Abstract. For smooth embeddings of an integral homology 3-sphere in the 6-sphere, we define an integer invariant in terms of their Seifert surfaces. Our invariant gives a bijection between the set of smooth isotopy classes of such embeddings and the integers. It also gives rise to a complete invariant for homology bordism classes of all embeddings of homology 3-spheres in the 6-sphere. As a consequence, we show that two embeddings of an oriented integral homology 3-sphere in the 6-sphere are isotopic if and only if they are homology bordant. We also relate our invariant to the Rohlin invariant and accordingly characterise those embeddings which are compressible into the 5-sphere.

1. Introduction

Haefliger’s seminal work [8] shows that in the smooth category, unlike in the piecewise linear and topological categories, knotting phenomena occur in codimension greater than three. For example, [8, 9] show that the group $C^{2k+1}_{4k-1}$ of smooth isotopy classes of smooth embeddings of the $(4k-1)$-sphere $S^{4k-1}$ in the $6k$-sphere $S^{6k}$ forms the infinite cyclic group for each $k \geq 1$. According to [1, 2, 24], the isotopy class of an embedding in $C^{2k+1}_{4k-1}$ can be read off from geometric characteristics of its Seifert surface.

The first occasion $C^3_3$, where smooth and piecewise linear isotopies differ, seems especially intriguing, since it lies in the interface between high- and low-dimensional topology. For instance, Montgomery and Yang [3] showed an isomorphic correspondence between the group $C^3_3$ and the group of diffeomorphism classes of all homotopy $CP^3$, that has provided various interesting applications (e.g. [14, 16, 17]). Boéchat and Haefliger [2] used the group $C^3_3$ to describe a necessary and sufficient condition for the embeddability of an orientable 4-manifold in 7-space (later their condition turned out to be always satisfied due to [5], so that this last case of the hard Whitney embedding theorems was settled). Our previous paper [25] also shows that the group $C^3_3$ is linked to various aspects of 4-dimensional topology.

In this paper, we study smooth embeddings of oriented integral homology 3-spheres in $S^6$. We first observe in Proposition 2.5 that for every smooth embedding $F: \Sigma^3 \rightarrow S^6$ of an oriented integral homology 3-sphere in $S^6$, there exists a smooth embedding (a Seifert surface) $\tilde{F}: V^4 \rightarrow S^6$ of a smooth oriented 4-manifold $V^4$ with $\partial V^4 = \Sigma^3$ such that $\tilde{F}|_{\partial V^4} = F$. For such a Seifert surface, we denote the signature of $V^4$ by $\sigma(V^4)$ and the normal Euler class of $\tilde{F}$ by $e_{\tilde{F}}$. In §4 we prove that

$$\Omega(F) = \frac{\sigma(V^4) - e_{\tilde{F}} \cdot e_{\tilde{F}}}{8},$$

does not depend on the choice of $\tilde{F}$ and is invariant up to isotopy of $F$, where $e_{\tilde{F}} \cdot e_{\tilde{F}}$ is the square of $e_{\tilde{F}}$, evaluated on the fundamental class. Then, Theorem 4.2 further shows that the invariant $\Omega$ gives a bijection:

$$\Omega: \operatorname{Emb}(\Sigma^3, S^6) \cong \mathbb{Z}$$

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between the set \(\text{Emb}(\Sigma^3, S^6)\) of smooth isotopy classes of smooth embeddings of \(\Sigma^3\) in \(S^6\) and the integers \(\mathbb{Z}\).

In the study of oriented integral homology 3-spheres, their bounding smooth compact oriented 4-manifolds (note that they always bound topological compact oriented acyclic 4-manifolds [4]) are important tools and give rise to many useful invariants in low-dimensional topology. In Corollary 4.3 we relate our invariant \(\Omega\) to the Rohlin invariant and consequently show that an embedding \(F\) of an integral homology sphere \(\Sigma^3\) in \(S^6\) is compressible into \(S^4\) if and only if \(\Omega(F)\) is modulo 2 equal to the Rohlin invariant \(\mu(\Sigma^3)\).

In [3] we show that when we assemble all embeddings of oriented integral homology 3-spheres in \(S^6\), the invariant \(\Omega\) proves to be invariant up to homology bordism (of embeddings). We say that two embeddings \(F_0\) and \(F_1\) of two integral homology spheres are homology bordant if there exists a proper embedding of a homology cobordism between the two homology spheres in \(S^6 \times [0, 1]\), whose restriction to the boundary coincides with the disjoint union of \(F_0 \times \{0\}\) and \(-F_1 \times \{1\}\) (see [3] for details). The collection of homology bordism classes forms an abelian group, denoted by \(\Gamma^3_3\), via connected sum. Then, we show that the invariant \(\Omega\) gives rise to the following isomorphism (Theorem 5.4):

\[
\bar{\Omega}: \Gamma^3_3 \xrightarrow{\cong} \mathcal{O}^3_2 \oplus \mathbb{Z}
\]

where \([\Sigma]\) denotes the homology cobordism class represented by the homology sphere \(\Sigma\) and \([F:\Sigma^3 \hookrightarrow S^6]\) denotes the homology bordism class represented by the embedding \(F:\Sigma^3 \hookrightarrow S^6\). As a corollary, we see that two embeddings of an oriented integral homology 3-sphere in \(S^6\) are isotopic if and only if they are homology bordant.

We work in the smooth category; all manifolds and mappings are supposed to be differentiable of class \(C^\infty\), unless otherwise explicitly stated. We will suppose all spheres and homology spheres are oriented. If \(M\) is an oriented manifold with non-empty boundary, then for the induced orientation of \(\partial M\) we adopt the outward vector first convention: we say an ordered basis of \(T_p(\partial M)\) \((p \in \partial M)\) is positively oriented if an outward vector followed by the basis is a positively oriented basis of \(T_p M\). For a closed \(n\)-dimensional manifold \(M\) we denote its punctured manifold by \(M_e\); i.e., \(M_e := M \setminus \text{Int} D^n\). The homology and cohomology groups are supposed to be with integer coefficients unless otherwise explicitly noted.

2. Seifert surfaces

The purpose of this section is to show that every embedding \(F:\Sigma^3 \hookrightarrow S^6\) of an oriented integral homology 3-sphere \(\Sigma^3\) has a Seifert surface.

