On Functional Formulation of the Statistical Theory of Homogeneous Turbulence and the Method of Skeleton Feynman Diagrams

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Abstract—The paper reviews the application of the formalism of a characteristic functional for statistical description of a random velocity field obeying the Navier–Stokes equation for incompressible fluids in the presence of regular and random external forces. The equation in functional derivatives for the characteristic functional is obtained using a representation of the characteristic functional in the form of a functional integral over two fields. From this equation one can obtain equations for various statistical characteristics of the velocity field such as the variance of velocity pulsations (the pair correlation function) or the mean response of velocity field to external forces (Green’s function). The method of skeleton Feynman diagrams is used in the analysis of the equations and of the solution structures. This fact follows directly from the functional formulation of the theory without referring to the commonly used methods of perturbation theory. The vertices of three types arising in the theory formulation appear to be linked. This enables considering the vertex of only one type and simplify the diagrammatic representations of various quantities.

Keywords: Navier–Stokes equation, statistical description of turbulence, skeleton diagram technique, characteristic functional, functional integral

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1. INTRODUCTION

In the description of the velocity field we consider the spatial-temporal distributions of velocities \( u(r, t) \), which we will call the velocity field implementations. The statistical description of the turbulent velocity field implies the prescription of the probability density function of various implementations of the field \( P[ u(r, t)] \). Knowledge of this function enables us to calculate all possible mean quantities characterizing the averaged behavior of the turbulent fluid and its characteristics such as the energy distribution over the wave-number spectrum and the averaged transfer coefficients of turbulent viscosity, diffusion, and thermal conductivity and so forth.

The appearance of stochasticity is usually associated with the development of instability of large-scale fluid flows. In the theory of random processes, the stochasticity is modeled by introducing the external random forces (the Langevin forces) through prescription of the distribution function of external random forces (the additive noise). In several cases the stochasticity is modeled by stochastic coefficients in the equations describing the processes (the multiplicative noise), for instance, the transfer of passive admixture in the stochastic velocity field. A prescribed quantity of external random force corresponds to each realization of the velocity field, and the averaging over realizations of the velocity field corresponds to the averaging over realizations of the external random force under a given distribution function of implementations of the random forces.

The functional formulation of the statistical theory of turbulence means that, instead of the statistical description in terms of the probability distribution function of the field implementations \( P[ u(r, t)] \) and instead of calculating various mean quantities with the formula

\[
\langle F[u] \rangle = \int d[u] P[u] F[u],
\]

where \( d[u] \) is the elementary volume in the space the velocity field implementations, we use the more convenient method of the characteristic functional (ChF) [1, Section 3.4].
The ChF method is a generalization of the method of characteristic functions, used in statistical description of stochastic quantities, to the case of stochastic fields. In the theory of stochastic quantities, the characteristic function is determined as the mean quantity from the Fourier-transform of the probability density function of stochastic quantities. Therefore, in the theory of stochastic fields, the ChF is determined as a functional analog of the Fourier-transform of the probability density function of stochastic fields

\[ \Phi[\eta] = \int d[u] P[u] \exp[i\eta \cdot u]; \quad \eta \cdot u = \int dr dt \eta(r,t) \cdot u(r,t). \]  

(1.1)

Here, the mean quantities are calculated by performing the operation of functional (variational) differentiation with the formula

\[ \langle F[u(r,t)] \rangle = F \left[ \frac{\delta}{i\delta \eta(r,t)} \right] \Phi[\eta] \bigg|_{\eta=0}. \]  

(1.2)

The concept of ChF is not only applied in statistical hydrodynamics, but it is also widely used in mathematics and various areas of physics [2]. Although the possibility for describing the stochastic fields in terms of ChF had been already formulated in 1935 by Kolmogorov [3]; in physics this concept was used somewhat later. In statistical mechanics, the so-called generating functional, corresponding to the ChF, was introduced in 1946 by Bogolyubov [4], and in the quantum field theory it was introduced in 1949 by J. Schwinger in order to formulate the problem beyond the framework of perturbation theory [5]. Later, he wrote the equations in functional (variational) derivatives for the generating functional (the Schwinger equations) [6]. In 1954, Gelfand and Minlos [7] and, independently, Edwards and Peierls [8] obtained the representation of the solution to the equations in functional derivatives of the field theory in the form of functional integrals (continual integrals, path integrals).

In the statistical theory of turbulence, the ChF method was first proposed by Hopf [9] for describing the correlation functions of the velocity field in different points and given time instance (the spatial ChF)

\[ \Phi[\eta(r), t] = \langle \exp[i\eta(r) \cdot u(r,t)] \rangle. \]  

(1.3)

Hopf derived the equation in functional derivatives specifying the temporal evolution of ChF under the given initial quantity of ChF.

