Entropic dynamics: reconstructing quantum field theory in curved space-time

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Abstract
The Entropic Dynamics reconstruction of quantum mechanics is extended to the quantum theory of scalar fields in curved space-time. The Entropic Dynamics framework, which derives quantum theory as an application of the method of maximum entropy, is combined with the covariant methods of Dirac, Hojman, Kuchař, and Teitelboim, which they used to develop a framework for classical covariant Hamiltonian theories. The goal is to formulate an information-based alternative to current approaches based on algebraic quantum field theory. One key ingredient is the adoption of a local notion of entropic time in which instants are defined on curved three-dimensional surfaces and time evolution consists of the accumulation of changes induced by local deformations of these surfaces. The resulting dynamics is a non-dissipative diffusion that is constrained by the requirements of foliation invariance and incorporates the necessary local quantum potentials. As applications of the formalism we derive the Ehrenfest for fields in curved-spacetime and briefly discuss the nature of divergences in quantum field theory.

Keywords: quantum field theory, quantum fields in curved space-time, Hamiltonian dynamics, quantum foundations, information theory

1. Introduction

Research on quantum field theory in curved space-time (QFTCS) has been the subject of a sustained effort for several decades. (See, for example, [1, 2] and the recent review [3].) The motivation is twofold. There is an intrinsic interest in QFTCS as an effective theory that is reliable except in those regions of space-time where the curvature is extremely high. But there is also a widespread belief that QFTCS is a necessary intermediate step on the way towards a more fundamental theory of interacting quantum and gravitational fields.

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As discussed in [3] QFTCS is usually formulated by merging classical general relativity with quantum field theory. Identifying the principles that govern the gravity sector does not appear to be problematic. Unfortunately, the choice of which principles of quantum field theory in flat space-time should be retained in a curved space-time is not nearly as clear. The lack of Poincare invariance implies the absence of a preferred vacuum state and forces one to revisit the notion of particle [4, 5]. The representation of quantum fields as operators, or better, as operator-valued distributions, is problematic on many levels. Prominent among them is the issue of the Hilbert space itself. A representation of the relevant commutation relations and field equations requires operators that act on a Hilbert space. The problem is that in curved space-time there is an infinite number of such representations that are unitarily inequivalent. Another problem is that representing fields as operators may itself be deeply misguided—it may turn out to be an attempt to preserve in curved space something that has already turned out to be questionable at the level of flat space-time, namely, the standard quantum theory of measurement.

But there is another direction from which this subject can be approached. The discoveries of black hole entropy and thermodynamics [6–8] suggest a deep connection between the fundamental laws of physics and information. Indeed, in recent decades the subject of foundations of quantum mechanics (FQM) has undergone a renaissance (for reviews see e.g. [9–11]) culminating in a variety of proposals for the reconstruction of quantum mechanics (see e.g. [12–18]). Notable among the latter are proposals that seek a deeper foundation based on information theory (see e.g. [19–29]). To a large extent the two subjects—QFTCS and FQM—have developed independently. In fact, it is somewhat surprising that progress in QFTCS has contributed so little towards clarifying the conceptual foundations of quantum mechanics and vice versa.

The purpose of this paper is to contribute to bridge this gap by extending one particular approach to the reconstruction of quantum mechanics—the entropic dynamics approach—to the reconstruction of quantum field theory.

Entropic dynamics (ED) is a framework for constructing dynamical theories on the basis of Bayesian and entropic principles of inference [30, 31]. (For a recent self-contained presentation and review see [32].) The goal is to develop models for the time evolution of the probability distributions of the positions of particles or the values of fields. ED differs from other information approaches to quantum theory by strictly adhering to Bayesian and entropic methods of inference, and by paying close attention to the notion of time.

One challenge that ED was designed to address is that a fully epistemic interpretation of the wave function \(|\Psi|^2\) is not achieved by merely declaring that the square of a wave function \(|\Psi|^2\) yields a Bayesian probability. To insure consistency one must also show that the rules for updating those probabilities—which include both the unitary evolution of the wave function \(\Psi\) and its ‘collapse’ during a measurement—are in strict accord with Bayesian and entropic methods [33–35]. Thus, in an entropic dynamics the evolution must be driven by information codified into constraints; it is through these constraints that the ‘physics’ is introduced.

An important early insight in the context of Nelson’s stochastic mechanics, was his realization that important aspects of quantum mechanics could be modelled as a non-dissipative Brownian motion [36]. This idea was suitably adapted to the ED setting and imposing the conservation of an appropriate energy functional became the main criterion for choosing the evolving constraints [30]. It eventually led to a fully Hamiltonian formalism with its attendant

\(^2\)Within a model such as ED a concept is referred as ‘epistemic’ when it is related to the state of knowledge, opinion, or belief of a rational agent. For example, Bayesian probabilities are epistemic—they are tools for reasoning with incomplete information [35].
action principle, a symplectic structure, and Poisson brackets [37] [32]. Unfortunately in curved space-times the energy criterion is not satisfactory because the notion of a global energy is not in general available. An alternative local criterion for evolving constraints is needed.

Another challenge that ED is designed to address concerns the concept of time. Entropic time is constructed as a scheme to keep track of the accumulation of small changes [30]. This involves identifying a suitable notion of ordered ‘instants’ and then introducing a convenient measure of the interval or separation between them. In ED an instant is defined through the information that is necessary to predict or ‘construct’ the next instant. This amounts to specifying a state of knowledge together with a purely kinematic criterion of simultaneity. In [38, 39] the concept of a global instant was used to generate an ED of quantum scalar fields in Minkowski space-time. However, although the model was fully relativistic, its relativistic invariance was not manifest, that is, the freedom to represent the relativity of simultaneity was not explicit.

The goal of this paper is to formulate an inference-based approach to QFTCS in which quantum fields are not represented as operators; there is no mention of a preferred vacuum state, nor of an associated Hilbert space3. We construct a manifestly relativistic quantum ED in curved space-time incorporating ideas developed in the classical field theories of Dirac, Hojman, Kuchař, and Teitelboim (DHKT) [41–46] which can themselves be traced to the earlier theories of Weiss, Tomonaga, Dirac, and Schwinger [47–50]. Drawing on the ideas of DHKT, we relax the assumption of a global time in favor of a notion of local time, and the non-covariant criterion of energy conservation is replaced by the covariant requirement of foliation invariance. In this view of ED, an instant is defined by a three-dimensional space-like surface embedded in space-time plus the fields and probability distributions defined on such surfaces. In a fully covariant theory, such surfaces are constructed by slicing or foliating space-time into a sequence of spacelike surfaces. The freedom to choose the foliation, which amounts to the local relativity of simultaneity, is implemented by a consistency requirement: the evolution of all dynamical quantities from an initial to a final surface must be independent of the choice of intermediate surfaces. We can refer to foliation invariance either as a requirement of consistency or, following Kuchař and Teitelboim, as a requirement of ‘path independence’ —if there are two alternative ways to evolve from an initial to a final instant, then the two ways must lead to the same result4.

The formalism is developed for a scalar field $\chi(x)$. First we find the transition probability for an infinitesimal change—the effect of an infinitesimal local deformation of the instant. The introduction of a local entropic time then leads to a set of functional ‘local-time Fokker–Planck’ diffusion equations for the evolution of probability distribution $\rho[\chi]$. Requiring that the evolution satisfy foliation invariance leads to a non-dissipative Hamiltonian diffusion. This is done by imposing that the generators that update the entropic constraints satisfy the same DHKT ‘algebra’ as the generators that describe the kinematics of surface deformations. The result is a covariant quantum entropic dynamics of scalar fields in curved space-time.

This paper focuses on quantun field theory but the ED approach has been applied to a variety of other topics in quantum theory. These include: the quantum measurement problem [51, 52]; momentum and uncertainty relations [53]; extensions to curved spaces [54]; the Bohmian limit [55, 56]; the classical limit [57]; and the ED of spin [58].

3 A brief presentation of some of these ideas appeared in [40].
4 Wheeler originally called this an ‘embeddability’ condition—the requirement that all these surfaces be embedded in the same space-time.
2. The Entropic dynamics of infinitesimal steps

2.1. The microstates

In an inference scheme such as ED the physics is introduced through the choice of dynamical variables and of the constraints that capture the dynamically relevant information. Here we consider the dynamics of a scalar field \( \chi(x) \). For notational convenience we will often write the \( x \)-dependence as a subscript, \( \chi(x) = \chi_x \). Unlike the standard Copenhagen interpretation of quantum theory where observables have definite values only when elicited through an experiment, in the ED approach these fields have definite values at all times. However, these values are unknown and the dynamics is indeterministic.

The field \( \chi \) lives on a 3-dimensional curved space \( \sigma \), the points of which are labeled by coordinates \( x_i \) \((i = 1, 2, 3)\). The space \( \sigma \) is endowed with a metric \( g_{ij} \) induced by the non-dynamical background space-time in which \( \sigma \) is embedded. Thus \( \sigma \) is an embedded hypersurface; for simplicity we shall refer to it as a ‘surface’. We assume that space-time is globally hyperbolic. It can be foliated by Cauchy surfaces and its topology is \( \mathbb{R} \times \Sigma \) where \( \Sigma \) stands for any Cauchy surface \( \sigma \) [59]. The field \( \chi_x \) is a scalar with respect to 3-diffeomorphisms on the surface \( \sigma \). The \( \infty \)-dimensional space of all possible field configurations is the configuration space \( C \). A single field configuration, labelled \( \chi \), is represented as a point in \( C \), and the uncertainty in the field is described by a probability distribution \( \rho[\chi] \) over \( C \).

2.2. Maximum entropy

Our goal here is to predict the evolution of the scalar field \( \chi \). To this end we make one major assumption: in ED, motion is continuous, the fields follow continuous trajectories in \( C \). This implies that any finite change can be analyzed as the accumulation of many infinitesimally short steps. Therefore our first goal is to calculate the probability \( P[\chi'|\chi] \) that the field undergoes a small change from an initial configuration \( \chi \) to a neighboring \( \chi' = \chi + \Delta \chi \) and later we calculate the probability of the finite change that results from a sequence of short steps. The transition probability \( P[\chi'|\chi] \) is found by maximizing the entropy functional,

\[
S[P, Q] = - \int D\chi' P[\chi'|\chi] \log \frac{P[\chi'|\chi]}{Q[\chi'|\chi]},
\]

relative to a prior \( Q[\chi'|\chi] \) and subject to appropriate constraints. It is through the prior and the constraints that the relevant physical information is introduced.

