STRINGs, BRANES and TWO-TIME PHYSICS

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Abstract

We generalize the ideas and formalism of Two-Time Physics from particle dynamics to some specific examples of string and p-brane ($p \geq 1$) dynamics. The two-time string or p-brane action can be gauge fixed to produce various one-time string or p-brane actions that are dual to each other under gauge transformations. We discuss the particular gauges that correspond to tensionless strings and p-branes in flat $(d-1)+1$ spacetime, rigid strings and p-branes in flat $(d-1)+1$ spacetime, and tensionless strings and p-branes propagating in the AdS$_{d-n}$ x $S^n$ backgrounds.

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1 Introduction

There exist by now many convincing examples of a remarkable connection between a global $SO(d, 2)$ symmetry (sometimes hidden and non-linearly realized) in one-time point particle dynamics and the two-time Lorentz $SO(d, 2)$ symmetry that is linearly realized in a two-time action with extra gauge symmetries [1]-[6]. The basic property is that the gauge symmetry reduces the two-time action to a one-time action, but in many possible gauge equivalent ways, thus establishing a “duality” between various one-time dynamical systems and unifying them in some sense. The motivation for this approach - called Two-Time Physics - comes from the algebraic approach to M-theory and its extensions [7, 8, 9, 10, 11] that provide various hints of hidden two timelike dimensions. There also exist indications that the formalism of Two-Time Physics is applicable to the full structure of M-theory [12].

The crucial observation of [1]-[6] is that many point particle systems (relativistic massless and massive particle, non-relativistic massless and massive particle, H-atom, harmonic oscillator, particle moving in the AdS background, etc.) in (d-1)-space and 1-time dimensions can be understood as different gauge choices of a 2-time point particle system endowed with a local $Sp(2, R)$ symmetry and manifestly invariant under the global $SO(d, 2)$ symmetry. Unlike Yang-Mills type gauge theories, the concept of “time” is gauge dependent in the Two-Time-Physics formalism (see also General Relativity although it has only one gauge dependent timelike dimension). Therefore the gauge fixed Lagrangian or Hamiltonian (spectrum) describes diverse systems that would appear unrelated in ordinary 1-time dynamics. However, there are gauge invariant quantities that can be computed in any one of these systems with identical results (for example Casimir eigenvalues of the $SO(d, 2)$ representation). Thus such systems display properties of their unification under a common gauge invariant action.

The particular $Sp(2, R)$ symmetry that is gauged in this case is nothing but the familiar automorphism of the Heisenberg commutation relations in quantum mechanics, or the symplectic group of the one-particle phase space of classical Hamiltonian dynamics. One can also think of the same $Sp(2, R) \sim SO(1, 2)$ as the conformal group in one (world-line) dimension. From this point of view many $0+1$ quantum gravity systems (relativistic massless and massive particle, non-relativistic massless and massive particle, etc.) can be viewed as different gauge choices of the same $0+1$ conformal quantum gravity theory [1].

In some specific instances (like the relativistic massless particle) the action of the global $SO(d, 2)$ can be interpreted as the natural action of the conformal group in d-dimensions. What is surprising is that the same group has a natural action, even though with a highly non-linear

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realization, in many other physical situations (the relativistic massive particle, for example) [2]. The Hamiltonians describing these two different physical systems (relativistic massless and massive particle) are related by a canonical $Sp(2, R)$ transformation, which is responsible for the appearance of the global $SO(d, 2)$ invariance of the action describing the relativistic massive particle. The evolution parameters (“times”) in the two cases, being canonically conjugate to the respective Hamiltonians, are different as well.

One can also interpret the same $SO(d, 2)$ as the Lorentz group in the space with one extra spatial and one extra time dimensions. The parameters in the various Hamiltonians related by the local $Sp(2, R)$ transformations, appear as moduli (values of fields) in this $(d, 2)$ dimensional space (e.g. mass of particle, coupling of H-atom, etc.). Note that the Two-Time Physics formalism works both classically and quantum-mechanically [1]-[6]. It turns out that in the full quantum theory, the $Sp(2, R)$ gauge invariant sector is completely described by a unique representation of the conformal group $SO(d, 2)$, which fixes the values of the $SO(d, 2)$ Casimirs.

