The Use of Geometric Quantities in the Tensor Description of a Euclidean Space

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Abstract

We present a tensor description of Euclidean spaces that emphasizes the use of geometric vectors. We demonstrate the effectiveness of the approach by proving a number of integral identities with vector integrands. Visit grinfeld.org for the latest version.

1 Introduction

Since the invention of coordinate systems in the middle of the 17th century, the subject of Geometry has followed the steady path of algebraization. This should not surprise us: the very idea of a coordinate system is to replace geometric objects with the coordinates of the constituent points, thus opening the problem up to algebraic – and, since the invention of Calculus, analytical – methods. This gives the method of coordinates a distinct advantage over its geometric peers: while geometric arguments typically require a unique insight into the problem, and therefore a certain degree of ingenuity, algebraic and analytical methods tend towards universality and robustness. The advent of computing has further cemented this advantage.

On the other hand, the use of coordinates comes with great costs. Chief among them is the loss of geometric insight. This pitfall is exemplified by the results of Leonhard Euler and Louis Lagrange in their foundational works on the Calculus of Variations. In 1744, in his search for a minimal shape of revolution, Euler introduced what we would now call a cylindrical coordinate system and described the profile of the minimal surface by an unknown function $r(z)$, as
illustrated in the following figure.

By making arguments based on manipulating geometric elements, Euler demonstrated that \( r(z) \) must satisfy the equation

\[
r''(z) r(z) - r'(z)^2 - 1 = 0. \tag{2}
\]

Euler then promptly solved this equation to reveal that

\[
r(z) = a \cosh \frac{z}{a}. \tag{3}
\]

In other words, the profile is a catenary, where the constant \( a \) represents the closest distance between the surface and the axis of revolution.

In 1755, the nineteen-year-old Lagrange took an even more unapologetically coordinate approach to the problem of minimal surfaces and represented the unknown surface by the graph of a function

\[
z = F(x, y) \tag{4}
\]

in Cartesian coordinates \( x, y, z \).

Reasoning analytically, Lagrange demonstrated that \( z(x, y) \) must satisfy the partial differential equation

\[
\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial x^2} \left( \frac{\partial F}{\partial y} \right)^2 + \frac{\partial^2 F}{\partial y^2} \left( \frac{\partial F}{\partial x} \right)^2 - 2 \frac{\partial F}{\partial x} \frac{\partial F}{\partial y} \frac{\partial^2 F}{\partial x \partial y} = 0. \tag{6}
\]
These works of Euler and Lagrange have been rightfully revered by later mathematicians for the seminal nature of the techniques developed in them and for the flight of creativity that their development required. However, to us, as we contemplate the advantages and disadvantages of coordinate systems, these works offer an additional insight that speaks to the potential loss of geometric insight that comes with the use of coordinates. It appears that neither Euler nor Lagrange found the geometric interpretation for their equations. It was only in 1774 that the French mathematician Jean Baptiste Meusnier discovered that a minimal surface is characterized by zero mean curvature. Critically, the loss of geometric insight is not some aesthetic lament: geometric insight, after all, serves as a guide for organizing analytical expressions into meaningful combinations. As a result, with the loss of geometric insight, we lose control over our calculations. The algebraic complexity in the analysis grows rapidly with every step until we are forced to retreat in the face of computational difficulties.

When it comes to combatting the loss of geometric insight, two distinct approaches have been developed: the tensor calculus approach and the dyadic approach associated with modern differential geometry. In tensor calculus, a geometric problem is analyzed by introducing a coordinate system and replacing all geometric objects by their components. Tensor calculus preserves the geometric insight by developing a framework of invariance that dictates precise rules for combing analytical expressions into geometrically meaningful combinations. By contrast, the dyadic approach eschews components altogether and operates only in terms of invariant objects and operators. Naturally, both approaches have their uses and misuses, their advantages and disadvantages, and their adherents and detractors. And, as always, the truth is that elements of both approaches are essential and the two schools of thoughts are complementary rather than in conflict.

The goal of this paper is to describe a vector tensor calculus, i.e. a particular style of tensor treatment of a Euclidean space that combines elements of both the tensor and dyadic approaches by emphasizing the use of geometric vectors. Some elements of this approach can be found in V.F. Kagan’s Foundations of the Theory of Surfaces in Tensor Terms [4]. However, Kagan typically uses geometric vector quantities only at the outset of any particular discussion only to abandon them in favor of working with scalar quantities. This is understandable – scalar quantities remain meaningful in the generalization to Riemannian spaces and are therefore more robust than geometric vector quantities in this sense. We take the analysis of vector quantities much further and discover their tremendous utility in simplifying the description of Euclidean spaces as well as revealing new and insightful relationships.

