frac{\kappa^2}{\sinh^2(q_j - q_k)} + \frac{\kappa^2}{\sinh^2(q_j + q_k)} \right) + \sum_{m < j < k \leq n} \left( \frac{\kappa^2}{\sinh^2(q_j - q_k)} + \frac{\kappa^2}{\sinh^2(q_j + q_k)} \right) \\
+ \frac{1}{2} \sum_{j=1}^{n} \frac{(x - y)^2}{\sinh^2(2q_j)} + \frac{1}{2} \sum_{j=1}^{m} \frac{xy}{\sinh^2(q_j)} - \frac{1}{2} \sum_{j=m+1}^{n} \frac{xy}{\cosh^2(q_j)},
\]

(1.1)

where \( \kappa > 0 \), \( x \) and \( y \) are real coupling constants. If \((x^2 - y^2) \neq 0\), then energy conservation ensures that the corresponding dynamics can be consistently restricted to the domain where \( q_1 > q_2 > \ldots > q_m > 0 \) and \( q_{m+1} > q_{m+2} > \ldots > q_n > 0 \).

Supposing also that \( xy > 0 \), the Hamiltonian (1.1) describes attractive-repulsive interactions between \( m \) ‘positively charged’ and \((n - m)\) ‘negatively charged’ particles influenced also by their mirror images and a positive charge fixed at the origin.
We shall derive the model (1.1) by reduction of the free geodesic motion on the group 
\( G = SU(n,n) \). Our reduction relies on a symmetry group of the form \( G_+ \times G^+ \), where \( G_+ \) is a maximal compact subgroup and \( G^+ \) is the (non-compact) fixed point subgroup of a \( G \)-involution that commutes with the Cartan involution fixing \( G_- \). Hashizume [7] reduced the geodesic motion on affine symmetric spaces such as \( G/G^+ \), which itself is the reduction of the free motion on \( G \) associated with the zero value of the moment map of the \( G^+ \) symmetry. He obtained the model (1.1) with two independent couplings, while we obtain it with three arbitrary couplings since we use non-trivial one-point coadjoint orbits of \( G^+ \) having a free parameter (corresponding to \( y \) in (1.1)) to define our reduction. In [5] (see also [9, 6]) the symmetry group \( G_+ \times G^+ \) was used in an analogous manner to describe the \( m = 0 \) case.

To be more precise regarding the comparison with Ref. [7], note that the model (1.1) with \( y = 0 \) was obtained in [7] by using \( G = U(n,n) \) and the model with a certain non-linear relation between the three couplings was obtained by using \( G = U(n + 1, n) \). The possible alternative reduction treatments of the model (1.1) are briefly discussed also in the concluding section.

Our derivation implies the Liouville integrability of the model (1.1) in the general case, and it also gives rise to a simple linear-algebraic algorithm for constructing the solutions. It could be interesting to analyze the dynamics of the model in the future by utilizing this algorithm, and to possibly relate it to special solutions in a field theory on the half-line. Further comments on open problems are offered at the end of the paper.

2 Group theoretic preliminaries

We here fix our notations and recall some group theoretic results that will be needed later.

To begin, we choose some integers

\[ 1 \leq m < n \] (2.1)

and define the matrices

\[ Q_{n,n} := \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix} \in gl(2n, \mathbb{C}), \quad I_m := \text{diag}(1_m, -1_{n-m}) \in gl(n, \mathbb{C}), \] (2.2)

where \( 1_n \) denotes the \( n \times n \) unit matrix. We also introduce

\[ D_m := \text{diag}(I_m, I_m) = \text{diag}(1_m, -1_{n-m}, 1_m, -1_{n-m}) \in gl(2n, \mathbb{C}). \] (2.3)

We adopt the convention in which the group \( G := SU(n,n) \) and its Lie algebra \( g := su(n,n) \) are given by

\[ SU(n,n) = \{ g \in SL(2n, \mathbb{C}) \mid g^\dagger Q_{n,n} g = Q_{n,n} \} \] (2.4)

and

\[ su(n,n) = \{ V \in sl(2n, \mathbb{C}) \mid V^\dagger Q_{n,n} + Q_{n,n} V = 0 \}. \] (2.5)

In the obvious \( n \times n \) block notation the elements \( V \in su(n,n) \) have the form

\[ V = \begin{bmatrix} X & Y \\ Z & -X^\dagger \end{bmatrix}, \quad Y^\dagger = -Y, \quad Z^\dagger = -Z, \quad \Re(\text{tr}(X)) = 0. \] (2.6)
We consider the commuting involutions of $SU(n,n)$ provided by the Cartan involution $\Theta$ and the involution $\Gamma$:

$$\Theta(g) := (g^\dagger)^{-1}, \quad \Gamma(g) := D_m \Theta(g) D_m, \quad \forall g \in G.$$  \hfill (2.7)

The fixed point subgroup of $\Theta$ is the maximal compact subgroup $G_+ < G$ and the (non-compact) fixed point subgroup of $\Gamma$ is denoted by $G^+$. Let $\theta$ and $\gamma$ be the corresponding involutions of $G = su(n,n)$. Using the $n \times n$ block notation, the Lie algebra $G_+$ of $G_+$ reads

$$G_+ = \left\{ \begin{bmatrix} X & Y \\
I_m & I_m \end{bmatrix} : X^\dagger = -X, \ Y^\dagger = -Y, \ \text{tr}(X) = 0 \right\},$$  \hfill (2.8)

and is isomorphic to $s(u(n) \oplus u(n))$ according to

$$s(u(n) \oplus u(n)) \ni \begin{bmatrix} \alpha & 0 \\
0 & \beta \end{bmatrix} \mapsto \psi(\alpha, \beta) := \frac{1}{2} \begin{bmatrix} \alpha + \beta & \alpha - \beta \\
\alpha - \beta & \alpha + \beta \end{bmatrix} \in G_+. \hfill (2.9)$$