R. M. Lewis and R. H. Kraichnan proposed the generalization of this method that allows describing the correlations of the velocity field at different points and at different time instances (the method of spatial—temporal ChF) [10] (see also [11]). The spatial—temporal ChF of Lewis and Kraichnan is determined by formula (1.1). The equation in functional derivatives was obtained [10] for the spatial-temporal ChF, and this equation, by authors’ opinion, allows determining the ChF. However, the issue about the way the averaging is performed and about the ensemble statistics over which the averaging is performed remains open, so the Lewis—Kraichnan equation contains an arbitrary rule and the procedures to eliminate this arbitrary rule are not formulated. Note that for the spatial Hopf ChF there is no such arbitrary rule, because the statistical properties of the ensemble are taken into account in the given initial quantity of ChF.

In the theory of turbulence, the use of the Langevin approach for modeling the appearance of stochasticity due to instability development in large-scale flows was first proposed by Wyld [12] (see also [13]). In contrast to the traditional approach to the statistical description of turbulence in terms of statistical moments, Wyld applied the perturbation method in the construction of solution to the Navier—Stokes equation (NSE) for the velocity field. In his theory, the nonlinear term in the NSE was considered the perturbation and the solution was represented as a series in terms of powers of the external random force. In calculation of the statistical moments of the velocity field, the expansions were multiplied and then term-by-term averaged using the rules for calculating the statistical moments in the case of centered normal distribution of external random forces. Such procedure requires laborious calculations which need thoroughness and attention. To simplify the calculations, it was proposed to use the technique of Feynman diagrams in the analysis of obtained expansions, which is a graphical representation of mathematical expressions (see, e.g., [14, 15]).

The main elements of the Wyld diagram technique are Green’s functions of the linear problem \( G^{(0)}(1,2) \), which we will depict by the segment of dashed lines between points 1 and 2 with an arrow incoming to point 1 (the straight line in the Wyld diagram technique), the so-called vertex \( V'(1,2,3) \), which we will indicate by a light circle with two incoming arrows 2 and 3 and one outgoing arrow 1 (the point in the Wyld notation), and the pair correlation function of external random forces \( D^{(0)}(1,2) \), which we will draw by an light circle with two outgoing arrows (the wavy line in the denotations of Wyld). The
correspondence rules between the Feynman diagrams of the perturbation theory and the analytical expressions are provided in Fig. 1.

Use of the perturbation theory and the corresponding Feynman skeleton diagram technique (SDT) supposes small perturbations and the possibility for judging the behavior of the system based on knowledge of the first few terms of the entire expansion. However, in the case of developed turbulence, the actual expansion parameter (the effective Reynolds number) is not small, and the question of the behavior of the entire expansion, i.e., of the summation of the diagrams of the perturbation theory, arises. This is achieved with the transition to the Feynman skeleton diagrams corresponding to the sum of some infinite subsequence of diagrams of the perturbation theory and with the system description in terms of skeleton diagrams.

A similar problem appeared in the theory of quantized fields in the formulation of the approach for describing strong interactions. In 1973, the theorem about the equivalence between the statistical problem for an arbitrary classical system and some quantum field theory (the so-called doubled formalism) was formulated in the study by Martin et al. [16]. Monograph [17] provided the formulation of the statement that in quantum field theory the representation of the generating functional for all possible Green’s functions may be obtained by averaging the exponent of action for a classical particle in the external field over the quantum fluctuations of the external field, which agrees with the equivalence theorem [16]. In the construction of the statistical theory of turbulence, the equivalence theorem allows applying the powerful mathematical apparatus developed in quantum field theory, which uses the improved perturbation theory, the renormalization concept, and the representation of the solution to the equation in functional derivatives for the generating functional (the Schwinger equation) in the form of a functional integral. In the statistical theory of turbulence, the transition to SDT is performed naturally in the functional formulation of the theory in terms of ChF and not based on the thorough analysis of the perturbation theory diagrams accompanied by mistakes [12–16].

In this paper, we describe the formulation of the statistical theory of turbulence in terms of ChF and provide the technique for representation of the spatial-temporal ChF following directly from the NSE with the presence of the external random and regular force. Using this representation, we may derive the equation in functional derivatives for ChF and use it for determining various relations between the statistical moments of the velocity field, Green’s functions, and other quantities characterizing the field of turbulent pulsations of velocity. We graphically present the derived equations using the SDT. We also show how the functional formulation of the statistical description of turbulence allows one to arrive at the SDT beyond the scope of perturbation theory.