The same formalism can be generalized to the case of particles with spin $n/2$ by using world-line supersymmetry with $n$ supercharges (by replacing $Sp(2, R)$ with $OSp(n|2)$) [3] and to target-space supersymmetry (by generalizing the action of the global $SO(d, 2)$ symmetry to various superconformal groups) [3]. In the latter case, the formalism of Two-Time Physics sheds new light on the origin of fermionic kappa supersymmetry. This provides an explicit way for constructing physical models invariant under global superconformal symmetry groups $OSp(8*/4^*)$, $SU(2, 2/4)$, $OSp(8*/4)$ which have usual space-time interpretations in $d = 3, 4, 6$ [6] and an expanded space-time interpretation including zero modes of p-brane gauge potentials in other dimensions [12]. The same approach explicitly establishes the existence of the novel bosonic counterparts of kappa supersymmetry [12].

In this letter we want to extend the formalism of Two-Time Physics for the case of strings and p-branes ($p \geq 2$). We find a natural action of the global $SO(d, 2)$ conformal symmetry in the case of many classical systems involving bosonic strings and branes in $(d − 1) + 1$ dimensions. The list includes the bosonic null string and branes, the null string moving in the AdS background and the bosonic rigid string and branes. The actions for these systems admit the extension to the two-time formalism by the addition of gauge degrees of freedom. The new gauge invariant actions can be gauge fixed to many other dual systems of branes in the same manner of particle dynamics.

The letter is organized as follows. In section 2 we briefly review the Two-time Physics formalism for the case of particle dynamics. Then in section 3 we extend this formalism and discuss tensionless strings and branes by working with configuration space variables only. We apply this approach in section 4 for the case of tensionless strings and branes moving in the AdS background. In section 5 we discuss the rigid bosonic strings and branes. Finally, we comment on possible generalizations of the results presented in this letter.
2 Basic Formalism of Two-Time Physics

We start with a brief summary of the Two-Time Physics formalism for the case of spinless particle dynamics [1]. The model is an $Sp(2, R)$ gauge theory described by the action (the various forms of the action and their utility will be explained below)

$$S_0 = \frac{1}{2} \int d\tau D_\tau X^M_i X^N_j \varepsilon^{ij} \eta_{MN}$$ (1)

$$\equiv \int d\tau (\partial_\tau X^M_1 X^N_2 - \frac{1}{2} A^{ij} X^M_i X^N_j) \eta_{MN}$$ (2)

$$= \int d\tau \left[ \partial_\tau X^M P^M - \frac{1}{2} e P^M P^M - A X^M P^M - \frac{1}{2} k X^M X^M \right] \eta_{MN}$$ (3)

$$\equiv \int d\tau \left[ \frac{1}{2e} (\partial_\tau X - AX)^2 - \frac{1}{2} k X^2 \right]$$ (4)

$$\equiv \int d\tau \left[ \frac{1}{2e} (\partial_\tau X)^2 - \frac{1}{2} K X^2 \right], \quad K = k - \frac{A^2}{e} - \partial_\tau \left( \frac{A}{e} \right)$$ (5)

Here $X^M_i(\tau)$ is an $Sp(2, R)$ doublet, consisting of the ordinary coordinate and its conjugate momentum ($X^M_1 \equiv X^M$ and $X^M_2 \equiv P^M$). The indices $i, j = 1, 2$ denote the doublet $Sp(2, R)$, they are raised and lowered by the antisymmetric Levi-Civita symbol $\varepsilon_{ij}$. The covariant derivative $D_\tau$ is defined as

$$D_\tau X^M_i = \partial_\tau X^M_i - \varepsilon_{ik} A^{kl} X^M_l.$$ (6)

The local $Sp(2, R)$ acts as $\delta X^M_i = \varepsilon_{ik} \omega^{kl} X^m_l$ and $\delta A^{ij} = \omega^{ij} \varepsilon_{kl} A^{kj} + \omega^{jk} \varepsilon_{kl} A^{ik}$, where $\omega^{ij}(\tau)$ is a symmetric matrix containing three gauge parameters. The second form of the action is obtained after an integration by parts so that only $X^M_1$ appears with derivatives. This allows the identification of $X, P$ by the canonical procedure, as indicated in the third form of the action.

