Fully Quantum Approach to Optomechanical Entanglement

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The radiation pressure induced coupling between an optical cavity field and a mechanical oscillator can create entanglement between them. In previous works this entanglement was treated as that of the quantum fluctuations of the cavity and mechanical modes around their classical mean values. This approach is limited to certain types of cavity drive. Here we provide a fully quantum approach to optomechanical entanglement, which is applicable to arbitrary cavity drive. We find the real-time evolution of optomechanical entanglement under continuous-wave drive of arbitrary detuning and highlight quantum noise effects that can cause entanglement sudden death and revival.

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The study of optomechanical systems (OMS) has undergone rapid development over the recent years. The quantum level of OMS has been reached in experiments. Entanglement is a particularly striking quantum feature. The coupling of the cavity field of an OMS to the mechanical oscillator under radiation pressure can lead to their entanglement. This mesoscopic or macroscopic entanglement possesses both fundamental interest and potential applications.

Theoretically an OMS is often approached via the expansion of its fluctuations about the mean values of the cavity and mechanical mode operators, where these mean values are determined by the classical equations of motion. This approximation of replacing a quantum system operator with the sum of a classical value and the accompanying quantum fluctuation has been widely applied to generic nonlinear quantum systems whose Heisenberg-Langevin equations are not analytically solvable. Most previous studies of optomechanical entanglement (see, e.g., ) concern the stationary entanglement of the fluctuations around the steady state solution of the classical Langevin equations under continuous-wave (CW) drive. Some other work has also considered the entanglement under periodic drive or pulsed drive. A common feature of these treatments is that the linearized dynamics about the fluctuations is based on the specific classical mean motion as the background. However, the classical motion of a nonlinear OMS can be chaotic. Under more general conditions it is therefore impossible to quantify the entanglement in this framework. On the other hand, the genuine entanglement between the cavity field and the mechanical oscillator themselves (rather than their fluctuations) is a fully quantum property independent of any classical background.

Very recently several quantum features of OMS have been studied in considerable detail. This research includes OMS dynamics under single photon drive, control and generation of OMS quantum states, enhancement of OMS nonlinearity for quantum information processing, and other quantum properties of OMS. These studies consider the quantum states associated with the cavity mode \( \hat{a} \) and mechanical mode \( \hat{b} \) themselves instead of those for their fluctuations. The previously unexplored entanglement of such quantum states is the theme to be discussed below.

We consider an OMS driven by a pulsed drive with the central frequency \( \omega_0 \) and arbitrary frequency distribution \( E(\omega - \omega_0) \). Its profile \( E(t)e^{i\omega_0 t} \) in time domain is related to \( E(\omega - \omega_0) \) by the Fourier transform. The drive reduces to a CW one when \( E(t) \) is constant. Without cavity and mechanical damping, one has the unitary evolution operator \( \hat{U}(t, 0) = \exp(-iH_0 t)T \exp(-i \int_0^t d\tau H_S(\tau)) \) for the OMS, where \( H_0 = \omega_c \hat{a}^\dagger \hat{a} + \omega_m \hat{b}^\dagger \hat{b} \) (\( \hbar = 1 \)) describes the cavity and mechanical oscillation with their frequency \( \omega_c \) and \( \omega_m \), respectively, and

\[
\begin{align*}
H_S(t) &= -\sqrt{2}g\left\{ \cos(\omega_m t) \dot{x}_m + \sin(\omega_m t) \dot{p}_m \right\} \hat{a}^\dagger \hat{a} \\
&+ iE(t) \hat{a}^\dagger e^{i\Delta_0 t} - \hat{a} e^{-i\Delta_0 t}
\end{align*}
\]

