Kondo Problem and Related One-Dimensional Quantum Systems:
Bethe Ansatz Solution and Boundary Conformal Field Theory

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We review some exact results on Kondo impurity systems derived from Bethe-ansatz solutions and boundary conformal field theory with particular emphasis on universal aspects of the phenomenon. The finite-size spectra characterizing the low-energy fixed point are computed from the Bethe-ansatz solutions of various models related to the Kondo problem. Using the finite-size scaling argument, we investigate their exact critical properties. We also discuss that a universal relation between the Kondo effect and the impurity effect in one-dimensional quantum systems usefully expedites our understanding of these different phenomena.

KEYWORDS: Kondo effect, Bethe ansatz, boundary conformal field theory

1. Introduction

A magnetic impurity introduced into metals drastically changes the local electronic structure of the host systems through quenching processes of the local magnetic moment; the phenomenon, known as the Kondo effect, is one of the most well-understood many-body problems in condensed matter physics.1,2 This old issue is still a topic of great interest in connection with modern subjects in the field such as the exploration of non-Fermi liquid states in heavy fermion compounds and the application to nanostructure devices like quantum dots.

In the development of theoretical investigations, the Bethe ansatz (BA) exact solutions have played an important role, yielding numerous non-perturbative results on thermodynamic quantities.3,4 The BA methods successfully elucidated strong-coupling aspects of the phenomenon. A decade ago, another important breakthrough was achieved by Affleck and Ludwig in the application of boundary conformal field theory (CFT) to the Kondo problem.5 CFT is a powerful tool to analyze exact low-energy critical properties including multipoint correlation functions, which are not obtained by the BA methods. In particular, the CFT analysis of the multi-channel Kondo problem revealed anomalous critical properties of the non-Fermi liquid state.5 These two theoretical methods have complementarily advanced our understanding on the Kondo effect and the related impurity problems in one-dimensional (1D) quantum systems. In this paper, we give a brief overview of some recent developments in these directions.

The CFT analysis is based upon an effective low-energy theory, which may be derived from the microscopic lattice Hamiltonians. In the Kondo problem, however, the derivation of the low-energy effective field theory is not straightforward because of its strong-coupling character. Affleck and Ludwig postulated that the universality class at the low-energy fixed point belongs to the SU(2) Kac-Moody CFT. The validity of this CFT analysis was established by the consistency with the results obtained by other methods. Using the BA solutions of various models related to the Kondo problem, we can directly calculate the exact finite size spectra (FSS), which characterize the universality class, and compare them with the results of CFT. In §2, we review some results related to this issue by exploiting the single-channel Anderson model, the multi-channel Kondo model, and the two-channel Anderson model. In §3, we then discuss various critical properties of these systems by applying the CFT analysis to the exact FSS obtained from the BA solutions.

Furthermore, these elaborated techniques revealed universal features of the phenomenon, clarifying a profound relation between the Kondo impurity problem and the issue of impurity effects in 1D quantum systems. For instance, in §4, we will discuss non-magnetic impurity effects in quantum spin chains with paying special attention to its similarity to the underscreening Kondo effect.

2. Exact Finite Size Spectra and Boundary CFT

As was elucidated by Affleck and Ludwig, the critical properties inherent in the Kondo problem are described by boundary CFT.5 Before discussing the Kondo problem, we briefly summarize the results of boundary CFT.

In general, critical behavior near the boundary is characterized by the surface critical exponent $x_s$ which controls the asymptotic behavior of the correlation function, $\langle \phi(t)\phi(0) \rangle \sim 1/t^{2x_s}$ ($t \to \infty$). The surface exponent $x_s$ is generally different from bulk exponents. We can regard the surface critical exponent $x_s$ as the conformal dimension of some boundary scaling operator. Decomposing a scaling field $\phi$ into the holomorphic part and the anti-holomorphic part, which is defined by the analytic continuation of the holomorphic part, $\phi(z, \bar{z}) = \phi_L(z)\phi_R(\bar{z})$, we have $\phi(z, \bar{z}) = \phi_L(x+iy)\phi_L(x-iy) \sim y^{-2\Delta + \Delta_b}\phi_L(x)$. Here $\Delta$ is the bulk conformal dimension of $\phi_L$, and $\Delta_b$ is the conformal dimension of a boundary operator $\phi_b$. Then in the vicinity of the boundary, we have the correlation function of the scaling operator,

$$\langle \phi(z, \bar{z})\phi(z', \bar{z'}) \rangle \sim \langle \phi_b(x)\phi_b(x') \rangle \sim |x - x'|^{-2\Delta_b}. \quad (1)$$

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According to the finite size scaling analysis in CFT, the boundary scaling dimension $\Delta_b$ for a given boundary operator in 1D critical systems with open ends enters in the FSS,

$$ E = E_0 - \frac{\pi v c}{2L} + \pi v \left(\Delta_b + n\right). $$

(2)

Here $l$ is the linear size of the finite system. Thus, critical exponents are derived from the FSS. In the case of the usual Kondo effect (the completely screened case), the exponents of correlation functions are characterized by those of the canonical Fermi liquid. In contrast, for the overscreening case realized in the multi-channel Kondo model, anomalous dimensions appear, signaling the non-Fermi liquid state. In the following, we show the results of the FSS for some exactly solvable models related to the Kondo problem.

