IMPROVING FAIRNESS VIA FEDERATED LEARNING

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ABSTRACT
Recently, lots of algorithms have been proposed for learning a fair classifier from decentralized data. However, many theoretical and algorithmic questions remain open. First, is federated learning necessary, i.e., can we simply train locally fair classifiers and aggregate them? In this work, we first propose a new theoretical framework, with which we demonstrate that federated learning can strictly boost model fairness compared with such non-federated algorithms. We then theoretically and empirically show that the performance tradeoff of FedAvg-based fair learning algorithms is strictly worse than that of a fair classifier trained on centralized data. To bridge this gap, we propose FedFB, a private fair learning algorithm on decentralized data. The key idea is to modify the FedAvg protocol so that it can effectively mimic the centralized fair learning. Our experimental results show that FedFB significantly outperforms existing approaches, sometimes matching the performance of the centrally trained model.

1 Introduction
Fair learning has recently received increasing attention. Various fairness notions have been introduced in the past few years [Dwork et al., 2012, Hardt et al., 2016, Zafar et al., 2017b,a, Kearns et al., 2018, Friedler et al., 2016]. Among various fairness notions, group fairness [Hardt et al., 2016, Zafar et al., 2017a] is the most studied one; it requires the classifier to treat different groups similarly, where groups are defined with respect to sensitive attributes such as gender and race. One of the most commonly used group fairness notions is demographic parity, which requires that different groups are equally likely to receive desirable outcomes.

There has been a large amount of work in training fair classifiers [Zafar et al., 2017c, Hardt et al., 2016, Roh et al., 2021], and almost all of these studies assume that the learner has access to the entire training data. Unfortunately, this is not the case in many critical applications. For example, the government may want to train a single classifier that is fair and accurate using datasets owned by different stakeholders across the country, but the stakeholders may not be able to share their raw data with the government due to the privacy act. This precisely sets the core question that we aim to answer in this paper – how can we privately train a fair classifier on decentralized data? To answer this, consider the following approaches: Local Fair Training + Ensemble and Local Fair Training + FedAvg.

Local Fair Training + Ensemble (LFT+Ensemble) One naïve yet the simplest way to learn fair classifiers from decentralized data without violating privacy policy is training locally fair classifiers and assembling them. We call this strategy LFT+Ensemble. This is more private than any data sharing schemes or federated learning approaches as only the fully trained models are shared just once.

Local Fair Training + FedAvg (LFT+FedAvg) LFT+FedAvg applies federated learning [Konečný et al., 2017] together with existing fair learning algorithms. Federated learning is a distributed learning framework, using which many data owners can collaboratively train a model under the orchestration of a central server while keeping their data decentralized. For instance, under FedAvg, the de facto aggregation protocol for federated learning, the central server periodically computes a weighted average of the locally trained model parameters. If each data owner runs a fair learning algorithm on its own data and these locally trained models are aggregated via FedAvg, then we call this approach Local Fair Training + FedAvg (LFT+FedAvg).
**Goal and Main Contributions** As LFT+FEDAVG makes more than one round of communications between the clients and the centralized aggregator, it is reasonable to expect a better performance (fairness and accuracy) with LFT+FEDAVG over LFT+Ensemble. However, rigorous performance analysis has not been made yet in the literature. Inspired by this, we first study the following question:

*Is federated learning needed for decentralized fair learning?*

In other words, it is not clear whether there is any strict performance (fairness and accuracy) improvement by using LFT+FEDAVG over LFT+Ensemble. If not, there is no reason to use LFT+FEDAVG as LFT+Ensemble is a more private scheme. Our first contribution is to theoretically compare their performances and show that federated learning is indeed necessary. Intuitively, if the data is highly heterogeneous, an ensemble model of locally trained models will not be fair even if the consisting models are locally fair. See Fig. 2 for a toy example illustrating this phenomenon. On the other hand, LFT+FEDAVG might find a globally fairer model by making frequent communications, of course at the cost of privacy loss. Another natural question follows:

*How good is LFT+FEDAVG compared to the case where the whole data is centralized?*

That is, can LFT+FEDAVG achieve similar performance with *Centralized Fair Learning (CFL)?* If not, can we develop a better federated learning approach? Our second contribution is to identify the performance gap between LFT+FEDAVG and CFL and to develop FedFB, a new federated learning algorithm that bridges the gap. Our major contributions are summarized as follows (see Fig. 1):

- We develop a theoretical framework for analyzing approaches for decentralized fair learning. Using this, we show the strict ordering between the existing approaches, i.e., under certain conditions, LFT+Ensemble < LFT+FEDAVG < CFL, w.r.t. their fairness-accuracy tradeoffs. To the best of our knowledge, it is the first rigorous comparison of these approaches. Our finding implies the necessity of federated learning, but at the same time, implies the limitation of the naive application of FedAvg together with local fair training.
- Leveraging the state-of-the-art algorithm for (centralized) fair learning [Roh et al., 2021], we design FedFB, a novel fairness-aware federated learning algorithm.
- Via extensive experiments, we show that (1) our theoretical findings hold under more general settings, and (2) FedFB significantly outperforms the existing approaches and achieves similar performance as CFL.

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**Figure 1:** A high-level illustration of various approaches to fair learning on decentralized data and our contributions. Assuming two data owners, from left to right, we show LFT+Ensemble (Local Fair Training + Ensemble), LFT+FEDAVG (Local Fair Training + FEDAVG), FedFB (ours), and CFL (Centralized Fair Learning). In LFT+Ensemble, each data owner trains a locally fair model on its own data and we assemble the local models to obtain a global model. LFT+FEDAVG applies FEDAVG together with off-the-shelf fair training algorithms for local training. Our proposed solution FedFB consists of a modified FEDAVG protocol with a custom-designed fair learning algorithm. CFL is the setting where a fair model is trained on the pooled data. In this work, we theoretically characterize the strict ordering between the existing approaches and empirically demonstrate the superior performance of FedFB.

**Figure 2:** An example illustrating the failure of LFT+Ensemble (Local Fair Training + Ensemble). Consider three clients with four samples of one-dimensional feature \(x\). F and M denote female and male, respectively, and pink boxes denote positive labels. Under LFT+Ensemble, every client first finds a locally fair classifier on their data, which are depicted in short blue dotted lines. The voting ensemble model is visualized in a long blue line. Note that this is strictly worse than the green line that is more fair and accurate.
2 Related work

**Model Fairness** Several works have studied the fundamental tradeoff between fairness and accuracy [Menon and Williamson 2018, Wick et al. 2019, Zhao and Gordon 2019]. Our analysis is highly inspired by the one by Menon and Williamson 2018, which characterized the tradeoff between accuracy and fairness under the centralized setting. Among various fair training algorithms [Zemel et al. 2013, Jiang and Nachum 2020, Zafar et al. 2017c, Hardt et al. 2016], our work is based on the current state of the art — FairBatch [Roh et al. 2021], which we will detail in Sec. 4.

**Federated Learning** Unlike traditional, centralized machine learning approaches, federated learning keeps the data decentralized throughout training, reducing the privacy risks involved in traditional approaches [Konečný et al., 2017 McMahan et al., 2017]. FEDAVG [McMahan et al., 2017] is the first and most widely used federated learning algorithm. The idea is to iteratively compute a weighted average of the local model parameters, with the weights proportional to the amount of data. We further assume a simple binary classification setting with a binary sensitive attribute, i.e., \( \{0, 1\} \) is the outcome and \( x \in \mathcal{X} = \mathbb{R} \) is the input feature, \( y \in \mathcal{Y} = \{1\} \) is the outcome and \( a \in \mathcal{A} = \{1\} \) is the binary sensitive attribute. Assume that \( x \) is a continuous variable. The algorithm we will develop later is applicable to general settings.

**Federated Fair Learning for Group Fairness** A few very recent studies [Ezzeldin et al., 2021, Rodríguez-Gálvez et al., 2021, Du et al., 2021, Cui et al., 2021] conducted concurrently with our work, aim at achieving group fairness under federated learning. In particular, Du et al. [2021], Rodríguez-Gálvez et al. [2021] and Chu et al. [2021] mimic the centralized fair learning setting by very frequently exchanging information for each local update, not for each round of local training. In contrast, FedFB requires much less frequent communications (one per round), resulting in higher privacy and lower communication costs. Moreover, Rodríguez-Gálvez et al. [2021] and Chu et al. [2021] are designed specifically for certain group fairness notions (such as equal opportunity and accuracy parity) and cannot be applied for other definitions such as demographic parity. On the other hand, FedFB is applicable for all popular group fairness notions including demographic parity, equalized odds, and equal opportunity. Similar to FedFB, Ezzeldin et al. [2021] employs FEDAVG and a reweighting mechanism, but it achieves a worse performance.

We visualize a high-level comparison of LFT+Ensemble, LFT+FedAvg, CFL, FedFB and AGNOSTICFAIR [Du et al. 2021] in terms of performance and privacy in Fig. 3(a). Most of the work in the literature and our work focus on satisfying a single fairness requirement on the “global” data distribution, but there is also a recent work that aims at finding a classifier that simultaneously satisfies multiple fairness requirements, each defined with respect to each client’s “local” data [Cui et al. 2021].

**Federated Fair Learning for Client Fairness** The definition of fairness used in most of the existing federated learning work is slightly different from the standard notion of group fairness. One popular definition of fairness in the federated setting is client parity (CP), which requires that all clients (i.e. data owners) achieve similar accuracies (or loss values). Several algorithms have been proposed to achieve this goal [Li et al. 2021, 2020a, Mohri et al. 2019, Yue et al. 2021, Zhang et al. 2020a]. Even though their goal is different from ours, we show that an adapted version of FedFB can also achieve comparable client parity, compared with the existing algorithms proposed specifically for CP.

3 Analysis of LFT+Ensemble & LFT+FedAvg

In this section, we first show the necessity of federated learning by proving that LFT+FedAvg achieves strictly higher fairness than LFT+Ensemble. We then prove the limitation of LFT+FedAvg by comparing its performance with an oracle bound of CFL (non-private). These two results together imply that federated learning is necessary, but there exists a limit on what can be achieved by FEDAVG-based approaches. We will present informal theoretical statements, deferring the formal versions and proofs to Sec. A.

**Problem Setting** Denote \( |N| := \{0, 1, \ldots, N - 1\} \) for any \( N \in \mathbb{Z}^+ \). We assume \( I \) clients, which have the same amount of data. We further assume a simple binary classification setting with a binary sensitive attribute, i.e., \( x \in \mathcal{X} = \mathbb{R} \) is the input feature, \( y \in \mathcal{Y} = \{1\} \) is the outcome and \( a \in \mathcal{A} = \{1\} \) is the binary sensitive attribute. Assume that \( x \) is a continuous variable. The algorithm we will develop later is applicable to general settings.
We denote by \( \delta > \min \) Corollary 2 (informal) is identical across the clients. Instead, Lemma 1 only requires the insensitive attribute distribution (Lemma 1 implies that LFT+Ensemble fails to achieve perfect fairness even when the sensitive attribute (constant). Under certain conditions, there exists \( \delta \) (informal) Achievable fairness range of LFT+Ensemble) since there always exists a perfectly fair classifier [Menon and Williamson, 2018], one question we can ask is if of local unfairness budgets. The blue plane is of perfect fairness, and the pink plane visualizes the value of \( \delta \) (Figure 4: Fundamental limitation of LFT+Ensemble in terms of fairness range. (a) (Example 1) On a distribution that satisfies the conditions of Corollary 2. (b, c) On distributions of local unfairness budgets. The blue plane is of perfect fairness, and the pink plane visualizes the value of \( \delta \).)

To measure how unfair a classifier is with respect to demographic parity (DP), we define DP disparity as:

\[
\text{DP Disp}(f) = |\mathbb{P}(|y = 1 | a = 0) - \mathbb{P}(|y = 1 | a = 1)|.
\]

3.1 LFT+FEDAVG strictly outperforms LFT+Ensemble

LFT+Ensemble Optimization We now present the optimization problem solved in the LFT+Ensemble scenario. Here, for analytical tractability, we will assume the population limit, i.e., the true data distribution is used in optimization. Recall that in this scenario, each client solves their own optimization problem to train a locally fair classifier. In particular, each client \( i \in [I] \) first sets its own local fairness constraint \( \epsilon_i \in [0, 1] \), and solves the following problem:

\[
\min_{f \in \mathcal{F}} \mathbb{P}(\hat{y} \neq y | i = i), \quad \text{(LFT+Ensemble}(i, \epsilon_i))
\]

s.t. \( \mathbb{P}(\hat{y} = 1 | a = 0, i = i) - \mathbb{P}(\hat{y} = 1 | a = 1, i = i) \leq \epsilon_i. \)

We denote by \( f^\epsilon_i \) the solution to LFT+Ensemble.(i, \epsilon_i) Let \( \epsilon = (\epsilon_0, \ldots, \epsilon_{i-1}) \). We consider the following ensemble of \( I \) locally trained classifiers:

\[
\hat{y} \mid x, a \sim \text{Bern} \left( \frac{\sum_{i \in [I]} f^\epsilon_i(x, a)/I}{1} \right) = \text{Bern}(f^\text{LFT+Ensemble}_\epsilon),
\]

where \( f^\text{LFT+Ensemble}_\epsilon := \sum_{i \in [I]} f^\epsilon_i(x, a)/I. \)

Since there always exists a perfectly fair classifier [Menon and Williamson, 2018], one question we can ask is if \( f^\text{LFT+Ensemble}_\epsilon \) can achieve an arbitrary level of fairness. The following lemma asserts that it cannot.

Lemma 1 (informal) Achievable fairness range of LFT+Ensemble. Let \( q_i = q \) for all \( i \in [I] \), where \( q \in (0, 1) \) is a constant. Under certain conditions, there exists \( \delta (\delta > 0) \) such that \( \min_{\epsilon \in [0, 1]} \text{DP Disp}(f^\text{LFT+Ensemble}_\epsilon) > \delta. \)

Lemma 1 implies that LFT+Ensemble fails to achieve perfect fairness even when the sensitive attribute (a) distribution is identical across the clients. Instead, Lemma 1 only requires the insensitive attribute distribution (x) to be highly heterogeneous across the clients. The following corollary provides a concrete example satisfying such conditions.

Corollary 2 (informal). Let \( I = 2 \) and \( q_0 = q_1 = 0.5, \eta(x) = \frac{1}{1 + |x - q|}, P_a(i) = N(\mu_a(i), \sigma_a(i)^2), \) where \( i = 0, 1. \)

If one client has much larger variance than the other; under certain assumptions, there exists \( \delta > 0 \) such that \( \min_{\epsilon_0, \epsilon_1 \in [0, 1]} \text{DP Disp}(f^\text{LFT+Ensemble}_\epsilon) > \delta. \)
We first present the optimization problem for CFL, whose achievable fairness region can serve as an upper bound on performance tradeoffs. Similar to Lemma 1, the technical assumptions include high enough heterogeneity of training data. Our finding on this untapped gain in fairness can be seen as a new justification for small-scale federated learning, i.e., cross-silo settings.

3.2 LFT+FedAvg is strictly worse than CFL

We first present the optimization problem for CFL, whose achievable fairness region can serve as an upper bound on that of all other decentralized algorithms. We then show the existence of data distributions on which LFT+FedAvg achieves a strictly worse fairness performance than CFL.

Note that the condition that one client has a much larger variance than the other contributes to the “high data heterogeneity” requirement. We provide the explicit form of $\delta$ in Corollary 1 in Sec. A.2.2. Corollary 2 implies that under a limiting case of Gaussian distribution, LFT+Ensemble cannot achieve high fairness requirements. In Sec. 5 we will numerically demonstrate the same claim holds for more general cases. Next, we give an example that satisfies the condition of Corollary 2 whose achievable DP disparity and lower bound are visualized in Fig. 4(a).

Example 1. Let $\mu_0^{(0)} = \mu_1^{(0)} = 0, \mu_0^{(1)} = 3, \mu_1^{(1)} = -1, \sigma_0^{(0)} = 70$ and $\sigma_0^{(1)} = 1$. Then, $\delta \approx 0.21$.

Remark 1. In this work, we classify local training followed by an ensemble method as a non-federated learning algorithm. However, one may also view it as an extreme instance of federated learning, called one-shot (single round) federated learning [Guha et al., 2019]. Depending on how one would classify this approach, our finding can be viewed as either the necessity of federated learning or the necessity of multi-round federated learning.

LFT+FedAvg Optimization The classifier obtained by LFT+FedAvg can be shown to be equivalent to $\min_{f_\varepsilon} \mathbb{P}(\hat{y} \neq y)$, the solution to the following problem.

$$\min_{f_\varepsilon} \mathbb{P}(\hat{y} \neq y) \quad \text{(LFT+FedAvg($\varepsilon$))}$$

s.t. $|\mathbb{P}(\hat{y} = 1 | a = 0, i = i) - \mathbb{P}(\hat{y} = 1 | a = 1, i = i)| \leq \varepsilon_i$.

The proof of the equivalence is provided in Sec. A.3.2. The following theorem asserts that LFT+FedAvg can achieve a strictly higher fairness than LFT+Ensemble.

Theorem 3 ((informal) Fundamental values of federated learning for fairness). Let $q_i = q$ for all $i \in [I]$, where $q \in (0, 1)$ is a constant. Under certain conditions, $\min_{x \in [0,1]} DP Disp(f^LFT+Ensemble_\varepsilon) > \min_{x \in [0,1]} DP Disp(f^LFT+FedAvg_\varepsilon)$.

Thm. 3 implies that LFT+Ensemble cannot match the highest fairness achieved by LFT+FedAvg. See Fig. 4 for the performance tradeoffs. Similar to Lemma 1 the technical assumptions include high enough heterogeneity of $x$ across the clients (see Sec. A.4).

Remark 2. Extending Thm. 3 (and the analysis of LFT+FedAvg) to general cases where $q_i$’s are different remains open. However, we conjecture that our lemma holds for more general cases, and we will numerically support our conjecture empirically.

Remark 3. Thm. 3 shows that even with just two clients ($I = 2$), a constant gap exists between non-federated algorithms and federated algorithms in their fairness performances. There is a stark difference between this phenomenon and the well-known gain of federated learning due to an increased sample size, which is generally negligible with a few number of clients. Our finding on this untapped gain in fairness can be seen as a new justification for small-scale federated learning, i.e., cross-silo settings.

3.2 LFT+FedAvg is strictly worse than CFL

We first present the optimization problem for CFL, whose achievable fairness region can serve as an upper bound on that of all other decentralized algorithms. We then show the existence of data distributions on which LFT+FedAvg achieves a strictly worse fairness performance than CFL.

![Figure 5: Accuracy-fairness tradeoff curves of CFL, LFT+FedAvg, and LFT+Ensemble for two clients cases.](image-url)
CFL Optimization We consider the same problem setting as last subsection. We now formulate the CFL problem as:

\[
\min_{f \in F} \mathbb{P}(\hat{y} \neq y),
\]

\[
\text{s.t. } |\mathbb{P}(\hat{y} = 1 | a = 0) - \mathbb{P}(\hat{y} = 1 | a = 1)| \leq \varepsilon.
\]

Denote the solution to \(f_{CFL}(\varepsilon)\) as \(f^{CFL}_{\varepsilon}\). It is clear that \(f^{CFL}_{\varepsilon}\) achieves the best accuracy-fairness tradeoff, at the cost of no privacy. The following lemma shows that there exists some distribution such that LFT+FedAvg is strictly worse than CFL when the distribution of sensitive attribute \(a\) is heterogeneous (\(q_i\) are not all the same).

Lemma 4 ((informal) A strict gap between LFT+FedAvg and CFL). When there exist \(i \neq j \in [I]\) s.t. \(q_i \neq q_j\), there exists a distribution such that \(\min_{\varepsilon \in [0,1]} DP Disp(f^{CFL}_{\varepsilon}) > \min_{\varepsilon} DP Disp(f^{LFT+FedAvg}_{\varepsilon}) = 0\).

Remark 4. A strict gap exists for certain distributions, but not for all distributions.

3.3 Numerical Comparisons of tradeoffs

One limitation of our current theoretical results is that they only compare the maximum achievable fairness. Extending our theoretical results to fully characterize two-dimensional tradeoffs is non-trivial, so we leave it as future work.

Shown in Fig. 5 are the tradeoff curves for two clients cases. We let \(x \mid a = 0, i = 0 \sim \mathcal{N}(3,1), x \mid a = 1, i = 0 \sim \mathcal{N}(5,1), x \mid a = 0, i = 1 \sim \mathcal{N}(1,1), x \mid a = 1, i = 1 \sim \mathcal{N}(-1,1)\), \(a \mid i = 0 \sim \text{Bern}(q_0), a \mid i = 1 \sim \text{Bern}(q_1)\), \(\eta(x) = \frac{1}{1+x}\), and vary the values of \(q_0\) and \(q_1\). Note that \((q_1 - q_0)\) captures the heterogeneity of the sensitive data \(a\), which increases from left to right. First, one can observe that LFT+Ensemble<\(\text{LFT+FedAvg}\)< CFL in terms of the achievable fairness range, as predicted by our theory. Beyond our theory, we also observe an increasing gap between the tradeoff curves as the data heterogeneity increases.

4 Proposed Algorithm: FedFB

Our findings in Sec. 3 imply that federated learning is necessary, but the current FedAvg-based approach might not be the best approach. Can we design a federated learning algorithm that is strictly better than FedAvg-based approaches? In this section, we propose a new federated learning algorithm for fair learning, which we dub FedFB (short for Federated FairBatch). Our approach is based on the state-of-the-art (centralized) fair learning algorithm FairBatch (FB for short) [Roh et al., 2021] and comes with a few desirable theoretical guarantees.

Centralized FB Let us first review how FB works in the centralized setting when demographic parity is considered. Assume there are \(A\) sensitive attributes. Let \(w\) be the model parameters, \(\ell(y, \hat{y}; w)\) be the loss function, \(n_{y,a} := |\{(x, y, a) : y = a, a = a\}|\) be the number of samples in group \(a\) with label \(y\), and \(n_{a} := |\{(x, y, a) : a = a\}|\) be the number of samples in group \(a\). Let \(L_{y,a}(w) = \sum_{y=a} \ell(y, \hat{y}; w), L'_{y,a}(w) := n_{y,a} L_{y,a}(w)/n_{a}\). Then, for \(a \in [A]\), we define

\[
F_a(w) := -L'_{0,a}(w) + L'_{1,a}(w) + \frac{n_{0,a}}{n_{a}} - \frac{n_{0,a}}{n_{a}}.
\]

Then, we can show the following proposition.