**Definition 2.1.** Let \(F:M^3 \hookrightarrow S^6\) be an embedding of a closed oriented 3-manifold. Then, a Seifert surface for \(F\) is an embedding \(\widetilde{F}:W^4 \hookrightarrow S^6\) of a compact connected oriented 4-manifold \(W^4\) with \(\partial W^4 = M^3\) such that \(\widetilde{F}|_{M^3} = F\).

Every closed oriented 3-manifold bounds a spin 4-manifold with only one 0-handle and 2-handles and such a 4-manifold can be embedded in \(S^6\), in fact, in \(S^5\) (e.g. see [12] Chapter VIII). Now let \(\Sigma^3\) be an oriented integral homology 3-sphere and take an embedding \(\widetilde{F}_0:W^4_0 \hookrightarrow S^6\) of a compact oriented 4-manifold \(W^4_0\) with \(\partial W^4_0 = \Sigma^3\). In this section, we fix this embedding \(\widetilde{F}_0\) and put \(F_0 := \partial \widetilde{F}_0 : \Sigma^3 \hookrightarrow S^6\). We can assume that \(F_0\) is standard on a 3-disc \(D^3 \subset M^3\) (i.e., coincides with the “northern hemisphere” \(D^3_+ \subset S^6\)) of the standard inclusion \(S^3 \subset S^6\) and \(F_0(\Sigma^3 \setminus \text{Int} D^3)\) lies in the southern hemisphere \(D^6_\text{south}\) of \(S^6\). Note that the complement \(C\) of a open neighbourhood of \(F_0(\Sigma^3 \setminus \text{Int} D^3)\) in \(S^6\) is contractible.

First, we need the following lemma.

**Lemma 2.2.** Let \(M^3\) be a closed oriented 3-manifold. Then, two embeddings \(M^3_0 \hookrightarrow S^6\) of the punctured manifold \(M^3_e\) in \(S^6\) are isotopic.
Proof. The punctured manifold $M^3_0$ has a handle decomposition

$$M^3_0 = D^3 \cup \{1\text{-handles}\} \cup \{2\text{-handles}\}.$$ 

For dimensional reasons, two embeddings $M^3_0 \hookrightarrow S^6$ are isotopic on the cores of the handles and we can extend the isotopy on the entire handles since $\pi_1(V_{2,2}) = 0$ and $\pi_2(V_{4,1}) = 0$, where $V_{n,k}$ denotes the Stiefel manifold of all $k$-frames in $n$-space. □

Remark 2.3. If $M^3 = \Sigma^3$ is an integral homology sphere, then an embedding $\Sigma^3_0 \hookrightarrow S^6$ of the punctured manifold can be compressed in $S^5$ [11, Theorem D] and is unique up to isotopy in $S^5$ [22, Corollary 4.10(1)].

Next, we prove the following.

**Proposition 2.4.** Any embedding $F : \Sigma^3 \hookrightarrow S^6$ of an oriented integral homology 3-sphere is isotopic to the connected sum $F^\circ \# g$ of $F_0$ with some embedding $g : S^3 \hookrightarrow S^6$ of the 3-sphere $S^3$.

**Proof.** By Lemma 2.2, $F : \Sigma^3 \hookrightarrow S^6$ can be isotoped to $F_0$ on $\Sigma^3_0 := \Sigma^3 \setminus \text{Int} D^3$. Then, we can assume that a small collar neighbourhood $\partial D^3 \times \{0, \varepsilon\}$ of $\partial D^3 \subset D^3$ is "standardly embedded" in a small collar neighbourhood $\partial D^3_+ \times \{0, \varepsilon\}$ of the northern hemisphere $D^3_+$ of $S^6$ (i.e., $F_0$ on $\partial D^3 \times \{0, \varepsilon\}$ is the product of the standard inclusion $S^2 \subset S^3 = \partial D^3_+$ by the interval).

The complement $C$ of an open neighbourhood of $F(\Sigma^3_0)$ in $S^6$ is contractible. Therefore, we can apply a lemma by Levine [14, Lemma 2] to $C$ and its two submanifolds, $D^6_+ \setminus \partial D^5_+ \times \{0, \varepsilon/2\}$ in codimension 0 and $F(\partial D^5_+ \times \{0, \varepsilon\})$ capped off by a 3-disc in $\partial D^5_+ \times \{\varepsilon\}$ (with corners smoothed) in codimension 3 (see Figure 1). This ensures that we can push $F(\partial D^3)$ into $D^3_+ \subset C$ by an isotopy of $C$ fixing a collar of $F(\partial D^3)$. Hence, $F$ is isotopic to the connected sum $F_0 \# g$ of $F_0$ with some embedding $g : S^3 \hookrightarrow S^6$. □

The following is now an easy corollary of Proposition 2.4.

**Proposition 2.5.** Any embedding $F$ of an oriented integral homology 3-sphere $\Sigma^3$ in $S^6$ has a Seifert surface.

**Proof.** By Proposition 2.4, $F$ is isotopic to the connected sum $F_0 \# g$ of $F_0$ with some embedding $g : S^3 \hookrightarrow S^6$. By [24], $g$ has a Seifert surface $\tilde{g} : V^4 \to S^6$. Thus, the boundary connected sum $\tilde{F}_0 \# \tilde{g} : W_0^4 \# V^4 \to S^6$ (composed with a suitable diffeomorphism on $S^6$) is a Seifert surface for $F = F_0 \# g$. □

**Remark 2.6.** In Proposition 2.5, we can choose both $W_0$ and $V^4$ to be simply connected (in fact, $V^4$ can be chosen as $S^2 \times S^2$ [24] or as the connected sum of some copies of the complex projective planes [6 25]). Therefore, we see that any embedding of an oriented integral homology 3-sphere has a simply connected Seifert surface.
3. The Hopf Invariant

In this section we associate to each embedding of a homology 3-sphere into $S^6$, which is equipped with a normal vector field, its Hopf invariant.

3.1. Embeddings with normal 1-fields. Let $F: M^3 \hookrightarrow S^6$ be an embedding of a closed oriented 3-manifold $M^3$. Then, the normal bundle of $F$ is trivial and homotopy classes of normal framings (i.e. trivialisations of the normal bundle) are classified by the homotopy set $[M^3, SO(3)]$, with respect to some fixed framing. Similarly, homotopy classes of normal 1-fields are classified by the set $[M^3, V_{3,1}] = [M^3, S^2]$, which has been computed by Pontrjagin [20].