2.1 Anderson model

We first discuss how critical properties of the Kondo impurity problem are derived from the FSS, by considering the single-impurity Anderson model as an example.\(^7\) The model describes free conduction electrons coupled with correlated $d$-electrons at the impurity site via the resonant hybridization $V$. The Hamiltonian is given by,

$$ H = \sum_{k\sigma} \varepsilon_k c_{k\sigma}^\dagger c_{k\sigma} + V \sum_{k\sigma} (c_{k\sigma}^\dagger d_{\sigma} + d_{\sigma}^\dagger c_{k\sigma}) $$

$$ + \varepsilon_d \sum_{\sigma} d_{\sigma}^\dagger d_{\sigma} + Ud_{\uparrow}^\dagger d_{\uparrow}d_{\downarrow}^\dagger d_{\downarrow} $$

(3)

with standard notations. The Hamiltonian is exactly solvable in terms of the BA method under the condition that the density of states for conduction electrons is constant around the Fermi energy.\(^3,4\) We can obtain the FSS by using standard techniques in BA,

$$ E = E_0 + \frac{1}{L} E_1 + \frac{1}{L^2} E_2 + O(1/L^3), $$

(4)

$$ \frac{1}{L} E_1 = \frac{2\pi v}{L} \frac{1}{4} \left[ (\Delta N_h - 2\Delta F)^2 + n_{c}^+ \right] $$

$$ + \frac{2\pi v}{L} \left( (\Delta S_h)^2 + n_{s}^+ \right), $$

(5)

$$ \frac{1}{L^2} E_2 = \frac{2\pi v}{L^2} \frac{\chi_{c_{\sigma}}^{imp}}{\chi_{c_{\sigma}}^h} \left[ (\Delta N_h)^2 + n_{c}^+ \right] $$

$$ + \frac{2\pi v}{L^2} \frac{\chi_{d_{\sigma}}^{imp}}{\chi_{d_{\sigma}}^h} \left[ (\Delta S_h)^2 + n_{s}^+ \right], $$

(6)

where $\chi_{c_{\sigma}}^{imp}$ and $\chi_{c_{\sigma}}^h$ are the charge (spin) susceptibility of impurity and host electrons, respectively. $\Delta N_h$ ($\Delta S_h$) is the deviation of the total number (magnetization) of host electrons from the ground-state values. $n_{c}^+$ is a non-negative integer, which features the conformal tower structure. The expression for the $1/L$ correction coincides with the fusion hypothesis, supporting the boundary CFT analysis. The $1/L^2$-term (6) characterizes non-universal correlation effects depending on the strength of $U$. Namely, the local spin and charge susceptibilities can be obtained from the $1/L^2$ corrections of the FSS.

To read off boundary scaling dimensions from eqs.(4) and (2), we replace $L$ with $2l$, since $L$ is defined as periodic length of the system. Note that the finite-size spectrum for bulk electrons in eq.(5) involves a non-universal phase shift $\delta F$. The phase shift $\delta F$ is regarded as the chemical-potential change due to the impurity. This means that the effect of the phase shift amounts to merely imposing twisted boundary conditions on conduction electrons.\(^5\) Therefore, when we derive the dimension of a scaling operator associated with conduction electrons, we should discard the $\delta F$ dependence in eq.(5) by redefining $\Delta N_h - 2\delta F/\pi \rightarrow \Delta N_h$. Hence the scaling dimension $x$ of the conduction electron field is obtained by taking the quantum numbers

$$ \Delta N_h = 1, \quad \Delta S_h = 1/2, $$

(7)

resulting in $x = 1/2$. Thus we can see that the single-electron Green function $\langle c_{\sigma}(t)c_{\sigma}^\dagger(0) \rangle \sim 1/t^\eta$ has the canonical exponent $\eta = 2x = 1$, in accordance with the fact that the system is described by the strong-coupling fixed point of the local Fermi liquid.\(^5,9,10\)

On the other hand, we can extract another interesting information from the FSS (5), i.e. the critical behavior related to the orthogonality catastrophe. To see this explicitly, let us consider the time-dependent Anderson model in which the hybridization $V = 0$ for $t < t_0$, and then $V$ is switched on at $t = 0$. This system shows the orthogonality catastrophe related to the Fermi edge singularity. We note that the model is defined on a semi-infinite plane with the boundary at $x = 0$. Using the conformal transformation $t + ix = \exp[(\pi/4)(t' + ix')]$, we map the semi-infinite plane to a strip with width $l$, in which the free boundary condition is imposed at $x' = l$, and, contrastively, the non-trivial Kondo boundary condition is imposed at $x' = 0$. We recall that the FSS (5) is computed under these boundary conditions. Thus, critical exponents which govern the long-time behavior of the correlation functions characterizing the orthogonality catastrophe can be obtained from eq.(5). In this case, the phase shift in (5) becomes a key quantity which controls the critical exponent. Actually, critical exponents related to the X-ray problem can be read off from the above FSS with keeping the dependence on the phase shift intact.\(^7,8\)

