Effective Lagrangian for the Polyakov line on a lattice\(^1\)

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ABSTRACT: We formulate a method for computing the effective Lagrangian of the Polyakov line on the lattice. Using mean field approximation we calculate the effective potential for high temperatures. The result agrees with recent lattice simulations. We reveal a new type of ultraviolet divergence (coming from longitudinal gluons) which dominates the effective potential and explains the discrepancy of the lattice simulations and standard perturbative calculations performed in covariant gauges.

KEYWORDS: Polyakov loop, effective Lagrangian, lattice gauge theory

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\(^1\)On December 26 2012 our esteemed colleague Dmitri Diakonov passed away untimely early. At that time he was working with one of us (V.P.) on a follow up project to a paper we had published in June of that year. This article is based on this cooperation and we dedicate it to the memory of our friend Dmitri Diakonov.
1 Introduction

The Polyakov line, defined as the path-ordered exponential of the time component of the Yang–Mills connection in Euclidian signature,

$$\mathcal{P}(\vec{x}) = P \exp \left( i \int_0^{1/T} dt A_4(\vec{x}, t) \right),$$

(1.1)

where $T$ is the temperature, is an order parameter for the confinement-deconfinement phase transition in pure gauge theory [1]. Its average over the gauge-field ensemble behaves as

$$\langle \text{Tr}\mathcal{P}(\vec{x}) \rangle \begin{cases} = 0 & \text{at } T < T_c, \text{ confinement,} \\ \neq 0 & \text{at } T > T_c, \text{ deconfinement.} \end{cases}$$

(1.2)

It would be helpful to know the precise effective Lagrangian for this important variable in order to understand better what physical mechanisms govern the deconfinement phase transition.

Quite recently the effective potential, i.e., the derivative-independent part of the effective Lagrangian has been found for the $SU(2)$ and $SU(3)$ gauge groups by direct numerical simulations on the lattice [2], with puzzling results: At high temperatures ($T \leq 5T_c$) the effective potential as function of the eigenvalues of $\mathcal{P}$ did not follow the expected long-known perturbative potential [3–5] but turned out to be a factor of 20 to 30 times larger, and only suitable ratios of observables could be matched to the perturbative results.
In this paper we first define accurately the effective Lagrangian for the Polyakov loop by using the lattice regularization in the ultraviolet. This is also necessary for a direct comparison with the lattice results \[2\]. Second, we develop a systematic mean-field method for computing the lattice-regularized effective Lagrangian. The method is in fact a well-defined loop expansion about the mean field with the expansion parameter being \(1/\beta\) where \(\beta\) is the lattice inverse gauge coupling. Special care is taken in dealing with the zero modes in that expansion. Third, we compare the results of our analytical calculations with the numerical lattice results \[2\]. We find good agreement between the two already at the one loop level. We thus resolve the above mentioned puzzle in the numerical results and discuss the reason for the large discrepancy between lattice results and the perturbative potential \[3–5\]. Fourth, our expansion about the mean field solution paves the way for computing terms with the spatial derivatives of the Polyakov loop eigenvalues in the effective Lagrangian, which can be also found on the lattice.

2 Lattice partition function as a path integral over the eigenvalues of the Polyakov line

The Yang–Mills partition function defined as a path integral over the connection \(A_\mu(\vec{x}, t)\) with periodic boundary conditions can be identically rewritten in such a way that the integration over \(\mathcal{P}(\vec{x})\) is performed last:

\[
Z_{\text{contin}} = \int D[A] \exp \left( -\frac{1}{2g^2} \int d^4x \text{Tr} F_{\mu \nu} F_{\mu \nu} \right) = \int D[\mathcal{P}] \exp \left( -S_{\text{eff}}[\mathcal{P}] \right).
\]

This equation formally defines what we call the effective action \(S_{\text{eff}}\) and the effective Lagrangian \(\mathcal{L}_{\text{eff}}\) for the Polyakov line, \(S_{\text{eff}}[\mathcal{P}] = \int \! d^3x \mathcal{L}_{\text{eff}}[\mathcal{P}]\).

Let us rewrite Eq. \((2.1)\) using the standard lattice regularization of pure Yang–Mills theory. The partition function is defined as the path integral over link variables \(U_\mu(x) \in SU(N)\),

\[
Z = \int D[U] e^{-\beta S[U]},
\]

where \(S[U]\) is the standard Wilson plaquette action, and \(\beta = \frac{2N}{g^2}\) is the inverse bare gauge coupling constant. The hypercubic lattice has \(N_s\) sites in the spatial directions and \(N_t\) sites in the Euclidean time direction. The corresponding coordinates assume values \(x_1, x_2, x_3 = 0, 1, ... N_s - 1\) and \(x_4 = t = 0, 1, ... N_t - 1\). We use \(x\) to denote the full vector with all 4 components, \(t\) is used for the time component (= 4-component) and \(\vec{x}\) for the spatial part. The lattice spacing is mostly set to \(a = 1\) in this paper and we display \(a\) only where necessary. The link variables \(U_\mu(x)\) are assumed to satisfy periodic boundary conditions in all four directions. The integration measure \(D[U]\) is the product of invariant Haar measures for all \(U_\mu(x)\) normalized to unity. We will use the standard Wilson action

\[
S[U] = \sum_{x, \mu < \nu} \left[ 1 - \frac{1}{N} \text{ReTr} U_{\mu \nu}(x) \right], \quad U_{\mu \nu}(x) = U_\mu(x)U_\nu(x + \hat{\mu})U^\dagger_\mu(x + \hat{\nu})U^\dagger_\nu(x),
\]

where the sum runs over all plaquettes. The temperature is given by \(T = \Lambda/N_t\) where \(\Lambda = \frac{1}{a} f(\beta)\) is a renormalization-group-invariant combination. Usually one uses the relative
temperature, say, the ratio of the temperature to the temperature of the phase transition $T_c$ which should be determined separately for a given setup. Then $\frac{T}{T_c} = \frac{N_t}{N_c}$, where $N_t^{(c)}$ is the critical temporal extent of the lattice.