2.3. The prior

We adopt a prior \( Q[\chi'|\chi] \) that incorporates the information that change happens by infinitesimally small amounts, but is otherwise maximally uninformative. In particular, as far as the prior is concerned, changes at different points are uncorrelated. Such a prior can itself be derived from the principle of maximum entropy. Indeed, maximize

\[
S[Q, \mu] = - \int d\chi' Q[\chi'|\chi] \log \frac{Q[\chi'|\chi]}{\mu(\chi')},
\]
relative to the measure $\mu(\chi')$ which we assume to be uniform and subject to appropriate constraints. The requirement that the field undergoes changes that are small and uncorrelated is implemented by imposing an infinite number of independent constraints, one at each point $x$,

$$
\langle \Delta \chi_x^2 \rangle = \int D\chi' \mathcal{Q}[\chi'|\chi] (\Delta \chi_x)^2 = \kappa_x,
$$

(3)

where $\Delta \chi_x = \chi'_x - \chi_x$ and, to enforce the continuity of the motion, we will eventually take the limit $\kappa_x \to 0$. The result of maximizing (2) subject to (3) and normalization is a product of Gaussians,

$$
\mathcal{Q}[\chi'|\chi] \propto \exp -\frac{1}{2} \int dx g_{1/2}^{1/2} \alpha_x (\Delta \chi_x)^2
$$

(4)

where $\alpha_x$ are the Lagrange multipliers associated to each constraint (3), and the scalar density $g_{1/2}^{1/2} = (\det g_0)^{1/2}$ is introduced so that $\alpha_x$ is a scalar field. The limit $\kappa_x \to 0$ is achieved by taking $\alpha_x \to \infty$. For notational simplicity we write $dx$ instead of $d^3x$.

2.4. The drift potential constraint

The dynamics induced by the prior (4) is a diffusion that is isotropic in configuration space. In order to introduce correlations, directionality, and such quintessential quantum effects as interference and entanglement, we impose one additional single constraint that is non-local in space but local in configuration space. This single constraint involves a functional on configuration space, the ‘drift’ potential $\phi[\chi]$. We impose that the expectation of the change of the drift potential $\Delta \phi = \phi'[\chi'] - \phi[\chi]$ is another small quantity $\kappa'$ that will eventually be taken to zero,

$$
\langle \Delta \phi \rangle = \kappa' \text{ or } \int D\chi' P[\chi'|\chi] \int dx \frac{\delta\phi[\chi]}{\delta \chi_x} = \kappa'.
$$

(5)

(Note that since $\chi_x$ and $\Delta \chi_x$ are scalars, in order for (5) to be invariant under coordinate transformations of the surface the derivative $\delta/\delta \chi_x$ must transform as a scalar density.) The physical meaning of the drift potential $\phi[\chi]$ will not be discussed here. As in so many other situations in physics the mere identification of forces and constraints can turn out to be useful even when their microscopic origins is not yet fully understood.

2.5. The transition probability

Next we maximize (1) subject to (5) and normalization. As discussed in [56] the multiplier $\alpha'$ associated to the global constraint (5) turns out to have no influence on the dynamics: it can be absorbed into the drift potential $\alpha' \phi \to \phi$ which means we can effectively set $\alpha' = 1$. The resulting transition probability is a Gaussian distribution,

$$
P[\chi'|\chi] = \frac{1}{Z[\alpha_x, g_x]} \exp -\frac{1}{2} \int dx g_{1/2}^{1/2} \alpha_x \left( \Delta \chi_x - \frac{1}{g_{1/2}^{1/2} \alpha_x} \frac{\delta\phi[\chi]}{\delta \chi_x} \right)^2.
$$

(6)

5 Since we deal with infinitesimally short steps the prior $Q$ turns to be quite independent of the choice of the underlying measure $\mu$.

6 Elsewhere we show that in order to describe single particle states with nonzero angular momentum $\phi$ needs to have the topological properties of an angle [60]. Additional evidence that $\phi$ must be interpreted as an angle is provided in the ED of non-relativistic particles with spin [58].
where $Z[\alpha_x, g]$ is the normalization constant. In previous work [38, 39] $\alpha_x$ was chosen to be a spatial constant $\alpha$ to reflect the translational symmetry of flat space. Here we make no such restriction and instead relax the global constant $\alpha$ in favor of a non-uniform spatial scalar $\alpha_x$ which will be a key element in implementing our scheme for a local entropic time.

The Gaussian form of (6) allows us to present a generic change,

$$\Delta \chi_x = \langle \Delta \chi_x \rangle + \Delta w_x,$$

as resulting from an expected drift $\langle \Delta \chi_x \rangle$ plus fluctuations $\Delta w_x$. At each $x$ the expected short step is

$$\langle \Delta \chi_x \rangle = \frac{1}{g^{1/2} x} \frac{\delta \phi [\chi]}{\delta \chi_x} \equiv \Delta \tilde{\chi}_x,$$

while the fluctuations $\Delta w_x$ satisfy,

$$\langle \Delta w_x \rangle = 0, \quad \text{and} \quad \langle \Delta w_x \Delta w_{x'} \rangle = \frac{1}{g^{1/2} x} \delta_{xx'}.$$

Thus we see that $\Delta \tilde{\chi}_x \sim 1/\alpha_x$ and $\Delta w_x \sim 1/\alpha_x^{1/2}$, so that for short steps, $\alpha_x \to \infty$, the fluctuations dominate the motion. The resulting trajectory is continuous but non-differentiable—a Brownian motion.

3. Entropic time

In ED the idea of time is derived from the idea of change. Time is introduced as a device to keep track of how the accumulation of many infinitesimal changes builds up into a finite change. Questions such as, ‘What is an instant?’ ‘How are they ordered?’ and ‘To what extent are they separated?’ are central to constructing any dynamical theory and ED is no exception.

3.1. Ordered instants

Of particular importance is the notion of an instant, which in ED involves several ingredients: (1) A foliation of spacelike surfaces $\sigma$ that codify spatial relations and provide a criterion of simultaneity and duration. (2) We must specify the ‘epistemic contents’ of the surfaces. This is a specification of a statistical state that is sufficient for the prediction of future states. It is given by a probability distribution $\rho[\chi]$ and a drift potential $\phi[\chi]$. And (3) an entropic step in which the statistical state at one instant is updated to generate the state at the next instant. This is the requirement that generates the sequence of ordered instants which makes the dynamics come alive.

3.2. Some space-time kinematics

We deal with a curved space-time; events are labeled by space-time coordinates $X^\mu$; and the metric is $g_{\mu\nu} \{X^\beta\}$. Space-time is foliated by a sequence of space-like surfaces $\{\sigma\}$. Points on the surface $\sigma$ are labeled by coordinates $x^i$ and the embedding of the surface within space-time is defined by four functions $X^\mu = X^\mu (x^i)$. The metric induced on the surface is

$$g_{\mu\nu} \{X^\beta\}.$$

We use Greek indices ($\mu, \nu, \ldots = 0, 1, 2, 3$) for space-time coordinates $X^\mu$ and latin indices ($a, b, \ldots, i, j, \ldots = 1, 2, 3$) for coordinates $x^i$ on the surface $\sigma$. The spacetime metric has signature $(- + + +)$.
\( g_{ij}(x) = X_i^\mu X_j^\nu g_{\mu\nu} \) where \( X_i^\mu = \frac{\partial X^\mu}{\partial x^i} \).

(10)

The metric \( g_{ij} \) will in general depend on the particular surface. In this work neither \( g_{\mu\nu}(X) \) nor \( g_{ij}(x) \) are themselves dynamical.

Following Teitelboim and Kuchař, we consider an infinitesimal deformation of the surface \( \sigma \) to a neighboring surface \( \sigma' \). This is specified by the deformation vector \( \delta \xi^\mu \) which connects the point in \( \sigma \) with coordinates \( x^i \) to the point in \( \sigma' \) with the same coordinates \( x^i \),

\[ \delta \xi^\mu = \delta \xi^\perp n^\mu + \delta \xi^i X_i^\mu, \]

(11)

where \( n^\mu \) is the unit normal to the surface (\( n_\mu n^\mu = -1 \), \( n_\mu X_i^\mu = 0 \)). The normal and tangential components are given by

\[ \delta \xi^\perp = -n_\mu \delta \xi^\mu \quad \text{and} \quad \delta \xi^i = X_i^\mu \delta \xi^\mu, \]

(12)

where \( X_i^\mu = g_\mu^\nu g_{ij} X_j^\nu \). They are known as lapse and shift respectively and are collectively denoted \( (\delta \xi^\perp, \delta \xi^i) \). As a matter of convention a deformation is identified by its normal \( \delta \xi^\perp \) and tangential \( \delta \xi^i \) components independently of the surface upon which it acts (i.e. independently of the normal \( n^\mu \)). This allows us to speak about applying the same deformation to different surfaces; a useful concept for our discussion of path independence.

3.3. Duration

In ED time is defined so that motion looks simple. The specification of the time interval between two successive instants is dictated purely by convenience, that is, the best choice is that which reflects the various symmetries of the physical situation. For example, in a non-relativistic QM one would adopt a Newtonian time \( \Delta t \) that reflects the translational symmetries of space so that "time flows equably everywhere and everywhen". In our case the most convenient choice is one that reflects the symmetries of the background curved space-time.

