A note on uncertainty relations of arbitrary $N$ quantum channels

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Abstract

Uncertainty principle plays a vital role in quantum physics. The Wigner-Yanase skew information characterizes the uncertainty of an observable with respect to the measured state. We generalize the uncertainty relations for two quantum channels to arbitrary $N$ quantum channels based on Wigner-Yanase skew information. We illustrate that these uncertainty inequalities are tighter than the existing ones by detailed examples. Especially, we also discuss the uncertainty relations for $N$ unitary channels, which could be regarded as variance-based sum uncertainty relations with respect to any pure state.
I. INTRODUCTION

As a fundamental characteristic of quantum theory, uncertainty principle has been widespread concerned since Heisenberg proposed the notions of uncertainties for measuring non-commuting observables [1]. The well-known Robertson uncertainty relation says that for arbitrary two observables $A$ and $B$ [2], $\Delta A \Delta B \geq \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle|$, where the commutator $[A, B] = AB - BA$ and $\Delta \Omega = \sqrt{\langle \Omega^2 \rangle - \langle \Omega \rangle^2}$ is the standard deviation of an observable $\Omega$ with respect to the measured state $|\psi\rangle$. With the development of quantum information theory, many kinds of characterizations and quantifications of uncertainty relations have been established, such as the ones based on entropy [3–10], Wigner-Yanase skew information [11–13], under successive measurements [14–18], and with majorization techniques [19–21].

In modern formalism of quantum theory, the most general description of quantum measurement is given in terms of quantum channels [22, 23]. Quantum channels play a pivotal role in quantum information processing. Many aspects related to the quantum channels have been extensively investigated, such as the coherence of quantum channels [24, 25], the operational resource theory of quantum channels [26, 27], the capacity of quantum channels [28, 29], and the abilities of quantum channels in producing or destroying quantum resources [30–33]. Recently, the uncertainty relations for quantum channels have been also widely studied in terms of the Wigner-Yanase skew information and variance [11, 13, 34, 35]. The Wigner-Yanase skew information $I_{\rho}(A)$ with respect to a quantum state $\rho$ and an arbitrary operator $A$ is defined by [36–39],

$$I_{\rho}(A) = \frac{1}{2} \text{Tr}([\sqrt{\rho}, A]^\dagger [\sqrt{\rho}, A]) = \frac{1}{2} \| [\sqrt{\rho}, A] \|^2,$$

(1)

where $\| \cdot \|$ denotes the Frobenius norm. Let $\Phi$ be a quantum channel with Kraus representation, $\Phi(\rho) = \sum_{i=1}^{n} K_i \rho K_i^\dagger$. The Wigner-Yanase skew information of $\rho$ with respect to the channel is given by

$$I_{\rho}(\Phi) = \sum_{i=1}^{n} I_{\rho}(K_i),$$

(2)

where $I_{\rho}(K_i) = \frac{1}{2} \text{Tr}([\sqrt{\rho}, K_i]^\dagger [\sqrt{\rho}, K_i])$[38]. The quantity $I_{\rho}(\Phi)$ is well-defined because it is independent of the choice of the Kraus of $\Phi$. It is demonstrated that $I_{\rho}(\Phi)$ can be regarded as a bona fide measure, for coherence as well as quantum uncertainty of $\rho$ with respect to quantum channel $\Phi$. 

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For a pure state, the skew information for channel has a similar physical meaning to the variance [40]. Pass through a given quantum channel $\Phi$ with Kraus operators $K_i$, a pure state $|\psi\rangle$ can be transformed into:

$$\rho = \Phi(\rho) = \sum_{i=1}^{n} K_i |\psi\rangle \langle K_i^\dagger.$$ 

The Fidelity between $|\psi\rangle$ and $\rho$ is defined as [23]:

$$F = \langle \psi | \rho | \psi \rangle = \sum_{i=1}^{n} |\langle \psi | K_i | \psi \rangle|^2.$$ 

The skew information of $|\psi\rangle$ for quantum channel $\Phi$

$$I_{|\psi\rangle}(\Phi) = \sum_{i} I_{|\psi\rangle}(K_i) = 1 - \sum_{i=1}^{n} |\langle \psi | K_i | \psi \rangle|^2 = 1 - F,$$

that is to say,

$$I_{|\psi\rangle}(\Phi) + F = 1.$$ 

The above equality shows a strict complementarity between fidelity and uncertainty of quantum channel [13]. The complementary relation reveals that any restriction on the uncertainty in the channel will impose a restriction on the fidelity between the input and output states.

