Fractional order sliding mode control based on single parameter adaptive law for nano-positioning of piezoelectric actuators

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Abstract
A fractional order sliding mode control (FOSMC) based on single parameter adaptive law for nano-positioning of Piezoelectric Actuators (PEAs) is proposed. First, the Bouc–Wen (B–W) model is used to describe the hysteresis of the nano-position platform based on PEAs, which provides a mathematical model for the subsequent controller design. Then, theoretical support is provided to design the FOSMC based on adaptive law of different parameters, which are proposed for the displacement tracking problem of PEAs, and the position error convergence is also proved. Moreover, the core parameters of FOSMC based on single parameter adaptive law are identified by hybrid differential evolution (HDE) and adaptive differential evolution (ADE), which require considering the relationship between the scaling factor and the cross-probability factor. Finally, experiments have been conducted with the displacement signals mixed with multiple frequencies and multiple amplitudes and the results obtained from them show that the proposed control scheme can produce a faster response and smaller tracking errors in PEAs system as compared to traditional control algorithms.

1 | INTRODUCTION

Piezoelectric actuators (PEAs) have been widely used in the manufacturing industry owing to their ability to provide high positioning accuracy [1, 2]. Nevertheless, there are still some downsides of PEAs that cannot be ignored. The hysteresis of PEAs is the main reason that restricts their use in industrial applications. Therefore, minimizing its influence is an emergent problem that needs to be solved.

1.1 | Literature review

Based on the establishment of an accurate hysteresis model of PEAs, the problem of suppressing the hysteresis in them has been proposed. The hysteresis model of PEAs can be roughly divided into a complete physical model and a fuzzy physical model [3]. The complete physical model is based on the strength of the measured hysteresis curve in PEAs and this hysteresis curve is fitted to the existing phenomenological models such as the Preisach model [4, 5], the Bouc–Wen (B–W) model [6, 7], and the Prandtl–Ishlinskii model [8], in which the Preisach model and the Prandtl–Ishlinskii model are integral hysteresis models, and the Bouc–Wen model is differential hysteresis model. It can be concluded that the complete physical models have good model accuracy under some conditions. However, the above models have poor self-adaptability. Hence, there will be a modified model that can adapt to the input voltage change and still provide high accuracy, which is the primary idea of the fuzzy physical models. The support vector machine model [9] and the neural network model [10, 11] are typical fuzzy physical models, but they need enough data to ensure the accuracy of the PEAs hysteresis and this poses a challenge for the real-time performance of the system. In general, a hysteretic model can be expressed in different forms, which are the basis for the control algorithm.

Generally, proportional-integral-derivative (PID) [12] control, mode predictive control (MPC) [13], and sliding mode control (SMC) [14–18] methods are used to suppress the hysteresis of PEAs. A PID method was proved to be effective in nano-positioning platforms [12]. However, the gain margin of the nano-positioning platform is low when the composite...
signal is positioned, which makes it difficult for PID to achieve good performance. A multivariable feedforward control [19, 20] method has also been designed for the control of the piezoelectric actuators. However, in this case, the feedback loop between the final output and the initial input is not applied and thus this control method has poor robustness. Li et al. [21] gave a modified repetitive control (MRC) method to alleviate the disturbance of the nano-positioning platform. Nevertheless, the task of choosing the appropriate repetitive control law to describe the platform is the key aspect in the MRC method. Jian et al. [22] proposed an iterative learning control (ILC) strategy to restrict the hysteresis of PEAs. The ILC method is similar to the MRC method and a suitable self-learning method can improve its control accuracy. On the basis of the segment similarity model, Liu et al. [23] proposed an adaptive inverse control to restrain the hysteresis of PEAs. This method relies on the segment similarity model to restrict the hysteresis accuracy of PEAs. Compared with other complex control structures such as MRC and ILC, SMC is similar to PID in that it not only has a simple control structure, but also ensures good control effect by introducing sliding mode surface and switching control law. In some earlier studies [14, 15, 17], different adaptive methods in combination with integer order SMC control were proposed. Furthermore, the discrete-time SMC [16, 18] has been applied to the position tracking. Xu [13] presented a new SMC method that integrates both SMC and MPC to optimize the SMC, while the accuracy of MPC depends on step forward prediction of the voltage. Hence, how to choose the step voltage might be a challenge. All the studies mentioned above [13–18] restrain the chattering problem by designing an integer order sliding mode surface. Deng et al. [24] proposed that the classical integer-order differential operator is a local operator, while the fractional-order differential operator is a non-local operator. The so-called non-locality means that the next state of the system is not only dependent on the current state, but also related to its historical state. Fractional order sliding mode control (FOSMC) has a weighted memory effect due to the existence of fractional order integral terms. That is to adjust the memory effect through fractional order parameters to better represent the actual situation. FOSMC has been proved to be effective among other controls [25–27]. Whereas some other studies [25, 26] realized a parameter adjustment through a neural network, whether a satisfactory parameter can be obtained quickly and reasonably depends on the degree of training of neurons, which may affect the real-time performance of the system. The stability of the Lyapunov function can be further simplified and the chattering phenomenon greatly reduced [27]. Sun et al.[28] presented an adaptive control method for precise control of pneumatic artificial muscle with hysteresis characteristics. Meanwhile, it was shown that the chattering suppression of precision control was better than that of traditional SMC. This paper provides a reference for the application of adaptive control method and fractional sliding mode control method in PEAs precise positioning control.

1.2 Contributions

In this paper, first, the standard B–W model is used to describe the hysteresis characteristics of nano-positioning platform. Then, the design of FOSMC based on single parameter adaptive law is considered to compensate the errors caused by the model so as to improve the positioning accuracy of the platform. In addition, this paper will design a FOSMC with multi-parameter adaptive law to discuss the influence of adaptive law with different parameters on FOSMC. Finally, sinusoidal displacement signals with different frequencies and different amplitudes are designed to verify the effectiveness of FOSMC with single parameter adaptive law in chattering suppression compared with other SMC. The main contributions of this paper are summarized as follows:

(i) The standard B–W model is used to describe the hysteresis characteristics of PEAs, and the stability of the standard B–W model is discussed in detail from the parameters of the standard B–W model. In other words, the parameters of the standard B–W model meet the conditions to correctly describe the hysteresis characteristics of PEAs.

(ii) This paper realizes PEAs precise positioning control based on FOSMC and standard B–W model. Then the parameter adaptive law is proposed to optimize FOSMC, and in the parameter adaptive law, the multi-parameter adaptive law and the single-parameter adaptive law are given respectively to optimize FOSMC. The Lyapunov-based stability of FOSMC and the parameter adaptive law FOSMC are proved.

(iii) Starting from the output error, the boundary conditions of multi-parameter adaptive FOSMC and single-parameter FOSMC, single-parameter FOSMC and FOSMC, and FOSMC and SMC are discussed in detail. At the same time, this paper uses hybrid differential evolution and adaptive differential evolution to compare and verify, and the parameters of the controller are optimized in a relatively short time, so as to meet the performance requirements of the controller.

The rest of this paper is presented as follows. A specific mathematical model of the PEAs based on the improved B–W model is described in Section 2. In Section 3, the arranged FOSMC based on adaptive law with different parameters for the positioning and tracking control of PEAs is developed, and the proof of the closed-loop stability is also provided. Section 4 presents the experimental setup and verifies the effectiveness of the proposed scheme in comparison with other control methods. Conclusions from the present work are given in Section 5.

