Simulation of motion of satellites after fixing the values of their accelerations

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Abstract. At the Department of Theoretical and Applied Mechanics of the Faculty of Mathematics and Mechanics of St. Petersburg State University [1] the theory of motion of nonholonomic systems with linear nonholonomic constraints of high order n>2 was created. The high-order constraints are considered as program and ideal ones, and their reaction force is considered as the required control force. A consistent system of differential equations with respect to unknown generalized coordinates and Lagrange multipliers is constructed to solve the problem. The report examines the motion of Soviet satellites of the systems “Cosmos”, “Molniya”, “Tundra” after fixing the values of their accelerations in apogees. This corresponds to imposing the nonlinear second-order nonholonomic constraints on the further motion of satellites [2, 3]. The equations of constraints are differentiated in time and presented as linear third-order constraints to make it possible to apply the above theory. The motions of satellites are studied in polar coordinates, the origin of this system coinciding with the center of the Earth. It turns out that after fixing the acceleration values in the apogees, the satellites begin to rotate between two concentric circles, alternately touching each of them.

1. Introduction
In the report we consider some problem of cosmonautics, when at some point of the orbit of an artificial Earth's satellite, its acceleration is fixed. Such problem is reduced to imposing a higher order constraint on the motion of the satellite.

There are just few examples of higher order constraints, one of them is an example by G. Hamel presented in his book [4] in 1949. G. Hamel considered the equation of a constraint arising while assuming that the vertical component of the acceleration of a point was equal to the product of horizontal components, that is, the following constraint was written:

$$\ddot{Z} = \dot{X} \cdot \dot{Y}$$

Despite the elegance of this equation, it has no physical meaning and at that, such constraint can be realized by means of some control system. A first example of a real higher order constraint seems to be one proposed in the papers by Sh.Kh. Soltakhanov and M.P. Yushkov [2, 3] published in journal «Vestnik Leningradskogo Universiteta» in 1990 and 1991. In these works, the authors considered the motion of an artificial Earth's satellite supposing that after some moment its acceleration is fixed. As a result, a nonlinear nonholonomic second order constraint begins to act on this satellite.

2. Formulation of problem
We proceed to the formulation of the problem. We consider the motion of an artificial Earth's satellite in polar coordinates r, ϕ. The origin is assumed to be at the Earth's center. As the satellite moves
around the Earth along an ellipse, the projections of its acceleration on the coordinate curves vary continuously according the following laws:

\[ pr_r w = \ddot{r} - r \dot{\varphi}^2, \quad pr_\varphi w = r \dddot{\varphi} + \dddot{r}. \]

We need to determine the motion of the satellite if after some moment of time \( t_0 = 0 \), its acceleration is fixed and remains constant all the time. We denote \( w(0) = w_0 \), then the above condition is equivalent to imposing the following nonlinear second order nonholonomic constraint on the motion of the satellite at the time \( t_0 \):

\[ f_2 = (\ddot{r} - r \dot{\varphi}^2)^2 + (r \dddot{\varphi} + 2i \dddot{r})^2 - w_0^2 = 0. \]

In monographs [1, 5], there were proposed two theory of motions of systems under linear nonholonomic constraints of higher order \( n \geq 3 \). We employ the first of them. To be able to apply this theory, the imposed constraint is to be linear. In order to have such property, we differentiate equation (2) in time:

\[ f_3 = (\ddot{r} - r \dot{\varphi}^2)(\dddot{r} - r \dddot{\varphi}^2 - 2i \dddot{r}) + (r \dddot{\varphi} + 2i \dddot{r})(i \dddot{\varphi} + r \dddot{\varphi} + 2i \dddot{r}) = 0. \]

This is a linear nonholonomic constraint of third order.

We rewrite (3) as follows:

\[ f_3 (q, \dot{q}, \ddot{q}, \dddot{q}) = a_{3,0} r + a_{3,0} + a_{3,0} = 0. \]

The coefficients in equation (4) depend on generalized coordinates, velocities and accelerations of these coordinates and are of the following form:

\[ a_{3,0} = \dddot{r} - r \ddot{\varphi}^2, \]

\[ a_{3,0} = (r \dddot{\varphi} + 2i \dddot{r})^2, \]

\[ a_{3,0} = (i \dddot{\varphi} + 2i \dddot{r})(i \dddot{\varphi} + 2i \dddot{r} - (\dddot{r} - r \dddot{\varphi}^2)(\dddot{r} \dddot{\varphi}^2 + 2r \dddot{\varphi} \dddot{r}). \]

3. Application of first theory of motion of systems with higher order constraints

In the first theory of motion of nonholonomic systems of higher order, one constructs a system of differential equations for generalized coordinates and the Lagrange multipliers by employing the equations of the constraints and the Newton equation for a non-free motion under higher order constraints.