Fu et al. established the uncertainty relation for two quantum channels $\Phi_1$ and $\Phi_2$ in terms of Wigner-Yanase skew information [41],

$$I_{\rho}(\Phi_1) + I_{\rho}(\Phi_2) \geq \max_{\pi \in S_n} \frac{1}{2} \sum_{i=1}^{n} I_{\rho}(K_i^1 \pm K_i^2),$$

where $\Phi_s = \sum_{i=1}^{n} K_i^s \rho (K_i^s)^\dagger$, $s = 1, 2$, $\pi \in S_n$ is an arbitrary $n$-element permutation.

Very recently, generalizing the results in [41] to the case of $N$ quantum channels, Zhang et al. provided two elegant uncertainty relations [13],

$$\sum_{s=1}^{N} I_{\rho}(\Phi_s) \geq \max_{\pi_s, \pi_t \in S_n} \frac{1}{N^2} \sum_{i=1}^{n} \left\{ \sum_{1 \leq s < t \leq N} I_{\rho}(K_{\pi_s(i)}^s + K_{\pi_t(i)}^t) \right\} - \frac{1}{(N - 1)^2} \left[ \sum_{1 \leq s < t \leq N} \sqrt{I_{\rho}(K_{\pi_s(i)}^s + K_{\pi_t(i)}^t)} \right]^2,$$

and

$$\sum_{s=1}^{N} I_{\rho}(\Phi_s) \geq \max_{\pi_s, \pi_t \in S_n} \frac{1}{N^n} \sum_{i=1}^{n} \left\{ I_{\rho} \left( \sum_{s} K_{\pi_s(i)}^s \right) \right\} + \frac{2}{N(N - 1)} \left[ \sum_{1 \leq s < t \leq N} \sqrt{I_{\rho}(K_{\pi_s(i)}^s - K_{\pi_t(i)}^t)} \right]^2.$$
where \( \pi_s, \pi_t \in S_n \) are arbitrary \( n \)-element permutations. For convenience, we denote the right hands of (4) and (5) as \( \overline{LB}_1, \overline{LB}_2 \), respectively.

In this paper, we formulate several new uncertainty relations based on Wigner-Yanase skew information for \( N \) quantum channels. The lower bounds of our uncertainty inequalities are tighter than the existing ones [13]. Detailed examples are presented to illustrate the superiority. Especially, we also discuss the uncertainty relations for unitary channels.

II. SKEW INFORMATION-BASED SUM UNCERTAINTY RELATIONS FOR QUANTUM CHANNELS

Let \( \Phi \) be a quantum channel with Kraus representation, \( \Phi(\rho) = \sum_{i=1}^{n} K_i \rho K_i^\dagger \). The skew information of the channel can be written as

\[
I_\rho(\Phi) = \frac{1}{2} tr(a^\dagger a) = \frac{1}{2} \|a\|^2,
\]

where \( a^\dagger = ([\sqrt{\rho}, K_1]^\dagger, [\sqrt{\rho}, K_2]^\dagger, \ldots, [\sqrt{\rho}, K_n]^\dagger) \). \( I_\rho(\Phi) \) characterizes some intrinsic features of both the quantum state and the quantum channel. For arbitrary \( N \) quantum channels, we have the following conclusion.