2 MATHEMATICAL MODE OF PEAS

The B–W hysteresis model is much simpler than other hysteresis models. As the B–W model is a differential hysteresis model,
compared with the integral hysteresis model, the description of the hysteresis term is simpler and it is easier to quickly solve the hysteresis term. The output displacement of PEAs nano-positioning system [29] is described by:

\[
\begin{align*}
    y(t) &= G(t) + H(t) \\
    G(t) &= k_0 u(t) \\
    H(t) &= k_1 b(t)
\end{align*}
\]  

(1)

where \(k_0\) and \(k_1\) are constants, \(y(t)\) and \(H(t)\) are functions of the input voltage \(u(t)\). The hysteresis term \(b(t)\) in the B–W model is given by [7]:

\[
b(t) = A_1 b(t) - B_1 |b(t)| b(t) |\dot{b}(t)|^{n-1} - C_1 |\dot{b}(t)|^{n} \]  

(2)

where \(u\) is the input voltage of the system, \(\dot{u}(t)\) is the derivative of \(u(t)\) and analogously, \(\dot{b}(t)\) is the derivative of \(b(t)\). \(A\), \(B\), \(C\) and \(n\) are parameters, deciding the loop of hysteresis in the system [7]. Li et al. [30] have discussed the stability of the B–W model in detail from the parameters of the B–W model, and the conditions were also analysed about the parameters of the B–W model. The hysteresis phenomenon can be effectively described by the above equation, and the hysteresis term of the above equation can be simplified when \(n=2\).

Regarding the inherent hysteresis of PEAs without ignoring its dynamic characteristics and combining B–W model, the PEAs nano-positioning system will be depicted in Figure 1, and it can be represented as follows [31]:

\[
n\ddot{y}(t) + \dot{y}(t) + k_s y(t) = k_s k_0 u(t) + k_s k_1 b(t) \]  

(3)

where \(m\) is the mass of the system, \(c\) is the viscous damping coefficient, \(k_s\) is the stiffness coefficient and \(\dot{y}(t)\) and \(\ddot{y}(t)\) mean velocity and acceleration, respectively.

There is external disturbance \(d\) in a real physical system and thus Equation (3) can be written as:

\[
n\ddot{y}(t) + \dot{y}(t) + k_s y(t) = k_s k_0 u(t) + k_s k_1 b(t) + d. \]  

(4)

where \(d\) satisfies |\(d\)| ≤ \(m\varepsilon\).

The above equation can be expressed by the state space equation as:

\[
\begin{align*}
    X'(t) &= A_1 X(t) + B_1 u(t) + D_1 b(t) + D_2 d \\
    Y(t) &= C_1 X(t)
\end{align*}
\]  

(5)

where

\[
X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \begin{aligned} (x_1(t) &= y(t), x_2(t) = \dot{y}(t)) \end{aligned}
\]

\[
B_1 = \begin{bmatrix} 0 \\ \frac{k_s k_0}{m} \end{bmatrix}
\]

\[
A_1 = \begin{bmatrix} 0 & 1 \\ -\frac{k_s}{m} & -\frac{c}{m} \end{bmatrix} \quad D_1 = \begin{bmatrix} 0 \\
\frac{k_s k_1}{m} \end{bmatrix}
\]

\[
C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}
\]

\[
D_2 = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}
\]

3. Design of Fractional Order Sliding Mode Controller Based on Parameter Adaptive Law

3.1 Discussion on the stability of FOSMC and FOSMC based on parameter adaptive law

In order to design a fractional order sliding mode controller based on single parameter adaptive law, a fractional order sliding surface based on single parameter adaptive law is drafted. Consequently, the state trajectories can accomplish the sliding surface and keep on it. Let us define the position error (\(e\)) is:

\[
e = y - y_d
\]

(7)

where \(y_d\) is the reference displacement.

The standard fractional order sliding surface is expressed as:

\[
s = k_3 \varepsilon + k_1 D_{\lambda}^\lambda \varepsilon + k_2 D_{\beta}^\beta \varepsilon
\]

(8)

where \(\lambda, \beta\) are the integral and differential operators of the fractional order sliding surface, respectively [32]. \(D\) is the fractional differential operator, and \(D_{\lambda}\) is the fractional differential operator from 0 to \(t\).

Using the model already established in Equation (3), the fractional sliding surface will be re-selected to simplify the controller design.

Selecting the fractional order sliding surface is:

\[
s = e + c_0 D_{-\lambda}^\lambda \varepsilon
\]

(9)

where \(c_0\) is a constant that directly determines the speed at which the sliding surface reaches zero. The ordinary sliding mode controller is obtained when \(\lambda = 1\). \(c_0\) and \(\lambda\) can be well designed in the sliding mode surface, and the error of tracking for an assigned displacement signal is smaller.
Taking a derivative of Equation (9), the following equation can be easily obtained,
\[ \dot{s} = \dot{\dot{s}} + \alpha_0 D^{1-\lambda} \varepsilon. \] (10)

Combining the above Equation (6) yields:
\[ \dot{s} = q_1 \left( A_1 X(t) + B_1 y(t) + D_1 b(t) + D_2 d \right) - \dot{y}_d + \alpha_0 D^{1-\lambda} \varepsilon \] (11)
where \( q_1 = [0 \ 1] \).

And setting \( \dot{s} = 0 \) to get:
\[ u_{eq} = (q_1^T B_1)^{-1} (\dot{y}_d - \alpha_0 D^{1-\lambda} \varepsilon) \]
\[ - q_1 A_1 X(t) - q_1 D_1 b(t) - q_1 D_2 d \]. (12)

Combined with Equation (12), fractional sliding mode control law can be chosen as:
\[ u_{fsmc} = (q_1^T B_1)^{-1} (\dot{y}_d - \alpha_0 D^{1-\lambda} \varepsilon) \]
\[ - q_1 A_1 X(t) - q_1 D_1 b(t) + u_{eq} \] (13)
where \( u_{pwm} = - (\varepsilon \text{sgn}(\dot{s}) + k_2 \hat{e}) \) means the switching control law for tolerating external disturbance. \( \varepsilon \) and \( k_2 \) are arbitrary positive constants and decide to approach the switching surface quickly and reduce chattering.

The fractional sliding surface must meet the following conditions:
\[ \lim_{t \to 0} \dot{s} = 0. \] (14)

Selecting the Lyapunov function as:
\[ V = \frac{1}{2} \dot{s}^2 \] (15)
\[ V' = \dot{s} \dot{s} \]
\[ = s(q_1 A_1 X(t) + B_1 u_{fsmc} + D_1 b(t) + D_2 d) - \dot{y}_d + \alpha_0 D^{1-\lambda} \varepsilon \]
\[ = s(q_1 D_2 d - (\varepsilon \text{sgn}(\dot{s}) + k_2 \hat{e})) \]
\[ = q_1 D_2 d - \varepsilon \dot{s} - k_2 \hat{e} \leq q_1 D_2 |d| - \varepsilon |\dot{s}| - k_2 \hat{e} \]
\[ = (q_1 D_2 |d| - \varepsilon |\dot{s}| - k_2 \hat{e}) \]
\[ = (q_1 D_2 |d| - \varepsilon |\dot{s}| - k_2 \hat{e}) \]
\[ = \left( \frac{|d|}{m} - \varepsilon \right) |\dot{s}| - k_2 \hat{e} \]
\[ (\varepsilon > 0, k_2 > 0) \] (16)

Obviously, when the external disturbance satisfies \( |d| \leq m \varepsilon \), we have \( V' < 0 \).

If the signals to be located are mixed with sinusoidal signals of different amplitudes and frequencies, then there will be greater jitter at the inflection point where the different sinusoidal signals are connected. It is a challenge for the selection of control parameters of FOSMC if the jitter of signal inflection point can be well suppressed. Therefore, it is necessary to design a fractional order sliding surface based on adaptive law with different parameters that can suppress the jitter at the signal inflection point. This paper will design a FOSMC with multi-parameter adaptive law and a fractional order sliding mode controller based on single parameter adaptive law, respectively, and the difference in output error between them will be also compared with. The fractional order sliding surface with multi-parameter adaptive law is defined as follows:
\[ s = \dot{s} + \hat{t}_0 D^{-\lambda} \varepsilon + \hat{\delta}_1 \hat{e}. \] (17)

Meanwhile, the multi-parameter adaptive laws are defined as:
\[ \dot{\hat{t}}_0 = k_0^T (\hat{e}) \]
\[ \dot{\hat{\delta}}_1 = - k_0 D^{-\lambda} \hat{e} (k_0 > 0) \] (18)
where \( \hat{t}_0 \) and \( \hat{\delta}_1 \) are adaptive parameters.

The fractional sliding surface with multi-parameter adaptive law also must meet the condition given by Equation (14).

