Chiral-scale perturbation theory $\chi PT_\sigma$ has been proposed as an alternative to chiral $SU(3)_L \times SU(3)_R$ perturbation theory which explains the $\Delta I = 1/2$ rule for kaon decays. It is based on a low-energy expansion about an infrared fixed point in three-flavor QCD. In $\chi PT_\sigma$, quark condensation $\langle \bar{q}q \rangle_{\text{vac}} \neq 0$ induces nine Nambu-Goldstone bosons: $\pi, K, \eta$ and a QCD dilaton $\sigma$ which we identify with the $f_0(500)$ resonance. Partial conservation of the dilatation and chiral currents constrains low-energy constants which enter the effective Lagrangian of $\chi PT_\sigma$. These constraints allow us to obtain new phenomenological bounds on the dilaton decay constant via the coupling of $\sigma/f_0$ to pions, whose value is known precisely from dispersive analyses of $\pi\pi$ scattering. Improved predictions for $\sigma \to \gamma\gamma$ and the $\sigma NN$ coupling are also noted. To test $\chi PT_\sigma$ for kaon decays, we revive a 1985 proposal for lattice methods to be applied to $K \to \pi$ on-shell.
1. Approximate Scale Invariance in Low-Energy QCD

In the low-energy regime of QCD with heavy quarks $t,b,c$ decoupled, the relevance of scale (dilatation) invariance is determined by the trace anomaly [1]–[4] of the resulting 3-flavor theory:

$$\theta^\mu_\mu = \frac{\beta (\alpha_s)}{4\alpha_s} G^\mu_\nu G^{\nu\mu} + (1 + \gamma_m (\alpha_s)) \sum_{q=u,d,s} m_q \bar{q} q.$$  \hspace{1cm} (1.1)

Depending on the infrared behaviour of $\beta$, there are only two realistic scenarios (Fig. 1 (A)):

1. If $\beta$ remains negative and non-zero, possibly diverging linearly at large $\alpha_s$, scale invariance is explicitly broken by $\theta^\mu_\mu$ being large as an operator. There is no hint of approximate scale invariance: quantities such as the nucleon mass $M_N = \langle N | \theta^\mu_\mu | N \rangle$ are generated almost entirely by the gluonic term in (1.1). Then conventional chiral $SU(3)_L \times SU(3)_R$ perturbation theory $\chi$PT$_3$ is the appropriate low-energy effective theory for QCD amplitudes expanded in powers of $O(m_K)$ external momenta and light quark masses $m_{u,d,s} = O(m_K^2)$.

2. If $\beta$ vanishes when $\alpha_s$ runs non-perturbatively to an infrared fixed point $\alpha_{IR}$, the gluonic term $\sim G^\mu_\nu G^{\nu\mu}$ in (1.1) is absent and the dilatation current $D_\mu = x^\nu \theta^\mu_\nu$ becomes conserved in the limit of vanishing quark masses:

$$\partial^\mu D_\mu \big|_{\alpha_s = \alpha_{IR}} = \theta^\mu_\mu \big|_{\alpha_s = \alpha_{IR}} = (1 + \gamma_m (\alpha_{IR})) \sum_{q=u,d,s} m_q \bar{q} q$$

$$\rightarrow 0, \qquad SU(3)_L \times SU(3)_R \text{ limit.}$$  \hspace{1cm} (1.2)