If $\Sigma^3$ is an integral homology sphere, then the absence of 2-dimensional obstructions allows us to identify $[\Sigma^3, S^2]$ with $H^3(\Sigma^3) = \mathbb{Z}$ in a natural way. In this case furthermore, the natural map $[\Sigma^3, SO(3)] \to [\Sigma^3, S^2]$ is a bijection (see [13 Proposition 2.2]). Therefore, a normal 1-field determines a normal framing uniquely up to homotopy and vice versa. Hence, in a similar way to [9, §5.1], we can consider the connected sum of two embeddings $\Sigma^3 \hookrightarrow S^6$ with normal 1-fields.

3.2. The Hopf Invariant of an embedding with a normal 1-field. Let $\Sigma^3$ be an oriented integral homology 3-sphere and $F: \Sigma^3 \hookrightarrow S^6$ be an embedding. Then, by Alexander duality, the exterior space $X := S^6 \setminus F(\Sigma^3)$ has the same homology groups as the 2-sphere $S^2$. We identify $H_2(X)$ with $\mathbb{Z}$ by assigning to $[z^2] \in H_2(X)$ the linking number $\text{lk}(\vec{z}^2, f(\Sigma^3)) \in \mathbb{Z}$ in $S^6$, which also gives an identification $H^2(X) = \mathbb{Z}$.

If an embedding $F: \Sigma^3 \hookrightarrow S^6$ is endowed with a normal 1-field $v$ of $F(\Sigma^3) \subset S^6$, we can define the Hopf invariant of $(F,v)$ in the following way.

**Definition 3.1.** Let $F: \Sigma^3 \hookrightarrow S^6$ be an embedding of an oriented homology 3-sphere and $v$ be a normal 1-field of $F(\Sigma^3) \subset S^6$. By using a small shift of $F(\Sigma^3)$ along $v$, we can define a map $\overline{v}: \Sigma^3 \to X := S^6 \setminus F(\Sigma^3)$. Then, we define the Hopf invariant $H_{(F,v)}$ of $(F,v)$ to be minus the functional cup product $L^\overline{v}(\omega, \omega) = R^\overline{v}(\omega, \omega) \in H^2(\Sigma^3) = \mathbb{Z}$ for the generator $\omega \in H^2(X)$. We use the definition of the functional cup product given in [26, p.368] and [24] §2.2. In our special case here, we can review it as follows.

Take a 2-cochain $\omega'$ of $X$ which represents $\omega \in H^2(X)$. Then, there exist a 3-cochain $a$ of $X$ such that $\delta a = \omega' \smile \omega'$ and a 1-cochain $b$ of $\Sigma^3$ such that $\delta b = \overline{v}^2 \omega'$, where $\overline{v}^2$ is the induced homomorphism on cochains. Finally, the 3-cochain

\[ z' = \overline{v}^2 a - b \smile \overline{v}^2 \omega' \tag{3.2} \]

actually becomes a cocycle of $\Sigma^3$. Thus, we define the functional cup product $L^\overline{v}(\omega, \omega)$ to be $[z'] \in H^2(\Sigma^3)$ and the Hopf invariant $H_{(F,v)}$ of $(F,v)$ to be $-[z'] \in H^2(\Sigma^3) = \mathbb{Z}$, where we identify $H^2(\Sigma^3)$ with the integers $\mathbb{Z}$ with respect to the orientation of $\Sigma^3$. Note that $H_{(F,v)}$ does not depend on the choice of the orientation of $X$.

**Definition 3.3.** Let $F: \Sigma^3 \hookrightarrow S^6$ be an embedding of an oriented integral homology 3-sphere and $\bar{F}: V^4 \hookrightarrow S^6$ be its Seifert surface. Then, we define the Hopf invariant $H_{\bar{F}}$ for $\bar{F}$ to be the Hopf invariant of $\bar{F}$, the outward normal field of $F(\Sigma^3) \subset \bar{F}(V^4)$.

**Theorem 3.4.** Let $F: \Sigma^3 \hookrightarrow S^6$ be an embedding of an oriented integral homology 3-sphere and $\bar{F}: V^4 \hookrightarrow S^6$ be its Seifert surface. Then,

\[ H_{\bar{F}} = -e_{\bar{F}} \smile e_{\bar{F}} \in \mathbb{Z}, \]

where $e_{\bar{F}} \in H^2(V^4) = H^2(V^4, \partial V^4)$ is the normal Euler class of $\bar{F}$ and $e_{\bar{F}} \smile e_{\bar{F}}$ is its square evaluated on the fundamental homology class of $V^4$. 


Proof. We calculate the Hopf invariant by translating the cup product operation on cohomology into intersection theory on homology using the duality theorems.

Put $S := F(\Sigma^3)$ and denote by $S' \in S^6 \setminus S$ a small shift of $S$ along the outward normal field of $F(\Sigma^3) \subset F(V^4)$. Let $\tilde{v} : S' \hookrightarrow S^6 \setminus S$ be the inclusion and

$$\omega = [\omega'] \in H^2(S^6 \setminus S) = \mathbb{Z}$$

denote the generator (see 3.2).

Let $N, N'$ be sufficiently small (i.e. $N \cap N' = \emptyset$) tubular neighbourhoods of $S$, $S'$ respectively, and put $X := S^6 \setminus \text{Int}N$. Then, consider the following diagram

$$\begin{array}{ccc}
H^*(S^6 \setminus S) & \xrightarrow{\nu^*} & H^*(S') \\
\approx & & \approx \\
H^*(X) & \xrightarrow{\nu^*} & H^*(N'),
\end{array}$$

where all homomorphisms are induced by the inclusion maps and the vertical arrows are isomorphisms. By this diagram, we can calculate the functional cup product

$$L_\nu(\omega, \omega) = R_\nu(\omega, \omega) \in H^3(N') = \mathbb{Z}$$

with respect to the inclusion $\tilde{v} : N' \hookrightarrow X$, instead of $L_\nu(\omega, \omega) = R_\nu(\omega, \omega) \in H^3(S') = \mathbb{Z}$, where we identify $\omega$ with its inverse image (also denoted by $\omega$) under the isomorphism $H^2(S^6 \setminus S) \rightarrow H^2(X)$.

Furthermore, by the duality and the excision theorems, we have the following:

$$\begin{array}{ccc}
H^i(X) & \xrightarrow{\phi^*} & H^i(N') \\
\approx & & \approx \\
H_{6-i}(X, \partial X) & \xrightarrow{\gamma_i} & H_{6-i}(N', \partial N') \\
\approx & & \\
H_{6-i}(S^6, \Sigma),
\end{array}$$

where all vertical arrows are isomorphisms. Now using the above diagram, we calculate the desired Hopf invariant in terms of intersections on homology groups.