Another noteworthy aspect of Kagan’s textbook that deserves to be mentioned is its penchant for omitting technical details in favor of transparency when communicating essential ideas. Interesting, and relevant to the goals of this paper, is the fact that Kagan’s approach did not sit right with many of his contemporaries. In an otherwise positive review [1], A.D. Alexandrov criticized Kagan’s lack of formalism:

...Other shortcomings of this book have to do with the prevalence of the ten-
sor framework. They manifest themselves in insufficient attention to the precise
definitions of concepts and to the specification of assumptions required for cor-
rectness of theorems.

It is not my goal to criticize the work of V.F. Kagan. Such deficiencies are
characteristic of an entire direction in differential geometry and can be found in
the majority of books devoted to this field. They have become a matter of style
that I find anachronistic, as our present notion of rigor is different from that
of, say, the middle of the nineteenth century.

With Alexandrov’s remarks duly noted and in the spirit of Kagan’s classic,
we too will favor clarity over rigor. We will generally assume that all surfaces
are infinitely smooth. The key takeaway will be that an emphasis on geometric
vectors provides greater insight into the structure of Euclidean spaces, offers
more elegant demonstrations of known results, and opens doors to new results.
For the more standard approach to the tensor description of Euclidean spaces,
see the classical textbooks [5], [6], [7].

2 Summary of demonstrated identities

As an illustration of the effectiveness of the proposed approach, we will demon-
strate a family of integral relationships for a smooth closed hypersurface \( S \)
with unit normal \( N \), mean curvature \( B_\alpha \), and Gaussian curvature \( K \) in an
\( n \)-dimensional Euclidean space. Naturally, the results involving the Gaussian
curvature are limited to \( n = 3 \). For higher-dimensional spaces, the scalar cur-
vature \( R \) takes the place of the Gaussian curvature \( K \).

First, we will prove the well-known fact that the surface integral of the unit
normal vanishes, i.e.
\[
\int_S N dS = 0.
\] (7)

Similarly, we will show that the surface integral of the combination \( NB_\alpha \)
known as the curvature normal, also vanishes, i.e.
\[
\int_S NB_\alpha dS = 0.
\] (8)

Finally, we will demonstrate that the surface integral of the combination \( NK \)
vanishes as well, i.e.
\[
\int_S NK dS = 0.
\] (9)

Let \( R \) be the position vector emanating from an arbitrary origin \( O \). For any
vector quantity \( U \) whose surface integral vanishes, i.e.
\[
\int_S U dS = 0,
\] (10)

it is natural to inquire as to the value of the integral
\[
\int_S R \cdot U dS
\] (11)
since it is independent of the arbitrary origin $O$. After all, for
\[ R' = R + d, \]
we have
\[
\int_S R' \cdot UdS = \int_S (R + d) \cdot UdS \tag{13}
\]
\[
= \int_S R \cdot UdS + d \cdot \int_S UdS \tag{14}
\]
\[
= \int_S R \cdot UdS. \tag{15}
\]
Independence from $O$ suggests that the integral
\[
\int_S R \cdot UdS \tag{16}
\]
represents a geometric characteristic of the surface $S$. Indeed, for each of the vector fields $N$, $NB_\alpha$, and $NK$, the surface integral of the dot product with the position vector $R$ yields an interested geometric quantity. Namely,
\[
\int_S R \cdot NdS = nV \tag{17}
\]
\[
\int_S R \cdot NB_\alpha dS = -(n - 1)A \tag{18}
\]
\[
\int_S R \cdot NKdS = -\frac{1}{2} \int_S B_\alpha dS, \tag{19}
\]
where $V$ is the volume of enclosed domain and $A$ is the surface area of $S$.

Note that the same logic applies to the integral
\[
\int_S R \times UdS \tag{20}
\]
and we would find that
\[
\int_S R \times NdS = \int_S R \times NB_\alpha dS = \int_S R \times NKdS = 0. \tag{21}
\]

3 A tensor description of a Euclidean space

In this Section, we will describe the fundamental tensor objects in a Euclidean space. As we have already mentioned above, the distinguishing characteristic of our description is its emphasis on geometric quantities. Since it is not possible to present a full account in the limited space, we will only give the definitions of the key objects, state their fundamental properties, and list the essential identities relating those objects. A detailed description of the presented approach can be found in [3].
Refer the ambient Euclidean space to arbitrary curvilinear coordinates \( Z^1, Z^2, Z^3 \)
or, collectively, \( Z^i \), and treat the position vector \( \mathbf{R} \) as a function of \( Z^i \), i.e.

