AN ORCHARD THEOREM

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Abstract. We describe a natural way to plant cherry- and plumtrees at prescribed generic locations in an orchard.

1. Main results

The main result of this paper may be paraphrased comprehensively as follows: Most people would agree that a natural way to plant trees of two species along a row is to alternate them. Our main result (the Orchard Theorem) generalises this to higher dimensions. In dimension 2 it implies that there is a natural way to plant cherrytrees and plumtrees in an orchard if the prescribed locations of the trees are generic (no alignments of three trees). Figure 1 shows such an orchard planted with 3 cherry- and 6 plumtrees (they play of course symmetric roles).

Figure 1: An orchard having 3 cherry- and 6 plumtrees in generic positions

A finite set \( \mathcal{P} = \{P_1, \ldots, P_n\} \) of \( n \) points in the oriented real affine space \( \mathbb{R}^d \) is a generic configuration if any subset of \( k + 1 \leq d + 1 \) points is affinely independent. Generic configurations of \( n \leq d + 1 \) points in \( \mathbb{R}^d \) are simply vertices of an \((n - 1)\)-dimensional simplex. For \( n \geq d + 1 \), genericity boils down to the fact that any set of \( d + 1 \) points in \( \mathcal{P} \) spans \( \mathbb{R}^d \) affinely.

Two generic configurations \( \mathcal{P}^1 \) and \( \mathcal{P}^2 \) are isomorphic if there exists a bijection \( \varphi : \mathcal{P}^1 \rightarrow \mathcal{P}^2 \) such that all corresponding \( d \)-dimensional simplices

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(with vertices \((P_0, \ldots, P_d) \subset \mathcal{P}^1\) and \((\varphi(P_0), \ldots, \varphi(P_d)) \subset \mathcal{P}^2)\) have the same orientation (given for instance for the first simplex by the sign of the determinant of the \(d \times d\) matrix with rows \(P_1 - P_0, \ldots, P_d - P_0\)).

Two generic configurations \(\mathcal{P}(-1)\) and \(\mathcal{P}(+1)\) are isotopic if there exists a continuous path (with respect to the obvious topology on \(\mathbb{C}^d\)) of generic configurations \(t \mapsto \mathcal{P}(t)\) which joins them. Isotopic configurations are of course isomorphic. I ignore to what extent the converse holds.

An affine hyperplane \(H \subset \mathbb{R}^d\) separates two points \(P, Q \in \mathbb{R}^d \setminus H\) if \(P, Q\) are not in the same connected component of \(\mathbb{R}^d \setminus H\). For two points \(P, Q\) of a generic configuration \(\mathcal{P} = \{P_1, \ldots, P_n\} \subset \mathbb{R}^d\) we denote by \(n(P, Q)\) the number of distinct hyperplanes separating \(P\) and \(Q\) which are affinely spanned by \(d\) distinct elements in \(\mathcal{P} \setminus \{P, Q\}\). The number \(n(P, Q)\) depends obviously only of the isomorphism type of \(\mathcal{P}\).

**Theorem 1.1 (Orchard Theorem).** The relation defined by \(P \sim Q\) if either \(P = Q\) or if

\[
n(P, Q) \equiv \binom{n - 3}{d - 1} \pmod{2}
\]

is an equivalence relation having at most 2 classes on a generic configuration \(\mathcal{P}\) of \(n\) points in \(\mathbb{R}^d\).

We call the equivalence relation of Theorem 1.1 the **Orchard relation** and the induced partition on \(\mathcal{P}\) the **Orchard partition**.

**Example 1.2.** Consider a configuration \(\mathcal{P} \subset \mathbb{S}^2 \subset \mathbb{R}^3\) consisting of \(n\) points contained in the Euclidean unit sphere \(\mathbb{S}^2\) and which are generic as a a subset of \(\mathbb{R}^3\) in the above sense, i.e. 4 distinct points of \(\mathcal{P}\) are never contained in a common affine plane of \(\mathbb{R}^3\). A stereographic projection \(\pi : S^2 \setminus \{N\} \rightarrow \mathbb{R}^2\) with respect to a point \(N \in S^2 \setminus \mathcal{P}\) sends the set \(\mathcal{P} \subset S^2\) into a set \(\mathcal{P}' = \pi(\mathcal{P}) \subset \mathbb{R}^2\) such that 4 points of \(\mathcal{P}'\) are never contained in a common Euclidean circle or line of \(\mathbb{R}^2\). The Orchard relation on \(\mathcal{P}\) can now be seen on \(\mathcal{P}'\) as follows: Given two distinct points \(\tilde{P} \neq \tilde{Q} \in \mathcal{P}'\) count the number \(n(\tilde{P}, \tilde{Q})\) of circles or lines determined by 3 points in \(\mathcal{P}' \setminus \{\tilde{P}, \tilde{Q}\}\) which separate them. The points \(P, Q \in \mathcal{P}\) are now Orchard-equivalent if and only if \(n(\tilde{P}, \tilde{Q}) \equiv \binom{n - 3}{2} \pmod{2}\).

A consequence of the Orchard Theorem is the fact that points of a generic configuration \(\mathcal{P} \subset \mathbb{R}^d\) carry a structure which can be encoded by a rooted binary tree \(\mathcal{T}\): Vertices of \(\mathcal{T}\) are suitable subsets of \(\mathcal{P}\) and define hence generic subconfigurations of \(\mathcal{P}\). The root corresponds to the complete set \(\mathcal{P}\). The remaining vertices of \(\mathcal{T}\) are defined recursively as follows: The two sons (if they exist) of a vertex \(V_{\tilde{P}} \in \mathcal{T}\) corresponding to a subset \(\tilde{P} \subset \mathcal{P}\) are the two non-empty Orchard classes (equivalence classes) of the generic configuration \(\tilde{P}\).

