Aspects of Born-Infeld Theory and String/M-Theory

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March 27, 2022

Abstract

1 Introduction

M/String Theory is the currently most popular approach to a unified quantum theory of gravity and the other interactions. We still lack a complete formulation of the theory, but there is a general consensus that whatever finally emerges it will involve in some way or to some degree of approximation, $p$–branes, i.e. $p+1$-dimensional Lorentzian submanifolds $\Sigma$ of a Lorentzian spacetime manifold $M$. In M-theory one supposes that $M$ is eleven dimensional. In string theory it is usually taken to be ten dimensional. Branes may crudely be sub-divided into two types Heavy and Light. In the former case one is usually thinking of many coincident branes whose gravitational field and hence the ambient spacetime metric is non-trivial. Semi-classically these may be studied using supergravity techniques. The other extreme is to study a single isolated brane moving in flat Minkowski spacetime as a solution of the Dirac-Born-Infeld equations of motion. This will be the approach taken in these lectures. It is well suited to newcomers to the subject because, as I will try to show, considerable insights into string theory can be gained by asking some of the simplest physical questions. There is little need for the full heavy technical machinery of supergravity or superstring theories. Thus the material is well suited for presentation at a School. I have deliberately tried to keep things simple. This runs the risk that experts may feel that I have not done full justice to the subject or indeed their contributions to it. If so, I apologize but I repeat my aim was to provide the beginner with a rapid survey of the subject. I will mainly assume that the brane is flat. It
is fairly straightforward to extend the present circle of ideas to the case of a curved background. In the case of Born-Infeld theory the reader is referred to [23]. The detailed material to be covered is given below.

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2 Classical Causality and the Dominant Energy Condition

A major pre-occupation of at least some of the lectures at this school are various issues concerning causality and locality in quantum and classical and in commutative and non-commutative field theories. Indeed they have been part of the motivation of much of the work reported in what follows. It therefore seems appropriate to begin by recapitulating the role of the dominant energy condition, particularly in the light of some recent papers either reporting or theoretically analysing experimental results on the speed of light and which will be described elsewhere in these proceedings.

2.1 The Language of Cones

The appropriate formal language for the discussion is that of convex cones. Since this seems to be playing an increasingly important role in M-theory, we shall pause to develop it a little. In fact the theory may be developed for general convex cones, but the most interesting case is that of homogeneous self-dual cones. The discussion below will be assuming that the cones are quadratic but is couched in such a way that it extends in a straightforward way to a more general setting.

We suppose that an n-dimensional vector space $X$, ultimately the tangent space $T_x M$ of a spacetime $M$ at some point $x$, is equipped with a Lorentzian metric $g$. In this section we use the mainly minus signature $+,−,−\ldots,−$. Picking a time orientation allows us to define the (solid) cone $C_g$ of future directed causal vectors. In a time oriented orthonormal basis or Lorentz frame such that $V = (V^0, V)$, this consists of vectors satisfying $V^0 \geq |V|$. Conversely a vector $V \in C_g$ iff $V^0 \geq |V|$ in all Lorentz frames. If $W$ is another member of $C_g$ then $g(V, W) \geq 0$ and conversely if $g(V, W) \geq 0$ for all $V \in C_g$ then $W \in C_g$. One deduces that $C_g$ is a convex cone homogeneous with respect to the Causal group Caus$(n−1, 1)$, i.e. the semi-direct product of Lorentz group $SO(n−1, 1)$ with $\mathbb{R}_+$ acting as dilations. Thus $C_g = \mathbb{R}_+ SO(n−1, 1)/SO(n−1)$. The set
$Y$ of Lorentzian metrics $g$ on a fixed vector space $X$ is the homogeneous space $GL(n, \mathbb{R})/SO(n-1, 1)$ and it admits a partial order $<$ (in fact an interesting type of causal structure) which corresponds to inclusion of cones $g < g'$ iff $C_g \subset C_{g'}$. The inclusion need not be strict, i.e. the two cones may touch.

Associated with any convex cone $C \in X$ is the dual cone-cone or co-cone $C^*$ in the dual space $X^*$. This is defined as the set of covectors $\omega \in X^*$ such that $\omega.V \geq 0$ $\forall$ $V \in C$. It is a simple exercise to convince one'self that duality reverses inclusion, $C \subset C'$ iff $C'^* \subset C^*$. For a Lorentzian cone $C_g$ the dual cone is given by the inverse metric $C^* = G_g^{-1}$, and because one may use the metric to set up an isosomorphism between $X$ and $X^*$, the cone is said to be self-dual and one does not normally distinguish between $C_g$ and $C_g^*$. However with more than one metric in the game it is essential to make the distinction.

The idea duality provides a duality between ray (or particle) and wave in all areas of physics. The basic observation of De Broglie’s Ph D Thesis [1] may be summarized by saying was the in Relativity the unique Einstein light cone demanded by the Equivalence Principle permits an identification of the se two dual concepts and hence leads to Quantum Mechanics.

### 2.2 The Energy Conditions

Given a metric may regard the energy momentum tensor as a bilinear form $T_{\mu\nu}$ or an endomorphism $T^\mu_{\phantom{\mu}\nu}$, according to taste. The various energy conditions depend on the metric. Thus

#### 2.2.1 The Weak Energy Condition

One regards the energy momentum tensor as a quadratic form and demands that $T_{\mu\nu}V^\mu V^\nu \geq 0$ $\forall$ $V \in C_g$. In other words the quadratic form is non-negative on the cone $C_g$.

Because it is equivalent to $T_{00} \geq 0$ in all Lorentz frames, the weak energy condition is regarded as a fairly minimal requirement but it is violated in gauged supergravity theories. This is because they may contain scalar fields with negative potentials. Note that the sign of $T_{00}$ is independent of spacetime signature.

#### 2.2.2 The Strong Energy Condition

This is similar and captures the idea that gravity is attractive. It is used to prove the singularity theorems but it also implies that any cosmological term must be negative and is inconsistent with inflation. It is satisfied by all supergravity models in all dimensions. Essentially it is incompatible with potentials for scalar fields which are positive.

One again regards the energy momentum tensor as a quadratic form and demands now that $(T_{\mu\nu} - \frac{1}{n-2}g_{\mu\nu}T^\rho_{\phantom{\rho}\rho})V^\mu V^\nu \geq 0$ $\forall$ $V \in C_g$.

By contrast
2.2.3 The Dominant Energy Condition

is most easily expressed by regarding the energy momentum tensor as an endomorphism and demanding that it maps $C_g$ into itself. That is if $V^\mu$ is causal and future directed then so is $T^\mu_\nu V^\nu$. Thus $T^\mu_\nu V^\nu W^\mu \geq 0 \quad \forall V,W \in C_g$. An equivalent requirent is that $T_{00} \geq T^\mu_\nu \forall \mu \nu$, hence the name.

Note that the set $C_{\text{condition},g}$ of energy momentum tensors satsifying any one of these conditions with respect to a fixed metric $g$ is itself a convex cone inside the $\frac{1}{2}n(n+1)$-dimensional vector space $T$ of symmetric tensors. This accords with one’s general prejudice that the state spaces of physical systems or substances are often convex cones. The structure of these cones, their boundaries and extreme points and mutual dispositions and their dependence on $g$ is an interesting topic which time does not permit us to pursue in detail here. We merely remark that one may classify the possible energy momentum tensors by bringing them to canonical form (see e.g. [5] in the case of four dimensions). Generically one may diagonalize $T^\mu_\nu$ with respect to the metric $g^\mu_\nu$ and we get simple conditions in terms of the energy density and principal pressures. Because the metric $g^\mu_\nu$ is not positive definite there are also some exceptional cases. In this way one classifies the orbits of $SO(n-1,1)$ on the space of symmetric tensors. Now one may identify the extreme points, faces etc of the relevant cone.

