Solitons in a six-dimensional super Yang-Mills-tensor system and non-critical strings

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ABSTRACT

In this letter we study a coupled system of six-dimensional $N = 1$ tensor and super Yang-Mills multiplets. We identify some of the solitonic states of this system which exhibit stringy behaviour in six dimensions. A discussion of the supercharges and energy for the tensor multiplet as well as zero modes is also given. We speculate about the possible relationship between our solution and what is known as tensionless strings.

MIRAMARE – TRIESTE
November 1997

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1. Introduction.

Six-dimensional supersymmetric systems emerge in many recent studies of superstring theories. They can be obtained either by compactifying a ten-dimensional theory or can be constructed as anomaly-free models in six dimensions. In this note we shall consider a coupled system of \( N = 1 \) supersymmetric tensor and Yang-Mills multiplets and study the properties of some of its supersymmetric solitonic configurations. The zero modes of the solitonic states constructed in this letter behave like scalars and spinors in a two-dimensional subspace of the six-dimensional spacetime and can be grouped together as the transverse bosonic and fermionic coordinates of a set of strings propagating in a six-dimensional spacetime. There are some intriguing features of our solution which are suggestive of the tensionless strings. We shall also construct supercharges which generate the correct supersymmetry transformation of the fields in the tensor multiplets. We shall then give a Hamiltonian which correctly yields the quantum equations of motion and show that it vanishes on the subset of static solitonic backgrounds constructed in this letter. Finally we shall speculate on possible connections with the non-critical \( E_8 \) strings.

2. The Model

The introduction of tensor multiplets is called for by the requirement of anomaly cancellation, namely, the presence of tensor fields in six dimensions enable us to make use of the Green-Schwarz anomaly cancellation mechanism [1]. The tensor multiplet by itself does not have nonsingular solitonic solutions. This is one more reason for considering the tensor multiplet coupled to Yang-Mills fields.

The Yang-Mills multiplet consists of the gauge field \( A_\mu \), the gaugino \( \lambda^i \) and an auxiliary field \( Y^{ij} \). The tensor multiplet involves a scalar \( \sigma \), a second rank antisymmetric tensor \( B_{ab} \) and a tensorino \( \chi^i \). All of the fermions are chiral. The index \( i \) is an \( SU(2) \)-index and the fermions satisfy a symplectic Majorana condition. The supersymmetry parameters will be taken to have positive six-dimensional chirality.
The supersymmetry transformations of the fields in our system are as follows [2].

\[
\begin{align*}
\delta A_a &= -\bar{\epsilon} \Gamma_a \lambda \\
\delta \lambda^i &= \frac{1}{8} \Gamma^{ab} F_{ab} \epsilon^i - \frac{1}{2} Y^{ij} \epsilon_j \\
\delta Y^{ij} &= -\epsilon^{(i} \Gamma^a D_a \lambda^{j)}
\end{align*}
\]

The corresponding rules for the tensor multiplet coupled to Yang–Mills are given by

\[
\begin{align*}
\delta \sigma &= \bar{\epsilon} \chi \\
\delta \chi^i &= \frac{1}{48} \Gamma^{abc} H^+_{abc} \epsilon^i + \frac{1}{4} \Gamma^a \partial_a \sigma \epsilon^i - \frac{\alpha'}{4} \text{Tr} \Gamma^a \lambda^i \epsilon \Gamma_a \lambda \\
\delta B_{ab} &= -\bar{\epsilon} \Gamma_{ab} \chi - \alpha' \text{Tr} A_{[a} \bar{\epsilon} \Gamma_{b]} \lambda \\
\end{align*}
\]

where

\[
H_{abc} = 3\partial_{[a} B_{bc]} + 3\alpha' \text{Tr} (A_{[a} \partial_{b} A_{c]} + \frac{2}{3} A_a A_b A_c)
\]

\[
H^\pm_{abc} = \frac{1}{2} \left( H_{abc} \pm \tilde{H}_{abc} \right).
\]

The closure of the supersymmetry algebra leads to the following field equations for various fields,

\[
\begin{align*}
H^-_{abc} &= -\frac{\alpha'}{2} \text{Tr} (\bar{\lambda} \Gamma_{abc} \lambda) \quad (4a) \\
\Gamma^a \partial_a \chi^i &= \alpha' \text{Tr} \left( \frac{1}{4} \Gamma^{ab} F_{ab} \lambda^i + Y^{ij} \lambda_j \right) \quad (4b) \\
\partial^2 \sigma &= \alpha' \text{Tr} \left( -\frac{1}{4} F^{ab} F_{ab} - 2\bar{\lambda} \Gamma^a D_a \lambda + Y^{ij} Y_{ij} \right) \quad (4c)
\end{align*}
\]

