Stability in a many-to-one job market with general increasing functions

Yasir ALI\textsuperscript{a}, Baqar ALI\textsuperscript{b}

\textsuperscript{a} National University of Sciences and Technology, College of Electrical and Mechanical Engineering, Peshawar Road, Rawalpindi, Pakistan
\textsuperscript{b} Riphah International University, I-14, Islamabad, Pakistan

Abstract

We consider an occupation market in which preferences of members are treated as non linear general increasing functions. The arrangement of members is separated into two non over-lapping sets; set of workers and set of firms. We consider that firms have vacant posts. Every worker needs a job and firms have opportunity to contract more than one workers. A worker can work for just in at most one firm. We demonstrate the existence of pairwise stability for such a business sector. Our model is the augmentation of the Ali and Farooq \cite{ali2013} model by considering non linear valuations and bounded side payments.

keyword: Stable matching, many-to-one matching, indivisible goods, increasing valuations

1 Introduction

Over the last few decades, a lot of research has been carried out in the two-sided matching problem by a large number of scholars. In a two-sided matching problem, the set of members is divided into two non over-lapping sets, where each member has a list of preferences which has the members of another set. Basically the matching is a mapping from one set of members to another set. More precisely the goal of two sided matching is to formulate the partnership between the members of two sets. This matching must be stable as well.

A matching $X$ is called stable matching if there is no blocking pair and all members currently matched are mutually acceptable. The concept of two-sided stable matching was introduced by Gale and Shapley \cite{gale1962} in their famous article “Marriage problem and College admission”. Another fundamental article in this area is due to Shapley and Shubik \cite{shapley1972}, that is known as assignment game. Gale and Shapley’s marriage model and Shapley and Shubik’s assignment game have been broadly studied and a number of extensions of these can be found in the literature. Many researchers studies the extension based one number of partners in matching. More specially, these models that involve many-to-many and many-to-one matchings. The other important direction of research is to look for the most general valuation function that includes these two basic models, Gale and Shapley \cite{gale1962} and Shapley

\footnote{corresponding author
E-mail address: cyasirali@gmail.com (Y. Ali).}
and Shubik [10] and well known generalization of these models like [6], Sotomayor [11] and [7], as special cases. In this context, Farooq [7] and Ali and Farooq [4] are particularly important. In Farooq [7], valuation function is considered as linear function of money where money in [7] is taken to be continuous variable. Eriksson and Karlander [6] and Sotomayor [11] both generalize the marriage model and assignment game. Both [6] and [11] are the special cases of Farooq [7]. Whereas in Ali and Farooq [4], many-to-one matching is considered with preferences of the players are represented as a linear function of money. Thus the models [8],[10], [7] become the special cases of Ali and Farooq [4]. We can find many articles with many-to-many and many-to-one matching. Many-to-one matching models due to Pycia [9] and Echenique and Yenmez [5] consider two different matching models. For the models when there complementarities and peer effects, Pycia [9] provides some important condition for the stability. Many-to-one matching model [5] provides solution for the matching market where the preferences are over the colleagues.

Here we highlight some related models that generalize valuation functions. The linear valuation are presented in Farooq [7] and Ali and Farooq [4] with continuous variable, but the valuation defined on discrete domain, is presented in Ali and Farooq [3]. The model [3] addresses to the situations where the side payments are indivisible.

In Ali [1], the set of members is classified into two non over-lapping sets: a set of workers and a set of firms. The set of workers is further classified into subsets, that represent different categories in everyday life. Firms are bounded to hire more than one worker from any category. A worker can work in only one category for at most one firm. This is a novel idea that is different from simple many-to-one matching that we have considered in our model.

Recently, Ali and Javaid [2] presented a model which involves general increasing functions. This model [2] is generalization of [3]. In this work we have considered a generalization of Ali and Javaid [2] model. In our model a job market is considered and the valuation of members are treated as general increasing functions which may be nonlinear with discrete side payments and the matching is many-to-one. The general nonlinear function for the valuations are $f_{ij}(z)$ and $f_{ji}(-z)$ for each $(i,j) \in E$. A firm can hire more than one worker and vacant positions are available for worker’s job. The number of workers that a firm can hire is known as the quota of that firm. Workers can work for only at most one firm in this model.