This is because a sudden potential change occurs in X-ray photoemission (or absorption) experiments. For example, the critical exponent $\eta$ of the single-particle Green function $\langle c_{\sigma}(t)c_{\sigma}^\dagger(0) \rangle \sim 1/t^\eta$ ($0 < t_0 \ll t$) for the model with a sudden potential change is obtained as\(^8\)

$$ \eta = 1 - 2\frac{\delta F}{\pi} + 2 \left( \frac{\delta F}{\pi} \right)^2. $$

(8)

This is just the exponent which governs the long-time behavior of the overlap integral between the initial and final states in the X-ray absorption problem. It is thus seen that the FSS (5) contains the information about orthogonality catastrophe in addition to the local Fermi-liquid properties.

2.2 Multi-channel Kondo model

The argument in the previous section is also applicable to the multi-channel Kondo model.\(^11,12\) In this model, conduction electrons with $n$-channels ($l = 1, 2, \cdots, n$) couple with the impurity spin $S$ antiferromagnetically.

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\(^5\) S. Fujimoto and N. Kawakami, J. Phys. Soc. Jpn. 59, 4391 (1990).

\(^7\) S. Fujimoto and N. Kawakami, Phys. Rev. Lett. 66, 3031 (1991).

\(^9\) S. Fujimoto and N. Kawakami, Phys. Rev. Lett. 67, 3160 (1991).

\(^10\) S. Fujimoto and N. Kawakami, J. Phys. Soc. Jpn. 61, 2481 (1992).

\(^11\) S. Fujimoto and N. Kawakami, J. Phys. Soc. Jpn. 57, 3024 (1988).

\(^12\) S. Fujimoto and N. Kawakami, J. Phys. Soc. Jpn. 58, 2420 (1989).

\(^13\) S. Fujimoto and N. Kawakami, J. Phys. Soc. Jpn. 59, 4391 (1990).
In the case of \( n > 2S \), the overscreening Kondo effect occurs, yielding the non-Fermi liquid ground state. The Hamiltonian reads

\[
H = \sum_{k}\varepsilon_{k}c_{k\sigma}^{\dagger}c_{k\sigma} + J\sum_{k}\sum_{\sigma,\sigma'}c_{k\sigma}^{\dagger}(\sigma\sigma' \cdot S)c_{k'\sigma'},
\]

where \( c_{k\sigma} \) is the annihilation operator for electrons with orbital index \( m \) and spin \( \sigma \). It is noted that although the multi-channel Kondo model is BA-solvable,\(^{12,13}\) we cannot apply standard techniques to the calculation of the FSS in this model, particularly, in the overscreening case \( n > 2S \). This is due to the fact that the BA solution to the multi-channel model takes the form of the so-called string solution even for the ground state.\(^{12,13}\) It has been known that the string solution is valid only in the thermodynamic limit. Thus, a naive application of finite-size techniques\(^{14}\) fails, giving only the Gaussian part of the spectrum for the spin sector. Since the spin sector of the multi-channel Kondo model is described by the level-\( n \) SU(2) Kac Moody theory with the central charge \( c_{\text{ZW}} = 3n/(2+n) \),\(^{5}\) the \( Z_{n} \) parafermion sector with \( c = c_{\text{ZW}} - c_{\text{Gaussian}} = (n-1)/(n+2) \) is not accessible by standard analytical techniques. To overcome the difficulty, we propose an alternative method to calculate the FSS analytically with paying a special attention to the nontrivial \( Z_{n} \) parafermion part.\(^{15}\) To this end, we extract the \( Z_{n} \) sector of the BA equations for the multi-channel model in the overscreening case \( n > 2S \),

\[
\tilde{f}_{m} + A_{m,k}^{(n)}f_{k} = A_{m,n-1}^{(n)}s\ast\tilde{f}_{n} + \frac{1}{L}A_{m,2}\ast s/(1+J), \tag{10}
\]

where \( \ast \) means the convolution. Here \( f_{m} \) and \( \tilde{f}_{m} \) are the distribution functions for rapidities of particle and hole excitations, respectively. The kernel is defined as \( A_{m,k}^{(n)} = (C_{m,k})^{-1}m \) with

\[
C_{m,k}^{(n)} = \delta_{m,k} - s(\lambda)(\delta_{m,k+1} + \delta_{m,k-1}), \tag{11}
\]