\[
\mathbf{R} = \mathbf{R}(Z),
\]

where the shorthand symbol \( \mathbf{R} \) represents the function \( \mathbf{R}(Z^1, Z^2, Z^3) \). Then \( \textit{covariant basis} Z_i \), the \( \textit{contravariant basis} Z^i \), the \( \textit{covariant metric tensor} \ Z_{ij} \), and the \( \textit{contravariant metric tensor} \ Z^{ij} \) are given by the identities

\[
Z_i = \frac{\partial \mathbf{R}(Z)}{\partial Z^i} \quad (23)
\]

\[
Z^i \cdot Z_j = \delta^i_j \quad (24)
\]

\[
Z_{ij} = Z_i \cdot Z_j \quad (25)
\]

\[
Z^{ij} Z_{jk} = \delta^i_k \quad (26)
\]

where \( \delta^i_k \) is the familiar \textit{Kronecker delta symbol}.

Suppose that the surface \( S \) is referred to the surface coordinates \( S^1, S^2 \) or,collectively \( S^\alpha \), and treat the surface restriction of the position vector \( \mathbf{R} \) as a function of \( S^\alpha \), i.e.

\[
\mathbf{R} = \mathbf{R}(S). \quad (27)
\]

Then \( \textit{covariant basis} S_\alpha \), the \( \textit{contravariant basis} S^\alpha \), the \( \textit{covariant metric tensor} \ S_{\alpha\beta} \), and the \( \textit{contravariant metric tensor} \ S^{\alpha\beta} \) are given by the identities

\[
S_\alpha = \frac{\partial \mathbf{R}(S)}{\partial S^\alpha} \quad (28)
\]

\[
S^\alpha \cdot S^\beta = \delta^\alpha_\beta \quad (29)
\]

\[
S_{\alpha\beta} = S_\alpha \cdot S^\beta \quad (30)
\]

\[
S^{\alpha\beta} S^{\beta\gamma} = \delta^\alpha_\gamma \quad (31)
\]

The components \( U^i \) of a vector \( \mathbf{U} \) are given by the dot product of \( \mathbf{U} \) with the contravariant basis \( Z^i \), i.e.

\[
U^i = Z^i \cdot \mathbf{U}. \quad (32)
\]

Similarly, the surface components \( U^\alpha \) of a vector \( \mathbf{U} \) is the plane tangential to surface \( S \) are given by the dot product of \( \mathbf{U} \) with the surface contravariant basis \( S^\alpha \), i.e.

\[
U^\alpha = S^\alpha \cdot \mathbf{U}. \quad (33)
\]

The shift tensor \( Z^i_\alpha \) represents the ambient coordinates of the surface covariant basis \( S_\alpha \), i.e.

\[
Z^i_\alpha = Z^i \cdot S_\alpha. \quad (34)
\]

The shift tensor is a critical object in the traditional approach to tensor Calculus, but will not figure in our analysis.

For a two-dimensional hypersurface, the unit normal \( \mathbf{N} \) is given by the identity

\[
\mathbf{N} = \frac{1}{2} \varepsilon^{\alpha\beta} S_\alpha \times S_\beta, \quad (35)
\]
where \( \varepsilon^{\alpha\beta} \) is the surface Levi-Civita symbol. It is a straightforward matter to generalize this identity to higher dimensions.

The covariant derivative \( \nabla_k \) is a differential operator that preserves the tensor property of its inputs. It satisfies product rule, the sum rule, and the metrinelic property with respect to all the ambient metrics, i.e.

\[
\begin{align*}
\nabla_k Z_{ij}, \nabla_k \delta^i_j, \nabla_k Z^{ij}, \nabla_k \varepsilon_{rst} &= 0 \quad (36) \\
\nabla_k Z_i, \nabla_k Z^i &= 0 
\end{align*}
\]

Furthermore, the covariant derivative \( \nabla_k \) coincides with the partial derivative \( \partial/\partial Z^k \) in affine coordinates as well as for tensors of order zero in arbitrary coordinates. In particular, the covariant basis \( Z_i \) can be expressed in terms of the covariant derivative, i.e.

\[
Z_i = \nabla_i R. \quad (38)
\]

The surface covariant derivative \( \nabla_\gamma \) is a differential operator that applies to objects defined on the surface. It is, too, distinguished by the property that it preserves the tensor property of its inputs. It satisfies the sum rule, the product rule, and the metrinelic property with respect to the metric tensors and the Levi-Civita symbols, i.e.