It would of course be interesting to understand the leaves (or atoms) of such trees. They correspond to configurations consisting only of equivalent points. Generic configurations of \(\mathbb{R}^d\) having at most \(d + 1\) elements are of course such leaves but there are many others (e.g. vertices of a convex plane polygon having an odd number of vertices).

A flip is a continuous path

\[
t \mapsto \mathcal{P}(t) = (P_1(t), \ldots, P_n(t)) \in (\mathbb{R}^d)^n, \ t \in [-1, 1]
\]
with $\mathcal{P}(t) = \{P_1(t), \ldots, P_n(t)\}$ generic except for $t = 0$ where there exists exactly one set $\mathcal{F}(0) = (P_{i_0}(0), \ldots, P_{i_d}(0))$, called the flipset, of $(d + 1)$ points contained in an affine hyperplane spanned by any subset of $d$ points in $\mathcal{F}(0)$. We require moreover that the simplices $(P_{i_0}(-1), \ldots, P_{i_d}(-1))$ and $(P_{i_0}(1), \ldots, P_{i_d}(1))$ carry opposite orientations. Geometrically this means that a point $P_{ij}(t)$ crosses the hyperplane spanned by $\mathcal{F}(t) \setminus \{P_{ij}(t)\}$ at time $t = 0$.

It is easy to see that two generic configurations $\mathcal{P}^1, \mathcal{P}^2 \subset \mathbb{R}^d$ having $n$ points can be related by a continuous path involving at most a finite number of flips.

The next result shows that flips modify the Orchard relation only locally.

**Proposition 1.3 (Flip Proposition).** Let $\mathcal{P}(-1), \mathcal{P}(+1) \subset \mathbb{R}^d$ be two generic configurations related by a flip with respect to a subset $\mathcal{F}(t)$ of $(d + 1)$ points.

- (i) If two distinct points $P(t), Q(t)$ are either both contained in $\mathcal{F}(t)$ or both contained in its complement $\mathcal{P}(t) \setminus \mathcal{F}(t)$ then we have
  \[ P(-1) \sim Q(-1) \text{ if and only if } P(+1) \sim Q(+1). \]

- (ii) For $P(t) \in \mathcal{F}(t)$ and $Q(t) \notin \mathcal{F}(t)$ we have
  \[ P(-1) \sim Q(-1) \text{ if and only if } P(+1) \not\sim Q(+1). \]

![Figure 2](image.png)

**Figure 2:** Two orchards with 6 trees related by a flip

The Flip Proposition bears bad news for a possible natural generalisation of the Orchard relation to non-generic configurations. The two equivalence classes play indeed a totally symmetric role with respect to the flipset and there seems no natural way to break this symmetry for the non-generic configuration $\mathcal{P}(0)$ involved in a flip. It is however possible to define the Orchard relation on *generic points* of arbitrary configurations: Call a point $P \in \mathcal{P} \subset \mathbb{R}^d$ of a subset of points *generic* if the affine span of $\{P, P_{i_1}, \ldots, P_{i_k}\}$ is $k$-dimensional (for $k \leq d$) for all subsets $\{P_{i_1}, \ldots, P_{i_k}\} \subset \mathcal{P} \setminus \{P\}$. A small generic perturbation $\bar{\mathcal{P}}$ of $\mathcal{P}$ allows then to compute the orchard relation on the set of generic points of $\mathcal{P}$ by considering the restriction of the Orchard relation on $\bar{\mathcal{P}}$ to the image of generic points.
However, the Flip Proposition suggests also perhaps interesting problems concerning generic configurations: Call two generic configurations of \( n \) points in \( \mathbb{R}^{2d+1} \) \textit{orchard-equivalent} if they can be related by a series of flips whose flipsets have always exactly \((d+1)\) points in each class.

More generally, flips are of different types according to the number of points of each class involved in the corresponding flipset. A very special type of flips are the \textit{monochromatic} ones, defined as involving only vertices of one class in their flipset.

Understanding isotopy (or more generally isomorphism) classes of generic configurations up to flips subject to some restrictions (e.g. only monochromatic flips or configurations up to orchard-equivalence in odd dimensions) might be an interesting problem.

Before describing a last consequence of the Flip Proposition we need a few notations:

Let \( P \) be a generic configuration with equivalence classes \( A(P) \) and \( B(P) \) consisting of \( a(P) \) and \( b(P) \) elements. Call a generic configuration \textit{pointed} if one of the equivalence classes, say \( A(P) \) has been selected and denote by \( a(P) \) the number of elements in the selected class. Let us moreover introduce the finite graph \( \Gamma^pC_n(\mathbb{R}^d) \) with vertices isomorphism classes of pointed generic configurations of \( n \) points in \( \mathbb{R}^d \), two edges being joined by an edge if the corresponding pointed configurations can be related by a flip (a flip of a pointed configuration does not change the selected equivalence class outside the flipset). The following result is then an easy consequence of the Flip Proposition.

**Corollary 1.4.** (i) If \( d \) is even then the graph \( \Gamma^pC_n(\mathbb{R}^d) \) is bipartite, the class of a pointed configuration \( P \) being given by \( a(P) \) (mod 2).

(ii) If \( d \) is odd and \( n = 2m + 1 \) is odd then the graph \( \Gamma^pC_n(\mathbb{R}^d) \) has two connected components, the function \( \overline{a}(P) \) (mod 2) being constant on each component.

(iii) If \( d \) is odd and \( n = 2m \) is even then there exist \( \pi(n,d) \in \{0,1\} \) such that \( \overline{a}(P) \equiv \overline{b}(P) \equiv \pi(n,d) \) (mod 2) for every generic configuration \( P \) of \( n = 2m \) points in \( \mathbb{R}^d \).

