H₀ Tension and the Phantom Regime: A Case Study in Terms of an Infrared f(T) Gravity

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Received 2018 September 25; revised 2018 December 13; accepted 2018 December 18; published 2019 February 1

Abstract

We propose an f(T) teleparallel gravity theory including a torsional infrared (IR) correction. We show that the governing Friedmann’s equations of a spatially flat universe include a phantom-like effective dark energy term sourced by the torsion IR correction. As has been suggested, this phantom phase does indeed act to reconcile the tension between local and global measurements of the current Hubble value H₀. The resulting cosmological model predicts an electron scattering optical depth τₑ ≈ 0.058 at reionization redshift zₑ ≈ 8.1, in agreement with observations. The predictions are, however, in contradiction with baryon acoustic oscillation (BAO) measurements, particularly the distance indicators. We argue that this is the case with any model with a phantom dark energy model that has effects significant enough at redshifts z < 2 as to be currently observable, the reason being that such a scenario introduces systematic differences in terms of distance estimates in relation to the standard model; e.g., if the angular diameter distance to the recombination era is to be kept constant while H₀ is increased in the context of a phantom scenario, the distances there are systematically overestimated to all objects at redshifts smaller than recombination. But no such discrepancies exist between ΛCDM predictions and current data for z > 2.

Key words: cosmological parameters – cosmology: observations – cosmology: theory – dark energy

1. Introduction

Cosmological observations clearly confirm that our universe has sped up its expansion as of a few billion years ago (Riess et al. 1998; Perlmutter et al. 1999), with transition redshift 0.67 ≲ z_tr ≲ 0.87 (Farooq et al. 2017). In the context of general relativity (GR), explaining this phenomenon requires the introduction of a cosmological constant or a negative pressure component in the field equations, referred to as the dark energy. Several recent analyses (see Planck Collaboration XIII 2016) show that this component represents ~69% of the energy density in the universe. The complementary components consist of ~26% pressureless dark matter and ~5% baryons.

Since the dark energy effects are felt on the cosmic scales, they are naturally tied to the gravitational interaction and its description. In GR this involves the field equations

\[ \frac{1}{k^2} \Phi_{\mu\nu} = \tau_{\mu\nu}, \]

where \( \Phi_{\mu\nu} \) is the Einstein tensor and \( \tau_{\mu\nu} \) is the energy-momentum tensor of the matter components; the coupling constant \( k \), in the natural units (\( c = h = k_B = 1 \)), can be related to the Newtonian constant \( G \) by \( k = \sqrt{8\pi G} \). To explain the late cosmic acceleration, one should represent the dark energy component \( \tau_{\mu\nu}^{DE} \) as an additional term in Einstein’s field equations as

\[ \frac{1}{k_{eff}^2} \Phi_{\mu\nu} = \tau_{\mu\nu} + \tau_{\mu\nu}^{DE} = \tau_{\mu\nu}^{eff}, \]

where the effective coupling constant \( k_{eff} \) reduces to the constant \( k \) at the GR limit and \( \tau_{\mu\nu}^{DE} \) is the total energy-momentum tensor. The additional term \( \tau_{\mu\nu}^{DE} \) in Equation (2) can be sourced by either matter (physical) or gravitational (geometrical) sectors. As noted by Sahni & Starobinsky (2006), the field Equation (2) put both physical and geometrical dark energies on equal footing. However, they provide different physical descriptions in some scenarios. For example, nonsingular bounce models have been viewed as alternatives to the big bang, whereas the null energy condition should be violated. These have been investigated using effective field theory techniques by introducing matter fields that violate the null energy condition (Cai et al. 2007, 2008). On the contrary, using the modified gravity, the null energy condition is violated effectively (gravitational sector), keeping the matter sector consistent with the null energy condition (see Cai et al. 2011; Bamba et al. 2012; El Hanafi & Nashed 2017b; see also the review by Nojiri & Odintsov 2006). In short, since the gravitational sector does not represent a physical matter field, we can exchange a particular exotic matter field with some modified gravity without worrying about the energy conditions.

(i) In the former case, the cosmological constant \( \Lambda \) is the simplest scenario for the dark energy. This constant is equivalent to a negative pressure term in Friedmann equations with the equation-of-state (EOS) parameter fixed to a value \( w_\Lambda = -1 \), allowing the universe to perform a transition from a decelerated expansion epoch dominated by cold dark matter (CDM) to an accelerated expansion dominated by \( \Lambda \)-dark energy, in agreement with observations. Although this \( \Lambda \)CDM model fits well with a wide range of observations, it lacks adequate theoretical underpinning. Indeed, it entails several puzzling issues, e.g., the cosmic coincidence problem and the enormous discrepancy between its theoretical and observational values (Weinberg 1989; Carroll 2001). On the other hand, an alternative to the cosmological constant consists of dynamical dark energy, akin to inflaton fields, which can be described as a canonical scalar field \( \phi \) minimally coupled to gravity with fixed or dynamical EOS parameter \(-1 < w_\phi < -1/3\).
(ii) From the geometrical point of view, there are three objects that can be used to describe deviations from Minkowski spacetime owing to the presence of a gravitational field, curvature $R$, torsion $T$, and nonmetricity $Q$. When $\Omega_{\text{DE}}$ is sourced by the gravitational sector, one needs to modify the GR equations, as in GR all geometrical terms but $\Omega_{\text{DE}}$ are collected on the right-hand side of the field Equation (2). Extensions proposed in order to fulfill this include those built on the basis of Riemannian geometry (curvature-based theories), such as Gauss–Bonnet and $f(R)$ theories (De Felice & Tsujikawa 2010; Clifton et al. 2012; Capozziello & De Laurentis 2011; Nojiri & Odintsov 2011), while others are constructed in the context of Weitzenböck geometry (torsion-based theories), e.g., new GR, teleparallel equivalent to general relativity (TEGR) gravity, and $f(T)$ theories (Cai et al. 2016; Nojiri et al. 2017). Also, some are constructed in the nonmetricity geometry (nonmetricity-based theories), e.g., symmetric teleparallel equivalent to general relativity (STEGR; Nester & Yo 1999) and its recent extension to $f(Q)$ theories (Järv et al. 2018).

As mentioned above, both physical and geometrical dark energies could have similar contributions to the field Equation (2). Nevertheless, they represent fundamentally different physical descriptions; exploration of alternative cosmological models based on modified gravity is thus motivated by theoretical considerations, as well as empirical anomalies listed in Di Valentino et al. (2016b).

Perhaps the most significant anomaly is embodied in the apparent inconsistency of the locally measured value of the Hubble parameter and that inferred from cosmic microwave background (CMB) observations. It is on this that we focus in this paper, showing how cosmological observations could be made consistent in terms of $f(T)$ theories of gravity with infrared corrections. The latest released data sets suggest that there is in fact no concordance value for the current Hubble value $H_0$. The local measurements (Type Ia supernovae, SNla, and HST) give $H_0 > 70 \text{ km s}^{-1} \text{Mpc}^{-1}$ (Anderson et al. 2014; Riess et al. 2016, 2018a, 2018b); on the contrary, the global (CMB) measurements give $H_0 < 70 \text{ km s}^{-1} \text{Mpc}^{-1}$ (Bennett et al. 2013; Planck Collaboration I 2016; Planck Collaboration XIII 2016). As the accuracy on both tracks has increased, the tension between these, instead of disappearing, has crossed over to 3.8 standard deviations (Riess et al. 2018a). So far no source of systematic uncertainty has been pinpointed to explain the discrepancy of the measurements of the Hubble constant. This being the case, it seems natural to investigate new physical inputs, which could restore consistency of the two tracks. However, major changes due to new physics are not supported by the CMB power spectrum.

Possible extensions to the ΛCDM scenario that have been suggested in order to resolve the aforementioned tension include invoking a larger neutrino effective number $N_{\text{eff}} \sim 3.5$, i.e., the possibility of a dark radiation component (the standard value is $N_{\text{eff}} = 3.04$). A second avenue involves a phantom dark energy component with an EOS $w_{\text{DE}} \leq -1.1$. This could bring the Planck constraint into better agreement with higher values of the Hubble constant. By varying both parameters simultaneously, it has been shown that there is no privilege for dark radiation if allowance is made for dynamical dark energy (Di Valentino et al. 2016a, 2017; Huang & Wang 2016; Zhao et al. 2017). Phantom energy can be shown to also ameliorate the age conflict (Cepa 2004).

Several analyses have in fact favored such a phantom dark energy scenario (e.g., Sahni et al. 2008, 2014; Di Valentino et al. 2016a, 2017, 2018; Huang & Wang 2016; Alam et al. 2017b; Wang et al. 2017; Zhao et al. 2017; Dutta et al. 2018). Notably, even a viable quintom behavior that allows a phantom phase can be achieved without ghost or gradient instabilities, if one extends $k$-essence to kinetic gravity braiding (Deffayet et al. 2010). However, if one insists on working within the GR framework and assumes the phantom dark energy to be sourced by the matter sector (e.g., ordinary scalar field), ghost instability would not be avoidable owing to violation of the dominant energy condition (Carroll et al. 2003, 2005; Ludwick 2017). This being the case, the choice of the gravity sector as a source of $\Omega_{\text{DE}}$ is preferable. In this paper, within the frame of the $f(T)$ modified gravity, we argue that the torsional IR correction is a good candidate to source the phantom-like dark energy. Subsequently, it could resolve the current tension in measuring the Hubble constant.

