The Einstein–null dust system in spherical symmetry with an inner mirror: structure of the maximal development and Cauchy stability

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Abstract

In this paper, we study the evolution of asymptotically AdS initial data for the spherically symmetric Einstein–massless Vlasov system for \( \Lambda < 0 \), with reflecting boundary conditions imposed on timelike infinity \( I \), in the case when the Vlasov field is supported only on radial geodesics. This system is equivalent to the spherically symmetric Einstein–null dust system, allowing for both ingoing and outgoing dust. In general, solutions to this system break down in finite time (independent of the size of the initial data); we highlight this fact by showing that, at the first point where the ingoing dust reaches the axis of symmetry, solutions become \( C^0 \) inextendible, although the spacetime metric remains regular up to that point.

One way to overcome this “trivial” obstacle to well-posedness is to place an inner mirror on a timelike hypersurface of the form \( \{ r = r_0 \} \), \( r_0 > 0 \), and study the evolution on the exterior domain \( \{ r \geq r_0 \} \). In this setting, we prove the existence and uniqueness of maximal developments for general smooth and asymptotically AdS initial data sets, and study the basic geometric properties of these developments. Furthermore, we establish the well-posedness and Cauchy stability of solutions with respect to a “rough” initial data norm, measuring the concentration of energy at scales proportional to the mirror radius \( r_0 \).

The above well-posedness and Cauchy stability estimates are used in our companion paper [32] for the proof of the AdS instability conjecture for the Einstein–null dust system. However, the results of the present paper might also be of independent interest.

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1 Introduction

In the presence of a cosmological constant \( \Lambda < 0 \), the simplest solution of the vacuum Einstein equations

\[
Ric_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0
\]

is Anti-de Sitter spacetime \((\mathcal{M}_{AdS}, g_{AdS})\) (see [29]). In the standard polar coordinate chart on \( \mathcal{M}_{AdS} \cong \mathbb{R}^{3+1}, g_{AdS} \) is expressed as:

\[
g_{AdS} = -\left(1 - \frac{1}{3} \Lambda r^2\right) dt^2 + \left(1 - \frac{1}{3} \Lambda r^2\right)^{-1} dr^2 + r^2 g_{S^2}.
\]

Solutions \((\mathcal{M}, g)\) to (1.1) which are asymptotically AdS, i.e. possess an asymptotic region where \( g \) is close to (1.2) for \( r \gg 1 \), fail to be globally hyperbolic: They are conformally equivalent to spacetimes \((\tilde{\mathcal{M}}, \tilde{g})\) with a timelike boundary \( \mathcal{I} \) (see [23]). Thus, the correct setting to study the dynamics of asymptotically AdS solutions to (1.1) is that of an initial-boundary value problem, with suitable boundary conditions prescribed asymptotically on the conformal boundary \( \mathcal{I} \). The well-posedness of the initial-boundary value problem for (1.1) is a fundamental question arising in the study of the long time dynamics of (1.1) which is whether \((\mathcal{M}_{AdS}, g_{AdS})\) is stable under perturbations of its initial data. In 2006, Dafermos and Holzegel [13, 12] proposed the so-called AdS instability conjecture, stating that there exist arbitrarily small perturbations to the initial data of \((\mathcal{M}_{AdS}, g_{AdS})\) which, under evolution by (1.1) with a reflecting boundary condition on \( \mathcal{I} \), lead to the formation of black hole regions (and, thus, \((\mathcal{M}_{AdS}, g_{AdS})\) is non-linearly unstable). This conjecture was also supported by results of Anderson [11].

Starting from the seminal work of Bizon and Rostworowski [7], a vast amount of numerical and heuristic works have been devoted to the study of the AdS instability conjecture (see, e.g., [16, 18, 21, 22, 30, 31, 19, 11, 10, 9, 17, 28]). The bulk of these works is dedicated to the study of this conjecture in the context of the spherically symmetric Einstein–scalar field system. This is a simpler model of (1.1), being a well-posed 1+1 hyperbolic system with non-trivial dynamics resembling the qualitative properties of (1.1) (see the discussion in [2]).

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An even simpler model for which the AdS instability conjecture can be addressed is the *Einstein–massless Vlasov* system (see \[2, 34\])

\[
\begin{align*}
\text{Ric}_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} &= 8\pi T_{\mu\nu}[f], \\
p^\nu \partial_\nu f - \Gamma^\nu_{\beta\gamma} p^\beta p^\gamma f &= 0, \\
\text{supp}(f) \subseteq \{ g_{\beta\gamma}(x) p^\beta p^\gamma = 0 \} \subset TM
\end{align*}
\]

(1.3)

in *spherical symmetry*. This system can be further reduced to the case when the Vlasov field \(f\) is supported only on *radial* null geodesics. The resulting system, which we will call the *spherically symmetric Einstein–radial massless Vlasov* system, is a singular reduction of (1.3) (see \[35\]) and is equivalent to the *spherically symmetric Einstein–null dust* system, allowing for both ingoing and outgoing null dust. This system was first studied by Poisson and Israel in \[33\].

The spherically symmetric Einstein–null dust system suffers from a severe break down occuring once the support of the ingoing null dust reaches the axis of symmetry (see the discussion in the next section for more details). This “trivial” obstacle to well-posedness can be overcome by restricting the evolution of the system in the exterior of an *inner mirror* with spherical radius \(r_0 > 0\).

In the present paper, we will establish a number of well-posedness results for the spherically symmetric Einstein–null dust system with reflecting boundary conditions on \(\mathcal{I}\) and an inner mirror, which are necessary for addressing the AdS instability conjecture in this setting. The proof of the AdS instability conjecture for this model is then established in our companion paper \[32\].

In particular, for any mirror radius \(r_0 > 0\) and any smooth, asymptotically AdS, spherically symmetric and radial initial data set \(S_{in}\) for (1.3), we will establish two types of results, under reflecting boundary conditions on both \(r = r_0\) and \(\mathcal{I}\):

1. We will establish the existence and uniqueness of the maximal development \((\mathcal{M}, g; f)\) of \(S_{in}\) (restricted to the domain \(\{ r \geq r_0 \}\) ) and study the basic geometric properties of \((\mathcal{M}, g; f)\). See Theorem [1].

2. We will establish a Cauchy stability result for \(S_{in}\) in a rough initial data norm measuring the concentration of the energy of \(S_{in}\) in spherical annuli of radius \(\sim r_0\). See Theorem [2]. In particular, this result will provide a Cauchy stability statement for \((\mathcal{M}_{AdS}, g_{AdS})\) uniformly in \(r_0\); this fact will be important for addressing the AdS instability conjecture in this setting in our companion paper \[32\].

We will now proceed to discuss these results in more detail.

### 1.1 Theorem 1: The maximal development for the Einstein–null dust system with reflecting boundary conditions on \(r = r_0\) and \(\mathcal{I}\)

The system (1.3), reduced to the case where \((\mathcal{M}, g)\) and \(f\) are spherically symmetric and \(f\) is supported only on radial null geodesics, is equivalent to the spherically symmetric Einstein–null dust system with two dusts (see \[2, 35, 33\]). In double null coordinates \((u, v)\), where the metric \(g\) takes the form

\[
g = -\Omega^2 du dv + r^2 ds^2,
\]

(1.4)

this system can be expressed in terms of \(r, \Omega\) and the renormalised components \(\bar{\tau} = r^2 T_{uv}, \bar{\tau} = r^2 T_{uu}\) of the energy momentum tensor \(T\) as:

\[
\begin{align*}
\partial_u \partial_v (r^2) &= -\frac{1}{2} (1 - \Lambda r^2) \Omega^2, \\
\partial_u \partial_v \log(\Omega^2) &= \frac{1}{2\Omega^2} (1 + 4\Omega^{-2} \partial_u r \partial_v r), \\
\partial_v (\Omega^{-2} \partial_u r) &= -4\pi^2 \Omega^{-2} \bar{\tau}, \\
\partial_u (\Omega^{-2} \partial_u r) &= -4\pi^2 \Omega^{-2} \bar{\tau}, \\
\partial_u \bar{\tau} &= 0, \\
\partial_v \bar{\tau} &= 0.
\end{align*}
\]

(1.5)
It is known that solutions to (1.5) with \( \tau = 0 \) (known in the literature as Vaidya spacetimes, see e.g. [24]) develop a curvature singularity beyond the first point when the support of the ingoing dust reaches the axis of symmetry \( \gamma \); see, e.g. [24, 30]. We will actually show an even stronger ill-posedness result for (1.5):

**Proposition 1.** Any spherically symmetric solution \((M, g; \tau, \bar{\tau})\) of (1.5) with non-empty axis \( \gamma \), arising from smooth initial data on \( \{u = 0\} \) with \( \mathbf{v}_{u=0} = 0 \), remains smooth up to the first point when a radial geodesic in the support of \( \bar{\tau} \) reaches \( \gamma \). However, beyond that point, \((M, g; \tau, \bar{\tau})\) is \(C^0\) inextendible as a spherically symmetric solution to (1.5). In particular, (1.5) is ill-posed in any “reasonable” initial data topology.

For a more detailed statement of Proposition 1 and a precise definition of a \(C^0\) solution of (1.5), see Proposition A.1 and Theorem A.1 in Section A.3 of the Appendix.

**Remark.** Notice that Proposition 1 yields a uniform upper bound on the time of existence \( u_* \) of solutions \((M, g; \tau, \bar{\tau})\) to (1.5) for any characteristic initial data set at \( \{u = 0\} \) for which \( \bar{\tau} \) is not identically equal to 0, with \( u_* \) depending only on the distance of \( \text{supp}(\bar{\tau}) \) from \( \gamma \) initially (and not on the proximity of the initial data to the trivial data, in any “reasonable” initial data norm). In the case when the initial data on \( \{u = 0\} \) are close to the trivial data, \((M, g)\) is globally \(C^\infty\) extendible as spherically symmetric Lorentzian manifold beyond \( \{u = u_*\} \), despite the fact that \((M, g; \tau, \bar{\tau})\) is \(C^0\) inextendible as a solution to (1.5) (see Proposition A.1).

We should also remark that Proposition 1 holds independently of the value of the cosmological constant \( \Lambda \). One way to overcome the obstacle to well-posedness raised by Proposition 1 is to restrict the evolution of the system (1.5) in the region \( \{ r \geq r_0 \} \), for some \( r_0 > 0 \), and impose a reflecting boundary condition on the portion \( \gamma_0 \) of the curve \( \{ r = r_0 \} \) which is timelike. Our first result concerns the well posedness and the structure of the maximal development of the characteristic initial-boundary value problem for the system (1.5) in this setting:

**Theorem 1** (rough version). For any \( r_0 > 0 \) and any smooth, asymptotically AdS initial data set \( (r, \Omega^2; \tau, \bar{\tau})|_{\tau=0} \) for (1.5), restricted to the region \( \{ r \geq r_0 \} \), there exists a unique, smooth, maximal future development \( (r, \Omega^2; \tau, \bar{\tau}) \) of \((r, \Omega^2; \tau, \bar{\tau})|_{\tau=0} \) on \( \{ r \geq r_0 \} \), solving (1.5) with reflecting boundary conditions on \( \mathcal{I} \) and \( \gamma_0 \), where \( r|_{\gamma_0} = r_0 \) and \( \gamma_0 \) coincides with the portion of the curve \( \{ r = r_0 \} \) which is timelike (fixing the gauge freedom by imposing a reflecting gauge condition on both \( \mathcal{I} \) and \( \gamma_0 \)). The conformal boundary \( \mathcal{I} \) is future complete, and the hypersurface \( \{ r = r_0 \} \) is timelike in the past \( J^- \) of \( \mathcal{I} \).

In the case when \( (r, \Omega^2; \tau, \bar{\tau}) \) has a non-empty future event horizon \( \mathcal{H}^+ \), \( \mathcal{H}^+ \) is smooth and future complete and the curve \( \{ r = r_0 \} \) has a spacelike portion which lies in the future of \( \mathcal{H}^+ \) (see Figure 1.1). A necessary condition for \( \mathcal{H}^+ \) to be non-empty is that the total mass \( \bar{m}|_{\mathcal{I}} \) satisfies

\[
\frac{2\bar{m}|_{\mathcal{I}}}{r_0} > 1 - \frac{1}{3} \Lambda r_0^2.
\]

For a more detailed statement of Theorem 1 see Section 4.1.

### 1.2 Theorem 2: Cauchy stability in a rough norm, uniformly in \( r_0 \)

In view of the fact that \( \mathcal{H}^+ = \emptyset \) when (1.6) holds, it can be readily deduced that, for fixed \( r_0 \), \((M_{\text{AdS}}, g_{\text{AdS}})\) is orbitally stable as a solution of (1.5) with reflecting boundary conditions on \( \mathcal{I} \) and \( \gamma_0 \) under perturbations which are sufficiently small with respect to \( r_0 \). Therefore, addressing the AdS instability conjecture in this setting (a task that will take place in our companion paper [32]) requires allowing \( r_0 \) to shrink to 0 with the size of the initial data. To this end, it will be necessary to obtain a Cauchy stability statement for (1.5) with reflecting boundary conditions on \( \mathcal{I} \) and \( \gamma_0 \) which is uniform in \( r_0 \).

---

2We do not examine the question of whether \((M, g; \tau, \bar{\tau})\) admits a low regularity extension as a solution of (the analogue of) (1.5) outside spherical symmetry. In general, the question of \(C^0\) extendibility outside the regime of surface symmetry is rather intricate; see [30].

3In the case \( \Lambda < 0 \), Proposition 1 implies that anti-de Sitter spacetime \((M_{\text{AdS}}, g_{\text{AdS}})\) is not Cauchy stable for (1.5) for any “reasonable” initial data topology. Similarly for Minkowski spacetime \((\mathbb{R}^{1+3}, \gamma)\) in the case \( \Lambda = 0 \), or de Sitter spacetime \((M_{\text{dS}}, g_{\text{dS}})\) in the case \( \Lambda > 0 \).
Figure 1.1: Schematic depiction of the domain on which the maximal future development \((r, \Omega^2, \tau, \bar{\tau})\) of a smooth initial data set on \(u = 0\) (with reflecting boundary conditions on \(\mathcal{I}\) and \(\gamma_0\)) is defined. Conformal infinity \(\mathcal{I}\) is always complete in this setting. In the case when \(J^- (\mathcal{I})\) does not cover all of the domain, the future event horizon \(\mathcal{H}^+\) is non-empty and has infinite affine length. In this case, in addition to the mirror \(\gamma_0\), there exists a spacelike piece of the boundary of the domain on which \(r = r_0\). This boundary piece necessarily lies in the future of \(\mathcal{H}^+\).

For any \(r_0 > 0\), let us define the distance function \(\text{dist}_{r_0}(\cdot, \cdot)\) on the set of smooth and asymptotically AdS initial data for (1.5) so that, for any two initial data sets \(S = (r, \Omega^2; \tau, \bar{\tau})|_{u=0}\) and \(S' = (r', \Omega'^2; \tau', \bar{\tau}')|_{u=0}\):

\[
\text{dist}_{r_0}(S, S') \triangleq \sup_{v} \int \frac{\dd v}{\rho(v)} \left( (|\varphi(v) - \varphi(\bar{v})| + \tan^{-1}(\sqrt{-\Lambda}r_0))\partial_v \varphi(v) - (|\varphi'(v) - \varphi'(\bar{v})| + \tan^{-1}(\sqrt{-\Lambda}r_0))\partial_v \varphi'(v) \right) (-\Lambda) dv + \sup_{u=0} \left| \frac{2\hat{m}}{r} - \frac{2\hat{m}'}{r'} \right| + \sqrt{-\Lambda} |\hat{m}|_{r \to +\infty} - |\hat{m}'|_{r' \to +\infty},
\]

where

\[
\varphi(v) \triangleq \tan^{-1}(\sqrt{-\Lambda}r(v)),
\]

and \(\hat{m}\) is the renormalised Hawking mass associated to \(S\), defined as

\[
\hat{m} \triangleq \frac{r}{2} \left(1 - 4\Omega^{-2} \partial_u r \partial_v r\right) - \frac{1}{6} \Lambda r^3
\]

(with \(\varphi', \hat{m}'\) defined similarly in terms of \(S'\)).

**Remark.** Denoting with \(S_0\) the trivial initial data set \((r_{\text{vac}}, \Omega_{\text{vac}}^2; 0, 0)\) in the standard gauge, where

\[
r_{\text{vac}}(v) \triangleq \sqrt{\frac{3}{\Lambda} \tan \left(\sqrt{\frac{\Lambda}{12}} v\right)}
\]

and

\[
\Omega^2_{\text{vac}} = 1 - \frac{1}{3} \Lambda r^2_{\text{vac}},
\]

the distance \(\text{dist}_{r_0}(S, S_0)\) of any initial data set \(S\) from \(S_0\) measures the concentration of the energy of \(S\) at scales comparable to \(r_0\).
Our second result is a Cauchy stability statement for (1.5) with reflecting boundary conditions on \( r = r_0 \) and \( I \), in the initial data topology defined by (1.7), independently of the precise value of \( r_0 \):

**Theorem 2** (rough version). Let \((r, \Omega^2; \tau, \bar{\tau})\) be a solution of (1.5) arising from a smooth asymptotically AdS characteristic initial data set \(S_{in} \), with reflecting boundary conditions on \(I\) and \(\gamma_0\). Let also \(u_\ast > 0\) be any value such that the domain \(U_\ast = \{0 \leq u \leq u_\ast\}\) lies in the past of \(I\), i.e.

\[
U_\ast \subset J^- (I).
\]

Then, for any \(\epsilon > 0\), there exists a \(\delta > 0\), depending only on \(U_\ast\), \(\epsilon\) and \(\text{dist}_{r_0}(S_{in}, S_0)\) (but not \(r_0\)), with the following property: For any other smooth asymptotically AdS initial data set \(S'_{in}\) for (1.5) satisfying

\[
\text{dist}_{r_0}(S_{in}, S'_{in}) < \delta,
\]

the maximal development \((r', \Omega')^2; \tau', \bar{\tau}'\) satisfies

\[
\sup_{0 \leq u \leq u_\ast} \text{dist}_{r_0}(S_u, S'_u) < \epsilon,
\]

where

\[
S_u = (r', (\Omega')^2; \tau', \bar{\tau}') |_{u = \bar{u}}
\]

(and similarly for \(S'_u\)).

For a more precise statement of Theorem 2, see Section 4.2.

**Remark.** Theorem 2 provides a Cauchy stability estimate for the domain of outer communications of solutions to (1.5). Restricting to the case when \(S_{in} = S_0\) in Theorem 2, we thus readily obtain a Cauchy stability estimate for the AdS spacetime \((M_{AdS}, g_{AdS})\) in the topology defined by (1.7), independently of the precise value of \(r_0\). This fact will be important for addressing the AdS instability conjecture in our companion paper [32].

In particular, for any \(u_\ast > 0\), the maximal development of any initial data set \(S'_{in}\) for (1.5) with reflecting boundary conditions on \(\gamma_0\) and \(I\) will not contain a trapped surface for \(0 \leq u \leq u_\ast\), provided \(\text{dist}_{r_0}(S_0, S'_{in})\) is small enough in terms of \(u_\ast\) (independently of \(r_0\)).

### 1.3 Outline of the paper

This paper is organised as follows:

In Section 2, we will formulate the spherically symmetric Einstein–radial massless Vlasov system in double null coordinates, and we will introduce the notion of reflecting boundary conditions for this system on timelike hypersurfaces.

In Section 3, we will introduce the basic definitions related to the characteristic initial-boundary value problem for the spherically symmetric Einstein–radial massless Vlasov system.

In Section 4, we will state in detail the main results of this paper, namely the existence and uniqueness of a maximal future development for the characteristic initial-boundary value problem introduced in Section 3, as well as a Cauchy stability statement in a rough norm. The proofs of these results will occupy sections 5 and 6, respectively.

Finally, in Section A of the Appendix, we will prove an ill-posedness result related to solutions of the spherically symmetric Einstein–null dust system with non-empty axis of symmetry.

### 1.4 Acknowledgements

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2 The spherically symmetric Einstein–massless Vlasov system: the null dust reduction

In this section, we will formulate the Einstein–massless Vlasov system, reduced to the spherically symmetric and radial case, in a double null coordinate gauge, following the conventions of [14]. We will also introduce the notion of reflecting boundary conditions for this system on spherically symmetric timelike hypersurfaces, possibly lying “at infinity”.

2.1 Spherically symmetric spacetimes in a double null gauge

In this paper, we will consider 3 + 1 dimensional smooth Lorentzian manifolds \((M, g)\) satisfying the following properties:

- \(M\) splits diffeomorphically as
  \[ M \cong \mathcal{U} \times \mathbb{S}^2 \]  
  where \(\mathcal{U}\) is an open domain of \(\mathbb{R}^2\) with piecewise Lipschitz boundary \(\partial \mathcal{U}\).

- In the Cartesian \((u, v)\) coordinates on \(\mathcal{U} \subset \mathbb{R}^2\), \(g\) takes the double-null form
  \[ g = -\Omega^2(u, v)du dv + r^2(u, v)g_{\mathbb{S}^2}, \]  
  where \(g_{\mathbb{S}^2}\) is the standard round metric on \(\mathbb{S}^2\) and \(\Omega, r : \mathcal{U} \to (0, +\infty)\) are smooth functions.

- The function \(r\) is bounded away from 0 on \(\mathcal{U}\):
  \[ \inf_{\mathcal{U}} r > 0. \]

Note that (2.1) and (2.2) imply that \((M, g)\) is time orientable. We will fix a time orientation by requiring that \(\partial_u + \partial_v\) is a future directed vector field on \(M\).

**Remark.** The form (2.2) of the metric \(g\) implies that the action of \(SO(3)\) on \((M, g)\) through rotations of the \(\mathbb{S}^2\) factor of (2.1) is an isometric action and the function \(r\) can be geometrically defined as

\[ r(p) = \sqrt{\frac{\text{Area}(\mathcal{S}(p))}{4\pi}}, \]

where \(\mathcal{S}(p)\) is the \(SO(3)\)-orbit of \(p \in M\). The condition (2.3) implies that this \(SO(3)\) action has no fixed points, i.e. \((M, g)\) does not have an axis of symmetry.

For a Lorentzian manifold \((M, g)\) as above, we will also define the Hawking mass \(m : M \to \mathbb{R}\) as

\[ m = \frac{r}{2} \left( 1 - g(\nabla r, \nabla r) \right). \]

Viewed as a function on \(\mathcal{U}\), \(m\) can be expressed as:

\[ m = \frac{r}{2} \left( 1 + 4\Omega^2 \partial_u r \partial_v r \right). \]

Note that (2.6) can be rearranged as

\[ \Omega^2 = \frac{4 \left( -\partial_u r \right) \partial_v r}{1 - \frac{2m}{r}}. \]
Any local coordinate chart \((g^1, g^2)\) on \(S^2\) yields a local \((u, v, g^1, g^2)\) coordinate chart on \(M\). In any such chart, the non-zero Christoffel symbols of (2.2) are computed as follows:

\[
\begin{align*}
\Gamma^u_{uu} &= \partial_u \log(\Omega^2), \quad \Gamma^u_v = \partial_v \log(\Omega^2), \\
\Gamma^u_{AB} &= \Omega^2 \partial_u (g^{AB})_B, \quad \Gamma^u_A = \Omega^2 \partial_u (g^{AB})_A, \\
\Gamma^A_{uB} &= \Omega^2 \partial_u (\delta_B^A) = \Omega^2 \partial_u r \delta_B^A, \\
\Gamma^A_{BC} &= (\Gamma^{AB})_C
\end{align*}
\]

where the latin indices \(A, B, C\) are associated to the spherical coordinates \(y^1, y^2\), \(\delta_B^A\) is Kronecker delta and \(\Gamma^{AB}\) are the Christoffel symbols of \((S^2, g_{AB})\) in the \((g^1, g^2)\) chart.

For any pair of smooth maps \(h_1, h_2 : \mathbb{R} \to \mathbb{R}\) which are strictly monotonic, we can introduce a new pair of double null coordinates \((\bar{u}, \bar{v})\) on \(M\) by defining

\[
(\bar{u}, \bar{v}) = (h_1(u), h_2(v)).
\]

In these new coordinates, the metric \(g\) takes the form

\[
\bar{g} = -\Omega^2(\bar{u}, \bar{v})d\bar{u}d\bar{v} + r^2(\bar{u}, \bar{v})g_{AB},
\]

where

\[
\bar{\Omega}^2(\bar{u}, \bar{v}) = \frac{1}{h_1' h_2'} \Omega^2(h_1^{-1}(\bar{u}), h_2^{-1}(\bar{v})),
\]

\[
r(\bar{u}, \bar{v}) = r(h_1^{-1}(\bar{u}), h_2^{-1}(\bar{v})).
\]

Throughout this paper, we will frequently make use of such coordinate transformations, without renaming the coordinates each time.

Note that \(m\) is invariant under coordinate transformations as above; that is to say, for \((\bar{u}, \bar{v})\) defined by (2.9):

\[
m(\bar{u}, \bar{v}) = m(h_1^{-1}(\bar{u}), h_2^{-1}(\bar{v})).
\]

### 2.2 The massless Vlasov equation: The radial reduction

Let \((M, g)\) be as in Section 2.1. For any local coordinate chart \((x^0, x^1, x^2, x^3)\) on \(M\), the associated momentum coordinate system \((p^0, p^1, p^2, p^3)\) on each fiber of \(TM\) is defined with respect to the coordinate frame \(\{\partial_x^a\}_{a=0}^3\). The geodesic flow on \(TM\) is then described by the following first order system:

\[
\begin{align*}
\frac{dx^a}{dt} &= p^a, \\
\frac{dp^a}{dt} &= -\Gamma^a_{\beta\gamma}(x)p^\beta p^\gamma,
\end{align*}
\]

where the Greek lowercase indices run from 0 to 3 and \(\Gamma^a_{\beta\gamma}\) are the Christoffel symbols associated to the \((x^a)_{a=0}^3\) coordinate chart.

The spherical symmetry of \((M, g)\) implies that the quantity

\[
\mathcal{L} = r^2 g_{AB} p^A p^B,
\]

evaluated in the \(p^A\)-momentum coordinates associated to the \(y^A\)-spherical coordinates in the double null coordinate chart \((u, v, g^1, g^2)\) on \(M\) (see Section 2.1), is constant along the geodesic flow. The quantity \(\mathcal{L}\) associated to any geodesic \(\gamma\) of \(M\) is called the angular momentum of the geodesic. Geodesics which are null, future directed and radial fall into two categories: the ingoing ones (for which \(p^\rho = 0\)) and the outgoing ones (for which \(p^\rho = 0\)).

Let \(f \geq 0\) be a measure on \(TM\) which is constant along the geodesic flow, i.e. satisfies in any local coordinate chart \((x^0, x^1, x^2, x^3)\) on \(M\) (with associated momentum coordinates \((p^0, p^1, p^2, p^3)\)):

\[
p^2 \partial_x f - \Gamma^a_{\rho\sigma}(x)p^\rho p^\sigma \partial_p f = 0,
\]
where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of $g$ in the chart $(x^0, x^1, x^2, x^3)$. We will call $f$ a massless Vlasov field if it is supported on the set $P \subset T\mathcal{M}$ of null vectors, i.e. on the set
\begin{equation}
(2.17) \quad g_{\beta\bar{\alpha}}(x)p^\alpha p^\beta = 0.
\end{equation}

The energy momentum tensor $T_{\beta\bar{\alpha}}$ of $f$ is a symmetric $(0, 2)$-tensor on $\mathcal{M}$ defined formally by the expression
\begin{equation}
(2.18) \quad T_{\beta\bar{\alpha}}(x) = \int_{\pi^{-1}(x)} p_\alpha p_\beta f,
\end{equation}
where $\pi^{-1}(x)$ denotes the fiber of $T\mathcal{M}$ over $x \in \mathcal{M}$ and
\begin{equation}
(2.19) \quad p_\gamma = g_{\gamma\bar{\delta}}(x)p^\bar{\delta}.
\end{equation}

Remark. In this paper, we will only consider distributions $f$ for which the expression (2.18) is finite and depends smoothly on $x \in \mathcal{M}$.

We will consider only distributions $f$ which are spherically symmetric, i.e. invariant under the action of $SO(3)$ on $\mathcal{M}$. In that case, in any $(u, v, y^1, y^2)$ local coordinate chart as in Section $2.1$, the energy-momentum tensor $T$ is of the form
\begin{equation}
(2.20) \quad T = T_{uu}(u, v)du^2 + 2T_{uv}(u, v)dudv + T_{vv}(u, v)dv^2 + T_{AB}(u, v)dx^A dx^B.
\end{equation}

A radial massless Vlasov field $f$ is a massless Vlasov field $f$ supported only on radial null geodesics. In view of the separation of radial null geodesics into ingoing and outgoing, a spherically symmetric, radial massless Vlasov field $f$ takes the following form in any $(u, v, y^1, y^2)$ local coordinate chart as in Section $2.1$ (with associated momentum coordinates $(p^u, p^v, p^1, p^2)$):
\begin{equation}
(2.21) \quad f(u, v, y^1, y^2; p^u, p^v, p^1, p^2) = (\bar{f}_{in}(u, v; p^u) + \bar{f}_{out}(u, v; p^v))\delta(\sqrt{(g_{\bar{\gamma}})_{AB}p^A p^B})\delta(\Omega^2 p^u p^v),
\end{equation}
where $\bar{f}_{in}, \bar{f}_{out} \geq 0$ and $\delta$ is the Dirac delta function on $\mathbb{R}^3$. In this case, we can compute the components of the energy momentum tensor (2.18) as follows:
\begin{equation}
(2.22) \quad T_{uu}(u, v) = \int_0^{+\infty} \Omega^4(p^v)\bar{f}_{out}(u, v; p^v) r^2 \frac{dp^v}{p^v},
\end{equation}
\begin{equation}
(2.23) \quad T_{vv}(u, v) = \int_0^{+\infty} \Omega^4(p^u)\bar{f}_{in}(u, v; p^u) r^2 \frac{dp^u}{p^u},
\end{equation}
and
\begin{equation}
(2.24) \quad T_{uv} = T_{AB} = 0.
\end{equation}

Remark. In this paper, we will only consider the case when $\bar{f}_{in}, \bar{f}_{out}$ are smooth functions which are compactly supported in their last argument.

The expression (2.21) implies that equation (2.16) is equivalent to the following system for $\bar{f}_{in}$ and $\bar{f}_{out}$:
\begin{equation}
(2.25) \quad \partial_u(\Omega^4 r^4 p^u \bar{f}_{in}) + p^u \partial_{p^u}(\Omega^4 r^4 p^u \bar{f}_{in}) = 0,
\end{equation}
\begin{equation}
(2.26) \quad \partial_v(\Omega^4 r^4 p^v \bar{f}_{out}) + p^v \partial_{p^v}(\Omega^4 r^4 p^v \bar{f}_{out}) = 0.
\end{equation}

The relations (2.25)–(2.26) imply that:
\begin{equation}
(2.27) \quad \partial_u(r^2 T_{uu}) = 0,
(2.28) \quad \partial_v(r^2 T_{vv}) = 0.
\end{equation}

\footnote{Note that the arguments of the $\delta$ functions in the right hand side of (2.21) are invariant under transformations of the spherically symmetric double null coordinate system.}
Remark. Under a double null coordinate transformation of the form (2.9), \( \bar{f}_{in} \) and \( \bar{f}_{out} \) transform as

\[
\bar{f}^{(new)}_{in}(h_1(u), h_2(v); h'_1(u)p) = \bar{f}_{in}(u, v; p)
\]

(2.29)

and

\[
\bar{f}^{(new)}_{out}(h_1(u), h_2(v); h'_2(v)p) = \bar{f}_{out}(u, v; p).
\]

(2.30)

respectively.

2.3 The spherically symmetric Einstein–radial massless Vlasov and Einstein–null dust system

The Einstein–Vlasov system with cosmological constant \( \Lambda < 0 \), for a smooth Lorentzian manifold \((\mathcal{M}, g)\) and a non-negative measure \( f \) on \( T\mathcal{M} \), takes the form

\[
\begin{cases}
Ric_{\mu\nu}(g) - \frac{1}{2} R(g) g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \\
p^2 \partial_{r^2} f - \Gamma^r_{\mu\nu} p^\mu p^\nu \partial_{r^2} f = 0,
\end{cases}
\]

(2.31)

where \( T_{\mu\nu} \) is expressed in terms of \( f \) by (2.18) (see [2]).

Reducing (2.31) to the case where \((\mathcal{M}, g)\) is a spherically symmetric spacetime as in Section 2.1 and \( f \) is a radial massless Vlasov field (i.e. has the form (2.21)), we obtain the following system for \((r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})\):

\[
\partial_u \Omega^2 = - \frac{1}{2} (1 - \Lambda r^2) \Omega^2,
\]

(2.32)

\[
\partial_u \partial_r \log(\Omega^2) = \frac{\Omega^2}{2r^2} \left( 1 + 4\Omega^{-2} \partial_u r \partial_r r \right),
\]

(2.33)

\[
\partial_u (\Omega^{-2} \partial_r r) = - 4\pi r T_{uu} \Omega^{-2},
\]

(2.34)

\[
\partial_u (\Omega^{-2} \partial_r r) = - 4\pi r T_{uu} \Omega^{-2},
\]

(2.35)

\[
\partial_u (\Omega^{-2} \partial_r r) = - 4\pi r T_{uu} \Omega^{-2},
\]

(2.36)

\[
\partial_u (\Omega^4 r^4 p^u \bar{f}_{in}) = - p^u \partial_r r (\Omega^4 r^4 p^u \bar{f}_{in}),
\]

(2.37)

where \( T_{uu}, T_{vv} \) are expressed in terms of \( \bar{f}_{out}, \bar{f}_{in} \) by (2.22), (2.23), respectively (for the spherically symmetric reduction of the massive Einstein–Vlasov system in double null coordinates, see [14]). Notice that the system (2.32)–(2.37) reduces to the following system for \((r, \Omega^2, T_{uu}, T_{vv})\):

\[
\partial_u \Omega^2 = - \frac{1}{2} (1 - \Lambda r^2) \Omega^2,
\]

(2.38)

\[
\partial_u \partial_r \log(\Omega^2) = \frac{\Omega^2}{2r^2} \left( 1 + 4\Omega^{-2} \partial_u r \partial_r r \right),
\]

(2.39)

\[
\partial_u (\Omega^{-2} \partial_r r) = - 4\pi r T_{vv} \Omega^{-2},
\]

(2.40)

\[
\partial_u (\Omega^{-2} \partial_r r) = - 4\pi r T_{uu} \Omega^{-2},
\]

(2.41)

\[
\partial_u (\Omega^{-2} \partial_r r) = - 4\pi r T_{uu} \Omega^{-2},
\]

(2.42)

\[
\partial_u (r^2 T_{uu}) = 0,
\]

(2.43)

\[
\partial_u (r^2 T_{vv}) = 0.
\]

Remark. The system (2.38)–(2.43) is the Einstein–null dust system with both ingoing and outgoing dust (see [39]).

Let us define the renormalised Hawking mass by the relation

\[
\tilde{m} \doteq m - \frac{1}{6} \Lambda r^3.
\]

(2.44)
From equations (2.38)–(2.43), we can formally obtain the following system for \((r, \tilde{m}, T_{uu}, T_{vv})\) (valid in the region where \(\partial_u r < 0 < \partial_v r\) and \(1 - \frac{2m}{r} > 0\)):

\[
(2.45) \quad \partial_u \log \left( \frac{\partial_u r}{1 - \frac{2m}{r}} \right) = -4\pi r^{-1} T_{uu}/\partial_u r,
\]

\[
(2.46) \quad \partial_v \log \left( \frac{\partial_v r}{1 - \frac{2m}{r}} \right) = 4\pi r^{-1} T_{vv}/\partial_v r,
\]

\[
(2.47) \quad \partial_u \partial_v r = -\frac{2\tilde{m} - \frac{2}{3} \Lambda r^3 (\partial_u r) \partial_v r}{r^2 / 1 - \frac{2m}{r}},
\]

\[
(2.48) \quad \partial_u \tilde{m} = -2\pi \left(1 - \frac{2m}{r}\right) r^2 T_{uu},
\]

\[
(2.49) \quad \partial_v \tilde{m} = 2\pi \left(1 - \frac{2m}{r}\right) r^2 T_{vv},
\]

\[
(2.50) \quad \partial_u (r^2 T_{vv}) = 0,
\]

\[
(2.51) \quad \partial_v (r^2 T_{uu}) = 0.
\]

### 2.4 Reflection of radial null geodesics and the reflecting boundary condition

Let \((\mathcal{M}, g)\) be as in Section 2.1. Recall that \(\mathcal{M} \cong \mathcal{U} \times \mathbb{S}^2\) (see (2.1)), where \(\mathcal{U} \subset \mathbb{R}^2\) has piecewise Lipschitz boundary \(\partial \mathcal{U}\). Let \(\partial_{\text{tim}} \mathcal{U}\) be the subset of \(\partial \mathcal{U}\) consisting of a union of connected, timelike Lipschitz curves with respect to the comparison metric

\[
(2.52) \quad g_{\text{comp}} = -du dv
\]
on \(\mathbb{R}^2\). Recall that a connected Lipschitz curve \(\gamma\) in \(\mathbb{R}^2\) is said to be timelike with respect to (2.52) if, for every point \(p = (u_0, v_0) \in \gamma\), we have

\[
(2.53) \quad \gamma(p) \in \Gamma^-(p) \cup \Gamma^+(p) \equiv \left(\{u < u_0\} \cap \{v < v_0\}\right) \cup \left(\{u > u_0\} \cap \{v > v_0\}\right).
\]

Let us also fix \(w : \mathcal{U} \cup \partial_{\text{tim}} \mathcal{U} \rightarrow \mathbb{R}\) to be a smooth boundary defining function of \(\partial_{\text{tim}} \mathcal{U}\), i.e. \(w|_{\partial_{\text{tim}} \mathcal{U}} = 0,\) \(dw|_{\partial_{\text{tim}} \mathcal{U}} \neq 0\) and \(w > 0\) on \(\mathcal{U}\). We can split \(\partial_{\text{tim}} \mathcal{U}\) into its “left” and “right” components as

\[
(2.54) \quad \partial_{\text{tim}} \mathcal{U} = \partial_{\text{tim}}^+ \mathcal{U} \cup \partial_{\text{tim}}^- \mathcal{U},
\]

where

\[
\partial_{\text{tim}}^+ \mathcal{U} = \{(u_0, v_0) \in \partial_{\text{tim}} \mathcal{U} : \partial_v w(u_0, v_0) > 0\},
\]

\[
\partial_{\text{tim}}^- \mathcal{U} = \{(u_0, v_0) \in \partial_{\text{tim}} \mathcal{U} : \partial_v w(u_0, v_0) < 0\}.
\]

**Remark.** Notice that any future directed radial null geodesic of \(\mathcal{M} = \mathcal{U} \times \mathbb{S}^2\) with a future limiting point on \(\partial_{\text{tim}} \mathcal{U} \times \mathbb{S}^2\) (in the ambient \(\mathbb{R}^2 \times \mathbb{S}^2\) topology of \(\text{clo}(\mathcal{U}) \times \mathbb{S}^2\)) is necessarily ingoing. Similarly, future directed radial null geodesics “terminating” at \(\partial_{\text{tim}}^- \mathcal{U} \times \mathbb{S}^2\) are necessarily outgoing.

