Radion Stabilization in Compact Hyperbolic Extra Dimensions

Salah Nasri\(^1\), Pedro J. Silva\(^1\), Glenn D. Starkman\(^2\) and Mark Trodden\(^1\)

\(^1\) Physics Department, Syracuse University, Syracuse, New York 13244-1130.

\(^2\) Department of Physics, Case Western Reserve University, Cleveland, OH 44106-7079.

Abstract

We consider radion stabilization in hyperbolic brane-world scenarios. We demonstrate that in the context of Einstein gravity, matter fields which stabilize the extra dimensions must violate the null energy condition. This result is shown to hold even allowing for FRW-like expansion on the brane. In particular, we explicitly demonstrate how one putative source of stabilizing matter fails to work, and how others violate the above condition. We speculate on a number of ways in which we may bypass this result, including the effect of Casimir energy in these spaces. A brief discussion of supersymmetry in these backgrounds is also given.
I. INTRODUCTION

Unification physics has traditionally been seen as the problem of reconciling wildly disparate mass scales, for example the weak scale ($\sim 10^2$ GeV) and the Planck scale ($\sim 10^{19}$ GeV). This exponential hierarchy is technically unnatural in particle physics, since in general, the effects of renormalization are to make the observable values of such scales much closer in size. Well-known attempts to address this issue, such as supersymmetry (SUSY) in which delicate cancellations between renormalization terms occur, or technicolor, in which the renormalization effects are much less dramatic than one might ordinarily expect, function by preventing dramatic corrections to an externally imposed mass hierarchy.

A fresh perspective on the problem of unification has received much attention in recent years [1, 2, 3, 4]. In this picture the hierarchy problem is is no longer a disparity between mass scales, and instead becomes an issue of length scales. The new approach is a superstring-inspired modification of the Kaluza-Klein idea that the universe may have more spatial dimensions than the three that we observe. The general hypothesis is that the universe as a whole is $3 + 1 + d$ dimensional, with gravity propagating in all dimensions, but the standard model fields are confined to a $3 + 1$ dimensional submanifold that comprises our observable universe. The primary motivation for this comes from Polchinski’s discovery [5] of D-branes in string theory. These extended objects have the property that open strings, the excitations of which correspond to standard model particles, may end on them, and thus are confined to the brane. However, closed string excitations, corresponding to gravitational degrees of freedom are free to occur anywhere in the space.

As in traditional Kaluza-Klein theories, it is necessary that all dimensions other than those we observe be compactified, so that their existence does not conflict with experimental data. The difference in the new scenarios is that, since standard model fields do not propagate in the extra dimensions, it is only necessary to evade constraints on higher-dimensional gravity, and not, for example, on higher-dimensional electromagnetism. As we shall see, this is important, since electromagnetism is tested to great precision down to extremely small scales, whereas microscopic tests of gravity are far less precise.

Since constraints on the new scenarios are less stringent than those on ordinary Kaluza-Klein theories, the corresponding extra dimensions can be significantly larger, which translates into a much larger allowed volume for the extra dimensions. It is the spreading of
gravitational flux into this large volume that allows gravity measured on our 3-brane to be so weak (parameterized by the Planck mass, $M_P$), while the fundamental scale of physics $M_{d+4}$ is parameterized by the weak scale, $M_W$ say. Thus, the problem of understanding the hierarchy between the Planck and weak scales now becomes that of understanding why extra dimensions are stabilized at a volume large in units of the fundamental length scale $M_{d+4}^{-1}$. This is the rephrasing of the hierarchy problem in these models. It constitutes a fundamental shift in thinking. Traditionally, these large compact extra dimensions have been conceived of as $d$-tori, or $d$-spheres. In this setting, one has the added bonus of requiring a linear tuning of length scales, compared to the usual exponential tuning of mass scales. Nevertheless, a significant tuning is still required, although now in an entirely different sector of the theory.

