Rainbow spacetime from a nonlocal gravitational uncertainty principle

Omar El-Refy1(a), Syed Masood2(b), Li-Gang Wang2(c) and Ahmed Farag Ali3(d)2

1 Department of Physics - Zewail City, Giza, Egypt
2 Department of Physics, Zhejiang University - Hangzhou 310027, China
3 Department of Physics, Faculty of Science, Benha University - Benha, 13518, Egypt

received 6 August 2020; accepted 7 September 2020
published online 17 December 2020

PACS 04.60.Bc – Phenomenology of quantum gravity

Abstract – The occurrence of spacetime singularities is one of the peculiar features of Einstein gravity, signalling limitation on probing short distances in spacetime. This alludes to the existence of a fundamental length scale in nature. On the contrary, the Heisenberg quantum uncertainty relation seems to allow for probing arbitrarily small length scales. To reconcile these two conflicting ideas in line with a well-known framework of quantum gravity, several modifications of Heisenberg algebra have been proposed. However, it has been extensively argued that such a minimum length would introduce nonlocality in theories of quantum gravity. In this letter, we analyze a previously proposed deformation of the Heisenberg algebra (i.e., \( p \rightarrow p(1 + \lambda p^{-1}) \)) for a particle confined in a box subjected to a gravitational field. For the problem in hand, such deformation seems to yield an energy-dependent behavior of spacetime in a way consistent with gravity’s rainbow, hence demonstrating a connection between non-locality and gravity’s rainbow.

It is very well known that a viable theory of quantum spacetime reconciling principles of quantum physics and Einstein gravity theory is one of the foremost goals of modern theoretical physics. Though both of these frameworks work very well in their respective regimes, however, a merger of principles of these two theories results in serious inconsistencies. One aspect of this contradiction can be seen readily from Heisenberg’s uncertainty relation and black hole phenomena. The famous position-momentum uncertainty relation \( \Delta x \Delta p \geq \hbar / 2 \) posits that there exists no lower limit on \( \Delta x \). This implies that arbitrarily small distances can be probed using sufficiently high energy scales. However, GR predicts that at sufficiently high energy, a black hole will be formed, which would inevitably prevent the probing process. In contradiction to quantum mechanics, a minimum length (given by the Schwarzschild radius of the black hole to be formed) must exist. Therefore, for any viable theory of quantum gravity reconciling both quantum mechanics and gravity, it must possess a minimum length. As a matter of fact, such a notion of minimum length manifests in almost all candidate theories of quantum gravity. For instance, in loop quantum gravity it is not possible to define an area below a certain minimum [1]. Also, in string theory, the fundamental string is the smallest possible probe. Hence, it is not possible to probe the geometry of spacetime below the string length scale, which introduces a minimum length to the theory [2–4]. In addition, a minimal length also manifests in many different approaches to quantum gravity such as asymptotically safe gravity [5], conformally quantized quantum gravity [6], and double field theory [7,8].

While such a minimum length is widely believed to be of the order of the Planck length, \( l_p \approx 10^{-35} \text{m} \), it may in fact be several orders of magnitude larger than \( l_p \) [9]. In case this minimum length is much larger than the Planck length, bounds could be imposed on its true value using current experimental data [10,11]. In fact, it has been previously suggested that Landau levels and the Lamb shift can be used to obtain bounds for such a length scale [12]. Also, it has been suggested that an optomechanical setup could be used to detect such a minimum length [13]. The existence of such a minimum length in theories of quantum gravity has many implications, the
most important of which is that it introduces nonlocality in the physical theories. In fact, it has been shown to be the case in almost all approaches to quantum gravity, including loop quantum gravity [14,15], perturbative and nonperturbative string theory [16–19] and the effective field theories based on it [19,20]. It is widely accepted that quantum gravitational effects would break locality at sufficiently high energy scales [21,22]. Such nonlocality can be incorporated into quantum gravitational theories via different deformations of the Heisenberg uncertainty principle, and hence the Heisenberg algebra [3]. A generic form of such an algebra is given by