We will define the reflection of radial null geodesics of \(\mathcal{M}\) on \(\partial_{\text{tim}} \mathcal{U} \times \mathbb{S}^2\) as follows:

**Definition 2.1.** Let \(\zeta_0 : [0, c) \rightarrow \mathcal{M}\), \(c \leq +\infty\), be a future directed radial null geodesic of \(\mathcal{M} = \mathcal{U} \times \mathbb{S}^2\) parametrised so that \(\nabla_{\zeta_0} \zeta_0 = 0\), such that, in the ambient \(\mathbb{R}^2 \times \mathbb{S}^2\) manifold, the limit \(\lim_{t \rightarrow +c} \zeta_0(t) = (u_0, v_0, y_0^1, y_0^2)\) exists and belongs to \(\partial_{\text{tim}} \mathcal{U} \times \mathbb{S}^2\). Then, the reflection of \(\zeta_0\) on \(\partial_{\text{tim}} \mathcal{U} \times \mathbb{S}^2\) at \((u_0, v_0, y_0^1, y_0^2)\) is defined as the unique inextendable, future directed and radial null geodesic \(\tilde{\zeta}_0 : (a, b) \rightarrow \mathcal{M}\) of \((\mathcal{M}, g)\), \(-\infty < a < b \leq +\infty\), parametrised so that \(\nabla_{\tilde{\zeta}_0} \tilde{\zeta}_0 = 0\), satisfying the following conditions:
Figure 2.1: Let $\mathcal{U} \subset \mathbb{R}^2$ be a domain as depicted above. The timelike part $\partial_{\text{tim}} \mathcal{U}$ of the boundary $\partial \mathcal{U}$ splits as the union of a "left" component $\partial_{\text{tim}}^{-} \mathcal{U}$ and a "right" component $\partial_{\text{tim}}^{+} \mathcal{U}$. While, in general, it is not necessary that $\partial_{\text{tim}}^{-} \mathcal{U}$ and $\partial_{\text{tim}}^{+} \mathcal{U}$ are straight line segments, in the rest of the paper we will only consider domains $\mathcal{U}$ with this property (see Definition 3.3).

1. $\bar{\zeta}_0$ emanates from $(u_0, v_0, x_0^1, x_0^2)$, i.e.:

$$\lim_{t \to a^+} \bar{\zeta}_0(t) = \lim_{t \to c^-} \zeta_0(t) = (u_0, v_0, x_0^1, x_0^2).$$

2. For any $0 < w_* \ll 1$, defining $t[w_*] \in [0, c)$ and $\bar{t}[w_*] \in (a, b)$ implicitly by

$$w(\zeta_0(t[w_*])) = w_*$$

and

$$w(\bar{\zeta}_0(\bar{t}[w_*])) = w_*,$$

the following relation holds:

$$\lim_{w_* \to 0^+} \frac{d}{dt}(w \circ \zeta_0)\big|_{t = i[w_*]} = -1.$$  

Note that the above definition is independent of the specific choice of the boundary defining function $w$ and the coordinate chart on $\mathcal{U}$, while, in view of (2.56), the parametrisation of the reflected geodesic is completely determined in the case $c < +\infty$, and determined up to a translation $t \to t + t_0$ in the case $c = +\infty$. Notice also that the reflection of an ingoing radial null geodesic is an outgoing one, and vice-versa.

Remark. In the next sections, we will only consider the reflection of radial null geodesics on parts of $\partial_{\text{tim}} \mathcal{U}$ for which either $r - r_0$ (for some $r_0 > 0$) or $1/r$ is a boundary defining function.

Definition 2.2. A radial massless Vlasov field $f$ on $T\mathcal{M}$ will be said to satisfy the reflecting boundary condition on $\partial_{\text{tim}} \mathcal{U} \times S^2$ if it is invariant under reflections of radial null geodesics on $\partial_{\text{tim}} \mathcal{U} \times S^2$.

It can be readily verified that a radial massless Vlasov field satisfies the reflecting boundary condition on $\partial_{\text{tim}} \mathcal{U} \times S^2$ if and only if its ingoing and outgoing components $\bar{f}_{\text{in}}, \bar{f}_{\text{out}}$ (see 2.21) satisfy the following boundary conditions (as a consequence of (2.55), (2.56)):
For any \((u_0, v_0) \in \partial^{{\ast}r}I^+\) and any \(p > 0\):

\[
\lim_{h \to 0^+} \left( \frac{\tilde{f}_{\text{out}}(u_0, v_0 + h; \cdot \nabla u_0 w(u_0, v_0) \cdot \Omega^{-2}(u_0, v_0 + h) \cdot p)}{\tilde{f}_{\text{in}}(u_0 - h, v_0; \Omega^{-2}(u_0 - h, v_0) \cdot p)} \right) = 1.
\]

For any \((u_1, v_1) \in \partial^{{\ast}r}I^+\) and any \(p > 0\):

\[
\lim_{h \to 0^+} \left( \frac{\tilde{f}_{\text{in}}(u_1 + h, v_1; \cdot \nabla u_1 w(u_1, v_1) \cdot \Omega^{-2}(u_1 + h, v_1) \cdot p)}{\tilde{f}_{\text{out}}(u_1, v_1 - h; \Omega^{-2}(u_1, v_1 - h) \cdot p)} \right) = 1.
\]

Note that the relations \((2.57)\) and \((2.58)\) for \(\tilde{f}_{\text{in}}, \tilde{f}_{\text{out}}\) imply the following boundary relations for the components \((2.22), (2.23)\) of the energy momentum tensor \(T\):

- For any \((u_0, v_0) \in \partial^{{\ast}r}I^+\):

\[
\lim_{h \to 0^+} \frac{r^2 T_{uw}(u_0, v_0 + h)}{r^2 T_{uv}(u_0 - h, v_0)} = \left( \frac{-\partial_u w(u_0, v_0)}{\partial_v w(u_0, v_0)} \right)^2.
\]

- For any \((u_1, v_1) \in \partial^{{\ast}r}I^+\):

\[
\lim_{h \to 0^+} \frac{r^2 T_{uv}(u_1 + h, v_1)}{r^2 T_{uu}(u_1, v_1 - h)} = \left( \frac{-\partial_u w(u_0, v_0)}{\partial_v w(u_0, v_0)} \right)^2.
\]

### 3 The boundary–characteristic initial value problem

In this Section, we will formulate the asymptotically AdS initial-boundary value problem for the system \((2.32) - (2.37)\) with reflecting boundary conditions on \(I\) and the timelike portion of \(\{r = r_0\}\) and \(I\).

#### 3.1 Asymptotically AdS characteristic initial data

We will define the notion of a boundary-characteristic initial data set for \((2.32) - (2.37)\) as follows:

**Definition 3.1.** For any \(v_1 < v_2\) and any \(r_0 > 0\), let \(r_j : [v_1, v_2) \to [r_0, +\infty), \Omega_j : [v_1, v_2) \to (0, +\infty)\) and \(\tilde{f}_{\text{in}j}, \tilde{f}_{\text{out}j} : \Omega(v, w) \times (0, +\infty) \to [0, +\infty)\) be smooth functions, such that

\[
\lim_{v \to v_2} r_j(v) = +\infty
\]

and

\[
r_j(v_1) = r_0.
\]

Let also \((\partial_v r_j) : [v_1, v_2) \to (-\infty, 0)\) be defined by

\[
(\partial_v r_j)(v) = \frac{1}{r_j(v)} \left( -r_0 \partial_v r_j(v_1) - \frac{1}{4} \int_{v_1}^{v} (1 - \Lambda r_j(v) \Omega_j(v)) \, dv \right).
\]

We will call \((r_j, \Omega_j, \tilde{f}_{\text{in}j}, \tilde{f}_{\text{out}j})\) an **asymptotically AdS boundary-characteristic initial data set** on \([v_1, v_2)\) for the system \((2.32) - (2.37)\) satisfying the reflecting gauge condition at \(r = r_0, +\infty\) if:

- The constraint equation \((2.34)\) is satisfied by \((r_j, \Omega_j^2)\), i.e.

\[
\partial_v (\Omega_j^2 \partial_v r_j) = -4\pi \Omega_j r_j / (T_{uv}) \Omega_j^{-2},
\]

where

\[
(T_{uv})_j(v) \equiv \int_{v_0}^{+\infty} \Omega_j^4(v) (p_v)^2 \tilde{f}_{\text{in}j}(v; p_u) r_j^2(v) \frac{dp_u}{p_u}.
\]
The functions \( \check{f}_{\text{out}} \), \( \check{f}_{\text{in}} \) satisfy the following conditions at \( v = v_1, v_2 \) for any \( p > 0 \):

\[
\frac{\check{f}_{\text{out}}(v_1; \Omega_f^{-2}(v_1) \cdot p)}{\check{f}_{\text{in}}(v_1; \Omega_f^{-2}(v_1) \cdot p)} = 1
\]

and

\[
\lim_{h \to 0^+} \left( \frac{\check{f}_{\text{in}}(v_2 - h; \Omega_f^{-2}(v_2 - h) \cdot p)}{\check{f}_{\text{out}}(v_2 - h; \Omega_f^{-2}(v_2 - h) \cdot p)} \right) = 1.
\]

The initial transversal derivative \( (\partial_v r)_j \) satisfies

\[
\lim_{v \to v_2} \frac{(\partial_v r)_j}{\partial_v r_j} = 1.
\]

The function \( \check{f}_{\text{out}} \) solves (2.37), i.e:

\[
\partial_v (\Omega_f^2(v) r_j^4(v) p^v \check{f}_{\text{out}}(v,p^v)) + p^v \partial_p^v (\Omega_f^2(v) r_j^4(v) p^v \check{f}_{\text{out}}(v,p^v)) = 0.
\]

Remark. The constraint equation (3.4) implies that

\[
\partial_v (\Omega_f^{-2} \partial_v r_j) \leq 0.
\]

Therefore, in view of (3.1), we can bound for all \( v \in [v_1, v_2] \):

\[
\partial_v r_j(v) > 0.
\]

In Section 5.1, we will introduce two more classes of characteristic initial data for the system (2.32)-(2.37).

We will also define the initial Hawking mass \( m_j \) and the initial renormalised Hawking mass \( \check{m}_j \), in accordance with (2.4), as follows:

**Definition 3.2.** Let \((r_j, \Omega_f^2, \check{f}_{\text{in}}, \check{f}_{\text{out}})\) be an asymptotically AdS boundary-characteristic initial data set on \([v_1, v_2]\) with reflecting gauge conditions at \( r = r_0, +\infty \). We will define the initial Hawking mass \( m_j \) and initial renormalised Hawking mass \( \check{m}_j \) on \([v_1, v_2]\) by the relations

\[
m_j = \frac{r_j}{2} \left( 1 - 4 \Omega_f^{-2} (\partial_v r)_j (\partial_v r_j) \right),
\]

and

\[
\check{m}_j = m_j - \frac{1}{6} \Lambda r_j^3.
\]

### 3.2 Developments with reflecting boundary conditions on \( \gamma_0 \) and \( \mathcal{I} \)

In the following sections, we will only consider solutions \((r, \Omega^2, \check{f}_{\text{in}}, \check{f}_{\text{out}})\) to (2.32)-(2.37) satisfying a reflecting gauge condition on \( \partial_{\text{in}} \mathcal{U} \). This condition fixes \( \partial_{\text{in}} \mathcal{U} \) to be a union of vertical straight lines in the \((u, v)\)-plane. This motivates defining the following class of domains \( \mathcal{U} \subset \mathbb{R}^2 \):
Figure 3.1: A typical domain $U \in \mathcal{U}_{v_0}$ would be as depicted above. In the case when the boundary set $\gamma$ is empty, it is necessary that both $\gamma_0$ and $I$ are unbounded (i.e. extend all the way to $u + v = \infty$).

**Definition 3.3.** For any $v_0 > 0$, let $\mathcal{U}_{v_0}$ be the set of all connected open domains $U$ of the $(u, v)$-plane with piecewise Lipschitz boundary $\partial U$, such that $\partial U$ splits as the following union of Lipschitz curves

$$\partial U = S_{v_0} \cup \gamma_0 \cup I \cup \text{clos}(\gamma),$$

where

$$S_{v_0} = \{0\} \times [0, v_0],$$

$$\gamma_0 = \{u = v\} \cap \{0 \leq u < u(\gamma_0)\},$$

$$I = \{u = v - v_0\} \cap \{0 \leq u < u(I)\}$$

(for some $0 < u(\gamma_0), u(I) \leq +\infty$) and $\gamma : (x_1, x_2) \rightarrow \mathbb{R}^2$ is an achronal (with respect to the reference Lorentzian metric (2.52)) curve, which is allowed to be empty. The closure $\text{clos}(\gamma)$ of $\gamma$ in (3.14) is considered with respect to the usual topology of $\mathbb{R}^2$.

Remark: Definition 3.3 implies that $U$ is necessarily contained in the future domain of dependence of $S_{v_0} \cup \gamma_0 \cup I$ (with respect to the comparison metric (2.52)). In the case when $\gamma$ in (3.14) is empty, it is necessary that both $\gamma_0$ and $I$ are unbounded in the future, i.e. extend to $u + v = +\infty$.

We will now proceed to define the notion of a future development of an asymptotically AdS boundary-characteristic initial data set for the system (2.32)–(2.37) with reflecting boundary conditions on $\gamma_0$ and $I$.

**Definition 3.4.** For any $v_0 > 0$ and $r_0 > 0$, let $(r_f, \Omega_f^2, \bar{f}_{in}, \bar{f}_{out})$ be a smooth, asymptotically AdS boundary-characteristic initial data set on $[0, v_0)$ for the system (2.32)–(2.37) satisfying the reflecting gauge condition at $r = r_0$, $+\infty$ (see Definition 3.1). A future development of $(r_f, \Omega_f^2, \bar{f}_{in}, \bar{f}_{out})$ will consist of a domain $U \in \mathcal{U}_{v_0}$ (see Definition 3.3) and a quadruple of smooth functions $r : U \rightarrow (r_0, +\infty)$, $\Omega^2 : U \rightarrow (0, +\infty)$ and $\bar{f}_{in}, \bar{f}_{out} : U \times (0, +\infty) \rightarrow [0, +\infty)$ satisfying the following conditions:
1. \((r, \Omega^2, \bar{f}_\text{in}, \bar{f}_\text{out})\) satisfy the given initial conditions on \(S_{v_0} = \{0\} \times [0, v_0)\), i.e.: \[(3.18) \quad (r, \Omega^2, \bar{f}_\text{in}, \bar{f}_\text{out})|_{S_{v_0}} = (r_f, \Omega_f^2, \bar{f}_\text{inf}, \bar{f}_\text{outf}).\]

2. \((r, \Omega^2, \bar{f}_\text{in}, \bar{f}_\text{out})\) solve \((2.32) - (2.37)\) on \(\mathcal{U}\).

3. The following gauge conditions are satisfied on \(\gamma_0\) and \(\mathcal{I}\): \[(3.19) \quad \partial_u r|_{\gamma_0} = -\partial_u r|_{\gamma_0}\] and \[(3.20) \quad \partial_u (1/r)|_{\mathcal{I}} = -\partial_u (1/r)|_{\mathcal{I}}.\]

4. \((r, \bar{f}_\text{in}, \bar{f}_\text{out})\) satisfy on \(\mathcal{I}\) the boundary conditions \[(3.21) \quad (1/r)|_{\mathcal{I}} = 0\] and \[(3.22) \quad \lim_{h \to 0^+} \frac{\bar{f}_\text{in}(u_*, v_*; \Omega^{-2}(u_*, v_*) \cdot p)}{\bar{f}_\text{out}(u_*, v_* - h; \Omega^{-2}(u_*, v_* - h) \cdot p)} = 1,\] for all \((u_*, v_*) \in \mathcal{I}\) and \(p > 0\).

5. \((r, \bar{f}_\text{in}, \bar{f}_\text{out})\) satisfy on \(\gamma_0\) the boundary conditions \[(3.23) \quad r|_{\gamma_0} = r_0\] and \[(3.24) \quad \bar{f}_\text{out}(u_*, v_*; p) = \bar{f}_\text{in}(u_*, v_*; p),\] for all \((u_*, v_*) \in \gamma_0\) and \(p > 0\).

**Remark.** Notice that the relations \[(3.19) \quad \text{and} \quad (3.20)\] follow from the boundary conditions \[(3.23) \quad \text{and} \quad (3.21),\] combined with the form \[(3.16) \quad \text{and} \quad (3.17)\] of \(\gamma_0\) and \(\mathcal{I}\), respectively. However, the relations \[(3.19) \quad \text{and} \quad (3.20)\] should be viewed as gauge conditions, fixing, in conjunction with \[(3.23) \quad \text{and} \quad (3.21),\] the form \[(3.16) \quad \text{and} \quad (3.17)\] of \(\gamma_0\) and \(\mathcal{I}\).

**Definition 3.5.** If \(\mathcal{D} = (\mathcal{U}; r, \Omega^2, \bar{f}_\text{in}, \bar{f}_\text{out})\) and \(\mathcal{D}' = (\mathcal{U}'; r', (\Omega')^2, \bar{f}'_\text{in}, \bar{f}'_\text{out})\) are two future developments of the same initial data \((r_f, \Omega_f^2, \bar{f}_\text{inf}, \bar{f}_\text{outf})\), we will say that \(\mathcal{D}'\) is an extension of \(\mathcal{D}\), writing \(\mathcal{D} \subseteq \mathcal{D}'\), if \(\mathcal{U} \subseteq \mathcal{U}'\) and the restriction \((r', (\Omega')^2, \bar{f}'_\text{in}, \bar{f}'_\text{out})|_{\mathcal{U}}\) of \((r', (\Omega')^2, \bar{f}'_\text{in}, \bar{f}'_\text{out})\) on \(\mathcal{U}\) satisfies \[(3.25) \quad (r', (\Omega')^2, \bar{f}'_\text{in}, \bar{f}'_\text{out})|_{\mathcal{U}} \equiv (r, \Omega^2, \bar{f}_\text{in}, \bar{f}_\text{out}).\]

**Remark.** It can be readily deduced from Propositions 5.1, 5.2, and 5.3 in Section 5.2 that, if \(\mathcal{D} = (\mathcal{U}; r, \Omega^2, \bar{f}_\text{in}, \bar{f}_\text{out})\) and \(\mathcal{D}' = (\mathcal{U}'; r', (\Omega')^2, \bar{f}'_\text{in}, \bar{f}'_\text{out})\) are two future developments of the same initial data \((r_f, \Omega_f^2, \bar{f}_\text{inf}, \bar{f}_\text{outf})\), then \[(3.26) \quad (r, \Omega^2, \bar{f}_\text{in}, \bar{f}_\text{out})|_{\mathcal{U} \cap \mathcal{U}'} = (r', (\Omega')^2, \bar{f}'_\text{in}, \bar{f}'_\text{out})|_{\mathcal{U} \cap \mathcal{U}'}.\]

4 **Precise statement of the main results**

In this section, we will provide a detailed statement of the main results of this paper. These results are used as a starting point in the proof of the instability of AdS for the system \([2.32] - [2.37]\) in our companion paper [32].
4.1 Existence, uniqueness and the basic properties of the maximal future development

Our first result concerns the existence, uniqueness and the basic properties of the maximal future development of any asymptotically AdS boundary-characteristic initial data set for (2.32)–(2.37):

**Theorem 1** (precise version). For any $v_0 > 0$ and any $r_0 > 0$, let $(r_j, \Omega_j^2, \tilde{f}_{in}, \tilde{f}_{out})$ be a smooth asymptotically AdS boundary-characteristic initial data set on $\{0, v_0\}$ for the system (2.32)–(2.37) satisfying the reflecting gauge condition at $r = r_0, +\infty$, according to Definition 3.1 such that the quantities $rac{\partial_r}{\Omega_j^2 - \Lambda r^2}r_J^2(T_{vv})$ and $\tan^{-1} r_J$ extend smoothly on $v = v_0$. Then, there exists a unique future development $(U; r', \Omega', \tilde{f}_{in}', \tilde{f}_{out}')$ of $(r_j, \Omega_j^2, \tilde{f}_{in}, \tilde{f}_{out})$ with reflecting boundary condition on $\gamma_0, \mathcal{I}$ (see Definition 3.4) which is maximal, that is to say, any other future development $(U'; r'', \Omega'', \tilde{f}_{in}'', \tilde{f}_{out}'')$ of $(r_j, \Omega_j^2, \tilde{f}_{in}, \tilde{f}_{out})$ with $r'' \geq r_0$ everywhere on $U'$ satisfies $U' \subseteq U$ and

$$r'(r, \Omega', \tilde{f}_{in}', \tilde{f}_{out}') \equiv (r, \Omega^2, \tilde{f}_{in}, \tilde{f}_{out})|_{\mathcal{U}}.$$  

The maximal future development $(U; r, \Omega, \tilde{f}_{in}, \tilde{f}_{out})$ satisfies the following properties (for the definition of the curves $\gamma_0, \mathcal{I}, \gamma$, see Definition 3.3):

1. We have

$$\partial_u r < 0,$$

$$1 - \frac{2m}{r}|_{J^{-}(\mathcal{I}) \cup J^{-}(\gamma_0)} > 0$$

and

$$\partial_v r|_{J^{-}(\mathcal{I}) \cup J^{-}(\gamma_0)} > 0,$$

where

$$J^{-}(\mathcal{I}) = \left\{ 0 \leq u < \sup_{\mathcal{I}} u \right\} \cap \mathcal{U}$$

is the causal past of $\mathcal{I}$ and

$$J^{-}(\gamma_0) = \left\{ 0 \leq v < \sup_{\gamma_0} v \right\} \cap \mathcal{U}$$

is the causal past of $\gamma_0$ (with respect to the reference Lorentzian metric (2.52)).

2. The renormalised Hawking mass $\tilde{m}$ is conserved on $\gamma_0$ and $\mathcal{I}$, i.e.:

$$\tilde{m}|_{\gamma_0} = \tilde{m}|_{\mathcal{I}}(0)$$

and

$$\tilde{m}|_{\mathcal{I}} = \lim_{v \to v_0} \tilde{m}|_{\gamma_0}(v).$$

3. The conformal infinity $\mathcal{I}$ is complete, i.e. $\Omega^2/(1 - \frac{4}{3} \Lambda r^2)$ has a finite limit on $\mathcal{I}$ and:

$$\int_{\mathcal{I}} \sqrt{\frac{\Omega^2}{1 - \frac{4}{3} \Lambda r^2}} \, du = +\infty.$$
4. In the case $\mathcal{U}\backslash J^-(I) \neq \emptyset$, the future event horizon

\begin{equation}
H^+ = \mathcal{U} \cap \partial J^-(I)
\end{equation}

has the following properties:

(a) $H^+$ has infinite affine length, i.e.:

\begin{equation}
\int_{H^+} \Omega^2 \, dv = +\infty.
\end{equation}

(b) All the matter falls inside the black hole, i.e.

\begin{equation}
\sup_{H^+} r = r_S
\end{equation}

and

\begin{equation}
\inf_{H^+} \left(1 - \frac{2m}{r}\right) = 0,
\end{equation}

where $r_S$ defined by the relation

\begin{equation}
1 - 2\lim_{v \to v_0^-} \tilde{m}_f(v) - \frac{1}{3} \Lambda r_S^2 = 0.
\end{equation}

5. In the case $H^+ \neq \emptyset$, the curve $\gamma_0$ is bounded and contains points lying to the future of $H^+$, i.e. satisfies

\begin{equation}
\gamma_0 \notin J^-(I).
\end{equation}

6. In the case $H^+ \neq \emptyset$, the curve $\gamma$ is non-empty and $r$ extends continuously on $\gamma$ with $r|_{\gamma_0} = r_0$. Moreover, there is no Cauchy horizon “emanating from timelike infinity”: for any point $(u_1, v_1) \in \gamma$, the line \{v = v_1\} intersects $I$. In other words, there is no point in $\gamma$ which lies on the curve \{v = v_I\}, where $(u_I, v_I)$ is the future limit point of $I$.

For the proof of Theorem 1 see Section 5.4.

Remark. Note that, in view of (4.12), (4.14) and the fact that $r > r_0$ on $\mathcal{U}$, a necessary condition for $H^+$ to be non-empty is that the total mass $\lim_{v \to v_0^-} \tilde{m}_f(v)$ and the mirror radius $r_0$ satisfy

\begin{equation}
2 \frac{\lim_{v \to v_0^-} \tilde{m}_f(v)}{r_0} > 1 - \frac{1}{3} \Lambda r_0^2.
\end{equation}

From the proof of Theorem 1 it follows that, in the case when $H^+ \neq \emptyset$, $(\mathcal{U}; r, \Omega^2, \tilde{f}_\text{in}, \tilde{f})$ approaches (in a suitable sense) the Schwarzschild–AdS solution near timelike infinity $(u_I, v_I)$.

**Definition 4.1.** The development $(\mathcal{U}; r, \Omega^2, \tilde{f}_\text{in}, \tilde{f}_\text{out})$ introduced by Theorem 1 will be called the maximal future development of the asymptotically AdS boundary-characteristic initial data set $(r_I, \Omega^2, \tilde{f}_\text{in}, \tilde{f}_\text{out})$.

Remark. In a similar way, we can uniquely define the maximal past development $(\mathcal{U}; r, \Omega^2, \tilde{f}_\text{in}, \tilde{f}_\text{out})$ of $(r_I, \Omega^2, \tilde{f}_\text{in}, \tilde{f}_\text{out})$, satisfying the properties outlined by Theorem 1 after performing a “time reversal” transformation $(u, v) \rightarrow (-v, -u)$. Notice that such a coordinate transformation turns an asymptotically AdS boundary-characteristic initial data set on $u = 0$ into an asymptotically AdS boundary-characteristic initial data set on $v = 0$. However, Theorem 1 also holds (with exactly the same proof) for such initial data sets.
Figure 4.1: Schematic depiction of the maximal future development \((U; r, \Omega^2, \tau, \bar{\tau})\) of a smooth asymptotically AdS boundary-characteristic initial data set \((r_j, \Omega_j^2, \tilde{f}_{\text{in}j}, \tilde{f}_{\text{out}j})\). In the case when \(\mathcal{J}^- (\mathcal{I})\) does not cover all of \(U\), the future event horizon \(H^+\) is non-empty and has infinite affine length. In this case, the curve \(\gamma\) is non-empty, and does not contain a Cauchy horizon component emanating from \(i^+\). The final mass of the event horizon is equal to the (conserved) renormalised Hawking mass on \(\mathcal{I}\).

4.2 Cauchy stability in a rough norm, uniformly in \(r_0\)

Our next result is a Cauchy stability statement for the domain of outer communications of solutions \((r, \Omega^2, \tilde{f}_{\text{in}}, \tilde{f}_{\text{out}})\) to (2.32)–(2.37) in a rough initial data topology:

**Theorem 2** (precise version). For any \(v_1 < v_2\) and any \(0 < r_0 < (-\Lambda)^{-1/2}\), let \(S_i = (r_{j_i}, \Omega_{j_i}^2, \tilde{f}_{\text{in}j_i}, \tilde{f}_{\text{out}j_i}), i = 1, 2\), be two smooth asymptotically AdS boundary-characteristic initial data sets on \([v_1, v_2)\) for the system (2.32)–(2.37) satisfying the reflecting gauge condition at \(r = r_0, +\infty\), according to Definition 3.1, such that the quantities \(r_{j_i}^2 \left( T_{\text{vu}} \right)_{j_i}\) and \(\tan^{-1} r_{j_i}\) extend smoothly on \(v = v_2\). Assume, furthermore, that the following conditions hold:

1. For some \(u_0 > 0\), the maximal future development \((U_1; r_1, \Omega_1^2, \tilde{f}_{\text{in}1}, \tilde{f}_{\text{out}1})\) of \(S_1\) satisfies

\[
\mathcal{W}_{u_0} = \{ 0 < u < u_0 \} \cap \{ u + v_1 < v < u + v_2 \} \subset U_1
\]
and

\begin{equation}
C_0 \leq \sup_{W_{u_0}} \left\{ \log \left( \frac{\Omega_1^2}{1 - \frac{2}{3} \Lambda r_1} \right) + \log \left( \frac{2 \partial_v r_1}{1 - \frac{2m_1}{r_1}} \right) + \log \left( \frac{1 - \frac{2m}{r_1}}{1 - \frac{2}{3} \Lambda r_1} \right) + \sqrt{-\Lambda} |\tilde{m}_1| \right\} + \\
+ \sup_u \int_{\{u = \bar{u}\} \cap W_{u_0}} r_1(T_{uu})_1 \, dv + \sup_v \int_{\{v = \bar{v}\} \cap W_{u_0}} r_1(T_{uu})_1 \, dv < + \infty.
\end{equation}

2. The pair of initial data \( S_1, i = 1, 2 \), satisfy

\begin{equation}
\sup_{v \in \{ v_1, v_2 \}} \left\{ \log \left( \frac{\Omega_1^2}{1 - \frac{2}{3} \Lambda r_1} \right) - \log \left( \frac{\Omega_2^2}{1 - \frac{2}{3} \Lambda r_2} \right) + \log \left( \frac{2 \partial_v r_1}{1 - \frac{2m_1}{r_1}} \right) - \log \left( \frac{2 \partial_v r_2}{1 - \frac{2m_2}{r_2}} \right) + \\
\log \left( \frac{1 - \frac{2m_1}{r_1}}{1 - \frac{1}{3} \Lambda r_1} \right) - \log \left( \frac{1 - \frac{2m_2}{r_2}}{1 - \frac{1}{3} \Lambda r_2} \right) + \sqrt{-\Lambda} |\tilde{m}_1 - \tilde{m}_2| \right\}(v) \leq \delta
\end{equation}

and

\begin{equation}
\sup_{v \in \{ v_1, v_2 \}} (-\Lambda) \int_{v_1}^{v_2} \left| \frac{r_1^2(T_{vv})_1(v_1)}{\left( |\dot{\rho}_{1/2}(v)| - \rho_{1/2}(v) \right) + \tan^{-1} \left( \frac{\sqrt{-\Lambda} r_1}{\frac{2}{3} r_1} \right) |\dot{\rho}_{1/2}(v)|} - \\
\frac{r_2^2(T_{vv})_2(v_2)}{\left( |\dot{\rho}_{2/2}(v)| - \rho_{2/2}(v) \right) + \tan^{-1} \left( \frac{\sqrt{-\Lambda} r_2}{\frac{2}{3} r_2} \right) |\dot{\rho}_{2/2}(v)|} \right| \, dv \leq \delta,
\end{equation}

where, for some fixed large absolute constant \( C_1 \), the parameter \( \delta \) satisfies

\begin{equation}
0 \leq \delta \leq \delta_0 = \exp \left( - \exp \left( C_1 (1 + C_0) \frac{u_0}{v_2 - v_1} \right) \right)
\end{equation}

and \( \rho_{1/2} \) is defined by the relation

\begin{equation}
\rho_{1/2}(v) = \tan^{-1} \left( \sqrt{-\Lambda} r_{1/2}^2(v) \right).
\end{equation}

Then, the maximal development \( (U_2; r_2, \Omega_2, f_{iu2}, f_{ou2}) \) of \( S_2 \) satisfies

\begin{equation}
W_{u_0} \subset U_2
\end{equation}

and

\begin{equation}
\sup_{W_{u_0}} \left\{ \log \left( \frac{\Omega_1^2}{1 - \frac{2}{3} \Lambda r_1} \right) - \log \left( \frac{\Omega_2^2}{1 - \frac{2}{3} \Lambda r_2} \right) + \log \left( \frac{2 \partial_v r_1}{1 - \frac{2m_1}{r_1}} \right) - \log \left( \frac{2 \partial_v r_2}{1 - \frac{2m_2}{r_2}} \right) + \\
\log \left( \frac{1 - \frac{2m_1}{r_1}}{1 - \frac{1}{3} \Lambda r_1} \right) - \log \left( \frac{1 - \frac{2m_2}{r_2}}{1 - \frac{1}{3} \Lambda r_2} \right) + \sqrt{-\Lambda} |\tilde{m}_1 - \tilde{m}_2| \right\} + \\
+ \sup_u \int_{\{u = \bar{u}\} \cap W_{u_0}} |r_1(T_{vv})_1 - r_2(T_{vv})_2| \, dv + \sup_v \int_{\{v = \bar{v}\} \cap W_{u_0}} |r_1(T_{uu})_1 - r_2(T_{uu})_2| \, du \leq \\
\leq \exp \left( \exp \left( C_1 (1 + C_0) \frac{u_0}{v_2 - v_1} \right) \right) \delta.
\end{equation}

For the proof of Theorem 2 see Section 6.1.
Remark. By repeating the proof of Theorem 2, the Cauchy stability estimate \((4.25)\) also holds in the case when \((U_i; r_i, \Omega_i^2, \tilde{f}_{\text{in}i}, \tilde{f}_{\text{out}i}), i = 1, 2\), are the maximal past developments of \(S_i\), i.e. when \(W_{u_0}\) is replaced by
\begin{equation}
W_{u_0}^{(c)} \doteq \{ -u_0 \leq u < 0 \} \cap \{ u + v_1 < v < u + v_2 \}
\end{equation}
and \((4.18)\) holds on \(W_{u_0}^{(c)}\) in place of \(W_{u_0}\) (see the remark below Definition 4.1).

For any \(r_0 > 0\) and any \(\nu_0 > 0\), let us define the following “norm” on the space of smooth asymptotically AdS boundary-characteristic initial data sets \(S = (r_j, \Omega_j^2, \tilde{f}_{\text{in}j}, \tilde{f}_{\text{out}j})\) on \([0, \nu_0]\) for the system \((2.32) - (2.37)\):
\begin{equation}
|| (r_j, \Omega_j^2, \tilde{f}_{\text{in}j}, \tilde{f}_{\text{out}j}) ||_{CS} \doteq \sqrt{-\Lambda} \sup_{0 \leq v < \nu_0} | \tilde{m}_j(v) | + \left( -\Lambda \right) \sup_{0 \leq v < \nu_0} \int_0^{\nu_0} \frac{1}{\varphi_j(v) - \varphi_j(\bar{v}) + \varphi_j(0)} r_j^2(T_v) R^j(\bar{v}) dv + \sup_{0 \leq v < \nu_0} \max \left( \frac{2\tilde{m}_j}{r_j}, 0 \right),
\end{equation}
where \(\varphi_j\) is defined by \((4.23)\).

**Remark.** Note that \((4.27)\) is invariant under gauge transformations, as well as scale transformations of the form \((u, v) \rightarrow (\lambda u, \lambda v), (r, \tilde{m}, \Lambda) \rightarrow (\lambda r, \lambda \tilde{m}, \lambda^{-2} \Lambda)\), \(r_0 \rightarrow \lambda r_0\), \((\tilde{f}_{\text{in}}, \tilde{f}_{\text{out}}) \rightarrow (\lambda^{-4} \tilde{f}_{\text{in}}, \lambda^{-4} \tilde{f}_{\text{out}})\). Furthermore, notice that \(|| (r_j, \Omega_j^2, \tilde{f}_{\text{in}j}, \tilde{f}_{\text{out}j}) ||_{CS} = 0\) if and only if \(\tilde{f}_{\text{in}} = 0\) and \(\tilde{f}_{\text{out}} = 0\), i.e. if \((r_j, \Omega_j^2, \tilde{f}_{\text{in}j}, \tilde{f}_{\text{out}j})\) is the initial data set for the pure AdS spacetime \((\mathcal{M}_{AdS}, g_{AdS})\) on \(\{ r \geq r_0 \}\). The dependence of \((4.27)\) in terms of the Vlasov fields \((\tilde{f}_{\text{in}j}, \tilde{f}_{\text{out}j})\) is only through the ingoing energy momentum component \((T_v)\).

Specialising to the case when \(S_1\) is the trivial initial data set \(S_{AdS}\),
\begin{equation}
S_{AdS} = (0, \Omega_{AdS}^2, 0, 0, 0),
\end{equation}
in Theorem 2, we obtain the following Cauchy stability statement for \((\mathcal{M}_{AdS}, g_{AdS})\) with respect to the topology defined by \((4.27)\), which is independent of the inner mirror radius \(l_0\):

**Corollary 1.** For any (possibly large) \(l_* > 0\), there exists a (small) \(c_0 > 0\) and a constant \(C_{l_*} > 0\) depending only on \(l_*\), so that the following statement holds: For any \(v_0 > 0\) and \(0 < r_0 < (-\Lambda)^{-1/2}\), if \((r_j, \Omega_j^2, \tilde{f}_{\text{in}j}, \tilde{f}_{\text{out}j})\) is a smooth asymptotically AdS boundary-characteristic initial data set on \([0, \nu_0]\) for the system \((2.32) - (2.37)\) satisfying the reflecting gauge AdS boundary condition at \(v = \nu_0, +\infty\), according to Definition 3.1, such that the quantities \(r_j^2(T_v)R^j\) and \(\tan^{-1} r_j\) extend smoothly on \(v = \nu_0\) and moreover
\begin{equation}
|| (r_j, \Omega_j^2, \tilde{f}_{\text{in}j}, \tilde{f}_{\text{out}j}) ||_{CS} < \epsilon
\end{equation}
for some \(0 < \epsilon \leq c_0\), then the maximal development \((U; r, \Omega^2, \tilde{f}_{\text{in}}, \tilde{f}_{\text{out}})\) satisfies
\begin{equation}
W_{l_*} \doteq \{ 0 < u < l_* \nu_0 \} \cap \{ u < v < u + \nu_0 \} \in U
\end{equation}
and
\begin{equation}
\sqrt{-\Lambda} \sup_{W_{l_*}} | m | + \max_{W_{l_*}} \left( 1 - \frac{1}{2} Ar^2 \right) \sup_{v \in \nu_0} \int_0^{\nu_0} \frac{rT_{uv}}{\partial_v r} dv + \sup_{v \in \nu_0} \int_{v = \nu_0}^\infty \frac{rT_{uv}}{\partial_u r} du < C_{l_*} \epsilon.
\end{equation}

For the proof of Corollary 1 see Section 6.2

## 5 Well-posedness and structure of the maximal development

The aim of this Section is the proof of Theorem 1. To this end, we will first introduce, in Section 5.1, a number of characteristic initial value problems for \((2.32) - (2.37)\), in addition to the asymptotically AdS boundary-characteristic initial value problem introduced by Definition 3.1. We will then establish the well-posedness of these initial value problems in Section 5.2. The results of Section 5.2, combined with a number of continuation criteria that will be established in Section 5.3, will allow us to construct the maximal future development of a general smooth, asymptotically AdS boundary-characteristic initial data set and complete the proof of Theorem 1. This will be achieved in Section 5.4.
5.1 Auxiliary types of characteristic initial data sets

In this Section, we will define some auxiliary types of characteristic initial data sets for \(2.32-2.37\), in addition to the one introduced by Definition 3.1.

