Entropy of higher dimensional nonrotating isolated horizons from loop quantum gravity

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Abstract
In this paper, we extend the calculation of the entropy of nonrotating isolated horizons in four-dimensional spacetime to that in a higher-dimensional spacetime. We show that the boundary degrees of freedom on an isolated horizon can be described effectively by an $SO(1, 1)$ BF theory. Then the entropy of the nonrotating isolated horizon can be calculated by counting the number of microstates. It satisfies the Bekenstein–Hawking law in its leading term and has a ‘zero-point entropy’ correction.

Keywords: loop quantum gravity, black hole entropy, BF theory

1. Introduction

Black holes have been attracting the attention of researchers for a long time. The pioneering works of Bekinstein [1], Hawking [2] and others [3] during the 1970s have suggested that black holes have temperature and entropy. The entropy is given by the famous Bekenstein–Hawking area law

$$ S = \frac{A}{4G\hbar}, \quad (1.1) $$

where $A$ is the area of the event horizon of a black hole. The entropy depends on both the Newtonian gravitational constant and the Planck constant, and indicates that its statistical description might tell us something profound about quantum gravity. There are many ways to explain the entropy of a black hole based on different theories, such as string theory [4] and loop quantum gravity [5]. For a brief review see [6].

Unlike the notion of the event horizon of a black hole, which is based on the global structure of a spacetime [7], an isolated horizon is defined quasilocally as a portion of an
event horizon [8]. As expected, the laws of black hole mechanics and thermodynamics can be generalized to the isolated horizons [8, 9]. In particular, the zeroth and first law of thermodynamics and the area law (equation (1.1)) for isolated horizons can also be set up. It is suggested that the microscopic degrees of freedom on an isolated horizon, which are responsible for the entropy, be described by the boundary Chern–Simons (CS) theory [5] in the framework of loop quantum gravity. An attempt at attributing the entropy of a black hole to the number of the bulk spin network states without invoking a boundary theory is also made [10, 11] based on Rovelli’s pioneering work [12].

General relativity (GR) in higher dimensional \((D > 4)\) spacetimes has been studied for almost a century. The original motivations for the study include Kaluza–Klein theory [13, 14], supergravity theory [15], string/M theory [16–18], brane-world scenarios [19, 20] and so on. Putting these motivations aside, in this paper we shall focus our attention on the Einstein theory on higher dimensional spacetimes. In higher dimensional Einstein theory, there exist many black hole solutions [21, 22]. Like their four-dimensional partners, black holes in higher dimensional spacetimes also have temperature and entropy. Similarly to GR, isolated horizons can be introduced in higher dimensional gravitational theories and the laws of black hole mechanics and thermodynamics can be generalized, as expected, to the isolated horizons [23–25]. How to explain the entropy of the isolated horizons in higher dimension in the framework of loop quantum gravity [26, 27] is an obvious issue. An immediate approach might be to invoke the CS theory again. However, the CS theory can only be defined on odd dimensional spacetimes, which limits its application in higher dimension.

In [28], we showed that in a four-dimensional spacetime, the boundary degrees of freedom on nonrotating isolated horizons can also be described effectively by another topological field theory—BF theory, whose Lagrangian is given by the wedge product of a \((\dim(M)-2)\)-form \(B\) taking values in the adjoint representation of given gauge group \(G\) and the curvature 2-form \(F\) of a connection \(A\) for \(G\). A BF theory can be defined on a spacetime with any dimension, which in this respect is the advantage over the CS theory. In the present paper, we extend the results in [28] to higher dimensional nonrotating isolated horizons.

This paper is organized as follows. In section 2, similarly to the four-dimensional case, we derive the symplectic structure for nonrotating isolated horizons. It can be seen that the boundary degrees of freedom can be described by a BF theory. In section 3, we quantize the BF theory and give the corresponding Hilbert space. In section 4, we set up the boundary condition to relate boundary fields to the bulk fields and calculate the entropy of nonrotating isolated horizons. The Bekenstein–Hawking law of nonrotating isolated horizons is obtained. Finally, discussions are presented in section 5. In the appendix, we give the detailed calculation of the solder field and connection. Throughout the paper, we use the units of \(\hbar = c = 1\).