In recent work [6, 7] two of us proposed a modification to the above picture, in which we argued that there exist attractive alternate choices of compactification. These compactifications employ a topologically non-trivial internal space – a $d$-dimensional compact hyperbolic manifold (CHM). They also throw into a new light the problem of explaining the large hierarchy $M_P/\text{TeV}$, since even though the volume of these manifolds is large, their linear size $L$ is only slightly larger than the new fundamental length scale ($L \sim 30M_{d+4}^{-1}$ for example), thus only requiring numbers of $\mathcal{O}(10)$. Further, cosmology in such spaces has interesting consequences for the evolution of the early universe [8, 9]. In the next section we provide a brief review of the relevant properties of CHMs.

The main purpose of this paper is to present a detailed analysis of radion stabilization in these models. It has recently been demonstrated [10, 11] that, in the context of general relativity in $4 + d$ dimensions, stabilization of large hyperbolic extra dimensions, leaving Minkowski space on our brane, requires a violation of the null dominant energy condition. In section [11] we extend this argument to the case in which our brane is allowed to exhibit standard FRW expansion, and comment on the regime of validity of this result. We then turn to possible ways in which stabilization may work due to a breakdown of the assumptions in the previous argument or through quantum stabilization effects. We provide an explicit example of this possibility through the Casimir force in CHMs.

For completeness we include some final comments on supersymmetry in compact hyperbolic backgrounds before concluding.
II. COMPACT HYPERBOLIC MANIFOLDS AND EXTRA DIMENSIONS

A $d$-dimensional compact hyperbolic manifold has spatial sections of the form $\Sigma = H^d/\Gamma$, where the fundamental group, $\Gamma$, is a discrete subgroup of $SO(d, 1)$ acting freely (i.e. without fixed points) and discontinuously (since it is discrete). The CHM can be obtained by gluing together the faces of a fundamental domain in hyperbolic space.

Hyperbolic space in $d$ dimensions can be viewed as the hyperboloid

$$-x_0^2 + x_1^2 + x_2^2 + \cdots + x_d^2 = -R_h^2,$$

(1)

embedded in $(d + 1)$-dimensional Minkowski space. In the simple case $d = 3$, of particular interest in this paper, we can use the coordinate identifications

$$x_0 = R_h \cosh \chi, \quad x_1 = R_h \sinh \chi \cos \alpha, \quad x_2 = R_h \sinh \chi \sin \alpha \cos \beta, \quad x_3 = R_h \sinh \chi \sin \alpha \sin \beta,$$

(2)

to relate this representation to the induced metric

$$ds^2 = R_h^2 \left[ d\chi^2 + \sinh^2(\chi) \left( d\alpha^2 + \sin^2(\beta)d\beta^2 \right) \right]$$

(3)
on $H^3$. From this perspective it is easy to understand why the isometries of $H^3$ are described by the orientation preserving homogeneous Lorentz group in 4-dimensions, $SO(3, 1)$.

To illustrate the features of compact hyperbolic spaces, we will consider the Thurston manifold [12] (see Fig. 1). One particularly useful way to study this and other compact hyperbolic spaces is to use the SnapPea [13] catalogue. The Thurston manifold $\Sigma_{Th}$ (m003(-2,3) in the SnapPea census) has fundamental group, $\Gamma = \pi_1(\Sigma_{Th})$, with presentation

$$\Gamma = \{a, b : a^2ba^{-1}b^3a^{-1}b, ababa^{-1}b^{-1}ab^{-1}a^{-1}b\}.$$  

(4)

Here $a$ and $b$ are the generators of the fundamental group, describing identifications in the faces of the fundamental cell shown in Fig. 1, and in usual group-theoretic notation the expressions following the colon in equation (4) are set equal to the identity.

The fundamental cell is drawn using Klein’s projective model for hyperbolic space. In this projection $H^3$ is mapped into an open ball in Euclidean 3-space $E^3$. Under this mapping hyperbolic lines and planes are mapped into their Euclidean counterparts. This is why the totally geodesic faces of the fundamental cell appear as flat planes. Thurston’s manifold has volume approximately $0.98 R_h^3$.
By acting on points lying on the symmetry axis of each group element it is possible to compile a list of the minimal geodesics. A typical isometry is a Clifford translation – a corkscrew type motion, consisting of a translation of length $L$ along a geodesic, combined with a simultaneous rotation through an angle $\omega$ about the same geodesic. The length and torsion can be found directly from the eigenvalues of the group element, and are conveniently listed by the SnapPea program [13].