Definition 5.1. For any \(u_1 < u_2, v_1 < v_2\) and any \(r_0 > 0\), let \(r_\lambda : [u_1, u_2] \to (r_0, +\infty), \Omega_\lambda : [u_1, u_2] \to (0, +\infty), f_{\text{in}}, f_{\text{out}} : [u_1, u_2] \times (0, +\infty) \to [0, +\infty), r_f : [v_1, v_2] \to (r_0, +\infty), \Omega_f : [v_1, v_2] \to (0, +\infty)\) and \(f_{\text{in},f}, f_{\text{out},f} : [v_1, v_2] \times (0, +\infty) \to [0, +\infty)\) be smooth functions, such that

\[
\begin{align*}
(5.1) & \quad r_\lambda(u_1) = r_f(v_1), \\
(5.2) & \quad \Omega_\lambda(u_1) = \Omega_f(v_1), \\
(5.3) & \quad f_{\text{in},f}(u_1, p) = \tilde{f}_{\text{in}}(u_1, p), \\
(5.4) & \quad \tilde{f}_{\text{out},f}(u_1, p) = \tilde{f}_{\text{out}}(u_1, p).
\end{align*}
\]

and, for all \(p \in (0, +\infty)\):

\[
\begin{align*}
(5.5) & \quad \partial_u(\Omega_\lambda^2 \partial_u r_\lambda) = -4\pi r_\lambda(T_{uu}); \Omega_\lambda^2, \\
(5.6) & \quad \partial_v(\Omega_f^2 \partial_v r_f) = -4\pi r_f(T_{vv}); \Omega_f^2,
\end{align*}
\]

where

\[
\begin{align*}
(5.7) & \quad (T_{uu})_\lambda(u) \doteq \int_0^{+\infty} \Omega_\lambda^2(u)(p^u)2 \tilde{f}_{\text{in},f}(u; p^u) r_\lambda^2(u) \frac{dp^u}{p^u}, \\
(5.8) & \quad (T_{vv})_f(v) \doteq \int_0^{+\infty} \Omega_f^2(v)(p^v)2 \tilde{f}_{\text{in},f}(v; p^v) r_f^2(v) \frac{dp^v}{p^v},
\end{align*}
\]

while \(f_{\text{in}}, f_{\text{out}}\) solve \(2.36\) and \(2.37\) along \(\{v = v_1\}\) and \(\{u = u_1\}\) respectively, i.e.

\[
\begin{align*}
(5.9) & \quad \partial_u(\Omega_\lambda^2(u) r_\lambda^2(u) p^u \tilde{f}_{\text{in}}(u, p^u)) + p^v \partial_{p^v} (\Omega_\lambda^2(u) r_\lambda^2(u) p^u \tilde{f}_{\text{in}}(u, p^u)) = 0, \\
(5.10) & \quad \partial_v(\Omega_f^2(v) r_f^2(v) p^v \tilde{f}_{\text{out}}(v, p^v)) + p^u \partial_{p^u} (\Omega_f^2(v) r_f^2(v) p^v \tilde{f}_{\text{out}}(v, p^v)) = 0.
\end{align*}
\]

Remark. Let \((r, \Omega^2, \tilde{f}_{\text{in}}, \tilde{f}_{\text{out}})\) be a solution of the system \(2.32-2.37\) on a closed subset of \(\mathcal{V}\) of \([u_1, u_2] \times [v_1, v_2]\) containing \((\{u_1, u_2\} \times \{v_1\}) \cup (\{u_2\} \times [v_1, v_2])\), such that

\[
(5.11) \quad (r, \Omega^2, \tilde{f}_{\text{in}}, \tilde{f}_{\text{out}})[[u_1, u_2] \times \{v_1\}] = (r_\lambda, \Omega_\lambda^2, \tilde{f}_{\text{in},f}, \tilde{f}_{\text{out},f})
\]

and

\[
(5.12) \quad (r, \Omega^2, \tilde{f}_{\text{in}}, \tilde{f}_{\text{out}})[(u_1) \times \{v_1, v_2\}] = (r_f, \Omega_f^2, \tilde{f}_{\text{in},f}, \tilde{f}_{\text{out},f}).
\]

Then, the transversal derivatives of \(r\) across \([u_1, u_2] \times \{v_1\}\) and \(\{u_2\} \times [v_1, v_2]\) can be computed in terms of \((r_\lambda, \Omega_\lambda^2)\) and \((r_f, \Omega_f^2)\) by integrating equation \(2.32\), i.e. for all \(u \in [u_1, u_2]\):

\[
\begin{align*}
(5.13) & \quad (r \partial_u r)(u, v_1) = r_f \partial_u r_f(v_1) - \frac{1}{4} \int_{u_1}^{u}(1 - \Lambda r^2(\bar{u})) \Omega_f^2(\bar{u}) d\bar{u}, \\
(5.14) & \quad (r \partial_u r)(u_1, v) = r_\lambda \partial_u r_\lambda(u_1) - \frac{1}{4} \int_{v_1}^{v}(1 - \Lambda r^2(\bar{v})) \Omega_\lambda^2(\bar{v}) d\bar{v}.
\end{align*}
\]
Definition 5.2. For any \( u_1 < u_2, v_1 < v_2 \) and any \( r_0 > 0 \), let \( r_1 : [u_1, u_2] \to [r_0, +\infty) \), \( \Omega_1 : [u_1, u_2] \to [0, +\infty) \), \( f_{in}, f_{out} : [u_1, u_2] \times (0, +\infty) \to [v_1, v_2] \) smooth functions, satisfying (5.1)–(5.4). We will call \((r_1, \Omega_1, f_{in}, f_{out})\) and \((r_1, \Omega_1, f_{in}, f_{out})\) a boundary-double characteristic initial data set for the system (2.32)–(2.37) with boundary at \( r = r_0 \) and satisfying the reflecting gauge condition at \( r = r_0 \) if they satisfy the assumptions (5.5)–(5.10) of Definition 5.1 and moreover

\[
(5.15) 
\partial_u r_1(u_2) = r_0, 
\]

\[
(5.16) 
\partial_u r_1(u_2) < 0 
\]

and

\[
(5.17) 
(\partial_v r_1(u_2)) = -\partial_u r_1(u_2), 
\]

where \((\partial_v r_1)\) is defined by (5.13), i.e., for any \( u_1 \leq u \leq u_2 \):

\[
(5.18) 
r_1(\partial_v r_1)(u) = r_1(\partial_u r_1)(v_1) - \frac{1}{4} \int_{u_1}^{u} (1 - \Lambda r_1^2(\bar{u})) \Omega_1^2(\bar{u}) d\bar{u}. 
\]

Remark. Note that (5.17) and (5.18) imply that

\[
(5.19) 
(\partial_v r_1)(u) > 0 
\]

for all \( u \in [u_1, u_2] \). In particular,

\[
(5.20) 
\partial_v r_1(v_1) > 0. 
\]

5.2 Local existence and uniqueness

In this Section, we will establish the local well-posedness of the initial value problems for (2.32)–(2.37) associated to the types of initial data sets introduced by Definitions 3.1, 5.1 and 5.2. We will in fact establish the well-posedness of these initial value problems in the rough topology defined by (4.27).

The next result is a well-posedness result for the initial data introduced by Definition 3.1.

Proposition 5.1. Let \( C_0 \gg 1 \) be a (large) constant. For any \( v_1 < v_2 \) and any \( r_0 > 0 \), let \((r_1, \Omega_1^2, f_{in}, f_{out})\) be a smooth asymptotically AdS boundary-characteristic initial data set on \([v_1, v_2]\) for the system (2.32)–(2.37) satisfying the reflecting gauge condition at \( r = r_0, +\infty \), according to Definition 3.1, such that the quantities

\[
(\frac{\Omega_1^2}{1 - \frac{2m_j}{r_j}}), r_1^2(T_{vv}), \tan^{-1} r_j \text{ extend smoothly on } v = v_2. 
\]

Let us also set

\[
(5.21) 
M \equiv \sup_{v \in [v_1, v_2]} \left\{ |\log(\frac{\Omega_1^2}{1 - \frac{2m_j}{r_j}})| + |\log(\frac{2\partial_v r_j}{1 - \frac{2m_j}{r_j}})| + |\log(\frac{1 - \frac{2m_j}{r_j}}{1 - \frac{2m_j}{r_j} + \sqrt{-\Lambda} |\tilde{m}_j|})| + \int_{v_1}^{v_2} r_j(T_{vv}) d\bar{v} \right\}, 
\]

and, for any \( 0 < \delta < 1 \):

\[
(5.22) 
v_{in}(\delta) \equiv \sup_{v \in [v_1, v_2]} \left\{ 0 < v_1 \leq v \leq v_2 : \sup_{v \in [v_1, v_2]} \int_{\max\{v - \delta v, v_1\}}^{\min\{v + \delta v, v_2\}} \frac{r_1^2(T_{vv})(\bar{v})}{|\bar{v} - v| + r_0} d\bar{v} < \delta \right\}, 
\]

where

\[
(5.23) 
m_j(v) = r_j^2 \left( 1 + 4\Omega_j^2(\partial_v r_j \partial_u r_j)(v) \right), 
\]

\[
(5.24) 
\tilde{m}_j(v) = m(v) - \frac{1}{6} \Lambda r_j^2(v). 
\]
and \((\partial_v r)\) is defined according to (3.3). Then, provided

\[
(5.25) \quad u_0 < \frac{\nu_{in}(2e^{-\frac{c_{in}^2 M}{2}})}{e^{\frac{c_{in}^2}{2}}(v_2-v_1)^2+1},
\]

the following holds: Setting

\[
(5.26) \quad \mathcal{I}_{u_0} \doteq \{ u = v - v_2 \} \cap \{ 0 < u < u_0 \},
\]

\[
(5.27) \quad \gamma_{0;u_0} \doteq \{ u = v - v_1 \} \cap \{ 0 < u < u_0 \}
\]

and

\[
(5.28) \quad \mathcal{W} \doteq \{ 0 < u < u_0 \} \cap \{ u + v_1 < v < u + v_2 \},
\]

there exist unique smooth functions \(r : \mathcal{W} \cup \gamma_{0;u_0} \to (r_0, +\infty), \Omega : \mathcal{W} \cup \gamma_{0;u_0} \to (0, +\infty)\) and \(\bar{f}_{in}, \bar{f}_{out} : \mathcal{W} \cup \gamma_{0;u_0} \times (0, +\infty) \to [0, +\infty)\) solving equations (2.32)–(2.37) on \(\mathcal{W}\) (with \(T_{uu}, T_{vv}\) expressed by (2.22), (2.23)), such that:

1. The functions \(r, \Omega, \bar{f}_{in}, \bar{f}_{out}\) satisfy the given initial conditions on \(\{0\} \times [v_1, v_2]\), i.e.:

\[
(5.29) \quad (r, \Omega, \bar{f}_{in}, \bar{f}_{out})|_{\{0\} \times [v_1, v_2]} = (r_0, \Omega_0, \bar{f}_{in0}, \bar{f}_{out0}).
\]

2. The functions \((r, \bar{f}_{in}, \bar{f}_{out})\) satisfy on \(\gamma_{0;u_0}\) the boundary conditions

\[
(5.30) \quad r|_{\gamma_{0;u_0}} = r_0
\]

and

\[
(5.31) \quad \bar{f}_{out}(u_*, v_*; p) = \bar{f}_{in}(u_*, v_*; p)
\]

for all \((u_*, v_*) \in \gamma_{0;u_0}\) and \(p > 0\), as well as the reflecting gauge condition

\[
(5.32) \quad \partial_u r|_{\gamma_{0;u_0}} = -\partial_v r|_{\gamma_{0;u_0}}.
\]

3. The functions \((r, \bar{f}_{in}, \bar{f}_{out})\) satisfy on \(\mathcal{I}_{u_0}\) the boundary conditions:

\[
(5.33) \quad (1/r)|_{\mathcal{I}_{u_0}} = 0
\]

and

\[
(5.34) \quad \lim_{h \to 0^+} \left( \frac{\bar{f}_{in}(u_* + h, v_*; \Omega^2(u_* + h, v_*) \cdot p)}{\bar{f}_{out}(u_* + v_* - h; \Omega^2(u_* + v_* - h) \cdot p)} \right) = 1
\]

for all \((u_*, v_*) \in \mathcal{I}_{u_0}\) and \(p > 0\), as well as the reflecting gauge condition

\[
(5.35) \quad \partial_u (1/r)|_{\mathcal{I}_{u_0}} = -\partial_v (1/r)|_{\mathcal{I}_{u_0}}.
\]

4. The function \(r\) satisfies

\[
(5.36) \quad \sup_{\mathcal{W}} \partial_u r < 0.
\]
5. The following estimates hold on $W$:

\[
(5.37) \quad \sup_{W} \left\{ \log \left( \frac{\Omega^2}{1 - \frac{3}{2} \Lambda r^2} \right) + | \log \left( \frac{2 \partial_r r}{1 - \frac{2 m}{r}} \right) | + | \log \left( \frac{1 - \frac{2 m}{r}}{1 - \frac{3}{2} \Lambda r^2} \right) + \sqrt{-\Lambda} \tilde{m} \right\} + \\
+ \sup_{\bar{u}} \int_{\{u=\bar{u}\} \cap W} r T_{uv} \ dv \leq C_0 M,
\]

\[
(5.38) \quad \sup_{\bar{v}} \int_{\{v=\bar{v}\} \cap W} r T_{uu} \ du + \sup_{\bar{u}} \sup_{v \in [u_1 + \bar{u}, v_2 + \bar{u}]} \int_{\min\{v-u_0, v_2+\bar{u}\}}^{\max\{v-u_0, v_1+\bar{u}\}} \frac{r^2(T_{uv})(\bar{u}, \bar{v})}{|0 - v| + r_0} \ dv \leq e^{-C_0 M} M
\]

and

\[
(5.39) \quad \sup_{W} r^2 T_{uu}, \sup_{W} r^2 T_{vv} = \sup_{[v_1, v_2]} r^2(T_{uv}).
\]

Remark. By repeating the proof of Proposition 5.1 without any significant change, one also infers the existence and of a smooth development of $(r, \Omega^2, \vec{f}_{in}, \vec{f}_{out})$ backwards in time, i.e. the existence and uniqueness of a smooth solution $(r, \Omega^2, \vec{f}_{in}, \vec{f}_{out})$ to the system (2.32)–(2.37) on

\[
(5.40) \quad W^* \triangleq \{-u_0 < u < 0\} \cap \{u + v_1 < v < u + v_2\}
\]
satisfying (the analogues of) (5.29)–(5.39).

\[
\text{Figure 5.1: Schematic depiction of the domain } W \text{ in the statement of Proposition 5.1.}
\]

Proof. It suffices to establish the existence and uniqueness of a smooth solution $(r, m, T_{uu}, T_{vv})$ to the system (2.45)–(2.51) on $W \cup \gamma_{0, u_0}$, satisfying initial conditions

\[
(5.41) \quad (r, T_{vv})|_{t=0, x \in [v_1, v_2]} = (r_0, T_{vv})
\]
and

\[ \tilde{m}(0, v_1) = \tilde{m}_f(0, v_1) \]

the gauge conditions (5.32), (5.35) and the boundary conditions (5.30), (5.33),

\[ \left( \frac{\partial_v r}{\partial_u r} \right)^2 T_{uu} \bigg|_{y_0, u_0} = 1 \]

and

\[ \left( \frac{\partial_v r}{\partial_u r} \right)^2 T_{uu} \bigg|_{u_0} = 1. \]

Given the existence and uniqueness of such a smooth solution \((r, m, T_{uu}, T_{vv})\) to the system (2.45)–(2.51), by solving equations (2.36) and (2.37) for \(\bar{f}_{\text{in}}, \bar{f}_{\text{out}}\) on \(\mathcal{W}\) with initial data

\[ (\bar{f}_{\text{in}}, \bar{f}_{\text{out}}) = (\bar{f}_{\text{in}}, \bar{f}_{\text{out}}) \]

and boundary conditions (5.31) and (5.34), and using the formula (2.7) for \(\Omega_2\), one then obtains the existence and uniqueness of a smooth solution \((r, \Omega_2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})\) of the system (2.32)–(2.37) satisfying (5.29), (5.32), (5.35), (5.30), (5.33), (5.31) and (5.34).

Let us introduce a new set of renormalised variables

\[ \rho = \tan^{-1} \sqrt{-\frac{A}{3}}, \]

\[ \chi = 2 \frac{\partial_v r}{1 - \frac{2m}{r}}, \]

\[ \chi = 2 \frac{-\partial_u r}{1 - \frac{2m}{r}}, \]

\[ \bar{\tau} = r^2 T_{vv}, \]

\[ \tau = r^2 T_{uu}. \]

We will also define, for convenience, the functions \(F_1, F_2 : (0, +\infty) \times \mathbb{R} \to \mathbb{R}\) by the relations

\[ F_1(x; y) = 8\pi \frac{1}{x(1 - \frac{2y}{x} + x^2)}, \]

and

\[ F_2(x; y) = -\frac{1}{2} \frac{y (1 + 3x^2)(1 - \frac{2y}{x} + x^2)}{x(1 + x^2)^2}. \]

Note that, for any \(x_0 > 0\), any \(y \in \mathbb{R}\) such that \(1 - \frac{2y}{x_0} + x_0^2 \geq 0\), any \(x \geq x_0\) and any integer \(k \geq 0\), we can bound:

\[ x^{k+1} |\partial_x^k F_1(x, y)| + x^{k+1} |\partial_x^k F_2(x, y)| \leq C_k \left( 1 + \frac{|y|^k}{x} (1 - \frac{2y}{x} + x^2)^{-k-1} + \frac{|y|^2}{x^2} \right) \]

Switching to the variables (5.46)–(5.50) and using the relations (2.6) and (2.44) for \(\tilde{m}\), the system (2.45)–(2.51)
transforms into:

\[(5.54) \quad \partial_u \log(x) = -\sqrt{-\frac{\Lambda}{3}} F_1(\tan \varphi; \sqrt{-\frac{\Lambda}{3}} \tilde{m}) x^{-1} \tau,\]

\[(5.55) \quad \partial_\tau \log(x) = \sqrt{-\frac{\Lambda}{3}} F_2(\tan \varphi; \sqrt{-\frac{\Lambda}{3}} \tilde{m}) x^{-1} \tilde{\tau},\]

\[(5.56) \quad \partial_u \partial_\varphi = (-\frac{\Lambda}{3}) F_2(\tan \varphi; \sqrt{-\frac{\Lambda}{3}} \tilde{m}) x \tilde{\varphi},\]

\[(5.57) \quad \partial_u \tilde{m} = -4\pi x^{-1} \tau,\]

\[(5.58) \quad \partial_\varphi \tilde{\varphi} = 0,\]

\[(5.59) \quad \partial_\tau \tilde{\varphi} = 0.\]

The initial condition (5.41), the gauge conditions (5.32), (5.35) and the boundary conditions (5.30), (5.33), (5.43) and (5.44) are then replaced by:

\[(5.60) \quad (\varphi, x, m, \tilde{\varphi})|_{(0) \times [v_1, v_2]} = \left( \tan^{-1}(\sqrt{-\frac{\Lambda}{3}} r_j), \frac{2\partial r_j}{1 - 2\rho/r_j} \tilde{m}_j, r_j^2 (T_{x,v})_j \right).\]

\[(5.61) \quad \varphi|_{\gamma_0 = u_0} = \varphi_0, \quad \varphi|_{x = u_0} = \frac{\pi}{2},\]

\[(5.62) \quad \frac{x}{\tilde{x}}|_{\gamma_0 = u_0} = 1, \quad \frac{x}{\tilde{x}}|_{x = u_0} = 1\]

and

\[(5.63) \quad \frac{\tau}{\tilde{\tau}}|_{\gamma_0 = u_0} = 1, \quad \frac{\tau}{\tilde{\tau}}|_{x = u_0} = 1,\]

where

\[(5.64) \quad \varphi_0 = \tan^{-1}(\sqrt{-\frac{\Lambda}{3}} r_0) > 0.\]

Remark: Note that equations (2.46) and (2.47) yield equation (2.49) for \(m\). The relations (2.49), (5.57) and (5.44) imply that \(m\) is conserved on \(I_{u_0}\), i.e.

\[(5.65) \quad m|_{I_{u_0}} = \lim_{v \to v_2} m(0, v_2) = \lim_{v \to v_2} m_f(0, v).\]

The proof of Proposition 5.1 will follow if we establish that the system (5.54)–(5.59) admits a unique smooth solution \((\varphi, x, \tilde{x}, \tau, m)\) on \(W \cup \gamma_{0,u_0}\) satisfying (5.60)–(5.65) and the estimates

\[(5.66) \quad \sup_{\gamma} \left\{ |\log(x)| + |\log(\tilde{x})| + \left| \log\left( \sqrt{-\frac{12}{\Lambda}} \partial_x \varphi \right) \right| + \left| \log\left( -\frac{\partial_x \varphi}{\sqrt{-\frac{12}{\Lambda}}} \right) \right| + \left| \log\left( \frac{1}{1 + \tan^2 \varphi} \right) \right| + \sqrt{-\Lambda} |\tilde{m}| \right\} + \sup_{\partial u \leq u_0} \int_{W \cap (u = \tilde{u})} \sqrt{-\Lambda} \tilde{\tau} dv \leq \frac{1}{2} C_0 M\]
(where $M$ is defined by (5.21),

$$
\sup_{v_1 \leq \bar{v} \leq v_2 + u_0} \int_{\mathcal{W} \cap \{v = \bar{v}\}} \sqrt{-\frac{\Lambda}{\rho}} \, du + \sup_{0 \leq \bar{v} \leq v_0} \sup_{v \in [v_1 + \bar{v}, v_2 + \bar{u}]} \int_{\min\{v - u_0, v_1 + \bar{u}\}}^{\max\{v - u_0, v_1 + \bar{u}\}} \frac{\bar{v}}{|\bar{v} - u| + r_0^2} \, dv \leq e^{-C_0^2 M M}
$$

and

$$
\sup_{v_1 \leq \bar{v} \leq v_2} \max\{\bar{v}, v\} \leq \sup_{v_1 \leq \bar{v} \leq v_2} r_1^2(T_{uv}) / f,
$$

as well as the bound

$$
\varphi_0 = \varphi\nu < \frac{\pi}{2}.
$$

Note that (5.69) readily follows once (5.66) has been established, in view of the fact that $\partial_u \varphi < 0$ (as a consequence of (5.66) and $r_1(\nu_1, \nu_2) < +\infty$).

The proof of the existence and uniqueness of a smooth solution to (5.54)–(5.59) satisfying (5.60)–(5.65) and the estimates (5.66)–(5.68) will consist of two steps:

**Step 1.** We will first show that there exists a (weak) $C^0$ solution $(\varphi, x, \bar{x}, \tau, \bar{\tau}, \bar{m})$ of the system (5.54)–(5.59) on $\mathcal{W}$ satisfying (5.60)–(5.65) and (5.66)–(5.68) and extending continuously on $\gamma_{0:u_0} \mathcal{I}_{u_0}$. Notice that, in view of our assumption that the quantities $\frac{v_1}{\gamma_1 + \nu_1^2}, r_j^2(T_{uv}) / f$ and $\tan^{-1} r_j / f$ extend smoothly up to $v = v_2$, the initial data (5.60) for $(\varphi, x, \bar{x}, \bar{m}, \bar{\tau})$ on $\{0\} \times [v_1, v_2]$ extend smoothly up to $v = v_2$. Using the bounds (5.66) and (5.68), it can be then readily inferred that $(\varphi, x, \bar{x}, \tau, \bar{m})$ is in fact a classical $C^\infty$ solution on $\gamma_{0:u_0}$, by commuting equations (5.54)–(5.59) with $\partial_u, \partial_x$ and treating the commuted equations as linear equations in the highest order terms.

The proof will follow by the usual iteration argument. Let us define inductively the sequence of $C^1$ functions $\{(\varphi_n, x_n, \bar{x}_n, \tau_n, \bar{\tau}_n, \bar{m}_n)\}_{n \in \mathbb{N}}$ on $\mathcal{W}$ by solving for each $n \in \mathbb{N}$:

$$
\partial_u \log(x_n) = -\frac{\Lambda}{3} F_1\left(\frac{\tan \varphi_{n-1}}{\sqrt{-\frac{\Lambda}{3} \bar{m}_{n-1}}} x_{n-1}^{-1} \tau_{n-1}\right),
$$

$$
\partial_x \log(x_n) = \sqrt{-\frac{\Lambda}{3}} F_1\left(\frac{\tan \varphi_{n-1}}{\sqrt{-\frac{\Lambda}{3} \bar{m}_{n-1}}} x_{n-1}^{-1} \bar{\tau}_{n-1}\right),
$$

$$
\partial_u \partial_x \varphi_n = \left(-\frac{\Lambda}{3}\right) F_2\left(\frac{\tan \varphi_{n-1}}{\sqrt{-\frac{\Lambda}{3} \bar{m}_{n-1}}} \bar{x}_{n-1} \bar{\tau}_{n-1}\right),
$$

$$
\partial_u \bar{m}_n = -4\pi \bar{x}_n^{-1} \tau_{n-1},
$$

$$
\partial_u \bar{\tau}_n = 0,
$$

$$
\partial_x \tau_n = 0,
$$

where $(\varphi_n, x_n, \bar{x}_n, \tau_n, \bar{\tau}_n, \bar{m}_n)$ satisfy

$$
(\varphi_n, x_n, \bar{m}_n, \bar{\tau}_n)|_{(0) \times [v_1, v_2)} = \left(\tan^{-1}\left(\frac{\sqrt{-\lambda}}{\nu_1} r_j / f, \frac{2\partial_x r_j}{1 - \frac{2\nu_1^2}{\nu_j} r_j}, \bar{m}_j, r_j^2(T_{uv})/f\right)\right),
$$

$$
\varphi_n|_{\gamma_0} = \varphi_0, \varphi_n|_{\gamma_{u_0}} = \frac{\pi}{2},
$$

$$
\frac{\bar{x}_n}{x_n}|_{\gamma_0} = 1, \frac{\bar{x}_n}{x_n}|_{\gamma_{u_0}} = 1,
$$

on $\mathcal{W}$. Using the bounds (5.66) and (5.68), it can be then readily inferred that $(\varphi, x, \bar{x}, \tau, \bar{\tau}, \bar{m})$ is in fact a classical $C^\infty$ solution on $\gamma_{0:u_0}$, by commuting equations (5.54)–(5.59) with $\partial_u, \partial_x$ and treating the commuted equations as linear equations in the highest order terms.
(5.79) \[ \frac{\tau_n}{\tilde{\tau}_n} \bigg|_{\gamma_0} = 1, \quad \frac{\tau_n}{\tilde{\tau}_n} \bigg|_{\iota_v} = 1 \]

and

(5.80) \[ \tilde{m}_n|_{\iota_v} = \lim_{v \to v_2} \tilde{m}_f(0,v). \]

We also use the convention that,

(5.81) \[ x_0 = \tilde{x}_0 = \tau_0 = \tilde{\tau}_0 = 0. \]

**Remark.** Notice that, under the assumption that \((\varphi_{n-1}, x_{n-1}, x_n, \tau_{n-1}, \tilde{\tau}_n, \tilde{m}_n)\) are \(C^1\) functions on \(W\) satisfying \(\inf_{W} \varphi_{n-1} > 0\), (5.53) and (5.68) imply that \((\varphi_n, x_n, x_n, \tau_n, \tilde{\tau}_n, \tilde{m}_n)\), obtained by solving (5.70)–(5.75) with (5.76)–(5.80), are \(C^1\) functions on \(W\) (despite the fact that \(\tan \varphi_{n-1}\), appearing as an argument of \(F_1, F_2, F_3\) in the right hand side of (5.70)–(5.75), is unbounded on \(W\)).

We will show that, as \(n \to \infty\), the sequence \((\varphi, x_n, x_n, \tau_n, \tilde{\tau}_n, \tilde{m}_n)_{n \in \mathbb{N}}\) converges in the \(C^0(W)\) norm to a solution \((\varphi, x, \tilde{x}, \tau, \tilde{\tau}, \tilde{m})\) of (5.54)–(5.59) on \(W\) satisfying (5.60)–(5.65) and (5.66)–(5.68). To this end, it suffices to show that, for any \(n \in \mathbb{N}\), under the assumption that for any \(1 \leq k \leq n - 1\)

(5.82) \[ \sup_{W} \left\{ \left| \log(x_k) \right| + \left| \log(\tilde{x}_k) \right| + \log \left( \sqrt{-\frac{12}{\Lambda}} \partial_u \varphi_k \right) + \left| \log \left( -\frac{12}{\Lambda} \partial_u \varphi_k \right) \right| + \right. \]

\[ + \left. \left| \log \left( \frac{1 - 2 \sqrt{\frac{5}{\tan \varphi_k}} \tan^2 \varphi_k}{1 + \tan^2 \varphi_k} \right) \right| + \sqrt{-\Lambda} \tilde{m}_k \right\} + \sup_{0 \leq u \leq u_0} \int_{W \cap \{u = \tilde{u}\}} \sqrt{-\Lambda} \tilde{m}_k \, dv \leq \frac{1}{2} C_0 M \]

(5.83) \[ \sup_{v_1 \leq u \leq v_2 + u_0} \int_{W \cap \{v = \tilde{v}\}} \sqrt{-\Lambda} \tilde{m}_k \, du \leq \sqrt{-\Lambda} \tilde{m}_n \]

(5.84) \[ \sup_{W} \max \{ \tau_k, \tilde{\tau}_k \} \leq 2 \sup_{v_1 \leq v \leq v_2} \rho_f(T_{vv}) \]

and

(5.85) \[ \varphi_k \geq \varphi_0, \]

the bounds (5.82)–(5.85) also hold for \((\varphi_n, x_n, x_n, \tau_n, \tilde{\tau}_n, \tilde{m}_n)\), i.e.:

(5.86) \[ \sup_{W} \left\{ \left| \log(x_n) \right| + \left| \log(\tilde{x}_n) \right| + \log \left( \sqrt{-\frac{12}{\Lambda}} \partial_u \varphi_n \right) + \left| \log \left( -\frac{12}{\Lambda} \partial_u \varphi_n \right) \right| + \right. \]

\[ + \left. \left| \log \left( \frac{1 - 2 \sqrt{\frac{5}{\tan \varphi_n}} \tan^2 \varphi_n}{1 + \tan^2 \varphi_n} \right) \right| + \sqrt{-\Lambda} \tilde{m}_n \right\} + \sup_{0 \leq u \leq u_0} \int_{W \cap \{u = \tilde{u}\}} \sqrt{-\Lambda} \tilde{m}_n \, dv \leq \frac{1}{2} C_0 M \]

(5.87) \[ \sup_{v_1 \leq u \leq v_2 + u_0} \int_{W \cap \{v = \tilde{v}\}} \sqrt{-\Lambda} \tilde{m}_n \, du \leq \sqrt{-\Lambda} \tilde{m}_n \]

(5.88) \[ \sup_{W} \max \{ \tau_n, \tilde{\tau}_n \} \leq 2 \sup_{v_1 \leq v \leq v_2} \rho_f(T_{vv}) \]
and

\[(5.89) \quad \varphi_n \geq \varphi_0\]

and, moreover, for \(n \geq 4:\)

\[(5.90) \quad D_n \leq \frac{1}{2} \max \{ D_{n-1}, D_{n-2} \},\]

where

\[(5.91) \quad D_n = \sup_{W} \max \left\{ |\varphi_n - \varphi_{n-1}|, |\log \chi_n - \log \chi_{n-1}|, |\log \bar{\varphi}_n - \log \bar{\varphi}_{n-1}|, \sqrt{\Lambda} |\bar{m}_n - \bar{m}_{n-1}|, |\tau_n - \tau_{n-1}|, |\bar{\tau}_n - \bar{\tau}_{n-1}| \right\}.

Notice that, when \(n = 2\), the bounds \((5.82)\)–\((5.89)\) can be readily obtained from \((5.70)\)–\((5.75)\), \((5.76)\)–\((5.80)\) and \((5.21)\)–\((5.25)\), provided \(C_0 \gg 1\).

Integrating equation \((5.74)\) along the lines \(\{v = \text{const}\}\) and \((5.75)\) along the lines \(\{u = \text{const}\}\), and using the boundary conditions \((5.79)\), we obtain for any \((u, v) \in W\)

\[(5.92) \quad \bar{\varphi}_n(u, v) = \bar{\varphi}_n(0, v_r[u, v]), \quad r^2(T_{uv})/(v_r[u, v]), \quad \tau_n(u, v) = \bar{\varphi}_n(0, v_r[u, v]), \quad r^2(T_{uv})/(v_r[u, v]),

where

\[(5.93) \quad v_r[u, v] = \begin{cases} v, & \text{if } v \leq v_2 \\ v - v_2 + v_1, & \text{if } v > v_2 \end{cases},

and

\[(5.94) \quad v_r[u, v] = v_1 + u,\]

In particular,

\[(5.95) \quad \bar{\varphi}_n, \tau_n \leq \sup_{v_1 \leq v < v_2} r^2(T_{uv})/\]

\[(5.96) \quad \bar{\varphi}_n = \bar{\varphi}_{n-1} \quad \text{and} \quad \tau_n = \tau_{n-1}.

Integrating equation \((5.70)\) along the lines \(\{0 \leq u \leq u_0\} \cap \{v = v_\ast\}\) for any \(v_1 \leq v_\ast \leq v_2\), we obtain for any point \((u, v_\ast) \in W \cap \{v \leq v_2\}\):

\[(5.97) \quad \log \left( \chi_n(u, v_\ast) \right) = \log \left( \chi_n(0, v_\ast) \right) - \int_0^u \sqrt{\Lambda} F_1 \left( \tan \varphi_{n-1}; \sqrt{\Lambda} \bar{m}_{n-1} \right) \bar{\varphi}^{\ast 1}_{n-1}(u, v_\ast) \, du.

In view of the bounds \((5.82)\), \((5.83)\) and \((5.53)\), using also \((5.21)\) and \((5.85)\), the relation \((5.97)\) yields for any point \((u, v) \in W \cap \{v \leq v_2\}\):

\[(5.98) \quad \left| \log \left( \chi_n(u, v) \right) \right| \leq M \left( 1 + e^{-\frac{1}{2} C_0^2 M} \right).