2. Higher dimensional nonrotating isolated horizons

The Einstein–Hilbert action can be generalized to \(D\)-dimensional spacetime \(\mathcal{M}\) [22]:

\[
I\bigg[\gamma_{\mu\nu}\bigg] = \frac{1}{16\pi G} \int_{\mathcal{M}} d^Dx \sqrt{-g} R. \tag{2.1}
\]

It can be written in the first-order form [25]:

\[
I[e, A] = -\frac{1}{2k} \int_{\mathcal{M}} F^{IJ} \wedge \Sigma_{IJ}, \tag{2.2}
\]
where \( \kappa = 8\pi G \), \( e^I \) are covielbein (1-form) fields,
\[
\Sigma_{IJ} = \frac{1}{(D-2)!} e^{IJK...N} e^K \wedge ... \wedge e^N
\]  
(2.3)
is a \((D-2)\)-form, \( A^I \) the \( SO(D - 1, 1) \) connection 1-form, \( F_{IJ} \) the curvature 2-form of \( A^I \), and \( I, J \) indices of the Lie algebra of \( so(D - 1, 1) \), running from 0 to \( D - 1 \). The spacetime region \( M \) is supposed to be bounded by the initial and final spacelike hypersurfaces \( M_1 \) and \( M_2 \) and an isolated horizon \( \Delta \) from the inner, and to extend to spatial infinity \( i^0 \). All fields are assumed to be smooth and satisfy the standard asymptotic boundary condition at spatial infinity, \( i^0 \).

From the first variation of the action (2.2) and variational principle, we get the vacuum field equations of gravitation:
\[
\varepsilon^{LMN} e^I \wedge ... \wedge e^K \wedge F_{MN} = 0,
\]  
(2.4)
and the symplectic potential density,
\[
\theta(\delta) = (-1)^{D-1} \frac{1}{2\kappa} \Sigma_{IJ} \wedge \delta A^I.
\]  
(2.6)
The second-order exterior variation will give the symplectic current,
\[
J(\delta_1, \delta_2) = (-1)^{D-1} \frac{1}{\kappa} \delta_2 \Sigma_{IJ} \wedge \delta_1 A^I.
\]  
(2.7)
The nilpotent of exterior variation, \( \delta^2 = 0 \), implies \( dJ = 0 \). Applying Stokes’ theorem to the integration \( \int_M dJ = 0 \), we can get the following equation:
\[
\frac{1}{\kappa} \left( \int_{M_2} \delta_1 \Sigma_{IJ} \wedge \delta_1 A^I - \int_{M_1} \delta_2 \Sigma_{IJ} \wedge \delta_1 A^I - \int_{\Delta} \delta_1 \Sigma_{IJ} \wedge \delta_1 A^I \right) = 0.
\]  
(2.8)
Note that the boundary integral at spatial infinity \( i^0 \) vanishes by suitable fall-off conditions [29]. We shall see that the last term in equation (2.8) is a pure boundary contribution, i.e., the symplectic flux across the isolated horizon \( \Delta \) can be expressed as an algebraic sum of two terms corresponding to the \( D - 2 \) dimensional compact manifold \( K_1 = \Delta \cap M_1 \) and \( K_2 = \Delta \cap M_2 \).

Now let’s consider the geometry near the isolated horizon. We adopt the Bondi-like coordinates \( x^\alpha = (u, r, \zeta^i) \) with coordinate indices \( i, j = 2, ..., D - 1 \) near the isolated horizon in [30]. The isolated horizon \( \Delta \) is characterized by \( r = 0 \). With the Bondi-like coordinates, the Bondi-like vielbein vector fields can be expressed as [30, 31]
\[
\begin{align*}
n^\alpha &= -\partial_u, \\
I^\alpha &= \partial_\alpha + U \partial_r + X^i \partial_i, \\
e^\alpha_k &= \omega_{\alpha} \partial_r + \xi^k \partial_\alpha, \quad \text{with vielbein indices } \alpha, \beta = 2, ..., D - 1,
\end{align*}
\]  
(2.9)
\[
\begin{align*}
\n_u &= -du, \\
i_a &= -Udu + dr - \xi^i\omega(X'du - d\xi^i), \\
e^i_a &= -\xi^i\omega(X'du - d\xi^i), \quad \text{with} \quad \xi^i \text{ being the inverse of } \xi^i_{\alpha}. 
\end{align*}
\]

which satisfy the condition: \(n^ai_a = l^in_i = -1\), \(e^i_a e^a_i = \delta^i_i\), and others vanish. The inverse metric reads
\[
g^{ab} = -l^a \otimes n^b - n^a \otimes l^b + \delta^{ab} e^a_i \otimes e^b_i. \tag{2.11}\]