**Theorem 1:** If Equation (19) given below is used to describe the control voltage of the PEAAs, the fractional sliding surface condition of Equation (14) is satisfied.
\[ u_{mas fmc} = (q_1 B_1)^{-1} (\dot{y}_d - \hat{t}_0 D^{1-\lambda} \varepsilon - \hat{\delta}_1 \hat{e} - q_1 A_1 X(t) - q_1 D_1 b(t) + u_{eq}) \]. (19)

**Proof:** Differentiating the fractional sliding surface equation yields:
\[ \dot{s} = \dot{s} + \hat{t}_0 D^{-\lambda} \varepsilon + \hat{\delta}_1 \hat{e} \] (20)
substituting relation Equation (5) into Equation (20) gives:
\[ \dot{s} = q_1 \left( A_1 X(t) + B_1 u_{mas fmc} + D_1 b(t) + D_2 d \right) \]
\[ - \dot{y}_d + \hat{t}_0 D^{-\lambda} \varepsilon + \hat{\delta}_1 \hat{e} \]
\[ (\varepsilon > 0, k_2 > 0) \] (21)
selecting the same Lyapunov function, same as before,
\[ V = \frac{1}{2} \dot{s}^2 \] (22)
and getting its derivative:

\[
\dot{V} = s \dot{\varepsilon} = s \left( Q_1 \left( A_1 \dot{x}(t) + B_1 u_{ad \text{ fosmc}} + D_1 b(t) + D_2 \dot{d} \right) \right.
\]

\[
- \varepsilon \ddot{\varepsilon} + \hat{c}_0 D^{-\lambda} \dot{\varepsilon} + \hat{\varepsilon}_0 D^{1-\lambda} \dot{\varepsilon} + \hat{\delta}_1 \dot{\varepsilon} + \hat{\delta}_2 \varepsilon
\]

\[
= s \left( \hat{c}_0 D^{-\lambda} \dot{\varepsilon} + \hat{\varepsilon}_0 D^{1-\lambda} \dot{\varepsilon} + \hat{\delta}_1 \dot{\varepsilon} + \hat{\delta}_2 \varepsilon \right)
\]

\[
= s \left( Q_1 D_2 d - (\varepsilon(\dot{\varepsilon}) + k_2 s) \right)
\]

\[
= (|d| - \varepsilon) |d| - k_2 s^2
\]

\[
= \left( \frac{|d|}{m} - \varepsilon \right) |d| - k_2 s^2 (\varepsilon > 0, k_2 > 0)
\]

(23)

According to the selected multi-parameter adaptive law \(\hat{c}_0 = k_0 \varepsilon, \hat{\varepsilon}_0 = -k_0 D^{-\lambda} \varepsilon\), under the condition that disturbance satisfies \(|d| \leq m \varepsilon, \dot{V} < 0\) is easily established.

Similarly, the fractional order sliding surface based on single parameter adaptive law is defined as follows:

\[
s = \dot{\varepsilon} + c_0 D^{-\lambda} \dot{\varepsilon} + \xi_1 \varepsilon + \xi_2.
\]

(24)

Meanwhile, the single-parameter adaptive law is defined as:

\[
\dot{\xi}_1 = \xi_0 \quad (\xi_0 > 0)
\]

\[
\dot{\xi}_2 = -\xi_0 \varepsilon
\]

(25)

where \(\xi_1\) and \(\xi_2\) are adaptive parameters.

The fractional sliding mode control law with \(\dot{\xi}_1 = \xi_0, \dot{\xi}_2 = -\xi_0 \varepsilon\) given below is used to describe the control voltage of the PEAs, and the fractional sliding surface condition of Equation (15) is also satisfied.

\[
u_{ad \text{ fosmc}} = (Q_1 B_1)^{-1} \left( \dot{\xi}_2 - \xi_0 D^{1-\lambda} \dot{\varepsilon} - \xi_1 \varepsilon \right.
\]

\[
- Q_1 A_1 \dot{x}(t) - Q_1 D_1 b(t) + u_{aw} \right).
\]

(26)

According to the selected single-parameter adaptive law \(\dot{\xi}_1 = \xi_0, \dot{\xi}_2 = -\xi_0 \varepsilon\), under the condition that disturbance satisfies \(|d| \leq m \varepsilon, \dot{V} < 0\) is also easily established. It can be proved in a similar way that SMC also satisfies the Lyapunov stability condition and hence the proof for SMC is omitted.

Thus, the proof is completed. The particular control methodology is illustrated in Figure 2.

3.2 Implementation details of FOSMC based on multi-parameter adaptive law and FOSMC based on single parameter adaptive law and the error analysis between them

The fractional order definition of Riemann–Liouville (R–L) is used to implement fractional sliding surface based on adaptive law with different parameters.

The fractional integral defined by R-L is expressed as [33]:

\[
\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau \quad (t > a, \alpha > 0)
\]

(27)

where \(\alpha\) is the fractional order, \(\Gamma(\alpha)\) is the Euler Gamma function and \(f(t)\) is an arbitrary function. Let \(f(t)\) be a function defined on interval \((a, b)\), Chen et al. [34] gives a method to find the initial condition of the R-L operator.

The \(n^{th}\) R-L fractional differential is given by [35]:

\[
\frac{d^n f(t)}{dt^n} = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau
\]

\[
\left. \frac{d^n f(t)}{dt^n} \right|_{t=0} = \frac{d^n f(0^+)}{dt^n} = \frac{1}{\Gamma(n-\alpha)} \int_a^b (\tau-0)^{n-\alpha-1} f(\tau) d\tau
\]

\[
\left. \frac{d^n f(t)}{dt^n} \right|_{t=a} = \frac{d^n f(a^-)}{dt^n} = \frac{1}{\Gamma(n-\alpha)} \int_a^b (\tau-a)^{n-\alpha-1} f(\tau) d\tau
\]

\[
\left. \frac{d^n f(t)}{dt^n} \right|_{t=b} = \frac{d^n f(b^-)}{dt^n} = \frac{1}{\Gamma(n-\alpha)} \int_a^b (\tau-b)^{n-\alpha-1} f(\tau) d\tau
\]

\[
\left. \frac{d^n f(t)}{dt^n} \right|_{t=b} = \frac{d^n f(b^-)}{dt^n} = \frac{1}{\Gamma(n-\alpha)} \int_a^b (\tau-b)^{n-\alpha-1} f(\tau) d\tau
\]

(28)

where the integral \(n\) verifies: \(n-1 < \alpha < n\).

Particle swarm optimization (PSO) [4, 5, 8] and differential evolution (DE) [34] have been proved to be effective in identifying the B–W model parameters. However, the PSO algorithm has a slower convergence rate than the DE algorithm in models that have a large number of parameters [36]. Under the same conditions, the calculation time of DE is shorter than that of PSO, but it is still too long. Therefore, the hybrid differential evolution (HDE) algorithm has been proposed in the present work. Meanwhile, in order to compare with HDE, adaptive differential evolution (ADE) will be introduced, the core of the ADE is the relationship between the scaling factor \(F\) and the cross-probability factor \(CR\), both of which are separately selected. It is expected that the adaptive fractional order sliding surface parameters are obtained and the calculation time can meet real-time requirements under smaller size and \(G\) conditions.

The HDE refers to the individuals in the dimension \(D\) that obtain an optimum value by the operations of mutation and crossover, whereas the value in the optimal individual dimension \(D\) is the optimal parameter of AFOSMC. The individual
variation and crossover operations are as follows:

$$\frac{dF}{dt} = F_0 \left( 1 - \frac{F}{F_{\text{min}}} \right) CR$$

$$F_{i=0} = F_{\text{max}}$$

$$\frac{dCR}{dt} = CR_0 \left( 1 - \frac{CR}{CR_{\text{max}}} \right) F$$

$$CR_{i=0} = CR_{\text{min}}$$

$$V_{i,G} = X_{ik,G} + F \times (X_{ij,G} - X_{ik,G}) \text{(29)}$$

$$U_j^{i,G} = \left\{ \begin{array}{ll}
    v_j^{i,G} \text{ (rand}(0, 1) < CR \text{ or } j = j_{\text{rand}}} & \\
    X_{i,G-1} & \text{otherwise}
\end{array} \right. \text{(30)}$$

where $F_0$ and $CR_0$ signify the initial decay rate and the initial growth rate, separately, $F, CR \in [0.5, 1]$. The fitting of FOSMC based on adaptive law with different parameters is expressed by the Root-Mean-Square error (RMSE):

$$RMSE = \sqrt{\frac{1}{N_{\text{FOSMC}}} \sum_{k=1}^{N_{\text{FOSMC}}} (\hat{y}_k - y_k)^2} \text{(31)}$$

where $y_k$ represents the $k_{\text{th}}$ true output value of the PEAs, $\hat{y}_k$ represents the $k_{\text{th}}$ predicted output value of the controller, and $N_{\text{FOSMC}}$ is the sum of the number of the controller outputs.