We consider such construction for the studied problem. Constraint (4) can be regarded as ideal and this implies that a force of constraint reaction has no component orthogonal to the vectors in the dual basis \( \mathbf{e}^r, \mathbf{e}^\varphi \) for the polar coordinates. Then under the action of the central gravitation force of the Earth

\[ F = -\frac{\mu m}{r^2} \mathbf{e}^r, \]

where \( m \) is the mass of the satellite, \( \mu \) is the Gauss constant for the gravitation field of the Earth, the Newton equation of AES under the presence of constraint (4) reads as

\[ m \ddot{w} = F + \Lambda \nabla' f_3 = F + \Lambda \left( \frac{\partial f_3}{\partial r} \mathbf{e}^r + \frac{\partial f_3}{\partial \varphi} \mathbf{e}^\varphi \right). \]

Here \( \nabla' \) stands for the Polyakhojv's operator or the generalized Hamilton operator [1, 5]:
\[ \nabla'' a = \sum_{\sigma=1}^{n} \frac{\partial a}{\partial q_{\sigma}} e_{\sigma}. \] (7)

We calculate the scalar product of (6) by the vector of the main bases \( e_{r}, e_{\phi} \) and we obtain two differential equations:

\[ m(\ddot{r} - r \dot{\phi}^2) = -\frac{\mu m}{r^2} + A(\ddot{r} - r \dot{\phi}^2), \quad m(\ddot{r} \dot{\phi} + 2\dot{r} \ddot{\phi}) = A(r \ddot{\phi} + 2\dot{r} \ddot{\phi}). \] (8)

It is convenient to introduce \( A_s \) depending linearly on \( A \) by the formula

\[ A_s = \frac{A}{m} - 1. \] (9)

Then equations (8) become

\[ A_s (\ddot{r} - r \dot{\phi}^2) = \frac{\mu}{r^3}, \quad A_s (r \ddot{\phi} + 2\dot{r} \ddot{\phi}) = 0. \] (10)

We resolve equations (10) with respect to the second derivatives and we arrive at the following system of differential equations

\[ \ddot{r} = r \dot{\phi}^2 + \frac{\mu}{A_s r^2}, \quad \ddot{\phi} = -\frac{2\dot{r} \ddot{\phi}}{r}. \] (11)

Let us show how we can obtain a differential for \( A_s \) in addition to system (11). We differentiate system (10) with respect to the time in order to introduce the third derivatives of generalized coordinates involved in constraint equation (4):

\[ \dot{A}_s (\ddot{r} - r \dot{\phi}^2) + A_s (r \ddot{\phi} + 2\dot{r} \ddot{\phi}) + 2 \frac{\mu}{r^3} \dot{r} = 0, \] (12)

\[ \dot{A}_s (r \ddot{\phi} + 2\dot{r} \ddot{\phi}) + A_s (3r \ddot{\phi} + r \dddot{\phi} + 2\dot{r} \ddot{\phi}) + 2 \frac{\mu}{r^3} \dot{r} = 0. \] (13)

We multiply (12) by \( a_{3r} \) and (13) by \( a_{3\phi} \) and we sum up the results. After selecting the constraint \( f_3 = 0 \), we obtain the following equation:

\[ \dot{A}_s = \frac{2a_{3r}(r \dddot{\phi} - \dddot{r})}{r^2((\ddot{r} - r \dot{\phi}^2)^2 + (r \dddot{\phi} + 2\dddot{r})^2)}. \] (14)

Employing the first equation in system (10) and constraint equation (2), we obtain the sought differential equation for \( A_s \):

\[ \dot{A}_s = \frac{2a_{3r} \dddot{r}}{w_{3} A_s r^5}. \] (15)

Integrating system of differential equations (11) and (15), we can find the needed motion in the polar coordinates and the Lagrange multiplier forming the controlling force.

4. **Study of motion of satellites**

In order to find the initial data for system of differential equations (11) and (15), we use the known formulae considered in details, for instance, in textbook [6]:
We study the motion of the artificial Earth's satellite after fixing its acceleration at the apogee of its orbit.

We consider the satellite of system «Kosmos» with a small eccentricity of the orbit. Hereinafter we employ numerical data on the satellites' motion taken from free sources in Internet.

