On regular genus and G-degree of PL 4-manifolds with boundary

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Abstract

In this article, we introduce two new PL-invariants: weighted regular genus and weighted G-degree for manifolds with boundary. We first prove two inequalities involving some PL-invariants which state that for any PL-manifold \(M\) with non spherical boundary components, the regular genus \(G(M)\) of \(M\) is at least the weighted regular genus \(\tilde{G}(M)\) of \(M\) which is again at least the generalized regular genus \(\bar{G}(M)\) of \(M\). Another inequality states that the weighted G-degree \(\tilde{D}(M)\) of \(M\) is always greater than or equal to the G-degree \(D(G(M)\) of \(M\). Let \(M\) be any compact connected PL 4-manifold with \(h\) number of non spherical boundary components. Then we compute the following:

\[
\tilde{G}(M) \geq 2\chi(M) + 3m + 2h - 4 + 2\hat{m} \quad \text{and} \quad \tilde{D}(M) \geq 12(2\chi(M) + 3m + 2h - 4 + 2\hat{m}),
\]

where \(m\) and \(\hat{m}\) are the ranks of the fundamental groups of \(M\) and the corresponding singular manifold \(\hat{M}\) (obtained by coning off the boundary components of \(M\)) respectively. As a consequence we prove that the regular genus \(G(M)\) satisfies the following inequality:

\[
G(M) \geq 2\chi(M) + 3m + 2h - 4 + 2\hat{m},
\]

which improves the previous known lower bounds for the regular genus \(G(M)\) of \(M\). Then we define two classes of gems for PL 4-manifold \(M\) with boundary: one consists of semi-simple gems and the other consists of weak semi-simple gems, and prove that the lower bounds for the weighted G-degree and weighted regular genus are attained in these two classes respectively.

Keywords PL-manifolds, Crystallizations, Regular genus, Gurau degree (G-degree).

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1 Introduction

A crystallization \((\Gamma, \gamma)\) of a PL \(d\)-manifold is a certain type of edge colored graph which represents the manifold (cf. Subsection 2.1 for details). The crystallization theory was introduced by Pezzana who gave the existence of a crystallization for every closed connected PL \(d\)-manifold (see [22]). Later, the existence of crystallization was shown for every compact PL \(d\)-manifold with boundary (see [14, 19]). The term regular genus for a closed connected PL \(d\)-manifold which extends the notion of genus in dimension 2, has been introduced in [20], which is related to the existence of regular embeddings of gems of the manifold into surfaces (cf. Subsection 2.2 for details). Later, in [18], the concept of

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regular genus has been extended for compact PL \(d\)-manifolds with boundary, for \(d \geq 2\). The regular genus of a closed connected orientable (resp., a non-orientable) surface equals the genus (resp., half of the genus) of the surface. Some beautiful results on regular genus of mapping tori can be found in [2]. In [5], the authors gave lower bound for regular genus of closed 4-manifolds. In [16], two lower bounds for the regular genus of connected compact PL \(d\)-manifolds with boundary have been computed. In [4], we gave several lower bounds for compact 4-manifolds with boundary. In [11], the authors provided a lower bound for both regular genus and G-degree for compact 4-manifolds with empty or connected boundary. In this article, we further improve the previous lower bounds for regular genus of compact PL 4-manifolds with boundary which is an easy consequence of the main theorem (cf. Theorem 19).

In [9], the authors have introduced the concept of generalized regular genus for compact PL \(d\)-manifolds with empty or non-spherical boundary components, which coincides with the regular genus for closed PL \(d\)-manifolds. A lower bound for the generalized regular genus of compact \(4\)-manifolds with at most one boundary component has been computed in [9, 11], which also gives a lower bound for the regular genus of compact 4-manifolds with at most one non-spherical boundary component. In this article, we introduce another new invariant weighted regular genus for compact 4-manifolds with several non-spherical boundary components, which coincides with the generalized regular genus when the manifold is a compact 4-manifold with at most one boundary component. First, we note that for any compact PL \(d\)-manifold \(M\) with boundary, the weighted regular genus of \(M\) lies between the regular genus and generalized regular genus of \(M\) such that the regular genus being the largest of these three. If \(M\) is closed, then all the three PL-invariants coincide. The main contribution of this article is that we give a lower bound for the weighted regular genus of compact PL 4-manifolds with several non-spherical boundary components. As a consequence, we give a lower bound for the regular genus of compact 4-manifolds with several boundary components, which improves the previous known estimates for the regular genus.

The notion of G-degree of closed connected PL \(d\)-manifold was first introduced in [10], which is a key concept in the approach to Quantum Gravity via tensor models. Later, it has been extended for singular manifolds in [12], and hence for compact PL 4-manifolds with empty or non-spherical boundary as there is a one-to-one correspondence between singular manifolds and compact 4-manifolds with empty or non-spherical boundary. Several combinatorial properties of G-degree have been studied recently in [13, 8, 9, 11] for compact 4-manifolds with at most one boundary component. In this article, we have introduced a new invariant weighted G-degree for compact 4-manifolds with several non-spherical boundary components, which coincide with the G-degree when the manifold is a compact 4-manifold with at most one boundary component. All the PL-invariants mentioned above are non-negative. Also, for any compact PL \(d\)-manifold \(M\) with boundary, weighted G-degree of the manifold is at least the G-degree of that manifold. In this article, we also give a lower bound for the weighted G-degree for compact 4-manifolds with non-spherical boundary components.

In [11], the class of semi-simple and weak semi-simple gems have been introduced for compact 4-manifolds with empty or connected boundary. In this paper, we extend the classes to compact 4-manifolds with several boundary components. Then we prove that a compact PL \(4\)-manifold \(M\) with boundary attains the lower bound for the weighted G-degree (resp., weighted regular genus) if and only if \(M\) admits a semi-simple (resp., weak semi-simple) gem.

2 Preliminaries

In this section, we shall give a brief overview of crystallization theory. It provides a combinatorial tool for representing piecewise-linear (PL) manifolds of arbitrary dimension via colored graphs and is used to study geometrical and topological properties of manifolds.

2.1 Crystallization

A multigraph is a graph where multiple edges are allowed but loops are restricted. For a multigraph \(\Gamma = (V(\Gamma), E(\Gamma))\), a surjective map \(\gamma : E(\Gamma) \rightarrow \Delta_d := \{0, 1, \ldots, d\}\) is called a proper edge-coloring.
if \( \gamma(e) \neq \gamma(f) \) for any adjacent edges \( e, f \). The elements of the set \( \Delta_d \) are called the *colors* of \( \Gamma \). A graph \( (\Gamma, \gamma) \) is called \((d + 1)\)-regular if degree of each vertex is \( d + 1 \) and is said to be \((d + 1)\)-regular with respect to a color \( c \) if after removing all the edges of color \( c \) from \( \Gamma \), the resulting graph is \( d \)-regular. We refer to [1] for standard terminology on graphs. All spaces and maps will be considered in PL-category.