In the case \( d = 1 \), it is easy to see that the function \( \pi(2m,1) \) of assertion (iii) is given by \( \pi(2m,1) \equiv m \) (mod 2). More generally (cf. Example 3.2 of Section 3), one has

\[
\pi(2m,2d+1) = \begin{cases} 
    m \pmod{2} & \text{if}\ (\frac{2m-3}{2d}) \text{ is odd} \\
    0 & \text{otherwise}
\end{cases}
\]

Call an antipodal subset \( \mathcal{P} = \{\pm P_1, \ldots, \pm P_n\} \subset \mathbb{S}^d \) of the \( d \)-dimensional unit sphere \( \mathbb{S}^d \subset \mathbb{R}^{d+1} \) \textit{generic} if the linear span in \( \mathbb{R}^{d+1} \) of any subset \( \pm P_{i_1}, \ldots, \pm P_{i_k} \) of \( k \) pairs of points is of dimension \( k \) for \( k \leq d+1 \).

**Theorem 1.5 (Spherical Orchard Theorem).** There exists a natural equivalence relation having at most 2 classes on the set \( \mathcal{P} = \{\pm P_1, \ldots, \pm P_n\} \subset \mathbb{S}^d \) of points of a generic antipodal spherical configuration.

If \( \binom{n-2}{d} \) is even, this relation satisfies

\[-P_i \sim P_i \text{ for } P_i \in \mathcal{P}\]
and if \( \binom{n-2}{d} \) is odd we have

\[ -P_i \not\sim P_i \text{ for } P_i \in \mathcal{P}. \]

We have of course also the obvious version (involving suitable subsets of \( d + 1 \) pairs of antipodal points) of the Flip proposition.

Let \( \mathcal{C} \) be a set of continuous real functions on \( \mathbb{R}^k \). Suppose \( \mathcal{C} \) is a \((d + 1)\)-dimensional vector space containing the constant functions. We denote by \( \mathcal{C}_0 \) the \( d \)-dimensional subspace

\[ \mathcal{C}_0 = \{ f \in \mathcal{C} | f(0) = 0 \}. \]

Call a set \( P \subset \mathbb{R}^k \) of \( n \) points \( \mathcal{C} \)-generic if for each subset \( S = \{P_{i_1}, \ldots, P_{i_d}\} \) of \( d \) distinct points in \( P \) if the set

\[ I(S) = \{ f \in \mathcal{C} | f(P_{i_j}) = 0, \; j = 0 \ldots, d \} \]

is 1-dimensional all \( \binom{n}{d} \) lines in \( \mathcal{C} \) of this form are distinct.

Given \( P, Q \in \mathcal{P} \), call a set \( S = \{P_{i_1}, \ldots, P_{i_d}\} \) of \( d \) points as above \( \mathcal{C} \)-separating (or separating for short) if \( f(P)f(Q) < 0 \) for any \( 0 \neq f \in I(S) \) and denote by \( n_C(P, Q) \) the number of \( \mathcal{C} \)-separating subsets of \( \mathcal{P} \).

**Corollary 1.6.** The relation \( P \sim_C Q \) if either \( P = Q \) or

\[ n_C(P, Q) \equiv \binom{n-3}{d-1} \pmod{2} \]

defines an equivalence relation having at most two classes on a set \( \mathcal{P} = \{P_1, \ldots, P_n\} \subset \mathbb{R}^k \) of \( n \) points in \( \mathbb{R}^k \) which are \( \mathcal{C} \)-generic.

**Examples 1.7.** (i) Considering the \((d + 1)\)-dimensional vector space of all affine functions in \( \mathbb{R}^d \), Corollary 1.6 boils down to Theorem 1.1.

(ii) Consider the vector space \( \mathcal{C} \) of all polynomial functions \( \mathbb{R}^2 \rightarrow \mathbb{R} \). A finite subset \( \mathcal{P} \subset \mathbb{R}^2 \) is \( \mathcal{C} \)-generic if and only if every subset of five points in \( \mathcal{P} \) defines a unique conic and all these conics are distinct.

(iii) Consider the vector space \( \mathcal{C} \) of all polynomials of degree \( < d \) in \( x \) together with the polynomials \( \lambda y, \lambda \in \mathbb{R} \). A subset \( \mathcal{P} = \{(x_1, y_1), \ldots, (x_n, y_n)\} \) with \( x_1 < x_2 < \ldots, < x_n \) is \( \mathcal{C} \)-generic if all \( \binom{n}{d} \) interpolation polynomials in \( x \) defined by \( d \) points of \( \mathcal{P} \) are distinct.

The sequel of this paper is organized as follows:

The next section contains the (elementary) proofs of Theorem 1.1, Proposition 1.3 and Corollaries 1.4, 1.6.

Section 3 contains a brief description for concrete computations of the Orchard relation.

Section 4 states the projective version of the Orchard Theorem and contains the proof (and construction of \( \sim \)) for Theorem 1.5.

Section 5 is devoted to simple arrangements of pseudolines in \( \mathbb{R}P^2 \).

2. Proofs

In this section we give a proof of Theorem 1.1, Proposition 1.3 and Corollary 1.4.

The case \( d = 1 \) (of both results) is of course trivial: A generic configuration of \( n \) points in \( \mathbb{R} \) is simply a strictly increasing sequence \( P_1 < P_2 < \cdots < P_n \) of \( n \) real numbers and the Orchard Theorem restates the obvious
fact that we get an equivalence relation by considering $P_i \sim P_j$ for $i \equiv j \pmod{2}$.

We suppose now $d \geq 2$ and introduce some useful notations.

Consider 3 points $P_i, P_j, P_k$ of a generic configuration $P = \{P_1, \ldots, P_n\} \subset \mathbb{R}^d$. If $d \geq 2$ which we will suppose in the sequel, the affine span of the points $P_i, P_j, P_k$ is a 2-dimensional affine plane. The three affine lines $L_{s,t}$ spanned by two points $P_s \neq P_t \in \{P_i, P_j, P_k\}$ contain three compact intervals denoted $[P_s, P_t]$ and subdivide the projective plane $\Pi \supset \Pi$ into four triangles $\Delta_0, \Delta_i, \Delta_j, \Delta_k$ as shown in Figure 3.