In Section 2 describes of our model briefly. Section 3 gives the chronological mechanisms for worker and firm. Section 4 describes the outcome and pairwise stability for our model. We devise an algorithm which finds a stable outcome in our model in Section 5. In Section 6, we discuss the main result of our model.

2 The Mathematical Model

We represent a model of matching market for job allocation of workers and firms. Let us consider a matching market in which there are two non over-lapping sets of worker and firms.
The requirement of each firm is the workers and the requirement of each worker is a job in a firm. A worker gives a financial benefit in form of revenue to the firm and firm pays a reward (money) to its workers, called *salary* of the worker. Firms hire workers to fill up their vacant positions.

Let us express our model mathematically, we take two sets $F$, a set of firms and $W$, a set of workers. Here $E = F \times W$ shows all possible pairs of firms and workers. Each firm hires some workers from all available workers. The maximum number of workers that a firm $j$ requires is denoted by $\mu(j)$ and is known as the quota of a firm $j$. A firm can not hire workers more than $\mu(j)$. Note that each worker can work for only at most one firm.

To increase or decrease the salary depends on the worker and firm consultancy. It is assumed that the salaries are bounded in this model i.e $\forall (i, j) \in E$, $a_{ij}$ and $b_{ij}$ shows the lower and upper bounds on salary, where $a, b \in \mathbb{Z}^E$. Let $p = (p_{ij} | (i, j) \in E) \in \mathbb{Z}^E$ shows the salary vector for all $(i, j) \in E$. It is feasible if $a \leq p \leq b$. We consider the preferences of players as the strictly general increasing functions, known as generalized increasing valuations. The valuation stand for estimation of real property or some asset.

For all $(i, j) \in E$, define $f_{ij}(z)$ and $f_{ji}(-z)$ from $\mathbb{Z}$ into $\mathbb{R}$, where $z \in \mathbb{Z}$. $f_{ij}(z)$ stands for the valuation of the worker $i$, at a salary $z$ from a firm $j$, when worker $i$ joins firm $j$ (that is, matched with $j$). Similarly, $f_{ji}(-z)$ stands for the valuation of the firm $j$ when it hires worker $i$ and pays it salary $z$. It is important to note here that valuations play a significant role in establishing the stability of matching.

### 3 The Firm-Worker Chronological Mechanism

Using valuations mentioned above, a comparison of the players of both sets can be made. A worker $i$ will prefer the firm $j$ to the firm $j^*$ at salary $z, z^* \in \mathbb{Z}$, if $f_{ij}(z) > f_{ij^*}(z^*)$. If $f_{ij}(z) = f_{ij^*}(z^*)$ then $i \in W$ is indifferent between firms $j$ and $j^*$ at salary $z, z^* \in \mathbb{Z}$.

Let us define the terms “indifferent” and “prefer” for a firm in the same sense. If a worker is agreed for a job in a firm then it means the firm is acceptable to that worker. Similarly, if a firm has willing to hire a worker then it means the worker is acceptable to that firm. By $f_{ij}(z) \geq 0$, represents that $j$ is acceptable to $i$ at salary $z \in \mathbb{Z}$ and $f_{ji}(-z) \geq 0$ represents $i$ is acceptable to $j$ at salary $z$.

### 4 Outcome and Pairwise Stability in Matching

In the two-sided stable matching theory, where firm and worker involves money plays an important role. In this section, we describe the characteristic of an outcome for which it would be stable.

---

2 The notation $\mathbb{Z}$ stand for set of integers and notation $\mathbb{R}$ stand for set of real numbers. The notation $\mathbb{Z}^E$ stands for integer lattice whose points are indexed by $E$.

3 For any $x, y \in \mathbb{Z}$, we define $[x, y] \mathbb{Z} = \{a \in \mathbb{Z} | x \leq a \leq y\}$.
Let $E$ be the set of all possible firm-worker pairs. A subset $X$ of a set $E$ is called matching if every member appear at most once in $X$. “A matching is called pairwise stable if it is not blocked by any worker-firm pair and all members of matched pairs are mutually acceptable.” We define $S = (S_j \mid j \in F)$, where $S_j$ is given below,

$$S_j = \{i \in W \mid i \text{ and } j \text{ are matched}\}. \quad (4.1)$$

If $S_j = \emptyset \ \forall j \in F$, we means all positions are vacant in the firm $j$. If $i \in S_j$ we means that a firm $j$ hires a worker $i$.