Correspondingly, the group $G_+$ is isomorphic to $S(U(n) \times U(n))$ via the formula

$$S(U(n) \times U(n)) \ni \begin{bmatrix} a & 0 \\
0 & b \end{bmatrix} \mapsto g(a,b) := \frac{1}{2} \begin{bmatrix} a + b & a - b \\
a - b & a + b \end{bmatrix} \in G_+, \hfill (2.10)$$

which can be written also as

$$g(a,b) = K(\text{diag}(a,b))K^{-1} \quad \text{with} \quad K := \frac{1}{\sqrt{2}} \begin{bmatrix} I_n & I_n \\
I_n & -I_n \end{bmatrix}. \hfill (2.11)$$

The Lie algebra $G^+$ of $G^+$ is furnished by

$$G^+ = \left\{ \begin{bmatrix} X & Y \\
I_m & I_m \end{bmatrix} : X^\dagger = -I_m X I_m, \ Y^\dagger = -Y, \ \text{tr}(X) = 0 \right\},$$  \hfill (2.12)

and is isomorphic to $s(u(m,n-m) \oplus u(m,n-m))$ via the map

$$s(u(m,n-m) \oplus u(m,n-m)) \ni \begin{bmatrix} \alpha & 0 \\
0 & \beta \end{bmatrix} \mapsto \chi(\alpha, \beta) := \frac{1}{2} \begin{bmatrix} \alpha + \beta & (\alpha - \beta)I_m \\
I_m(\alpha - \beta) & I_m(\alpha + \beta) \end{bmatrix} \in G^+. \hfill (2.13)$$

In the above formula $u(m,n-m)$ is realized as the Lie algebra of the $n \times n$ matrices satisfying the relation

$$\alpha^\dagger I_m + I_m \alpha = 0$$  \hfill (2.14)

and it holds that

$$\chi(\alpha, \beta) = \tilde{K}(\text{diag}(\alpha, \beta))\tilde{K}^{-1} \quad \text{with} \quad \tilde{K} := \frac{1}{\sqrt{2}} \begin{bmatrix} I_n & I_n \\
I_n & -I_n \end{bmatrix}. \hfill (2.15)$$

Similarly, $G^+$ is isomorphic to $S(U(m,n-m) \times U(m,n-m))$ by means of conjugation by $\tilde{K}$.

The eigensubspaces $G_-$ of $\theta$ and $G^-$ of $\gamma$ having eigenvalue $-1$ can be displayed as

$$G_- = \left\{ \begin{bmatrix} X & Y \\
-Y & -X \end{bmatrix} : X^\dagger = X, \ Y^\dagger = -Y \right\},$$  \hfill (2.16)
Introducing $G_s := G_s \cap G^r$ for any signs $s, r \in \{\pm\}$, we can decompose $G$ as the direct sum of disjoint subspaces,

$$G = G^- \oplus G^+ \oplus G^+_+ \oplus G^+_+,$$

which are pairwise perpendicular to each other with respect to the invariant scalar product on $G$ defined by

$$\langle V, W \rangle := \frac{1}{2} \text{tr}(VW), \quad \forall V, W \in G.$$

We have

$$G^+_+ = \left\{ \begin{bmatrix} X & Y \\ Y & X \end{bmatrix} : X = I_m X I_m = -X^\dagger, \ Y = I_m Y I_m = -Y^\dagger, \ \text{tr}(X) = 0 \right\},$$

and the appropriate restriction of the map (2.9) gives rise to an isomorphism

$$G^+_+ \simeq s((u(m) \oplus u(n - m)) \oplus (u(m) \oplus u(n - m))).$$

We shall also use the explicit form of the other subspaces:

$$G^- = \left\{ \begin{bmatrix} X & Y \\ -Y & -X \end{bmatrix} : X = I_m X I_m = -X^\dagger, \ Y = I_m Y I_m = -Y^\dagger \right\},$$

$$G^+_+ = \left\{ \begin{bmatrix} X & Y \\ -Y & -X \end{bmatrix} : X = I_m X I_m = X^\dagger, \ Y = I_m Y I_m = -Y^\dagger \right\},$$

$$G^- = \left\{ \begin{bmatrix} X & Y \\ -Y & -X \end{bmatrix} : X = I_m X I_m = X^\dagger, \ Y = I_m Y I_m = -Y^\dagger \right\}.$$

Next, for our purpose we choose a maximal Abelian subspace $A$ of $G^-$, i.e., an Abelian subalgebra of $G$ which lies in $G^-$ and is not properly contained in any Abelian subalgebra of the same kind. It is known [16] that any two choices are equivalent by the conjugation action of $G^+_+ \oplus G^- \oplus G^+_+ \oplus G^+_+$ on $G^-$, and concretely we choose

$$A := \left\{ q := \begin{bmatrix} q & 0 \\ 0 & -q \end{bmatrix} : q = \text{diag}(q_1, \ldots, q_n), \ q_k \in \mathbb{R} \right\}.$$ 