Define
\[
\pi_{\alpha} := e^a_i l^b V_b n_a, \tag{2.12}\]

which is related to the angular momentum of the isolated horizon [24]. For nonrotating isolated horizons, \(\pi_{\alpha} \equiv 0\). The unknown functions near the nonrotating isolated horizon may be expanded as [31]
\[
\begin{align*}
U &= \kappa_r r + \frac{1}{2}R_{\text{min}}^{(0)} r^2 + O(r^3), \\
\omega_\alpha &= \frac{1}{2}R_{\text{min}}^{(0)} r^2 + O(r^3), \\
X^i &= \frac{1}{2}R_{\text{min}}^{(0)} \varepsilon^i_{\theta} r^2 + O(r^3), \\
\varepsilon^i_\alpha &= \varepsilon^i_{\theta} - \partial_i^{(0)} r^2 + O(r^2), \\
\end{align*}
\]

where \(\kappa_r\) is the surface gravity of the isolated horizon, which is dependent on the choice of \(l\), and \(\partial_i^{(0)} := e^a_i e^b_j V_b n_a\). Due to the zero law of isolated horizon, \(\kappa_r\) is a constant on the horizon. The asymptotic expansion of the inverse metric near the nonrotating isolated horizon is then
\[
g^{ar} = 1, \quad g^{ai} = 0, \\
g^{rr} = 2U + \delta^{ab} \omega_\alpha \omega_\beta = 2\kappa_r r + R_{\text{min}}^{(0)} r^2 + O(r^3), \\
g^{ri} = X^i + \delta^{ab} \omega_\alpha \varepsilon^i_\beta = \delta^{ab} R_{\text{min}}^{(0)} \varepsilon^i_{\theta} r^2 + O(r^3), \\
g^{ij} = \delta^{ab} e^i_\alpha e^j_\beta + O(r). \tag{2.14}\]

We may choose a set of orthogonal vielbein fields, as in [28], which are compatible with the metric (2.11):
\[
e^a_0 = -\frac{1}{\sqrt{2}} \left( an^a + \frac{1}{\alpha} l^a \right), \quad e^a_1 = \frac{1}{\sqrt{2}} \left( an^a - \frac{1}{\alpha} l^a \right), \quad e^a_2. \tag{2.15}\]

Here \(\alpha(x)\) is an arbitrary function of the coordinates. \((e_0, e_1)\) with different choices \(\alpha(x)\) are related by a Lorentz transformation. So we make a gauge fixing from \(SO(D - 1, 1)\) to \(SO(1, 1) \otimes SO(D - 2)\) which is similar to the analysis performed in section 10 of [26]. The covielbein fields are given by
\[
e^a_0 = \frac{1}{\sqrt{2}} \left( an^a + \frac{1}{\alpha} l^a \right), \quad e^a_1 = \frac{1}{\sqrt{2}} \left( an^a - \frac{1}{\alpha} l^a \right), \quad e^a_2. \tag{2.16}\]

Restricted on the isolated horizon \(\Delta\), the 1-form \(l\) vanishes, so we have \(e^0 = e^1 \) (hereafter, we omit the abstract subscript \(a\) for 1-form). Then the non-zero solder fields on the horizon \(\Delta\) satisfy
\[ \Sigma_{01} = e^2 \wedge e^3 \wedge \cdots \wedge e^{D-1}, \quad \Sigma_{0\alpha} = - \Sigma_{1\alpha}. \]  

(2.17)

After some straightforward calculation (see appendix), we can get the following properties for the \( SO(D - 1, 1) \) connections:

\[ A^{01} \triangleq \kappa \, du + d(\ln a), \quad A^{0\alpha} \triangleq A^{1\alpha}. \]  

(2.18)

By equations (2.17) and (2.18) the integral on the horizon can be reduced to

\[ \frac{1}{2\kappa} \int_\Delta \delta_\Sigma \wedge \delta_{11} A^{01} = \frac{1}{\kappa} \int_\Delta \delta_\Sigma \wedge \delta_{11} A^{01}, \]  

(2.19)

since other terms either vanish or cancel each other out.

On the isolated horizon \( \Delta \), \( \Sigma_{01} \) is just the volume form of its spatial section. From the field equation (2.5) and the properties (2.17) and (2.18), it is easy to show that \( d\Sigma_{01} \triangleq 0 \), so \( \Sigma_{01} \) is a closed \((D - 2)\)-form on the horizon \( \Delta \). Locally we can define a \((D - 3)\)-form \( \tilde{B} \) which satisfies

\[ d\tilde{B} \triangleq \Sigma_{01}. \]  

(2.20)

In addition, the \( \tilde{B} \) field must satisfy the following global condition

\[ \oint_K d\tilde{B} \triangleq \oint_K |\Sigma_{01}| = a_K, \]  

(2.21)

where \( a_K \) is the ‘area’ of the horizon, and the absolute value is to ensure that when choosing some patches to cover the orientable manifold \( K \), they should have the same orientation. The second equality of equation (2.21) comes from the flux-area relation [32].