In Section 2, we revisit the teleparallel geometry and briefly discuss $f(T)$ gravity. In Section 3, we derive the modified Friedmann’s equations of the torsional IR correction obtaining the Hubble–redshift relation. In Section 4, we adopt the dynamical system approach, showing that the governing equation is a 1D autonomous system. This allows us to analyze its phase portrait and extract some useful information. We show that the model predicts a transitional redshift compatible with observations. Also, we determine the phantom-like nature of the torsional counterpart. Moreover, we find that the model predicts an age of the universe compatible with observations. In Section 5, we fix the model parameters. We show that the torsional IR model reconciles CMB with the local value of $H_0$. In addition, we confront the model with other measured parameters, such as the electron scattering optical depth $\tau_e$. However, the model is in serious tension with the baryon acoustic oscillation (BAO) observations, in particular the angular distance measures. We argue that phantom/phantom-like DE models, in principle, cannot solve the conflict with BAO observations. In Section 6, we conclude the present work. We add Appendix A, for some particular values of the model parameters, to give explicit forms of some useful cosmological parameters, such as time–redshift relation, density parameters, and comoving volume element. Also, we show that the scalar fluctuation propagates with a sound speed $0 \leq c_s \leq 1$ at all times.

2. $f(T)$ Teleparallel Gravity

In general, one requires the manifold to be differentiable in order to describe dynamical evolution of a physical system under gravity. This can be achieved by defining a compatible differential structure on the manifold, in other words, installing a connection. Let us focus on linear (affine) connections, which are used to transport tangent vectors to a manifold between two points along some curve in a covariant way. In modern literature it can be viewed as a differential operator $\nabla$ and is known as the Koszul Connection,

$$\nabla_{\mu} \partial_{\nu} := \Gamma_{\mu \nu}^{\alpha} \partial_{\alpha},$$

where $\Gamma_{\mu \nu}^{\alpha}$ are $d^3$ functions ($d$ is the dimension of the manifold) called the connection coefficients of $\nabla$, or simply an affine connection. It is related to the metric by the nonmetricity...
tensor (Ortín 2007)

\[ Q_{\mu\nu} := -\nabla_\mu g_{\nu\rho} \]

(3)

By taking the combination \( \nabla_\mu g_{\nu\rho} + \nabla_\nu g_{\mu\rho} - \nabla_\rho g_{\mu\nu} \), one can write a generalized form of the affine connection as

\[ \Gamma_{\mu\nu}^\rho = \Gamma_{\mu\nu}^\rho + K_{\mu\nu}^\rho + L_{\mu\nu}^\rho, \]

where \( \Gamma_{\mu\nu}^\rho = \frac{1}{2\epsilon}\delta^\rho_{\nu}(\partial_\mu g_{\epsilon\rho} + \partial_\epsilon g_{\mu\rho} - \partial_\rho g_{\mu\epsilon}) \) is the Levi–Civita symmetric connection, \( K_{\mu\nu}^\rho \) is called the contortion tensor, and \( L_{\mu\nu}^\rho \) is defined in terms of the nonmetricity tensor (3); more geometrical constructions with physical aspects have been reviewed in Hehl et al. (1995). Notably, the GR has been formulated by requiring a vanishing torsion (contortion) and nonmetricity, and then all gravitational effects are encoded in terms of the Riemannian curvature of the Levi–Civita connection.

In the TTEGR gravity, it is required to dispel the curvature and the nonmetricity that defines the Weitzenböck connection, and then all gravitational effects are encoded in terms of the torsion tensor of that connection (Maluf 1994, 2013). In STEGR gravity, on the other hand, it is required to have a flat connection (null curvature and null torsion tensors), and then all gravitational effects are encoded in terms of the nonmetricity tensor (Nester & Yo 1999). It has been shown that three equivalent variants of GR can be obtained in these three geometries.

In this section, we give a brief description of teleparallel geometry (for more details see Aldrovandi & Pereira 2013) and summarize some of the modifications of the Friedmann equations that can come about in the context of \( f(T) \) gravity generalization.

In a 4D \( C^\infty \)-manifold \((\mathcal{M}, \epsilon_a)\), where \( \epsilon_a \) (\( a = 0, 1, 2, 3 \)) are four linear independent vector (tetrads, vierbein) fields defined on \( \mathcal{M} \), the vierbein fields fulfill the conditions \( e^a_\nu e^\nu_\mu = \delta^a_\mu \) and \( e^a_\mu e^\mu_\nu = \delta^a_\nu \), where \( \mu = 0, 1, 2, 3 \) denotes the coordinate components. The Einstein summation convention is applied to both Latin (tangent space coordinates) and Greek (spacetime coordinates) indices.

One can straightforwardly construct the spacetime metric tensor

\[ g_{\mu\nu} \equiv \eta_{ab}e^a_\mu e^b_\nu, \]

where \( \eta_{ab} \) is the flat Minkowski metric on the tangent space of \( \mathcal{M} \). Consequently, one can define the Levi–Civita symmetric connection \( \Gamma^a_{\mu\nu} \) and in fact the full machinery of the Riemannian geometry. As can be noticed from Equation (5), the vierbein has 16 components, while the associated metric has only 10 components, which leaves 6 extra degrees of freedom in the vierbein formalism unfixed. In other words, for a given spacetime metric one cannot define a unique vierbein, which is the local Lorentz invariance problem of teleparallel formalism. However, it has been shown that this problem can be alleviated if one allows for flat but nontrivial spin connection (Krššák & Saridakis 2016; see also Hohmann et al. 2018).

In the teleparallel geometry one can construct the non-symmetric (Weitzenböck) linear connection directly from the vierbein, \( \Gamma^a_{\mu\nu} \equiv e^a_\alpha \partial_\mu e^\alpha_\nu = -e^a_\mu \partial_\nu e^a_\alpha \), where the vierbein are parallel with respect to this connection \( \nabla_\nu e^a_\mu = 0 \), and the differential operator \( \nabla_\nu \) denotes the covariant derivative associated with the Weitzenböck connection. Since \( \Gamma^a_{\mu\nu} \) is non-symmetric, it defines the torsion tensor \( T^a_{\mu\nu} \equiv \Gamma^a_{\rho\nu} - \Gamma^a_{\rho\mu} = e^a_\delta (\partial_\mu e^\delta_\nu - \partial_\nu e^\delta_\mu) \). However, its curvature vanishes identically. Also, the contortion tensor is given by \( K^a_{\mu\nu} = e^a_\sigma \nabla_\nu e^\sigma_\mu \), where the differential operator \( \nabla_\nu \) denotes the covariant derivative associated with the Levi–Civita connection.

In teleparallel geometry, the teleparallel torsion scalar, \( T \equiv T^a_{\mu\nu}S_a^{\mu\nu} \), is equivalent to the Ricci scalar \( R \) up to a total derivative term. In the above, the superpotential tensor \( S_a^{\mu\nu} \) is defined as

\[ S_a^{\mu\nu} = \frac{1}{2}(K_{\mu\nu}^a + \delta^a_{\alpha}T^{\kappa\beta}_{\mu\nu} - \delta^a_{\nu}T^{\kappa\beta}_{\mu\nu}). \]

Use the action

\[ S = S_m + S_g = \int d^4x|e|(\mathcal{L}_m + \mathcal{L}_g), \]

where \( |e| = \sqrt{-g} = \det(\epsilon_{\mu\nu}) \). Also, we use \( S_m \) (\( \mathcal{L}_m \)) and \( S_g \) (\( \mathcal{L}_g \)) to represent the actions (Lagrangians) of matter and gravity, respectively. Since the teleparallel torsion scalar (6) differs from the Ricci scalar \( R \) by a total derivative term, the field equations that transpire when using \( T \) (in the Einstein–Hilbert action as the gravitational Lagrangian) are just equivalent to those with \( R \). This is the TTEGR theory of gravity.

2.1. The Matter Sector

Varying \( S_m \) with respect to the tetrads fields (which has been shown to be equivalent to varying with respect to the metric; de Andrade & Pereira 1998) enables one to define the stress–energy tensor of a perfect fluid as

\[ \tau_{\mu\nu} = e_{\mu\nu}
\]

\[ \left(-\frac{1}{e}\frac{\delta S_a}{\delta \epsilon^a_\nu}\right) = \rho u_\mu u_\nu + p(u_\mu u_\nu + g_{\mu\nu}), \]

where \( u^\mu \) is the 4-velocity unit vector of the fluid.