An important application of the dominant energy condition is to the Positive Energy Theorem of Classical General Relativity which states that if locally the stress tensor lies everywhere in the dominant energy cone then the the ADM energy momentum vector $P_{ADM}^\mu$ of a regular asymptotically flat spacetime lies in the cone of future directed causal vectors.

2.3 Ex nihilo nihil fit

We now outline an elegant argument of Hawking [4, 5] which shows that even if the background metric is time dependent the dominat energy condition implies causal propagation. If the metric is time independent this follows form energy conservation but energy conservation fails if the metric is time dependent and one might worry that that classically matter might appear ”out of nowhere” that is it might travel at superluminal speeds. The point of Hawking’s argument is that this cannot happen classically. Of course quantum mechnically things are different, a point made strenuously by Zel’dovich [6, 7]. Pair-creation processes in external fields often give the appearance of a-causality because, thought of as a tunnelling process, especially using the semi-classical or instanton approximation the particles suddenly materialize at spacelike separations. In this context the instanton is often called a bounce. Think of an electron positron pair in an external electric field for example. The instanton is a closed circle in Euclidean spacetime which analytically continues to a pair of causally disjoint timelike hyperbolae in Minkowski spacetime [8]. If, rather than using the instanton, one considered a smooth world line in spacetime with continous tangent vector it is clear that in neighbourhood of a creation event the tangent vector must be
A related point is that the quantum mechanical Feynman propagator in contrast to the classical retarded or advanced propagator has support both inside and outside the light cone. As Feynman has pointed out, this is because while one may not be able to join two spacelike points by a smooth timelike curve one may join them by one which is piecewise smooth and consisting of some past directed and some future directed intervals. Where the past directed and future directed intervals join is the site of a pair-annihilation of or pair-creation event.

Now one might try to describe the pair creation process using a regularized expectation value of energy momentum tensor operator, that is

\[ T^{\mu \nu} = \langle \hat{T}^{\mu \nu} \rangle. \]  

Zel’dovich and Pitaevsky [7] pointed out that the dominant energy property cannot and does not survive the regularization process.

Here is Hawking’s argument. Let \( U \) be a compact region of a spacetime \( M \) admitting a time function \( t \) whose gradient \( \partial_\mu t = V_\mu \). Let the level surfaces of the time function be called \( \Sigma_t \) and the part of \( U \) earlier than \( \Sigma_t \) , i.e. that part containing events at which the time function is less than \( t \) is called \( U_t \). The boundary \( \partial U \) decomposes into three components \( \partial U_1 \) and \( \partial U_2 \) on which the normal is non-spacelike and time function is decreasing or increasing along the outward normal , and \( \partial U_3 \) with spacelike normal. We note that

\[ J_\mu \Sigma^{\mu} = T^{\mu \nu} V_\mu V_\nu, \]  

where, by the dominant energy condition, \( J^\mu = T_\mu^\nu V^\nu \) is a future directed timelike vector field. By the dominant energy condition and the compactness of of \( U \) that there exists a positive constant \( P \) such that

\[ T^{\mu \nu} V_{\mu \nu} \leq PT^{\mu \nu} V_{\mu \nu}. \]  

Let

\[ E(t) = \int_{\Sigma_t} J_\mu \Sigma^{\mu}. \]  

Clearly \( E(t) \geq 0 \) and \( E(t) = 0 \) implies that \( T^{\mu \nu} \) vanishes on \( \Sigma_t \).

Integration of (4) over \( U(t) \) gives

\[ E(t) \leq -\int_{U(t) \cap \partial U_1} J_\mu d\Sigma^\mu + \int_{U(t) \cap \partial U_3} J_\mu d\Sigma^\mu + P \int dt E(t'). \]  

We have used the fact that by the dominant energy condition

\[ \int_{U(t) \cap \partial U_2} J_\mu d\Sigma^\mu \geq 0. \]  

Now suppose that \( T^{\mu \nu} \) vanishes on \( \partial U_3 \), the timelike component of the boundary \( \partial U \) and so nothing flows into the region \( U \). We deduce that

\[ \frac{dE}{dt} \leq PE(t). \]  

6
Integration of this simple differential identity implies that

$$E(t) \leq E(t') \exp P(t - t').$$

(8)

and hence that if $E(t)$ vanishes at some time $t'$, then it must vanish for all times. This despite the fact that we have allowed for the possibility of a time dependent metric doing work on the matter. One cannot get something from nothing. To get a statement about causality we apply this result to the case when $U = D^+(S)$ the future Cauchy development of some set $S$. This is the set of all points $p$ such that every past directed causal curve through $p$ intersects $S$. If $T^{\mu\nu}$ vanishes on $S$ then it vanishes everywhere in $S$.

The results just given, and obvious generalizations show clearly that according to Maxwell’s equations, electromagnetic waves can never, in the sense defined above, travel faster than light.

3 Open Strings and D-branes

Branes may be incorporated in string theory if one contemplates opens strings whose ends are constrained (by Dirichlet boundary conditions) to lie on a $(p+1)$-dimensional submanifold $\Sigma_{p+1}$. Now open strings can couple minimally to vector $A_\mu$ at the ends of the strings. In the Polyakov approach one has an action of the form

$$-\frac{1}{2} \int_{\Sigma_1} d^2\sigma (G_{ab} + B_{ab}) \partial y^a \partial y^b + \int_{\partial \Sigma_1} A_a dy^a,$$

(9)

where the embedding of the string world sheet $\Sigma_1 \to M$ is given by $y^a = y^a(\sigma^A)$, $A = 1, 2$ and $a = 1, 2, \ldots, n = \dim M$, and $G_{ab}$ and $B_{ab}$ are the spacetime metric and Neveu-Schwarz two-form respectively.

One obtains an effective action for a D-brane if one “integrates out” all possible string motions subject to the Dirichlet boundary condition. The resulting action depends on the position of the D-brane and the pullback to the D-brane of the metric and Neveu-Schwarz two-form. It also contains the vector field $A_\mu$.

4 Dirac-Born-Infeld Actions

This is governed by the embedding $y : \Sigma_{p+1} \to M$ given in local coordinates by $y^a = y^a(x^\mu)$, where $a = 1, 2, \ldots, n = \dim M$ and $\mu = 0, 1, 2, \ldots, p$. It is

$$-T_p \int dx^{p+1} \sqrt{\det (g_{\mu\nu} + (2\pi\alpha') F_{\mu\nu} + B_{\mu\nu})},$$

(10)

where

$$g_{\mu\nu} = \eta_{ab} \partial_\mu y^a \partial_\nu y^b,$$

(11)

and

$$B_{\mu\nu} = B_{ab} \partial_\mu y^a \partial_\nu y^b,$$

(12)
are the pull-backs of the metric $\eta_{ab}$ and Neveu-Schwarz two-form $B_{ab}$ to the world volume $\Sigma_{p+1}$ of the $p$-brane.

The world-volume field $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  

(13)

One often defines

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + B_{\mu\nu}.$$  

(14)

This is invariant under a Neveu-Schwarz gauge transformation $B \rightarrow B - dC$ where $C$ is a one-form, if we transform $F \rightarrow F + dC$. One may check that this is consistent with the behaviour of the open string metric.

4.1 Monge Gauge

To proceed we fix some of the gauge-invariance associated with world sheet diffeomorphisms of the coordinates $X^\mu$ by using what is usually, and misleadingly called static gauge (since it applies in non-static situations) and which is more accurately and with more justice called Monge gauge. In effect we project onto a $p+1$ plane by setting $y^a = x^\mu, y^i$ and use the $n-p-1$ height functions $y^i, i = 1, 2, \ldots, n-p-1$ as scalar fields on the world volume. In the theory of minimal surfaces this is called a non-parametric representation. For Monge’s work see [8]. Of course there may not be a global Monge gauge, and we shall encounter this situation later.