Since for short steps the dynamics is dominated by fluctuations, equation (9), the choice of time interval is achieved through an appropriate choice of the multipliers \( \alpha_x \). So far the present development of ED has followed closely along the lines of the non-covariant models discussed in [37] and [39]). The important point of departure is that here we are concerned with instants defined on the curved embedded surfaces \( \sigma \) and \( \sigma' \). It is then natural to define a local notion of duration in terms of an invariant—the proper time. The idea is the familiar one: at the point \( x \) in \( \sigma \) draw a normal segment reaching out to \( \sigma' \). The proper time \( \delta \xi^\perp \) along this normal segment provides us with the local measure of duration between \( \sigma \) and \( \sigma' \) at \( x \). More specifically, let

\[ \alpha_x = \frac{1}{\delta \xi^\perp} \quad \text{so that} \quad \langle \Delta w_x, \Delta w_x' \rangle = \frac{\delta \xi^\perp}{g^1/2} \delta x'. \]

(13)

3.4. The statistical state and its evolution

Entropic dynamics is generated by the short-step transition probability \( P[\chi' | \chi] \). In a generic short step both the initial \( \chi \) and the final \( \chi' \) are unknown. Integrating the joint probability, \( P[\chi', \chi] \), over \( \chi \) gives

\[ P[\chi'] = \int d\chi P[\chi', \chi] = \int d\chi P[\chi' | \chi] P[\chi]. \]

(14)
These equations are true by virtue of the laws of probability; they involve no assumptions. However, if \( P [\chi] \) happens to be the probability of \( \chi \) at an 'instant' labelled \( \sigma \), then we can interpret \( P [\chi'] \) as the probability of \( \chi' \) at the 'next instant,' which we will label \( \sigma' \). Accordingly, we write \( P [\chi] = \rho_\sigma [\chi] \) and \( P [\chi'] = \rho_{\sigma'} [\chi] \) so that

\[
\rho_{\sigma'} [\chi'] = \int D\chi P [\chi'] | \rho_\sigma [\chi]. \tag{15}
\]

This is the basic dynamical equation; it allows one to update the statistical state \( \rho_\sigma [\chi] \) from one instant to the next. Note that since \( P [\chi'] | \rho_\sigma [\chi] \) is found by maximizing entropy not only are these instants ordered but there is a natural entropic arrow of time: \( \sigma' \) occurs after \( \sigma \). But we are not done yet. With the definition (13) of duration, the dynamics given by (15) and (6) describes a Wiener process evolving along a given foliation of space-time. To obtain a fully covariant dynamics we require that the evolution of any dynamical quantity such as \( \rho_\sigma [\chi] \) from an initial \( \sigma \) to a final \( \sigma_f \) must be independent of the intermediate choice of surfaces. This ‘foliation invariance’ or ‘path independence’, which amounts to the local relativity of simultaneity, is a consistency requirement: if there are different ways to evolve from a given initial instant into a given final instant, then all these ways must agree. The conditions to implement this consistency are the subject of the next section.

3.5. The local-time diffusion equations

The dynamics expressed in integral form by (15) with (6) and (13) can be rewritten in differential form as an infinite set of local equations, one for each spatial point,

\[
\frac{\delta \rho_\sigma}{\delta \xi_\perp} = -g_{\perp}^{-1/2} \frac{\delta}{\delta \chi} \left( \frac{\delta \Phi_\sigma}{\delta \chi} \right) \quad \text{with} \quad \Phi_\sigma [\chi] = \phi_\sigma [\chi] - \log \rho_\sigma^{1/2} [\chi]. \tag{16}
\]

(The derivation is given in appendix.) This set of equations describes the flow of the probability \( \rho_\sigma [\chi] \) in the configuration space \( \mathcal{C} \) as the surface \( \sigma \) is deformed. More explicitly, the actual change in \( \rho_\sigma [\chi] \) as \( \sigma \) is infinitesimally deformed to \( \sigma' \) is

\[
\delta_{\perp} \rho_\sigma [\chi] = \int dx \frac{\delta \rho_\sigma [\chi]}{\delta \xi_\perp} \delta \xi_\perp = - \int dx \frac{\delta \xi_\perp^2}{2g_{\perp}^{1/2}} \frac{\delta}{\delta \chi} \left( \rho_\sigma [\chi] \frac{\delta \Phi_\sigma [\chi]}{\delta \chi} \right). \tag{17}
\]

In the special case when both surfaces \( \sigma \) and \( \sigma' \) happen to be flat then \( g_{\perp}^{1/2} = 1 \) and \( \delta \xi_\perp^2 = dt \) are constants and equation (17) becomes

\[
\frac{\partial \rho_\sigma [\chi]}{\partial t} = - \int dx \frac{\delta}{\delta \chi} \left( \rho_\sigma [\chi] \frac{\delta \Phi_\sigma [\chi]}{\delta \chi} \right), \tag{18}
\]

which we recognize as a diffusion or Fokker–Planck equation written as a continuity equation for the flow of probability in configuration space \( \mathcal{C} \). Accordingly we will refer to (16) as the ‘local-time Fokker–Planck’ equations (LTFP). These equations describe the flow of probability with a current velocity \( v_\chi [\chi] = \delta \Phi_\sigma / \delta \chi \). Eventually, the functional \( \Phi \) will be identified as the Hamilton–Jacobi functional, or the phase of the wave functional in the quantum theory.

Anticipating later developments we note that the LTFP equation (16) can be rewritten in an alternative and very suggestive form involving the notion of an ensemble functional or e-functional. Just as a regular functional such as \( \rho [\chi] \) maps a field distribution \( \chi \) into a real number (a probability in this case), an e-functional maps a functional, such as \( \rho [\chi] \) or \( \Phi [\chi] \),...
into a real number. Then, just as one can define functional derivatives, one can also define e-functional derivatives. Introduce an e-functional $\tilde{H}_{\perp x}[\rho, \Phi, \sigma]$ such that at each point $x$

$$\frac{\delta \rho_e[\chi]}{\delta \xi_x^A} = \frac{\delta \tilde{H}_{\perp x}[\rho, \Phi, \sigma]}{\delta \Phi[\chi]}$$

(19)

reproduces (16). In what follows we denote all ensemble quantities such as $\tilde{H}_{\perp x}$ with a tilde: $\tilde{\delta}/\tilde{\delta} \Phi[\chi]$ is the e-functional derivative with respect to $\Phi[\chi]$. We stress that writing (16) in the form (19) does not involve any new assumptions; an appropriate $\tilde{H}_{\perp x}$ can always be found. Indeed, substitute (16) into the left of (19), then an easy integration gives

$$\tilde{H}_{\perp x}[\rho, \Phi, \sigma, \chi] = \int Dx \frac{1}{2\sqrt{g}} \left( \frac{\delta \Phi[\chi]}{\delta \chi_x} \right)^2 + F_x[\rho; \sigma, \chi],$$

(20)

where the integration constants $F_x = F_x[\rho; \sigma, \chi]$ are independent of $\Phi$; they may depend on $\rho$, on the geometry of the surface $\sigma$, and also on the fields $\chi_x$. In later sections we will see that $\tilde{H}_{\perp x}$ captures dynamical information about the evolution of $\Phi$ as well as $\rho$ and can be cast as a Hamiltonian generator.

4. Kinematics of surface deformations and their generators

Dynamics in local time must reflect the kinematics of surface deformations, and this kinematics can be studied independently of the particular dynamics being considered. As a surface is deformed, its geometry and, more generally, the statistical state associated with it is also subject to change. Consider a generic functional $T[X(x)]$ that assigns a number to every surface $X^\mu(x)$. The change in the functional $\delta T$ resulting from an arbitrary deformation $\delta \xi^A$ has the form

$$\delta T = \int dx \frac{\delta T}{\delta \xi_x^A} \delta \xi_x^A = \int dx \left( \delta \xi_x^A G_{\perp x} + \delta \xi_x^A G_{\parallel x} \right) T,$$

(21)

where

$$G_{\perp x} = \frac{\delta}{\delta \xi_x^A} = n_\mu^A \frac{\delta}{\delta X_\mu^x} \quad \text{and} \quad G_{\parallel x} = \frac{\delta}{\delta \xi_x^A} = X_\mu^x \frac{\delta}{\delta X_\mu^x}$$

(22)

are the generators of normal and tangential deformations respectively. Unlike the vectors, $\delta/\delta X_\mu^x$, which form a coordinate basis in the space of hypersurfaces and therefore commute, the generators of deformations $\delta/\delta \xi^A$ form a non-holonomic basis. Their non-vanishing commutator

$$\frac{\delta}{\delta \xi_x^A} \frac{\delta}{\delta \xi_{x'}^B} = \frac{\delta}{\delta \xi_x^B} \frac{\delta}{\delta \xi_{x'}^A} = \int dx' \kappa_{BA}(x'; x, x) \frac{\delta}{\delta \xi_{x'}^C}$$

(23)

where $\kappa_{BA}$ are the ‘structure constants’ of the ‘group’ of deformations.

The calculation of $\kappa_{BA}$ is given in [43] [45]. The basic idea is embeddability: When we perform two successive infinitesimal deformations $\delta \xi^A$ followed by $\delta \eta^B, \sigma \rightarrow \sigma_1 \rightarrow \sigma'$, the

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8 An excellent brief review of the ensemble calculus is given in the appendix of [61].

9 The notation $F_x[\rho; \sigma, \chi]$ indicates that $F_x$ is a functional of various dynamical and non-dynamical variables. To unclutter the notation some of these dependencies will not be explicitly displayed.

10 Equation (19) provides the criterion that allows us to identify the momentum $\Phi[\chi]$ that is canonically conjugated to the generalized coordinate $\rho[\chi]$. 
three surfaces are all embedded in the same space-time. The same happens when we execute them in the opposite order, $\sigma \xrightarrow{\delta \xi} \sigma_2 \xrightarrow{\delta \xi} \sigma''$. The key point is that the since the surfaces $\sigma'$ and $\sigma''$ are embedded in the same space-time there must exist a third deformation $\delta \zeta''$ that takes $\sigma'$ to $\sigma''$: $\sigma' \xrightarrow{\delta \zeta} \sigma''$. Thus the deformation from $\sigma$ to $\sigma''$ can be attained by following two different paths: either we follow the direct path $\sigma \xrightarrow{\delta \xi} \sigma_2 \xrightarrow{\delta \xi} \sigma''$ or we follow the indirect path $\sigma \xrightarrow{\delta \zeta} \sigma_1 \xrightarrow{\delta \xi} \sigma''$. Then, as shown in [43] [45], equation (23) leads to the ‘algebra’,

$$[G_{ix}, G_{ix'}] = -(g_{ix}^i G_{ix'} + g_{ix'}^i G_{ix'}) \partial_0 \delta(x,x'), \quad \tag{24}$$

$$[G_{ix}, G_{ix'}] = -G_{ix} \partial_0 \delta(x,x') \quad \tag{25}$$

$$[G_{ix}, G_{ix'}] = -G_{ix} \partial_x \delta(x,x') - G_{ix'} \partial_x \delta(x,x'). \quad \tag{26}$$

The previous quotes in ‘group’ and ‘algebra’ are a reminder that strictly, the set of deformations do not form a group. The composition of two successive deformations is itself a deformation, of course, but it also depends on the surface to which the first deformation is applied. Thus, the ‘structure constants’ on the right hand sides of (24)–(26) are not constant, they depend on the surface $\sigma$ through its metric $g_{ij}$ which appears explicitly on the right hand side of (24).