The steps of the HDE are as follows:

Step 1: Initially set up size, $G$, and individual $X$ upper and lower limits.

Step 2: Substitute the individual $X$ into Equation (29) and perform the mutation operation to obtain the variant individual $V$.

Step 3: Bring the variant individual $V$ and the optimal individual $X$ of the previous generation into Equation (30) for the cross operation to obtain the new individual $U$. Further, substitute $U$ into Equation (31) to get $RMSE_U$; and substitute $X$ into Equation (31) to obtain $RMSE_X$. The individuals with smaller $RMSE$ are retained after comparison the value of $RMSE_U$ with $RMSE_X$.

Step 4: Update the population and return to Step 2 until a satisfied error accuracy.

The stabilities of the FOSMC with multi-parameter adaptive law and FOSMC are theoretically proved by Equations (23) and (16), respectively. In terms of tracking error, the FOSMC and FOSMC with multi-parameter adaptive law will converge in a finite time.

Theorem 2: Define the FOSMC output error with multi-parameter adaptive law $e_2$ is $y_{\text{ref}}(\text{multi-fosmc}) - y_d$, where $e_2 \leq \xi_0$ and $\xi_0$ is a constant.

Proof:

$$e_2 = k_0(Q(t_1)B_i)^{-1}(\hat{j}_d - \hat{\xi}_0D^1\hat{\xi}_0 - \hat{\delta}_1\hat{e})$$

$$-Q(t_1)A_1X(t) - Q_1D_1b(t) + n_{qw} + k_1b(t) - y_d$$

$$= ((m(\hat{j}_d - \hat{\xi}_0D^1\hat{\xi}_0 - \varepsilon \text{sgn}(\hat{\delta}_1\hat{e} - k_2\varepsilon) + \hat{c})/k_1) + \varepsilon$$

(32)

Arbitrary complex signals can be viewed as consisting of a finite number of different frequencies and amplitudes of sinusoidal [37]. Hence, taking a sinusoidal signal in an arbitrary complex signal can be an example.

Let

$$\left\{ \begin{array}{ll}
e = \sum_{i=1}^{n} e_i = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} y_{di} & (\omega_i = \omega_{di}, a_i > 0, a_{di} > 0) \\
y_i = a_i \sin(\omega_{di}t + \varphi_{di}) + d_i \\
y_{di} = a_{di} \sin(\omega_{di}t + \varphi_{di}) + d_{di}
\end{array} \right. \text{(33)}$$

where $y_i, y_{di}$ are any sinusoidal component in the real output displacement and the sinusoidal component corresponding to the reference displacement, respectively.

$$\left\{ \begin{array}{ll}
\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} (a_i \sin(\omega_{di}t + \varphi_{di}) + d_i - a_{di} \sin(\omega_{di}t + \varphi_{di}) - d_{di})
\end{array} \right. \text{(34)}$$

the upper limit $\sum_{i=1}^{n} e_i$ of $\sum_{i=1}^{n} (a_i + d_i - a_{di} - d_{di})$. Similarly,

$$\left\{ \begin{array}{ll}
\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} (a_i \sin(\omega_{di} \cos(\omega_{di}t + \varphi_{di})) - a_{di} \sin(\omega_{di} \cos(\omega_{di}t + \varphi_{di}))
\end{array} \right. \text{(35)}$$

$$\sum_{i=1}^{n} e_i \leq \sum_{i=1}^{n} (\omega_{di} a_i - \omega_{di} a_{di}) \leq \sum_{i=1}^{n} (\omega_{di} a_i + \omega_{di} a_{di})$$

$\sum_{i=1}^{n} e_i$ also has the upper limit value. Thus,

$$\left\{ \begin{array}{ll}
D^{\lambda} e_i = \frac{1}{\Gamma(\lambda)} \int_{0}^{\Delta t} (\Delta t - \tau)^{\lambda-1} e_i(\tau) d\tau
\end{array} \right.$$

$$\frac{1}{\Gamma(\lambda)} \left( \frac{1}{X} (\Delta t)^{\lambda} \right) \sum_{i=1}^{n} (d_i - a_i - a_{di} - d_{di}) \leq \sum_{i=1}^{n} D^{\lambda} e_i$$

$$\sum_{i=1}^{n} D^{\lambda} e_i \leq \sum_{i=1}^{n} \left( \frac{1}{\Gamma(\lambda)} \left( \frac{1}{X} (\Delta t)^{\lambda} \right) (a_i + d_i - a_{di} - d_{di}) \right)$$

(36)
where $1/\Gamma(\lambda) \subset [0, 1]$ is shown in Figure 3. The same is available to get:

$$
D^{1-\lambda} e_i = \frac{1}{\Gamma(1-\lambda)} \int_0^\Delta (\Delta t - r)^{-\lambda} e_i(r) dr
$$

$$
\frac{(\Delta t)^{-\lambda}}{\Gamma(1-\lambda)} \sum_{i=1}^n (d_i - a_i - a_\delta - d_\delta) \leq \sum_{i=1}^n D^{1-\lambda} e_i
$$

$$
\sum_{i=1}^n D^{1-\lambda} e_i \leq \sum_{i=1}^n \frac{(\Delta t)^{-\lambda}}{\Gamma(1-\lambda)} (a_i + d_i - a_\delta - d_\delta)
$$

where $1/\Gamma(1-\lambda) \subset [0, 1]$ is also shown in Figure 3. The process of directly obtaining $1/\Gamma(\lambda)$ and $1/\Gamma(1-\lambda)$ are complex when $\lambda \in (0, 1)$, and we can consider the MATLAB to draw $1/\Gamma(\lambda)$ and $1/\Gamma(1-\lambda)$ curve to intuitively describe $1/\Gamma(\lambda)$ and $1/\Gamma(1-\lambda)$ value. In addition, Chen et al. [38] discusses the robust stability of fractional order (fractional order parameters between 0 and 2) uncertain linear systems, and gives theoretical and numerical simulation proof. Therefore, it’s easy to get $1/\Gamma(\lambda)$ and $1/\Gamma(1-\lambda)$ are bounded functions when $\lambda \in (0, 1)$. On the one hand, the values of $\epsilon$ and $k_2$ are very small, that to say,

$$
\sum_{i=1}^n e_{2i} = \left( m \sum_{i=1}^n \frac{\sum_{i=1}^n \tilde{y}_i - \tilde{y}_0 \sum_{i=1}^n D^{1-\lambda} e_i}{k_2} - \delta_i \sum_{i=1}^n e_i \right) / k_2
$$

$$
+ \sum_{i=1}^n e_i = \left( -\frac{m \sum_{i=1}^n \tilde{y}_i - \tilde{y}_0 \sum_{i=1}^n D^{1-\lambda} e_i}{k_2} - \frac{m \delta_i}{k_2} \sum_{i=1}^n e_i \right)
$$

$$
+ \sum_{i=1}^n e_i + \frac{m}{k_2} \sum_{i=1}^n \tilde{y}_i + \frac{\sum_{i=1}^n \tilde{y}_i}{k_2}
$$

Combining with the condition

$$
\frac{-m}{k_2} e_i = \frac{-m k_0}{k_2} \sum_{i=1}^n \left( \int e_i d\tau \right) \leq \frac{-m k_0}{k_2} \sum_{i=1}^n (d_i - a_i - a_\delta - d_\delta) \Delta t
$$

$$
\frac{-m}{k_2} e_i = \frac{-m k_0}{k_2} \sum_{i=1}^n \left( \int e_i d\tau \right) \leq \frac{-m k_0}{k_2} \sum_{i=1}^n (d_i - a_i - a_\delta - d_\delta) \Delta t
$$

$$
\frac{m}{k_2} \sum_{i=1}^n (d_i - a_i - a_\delta - d_\delta) \Delta t
$$

$$
\sum_{i=1}^n \frac{m}{k_2} \sum_{i=1}^n (d_i - a_i - a_\delta - d_\delta) \Delta t
$$

$$
\sum_{i=1}^n (d_i - a_i - a_\delta - d_\delta) \Delta t
$$

where $\sum_{i=1}^n \tilde{y}_i, \sum_{i=1}^n \tilde{y}_i, \sum_{i=1}^n \tilde{y}_i$ and $\sum_{i=1}^n \tilde{y}_i$ are bounded, so $\sum_{i=1}^n e_{2i}$ is bounded. On the other hand, $e_2$ consists of a finite number of $e_2$. Thus $e_2 = \sum_{i=1}^n e_2$ is also bounded, that is $e_2 \leq \xi_0$.