inside the time-ordered exponential is the system Hamiltonian in the interaction picture with respect to \( H_0 \). In the above equation, \( g \) is the optomechanical coupling constant, and \( \Delta_0 = \omega_c - \omega_0 \) is the detuning of the drive’s central frequency from the cavity frequency. The dimensionless mechanical coordinate operator and mechanical momentum operator are defined as \( \dot{x}_m = (\hat{b} + \hat{b}^\dagger)/\sqrt{2} \) and \( \dot{p}_m = -i(\hat{b} - \hat{b}^\dagger)/\sqrt{2} \), respectively. The cavity (mechanical) damping at the rate \( \kappa \) (\( \gamma_m \)) can be described in terms of a linear coupling between the cavity (mechanical) mode with the stochastic Langevin noise operator \( \xi_c \) (\( \xi_m \)) of the reservoir:

\[
H_D(t) = i\sqrt{\kappa} \hat{a}^\dagger \xi_c(t) + \sqrt{\gamma_m b} \hat{c}_m(t) + H.c.
\]

The zero temperature cavity reservoir has the correlation \( \langle \xi_c(t) \xi_c^\dagger(\tau) \rangle_R = \delta(t - \tau) \) for its noise operator, while the noise operator of the mechanical reservoir at the temperature \( T \) corresponding to the thermal phonon number \( n_m = 1/(e^{\omega_m/(k_B T)} - 1) \) (\( k_B \) is the Boltzmann constant) satisfies the relation \( \langle \xi_m(t) \xi_m^\dagger(\tau) \rangle_R = \ldots \)
displacement term for $\hat{A}$ describes the evolution solely from the cavity drive, with the optomechanical coupling Hamiltonian and the cavity interaction picture to construct the evolution operator $U_S(t, 0) = T e^{-i \int_0^t d\tau H_S(\tau)}$ for the combination of the OMS and its associated reservoirs (its momentumary action $U_S(t + dt, t)$ gives the exact Langevin equation and master equation of the OMS) [13].

The development of the entanglement between the cavity and mechanical mode is closely connected to the dynamical evolution of these modes. Their evolution under $U_S(t, 0)$ involves three non-commutative processes—cavity drive, optomechanical coupling and dissipation, so it is impossible to solve the system dynamics directly from this joint evolution operator. Our method to reduce the intricacy is factorizing it into numerous ones corresponding to relatively tractable processes [27]. Here we apply the technique to find a factorization that is suitable to study the dynamically evolving Gaussian states. Our factorization is obtained as $U_S(t, 0) = U_E(t, 0)U_{OM}(t, 0)U_K(t, 0)U_D(t, 0)$ [13]. The first operator

$$U_E(t, 0) = T \exp\{-i \int_0^t d\tau E(\tau) e^{i\Delta_0\tau} \hat{A}^\dagger(t, \tau) - H.c.\}$$

(3) describes the evolution solely from the cavity drive, with $\hat{A}(t, \tau) = e^{-\frac{i}{2} (t-\tau) } \hat{a} + \hat{n}_c(t, \tau)$ being the sum of the decayed cavity mode operator and the colored cavity noise operator $\hat{n}_c(t, \tau) = \sqrt{\kappa} \int_0^\tau d\tau' e^{-\kappa (\tau-\tau')} / 2 \xi_c(\tau')$. The second and third operator involving optomechanical coupling are respectively given as

$$U_{OM}(t, 0) = T \exp\{ig \int_0^t d\tau \left[ \hat{K}_m(t, \tau) - g \hat{N}(t, \tau) \right] \times \left( D^\dagger(\tau) \hat{P}(\tau) \hat{A}(t, \tau) + D(\tau) \hat{A}^\dagger(t, \tau) \hat{P}(\tau) + |D(\tau)|^2 \right) \},$$