5. Consistent entropic dynamics: path independence

To obtain a fully covariant dynamics we require that the evolution of any dynamical quantity such as $\rho_\sigma[\chi]$ from an initial $\sigma_i$ to a final $\sigma_f$ be consistent with the kinematics of surface deformations. Thus, the requirement of embeddability translates into a consistency requirement of path independence: if there are different paths to evolve from an initial instant into a final instant, then all these paths must lead to the same final values for all quantities.

In ED the relevant physical information—supplied through the prior (4) and the constraint (5)—have led us to a diffusive dynamics in which the probability $\rho_\sigma[\chi]$ evolves under the action of the externally prescribed drift potential $\phi[\chi]$. It is a curious diffusion in a curved background space-time, but it is a diffusion nonetheless. This, however, is not what we seek: it is not a quantum dynamics.

A quantum dynamics requires a different choice of constraints. Specifically, in the ED developed in the previous sections there is one basic dynamical variable, the distribution $\rho_\sigma[\chi]$. The drift potential $\phi[\chi]$, being externally prescribed, is not a dynamical variable. In contrast, in a quantum dynamics there are two dynamical variables, the magnitude and the phase of the wave function. An additional degree of freedom must be introduced into ED. Perhaps the simplest way is to replace the fixed prescribed potential $\phi[\chi]$ in constraint (5) by an evolving drift potential $\phi_\sigma[\chi]$ that is updated at every step in local time in response to the evolving $\rho_\sigma[\chi]$. In such an ED every infinitesimal step in local time involves two updates: one is the entropic update of $\rho_\sigma[\chi]$, the other is updating the constraint represented by $\phi_\sigma[\chi]$. The resulting ED describes the coupled evolution of $\rho_\sigma[\chi]$ and $\phi_\sigma[\chi]$.

The obvious question is how should the potential $\phi_\sigma[\chi]$ be updated? The LTFP equations, particularly when written in the form (19), suggest that the rules for updating $\phi_\sigma$ are more conveniently expressed in terms of the transformed variable $\Phi_\sigma$. In previous work on ED [30, 37, 39] the basic criterion involved imposing the conservation of a global energy functional. This non-dissipative diffusion led to a fully Hamiltonian formalism with $\rho_\sigma[\chi]$ and $\Phi_\sigma[\chi]$ as conjugate variables: one Hamilton equation describes the entropic evolution of $\rho_\sigma[\chi]$, while the conjugate Hamilton equation describes the evolving constraints through $\Phi_\sigma[\chi]$. 

10
In a covariant ED involving local surface deformations this is not satisfactory because the notion of a global energy is not available. Here we propose instead that the update of $\Phi_\sigma$ must reflect the kinematics of surface deformations. We require path independence; the update must be independent of the selected foliation. Next we tackle the problem of implementing this proposal.

We saw that in an inference-based framework such as ED the concept of time is designed so that each instant—which includes a specification of both the surface $\sigma$ and the statistical state $\rho_\sigma[\chi]$ and $\Phi_\sigma[\chi]$—contains the relevant information to construct the next instant. This means that by design in ED time is constructed so that ‘given the present, the future is independent of the past.’ Thus in ED the equations of motion will necessarily be first order in time. We have also seen that in the limit of flat space limit—whether relativistic [39], or not [32, 37]—the non-dissipative ED is Hamiltonian. Therefore it is natural to adopt a Hamiltonian formalism in which $\Phi_\sigma$ is the momentum canonically conjugate to $\rho_\sigma$.

The assumption of an underlying symplectic structure is a strong one that demands justification. In the context of non-relativistic quantum mechanics the symplectic and complex structures characteristic of quantum mechanics can be motivated using arguments from information geometry [32].

Once a Hamiltonian framework is adopted, we can follow Dirac and treat the surface variables as if they were dynamical variables too. This allows a Hamiltonian formalism that treats dynamical and kinematical variables in a unified way. To do this one formally introduces auxiliary variables $\pi_\mu(x) = \pi_\mu^\sigma$ that will play the role of momenta conjugate to $X_\mu^\sigma$. These $\pi'$s are defined through the Poisson bracket (PB) relations,

$$[X_\mu^\sigma, X_\nu^\sigma] = 0, \quad [\pi_\mu^\sigma, \pi_\nu^\sigma] = 0, \quad [X_\mu^\sigma, \pi_\nu^\sigma] = \delta^\mu_\nu \delta(x,x').$$

The canonical pairs $(X_\mu^\sigma, \pi_\mu^\sigma)$ represent the geometry of the surfaces and how they change along the foliation.

The change of a generic functional $T[X, \pi, \rho, \Phi]$ resulting from an arbitrary deformation $\delta \xi^\alpha = (\delta \xi^\mu_+, \delta \xi^\mu_-)$ is expressed in terms of PBs,

$$\delta T = \int dx \delta \xi^\mu_+ [T, H_{\mu\nu}] = \int dx \left( \delta \xi^\mu_+ [T, H_\parallel] + \delta \xi^\mu_- [T, H_\perp] \right),$$

(28)

where $H_\parallel[X, \pi, \rho, \Phi]$ and $H_\perp[X, \pi, \rho, \Phi]$ are the generators of normal and tangential deformations respectively, and the generic PB of two arbitrary functionals $U$ and $V$ is

$$[U, V] = \int dx \left( \frac{\delta U}{\delta \pi_\mu^\sigma} \frac{\delta V}{\delta \pi_\mu^\sigma} - \frac{\delta U}{\delta \pi_\mu^\sigma} \frac{\delta V}{\delta \pi_\mu^\sigma} \right) + \int D\chi \left( \frac{\delta U}{\delta \rho} \frac{\delta V}{\delta \Phi} - \frac{\delta U}{\delta \rho} \frac{\delta V}{\delta \Phi} \right).$$

(29)

Thus, the PBs perform a double duty: on one hand they reflect the kinematics of deformations of surfaces embedded in a background space-time, and on the other hand they express the genuine entropic dynamics of $\rho$ and $\Phi$.

To comply with the requirement of path independence we follow Teitelboim and Kuchař [43–45] to seek generators $H_{\perp \chi}$ and $H_{\parallel \chi}$ that provide a canonical representation of the DHKT ‘algebra’ of surface deformations. Unlike DHKT who developed a classical formalism based on choosing the field $\chi(x)$ and its momentum as canonical variables, here we develop a quantum formalism. We choose the functionals $\rho[\chi]$ and $\Phi[\chi]$ as the pair of canonical variables.

The idea then is that in order for the dynamics to be consistent with the kinematics of surface deformations the PBs of $H_{\perp \chi}$ and $H_{\parallel \chi}$ must close in the same way as the ‘group’
of deformations (24)–(26)—that is, they must provide a ‘representation’ involving the same ‘structure constants’

\[ [H_{\perp x}, H_{\perp x'}] = (g^I_{\mu} H_{\mu} + g^I_{\nu} H_{\nu'}) \partial_{x'} \delta(x, x'), \]  

(30)

\[ [H_{\iota x}, H_{\perp x'}] = H_{\perp} \partial_{x'} \delta(x, x') + H_{\mu} \partial_{\mu} \delta(x, x'). \]  

(31)

\[ [H_{\iota x}, H_{\iota x'}] = H_{\iota} \partial_{x'} \delta(x, x') + H_{\mu} \partial_{\mu} \delta(x, x'). \]  

(32)

It may be worth noting that these equations have not been derived; it is more appropriate to say that imposing (30)–(32) as strong constraints constitutes our definition of what we mean by a ‘representation’. To complete the definition, we add that, as shown in [43, 44], the requirement that the evolution of an arbitrary functional \( T[X, \pi, \rho, \Phi] \) satisfy path independence implies that the initial values of the canonical variables must be restricted to obey the weak constraints

\[ H_{\perp x} \approx 0 \quad \text{and} \quad H_{\iota x} \approx 0. \]  

(33)

Furthermore, once satisfied on an initial surface \( \sigma \) the dynamics will be such as to preserve (33) for all subsequent surfaces of the foliation.

6. The canonical representation

Next we seek explicit expressions for \( H_{\perp x} \) and \( H_{\iota x} \). A surface deformation is described by (11),

\[ \delta X^\mu_{i x} = \delta \xi^\perp_{i x} n^\mu_{i x} + \delta \xi^i_{i x} X^\mu_{i x}. \]  

(34)

On the other hand, we can evaluate \( \delta X^\mu_{i x} \) using (29),

\[ \delta X^\mu_{i x} = \int dx' \left( [X^\mu_{i x}, H_{\perp x'}] \delta \xi^\perp_{i x'} + [X^\mu_{i x}, H_{\iota x'}] \delta \xi^i_{i x'} \right). \]  

(35)

Since

\[ [X^\mu_{i x}, H_{\perp x'}] = \frac{\delta H_{\perp x'}}{\delta \pi_{\mu x}} \quad \text{and} \quad [X^\mu_{i x}, H_{\iota x'}] = \frac{\delta H_{\iota x'}}{\delta \pi_{\mu x}}, \]  

(36)

comparing (34) and (35) leads to

\[ \frac{\delta H_{\perp x'}}{\delta \pi_{\mu x}} = n^\mu_{i x} \delta(x, x') \quad \text{and} \quad \frac{\delta H_{\iota x'}}{\delta \pi_{\mu x}} = X^\mu_{i x} \delta(x, x'). \]  

(37)

These equations can be integrated to give,

\[ H_{\perp x} = \pi_{\perp x} + \tilde{H}_{\perp} \quad \text{and} \quad H_{\iota x} = \pi_{\iota x} + \tilde{H}_{\iota}, \]  

(38)

where

\[ \pi_{\perp x} = n^\mu_{i x} \pi_{\mu x} \quad \text{and} \quad \pi_{\iota x} = X^\mu_{i x} \pi_{\mu x}, \]  

(39)

and \( \tilde{H}_{\perp} \) and \( \tilde{H}_{\iota} \) are constants of integration that are independent of the surface momenta \( \pi_{\mu} \) but can in principle depend on the other canonical variables, \( X, \rho, \) and \( \Phi \).

\[ The \ difference \ in \ sign \ in \ the \ Poisson \ brackets \ (30)–(32) \ relative \ to \ the \ Lie \ brackets \ (24)–(26) \ arises \ from \ the \ change \ \delta T \ in \ (28) \ being \ written \ in \ terms \ of \ [T, H_{\mu x}] \ rather \ than \ [H_{\mu x}, T]. \]
Thus, the generators \( H_{\perp x} \) and \( H_\alpha \) separate into two components: one pair, \( \pi_{\perp x} \) and \( \pi_{ix} \), that acts only on the geometry and another pair, \( \tilde{H}_{\perp x} \) and \( \tilde{H}_\alpha \), that acts both on the matter variables\(^{12}\) and the geometry. The latter, \( H_{\perp x} \) and \( H_\alpha \), are called the ensemble Hamiltonian and the ensemble momentum. In what follows these will be abbreviated to e-Hamiltonian and e-momentum respectively.