In addition, the proof for FOSMC and FOSMC based on single parameter adaptive law can also be done in a similar way.

Similarly, let $\sum_{i=1}^n \epsilon_i$ be output error of the FOSMC based on single parameter adaptive law and using Theorem 2 in the paper, the output error $\sum_{i=1}^n \epsilon_i$ can be expressed...
below:
\[
\sum_{i=1}^{n} c_{1i} = \left( m \left( \sum_{j=1}^{n} \hat{y}_{ij} - \alpha_{i} \right) \sum_{j=1}^{n} D^{1-\lambda} e_{ij} - \xi_{j} \sum_{j=1}^{n} \hat{e}_{ij} \right) + \epsilon \sum_{j=1}^{n} \hat{y}_{ij} / k_{i} + \sum_{j=1}^{n} \hat{e}_{ij}.
\]

(41)

It can be known that \( e_{1} = \sum_{j=1}^{n} c_{1j} \) is also bounded. The difference between the output errors of \( e_{2} \) and \( e_{1} \) will be discussed in Theorem 3.

Theorem 3: The output error of the FOSMC with multi-parameter adaptive law is not less than the output error of the FOSMC based on single parameter adaptive law, that is \( e_{2} \geq e_{1} \).

Proof:
\[
\sum_{j=1}^{n} \left( e_{2j} - e_{1j} \right) = \frac{m}{k_{i}} \left( (e_{0} - \hat{e}_{0}) \sum_{j=1}^{n} D^{1-\lambda} \xi_{j} + (\hat{\xi}_{j} - \delta_{j}) \sum_{j=1}^{n} \epsilon_{ij} \right)
\]
\[
= \frac{m}{k_{i}} \left( \left( e_{0} - \hat{e}_{0} \right) \sum_{j=1}^{n} \left( \int \epsilon \, dr \right)^{T} \left( \sum_{j=1}^{n} \frac{(\Delta_{j})^{-1}}{\Gamma(1-\lambda)} \left( a_{ij} + d_{i} - a_{ij} - d_{ij} \right) \right) \right)
\]
\[
+ \left( \int \zeta_{0} \, dr + k_{0} \sum_{j=1}^{n} \left( \int D^{1-\lambda} \epsilon \, dr \right) \sum_{j=1}^{n} \hat{e}_{ij} \right)
\]
\[
= \frac{m}{k_{i}} \left( \left( e_{0} - \hat{e}_{0} \right) \sum_{j=1}^{n} \left( \int \epsilon \, dr \right)^{T} \right)
\]
\[
\times \left( \sum_{j=1}^{n} \frac{(\Delta_{j})^{-1}}{\Gamma(1-\lambda)} \left( a_{ij} + d_{i} - a_{ij} - d_{ij} \right) \right)
\]
\[
+ \left( \int \zeta_{0} \, dr + k_{0} \sum_{j=1}^{n} \left( \int D^{1-\lambda} \epsilon \, dr \right) \sum_{j=1}^{n} \hat{e}_{ij} \right).
\]

(42)

If \( \int \zeta_{0} \, dr + k_{0} \sum_{j=1}^{n} \left( \int D^{1-\lambda} \epsilon \, dr \right) \sum_{j=1}^{n} \hat{e}_{ij} \leq \sum_{j=1}^{n} \sum_{i=1}^{n} \hat{e}_{ij} \leq \sum_{j=1}^{n} \sum_{i=1}^{n} \omega_{ji} - \omega_{ji} a_{ij} \),

then
\[
\sum_{j=1}^{n} \left( e_{2j} - e_{1j} \right) = \frac{m}{k_{i}} \left( (e_{0} - \hat{e}_{0}) \sum_{j=1}^{n} D^{1-\lambda} \xi_{j} + (\hat{\xi}_{j} - \delta_{j}) \sum_{j=1}^{n} \epsilon_{ij} \right)
\]
\[
= \frac{m}{k_{i}} \left( \left( e_{0} - \hat{e}_{0} \right) \sum_{j=1}^{n} \left( \int \epsilon \, dr \right)^{T} \left( \sum_{j=1}^{n} \frac{(\Delta_{j})^{-1}}{\Gamma(1-\lambda)} \left( a_{ij} + d_{i} - a_{ij} - d_{ij} \right) \right) \right)
\]
\[
+ \left( \int \zeta_{0} \, dr + k_{0} \sum_{j=1}^{n} \left( \int D^{1-\lambda} \epsilon \, dr \right) \sum_{j=1}^{n} \hat{e}_{ij} \right)
\]
\[
= \frac{m}{k_{i}} \left( \left( e_{0} - \hat{e}_{0} \right) \sum_{j=1}^{n} \left( \int \epsilon \, dr \right)^{T} \right)
\]
\[
\times \left( \sum_{j=1}^{n} \frac{(\Delta_{j})^{-1}}{\Gamma(1-\lambda)} \left( a_{ij} + d_{i} - a_{ij} - d_{ij} \right) \right)
\]
\[
+ \left( \int \zeta_{0} \, dr + k_{0} \sum_{j=1}^{n} \left( \int D^{1-\lambda} \epsilon \, dr \right) \sum_{j=1}^{n} \hat{e}_{ij} \right).
\]

(43)

Obviously if:
\[
\int \zeta_{0} \, dr + k_{0} \sum_{j=1}^{n} \left( \int D^{1-\lambda} \epsilon \, dr \right) \sum_{j=1}^{n} \hat{e}_{ij} \leq 0.
\]

then \( \sum_{j=1}^{n} \left( e_{2j} - e_{1j} \right) \geq 0. \)

Sorting Equation (44) shows that if condition
\[
\begin{cases}
\int \zeta_{0} \, dr + k_{0} \sum_{j=1}^{n} \left( \int D^{1-\lambda} \epsilon \, dr \right) \sum_{j=1}^{n} \hat{e}_{ij} \leq 0 \text{ is satisfied, then } \\
\sum_{j=1}^{n} \sum_{i=1}^{n} \left( \int \epsilon \, dr \right)^{T} \left( \sum_{j=1}^{n} \frac{(\Delta_{j})^{-1}}{\Gamma(1-\lambda)} \left( a_{ij} + d_{i} - a_{ij} - d_{ij} \right) \right) \leq 0
\end{cases}
\]

(44)

is satisfied. That is \( \sum_{j=1}^{n} \left( e_{2j} - e_{1j} \right) \geq 0. \)

In addition, if \( \int \zeta_{0} \, dr + k_{0} \sum_{j=1}^{n} \left( \int D^{1-\lambda} \epsilon \, dr \right) \sum_{j=1}^{n} \hat{e}_{ij} \geq 0 \).

\[
\sum_{j=1}^{n} \sum_{i=1}^{n} \left( \int \epsilon \, dr \right)^{T} \left( \sum_{j=1}^{n} \frac{(\Delta_{j})^{-1}}{\Gamma(1-\lambda)} \left( a_{ij} + d_{i} - a_{ij} - d_{ij} \right) \right) \leq 0
\]

(45)
Obviously if,
\[
\frac{m}{k_i} \left( e_0 - k_i \sum_{j=1}^m \left( \iint \epsilon_j \, dt \right)^T \right) > 0
\]
\[
\sum_{i=1}^n \left( (\Delta t)^{-\lambda} \right) \frac{\Gamma(1-\lambda)}{\Gamma(1-\lambda)} (a_i + d_i - a_{di} - d_{di}) \leq 0
\]
\[
\iint \epsilon_i \, dt + k_i \sum_{j=1}^m \left( \iint D^{\lambda} \epsilon_j \, dt \right)^T > 0
\]
\[
\sum_{i=1}^n ( \omega_i a_i - \omega_i d_i a_{di} ) \leq 0
\]
is satisfied. However, those conditions are contradictory. It can be seen that \( \sum_{j=1}^m (c_j - e_j) \leq 0 \) cannot be established, so \( \sum_{j=1}^m (c_j - e_j) \geq 0 \). That is \( e_i - c_i \geq 0 \).

