Cogenerators of 2-isometric Hilbert space endomorphism semigroups

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Abstract

We observe a necessary and sufficient condition for a bounded linear Hilbert space operator to be the cogenerator of a $C_0$-semigroup of 2-isometries.

1 Introduction

Let $\mathcal{H}$ denote a complex Hilbert space, not necessarily separable, and $\mathcal{L} = \mathcal{L}(\mathcal{H})$ the corresponding space of bounded linear transformations. We also use $\mathcal{E}$ to denote an auxiliary Hilbert space. We are typically concerned with a situation where $\mathcal{E}$ is a certain subspace of $\mathcal{H}$.

We define a few subclasses of $\mathcal{L}$ as follows: For $T \in \mathcal{L}$, let

$$\beta_1(T) := T^*T - I, \quad \text{and} \quad \beta_2(T) := T^{*2}T^2 - 2T^*T + I.$$

We say that $T$ is isometric if $\beta_1(T) = 0$, contractive if $\beta_1(T) \leq 0$, expansive if $\beta_1(T) \geq 0$, 2-isometric if $\beta_2(T) = 0$, and concave if $\beta_2(T) \leq 0$. We also define the corresponding classes of $C_0$-semigroups by considering their element-wise properties, i.e. an isometric semigroup is a $C_0$-semigroup $(T_t)_{t \geq 0} \in \mathcal{L}$ such that each $T_t$ is isometric, etc. The primary concern of the present paper is the class of 2-isometric semigroups.

The study of 2-isometries was initiated by Agler [1]. Recently, Gallardo-Gutierrez and Partington have considered $C_0$-semigroups of 2-isometries [6]. A few basic facts about these notions are reviewed in Section 2 together with notation, and some other preliminary material.

The relation between contractive semigroups and their cogenerators is classical, see [16, Chapter III, Section 8]. Similar investigations for other classes of semigroups are more recent [13, 7, 6, 3]. The next result can essentially be traced back to an early preprint of [7]. We refer to [6, Proposition 2.2], and [3, Section 4.2] for details:

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Proposition 1.1. Suppose that \((T_t)_{t \geq 0}\) is a 2-isometric (concave) semigroup. Then \((T_t)_{t \geq 0}\) admits a cogenerator \(T\), which is 2-isometric (concave). Conversely, if \(T\) is the cogenerator of a \(C_0\)-semigroup \((T_t)_{t \geq 0}\), and \(T\) is 2-isometric (concave), then \((T_t)_{t \geq 0}\) is 2-isometric (concave).

Remark 1.2. The given references assume that the underlying space \(H\) is separable. However, this hypothesis is never utilized in the proofs of the above result.

Note how the converse statement in Proposition 1.1 is conditioned on the fact that \(T\) is a cogenerator. In the present paper we extend the 2-isometric case of Proposition 1.1. In particular, we give necessary and sufficient conditions for a 2-isometry to be the cogenerator of a \(C_0\)-semigroup. In order to make the statement more manageable, we give separate formulations for the direct and converse results:

Theorem 1.3. Let \((T_t)_{t \geq 0}\) be a 2-isometric semigroup. Then \((T_t)_{t \geq 0}\) admits a 2-isometric cogenerator \(T\), with \(1/\in \sigma_p(T)\). Moreover, \((T_t)_{t \geq 0}\) is quasicontractive for some parameter \(w \geq 0\), and \(T\) satisfies
\[
\langle \beta_1(T)x, x \rangle_H \leq w\|x\|^2_H, \quad x \in H.
\]

The optimal (smallest) value of the parameter \(w\) is given by Proposition 3.2.

Theorem 1.4. Let \(T \in L(H)\) be a 2-isometry. Assume further that \(1/\in \sigma_p(T)\), and that there exists \(w \geq 0\) for which (1.1) is satisfied. Then \(T\) is the cogenerator of a \(C_0\)-semigroup \((T_t)_{t \geq 0}\). This is 2-isometric, and quasicontractive with parameter \(w\).

In Theorem 1.3, the fact that \(T\) exists and is 2-isometric is of course already contained in Proposition 1.1. An analogous statement holds for Theorem 1.4. We also remark that the condition \(1/\in \sigma_p(T)\) is a mere algebraic consequence of how cogenerators are defined. Hence, the essential condition in the above theorems is (1.1).

Once we have established that any 2-isometric semigroup is quasicontractive (Proposition 3.2), the necessity of (1.1) is immediate from a Hille–Yosida-type theorem (Theorem 2.1). This proves Theorem 1.3. The details are carried out in Section 3.

In order to prove Theorem 1.4, we need three main ingredients: A Wold-type decomposition for 2-isometries (Theorem 2.9), a functional model for analytic 2-isometries (Theorem 2.10), and some theory for harmonically weighted Dirichlet spaces of analytic functions.

The first of these results, due to Shimorin [15], states that any 2-isometry \(T\) may be written as a direct sum of a unitary operator \(T_u\), and an analytic 2-isometry \(T_a\). By some classical properties of unitary cogenerators, this allows us to reduce the proof of Theorem 1.4 to the case where \(T\) is analytic.

Suppose that \(T\) is an analytic 2-isometry, and let \(E = H \ominus TH\). A result conceived by Richter [11] (the case where \(\dim E = 1\), and extended by Olofsson [10]...
(the general case), states that \( T \) is unitarily equivalent to the operator \( M_z \), multiplication by the identity function \( z \mapsto z \), acting on the harmonically weighted Dirichlet space \( D^2_{\mu}(\mathcal{E}) \). This is a space of \( \mathcal{E} \)-valued analytic functions on \( \mathbb{D} \). The parameter \( \mu \) is an \( \mathcal{L}_+(\mathcal{E}) \)-valued measure on \( T \). The correspondence between \( T \) and \( \mu \) is essentially bijective. In terms of this functional model, condition (1.4) becomes

\[
\frac{1}{2\pi} \int_T \langle d\mu(\zeta), f(\zeta) \rangle E \leq w \| (I - M_z)f \|_{D^2_{\mu}(\mathcal{E})}^2, \quad f \in \mathcal{P}_a(\mathcal{E}),
\]

where \( \mathcal{P}_a(\mathcal{E}) \) denotes the space of analytic polynomials with coefficients in \( \mathcal{E} \).

Now let \( M_{\phi_t} \) denote the operator of multiplication by the analytic function \( \phi_t : z \mapsto \exp \left( \frac{\zeta - 1}{\zeta + 1} \right) \). The main idea of this paper is that an analytic \( 2 \)-isometry \( T \) is the cogenerator of a \( C_0 \)-semigroup on \( \mathcal{H} \) if and only if \( (M_{\phi_t})_{t \geq 0} \) is a \( C_0 \)-semigroup on the corresponding Dirichlet space \( D^2_{\mu}(\mathcal{E}) \). After reduction to the analytic case, Theorem 1.4 follows from:

**Theorem 1.5.** Let \( \mu \) be an \( \mathcal{L}_+(\mathcal{E}) \)-valued measure on \( T \). Then the following are equivalent:

(i) The family \( (M_{\phi_t})_{t \geq 0} \subset \mathcal{L}(D^2_{\mu}(\mathcal{E})) \) is a \( C_0 \)-semigroup.

(ii) The operator \( M_z \), acting on \( D^2_{\mu}(\mathcal{E}) \), is the cogenerator of a \( C_0 \)-semigroup.

(iii) There exists \( w_1 \geq 0 \) such that

\[
\frac{1}{2\pi} \int_T \langle d\mu(\zeta), f(\zeta) \rangle E \leq w_1 \| (I - M_z)f \|_{D^2_{\mu}(\mathcal{E})}^2, \quad f \in \mathcal{P}_a(\mathcal{E}).
\]  

(iv) There exists \( w_2 > 0 \) such that, for each \( x \in \mathcal{E} \), the measure \( \mu_{x,x} : E \mapsto \langle \mu(E)x, x \rangle E \) satisfies \( \int_T \frac{d\mu_{x,x}(\zeta)}{1 - \zeta^2} \leq w_2 \| x \|_E^2 \). Moreover, the set function \( E \mapsto \int_E \frac{d\mu(\zeta)}{1 - \zeta^2} \) is an \( \mathcal{L}_+(\mathcal{E}) \)-valued measure, and

\[
\frac{1}{2\pi} \int_T \langle d\mu(\zeta), f(\zeta), f(\zeta) \rangle E \leq w_2 \| f \|_{D^2_{\mu}(\mathcal{E})}^2, \quad f \in \mathcal{P}_a(\mathcal{E}).
\]

If either of the above conditions is satisfied, then the semigroup \( (M_{\phi_t})_{t \geq 0} \) is \( 2 \)-isometric, has cogenerator \( M_z \), and is quasicontractive with some parameter \( w \geq 0 \). The optimal values for \( w \), \( w_1 \) and \( w_2 \) coincide.