One can verify that the centralizer of $A$ in $G$ is given by the direct sum

$$\mathcal{C} = A \oplus \mathcal{M}, \quad \mathcal{M} = \left\{ d := i \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} : d = \text{diag}(d_1, \ldots, d_n), \ d_k \in \mathbb{R}, \ \text{tr}(d) = 0 \right\} < G^+_+.$$

Denote by $A$ and $\mathcal{M}$ the connected subgroups of $G$ corresponding to the Abelian subalgebras $A$ and $\mathcal{M}$, respectively. In fact, $\mathcal{M}$ is precisely the subgroup of $G^+_+$ whose elements $g$ satisfy

$$g q g^{-1} = q, \quad \forall q \in A.$$
Furthermore, we call an element $q \in \mathcal{A}$ regular if the elements $g \in G^+$ satisfying the relation $g q g^{-1} = q$ all belong to $M$. It is not difficult to check that $q \in \mathcal{A}$ is regular in this sense if and only if the following conditions hold:

$$q_i \neq 0 \quad i = 1, \ldots, n, \quad (q_j - q_k)(q_j + q_k) \neq 0 \quad 1 \leq j < k \leq m \quad \text{and} \quad m < j < k \leq n. \quad (2.26)$$

Choose a connected component $\mathcal{A}_c$ of the open set formed by regular elements of $\mathcal{A}$, and denote the closure of this ‘open Weyl chamber’ by $\bar{\mathcal{A}}_c$. According to general results [16], every element $g \in G$ can be decomposed in the form

$$g = g_+e^s g^+ \quad \text{with} \quad q \in \bar{\mathcal{A}}_c, \; g_+ \in G_+, \; g^+ \in G^+.$$

The constituent $q$ that enters this decomposition is unique, and if $q$ is regular then the ambiguity of the pair $(g_+, g^+)$ is exhausted by the replacement

$$(g_+, g^+) \rightarrow (g_+\mu, \mu^{-1}g^+) \quad \forall \mu \in M. \quad (2.28)$$

In the generalized Cartan decomposition (2.27) the open Weyl chamber $\mathcal{A}_c$ can be taken to consist of the elements $q$ in (2.23) that are subject to the condition

$$q_1 > q_2 > \ldots > q_m > 0 \quad \text{and} \quad q_{m+1} > q_{m+2} > \ldots > q_n > 0. \quad (2.29)$$

Both $G_+$ and $G^+$ possess one-dimensional centres. The centre of $G_+$ is generated by

$$C^l := iQ_{n,n} = i \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix} \quad (2.30)$$

and the centre of $G^+$ is spanned by

$$C^r := i \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}. \quad (2.31)$$

These elements enjoy the property

$$C^\lambda \in \mathcal{M}^\perp \cap G_+^\perp \quad \text{for} \quad \lambda = l, r. \quad (2.32)$$

The decomposition (2.27) and the property (2.32) will be important for us in Section 3.

By means of the invariant scalar product (2.19), we can regard $\mathcal{G}, \mathcal{G}_+$ and $\mathcal{G}^+$ as their own dual spaces, respectively. This then also identifies the respective coadjoint actions with the adjoint actions. In the next section we shall utilize particular coadjoint orbits of $G_+$. To describe them, for any non-zero column vector $u \in \mathbb{C}^n$ define the matrices

$$X(u) := i(uu^\dagger - \frac{u^\dagger u}{n}1_n) \quad \text{and} \quad \xi(u) := \frac{1}{2} \begin{bmatrix} X(u) & X(u) \\ X(u) & X(u) \end{bmatrix}. \quad (2.33)$$

Fixing arbitrary real constants $\kappa > 0$ and $x \neq 0$, it is easy to see (cf. (2.9)-(2.11)) that the set

$$\mathcal{O}_{\kappa, x} := \{xC^l + \xi(u) \mid u \in \mathbb{C}^n, \; u^\dagger u = 2\kappa n\} \quad (2.34)$$

is a coadjoint orbit of $G_+$ of minimal non-zero dimension. The action of $g(a, b) \in G_+$ on $\mathcal{O}_{\kappa, x}$ takes the form

$$g(a, b)(xC^l + \xi(u))g(a, b)^{-1} = (xC^l + \xi(au)). \quad (2.35)$$

Since $\xi(u)$ determines $u$ up to an overall $U(1)$ phase, the orbit $\mathcal{O}_{\kappa, x}$ can be identified with the complex projective space $\mathbb{C}P_{n-1}$.

We remark that, for any real constants $x$ and $y$, $xC^l$ and $yC^r$ represent one-point coadjoint orbits of $G_+$ and $G^+$, respectively.
3 Hamiltonian reduction

We shall reduce the free geodesic motion on the group $G = SU(n,n)$ formulated as a Hamiltonian system on the cotangent bundle $T^*G$. We find it convenient to analyze the reduction by using the so-called shifting trick of symplectic geometry, which amounts to extending the phase space by a coadjoint orbit before reduction \[11\]. Specifically, trivializing $T^*G$ by right-translations and identifying $\mathcal{G}^*$ with $\mathcal{G}$ by means of the invariant scalar product, we consider the phase space

$$P := T^*G \times O_{\kappa,x} \simeq (G \times \mathcal{G}) \times O_{\kappa,x} \equiv \{(g, J, \zeta) \mid g \in G, J \in \mathcal{G}, \zeta \in O_{\kappa,x}\}. \quad (3.1)$$