**Theorem 1** Let \( \Phi_1, \Phi_2, \ldots, \Phi_N \) be \( N \) quantum channels with Kraus representations \( \Phi_s(\rho) = \sum_{i=1}^{n} K_i^s \rho(K_i^s)^\dagger, s = 1, 2, \ldots, N \). We have

\[
\sum_{s=1}^{N} I_\rho(\Phi_s) \geq \max\{\overline{LB}_1, \overline{LB}_2, \overline{LB}_3\},
\]

where

\[
\overline{LB}_1 = \max_{\pi_s, \pi_t \in S_n} \frac{1}{N} \left\{ \sum_{1 \leq s < t \leq N} \sum_{i=1}^{n} I_\rho(K_{\pi_s(i)}^s + K_{\pi_t(i)}^t) \right\}
\]

\[
- \frac{1}{(N-1)^2} \left\{ \sum_{1 \leq s < t \leq N} \sqrt{\sum_{i=1}^{n} I_\rho(K_{\pi_s(i)}^s + K_{\pi_t(i)}^t)} \right\}^2,
\]

\[
\overline{LB}_2 = \max_{\pi_s, \pi_t \in S_n} \frac{1}{N} \left\{ \sum_{i=1}^{n} I_\rho(\sum_{s=1}^{N} K_{\pi_s(i)}^s) \right\}
\]

\[
+ \frac{2}{N(N-1)} \left\{ \sum_{1 \leq s < t \leq N} \sqrt{\sum_{i=1}^{n} I_\rho(K_{\pi_s(i)}^s - K_{\pi_t(i)}^t)} \right\}^2,
\]

In this paper, we formulate several new uncertainty relations based on Wigner-Yanase skew information for \( N \) quantum channels. The lower bounds of our uncertainty inequalities are tighter than the existing ones [13]. Detailed examples are presented to illustrate the superiority. Especially, we also discuss the uncertainty relations for unitary channels.
\[ LB_3 = \max_{\pi_s, \pi_t \in S_n} \frac{1}{2N - 2} \left\{ \sum_{1 \leq s < t \leq N} \sum_{i=1}^{n} I_{\rho}(K_{\pi_s}(i) \mp K_{\pi_t}(i)) \right\} + \frac{2}{N(N - 1)} \left[ \sum_{1 \leq s < t \leq N} \sqrt{\sum_{i=1}^{n} I_{\rho}(K_{\pi_s}(i) \pm K_{\pi_t}(i))} \right]^2, \]  

(10)

\( \pi_s, \pi_t \in S_n \) are arbitrary \( n \)-element permutations.

[Proof] To prove the inequality (7), we employ the following equality,

\[ \| \sum_{s=1}^{N} a_s \|^2 + (N - 2) \sum_{s=1}^{N} \| a_s \|^2 = \sum_{1 \leq s < t \leq N} \| a_s + a_t \|^2. \]

Note that

\[ \| \sum_{s=1}^{N} a_s \| = \| \frac{1}{N - 1} \sum_{1 \leq s < t \leq N} (a_s + a_t) \| \leq \frac{1}{N - 1} \sum_{1 \leq s < t \leq N} \| a_s + a_t \|, \]

we get

\[ \sum_{s=1}^{N} \| a_s \|^2 \geq \frac{1}{N - 2} \left[ \sum_{1 \leq s < t \leq N} \| a_s + a_t \|^2 - \frac{1}{(N - 1)^2} \left( \sum_{1 \leq s < t \leq N} \| a_s + a_t \| \right)^2 \right]. \]

From (6), we have \( \| a_s \|^2 = 2I_{\rho}(\Phi_s) \) and \( \| a_s + a_t \|^2 = 2\sum_{i=1}^{N} I_{\rho}(K_i^s + K_i^t) \), which proves \( \sum_{s=1}^{N} I_{\rho}(\Phi_s) \geq LB_1 \).