Proposition 5 (Necessary and sufficient condition for demographic parity). Consider 0-1 loss: \(\ell(y, \hat{y}; w) = [y \neq \hat{y}(w)]\), where \([\cdot]\) is the indicator function. Then,

\[
F_a(w) = 0, \forall a \in [A]
\]

is a necessary & sufficient condition for demographic parity.

Note that Proposition 5 allows us to measure demographic disparity using subgroup-specific losses when the 0-1 loss is used. Inspired by this observation, FB uses \(F_a(w)\) as a surrogate of the unfairness measure even for non-0 – 1 loss functions. With this surrogate, FB solves the following bi-level optimization.

\[
\begin{align*}
\min_{w} & \sum_{a=0}^{A-1} \left( F_a(w) \right)^2, \\
\text{subject to} \quad w & = \arg\min_{w} \sum_{a=0}^{A-1} \left[ \lambda_a L'_{0,a}(w) + \frac{2n_{a}}{n} - \lambda_a \right] L'_{1,a}(w).
\end{align*}
\]
The bi-level structure of FB

All together, FB alternates this update equation (for the outer optimization) and the inner-loop updates (e.g., via SGD)

Algorithm 1: FedFB\( (k, t) \)

ClientUpdate\((i, w, \lambda)\):
- Update local \(w^{(i)}\) according to sample weights \(\lambda\);
- Compute local \(F_a^{(i)}\) for all \(a \in [A]\);

ServerExecutes:
- for each round \(t\) do
  - Wait until clients perform their updates;
  - \(w \leftarrow \text{SecAgg}\{w^{(i)}\}\);
  - \(F_a \leftarrow \text{SecAgg}\{F_a^{(i)}\} - (I - 1)(\frac{n_{a,0}}{n_{*0}} - \frac{n_{a,a}}{n_{*a}})\);
  - if \(t \% k = 0\) then
    - \(\lambda \leftarrow \text{Update}(\lambda, F_a)\);
    - Broadcast \(\lambda\) to clients;
  - Broadcast \(w\) to clients;
- output : \(w, \lambda\)

Here, \(\lambda = (\lambda_0, \ldots, \lambda_{A-1})\) are the outer optimization variables, which denote adaptive coefficient to loss occurring from each subgroup. Given \(\lambda\), the inner optimization problem minimizes a reweighted loss function. The intuition here is that if one carefully chooses the coefficients for each subpopulation loss term, the weighted risk minimization will result in a fair classifier, even if it is unconstrained.

In particular, Roh et al. \cite{Roh} proposed the following iterative algorithms to solve the outer optimization problem:

\[
\mu_0(\lambda) = - \sum_{a=1}^{A-1} F_a(w_\lambda), \quad \mu_a(\lambda) = F_a(w_\lambda), \quad \text{for } a \neq 0, \tag{2}
\]

\[
\lambda_a \leftarrow \lambda_a + \frac{\alpha}{\|\mu(\lambda)\|_2} \mu_a(\lambda), \quad \text{for } a \in [A], \alpha \text{ is the step size.}
\]

All together, FB alternates this update equation (for the outer optimization) and the inner-loop updates (e.g., via SGD) to solve the bi-level optimization problem\(^1\)

FedFB: Federated Learning + FB

Recall that under the FedAVG protocol, clients periodically share their locally trained model parameters with the server. Our proposed algorithm is based on the following simple observation:

The bi-level structure of FB naturally fits the hierarchical structure of federated learning.

To see this, observe that the update equation given in (2) only requires the knowledge about all \(\mu_a\), which is the function of \(F_a\)’s. By definition of \(F_a\), it is the sum of the locally computed \(F_a\) values across all the clients. Thus, this can be securely collected by the central aggregator via secure aggregation. Once the \(F_a\) values are securely aggregated at the central server, the central aggregator can update the per-group coefficients \(\lambda\) using the exact same equation \(2\).

This allows us to run the FB algorithm even on decentralized data. More specifically, we modify the FedAvg protocol so that each client shares not only its intermediate model parameters but also its \(\{F_a\}_{a \in [A]}\) with the central coordinator. Once the central server receives the securely aggregated model parameters and \(F_a\) values, it performs both (1) model averaging and (2) reweighting coefficients updates (\(\lambda\)). The server then broadcasts the averaged model parameter together with the updated coefficients, which are then used for the subsequent round of local training with a reweighted loss function. Note that we update \(\lambda\) every \(k\) communication rounds.

This algorithm, which we dub FedFB, is formally described in Alg. 1. While this description is specific to the demographic parity, FedFB can be used for other fairness notions including equal opportunity, equalized odds, and client parity. The only difference would be how to design the \(F_a\) function, which is the surrogate for the unfairness measure. See Sec. \(B\) for more details.

Convergence of FedFB

The following theorem shows that FedFB converges to the optimal reweighting coefficients. The proof is based on the analysis tools for federated learning and FB – See Thm. \(2\) for more details.

\(^1\)Technically speaking, the algorithm shown here is a slightly generalized version of the original FB. In particular, it works for more than two sensitive groups, while the original FB algorithm only works for binary groups. However, we will still call ours FB.
Theorem 6 (informal) Partial convergence guarantee of FedFB). Let the output of FedFB\((k, t)\) be \(\lambda_{k,t}\). Assume \(A = 2\). Then, under certain conditions, \(\lim_{t \to \infty} \lim_{k \to \infty} \lambda_{k,t} = \lambda^*\) for some \(\lambda^* \in \text{arg min}_{\lambda} \sum_a [F_a(w_\lambda)]^2\).

Additional privacy leakage Under FedFB, clients exchange real-valued \(F_a\)'s with the server in addition to the model parameters. To limit the information leakage, we develop a variant of FedFB, which exchanges the quantized loss values. For instance, “FedFB(10bits)” means each loss value is uniformly quantized using 10 bits. Such a quantization scheme limits the amount of additional information shared in communication round, at the cost of perturbed coefficient updates.

5 Experiments

In this section, we numerically study the performance of LFT+Ensemble, LFT+FEDAVG, and CFL for more general cases, and evaluate the empirical performance of FedFB. In each simulation study, we report the summary statistics across five replications. Similar to the experimental settings used in [Roh et al., 2020], we train all algorithms using a two-layer ReLU neural network with four hidden neurons to evaluate the performance of FedFB for the non-convex case. Due to the space limit, we defer (1) logistic regression results, (2) the performance tradeoffs of FedFB as a function of the number of clients, and (3) more implementation details to Sec. C.

5.1 Limitation of LFT+Ensemble on general cases

The first experiment examines the fairness range of LFT+Ensemble under various Gaussian distributions, which does not satisfy the conditions of Corollary 2. To do so, we numerically solve the optimization problems \[ LFT+\text{Ensemble}(i, \xi, \epsilon) \] See Sec. A for more details. We conjecture that the same phenomenon holds for more general distributions, and corroborate our conjecture with numerical experiments. Shown in Fig. 4(b,c) are the numerically computed lower bound on LFT+Ensemble’s achievable fairness. In particular, for (b), we let \(x | a = 0, i = 0 \sim N(10, 0.2^2)\), \(x | a = 1, i = 0 \sim N(9.8, 0.2^2)\), \(x | a = 1, i = 1 \sim N(0, 0.2^2)\), \(a \sim \text{Bern}(0.2)\), and for (c), we let \(x | a = 0, i = 0 \sim N(3, 1)\), \(x | a = 1, i = 0 \sim N(5, 1)\), \(x | a = 0, i = 1 \sim N(1, 1)\), \(x | a = 1, i = 1 \sim N(1, 0.5)\). For both cases, we set \(\eta(x) = \frac{1}{1 + e^{-x}}\). It is easy to check that these distributions do not satisfy the conditions of Corollary 2. In particular, the distribution (b) corresponds to the case that the same group is favored on both clients, and the positive rates of each group in different clients are distinctive. The distribution (c) represents the case that different groups are favored on two clients. In both cases, LFT+Ensemble fails to achieve perfect fairness, i.e., \(\delta > 0\). We also observe that \(\delta\) is large on the distribution (c). This supports our conjecture that LFT+Ensemble’s fairness performance is strictly limited not only on certain data distributions but also on more general ones.

5.2 Accuracy-fairness tradeoffs of LFT+Ensemble, LFT+FEDAVG and CFL

The second experiment extends the experiments conducted in Sec. 5.1 We assess the relationship between the data heterogeneity and the gap between the three fair learning scenarios with three clients. As shown in Fig. 9, LFT+FEDAVG is observed to achieve a strictly worse tradeoff than CFL and a strictly higher maximum fairness value than LFT+Ensemble. The results corroborate the benefit and limitation of FEDAVG-based federated learning in improving fairness. A very interesting observation is that LFT+Ensemble is observed to obtain a strictly higher accuracy than LFT+FEDAVG. Indeed, this could be attributed to the fact that the average of locally fair models might not be fair to any sub-distribution, while LFT+Ensemble at least ensures that each component of the mixture classifier is fair on some sub-distribution.

5.3 FedFB evaluation on demographic parity

We assess the empirical performance of FedFB for non-convex cases on four datasets and the performance of FedFB under different data heterogeneity. We focus on demographic parity and report DP disparity = max_{\alpha \in [A]} |P(\hat{y} = 1 | a = \alpha) - P(\hat{y} = 1)|, where \(A\) is the number of groups. Note that this is slightly different from the definition we used in the previous sections, which was specific for the binary sensitive attribute.

Baselines We employ three types of baselines: (1) decentralized non-fair training (FEDAVG); (2) decentralized fair training (LFT+Ensemble, LFT+FEDAVG, FAIRFED, AGNOSTICFAIRS); (3) centralized fair training (CFL). Here, for the fairer comparison and better performance, LFT+Ensemble, LFT+FEDAVG, and CFL are implemented based on FB.  

\footnote{Our implementation is available in https://github.com/yzeng58/Improving-Fairness-via-Federated-Learning}
Improving Fairness via Federated Learning

Table 1: Comparison of accuracy and DP disparity on the synthetic, Adult, COMPAS, and Bank datasets. FedFB outperforms the other approaches on most of the tested datasets, sometimes nearly matching the performance of CFL. Note that LFT+FEDAVG sometimes gets a strictly worse performance than FEDAVG. This is because the average of fair models may not be fair at all.

| PROPERTY | PRIVATE | FAIR | METHOD | SYNTHETIC | ADULT | COMPAS | BANK |
|----------|---------|------|--------|-----------|-------|--------|-----|
|          |         |      |        | ACC.(↑)   | DP Disp.(↓) | ACC.(↑) | DP Disp.(↓) | ACC.(↑) | DP Disp.(↓) |
|          |         |      |        |          |        |        |      |
| ±        |         |      | FedAvg | 886.0±0.03 | 0.046±0.09 | 829.0±0.12 | 153.0±012 | 655.0±0.09 | 167.0±0.03 | 899.0±0.01 | 0.028±0.03 |
|          |         |      | LFT+ENSEMBLE | 727.0±194 | 0.248±194 | 825.0±0.088 | 0.034±0.028 | 620.0±0.019 | 0.008±0.055 | 892.0±0.002 | 0.014±0.006 |
|          |         |      | LFT+FEDAVG | 823.0±102 | 0.305±131 | 801.0±0.043 | 0.123±0.71 | 595.0±0.05 | 0.059±0.009 | 893.0±0.00 | 0.017±0.001 |
|          |         |      | FedFB (OURS) | 725.0±12 | 0.051±0.18 | 804.0±0.001 | 0.028±0.001 | 606.0±0.019 | 0.086±0.029 | 883.0±0.00 | 0.000±0.00 |
| ±        |         |      | CFL | 726.0±0.09 | 0.028±0.016 | 814.0±0.02 | 0.010±0.004 | 616.0±0.033 | 0.036±0.028 | 883.0±0.00 | 0.000±0.00 |

Table 2: Comparison of accuracy and DP disparity on the synthetic dataset with varying heterogeneity. FedFB achieves good performance on all the tested levels of heterogeneity. This is because FedFB is designed to mimic the operation of CFL, whose performance is independent of data heterogeneity.

| METHOD | LOW DATA HETEROREGENCE | MEDIUM DATA HETEROREGENCE | HIGH DATA HETEROREGENCE |
|--------|------------------------|---------------------------|------------------------|
|        | ACC.(↑)   | DP Disp.(↓) | ACC.(↑) | DP Disp.(↓) | ACC.(↑) | DP Disp.(↓) |
|        |          |        |          |        |          |        |
| FedFB (OURS) | 0.669±0.040 | 0.58±0.042 | 0.725±0.012 | 0.051±0.018 | 0.703±0.012 | 0.013±0.008 |
| CFL | 0.726±0.099 | 0.028±0.016 | 0.726±0.009 | 0.028±0.016 | 0.726±0.009 | 0.028±0.016 |

which is the state-of-the-art fair learning algorithm on centralized data. Note that LFT+Ensemble is the most private, CFL is the least private, and FEDAVG, LFT+FEDAVG, FAIRFED and FedFB are somewhere in-between as they share some statistics at each communication round but not directly reveal the data. AGNOSTICFAIR is close to CFL, as it exchanges information every local update.

Datasets (synthetic) We follow [Roh et al., 2021] for data generation, but with a slight modification to make the dataset more unbalanced. To study the empirical relationship between accuracy, fairness, and data heterogeneity, we split the dataset in different ways to obtain desired levels of data heterogeneity. More details are given in Sec. C.3 (real) We use three benchmark datasets: Adult [Dua and Graff, 2017] with 48,842 samples, COMPAS [ProPublica, 2021] with 7,214 samples, and Bank [Moro et al., 2014] with 45,211 samples. We follow [Du et al., 2021]'s method to preprocess and split Adult into two clients and [Jiang and Nachum, 2020]'s method to preprocess COMPAS and Bank. Then, we split COMPAS into two clients based on age and split Bank into three clients based on the loan decision. Note that all the datasets are split in heterogeneous ways.

Results Table[1] reports the test accuracy and DP disparity of four base-lines and FedFB. As expected, the resulting fairness level of FedFB is close to that of CFL. Besides, we observe the poor performance of LFT+Ensemble and LFT+FEDAVG, which is due to the fundamental limitation of LFT+Ensemble and LFT+FEDAVG. Table[2] reports the accuracy and fairness of each method under different data heterogeneity. FedFB is observed to be robust to data heterogeneity. This agrees with our expectation as FedFB closely mimics the operation of the centralized FB, which is not affected by data heterogeneity. We make a more thorough comparison between CFL and FedFB by plotting the accuracy-fairness tradeoff curves in Fig. [3] (b). To demonstrate the performance gain does not come at the cost of privacy loss, we restrict FedFB to only exchange 10 bits of information per communication round. Fig. [3] (b) shows that FedFB still performs well with quantized communications.

As suggested in [Ezzeldin et al., 2021], we use logistic regression to compare our approach with FAIRFED and AGNOSTICFAIR. Since FAIRFED is only applicable to single binary sensitive attribute cases, we report the performance on synthetic and Adult datasets. Fig. [4] shows that FedFB achieves the best accuracy-fairness performance among the three methods. As shown in Fig. [5], FedFB outperforms the other approaches on most of the tested datasets, sometimes nearly matching the performance of CFL. Note that LFT+FEDAVG sometimes gets a strictly worse performance than FEDAVG. This is because the average of fair models may not be fair at all.

Figure 6: Performance comparison in terms of accuracy and DP disparity on logistic regression. FedFB achieves good accuracy and fairness, compared to FAIRFED and AGNOSTICFAIR.
5.4 FedFB evaluation on client parity

We evaluate the performance of FedFB in achieving client parity (CP) and compare it with the state-of-the-art algorithms for CP. We will measure CP disparity $\Delta L = \max_{i \neq j} |L(i) - L(j)|$, where $L(i)$ is the loss in the $i$th client.

**Baselines** We consider GIFAIR [Yue et al., 2021], Q-FFL [Li et al., 2020a], Ditto [Li et al., 2021], and the unconstrained baseline FEDAVG [McMahan et al., 2017]. Similar to FedFB, both GIFAIR and Q-FFL propose a modified aggregation protocol, under which clients share some additional information with the central server, which then accordingly adjusts the objective function used for the next round of training. The key difference is that while FedFB optimizes the coefficients for the primary objective terms (i.e., sample reweighting) by solving a bi-level optimization problem, GIFAIR updates the coefficient for the penalty terms, and Q-FFL implicitly updates the weights on each objective term based on nonlinear behaviors of polynomial functions, which is equivalent to the $\alpha$-fairness algorithm used in networking [Mo and Walrand, 2000]. The Ditto algorithm combines multitask learning with federated learning to learn a personalized classifier for each client, improving the accuracy of the clients with low accuracy.

**Datasets** We use the same datasets as before, but split the datasets according to their sensitive attributes to simulate the same setting as assumed by GIFAIR, Q-FFL, and Ditto.

**Results** Fig. 7 shows that FedFB offers competitive and stable performances in mitigating the model bias, especially in the high fairness region. Although Q-FFL achieves better accuracy and fairness on the synthetic data, under strict fairness constraint, FedFB and its private variant nearly achieve the highest accuracy on the other three datasets.

6 Conclusions

**Summary** We have investigated how to achieve group fairness under a decentralized setting. We developed a theoretical framework for decentralized fair learning algorithms and analyzed the performance of LFT+Ensemble, LFT+FEDAVG, and CFL. As a result, we showed that (1) federated learning can significantly boost model fairness even with only a handful number of participating clients, and (2) FEDAVG-based federated fair learning algorithms are strictly worse than the oracle upper bound of CFL. To close the gap, we propose FedFB. Our extensive experimental results demonstrate that our proposed solution FedFB achieves state-of-the-art performances.

**Open questions** (Theory) First, full theoretical characterization of accuracy-fairness tradeoff still remains open. Furthermore, a three-way tradeoff between accuracy, fairness, and privacy remains widely open. (Algorithm) It remains open whether or not FedFB can handle different fairness notions such as proportional fairness [Zhang et al., 2020b, Lyu et al., 2020a,b].

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A Appendix - LFT+Ensemble, LFT+FEDAVG, CFL analysis

In this section, we provide the concrete analysis for LFT+Ensemble, LFT+FEDAVG, and CFL. For illustration purposes, we will start by discussing two clients cases, and then extend the analysis into more clients cases in Sec. A.4. We begin with the analysis of CFL in Sec. A.1. Then we analyze LFT+Ensemble and LFT+FEDAVG. To be specific, in Sec. A.2, we analyze the limitation of LFT+Ensemble and present the formal version of Lemma 7 under two clients and Corollary 2. In Sec. A.3, we analyze LFT+FEDAVG and compare it with LFT+Ensemble and CFL, then we present the formal two-client version of Thm. 3 and Lemma 4. All the multi-client statements are included in Sec. A.4. We summarize the commonly used notations in Table 3.

### A.1 CFL analysis

In this section, we analyze the CFL classifier $f_{CFL}^{\epsilon}$ given in CFL($\epsilon$). We mainly derive the solution of CFL($\epsilon$) in Lemma 7. In Lemma 8 and Lemma 9, we summarize the properties of $f_{CFL}^{\epsilon}$. For a given classifier $f$, we define the mean difference to be

$$
MD(f) = \mathbb{P}(\hat{y} = 1 \mid a = 0) - \mathbb{P}(\hat{y} = 1 \mid a = 1).
$$

**Lemma 7.** Let $q \in (0, 1)$. Define $g(\cdot) : [-\infty, q] \rightarrow [-1, 1]$ as

$$
g(\lambda) = \int_{1-(1-\frac{a}{q})}^{1-(1-\frac{1}{q})} \mathbb{P}_0 - \int_{1-(1-\frac{a}{q})}^{\infty} \mathbb{P}_1.\]

then $f_{CFL}^{\epsilon} = \{[s(x, a) > 0] + \alpha [s(x, a) = 0] : \alpha \in [0, 1]\}$, where $s(x, 0) = 2\eta(x) - 1 + \frac{\lambda}{q}$, $s(x, 1) = 2\eta(x) - 1 - \frac{\lambda}{q}$, $\lambda = g^{-1}(\text{sign}(g(0)) \min\{\varepsilon, |g(0)|\})$. Here we denote the indicator function as $[E] : [E] = 1$ if $E$ is true, zero otherwise.

**Proof.** The proof is similar as Menon and Williamson [2018]. To solve CFL($\epsilon$), we first write the error rate and the fairness constraint as a linear function of $f$. Let $p_a(\cdot)$ be the pdf of $P_a$, where $a = 0, 1$. Denote the joint distribution of $x$ and $a$ as $p_{x,a}(x, a)$. Note that

$$
\mathbb{P}(\hat{y} \neq y) = \int_X \sum_{a \in A} [f(x, a)(1 - \eta(x)) + (1 - f(x, a))\eta(x)] p_{x,a}(x, a) \, dx
$$

$$
= \mathbb{E}_{x,a} f(x, a)(1 - 2\eta(x)) + \mathbb{P}(y = 1)
$$

and

$$
\mathbb{P}(\hat{y} = 1 \mid a = 0) - \mathbb{P}(\hat{y} = 1 \mid a = 1)
$$

$$
= \int_X f(x, 0)p_0(x) \, dx - \int_X f(x, 1)p_1(x) \, dx
$$

$$
= \int_X \sum_{a \in A} [a = 0] f(x, 0) \frac{p_{x,a}(x, a)}{\mathbb{P}(a = 0)} \, dx - \int_X \sum_{a \in A} [a = 1] f(x, 1) \frac{p_{x,a}(x, a)}{\mathbb{P}(a = 1)} \, dx
$$

$$
= \mathbb{E}_{x,a} \left[ f(x, 0) \frac{[a = 0]}{1-q} - f(x, 1) \frac{[a = 1]}{q} \right].
$$

Consequently, our goal becomes solving

$$
\min_{f \in \mathcal{F}} \mathbb{E}_{x,a} f(x, a)(1 - 2\eta(x)) + \mathbb{P}(y = 1)
$$

s.t. $|\mathbb{E}_{x,a} \left[ f(x, 0) \frac{[a = 0]}{1-q} - f(x, 1) \frac{[a = 1]}{q} \right]| \leq \varepsilon. \quad (3)$
Denote the function that minimizes the error rate (ERM) as $\tilde{f} \in \mathcal{F}$. It is easy to see that,

$$\tilde{f}(x) \in \left\{ [\| \eta(x) > 1/2 \| + \alpha [\| \eta(x) = 1/2 \|] : \alpha \in [0, 1] \right\}.$$ 

Next, consider the following three cases. In particular, we provide the proof for $|\text{MD}(\tilde{f})| \leq \varepsilon$ and $\text{MD}(\tilde{f}) > \varepsilon$. The proof for $\text{MD}(\tilde{f}) < -\varepsilon$ is similar as the proof for $\text{MD}(\tilde{f}) > \varepsilon$.