Although the Hamiltonian preserves dilatations in this limit, the vacuum state is not scale invariant due to the formation of a quark condensate $\langle \bar{q} q \rangle_{vac} \neq 0$. As a result, both chiral $SU(3)_L \times SU(3)_R$ and scale symmetry are realized in the Nambu-Goldstone (NG) mode and the spectrum contains nine massless bosons: $\pi, K, \eta$ and a $0^{++}$ QCD dilaton $\sigma$. Non-NG bosons remain massive despite the vanishing of $\theta^\mu_\mu$ and have their scale set by $\langle \bar{q} q \rangle_{vac}$. The relevant low-energy expansion involves a combined limit

$$m_{u,d,s} \sim 0 \quad \text{and} \quad \alpha_s \lesssim \alpha_{IR},$$  \hspace{1cm} (1.3)

and leads to a new effective theory $\chi$PT$_\sigma$ of approximate chiral-scale symmetry [5, 6]. In this scenario, the dilaton mass is set by $m_\sigma$, so the natural candidate for $\sigma$ is the $f_0(500)$ resonance, a broad $0^{++}$ state whose complex pole mass has real part $\lesssim m_K$ [7, 8, 9].

Until now, scenario 1 has been the generally accepted view, but we have observed [5, 6] that $\chi$PT$_\sigma$ offers several advantages over $\chi$PT$_3$: it explains the mass and width of $f_0(500)$, produces convergent chiral expansions as a result of $\sigma/f_0$ being promoted to the NG sector, and most importantly, explains the $\Delta I = 1/2$ rule for non-leptonic $K$ decays (Fig. 1 (B)).

Because approximate scale symmetry is included, the effective Lagrangian for $\chi$PT$_\sigma$ (Sec. 2) contains several new low-energy constants (LECs) yet to be determined precisely from data. Of particular interest is the dilaton decay constant $F_\sigma$ given by $m_\sigma^2 F_\sigma = -\langle \sigma | \theta^\mu_\mu | \text{vac} \rangle$. If $F_\sigma$ is roughly 100

$^{1}$Here, $G^\mu_\nu$ is the gluon field strength, $\alpha_s = g^2_s/4\pi$ is the strong running coupling, and $\beta = \mu^2 \partial \alpha_s / \partial \mu^2$ and $\gamma_m = \mu^2 \partial \ln m_q / \partial \mu^2$ refer to a mass-independent renormalization scheme with scale $\mu$. 

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Chiral-Scale Perturbation Theory
Figure 1: (A) Scenarios for the $\beta$ function in three-flavor QCD, with corresponding low-energy expansions. In the absence of an infrared fixed point $\alpha_{IR}$ (top diagram), there is no approximate scale invariance and chiral $SU(3)_L \times SU(3)_R$ perturbation theory $\chi PT_3$ is relevant at low-energies. If $\alpha_{IR}$ exists (bottom diagram), quark condensation $\langle \bar{q}q \rangle_{\text{vac}} \neq 0$ implies that the NG spectrum contains a QCD dilaton $\sigma$, and $\chi PT_3$ must be replaced by chiral-scale perturbation theory $\chi PT_\sigma$. (B) Diagrams for $K \rightarrow \pi \pi$ decay in lowest-order $\chi PT_\sigma$. The dilaton pole diagram is responsible for the dominant $\Delta I = 1/2$ amplitude.

MeV, scale breaking by the vacuum can generate large masses such as $m_N \approx F_\sigma g_{\sigma NN}$ (Goldberger-Treiman relation for dilatons [10]) for $m_\sigma$ small. The imprecise value of $F_\sigma$ in our previous work [5, 6] arose from large uncertainties in the phenomenological value of $g_{\sigma NN}$ [11, 12].

We circumvent this difficulty in Secs. 3 and 4. First, we find new constraints on LECs in the $\chi PT_\sigma$ effective Lagrangian by requiring full consistency with the dilatation and chiral currents being conserved in the limit (1.2). These constraints allow us to determine $F_\sigma$ from the $\sigma \pi \pi$ coupling, whose value is known to remarkable precision from dispersive analyses [7, 8, 9] of $\pi \pi$ scattering. Then we obtain improved predictions for the non-perturbative Drell-Yan ratio

$$R = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)}{\sigma_{IR}}, \quad (1.4)$$

as well as the $\sigma NN$ coupling.

In Sec. 5, we resurrect an old proposal [13] to apply lattice QCD for $K \rightarrow \pi$ on-shell to determine the couplings $g_{8,27}$ in Fig. 1 (B). Comments on the validity of $\chi PT_\sigma$ are reviewed in Sec. 6.