### 4 | EXPERIMENTAL DETAILS AND RESULTS

On the one hand, the effectiveness of the B-W model to describe the hysteresis characteristics of PEAs will be verified. On the other hand, there will be a large disturbance at the signal inflection point of the SMC and the inherent characteristic of SMC is the chattering phenomenon. In addition, SMC is also a back-stepping controller, which has differential expansion problems. The presence of a derivative term in the SMC will amplify the error of the inflection point, which will cause a large disturbance at the signal inflection point. In this paper, the reference displacement \( y_{ref} \) mixed with different frequencies and different amplitudes sinusoidal signal will be used to test the performance of the FOSMC based on single parameter adaptive law, FOSMC and SMC at \( y_{ref} \) inflection point, and the boundary conditions of the \( y_{ref} \) will be given later. Moreover, many literatures have proved that the piezoelectric driven platform has rate-dependent hysteresis characteristics, and it is difficult for the controller to suppress hysteresis characteristics of the platform. Therefore, the difference between the FOSMC based on single parameter adaptive law proposed in this paper and the conventional PID for the positioning accuracy of the output displacement is tested by the sinusoidal displacement signal of 120 Hz. The FOSMC based on single parameter adaptive law is composed of the adaptive differential evolution (ADE).

#### 4.1 | Experiments setup

A sample rate of 20 kHz was used for all experiments. The dynamics model parameters of the PEAs nano-positioning system can be derived as: \( m = 1.45 kg \), \( c = 11 Ns/m \), and \( k_i = 9.998 \times 10^5 N/m \).

These experiments are conducted on the PEAs nano-positioning system, which is composed of a PEAs product (P-756.3CD, Physik Instrument) and digital controller E-725. A built-in capacitive displacement sensor is provided for measurement. The frequency bandwidth of the capacitive displacement sensor is 5.6 kHz, and its sampling accuracy is controlled by 18-bit A/D. The digital controller consists of a floating-point DSP, a voltage amplifier which is operating voltage range from −30 to 135 V, a signal conditioner for capacitive sensors. Then the identification and control algorithms are implemented in MATLAB, data transmission between host computer and E-725 is completed through dSPACE to verify the real-time performance.
4.2 Identify the B–W model by HDE and ADE algorithms

In order to accurately describe the rate-development of the B–W model, the standard B–W model is considered to be modified. Meanwhile, in order to facilitate the solution of the hysteresis term $H_1$, $n$ is set as 2 just to make $H_1$ easy to calculate. Then an improved B–W model is proposed as follows:

\[
\begin{align*}
    y(t) &= G(t) + H_1(t) + d_1 \\
    u_1 &= u(t + \varphi) \\
    b_1 &= A\dot{u}_1 - B|\dot{h}_1| b_1|\dot{b}_1| - C\dot{h}_1|\dot{b}_1|^2 \\
    G(t) &= k_0 u(t) \\
    H_1(t) &= k_1 b_1(t) 
\end{align*}
\]

(49)

where the phase compensation factor $\varphi$ compensates the phase of the input voltage to optimize the input-output asymmetry relationship. In addition, the initial value compensation factor $d_1$ is closely related to the initial state of the real system.

The hysteresis characteristic curves of input voltage and output displacement are shown in Figures 5 and 6. The hysteresis model parameters are shown in Tables 1 and 2.

It can be concluded that the parameters of the improved B–W model meet the hysteresis characteristics of PEAs. In addition, ADE and HDE algorithms were used to compare and verify the rate characteristics of improved B–W.

4.3 FOSMC based on single parameter adaptive law with HDE and ADE algorithms

The optimization algorithm is designed and ADE and HDE are inserted into it as sub-modules. In this way, the performance between ADE and HDE can be distinguished well. The initial settings of ADE and HDE are kept the same, $size = 20$ and $G = 20$ can guarantee the real-time performance and reflect the difference between them. The displacement tracking experiments are conducted with $y_d$.

In order to further analyse the influence of HDE and ADE on the control accuracy of FOSMC based on single parameter adaptive law, the experimental results of the tracking composite displacement signal, the error of tracking between HDE and ADE and the adaptability of ADE and HDE to the $y_d$ are plotted in Figure 7 for a comprehensive comparison. In the figures, Figure 7(a) corresponds to the plots of experimental results of the tracking composite displacement signal diagram, Figure 7(b) is the plots for the error of tracking in both algorithms and

| $A$ | $B$ | $C$ | $\varphi$ | $d_1$ | $k_0$ | $k_1$ | RMSE (\(\mu m\)) |
|-----|-----|-----|-----------|-------|-------|-------|---------------------|
| HDE | 0.0653 | 3.4017 | 5.1231 | 0.0515 | 1.1903 | 0.9241 | 2.5125 | 0.0136 |
| ADE | 0.0238 | 4.3046 | 0 | 0 | 0.4647 | 0.2009 | 7.4047 | 0.2006 |

TABLE 2 Identification parameters of improved B–W model with 8 Hz
Figure 7 corresponds to the plots showing their adaptability respectively.

Figure 7(a) shows that both of ADE and HDE algorithms can track the composite displacement signal well. In addition, Figure 7(b) shows that FOSMC based on single parameter adaptive law has a good positioning effect at inflection points (such as tag2), but the positioning error at tag1 is relatively large. On one hand, because the reference displacement $y_d$ is mixed with a sinusoidal signal (100–120 Hz), the error at the inflection point becomes larger. On the other hand, FOSMC based on single parameter adaptive law is also a back-stepping controller, which will also further amplify the error at the inflection point. Therefore, the positioning experiment of 120 Hz sine signal with FOSMC based on single parameter adaptive law will be supplemented for analysis. Figure 7(c) shows that single-parameter identification FOSMC based on HDE is not only faster than single-parameter identification FOSMC based on ADE in calculation speed ($G = 5$), but also more accurate than single-parameter identification FOSMC based on ADE in calculation ($G = 20$).

Define MAXE as:

$$\text{MAXE} = \epsilon_{\text{max}} - \epsilon_{\text{min}} \quad (50)$$

It can be known that MAXE is closely related to the maximum error at the inflection point. Therefore, the maximum error at the inflection point is larger and the MAXE will also be larger. In order to facilitate the comparison of the results, the experimental parameters, MAXE values and the RMSE values obtained by using HDE and ADE algorithms are given in Table 3.

Among the data in Table 3, comparing single-parameter identification FOSMC based on HDE with single-parameter identification FOSMC identified by ADE, the former not only has a smaller maximum error at the inflection point, but also has a smaller overall error.