2.2. The Gravity Sector

In the Einstein–Hilbert action, the TTEGR has been generalized by replacing \( T \) by an arbitrary \( f(T) \) function (Bengochea & Ferraro 2009; Linder 2010; Bamba et al. 2010, 2011) similar to the \( f(R) \) generalization. The \( f(T) \) Lagrangian is

\[ \mathcal{L}_g = \frac{1}{2\kappa^2}f(T). \]

By varying the action \( S_g \) with respect to the tetrads fields, we obtain the tensor

\[ S_{\mu\nu} = \epsilon_{\mu\nu}
\]

\[ \left(1, \frac{\delta S_g}{\delta \epsilon^a_\nu}\right). \]

(11)

Remarkably, other linear connections in vierbein space are discussed in detail in Youssef & Sid-Ahmed (2007) (for applications, see Mikhail et al. 1995).
where the Minkowskian signature is $\eta_{ab} = (+; - , - , - )$. We note that this choice of the vierbein (16) leads to consistent field equations without involving any unphysical degrees of freedom for any $f(T)$ theory (Ferraro & Fiorini 2011; Krššák & Saridakis 2016). The diagonal vierbein (16) directly relates the teleparallel torsion scalar (6) to Hubble rate as follows:

$$T = -6H(t)^2,$$

(18)

where $H(t) \equiv \dot{a}/a$ is the Hubble parameter, and the “dot” denotes differentiation with respect to the cosmic time $t$. Inserting the vierbein (16) into the field equations (14) for the matter fluid (9), the modified Friedmann equations of the $f(T)$ gravity are

$$\frac{3}{k^2}H^2 = \rho + \rho_T \equiv \rho_{\text{eff}},$$

(19)

$$-\frac{1}{k^2}(3H^2 + 2H) = p + p_T \equiv p_{\text{eff}},$$

(20)

where $\rho$ and $p$ are the energy density and pressure of the matter sector, respectively, considered to correspond to a perfect fluid. Independently of the above equations, one should choose an EOS to relate $\rho$ and $p$. Here we choose the simple linear barotropic case $p = w\rho$, where $w$ is the EOS parameter. We are interested in evolution during (pressureless) matter domination; we therefore in practice set $w = 0$. Additionally, the torsional density and pressure in the above equations are

$$\rho_T = \frac{1}{2k^2}[2f_T T - f(T)],$$

(21)

$$p_T = \frac{1}{2k^2}[ \frac{f(T) - T + f(T)}{f_T + 2f_T T} ].$$

(22)

By acquiring the standard matter conservation, we write the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0.$$  

(23)

This in turn implies the continuity equation of the torsional fluid

$$\dot{\rho}_T + 3H(\rho_T + p_T) = 0,$$  

(24)

in order to have a conservative universe. We additionally take an EOS parameter $w_T \equiv p_T/\rho_T$ of the torsional fluid, which incorporates the dark energy sector. Hence, we write

$$w_{\text{DE}} \equiv w_T = \frac{p_T}{\rho_T} = -1 + \frac{[f(T) - 2f_T T - f_T T]}{[f(T) + T - 2f_T T](f_T + 2f_T T)}.$$  

(25)

It is useful to define the effective EOS parameter

$$w_{\text{eff}} \equiv \frac{\rho_{\text{eff}}}{\rho_{\text{eff}}} = -1 - \frac{2H}{3H^2},$$  

(26)

which can be related to the deceleration parameter $q$ by the following expression:

$$q \equiv -1 - \frac{H}{H^2} = \frac{1}{2}(1 + 3w_{\text{eff}}).$$  

(27)

Thanks to the nice feature of the $f(T)$ theory being that its field equations are of second order, currently there are viable $f(T)$ theories of gravity that give good results with a wide range of cosmological observations (Nunes et al. 2016; Xu et al. 2018; Nunes 2018). In the following sections, we focus on a specific
model with IR torsional gravity, which is in principle an alternative to phantom dark energy.

3. Torsional IR Correction Model

In this section, we explore the cosmic evolution that arises as a consequence of the $f(T)$ teleparallel gravity

$$f(T) = T + \alpha \frac{T_0^{1+n}}{T^n},$$  

(28)

where $\alpha$ and $n$ are dimensionless parameters. We denote the present value of a quantity by a subscript “0”, so $T_0$ is the present value of the teleparallel torsion scalar (using Equation (18), we have $T_0 = -6H_0^2$). As a matter of fact, the additional $1/T^n$-term will be effective in the small torsion (i.e., Hubble) regime on the large scale, so we would refer to this term as torsional IR correction. As is clear, the model recovers the GR limit at $\alpha = 0$ or in the large-$T$ regime, where the orders of magnitude $O(T^n) \gg O(T_0^{1+n})$. On the other hand, it reduces to $\Lambda$CDM at $n = 0$ or in the small-$T$ regime, as the magnitudes $O(T^n) \sim O(T_0^{1+n})$, where the quantity $O(\alpha T_0) \approx O(\Lambda)$. Using relation (18), we write the torsional density and pressure (21) and (22) in terms of the Hubble parameter as

$$\rho_T = \frac{3\alpha}{k^2(2n+1)}H_0^{2n}\left(\frac{H_0^2}{H}\right)^n,$$  

(29)

$$p_T = -\frac{3\alpha}{k^2}H_0^{2n+2} + \alpha(2n + 1)H_0^{2n+2}.$$

(30)

Inserting Equation (29) into Friedmann Equation (19) at current time, we write

$$\alpha = \frac{1 - \Omega_{m,0}}{2n + 1},$$

(31)

where $\Omega_{m,0} = \frac{\rho_0}{3H_0^2/k^2}$ is the current matter density parameter. The above equation shows that only one of the parameters $\alpha$ and $n$ is independent. In addition, we use the constraint $\Omega_{m,0}H^2 = 0.1417$, where $h = H_0/100$, as estimated by the CMB measurements (Planck Collaboration XIII 2016), in the rest of our calculations. We will discuss this condition in more detail later on.

The continuity equation of the CDM gives $\rho(H) = \rho_0/a(H)^3$, where $\rho_0 = 3\Omega_{m,0}H_0^2/k^2$ is the current density. Then, the scale factor reads

$$a^3 = \frac{\Omega_{m,0}H_0^2H_0^{2n}}{H_0^{2n+2} - (1 - \Omega_{m,0})H_0^{2n+2}}.$$  

(32)

Using the scale factor–redshift relation, $1 + z = \frac{a_0}{a}$, where $a_0 = 1$ at present, we write

$$z = \left(\frac{E^2 - (1 - \Omega_{m,0})E^{-2n}}{\Omega_{m,0}}\right)^{1/2} - 1,$$  

(33)

where $E = H/H_0$. The inverse relation of Equation (33) gives $H(z)$; however, for a particular $n$ this could be expressed explicitly but complicated. Later in Section 5, we show that the value $n = 1/3$ is preferable by observations. For the $n = 1$ case, a simpler form of $H(z)$ is given in Appendix A as an example with other features to show the cosmic history according to the torsional IR correction model. Hence, in the following we focus our discussion on case $n = 0$, which reduces to the $\Lambda$CDM scenario, in addition to the $n = 1/3$ and $n = 1$ models.

4. Cosmic History and the Phantom Regime

In this section we describe cosmic history in the context of torsional gravity models with IR corrections of the form described in the previous section. We show that these corrections can provide a mechanism for an accelerated phase of cosmic expansion. Prior to this, the evolution is essentially equivalent to that of the standard model; thermal history and structure formation are therefore not expected to be affected. We evaluate the transition to the accelerated phase and show that this eventually involves a phantom regime. The measurements of the deceleration-to-acceleration transition and dark energy EOS are not currently precise enough to distinguish our models from the standard one. Current observations that can be examined in the next section.

4.1. Phase Portrait Analysis and Deceleration-to-acceleration Transition

In a recent study (Awad et al. 2018b; see also Hohmann et al. 2017), the dynamical system approach was applied to the ordinary differential equations arising in the context of $f(T)$ cosmologies. This showed that the modified Friedmann equations can be reduced to a 1D autonomous system, where $H = \mathcal{F}(H)$. This allows us to utilize some geometrical procedures to analyze the dynamical behavior of the set of all solutions and its stability just by visualizing it as trajectories in an $(H, \dot{H})$ phase space. As seen in Figure 1, the $(H, \dot{H})$ phase space has a Minkowskian origin at $(0, 0)$, while by identifying the zero acceleration boundary curve $q \equiv -1 - \frac{\dot{H}}{H^2} = 0$ (given as a dotted curve) the phase space is divided into four dynamical regions according to the values of $H$ and $q$ in each region: The unshaded region (I) represents an accelerated contraction, since $H < 0$ and $q < 0$. The shaded region (II) represents a decelerated contraction, since $H < 0$ and $q > 0$. The shaded region (III) represents a decelerated expansion, since $H > 0$ and $q < 0$. The unshaded region (IV) represents an accelerated expansion, since $H > 0$ and $q > 0$. Notably, one can thus examine and evaluate complicated cosmological models by following their phase trajectories and studying their qualitative behavior, such as their ability to cross between different regions of the phase space.