By using the identity,

\[ N \sum_{s=1}^{N} \| a_s \|^2 = \| \sum_{s=1}^{N} a_s \|^2 + \sum_{1 \leq s < t \leq N} \| a_s - a_t \|^2, \]

and the Cauchy-Schwarz inequality, we obtain

\[ \sum_{1 \leq s < t \leq N} \| a_s - a_t \|^2 \geq \frac{2}{N(N - 1)} \left( \sum_{1 \leq s < t \leq N} \| a_s - a_t \| \right)^2, \]

and

\[ \sum_{s=1}^{N} \| a_s \|^2 \geq \frac{1}{N} \left[ \| \sum_{s=1}^{N} a_s \|^2 + \frac{2}{N(N - 1)} \left( \sum_{1 \leq s < t \leq N} \| a_s - a_t \| \right)^2 \right]. \]

Taking account into that \( \| \sum_{s=1}^{N} a_s \|^2 = 2\sum_{i=1}^{n} I_{\rho}(\sum_{s} K_i^s) \) and \( \| a_s - a_t \|^2 = 2\sum_{i=1}^{N} I_{\rho}(K_i^s - K_i^t) \), we prove the inequality \( \sum_{s=1}^{N} I_{\rho}(\Phi_s) \geq LB_2 \).

At last, by using the parallelogram law,

\[ (2N - 2) \sum_{s=1}^{N} \| a_s \|^2 = \sum_{1 \leq s < t \leq N} \| a_s - a_t \|^2 + \sum_{1 \leq s < t \leq N} \| a_s + a_t \|^2 \]
and the Cauchy-Schwarz inequality, we get
\[
\sum_{1 \leq s < t \leq N} \|a_s - a_t\|^2 \geq \frac{2}{N(N-1)} \left( \sum_{1 \leq s < t \leq N} \|a_s - a_t\| \right)^2
\]
and
\[
\sum_{1 \leq s < t \leq N} \|a_s + a_t\|^2 \geq \frac{2}{N(N-1)} \left( \sum_{1 \leq s < t \leq N} \|a_s + a_t\| \right)^2.
\]
Therefore we have
\[
\sum_{s=1}^{N} \|a_s\|^2 \geq \frac{1}{2N-2} \left[ \frac{2}{N(N-1)} \left( \sum_{1 \leq s < t \leq N} \|a_s \mp a_t\| \right) \right] + \sum_{1 \leq s < t \leq N} \|a_s \pm a_t\|^2,
\]
which proves the inequality \(\sum_{s=1}^{N} I_{\rho}(\Phi_s) \geq LB3\). \(\square\)

As examples, let us consider the mixed state given by Bloch vector \(\vec{r} = (\frac{\sqrt{3}}{2} \cos \theta, \frac{\sqrt{3}}{2} \sin \theta, 0)\) [13],
\[
\rho = \frac{I_2 + \vec{r} \cdot \vec{\sigma}}{2}, \tag{11}
\]
where \(\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)\) is given by the standard Pauli matrices, \(I_2\) is the \(2 \times 2\) identity matrix.

We respectively consider three quantum channels: the phase damping channel \(\phi\),
\[
\phi(\rho) = \sum_{i=1}^{2} A_i \rho(A_i)^\dagger, \quad A_1 = |0\rangle\langle 0| + \sqrt{1-q}|1\rangle\langle 1|, \quad A_2 = \sqrt{q}|1\rangle\langle 1|,
\]
the amplitude damping channel \(\epsilon\),
\[
\epsilon(\rho) = \sum_{i=1}^{2} B_i \rho(B_i)^\dagger, \quad B_1 = |0\rangle\langle 0| + \sqrt{1-q}|1\rangle\langle 1|, \quad B_2 = \sqrt{q}|0\rangle\langle 1|
\]
and the bit flip channel \(\Lambda\),
\[
\Lambda(\rho) = \sum_{i=1}^{2} C_i \rho(C_i)^\dagger, \quad C_1 = \sqrt{q}|0\rangle\langle 0| + \sqrt{q}|1\rangle\langle 1|, \quad C_2 = \sqrt{1-q}(|0\rangle\langle 1| + |1\rangle\langle 0|)
\]
with \(0 \leq q < 1\).