The proofs of Theorem 1.4 and Theorem 1.5 are presented in Section 4. However, the following observation seems to be of independent interest:

A key component in the proof of Theorem 1.4 is Lemma 1.7. In the scalar-valued case, the hypothesis of this lemma may be relaxed; if \( \mu \) is any scalar-valued positive finite regular Borel measure, then the multiplication formula (1.5) holds. This implies that the family \( (M_{\phi_t})_{t \geq 0} \) defines a \( C_0 \)-semigroup on \( D^2_{\mu} \) if and only if there exists \( t > 0 \) for which \( M_{\phi_t} \in \mathcal{L}(D^2_{\mu}) \). It does not seem evident whether or not this is true in the general case.
Theorem 1.5 allows one to obtain plenty of examples of 2-isometric semi-groups by choosing appropriate measures $\mu$. It also yields non-examples. Consider for instance the classical Dirichlet space, i.e. the class of analytic functions $f : D \to \mathbb{C}$ such that $\int_D |f'(z)|^2 dA(z) < \infty$. Equipped with the appropriate norm, this is the space $D^2_\lambda$, where $\lambda$ denotes Lebesgue measure on $\mathbb{T}$. $\lambda$ clearly violates condition $(iv)$ of Theorem 1.5. Hence, $M_\pi$ acting on $D^2_\lambda$ is not the cogenerator of a $C_0$-semigroup.

2 Notation and preliminaries

We use the standard notation $D := \{z \in \mathbb{C}; |z| < 1\}$ for the open unit disc, and $T := \{z \in \mathbb{C}; |z| = 1\}$ for the unit circle of the complex plane $\mathbb{C}$. By $\lambda$ we denote Lebesgue (arc length) measure on $\mathbb{T}$, while $dA$ will signify integration with respect to area measure on $\mathbb{C}$. We also use $\partial \Omega$ and $\Omega$ to denote the boundary and closure of $\Omega \subset \mathbb{C}$, respectively.

Given a linear operator $A$ with domain $D(A) \subset \mathcal{H}$, we let $\sigma(A)$, $\sigma_p(A)$, $\sigma_{ap}(A)$, and $W(A)$ respectively denote the spectrum, point spectrum, approximate point spectrum, and numerical range of $A$, i.e.

$$\sigma(A) := \{z \in \mathbb{C}; z - A : D(A) \to \mathcal{H} \text{ is not bijective}\},$$

$$\sigma_p(A) := \{z \in \mathbb{C}; z - A : D(A) \to \mathcal{H} \text{ is not injective}\},$$

$$\sigma_{ap}(A) := \{z \in \mathbb{C}; z - A : D(A) \to \mathcal{H} \text{ is not bounded below}\},$$

$$W(A) := \{(Ay, y)_{\mathcal{H}} \in \mathbb{C}; y \in D(A)\}.$$

We also let $\rho(A) := \mathbb{C} \setminus \sigma(A)$.

Given a family $\mathcal{S}$ of subsets of $\mathcal{H}$, we let $\bigvee_{S \in \mathcal{S}} S$ denote the smallest closed subspace of $\mathcal{H}$ that contains each $S \in \mathcal{S}$.

2.1 $C_0$-semigroups

By a semigroup we mean a family $(T_t)_{t \geq 0} \subset \mathcal{L}$ such that $T_0 = I$, and $T_{s+t} = T_s T_t$ for $s, t \geq 0$. A $C_0$-semigroup (or strongly continuous semigroup) is a semigroup which is continuous with respect to the strong operator topology, i.e. for any $x \in \mathcal{H}$, the function $\varphi_x : t \mapsto T_t x$ is continuous. By its defining properties, a semigroup is strongly continuous if and only if each $\varphi_x$ is continuous at $t = 0$.

Throughout this note, we let $(T_t)_{t \geq 0}$ denote a $C_0$-semigroup. For a detailed treatment of the facts outlined below, we refer to Chapter II.

Strong continuity together with the principle of uniform boundedness implies that the family $(T_t)_{0 \leq t \leq 1} \subset \mathcal{L}$ is bounded. The semigroup property implies that there exists $M \geq 1$ and $w \in \mathbb{R}$ such that $\|T_t\|_{\mathcal{L}} \leq Me^{wt}$. We refer to this as the exponential boundedness property.

The (infinitesimal) generator of $(T_t)_{t \geq 0}$ is the operator $A$ given by

$$Ax = \lim_{t \to 0^+} \frac{T_t x - x}{t}.$$
Its domain $D(A)$ is the subspace of all $x \in \mathcal{H}$ for which the above limit exists. The operator $A$ is closed, densely defined, and uniquely determines the semigroup.

For $x \in \mathcal{H}$, and $z \in \mathbb{C}$, consider the (formal) integral $\int_0^\infty T_t x e^{-zt} \, dt$. If $z$ has sufficiently large real part, then the integral is absolutely convergent. Since the function $\varphi_x : t \mapsto T_t x$ is continuous, the integral is then well-defined in the sense of a generalized Riemann integral. Moreover, it is easily seen to be a bounded right inverse of the operator $z - A$. We express these considerations as

$$(z - A)^{-1} = \int_0^\infty T_t e^{-zt} \, dt. \quad (2.1)$$

Suppose now that $1 \not\in \sigma(A)$. This allows us to define the \textit{cogenerator} of $(T_t)_{t \geq 0}$ as the (negative) Cayley transform of $A$, i.e.

$$T = (A + I)(A - I)^{-1} = I + 2(A - I)^{-1}.$$  

$T$ is a bounded operator, and $T - I = 2(A - I)^{-1}$. Since $D(A) = (I - A)^{-1} \mathcal{H}$, one obtains that $D(A) = (T - I)\mathcal{H}$. Moreover, $(T - I)^{-1} = \frac{1}{2}(A - I)$ exists in the sense of a left inverse, so $1 \not\in \sigma_p(T)$.

If $T$ is the cogenerator of $(T_t)_{t \geq 0}$, then we may recover the generator $A$ by another application of the Cayley transform. Generally speaking, suppose that $T \in \mathcal{L}$ and $1 \not\in \sigma_p(T)$. We may then define the operator

$$A = (T + I)(T - I)^{-1} = I + 2(T - I)^{-1}, \quad D(A) = (T - I)\mathcal{H}.$$  

Since $A - I = 2(T - I)^{-1}$, we conclude that $A - I : D(A) \to \mathcal{H}$ is a bijection, i.e. $1 \not\in \sigma(A)$.

An operator $T \in \mathcal{L}$ with $1 \not\in \sigma_p(T)$ is the cogenerator of a $C_0$-semigroup if and only if its Cayley transform $A$ is the generator of a $C_0$-semigroup. There is a multitude of theorems that characterize generators corresponding to certain classes of $C_0$-semigroups. One such result is the following theorem by Hille and Yosida [5, Chapter II, 3.6 Corollary]:

\textbf{Theorem 2.1.} Given $w \in \mathbb{R}$, and a linear Hilbert space operator $A$, with domain $D(A)$, the following are equivalent:

\begin{enumerate}[(i)]
  \item $A$ is the generator of a $C_0$-semigroup $(T_t)_{t \geq 0}$ satisfying
  \begin{equation}
  \|T_t\|_{\mathcal{L}} \leq e^{wt}. \quad (2.2)
  \end{equation}
  \item $A$ is closed, densely defined, and for each $v > w$ the operator $(v - A)^{-1}$ is well-defined and satisfies
    \begin{equation}
    \|(v - A)^{-1}\|_{\mathcal{L}} \leq \frac{1}{v - w}. \quad (2.3)
    \end{equation}
\end{enumerate}
Remark 2.2. The above statement is also true in the context of Banach spaces, but this is beyond what we need.

The quantitative condition (2.2) is significantly stronger than the general exponential boundedness property. A $C_0$-semigroup satisfying this stronger condition is called \textit{quasicontractive with parameter $w$}.

Theorem 2.1 has an equivalent formulation in terms of cogenerators. We include a proof, even though it is probably known among experts:

\textbf{Theorem 2.3.} Let $w \in \mathbb{R}$, and $T \in \mathcal{L}$. The following are equivalent:

(i) $T$ is the cogenerator of a quasicontractive $C_0$-semigroup with parameter $w$.