It has been proven that the $f(T)$ phase portraits can be analyzed easily and information can be extracted in a clear way (for more applications of this approach to $f(T)$ gravity cosmology see Bamba et al. 2016; El Hanafy & Nashed 2017a, 2017b; Awad et al. 2018a). In particular, the governing equation is given by

$$\dot{H} = 3(1 + w)\frac{f - Hf_H}{f_{HH}} = \mathcal{F}(H),$$  

(34)

where $f = f(H)$, $f_H = \frac{df}{dH}$, and $f_{HH} = \frac{d^2f}{dH^2}$. Inserting the torsional IR correction (28) into the governing Equation (34), we can determine the phase portrait equation of the model:

$$\dot{H} = -\frac{3}{2}(1 + w)H^2\left[\frac{H/H_0^{2n+1} - (1 - \Omega_{m,0})}{(H/H_0)^{2n+1} + n(1 - \Omega_{m,0})}\right].$$  

(35)
Following the drawing codes in Awad et al. (2018b), the phase portrait (35) of the torsional IR correction matches the sCDM portrait at \( \dot{H} = H_0 \), while it intersects the zero acceleration curve at \( H_0 \) and evolves toward a fixed point \( H_0 \). Thus, the model is in agreement with standard cosmology in the past and can perform late acceleration in agreement with observations. We use \( w = 0.798 \), \( \Omega_{m,0} = 0.262 \), and \( H_0 = 73.5 \text{ km s}^{-1} \text{ Mpc}^{-1} \) for \( n = 1/3 \) and \( n = 1 \) models, while in the \( n = 0 \) model we use Planck parameters.

At large Hubble regime, the above equation reduces to

\[
\dot{H} \approx -\frac{3}{2} (1 + w) H^2,
\]

which characterizes the phase portrait in GR. The torsional IR correction model thus matches standard predictions of matter domination \( (w = 0) \), prior to cosmic acceleration, as well as the earlier radiation domination era, \( w = 1/3 \). Indeed, from Equations (29) and (30), it is not difficult to show that \( \rho_r \to 0 \) and \( p_r \to 0 \) as \( z \to \infty (H \to \infty) \). We thus expect that our torsion correction to the teleparallel equivalent to GR will not affect the thermal history and structure formation up to the transition to cosmic acceleration.

In Figure 1, we visualize the phase portrait (34) for different values of \( n \) versus the \( \Lambda \)CDM \((n = 0)\) using Planck parameters. As is clear, the portrait is unbounded from below, where \( \dot{H} \to -\infty \) as \( H \to \infty \), which indicates an initial singularity (big bang); asymptotically the portrait matches the sCDM one in the shaded region III (decelerated expansion). However, it cuts the zero acceleration curve, \( q = 0 \) (i.e., \( H = -H^2 \)), which determines the value of the Hubble parameter at transition \( H_{tr} \).

Using Equation (34), we find

\[
H_{tr} = [(2n + 3)(1 - \Omega_{m,0})\frac{1}{\Omega_{r,0}^{2/3}}] H_0.
\]

For \( n = 0 \) and Planck parameters (\( \Lambda \)CDM), we find \( H_{tr} = 98 \text{ km s}^{-1} \text{ Mpc}^{-1} \). Plugging these results into Equation (33), we determine the transitional redshift \( z_{tr} \approx 0.797 \) (\( \sim 0.798 \)) for \( n = 1/3 \) \((n = 1)\). However, for \( \Lambda \)CDM with Planck parameters, it is \( z_{tr} \approx 0.649 \).

For comparison, we can also pin the predicted transition to cosmic acceleration directly from the evolution of the deceleration parameter. Plugging Equation (35) into Equation (27), we write the deceleration parameter

\[
q(z) = -1 + \frac{3}{2} E(z)^{2n+2} - (1 - \Omega_{m,0}) \frac{E(z)^{2n+2} + n(1 - \Omega_{m,0})}{E(z)^{2n+2}}.
\]

In Figure 2, we plot the evolution of the deceleration parameter for different values of \( n \) versus the Planck parameters. The plots show that the deceleration parameter \( q \to 0.5 \) \((w_{eff} \to 0)\) at high redshift, which is in agreement with the \( \Lambda \)CDM domination. In addition, the transition from deceleration to acceleration occurred at redshift \( z_{tr} \approx 0.8 \), which is in agreement with the measured value (Faroq et al. 2017). Also, the current value of the deceleration parameter \( q_0 \approx -0.68 \) \((w_{eff} \approx -0.79)\).

The portrait crosses the zero acceleration curve to the unshaded region IV (accelerated expansion) and evolves toward a fixed point \( H_f \) at \( H = 0 \). This determines the Hubble value at the fixed point

\[
H_f = (1 - \Omega_{m,0})^{\frac{1}{2}} H_0.
\]

Notably, this fixed point cannot be reached in finite time, i.e., \( H \to H_f = \text{constant} \) as \( t \to \infty \); this indicates a pseudo-rip fate (Frampton et al. 2012). In the following we show that this is associated with a phantom regime.

4.2. Phantom-like Effective DE

In order to investigate the physics of the torsional IR correction, we define its EOS, Equation (25). Substituting from Equations (29) and (30), we obtain

\[
w_T(z) = -1 - n \left( \frac{E(z)^{2n+2} - (1 - \Omega_{m,0})}{E(z)^{2n+2} + n(1 - \Omega_{m,0})} \right).
\]

The inverse relation of Equation (33) allows us to express the torsional EOS in terms of redshift, \( w_T(z) \), explicitly. In Figure 3, we plot the torsional EOS for different choices of the parameter \( n \).

We thus determine the current value of the torsional EOS, \( w_T(z = 0) = -1.07 \) \((-1.15)\) when \( n = 1/3 \) \((1)\), which is in
agreement with observations (Sahni et al. 2014; Di Valentino et al. 2016a, 2017). We find that the torsional fluid in the past fixed to \( w_T \rightarrow -1 \) is also avoided in those models. The current value is \( w_T = -1.07 \) (1.15) where \( n = 1/3 \) (1), in agreement with observational constraints. For \( n = 0 \), the torsional fluid gives a fixed EOS \( w_T = -1 \), i.e., a cosmological constant.

As mentioned in the Introduction, dynamical phantom-like dark energy is in fact favored by recent observations. Modified gravity can provide for a framework for such scenarios without introducing ghost instabilities associated with scalar field models of phantom dark energy.

Finally, it is worth noting that the invoked phantom regime does not violate age constraints (e.g., those derived from old globular clusters) even while using the locally measured value of \( H_0 \). For the proposed model (28), the age of the universe is

\[
t_{\text{age}} = -\int_{H_0}^{\infty} \frac{dH}{H} = \frac{2}{3H_0} \int_1^{\infty} E^{-2(1/n-1)} - n(1 - \Omega_{m,0})/E^{2(1/n-1)} - (1 - \Omega_{m,0}) dE.
\]

Even for a large Hubble constant, e.g., \( H_0 = 73.5 \text{ km s}^{-1} \text{ Mpc}^{-1} \) as measured by local observations (Riess et al. 2018a, hereafter referred to as R18) and \( \Omega_{m,0} \sim 0.262 \) (so as to keep \( \Omega_m h^2 = 0.1417 \) constant as we discuss below), the model predicts an age \( t_{\text{age}} \sim 13.6 (13.9) \) billion years for \( n = 1/3 \) (1). In conclusion, the model predicts an age of the universe compatible with current observations.

5. Confrontation with Observations

In this section, we fix the free parameters of the torsional IR gravity model, \( n \) and \( \alpha \) (alternatively \( \Omega_{m,0} \)). We use Planck measurement of the CMB shift parameter at recombination to constrain the value of \( n \) according to the \( H_0 \) value. In addition, we use the Planck constraint fixing \( \Omega_{m,0} h^2 \sim 0.1417 \) so that we do not have any deviation from the CMB Planck results. Also, we confront the model predictions of the CMB Planck results. Also, we confront the model predictions of the electron scattering optical depth at reionization with the Planck measurements. Next, we use cosmic chronography (CC) and radial and transverse BAO measurements including Ly\( \alpha \) observations to examine the model.

5.1. Distance to CMB and Shift Parameter: Resolving the \( H_0 \) Tension

As is now well known, there exists significant tension between the locally measured value of the Hubble constant and that inferred from the CMB. For example, Riess et al. (2018a) recently measured \( H_0 = 73.52 \pm 1.62 \text{ km s}^{-1} \text{ Mpc}^{-1} \), while Planck Collaboration XIII (2016) estimate \( H_0 = 67.8 \pm 0.9 \text{ km s}^{-1} \text{ Mpc}^{-1} \). While the debate continues as to whether the discrepancy is due to new physics or simply observational systematics, it is straightforward to show that the values can in principle be reconciled by invoking a phantom acceleration regime, as we now outline.