From (2.18) we have

\[ dA^{01} \triangleq 0. \]  

(2.22)

Thus, the integral in (2.19) can be written as

\[ \int_\Delta \delta_\Sigma \wedge \delta_{11} A^{01} \triangleq \int_{K_2} \delta_\Sigma \wedge \delta_{11} A^{01} - \int_{K_0} \delta_\Sigma \wedge \delta_{11} A^{01}, \]  

(2.23)

as claimed before, where

\[ \int_{K_2} \delta_\Sigma \wedge \delta_{11} A^{01} := \sum_\alpha \int_{P_\alpha} \delta_\Sigma \wedge \delta_{11} A^{01} - \sum \int_{P_\alpha \cap \tilde{P}_\beta} \delta_\Sigma \wedge \delta_{11} A^{01} \]  

\[ + \sum \int_{P_\alpha \cap \tilde{P}_\beta \cap \tilde{P}_\gamma} \delta_\Sigma \wedge \delta_{11} A^{01} - \cdots, \]  

(2.24)

where \( \{P_\alpha\} \) are the set of patches covering \( K \). Then, (2.8) implies

\[ \frac{1}{\kappa} \left( \int_M \delta_\Sigma \wedge \delta_{11} A^{01} - \int_K \delta_\Sigma \wedge \delta_{11} A^{01} \right) \text{ is independent of } u. \]  

(2.25)

Consider an \( SO(1, 1) \) boost for \((e_0, e_1)\) with \( g = \exp(\varphi) \). Under the transformation, \( A^{01} \wedge \delta_{11} \rightarrow A^{01} - d\zeta, \) and \( \Sigma_{01} \) remains unchanged. So \( A^{01} \) is an \( SO(1, 1) \) connection, and \( \Sigma_{01} \) is in its adjoint representation. Equation (2.23) is the symplectic flux of an \( SO(1, 1) \) BF theory across the sections of the isolated horizon. Such an \( SO(1, 1) \) BF theory is what we need to supplement GR to explain the statistical origin of the area entropy of a horizon, if the following replacements are made,

\[ B \leftrightarrow \frac{\tilde{B}}{\kappa}, \quad A \leftrightarrow A^{01}, \]  

(2.26)

as in [28].
3. \((D-1)\)-dimensional \(SO(1,1)\) BF theory

In \((D-1)\)-dimensional spacetime \(\Delta\), the action of an ordinary \(SO(1,1)\) BF theory can be written as [33]

\[
S[B,A] = \int_{\Delta} \text{Tr} (B \wedge F(A)) = \int_{\Delta} B \wedge dA.
\]  

(3.1)

where \(A\) is an \(SO(1,1)\) connection field, \(F\) its field strength 2-form, and \(B\) a \((D-3)\)-form field in the adjoint representation of \(SO(1,1)\). From the action (3.1), we can easily obtain the field equations as

\[
F := dA = 0, \quad \tilde{F} := dB = 0.
\]  

(3.2)

In the ordinary BF theory, \(A\) is a flat connection and \(B\)-field is closed. So there are no local degrees of freedom in this theory and only global ones associated with non-trivial topology.

However, the field equations for the BF theory we need on the isolated horizon are

\[
F = dA = 0, \quad \tilde{F} = dB \equiv \frac{\Sigma_{01}}{\kappa}.
\]  

(3.3)

The field equation can be obtained from the action

\[
S = \int_{\Delta} B \wedge dA - \frac{1}{2\kappa} \int_{M} F_{IJ} \wedge \Sigma_{IJ}
\]  

(3.4)

with identify \(SO(1,1)\) connection \(A\) in BF theory with the 0–1 component of \(SO(D-1,1)\) connection \(A_{IJ}\) on the horizon. Compared with equations (3.2), (3.3) shows that the bulk field \(\Sigma_{01}\) serves as the source of the \(B\) field and \(A\) remains a flat connection. After quantization, the bulk spin network intersects the section of the horizon \(K\) with punctures, at which \(\Sigma_{01}\) is nonzero. So the \(dB\) is otherwise nonzero, which gives local degrees of freedom associated with those punctures besides the global degrees of freedom.