\[ [x^i, p_j] = i\hbar [\delta^i_j + f(p^i_j)], \]

where \( f(p^i_j) \) is the suitable tensorial function that can be chosen from a specific type of generalized uncertainty principle (variously known as Gravitational Uncertainty Principle or Extended Uncertainty Principle). This results in modifications in the coordinate representation of the momentum operator. For example, for a quadratic generalized uncertainty principle (GUP), the modified operator representation for momentum \( p \) is

\[ \hat{p} \rightarrow i\hbar \delta_i^j (1 - \lambda\hbar^2 \partial_j \partial_i) \]

with \( \lambda \) as the deformation parameter arising from minimal length considerations. In other words, \( \hat{p} \) is the momentum representation at ultrahigh energy scales which reduces to the standard quantum representation \( p = i\hbar \partial \) in the low energy limit when \( \lambda \rightarrow 0 \). By modifying Heisenberg algebra via this momentum operator representation, we can study the effect of nonlocality using a semi-classical approach, where the gravitational field is treated classically and the matter is treated quantum mechanically. Here we use the specific deformation of the Heisenberg algebra where the aditional term on the right-hand side of the uncertainty relation is a properly scaled linear inverse momentum proposed in [23]. We demonstrate our result by invoking this modified momentum representation in the Schrödinger-Newton equation.

Let us consider an astrophysical body of mass \( M \) and radius \( R \). Let a probe in the form of a quantum particle (test particle) with energy \( E \) and mass \( m \) move on the surface of that astrophysical body. The Schrödinger-Newton equation for this system can be written as

\[ -\hbar^2 \frac{d^2 \psi}{2m \, dx^2} + V(x)\psi(x) = E\psi(x). \]

We construct the system in such a configuration that this situation can be approximated as a particle trapped in an infinite potential well, with gravitational force acting between the walls of that potential well. One can imagine this situation by choosing a particular region of spacetime near that astrophysical body. Further, this spacetime has nonlocal features embodied in it, manifested through the algebraic structure of the modified Heisenberg algebra. Thus, for this system, we can write the potential as

\[ V(x) = \begin{cases} -kx, & 0 \leq x \leq L, \\ 0, & \text{elsewhere}. \end{cases} \]

Here \( k = \frac{GmM}{R^2} \) and \( L \) is the width of the infinite potential well. Now this system can be deformed using the nonlocal deformation of the Heisenberg algebra as [23]

\[ p \rightarrow p(1 + \lambda p^{-1}) \rightarrow (p + \lambda), \]

where the parameter \( \lambda \) signifies the energy scale or the extent of quantum gravity effects. Using this nonlocal deformation, the modified Schrödinger-Newton equation (Schrödinger equation with gravitational coupling) for this system can be written as

\[ \frac{d^2 \psi}{dx^2} + 2i\alpha \frac{d\psi}{dx} + \beta x + E \psi = 0, \]

where \( \alpha = \frac{\lambda}{k} \) and \( \beta = \frac{2mk}{\hbar^2} \). Now we introduce \( u(x) \), which is related to \( \psi(x) \) by

\[ \psi(x) = e^{-i\alpha x} u(x). \]

Thus, we can obtain the following equation for the system;

\[ \frac{d^2 u(x)}{dx^2} + \beta x u(x) = 0. \]

Furthermore, we let \( s = -\beta x + E \) and obtain

\[ \frac{d^2 u(s)}{ds^2} - su(s) = 0. \]

Thus, we can write the solution to this equation as

\[ u(s) = C_1 Ai(s) + C_2 Bi(s), \]

where \( Ai(s) \) and \( Bi(s) \) are the Airy functions of first and second kinds, respectively, and \( C_1 \) and \( C_2 \) are two constants. Now we obtain the following solution for \( \psi(x) \):

\[ \psi(x) = e^{-i\alpha x} \left\{ C_1 Ai \left[ -\beta x + E \right] \right\} + C_2 Bi \left[ -\beta x + E \right]. \]