We can make some general observations about the existence of long wavelength modes on $H^3/\Gamma$. In large volume CHM’s there is generically a gap in the spectrum of the Laplacian (more specifically, the Laplace-Beltrami operator) between the zero mode and the next lowest mode. A theorem due to Sarnak in $d = 2$ and a conjecture due to Brooks in $d \geq 3$ state (approximately) that for large volumes characteristically $m_{\text{gap}} = \mathcal{O}(R_h^{-1})$. This puts an upper limit on the wavelength of modes.

We are interested in CHMs as extra-dimensional manifolds. Because they are locally negatively curved, CHM’s exist only for $d \geq 2$. Their properties are well understood only for $d \leq 3$, however, it is known that CHM’s in dimensions $d \geq 3$ possess the important property of rigidity [14]. As a result, these manifolds have no massless shape moduli. Hence, the stabilization of such internal spaces reduces to the problem of stabilizing a single modulus, the curvature length or the “radion.”

The primary reason for considering such manifolds for compactification is the behavior of
their volume as a function of linear size. In general, the total volume of a smooth compact hyperbolic space in any number of dimensions is

\[ \text{Vol}_{\text{new}} = R_h^d e^\alpha, \quad (5) \]

where \( R_h \) is the curvature radius and \( \alpha \) is a constant determined by topology. (For \( d = 3 \) it is known that there is a countable infinity of orientable CHM’s, with dimensionless volumes, \( e^\alpha \), bounded from below, but unbounded from above; moreover the \( e^\alpha \) do not become sparsely distributed with large volume.) In addition, because the topological invariant \( e^\alpha \) characterizes the volume of the CHM, it is also a measure of the largest distance \( L \) around the manifold. CHM’s are globally anisotropic; however, since the largest linear dimension gives the most significant contribution to the volume, there exists an approximate relationship between \( L \) and \( \text{Vol}_{\text{new}} \). For \( L \gg R_h/2 \) the appropriate asymptotic relation, dropping irrelevant angular factors, is

\[ \alpha \simeq \frac{(d_{\text{eff}} - 1)L}{R_h}, \quad (6) \]

where \( 1 < d_{\text{eff}} \leq d \).

Thus, in strong contrast to the flat case, the expression for \( M_P \) depends exponentially on the linear size,

\[ M_P^2 = M_{d+4}^2 R_h^d e^\alpha \simeq M_{d+4}^2 R_c^d \exp \left[ \frac{(d_{\text{eff}} - 1)L}{R_h} \right]. \quad (7) \]

The most interesting case (and as we will see later, most reasonable) is the smallest possible curvature radius, \( R_h \sim M_{d+4}^{-1} \). Taking \( M_{d+4} \sim \text{TeV} \) then yields (with \( d_{\text{eff}} = d = 3 \))

\[ L \simeq 35 M_{d+4}^{-1} = 10^{-15} \text{mm}. \quad (8) \]

Therefore, one of the most attractive features of CHM’s is that to generate an exponential hierarchy between \( M_{d+4} \sim \text{TeV} \), and \( M_P \) only requires that the linear size \( L \) be very mildly tuned if the internal space is a CHM.

### III. HYPERBOLIC BRANE WORLD COSMOLOGY

Our starting point is the action for Einstein gravity in a \( 4 + d + n \)-dimensional space-time, with bulk matter.

\[ S = \int d^{4+d+n}x \sqrt{-G} \left[ M^{d+n+2} R(G) - \mathcal{L}_{\text{bulk}} \right], \quad (9) \]
where $M$ is the $4+d+n$-dimensional Planck mass and $\mathcal{L}_{\text{bulk}}$ is the Lagrangian density. We will assume that the geometry of the bulk space $\Sigma^{4+d+n}$ is factorizable into the form

$$\Sigma^{4+d+n} = \mathcal{F}^{3+1} \times H^d / \Gamma \times S^n,$$

(10)

where $\mathcal{F}^{3+1}$ denotes a $3+1$-dimensional Friedmann, Robertson-Walker (FRW) space, $H^d / \Gamma$ is a $d$-dimensional compact hyperbolic manifold and $S^n$ is the $n$-sphere, with volume $\Omega_n$.