A regular \((d + 1)\)-colored graph is a pair \((\Gamma, \gamma)\), where \( \Gamma \) is \((d + 1)\)-regular and \( \gamma \) is a proper edge-coloring. A \((d + 1)\)-colored graph with boundary is a pair \((\Gamma, \gamma)\), where \( \Gamma \) is not a \((d + 1)\)-regular graph but a \((d + 1)\)-regular with respect to a color \( c \) and \( \gamma \) is a proper edge-coloring. If coloration is understood, one can use \( \Gamma \) for \((d + 1)\)-colored graphs instead of \((\Gamma, \gamma)\). For each \( B \subseteq \Delta_d \) with \( h \) elements, the graph \( \Gamma_B = (V(\Gamma), \gamma^{-1}(B)) \) is an \( h \)-colored graph with edge-coloring \( \gamma|_{\gamma^{-1}(B)} \).

For a color set \( \{j_1, j_2, \ldots, j_k\} \subseteq \Delta_d \), \( g(\Gamma_{\{j_1, j_2, \ldots, j_k\}}) \) or \( g_{j_1, j_2, \ldots, j_k} \) denotes the number of connected components of the graph \( \Gamma_{\{j_1, j_2, \ldots, j_k\}} \). Let \( g_{\hat{j}_1, \hat{j}_2, \ldots, \hat{j}_k} \) denote the number of regular components of \( \Gamma_{\{\hat{j}_1, \hat{j}_2, \ldots, \hat{j}_k\}} \). A graph \( (\Gamma, \gamma) \) is called contracted if subgraph \( \hat{\Gamma}_c := \Gamma_{\Delta_d \setminus c} \) is connected for all \( c \).

Let \( G_d \) denote the set of graphs \( (\Gamma, \gamma) \) which are \((d + 1)\)-regular with respect to the fixed color \( d \). Thus \( G_d \) contains all the regular \((d + 1)\)-colored graphs as well as all \((d + 1)\)-colored graphs with boundary. We sometimes call a regular \((d + 1)\)-colored graph or a \((d + 1)\)-colored graphs with boundary simply by a \((d + 1)\)-colored graph if it is understood that the graph is regular or not. If \( (\Gamma, \gamma) \in G_d \) then the vertex with degree \( d + 1 \) is called an internal vertex and the vertex with degree \( d \) is called a boundary vertex. Let \( 2\overline{p} \) and \( 2\hat{p} \) denote the number of boundary vertices and internal vertices respectively. Then the number of vertices is denoted by \( 2p \), i.e., \( p = \overline{p} + \hat{p} \). For each graph \( (\Gamma, \gamma) \in G_d \), we define its boundary graph \( (\partial \Gamma, \partial \gamma) \) as follows:

- there is a bijection between \( V(\partial \Gamma) \) and the set of boundary vertices of \( \Gamma \);
- \( u_1, u_2 \in V(\partial \Gamma) \) are joined in \( \partial \Gamma \) by an edge of color \( j \) if and only if \( u_1 \) and \( u_2 \) are joined in \( \Gamma \) by a path formed by \( j \) and \( d \) colored edges alternatively.

Note that, if \( (\Gamma, \gamma) \) is \((d + 1)\)-regular then \( (\Gamma, \gamma) \in G_d \) and \( \partial \Gamma = \emptyset \). For each \( (\Gamma, \gamma) \in G_d \), a corresponding \( d \)-dimensional simplicial cell-complex \( \mathcal{K}(\Gamma) \) is determined as follows:

- for each vertex \( u \in V(\Gamma) \), take a \( d \)-simplex \( \sigma(u) \) and label its vertices by \( \Delta_d \);
- corresponding to each edge of color \( j \) between \( u, v \in V(\Gamma) \), identify the \((d - 1)\)-faces of \( \sigma(u) \) and \( \sigma(v) \) opposite to \( j \)-labeled vertices such that the vertices with same label coincide.

The geometric carrier \( |\mathcal{K}(\Gamma)| \) is a \( d \)-pseudo-manifold and \( (\Gamma, \gamma) \) is said to be a gem (graph encoded manifold) of any \( d \)-pseudo-manifold homeomorphic to \( |\mathcal{K}(\Gamma)| \) or simply is said to represent the \( d \)-pseudo-manifold. We refer to [8] for CW-complexes and related notions. From the construction it is easy to see that, for \( B \subset \Delta_d \) of cardinality \( h + 1 \), \( \mathcal{K}(\Gamma) \) has as many \( h \)-simplices with vertices labeled by \( B \) as the number of connected components of \( \Gamma_{\Delta_d \setminus B} \) (cf. [12]).

**Definition 1.** A closed connected PL \( d \)-manifold is a compact \( d \)-dimensional polyhedron which has a simplicial triangulation such that the link of each vertex is \((d - 1)\)-sphere.

A connected compact PL \( d \)-manifold with non-empty boundary is a compact \( d \)-dimensional polyhedron which has a simplicial triangulation where the link of at least one vertex is a \((d - 1)\)-ball and the link of each of the other vertices is either a \((d - 1)\)-sphere or a \((d - 1)\)-ball.

A singular PL \( d \)-manifold is a compact \( d \)-dimensional polyhedron which has a simplicial triangulation where the links of vertices are closed connected \((d - 1)\) manifolds while, for each \( h \geq 1 \), the link of any \( h \)-simplex is a PL \((d - h - 1)\)-sphere. A vertex whose link is not a sphere is called a singular vertex. Clearly, a closed (PL) \( d \)-manifold is a singular (PL) \( d \)-manifold with no singular vertices.

From the correspondence between \((d + 1)\)-regular colored graphs and \( d \)-pseudomanifolds, it is easy to visualise that:

1. \( |\mathcal{K}(\Gamma)| \) is a closed connected PL \( d \)-manifold if and only if for each \( c \in \Delta_d \), \( \hat{\Gamma}_c \) represents \( S^{d - 1} \).
(2) $|K(\Gamma)|$ is a connected compact PL $d$-manifold with boundary if and only if for each $c \in \Delta_d$, $\Gamma_c$ represents either $S^{d-1}$ or $B^{d-1}$, and there exists a vertex $c \in \Delta_d$ such that $\Gamma_c$ represents $B^{d-1}$.

(3) $|K(\Gamma)|$ is a singular (PL) $d$-manifold if and only if for each $c \in \Delta_d$, $\Gamma_c$ represents a closed connected PL $(d - 1)$-manifold.