The determinant then becomes (we use units in which $2\pi\alpha' = 1$),

$$\det(\eta_{ab} + \partial_\mu y^i \partial_\nu y^j + F_{\mu\nu}).$$  

(15)

It is evidently consistent to set the scalars to zero $y^i = 0$ and we then obtain the Lagrangian of Born and Infeld which is a special form of Non-Linear Electrodynamics.

4.2 Dimensional Reduction

The previous section result has a sort of converse. We could start with a pure Born-Infeld action in $n$ flat dimensions and dimensionally reduce to $p+1$ dimensions. We begin with

$$- \int d^n x \sqrt{-\det(\eta_{ab} + F_{ab})}.$$  

(16)

We make the ansatz $A_a = (A_\mu(x^\lambda), y^i(x^\lambda))$ and obtain the Monge-gauge-fixed Dirac-Born-Infeld action

$$- \int d^{p+1} \sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu} + \partial_\mu y^i \partial_\nu y^j)}.$$  

(17)

Thus all solutions of the Dirac Born-Infeld action are solutions of the Born-Infeld action. Interestingly in the case $p = 1$ we get a string action from the pure Born-Infeld action.
5 Non-Linear Electrodynamics

There are advantages in viewing the theory in this context. An excellent account of the theory is given in [11]. The general theory in four-spacetime dimensions ($p=3$) has equations

\begin{align*}
\text{curl} \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} : \quad \text{div} \mathbf{B} = 0 \\
\text{curl} \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} : \quad \text{div} \mathbf{D} = 0.
\end{align*}

5.1 Constitutive Relations

To close the system one needs constitutive relations $\mathbf{H} = \mathbf{H} (\mathbf{E}, \mathbf{B})$ and $\mathbf{D} = \mathbf{D} (\mathbf{E}, \mathbf{B})$ which, if one has a Lagrangian $L = L(\mathbf{E}, \mathbf{B})$, take the form

\begin{align*}
\mathbf{H} &= -\frac{\partial L}{\partial \mathbf{B}} \\
\mathbf{D} &= \frac{\partial L}{\partial \mathbf{E}}.
\end{align*}

Because

\begin{equation}
\mathbf{D} = \frac{\partial L}{\partial \dot{\mathbf{A}}},
\end{equation}

$\mathbf{D}$ is the canonical momentum density. Note also that the conserved electric charge is given by the flux of $\mathbf{D}$ and not as is often assumed, the flux of $\mathbf{E}$.

In what follows we shall denote by $K_{\mu\nu}$ the Ampère 2-form with components $(\mathbf{D}, \mathbf{H})$ and refer to $F_{\mu\nu}$ as the Faraday 2-form. Thus the equations of motion without sources are

\begin{equation}
d\mathbf{F} = 0 \quad d\star \mathbf{K} = 0.
\end{equation}

5.2 Lorentz-Invariance

The symmetry of the energy momentum tensor $T_{0i} = T_{i0}$ and hence the uniqueness of the Poynting vector requires that the latter be given by

\begin{equation}
\mathbf{E} \times \mathbf{H} = \mathbf{D} \times \mathbf{B}.
\end{equation}

This will follow if $L$ is constructed from the two Lorentz invariants

\begin{align*}
x &= \frac{1}{2} (\mathbf{B}^2 - \mathbf{E}^2) \\
y &= \mathbf{E} \cdot \mathbf{B}.
\end{align*}
5.3 Duality Invariance

The constitutive relations will permit the obvious rotation needed to rotate the two sets of equations \([8, 9]\) into themselves

\[
\begin{align*}
E + iH &\rightarrow e^{i\theta}(E + iH), \\
D + iB &\rightarrow e^{i\theta}(D + iB),
\end{align*}
\]

with \(\theta\) constant if

\[
E \cdot B = D \cdot H.
\]

Note that what we are encountering here is a non-linear form of the familiar linear Hodge duality. This gives a constraint on possible theories. For example if the Lagrangian depends arbitrarily on the invariants \(x\) and \(y\) it gives rise to a Lorentz-invariant theory. Imposing duality invariance reduces this freedom to that of a function of a single variable. For more details on duality invariance see \([19, 20]\) and \([11]\) which was not known to the authors of \([19, 20]\) when they were written.

5.4 Hamiltonian density

One has

\[
\mathcal{H} = T_{00} = E \cdot D - L.
\]

one may think of \(\mathcal{H} = \mathcal{H}(B, D)\) as the Legendre transform of the Lagrangian and is thus expressed in terms of the canonical variables \(B\) and \(D\) whose Poisson Brackets are

\[
\{B_i(x), D_j(y)\} = -\epsilon_{ijk}\partial_k\delta(x - y).
\]

5.5 Born-Infeld

We have

\[
L = 1 - \sqrt{1 - E^2 + B^2 - (E \cdot B)^2}
\]

and

\[
\mathcal{H} = \sqrt{1 + B^2 + D^2 + (B \times D)^2} - 1.
\]

A constant has been added to make the zero field have zero energy. This is not strictly necessary in the theory of branes since the notion of world volume energy is not well defined because there are no privileged coordinates on the brane. However it is convenient when making comparisons with standard flat space field theory. To do so we must however use Monge gauge.

Lorentz and Duality invariance are clear. Before the advent of String/M-theory the latter was rather mysterious. Nowadays it may be thought of as a manifestation of S-duality. In this way we see how Born-Infeld theory considered \(sui generis\) has important lessons for M/String theory. Conversely M/String theory throws light on Born-Infeld theory. We shall see more examples of this mutually symbiotic behaviour later.
6 The Maximal Electric Field Strength

If $B = 0$, the Born-Infeld Lagrangian is

$$L = 1 - \sqrt{1 - E^2}. \quad (33)$$

If we use a gauge in which $A_0 = 0$, we have

$$L = 1 - \sqrt{1 - A^2}. \quad (34)$$

The analogy with special relativity is clear. There will be an upper bound to the electric field strength. The special relativistic analogy may also be understood from the point of T-duality.

In string theory the existence of a maximal electric field strength may be understood dynamically as follows. A stretched open string of length $L$ has, in our units, elastic energy $L$. If it has charges +1 at one end and −1 at the other it will, in an electric field have energy $-EL$. This if $E > 1$ one may gain energy from the background electric field by creating open strings. This an electric field with strength greater than 1 will quickly breakdown and the electric field will be reduced to a value less than one.

Note that if one restores dimensions and units the critical field strength $E_c$ is given by

$$E_c = \frac{1}{2\pi\alpha'}. \quad (35)$$

In the zero slope limit $\alpha' \to 0$ there is no upper bound and in the strong coupling limit $\alpha' \to \infty$ the critical field goes down to zero. Later we will investigate the behaviour of the theory in this limit.

7 BIons

The maximal electric field was originally invoked to ensure the existence of a classical solution representing a charged object with finite total energy

$$\int_{B^3} d^3x T_{00} < \infty \quad (36)$$

This can be achieved by setting

$$D = \frac{q}{r^2} \hat{r}. \quad (37)$$

Because

$$E = \frac{D}{\sqrt{1 + D^2}} \quad (38)$$

the electric field achieves its maximal value at the centre. Note that

$$D = \frac{D}{\sqrt{1 - D^2}}. \quad (39)$$
the electric induction \( \mathbf{D} \) diverges at the origin and so does the energy density

\[
T_{00} = \mathcal{H} = \mathbf{E} \cdot \mathbf{D} - L. \tag{40}
\]

Thus this solution is not a smooth soliton solution without sources. In fact there is a distributional source

\[
\text{div}\, \mathbf{D} = 4\pi \eta \delta(\mathbf{r}). \tag{41}
\]

Finite energy but singular solutions like this of non-linear theories with distributional sources are a sufficiently distinct phenomenon from the familiar finite energy non-singular lump solutions without sources as to deserve a different name. The suggestion has been made \[15\] that they be called BIons. From the string point of view the source has a natural interpretation as being associated with a string ending on a three-brane. In fact one returns in this way to a picture very close to late nineteenth century speculations in which an electron is regarded as an "ether-squirt" on a 3-surface embedded in four dimensional space \[9\]. The application to strings is contained in \[15\] and \[22\]. The present account is largely based on \[15\].