![Figure 3: The projective plane $\Pi = \Delta_0 \cup \Delta_i \cup \Delta_j \cup \Delta_k$](image)

Let $\alpha_i$ denote the number of hyperplanes containing $d$ distinct points in $P \setminus \{P_i, P_j, P_k\}$ which intersect both segments $[P_i, P_j]$ and $[P_i, P_k]$. Introduce $\alpha_j$ and $\alpha_k$ similarly.

For a subset

$$S = \{P_1, \ldots, P_{d-1}\} \subset P \setminus \{P_i, P_j, P_k\}$$

of $(d-1)$ distinct points in $P \setminus \{P_i, P_j, P_k\}$, we denote by $P_S$ the intersection (which might be at infinity) of the projective span of $S$ with the projective plane $\Pi$. We claim that $P_S$ consists of exactly one point contained in the interior of exactly one of the four triangles $\Delta_0, \Delta_i, \Delta_j, \Delta_k$: Indeed, $P_S$ is non-empty by a dimension argument. If $P_S$ is of dimension $> 0$ or if $P_S$ is included in the projective line $\overline{L_{i,j}}$, the set $S \cup \{P_i, P_j\}$ of $(d + 1)$ distinct points in $P$ is affinely dependent. The same argument holds of course for the projective lines $\overline{L_{i,k}}$ and $\overline{L_{j,k}}$.

For $* \in \{0, i, j, k\}$, we can hence introduce the number $\sigma_*$ of subsets

$$S = \{P_1, \ldots, P_{d-1}\} \subset P \setminus \{P_i, P_j, P_k\}$$

of $(d-1)$ distinct points in $P \setminus \{P_i, P_j, P_k\}$ whose projective span intersects the projective plane $\Pi$ in an interior point $P_S$ of the triangle $\Delta_*$. 

Recall that $n(P_i, P_j)$ (and similarly $n(P_j, P_k), n(P_i, P_k)$) denotes the number of hyperplanes spanned by $d$ points in $P \setminus \{P_i, P_j\}$ which separate $P_i$ from $P_j$ in the affine space $\mathbb{R}^d$.

**Lemma 2.1.** We have

$$n(P_i, P_j) = \alpha_i + \alpha_j + \sigma_0 + \sigma_k,$$

$$n(P_j, P_k) = \alpha_j + \alpha_k + \sigma_0 + \sigma_i,$$

$$n(P_i, P_k) = \alpha_i + \alpha_k + \sigma_0 + \sigma_j.$$ 

**Proof of Lemma 2.1.** We prove the formula for $n(P_i, P_j)$. The remaining cases follow by symmetry.
A hyperplane $H$ separating $P_i$ from $P_j$ intersects the interior of the segment $[P_i, P_j]$ and we have two subcases depending on the position of $P_k$ with respect to $H$.

If $P_k \notin H$, the line $H \cap \Pi$ cuts the interior of the edge $[P_i, P_j] \subset \Delta_0$ and leaves the triangle $\Delta_0$ by crossing the interior of either the edge $[P_i, P_k]$ or of the edge $[P_j, P_k]$. Such a hyperplane $H$ contributes hence 1 to either $\alpha_i$ or $\alpha_j$. The line $H_3$ of Figure 3 shows the intersection of such a hyperplane with the plane $\Pi$. It yields a contribution of 1 to either $\alpha_i$ or $\alpha_j$.

In the remaining case $P_k \in H$, the projective hyperplane $H$ is spanned by $P_k$ and by a subset $S$ consisting of exactly $(d-1)$ points in $\mathcal{P} \setminus \{P_i, P_j, P_k\}$. The projective line $\Pi \cap H$ is hence defined by the point $P_k \in \Pi$ and by the point $P_S$ (which might very well be located at infinity), defined as above as the intersection of the projective span of $S$ with the projective plane $\Pi$. Since the line $\Pi \cap H$ crosses $[P_i, P_j]$, the point $P_S \in \Pi$ belongs either to the interior of $\Delta_0$ or $\Delta_k$ (it cannot be on the boundary by genericity, see above) and such a hyperplane $H$ yields hence a contribution of 1 to either $\sigma_0$ or $\sigma_k$.

Figure 4 shows two such situations giving rise to hyperplanes intersecting $\Pi$ along the lines $H_1$ and $H_2$. The corresponding points $P_{S_1}$ and $P_{S_2}$ belong respectively to $\Delta_k$ and $\Delta_0$.

Proof of Theorem 1.1 in the case $d \geq 2$. It is obvious that the Orchard relation $\sim$ is reflexive and symmetric. Transitivity is however not completely obvious. We consider hence three points $P_i, P_j$ and $P_k$ satisfying $P_i \sim P_j$ and $P_j \sim P_k$.

We have then by Lemma 2.1

$$n(P_i, P_j) = \alpha_i + \alpha_j + \sigma_0 + \sigma_k \equiv \frac{n-3}{d-1} \pmod{2}$$

$$n(P_j, P_k) = \alpha_j + \alpha_k + \sigma_0 + \sigma_i \equiv \frac{n-3}{d-1} \pmod{2}$$

(where $\alpha_*$ and $\sigma_*$ are of course as above) and adding these two equalities we get

$$\alpha_i + \alpha_k + \sigma_i + \sigma_k \equiv 0 \pmod{2}.$$
The identity 
\[ \sigma_0 + \sigma_i + \sigma_j + \sigma_k = \left(\frac{n-3}{d-1}\right) \]

yields now 
\[ n(P_i, P_k) = \alpha_i + \alpha_k + \sigma_j + \sigma_0 \equiv \left(\frac{n-3}{d-1}\right) \pmod{2}. \]

This shows \( P_i \sim P_k \) and establishes the transitivity of \( \sim \).