\[
\nabla_\gamma S_{\alpha\beta}, \nabla_\gamma \delta^\alpha_\beta, \nabla_\gamma S^{\alpha\beta} = 0, \quad (39)
\]

but not with respect to the surface bases \( S_\alpha \) and \( S^\alpha \). It coincides with the partial derivative \( \partial/\partial S^\gamma \) in affine coordinates, provided that the surface admits such coordinates. It also coincides with the partial derivative for tensors of order zero in arbitrary coordinates. In particular,

\[
S_\alpha = \nabla_\alpha R. \quad (40)
\]

The ambient version of the divergence theorem states that the volume integral of an invariant quantity \( \nabla_i T^i \) equals the surface integral of the invariant quantity \( N_i T^i \), i.e.

\[
\int_\Omega \nabla_i T^i dZ = \int_S N_i T^i dS, \quad (41)
\]

where \( \Omega \) is the domain enclosed by the surface \( S \), and \( N_i \) are the components of the external normal \( N \). The theorem is valid for a tensor \( T^i \) with either scalar or vector elements.

Similarly, according to the surface divergence theorem for a patch \( S \) with a boundary \( L \), the surface integral of the invariant quantity \( \nabla_\alpha T^\alpha \) equals the boundary integral of the invariant \( n_\alpha T^\alpha \), i.e.

\[
\int_S \nabla_\alpha T^\alpha dS = \int_L n_\alpha T^\alpha dS, \quad (42)
\]

where \( n_\alpha \) are the surface components of the exterior unit vector \( n \) that is normal to the boundary \( L \) and lies in the plane tangent to the surface. Once again, the statement is valid for a tensor \( T^\alpha \) with either scalar or vector elements.
The concept of curvature arises in the analysis of the vectors $\nabla_\alpha S_\beta$. While $\nabla_\alpha S_\beta$ do not vanish, they are shown to be orthogonal to the surface, i.e.

$$\nabla_\alpha S_\beta = NB_{\alpha\beta}. \quad (43)$$

where the system $B_{\alpha\beta}$, which consists of the coefficients of proportionality between $\nabla_\alpha S_\beta$ and the unit normal $N$, is known as the curvature tensor. Thanks to equation (40), the vector quantity $NB_{\alpha\beta}$, which we will refer to as the vector curvature tensor, is given in terms of the position vector $R$ by the identity

$$NB_{\alpha\beta} = \nabla_\alpha \nabla_\beta R. \quad (44)$$

From this equation, it immediately follows that the curvature tensor $B_{\alpha\beta}$ is symmetric, i.e.

$$B_{\alpha\beta} = B_{\beta\alpha}. \quad (45)$$

Raising the subscript $\beta$ in equation (44) and contracting with $\alpha$ yields

$$NB^\alpha_{\alpha} = \nabla_\alpha \nabla^\alpha R. \quad (46)$$

The vector quantity $NB^\alpha_{\alpha}$ is known as the curvature normal by analogy with the curvature normal characteristic of a curve. In words, the above identity says that the curvature normal is the surface Laplacian of the position vector. The invariant $B^\alpha_{\alpha}$ is known as the mean curvature.

The covariant derivative of the unit normal $N$ is given by the Weingarten equation

$$\nabla^\alpha N = -S^\beta B^\gamma_{\beta}. \quad (47)$$

One of the most elegant identities involving the curvature tensor is the Gauss equations of the surface which read

$$B_{\alpha\gamma} B_{\beta\delta} - B_{\alpha\delta} B_{\beta\gamma} = R_{\alpha\beta\gamma\delta}, \quad (48)$$

where $R_{\alpha\beta\gamma\delta}$ is the Riemann-Christoffel tensor. For a two-dimensional hypersurface in a three-dimensional Euclidean space, the Gauss equations reduce to the form

$$B_{\alpha\gamma} B_{\beta\delta} - B_{\alpha\delta} B_{\beta\gamma} = K \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta}, \quad (49)$$

where $K$ is, once again, the Gaussian curvature and $\varepsilon_{\alpha\beta}$ is the Levi-Civita symbol. Raising the subscripts $\alpha$ and $\beta$ yields

$$B^\alpha_{\gamma} B^\beta_{\delta} - B^\alpha_{\delta} B^\beta_{\gamma} = K \delta^\alpha_{\gamma\delta}, \quad (50)$$

where $\delta^\alpha_{\gamma\delta} = \varepsilon^\alpha_{\gamma\delta} \varepsilon_{\gamma\delta}$ is known as the second-order delta system.