and \( s(\lambda) = 1/(2\cosh(\pi\lambda)) \). Note that the second term of the right-hand side of eq.(10) reflects the existence of the impurity, and appears only for \( n > 2S \). The bulk part of eq.(10) gives the exact spectrum of the \( Z_{n} \) parafermion theory. We recall the fact that this BA equation is also derived from the S-matrix for “physical particles” of the \( Z_{n} \) model.\(^{16}\) Since the \( Z_{n} \) model can be described by the restricted solid-on-solid (RSOS) model in the regime I/II,\(^{17}\) the corresponding S-matrix is given by the face weight in the RSOS model.\(^{18}\) A remarkable point for the overscreening model is that the interaction between “physical particles” and the impurity is described by the S-matrix of multi-kinks,\(^{16}\) which is given by the fusion of the face weights of the RSOS model with the fusion level \( p = n - 2S \). Therefore, the spectrum of eq.(10) is essentially determined by the RSOS model coupled with the impurity, of which the FSS can be obtained analytically by using the functional equation method.\(^{15}\) By some technical reasons, we restrict our analysis to the case of \( p \equiv n - 2S = 1 \) here. After some manipulations, we end up with the FSS of the RSOS model coupled with the impurity,\(^{15}\)

\[
E_{\text{RSOS}} = \frac{2\pi v}{L} \left( \frac{j(j+1)}{n+2} - \frac{(m+p)^2}{4n} \right) \quad \text{const.} \quad \tag{12}
\]

where \( m = 2j \mod 2 \), and \( j = 0,1/2,1,...,n/2 \). It is seen that the spectrum fits in with \( Z_{n} \) parafermion theory, and only the selection rule for quantum numbers is changed by the impurity effect, \( m \rightarrow (m+p) \). Adding eq.(12) to the Gaussian spectrum, we arrive at the total FSS,

\[
E = \frac{2\pi v}{L} \left( \frac{(Q-n)^2}{4n} + \frac{j(j+1)}{n+2} \right) \quad \text{(orbital part)}
+ n_{Q} + n_{s} + n_{f} \quad \tag{13}
\]

where \( j = |j-p/2| \) is the new quantum number. Here \( Q \) and \( j \) are the charge and spin quantum numbers for free electrons without the Kondo impurity. \( n_{Q}, n_{s}, \) and \( n_{f} \) are non-negative integers characterizing the conformal tower. We can say that the effect due to the Kondo impurity is merely to modify the selection rule for quantum numbers of spin excitations by \( j \rightarrow j \), which indeed results in non-Fermi liquid state. This is the essence of the Kac-Moody fusion hypothesis proposed by Affleck and Ludwig.\(^{5}\) The above result may be a microscopic description of the fusion hypothesis for the multi-channel Kondo model.

2.3 Two-channel Anderson model

Recently, the BA exact solution of the two-channel Anderson model was obtained by Bolech and Andrei.\(^{20}\) This model was proposed to describe a mixed valence regime of the Uranium compounds such as UBe\(_{13}\) that have both magnetic and quadrupolar degrees of freedom.\(^{21}\) The model consists of the impurity \( f \)-electrons, which are assumed to be either in the 5f\(^{3}\) configuration or in the 5f\(^{2}\) configuration. The former state is the \( \Gamma_{6} \) magnetic spin doublet, and the latter state is the \( \Gamma_{3} \) quadrupolar doublet. Conduction electrons, which carry both magnetic and quadrupolar quantum numbers, hybridize with the \( f \)-electrons. In the strong coupling limit, the model Hamiltonian reads,

\[
H = \sum_{\alpha}\int dx\varepsilon_{\alpha}(x)(-\frac{\partial}{\partial x})c_{\alpha}(x) + \varepsilon_{s}f_{\sigma}^{\dagger}f_{\sigma} + \varepsilon_{q}\sum_{\alpha}b_{\alpha}^{\dagger}b_{\alpha} + V\sum_{\alpha}\int dx\delta(x)(f_{\sigma}^{\dagger}b_{\alpha}c_{\alpha}(x) + c_{\alpha}^{\dagger}(x)b_{\alpha}^{\dagger}f_{\sigma}) \tag{14}
\]

Here \( c_{\alpha}(x) \) (\( c_{\alpha}^{\dagger}(x) \)) is the annihilation (creation) operator of a conduction electron with spin \( \sigma = \uparrow \downarrow \) and quadrupolar quantum number \( \alpha = \pm \). \( \bar{\alpha} \) is the conjugate representation of \( \alpha \). \( f_{\sigma} \) (\( f_{\uparrow}^{\dagger} \)) is the annihilation (creation) operator for the \( \Gamma_{6} \) magnetic spin doublet on the impurity site, and \( b_{\alpha} \) (\( b_{\alpha}^{\dagger} \)) is the annihilation (creation) operator for the \( \Gamma_{3} \) quadrupolar doublet. Under the assumption that the Hilbert space is restricted to these two configurations, the constraint \( \sum_{\alpha}f_{\alpha}^{\dagger}f_{\alpha} + \sum_{\alpha}b_{\alpha}^{\dagger}b_{\alpha} = 1 \) is
imposed. The system exhibits crossover between magnetic and quadrupolar Kondo effects depending on the parameter \( \varepsilon = \varepsilon_s - \varepsilon_q \); for \( \varepsilon - \mu \ll -\Delta = \pi \rho V^2 \), the magnetic overscreened fixed point appears, and for \( \varepsilon - \mu \gg \Delta = \pi \rho V^2 \) the low-energy properties are characterized by the quadrupolar overscreening Kondo effect.