A set $X = \{(S_j, j) \mid j \in F\}$ is known as job allocation if

(i) $|S_j| \leq \mu(j) \ \forall j \in F$.

(ii) $S_j \cap S_{j^*} = \emptyset \ \forall j, j^* \in F$ with $j \neq j^*$.

First condition may be considered as quota requirement for all firms and second condition tells that no worker can work for more than one firm. This may be considered as quota condition for workers. For any feasible salary $p_{ij}$, for all $(i, j) \in E$ define $q \in \mathbb{R}^W$ as follows:

$$q_i = \begin{cases} f_{ij}(p_{ij}) & \text{if } i \in S_j \text{ for any } j \in F \quad (\forall i \in W). \\ 0 & \text{otherwise} \end{cases} \quad (4.2)$$

We have $r = (r_j \mid j \in F) \in \mathbb{R}^F$, is defined as

$$r_j = \begin{cases} \min\{f_{ji}(-p_{ij}) \mid i \in S_j\} & \text{if } |S_j| = \mu(j) \quad (\forall i \in W), \\ 0 & \text{otherwise} \end{cases} \quad (4.3)$$

where minimum over a null set is defined to be 0.

If $X$ is a job allocation then $(X; p, q, r)$ is known as an outcome, $p$ is feasible salary vector, and $q$ and $r$ are given by (4.2) and (4.3), respectively.

For convenience, we write that $S_j \in X$ (or $j \in X$), it always means that $(S_j, j) \in X$ and by $(i, j) \in X$, we always mean that $i \in S_j$.

An outcome will be blocked by a worker-firm pair in which firm and worker are not matched to each other but both of these unmatched members prefer each other to their current partners. Note here that there may be some members who are not matched to any member from the opposite set, such a member is called self matched. In mathematical language, the outcome $(X; p, q, r)$ has a blocking pair $(i, j) \in E$ if $\exists \theta \in \mathbb{Z}$ with $a_{ij} \leq \theta \leq b_{ij}$ such that $i \notin S_j$ and $f_{ij}(\theta) > q_i$, $f_{ji}(-\theta) > r_j$.

### 4.1 Pairwise Stability

Here for outcome $(X; p, q, r)$, the pairwise stability is defined as:

(ps1) $f_{ij}(p_{ij}) \geq 0$ and $f_{ji}(-p_{ij}) \geq 0 \ \forall (i, j) \in X$.

(ps2) $f_{ij}(\theta) \leq q_i$ or $f_{ji}(-\theta) \leq r_j, \forall \theta \in \mathbb{Z}$ with $a_{ij} \leq \theta \leq b_{ij}$ and $\forall (i, j) \in E$. 

4
5 Existence of a Stable Outcome in This Model

In this part, it will be shown that the pairwise stability always exist for the model given in previous segment. We develop an algorithm to show the existence of pairwise stable outcome. Initially, the highest feasible salary is set out in the algorithm in such a way that firms are acceptable to workers at that salary. It is important to note that at this initial value of the salary workers may not be acceptable to some firms. In other words, we can say that at initially fixed salary firms and workers may not be mutually acceptable. Then the mutually acceptable pairs of workers and firms are found. After this workers are to be engaged to the firms that the workers prefer most. At this point firms accept proposals by considering their quota and gain. Due to these constraints there may be some rejections. These rejections induce the modification in salary vector. The salary is adjusted for a pair if worker is rejected by the firm he or she prefers most. Salary is adjusted in every iteration conserving the feasibility, till the pairwise stability is obtained. The algorithm will stop if there are no rejections. Lastly, we will show that when the algorithm terminates, the output is a stable matching.

Now \( \forall (i, j) \in E, p_{ij} \in \mathbb{Z} \) is given as below:

\[
p_{ij} = \begin{cases} b_{ij} & \text{if } f_{ji}(-b_{ij}) \geq 0 \text{ and } \max\{a_{ij}, -f_{ji}^{-1}(0)\} \\ \text{elsewhere} & \end{cases} \quad (\forall (i, j) \in E). \tag{5.4}
\]

Equation (5.4) guarantees that the salary vector \( p \) is a feasible. Also, note that \( p \), defined by (5.4), is the maximum integer in \([a, b]\) for which \( f_{ji}(-p_{ij}) \geq 0 \) \( \forall (i, j) \in E \).