7.1 Maximal Spacelike Hypersurfaces

Another interpretation of the static solutions may be obtained as follows. One introduces the electrostatic potential \( \phi = A_0 \) and finds the Lagrangian density to be given by

\[
1 - \sqrt{1 - (\nabla \phi)^2}. \tag{42}
\]

The Euler-Lagrange equation

\[
\text{div} \left( \frac{\nabla \phi}{\sqrt{1 - (\nabla \phi)^2}} \right) = 0, \tag{43}
\]

is just that which would be obtained if one sought a maximal spacelike hypersurface of minkowski spacetime where \( \phi \) is now thought of as a time function

\[
x^0(\mathbf{x}) = \phi(\mathbf{x}). \tag{44}
\]

The maximal hypersurface becomes null at the critical field strength.

7.2 Catenoids and \( D - \bar{D} \) solutions

Rather than exciting the electric field we can excite a single scalar \( y \). We get as Lagrangian density

\[
1 - \sqrt{1 + (\nabla y)^2}. \tag{45}
\]

The Euler-Lagrange equation

\[
\text{div} \left( \frac{\nabla y}{\sqrt{1 + (\nabla y)^2}} \right), \tag{46}
\]

12
is that governing the height function of a minimal surface in four space-like dimensions. One readily checks that Monge gauge is not global. In the spherically symmetric case there is a branch 2-surface at a finite radius. One needs two Monge patches. The resulting two sheeted worm-hole or better Einstein-Rosen bridge type surface looks like two parallel three planes with finite separation joined by a neck. The solution is not stable and therefore one thinks of it as a Brane-Anti-Brane pair.

7.3 Charged Catenoids: $O(1, 1)$ symmetry relating Catenoids and Bions

Including both electric and scalar fields gives a Lagrangian

$$1 - \sqrt{1 + (\nabla y)^2 - (\nabla \phi)^2}.$$ \hspace{1cm} (47)

It and the Euler-Lagrange equations

$$\text{div}\left(\frac{\nabla y}{\sqrt{1 + (\nabla y)^2 - (\nabla \phi)^2}}\right),$$ \hspace{1cm} (48)

and

$$\text{div}\left(\frac{\nabla \phi}{\sqrt{1 + (\nabla y)^2 - (\nabla \phi)^2}}\right) = 0,$$ \hspace{1cm} (49)

which are manifestly invariant under an obvious $O(1, 1)$ action analogous to the well-known Harrison transformation of static Einstein Maxwell theory. Using this action one may construct everywhere smooth charged catenoids, the electric field lines passing through the neck or throat in a way similar to that discussed by Wheeler in the case of Einstein-Maxwell theory. This family I call under-extreme. They are obviously analogous to under extreme Reissner-Nordstrom solutions. One may also excite the scalar field of the BIon solution. The original flat three-brane acquires a cusp as if it were being pulled. All of these solutions are singular. I call them over-extreme. They are obviously analogous to over extreme Reissner-Nordstrom solutions.

7.4 The BPS solution: S-Duality

In the limit of infinite $O(1, 1)$ parameter one obtains an extreme solution analogous to extreme Reissner-Nordstrom. This solution is in fact supersymmetric. It may be interpreted as a fundamental (F-) or (1, 0) string ending on a three-brane. Using the electric-magnetic duality one may easily obtain a magnetic monopole solution which represents a D-string or (0, 1) string ending on a three-brane. In fact using $SL(2, \mathbb{Z})$ and the Dirac quantization condition we can get dyon or $(p, q)$ strings ending on a three-brane.
8 Open String Causality

In string theory, open string states propagating in a background $F_{\mu \nu}$ field do so according to a different metric from the Einstein metric $g_{\mu \nu}$ felt by closed strong states.

One has
\[
\left( \frac{1}{g + F} \right)^{\mu \nu} = G^{\mu \nu} + \theta^{\mu \nu},
\]
(50)
where $G^{\mu \nu} = G^{(\mu \nu)}$ and $\theta^{\mu \nu} = \theta^{[\mu \nu]}$. If
\[
G_{\mu \lambda} G^{\lambda \mu} = \delta^\nu_{\mu},
\]
(51)
then
\[
G = g_{\mu \nu} - F_{\mu \lambda} g^{\lambda \rho} F_{\rho \nu}.
\]
(52)

Note that even if $B_{\mu \nu} = 0$ so that $F_{\mu \nu} = F^{\mu \nu}$, the metric $G_{\mu \nu}$ is not invariant under electric-magnetic duality.

8.1 Boillat metrics

One may investigate the propagation of small disturbances of vectors, $A_\mu$ scalars $y$ and spinors $\psi$ around a Born-Infeld background using the method of characteristics. This was done in great detail by Boillat for a general non-linear electrodynamic theory. He found that in general, because of bi-refringence, there are a pair of characteristic surfaces $S = \text{constant}$ satisfying
\[
\left( T_{\mu \nu}^{\text{Maxwell}} + \mu g^{\mu \nu} \right) \partial_\mu S \partial_\nu S = 0,
\]
(53)
where $T_{\mu \nu}^{\text{Maxwell}}$ is the Maxwell stress tensor constructed form $F_{\mu \nu}$. Of course the stress tensor $T^{\mu \nu}$ of the non-linear electrodynamic theory is different from $T_{\mu \nu}^{\text{Maxwell}}$. The quantity $\mu = \mu(x, y)$ satisfies a quadratic equation whose coefficients depend upon first and second derivatives of the Lagrangian $L(x, y)$ with respect to $x$ and $y$. Boillat finds it convenient to fix the arbitrary conformal rescaling freedom in the characteristic co-metric by setting
\[
C^{\mu \nu} = \frac{1}{\sqrt{\mu^2 - x^2 - y^2}} \left( \mu g^{\mu \nu} + T_{\mu \nu}^{\text{Maxwell}} \right)
\]
(54)
with inverse or metric
\[
C^{-1}_{\mu \nu} = \frac{1}{\sqrt{\mu^2 - x^2 - y^2}} \left( \mu g^{\mu \nu} - T_{\mu \nu}^{\text{Maxwell}} \right)
\]
(55)

In general the boundaries of the two Boillat cones $C_{\text{Boillat}} : C^{-1}_{\mu \nu} v^\mu v^\nu \geq 0, v^0 > 0$ and the Einstein cone $C_{\text{Boillat}} : g_{\mu \nu} v^\mu v^\nu \geq 0, v^0 > 0$ will touch along the two principle null directions of $F_{\mu \nu}$. One sometimes find that one at least of the Boillat cones lies outside the Einstein cone. In other words small fluctuations can travel faster than gravitational waves whose speed is governed by $g_{\mu \nu}$.
To check causality we examine the Boillat co-cones $C^*_\text{Boillat} : C^{\mu\nu} p_\mu p_\nu \geq 0, p_0 > 0$ and the Einstein co-cone $C^*_\text{Einstein} : C^{\mu\nu} p_\mu p_\nu \geq 0, p_0 > 0$ in the cotangent space $T^*\Sigma_{\mu+1}$. Suppose that $l_\mu$ is the co-normal the Einstein co-cone

$$g^{\mu\nu} l_\mu l_\nu = 0.$$  \hspace{1cm} (56)

The weak energy condition implies

$$T^{\mu\nu}_{\text{Maxwell}} l_\mu l_\nu \geq 0.$$  \hspace{1cm} (57)

Thus

$$C^{\mu\nu} l_\mu l_\nu \geq 0.$$  \hspace{1cm} (58)

This means that if $\mu$ is positive then $C^*_\text{Einstein}$ lies inside or touches $C^*_\text{Boillat}$. Remembering that duality reverses inclusions one finds then that the Einstein cone $C^*_\text{Einstein}$ lies outside or touches the Boillat cone $C^*_\text{Boillat}$. Note that what we are calling a cone here is the solid cone. The light cone is the boundary of this solid cone.