The gauge fields $A^{11} \equiv k, A^{12} = A^{21} \equiv A, A^{22} \equiv e$ also act as Langrange multipliers for the following three first class constraints (that form the Sp(2, R) algebra)

$$X_i \cdot X_j = 0 \rightarrow X^2 = P^2 = X \cdot P = 0,$$ (7)

as implied by the local $Sp(2, R)$ invariance. It is precisely the solution of these constraints that require that the global metric $\eta_{MN}$ has a signature with two-time like dimensions. Thus, $\eta_{MN}$ stands for the flat metric on a $(d, 2)$ dimensional space-time, which is the only signature consistent with the equations of motion for the $Sp(2, R)$ gauge field $A^{kl}$, leading to a non-trivial dynamics that can be consistently quantized. Hence the global two-time $SO(d, 2)$ is implied by the local $Sp(2, R)$ symmetry. It is possible to extend $\eta_{MN}$ to curved spacetime $G^{MN}(X)$ with suitable modification of the transformation laws and certain conditions on the metric. Likewise, it is possible to extend the action to include background gauge fields. Such generalizations will be discussed in [13].

The fourth form of the action [1] is obtained by integrating out $P^M$. Note that the ordinary $\tau$ reparametrization $\tau \rightarrow \tau + \delta \tau(\tau)$ corresponds to a $U(1)$ subgroup of $Sp(2, R)$, with the gauge
potential $A^{22} = e$ and gauge parameter $\omega^{22} = e(\tau) \delta \tau (\tau)$. Recalling that $\text{Sp}(2, R) = SO (1, 2)$ is the conformal group on the worldline, this form of the action can be viewed as a generalization of $0 + 1$ gravity ($\tau$-reparametrization) to $0 + 1$ conformal gravity. The gauge potentials $e, A, k$ and parameters $\omega^{22}, \omega^{12}, \omega^{11}$ are associated with the $SO(1, 2)$ generators for local translations, dilatations and conformal transformations respectively.

The fifth form of the action (5) is obtained after integrating by parts the cross term $-\frac{d}{e} X \cdot \partial_\tau X$ and defining $K = k - \frac{d^2}{e} - \partial_\tau \left( \frac{d}{e} \right)$ as the total coefficient of $X^2/2$. This form of the action is still invariant under the same $\text{Sp}(2, R) = SO (1, 2)$ local symmetries. The simplest form of the symmetries are the $\tau$ reparametrization $\delta \tau (\tau)$ and the local scaling $\omega^{12} \equiv \alpha (\tau)$

$$\delta X^M = \delta \tau \partial_\tau X^M, \quad \delta K = \partial_\tau (K \delta \tau)$$

$$\delta_\alpha X^M = \alpha X^M, \quad \delta_\alpha e = 2\alpha e, \quad \delta_\alpha K = -2\alpha K + \partial_\tau \left( \frac{\partial_\tau \alpha}{e} \right),$$

while $X^M, e$ and $K$ are all invariant under the local conformal transformations $\omega^{11}$. This is the simplest form of the action, and it is this form that we will explore below in discussing strings and p-branes.

All forms of the action (1-5) have an explicit global $SO(d, 2)$ invariance, generated by the Lorentz generators $L^{MN} = X^M P^N - X^N P^M = \varepsilon^{ij} X^M_i X^N_j$ that are gauge invariant. As we mentioned above, different gauge choices lead to different $0 + 1$ theories of $0 + 1$ gravity (the relativistic massless and massive particles, H-atom, harmonic oscillator, etc.) all of which have $SO(d, 2)$ invariant actions that are directly obtained from (1-5) by gauge fixing. Since the action (1-5) and the generators $L^{MN}$ are gauge invariant, the global symmetry $SO(d, 2)$ is not lost by gauge fixing. This explains why one should expect a hidden (previously unnoticed) global symmetry $SO(d, 2)$ for each of the $0 + 1$ systems that result by gauge fixing. Note that the equations of motion, not only the action, of the manifestly $SO(d, 2)$ invariant particle system reduce to the expected equations of motion of the $(d - 1, 1)$ gauge fixed systems, indicating the validity of the gauge-fixing procedure.