Given a primordial fluctuation spectrum and an FLRW cosmology, the relative height of the CMB peaks is essentially determined by the dimensionless physical dark matter and baryon densities \( \Omega_m h^2 \) and \( \Omega_b h^2 \), respectively. Fixing, in addition, the number of effective relativistic degrees of freedom in turn fixes the era of matter radiation equality and recombination, and with these the intrinsic physical scale of the CMB peaks (e.g., Hu & Sugiyama 1995; Percival et al. 2002), as well as light-element production in the context of big bang nucleosynthesis (BBN). We will assume that all these parameters are fixed to standard values (namely, as quoted in Planck Collaboration XIII 2016) and that the cosmological evolution is practically indistinguishable from the standard scenario up to late times, when the dark-energy-like component becomes significant. For the specific case of the modified gravity models used here, the latter assumption is justified by the fact that the IR correction theory tends to the teleparallel equivalent to GR at such redshifts; we thus expect the evolution, including the growth of perturbations, to be similar.

In this context, a measurement of the angular diameter (transverse) distance to the CMB,

\[
D_A(z) = \frac{1}{1+z} \int_0^z \frac{dz'}{H(z')},
\]

with \( z = z_{\text{lsr}} \) referring to the redshift of the last scattering surface, determines \( H_0 \), given a cosmological model (i.e., \( H(z) \)). In the standard \( \Lambda \)CDM model, such a measurement should yield a value that is smaller than locally measured values (similar to the one obtained by Planck Collaboration XIII 2016 by fitting the full CMB spectrum). Nevertheless, as \( H(z) \) can be written as \( H(z) = H_0 E(z) \), it is easy to see that one can increase \( H_0 \), while keeping the angular distance constant, by choosing a model where \( E(z) \) is smaller than that associated with \( \Lambda \)CDM in the redshift range \( 0 \leq z \leq z_{\text{lsr}} \). This state of affairs would also be reflected in the invariance of the “shift parameter” of Efstathiou & Bond (1999),

\[
R_{\text{lsr}} = \sqrt{\Omega_{m,0} h^2} \int_0^{z_{\text{lsr}}} \frac{dz}{H(z)} = \sqrt{\Omega_{m,0}} \int_0^{z_{\text{lsr}}} \frac{dz}{E(z)}.
\]
below its ΛCDM value $E_\Lambda(z)$ for $0 \leq z \leq z_{\text{ls}}$ for it to be possible to keep $R_{\text{ls}}$ constant. We now show that it is not only possible but necessary, in the phantom regime, that $E_p(z) \leq E_\Lambda(z)$.

Friedmann evolution in a flat universe with matter and DE (or DE-like, as in torsion gravity) components implies

$$\frac{E_p^2(z)}{E_\Lambda^2(z)} = \frac{\Omega_{mp}a^{-3} + \Omega_p(z)}{\Omega_{m\Lambda}a^{-3}} + \Omega_\Lambda,$$  \hspace{1cm} (43)

where $\Omega_{mp}$ and $\Omega_{m\Lambda}$ refer to the contributions of the matter densities to the critical density at $z = 0$ in the phantom and ΛCDM cases, respectively. If one requires a larger value for $H_0$ in the phantom case, while keeping $\Omega_{m\Lambda}h^2$ the same in the two cases, then $\Omega_{mp} < \Omega_{m\Lambda}$. The contribution $\Omega_\Lambda$ to the current critical density is constant, while $\Omega_p = \Omega_p(z)$, being a phantom DE contribution, necessarily increases in time (with decreasing $z$).

By definition $E_p(z = 0)/E_\Lambda(z = 0) = 1$. But since $\frac{E_p(z)}{E_\Lambda(z)} < 1$ for $z > 0$, if $\Omega_{mp} = \Omega_{m\Lambda}$. If we require that $\Omega_{mp} < \Omega_{m\Lambda}$, so as to keep $\Omega_{m\Lambda}h^2$ the same while increasing $H_0$, then the ratio $\frac{E_p(z)}{E_\Lambda(z)}$ becomes smaller still. It is thus apparent that in the presence of phantom-like dark energy, it is not only possible but necessary to decrease $E(z)$ relative to the standard case, which in turn necessitates an increase in $H_0$ if CMB angular distance, shift parameter, and physical matter densities are to be kept constant.

Figure 4 illustrates this in the context of our torsion gravity models. Here we vary $H_0$, keeping $\Omega_{m\Lambda}h^2 = 0.1417$ (as measured in Planck Collaboration XIII 2016), and evaluate the shift parameter $R_{\text{ls}}$, substituting $H(z)$, namely, the inverse of Equation (33), into Equation (42), where $z_{\text{ls}} = 1089.9$ (Planck Collaboration XIII 2016). We then subtract this from the measured value of $R_{\text{ls}} = 1.7488$, retrieved from PlanckTT +lowP (Planck Collaboration XIV 2016), and divide by the error estimate quoted therein ($\pm 0.0074$). As can be seen, as one deviates from the cosmological constant scenario ($n = 0$) and further into the phantom regime, the lines intersect the zero error horizontal at larger values of $H_0$, as expected. These larger values are thus necessary in order to fit the CMB data embodied in the shift parameter. In particular, a value of about $n = 1/3$ fits the shift parameter with $H_0 = 73.5 \text{ km s}^{-1} \text{ Mpc}^{-1}$, as locally measured by Riess et al. (2018a).

5.2. Reionization Redshift

The electron scattering optical depth $\tau_e$ of the CMB provides a direct probe of the reionization epoch and its redshift $z_{\text{re}}$; it places constraints on the cosmological model, as it depends on $H(z)$ at redshifts intermediate between $z_{\text{ls}}$ and local measurements. The optical depth can be evaluated from

$$\tau_e(z_{\text{re}}) = c \int_0^{z_{\text{re}}} n_e(z) \sigma_T \frac{dz}{(1 + z)H(z)},$$  \hspace{1cm} (44)

where $n_e(z)$ is the electron density and $\sigma_T$ is the Thomson cross section describing scattering between electrons and CMB photons. Here we take the densities of hydrogen, helium, and electrons, respectively, as $n_H = [(1 - Y_p)\Omega_{bH_0}/m_H](1 + z)^3$, $n_{He} = n_H$, and $n_e = (1 + y)n_{He}$, where $y = 4(1 - Y_p)$ and $m_H$ is the hydrogen mass (Shull & Venkatesan 2008). We use the Planck constraint $\Omega_{bH_0}h^2 = 0.02230$ (Planck Collaboration XIII 2016), which gives the baryon density parameter $\Omega_{bH_0} = 0.0413$ for $H_0 = 73.5 \text{ km s}^{-1} \text{ Mpc}^{-1}$, the helium mass fraction $Y_p = 0.247$ (Peimbert et al. 2007), and the current critical density $\rho_c = 1.88 \times 10^{-29} h^2 \text{ g cm}^{-3}$. Then, using the inverse function of Equation (33) and by evaluating the integral (44), we get $\tau_e(z_{\text{re}}) \approx 0.058$ at $z_{\text{re}} = 8.1$, which is in agreement with Planck Collaboration XLVII (2016) (lollipop + PlanckTT + lensing) observations, where $\tau_e(z_{\text{re}}) = 0.058 \pm 0.012$, where $7.8 < z_{\text{re}} < 8.8$.

5.3. Local Hubble Parameter Evolution

As the resolution of the $H_0$ tension in terms of phantom dark energy described above involves changing the evolution of $H(z)$—through changing $E(z)$—it is natural to inquire whether this change can be actually distinguished directly from local $H(z)$ measurements. Figure 5 collects such measurements. These include the 43 Hubble measurements given in Cao et al. (2018), which lists a number of CC and BAO measurements (including two Lyα observations). Also, we include four BAO measurements $H(z = 0.978) = 113.72 \pm 14.63$, $H(z = 1.23) = 131.44 \pm 12.42$, $H(z = 1.526) = 148.11 \pm 12.75$, and $H(z = 1.944) = 172.63 \pm 14.79 \text{ km s}^{-1} \text{ Mpc}^{-1}$ (Zhao et al. 2018) and one BAO Lyα observation $H(z = 2.33) = 224 \pm 8 \text{ km s}^{-1} \text{ Mpc}^{-1}$ (du Mas des Bourboux et al. 2017). In addition to these observations we include the R18 observation of $H_0 = 73.52 \pm 1.62 \text{ km s}^{-1} \text{ Mpc}^{-1}$ as measured by Riess et al. (2018a). Figure 5 shows clearly the capability of the torsional IR gravity (with $n = 1/3$) to fit with R18 and Lyα better than the ΛCDM model.