The class in $H_4(X, \partial X) \approx H_4(S^6, \Sigma)$ dual to the generator $\omega \in H^2(X)$ is represented by $V := F(V^4)$. Extend the outward normal field of $F(\Sigma^3) \subset F(V^4)$ to a vector field in $S^6$ and perturb $V$ by this field into $V'$. We can assume that the intersection $\Delta := V \cap V'$ of $V$ and $V'$ lies in their interior; $\Delta = \text{Int}V \cap \text{Int}V'$ (since the normal bundle of $F$ is trivial). Then, $\tilde{F}^{-1}(\Delta) \subset V^4$ represents the integral dual to the normal Euler class $e_F$ of $\tilde{F}$.

Since a sufficiently small tubular neighbourhood $N'$ of $S' \subset S^6$ does not intersect $V$, the relative 4-chain of $(N', \partial N')$ corresponding to $\tilde{v}^*\omega'$ in the second term of the right-hand side of 3.2 in Definition 3.1 vanishes.

Next we check the first term of the right-hand side in 3.2. Since the relative 2-chain dual to $\omega' \sim \omega'$ is represented by the intersection $\Delta := V \cap V'$, the term corresponding to $\tilde{v}^*\omega$ (in 3.2) is represented by the intersection of $S$ and a relative 3-chain bounded by $\Delta$. Therefore, the homology class dual to the functional cup product $L_\nu(\omega, \omega)$ is equal to

$$-(\text{a relative 3-chain bounded by } \Delta) \cap S = -\text{lk}(\Delta, S) = -[\Delta] \in H_2(X) = \mathbb{Z},$$

where the minus sign is due to the change of the order of the intersection. Thus, we obtain $H_{\tilde{F}} = [\Delta]$. 

5
If we take another copy $V'$ of $V = \tilde{F}(S^4)$ perturbed in an appropriate manner and put $\Delta' := V \cap V'$, then we have

$$\begin{align*}
|\Delta| &= |\Delta'| \quad (\in H_2(X) = \mathbb{Z}) \\
= &\operatorname{lk}(\Delta', S) \quad (\in H_0(S^6) = \mathbb{Z}) \\
= &-\operatorname{lk}(S, \Delta') \\
= &-[V \cap \Delta'] \\
= &-[\Delta \cap \Delta'] \quad (\in H_0(V) = H_0(V^4) = \mathbb{Z}) \\
= &-e_{\tilde{F}} \sim e_{\tilde{F}} \quad (\in H^4(\tilde{V}^4) = \mathbb{Z})
\end{align*}$$

This completes the proof.

\[\square\]

Remark 3.5. Theorem 3.4 is very similar to [24, Theorem 5.1], which however contained a sign error. To be precise, the error occurs in the last sentence of the proof of [24, Lemma 5.2]. It reads “Thus, we have $H_{\tilde{F}} = -[\tilde{F}(\Sigma^2)]$,” but should be “Thus, the desired functional product is computed to be equal to $-\tilde{F}(\Sigma^2)$ and hence the Hopf invariant $H_{\tilde{F}}$ is equal to $\tilde{F}(\Sigma^2) \in H_{2k}(X) = \mathbb{Z}$.” With this correction, each term involving the square of the normal Euler class, appearing in [24, Theorem 5.1, Corollaries 6.2, 6.3(a) and 6.5], should change its sign.

3.3. The case of embeddings of the 3-sphere. Haefliger [8] and [9, §5.16] proved that the group $C_3$ of isotopy classes of embeddings $S^3 \hookrightarrow S^6$ is isomorphic to the integers $\mathbb{Z}$. In fact, he gave a complete invariant $\Omega: C_3 \to \mathbb{Z}$ and also an explicit construction of an embedding representing a generator of $C_3$.

The following is given in [6, 24] (see also [6, 25]).

Theorem 3.6. Every embedding $F: S^3 \hookrightarrow S^6$ has a Seifert surface $\bar{F}: V^4 \hookrightarrow S^6$ and

$$\Omega(F) = -\frac{1}{8}(\sigma(V^4) + H_{\bar{F}})$$

$$= -\frac{1}{8}(\sigma(V^4) - e_{\tilde{F}} \sim e_{\tilde{F}})$$

gives the isomorphism $\Omega: C_3 \cong \mathbb{Z}$.

The following two examples of Seifert surfaces for the generator in $C_3$, given in [25, §2.3], will be important in our argument. Let $CP^2_\circ$ be the punctured complex projective plane and $\overline{CP^2_\circ}$ the punctured complex projective plane with the reversed orientation.

Proposition 3.7. The standard inclusion $S^3 \subset S^6$ has a Seifert surface $\bar{P}: CP^2_\circ \hookrightarrow S^6$ with Hopf invariant $\bar{H}_{\bar{P}} = -1$.

Proposition 3.8. The standard inclusion $S^3 \subset S^6$ has a Seifert surface $\bar{Q}: \overline{CP^2_\circ} \hookrightarrow S^6$ with Hopf invariant $\bar{H}_{\bar{Q}} = 1$.

Remark 3.9. Propositions 3.7 and 3.8 imply that by taking the boundary connected sums with a suitable number of copies of $\bar{P}$ or of $\bar{Q}$, we can alter a given Seifert surface for an embedding $F: S^3 \hookrightarrow S^6$ so that it has an arbitrarily preassigned Hopf invariant (without changing the isotopy class of $F$). This means further that for a given $F: S^3 \hookrightarrow S^6$ we can choose a Seifert surface so that near the boundary it extends $F$ in an arbitrarily prescribed normal direction, since the natural homomorphism $\pi_1(SO(3)) \to \pi_3(S^2)$ and the Hopf invariant $\pi_3(S^2) \to \mathbb{Z}$ are both isomorphisms (see [25]).

3.4. The additivity of the Hopf invariant. Let $(F_i, v_i): \Sigma^3_i \hookrightarrow S^6$ $(i = 1, 2)$ be embeddings with normal 1-fields of homology 3-spheres. Then, we can consider the connected sum $(F_1, v_1)^\# (F_2, v_2)$ in a natural way (see Figure 2) and we have the following.
Proposition 3.10. Let \((F_i, \nu_i) : \Sigma^3 \hookrightarrow S^6 \ (i = 1, 2)\) be two embeddings with normal 1-fields of homology 3-spheres. Then,

\[ H_{(F_1, \nu_1) \sharp (F_2, \nu_2)} = H_{(F_1, \nu_1)} + H_{(F_2, \nu_2)} \in \mathbb{Z}. \]

To prove Proposition 3.10 we use the following lemma, which will be also needed later.