Critical properties of this model were extensively discussed by Johannesson et al. on the basis of boundary CFT.\(^{22}\) Here we briefly summarize their results. The low-energy universality class of the model (14) is classified by examining the FSS, which is obtained exactly by using the method explained in the previous section:

\[
E = \frac{2\pi v}{L} \left( \frac{1}{8} (Q - 4 \frac{\delta_F}{\pi})^2 + \frac{j_s (j_s + 1)}{4} + \frac{j_f (j_f + 1)}{4} \right). \quad (15)
\]

Here \( Q, j_s, \) and \( j_f \) are, respectively, the charge, spin, and quadrupole quantum numbers. \( \delta_F \) is the phase shift related to the charge number of the impurity site \( \sum_j f_j f_j^\dagger \). The quantum numbers obey the selection rule, \( Q = 0 \text{ (mod 2)} \), \( j_s = 0 \text{ or 1} \), \( j_f = 1/2 \), or, alternatively, \( Q = 1 \text{ (mod 2)} \), \( j_s = 1/2 \), \( j_f = 0 \text{ or 1} \). Note that this FSS depends on the parameter \( \varepsilon \) only through the phase shift \( \delta_F \). In the magnetic limit \( \varepsilon - \mu \ll -\Delta, \delta_F \rightarrow 0 / 2 \), and in the quadrupolar limit \( \varepsilon - \mu \gg \Delta, \delta_F \rightarrow 0 \). This observation implies that even in the intermediate mixed valence regime, the low-energy fixed point is governed by the Kac-Moody fusion rules for two-channel systems with impurity spin \( s = 1/2 \) proposed by Affleck and Ludwig.\(^{8}\) The critical properties of the model are derived from the FSS (15).

Before closing this section, a brief comment is in order for the dynamically-induced multi-channel Kondo effect.\(^{23}\) Let us imagine a completely screened Kondo impurity with \( S > 1/2 \). If one of the core electrons, which compose the impurity spin-\( S \), is emitted via photoemission process, the screening Kondo effect is induced in the excited state, although the complete screening is realized in the initial state. Therefore, anomalous power-law behavior inherent in the overscreening model may be observed in the photoemission spectrum at low energies. This exemplifies the idea of dynamically-induced (or photo-induced) multi-channel Kondo effect.

3. Critical Exponents of Pseudo-Particles in the Kondo Problem

The exponents related to the orthogonality catastrophe mentioned in §2.1 also manifest themselves in the dynamical behavior of pseudo-particles, which are introduced to describe the strong-coupling limit (\( U \rightarrow \infty \)) of the Anderson model. In this limit, the double occupancy of electrons is forbidden, and thus the Fock space of the impurity electron can be mapped to that spanned by the slave-boson field and the pseudo-fermion field which represent an empty site and a singly occupied site, respectively. The long-time asymptotic behavior of dynamical correlation functions of these pseudo-particles can be derived exactly from a boundary CFT analysis.

For example, we consider the \( U \rightarrow \infty \text{ SU}(N) \) Anderson model.\(^{24}\) The Hamiltonian is given by

\[
H = \sum_{m=1}^{N} \int dx c^\dagger_m(x) \left( -i \frac{\partial}{\partial x} \right) c_m(x) + \varepsilon_f \sum_{m=1}^{N} f^\dagger_m f_m + V \sum_{m=1}^{N} \int dx \delta(x) (f^\dagger_m b c_m(x) + c^\dagger_m(x) b^\dagger f_m), \quad (16)
\]

where \( b \) and \( b^\dagger \) are the annihilation and creation operators of the slave-boson, and \( f_m \) and \( f^\dagger_m \) \( (m = 1, 2, \ldots, N) \) are the annihilation and creation operators of the pseudo-fermion. Since the double occupancy is forbidden, the constraint \( b^\dagger b + \sum_{m=1}^{N} f^\dagger_m f_m = 1 \) is imposed. The FSS of this model is readily obtained as,

\[
\frac{1}{L} E_1 = \frac{2\pi v}{L} \frac{1}{2} M^T C_f \Delta M - \frac{\pi v}{L} N \left( \frac{\delta_F}{\pi} \right)^2, \quad (17)
\]

where \( M^T = (\Delta M^{(1)}_h, \ldots, \Delta M^{(N-1)}_h, \Delta N_h - N \delta_F / \pi) \). \( \Delta N_h \) and \( \Delta M^{(i)}_h \) are the number of charge excitations and spin excitations, respectively. \( C_f \) is the Cartan matrix for the \( \text{OSP}(N,1) \) Lie superalgebra.\(^{24}\) It is intriguing that the Cartan matrix in (17) reflects the hidden higher symmetry of the impurity model (16).