Further, by virtue of its definition, \( H_{abc} \) satisfies the identity

\[
\partial_{[a} H^+_{bcd]} = \alpha' \text{tr} \left( \frac{3}{4} F_{[ab} F_{cd]} - \bar{\lambda} \Gamma_{[abc} D_{d]} \lambda \right) \quad (5)
\]

3. The Solution.

We shall look for a bosonic background configuration in which all the fermions as well as the auxiliary field \( Y^{ij} \) will vanish. The six-dimensional coordinates will be chosen as \( x^+, x^- \) and \( x^\mu \) where \( \mu = 1, \ldots, 4 \). We shall consider a multi instanton-type configuration in the \( \mathbb{R}^4 \) spanned by \( x^\mu \). We shall show that the moduli of this instanton can depend on
\( x^+ \). This will require that the \( A_+ \)-component of the vector potential is different from zero. In this sense our solution looks like a static monopole configuration in the six-dimensional spacetime in which \( x^- \) is taken to be the time coordinate. This configuration will preserve half the six-dimensional \( N = 1 \) supersymmetry.

It follows from (4a) that if \( \lambda = 0 \), then \( H \) is selfdual.\(^*\). Now setting \( \delta \lambda \) and \( \delta \chi \) equal to zero we obtain

\[
\Gamma^{ab} F_{ab} \epsilon = 0, \quad \left( \Gamma^a \partial_a \sigma + \frac{1}{12} \Gamma^{abc} H_{abc} \right) \epsilon = 0 \tag{6}
\]

To satisfy these equations, we can choose \( \epsilon = \left( \begin{array}{c} \varepsilon \\ 0 \end{array} \right) \), where, \( \gamma_5 \varepsilon = \pm \varepsilon \), and \( \gamma_5 \) gives the four-dimensional chirality. We shall first discuss the case of positive chirality; the case of negative chirality can be obtained by essentially trivial change of some selfduality conditions. With this choice the fields must obey the equations

\[
H_{05\mu} = -\partial_{\mu} \sigma \tag{7a}
\]

\[
H_{0\mu\nu} = \tilde{H}_{0\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} H_{0\alpha\beta} \tag{7b}
\]

\[
F_{\mu\nu} = \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{7c}
\]

together with \( F_{+-} = 0, \ F_{-\mu} = 0, \ \partial_- \sigma = 0 \). Choosing the gauge \( A_- = 0 \), these reduce to \( \partial_- A_\mu = \partial_- A_+ = \partial_- \sigma = 0 \).

The constraint (5) for \( H_{abc} \), expressing its coupling to the Yang-Mills fields via the Chern-Simons 3-form, now gives the following conditions,

\[
\partial_+ H_{+\mu\nu} = 0 \tag{8a}
\]

\[
\partial_\lambda H_{+\lambda\alpha} = \partial_+ \partial_\alpha \sigma - 2c \text{Tr}(F_{\lambda\alpha} F_{+\lambda}) \tag{8b}
\]

\[
\partial_\mu \partial_\mu \sigma = -\frac{c}{2} \text{Tr}(F_{\mu\nu} \tilde{F}_{\mu\nu}) \tag{8c}
\]

where \( c = 3a'/4 \). Further, \( H_{-\mu\nu} = 0 \) and \( H_{+\mu\nu} = \tilde{H}_{+\mu\nu} \). Setting the auxiliary field \( Y^{ij} \) to zero implies \( D_a F^{ab} = 0 \) \([2]\). The only nontrivial surviving component of this equation is

\[
D_\lambda (D_\lambda A_- - \partial_+ A_\lambda) = 0 \tag{9}
\]

\(^*\) We shall henceforth drop the superscript + from H
where \( D_\lambda A_+ = \partial_\lambda A_+ + [A_\lambda, A_+] \).

The strategy for solving these equations is as follows. We first choose \( F_{\mu\nu} \) to be a multi-instanton configuration in \( \mathbb{R}^4 \). Then equation (8c) gives \( \sigma \), and (7a) gives \( H_{05\mu} \). Since \( D_\lambda D_\lambda \) is invertible in the instanton background, (9) can be uniquely solved for \( A_+ \). Finally, equation (8b) can be solved, consistently with its selfduality, to get \( H^{+\mu\nu} \). As a consequence of \( \partial_- A_\mu = \partial_- A_+ = 0 \), the instanton parameters, collectively denoted by \( \xi \), obey the condition \( \partial_- \xi = 0 \), but they can, of course, depend on \( x^+ \). (They are thus left-moving modes in the \((x^0, x^5)\)-subspace.)