Furthermore, subtracting from equation \((5.97)\) the same equation for \(n - 1\) in place of \(n\), and using \((5.76)\), \((5.53)\), \((5.82)\), \((5.83)\), \((5.85)\) and \((5.96)\), we infer that for any point \((u, v) \in W \cap \{v \leq v_2\}\):

\[(5.99) \quad \left| \log \left( \chi_n(u, v) \right) - \log \left( \chi_{n-1}(u, v) \right) \right| \leq C_0 e^{-\frac{1}{2} C_0^2 M} D_n \]
In view of (5.70), (5.73) and (5.74) for \( u_*, v \leq v \leq v_2 + u_* \) for any \( 0 \leq u_* \leq u_0 \), we obtain for any point \((u_*, v) \in \mathcal{W}\):

\[
(5.100)
\]

\[
\log(\tilde{\varphi}(u_*, v)) = \log(\tilde{\varphi}(u_*, v_1 + u_*)) + \int_{v_1 + u_*}^{u} \sqrt{-\frac{\Lambda}{3}} F_1\big(\tan \varphi_{n-1} ; \sqrt{-\frac{\Lambda}{3} \bar{m}_{n-1}} \big) x_{n-1}^{-1} \bar{x}_{n-1}(u_*, \bar{v}) \, d\bar{v} = 
\]

\[
= \log(\tilde{\varphi}(u_*, v_1 + u_*)) + \int_{v_1 + u_*}^{\min\{v_2, v\}} \sqrt{-\frac{\Lambda}{3}} F_1\big(\tan \varphi_{n-1} ; \sqrt{-\frac{\Lambda}{3} \bar{m}_{n-1}} \big) x_{n-1}^{-1} \bar{x}_{n-1}(u_*, \bar{v}) \, d\bar{v} + 
\]

\[
+ \int_{\min\{v_2, v\}}^{u} \sqrt{-\frac{\Lambda}{3}} F_1\big(\tan \varphi_{n-1} ; \sqrt{-\frac{\Lambda}{3} \bar{m}_{n-1}} \big) x_{n-1}^{-1} \bar{x}_{n-1}(u_*, \bar{v}) \, d\bar{v} =
\]

\[
= \log(\tilde{\varphi}(u_*, v_1 + u_*)) + \int_{v_1 + u_*}^{\min\{v_2, v\}} \int_{0}^{u_*} \partial_\bar{v} \left\{ \sqrt{-\frac{\Lambda}{3}} F_1\big(\tan \varphi_{n-1} ; \sqrt{-\frac{\Lambda}{3} \bar{m}_{n-1}} \big) x_{n-1}^{-1} \bar{x}_{n-1}(\bar{u}, \bar{v}) \right\} d\bar{d} d\bar{v} +
\]

\[
+ \int_{v_1 + u_*}^{\min\{v_2, v\}} \sqrt{-\frac{\Lambda}{3}} F_1\big(\tan \varphi_{n-1} ; \sqrt{-\frac{\Lambda}{3} \bar{m}_{n-1}} \big) x_{n-1}^{-1} \bar{x}_{n-1}(0, \bar{v}) \, d\bar{v} +
\]

\[
+ \int_{\min\{v_2, v\}}^{u} \sqrt{-\frac{\Lambda}{3}} F_1\big(\tan \varphi_{n-1} ; \sqrt{-\frac{\Lambda}{3} \bar{m}_{n-1}} \big) x_{n-1}^{-1} \bar{x}_{n-1}(u_*, \bar{v}) \, d\bar{v}.
\]

In view of (5.78) and (5.98), we can bound

\[
(5.101)
\]

\[
\left| \log(\tilde{\varphi}(u_*, v_1 + u_*)) \right| \leq M(1 + e^{-\frac{\epsilon}{2} C_0^2 \mathcal{M}}).
\]

In view of (5.70), (5.73) and (5.74) for \( n-1 \) in place of \( n \), as well as (5.82), (5.83), (5.85) and (5.53), we can readily
bound for all \((u, v) \in \mathcal{W}\):

\[
(5.102) \quad \int_{v_1 + u_s}^{u_s} \int_0^{u_s} \partial_u \left\{ \sqrt{-\frac{\Lambda}{3}} F_1(\tan \varphi_{n-1} ; -\frac{\Lambda}{3} \tilde{u} \tilde{v}) \right\} d\tilde{u} d\tilde{v} \leq \]

\[
\leq C_1 e^{10C_0M} (-\Lambda) \int_{v_1 + u_s}^{u_s} \int_0^{u_s} \frac{\partial \tilde{u}}{\tan^2 \varphi_{n-1} + \frac{\partial \tilde{v}}{\tan \varphi_{n-1}}} d\tilde{u} d\tilde{v} \leq \]

\[
\leq C_1 e^{20C_0M} (-\Lambda) \left( \int_{v_1 + u_s}^{u_s} \int_0^{u_s} \frac{1}{\partial \tilde{u}} \frac{\partial \tilde{v}}{\partial \tilde{v}} d\tilde{u} d\tilde{v} + \sup_{0 \leq \tilde{u} \leq u_s} \int_{v_1 + u_s}^{u_s} \frac{\partial \tilde{u}}{\partial \tilde{v}} d\tilde{v} \right) \leq \]

\[
\leq C_1 e^{20C_0M} \left( e^{10C_0M} \int_{v_1 + u_s}^{u_s} \frac{u_s}{(r_0 + \tilde{v} - v_1 - u_s)(r_0 + \tilde{v} - v_1)} \frac{\partial \tilde{v}}{\partial \tilde{v}} d\tilde{v} + e^{-C_0^2 M} \right) \leq \]

\[
\leq C_1 e^{30C_0M} \left( \int_{v_1 + u_s}^{u_s + c_0^{3/2} u_s} \frac{u_s}{(r_0 + \tilde{v} - v_1 - u_s)(r_0 + \tilde{v} - v_1)} \frac{\partial \tilde{v}}{\partial \tilde{v}} d\tilde{v} + e^{-C_0^2 M} \right) \leq \]

for some absolute constant \(C_1 > 0\).\(^6\) Thus, in view of (5.22) and (5.25), (5.102) implies (provided \(C_0 \gg C_1\)) that

\[
(5.103) \quad \left\| \int_{v_1 + u_s}^{u_s} \int_0^{u_s} \partial_u \left\{ \sqrt{-\frac{\Lambda}{3}} F_1(\tan \varphi_{n-1} ; -\frac{\Lambda}{3} \tilde{u} \tilde{v}) \right\} d\tilde{u} d\tilde{v} \right\| \leq 10 e^{-\frac{3}{4} C_0^2 M} M. \]

In view of (5.76), (5.21) and (5.71) for \(n-1\) in place of \(n\) and \(u = 0\), we can estimate

\[
(5.104) \quad \left\| \int_{v_1 + u_s}^{u_s} \sqrt{-\frac{\Lambda}{3}} F_1(\tan \varphi_{n-1} ; -\frac{\Lambda}{3} \tilde{u} \tilde{v}) \frac{\partial \tilde{v}}{\partial \tilde{v}} d\tilde{v} \right\| \leq 2 \sup_{v_1 \leq \tilde{v} \leq u_s} \left| \log (\tilde{x}(0, v)) \right| \leq 2M. \]

Furthermore, inequality

\[
(5.105) \quad v - \min \{v_2, v\} \leq u_s \leq u_0 \]

combined with (5.83), (5.82), (5.92) and (5.93) yields

\[
(5.106) \quad \left\| \int_{v_1 + u_s}^{u_s} \sqrt{-\frac{\Lambda}{3}} F_1(\tan \varphi_{n-1} ; -\frac{\Lambda}{3} \tilde{u} \tilde{v}) \frac{\partial \tilde{v}}{\partial \tilde{v}} d\tilde{v} \right\| \leq 10 e^{-\frac{3}{4} C_0^2 M} M. \]

Combining (5.101), (5.103), (5.104) and (5.106), the relation (5.100) readily yields for any \((u, v) \in \mathcal{W}\):

\[
(5.107) \quad \left| \log (\tilde{x}(u, v)) \right| \leq 3M \left( 1 + 10 e^{-\frac{3}{4} C_0^2 M} \right). \]

\(^6\)Note that in passing from the second to the third line of (5.102), we have used the fact that \(\partial_u \xi = 0\) and \(\partial_v \tau = 0\).
Subtracting from \(5.100\) the same relation for \(n - 1\) in place of \(n\) and using (similarly as before) \((5.22), (5.25), (5.82), (5.83), (5.53)\) and \((5.99)\) (in some instances for \(n - 1\) in place of \(n\)), we obtain for any \((u, v) \in W:\)

\[
\log \left( \bar{x}_n(u, v) \right) - \log \left( \bar{x}_{n-1}(u, v) \right) \leq C_0 M e^{-C_0^2 M} D_{n-1}.
\]

Integrating equation \((5.70)\) along \(\{v_* - v_2 \leq u \leq u_0\} \cap \{v = v_*\}\) for any \(v_2 \leq v_* \leq v_2 + u_0\), we obtain for any point \((u, v) \in W \cap \{v_2 \leq v \leq v_2 + u\}:

\[
\log \left( x_n(u, v) \right) = \log \left( x_n(v - v_2, v) \right) - \int_{v - v_2}^u \sqrt{-\frac{A}{3}} F_1 \left( \tan \varphi_{n-1}; \sqrt{-\frac{A}{3} \bar{m}_{n-1}} \bar{x}_{n-1} \tau_{n-1}(\bar{u}, \bar{v}) \right) du.
\]

Thus, in view of \((5.107)\) for \(n - 1\) in place of \(n\), \((5.25), (5.82), (5.83)\) and \((5.53)\), using also \((5.78)\) and \((5.107)\) for the first term of the right hand side of \((5.109)\), the relation \((5.109)\) yields for any point \((u, v) \in W \cap \{v_2 \leq v \leq v_2 + u\}:

\[
\log \left( x_n(u, v) \right) \leq 3 M (1 + 10 e^{-\frac{1}{2} C_0^2 M}).
\]

Furthermore, subtracting from equation \((5.109)\) the same equation for \(n - 1\) in place of \(n\), and using \((5.92), (5.82), (5.83), (5.53), (5.85), (5.78)\) and \((5.108)\) (in some instances for \(n - 1\) in place of \(n\)), we infer for any point \((u, v) \in W \cap \{v_2 \leq v \leq v_2 + u\}:

\[
\log \left( x_n(u, v) \right) \leq C_0 M e^{-C_0^2 M} (D_{n-1} + D_{n-2}).
\]

Combining \((5.98), (5.99), (5.110)\) and \((5.111)\), we thus deduce that, for any \((u, v) \in W:\)

\[
\log \left( x_n(u, v) \right) \leq 3 M (1 + 10 e^{-\frac{1}{2} C_0^2 M})
\]

and

\[
\log \left( x_n(u, v) \right) - \log \left( x_{n-1}(u, v) \right) \leq C_0 M e^{-C_0^2 M} (D_{n-1} + D_{n-2}).
\]

Setting

\[
\varrho_n = \partial_v \varphi_n
\]

and

\[
\bar{\varrho}_n = -\partial_u \varphi_n,
\]

equation \((5.72)\) is equivalent to the system

\[
\partial_u \varrho_n = \left( -\frac{A}{3} \right) F_2 \left( \tan \varphi_{n-1}; \sqrt{-\frac{A}{3} \bar{m}_{n-1}} \bar{x}_{n-1} \bar{x}_{n-1}, \right),
\]

\[
\partial_v \bar{\varrho}_n = \left( -\frac{A}{3} \right) F_2 \left( \tan \varphi_{n-1}; \sqrt{-\frac{A}{3} \bar{m}_{n-1}} \bar{x}_{n-1} \bar{x}_{n-1}, \right),
\]

while the initial data \((5.76)\) and the boundary conditions \((5.77)\) gives rise to the conditions

\[
\sqrt{-\frac{12}{A}} \varrho_n(0, v) = \frac{2 \partial_v r_f}{1 - \frac{1}{3} Ar_f^2}(v)
\]

and

\[
\left. \frac{\bar{\varrho}_n}{\varrho_n} \right|_{\gamma_o} = 1, \left. \frac{\varrho_n}{\bar{\varrho}_n} \right|_{\gamma_o} = 1.
\]
Notice the similarity of (5.110), (5.117) and (5.119) with (5.70), (5.71) and (5.78), respectively. In particular, by repeating the same arguments that led to (5.107), (5.108), (5.112) and (5.113) (treating the regions \( \mathcal{W} \cap \{ v \leq v_2 \} \) and \( \mathcal{W} \cap \{ v_2 \leq v \leq v_2 + u \} \) separately, as was done in the case of \( \chi_n \)), we readily obtain for any \((u, v) \in \mathcal{W}\) (provided \( \delta_M \) is small enough in terms of \( \rho_0, M \)):

\[
\log \left( \sqrt{-\Lambda} \theta_n(u, v) \right) \leq 3M(1 + 10e^{-\frac{1}{2}C_0^2 M}),
\]

(5.120)

\[
\log \left( \sqrt{-\Lambda} \tilde{\theta}_n(u, v) \right) \leq 3M(1 + 10e^{-\frac{1}{2}C_0^2 M}),
\]

(5.121)

\[
\left| \log \left( \sqrt{-\Lambda} \theta_n(u, v) \right) - \log \left( \sqrt{-\Lambda} \theta_{n-1}(u, v) \right) \right| \leq C_0 M e^{-C_0^2 M} (\mathcal{D}_{n-1} + \mathcal{D}_{n-2}),
\]

(5.122)

and

\[
\left| \log \left( \sqrt{-\Lambda} \tilde{\theta}_n(u, v) \right) - \log \left( \sqrt{-\Lambda} \tilde{\theta}_{n-1}(u, v) \right) \right| \leq C_0 M e^{-C_0^2 M} (\mathcal{D}_{n-1} + \mathcal{D}_{n-2}).
\]

(5.123)

Integrating (5.115) in \( u \) and using (5.25), (5.76), (5.121) and (5.123), we readily obtain that, for any \((u, v) \in \mathcal{W} \cap \{ v \leq v_2 \} \):

\[
|\varphi_n(u, v) - \varphi_n(0, v)| \leq 4M e^{-C_0^2 M}
\]

(5.124)

and

\[
|\varphi_n(u, v) - \varphi_{n-1}(u, v)| \leq C_0 M e^{-C_0^2 M} (\mathcal{D}_{n-1} + \mathcal{D}_{n-2}).
\]

(5.125)

Integrating (5.114) in \( v \) in the region \( \mathcal{W} \cap \{ v_2 \leq v \leq v_2 + u \} \) and using (5.25), (5.124) and (5.125) (for \( v = v_2 \)), we readily obtain for any \((u, v) \in \mathcal{W} \cap \{ v_2 \leq v \leq v_2 + u \} \):

\[
|\varphi_n(u, v) - \varphi_n(0, v_2)| \leq 4M e^{-C_0^2 M}
\]

(5.126)

and

\[
|\varphi_n(u, v) - \varphi_{n-1}(u, v)| \leq C_0 M e^{-C_0^2 M} (\mathcal{D}_{n-1} + \mathcal{D}_{n-2}).
\]

(5.127)

By integrating equation (5.73) in \( u \) and using (5.76), (5.80), (5.92), (5.21), (5.82), (5.112) and (5.113), we readily obtain that

\[
\sqrt{-\Lambda} |\tilde{\varphi}_n| \leq M(1 + 10e^{-C_0^2 M})
\]

(5.128)

and

\[
\sqrt{-\Lambda} |\tilde{\varphi}_n - \tilde{\varphi}_{n-1}| \leq C_0 M e^{-C_0^2 M} (\mathcal{D}_{n-1} + \mathcal{D}_{n-2}).
\]

(5.129)

From (5.76), (5.21), (5.124), (5.126) and (5.128), we also obtain for all \((u, v) \in \mathcal{W}\):

\[
\left| \log \left( \frac{1 - \frac{2\sqrt{-\Lambda}\tilde{\varphi}_{n-1}}{\tan \varphi_n} + \frac{1}{3} \tan^2 \varphi_n}{1 + \frac{2}{3} \tan^2 \varphi_n} \right) \right| \leq M(1 + 10e^{-C_0^2 M})
\]

(5.130)
Finally, from (5.96), (5.21), (5.22), (5.77), (5.120) and (5.121) we can readily estimate

\[
(5.131) \quad \sup_{0 \leq s \leq u_0} \int_{W \cap \{u = v\}} \frac{1}{\tau_n} dv + \sup_{0 \leq s \leq u_0} \sup_{v \in [v_1, v_1 + \hat{u}]} \int_{\min(v - u_0, v_1 + \hat{u})}^{\max(v - u_0, v_1 + \hat{u})} \left| \frac{\tau_n}{\bar{\epsilon} - v} + r_0 \right| dv \leq 10M(1 + 10e^{-C_\sigma^2 M}).
\]

The bound (5.86) readily follows from (5.107), (5.112), (5.120), (5.121), (5.128), (5.130) and (5.131), while the bound (5.90) follows from (5.96), (5.108), (5.113), (5.122), (5.123), (5.125), (5.127) and (5.129). The estimate (5.87) follows from (5.22), (5.23), (5.96), (5.77), (5.120) and (5.121). Finally, the bound (5.88) follows from (5.95), while the lower bound (5.89) follows immediately from the fact that \( \varrho_n > 0 \) (in view of (5.121)) and the boundary condition (5.77).

**Step 2.** The second step of the proof will consist of showing that the smooth solution constructed in the previous step is actually unique. Let \((\varphi_1, \chi_1, \tau_1, \bar{\tau}_1, \bar{m}_1)\) and \((\varphi_2, \chi_2, \tau_2, \bar{\tau}_2, \bar{m}_2)\) be two \(C^1\) solutions of the system (5.54)–(5.59) on \(W\) satisfying (5.66)–(5.68), such that \((\varphi_1, \chi_1, \tau_1, \bar{\tau}_1, \bar{m}_1)\) is the solution constructed in the previous step. In particular, \((\varphi_1, \chi_1, \tau_1, \bar{\tau}_1, \bar{m}_1)\) satisfies the bounds (5.66)–(5.68), and the two solutions satisfy the same initial data, i.e.

\[
(5.132) \quad (\varphi_1, \chi_1, \bar{m}_1, \bar{\tau}_1)(0) \times \{v_1, v_2\} = (\varphi_2, \chi_2, \bar{m}_2, \bar{\tau}_2)(0) \times \{v_1, v_2\}.
\]

Let \(\mathcal{B}\) be the set of all subsets \(\mathcal{B}\) of \(W\) satisfying the following properties:

1. The closure \(\text{clos}(\mathcal{V})\) of \(\mathcal{V}\) (in the ambient topology of \(\mathbb{R}^2\)) contains \(\{0\} \times \{v_1, v_2\}\).
2. For any \((u_*, v_*) \in \mathcal{V}\), the line segments \(\{u = u_*\} \cap \{v_1 + u_* \leq v \leq v_2 + u_*\}\) and \(\{0 \leq u \leq u_*\} \cap \{v = v_*\}\) are contained in \(\mathcal{V}\).

Let \(W_0 \in \mathcal{B}\) be a subset of \(W\), which is closed in the induced topology of \(W \subset \mathbb{R}^2\), such that

\[
(5.133) \quad (\varphi_1, \chi_1, \bar{m}_1, \bar{\tau}_1)(W_0) = (\varphi_2, \chi_2, \bar{m}_2, \bar{\tau}_2)(W_0),
\]

Since (5.133) implies automatically that \((\varphi_2, \chi_2, \bar{m}_2, \bar{\tau}_2, \bar{m}_2)\) satisfies the bounds (5.66)–(5.68) on \(W_0\), we can always find an set \(\mathcal{V} \supset W_0\) with \(\mathcal{V} \in \mathcal{B}\), such that \(\mathcal{V}\) is open in the induced topology of \(W \subset \mathbb{R}^2\) and \((\varphi_2, \chi_2, \bar{m}_2, \bar{\tau}_2, \bar{m}_2)\) satisfies (5.66)–(5.68) on \(\mathcal{V}\). Therefore, it suffices to show that (5.133) holds on any such set \(\mathcal{V}\), since this will imply that \(W_0 \subset \mathcal{V}\) and, therefore, \(W_0 = \mathcal{V}\).

Let \(\mathcal{V} \in \mathcal{B}\) such that \((\varphi_2, \chi_2, \bar{m}_2, \bar{\tau}_2, \bar{m}_2)\) satisfies the bounds (5.66)–(5.68). Subtracting equations (5.54)–(5.59) for \((\varphi_2, \chi_2, \bar{m}_2, \bar{\tau}_2, \bar{m}_2)\) from the same equations for \((\varphi_1, \chi_1, \bar{m}_1, \bar{\tau}_1, \bar{m}_1)\), we obtain:

\[
(5.134) \quad \partial_u \left( \log(x_2) - \log(x_1) \right) = -\left( -\frac{\Lambda}{3} \right) \left( F_1(\tau_2; \sqrt{-\frac{\Lambda}{3} \bar{m}_2}) x_2^{-1} \tau_2 - F_1(\tau_1; \sqrt{-\frac{\Lambda}{3} \bar{m}_1}) x_1^{-1} \tau_1 \right),
\]

\[
(5.135) \quad \partial_u \left( \log(x_2) - \log(x_1) \right) = \left( -\frac{\Lambda}{3} \right) \left( F_1(\tau_2; \sqrt{-\frac{\Lambda}{3} \bar{m}_2}) x_2^{-1} \tau_2 - F_1(\tau_1; \sqrt{-\frac{\Lambda}{3} \bar{m}_1}) x_1^{-1} \tau_1 \right),
\]

\[
(5.136) \quad \partial_u \partial_v (\varphi_2 - \varphi_1) = \left( -\frac{\Lambda}{3} \right) \left( F_2(\tau_2; \sqrt{-\frac{\Lambda}{3} \bar{m}_2}) x_2 \tau_2 - F_2(\tau_1; \sqrt{-\frac{\Lambda}{3} \bar{m}_1}) x_1 \tau_1 \right),
\]

\[
(5.137) \quad \partial_u (\bar{\tau}_2 - \bar{\tau}_1) = 4\pi \left( x_2^{-1} \tau_2 - x_1^{-1} \tau_1 \right),
\]

\[
(5.138) \quad \partial_u (\bar{m}_2 - \bar{m}_1) = 0,
\]

\[
(5.139) \quad \partial_v (\bar{\tau}_2 - \bar{\tau}_1) = 0.
\]

Integrating equations (5.134), (5.136), (5.137) and (5.138) in \(u\) and equations (5.135), (5.136) and (5.139) in \(v\), using also the initial condition (5.132), the boundary conditions (5.61)–(5.65) (noticing that (5.61) implies that \(\partial_u \varphi_i = -\partial_v \varphi_i\) on \(\gamma_0, u_0 \cap \text{clos}(\mathcal{V})\) and \(\bar{T}_u \cap \text{clos}(\mathcal{V})\), \(i = 1, 2\)) and the bounds (5.66)–(5.68) and (5.53), we obtain for
any \((u, v) \in \mathcal{V}\)

\[
\left| \log (x_2(u, v)) - \log (x_1(u, v)) \right| \leq C_{M, \rho_0, T} \sqrt{\Lambda} \left\{ \int_{\max\{0, v-v_2\}}^{u} \mathcal{D}(u, v) \, dv + \chi_{v>v_2}(v) \int_{v-v_2}^{v} \mathcal{D}(u-v_2, \bar{v}) \, d\bar{v} + + \chi_{v<v_2}(v) \int_{v}^{v-v_2} \mathcal{D}(u, v-v_v) \, dv \right\},
\]

\[
\left| \log (\bar{x}_2(u, v)) - \log (\bar{x}_1(u, v)) \right| \leq C_{M, \rho_0, T} \sqrt{\Lambda} \left\{ \int_{\max\{0, v-v_2\}}^{u} \mathcal{D}(u, v) \, dv + \int_{v}^{u} \mathcal{D}(\bar{u}, v_{1} + u) \, du \right\},
\]

\[
\left| \partial_u \bar{x}_2(u, v) - \partial_u \bar{x}_1(u, v) \right| \leq C_{M, \rho_0, T} \sqrt{\Lambda} \left\{ \int_{\max\{0, v-v_2\}}^{u} \mathcal{D}(u, v) \, dv + \int_{v}^{u} \mathcal{D}(\bar{u}, v_{1} + u) \, du \right\},
\]

\[
\left| \bar{v}_2(u, v) - \bar{v}_1(u, v) \right| = 0,
\]

\[
\left| v_2(u, v) - \bar{v}_1(u, v) \right| = 0,
\]

where

\[
\mathcal{D}(u, v) \doteq \left| \log (x_2(u, v)) - \log (x_1(u, v)) \right| + \left| \log (\bar{x}_2(u, v)) - \log (\bar{x}_1(u, v)) \right| + + \left| \partial_u \bar{x}_2(u, v) - \partial_u \bar{x}_1(u, v) \right| + + \sqrt{\Lambda} \left| \tilde{m}_2(u, v) - \tilde{m}_1(u, v) \right| + + \left| \tilde{v}_2(u, v) - \tilde{v}_1(u, v) \right| + + \left| v_2(u, v) - \bar{v}_1(u, v) \right|,
\]

\[
\chi_{v>v_2}(v) = \begin{cases} 1, & \text{if } v > v_2, \\ 0, & \text{if } v \leq v_2 \end{cases}
\]

and \(C_{M, \rho_0, T}\) depends on \(M, \rho_0\) and

\[
T_* \doteq \sup_{v_1 < v \leq v_2} \nu^2(T_{v_1})(\bar{v}).
\]

Inequalities \((5.140)-(5.146)\) now readily yield \((5.133)\).

Our next result is well-posedness result for the initial data introduced by Definition 5.2.

**Proposition 5.2.** Let \(C_0 > 1\) be a (large) absolute constant. For any \(u_1 < v_2, v_1 < v_2\) and \(r_0 > 0\), let \(r_1 : [u_1, v_2) \to [r_0, +\infty), \Omega_1 : [u_1, v_2) \to (0, +\infty), \tilde{f}_{\text{in}}, \tilde{f}_{\text{out}} : [u_1, v_2] \times (0, +\infty) \to [0, +\infty), r_f : [v_1, v_2] \to (r_0, +\infty), \Omega_f : [v_1, v_2] \to (0, +\infty) \text{ and } f_{\text{in}}, f_{\text{out}} : [v_1, v_2] \times (0, +\infty) \to [0, +\infty)\) be smooth functions, so that \((r_1, \Omega_1, \tilde{f}_{\text{in}}, \tilde{f}_{\text{out}}) \text{ and } (r_f, \Omega_f, f_{\text{in}}, f_{\text{out}})\) consist a boundary-double characteristic initial data set for the system \((2.32)-(2.37)) satisfying the reflecting boundary condition at \(r = r_0\), according to Definition 5.2. We will assume, in addition, that \(r_1\) satisfies for all \(u \in [u_1, v_2]\)

\[
\partial_u r_1(u) < 0
\]

and \(r_f\) satisfies for all \(v \in [v_1, v_2]\)

\[
\partial_v r_f(v) > 0.
\]
Note that (2.7), (5.19) and (5.149) imply that

\[ 1 - \frac{2m_{\lambda}}{r_{\lambda}} > 0, \]

while (2.7), (5.6), (5.150) and (5.149) imply that

\[ 1 - \frac{2m_{f}}{r_{f}} > 0, \]

where

\[ m_{\lambda}(v) = \frac{r_{\lambda}}{2} \left( 1 + 4 \Omega_{\lambda}^{-2} \partial_{v} r_{\lambda} (\partial_{v} r_{\lambda}) (v) \right), \]

\[ m_{f}(v) = \frac{r_{f}}{2} \left( 1 + 4 \Omega_{f}^{-2} (\partial_{v} r_{f}) (\partial_{v} r_{f}) (v) \right) \]

and \((\partial_{v} r_{\lambda})(\partial_{v} r_{f})\), are defined according to the formulas (5.13) and (5.14), respectively. Let us also set

\[ M \doteq \max_{u \in [u_{1}, u_{2}]} \left\{ \log \left( \frac{\Omega_{\lambda}^{2}}{1 - \frac{2m_{\lambda}}{r_{\lambda}}} \right) + \left| \log \left( \frac{-2 \partial_{v} r_{\lambda}}{1 - \frac{2m_{\lambda}}{r_{\lambda}}} \right) \right| + \left| \log \left( \frac{1 - \frac{2m_{\lambda}}{r_{\lambda}}}{1 - \frac{1}{3} \Lambda_{\lambda}^{2}} \right) \right| \right\}(u) + \int_{u_{1}}^{u_{2}} r_{\lambda}(T_{uu}) \, d\bar{u} + \]

\[ + \max_{v \in [v_{1}, v_{2}]} \left\{ \log \left( \frac{\Omega_{f}^{2}}{1 - \frac{2m_{f}}{r_{f}}} \right) + \left| \log \left( \frac{-2 \partial_{v} r_{f}}{1 - \frac{2m_{f}}{r_{f}}} \right) \right| + \left| \log \left( \frac{1 - \frac{2m_{f}}{r_{f}}}{1 - \frac{1}{3} \Lambda_{f}^{2}} \right) \right| + \sqrt{-\Lambda_{f}} |\hat{m}_{f}| \left( v \right) \right\}(v) \]

and, for any \(0 < \delta < 1\):

\[ u_{in}(\delta) \doteq \sup \left\{ 0 \leq u_{\ast} \leq u_{2} - u_{1} : \sup_{u \in [u_{1}, u_{2}]} \int_{\min\{u+u_{\ast}, u_{2}\}}^{\max\{u-u_{\ast}, u_{1}\}} \frac{r_{\lambda}^{2}(T_{uu}) \left( \bar{u} \right)}{\left| \bar{u} - u \right| + r_{0}} \, d\bar{u} < \delta \right\}, \]

where

\[ m_{f}(v) = \frac{r_{f}}{2} \left( 1 + 4 \Omega_{f}^{-2} (\partial_{v} r_{f}) \partial_{v} r_{f} \right)(v), \]

\[ \hat{m}_{f}(v) = m(v) - \frac{1}{6} \Lambda_{r}^{2}(v) \]

and \((\partial_{v} r_{f})\) is defined according to (3.3). Then, provided \(v_{2} - v_{1}\) is sufficiently small so that

\[ v_{2} - v_{1} < \frac{u_{in}(2e^{-C_{0}^{2}((-\Lambda)(u_{2} - u_{1})^{2} + 1)M \bar{M}}}{e^{C_{0}^{2}((-\Lambda)(u_{2} - u_{1})^{2} + 1)M \bar{M}}}, \]

and

\[ \int_{v_{1}}^{v_{2}} r_{f}(T_{vv}) \, d\bar{v} < 2e^{-C_{0}^{2}((-\Lambda)(u_{2} - u_{1})^{2} + 1)M \bar{M}}, \]

the following holds: Setting

\[ \mathcal{V} = \left\{ u_{1} < u < v - v_{1} + u_{2} \right\} \cap \left\{ v_{1} < v < v_{2} \right\} \]

and

\[ \gamma_{0;\mathcal{V}} = \left\{ u - u_{2} = v - v_{1} \right\} \cap \left\{ v_{1} < v < v_{2} \right\}, \]

there exist smooth functions \(r: \mathcal{V} \cup \gamma_{0;\mathcal{V}} \to (r_{0}, +\infty)\), \(\Omega: \mathcal{V} \cup \gamma_{0;\mathcal{V}} \to (0, +\infty)\) and \(\bar{f}_{in}, \bar{f}_{out}: \mathcal{V} \cup \gamma_{0;\mathcal{V}} \times (0, +\infty) \to [0, +\infty)\)

solving equations (2.33), (2.37) on \(\mathcal{V}\) (with \(T_{uu}, T_{vv}\) given by the formulas (2.22), (2.23)), such that:
1. The functions \( r, \Omega^2, \tilde{f}_{\text{in}}, \tilde{f}_{\text{out}} \) satisfy the given initial conditions on \([u_1, u_2] \times \{v_1\} \) and \( \{u_1\} \times [v_1, v_2] \), i.e.: 
\[
(5.163) \quad (r, \Omega^2, \tilde{f}_{\text{in}}, \tilde{f}_{\text{out}})|_{[u_1, u_2] \times \{v_1\}} = (r|_{\gamma_0}, \Omega^2|_{\gamma_0}, \tilde{f}_{\text{in}}, \tilde{f}_{\text{out}})
\]
and
\[
(5.164) \quad (r, \Omega^2, \tilde{f}_{\text{in}}, \tilde{f}_{\text{out}})|_{\{u_1\} \times [v_1, v_2]} = (r|_{\gamma_0}, \Omega^2|_{\gamma_0}, \tilde{f}_{\text{in}}, \tilde{f}_{\text{out}}).
\]

2. The functions \( (r, \tilde{f}_{\text{in}}, \tilde{f}_{\text{out}}) \) satisfy on \( \gamma_0 \backslash \mathcal{V} \) the boundary conditions
\[
(5.165) \quad r|_{\gamma_0 \backslash \mathcal{V}} = r_0
\]
and
\[
(5.166) \quad \tilde{f}_{\text{out}}(u_*, v_*; p) = \tilde{f}_{\text{in}}(u_*, v_*; p)
\]
for all \((u_*, v_*) \in \gamma_0 \backslash \mathcal{V} \) and \( p > 0 \), as well as the reflecting gauge condition
\[
(5.167) \quad \partial_{u_0} r|_{\gamma_0 \backslash \mathcal{V}} = -\partial_{v_0} r|_{\gamma_0 \backslash \mathcal{V}}.
\]

3. The function \( r \) satisfies
\[
(5.168) \quad \sup_{\mathcal{V}} \partial_{u_0} r < 0.
\]

4. The following estimates hold on \( \mathcal{V} \):
\[
(5.169) \quad \sup_{\mathcal{V}} \left\{ \log \left( \frac{\Omega^2}{1 - \frac{1}{3} \Lambda r^2} \right) + \left| \log \left( \frac{2\partial_{u_0} r}{1 - \frac{2m}{r}} \right) \right| + \left| \log \left( \frac{1 - \frac{2m}{r}}{1 - \frac{1}{3} \Lambda r^2} \right) \right| + \sqrt{-\Lambda} |\vec{m}| \right\} + \\
\quad + \sup_{\mathcal{V}} \int_{\{x = v\} \cap \mathcal{V}} r T_{uu} du \leq C_0 M
\]
and
\[
(5.170) \quad \sup_{\mathcal{V}} \int_{\{u = \bar{u}\} \cap \mathcal{V}} r T_{uv} dv \leq e^{-C_0 M} M.
\]

Proof. As in the proof of Proposition 5.1 by introducing the new variables \((5.46) - (5.50)\), the proof of Proposition 5.2 will follow by establishing the existence and uniqueness of a smooth solution to the system \((5.54) - (5.59)\) on \( \mathcal{V} \cup \gamma_0 \backslash \mathcal{V} \) satisfying the initial conditions
\[
(5.171) \quad (\varphi, \chi, \vec{m}, \vec{r})|_{\{u_1\} \times [v_1, v_2]} = \left( \tan^{-1} \left( \sqrt{-\frac{\Lambda}{3} r^2} \right), \frac{2\partial_{u_0} r}{1 - \frac{2m}{r}}, \vec{m}, r \right)^2 (T_{uv})
\]
and
\[
(5.172) \quad (\varphi, \chi, \tau)|_{[u_1, u_2] \times \{v_1\}} = \left( \tan^{-1} \left( \sqrt{-\frac{\Lambda}{3} r^2} \right), \frac{-2\partial_{u_0} r}{1 - \frac{2m}{r}}, r \right)^2 (T_{uv})
\]
as well as the boundary conditions
\[
(5.173) \quad \varphi|_{\gamma_0 \backslash \mathcal{V}} = \varphi_0.
\]
\( \frac{\gamma}{\gamma|_{\gamma_0 \cdot \gamma}} = 1 \)

and

\( \frac{\tau}{\tau|_{\tau_0 \cdot \gamma}} = 1, \)

where \( \tau_0 \) is defined by (5.64), such that, moreover, the estimates (5.169)–(5.170) are satisfied. The proof follows in the same way as the proof of Proposition 5.1, and hence the details will be omitted.

**Proposition 5.3.** For any \( u_1 < u_2, v_1 < v_2 \) and \( 0 < r_1 < R_1 < +\infty \), let \( r_i : [u_1, u_2] \to [r_1, R_1], \Omega_i : [u_1, u_2] \to (0, +\infty), \hat{f}_{\text{in}}, \hat{f}_{\text{out}} : [u_1, u_2] \times (0, +\infty) \to [0, +\infty), r_j : [v_1, v_2] \to [r_1, R_1], \Omega_j : [v_1, v_2] \to (0, +\infty) \) and \( \tilde{f}_{\text{in}}, \tilde{f}_{\text{out}} : [v_1, v_2] \times (0, +\infty) \to (0, +\infty) \) be smooth functions consisting a characteristic initial data set for the system (2.32)–(2.37), according to Definition 5.1, satisfying, in addition, for all \( u \in [u_1, u_2] \), the sign condition

\( \partial_u r_i(u) < 0. \)

Let us also set

\[
M \triangleq \max_{u \in [u_1, u_2]} \left\{ \left| \log(-\partial_u r_i) \right| + \left| \log(\Omega_i^2) \right| + \sqrt{-\Lambda} |\tilde{m}_i| \right\}(u) + \int_{u_1}^{u_2} r_i(T_{uu}) \, d\bar{u} + \max_{v \in [v_1, v_2]} \left\{ \left| \log(-\partial_v r_j) \right| + \left| \log(\Omega_j^2) \right| + \sqrt{-\Lambda} |\tilde{m}_j| \right\}(v)
\]

where

\[
\tilde{m}_i(u) = \frac{r_i}{2} \left( 1 + 4\Omega_i^2 \partial_u r_i(\partial_u r_i)(u) - \frac{1}{6} \Lambda r_i^3(u) \right),
\]

\[
\tilde{m}_j(v) = \frac{r_j}{2} \left( 1 + 4\Omega_j^2 (\partial_v r_j)(\partial_v r_j)(v) - \frac{1}{6} \Lambda r_j^3(v) \right),
\]
and the transversal derivatives \((\partial_c r)_i, (\partial_a r)_j\) are computed according to (5.13)–(5.14), i.e.: 

\[
(\partial_c r)_i(u) = \partial_\nu r_j(v_1) - \frac{1}{4r_i(u)} \int_{v_1}^\nu (1 - \Lambda r^2(\hat{u})) \Omega^2(\hat{u}) \, d\hat{u},
\]

\[
(\partial_a r)_j(v) = \partial_\nu r_i(u_1) - \frac{1}{4r_j(v)} \int_{v_1}^\nu (1 - \Lambda r^2(\hat{v})) \Omega^2(\hat{v}) \, d\hat{v}.
\]

We will also define for any \(0 < \delta < 1\):

\[
u_2 - \nu_1 < \frac{u_{in}(2e^{-C_{r_1, r_1}((-\Lambda)(u_2 - u_1)^2 + 1)}M)}{e^{-C_{r_1, r_1}((-\Lambda)(u_2 - u_1)^2 + 1)}M}
\]

and

\[
\int_{v_1}^{\nu_2} r_j(T_{v_1}) \, dv < 2e^{-C_{r_1, r_1}((-\Lambda)(u_2 - u_1)^2 + 1)}M,
\]

where \(C_{r_1, R_1} > 1\) is a constant depending only on \(r_1\) and \(R_1\), there exist unique smooth functions \(r : [u_1, u_2] \times [v_1, v_2] \rightarrow \mathbb{R}_+\), \(\Omega : [u_1, u_2] \times [v_1, v_2] \rightarrow (0, +\infty)\) and \(f_{in}, f_{out} : [u_1, u_2] \times [v_1, v_2] \times (0, +\infty) \rightarrow (0, +\infty)\) solving equations \(2.33\)–\(2.37\) (with \(T_{uu}, T_{uv}\) given by the formulas \(2.22\), \(2.23\)), such that:

1. The functions \(r, \Omega, f_{in}, f_{out}\) satisfy the initial conditions on \([u_1, u_2] \times \{v_1\}\) and \(\{u_1\} \times [v_1, v_2]\), i.e.:

\[
(r, \Omega^2, f_{in}, f_{out})|_{[u_1, u_2] \times \{v_1\}} = (r_1, \Omega_1^2, \hat{f}_{in}, \hat{f}_{out})
\]

and

\[
(r, \Omega^2, f_{in}, f_{out})|_{\{u_1\} \times [v_1, v_2]} = (r_1, \Omega_1^2, \hat{f}_{in}, \hat{f}_{out}).
\]

2. The function \(r\) satisfies

\[
\sup_{[u_1, u_2] \times [v_1, v_2]} \partial_a r < 0.
\]

3. The following estimates hold on \([u_1, u_2] \times [v_1, v_2]\):

\[
\sup_{[u_1, u_2] \times [v_1, v_2]} \left\{ \log(1 + |\partial_c r|) + |\log(\Omega^2)| + \left| \log(-\partial_a r) \right| + \sqrt{-\Lambda} |\tilde{m}| \right\} + \sup_{\tilde{u} \in [v_1, v_2]} \int_{u_1}^{u_2} r_{T_{uu}}(\tilde{u}, \tilde{v}) \, du < C_{r_1, R_1} M
\]

and

\[
\sup_{\tilde{u} \in [u_1, u_2]} \int_{v_1}^{\nu_2} r_{T_{vv}}(\tilde{u}, \tilde{v}) \, dv \leq e^{-C_{r_1, R_1} M} M.
\]

Proof. Using the expression

\[
\frac{\partial_c r_j}{1 - \frac{2m_l}{r_j} - \frac{1}{3} \Lambda r_j^2} \leq 1 - \frac{\Omega_j^2}{4 - (\partial_a r)_j},
\]

40
the relation (5.177) implies that
\[
\max_{v \in [v_1, v_2]} \log \left(1 + |\partial_v r|\right) \leq 4M.
\]

Equations (2.32)–(2.37) for \((r, \Omega^2, \bar{f}_\text{in}, \bar{f}_\text{out})\) on \([u_1, u_2] \times [v_1, v_2]\) (together with the initial constraint (5.6)) are equivalent to the system (2.38)–(2.43) for \((r, \Omega^2, T_{uu}, T_{vv})\). In particular, if \((r, \Omega^2, T_{uu}, T_{vv})\) is a solution to the system (2.38)–(2.43) on \([u_1, u_2] \times [v_1, v_2]\) satisfying the initial conditions
\[
(r, \Omega^2, T_{uu})|_{[u_1, u_2] \times \{v_1\}} = (r, \Omega^2, (T_{uu}))
\]
and
\[
(r, \Omega^2, T_{vv})|_{[u_1, u_2] \times \{v_1, v_2\}} = (r, \Omega^2, (T_{vv}))
\]
(which are assumed to satisfy the constraints (5.5)–(5.6)), then, by solving equations (2.36) and (2.37) for \(\bar{f}_\text{in}, \bar{f}_\text{out}\) on \([u_1, u_2] \times [v_1, v_2]\) with initial data
\[
(\bar{f}_\text{in}, \bar{f}_\text{out})|_{[u_1, u_2] \times \{v_1\}} = (\bar{f}_\text{in}, \bar{f}_\text{out})
\]
and
\[
(\bar{f}_\text{in}, \bar{f}_\text{out})|_{[u_1, u_2] \times \{v_1, v_2\}} = (\bar{f}_\text{in}, \bar{f}_\text{out})
\]
one obtains a solution \((r, \Omega^2, \bar{f}_\text{in}, \bar{f}_\text{out})\) of the system (2.32)–(2.37) satisfying (5.183) and (5.184). Therefore, it suffices to establish the existence and uniqueness of a smooth solution \((r, \Omega^2, T_{uu}, T_{vv})\) to the system (2.38)–(2.43) on \([u_1, u_2] \times [v_1, v_2]\) satisfying (5.189) and (5.190).