If the system is quantized in a fixed gauge the absence of ghosts is evident. The system can also be quantized covariantly; then the $Sp(2, R)$ gauge symmetry is just enough to remove all negative-norm states (“ghosts”) introduced by two timelike dimensions so that the resulting quantum theory is unitary. In the quantum version of the theory the unifying feature of the system is displayed by the fact that all the gauge fixed systems, or the covariantly quantized system, share the same unitary representation of $SO(d, 2)$. For example the quadratic Casimir eigenvalue is $C_2 (SO (d, 2)) = 1 - d^2/4$ for the free massless particle, the H-atom, harmonic oscillator (in one less dimension), the particle moving on $\text{AdS}_{d-n} \times S^n$ (any $n$), etc., and similarly for all Casimir eigenvalues $C_n (SO (d, 2))$.

As mentioned before this action has been generalized to particles with spin $n/2$ (via local $\text{OSp}(n/2)$) to spacetime supersymmetry [6] [12] and to interactions with background fields [13].
3 Extensions to Tensionless Strings and Branes

In this section we want to extend the formalism of Two-Time Physics reviewed in the previous section for the case of tensionless strings and branes. We do not start from a general formalism as in the case of particle mechanics (the general formalism which is still being developed will be given elsewhere). Rather, we discuss specific cases, which point toward a more general formulation. We will work with configuration space variables as in the the last form of the action (5).

We begin with a string analog of the massless relativistic particle. We rotate the extra space $X^1'$ and time $X^0'$ dimension into a pair of “light-cone-like” coordinates $X^+ = X^0 + X^1$ and $X^- = X^0 - X^1$. We recall that [1]-[6] in this basis $M = (+', -', \mu)$, fixing two gauges $X^+ = 1$, $P^+ = 0$, and solving two constraints $X^2 = X \cdot P = 0$, gives $X^- = x^2/2$ after identifying $X^\mu = x^\mu$.

Then the action (1-5) reduces to the action for a massless particle in $d$-dimensions

$$S = \int d\tau \dot{x}_\mu^2.$$ 

This action has a well known global non-linearly realized conformal symmetry $SO(d, 2)$ which is a reflection of the $SO(d, 2)$ two-time Lorentz symmetry of the original gauge invariant action as as explained in the previous section. This action implies the familiar equations of motion $\ddot{x}_\mu = 0$ and $\dot{x}_2^2 = 0$. The stringy counterpart of the massless particle is the null string [14] in $d$-space-time dimensions

$$S_{ns} = \int d^2\sigma \frac{\{x^\mu, x^\nu\}^2}{2e}, \quad (10)$$

where we used the Poisson bracket $\{x^\mu, x^\nu\} = \epsilon^{ab} \partial_a x^\mu \partial_b x^\nu$ with $a$ or $b = 0, 1$ denoting the worldsheet indices for $\tau$ and $\sigma$. The equations of motion for $x^\nu(\tau, \sigma)$ and $e(\tau, \sigma)$ are respectively

$$\{x_\mu, \{x^\mu, x^\nu\}\} = 0, \quad \{x^\mu, x^\nu\}^2 = 0.$$

The last equation says that the determinant $g = \det g_{ab}$ of the induced metric $g_{ab} = \partial_a x \cdot \partial_b x$ is zero, i.e. $g \equiv \frac{1}{2} \{x^\mu, x^\nu\}^2 = 0$. The fact that the metric is degenerate implies that the worldsheet of the string is a null surface. Also, the tension of the null string is zero. We note that a solution of the equations of motion is given by strings that can be excited purely for left or purely right movers (keep either $+$ or $-$) with the following constraints

$$x^\mu(\tau, \sigma) = q^\mu + p^\mu \tau - i \sum_n \frac{1}{n} \alpha_n^\mu e^{in(\tau \pm \sigma)}, \quad (11)$$

$$\alpha_n^\mu = \alpha_n p^\mu \quad \text{if} \quad p^2 \neq 0, \quad p \cdot \alpha_n = 0 \quad \text{if} \quad p^2 = 0, \quad \alpha_n^\mu = \text{any} \quad \text{if} \quad p^\mu = 0. \quad (12)$$