The partition function \( (2.2) \) implies the integration over all possible gauges and therefore contains the volume of the gauge group. The volume, however, is unity according to our definition of the Haar measure. Let us rewrite eq. \((2.2)\) in the form which is close to the physical gauge $A_4 = 0$ in the continuum. Using standard arguments (see, e.g., \cite{6}) one can fix the gauge links on a tree (a structure of links on the lattice without closed loops) to arbitrary values. This allows us to fix all temporal gauge links to $U_4(\vec{x}, t) = 1$ with $t = 0, 1, ...N_t - 2$, except for the last time-slice, i.e., the temporal links $U_4(\vec{x}, N_t - 1)$. The Polyakov line at $\vec{x}$ then reduces to the temporal link at this last timeslice

$$P(\vec{x}) \equiv \prod_{t=0}^{N_t-1} U_4(\vec{x}, t) = U_4(\vec{x}, N_t - 1).$$

We arrive at the following representation for the statistical sum:

$$Z = \int D[P] \int D[U_j] e^{-\beta S[U_j, P]}. \quad (2.4)$$

The action $S[U_j, P]$ is the Wilson action in the gauge where all temporal links are trivial except those on the last time slice where they are given by $P(\vec{x})$.

The Polyakov lines $P(\vec{x}) \in SU(N)$ are diagonalized by matrices $Q(\vec{x})$ such that

$$Q(\vec{x}) P(\vec{x}) Q(\vec{x})^\dagger = \text{diag} \left(e^{i\varphi_1(\vec{x})}, \ldots, e^{i\varphi_r(\vec{x})}, e^{-i \sum_{n=1}^r \varphi_n(\vec{x})}\right), \quad (2.5)$$

where $r = N - 1$ is the rank of the group. With a gauge transformation independent of time, which leaves the already trivial temporal links unchanged, diagonalizes the $P(\vec{x})$, and transforms the spatial link variables as

$$U_j(\vec{x}, t) \rightarrow Q(\vec{x}) U_j(\vec{x}, t) Q(\vec{x} + \vec{j})^\dagger,$$  \quad (2.6)

one can reduce the matrices $P(\vec{x})$ to diagonal form in the partition sum, i.e., eq. \((2.4)\) depends only on the eigenvalues of the Polyakov lines. Hence, the integration measure $D[P]$ in eq. \((2.4)\) can be substituted by the integration over phases $\varphi_1(\vec{x}) \ldots \varphi_r(\vec{x})$ with the corresponding measure. Note that the eigenvalues of the Polyakov lines $P(\vec{x})$ (i.e., the phases $\varphi_n(\vec{x})$) are gauge invariant.

We thus define the effective action for the Polyakov line on the lattice as:

$$e^{-\beta S_{\text{eff}}(P)} = \int D[U_j] e^{-\beta S[U_j, P]}, \quad (2.7)$$

which is a path integral over only the spatial links $U_j$. The effective action defined in eq. \((2.7)\) depends only on the eigenvalues of the matrices $P(\vec{x})$. The full partition sum and moments of Polyakov lines can be found by integrating this action over the $\varphi_n(\vec{x})$ (including the necessary measure factors).
Yet another gauge transformation $U_\mu(x) \rightarrow S(x) U_\mu(x) S(x + \hat{\mu})$ with
\[
S(\vec{x}, t) = \text{diag} \left( e^{i\frac{N_{t-1}}{N} \varphi_1(\vec{x})}, \ldots, e^{i\frac{N_{t-1}}{N} \sum_{n=1}^{r_n \phi_n(\vec{x})}} \right), \quad t = 0, \ldots, N_t - 1, \quad (2.8)
\]
is used to make all temporal gauge links independent of time,
\[
U_4(\vec{x}, t) = \text{diag} \left( e^{i\frac{\phi_1(\vec{x})}{N_t}}, \ldots e^{-i\frac{\phi_1(\vec{x})}{N_t}} \sum_{n=1}^{r_n \phi_n(\vec{x})} \right), \quad t = 0, N_t - 1. \quad (2.9)
\]
This gauge is the one most suitable for our purposes.

In this paper we will consider the effective action only for phases $\varphi_n$ that are constant in space. In other words, we are going to calculate an effective potential for spatially constant Polyakov lines. We will restrict ourselves to the case of gauge group $SU(2)$. It is straightforward to generalize these calculations to other gauge groups.

Numerical studies of the effective action $S_{\text{eff}}[\mathcal{P}]$ for spatially constant $\varphi_n$ were presented in Ref. [2] for both gauge groups $SU(2)$ and $SU(3)$. Recent results for the effective potential from other approaches can be found in [7].

Another study of an effective potential was presented recently in Ref. [8], where the per-site potential for the Polyakov line was calculated. This object is different from the effective potential discussed here and aims at analyzing gradient terms. In principle, the per-site quantity can be found in our setting by integrating eq. (2.7) over all Polyakov lines $\mathcal{P}(x)$ except for one spatial site.

3 Mean field approximation

To compute the statistical sum $Z$ and the effective action $S_{\text{eff}}[\mathcal{P}]$ one can use mean field approximation which is known to be applicable at sufficiently large $\beta$. Of course, mean field approximation is not able to describe confinement – this approximation is nothing more than a modification of perturbation theory. Nevertheless, it can provide an estimate for the effective action valid at large $\beta$ and/or at large temperatures.

For our mean field calculation we will follow the approach of [9] where mean field theory for lattice fields theories was considered as a version of saddle point approximation.

We identically rewrite the expression eq. (2.7) for the effective action as:
\[
e^{-\beta S_{\text{eff}}[\mathcal{P}]} = \int D[H_j] D[V_j] \int D[U_j] \exp \left( -\beta S[V_j, \mathcal{P}] + \text{Re} \sum_{x,j} \text{Tr}[H_j^\dagger(x)(U_j(x) - V_j(x))] \right). \quad (3.1)
\]
Here $H_j$ and $V_j$ are suitable $N \times N$ matrices defined on the spatial links, which we will specify in more detail below. The $H_j$ are chosen such that integration $\int D[H_j]$ produces the $\delta$-function $\delta(U - V)$. Subsequent integration over the $V_j$ returns one to the original expression (2.7).