### 8.2 Hooke’s’ Law

Born-Infeld is exceptional in that there is just one solution for $\mu$:

$$\mu = 1 + x.$$  \hspace{1cm} (59)

Thus there is no bi-refringence. Moreover one finds that the Boillat co-metric satisfies the remarkable identity

$$C^{\mu\nu}_{\text{BI}} = g^{\mu\nu} + T^{\mu\nu}.$$  \hspace{1cm} (60)

I call this identity Hooke’s Law for reasons which will be explained below. Another striking identity is

$$\det(\delta^\nu_\nu + T^{\nu}_\nu) = 1.$$  \hspace{1cm} (61)

This follows form another useful identity is

$$\det C^{\mu\nu} = \det g^{\mu\nu}.$$  \hspace{1cm} (62)

From Hooke’s Law it is easy to see, since $T^{\mu\nu}$ for Born-Infeld theory satisfies the Weak Energy Condition, that the Boillat cone lies inside or touches the Einstein cone. In other words small fluctuations travel with a speed no greater than gravitational waves. Because the Born-Infeld energy momentum tensor is invariant under electric-magnetic duality rotations, the Boillat metric, unlike the open string metric $G_{\mu\nu}$ is also invariant. One has

$$C^{-1}_{\mu\nu} = \frac{1}{\sqrt{1 + 2x - y^2}} G_{\mu\nu}.$$  \hspace{1cm} (63)

The conformal factor is related to the Lagrangian:

$$\sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu})} = \sqrt{1 + 2x - y^2}.$$  \hspace{1cm} (64)

15
The reason for the name Hooke’s law is that Hooke asserted, in the days when
the archive was in Latin that *ceiinosssttuu*. In this way he hoped to make both
a priority claim and preserve his discovery for his own later use. Bearing in
mind that *u v* are not distinguished in Latin, the earth shattering discovery
that he wished to hide was that *ut tensio sic vis* [10]. In other words stress
is proportional to strain. A standard measure of strain in non-linear elasticity
theory is the difference of two metrics. More precisely, the configuration space
of an elastic medium is a map from an elastic manifold to an embedding space.
There is usually a rest or un-deformed configuration and one takes as a measure
of stress the difference between the pullbacks from the embedding space to the
elastic manifold in the strained and unstrained configuration.

What we have is an expression involving the difference of two co-metrics but
the idea is similar. One is the co-metric induced on the brane from the Einstein
co-metric and the other is a measure of the vector field excitations.

### 8.3 Hooke’s Law, the Monge-Ampère Equation and Pulse
interactions

The striking determinantal identity has an interesting application to the prop-
agation of pulses in Born-Infeld theory.

In flat two dimensional spacetime, the conservation law for the stress tensor
implies that it is given by a single free function, the Airy stress function \( \psi \), such that

\[
T_{tt} = \psi_{zz}, \quad T_{zz} = \psi_{tt}, \quad T_{tz} = \psi_{zt}. \tag{65}
\]

Written in terms of the Airy stress function, the determinantal identity becomes
the Monge-Ampère equation

\[
\psi_{zz} \psi_{tt} - \psi_{zt}^2 = \psi_{zz} - \psi_{tt} \tag{66}
\]

This can be solved exactly (see [16] and references therein) by a Legendre trans-
form under which it becomes D’Alembert’s equation with respect to a new set
of variables \( T \) and \( Z \).

One has

\[
T^{tt} = \frac{A + B + 2AB}{1 - AB}, \tag{67}
\]

\[
T^{zz} = \frac{A + B - 2AB}{1 - AB}, \tag{68}
\]

\[
T^{zt} = \frac{B - A}{1 - AB}, \tag{69}
\]

where \( A = A(T + Z) \) and \( B = B(Z - T) \) are arbitrary functions of their
arguments. The relation between the new coordinates \( (T, Z) \) and usual co-
dinates \( (t, z) \) is most conveniently expressed using null coordinates. Let
\( v + t + z, u = t - z, \xi = Z - T, \eta = Z - T \). The asymmetrical definition of
\( \eta \) is so as to agree with previous work cited in [16]. One has

\[
dv = d\xi - B d\eta, \quad du = -d\eta + A d\xi. \tag{70}
\]
Thus

\[(1 - AB)d\eta = A\nu - du, \quad (1 - AB)d\xi = dv - Bdu. \quad (71)\]

one checks that

\[dT^2 - dZ^2 = dt^2(1 - A - B + AB) - dz^2(1 + A + B + AB) - 2dtdz(A - B) = C_{\mu\nu}^{-1}dx^\mu dx^\nu, \quad (72)\]

where

\[C_{\mu\nu} = \eta_{\mu\nu} + T_{\mu\nu}. \quad (73)\]

Thus we see that the Legendre transformation to the new coordinates \((T, Z)\) used to solve the Monge-Ampère equation in effect passes to flat inertial coordinates with respect to the Boillat metric. It should be noted that one does not expect the Boillat metric to be flat in general.

The general solution consists of two pulses, one right-moving and one left moving which pass through another without distortion. In terms of the usual coordinates \((t, z)\) they two pulses experience a delay. That is measured with respect to the closed string metric. However with respect to the Boillat coordinates, that is measured with respect to the Boillat metric, there is no delay.

## 8.4 Scalars and fermions: Open String Equivalence Principle

The coupling of scalars has already been given above. It is easy to check that the Boillat co-metric determines their fluctuations around a background, they are in fact governed by the D’Alembert equation constructed from the co-metric \(C_{\mu\nu}\). One may also consider fermion fields \(\psi\). Omitting four-fermion terms, they couple in a typical Volkov-Akulov fashion.

\[-\int dx^{p+1}\sqrt{\det(g_{\mu\nu} + i\bar{\psi}\gamma_{\mu}\nabla_{\nu}\psi + F_{\mu\nu} + B_{\mu\nu})}, \quad (74)\]

where

\[\gamma_{\mu}\gamma_{\nu} + \gamma_{\nu}\gamma_{\mu} = 2g_{\mu\nu}. \quad (75)\]

Let’s define Boillat gamma matrices by

\[a^{\mu\nu} = G^{\mu\nu} + \theta^{\mu\nu}, \quad (76)\]

and

\[\tilde{\gamma}^{\mu} = a^{\mu\nu}\gamma_{\nu}. \quad (77)\]

One has

\[\tilde{\gamma}^{\mu}\tilde{\gamma}^{\nu} + \tilde{\gamma}^{\nu}\tilde{\gamma}^{\mu} = 2G^{\mu\nu}. \quad (78)\]

Because the leading derivative term in the action is

\[i\bar{\psi}\tilde{\gamma}^{\mu}\partial_{\mu}\psi, \quad (79)\]

It is clear that the characteristics of the fermions are also given by the Open String metric or equivalently the Boillat metric.
Thus we have a sort of world sheet equivalence principle or universality holding: all open string fields have the same characteristics and hence the same maximum speed.