Introducing the new set of variables
\[
\lambda = \partial_v r,
\]
\[
\tau = r^2 T_{uu},
\]
\[
\tau = r^2 T_{vv},
\]
the system (2.38)–(2.43) yields:
\[
\partial_v \lambda = -\frac{1}{2} \frac{\tilde{m}}{r} - \frac{3}{4} \Lambda r^3 \Omega^2,
\]
\[
\partial_v \lambda = \frac{\Omega^2}{2r^2} \left(1 + x^{-1}\lambda\right),
\]
\[
\partial_v \lambda = -4 \pi r^{-1} x \Omega^{-2},
\]
\[
\partial_v \lambda = 0,
\]
\[
\partial_v \tau = 0,
\]
\[
\partial_v \tau = 0,
\]
\[
\partial_v \tilde{m} = -8 \pi \Omega^{-2} \lambda \tau,
\]
\[
\partial_v r = \lambda,
\]
while the initial conditions (5.189)–(5.190) give rise to the following initial conditions for \((\lambda, \Omega^2, x, \bar{z}, \bar{\tau}, \tilde{m}, r)\):
\[
(r, \Omega^2, \tau)|_{[u_1, u_2] \times \{v_1\}} = (r, \Omega^2, (T_{uu}) )
\]
and
\[
(\lambda, \Omega^2, x, \bar{z}, \tilde{m})|_{[u_1] \times [v_1, v_2]} = \left(\partial_v r, \Omega^2, \frac{-(\partial_v r)}{\Omega^2}, r^2 (T_{vv}), \tilde{m}\right).
\]
The proof of the fact that the system (5.197)–(5.204) admits a unique smooth solution on \([u_1, u_2] \times [v_1, v_2]\) satifying the initial conditions (5.204)–(5.205), such that, moreover,

\[
\sup_{[u_1, u_2] \times [v_1, v_2]} \left\{ \log \left( 1 + |\lambda| \right) + \log(\Omega^2) \right\} + \sup_{[u_1, u_2] \times [v_1, v_2]} \int_{u_1}^{u_2} \frac{\bar{r}(u, \bar{v})}{r} du < C_{r_1 R_1} M
\]

and

\[
\sup_{\bar{u} \in [u_1, u_2]} \int_{v_1}^{v_2} \frac{\bar{r}(\bar{u}, v)}{r} dv \leq e^{-C_{r_1 R_1} M M}
\]

hold, follows by similar arguments as in the proof of Proposition 5.1. We will omit the details.

\[\square\]

### 5.3 Continuation criteria

In this Section, we will establish two continuation criteria, i.e. conditions under which smooth solutions \((r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})\) of (2.32)–(2.37) on open regions \(U \subset \mathbb{R}^2\) admit a smooth extension across \(\partial U\).

The following lemma is a continuation criterion away from \(r = 0\) and \(r = +\infty\) (cf. the extension principle for the spherically symmetric Einstein–massive Vlasov system, not reduced to the radial case, in [14] and the generalised extension principle for strongly tame matter models in [29]).

**Lemma 5.1.** For any \(u_1 < u_2\) and \(v_1 < v_2\), let \((r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})\) be a smooth solution of the system (2.32)–(2.37) on \([u_1, u_2] \times [v_1, v_2]\), such that \((r, \log(\Omega^2))|[u_1, u_2] \times [v_1]\) and \((r, \log(\Omega^2))|[u_1, u_2] \times [v_2]\) extend smoothly on \([u_2] \times [v_1]\) and \([u_1] \times [v_2]\), respectively, and \((\bar{f}_{in}, \bar{f}_{out})|[u_1, u_2] \times [v_1]\times(0, +\infty)\) and \((\bar{f}_{in}, \bar{f}_{out})|[u_1, u_2] \times [v_2]\times(0, +\infty)\) extend smoothly on \([u_2] \times [v_1] \times (0, +\infty)\) and \([u_1] \times [v_2] \times (0, +\infty)\), respectively. Assume, moreover, that

\[
\inf_{[u_1, u_2] \times [v_1, v_2]} r > 0
\]

Figure 5.3: Schematic depiction of the domain of definition \([u_1, u_2] \times [v_1, v_2]\) of \((r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})\) in Lemma 5.1. The functions \((r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})\) are assumed to extend smoothly on \([u_2] \times [v_1]\) and \([u_1] \times [v_2]\) and, under the conditions (5.208) and (5.209), it is shown that they extend smoothly on \([u_2] \times [v_1, v_2]\) and \([u_1, u_2] \times [v_2]\).
and

\[(5.209)\]

\[\sup_{[u_1,u_2] \times [v_1,v_2]} r < +\infty.\]

Then, \((r, \log(\Omega^2))\) extend smoothly on the whole of \([u_1,u_2] \times [v_1,v_2]\) and \((\bar{f}_{in}, \bar{f}_{out})\) extend smoothly on the whole of \([u_1,u_2] \times [v_1,v_2] \times (0, +\infty)\).

**Proof.** It suffices to show that \((r, \log(\Omega^2))\) extend smoothly on the whole of \([u_1,u_2] \times [v_1,v_2]\), since then the smooth extension of \((\bar{f}_{in}, \bar{f}_{out})\) will readily follow by integrating equations \((2.36)–(2.37)\). In fact, we will only show that \((r, \log(\Omega^2))\) extend continuously on the whole of \([u_1,u_2] \times [v_1,v_2]\). Assuming that \((r, \log(\Omega^2)) \in C^0([u_1,u_2] \times [v_1,v_2])\), equation \((2.32)\), combined with the fact that \(r|_{[u_1,u_2] \times [v_1]}\) and \(r|_{[u_1] \times [v_1,v_2]}\) extend smoothly on \([u_2] \times [v_1]\) and \([u_1] \times [v_2]\), implies that \(\partial_r, \partial_v\) are continuous on \([u_1,u_2] \times [v_1,v_2]\). Similarly, equation \((2.33)\) yields, in turn, that \(\partial_r, \log(\Omega^2), \partial_v, \log(\Omega^2)\) are continuous on \([u_1,u_2] \times [v_1,v_2]\). Commuting \((2.32)–(2.33)\) successively with \(\partial_u, \partial_v\) and treating the commuted equations as linear equations in the highest order terms, the smoothness of \((r, \log(\Omega^2))\) then follows readily.

In view of \((2.6), (2.44), (2.22), (2.23)\), the system \((2.32)–(2.37)\) yields

\[(5.210)\]

\[\partial_u \partial_v (r^2) = -\frac{1}{2} (1 - \Lambda r^2) \Omega^2,\]

\[(5.211)\]

\[\partial_u \partial_v \log(\Omega^2) = (\check{m} r^{-3} + \frac{1}{6} \Lambda) \Omega^2,\]

\[(5.212)\]

\[\partial_u \partial_v \check{m} = 8 \pi^2 r^{-1} \Omega^{-2} (r^2 T_{uu})(r^2 T_{vv}),\]

\[(5.213)\]

\[\partial_u (\Omega^2 \partial_u r) = 4 \pi r T_{v\nu} \Omega^{-2},\]

\[(5.214)\]

\[\partial_u (\Omega^2 \partial_v r) = 4 \pi r T_{u\nu} \Omega^{-2},\]

\[(5.215)\]

\[\partial_u (r^2 T_{uu}) = 0,\]

\[(5.216)\]

\[\partial_u (r^2 T_{vv}) = 0.\]

Integrating equations \((5.210)–(5.212)\) we obtain for any \((u, v) \in [u_1,u_1] \times [v_1,v_2]\):

\[(5.217)\]

\[r^2(u, v) = r^2(u_1, v_1) + r^2(u, v_1) - r^2(u_1, v_1) - \frac{1}{2} \int_{u_1}^{u} \int_{v_1}^{v} (1 - \Lambda r^2) \Omega^2 d\bar{v}d\bar{u},\]

\[(5.218)\]

\[\log(\Omega^2(u, v)) = \log(\Omega^2(u_1, v_1)) + \log(\Omega^2(u, v_1)) - \log(\Omega^2(u_1, v_1)) + \int_{u_1}^{u} \int_{v_1}^{v} (\check{m} r^{-3} + \frac{1}{6} \Lambda) \Omega^2 d\bar{v}d\bar{u},\]

\[(5.219)\]

\[\check{m}(u, v) = \check{m}(u_1, v_1) + \check{m}(u, v_1) - \check{m}(u_1, v_1) + 8 \pi^2 \int_{u_1}^{u} \int_{v_1}^{v} r^{-1} \Omega^{-2} (r^2 T_{uu})(r^2 T_{vv}) d\bar{v}d\bar{u}.\]

Since \((r, \log(\Omega^2))|_{[u_1,u_2] \times [v_1]}\) and \((r, \log(\Omega^2))|_{[u_1] \times [v_1,v_2]}\) extend smoothly on \([u_2] \times [v_1]\) and \([u_1] \times [v_2]\), the functions \(r(u, v_1), \log(\Omega^2(u_1, v))\) and \(\check{m}(u_1, v)\) extend continuously to \(v = v_2\), while the functions \(r(u_1, v), \log(\Omega^2(u, v_1))\) and \(\check{m}(u, v_1)\) extend continuously to \(u = u_2\). Therefore, in view of \((5.208), (5.209), (5.217)–(5.219)\), the continuous extension of \(r, \log(\Omega^2), \check{m}\) on the whole of \([u_1,u_2] \times [v_1,v_2]\) will follow if we establish

\[(5.220)\]

\[\sup_{[u_1,u_2] \times [v_1,v_2]} \int_{u_1}^{u} \int_{v_1}^{v} \Omega^2 d\bar{v}d\bar{u} < +\infty,\]

\[(5.221)\]

\[\sup_{[u_1,u_2] \times [v_1,v_2]} |\check{m}(u, v)| < +\infty\]

and

\[(5.222)\]

\[\sup_{[u_1,u_2] \times [v_1,v_2]} (T_{uu} + T_{vv}) < +\infty.\]
Since \((r, \log(\Omega^2))|_{(u_1, u_2) \times \{v_1\}}\) and \((r, \log(\Omega^2))|_{(u_1) \times \{v_1, v_2\}}\) extend smoothly on \(\{u_2\} \times \{v_1\}\) and \(\{u_1\} \times \{v_2\}\), from \(5.208\), \(5.213\) and \(5.214\) we infer that

\[
(5.223) \quad \sup_{(u_1, u_2) \times \{v_1\}} T_{uu} + \sup_{(u_1) \times \{v_1, v_2\}} T_{vv} < +\infty,
\]

Integrating \(5.215\)–\(5.216\) and using \(5.223\) and \(5.220\), we readily infer that \(r^2 T_{uu}\) and \(r^2 T_{vv}\) extend continuously on the whole of \([u_1, u_2] \times [v_1, v_2]\) and satisfy \(5.222\). In view of \(5.220\), equation \(5.217\) yields

\[
(5.224) \quad \sup_{(u_1, u_2) \times \{v_1, v_2\}} \int_{u_1}^{u} \int_{v_1}^{v} (1 - \Lambda r^2) \Omega^2 \, d\tilde{u} < +\infty
\]

and, therefore, \(5.220\). Thus, it only remains to establish \(5.221\).

In view of the fact that \(T_{uu}, T_{vv} \geq 0\) and \(\tilde{m}\) extends smoothly on \(\{u_2\} \times \{v_1\}\) and \(\{u_1\} \times \{v_2\}\), equation \(5.219\) implies that

\[
(5.225) \quad \inf_{(u_1, u_2) \times \{v_1, v_2\}} \tilde{m} > -\infty.
\]

Equation \(5.218\) then yields that

\[
(5.226) \quad \inf_{(u_1, u_2) \times \{v_1, v_2\}} \left( \log(\Omega^2(u,v)) \right) \geq \inf_{(u_1, u_2) \times \{v_1, v_2\}} \left( \log(\Omega^2(u,v)) + \log(\Omega^2(u_1, v)) - \log(\Omega^2(u_1, v_1)) + \left( \left( \inf_{(u_1, u_2) \times \{v_1, v_2\}} \tilde{m} \right) \left( \inf_{(u_1, u_2) \times \{v_1, v_2\}} r \right)^{-3} + \frac{1}{6} \Lambda \right) \sup_{(u_1, u_2) \times \{v_1, v_2\}} \int_{u_1}^{u} \int_{v_1}^{v} \Omega^2 \, d\tilde{u} \right) > -\infty
\]

in view of \(5.208\), \(5.220\) and \(5.225\). Returning to equation \(5.219\) and considering the supremum of the right hand side, we infer:

\[
(5.227) \quad \sup_{(u_1, u_2) \times \{v_1, v_2\}} \tilde{m}(u,v) \leq \sup_{(u_1, u_2) \times \{v_1, v_2\}} \left( \tilde{m}(u_1,v) + \tilde{m}(u,v_1) - \tilde{m}(u_1,v_1) \right) + 32\pi^2 \left( \inf_{(u_1, u_2) \times \{v_1, v_2\}} r \right)^{-1} e^{-\inf_{(u_2, u_2) \times \{v_1, v_2\}} \log(\Omega^2(u_2 - u_1, v_2 - v_1))}.
\]

Thus, \(5.208\), \(5.209\), \(5.222\), \(5.226\) and \(5.227\) imply that

\[
(5.228) \quad \sup_{(u_1, u_2) \times \{v_1, v_2\}} \tilde{m}(u,v) < +\infty.
\]

Therefore, \(5.221\) follows from \(5.225\) and \(5.228\). \(\square\)

The following lemma is a continuation criterion on the mirror \(\gamma_0\).

**Lemma 5.2.** For any \(u_1 < u_2\) and \(v_1 < v_2\) such that \(u_2 - u_1 = v_2 - v_1\), let

\[
(5.229) \quad \gamma_{0;u_1 u_2} \doteq \{ u_1 \leq u < u_2 \} \cap \{ v = u \}
\]

and

\[
(5.230) \quad D \doteq \{ u_1 \leq u \leq u_2 \} \cap \{ v_1 \leq v \leq u \}.
\]

For any \(r_0 > 0\), let \((r, f^2, \bar{f}_{in}, \bar{f}_{out})\) be a smooth solution of the system \((2.32)\)–\((2.37)\) on \(D \setminus \{u_2\} \times \{ v_2\}\), such that

\[
(5.231) \quad r|_{\gamma_{0;u_1 u_2}} = r_0,
\]

and
Figure 5.4: Schematic depiction of the domain $D$ in Lemma 5.2. The functions $(r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ are assumed to be smooth on $D \setminus \{u_2\} \times \{v_2\}$ and, under the condition $\delta > 0$, it is shown that they extend smoothly on $\{u_2\} \times \{v_2\}$.

(5.232) \[
\partial_u r|_{\gamma_0; u_1, u_2} = -\partial_u r|_{\gamma_0; u_1, u_2}
\]

and $\bar{f}_{\text{in}}, \bar{f}_{\text{out}}$ satisfy the reflecting boundary condition (2.57) on $\gamma_{0; u_1, u_2}$ with $w = r - r_0$. Then, provided

(5.233) \[
\inf_{\gamma_{0; u_1, u_2}} \partial_u r > 0,
\]

the pair $(r, \log(\Omega^2))$ extends smoothly on the whole of $D$ and the pair $(\bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ extends smoothly on the whole of $D \times (0, +\infty)$.

Proof. It suffices to show that

(5.234) \[
M \geq \sup_{D \setminus \{u_2\} \times \{v_2\}} \left\{ \left| \log \Omega^2 \right| + \left| \log \left( \frac{r}{r_0} \right) \right| + \left| \log(-\partial_u r) \right| + \left| \log(\partial_u r) \right| + \left| \log \left( 1 - \frac{2m}{r} \right) \right| + \sqrt{\Lambda} |\bar{m}| + r^2 T_{uv} + r^2 T_{uu} \right\} < +\infty.
\]

Provided (5.234) has been established, in view of the fact that $(r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ are smooth on $D \setminus \{u_2\} \times \{v_2\}$, choosing some $\delta > 0$ small enough in terms of $M, r_0$ and $v_2 - v_1$, we can smoothly extend the boundary-double characteristic initial data set

\[
(r_j, \Omega_j^2, \bar{f}_{\text{in}}^j, \bar{f}_{\text{out}}^j) \mid_{x = u_1} = (r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})
\]

and

\[
(r_\gamma, \Omega_\gamma^2, \bar{f}_{\text{in}}^\gamma, \bar{f}_{\text{out}}^\gamma) \mid_{x = v_2 - \delta} = (r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})
\]

induced by $(r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ on $\{(u_1, u_2 - \delta) \times \{v_2 - \delta\}\} \cup \{(v_1) \times \{v_2 - \delta, v_2\}\}$ (see Definition 5.2), to a boundary-double characteristic initial data set $(r_j, \Omega_j^2, \bar{f}_{\text{in}}^j, \bar{f}_{\text{out}}^j), (r_\gamma, \Omega_\gamma^2, \bar{f}_{\text{in}}^\gamma, \bar{f}_{\text{out}}^\gamma)$ on $\{(u_1, u_2 - \delta) \times \{v_2 - \delta\}\} \cup \{(u_1) \times \{v_2 - \delta, v_2 + \delta\}\}$.
satisfying

\[
\max_{u \in [u_1, u_2-\delta]} \left\{ \left( \log \left( \frac{\Omega^2_{\gamma}}{1 - \frac{2m_{\gamma}}{r}} \right) \right) + \left( \log \left( \frac{1 - \frac{2m_{\gamma}}{r}}{1 - \frac{2m_{\gamma}}{r}} \right) \right) \right\}(u) + \int_{u_1}^{u_2-\delta} r_1(T_{uu}) \, d\tilde{u} + \\
+ \max_{v \in [v_2-\delta, v_2+\delta]} \left\{ \left( \log \left( \frac{\Omega^2_{\gamma}}{1 - \frac{2m_{\gamma}}{r}} \right) \right) + \left( \log \left( \frac{1 - \frac{2m_{\gamma}}{r}}{1 - \frac{2m_{\gamma}}{r}} \right) \right) \right\}(v) \leq 2M,
\]

(5.235)

\[
(v_2 + \delta) - (v_2 - \delta) < \frac{u_{in}(2e^{-C^2_{\gamma}((\Lambda)(u_2-u_1)^2+1)M M})}{cC^2_{\gamma}((\Lambda)(u_2-u_1)^2+1)} \]

and

(5.236)

\[
\int_{v_2-\delta}^{v_2+\delta} r_{f}(T_{uv}) \, d\tilde{v} < 2e^{-C^2_{\gamma}((\Lambda)(u_2-u_1)^2+1)M M}.
\]

(5.237)

Therefore, by applying Proposition 5.2 for the boundary-double characteristic initial data set \((r_1, \Omega^2_{\gamma}, f_{in}, f_{out})\), \((r_1, \Omega^2_{\gamma}, f_{in}, f_{out})\) on \(\{u_1, u_2-\delta\} \times \{v_2-\delta\} \cup \{u_1\} \times \{v_2-\delta, v_2+\delta\}\), we readily infer that this initial data set admits a smooth development on \(\{u_1 < u < u_2 - \delta\} \cap \{\ell \leq \ell\} \cap \{v_2 - \delta < v < v_2 + \delta\}\), which coincides with \((r, \Omega^2, f_{in}, f_{out})\) on \(\{u_1\} \times \{v_2\}\). This fact then implies the statement of Lemma 5.2.

Let us set

(5.238)

\[\lambda_0 = \inf_{y_0, u_1, u_2} \partial_u r.\]

Note that \(\lambda_0 > 0\), in view of (5.233).

By integrating equation (2.32) in \(v\) and using (5.231), (5.232) and (5.238), we readily obtain that

(5.239)

\[r \partial_u r \leq -r_0 \lambda_0 < 0\]

on \(\{u_2\} \times \{v_2\}\). Therefore, (5.239) and (5.231) imply that

(5.240)

\[r_0 \leq r \leq \max_{\{u_1\} \times \{v_1, v_2\}} r \pm r_+ < +\infty\]

on \(\{u_2\} \times \{v_2\}\). Furthermore, integrating (2.32) in \(u\) and using (5.231) and (5.238), we obtain for all points in \(\{u_2\} \times \{v_2\}\):

(5.241)

\[0 < r_0 \lambda_0 \leq r \partial_u r \leq \max_{\{u_1\} \times \{v_1, v_2\}} r \partial_u r.\]

In view of the (2.7), the fact that \(\Omega^2 > 0\) on \(\{u_2\} \times \{v_2\}\) combined with (5.239), (5.240) and (5.241), implies that

(5.242)

\[1 - \frac{2m}{r} > 0\]

everywhere on \(\{u_2\} \times \{v_2\}\). Setting

(5.243)

\[\delta_0 = \min_{\{u_1\} \times \{v_1, v_2\}} \left( 1 - \frac{2m}{r} \right),\]

inequality (5.242) implies that \(\delta_0 > 0\). In view of (5.239) and the fact that \(T_{uu} \geq 0\), the relation (2.46) (which is well defined on \(\{u_2\} \times \{v_2\}\) in view of (5.241) and (5.242)) yields

(5.244)

\[\partial_u \log \left( \frac{\partial_u r}{1 - \frac{2m}{r}} \right) \leq 0.\]
Integrating (5.244) in $u$ starting from $u = u_1$, we obtain for all points in $\mathcal{D}\setminus\{u_2\} \times \{v_2\}$:

$$
(5.245) \quad \frac{\partial_v r}{1 - \frac{2m}{r}} \leq \max_{\{u_1\} \times \{v_1, v_2\}} \frac{\partial_v r}{1 - \frac{2m}{r}} \pm x_0 < +\infty.
$$

The bounds (5.240), (5.241), (5.243) and (5.244) yield for all points in $\mathcal{D}\setminus\{u_2\} \times \{v_2\}$:

$$
(5.246) \quad 1 - \frac{2m}{r} \geq \frac{1}{x_0 r_0 - 1} > 0.
$$

Integrating (2.50) in $u$, we obtain for all points in $\mathcal{D}\setminus\{u_2\} \times \{v_2\}$:

$$
(5.247) \quad r^2 T_{uv} \leq \max_{\{u_1\} \times \{v_1, v_2\}} (r^2 T_{uv}) < +\infty.
$$

Since $\tilde{f}_{in}, \tilde{f}_{out}$ satisfy the boundary condition (2.51) on $\gamma_{0, u_1 u_2}$ with $w = r - r_0$, $T_{uu}|_{\gamma_{0, u_1 u_2}}$ and $T_{uv}|_{\gamma_{0, u_1 u_2}}$ are related by (2.59), i.e. (in view of (5.231) and (5.232)):

$$
(5.248) \quad \frac{T_{uu}}{T_{uv}}|_{\gamma_{0, u_1 u_2}} = 1.
$$

Therefore, integrating (2.51) in $v$ and using (5.248), (5.233) and (5.247), we infer that, for all points in $\mathcal{D}\setminus\{u_2\} \times \{v_2\}$:

$$
(5.249) \quad r^2 T_{uu} \leq \max_{\{u_1\} \times \{v_1, v_2\}} (r^2 T_{uv}) < +\infty.
$$

Integrating (2.46) in $u$ starting from $u = u_1$ and using (5.239), (5.249) and (5.240), we obtain:

$$
(5.250) \quad \sup_{\mathcal{D}\setminus\{u_2\} \times \{v_2\}} \left| \log \left( \frac{\partial_u r}{1 - \frac{2m}{r}} \right) \right| < +\infty.
$$

The relation (2.46) is well defined on $\mathcal{D}\setminus\{u_2\} \times \{v_2\}$, in view of (5.239) and (5.242). Integrating (2.46) in $v$ starting from $\gamma_{0, u_1 u_2}$ and using (5.232), (5.240), (5.241), (5.250) and (5.247), we infer that

$$
(5.251) \quad \sup_{\mathcal{D}\setminus\{u_2\} \times \{v_2\}} \left| \log \left( \frac{-\partial_u r}{1 - \frac{2m}{r}} \right) \right| < +\infty.
$$

Thus, (2.7), (5.241) and (5.251) yield:

$$
(5.252) \quad \sup_{\mathcal{D}\setminus\{u_2\} \times \{v_2\}} \left| \log (\Omega^2) \right| < +\infty.
$$

Finally, by integrating equation

$$
(5.253) \quad \partial_u \tilde{m} = -2\pi \left( 1 - \frac{2\tilde{m}}{r} - \frac{1}{3} \Lambda r^2 \right) \frac{r^2 T_{uu}}{-\partial_u r}
$$

in $u$ starting from $u = u_1$ and using (5.239), (5.240) and (5.249), we infer that:

$$
(5.254) \quad \sup_{\mathcal{D}\setminus\{u_2\} \times \{v_2\}} \left| \tilde{m} \right| < +\infty.
$$

The bound (5.234) now readily follows from (5.240), (5.241), (5.242), (5.247), (5.249), (5.251), (5.252) and (5.254).
5.4 Proof of Theorem 1

The construction of the maximal future development \((U; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})\) will be performed in two steps: In the first step, we will construct the domain of outer communications \((I^- \cap U; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})\) of the maximal future development, using Proposition 5.1 as the main tool. In the second step, we will construct the rest of the maximal future development, i.e. \((U \setminus J^-(I); r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})\), using Propositions 5.2 and 5.3 as the main tool. Notice that \(U \setminus J^-(I)\) can possibly be empty; in the proof, we will actually consider the case \(U \setminus J^- (I) = \emptyset\) separately.

The uniqueness and maximality of \((U; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})\) will follow readily from our construction, in conjunction with the uniqueness statements of Propositions 5.1, 5.2 and 5.3. The properties 1–6 of \((U; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})\) stated in Theorem 1 will also be established during the construction of \((U; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})\).

In order to better keep track of the notations introduced throughout the proof, the reader is advised to refer to Figure 5.5.

Figure 5.5: The construction of the maximal development \((U; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})\) of \((r; \Omega^2, \bar{f}_{in}, \bar{f}_{out})\) will proceed in two steps: In the first step, we will construct the domain of outer communications \(U \setminus J^- (I)\) corresponding to the region \(U = \{0 < u < u_\ast\}\) depicted above. In the case when \(u_\ast < +\infty\) (which is the case depicted), the solution will have a non-empty future event horizon \(H^+ = \{u = u_\ast\}\), and the second step of the construction will consist of constructing the part of the solution lying to the future of \(H^+\). In the figure, this corresponds to the domains \(W_\ast\) and \(V_\ast\). The construction of \(U\) will require the use of Proposition 5.1 in the region \(U_\ast\), Proposition 5.2 in the region \(W_\ast\) and Proposition ?? in the region \(V_\ast\).
Step 1: Construction of \( J^-(\mathcal{I}) \cap \mathcal{U} \)

We will first construct the domain of outer communications (\( J^-(\mathcal{I}) \cap \mathcal{U}; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out} \)) of the maximal future development (\( \mathcal{U}; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out} \)) of \( (r_j, \Omega^2_j, \bar{f}_{inj}, \bar{f}_{outj}) \). (according to Definition 3.4)

Let \( \mathcal{W}_\mathcal{I} \) be the set of all developments \( \mathcal{D} = (\mathcal{U}_\mathcal{D}; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out}) \) of \( (r_j, \Omega^2_j, \bar{f}_{inj}, \bar{f}_{outj}) \) (such that, for some \( 0 < u_\mathcal{D} < +\infty \)):

\[
\mathcal{D} = \{0 < u < u_\mathcal{D}\} \cap \{u < v < u + v_0\}.
\]

In view of Proposition 5.1, \( \mathcal{W}_\mathcal{I} \neq \emptyset \). Furthermore, in view of the remark below Definition 3.4 and the form (5.255) of the domain of the developments belonging to \( \mathcal{W}_\mathcal{I} \), any two developments \( \mathcal{D}_1, \mathcal{D}_2 \in \mathcal{W}_\mathcal{I} \) will satisfy \( \mathcal{D}_1 \subseteq \mathcal{D}_2 \) or \( \mathcal{D}_2 \subseteq \mathcal{D}_1 \). Therefore, there exists a unique \( 0 < u_* \leq +\infty \) and a unique development \( \mathcal{D}_* = (\mathcal{U}_*; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out}) \) of \( (r_j, \Omega^2_j, \bar{f}_{inj}, \bar{f}_{outj}) \), where

\[
\mathcal{U}_* = \{0 < u < u_*\} \cap \{u < v < u + v_0\},
\]

such that any \( \mathcal{D} \in \mathcal{W}_\mathcal{I} \) satisfies \( \mathcal{D} \subseteq \mathcal{D}_* \).

Let us set

\[
\mathcal{I} = \{u = v - v_0\} \cap \{0 < u < u_*\}
\]

and

\[
\gamma_0^* = \{u = v\} \cap \{0 \leq u < u_*\}.
\]

We will now establish that \( \mathcal{D}_* \) has the property that, for any \( u' < u_* \):

\[
\sup_{\mathcal{U}_* \cap \{u \leq u'\}} \max \left\{ \left| \log \left( \frac{\Omega^2}{1 - \frac{3}{5} \Lambda r^2} \right) \right|, \left| \log \left( \frac{2 \partial_r r}{1 - \frac{2m}{r}} \right) \right|, \left| \log \left( \frac{1 - \frac{2m}{r}}{1 - \frac{3}{5} \Lambda r^2} \right) \right|, \sqrt{-\Lambda} |\bar{m}|, r^2 T_{vv} \right\} < +\infty
\]

and

- \( u_* = +\infty \) or
- \( u_* < +\infty \) and

\[
\sup_{\mathcal{U}_*} \max \left\{ \left| \log \left( \frac{\Omega^2}{1 - \frac{3}{5} \Lambda r^2} \right) \right|, \left| \log \left( \frac{2 \partial_r r}{1 - \frac{2m}{r}} \right) \right|, \left| \log \left( \frac{1 - \frac{2m}{r}}{1 - \frac{3}{5} \Lambda r^2} \right) \right|, \sqrt{-\Lambda} |\bar{m}|, r^2 T_{vv} \right\} = +\infty.
\]

This can be inferred as follows: Let \( \mathcal{D} = (\mathcal{U}_\mathcal{D}; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out}) \) be any development of \( (r_j, \Omega^2_j, \bar{f}_{inj}, \bar{f}_{outj}) \) belonging to the set \( \mathcal{W}_\mathcal{I} \). Provided that

\[
M = \sup_{\mathcal{U}_\mathcal{D}} \max \left\{ \left| \log \left( \frac{\Omega^2}{1 - \frac{3}{5} \Lambda r^2} \right) \right|, \left| \log \left( \frac{2 \partial_r r}{1 - \frac{2m}{r}} \right) \right|, \left| \log \left( \frac{1 - \frac{2m}{r}}{1 - \frac{3}{5} \Lambda r^2} \right) \right|, \sqrt{-\Lambda} |\bar{m}|, r^2 T_{vv} \right\} < +\infty,
\]

by applying Proposition 5.1 for the initial data induced by \( (r, \Omega^2, \bar{f}_{in}, \bar{f}_{out}) \) on \( \{u = u_\mathcal{D} - u'\} \times \{u \leq v \leq v_0 + u\} \) for some \( u' \) small enough in terms of \( r_0, v_0 \) and \( M \), we infer that there exists some \( \mathcal{D}' \in \mathcal{W}_\mathcal{I} \) strictly extending \( \mathcal{D} \), i.e. \( \mathcal{D} \subseteq \mathcal{D}' \). Therefore, in view of the inextendibility of \( \mathcal{D}_* \) in \( \mathcal{W}_\mathcal{I} \), either \( u_* = +\infty \), or (5.260) holds. Moreover, it can be readily verified that (5.261) always holds if \( \mathcal{D} \) has a strict extension \( \mathcal{D}' \) in \( \mathcal{W}_\mathcal{I} \). Therefore, (5.259) holds.
Some basic estimates on $\mathcal{I}$ and $\mathcal{U}_*$. It follows readily from the proof of Proposition 5.1 that the quantities (5.46)–(5.50) and $\tilde{m}$ extend smoothly on $\mathcal{I}$. The relations (2.48)–(2.49) and the conditions (3.23), (3.19), (3.21) and (3.20) imply that $\tilde{m}$ is constant on $\gamma_0^*$ and $\mathcal{I}$, i.e.

\begin{equation}
\tilde{m}|_{\gamma_0^*} = \tilde{m}_f(0) \text{ and } \tilde{m}|_{\mathcal{I}} = \lim_{v \to v_0^*} \tilde{m}_f(v).
\end{equation}

The relations (2.48)–(2.49) and the bound (5.259) imply that

\begin{equation}
\partial_u \tilde{m} \leq 0 \text{ and } \partial_v \tilde{m} \geq 0 \text{ on } \mathcal{U}_*
\end{equation}

and, hence

\begin{equation}
\tilde{m}_f(0) \leq \tilde{m} \leq \lim_{v \to v_0^*} \tilde{m}_f(v) \text{ on } \mathcal{U}_*.
\end{equation}

Moreover, the relations (2.7) and (2.44) imply, in view of the fact that (5.47)–(5.48) extend smoothly on $\mathcal{I}$, that the quantity

\begin{equation}
\omega = \sqrt{-\frac{\Omega^2}{1 - \frac{1}{3} \Lambda r^2}}
\end{equation}

extends smoothly on $\mathcal{I}$, with

\begin{equation}
\omega^2 = 4 \left( \frac{\partial_v \tilde{r}}{1 - \frac{2m}{r}} \right)^2 |_{\mathcal{I}}
\end{equation}

in view of (2.7), (2.44) and (3.20).

End of the proof in the case $u_* = +\infty$

In the case $u_* = +\infty$, we will set

\begin{equation}
\mathcal{U} = \mathcal{U}_*.
\end{equation}

Note that, in this case, we necessarily have

\begin{equation}
\mathcal{U} \setminus J(\mathcal{I}) = \emptyset
\end{equation}

and, thus,

\begin{equation}
\mathcal{H}^+ = \emptyset
\end{equation}

and

\begin{equation}
\gamma_0 = \gamma_0^*.
\end{equation}

In order to complete the proof of Theorem 1 in this case, it remains to establish (4.2)–(4.4), (2.48)–(2.49) and (4.9). The bounds (4.2)–(4.4) follow readily from (5.259), while (2.48)–(2.49) follow immediately from (5.262) and the fact that $\gamma_0 = \gamma_0^*$.

The proof of (4.9) will follow by showing that

\begin{equation}
\int_{\mathcal{I}} \omega \, du = \lim_{u \to u_*} \int_0^u \omega(\bar{u}, \bar{u} + v_0) \, d\bar{u} = +\infty.
\end{equation}
In view of (5.266), it suffices to show that, for any \( u \geq v_0 \):

\[
(5.272) \int_{u-v_0}^{u} \left( \frac{\partial v}{1 - \frac{2m}{r}} \right) \bigg|_{x} (\bar{u}, \bar{u} + v_0) \, d\bar{u} \geq c_1 > 0
\]

for some absolute constant \( c_1 \) depending only on the initial data \((r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})\).