In order to prove that the Orchard relation has at most two classes, consider \( P_i \not\sim P_j \) and \( P_k \not\sim P_j \). The above computation shows that \( P_i \sim P_k \).

QED

**Proof of Proposition 1.3.** Consider \( P(t), Q(t) \in \mathcal{P}(t) \). A hyperplane \( H(t) \) determined by \( d \) points \( P_1(t), \ldots, P_d(t) \) of \( \mathcal{P}(t) \setminus \{P(t), Q(t)\} \) changes its incidence with the interval \([P(t), Q(t)]\) at time \( t = 0 \) if and only if \( P_1(t), \ldots, P_d(t) \subset F(t) \) and exactly one of the points \( P(t), Q(t) \) is the last remaining point of \( F(t) \). This proves the equality 
\[ n(P(1), Q(1)) = n(P(-1), Q(-1)) \]

if \( P(t), Q(t) \in F(t) \) or if \( P(t), Q(t) \in \mathcal{P}(t) \setminus F(t) \). In the remaining case where \( P(t) \in F(t) \) and \( Q(t) \in \mathcal{P}(t) \setminus F(t) \) (up to permutation of \( P \) and \( Q \)) we have 
\[ n(P(1), Q(1)) = n(P(-1), Q(-1)) \pm 1 \]

(the difference coming of course from the unique hyperplane spanned by \( F(t) \setminus P(t) \)). Proposition 1.3 follows now easily. QED

**Proof of Corollary 1.4.** A flip induces an exchange of exactly \((d+1)\) points between the two equivalence classes. This implies assertions (i) and (ii) at once. For assertion (iii) we need also the fact that every pair of generic configurations having \( n \) points in \( R^d \) can be joined by a finite number of flips.

QED

**Proof of Corollary 1.6.** Given a \((d+1)\)-dimensional vectorspace \( \mathcal{C} = \mathcal{C}_0 + R \) of continuous functions \( R^k \to R \) and a \( \mathcal{C} \)-generic set \( \mathcal{P} = \{P_1, \ldots, P_n\} \) in \( R^k \), consider the **generalised Veronese-map** \( V : R^k \to R^d \) defined by \( V(P) = (b_1(P), \ldots, b_d(P)) \) where \( b_1, \ldots, b_d \in \mathcal{C}_0 \) form a basis of the vectorspace \( \mathcal{C}_0 \). This map is of course well-defined up to a linear automorphism of \( R^d \) and sends \( \mathcal{P} \) into a subset \( V(\mathcal{P}) \subset R^d \) which is generic in the sense of Theorem 1.1: A point \( V(P) \in R^d \) belongs to a hyperplane \( H \) spanned by \( V(P_1, \ldots, V_d) \) if and only if \( f(P) = 0 \) for any function \( f \in I(P_1, \ldots, P_d) \). Such a non-zero function \( f \) changes the sign according to the two connected components of \( R^k \setminus H = H_+ \cup H_- \). One has \( f(P) < 0 \) for \( P \in R^k \) if and only if \( V(P) \in H_- \). Theorem 1.1 implies now obviously the result.

QED

### 3. Computational Aspects

Given a finite, totally ordered set \( S \), we denote by \( \binom{S}{k} \) the set of all distinct strictly increasing sequences 
\[ s_{i_1} < s_{i_2} < \cdots < s_{i_k} \]
of length \( k \) in \( S \).

Order the points \( P = \{P_1, \ldots, P_n\} \subset \mathbb{R}^d \) of a generic configuration totally (for instance by setting \( P_i < P_j \) if \( i < j \)). For \( X \in \mathbb{R}^d \) and \( S = \{P_{i_1}, \ldots, P_{i_d}\} \in \binom{P}{d} \) we define \( \det(S - X) \) as the determinant of the square \( d \times d \) matrix with rows \( P_{i_1} - X, \ldots, P_{i_d} - X \).

**Proposition 3.1.** Let \( P = \{P_1, \ldots, P_n\} \subset \mathbb{R}^d \) be a generic configuration of \( n \) points. We have then \( P \sim Q \) if and only if

\[
(-1)^{\binom{n-1}{2}} \prod_{S \in \binom{P \setminus \{P, Q\}}{d}} \det(S - P) \det(S - Q) > 0.
\]

**Proof.** The set \( \binom{P \setminus \{P, Q\}}{d} \) corresponds to the set of all hyperplanes spanned by \( d \) points in \( P \setminus \{P, Q\} \). Such a hyperplane \( H = H_S \) separates the points \( P \) and \( Q \) if and only if \( \det(S - P) \det(S - Q) < 0 \). QED

In particular, the Orchard relation \( \sim \) on a generic configuration \( P \) of \( n \) points in \( \mathbb{R}^d \) can be constructed by computing

\[
(n - 1) \begin{pmatrix} n - 2 \\ d - 1 \end{pmatrix}
\]
determinants of \( d \times d \) matrices. The computational cost of determining \( \sim \) is hence of order \( O(n^d) \) in \( n \) (for fixed \( d \)).

**Example 3.2.** Choose \( 0 < a_1 < a_2 < \ldots, < a_n \) and set

\[
P_k = (a_k^1, a_k^2, \ldots, a_k^d) \in \mathbb{R}^d
\]
for \( 1 \leq k \leq n \). This yields a generic configuration \( P \) of \( n \) points in \( \mathbb{R}^d \).