Let us now study the long-time behavior of the Green functions for pseudo-particles; \( \langle f^\dagger_m(t) f_m(0) \rangle \sim t^{-\alpha_f} \), and \( \langle b^\dagger(t) b(0) \rangle \sim t^{-\alpha_b} \). As explained before, when determining canonical exponents for the local Fermi liquid, we can neglect the phase shift in eq.(4). In the derivation of critical exponents for pseudo-particles, however, we must regard the number of impurity electrons (or phase shift) \( n_l = \delta_l / \pi \) as a quantum number. Therefore the phase shift plays an essential role to determine the critical exponents. For instance, in order to obtain the Green function of pseudo-fermions, we take \( \Delta N_h = 1 \) and \( \Delta M^{(i)}_h = 0 \) as quantum numbers. We thus obtain the corresponding critical exponent as,

\[
\alpha_f = 1 - \frac{2 \delta_f}{\pi} + N \left( \frac{\delta_F}{\pi} \right)^2. \quad (18)
\]

In a similar way, the critical exponent \( \alpha_b \) for the slave-boson Green function can be obtained. Since the slave-boson expresses a vacancy, it carries neither charge nor spin. Putting \( \Delta N_h = \Delta M^{(i)}_h = 0 \), one gets

\[
\alpha_b = N \left( \frac{\delta_F}{\pi} \right)^2 \quad (19)
\]

These exact expressions for \( \alpha_f \) and \( \alpha_b \) agree with those obtained for the \( N = 1 \) and \( N = 2 \) cases,\(^{25,26}\) and take the same form as those in the X-ray problem: the exponent of pseudo-fermion corresponds to the X-ray absorption exponent, and that of slave-boson to the X-ray photoemission exponent. In the Kondo effect, the Fermi edge singularity shows up in the intermediate state as pointed out in the Anderson-Yuval approach.\(^{27}\) The anomalous exponents discussed here reflect this singularity.

The above argument is also applicable to the two-channel Anderson model (14), of which the low-energy fixed point is a non-Fermi liquid. The exponents for the pseudo-fermion \( f \) for this system is easily read off from eq.(15) with \( Q = 1, j_s = 1/2, j_f = 0 \) and the phase shift
retained in the charge sector,\textsuperscript{22}
\[ \alpha_{f}^{TA} = \frac{5}{8} - \frac{2\delta_{F}}{\pi} + 4\left(\frac{\delta_{F}}{\pi}\right)^{2}. \]
Similarly, the exponent for the slave boson \( b \) is obtained as,\textsuperscript{22}
\[ \alpha_{b}^{TA} = \frac{3}{8} + 4\left(\frac{\delta_{F}}{\pi}\right)^{2}. \]
These results present a striking contrast to the exponents for the Fermi liquid state (18) and (19).

4. “Underscreening Kondo effect” due to Non-Magnetic Impurities in 1D Systems

In this section, we would like to discuss the analogy between the Kondo problem and the impurity effects in quantum 1D systems. The Kondo effect is deeply related with the effects of spinless impurities in 1D quantum spin systems. In 1D systems, non-magnetic impurities typically cut the chains, and play the role of open boundary conditions. Remarkably, magnetic correlations in the vicinity of the open ends show anomalous behavior similar to “underscreening Kondo effect”.\textsuperscript{29-35}

For example, we consider the \( s = 1/2 \) Heisenberg spin chain with open boundaries, of which the Hamiltonian is,
\[ H_{XXZ} = J \sum_{i=1}^{N} [S_{i}^{x} S_{i+1}^{x} + S_{i}^{y} S_{i+1}^{y} + \Delta S_{i}^{z} S_{i+1}^{z}]. \]
(22)

The system is exactly solvable in terms of the BA method. For the isotropic case \( \Delta = 1 \), the BA equation for rapidities \( \lambda_{j} \) reads,
\[ \left( \frac{\lambda_{j} + i}{\lambda_{j} - i} \right)^{2N+1} \lambda_{j} + i = -\prod_{j=1}^{M} \frac{\lambda_{j} - \lambda_{j} + i}{\lambda_{j} - \lambda_{j} - i}, \]
(23)
As was pointed out by de Sa and Tsvelik, this equation has an analogy to the BA equation for the \( s-d \) exchange model with arbitrary spin \( S \).
\[ \left( \frac{\lambda_{j} + i}{\lambda_{j} - i} \right)^{N} \lambda_{j} + 1 + iS = -\prod_{j=1}^{M} \frac{\lambda_{j} - \lambda_{j} + i}{\lambda_{j} - \lambda_{j} - i}, \]
(24)
where \( c = 2J/[1 - S(S + 1)J^{2}] \). The BA equation for the \( s = 1/2 \) Heisenberg chain with open boundaries is equivalent to that for the Kondo problem with the impurity spin \( S = 1 \) in the strong coupling case \( c \rightarrow \infty \).

We can easily check that this similarity is also found in the anisotropic case \( \Delta \neq 1 \). Thus, it is expected that the underscreening Kondo effect may emerge in magnetic properties of eq.(22). However, this analogy is not complete. In the underscreening Kondo effect, the ground state is not a spin singlet, because of the presence of the partially unquenched impurity spin. On the other hand, the ground state of the Heisenberg spin chain with open boundaries is a spin singlet in the thermodynamic limit \( N \gg J/T \). It is important that even in this singlet case, the boundary effect on eq.(22) gives rise to non-trivial magnetic behavior as suggested from the above BA equation. In fact, the boundary spin susceptibility at zero temperature for the isotropic case obtained from eq.(23) shows the singular dependence on an external magnetic field,\textsuperscript{34,35}
\[ \chi_{B}(T = 0) = \frac{1}{4N\hbar(\ln(2\pi\alpha/h))^{2}} \left( 1 - \frac{\ln \ln(2\pi\alpha/h)}{\ln(2\pi\alpha/h)} + ... \right). \]
(25)
Here \( \alpha = \sqrt{\pi/2} \exp(1/4 + \gamma) \) with \( \gamma \) the Euler constant.