2. MATHEMATICAL PROBLEM FORMULATION

To simplify the notation, we will further use the digital notation \( \{\mathbf{r}_i, t_i, \alpha_i\} \equiv 1 \), according to which \( u_{\alpha i}(\mathbf{r}_i, t_i) \equiv u(1) \), for all coordinates of the spatial-temporal point and for the indexes of vector components. We will also mean the integration over the spatial-temporal variables and summation over repeated indices (the Einstein rule), that is,

\[
u(1) \psi(1) \equiv \int u(\mathbf{r}_i, t_i) \psi(\mathbf{r}_i, t_i) d\mathbf{r}_i dt_i.
\]

When the vector or tensor indices are written explicitly, the number notation will only refer to the spatial-temporal variables.

The current consideration is based on the NSE for an incompressible fluid with the presence of given statistically external random force \( x(1) \) and regular force \( f(1) \). In this notation, the NSE is given by (more details are given in [18])
With no loss of generality, we may assume that the random force $X(1)$ is solenoidal (nondivergent) and obeys the centered normal distribution with the pair correlation function given by

$$P_{ab}(1, 2) = \langle X_a(1)X_b(2) \rangle = P_{ab}(1, 1')D^{(0)}(1' - 2),$$

where $P_{ab}(1, 2)$ is the operator of transverse projection whose Fourier-transform has the form $P_{ab}(k) = \delta_{ab} - k_\alpha k_\beta / k^2$, where $k_\alpha P_{ab}(k) = 0$.

According to definition (1.1), to obtain the representation of the ChF, we need to know the probability distribution function of different implementations of the velocity field $\phi(1)$, which may be represented in the form of the mean quantity of the functional

$$\Phi[\eta, f] = \langle \hat{\delta}[\hat{\phi}(1) - \hat{\phi}(1, X)] \rangle, \quad \delta[\phi - \bar{\phi}] = \delta(\phi - \bar{\phi})/|\delta[\phi - \bar{\phi}]|,$$

and of the functional analog of the Fourier expansion for $\delta$ function

$$\delta[L[\phi] - f - X] = \int \hat{\delta}L[\phi]/\hat{\delta}\phi \exp [i\hat{\phi}L[\phi] - f - X] \rangle$$

and obtain the representation for the ChF in the form of a double functional integral over the fields $\phi$ and $\bar{\phi}$

$$\Phi[\eta, f] = \langle \int \hat{\delta}L[\phi]/\hat{\delta}\phi \exp [i\hat{\phi}\phi - f - X] \rangle$$

It is usually assumed that the contribution of Jacobian of the functional mapping $L[\phi] \rightarrow \phi$ (of the Berezinian) $\delta L[\phi]/\delta \phi$ may be reduced to the redefinition of the integral measure by including it in $d[\phi]$, which allows taking $|\delta L[\phi]/\delta \phi| = 1$ in Eq. (2.3). However, since we are interested not in integral (2.3), but in the equation for the ChF followed from that, the explicit form of this quantity appears to be insignificant.

In the case of Gaussian statistics of the field $X(1)$, the averaging over implementations of the external random force gives

$$\langle \exp(-i\hat{\phi}(1)X(1)) \rangle = \exp \left\{-\frac{1}{2} \hat{\phi}(1)D^{(0)}(1, 1')\hat{\phi}(1') \right\},$$

where

$$D^{(0)}(1, 1') = \langle X(1)X(1') \rangle.$$
Using the invariance of the functional integral with respect to the shift of the functional variable \( \hat{u} \to \hat{u} + \delta \hat{u} \), we arrive at the equation in functional derivatives for the spatial-temporal ChF

\[
L^{(0)}(1,2,3) = \frac{1}{2} \left\{ V(1 \mid 2,3) + \frac{\delta^2 \ln \Phi}{\delta \eta(2) \delta \eta(3)} \right\} 
- \frac{\delta \ln \Phi}{\delta \eta(2)} \Phi[\eta,f] 
+ \frac{\delta^2 \ln \Phi}{\delta \eta(2) \delta \eta(3)} 
= f(1) \Phi[\eta,f] + D^{(0)}(1,2) \frac{\delta \Phi[\eta,f]}{\delta \eta(2)}. \tag{2.5}
\]

It differs from the equation for the spatial-temporal ChF [10] in account for the external regular force \( f \) and by the presence of the term containing the variance of the external stochastic forces \( D(0) \) determining the statistics of the ensemble of velocity field implementations.

