Propagation of microlocal singularities for stochastic partial differential equations

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Abstract

Microlocal analysis techniques are extended and applied to stochastic partial differential equations (SPDEs). In particular, the Hörmander propagation of singularities theorem is shown to be valid for hyperbolic SPDEs driven by a standard Brownian motion. In this case the wave front set of the solution is invariant under the stochastic Hamiltonian flow associated to the principal symbol of the SPDE. This study leads to the introduction of a class of random pseudodifferential operators.

Keywords: Stochastic partial differential equations, Hyperbolic systems, pseudodifferential operators, Microlocal analysis, Propagation of singularities.

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1 Introduction

The singularities of the solutions to linear partial differential equations (PDEs) have been studied for a long time and these studies culminated in the early 1970s with microlocal analysis; this framework has provided a refined mathematical formulation of the concept of singularities and the corresponding propagation theorem is one of the remarkable results obtained in this context.

When the partial differential equations govern some systems involving random media or random perturbations, one is led to consider stochastic partial differential equations (SPDEs) models, a subject that has been extensively studied during the last three decades.

The investigation of singularities of SPDEs together with the possible applicability of microlocal analysis techniques is a natural issue in this perspective. Several problems may be addressed: for what type of SPDEs may these techniques be relevant? how to deal with randomness when considering pseudodifferential operators? is it possible

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define singularities in a way similar to the deterministic linear PDEs and is there a propagation phenomenon and theorem for SPDEs? etc.

The purpose of this paper is to address such issues for hyperbolic stochastic partial differential equations of the type:

\[
\begin{cases}
  du(t) = \sum_{i=1}^{n} [a_i(t, x, D)u(t) \circ dw^i(t) + f_i(t) \circ dw_i(t)] + b(t, x, D)u(t)dt + g(t)dt, \\
  u(0) = u_0 \in (H^s(\mathbb{R}^d))^{d'},
\end{cases}
\]

This is the stochastic counterpart of linear hyperbolic systems which were studied in particular by Friedrichs [12]. These systems give rise to a large class of hyperbolic PDEs, such as second order linear hyperbolic PDEs, Maxwell equations, Dirac equations, etc. The existence and uniqueness of the solution to the stochastic hyperbolic systems \((E)\) were studied in [1], where \(a_i(t, x, D), b(t, x, D)\) are smooth families of \(d' \times d'\)-matrices of first order pseudodifferential operators (PDO), \(w^i(t), t \in I\) are standard Wiener processes, \(f_i, g\) are continuous, possibly random, functions from \(I = [0, T], T > 0\) to \((H^s(\mathbb{R}^d))^{d'}\), \(\circ\) corresponds to the Fisk-Stratonovich integral or differential, and \(H^s(\mathbb{R}^d)\) is a Sobolev space \((s \in \mathbb{R}, d, d' \geq 1)\).

There has been a wide interest in the topic of singularities of PDEs solutions during the 20th century in accordance with the concepts and techniques available. At the beginning, singularities were associated with the discontinuities of the solutions or their derivatives and were studied for wave and Maxwell equations. It was shown that the discontinuities of the solutions of wave equations in inhomogeneous media propagate along the bi-characteristic curves which correspond to the paths given by the geometrical optics, see, e.g., Luneberg [31], Kline[24]. This fact was important not only because it establishes a formal link between wave and geometric optics, viewing the later as an approximation of the former and providing a tool to justify the practical calculations based on geometrical optics, but also because of its conceptual side: this correspondence can be applied to other areas such as acoustics (see [23]) and mechanics. This result was subsequently generalized to other types of PDEs, possibly non-linear, see Courant and Hilbert [6], and John [21]. It was also extended to linear symmetric systems (Courant and Lax [7]).

On the other hand, the fact that geometrical optics (Fermat’s principle, eikonal equation) forms an approximation of wave optics was noticed before from a different point of view, that is to show that the solutions to geometrical optics equations correspond to those of wave optics when the wave lengths become very small (Runge - Sommerfeld [37]). Let us recall in passing that this fact was actually a fundamental starting point of quantum mechanics in Shrödinger’s work: classical mechanics whose equations are similar to those of geometrical optics, may be considered as an approximation of a wave mechanics for which we have to find the equation, and based on de Broglie’s work, Shrödinger was able to find his famous equation. This point a view was subsequently clarified, starting with Birkhoff’s work [2] and extensively studied in the framework of semi-classical approximation or analysis (see, e.g., [33], [42], [14]).

The description of singularities was refined in the late 1960s by Sato and Hörmander with the concept of singular spectrum or wave front set of a distribution; this set
$WF(u)$ provides the couples $(x, \xi)$ such that $u$ is singular in the point $x$ and in the frequency direction given by $\xi$, which means that $\hat{u}$ is not rapidly decreasing in the direction $\xi$. In fact the thus defined wave front provides the space points $x$ where $u$ is not smooth, which are viewed as singularity points whereas the frequency direction $\xi$ can give information about the form of the set of singularities or the direction of propagation (see §2 below), whence a link with the intuitive term wave front used by the physicists to refer to wave propagation phenomena. With this concept, Hörmander obtained the propagation of singularities theorem of the solution to the hyperbolic equation $\partial_t u = A(x, D) u$, where $A(x, D)$ is a first order pseudodifferential operator with a classical symbol; it states that the wave front set of the solution $u(t, \cdot)$ is invariant under the bicharacteristic curves $\chi_t$ of the principal symbol $a_1$ of the operator $A$, i.e. the integral curves of the flow $(\text{grad}_x a_1, -\text{grad}_\xi a_1)$, see Hörmander [18]. This result was also established for more general equations of the form $Pu = 0$.

The goal of this paper is to show that this result is still valid for the hyperbolic stochastic differential equations $(E)$, that is $WF(u(t, \cdot)) = \chi_t WF(u(0, \cdot))$, where this time $\chi_t$ is the stochastic integral Hamiltonian flow associated to the principal symbols of the operators $a_i(x, D)$ which give the stochastic part with Stratonovich differential equations and $b(x, D)$ which gives the deterministic part.

For this purpose, we shall be led to introduce a class of random pseudodifferential operators which is more general than the ones used in some previous works (see Dedik, Shubin [9], Pankov [34], Fedosov-Shubin [11]). Some properties of these random PDOs will be established; let us mention that Liu and Zhang [30] studied a class of random PDO close to the one considered in this paper.

In the case of SPDEs, hyperbolic equations are often studied with a space-time white noise, in which case the solutions are at most Hölder continuous. Despite this lack of regularity, the problem of propagation of singularities was considered by Walsh ([40], [41]) for the Brownian sheet which actually corresponds to the solution of the wave equation with a space-time white noise source; this study was extended to a nonlinear stochastic wave equation by Carmona and Nualart[4]. In this case singularities mean a failure to have some modulus of continuity. See also Blath and Martin [3] for an extension of this kind of study to semi-fractional Brownian sheets.

This paper is organized as follows: Notations and preliminaries on linear hyperbolic SPDEs are presented in section 2. In order to position the problem addressed in this paper, it is relevant to revisit the results so far obtained on propagation of singularities for PDEs and SPDEs; section 3 provides a quick overview of these results. The notion of random pseudodifferential operators is introduced in section 4, with some results on the stochastic integration of random symbols that will be needed later. The definition of these random PDOs requires a pathwise bound of the derivatives of random symbols. One of the complications encountered is the verification of these pathwise bounds when the random symbols used are defined by stochastic integrals. This difficulty is overcome with the use of a device based on the Kolmogorov-Centsov continuity criterion. Section 5 contains the statement and proof the main result, that is the propagation of singularities of the hyperbolic SPDEs $(E)$ along its stochastic Hamilton flow. The proof is based on the pathwise approximation of the solutions established in [1] and the properties of random symbols obtained in section 4.
2 Notations and preliminaries

2.1 Notations and settings

We consider SPDEs of type (E), driven by a finite dimensional Brownian motion. Let \((w_t), t ∈ I := [0, T]\) be a one-dimensional Brownian motion defined on a filtered probability space \((Ω, F, (F_t)_t, P)\) with \(F_t = σ(w_τ, τ ≤ t)\) and \(T > 0\) fixed throughout this paper. We also fix \(d, d' ≥ 1\). The solutions to (E) will be considered as \((H^s)^{d'}\)-valued process, where \(H^s\) is the Sobolev space of order \(s\); \(\{·, ·, ·, ·\}\) will be the scalar product on the Sobolev space \(H^s := H^s(\mathbb{R}^d), s ∈ \mathbb{R}\) and the same notation is used for the scalar product on \((H^s)^{d'}\). We refer to [1] for a detailed study of the stochastic hyperbolic systems (E) and their properties.

For \(p > 0\), \(M^s_p(I, (H^s)^{d'})\) denotes the set of adapted \((H^s)^{d'}\)-valued processes \(u(t), t ∈ I\) such that \(\|u\|_{s,p} := (E \sup_{t≤T} |u(t)|^p)^{1/p} < +∞\), this quantity being the norm of \(u\) in \(M^s_p\).

We use the standard PDO notations: for \(α = (α_1, · · · , α_d) ∈ \mathbb{N}^d\), \(|α| = \sum_{j=1}^d α_j\) and \(D^α = D_1^{α_1} · · · D_d^{α_d}\), where \(D_j^{α_j} = (−i)^{α_j}∂_{x_j}^{α_j}\) with \(i = √−1\) and \(∂_{x_j} = ∂_{x_j}\), for \(x = (x_1, · · · , x_d)\).

As usual \(S^m\) is the set of symbols \(a(x, ξ)\) of order \(m\) on \(\mathbb{R}^d\), i.e. \(a ∈ C^∞(\mathbb{R}^d × \mathbb{R}^d)\) and for all \(α, β ∈ \mathbb{N}^d\) there is a constant \(C(α, β)\) such that \(|D_ξ^α D_x^β a(x, ξ)| ≤ C(α, β)(1 + |ξ|)^{m−|α|}\). For such a symbol, \(a(x, D)\) denotes the associated pseudodifferential operator defined by \(a(x, D)u(x) = ∫ a(x, ξ) u(ξ)e^{ix.ξ}dξ\) for \(u ∈ C^∞_0(\mathbb{R}^d)\), \(a^∗(x, D)\) is the adjoint of \(a(x, D)\) and \(OPS^m\) will denote the set of such operators. As we consider symmetric systems, the solutions are vector valued (in \(\mathbb{R}^{d'}\)) and the operators \(a(x, D)\) in (E) are matrices of pseudodifferential operators: \(a(x, D) = (a^{ij}(x, D), i, j = 1, · · · , d')\) with \(a^{ij}(x, D) ∈ OPS^m\) for some \(m\); the set of such matrices of PDOs is also denoted by \(OPS^m\).