(4)

and $U_K(t, 0) = T \exp\{ig \int_0^t d\tau \hat{K}_m(t, \tau) \hat{A}^\dagger(t, \tau) \}$, where $\hat{K}_m(t, \tau) = \cos(\omega_m \tau) \hat{X}_m(t, \tau) + \sin(\omega_m \tau) \hat{P}_m(t, \tau)$ is a linear combination of the mechanical operators $\hat{X}_m(t, \tau) = \hat{B}(t, \tau) + \hat{B}^\dagger(t, \tau)$ and $\hat{P}_m(t, \tau) = -i \hat{B}(t, \tau) + i \hat{B}^\dagger(t, \tau)$ from $\hat{B}(t, \tau) = e^{-\frac{i}{2} (t-\tau) } \hat{b} + \hat{n}_m(t, \tau)$ and $\hat{n}_m(t, \tau) = \sqrt{\kappa_m} \int_0^\tau d\tau' e^{-\gamma_m (\tau-\tau')} / 2 \xi_m(\tau')$, and the term

$$D(\tau) = e^{-\frac{i}{2} (t-\tau) } \int_0^\tau dt' E(t') e^{i\Delta_0 t'} e^{-\frac{i}{2} (t-t')} + \int_0^\tau dt' \left[ \hat{n}_c(t, t'), \hat{n}_c^\dagger(t, \tau) \right] E(t') e^{i\Delta_0 t'},$$

(5)

in $U_{OM}(t, 0)$ arises from the non-commutativity between the optomechanical coupling Hamiltonian and the cavity drive Hamiltonian proportional to $E(t)$. In [13] the displacement term for $\hat{K}_m(t, \tau)$ is defined as $\hat{N}(t, \tau) = 2 \int_0^\tau dt' e^{-\gamma_m (\tau-\tau')} / 2 \sin(\omega_m (\tau-\tau')) \hat{A}^\dagger(t, \tau') \hat{A}(t, \tau')$, and $\hat{P}(\tau) = T e^{ig \int_0^\tau dt' e^{-\gamma_m (\tau-\tau')} / 2 U_K(t, \tau) K_m(t, \tau) \hat{U}_K(t, \tau)}$ is a non-Abelian phase factor. Meanwhile, the last operator $U_D(t, 0) = T \exp\{-i \int_0^t d\tau H_D(\tau)\}$ in the factorization is only concerned with the dissipation process. To the first order of the optomechanical coupling constant $g$, the Hamiltonian in $U_{OM}(t, 0)$ becomes

$$H_{OM}(\tau) = g \hat{K}_m(t, \tau) \left( \hat{A}^\dagger(t, \tau) D(\tau) + \hat{A}(t, \tau) D^\dagger(\tau) + |D(\tau)|^2 \right).$$

(6)

Under a CW drive this Hamiltonian takes a similar form to the linearized one from the fluctuation expansion approach [1, 3] but with the different operators (their fluctuation $\delta \hat{a}$ is replaced by our $\hat{A}(t, \tau)$, and the mechanical operator $\hat{b}$ there is replaced by $\hat{B}(t, \tau)$). Such difference, however, leads to contrasting quantum noise effect.

Next we start with an initial OMS state $\rho(0)$ in thermal equilibrium with the environment, i.e. $\rho(0)$ is a Gaussian state as the product of a cavity vacuum and a finite temperature mechanical thermal state. This initial state becomes entangled under optomechanical coupling. Its evolution can be studied by successively acting each factor in the factorized form $U_E(t, 0)U_{OM}(t, 0)U_K(t, 0)U_D(t, 0)$ of the joint evolution operator $U_S(t, 0)$ on the total initial state $\chi(0) = \rho(0)\rho(0)$, in which $\rho(0)$ denotes the reservoir state in thermal equilibrium with $\rho(0)$. One has $U_D(t, 0)\chi(0)U_D^\dagger(t, 0) = \chi(0)$ since, under thermal equilibrium, the system-reservoir coupling in (2) does not change the state $\rho(0)$ [43], and $U_K(t, 0)$ always keeps $\chi(0)$ invariant because $\hat{A}(t, \tau) |0\rangle_C = 0$ for the combined initial vacuum state $|0\rangle_C$ of the cavity and its reservoir. Thus the expectation values of system operators $\hat{O}(t)$ reduce to the following trace over system and reservoir degrees of freedom:

$$\langle \hat{O}(t) \rangle = Tr_{S,R}\{U_{OM}^\dagger(t, 0)U_E^\dagger(t, 0)\hat{O}U_E(t, 0)U_{OM}(t, 0) \times \chi(0) \}. \quad \text{(7)}$$