**Case 1.** $|\text{MD}(\tilde{f})| \leq \varepsilon$: ERM is already fair.

The solution to (3) and $\text{CFL}(\varepsilon)$ is $\tilde{f}$.

**Case 2.** $\text{MD}(\tilde{f}) > \varepsilon$: ERM is favoring group 0 over 1.

We will show that solving (3) is equivalent to solving an unconstrained optimization problem.

First, we will prove by contradiction that the solution $f^* \in \mathcal{F}$ to (3) satisfies $\text{MD}(f^*) = \varepsilon$. We use $f^* \in \mathcal{F}$ to denote the solution of (3). Suppose the above claim does not hold. Then we have $\text{MD}(f^*) < \varepsilon$. To show the contradiction, we construct a $f' \in \mathcal{F}$ that satisfies the fairness constraint and has a lower error rate than that of $f^*$. Let $f'$ be a linear combination of $f^*$ and $\tilde{f}$:

$$f' = af^* + (1-a)\tilde{f},$$

where $a = \frac{\text{MD}(\tilde{f}) - \varepsilon}{\text{MD}(f') - \text{MD}(\tilde{f})} \in (0, 1)$. Then we obtain

$$\text{MD}(f') = a\text{MD}(f^*) + (1-a)\text{MD}(\tilde{f}) = \frac{\varepsilon(\text{MD}(\tilde{f}) - \text{MD}(f^*))}{\text{MD}(f') - \text{MD}(\tilde{f})} = \varepsilon.$$ 

Denote the error rate $P \{ \tilde{y} \neq y \}$ as $e : \mathcal{F} \rightarrow [0, 1]$. Then the error rate of $f'$ is

$$e(f') = ae(f^*) + (1-a)e(\tilde{f}) < e(f^*),$$

which is inconsistent to the optimality assumption of $f^*$. Therefore, $\text{MD}(f^*) = \varepsilon$.

Now, solving $\text{CFL}(\varepsilon)$ is equivalent to solving

$$\min_{f \in \mathcal{F}} \mathbb{E}_{x,a}f(x,a)(1-2\eta(x)) + P(y = 1) s.t. \left| \mathbb{E}_{x,a} \left[ f(x, 0) \left[ a = 0 \right] - f(x, 1) \left[ a = 1 \right] \right] \right| = \varepsilon.$$ 

Furthermore, the optimization problem above is also equivalent to

\begin{equation}
\min_{f \in \mathcal{F}} \mathbb{E}_{x,a}f(x,a)(1-2\eta(x)) - \lambda \mathbb{E}_{x,a} \left( f(x, 0) \left[ a = 0 \right] - f(x, 1) \left[ a = 1 \right] \right) \end{equation} \tag{4}

$$s.t. \left| \mathbb{E}_{x,a} \left[ f(x, 0) \left[ a = 0 \right] - f(x, 1) \left[ a = 1 \right] \right] \right| = \varepsilon, \tag{5}$$

for all $\lambda \in \mathbb{R}$.

Next, our goal is to select a suitable $\lambda$ such that the constrained optimization problem above becomes an unconstrained problem, i.e., we will select a suitable $\lambda$ such that the minimizer to the unconstrained optimization problem (4) satisfies equality constraint (5).

Note that

$$\mathbb{E}_{x,a}f(x,a)(1-2\eta(x)) - \lambda \mathbb{E}_{x,a} \left( f(x, 0) \left[ a = 0 \right] - f(x, 1) \left[ a = 1 \right] \right)$$

$$= \mathbb{E}_{x,a}f(x,a) \left( 1-2\eta(x) - \lambda \left[ a = 0 \right] + \lambda \left[ a = 1 \right] \right),$$

then the solution to unconstrained optimization problem (4) is

$$\tilde{f} \in \left\{ [s(x,a) > 0] + \alpha [s(x,a) = 0] : \alpha \in [0, 1] \right\},$$

where

$$s(x,a) = \begin{cases} -1 + 2\eta(x) + \frac{\lambda}{1-q} & a = 0 \\ -1 + 2\eta(x) - \frac{\lambda}{q} & a = 1 \end{cases}.$$
Therefore, Lemma 7 implies that the best classifier of the CFL problem is equivalent to simply applying a constant threshold to the class-probabilities for each value of the sensitive feature.

Remark 5. Select $\alpha = 0$, then the solution to $\text{CFL}(\varepsilon)$ can be written as

$$f(x, a) = \begin{cases} \|\eta(x) > \frac{1}{2} - \frac{\lambda}{2(1-q)}\| & a = 0 \\ \|\eta(x) > \frac{1}{2} + \frac{\lambda}{2q}\| & a = 1 \end{cases}$$

(6)

Therefore, Lemma 7 implies that the best classifier of the CFL problem is equivalent to simply applying a constant threshold to the class-probabilities for each value of the sensitive feature.

Lemma 7 suggests the following property of the solution to $\text{CFL}(\varepsilon)$.
Lemma 8. If $f$ and $g$ are two solutions to $\text{CFL}^{(\varepsilon)}$, then $f = g$ almost everywhere.

For illustration purposes, we denote

$$
\lambda^{\text{CFL}}_{\varepsilon} = g^{-1}(\text{sign}(g(0))) \min \{\varepsilon, |g(0)|\}.
$$

(7)

Below we summarize some useful properties of $\lambda^{\text{CFL}}_{\varepsilon}$.

Lemma 9. The sign of $\lambda^{\text{CFL}}_{\varepsilon}$ and $\text{MD}^{(0)}(f^{\text{CFL}}_{\varepsilon})$ are determined by $g(0)$.

1. If $\varepsilon < |g(0)|$, then $\text{MD}^{(0)}(f^{\text{CFL}}_{\varepsilon}) = \lambda^{\text{CFL}}_{\varepsilon} \neq 0$, and $g(\lambda^{\text{CFL}}_{\varepsilon}) = \text{sign}(g(0)) \varepsilon$. If $|g(0)| \leq \varepsilon$, then $\lambda^{\text{CFL}}_{\varepsilon} = 0$ and $\text{MD}^{(0)}(f^{\text{CFL}}_{\varepsilon}) = g(0)$.

2. If $g(0) > 0$ or $g(0) < 0$, then for any $\varepsilon \geq 0$, we have $\lambda \leq 0$ or $\lambda \geq 0$, respectively.

Proof. The first property follows directly from the definition of $\lambda^{\text{CFL}}_{\varepsilon}$. Next, we prove the second property.

When $\varepsilon > |g(0)|$, we have $\lambda = 0$ and the first property holds. When $g(0) > \varepsilon > 0$, by the definition of $\lambda^{\text{CFL}}_{\varepsilon}$, we have $\lambda = g^{-1}(\varepsilon) < g^{-1}(g(0)) = 0$. When $g(0) < -\varepsilon < 0$, we have $\lambda = g^{-1}(-\varepsilon) > g^{-1}(g(0)) = 0$. Combining all the cases above yields the desired conclusion.

\[ \square \]

A.2 LFT+Ensemble analysis

With the analysis of CFL in Sec. 4.1, in this section, we analyze the LFT+Ensemble classifier $\text{LFT+Ensemble}(\varepsilon_i, \varepsilon)$ for the case of two clients. For illustration purpose, with $\ell = 2$, we denote $f^\varepsilon_{\varepsilon_i, \varepsilon} = f^\varepsilon_{\varepsilon_i} + f^\varepsilon_{\varepsilon_f}/2$.

In Sec. A.2.1, we introduce some notations for the LFT+Ensemble classifier $f^\varepsilon_{\varepsilon_i, \varepsilon}$ that follows from Lemma 7. In Sec. A.2.2, we analyze the limitation of $f^\varepsilon_{\varepsilon_i, \varepsilon}$ as stated in Sec. 4.1. To be more specific, we present the two clients' version of Lemma 7, a formal version of Corollary 2, and conclude their proof. In Sec. A.2.3, we analyze the performance gap between $f^\varepsilon_{\varepsilon_i, \varepsilon}$ and CFL classifier $f^{\text{CFL}}_{\varepsilon}$.

A.2.1 Problem setting

By Lemma 7, the solution to $f^{\varepsilon}_{\varepsilon_i, \varepsilon}$ is

$$
f^{\varepsilon}_{\varepsilon_i, \varepsilon} = \begin{cases} 
\frac{1}{2} \left[ \eta(x) > \frac{1}{2} - \frac{1}{2} \lambda^{\text{LFT+Ensemble}}_{\varepsilon_i} \right] & a = 0 \\
\frac{1}{2} \left[ \eta(x) > \frac{1}{2} + \frac{1}{2} \lambda^{\text{LFT+Ensemble}}_{\varepsilon_i} \right] & a = 1
\end{cases},
$$

and

$$
\lambda^{\text{LFT+Ensemble}}_{\varepsilon_i} = g^{-1}_i(\text{sign}(g_i(0))) \min \left\{ \varepsilon_i, |g_i(0)| \right\}.
$$

(8)

where the associated $\lambda^{\text{LFT+Ensemble}}_{\varepsilon_i}$ is defined as

$$
\lambda^{\text{LFT+Ensemble}}_{\varepsilon_i} = g^{-1}_i(\text{sign}(g_i(0))) \min \left\{ \varepsilon_i, |g_i(0)| \right\}.
$$

and

$$
g_i(\lambda) = \int_{[\lambda^{-1}(\frac{1}{2} - 2(1 - q)), +\infty]} d\mathcal{P}^{(0)}_i - \int_{[\lambda^{-1}(\frac{1}{2} + 2q), +\infty]} d\mathcal{P}^{(1)}_i.
$$

Note that $g_i(\lambda)$ is the mean difference on $i$th client $E_{x \sim \mathcal{P}^{(0)}_i} f(x, 0) - E_{x \sim \mathcal{P}^{(1)}_i} f(x, 1)$ of the classifier of the form $(6)$. Now, the demographic disparity for $f^{\varepsilon}_{\varepsilon_i, \varepsilon}$ can be written as

$$
\text{DP Disp}(f^{\varepsilon}_{\varepsilon_i, \varepsilon}) = \left| \frac{1}{4} \sum_{i=0}^{1} \left[ E_{x \sim \mathcal{P}^{(0)}_i} f^{\varepsilon}_{i, \varepsilon}(x, 0) - E_{x \sim \mathcal{P}^{(1)}_i} f^{\varepsilon}_{i, \varepsilon}(x, 1) \right] \right|
$$

$$
= \left| \frac{1}{4} \left( g_0(\lambda^{\text{LFT+Ensemble}}_{\varepsilon_0}) + g_0(\lambda^{\text{LFT+Ensemble}}_{\varepsilon_1}) + g_1(\lambda^{\text{LFT+Ensemble}}_{\varepsilon_0}) + g_1(\lambda^{\text{LFT+Ensemble}}_{\varepsilon_1}) \right) \right|.
$$

For ease of notation, we define local mean difference on $i$th client as $\text{MD}_i(f) = E_{x \sim \mathcal{P}_i} f(x, 0) - E_{x \sim \mathcal{P}_i} f(x, 1)$ and local demographic disparity on $i$th client as $\text{DP Disp}_i(f) = |\text{MD}_i(f)|$, where $i = 0, 1$. Since the overall distribution of the data samples is $x \sim \mathcal{P}_0 = \mathcal{P}_a^{(0)}/2 + \mathcal{P}_a^{(1)}/2$, $a = 0, 1, g$ (see the definition of $g$ in Lemma 7) and $g_0, g_1$ has the following relation:

$$
g(\lambda) = \frac{1}{2} g_0(\lambda) + \frac{1}{2} g_1(\lambda).
$$
A.2.2 Limitation of LFT+Ensemble

In this section, we mainly analyze the limitation of LFT+Ensemble in Lemma 10, which shows that $\ell_{\text{LFT+Ensemble}}$ cannot achieve 0 demographic disparity in certain cases. Corollary 11 is a specific example of Lemma 10.

**Lemma 10** (Formal version of Lemma 1 under two clients cases). Let $q \in (0, 1)$. Let $c = \min \{|g_0(0)|, |g_1(0)|\}$. Consider five different cases to derive the desired result.

First, we will prove that LFT+Ensemble achieves its lowest mean difference when $g_0(0), g_1(0) > 0$. We will discuss $g_1(0) > 0$ later.

By (8), we have $g_1(0) > 0$, and applying Lemma 9 to have $g_0(0) + 1 > 0$. This allows us to write

$$\psi(0, 0, 0, 0) = 2 \psi(0, 0, 0, 0) \geq \delta.$$  

Case 1. $\varepsilon_0 > |g_0(0)|, \varepsilon_1 < |g_1(0)|$: ERM is fair on both clients.

By (8), we have $\ell_{\text{LFT+Ensemble}} = 0$. Recall $g_0(0)$ is a monotone increasing function, we combine $g_1(0) > 0$ and Lemma 9 to have $g_0(g_1^{-1}(0)) < g_0(0) > 0$. Applying the above conclusion yields

$$\psi(0, 0, 0, 0) = 2 \psi(0, 0, 0, 0) \geq \delta.$$  

Case 2. $\varepsilon_0 \leq |g_0(0)|, \varepsilon_1 > |g_1(0)|$: ERM is unfair on client 0, but fair on client 1.

Applying (8) results in $\ell_{\text{LFT+Ensemble}} = 0$. By the fact that $g_0(\cdot)$ is a strictly monotone increasing function, we have $\ell_{\text{LFT+Ensemble}} = g_0^{-1}(0) > g_0^{-1}(0) = 0$. Applying the above conclusion yields

$$\psi(0, 0, 0, 0) = 2 \psi(0, 0, 0, 0) \geq \delta.$$  

Case 3. $\varepsilon_0 \leq |g_0(0)|, \varepsilon_1 \leq |g_0(0)|$: ERM is unfair on both client 0 and client 1.

Applying (8) we have $\ell_{\text{LFT+Ensemble}} = g_0^{-1}(\varepsilon_0) > g_0^{-1}(0) = 0$. Then we have

$$\psi(0, 0, 0, 0) = 2 \psi(0, 0, 0, 0) \geq \delta.$$  

Case 4. $\varepsilon_0 > |g_0(0)|, \varepsilon_1 \leq |g_0(0)|$: ERM is fair on client 0 and very unfair on client 1.

By (8), we have $\ell_{\text{LFT+Ensemble}} = 0$. Then we obtain

$$\psi(0, 0, 0, 0) = 2 \psi(0, 0, 0, 0) \geq \delta.$$  

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Applying (8) implies $\lambda^{\text{LFT+Ensemble}}_1 = g^{-1}_1(\varepsilon_1) > g^{-1}_1(0)$. Therefore,

$$
\tag{10} \frac{1}{4} (g_0(0) + g_0(g_1^{-1}(\varepsilon_1)) + \varepsilon_1 + g_1(0)) > (g_0(0) + g_1(0) + g_0(g_1^{-1}(0)) + \varepsilon_1)/4 > \psi(g_0(0), 0) \geq \delta.
$$

Combining all the cases above, we conclude that when $g_1(0) > 0$ DP Disp$(\psi^{\text{LFT+Ensemble}}_{\varepsilon_0, \varepsilon_1}) \geq \delta = \min\{|\psi(\varepsilon_0, \varepsilon_1)| : \varepsilon_0, \varepsilon_1 \in [0, c]\} > 0$ for all $\varepsilon_0, \varepsilon_1 \in [0, 1]$.

Now we consider $g_1(0) < 0$, by the setting $|g_0(0)| < |g_1(0)|$ and the assumption $g_0(0)g_1(0) < 0$ and $\psi(\varepsilon_0, \varepsilon_1)(g_0(0) + g_1(0)) > 0$, we have $g_0(0) > 0$ and $\psi(\varepsilon_0, \varepsilon_1) < 0$ for all $\varepsilon_0, \varepsilon_1 \in [0, c]$. Following similar computation above, Case 1 - Case 5 become:

Case 1. $\varepsilon_0 > |g_0(0)|, \varepsilon_1 > |g_1(0)|$. Now we have $0 < g_0(0) < g_0(g_1^{-1}(0))$, thus

$$
\tag{10} \frac{1}{4} \left( \frac{1}{4} g_0(0) + \frac{1}{4} g_1(0) + \frac{1}{4} g_0(g_1^{-1}(0)) \right) = 2\psi(g_0(0), 0) \leq -\delta.
$$

Case 2. $\varepsilon_0 \leq |g_0(0)|, \varepsilon_1 > |g_1(0)|$. Now we have $g_0(0) > \varepsilon_0, g_1(\lambda^{\text{LFT+Ensemble}}_{\varepsilon_0}) < g_1(0)$, thus

$$
\tag{10} \frac{1}{2} g_0(0) + \frac{1}{2} g_1(0) < 2\psi(g_0(0), 0) \leq -\delta.
$$

Case 3. In this case we have

$$
\tag{10} \frac{1}{4} (\varepsilon_0 - \varepsilon_1 + g_0(g_1^{-1}(\varepsilon_1)) + g_1(g_0^{-1}(\varepsilon_0))) = \psi(\varepsilon_0, \varepsilon_1) \leq -\delta.
$$

Case 4. Now we have $\lambda^{\text{LFT+Ensemble}}_1 = g_1^{-1}(-\varepsilon_1) < g_1^{-1}(0)$, thus

$$
\tag{10} \frac{1}{4} (\varepsilon_0 + \varepsilon_1 + g_0(g_1^{-1}(\varepsilon_1)) + g_1(g_0^{-1}(\varepsilon_0))) = \psi(\varepsilon_0, \varepsilon_1) \leq -\delta.
$$

Case 5. Now we have $\lambda^{\text{LFT+Ensemble}}_1 = g_1^{-1}(\varepsilon_1) < g_1^{-1}(0)$, thus

$$
\tag{10} \frac{1}{4} \left( \frac{1}{4} g_0(0) + \frac{1}{4} g_1(0) + \frac{1}{4} g_0(g_1^{-1}(0)) \right) = \psi(g_0(0), 0) \geq -\delta.
$$

Then we conclude the proof.

**Remark 6.** Note that $c$ is the smallest local demographic disparity the ERM achieves on clients. The condition $g_0(0)g_1(0) < 0$ implies that the ERM is favoring different groups in different clients. The condition $\psi(\varepsilon_0, \varepsilon_1)(g_0(0) + g_1(0)) > 0$ for all $\varepsilon_0, \varepsilon_1 \in [0, c]$ implies that $\psi^{\text{LFT+Ensemble}}_{\varepsilon_0, \varepsilon_1}$ favors the same group as ERM when the constraint is very tight. If the conditions above hold, Lemma [10] suggests that there exists a lower bound of all the demographic disparity that LFT+Ensemble can achieve. In particular, if the conditions above hold, LFT+Ensemble fails to achieve perfect demographic parity.

Among the conditions of Lemma [10], $g_0(0)g_1(0) < 0$ can be satisfied by the distribution with high data heterogeneity. To demonstrate the condition $\psi(\varepsilon_0, \varepsilon_1)(g_0(0) + g_1(0)) > 0$ for all $\varepsilon_0, \varepsilon_1 \in [0, c]$ can be satisfied, we consider a limiting Gaussian case. The following corollary serves as an example that satisfies the conditions of Lemma [10] and provides a more explicit expression of the lowest demographic disparity LFT+Ensemble can reach in the Gaussian case.

**Corollary 11** (Formal version of Corollary 2). Let $q = 0.5, \eta(x) = \frac{1}{1 + e^{-x}}, \mathcal{P}_a(i) = N(\mu_a(i), \sigma^{(i2)})$ where $(\mu_0 - \mu_1^{(0)}), (\mu_0^{(1)} - \mu_1^{(1)}) < 0$. Then DP Disp$(\psi^{\text{LFT+Ensemble}}_{\varepsilon_0, \varepsilon_1}) \geq \delta = \frac{1}{4}|g_0(0) + g_1(0)| > 0$ for all $\varepsilon_0, \varepsilon_1 \in [0, 1]$ if one of the following condition holds:

1. $\sigma^{(0)} > \sigma^{(1)}, |\mu_0^{(0)}|, |\mu_1^{(0)}|, |\mu_0^{(1)}|, |\mu_1^{(1)}|$ and $\mu_0 > \mu_1$: client 0 has much larger variance than client 1, and client 1 is favoring group 0;
2. $\sigma^{(1)} > \sigma^{(0)}, |\mu_0^{(0)}|, |\mu_1^{(0)}|, |\mu_0^{(1)}|, |\mu_1^{(1)}|$ and $\mu_0 > \mu_1$: client 1 has much larger variance than client 0, and client 0 is favoring group 0.
Proof. In this example, note that local mean difference function of $\lambda$ can be written as:

$$g_i(\lambda) = \Phi\left(\frac{\eta^{-1}\left(\frac{1}{2} + \lambda\right) - \mu_1^{(i)}}{\sigma^{(i)}}\right) - \Phi\left(\frac{\eta^{-1}\left(\frac{1}{2} - \lambda\right) - \mu_0^{(i)}}{\sigma^{(i)}}\right),$$  \hspace{1cm} (11)

where $\Phi(\cdot)$ is the CDF of the standard Gaussian distribution.

We only provide the proof for condition 1, and the proof for condition 2 is similar. Assume condition 1 holds. By $(\mu_0^{(i)} - \mu_1^{(i)})(\mu_0^{(i)} - \mu_1^{(i)}) < 0$ and $\mu_0^{(i)} - \mu_1^{(i)} > 0$, we have $\mu_0^{(i)} - \mu_1^{(i)} < 0$. By (11), we have $g_0(0) < 0$ and $g_1(0) > 0$. Consequently, combining Lemma 9 and (9) yields

$$\psi(\varepsilon_0, \varepsilon_1) = \frac{1}{4} \left( g_0(g_1^{-1}(\varepsilon_1)) + g_1(g_0^{-1}(\varepsilon_0)) - \varepsilon_0 + \varepsilon_1 \right).$$

First we show that $\psi(\varepsilon_0, \varepsilon_1)$ reaches its minimum either at $(c, 0)$ or $(0, c)$ by taking the derivative of $\psi$, where $c = \min\{|g_0(0)|, |g_1(0)|\}$ is the smallest local demographic disparity. And then, we will estimate the minimum of $\psi$ on $[0, 1] \times [0, 1]$.