Let us now first integrate out the original spatial link variables. This integration leads to local functionals $W[H_j(x)]$ for each spatial link,
\[
e^{W[H_j(x)]} = \int dU_j(x) \exp \left( \text{Re} \text{Tr}[H_j^\dagger(x) U_j(x)] \right), \quad (3.2)
\]
which can be calculated explicitly for the given gauge group \((dU_j(x)\) denotes Haar measure integration for a single element of the gauge group). The remaining integral,

\[
e^{-\beta S_{\text{eff}}[\mathcal{P}]} = \int \frac{D[H_j]D[V_j]}{2\pi} \exp \left( -\beta S[V_j,\mathcal{P}] + \sum_{x,j} \left[ W[H_j(x)] - \text{ReTr}[H_j^\dagger(x)V_j(x)] \right] \right)
\]

(3.3)
can be calculated in the saddle point approximation. The saddle point \(\overline{H}_j, \overline{V}_j\) can be found from the “equations of motion”,

\[
\frac{\delta S}{\delta V_j(x)} = H_j^\dagger(x), \quad \frac{\delta W}{\delta H_j(x)} = V_j(x).
\]

(3.4)

We are looking for the translationally invariant and isotropic solution proportional to the unit matrix:

\[
\overline{H}_j(x) = h \cdot \mathbb{1}, \quad \overline{V}_j(x) = v \cdot \mathbb{1}.
\]

(3.5)

For such configurations the Wilson action eq. (2.3) reduces to

\[
\frac{S[\overline{V}_j, \mathcal{P}]}{N_3^2 N_t} = \frac{d(d-1)}{2} - (d-1) \left[ v^2 + \frac{(d-2)}{2} v^4 \right],
\]

(3.6)

where \(N_3^2 N_t\) is the total number of lattice sites and \(d = 4\) is the dimension of Euclidean space-time. Let us note that for the case of constant phases \(\varphi_n\) the dependence on the Polyakov line disappears from the saddle point action. It will re-appear only in the 1-loop correction.

The calculation of \(W[H_j(x)]\) is possible for any gauge group. However, here we will restrict ourselves to the group \(SU(2)\) to keep things as simple as possible. In this case we need only one phase in (2.9) which we denote as \(\varphi\). The group \(SU(2)\) has some specifics [9]: The matrices \(H_j(x)\) and \(V_j(x)\) are Hermitian, which is related to the fact that \(SU(2)\) is a self-conjugated group. We parameterize them as \((\alpha = 1, 2, 3, 4, \text{summed}), U = u_\alpha \sigma^-_\alpha, V = v_\alpha \sigma^-_\alpha, H = h_\alpha \sigma^\pm_\alpha, \sigma^\pm = (\mp i \sigma, 1)\).

\[
(3.7)
\]

For the unitary matrix \(U\) we have the additional constraint \(u_\alpha^2 = 1\). The integral determining \(W[H_j(x)] = W[h]\) becomes:

\[
e^{W[h]} = \int \frac{d^4 u}{2\pi^2} \delta(u_\alpha^2 - 1) e^{hu_4}, \quad \text{such that } W[h] = \log \frac{2I_1(h)}{h}.
\]

(3.8)

Here \(2\pi^2\) is the volume of \(SU(2)\), and \(I_1\) a modified Bessel function. The equations of motion (3.4) assume the form

\[
h = 2\beta(v + 2v^3), \quad v = \frac{I_2(h)}{I_1(h)}.
\]

(3.9)

They have non-trivial solutions for \(\beta > \beta_c \approx 1.6817\). At large \(\beta \gg 1\) the mean field \(v\) tends towards 1 and \(h\) becomes large. The statistical sum in the saddle point approximation is given by

\[
\log Z_0 = 3N_3^2 N_t \left[ \log \left( \frac{2I_1(h)}{h} \right) - \beta v^2(1 + 3v^2) - 6\beta \right].
\]

(3.10)
At the critical point $\beta_c$ the mean field approximation predicts a phase transition. For gauge group $SU(2)$ this transition is fictitious – it appears due to the fact that our approximation is rather crude. However, at larger $\beta$ mean field approximation describes the lattice data (e.g., for the average plaquette energy) quite accurately (see, e.g., [9]).

## 4 Loop corrections to the mean field solution

As already remarked, in the mean field approximation the free energy of $SU(2)$ lattice gauge theory does depend neither on temperature nor on the Polyakov line. One needs to compute 1-loop corrections in order to study this dependence.

To obtain the first correction to the saddle point approximation we consider quantum fluctuations for $V_j$ and $H_j$ around the saddle points and parameterize $V_j$ and $H_j$ entering eq. (3.3) as $(\alpha = 1, 2, 3, 4$ is summed, $\sigma^\pm_\alpha = (\mp i\sigma, 1)$):

$$V_j(x) = v \cdot 1 + w_j^\alpha(x)\sigma^-_\alpha, \quad H_j(x) = h \cdot 1 + \rho_j^\alpha(x)\sigma^-_\alpha, \quad (4.1)$$

and in the path integrals the measures $D[V_j]$ and $D[H_j]$ are replaced by the corresponding measures for the parameters $w_j^\alpha(x)$, $h_j^\alpha(x)$, which we denote as $D[w_j]$ and $D[\rho_j]$. We assume that the quantum fluctuations $w_j^\alpha(x)$ and $h_j^\alpha(x)$ are small and expand the action in these quantum fluctuations. The linear term disappears due to the equations of motion.

For computing the quadratic terms we begin with the corrections to $W[H_j(x)]$:

$$W[H_j(x)] = W_0[h] + \frac{I_2(h)}{2hI_1(h)} \left[ \frac{(\rho_j^1(x))^2}{\kappa^2} + (\bar{\rho}_j(x))^2 \right], \quad \kappa^{-2} = 1 + \frac{hI_3(h)}{I_2(h)} - \frac{hI_2(h)}{I_1(h)}, \quad (4.2)$$

where $W_0[h]$ is the saddle point value. It can be seen that $\kappa$ is always larger than unity and monotonically increases with increasing $h$.

The integrals over the $\rho_j^\alpha(x)$ are Gaussian and can be performed easily, such that we end up with:

$$e^{-\beta S_{\text{eff}}[\mathcal{P}]} = Z_0 \left[ \sqrt{\kappa} \frac{\beta(1 + 2\nu^2)}{\pi} \right]^{2N_l} \int D[w_j] \exp \left( -\frac{\beta}{2} \sum_{x,j} w_j^\alpha(x) S^{(2)}_{\alpha\beta}[\mathcal{P}] w_j^\beta(x) \right. - \left. \beta(1 + 2\nu^2) \sum_{x,j} \left[ \kappa^2(w_j^\alpha(x))^2 + (\bar{w}_j(x))^2 \right] \right), \quad (4.3)$$

where $N_l = 3N_s^2N_t$ is the number of links, $S^{(2)}_{\alpha\beta}[\mathcal{P}]$ is a quadratic form from the Wilson action and we used the equations of motion (3.9). We see that the result of integrating the variables $\rho_j^\alpha(x)$ can be formulated as a specific change of the Wilson quadratic form.