If $(d + 1)$-colored graph $(\Gamma, \gamma)$ which is a gem of a closed (PL) $d$-manifold $M$, is contracted then it is called a crystallization of $M$. In this case, the number of vertices of $K(\Gamma)$ is exactly $d + 1$ (which is the minimal). It is not hard to see that, for any $(d + 1)$-colored graph (with boundary) $(\Gamma, \gamma)$, $|K(\Gamma)|$ is orientable if and only if $\Gamma$ is a bipartite graph. If $\Gamma$ represents a PL $d$-manifold with boundary then we can define its boundary graph $(\partial \Gamma, \partial \gamma)$, and each component of the boundary-graph $(\partial \Gamma, \partial \gamma)$ represents a component of $\partial M$.

On the other hand, if $(\Gamma, \gamma) \in G_d$ represents a $d$-manifold with non-connected boundary then $\Gamma$ cannot be contracted. Let $(\Gamma, \gamma) \in G_d$ represent a $d$-manifold with boundary. Then, it is easy to see that $\Gamma$ is connected and each component of $\partial \Gamma$ is contracted if and only if $K(\Gamma)$ has exactly $dh + 1$ vertices (which is the minimal), where $h$ is the number of components of $\partial M$.

Let $M$ be a compact $d$-manifold with $h$ boundary components. A $(d + 1)$-colored gem $(\Gamma, \gamma) \in G_d$ of $M$ is said to be a crystallization of $M$ if $g(\Gamma_d) = 1$ and $g(\Gamma_\xi) = h$, for each $c \in \Delta_{d-1}$.

The initial point of the crystallization theory is the Pezzana's existence theorem (cf. [22]) which gives the existence of a crystallization for every closed connected PL $d$-manifold. Later, it has been extended to the boundary case (cf [11]). Further, the existence of crystallizations/gems has been extended for singular (PL) $d$-manifolds (cf. [12]). It is known that a PL $d$-manifold with boundary can always be represented by a $(d + 1)$-colored graph $(\Gamma, \gamma)$ which is regular with respect to a fixed color $k$, for some $k \in \Delta_d$. Without loss of generality, we can assume that $k = 0$, i.e., $(\Gamma, \gamma) \in G_d$.

A 1-dipole of color $j \in \Delta_d$ of a $(d + 1)$-colored graph (possibly with boundary) $(\Gamma, \gamma) \in G_d$ is a subgraph $\theta$ of $\Gamma$ consisting of two vertices $x, y$ joined by color $j$ such that $\Gamma_j(x) \neq \Gamma_j(y)$, where $\Gamma_j(u)$ denotes the component of $\Gamma$ containing $u$. The cancellation of a 1-dipole from $\Gamma$ consists of two steps: first deleting $\theta$ from $\Gamma$ and second welding the same colored hanging edges.(see [1] for more details)

### 2.2 Regular Genus of PL $d$-manifolds

Let $(\Gamma, \gamma) \in G_d$ be a bipartite (resp. non bipartite) $(d + 1)$-colored graph (with boundary) which represents a connected compact PL $d$-manifold $M$ with boundary (possibly $\partial M = \emptyset$). Add a new vertex $u'$ for each boundary vertex $u$ and join these two vertices by a $d$-colored edge and we get a new graph $(\Gamma', \gamma')$. Then for each cyclic permutation $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_d)$ of $\Delta_d$, there exists a regular imbedding of $\Gamma'$ into an orientable (resp. non orientable) surface $F$ such that the intersection of vertices of $\Gamma'$ and $\partial F$ is the set of new added vertices and the regions are bounded either by a cycle (internal region) or by a walk (boundary region) of $\Gamma'$ with $\varepsilon_j, \varepsilon_{j+1}(j \mod d+1)$ colored edges alternatively.

Using Gross ‘voltage theory’ (resp. Stahl ‘embedding schemes’) in (21) (resp., 23), it is easy to show that for every cyclic permutation $\varepsilon$ of $\Delta_d$, a regular imbedding of bipartite graph (resp. non bipartite) $\Gamma$ into an orientable (resp. non orientable) surface $F_\varepsilon$ exists. Moreover, genus (resp. half of genus) $\rho_\varepsilon$ of $F_\varepsilon$ satisfies

$$2 - 2\rho_\varepsilon(\Gamma) = \sum_{i \in \mathbb{Z}_{d+1}} \tilde{g}_{\varepsilon_i \varepsilon_{i+1}} + (1 - d) \tilde{p} + (2 - d) \tilde{p} + \partial g_{\varepsilon_i \varepsilon_{d+1}}$$

where $2\tilde{p}$ and $2\tilde{p}$ denote the number of internal vertices and boundary vertices in $\Gamma$, and $\partial g_{ij}$ denote the number of $\{i, j\}$-colored cycles in $\partial \Gamma$. For more details we refer [1] [13].

The regular genus $\rho(\Gamma)$ of $(\Gamma, \gamma)$ is defined as

$$\rho(\Gamma) = \min\{\rho_\varepsilon(\Gamma) \mid \varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_d) \text{ is a cyclic permutation of } \Delta_d\}.$$ 

In dimension two, it is easy to see that if $(\Gamma, \gamma)$ represents a surface $F$, then the corresponding $(\Gamma', \gamma')$ regularly imbeds into $F$ itself. Hence, for each surface $F$,
\[ G(F) = \begin{cases} \text{genus}(F) & \text{if } F \text{ is orientable}, \\ \frac{1}{2} \times \text{genus}(F) & \text{if } F \text{ is non-orientable}. \end{cases} \]

Further on similar steps, if \((\Gamma, \gamma) \in G_d\) is a regular bipartite (resp. non-bipartite) \((d+1)\)-colored graph of order \(2p\) which represents a singular \(d\)-manifold, then the genus (resp. half of genus) \(\rho_{\varepsilon}\) of surface \(F_\varepsilon\) satisfies
\[ 2 - 2\rho_{\varepsilon}(\Gamma) = \sum_{\varepsilon \in \mathbb{Z}_{d+1}} g_{\varepsilon, \varepsilon+1} + (1 - d)p. \]

The regular genus \(\rho(\Gamma)\) of \((\Gamma, \gamma)\) is defined as
\[ \rho(\Gamma) = \min\{\rho_{\varepsilon}(\Gamma) \mid \varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_d) \text{ is a cyclic permutation of } \Delta_d\}. \]

**Definition 2.** Let \(M\) be a connected compact PL \(d\)-manifold with boundary (possibly empty). Then, the regular genus of \(M\) is defined as
\[ G(M) = \min\{\rho(\Gamma) \mid (\Gamma, \gamma) \in G_d \text{ represents } M\}. \]

### 3 Main results

**Proposition 3.** Any compact orientable (resp., non-orientable) (PL) \(d\)-manifold \(M\) with boundary (possibly empty) admits a bipartite (resp., non-bipartite) \((d+1)\)-colored graph with boundary (possibly empty) representing it. Moreover, \(M\) admits a crystallization.