The $\chi^2$ statistics of these 49 Hubble measurements are

$$\chi^2(n, \alpha) = \sum_i \frac{(H_i - H^\alpha_i)^2}{\sigma_i^2 H^\alpha_i^2},$$  \hspace{1cm} (45)

where $H_i$ is the Hubble measurement, $H^\alpha$ is the theoretical $H(z)$, and $\sigma_i$ is the corresponding uncertainty. The $\chi^2$ values are shown in Figure 5. The best fit is obtained with $n = 1/3$, which gives $\chi^2 = 7.82 \pm 0.71$, with $\tau_e(z_{\text{re}}) \approx 0.0553$ at $z_{\text{re}} = 8.1$, which is in agreement with Planck Collaboration VI (2018) observations (TT,TE,EE+lowE+lensing+BAO). The $\tau_e(z_{\text{re}}) \approx 0.0553$ at $z_{\text{re}} = 8.1$, which is in agreement with Planck Collaboration VI (2018) observations (TT,TE,EE+lowE+lensing+BAO).
where the subscript $i = 1, 2, \ldots, 49$, the superscripts $t$ and $o$ denote the theoretical and observed values of $H(z)_i$, respectively, and $\sigma_{H_i^o}$ denote the one standard deviations in the measured values. As can be inferred from Table 1, both $\Lambda$CDM and the model with $n = 1/3$ and $H_0 = 73.5\text{ km s}^{-1}\text{ Mpc}^{-1}$ display $\chi^2/\text{dof} \lesssim 1$, where “dof” is the number of degrees of freedom given that there are two model parameters. But the model associated with a phantom-like effective dark energy component performs better than that invoking the cosmological constant. This remains the case as long as the Ly$\alpha$ and R18 data are included. The BAO data on their own, on the other hand, favor the standard model. As we see below, this conclusion is definitely consolidated by BAO distance measurements, combined with the CMB.

We note that the Hubble function is related to the luminosity–distance $D_L(z)$ by

$$\frac{d}{dz} \left( \frac{D_L(z)}{1+z} \right)^{-1}. \quad (46)$$

Since the Hubble function is related to the first derivative of $D_L$, one expects the measured values of $H(z)$ to be much noisier than $D_L(z)$ measurements. In other words, distances are in principle integrable quantities, which makes them relatively more precise. In the following we confront the torsional IR gravity with the BAO angular distance measurements.

### 5.4. BAO Distance Measurements

BAO can be used as standard rulers; from isotropic measurements one can infer $D_V = [cz(1 + z)^2D_L^2(z)/H(z)]^{1/3}$ and from anisotropic measurements $D_A$ itself (given the sound horizon at the baryon drag epoch $r_d$). These measurements rely on the same principle as that used to infer the angular diameter distance to the CMB (as once the physical densities and eras of recombination and matter radiation equality are determined, the physical scale of the peaks is fixed). It turns out that such measurements are highly constraining and essentially rule out solutions of the $H_0$ tension invoking phantom-like dark energy.

We show this by using the observations from various independent data sets: Beutler et al. (2011), which use 6dFGRS data, Kazin et al. (2014) for reconstructed WiggleZ, Ross et al. (2015) for the SDSS MGS data, Alam et al. (2017a) for BOSS, Ata et al. (2018) for eBOSS quasar data, and Abbott et al. (2019) for the DES survey. We use $r_d = 147.5\text{ Mpc}$ when the results are given as ratios involving $r_d$. We calculate the relevant distances to the observed redshift of the BAO peaks of each observation for our torsion models, deriving $D_A$ and $D_V$ for values of $0 \lesssim n \lesssim 1$. For each such value, we vary $\Omega_m$ to search for the minimum of

$$\chi^2(n, H_0) = \sum_i \frac{(D_i^t - D_i^o)^2}{\sigma_i^2} + \frac{(R_{i,\text{fs}} - R_{i,\text{obs}})^2}{\sigma_{i,\text{fs}}^2}, \quad (47)$$

where $D_i$ refers to the different BAO measurements (either $D_V$ or $D_A$, depending on the particular set of observations), $\sigma$ denotes the one standard deviations in measurements, and the superscripts $t$ and $o$ refer to the theoretical and observed values of the different quantities. For each $n$ there is then a unique $\Omega_m$ that minimizes the $\chi^2$. Assuming, as we do, that $\Omega_mh^2$ is held fixed (at 0.1417), one can also associate a unique $H_0$ with each $\Omega_m$, and hence for each $\Omega_m$ that minimizes $\chi^2$ at each $n$.

The results are shown in Figure 6. As can be seen, the deviation between the observed and inferred distances, as measured using Equation (47), is smallest for $n = 0$. Values of $n \gtrsim 1/5$ are ruled out at the 99% confidence level. Moreover, for $n = 1/3$, the corresponding $H_0$ that minimizes $\chi^2$ is significantly smaller than that inferred when fitting the CMB alone in Section 5.1. The reason for this failure is discussed in the next section.

### 5.5. BAO Distance Measurements and the Failure of Phantom Models

In Section 5.1, we argued that the angular diameter distance to the CMB and the shift parameter can be kept constant if one increases the value of $H_0$ while invoking cosmic evolution in the phantom regime. This was because $E(z) = H(z)/H_0$ is smaller up to $z = 0$ for such scenarios than in the case when the DE contribution comes from a cosmological constant. 

![Figure 5](image)

Figure 5. Evolution of Hubble function in terms of redshift. For the $\Lambda$CDM model with $n = 0$, we take $H_0 = 68\text{ km s}^{-1}\text{ Mpc}^{-1}$. For the torsional IR gravity model with $n = 1/3$, we take $H_0 = 73.5\text{ km s}^{-1}\text{ Mpc}^{-1}$.
Requiring a larger value of $H_0$ evidently implies that the $H(z)$ associated with phantom dark energy becomes larger than that of $\Lambda$CDM at some redshift $z_c > 0$. From the Friedmann equations

$$\frac{H_p^2(z)}{H(z)^2} = \frac{\rho_{mp}(z) + \rho_p(z)}{\rho_{m\Lambda}(z) + \rho_{\Lambda}},$$

one can find this redshift. The physical matter densities remain such that $\rho_{mp} = \rho_{m\Lambda}$ if we keep $\Omega_{m,0}h^2$ fixed, so that the ratio in Equation (48) is smaller than 1 when the phantom dark energy density is less than that associated with a cosmological constant: $\rho_p(z) < \rho_{\Lambda}$. The ratio then increases to finally reach $H_p^2(z = 0)/H(z)^2 = 1$ at $z = 0$, which is greater than unity if we assume a larger value of the Hubble constant to be associated with the phantom case. The critical value $z_c$ corresponds to a ratio of 1. If this occurs during matter domination, then the epoch where $H_p^2(z)/H(z)^2 < 1$ has negligible effect on the evolution and $H_p^2(z) > H(z)^2$ for all practical purposes (that is, during DE domination). In this case, the angular diameter distance to the CMB will increase. If this distance (and shift parameter) is to be kept in line with observed values, then $H_p^2(z) > H(z)^2$ should become unity at $z_c \sim 1$.

In the case of torsion gravity models studied here this is illustrated in Figure 7, where we plot $1/H(z)$—the integrand in the formula for the angular diameter distance—for the standard model with $n = 0$ and for $n = 1$. If $H_0$ is assumed to be the same in the two cases, $H(n = 0)$ is always smaller than or equal to $H(z = 1)$, which simply reflects the fact that $E_p \leq E_\Lambda$ up to $z = 0$ as expected from the discussion following Equation (43). When $H_0$ associated with the $n = 1$ case is larger, the lines cross at $z = z_c$. What this implies is that the angular diameter distance will be smaller than in the standard case for objects at $z < z_c$. And if the distance to the CMB $D_A(z = z_{\text{obs}})$ is to be kept fixed, while increasing $H_0$ and invoking the phantom regime, then $D_A(z)$ to any object at $0 \leq z \leq z_{\text{obs}}$ will be larger than or

---

**Figure 6.** Combined $\chi^2$ per degree of freedom of CMB and BAO distances, with horizontal lines showing the associated confidence levels. Left panel: as a function of $n$ (measuring deviation from cosmological constant at $n = 0$). Right panel: in terms of the value of $H_0$ that minimizes $\chi^2$ for each value of $n$ (assuming $\Omega_{m,0}h^2 = 0.1417$).

**Figure 7.** Evolution of the radial and angular distances. Left panel: redshift evolution of the integrand in Equation (41). Note that for $n = 0$—i.e., deviation from a cosmological constant and into the phantom regime—$1/H(z)$ is invariably larger if $H_0$ is kept fixed. This implies larger distances for all $z$. For larger $H_0$ the curves of $n = 0$ and $n = 1$ cross. This means that radial distances can be underestimated or overestimated, depending on their location relative to the crossing point. However, if the distance to the CMB is fixed, in both models, then the $n > 0$ distances are again invariably smaller. Right panel: comparison of $\Lambda$CDM (horizontal line) and phantom model with $n = 1/3$ and $H_0 = 73.5\,\text{km/s/Mpc}^{-1}$ with BAO transverse distance estimates. As can be seen, phantom models systematically underestimate the transverse distance compared to the $\Lambda$CDM, while no such systematics in the data at redshift $z \lesssim 1.5$. 