Let \(X ⊂ \mathbb{R}^d\) be an open set and \(u ∈ D'(X)\). We denote by \(WF(u) ⊂ X × \mathbb{R}^d - \{0\}\) the wave front set of the distribution \(u\) defined by: \((x_0, ξ_0) ∈ WF(u)\) if there is a \(φ ∈ C_0^∞(\mathbb{R}^d)\) with \(φ ≡ 1\) near \(x_0\) such that \(φu(ξ)\) is rapidly decreasing as \(|ξ| → ∞\) in an open cone \(Γ\) containing \(ξ_0\); \(Γ ⊂ X × \mathbb{R}^d\) is said to be a cone if for all \((x, ξ) ∈ Γ\) we have \((x, tξ) ∈ Γ, ∀t ∈ [0, +∞]\).

The wave front \([15]\), also called the singular spectrum of a distribution \(u ∈ D'(X)\) is a refinement of the singular support \(sing\ supp(u)\) defined by \(x_0 \notin sing\ supp(u)\) if there exists a \(φ ∈ C_0^∞(\mathbb{R}^d)\) with \(φ(x_0) ≠ 0\) and \(φu ∈ C_0^∞(X)\). We have indeed: \(sing\ supp(u) = \text{Pr}(WF(u))\) where \(\text{Pr} : (x, ξ) ↦ x\) is the projection on the \(x\)-space \(X\).

The wave front set provides the frequency directions \(ξ\) in which the singularities occur. At a first sight, there is no direct link with the physical intuitive notion of wave fronts which gives the form of curve or surface of singularities and the direction of their propagation. However these elements of the frequency space, the \(ξ\), give indeed such information: for instance let \(u(x_1, x_2)\) be a function on \(\mathbb{R}^2\) which is smooth except a singularity at the line \(x_1 = a\); then its wave front is the set \(\{((x_1, x_2), (ξ_1, 0)) : x_2 ∈ \mathbb{R}, ξ_1 ∈ \mathbb{R}^*\}\):

one can indeed see that:
\[
\hat{\phi}u(0, \xi_2) = \int e^{ix_2\xi_2}dx_2 \left( \int u(x_1, x_2)dx_1 \right),
\]

and from the assumptions on \(u\) it follows that \(\int u(x_1, x_2)dx_1\) is smooth as a function of \(x_2\), so that \(\hat{\phi}u(0, \xi_2)\) is rapidly decreasing as a function of \(\xi_2\). In this example the direction \((\xi_1, 0)\) corresponds indeed to vectors perpendicular to the set of singularities \(x_1 = a\) and hence provides an information about the direction of singularities. This elementary example can be extended to more general situations with singularities given by curves or surfaces and with a time variable representing the evolution which will be used to see the direction of propagation of singularities.

Equivalently, we have the following characterization of the wave front set, which is usually used in the study of singularities and their propagation: \(WF(u) = \bigcap \{\text{Char}(p) : p(x, D)u \in \mathcal{C}_0^{\infty}, p \in S^0_{ph}\}\), where \(S^0_{ph}\) is the set of polyhomogeneous symbols in \(S^0\), that is \(p(x, \xi) \sim \sum \alpha \phi_{\alpha}(x, \xi)\) with \(p_{\alpha}(x, \xi)\) homogeneous of degree \(-\alpha\) in \(\xi\) for \(|\xi| > 1\) and the characteristic set \(\text{Char}(p)\) is defined by \(\text{Char}(p) = \{(x, \xi) : p_0(x, \xi) = 0\}\). This means that \((x, \xi) \notin WF(u)\) iff we can find \(p \in S^0_{ph}\) with \(p_0(x, \xi) \neq 0\), and \(p\) vanishes outside some conic neighborhood of \((x, \xi)\) and \(p(x, D)u \in \mathcal{C}_0^{\infty}\).

## 3 Propagation of singularities for PDEs and SPDEs

### 3.1 The case of partial differential equations

Before the development of microlocal analysis, the singularities of a solution \(u\) of a PDE meant discontinuities of \(u\) or its first or higher order derivatives. The corresponding results on propagation of singularities concerned thus the time evolution of these discontinuities. We start this overview by a digression on the arguments used to show how the bicharacteristic curves carry the discontinuities for hyperbolic equations (see Luneburg [31], Courant, Hilbert [6], John [21], Lax [29]). We consider the simplest hyperbolic equation in a two dimension space:

\[
a(x, y) \frac{\partial^2 u}{\partial x^2} + b(x, y) \frac{\partial^2 u}{\partial xy} + c(x, y) \frac{\partial^2 u}{\partial y^2} + d(x, y) = 0
\]

Let \(\mathcal{C}\) be a curve defined by \(\phi(x, y) = 0\) which divides the plane \(x, y\) into two regions \(O_-, O_+\) where \(\phi < 0\) and \(\phi > 0\) respectively. We suppose that the derivatives \(\partial_x u, \partial_y u\) are continuous on \(\mathcal{C}\) but the 2d derivatives have jumps across this curve: we suppose that \(u\) is \(C^2\) in the region \(O_-, O_+\). If \(s\) is a parametrization of the curve \(\mathcal{C}_c\): \(\phi(x, y) = c = \text{Constant}\), that is: \(\phi(x(s), y(s)) = c\) then we have \(\phi_x(x(s), y(s))x'(s) + \phi_y(x(s), y(s))y'(s) = 0\) and by a suitable choice of \(s\) the interior derivatives along a curve \(\mathcal{C}_c\), which we denote by \(u'_x\), is given by: \(u'_x = u_{xx}\phi_x - u_{xy}\phi_y\); this expression is taken at the points \((x(s), y(s))\). Similarly we have \(u'_y = u_{yx}\phi_x - u_{yy}\phi_y\). If we suppose that \(u'_x, u'_y\) are continuous, then their jumps \([u'_x], [u'_y]\) across \(\mathcal{C}\) will be \(0\) and we will have:

\[
[\partial^2 u]_{\partial x\phi} - [\partial^2 u]_{\partial y\phi} = [\partial^2 u]_{\partial x\phi} - [\partial^2 u]_{\partial y\phi} = 0
\]
and \([\partial^2_{yy}u] \partial_x \phi - [\partial^2_{yx}u] \partial_y \phi\), which implies that:
\[
[\partial^2_x u] = k(\partial_x \phi)^2, \quad [\partial^2_y u] = k(\partial_y \phi)^2, \quad [\partial^2_{xy} u] = k\partial_x \phi \partial_y \phi
\]

(3.3)

Now let \(P \in C\) and \(P_1 \in O_-, P_2 \in O_+\) be 3 points close to each other and consider the value of a solution to (3.1) when \(P_1, P_2\) tend to \(P\); then we will have:
\[
a[\partial^2_x u] + b[\partial^2_{yx} u] + c[\partial^2_y u] = 0
\]

(3.4)

Using (3.3) we see that \(\phi\) must satisfy the following equation:
\[
a\partial_x^2 \phi^2 + b\partial_x \phi \partial_y \phi + c\partial_y^2 \phi^2 = 0
\]

(3.5)

which is the characteristic equation of (3.1). From this we deduce that if we call “wave front surface” the surface \(C\) across which the solution \(u\) to (3.1) has discontinuities for some higher order derivatives (and \(C\) separates the space into two regions on which the solution \(u\) is regular, see Luneberg [31], [6]), then this wave front must be a characteristic surface of (3.1). Now since (3.5) is a 1st order partial differential equation, one can see that the discontinuities are carried on the integral curves of this PDE, which are called characteristics, and since the PDE (3.5) is already a characteristic equation, these curves are called the bicharacteristics of the PDE (3.1), see [31]. In the case of a wave equation where \(y = t, a = 1/c^2, b = 0, c = 1\), we recover the eikonal equation:
\[
\partial_t \phi^2(t, x) - \frac{1}{c^2} \partial_x \phi^2(t, x) = 0
\]

(3.6)

The property of propagation of discontinuities along the bicharacteristics was extended by Courant and Lax to the hyperbolic systems \(\sum_{j=1}^p A_j(x) \partial_j u + B(x) u = 0\) with some additional conditions. Luneburg [31] was probably the first to study this propagation of discontinuities in a clear framework for the Maxwell equations, considering the rays as the orthogonal curves to characteristic surfaces. He showed that they correspond to the geometrical optics rays which are given by Fermat’s principle. This property represents the second aspect of the links between wave and geometrical optics.

The first aspect of this connection is the fact that the solutions of wave optics equations tend to those of geometrical optics when the wave lengths become small or, equivalently, in the high frequency limit. This remark went back to the works of Kirchhoff [25], Debye, Sommerfeld and Runge [37]. We recall that the idea of this approach is to express the solution to an equation of the type:
\[
\frac{\partial^2 u}{\partial t^2}(t, x) - \frac{1}{c^2(x)} \frac{\partial^2 u}{\partial x^2}(t, x) = 0
\]

(3.7)
as: \(u(t, x) = e^{i\omega t} A(x)\); here the velocity can be written as \(c(x) = c_0/n(x)\), where \(c_0\) is the velocity in the vacuum or in an homogeneous medium and \(n(x)\) is the index of the inhomogeneous medium. Then \(A(x)\) verifies:
\[
A''(x) + k^2(x) A(x) = 0 \quad \text{with} \quad k(x) = \frac{\omega n(x)}{c_0}
\]

(3.8)
and if we consider a high frequency expansion of the form:

\[ A(x) = e^{iωφ(x)}(A_0(x) + \frac{1}{ω}A_1(x) + \frac{1}{ω^2}A_2(x) + \cdots), \quad (3.9) \]

then the phase \( φ(x) \) will satisfy the eikonal equation:

\[ \left| \frac{dφ}{dx}(x) \right| = \frac{n(x)}{c_0}, \quad (3.10) \]

while the amplitudes verify transport equations of the type: \( φ''(x)A_0(x) + 2φ'(x)A_0'(x) = 0 \), etc. This point of view was also studied in order to give a satisfactory mathematical basis to the assertion that geometrical optics corresponds to a high frequency limit of wave optics. The same idea was applied to mechanics in order to establish that the limit as the Plank constant \( h \to 0 \) of the quantum mechanical description (via Shrödinger equation) corresponds to the classical mechanic equation (Birkhoff [2]).