The lower bound (5.272) is deduced as follows: Inequality (2.45) and the bound (5.259) imply that

\[
(5.273) \quad \partial u \left( \frac{\partial v}{1 - \frac{2m}{r}} \right) \leq 0
\]
on \( U_* \) and, thus, for any \( u \geq v_0 \) and \( u \leq v \leq u + v_0 \):

\[
(5.274) \quad \left( \frac{\partial v}{1 - \frac{2m}{r}} \right) (u, v) \geq \frac{\partial v}{1 - \frac{2m}{r}} (u, v).
\]

For any \( u \geq 0 \), we compute:

\[
(5.275) \quad \sqrt{-\frac{\Lambda}{3}} \int_{u}^{u+v_0} \frac{\partial v}{1 - \frac{2m}{r}} (u, v) \, dv = \tan^{-1} \left( \sqrt{-\frac{\Lambda}{3}} \right) \bigg|_{v_0} - \tan^{-1} \left( \sqrt{-\frac{\Lambda}{3}} \right) \bigg|_{v_0} = \frac{\pi}{2} - \tan^{-1} \left( \sqrt{-\frac{\Lambda}{3}} r_0 \right).
\]

Therefore, (5.274), (5.275) and (5.264) readily yield that, for any \( u \geq 0 \)

\[
(5.276) \quad \int_{u-v_0}^{u} \left( \frac{\partial v}{1 - \frac{2m}{r}} \right) \bigg|_{x} (\bar{u}, \bar{u} + v_0) \, d\bar{u} = \int_{u}^{u+v_0} \left( \frac{\partial v}{1 - \frac{2m}{r}} \right) \bigg|_{x} (v-v_0, v) \, dv \geq \int_{u}^{u+v_0} \left( \frac{\partial v}{1 - \frac{2m}{r}} \right) (u, v) \, dv \geq \sqrt{-\frac{3}{\Lambda}} \left( 1 + \frac{2\max\{0, -\tilde{m}_f(0)\}}{r_0} \right)^{-1} \left( \frac{\pi}{2} - \tan^{-1} \left( \sqrt{-\frac{\Lambda}{3}} r_0 \right) \right)
\]

and, thus, (5.272) holds.

**The case** \( u_* < +\infty \)

For the rest of the proof, we will assume without loss of generality that

\[
(5.277) \quad u_* < +\infty.
\]

We will set

\[
(5.278) \quad \mathcal{H}^+ = \{ u = u_* \} \cap \{ u_* \leq v < u_* + v \}.
\]

We will show that \((r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})\) extend smoothly beyond \( \mathcal{H}^+ \) and, therefore, in the case (5.277), we have

\[
(5.279) \quad U \setminus J^-(I) \neq \emptyset
\]

and \( \mathcal{H}^+ \) will actually be the future evnt horizon defined by (4.10).

Notice that, equations (2.50)–(2.51) and the reflecting boundary conditions

\[
(5.280) \quad r^2 T_{uu} \bigg|_{v_0} = 1 \quad \text{and} \quad r^2 T_{vv} \bigg|_{v_0} = 1
\]

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imply that

\[(5.281) \quad r^2 T_{uu}, r^2 T_{vv} \leq \sup_{[0,v_0)} r^2 (T_{vv})_f\]

and, thus, in view of (2.7), (5.259), (5.264) and (5.281), the condition (5.260) is equivalent to

\[(5.282) \quad \limsup_{u \to u_*} \left\{ \sup_{u \to u} \left( |\log \left( \frac{\partial_v r}{1 - \frac{2m}{r}} \right) + |\log \left( -\frac{\partial_u r}{1 - \frac{2m}{r}} \right) + \left( 1 - \frac{2m}{r} \right)^{-1} \right) \right\} = +\infty.\]

**Smooth extension across \( \mathcal{H}^+ \).** We will now show that \((r, \log \Omega^2)\) extend smoothly on the whole of \( \mathcal{H}^+ \), and \((f_{in}, f_{out})\) extends smoothly on the whole of \( \mathcal{H}^+ \times (0, +\infty) \) and, thus, (5.279) holds.

Assuming, first, that \((r, \log \Omega^2)\) extend smoothly on \( \{u_*\} \times \{u_*\} \) and \((f_{in}, f_{out})\) extend smoothly on \( \{u_*\} \times \{u_*\} \times (0, +\infty) \), we can readily deduce that \((r, \log \Omega^2)\) extends smoothly on the whole of \( \mathcal{H}^+ \), and \((f_{in}, f_{out})\) extends smoothly on the whole of \( \mathcal{H}^+ \times (0, +\infty) \), by applying Lemma 5.1 for \( u_1 = u_* - \delta, u_2 = u_*, v_1 = u_* , v_2 = u_* + v_0 - 2\delta \) for any \( 0 < \delta \ll 1 \).

Thus, it remains to show that \((r, \log \Omega^2)\) extend smoothly on the point \( \{u_*\} \times \{u_*\} \), and \((f_{in}, f_{out})\) extend smoothly on \( \{u_*\} \times \{u_*\} \times (0, +\infty) \). Provided

\[(5.283) \quad \limsup_{u \to u_*} \partial_v r(u, u) > 0,\]

the smooth extension of \((r, \log \Omega^2)\) on \( \{u_*\} \times \{u_*\} \) and \((f_{in}, f_{out})\) on \( \{u_*\} \times \{u_*\} \times (0, +\infty) \) follows readily from the continuation criterion of Lemma 5.2 with \( u_1 = v_1 = 0 \) and \( u_2 = v_2 = v_0 \). Hence, it suffices to establish (5.283).

**Proof of (5.283).** Assume, for the sake of contradiction, that (5.283) is false, i.e.

\[(5.284) \quad \limsup_{u \to u_*} \partial_v r(u, u) = 0.\]

In view of (2.34) and (5.259), we can bound

\[(5.285) \quad \partial_v r > 0\]

and

\[(5.286) \quad \partial_v (\Omega^2 \partial_v r) = -4\pi r T_{vv} \Omega^{-2} \leq 0\]

everywhere on \( \mathcal{U}_* \). Thus, (3.23), (5.284), (5.285) and (5.286) imply that \( r \) extends continuously on \( \mathcal{H}^+ \) so that

\[(5.287) \quad r|_{\mathcal{H}^+} = r_0\]

and, for any \( u_* < v < u_* + u_0 \):

\[(5.288) \quad \lim_{u \to u_*} T_{vv}(u, v) = 0.\]

Equations (2.50)–(2.51) and the reflecting boundary conditions (5.280) imply, in view of (5.288), that

\[(5.289) \quad T_{uu}(u, v) = T_{vv}(u, v) = 0\]

for all points \((u, v) \in \mathcal{U}_* \gamma_\gamma^-(u_*, u_*)\), where \( \gamma^-(u_*, u_*) \) is the past directed null geodesic emanating from \((u_*, u_*)\), reflected successively on \( \mathcal{I}^+ \) and \( \gamma^*_{0+} \), i.e.

\[\gamma^-(u_*, u_*) = \cup_{n \in \mathbb{N}} \mathcal{Y}_\gamma(n)(u_*, u_*)\]
with

\[ \gamma_r^{(2n)}(u_*, u_*) = \{ v = u_* - nv_0 \} \cap \{ u_* - (n + 1)v_0 \leq u \leq u_* - nv_0 \}, \quad n \geq 0, \]

\[ \gamma_r^{(2n+1)}(u_*, u_*) = \{ u = u_* - (n + 1)v_0 \} \cap \{ u_* - (n + 1)v_0 \leq v \leq u_* - nv_0 \}, \quad n \geq 0. \]

Since \( T_{uu}, T_{uv} \) are smooth on \( \mathcal{U}_* \), we infer that (5.29) holds on the whole of \( \mathcal{U}_* \). Thus, \( \tilde{m} \) is constant on \( \mathcal{U}_* \). Since \( \partial_v |_{S_{v_0}} > 0 \), we infer that

\[ 1 - \frac{2M}{r_0} - \frac{1}{3} \lambda r_0^2 > 0. \]

Equations (2.45) and (2.46), combined with (3.19) and (3.20), imply in this case that

\[ \left| \log \left( \frac{\partial_v r}{1 - \frac{2m}{r}} \right) \right| \leq \sup_{\{0\} \times [0,v_0]} \left| \log \left( \frac{\partial_v r}{1 - \frac{2m}{r}} \right) \right| \]

everywhere on \( \mathcal{U}_* \). Therefore, (5.291) and (5.292) imply that (5.28) is false, which is a contradiction.

**Useful bounds on** \( \mathcal{H}^+ \). Notice that, in view of (5.25):

\[ \inf_{\mathcal{U}_* \cup \mathcal{H}^+} (-\partial_v r) > 0. \]

**Proof of (4.12) and (4.13)**. In order to establish (4.12), we will first establish

\[ \sup_{\mathcal{H}^+} r \leq r_S \]

and (4.13), and then show that

\[ \sup_{\mathcal{H}^+} r \geq r_S. \]

**Proof of (4.12) and (4.13)**. The upper bound (4.12) follows by a contradiction argument: Assuming that (5.29) is false, in view of (5.29) and (5.295) we infer that there exists some (possibly small) \( \delta > 0 \) such that

\[ \inf_{\mathcal{V}_b} r \geq (1 + \delta) r_S, \]

where

\[ \mathcal{V}_b = \{ u_* - \delta \leq u \leq u_* \} \cap \{ u_* + v_0 - \delta \leq v < u + v_0 \}. \]

In view of (5.26) and (4.14), the lower bound (5.29) implies that

\[ C_{\mathcal{H}^+} \geq \sup_{\mathcal{V}_b} \left\{ (1 - \frac{2m}{r})^{-1} \right\} < +\infty. \]

By integrating (2.45) in \( u \) and (2.46) in \( v \) and using condition (3.20) on \( I \), we infer that, for any \( u \in [u_* - \delta, u_*] \):

\[ \log \left( \frac{\partial_v r}{1 - \frac{2m}{r}} \right)(u_* + v_0) = \log \left( \frac{-\partial_u r}{1 - \frac{2m}{r}} \right)(u, u_* + v_0 - \delta) + 4\pi \int_{u_* + v_0 - \delta}^{u_*} \frac{T_{uv}}{\partial_u r}(u, v) \, dv - 4\pi \int_{u_*}^{u_* + v_0} r \left( \frac{T_{uu}}{-\partial_u r}(u, u_* + v_0) \right) \, du. \]
In view of (2.48), (2.49), (5.285) and (5.300), we can estimate:

\begin{equation}
(5.302) \quad 4\pi \int_{u^+ + v_0 - \delta}^{u^+ + v_0} r_0 \frac{T_{uv}}{\partial_v r}(u, \tilde{v}) \, d\tilde{v} - 4\pi \int_{u}^{u^+} \frac{T_{uu}}{-\partial_u r}(\tilde{u}, u + v_0) \, d\tilde{u} \leq \leq 2C r_0^{-1} \left( \tilde{m}(u, u + v_0) - \tilde{m}(u, u_0 + v_0 - \delta) + \tilde{m}(u, u + v_0) - \tilde{m}(u_0, u + v_0) \right).
\end{equation}

Thus, in view of (5.264), (5.295), (5.300) and (5.302), from (5.301) we infer that

\begin{equation}
(5.303) \quad \sup_{\nu_0} \left| \log \left( \frac{\partial_u r}{1 - \frac{2m}{r}} \right) \right| < +\infty.
\end{equation}

By integrating equation (2.46) in \( v \) starting from the point \((u, u_0 + v_0 - \delta)\) for any \( u \in [u_0 - \delta, u_0] \) and using (5.295), (5.281), (5.300) and (5.303), we also infer that

\begin{equation}
(5.304) \quad \sup_{\nu_0} \left| \log \left( - \frac{\partial_u r}{1 - \frac{2m}{r}} \right) \right| < +\infty.
\end{equation}

The bounds (5.300), (5.303) and (5.304) combine into:

\begin{equation}
(5.305) \quad \sup_{\nu_0} \left( \log \left( \frac{\partial_u r}{1 - \frac{2m}{r}} \right) \right) + \left| \log \left( - \frac{\partial_u r}{1 - \frac{2m}{r}} \right) \right| + \left( 1 - \frac{2m}{r} \right)^{-1} < +\infty.
\end{equation}

In view of (5.259), (5.294) and (5.305), we infer that

\begin{equation}
(5.306) \quad \sup_{\{u, u_0 + v_0 - \delta\} \cap \{u_0 - \delta \leq u \leq u_0\}} \left| \log(\partial_u r) \right| < +\infty.
\end{equation}

In view of (2.7), (5.295), (5.306) and the fact that \( \log(\Omega^2) \) extends continuously on \( \mathcal{H}^+ \), we also infer that

\begin{equation}
(5.307) \quad \sup_{\{u, u_0 + v_0 - \delta\} \cap \{u_0 - \delta \leq u \leq u_0\}} \left( 1 - \frac{2m}{r} \right)^{-1} < +\infty.
\end{equation}

Therefore, (5.295), (5.306) and (5.307) imply that

\begin{equation}
(5.308) \quad \sup_{\{u, u_0 + v_0 - \delta\} \cap \{u_0 - \delta \leq u \leq u_0\}} \left( \log \left( \frac{\partial_u r}{1 - \frac{2m}{r}} \right) \right) + \left| \log \left( - \frac{\partial_u r}{1 - \frac{2m}{r}} \right) \right| + \left( 1 - \frac{2m}{r} \right)^{-1} < +\infty.
\end{equation}

Combining (5.259), (5.305) and (5.308), we thus obtain

\begin{equation}
(5.309) \quad \sup_{u_0} \left( \log \left( \frac{\partial_u r}{1 - \frac{2m}{r}} \right) \right) + \left| \log \left( - \frac{\partial_u r}{1 - \frac{2m}{r}} \right) \right| + \left( 1 - \frac{2m}{r} \right)^{-1} < +\infty.
\end{equation}

This is a contradiction, in view of (5.282). Thus, (4.12) holds.

The relation (4.13) also follows by a similar argument: Assuming that (4.13) is false, i.e.

\begin{equation}
(5.310) \quad \inf_{\mathcal{H}^+} \left( 1 - \frac{2m}{r} \right) > 0,
\end{equation}

the relation (2.7) and the fact that \( \log(\Omega^2) \) extends continuously on \( \mathcal{H}^+ \) implies that

\begin{equation}
(5.311) \quad \partial_u r|_{\mathcal{H}^+} > 0.
\end{equation}

Thus, the inequality

\begin{equation}
(5.312) \quad \partial_u \log \left( \frac{\partial_u r}{1 - \frac{2m}{r}} \right) \leq 0
\end{equation}

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(following from (2.45) and (5.295)), combined with (5.285) (on $\mathcal{U}_s$), (5.310) and (5.311), implies (5.300). Therefore, repeating the same arguments as for the proof of (5.296), we reach a contradiction.

Proof of (5.297). In view of (5.296) and (4.13), it suffices to show that
\begin{equation}
(5.313) \quad \lim_{{v \to \bar{u} + v_0}} \tilde{m}|_{\mathcal{H}^* (v = \bar{v})} = \lim_{{v \to \bar{v}}_0} \tilde{m}(v).
\end{equation}

Integrating (2.48) in $u$, we calculate for any $u_* + v_0 - \delta \leq \bar{v} < u_* + v_0$:
\begin{equation}
(5.314) \quad \tilde{m}|_{\mathcal{H}^* (v = \bar{v})} = \tilde{m}|_{\mathcal{I} (v = \bar{v})} + 2\pi \int_{\bar{v} - v_0}^{u_*} \frac{1 - 2m}{r - \partial_u r} r T_{u\bar{u}}(u, \bar{v}) \, du.
\end{equation}

The relation
\begin{equation}
(5.315) \quad \partial_u \left( \frac{1 - 2m}{r - \partial_u r} \right) \geq 0
\end{equation}
(following from (2.46) and (5.285)), combined with the bound (5.281) (and the fact that $r \geq r_0$ on $\mathcal{U}_s$) implies that
\begin{equation}
(5.316) \quad \sup_{(u_* - \delta \leq u \leq u_*) \cap \{v_* + v_0 - \delta \leq v \leq v_0 + v_0\}} \frac{1 - 2m}{r - \partial_u r} r T_{u\bar{u}} < +\infty.
\end{equation}

Thus, in view of (5.262) and (5.316), the relation (5.313) is obtained by taking the limit $\bar{v} \to u_* + v_0$ in (5.314).

Proof of (4.9) In view of (2.7), (5.264) and (3.20), (4.9) will follow from
\begin{equation}
(5.317) \quad \int_{\mathcal{I}} \frac{-\partial_u r}{1 - 2m} \, du = +\infty.
\end{equation}

Assume, for the sake of contradiction, that (5.317) is false. Then
\begin{equation}
(5.318) \quad \lim_{{\bar{u} \to u_*}} \int_{\bar{u}}^{u_*} \frac{-\partial_u r}{1 - 2m} (u, u + v_0) \, du = \lim_{{\bar{u} \to u_*}} \int_{\mathcal{I} \cap \{u \geq \bar{u}\}} \frac{-\partial_u r}{1 - 2m} \, du = 0.
\end{equation}

The relation (5.315) implies, in view of (5.318) that:
\begin{equation}
(5.319) \quad \lim_{{\bar{u} \to u_*}} \int_{\bar{u}}^{u_*} \frac{-\partial_u r}{1 - 2m} (u, \bar{u} + v_0) \, du = 0.
\end{equation}

Thus, (5.264) and (5.319) imply that
\begin{equation}
(5.320) \quad 0 = \lim_{{\bar{u} \to u_*}} \int_{\bar{u}}^{u_*} \frac{-\partial_u r}{1 - 2m} (u, \bar{u} + v_0) \, du = \frac{\sqrt{3}}{\Lambda} \lim_{{\bar{u} \to u_*}} \left( \tan^{-1} \left( \frac{\Lambda}{r} \right) \big|_{\mathcal{I} \cap \{u = \bar{u} + v_0\}} - \tan^{-1} \left( \frac{\Lambda}{3} \right) \big|_{\mathcal{H}^* \cap \{v = \bar{u} + v_0\}} \right) = \frac{\sqrt{3}}{2} \lim_{{\bar{u} \to u_*}} \tan^{-1} \left( \frac{\Lambda}{3} \right) \big|_{\mathcal{H}^* \cap \{v = \bar{u} + v_0\}},
\end{equation}
which is a contradiction in view of (5.296).
Proof of (4.11). In view of (2.7), (5.295), (5.296) and the fact that
\begin{equation}
\partial_v(-r \partial_u r) \geq 0
\end{equation}
(following from (2.32)), in order to establish (4.11) it suffices to show that
\begin{equation}
\int_{\mathcal{H}^+} \frac{\Omega^2}{-\partial_u r} \, dv = +\infty.
\end{equation}

In view of (5.295), (5.296), (5.321) and (5.281), we can bound
\begin{equation}
C_T \sup_{\mathcal{U}_t} \frac{r T_{uv}}{-\partial_u r} < +\infty.
\end{equation}

Integrating equation
\begin{equation}
\partial_u \log \left( \frac{\Omega^2}{-\partial_u r} \right) = -4\pi r \frac{T_{uv}}{-\partial_u r}
\end{equation}
in $u$ and using (5.323)\footnote{Equation (5.324) is readily obtained from (2.36).} we obtain for any $v_0 \leq \bar{v} < u_0 + v_0$:
\begin{equation}
\left| \log \left( \frac{\Omega^2}{-\partial_u r} \right) \right|_{\mathcal{H}^+ \cap \{v=\bar{v}\}} - \log \left( \frac{\partial_v r}{1 - \frac{2m}{r}} \right)_{\mathcal{I} \cap \{v=\bar{v}\}} \leq C_T (u_0 + v_0 - \bar{v}).
\end{equation}

Integrating (5.325) in $\bar{v} \in [v_0, u_0 + v_0]$ and using (2.7) on $\mathcal{I}$, (5.317) and (3.20), we thus infer (5.322).

Step 2: Construction of $\mathcal{U}\setminus J^- (\mathcal{U})$ in the case $u_0 < +\infty$

By applying Proposition 5.2 for the initial data induced by $(r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$ on $[u_0 - \delta_1, u_0] \times \{u_0\} \cup (u_0 - \delta_1) \times [u_0, u_0 + \bar{\delta}_2]$ for some $0 < \delta_2 \ll \delta_1 \ll 1$, we infer that $(r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$ extends smoothly as a solution to (2.32)–(2.37) (satisfying (3.23)–(3.19)) on $u = v$ on
\begin{equation}
\mathcal{W}_{\delta_2} \doteq \{u_0 \leq u \leq v\} \cap \{v \leq u_0 + \bar{\delta}_2\}.
\end{equation}

Repeating the same procedure as for the construction of $\mathcal{U}$, with the use of Proposition 5.1 in the previous step, by using Proposition 5.2 for the initial data induced by $(r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$ on sets of the form $[u_0, u_0 + \bar{u}] \times \{u_0 + \bar{u}\} \cup \{u_0\} \times [u_0 + \bar{u}, u_0 + \bar{u} + \bar{\delta}]$ (starting from $\bar{u} = \bar{\delta}_2$), we infer that there exists a $v_0 \leq u_0 + v_0$ with the following properties:

1. Setting
\begin{equation}
\mathcal{W}_s \doteq \{u_0 \leq u < v\} \cap \{v < v_0\},
\end{equation}
\begin{equation}
\mathcal{C}^- \doteq \{v = v_0\} \cap \{u < u_0 < v_0\}
\end{equation}
and
\begin{equation}
\gamma_0 \doteq \{u = v\} \cap \{0 < u < u_0\},
\end{equation}

the functions $(r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$ extend smoothly on $\mathcal{W}_s \cup \mathcal{C}^- \cup \gamma_0$, so that $(\mathcal{U}, \mathcal{U} \cup \mathcal{W}_s; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$ is a development of the initial data $(r/\Omega^2, \bar{f}_{in}/\bar{f}_{out})$ according to Definition 3.4.
In view of (2.34) and the fact that hyperbolic with respect to the reference metric (2.52). Therefore, the domain can intersect the level set (5.335) on $V$ for any $0 \leq \delta < r_0$.

(5.334) 
\[ \mathcal{Y}' \subseteq (u_*, v_*) \times (v_*, u_* + v_0) \]

which is globally hyperbolic (with respect to the reference metric (2.52)), such that $(r, \Omega^2, f_{in}, f_{out})$ extend smoothly on $\mathcal{V}'$ as solutions to the system (2.32)–(2.37).

Let us set

(5.335) 
\[ \mathcal{V}_* = \mathcal{V}' \cap \{ r > r_0 \} \]

In view of (2.34) and the fact that $T_{uv} \geq 0$, any $\{ u = const \}$ line in the region

\[ \mathcal{U}' = \mathcal{U}_* \cup \mathcal{H}^+ \cup \mathcal{W}_* \cup \mathcal{C}^- \cup \mathcal{V}' \]

can intersect the level set $\{ r = r_0 \}$ at most two times. Thus, since $r|_{Y_0} = r_0$, we readily infer that the boundary of $\mathcal{V}_*$ in $\mathcal{V}'$ (which consists of a subset of the level set $\{ r = r_0 \}$) is a smooth achronal curve and $\mathcal{V}_*$ is globally hyperbolic with respect to the reference metric (2.52). Therefore, the domain $\mathcal{U}_* \cup \mathcal{H}^+ \cup \mathcal{W}_* \cup \mathcal{V}'$ belongs to the set $\mathcal{Y}_{r_0}^*$ introduct in Definition 3.3. In this case, we will set

(5.336) 
\[ \mathcal{U} = \mathcal{U}_* \cup \mathcal{H}^+ \cup \mathcal{W}_* \cup \mathcal{C}^- \cup \mathcal{V}_*. \]

In both the case $v_0 = u_* + v_0$ and the case $v_0 < u_* + v_0$, the boundary $\partial \mathcal{U}$ of $\mathcal{U}$ splits as (3.14), for a continuous achronal curve $\gamma$ with a parametrization $\gamma : (0, v_0) \rightarrow \mathbb{R}^2$ of the form $\gamma(v) = (u_*(x), v_*(x))$, where

(5.337) 
\[ u_*(x) = \begin{cases} 
    v_0, & 0 < x \leq v_1 \\
    f_1(x), & v_1 < x < v_2 \\
    u_* + v_0 - x, & v_2 \leq x \leq v_0 
\end{cases} \]
and

\[(5.338) \quad v_\gamma(x) = \begin{cases} v_* + x, & 0 < x \leq v_1 \\ f_2(x), & v_1 < x < v_2 \\ u_* + v_0, & v_2 \leq x < v_0 \end{cases}\]

for some \(0 \leq v_1 \leq v_2 \leq v_0\), where

\[(5.339) \quad u_* < f_1(x) < v_*\]

and

\[(5.340) \quad v_* < f_2(x) < u_* + v_0\]

for all \(x \in (v_1, v_2)\). Note that the properties \((5.339), (5.340)\) of \(f_1, f_2\) imply, by an application of Lemma 5.1 that \((r, \log \Omega^2, f_{in}, f_{out})\) extends smoothly across \(\gamma((v_1, v_2))\).

We will now proceed to show that

\[(5.341) \quad \nu_2 = v_0\]

in \((5.337)-(5.338)\), and that \(r\) extends continuously on \(\gamma\) with

\[(5.342) \quad r|_\gamma = r_0.\]

Since in the case \(v_* = u_* + v_0\) it is necessary that \(v_2 = 0\), \((5.341)\) will imply that \(v_* < u_* + v_0\) always. Furthermore, since \(r\) extends smoothly across \(\gamma((v_1, v_2))\), we will also infer from \((5.342)\) that \(\gamma((v_1, v_2))\) is smooth. Note that, since \(\gamma\) is continuous and \(\partial \Omega\) has the form \((5.14)\), the relation \((5.341)\) also implies that, necessarily, \(v_1 < v_2 = v_0\).

**Proof of \((5.341)\).** Integrating equation \((2.32)\) in \(v\) starting from \(\nu_0\), we readily obtain that

\[(5.343) \quad \sup_{\Omega} \partial_u r < 0.\]

Integrating \((2.50)\) in \(u\) and \((2.51)\) in \(v\) and using the boundary conditions \((5.280)\), we readily infer that \((5.281)\) holds on the whole of \(\Omega\). In view of \((2.48), (5.281), (5.343), (5.313)\) and the fact that \(\lim_{u \to u_0} m_t(v) > 0\) in the case \(\partial \Omega \neq \emptyset\) (otherwise \((4.13)\) is violated), we infer that there exists some \(0 < \delta < 1\) so that

\[(5.344) \quad \inf_{[u_* + u_* - \delta, u_* + v_0]} \tilde{m} > 0.\]

Integrating equation

\[(5.345) \quad \partial_u \log(-\partial_u r) = \frac{1}{2} \frac{\tilde{m} - \frac{1}{2} \Omega^2}{\partial_u r}\]

(which is readily obtained from \((2.32)\)) in \(v\) starting from \(v = u_* + v_0 - \delta\) and using \((5.344)\), we obtain for any point \((u, v) \in [u_* + u_* - \delta, u_* + v_0] \cap \Omega:\)

\[(5.346) \quad \log(-\partial_u r)(u, v) \geq \log(-\partial_u r)(u_* + v_0 - \delta) + c_0 \int_{u_* + v_0 - \delta}^{v} \frac{\Omega^2}{\partial_u r}(u, \tilde{v}) d\tilde{v}.\]

for some \(c_0 > 0\) depending on \(r_0\) and \((5.344)\). Integrating \((5.324)\) in \(u\) starting from \(u = u_*\) and using \((5.281)\) and \((5.343)\), we can also estimate

\[(5.347) \quad \int_{u_* + v_0 - \delta}^{v} \frac{\Omega^2}{\partial_u r}(u, \tilde{v}) \geq c_1 \int_{u_* + v_0 - \delta}^{v} \frac{\Omega^2}{\partial_u r}(u_*, \tilde{v}).\]
for some \( c_1 > 0 \) depending on \( r_0, 5.281, 5.343 \) and \( 5.344 \). Thus, from \( 5.346 \) and \( 5.347 \) we can bound for any \((u, v) \in [u_*, u_* + \delta] \times [u_* + v_0 - \delta, u_* + v_0] \cap \mathcal{U}:\)

\[
(5.348) \quad r(u_*, v) - r(u, v) = \int_{u_*}^{u} \left( - \partial_u r(\bar{u}, \bar{v}) \right) d\bar{u} \geq \exp \left( c_0 \sup_{\bar{u} \in [u_*, u]} \int_{u_* + v_0 - \delta}^{u} \frac{\Omega^2}{\partial_u r} (\bar{u}, \bar{v}) d\bar{v} \right) \cdot \int_{u_*}^{u} \left( - \partial_u r(\bar{u}, u_* + v_0 - \delta) \right) d\bar{u} \geq \exp \left( c_0 c_1 \int_{u_* + v_0 - \delta}^{u} \frac{\Omega^2}{\partial_u r} (\bar{u}, \bar{v}) d\bar{v} \right) \cdot \left( r(u_*, u_* + v_0 - \delta) - r(u, u_* + v_0 - \delta) \right).
\]

Assuming, for the sake of contradiction, that \( v_2 < v_0 \) in \( 5.337 \), for any fixed \( u_0 \in (u_*, u_* + v_0 - v_2) \) and any \( v \in [u_* + v_0 - \delta, u_* + v_0] \), the point \((u_0, v)\) belongs to \([u_*, u_* + \delta] \times [u_* + v_0 - \delta, u_* + v_0] \cap \mathcal{U}\). Therefore, applying \( 5.348 \) for \((u, v) = (u_0, v)\) and considering the limit \( v \to u_* + v_0\), we obtain

\[
(5.349) \quad \int_{u_* + v_0 - \delta}^{u_* + v_0} \frac{\Omega^2}{\partial_u r} (u, v) d\bar{v} \leq C_0 \lim_{v \to u_* + v_0} \sup_{v' \in [u_*, u_* + \delta]} \log \left( \frac{r(u_*, v) - r(u_0, v)}{r(u_*, u_* + v_0 - \delta) - r(u_0, u_* + v_0 - \delta)} \right),
\]

where \( C_0 > 0 \) depends on \( r_0, 5.281, 5.343 \) and \( 5.344 \). In view of \( 5.343 \) and the fact that \( r_0 \leq r \leq r_S\) on \([u \geq u_*] \cap \mathcal{U}\), the right hand side of \( 5.349\) is finite, while the left hand side is infinite in view of \( 5.322\), which is a contradiction. Therefore, \( 5.344\) holds.

**Proof of 5.342.** In view of Lemma 5.2 \( 5.331\) and the fact that \((r, \log \Omega^2, \tilde{f}_{in}, \tilde{f}_{out})\) are smooth on \([u = u_*] \cap \{u_* \leq v \leq v_*\}\), we infer that, necessarily

\[
(5.350) \quad \lim_{v \to u_* + \delta} \partial_v r|_{\gamma \cap \{v = v_1\}} = 0.
\]

In view of the inequality \( 5.286\), the relation \( 5.350\) implies that, for any \( v \in [v_*, v_* + v_1] \) (with \( v_1 \) as in \( 5.338\))

\[
(5.351) \quad \lim_{u \to v_*} \partial_u r(u, v) \leq 0.
\]

Since \( \mathcal{V}_* \) was defined by \( 5.335\), we necessarily have for any point \( p \in \gamma:\)

\[
(5.352) \quad \liminf_{p \to v_{10}} r \geq r_{0}.
\]

Thus, since \( r|_{\gamma_{10}} = r_{0}\), the relations \( 5.352\) and \( 5.351\) imply that \( r \) extends continuously on \( \gamma((0, v_1))\), with

\[
(5.353) \quad r|_{\text{clos}((0, v_1))} = r_{0}.
\]

Since

\[
(5.354) \quad \gamma((v_1, v_{0}) \subset (u_*, v_*) \times (v_*, u_* + v_0)),
\]

an application of Lemma 5.1 yields that \((r, \log \Omega^2)\) extend smoothly across \( \gamma\). Since \( \mathcal{V}_* \) was defined by \( 5.335\), and \( \mathcal{V}'\) is the maximal globally hyperbolic development of the characteristic initial data induced by \((r, \Omega^2, \tilde{f}_{in}, \tilde{f}_{out})\) on \([u_*, v_*) \times \{v_*\} \cup \{u_*\} \times [v_*, u_* + v_0]\), we infer that \(\gamma((v_1, v_{0})\) is the boundary of \(\mathcal{V}_*\) in \(\mathcal{V}'\) and, thus:

\[
(5.355) \quad r|_{\gamma((v_1, v_{0})} = r_{0}.
\]

The relation \(5.342\) now follows from \(5.353\) and \(5.355\).
End of the proof. In order to finish the proof of Theorem 1 in the case when (5.277) holds, it remains to establish (4.2), (4.4), (4.7)–(4.8), (4.15), as well as Property 6.

The bound (4.2) has been already established in (5.343). The bounds (4.3)–(4.4) follow readily from (5.259) (on $J^-$($T$)) and (5.330) (on $J^-$($\gamma_0$)).

The conservation of $\tilde{m}$ on $\gamma_0$ and $\mathcal{L}$, i.e. (4.7)–(4.8), follows readily from relations (2.48)–(2.49) and the conditions (3.23), (3.19), (3.21) and (3.20).

The relation (4.15) follows immediately from the fact that $\gamma_0 \cap \{u > u_*\} \neq \emptyset$ (see (5.329)). Finally, Property 6 is an immediate consequence of (5.341).

6 Cauchy stability in a rough norm, uniformly in $r_0$

In this Section, we will establish Theorem 2 and Corollary 4.

6.1 Proof of Theorem 2

Let $C_1 \gg 1$ be a large, fixed constant. Using a standard continuity argument, it suffices to show that, for any $0 < u_* \leq u_0$ such that

$$\mathcal{W}_{u_*} = \{0 < u < u_*\} \cap \{u + v_1 < v < u + v_2\} \subset \mathcal{U}_2$$

and

$$\sup_{\mathcal{W}_{u_*}} \left\{ \log \left( \frac{\Omega^2_1}{1 - \frac{1}{3} \Lambda r_1^2} \right) - \log \left( \frac{\Omega^2_2}{1 - \frac{1}{3} \Lambda r_2^2} \right) \right\} + \left| \log \left( \frac{2\partial_x r_1}{1 - \frac{2m_1}{r_1}} \right) - \log \left( \frac{2\partial_x r_2}{1 - \frac{2m_2}{r_2}} \right) \right| +$$

$$+ \left| \log \left( \frac{1 - \frac{2m_1}{r_1}}{1 - \frac{1}{3} \Lambda r_1^2} \right) - \log \left( \frac{1 - \frac{2m_2}{r_2}}{1 - \frac{1}{3} \Lambda r_2^2} \right) \right| + \sqrt{\Lambda} |\bar{m}_1 - \bar{m}_2| \right\} +$$

$$+ \sup_{\tilde{u}} \int_{\{u = \tilde{u}\} \cap \mathcal{W}_{u_*}} |r_1(T_{uv})_1 - r_2(T_{uv})_2| \, dv + \sup_{v} \int_{\{v = \tilde{v}\} \cap \mathcal{W}_{u_*}} |r_1(T_{uu})_1 - r_2(T_{uu})_2| \, du \leq$$

$$\leq 2 \exp \left( \exp \left( C_1(1 + C_0) \frac{u_0}{v_2 - v_1} \right) \right) \delta,$$

the following improvement of (6.1) actually holds:

$$\sup_{\mathcal{W}_{u_*}} \left\{ \log \left( \frac{\Omega^2_1}{1 - \frac{1}{3} \Lambda r_1^2} \right) - \log \left( \frac{\Omega^2_2}{1 - \frac{1}{3} \Lambda r_2^2} \right) \right\} + \left| \log \left( \frac{2\partial_x r_1}{1 - \frac{2m_1}{r_1}} \right) - \log \left( \frac{2\partial_x r_2}{1 - \frac{2m_2}{r_2}} \right) \right| +$$

$$+ \left| \log \left( \frac{1 - \frac{2m_1}{r_1}}{1 - \frac{1}{3} \Lambda r_1^2} \right) - \log \left( \frac{1 - \frac{2m_2}{r_2}}{1 - \frac{1}{3} \Lambda r_2^2} \right) \right| + \sqrt{\Lambda} |\bar{m}_1 - \bar{m}_2| \right\} +$$

$$+ \sup_{\tilde{u}} \int_{\{u = \tilde{u}\} \cap \mathcal{W}_{u_*}} |r_1(T_{uv})_1 - r_2(T_{uv})_2| \, dv + \sup_{v} \int_{\{v = \tilde{v}\} \cap \mathcal{W}_{u_*}} |r_1(T_{uu})_1 - r_2(T_{uu})_2| \, du \leq$$

$$\leq 2 \exp \left( \exp \left( C_1(1 + C_0) \frac{u_0}{v_2 - v_1} \right) \right) \delta.$$

In order to establish (6.2), it suffices to show that, for any $u_1 < u_2 < u_*$ such that

$$u_2 - u_1 \leq v_2 - v_1,$$

setting

$$\mathcal{W}_{u_1, u_2} = \{u_1 < u < u_2\} \cap \{u < v < u + v_0\},$$
we can bound on $W_{u_1,u_2}$

\begin{align}
(6.5) \quad \sup_{W_{u_1,u_2}} \left\{ \log \left( \frac{\Omega_1^2}{1 - \frac{1}{3} \Lambda r_1^2} \right) - \log \left( \frac{\Omega_2^2}{1 - \frac{1}{3} \Lambda r_2^2} \right) + \log \left( \frac{2 \partial_u r_1}{1 - \frac{2m_1}{r_1}} \right) - \log \left( \frac{2 \partial_u r_2}{1 - \frac{2m_2}{r_2}} \right) \right. + \\
+ \log \left( \frac{1 - \frac{2m_1}{r_1}}{1 - \frac{1}{3} \Lambda r_1^2} \right) - \log \left( \frac{1 - \frac{2m_2}{r_2}}{1 - \frac{1}{3} \Lambda r_2^2} \right) + \sqrt{-\Lambda} |\tilde{m}_1 - \tilde{m}_2| + \\
+ \sup \int_{\{u=\tilde{u}\cap W_{u_1,u_2}\}} |r_1(T_{uv})_1 - r_2(T_{uv})_2| dv + \sup \int_{\{v=\tilde{v}\cap W_{u_1,u_2}\}} |r_1(T_{uu})_1 - r_2(T_{uu})_2| du \leq \\
\leq \exp \left( \exp \left( C_1(1 + C_0) \right) \right) \delta_{u_1},
\end{align}

where

\begin{align}
(6.6) \quad \delta_{u_1} \leq \sup_{v \in (u_1 + v_1, u_1 + v_2)} \left\{ \log \left( \frac{\Omega_1^2}{1 - \frac{1}{3} \Lambda r_1^2} \right) \right. - \log \left( \frac{\Omega_2^2}{1 - \frac{1}{3} \Lambda r_2^2} \right) + \log \left( \frac{2 \partial_u r_1}{1 - \frac{2m_1}{r_1}} \right) - \log \left( \frac{2 \partial_u r_2}{1 - \frac{2m_2}{r_2}} \right) \left( u_1, v \right) + \\
+ \left. \log \left( \frac{1 - \frac{2m_1}{r_1}}{1 - \frac{1}{3} \Lambda r_1^2} \right) \right|_{(u_1, v)} - \log \left( \frac{1 - \frac{2m_2}{r_2}}{1 - \frac{1}{3} \Lambda r_2^2} \right) \left|_{(u_1, v)} \right. + \sqrt{-\Lambda} |\tilde{m}_1 - \tilde{m}_2|_{(u_1, v)} + \\
+ \sup_{v \in (u_1 + v_1, u_1 + v_2)} (-\Lambda) \int_{u_1 + v_1}^{u_1 + v_2} \left. \frac{r_1^2(T_{uv})_1 \left( u_1, \tilde{u} \right)}{\left( \|\tilde{e}_1(u_1, \tilde{u}) - \tilde{e}_1(u_1, v) \| + \| \tilde{e}_1(u_1, u_1 + v_1) \| \partial_\tilde{e}_1(u_1, \tilde{u}) \right) \left( u_1, \tilde{u} \right) - \\
- \frac{r_2^2(T_{uv})_2 \left( u_1, \tilde{u} \right)}{\left( \|\tilde{e}_2(u_1, \tilde{u}) - \tilde{e}_2(u_1, v) \| + \| \tilde{e}_2(u_1, u_1 + v_1) \| \partial_\tilde{e}_2(u_1, \tilde{u}) \right) \left( u_1, \tilde{u} \right) \left( u_1, \tilde{u} \right) \right| d\tilde{v}
\right.
\end{align}

(with $\tilde{e}_i$ defined in terms of $r_i$ by (5.40)) measures the distance of the initial data induced by $(r_1, \Omega_1^2, \tilde{f}_inl, \tilde{f}_outl)$ and $(r_2, \Omega_2^2, \tilde{f}_in2, \tilde{f}_out2)$ on $u = u_1$. Note that, in the general case when $u_*$ does not satisfy (6.3), the bound (6.2) follows by applying (6.5) successively on intervals of the form $\{u_1(n) \leq u \leq u_2(n)\}$, where $u_2(n) = n \cdot (v_2 - v_1)$ and $u_1(n) = u_1^{(n-1)}$.