We take the derivative of $\psi$ with respect to $\varepsilon_i$ and get

$$\frac{\partial \psi}{\partial \varepsilon_i}(\varepsilon_0, \varepsilon_1) = \text{sign}(g_i(0)) \left( 1 + \frac{g_1^{-1}(\text{sign}(g_i(0))\varepsilon_i))}{g_0^{-1}(\text{sign}(g_i(0))\varepsilon_i))} \right) / 4.$$

By condition 1, we have $g_0(0) = \Phi(-\frac{\mu_0^{(i)}}{\sigma^{(i)}}) - \Phi(-\frac{\mu_0^{(i)}}{\sigma^{(i)}}) \approx 0$, thus $|g_0(0)| \ll |g_1(0)|$ and $c = |g_0(0)|$. Since $g_i$ are increasing function, $i = 0, 1$, we have $g_i'(c) > 0$, and thus $\frac{\partial \psi}{\partial \varepsilon_0} < 0, \frac{\partial \psi}{\partial \varepsilon_1} > 0$. Therefore, $\psi$ reaches its extreme value at $(0, c)$ and $(c, 0)$.

Now, we evaluate

$$\psi(0, c) = (g_0(g_1^{-1}(c)) + g_1(g_0^{-1}(0)) - g_0(0))/4 > \frac{g_1(0) - g_0(0) + g_0(g_1^{-1}(c))}{4},$$

where the inequality comes from $g_0(0) < 0$ to have $g_1(g_0^{-1}(0)) > g_0(0)$. And at $(c, 0)$, we have

$$\psi(c, 0) = (g_0(g_1^{-1}(0)) + g_1(0) + g_0(0))/4.$$

In what follows, we will show that $\min\{\psi(c, 0), \psi(0, c)\} \approx \frac{\sigma(0) + g_0(0)}{4}$ by proving that $0 > g_0(g_1^{-1}(c)) > g_0(g_1^{-1}(0)) \approx 0$.

Consider $\psi(c, 0)$ and $\psi(0, c)$. Since $g_1(0) > c, g_0(0) < 0$, we have

$$g_0(g_1^{-1}(0)) < g_0(g_1^{-1}(c)) < g_0(0) < 0.$$

Therefore, the only thing left is to show $g_0(g_1^{-1}(0)) \approx 0$. We divide the rest of the proof into the following three cases.

Case 1. $\mu_1^{(i)} < \mu_0^{(i)} < 0$: on client 1, the local classifier is favoring group 0 over group 1, and the positive rate of both groups are under $\frac{1}{2}$.

Clearly, under this case, we have $\int_{\eta^{-1}(1/2, \infty)} dP_0^{(i)} = \int_{(0, \infty)} dP_0^{(i)} < 1/2$. We select $\lambda' < 0$ such that $\eta^{-1}(\frac{1}{2} + \lambda') = \mu_1^{(i)}$. Then we have $\int_{\eta^{-1}(\frac{1}{2} + \lambda', \infty)} dP_1^{(i)} = \int_{\mu_1^{(i)}, \infty} dP_1^{(i)} = 1/2$, while $\int_{\eta^{-1}(\frac{1}{2} - \lambda', \infty)} dP_0^{(i)} < 0$. Thus we get $g_1(\lambda') < 0$. Combining $g_1(0) > 0$ and intermediate value theorem results in $\lambda' < g_1^{-1}(0) < 0$. Then we obtain

$$\mu_1^{(i)} = \eta^{-1}\left(\frac{1}{2} + \lambda'\right) < \eta^{-1}\left(\frac{1}{2} + g_1^{-1}(0)\right) < 0$$

$$-\mu_1^{(i)} = \eta^{-1}\left(\frac{1}{2} - \lambda'\right) > \eta^{-1}\left(\frac{1}{2} - g_1^{-1}(0)\right) > 0 \quad (\eta(x) - \frac{1}{2} \text{ is odd}).$$  \hspace{1cm} (12)

By plugging (12) into (11), we have the other side of $g_0(g_1^{-1}(0)) < 0$ is bounded by $g_0(\lambda') = \Phi\left(\frac{\mu_1^{(i)} - \mu_0^{(i)}}{\sigma^{(i)}}\right) - \Phi\left(\frac{-\mu_1^{(i)} - \mu_0^{(i)}}{\sigma^{(i)}}\right)$. Since $\sigma^{(i)} \gg |\mu_1^{(i)}|, |\mu_0^{(i)}|$, we get $g_0(g_1^{-1}(0)) \approx 0$.  

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Case 2. $0 < \mu_1^{(1)} < \mu_0^{(1)}$: on client 1, the local classifier is favoring group 0 over group 1, and the positive rate of both groups are above $\frac{1}{2}$.

This proof of this case is similar to Case 1.

Case 3. $\mu_1^{(1)} < 0 < \mu_0^{(1)}$: with respect to the local classifier trained by client 1, the positive rate of group 0 is above $\frac{1}{2}$ while that of group 1 is under $\frac{1}{2}$.

Without any loss of generality, we assume $|\mu_1^{(1)}| < |\mu_0^{(1)}|$. Select $\lambda'' < 0$ such that $\eta(\frac{1}{2} - \lambda'') = \mu_0^{(1)}$. Clearly $\int_{[y^{-1}(\frac{1}{2} - \lambda''), +\infty)} dP_0^{(1)} = \int_{[\mu_0^{(1)}, +\infty)} dP_0^{(1)} = \frac{1}{2}$, while

$$\int_{[y^{-1}(\frac{1}{2} + \lambda''), +\infty)} dP_1^{(1)} = \int_{[-\mu_1^{(1)}, +\infty)} dP_1^{(1)} > \int_{[-\mu_1^{(1)}, +\infty)} dP_1^{(1)} > \frac{1}{2}.$$ 

Consequently, we get $g_1(\lambda'') < 0$ and $\lambda'' < g_1^{-1}(0) < 0$. Then we draw the same conclusion as (12). Therefore, $g_0(g_1^{-1}(0)) \approx 0$.

Combining all three cases above, we get $\delta > 0$. Then applying Lemma [10] we complete the proof.

A.2.3 Comparison between LFT+Ensemble and CFL

In this section, we compare the performance of LFT+Ensemble and CFL. In Lemma 12 and Lemma 13 we illustrate the conditions for LFT+Ensemble to have the same performance as CFL. In Lemma 14 Lemma 15 and Lemma 16 we illustrate the scenarios when CFL outperforms LFT+Ensemble.

To do the comparison, first we introduce some additional notations. Define the accuracy of a classifier $f$ as

$$\text{Acc}(f) = \mathbb{P}(\hat{y} = y) = \mathbb{P}(y = 0)\mathbb{E}_{x, a|y=0}[1 - f(x, a)] + \mathbb{P}(y = 1)\mathbb{E}_{x, a|y=0}f(x, a).$$

Given the required global demographic disparity $\varepsilon$, define the performance of $f^{LFT+Ensemble}_{\varepsilon_0, \varepsilon_1}$ as:

$$LFT+Ensemble(\varepsilon_0, \varepsilon_1; \varepsilon) = \begin{cases} \text{Acc}(f^{LFT+Ensemble}_{\varepsilon_0, \varepsilon_1}) & \text{DP Disp}(f^{LFT+Ensemble}_{\varepsilon_0, \varepsilon_1}) \leq \varepsilon, \\ 0 & \text{otherwise} \end{cases},$$

and define performance of $f^{CFL}_{\varepsilon}$ as:

$$\text{CFL}(\varepsilon) = \text{Acc}(f^{CFL}_{\varepsilon}).$$

Now we are able to compare the performance between LFT+Ensemble and CFL with the metric $LFT+Ensemble(\varepsilon_0, \varepsilon_1; \varepsilon)$ and $\text{CFL}(\varepsilon)$. In particular, we will show that, under some mild conditions, $\max_{\varepsilon_0, \varepsilon_1} LFT+Ensemble(\varepsilon_0, \varepsilon_1; \varepsilon) < \text{CFL}(\varepsilon)$, which implies the gap between LFT+Ensemble and CFL is inevitable.

We begin with the following two lemmas, which describe the cases that $LFT+Ensemble(\varepsilon_0, \varepsilon_1; \varepsilon) = \text{CFL}(\varepsilon)$.

**Lemma 12.** Let $\varepsilon \in (0, 1)$. Given an LFT+Ensemble classifier $f^{LFT+Ensemble}_{\varepsilon_0, \varepsilon_1}$ such that DP Disp$(f^{LFT+Ensemble}_{\varepsilon_0, \varepsilon_1}) \leq \varepsilon$ and a CFL classifier $f^{CFL}_{\varepsilon}$, we have $LFT+Ensemble(\varepsilon_0, \varepsilon_1; \varepsilon) \leq \text{CFL}(\varepsilon)$. The equality holds if and only if $\lambda^{LFT+Ensemble}_{\varepsilon_0} = \lambda^{CFL}_{\varepsilon_1}$, where $\lambda^{CFL}_{\varepsilon}$ is defined in (8).

**Proof.** The proof is straightforward. Clearly, since $f^{CFL}_{\varepsilon}$ is the optimizer to $\text{CFL}(\varepsilon)$, we have $LFT+Ensemble(\varepsilon_0, \varepsilon_1; \varepsilon) \leq \text{CFL}(\varepsilon)$. By Lemma 7 according to the form of the solution to $\text{CFL}(\varepsilon)$, if $f^{LFT+Ensemble}_{\varepsilon_0, \varepsilon_1}$ is the solution to $\text{CFL}(\varepsilon)$ if and only if $\lambda^{LFT+Ensemble}_{\varepsilon_0} = \lambda^{LFT+Ensemble}_{\varepsilon_1} = \lambda^{CFL}_{\varepsilon}$. Thus complete the proof.

**Lemma 13.** Let $\varepsilon \in (0, 1)$. If the ERM is already fair, i.e., $\varepsilon \geq |g(0)|$, then

$$\max_{\varepsilon_0, \varepsilon_1} LFT+Ensemble(\varepsilon_0, \varepsilon_1; \varepsilon) = \text{CFL}(\varepsilon).$$

**Proof.** Since the ERM is already fair, $f^{CFL}_{\varepsilon}$ is the solution to $\text{ERM}=\{\eta(x) > 1/2\}$. Therefore, we take $\varepsilon_0 = \varepsilon_1 = 1$, and $f^{LFT+Ensemble}_{\varepsilon_0, \varepsilon_1}$ also equals to ERM. Thus we conclude the lemma.
The next two lemmas describe the cases that LFT+Ensemble(\(\varepsilon_0, \varepsilon_1; \varepsilon\)) < CFL(\(\varepsilon\)).

**Lemma 14.** Let \(q \in (0, 1)\). If \(g_0(0)g_1(0) < 0\), \(\max_{\varepsilon_0, \varepsilon_1} LFT+\text{Ensemble}(\varepsilon_0, \varepsilon_1; \varepsilon) < \text{CFL}(\varepsilon)\) for all \(\varepsilon < |g(0)|\).

**Proof.** In this proof, we only consider the case that \(g_1(0) > 0\), \(|g_1(0)| \geq |g_0(0)|\). The proof for \(g_1(0) > 0\) or \(|g_1(0)| \geq |g_0(0)|\) is similar. Next, we divide the proof into two cases.

**Case 1.** \(\max_{\varepsilon_0, \varepsilon_1} LFT+\text{Ensemble}(\varepsilon_0, \varepsilon_1; \varepsilon) = 0\): LFT+Ensemble cannot achieve \(\varepsilon\) global demographic disparity.

The conclusion holds.

**Case 2.** \(\max_{\varepsilon_0, \varepsilon_1} LFT+\text{Ensemble}(\varepsilon_0, \varepsilon_1; \varepsilon) > 0\): LFT+Ensemble can achieve \(\varepsilon\) global demographic disparity.

Since \(\varepsilon < |g(0)|\), by Lemma 9, we have \(\lambda_{\text{CFL}}^{0} \neq 0\). Next, we solve \(f_{0, \varepsilon_0}^{LFT+\text{Ensemble}}\) by solving the local version of CFL(\(\varepsilon\)). Combining Lemma 9, \(g_1(0) > 0\) and \(g_0(0) < 0\) yields \(\lambda_{\varepsilon_0}^{LFT+\text{Ensemble}} \geq 0\), \(\lambda_{\varepsilon_1}^{LFT+\text{Ensemble}} = 0\). If \(\lambda_{\varepsilon_0}^{LFT+\text{Ensemble}} = \lambda_{\varepsilon_1}^{LFT+\text{Ensemble}}\), then \(\lambda_{\varepsilon_0}^{LFT+\text{Ensemble}} = \lambda_{\varepsilon_1}^{LFT+\text{Ensemble}} = 0 \neq \lambda_{\varepsilon}^{\text{CFL}}\). Thus, we conclude the lemma by applying Lemma 12.

**Remark 7.** Lemma 14 implies that if ERM is favoring different groups in different clients, there exists an inevitable gap between the performance of LFT+Ensemble and that of CFL.

**Lemma 15.** Let \(q \in (0, 1)\). Let \(\tau = \min \{|g(0)|, \max\{\text{sign}(g(0))g(g_0^{-1}(0)), \text{sign}(g(0))g(g_1^{-1}(0))\}\}\).

If \(g_0(0)g_1(0) > 0\), we have

\[
\max_{\varepsilon_0, \varepsilon_1} LFT+\text{Ensemble}(\varepsilon_0, \varepsilon_1; \varepsilon) \begin{cases} = \text{CFL}(\varepsilon) & \text{for all } \varepsilon \geq \tau \\ < \text{CFL}(\varepsilon) & \text{o.w.} \end{cases}
\]

**Proof.** Without any loss of generality, assume \(g_0(0), g_1(0) > 0\). Then by Lemma 9, we have \(\lambda_{\varepsilon_0}^{LFT+\text{Ensemble}} \leq 0\), \(\lambda_{\varepsilon_1}^{LFT+\text{Ensemble}} \leq 0\) for all \(\varepsilon_0, \varepsilon_1 \in [0, 1]\), and \(g(0) = (g_0(0) + g_1(0))/2 > 0\).

To study the performance of LFT+Ensemble when \(g_0(0)g_1(0) < 0\), recall that we use \(f_i^{\varepsilon}\) to denote the local classifier trained by client \(i\) in LFT+Ensemble analysis. Therefore, \(f_0^{\varepsilon}\) is the local classifier trained by client \(i\) that achieves perfect local fairness.

Next, we discuss two cases to prove the result. In Case 1, we will show that \(\tau = 0\) and then prove that \(\max_{\varepsilon_0, \varepsilon_1} LFT+\text{Ensemble}(\varepsilon_0, \varepsilon_1; \varepsilon) = \text{CFL}(\varepsilon)\); in Case 2, we will show that \(\tau > 0\), and then prove that \(\max_{\varepsilon_0, \varepsilon_1} LFT+\text{Ensemble}(\varepsilon_0, \varepsilon_1; \varepsilon) < \text{CFL}(\varepsilon)\) when \(\varepsilon < \tau\).

**Case 1.** \(f_0^{\varepsilon} = f_1^{\varepsilon}\): the two local classifiers that achieve perfect local fairness are equal.

When \(\varepsilon \geq g(0)\), the conclusion holds by directly applying Lemma 13. Therefore, in what follows, we focus on the case that \(\varepsilon < g(0)\).

Next, we will first show that when \(\varepsilon = 0\), \(\lambda_0^{LFT+\text{Ensemble}} = \lambda_0^{LFT+\text{Ensemble}_1} = \lambda_0^{\text{CFL}}\), which implies that \(f_0^{\varepsilon} = f_1^{\varepsilon} = f_0^{\text{CFL}}\).

Since \(f_0^{\varepsilon} = f_1^{\varepsilon}\), we have \(\lambda_0^{LFT+\text{Ensemble}} = \lambda_0^{LFT+\text{Ensemble}_1}\), and thus \(g_0^{-1}(0) = g_1^{-1}(0)\). Consequently, we get

\[
g(\lambda_0^{LFT+\text{Ensemble}}) = g(\lambda_0^{LFT+\text{Ensemble}_1}) = g(g_0^{-1}(0)) = \frac{g_0(g_0^{-1}(0)) + g_1(g_1^{-1}(0))}{2} = 0 = g(\lambda_0^{\text{CFL}}).
\]

By the monotonicity of \(g\), we have \(\lambda_0^{LFT+\text{Ensemble}} = \lambda_0^{LFT+\text{Ensemble}_1} = \lambda_0^{\text{CFL}}\) and \(\tau = 0\). Next, we will show that, for \(\varepsilon \neq 0\), there also exists \(\varepsilon_0, \varepsilon_1 \in [0, 1]\) such that \(\lambda_{\varepsilon_0}^{LFT+\text{Ensemble}} = \lambda_{\varepsilon_1}^{LFT+\text{Ensemble}} = \lambda_{\varepsilon}^{\text{CFL}}\), which implies \(f_0^{LFT+\text{Ensemble}} = f_1^{LFT+\text{Ensemble}} = f_\varepsilon^{\text{CFL}}\).

Consider \(\varepsilon \neq 0\). Select \(\varepsilon_i = g_i(\lambda_{\varepsilon}^{\text{CFL}}), i = 0, 1\). By Lemma 7 and the monotonicity of \(g\), we have \(\lambda_{\varepsilon}^{\text{CFL}} < \lambda_{\varepsilon_1}^{\text{CFL}} < 0\). Therefore, \(\varepsilon_i = g_i(\lambda_{\varepsilon}^{\text{CFL}}) > g_i(\lambda_{\varepsilon_1}^{\text{CFL}}) = 0\). By Lemma 7, we get \(\lambda_{\varepsilon_i}^{LFT+\text{Ensemble}} = g_i^{-1}(\varepsilon_i)\) for \(i = 0, 1\). By the selection of \(\varepsilon_i\), we have \(\lambda_{\varepsilon}^{\text{CFL}} = g_i^{-1}(\varepsilon_i)\). Therefore, \(\lambda_{\varepsilon}^{\text{CFL}} = \lambda_{\varepsilon_0}^{LFT+\text{Ensemble}} = \lambda_{\varepsilon_1}^{LFT+\text{Ensemble}}\) when \(\varepsilon \neq 0\).

Combining all the discussion above yields \(f_0^{LFT+\text{Ensemble}} = f_\varepsilon^{\text{CFL}}\) for all \(\varepsilon < g(0)\), thus we conclude \(\max_{\varepsilon_0, \varepsilon_1} LFT+\text{Ensemble}(\varepsilon_0, \varepsilon_1; \varepsilon) = \text{CFL}(\varepsilon)\) for all \(\varepsilon < g(0)\). Consequently, the lemma holds under Case 1.
Case 2. \( f_0^{g_1} \neq f_1^{g_1} \): the two local classifiers that achieve perfect local fairness are different.

The key idea of this proof is: when \( MD_0(f_{\epsilon, 0}^{CFL}), MD_1(f_{\epsilon, 1}^{CFL}) \geq 0 \), then we can always select \( \epsilon_0 = g_0(\lambda_{\epsilon, 0}^{CFL}) = MD_0(f_{\epsilon, 0}^{CFL}) > 0 \), \( \epsilon_1 = g_1(\lambda_{\epsilon, 1}^{CFL}) = MD_1(f_{\epsilon, 1}^{CFL}) > 0 \) such that \( \lambda_{\epsilon, 0}^{LFT+Ensemble} = g_0(\epsilon_0) = \lambda_{\epsilon, 0}^{CFL} \) and \( \lambda_{\epsilon, 1}^{LFT+Ensemble} = g_1^{-1}(\epsilon_1) = \lambda_{\epsilon, 1}^{CFL} \), thus \( f_{\epsilon, 0}^{LFT+Ensemble} = f_{\epsilon, 0}^{CFL} \); when \( MD_0(f_{\epsilon, 0}^{CFL})/MD_1(f_{\epsilon, 1}^{CFL}) = g_0(\lambda_{\epsilon, 0}^{CFL})g_1(\lambda_{\epsilon, 1}^{CFL}) < 0 \), however, by Lemma 4 for all \( \epsilon_0, \epsilon_1 \in [0, 1] \times [0, 1] \), we have \( g_0(\lambda_{\epsilon, 0}^{LFT+Ensemble})g_1(\lambda_{\epsilon, 1}^{LFT+Ensemble}) > 0 > g_0(\lambda_{\epsilon, 0}^{CFL})g_1(\lambda_{\epsilon, 1}^{CFL}) \), thus there exist \( i \in \{0, 1\} \) such that \( \lambda_{\epsilon, i}^{LFT+Ensemble} \neq \lambda_{\epsilon, i}^{CFL} \) and \( f_{\epsilon, i}^{LFT+Ensemble} \neq f_{\epsilon, i}^{CFL} \). Next, we will give rigorous proof.

Since \( f_0^{g_1} \neq f_1^{g_1} \), we have \( \lambda_{\epsilon, 0}^{LFT+Ensemble} \neq \lambda_{\epsilon, 1}^{LFT+Ensemble} \). By Lemma 7 we get \( g_0^{-1}(0) \neq g_1^{-1}(0) \). Without any loss of generality, assume \( g_0^{-1}(0) < g_1^{-1}(0) \), which implies \( g_1(g_0^{-1}(0)) < g_1(g_1^{-1}(0)) = 0 \) and \( g_0(g_0^{-1}(0)) = 0 < g_0(g_1^{-1}(0)) \). Thus we get

\[
g_1(g_0^{-1}(0)) < 0 < g_0(g_1^{-1}(0)).
\]

Combining the inequality above and \( g = \frac{2\epsilon_0 + \epsilon_1}{2} \) yields

\[
g_0(g_0^{-1}(0)) = g_1(g_0^{-1}(0)) < 0 = g_1(g_0^{-1}(0)) < g_0(g_1^{-1}(0)) = g_0(g_0^{-1}(0)).
\]

Thus we have

\[
\tau = \max\{\text{sign}(g_0)g(g_0^{-1}(0)), \text{sign}(g_0)g(g_0^{-1}(0))\} = g_1(g_0^{-1}(0)).
\]

When \( \epsilon \geq |g(0)| \), by Lemma 13 clearly we have \( \max_{\epsilon_0, \epsilon_1} LFT+Ensemble(\epsilon_0, \epsilon_1; \epsilon) = CFL(\epsilon) \).