The symplectic form on $P$ is given by

$$\Omega = \Omega_{T^*G} + \Omega_{O_{\kappa,x}}, \quad (3.2)$$

where $\Omega_{T^*G}$ can be written explicitly as

$$\Omega_{T^*G} = d\langle J, dgg^{-1} \rangle \quad (3.3)$$

while the explicit form of the Kirillov-Kostant-Souriau symplectic form $\Omega_{O_{\kappa,x}}$ of the coadjoint orbit $O_{\kappa,x}$ \[2.34\] will not be needed. The phase space $P$ carries the commuting family of Hamiltonians provided by

$$H_k(g, J, \zeta) := \frac{1}{4k} \text{tr}(J^{2k}), \quad k = 1, 2, \ldots, n, \quad (3.4)$$

the first member of which is responsible for the geodesic motion. These Hamiltonians are explicitly integrable; the flow of $H_k$ with initial value $(g_0, J_0, \zeta_0)$ is readily verified to be

$$(g(t), J(t), \zeta(t)) = (e^{tv_k}g_0, J_0, \zeta_0) \quad \text{with} \quad V_k := J_0^{2k-1} - \frac{1}{2n} \text{tr}(J_0^{2k-1})1_{2n}. \quad (3.5)$$

Note that $H_k$ is real since $J^{2k}$ satisfies $(J^{2k})^\dagger = Q_{n,n}J^{2k}(Q_{n,n})^{-1}$, and $V_k$ in \[3.5\] belongs to $\mathcal{G} = su(n,n)$.

We introduce an action of the group $G_+ \times G^+$ on $P$ by sending the pair $(\eta, h) \in G_+ \times G^+$ to the symplectomorphism $\Psi_{\eta,h}$ of $P$ operating as follows:

$$\Psi_{\eta,h}(g, J, \zeta) := (\eta g h^{-1}, \eta J \eta^{-1}, \eta \zeta \eta^{-1}). \quad (3.6)$$

The Hamiltonians $H_k$ \[3.4\] are invariant under this group action, which is generated by the equivariant moment map

$$\Phi = (\Phi_+, \Phi^+) : P \to (\mathcal{G}_+, \mathcal{G}^+), \quad (3.7)$$

$$\Phi_+(g, J, \zeta) = \pi_+(J) + \zeta, \quad \Phi^+(g, J, \zeta) = -\pi^+(g^{-1}Jg), \quad (3.8)$$

where the projection $\pi_+ : \mathcal{G} \to \mathcal{G}_+$ is given by means of the decomposition $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$ and $\pi^+ : \mathcal{G} \to \mathcal{G}^+$ by $\mathcal{G} = \mathcal{G}^+ \oplus \mathcal{G}^-$. We are interested in the reduction defined by imposing the moment map constraint

$$\Phi = \nu \quad \text{with} \quad \nu := (0, -yC^n), \quad (3.9)$$
where \( y \neq 0 \) is a real constant and we refer to (3.11). The action of the symmetry group \( G_+ \times G^+ \) preserves the ‘constraint surface’

\[
P_c := \Phi^{-1}(\nu) \subset P. \tag{3.10}
\]

We require that the constants \( x \) and \( y \) verify

\[
(x^2 - y^2) \neq 0. \tag{3.11}
\]

Then the corresponding space of orbits,

\[
P_{\text{red}} := P_c / (G_+ \times G^+), \tag{3.12}
\]

will turn out to be a smooth manifold. According to the general theory [11], \( P_{\text{red}} \) inherits the symplectic form \( \Omega_{\text{red}} \) and the reduced Hamiltonians \( H^\text{red}_k \) defined by the formulas

\[
\pi^* \Omega_{\text{red}} = \Omega|_{P_c}, \quad \pi^* H^\text{red}_k = H_k|_{P_c}, \tag{3.13}
\]

where \( \pi : P_c \to P_{\text{red}} \) is the natural projection and \( \Omega|_{P_c} \) is the restriction of \( \Omega \) (3.2) on \( P_c \subset P \).

**Remark 3.1.** In this technical remark we explain why the space of orbits (3.12) is a smooth manifold. First, we note that the action (3.6) of \( G_+ \times G^+ \) on the manifold \( P \) is proper and free and proper action on \( P_c \). Moreover, \( P_c \) is a closed, embedded submanifold of \( P \). Then, by considering the convergent sequences \( (\eta_n, h_n) \) in \( G_+ \times G^+ \) and \( (g_n, J_n, \zeta_n) \) in \( P \) (with \( n \in \mathbb{N} \)) for which \( (g_n, J_n, \zeta_n) \) and \( \Psi_{(\eta_n, h_n)}(g_n, J_n, \zeta_n) \) are both convergent, there exists a convergent subsequence of the sequence \( (\eta_n, h_n) \). To show this, choose a convergent subsequence \( \eta_{n_i} \) of the sequence \( \eta_n \) in \( G_+ \). This is always possible since \( G_+ \) is compact. Then, by considering the converging sequences \( \eta_{n_i} g_{n_i} (h_{n_i})^{-1} \) and \( g_{n_i} \) one can immediately conclude that \( h_{n_i} \) must be a convergent sequence in \( G^+ \), which proves the claim. To continue, notice from (3.6) that the effectively acting symmetry group is the factor group \( (G_+ \times G^+)/(\mathbb{Z}_n)_{\text{diag}} \), where \( (\mathbb{Z}_n)_{\text{diag}} \) is formed by the pairs \((z1_{2n}, z1_{2n}) \in G_+ \times G^+\) with \( z \) running over the \( n^{\text{th}} \) roots of unity. We shall demonstrate in the proof of Theorem 3.4 that the action of \( (G_+ \times G^+)/(\mathbb{Z}_n)_{\text{diag}} \) on \( P_c \) is a free action. Moreover, \( P_c \) is a closed, embedded submanifold of \( P \), as it follows from the definition (3.10) of \( P_c \) and from the locally free character of the \( (G_+ \times G^+)-\text{action} \) on it. Since we have a free and proper action on the manifold \( P_c \), the general theory [11] guarantees that \( P_{\text{red}} \simeq P_c / ((G_+ \times G^+)/(\mathbb{Z}_n)_{\text{diag}}) \) is a smooth symplectic manifold. This is manifest by the model of \( P_{\text{red}} \) constructed below.