A simple computation using Vandermonde’s formula shows then that the number \( n(P_k, P_{k+1}) \) of hyperplanes separating \( P_k \) from \( P_{k+1} \) is always zero. The Orchard relation on \( P \) is hence trivial if \( \begin{pmatrix} n - 3 \\ d - 1 \end{pmatrix} \equiv 0 \pmod{2} \) and has two non-empty classes (for \( n \geq 2 \)) otherwise. This implies easily the formula for \( \pi(2n, 2d + 1) \) given after Corollary 1.4.

**Remark 3.3.** For practical purposes, the Orchard relation (for a huge number \( n \) of generic points \( P_1, \ldots, P_n \in \mathbb{R}^d \)) is probably best determined as follows: For each affine hyperplane \( H \) spanned by \( d \) points of \( \{P_2, \ldots, P_n\} \) compute an affine function \( f_H \) satisfying \( f_H|_H \equiv 0 \) and \( f_H(P_1) = -1 \). The hyperplane \( H \) contributes then 1 to \( n(P_1, P_i) \) if and only if \( f_H(P_1) > 0 \) and the knowledge of the parity of all numbers \( n(P_1, P_i) \) determines of course the Orchard relation by transitivity.

The Orchard relation \( \sim \) (or the whole binary rooted tree obtained by recursive iterations of \( \sim \) on equivalence classes) is probably one of the simplest invariants of generic configurations. It can of course be combined with other invariants (e.g. the subset of vertices forming the convex hull) or used to define other new invariants.

Such a new invariant is for instance the function

\[
\varphi(P) = \text{sign} \left( \prod_{1 \leq i_1 < i_2 < \cdots < i_d \leq n, \ P_{i_j} \neq P} \det(P_{i_1} - P, \ldots, P_{i_d} - P) \right) \in \{\pm 1\}
\]
on an equivalence class \( A \subset P = \{P_1, \ldots, P_n\} \). This function is well-defined if \( \begin{pmatrix} n - 2 - |A| \\ d - 2 \end{pmatrix} \equiv 0 \pmod{2} \) and is defined up to a global sign change otherwise.
Similarly, the function
\[
\omega(P, Q) = \text{sign} \left( \prod_{1 \leq i_1 < i_2 < \cdots < i_{d-1} \leq n, \ P_{i_j} \neq P, Q} \det(P - Q, P_{i_1} - Q \ldots, P_{i_{d-1}} - Q) \right)
\]
is antisymmetric on an equivalence class \( A \subset P \). It is well-defined if \( n - 3 - |A| d \equiv 0 \pmod{2} \) and is defined up to a global sign otherwise.

**Remark 3.4.** The formula of Proposition 3.1 suggests perhaps a homological origin for the Orchard relation. Indeed, the configuration space of \( n \) points in generic position in \( \mathbb{C}^d \) is connected but (generally) not simply connected. One gets hence fundamental groups \( B_n(\mathbb{C}^d) \) (respectively \( P_n(\mathbb{C}^d) \)) by considering loops up to isotopy in this space (respectively loops not permuting the points). For \( d = 1 \) we get of course the braid and the pure braid groups.

The abelianisation of the pure group \( P_n(\mathbb{C}^d) \) is then isomorphic to \( \mathbb{Z}^{n_{d+1}} \).

Indeed, each subset \( P_{i_0}, \ldots, P_{i_d} \) of \( d + 1 \) distinct points in \( \mathbb{P}^d \) determines a homorphism onto \( \mathbb{Z} \) by considering the winding number
\[
\frac{1}{2 \pi \sqrt{-1}} \int_0^1 \ln \left( \prod_{1 \leq i_0 < i_1 < \cdots < i_{d-1} \leq n} \left( \det(P_{i_1}(t) - P_{i_0}(t), \ldots, P_{i_{d-1}}(t) - P_{i_0}(t)) \right)^2 \right)
\]
along a loop \( \mathcal{P}(t) = (P_1(t), \ldots, P_n(t)), \ t \in [0, 1] \), in the space of generic configurations. These homorphisms are linearly independent and the intersection of all their kernels is the derived group.

In the case of \( B_n(\mathbb{C}^d) \) we get a homomorphism into \( \mathbb{Z} \) by considering the winding number
\[
\frac{1}{2 \pi \sqrt{-1}} \int_0^1 \ln \left( \prod_{1 \leq i_0 < i_1 < \cdots < i_{d-1} \leq n} \left( \det(P_{i_1}(t) - P_{i_0}(t), \ldots, P_{i_{d-1}}(t) - P_{i_0}(t)) \right)^2 \right)
\]

4. Generic configurations in real projective spaces

A configuration of \( n \) points \( \mathcal{P} = \{P_1, \ldots, P_n\} \subset \mathbb{R} P^d \) in real projective space is generic if no subset of \( (k + 1) \leq (d + 1) \) points in \( \mathcal{P} \) is contained in a projective subspace of dimension \( < k \). The injection \( i : \mathbb{R}^d \rightarrow \mathbb{R} P^d \) constructed by gluing an \( \mathbb{R} P^{d-1} \) along the boundary of \( \mathbb{R}^d \) yields a surjection (which is generally not injective) from the set of generic configuration in \( \mathbb{R}^d \) (up to isomorphism) onto the set of generic configurations in \( \mathbb{R} P^d \) (up to the obvious natural notion of isomorphism obtained by allowing the hyperplane at infinity of affine configurations to move).

The exact statement of the projective counterpart of the Orchard Theorem depends unfortunately on the parity of the binomial coefficient \( \binom{n-2}{d} \) enumerating all relevant hyperplanes not containing two given points of a generic configuration. For the easy case we have:

**Theorem 4.1A.** (Projective Orchard Theorem.) If \( \binom{n-2}{d} \) is even then there exists a natural partition of the points \( \mathcal{P} = \{P_1, \ldots, P_n\} \subset \mathbb{P} \mathbb{R}^d \) of a generic projective configuration into two classes.