In weak coupling regime where the Hamiltonian of $U_{OM}(t, 0)$ takes the form in [10], the evolved OMS is preserved to be in Gaussian state [44]. The higher order terms of $g$ relevant to OMS of stronger coupling deviate the evolved system out of Gaussian state, and their effect can also be discussed [27].

The reason for us to focus on the evolved Gaussian states is that their entanglement can be quantified by the logarithmic negativity $E_N$ [43]. One should consider the correlation matrix (CM) with the elements $V_{ij}(t) = 1/2 \langle \hat{u}_i(t) \hat{u}_j(t) + \hat{u}_i^\dagger(t) \hat{u}_j^\dagger(t) - (\hat{u}_i^\dagger)(\hat{u}_j) \rangle$, where $\hat{u}_i = (\hat{x}_i(t), \hat{p}_i(t), \hat{x}_m(t), \hat{p}_m(t))^T$. For the calculation of $E_N$ [43]. Each entry of the CM can be calculated following (7) with $\hat{O} = \hat{u}_i \hat{u}_j + \hat{u}_j \hat{u}_i$, etc. The contribution to the matrix elements is from the optomechanical coupling in [13], which evolves the cavity and mechanical mode in
drives. The entanglement values measured by
is the entanglement evolution under blue detuned CW
different detuning. The first example we present in Fig. 1
solution of OMS entanglement with the CW drives of dif-
ferent detuning. The first example we present in Fig. 1
is obtained with the drive intensity \( E_\kappa = 3 \times 10^5 \)
while (b) is for a drive of \( E_\kappa = 2 \times 10^6 \). The long dashed
(red) curve is for \( \Delta_0 = -0.5\omega_m \), the short dashed (blue) curve
for \( \Delta_0 = -\omega_m \), the thin solid (black) curve for \( \Delta_0 = -1.5\omega_m \),
and the thick solid (purple) curve for \( \Delta_0 = -2\omega_m \). Here \( g/\kappa = 10^{-6} \), \( \omega_m/\kappa = 2.5 \), \( \omega_m/\gamma_m = 10^7 \), and \( T = 0 \).

The entanglement measure by \( E_N \) for these blue detuned drives
tends to a stable value with time. The maximum entangle-
ment is reached at the SQ resonant point \( \Delta_0 = -\omega_m \). For
the stronger drive, the entanglement at the smaller detuning
\( \Delta_0 = -0.5\omega_m \) dies a sudden death after a finite time period.

terms of the following differential equations:

\[
\frac{i}{\hbar} \frac{d\hat{a}}{dt} = g e^{-i(\kappa+\gamma_m)(t-\tau)/2} D(\tau)(e^{-i\omega_m \tau} \hat{b} + e^{i\omega_m \tau} \hat{b}^\dagger)
+ g e^{-i(\kappa+\gamma_m)(t-\tau)/2} D(\tau) \cos(\omega_m \tau)(\hat{n}_m(t, \tau) + \hat{n}_m^\dagger(t, \tau))
+ g e^{-i(\kappa+\gamma_m)(t-\tau)/2} D(\tau) \sin(\omega_m \tau)(i\hat{n}_m(t, \tau) - i\hat{n}_m^\dagger(t, \tau)),
\]

\[
\frac{i}{\hbar} \frac{d\hat{b}}{dt} = g e^{-i(\kappa+\gamma_m)(t-\tau)/2} D(\tau) \sin(\omega_m \tau)(\hat{n}_m^\dagger(t, \tau) + \hat{n}_m(t, \tau)),
\]

while the operation \( U_E(t, 0) \) on the cavity alone does not
contribute to entanglement. The \( \hat{a} (\hat{b}) \) terms on the right
side of \( \mathfrak{S} \) are due to the beam-splitter (BS) action in the
quadratic Hamiltonian \( \mathfrak{H} \), and the \( \hat{a}^\dagger (\hat{b}^\dagger) \) terms reflect
the coexisting squeezing (SQ) action.