For the given satellite with the heights \(H_\pi = 183\) km at the perigee and \(H_\alpha = 244\) km at the apogee over the Earth's surface of the radius \(R = 6371\) km, and under the free fall acceleration \(g_0 = 9.8210^{-3}\) km/sec\(^2\), we have the following parameters:

\[
    r_\pi = 6554\ km, \quad r_\alpha = 6615\ km, \quad \mu = g_0 R^2 = \frac{398590 \ km^3}{\sec^2},
\]

\[
    e = \frac{r_\alpha - r_\pi}{r_\alpha + r_\pi} = 0.004632, \quad p = r_\pi (1 + e) = 6584.36\ km,
\]

where \(e\) is the eccentricity of the orbit and \(p\) is the focal parameter.

To have the satellite at the apogee at the time \(t_0 = 0\), the initial data should read as follows:

\[
    r(0) = 6615, \quad \varphi(0) = \pi, \quad \dot{r}(0) = 0, \quad \dot{\varphi}(0) = 0.001171, \quad \ddot{r}(0) = -0.000038, \quad \ddot{\varphi}(0) = 0,
\]

\[
    \Lambda_s = -1.
\]

By integrating system (11), (15) with initial date (18), the trajectory of the satellite motion and the graph of the controlling force are of the form presented in figure 1.

![Figure 1](image)

We observe that under the constraint, the trajectory of the motion of the satellite is located between two concentric circumferences centered at the origin. To guide the eye, in figure 2 we provide some parts of the motion of the satellite in third quadrant; the solid curve shows the trajectory of the satellite, while dashed lines show circumferences bounding the trajectory.
Let us consider the motion of the satellites with a large eccentricity of the orbit. For the satellites of system «Molnia» moving along a high elliptic orbit, we have the following parameters and initial data:

\[ r_\pi = 6871 \text{ km}, \quad r_a = 46371 \text{ km}, \quad \mu = \frac{\mu_0 R^2}{\varepsilon e c^2} = \frac{398590 \text{ km}^3}{\varepsilon e c^2}, \]

\[ e = \frac{r_a - r_\pi}{r_a + r_\pi} = 0.741895, \quad p = r_\pi (1 + e) = 11968.6 \text{ km}, \]

\[ r(0) = 46871, \quad \varphi(0) = \pi, \quad \dot{r}(0) = 0, \quad \dot{\varphi} = 0.000032, \quad \ddot{r}(0) = -0.0001375, \quad \ddot{\varphi}(0) = 0. \]  

The results of numerical integration of the system with initial data (20) are presented in figure 3.
Without constraints, the satellite moved along an elliptic orbit (curve with short strokes). Then at the apogee a controlled force started acting on it, and after this the satellite goes along a new trajectory indicated by the solid curve. As we see in the graph, now the trajectory of the motion of the satellite is bounded by two concentric circumferences centered as the origin (curves with long strokes) touching in turn the external and internal circumference. We note that this motion is not periodic in the sense that the satellite does not return back to the initial position in an integer number of rotations.

The graph of the dependence of the controlling force on the time is presented in figure 4.
At the same time, $A$ is a periodic function in time. The extremal points of this function correspond to the touching of the trajectory with the circumference. At that, $A = 0$ for touching with the external circumference and is maximal by the absolute value for touching with the internal circumference.

For the satellite «Tundra» we have the parameters:

$$
r_\pi = 26371 \text{ km}, \quad r_\alpha = 56371 \text{ km}, \quad \mu = g_0 R^2 = \frac{398590 \text{ km}^3}{\sec^2},
$$

$$
e = \frac{r_\alpha - r_\pi}{r_\alpha + r_\pi} = 0.362573, \quad p = r_\pi(1 + e) = 35932.4 \text{ km},
$$

$$
r(0) = 56371, \quad \varphi(0) = \pi, \quad \dot{r}(0) = 0, \quad \dot{\varphi} = 0.000038, \quad \ddot{r}(0) = -0.0000455, \quad \ddot{\varphi}(0) = 0.
$$

By the numerical integration we obtain the following graphs.
5. Conclusion
In the paper we consider the motion of a series of satellites used in the last century to study their motion after fixing their accelerations. Such problem is equivalent to imposing nonlinear second order nonholonomic constraints on the motion of the satellites. By the differentiation in time, these constraints are transformed to linear nonholonomic constraints of third order. We construct a compatible system of differential equations for the polar coordinates and the Lagrange multiplier. Numerical integration of this system for three different satellites show that once their accelerations at the apogee are fixed, the satellites start moving between two concentric orbits touching them in turn.

References
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