For the other case \( \epsilon < |g(0)| \), by Lemma 9 we have \( \lambda = g_1^{-1}(\epsilon) \). Similar to Case 1, in order to achieve \( \lambda_{\epsilon, 0}^{LFT+Ensemble} = \lambda_{\epsilon, 1}^{LFT+Ensemble} = \lambda_{\epsilon, 1}^{CFL} \), we select \( \epsilon_1 = g_1(\lambda_{\epsilon, 1}^{CFL}) \).

When \( \epsilon < g(g_1^{-1}(0)) \),

\[
g_1(\lambda_{\epsilon, 1}^{CFL}) = g_1(g_1^{-1}(\epsilon)) < g_1(g_1^{-1}(g_1^{-1}(0))) = 0.
\]

Since \( g_1(0) > 0 \), by Lemma 9 we have \( g_1(\lambda_{\epsilon, 1}^{LFT+Ensemble}) \geq 0 \), from the monotonicity of \( g_1 \) we conclude \( \lambda_{\epsilon, 1}^{LFT+Ensemble} \neq \lambda_{\epsilon, 1}^{CFL} \). By Lemma 12 we have \( \max_{\epsilon_0, \epsilon_1} LFT+Ensemble(\epsilon_0, \epsilon_1; \epsilon) < CFL(\epsilon) \) for all \( \epsilon \leq \tau \).

When \( \epsilon \geq g(g_1^{-1}(0)) \), applying \( 13 \) we have

\[
g_1(0) > \epsilon_1 = g_1(\lambda) = g_1(g_1^{-1}(\epsilon)) \geq g_1(g_1^{-1}(g_1^{-1}(0))) = 0,
\]

where the first inequality comes from Case 1. Thus, \( \lambda_1 = g_1^{-1}(\epsilon_1) = \lambda \) as desired, and we obtain \( f_0^{LFT+Ensemble} = f_\epsilon^{CFL} \).

Combining both cases above yields the desired conclusion.

**Remark 8.** Recall that we use \( f_i^{g_i} \) to denote the local classifier trained by client \( i \) in LFT+Ensemble analysis. In the expression of \( \tau \):

\[
\min\{\{g(0), \max\{\text{sign}(g(0))g(g_0^{-1}(0)), \text{sign}(g(0))g(g_1^{-1}(0))\}\}\},
\]

\( g(0) \) is the demographic disparity of ERM, \( \text{sign}(g(0))g(g_0^{-1}(0)) \) is the demographic disparity of local classifier \( f_0^{g_1} \), and \( \text{sign}(g(0))g(g_1^{-1}(0)) \) is the demographic disparity of local classifier \( f_1^{g_1} \). According to the proof of Lemma 13 we obtain \( \max\{\text{sign}(g(0))g(g_0^{-1}(0)), \text{sign}(g(0))g(g_1^{-1}(0))\} > 0 \) if and only if two local classifiers which achieves perfect local fairness is equal, i.e., \( f_0^{g_1} = f_1^{g_1} \). Therefore, Lemma 13 implies that, if the ERM is favoring the same group on different clients and the two local classifiers which achieve perfect local fairness are unequal, then LFT+Ensemble performs strictly worse than CFL when the required demographic disparity is smaller than a certain value.

So far we assume a \( \sim Bern(\eta) \) for both client 0 and client 1. When both clients do not share the same \( q \), we can conclude that CFL outperforms LFT+Ensemble in the following lemma.

**Lemma 16.** Assume a \( \sim Bern(q_i) \) in client \( i \), and \( q_0 \neq q_1 \in (0, 1) \). Then \( \max_{\epsilon_0, \epsilon_1} LFT+Ensemble(\epsilon_0, \epsilon_1; \epsilon) < CFL(\epsilon) \) for all \( \epsilon < |g(0)| \).

**Proof.** We assemble the dataset from two clients to have \( x | a = a \sim \mathcal{P}_a = \frac{q_0}{q_0 + q_1}\mathcal{P}_a^{(0)} + \frac{q_1}{q_0 + q_1}\mathcal{P}_a^{(1)} \), \( a = 0, 1 \). We let \( f_{\epsilon, q}^{CFL} \) be the solution to CFL(\( \epsilon \)) with \( q = \frac{2\epsilon_0 + \epsilon_1}{2} \):

\[
f_{\epsilon, q}^{CFL} = \left\lfloor \frac{\epsilon x, a > 0}{\lambda_{\epsilon, q}^{CFL}} \right\rfloor,
\]

where \( s(x, 0) = 2\eta(x) - 1 + \frac{\lambda_{\epsilon, q}^{CFL}}{\Gamma - q} \), \( s(x, 1) = 2\eta(x) - 1 - \frac{\lambda_{\epsilon, q}^{CFL}}{q} \).
Given an LFT+Ensemble classifier \( f_{\text{LFT+Ensemble}}^{\epsilon} = (f_0^{\epsilon_0} + f_1^{\epsilon_1})/2 \) such that \( \text{DP Disp}(f_{\text{LFT+Ensemble}}^{\epsilon}) \leq \epsilon \), the solution reads
\[
f_1^{\epsilon_1} = \left[ s_1(x, a) > 0 \right],
\]
where \( s_1(x, 0) = 2\eta(x) - 1 - \frac{\lambda_{\text{LFT+Ensemble}}^{\epsilon}}{1 - q_1} \), \( s_1(x, 1) = 2\eta(x) - 1 - \frac{\lambda_{\text{LFT+Ensemble}}^{\epsilon}}{q_1} \).

We prove the lemma by contradiction argument. If \( \text{Acc}(f_{\text{LFT+Ensemble}}^{\epsilon}) = \text{CFL}(\epsilon) \), then \( f_{\text{LFT+Ensemble}}^{\epsilon} \) is a solution to \( \text{CFL}(\epsilon) \) with \( q = \frac{q_0 + q_1}{2} \). Since \( \epsilon < q(0) \), by Lemma 9 we have \( \lambda_{\epsilon}^{\text{CFL}} = 0 \). Without any loss of generality, assume \( \lambda_{\epsilon}^{\text{CFL}} < 0 \). Below we discuss three cases.

**Case 1.** \( \lambda_{\epsilon_0}^{\text{LFT+Ensemble}} = \lambda_{\epsilon_1}^{\text{LFT+Ensemble}} = 0 \): the LFT+Ensemble classifier is ERM.

In this case, \( f_0^{\epsilon_0} = f_1^{\epsilon_1} \). We have \( f_{\text{LFT+Ensemble}}^{\epsilon_0}(0, 1) = 1 \) for \( \eta(x) > \frac{1}{2} \), and \( f_{\text{CFL}}^{\epsilon_0} = 0 \) for \( \eta(x) < (1 - \lambda_{\epsilon_0}^{\text{CFL}})/(1 - q_1)/2 \). By Lemma 8 \( f_{\text{LFT+Ensemble}}^{\epsilon_0} ) is not a solution to \( \text{CFL}(\epsilon) \).

**Case 2.** \( \lambda_{\epsilon_0}^{\text{LFT+Ensemble}} \neq 0 \) or \( \lambda_{\epsilon_1}^{\text{LFT+Ensemble}} \neq 0 \), and \( \lambda_{\epsilon_0}^{\text{LFT+Ensemble}}\lambda_{\epsilon_1}^{\text{LFT+Ensemble}} = 0 \): the LFT+Ensemble classifier is not ERM, but one of the local classifier is ERM.

Without any loss of generality, let \( \lambda_{\epsilon_0}^{\text{LFT+Ensemble}} = 0 \), \( \lambda_{\epsilon_1}^{\text{LFT+Ensemble}} < 0 \). Then \( f_0^{\epsilon_0}(x, 0) = 1 \) for \( \eta(x) > \frac{1}{2} \), while \( f_1^{\epsilon_1}(x, 0) = 0 \) for \( \eta(x) < (1 - \lambda_{\epsilon_1}^{\text{LFT+Ensemble}})/(1 - q_1)/2 \). Thus we get
\[
f_{\text{LFT+Ensemble}}^{\epsilon_0}(x, 0) = \frac{1}{2} \text{ for } \frac{1}{2} < \eta(x) < (1 - \lambda_{\epsilon_1}^{\text{LFT+Ensemble}})/(1 - q_1)/2.
\]
By Lemma 8 \( f_{\text{LFT+Ensemble}}^{\epsilon_0} ) is not a solution to \( \text{CFL}(\epsilon) \).

**Case 3.** \( \lambda_{\epsilon_0}^{\text{LFT+Ensemble}}\lambda_{\epsilon_1}^{\text{LFT+Ensemble}} \neq 0 \): the local classifiers are not ERM.

When \( \frac{\lambda_{\epsilon_0}^{\text{LFT+Ensemble}}}{1 - q_0} \neq \frac{\lambda_{\epsilon_1}^{\text{LFT+Ensemble}}}{1 - q_1} \), without loss of generality, let \( \frac{\lambda_{\epsilon_0}^{\text{LFT+Ensemble}}}{1 - q_0} > \frac{\lambda_{\epsilon_1}^{\text{LFT+Ensemble}}}{1 - q_1} \). Then by the same argument in Case 2, we have
\[
f_{\text{LFT+Ensemble}}^{\epsilon_0}(x, 0) = \frac{1}{2} \text{ for } \frac{1}{2} < \eta(x) < \frac{1 - \lambda_{\epsilon_0}^{\text{LFT+Ensemble}}(1 - q_0)/(1 - q_1)}{2}.
\]
By Lemma 8 \( f_{\text{LFT+Ensemble}}^{\epsilon_0} ) is not a solution to \( \text{CFL}(\epsilon) \).

When \( \frac{\lambda_{\epsilon_0}^{\text{LFT+Ensemble}}}{q_0} \neq \frac{\lambda_{\epsilon_1}^{\text{LFT+Ensemble}}}{q_1} \), similarly, \( f_{\text{LFT+Ensemble}}^{\epsilon_0} ) is not a solution to \( \text{CFL}(\epsilon) \).

When \( \lambda_{\epsilon_0}^{\text{LFT+Ensemble}} = \frac{\lambda_{\epsilon_0}^{\text{LFT+Ensemble}}}{q_0} = \frac{\lambda_{\epsilon_1}^{\text{LFT+Ensemble}}}{q_1} \), and \( \lambda_{\epsilon_0}^{\text{LFT+Ensemble}} = \frac{\lambda_{\epsilon_1}^{\text{LFT+Ensemble}}}{1 - q_1} \), since \( \lambda_{\epsilon_0}^{\text{LFT+Ensemble}}\lambda_{\epsilon_1}^{\text{LFT+Ensemble}} = 0 \), we have
\[
\frac{q_0}{\lambda_{\epsilon_0}^{\text{LFT+Ensemble}}} = \frac{1 - q_0}{\lambda_{\epsilon_1}^{\text{LFT+Ensemble}}}, \quad \frac{1 - q_0}{\lambda_{\epsilon_0}^{\text{LFT+Ensemble}}} = \frac{1 - q_1}{\lambda_{\epsilon_1}^{\text{LFT+Ensemble}}},
\]
which leads to \( \frac{1}{\lambda_{\epsilon_0}^{\text{LFT+Ensemble}}} = \frac{1}{\lambda_{\epsilon_1}^{\text{LFT+Ensemble}}} \) and thus \( q_0 = q_1 \). This contradicts with the assumption that \( q_0 \neq q_1 \).

Combining all three cases yields desired conclusion.

A.3 LFT+FEDAVG analysis

In this section, we analyze LFT+FEDAVG for the case of two clients. For purpose of illustration, with \( I = 2 \), we denote \( f_{\epsilon_{i_{\text{FEDAVG}}}} = f_{\epsilon}^{\text{FEDAVG}} \) to be the solution to \( \text{LFT+FEDAVG}(\epsilon) \). In Sec. A.3.1, we present a formal version of Thm. 3 and show that compared to LFT+Ensemble, LFT+FEDAVG has strictly higher fairness. In Sec. A.3.2, we derive the solution to \( \text{LFT+FEDAVG}(\epsilon) \) and show it is equivalent to LFT+FEDAVG. In Sec. A.3.3, we analyze the limitation of LFT+FEDAVG and present a formal version of Lemma 4.

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[Note: The above text is a revised version of the original document, converted to a plain text representation to facilitate easier reading.]
A.3.1 Improve fairness via federated learning

Different to LFT+Ensemble, LFT+FedAvg can reach any $\varepsilon$ demographic disparity:

**Theorem 17** (Formal version of Thm. 3 under two clients cases). Let $q \in (0, 1)$. For all $\varepsilon \in [0, 1]$, there exists $\varepsilon_0, \varepsilon_1 \in [0, 1]$ such that $DP Disp(f_{\varepsilon_0, \varepsilon_1}^{\text{LFT+Ensemble}}) \leq \varepsilon$. Thus under the condition in Lemma 10 we have

$$\min_{\varepsilon_0, \varepsilon_1 \in [0, 1]} DP Disp(f_{\varepsilon_0, \varepsilon_1}^{\text{LFT+Ensemble}}) \geq \min_{\varepsilon_0, \varepsilon_1 \in [0, 1]} DP Disp(f_{\varepsilon_0, \varepsilon_1}^{\text{LFT+FedAvg}}) = 0.$$

**Proof.** For any $\varepsilon \in [0, 1]$, let $\varepsilon_0 = \varepsilon_1 = \varepsilon$. Then the global DP disparity becomes

$$DP Disp(f_{\varepsilon_0, \varepsilon_1}^{\text{LFT+FedAvg}}) = |\mathbb{E}_{x_0|a=0} f(x_0, 0) - \mathbb{E}_{x_1|a=1} f(x_1, 1)|$$

$$= |(\int f(x, 0) d\mathcal{P}^{(0)}_0 + \int f(x, 0) d\mathcal{P}^{(1)}_0)/2 - (\int f(x, 1) d\mathcal{P}^{(0)}_1 + \int f(x, 1) d\mathcal{P}^{(1)}_1)/2|$$

$$= |MD_0(f_{\varepsilon_0, \varepsilon_1}^{\text{LFT+FedAvg}})/2 + MD_1(f_{\varepsilon_0, \varepsilon_1}^{\text{LFT+FedAvg}})/2|$$

$$\leq DP Disp_q(f_{\varepsilon_0, \varepsilon_1}^{\text{LFT+FedAvg}})/2 + DP Disp_1(f_{\varepsilon_0, \varepsilon_1}^{\text{LFT+FedAvg}})/2$$

$$\leq (\varepsilon_0 + \varepsilon_1)/2 = \varepsilon.$$

\[ \square \]

A.3.2 The best classifier of LFT+FedAvg

For LFT+FedAvg, we directly consider multi-client cases. To visualize the gap between LFT+Ensemble, LFT+FedAvg, and CFL, in our numerical experiments, we draw finite samples from Gaussian distribution, and then we optimize the empirical risk with the fairness constraint to obtain the classifier trained by LFT+FedAvg. The following lemma provides the solution to $LFT+FedAvg(\varepsilon)$ when $\mathcal{X}$ is finite.

**Lemma 18.** For finite $\mathcal{X}$, the solution to $LFT+FedAvg(\varepsilon)$ is given by

$$f(x, a) = \| \sum_{i \in [l]} s_i(x, a)p_i^{(1)}(x) > 0 \|,$$

where $s_i(x, a) = 2\eta(x) - 1 + I\lambda_1|\{a=0\} - \frac{1}{q} - I\lambda_1|\{a=1\} - \frac{1}{q}$, for certain $\lambda_0, \ldots, \lambda_{l-1} \in \mathbb{R}$.

**Proof.** This proof is based on Menon and Williamson [2018]. The key idea of this proof is to use the Lagrangian approach. Before we applying the Lagrangian approach, we will show that $LFT+FedAvg(\varepsilon)$ is expressible as a linear program, and thus the strong duality holds.

Since $\mathcal{X}$ is finite, $f$ is a vector of finite dimension. Based on the proof of Lemma 7 the error rate can be written as

$$\mathbb{P}(\hat{y} \neq y) = \sum_{x \in \mathcal{X}, a \in A} f(x, a)(1 - 2\eta(x))\mathbb{P}(x = x, a = a) + \mathbb{P}(y = 1),$$

and the fairness constraints can be written as

$$\mathbb{P}(\hat{y} = 1 \mid a = 0, i = i) - \mathbb{P}(\hat{y} = 1 \mid a = 1, i = i)$$

$$= \sum_{x \in \mathcal{X}} \left[ f(x, 0) \frac{\mathbb{P}(x = x \mid a = 0, i = i)}{\mathbb{P}(a = 0)} - f(x, 1) \frac{\mathbb{P}(x = x \mid a = 1, i = i)}{\mathbb{P}(a = 1)} \right],$$

for $i \in [l]$. Let $u(x, a) = (1 - 2\eta(x))\mathbb{P}(x = x, a = a)$, $u' = \mathbb{P}(y = 1)$, and $v_i(x, a) = ([a = 0] - [a = 1])\mathbb{P}(x = x \mid a = a, i = i)/\mathbb{P}(a = a)$, for $x \in \mathcal{X}, a \in A, i \in [l]$. Note that $u_0, v_0, v_1$ are vectors of the same dimension of $f$. For ease of notation, we allow $\leq$ to be applied to pairs of vectors in an element-wise manner. Therefore, the optimization is

$$\min_f u^T f + u'$$

s.t.

$$u_i^T f \leq \varepsilon_i$$

$$0 \leq f \leq 1,$$
which is a linear objective with linear constraint. Therefore, the strong duality holds for \( \text{LFT+FedAvg}(\varepsilon) \). Next, we apply Lagrangian approach to solve the \( \text{LFT+FedAvg}(\varepsilon) \).

Recall that
\[
\mathbb{P}(\hat{y} \neq y) = \frac{1}{I} \mathbb{E}_a \mathbb{E}_{x \sim p_a^{(i)}} f(x, a) (1 - 2\eta(x)) + \mathbb{P}(y = 1),
\]
and
\[
\mathbb{P}(\hat{y} = 1 | a = 0, i = i) - \mathbb{P}(\hat{y} = 1 | a = 1, i = i)
= \mathbb{E}_a \mathbb{E}_{x \sim p_a^{(i)}} \left[ f(x, 0) \frac{[a = 0]}{1 - q} - f(x, 1) \frac{[a = 1]}{q} \right].
\]

By strong duality, for \( \lambda_0', \ldots, \lambda_{2I-1}' \geq 0 \), the corresponding Lagrangian version of \( \text{LFT+FedAvg}(\varepsilon) \) is
\[
\min_{f \in \mathcal{F}} \mathbb{P}(\hat{y} \neq y) - \sum_{i \in [I]} \left[ \lambda_{2i}' \mathbb{P}(\hat{y} = 1 | a = 0, i = i) - \mathbb{P}(\hat{y} = 1 | a = 1, i = i) - \varepsilon_i \right]
\]
\[
+ \lambda_{2i+1}' (\varepsilon_i - \mathbb{P}(\hat{y} = 1 | a = 0, i = i) - \mathbb{P}(\hat{y} = 1 | a = 1, i = i))
\]

Let \( \lambda_i = \lambda_{2i}' - \lambda_{2i+1}' \), \( i \in [I] \), then we get
\[
(14) = \min_{f \in \mathcal{F}} \mathbb{E}_a \mathbb{E}_{x \sim p_a^{(i)}} \left[ \frac{I}{I} f(x, a) (1 - 2\eta(x)) - \sum_{i \in [I]} \left( \lambda_i f(x, 0) \frac{[a = 0]}{1 - q} - \lambda_i f(x, 1) \frac{[a = 1]}{q} \right) \right]
\]
\[
= \min_{f \in \mathcal{F}} \int \sum_{a \in A} \frac{I}{I} f(x, a) \left[ \sum_{i \in [I]} s_i(x, a) p_a^{(i)}(x) \right] dx.
\]
where \( s_i \) is defined in Lemma 18. Thus the above equation reaches its minimum at
\[
f(x, a) = \left[ \sum_{i \in [I]} s_i(x, a) p_a^{(i)}(x) > 0 \right].
\]

\[\square\]

**Remark 9.** Based on the proof of Lemma 18, \( \text{LFT+FedAvg}(\varepsilon) \) is equivalent to solving
\[
\min_{f \in \mathcal{F}} \mathbb{P}(\hat{y} \neq y) - \sum_{i = 0}^{I-1} \lambda_i (\mathbb{P}(\hat{y} = 1 | a = 0, i = i) - \mathbb{P}(\hat{y} = 1 | a = 1, i = i)).
\]

Under certain conditions (assumptions 1 to 4 in [Li et al., 2020b]), we have solving \( \text{LFT+FedAvg}(\varepsilon) \) is equivalent to minimizing
\[
\mathbb{P}(\hat{y} \neq y | i = i) - \lambda_i (\mathbb{P}(\hat{y} = 1 | a = 0, i = i) - \mathbb{P}(\hat{y} = 1 | a = 1, i = i))
\]
locally and applying FedAvg (Theorem 1 in [Li et al., 2020b]).

### A.3.3 Comparison of LFT+FedAvg and CFL

**Lemma 19** (Formal version of Lemma 4 under two clients cases). When \( a | i = 0 \sim \text{Bern}(0) \), \( a | i = 1 \sim \text{Bern}(1) \) and \( \text{DP Disp}(f_{1,\varepsilon}^{\text{CFL}}) > 0 \), we have
\[
\min_{\varepsilon_0, \varepsilon_1} \text{DP Disp}(f_{\varepsilon_0, \varepsilon_1}^{\text{LFT+FedAvg}}) = \text{DP Disp}(f_1^{\text{CFL}}) > \min_{\varepsilon} \text{DP Disp}(f_{\varepsilon}^{\text{CFL}}) = 0.
\]

**Proof.** Since \( a | i = 0 \sim \text{Bern}(0) \), \( a | i = 1 \sim \text{Bern}(1) \), the constraints in \( \text{LFT+FedAvg}(\varepsilon) \) vanish. When \( \varepsilon = 1 \), the constraint in \( \text{CFL}(\varepsilon) \) always holds and thus also vanishes. Thus in such scenario the solution to \( \text{LFT+FedAvg}(\varepsilon) \) becomes \( f_1^{\text{CFL}} \). Then from the assumption we have
\[
\text{DP Disp}(f_{\varepsilon_0, \varepsilon_1}^{\text{LFT+FedAvg}}) = \text{DP Disp}(f_1^{\text{CFL}}) > \text{DP Disp}(f_0^{\text{CFL}}) = 0.
\]

\[\square\]
A.4 Extension to multi-client cases

In this subsection, we perform the analysis of LFT+Ensemble and LFT+FEDAVG for the multi-client cases. We present a more general version of Lemma \[10\] Thm. \[17\] and Lemma \[19\].