We also write the equation for the logarithm of a ChF, whose functional derivatives give the representation for various irreducible statistical moments (cumulative mean quantities) of the velocity field. In the language of the Feynman diagrams of perturbation theory, the functional derivatives of this functional correspond to accounting only for the one-particle irreducible diagrams, that is, the diagrams which cannot be decomposed into two independent parts by breaking one line

\[
L^{(0)}(1,2) \frac{\delta \ln \Phi}{\delta \eta(1)} + \frac{1}{2} V(1 \mid 2,3) \left[ \frac{\delta \ln \Phi}{\delta \eta(2)} \delta \eta(3) \right] 
+ \frac{\delta^2 \ln \Phi}{\delta \eta(2) \delta \eta(3)} 
= f(1) + D^{(0)}(1,2) \frac{\delta \ln \Phi}{\delta \eta(1)}. \tag{2.6}
\]

Following the early proposed approach [18,20] we make a transit to the new functional variables \( \hat{\eta} \) and \( \hat{f} \):

\[
\hat{\eta}(1) = \frac{\delta \ln \Phi[\eta,f]}{\delta \eta(1)}, \quad \hat{f}(1) = \frac{\delta \ln \Phi[\eta,f]}{\delta \eta(1)}. \tag{2.7}
\]

For \( \eta \to 0 \) we obtain \( \hat{\eta}(1) \to \left\langle u(1) \right\rangle \), that is, this quantity tends to the averaged quantity of the velocity component \( \alpha_i \) at the spatial-temporal point \( (\eta_i, t) \) in the field of external regular force \( f_{\alpha}(r, t) \).

The transition to the new functional variables is done using the functional Legendre transformation by introducing the new functional

\[
\Psi[\hat{\eta}, \hat{f}] = -\ln \Phi[\eta,f] + i \eta \hat{\eta} + i \hat{f}. \tag{2.8}
\]

In this case,

\[
\frac{\delta \Psi}{\delta \hat{\eta}(1)} = \eta(1), \quad \frac{\delta \Psi}{\delta \hat{f}(1)} = f(1),
\]

\[
\frac{\delta^2 \ln \Phi}{\delta \eta(1) \delta \eta(2)} = C(1,2), \quad \frac{\delta \hat{f}(2)}{\delta \hat{f}(1)} = \frac{\delta \hat{\eta}(1)}{\delta \hat{f}(1)} = G(1,2). \tag{2.9}
\]

In the limit \( \eta \to 0 \), the functional derivatives of \( \ln \Phi \) in Eqs. (2.9) admit a simple interpretation, namely: \( C(1,2) \) is the pair correlation function (variance) of the velocity field, the function \( G(1,2) \) describes the response of the averaged velocity field at spatial-temporal point \( I \) to the action of the force source localized at point \( I \), in other words, it is Green’s function (tensor). In SDT we will depict the function \( C(1,2) \) (also called the correlator) by the solid line connecting points \( I \) and \( 2 \) with the arrows incoming to these points and the function \( G(1,2) \) (called the propagator) by the solid line with the arrow incoming at point \( I \) (Fig. 2).

The equation in functional derivatives for the functional \( \Psi[\hat{\eta}, \hat{f}] \) becomes

\[
L^{(0)}(1,2) \hat{\eta}(2) + \frac{1}{2} V(1 \mid 2,3) \left[ \hat{\eta}(2) \hat{\eta}(3) + \frac{\delta \hat{\eta}(3)}{\delta \hat{\eta}(2)} \right] 
= \frac{\delta \Psi}{\delta \hat{f}(1)} + i D^{(0)}(1,2) \hat{f}(2). \tag{2.10}
\]

In the diagram technique of perturbation theory, the functional derivatives of the functional \( \Psi[\hat{\eta}, \hat{f}] \) correspond to the (infinite) set of one-particle irreducible diagrams with the external lines taken off. In the SDT (we take the sum of some infinite subsequence of perturbation theory diagrams) we will denote the functional \( \Psi[\hat{\eta}, \hat{f}] \) by a dark circle and the action upon it by the operators of functional differentiation.
\[ \frac{\delta^2 \Psi}{\delta \hat{\eta}(1) \delta \hat{\eta}(2)} = G^{-1}(1, 2) \]

And

\[ \frac{\delta^2 \Psi}{\delta \hat{\eta}(1) \delta \hat{\eta}(2)} = D(1, 2) \]

For the following analysis we consider the mixed functional derivatives of the functional \( \Psi \)

\[ \frac{\delta^3 \Psi}{\delta \eta(2) \delta \hat{\eta}(1)} = \frac{\delta \hat{\eta}(3)}{\delta \eta(2)} \frac{\delta^2 \Psi}{\delta \hat{\eta}(1) \delta \hat{\eta}(1)} + \frac{\delta \hat{\eta}(3)}{\delta \hat{\eta}(1)} \frac{\delta^2 \Psi}{\delta \eta(2) \delta \hat{\eta}(1)} = \delta(1 - 2). \tag{2.11} \]

And

\[ \frac{\delta^3 \Psi}{\delta \eta(2) \delta \hat{\eta}'(1)} = \frac{\delta \hat{\eta}(3)}{\delta \eta(2)} \frac{\delta^2 \Psi}{\delta \hat{\eta}(1) \delta \hat{\eta}'(1)} + \frac{\delta \hat{\eta}(3)}{\delta \hat{\eta}(1)} \frac{\delta^2 \Psi}{\delta \eta(2) \delta \hat{\eta}'(1)} = 0. \tag{2.12} \]