---

\[
\frac{H_p^2(z)}{H(z)^2} = \frac{\rho_{mp}(z) + \rho_p(z)}{\rho_{m\Lambda}(z) + \rho_{\Lambda}},
\]
equal to that predicted by $\Lambda$CDM. If the distance to the CMB is overestimated, then the distances to objects can be either overestimated or underestimated depending on its redshift.

This implies that, in order to fit CMB and BAO distances simultaneously using a larger $H_0$ and $n > 0$, the standard model should systematically overestimate distances to BAO measurements, with the discrepancy being maximal for redshifts around $z$. This is not observed, as can be seen from Figure 7 (right panel). To further illustrate the point, we plot the errors associated with the different observations, which were used to estimate the $\chi^2$ in Figure 8. As can be seen, at $n=0$ some distances are overestimated and some underestimated, with no clear trend in terms of redshift dependence. As $n$ is varied, the critical redshift $z_c$ changes, and the $\chi^2$ minimization procedure causes the distance to the CMB to also shift. As a result, there is another critical redshift below which BAO measurements are underestimated relative to the standard case and beyond which they are overestimated. Since there is no systematic deviation with respect to $\Lambda$CDM predictions in the BAO data used, this process means that some distances that were initially underestimated at $n=0$ become even more so for $n > 0$, and conversely some overestimates are increased.

Current CMB and BAO measurements seem to therefore rule out a significant phantom-like regime in the redshift range of the BAO data included here. This is the case even if one keeps $H_0$ at a small value, for this would shift the distance to the CMB and also the BOA points owing to the smaller $E(z)$ and hence larger associated $1/H(z)$ (as discussed in Section 5.1 and reflected in Figure 7). We note, nevertheless, that there seems to be a systematic underestimate of the BAO distances inferred from $\Lambda$CDM measurements in the context of $\Lambda$CDM. We have not included these points here, as they lead to worse $\Lambda$CDM fits and do not lead to much improvement for the cases with $n=0$, given that the models studied here are close to $\Lambda$CDM for the relevant redshifts ($z \gtrsim 2$) and the relatively large observational errors. Possible modest phantom evolution confined to redshifts $z \gtrsim 2$ is therefore not ruled out and can be tested by upcoming data.

6. Conclusion

The results presented here suggest that the torsional IR corrections to teleparallel gravity lead to a phantom-like effective dark energy term in the Friedmann equations. Given the current matter density, our family of models contain only one free parameter. A phantom-like dark energy evolution, sourced by the gravitational sector, can be derived for positive values of this parameter without invoking a canonical scalar field that suffers from ghost instabilities. We perform a dynamical system analysis that elucidates the basic qualitative evolution of the system, including the transition to the accelerated regime.

As has recently been noted, the phantom regime provides a basis for resolving the tension between local and global measurements of the Hubble constant $H_0$. We find that these can indeed be reconciled by our model. For values of the parameter that completely reconcile the two values, the phantom regime comes with a dynamical EOS $-1 \leq w_\text{eff}(z) \leq -1$ with $w_\gamma = -1.07$ at present. These correspond to deceleration parameter $q_0 = -0.68$ and effective EOS $w_{\text{eff},0} = -0.79$ at present, with transition redshift $z_\text{tr} \approx 0.8$. The model also predicts an electron scattering optical depth $\tau_e \approx 0.058$ at reionization redshift $z_{\text{re}} \approx 8.1$, which is in agreement with observations.

The model, however, faces serious problems when BAO data are included. This is true for both line-of-sight measurements, from which the Hubble parameter can be inferred, and transverse ones yielding measures of the distances to the BAO peaks at different redshifts. The latter case is the most severe, with the model parameter $n$ that corresponds to the reconciliation of the local and CMB values being ruled out to more than 99.99% confidence by these data.

We argue that this failure should be a generic feature of phantom dark energy models, particularly ones that may solve the $H_0$ tension by predicting currently observable deviations from $\Lambda$CDM evolution at $z \lesssim 2$. For, assuming that $\Omega_{m,h}^2$ is held constant, so as not to modify the heights of the CMB acoustic peaks, one finds that in fact distances to objects in the whole redshift range to the CMB last scattering surface are necessarily overestimated, if the angular diameter distance and associated shift parameter are to be kept fixed to current observations. Therefore, if the model predicts currently observable deviations from $\Lambda$CDM evolution at $z \lesssim 2$, then it necessarily contradicts the BAO measurements at these redshifts, which do not show any such systematic discrepancies with the standard model. If the distance to the CMB is allowed to shift, then the distance to some objects (beyond some critical redshift) will be underestimated and the distance to some (at lower redshift) will be underestimated, again in a systematic way that is not in line with observations. In this case, we mention some scenarios that possibly resolve the conflict with the angular distance measurements: (i) Phantom models with
a sudden ripping behavior at low redshift. As seen from Figure 7 (right panel), the nonsystematics of the data in fact fit well with models similar to \(\Lambda\)CDM at low redshifts \(z \lesssim 1.5\); however, in order to fit with large \(H_0\) the model needs to suddenly evolve to phantom regime at \(z \lesssim 0.07\); such models may evolve to big rip singularity or in the best scenario toward a pseudo-rip. In the latter one should calculate the ripping inertial force. (ii) Oscillating DE models with quintom behavior (i.e., oscillating about \(\Lambda\)CDM), where phantom behavior should show up at law redshifts \(0 < z \lesssim 0.1\) and \(z \lesssim 1.5\), quintessence behavior at an intermediate region \(0.1 \lesssim z \lesssim 1.5\), and matching \(\Lambda\)CDM at larger redshifts. (iii) Nonflat models, where the contribution of the curvature density parameter \(\Omega_k\) to the angular distance could provide a correction for better matching with the measured values.

We note that Ly\(\alpha\) BAO observations at \(z \gtrsim 2\) do indeed currently suggest a systematic underestimate on the part of the standard \(\Lambda\)CDM of the distances involved. If these persist with incoming measurements, they could in principle be explained by a phantom regime confined to a range around that redshift.

We would like to thank Adi Nusser and Joe Silk for helpful communication. This work was supported by grant no. 25859 from the Egyptian Science and Technology Development Fund Basic and Applied Research Grants.

**Appendix A**

**Example: \(n = 1\) Case**

As mentioned earlier, Equation (33) can be inverted to give an explicit Hubble–redshift relation for a particular choice of \(n\). However, this form is complicated to give in detail. For the \(n = 1\) model, the formulae are not complicated and can be given explicitly. Since qualitative features are similar to those discussed for smaller \(n\) values, we present the \(n = 1\) model in detail. In addition, we examine the torsional IR gravity on the perturbation level of the theory by investigating the sound speed \(c_s\) of the scalar fluctuations.

A.1. **Cosmological Parameters**

For the \(n = 1\) case, the torsion gravity model (28) reads

\[
f(T) = T + \alpha \frac{T_0^2}{T}.\tag{49}\]

The modified Friedmann's equations, Equations (19) and (20), become

\[
\rho = \frac{3}{\kappa^2} \left[ H^2 - 3\alpha H_0^2 \left( \frac{H_0}{H} \right)^2 \right],\tag{50}\]

\[
p = -\frac{2}{\kappa^2} H \left[ 1 + 3\alpha \left( \frac{H_0}{H} \right)^2 \right] - \rho.\tag{51}\]

By constraining the above to the linear EOS choice \(p = w\rho\), the solution is given as

\[
t = t_0 + \frac{2}{3(1 + w)H} \left[ \ln \left( \frac{H + (3\alpha)^{1/2}H_0}{H - (3\alpha)^{1/2}H_0} \right) - 2\arctan \left( \frac{H}{(3\alpha)^{1/2}H_0} \right) \right] + \frac{3t_0^2}{9} \ln \left( \frac{H + (3\alpha)^{1/2}H_0}{H - (3\alpha)^{1/2}H_0} \right),\tag{52}\]

where \(t_0\) is an integration constant. Although the above solution is exact, it is hard to extract information about the system from Equation (52). For example, its not clear how the system could behave at \(t \to \infty\), or how sensitive it is to the choice of initial conditions. On the contrary, as we have shown, the graphical analysis of its phase portrait represents an adequate description of the qualitative features of the global dynamics. For the \(n = 1\) model, the phase portrait (35) reads

\[
\dot{H} = -\frac{3}{2} (1 + w)H^2 \left[ \frac{(H/H_0)^3 - 3\alpha}{(H/H_0)^3 + 3\alpha} \right],\tag{53}\]

which has been drawn in Figure 1. As clear from Equations (50) and (51), the torsional counterpart has density and pressure,

\[
\rho_T = \frac{9\alpha H_0^4}{\kappa^2 H^2},\tag{54}\]

\[
p_T = -\frac{18\alpha H_0^4 H^2}{\kappa^2 (H^4 + 3\alpha H_0^4)}.\tag{55}\]