This partition is given by considering the equivalence classes of the affine configuration obtained after erasing a generic \( \mathbb{R} P^{d-1} \) not intersecting \( \mathcal{P} \).
Before stating the result if the binomial coefficient \( \binom{n-2}{d} \) is odd we need to introduce some notations: Given two points \( P \neq Q \) of a generic configuration \( \mathcal{P} \subset \mathbb{R}P^d \) we denote by \( L_{P,Q} \) the projective line spanned by them. Denote by \( \alpha \), respectively \( \beta = \binom{n-2}{d} - \alpha \), the number of projective \((d - 1)\)-dimensional subspaces spanned by \( d \) points of \( \mathcal{P} \setminus \{P, Q\} \) which intersect the first, respectively the second, of the two connected components in \( L_{P,Q} \setminus \{P, Q\} \). Let \( \gamma \in \{\alpha, \beta\} \) be the unique integer such that

\[
\gamma \equiv \binom{n-3}{d-1} \pmod{2}
\]

and denote by \( I(P, Q) \) the corresponding connected component of \( L_{P,Q} \setminus \{P, Q\} \). Denote by \( \Gamma \) the immerged complete graph with vertices \( \mathcal{P} \) and edges \( I(P, Q) \) for \( P \neq Q \in \mathcal{P} \). We call a continuous application of a connected graph \( G \) into \( \mathbb{R}P^d \) (for \( d \geq 2 \)) homologically trivial if the induced group homomorphism \( i_* : \pi_1(G) \to \mathbb{Z}/2\mathbb{Z} = \pi_1(\mathbb{R}P^d) \) is trivial.

**Theorem 4.1B.** The immersion of the complete graph \( \Gamma \) into \( \mathbb{R}P^d \) is homologically trivial.

Before sketching proofs, let us remark that these projective versions can be applied (modulo point-hyperplane duality in projective space) to generic arrangements of hyperplanes in \( \mathbb{R}P^d \). (A finite set \( \mathcal{H} = \{H_1, \ldots, H_n\} \) of \( n \) distinct hyperplanes in \( \mathbb{R}P^d \) is generic if the intersection of any subset of \( k \leq d \) hyperplanes in \( \mathcal{H} \) is of codimension \( k \).) In the case of a projective generic line arrangement \( \mathcal{L} \) in the projective plane \( \mathbb{R}P^2 \), the statement of Theorem 4A can visually be seen as follows: Two distinct lines \( L_1, L_2 \in \mathcal{L} \) define two connected components in \( \mathbb{R}P^2 \setminus \{L_1, L_2\} \) containing all the remaining \( \binom{n-2}{2} \) intersections of distinct lines in \( \mathcal{L} \setminus \{L_1, L_2\} \). If the binomial coefficient \( \binom{n-2}{2} \) is even, the numbers of intersections \( L_i \cap L_j, \ L_i, L_j \in \mathcal{L} \setminus \{L_1, L_2\} \) contained in each connected component of \( \mathbb{R}P^2 \setminus \{L_1, L_2\} \) have the same parity. The two lines \( L_1 \) and \( L_2 \) are equivalent if and only if the above parity is given by \( \binom{n-3}{2-1} \equiv n + 1 \pmod{2} \). An interesting feature of this construction is the fact that it generalises also to generic configurations of pseudolines (cf. \[\square\] for the definition), see the next section for details. A still more general setting for considering the (projective) Orchard relation seems to be given by a suitable subset of oriented matroids (cf. \[\square\]), perhaps the set of arrangements of pseudohyperplanes in (projective or affine) real space of dimension \( d \) which are generic in the sense that \( k \) distinct pseudohyperplanes have an intersection of codimension \( k \) for \( k \leq d \). Corollary 1.6 is a step in this direction.

**Sketch of proof for Theorem 4.1A.** Erase a projective subspace \( \mathbb{P} \mathbb{R}^{d-1} \) containing no point of \( \mathcal{P} \subset \mathbb{P} \mathbb{R}^d \). The equivalence relation of the resulting affine generic configuration is independent of the choice of the above subspace.

**Sketch of proof of Theorem 4.1B.** Suppose by contradiction that \( \Gamma \) contains a loop \( \lambda \) which is homologically non-trivial in \( \mathbb{R}P^2 \). Use isotopies and the Orchard Theorem (after suitable affine projections) to reduce the number of intersections of \( \lambda \) with \( \mathcal{P} \). The result follows then by induction.

QED
Proof of Theorem 1.5. Consider the 2–fold cover $S^d \rightarrow \mathbf{R}P^d$ sending a generic antipodal spherical configuration $\mathcal{P} = \{\pm P_1, \ldots, \pm P_n\}$ into a generic projective configuration $\overline{\mathcal{P}} = \{\overline{P}_1, \ldots, \overline{P}_n\}$ of $\mathbf{R}P^d$. If $\binom{n-2}{d}$ is even, lift the equivalence relation on $\overline{\mathcal{P}}$ onto $\mathcal{P}$ in the obvious way. If $\binom{n-2}{d}$ is odd, choose an equatorial circle $S^{d-1} \subset S^d$ which avoids all points of $\mathcal{P}$. Up to a sign choice, we can now suppose that the points $P_1, \ldots, P_n$ of $S^d$ belong all to the same hemisphere (connected component of $S^d \setminus S^{d-1}$) which we identify with the affine space $\mathbf{R}^d$ via the central projection $\pi$ sending $S^{d-1}$ at infinity. The set $\{\pi(P_1), \ldots, \pi(P_n)\}$ is now a generic set of $n$ points in $\mathbf{R}^d$. Set now $P_i \sim P_j$ (for $i \neq j$) if and only if $\pi(P_i) \sim \pi(P_j)$ and extend this by $P_i \not\sim -P_i$. Theorem 4.1B above implies that this defines an equivalence relation with 2 classes which is independent of the choice of the equatorial circle $S^{d-1} \subset S^d \setminus \mathcal{P}$. QED

5. Simple arrangements of pseudolines in $\mathbf{R}P^2$

The aim of this section is to work out some consequences of the Orchard Theorem for simple arrangements of pseudolines in the projective plane $\mathbf{R}P^2$.