Before presenting the algorithm in mathematical form, let us define some subsets of set \( E \) which are useful to obtain a matching \( X \) that satisfies the condition (ps1). First, the subset \( W_0 \) and \( F_0 \) of set \( E \), containing those worker-firm pairs that are not mutually acceptable are defined as:

\[
W_0 = \{(i, j) \in E \mid f_{ij}(p_{ij}) < 0\}. \tag{5.5}
\]

\[
F_0 = \{(i, j) \in E \mid f_{ji}(-p_{ij}) < 0\}. \tag{5.6}
\]

\( W_0 \) is the set containing those pairs where worker is not willing to work in firm and \( F_0 \) is the set containing those pairs where firm is not willing to hire the worker at \( p \) given by \( \text{[5.4]} \).

Next, we define \( \tilde{E} \) by

\[
\tilde{E} = E \setminus \{W_0 \cup F_0\},
\tag{5.7}
\]

or

\[
\tilde{E} = \{(i, j) \in E \mid f_{ij}(p_{ij}) \geq 0 \text{ and } f_{ji}(-p_{ji}) \geq 0\}. \tag{5.8}
\]

\( \tilde{E} \) contains the set of mutually acceptable players.
For each $i \in W$ define $\tilde{q}_i$ as follows:

$$\tilde{q}_i = \max \{ f_{ij}(p_{ij}) \mid (i, j) \in \tilde{E} \} \quad (\forall i \in W). \quad (5.9)$$

In equation (5.9) $\tilde{q}_i$ represents the valuation of most preferred firm for $i \in W$. Also, define $\tilde{E}_W$ as follows:

$$\tilde{E}_W = \{(i, j) \in \tilde{E} \mid f_{ij}(p_{ij}) = \tilde{q}_i \}. \quad (5.10)$$

Equation (5.10) shows that $\tilde{E}_W$ contains those worker-firm pairs where firm is most preferred for the worker. Since $\tilde{E}_W \subseteq \tilde{E}$ this means that these pairs are acceptable. Initially consider vector $r = 0$ and the subset $\hat{E}_W$ of $\tilde{E}_W$ is given by:

$$\hat{E}_W = \{(i, j) \in \tilde{E}_W \mid f_{ji}(-p_{ij}) \geq r_j \}. \quad (5.11)$$

It can be noted that $\hat{E}_W = \tilde{E}_W$ where $r = 0$. But in the further iterations of algorithm $\tilde{E}_W$ may become a proper subset of $\hat{E}_W$.

At the start of the algorithm, there is no matching $X$, so $\tilde{F} = \emptyset$, where $\tilde{F}$ shows the set of matched firms in $X$, and defined by

$$\tilde{F} = \{ j \in F \mid j \text{ is matched in } X \}. \quad (5.12)$$

Now, we find a job allocation $X = \{(S_j, j) \mid j \in F \}$, in the bipartite graph $(W, F; \tilde{E}_W)$ that satisfies the conditions given below:

- $X$ matches all members of $\tilde{F}$, \quad (5.13)
- $X$ minimizes $||S_j| - \mu(j)||$, among the matchings that satisfy (5.13), \quad (5.14)
- $X$ maximizes $\sum_{(i, j) \in X} f_{ji}(-p_{ij})$ among the matchings that satisfy (5.13) and (5.14), \quad (5.15)

Initially $\tilde{F} = \emptyset$, which follows that any matching satisfies (5.13). The outcome $(X; p, q, r)$ up till these steps clearly satisfies the (ps1). For the satisfaction of (ps2), next we define a set $U$ having all mutually acceptable worker-firm pairs. Also firm is most preferred for the worker but it reject the workers and consequently is unmatched in the matching $X$ by:

$$U = \{(i, j) \in \tilde{E}_W \mid i \notin S_j \}. \quad (5.16)$$

$U$ has worker-firm pairs in which worker is not matched to his or her most preferred firm. As the set $\tilde{E}_W$ has such mutually acceptable pairs and the firm is most preferred for the worker out of all firms.

If $U = \emptyset$, then do not modify the salary vector $p$ but if $U \neq \emptyset$, then salary vector will be modified at each iteration of STEP[3] by satisfying (ps1). New salary vector $\tilde{p}$ must be
feasible, that is \( a_{ij} \leq \bar{p}_{ij} \leq b_{ij} \).