The following lemma shows the fundamental limitation of LFT+Ensemble:

**Lemma 20** (Formal version of Lemma \[1\].) Let \( q \in (0, 1) \). Consider a partition which divides \( I \) clients into two subsets. Denote the mixture distribution of each subset as \( x | x = a, j = j \sim \hat{P}^{(j)}_a \), where \( j \) is the index of the subset, and \( \hat{P}^{(j)}_a \) is a distribution, for \( a, j = 0, 1 \). Similar to two clients case, define \( \tilde{g}_j(\lambda) = \int_{[\frac{q}{2} - \frac{1}{4}q, + \infty)} d\hat{P}^{(j)}_0 - \int_{[\frac{q}{2} - \frac{1}{4}q, + \infty)} d\hat{P}^{(j)}_1 \). Consider the case that \( q_i = q \) for all \( i \in [I] \) and \( q \in (0, 1) \). Denote the proportion of the two subset as \( J_0 \) and \( J_1 \), where \( J_0, J_1 > 0 \) and \( J_0 + J_1 = 1 \). Let \( c = \min \{ |\tilde{g}_0(0)|, |\tilde{g}_1(0)| \} \). Define \( \tilde{\psi} : [0, c] \times [0, c] \rightarrow [-1, 1] \) as

\[
\tilde{\psi}(\tilde{z}_0, \tilde{z}_1) = J_0 J_1 \tilde{g}_0(\tilde{g}_1(0)\tilde{z}_0) + J_0 J_1 \tilde{g}_1(\tilde{g}_0^{-1}(\text{sign}(\tilde{g}_0(0))\tilde{z}_1)) + J_0 J_1 \tilde{g}_1(\tilde{g}_0^{-1}(\text{sign}(\tilde{g}_0(0))\tilde{z}_0)) + J_0 J_1 \text{sign}(\tilde{g}_0(0)\tilde{z}_0) + J_0 J_1 \text{sign}(\tilde{g}_1(0)\tilde{z}_1).
\]

If there exists a partition such that \( \tilde{g}_0(0)\tilde{g}_1(0) < 0 \) and \( \tilde{\psi}(\tilde{z}_0, \tilde{z}_1)(\tilde{g}_0(0) + \tilde{g}_1(0)) > 0 \) for all \( \tilde{z}_0, \tilde{z}_1 \in [0, c] \), then for all \( \tilde{z}_0, \tilde{z}_1 \in [0, c] \), DP Disp \( \{f_\mu^{LFT+Ensemble}\} \geq \delta = \min \{ \tilde{\psi}(\tilde{z}_0, \tilde{z}_1) : \tilde{z}_0, \tilde{z}_1 \in [0, c] \} > 0 \).

**Proof.** By Lemma \[10\] we conclude the achievable fairness range of LFT+Ensemble is strictly smaller than that of CFL. Therefore, pooling the datasets in one subset and perform fair learning can clearly achieve a wider range of fairness than perform fair learning on each client individually. Thus, we can consider the two subsets as two clients with uneven amounts of data, which is almost the same case Lemma \[10\] considers. Therefore, we follow the same proof idea as Lemma \[10\] to prove our claim.

Denote the assembled classifier trained from two pooled datasets as \( \tilde{f}_{\tilde{z}_0, \tilde{z}_1}^{LFT+Ensemble} \). Note that the mean difference can be expressed as

\[
\text{MD}(\tilde{f}_{\tilde{z}_0, \tilde{z}_1}^{LFT+Ensemble}) = J_0^2 \tilde{g}_0(\tilde{L}_0^{LFT+Ensemble}) + J_0 J_1 \tilde{g}_1(\tilde{L}_1^{LFT+Ensemble}) + J_0 J_1 \tilde{g}_0(\tilde{L}_1^{LFT+Ensemble}) + J_0 J_1 \tilde{g}_1(\tilde{L}_1^{LFT+Ensemble}).
\]

In the following proof, we will show the mean difference cannot reach 0.

Without any loss of generality, assume \( |\tilde{g}_0(0)| < |\tilde{g}_1(0)| \) and \( \tilde{g}_0(0) > 0 \). By \( \tilde{g}_0(0)\tilde{g}_1(0) < 0 \) and \( \tilde{\psi}(\tilde{z}_0, \tilde{z}_1)(\tilde{g}_0(0) + \tilde{g}_1(0)) > 0 \) for all \( \tilde{z}_0, \tilde{z}_1 \in [0, c] \), we have \( \tilde{g}_0(0) < 0 \) and \( \tilde{\psi}(\tilde{z}_0, \tilde{z}_1) > 0 \) for all \( \tilde{z}_0, \tilde{z}_1 \in [0, c] \). Without any loss of generality, assume \( J_0 \leq J_1 \).

First, we will prove that LFT+Ensemble achieves its lowest mean difference when \( \tilde{z}_0, \tilde{z}_1 \in [0, c] \). In what follows, we consider five different cases to derive the desired result.

**Case 1.** \( \tilde{z}_0 \geq |\tilde{g}_0(0)|, \tilde{z}_1 \geq |\tilde{g}_1(0)| \): ERM is fair on both clients.

By \( 0 \), we have \( \tilde{L}_0^{LFT+Ensemble} = \tilde{L}_1^{LFT+Ensemble} = 0 \). Recall \( \tilde{g}_1(\cdot) \) is a monotone increasing function, combine \( \tilde{g}_1(0) > 0 \) and Lemma \[0\] and thus \( \tilde{g}_0(\tilde{g}_1^{-1}(0)) < \tilde{g}_0(0) < 0 \). Applying the above conclusion yields

\[
J_0 \tilde{g}_0(0) + J_1 \tilde{g}_1(0) > \frac{1}{J_1} (J_0^2 \tilde{g}_0(0) + J_0 J_1 \tilde{g}_1(0) + J_0 J_1 \tilde{g}_0(\tilde{g}_1^{-1}(0))) = \frac{1}{J_1} \tilde{\psi}(\tilde{g}_0(0), 0) \geq \delta.
\]

**Case 2.** \( \tilde{z}_0 \leq |\tilde{g}_0(0)|, \tilde{z}_1 > |\tilde{g}_1(0)| \): ERM is unfair on client 0, but fair on client 1.

Applying \( 0 \) results in \( \tilde{L}_1^{LFT+Ensemble} = 0 \). By the fact that \( \tilde{g}_1(\cdot) \) is a strictly monotone increasing function, we have \( \tilde{L}_0^{LFT+Ensemble} = \tilde{g}_0^{-1}(-\tilde{z}_0) > \tilde{g}_0^{-1}(\tilde{g}_0(0)) = 0 \). Applying the above conclusion yields

\[
J_0 \tilde{g}_0(0) + J_1 \tilde{g}_1(0) > \frac{1}{J_1} \tilde{\psi}(\tilde{g}_0(0), 0) \geq \delta.
\]

In the first equality we used \( \tilde{L}_0^{LFT+Ensemble} > 0, \tilde{g}_1(\tilde{L}_0^{LFT+Ensemble}) > \tilde{g}_1(0), \tilde{g}_0(0) < -\tilde{z}_0 \).

**Case 3.** \( \tilde{z}_0 \leq |\tilde{g}_0(0)|, \tilde{z}_1 \leq |\tilde{g}_1(0)| \): ERM is unfair on both client 0 and client 1.
Applying (8) we have $\tilde{\lambda}_{\xi_1}^\text{LFT+Ensemble}_0 = \tilde{g}_0^{-1}(\varepsilon_0)$, $\tilde{\lambda}_{\xi_1}^\text{LFT+Ensemble}_1 = \tilde{g}_1^{-1}(\varepsilon_1)$. Then we have

$$i = -J_0^2 \tilde{\varepsilon}_0 + J_0^2 \tilde{\varepsilon}_1 + J_0 J_0 \tilde{g}_0(\tilde{g}_1^{-1}(\varepsilon_1)) + J_0 J_0 \tilde{g}_1(\tilde{g}_1^{-1}(\varepsilon_0))$$

$$\geq -J_0^2 \tilde{\varepsilon}_0 + J_0^2 \tilde{\varepsilon}_1 + J_0 J_0 \tilde{g}_0(\tilde{g}_1^{-1}(\varepsilon_0)) = \tilde{\psi}(\varepsilon_0, \varepsilon_0) \geq \delta.$$

**Case 4.** $\varepsilon_0 > |\tilde{g}_0(0)|, \varepsilon_1 \leq |\tilde{g}_0(0)|$: ERM is fair on client 0 and very unfair on client 1.

By (8), we have $\tilde{\lambda}_{\xi_1}^\text{LFT+Ensemble}_0 = 0$, $\tilde{\lambda}_{\xi_1}^\text{LFT+Ensemble}_1 = \tilde{g}_1^{-1}(\varepsilon_1) > \tilde{g}_1^{-1}(0)$. Then we obtain

$$i = J_0^2 \tilde{g}_0(0) + J_0 J_0 \tilde{g}_0(\tilde{\lambda}_{\xi_1}^\text{LFT+Ensemble}_1) + J_0 J_0 \tilde{g}_1(0) + J_0^2 \tilde{\varepsilon}_1$$

$$> J_0^2 \tilde{g}_0(0) + J_0 J_0 \tilde{g}_0(\tilde{g}_1^{-1}(0)) + J_0^2 \tilde{\varepsilon}_1 = \tilde{\psi}(\tilde{g}_0(0), 0) \geq \delta.$$

**Case 5.** $\varepsilon_0 > |\tilde{g}_0(0)|, |\tilde{g}_0(0)| \leq \varepsilon_1 < |\tilde{g}_1(0)|$: ERM is fair on client 0 and unfair on client 1.

Applying (8) implies $\tilde{\lambda}_{\xi_1}^\text{LFT+Ensemble}_1 = \tilde{g}_1^{-1}(\varepsilon_1) > \tilde{g}_1^{-1}(0)$. Therefore,

$$i = J_0^2 \tilde{g}_0(0) + J_0 J_0 \tilde{g}_0(\tilde{\lambda}_{\xi_1}^\text{LFT+Ensemble}_1) + J_0 J_0 \tilde{g}_1(0) + J_0^2 \tilde{\varepsilon}_1$$

$$> J_0^2 \tilde{g}_0(0) + J_0 J_0 \tilde{g}_0(\tilde{g}_1^{-1}(0)) + J_0^2 \tilde{\varepsilon}_1 = \tilde{\psi}(\tilde{g}_0(0), 0) \geq \delta.$$

Combining all the cases above, we conclude that when $\tilde{g}_1(0) > 0$, $\text{DP Disp}(\tilde{\lambda}_{\varepsilon_0, \varepsilon_1}^\text{LFT+Ensemble}) \geq \delta = \min\{|\tilde{\psi}(\varepsilon_0, \varepsilon_1)| : \varepsilon_0, \varepsilon_1 \in [0, \varepsilon]\} > 0$ for all $\varepsilon_0, \varepsilon_1 \in [0, 1]$.

**Remark 10.** Based on the proof above, we can conclude, for the cases with multiple clients, the fundamental limitation of LFT+Ensemble still exists.

The following theorem shows that LFT+FEDAVG can reach any ε DP disparity:

**Theorem 21** (Generalized version of Thm. 17). Let $g_i = q \in (0, 1)$ for all $i \in [I]$. For all $\varepsilon \in [0, 1]$, let $\varepsilon_i \leq \varepsilon$ for all $i \in [I]$, then $\text{DP Disp}(\tilde{f}_\varepsilon^\text{LFT+FEDAVG}) \leq \varepsilon$. Thus under the condition in Lemma 20 we have

$$\min_{\varepsilon \in [0, 1]^I} \text{DP Disp}(\tilde{f}_\varepsilon^\text{LFT+Ensemble}) > \min_{\varepsilon \in [0, 1]^I} \text{DP Disp}(\tilde{f}_\varepsilon^\text{LFT+FEDAVG}) = 0.$$

**Proof.** When $\varepsilon_i \leq \varepsilon$ the global DP disparity becomes

$$\text{DP Disp}(\tilde{f}_\varepsilon^\text{LFT+FEDAVG}) = |E_{x_i=0} f(x, 0) - E_{x_i=1} f(x, 1)|$$

$$= |\left(\sum_{i=0}^{I-1} \int_X f(x, 0) dP_0^{(i)}\right)/I - \sum_{i=0}^{I-1} \int_X f(x, 1) dP_1^{(i)}/I|$$

$$= |\sum_{i=0}^{I-1} \text{MD}(f_\varepsilon^\text{LFT+FEDAVG})/I| \leq \sum_{i=0}^{I-1} \text{DP Disp}(f_\varepsilon^\text{LFT+FEDAVG})/I \leq \sum_{i=0}^{I-1} \varepsilon_i/I = \varepsilon.$$

The following theorem shows the limitation of LFT+FEDAVG:

**Lemma 22** (Generalized version of Lemma 19). Let $a \mid i - i \sim \text{Bern}(0)$ or $a \mid i = i \sim \text{Bern}(1)$ for all $i \in [I]$. When $\text{DP Disp}(f_1^\text{CFL}) > 0$, we have

$$\min_{\varepsilon \in [0, 1]^I} \text{DP Disp}(f_\varepsilon^\text{LFT+FEDAVG}) = \text{DP Disp}(f_1^\text{CFL}) > \min_{\varepsilon} \text{DP Disp}(f_\varepsilon^\text{CFL}) = 0.$$

**Proof.** Since $a \mid i = i \sim \text{Bern}(0)$ or $a \mid i = i \sim \text{Bern}(1)$, the constraints in $f_\varepsilon^\text{CFL}(\varepsilon)$ always holds and thus also vanishes. Thus in such scenario the solution to $f_\varepsilon^\text{CFL}(\varepsilon)$ becomes $f_1^\text{CFL}$. Then from the assumption we have

$$\text{DP Disp}(f_\varepsilon^\text{LFT+FEDAVG}) = \text{DP Disp}(f_1^\text{CFL}) > \text{DP Disp}(f_0^\text{CFL}) = 0.$$
B  Appendix - FedFB analysis and algorithm description

In this section, we provide our bi-level optimization formulation for FedFB for four fairness notions: demographic parity, equal opportunity, equalized odds and client parity, and design the corresponding update rule. This development can also be applied to centralized case. Then, we provides more details of how we incorporate FB with federated learning.

To explain how to optimize the weights of different groups, we introduce some necessary notations first. Recall that in Lemma 23, we have update rule (2) of $\lambda$. Recall the bi-level optimization problem in (1), we denote the outer objective function to be $L$. If $|A|$ is the number of samples belong to client $i$ of label $y$ and sensitive attribute $a$. And we further define the local version of $L_{y,a}$ as $L_{y,a}(w) := \sum_{(x,y,a,i):y=y,a=a,i=i} \ell(y,\hat{y};w)/n_{y,a}$.

B.1  FedFB w.r.t demographic parity

We first provide the proof for Proposition 5.

\textbf{Proof of Proposition 5}  We denote by $P$ the empirical probability. The demographic parity is satisfied when $P(\hat{y} = 1 \mid a = 0) = P(\hat{y} = 1 \mid a = a)$ holds for all $a \in [A]$. Thus, 
\begin{align*}
P(\hat{y} = 1, y = 0 \mid a = 0) + P(\hat{y} = 1, y = 1 \mid a = 0) = & P(\hat{y} = 1, y = 0 \mid a = a) + P(\hat{y} = 1, y = 1 \mid a = a)
\end{align*}
For 0-1 loss, we have $\ell(1 - y, \cdot) = 1 - \ell(y, \cdot)$, thus
\begin{align*}
\frac{1}{n_{*0}} \sum_{(x,y,a):y=0,a=0} (1 - \ell(y,\hat{y};w)) + \frac{1}{n_{*0}} \sum_{(x,y,a):y=1,a=0} \ell(y,\hat{y};w)
= & \frac{1}{n_{*,a}} \sum_{(x,y,a):y=a,a=a} (1 - \ell(y,\hat{y};w)) + \frac{1}{n_{*,a}} \sum_{(x,y,a):y=1,a=a} \ell(y,\hat{y};w).
\end{align*}
By replacing $\sum_{(x,y,a):y=a,a=a} \ell(y,\hat{y};w) = n_{y,a} L_{y,a}(w) $, we have
\begin{align*}
\frac{n_{0,0}}{n_{*,0}} (L_{0,0}(w)) + \frac{n_{1,0}}{n_{*,0}} L_{1,0}(w) = & \frac{n_{0,a}}{n_{*,a}} (1 - L_{0,a}(w)) + \frac{n_{1,a}}{n_{*,a}} L_{1,a}(w).
\end{align*}
\hfill \square

We make the following assumption to our loss function for showing the partial convergence guarantee of FB.

\textbf{Assumption 1}. $L_{y,a}'(\cdot)$ is twice differentiable for all $y \in \{0, 1\}$, $a \in [A]$, and
\begin{align}
\sum_{a=0}^{A-1} \left[ \lambda_a \nabla^2 L_{0,a}'(w) + \frac{2n_{*,a}}{n} - \lambda_a \nabla^2 L_{1,a}'(w) \right] > 0 \text{ for all } \lambda \in \Lambda. \tag{16}
\end{align}
If $L_{y,a}'(w_{\lambda})$ is convex for all $y \in \{0, 1\}$, $a \in [A]$, the condition (16) holds unless for all $a$, $L_{0,a}'(\cdot)$, $L_{1,a}'(\cdot)$ share their stationary points, which is very unlikely (see Remark 1 in [Roh et al., 2021]).

Recall the bi-level optimization problem in (1), we denote the outer objective function to be $F_{dp}(\lambda) = \sum_{a=1}^{A-1} (F_a(w_{\lambda}))^2$. The following lemma provides a decreasing direction of $F_{dp}$, which inspired us to design the update rule (2) of $\lambda$.

\textbf{Lemma 23} (Decreasing direction of $F_{dp}$). If Assumption 2 holds, then on the direction $\mu(\lambda) = (\mu_0(\lambda), \ldots, \mu_{A-1}(\lambda))$ where
\begin{align}
\mu_0(\lambda) = - \sum_{a=1}^{A-1} F_a(w_{\lambda}) \tag{17}
\end{align}
\begin{align*}
\mu_a(\lambda) = F_a(w_{\lambda})
\end{align*}
for all $a \in \{1, \ldots, A - 1\}$, we have $\mu(\lambda) \cdot \nabla F_{dp}(\lambda) \leq 0$, and the equality holds if only if $\mu(\lambda) = 0$. 

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We take the derivative to the above equation and have
\[
\partial F_{dp}(\lambda) = 2 \sum_{a=1}^{A-1} \left[ \left( - L'_{0,0}(w_\lambda) + L'_{1,0}(w_\lambda) + L'_{0,a}(w_\lambda) - L'_{1,a}(w_\lambda) + \frac{n_{0,0}}{n_{s,0}} - \frac{n_{0,a}}{n_{s,a}} \right) \right.
\]
\[
\left. \left( - \nabla L'_{0,0}(w_\lambda) + \nabla L'_{1,0}(w_\lambda) + \nabla L'_{0,a}(w_\lambda) - \nabla L'_{1,a}(w_\lambda) \right) \right] \frac{\partial w_\lambda}{\partial \lambda_j}. \tag{18}
\]

Note that \( w_\lambda \) is the minimizer to \([1]_{\text{inner}}\), we have
\[
\sum_{a=0}^{A-1} \left[ \lambda_a \nabla^2 L'_{0,a}(w_\lambda) + (2 \frac{n_{s,a}}{n} - \lambda_a) \nabla^2 L'_{1,a}(w_\lambda) \right] = 0.
\]

We take the \( \lambda_j \) derivative to the above equation and have
\[
\nabla L'_{0,j}(w_\lambda) - \nabla L'_{1,j}(w_\lambda) + \sum_{a=0}^{A-1} \left[ \lambda_a \nabla^2 L'_{0,a}(w_\lambda) + (2 \frac{n_{s,a}}{n} - \lambda_a) \nabla^2 L'_{1,a}(w_\lambda) \right] \frac{\partial w_\lambda}{\partial \lambda_j} = 0.
\]

Thus we get
\[
\frac{\partial w_\lambda}{\partial \lambda_j} = \left( \sum_{a=0}^{A-1} \left[ \lambda_a \nabla^2 L'_{0,a}(w_\lambda) + (2 \frac{n_{s,a}}{n} - \lambda_a) \nabla^2 L'_{1,a}(w_\lambda) \right] \right)^{-1} \left[ \nabla L'_{1,j}(w_\lambda) - \nabla L'_{0,j}(w_\lambda) \right]. \tag{19}
\]

Then on the direction \( \mu(\lambda) \) given by \([17]\), we combine \([18]\) and \([19]\) to have
\[
\mu(\lambda) \cdot \nabla F_{dp}(\lambda)
\]
\[
= 2 \left( \sum_{a=1}^{A-1} \left[ \left( - L'_{0,0}(w_\lambda) + L'_{1,0}(w_\lambda) + L'_{0,a}(w_\lambda) - L'_{1,a}(w_\lambda) + \frac{n_{0,0}}{n_{s,0}} - \frac{n_{0,a}}{n_{s,a}} \right) \right.
\]
\[
\left. \left( - \nabla L'_{0,0}(w_\lambda) + \nabla L'_{1,0}(w_\lambda) + \nabla L'_{0,a}(w_\lambda) - \nabla L'_{1,a}(w_\lambda) \right) \right] \right)
\]
\[
\left( \sum_{a=0}^{A-1} \left[ \lambda_a \nabla^2 L'_{0,a}(w_\lambda) + (2 \frac{n_{s,a}}{n} - \lambda_a) \nabla^2 L'_{1,a}(w_\lambda) \right] \right)^{-1}
\]
\[
- \sum_{a=1}^{A-1} \left[ \left( - L'_{0,0}(w_\lambda) + L'_{1,0}(w_\lambda) + L'_{0,a}(w_\lambda) - L'_{1,a}(w_\lambda) + \frac{n_{0,0}}{n_{s,0}} - \frac{n_{0,a}}{n_{s,a}} \right) \right.
\]
\[
\left. \left( - \nabla L'_{0,0}(w_\lambda) + \nabla L'_{1,0}(w_\lambda) + \nabla L'_{0,a}(w_\lambda) - \nabla L'_{1,a}(w_\lambda) \right) \right] \leq 0
\]

where we have used Assumption\([1]\) and the equality holds only when \( \mu(\lambda) = 0 \). \hfill \Box

Now we introduce how clients collaborate to solve the bi-level optimization problem \([1]\). First, we focus on the outer objective function in \([1]\) and introduce how clients collaborate to update the weight \( \lambda \). Note that the central server can compute \( L'_{y,a}(w_\lambda) \) by weight-averaging the local group loss \( L^{(i)}_{y,a}(w_\lambda) \) sent from clients at communication rounds as
\[
L'_{y,a}(w_\lambda) = \sum_{i \in [I]} \frac{n_{y,a}^{(i)}}{n_{s,a}} L^{(i)}_{y,a}(w_\lambda),
\]
thereby obtain \( \mu(\lambda) \) and update \( \lambda \) by \([2]\).