A pseudoline in $\mathbf{R}P^2$ is a simple, smooth curve isotopic to a projective line in $\mathbf{R}P^2$. An arrangement of $n$ pseudolines is a finite set $\mathcal{L} = \{L_1, \ldots, L_n\}$ of $n$ pseudolines intersecting each other transversally exactly once. Such an arrangement is simple if no triple intersections occur. If a simple arrangement is stretchable (isotopic in the obvious sense to a simple arrangement of projective lines), then Theorem 4.1A, respectively Theorem 4.1B, of the preceding section imply that there is a natural equivalence relation on the set of pseudolines if $\binom{n-2}{2}$ is even (i.e. if $n \equiv 2 \pmod{4}$ or $n \equiv 3 \pmod{4}$), respectively that the (pseudo)lines carry a natural orientation (up to global reversion of all orientations) if $\binom{n-2}{2}$ is odd.

All this remains valid for simple arrangements of pseudolines and we have of course an analogue of the Flip Proposition. As a consequence, the polygons (connected components of $\mathbf{R}P^2 \setminus \mathcal{L}$) have extra structures according to the classes or orientations of their sides.

Two pseudolines $L_i, L_j$ of a simple pseudoline arrangement $\mathcal{L} = \{L_1, \ldots, L_n\}$ define two open digons by considering the two connected components of $\mathbf{R}P^2 \setminus \{L_i \cup L_j\}$. The interior of these digons contain $\alpha$ respectively $\beta$ intersections $L_s \cap L_t$ of distinct pseudolines $L_s \neq L_t$ in $\mathcal{L} \setminus \{L_i, L_j\}$.

For $\binom{n-2}{2} = \alpha + \beta$ even, set $L_i \sim L_j$ if $\alpha \equiv \binom{n-3}{2} \pmod{2}$ (or if $L_j = L_i$).

Theorem 5.1A. For $\binom{n-2}{2}$ even, the relation $\sim$ defined above is an equivalence relation on $\mathcal{L}$ into at most two classes.

Consider now two oriented pseudolines $L_i^o, L_j^o$. They induce an orientation on the boundary of exactly one digon $D^o$ in $\mathbf{R}P^2 \setminus \{L_i \cup L_j\}$ and they have disagreeing orientations on the boundary of the remaining digon. We call the orientations of the oriented pseudolines $L_i^o$ and $L_j^o$ compatible at $L_i \cap L_j$ if the number $\alpha^o$ of intersections $L_s \cap L_t$ (for $L_s \neq L_t \in \mathcal{L} \setminus \{L_i, L_j\}$) contained in the digon $D^o$ satisfies

$$\alpha^o \equiv \binom{n-3}{2} \pmod{2}.$$
An orientation of all pseudolines in $\mathcal{L}$ is compatible if it is compatible at $L_i \cap L_j$ for all $L_i \neq L_j$.

**Theorem 5.1B.** If $\binom{n-2}{2}$ is odd, then there exist exactly two compatible orientations of all pseudolines in a simple pseudoline arrangement $\mathcal{L} = \{L_1, \ldots, L_n\} \subset \mathbb{RP}^2$. These two compatible orientations induce opposite orientations on all pseudolines.

We call the partition, respectively compatible orientation, on the set of pseudolines of a simple pseudoline arrangement $\mathcal{L}$ the Orchard partition, respectively Orchard orientation, on $\mathcal{L}$.

**Proof of Theorem 5.1A** Consider three pseudolines $L_i, L_j, L_k$ decomposing the projective plane into four triangles $\Delta_0, \Delta_i, \Delta_j, \Delta_k$ as in Figure 5 (here we do not care about orientations).

![Figure 5](image-url)

Denote by $\alpha_{i,j}$ the number of pseudolines intersecting $L_i$ and $L_j$ on the boundary of $\Delta_0$ and define $\alpha_{i,k}, \alpha_{j,k}$ similarly. For $* \in \{0, i, j, k\}$ denote by $\sigma_*$ the number of intersections $L_s \cap L_t$ inside $\Delta_*$ of distinct pseudolines $L_s, L_t \in \mathcal{L} \setminus \{L_i, L_j, L_k\}$. An argument similar to Lemma 2.1 shows that we have $L_i \sim L_j$ if and only if

$$\sigma_0 + \sigma_k + \alpha_{i,k} + \alpha_{j,k} \equiv \binom{n-3}{2} \pmod{2}$$

and $L_j \sim L_k$ if and only if

$$\sigma_0 + \sigma_i + \alpha_{i,j} + \alpha_{i,k} \equiv \binom{n-3}{3} \pmod{2}.$$ 

The equality

$$\sigma_0 + \sigma_i + \sigma_j + \sigma_k = \binom{n-3}{3}$$

and arguments similar to those used in the proof of Theorem 1.1 imply now the result.

QED

The proof of Theorem 5.1B is similar and left to the reader.

The analogue of flips are triangle-moves (corresponding to Reidemeister III-moves for knots) and they change the Orchard-equivalence relation (respectively the Orchard-orientation) only on the 3 pseudolines involved in the triangle move. Figure 6 shows two simple pseudoline arrangements (with arrows indicating one of the two Orchard-orientations) which are related by a triangle-move.
Orchard-partitions and Orchard orientations yield of course nice invariants of pseudoline arrangements. Two such particularly nice invariants induced by an Orchard orientation (hence in the case where \(\binom{n-2}{2}\) is odd) are given by desingularising all crossings in one of the two generic ways (either by respecting all orientations or in the other way) and by analysing the resulting pattern of noncrossing closed curves in \(\mathbb{R}P^2\).

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