Corresponding to the initial quantum state in ther-
nal equilibrium, the initial CM takes the form \( \hat{V}(0) = \text{diag}(1/2, 1/2, n_m + 1/2, n_m + 1/2) \). The evolved CM ac-

FIG. 1: Evolution of entanglement for blue detuned CW
FIG. 2: Evolution of entanglement for red detuned CW

drives. (a) is obtained with the same system parameters as
in Fig.1(a), while (b) is found with the same parameters as in
Fig.1(b). The thin solid (blue) curve is for \( \Delta_0 = 0.5\omega_m \), and the long dashed (purple) curve for \( \Delta_0 = \omega_m \), the solid (red)
curve for \( \Delta_0 = 1.5\omega_m \), and the short dashed (black) curve
for \( \Delta_0 = 2\omega_m \). The entanglement dies earlier for a detun-
ing closer to the BS resonant point \( \Delta_0 = \omega_m \) or the stronger
drive. Given the stronger drive in (b), the entanglement at
\( \Delta_0 = 1.5\omega_m \) exhibits sudden death and revival.

tivity mode \( \hat{a} \) and mechanical mode \( \hat{b} \) themselves can be
well beyond this limit (see Fig. 1(b)). Compared with
the blue detuned regime, the entanglement of the red det-
unregime shown in Fig. 2 is lower. This reflects the
difference of the BS action from the SQ action in creating
the optomechanical entanglement.

The exact degree of entanglement is determined by
two competitive factors—the direct BS and SQ action
on the initial quantum state \( \rho(0) \) of OMS, and the noise
drives depending on the drive detuning and intensity.

Given a CW drive, the noise drive terms in \( \mathfrak{S} \) are
magnified by the functions with the modulo \( |D(\tau)| = E/\sqrt{0.25\omega_n^2 + \Delta_0^2} \), indicating their more significant effect
at a small detuning \( \Delta_0 \) or with a stronger drive intensity
\( E \). In what follows, we illustrate the noise effects as a
function of time and of different system parameters.

First, the entanglement for some values of detuning
in Fig.1 and 2 will die at a finite time. The phenomenon
that entanglement is killed by noise in this way is known
as entanglement sudden death (ESD) \([46, 47]\). The sys-
tem evolution according to \( \mathfrak{S} \) provides a model in which
the ESD for the continuous variable states is caused by
the colored noises \( i\hat{n}_m(t, \tau), \hat{n}_m(t, \tau) \) and their con-
jugates on the right side of \( \mathfrak{S} \) rather than the white noises in
many other examples (see the reference in \([47]\)). In
this situation the noise effect can be so significant that
this type of ESD happens while the optomechanical coupling
exists all the time. Interestingly, the entanglement under
some drives, e.g. \( \Delta_0 = 1.5\omega_m \) in Fig. 2(b), could also
revive from time to time during evolution. In the ab-

ence of the quantum noise drives there will be no ESD
phenomenon, c.f. Fig. (C-1) of Supplementary Material.

Fig. 3 shows the magnitude of the noise correction
to optomechanical entanglement in the system parameter
space. Given the same drive intensities, the relations
between the entanglement and drive detuning after suffi-
ciently long interaction time are shown in Fig.3(a)-3(b).