### 4.4 The comparison of FOSMC based on single parameter adaptive law and FOSMC

In order to discuss the effect of the single parameter adaptive law on the controller in this paper, the reference displacement $y_d$ mixed with triangular signals and sinusoidal signals is used for positioning verification by FOSMC. The influence of single parameter adaptive law on the controller is amplified by keeping the single-parameter identification FOSMC other coefficients and FOSMC coefficients the same, and then analyse from the output error. Hence, we can get the control law of fractional order sliding mode control as follows:

$$u_{\text{fosmc}} = \left(Q_1B_1\right)^{-1} \times \left(\hat{y}_d - \epsilon_{\text{fosmc}}D^{1-\lambda}e - Q_1A_1X(t) - Q_1D_1b(t) + u_w\right). \quad (51)$$

Define FOSMC output error $\sum_{i=1}^{n} \epsilon_{\text{fosmc}} = \epsilon_{\text{fosmc}}$ as follows:

$$\sum_{i=1}^{n} \epsilon_{\text{fosmc}} = \left(\left(\sum_{i=1}^{n} \hat{y}_d - \epsilon_{\text{fosmc}} \sum_{i=1}^{n} D^{1-\lambda}e_i + e \sum_{i=1}^{n} \gamma_i / k\right) + \sum_{i=1}^{n} \epsilon_i\right) \quad (52)$$

then,

$$\sum_{i=1}^{n} \left(\epsilon_{\text{fosmc}} - \epsilon_{1i}\right) = \frac{m}{k_e} \left(\epsilon_{a_{\text{fosmc}}} - \epsilon_{\text{fosmc}} \sum_{i=1}^{n} D^{1-\lambda}e_i + \sum_{i=1}^{n} \epsilon_i\right) \quad (53)$$

### Table 3 Comparison of ADE and HDE

| Algorithm | $\lambda$ | $e_0$ | $\xi_0$ | RMSE ($\mu m$) | MAXE ($\mu m$) |
|-----------|----------|-------|---------|----------------|----------------|
| ADE       | 0.1813   | 0.3046| 0.9996  | 0.0062         | 0.1489         |
| HDE       | 0.5364   | 0.2875| 0.3230  | 0.0060         | 0.1439         |
TABLE 4 The experimental parameters of FOSMC and the MAXE and RMSE of FOSMC.

| Algorithm | $\lambda$ | $\epsilon_0$ | RMSE ($\mu$m) | MAXE ($\mu$m) |
|-----------|-----------|-------------|---------------|--------------|
| FOSMC     | 0.5364    | 0.2532      | 0.0204        | 0.2147       |

The comparison of FOSMC and SMC

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Then there is

$$\sum_{i=1}^{n} (d_i - a_i - a_{di} - d_{di}) > 0$$

If Equation (53) satisfies the condition:

$$\begin{cases} 
\epsilon_{faosmc} - \epsilon_{fosc} > 0 \\
\epsilon_{faosmc} - \epsilon_{fosc} \geq (\omega_{faosmc})^{(1 - 1/\alpha)} \sum_{i=1}^{n} (d_i - a_i - a_{di} - d_{di}) 
\end{cases}$$

then there is $\sum_{i=1}^{n} (\epsilon_{fosc} - \epsilon_{i}) \geq 0$. In addition, if Equation (53) satisfies the condition:

$$\begin{cases} 
\epsilon_{faosmc} - \epsilon_{fosc} > 0 \\
\sum_{i=1}^{n} (d_i - a_i - a_{di} - d_{di}) < 0 
\end{cases}$$

then there is $\sum_{i=1}^{n} (\epsilon_{fosc} - \epsilon_{i}) \leq 0$. Since $\sum_{i=1}^{n} (d_i - a_i - a_{di} - d_{di}) > 0$ is the basis of subsequent analysis, so it is advisable to assume that $\gamma$ meets this condition.

The experimental parameters, MAXE values and RMSE values obtained for FOSMC are given in Table 4.

It can be seen from Table 4 that the control parameters of FOSMC, combined with the control parameters of FOSMC based on single parameter adaptive law and connected with above mentioned theories, we have $\epsilon_{fosc} - \epsilon_{i} \geq 0$. In addition, in order to explain more intuitively that the single parameter adaptive law can improve the positioning accuracy of FOSMC, the experimental results of the tracking composite displacement signal, the error of tracking between single-parameter identification FOSMC and FOSMC are plotted in Figure 8. In the figures, Figure 8(a) corresponds to the plots of experimental results of the tracking composite displacement signal diagram and Figure 8(b) is the plots for the error of tracking in FOSMC based on single parameter adaptive law and FOSMC.

Figure 8(b) shows that the single parameter adaptive law designed in this paper can not only reduce the maximum error of FOSMC at $\gamma$ inflection point, but also improve the accuracy of $\gamma$ positioning by FOSMC.

4.5 The comparison of FOSMC and SMC

For discussing the effect of the fractional parameter $\lambda$ on the controller in this paper, the reference displacement $\gamma$ mixed with triangular signals and sinusoidal signals is used for positioning verification by SMC. The influence of fractional order parameters on the controller is amplified by keeping the SMC coefficient and FOSMC coefficient the same, and then analysed from the output error. Hence, we can get the control law of sliding mode control as follows:

$$u_{soc} = (Q_1 B_1)^{-1} \left( y_{di} - \epsilon_{soc} - Q_1 A_1 x(t) - Q_1 D_1 b(t) + u_{soc} \right).$$

Define SMC output error $\sum_{i=1}^{n} \epsilon_{soc} = \epsilon_{soc}$ as:

$$\sum_{i=1}^{n} \epsilon_{soc} = \left( m \sum_{i=1}^{n} \left( \gamma_{di} - \epsilon_{soc} \sum_{j=1}^{n} \epsilon_{j} \right) + c \sum_{j=1}^{n} \gamma_{j} \right) / k_{s} + \sum_{i=1}^{n} \epsilon_{i}$$

then,

$$\sum_{i=1}^{n} (\epsilon_{fosc} - \epsilon_{soc}) = \frac{m}{k_{s}} \left( \epsilon_{soc} \sum_{i=1}^{n} \epsilon_{i} - \epsilon_{fosc} \sum_{i=1}^{n} D^{-\frac{1}{\alpha}} \epsilon_{i} \right)$$

According to the above, if Equation (57) satisfies the condition:

$$\epsilon_{soc} \sum_{i=1}^{n} (\omega_{fosc} + \omega_{soc} d_{soc}) = \frac{1}{\alpha} \sum_{i=1}^{n} (d_i - a_i - a_{di} - d_{di})$$

then there is $\sum_{i=1}^{n} (\epsilon_{fosc} - \epsilon_{soc}) \leq 0$. In addition, if Equation (57) satisfies the condition:

$$\begin{cases} 
\sum_{i=1}^{n} (d_i - a_i - a_{di} - d_{di}) < 0 \\
\epsilon_{soc} (\sum_{i=1}^{n} (\omega_{fosc} d_{soc} \Gamma(1 - 1/\alpha) \sum_{i=1}^{n} (d_i - a_i - a_{di} - d_{di}) = \sum_{i=1}^{n} (\omega_{soc} a_i + \omega_{soc} d_{soc}) 
\end{cases}$$

then there is $\sum_{i=1}^{n} (\epsilon_{fosc} - \epsilon_{soc}) \leq 0$. Because the experiment uses the same $\gamma_{di}$ that is, $\sum_{i=1}^{n} (d_i - a_i - a_{di} - d_{di}) > 0$. The experimental parameters, MAXE values and RMSE values obtained for SMC are given in Table 5.

It can be known from Table 5 that the control parameters of SMC, connected with above mentioned theories, there is $\sum_{i=1}^{n} (\epsilon_{fosc} - \epsilon_{soc}) \leq 0$. The fractional order parameter $\lambda$
### TABLE 5  The experimental parameters of SMC and the MAXE and RMSE of SMC

| Algorithm | $c_0$ | RMSE ($\mu m$) | MAXE ($\mu m$) |
|-----------|-------|----------------|----------------|
| SMC       | 0.2532| 0.1043         | 0.5840         |

**FIGURE 9**  The displacement tracking results of SMC

designed in this paper can improve the positioning accuracy of SMC. In addition, in order to demonstrate more intuitively that fractional order parameters can improve the positioning accuracy of SMC, the experimental results of the tracking composite displacement signal, the error of tracking between FOSMC and SMC are plotted in Figure 9. In the figures, Figure 9(a) corresponds to the plots of experimental results of the tracking composite displacement signal diagram and Figure 9(b) is the plots for the error of tracking in FOSMC and SMC.

Figure 9 illustrates that fractional order parameters designed in this paper can improve the accuracy of $y_d$ positioning by SMC.