As here general non linear strictly increasing functions are considered to represent the valuations, thus to modify salary vector \( p \), a real number \( m_{ij}^* \in \mathbb{R}^{++} \) can be found for each \((i, j) \in U\), such that

\[
f_{ji}(-(p_{ij} - m_{ij}^*)) = r_j. \tag{5.17}
\]

As we have considered discrete salaries, so we define an integer \( m_{ij} \forall (i, j) \in U \) as follows:

\[
m_{ij} = \max \{1, \lceil m_{ij}^* \rceil \} \forall (i, j) \in U. \tag{5.18}
\]

The number \( m_{ij} \forall (i, j) \in U \) is an integer such that

\[
f_{ji}(-(p_{ij} - m_{ij})) \geq r_j, \tag{5.19}
\]

Note that \( \forall (i, j) \in U \) \( p_{ij} - m_{ij} \) is an integer and \( m_{ij} \) is the minimum positive integer, which satisfies the inequality given by the equation (5.19).

Here the integer \( m_{ij} \) for all \((i, j) \in U\) helps us to find the new salary vector such that condition (ps2) is also satisfied. Now we define a subset \( L \) of \( U \) that has the pairs from the \( U \) for which modified salary will not be feasible.

\[
L = \{(i, j) \in U \mid p_{ij} - m_{ij} < a_{ij}\}. \tag{5.20}
\]

The modified salary vector \( \tilde{p} \) must also be feasible and is given as:

\[
\tilde{p}_{ij} := \begin{cases} 
\max \{a_{ij}, p_{ij} - m_{ij}\} & \text{if } (i, j) \in U \\
p_{ij} & \text{otherwise}
\end{cases} \quad (i, j) \in E. \tag{5.21}
\]

We also define a subset \( \tilde{W}_0 \) of \( U \) by:

\[
\tilde{W}_0 := \{(i, j) \in U \mid f_{ij}(\tilde{p}_{ij}) < 0\}. \tag{5.22}
\]

In the algorithm the modified salary vector will always decrease and the size of matching \( X \) will increase. Also, the participants can change their preferences according to new salary vector. Now, we suggest the algorithm.

**Job Allocation Algorithm**

**STEP[0a]:** First define \( p, W_0, F_0, \tilde{E}_0, \tilde{q}, \tilde{E}_W \) using equation (5.4)-(5.7), (5.9), and (5.10), respectively. Set \( X = \emptyset \) which gives \( r = 0, q = 0 \) and \( \tilde{F} = \emptyset \) by (4.3), (4.2) and (5.12). Also define \( \hat{E}_W \) by (5.11).

**STEP[0b]:** Form the bipartite graph \((W, F, \hat{E}_W)\) find a matching that satisfies (5.13) to (5.15). Define \( U \) by (5.16) and update \( S \) and \( r \) by (4.1) and (4.3) respectively.

**STEP[1]:** If \( U \neq \emptyset \) go to STEP[2], otherwise define \( q \) by (4.2) and stop.
STEP[2]: To update \( \tilde{p} \) compute \( m_{ij} \) for all \((i, j) \in U \) by (5.18) and update \( \tilde{p} \) by (5.21) and set \( p = \tilde{p} \).

STEP[3]: Define \( L \) by (5.20) and \( \tilde{W}_0 \) by (5.22). Modify \( W_0 : W_0 \cup \tilde{W}_0 \), and \( F_0 : F_0 \cup L \) and update \( \tilde{E} \) by (5.3). Modify \( \tilde{q}, \tilde{E}_W \) and \( \tilde{E}_W \) by (5.8), (5.10) and (5.11), respectively.

STEP[4]: Form the bipartite graph \((W, F, \tilde{E}_W)\) find a matching that satisfies (5.13) to (5.15). Define \( U \) by (5.16) and update \( S \) and \( r \) by (4.1) and (4.3) respectively. Go to STEP[1].

6 The Main Results

In this segment we will express that we developed a model by taking the preferences of members as non linear strictly increasing general function. Also this is many-to-one matching model and money is taken as discrete variable. In this section, we will prove some important lemmas and theorems. We will put prefixes \((old)\star \) and \((new)\star \) before and after the updating the integers/sets/vectors in any iteration of algorithm. The main result is the Lemma 6.1 that we prove here using the assumption given in equation (5.18). The proof of Lemma 6.2 is the direct consequence of Lemma 6.1.