**Remark 4.** There is a bijection between the class of connected singular (PL) \(d\)-manifolds and the class of connected closed (PL) \(d\)-manifolds union with the class of connected compact (PL) \(d\)-manifolds with non-spherical boundary components. For, if \(M\) is a singular \(d\)-manifold then removing small open neighborhood of each of its singular vertices (if possible), a compact \(d\)-manifold \(\tilde{M}\) (with non spherical boundary components) is obtained. It is obvious that \(M = \tilde{M}\) if and only if \(M\) is a closed \(d\)-manifold.

Conversely, if \(M\) is a compact \(d\)-manifold with non spherical boundary components then a singular \(d\)-manifold \(\tilde{M}\) is obtained by coning off each component of \(\partial M\). If \(M\) is a closed \(d\)-manifold then \(M = \tilde{M}\).

From now onwards, we mean ‘a connected compact PL \(d\)-manifold with non spherical boundary components’ by the term ‘a manifold with boundary’. The boundary component can be empty as well.

**Definition 5.** A regular \((d+1)\)-colored graph \((\Gamma, \gamma)\) is said to be a regular gem of a compact \(d\)-manifold \(M\) with boundary if \(\Gamma\) represents the associated singular \(d\)-manifold \(\tilde{M}\).

For a colored graph \((\Gamma, \gamma)\), a color \(c \in \Delta_d\) is said to be singular, if the vertices labeled by color \(c\) in \(K(\Gamma)\) are singular. Let \(M\) be a connected compact \(d\)-manifold with (non-empty) boundary. Then by Proposition 3 there exists a \((d+1)\)-colored graph \((\Gamma, \gamma) \in G_d\) representing \(M\). For each component of \(\partial \Gamma\), choose \(c \in \Delta_{d-1}\). Join an edge of color \(d\) in \(\Gamma\) between the two boundary vertices which lie on \(\{c, d\}\)-colored path in \(\Gamma\). The resulting \((d+1)\)-regular colored graph represents the singular \(d\)-manifold \(\tilde{M}\) which is obtained by coning off each boundary component of \(M\). The colors \(c\) which were chosen for different components of \(\partial \Gamma\) are singular. If we choose a fixed color \(c \in \Delta_{d-1}\) for each boundary component then the resulting \((d+1)\)-regular colored graph has exactly one color \(c\) as a singular color. Replace \(c\) and \(d\) with each other. Then, we get a \((d+1)\)-colored graph \((\tilde{\Gamma}, \tilde{\gamma})\) with \(d\) as a singular color and the colors in \(\Delta_{d-1}\) are non-singular.

Let \(\tilde{G}_d\) denote the set of all \((d+1)\)-regular colored graphs \((\Gamma, \gamma)\) such that vertices in \(K(\Gamma)\) labeled by color \(c \in \Delta_{d-1}\) are non-singular and the vertices labeled by color \(d\) are singular. Thus, from Proposition 3 we have the following result.
Corollary 6. Any connected compact orientable (resp., non-orientable) PL \(d\)-manifold \(M\) with boundary admits a regular bipartite (resp., non-bipartite) \((d+1)\)-colored gem of \(M\). Moreover, there is a regular \((d+1)\)-colored gem of \(M\) which lies in \(\mathbb{G}_d\).

With this result, we have two representations for a connected compact PL \(d\)-manifold \(M\) with non spherical boundary components: one is a non-regular \((d+1)\)-colored graph in \(\mathbb{G}_d\) and the other is a regular \((d+1)\)-colored graph (may not lie in \(\mathbb{G}_d\)). Moreover, we have another regular \((d+1)\)-colored gem of \(M\), which lies in \(\mathbb{G}_d\).

Definition 7. Let \((\Gamma, \gamma)\) be a regular \((d+1)\)-colored graph. If \(\{e^{(j)}|j = 1, \ldots, d\}\) is the set of all permutations of \(\Delta_d\) then the Gurau degree (or in short \(G\)-degree) \(\omega_G(\Gamma)\) of \(\Gamma\) is defined as
\[
\omega_G(\Gamma) = \sum_{j=1}^{d} \rho_{e^{(j)}}(\Gamma).
\]

Now, we define some PL-invariants combinatorially.

Definition 8. Let \(M\) be a compact connected PL \(d\)-manifold with boundary \((d \geq 2)\). Then the generalized regular genus \(\mathcal{G}(M)\) of \(M\) is defined as
\[
\mathcal{G}(M) = \min\{\rho(\Gamma) | (\Gamma, \gamma) \text{ is a regular gem of } M\};
\]
the weighted regular genus \(\hat{\mathcal{G}}(M)\) is defined as
\[
\hat{\mathcal{G}}(M) = \min\{\rho(\Gamma) | (\Gamma, \gamma) \in \hat{\mathcal{G}}_d \text{ is a regular gem of } M\};
\]
the Gurau degree (\(G\)-degree in short) \(D_G(M)\) of \(M\) is defined as
\[
D_G(M) = \min\{\omega_G(\Gamma) | (\Gamma, \gamma) \text{ is a regular gem of } M\};
\]
the weighted Gurau degree (weighted \(G\)-degree in short) \(\hat{D}_G(M)\) of \(M\) is defined as
\[
\hat{D}_G(M) = \min\{\omega_G(\Gamma) | (\Gamma, \gamma) \in \hat{\mathcal{G}}_d \text{ is a regular gem of } M\};
\]

If \(M\) is a closed \(d\)-manifold then it is clear that \(\mathcal{G}(M) = \hat{\mathcal{G}}(M) = \mathcal{G}(M)\) and \(\hat{D}_G(M) = D_G(M)\). Also, if \(M\) is a \(d\)-manifold with at most one boundary component then it is easy to see that \(\mathcal{G}(M) = \hat{\mathcal{G}}(M)\) and \(\hat{D}_G(M) = D_G(M)\). In general, \(\hat{\mathcal{G}}(M) \geq \mathcal{G}(M)\) and \(\hat{D}_G(M) \geq D_G(M)\).

Lemma 9. Let \((\Gamma, \gamma) \in \mathcal{G}_d\) represent a compact connected \(d\)-manifold \(M\) with boundary. Choose an arbitrary fixed color \(c \in \Delta_{d-1}\). Let \((\tilde{\Gamma}, \tilde{\gamma})\) be the corresponding regular gem for \(M\) with \(c\) as a singular color. Let \(\tilde{g}_{ij}\) and \(\partial g_{kl}\) be the number of connected components of \(\tilde{\Gamma}_{\{i,j\}}\) and \(\partial \Gamma_{\{k,l\}}\) respectively. Then
\[
\tilde{g}_{id} = \tilde{g}_{id} + \partial g_{ic}, \ i \neq c \in \Delta_{d-1},
\]
and
\[
\tilde{g}_{cd} = g_{cd} = \tilde{g}_{cd} + \tilde{p},
\]
where \(2\tilde{p}\) denotes the number of boundary vertices in \((\Gamma, \gamma)\).