With the increase of cavity drive intensity, the entangle-
ment in a more extended detuning range around the BS
FIG. 3: (a)-(b): Entanglement versus detuning. (a) is under the same conditions as in Fig.1(a) and 2(a), and (b) corresponds to the situation in Fig. 1(b) and 2(b). The long dashed (blue) curves show the entanglement obtained at zero temperature without the noise drive terms in $\hat{H}_N$, while the thick solid (red) curves include the effect of the noise drive terms at zero temperature. In (a) the zero temperature entanglement is eliminated around $\Delta_0 = \omega_m$. The thin solid (purple) curves in (a) and (b) give the exact degree of entanglement at the temperature corresponding to $\kappa T = 10^5$. (c)-(d): Entanglement versus cavity drive intensity. In (c), the long-dashed curve (blue) stands for $\Delta_0 = -\omega_m$, and the short-dashed curve (red) for $\Delta_0 = -0.5\omega_m$, and the solid (purple) curve for $\Delta_0 = -2\omega_m$. In (d), the long-dashed curve (green) stand for $\Delta_0 = 0$, the short-dashed (purple) curve for $\Delta_0 = \omega_m$, and the solid (blue) curve for $\Delta_0 = 0.5\omega_m$. Except for the thin solid (purple) curves about finite temperature entanglement in (a) and (b), all plots are found for the initial temperature $T = 0$ at the moment $\kappa t = 15$, when the concerned entanglement has stabilized.

resonant point $\Delta_0 = \omega_m$ will be eliminated by the quantum noises. The overall tendency of the entanglement change with the drive intensity for various drive detuning values is described in Fig. 3(c)-3(d). The plots in these figures show a competition between the effective coupling $gD(t)$ and the noise drives (see the respective terms in $\hat{H}_N$) in affecting the degree of entanglement. The entanglement reaches the maximum at a certain drive intensity $E$ determined by the system parameters, instead of monotonically increasing with $E$ which enhances the effective optomechanical coupling. The cavity noise associated with the operator $\hat{n}$ dominates in our calculation. The mechanical noise effect of $\hat{n}_m$, which is more obvious at high temperature, becomes less significant with increasing quality factor $Q = \omega_m/\gamma_m$. At low temperature the entanglement is insensitive to $Q$. Despite the existence of the noises, the entanglement in the blue detuned regime can be high. The SQ generated entanglement is also rather robust against temperature; see the comparison in Fig. 3(a) and 3(b).

We are concerned with the regime of strong drive ($E/\kappa \gg 1$) and weak optomechanical coupling ($g/\kappa \ll 1$) in the study of Gaussian state entanglement for OMS.

Starting from our initial OMS quantum state (the cavity in a vacuum state and the mechanical oscillator in a thermal state), such entanglement for the evolved quantum state develops as the optomechanical coupling starts with the cavity field being built up by an external drive. Meanwhile, the generated entanglement is also weakened or even destroyed by the noise drives. The entanglement in the same regime was well studied in the fluctuation expansion approach [11–18]. As we mentioned at the beginning, this previously adopted approach works with approximating the OMS operators with the sum of their mean values following classical dynamics without noise drives and the fluctuations evolving according to quantum mechanics. Then the system operators in the system-reservoir coupling of (2) are replaced by their fluctuations, so that only delta-function correlated Langevin noises $\xi$ and $\xi_m$, independent of cavity drive detuning and intensity are relevant to the linearized dynamics about the fluctuations and their entanglement. Instead, in our fully quantum approach, the linearized dynamics for the system operators in weak coupling regime (see Eqs. (6) and (5)) involves the magnified noise drives due to the cubic term $-g(\hat{b} + b^\dagger)\hat{a}\hat{a}\hat{a}$ of the original OMS Hamiltonian. The difference of the quantum noise effects is expected to be experimentally tested by the measurement of cavity fluctuation amplitude. As illustrated in Fig. 4, the cavity fluctuations found in the different approaches drastically deviate with drive intensity. Such phenomenon also indicates the distinct quantum states due to the different linearized dynamics. Our concerned OMS quantum states and those of the fluctuations around classical steady states can be seen to be different from their CMs, which are in one-to-one correspondence to the respective Gaussian states. The entanglement for the different quantum states obtained in the different approaches is therefore not the same.