This high frequency analysis was extensively studied afterward within the framework of semi-classical approximations. The two points of views (propagation of singularities and high frequency limit) have received modern formulations through microlocal analysis, see Garding [13] for a detailed account of this subject. Hormander’s theorem on propagation of singularities appeared in several papers (see, e.g., [16], [17], [10]); for the case of hyperbolic PDEs it is formulated as follows: let \( u \) be the solution to:

\[ \frac{∂u}{∂t} = a(t, x, D)u + f, \quad u(0, x) = u_0 \in H^s \quad (3.11) \]

where \( a(t, x, D) \) is a first order family of PDOs whose symbols \( a(t, x, ξ) \) form a bounded family in \( S^1 \), with \( t \mapsto a_t \) continuous and \( a(t, x, ξ) ∼ \sum_{j=0}^{∞} a_{1-j}(t, x, ξ) ∈ S^1_{ph} \). Then we have \( WF(u(t, ·)) = χ_t WF(u(0, ·)) \), that is, the wave front of \( u(t, 0) \) is invariant under the Hamiltonian flow \( χ_t \) of the principal symbol of \( a(t, x, D) \) given by: \( χ_t(x, ξ) = (x_t, ξ_t) \) with:

\[ \frac{dx_t}{dt} = -\frac{1}{i} \frac{∂a_1}{∂ξ_t}(t, x(t), ξ(t)) \frac{∂}{∂x}, \quad \frac{dξ_t}{dt} = \frac{1}{i} \frac{∂a_1}{∂x}(t, x(t), ξ(t)) \frac{∂}{∂ξ_t} \]

and \( x_0 = x, ξ_0 = ξ \). This theorem was extended to other PDEs of the form \( Pu = f \) where \( P \) is a pseudodifferential operator of order \( m \) such that its principal symbol \( p_m \) is homogeneous of degree \( m \); it is also assumed that \( P \) is properly supported and \( p_m \) is real. Let

\[ H_{p_m} = \sum_{j=1}^{d} \frac{∂p_m}{∂x_j} \frac{∂}{∂ξ_j} - \frac{∂p_m}{∂ξ_j} \frac{∂}{∂x_j} \]

be the Hamiltonian vector field of \( p_m \); its integral curves \( γ_t(x_0, ξ_0) = (x_t, ξ_t) \) are called the bicharacteristics and among these curves, those for which we have: \( p_m(x_t, ξ_t) = 0 \) are called the zero bicharacteristics. The propagation of singularities theorem in this case asserts that if \( Pu = f \) with \( f ∈ D'(U), U ⊂ \mathbb{R}^d \) being an open set, then:

\( WF(u) - WF(f) ⊂ \text{Char}(P) \) and is invariant under the zero bicharacteristics. In particular, if \( f \) is smooth then \( (x_0, ξ_0) ∈ WF(u) \) if \( (x_t, ξ_t) := γ_t(x_0, ξ_0) ∈ WF(u) \). This result has been reformulated and studied in several situations, including the problems of reflection, diffraction, etc., see, e.g., [39], [35].
3.2 The case of stochastic partial differential equations

In the case of SPDEs, we have two situations: in the first one, these equations are driven by standard or cylindrical Brownian motion and the solutions may have sufficient regularity properties, as being $C^\infty$ or being elements of Sobolev spaces $H^s$. In this case one can hope for a propagation of singularities result similar to deterministic PDEs, provided we deal with hyperbolic type equations, since the parabolic type equations do not lead to propagation phenomena. This is precisely the purpose of the results presented in the next sections.

In the second situation, one considers SPDEs driven by a space-time white noise, in which case the solutions are not regular and they are at most Hölder continuous. Singularities are then defined as a failure to have a local modulus of continuity, which is related to the law of iterated logarithm (LIL) for the models so far studied. More precisely, Walsh [40], [41] considered the wave equation in two space-time dimension:

$$\frac{\partial^2 X}{\partial t^2}(t, x) - \frac{\partial^2 X}{\partial x^2}(t, x) = \xi(t, x),$$

(3.12)

whose solution is the Brownian sheet $w(s, t)$. For a Brownian motion $w_t$ we have the following (LIL):

$$\limsup_{h \to 0} \frac{w_{t+h} - w_t}{\sqrt{2h \log \log h}} = 1, \text{ a.e.}$$

(3.13)

for each $t$ fixed. The extension of this estimate to the Brownian sheet is: for each $s$ fixed:

$$\forall t \geq 0 \limsup_{h \to 0} \frac{w_{s+h,t} - w_{s,t}}{\sqrt{2h \log \log h}} = \sqrt{t}, \text{ a.e.}$$

(3.14)

Then a singularity is defined when we take $s$ as a random variable $S(\omega)$ and when the modulus of continuity (or the LIL) given by (3.13) fails, i.e. when we have:

$$\limsup_{h \to 0} \frac{w_{S+h,t} - w_{S,t}}{\sqrt{2h \log \log h}} = \infty, \text{ a.e.,}$$

(3.15)

and the propagation of singularities result of Walsh [40] states that that if (3.15) holds for some $(S, t_0)$ (supposing that $S$ is $\sigma(w_{s,t}, t \leq t_0)$-measurable) then it holds for all $(S, t)$ with $t \geq t_0$. This fact means that the singularities propagates in the $t$ direction (vertically) and by symmetry also horizontally in the $s$ direction if we take points $(s, T)$ with a random time $T$. This result was extended by Carmona and Nualart [4] who considered a non-linear generalization of (3.12):

$$\frac{\partial^2 X}{\partial t^2}(t, x) - \frac{\partial^2 X}{\partial x^2}(t, x) = a(X(t, x))\xi(t, x) + B(X(t, x)),$$

(3.16)

and these authors studied the existence of the above mentioned kind of singularities for (3.16) and their propagation and reflexion with some other probabilistic properties. The results of Walsh were also extended by Blath and Martin [3] to semi-fractional Brownian sheets $X_{s,t}$ which are two-dimensional Gaussian random fields with a covariance of the form $EX_{s,t}X_{s',t'} = t \wedge t' (s^\alpha + s'^\alpha - |s - s'|^\alpha)/2, \alpha \in ]0, 2[$.
4 Random symbols and random pseudodifferential operators

For the purpose of the next section, we introduce a class of random symbols and random pseudodifferential operators. By a random symbol, we mean a symbol $p(x, \xi, \omega)$ which depends on a parameter $\omega \in \Omega$ and such that for each $x, \xi \in \mathbb{R}^d \omega \mapsto p(x, \xi, \omega)$ is measurable.

We shall use the following classes of random symbols:

- The class $RS^m_\omega$ is formed by the set of random symbols $p(x, \xi, \omega)$ such that for each compact $K \subset \mathbb{R}^d$ and $\alpha, \beta \geq 1$ there is a constant $C(\alpha, \beta, K, \omega)$ (measurable w.r.t $\omega$) that satisfies:
  \begin{equation}
  \forall x \in K, \xi \in \mathbb{R}^d : |D^\alpha_\xi D^\beta_x a(x, \xi, w)| \leq C(\alpha, \beta, K, \omega)(1 + |\xi|)^{m-|\alpha|} \text{ a.e.} \quad (4.17)
  \end{equation}
  We note here that the bounds of the derivatives of the symbols are allowed to be pathwise ($\omega$) dependent.

N.B. In the following, the letter $\omega$ may be omitted in the notation of random symbols.

- The class $RS^m_{h_j}$ is formed by the set of random symbols $p(x, \xi, \omega) \in RS^m_{h_j}$ which are homogeneous of degree $m$ in $\xi$ (at least for $\xi \geq 1$).

- The class $RS^m_{cl}$ that we call the classical random symbols (as in the deterministic case), is formed by the set of random symbols $p(x, \xi, \omega) \in RS^m$ such that there exist a sequence of random symbols $a_j(x, \xi, \omega) \in RS^m_{h_j}$, $j = 0, 1, 2, \ldots$ with:
  \begin{equation}
  p(x, \xi, \omega) = \sum_{j=0}^{n} a_j(x, \xi, \omega) \in RS^{m-(n+1)} \forall N \geq 0, \text{ a.e.} \quad (4.18)
  \end{equation}

When a random symbol $a(t, x, \xi)$ depends on the time parameter $t$, the family $(a(t, x, \xi)), t \in \mathbb{R}$ will be assumed to be $\mathcal{F}_t$ adapted.

As mentioned in the introduction, random symbols and random PDOs have already been used by Dedik, Shubin [9], Pankov [34], Fedosov-Shubin [11]. These authors considered symbols $a(x, \xi, \omega)$ which depend on a random parameter $\omega$, satisfying the inequalities (4.17). However, the constants $C(\alpha, \beta, K, \omega)$ do not depend on $\omega$ in these works. On the other hand, Liu and Zhang [30] introduced a class of random PDOs and random symbols satisfying the inequalities (4.17) and depending on a time parameter $t \in [0, T]$ with constants $C(t, \alpha, \beta, K, \omega)$ that depend on $\omega$ and verify conditions like:
  \begin{equation}
  E \int_0^T |C(t, \alpha, \beta, K)|^p < \infty \quad (4.19)
  \end{equation}

These authors transposed some of the usual properties of symbols and PDOs to this stochastic settings. In our case, we will be led to use symbols which are defined as a stochastic integral of random symbols:
  \begin{equation}
  q(t, x, \xi) := \int_0^t p(\tau, x, \xi)dw(\tau) \quad (4.20)
  \end{equation}
However, the condition (4.17) is not trivial for \( q(t,x,\xi) \) as it involves a pathwise bound for stochastic integrals. Nonetheless, this condition can be verified if we consider homogeneous random symbols; this is the subject of the following proposition:

**Proposition 4.1** Let \((p(t,x,\xi) \in R S^m_h, t \in [0,T])\) be a family of random homogeneous symbols such that for each \((x,\xi)\), \(p(t,x,\xi)\) is a semimartingale and there exists a constant \(\gamma > d\) with:

1. \(\forall \alpha, \beta \geq 0, \exists C_1 : \mathbb{E}|D^\alpha_\xi D^\beta_x p(t,x,\xi)|^\gamma \leq C_1 < +\infty\)
2. \(\forall \alpha, \beta \geq 0, \exists C_2 : \mathbb{E}|\langle D^\alpha_\xi D^\beta_x p(t,x,\xi), w \rangle|^\gamma \leq C_2 < +\infty\)

Then:

\[ q(t,x,\xi) := \int_0^t p(\tau,x,\xi) \circ dw(\tau) \]

(which is well defined) is a random symbol in \( R S^m_h \).