Proof. Let \(i \neq c \in \Delta_{d-1}\). From the construction of \((\tilde{\Gamma}, \tilde{\gamma})\), it is clear that any \(\{i,d\}\)-colored cycle in \((\Gamma, \gamma)\) is also a \(\{i,d\}\)-colored cycle in \((\Gamma, \gamma)\). But, there will be some additional \(\{i,d\}\)-colored cycles in \((\Gamma, \gamma)\), and each of them is obtained by connecting some \(\{i,d\}\)-colored walks in \((\Gamma, \gamma)\) by \(d\)-colored edges. We claim that these additional cycles in \((\tilde{\Gamma}, \tilde{\gamma})\) have one-one correspondence with the \(\{i,c\}\)-colored cycles in \((\partial \Gamma, \partial \gamma)\), i.e., \(\tilde{g}_{id} - \tilde{g}_{id} = \partial g_{ic}\). From the construction of \((\tilde{\Gamma}, \tilde{\gamma})\), it is easy to see that any \(c\)-colored edge between two vertices \(u_1\) and \(u_2\) in \(\partial \Gamma\) corresponds to a \(d\)-colored edge between the vertices \(u_1\) and \(u_2\) in \(\tilde{\Gamma}\) and any \(i\)-colored edge between two vertices \(v_1\) and \(v_2\) in \(\partial \Gamma\) corresponds to a \(\{i,d\}\)-colored walk between the vertices \(u_1\) and \(v_2\) in \(\tilde{\Gamma}\). Thus, there is a one-one correspondence between the \(\{i,c\}\)-colored cycles in \((\partial \Gamma, \partial \gamma)\) and the \(\{i,d\}\)-colored cycles in
which is obtained by connecting some \{i, d\}-colored walks in \((\Gamma, \gamma)\) by \(d\)-colored edges. Thus, \(\tilde{g}_{id} = \tilde{g}_{id} + \partial g_{ic}\).

From the construction of \((\tilde{\Gamma}, \tilde{\gamma})\), it is clear that a \{c, d\}-colored cycle in \((\tilde{\Gamma}, \tilde{\gamma})\) is either a \{c, d\}-colored cycle in \((\Gamma, \gamma)\) or is obtained from a \{c, d\}-colored walk in \((\Gamma, \gamma)\) by adding an edge of color \(d\) between the boundary vertices. Therefore, \(\hat{g}_{cd} = g_{cd}\). Further, each \{c, d\}-colored walk in \((\Gamma, \gamma)\) corresponds to a \(c\)-colored edge in \((\partial \Gamma', \partial \gamma)\). Since the number of boundary vertices in \((\Gamma, \gamma)\) is \(2\tilde{p}\), the number of \(c\)-colored edge in \((\partial \Gamma', \partial \gamma)\) is \(\tilde{p}\). Therefore, \(\hat{g}_{cd} = g_{cd} = \tilde{g}_{cd} + \tilde{p}\).

**Corollary 10.** Let \((\Gamma, \gamma) \in G_d\) represent a compact connected \(d\)-manifold \(M\) with boundary. Choose an arbitrary fixed color \(c \in [\Delta_d - 1]\). Let \((\tilde{\Gamma}, \tilde{\gamma})\) be the corresponding regular gem for \(M\) with \(c\) as a singular color. Then for a permutation \(\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_d = d)\) of \(\Delta_d\), we have

1. \(\rho_c(\tilde{\Gamma}) = \rho_c(\Gamma)\) if \(c \in \{\varepsilon_0, \varepsilon_d - 1\}\),
2. \(\rho_c(\tilde{\Gamma}) = \rho_c(\Gamma) + \partial g_{\varepsilon c} - \partial g_{\varepsilon d - 1 - c}\) if \(c \notin \{\varepsilon_0, \varepsilon_d - 1\}\), where \(\partial g_{\varepsilon_0 \cdots i_k} = g(\partial \Gamma_{i_0 \cdots i_k})\).

**Proof.**

\[
2\rho_c(\Gamma) = 2 - \sum_{i \in \mathcal{Z}_d + 1} \tilde{g}_{\varepsilon_i, \varepsilon_{i + 1}} - (1 - d) \tilde{p} - (2 - d) p - \partial g_{\varepsilon d - 1}
\]

\[
= 2 - \left( \sum_{i \in \mathcal{Z}_d + 1} \tilde{g}_{\varepsilon_i, \varepsilon_{i + 1}} - \partial g_{\varepsilon c} - \partial g_{\varepsilon d - 1} \right) - (1 - d) \tilde{p} - (2 - d) p - \partial g_{\varepsilon d - 1}
\]

\[
= 2 - \sum_{i \in \mathcal{Z}_d + 1} \tilde{g}_{\varepsilon_i, \varepsilon_{i + 1}} - (1 - d) \tilde{p} - (2 - d) p = 2\rho_c(\tilde{\Gamma}).
\]

This proves Part (i). Further for any \(i, j, k \in \Delta_d - 1\), \(\partial g_{ij} + \partial g_{jk} + \partial g_{ik} = 2 + \tilde{p}\). Thus,

\[
2\rho_c(\Gamma) = 2 - \sum_{i \in \mathcal{Z}_d + 1} \tilde{g}_{\varepsilon_i, \varepsilon_{i + 1}} - (1 - d) \tilde{p} - (2 - d) p - \partial g_{\varepsilon d - 1}
\]

\[
= 2 - \left( \sum_{i \in \mathcal{Z}_d + 1} \tilde{g}_{\varepsilon_i, \varepsilon_{i + 1}} - \partial g_{\varepsilon c} - \partial g_{\varepsilon d - 1} \right) - (1 - d) p - \partial g_{\varepsilon d - 1}
\]

\[
= 2 - \sum_{i \in \mathcal{Z}_d + 1} \tilde{g}_{\varepsilon_i, \varepsilon_{i + 1}} + \partial g_{\varepsilon c} + \partial g_{\varepsilon d - 1} - \partial g_{\varepsilon d - 1} - (1 - d) p - \partial g_{\varepsilon d - 1}
\]

\[
= 2 + 2\partial g_{\varepsilon d - 1} - 2\partial g_{\varepsilon d - 1}.
\]

This proves Part (ii). 

**Theorem 11.** Let \(M\) be a compact connected \(d\)-manifold with boundary. Then, \(\mathcal{G}(M) \geq \tilde{G}(M) \geq \mathcal{G}_G(M) \geq \mathcal{D}_G(M)\).