It is convenient to write the complete quadratic form in the momentum representation. We expand the quantum fluctuations $w_j^\alpha(x)$ in plane waves:

$$w_j^\alpha(x) = \frac{1}{\sqrt{N_s^2N_t}} \sum_p e^{ip_4t + ip_\rho \cdot \vec{x}} \tilde{w}_j^\alpha(p), \quad p_4 = \frac{2\pi k_4}{N_t}, \quad p_j = \frac{2\pi k_j}{N_s}, \quad (4.4)$$
where \( k_4, k_j \) are integers: \( k_4 = 0, 1, \ldots N_t - 1, k_j = 0, 1, \ldots N_s - 1 \). The kernel \( W^{\alpha\beta}_{ij}(p) \) of the resulting quadratic form has indices related to color \( (\alpha, \beta = 1, 2, 3, 4) \) and for the direction of the given spatial link \((i, j = 1, 2, 3)\). For the trivial Polyakov line \((\varphi = 0)\) it is diagonal in the color indices:

\[
W^{\alpha\beta}_{ij}(p) = \delta^{\alpha\beta} \left[ W^{(G)}(\delta_{\alpha k} \delta^{\beta l}) + W^{(M)}_{ij}(p) \delta^{\alpha 4} \delta^{\beta 4} \right],
\]

\((k, l = 1, 2, 3)\). The first term describes 3 massless degrees of freedom which correspond to 3 gluons, the second one is related to a massive excitation which is a lattice artifact,

\[
W^{(G)}_{ij}(p) = (1 - \cos p_4) \delta_{ij} + v^2 \left[ \delta_{ij} \left( \cos p_i - \sum_l \cos p_l \right) - 2(1 - \delta_{ij}) e^{i(p_i - p_j)} \sin p_i \sin p_j \right] + 2v^2 \delta_{ij},
\]

\[
W^{(M)}_{ij}(p) = (1 - \cos p_4) \delta_{ij} + v^2 \left[ \delta_{ij} \left( \cos p_i - \sum_l \cos p_l \right) - 2(1 - \delta_{ij}) e^{i(p_i - p_j)} \cos p_i \cos p_j \right] + 2\kappa v^2 \delta_{ij}.
\]

In these expressions the first term comes from the expansion of time-like plaquettes, the second one is related to a massive excitation which is a lattice artifact, the third is due to the expansion of space-like plaquettes and the third is the result of integrating over the \( \rho_j^\dagger(x) \). The integral \((4.3)\) turns into a Gaussian one,

\[
e^{-\beta S_{\text{eff}}[\rho=1]} = Z_0 \left[ \sqrt{\frac{\beta(1 + 2v^2)}{\pi}} \right]^{2N_t} \int D[\tilde{\omega}] \exp \left( -\frac{\beta}{2} \sum_p \left[ \bar{\tilde{\omega}}_i^\dagger(p) W^{(M)}_{ij}(p) \tilde{\omega}_j(p) - \bar{\tilde{\omega}}_i^\dagger(p) W^{(G)}_{ij}(p) \tilde{\omega}_j^\dagger(-p) \right] \right).
\]

The kernel \( W^{(G)} \) of the quadratic form for the gluons has 3 eigenvalues,

\[
\lambda_{1,2} = 2 - 2 \cos p_4 + 2v^2(3 - \cos p_1 - \cos p_2 - \cos p_3), \quad \lambda_3 = 2 - 2 \cos p_4.
\]

The first two correspond to two transverse gluons, the third one to a longitudinal gluon. The eigenvalue of \( W^{(M)} \) cannot be determined analytically, but it is seen to correspond to a massive particle (the eigenvalue does not vanish at \( p_\mu = 0 \)) with real mass (the eigenvalue is positive for all \( p \)).

The longitudinal eigenvalue \( \lambda_3 \) vanishes for all momenta with \( p_4 = 0 \) and therefore the integral \((4.8)\) diverges. As always, zero modes of the quadratic form correspond to some continuous symmetry of the problem. In our case it is the symmetry under gauge transformations independent of time. Indeed, the solution \((3.5)\) of the saddle point equations is not unique, since any function of the form

\[
H_j(x) = h S(\bar{x}) S^\dagger(\bar{x} + \hat{j}) , \quad V_j(x) = v S(\bar{x}) S^\dagger(\bar{x} + \hat{j}) ,
\]

with time independent gauge matrices \( S(\bar{x}) \), is also a solution of the saddle point equations with the same action. For this reason fluctuations \( w^{\alpha}_{ij}(x) \) in the directions corresponding to \( S(\bar{x}) \) give rise to zero modes. Of course this does not mean that the complete integral is
divergent, but the expansion in \( w(x)^p \), which we use, breaks down and we have to modify it.

The eigenfunctions corresponding to the zero modes of the quadratic form for \( w(x)^p \) are:

\[
\eta_j^p(x) = \frac{\xi_j(p)}{\sqrt{N_t N_s^3}} e^{i \vec{p} \cdot \vec{x}}, \quad \xi_j(p) = \frac{\{e^{ip_1} - 1, e^{ip_2} - 1, e^{ip_3} - 1\}}{\sqrt{4(\sin^2 \frac{p_1}{2} + \sin^2 \frac{p_2}{2} + \sin^2 \frac{p_3}{2})}}. \tag{4.11}
\]

In the continuum limit \( \xi_j(p) = \frac{i p_j}{\eta_{\bar{p}}(p)} \) is the vector of the longitudinal gluon polarization. Let us introduce unity (a la Faddeev-Popov):

\[
1 = \int D[S] D[w_j] J_{SU(2)}(V_j) \delta\left(V_j(x) - S(x) \left[\epsilon \mathbb{1} + \bar{w}_j(x) \cdot \vec{\sigma}\right] S^\dagger(x + \vec{\eta})\right) \delta\left(\sum_x \bar{w}_j(x) \eta_j^p(x)\right). \tag{4.12}
\]

Here \( J_{SU(2)}(V_j) \) is the Jacobian for changing from variables \( V_j(x) \) to variables \( S(x) \) and \( w_j(x) \). The second \( \delta \)-function restricts the integration over the \( w_j(x) \) to be orthogonal to the zero modes of the quadratic form. Direct calculation gives the following result:

\[
\log J_{SU(2)}(V_j) = N_s^3 \log(2\pi^2) + \frac{3}{2} \sum_p \log \left(4v^2 N_t \left[\sin^2 \frac{p_1}{2} + \sin^2 \frac{p_2}{2} + \sin^2 \frac{p_3}{2}\right]\right). \tag{4.13}
\]

The first term is the volume of \( SU(2) \) to which the measure \( D[S] \) is normalized.