(ii) The operator $T - I$ is injective, has dense range, and

$$\langle \beta_1(T)x, x \rangle_H = \|Tx\|^2 - \|x\|^2 \leq w\|(T - I)x\|^2, \quad x \in H. \quad (2.4)$$

\textbf{Proof.} Assuming that (i) holds, there exists a $C_0$-semigroup $(T_t)_{t \geq 0}$, which is quasicontractive with parameter $w \geq 0$, and has a cogenerator $T$. We already noted that $T - I$ is injective, and that the corresponding generator is given by

$$A = (T + I)(T - I)^{-1} = I + 2(T - I)^{-1}, \quad D(A) = (T - I)H.$$

By Theorem 2.1 we know that $A$ is densely defined, so $T - I$ must have dense range. Furthermore, if $v > w$, then $v - A : D(A) \to H$ is a bijection, so (2.3) holds if and only if

$$(v - w)^2\|y\|^2_H \leq \|(v - A)y\|^2_H, \quad y \in D(A).$$

By elementary properties of the Hilbert space metric, this is equivalent to

$$2v \operatorname{Re}(Ay, y)_H + w^2\|y\|^2_H \leq \|Ay\|^2_H + 2vw\|y\|^2_H, \quad y \in D(A).$$

The former equality implies that the latter one is satisfied for $v = w$. Since both sides of the latter one depend linearly on $v$, this is satisfied for every $v > w$ if and only if

$$\operatorname{Re}(Ay, y)_H \leq w\|y\|^2_H \quad \text{for} \quad y \in D(A).$$

We now use that any $y \in D(A)$ satisfies $y = (T - I)x$ for some $x \in H$. Under this correspondence, $Ay = (T + I)x$. In terms of the cogenerator, the above condition therefore becomes

$$\|Tx\|^2_H - \|x\|^2_H \leq w\|(T - I)x\|^2_H, \quad x \in H,$$

which shows that (2.3) implies (2.4). This establishes that (i) $\Rightarrow$ (ii).

Assuming (ii), we may define

$$A = (T + I)(T - I)^{-1} = I + 2(T - I)^{-1}, \quad D(A) = (T - I)H.$$
Clearly, \( D(A) \) is dense. The steps showing that (2.3) implies (2.4) are easily reversed. We now show that \( A \) is closed.

Given \( y \in D(A) \), we use again that \( y = (T-I)x \), and \( Ay = (T+I)x \) for some \( x \in \mathcal{H} \). Suppose that \( y_n = (T-I)x_n \in D(A) \), \( y_n \to y \), and \( Ay_n \to z \). Then \( \frac{1}{2}(Ay_n - y_n) = x_n \to x := \frac{1}{2}(z - y) \), and \( Tx = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} \frac{1}{2}(Ay_n + y_n) = \frac{1}{2}(z + y) \). It follows that \( y = (T-I)x \in (T-I)\mathcal{H} \), and \( Ay = (T+I)x = z \).

Since \( A \) is a closed, densely defined operator, and satisfies (2.3), Theorem 2.1 implies that \( A \) is the generator of a quasicontractive \( C_0 \)-semigroup with parameter \( w \). Clearly, the cogenerator of this semigroup is \( T \), which shows that (ii) \( \Rightarrow \) (i).

Consider for a moment the case where \( T \in \mathcal{L} \) is a contraction. Then (2.4) is trivially satisfied with \( w = 0 \). It seems less obvious that if \( T \) is a contraction and \( T-I \) is injective, then \( T-I \) also has dense range. Nevertheless, this is implied by the following result:

\textbf{Theorem 2.4} ([16, Chapter III, Theorem 8.1]). A contraction \( T \in \mathcal{L} \) is the cogenerator of a \( C_0 \)-semigroup if and only if \( 1 \notin \sigma_p(T) \).

In the above reference, the proof of this result is based on a sophisticated functional calculus for contractions. While the same can of course be said about the Hille-Yosida theorem, it still seems worthwhile to present the following alternative proof based on Theorem 2.3:

\textbf{Proof of Theorem 2.4} We need to show that if \( T \) is a contraction, and \( T-I \) is injective, then \( T-I \) has dense range.

For the sake of contradiction, suppose that it does not. By duality, this means that \( T^*-I \) is not injective, i.e. there exists \( x \in \mathcal{H} \setminus \{0\} \) such that \( T^*x = x \). But then \( \langle Tx, x \rangle = \langle x, T^*x \rangle = \|x\|^2 \), so

\[
\|Tx - x\|^2 = \|Tx\|^2 + \|x\|^2 - 2 \text{Re}\langle Tx, x \rangle = \|Tx\|^2 - \|x\|^2 \leq 0.
\]

Hence, \( Tx = x \), and \( T-I \) is not injective. \( \square \)

To conclude this subsection, we remark that several other properties carry over from a cogenerator to its semigroup, and vice versa. We will use the following instance, taken from [16, Chapter III, Proposition 8.2]:

\textbf{Proposition 2.5.} If \( T \) is the cogenerator of a \( C_0 \)-semigroup, and \( T \) is unitary, then the semigroup is also unitary.

\subsection*{2.2 Operator measures}

Let \( \mathcal{S} \) denote the Borel \( \sigma \)-algebra of subsets of \( \mathcal{T} \), and \( \mathcal{L}_+ = \mathcal{L}_+(\mathcal{E}) \) the convex cone of positive operators on the Hilbert space \( \mathcal{E} \). An \( \mathcal{L}_+ \)-valued measure is a finitely additive set function \( \mu: \mathcal{S} \to \mathcal{L}_+ \) with the property that for every
Let \( x, y \in \mathcal{H} \), the set function \( \mu_{x,y} : E \mapsto \langle \mu(E)x, y \rangle_{\mathcal{E}} \) defines a complex regular Borel measure. For each \( E \in \mathcal{S} \), it holds that
\[
|\mu_{x,y}(E)| \leq \mu_{x,x}(E)^{1/2} \mu_{y,y}(E)^{1/2}. \tag{2.5}
\]
This implies in turn that
\[
\sup_{\|x\|, \|y\| \leq 1} |\mu_{x,y}(E)| = \sup_{\|x\| \leq 1} \langle \mu(E)x, x \rangle_{\mathcal{E}} = \|\mu(E)\|_{\mathcal{L}(\mathcal{E})}.
\]
We refer to [9, Proposition 1.1 et seq.].

Given a bounded (Borel) measurable function \( f : T \to \mathbb{C} \), we can define the sesquilinear form \( I_f : (x, y) \mapsto \int_T f \, d\mu_{x,y} \). From the previous equalities, it is clear that
\[
|I_f(x, y)| \leq \|f\|_{\infty} \|\mu(T)\|_{\mathcal{L}(\mathcal{E})} \|x\|_{\mathcal{E}} \|y\|_{\mathcal{E}}.
\]
By standard functional analytic considerations, the above inequality implies the existence of a uniquely determined operator \( J \in \mathcal{L}(\mathcal{E}) \) such that \( \langle Jx, y \rangle_{\mathcal{E}} = \int_T f \, d\mu_{x,y} \). We denote the operator \( J \) by \( \int_T f \, d\mu \). The integral thus defined satisfies the triangle type inequality
\[
\left\| \int_T f \, d\mu \right\|_{\mathcal{L}(\mathcal{E})} \leq \|f\|_{\infty} \|\mu(T)\|_{\mathcal{L}(\mathcal{E})}.
\]

We will need the following version of Cauchy–Schwarz inequality:

**Lemma 2.6.** Let \( \mu \) be an \( \mathcal{L}^+ \)-valued measure. If \( x, y \in \mathcal{E} \), and \( f, g : T \to \mathbb{C} \) are Borel measurable functions, then
\[
\int_T |fg| \, d|\mu_{x,y}| \leq \left( \int_T |f|^2 \, d\mu_{x,x} \right)^{1/2} \left( \int_T |g|^2 \, d\mu_{y,y} \right)^{1/2}. \tag{2.6}
\]

**Proof.** When \( f \) and \( g \) are simple functions, (2.6) is just the Cauchy-Schwarz inequality for finite sums, by virtue of (2.5). General functions are approximated in the standard fashion. \( \square \)

Two instances of the above integral will be particularly interesting to us, namely the Fourier coefficients
\[
\hat{\mu}(n) = \frac{1}{2\pi} \int_T \overline{\zeta^n} \, d\mu(\zeta), \quad n \in \mathbb{Z},
\]
and the Poisson extension
\[
P_{\mu}(z) = \frac{1}{2\pi} \int_T \frac{1 - |z|^2}{|\zeta - z|^2} \, d\mu(\zeta), \quad z \in \mathbb{D}.
\]
Given \( z \in \mathbb{D} \), we have the two-sided power series expansion
\[
\frac{1 - |z|^2}{|\zeta - z|^2} = \sum_{n=0}^{\infty} z^n \overline{\zeta^n} + \sum_{n=1}^{\infty} \frac{z^n}{\overline{\zeta^n}},
\]
which converges uniformly for $\zeta \in \mathbb{T}$. Using term by term integration, we conclude that

$$P_\mu(z) = \sum_{n=0}^\infty \hat{\mu}(n)z^n + \sum_{n=1}^\infty \hat{\mu}(-n)\overline{z}^n.$$ 