We have included a spherical factor here because of the hope that its curvature will play a role in cancelling that of the hyperboloid. As we shall see this is not sufficient. In demonstrating this it will become clear that adding other factors will not help the situation. Therefore this choice of manifold seems sufficiently general to prove our result.

The metric ansatz consistent with this factorization is

$$ds^2 = G_{AB} dx^A dx^B = \bar{g}_{\mu\nu} dx^\mu dx^\nu + r_h^2 \gamma_{ij} dy^i dy^j + r_s^2 \omega_{ab} dz^a dz^b.$$  

(11)

Here $A, B, \ldots = 0 \ldots 3 + d + n$ are indices on the whole bulk space-time, $\mu, \nu, \ldots = 0 \ldots 3$ are indices on the $3+1$ dimensional brane, $i, j, \ldots = 4 \ldots d + 3$ are indices on the CHM and $a, b, \ldots = d + 4 \ldots d + n + 3$ are indices on the sphere. The metric on the brane is denoted by $\bar{g}_{\mu\nu}$, that on the unit $d$-hyperboloid by $\gamma_{ij}$ and the metric on the unit $n$-sphere by $\omega_{ab}$. There are therefore two radion fields in the problem, $r_h$, the curvature radius of the CHM, and $r_s$, the curvature radius of the sphere. We will denote the values of these radii at their putative stable points by $R_h$ and $R_s$ respectively. By the volume considerations of the previous section, the effective $3 + 1$-dimensional Planck mass will then be given by

$$M_4^2 = M_4^{2+d+n} (R_h^d e^\phi)(R_s^n \Omega_n).$$

(12)

There are two ways to analyze the issue of stabilization in extra-dimensional theories. We may consider the full $4+d+n$-dimensional equations, or those of the dimensionally reduced $3+1$ dimensional theory. For completeness we will express the problem in an effective theory setting and then demonstrate our main result in the full theory.

To derive our effective theory, let us define the fields $\phi$ and $\psi$ by

$$r_h = R_h \exp \left[ \frac{1}{d(d+2) M_4} \phi \right],$$

and

$$r_s = R_s \exp \left[ \frac{1}{n(n+2) M_4} \psi \right],$$

(13)
and perform a conformal rescaling of the brane metric

\[ \bar{g}_{\mu\nu} = g_{\mu\nu} \exp \left[ -\sqrt{\frac{d}{d + 2}} \frac{\phi}{M_4} - \sqrt{n} \frac{\psi}{n + 2 M_4} \right]. \]  

This decouples \( \phi \) and \( \psi \) from the reduced Einstein tensor. Integrating over the compact manifolds, we may now define an effective action \( S_{\text{eff}} \) by

\[ S_{4+d+n} = \int d^{d+n}x \sqrt{\Omega_n} \sqrt{\omega} S_{\text{eff}}, \]

with

\[ S_{\text{eff}} = \int d^4x \sqrt{-g} \left[ \frac{M_4^2}{2} R(g) - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} (\nabla \psi)^2 
- 2 \sqrt{\frac{n d}{(d + 2)(n + 2)}} \nabla \phi \nabla \psi - W(\phi, \psi, g) \right]. \]

Here, the effective rescaled potential is

\[ W(\phi, \psi, g) = \left( \frac{M_4^2}{M_{n+d+2}} \right) \mathcal{L}_{\text{bulk}} \exp \left[ -\sqrt{\frac{d}{d + 2}} \left( \frac{\phi}{M_4} \right) - \sqrt{n} \left( \frac{\psi}{M_4} \right) \right] + \left[ \frac{d(d - 1)}{R_h^2} - \frac{n(n - 1)}{R_s^2} \right] M_4^2 \exp \left[ -\sqrt{\frac{d + 2}{d}} \left( \frac{\phi}{M_4} \right) - \sqrt{n + 2} \left( \frac{\psi}{M_4} \right) \right] \]

where we have used that \( R(\omega) = n(n - 1) \).