It is a lengthy but straightforward algebraic exercise to check that \( \pi_{\perp x} \) and \( \pi_{ix} \) satisfy the DHKT ‘algebra’, equations (30)–(32),

\[
\begin{align*}
[\pi_{\perp x}, \pi_{\perp x}'] &= (g_{ij}^x \pi_{ix} + g_{ij}^x' \pi_{ijx'}) \partial_i \delta(x,x'), \\
[\pi_{ix}, \pi_{\perp x}'] &= \pi_{\perp x} \partial_i \delta(x,x'), \\
[\pi_{ix}, \pi_{jx}'] &= \pi_{ix'} \partial_j \delta(x,x') + \pi_{jx} \partial_i \delta(x,x').
\end{align*}
\]

### 6.1. The e-momentum generators

The generators of tangential deformations are the simpler ones: they induce translations of the dynamical variables parallel to the surface. The change in \( \rho \) and \( \Phi \) (and functionals thereof) under a tangential deformation \( \delta \chi_x \) is\(^{13}\)

\[
\delta \rho = \frac{\delta \rho}{\delta \chi_x} (\partial_\alpha \chi_x) \quad \text{and} \quad \delta \Phi = \frac{\delta \Phi}{\delta \chi_x} (\partial_\alpha \chi_x).
\]

This change is generated by the e-momentum \( \tilde{H}_\alpha \) according to

\[
\begin{align*}
\frac{\delta \rho}{\delta \chi_x} &= [\rho, \tilde{H}_\alpha] = \frac{\delta \tilde{H}_\alpha}{\delta \rho}, \\
\frac{\delta \Phi}{\delta \chi_x} &= [\Phi, \tilde{H}_\alpha] = -\frac{\delta \tilde{H}_\alpha}{\delta \rho}.
\end{align*}
\]

One can easily check that the required e-momentum is

\[
\tilde{H}_\alpha[\rho, \Phi] = -\int D\chi \rho[\chi] \frac{\delta \Phi[\chi]}{\delta \chi_x} \partial_\alpha \chi_x,
\]

which shows that \( \tilde{H}_\alpha[\rho, \Phi, \chi] \) has an explicit dependence on \( \chi_x \) but is independent of the surface variables \( X^\mu \). It is also straightforward to check that \( \tilde{H}_\alpha \) satisfies the condition (32),

\[
[H_{\alpha x}, \tilde{H}_{\alpha'}] = H_{\alpha x} \partial_{\alpha} \delta(x,x') + H_{\alpha} \partial_{\alpha} \delta(x,x'),
\]

so that the tangential generators \( \pi_\alpha \) and \( \tilde{H}_\alpha \) satisfy (32) separately.

\(^{12}\) We call ‘matter’ any quantity that is not ‘geometry’. It is an abuse of language to refer to the epistemic quantities \( \rho \) and \( \Phi \) as ‘matter’ but it is nevertheless convenient to do so.

\(^{13}\) We are comparing new \( \chi' \) and old \( \chi \) fields located at different points with the same coordinate \( x \). Under a tangential deformation \( \delta \chi \) the new field is \( \chi'(x) = \chi(x + \delta \chi) \) so that

\[
\delta \chi_x = \chi(x + \delta \chi) - \chi(x) = (\partial_\alpha \chi_x) \delta \chi_x.
\]
6.2. The $e$-Hamiltonian generators

The mixed PB relations, equation (31), are the easiest to satisfy and therefore the least informative. Using (41) and $[\tilde{H}_{ix}, \pi_{\perp x}] = 0$ we have

$$[\pi_{ix} + \tilde{H}_{ix}, \tilde{H}_{\perp x}] = H_{\perp x} \partial_x \delta(x, x'),$$

which tells us that $\tilde{H}_{\perp x}$ is a scalar density. In contrast, the normal PB relations, equation (30) which we use (40) to re-write as

$$[\tilde{H}_{\perp x}, \tilde{H}_{\perp x}] + [\pi_{\perp x}, \tilde{H}_{\perp x}] + [\tilde{H}_{\perp x}, \pi_{\perp x}] = (g^{ij}_x \tilde{H}_j + g^{ij}_x \tilde{H}_j) \partial_x \delta(x, x'),$$

are crucial. They provide the criteria for updating the constraints that define the entropic dynamics. Thus, our goal is to find a functional $\tilde{H}_{\perp x}[X, \rho, \Phi; \chi]$ that generates a path-independent entropic dynamics—that is, it reproduces the local time Fokker–Planck equation (16), while remaining consistent with the algebra of deformations.

The desired $\tilde{H}_{\perp x}$ is given by (20), and the relation (53) will serve to determine the so-far unknown e-functional $F_i[X, \rho; \chi]$. Finding the most general solution of (49) lies beyond the scope of this paper; what we will do is to identify a sufficiently large class of solutions that proves to be of physical interest.

Next we turn out attention to finding a class of physically interesting e-functionals $F_i$ depend on the geometric variables $X_i^\mu$ only through the metric $g_{ij}$, and not through any of its derivatives. Then, using

$$\frac{\delta g_{ij}}{\delta \xi^c} = n^c_i \frac{\delta g_{ij}}{\delta X_i^\mu} = 2K_{ij} \delta(x', x)$$

where $K_{ij}$ is the extrinsic curvature, we find that

$$[\pi_{\perp x}, \tilde{H}_{\perp x}] = 2K_{ij} \frac{\partial H_{\perp x}}{\partial g_{ij}} \delta(x', x)$$

is symmetric in $(x', x)$. Therefore

$$[\pi_{\perp x}, \tilde{H}_{\perp x}] + [\tilde{H}_{\perp x}, \pi_{\perp x}] = 0,$$

and (49) simplifies to

$$[\tilde{H}_{\perp x}, \tilde{H}_{\perp x}] = (g^{ij}_x \tilde{H}_j + g^{ij}_x \tilde{H}_j) \partial_x \delta(x, x').$$

Thus, the normal generators $\pi_{\perp x}$ and $\tilde{H}_{\perp x}$ satisfy (30) separately [44].

Next we turn out attention to finding a class of physically interesting e-functionals $F_i[X, \rho; \chi]$. We proceed in steps. First we re-write (20) in the form,

$$\tilde{H}_{\perp x}[X, \rho, \Phi; \chi] = \tilde{H}^0_{\perp x}[X, \rho, \Phi; \chi] + F^0_i[X, \rho; \chi]$$

where

$$\tilde{H}^0_{\perp x} = \int D\chi \left\{ \frac{1}{2} \rho \frac{\delta \phi}{\delta \chi^i} \left( \frac{\delta \Phi}{\delta \chi^i} \right)^2 + \rho \frac{g^1/2}{2} g^{ij} \partial \chi^i \partial \chi^j \right\}.$$  

This amounts to a mere definition of a new $F^0_i$ in terms of the old $F_i$ so there is no loss of generality. The reason that adding the second term in (55) turns out to be convenient is that the new $\tilde{H}^0_{\perp x}$ satisfies (53),

$$[\tilde{H}^0_{\perp x}, \tilde{H}^0_{\perp x}] = (g^{ij}_x \tilde{H}_j + g^{ij}_x \tilde{H}_j) \partial_x \delta(x, x').$$
Then, substituting (54) into (53), and noting that
\[ [F_0^x, F_0^{x'}] = 0, \]  
leads to
\[ [\tilde{F}_0^x, F_0^{x'}] = [\tilde{H}_0^{x}, F_0^{x'}], \]  
which is a homogeneous and linear equation for $F_0^x$. Thus, the condition for a functional $F_0^x$ to be acceptable is that the PB $[\tilde{H}_0^{x}, F_0^{x'}]$ be symmetric under the exchange of $x$ and $x'$.

The next step is to calculate the PB on the left,
\[ [\tilde{F}_0^x, F_0^{x'}] = -\int \tilde{D} \chi \tilde{\delta} \tilde{\delta} \tilde{H}_0^{x} \delta \frac{\delta \Phi}{\delta \chi_x} \left( \frac{\delta \rho}{\delta \chi_x} \right) \]  
and note that $\tilde{\delta} \tilde{H}_0^{x} / \tilde{\delta} \Phi$ reproduces the LTFP equations (16) and (19),
\[ \tilde{\delta} \tilde{H}_0^{x} = -g_x^{-1/2} \delta \left( \rho \frac{\delta \Phi}{\delta \chi_x} \right). \]  
We restrict our search further by considering e-functionals $F_0^x[X, \rho; \chi]$ of the form,
\[ F_0^x[X, \rho; \chi] = \int \tilde{D} \chi f_x \left( X_x, \rho, \delta \rho/\delta \chi_x \right), \]  
where $f_x$ is a function (not a functional) of its arguments. For such a special $F_0^x$ we have
\[ \tilde{\delta} F_0^{x'} = \frac{\partial f_x}{\partial \rho} \delta - \frac{\partial f_x}{\partial \chi_x} \frac{\partial (\delta \rho/\delta \chi_x)}{\partial (\delta \rho/\delta \chi_x')}. \]  
Substituting (60) and (62) into (59) gives
\[ [\tilde{F}_0^x, F_0^{x'}] = -\int \tilde{D} \chi g_x \delta \frac{\delta \Phi}{\delta \chi_x} \left( \frac{\partial f_x}{\partial \rho} - \frac{\partial f_x}{\partial \chi_x} \frac{\partial (\delta \rho/\delta \chi_x)}{\partial (\delta \rho/\delta \chi_x')} \right). \]  
Any $F_0^x$, whether of type (61) or not, must be a scalar density which means that $f_x$ must be a scalar density too. Since the available scalar densities are $g_x^{1/2}$ and $\delta \Phi / \delta \chi_x$, some natural proposals are
\[ f_x \sim g_x^{1/2} \rho \chi_x^n \quad \text{(integer } n) \quad \text{and} \quad f_x \sim g_x^{1/2} \left( \frac{\delta \rho}{\delta \chi_x} \right)^2. \]  
A straightforward substitution into (63) shows that all these trials satisfy (58)—indeed, the PB $[\tilde{H}_0^{x}, F_0^{x'}]$ is symmetric in $(x, x')$. Finally, since (58) is linear we can also consider linear combinations of these trial forms which leads to a generic potential describing self-interactions and interactions with the background geometry.