Our goal in what follows is to exhibit a global cross section (a global ‘gauge slice’) of the orbits of \( G_+ \times G^+ \) in \( P_c \), which will yield a concrete model of the reduced Hamiltonian systems \((P_{\text{red}}, \Omega_{\text{red}}, H^\text{red}_k)\). We first present the following lemma, whose proof will also show how to construct a convenient global gauge slice.

**Lemma 3.2.** The element \( e^q, q \in A \) in (2.29), and \( u \in \mathbb{C}^n \) enter a triple \((e^q, J, xC^l + \xi(u)) \in P_c \) (3.10) if and only if \(|u_j|^2 = 2k \) for all \( j = 1, \ldots, n \) and \( q \) is regular in the sense of Eq. (2.20).

**Proof.** Let us inspect the moment map constraint for an element of \( P \) of the form

\[
(e^q, J, xC^l + \xi(u)) \quad \text{with some} \quad q \in A. \tag{3.14}
\]

Denoting the projections associated to the decomposition (2.18) as \( \pi^*_k \) and decomposing \( J \) as

\[
J = J^+_++J^-+J^+_-, \tag{3.15}
\]
we can spell out the moment map constraint as the conditions
\[ J_+^1 = -xC^d - \pi_+^1(\xi(u)), \quad J_+^1 = -\pi_+^1(\xi(u)), \] (3.16)
and
\[ \pi^+_1(e^{-ad_q}(J)) \equiv (\cosh ad_q)(J_+^1 + J_-^1) - (\sinh ad_q)(J_+^1 - J_-^1) = yC^r. \] (3.17)
Since \( C^r \in G^+_s \), the \( \pi^+_1 \) projection of equation (3.17) says that
\[ (\cosh ad_q)(J_+^1) - (\sinh ad_q)(J_-^1) = 0, \] (3.18)
and its \( \pi^+_1 \) projection requires that
\[ (\cosh ad_q)(J_+^1) - (\sinh ad_q)(J_-^1) = yC^r. \] (3.19)
We here used that \( \cosh ad_q \) maps \( G^+_s \) to \( G^+_s \) and \( \sinh ad_q \) maps \( G^+_s \) to \( G^-_s \) (with \(-s = \mp \) for \( s = \pm \)). By substituting \( J_+^1 \) from (3.16) into (3.19) and then taking the scalar product of both sides of equation (3.19) with an arbitrary \( T \in M \) (2.21), we obtain the requirement
\[ \langle T, \xi(u) \rangle = 0 \quad \forall T \in M, \] (3.20)
where we also took into account that \( C^d \) and \( C^r \) belong to \( M^\perp \) (2.32). By using the form of \( M \) (2.21) and that of \( \xi(u) \) (2.33), we can rewrite (3.20) as the condition
\[ |u_j|^2 = 2\kappa, \quad \forall j = 1, \ldots, n. \] (3.21)
If (3.21) holds, then we can apply the action of the subgroup \( M_{\text{diag}} \) of \( G_+ \times G^+ \),
\[ M_{\text{diag}} := \{ (\mu, \mu) \in G_+ \times G^+ \mid \mu \in M \} \] (3.22)
to replace (without changing \( q \)) the element in (3.14) by an element of the form
\[ (e^q, J, xC^d + \xi(a^e)) \text{ with the vector } u^e_j := \sqrt{2\kappa}, \quad j = 1, \ldots, n. \] (3.23)
We further inspect the moment map constraint for the element (3.28). First looking at the block-diagonal components of (3.19), we see that the matrix elements \((J^-)_{k,k}\) are arbitrary real numbers for all \( k = 1, \ldots, n \), and that we must have
\[ -i\kappa \cosh(q_j - q_k) - (J^-)_{j,k} \sinh(q_j - q_k) = 0 \quad \text{for} \quad 1 \leq j < k \leq m \quad \text{and} \quad m < j < k \leq n. \] (3.24)
The last equation can be solved for \((J^-)_{j,k}\) if and only if \((q_j - q_k) \neq 0\) for the pertinent indices. Next, the block off-diagonal components of (3.19) can be spelled out as the conditions
\[ -i\kappa \cosh(q_j + q_k) - (J^-)_{j,n+k} \sinh(q_j + q_k) = 0 \quad \text{for} \quad 1 \leq j < k \leq m \quad \text{and} \quad m < j < k \leq n, \] (3.25)
and
\[ -ix \cosh(2q_k) - (J^-)_{k,n+k} \sinh(2q_k) = y(C^r)_{k,n+k} \quad \text{for} \quad k = 1, \ldots, n. \] (3.26)
Equation (3.25) can be solved for \((J^-)_{j,n+k}\) if and only if \((q_j + q_k) \neq 0\) for the relevant indices. Taking into account the assumption \((x^2 - y^2) \neq 0\) (3.11) and the formula (2.51) of \( C^r \), equation (3.26) can be solved for \((J^-)_{k,n+k}\) if and only if \( q_k \neq 0\) for all \( k \).