Using eq. (4.12) in the integral (4.8) we see that the integrand does not depend on \( S \) and hence the integral over \( D[S] \) gives unity. Performing the integral over the \( w_j^p(x) \) we arrive at

\[
-\beta S_{eff}[\mathcal{P} = 1] = \log Z_0 + 3N_s^3 N_t \log(4[1 + 2v^2]^{3/2}) - \frac{3}{2} \log \det \tilde{W}^{(M)} + \log J_{SU(2)} + \frac{3}{2} N_s^3 \log \frac{\beta}{2\pi}
- 3 \sum_p \log \left(4\sin^2 \frac{p_1^2}{2} + 4v^2 \left[\sin^2 \frac{p_1^2}{2} + \sin^2 \frac{p_2^2}{2} + \sin^2 \frac{p_3^2}{2}\right]\right) - \frac{3}{2} \sum_{p_4 \neq 0} \log \left(4\sin^2 \frac{p_4^2}{2}\right), \tag{4.14}
\]

where we introduced the renormalized quadratic form \( \tilde{W}^{(M)} = W^{(M)}/[(1 + 2v^2)\kappa^2] \). The remaining contribution from massive gluons (third term in (4.14)) appears to be small in a wide region of \( \beta \) values. The second term in (4.14) is the one loop correction to the leading order vacuum energy (\( \log Z_0 \)), the 4-th and the 5-th are the result of fixing the longitudinal zero modes, the 6-th is the contribution of transverse gluons, and the 7-th from longitudinal gluons.

### 5 Effective potential at 1-loop

For \( SU(2) \) we parameterize the constant temporal link variables by the matrix \( U_4(x) = e^{i \vec{x} \cdot \sigma_3} \) such that the Polyakov line is given by \( \mathcal{P} = e^{i \sigma_3} \). A non-trivial Polyakov line \( (\varphi \neq 0) \) modifies the time-like plaquettes and the modification is equivalent to a shift of
the temporal momenta $p_4$ entering the momentum space kernel,

$$W_{ij}^{\mu\nu}(p) = \delta^{\mu\nu} \left[ \delta^2 W(G) \left( p_4 + \frac{\varphi}{N_t}, \vec{p} \right) + \delta^2 W(G) \left( p_4 - \frac{\varphi}{N_t}, \vec{p} \right) + \delta^2 W(M) \left( p_4, \vec{p} \right) \right]. \quad (5.1)$$

The quadratic form for the massive particles remains unchanged.

We see that in the presence of a non-trivial Polyakov line the gluon quadratic form has only one zero mode. The reason is clear: The Polyakov line explicitly breaks the residual gauge invariance down to $U(1)$. Correspondingly, at $\varphi \neq 0$ we have to fix only one zero mode. For this purpose we introduce unity similar to eq. (4.12), but only for the third color component of $w^\alpha_j(x)$ and change the integration in $D[S]$ from $SU(2)$ to $U(1)$. The effect of fixing the zero mode reduces for the statistical sum to:

$$\log Z_{\text{longitudinal}} = -\sum_{\vec{p}} \left[ \log(4 \sin^2 \varphi) + \frac{1}{2} \log \left( \frac{N_t}{2\pi \beta \cdot 4 \sinh^2(N_t \alpha)} \right) \right]. \quad (5.6)$$

The complete expression for the statistical sum is obtained as a sum of $\log Z_{\text{transverse}}$, $\log Z_{\text{longitudinal}}$ and the first 3 terms from eq. (4.14).

Lattices accessible for Monte Carlo simulations have become quite large in recent years. For this reason one can use the approximation $N_s \to \infty$ and change the sums over the spatial momenta to integrals. We introduce new continuous variables:

$$q_j = \frac{2}{a} \sin \frac{p_j}{2}, \quad \sinh^2 \alpha = v^2 \left( \frac{q_j^2 a^2}{4} \right).$$
Then expression (5.4) becomes:

$$
\frac{\log Z^{\text{transverse}}}{N_s^3} = -\prod_j \int_{-2/a}^{2/a} \frac{a \, dq_j}{2\pi \sqrt{1 - \frac{q_j^2 a^2}{4}}} \left[ \log \left( 4 \sinh^2 N_t \alpha + 4 \sin^2 \varphi \right) + \log(4 \sinh^2 N_t \alpha) \right],
$$

(5.7)

and analogously for $\log Z^{\text{longitudinal}}$, eq. (5.6). These expressions correspond to the continuum limit for spatial momenta. At $q \sim 1/a$ they are regularized by our (cubic) lattice.

The continuum limit also requires that $N_t \to \infty$. Usually this condition is fulfilled much worse in a typical lattice simulation than $N_s \to \infty$. Nevertheless, let us calculate the expansion of eq. (5.7) in this limit. We rewrite $\log Z^{\text{transverse}}$ in the form

$$
\frac{\log Z^{\text{transverse}}}{N_s^3 N_t} = -2 \prod_i \int_{-2/a}^{2/a} \frac{a \, dq_i}{2\pi \sqrt{1 - \frac{q_i^2 a^2}{4}}} \left[ \log \left( 1 - 2 \cos 2 \varphi e^{-2N_t \alpha} + e^{-4N_t \alpha} \right) + \log \left( 1 + e^{-4N_t \alpha} \right) + 3N_t \alpha \right].
$$

(5.8)

For large $N_t$ the main contribution to the integrals over the first two terms in the parenthesis comes from small $q_i a \ll 1$. Performing the integrals we obtain:

$$
\frac{\log Z^{\text{transverse}}}{N_s^3 N_t} = -6c_1 - \frac{\pi^2}{3N_t^4} \left[ -\frac{1}{5} + 4 \left( \frac{\varphi}{\pi} \right)^2 \left( 1 - \frac{\varphi}{\pi} \right)^2 \right] + \ldots ,
$$

(5.9)

where $c_1$ is the constant

$$
c_1 = \prod_i \int_{-2/a}^{2/a} \frac{a \, dq_i}{2\pi \sqrt{1 - \frac{q_i^2 a^2}{4}}} \alpha(q) \approx 0.90003 .
$$

(5.10)

Expression (5.8) is the celebrated perturbative potential in $\varphi$ coming from transverse gluons, which is well-known in the continuum [3] (for a lattice analogue of these calculations see [4] and [10]). This potential is defined only for $0 < \varphi < \pi$ and outside this interval should be continued according to periodicity. The constant in square brackets is precisely the Stefan-Boltzmann energy of the transverse gluons. Expression (5.8) is only an approximation of eq. (5.4) for large $N_t$. Nevertheless, it works rather well even for $N_t = 2$.