Using the boundary conditions, \( \psi(0) = 0 \) and \( \psi(L) = 0 \) (as it is approximated by an infinite potential well, with gravitational potential inside it), we obtain from above

\[ \psi(x) = Ce^{-i\alpha x} \left\{ Bi \left[ -\beta x + E \right] Ai \left[ -\beta x + E \right] \right\} - Ai \left[ -\beta x + E \right] Bi \left[ -\beta x + E \right], \]

where \( C = \frac{C_1}{B_1 |\beta|^{-\frac{1}{3}}}. \)
We now consider the limiting case, where \( E \to \infty \) (i.e., for extremely high energies), which would imply that the arguments of the Airy functions would approach \(-\infty\). In this limit, the two functions \( Ai(x) \) and \( Bi(x) \) exhibit a sinusoidal behavior, namely

\[
Ai(x) \sim \frac{1}{\sqrt{2\pi x^3}} \sin \left( \frac{2}{3} |x|^\frac{3}{2} + \frac{\pi}{4} \right), \quad (11)
\]

\[
Bi(x) \sim \frac{1}{\sqrt{2\pi x^3}} \cos \left( \frac{2}{3} |x|^\frac{3}{2} + \frac{\pi}{4} \right), \quad (12)
\]
as \( x \to -\infty \). Using this asymptotic behavior, we obtain

\[
\psi(x) \sim Ce^{-i\alpha x} \frac{1}{2\pi} \left[ \frac{k}{\beta^2 E(x + \frac{\beta}{k})} \right]^{\frac{1}{4}} \times \left[ \cos \left( \frac{2}{3} \beta E \frac{2}{k} - \frac{\pi}{4} \right) \sin \left( \frac{2}{3} \beta \left( x + \frac{E}{k} \right) \right)^{\frac{1}{4}} + \frac{\pi}{4} \right] - \sin \left( \frac{2}{3} \beta E \frac{2}{k} - \frac{\pi}{4} \right) \cos \left( \frac{2}{3} \beta \left( x + \frac{E}{k} \right) \right) + \frac{\pi}{4} \right]. \quad (13)
\]

Now, applying the second boundary condition (i.e., \( \psi(L) = 0 \)) to the above asymptotic expression and re-arranging, we obtain

\[
\cos \left( \frac{2}{3} \beta E \frac{2}{k} + \frac{\pi}{4} \right) \sin \left( \frac{2}{3} \beta L + \frac{2}{3} \beta E \frac{2}{k} + \frac{\pi}{4} \right) = \sin \left( \frac{2}{3} \beta E \frac{2}{k} + \frac{\pi}{4} \right) \cos \left( \frac{2}{3} \beta L + \frac{2}{3} \beta E \frac{2}{k} + \frac{\pi}{4} \right) \quad (14)
\]
in the limit \( E \to \infty \). Now, since the above equation holds for all \( \beta, E, k \) and \( L \) that lie in the asymptotic region we assume, we must have

\[
\frac{2}{3} \beta E \frac{2}{k} + \frac{\pi}{4} = \frac{2}{3} \beta L + \frac{2}{3} \beta E \frac{2}{k} + \frac{\pi}{4} + 2\pi n, \quad (15)
\]
where \( n = 0, \pm 1, \pm 2, \pm 3, \ldots \). Now, noting that \( \beta, E, k \) and \( L \) are all positive, we can express \( L \) as

\[
L = \left[ \left( \frac{E}{k} \right)^{\frac{2}{3}} + \frac{3\pi n}{\beta^2} \right]^{\frac{2}{3}} - \frac{E}{k}, \quad (16)
\]
where \( n \in \mathbb{Z} \), an integer. Next, we note that, for \( L \) to be of positive-definite value, one must have

\[
\left[ \left( \frac{E}{k} \right)^{\frac{2}{3}} + \frac{3\pi n}{\beta^2} \right]^{\frac{2}{3}} > \frac{E}{k}, \quad (17)
\]
which, upon using \( \beta = \frac{2mk}{\hbar^2} \) and \( \hbar = 1 \), simplifies to

\[
\left( 1 + \frac{3\pi n}{(2m)^2 E^2} \right)^{\frac{2}{3}} > 1. \quad (18)
\]