3. EQUATIONS FOR PAIR CORRELATION FUNCTION AND GREEN’S FUNCTION

Formula (2.11) and definitions of Green’s function (2.9) imply

\[ \frac{\delta^2 \Psi}{i \delta \hat{\eta}(1) \delta \hat{\eta}(2)} = G^{-1}(1, 2), \tag{3.1} \]

where \( G^{-1} \) is the inverse Green’s function determined by

\[ G(1, 1') G^{-1}(1', 2) = \delta(1 - 2). \]

When accounting for Eq(3.1), the Eq.(2.12) takes the form:

\[ C(1, 1') G^{-1}(1', 2) - G(1, 1') D(1', 2) = 0. \tag{3.2} \]
or

\[ C(1,2) = G(1,1')G(2,2')D(1',2), \quad D(1,2) = \frac{\delta^2 \Psi}{i \delta \hat{f}(1) \delta \hat{\eta}(2)}. \]  

(3.3)

In the statistical theory of turbulence Eq. (3.3) was first obtained by Wyld [12] by analyzing the series of perturbation theory using the SDT. We should interpret the function \( D(1,2) = D(0)(1,2) + D(1)(1,2) \) in Wyld Eq. (3.3) as the pair correlation function of effective random forces, \( D(1) \) as the correction to the original (bare) correlation function of random forces \( D(0) \) caused by the intermodal interactions. Note that in the derivation of the Wyld equation, the NSE and the equation for the characteristic functional followed from that have not been used, and we may treat this equation as the introduction of a new statistical characteristic of the turbulent velocity field \( D(1,2) \) determined by formula (3.3).

To derive the equations for the functions \( G^{-1} \) and \( D \), we act with the operator \( \delta / \delta \hat{\eta}(2) \) on Eq. (2.10) for the functional \( \Psi \). Using relations (2.9) and (3.1), we obtain

\[ \frac{\delta^2 \Psi}{i \delta \hat{f}(1) \delta \hat{\eta}(2)} = G^{-1}(1,2) = L(0)(1,2) - \Sigma(1,2), \]  

(3.4)

where

\[ \Sigma(1,2) = \Sigma(0)(1,2) + \Sigma(1)(1,2), \]

\[ \Sigma(0)(1,2) = -V(1 \mid 2,3)\hat{\eta}(3), \quad \Sigma(1)(1,2) = -\frac{1}{2} V(1 \mid 3,4) \frac{\delta C(3,4)}{\delta \hat{\eta}(2)}, \]  

(3.5)

and, by acting by the operator \( \delta / \delta \hat{\eta}(2) \) on Eq. (2.10) and accounting for Eq. (3.2), we obtain

\[ \frac{\delta^2 \Psi}{i \delta \hat{f}(2) \delta \hat{f}(1)} = D(1,2) = D(0)(1,2) + D(1)(1,2), \]  

(3.6)

\[ D(1)(1,2) = -\frac{1}{2} V(1 \mid 3,4) \frac{\delta C(3,4)}{i \delta \hat{\eta}(2)}. \]  

(3.7)

From Eqs. (3.4) and (3.6) it follows that the quantity \( \Sigma(1,2) \) and \( D(1)(1,2) \) determined by formulas (3.5) and (3.7) should be interpreted as the corrections to the fluid viscosity and to the correlation function of external random forces that are caused by the influence of the turbulent mixing.

Equation (3.4) may be rewritten in the form

\[ G(1,2) = G(0)(1,2) + G(0)(1,1')\Sigma(1',2')G(2',2), \]  

(3.8)

where \( G(0) \) is Green’s function of the linear equation

\[ L(0)(1,1')G(0)(1',2) = \delta(1 - 2), \quad [G(0)(1,2)]^{-1} = L(0)(1,2). \]

Equation (3.8) is the analog of the Dyson equation in quantum field theory, and the quantity \( \Sigma \) is the analog of the self-energy operator. Next, let us put in correspondence the pair correlation function (PCF) of effective random forces \( D(1,2) \) to a dark circle with two outgoing arrows, the correction to the correlation function of effective random forces to a rectangle with two outgoing arrows and the quantity \( \Sigma(1,2) \), which describe a correction to the viscous term in NSE due to momentum transfer by turbulent velocity pulsations, to a rectangle with the arrow incoming in the point 2 and with the arrow outgoing from the point 1. In the SDT, Eqs. of Dyson (3.8) and Wyld (3.3) are depicted by the diagrams presented in Fig. 3.