Note that for any second-order system $A_\gamma^\alpha$ in two dimensions we have

$$A_\gamma^\alpha A_\delta^\beta - A_\delta^\alpha A_\gamma^\beta = A_\delta^\alpha \delta^\alpha_{\gamma\delta}, \quad (51)$$
where \( A \) is the determinant of \( A^2 \). Therefore, the Gauss equations are equivalent to the statement that the Gaussian curvature equals the determinant \( B \) of \( B^2 \), i.e.

\[
K = B.
\] (52)

In fact, an interesting generalization of the Gauss-Bonnet theorem to arbitrary dimension is the statement that the surface integral of \( B \), i.e.

\[
\int_S B dS,
\] (53)

depends on the topology of \( S \) but not its shape.

In \( n \) dimensions, write the Gauss equations

\[
B_{\alpha\gamma} B_{\beta\delta} - B_{\alpha\delta} B_{\beta\gamma} = R_{\gamma\delta},
\] (48)

with \( \alpha \) and \( \beta \) as superscripts, i.e.

\[
B^\alpha_\gamma B^\beta_\delta - B^\alpha_\delta B^\beta_\gamma = R_{\gamma\delta}^\alpha\beta,
\] (54)

and contract \( \alpha \) with \( \gamma \) and \( \beta \) with \( \delta \), i.e.

\[
B^\alpha_\alpha B^\beta_\beta - B^\alpha_\beta B^\beta_\alpha = R_{\alpha\beta}.
\] (55)

The invariant

\[
R = R_{\gamma\delta}^\alpha\beta,
\] (56)

is known as the scalar curvature. In terms of \( R \), we have

\[
B^\alpha_\alpha B^\beta_\beta - B^\alpha_\beta B^\beta_\alpha = R.
\] (57)

In words, the difference between the square of the trace of the curvature tensor \( B\gamma_\delta \) and the trace of the third fundamental form \( B^2 \) equals the scalar curvature.

Closely related to the Gauss equations, are the Codazzi equations

\[
\nabla_\alpha B_{\beta\gamma} = \nabla_\beta B_{\alpha\gamma},
\] (58)

which, in combination with the symmetry of \( B_{\beta\gamma} \), imply that the tensor \( \nabla_\alpha B_{\beta\gamma} \) is symmetric in all of its subscripts. Below, we will use the following immediate consequence of the Codazzi equations:

\[
\nabla_\alpha B^\alpha_\beta = \nabla_\beta B^\alpha_\alpha.
\] (59)

4 Demonstrations of the integral identities

4.1 The integral \( \int_S N dS \)

With the help of the two flavors of the divergence theorem – ambient and surface – we can prove, in two different ways, the fact that the integral of the unit normal \( N \) vanishes, i.e.

\[
\int_S N dS = 0.
\] (7)
Indeed, since
\[ \mathbf{N} = N^i \mathbf{Z}_i, \]  
we have, by the divergence theorem, that
\[ \int_S \mathbf{N} dS = \int_S N^i \mathbf{Z}_i dS = \int_\Omega \nabla^i \mathbf{Z}_i dZ, \]  
where the integrand in the last integral vanishes by the metrinilic property. Thus, indeed,
\[ \int_S \mathbf{N} dS = 0. \]  
as we set out to show.

One dissatisfying aspect of this proof is the fact that it engages the ambient space even though the elements of the integral identity
\[ \int_S \mathbf{N} dS = 0. \]  
are defined exclusively on the surface \( S \). Therefore, we will now construct a proof that involves only quantities defined on \( S \). The presented proof applies only to a two-dimensional hypersurface, but can be easily generalized to arbitrary dimension.