It is useful to represent the Friedmann Equation (50) in dimensionless form:

\[
\Omega_m + \Omega_T = 1,\tag{56}\]

where \(\Omega_m = \rho/\rho_{\text{crit}}\) and \(\Omega_T = \rho_T/\rho_{\text{crit}}\) are the matter and the torsion density parameters, respectively. Also, we note that the model parameter \(\alpha\), namely, Equation (31), is related to the current matter density parameter,

\[
\alpha = \frac{1}{3} (1 - \Omega_{m,0}) = \frac{1}{3} \Omega_{T,0}.\tag{57}\]

Using the above equation and the useful relation

\[
\dot{H} = - (1 + z)H(z) \frac{dH}{dz},\tag{58}\]

one can solve Equation (35) for Hubble,

\[
H(z) = \frac{H_0}{\sqrt{2}} \sqrt{\Omega_{m,0}(1 + z)^3 + \sqrt{\Omega_{m,0}^2(1 + z)^6 + 4\Omega_{T,0}}}.	ag{59}\]

One of the important results that can be directly extracted from Equation (59) is the age–redshift relation,

\[
t(z) = \frac{\sqrt{2}}{H_0} \int_z^{\infty} \frac{dz'}{(1 + z')^3} \sqrt{\Omega_{m,0}(1 + z')^3 + \sqrt{\Omega_{m,0}^2(1 + z')^6 + 4\Omega_{T,0}}}.	ag{60}\]

Next, we evaluate the matter density parameter by substituting from Equation (59) into Equation (50), which yields

\[
\Omega_m(z) = \frac{2 \Omega_{m,0}(1 + z)^3}{\Omega_{m,0}(1 + z)^3 + \sqrt{\Omega_{m,0}^2(1 + z)^6 + 4\Omega_{T,0}}}.	ag{61}\]

Thus, the torsional density parameter is

\[
\Omega_T(z) = 1 - \frac{2 \Omega_{m,0}(1 + z)^3}{\Omega_{m,0}(1 + z)^3 + \sqrt{\Omega_{m,0}^2(1 + z)^6 + 4\Omega_{T,0}}}.	ag{62}\]

We plot the evolution of $\Omega_m(z)$ and $\Omega_T(z)$ in Figure 9 (left panel). It shows that $\Omega_m \rightarrow 1$ at large $z$ while $\Omega_T \rightarrow 0$, which indicates the CDM domination. On the contrary, $\Omega_m$ drops to zero and $\Omega_T \rightarrow 1$ at $z \rightarrow -1$ ($t \rightarrow \infty$), where the evolution is dominated by the dark torsion with a pseudo-rip cosmology as a final fate. The pattern shown in Figure 9 (left panel) is in agreement with basic requirements of the viable scenario.

Using Equations (58) and (59), the deceleration parameter of the torsional IR model is given by

$$q(z) = -1 + \frac{3\Omega_m,0(1 + z)^3}{2\sqrt{\Omega^2_m,0(1 + z)^6 + 4\Omega_{T,0}}}.$$  \hspace{1cm} (63)

Alternatively, using Equation (27), we write the effective (total) EOS

$$w_{eff}(z) = -1 + \frac{\Omega_m,0(1 + z)^3}{\sqrt{\Omega^2_m,0(1 + z)^6 + 4\Omega_{T,0}}}.$$ \hspace{1cm} (64)

which is plotted as in Figure 9 (middle panel), showing that $-1 \geq w_{eff} \leq -1/3$ at $-1 \geq z \leq 0.7$ in agreement with observations. However, to express the torsional counterpart EOS in terms of redshift, $w_T(z)$, we substitute Equation (59) into Equations (29) and (30), to write its density and pressure

$$\rho_T(z) = \frac{6\Omega_{T,0}H_0^2}{\kappa^2[\Omega_m,0(1 + z)^3 + \sqrt{\Omega^2_m,0(1 + z)^6 + 4\Omega_{T,0}}]},$$

$$p_T(z) = -\frac{6\Omega_{T,0}H_0^2}{\kappa^2\sqrt{\Omega^2_m,0(1 + z)^6 + 4\Omega_{T,0}}}.$$ \hspace{1cm} (65)

Hence, we obtain the torsional EOS

$$w_T(z) = -1 - \frac{\Omega_m,0(1 + z)^3}{\sqrt{\Omega^2_m,0(1 + z)^6 + 4\Omega_{T,0}}}.$$ \hspace{1cm} (66)

At present, $z = 0$, the above equation reduces to

$$w_{T,0} = -1 - \frac{\Omega_m,0}{2 - \Omega_{m,0}}.$$  

For any value $\Omega_{m,0} > 0$, the torsional EOS goes below $-1$. This clarifies the phantom-like nature of the torsional IR corrections. Also, we note that the angular distance, namely, Equation (41), allows us to perform an important qualitative test, that is, the evolution of the comoving volume element within solid angle $d\Omega$ and redshift $dz$,

$$dV = \frac{(1+z)^2D_A^2}{H(z)}d\Omega dz.$$ \hspace{1cm} (67)

This quantity provides a useful test for computing the source counts (Newman & Davis 2000). Using Equations (59) and (41), the evolution of the volume element (up to a factor of Hubble volume $H_0^3$) is plotted in Figure 9 (right panel). The plot shows that the comoving volume element reaches a maximum value at $z \gtrsim 2$ very similar to the CDM pattern.

**A.2. Physical Viability**

In addition, we perform a basic test on the perturbation level of the theory that should be carried out for any modified gravity theory, that is, the propagation of the sound speed of the scalar fluctuations. As a matter of fact, a considerable array of modified gravity theories can describe the late transition of the cosmic acceleration fulfilling the basic requirements on the background level. However, any such theory remains at risk until its description on the perturbation level also fulfills some physical conditions. A necessary condition is for the sound speed of scalar fluctuations to be constrained between $0 \leq c_s^2 \leq 1$. This is required in order to have a stable and causal theory.

To calculate the sound speed, we take the longitudinal gauge with two-scalar metric fluctuation, that is,

$$ds^2 = (1 + 2\Phi)dt^2 - a^2(1 - 2\Psi)dx^2.$$ \hspace{1cm} (68)

This leads to a fluctuation in the teleparallel torsion scalar (Cai et al. 2011)

$$\delta T = 12H(\dot{\Phi} + H\Psi).$$

Just as in GR theory, the weak-field limit about Minkowski space clarifies that the scalar metric fluctuation $\Phi$ plays the role of the gravitational potential. We follow the perturbation
equations (Cai et al. 2011) up to the linear order, assuming that the matter sector is a canonical scalar field $\phi$ with a Lagrangian

$$\mathcal{L}_m \to \mathcal{L}_\phi = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi).$$

For the choice of the vierbein (16), it has been shown that (see Cai et al. 2011; Chen et al. 2011), in the $f(T)$ gravity, we have only a single degree of freedom minimally coupled to a canonical scalar field $\phi$, since the scalar field fluctuation $\delta \phi$ can fully determine the gravitational potential $\Phi$ in the absence of anisotropic stress, i.e., $\Phi = \Psi$. Using relation (18), we find that the square of the sound speed for the general form of the IR $f(T)$ theory is

$$c_s^2 = \frac{H_{	ext{eff}}}{H_{\text{eff}}} = 1 - \frac{2n(1 + n)(1 - \Omega_{m,0})}{(1 + 2n)[H^{2(1 + n)} + n(1 - \Omega_{m,0})]}.$$  

As is clear, for $n = 0$, the model reduces to $\Lambda$CDM where the speed of sound is fixed to the value $c_s = 1$. For the $n = 1$ case, substituting from Equation (59) into Equation (70), we write the square of the sound speed in terms of the redshift,

$$c_s^2(z) = 1 - \frac{8\Omega_{r,0}/(\Omega_{m,0}(1 + z)^6 + 4\Omega_{r,0})}{3(\Omega_{m,0}(1 + z)^3 + \sqrt{\Omega_{m,0}(1 + z)^6 + 4\Omega_{r,0}})}.$$  

We thus can verify that the square of the sound speed of the primordial scalar fluctuation $c_s^2 \to 1$ in the past as $z \to \infty$, while its current value $c_s^2(z = 0) \sim 0.43$. However, in the far future $c_s^2 \to \frac{1}{3}$ as $z \to -1$. The detailed evolution is given in Figure 10, which shows that the square of the sound speed is $\frac{1}{3} \leq c_s^2 \leq 1$. Also, we include the evolution of $c_s^2(z)$ in the $n = 1/3$ case for completeness. This result confirms that the torsional IR correction theory is free from ghost/gradient instabilities and acausality problems at all times.

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5 Usually the square of the sound speed of the scalar fluctuations is given in the form $c_s^2 = \frac{\partial^2 H}{\gamma + 2\delta H^2}$ (see Cai et al. 2011; Chen et al. 2011). We re-express it in terms of $H$ as given in Equation (70), which is more appropriate for our analysis.