- The condition (2) is verified in particular if the martingale part of \( p(t,x,\xi) \) is of the form:

\[ \int_0^t r(\tau,x,\xi) dw(\tau), \text{ with } r \in R S^m_h \text{ and } \mathbb{E}|D^\alpha_\xi D^\beta_x r(t,x,\xi)|^\gamma < +\infty \]

NB. The brackets \( \langle U,V \rangle_t \) in the point (2) of the proposition denotes here the quadratic variation of the processes \( U, V \), see [1]. The proof of this proposition relies on the following lemmas:

**Lemma 4.1** Let \( g(x) \) be a random field indexed by \( x \in \mathbb{R}^d \) and suppose that there exist constants \( \gamma > d \) and \( C > 0 \) such that:

\[ \mathbb{E}|\frac{\partial g}{\partial x_i}(x)|^\gamma \leq C < +\infty \ \forall x \in K, i = 1, \cdots, d \]  

(4.21)

where \( K \subset \mathbb{R}^d \) is a compact set. Then \( g \) has a modification (still denoted by \( g \)) which is a.e. continuous and for almost all \( \omega \in \Omega \) there exists a constant \( C(K,\omega) \) such that:

\[ \sup_{x \in K} |g(x)| \leq C(K,\omega) < +\infty. \]

**Proof.** To prove the continuity of \( g \) we use the multidimensional Kolmogorov-Centsov criterion (see Kallenberg [22]) that we recall here: if \((E,d)\) be a complete metric space, and \( g(x), x \in K \subset \mathbb{R}^d \) an \( E \)-valued process such that there exist \( C, \alpha, \beta > 0 \):

\[ \mathbb{E}(d(g(x),g(y))^\alpha \leq C|x-y|^{d+\beta} \ \forall x,y \in I, \]

(4.22)

then \( g(x) \) has a modification which is almost-surely \( \lambda \)-Hölder continuous for all \( \lambda \in [0, \beta/\alpha] \) and this modification verifies (4.22).

For the function \( g \) of the lemma, we have:

\[ g(y) - g(x) = \int_0^1 \sum_{k=1}^d \frac{\partial g}{\partial x_k}(x + t(y - x))(y_k - x_k)dt \]
By the Hölder inequality we have for $x_k \geq 0, p > 1$:

$$\left( \sum_{k=1}^{N} |x_k|^p \right)^{\frac{1}{p}} \leq N^{p-1} \sum_{k=1}^{N} |x_k|^p$$  \hspace{1cm} (4.23)

Then, for $\alpha \geq 1$:

$$|g(y) - g(x)|^\alpha \leq d^{\alpha-1} \sum_{k=1}^{d} \int_0^1 \left| \frac{\partial g}{\partial x_k} (x + t(y-x)) \right| |y_k - x_k|^\alpha dt,$$

which implies by the assumption of the lemma that:

$$\mathbb{E}|g(y) - g(x)|^\alpha \leq d^\alpha C |y_k - x_k|^\alpha$$

If we choose $\alpha = \gamma > d, \beta = \gamma - d > 0$ then the criterion (4.22) will be verified and there is a modification of $g$ which is a.e. continuous; this modification is therefore almost everywhere bounded on every compact $K$, which means that

$$\exists C(K, \omega) : \sup_{x \in K} |g(x)| \leq C(K, \omega) < +\infty$$

for almost all $\omega$. □

**Lemma 4.2** Let $f(\xi)$ be a random field indexed by $\xi \in \mathbb{R}^d$ and suppose that there exist constants $\gamma > d$ and $C > 0$ such that:

$$\mathbb{E} |\frac{\partial f}{\partial x_k}(x)|^\gamma \leq C < +\infty \quad \forall \xi \in U_1, k = 1, \ldots, d$$  \hspace{1cm} (4.24)

where $U_1 = \{ \xi : |\xi| = 1 \}$. Then $f$ has a modification (still denoted by $f$) which is a.e. continuous and for almost all $\omega \in \Omega$ there exists a constant $C(K, \omega)$ such that:

$$\sup_{\xi \in \mathbb{R}^d} |f(\xi)| \leq C(\omega)|\xi|^m$$

**Proof.** We make the change of variable $x = \xi/|\xi|$ and set $g(x) = f(x) = f(\xi/|\xi|) = f(\xi)/|\xi|^m$. Then $\mathbb{E} |\partial_k g(x)|^\gamma = \mathbb{E} |\partial_k f(x)|^\gamma \leq C$ for all $\xi \in U_1$ and by Lemma 4.1, there exists a modification of $g$ (and $f$), still denoted by the same letters, and a constant $C(\omega)$ such that $|g(x)| \leq C(\omega)$ for almost all $\omega$, i.e., $|f(\xi)| \leq C(\omega)|\xi|^m$ for all $\xi \in \mathbb{R}^d$. □

**Proof of Proposition 4.1.** First, we shall show that $q$ has a modification for which the derivatives $D_\xi^\alpha D_x^\beta q(t, x, \xi)$ exist and

$$D_\xi^\alpha D_x^\beta \int_0^t p(\tau, x, \xi) \circ dw(\tau) = \int_0^t D_\xi^\alpha D_x^\beta p(\tau, x, \xi) \circ dw(\tau)$$  \hspace{1cm} (4.25)
Since this assertion is of local character \((\text{in } x \text{ and } \xi)\) we may suppose that all the derivatives \(D_\xi^\alpha D_x^\beta p\) are bounded and Hölder continuous. We can thus apply the theorem 1.2 of Kunita \([26]\) which gives (4.25). We also have:

\[ D_\xi^\alpha D_x^\beta q(t, x, \xi) = \int_0^t D_\xi^\alpha D_x^\beta p(\tau, x, \xi)dw(\tau) + \frac{1}{2} \int_0^t D_\xi^\alpha D_x^\beta \langle p(\cdot, x, \xi), w(\cdot) \rangle_\tau d\tau, \]

which yields:

\[
E|D_\xi^\alpha D_x^\beta q(t, x, \xi)|^\gamma \leq 2\gamma^{-1}|E| \int_0^t D_\xi^\alpha D_x^\beta p(\tau, x, \xi)dw(\tau)|^\gamma
\]

\[ + E|\frac{1}{2} \int_0^t D_\xi^\alpha D_x^\beta \langle p(\cdot, x, \xi), w(\cdot) \rangle_\tau d\tau|^\gamma, \tag{4.27} \]

where we have used the Hölder inequality \((\sum_{k=1}^N |x_k||y_k| \leq (\sum_{k=1}^N |x_k|^p)(\sum_{k=1}^N |y_k|^q)^{1/q}\) with \(p = \gamma, N = 2, y_k = 1\). By the martingale moment inequalities and Hölder’s inequality we get:

\[
E|\int_0^t D_\xi^\alpha D_x^\beta p(\tau, x, \xi)dw(\tau)|^\gamma \leq C_\gamma E(\int_0^t |D_\xi^\alpha D_x^\beta p(\tau, x, \xi)|^2 d\tau)^{\gamma/2}
\]

\[ \leq C_\gamma E(\int_0^T |D_\xi^\alpha D_x^\beta p(\tau, x, \xi)|^\gamma d\tau)T^{\gamma/2-1} \]

As for the second term of the r.h.s. of (4.26), we use the Hölder inequality to get:

\[
E|\frac{1}{2} \int_0^t D_\xi^\alpha D_x^\beta \langle p(\cdot, x, \xi), w(\cdot) \rangle_\tau d\tau|^\gamma \leq \frac{1}{2\gamma} T^{\gamma-1} \int_0^T E|D_\xi^\alpha D_x^\beta \langle p(\cdot, x, \xi), w(\cdot) \rangle_\tau |^\gamma d\tau
\]

Hence, by the assumptions (1) and (2) of the proposition, the last inequalities imply that for all \(\alpha, \beta \geq 0\) we have:

\[
E|D_\xi^\alpha D_x^\beta q(t, x, \xi)|^\gamma \leq T^{\gamma/2-1}TC_\gamma C_1 + T^{\gamma-1}TC_2
\]

Now we use lemma 4.1 to control the derivatives w.r.t. \(x \in K\) and Lemma 4.2 in order to control the derivatives w.r.t. \(\xi \in \mathbb{R}^d\). For the later case, we also use the fact that if a differentiable function \(f(\xi)\) is homogeneous of degree \(m\), then the derivative \(D_\alpha f(\xi)\) is homogeneous of degree \(m - \alpha\). We will thus have: for any compact \(K \subset \mathbb{R}^d\) and for all \(\alpha, \beta \geq 0\) and for almost all \(\omega \in \Omega\), there exists a constant \(C(\alpha, \beta, K, \omega)\) such that:

\[
|D_\xi^\alpha D_x^\beta p(\tau, x, \xi)| \leq C(\alpha, \beta, K, \omega)(1 + |\xi|)^{m-|\alpha|},
\]

which means \(q(t, x, \xi) := \int_0^t p(\tau, x, \xi) \circ dw(\tau)\) is a random symbol in \(RS_h^m\). As for the second part of the proposition, we note that \(\langle p(\cdot, x, \xi), w(\cdot) \rangle_\tau = r(\tau, x, \xi)\) and condition (2) of the proposition is verified by the assumption on \(r\). □

A classical symbol can be constructed from a sequence of homogeneous symbols; we shall need the following extension of this result to the random symbols:
Proposition 4.2 Let \( q_j(x, \xi, \omega) \in RS_h^{m-j} \), \( j = 0, 1, \ldots \) be a sequence of random symbols homogenous of degree \( m-j \). Then there exists a random symbol \( q(x, \xi, \omega) \in RS_h^m \) such that \( q(x, \xi, \omega) \sim \sum_{j=0}^{\infty} q_{m-j}(x, \xi, \omega) \), a.e.

Unlike the situation of Proposition 4.1, the proof in the deterministic case can be adapted to the case of random symbols without difficulties (see, e.g. [38]). For completeness, this proof will be outlined in §6.