The divergent behavior of the boundary spin susceptibility is also seen in its temperature dependence. Since it is difficult to calculate the low temperature behavior from the BA equation, we carry out a perturbative expansion for the effective field theory, which gives asymptotically exact low-energy properties. For the boundary problem, it is convenient to express the partition function by using the transfer matrix exp(\( -LH^{c} \)), where \( H^{c} \) is the Hamiltonian defined on the imaginary time axis. The effective Hamiltonian in the isotropic case \( \Delta = 1 \) is,\textsuperscript{36}
\[ H = H_{WZW} + H_{m}, \]
(26)
\[ H_{m} = \beta \int_{0}^{1/T} \frac{d\tau}{2\pi} \sum_{a=1}^{3} J^{a}(\tau) \bar{J}^{a}(\tau). \]
(27)

Here \( H_{WZW} \) is the Hamiltonian of the level \( k = 1 \) SU(2) Wess-Zumino-Witten model, and \( J^{a}(\tau) \) is the left (right) moving current of the level \( k = 1 \) SU(2) Kac-Moody algebra. The running coupling constant \( g \) depends on temperature \( T \) and an external magnetic field \( h \) through the scaling equation,\textsuperscript{36}
\[ g^{-1} + \frac{1}{2} \ln(g) = -\Re[\psi(1 + \frac{ih}{2\pi T})] + \ln(\sqrt{\pi/2eq^{1/2}})/J/T, \]
(28) with \( \psi(x) \) the di-gamma function. The boundary contributions are obtained from a perturbative expansion of the free energy,
\[ F = \frac{T}{N} \ln(\hbar e^{-LH^{c}} |B\rangle), \]
(29)
with respect to \( H_{m} \). Here \( |B\rangle \) and \( |0\rangle \) are the boundary and (bulk) vacuum states, respectively. Up to the lowest order, it is sufficient to consider the non-perturbative state of \( |B\rangle \), i.e. the conformally invariant boundary state of \( H_{WZW} \), of which the expression in terms of free boson fields is well-known.\textsuperscript{37} The perturbation term of (29) is non-vanishing only for the Dirichlet boundary condition, which corresponds to the situation where the local magnetization at the boundary is not fixed to a particular value. Eventually, we obtain the leading term of the boundary spin susceptibility,\textsuperscript{34,35}
\[ \chi_{B} = \frac{1}{12NT\ln(\alpha/T)} \left( 1 - \frac{\ln \ln(\alpha/T)}{\ln(\alpha/T)} + ... \right), \]
(30)
Also, a similar temperature dependence is found in the boundary contribution of the specific heat coefficient,\textsuperscript{34,35}
\[ C_{B} = \frac{T}{2NT(\ln(\alpha/T))^{2}} \left( 1 - \frac{\ln \ln(\alpha/T)}{\ln(\alpha/T)} + ... \right), \]
(31)
We would like to stress that the divergent behavior at low temperatures appears even in the thermodynamic limit \( N \gg J/T \); i.e. irrespective of whether \( N \) is even or odd. These results are also confirmed by the numerical
transfer matrix renormalization group method."  

The above singular temperature dependence implies that in the vicinity of open boundaries, spin excitations are very sensitive to thermal perturbations. This sensitivity partially stems from a boundary entropy perturbed by bulk interactions. The presence of the ground state degeneracy caused by open boundaries is a universal feature of boundary critical phenomena in 1+1 dimensional systems. For the s = 1/2 Heisenberg spin chain with free open boundaries, the boundary entropy is equal to \( \ln(|B|) = \ln(K/2)^{1/4} \). Here K is related to the anisotropic parameter \( \Delta \), \( K = [1 - \cos^{-1}(\Delta)/\pi]^{-1} \). K = 1 for the isotropic case. This residual entropy per site degeneracy caused by open boundaries is a universal feature of boundary critical phenomena in 1+1 dimensional systems. We have also discussed an intriguing relation between the Kondo problems and some impurity problems in one-dimensional quantum systems. It has been shown that a non-magnetic impurity in quantum spin chains gives rise to a phenomenon similar to the underscreening Kondo effect, the underlying physics is a bit different.

5. Summary

In this paper, we have given a brief overview of universal aspects of the Kondo problem on the basis of the BA exact solution and boundary conformal field theory. The finite-size spectra for various solvable models related to the Kondo problem obtained from the BA method are consistent with the results of the CFT analysis. The exact FSS and the finite-size scaling argument provide the exponents of boundary scaling fields characterizing critical properties of the Kondo problems. We have also discussed an intriguing relation between the Kondo problems and some impurity problems in one-dimensional quantum systems. It has been shown that a non-magnetic impurity in quantum spin chains gives rise to a phenomenon similar to the underscreening Kondo effect.