Recall that the normal \( \mathbf{N} \) is given by the identity
\[ \mathbf{N} = \frac{1}{2} \varepsilon^{\alpha\beta} S_\alpha \times S_\beta. \]  
Since
\[ S_\alpha = \nabla_\alpha \mathbf{R}, \]  
we have
\[ \mathbf{N} = \frac{1}{2} \varepsilon^{\alpha\beta} \nabla_\alpha \mathbf{R} \times S_\beta. \]  
By the combination of the product rule and the metrinilic property of \( \nabla_\alpha \) with respect to \( \varepsilon^{\alpha\beta} \), we have
\[ \mathbf{N} = \frac{1}{2} \nabla_\alpha \left( \varepsilon^{\alpha\beta} \mathbf{R} \times S_\beta \right) - \frac{1}{2} \varepsilon^{\alpha\beta} \mathbf{R} \times \nabla_\alpha S_\beta \]  
Since \( \nabla_\alpha S_\beta = \mathbf{N} B_{\alpha\beta} \), the normal \( \mathbf{N} \) is given by
\[ \mathbf{N} = \frac{1}{2} \nabla_\alpha \left( \varepsilon^{\alpha\beta} \mathbf{R} \times S_\beta \right) - \frac{1}{2} \mathbf{R} \times \mathbf{N} \varepsilon^{\alpha\beta} B_{\alpha\beta}. \]  
Next, note that since \( B_{\alpha\beta} \) is symmetric and \( \varepsilon^{\alpha\beta} \) is skew-symmetric, the combination \( \varepsilon^{\alpha\beta} B_{\alpha\beta} \) vanishes and we end up with the identity
\[ \mathbf{N} = \frac{1}{2} \nabla_\alpha \left( \varepsilon^{\alpha\beta} \mathbf{R} \times S_\beta \right). \]
in which the normal \( \mathbf{N} \) is expressed as the surface divergence of the combination \( \varepsilon^{\alpha \beta} \mathbf{R} \times \mathbf{S}_\beta \).

Integrating both sides of the above identity over the surface \( S \), i.e.

\[
\int_S \mathbf{N} dS = \frac{1}{2} \int_S \nabla_\alpha (\varepsilon^{\alpha \beta} \mathbf{R} \times \mathbf{S}_\beta) dS. \tag{67}
\]

By an application of the surface divergence theorem – recognizing that a closed surface has no boundary \( L \) – we once again arrive at

\[
\int_S \mathbf{N} dS = 0. \tag{7}
\]

as we set out to do. Furthermore, if \( S \) is a patch with a boundary \( L \), we have

\[
\int_S \mathbf{N} dS = \frac{1}{2} \int_L n_\alpha \varepsilon^{\alpha \beta} \mathbf{R} \times \mathbf{S}_\beta dL \tag{68}
\]

Since the combination \( n_\alpha \varepsilon^{\alpha \beta} \) equals the components \( T^\beta \) of the unit tangent vector to the boundary \( L \), we discover that

\[
\int_S \mathbf{N} dS = \frac{1}{2} \int_L \mathbf{R} \times \mathbf{T} dL. \tag{69}
\]

The key elements in this identity are illustrated in the following figure.

4.2 The integral \( \int_S \mathbf{N} B_\alpha^\alpha dS \)

Let us now turn our attention to the identity

\[
\int_S \mathbf{N} B_\alpha^\alpha dS = 0. \tag{8}
\]

Recall the identity

\[
\mathbf{N} B_\alpha^\alpha = \nabla_\alpha \nabla^\alpha \mathbf{R}. \tag{46}
\]
Integrating both sides over a closed surface $S$, we find

$$\int_S N B_\alpha^{\alpha} dS = \int_S \nabla_\alpha \nabla^{\alpha} R dS,$$

(71)

where the integral on the right vanishes by the surface divergence theorem since a closed surface has no boundary. In other words, we indeed find that

$$\int_S N B_\alpha^{\alpha} dS = 0$$

(8)

as we set out to show.

Meanwhile, for a patch $S$ with a boundary $L$, the divergence theorem yields

$$\int_S N B_\alpha^{\alpha} dS = \int_L n_\alpha \nabla^{\alpha} R dL.$$

(72)

Since

$$n_\alpha \nabla^{\alpha} R = n_\alpha S^{\alpha} = n$$

(73)

we arrive at the final identity

$$\int_S N B_\alpha^{\alpha} dS = \int_L dL.$$  

(74)

Two proofs of a special case of this identity can be found in [2]. The contrast in complexity between those proofs and the one presented here speaks to the effectiveness of our approach.

The key elements in the above identity are illustrated in the following figure.

4.3 The integral $\int_S N K dS$

Finally, let us demonstrate the identity

$$\int_S N K dS = 0.$$ 

(9)
Apply the covariant derivative $\nabla_\alpha$ to both sides of the Weingarten equation

$$\nabla^\alpha N = -S^\beta B^\alpha_\beta,$$

i.e.

$$\nabla_\alpha \nabla^\alpha N = -\nabla_\alpha \left( S^\beta B^\alpha_\beta \right),$$

(76)

to produce the surface Laplacian $\nabla_\alpha \nabla^\alpha N$ of the unit normal $N$. An application of the product rule on the right yields

$$\nabla_\alpha \nabla^\alpha N = -\nabla_\alpha S^\beta B^\alpha_\beta - S^\beta \nabla_\alpha B^\alpha_\beta.$$  

(77)