2. Chiral-Scale Lagrangian

For strong interactions, the most general effective Lagrangian of $\chi PT_\sigma$ is of the form

$$\mathcal{L}_{\chi PT_\sigma} = \mathcal{L}_{\text{inv}}^{d=4} + \mathcal{L}_{\text{anom}}^{d>4} + \mathcal{L}_{\text{mass}}^{d<4}, \quad (2.1)$$

where

$$d_{\text{anom}} = 4 + \gamma^d(\alpha_s) \quad \text{and} \quad d_{\text{mass}} = 3 - \gamma_m(\alpha_s) \quad (2.2)$$
are the respective scaling dimensions of \( G^\mu_\nu G^{\mu\nu} \) and \( \bar{q}q \). In lowest order (LO) of the chiral-scale expansion, we have \( \gamma_m = \gamma_m(\alpha_{IR}) \) and

\[
\gamma_G(\alpha_s) = \beta'(\alpha_s) - \beta(\alpha_s)/\alpha_s = \beta'(\alpha_{IR}) + O(\alpha_s - \alpha_{IR}),
\]

so the resulting terms in (2.1) are

\[
\mathcal{L}_{anom,LO}^{d<4} = \{(1 - c_1) + (1 - c_2)\mathcal{K}_m + c_4 e^{2\beta'/\alpha_s}\} e^{2\alpha'/\alpha_s},
\]

\[
\mathcal{L}_{mass,LO} = \text{Tr}(\mu^\dagger U^\dagger + U^\dagger \mu + \mathcal{L}) e^{(3-\gamma_m)/\alpha_s}.
\]

As \( \alpha_s \to \alpha_{IR} \), the gluonic anomaly vanishes, so \( \mathcal{L}_{anom} = O(\alpha^2, M) \) and we must set \( c_4 = O(M) \). Vacuum stability in the \( \sigma \) direction about \( \sigma = 0 \) (no tadpoles) implies

\[
4c_3 + (4 + \beta')c_4 = -3 - \gamma_m(\text{Tr}(\mu^\dagger U^\dagger + U^\dagger \mu))_{\text{vac}} = -3 - \gamma_m f_\pi^2 (m_\pi^2 + \frac{2}{3} m_F^2),
\]

so \( c_3 \) is also \( O(M) \). Expanding (2.4) about \( \sigma = 0 \) and \( U = 1 \) yields the \( \sigma\pi\pi \) coupling

\[
\mathcal{L}_{\sigma\pi\pi} = \left\{ \frac{1}{2} (2 + (1 - c_1) \beta') |\partial \pi|^2 - (3 - \gamma_m) m_\pi^2 |\pi|^2 \right\} \sigma/(2f_\sigma),
\]

while the corresponding \( \sigma\pi\pi \) vertex for an on-shell dilaton is

\[
g_{\sigma\pi\pi} = -\frac{4}{f_\sigma} \left\{ \left( 2 + (1 - c_1) \beta' \right) m_\sigma^2 + 2 \left( 1 - \gamma_m - (1 - c_1) \beta' \right) m_\pi^2 \right\}.
\]

3. Effective Energy-Momentum Tensor and its Trace

In any field theory, the energy-momentum tensor can be identified by adding a gravitational source field \( g_{\mu\nu}(x) \) coupled to matter fields in a generally covariant fashion. In \( \chi\PT \), this amounts to the substitution

\[
\mathcal{L}_{\chi\PT}[U,U^\dagger,\sigma] \to \mathcal{L}_{\chi\PT}[U,U^\dagger,\sigma,g_{\mu\nu}],
\]

where the new effective Lagrangian must be constructed in terms of generally covariant operators. Then the energy-momentum tensor is defined via the variation