Obviously, (18) is not satisfied for zero and negative values of \( n \), which rules them out as nonphysical states. Thus, only \( n = 1, 2, 3, \ldots \) are allowed. Now, since we are addressing an ultra-high energy limit, (16) could take a much simpler form. To elucidate, using \( \beta = \frac{2mk}{\hbar^2} \) and \( \hbar = 1 \), we re-write it as

\[
L = \frac{E}{k} \left( 1 + \frac{3\pi n}{(2m)^2 E^2} \right)^{\frac{2}{3}} - \frac{E}{k}, \quad (19)
\]

Obviously, the second term inside the brackets is extremely small, which enables us to Taylor-expand the whole bracket up to first order, which, upon inserting the value of \( k \) further simplifies to

\[
L = \frac{n\pi R^2}{GM\sqrt{2m^3E}}. \quad (20)
\]
The above relation indicates that the box length is quantized in an energy-dependent way, whilst the gravity constant \( G \) enters this quantization scheme. It puts a limitation on the way we find the particle in the box. For a box length different than what the above relation implies, there is no meaning to the existence of a particle. It must be emphasized here that the box length is purely a geometrical aspect and the box under consideration here is a hypothetical region of spacetime. If this length is explicitly energy dependent as seen in the above relation, then the geometry of spacetime as seen by the probe depends on its energy.

In view of the above result, it can be shown that the value of Newton’s constant becomes dependent on the energy of the spacetime probe through the relation (20). Hence, any metric geometry shall pick up the energy dependence. This energy-dependent Newton’s constant signifies the conjecture that the effective gravitational coupling might depend on the energy scale and satisfy a condition of renormalization group flow. This is what is essentially entailed by the theory of rainbow gravity. To put it simply, a particle of energy \( E \) propagating in such a spacetime sees a geometry that depends upon its own energy \( E \). This results in a spacetime geometry characterized by the metric [24]

\[
g_{\mu\nu}(E) = \eta^{\mu\nu} e_{\mu}(E) \otimes e_{\nu}(E), \quad (21)
\]
where \( e_{\mu} \) and \( e_{\nu} \) are orthonormal frame fields and \( E \) is the energy of the probing particle. In fact, such a behavior of spacetime geometry can be motivated from the extension of the doubly special relativity [25] on curved spacetime background, where the energy-momentum dispersion relation modifies as

\[
E^2 f^2(l_{pl} E) - p^2 g^2(l_{pl} E) = m_0^2. \quad (22)
\]
Here \( f(l_{pl} E) \) and \( g(l_{pl} E) \) are called rainbow functions that depend on the energy of the probing particle with \( l_{pl} \) as Planck length. In this formulation, Einstein field equations of general relativity (GR) (in \( c = 1 \) units) read as

\[
G_{\mu\nu}(E) - g_{\mu\nu} \Lambda(E) = 8\pi G(E)T_{\mu\nu}. \quad (23)
\]
Note the energy dependence of Newton’s constant, $G(E)$, and of the cosmological constant, $\Lambda(E)$. Such effects are much more pronounced near the quantum gravity scale, for $\lambda$ in (2) is a deformation parameter, which essentially deforms the theory significantly near this energy scale, while preserving the familiar classical limit of GR at low energies, implying both $f(l_{pl}E) \to 1$ and $g(l_{pl}E) \to 1$ at low energies $E$. However, as the minimal measurable length in string theory can be several orders of magnitude above the Planck scale, $l_{pl} = g_1^{-1/4} l_s^s[9]$, this energy can also be several orders of magnitude below the Planck energy (as here $\beta$ would also be several magnitudes above the Planck scale). This energy could be bounded by astrophysical observations, and it would be interesting to analyze such observations, and obtain certain bounds on the parameters that signify the energy dependence of the metric.