In this language the stabilization of the two radii translates into the following obvious system of equations:

\[ \partial_\phi W|_{(\phi, \psi) = 0} = 0 \quad , \quad \partial_\phi^2 W|_{(\phi, \psi) = 0} > 0 \]
\[ \partial_\psi W|_{(\phi, \psi) = 0} = 0 \quad , \quad \partial_\psi^2 W|_{(\phi, \psi) = 0} > 0 , \]

plus the condition that the effective four-dimensional cosmological constant vanish,

\[ \frac{\delta S_{\text{eff}}}{\delta g^{\mu\nu}} = \left( g_{\mu\nu} W - 2 \frac{\partial W}{\partial g^{\mu\nu}} \right)|_{(\phi, \psi) = 0} = 0 . \]

In the full theory we shall adopt a cosmological approach and assume that, whatever bulk matter is present, its energy-momentum tensor can be expressed in perfect fluid form on each of the submanifolds

\[ T_{00} = \rho \]
\[ T_{\alpha\beta} = p \bar{g}_{\alpha\beta} \]
\[ T_{ij} = qr_{\gamma ij} \]
\[ T_{ab} = sr_{\omega ab} , \]
where $\alpha$, $\beta = 1..3$. The Einstein equations then become

$$\rho = 3 \left[ \frac{k}{a^2} + \left( \frac{\dot{a}}{a} \right)^2 \right] + \frac{1}{2} \left[ \frac{n(n-1)}{R_s^2} - \frac{d(d-1)}{R_h^2} \right] M^{2+d+n} \tag{21}$$

$$p = -\frac{k}{a^2} + \left( \frac{\dot{a}}{a} \right)^2 + 2 \frac{\ddot{a}}{a} - \frac{1}{2} \left[ \frac{n(n-1)}{R_s^2} - \frac{d(d-1)}{R_h^2} \right] M^{2+d+n}$$

$$q = -3 \left[ \frac{k}{a^2} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} \right] - \frac{1}{2} \left[ \frac{2n(n-1)}{R_s^2} - \frac{(d-1)(d-2)}{R_h^2} \right] M^{2+d+n}$$

$$s = -3 \left[ \frac{k}{a^2} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} \right] - \frac{1}{2} \left[ \frac{(n-1)(n-2)}{R_s^2} - \frac{2d(d-1)}{R_h^2} \right] M^{2+d+n}.$$ 

The null energy condition, $T_{AB} N^A N^B \geq 0$ for all null 4 + $d + n$-vectors $N^A$, in conjunction with the ansatz (20) yields

$$\rho + p \geq 0 \tag{22}$$

$$\rho + q \geq 0$$

$$\rho + s \geq 0,$$

which then implies

$$\frac{k}{a^2} + \left( \frac{\dot{a}}{a} \right)^2 - \frac{\ddot{a}}{a} \geq 0 \tag{23}$$

$$\frac{\ddot{a}}{a} \leq -\frac{1}{6} \left[ \frac{n(n-1)}{R_s^2} + \frac{2(d-1)}{R_h^2} \right] \tag{24}$$

$$\frac{\ddot{a}}{a} \leq \frac{1}{6} \left[ \frac{2(n-1)}{R_s^2} + \frac{d(d-1)}{R_h^2} \right]. \tag{25}$$

It is relatively straightforward to see how these inequalities are incompatible with a reasonable cosmological evolution on the brane. Note first that (24) implies (23). Thus (24) is the important inequality to deal with. Successful 3 + 1-dimensional cosmology requires a radiation dominated phase (in order that successful nucleosynthesis occur), followed by a matter dominated phase. Focusing on the radiation dominated phase, the scale factor evolves as $a(t) \propto t^{1/2}$, which means that

$$\left. \frac{\ddot{a}}{a} \right|_{\text{radiation}} = -\frac{1}{4} \frac{1}{t^2}. \tag{26}$$

Therefore, certainly when the universe is older than than $R_h^{-1} \sim (\text{TeV})^{-1}$, the null energy condition is violated. Since nucleosynthesis occurs at a later time than this, it is clear that, even in the most optimistic case the model is not cosmologically viable.
Therefore we conclude that any bulk matter that stabilizes the radion and gives rise to an acceptable cosmology on the brane must violate the null energy condition.