**Proof.** From Definition \[8\] it is clear that \(\tilde{G}(M) \geq \tilde{G}(M)\) and \(\mathcal{D}_G(M) \geq \mathcal{D}_G(M)\). Let \((\Gamma, \gamma) \in G_d\) be a non-regular graph such that the regular genus of \(M\) is assumed by the regular genus of \(\Gamma\). Let the regular genus \(\rho(\Gamma)\) of \(\Gamma\) be attained with respect to the cyclic permutation \(\varepsilon = (\varepsilon_0, \ldots, \varepsilon_d = d)\) of \(\Delta_d\). Let \((\tilde{\Gamma}, \tilde{\gamma})\) be the corresponding regular gem for \(M\) with \(c \in \{\varepsilon_0, \varepsilon_d - 1\}\) as a singular color. Then by Corollary \[10\] \(\rho_c(\tilde{\Gamma}) = \rho_c(\Gamma) = G(M)\). Let \(\varepsilon'\) be the permutation by interchanging the colors \(c\) and \(d\) in \(\varepsilon\) and \((\Gamma', \gamma')\) be the graph by interchanging the colors \(c\) and \(d\) in \((\tilde{\Gamma}, \tilde{\gamma})\). Then \((\Gamma', \gamma')\) is a regular gem of \(M\) in \(\mathcal{G}_d\), and \(\rho_c(\Gamma') = G(M)\). Therefore, \(\mathcal{G}(M) \geq \tilde{G}(M)\).

**Proposition 12.** \(\mathcal{G}(M) \leq \mathcal{G}(M)\). Let \((\Gamma, \gamma)\) be a \((d + 1)\)-colored graph representing the singular \(d\)-manifold \(M\) and the associated compact \(d\)-manifold \(\tilde{M}\). Let \(X_{ij}\) and \(R_{ij}\) be the sets which are in bijective correspondence to the connected components of \(\Gamma_{\Delta_d \setminus \{i, j\}}\) and \(\Gamma_{ij}\) respectively. Let \(R_{ij} \subset X_{ij}\) corresponds to a maximal tree of the subcomplex \(K_{ij}\) of \(K(\Gamma)\) (consisting only of vertices labeled by \(i\) and \(j\), and edges connecting them). Then,
(a) if \( i, j \in \Delta_d \) are not singular in \( \Gamma \),
\[
\pi_1(M) = \langle X_{ij}/R_{ij} \cup \bar{R}_{ij} \rangle.
\]

(b) if no color in \( \Delta_d \backslash \{i, j\} \) is singular in \( \Gamma \). Then
\[
\pi_1(M) = \langle X_{ij}/R_{ij} \cup \bar{R}_{ij} \rangle.
\]

From now onwards, we shall focus on connected compact 4-manifolds with non-spherical boundary components.

**Remark 13.** Let \((\Gamma, \gamma)\) be a regular 5-colored graph. Then the Euler characteristic of the corresponding simplicial cell complex does not change by canceling 1-dipole in \( \Gamma \). Because with the cancellation of one 1-dipole in \( \Gamma \), the number of \( f \)-vectors in the corresponding simplicial cell complex are reduced by
\[
f_0 = 1, f_1 = 4, f_2 = 6, f_3 = 5, f_4 = 2,
\]
and hence the result follows.

**Remark 14.** Let \( M \) be a compact 4-manifold with boundary such that \( \partial^1 M, \ldots, \partial^h M \) are components of \( \partial M \). Let \( \hat{M} \) be the singular 4-manifold obtained by coning off each component of \( \partial M \). Let \( \mathbb{H}_{ji^r}M \) denote the cone over \( \partial^i M \) for \( i \in \{1, 2, \ldots, h\} \). Then \( \chi(\mathbb{H}_{ji^r}M) = 1 \). Since \( \partial M \) is a 3-manifold, \( \chi(\partial M) = 0 \). Therefore, \( \chi(\hat{M}) = \chi(M) + \sum_{i=1}^{h} \chi(\mathbb{H}_{ji^r}M) = \chi(M) + h \).

**Remark 15.** Let \((\Gamma, \gamma)\) be a regular 5-colored graph then the regular genus of the graph does not change by canceling 1-dipoles in \( \Gamma \).

**Remark 16.** Let \((\Gamma, \gamma)\) be a 5-colored graph (possibly with boundary) representing the simplicial cell complex \( K(\Gamma) \). Let \((\Gamma', \gamma')\) be the 5-colored graph after canceling of 1-dipole from \((\Gamma, \gamma)\). Let \( K(\Gamma') \) be the simplicial cell-complex corresponding to \((\Gamma', \gamma')\). Then \( |K(\Gamma')| \) is a deformation retract of \( |K(\Gamma)| \). Thus, \(|(\Gamma', \gamma')|\) and \(|K(\Gamma')|\) have the same fundamental group.

Let \( (\hat{\Gamma}, \hat{\gamma}) \in \hat{G}_4 \) be a regular gem of a compact connected 4-manifold \( M \) with empty or connected boundary. Let \( m \) and \( \hat{m} \) be the rank of the fundamental groups of \( M \) and the corresponding singular manifold \( \hat{M} \) respectively. Let \( \hat{g}_{ijk} = g(\hat{\Gamma}_{\{i,j,k\}}) \). It follows from Proposition 12 that \( \hat{g}_{ijk} \geq \hat{m} + 1 \) and \( \hat{g}_{ijkl} \geq m + 1 \) for \( \{i, j, k, l\} \in \Delta_4 \). Let \( \hat{g}_{ijk} = (\hat{m} + 1) + t_{ijk} \) and \( \hat{g}_{ijkl} = (m + 1) + t_{ijkl} \) where \( t_{ijk} \geq 0 \).

**Proposition 17** \((\Delta_4)\). Let \( (\hat{\Gamma}, \hat{\gamma}) \in \hat{G}_4 \) be a regular gem of a compact connected 4-manifold \( M \) with empty or connected boundary. Let \( \hat{M}, m, \hat{m}, t_{ijk} \) be as above. Then for any cyclic permutation \( \varepsilon = (\varepsilon_0, \ldots, \varepsilon_4) \),
\[
\rho_\varepsilon(\hat{\Gamma}) = 2\chi(\hat{M}) + 5m - 2(m - \hat{m}) - 4 + \sum_{i \in \mathbb{Z}_5} \hat{t}_{\varepsilon, \varepsilon_{i+2}, \varepsilon_{i+4}}.
\]

**Proposition 18** \((\Delta_4)\). Let \( (\hat{\Gamma}, \hat{\gamma}) \) be a regular 5-colored graph in \( \hat{G}_4 \). For any cyclic permutation \( \varepsilon = (\varepsilon_0, \ldots, \varepsilon_4 = 4) \) of \( \Delta_4 \), let \( \varepsilon' = (\varepsilon_1, \varepsilon_3, \varepsilon_0, \varepsilon_2, \varepsilon_4 = 4) \). Then,
\[
\omega_G(\hat{\Gamma}) = 6(\rho_\varepsilon(\hat{\Gamma}) + \rho_{\varepsilon'}(\hat{\Gamma})).
\]