Meanwhile, to appreciate the impact of nonlocality on particle dynamics and spacetime geometry, wave functions for different values of $n$ and $\lambda$ are plotted in fig. 1 and fig. 2. As evident from fig. 2, the deformation parameter, $\lambda$, controls the allowable box lengths (acting as the probing energy scale indicator). However, in the scenario of ultra-high energy regime, the energy dependence of the box length $L$ is absorbed into the particle energy $E$ as depicted in (20). Though the form of sinusoidal nature of the box wave functions remain unaffected compared to the conventional box physics, as can be seen from fig. 1, we however observe the full of the amplitude for the particle wave functions as the parameter $\lambda$ increases. This is tantamount to saying that a particle becomes more and more confined in a particular region of spacetime as the parameter $\lambda$ increases. However, beyond a certain value of $\lambda$, the wave function might lose its meaning, given the occurrence of spacetime singularities or the existence of a minimal measurable length scale. The above result is potentially alluding to the breakdown of our present theories to explain the nature of spacetime at the quantum gravity scale.

Several quantum gravity approaches can be taken to justify gravity’s rainbow emergence from nonlocality, such as the string theory. It is very known that due to renormalization group flow in a quantum field theory, the coupling constants flow and thus depend on the scale at which the theory is probed [26,27]. However, the scale at which a theory is probed depends on the energy of the probe. Therefore, the coupling constants depend explicitly on the scale at which the theory is probed and implicitly on the energy of the probe used to probe the theory. Since string theory can be analyzed as a two-dimensional conformal field theory, where the target space metric is regarded as a matrix of coupling constants of the theory, coupling constants should in turn depend on the scale at which the theory is being probed. And since the scale at which the theory is being probed is equivalent to the energy of the probe, it can be argued from string theory that the geometry of spacetime should depend on the energy of the probe [24,28]. Arguably, such energy-dependent spacetime deformation should be manifested in various other approaches to quantum gravity. For instance, gravity’s rainbow has been motivated from the results obtained in loop quantum gravity and $\kappa$-deformed Minkowski spacetime [29,30]. In addition, deformations of the energy-momentum dispersion relation appear in the Horava-Lifshitz gravity [31,32], discrete spacetime [33], models based on the string field theory [34], spacetime foam [35], spin-network [36], noncommutative geometry [37,38], and ghost condensation [39]. It may also be noted that the Greisen-Zatsepin-Kuzmin limit (GZK limit), with which the Pierre Auger Collaboration and the High Resolution Fly’s Eye (HiRes) experiment are consistent [40], has been used to argue for such a deformation of the energy-momentum dispersion relation [41,42]. In fact, several different tests have been proposed to experimentally verify this idea [43]. In addition, an explanation of the hard spectra of gamma ray bursts has been previously proposed [35]. Hence, there are strong motivations...
of both gravity’s rainbow and nonlocality from numerous approaches to quantum gravity. In this work, we have briefly demonstrated one such way by which gravity’s rainbow could emerge from the nonlocality in spacetime.

***

L-GW would like to acknowledge support from the Zhejiang Provincial Natural Science Foundation of China under Grant No. LD18A040001 and the National Natural Science Foundation of China under Grant Nos. 11674284 and 11974309.