**Lemma 6.1.** For each \((i, j) \in U \), \( f_{ji}(-\tilde{p}_{ij}) \geq r_j \) holds at STEP[2] in each iteration of the algorithm. Also for any \((i, j) \in U \), \( p_{ij} - m_{ij} \) is the maximum integer, whenever \( f_{ji}(-(p_{ij} - m_{ij})) > r_j \).

**Proof.** From equation (5.21) we have \( \forall (i, j) \in U \)

\[
\tilde{p}_{ij} = \max \{a_{ij}, p_{ij} - m_{ij}\}.
\]

Without loss of generality we assume that \( p_{ij} - m_{ij} > a_{ij} \), that is, \( \tilde{p}_{ij} = p_{ij} - m_{ij} \). Then by the definition of \( m^*_{ij} \) given in (5.17), we have

\[
\tilde{p}_{ij} = p_{ij} - m_{ij} \leq p_{ij} - m^*_{ij}.
\]

We can easily write it as \( -\tilde{p}_{ij} = -(p_{ij} - m_{ij}) \geq -(p_{ij} - m^*_{ij}) \). Now by the nature of valuation functions we obtain the following relation

\[
f_{ji}(-\tilde{p}_{ij}) = f_{ji}(-(p_{ij} - m_{ij})) \geq f_{ji}(-(p_{ij} - m^*_{ij})) = r_j \quad \text{by (5.17)}.
\]

Thus we have

\[
f_{ji}(-\tilde{p}_{ij}) \geq r_j.
\]

This completes the proof of first part.

For the second part of the lemma, suppose that \( f_{ji}(-(p_{ij} - m_{ij})) > r_j \), on contrary suppose that \( p_{ij} - m_{ij} \) is not maximum for which the above inequality holds. This means
that there exists another integer \( p_{ij} - m'_{ij} \) such that
\[
p_{ij} - m_{ij} < p_{ij} - m'_{ij}, \quad \text{where } m'_{ij} \in \mathbb{Z}^+.
\] (6.23)

and
\[
f_{ji}(-(p_{ij} - m'_{ij})) > r_j.
\] (6.24)

It follows equation (6.23) that \( m_{ij} > m'_{ij} \) this means that
\[
m_{ij} = \max \{ 1, \lceil m^*_ij \rceil \} > m'_{ij}.
\]

Suppose \( m_{ij} = 1 \) then \( m'_{ij} < 1 \), which is not possible as \( m'_{ij} \) must be an integer. Now suppose that
\[
m_{ij} = \lceil m^*_ij \rceil > m'_{ij}.
\]

By definition of ceiling function
\[
m_{ij} \geq m^*_ij > m'_{ij},
\]
which implies that
\[
-(p_{ij} - m^*_ij) > -(p_{ij} - m'_{ij}),
\]
\[
f_{ji}(-(p_{ij} - m^*_ij)) > f_{ji}(-(p_{ij} - m'_{ij})).
\]

Using equation (5.17), we have
\[
f_{ji}(-(p_{ij} - m'_{ij})) < r_j,
\]
which is contradiction to (6.24). Hence \( p_{ij} - m_{ij} \) is the maximum integer. \( \square \)

The base of our model is Lemma 6.1 that allows us to calculate a new salary vector by preserving the of mutually acceptability and feasibility condition for the salary vector.

**Lemma 6.2.**

At STEP[3] \( \tilde{p}_{ij} \) is feasible and \( f_{ji}(-\tilde{p}_{ij}) \leq (\text{old})r_j \) for each \((i, j) \in L\).

*Proof.* To show the feasibility of \( \tilde{p}_{ij} \), \( \forall (i, j) \in L \) it would be enough to show that \( \tilde{p}_{ij} = a_{ij} \), \( \forall (i, j) \in L \). As \( L \subseteq U \) and for each \((i, j) \in U \), we have from (5.21)
\[
\tilde{p}_{ij} = \max \{ a_{ij}, p_{ij} - m_{ij} \}.
\]

By equation (5.20) we know \( \forall (i, j) \in L \),
\[
p_{ij} - m_{ij} < a_{ij},
\]
So by equation (5.20) and (5.21), and fact that \( L \subseteq U \), it is obvious that \( \tilde{p}_{ij} = a_{ij}, \forall (i, j) \in L \).
By first part of the lemma we know that

\[ p_{ij} - m_{ij} < a_{ij} = \tilde{p}_{ij}, \]
\[ -\tilde{p}_{ij} < -(p_{ij} - m_{ij}), \]
\[ f_{ji}(-\tilde{p}_{ij}) < f_{ji}(-(p_{ij} - m_{ij})). \]