Lemma 4.3 Let \( p(t, x, \xi) \) be a random symbol satisfying the assumptions of Proposition 4.1 and \( v \in M^2(I, H^s) \). Define \( u(t) = \int_0^t v(\tau) \circ dw(\tau) \). Then

\[
q(t, x, D)u(t) = \int_0^t p(\tau, x, D)u(\tau) \circ dw(\tau) + \int_0^t q(\tau, x, D)v(\tau) \circ dw(\tau). \tag{4.28}
\]

Proof. Suppose first that \( v(t) \in S \) a.s. for all \( t \). Then a stochastic Fubini theorem (see, e.g., [8]) yields:

\[
\hat{u}(t, \xi) = \int_0^t \hat{v}(\tau, \xi) \circ dw(\tau).
\]

where \( \hat{u}(t, \cdot) \) means the Fourier transform of \( u(t, \cdot) \) (w.r.t. \( x \)), and

\[
q(t, x, D)u(t) = \int_{\mathbb{R}^d} q(\tau, x, \xi) [\int_0^\tau \hat{v}(\theta, \xi) \circ dw(\theta)] e^{ix\cdot \xi} d\xi.
\]

But the Itô formula gives

\[
q(\tau, x, \xi) \int_0^\tau \hat{v}(\theta, \xi) \circ dw(\theta) = \int_0^\tau q(\tau, x, \xi) \hat{v}(\theta, \xi) \circ dw(\theta) + \int_0^\tau p(\tau, x, \xi) \hat{u}(\theta, \xi) \circ dw(\theta).
\]

Using again the stochastic Fubini theorem we deduce that (4.28) holds for \( v(t) \in S \). The case where \( v \in M^2(I, H^s) \) follows by a density argument. \( \square \)

The next proposition concerns the dependance of a particular stochastic flow with respect to the initial conditions:

Proposition 4.3 Let \( a(t, x, \xi), t \in [0, T] \) be a bounded family in \( S^1 \) and consider the stochastic equations

\[
(C) : \begin{cases}
\frac{dx(t)}{dt} = \frac{\partial a}{\partial \xi}(t, x(t), \xi(t)) \circ dw(t) \\
\frac{d\xi(t)}{dt} = -\frac{\partial a}{\partial x}(t, x(t), \xi(t)) \circ dw(t)
\end{cases}
\]

with \( x(0) = x_0, \xi(0) = \xi_0 \). Then (C) has a global solution defined on \([0, T]\). Furthermore, if we denote \( \phi_t(x_0, \xi_0) = (x(t), \xi(t)) \) the stochastic flow of diffeomorphisms associated to (C) and \( (\bar{x}(t), \bar{\xi}(t)) = \phi_t^{-1}(x_0, \xi_0) \), then we have for \( n \geq 1 \):

- The quantities \( E|x(t, x_0, \xi_0)|^n, E|\xi(t, x_0, \xi_0)|^n, E|\bar{x}(t, x_0, \xi_0)|^n, E|\bar{\xi}(t, x_0, \xi_0)|^n \) are bounded.
- For all \( \alpha, \beta \geq 1 \), the following quantities are bounded:

\[
E|D^\alpha_{\xi_0} D^\beta_x x(t, x_0, \xi_0)|^n, E|D^\alpha_{\xi_0} D^\beta_x \bar{x}(t, x_0, \xi_0)|^n, E|D^\alpha_{\xi_0} D^\beta_x \xi(t, x_0, \xi_0)|^n, E|D^\alpha_{\xi_0} D^\beta_x \bar{\xi}(t, x_0, \xi_0)|^n.
\]

- If \( a(t, x, \xi) \) is homogeneous of degree 1 in \( \xi \), then \( \phi_t(x_0, \xi_0) \) is homogeneous of degree 1 in \( \xi_0 \).
This proposition and the following corollary will be used in the proof of the main result. Their proofs are given in the section 6.

**Corollary 4.4** Let \( p(x, \xi) \in S^m_h \) and \( \phi_t \) be the flow associated to (C). Then \( q_t(x, \xi) := p(\phi_t(x, \xi)), t_t(x, \xi) = p(\phi_t^{-1}(x, \xi)) \) define a family of random symbols in \( RS^m_h \).

### 5 Propagation of singularities for hyperbolic stochastic partial differential equations

In this section we study the singularities of the solutions to Eq. (E), at the wave front level. We prove a propagation of singularities result which is similar to the deterministic case. For the sake of simplification we state and prove this result in the scalar case \((d' = 1)\).

Let us consider the stochastic equation:

\[
du(t) = A(t, x, D)u(t) \circ dw(t) + B(t, x, D)u(t)dt, \quad u(0) = u_0 \in (H^s(\mathbb{R}^{d'})).
\]

(5.29)

We set \( A(t, x, \xi) = ia(t, x, \xi), b(t, x, \xi) = ib(t, x, \xi) \) and we suppose in this section that \( a(t, x, \xi), b(t, x, \xi) \) satisfy the following additional condition:

(v) The principal symbols of \( a(t, x, \xi) \) and \( b(t, x, \xi) \) are real and

\[
a(t, x, \xi) \sim \sum_{j=0}^{\infty} a_{1-j}(t, x, \xi), \quad b(t, x, \xi) \sim \sum_{j=0}^{\infty} b_{1-j}(t, x, \xi),
\]

with \( a_{1-j}(t, x, \xi), b_{1-j}(t, x, \xi) \) homogeneous in \( \xi \) of degree \( 1 - j \). We are interested in the relationship between the singularities of \( u(t, \cdot) \) and those of \( u_0 \). In the deterministic case \((a = 0)\) the singularities of the solution of the hyperbolic PDE: \( \partial_t u + b(t, x, D)u = 0 \)

\( u(t, \cdot) \) propagate along the bi-characteristic curves of the principal symbol \( b_1 \) of \( b \), i.e. the integral curves of the Hamilton vector field

\[
H_{b_1} = \frac{\partial b_1}{\partial x^i} \frac{\partial}{\partial x^i} - \frac{\partial b_1}{\partial \xi^i} \frac{\partial}{\partial \xi^i}.
\]

In other words, if \( \chi(t, x, \xi) \) represents the integral curves (or flow) associated to \( H_{b_1} \), then we have \( WF(u(t, \cdot)) = \chi_t(WF(u(0, \cdot))) \). This theorem has been proved in several ways, see, e.g., Duistermaat-Hörmander [10], Hörmander [16]-[19], Sogge [?], Rauch[36], Raltson[35], Taylor[39].

In the case of hyperbolic SPDEs (5.29), we have a similar result. We shall use random symbols and random PDOs to adapt the proof given in Hörmander[19], Raltson[35], and it is instructive to recall the idea of this proof: given a couple \((x_0, \xi_0) \notin WF(u_0)\), we want to prove that \((x_t, \xi_t) := (\chi(t(x_0, \xi_0)) \notin WF(u_{t, \cdot}), where \((x_t, \xi_t) \) is the integral curve of the vector field \( H_{b_1} \) (or Hamiltonian flow). We seek a family of operators \( Q(t, x, D) \) which nearly commute with \( iD_tu + b(t, x, D)u \), so that:

\[
(iD_t + b(t, x, D))Q(t, x, D)u - Q(t, x, D)(iD_t + b(t, x, D))u \in S^{-\infty}
\]

(5.30)
If \( u(t, x) \) is a solution of \( (iD_t + b(t, x, D))u = 0 \), then we will have

\[
(iD_t + b(t, x, D))Q(t, x, D)u \in S^{-\infty}
\] (5.31)

This implies that \( Q(t, x, D)u(t, x) \in C^\infty \). Now if we find \( Q(t, x, D) \) as operators with classical symbols \( Q(t, x, \xi) \sim \sum_0^\infty Q_j(t, x, \xi) \) then we can conclude that \( (x, \xi) \notin WF(u_{t_i}) \) provided that \( Q_0(t, x, \xi) \neq 0 \). The point is that when such \( Q_j \) are constructed, it turns out that they verify:

\[
Q_0(t, x_t, \xi_t) := Q_0(t, \chi_t(x_0, \xi_0)) = q_0(x_0, \xi_0).
\]

Since \( q_0(x_0, \xi_0) \neq 0 \), this means that we will have \( Q_0(t, x, \xi) \neq 0 \) as soon as we have \( (x, \xi) = \chi_t(x_0, \xi_0) \), i.e., when \( (x, \xi) \) belongs to the integral curves of \( H_{b_i} \). This proves that \( WF(u_{t_i}) \subset \chi_t(WF(u_0)) \); the equality of the two sets is obtained by time reversal.

**Theorem 5.1** Let \( u_0 \in H^s \) and \( u \in M^s(I, H^s) \) be the solution to (5.29). Then \( WF(u(t, \cdot)) = \Phi_t(WF(u_0)) \) a.e. where the transformation \( \Phi_t(x_0, \xi_0) = (x_t, \xi_t) \) is given by

\[
\begin{align*}
(\partial_x dx(t) &= \frac{\partial a_1}{\partial \xi_i}(t, x(t), \xi(t)) \frac{\partial}{\partial x^i} \circ dw(t) + \frac{\partial b_1}{\partial \xi_i}(t, x(t), \xi(t)) \frac{\partial}{\partial x^i} dt, \\
\partial_\xi d\xi(t) &= -\frac{\partial a_1}{\partial x^i}(t, x(t), \xi(t)) \frac{\partial}{\partial \xi_i} \circ dw(t) - \frac{\partial b_1}{\partial x^i}(t, x(t), \xi(t)) \frac{\partial}{\partial \xi_i} dt
\end{align*}
\]

with \( x(0) = x_0, \xi(0) = \xi_0 \) and \( a_1, b_1 \) are the principal symbols of order 1 of \( a \) and \( b \).

**Proof.** To simplify the notations we suppose that \( b \equiv 0 \), as we are mainly interested in the proof related to the stochastic part. Let \( (x_0, \xi_0) \notin WF(u_0) \). Then there is a symbol \( q(x, \xi) \sim \sum_0^\infty q_j(x, \xi) \in S^0 \) with \( q_j(x_0, \xi_0) \neq 0 \) and \( q(x, D)u_0 \in C^\infty \).

To prove the theorem we shall construct a family of random symbols such that \( Q(t, x, \xi) \sim \sum_0^\infty Q_j(t, x, \xi) \in S^0 \) a.e. with \( Q(t, x, D)u_t \in C^\infty \) and \( Q_0(t, x_t, \xi_t) = Q_0(t, \Phi_t(x_0, \xi_0)) \neq 0 \).