Next, we focus on the inner objective function in \([1]\) and introduce how clients collaborate to update the model parameters \( w_\lambda \) using FedAVG. Note that we can decompose the objective function \( L(w, \lambda) \) into \( L(w, \lambda) = \sum_{i \in [I]} L^{(i)}(w, \lambda) \), where \( L^{(i)}(w, \lambda) := \sum_{a=0}^{A-1} \left[ \lambda_a L^{(i)}_{0,a}(w_\lambda) + (2 \frac{n_{s,a}}{n} - \lambda_a) L^{(i)}_{1,a}(w_\lambda) \right] \) is the client objective function of client \( i \). The global objective can be seen as a weighted sum of the client objective function. Therefore, we can use FedAVG to solve the inner optimization problem.
Algorithm 2: FedFB ($k, t$) w.r.t Demographic Parity

Server executes:

input : Learning rate $\{\alpha_t\}_{t \in \mathbb{N}}$;
Initialize $\lambda_0$ as $\frac{n_0}{n}$ for all $a \in [A] \setminus \{0\}$;
for each iteration $t = 1, 2, \ldots$ do

Clients perform updates;
$w_\lambda \leftarrow \text{SecAgg}(\{w^{(i)}\})$ for all $i$;
$F_a \leftarrow \text{SecAgg}(\{F_a^{(i)}\}) - (I - 1)\frac{n_{0,a}}{n_{i,a}} - \frac{n_{0,a}}{n_{i,a}}$ for all $a, i$;
$\mu_0 \leftarrow -\sum_{a=1}^{A-1} F_a$;
$\mu_a \leftarrow F_a$, for all $a \in [A] \setminus \{0\}$;
if $t \% k = 0$ then
$\lambda_a \leftarrow \lambda_a + \frac{\alpha_t}{\|w\|_2^2} \mu_a$, for all $a \in [A]$;
Broadcast $\lambda$ to clients
Broadcast $w_\lambda$ to clients;
end
output : $w_\lambda, \lambda$

ClientUpdate($i, w, \lambda$):

$w^{(i)} \leftarrow$ Gradient descent w.r.t objective function $\sum_{a=0}^{A-1} \left[ \lambda_a L_{0,a}^{(i)}(w) + (2\frac{n_{0,a}}{n} - \lambda_a) L_{1,a}^{(i)}(w) \right]$;
$F_a^{(i)} \leftarrow L_{0,a}^{(i)} + L_{1,a}^{(i)} - L_{0,a}^{(i)} + \frac{n_{0,a}}{n_{i,a}} - \frac{n_{0,a}}{n_{i,a}}$;
Send $w^{(i)}, F_a^{(i)}(w)$ for all $a \in [A]$ to server via a SecAgg protocol;

Denote $L_{y,a}^{(i)}(w) := \frac{n_{y,a}^{(i)}}{n_{i,a}} L_{y,a}^{(i)}(w)$, we present the pseudocode of FedFB w.r.t demographic parity in Algorithm 2.

Next, we analyze the convergence performance of FedFB. We need to make the following assumptions on the objective function $L^{(i)}(w, \lambda), i \in [I]$. For simplicity, we drop the $\lambda$ here and use the notations $L^{(i)}(w)$ and $L(w)$ instead. We use $w^{(i)}$ to denote the model parameters at $t$-th iteration in $i$-th client. The assumptions below are proposed by work Li et al. (2020b).

Assumption 2 (Strong convexity, Assumption 1 in Li et al. (2020b)). $L^{(i)}(w)$ is $\mu$-strongly convex for $i \in [I]$, i.e., for all $v$ and $w$, $w, L^{(i)}(v) \geq L^{(i)}(w) + \langle v - w, \nabla L^{(i)}(w) \rangle + \frac{\mu}{2} \|v - w\|^2$.

Assumption 3 (Smoothness, Assumption 2 in Li et al. (2020b)). $L^{(i)}(w)$ is $L$-smooth for $i \in [I]$, i.e., for all $v$ and $w$, $L^{(i)}(v) \leq L^{(i)}(w) + \langle v - w, \nabla L^{(i)}(w) \rangle + \frac{L}{2} \|v - w\|^2$.

Assumption 4 (Bounded variance, Assumption 3 in Li et al. (2020b)). Let $\xi^{(i)}_t$ be sampled from $i$-th device’s local data uniformly at random, where $t \in [T]$, and $T$ is the total number of every client’s SGDs. The variance of stochastic gradients in each device is bounded: $E \left\| \nabla L^{(i)}(w^{(i)}_t, \xi^{(i)}_t) - \nabla L^{(i)}(w^{(i)}_t) \right\|^2 < \infty$ for $i \in [I]$.

Assumption 5 (Bounded gradients, Assumption 4 in Li et al. (2020b)). The expected squared norm of stochastic gradients is uniformly bounded, i.e., $E \left\| \nabla L^{(i)}(w^{(i)}_t, \xi^{(i)}_t) \right\|^2 < \infty$ for all $i \in [I], t \in [T]$.

In FedAvg, first, the central server broadcasts the latest model to all clients, then, every client performs local updates for a number of iterations, last, the central server aggregates the local models to produce the new global model (see Algorithm Description in Li et al. (2020b) for more detailed explanation).

Note that we assume $\lambda$ is not updated at each communication round. With the above assumptions, the following theorem shows the convergence of FedFB in the case of two clients.

Theorem 24. Consider the case of $A = 2$. Let Assumption 2, 3, 4, 5 on $\{L^{(i)}(\cdot, \lambda)\}_{i \in [I]}$ and Assumption 1 on $\{L_{y,a}^{(i)}(\cdot)\}_{y,a \in [0,1]}$ hold. Choose $\{\alpha_t\}_{t=1}^\infty$ such that $\lim_{t \to \infty} \alpha_t = 0, \sum_{t=1}^\infty \alpha_t = \infty$. Suppose we have sufficiently large number of communication rounds to solve the inner optimization problem between two $\lambda$ update rounds, i.e., $k \to \infty$ in Algorithm 2. Denote the number of $\lambda$ update as $T$, $\lambda^{(i)} = (\lambda_0^{(i)}, \cdots, \lambda_{A-1}^{(i)})$ as the weight at $t$-th updated round. We can find sufficiently large $T$ such that with high probability, applying update rule 2, $\lambda^{(i)} \to \lambda^*$ leads to $|\lambda_0^{(T)} - \lambda_0^*| \leq \max \left\{ |\lambda_0^{(0)} - \lambda_0^*| - \sum_{t=1}^T \alpha_t, \alpha_T \right\} \to 0$, where $\lambda^*$ is the global minimizer: $\lambda^* \in \arg \min_\lambda \sum_a |F_a(w_\lambda)|^2$. 

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**Proof.** We first derive the update rule in the case of $A = 2$. From (2), we have $\mu_0 = -\mu_1$. Denote

$$f(\lambda(t)) = -L'_{0,0}(w_{\lambda(0)}) + L'_{1,0}(w_{\lambda(t)}) + L'_{0,1}(w_{\lambda(t)}) - L'_{1,1}(w_{\lambda(t)}) + \frac{n_{0,0}}{n_{*,0}} - \frac{n_{0,1}}{n_{*,1}}.$$ 

Then the update rule becomes

$$\lambda_0(t+1) = \lambda_0(t) - \frac{\sqrt{2}}{2} \alpha_t \text{sign}(f(\lambda(t)))$$

$$\lambda_1(t+1) = \lambda_1(t) + \frac{\sqrt{2}}{2} \alpha_t \text{sign}(f(\lambda(t))).$$

We denote $E$ as the number of local epochs performed between two communication rounds, and $R$ be the total number of iterations between two $\lambda$ update rounds. Then we apply FEDAVG with $R$ number of total iterations to solve $\min_w L(w; \lambda(t))$, and obtain $w_R(t)$. Then in the update round from $\lambda(t)$ to $\lambda(t+1)$, by Thm. 1 in Li et al. [2020b], we have

$$\mathbb{E}[L(w_R(t); \lambda(t))] - L(w_{\lambda(0)}; \lambda(t)) = O\left(\frac{E^2}{R}\right),$$

$$\mathbb{E}\left\|w_R(t) - w_{\lambda(0)}\right\|^2 = O\left(\frac{E^2}{R}\right),$$

where $w_{\lambda(0)} = \arg \min_w L(w; \lambda(t))$. By Markov’s inequality, with probability $1 - \delta$,

$$L(w_R(t); \lambda(t)) - L(w_{\lambda(0)}; \lambda(t)) = O\left(\frac{E^2}{\delta R}\right),$$

$$\left\|w_R(t) - w_{\lambda(0)}\right\|^2 = O\left(\frac{E^2}{\delta R}\right).$$

Then taking the union bound over $T$ updating iterations of $\lambda(t)$, the conclusions above hold with probability at least $1 - T \delta$ for all $\lambda(t), t = 1, 2, \cdots, T$. Therefore, with sufficiently large $R = R(\delta)$ such that $\frac{E^2}{\delta R} \to 0$, for all $\lambda(t)$ in the $T$ iteration, $|L(w_R(t); \lambda(t)) - L(w_{\lambda(0)}; \lambda(t))| \ll 1$, $\left\|w_R(t) - w_{\lambda(0)}\right\| \ll 1$.

By Lemma 23, $F_{a}(\lambda)$ has minimizer $\lambda^*$ on direction $\mu(\lambda)$ such that $\mu(\lambda^*) = 0$, which implies $F_{d}(\lambda^*) = 0$. Therefore, $\lambda^*$ is a global minimizer. By update rule (20), we can find large $T > 0$ to have

$$|\lambda_0^0 - \lambda_0^*| \leq \max \left\{ |\lambda_0^0 - \lambda_1^*| - \sum_{t=1}^{T} \alpha_t, \alpha_T \right\} \to 0,$$

where $a \in \{0, 1\}$. 

**Remark 11.** Note that Thm. 24 assumes FEDFB does not update $\lambda$ in each communication round and there are infinite rounds of aggregations between two $\lambda$ updating round. However, for computation efficiency, we update $\lambda$ at every communication round in practice.

### B.2 FEDFB w.r.t equal opportunity

Similar to Proposition 5, we design the following bi-level optimization problems to capture equal opportunity:

$$\min_{\lambda \in \Lambda} F_{a}(\lambda) = \min_{\lambda \in \Lambda} \sum_{a=1}^{A-1} (L_{1,a}(w_{\lambda}) - L_{1,0}(w_{\lambda}))^2$$

$$w_{\lambda} = \arg \min_{w} \sum_{a=1}^{A-1} \lambda_a L_{1,a}(w) + \left(\frac{n_{1,a}}{n} - \sum_{a=1}^{A-1} \lambda_a\right)L_{1,0}(w) + \frac{n_{0,a}}{n} L_{0,a}(w).$$

Here $\Lambda = \{(\lambda_1, \ldots, \lambda_{A-1}) : \lambda_1 + \cdots + \lambda_{A-1} \leq \frac{n_{1,a}}{n}, \lambda_a \geq 0 \text{ for all } a = 1, \ldots, A-1\}$. For equal opportunity, we make the following assumption:
Assumption 6. $L_{y,a}(\cdot)$ is twice differentiable for all $y \in \{0, 1\}$, $a \in [A]$, and
\[
\sum_{a=1}^{A-1} \lambda_a \nabla^2 L_{1,a}(w_\lambda) + \left(\frac{n_{1,a}}{n} - \sum_{a=1}^{A-1} \lambda_a\right)\nabla^2 L_{1,0}(w_\lambda) + \frac{n_{0,a}}{n} \nabla^2 L_{0,\ast}(w_\lambda) > 0
\]
for all $\lambda \in \Lambda$.

With the above assumption, the following lemma provides the update rule:

**Lemma 25.** If Assumption 6 holds, then on the direction
\[
\mu(\lambda) = \left(L_{1,1}(w_\lambda) - L_{1,0}(w_\lambda), \ldots, L_{1,A-1}(w_\lambda) - L_{1,0}(w_\lambda)\right),
\]
we have $\mu(\lambda) \cdot \nabla F_{\text{co}}(\lambda) \leq 0$, and the equality holds if only if $\mu(\lambda) = 0$.

**Update rule for equal opportunity:**
\[
\lambda^{(t+1)}_a = \lambda^{(t)}_a + \frac{\alpha_t}{\|\mu^{(t)}(\lambda)|^2} (L_{1,a}(w^{(t)}_\lambda) - L_{1,0}(w^{(t)}_\lambda)).
\]

**Proof of Lemma 25** We compute the derivative as
\[
\frac{\partial F_{\text{co}}(\lambda)}{\partial \lambda_j} = 2 \sum_{a=1}^{A-1} (L_{1,a}(w_\lambda) - L_{1,0}(w_\lambda)) (\nabla L_{1,a}(w_\lambda) - \nabla L_{1,0}(w_\lambda)) \frac{\partial w_\lambda}{\partial \lambda_j}.
\]

Note that $w_\lambda$ is the minimizer to (21), we have
\[
\sum_{a=1}^{A-1} \lambda_a \nabla L_{1,a}(w_\lambda) + \left(\frac{n_{1,a}}{n} - \sum_{a=1}^{A-1} \lambda_a\right)\nabla L_{1,0}(w_\lambda) + \frac{n_{0,a}}{n} \nabla L_{0,\ast}(w_\lambda) = 0.
\]

We take the $\lambda_j$ derivative to the above equation and have
\[
\sum_{a=1}^{A-1} \lambda_a \nabla^2 L_{1,a}(w_\lambda) + \left(\frac{n_{1,a}}{n} - \sum_{a=1}^{A-1} \lambda_a\right)\nabla^2 L_{1,0}(w_\lambda) + \frac{n_{0,a}}{n} \nabla^2 L_{0,\ast}(w_\lambda)
\]
\[
\nabla L_{1,j}(w_\lambda) - \nabla L_{1,0}(w_\lambda) + \frac{\partial w_\lambda}{\partial \lambda_j} = 0.
\]

Thus we get
\[
\frac{\partial w_\lambda}{\partial \lambda_j} = \left[\sum_{a=1}^{A-1} \lambda_a \nabla^2 L_{1,a}(w_\lambda) + \left(\frac{n_{1,a}}{n} - \sum_{a=1}^{A-1} \lambda_a\right)\nabla^2 L_{1,0}(w_\lambda) + \frac{n_{0,a}}{n} \nabla^2 L_{0,\ast}(w_\lambda)\right]^{-1}
\]
\[
\left[\nabla L_{1,0}(w_\lambda) - \nabla L_{1,j}(w_\lambda)\right].
\]

Then on the direction $\mu(\lambda)$ given by (22), we combine (23) and (24) to have
\[
\mu(\lambda) \cdot \nabla F_{\text{co}}(\lambda)
\]
\[
= 2 \left[\sum_{a=1}^{A-1} (L_{1,a}(w_\lambda) - L_{1,0}(w_\lambda)) (\nabla L_{1,a}(w_\lambda) - \nabla L_{1,0}(w_\lambda))\right]
\]
\[
\left[\sum_{a=1}^{A-1} \lambda_a \nabla^2 L_{1,a}(w_\lambda) + \left(\frac{n_{1,a}}{n} - \sum_{a=1}^{A-1} \lambda_a\right)\nabla^2 L_{1,0}(w_\lambda) + \frac{n_{0,a}}{n} \nabla^2 L_{0,\ast}(w_\lambda)\right]^{-1}
\]
\[
\left[\sum_{a=1}^{A-1} (L_{1,a}(w_\lambda) - L_{1,0}(w_\lambda)) (\nabla L_{1,0}(w_\lambda) - \nabla L_{1,a}(w_\lambda))\right] \leq 0,
\]
where we have used Assumption 6, the equality holds only when $\mu(\lambda) = 0$.

We present the FedFB algorithm w.r.t equal opportunity in Algorithm 3.
Algorithm 3: FedFB($k, t$) w.r.t Equal Opportunity

Server executes:

**input**: Learning rate $\{\alpha_i\}_{i \in \mathbb{N}}$; Initialize $\lambda_a$ as $\frac{n_0}{n}$ for all $a \in [A] \setminus \{0\}$;

**for each iteration** $t = 1, 2, \ldots$ **do**

Clients perform updates;

$w_\lambda \leftarrow \text{SecAgg}\{w^{(i)}\}$ for all $i$;

$\mu_a \leftarrow \text{SecAgg}\{\mu_a^{(i)}\}$ for all $a \in [A] \setminus \{0\}$;

**if** $t \% k = 0$ **then**

$\lambda_a \leftarrow \lambda_a + \frac{n_0}{|a|} \mu_a$, for all $a \in [A] \setminus \{0\}$;

Broadcast $\lambda$ to clients;

Broadcast $w_\lambda$ to clients;

end

**output**: $w_\lambda, \lambda$

ClientUpdate($i, w, \lambda$):

$w^{(i)} \leftarrow$ Gradient descent w.r.t objective function

$\sum_{a=0}^{A-1} \lambda_a L_{1,a}(w) + (\frac{n_0}{n} - \sum_{a=0}^{A-1} \lambda_a) L_{0,0}(w) + \frac{n_0}{n} L_{0,0}(w)$;

Send $w^{(i)}, \mu_a^{(i)}$ for all $a \in [A] \setminus \{0\}$ to server via a SecAgg protocol;

B.3 FedFB w.r.t equalized odds

For equalized odd, we design the following bi-level optimization problem:

$$
\min_{\lambda \in \Lambda} F_{\text{eod}}(\lambda) = \min_{\lambda \in \Lambda} \sum_{a=1}^{A-1} \left[ (L_{1,a}(w_\lambda) - L_{1,0}(w_\lambda))^2 + (L_{0,a}(w_\lambda) - L_{0,0}(w_\lambda))^2 \right]
$$

$$
w_\lambda = \arg \min_w \sum_{a=1}^{A-1} (\lambda_{0,a} L_{0,0}(w) + \lambda_{1,a} L_{1,a}(w)) + (\frac{n_0}{n} - \sum_{a=1}^{A-1} \lambda_{0,a}) L_{0,0}(w) + (\frac{n_1}{n} - \sum_{a=1}^{A-1} \lambda_{1,a}) L_{1,0}(w).
$$

(25)

Here

$$
\Lambda = \{ (\lambda_{0,1}, \ldots, \lambda_{0,A-1}, \lambda_{1,1}, \ldots, \lambda_{1,A-1}) : \sum_{a=1}^{A-1} \lambda_{0,a} \leq \frac{n_0}{n}, \sum_{a=1}^{A-1} \lambda_{1,a} \leq \frac{n_1}{n}, \lambda_{0,a}, \lambda_{1,a} \geq 0, \text{ for all } a = 1, \ldots, A-1 \}.
$$

For equalized odds, we make the following assumption:

**Assumption 7.** $L_{y,a}(\cdot)$ is twice differentiable for all $y \in \{0, 1\}$, $a \in [A]$, and

$$
\sum_{a=1}^{A-1} \left[ (\lambda_{1,a} \nabla^2 L_{1,a}(w_\lambda) + \lambda_{0,a} \nabla^2 L_{1,a}(w_\lambda)) + (\frac{n_0}{n} - \sum_{a=1}^{A-1} \lambda_{0,a}) \nabla^2 L_{0,0}(w_\lambda) + \frac{n_1}{n} - \sum_{a=1}^{A-1} \lambda_{1,a}) \nabla^2 L_{1,0}(w_\lambda) \right] \geq 0
$$

for all $\lambda \in \Lambda$.

With the above assumption, the following lemma provides the update rule:

**Lemma 26.** If Assumption 7 holds, then on the direction

$$
\mu(\lambda) = (\mu_{0,1}(\lambda), \ldots, \mu_{0,A-1}(\lambda), \mu_{1,1}(\lambda), \ldots, \mu_{1,A-1}(\lambda)), \quad \mu_{y,a}(\lambda) = L_{y,a}(w_\lambda) - L_{y,0}(w_\lambda), \quad y \in \{0, 1\}, \quad a \in [A],
$$

we have $\mu(\lambda) \cdot \nabla F_{\text{eod}}(\lambda) \leq 0$, and the equality holds if only if $\mu(\lambda) = 0$. 