Just like the massless particle action, the null string action is invariant under the global

The string with tension $1/\alpha'$ is obtained by adding a cosmological constant by analogy to the massive particle action $S = \int d^2\sigma \left[\frac{(x^\mu, x^\nu)^2}{2e} - \frac{e}{2\alpha'}\right] = \frac{1}{\alpha'} \int d^2\sqrt{-g}$ as seen by integrating out $e$. Recall that the massive particle action is obtained as a gauge choice of the gauge invariant action (5) and the mass is a modulus (value of a field [5]). We expect the tension to arise in a similar way in the formalism of Two-Time-Physics.

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\[ d\text{-dimensional conformal group } SO(d, 2) \text{ for which} \]
\[
\delta x^\mu = \lambda x^\mu + a^\mu + \left( \frac{1}{2} b^\mu x^2 - b \cdot x x^\mu \right) + \frac{1}{2} \omega^{\mu \nu} x_\nu
\]
\[ \delta e = 2 \left( \lambda - b \cdot x \right) e \]  \hspace{1cm} (13)

where \( \lambda, a^\mu, b^\mu, \omega^{\mu \nu} \) are the infinitesimal global parameters for dilatations, translations, conformal transformations and Lorentz transformations in Minkowski spacetime. We can reinterpret this \( SO(d, 2) \) conformal symmetry as the two-time Lorentz symmetry if we start from an action that is manifestly \( SO(d, 2) \) invariant in \( d + 2 \) dimensions, but has extra local symmetries, so that the gauge fixed version of such an action reduces to a \( d \)-space-time dimensional action for the null string (14). Thus consider the following manifestly \( SO(d, 2) \) invariant action as a generalization of the particle action (10)
\[
S_1 = \int d^2 \sigma \left[ \left\{ X^M, X^N \right\}^2 - \frac{1}{2} K X \cdot X \right]
\]  \hspace{1cm} (14)

In addition to \( \tau, \sigma \) reparametrization invariance with parameter \( \varepsilon^a (\tau, \sigma) \)
\[
\delta_\varepsilon X^M = \varepsilon^a \partial_a X^M, \quad \delta_\varepsilon e = \partial_a (\varepsilon^a e), \quad \delta_\varepsilon K = \partial_a (\varepsilon^a K),
\]  \hspace{1cm} (15)

the action is also invariant under the local scale transformations with parameter \( \alpha(\tau, \sigma) \)
\[
\delta_\alpha X^M = \alpha X^M, \quad \delta_\alpha e = 4 \alpha e, \quad \delta_\alpha K = -2 \alpha K + \left\{ X^M, \frac{1}{e} \{ \alpha, X_M \} \right\}
\]  \hspace{1cm} (16)

These generalize the symmetries (8, 9) from particles to strings.

Using the local scale invariance we can choose again the gauge \( X^+ (\tau, \sigma) = 1 \). Then the equation of motion for the gauge potential \( K \) (which also acts as a lagrange multiplier for the condition \( X^2 = 0 \)) can be solved for \( X^- = x^2/2 \). Plugging back this particular gauge choice into the \( SO(d, 2) \) invariant action (14), we immediately get the ordinary null string action in \( d \)-space-time dimensions (10).

One can also examine the equations of motion for the components \( X^+ \), \( X^- \) and \( X^\mu \) to see that they indeed reduce to the equations of motions of the \( d \)-space-time dimensional null string. Thus, the \( d \)-space-time dimensional action (10) is a genuine gauge fixed version of the manifestly \( SO(d, 2) \) invariant action (13)!