Since there are similarities between CS theory and BF theory, the BF theory coupled to the Palatini action could be supposed to be used to describe the boundary degrees of freedom, just as the CS theory used before [5]. The quantization of the BF theory in \((D-1)\)-dimensions is similar to that in three-dimensions [28]. Let us assume that on the \((D-2)\)-dimensional spatial slice \(K\) there are \(n\) punctures denoted by \(P = \{ p_{\alpha} | \alpha = 1, \cdots, n \}\). For every puncture \(p_{\alpha}\) we associate a \((D-2)\)-dimensional bounded neighborhood \(s_{\alpha}\) which contains it and does not intersect any other. Denote the boundary of \(s_{\alpha}\) by \(\eta_{\alpha}\). Define the gauge-invariant functions of the \(B\) field

\[
f_{\alpha} = \int_{s_{\alpha}} dB = \oint_{\eta_{\alpha}} B.
\]  

(3.5)

The common eigenstates of the corresponding quantum operators \(\hat{f}_{\alpha}\) are the Dirac distributions \((\{ a_{\alpha} \}, \mathcal{P}) \equiv (a_{1}, a_{2}, \cdots, a_{n})\) characterized by \(n\) real numbers \(\{ a_{\alpha}, \alpha = 1, \cdots, n \}\). As unbounded self-adjoint operators, the collection \(\{ \hat{f}_{\alpha} | \alpha = 1, \cdots, n \}\) comprises a complete set of observables in \(\mathcal{H}_{K}^{P} \equiv L^{2}(\mathbb{R}^{n})\). There is a spectral decomposition of \(\mathcal{H}_{K}^{P}\) with respect to each \(\hat{f}_{\alpha}\), i.e,

\[
(\{ a_{\beta} \}, \mathcal{P}) \hat{f}_{\alpha} (\{ a_{\beta} \}, \mathcal{P}) a_{\alpha}.
\]  

(3.6)
4. The entropy of an isolated horizon

In order to calculate the entropy of an isolated horizon, we consider the system described by the action (2.2) for the bulk $\mathcal{M}$ plus an $SO(1, 1)$ BF theory for the isolated horizon as the internal boundary of $\mathcal{M}$, whose $B$-field has nontrivial topology. In the loop quantum gravity approach, only the horizon degrees of freedom contribute to the black hole entropy. Hence, the bulk degrees of freedom need to be traced out. We can construct a density matrix $\rho_{BH}$ for such a system and assume the system is in a maximally mixed state. The entropy can be given by the von Neumann formula

$$ S_{BH} = -\text{Tr} \left( \rho_{BH} \ln \rho_{BH} \right), $$

or equivalently,

$$ S_{BH} = \ln N_{BH}, $$

where $N_{BH}$ is the total number of the states in the horizon Hilbert space that satisfy some constraints. The goal of this section is to compute this number to give the entropy.

The second field equation of the BF theory (equation (3.3)) serves as the classical boundary condition relating the boundary and bulk fields:

$$ \Sigma_{\hat{0}1} \approx F_k(x) \big|_{\partial K} = F_k(x), $$

where $\approx$ means that the equality is valid when the limit of $\Sigma_{\hat{0}1}$ at the spatial slice $K$ of the isolated horizon is taken. The right hand side of the equality is proportional to the flux of $\Sigma_{\hat{0}1}$ across $K$, $F_{\hat{0}1}(K) := \oint K \Sigma_{\hat{0}1}(x)$, which is well defined in loop quantum gravity [34]. The left hand side is the volume integral of $F$ over $K$, which is just the sum of the observables (equation (3.5)) in the loop-quantized version of the BF theory.

The Hilbert space for the bulk theory in higher dimension can be constructed in the approach proposed by Bodendorfer and his collaborators [34–37]. The key point is that one can proceed in the phase space for the $D$-dimensional spacetime, which consists of an $SO(D)$ connection $A_a^{IJ}$ and its conjugate momentum $\pi^{IJ}_a$. Besides the usual Gauss constraint, the spatial diffeomorphism constraint and Hamiltonian constraint, there is a simplicity constraint. In quantum theory, the simplicity constraint can be implemented on the links of a spin network by restricting the representations of the $SO(D)$ to be of class 1, so that their highest weight vector $\lambda$ is determined by a single non-negative integer $\lambda$ as $\lambda = (\lambda, 0, \cdots, 0)$ [38]. Under the representation, the ‘area’ $Ar[S]$ for a $(D - 2)$- hypersurface $S$ can be constructed as “flux squared”, i.e. [34],

$$ Ar[S] := \sum_U \left( \frac{1}{2} F_{ij}(S_U)F^{ij}(S_U) \right), $$

where $S = \bigcup S_U$ is a partition of the hypersurface $S$ by a set of closed sets $\{S_U\}$ with each $S_U$ containing, at most, one puncture, and $F_{ij}(S_U)$ is the flux through $S_U$ which can be quantized properly.

Such a construction has been used in the study of isolated horizons in higher dimensional spacetime [26]. For $2n$-dimensional spacetime, the degrees of freedom on the $(2n - 1)$-dimensional isolated horizon $\Delta$ can be described by an $SO(2n)$ CS theory. Unfortunately, the non-Abelian CS theory with $n > 1$ has local degrees of freedom, which would result in the
divergence of the entropy. In order to avoid the problem, a stronger boundary condition is proposed. The stronger boundary condition relates the CS connection on the boundary to a hybrid connection on the bulk. In contrast, there is no such a problem with BF theory, since a BF theory has no local degree of freedom even in higher dimension [33].