### 4.6  The comparison of FOSMC based on single parameter adaptive law and PID

In this part, a standard displacement of a 120 Hz sinusoidal signal is used to verify the positioning differences between single-parameter identification FOSMC and PID. Define the output displacement of the PID as

$$
\begin{align*}
    y_{pid} &= k_0 \left( u_{pid} - \frac{k_1}{k_0} b(t) \right) + k_1 b(t) + d_0 \\
    u_{pid} &= k_p \dot{e} + k_i \int e dt + k_d e
\end{align*}
$$

(60)

where $k_p$, $k_i$, and $k_d$ are controller parameters, and $d_0$ is correction coefficient.

According to the observation, it is difficult to analyse the relationship between single-parameter identification FOSMC and PID from the output error, so consider analysing the output error of single-parameter identification FOSMC and PID under appropriate parameters. Hybrid differential evolution algorithm and adaptive differential evolution algorithm are used to identify the controller parameters ($size = 20$, $G = 20$). The control parameters of single-parameter identification FOSMC and PID, the RMSE and MAXE of single-parameter identification FOSMC and PID are sorted out, in which the correction coefficient of 120 Hz sine signal is 5.0003. As shown in Tables 6 and 7, respectively.

Tables 6 and 7 indicate that the positioning accuracy of single-parameter identification FOSMC is higher than PID under their respective appropriate control parameters, besides, it can be seen from Table 7 that $k_d = 0$, indicating that PID controller has only $k_p$ and $k_i$ as effective control parameters. It is also reflected from the side that PID controller takes a long time to locate sinusoidal displacement with relatively high frequency, and its dynamic performance is relatively poor. In addition, in order to more intuitively explain the difference in positioning accuracy between single-parameter identification FOSMC and PID, the experimental results of the tracking composite displacement signal, the error of tracking between single-parameter identification FOSMC and PID are plotted in Figure 10. In the figures, Figure 10(a) corresponds to the plots of experimental results of the tracking composite displacement signal diagram, and Figure 10(b) is the plots for the error of tracking in single-parameter identification FOSMC and PID.

From Figure 10, it can be seen that the single-parameter identification FOSMC is effective compared with PID in the positioning accuracy of 120 Hz sine signal.

**FIGURE 10**  The displacement tracking results of FOSMC based on single parameter adaptive law and PID at 120 Hz
5 | CONCLUSIONS

In this paper, a FOSMC based on single parameter adaptive law is proposed for precise positioning of platform driven PEAs. Different from the traditional SMC applied in the piezoelectric driving platform, this paper proposes a simple adaptive law multi-parameter identification adaptive method, upgrades FOSMC to multi-parameter identification FOSMC, and proves the Lyapunov stability and error convergence of multi-parameter identification FOSMC. Then, considering the optimization of multi-parameter identification FOSMC, a single-parameter identification FOSMC and the single-parameter identification FOSMC is given, and a judgment condition is provided to distinguish the control performance of the multi-parameter identification FOSMC and the single-parameter identification FOSMC. Finally, the multi-frequency and multi-amplitude complex displacement signal $y_j$ is used for experiments to verify the difference between the comprehensive performance of AFOSMC proposed in this paper and SMC and FOSMC, and the difference determination conditions are given. Meanwhile, compared with the FOSMC analysis of the positioning performance of a single frequency signal using a PID controller, it can be seen that the positioning accuracy of FOSMC is higher than PID.

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APPENDICES

A1 The standard Bouc–Wen hysteresis model is used to describe the convergence analysis of dynamic equations

Proof of convergence of the new dynamic equation

The stability of
\[ \dot{X}(t) = A_1 X(t) + B_1 u(t) + D_1 b(t) + D_2 d \]
will be discussed through the B–W model. By treating \( b(t) \) and \( d \) as nonlinear term \( H \),
\[ \dot{X}(t) = A_1 X(t) + B_1 u(t) + D_1 b(t) + D_2 d \]
can be rewritten as new dynamic equation:
\[ \dot{X}(t) = A_1 X(t) + B_1 u(t) + H \\
Y(t) = C_1 X(t) \] (A1)

where,
\[ H = D_1 b(t) + D_2 d \]
\[ X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \]
\[ x_1(t) = y(t), x_2(t) = y(t) \]
\[ A_1 = \begin{bmatrix} 0 & 0 \\ k_0 & -1 \end{bmatrix} \]
\[ B_1 = \begin{bmatrix} 0 \\ k_0 \end{bmatrix} \]
\[ C_1 = [1 \ 0] \]

Under the sampling time \( \Delta t \), and system (A1) is described as:
\[ X_{k+1} = A_2 X_k + B_2 u_k + H_k \]
\[ Y_k = C_1 X_k \] (A3)

\[ A_2 = e^{A \Delta t} \]
where
\[ B_2 = B_1 \int_0^{\Delta t} e^{A \tau} d\tau \] (A4)
\[ d_k = D_1 \int_0^{\Delta t} e^{A \tau} d((k+1)\Delta t - \tau) d\tau \]

\( H_k \) is a discretization of \( H \), \( A_2 \) and \( B_2 \) are constant matrices. Then define \( \hat{H}_k = H_k - H_{k-1} \) and integrate Equation (A3) to get Equation (A5):
\[ \hat{H}_k = A_2 (X_{k-1} - X_k) + B_2 (u_{k-1} - u_k) + X_{k+1} - X_k. \] (A5)

In the dynamic model described by the B–W model, the \( u \) and the \( y \) are bounded, so \( \hat{H}_k \) is bounded and this indicates that the dynamic model described by the B–W always converges even in the presence of disturbance.

A2 Discussion of initial conditions for Riemann–Liouville operators

Let \( u : N_{a_0} \rightarrow R \), and \( \lambda > 0 \) be given. Then the fractional sum of \( \lambda \) order is defined by [34]:
\[ \Delta_{a}^{\lambda} u(t) = \frac{1}{\Gamma(\lambda)} \sum_{i=a_0}^{t-\sigma} (t - \sigma(i))^{\lambda-1} u(i), t \in N_{a_0} + \lambda \] (A6)

where \( a_0 \) is the starting point, \( \sigma(i) = s + 1, N_{a_0} = \{a_0, a_0 + 1, a_0 + 2, \ldots\} \) and \( f^{(\lambda)} \) is the falling factorial function defined as [34]:
\[ f^{(\lambda)} = \frac{\Gamma(t + 1)}{\Gamma(t + 1 - \lambda)} \] (A7)

where \( \Gamma(\cdot) \) is Gamma function, and \( \Gamma(i) = \int_0^{\infty} t^{-i-1} e^{-t} dt \).

For \( \lambda > 0, \lambda \notin N \) and \( u(t) \) defined on \( N_{a_0} \), the Caputo-like delta difference is defined by
\[ C^{\lambda} \Delta_{a_0}^{\lambda} u(t) = \Delta_{a_0}^{\lambda - (m-\lambda)} \Delta_{a_0}^{m} u(t) \]
\[ = \frac{1}{\Gamma(m-\lambda)} \sum_{s=a_0}^{t-\sigma} (t - \sigma(s))^{-(m-\lambda)} \Delta_{a_0}^{m} u(s) \] (A8)

where \( t \in N_{a_0} + m-\lambda, m = [\lambda] + 1, \Delta_{a_0}^{m}, \Delta_{a_0}^{m} \) means the integer order difference with starting point 0 and \( a_0 \), respectively.

So, for the delta fractional difference equation:
\[ C^{\lambda} \Delta_{a_0}^{\lambda} u(t) = f(t + \lambda - 1, u(t + \lambda - 1)) \]
\[ \Delta_{a_0}^{k} u(a_0) = u_k, m = [\lambda] + 1, k = 0, \ldots, m - 1 \] (A9)
The equivalent discrete integral equation can be obtained as:

\[ u(t) = u_0(t) + \frac{1}{\Gamma(\lambda)} \sum_{s=a_0+m-\lambda}^{t-\lambda} (t - \sigma(s))^{(\lambda-1)} \times f(s + \lambda - 1, u(s + \lambda - 1)) \]  

(A10)

where the initial iteration \( u_0(t) \):

\[ u_0(t) = \sum_{k=0}^{m-1} \frac{(t - a_0)^k}{k!} \Delta^k u(a), \quad t \in N_{a_0+m} \]  

(A11)