Since $\|\hat{\mu}(n)\|_{\mathcal{L}(E)} \leq \frac{\|\mu(T)\|_{\mathcal{L}(E)}}{2\pi}$, we may apply the triangle inequality, and geometric summation to the above power series to conclude that

$$\|P_\mu(z)\|_{\mathcal{L}(E)} \leq \frac{1 + |z| \|\mu(T)\|_{\mathcal{L}(E)}}{1 - |z|} \frac{2\pi}{2\pi}. \tag{2.7}$$

With the above construction, the integral $\int f \, d\mu$ is only defined when $f$ is bounded. As a remedy for this, adequate for our purposes, we use the following construction: Let $\mu$ be an $\mathcal{L}_+-$valued measure, and $h$ a non-negative scalar-valued function. If there exists $C > 0$ such that

$$\int_T h \, d\mu_{x,x} \leq C \|x\|_E^2,$$

then one can define a new set function $\tilde{\mu}$ by

$$\langle \tilde{\mu}(E)x, y \rangle_E = \int_E h(\zeta) \, d\mu_{x,y}(\zeta).$$

By Lemma 2.6, the above right-hand side has modulus less than

$$\left( \int_T h \, d\mu_{x,x} \right)^{1/2} \left( \int_T h \, d\mu_{y,y} \right)^{1/2} \leq C \|x\|_E \|y\|_E.$$

This estimate implies that $\tilde{\mu}$ is another $\mathcal{L}_+-$valued measure. If $f$ is bounded, then we may take $\int f \, d\tilde{\mu}$ as a definition of $\int f h \, d\mu$.

### 2.3 Function spaces

The space of analytic polynomials $\sum_{k=0}^N a_k z^k$ with coefficients $a_k \in \mathcal{E}$ is denoted by $\mathcal{P}_a(\mathcal{E})$. As a notational convention, we write $\mathcal{P}_a$ in place of $\mathcal{P}_a(\mathbb{C})$. The same principle applies to all function spaces described below.

Let $f$ be a function which is analytic in a neighbourhood of the origin. The $k$th Maclaurin coefficient of $f$ is denoted by $\hat{f}(k)$. By $\mathcal{D}_a(\mathcal{E})$, we denote the space of $\mathcal{E}$-valued analytic functions whose Maclaurin coefficients $(\hat{f}(k))_{k=0}^\infty$ decay faster than any power of $k$. These are precisely the analytic functions on $\mathbb{D}$ which extend to smooth functions on $\overline{\mathbb{D}}$.

The Hardy space $H^2(\mathcal{E})$ consists of all functions $f : z \mapsto \sum_{k=0}^\infty \hat{f}(k)z^k$, where $f \in \mathcal{E}$ and $\|f\|_{H^2(\mathcal{E})} := \sum_{k=0}^\infty \|\hat{f}(k)\|_E^2 < \infty$. Using a standard radius of convergence formula, functions in $H^2(\mathcal{E})$ are seen to be analytic on $\mathbb{D}$. The subspaces $\mathcal{P}_a(\mathcal{E})$ and $\mathcal{D}_a(\mathcal{E})$ are dense in $H^2(\mathcal{E})$. A direct application of
Proposition 2.7

function $f$ on $\mathbb{D}$, and $f \in H^2(\mathcal{E})$.

We refer to this fact as the reproducing property. If $f \in H^2(\mathcal{E})$, then the radial boundary value $f(\zeta) := \lim_{r \to 1^-} f(r\zeta)$ exists for $\lambda$-a.e. $\zeta \in \mathbb{T}$. If $f \in H^2(\mathcal{E})$, and $\|f(\zeta)\|_\mathcal{E} = 1$ for $\lambda$-a.e. $\zeta \in \mathbb{T}$, then we say that $f$ is inner. For a general function $f \in H^2(\mathcal{E})$, the radial boundary value function satisfies

$$\|f\|_{H^2(\mathcal{E})}^2 = \frac{1}{2\pi} \int_\mathbb{T} \|f(\zeta)\|^2_\mathcal{E} \, d\lambda(\zeta). \quad (2.8)$$

In the case where $\mathcal{E} = \mathbb{C}$, these facts will be included in any reasonable introduction to Hardy spaces. Although the adaptation to general $\mathcal{E}$ is not very difficult, we still refer to either [8, Chapter III] or [12, Chapter 4] for beautiful presentations of the details.

Given an $L_+(\mathcal{E})$-valued measure $\mu$, and an analytic function $f : \mathbb{D} \to \mathcal{E}$, we define the corresponding Dirichlet integral

$$D_\mu(f) := \frac{1}{\pi} \int_\mathbb{D} \langle P_\mu(z), f'(z) \rangle_\mathcal{E} \, dA(z).$$

We define the Dirichlet space $D^2_\mu(\mathcal{E})$ as the space of functions $f \in H^2(\mathcal{E})$ for which $D_\mu(f) < \infty$. We equip $D^2_\mu(\mathcal{E})$ with the norm $\|f\|_{D^2_\mu(\mathcal{E})}$ given by

$$\|f\|^2_{D^2_\mu(\mathcal{E})} := \|f\|_{H^2(\mathcal{E})}^2 + D_\mu(f).$$

**Proposition 2.7** ([10] Corollary 3.1). Let $\mu$ be an $L_+(\mathcal{E})$-valued measure. Then $\mathcal{P}_\mu(\mathcal{E})$ is dense in the corresponding Dirichlet space $D^2_\mu(\mathcal{E})$.

Given an analytic function $f : \mathbb{D} \to \mathbb{C}$, and $\zeta \in \mathbb{T}$, we define the corresponding local Dirichlet integral

$$D_\zeta(f) := \frac{1}{\pi} \int_\mathbb{D} |f'(z)| \frac{1 - |z|^2}{|\zeta - z|^2} \, dA(z). \quad (2.9)$$

This is a convenient shorthand for the Dirichlet integral $D_{\delta_\zeta}(f)$, where $\delta_\zeta$ denotes a (scalar) unital point mass at $\zeta$. If $\mu$ is a positive scalar-valued measure, then it is immediate from Fubini’s theorem that

$$D_\mu(f) = \frac{1}{2\pi} \int_\mathbb{T} D_\zeta(f) \, d\mu(\zeta). \quad (2.10)$$

In particular, if $D_\mu(f) < \infty$, then $D_\zeta(f) < \infty$ for $\mu$-a.e. $\zeta \in \mathbb{T}$.

If $f, g \in D^2_\mu(\mathcal{E})$, then we define the polarized Dirichlet integral

$$D_\mu(f, g) := \frac{1}{\pi} \int_\mathbb{D} \langle P_\mu(z), f'(z), g'(z) \rangle_\mathcal{E} \, dA(z).$$
2.4 2-isometries

We observe the simple algebraic identity $\beta_2(T) = T^*\beta_1(T)T - \beta_1(T)$. If $T$ is a 2-isometry, i.e. $\beta_2(T) = 0$, then one obtains by induction that $T^{*k}\beta_1(T)T^k = \beta_1(T)$, and in particular that $\|T^{k+1}x\|^2 - \|T^kx\|^2 = \langle \beta_1(T)x, x \rangle$ for $x \in \mathcal{H}$. Summing over $k \in \{0, \ldots, n - 1\}$ yields:

**Proposition 2.8.** If $T \in \mathcal{L}$ is a 2-isometry, then

$$\|T^n x\|^2_H = \|x\|^2_H + n \langle \beta_1(T)x, x \rangle_H, \quad x \in \mathcal{H}, n \geq 0.$$  

Since $\|T^n x\|^2_H \geq 0$ for every $n$, this simple formula implies that 2-isometries are expanding. Proposition 2.8 also implies that the sequence $\{\|T^n\|_{\mathcal{L}}\}_{n \in \mathbb{Z}}$ has polynomial growth. By Gelfand’s formula for the spectral radius of an operator, $\sigma(T) \subseteq \overline{B}$. A slightly more careful analysis shows that $\sigma_{ap}(T) \subset T$. In particular, any eigenvalue of a 2-isometry has unit modulus. By the general fact that $\partial \sigma(T) \subset \sigma_{ap}(T)$, we also have that a 2-isometry $T$ is either not invertible, in which case $\sigma(T) = \overline{B}$, or it is invertible, and $\sigma(T) \subset T$. In the latter case, $T$ is in fact unitary. We refer to [2, Lemma 1.21 et seq.] for details.

An operator $T \in \mathcal{L}$ is called analytic if $\cap_{n \geq 0} T^n \mathcal{H} = \{0\}$. It is clear that an analytic operator cannot have any eigenvalues of unit modulus. In particular, an analytic 2-isometry does not have eigenvalues.