For a simple representation of the $SO(D)$ group with highest weight vector $(\lambda, 0, \cdots, 0)$, an orthonormal basis of the space of harmonic homogeneous polynomials of degree $\lambda$ can be characterized by an integral sequence $K = (K_1, \ldots, K_{D-3}, \pm K_{D-2})$ with $\lambda \geq K_1 \geq \cdots \geq K_{D-3} \geq 0$ [39]. In terms of the $(D-2)$-dimensional spherical functions, they are characterized by $|m, \cdots >$.

Then the eigenvalues of the flux operator have the form

$$\oint S |\Sigma_{01}(x)| m_a, \cdots > = 8\pi G \beta \sum_{a} m_a \left| m_a, \cdots \right>.$$  \hfill (4.6)

for an arbitrary $(D-2)$-dimensional closed, connected, oriented, spacelike hypersurface $S$ in $M$. The $\beta$ is a parameter analogous to the Barbero–Immirzi parameter in four dimensions. The index $a$ indicates the $a$th puncture on $S$ for a spin-network eigenstate, which will coincide with the punctures in the boundary BF theory, $m_a \in \{-\lambda_a, -\lambda_a + 1, \cdots, \lambda_a\} \subset \mathbb{Z}$ is the quantum number associated with the flux operator, $|m_a, \cdots>$ represents an $SO(D)$ spin network state in the bulk, and $\cdots$ represents other quantum numbers that characterise the bulk state. The bulk degree of freedom, including the degeneracy of $m$, is not very important, since they should be traced out. The important fact for the later calculation is that $m$ is an integer but not its range.

The quantum version of the boundary condition (4.4) reads

$$\left( \text{Id} \otimes \oint_{s_a} \hat{F} - \frac{1}{8\pi G} \oint_{s_a} \hat{\Sigma}_{01} \otimes \text{Id} \right) (\Psi_b \otimes \Psi_b) = 0,$$  \hfill (4.7)

where $\text{Id}$ means the identity operator, $\Psi_b$ and $\Psi_b$ bulk and boundary states, respectively, and $s_a$ the $(D-2)$-dimensional bounded neighborhood associated to the puncture $p_a$ as before.

From (4.7), we can get the relation between the eigenvalues of $\hat{F}$ and $\hat{\Sigma}_{01}$:

$$a_a = \beta m_a, \quad m_a \in \mathbb{Z}.$$  \hfill (4.8)

In other words, $a_a$ is no longer any real number, but takes discrete values.

The eigenvalues of the flux-area operator which appears in the quantum version of (2.21) has the form

$$\oint S |\Sigma_{01}(x)| m_a, \cdots > = 8\pi G \beta \sum_{a} m_a \left| m_a, \cdots \right>.$$  \hfill (4.9)

Then, a global constraint appears from the equations (2.21) and (2.26):

$$\sum_{a} |a_a| = a_K/(8\pi G).$$  \hfill (4.10)

Similar to the ‘flux-area’ constraint in [32], (4.10) is called flux constraint. With (4.8), the constraint can be reformulated as

$$\sum_{a} |m_a| = \left[ a_K/(8\pi G) \right] = a, \quad m_a \in \mathbb{Z}, \quad a \in \mathbb{N}.$$  \hfill (4.11)
So the number of compatible states is given by
\[ \mathcal{N}_{BH} = \sum_{n=1}^{na} C_{a-1}^{n-1} 2^n = 2 \times 3^{n-1} \] (4.12)
where \( a = a_K/(8\pi\beta) \in \mathbb{N} \). So the entropy is given by
\[ S_{BH} = \ln \mathcal{N}_{BH} = a \ln 3 + \ln \frac{2}{3} = \frac{\ln 3}{2\pi\beta} \frac{a_K}{4G} + \ln \frac{2}{3}. \] (4.13)

In the above calculation, we omit the global degrees of freedom since compared with the local degrees of freedom associated with punctures, those global ones contribute small numbers and may affect the sub-leading term, as shown in [40]. The leading term of (4.13) just gives the area law. Besides, if we set \( \beta = \ln 3/(2\pi) \), we also get the famous coefficient 1/4. Note that this value is dimension-independent. Compared with the case in four dimensions [28], this parameter has an additional factor two. This is due to the fact that in higher dimension we use the group \( SO(D) \) which only has integer representation, and in four dimensions we use \( SU(2) \) which can have half-integer representation. If the present procedure is applied to the \( D = 4 \) case, in which the \( SO(4) \) group instead of the \( SU(2) \) group is used, the \( \beta \) parameter will be different, but the spectrum of the area operator remains the same. Whether there exists any physical process which can distinguish the two approaches is an interesting problem.