An analogous calculation of the longitudinal gluons results in

$$
\frac{\log Z^{\text{longitudinal}}}{N_s^3} = c_2 - \frac{1}{2} \log \left( \frac{N_t v^2}{2\pi^2} \right) - \log(4 \sin^2 \varphi) ,
$$

$$
c_2 = \prod_i \int_{-2/a}^{2/a} \frac{a \, dq_i}{2\pi \sqrt{1 - \frac{q_i^2 a^2}{4}}} \log \left( \frac{4 \sinh^2 \alpha}{v^2} \right) \approx 1.67339 .
$$

(5.11)

We are not so much interested in the statistical sum, but in the free energy $\mathcal{F}$ given as

$$
\mathcal{F} = -T \log Z = \beta T S_{\text{eff}} = V_3 \frac{1}{a^4} f .
$$

(5.12)
Being an extensive quantity, the free energy is proportional to the volume $V_3 = (N_s a)^3$ of the system. It is convenient to introduce the dimensionless quantity $f$, which is a specific free energy in lattice units. Using $T = 1/(N_t a)$ it is simply given by

$$f = -\frac{\log Z}{N_s^3 N_t}.$$ (5.13)

We are interested in this quantity as a function of $\varphi$ and temperature $T/T_c = N_t^{(c)}/N_t$ in units of the critical temperature $T_c$ (here $N_t^{(c)}$ is the value of $N_t$ at which the phase transition takes place). Collecting all terms we arrive at:

$$f = \varepsilon_{\text{vac}} + \frac{\log \left(\frac{N_t}{2\pi^2}\right) - 2c_2}{2N_t} + \frac{\log(4 \sin^2 \varphi)}{N_t} + \frac{\pi^2}{3N_t^3} \left[ -\frac{1}{5} + 4 \left(\frac{\varphi}{\pi}\right)^2 \left(1 - \frac{\varphi}{\pi}\right)^2 \right].$$ (5.14)

Here $\varepsilon_{\text{vac}}$ is the vacuum energy density in the 1-loop approximation:

$$\varepsilon_{\text{vac}} = 3\beta \left[ 6 + v^2 (1 + 3v^2) - \frac{1}{\beta} \log \left(\frac{2I_1(h)}{h}\right) \right] - 3 \log \left(4(1 + 2v^2)^{3/2}\right) + 6c_1 + \frac{3}{2N_s^3 N_t} \log \det \tilde{W}^m + \ldots$$ (5.15)

The first term here is the mean field approximation, eq. (3.10), and the subsequent terms represent the 1-loop correction. As it was already said, the last term is rather small and one can neglect it.

Other terms in eq. (5.14) are corrections in $1/N_t$ and hence depend on the temperature. We see that the effective potential on the lattice contains not only the well-known perturbative potential [4, 10] and the Stefan-Boltzmann energy, but also additional terms coming from longitudinal gluons. These terms are linear in the temperature (or depend as $T \log T$) and the coefficient in front of them is ultraviolet divergent (it is proportional to $a^{-3}$). This contribution disguises the contribution of transverse gluons to the energy and the effective potential, as at all $N_t$ (even for $N_t = 2$) it appears to be much larger than the perturbative potential and the Stefan-Boltzmann energy.

The effective potential eq. (5.14) diverges at $\varphi \to 0$. The reason is obvious: at $\varphi \to 0$ we return to the situation where the symmetry of the saddle point field restores back to $SU(2)$ and we have to fix 3 zero modes instead of one. At $\varphi = 0$ the statistical sum is given by eq. (4.14). As compared to eq. (5.14) only the contribution of the longitudinal gluons changes

$$\frac{\log Z_{\text{longitudinal}}(\varphi = 0)}{N_s^3 N_t} = \frac{3c_2}{2N_t} - \frac{3}{2N_t} \log \left(\frac{2\pi N_t}{\beta}\right) + \frac{\log(2\pi^2)}{N_t}.$$ (5.16)

In fact, it is clear that there is a transitional region at small $\varphi$ where neither eq. (5.11) nor eq. (5.16) are applicable.

In this region one can still try to fix 3 modes (one zero mode and two with non-zero but small eigenvalues) introducing unity according to eq. (4.12). However, the action at non-zero $\varphi$ will depend on a $SU(2)$ matrix $S$. This dependence comes from time-like plaquettes.
and leads to the following integrals over unitary matrices $S$ located on spatial sites,

$$Z_S(\varphi) = \int DS(\vec{x}) \exp \left( \beta v^2 N_t \sum_{\vec{x}, \vec{y}} \left[ \frac{1}{2} \text{Tr} \left[ S(\vec{x}) e^{i \frac{\beta}{2\pi} S(\vec{x}) S(\vec{x} + \vec{j})} \right] - 1 \right] \right),$$

(5.17)

(we here neglect the quantum fluctuations $w$). Another modification necessary for the contribution of longitudinal gluons is that in the sum of logarithms of longitudinal eigenvalues $\lambda_3$ the value $p_4 = 0$ should be omitted for all 3 colors. Finally also the Jacobian $J_{SU(2)}$ should be taken into account. We find

$$\log \frac{Z_{\text{longitudinal}}(\varphi)}{N_s^3} = -\log \left( \frac{4 \sin^2 \varphi}{4 \sin^2 \frac{\varphi}{N_t}} \right) + \frac{1}{2} \log N_t + \frac{3}{2} \log \frac{\beta}{2\pi} + \frac{3}{2} c_2 + \log 2 \pi^2 + \log Z_S(\varphi),$$

(5.18)

At $\varphi = 0$ when $Z_S = 1$ this expression reduces to eq. (5.16). Let us pay attention to the fact that it contains terms which are not periodic in $\varphi$ with period $\pi$. This means that $\log Z_S(\varphi)$ is also not periodic to cancel this non-periodicity.