**Theorem 19.** Let \( M \) be a compact connected PL 4-manifold with \( h \) boundary components. Let \( \hat{M} \) be the singular 4-manifold obtained from \( M \) by coning off each boundary component of \( M \). Let \( m \) and \( \hat{m} \) be the rank of fundamental groups of \( M \) and \( \hat{M} \) respectively. Then,
\[
\hat{G}(M) \geq 2\chi(M) + 3m + 2h - 4 + 2\hat{m}
\]
and
\[
\hat{D}_G(M) \geq 12(2\chi(M) + 3m + 2h - 4 + 2\hat{m}).
\]
Proof. Let $\partial^1 M, \partial^2 M, \ldots, \partial^h M$ be the boundary components of $M$. Let $(\hat{\Gamma}, \hat{\gamma}) \in \hat{G}_4$ be any regular gem of $M$, i.e., it represents the singular manifold $\hat{M}$. Let $(\hat{\Gamma}', \hat{\gamma}') \in \hat{G}_4$ be the regular 5-colored graph obtained from $(\hat{\Gamma}, \hat{\gamma})$ by canceling all possible 1-dipoles of each color labeled by $i \in \Delta$. Then, $K(\hat{\Gamma}')$ has 5 vertices such that the vertices labeled by color $i \in \Delta$ are non-singular and the vertex labeled by color 4 is singular. Further, the link of the vertex labeled by color 4 in $\hat{M}$ bounds for the regular genus of $\hat{M}$. Thus, for any $(\hat{\Gamma}, \hat{\gamma}) \in \hat{G}_4$ we have $\chi(\hat{M}) = \chi(M) + h$. Let $\hat{M}'$ be the manifold with connected boundary obtained by removing a small neighborhood of vertex 4 from $\hat{M}$. Then from Remarks 13 and 14 we have $\chi(\hat{M}') = \chi(M)$. Let $\hat{\Gamma}'$ be the graph $\hat{\Gamma}$ obtained from $(\hat{\Gamma}, \hat{\gamma})$ by canceling all possible 1-dipoles of each color labeled by $i \in \Delta$. Then by Proposition 18 we have $\chi(\hat{M}') = \chi(M)$. Further, it follows from Eq. (1) that, for any regular gem $(\hat{\Gamma}, \hat{\gamma})$, \[ \rho(\hat{\Gamma}) = 2\chi(\hat{M}') + 5m - 2(m - \hat{m}) - 4 + \sum_{i \in Z_5} \hat{t}_{i\hat{\gamma}+2\hat{\gamma}+2}. \] and \[ \rho(\hat{\Gamma}) = 2\chi(\hat{M}') + 5m - 2(m - \hat{m}) + 4 + \sum_{i \in Z_5} \hat{t}_{i\hat{\gamma}+2\hat{\gamma}+2}. \] Then by Proposition 18 we have \[ \omega_G(\hat{\Gamma}) = 6(\rho(\hat{\Gamma}) + \rho(\hat{\Gamma}')) = 6(4\chi(\hat{M}') + 10m - 4(m - \hat{m}) - 8 + \sum_{0 \leq i < j < k \leq 4} \hat{t}_{ijk}) \] = \[ 6(4\chi(M) + 4h + 10m - 4(m - \hat{m}) - 8 + \sum_{0 \leq i < j < k \leq 4} \hat{t}_{ijk}). \] Thus, for any $(\hat{\Gamma}, \hat{\gamma}) \in \hat{G}_d$, \[ \omega_G(\hat{\Gamma}) \geq 12(2\chi(M) + 3m + 2h - 4 + 2\hat{m}). \] Since \[ D_G(M) = \min \{ \omega_G(\hat{\Gamma}) \mid (\hat{\Gamma}, \hat{\gamma}) \in \hat{G}_d \text{ is a regular gem of } M \}, \] we have \[ D_G(M) \geq 12(2\chi(M) + 3m + 2h - 4 + 2\hat{m}). \] Further, it follows from Eq. (11) that, for any regular gem $(\hat{\Gamma}, \hat{\gamma}) \in \hat{G}_d$ of $M$, \[ \rho(\hat{\Gamma}) \geq \rho(\hat{\Gamma}) \geq 2\chi(M) + 3m + 2h - 4 + 2\hat{m}. \] Since \[ \hat{G}(M) = \min \{ \rho(\hat{\Gamma}) \mid (\hat{\Gamma}, \hat{\gamma}) \in \hat{G}_d \text{ is a regular gem of } M \}, \] we have \[ \hat{G}(M) \geq 2\chi(M) + 3m + 2h - 4 + 2\hat{m}. \] The above result in combination with Theorem 11 gives a lower bound for the regular genus of compact connected 4-manifolds $M$ with boundary, which is stronger than the previous known lower bounds for the regular genus of $M$. \[ \square \]
Corollary 20. Let $M$ be a connected compact 4-manifold with $h$ boundary components. Then, the regular genus of $M$ satisfies the following inequality:

$$G(M) \geq 2\chi(M) + 3m + 2h - 4 + 2\hat{m},$$

where $m$ and $\hat{m}$ are the rank of fundamental groups of $M$ and the corresponding singular manifold $\hat{M}$ respectively.

Definition 21. Let $M$ be a connected compact 4-manifold with $h$ boundary components and $\hat{M}$ be the corresponding singular 4-manifold. Let $(\hat{\Gamma}, \hat{\gamma}) \in \hat{G}_4$ be the regular gem of $M$ such that $g(\hat{\Gamma}_c) = h$ and $g(\hat{\Gamma}) = 1$ for each $c \in \Delta_3$. For $\{i_0, \ldots, i_k\} \subset \Delta_3$, let $g_{i_0 \ldots i_k}$ denote the number of connected components of $\hat{\Gamma}_{\{i_0, \ldots, i_k\}}$. Let $m$ and $\hat{m}$ be the rank of fundamental groups of $M$ and $\hat{M}$ respectively. Then, $\hat{\Gamma}$ is said to be semi-simple if

$$g_{ijk} = \hat{m} + h \quad \text{and} \quad g_{ij4} = m + 1,$$

where $i, j, k \in \Delta_3$

and is said to be the weak semi-simple if, for a cyclic permutation $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_4)$ of $\Delta_4$,

$$g_{\varepsilon_1\varepsilon_2\varepsilon_4} = \hat{m} + h, \quad \text{where} \quad i \in \{1, 3\} \quad \text{and} \quad g_{\varepsilon_1\varepsilon_2\varepsilon_4} = m + 1, \quad \text{where} \quad i \in \{0, 2, 4\}.$$

Definition 22. Let $M$ be a connected compact 4-manifold with boundary. Then $M$ is said to admit a semi-simple gem if there exists a regular gem $(\hat{\Gamma}, \hat{\gamma}) \in \hat{G}_4$ of $M$, which is semi-simple, and $M$ is said to admit a weak semi-simple gem if there exists a regular gem $(\hat{\Gamma}, \hat{\gamma}) \in \hat{G}_4$ of $M$, which is weak semi-simple.