Using the selfduality of \( H^{+\mu\nu} \), we can rewrite (8b) as

\[
\partial_\lambda \partial_\lambda H^{+\mu\nu} = (\partial_\mu J_\nu - \partial_\nu J_\mu) + \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} (\partial_\alpha J_\beta - \partial_\beta J_\alpha)
\]

where \( J_\alpha = \partial_+(\partial_\alpha \sigma) - 2c \text{Tr}(F_{\lambda\alpha}F_{+\lambda}) \). It is easy to see that \( \partial_\alpha J_\alpha = 0 \), as required by the consistency of the equations. Since the four-dimensional Laplacian is invertible, the above equation can be easily solved, once we have \( J_\alpha \). For gauge group \( SU(2) \), \( A_+ \) is given by

\[
A^a_+ = \int d^4y \Delta^{ab}(x, y) e^{bkl}(A^k_\lambda \partial_+ A^l_\lambda)(y)
\]

where the Green’s function \( \Delta^{ab}(x, y) \) for \( D_\lambda D_\lambda \) in the instanton background is given in reference [3]. To make the above solutions explicit, we can, for example, take the ’t Hooft ansatz for instantons, viz., \( A^a_\mu = \bar{\eta}^a_{\mu\nu} \partial_\nu (\log \phi) \) where \( \phi = 1 + \sum_1^N \rho_i^2/(x - a_i)^2 \) and insert it in various equations above. In this case, \( \sigma \), for example, becomes \( 2c \partial_\mu \phi \partial_\mu \phi/\phi^2 \).

4. Supercharges.

The condition that half of the supersymmetry is unbroken leads to BPS-like constraints. For the theory we are considering, there is no known action or Hamiltonian formulation without introducing additional degrees of freedom. For the free tensor multiplet, possible Hamiltonian formulations have been given in [4, 5]. It is possible to extend the formulation of [4] to the case that the Yang-Mills fields are treated as background fields. Specifically, we can choose \( \epsilon^1 = (\alpha, 0) \), \( \chi^1 = (0, v) \) where \( \alpha, v \) are four-component spinors, with \( \epsilon^2, \chi^2 \) given by the symplectic Majorana condition. The supersymmetry variation for
the field $v$ is then given as

$$\delta v = \left[ -\frac{1}{8} \gamma^a \gamma^b H^*_{ab} + \frac{1}{4} (-\pi + \gamma^a \partial_a \sigma) \right] \alpha$$  \hspace{1cm} (12)

where $H^*_{ab} = (1/6) \epsilon_{abcde} H^{cde}$, $\pi = \partial_0 \sigma$. The indices $a, b$ now range from 1 to 5. The bosonic fields obey the commutation rules

$$[\sigma(x), \pi(y)] = 4i \delta(x - y)$$

$$[B_{ab}(x), B_{cd}(y)] = 4i \epsilon_{abcde} \partial_e G(x, y)$$  \hspace{1cm} (13)

where $G(x, y)$ is the five-dimensional Coulomb Green’s function. The supercharge for the tensor multiplet is given by

$$Q_r = \int v_s^\dagger \left[ -\frac{1}{8} \gamma^a \gamma^b H^*_{ab} + \frac{1}{4} (-4 \pi + \gamma^a \partial_a \sigma) \right]_{sr}$$  \hspace{1cm} (14)

(This generates the correct transformations in the gauge $B_{0a} = 0$, $\partial_a B^{ab} = 0$.) The supercharges obey the algebra $\{Q_r, \bar{Q}_s\} = \frac{1}{2} (\tilde{H} \delta_{rs} + \gamma^a_{rs} \tilde{P}_a)$ with

$$\tilde{H} = \frac{1}{8} \int \left[ \pi^2 + (\partial \sigma)^2 + \frac{1}{2} (H^*)^2 \right]$$

$$\tilde{P}_a = -\frac{1}{8} \int \left[ (\pi \partial_a \sigma + \partial_a \sigma \pi) + \frac{1}{4} \epsilon_{amnpqr} H^*_{mn} H^*_{pq} + 2 \partial_b \sigma H^*_b \right]$$  \hspace{1cm} (15)