Note further that, in the special case that we restrict our brane to be 3 + 1-dimensional Minkowski space-time and restrict the extra-dimensional manifold to be purely a CHM (no $S^n$ factor), our constraints simplify (see [10]) considerably to become

\[
\rho = \frac{-d(d-1)M^{2+d}}{2R_h^2}, \\
p = \frac{d(d-1)M^{2+d}}{2R_h^2}, \\
q = \frac{(d-1)(d-2)M^{2+d}}{4R_h^2}.
\]

In this case it is simple to see that the required matter field cannot obey the null energy condition. Consider a general null vector with spatial components in the direction of the hyperboloid, for example

\[
N = \partial_t + e_i,
\]

where \(\{e_i\}\) is an orthogonal vector basis on the hyperboloid. Contracting the energy-momentum tensor twice with this vector yields

\[
T_{AB}N^AN^B = -\frac{(d-1)(d+2)M^{2+d}}{4R_h^2} < 0.
\]

Hence the required matter must violate the null energy condition.

To clarify the above general statement, let us consider a simple example of bulk matter that one might expect to stabilize our CHM. For this example we’ll stay within the simplified context of requiring Minkowski space to be the solution on our brane, as above. Our bulk matter will consist of a cosmological constant $\Lambda = M^{d+4} \lambda$ and a $d$-form $F_{[d]}$ over the hyperbolic manifold. The bulk Lagrangian density is therefore

\[
\mathcal{L}_{\text{bulk}} = \Lambda + \frac{F_{[d]}^2}{2d!}.
\]

The resulting field equations for $F_{[d]}$ can be solved by the ansatz

\[
F_{\mu ABC} = 0, \\
F_{45..d+4} = B(x^\mu),
\]

so that $B$ is independent of the hyperbolic coordinates. With this ansatz there results a trapped magnetic flux on the compact hyperbolic space and the constant $B$ is related to the
radion $r_h$ by the equation
\begin{equation}
B = bM^{(d+4)/2} \left( \frac{R_h}{r_h} \right)^d.
\end{equation}
Thus the on-shell form of the bulk Lagrangian density becomes
\begin{equation}
\mathcal{L}_{\text{bulk}} = M^{d+4} \left[ \Lambda + \frac{b^2}{2} \left( \frac{R_h}{r_h} \right)^{2d} \right].
\end{equation}
Defining $\beta^{-1} = MR_h$, equations (18) and (19) reduce to
\begin{align}
\Lambda + \frac{b^2}{2} + d(d-1)\beta^2 &= 0, \\
b^2d + d(d-1)(d+2)\beta^2 &= 0.
\end{align}
The corresponding solution is
\begin{align}
\Lambda &= -(d-1)(d-2)\beta^2 \\
b^2 &= -(d-1)(d+2)\beta^2.
\end{align}
Using these solutions, the on-shell stress energy tensor of the d-form becomes
\begin{equation}
T_{AB} = \frac{(d-1)(d-2)\beta^2}{8} (g_{AB} - \delta_{AB}g_{ij}g^{ij}),
\end{equation}
therefore when contracted twice with a null vector along one of the hyperbolic directions (i.e. $N = N^0e_0 + N^i e_i$), we obtain
\begin{equation}
T_{AB}N^A N^B = \frac{(d-1)(d-2)\beta^2}{8} g_{ii}N^iN^i < 0,
\end{equation}
clearly violating the null energy condition.

IV. GOING BEYOND CLASSICAL MATTER: THE CASIMIR FORCE

Although classical fluids are expected to respect the null energy condition, it is well known that such conditions are violated quantum mechanically. A specific example of such violation is Casimir energy density – the zero point energy density of the quantum fields on the space time. The magnitude and sign of the Casimir energy depend in complicated ways on the topology and geometry of the underlying manifold, and on the specific field content (for an example of the use of the Casimir force in stabilizing extra-dimensional models see [15]).
Let us continue to consider the simplified example space-time and further focus on the case in which the extra-dimensional manifold is 3-dimensional,

\[ M = M^{(4)} \times H^3/\Gamma \, . \]  

(38)

The Casimir energy density will depend on both the details of \( \Gamma \), and on the curvature \( R_h \) of the \( H^3 \),

\[ \rho_C \propto R_h^{-p} , \]  

(39)

with \( p > 0 \). Thus the Casimir pressure in the \( H^3/\Gamma \) directions will be proportional to \( R_h^{-p-1} \), while the pressure will be zero in the Minkowski space directions. Since the null energy condition would require that when the Casimir energy density, \( \rho_C \) is negative the associated pressure is isotropic and equal to \(-\rho_C\), the null energy condition is therefore violated. Interestingly, with \( \rho_C < 0 \), the sign of the energy density and the pressure are precisely right to help stabilize \( H^3/\Gamma \).