Let $M$ be a connected compact 4-manifold with $h$ boundary components. Then we always have a regular gem $(\hat{\Gamma}, \hat{\gamma}) \in \hat{G}_4$ of $M$ such that $g(\hat{\Gamma}_c) = h$ and $g(\hat{\Gamma}) = 1$ for each $i \in \Delta_3$. Let $(\Gamma, \gamma)$ be a crystallization of $M$. Let $(\hat{\Gamma}, \hat{\gamma}) \in \hat{G}_4$ be the corresponding regular gem of $M$. Then by removing all 1-dipoles of color $i \in \Delta_3$, let $(\tilde{\Gamma}, \tilde{\gamma})$ be the resulting graph. Then $(\hat{\Gamma}, \hat{\gamma}) \in \hat{G}_4$, $g(\hat{\Gamma}_c) = h$ and $g(\hat{\Gamma}) = 1$ for each $i \in \Delta_3$.

Theorem 23. Let $M$ be a connected compact 4-manifold with $h$ boundary components. Let $m$ and $\hat{m}$ be the rank of the fundamental groups of $M$ and the corresponding singular manifold $\hat{M}$ respectively. Then, $M$ admits a semi-simple gem if and only if $D_G(M) = 12(2\chi(M) + 3m + 2h - 4 + 2\hat{m})$ and $M$ admits a weak semi-simple gem if and only if $G(M) = 2\chi(M) + 3m + 2h - 4 + 2\hat{m}$.

Proof. Let $(\hat{\Gamma}, \hat{\gamma}) \in \hat{G}_4$ be a regular gem of $M$ such that $g(\hat{\Gamma}_c) = h$ and $g(\hat{\Gamma}) = 1$ for each $i \in \Delta_3$. Let $(\tilde{\Gamma}, \tilde{\gamma}) \in \tilde{G}_4$ be the graph obtained from $(\hat{\Gamma}, \hat{\gamma})$ by canceling all the $(h - 1)$ number of 1-dipoles of color 4, and let $\tilde{M}' = [K(\tilde{\Gamma})]$. Let $M'$ be the manifold with connected boundary obtained by removing a small neighborhood of vertex 4 from $\tilde{M}'$. It follows from Remark 10 that $\text{rank}(\pi_1(\tilde{M}')) = \text{rank}(\pi_1(\tilde{M})) = \hat{m}$. Since $M$ is a deformation retract of $M'$, $\text{rank}(\pi_1(M')) = \text{rank}(\pi_1(M)) = m$. For $\{i_0, \ldots, i_k\} \subset \Delta_3$, let $\hat{g}_{i_0 \ldots i_k}$ and $\hat{g}'_{i_0 \ldots i_k}$ denote the number of components of $\hat{\Gamma}_{\{i_0, \ldots, i_k\}}$ and $\hat{\Gamma}'_{\{i_0, \ldots, i_k\}}$ respectively. Further, Proposition 12 implies, $\hat{g}_{ijk} - h + 1 = \hat{g}'_{ijk} \geq \hat{m} + 1$ and $\hat{g}_{ij4} = \hat{g}'_{ij4} \geq m + 1$, for $i, j, k \in \Delta_3$. We write, $\hat{g}_{ijk} - h + 1 = \hat{g}'_{ijk} = (\hat{m} + 1) + \hat{t}_{ijk}$ and $\hat{g}_{ij4} = \hat{g}'_{ij4} = (m + 1) + \hat{t}_{ij4}$. Therefore,

$$M \text{ admits a semi-simple gem } \iff \text{ there exists a regular gem } (\tilde{\Gamma}, \tilde{\gamma}) \in \tilde{G}_4 \text{ of } M \text{ such that }$$

$$\sum_{0 \leq i < j < k \leq 4} \hat{t}_{ijk} = 0$$

$$\iff \omega_G(\tilde{\Gamma}) = 12(2\chi(M) + 3m + 2h - 4 + 2\hat{m})$$

$$\iff \hat{D}_G(M) = 12(2\chi(M) + 3m + 2h - 4 + 2\hat{m}).$$
On the other hand,

\[ M \text{ admits a weak semi-simple gem } \iff \exists \text{ a regular gem } (\hat{\Gamma}, \hat{\gamma}) \in \hat{G}_4 \text{ of } M \text{ such that } \sum_{i \in \mathbb{Z}_5} \hat{t}'_{i,i+2i+4} = 0 \text{ for some } \varepsilon = (\varepsilon_0, \ldots, \varepsilon_4) \]
\[ \iff \rho_{\varepsilon}(\hat{\Gamma}) = 2\chi(M) + 3m + 2h - 4 + 2\hat{m} \]
\[ \iff \hat{G}(M) = 2\chi(M) + 3m + 2h - 4 + 2\hat{m}. \]

For a connected compact 4-manifold \( M \) with boundary, let us define its \textit{weighted gem-complexity} as the non-negative integer \( \hat{k}(M) = p - 1 \), where \( 2p \) is the minimum order of a regular gem of \( M \).

**Corollary 24.** Let \( M \) be a connected compact 4-manifold with \( h \) boundary components. Let \( m \) be the rank of the fundamental group of \( M \). Then
\[
\hat{D}_G(M) = 6(\chi(M) - 1 + \hat{k}(M)).
\]

**Proof.** Let \( (\hat{\Gamma}, \hat{\gamma}) \in \hat{G}_4 \) be a regular of \( M \) such that \( g(\hat{\Gamma}, i) = h \) and \( g(\hat{\Gamma}, i) = 1 \) for each \( i \in \Delta_3 \). Let \( 2p \) be the number of vertices of \( \hat{\Gamma} \). Let \( \hat{\Gamma}', \hat{M}', \hat{t}', \hat{g}', \hat{t}'_{ijk} \) be as in Theorem 23. Let \( 2p' \) be the number of vertices of \( \hat{\Gamma}' \). Then \( 2p = 2p' + 2(h - 1) \). Then, from the proof of Theorem 19, we have
\[
\omega_G(\hat{\Gamma}) = 6(4\chi(M) + 4h + 6m + 4\hat{m} - 8 + \sum_{0 \leq i < j < k \leq 4} \hat{t}'_{ijk}).
\]

Further by the Dehn–Sommerville equations in dimension four (cf. [3, Lemma 6]) we have
\[
2p' = 6\chi(\hat{M}') + 2\sum_{0 \leq i < j < k \leq 4} \hat{g}'_{ijk} - 30 = 6(\chi(M) + h) + 12m + 8\hat{m} - 10 + 2\sum_{0 \leq i < j < k \leq 4} \hat{t}'_{ijk}.
\]

Therefore,
\[
\omega_G(\hat{\Gamma}) = 6(4\chi(M) + 4h + 6m + 4\hat{m} - 8 + p - h + 1 - 3\chi(M) - 3h - 6m - 4\hat{m} + 5)
\]
\[ = 6(\chi(M) - 1 + p - 1).
\]

This proves the result.

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