\[
\theta_{\mu\nu}(x) = 2 \left[ \frac{\delta}{\delta g_{\mu\nu}(x)} \sqrt{-g} \mathcal{L}[U,U^\dagger,\sigma,g_{\mu\nu}] \right]_{g_{\mu\nu} = \eta_{\mu\nu}},
\]

where \( g = \det(g_{\mu\nu}) \) is the determinant of the metric tensor and \( \eta_{\mu\nu} \) is the flat Minkowski metric. Generalising Donoghue and Leutwyler [14], we obtain the lowest order result

\[
\theta_{\mu\nu} = \left[ \frac{1}{2} f_\pi^2 \text{Tr}(\partial_\nu U \partial_\mu U^\dagger) - g_{\mu\nu} \mathcal{K} \right] c_1 e^{2\alpha'/\alpha_s} + (1 - c_1) e^{(2 + \beta')/\alpha_s}
\]

\[
+ (\partial_\nu \sigma \partial_\mu \sigma - g_{\mu\nu} \mathcal{K}_m) c_2 e^{2\alpha'/\alpha_s} + (1 - c_2) e^{(2 + \beta')/\alpha_s}
\]

\[
- g_{\mu\nu} \text{Tr}(\mu^\dagger U^\dagger + U^\dagger \mu) e^{(3 - \gamma_m)/\alpha_s} - g_{\mu\nu} e^{4\alpha'/\alpha_s} (c_3 + c_4 \beta'/\alpha_s).
\]
The trace of (3.3) involves scale invariant operators like $\text{Tr}(\partial_\mu U \partial^\mu U^\dagger) e^{2\sigma/F}$ which obscure the connection between the scale invariance and a conserved dilatation current $D_\mu$. To remedy this, we “improve” $\theta_{\mu\nu}$ [15] by adding a term

$$I_{\mu\nu} = \frac{F_\sigma^2}{6} (g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) \left[ c_2 e^{2\sigma/F} + \frac{2(1-c_2)}{2+\beta'} e^{(2+\beta')\sigma/F} \right],$$

such that the trace of

$$\theta_{\mu\nu}|_{\text{eff}} = \theta_{\mu\nu} + I_{\mu\nu},$$

is given entirely in terms of explicit scale-breaking operators $L_d$ of scale dimension $d$:

$$\partial^\mu D_\mu|_{\text{eff}} = \theta_{\mu\mu}|_{\text{eff}} = \sum_d (d-4) L_d.$$ (3.6)

Explicitly, the improved trace is

$$\theta_{\mu\mu}|_{\text{eff}} = \beta' \mathcal{L}^{d>4}_{\text{anom}} - (1 + \gamma_m) \mathcal{L}^{d<4}_{\text{mass}}$$

$$= \beta' \left\{ (1-c_1) \mathcal{X} + (1-c_2) \mathcal{X}_\sigma + c_4 e^{2\sigma/F} \right\} e^{(2+\beta')\sigma/F}$$

$$- (1 + \gamma_m) \text{Tr}(MU^\dagger + UM^\dagger) e^{(3-\gamma_m)\sigma/F}.$$ (3.7)

It vanishes in the chiral-scale limit (1.2) only if the low-energy constants associated with $d > 4$ operators satisfy

$$c_1 = c_2 = 1,$$

for $m_{u,d,s} \to 0$ and $\alpha_i \to \alpha_{\text{IR}},$ (3.8)

in addition to the condition $c_4 = O(M)$ required by tadpole cancellation (2.6). Note that the condition $c_1 \to 1$ in (3.8) ensures that chiral currents have vanishing anomalous dimensions. We can summarise these LO conditions by writing

$$c_i = 1 + O(M),$$

where the $O(M)$ term is a linear superposition of $O(p^2,M)$ operators and associated LECs.