The generators of the Lorenz transformations in the \((d, 2)\)-dimensional space as derived from the action (14) are \( L^{MN} = \int_0^{2\pi} d\sigma \left( X^M P_0^N - X^N P_0^M \right) \), where now \( P_{0,1}^M = \partial_{0,1} X^M \). It is a simple matter to check that by using the gauge \( X^+ = 1 \), this reduces to the ordinary generators of the \( d \)-dimensional conformal group (13) as they would be derived from the action (10).

We can immediately extend our discussion to the case of null branes. The manifestly \( SO(d, 2) \) invariant action for a null \( p \)-brane \( (p \geq 2) \) is given by
\[
S_p = \int d^{p+1} \sigma \left[ \frac{A^{M_1 \ldots M_{p+1}} A_{M_1 \ldots M_{p+1}}}{2e} - \frac{1}{2} K X^2 \right]
\]  \hspace{1cm} (17)
where \( \det (g_{ab}) = A^{M_1 \ldots M_{p+1}} \), with the following definition:

\[
A^{M_1 \ldots M_{p+1}} = \varepsilon^{a_1 \ldots a_{p+1}} \partial_{a_1} X^{M_1} \ldots \partial_{a_{p+1}} X^{M_{p+1}} = \{ X^{M_1}, X^{M_2}, \ldots, X^{M_{p+1}} \},
\]

(18)

where now \( a_k \) denote the world-volume indices. As before, by using the local scale invariance, and going to the gauge \( X^{+'} = 1 \), one derives the standard d-space-time dimensional action for the null p-brane. The equations of motion of the manifestly \( SO(d, 2) \) invariant tensionless p-brane reduce in the same gauge to the equations of motion in d-space-time dimensions.

Thus we conclude that the physics of both null strings and branes are invariant under the action of the d-dimensional conformal group \( SO(d, 2) \) and can be viewed from a \((d, 2)\)-dimensional point of view.

We have also gained a new perspective for these systems. The action (14) can now be gauge fixed in many ways as in the particle case [1]-[6]. This produces various field theories in a \( p+1 \) worldvolume with \( (d-1)+1 \) degrees of freedom. Since these have have a common origin, a duality exists between them, which amounts to gauge transformations involving the local symmetries. We examine one class of such gauges in the next section.

4 Tensionless p-Branes in AdS\(_{d-n} \times S^n\) Backgrounds

In this section we briefly discuss a class of gauges for the theory defined by the action (14) in analogy to the particle case in [5]. We work in the \( d+2 \) dimensional coordinate system labelled by \( M = (+', -', \mu, i) \) where the \( (d-1)+1 \) dimensions are divided into two sets: the first labelled by \( \mu \) describes \( (d-n-2)+1 \) dimensions including one time, and the second labelled by \( i \) describes \( n+1 \) spacelike dimensions. We choose the gauge that makes the length of the vector \( X^i(\tau, \vec{\sigma}) \) a constant \( R \) independent of \( \tau, \vec{\sigma} \), i.e. \( |X^i(\tau, \vec{\sigma})| = R \), and then solve the constraint \( X^2 = 0 \). The solution is conveniently parametrized by the two vectors \( \mathbf{u}^i(\tau, \vec{\sigma}) \) and \( x^\mu(\tau, \vec{\sigma}) \)

\[
X^i = R \frac{\mathbf{u}^i}{|\mathbf{u}|}, \quad X^\mu = \frac{R}{|\mathbf{u}|} x^\mu, \quad X^{+'} = |\mathbf{u}|, \quad X^{-'} = \frac{R^2 + \mathbf{u}^2 x^2}{2R^2 |\mathbf{u}|}.
\]

(19)

The induced metric is

\[
g_{ab} = \partial_a X^M \partial_b X^N \eta_{MN} = \frac{R^2}{u^2} \partial_a \mathbf{u}^i \partial_b \mathbf{u}^i + \frac{u^2}{R^2} \partial_a x^\mu \partial_b x^\mu
\]

(20)

\[
= \frac{R^2}{u^2} \partial_a x^\mu \partial_b x^\mu + \partial_a \mathbf{U}^i \partial_b \mathbf{U}^i
\]