Because of the chaotic nature of flows on CHMs, we do not present any generic results for the value of \( \rho_C \) in specific CHMs, except to note that the case of a minimally coupled scalar field on the Thurston manifold has been studied numerically by Miller, Fagundes and Opher [16]. In this study they discovered that the Casimir energy density can be negative, as needed for stabilizing the extra-dimensional manifold in our model. An understanding of the Casimir energy in the specific highly topologically complex manifolds used in our model would be extremely useful, but seems calculationally formidable and is beyond the scope of this paper.

V. SUPERSYMMETRY IN HYPERBOLIC BACKGROUNDS

A question of great interest when suggesting any compactification scheme is that of low-energy supersymmetry. For the case of interest in this paper, we may begin with an explicit construction of the Killing spinors of maximally symmetric spaces with negative cosmological constant [17, 18]. For the space \( H^d \) we choose coordinates in the horospherical frame in which the metric takes the form,

\[ ds^2 = e^{2r} \delta_{\alpha \beta} dx^\alpha dx^\beta + dr^2 \, . \]  

(40)
In this frame, the Killing spinors are given by,

$$\xi = e^{\frac{1}{2}r \Gamma_r} \left[ 1 + \frac{1}{2} x^\alpha \Gamma_\alpha (1 - \Gamma_r) \right] \epsilon ,$$  \hspace{1cm} (41)

where \( \epsilon \) is an arbitrary constant spinor (the cases \( d=2,3 \) are special and expressions can be found in ref \[19\]).

We can see from this expression that the number of supersymmetries of \( H^d \) is equal to the number of independent spinor components. Now, recall that the isometry group is \( SO(1, d) \) and that compact hyperbolic manifolds are obtained by quotient of \( H^d \) by a discrete subgroup \( \Gamma \) of \( SO(1, d) \), with no fixed points. Whether or not any Killing spinors survive this quotienting process depends on \( \Gamma \) \[20\].

1. If \( d \) is even then the spinors are in an \( SO(1, d-1) \) representation, and all supersymmetries are broken.

2. If \( d \) is odd, then the spinors are in an \( SO(1, d) \) representation. In this case there are several possibilities.

   (a) If \( \Gamma \) is a subgroup of \( SO(1, d-1) \) some Killing spinors may survive, since we can decompose the original Killing spinors into Weyl spinors on the representation of this group.

   (b) If \( \Gamma \) is a not subgroup of \( SO(1, d-1) \) then all supersymmetries are broken.

   (c) In the special case \( d = 3 \) there also remain no supersymmetries.

VI. CONCLUSIONS

The initial motivation for the idea of large extra dimensions was to address the hierarchy problem. Allowing the internal space to be a compact hyperbolic manifold \[3\] introduces the attractive new feature that the hierarchy problem becomes a truly mild tuning of the length scales in the theory. In addition, such models have been shown to have interesting cosmological consequences \[8, 9\]. Of course, in all extra-dimensional models, the stabilization of the radion mode is an essential component, since the effective four-dimensional gravitational constant does not change today. One advantage of CHMs as internal spaces is that in three
or more dimensions the property of rigidity means that there is a single radion – the curvature radius of the manifold. Nevertheless, stabilization of this single mode remains an important issue in these models.

In this paper we have analyzed the constraints on the bulk matter that is required to achieve this necessary stabilization. We have allowed for the metric on the brane to be cosmological, of FRW form, and in this sense we have expanded on the elegant results of [10] which apply to the case that the brane is Minkowski. Our results apply to extra dimensions large enough that general relativity is a valid theory, and show that stabilization may only be achieved by bulk matter that violates the weak energy condition. We have demonstrated this result explicitly, and have provided an example of how a specific example of matter violates this condition.