Let \( p(t, x, \xi) \sim \sum_0^\infty p_j(t, x, \xi) \in S^0 \) be a random symbol in \( RS^0 \) which satisfies the conditions of Proposition 4.1; we shall seek \( Q \) of the form:

\[
Q(t, x, \xi) = q(x, \xi) + \int_0^t p(\tau, x, \xi) \circ dw(\tau).
\]

Then by Lemma 4.3 we have:

\[
Q(t)u(t) - Q(0)u_0 = \int_0^t p(\tau)u(\tau) \circ dw(\tau) + \int_0^t Q(\tau)a(\tau)u(\tau) \circ dw(\tau) \nonumber \\
= \int_0^t a(\tau)Q(\tau)u(\tau) \circ dw(\tau) \nonumber \\
+ \int_0^t (Q(\tau)p(\tau) + [Q, a](\tau))u(\tau) \circ dw(\tau).
\]

15
We shall choose \( p \) such that \( c(t, x, \xi) := p(t, x, \xi) + [Q, a](t, x, \xi) \in S^{-\infty} \) a.e. We have:

\[
[Q, a](t, x, \xi) \sim \sum_{|\alpha| \geq 1} \frac{i^\alpha}{|\alpha|!} (D^\alpha_\xi Q D_\xi^\alpha a - D^\alpha_\xi a D_\xi^\alpha Q) \tag{5.32}
\]

\[
\sim \{a_1, Q\} + \{a - a_1, Q\} + \sum_{|\alpha| \geq 2} \frac{i^\alpha}{|\alpha|!} (D^\alpha_\xi a D_\xi^\alpha Q - D^\alpha_\xi Q D_\xi^\alpha a)
\]

In this sum, the term of order 0 is \( \{Q, a_1\} = H_{a_1}Q_0(t, x, \xi) \), where \( H_{a_1} = (\partial a_1/\partial x)\partial/\partial \xi - (\partial a_1/\partial \xi)\partial/\partial x \) and the principal symbol of \( c(t, x, \xi) \) (order 0) is:

\[
c_0(t, x, \xi) = p_0(t, x, \xi) + H_{a_1}Q_0(t, x, \xi),
\]

The symbol of order \( -j \) of \( [Q, a](t, x, \xi) \) contains the terms \( D^\alpha_\xi Q D_\xi^\alpha a - D^\alpha_\xi a D_\xi^\alpha Q \) with \( |\alpha| \leq j + 1 \), and when taking the terms \( Q_j, a_j \) of the expansion of \( Q \) and \( a \), the terms that contribute to the order \( -j \) of \( [Q, a](t, x, \xi) \) should satisfy: \( j', j'' \leq j - 1 \). Hence the symbol of order \( -j, j \geq 1 \) of \( c(t, x, \xi) \) can be written as:

\[
c_j(t, x, \xi) = p_j(t, x, \xi) + H_{a_j}Q_j(t, x, \xi) + R_j(t, x, \xi),
\]

where the symbols \( R_j(t, x, \xi) \) are determined by \( Q_0, ..., Q_{j-1} \), this can be seen by expanding (5.32), see Raltson [35] p. 96 for explicit expressions of similar quantities.

Let us remark that, in order to have \( c(t, x, \xi) \in S^{-\infty} \) a.e., the condition \( c_j(t, x, \xi) = 0 \) for all \( t \) is equivalent to \( \int_0^t c_j(\tau, x, \xi) \circ dw(\tau) = 0 \) a.e. for all \( t \). Thus, we have to determine the \( (Q_j) \) which satisfy

\[
dQ_0(t, x, \xi) = -H_{a_1}Q_0(t, x, \xi) \circ dw(t), \tag{5.33}
\]

\[
dQ_j(t, x, \xi) = -H_{a_1}Q_j(t, x, \xi) \circ dw(t) + R_j(t, x, \xi) \circ dw(t). \tag{5.34}
\]

The stochastic characteristic equations associated to (5.33) are the Eqs. (C), which admit a global solution by Proposition 4.3. Hence, by Kunita [26], the equation (5.33) has a global solution which is given by \( Q_0(t, x, \xi) = q_0(\Phi_t^{-1}(x, \xi)) \). By the same arguments, (5.34) has a global solution given by:

\[
Q_j(t, x, \xi) = Q_j((\Phi_t^{-1}(x, \xi)) + \int_0^t R_j(\tau, \Phi_t \circ \Phi_t^{-1}(x, \xi)) \circ dw(\tau). \tag{5.35}
\]

Now let us check the properties of the random symbols \( Q_j(t, x, \xi) \) constructed by the above formula. First, observe that the random maps \( \Phi_t(x, \xi) \) and \( \Phi_t^{-1}(x, \xi) \) are homogeneous of degree 0 and 1 respectively in \( x \) and \( \xi \). On the other hand an inspection of the terms involved in the random symbols \( R_j(t, x, \xi) \) shows that these symbols are homogeneous of degree \( -j \) in \( \xi \). By corollary 4.4, the first terms of the r.h.s. of (5.35), \( Q_j((\Phi_t^{-1}(x, \xi)) \) form a set of random symbols in \( RS_h^{-j} \). As for the second term \( \int_0^t R_j(\tau, \Phi_t \circ \Phi_t^{-1}(x, \xi)) \circ dw(\tau), \) since \( R_j(t, x, \xi) \in RS_h^{-j} \) as we have just seen, we can apply Proposition 4.1; its conditions are indeed verified: for Condition (1) we have to verify that there is a constant \( \gamma > d \) such that:

\[
\forall \alpha, \beta \geq 0, \exists C_1 : E|D^\alpha_\xi D_\xi^\beta R_j(t, \Phi_t \circ \Phi_t^{-1}(x, \xi))|^\gamma \leq C_1 < +\infty \tag{5.36}
\]
To see this, we remark that (5.36) is verified with \( \gamma = d + 1 \) when \( R_j = Q_0 \); by Proposition 4.2 and Corollary 4.4, we see that this condition holds for \( R_1 \) which depends on \( Q_0 \), and the same argument can be applied by induction to show that (5.36) is verified for \( R_j \) which depends on \( Q_{j'}, j' = 0, ..., j - 1 \).

As for the condition (2) of Proposition 4.1, we use the Itô formula to expand \( R_j(t, \Phi_t \circ \Phi_t^{-1}(x, \xi)) \) and we note that its martingale part is of the form \( \int_0^t r_j(\tau, x, \xi) d\tau \) with \( r_j \in RS^{-j}_h \) and by the same arguments used for condition (5.36) we will have \( E|D_{\xi}^a D_{\eta}^b r_j(t, x, \xi)|^{d+1} < +\infty \).

Now since \( Q_j(t, x, \xi) \in RS^{-j}_h \), by Proposition 4.2 there exists a random symbol \( Q(t, x, \xi) \in RS_0^{\infty} \) such that \( Q(t, x, \xi) \sim \sum_{j=0}^{\infty} Q_j(t, x, \xi) \). With this choice of \( Q \) we have:

\[
Q(t)u(t) - Q(0)u_0 = \int_0^t a(\tau)Q(\tau)u(\tau) \circ d\tau + \int_0^t c(\tau)u(\tau) \circ dw(\tau),
\]

with \( c(\tau) \in S^{-\infty} \) for all \( \tau \), which implies that \( q(t, x, D)u(t) \in H^s \) a.e. for all \( s \) by Theorem 2.1 of [1]. To summarize, given \( (x_0, \xi_0) \notin WF(u_0) \) and \( q(x, \xi) \in S^\infty_{ph} \) with \( q_0(x_0, \xi_0) \not= 0 \) and \( q(x, D)u_0 \in C_0^\infty \), we have constructed a symbol \( q(t, x, \xi) \in RS^0_{ph} \) such that \( q(t, x, D)u(t) \in C_0^\infty \) and \( q_0(t, \Phi_t(x_0, \xi_0)) = q_0(x_0, \xi_0) \not= 0 \) which implies that \( \Phi_t(x_0, \xi_0) \notin WF(u(t, \cdot)) \) and \( WF(u(t, \cdot)) \subset \Phi_t(WF(u_0)) \). To prove the converse, let \( t \in [0, T] \) be fixed and let us denote by \( U_b(t, \phi, s) \in [0, t] \) the solution of the backward equation

\[
u(s) = \phi - \int_s^t a_r(x, D)u(\tau) \circ d\tau.
\]

By proposition 4.5 of [1] we have \( U_b(t, 0)u(t) = u_0 \) a.s. On the other hand, given a random symbol \( q(t) = q(t, x, \xi) \) one can construct a family of random symbols \( \tilde{q}(\tau), 0 \leq \tau \leq t \) such that

\[
\tilde{q}(t)\phi - \tilde{q}(s)u(s) - \int_s^t a(\tau)\tilde{q}(\tau)u(\tau) \circ d\tau = \int_s^t c(\tau)u(\tau) \circ d\tau,
\]

with \( c(\tau) \in S^{-\infty} \). This can be done as above by using backward equations. From this we conclude in the same way that for each (deterministic) \( \phi \) in some \( H^s \) we have \( WF(U(t, 0)\phi) \subset \Phi_{-t}(WF(\phi)) \) for almost all \( \omega \). Now let \( \omega \) be given and fix \( \phi = u(t, \omega) \) in some \( H^s \), then we have \( WF(u_0) = WF(U(t, 0, \omega)\phi) \subset \Phi_{-t}WF((u(t, \omega))) \) i.e. \( \Phi_t(WF(u_0)) \subset WF(u(t, \omega)) \). \( \square \)

**Remark:** The case of differential operators. We shall consider the case where \( a_t(x, D), b_t(x, D) \) are differential operators to obtain simply a ‘majorization’ of the wave front set of the solution to \( (\mathcal{E}) \) as in the deterministic case. For notational convenience we only consider the equation

\[
du(t) = \alpha(t, x)\frac{\partial u}{\partial x} \circ dw(t) + \beta(t, x)\frac{\partial u}{\partial x} dt, \quad u(0) = u_0,
\]

whose solution is \( u(t, x) = u_0(\phi_t^{-1}(x)) \) where \( \phi_t(x) \) is the stochastic flow associated to the eq. \( dX(t) = \alpha(t, X(t)) \circ dw(t) + \beta(t, X(t)) dt \).
Then for each $w, u(t, x)$ can be written as a Fourier integral distribution

$$u(t, x) = K_t u_0(x) := \langle K_t(x, \cdot), u_0(\cdot) \rangle,$$

with

$$K_t(x, y) = (2\pi)^{-n} \int e^{i\phi^{-1}(x) - y \cdot \xi} d\xi.$$

It is known that the singularities of $K_t u_0$ are given by

$$WF(K_t u_0) = WF'(K_t) \circ WF(u_0) := \{ (x, \xi) : \exists (y, \eta) \in WF(u_0) : (x, y, \xi, \eta) \in WF(K_t) \}.$$

On the other hand the wave front set of an oscillatory integral of the form $I(x) = \int a(x, \xi) \exp(i\phi(x, \xi))d\xi$ satisfies $WF(I) \subset \{ (y, \eta) : \partial\phi/\partial\xi(y, \eta) = 0, \eta = \partial\phi/\partial x(y, \eta) \}$. Hence:

$$WF(K_t) \subset \{ (x, \phi_t^{-1}(x), \partial_x \phi_t^{-1}(x), -\xi), (x, \xi) \in (\mathbb{R}^d)^2 \}.$$

From this, it follows easily that $WF(K_t u_0) \subset \{ (x(t), \xi(t)) = (\phi_t(x), \partial_x \phi_t^{-1}(x)\xi) : (x, \xi) \in WF(u_0) \}$. Now we can verify that the above curves $(x(t), \xi(t))$ are the solutions to the stochastic bicharacteristic equations of the theorem. Indeed we have:

$$dx(t) = d\phi_t(x) = dx(t) = -\partial a_1/\partial \xi(t, x(t), \xi(t)) \circ dw(t) - \partial b_1(t, x(t), \xi(t)) dt$$

since in this case $ia(t, x, \xi) = -i\xi \alpha(t, x), \quad ib(t, x, \xi) = -i\xi \beta(t, x)$. On the other hand, $Z_t := \partial_x \phi_t(x)^{-1}$ satisfies (see, e.g. [27], [20]):

$$dZ(t) = Z(t) \partial_x \alpha(t, x(t)) \circ dw(t) - Z(t) \partial_x \beta(t, x(t)) dt,$$

which implies that $\xi(t) = \partial_x \phi_t(x)^{-1}\eta$ verifies the bicharacteristic equation of the theorem.