$\tilde{H}, \tilde{P}_a$ are the Hamiltonian and the momentum for the bosonic fields of the free tensor multiplet. With a Yang-Mills background, one cannot expect a super-Poincaré algebra and one cannot read off the Hamiltonian from this algebra. Rather, the Hamiltonian, for the bosonic fields, is now given by

$$\mathcal{H} = \tilde{H} + \int \left[ \sigma J - \frac{c}{8} \epsilon_{mpqrt} \omega_{0mn} \partial_r B_{pq} \right]$$  \hspace{1cm} (16)

where $J = -(\alpha'/4) \text{Tr} \left( -\frac{1}{4} F^{ab} F_{ab} - 2 \lambda \Gamma^a D_a \lambda + Y_{i\bar{j}} Y_{\bar{i}j} \right)$ and $H^*$ entering $\mathcal{H}$ now does contain the Yang-Mills Chern-Simons contribution. The condition of half-supersymmetry gives the saturation condition $\tilde{H} = |\tilde{P}|$, where, for our solution in the static limit, $|\tilde{P}| = (1/4) \int (\partial \sigma)^2$. Using the equation of motion for $\sigma$, we then find $\mathcal{H} = 0$ for our solution in
the static limit. Of course, the Hamiltonian (16) treats the gauge fields as a background. One cannot simply add the Yang-Mills Hamiltonian to this since the equations of motion for the latter are unaffected by the tensor multiplet. Nevertheless, the fact that $\mathcal{H} = 0$ is indicative of a possible connection to the tensionless string, for which we shall give more evidence in the next section.

5. String Interpretation.

To see the stringy interpretation of our solution, we need to analyze its moduli or zero mode structure. From the above equations, we see that, given the gauge field $F_{\mu\nu}$, all the fields are uniquely determined up to the addition of the freely propagating six-dimensional waves for the tensor multiplet. Therefore only the zero modes correspond to the moduli of the instantons.

In order for our models to be mathematically meaningful they should be free from local and global gauge anomalies. In the absence of hypermatter, the gauge groups $SU(2)$, $SU(3)$, $G_2$, $F_4$, $E_6$, $E_7$ and $E_8$ can be made perturbatively anomaly-free with the help of the Green-Schwarz prescription [1]. However, since the homotopy group $\Pi_6$ of the first three groups in this list are nontrivial, these theories will harbour global gauge anomalies [6]. To make them consistent we need to introduce hypermatter for these theories [7]. The allowed matter contents for the cancellation of the global [7] as well as the local [8] anomalies in the presence of one tensor multiplet are $n_2 = 4, 10$ for $SU(2)$, $n_3 = 0, 6, 12$ for $SU(3)$ and $n_7 = 1, 4, 7$ for $G_2$, where $n_2, n_3$ and $n_7$ represent the number of the doublets for $SU(2)$, triplets for $SU(3)$ and 7-dimensional representation of $G_7$, respectively. All other gauge groups are free from global anomalies and they can be made free from perturbative anomalies (using the Green-Schwarz prescription) if appropriate amount of hypermultiplets are taken together with the gauge and the tensor multiplets [7, 8].

For the gauge group $SU(2)$, for the four-dimensional space being $\mathbb{R}^4$ and for instanton number $k$, we have $8k$ bosonic moduli corresponding to the instanton positions, scale sizes and group orientations. (The equations of motion, despite the appearance of the

* Note the soliton does not modify their propagation.
dimensional parameter $c$, have scale invariance and give the scale size parameter in the solutions.) These moduli appear in the solution for the fields $B_{ab}$ as well.

The surviving supersymmetry has $\gamma_5 \varepsilon = \varepsilon$, i.e., left-chirality in the four-dimensional sense corresponding to a $(4,0)$ world-sheet supersymmetry for the solitonic string. There must necessarily be fermionic zero modes. For the gauginos, we have $4k$ zero modes for the gauge group $SU(2)$, which are of right-chirality in the four-dimensional sense and are in the right-moving sector. The Dirac equation for the gauginos along with the half-supersymmetry condition shows that the gaugino zero mode parameters are constants; the bosonic parameters are constant as well, by supersymmetry. The fermionic zero mode parameters are complex, i.e., we have $8k$ real Grassman parameters which balance the $8k$ bosonic parameters. Some of the fermionic zero modes correspond to the supersymmetries which are broken by the background and can be obtained by such supersymmetry variations. With hypermatter, there are also hyperino zero modes, which are in the left-moving sector. There is no supersymmetry for these modes and generically there are no hyperscalar zero modes.