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1) J. Kondo, Prog. Theor. Phys. 32 (1964) 37.
2) K. Yosida, Theory of Magnetism, (Springer-Verlag, Berlin, 1996).
3) N. Andrei, K. Furuya and J. H. Lowenstein: Rev. Mod. Phys. 55 (1983) 331; A. M. Tsvelick and P. B. Wiegmann: Adv. Phys. 32 (1983) 453.
4) N. Kawakami and A. Okiji: Phys. Lett. 86A (1981) 483; A. Okiji and N. Kawakami: Springer Series in Solid State Sciences 62 (1985) 46, 57.
5) I. Affleck: Nucl. Phys. B336 (1990) 517; I. Affleck and A. W. W. Ludwig: Nucl. Phys. B352 (1991) 849; ibid B360 (1991) 641; I. Affleck and A. W. W. Ludwig: Phys. Rev. Lett. 67 (1991) 161; ibid 67 (1991) 3160; Phys. Rev. B48 (1993) 7297.
6) J. L. Cardy: Nucl. Phys. B240 514 (1984), ibid B324 581 (1989); J. L. Cardy and D. C. Lewellen: Phys. Lett. B259 (1991) 274.
7) S. Fujimoto, N. Kawakami, and S.-K. Yang: Phys. Rev. B50 (1994) 1046.
8) I. Affleck and A. W. W. Ludwig: J. Phys. A27 (1994) 5375.
9) P. Nozi`eres: J. Low. Temp. Phys. 17 (1974) 31.
10) K. Yosida and K. Yamada,Prog. Theor. Phys. 53, 1286 (1975); K. Yamada, Prog. Theor. Phys. 53, 970 (1975), ibid 54, 316 (1975).
11) P. Nozieres and A. Blandin: J. Phys. (Paris) 41 (1980) 193.
12) A. M. Tsvelick and P. B. Wiegmann: Phys. Rev. B54 (1985) 201; J. Stat. Phys. 38 (1985) 125.
13) N. Andrei and C. Destri: Phys. Rev. Lett. 52 (1984) 364.
14) H. J. de Vega and F. Woynarovich: Nucl. Phys. B251 (1985) 439.
15) S. Fujimoto and N. Kawakami: Phys. Rev. B52 (1995) 13102.
16) P. Fendley: Phys. Rev. Lett. 71 (1993) 2485.
17) G. E. Andrews, R. J. Baxter, and P. J. Forrester: J. Stat. Phys. 35 (1984) 193.
18) N. Yu. Reshetikhin: J. Phys. A24 (1991) 3299.
19) A. Klümper and P. A. Pearce: Physica A183 (1992) 304.
20) C. J. Bolech and N. Andrei: Phys. Rev. Lett. 88 (2002) 237206.
21) A. Schiller, F. B. Anders, and D. L. Cox: Phys. Rev. Lett. 81 (1998) 3235.
22) H. Johannesson, N. Andrei, and C. J. Bolech: Phys. Rev. B68 (2003) 075112.
23) T. Fujii and N. Kawakami: Phys. Rev. B63 (2001) 064414.
24) S. Fujimoto, N. Kawakami, and S.-K. Yang: J. Phys. Soc. Jpn. 64 (1995) 4552.
25) B. Menge and E. Müller-Hartmann: Z. Phys. B73 (1988) 225.
26) T. A. Costi, P. Schmelleckert, and J. Kroha, and P. Wölfle: Phys. Rev. Lett. 73 (1994) 1275.
27) P. W. Anderson and G. Yuval: Phys. Rev. Lett. 23 (1969) 89.
28) F. C. Alcaraz, M. N. Barber, M. T. Batchelor, R. J. Baxter, and G. R. W. Quispel: J. Phys. A20 (1987) 6397.
29) P. de Sa and A. M. Tsvelik: Phys. Rev. B52 (1995) 3067.
30) F.H.L. Essler: J. Phys. A 29 (1996) 6183.
31) H. Asakawa and M. Suzuki: J. Phys. A 29, 225 (1996); ibid 29 (1996) 7811.
32) H. Fröhlich and A. A. Zyvagin: J. Phys.: Condens. Matter 9 (1997) 9939.
33) S. Fujimoto: Phys. Rev. B 63 (2000) 024406.
34) S. Fujimoto and I. Affleck: Phys. Rev. Lett. 92 (2004) 037206.
35) A. Furusaki and T. Hikihara: Phys. Rev. B69 (2004) 094429.
36) S. Lukyanov: Nucl. Phys. B 522 (1998) 533.
37) C. G. Callan, C. Lovelace, and C. R. Nappi, and S. A. Yost: Nucl. Phys. B 293 (1987) 83; M. Oshikawa and I. Affleck: Nucl. Phys. B 495 (1997) 533.
38) S. Eggert and I. Affleck: Phys. Rev. B 46 (1992) 10866 ; Phys. Rev. Lett. 75 (1995) 934.