9 Tolman Redshifting of the Hagedorn temperature

As an application of the equivalence principle it is interesting to consider open strings at finite temperature in a background electromagnetic field. This was done for the neutral bosonic string in [13]. If the free energy density in the absence of a background is \( F = F(\beta) \) where \( \beta \) is the inverse temperature, then the free energy in a background is obtained by the replacement

\[
F \to \sqrt{G_{00}} \sqrt{-G_{ij}} F(\beta \sqrt{G_{00}}),
\]

where \( G_{\mu\nu} \) is the open strong metric. The first factor may be thought of as a redshift and volume contraction factor. The rescaling of the argument is essentially the Tolman effect whereby in order to retain local equilibrium in an external static or stationary metric \( G_{\mu\nu} \), the local temperature must vary as \( \frac{1}{\sqrt{G_{00}}} \). Note that \( G_{00} = 1 - E^2 \) and so the redshifting is indeed redshifting and it depends only on the electric field, the effect diverging at the critical electric field strength.

Alternatively, one may regard the effect as being due to the fact that finite temperature physics corresponds to working in imaginary time with a period given by the inverse temperature. If the global time variable is identified with period \( \beta \), the local period will be \( \beta \sqrt{G_{00}} \). Thus the locally measured temperature will be higher. If more than one metric is involved, then the temperature of states in local equilibrium may differ, since each will be redshifted by the appropriate Tolman factor. In the present case one has closed string states at temperature \( \frac{1}{\beta} \) and open strong states at temperature \( \frac{1}{\beta \sqrt{G_{00}}} \). The redshifting of open string states is universal, was confirmed in [14].

In the absence of a background field the open string has, in perturbation theory, the free energy has a singularity at the Hagedorn temperature \( T_{\text{Hagedorn}} = \frac{1}{\beta_{\text{Hagedorn}}} \). This is represents a maximum possible temperature because above it there are so many massive string states that thermal equilibrium becomes impossible. In a background electric field the maximum temperature is reduced to

\[
T_{\text{Hagedorn}} \sqrt{1 - E^2}.
\]

This effect has been interpreted as being due to a reduction in the effective string tension in an electric field. This is certainly true but one cannot derive the exact formulae from that assumption alone whereas everything follows rather naturally by an application of the equivalence principle, as long as one uses the open string metric.
9.1 Shocks and Exceptionality

Loosely speaking, shocks can occur of the speed of waves depends on the phase or amplitude in such a way that different waves surfaces \( S = \text{constant} \) can catch up and form caustics. More precisely one assumes the ansatz

\[
F_{\mu\nu} = F^0_{\mu\nu}(f(S))
\]

with

\[
S = \mathbf{n} \cdot \mathbf{x} - v(\mathbf{n}, S)t.
\]

and \( f(S) \) an arbitrary function. The surfaces \( S = \text{constant} \) are hyperplanes and are to be thought of as surfaces of constant phase. If the phase speed \( v \) depends non-trivially on the phase \( S \) there will be shocks along the envelope of the hyperplanes. Theories without shocks for which \( v \) is independent of \( S \) are called exceptional.

Boillat has shown that the only form of non-linear electrodynamics with a sensible weak field limit is that of Born-Infeld.

Theories with shocks are essentially incomplete. In a sense, like General Relativity they predict their own demise. By contrast Born-Infeld, like classical Non-Abelian Yang-Mills theory seems to be a perfect example of a classical theory. As far as one can tell it appears to possess the property which is known to be true for Yang-Mills theory, that regular Born-Infeld initial data with finite energy may evolved for all time to give everywhere non-singular solutions of the field equations. For a more detailed discussion and references to the original literature see [18]

10 Strong Coupling Behaviour of Born-Infeld

There are (at least) two interesting strong coupling limits of Born-Infeld theory.

- A Weyl-invariant duality invariant theory which appears to be related to a fluid of massless magnetic Schild type strings and may describe string theory near critical electric field strengths.

- A massive theory which is related to a fluid of massive strings and may be related to current ideas about \( D-\bar{D} \) annihilation and tachyon condensates.

In both cases the key to understanding these limiting theories is passing to the Hamiltonian formulation. It also helps to bear in mind some facts about:

10.1 Simple 2-forms, Distributions and String Fluids

A 2-form \( \Omega \) is simple iff

\[
\Omega = \alpha \wedge \beta,
\]
equivalently
\[ \Omega \wedge \Omega = 0. \tag{85} \]

In particular since the matrix of components has \( \Omega_{\mu\nu} \) has rank two:
\[ \det \Omega_{\mu\nu} = 0. \tag{86} \]

In four spacetime dimensions \( \Omega \) is simple iff
\[ \Omega_{\mu\nu} \star \Omega^{\mu\nu} = 0. \tag{87} \]

Of course \( \alpha \) and \( \beta \) are not unique but a field of simple 2-forms defines the unique two-dimensional sub-space which they span in the cotangent space \( T_x M \) at every point of spacetime. Hence, given a metric, a simple two form is equivalent to a simple bi-vector \( \Omega_{\mu\nu} \) which defines a distribution \( D \) of 2-planes in the tangent space \( TM \). Raising indices with the metric, the simplicity condition becomes in terms of the bi-vector
\[ \Omega^{[\mu\nu} \Omega^{\alpha]\beta} = 0. \tag{88} \]

One may think of the distribution \( D \) as a sub-bundle of the tangent bundle with two-dimensional fibres. The 2-planes will be timelike, null or spacelike depending upon whether \( \Omega_{\mu\nu} \Omega^{\mu\nu} \) is negative, zero or positive respectively. (Note that this statement is signature independent.) In the timelike case one may chose \( \alpha \) to be timelike and \( \beta \) to be spacelike. In the null case one may choose \( \alpha \) to be null and \( \beta \) to be spacelike.

In general the distribution \( D \) will not be integrable. That is neighbouring 2-planes will not mesh together to form the tangent spaces of a co-dimension two family of 2-dimensional surfaces. If it is, then if two vector fields \( X \) and \( Y \) belong to \( D \) then their Lie bracket \( [X, Y] \) must belong to \( D \). Such an integrable distribution may be identified as a gas or soup, perhaps more accurately a spaghetti of strings. The condition for integrability may be expressed in various ways. For us the simplest condition is in terms of the bi-vector and is
\[ \Omega^{\mu\nu} \delta \nu^{\mu\nu} = 0. \tag{89} \]

Note that if \( f \) is a smooth function, then \( \Omega \) and \( f \Omega \) define the same distribution and if the first is integrable then so is the second. Moreover, the partial derivative in (89) may be replaced by a torsion free covariant derivative. In four spacetime dimensions we may re-express the integrability condition as
\[ \star \Omega_{\mu\nu} \nabla_{\kappa} \Omega^{\mu\nu} = 0. \tag{90} \]

We may re-write this as
\[ \Omega \wedge \delta \Omega = 0. \tag{91} \]

where \( \delta \Omega = \star d \star \Omega \).

Now if we take for \( \Omega \) the Ampère tensor \( K_{\mu\nu} \) of any non-linear electrodynamic theory. We see that any simple solution of the equations of motion
\[ \nabla K^{\mu\nu} = 0 \tag{92} \]
automatically defines an integrable distribution. In other words non-linear electrodynaminc theory supplemented with the constraint
\[ F \wedge K = 0, \] (93)
may be re-interpreted as a (vorticity free) string fluid. Different Lagrangians correspond to different equations of state.

### 10.2 0-brane fluids

This section is based on part on [16] The situation described above should be compared with the familiar case of a non-linear scalar field theory with a Lagrangian \( L(\partial \phi) \) containing no explicit dependence on the scalar field \( \phi \). The equations of motion may be cast in the form
\[ \nabla_{\mu} (s U^{\mu}) = 0 \] (94)
where \( U^{\mu} \) is a normalized timelike vector given by
\[ U^{\mu} = \frac{\partial_{\mu} \phi}{\sqrt{(\partial_{\phi})^2}}, \] (95)
and
\[ s U^{\mu} = \frac{\partial L}{\partial (\partial_{\mu} \phi)} \] (96)
may be interpreted as a conserved entropy current. The quantity \( s \) corresponds to the entropy density and \( \rho \) to the local energy density. The energy momentum tensor takes the perfect fluid form
\[ T^{\mu\nu} = (\rho + P) U^{\mu} U^{\nu} - P g^{\mu\nu} \] (97)
One has
\[ P = L. \] (98)
If one defines
\[ T^{2} = (\partial \phi)^2, \] (99)
it is natural to regard the pressure as a function of the temperature \( T \) but the energy density as a function of the entropy density \( s \). In fact they are related by a Legendre transform. One finds that
\[ \rho + P = s T \] (100)
and
\[ s = \frac{\partial P}{\partial T} \quad T = \frac{\partial \rho}{\partial s} \] (101)

It is an illuminating exercise to convince oneself that finding the speed of small fluctuations by the calculating the sound speed
\[ c_s = \sqrt{\frac{\partial P}{\partial \rho}} \] (102)
is equivalent to calculating the characteristics, that is the Boillat metric.