If we act on Eq. (2.6) for \( \ln \Phi \) with the operator \( \delta / \delta \eta(4) \), then we may derive the equation for the PCF

\[ L(0)(1,2)C(2,4) + \frac{1}{2} V(1 \mid 2,3)C(2,3,4) - G(4,2)D(0)(1,2) = 0, \]  

(3.9)

where \( C(2,3,4) \) is the three-point third-order statistical moment;

\[ C(2,3,4) = -\frac{\delta}{i \delta \eta(4) \delta \eta(2) \delta \eta(3)} \delta^2 \ln \Phi = \langle u(2)u(3)u(4) \rangle. \]  

(3.10)
To obtain the closed equation for the PCF, which is a problem of direct interest, various phenomenological closing hypotheses of the turbulent viscosity type are traditionally used; according to this hypothesis, the contribution of the nonlinear term in Eq. (3.8) is taken proportional to the pair correlation function and is written as $V_4 \{\nu(T, 1, 4) V_2(C(2, 4))\}$, where the integral kernel $\nu(T, 1, 2)$ is interpreted as the turbulent viscosity coefficient. Note that the third statistical moment enters Eq. (3.6) in form of the combination $\delta \langle \hat{\eta} \rangle$, which is the second-rank tensor also referred to as the inertia tensor. It is exactly this quantity that is needed for deriving the closed equation for the PCF. It appears that this quantity may be expressed in terms of the quantities related with the PCF and with Green’s function.

The third-order moment with account for formulas (2.9) may be written as

$$C(2, 3, 4) = \frac{\delta C(2, 3)}{i \delta \eta(4)} = \frac{\delta \hat{\eta}(4') \delta C(2, 3)}{i \delta \eta(4')} + \frac{\delta \hat{\eta}(4') \delta C(2, 3)}{i \delta \eta(4')}$$

$$= C(1', 1) \frac{\delta C(2, 3)}{i \delta \eta(4')} + G(4, 4') \frac{\delta C(2, 3)}{i \delta \eta(4')}.$$

The use of formulas (3.5) and (3.7) for the second-rank tensor $T(1, 4)$ leads to the representation

$$T(1, 4) = \frac{1}{2} V(1 \mid 2, 3) C(2, 3, 4) = -\frac{1}{2} \Sigma^{(1)}(1', 1') C(1', 4) - \frac{1}{2} G(1, 1') D^{(1)}(1', 4),$$

and the problem is reduced to the determination of the quantities $\Sigma^{(1)}$ and $D^{(1)}$ given by formulas (3.5) and (3.7).

To find the explicit form of the quantities $\Sigma^{(1)}$ and $D^{(1)}$, according to (3.5) and (3.7), we should at first determine the functional derivatives $C$ with respect to the fields $\hat{\eta}$ and $\hat{f}$.

After differentiating the identity $G(1, 1') G^{-1}(1', 2) = \delta(1, 2)$ with respect to the functional variable $\hat{\eta}(3)$ and using the definition of the inverse Green’s function, Eq. (3.1), one finds

$$\frac{\delta G(1, 2)}{\delta \hat{\eta}(3)} = -G(1, 1') \frac{\delta \Psi}{i \delta \hat{f}(1') \delta \hat{\eta}(2') \delta \hat{\eta}(3)} G(2', 2) = -G(1, 1') \Gamma(1' \mid 2', 3) G(2', 2)$$

(3.12)

and, similarly,

$$\frac{\delta G(1, 2)}{i \delta \hat{f}(3)} = -G(1, 1') \frac{\delta \Psi}{i \delta \hat{f}(1') \delta \hat{\eta}(2') \delta \hat{\eta}(3)} G(2', 2) = -G(1, 1') \Gamma(1', 3 \mid 2') G(2', 2).$$

(3.13)

The differentiation of the Wyld Eq. (3.3) with respect to $\hat{\eta}(3)$ and $\hat{f}(3)$ with the use of Eqs. (3.12) and (3.13) gives

$$\frac{\delta C(1, 2)}{\delta \hat{\eta}(3)} = -2G(1, 1') \Gamma(1' \mid 2', 3) C(2', 2) + G(1, 1') G(2, 2') \Gamma(1', 2' \mid 3),$$

(3.14)

$$\frac{\delta C(1, 2)}{i \delta \hat{f}(3)} = -2G(1, 1') \Gamma(1', 3 \mid 2') C(2', 2) + G(1, 1') G(2, 2') \Gamma(1', 2', 3),$$

(3.15)

where the three new functions have been introduced.
called the vertices in the diagram technique and interpreted in quantum field theory as the quantities describing the merger of two quanta into a single one (the vertex of the first type), the decay of a single quantum into two (the vertex of the second type), and the generation of three quanta by an external field (the vertex of the third type). The necessity of accounting for the vertices of all three types was first pointed in the paper [16] (see, e.g., [18, 19]).