### 4. Improved Predictions

An immediate consequence of the constraint (3.9) is that the $\sigma\pi\pi$ coupling for an on-shell dilaton (2.8) takes a particularly simple form

$$g_{\sigma\pi\pi} = -\frac{1}{F_\sigma} \left[ m_\sigma^2 + (1-\gamma_m)m_\pi^2 \right],$$

where $-1 \leq 1 - \gamma_m < 2.$ (4.1)

Since the narrow-width approximation is valid in lowest order $\chi PT_\sigma$ [6], we have

$$\Gamma_{\sigma\pi\pi} = \frac{|g_{\sigma\pi\pi}|^2}{16\pi m_\sigma} \sqrt{1 - 4m_\pi^2/m_\sigma^2},$$

and this allows us to obtain bounds on $F_\sigma$ from dispersive analyses of $\pi\pi$ scattering based on the Roy equations. For example, the $f_0/\sigma'$s mass and width from [7]

$$m_\sigma = 441^{+16}_{-8} \text{ MeV}, \quad \Gamma_{\sigma\pi\pi} = 544^{+18}_{-25} \text{ MeV},$$ (4.3)
constrain $F_\sigma$ to lie within the interval $44 \text{ MeV} \leq F_\sigma \leq 61 \text{ MeV}$, where we have allowed $1 - \gamma_m$ to vary according to (4.1). For the moment, we assume that NLO corrections are not a problem.

With $F_\sigma$ fixed in this manner, we can now use the Golberger-Treiman relation for dilatons [10] to predict the value for the $\sigma NN$ coupling. We find $16 \leq g_{\sigma NN} \leq 21$, which is somewhat larger than previous phenomenological determinations [11, 12]. Another important application concerns $\sigma \rightarrow \gamma\gamma$, where an analysis [5, 6] of the electromagnetic trace anomaly in $\chi$PT$_\sigma$ relates the $\sigma\gamma\gamma$ coupling to (1.4):

$$g_{\sigma\gamma\gamma} = \frac{2\alpha}{3\pi F_\sigma} \left(R_{IR} - \frac{1}{2}\right).$$

By fixing $g_{\sigma\gamma\gamma}$ from the di-photon width $\Gamma_{\sigma\gamma\gamma} = 2.0 \pm 0.2 \text{ keV}$ [16], we find $2.4 \leq R_{IR} \leq 3.1$, which is to be compared with our previous estimate $R_{IR} \approx 5$ [5, 6].

5. Proposal to test $K \rightarrow \pi$ on the Lattice

The key idea [13] is to keep both $K$ and $\pi$ on shell and allow $O(m_K)$ momentum transfers.

The lowest-order diagrams for the decay $K \rightarrow \pi\pi$ in Fig. 1 (B) are derived from an effective weak $\chi$PT$_\sigma$ Lagrangian [5, 6]

$$\mathcal{L}_{\text{weak}} = \mathcal{L}_8 + \mathcal{L}_{27} + \mathcal{L}_{\text{mass}},$$

which reduces to the standard $\chi$PT$_3$ Lagrangian

$$\mathcal{L}_{\text{weak}}|_{\sigma=0} = g_8 \mathcal{L}_8 + g_{27} \mathcal{L}_{27} + \mathcal{L}_{\text{mass}}.$$  

in the limit $\sigma \rightarrow 0$. Eqs. (5.1) and (5.2) contain an octet operator [17]

$$\mathcal{L}_8 = J_{13}^\mu J_{21}^\mu - J_{23}^\mu J_{11}^\mu, \quad J_{ij}^\mu = (U\partial\mu U^\dagger)_{ij}$$

the $U$-spin triplet component [13, 18] of a $27$ operator

$$\mathcal{L}_{27} = J_{13}^\mu J_{21}^\mu + \frac{1}{2} j_{23}^\mu J_{11}^\mu$$

and a weak mass operator [19]

$$\mathcal{L}_{\text{mass}} = \mathcal{L}(\sigma - i\lambda_2) (g_MMU^\dagger + \bar{g}_MUM^\dagger).$$

Powers of $e^{\sigma/F_\sigma}$ are used to adjust the operator dimensions of $\mathcal{L}_8$, $\mathcal{L}_{27}$, and $\mathcal{L}_{\text{mass}}$ in (5.1), with octet quark-gluon operators allowed to have differing dimensions at $\alpha_{IR}$.