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Update rule for equalized odd:

\[
\lambda_{y,a}^{(t+1)} = \lambda_{y,a}^{(t)} + \frac{\alpha_t}{\|\mu(\lambda)^{(t)}\|_2} (L_{y,a}(w_{\lambda^{(t)}}) - L_{y,0}(w_{\lambda^{(t)}})) \quad \text{for} \quad y \in \{0,1\}, a \in [A].
\]

**Proof of Lemma 26** We compute the derivative as

\[
\frac{\partial F_{\text{cod}}(\lambda)}{\partial \lambda_{y,j}} = 2 \sum_{a=1}^{A-1} \left[ (L_{1,a}(w_{\lambda}) - L_{1,0}(w_{\lambda}))(\nabla L_{1,a}(w_{\lambda}) - \nabla L_{1,0}(w_{\lambda})) + (L_{0,a}(w_{\lambda}) - L_{0,0}(w_{\lambda}))(\nabla L_{0,a}(w_{\lambda}) - \nabla L_{0,0}(w_{\lambda})) \right] \frac{\partial w_{\lambda}}{\partial \lambda_{y,j}}.
\]

Note that \( w_{\lambda} \) is the minimizer to (25), we have

\[
\sum_{a=1}^{A-1} (\lambda_{0,a} \nabla L_{0,a}(w_{\lambda}) + \lambda_{1,a} \nabla L_{1,a}(w_{\lambda})) + \left( \frac{n_0}{n} - \sum_{a=1}^{A-1} \lambda_{0,a} \right) \nabla L_{0,0}(w_{\lambda}) + \left( \frac{n_1}{n} - \sum_{a=1}^{A-1} \lambda_{1,a} \right) \nabla L_{1,0}(w_{\lambda}) = 0.
\]

We take the \( \lambda_{y,j} \) derivative to the above equation and have

\[
\nabla L_{y,j}(w_{\lambda}) - \nabla L_{y,0}(w_{\lambda}) + \left[ \sum_{a=1}^{A-1} (\lambda_{0,a} \nabla^2 L_{0,a}(w_{\lambda}) + \lambda_{1,a} \nabla^2 L_{1,a}(w_{\lambda})) + \left( \frac{n_0}{n} - \sum_{a=1}^{A-1} \lambda_{0,a} \right) \nabla^2 L_{0,0}(w_{\lambda}) + \left( \frac{n_1}{n} - \sum_{a=1}^{A-1} \lambda_{1,a} \right) \nabla^2 L_{1,0}(w_{\lambda}) \right] \frac{\partial w_{\lambda}}{\partial \lambda_{y,j}} = 0.
\]

Thus we get

\[
\frac{\partial w_{\lambda}}{\partial \lambda_{y,j}} = \left[ \sum_{a=1}^{A-1} (\lambda_{0,a} \nabla^2 L_{0,a}(w_{\lambda}) + \lambda_{1,a} \nabla^2 L_{1,a}(w_{\lambda})) + \left( \frac{n_0}{n} - \sum_{a=1}^{A-1} \lambda_{0,a} \right) \nabla^2 L_{0,0}(w_{\lambda}) + \left( \frac{n_1}{n} - \sum_{a=1}^{A-1} \lambda_{1,a} \right) \nabla^2 L_{1,0}(w_{\lambda}) \right]^{-1} \nabla L_{y,0}(w_{\lambda}) - \nabla L_{y,j}(w_{\lambda}).
\]

Then on the direction \( \mu(\lambda) \) given by (26), we combine (27) and (28) to have

\[
\mu(\lambda) \cdot \nabla F_{\text{cod}}(\lambda)
\]

\[
= 2 \left[ \sum_{a=1}^{A-1} \left( (L_{1,a}(w_{\lambda}) - L_{1,0}(w_{\lambda}))(\nabla L_{1,a}(w_{\lambda}) - \nabla L_{1,0}(w_{\lambda})) + (L_{0,a}(w_{\lambda}) - L_{0,0}(w_{\lambda}))(\nabla L_{0,a}(w_{\lambda}) - \nabla L_{0,0}(w_{\lambda})) \right) \right]
\]

\[
\sum_{a=1}^{A-1} (\lambda_{0,a} \nabla^2 L_{0,a}(w_{\lambda}) + \lambda_{1,a} \nabla^2 L_{1,a}(w_{\lambda})) + \left( \frac{n_0}{n} - \sum_{a=1}^{A-1} \lambda_{0,a} \right) \nabla^2 L_{0,0}(w_{\lambda}) + \left( \frac{n_1}{n} - \sum_{a=1}^{A-1} \lambda_{1,a} \right) \nabla^2 L_{1,0}(w_{\lambda}) \right]^{-1}
\]

\[
\sum_{a=1}^{A-1} \left( (L_{1,a}(w_{\lambda}) - L_{1,0}(w_{\lambda}))(\nabla L_{1,0}(w_{\lambda}) - \nabla L_{1,a}(w_{\lambda})) + (L_{0,a}(w_{\lambda}) - L_{0,0}(w_{\lambda}))(\nabla L_{0,0}(w_{\lambda}) - \nabla L_{0,a}(w_{\lambda})) \right) \leq 0
\]

where we have used Assumption 7 and the equality holds only when \( \mu(\lambda) = 0 \).

The full procedure is described in Algorithm 4.
Algorithm 4: FedFB\((k, t)\) w.r.t Equalized Odds

Server executes:

- **input:** Learning rate \(\{\alpha_t\}_{t \in \mathbb{N}}\); Initialize \(\lambda_{y,a} = \frac{n_{y,a}}{n}\) for all \(y \in \{0, 1\}, a \in [A] \setminus \{0\}\);
- **for each iteration** \(t = 1, 2, \ldots\) **do**
  - Clients perform updates;
    - \(w_\lambda \leftarrow \text{SecAgg}\left(\{w_y^{(i)}\}\right)\) for all \(i\);
    - \(\mu_{y,a} \leftarrow \text{SecAgg}\left(\{\mu_{y,a}^{(i)}\}\right)\) for all \(y \in \{0, 1\}, a \in [A] \setminus \{0\}\);
  - if \(t \% k = 0\) then
    - \(\lambda_{y,a} \leftarrow \lambda_{y,a} + \lambda_{y,a}^{(i)}\mu_{y,a}\) for all \(a \in [A] \setminus \{0\}\);
    - Broadcast \(\lambda\) to clients;
  - Broadcast \(w_\lambda\) to clients;
- **end**

- **output:** \(w_\lambda\)

**ClientUpdate\((i, \omega, \lambda)\):**

- \(w^{(i)} \leftarrow \text{Gradient descent w.r.t objective function}\)
- \[\sum_{a=1}^{A-1} \left(\lambda_{0,a}L^{(i)}_{0,a}(w) + \lambda_{1,a}L^{(i)}_{1,a}(w)\right) + \left(\frac{n_{0,a}}{n} - \sum_{a=1}^{A-1} \lambda_{0,a}\right)\lambda_{0,a}L^{(i)}_{0,0}(w) + \left(\frac{n_{1,a}}{n} - \sum_{a=1}^{A-1} \lambda_{1,a}\right)\lambda_{1,a}L^{(i)}_{1,0}(w)\]
- Send \(w^{(i)}, \mu_{y,a}^{(i)}\) for all \(y \in [1], a \in [A]\) to server via a SecAgg protocol;

B.4 FedFB w.r.t client parity

For client parity, we slightly abuse the notation and define the loss over client \(i\) as \(L^{(i)}(w) := \sum_{(x,y,a,i):i=1} \ell(y, \hat{y}; w)/n\), with \(n := |\{(x, y, a) : i = i\}|\). Note that the \(L^{(i)}(w)\) here is different from the one in Sec. [B.1]. We design the following bi-level optimization problem:

\[
\begin{align*}
\min_{\lambda \in \Lambda} F_{cp}(\lambda) &= \min_{\lambda \in \Lambda} \sum_{i=1}^{I-1} \left( L^{(i)}(w_\lambda) - L^{(0)}(w_\lambda) \right)^2, \\
&= \min_{\lambda \in \Lambda} \sum_{i=1}^{I-1} \left( L^{(i)}(w_\lambda) + (1 - \sum_{i=1}^{I-1} \lambda(i))L^{(0)}(w_\lambda) \right) + (1 - \sum_{i=1}^{I-1} \lambda(i))L^{(0)}(w_\lambda).
\end{align*}
\]

Here

\[
\Lambda = \{(\lambda^{(1)}, \ldots, \lambda^{(I-1)}) : 0 \leq \lambda^{(i)} \leq 1, \ \sum_{i=1}^{I-1} \lambda^{(i)} \leq 1\}.
\]

For client parity, we make the following assumption:

**Assumption 8.** \(L^{(i)}(\cdot)\) is twice differentiable for all \(i = 1, \ldots, I-1\), and

\[
\sum_{i=1}^{I-1} \lambda^{(i)}\nabla^2 L^{(i)}(w_\lambda) + (1 - \sum_{i=1}^{I-1} \lambda^{(i)})\nabla^2 L^{(0)}(w_\lambda) \succ 0.
\]

With the above assumption, the following lemma provides the update rule:

**Lemma 27.** If Assumption [8] holds, then on the direction

\[
\mu(\lambda) = (L^{(1)}(w_\lambda) - L^{(0)}(w_\lambda), \ldots, L^{(I-1)}(w_\lambda) - L^{(0)}(w_\lambda)),
\]

we have \(\mu(\lambda) \cdot \nabla F_{cp}(\lambda) \leq 0\), and the equality holds if and only if \(\mu(\lambda) = 0\).

Then we update \(\lambda^{(i)} = (\lambda^{(1)}^{(i)}, \ldots, \lambda^{(I-1)}^{(i)})\) as follows:

**Update rule for client parity:**

\[
\lambda^{(i)(t+1)} = \lambda^{(i)(t)} + \frac{\alpha_t}{\|\mu(\lambda^{(i)(t)})\|^2} (L^{(i)}(w_\lambda^{(t)}) - L^{(0)}(w_\lambda^{(t)})) \text{ for } i = 1, \ldots, I-1.
\]
We continue from Sec. 5 and provide more details on experimental settings, and other supplementary experiment results.

We randomly split data into a training set and a testing set at a ratio of 7:3. The batch size is set to be 128. For \( \mu \) where we have used Assumption 8, and the equality holds only when \( \mu \).

Thus we get

\[
\text{Fig. 9 shows the accuracy versus fairness violation for the 3 clients' cases. We see a clear advantage of LFT+FEDAVG over the LFT+Ensemble in both 2 clients' cases and 3 clients' cases. The achievable fairness range of LFT+FEDAVG is}
\]

The Algorithm gives the full description.

C Appendix - Experiments

We continue from Sec. 5 and provide more details on experimental settings, and other supplementary experiment results.

C.1 Experiment setting

The reported statistics are computed on the test set, and we set 10 communication rounds and 30 local epochs for all federated learning algorithms except AGNOSTICFAIR. We run 300 epochs for other methods. For all tasks, we randomly split data into a training set and a testing set at a ratio of 7:3. The batch size is set to be 128. For all methods, we choose learning rate from \( \{0.001, 0.005, 0.01\} \). We solve FEDFB with sample weight learning rates \( \alpha \in \{0.001, 0.05, 0.08, 0.1, 0.2, 0.5, 1, 2\} \) in parallel and select the \( \alpha \) which achieves the highest fairness. All benchmark models are tuned according to the hyperparameter configuration suggested in their original works. We perform cross-validation on the training sets to find the best hyperparameter for all algorithms.

C.2 Supplementary experiment results

Fig. 9 shows the accuracy versus fairness violation for the 3 clients’ cases. We see a clear advantage of LFT+FEDAVG over the LFT+Ensemble in both 2 clients’ cases and 3 clients’ cases. The achievable fairness range of LFT+FEDAVG is
We split the data into three clients in a non-iid way. We randomly assign \( \frac{n}{4} \) for all \( i \in [I]\backslash\{0\} \).

**Algorithm 5: FedFB \( w.r.t \) Client Parity**

**Server executes:**
- **input**: Learning rate \( \{\alpha_t\}_{t\in\mathbb{N}} \); Initialize \( \lambda^{(i)} \) as \( \frac{n}{5} \) for all \( i \in [I]\backslash\{0\} \);
- **for each iteration** \( t = 1, 2, \ldots \) do
  - Clients perform updates;
  - \( \mathbf{w}_{\lambda} \leftarrow \text{SecAgg} \{\mathbf{w}^{(i)}\} \) for all \( i \);
  - \( \mu^{(i)} \leftarrow L^{(i)} - L^{(0)} \), \( i \in [I]\backslash\{0\} \);
  - if \( t\%k = 0 \) then
    - \( \lambda^{(i)} \leftarrow \lambda^{(i)} + \frac{\alpha^{(i)}}{\|\mu^{(i)}\|} \mu^{(i)} \), for all \( i \in [I]\backslash\{0\} \);
    - Broadcast \( \lambda \) to clients;
    - Broadcast \( \mathbf{w}_{\lambda} \) to clients;
  - end
- **output**: \( \mathbf{w}_{\lambda} \)

**ClientUpdate** \( (i, \mathbf{w}, \lambda) \):
- \( \mathbf{w}^{(i)} \leftarrow \text{Gradient descent} \ w.r.t \text{ objective function} \ [i \neq 0] \lambda^{(i)} L^{(i)}(\mathbf{w}) + [i = 0] (1 - \sum_{j=1}^{t-1} \lambda^{(i)} L^{(0)}(\mathbf{w})) \);
- Send \( \mathbf{w}^{(i)} \) to server via a SecAgg protocol; Send \( L^{(i)}(\mathbf{w}) \) to server;

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**Figure 9**: Accuracy-Fairness tradeoff curves of CFL, LFT+FedAvg, and LFT+Ensemble for three clients cases. The data heterogeneity is increasing from left to right. The green dotted vertical line describes the lower bound of unfairness LFT+FedAvg can achieve, and the orange dotted vertical line describes the lower bound of unfairness LFT+Ensemble can achieve. Here the distribution setting is \( x \mid a = 0, i = 0 \sim \mathcal{N}(3, 1), x \mid a = 1, i = 0 \sim \mathcal{N}(5, 1), x \mid a = 0, i = 1 \sim \mathcal{N}(1, 1), x \mid a = 1, i = 1 \sim \mathcal{N}(-1, 1), x \mid a = 0, i = 2 \sim \mathcal{N}(1, 1), x \mid a = 1, i = 2 \sim \mathcal{N}(2, 1), a \mid i = 0 \sim \text{Bern}(q_0) \) for \( i = 0, 1, 2 \). The data heterogeneity here is captured by \( \{q_2 - q_0\} \). (a) \( q_0 = q_1 = q_2 = 0.5 \). (b) \( q_0 = 0.4, q_1 = 0.5, q_2 = 0.6 \). (c) \( q_0 = 0.3, q_1 = 0.5, q_2 = 0.7 \). (d) \( q_0 = 0.2, q_1 = 0.5, q_2 = 0.8 \). For the sensitive attribute \( a \), we generate biased data using an unfair scenario \( p_{x_1, x_2}((x_1', x_2') \mid y = 1) / p_{x_1, x_2}((x_1', x_2') \mid y = 0) + p_{x_1, x_2}((x_1', x_2') \mid y = 1) \), where \( p_{x_1, x_2} \) is the pdf of \( (x_1, x_2) \).

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much wider than that of LFT+Ensemble, though the accuracy of FedAvg is not guaranteed when the data heterogeneity increases.

We also compare FedAvg, LFT+Ensemble, LFT+FedAvg, FedFB, AGNOSTICFair and CFL on logistic regression. Table 4 shows that our method outperforms LFT+Ensemble and LFT+FedAvg, while achieving similar performance as AGNOSTICFair and CFL but ensuring higher privacy.

### C.3 Synthetic dataset generation

We generate a synthetic dataset of 5,000 examples with two non-sensitive attributes \((x_1, x_2)\), a binary sensitive \( a \), and a binary label \( y \). A tuple \((x_1, x_2, y)\) is randomly generated based on the two Gaussian distributions: \((x_1, x_2) \mid y \sim \mathcal{N}([-2, -2], [10, 1, 1, 3])\) and \((x_1, x_2) \mid y = 1 \sim \mathcal{N}([2, 2], [5, 1])\), where \( y \sim \text{Bern}(0.6) \). For the sensitive attribute \( a \), we generate biased data using an unfair scenario \( p_{x_1, x_2}((x_1', x_2') \mid y = 1) / p_{x_1, x_2}((x_1', x_2') \mid y = 0) + p_{x_1, x_2}((x_1', x_2') \mid y = 1) \), where \( p_{x_1, x_2} \) is the pdf of \((x_1, x_2)\).

We split the data into three clients in a non-iid way. We randomly assign 50%, 30%, 20% of the samples from group 0 and 20%, 40%, 40% of the samples from group 1 to 1st, 2nd, and 3rd client, respectively. To study the empirical relationship between the performance of FedFB and data heterogeneity, we split the dataset into other ratios to obtain
We investigate the empirical relationship between accuracy, fairness, and the number of clients. We generate three synthetic datasets of 3,333, 5,000, and 6,667 samples, and split them into two, three, and four clients, respectively. Table 5 shows that our method outperforms under the three cases.

Table 6: Comparison of accuracy and fairness under the same setting as Ezzeldin et al. [2021]. The implementation of LFT+Ensemble, LFT+FEDAVG, and CFL are all based on FB.

| Property | Fair | Method | Synthetic Acc. (%) | DP Disp. (↓) | Adult Acc. (%) | DP Disp. (↓) | COMPAS Acc. (%) | DP Disp. (↓) | Bank Acc. (%) | DP Disp. (↓) |
|----------|------|--------|--------------------|--------------|---------------|--------------|----------------|--------------|---------------|--------------|
| ✓        | ✓    | FedAVG | .884 ± 0.001      | .419 ± 0.06  | .837 ± 0.07  | 1.44 ± 0.15  | .658 ± 0.06  | 1.49 ± 0.22  | .908 ± 0.00  | .026 ± 0.01  |
| ✓        | ✓    | LFT+ENSEMBLE | .712 ± 0.19  | .266 ± 0.15  | .319 ± 0.06  | .032 ± 0.31  | .606 ± 0.18  | .089 ± 0.08  | .888 ± 0.00  | .008 ± 0.07  |
| ✓        | ✓    | FedFB (OURS) | .756 ± 0.01  | .085 ± .001  | .820 ± 0.00  | .002 ± 0.01  | .600 ± 0.03  | .059 ± 0.03  | .890 ± 0.01  | .014 ± 0.02  |
| ✓        | ✓    | CFL     | .709 ± 0.01  | .011 ± 0.01  | .800 ± 0.03  | .016 ± 0.03  | .587 ± 0.03  | .032 ± 0.02  | .883 ± 0.00  | .000 ± 0.00  |

To make fairer comparison, we follow the exact same setting as Ezzeldin et al. [2021] to re-split Adult and COMPAS datasets, employ |$SPD|$ = $|P(\hat{y} = 1 \mid a = 0) - P(\hat{y} = 1 \mid a = 1)|$ as unfairness metric and present the results in Table 6. We observe that FedFB achieves higher fairness than FAIRFED while being robust to data heterogeneity.

C.4 Empirical relationship between accuracy, fairness, and the number of clients

We investigate the empirical relationship between accuracy, fairness, and the number of clients. We generate three synthetic datasets of 3,333, 5,000, and 6,667 samples, and split them into two, three, and four clients, respectively. Table 5 shows that our method outperforms under the three cases.

Table 5: Comparison of accuracy and fairness in the synthetic, Adult, COMPAS, and Bank datasets w.r.t demographic parity (DP) on logistic regression. The implementation of LFT+Ensemble, LFT+FEDAVG, and CFL are all based on FB.

| Number of Clients | Method      | Synthetic Acc. (%) | Synthetic DP Disp. (↓) | Adult Acc. (%) | Adult DP Disp. (↓) | COMPAS Acc. (%) | COMPAS DP Disp. (↓) | Bank Acc. (%) | Bank DP Disp. (↓) |
|-------------------|-------------|--------------------|------------------------|---------------|-------------------|----------------|---------------------|---------------|-----------------|
| Two clients       | LFT+ENSEMBLE | .780 ± 0.090      | .161 ± 0.157           | .727 ± 0.194  | .248 ± 0.194      | .720 ± 0.192  | .246 ± 0.180       |               |                 |
|                   | LFT+FEDAVG  | .866 ± 0.002      | .429 ± 0.017           | .823 ± 1.02   | .305 ± 1.31       | .608 ± 0.367  | .491 ± 1.02        |               |                 |
|                   | FedFB       | .679 ± 0.007      | .047 ± 0.012           | .725 ± 0.012  | .051 ± 0.016      | .705 ± 0.004  | .005 ± 0.003       |               |                 |
| Three clients     | CFL         | .668 ± 0.028      | .063 ± 0.035           | .726 ± 0.009  | .028 ± 0.016      | .670 ± 0.035  | .064 ± 0.020       |               |                 |
| Four clients      |             |                    |                        |               |                   |               |                     |               |                 |

C.5 Comparison with FAIRFED

| Property | Fair | Method | Synthetic Acc. (%) | Synthetic DP Disp. (↓) | Adult Heterogeneity Level | Adult Heterogeneity Level | COMPAS Heterogeneity Level | COMPAS Heterogeneity Level | Bank Heterogeneity Level | Bank Heterogeneity Level |
|----------|------|--------|--------------------|------------------------|--------------------------|--------------------------|---------------------------|--------------------------|--------------------------|--------------------------|
| ✓        | ✓    | FAIRFED | .775               | .794                   | .819                     | .824                     | .824                      | .594                     | .586                     | .608                     |
| ✓        | ✓    | FedFB (OURS) | .799               | .769                   | .772                     | .822                     | .769                      | .668                     | .655                     | .665                     |
| ✓        | ✓    | FAIRFED | .021               | .037                   | .061                     | .065                     | .065                      | .048                     | .040                     | .072                     |
| ✓        | ✓    | FedFB (OURS) | .042               | .034                   | .038                     | .063                     | .060                      | .006                     | .011                     | .026                     |

Table 7: Comparison of accuracy and fairness in the synthetic, Adult, COMPAS, and Bank datasets w.r.t demographic parity (DP) on multilayer perceptron.

| Method           | Synthetic Acc. (%) | Synthetic DP Disp. (↓) | Adult Acc. (%) | Adult DP Disp. (↓) | COMPAS Acc. (%) | COMPAS DP Disp. (↓) | Bank Acc. (%) | Bank DP Disp. (↓) |
|-------------------|--------------------|------------------------|---------------|-------------------|----------------|---------------------|---------------|-----------------|
| FedFB (OURS)      | .725 ± 0.012       | .051 ± 0.018           | .804 ± 0.001  | .028 ± 0.001      | .616 ± 0.033  | .036 ± 0.028         | .883 ± 0.000  | .000 ± 0.000    |
| AGNOSTICFAIR      | .655 ± 0.028       | .028 ± 0.013           | .790 ± 0.012  | .032 ± 0.016      | .612 ± 0.037  | .096 ± 0.046         | .883 ± 0.000  | .000 ± 0.000    |

To make fairer comparison, we follow the exact same setting as Ezzeldin et al. [2021] to re-split Adult and COMPAS dataset, employ |$SPD|$ = $|P(\hat{y} = 1 \mid a = 0) - P(\hat{y} = 1 \mid a = 1)|$ as unfairness metric and present the results in Table 6. We observe that FedFB achieves higher fairness than FAIRFED while being robust to data heterogeneity.
Lastly, we compare our method with AGNOSTICFAIR [Du et al., 2021], which exchanges the model parameters and the other information after every gradient update to mimic the performance of CFL with FAIRNESSCONSTRAINT implementation [Zafar et al., 2017c]. Table 7 shows that our FEDFB achieves similar performance as AGNOSTICFAIR, at a much lower cost of privacy.