Analytic operators are important in the study of 2-isometries. A reason for this is the existence of a so-called Wold decomposition: For $T \in \mathcal{L}$, we let $\mathcal{E} = \mathcal{H} \ominus T\mathcal{H}$. The dimension of $\mathcal{E}$ is called the multiplicity of $T$. Furthermore, define the sets $\mathcal{H}_u = \cap_{n \geq 0} T^n \mathcal{H}$, and $\mathcal{H}_a = \cup_{n \geq 0} T^n \mathcal{E}$. The following is a special case of [15, Theorem 3.6]:

**Theorem 2.9.** If $T$ is a 2-isometry, then $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_a$. Moreover, the spaces $\mathcal{H}_u$ and $\mathcal{H}_a$ are invariant under $T$, $T_u := T|_{\mathcal{H}_u}$ is unitary, and $T_a := T|_{\mathcal{H}_a}$ is analytic.

The class of analytic 2-isometries with multiplicity 1 can be completely described in terms of $M_z$, multiplication by the function $z \mapsto z$, acting on harmonically weighted Dirichlet spaces [11]. This result was later generalized to arbitrary multiplicity by Olofsson [10]:

**Theorem 2.10.** Let $\mathcal{E}$ be a Hilbert space, and suppose that $\mu$ is an $\mathcal{L}_+(\mathcal{E})$-valued measure. Then $M_z$ acts as an analytic 2-isometry on the space $\mathcal{D}_\mu^2(\mathcal{E})$.

Conversely, suppose that $T \in \mathcal{L}(\mathcal{H})$ is an analytic 2-isometry, and let $\mathcal{E} = \mathcal{H} \ominus T\mathcal{H}$. Then there exists an $\mathcal{L}_+(\mathcal{E})$-valued measure $\mu$, and a unitary map $V : \mathcal{H} \to \mathcal{D}_\mu^2(\mathcal{E})$, such that $T = V^* M_z V$.

The above correspondence is unique, in the sense that $(M_z, \mathcal{D}_\mu^2(\mathcal{E}_1))$ and $(M_z, \mathcal{D}_\mu^2(\mathcal{E}_2))$ are unitarily equivalent if and only if there exists a unitary map $U : \mathcal{E}_1 \to \mathcal{E}_2$ such that $\mu_1(E) = U^* \mu_2(U) U$ whenever $E \in \mathcal{G}$.

Let $f \in \mathcal{D}_\mu(\mathcal{E})$, and $0 < r < 1$. Then

$$\frac{1}{2\pi} \int_T \langle P_\mu(r\zeta)f(r\zeta), f(r\zeta) \rangle_\mathcal{E} \, d\lambda(\zeta) = \sum_{k,l} r^{2\max(k,l)} \langle \hat{\mu}(l-k)\hat{f}(k), \hat{f}(l) \rangle_\mathcal{E}.$$

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Similarly,

\[
\frac{1}{\pi} \int_{\mathbb{R}^d} \langle P_\mu(z)f(z), f(z) \rangle_E \, dA(z) = \sum_{k,l=0}^{\infty} \min(k,l) r^{2 \max(k,l)} \langle \hat{\mu}(l-k) \hat{f}(k), \hat{f}(l) \rangle_E.
\]

The above formulae are obtained through term-wise integration of the appropriate power series, c.f. the proof of [14, Lemma 3.2]. By dominated convergence, the above right-hand sides both have limits as \( r \to 1^- \). This motivates us to define

\[
\frac{1}{2\pi} \int_T \langle d\mu f, f \rangle_E := \sum_{k,l=0}^{\infty} \langle \hat{\mu}(l-k) \hat{f}(k), \hat{f}(l) \rangle_E.
\]

It also shows that

\[
\mathcal{D}_\mu(f) = \sum_{k,l=0}^{\infty} \min(k,l) \langle \hat{\mu}(l-k) \hat{f}(k), \hat{f}(l) \rangle_E.
\]

In particular, \( \mathcal{D}_\mu(E) \subseteq \mathcal{D}_2^2(\mathcal{E}) \) for any \( \mu \).

The following result is a simple consequence of the above formulae:

**Proposition 2.11.** Let \( \mu \) be an \( \mathcal{L}_+(\mathcal{E}) \)-valued measure, and \( f \in \mathcal{D}_\mu(\mathcal{E}) \). Then

\[
\langle \beta_1(M_z)f, f \rangle_{\mathcal{D}_2^2(\mathcal{E})} = \frac{1}{2\pi} \int_T \langle d\mu f, f \rangle_E.
\]

Let \( P_N \) denote the orthogonal projection of \( \mathcal{E} \) onto \( \bigvee_{k=0}^N \{ \hat{f}(k) \} \), and \( f_N = P_N \circ f \). Together with the double series expressions for the above integrals, dominated convergence implies that

\[
\int_T \langle d\mu f, f \rangle_E = \lim_{N \to \infty} \int_T \langle d\mu f_N, f_N \rangle_E, \quad \text{and} \quad \mathcal{D}_\mu(f) = \lim_{N \to \infty} \mathcal{D}_\mu(f_N). \quad (2.11)
\]

## 3 2-isometric semigroups

We begin our investigation with a \( C_0 \)-semigroup analogue of Proposition 2.8:

**Proposition 3.1.** Let \( (T_t)_{t \geq 0} \) be a 2-isometric semigroup with generator \( A \). For \( y \in D(A) \) and \( t \geq 0 \), it holds that

\[
\| T_t y \|^2_E = \| y \|^2_E + 2t \text{Re} \langle Ay, y \rangle_E.
\]

**Proof.** Since \( T_t \) is 2-isometric for any \( t \geq 0 \), Proposition 2.8 implies

\[
\| T_t y \|^2 = \| T_{t/n}^n y \|^2 = \| y \|^2 + n \langle \beta_1(T_{t/n})y, y \rangle.
\]

Leibniz’s rule for inner products yields

\[
\lim_{n \to \infty} n \langle \beta_1(T_{t/n})y, y \rangle = t \lim_{n \to \infty} \frac{\| \beta_1(T_{t/n})y \|^2 - \| y \|^2}{t/n} = t (\langle Ay, y \rangle + \langle y, Ay \rangle),
\]

because \( y \in D(A) \).

\( \square \)
We now list a few consequences of Proposition 3.1. Some of these are already known from [6], but are included for the sake of completeness.

**Proposition 3.2.** Let \((T_t)_t \geq 0\) be a 2-isometric semigroup with generator \(A\).

(i) The bounded positive operator \(B_t := \frac{1}{t} \beta_1(T_t)\) is independent of \(t > 0\).

(ii) For \(x \in \mathcal{H}\) and \(t \geq 0\),
\[
\|T_t x\|_\mathcal{H}^2 = \|x\|_\mathcal{H}^2 + t \langle Bx, x \rangle_\mathcal{H},
\]
and for \(y \in D(A)\),
\[
\langle By, y \rangle_\mathcal{H} = 2 \operatorname{Re} \langle Ay, y \rangle_\mathcal{H}.
\]

(iii) \(\|T_t\|_2^2 = 1 + t \|B\|_\mathcal{L}\).

(iv) \(\sigma(A) \subseteq \{z \in \mathbb{C}; \operatorname{Re} z \leq 0\}\), and \(W(A) \subseteq \{z \in \mathbb{C}; 0 \leq \operatorname{Re} z \leq \frac{\|B\|_\mathcal{L}}{2}\}\).

(v) \((T_t)_{t \geq 0}\) is quasicontractive with parameter \(\frac{\|B\|_\mathcal{L}}{2}\). Moreover, this is the optimal parameter of quasicontractivity.

**Proof.** For given \(t > 0\), the operator \(B_t\) is positive and bounded. A polarizaiton argument together with Proposition 3.1 leads to the conclusion that \(\langle By_1, y_2 \rangle_\mathcal{H} = 2 \operatorname{Re} \langle Ay_1, y_2 \rangle_\mathcal{H}\) for \(y_1, y_2 \in D(A)\). Since \(D(A)\) is dense in \(\mathcal{H}\), this completely determines \(B_t\). In particular, \(B_t\) does not depend on \(t\), which proves (i). The second part of (ii) is obvious, while the first part follows by approximating a general \(x \in \mathcal{H}\) with elements of \(D(A)\). (iii) follows from (ii), because \(\|B\|_\mathcal{L} = \sup_{\|x\|_\mathcal{H} \leq 1} \langle Bx, x \rangle_\mathcal{H}\). The statement about \(\sigma(A)\) follows from (iii) and (2.1). The statement about \(W(A)\) follows similarly from (ii). It follows from (iii) that \((T_t)_{t \geq 0}\) is quasicontractive with parameter \(\frac{\|B\|_\mathcal{L}}{2}\), since \(t \mapsto 1 + t \|B\|_\mathcal{L}\) is tangent at \(t = 0\) to the convex function \(t \mapsto e^{t \|B\|_\mathcal{L}}\). This observation also shows that \((T_t)_{t \geq 0}\) fails to be quasicontractive for any smaller parameter value. \(\square\)

Assume now that \((T_t)_{t \geq 0}\) is a 2-isometric \(C_0\)-semigroup. By Proposition 1.1, we know that \((T_t)_{t \geq 0}\) has a well-defined cogenerator \(T\), and that this is 2-isometric. From Proposition 3.2, we conclude that \((T_t)_{t \geq 0}\) is quasicontractive for some parameter \(w \geq 0\). By Theorem 2.3, \(T - I\) is injective, has dense range, and there exists \(w \geq 0\) such that
\[
\langle \beta_1(T)x, x \rangle_\mathcal{H} \leq w \| (T - I)x \|_\mathcal{H}^2, \quad x \in \mathcal{H}.
\]
This concludes the proof of Theorem 1.3.