In the SDT, we will put in correspondence the vertices \( \Gamma(1 \mid 2, 3) \), \( \Gamma(1, 2 \mid 3) \), and \( \Gamma(1, 2, 3) \) to the dark circle with two incoming and one outgoing arrows, with one incoming and two outgoing arrows and with three outgoing arrows (Fig. 2). According to definitions (3.16), within the SDT the vertices are are depicted by the arrows incoming or outgoing to the rectangles (Fig. 4). Using relations (3.14) and (3.15), for the quantities \( \Sigma(1) \) and \( \eta(1) \) we obtain

\[
\Sigma^{(1)}(1, 2) = V(1 \mid 3, 4)G(3, 3') \left[ C(4, 4')\Gamma(3' \mid 4', 2) - \frac{1}{2} G(4, 4')\Gamma(3', 4' \mid 2) \right],
\]

(3.17)

\[
D^{(1)}(1, 2) = V(1 \mid 3, 4)G(3, 3') \left[ C(4, 4')\Gamma(2, 3' \mid 4') - \frac{1}{2} G(4, 4')\Gamma(2, 3', 4') \right].
\]

(3.18)

Within the SDT the operations of functional differentiation will be graphically shown as the insert of incoming and outgoing arrows in the lines of propagators, correlators, and vertices. The rules of action of the operators \( \delta / \delta \hat{\eta} \) and \( \delta / \delta \hat{f} \) upon the functions of the propagator \( G \) and the correlator \( C \) following from formulas (3.12)—(3.15) are provided in Fig. 4, and the diagrams for the quantities \( \Sigma^{(1)}(1, 2) \) and \( D^{(1)}(1, 2) \), depicted by rectangles, which are with formulas (3.5) and (3.7) expressed through the functional derivatives of the correlator \( C(3, 4) \) with respect to the fields \( \hat{\eta}(2) \) and \( \hat{f}(2) \), which are in their turn depicted by the insert of the incoming and outgoing arrows in the line of the correlator, which leads to the diagrammatic representation for \( \Sigma^{(1)}(1, 2) \) and \( D^{(1)}(1, 2) \) provided in Fig. 4 and to corresponding formulas (3.17) and (3.18).

The system of Eqs. (3.3), (3.8), (3.17), and (3.18) is the exact consequence of the equation for ChF (2.6) and of Eq. (2.10) followed from that.

When deriving these equations, we have used neither approximations nor additional phenomenological assumptions. However, this system is not complete since it contains three unknown functions \( \Gamma \) for which we can obtain the equations containing the higher-order moments, that is, the chain of equations arises in something similar to the chain of Friedman—Keller equations in the traditional formulation of the statistical theory of turbulence in terms of statistical moments. Note that the unknown functions \( \Gamma \) enter only the expressions for \( \Sigma \) and \( D^{(1)} \) determined by formulas (3.17) and (3.18) and following from theNSE with the presence of external random force.

A scheme for the approximate closing the chain was proposed in [21], where perturbation theory was used at some stage that allows finding at least some infinite subsequence of the complete series of perturbation theory. In particular, in the calculation of the quantities \( \Sigma^{(1)}(1, 2) \) and \( D^{(1)}(1, 2) \) it was proposed to use the lowest approximation of perturbation theory for the vertices by taking

\[
\Gamma(1 \mid 2, 3) = V(1 \mid 2, 3), \quad \Gamma(1, 2 \mid 3) = \Gamma(1, 2, 3) = 0,
\]

(3.19)

which, from the viewpoint of the Friedman—Keller chain, in something corresponds to the closing scheme of the equations for the statistical moments. However, in the considered case the closing is performed without invoking the phenomenological hypotheses of the type of the Millionshchikov quasi-normality approximation [22] about the relation between the higher statistical moments and the lowest ones or of the Kraichnan direct interactions approximation [23]. Approximation (3.19) corresponds to the one-loop approximation well known in quantum field theory and successfully used in several other areas,
where the contribution of all the perturbation theory diagrams not containing the intersecting loops is taken into account.

The solution to the derived set of equations may be constructed by the iteration method, where, as the zeroth approximation, we use the expressions for $\Sigma$ and $D^{(1)}$ calculated in the lowest approximation of perturbation theory. The substitution of these expressions in the Wyld Eqs. (3.3) and in the Dyson Eqs. (3.8) with their subsequent solution corresponds to the summation of some infinite subsequence of the series of perturbation theory for Green’s function and for the pair correlation function.

We may improve the one-loop approximation, Eq. (3.19), calculating the results of functional differentiation operator actions to expressions for $\Sigma$ and $D^{(1)}$ with respect to the fields $\hat{\eta}$ and $\hat{f}$ according to formulas (3.16). In the SDT, the operations of functional differentiation will be drawn by the insertion of incoming and outgoing arrows in the lines of propagators, correlators, and vertices.