REFERENCES

[1] Rovelli C., Living Rev. Relativ., 1 (1998) 1.
[2] Amati D., Ciafaloni M. and Veneziano G., Phys. Lett. B, 216 (1989) 41.
[3] Kempf A., Mangano G. and Mann R. B., Phys. Rev. D, 52 (1995) 1108.
[4] Maggiore M., Phys. Lett. B, 304 (1993) 65.
[5] Percacci R. and Vacca G. P., Quantum Grav., 27 (2010) 245026.
[6] Padmanabhan T., Class. Quantum Grav., 4 (1987) L107.
[7] Hull C. and Zwiebach B., JHEP, 09 (2009) 099.
[8] Marotta V. E., Pezzella F. and Vitale P., J. High Energy Phys., 2018 (2018) 185.
[9] Hossenfelder S., Living Rev. Relativ., 16 (2013) 2.
[10] Das S. and Vagenas E. C., Phys. Rev. Lett., 101 (2008) 221301.
[11] Das Saurya and Vagenas Elias C., Phys. Rev. Lett., 104 (2010) 119002.
[12] Ali A. F., Das S. and Vagenas E. C., Phys. Rev. D, 84 (2011) 044013.
[13] Pikovski I. et al., Nat. Phys., 8 (2012) 393.
[14] Calcagni G. et al., Class. Quantum Grav., 29 (2012) 105005.
[15] Dzierzak P. et al., The minimum length problem of loop quantum cosmology, arXiv:0810.3172.
[16] Chang L. N. et al., Phys. Rev. D, 65 (2002) 125028.
[17] Sándor Benczik et al., Phys. Rev. D, 66 (2002) 026003.
[18] Douglas M. R. et al., Nucl. Phys. B, 485 (1997) 85.
[19] Dodelson M. and Silverstein E., Phys. Rev. D, 96 (2017) 066010.
[20] Calcagni G. and Modesto L., J. Phys. A: Math. Theor., 47 (2014) 355402.
[21] Ghosh S. and Raju S., Phys. Rev. D, 96 (2017) 066033.
[22] Papadodimas K. and Raju S., Phys. Rev. Lett., 112 (2014) 051301.
[23] Masood S. et al., Phys. Lett. B, 763 (2016) 218.
[24] Magueljo J. and Smolin L., Class. Quantum Grav., 21 (2004) 1725.
[25] Amelino-Camelia G., Int. J. Mod. Phys. D, 11 (2002) 35.
[26] Rosten O. J., Phys. Rep., 511 (2012) 177.
[27] Warner N. P., Class. Quantum Grav., 17 (2000) 1287.
[28] Ali A. F., Faizal M. and Khalil M. M., Phys. Lett. B, 743 (2015) 295.
[29] Amelino-Camelia G. et al., Int. J. Mod. Phys. A, 12 (1997) 607.
[30] Amelino-Camelia G., Living Rev. Relativ., 16 (2013) 5.
[31] Hořava P., Phys. Rev. D, 79 (2009) 084008.
[32] Hořava P., Phys. Rev. Lett., 102 (2009) 161301.
[33] ’t Hooft G., Class. Quantum Grav., 13 (1996) 1023.
[34] Kostelecký V. A. and Samuel S., Phys. Rev. D, 39 (1989) 683.
[35] Amelino-Camelia G. et al., Nature, 393 (1998) 763.
[36] Gambini R. and Pullin J., Phys. Rev. D, 59 (1999) 124021.
[37] Carroll S. M. et al., Phys. Rev. Lett., 87 (2001) 141601.
[38] Faizal M., Mod. Phys. Lett. A, 27 (2012) 1250075.
[39] Faizal M., J. Phys. A: Math. Theor., 44 (2011) 402001.
[40] Abraham J. et al., Phys. Lett. B, 685 (2010) 239.
[41] Greisen K., Phys. Rev. Lett., 16 (1966) 748.
[42] Zatsepin G. T. and Kuzmin V. A., JETP Lett., 4 (1966) 78.
[43] Ali A. F. and Khalil M. M., EPL, 110 (2015) 20009.