In 1985, it was observed [13] that the isospin-$\frac{1}{2}$ term $\mathcal{L}_{\text{mass}}$ in Eq. (5.2), when combined with the strong mass term, would be removed by vacuum realignment and therefore could not help solve the $\Delta l = 1/2$ puzzle. In $\chi$PT$_\sigma$, the outcome is different [5, 6] due to the $\sigma$ dependence of the $\mathcal{L}_{\text{mass}}$ term in Eq. (5.1). Provided there is a mismatch between the weak mass operator’s dimension $(3 - \gamma_m)$ and the dimension $(3 - \gamma_m)$ of $\mathcal{L}_{\text{mass}}$, the $\sigma$ dependence of $\mathcal{L}_{\text{mass}}$ cannot be eliminated by a chiral rotation. As a result, there is a residual interaction $\mathcal{L}_{K_3\sigma} = g_{K_3\sigma} K_3 \sigma$ which mixes $K_3$ and $\sigma$ in lowest $O(p^2)$ order:

$$g_{K_3\sigma} = (\gamma_m - \gamma_m) \Re \{ (2m_K^2 - m_\pi^2)\bar{g}_M - m_\pi^2 g_M \} F_\pi/F_\sigma$$

We have corrected a factor of 2 in the formula for the $K_3\sigma$ coupling in our original papers [5, 6].
and produces the $\Delta I = 1/2$ $\sigma$-pole amplitude of Fig. 1 (B).

The $\chi$PT$_3$ analysis of 1985 [13] included a suggestion that kaon decays be tested by applying lattice QCD to the weak process $K \rightarrow \pi$, with both $K$ and $\pi$ on shell. It was made at a time when low-lying scalar resonances ($\epsilon(700)$ before 1974, $f_0(500)$ since 1996) were thought not to exist.

This proposal now needs to be taken seriously because:

- Lattice calculations are much easier with only two particles on shell instead of the three in $K \rightarrow \pi \pi$ (all on shell) being analysed by the RBC/UKQCD collaborations [20, 21].
- The 1985 analysis is easily extended to $\chi$PT$_\sigma$ by including $\sigma/f_0$ pole amplitudes in chiral Ward identities connecting on-shell $K \rightarrow \pi \pi$ to $K \rightarrow \pi$ on shell. The no-tadpoles theorem
  \begin{equation}
  \langle K|,H_{\text{weak}}|\text{vac}\rangle = O(m_k^2 - m_D^2), \ K \text{ on shell},
  \end{equation}
  remains valid.
- The lattice result for $K \rightarrow \pi \pi$ on-shell will not distinguish $\Delta I = 1/2$ contributions from the $g_8$ contact diagram and the $\sigma/f_0$ pole diagram in Fig. 1 (B). A lattice calculation of $K \rightarrow \pi$ on shell would measure $g_8$ (and $g_{27}$) directly, with no interference from $\sigma/f_0$ poles. Then we would finally learn whether $g_8$ is unnaturally large or not.

A key feature of the proposal is that the operator in the on-shell amplitude $\langle \pi|[F_3,H_{\text{weak}}]|K\rangle$ necessarily carries non-zero momentum $q^\mu = O(m_K)$. For either $\chi$PT$_\sigma$ or $\chi$PT$_3$, the $K \rightarrow \pi$ amplitude can be evaluated in the range
  \begin{equation}
  -m_K^2 \lesssim q^2 \lesssim (m_K - m_\pi)^2.
  \end{equation}

We highlight the point $q^\mu \neq 0$ because since 1985, there has been a widespread misconception in the literature\(^3\) that the analysis [13] involved setting $q^\mu = 0$ as in [19], with the pion in $K \rightarrow \pi$ sent off shell via an interpolating operator. There was and is no reason for this. For example, when writing a soft meson theorem for $\Sigma \rightarrow p\pi$, it is not necessary to force one of the baryons off shell.