As already noted above, the integrand in eq. (5.17) does not depend on the matrices $S$ belonging to the $U(1)$ subgroup of $SU(2)$: $S = e^{i \delta \tau_3}$. At $\varphi \sim 1$ eq. (5.18) should reduce to eq. (5.11). Parameterizing an $SU(2)$ matrix as $S = e^{i \delta \tau_3} e^{i \delta_1 \tau_1 + i \delta_2 \tau_2}$ we see that at $\varphi \sim 1$ the integral over $s_1, s_2$ can be calculated by the saddle point method. Expanding in $s_{1,2}$ we obtain

$$Z_S = \frac{1}{\pi} \int ds_1(\vec{x}) ds_2(\vec{x}) \exp \left( -2 \beta v^2 \sin^2 \frac{\varphi}{N_t} \sum_{\vec{x}, \vec{y}} \left[ (s_1(\vec{x}) - s_1(\vec{x} + \vec{j}))^2 + (s_2(\vec{x}) - s_2(\vec{x} + \vec{j}))^2 \right] \right).$$

(5.19)

This integral can be easily calculated in the momentum representation

$$Z_S = \exp \left( 8 \beta v^2 \sin^2 \frac{\varphi}{N_t} \sum_{\vec{p}} \left[ s_1(\vec{p}) s_1(-\vec{p}) + s_2(\vec{p}) s_2(-\vec{p}) \right] \left[ \sin^2 \frac{p_1}{2} + \sin^2 \frac{p_2}{2} + \sin^2 \frac{p_3}{2} \right] \right).$$

(5.20)

Integration results in:

$$\frac{Z_S}{N_s^3} = -c_2 - \log 4 \sin^2 \frac{\varphi}{N_t} - \log \frac{\beta}{2\pi} - \log \pi$$

(5.21)

This expression is also not periodic in $\varphi$ as expected. Substituting eq. (5.21) into eq. (5.18) we return to eq. (5.11) as it should be. However, we see that eq. (5.11) is valid for not too small $\varphi$ where the coupling constant in the Gaussian integration in $s_{1,2}$ is small enough.

To cover also the region of small $\varphi$, let us change in the integral determining $Z_S$ to the new variables:

$$Z_S(\varphi) = \int D[\vec{n}] \delta(\vec{n}^2 - 1) \exp \left( \frac{1}{2} N_t \beta v^2 \sin^2 \frac{\varphi}{N_t} \sum_{\vec{x}, \vec{y}} \left[ \vec{n}(\vec{x}) \cdot \vec{n}(\vec{x} + \vec{j}) - 1 \right] \right), \quad n_a = \frac{1}{2} \text{Tr}[S^a \tau_a \tau_a].$$

(5.22)

---

1. The factor $\pi^{-1}$ in the measure is the ratio of $(2\pi)$, which is volume of $U(1)$ and $2\pi^2$ which is the volume of $SU(2)$. 
Expression (5.22) is the statistical sum for a classical lattice theory – the $O(3)$ $\sigma$-model. The role of the inverse coupling constant is played by:

$$\beta_{O(3)} = N_t \beta v^2 \sin^2 \left( \frac{\varphi}{N_t} \right) \approx \beta \frac{v^2 \varphi^2}{N_t}.$$  \hspace{1cm} (5.23)

The approximation eq. (5.11) for the contribution of the longitudinal gluons corresponds to the weak coupling limit of eq. (5.22). We can also construct the strong coupling expansion for the statistical sum:

$$\log \frac{Z_S}{N_3^4} = -3\beta_{O(3)} + \frac{1}{2} \beta_{O(3)}^2 + \ldots$$ \hspace{1cm} (5.24)

This theory was the subject of many lattice studies (see, e.g. [11]) but we were not able to find results for its free energy as a function of the coupling constant which are needed for our purposes.

Still this calculation is a relatively easy task for lattice simulations and we performed a corresponding simulation on $64^3$ lattices. In Fig. 1 we display the logarithm of the statistical sum for 3-dimensional $SO(3)$ theory together with the strong and weak coupling expansion results. It appears that the weak coupling approximation works well in a wide range of $\beta_{O(3)}$ values (see Fig. 1).

6 Comparison with numerical simulations

A numerical evaluation of the effective action $S_{\text{eff}}(P)$ for gauge groups $SU(2)$ and $SU(3)$ was presented recently in [2]. Relatively large lattices with sizes up to $40^3 \times N_t$ for $N_t =
2, . . . 20 were used in a fixed scale approach (i.e., fixed inverse gauge coupling $\beta$ and thus fixed lattice spacing $a$, with the temperature driven by varying $N_t$). All temporal links, except those on the last time slice, were fixed to be $U_4(x) = 1$. At the final time slice the temporal link variables were suitably parameterized, using, e.g., $U_4(x, N_t - 1) = e^{i\epsilon_3}$. In other words, the setup of [2] precisely corresponds to the one described in our Section II.

![Figure 2](image-url)

**Figure 2.** Free energy for $\varphi = 0$ as a function of temperature $T/T_c$ (vacuum energy subtracted). The value of $\epsilon_{vac} - f$ is plotted. The data points are from [2] and the full curve corresponds to eq. (5.14).

The free energy of the theory was calculated by numerically integrating the averaged plaquette expectation value over $\beta$ starting from $\beta = 0$ up to the value of $\beta$ one wants to work at and which sets the lattice spacing $a$, i.e., the cutoff. The corresponding values of the inverse coupling were $\beta = 2.6$ for $SU(2)$ and $\beta = 6.2$ for $SU(3)$. Results were presented as a function of $\varphi$ and $T/T_c$. The overall magnitude for the free energy was off by the aforementioned factors of 20 to 30 for the two gauge groups, but suitable ratios were in very good agreement with perturbative results.

For $SU(2)$ the phase transition was observed near $N_t = 11$, so that the maximal temperature studied with $N_t = 2$ was $T/T_c \sim 11/2 = 5.5$. Corrections from the discretization in the spatial directions can easily be estimated and were found [2] to be very small for $N_s = 40$.

In order to describe data obtained in [2] without forming ratios and to account for the missing factors, we now apply the mean field approximation developed above. This will be done only for the $SU(2)$ case.