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14 A more systematic study, carried out in forthcoming work, shows that trials of the form
\[ f_x \sim g_x^{1/2} \left( \frac{\delta \rho}{\delta \chi_x} \right)^k \rho \chi_x^m (\partial \chi_x)^n, \]
where $k$, $\ell$, $m$, and $n$ are integers, are ruled out except for (64).
\[ V(\chi, X_i) = \sum_{n \ell} \lambda_n \chi_n, \]  
\[ (65) \]
where \( \lambda_n \) are coupling constants. We have therefore shown that the family of Hamiltonians
\[ \tilde{H}_x = \int D\chi \rho \left\{ \frac{1}{2} g_{ij}^{1/2} \left( \frac{\delta \Phi}{\delta \chi} \right)^2 + g_{ij}^{1/2} g^{\mu \nu} \partial_i \chi_\mu \partial_j \chi_\nu + \frac{\lambda}{g_{ij}^{1/2}} \right\}, \]
\[ (66) \]
generates a path-independent entropic dynamics. To interpret the last term in (66) we recall that in flat space-time the quantum potential is given by \[ Q = \int D\chi \int d^3x \frac{\lambda}{\rho} \left( \frac{\delta \rho}{\delta \chi} \right)^2. \]
\[ (67) \]
The transition to curved coordinates and to curved space-time is achieved by setting \[ d^3x \rightarrow g_{ij}^{1/2} d^3x \quad \text{and} \quad \frac{\delta}{\delta \chi} \rightarrow g_{ij}^{1/2} \frac{\delta}{\delta \chi} \]
\[ (68) \]
which gives
\[ Q_\sigma = \int D\chi \int d^3x \frac{\lambda}{g_{ij}^{1/2} \rho} \left( \frac{\delta \rho}{\delta \chi} \right)^2. \]
\[ (69) \]
Therefore the last term in (66) may be called the ‘local quantum potential’. Its contribution to the energy is such that those states that are more smoothly spread out in configuration space tend to have lower energy. The corresponding coupling constant \( \lambda > 0 \) controls the relative importance of the quantum potential; the case \( \lambda < 0 \) is excluded because it leads to instabilities.

7. The evolution equations

We will now summarize the main results of the previous sections by writing down the equations that describe how the probability distribution \( \rho[\chi] \) evolves in a curved space-time.

7.1. Entropic dynamics in a curved space-time

Given a space-time with metric \( g_{\mu \nu}(X) \) we start by specifying a foliation of surfaces \( \sigma_t \) labeled by a time parameter \( t \), \( X^\mu = X^\mu(x, t) \), where \( x^i \) are coordinates on the surface. The metric induced on the surface is given by (10). The deformation of \( \sigma_t \) to \( \sigma_{t+\delta t} \) is given by (11),
\[ \delta \xi^\mu = \delta \xi^i n^i + \delta \xi^i X^\mu_i = [N_\mu n^\mu + N_\mu^i X^\mu_i] d\delta t, \]
\[ (70) \]
where we introduced the scalar \( N_\sigma = \frac{\delta \xi^i}{\delta t} \), and the vector shift, \( N_\mu^i = \frac{\delta \xi^i}{\delta t} \).

The goal is to determine the evolution of the probability distribution \( \rho_t[\chi] \) with time \( t \). This requires finding the evolution of the phase functional, \( \Phi_t[\chi] \). The evolution of \( \rho_t \) and \( \Phi_t \) is given by (28),
\[ \frac{\partial \rho_t}{\partial t} = [\rho_t, H] \quad \text{and} \quad \frac{\partial \Phi_t}{\partial t} = [\Phi_t, H], \]
\[ (71) \]
where $H$ is the smeared Hamiltonian,

$$H[N,N'] = \int dx \left( N_u H_{\perp x} + N'_u H_{\perp x} \right).$$

(72)

The result is

$$\frac{\partial \rho_t}{\partial t} = \int d^3 x \left( \frac{\delta}{\delta \xi^\perp_{\perp x}} N_u + \frac{\delta}{\delta \xi^\perp_{\perp x}} N'_u \right).$$

(73)

and

$$\frac{\partial \Phi_t}{\partial t} = \int d^3 x \left( \frac{\delta}{\delta \xi^\perp_{\perp x}} N_u + \frac{\delta}{\delta \xi^\perp_{\perp x}} N'_u \right).$$

(74)

The tangential derivatives, $\delta \rho_t / \delta \xi^\perp_{\perp x}$ and $\delta \Phi_t / \delta \xi^\perp_{\perp x}$, are given by equations (43–45). The normal derivatives, $\delta \rho_t / \delta \xi^\parallel_{\perp x}$ and $\delta \Phi_t / \delta \xi^\parallel_{\perp x}$, are given by

$$\frac{\delta \rho_t}{\delta \xi^\parallel_{\perp x}} = [\rho_t, \tilde{H}_{\perp x}] = \frac{\tilde{\delta}}{\delta \Phi_t},$$

(75)

$$\frac{\delta \Phi_t}{\delta \xi^\parallel_{\perp x}} = [\Phi_t, \tilde{H}_{\perp x}] = -\frac{\tilde{\delta}}{\delta \rho_t}.$$  

(76)

Substituting $\tilde{H}_{\perp x}$ from (66) gives the local-time Fokker–Planck equation (16),

$$\frac{\delta \rho_t}{\delta \xi^\perp_{\perp x}} = -\frac{1}{g^1/2} \frac{\delta}{\delta \chi^\perp_{\perp x}} \left( \rho_t \frac{\delta \Phi_t}{\delta \chi^\perp_{\perp x}} \right),$$

(77)

and the local time generalization of the Hamilton–Jacobi equations,

$$-\frac{\delta \Phi_t}{\delta \xi^\perp_{\perp x}} = \frac{1}{2g^1/2} \left( \frac{\delta \Phi_t}{\delta \chi^\perp_{\perp x}} \right)^2 + \frac{g^1/2}{2} g^{1/2} \partial_{\chi^\perp_{\perp x}} \partial_{\chi^\perp_{\perp x}} + g^{1/2} V(\chi^\perp_{\perp x}, \chi^\parallel_{\perp x}) - \frac{4\lambda}{g^{1/2} \rho^{1/2}} \frac{\delta^2 \rho_t^{1/2}}{\delta \chi^\perp_{\perp x}^2}.$$  

(78)

The formulation of the ED of fields in curved space-time is thus completed. However, it may not yet be obvious that this is a quantum theory; to make it explicit is the next task.

7.2. The local-time Schrödinger functional equation

The relation of the ED formalism to quantum theory can made explicit by making a canonical transformation (often called a Madelung transformation) from the dynamical variables $\rho$ and $\Phi$ into a pair of complex variables $\Psi$,

$$\Psi[\chi] = \rho^{1/2} e^{i\Phi}.$$  

(79)

The equation of evolution for the new variable $\Psi[\chi]$ is then given by

$$\frac{\partial \Psi_t}{\partial t} = \int d^3 x \left( \frac{\delta \Psi_t}{\delta \xi^\perp_{\perp x}} N_u + \frac{\delta \Psi_t}{\delta \xi^\perp_{\perp x}} N'_u \right).$$  

(80)

15 For non-relativistic particles the underlying symplectic and complex structures and their relation to information geometry are derived in [32].
The tangential derivative, $\delta \Psi_t / \delta \xi^i$, is obtained from equation (43),

$$\frac{\delta \Psi_t}{\delta \xi^i} = (\partial_i \chi_x) \frac{\delta \Psi_t}{\delta \chi_x}.$$  \hspace{1cm} (81)

Using (66) the normal derivative,

$$\frac{\delta \Psi_t}{\delta \xi^\perp} = [\Psi_t, \tilde{H}_\perp],$$  \hspace{1cm} (82)

gives the local time version of Schrödinger equation for the wave functional $\Psi_t[\chi]$, \hspace{1cm} (83)

$$i\hbar \frac{\delta \Psi_t}{\delta \xi^\perp} = -\frac{\hbar^2}{2g_{1/2}^2} \frac{\delta^2 \Psi_t}{\delta \chi_x^2} + \frac{1}{2} g_{ij} \partial_i \chi_x \partial_j \chi_x \Psi_t + g_{1/2} V \Psi_t,$$ \hspace{1cm} (83)

where we have introduced $\hbar$ by setting $\lambda = \hbar^2 / 8$ and rescaling $\Phi_t \rightarrow \Phi_t / \hbar$, so that $\Phi_t$ has units of action$^{16}$. The limit of flat space-time is obtained setting $g_{1/2}^2 = 1$, $\delta \xi^\perp = dt$, $N = 1$, and $N' = 0$. Then equations (80) and (83) become the Schrödinger functional equation, \hspace{1cm} (84)

$$i\hbar \frac{\partial \Psi_t}{\partial t} = \int d^3x \left\{ -\frac{\hbar^2}{2} \frac{\delta^2 \Psi_t}{\delta \chi_x^2} + \frac{1}{2} g_{ij} \partial_i \chi_x \partial_j \chi_x \Psi_t + V \Psi_t \right\}.$$  \hspace{1cm} (84)

This is quantum field theory in the Schrödinger functional representation [64]. From here one can proceed to introduce a Hilbert space, operators, and the standard machinery of quantum field theory. Equation (84) justifies identifying the expression $(8\lambda)^{1/2}$ with Planck's constant $\hbar$. We thus see that the coupling $\lambda = \hbar^2 / 8$ in (66) plays a crucial role: it defines the numerical value of $\hbar$ and sets the scale that separates quantum from classical regimes.

8. Some applications

As laid out above, the ED that we have developed here is formally identical to the standard quantum field theory in the Schrödinger functional representation. This means that predictions made on the basis of equations (80)–(83) are identical to those obtained using the standard methods. And indeed, there have been a wide range of topics pursued within this formalism; including, studies of vacuum states in curved space-time [62], research on the Hawking effect [65], applications in cosmology [66], and investigations into symmetries [67, 68], just to name a few. In other words: the ED developed here, while being fully consistent with all of these developments, it does not go beyond them. And, indeed, the purpose of the ED developed here is not to generate better techniques for calculation, but to put QFTCS on a firm conceptual foundation.