For higher gauge groups, there will be more moduli. Thus, for example, for $SU(3)$, with the standard embedding of the instanton and $n_3 = 0$, we have $12k$ bosonic parameters and $6k$ fermionic parameters. It is easy to see that the number of moduli for all of the anomaly-free gauge groups listed above is always a multiple of 4. We may thus interpret these solutions as six-dimensional strings with 4 tranverse coordinates corresponding to the zero modes for the broken translational symmetries. The remaining zero modes can be regarded as additional world-sheet degrees of freedom. In this way for instanton number $k$, we have $k$ strings with $(4,0)$ world-sheet supersymmetry.

As an example, consider an $SU(2)$ theory with 10 hypermatter doublets [9]. In this case, for instanton number equal to one, we have eight instanton moduli, eight gaugino zero modes for the right-moving sector and 20 hypermatter zero modes for the left-moving sector. The $SU(2)$ symmetry can be spontaneously broken by vacuum expectation values of the scalars originating from the moduli corresponding to the global $SU(2)$ rotations and
the scale size of the instanton. By supersymmetry this should remove four of the gaugino zero modes from the right moving sector by giving them a non zero mass, which will also eat up four hyperino zero modes in the left moving sector. One is left with four moduli for the instanton, four gaugino modes in the right-moving sector and 16 hyperino zero modes in the left-moving sector. These 16 hyperino zero modes presumably generate a left moving $E_8$ current algebra. This looks like the spectrum of the non critical string which lives in the boundary of a membrane joining a 5-brane to a 9-brane in $M$-theory and which becomes tensionless as the 5-brane approaches the 9-brane \[10\]. It has been argued in \[11\] that the same model corresponds to one of the phases of the $F$-theory.

There are also independent solutions with the opposite chirality. The choice $\gamma_5 \epsilon = -\epsilon$ leads to antiselfdual $H_{+\mu\nu}$, $F_{\mu\nu}$ with $A_+ = 0$ and $\partial_+ \xi = 0$.

The solution we have obtained is a static one. The choice of four-dimensional chirality as $\gamma_5 \epsilon = \pm \epsilon$ leads to static solitons. By Lorentz boosts, it is possible to obtain a solution whose center of mass is moving at a constant velocity. For a moving soliton, the condition $\gamma_5 \epsilon = \pm \epsilon$ must be modified. Consider, for example, the one-soliton (one-instanton) solution. We choose the supersymmetry parameters $\epsilon$ as $S \epsilon(0)$ where $S = \exp(-\frac{1}{2} \omega^\mu \gamma_\mu) \approx 1 - \frac{1}{2} \omega^\mu \gamma_\mu$ and $\epsilon(0)$ obeys $\gamma_5 \epsilon(0) = \epsilon(0)$. (For small velocities, the parameter $\omega^\mu \approx v^\mu$, the velocity.) The vanishing of the gaugino variation, viz., $\Gamma^{ab} F_{ab\epsilon} = 0$, now gives, to first order in $v^\mu$,

\[ F_{\mu\nu} - \tilde{F}_{\mu\nu} = 0 \]
\[ F_{-\mu} + \frac{1}{\sqrt{2}} F_{\mu\nu} v^\nu = 0 \]
\[ F_{+-} - \frac{1}{\sqrt{2}} F_{+\nu} v^\nu = 0 \]

To this order, $F_{\mu\nu}$ is still selfdual. The other two equations are seen to be satisfied if we take the instanton position $a^\alpha$ to move with velocity $v^\alpha$, i.e., $\partial_0 a^\alpha = v^\alpha$. (We can make a gauge transformation $A_- \to A_- - (1/\sqrt{2}) A_\mu v^\mu$ to restore the $A_- = 0$ gauge.) There is a similar set of statements for the vanishing of the tensorino variation. What we have shown is that a soliton whose center of mass is moving at a constant velocity $v^\alpha$ is also a
supersymmetric solution with supersymmetry parameters being $S\varepsilon(0), \varepsilon(0)$ having definite four-dimensional chirality.

After completion of this paper there appeared ref. [12] which overlaps with our section 3.

Acknowledgments. We are grateful to E. Gava, K.S. Narain, E. Sezgin and G. Thompson for useful discussions. We are particularly indebted to C. Vafa for many helpful discussions, for drawing our attention to references 7 and 8 and his criticisms of our earlier treatment of the zero modes. VPN also thanks J.T. Liu for useful comments.

The work of VPN was supported in part by NSF Grant Number PHY-9322591.

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