Lemma 3.11. Let \(F : \Sigma^3 \hookrightarrow S^6\) be an embedding of a homology 3-sphere \(\Sigma^3\) and \(\nu\) be a normal 1-field of \(F(\Sigma^3) \subset S^6\). Then, there exists a Seifert surface \(\tilde{F} : W^4 \hookrightarrow S^6\) for \(F\) such that the outward normal field of \(F(\Sigma^3) \subset \tilde{F}(W^4)\) coincides with \(\nu\).

Proof. Let \(\tilde{F}' : W' \hookrightarrow S^6\) be a Seifert surface for \(F\) by Proposition 2.5 and denote by \(\nu'\) the outward normal vector field of \(F(\Sigma^3) \subset \tilde{F}'(W')\). Then, \(\nu'\) and the given \(\nu\) are homotopic on \(\Sigma_o = \Sigma^3 \setminus \text{Int}D^3\), since \(H^2(\Sigma^3) = 0\) (cf. [7, Corollary 4.9]). We can resolve the difference between \(\nu'\) and \(\nu\) on the final 3-cell by taking the boundary connected sum of \(\tilde{F}'\) with a suitable number of copies of \(\tilde{P}\) or of \(\tilde{Q}\) (in Propositions 3.7 and 3.8). Thus, we have a new Seifert surface for \(F\) for which the outward vector field along the boundary is homotopic to \(\nu\). After composing a diffeomorphism of \(S^6\) if necessary, we obtain a desired Seifert surface.

Remark 3.12. In view of Remark 2.6 the Seifert surface in Lemma 3.11 can always be assumed to be simply connected.

Now Proposition 3.10 is an easy corollary of Theorem 3.4 and Lemma 3.11.

Proof of Proposition 3.10. By Lemma 3.11 for each \((F_i, \nu_i) : \Sigma^3 \hookrightarrow S^6\) we can consider a Seifert surface \(\tilde{F}_i : W_i^4 \hookrightarrow S^6\) so that the outward normal vector field of \(F_i(\Sigma_i) \subset \tilde{F}_i(W_i^4)\) coincides with \(\nu_i\).

Then, by Theorem 3.4

\[ H_{(F_1, \nu_1) \sharp (F_2, \nu_2)} = -\varepsilon_{\tilde{F}_1 \sharp \tilde{F}_2} - \varepsilon_{\tilde{F}_1 \sharp \tilde{F}_2}, \]

where \(\varepsilon_{\tilde{F}_1 \sharp \tilde{F}_2}\) is the normal Euler class of \(\tilde{F}_1 \sharp \tilde{F}_2\). The homology class dual to \(\varepsilon_{\tilde{F}_1 \sharp \tilde{F}_2}\) is represented by the intersection \(\Delta\) of \((\tilde{F}_1^i \sharp \tilde{F}_2^i)(W_1^4 \sharp W_2^4)\) with its small perturbation in \(S^6\).

Since \(\tilde{F}_i \sharp \tilde{F}_2\) is an embedding and \(F_i \sharp F_2\) has trivial normal bundle, \(\Delta\) can be assumed to be the disjoint union \(\Delta_1 \sqcup \Delta_2\) of \(\Delta_i \subset \tilde{F}_i(\text{Int}W_i^4)\) dual to \(\varepsilon_{\tilde{F}_i}(i = 1, 2)\). Thus,

\[ H_{(F_1, \nu_1) \sharp (F_2, \nu_2)} \equiv -[\Delta] \cdot [\Delta] \equiv -[\Delta_1] \cdot [\Delta_1] - [\Delta_2] \cdot [\Delta_2] \equiv -\varepsilon_{\tilde{F}_1} - \varepsilon_{\tilde{F}_2} = H_{\tilde{F}_1} + H_{\tilde{F}_2} = H_{(F_1, \nu_1)} + H_{(F_2, \nu_2)}, \]

where \(\cdot\) means the intersection pairing of homology.

Furthermore, we have the following.

Corollary 3.13. Let \(F : \Sigma^3 \hookrightarrow S^6\) be an embedding of a homology 3-sphere \(\Sigma^3\). Then, two normal 1-fields \(\nu_1\) and \(\nu_2\) of \(F(\Sigma^3) \subset S^6\) are homotopic if and only if \(H_{(F, \nu_1)} = H_{(F, \nu_2)}\).
Proof. Since $H^2(\Sigma^3; \pi_2(S^2)) = 0$, the 1-fields $v_1$ and $v_2$ are homotopic on $\Sigma^3 \setminus \text{Int}D^3$. Therefore, their first difference may be in $H^3(\Sigma^3; \pi_3(S^2)) = \mathbb{Z}$ and we can assume that $(F, v_2) = (F, v_1) \sharp (j, \nu)$ for the standard inclusion $j : S^3 \hookrightarrow S^6$ and a normal 1-field $\nu$ of $j(S^3) \subset S^6$. Then, $v_1$ and $v_2$ are homotopic if and only if $\nu$ is standard, i.e., is homotopic to the first vector field to the standard normal framing of the standard inclusion $j$. Since the Hopf invariant $\pi_3(S^2) \to \mathbb{Z}$ is an isomorphism, we have further
\[ v_1 \text{ and } v_2 \text{ are homotopic } \iff \nu \text{ is standard} \]
\[ \iff H_{(j, \nu)} = 0 \]
\[ \iff H_{(F, v_2)} = H_{(F, v_1)} + H_{(j, \nu)} = H_{(F, v_1)}. \]
This completes the proof. \qed

4. The Invariant

In the section, we prove the analogue of Theorem 3.6 for embeddings of an integral homology sphere in $S^6$.

First, we prove the following.

Proposition 4.1. Let $F : \Sigma^3 \hookrightarrow S^6$ be an embedding of an oriented integral homology 3-sphere and $\bar{F} : \bar{W}^4 \hookrightarrow S^6$ be its Seifert surface. Then, the integer
\[ \Omega(F) := -\frac{1}{8}(\sigma(W^4) + H_{\bar{F}}) \]
\[ = -\frac{1}{8}(\sigma(W^4) - e_{\bar{F}} \sim e_{\bar{F}}). \]
does not depend on the choice of $\bar{F}$ and depends only on the isotopy class of $F$.