The ED formulation of QFTCS, nonetheless, is sufficiently different from the usual approaches that a demonstration of the framework is, in fact, warranted. We do this with two examples. One example illustrates the formal flexibility of the ED formalism, while the other demonstrates the conceptual clarity that an entropic framework supplies to QFTCS. Both insights may have important implications as ED moves beyond QFTCS and begins to incorporate dynamical gravity.

$^{16}$ Note also that this modification means that we have adjusted the units of $\chi$ so that $|\chi| = [\hbar^{1/2}] / \text{length}.$
8.1. The Ehrenfest equations in Entropic dynamics

We begin by obtaining the Ehrenfest equations for a quantum scalar field in ED. The derivation highlights many of the novel features of the ED approach; in particular, the utilization of Hamiltonians, Poisson brackets, and so on, rather than the conventional quantum tools.

8.1.1. Some background. In ED, the focus of our inquiries are the field variables $\chi_x$ which have definite, but unknown values. Thus, in the absence of such definite information, our goal is to obtain an estimate of these values. One such estimate is provided by the expected value of the field variables

$$\tilde{\chi}_x = \int D\chi \rho \chi_x.$$  \hspace{1cm} (85)

This, in turn, defines a functional $\tilde{\chi}_x$ on the ensemble phase space, which makes it amenable to treatment through the canonical formalism.

However, this expected value is not static and we wish to know how it changes in time. To determine this evolution, we carry over the formalism of the previous section and introduce a Hamiltonian $H[N_i, N_i']$, as in equation (72), adapted to a particular foliation with lapse $N$, shift $N_i$, and parameter $t$. Moreover, we are concerned with a quantum dynamics so we choose $H_{\perp x}$ and $H_{ix}$ in equation (72) to match that of equations (46) and (66).

8.1.2. Evolution of the expected values. Since $\tilde{\chi}_x = \tilde{\chi}_x[\rho]$ is a Hamiltonian functional, its update can be obtained by taking the appropriate Poisson brackets for a suitably chosen Hamiltonian $H$. Indeed, the velocity of $\tilde{\chi}_x$ is given by

$$\partial_t \tilde{\chi}_x = \int dx' \left( N_{x'} \{ \tilde{\chi}_x, H_{\perp x'} \} + N_{ix'} \{ \tilde{\chi}_x, H_{ix'} \} \right)$$  \hspace{1cm} (86)

with

$$\{ \tilde{\chi}_x, H_{\perp x'} \} = \delta(x, x') \int D\chi \rho \frac{1}{g_{x'}} \frac{\delta \Phi}{\delta \chi_{x'}}$$  \hspace{1cm} (87)

and

$$\{ \tilde{\chi}_x, H_{ix'} \} = -\delta(x, x') \int D\chi \rho \partial_{ix'} \chi_{x'}.$$  \hspace{1cm} (88)

Taken together, equations (86)–(88) result in

$$\partial_t \tilde{\chi}_x = \int D\chi \rho \left( \frac{N}{g_{x'}^{1/2}} \frac{\delta \Phi}{\delta \chi_{x'}} - N' \partial_{ix} \chi_{x'} \right),$$  \hspace{1cm} (89)

which contains two contributions. The latter contribution in equation (89) is just due to the shift, while the former is due to the flow of probability, which is characterized by the appearance of the current velocity

$$v_x = \frac{1}{g_{x'}^{1/2}} \frac{\delta \Phi}{\delta \chi_{x'}}.$$  

first introduced in the context of the LTFP equation (16).
Consider here a related quantity, the *current momentum* and its expectation, the *ensemble current momentum*\(^{17}\),

\[ P_x = \frac{\delta \Phi}{\delta \chi_x} = g_s^{1/2} v_x \quad \text{and} \quad \tilde{P}_x = \int D\chi \rho_P \chi_x, \]  

respectively\(^{18}\). We can now conveniently rewrite the velocity \( \partial_t \tilde{\chi}_x \) in terms of the current momentum, yielding

\[ \partial_t \tilde{\chi}_x = \frac{N}{g_s^{1/2}} \tilde{P}_x - N^i \langle \partial_i \chi_x \rangle, \]  

where we have used the notation \( \langle A \rangle = \int \rho A \) to denote expectation.

The advantage of introducing the current momentum (as opposed to the current velocity) is that the expected current momentum \( \tilde{P}_x \), together with the expected field value \( \tilde{\chi}_x \), satisfy the canonical Poisson bracket relations

\[ \{ \tilde{\chi}_x, \tilde{P}_{x'} \} = \delta(x, x'). \]  

This seems to suggest that \( \tilde{P}_x \) plays the role of a momentum *conjugate* to \( \tilde{\chi}_x \).

Indeed, let us take this hint seriously and compute the corresponding Hamilton’s equations for this canonical pair. The time derivative of \( \tilde{\chi}_x \) was provided earlier, in equation (91). The velocity of \( \tilde{P}_x \), on the other hand, is given by

\[ \partial_t \tilde{P}_x = \int d\chi' \left( N_{x'} \{ \tilde{P}_x, H_{L,x'} \} + N^i_{x'} \{ \tilde{P}_x, H_{i,x'} \} \right). \]  

To compute this we need the two Poisson brackets in equation (93). A quick calculation gives

\[ \{ \tilde{P}_x, H_{L,x'} \} = -\int D\chi \rho g_s^{1/2} \left( \delta(x, x') \frac{\partial V_{L}}{\partial \chi_{x'}} + g^0_{i'} \partial_i \chi_{x'} \partial_j \delta(x', x) \right) \]  

and

\[ \{ \tilde{P}_x, H_{i,x'} \} = \int D\chi \rho \frac{\delta \Phi}{\delta \chi_{x'}} \delta(x', x) \]  

so that from equation (93) we obtain

\[ \partial_t \tilde{P}_x = \partial_i \left( N g_s^{1/2} \delta \frac{\partial \chi_x}{\partial \chi_{x'}} \right) - \partial_i \left( N^i \tilde{P}_x \right) - N g_s^{1/2} \left\langle \frac{\partial V}{\partial \chi_x} \right\rangle. \]  

Thus an initial assignment of \( \tilde{\chi}_x, \tilde{P}_x \), and higher statistical moments of \( \chi_x \), will be sufficient to determine the evolution of \( \tilde{\chi}_x \), i.e. no further derivatives are required.

### 8.1.3. Ehrenfest equations.

The equations (89) and (96) taken together have the character of classical field equations for some ‘classical’ field variables \( \tilde{\chi}_x \) and \( \tilde{P}_x \). In fact, it is not difficult to show that these are nothing but the Ehrenfest equations (see e.g. [69]).

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\(^{17}\) Note that while the current velocity is a scalar valued quantity, the current momentum is a scalar *density* of weight one.

\(^{18}\) Alternatively, introduce the differential operator \( \hat{P}_x = -i\delta/\delta \chi_x \). Then the ensemble current momentum translates in the conventional language to the expected value of this operator. That is, \( \tilde{P}_x = \int \Psi^* \hat{P}_x \Psi \), using the complex functionals \( \langle \Psi^*, \Psi \rangle \) introduced above.
To see this, note that the velocity $\partial_t \tilde{\chi}_x$ is linear in the current momentum $\tilde{P}_x$. Now, invert this relation for $\tilde{P}_x$ in terms of the velocity $\partial_t \tilde{\chi}_x$ and substitute into equation (96). The result is that $\tilde{\chi}_x$ evolves according to the equation

$$\Box \tilde{\chi}_x = \left\langle \frac{\partial V(\chi_x)}{\partial \chi_x} \right\rangle,$$

(97)

where

$$\Box = -\frac{1}{Ng^{1/2}} \hat{\partial}_i \left[ \frac{g^{1/2}}{N} \hat{\partial}_i \right] - \frac{\partial_i N^i}{N^2} \hat{\partial}_i + \frac{1}{Ng^{1/2}} \hat{\partial}_i \left[ Ng^{1/2} g^{ij} \hat{\partial}_j \right],$$

(98)

is the wave operator in curved space-time in foliation-adapted coordinates [62, 63], and where we have introduced the operator $\hat{\partial}_i = \partial_i + N^i \partial_i$.

Equation (97) comprises the Ehrenfest relations that we seek. An appealing feature of such relations is that they are exact, and thus contain complete information about the underlying quantum dynamics. This makes the Ehrenfest relations ideal for probing the behavior of quantum fields and their deviations from classical behavior. For instance, it is not difficult to see that $\tilde{\chi}_x$ follows a classical evolution only when

$$\left\langle \frac{\partial V(\chi_x)}{\partial \chi_x} \right\rangle = \frac{\partial V(\langle \chi_x \rangle)}{\partial \chi_x}.$$

(99)

But this, of course, only occurs when the potential is itself quadratic in the field.

Indeed, choose for $V_x$ the potential $V_x = \frac{1}{2} m^2 \chi_x^2$ so that $\partial V / \partial \chi_x = m^2 \chi_x$. Equation (97) then reduces to

$$\left( \Box - m^2 \right) \tilde{\chi}_x = 0,$$

(100)

which is a classical Klein–Gordon equation in curved space-time (see e.g. [3]) for the expected field configuration. Thus $\tilde{\chi}_x$ follows—exactly—the classical equations of motion, which is precisely the content of Ehrenfest’s theorem, familiar from non-relativistic quantum mechanics (see e.g. [69]). For potentials that are not quadratic, however, we can expect deviations from classical behavior and it is legitimate to obtain quantum corrections via an approximation scheme.

8.1.4. Comments. As opposed to standard formulations of equations of this type, our derivation of the Ehrenfest relations was performed entirely within the framework of ED, using Hamiltonians, Poisson brackets, etc, rather than commutators and the standard quantum machinery[19]. These methods, which are geometric in nature, are thus of more general applicability than the standard techniques, which rely heavily on the linearity of quantum theory. Indeed, while the assumption of linearity has thus far proved quite robust, it is not immediately obvious that quantum gravity needs to follow suit (see e.g. [71–75]). In such cases the ED approach might provide a viable alternative framework.