To begin with, we compare the vacuum energy densities $\epsilon_{vac}$ to get a first estimate for the accuracy of the mean field approach. Extrapolating the data to low temperatures we obtain:

$$\epsilon_{vac}^{(exp)} = 10.52 .$$

(6.1)
At the same time, for $\beta = 2.6$ eq. (5.15) gives
\[
\varepsilon_{\text{vac}}^{(0)} = 12.51, \quad \varepsilon_{\text{vac}}^{(1)} = 9.63,
\]
for the leading order and the 1-loop approximation. Based on these numbers we estimate the accuracy of the mean field 1-loop approximation to be at the 10% level.

Next we inspect the temperature dependence of the free energy at different values of $\varphi$. In particular the values $\varphi = 0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{\pi}{2}$ were studied in [2] and are compared to our calculations in Figs. 2, 3 and 4. In the plots we subtract the vacuum energy and start at temperatures $T > T_c$, since below $T_c$ mean field approximation cannot be expected to apply.

At $\varphi = 0$ (see Fig. 2) the data are described rather well by our results. The small discrepancy at the largest $T$ may be understood from the fact that the corresponding $N_t = 2$ is too small for using the expansion in $1/N_t$. 

\hspace{1cm}
The results at $\varphi \neq 0$ are more ambiguous, as they should include the knowledge of the free energy for the $O(3)$ sigma-model. Dotted blue curves represent the approximation eq. (5.14) which should be valid at sufficiently large $\varphi$ and in red the answer corresponding to the full $O(3)$ model in $d = 3$. We see that the numerical simulations have sufficient accuracy to distinguish these two cases, though the weak coupling expansion is not bad for the whole range of temperatures and $\varphi$. The complete mean field theory works very well in all cases above the deconfinement transition. The contribution of the perturbative potential (and the Stefan-Boltzmann energy) is rather small on the scale used in the above pictures.

7 Conclusions

The effective potential for the Polyakov line $P(\vec{x})$ is an interesting quantity which easily can be measured on the lattice. A detailed analysis of its properties should help with understanding quark confinement since the Polyakov line is the order parameter for the deconfinement transition.

Unfortunately, as is well-known, the Polyakov line is ill-defined due to ultraviolet divergences. In particular the Coulomb energy of the static charge, which is linearly divergent, contributes to the Polyakov line. Correspondingly, its average should be exponentially suppressed for small lattice spacing in both, the confining and deconfined phases. It is thus not surprising that the effective potential for such a quantity is also ultraviolet divergent. In this paper we identified a corresponding singularity originating in the contribution of longitudinal gluons. Taking into account this singularity, the effective potential has a very deep minimum at $P(\vec{x}) = 1$ which always dominates. This ultraviolet divergence completely obscures the perturbative potential even at very high temperatures and strongly distorts the effective Lagrangian for the Polyakov line. It is this singularity which gives the main contribution to the lattice simulations of [2], such that in [2] only suitable ratios of observables could be matched to perturbative results.

On the other hand it is clear that this problem is completely due to the poor definition of the Polyakov line as an order parameter in Yang-Mills theory. One has to introduce some other quantity which is ultraviolet stable (see, e.g. [12] and references therein) and try to investigate the corresponding effective Lagrangian. The simplest way out is to subtract the mentioned UV divergence (calculated according to the formulae of this paper) from the data and identify the remaining piece as the effective potential for the Polyakov line. However, it seems that the current accuracy of the performed simulations is not yet sufficient for this procedure.

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A Summation formulae

In the text we need the summation of series of the form

$$\sum_{p} f(\cos p) ,$$  \hspace{1cm} (A.1)

where the sum is running over momenta $p_k = \frac{2\pi k}{N_t}$ with integers $k = 0,1,\ldots N_t - 1$. To calculate such sums let us consider the integral,

$$\int_{\Gamma} \frac{dz}{2\pi i} \frac{N_t z^{N_t-1}}{z^{N_t} - 1} f(\frac{1}{2}[z + z^{-1}]) ,$$  \hspace{1cm} (A.2)

where the contour $\Gamma$ envelops the unit circle. The integrand in eq. (A.2) has poles at $z = e^{i2\pi k/N_t}$, and maybe additional ones from the function $f$. The residues of the poles $z = e^{i2\pi k/N_t}$ coincide with the terms of the sum eq. (A.1). According to Cauchy’s theorem the integral can be evaluated in two ways: Taking the residues inside the contour $\Gamma$ (the sum eq. (A.1) and possibly singularities of $f$) and outside the contour (singularities of $f$ only). Comparison of these two ways of computing the integral gives suitable summation formulas for eq. (A.1).

By this method one can, e.g., obtain the general formula,

$$\sum_{p} \frac{1}{\sin^2\left(\frac{p}{2} + \frac{\varphi}{N_t}\right) + \sinh^2\alpha} = \frac{N_t}{\sinh 2\alpha} \frac{\sinh 2N_t\alpha}{\sinh^2\alpha + \sin^2\varphi} .$$  \hspace{1cm} (A.3)

For $\varphi = 0$ this formula (without derivation) was given in [4]. Next, we consider the sum,

$$\sum_{p} \log \left(1 + \gamma \sin^2\left(\frac{p}{2} + \frac{\varphi}{N_t}\right)\right) = \int_{0}^{\infty} \frac{d\gamma'}{\gamma'} \left[ N_t - \sum_{p} \frac{1}{1 + \gamma' \sin^2\left(\frac{p}{2} + \frac{\varphi}{N_t}\right)} \right] .$$  \hspace{1cm} (A.4)

Using eq. (A.3) to calculate the sum and changing variables according to $(\gamma')^{-1} = \sinh^2\alpha'$ we obtain a calculable integral and arrive at:

$$\sum_{p} \log \left[\sin^2\left(\frac{p}{2} + \frac{\varphi}{N_t}\right) + \sinh^2\alpha\right] = \log \left(\frac{\sin^2\varphi + \sinh^2 N_t\alpha}{2\sinh^2\gamma'}\right) .$$  \hspace{1cm} (A.5)

This is the main formula which we use to perform all summations in the text. We use also particular cases of this formula:

$$\sum_{p \neq 0} \log \left(\sin^2\left(\frac{p}{2} + \frac{\varphi}{N_t}\right)\right) = \log \left(\frac{\sin^2\varphi}{\sin^2\frac{\varphi}{N_t}}\right) , \quad \sum_{p \neq 0} \log \sin^2\frac{p}{2} = 2 \log N_t .$$  \hspace{1cm} (A.6)

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