### 4 2-isometric cogenerators

The purpose of this section is to prove Theorem 1.4. This essentially amounts to proving Theorem 1.5.
Suppose that $T$ is a 2-isometry satisfying (1.1), and $1 \notin \sigma_p(T)$. We wish to prove that $T$ is the cogenerator of a $C_0$-semigroup.

The first step is a reduction to the case of analytic operators. By the Wold-decomposition (Theorem 2.4), $T = T_u \oplus T_a$, where $T_u$ is unitary and $T_a$ is analytic. Since $1 \notin \sigma_p(T)$, we know that $1 \notin \sigma_p(T_u)$. By Theorem 2.4, $T_u$ is the cogenerator of a contractive $C_0$-semigroup on $\mathcal{H}_a$, and by Proposition 2.3, the semigroup is even unitary, hence 2-isometric. Therefore, it suffices to show that $T_a$ is the cogenerator of a $C_0$-semigroup on $\mathcal{H}_a$.

By orthogonality, the invariance of subspaces, and the fact that $T_u$ is unitary, Theorem 2.3 further implies that (1.1) is equivalent to

$$||T_a x_a||^2 - ||x_a||^2 \leq ||(T_u - I)x_a||^2 + ||(T_a - I)x_a||^2, \quad (x_a, x_a) \in \mathcal{H}_u \oplus \mathcal{H}_a.$$  

Consequently, $T$ satisfies (1.1) if and only if $T_u$ does. Together with the previous paragraph, this reduces the proof of Theorem 1.4 to the case where $T$ is an analytic operator.

Now let $T$ be an analytic 2-isometry which satisfies (1.1). The condition $1 \notin \sigma_p(T)$ is superfluous, since $T$ does not have any eigenvalues. To complete the proof of Theorem 1.4, we need to show that $T$ is the cogenerator of a $C_0$-semigroup.

Let $\mathcal{E} = \mathcal{H} \ominus \mathcal{T} \mathcal{H}$. By Theorem 2.10, $T = V^*M_zV$, where $V : \mathcal{H} \to \mathcal{D}_\mu^2(\mathcal{E})$ is a unitary map, and $\mu$ is some $\mathcal{L}_1(\mathcal{E})$-valued measure. It then holds that $\beta_1(T) = V^*\beta_1(M_z)V$, and (1.1) becomes

$$\langle \beta_1(M_z)f, f \rangle_{\mathcal{D}_\mu^2(\mathcal{E})} \leq w\| (I - M_z)f \|_{\mathcal{D}_\mu^2(\mathcal{E})}^2, \quad f \in \mathcal{D}_\mu^2(\mathcal{E}).$$

It is sufficient to verify this for $f \in \mathcal{P}_a(\mathcal{E})$, so by Proposition 2.11, $T$ satisfies (1.1) if and only if

$$\frac{1}{2\pi} \int_T \langle d\mu f, f \rangle_{\mathcal{E}} \leq w\| (I - M_z)f \|_{\mathcal{D}_\mu^2(\mathcal{E})}^2, \quad f \in \mathcal{P}_a(\mathcal{E}).$$

If we accept Theorem 1.5 for a moment, then we may conclude that $(M_{\phi_\tau})_{\tau \geq 0}$ is a $C_0$-semigroup on $\mathcal{D}_\mu^2(\mathcal{E})$. Furthermore, this is 2-isometric, and quasicontractive with parameter (at most) $w$. But then $(V^*M_{\phi_\tau}V)_{\tau \geq 0}$ is a $C_0$-semigroup on $\mathcal{H}$, and this semigroup clearly enjoys the same properties. Since $M_z$ is the cogenerator of $(M_{\phi_\tau})_{\tau \geq 0}$, the cogenerator of $(V^*M_{\phi_\tau}V)_{\tau \geq 0}$ is given by $V^*M_zV = T$.

### 4.1 Proof of Theorem 1.5

We first prove that $(iii) \iff (iv)$. In the process, it becomes apparent that the optimal values of $w_1$ and $w_2$ coincide. The second major step is to prove that $(iv) \implies (i)$. The steps $(i) \implies (ii)$ and $(ii) \implies (iii)$ are easy. We then discuss the final assertions of the theorem.

$(iii) \iff (iv)$. We will prove the slightly more precise statement that

$$\frac{1}{2\pi} \int_T \langle d\mu f, f \rangle_{\mathcal{E}} \leq w\| (I - M_z)f \|_{\mathcal{D}_\mu^2(\mathcal{E})}^2, \quad f \in \mathcal{D}_a(\mathcal{E}),$$

\[ (4.1) \]
Fourier coefficients, in the scalar-valued case

\[
\frac{1}{2\pi} \int_T \left\langle \frac{d\mu(\zeta)}{1-\zeta^2}, f(\zeta) \right\rangle_E \leq \|f\|_{\mathcal{H}_2^2(E)}^2, \quad f \in \mathcal{D}_a(E). \tag{4.2}
\]

Assume (iv). The uniform integrability condition implies that the set function \( \tilde{\mu} : E \mapsto \int_E \frac{d\mu(\zeta)}{1-\zeta^2} \) is an \( \mathcal{L}_+(E) \)-valued measure. By a simple comparison of Fourier coefficients,

\[
\frac{1}{2\pi} \int_T \langle \tilde{\mu} (I-M_z), f(I-M_z) \rangle_E = \frac{1}{2\pi} \int_T \langle d\mu, f \rangle_E.
\]

Hence, (4.1) follows if we replace \( f \in \mathcal{D}_a(E) \) with \( (I-M_z)f \) in (4.2).

For the converse, it will be useful to have the notation \( f_r : z \mapsto \frac{1-r}{1-\bar{z}r}f(z) \), whenever \( f : \mathbb{D} \to \mathcal{E} \) is analytic.

**Lemma 4.1.** If \( f \in H^2(E) \), then \( \|f_r\|_{H^2(E)} \leq \frac{2}{1+r^2} \|f\|_{H^2(E)} \), and

\[
\|f - f_r\|_{H^2(E)} \to 0 \quad \text{as} \quad r \to 1^-.
\]

In the scalar-valued case \( \mathcal{E} = \mathbb{C} \), if \( \zeta \in T \), then the corresponding local Dirichlet integral defined by (2.8) satisfies

\[
\mathcal{D}_\zeta(f_r) \leq \frac{8}{(1+r)^2} \mathcal{D}_\zeta(f) + 2 \frac{1-r}{1+r} \frac{|f(\zeta)|^2}{|1-r\zeta|^2}.
\]

If \( \zeta \in T \setminus \{1\} \), and \( \mathcal{D}_\zeta(f) < \infty \), then \( \mathcal{D}_\zeta(f - f_r) \to 0 \) as \( r \to 1^- \).

**Proof.** We note that \( \left| \frac{1-r}{1-\bar{z}r} \right| \leq \frac{1}{1+r} < 2 \) for \( z \in \overline{\mathbb{D}} \) and \( 0 < r < 1 \). By (2.8), it is immediate that \( \|f_r\|_{H^2(E)} \leq \frac{2}{1+r^2} \|f\|_{H^2(E)} \), and \( \|f - f_r\|_{H^2(E)} \to 0 \) follows from dominated convergence.