For the case \(q = 0.1\) and \(\theta = \pi/2\), we have \(I_{\rho}(\phi) + I_{\rho}(\epsilon) + I_{\rho}(\Lambda) = 0.475658\). The lower bound \(LB1\) is 0.449135, the lower bounds \(\overline{LB}B1\) and \(\overline{LB}B2\) are 0.475658 and 425827, respectively. Obviously, \(LB1\) is tighter than \(\overline{LB}B2\) in [13]. Here the lower bound \(LB1\) is also greater than 0.42873 from \(LB2\) and 0.440462 from \(LB3\).

We also consider the case \(q = 0.5\), the comparison among the lower bounds \(\overline{LB}B1\), \(\overline{LB}B2\), \(LB1\), \(LB2\) and \(LB3\) is shown in Figure. 1. Especially, we take some special \(\theta\), see Table. 1. These results show that our Theorem 1 improve the existing ones given in [13].
FIG. 1: The comparison among the lower bounds $\overline{\text{LB}1}$, $\overline{\text{LB}2}$, LB1, LB2 and LB3 for the state $\rho$ with Bloch vector $\vec{r} = (\sqrt{3}/2 \cos \theta, \sqrt{3}/2 \sin \theta, 0)$, and three quantum channels, the phase damping channel, the amplitude damping channel and the bit flip channel. Let $\text{Sum} = I_{\rho}(\phi) + I_{\rho}(\epsilon) + I_{\rho}(\Lambda)$.

| $\theta$ | $\overline{\text{LB}1}$ | $\overline{\text{LB}2}$ | LB1 | LB2 | LB3 | $I_{\rho}(\phi) + I_{\rho}(\epsilon) + I_{\rho}(\Lambda)$ |
|---------|-----------------|-----------------|-----|-----|-----|---------------------|
| $\pi/6$ | 0.133979        | 0.204181        | 0.127677 | 0.208898 | 0.20891 | 0.208947           |
| $\pi/4$ | 0.194803        | 0.264726        | 0.182753 | 0.271447 | 0.271447 | 0.271447           |
| $\pi/2$ | 0.342466        | 0.383224        | 0.324177 | 0.393068 | 0.393913 | 0.396447           |

In addition, by using the generalized Hlawka’s inequality [12, 42, 43],

$$\sum_{s=1}^{N} \|a_s\| \geq \frac{1}{N-2} \left( \sum_{1 \leq s < t \leq N} \|a_s + a_t\| - \| \sum_{i=1}^{N} a_s \| \right),$$

we can similarly prove the following theorem.

**Theorem 2** Let $\Phi_1, \Phi_2, \ldots, \Phi_N$ be $N$ quantum channels with Kraus representations $\Phi_s(\rho) = \sum_{i=1}^{n} K^{s}_i \rho (K^{s}_i)^\dagger$, we have

$$\sum_{s=1}^{N} \sqrt{I_{\rho}(\Phi_s)} \geq \max_{\pi_s, \pi_t \in S_n} \frac{1}{N-2} \left\{ \sum_{1 \leq s < t \leq N} \left( \sum_{i=1}^{n} I_{\rho}(K^{s}_{\pi_s(i)} + K^{t}_{\pi_t(i)}) - \sum_{i=1}^{n} I_{\rho}(\sum_{s=1}^{N} K^{s}_{\pi_s(i)}) \right) \right\},$$

(12)

where $\pi_s, \pi_t \in S_n$ are arbitrary $n$-element permutations.

Unitary channels are also used a lot in quantum computation and quantum information theory[23]. Consider an arbitrary channel $U(\rho) = U\rho U^\dagger$, the Wigner-Yanase skew informa-
tion of $\rho$ with respect to the channel is given by
\[
I_\rho(U) = \frac{1}{2} tr([\sqrt{\rho}, U]^{\dagger} [\sqrt{\rho}, U]).
\]