### 6 Proofs of technical results

**Proof Proposition 4.2.**

Let $\chi \in C^\infty(\mathbb{R})$ such that: $\chi(t) = 1$ if $|t| \leq 1$ and $\chi(t) = 0$ if $|t| \geq 2$. We introduce

$$q_j(x, \xi, \omega) = (1 - \chi(\epsilon_j |\xi|))q_j(x, \xi, \omega),$$

with $\epsilon_j$ a decreasing sequence whose limit is 0, that will be determined later (it will be $\omega$-dependent), and we set:

$$q(x, \xi, \omega) = \sum_{j=0}^{\infty} q_{m-j}(x, \xi, \omega).$$

Then $q(\ldots, \omega) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ because for each $\xi$ the previous sum is finite, and the fact that $\tilde{q}_j - q_j = 0$ when $|\xi| > 1/2\epsilon_j$ implies that $\tilde{q}_j - q_j \in RS^{-\infty}$ and $\tilde{q}_j \in RS^{m-j}$ like $q_j$. To control the derivatives of $\tilde{q}$ we begin with those of $\tilde{q}_j$ and we suppose in the following that $|\xi| \geq 1$; we have:

$$|\partial_\xi^\alpha \partial_\omega^\beta q_j| \leq \sum_{\gamma \leq \alpha} \frac{\alpha!}{\gamma!(\alpha - \gamma)!} \epsilon_j^\gamma |\partial_\xi^{\alpha-\gamma} \partial_\omega^\beta q_j| \leq C(\alpha, \beta, j, \omega) |\xi|^{m-j-|\alpha|}.$$
To obtain the last inequality we use the following facts: \(|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} q_{j}| \leq C_{j}(\alpha, \beta, \omega) |\xi|^{m-j-|\alpha|+|\beta|}\) and for \(|\xi| \leq 2/\epsilon_{j}\) we have \(|\xi|^{\alpha-j} \leq 2^{\gamma}\) while for \(|\xi| > 2/\epsilon_{j}\) we have \(q_{j} = 0\).

On the other hand if \(\epsilon_{j}|\xi| < 1\) then \(q_{j} = 0\); hence we may suppose that \(\epsilon_{j}|\xi| \geq 1\) as far as we consider \(\tilde{q}_{j}\) and its derivatives and we have:

\[
|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} \tilde{q}_{j}| \leq C(\alpha, \beta, j, \omega) \epsilon_{j} |\xi|^{m-j-|\alpha|}
\]

and with the choice of an \(\epsilon_{j}(\omega) = \min\{1/C(\alpha, \beta, j, \omega), \text{ with } |\alpha + \beta| \leq j\}\) we have:

\[
|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} \tilde{q}_{j}| \leq |\xi|^{1-j} |\xi|^{\alpha-m} \text{ for } |\alpha + \beta| \leq j.
\]

Now for \(k \geq 0\), to prove that \(q - \sum_{j=0}^{k-1} q_{j} \in RS^{m-k}\) we set:

\[
q - \sum_{j=0}^{k-1} q_{j} = r_{k}^{(1)} + r_{k}^{(2)} + \sum_{j=0}^{k-1} \tilde{q}_{j}
\]

with \(r_{k}^{(1)} = \sum_{j=0}^{k-1} (\tilde{q}_{j} - q_{j})\), \(r_{k}^{(2)} = \sum_{j=k}^{N} \tilde{q}_{j}\), where \(N = \max(|\alpha + \beta|, k + 1)\). We have \(r_{k}^{(1)} \in RS^{-\infty}\) and \(r_{k}^{(2)} \in RS^{m-k}\) as a finite sum of terms in \(RS^{m-k}\). For the last term \(r_{k}^{(3)}\), since \(j \geq N \geq \max(|\alpha + \beta|, k + 1)\) in its sum, we have:

\[
|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} r_{k}^{(3)}| \leq \sum_{j=k+1}^{\infty} |\xi|^{\alpha-m} |\xi|^{1-j} \leq |\xi|^{\alpha-m-k} \sum_{j=0}^{\infty} |\xi|^{-j}
\]

Hence for \(|\xi| \leq 2\) we have \(|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} r_{k}^{(3)}| \leq |\xi|^{\alpha-m-k} \sum_{j=0}^{\infty} 2^{-j}\). This shows that \(\sum_{j=0}^{k-1} q_{j}(x, \xi, \omega) \in RS^{m-k}\) a.e. \(\Box\)

**Proof of Proposition 4.3.**

- Let \(X(t) = (x(t), \xi(t))\). If we write (C) in the form \(dX(t) = f(X(t)) \circ dw(t)\), then from the assumptions on \(a^{1}\), we deduce that \(f\) is locally Lipschitz and has at most a linear growth as \(x, \xi \to \infty\). By standard results on stochastic differential equations (cf., e.g., [20]), (C) has a global solution.

- First, we write \(x(t) := x(x(t), \xi(t)) = \xi(t, x(0), \xi(0))\) in the Itô form:

\[
dx(t) = D_{\xi}a(t)dw(t) + \frac{1}{2}[D_{x}D_{\xi}a(t)D_{\xi}a(t) - D_{\xi}^{2}a(t)D_{x}a(t)]dt
\]

\[
d\xi(t) = - D_{x}a(t)dw(t) - \frac{1}{2}[D_{x}^{2}a(t)D_{\xi}a(t) - D_{x}D_{\xi}a(t)D_{x}a(t)]dt
\]

By the Itô formula we have:

\[
|x(t)|^{n} = |x(0)|^{n} + \int_{0}^{t} n|x(\tau)|^{n-1}D_{\xi}a(\tau)dw(\tau) + \frac{1}{2} \int_{0}^{t} n(n-1)|x(\tau)|^{n-2}|x(\tau)|^{n-2}(D_{\xi}a(\tau))^{2}d\tau + \frac{1}{2} \int_{0}^{t} n|x(\tau)|^{n-1}[D_{x}D_{\xi}a(\tau)D_{\xi}a(\tau) - D_{\xi}^{2}a(\tau)D_{x}a(\tau)]d\tau.
\]

We have a similar formula for \(|\xi(t)|^{n}\). Let us set \(\tilde{x}(t) := E \sup_{\tau \leq t} |x(\tau)|^{2n}\) and \(\tilde{\xi}(t) := E \sup_{\tau \leq t} |\xi(\tau)|^{2n}\). Using the fact that \(|x(\tau)|^{n-1} and |x(\tau)|^{n-2}\) are \(\leq 1 + |x(\tau)|^{n}\) and the fact that \(|D_{\xi}a(\tau)|, |D_{x}D_{\xi}a(\tau)|\) are bounded and applying martingale and Hölder inequalities we get:

\[
\tilde{x}(t) \leq |x(0)|^{n} + C_{2} \int_{0}^{t} \tilde{x}(\tau)d\tau.
\]
which yields \( \dot{x}(t) \leq |x_0|^n e^{C_2 T} \) by the Gronwall lemma. Hence \( E \sup_{\tau \leq t} |x(\tau)|^{2n} \) is bounded. We use this result and the same previous argument to get the boundedness of \( E \sup_{\tau \leq t} |\xi(\tau)|^{2n} \).

- Now, we shall sketch the proof of the estimates concerning \( E|D_{\xi_0}^\alpha D_{x_0}^\beta x(t, x_0, \xi_0)| \) and \( E|D_{\xi_0}^\alpha D_{x_0}^\beta x(t, x_0, \xi_0)| \). This will be done by induction on \( \alpha, \beta \). For simplicity we consider only the quantities \( E|D_{\xi_0}^\alpha x(t, x_0, \xi_0)|, E|D_{\xi_0}^\beta x(t, x_0, \xi_0)| \) (the proof is similar if we take into account the derivatives w.r.t. \( x \)). We use the notation:

\[
a(t) = a(t, x(t), \xi(t))
\]

For \( \alpha = 1 \), we use Theorem 1.1 in Kunita [26] to get:

\[
D_{\xi_0} x(t) = D_{\xi_0} x(0) + \int_0^t [D_x D_{\xi} a(\tau) D_{\xi_0} x(\tau) + D_{\xi}^2 a(\tau) D_{\xi_0} \xi(\tau)] dw(\tau)
\]

\[
+ \frac{1}{2} \int_0^t [D_x D_{\xi_0}^2 a(\tau) D_{\xi} a(\tau) + (D_x D_{\xi_0} a(\tau))^2]
- D_x^2 D_x a(\tau) D_x a(\tau) - D_{\xi_0}^2 a(\tau) D_x^2 a(\tau) D_{\xi_0} x(\tau) d\tau
+ \frac{1}{2} \int_0^t [D_x^2 D_x a(\tau) D_{\xi_0} a(\tau) + D_x D_{\xi_0} a(\tau) D_x^2 a(\tau)]
- D_{\xi_0}^2 a(\tau) D_x a(\tau) - D_x^2 a(\tau) D_x D_x a(\tau) D_{\xi_0} \xi(\tau) d\tau,
\]

\[
D_{\xi_0} \xi(t) = D_{\xi_0} \xi(0) - \int_0^t [D_x^2 a(\tau) D_{\xi_0} x(\tau) + D_x D_{\xi_0} a(\tau) D_{\xi_0} \xi(\tau)] dw(\tau)
- \frac{1}{2} \int_0^t [D_x D_{\xi_0}^2 a(\tau) D_x a(\tau) + (D_x D_{\xi_0} a(\tau))^2]
- D_x D_x a(\tau) D_x a(\tau) - D_{\xi_0} D_x a(\tau) D_x^2 a(\tau) D_{\xi_0} x(\tau) d\tau
- \frac{1}{2} \int_0^t [D_x^2 D_x a(\tau) D_{\xi_0} a(\tau) D_x^2 a(\tau) D_{\xi_0} a(\tau)]
- D_{\xi_0} D_x a(\tau) D_x a(\tau) - (D_x D_{\xi_0} a(\tau))^2] D_{\xi_0} \xi(\tau) d\tau.
\]

Let \( n \geq 1 \). It is well known that \( E|D_{\xi_0} x(0)|^n \leq K_1, E|D_{\xi_0} \xi(0)|^n \leq K_2 \), for some constants \( K_1, K_2 \), see Kunita[26].