By the so-called Local Douglas formula, e.g. [4] Theorem 7.2.5], \( \mathcal{D}_\zeta(f) = \|F(z) - f(\zeta)\|_{\mathcal{H}_2^2} \), where \( F(z) = \frac{f(z) - f(\zeta)}{z-\zeta} \). Applying the same formula to \( f_r \), one obtains

\[
\mathcal{D}_\zeta(f_r) = \|F_r(z) - \frac{1-r}{1-r\bar{\zeta}}f(\zeta)\|_{\mathcal{H}_2^2}^2
\]

\[
\leq 2 \left( \|F_r\|_{\mathcal{H}_2^2}^2 + \left| \frac{1-r}{1-r\bar{\zeta}}f(\zeta) \right|_{\mathcal{H}_2^2}^2 \right)
\]

\[
\leq 2 \left( \frac{4}{(1+r)^2} \|F\|_{\mathcal{H}_2^2}^2 + (1-r^2) \left| \frac{f(\zeta)}{1-r\zeta} \right|_{\mathcal{H}_2^2}^2 \frac{1}{1+r} \right)
\]

\[
= \frac{8}{(1+r)^2} \mathcal{D}_\zeta(f) + 2 \frac{1-r}{1+r} \frac{|f(\zeta)|^2}{|1-r\zeta|^2}.
\]

Here we have used geometric summation to compute \( \| \frac{1}{1-rz} \|_{\mathcal{H}_2^2} = \frac{1}{1-r^2} \).

By a similar calculation,

\[
\mathcal{D}_\zeta(f - f_r) \leq 2 \left( \|F - F_r\|_{\mathcal{H}_2^2} + \frac{1-r}{1+r} \frac{|f(\zeta)|^2}{|1-r\zeta|^2} \right).
\]

If \( \zeta \neq 1 \), then the above right-hand side tends to 0 as \( r \to 1^- \). \(\square\)
Lemma 4.2. Let μ be an $L_+\text{-valued measure}$ that satisfies (4.1). Then

$$\frac{1}{2\pi} \int_{\mathbb{T}} \frac{|f(\zeta)|^2}{|1 - \zeta|^2} d\mu_{x,x}(\zeta) \leq w \left( \|f\|^2_{L^2} \|x\|^2_{\mathcal{H}} + 2D_{\mu_{x,x}}(f) \right), \quad x \in \mathcal{H}, f \in \mathcal{D}_a.$$  

In particular, the set function $E \mapsto \int_E \frac{d\mu(\zeta)}{|1 - \zeta|^2}$ is an $L_+\text{-valued measure}$, and $\mu(\{1\}) = 0$.

Proof. Applying (4.1) to the function $z \mapsto \frac{f(z)}{1 - r^2} x$ yields

$$\frac{1}{2\pi} \int_{\mathbb{T}} \frac{|f(\zeta)|^2}{|1 - r\zeta|^2} d\mu_{x,x}(\zeta) \leq \frac{w}{(1 + r)^2} \left( \|f\|^2_{L^2} \|x\|^2_{\mathcal{H}} + D_{\mu_{x,x}}(f) \right). \quad (4.3)$$

Using Proposition 4.1 together with Fubini’s theorem (2.10),

$$D_{\mu_{x,x}}(f_r) \leq \frac{8}{(1 + r)^2} D_{\mu_{x,x}}(f) + \frac{1 - r}{\pi(1 + r)} \int_{\mathbb{T}} \frac{|f(\zeta)|^2}{|1 - r\zeta|^2} d\mu_{x,x}(\zeta).$$

Subtracting $w \frac{1 - r}{\pi(1 + r)} \int_{\mathbb{T}} \frac{|f(\zeta)|^2}{|1 - r\zeta|^2} d\mu_{x,x}(\zeta)$ from both sides of (4.3), and letting $r \rightarrow 1^-$, Fatou’s lemma implies that

$$\frac{1}{2\pi} \int_{\mathbb{T}} \frac{|f(\zeta)|^2}{|1 - \zeta|^2} d\mu_{x,x}(\zeta) \leq \liminf_{r \rightarrow 1^-} \frac{1}{2\pi} \int_{\mathbb{T}} \frac{|f(\zeta)|^2}{|1 - r\zeta|^2} d\mu_{x,x}(\zeta) \leq w \left( \|f\|^2_{L^2} \|x\|^2_{\mathcal{H}} + 2D_{\mu_{x,x}}(f) \right).$$

Choosing $f$ to be constant, we obtain in particular that

$$\frac{1}{2\pi} \int_{\mathbb{T}} \frac{1}{|1 - \zeta|^2} d\mu_{x,x}(\zeta) \leq w \|x\|^2_{\mathcal{H}}, \quad x \in \mathcal{H}.$$  

As was noted in Subsection 2.2, this uniform integrability condition implies that $E \mapsto \int_E \frac{d\mu(\zeta)}{|1 - \zeta|^2}$ is an $L_+\text{-valued measure}$. It is also clear that $\zeta \mapsto \frac{1}{|1 - \zeta|^2}$ is finite for $\mu_{x,x}$-a.e. $\zeta \in \mathbb{T}$. As this holds for every $x \in \mathcal{H}$, $\mu(\{1\}) = 0$.  

Lemma 4.3. Let $\mu$ be an $L_+(\mathcal{E})\text{-valued measure}$, and $x, y \in \mathcal{E}$. Then

$$P_{\mu_{x,y}}(z) \leq P_{\mu_{x,x}}(z)^{1/2} P_{\mu_{y,y}}(z)^{1/2}. \quad (4.4)$$

Proof. Recalling (2.5), we may apply (2.6), with $f(\zeta) = g(\zeta) = \left( \frac{1 - |z|^2}{|\zeta - z|^2} \right)^{1/2}$, $\mu_0 = \mu_{x,y}$, $\mu_1 = \mu_{x,x}$, and $\mu_2 = \mu_{y,y}$.

This allows us to prove an $L_+(\mathcal{E})\text{-valued substitute}$ for (2.10).
Lemma 4.4. Let \( \mu \) be an \( \mathcal{L}_+(\mathcal{E}) \)-valued measure, and \( x, y \in \mathcal{E} \). If \( f \in \mathcal{D}^2_{\mu_{x,x}} \) and \( g \in \mathcal{D}^2_{\mu_{y,y}} \), then the integral

\[
\mathcal{D}_{\mu_{x,y}}(f, g) = \frac{1}{2\pi^2} \int_{\mathcal{T}} \int_{\mathcal{D}} f'(z)g''(z) \frac{1 - |z|^2}{|\zeta - z|^2} d\mu_{x,y}(\zeta) dA(z)
\]

is absolutely convergent. In particular,

\[
\mathcal{D}_{\mu_{x,y}}(f, g) = \frac{1}{2\pi} \int_{\mathcal{T}} \mathcal{D}_z(f, g) d\mu_{x,y},
\]

and \( |\mathcal{D}_{\mu_{x,y}}(f, g)| \leq \|f\|_{\mathcal{D}^2_{\mu_{x,x}}} \|g\|_{\mathcal{D}^2_{\mu_{y,y}}} \).

Proof. We need to show that

\[
\int_{\mathcal{T}} \int_{\mathcal{D}} |f'(z)g''(z)| \frac{1 - |z|^2}{|\zeta - z|^2} d|\mu_{x,y}|(\zeta) dA(z) < \infty.
\]

The inner integral equals \( P_{\mu_{x,y}}(z) \). By \([4.4]\), and the Cauchy–Schwarz inequality, the above integral is less than \( \|f\|_{\mathcal{D}^2_{\mu_{x,x}}} \|g\|_{\mathcal{D}^2_{\mu_{y,y}}} \). Fubini’s theorem allows one to change the order of integration in \( \mathcal{D}_{\mu_{x,y}}(f, g) \). \( \square \)

Lemma 4.5. Let \( \mu \) be an \( \mathcal{L}_+ \)-valued measure that satisfies \([4.1]\). If \( f \in \mathcal{D}_a(\mathcal{E}) \) has finite dimensional range, then \( \lim_{r \to 1^-} \mathcal{D}_\mu(f_r) = \mathcal{D}_\mu(f) \).