We have seen that equation (3.19) admits a solution if and only if \( u \) satisfies (3.21) and \( q \) is regular (2.26). The proof is finished by noting that the remaining equation (3.18) can always be solved for \( J_+^1 \) if \( J_+^1 = -\pi_+^1(\xi(u)) \) is given, since \( \cosh ad_q \) yields an invertible map on \( G^+_s \).
Definition 3.3. Suppose that \( \kappa > 0 \) and \( x, y \) satisfy (3.11). For any \( q \in A_c \) and \( p \in A \) define the function \( J(q, p) \) by the formula

\[
J(q, p) := -xC^d - \xi(u^c) + L(q, p),
\]

(3.27)

where \( (u^c)_j = \sqrt{2\kappa} \) (\( j = 1, ..., n \)) and the matrix elements of \( L(q, p) = \pi(J(q, p)) \) are the following. Firstly, if \( 1 \leq j < k \leq m \) or \( m < j < k \leq n \), then

\[
L_{j,k} = -L_{k,j} = -L_{j+n,k+n} = L_{k+n,j+n} = -i\kappa \coth(q_j - q_k),
\]

(3.28)

\[
L_{j,k+n} = L_{k,j+n} = -L_{j+n,k} = -L_{k+n,j} = -i\kappa \coth(q_j + q_k).
\]

(3.29)

Secondly, if \( 1 \leq j \leq m \) and \( m < k \leq n \), then

\[
L_{j,k} = -L_{k,j} = -L_{j+n,k+n} = L_{k+n,j+n} = -i\kappa \tanh(q_j - q_k),
\]

(3.30)

\[
L_{j,k+n} = L_{k,j+n} = -L_{j+n,k} = -L_{k+n,j} = -i\kappa \tanh(q_j + q_k).
\]

(3.31)

Finally, for any \( 1 \leq j \leq m \), \( m < k \leq n \), and \( 1 \leq l \leq n \), we have

\[
L_{j,j+n} = -L_{j+n,j} = -\frac{iy}{\sinh(2q_j)} - i\kappa \coth(2q_l),
\]

(3.32)

\[
L_{k,k+n} = -L_{k+n,k} = \frac{iy}{\sinh(2q_k)} - i\kappa \coth(2q_k),
\]

(3.33)

\[
L_{l,l} = -L_{l+n,l+n} = p_l.
\]

(3.34)

Theorem 3.4. By using the above definition of \( J(q, p) \), consider the set

\[
S = \{ (e^q, J(q, p), xC^d + \xi(u^c)) \mid q \in A_c, p \in A \}.
\]

(3.35)

The submanifold \( S \subset P \) lies in the constraint surface \( P_c \) (3.10) and intersects every orbit of \( G_+ \times G^+ \) in \( P_c \) precisely in one point. The pull-back \( \Omega_S \) of the symplectic form \( \Omega \) (3.2) on \( S \) is given by

\[
\Omega_S = \sum_{k=1}^n dp_k \wedge dq_k.
\]

(3.36)

Thus the symplectic manifold \((S, \Omega_S)\) provides a model of the reduced phase space \((P_{\text{red}}, \Omega_{\text{red}})\) (3.12), which can be identified with the cotangent bundle \( T^*A_c \).

Proof. We know that every \( g \in G \) can be decomposed according to (2.27), and Lemma 3.2 implies that every gauge orbit (i.e. \( G_+ \times G^+ \) orbit) in \( P_c \) admits a representative of the form

\[
(e^q, J, xC^d + \xi(u^c)) \quad \text{with} \quad q \in A_c.
\]

(3.37)

where \( A_c \) is an open Weyl chamber (for example the one defined in (2.26)). Following the proof of Lemma 3.2, it is easy to check that \( J \) in (3.37) can be written as \( J = J(q, p) \) in (3.27) with some \( p \in A \). Indeed, the formula (3.27) was obtained by directly solving the constraint
equations listed in the proof of Lemma 3.2. To check that \( S \) intersects every gauge orbit only once, suppose that we have
\[
(\eta e^q h^{-1}, \eta J(q, p) h^{-1}, x C^d + \eta \xi(u^c) h^{-1}) = (e^{q'}, J(q', p'), x C^d + \xi(u^c)), \quad (\eta, h) \in G_+ \times G_+, \quad (3.38)
\]
for two triples in \( S \). The uniqueness property of the decomposition \((2.27)\) entails that \( e^q = e^{q'} \), which is equivalent to \( q = q' \), and \( (\eta, h) = (\mu, \mu) \) for some \( \mu \in M \). Then it follows from the second component of the equality in \((3.38)\) that \( p = p' \) holds, i.e., the two representatives of the orbit coincide. Incidentally, the equality \( \mu \xi(u^c) \mu^{-1} = \xi(u^c) \) implies that \( \mu \in M \) must belong to the centre of \( M \), which is isomorphic to the group \( \mathbb{Z}_n \) and equals the centre of \( G \). The corresponding subgroup \((\mathbb{Z}_n)_{\text{diag}} < M_{\text{diag}}\) acts trivially on \( P \), and hence we can also conclude that the factor group \((G_+ \times G_+)/(\mathbb{Z}_n)_{\text{diag}}\) acts freely on the constraint surface \( P_c \).