Remark. Notice that the bound (4.18) yields the following bound for $\partial_\theta \tan^{-1} \left( \sqrt{-\frac{3}{\Lambda}} \right)$:

\[ \left| \log \left( \sqrt{-\frac{3}{\Lambda}} \right) \right|_{\gamma_0} = \tan^{-1} \left( \sqrt{-\frac{3}{\Lambda}} r_0 \right) \text{ and } \tan^{-1} \left( \sqrt{-\frac{3}{\Lambda}} r \right) \left( \gamma_0 \right) = \frac{n}{2}, \] 

we readily infer that $|v_2 - v_1|$ must necessarily satisfy the bound:

\begin{align}
(6.7) \quad \left| \log \left( \sqrt{-\Lambda} |v_2 - v_1| \right) \right| \leq 2C_0.
\end{align}

Let us define the variables $\tilde{e}_i, \tilde{e}_i, \tilde{e}_i, i = 1, 2, \tilde{e}_i$, by (5.40)-(5.50). Recall that these variables satisfy equations (5.54)-(5.59) and the boundary conditions (5.61)-(5.66). In view of (2.7), the bounds (4.18) and (6.1) yield that:

\begin{align}
(6.8) \quad \max \sup_{i=1,2} W_{u_1,u_2} \left\{ \left| \log (\tilde{e}_i) \right| + \left| \log (\tilde{e}_i) \right| + \left| \log (-\Lambda)^{-\frac{1}{2}} \partial_\theta \tilde{e}_i \right| + \left| \log \left( -\Lambda \right)^{-\frac{1}{2}} \partial_\theta \tilde{e}_i \right| + \left| \log \left( \frac{1 - \frac{2m}{r}}{1 + \frac{2m}{r}} \right) \right| + \sqrt{-\Lambda} |\tilde{m}_i| \right\} \leq 10C_0.
\end{align}

Furthermore, using the fact that, for $i = 1, 2, \tilde{e}_i$ are functions of $v$ and $\gamma_i$ are functions of $u$ (in view of equations 5.58 and 5.59, respectively) from (4.18), (6.1) and (6.8) (using also (6.7)) we obtain the following bound for
\( \tilde{\tau}_i, \tau_i \) on \( \mathcal{W}_{u_1; u_2} \):

\[
\sup_{\tilde{u} \in (u_1, u_2)} \sup_{v \in (u_1, v_2 + u_2 - u_1)} \int_{u + v}^{u + v_2} \tilde{\tau}_i(u, v) \, dv + \sup_{\tilde{u} \in (v_1, v_2 + u_2 - u_1)} \sup_{u \in (u_1, u_2)} \int_{u}^{u_2} \tilde{\tau}_i(u, \tilde{v}) \, du \leq e^{10(C_0 + 1)}.
\]

Moreover, from (6.6), (6.8) and (6.9) we can estimate on \( u = u_1 \):

\[
\sup_{v \in (u_1 + v_1, u_1 + v_2)} \left\{ \left| \chi_1(u_1, v) - \chi_2(u_1, v) \right| + \left| \chi_1^{-1}(u_1, v) - \chi_2^{-1}(u_1, v) \right| + \left| \frac{\partial \varphi_1(u_1, v)}{\partial v} - \frac{\partial \varphi_2(u_1, v)}{\partial v} \right| + \left| (\partial \varphi_1)^{-1}(u_1, v) - (\partial \varphi_2)^{-1}(u_1, v) \right| + \left| \frac{1 - \frac{2\sqrt{\frac{2}{2} m_1}}{\tan \varphi_1} + \tan^2 \varphi_1}{1 + \tan^2 \varphi_1} (u_1, v) - \left( \frac{1 - \frac{2\sqrt{\frac{2}{2} m_2}}{\tan \varphi_2} + \tan^2 \varphi_2}{1 + \tan^2 \varphi_2} \right) (u_1, v) \right| \right.
\]

\[
+ \left( \frac{1 - \frac{2\sqrt{\frac{2}{2} m_1}}{\tan \varphi_1} + \tan^2 \varphi_1}{1 + \tan^2 \varphi_1} \right)^{-1} (u_1, v) - \left( \frac{1 - \frac{2\sqrt{\frac{2}{2} m_2}}{\tan \varphi_2} + \tan^2 \varphi_2}{1 + \tan^2 \varphi_2} \right)^{-1} (u_1, v) \right| + \left| \sqrt{-\Lambda} \left[ m_1 |(u_1, v) - m_2|/(u_1, v) \right] \right| + \sup_{v \in (u_1 + v_1, u_1 + v_2)} \int_{v_1}^{v_2} \left| \tilde{\tau}_1(u_1, v) - \tilde{\tau}_2(u_1, v) \right| \, dv \right) \leq e^{100(C_0 + 1)} \delta u_1.
\]

Thus, the proof of (6.5) (and, therefore, the proof of Theorem 2) will follow (in view of (6.8) and the boundary conditions (6.61) for \( \varphi_1 \)) by showing that

\[
\sup_{\mathcal{W}_{u_1; u_2}} \left\{ \log(\chi_1) - \log(\chi_2) + \log(\chi_1) - \log(\chi_2) + \log((-\Lambda)^{-\frac{1}{2}} \partial \varphi_1) - \log((-\Lambda)^{-\frac{1}{2}} \partial \varphi_2) \right. + \\
\left. + \log((-\Lambda)^{-\frac{1}{2}} \partial \varphi_1) - \log((-\Lambda)^{-\frac{1}{2}} \partial \varphi_2) \right\} + \sqrt{-\Lambda} \left[ m_1 - m_2 \right] + \\
+ \sup_{\tilde{u} \in (u_1, u_2)} \int_{(u = \tilde{u}) \cap \mathcal{W}_{u_1; u_2}} \left| \tilde{\tau}_1(u, \tilde{v}) - \tilde{\tau}_2(u, \tilde{v}) \right| \, d\tilde{v} + \sup_{\tilde{u} \in (u_1, u_1 + u_2 - u_1) \cap \mathcal{W}_{u_1; u_2}} \int_{(u = \tilde{u}) \cap \mathcal{W}_{u_1; u_2}} \left| \tilde{\tau}_1(u, \tilde{v}) - \tilde{\tau}_2(u, \tilde{v}) \right| \, d\tilde{u} \leq \exp\left( \exp(C_1^2 (1 + C_0)) \right) \delta u_1.
\]

The differences \( \varphi_1 - \varphi_2, \chi_1 - \chi_2, \tilde{\tau}_1 - \tilde{\tau}_2, \tau_1 - \tau_2, \tilde{\tau}_1 - \tilde{\tau}_3 \) satisfy on \( \mathcal{W}_u \), the system (5.134)-(5.139), i.e.:

\[
\partial_u \left( \log(\chi_2) - \log(\chi_1) \right) = -\sqrt{-\Lambda} \left( F_1(\tan \varphi_2; \sqrt{-\Lambda} \tilde{m}_2) \tilde{x}_1^{-1} \tau_2 - F_1(\tan \varphi_1; \sqrt{-\Lambda} \tilde{m}_1) \tilde{x}_1^{-1} \tau_1 \right),
\]

\[
\partial_u \left( \log(\tilde{x}_2) - \log(\tilde{x}_1) \right) = \sqrt{-\Lambda} \left( F_1(\tan \varphi_2; \sqrt{-\Lambda} \tilde{m}_2) \tilde{x}_2^{-1} \tau_2 - F_1(\tan \varphi_1; \sqrt{-\Lambda} \tilde{m}_1) \tilde{x}_2^{-1} \tau_1 \right),
\]

\[
\partial_u (\varphi_2 - \varphi_1) = \left( -\frac{\Lambda}{3} \right) \left( F_2(\tan \varphi_2; \sqrt{-\Lambda} \tilde{m}_2) \tilde{x}_2 \tilde{x}_2 - F_2(\tan \varphi_1; \sqrt{-\Lambda} \tilde{m}_1) \tilde{x}_1 \tilde{x}_1 \right),
\]

\[
\partial_u (\tilde{m}_2 - \tilde{m}_1) = -4\pi (\tilde{x}_2^{-1} \tau_2 - \tilde{x}_1^{-1} \tau_1),
\]

\[
\partial_u (\tilde{\tau}_2 - \tilde{\tau}_1) = 0,
\]

\[
\partial_u (\tau_2 - \tau_1) = 0.
\]

Integrating equations 6.16, 6.17 and using the boundary conditions (5.63) for \( \tau_1, \tau_2 \) and \( \tilde{\tau}_1, \tilde{\tau}_2 \), we infer that, for any \((u, v) \in \mathcal{W}_{u_1; u_2} \):

\[
(\tau_1 - \tau_2)(u, v) = (\tilde{\tau}_1 - \tilde{\tau}_2)(u_1, u + v_1)
\]

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(6.19) \[(\bar{\tau}_1 - \tilde{\tau}_2)(u, v) = \begin{cases} (\bar{\tau}_1 - \tilde{\tau}_2)(u_1, v), & v \leq u_1 + v_2, \\ (\bar{\tau}_1 - \tilde{\tau}_2)(u_1, v - v_2 + v_1), & v \geq u_1 + v_2. \end{cases}\]

In view of (6.8) and the initial bound (6.10), from (6.18)–(6.19) we can readily estimate:

(6.20) \[\sup_{\tilde{u}e(u_1, u_2) \cap W_{u_1, u_2}} \int_{(u-\tilde{u})} |\bar{\tau}_1(\tilde{u}, v) - \tilde{\tau}_2(\tilde{u}, v)| \, dv + \sup_{\tilde{v}e(u_1 + v_1, u_2 + v_2) \cap (v - \tilde{v})} \int_{(v-\tilde{v})} |\bar{\tau}_1(u, \tilde{v}) - \tilde{\tau}_2(u, \tilde{v})| \, du \leq e^{C_{(1+C_0)}\delta_0}.\]

Integrating equations (6.12), (6.14) and (6.15) in \(u\) and equations (6.13) and (6.14) in \(v\), using the boundary conditions (6.61), (6.62) and (5.55) on \(\Gamma_0\) and \(Z\) as well as the bounds (5.53) for \(F_1, F_2\) and (6.8), (6.9) and (6.10) for \(x_1, \tilde{x}_1, \tilde{\tau}_1, \bar{\tau}_1, \bar{x}_1, \bar{\tau}_1\), we readily infer that, for any \((u, v) \in W_{u_1, u_2}\):

(6.21) \[|\log(x_2(u, v)) - \log(x_1(u, v))| \leq \sqrt{-\Lambda} e^{C_{(1+C_0)}} \left\{ \int_{\max\{u_1, u-v_2-v_1\}}^{u} \left( \mathfrak{M} \cdot \mathfrak{D} + |\bar{\tau}_2 - \tilde{\tau}_1| \right)_{(u,v)} \, du \right. + \chi_{u \geq v_2}(v) \int_{u-v_2+v_1}^{v} \left( \mathfrak{M} \cdot \mathfrak{D} + |\bar{\tau}_2 - \tilde{\tau}_1| \right)_{(v-v_2-u_1, \tilde{v})} \, dv \\
+ \chi_{u \geq v_2}(v) \int_{u-v_2-u_1}^{v} \left( \mathfrak{M} \cdot \mathfrak{D} + |\bar{\tau}_2 - \tilde{\tau}_1| \right)_{(u,v)} \, du \right\} + e^{C_{(1+C_0)}\delta_{u_1}},\]

(6.22) \[|\log(\tilde{x}_2(u, v)) - \log(\tilde{x}_1(u, v))| \leq \sqrt{-\Lambda} e^{C_{(1+C_0)}} \left\{ \int_{\max\{u_1, u-v_2-v_1\}}^{v} \left( \mathfrak{M} \cdot \mathfrak{D} + |\bar{\tau}_2 - \tilde{\tau}_1| \right)_{(u,v)} \, dv \right. + \chi_{u \geq v_2}(v) \int_{u-v_2+v_1}^{v} \left( \mathfrak{M} \cdot \mathfrak{D} + |\bar{\tau}_2 - \tilde{\tau}_1| \right)_{(v-v_2-u_1, \tilde{v})} \, dv \\
+ \chi_{u \geq v_2}(v) \int_{u-v_2-u_1}^{v} \left( \mathfrak{M} \cdot \mathfrak{D} + |\bar{\tau}_2 - \tilde{\tau}_1| \right)_{(u,v)} \, du \right\} + e^{C_{(1+C_0)}\delta_{u_1}},\]

(6.23) \[|\partial_u \bar{\tau}_2(u, v) - \partial_v \bar{\tau}_1(u, v)| \leq (-\Lambda) e^{C_{(1+C_0)}} \left\{ \int_{\max\{u_1, u-v_2-v_1\}}^{u} \left( \mathfrak{M} \cdot \mathfrak{D} + |\bar{\tau}_2 - \tilde{\tau}_1| \right)_{(u,v)} \, du \right. + \chi_{v \geq v_2}(v) \int_{u-v_2+v_1}^{v} \left( \mathfrak{M} \cdot \mathfrak{D} + |\bar{\tau}_2 - \tilde{\tau}_1| \right)_{(v-v_2-u_1, \tilde{v})} \, dv \\
+ \chi_{v \geq v_2}(v) \int_{u-v_2-u_1}^{v} \left( \mathfrak{M} \cdot \mathfrak{D} + |\bar{\tau}_2 - \tilde{\tau}_1| \right)_{(u,v)} \, du \right\} + \sqrt{-\Lambda} e^{C_{(1+C_0)}\delta_{u_1}},\]

(6.24) \[|\partial_u \tilde{\tau}_2(u, v) - \partial_v \tilde{\tau}_1(u, v)| \leq (-\Lambda) e^{C_{(1+C_0)}} \left\{ \int_{\max\{u_1, u-v_2-v_1\}}^{v} \left( \mathfrak{M} \cdot \mathfrak{D} + |\bar{\tau}_2 - \tilde{\tau}_1| \right)_{(u,v)} \, dv \right. + \chi_{v \geq v_2}(v) \int_{u-v_2+v_1}^{v} \left( \mathfrak{M} \cdot \mathfrak{D} + |\bar{\tau}_2 - \tilde{\tau}_1| \right)_{(v-v_2-u_1, \tilde{v})} \, dv \\
+ \chi_{v \geq v_2}(v) \int_{u-v_2-u_1}^{v} \left( \mathfrak{M} \cdot \mathfrak{D} + |\bar{\tau}_2 - \tilde{\tau}_1| \right)_{(u,v)} \, du \right\} + \sqrt{-\Lambda} e^{C_{(1+C_0)}\delta_{u_1}},\]

(6.25) \[|\tilde{m}_2(u, v) - \tilde{m}_1(u, v)| \leq e^{C_{(1+C_0)}} \left\{ \int_{\max\{u_1, u-v_2-v_1\}}^{u} \left( \mathfrak{M} \cdot \mathfrak{D} + |\bar{\tau}_2 - \tilde{\tau}_1| \right)_{(u,v)} \, du \right. + (-\Lambda)^{-\frac{1}{2}} e^{C_{(1+C_0)}\delta_{u_1}},\]

where

(6.26) \[\chi_{v \geq v_2}(v) = \begin{cases} 1, & \text{if } v > v_2, \\ 0, & \text{if } v \leq v_2. \end{cases}\]

(6.27) \[\mathfrak{D}(u, v) = \sqrt{-\Lambda} \left| \log(x_2) - \log(x_1) \right|(u, v) + \left| \log(\tilde{x}_2) - \log(\tilde{x}_1) \right|(u, v) + \left| \partial_u \bar{\tau}_2 - \partial_v \bar{\tau}_1 \right|(u, v) + \sqrt{-\Lambda} |\tilde{m}_2 - \tilde{m}_1|(u, v)\]

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and the functions $\mathcal{M}(u, v) \geq 0$, $\mathcal{M}(u, v) \geq 0$ satisfy the bound:

\begin{equation}
(6.29) \quad \sup_{\tilde{v} \in (u_1, u_2 + v_2)} \int_{\tilde{v} = \tilde{v}} \mathcal{M}(u, \tilde{v}) \, du + \sup_{\tilde{v} \in (u_1, v_2)} \int_{\tilde{v} = \tilde{v}} \mathcal{M}(\tilde{u}, \tilde{v}) \, dv \leq 1.
\end{equation}

Defining the function $\mathcal{X} : (u_1 + v_1, 2u_2 + v_2) \to [0, +\infty)$ by the relation

\begin{equation}
(6.30) \quad \mathcal{X}(t) = \sup_{u + v \leq t} \left\{ \left| \log(x_2) - \log(x_1) \right| (u, v) + \left| \log(x_2) - \log(x_1) \right| (u, v) + (-\Lambda) \frac{1}{2} \left| \partial_u \varphi_2 - \partial_u \varphi_1 \right| (u, v) + (-\Lambda) \frac{1}{2} \left| \partial_u \varphi_2 - \partial_u \varphi_1 \right| (u, v) + \left| \varphi_2 - \varphi_1 \right| (u, v) + \sqrt{-\Lambda} |\tilde{v}_2 - \tilde{v}_1| (u, v) \right\},
\end{equation}

from (6.21)–(6.26) (using also (6.10), (6.18), (6.19) and (6.29)) we readily obtain that, for all $t \in (2u_1 + v_1, 2u_2 + v_2)$:

\begin{equation}
(6.31) \quad \mathcal{X}(t) \leq e^{C_1 (1 + C_0)} \int_{2u_1 + v_1}^{t} \mathcal{Y}(\tilde{t}) \mathcal{X}(\tilde{t}) \, d\tilde{t} + e^{C_1 (1 + C_0)} \delta_{u_1},
\end{equation}

where the function $\mathcal{Y} \geq 0$ satisfies

\begin{equation}
(6.32) \quad \int_{2u_1 + v_1}^{2u_2 + v_2} \mathcal{Y}(t) \, dt \leq 1.
\end{equation}

From (6.31), an application of Gronwall’s inequality readily yields that

\begin{equation}
(6.33) \quad \sup_{t \in (2u_1 + v_1, 2u_2 + v_2)} \mathcal{X}(t) \leq \exp \left( \exp(C_1 (1 + C_0)) \right) \delta_{u_1}.
\end{equation}

The bound (6.11) follows readily from (6.20) and (6.33). Therefore, the proof of the Theorem 2 is complete.

6.2 Proof of Corollary [1]

By possibly applying a suitable gauge transformation of the form $(u, v) \to (U(u), V(v))$, such that $U(0) = 0$, $V(0) = 0$, $V = \bar{U}$ when $v = u$ (i.e. on $v_0$) and $V = U + V_0$ when $v = u + v_0$ (i.e. on $\bar{U}$), we will assume without loss of generality that the initial data satisfy the gauge condition

\begin{equation}
(6.34) \quad \frac{\partial_u r_i}{1 - \frac{2m_i}{r_i}} = \frac{1}{2}.
\end{equation}

In view of (6.34), equation (2.46), combined with the boundary condition

\begin{equation}
(6.35) \quad - \left( \partial_u r_i \right)(0) = \partial_v r_i(0),
\end{equation}

imply that

\begin{equation}
(6.36) \quad - \frac{2}{1 - \frac{2m_i}{r_i}} \left( \partial_u r_i \right)(v) = \exp \left( 4\pi \int_{0}^{v} \frac{r_i(T_{vv})}{\left( \partial_u r_i \right)} \, d\tilde{v} \right).
\end{equation}

Therefore, (2.7) and (6.34) yield

\begin{equation}
(6.37) \quad \frac{1}{1 - \frac{1}{3} \Lambda r_i} = \frac{1 - \frac{2m_i}{r_i}}{1 - \frac{1}{3} \Lambda r_i^2} \exp \left( 4\pi \int_{0}^{v} \frac{r_i(T_{vv})}{\left( \partial_u r_i \right)} \, d\tilde{v} \right),
\end{equation}

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Applying Theorem 2 for

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A.1 The class of admissible

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the spherically symmetric Einstein–null dust system on such spacetimes in Section A.2. Finally, in Section A.3,
spherically symmetric spacetimes admitting a double-null foliation. We will then examine the basic properties of

system in the presence of a regular axis of symmetry. In order to state this result in its strongest form, we will

A Ill-posedness of the spherically symmetric Einstein–null dust system

at \( r = 0 \)

The aim of this Section is to establish a general ill-posedness result for the spherically symmetric Einstein–null dust system in the presence of a regular axis of symmetry. In order to state this result in its strongest form, we will first introduce the notion of admissible \( C^0 \) spherically symmetric spacetimes \( (\mathcal{M}, g) \) in Section A.1. These are \( C^0 \) spherically symmetric spacetimes admitting a double-null foliation. We will then examine the basic properties of the spherically symmetric Einstein–null dust system on such spacetimes in Section A.2. Finally, in Section A.3, we will establish that smooth solutions to the spherically symmetric Einstein–null dust system with a non-trivial axis of symmetry break down (as admissible \( C^0 \) spherically symmetric solutions of the Einstein–null dust system) in finite time.

A.1 The class of admissible \( C^0 \) spherically symmetric spacetimes

In this section, we will introduce the notion of \( C^0 \) spherically symmetric spacetimes and state their basic properties. While we will only restrict to the case of 3 + 1 dimensional spacetimes, the definitions and results of this section can be immediately extended to arbitrary dimensions. We will also introduce the notion of an admissible \( C^0 \) spherically symmetric spacetime, which are the \( C^0 \) spacetimes on which the spherically symmetric Einstein–null dust can be rigorously formulated.

A Lorentzian manifold \( (\mathcal{M}^{3+1}, g) \) will be called a \( C^0 \) spacetime if \( \mathcal{M} \) is a \( C^1 \) manifold and \( g \) is a \( C^0 \) Lorentzian metric on \( \mathcal{M} \). We will define the notion of spherical symmetry in the class of \( C^0 \) spacetimes as follows:

Definition A.1. A \( C^0 \) spacetime \( (\mathcal{M}^{3+1}, g) \) will be called \( C^0 \) spherically symmetric if there exists a \( C^1 \) action

\[
\mathcal{A} : SO(3) \times \mathcal{M} \to \mathcal{M}
\]

with the following properties:

1. For any \( a \in SO(3) \), the map

\[
\mathcal{A}(a, \cdot) : \mathcal{M} \to \mathcal{M}
\]

defined by [A.1] is an isometry with respect to \( g \).

2. For any \( p \in \mathcal{M} \), the orbit \( Orb(p) \) of \( p \) under the action (A.1) is either the single point \( \{ p \} \) or is a 2-dimensional surface homeomorphic to \( S^2 \).

3. For any \( p \in \mathcal{M} \), there exists an open \( SO(3) \)-invariant neighborhood \( \mathcal{V}_p \) of \( p \) such that:

(a) In the case \( Orb(p) \neq \{ p \} \), there exists a \( C^1 \)-diffeomorphism

\[
\mathcal{F} : \mathcal{V}_p \to U \times S^2
\]

for some domain \( U \) in \( \mathbb{R}^2 \), such that \( \mathcal{F} \) commutes the action (A.1) of \( SO(3) \) on \( \mathcal{V}_p \) with the natural action of \( SO(3) \) on \( U \times S^2 \) by rotations of \( S^2 \).

From (4.29), (6.34) and (6.37) we thus infer that, for some absolute constant \( C > 0 \):

\[
\sup_{\nu \in \{ v_1, v_2 \}} \left\{ \log \left( \frac{\Omega_j^2}{1 - \frac{2m_j}{r_j}} \right) + \left| \frac{2\delta\nu r_j}{1 - \frac{2m_j}{r_j}} \right| + \left| \left( \frac{1 - \frac{2m_j}{r_j}}{1 - \frac{2m_j}{r_j}} \right)^{1/2} \right| + \sqrt{\Lambda \nu} \right\} (v) + \int_{v_1}^{v_2} r_j(T_{uv})_j \, d\overline{v} \leq C \varepsilon.
\]

Applying Theorem 2 for \( v_1 = 0 \), \( v_2 = v_0 \), \((r_{j2}, \Omega_j^2, f_{in/2}, f_{out/2}) = (r_j, \Omega_j, f_{in}, f_{out})\) and \((r_{j1}, \Omega_j, \tilde{f}_{in}, \tilde{f}_{out}) = (r_{AdS}, \Omega_{AdS}, 0, 0)\), where \((r_{AdS}, \Omega_{AdS}, 0, 0)\) are the trivial initial data renormalised by the gauge condition (6.34), and noting that, in this case, \( \mathcal{U}_1 = \{ 0 < u < +\infty \} \cap \{ u < v < u + v_0 \} \) and \( C_0 = 0 \), we readily obtain (4.30) and (4.31), in view of (4.24) and (4.25).
(b) For any \( p \in \mathcal{M} \) such that \( \text{Orb}(p) = \{p\} \), there exists a \( C^1 \)-diffeomorphism
\[
\mathcal{F} : \mathcal{V}_p \to \mathbb{D}^3 \times (-1, 1),
\]
where \( \mathbb{D}^3 \) is the unit 3-disc, such that \( \mathcal{F} \) commutes the action \([A.1]\) of \( \text{SO}(3) \) on \( \mathcal{V}_p \) with the natural action of \( \text{SO}(3) \) on \( \mathbb{D}^3 \times (0, 1) \) by rotations of \( \mathbb{D}^3 \).

We will also define the axis of a \( C^0 \) spherically symmetric spacetime \((\mathcal{M}, g)\) as follows:

**Definition A.2.** The *axis* of the action \([A.1]\) is the set
\[
\mathcal{Z} = \{ p \in \mathcal{M} : \text{Orb}(p) = \{p\}\}.
\]

Definition \([A.1]\) implies that \( \mathcal{Z} \) (if non-empty) is a 1-dimensional \( C^1 \) submanifold of \( \mathcal{M} \).

Remark. For the rest of this section, we will only work on \( C^0 \) spherically symmetric spacetimes \((\mathcal{M}, g)\) with \( \mathcal{Z} \neq \emptyset \).

Let us define the continuous function \( r : \mathcal{M} \to [0, +\infty) \) by the relation
\[
r(p) = \sqrt{\frac{\text{Area}(\text{Orb}(p))}{4\pi}}.
\]

In view of the properties 3.a and 3.b of \( C^0 \) spherically symmetric spacetimes (see Definition \([A.1]\)) and the fact that \( g \) is non-degenerate on \( \mathcal{M} \), we infer that
\[
r(p) = 0 \iff p \in \mathcal{Z}.
\]

For any \( p \in \mathcal{Z} \), let \( \mathcal{V}_p \) be the open neighborhood of \( p \) in \( \mathcal{M} \) appearing in Definition \([A.1]\). According to 3.b in Definition \([A.1]\) \( \mathcal{V}_p \) is identified with \( \mathbb{D}^3 \times (-1, 1) \) through a \( C^1 \)-diffeomorphism. In the natural \( \text{SO}(3) \)-invariant coordinate chart \( \mathcal{U}_p \times \mathbb{S}^2 \) on \( \mathcal{V}_p \) \( \mathcal{Z} \cong (\mathbb{D}^3 \setminus \{0\}) \times (-1, 1) \) (where \( \mathcal{U}_p \subset \mathbb{R}^2 \) is naturally identified with a radial slice of \( (\mathbb{D}^3 \setminus \{0\}) \times (-1, 1) \)), the metric \( g \) splits as
\[
g = \bar{g} + r^2 \delta_{\mathbb{S}^2},
\]
where \( \bar{g} \) is a \( C^0 \) Lorentzian metric on \( \mathcal{U}_p \), extending continuously on \( \partial \mathcal{U}_p \). Note that the resulting \( C^1 \) projection \( \pi : \mathcal{V}_p \setminus \mathcal{Z} \to \mathcal{U}_p \times \mathbb{S}^2 \to \mathcal{U}_p \) admits a \( C^1 \) extension on \( \mathcal{Z} \), mapping \( \mathcal{Z} \) into \( \partial \mathcal{U}_p \). We will denote
\[
\gamma_{\mathcal{Z}, p} \cong \pi(\mathcal{Z}) \subset \partial \mathcal{U}_p \subset \mathbb{R}.
\]

In view of the properties 1 and 3.b of \( C^0 \) spherically symmetric spacetimes (see Definition \([A.1]\)) and the fact that \( g \) is non-degenerate, every connected component of \( \mathcal{Z} \) is a timelike curve in \( \mathcal{M} \). Thus, the curve \( \gamma_{\mathcal{Z}, p} \) in \( \mathbb{R}^2 \) is timelike with respect to \( \bar{g} \).

**Definition A.3.** A \( C^0 \) spherically symmetric spacetime \((\mathcal{M}, g)\) with non-empty axis \( \mathcal{Z} \) will be called *admissible* if all of the following conditions are satisfied:

1. \( \mathcal{Z} \) is connected.
2. \((\mathcal{M}, g)\) has a "simple" topology in the following sense: There exists an \( \text{SO}(3) \)-invariant \( C^1 \)-diffeomorphism \( \mathcal{F} : \mathcal{M} \setminus \mathcal{Z} \to \mathcal{U} \times \mathbb{S}^2, \mathcal{U} \subset \mathbb{R}^2 \), with \( \pi : \mathcal{M} \setminus \mathcal{Z} \xrightarrow{\mathcal{F}} \mathcal{U} \times \mathbb{S}^2 \to \mathcal{U} \) extending as a \( C^1 \) map on \( \mathcal{Z} \) with
\[
\gamma_{\mathcal{Z}} \cong \pi(\mathcal{Z}) \subset \partial \mathcal{U}.
\]
3. There exists a pair of \( C^0 \) functions \( u, v : \mathcal{U} \cup \gamma_{\mathcal{Z}} \to \mathbb{R} \) such that:
(a) For any $u_0, v_0$ in the image of $u, v$, respectively, the level curves $\{u = u_0\}$ and $\{v = v_0\}$ are $C^1$ curves in $\mathcal{U} \cup \gamma_Z$, either intersecting transversally or not intersecting at all. In particular, $(u, v)$ constitute a $C^0$ coordinate chart on $\mathcal{U} \cup \gamma_Z$ and the coordinate vector fields $\partial_u, \partial_v$ are well defined $C^0$ vector fields on $\mathcal{U} \cup \gamma_Z$.

(b) The vector fields $\partial_u, \partial_v$ satisfy
\[(A.11) \quad \bar{g}(\partial_u, \partial_u) = \bar{g}(\partial_v, \partial_v) = 0\]
everywhere on $\mathcal{U} \cup \gamma_Z$.

4. Any other pair $(\bar{u}, \bar{v})$ of $C^0$ functions on $\mathcal{U} \cup \gamma_Z$ satisfying the above property is related to $(u, v)$ by a transformation of the form $\bar{u} = U(u)$, $\bar{v} = V(v)$, for some unique and strictly monotonic, locally bi-Lipschitz functions $U, V$.

Remark. The pair $(u, v)$ will be called a double null foliation. The existence of a double-null foliation $(u, v)$ locally around each point on $Z$ can be readily established in the case when $\bar{g}$ (see (A.8)) is assumed, in addition, to be of $C^{0,1}$ regularity. In general, however, when $\bar{g}$ is merely $C^0$, the integral curves of a congruence of null vectors for $\bar{g}$ passing through a given point are not necessarily unique, and it is not necessary that a continuous foliation of $\mathcal{U}$ by such curves exists (even if one restricts to open subsets of $\mathcal{U}$).

In the case when $\bar{g}$ is assumed to be $C^{0,1}$, condition 4 in Definition [A.3] is also automatically satisfied.

Note that, if $(\mathcal{M}, g)$ is an admissible $C^0$ spherically symmetric spacetime, for any $p \in Z$, in a $(u, v)$ coordinate chart as in Definition [A.3] the metric $g$ takes the form
\[(A.12) \quad g = -\Omega^2 du dv + r^2 g_{\mathbb{S}^2},\]
where $\Omega > 0$ is a $C^0$ function on $\mathcal{U}$, extending continuously on $\gamma_Z$ so that
\[(A.13) \quad \Omega|_{\gamma_Z} > 0.\]

### A.2 The spherically symmetric Einstein–null dust system on admissible $C^0$ spherically symmetric spacetimes

Let $(\mathcal{M}^{3+1}, g)$ be an admissible $C^0$ spherically symmetric spacetime as in Section [A.1] with non-empty axis $Z$. Recall that the metric $g$ is expressed on $\mathcal{M} \setminus Z$ as \[(A.12),\] where $\Omega, r \in C^0(\mathcal{U})$ extend continuously on $\gamma_Z$ such that
\[(A.14) \quad \Omega > 0 \text{ on } \mathcal{U} \cup \gamma_Z\]
and
\[(A.15) \quad r|_{\gamma_Z} = 0, \quad r|_{\mathcal{U}} > 0.\]

By possibly reparametrising the functions $u, v$ as $u \to U(u)$, $v \to V(v)$ for some $C^1$ functions $U, V : \mathbb{R} \to \mathbb{R}$, we will assume that
\[(A.16) \quad u = v \text{ on } \gamma_Z.\]

The spherically symmetric Einstein–null dust system for $(r, \Omega^2; \tau, \bar{\tau})$ on $\mathcal{U}$, where $\tau, \bar{\tau}$ are regular Borel measures on $\mathcal{U}$, is the following system:
\[(A.17) \quad \partial_u \partial_v (r^2) = -\frac{1}{2} (1 - \Lambda r^2) \Omega^2,\]
\[(A.18) \quad \partial_u \partial_v \log(\Omega^2) = \frac{\Omega^2}{2r^2} \left( 1 + 4 \Omega^{-2} \partial_u r \partial_v r \right),\]
\[(A.19) \quad \partial_v (\Omega^{-2} \partial_u r) = -4 \pi r^{-1} \Omega^{-2} \bar{\tau},\]
\[(A.20) \quad \partial_u (\Omega^{-2} \partial_v r) = -4 \pi r^{-1} \Omega^{-2} \tau,\]
\[(A.21) \quad \partial_u \tau = 0,\]
\[(A.22) \quad \partial_v \bar{\tau} = 0,\]
where $\Lambda \in \mathbb{R}$ is fixed. Notice that, for equations (A.17)–(A.22) to be well-defined in the sense of distributions, it suffices that $\Omega, r \in C^0$, $\Omega, r > 0$. The system (A.17)–(A.22) is also supplemented by the following boundary conditions on $\gamma_Z$:

(A.23) \[ r|_{\gamma_Z} = 0 \]

and:

(A.24) \[ \tau|_{\gamma_Z} = \bar{\tau}|_{\gamma_Z} \]

Note that the condition (A.24) is well defined because of (A.21) and (A.22).

Remark. The condition (A.24) arises naturally by requiring that the energy momentum tensor $T_{\mu\nu}$ on $\mathcal{M}$, defined in the $(u, v, y^1, y^2)$ coordinate chart on $\mathcal{M} \setminus \mathcal{Z} \simeq \mathcal{U} \times S^2$ by

(A.25) \[ T_{\mu\nu} = r^{-2}\tau, \quad T_{uv} = r^{-2}\gamma, \quad T_{Au} = T_{Av} = T_{AB} = 0, \]

satisfies (in the weak sense) the conservation-of-energy condition $\nabla^\mu T_{\mu\nu} = 0$ everywhere on $\mathcal{M}$ (using also the gauge condition (A.16)).

Notice that the system (A.17)–(A.22) is gauge invariant in the following sense: If $(r, \Omega^2, \tau, \bar{\tau})$ is a solution to (A.17)–(A.22), then, for any double null coordinate transformation of the form

(A.26)

\[
\begin{align*}
  u' &= U(u), \\
  v' &= V(v),
\end{align*}
\]

for some strictly monotonic, locally bi-Lipschitz functions $U, V$, the set of functions

(A.27) \[ \left( r', (\Omega')^2, \tau', \bar{\tau}' \right) \big|_{(u', v')} \equiv \left( r, \Omega^2, \frac{\tau}{(dU/du)^2}, \frac{\bar{\tau}}{(dV/dv)^2} \right) \big|_{(U^{-1}(u'), V^{-1}(v'))} \]

is a solution of (A.17)–(A.22) in the new coordinates. In view of condition 4 in Definition A.3, any double null foliation $(u', v')$ on an admissible $C^0$ spherically symmetric spacetime $(M, g)$ is related to $(u, v)$ by a transformation of the form (A.26). Furthermore, if the new coordinates $u', v'$ also satisfy the gauge condition (A.16), then $\tau', \bar{\tau}'$ satisfy (A.24) (provided (A.24) is satisfied by $\tau, \bar{\tau}$).