Since

$$\nabla_\alpha S^\beta = NB^\beta_\alpha$$

(78)
and, by the Codazzi equations,

$$\nabla_\alpha B^\alpha_\beta = \nabla_\beta B^\alpha_\alpha,$$

(59)
we have

$$\nabla_\alpha \nabla^\alpha N = -NB^\beta_\alpha B^\alpha_\beta - S^\beta \nabla_\beta B^\alpha_\alpha.$$  

(79)

Apply the "reverse" product rule to the second term on the right, i.e.

$$\nabla_\alpha \nabla^\alpha N = -NB^\beta_\alpha B^\alpha_\beta - \nabla_\beta \left( S^\beta B^\alpha_\alpha \right) + \nabla_\beta S^\beta B^\alpha_\alpha.$$  

(80)

Since $\nabla_\beta S^\beta = NB^\beta_\alpha$, we have

$$\nabla_\alpha \nabla^\alpha N = N \left( B^\alpha_\alpha B^\beta_\beta - B^\beta_\alpha B^\alpha_\beta \right) - \nabla_\beta \left( S^\beta B^\alpha_\alpha \right).$$  

(81)

Now, recall that, by the Gauss equations of the surface, the quantity $B^\alpha_\alpha B^\beta_\beta - B^\beta_\alpha B^\alpha_\beta$ corresponds to the scalar curvature $R$, i.e.

$$B^\alpha_\alpha B^\beta_\beta - B^\beta_\alpha B^\alpha_\beta = R.$$  

(57)

Thus, the surface Laplacian of $N$ is given by

$$\nabla_\alpha \nabla^\alpha N = NR - \nabla_\beta \left( S^\beta B^\alpha_\alpha \right).$$  

(82)

Solving for $NR$, we find

$$NR = \nabla_\alpha \left( \nabla^\alpha N + S^\alpha B^\beta_\beta \right).$$  

(83)

Next, integrate both sides of the above equation over the surface $S$, i.e.

$$\int_S NRdS = \int_S \nabla_\alpha \left( \nabla^\alpha N + S^\alpha B^\beta_\beta \right) dS.$$  

(84)

If the surface $S$ is closed then, by the surface divergence theorem, we have

$$\int_S NRdS = 0.$$  

(85)
If $S$ is a surface patch with a boundary $L$ then, by the same divergence theorem,

$$
\int_S N RdS = \int_L \left( n_\alpha \nabla \alpha N + n_\alpha S^\alpha B^\beta_\beta \right) dL,
$$

or, equivalently,

$$
\int_S N RdS = \int_L \left( n_\alpha \nabla \alpha N + n B^\alpha_\alpha \right) dL,
$$

(86)

With the help of Weingarten’s equation, this identity can also be written in the form

$$
\int_S N RdS = \int_L \left( n_\beta B^\alpha_\alpha - n_\alpha B^\beta_\beta \right) S^\beta dL.
$$

(87)

For a two-dimensional surface, the scalar curvature is twice the Gaussian curvature, i.e.

$$
R = 2K.
$$

(89)

Thus, for a closed two-dimensional surface,

$$
\int_S N K dS = 0,
$$

(90)

as we set out to show. Furthermore, for a patch with a boundary $L$, we have

$$
\int_S N K dS = \frac{1}{2} \int_L \left( n_\beta B^\alpha_\alpha - n_\alpha B^\beta_\beta \right) S^\beta dL.
$$

(91)

### 4.4 The integral $\int_S R \cdot NdS$

Let us now prove the related integral identities involving dot products with the position vector $R$. Since the following demonstrations rely on the very same elements that we used extensively above, we will present the demonstrations in a compressed format.

For the integral

$$
\int_S R \cdot NdS
$$

(91)
we have

\[
\int_S \mathbf{R} \cdot \mathbf{N} dS = \int_S \mathbf{R} \cdot \mathbf{N}^i \mathbf{Z}_i dS = \int_{\Omega} \nabla^i (\mathbf{R} \cdot \mathbf{Z}_i) d\Omega = \int_{\Omega} \mathbf{R} \cdot \mathbf{Z}_i d\Omega = \int_{\Omega} \mathbf{Z}_i \cdot \mathbf{Z}_i d\Omega = \int_{\Omega} \delta_i d\Omega = n \int_{\Omega} d\Omega = nV.
\] (92) (93) (94) (95) (96) (97) (98)

In the above chain, \( n \) is the dimension of the ambient space and \( V \) is the volume of the enclosed domain. In summary,

\[
\int_S \mathbf{R} \cdot \mathbf{N} dS = nV,
\] (99)

as we set out to show.