(22)

where in the last line we have used the definition of the the length of the vector \( u = |\mathbf{u}| \) and the unit vector \( \mathbf{U}^i = \frac{\mathbf{u}^i}{|\mathbf{u}|} \) in \( n+1 \) dimensions that describes the sphere \( S^n \). The form of the induced metric \( g_{ab} \) shows that the background metric describes the curved space \( \text{AdS}_{d-n} \times S^n \) given by

\[
ds^2 = \frac{R^2}{u^2} (du)^2 + \frac{u^2}{R^2} (dx^\mu)^2 + (d\mathbf{U})^2.
\]

(23)
Therefore this choice of gauge corresponds to a null p-brane propagating in an AdS_{d-n} \times S^n background described by the lagrangian density

\[ L = \frac{1}{2e} \det(g_{ab}). \]  

This is of interest in the recent literature on the AdS/CFT duality [15, 17, 16]. Our form shows that the global symmetry is larger than the Killing symmetries of the background SO(d−n−1, 2) \otimes SO(n + 1); the full symmetry is SO(d, 2) which contains more than the Killing symmetries.

The important point is that both the flat null p-branes of (10) and the null AdS \times S p-branes for every n, appear as particular gauge choices of the same manifestly SO(d, 2) invariant action! This provides some examples of dual p-brane theories. We expect that the further exploration of other gauges and dualities would be quite interesting.

5 Higher Order Terms in the Lagrangian: the Rigid String and Branes

We can also discuss the presence of higher order terms in the manifestly SO(d, 2) invariant lagrangian we have used in the discussion of the null string. It turns out that the rigid string [18] naturally appears in this formulation.

First we recall that the action for the rigid string in d-space-time dimensions is given by

\[ S_{\text{rigid}} = \int d^2 \sigma \sqrt{-g} (K^i_{ab} K^{ia})^2. \]  

Here the second fundamental form \( K^i_{ab} \) is defined by the equation

\[ \partial_a \partial_b x^\mu = \Gamma^c_{ab} \partial_c x^\mu + K^i_{ab} n^\mu_i, \]  

where \( \Gamma^c_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{db} + \partial_b g_{da} - \partial_d g_{ab}) \) is the Christoffel symbol for the induced metric \( g_{ab} = \partial_a x^\mu \partial_b x^\mu \). Also, the vectors \( n^\mu_i \) satisfy

\[ n^\mu_i n^\mu_j = \delta_{ij}, \quad n^\mu_i \partial_a x^\mu = 0, \]  

where \( i = 1, \ldots, (d-2) \). The above action can be rewritten in many ways up to a total derivative [18]. For example

\[ S_{\text{rigid}} = \int d^2 \sigma \sqrt{-g} (\Delta(g) x^\mu)^2, \]  

where \( \Delta(g) x^\mu = -\frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{ab} \partial_b x^\mu) \) is the Laplacian defined with respect to the induced metric \( g_{ab} \). Another rewriting of the same action, which we find particularly useful for our purposes, is [18]

\[ S_{\text{rigid}} = \int d^2 \sigma \sqrt{-g} g^{ab} \partial_a t^{\mu \nu} \partial_b t^{\mu \nu}, \]
where \( t^{\mu \nu} = \frac{1}{\sqrt{-g}} \{ x^\mu, x^\nu \} \).

Now we can easily see that (29) is invariant under the action of the d-dimensional conformal group \( SO(d, 2) \). We can directly prove the invariance under dilatations (as originally observed in [18]) and the special conformal transformations (as recently observed in [19]) for the case of the action (28), if we recall the results from section 3.

Consequently, we can extend the action (29) to a space-time with one extra space and one extra time dimension to the manifestly \( SO(d, 2) \) invariant action

\[
S_{R1} = \int d^2 \sigma \left[ \sqrt{-g} g^{ab} \partial_a t^{MN} \partial_b t^{MN} - \frac{1}{2} K X^2 \right],
\]

with \( t^{MN} = \frac{1}{\sqrt{-g}} \{ X^M, X^N \} \) and \( g_{ab} = \partial_a X^M \partial_b X^N \eta_{MN} \). In other words, we claim that when (30) is evaluated in the gauge \( X^+ = 1, X^- = x^2/2 \) and \( X^\mu = x^\mu \) it reduces to the d-space-time dimensional action (29). The usual conformal transformations are generated again by \( L^{MN} \) of section 3. Again \( L^{MN} \) are naturally interpreted as Lorentz generators in \((d,2)\)-dimensional space-time.