We know by lemma 6.1 that \( f_{ji}(-(p_{ij} - m_{ij})) \geq r_j \). Result is trivial for \( f_{ji}(-(p_{ij} - m_{ij})) = r_j \). The inequality is true for a maximum integer \( p_{ij} - m_{ij} \) by lemma 6.1. As \( p_{ij} - m_{ij} < \tilde{p}_{ij} \) that is why \( f_{ji}(-\tilde{p}_{ij}) \leq (old)r_j \). Hence the result is proved.

The next lemma shows that we can find a matching in the bipartite graph \((W, F; \hat{E}_W)\) which satisfies the equations (5.13) to (5.15).

**Lemma 6.3.** At STEP[4] of algorithm, we can always find a matching \(X\) satisfying the equations (5.13) to (5.15).

**Proof.** If it is proved that \((old)X \subseteq (old)\hat{E}_W\) in each iteration of the STEP[4]. It completes the proof. We update the salary vector \(p\) and \(\hat{E}\) in each iteration at STEP[2] and STEP[3] by equation (5.4) and (5.7). It is clear from these two equations that these modifications are only element and subsets of \(U\). Since \(U \cap (old)X = \emptyset\). Therefore it implies that \((old)X \subseteq (old)\hat{E}_W\).

**Lemma 6.4.** The properties:

(i) For each element of \(U \setminus \{L \cup \tilde{W}_0\} \neq \emptyset\), salary \(p\) decreases at STEP[3], otherwise salary vector remains the same.

(ii) For each element of \(L \cup \tilde{W}_0\), at STEP[3], \(\hat{E}\) reduces otherwise remains same.

(iii) The vector \(r\) increases or remains the same.

**Proof.** (i) Initially the salary vector \(p\) is given by (5.4) and updated in each iteration by (5.21). From (5.21), it follows that for each \((i, j) \in U\), \(\tilde{p}_{ij} \leq p_{ij}\), the inequality may be true for \((i, j) \in L \cup \tilde{W}_0\).

(ii) Initially \(\hat{E}\) is given by (5.7) and it is updated at STEP[3] in each iteration. At STEP[3], \(F_0 = F_0 \cup L\) and \(W_0 = \tilde{W}_0 \cup W_0\), by equation (5.7) \(\hat{E}\) reduces if \(L \neq \emptyset\) and \(\tilde{W}_0 \neq \emptyset\) at STEP[3], if \(L = \emptyset\) and \(\tilde{W}_0 = \emptyset\), \(\hat{E}\) will not be changed by equation (5.7).

(iii) At the start Step[0], we set \(r = 0\). Later on we modify \(r\) by (4.3). The matching \(X\) satisfies condition (5.13). In each iteration, this means that \(\tilde{F} \subseteq \tilde{F}\). Also \((new)p \leq (old)p\) by part (ii) of Lemma 6.4. Thus \((new)r_j = f_{ji}(-(new)p_{ij}) \geq (old)r_j\). For \(j \in (old)\tilde{F}\), as matching \(X\) also satisfies (5.15). Moreover, \((new)r_j = (old)r_j = 0 \ \forall j \in F \setminus (new)\tilde{F}\). Hence, the vector \(r\) increases or remains same.

**Theorem 6.5.** If the algorithm terminates it produces a stable outcome.
Proof. We know that \( X \subseteq \tilde{E} \). Initially \( \tilde{E} \) is defined by (5.7) and afterwards it is updated at STEP[3] in each iteration. Thus \( f_{ij}(p_{ij}) \) and \( f_{ji}(-p_{ij}) \) are non-negative \( \forall (i, j) \in \tilde{E} \). Therefore, \( f_{ij}(p_{ij}) = 0 \) and \( f_{ji}(-p_{ij}) = 0 \ \forall (i, j) \in X \). This shows that the \( X \) satisfies (ps1) at termination. On contrary to (ps2), assume that there exist a \( c \in [a, b] \) and \( (i, j) \in E \) such that

\[ f_{ij}(c) > q_i \text{ and } f_{ji}(-c) > r_j. \]