We set: \( \phi(t) := E \sup_{\tau \leq t} |D_{\xi} x(\tau)|^n \) and \( \psi(t) := E \sup_{\tau \leq t} |D_{\xi} \xi(\tau)|^n \). Using the estimates of \( D_{\xi}^\alpha D_x^\beta a(t) \), martingale and Hölder inequalities (we could also use the Itô formula for the expressions of \( |D_{\xi} x(t)|^n, |D_{\xi} \xi(t)|^n \), it follows from the above equations that:

\[
\phi(t) \leq K_n + C_n \int_0^t (\phi(\tau) + (1 + |\xi|)^{-n} \psi(\tau)) d\tau,
\]

\[
\psi(t) \leq K'_n + C_n \int_0^t (\psi(\tau) + \phi(\tau)) d\tau,
\]

By the Gronwall lemma used in (6.38) we get: \( \psi(t) \leq K'_n T \phi(t) e^{C_n T} =: C'_n \phi(t) \). We substitute this in (6.37) and use the Gronwall Lemma again for this inequality, we
Now, we shall show how to get the estimates for \( \alpha \) and:

Next, let \( \varphi \) be the estimates of the proposition, where (\( \bar{C} \)). Then we can write:

\[
D_{\xi}^{\alpha+1} x(t) = D_{\xi_0}^{\alpha+1} x(0) + \int_{0}^{t} [D_{\xi} D_{\xi} a(\tau) D_{\xi}^{\alpha+1} x(\tau) + D_{\xi}^{2} D(\tau) D_{\xi}^{\alpha+1} \xi(\tau)]d\tau
\]

\[
+ \frac{1}{2} \int_{0}^{t} [D_{\xi} D_{\xi}^{2} a(\tau) D_{\xi} a(\tau) + (D_{\xi} D_{\xi}^{2} a(\tau))]^2 d\tau
\]

\[
- D_{\xi}^{2} D_{\xi} a(\tau) D_{\xi} a(\tau) - D_{\xi}^{2} D_{\xi}^{2} a(\tau)] D_{\xi}^{\alpha+1} x(\tau) d\tau
\]

\[
+ \frac{1}{2} \int_{0}^{t} [D_{\xi} D_{\xi} a(\tau) D_{\xi} a(\tau) + D_{\xi} D_{\xi} a(\tau) D_{\xi}^{2} a(\tau)] D_{\xi}^{\alpha+1} \xi(\tau) d\tau
\]

\[
+ \int_{0}^{t} k_1(\tau) d\tau(\tau) + \int_{0}^{t} k_2(\tau) d\tau,
\]

and:

\[
D_{\xi}^{\alpha+1} \xi(t) = D_{\xi_0}^{\alpha+1} \xi(0) - \int_{0}^{t} [D_{\xi}^{2} a(\tau) D_{\xi}^{\alpha+1} x(\tau) + D_{\xi} D_{\xi} a(\tau) D_{\xi}^{\alpha+1} \xi(\tau)]d\tau
\]

\[
- \frac{1}{2} \int_{0}^{t} [D_{\xi} D_{\xi}^{2} a(\tau) D_{\xi} a(\tau) + D_{\xi}^{2} a(\tau) D_{\xi} D_{\xi} a(\tau)] D_{\xi}^{\alpha+1} x(\tau) d\tau
\]

\[
- D_{\xi} D_{\xi}^{2} a(\tau) D_{\xi} a(\tau) - D_{\xi} D_{\xi} a(\tau) D_{\xi}^{2} a(\tau)] D_{\xi}^{\alpha+1} \xi(\tau) d\tau
\]

\[
- \frac{1}{2} \int_{0}^{t} [D_{\xi} D_{\xi}^{2} a(\tau) D_{\xi} a(\tau) + D_{\xi}^{2} a(\tau) D_{\xi}^{2} a(\tau)] D_{\xi}^{\alpha+1} \xi(\tau) d\tau
\]

\[
+ \int_{0}^{t} k_1(\tau) d\tau(\tau) + \int_{0}^{t} k_2(\tau) d\tau,
\]

where \( h_i, k_i \) satisfies \( E|h_i(\tau)|^n \leq C_n, E|h_i(\tau)|^n \leq C'_n \). Using the same arguments as for \( \alpha = 1 \) we get the boundedness of \( E|D_{\xi}^{\alpha+1} x(t)|^n \) and \( E|D_{\xi}^{\alpha} \xi(t)|^n \).

Now, we shall show how to get the estimates for \( E|D_{\xi}^{\alpha} \bar{x}(t)|, E|D_{\xi}^{\alpha} \bar{\xi}(t)| \). Let us rewrite Eq. (C) as \( dX(t) = f(t, X(t)) \circ dw(t) \) and consider the associated flow of diffeomorphisms \( \phi_{s,t}(Y) \) which satisfies:

\[
\phi_{s,t}(Y) = Y - \int_{s}^{t} f(\tau, \phi_{s,\tau}^{-1}(Y)) \circ dw(\tau).
\]

Then, by Kunita ([27], Theorem 7.3), the inverse map \( \phi_{s,t}^{-1} \) satisfies the backward Stratonovich equation

\[
\phi_{s,t}^{-1}(Y) = Y + \int_{s}^{t} f(r, \phi_{r,t}(Y)) \circ dw(r).
\]

Hence, by the same arguments as above, one can prove that \( E|D_{\xi}^{\alpha} \bar{x}_{s,t}|, E|D_{\xi}^{\alpha} \bar{\xi}_{s,t}| \) satisfy the estimates of the proposition, where \( (\bar{x}_{s,t}, \bar{\xi}_{s,t}) := \phi_{s,t}^{-1} \) for each \( 0 \leq s \leq t \leq T \), which
completes the proof because $(\bar{x}(t), \bar{\xi}(t)) = \phi_{0,1}^{-1}$.

• If $a(t, x, \xi)$ is homogeneous of degree 1 in $\xi$, then the fact that the solution $\phi_t(x_0, \xi_0)$ to $C$ is homogeneous of degree 1 in $\xi_0$ can be seen as follows: $\xi(t)$ verifies the equation:

$$d\xi(t) = -\xi(t) \frac{\partial a}{\partial x}(t, x(t), 1) \circ dw(t)$$

whose solution is of the form $\xi(t) = \xi(0) \exp[\int_0^t -\partial_x a(\tau, x(\tau), 1) \circ dw(\tau)]. \quad \square$

**Proof of Corollary 4.4.**

First, we note that the flow $\phi_t(x, \xi)$) is homogeneous of degree 0 and 1 in $x$ and $\xi$ respectively. Next we shall prove that given a compact set $K \subset \mathbb{R}^d$ there exist constants $C(\alpha, \beta, K, w)$ such that: $\forall x \in K, \xi \in \mathbb{R}^d : |D_\xi^\alpha D_x^\beta q(x, \xi, w)| \leq C(\alpha, \beta, K, w)(1 + |\xi|)^{m-|\alpha|} \ a.e.$ We shall apply Lemma 4.2 to $f(\xi, x) = D_\xi^\alpha D_x^\beta q(x, \xi, w)$. As $q(x, \xi, w)$ is homogeneous of degree $m$ in $\xi$ we have that $f(\xi, x)$ is homogeneous of degree $m-\alpha$ in $\xi$. Now we have to show that there is a constant $\gamma > 0$ such that $E[|\partial_\xi f(\xi, x)|^\gamma, j = 1, \cdot \cdot \cdot, d$ are bounded. We take $\gamma = d + 1$. Given the expression of $f$ we see that the derivatives $\partial_\xi f(\xi, x)$ contain terms of the form:

$$S_{\alpha', \beta'} \times T_{\alpha'', \beta''} := \frac{\partial_\xi^\alpha \partial_x^\beta p}{\partial_\xi^\alpha \partial_x^\beta}(\phi(\xi, x)) \times \frac{\partial_\xi^\gamma \partial_x^\beta \phi}{\partial_\xi^\alpha \partial_x^\beta}(x, \xi)$$

To estimate $E[|\partial_\xi f(\xi, x)|^\gamma$, by using Schwarz’s inequality we will have to show that the quantities: $E|S_{\alpha', \beta'}|^n$ and $E|T_{\alpha'', \beta''}|^n$ are bounded with $n = 2\gamma$ when $|\xi| = 1$ and $x \in K$. For the terms $S$, we note that since $p \in S_h^\gamma$, we have $|\partial_\xi^\alpha \partial_x^\beta p(x', \xi')| \leq C(\alpha', \beta')(1 + |\xi|)^{m-|\alpha'}| \alpha, \beta', K| = 1$ and therefore: $S_{\alpha', \beta'} \leq C(\alpha', \beta')(1 + |\xi|)^{m-|\alpha'}|$ where $(x_t, \xi_t) = \phi_t(x, \xi))$. According to Proposition 4.3 we have: $E|\xi_t(x, \xi)|^n \leq C|\xi|^n$, hence $E|S_{\alpha', \beta'}|^n \leq C_2(\alpha', \beta')(1 + |\xi|)^{n(1-|\alpha'|)}$ for some constant $C_2$, which implies that:

$$E|S_{\alpha', \beta'}|^n \leq C_3 := C_2(\alpha', \beta')2^n(m-|\alpha'|) \quad \forall \xi : |\xi| = 1$$

As for the terms $T$, we apply Proposition 4.3 and there is a constant $C_n'$ such that:

$$E|\partial_\xi^\alpha \partial_x^\beta \phi(x, \xi)|^n \leq C_n'(\alpha'', \beta')2^n(m-|\alpha|),$$

which implies that $E|T_{\alpha'', \beta'}|^n \leq C_n''(\alpha'', \beta'')2^n(m-|\alpha''|)$ for $|\xi| = 1$. We can therefore apply Lemma 4.2 to $f$ with $\gamma = d + 1$ to deduce that $q_t \in RS_h^\gamma$. \quad \square
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