4.5 The integral \( \int_S \mathbf{R} \cdot \mathbf{N} B^\alpha_\alpha dS \)

Let us now turn our attention to the integral

\[
\int_S \mathbf{R} \cdot \mathbf{N} B^\alpha_\alpha dS.
\] (99)

Since

\[
\mathbf{N} B^\alpha_\alpha = \nabla_\alpha \nabla^\alpha \mathbf{R},
\] (99)

we have, by the product rule,

\[
\mathbf{R} \cdot \mathbf{N} B^\alpha_\alpha = \nabla_\alpha (\mathbf{R} \cdot \nabla^\alpha \mathbf{R}) - \nabla_\alpha \mathbf{R} \cdot \nabla^\alpha \mathbf{R}.
\] (100)

Thus, by the surface divergence theorem,

\[
\int_S \mathbf{R} \cdot \mathbf{N} B^\alpha_\alpha dS = - \int_{\Omega} \nabla_\alpha \mathbf{R} \cdot \nabla^\alpha \mathbf{R} d\Omega.
\] (101)
Continuing,
\[
\int_S \mathbf{R} \cdot \mathbf{N} B_\alpha^a dS = - \int_S \nabla_\alpha \mathbf{R} \cdot \nabla^\alpha \mathbf{R} dS
\]  
(102)
\[
= - \int_S S_\alpha \cdot S^\alpha dS
\]  
(103)
\[
= - \int_S \delta_\alpha^a dS
\]  
(104)
\[(\delta_\alpha^a = n - 1) = - (n - 1) \int_S dS
\]  
(105)
\[= - (n - 1) A.
\]  
(106)

In the above chain of identities, \( n \) is, once again, the dimension of the ambient space and \( A \) is the area of the surface patch \( S \). In summary,
\[
\int_S \mathbf{R} \cdot \mathbf{N} B_\alpha^a dS = - (n - 1) A,
\]  
(108)
as we set out to show.

4.6 The integral \( \int_S \mathbf{R} \cdot \mathbf{N} K dS \)

For greater generality, let us perform our analysis for a hypersurface in \( n \) dimensions where the scalar curvature \( R \) takes the place of \( K \) – or, more precisely, of \( 2K \).

Recall that the expression for \( \mathbf{N} R \) in divergence form reads
\[
\mathbf{N} R = \nabla_\alpha \left( \nabla^\alpha \mathbf{N} + S^\alpha B^\beta \right).
\]  
(83)

Thus, by the ”reverse” product rule,
\[
\mathbf{R} \cdot \mathbf{N} R = \nabla_\alpha \left( \mathbf{R} \cdot \left( \nabla^\alpha \mathbf{N} + S^\alpha B^\beta \right) \right) - \nabla_\alpha \mathbf{R} \cdot \left( \nabla^\alpha \mathbf{N} + S^\alpha B^\beta \right)
\]  
(107)
and, by the surface divergence theorem,
\[
\int_S \mathbf{R} \cdot \mathbf{N} R dS = - \int_S \nabla_\alpha \mathbf{R} \cdot \left( \nabla^\alpha \mathbf{N} + S^\alpha B^\beta \right) dS
\]  
(108)

Continuing,
\[
\int_S \mathbf{R} \cdot \mathbf{N} R dS = - \int_S \nabla_\alpha \mathbf{R} \cdot \left( \nabla^\alpha \mathbf{N} + S^\alpha B^\beta \right) dS
\]  
(109)
\[
= - \int_S S_\alpha \cdot \left( \nabla^\alpha \mathbf{N} + S^\alpha B^\beta \right) dS
\]  
(110)
\[
= - \int_S \left( - \delta_\alpha^3 B^3_\beta + \delta_\alpha^\beta B^\beta_\beta \right) dS
\]  
(111)
\[
(\delta_\alpha^3 B^3_\beta = B^\beta_\beta) = - \int_S (\delta_\alpha^\beta - 1) B^\beta_\beta dS
\]  
(112)
\[
(\delta_\alpha^\beta = n - 1) = - (n - 2) \int_S B^\alpha_\beta dS.
\]  
(113)
In summary,
\[ \int_S \mathbf{R} \cdot \mathbf{N} dS = -(n-2) \int_S B^\alpha dS, \quad (114) \]

For the special case of a two-dimensional hypersurface, i.e. \( n = 3 \), we have \( n - 2 = 1 \) and
\[ R = 2K, \quad (89) \]
and therefore
\[ \int_S \mathbf{R} \cdot \mathbf{N} K dS = -\frac{1}{2} \int_S B^\alpha dS, \quad (18) \]
as we set out to show.

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