The most interesting case from the present point of view arises when one takes the scalar Born-Infeld Lagrangian

\[ L = 1 - \sqrt{1 - (\partial \phi)^2}. \] (103)

One has

\[ P = 1 - \sqrt{1 - T^2} \] (104)

and

\[ \rho = \frac{1}{\sqrt{1 + s^2} - 1} \] (105)

which has a maximum temperature reminiscent of the Hagedorn temperature. However the detailed equation of state is different. One has the equation of state

\[ P = \frac{\rho}{1 + \rho} \] (106)

and hence

\[ c_s = \frac{1}{1 + \rho}. \] (107)

Note that one need not regard the conserved current as an entropy current if one does not wish to. One could regard it as a conserved particle number.

### 10.3 The Weyl-invariant Bialynicki-Birula limit

The Hamiltonian density, with units restored is

\[ \mathcal{H} = T^2 \sqrt{1 + \frac{B^2 + D^2}{T^2} + \frac{(D \times B)^2}{T^4}} - T^2. \] (108)

One can take the limit \( T \downarrow 0 \) to get

\[ \mathcal{H} = |D \times B|. \] (109)

This gives the constitutive relations

\[ E = -n \times B, \quad H = n \times D, \] (110)

where we have defined a unit vector in the direction of the Poynting vector

\[ n = \frac{D \times B}{|D \times B|}. \] (111)

Remarkably these constitutive relations (which arise as the limiting form of the constitutive relations of the full theory) imply the constraints

\[ E^2 - B^2 = 0 \quad E \cdot B = 0. \] (112)

Defining a null vector \( l^\mu = (1, n) \), the energy momentum tensor becomes

\[ T^{\mu\nu} = \mathcal{H} l^\mu l^\nu. \] (113)
It follows that the trace vanishes
\[ T^\mu_\mu = 0, \quad (114) \]
and hence the limiting theory is Weyl-invariant. It may be checked that is Lorentz-invariant and invariant under electric-magnetic duality rotations. One may also check from the equation of motion that there are infinitely many conserved symmetric tensors
\[ T^\mu_1^\mu_2^...^\mu_k = \mathcal{H}^i_1^i_2^...^i_k. \quad (115) \]

The constraints (112) tell us that the Faraday tensor \( F_{\mu \nu} \) is simple
\[ \det F_{\mu \nu} = 0, \quad (116) \]
and null,
\[ F_{\mu \nu} F^{\mu \nu} = 0. \quad (117) \]
Thus \( F_{\mu \nu} \) defines a two plane which is null, that is, the two-plane is tangent to the light cone along the lightlike vector \( l^\mu \) and
\[ F_{\mu \nu} l^\mu = 0. \quad (118) \]

The equations of motion tell us that the two-plane distribution in the tangent space defined by the Faraday two-form \( F_{\mu \nu} \) is integrable, that is surface forming, and hence that spacetime is foliated by two-dimensional lightlike surfaces which may be interpreted as the world sheets of magnetic null or Schild strings. In other words, in this critical limit which may be interpreted as describing Born-Infeld theory near critical field strength, the system dissolves into a gas or fluid of Schild strings.

Since electric-magnetic duality is maintained in the limit, one can of course pass to a dual description in terms of \( K_{\mu \nu} \). This amounts to the observation that \( \mathbf{H}^2 = \mathbf{D}^2 \) and \( \mathbf{H} \cdot \mathbf{D} = 0 \), i.e. \( K_{\mu \nu}^2 = 0 \) and \( K_{\mu \nu} \star K_{\mu \nu}^\star = 0 \).

### 10.4 Covariant formulation of UBI using auxiliary fields

The Weyl-invariant limit was called by Bialynicki-Birula [11, 12], Ultra-Born-Infeld. Let us follow him and consider
\[ L = -\frac{\mu}{4} F_{\mu \nu} F^{\mu \nu} + \frac{\nu}{4} F_{\mu \nu} \star F^{\mu \nu}, \quad (119) \]
where \( \mu \) and \( \nu \) are dimensionless auxiliary fields, variation with respect to which gives the constraints
\[ F_{\mu \nu} F^{\mu \nu} = 0 = F_{\mu \nu} \star F^{\mu \nu}. \quad (120) \]

Variation with respect to \( A_\mu \) gives the field equation.

Note that in axion-dilaton Maxwell theory, the auxiliary fields could be functions of the dimensionless dilaton \( \Phi \) and axion \( \chi \), \( \mu = \mu(\Phi, \chi) \), \( \nu(\Phi, \chi) \) chosen in such a way that the system was \( SL(2, \mathbb{R}) \) invariant. The dilaton and
axion provide a map from spacetime into $SL(2,\mathbb{R})/SO(2)$ and one would, in general, have a non-linear sigma model type kinetic term for them (see e.g. [21]). For dimensional reasons it must be multiplied by $T$. In the limit we are considering the kinetic term vanishes and the axion and dilaton become auxiliary fields.

### 10.5 Tachyon Condensation

This subsection is based on [21] where references to the string literature may be found. The basic idea goes back to Ashoke Sen. In the presence of a tachyon field the Born-Infeld Lagrangian density is believed to be modified by the tachyon potential $V$ to take the form

$$L = V - V\sqrt{1 - E^2 + B^2 - (E \cdot B)^2}.$$  \hfill (121)

We are now keeping $\alpha'$ fixed and using units in which $2\pi\alpha' = 1$. Now it is believed that $V$ has a critical point away from zero at which $V$ vanishes. It is also believed that dynamically the system will relax to the state with $V = 0$, a so-called tachyon condensate. One may thus ask, what happens to the Born-Infeld vector in this limit. Again the Lagrangian density causes confusion: it vanishes identically in the limit. However the Hamiltonian density is

$$\mathcal{H} = \sqrt{V^2(1 + B^2) + D^2 + (D \times B)^2} - V.$$  \hfill (122)

and the limiting form is

$$\mathcal{H} = \sqrt{D^2 + (D \times B)^2}.$$  \hfill (123)

The resulting constitutive relations are

$$D = \frac{BD^2 - D(B \cdot D)}{\sqrt{D^2 + (B \times D)^2}},$$

$$H = \frac{B + DB^2 - B(B \cdot D)}{\sqrt{D^2 + (B \times D)^2}}.$$  \hfill (124, 125)

They tell us that $E \times H = D \times B$, and therefore the theory is Lorentz-invariant. One may check that electric-magnetic duality invariance is lost in this limit. The constitutive relations also imply that

$$D \cdot H = 0.$$  \hfill (126)

but

$$D^2 - H^2 > 0.$$  \hfill (127)

It follows that the Ampère tensor $K_{\mu\nu}$ with components $D, H$, is simple but timelike. Thus the two-form $K_{\mu\nu}$ it defines a 2-plane distribution in the tangent space. As discussed above the equation of motion for $K$ implies that the distribution is integrable.
The limiting theory maybe expressed in terms of the Ampère tensor \( K_{\mu\nu} \). One way to proceed is to consider a dual Lagrangian. We define \( G = \star K \). The field equation \( d \star K = 0 \) becomes the Bianchi-Identity \( dG = 0 \). We now set \( G = dC \) and consider the Lagrangian
\[
\hat{L} = \sqrt{\frac{1}{2} G_{\mu\nu} G^{\mu\nu}} = \sqrt{-\frac{1}{2} K_{\mu\nu} K^{\mu\nu}}. \tag{128}
\]

The action is now varied with respect to \( C \) but subject to the constraint that
\[
K_{\mu\nu} \star K^{\mu\nu} = 0. \tag{129}
\]
The resulting energy momentum tensor is given by
\[
T_{\mu\nu} = -\frac{K_{\mu\lambda} K_{\nu}^{\lambda}}{\sqrt{-\frac{1}{2} K_{\sigma\tau} K^{\sigma\tau}}} \tag{130}
\]
The trace is given by
\[
T_\mu = -\sqrt{-2 K_{\sigma\tau} K^{\sigma\tau}}. \tag{131}
\]
and therefore this is certainly not a conformally invariant theory.