8.2. On divergences in Entropic dynamics

Some of the principal benefits of the ED approach are conceptual. For instance, a central difficulty of any quantum field theory, one that is also shared by the ED formalism, is the...
problem of infinities. The nature of the infinities is, however, very different [39]. To see this consider the limit of flat space-time, i.e. \( N = 1, N' = 0, g^{1/2} = 1 \). Setting \( V = m^2 \chi^2 / 2 \) in the Schrödinger equation (84) leads to the quantum theory of free real scalar fields [64] from which all the standard results can be recovered (see e.g. [62]). For example, in units such that \( \hbar = c = 1 \), the wave functional of the ground state is

\[
\Psi_0[\chi] = \frac{1}{Z_0^{1/2}} e^{-iE_0t} \exp \left[ -\frac{1}{2} \int d^3x \int d^3y \chi_x G_{xy} \chi_y \right].
\]  

(101)

where

\[
G_{xy} = \int \frac{d^3k}{(2\pi)^3} \omega_k e^{i\vec{k} \cdot (\vec{x} - \vec{y})}, \quad \text{with} \quad \omega_k = (k^2 + m^2)^{1/2},
\]

(102)

and the energy of the ground state,

\[
E_0 = \int d^3x \tilde{H}_{\perp x}[\Psi_0],
\]

(103)

is obtained from (66) and (101),

\[
\tilde{H}_{\perp x}[\Psi_0] = \frac{1}{2} \int D\chi \rho_0 \left[ \left( \frac{\delta \Phi_0}{\delta \chi_x} \right)^2 + g^{ij} \partial_i \chi_x \partial_j \chi_x + m^2 \chi_x^2 + \left( \frac{1}{2} \frac{\delta \rho_0}{\delta \chi_x} \right)^2 \right].
\]

(104)

The result,

\[
E_0 = \frac{1}{2} \int d^3x G_{xx} = \int d^3x \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k},
\]

(105)

is both infrared and ultraviolet divergent. Similarly, the expected value of the field at any point \( \vec{x} \) vanishes but its variance diverges,

\[
\langle \chi_x \rangle = 0 \quad \text{and} \quad \text{Var}[\chi_x] = \langle \chi_x^2 \rangle_0 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k}.
\]

(106)

The point we want to stress in repeating these well-known results is that in the ED framework the divergent quantities in (105) and (106) are not physical, ontic quantities. Both the e-Hamiltonian \( \tilde{H}_{\perp x}[\Psi_0] \) in (104) and the variance in (106) are expected values. The infinities are not real; they are epistemic.

ED recognizes the role of incomplete information: the interpretation of the diverging \( \text{Var}[\chi_x] \) is not that the field \( \chi_x \) undergoes fluctuations of infinite magnitude, but rather that with the information that is available to us we are completely unable to predict the value of the field \( \chi_x \) at the sharply localized point \( \vec{x} \). The infinities are epistemic: what diverges are not physical quantities but our uncertainty about them. However, this does not mean the theory is useless. It may be incapable of predicting some quantities but it can provide useful predictions for many others. For example, the equal time correlations between two field variables at different locations are perfectly finite,

\[
\langle \chi_x \chi_y \rangle_0 = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot (\vec{x} - \vec{y})}}{2\omega_k} = \frac{m}{4\pi^2 |\vec{x} - \vec{y}|} K_1 \left( m |\vec{x} - \vec{y}| \right),
\]

(107)

where \( K_1 \) is a modified Bessel function [62].
9. Discussion

Entropic dynamics provides an inferential alternative to the standard methods of quantization. The ED approach avoids a representation of fields as operators and any reference to the Hilbert or Fock spaces on which they presumably act. In effect this eliminates the issue of choosing among many inequivalent representations. Consequently, the operator ordering ambiguities that are characteristic of conventional quantization methods are avoided too. Indeed, many of the problems associated with the Dirac quantization method [42] and the laborious techniques necessary to implement it (such as the identification and elimination of second-class constraints etc) are completely sidestepped.

The ED approach to QFTCS, however, also offers new insights to problems such as the Unruh effect at the interface between QFTCS and the quantum measurement problem 20. Indeed, the fact that in the ED approach fields are physical entities which at all times have definite but possibly unknown values, while ‘particles are whatever particle detectors detect,’ immediately raises the question of what is a particle within the ED framework.

Moreover, the ED approach to QFTCS leads to a theory that is Hamiltonian in character, thus retaining the powerful tools and intuitive appeal of the classical Hamiltonian framework21 but now in the context of a fully quantum theory. This is particularly attractive for a couple reasons. First is that the ED approach proceeds without ever invoking the use of linearity or of Hilbert spaces, thus leaving open the possibility that deeper theories will introduce non-linearities—something the standard approaches cannot readily account for. Another is that the scheme introduced here allows us to borrow the methods of DHKT [46], which were used to develop classical covariant Hamiltonian theories, but instead apply them to a theory that is inherently statistical and quantum.

This is not insignificant. A primary difficulty in formulating a theory of quantum gravity is that general relativity and quantum field theory are couched in completely different formal languages. The ED that we have developed here, however, actually helps to bridge this divide. Indeed, while a common approach to quantum gravity is to incorporate gravitation into the linear, algebraic framework of quantum theory, ED opens the door to an alternative approach wherein quantum theory more resembles general relativity. The key is to recognize the central role played by the Hamiltonian formalism, not only in the ED formulation of QFTCS here, but also in the geometrodynamics approach to general relativity. Therefore, in contrast to many other approaches to QFTCS, ED seems to offer the possibility of extending the scheme developed here towards deriving, from first principles, a fully dynamical theory of quantum fields interacting with classical gravity [78].

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20 Being an inference theory, ED is particularly well suited to tackling the quantum problem of measurement [51, 52].
21 That quantum theory can be formulated as a Hamiltonian theory has been explored by many [26, 27] [70] [76, 77]. The connection between the ED framework and information geometry with the symplectic/Hamiltonian structure is explored in [32].
Appendix. The local-time Fokker–Planck equations

To rewrite the dynamical equation (15) in differential form consider the probability $P[\chi', \sigma'|\chi_0, \sigma_0]$ of a finite transition from a field configuration $\chi_0$ at some early surface $\sigma_0$ to a configuration $\chi$ at a later $\sigma$. The result of a further evolution from $\sigma$ to a neighboring $\sigma'$ obtained from $\sigma$ by an infinitesimal normal deformation $\delta \xi^\perp_\sigma$ is given by (15),

$$P[\chi', \sigma'|\chi_0, \sigma_0] = \int D\chi P[\chi', \sigma'|\chi, \sigma] P[\chi, \sigma|\chi_0, \sigma_0].$$  \hspace{1cm} (A.1)

To obtain a differential equation one cannot just Taylor expand as $\delta \xi^\perp_\sigma \to 0$ because $P[\chi', \sigma'|\chi, \sigma]$ becomes a very singular object—a delta functional. Instead, we multiply by an arbitrary smooth test functional $T[\chi']$ and integrate

$$\int D\chi' P[\chi', \sigma'|\chi_0, \sigma_0] T[\chi'] = \int D\chi P[\chi, \sigma|\chi_0, \sigma_0] \int D\chi' T[\chi'] P[\chi', \sigma'|\chi, \sigma].$$  \hspace{1cm} (A.2)

Next expand the test function $T[\chi'] = T[\chi + \Delta \chi]$ in powers of $\Delta \chi = \chi' - \chi$. Since $\chi$ is Brownian to obtain $T[\chi']$ to first order in $\delta \xi^\perp_\sigma$ we need to keep second order in $\Delta \chi$.\hspace{1cm} (A.3)

Use this expansion together with (8) and (9) to obtain

$$\int D\chi' T[\chi'] P[\chi', \sigma'|\chi, \sigma] = T[\chi] + \int d\chi \frac{\eta \delta \xi^\perp_\sigma}{g^\perp_\chi} \left\{ \frac{\delta T[\chi]}{\delta \chi} \frac{\delta \phi[\chi]}{\delta \chi} + \frac{1}{2} \frac{\delta^2 T[\chi]}{\delta \chi^2} \right\}. $$

Substituting back into equation (A.2), leads to

$$\int D\chi \{ P[\chi, \sigma'|\chi_0, \sigma_0] - P[\chi, \sigma|\chi_0, \sigma_0] \} T[\chi]$$

$$= \int d\chi \frac{\eta \delta \xi^\perp_\sigma}{g^\perp_\chi} \int D\chi P[\chi, \sigma|\chi_0, \sigma_0] \left\{ \frac{\delta T[\chi]}{\delta \chi} \frac{\delta \phi[\chi]}{\delta \chi} + \frac{1}{2} \frac{\delta^2 T[\chi]}{\delta \chi^2} \right\}. $$  \hspace{1cm} (A.4)

Since $T[\chi]$ is arbitrary, after some integrations by parts we get

$$P[\chi, \sigma'|\chi_0, \sigma_0] - P[\chi, \sigma|\chi_0, \sigma_0] = \int d\chi \frac{\delta P[\chi, \sigma|\chi_0, \sigma_0]}{\delta \xi^\perp_\sigma} \delta \xi^\perp_\sigma$$

$$= \int d\chi \frac{\eta \delta \xi^\perp_\sigma}{g^\perp_\chi} \left\{ \frac{\delta}{\delta \chi} \left( P[\chi, \sigma|\chi_0, \sigma_0] \frac{\delta \phi[\chi]}{\delta \chi} \right) + \frac{1}{2} \frac{\delta^2}{\delta \chi^2} P[\chi, \sigma|\chi_0, \sigma_0] \right\}. $$  \hspace{1cm} (A.5)

Finally, for a finite evolution from $\sigma_0$ to $\sigma$, (15) reads,

$$\rho_\sigma[\chi] = \int D\chi_0 P[\chi, \sigma|\chi_0, \sigma_0] \rho_\sigma[\chi_0].$$  \hspace{1cm} (A.6)
A further infinitesimal normal deformation $\sigma \to \sigma'$ by $\delta \xi^\perp_i$ gives
\[
\rho_{\sigma'}[\chi] - \rho_\sigma[\chi] = \int dx \frac{\delta \rho_\sigma[\chi]}{\delta \xi^\perp_i} \delta \xi^\perp_i
\]
\[
= \int D\chi_0 \left( \int dx \frac{\delta P[\chi, \sigma|\chi_0, t_0]}{\delta \xi^\perp_i} \delta \xi^\perp_i \right) \rho_{\sigma_0}[\chi_0]
\]
(A.7)
which, using (A.5) and the fact that $\sigma'$ (or $\delta \xi^\perp_i$) can be freely chosen leads to the local Fokker–Planck equations,
\[
\frac{\delta \rho_\sigma[\chi]}{\delta \xi^\perp_i} = \frac{\eta}{g^2_s \chi^4} \left\{ -\frac{\delta}{\delta \chi^4} \left( \rho_\sigma[\chi] \frac{\delta \phi[\chi]}{\delta \chi^4} \right) + \frac{1}{2} \frac{\delta^2}{\delta \chi^4} \rho_\sigma[\chi] \right\},
\]
(A.8)
which is equation (16).

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