Figure 4 shows the graphical representations of the results of the functional differentiation operator action upon the lines of propagators and correlators corresponding to formulas (3.12)–(3.15). According to formulas (3.5) and (3.7), the diagrams for $\Sigma^{(1)}$ and $D^{(1)}$ contain the insert in the line of the correlator with incoming and outgoing arrows, and the use of formulas (3.16) allows graphically presenting the vertices as rectangles with the inserts of incoming and outgoing arrows (Fig. 4).

To obtain the representations for the vertices by applying the functional differentiation operation in the expressions for $\Sigma^{(1)}$ and $D^{(1)}$, according to formulas (3.16), we need to perform the long calculations with the use of cumbersome formulas. We may considerably simplify this procedure in the SDT by using the rules of graphical execution of the operations of functional differentiation formulated above.
Figure 5 provides the result of the SDT application in calculating the vertex $\Gamma(1|2,3)$. The graphical representation of the vertices of two other types may be obtained by replacing one or two incoming arrows with outgoing ones. Note that the graphical representations for the vertices $\Gamma$ contain even more vertices with four incoming and outgoing arrows (four-tailed vertices); for them we may obtain the representation containing the five-tailed vertices by applying the graphical representation of operator action. Thus, the chain of equations appears, which, in some sense, is similar to the chain of the Friedman–Keller equation in the statistical description of turbulence in terms of statistical moments. Note that, in the graphical representation, the left vertex is always bare one.

However, in the considered case the approximation is constructed not in the form of expansion in powers of some parameter as in the perturbation theory, but with another scheme. We have already said that approximation (3.19) corresponds to accounting for all the diagrams of the perturbation theory not containing the intersecting loops. If we ignore all four-tailed vertices in the diagrammatic representation for vertices, then the derived system of diagrammatic equations will reflect accounting for all the diagrams of perturbation theory not containing the twice-intersecting loops.

From the one-loop approximation, Eq. (3.19), it follows that the corrections to the correlation function of external random forces $D^{(1)}$, which, according to formula (3.18) are expressed through the vertices $\Gamma(1,2|3)$ and $\Gamma(1,2,3)$ containing two and three outgoing arrows, are equal to zero. However, it appears that we can express the vertices with two and three outgoing arrows through the vertex with one outgoing arrow, $\Gamma(1|2,3)$.

To find the additional relations linking the vertices of various types, we act on Eq. (2.11) with the operator of functional differentiation $\delta / \delta \hat{\varphi}(4)$ and obtain

$$C(2,3)\Gamma(4|3,1) + G(2,3)\Gamma(4,3|1) = 0.$$  

Similarly, by acting on Eq. (2.12) with this operator, we find

$$C(2,3)\Gamma(4,1|3) + G(2,3)\Gamma(4,1,3) = 0.$$  

Using the Wyld Eq. (3.3), we arrive at the representation of vertices with two and three outgoing arrows through the vertex with two incoming and one outgoing arrow.
4. DISCUSSION

Since the functional formulation of the statistical theory of turbulence and the SDT followed from that is an alternative to perturbation theory and the corresponding diagram technique, it seems appropriate to compare these two approaches. In perturbation theory the solution for the velocity in the form of power series is constructed with the iteration procedure, where the solution of the linearized NSE in the field of external random force \( u^{(0)}(1) = G^{(0)}(1,2)X(2) \) is used as the zeroth approximation; this approximation is substituted in the nonlinear term of the NSE, and the resulting equation is solved taking into account the correction to the external random force to obtain the next approximation. The iteration procedure leads to the representation of the solution for the velocity in the series form in powers of the external random force.
force. The execution of the first iteration is already a very cumbersome procedure; therefore, the Feynman diagram technique is used to simplify the record and analysis of subsequent approximations. The account for the higher approximations is done with the renormalization procedure which is reduced to the replacement of basic elements of the perturbation theory, $G^{(0)}, V(1|2,3), \text{ and } D^{(0)}$, with their renormalized quantitie $G, \Gamma(1|2,3), \text{ and } D$, which, in the language of diagrams, means the transition from the diagram technique of perturbation theory to the SDT. Such procedure is nonunique, and this is manifested, for instance, in counting the number of topologically equivalent diagrams or in the fact that there are the vertices of three types in the SDT (see formulas (3.16)) whereas in the diagram technique of perturbation theory there is only one vertex.

In the functional formulation of the theory based on the ChF method and on the skeleton Feynman diagram method, it follows that the mistakes at the transition from perturbation theory to the skeleton diagrams are automatically avoided [20] and the Wyld and Dyson equations follow directly from the ChF formalism outside the scope of perturbation theory. The relation between the vertices of various types obtained in the functional formulation allows expressing the vertices of the second and third types (with two and three outgoing arrows) through the vertex of the first type (with two incoming and one emerging arrow). In addition, it becomes clear what diagrams are not taken into account in using the one-loop and two-loop approximations. We may hope that, as the entanglement of the skeleton diagrams increases, the contribution of the corresponding diagrams decreases.

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