We can immediately extend this result for the case of rigid branes. The manifestly \( SO(d, 2) \) invariant action for a rigid p-brane is

\[
S_{Rp} = \int d^{p+1} \sigma \left[ \sqrt{-g} g^{ab} \partial_a t^{M_1 \ldots M_{p+1}} \partial_b t^{M_1 \ldots M_{p+1}} - \frac{1}{2} K X^2 \right],
\]

with

\[
t^{M_1 \ldots M_{p+1}} = (-g)^{\frac{p+3}{2p+1}} \epsilon^{a_1 \ldots a_{p+1}} \partial_{a_1} X^{M_1} \ldots \partial_{a_{p+1}} X^{M_{p+1}}
\]

(32)

\[
= (-g)^{\frac{p+3}{2p+1}} \{ X^{M_1}, X^{M_2}, \ldots, X^{M_{p+1}} \}.
\]

(33)

Once again, in the gauge \( X^+ = 1, X^- = x^2/2 \) and \( X^\mu = x^\mu \) this action reduces to the d-space-time dimensional action.

Obviously the same observations can be made if we consider a linear combination of the manifestly \( SO(d, 2) \) invariant classical null string (or brane) and rigid string (or brane) actions in the same gauge.

It is tempting to speculate (as in [19]) that a supersymmetric version of these tensionless strings is relevant for the understanding of the dynamics of tensionless strings in 6-space-time dimensions. To make this speculation more concrete we need to extend our formalism to target-space supersymmetry, perhaps along the lines of [6, 12].

6  Outlook

In this letter we have made a few preliminary steps towards extending the formalism of Two-Time Physics for the case of string and p-brane \((p \geq 2)\) dynamics. We have found a natural action of the global \( SO(d, 2) \) symmetry in the case of several classical systems involving bosonic
strings and branes. The list includes the bosonic null string and branes, the null string moving in the AdS×S background and the bosonic rigid string and branes.

The next natural question is to ask whether we can extend our results to other examples of string and brane dynamics. For example, in the case of particle dynamics it is possible to obtain the massive relativistic particle as a gauge choice [5]. Is it analogously possible to relate the dynamics of strings and branes with tension (see footnote 3) to tensionless strings and branes such that they are all derived from the same two-time physics action? One positive indicator in this direction is the observation made in [20]: the tension of a bosonic string or brane can be generated if a kind of bosonic Wess-Zumino term is added to the action of the tensionless string. This mechanism has a natural supersymmetric extension. It would be interesting to see if this mechanism has a counterpart in the formalism of Two-Time Physics.

As we have pointed out in the introduction and in eqs.(1-5), one way to think about the gauging of the canonical \( Sp(2, R) \) group in the case of particle dynamics, is to identify \( Sp(2, R) \sim SO(1, 2) \) as the conformal group on the world-line (similarly \( OSp(n/2) \) for conformal supergravity [3]). Then various 0+1 (super) quantum gravity systems emerge as gauge fixed versions of the same (super)conformal gravity system with the manifest global \( SO(d, 2) \) symmetry. What is the analog of this viewpoint in the case of strings and p-branes? It would be natural to extend gravity to conformal gravity on the world-sheet (or world p+1-volume). For strings the world-sheet conformal group is \( SO(2, 2) \sim SO(1, 2)_l \times SO(1, 2)_r \sim Sp(2, R)_l \times Sp(2, R)_r \) which is tantalizing. More generally, for a p-brane conformal gravity would correspond to the gauging of \( SO(p + 1, 2) \). A general formulation of various string and p-brane models along these lines, with the aim of obtaining a general description of Two-Time Physics is currently under study.

Finally, it is of crucial importance to consider the space-time supersymmetric extensions of our results, perhaps by generalizing the novel formalism of [6, 12] to strings and branes.

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