Since such violations are problematic for classical matter, quantum effects may be crucial in stabilizing these manifolds. In fact, since the essential feature that makes CHMs so attractive is their large volume without large linear spatial extent, the assumption that general relativity is completely valid may break down. As a specific example we have suggested the Casimir effect as such a quantum contribution. While simple estimates indicate that this may be a successful way to stabilize CHMs, we caution that to understand the effect in the manifolds that are useful for brane-world models is a much more difficult task.

Finally, although independent from the issue of stabilization, for completeness we have briefly discussed the ways in which supersymmetry in compact hyperbolic extra dimensions can be broken due to the obstructions to defining covariantly constant spinors on these spaces.

**Acknowledgments**
The work of SN and GDS is supported by grants from the U.S. Department of Energy. The work of PS is supported in part by the National Science Foundation (NSF) under grant PHY-0098747 and by funds from Syracuse University. The work of MT is supported by the NSF under grant PHY-0094122. GDS thanks Nemanja Kaloper for extensive discussions of the Casimir energy density and thanks the Aspen Center for Physics, where some of this
work was completed. MT thanks Sean Carroll and Mark Hoffman for useful discussions.

[1] I. Antoniadis, Phys. Lett. B246, 377 (1990); J. Lykken Phys. Rev. D54 3693 (1996).
[2] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B429, 263 (1998); I. Antoniadis, et al, Phys. Lett. B436, 257 (1998).
[3] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999); ibid 4690 (1999); N. Arkani-Hamed, S. Dimopoulos, G. R. Dvali and N. Kaloper, Phys. Rev. Lett. 84, 586 (2000) arXiv:hep-th/9907209; J. Lykken and L. Randall, JHEP 0006, 014 (2000) arXiv:hep-th/9908076.
[4] I. Antoniadis and K. Benakli, Phys. Lett. B326, 69 (1994); K. Dienes, E. Dudas, and T. Gherghetta, Phys. Lett. B436, 55 (1998), Nucl. Phys. B537, 47 (1999).
[5] J. Polchinski, Phys. Rev. Lett. 75, 4724 (1995).
[6] N. Kaloper, J. March-Russell, G. D. Starkman and M. Trodden, Phys. Rev. Lett. 85, 928 (2000) arXiv:hep-ph/0002001.
[7] M. Trodden, arXiv:hep-th/0010032.
[8] G. D. Starkman, D. Stojkovic and M. Trodden, Phys. Rev. Lett. 87, 231303 (2001) arXiv:hep-th/0106143.
[9] G. D. Starkman, D. Stojkovic and M. Trodden, Phys. Rev. D 63, 103511 (2001) arXiv:hep-th/0012226.
[10] S. M. Carroll, J. Geddes, M. B. Hoffman and R. M. Wald, arXiv:hep-th/0110149.
[11] M. Rainer and A. Zhuk, Phys. Rev. D54, 6186 (1996); U. Guenther and A. Zhuk, Phys. Rev. D56, 6391 (1997); ibid D61, 124001 (2000); U. Guenther and A. Zhuk, Class. Quan. Grav. 18, 1441 (2001); V. Ivshchuk, V. Melnikov and A. Zhuk, Nuovo Cimento B104, 575 (1989).
[12] W. P. Thurston, Bull. Am. Math. Soc. 6, 357 (1982).
[13] J. Weeks, SnapPea: A Computer Program for Creating and Studying Hyperbolic 3-Manifolds, available at http://www.geom.umn.edu:80/software.
[14] G. Mostow, Ann. Math. Stud.78 (Princeton UP, Princeton 1973); G. Prasad, Invent. Math. 21 255 (1973).
[15] E. Ponton and E. Poppitz, JHEP 0106, 019 (2001) arXiv:hep-ph/0105021.
[16] D. Muller, H. V. Fagundes and R. Opher, Phys. Rev. D 63, 123508 (2001) arXiv:gr
qc/0103014].

[17] H. Lu, C. N. Pope and P. K. Townsend, Phys. Lett. B391 (1997) 39-46, hep-th/9607164.

[18] H. Lu, C. N. Pope and J. Rahmfeld, J. Math. Phys. 40 (1999) 4518-4526, hep-th/9805151.

[19] Y. Fujii and K. Yamagishi, J. Math. Phys. 27 (1986) 979.

[20] A. Kehagias and J. G. Russo, JHEP 0007, 027 (2000) arXiv:hep-th/0003281.