### 6. Issues

When considering the validity of $\chi$PT$_\sigma$, it is important to avoid any presumption that dimensional transmutation necessarily implies that $\theta_{\mu}^\mu$ is large and $\neq 0$. Implicit in this intuition is a prejudice that scale invariance cannot be strongly broken via the vacuum when $\theta_{\mu}^\mu \rightarrow 0$. If the dilaton is a true NG boson, i.e. $m_\sigma \rightarrow 0$ with $F_\sigma \neq 0$ for $\theta_{\mu}^\mu \rightarrow 0$, it can couple to mass insertion terms in Callan-Symanzik equations and cause them to be non-zero in the zero-mass limit. Then Green’s functions do not exhibit the power-law scaling expected for manifestly scale-invariant field theories.

This point is illustrated for the quark condensate in Fig. 1 (A). In scenario 1 (top diagram), the running of $\tilde{\alpha}_Q$ is driven by the presence of quantities like $\langle \bar{q}q \rangle_{\text{vac}}$ (a mechanism often cited in papers on walking gauge theories [22]). In scenario 2 (bottom diagram), the running coupling freezes at $\alpha_{\text{IR}}$, where the condensate is a scale-breaking property of the vacuum.

\(^3\)We thank the final referee of our long paper [6] for drawing our attention to this.
Lattice investigations of IR fixed points inside the conformal window $8 \lesssim N_f \leqslant 16$ all depend on naive scaling of Green’s functions [22], so they correspond to \textit{scale-invariant vacua}. A recent lattice study [23] of the running of $\alpha_s$ for two flavors with \textit{no} naive scaling suggests that it freezes: the fixed point realises scale invariance in NG mode, i.e. with a scale-breaking vacuum. That is what $\chi_{PT}\sigma$ assumes for three flavors.

The term “dilaton” often refers to a spin-0 particle or resonance which couples to $\theta_{\mu \nu}$ and acquires its mass “spontaneously” due to self interactions. Originally, this idea concerned a scalar component of gravity [24], but now it is a key ingredient of dynamical electroweak symmetry breaking (pp. 198 and 1622-3, PDG tables [9]). This approximates theories with \textit{scale-invariant vacua}, as is evident in walking technicolor. Therefore it has \textit{nothing} to do with our dilaton [25].

It is well known that a resonance cannot be represented by a local interpolating operator, so is the fact that $\sigma/f_0(500)$ has a finite width a problem for $\chi_{PT}\sigma$? The answer is “no” because $\chi_{PT}\sigma$ is an expansion in powers and logarithms of $m_{\pi,K,\eta,\sigma}$ with coefficients determined in the \textit{exact} chiral-scale limit (1.2) where $\sigma$ has zero width [6]. In any perturbation theory, decay rates are calculated that way.

A related remark concerns what is current best practice for scenario 1. The resonance $f_0(500)$ is treated as a member of the non-NG sector with an accidentally small mass. It causes $\chi_{PT}3$ to produce divergent expansions for amplitudes involving $f_0(500)$ poles: the radius of convergence is too small. Instead, these amplitudes are approximated dispersively via contributions from the dominant $f_0(500)$ poles with corrections from nearby thresholds, subject to exact chiral $SU(3) \times SU(3)$ constraints such as Adler zeros. One would certainly not use local fields in this framework.

However $\chi_{PT}\sigma$ is a more ambitious theory. Having promoted $\sigma/f_0$ to the NG sector, we expect convergent asymptotic expansions for \textit{all} mesonic amplitudes (scenario 2). The NLO corrections are still being worked out, but a first guess is to set all multi-dilaton vertices to zero. That is equivalent to adding the simplest dilaton diagrams to all $\chi_{PT}3$ diagrams. It seems to produce amplitudes very similar to those of the dispersive approximations of scenario 1.

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