Next we consider the skew information-based uncertainty relations for $N$ unitary channels $U_1, U_2, \ldots, U_N$. Directly from Theorem 1, the following uncertainty relations hold:
\[
\sum_{s=1}^{N} I_\rho(U_s) \geq \frac{1}{N} \left\{ \sum_{1 \leq s < t \leq N} I_\rho(U_s + U_t) - \frac{1}{(N-1)^2} \left[ \sum_{1 \leq s < t \leq N} \sqrt{I_\rho(U_s + U_t)} \right]^2 \right\}, \quad \text{(13)}
\]
\[
\sum_{s=1}^{N} I_\rho(U_s) \geq \frac{1}{N} \left\{ I_\rho(\sum_{s=1}^{N} U_s) + \frac{2}{N(N-1)} \left[ \sum_{1 \leq s < t \leq N} \sqrt{I_\rho(U_s - U_t)} \right]^2 \right\}, \quad \text{(14)}
\]
\[
\sum_{s=1}^{N} I_\rho(U_s) \geq \frac{1}{2N-2} \left\{ \sum_{1 \leq s < t \leq N} I_\rho(U_s \mp U_t) + \frac{2}{N(N-1)} \left[ \sum_{1 \leq s < t \leq N} \sqrt{I_\rho(U_s \pm U_t)} \right]^2 \right\}. \quad \text{(15)}
\]
For convenience, we denote the right hands of (13), (14) and (15) as Lb1, Lb2 and Lb3, respectively.

Theorem 2 implies the following inequality holds for $N$ Unitary channels:
\[
\sum_{s=1}^{N} \sqrt{I_\rho(U_s)} \geq \frac{1}{N} \left\{ \sum_{1 \leq s < t \leq N} \sqrt{I_\rho(U_s + U_t)} - \sqrt{I_\rho(\sum_{s=1}^{N} U_s)} \right\}. \quad \text{(16)}
\]
Noticed that quantum variance is defined as: $(\Delta_{|\psi\rangle} U)^2 = \frac{1}{2} \langle U U^\dagger + U^\dagger U \rangle - \langle U \rangle \langle U^\dagger \rangle$ with any quantum pure state $|\psi\rangle$, then the following equality holds:
\[
(\Delta_{|\psi\rangle} U)^2 = I_{|\psi\rangle}(U). \quad \text{(17)}
\]
The above inequalities (13) to (16) can be regarded as variance-based sum uncertainty relations for $N$ unitary operators.

We take an example to illustrate these uncertainty relations. Let us consider the pure state $\rho = \frac{1}{2} (I + \vec{r} \cdot \vec{\sigma})$ with $\vec{r} = (\frac{1}{\sqrt{2}} \cos \theta, \frac{1}{\sqrt{2}} \sin \theta, \frac{1}{\sqrt{2}})$, where $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrices.

Consider three unitary operators,
\[
U_1 = e^{i\sigma_z \frac{\pi}{8}} = \begin{pmatrix} \cos \frac{\pi}{8} & i \sin \frac{\pi}{8} \\ i \sin \frac{\pi}{8} & \cos \frac{\pi}{8} \end{pmatrix},
\]
\[
U_2 = e^{i\sigma_y \frac{\pi}{8}} = \begin{pmatrix} \cos \frac{\pi}{8} & \sin \frac{\pi}{8} \\ -i \sin \frac{\pi}{8} & \cos \frac{\pi}{8} \end{pmatrix},
\]
\[
U_3 = e^{-i\sigma_z \frac{\pi}{8}} = \begin{pmatrix} e^{i\frac{\pi}{8}} & 0 \\ 0 & e^{-i\frac{\pi}{8}} \end{pmatrix},
\]
which correspond to Bloch sphere rotations of $-\pi/4$ about the x axis, the y axis and z axis, respectively. Then the lower bounds of inequalities (13), (14) and (15) associated with $\rho$ can be computed. Figure 2 shows that lower bound of (15) is strictly greater than (13) and (14) in this case.

III. CONCLUSION

Based on Wigner-Yanase skew information for quantum channels, we have derived several uncertainty relations for arbitrary $N$ quantum channels. By detailed examples we have shown that our uncertainty relations improve the existing ones. We also get several uncertainty relations for $N$ unitary channels. It can be regarded as variance-based sum uncertainty relations for $N$ unitary operators as we take the pure state. These results and the simple approaches used in this article may highlight further investigations on related uncertainty relations.

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