We can compute the pull-back of \( \Omega \ (3.2) \) on the submanifold \( S \subset P \), which gives the formula \((3.36)\). Since we have seen that \( S \) is a global cross section of the gauge orbits in \( P_c \), it follows that \((S, \Omega_S)\) represents a model of the reduced phase space \((P_{\text{red}}, \Omega_{\text{red}})\). Referring to the identification \( \mathcal{A} \simeq \mathcal{A}^* \) defined by the scalar product \((2.19)\) of \( \mathcal{G} \), \((S, \Omega_S)\) is symplectomorphic to the cotangent bundle \( T^* \mathcal{A}_c \simeq \mathcal{A}_c \times \mathcal{A}^* \) equipped with the Darboux symplectic form.

\[ \square \]

Let us recall that a Hamiltonian given by a smooth function on a \( 2n \) dimensional symplectic manifold is called \textit{Liouville integrable} if it is contained in a family of \( n \) functionally independent, globally smooth functions on the phase space whose mutual Poisson brackets vanish and their Hamiltonian flows are complete. Now the following result is an immediate consequence of the Hamiltonian reduction.

\textbf{Corollary 3.5.} A family of functionally independent Hamiltonians that are in involution with respect to the canonical Darboux Poisson structure on \( T^* \mathcal{A}_c \) is provided by
\[
H_{k}^{\text{red}} = \frac{1}{4k} \text{tr}(J(q, p)^{2k}), \quad k = 1, \ldots, n. \quad (3.39)
\]
The generalized Sutherland Hamiltonian \( H(q, p) \ (1.1) \) is Liouville integrable, since it obeys
\[
H(q, p) = \frac{1}{4} \text{tr}(J(q, p)^2) = H_1^{\text{red}}(q, p). \quad (3.40)
\]

\textbf{Proof.} The reduced Hamiltonians \((3.39)\) are in involution with respect to the canonical Poisson structure derived from \( \Omega_S \ (3.36) \) since the original Hamiltonians \( H_k \ (3.4) \) are in involution with respect to the Poisson structure on \((P, \Omega)\). By using Definition 3.3, the identity \((3.40)\) is a matter of direct verification. At generic points of the phase space, the Hamiltonians \((3.39)\) are independent, since they start with independent ‘leading terms’ given by respective homogeneous polynomials in \( p_1, \ldots, p_n \). The reduction guarantees that the corresponding Hamiltonian flows are complete, and thus \( H_k^{\text{red}} \) (and in particular \( H = H_1^{\text{red}} \)) is Liouville integrable. \[ \square \]
Finally, let us describe how the flows of the reduced Hamiltonians $H^\text{red}_k$ can be constructed from the ‘free flows’ given in (3.3). Take an arbitrary initial value $(q(0), p(0))$. As a consequence of the Hamiltonian reduction, the corresponding solution $(q(t), p(t))$ of Hamilton’s equation for $H^\text{red}_k$ can be read off from the equality

$$
(e^{q(t)}, J(q(t), p(t)), xC^d + \xi(u^\kappa)) =
$$

$$
= (\eta(t)e^{V_k}e^{q(0)}h(0)^{-1}, \eta(t)J(q(0), p(0))\eta(t)^{-1}, \eta(t)(xC^d + \xi(u^\kappa))\eta(t)^{-1}),
$$

where

$$
V_k = J(q(0), p(0))^{2k-1} - \frac{1}{2n}\text{tr}(J(q(0), p(0))^{2k-1}) \mathbf{1}_{2n}
$$

(3.42)

and $(\eta(t), h(t)) \in G_+ \times G^+$ is determined by the condition that the left-hand-side of (3.41) must belong to the gauge slice $S$ (3.35). Thus, finding the solution requires the determination of the generalized Cartan decomposition

$$
\eta(t) = \pi_- (\eta(t))J(q(0), p(0))\eta(t)^{-1} = \pi_- (\eta(t))L(q(0), p(0))\eta(t)^{-1}.
$$

(3.44)

This means that the entries of the diagonal matrix $e^{2q(t)}D_m$ are the eigenvalues of the Hermitian matrix

$$
e^{V_k}e^{q(0)}D_m e^{V_k} = e^{q(t)}D_m e^{q(t)}
$$

(3.43)

and the Weyl alcove (2.29) that the eigenvalues of the above Hermitian matrix are all different, and therefore finding $q(t)$ boils down to an ordinary diagonalization problem.

The above algorithm could be particularly useful to analyze the generalized Sutherland dynamics, which arises as the $k = 1$ special case of the reduced systems (3.3). The formula (1.1) entails that in this case $p(t) = \dot{q}(t)$. If desired, one could also use the above derivation to obtain a Lax representation for the equations of motion, taking $J(q, p)$ in (3.27), or alternatively $L(q, p)$, as the Lax matrix. However, the Lax pair would not give anything substantial to our knowledge about the generalized Sutherland model, whose main features follow from its realization as a reduction of the free geodesic motion on $SU(n, n)$. 