The following regularity result for (A.17)–(A.22) can be readily established:

Lemma A.1. Let $(\mathcal{M}, g)$ be a $C^0$ spherically symmetric spacetime, and let $\mathcal{U}, \gamma_Z$, $(u, v)$ and $r, \Omega^2 \in C^0(\mathcal{U} \cup \gamma_Z)$ be as above. Let also $\tau, \bar{\tau}$ be regular (and non-negative) Borel measures on $\mathcal{U}$. Assume that $(r, \Omega^2; \tau, \bar{\tau})$ satisfy (in the weak sense) (A.17)–(A.22) on $\mathcal{U}$. Then, for any $u_0, v_0 \in \mathbb{R}$ such that \{ $u = u_0$ \} and \{ $v = v_0$ \} are non-trivial curves on $\mathcal{U}$, the derivatives $\partial_u r$ and $\partial_v r$ are defined almost everywhere on \{ $u = u_0$ \} $\cap$ $\mathcal{U}$ and \{ $v = v_0$ \} $\cap$ $\mathcal{U}$, respectively, with

(A.28) \[ \partial_u r \in L^\infty_{\text{loc}}(\{ u = u_0 \} \cap \mathcal{U}), \]

(A.29) \[ \partial_v r \in L^\infty_{\text{loc}}(\{ v = v_0 \} \cap \mathcal{U}). \]

If, in addition, $\tau, \bar{\tau}$ are non-negative $L^1_{\text{loc}}$ functions on $\mathcal{U}$ (and not merely measures), then

(A.30) \[ \partial_u r, \partial_v r \in C^0(\mathcal{U}). \]

Proof. In view of (A.21) and (A.22), we can write

(A.31) \[ \tau(u, v) = \tau(u) \]

and

(A.32) \[ \bar{\tau}(u, v) = \bar{\tau}(v) \]

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and \( r \), that (in the sense of distributions). Therefore, (A.28) and (A.29) follow readily from (A.19) and (A.20), using the fact that \( r, \Omega \in C^0(\mathcal{U}) \) and that \( \tau, \bar{\tau} \) are non-negative regular Borel measures.

In the case when \( \tau, \bar{\tau} \) are also \( L^1_{loc} \) functions, the same procedure yields that, for any \( u_0, v_0 \in \mathbb{R} \) such that \( \{ u = u_0 \} \) and \( \{ v = v_0 \} \) are non-trivial curves on \( \mathcal{U} \)

\[
\begin{align*}
\partial_u r & \in C^0(\{ u = u_0 \} \cap \mathcal{U}), \\
\partial_v r & \in C^0(\{ v = v_0 \} \cap \mathcal{U}).
\end{align*}
\]

Integrating equation (A.17) in \( u \) and in \( v \), using the fact that the right hand side of (A.17) is continuous in \( u \), we obtain (A.30).

We will also need to define the notion of a \( C^0 \) future extension of a solution to (A.17)–(A.22).

**Definition A.4.** Let \((\mathcal{M}, g)\) be an admissible \( C^0 \) spherically symmetric spacetime with non-empty axis \( \mathcal{Z} \), and let \( \mathcal{U}, \gamma_\mathcal{Z}, (u, v) \) and \( r, \Omega^2 \in C^0(\mathcal{U} \cup \gamma_\mathcal{Z}) \) be as above. Let us fix a time orientation on \((\mathcal{M}, g)\) by requiring that the timelike vector field \( N = \partial_u + \partial_v \) in \( \mathcal{U} \) is future directed. Let also \( \tau, \bar{\tau} \) be regular (and non-negative) Borel measures on \( \mathcal{U} \) and assume that \((r, \Omega^2; \tau, \bar{\tau})\) satisfy (in the weak sense) (A.17)–(A.22) on \( \mathcal{U} \).

An admissible \( C^0 \) spherically symmetric and time oriented spacetime \((\bar{\mathcal{M}}, \bar{g})\) will be called a \( C^0 \) spherically symmetric future extension of \((\mathcal{M}, g)\) as a solution of (A.17)–(A.22) if the following conditions are satisfied:

1. There exists a \( C^1 \) embedding \( i: \mathcal{M} \to \bar{\mathcal{M}} \) which is an isometry, i.e. \( i^* \bar{g} = g \), and preserves time orientation.
2. There exists a point \( p \in \bar{\mathcal{M}} \setminus \mathcal{M} \) lying to the future of \( i(\mathcal{M}) \).
3. There exists a double null foliation \((\bar{u}, \bar{v})\) on \((\bar{\mathcal{M}}, \bar{g})\), not necessarily coinciding with \((u, v) \circ i^{-1} \) on \( \mathcal{M} \) (see Definition A.3), with the following property: Denoting with \( \bar{\mathcal{Z}} \) the axis of \( \bar{\mathcal{M}} \), with \( \bar{\mathcal{U}}, \gamma_{\bar{\mathcal{Z}}} \subset \mathbb{R}^2 \) the sets related to \((\bar{u}, \bar{v})\) according to Definition A.3 and with \( \bar{r}, \bar{\Omega}^2 \) the metric components defined for \( \bar{g} \) by (A.12), assuming also that (A.16) holds, there exists a pair of regular, non-negative Borel measures \( \bar{\tau}, \bar{\bar{\tau}} \) on \( \mathcal{U} \) such that:

   a) \((\bar{r}, \bar{\Omega}^2; \bar{\tau}, \bar{\bar{\tau}})\) satisfy (in the weak sense) (A.17)–(A.22) on \( \bar{\mathcal{U}} \) and (A.23)–(A.24) on \( \gamma_{\bar{\mathcal{Z}}} \).

   b) Restricted to \( i(\mathcal{M}) \), the pair \((\bar{\tau}, \bar{\bar{\tau}})\) satisfies:

\[
\left( \bar{\tau}, \bar{\bar{\tau}} \right)_{(\bar{u}, \bar{v})} = \left( \frac{\tau}{(dU/du)^2}, \frac{\bar{\bar{\tau}}}{(dV/dv)^2} \right)_{(U^{-1}(\bar{u}_0), V^{-1}(\bar{v}_0))},
\]

where the functions \( U, V \) are strictly increasing, locally bi-Lipschitz functions defining the following coordinate transformation between \((u, v)\) and \((\bar{u} \circ i, \bar{v} \circ i)\) on \( \mathcal{M} \):

\[
\begin{align*}
\bar{u} \circ i &= U(u), \\
\bar{v} \circ i &= V(v),
\end{align*}
\]

(such \( U, V \) exist and are unique according to Definition A.3)\footnote{The fact that \( U, V \) are increasing follows from the fact that \( i \) preserves time orientation.}

Finally, in the next section, we will need the notion of a smooth characteristic initial value problem for (A.17)–(A.22).

**Definition A.5.** Let \( v_0 > 0 \). A smooth characteristic initial data set for (A.17)–(A.22) on \( u = 0 \) for \( v \in [0, v_0] \) consists of a set of smooth functions \((r_f, \Omega_f^2; \gamma_f, \tau_f)\) on \((0, v_0)\) satisfying the following properties:

1. \( r_f(0) = 0 \) and \( r_f|_{(0,v_0)} > 0 \),
2. \( \Omega_f > 0 \),
3. $\bar{\tau}_f, \tau_f \geq 0$.
4. $r_f, \Omega_f, \bar{\tau}_f$ satisfy the constraint equation
   \begin{equation}
   \partial_v (\Omega_f^{-2} \partial_v r_f) = -4\pi r_f^{-1} \Omega_f^{-2} \bar{\tau}_f.
   \end{equation}
5. $\tau_f$ is constant, i.e. satisfies
   \begin{equation}
   \partial_v \tau_f = 0.
   \end{equation}

A.3 Break down for the system (A.17)–(A.22)

In this section, we will establish two results: one related to the well posedness of the system (A.17)–(A.22) up to the first point when the null dust reaches the axis, and one related to the break down of (A.17)–(A.22) beyond that point.

Our first result is the following:

**Proposition A.1.** For any $0 < v_* < v_0$, let $(r_f, \Omega_f, \tau_f, \bar{\tau}_f)$ be a smooth characteristic initial data set for (A.17)–(A.22) on $u = 0$, according to Definition A.5, satisfying the following properties:

- $(r_f, \Omega_f^2, \bar{\tau}, \bar{\tau}_f)$ is purely ingoing, i.e.:
  \begin{equation}
  \tau_f = 0.
  \end{equation}

- $(r_f, \Omega_f^2, \bar{\tau}, \bar{\tau}_f)$ is trivial on $[0, v_*)$, i.e.:
  \begin{equation}
  \bar{\tau}_f|_{[0,v_*)} = 0.
  \end{equation}
• In the case \( \Lambda > 0 \):

\[
\min_{[0, v^*_0]} \left(1 - \frac{1}{3} \Lambda r^2 \right) > 0.
\]

Then, there exists some (possibly small) \( 0 < \delta < v_0 - v_* \) depending on \( (r_1, \Omega^2, \bar{\tau}, \bar{\tau}) \), such that, setting

\[
U = \{0 < u < v_*\} \cap \{u < v < v_* + \delta\},
\]

\[
\gamma_Z \doteq \{u = v\} \cap \{0 < u < v_*\}
\]

and

\[
S_{v_* + \delta} \doteq \{u = 0\} \cap \{0 < v < v_* + \delta\},
\]

there exists a unique \( C^\infty \) quadruple \( (r, \Omega^2, \tau, \bar{\tau}) \) on \( U \cup \gamma_Z \cup S_{v_* + \delta} \) solving (A.17)–(A.22) on \( U \) and satisfying the initial data

\[
(r, \Omega^2, \tau, \bar{\tau})|_{u=0} = (r_1, \Omega^2_1, \tau_1, \bar{\tau}_1).
\]

on \( S_{v_* + \delta} \) and the boundary conditions (A.15), (A.24) on \( \gamma_Z \). Furthermore, \( (r, \Omega^2, \tau, \bar{\tau}) \) satisfy the following properties:

1. \( (r, \Omega^2, \tau, \bar{\tau}) \) extend as \( C^\infty \) functions on \( \partial U \), satisfying moreover

\[
\inf_{U} \Omega > 0,
\]

\[
\sup_{U} \partial_u r < 0 < \inf_{U} \partial_v r
\]

and, for any \( k \in \mathbb{N} \)

\[
\sup_{U} \frac{m}{r^k} < +\infty
\]

(where \( m \) is defined by (2.6)).

2. The solution is purely ingoing, i.e. the function \( \tau \) satisfies

\[
\tau \equiv 0.
\]

3. The solution is vacuum in the region

\[
D_{vac} = \{0 \leq u \leq v_*\} \cap \{u \leq v \leq v_*\},
\]

i. e.:

\[
\tilde{\Omega}^{(0\leq u\leq v_*) \cap \{u\leq v \leq v_*\}} = 0
\]

Remark. Let \( (\mathcal{M}, g) \) be the smooth, spherically symmetric spacetime with boundary, obtained by equipping \( (\text{clos}(U) \setminus \gamma_Z) \times S^2 \) with the metric

\[
g = -\Omega^2 du dv + r^2 \hat{g}_{S^2}
\]

and then attaching an axis \( Z \) corresponding to \( \gamma_Z \). Then, it follows from Proposition A.1 that \( (\mathcal{M}, g) \) is \( C^\infty \) extendible as a Lorentzian manifold beyond \( \partial \mathcal{M} = (\partial U \setminus \gamma_Z) \times S^2 \).
Proof. Let us define the Hawking mass \( m \) and the renormalised Hawking mass \( \tilde{m} \) by the relations (2.6) and (2.44), respectively. The construction of \((r, \Omega^2, \tau, \bar{\tau})\) will proceed by solving, instead of (A.17)–(A.22), the following (equivalent) system for \( r, m, \tau, \bar{\tau} \) on \( U \):

\[
\begin{align*}
\partial_u \log \left( \frac{\partial_v r}{1 - \frac{2m}{r}} \right) &= -4\pi r^{-1} \frac{\tau}{\partial_u r}, \\
\partial_v \log \left( \frac{\partial_u r}{1 - \frac{2m}{r}} \right) &= 4\pi r^{-1} \frac{\bar{\tau}}{\partial_v r}, \\
\partial_u \tilde{m} &= -\frac{2\pi}{1 - \frac{2m}{r}} \frac{\tau}{\partial_u r}, \\
\partial_v \tilde{m} &= \frac{2\pi}{1 - \frac{2m}{r}} \frac{\bar{\tau}}{\partial_v r}, \\
\partial_u \bar{\tau} &= 0, \\
\partial_v \tau &= 0.
\end{align*}
\]

The proof of Proposition [A.1] will conclude by showing that there exists a \( 0 < \delta < v_0 - v_* \) and a unique smooth quadruple \((r, m, \tau, \bar{\tau})\) on \( U \cup \gamma_Z \cup S_{v_*+\delta} \), solving (A.53)–(A.58) on \( U \) with the boundary conditions (A.15) and (A.37) on \( \gamma_Z \) and the initial conditions

\[
(r, m, \bar{\tau})|_{S_{v_*+\delta}} = (r_f, m_f, \bar{\tau}_f)
\]

on \( S_{v_*+\delta} \) and such that, moreover:

1. \((r, m, \tau, \bar{\tau})\) extend smoothly on \( \partial U \), satisfying (A.47),

\[
\inf_U \left( 1 - \frac{2m}{r} \right) > 0,
\]

and (A.48) (for any \( k \in \mathbb{N} \)).

2. The purely ingoing condition (A.49) holds.

3. The relation (A.51) holds.

Remark. In view of (2.6), the smooth extension of \( \Omega \) on \( \partial U \) follows from the smooth extension of \( r, m \) on \( \partial U \), (A.60) and (A.48).

Since \( \bar{\tau}_f \in C^\infty([0, v_0]) \) satisfies (A.40), we can bound for any \( k \in \mathbb{N} \):

\[
\sup_{[0, v_0]} \frac{\bar{\tau}_f}{|r - v_*|} < +\infty.
\]

Since \( r_f, \Omega_f \in C^\infty([0, v_0]) \) satisfy the constraint equation (A.37) with \( r_f(0) = 0, r_f'(0, v_0) > 0 \) and \( \Omega_f > 0 \), there exists some \( 0 < \delta < v_0 - v_* \) such that

\[
\min_{[0, v_*+\delta]} \partial_r r_f > \varepsilon_\Lambda,
\]

where \( 0 < \varepsilon_\Lambda \ll 1 \) is a fixed small parameter depending on \( \Lambda \) and (A.41). Furthermore, in view of (A.41) and (A.40), we can assume that \( \delta > 0 \) is small enough so that

\[
\inf_{[0, v_*+\delta]} \left( 1 - \frac{2m_f}{r_f} \right) > \varepsilon_\Lambda
\]
where the initial Hawking mass \( m_f \) is defined by the relation
\[
\partial_v (m_f - \frac{1}{6} \Lambda r_f^3) = 2\pi \frac{(1 - \frac{2m_f}{r_f})}{\partial_v r_f} \tau_f
\]
under the condition \( m_f(0) = 0 \); note that, equivalently, \( m_f \) can be defined by
\[
m_f(v) = \frac{1}{2} \left( r_f(v) - \Omega_f^2(v) \partial_v r_f(v) \int_0^v (1 - \Delta r_f^2(\bar{v})) \Omega_f^2(\bar{v}) \, d\bar{v} \right).
\]
Finally, in view of (A.40), we can assume that \( \delta \) is small enough so that
\[
\frac{\max_{[0,v_\ast+\delta]} \partial_v \tilde{m}}{\min_{[0,v_\ast]} \partial_v r_f} + \frac{\partial_v \tilde{m}}{|v - v_\ast|} < \varepsilon_{\Lambda}.
\]
From now on, we will assume that \( \delta \) has been fixed as above.

A priori, if a smooth smooth solution \((r, m, \tau, \bar{\tau})\) to (A.53)-(A.58) on \( \mathcal{U} \) satisfying (A.15), (A.37) and (A.59) exists, then (A.51) follows immediately from (A.40) and (A.57). Moreover, (A.49) follows readily from the initial condition (A.40) for \( \tau \), the conservation equations (A.57)-(A.58) and the form of the domain \( \mathcal{U} \). Thus, the existence and uniqueness of a smooth solution \((r, m, \tau, \bar{\tau})\) to (A.53)-(A.58) on \( \mathcal{U} \) satisfying (A.15), (A.37) and (A.59) is reduced to the existence and uniqueness of a smooth solution \((r, m, \bar{\tau})\) of the system
\[
\frac{\partial_u (\frac{\partial_v r}{1 - \frac{2m}{r}})}{\partial_{u} r} = 0,
\]
\[
\partial_v \log \left( \frac{-\partial_u r}{1 - \frac{2m}{r}} \right) = 4\pi r^{-1} \frac{\bar{\tau}}{\partial_v r},
\]
\[
\partial_u \tilde{m} = 0,
\]
\[
\partial_v \tilde{m} = 2\pi \frac{1 - \frac{2m}{r}}{\partial_v r} \bar{\tau},
\]
\[
\partial_u \bar{\tau} = 0
\]
on the domain
\[
\mathcal{U}_{\bar{\tau}} = (0, v_\ast) \times (v_\ast, v_\ast + \delta)
\]
satisfying the characteristic initial conditions
\[
(r, m, \bar{\tau})_{t=0} = (r_f, m_f, \bar{\tau}_f)_{[v_\ast, v_\ast + \delta)}
\]
and
\[
r_{v_{\text{vac}}} |_{(0, v_\ast) \times \{v_\ast\}} = r_{v_{\text{vac}}}, [0, v_\ast) \times \{v_\ast\},
\]
where the function \( r_{v_{\text{vac}}} \) in the vacuum region (A.50) is determined by solving
\[
\left\{
\begin{aligned}
\partial_u \frac{\partial_{v} r_{v_{\text{vac}}}}{1 - \frac{2m_{v_{\text{vac}}}}{r_{v_{\text{vac}}}}} &= 0, \quad \text{on } \text{clos}(\mathcal{D}_{v_{\text{vac}}}) \\
r_{v_{\text{vac}}} = r_f & \quad \text{on } \{0\} \times [0, v_\ast] \\
r_{v_{\text{vac}}} |_{u=v} &= r_f
\end{aligned}
\right.
\]
(note that (A.41) is necessary for (A.75) to admit a smooth solution on the whole of \( \text{clos}(\mathcal{D}_{v_{\text{vac}}}) \)). Given a smooth solution \((r, m, \bar{\tau})\) to (A.67)-(A.71) on \( \mathcal{U} \) satisfying (A.73)-(A.74), we can then obtain a smooth solution \((r, m, \tau, \bar{\tau})\) of (A.53)-(A.58) on \( \mathcal{U} \) satisfying (A.15), (A.37) and (A.59) by extending \( r, m, \bar{\tau} \) smoothly \( \mathcal{D}_{v_{\text{vac}}} \) by the relations
\[
r_{v_{\text{vac}}} |_{\mathcal{D}_{v_{\text{vac}}} = r_{v_{\text{vac}}}}.
\]

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and setting $\tau \equiv 0$.

The existence and uniqueness of a smooth solution $(r, m, \bar{\tau})$ to (A.67)–(A.71) on $U_\#$ satisfying (A.73)–(A.74) follows readily: Equations (A.69) and (A.71) imply that

(A.78) \[ \tilde{m}(u, v) = \tilde{m}_f(v) \]

and

(A.79) \[ \bar{\tau}(u, v) = \bar{\tau}_f(v) \]

and, thus, equation (A.67) is equivalent to

\[ \partial_u \left( \frac{\partial_r r}{1 - \frac{2m}{r} - \frac{1}{3} \Lambda r^2} \right) = 0. \] (A.80)

Note that, in view of the initial bounds (A.62)–(A.66), applying a standard bootstrap argument for equation (A.80) (the details of which will be omitted), we infer that there exists a unique smooth function $r$ on $U_\#$ solving (A.80) and satisfying (A.73)–(A.74), such that, in addition:

(A.81) \[ \inf_{U_\#} \left( 1 - \frac{2m}{r} \right) > \frac{1}{2} \varepsilon \Lambda \]

and

(A.82) \[ \inf_{U_\#} \partial_v r > \frac{1}{2} \varepsilon \Lambda. \]

The bound (A.60) on $U$ follows readily from (A.81), (A.76) and (A.77). From (A.61) and the transport equations (A.67), (A.69) and (A.71), we readily infer (A.48).

Finally, in view of the fact that $(r, m, \bar{\tau})$ satisfy equations (A.67)–(A.71) everywhere on $U$, from (A.60) and (A.48) it readily follows that $(r, m, \bar{\tau}, \tau)$ extend smoothly on $\partial U$.

Our second result is the following ill-posedess theorem:

**Theorem A.1.** Let $0 < v_* < v_0$, $(r_f, \Omega_f^2, \tau_f, \bar{\tau}_f)$, $\delta$, $\mathcal{U}$ and $\gamma_Z$ be as in Proposition A.1. Let also $(\mathcal{M}, g)$ be the $C^\infty$ spacetime with boundary, obtained by equipping $(\text{clos}(\mathcal{U}) \setminus \gamma_Z) \times S^2$ with the metric

(A.83) \[ g = -\Omega^2 du dv + r^2 g_{S^2} \]

and then attaching an axis $Z$ corresponding to $\gamma_Z$ (see the remark below Proposition A.1). Assume also that there exists a $0 < \delta_1 \leq \delta$ such that $\bar{\tau}_f$ satisfies

(A.84) \[ \bar{\tau}_f|_{(v_*, v_0 + \delta_1)} > 0. \]

Then, there exists no globally hyperbolic, admissible $C^0$ spherically symmetric spacetime $(\tilde{\mathcal{M}}, \tilde{g})$ (see Section A.1 for the relevant definitions) which is a $C^0$ spherically symmetric future extension of $(\mathcal{M}, g)$ as a solution of (A.17)–(A.22) according to Definition A.4.

**Proof.** We will argue by contradiction, assuming that there exists an admissible $C^0$ spherically symmetric spacetime $(\tilde{\mathcal{M}}, \tilde{g})$ which is globally hyperbolic and at the same time is a $C^0$ spherically symmetric future extension of $(\mathcal{M}, g)$ as a solution of (A.17)–(A.22).
Let \( i : \mathcal{M} \to \tilde{\mathcal{M}}, \tilde{Z}, \tilde{\mathcal{U}}, \tilde{\gamma}_Z, (\bar{u}, \bar{v}), (\check{r}, \check{\Omega}^2, \check{\bar{\tau}}, \check{\bar{\mathcal{V}}}) \) and \( U, V \in C^0 \) be as in definition \( \text{A.4} \). Recall that the double null foliations \( (u, v) \) and \((\bar{u}, \bar{v}) \circ i \) on \( \mathcal{M} \) are related by

\[
\begin{align*}
\check{\bar{u}} \circ i &= U(u), \\
\check{\bar{v}} \circ i &= V(v),
\end{align*}
\]

where \( U, V \) are strictly increasing, locally bi-Lipschitz functions. Furthermore, \( \bar{u}, \bar{v} \) are assumed to satisfy

\[ \bar{u} = \check{\bar{v}} \text{ on } \gamma_Z. \]

In view of \( \text{A.42} \) and \( \text{A.85} \), the projection (under spherical symmetry) of \( i(\mathcal{M}) \) on \( \tilde{\mathcal{U}} \cup \gamma_Z \) is of the form

\[
\tilde{\pi} \circ i(\mathcal{M}\backslash \bar{Z}) = \{ U(0) < \bar{u} < V(v_*) \} \cap \{ V^{-1}(\bar{v}) < V(v_* + \bar{\delta}) \}.
\]

Since \( i(\mathcal{Z}) \subseteq \tilde{Z} \), it is necessary, in view of \( \text{A.86} \), that

\[ V \circ U^{-1}(\bar{u}) = \bar{u} \]

in \( \text{A.87} \). Hence, by passing to a new double null coordinate chart \((u', v')\) on \( (\tilde{\mathcal{M}}, \check{\bar{g}}) \) through a gauge transformation of the form

\[ \begin{align*}
u' &= \tilde{U}(\check{u}), \\
v &= \tilde{V}(\check{v})
\end{align*} \]

for some strictly increasing, locally bi-Lipschitz functions \( \tilde{U}, \tilde{V} \) so that the condition

\[ u' = v' \text{ on } \gamma_Z \]

holds, we will assume without loss of generality that \( \text{A.87} \) is of the form

\[ \tilde{\pi} \circ i(\mathcal{M}\backslash \bar{Z}) = \{ 0 < u' < v_* \} \cap \{ u' < v' < v_* + \bar{\delta} \}. \]

**Remark.** Note that \((u', v')\) do not necessarily coincide with \((u, v)\) on \( \tilde{\pi} \circ i(\mathcal{M}\backslash \bar{Z}) \).

The components \( r', (\Omega')^2 \) of the metric \( \check{\bar{g}} \) (according to the splitting \( \text{A.12} \)) satisfy

\[ r'|_{\gamma_Z} = 0, \ r'|_{\tilde{\mathcal{U}}} > 0 \]

and

\[ \Omega' > 0 \text{ on } \tilde{\mathcal{U}} \cup \gamma_Z \]

(see the properties \( \text{A.7} \) and \( \text{A.13} \) associated to a double null foliation of a general admissible \( C^0 \) spherically symmetric spacetime). We will also denote with \( \bar{\tau}', \bar{\bar{\tau}}' \) the transformed quantities

\[ \left. \left. \left( \bar{\tau}', \bar{\bar{\tau}}' \right) \right|_{(u', v')} \right| = \left( \frac{\check{\bar{\tau}}}{(d\tilde{U}/d\check{u})^2}, \frac{\check{\bar{\tau}}}{(d\tilde{V}/d\check{v})^2} \right) \right| \left. \left. (\tilde{\pi}^{-1}(u'), \tilde{\pi}^{-1}(v')) \right. \right|. \]

Recall that \((r', (\Omega')^2, \bar{\tau}', \bar{\bar{\tau}}')\) satisfy \( \text{A.17} \) \( \text{A.22} \) and \( \text{A.24} \) on \( \gamma_Z \).

Since \( (\tilde{\mathcal{M}}, \check{\bar{g}}) \) is globally hyperbolic and contains a point \( p \in \tilde{\mathcal{M}}\backslash \mathcal{M} \) lying in the future of \( i(\mathcal{M}) \) (see Definition \( \text{A.4} \), \( \tilde{\mathcal{U}} \cup \gamma_Z \cup \mathcal{U} \cup \gamma_Z \) contains a point \( q \) lying in the future of \( \mathcal{U} \cup \gamma_Z \). Therefore, in view of \( \text{A.90} \) and the form \( \text{A.91} \) of \( \tilde{\pi} \circ i(\mathcal{M}\backslash \bar{Z}) \), we infer that \( \tilde{\mathcal{U}} \) contains a set of the form

\[ \mathcal{V} = \{ 0 < u' < v_* + \bar{\delta}_2 \} \cap \{ u' < v' < v_* + \bar{\delta}_2 \} \]
for some fixed $\delta$ satisfying

\[(A.96) \quad 0 < \delta_2 < \frac{1}{2} \min\{\delta, \delta_1\} \]

and moreover:

\[(A.97) \quad \gamma_Z' = \{0 < u' < v_\ast + \delta_2 \} \cap \{u' = v'\} \subset \gamma_Z. \]

For the rest of the proof, we will restrict to $\mathcal{V} \cup \gamma_Z'$.

In view of (A.35) and (A.94), from the fact that $\tau, \bar{\tau}$ are smooth functions of $(u, v)$ on $\mathcal{U}$ and the coordinates $u', v'$ are strictly increasing, locally bi-Lipschitz functions of $u, v$, respectively, we infer that

\[(A.98) \quad \tau', \bar{\tau} \in L^1_{\text{loc}}(\mathcal{V}). \]

Therefore, as a consequence of Lemma A.1:

\[(A.99) \quad \partial_{\nu} r', \partial_{\bar{\nu}} r' \in C^0(\mathcal{V}). \]

In view of the bound (A.47) of Proposition A.1 and the fact that, on $\bar{\pi} \circ i(\mathcal{M})$, the coordinates $u', v'$ are strictly increasing, locally bi-Lipschitz functions of $u, v$, respectively, we infer that, on $\bar{\pi} \circ i(\mathcal{M}\backslash \mathcal{Z})$:

\[(A.100) \quad \inf_{\bar{\pi} \circ i(\mathcal{M}\backslash \mathcal{Z})} \partial_{\nu} r' > 0 \]

and

\[(A.101) \quad \sup_{\bar{\pi} \circ i(\mathcal{M}\backslash \mathcal{Z})} \partial_{\nu} r' < 0. \]

In view of the constraint equation (A.20), we have (in the sense of distributions)

\[(A.102) \quad \partial_{\nu} (\Omega')^{-2} \partial_{\nu} r' \leq 0. \]

Thus, from (A.101) and (A.102), using also the fact that $\Omega'$ is continuous on $\mathcal{V} \cup \gamma_Z$, and satisfies (A.93), we infer that:

\[(A.103) \quad \sup_{\bar{\pi} \circ i(\mathcal{M}\backslash \mathcal{Z})} \sup_{\{\nu \in [v_\ast, v_\ast + \frac{3}{8} \delta_2] \cap \mathcal{V} \}} \partial_{\nu} r' < 0. \]

In view of the transport equations (A.21) and (A.22) for $\tau', \bar{\tau}'$ and the boundary condition (A.24), we obtain, for any $(u', v') \in \mathcal{V}$:

\[(A.104) \quad \tau'(u', v') = \overline{\tau'}(0, v' - u'). \]

Hence, it follows from the assumption (A.84) on $\overline{\tau_j}$ and the bound (A.96) on $\delta_2$ that

\[(A.105) \quad c_0 \leq \inf_{\{v_\ast + \frac{1}{4} \delta_2 \leq v \leq v_\ast + \frac{1}{4} \delta_2\} \cap \mathcal{V}} \tau' > 0. \]

The following relations hold for all $\bar{\nu} \in [v_\ast + \frac{3}{8} \delta_2, v_\ast + \frac{5}{8} \delta_2]$

\[(A.106) \quad \limsup_{u' \to \bar{\nu}^-} (-\partial_{u'} r')(u', \bar{\nu}) = +\infty \]

and

\[(A.107) \quad \lim_{u' \to \bar{\nu}^-} (-r' \partial_{u'} r')(u', \bar{\nu}) = 0. \]

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We will establish (A.106) and (A.107) later. Assuming, for now, that (A.107) holds, we will finish the proof of Theorem A.1 by reaching a contradiction with (A.106).

Let us define the function

\[ y \doteq (r')^2 \]

on

\[ D_{v_\ast} \doteq [v_\ast + \frac{3}{8} \delta_2, v_\ast + \frac{1}{2} \delta_2] \times [v_\ast + \frac{3}{8} \delta_2, v_\ast + \frac{1}{2} \delta_2] \cap \mathcal{V}. \]

Since \( r' \in C^0(\overline{U} \cup \mathcal{Y}_Z) \), we infer that \( y \in C^0(\overline{V} \cup \mathcal{Y}_Z) \).

In view of (A.17), (A.15) and (A.107), \( y \) satisfies the following initial value problem on \( D_{v_\ast} \), with initial data on \( \mathcal{Y}_Z \cap \text{clos}(D_{v_\ast}) \):

\[
\begin{cases}
\partial_u \partial_v y = -\frac{1}{2} (1 - \Lambda(r')^2)(\Omega')^2 & \text{on } D_{v_\ast}, \\
\lim_{u' \to v' -} y|_{(u', v')} = 0 & \text{for all } v' \in [v_\ast + \frac{3}{8} \delta_2, v_\ast + \frac{1}{2} \delta_2], \\
\lim_{u' \to v' -} \partial_u y|_{(u', v')} = 0 & \text{for all } v' \in [v_\ast + \frac{3}{8} \delta_2, v_\ast + \frac{1}{2} \delta_2].
\end{cases}
\]

Therefore, for all \((u', v') \in D_{v_\ast}, y|_{(u', v')} \) can be uniquely represented as

\[
y|_{(u', v')} = -\frac{1}{2} \int_{(u' \leq \bar{u}) \cap (v' \geq \bar{v}) \cap D_{v_\ast}} (1 - \Lambda(r')^2)(\Omega')^2(\bar{u}, \bar{v}) d\bar{u} d\bar{v}.
\]

From (A.110), we infer that there exists some \( C > 0 \) depending only on \( ||r'||_{C^0(V)} \) and \( ||\Omega'||_{C^0(V)} \), such that, for all \((u', v') \in D_{v_\ast}, y|_{(u', v')} \)

\[
|y|_{(u', v')} \leq C^2 \cdot (v' - u')^2
\]
or, equivalently, in view of the definition (A.108) of \( y \):

\[
r'|_{(u', v')} \leq C \cdot (v' - u').
\]

The bound (A.112), combined with (A.92), (A.103) and the fact that \((\Omega')^{-2} \in C^0(V \cup \mathcal{Y}_Z) \) and \((\Omega')^{-2} \partial_u r' \) is decreasing in \( u' \) (in view of (A.20)), yields that

\[
\lim_{u' \to v'} \sup (\partial_u r')|_{(u', v')} < +\infty,
\]

which is a contradiction in view of (A.106).

In order to complete the proof of Theorem A.1 it only remains to establish (A.106) and (A.107).

**Proof of (A.106) and (A.107).** Let us set:

\[
M_0 \doteq \max_{v' \in [v_\ast + \frac{3}{8} \delta_2, v_\ast + \frac{1}{2} \delta_2]} \frac{(\Omega')^2}{-\partial_u r'}(0, v')
\]

\[
\gamma_0 \doteq \min_{v' \in [v_\ast + \frac{3}{8} \delta_2, v_\ast + \frac{1}{2} \delta_2]} \frac{(\Omega')^2}{-\partial_u r'}(0, v')
\]

\[
\varphi_0 \doteq \min_{v' \in [v_\ast + \frac{3}{8} \delta_2, v_\ast + \frac{1}{2} \delta_2]} r'|_{(v_\ast + \frac{1}{2} \delta_2, v')}
\]

and

\[
\tau_0 \doteq \sup_{v' \in [v_\ast + \frac{3}{8} \delta_2]} \bar{\tau}'|_{(0, v')}
\]
Note that, in view of \( (A.103), (A.103), (A.105), (A.105), (A.105) \) and the fact that \( \Omega', r', \partial_{u'} r' \in C^0(\mathcal{U}) \), we have 
\( 0 < M_0 < +\infty, 0 < \mu_0 < +\infty, 0 < \varepsilon_0 < +\infty \) and \( 0 < \delta_0 < +\infty \).

We will first establish \( (A.106) \). We will argue by contradiction, assuming that there exists a \( \bar{v} \in \left[ v_* + \frac{3}{2} \delta_2, v_* + \frac{1}{2} \delta_2 \right] \) such that
\[ (A.126) \]
\[
\liminf_{u' \to \bar{v}^-} (-\partial_u r')(u', \bar{v}) < +\infty.
\]
In view of \( (A.103) \) and \( (A.118) \), the quantity
\[ (A.119) \]
\[
M_1 \doteq \sup_{u' \in [0, \bar{v}]} (-\partial_u r')(u', \bar{v})
\]
satisfies
\[ (A.120) \]
\[
0 < M_1 < +\infty.
\]

The constraint equation \( (A.20) \) can be rewritten as:
\[ (A.121) \]
\[
\partial_{u'} \log \left( \frac{(\Omega')^2}{-\partial_u r'} \right) = \frac{4\pi}{r'} \frac{\tau'}{(-\partial_u r')}.
\]
Therefore, integrating \( (A.121) \) in \( u' \) for \( v' = \bar{v} \) and using \( (A.114), (A.116), (A.103), (A.105) \) and \( (A.119) \), we obtain for any \( 0 \leq \bar{u} < \bar{v} \):
\[ (A.122) \]
\[
\log \left( \frac{(\Omega')^2}{-\partial_u r'} \right)_{(\bar{u}, \bar{v})} - \int_{\bar{u}}^{\bar{v}} \frac{4\pi}{r'} \frac{\tau'}{(-\partial_u r')} (u', \bar{v}) \, du = \]
\[
\leq \log M_0 - 4\pi M_1^{-2} c_0 \int_{v_* + \frac{1}{2} \delta_2}^{\bar{v}} \frac{1}{r'} (-\partial_u r') (u', \bar{v}) \, du = \]
\[
\leq \log M_0 - 4\pi M_1^{-2} c_0 \log \left( \frac{\rho_0}{r(\bar{u}, \bar{v})} \right).
\]
Notice that the right and side of \( (A.122) \) goes to \( -\infty \) as \( \bar{u} \to \bar{v}^- \), which is a contradiction in view of \( (A.118) \). Therefore, \( (A.106) \) holds.

We now proceed to establish \( (A.107) \). We will argue again by contradiction. To this end, since \( r', \Omega' \in C^0(\mathcal{V} \cup \gamma_{x'}) \) and \( (\Omega')^{-2} \partial_{u'} r' \) is decreasing in \( u' \) (in view of \( (A.20) \)), it suffices to reach a contradiction based on the assumption that there exists a \( \bar{v} \in \left[ v_* + \frac{3}{2} \delta_2, v_* + \frac{1}{2} \delta_2 \right] \) such that
\[ (A.123) \]
\[
\liminf_{u' \to \bar{v}^-} (-r' \partial_u r')(u', \bar{v}) \neq 0.
\]
In view of \( (A.103), (A.123) \) is equivalent to
\[ (A.124) \]
\[
c_1 \doteq \inf_{u' \in [0, \bar{v}]} (-r' \partial_u r')(u', \bar{v}) > 0.
\]
Integrating \( (A.121) \) in \( u' \) for \( v' = \bar{v} \) and using \( (A.104), (A.115), (A.117), (A.105) \) and \( (A.124) \), we obtain for any \( 0 \leq \bar{u} < \bar{v} \):
\[ (A.125) \]
\[
\log \left( \frac{(\Omega')^2}{-\partial_u r'} \right)_{(\bar{u}, \bar{v})} - \int_{\bar{u}}^{\bar{v}} \frac{4\pi}{r'} \frac{\tau'}{(-\partial_u r')} (u', \bar{v}) \, du \geq \]
\[
\geq \log \mu_0 - \frac{4\pi \tau_0}{c_1} \int_{v_* + \frac{1}{2} \delta_2}^{\bar{v}} \, du.
\]
Thus,
\[ (A.126) \]
\[
\liminf_{\bar{u} \to \bar{v}^-} \log \left( \frac{(\Omega')^2}{-\partial_u r'} \right)_{(\bar{u}, \bar{v})} > -\infty,
\]
which is a contradiction, in view of \( (A.106) \). Thus, \( (A.107) \) holds, and the proof of Theorem A.1 is complete. \( \square \)
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