Proof. First, notice that since the modulo two reduction of $e_{\bar{F}}$ equals the second Stiefel-Whitney class $w_2(W^4)$ of $W^4$, the integer $\sigma(W^4) - e_{\bar{F}} \sim e_{\bar{F}}$, and hence $\sigma(W^4) + H_{\bar{F}}$ (by Theorem 3.4) are divisible by 8.

Consider two mutually isotopic embeddings $F_i : \Sigma^3 \hookrightarrow S^6$ and their Seifert surfaces $\bar{F}_i : W^4 \hookrightarrow S^6$ ($i = 1, 2$). Then, there exists a diffeomorphism $h$ on $S^6$ such that $F_1 = h \circ F_2$. We abuse the symbols $F_2$ and $\bar{F}_2$ to denote respectively $h \circ F_2(= F_1)$ and $h \circ \bar{F}_2$.

Put $k := H_{\bar{F}_1} - H_{\bar{F}_2}$ and consider the boundary connected sum $\bar{F}_1 \sharp k\bar{P}$, where $k\bar{P}$ means
\[ \begin{cases} 
\text{the standard inclusion} & \text{if } k = 0, \\
\text{the boundary connected sum of } k \text{ copies of } \bar{P} & \text{if } k > 0, \\
\text{the boundary connected sum of } |k| \text{ copies of } \bar{Q} & \text{if } k < 0
\end{cases} \]
(see Propositions 3.7 and 3.8). Then, $\bar{F}_1 \sharp k\bar{P}$ is still a Seifert surface for $F_1$ and has the Hopf invariant $H_{\bar{F}_1 \sharp k\bar{P}} = H_{\bar{F}_1} - k = H_{\bar{F}_2}$ by Proposition 3.10. This means that the outward normal fields along the boundaries of $\bar{F}_1 \sharp k\bar{P}$ and of $\bar{F}_2$ are homotopic by Corollary 3.13.

Since a map from $\Sigma^3$ to $S^2$ and its composition with the antipodal map on $S^2$ are homotopic (see [3,2]), Corollary 3.13 implies that we can isotope $\bar{F}_1 \sharp k\bar{P}$ so that near the boundary it is prolonged in the outward normal direction of $F_2(\Sigma^3) \subset F_2(W^4)$.

Thus, by using $\bar{F}_1 \sharp k\bar{P}$ and $\bar{F}_2$, we obtain an immersion
\[ (\bar{F}_1 \sharp k\bar{P}) \cup F_2 : (W^4 \sharp kCP^2_o) \cup_{\partial} (-W^4) \hookrightarrow S^6, \]
where $(W^4 \sharp kCP^2_o) \cup_{\partial} (-W^4)$ denotes the closed 4-manifold obtained by gluing $W^4 \sharp kCP^2_o$ and $W^4$ (with the reversed orientation) via the orientation-reversing diffeomorphism on the common boundary $\Sigma^3$.

Clearly, this immersion $(\bar{F}_1 \sharp k\bar{P}) \cup F_2$ has no triple points, since its multiple points consist only of the intersection of the two embeddings $\bar{F}_1 \sharp k\bar{P}$ and $\bar{F}_2$. Hence, by (27) (see also
Then, we can characterise compressible embeddings of the 3-sphere:\footnote{p.44)}
\begin{align*}
0 &= \text{the algebraic number of triple points} \\
&= \sigma((W_4^4 \natural kCP^2_o) \cup \partial (-W_4^4)) \\
&= \sigma(W_4^4) + k\sigma(CP^2_o) - \sigma(W_4^4) \\
&= \sigma(W_4^4) - \sigma(W_4^4) + k \\
&= (\sigma(W_4^4) + H_{F_1}) - (\sigma(W_4^4) + H_{F_2}).
\end{align*}
This implies that for an embedding \( F: \Sigma^3 \hookrightarrow S^6, \Omega(F) \) does not depend on the choice of its Seifert surface and clearly is invariant up to isotopy. \qed

The following Theorem 4.2 shows that the invariant \( \Omega \) is a complete invariant up to isotopy. Let \( \text{Emb}(\Sigma^3, S^6) \) be the set of isotopy classes of embeddings of an oriented integral homology 3-sphere \( \Sigma^3 \) in \( S^6 \).

**Theorem 4.2.** Let \( F: \Sigma^3 \hookrightarrow S^6 \) be an embedding of an oriented integral homology 3-sphere and \( \tilde{F}: V^4 \hookrightarrow S^6 \) be its Seifert surface. Then, \( \Omega \) gives a bijection
\[
\Omega: \text{Emb}(\Sigma^3, S^6) \xrightarrow{\cong} \mathbb{Z}.
\]

**Proof.** Since the signature and the Hopf invariant are both additive under the boundary connected sum of Seifert surfaces, the surjectivity of \( \Omega \) follows from the fact that in the case of embeddings of the 3-sphere \( S^3 \), the same formula gives the bijection \( \Omega: C^3 \rightarrow \mathbb{Z} \) (see Theorem 3.6).

We prove injectivity. Let \( F_i: \Sigma^3 \hookrightarrow S^6 \) \((i = 0, 1)\) be two embeddings. By an argument similar to that of Proposition 3.4, \( F_1 \) is isotopic to \( F_0 \sharp g \) for some embedding \( g: S^3 \hookrightarrow S^6 \). Clearly, if \( F_0 \) and \( F_1 \) are not isotopic to each other, then \( g \) is not isotopic to the standard embeddings (see \cite{2} Lemma 1.3); \( \Omega(g) \neq 0 \). Hence, \( \Omega(F_1) = \Omega(F_1 \sharp g) = \Omega(F_0) + \Omega(g) \neq \Omega(F_0) \). \qed

**Remark 4.3.** Hausmann \cite{10} stated (with a short outline of proof) that there is a bijective correspondence between \( \text{Emb}(\Sigma^3, S^6) \) and \( C^3 \).

Now recall that the Rohlin invariant \( \mu(\Sigma^3) \) of an integral homology 3-sphere \( \Sigma^3 \) is defined, by choosing a smooth compact oriented spin 4-manifold \( W^4 \) with \( \partial W^4 = \Sigma^3 \), to be
\[
\mu(\Sigma^3) := \sigma(W^4)/8 \mod 2.
\]
Then, we can characterise compressible embeddings \( \Sigma^3 \hookrightarrow S^6 \) in the relation of our invariant \( \Omega \) and the Rohlin invariant. Here, we say that an embedding \( F: \Sigma^3 \hookrightarrow S^6 \) is compressible in \( S^6 \) if it is isotopic to an embedding in \( S^6 \) composed with the inclusion \( j: S^5 \subset S^6 \).