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4 Conclusion

In this paper we applied the Hamiltonian reduction approach to a particular many-body system. Our derivation of the generalized Sutherland model (1.1) by reduction of the free geodesic motion on the group $SU(n,n)$ proves the integrability of the model in the new case of three independent coupling constants, and provides a simple algorithm for constructing the solutions. This potentially paves the way for future work to analyze the scattering characteristics of the model along the lines of the papers [14, 13]. The investigation of the quantum mechanics of the model (for example by quantum Hamiltonian reduction) is also a challenging problem.

Another interesting problem is to find duality properties for the generalized Sutherland model, which would extend the action-angle dualities of the integrable many-body systems studied by Ruijsenaars (see e.g. the review [15]). This problem exists in general for the Sutherland models with two types of particles, whose duality properties are not even known in the $A_n$ case. For the description of dualities in the reduction approach, see also [3, 4] and references therein.

Recently [1] new integrable random matrix models have been constructed in association with certain integrable many-body systems of Calogero-Sutherland type. It could be feasible to extend this correspondence between random matrix models and integrable many-body systems to other cases, possibly including generalized Sutherland models with two types of particles.

We end with a remark on the Lax matrices that can be associated to the model (1.1). Namely, we note that our usage of $SU(n,n)$ as the starting point leads to a $2n \times 2n$ Lax matrix, but it should be also possible to derive a $(2n+1) \times (2n+1)$ Lax matrix for the same model, with 3 independent couplings, by reduction of the free motion on $SU(n+1,n)$ (cf. Ref. [5]). In the case of equal couplings ($x = 2y = 2\sqrt{2}\kappa$ in (1.1)), it is this latter Lax matrix that one may expect to obtain directly as well from the standard Lax matrix of the original Sutherland model of $2n+1$ particles by applying imaginary shifts and restriction to ‘mirror symmetric’ configurations. Although in the reduction approach the role of the Lax matrices is somewhat secondary, they are central in other approaches [15, 17]. For this reason, we plan to describe the alternative Lax matrices of different size and their relationship elsewhere.

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References

[1] E. Bogomolny, O. Giraud and C. Schmit, Integrable random matrix ensembles, arXiv:1104.3777 [nlin.CD]

[2] F. Calogero, Exactly solvable one-dimensional many-body problems, Lett. Nuovo Cim. 13, 411-416 (1975)

[3] L. Fehér and V. Ayadi, Trigonometric Sutherland systems and their Ruijsenaars duals from symplectic reduction, J. Math. Phys. 51, 103511 (2010), arXiv:1005.4531 [math-ph]
[4] L. Fehér and C. Klimčík, Poisson-Lie interpretation of trigonometric Ruijsenaars duality, Commun. Math. Phys. 301, 55-104 (2011), arXiv:0906.4198 [math-ph]

[5] L. Fehér and B.G. Pusztai, A class of Calogero type reductions of free motion on a simple Lie group, Lett. Math. Phys. 79, 263-277 (2007), arXiv:math-ph/0609085

[6] L. Fehér and B.G. Pusztai, Derivations of the trigonometric BC(n) Sutherland model by quantum Hamiltonian reduction, Rev. Math. Phys. 22, 699-732 (2010), arXiv:0909.5208 [math-ph]

[7] M. Hashizume, Geometric approach to the completely integrable Hamiltonian systems attached to the root systems with signature, Adv. Stud. Pure Math. 4, 291-330 (1984)

[8] N. Nekrasov, Infinite-dimensional algebras, many-body systems and gauge theories, pp. 263-299 in: Moscow Seminar in Mathematical Physics, AMS Transl. Ser. 2, Vol. 191, 1999

[9] A. Oblomkov, Heckman-Opdam’s Jacobi polynomials for the BC_n root system and generalized spherical functions, Adv. Math. 186, 153-180 (2004), arXiv:math/0202076 [math.RT]

[10] M.A. Olshanetsky and V.-B. K. Rogov, Bound states in completely integrable systems with two types of particles, Ann. Inst. H. Poincaré XXIX, 169-177 (1978)

[11] J.-P. Ortega and T.S. Ratiu, Momentum Maps and Hamiltonian Reduction, Progress in Mathematics 222, Birkhäuser, 2004

[12] A.M. Perelomov, Integrable Systems of Classical Mechanics and Lie Algebras, Birkhäuser, 1990

[13] B.G. Pusztai, On the scattering theory of the classical hyperbolic C_n Sutherland model, J. Phys. A: Math. Theor. 44, 155306 (2011), arXiv:1010.4663 [math-ph]

[14] S.N.M. Ruijsenaars, Action-angle maps and scattering theory for some finite-dimensional integrable systems II. Solitons, antisolitons and their bound states, Publ. RIMS 30, 865-1008 (1984)

[15] S.N.M. Ruijsenaars, Systems of Calogero-Moser type, pp. 251-352 in: Proceedings of the 1994 CRM–Banff Summer School ‘Particles and Fields’, Springer, 1999

[16] H. Schlichtkrull, Harmonic analysis on semisimple symmetric spaces, pp. 91-225 in: G. Heckman and H. Schlichtkrull, Harmonic Analysis and Special Functions on Symmetric Spaces, Perspectives in Mathematics 16, Academic Press, 1994

[17] B. Sutherland, Beautiful Models, World Scientific, 2004