Proof. Let \( \{e_n\}_n \) be an orthonormal basis of a finite dimensional subspace containing the range of \( f \). Then \( f = \sum_n f_n e_n \), where each \( f_n = \langle f, e_n \rangle_\mathcal{E} \in \mathcal{D}_a \). If we let \( \mu_{m,n} = \mu_{e_m, e_n} \), then

\[
\mathcal{D}_\mu(f_r) = \sum_{m,n} \mathcal{D}_{\mu_{m,n}}((f_m)_r, (f_n)_r).
\]

By Lemma [4.4] each one of these Dirichlet integrals can be computed as

\[
\mathcal{D}_{\mu_{m,n}}((f_m)_r, (f_n)_r) = \frac{1}{2\pi} \int_{\mathcal{T}} \mathcal{D}_z((f_m)_r, (f_n)_r) d\mu_{m,n}.
\]

By a polarized version of Lemma [4.1] \( \mathcal{D}_z((f_m)_r, (f_n)_r) \to \mathcal{D}_z(f_m, f_n) \) for \( \zeta \in \mathcal{T} \setminus \{1\} \), i.e. \( |\mu_{m,n}| \)-a.e. Moreover,

\[
\mathcal{D}_z((f_m)_r, (f_n)_r) \leq \mathcal{D}_z((f_m)_r)^{1/2} \mathcal{D}_z((f_n)_r)^{1/2}
\]

\[
\leq \left( 8\mathcal{D}_z(f_m) + \frac{|f_m(\zeta)|^2}{|1-\zeta|^2} \right)^{1/2} \left( 8\mathcal{D}_z(f_n) + \frac{|f_n(\zeta)|^2}{|1-\zeta|^2} \right)^{1/2}.
\]

By Lemma [2.10] and Lemma [1.2] the right-hand side is \( |\mu_{m,n}| \)-integrable. By the dominated convergence theorem,

\[
\mathcal{D}_{\mu_{m,n}}((f_m)_r, (f_n)_r) \to \mathcal{D}_{\mu_{m,n}}(f_m, f_n).
\]

Summing over \( m \) and \( n \) completes the proof. \( \square \)
Note that $\varphi$ has finite dimensional range, then

$$\int_\mathbb{T} \left\langle \frac{d\mu(\zeta)}{|1 - \zeta|^2} f(\zeta), f(\zeta) \right\rangle_{\mathcal{E}} = \lim_{r \to 1^+} \int_\mathbb{T} \left\langle \frac{d\mu(\zeta)}{1 - r\zeta}, f(\zeta) \right\rangle_{\mathcal{E}}$$

Proof. We reuse our notation from the proof of Lemma 4.5, i.e.

$$\text{Lemma 4.7.}$$

By the previous two lemmas, we obtain (4.2) by letting $\mu_{rM_2} \to 1^+ f$ yields

$$\frac{1}{2\pi} \int_\mathbb{T} \left\langle \frac{d\mu(\zeta)}{|1 - r\zeta|^2} f(\zeta), f(\zeta) \right\rangle_{\mathcal{E}} \leq \|f\|_{D_2^\mu(\mathcal{E})}^2.$$  

By the previous two lemmas, we obtain (4.2) by letting $r \to 1^+$.  

(iv) $\Rightarrow$ (i). The proof is based on the following formula:

**Lemma 4.7.** Let $\mu$ be an $\mathcal{L}_+(\mathcal{E})$-valued measure that satisfies (4.2). If $f \in D_\alpha(\mathcal{E})$, and $t > 0$, then

$$\|\phi_t f\|_{D_2^\mu(\mathcal{E})}^2 = \|f\|_{D_2^\mu(\mathcal{E})}^2 + \frac{t}{\pi} \int_\mathbb{T} \left\langle \frac{d\mu(\zeta)}{|1 - \zeta|^2} f(\zeta), f(\zeta) \right\rangle_{\mathcal{E}}.$$  

Proof. By (2.11), it suffices to consider the case where $f$ has finite dimensional range. We reuse our notation from the proof of Lemma 4.5, i.e. $f = \sum f_n e_n$, where each $f_n$ is scalar-valued, and $\{e_n\}$ is a finite orthonormal set.

First, we show that $\phi_t f_n \in D_{\mu_n,n}$. Indeed,

$$D_{\mu_n,n}(\phi_t f_n) = \frac{1}{2\pi} \int_\mathbb{T} D_\zeta(\phi_t f_n) d\mu_n,n(\zeta).$$

Note that $\phi_t$ is inner. According to a well-known formula, e.g [3, Theorem 7.6.1],

$$D_\zeta(\phi_t f_n) = D_\zeta(f_n) + |f_n(\zeta)|^2 D_\zeta(\phi_t),$$  

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and
\[ D_\zeta(\phi_t) = |\phi_t'(\zeta)| = \frac{2t}{|1 - \zeta|^2}, \]
whenever \( \phi_t \) is analytic in a neighbourhood of \( \zeta \). In particular, the last identity holds \( \mu_{n,n} \)-a.e., so
\[ D_{\mu_{n,n}}(\phi_t f_n) = D_{\mu_{n,n}}(f_n) + \frac{t}{\pi} \int_\Omega \frac{|f(\zeta)|^2}{|1 - \zeta|^2} \, d\mu_{n,n}(\zeta). \]
By Lemma 4.2, the last integral is finite, so \( \phi_t f_n \in D^{2}_{\mu_{n,n}} \).

By Lemma 4.4, \( D_\zeta(\phi_t f_m, \phi_t f_n) \) is analytic in a neighbourhood of \( \zeta \). By polarization of (4.6),
\[ D_\zeta(\phi_t f_m, \phi_t f_n) = D_\zeta(f_m, f_n) + \frac{2t}{|1 - \zeta|^2} f_m(\zeta) \overline{f_n(\zeta)}. \]
By Lemma 4.2 and Cauchy–Schwarz inequality, the right-hand side is \( \mu_{m,n} \)-integrable, so
\[ D_\mu(\phi_t f) = \sum_{m,n} D_{\mu_{m,n}}(\phi_t f_m, \phi_t f_n) \]
\[ = \sum_{m,n} \left[ D_\mu(f_m, f_n) + \frac{t}{\pi} \int_\Omega \frac{f_m(\zeta) \overline{f_n(\zeta)}}{|1 - \zeta|^2} \, d\mu_{m,n}(\zeta) \right] \]
\[ = D_\mu(f) + \frac{t}{\pi} \int_\Omega \left\langle \frac{d\mu(\zeta)}{|1 - \zeta|^2}, f(\zeta), \overline{f(\zeta)} \right\rangle. \]
Adding \( \|\phi_t f\|_{H^2(\mathcal{E})}^2 = \|f\|_{H^2(\mathcal{E})}^2 \) to both sides of the obtained identity completes the proof.

Let \( t > 0 \). By Lemma 4.7, \( M_{\phi_t} \) is bounded, and densely defined on \( D^2_\mu(\mathcal{E}) \). As such, it has a unique bounded extension to the whole space. By the reproducing property, if \( f_n \to f \) in \( D^2_\mu(\mathcal{E}) \), then \( f_n(z) \to f(z) \) for each \( z \in \mathbb{D} \). This implies that the bounded extension is indeed given by multiplication by \( \phi_t \). Hence, \( (M_{\phi_t})_{t \geq 0} \subset \mathcal{L}(D^2_\mu(\mathcal{E})) \). Moreover, \( (M_{\phi_t})_{t \geq 0} \) clearly is a semigroup. As for strong continuity, let \( z \in \mathbb{D} \), \( x \in \mathcal{H} \), and consider the linear functional \( f \mapsto \langle f(z), x \rangle \). By the reproducing property, this functional is bounded on \( D^2_\mu(\mathcal{E}) \). If \( f \in D^2_\mu(\mathcal{E}) \), then (4.5) implies that the family \( (\phi_t f)_{0 \leq t \leq 1} \) is bounded in \( D^2_\mu(\mathcal{E}) \). As we let \( t \to 0 \), a subsequence of \( (\phi_t f)_{0 \leq t < 1} \) will converge weakly to some \( g \in D^2_\mu(\mathcal{E}) \). In particular, \( \langle \phi_t f(z), x \rangle \to \langle g(z), x \rangle \) for this subsequence. But the pointwise limit of \( \phi_t f \) is \( f \), so \( g = f \). As this uniquely determines the limit of any subsequence, the entire family converges weakly to \( f \). Since (4.5) also implies that \( \lim_{t \to 0} \|\phi_t f\|_{D^2_\mu(\mathcal{E})}^2 = \|f\|_{D^2_\mu(\mathcal{E})}^2 \), we conclude that \( \lim_{t \to 0} \phi_t f = f \) with convergence in norm. This establishes that \( (M_{\phi_t})_{t \geq 0} \) is a \( C_0 \)-semigroup.
(i) ⇒ (ii). It is a straightforward calculation that $M_z$ is the cogenerator of $(M_{\phi_t})_{t \geq 0}$, provided that the latter is a $C_0$-semigroup.

(ii) ⇒ (iii). Suppose the $M_z$ is the cogenerator of a $C_0$-semigroup $(T_t)_{t \geq 0}$. Since $M_z$ acts as a 2-isometry on $D_2^T(E)$, the same is true for each operator $T_t$ (Proposition 1.1). Hence, $(T_t)_{t \geq 0}$ is quasiscontractive with some parameter $w$ (Proposition 3.2), so its cogenerator $M_z$ satisfies (2.4), which translates into (iii), with $w_1 = w$.

Assume now that (i) − (iv) hold. We have already seen that the optimal values of $w_1$ and $w_2$ coincide. Moreover, $M_z$ is the cogenerator of $(M_{\phi_t})_{t \geq 0}$, and this semigroup is quasiscontractive with some parameter $w$. Since (1.2) is just a reformulation of (2.4), Theorem 2.3 yields that the optimal values of $w$ and $w_1$ coincide. This concludes the proof of Theorem 1.5.

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