**Corollary 4.4.** An embedding \( F: \Sigma^3 \hookrightarrow S^6 \) of an integral homology 3-sphere \( \Sigma^3 \) in \( S^6 \) is compressible into \( S^6 \) if and only if \( \Omega(F) \equiv \mu(\Sigma^3) \) (mod 2).

**Proof.** Assume that \( F: \Sigma^3 \hookrightarrow S^6 \) is isotopic to an embedding \( f: \Sigma^3 \hookrightarrow S^6 \). Let \( \tilde{f}: V^4 \hookrightarrow S^6 \) be a Seifert surface for \( f \). Then, \( \tilde{f} \) has trivial normal bundle and \( V^4 \) is necessarily a spin manifold (since it is embedded in codimension 1). Therefore, we have \( \Omega(F) = -(\sigma(V^4) - 0)/8 \), which is modulo 2 equal to the Rohlin invariant \( \mu(\Sigma^3) \).

Conversely, assume that \( \Omega(F) \equiv \mu(\Sigma^3) \) (mod 2). By the same argument of Proposition 3.4, \( F \) is isotopic to \( j \circ (f_0 \natural g) \) for an embedding \( f_0: W_0^4 \hookrightarrow S^6 \) of a spin 4-manifold \( W_0^4 \) with \( \partial W_0^4 = \Sigma^3 \) and an embedding \( g: \Sigma^3 \hookrightarrow S^6 \) of the 3-sphere. Then, we have
\[
\Omega(F) = \Omega(j \circ f_0 \natural g) + \Omega(g) \\
\equiv \mu(\Sigma^3) + \Omega(g) \mod 2.
\]
Thus, we have $\Omega(g) \equiv 0 \pmod{2}$, which implies by [9] Theorem 5.17 (see also [24] Theorem 7.1) that $g: S^3 \hookrightarrow S^6$ can be isotoped to an embedding $g'$ of $S^3$ into $S^6$. Hence, $F$ is isotopic to $\tilde{f}_0 \circ \tilde{g}' : \Sigma^3 \hookrightarrow S^6$ (composed with the inclusion $f$).

In view of the generalisation of Rohlin’s theorem by Guillou and Marin [6] (see also [18]), the following is a direct consequence of Corollary 4.4.

**Corollary 4.5.** Let $\tilde{F} : V^4 \hookrightarrow S^6$ be an embedding of a compact orientable 4-manifold $V^4$ whose boundary $\partial V^4$ is diffeomorphic to an integral homology 3-sphere and let $\Delta \subset V^4$ be an oriented surface dual to the normal Euler class of $\tilde{F}$. Then, $\tilde{F}|_{\partial V^4} : \partial V^4 \hookrightarrow S^6$ is compressible into $S^5$ if and only if Arf($V^4, \Delta$) $\equiv 0 \pmod{2}$.

### 5. The homology bordism group

In this section, we show that the invariant $\Omega$ gives a homomorphism from the homology bordism group of embeddings of integral homology 3-spheres in $S^6$.

We say that two oriented integral homology 3-spheres $\Sigma^3_0$ and $\Sigma^3_1$ are homology cobordant if there exists a smooth compact oriented 4-manifold $W^4$ with $\partial W^4 = \Sigma^3_0 \cup (-\Sigma^3_1)$ (the disjoint union) such that the inclusions induce isomorphisms $H_i(\Sigma^3_0) \to H_i(W^4)$ for $i = 0, 1$. A such $W^4$ is called a homology cobordism between $\Sigma^3_0$ and $\Sigma^3_1$. All homology cobordism classes of oriented integral homology 3-spheres form an abelian group $\Theta_Z^3$ with the group operation induced by the connected sum. The group $\Theta_Z^3$ is infinite [3] and is infinitely generated [3].

For embeddings of oriented integral homology 3-spheres, we can naturally consider the notion of homology bordism (see [24]).

**Definition 5.1.** Let $F_i (i = 0, 1)$ be embeddings of oriented integral homology 3-spheres $\Sigma^3_i$ in $S^6$ (or in $R^6$). Then, $F_0$ and $F_1$ are said to be homology bordant if there exist a homology cobordism $W^4$ between $\Sigma^3_0$ and $\Sigma^3_1$ and a proper embedding $H$ of $W^4$ in $S^6 \times [0, 1]$ (or in $R^6 \times [0, 1]$) such that the embedding $H|_{\Sigma^3_i}$ of $\Sigma^3_i$ in $S^6 \times \{i\} = S^6$ (or in $R^6 \times \{i\} = R^6$) coincides with $F_i$ for $i = 0, 1$. Such an $H$ is called a homology bordism between $F_0$ and $F_1$.

Now we prove that our invariant given in Proposition 4.1 (and in Theorem 4.2) turns out to be invariant up to homology bordism.

**Proposition 5.2.** If two embeddings $F_i : \Sigma^3_i \hookrightarrow S^6$ of (distinct) homology 3-spheres $(i = 0, 1)$ are homology bordant, then $\Omega(F_0) = \Omega(F_1)$.

**Proof.** We consider each embedding $F_i$ as an embedding $F_i : \Sigma^3_i \hookrightarrow R^6$ in $R^6$ and let $H : (W^4, \partial W^4) \hookrightarrow (R^6 \times [0, 1], R^6 \times \{0\} \sqcup R^6 \times \{1\})$ be a homology bordism between $F_0$ and $F_1$. We can assume that $H$ is perpendicula to $R^6 \times \{0\} \sqcup R^6 \times \{1\}$ along the boundary $\partial W^4$. Then, since $H^4(W^4) = H^2(W^4; \pi_2(\text{SO}(3))) = H^2(W^4) = H^1(W^4) = 0$, the normal bundle of $H$ is trivial and a normal framing $v = (v_1, v_3)$ for $H(W^4) \subset R^6 \times [0, 1]$ determines normal framings for $F_i(\Sigma^3_i) \subset R^6 \times \{i\} (i = 0, 1)$.

According to Lemma 3.11 for each $i \in \{0, 1\}$, consider a Seifert surface $\tilde{F}_i : \Sigma^3_i \hookrightarrow R^6$ for $F_i$ such that the outward normal vector field of $F(\Sigma^3_i) \subset \tilde{F}_i(\Sigma^3_i)$ coincides with the first vector field $v_1$ of $v$.