As in four dimensional spacetime, the entropy we get has a constant correction term besides the leading area term. The ‘zero-point entropy’ first appears as the quantum correction for the area law of entropy in [41, 42].

5. Discussion

In this paper, we calculate the entropy of nonrotating isolated horizons in a higher dimensional spacetime, following the boundary CS theory approach [5]. From the first-order action (2.2), the presymplectic current is obtained. The current through the isolated horizon \( \Delta \) can be reformulated as the difference across its final and initial spatial sections. Then the symplectic form on a \( (D-2) \)-dimensional hypersurface is acquired. The degrees of freedom on the cross section of the isolated horizon can be described by an \( SO(1, 1) \) BF theory with the symplectic form. The entropy, as in four-dimensional spacetime, obeys the area law in its leading term and has a ‘zero-point entropy’.

It is worthwhile to compare our approach with the approaches in the framework of loop quantization in the literature for four-dimensional spacetime. First, the approach adopted by [10–12] counts the number of spin network states just from the gravitational theory, without using a boundary theory. In the approach the full spectrum of the area operator is used, so it seems to contain the full information of the boundary. However, the area operator is valid for any arbitrary surface, so that the approach can be applied to a generic surface, but not limited for horizons. Then, one has to explain why an arbitrary surface (or hypersurface) has the area-entropy just as an horizon and what the physical meaning of the entropy is for an arbitrary surface or hypersurface. Also, this method has not been applied to higher dimensions. However, the system in the approaches with a boundary CS or BF theory consists of the bulk and the isolated horizon as its inner boundary. On the isolated horizon the simplified area spectrum can be used. The meaning of the entropy for the horizon may be easily understood. But an additional topological field theory on the isolated horizon is invoked. The CS approach has been extended to higher dimensional isolated horizons [26, 27].
Secondly, also in four-dimensions, comparing the approaches in which a boundary CS theory or a boundary BF theory is used, we would like first to stress that the $SO(1, 1)$ symmetry, used for the boundary BF theory, comes from $e^0$ and $e^1$, which is normal to section $K$. However, the $U(1)$ symmetry [5] used in the boundary CS theory is the rotation symmetry on the $K$.

In higher dimension, the main difference between the boundary CS theory approach and the boundary BF theory approach is that in the former the area constraint is used for the calculation of the entropy of isolated horizons, but in the latter the flux constraint (4.10) is used instead. Classically, the area of the horizon equals its flux area, which can be seen from equation (2.21). But at quantum level, they correspond to two different operators: (4.5) and (4.9), respectively. The eigenvalues of the ‘area’ operator are given by [34]

$$8\pi G\hbar \sum_{\lambda_a} \sqrt{\lambda_a (\lambda_a + D - 2)}, \quad \lambda_a \in \mathbb{N}.$$ 

Those eigenvalues are obviously different from that of the flux-area operator (4.9). We use the flux constraint because in a loop quantization of a generalized gravity the flux-area operator turns out to measure the Wald entropy [43].

When the boundary CS theory approach is generalized to a higher dimensional space-time, a non-Abelian CS theory is needed, which has local degrees of freedom. To overcome the problem, a gauge fixing from the $SO(D)$ group to the $SO(2)$ group is suggested. Even so, the dimension restriction limits its application. To break through the limitation on the dimension, the flux-like boundary variables are used in [27]. But from those variables it is difficult to tell which boundary theory is behind those variables. In contrast, these two problems do not exist in the boundary BF theory approach. The boundary BF theory approach can only be applied to nonrotating isolated horizons at the present stage while the boundary CS theory approach can be applied to generic boundaries [44], which has the problem mentioned above.

An important problem in the boundary BF theory approach is whether the group $SO(1, 1)$ for the boundary matches $SO(D)$ for the bulk. On the one hand, the gauge group $SO(D - 1, 1)$ in the bulk reduces to its subgroup $SO(1, 1)$ on the isolated horizon. So, a BF theory with gauge group $SO(1, 1)$ is chosen to describe the boundary degrees of freedom. On the other hand, the quantum states in the bulk are well presented only by the Hamiltonian framework with gauge group $SO(D)$ [35, 36]. The following reasons may support the match. Firstly, the kinematical state spaces have been well defined in both bulk theories with local $SO(D - 1, 1)$-symmetry and with local $SO(D)$-symmetry, in which each state is invariant under the diffeomorphism on a $D - 1$-dimensional space-like hypersurface. Roughly speaking, the dynamical state space of quantum gravity should be a subset of the kinematical state space which satisfies the Hamiltonian constraint. Secondly, what we are concerned with are the entropy of isolated horizons. The isolated horizons, by definition, are stationary within the defined region. Thirdly, the bulk degrees of freedom are not very important since they are traced out in the calculation of the entropy. We need just the spectrum of the flux operator (4.9). Therefore, the above match is reasonable.