Locally one may pass to a rest frame in which \( B = 0 \). Then
\[
\mathcal{H} = |\mathbf{D}|. \tag{132}
\]
This is precisely what one expects of electric flux tubes with an energy proportional to the length and to the total flux carried by the tube.

In this rest frame one finds that
\[
T_{\mu\nu} = \begin{pmatrix}
\tau & 0 & 0 & 0 \\
0 & -\tau & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \tag{133}
\]
This is just what one expects for a string fluid.

11 The M5-brane

In this concluding section I will indicate how many of the ideas described above extend to the theory of the M5-brane. To paraphrase Hooke ut D3-brane sic M5-brane. Indeed from the M-Theory point of view one should perhaps have reversed the logic, since one may regard the equations of Born-Infeld theory as the dimensional reduction of the M5-brane equations. The theory and it’s equation have a reputation for complexity and so I will try to present them in as direct a way as possible. The interested reader may find references to the original papers and the statements made below the paper on which section is based [17].
One is of course considering a 6-dimensional non-linear theory involving scalar and spinor fields and in addition and closed 3-form \( H_{\alpha\beta\gamma} \). In what follows I shall follow the original papers except that \( \mu = 0, 1, \ldots, 5 \). In particular in this section I shall follow their lead in this section be using the mainly positive signature convention.

### 11.1 Bianchi Identity

Consider the simplest situation: just the 3-form in a fixed background Einstein-metric \( g_{\mu\nu} \). Thus

\[
    dH = 0. \tag{134}
\]

Locally therefore one has \( H = dA \), for some 2-form \( A \).

### 11.2 Non-Linear self-duality

Now in six-dimensional Minkowski spacetime and acting on three-forms the standard linear Hodge duality is an involution of order two: \( * * = 1 \) and in linear theory self-duality is a consistent field equation. In other words closure of \( H \) and the self-duality condition give the complete set of equations of motion. For the M5-brane a very remarkable non-linear self-duality condition is possible which fulfills the same purpose. This was first discovered by Perry and Schwarz and its covariant form written down by Howe Sezgin and West. One does not seem to be able to construct a covariant Lagrangian just using the 2-form \( A \). Non-covariant variational principles exist and a covariant action principle has been written down using an additional scalar field which acts as a time function. For the time being we need in these lectures only the equations of motion.

This remarkable condition is perhaps most expeditiously written as

\[
    * H_{\alpha\beta\gamma} = \frac{1}{\sqrt{1 + \frac{2}{3} H^2}} \left[ (1 + \frac{4}{3} H^2) \delta^\epsilon_\alpha - 4( H^2)^\epsilon_\alpha \right] H_{\epsilon\beta\gamma}. \tag{135}
\]

Of course in the limit of small \( H \) we have \( H \approx * H \). As stated above, if one reduces to five spacetime dimensions the equations reduce to the standard Born-Infeld equations.

### 11.3 Boillat Cone and Hooke’s Law

One may introduce the analogue of the Boillat co-metric:

\[
    C^{\alpha\gamma} = \frac{Q}{(2 - Q)} \left( g^{\alpha\gamma}(1 + \frac{4}{3} H^2) - 4( H^2)^{\alpha\gamma} \right), \tag{136}
\]

where

\[
    Q = -\frac{3}{H^2}(1 + \frac{2}{3} H^2). \tag{137}
\]
The characteristics of the scalar, spinor and 3-form equations of motion are determined by the Boillat metric \( C^{\mu\nu} \). Moreover one may introduce an energy momentum tensor \( T^{\mu\nu} \) which satisfies Hooke’s Law:

\[
T^{\alpha\beta} = g^{\alpha\beta} - C^{\alpha\beta},
\]

and is conserved

\[
T^{\mu\nu}_{\ :\nu} = 0.
\]

Note the sign change in Hooke’s law because of the signature change. (However \( T^{00} \geq 0 \) in both conventions.)

One may prove that \( T^{\mu\nu} \) satisfies the Dominant Energy Condition and hence, as with Born-Infeld theory, that the Einstein cone never lies inside the Boillat cone. In general the two cones touch along a circle of directions.

### 11.4 Weyl-invariant strong coupling limit

In general the trace of the energy momentum tensor \( T^\mu_\mu \) does not vanish. The theory is not Weyl-invariant except at vanishing field strength. However it becomes Weyl invariant in the limit of strong coupling. As with Born-Infeld, the most direct route to this result is the non-covariant (in our case \( SO(5) \subset SO(5,1) \) symmetric) form of the equations. One defines a pair of two-forms \( E_{ij} = H_{0ij} \) and \( B_{ij} = -\frac{1}{6} \epsilon_{ijpqr} H^{pqr} \). The Bianchi identity may be written in an obvious notation as

\[
\frac{\partial B}{\partial t} + \text{curl} E = 0, \quad \text{div} B = 0.
\]

To close the system one need a constitutive relation. To this end one defines

\[
\mathcal{H} = \sqrt{\det(\delta_{ij} + 4B_{ij})} - 1.
\]

The full non-linear self-duality constraint has as solution

\[
E_{ij} = \frac{1}{16} \frac{\partial \mathcal{H}}{\partial B_{ij}}.
\]

The quantity \( \mathcal{H} \) is the energy density \( T_{00} \). One may now restore dimensions by setting

\[
\mathcal{H} = T^2 \sqrt{\det(\delta_{ij} + 4 \frac{B_{ij}}{T})} - T^2.
\]

where \( T \) has dimension mass cubed. The limit \( T \downarrow \) is now easily taken. More interesting than the general formulae for \( E_{ij} \) are the results for the energy momentum tensor. It takes the null matter form

\[
T^{\mu\nu} = \mathcal{H} h^{\mu\nu},
\]

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where, \( l^\mu \) is again a null vector in the direction of the Poynting flux. Thus, just as is the case with Born-Infeld in fours spacetime dimensions, we attain Weyl-invariance in this limit and the theory has infinitely many conservation laws.

Point wise, one may skew diagonalize \( B_{ij} \). In general it has rank four and two distinct skew eigenvalues \( B_1 \) and \( B_2 \) respectively. Of course the basis in which \( B_{ij} \) is skew diagonalized will in general vary with position. Pointwise one finds that if \( l^\mu = (1, 0, 0, 0, 0, 1) \)

\[
H = (dt - dx^5) \wedge (B_2 dx^2 \wedge dx^2 + B_1 dx^4 \wedge dx^5). \tag{145}
\]

The three form \( H \) is in general not self-dual and is the sum of two totally simple three-forms. One factor is the null one form \( L_\mu dx^\mu \), and one has \( H_{\mu \nu \sigma} l^\mu = 0, \star H_{\mu \nu \sigma} l^\mu = 0 \).

The quantum mechanical nature of this mysterious conformally invariant theory is an interesting challenge for the future.

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