Take collar neighbourhoods $C_i := \Sigma^3_i \times [0, \varepsilon]$ of the boundary $\partial V^4_i = \Sigma^3_i$. Then, by using $\tilde{F}_0, \tilde{F}_1, H$ and an appropriate “smoothing function” $h : [0, \varepsilon] \to [0, \varepsilon]$ as in Figure 3, we can define a smooth embedding

$$G : V^4 \cup W^4 \cup (-V^4) \hookrightarrow R^6 \times [-\varepsilon, 1 + \varepsilon] \subset R^7$$
of the closed 4-manifold $M^4 := V^4_0 \cup W^4 \cup (-V^4_0)$ by

$$G(x) = \begin{cases} (\tilde{F}_0(x), -\varepsilon) & \text{if } x \in V^4_0 \setminus C_0 \\ (\tilde{F}_0(x), -h(t)) & \text{if } (x, t) \in \Sigma^3 \times [0, \varepsilon] = C_0 \\ H(x) & \text{if } x \in W^4 \\ (\tilde{F}_1(x), 1 + \varepsilon) & \text{if } x \in V^4_1 \setminus C_1. \end{cases}$$

Furthermore, with $\theta_i \in [0, \pi/2]$ such that $h'(t) = \tan \theta_i$, we can define a non-zero normal vector field on $G(M^4) \subset \mathbb{R}^7 = \{(y_1, y_2, \cdots, y_7)\}$ as:

$$\begin{cases} -\partial / \partial y_7 & \text{on } G(V^4_0 \setminus C_0) \\ -\cos \theta_i (\partial / \partial y_7) + \sin \theta_i \cdot v_1 & \text{on } G(\ast, t) \text{ where } (\ast, t) \in \Sigma^3 \times [0, \varepsilon] = C_0 \\ v_1 & \text{on } W^4 \\ \cos \theta_i (\partial / \partial y_7) + \sin \theta_i \cdot v_1 & \text{on } G(\ast, t) \text{ where } (\ast, t) \in \Sigma^3 \times [0, \varepsilon] = C_1 \\ \partial / \partial y_7 & \text{on } G(V^4_1 \setminus C_1). \end{cases}$$

Denote the push-off of $G(M^4)$ along the normal field $n$ by $G(M^4)'$. Then, for any $c = (c_1, c_2) \in H_2(M^4) \approx H_2(V_0) \oplus H_2(-V_1)$, the linking number $\text{lk}(c, G(M^4)')$ in $\mathbb{R}^7$ is equal to $\text{lk}(c_1, G(M^4)') + \text{lk}(c_2, G(M^4)')$, which vanishes since $c_0$ and $c_1$ bound 3-chains respectively in $\mathbb{R}^6 \times \{-\varepsilon\}$ and in $\mathbb{R}^6 \times \{1 + \varepsilon\}$. Therefore, the normal vector field $n$ is unlinked (**non enlace**) in the sense of Boéchat and Haefliger [2, Definition 1.1].

Now, we want to apply Boéchat and Haefliger’s argument [2] to the embedding $G$ with the normal field $n$. Let $e$ be the Euler class of the 2-plane bundle complementary to $n$ in the normal bundle of $G(M^4) \subset \mathbb{R}^7$. Then, by [2, Théorème 2.1], we have

$$\sigma(M^4) - e \sim e = 0.$$

Since $\sigma(M^4) = \sigma(V_0) - \sigma(V_1)$ and $e = e_{F_0} - e_{F_1} \in H_2(M^4) \approx H_2(V_0) \oplus H_2(-V_1)$,

$$0 = (\sigma(V_0) - \sigma(V_1)) - (e_{F_0} \sim e_{F_0} - e_{F_1} \sim e_{F_1}).$$

Thus we have $\sigma(V_0) - e_{F_0} \sim e_{F_0} = \sigma(V_1) - e_{F_1} \sim e_{F_1}$. This completes the proof. □

**Remark 5.3.** In the above proof of Proposition 5.2, the Seifert surfaces $F_i : V^4_i \to \mathbb{R}^6$ ($i = 0, 1$) are chosen in such a way that the normal field $n$ is actually non-zero on the whole $G(M^4)$, so that we actually have also $3\sigma(M^4) + e \sim e = 0$, and hence $\sigma(M^4) = e \sim e = 0$ (see [6, p94]).

All homology bordism classes of embeddings of oriented integral homology 3-spheres in $S^6$ form an abelian group, denoted by $\Gamma^3_3$, with group structure via connected sum. Then, Proposition 5.2 implies that $\Omega$ induces the homomorphism $\Omega : \Gamma^3_3 \to \mathbb{Z}$, since Hopf invariants and signatures of Seifert surfaces are additive under boundary connected sum (by Proposition 3.10 and by Novikov additivity). Furthermore, the invariant $\Omega$ leads to the following characterisation of $\Gamma^3_3$. 

\[\text{Figure 3. “Smoothing function } h'\text{”}\]
Theorem 5.4. The following is an isomorphism:
\[
\Omega: \Gamma^3 \to \bigoplus \mathbb{Z}
\]
\[
[F: \Sigma^3 \to S^6] \mapsto [\Sigma^3, \Omega(F)]
\]
where \([\Sigma^3]\) is the homology cobordism class represented by \(\Sigma^3\) and \([F: \Sigma^3 \to S^6]\) is the homology bordism class represented by \(F: \Sigma^3 \to S^6\).

Proof. Since any integral homology 3-sphere can be embedded in \(S^6\), the surjectivity follows from Theorem 5.4 and the additivity of \(\Omega\) under connected sum.

Now, let \(\Sigma^3\) be an integral homology sphere homology cobordant to zero (i.e., bounding an bounding acyclic 4-manifold \(V^4\)) and \(F: \Sigma^3 \to S^6\) be an embedding with \(\Omega(F) = 0\). Then, \(V^4\) can be embedded in \(S^6\); let \(f: V^4 \to S^5\) be an embedding. Clearly, \(\Omega((j \circ f)|_{\partial V^4}) = -\sigma(V^4)/8 = 0\), where \(j: S^5 \subset S^6\) is the inclusion. Therefore, by Theorem 4.2, \(F\) is isotopic to \((j \circ f)|_{\partial V^4}\) and hence is homology bordant to zero. Since \(\Omega\) is clearly a homomorphism, it should be an isomorphism. \(\square\)

Corollary 5.5. Two embeddings of an oriented integral homology 3-sphere in \(S^6\) are isotopic if and only if they are homology bordant.

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