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Appendix A. Solder field and connection on the isolated horizon

The connections $A^I_j$ which are adapted with the vielbein $e_I$ are given by

$$
(A^I_j)_a = (e_j)^a \left[ \partial_a (e^I) - \Gamma^b \left( e^j \right)_b \right].
$$

Since what concerns us is the connection restricted on the isolated horizon with $r = 0$, which has no $dr$ component, we only need to calculate it in the infinitesimal neighborhood of the horizon, so that the $e^I$ and the Christoffel symbol $\Gamma^a_{bc}$ are kept to the zero order of $r$ and metric to the first order of $r$.

First we choose the parameter $\alpha(x) = 1$, and other cases can change into this case through a Lorentz transformation. Thus, from (2.15), (2.9) and (2.10), one has

$$
e_0 = -\frac{1}{\sqrt{2}} \left( \partial_u + \partial_r \right) + O(r), \quad e_1 = \frac{1}{\sqrt{2}} \left( -\partial_u + \partial_r \right) + O(r),$$

$$e_2 = \xi_i \partial_i + O(r), \quad (A.2)
$$

and

$$
e^0 = -\frac{1}{\sqrt{2}} (du + dr) + O(r), \quad e^1 = \frac{1}{\sqrt{2}} (-du + dr) + O(r),$$

$$e^\lambda = \xi^\lambda dx^i + O(r). \quad (A.3)
$$

On the horizon $r = 0$,

$$e^0 \triangleq e^1 \triangleq -\frac{\sqrt{2}}{2} du \quad (A.4)
$$

and

$$\Sigma_{01} = e^2 \wedge e^3 \wedge \cdots \wedge e^{D-1}, \quad \Sigma_{03} \triangleq -\Sigma_{13}. \quad (A.5)
$$

Next we calculate the connection $(A_0)^I_j = (e_0)^I \left[ \partial_a (e^I) - \Gamma^b \left( e^j \right)_b \right]$ or $(A_0)^I_j dx^\nu = e_0^\nu \left( \partial_a (e^I) - \Gamma^b \left( e^j \right)_b \right) dx^a$ restricted on the horizon $r = 0$. From the expressions (A.2) and (A.3) we can see that $\nu, \sigma$ can only take values corresponding to $u, r$. For $\mu = u$, the nonzero Christoffel symbols are $\Gamma^u_{ur} \triangleq \Gamma^u_{ur} \triangleq \kappa_i$, so we can get $(A_0)^i_j \triangleq -\kappa_i$. For $\mu = i$, using the same method, it can be shown that $(A_0)^i_j \triangleq 0$. So in this case, we can get

$$A^{01} \triangleq \kappa_i du. \quad (A.6)
$$

For $\alpha(x) \neq 1$, which relates to our case with a Lorentz transformation, the connection transforms into $A^{01} \triangleq \kappa_i du + d(\ln \alpha)$, which is the first result of (2.18).

Finally, the second result of (2.18) is equivalent to $(A_0)^i_j \triangleq -(A_0)^i_j$, or

$$\left[ (A_0^\lambda^I)_\mu + (A_1^\lambda^I)_\mu \right] dx^\mu = (e^0_\mu + e^I_\mu) \left( \partial_\mu e^\lambda_i - \Gamma^b_{\mu\sigma} e^\lambda_i \right) dx^\nu \triangleq 0. \quad (A.7)
$$

Again, $\sigma$ can only take values corresponding to $u, r$. For $\sigma = r$, we already have $e^0_r + e^I_r \triangleq 0$. For $\sigma = u$, we have

$$\left( e^0_u + e^I_u \right) \left( \partial_\mu e^\lambda_i - \Gamma^b_{\mu\sigma} e^\lambda_i \right) \triangleq \sqrt{2} \Gamma^i_{\mu\sigma} e^\lambda_i \triangleq 0. \quad (A.8)
$$
since
\[
\Gamma^i_{\mu
u} = \frac{1}{2} g^{iu} \left( g_{\mu\nu,u} + g_{\mu,u,\nu} - g_{\mu,u,\nu} \right) \triangleq \frac{1}{2} g^{ij} g_{k,\mu,\nu,} \triangleq 0. \tag{A.9}
\]
In the last step, \(d\Sigma_0 \triangleq 0\) has been used. So, we complete the proof of the results (2.18).

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