If we take \( p_{ij} < c \) it yields \( f_{ji}(-p_{ij}) > f_{ji}(-c) > r_j \). But according to Lemma 6.4(1), \( p_{ij} \) is the maximum integer for which this inequality holds. Thus \( p_{ij} < c \) is not true. Let us consider that \( p_{ij} = c \), which implies that

\[ f_{ij}(c) = f_{ij}(p_{ij}) > q_i. \tag{6.25} \]

However, at termination we have \( U = \emptyset \) means that \( (i, j) \in U \) and since \( (i, j) \) are not matched, therefore, \( f_{ij}(p_{ij}) < \tilde{q}_i = q_i \). A contradiction to (6.25). Thus (ps2) holds when the algorithm terminates.

Theorem 6.6. The algorithm terminates after finite number of iterations.

Proof. Termination of the algorithm depends upon set of mutually acceptable pairs and salary vector \( p \). By the Lemma 6.4 part (ii), \( \tilde{E} \), reduces either \( L \neq \emptyset \), and \( \tilde{W}_0 \neq \emptyset \), or remain unchanged. This case is true at most \( |E| \), times.

If \( L = \tilde{W}_0 = \emptyset \), then, by part (i) of Lemma 6.4 \( p_{ij} \) decreases for each \( (i, j) \in U \). Otherwise, \( p \) remains unchanged. As we know that \( p \) is bounded and discrete, therefore, it can be decreased a finite number of times. This proves that in either case our algorithm terminates after a finite number of iterations.

This is the most important result which establishes the existence of pairwise stability for our model.

7 Open Problem

- We generalized a one-to-one matching model of Ali and Farooq [3] in two directions. Firstly, our model is many-to-one model, where as the model given by Ali and Farooq [3] is one-to-one. Secondly the we the valuations of the players in our model are represented by strictly increasing general functions of money. Money is not a continuous variable here. We can establish similar results by taking money as continuous variable, it will include a large number of well known models as special cases. A many-to-many version of our model would be an interesting problem.

- A slightly harder but worth while problem would to design a polynomial time algorithm for these models.
8 Concluding Remarks

We have represented a many-to-one matching market in which valuations are represented by general non linear increasing functions. Each firm can hire the workers according to its demand, but workers can not work for more than one firms. In this section we represent some key functions and important remarks about the model presented in this chapter.

1. As proved in theorem (3.6.5), the algorithm always produces a stable many-to-one matching, at termination. This guarantees the existence of stable outcome for our model.

2. The algorithm presented here outputs a worker-optimal matching. This work is obvious from the result discussed in Lemma [6.1] and the due increasing nature of valuation function.

3. Marriage model by Gale and Shapley [8], and model presented by Ali and Farooq [3], Ali and Javid model [2], are the special cases of this model.

References

[1] Y. Ali: Stability in a job market with linearly increasing valuations and quota system. Turk J Math (2015); 39: 427-438.

[2] Y. Ali and A. Javid: Pairwise Stability in Two Sided Market with Strictly Increasing Valuation Functions. Discret math theor comput sci, (To appear).

[3] Y. Ali and R. Farooq: Pairwise stability in a two-sided matching market with indivisible goods and money. J Oper Res Soc Jpn (2011); 54: 1-11.

[4] Y. Ali and R. Farooq: Existence of the stable outcome for linear valuations and possibly bounded salaries. Pac J Optim (2011); 7: 531-550.

[5] F. Echenique and M.B. Yenmez: A solution to matching with preferences over colleagues. Games Econ Behav (2007); 59: 46 – 71.

[6] K. Eriksson and J. Karlander: Stable matching in a common generalization of the marriage and assignment Models. Discrete Math (2000); 217: 135 – 156.

[7] R. Farooq: A polynomial-time algorithm for a stable matching problem with linear valuations and bounded side payments. Jpn J Ind Appl Math (2008); 25: 83-98.

[8] D. Gale and L.S. Shapley: College admissions and the stability of marriage. Amer Math Monthly (1962); 69: 9-15.
[9] M. Pycia: Many-to-One Matching without Substitutability. MIT Industrial Performance Center Working Paper 008/2005

[10] L. S. Shapley and M. Shubik: The assignment game I: The Core. Internat J Game Theory (1972); 1: 111-130.

[11] M. Sotomayor: Existence of stable outcomes and the lattice property for a unified matching market. Math Social Sci (2000); 39: 119-132.