In conclusion, we have studied the dynamically generated entanglement of quantum OMSs that are initially
in thermal equilibrium with their environment. The approach we present allows a complete description of the entanglement in terms of its real-time evolution under arbitrary cavity drive. Here we have shown the entanglement evolution patterns for CW drives of arbitrary detuning. High and robust entanglement can be achieved with blue detuned drives. Our fully quantum dynamical approach also predicts a non-trivial quantum noise effect on the optomechanical entanglement. In the regime where the effect is significant, complicated evolution patterns such as entanglement sudden death and revival exist for this macroscopic entanglement. Such quantum noise effect can also exist in other quantum nonlinear systems.

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Note added.—During the revision of the present work, we noticed another study of OMS entanglement by Paternostro et al. (2009). This work was supported by AITF and NSERC. Q. L. thanks M. Hillery for helpful conversations.
A. Factorization of Joint System-Reservoir Evolution Operator

Our discussion is based on the two following factorizations for a unitary evolution operator $U(t,0) = T \exp\{-i \int_0^t dt' \{H_1(t') + H_2(t')\}\}$ involving two processes described by $H_1(t)$ and $H_2(t)$, respectively:

$$T e^{-i \int_0^t d\tau (H_1(\tau) + H_2(\tau))} = T e^{-i \int_0^t d\tau H_1(\tau)} T e^{-i \int_0^t d\tau V_1^\dagger(\tau,0)H_2(\tau)V_1(\tau,0)},$$  \hspace{1cm} (A-1)

and

$$T e^{-i \int_0^t d\tau (H_1(\tau) + H_2(\tau))} = T e^{-i \int_0^t d\tau V_2(t,\tau)H_1(\tau)V_2(\tau,t)} T e^{-i \int_0^t d\tau H_2(\tau)},$$  \hspace{1cm} (A-2)

where $V_k(t,\tau) = T \exp\{-i \int_0^t dt' H_k(t')\}$ for $k = 1, 2$. The operator $U(t,0)$ is the solution to the differential equations $dU/dt = -i (H_1(t) + H_2(t))U(t)$, while $V_1(t,0) = T \exp\{-i \int_0^t d\tau H_1(\tau)\}$ is the solution to $dV_1/dt = -i H_1(t)V_1(t)$. The initial condition for the differential equations is $U(0,0) = V_1(0,0) = I$, the identity operator. The differential of $W(t,0) = V_1^\dagger(t,0)U(t,0)$ with respect to $t$ gives

$$\frac{dW}{dt} = -V_1 \frac{dV_1}{dt} V_1^\dagger U + V_1^\dagger \frac{dU}{dt} = i V_1^\dagger H_1 V_1^\dagger \dot{U} - i V_1^\dagger (H_1 + H_2) \dot{U} = -i V_1^\dagger H_2 V_1 W.$$  \hspace{1cm} (A-3)

One has the solution to the above differential equation as $W(t,0) = T \exp\{-i \int_0^t d\tau V_1^\dagger(\tau,0)H_2(\tau)V_1(\tau,0)\}$, thus proving the factorization in (A-1). By exchanging $H_1(t)$ and $H_2(t)$ in (A-1), one has the factorization of the operator $U(t,0)$ as

$$V_2(t,0) T e^{-i \int_0^t d\tau V_2^\dagger(\tau,0)H_1(\tau)V_2(\tau,0)} = V_2(t,0) T e^{-i \int_0^t d\tau V_1^\dagger(\tau,0)H_2(\tau)V_2(\tau,0)} V_2^\dagger(t,0)V_2(0).$$

Because $V_2(t,0)$ is a unitary operation, one can rewrite the right side of the above as $T e^{-i \int_0^t d\tau V_2^\dagger(\tau,0)H_1(\tau)V_2(\tau,0)} V_2(t,0)$, giving the form in Eq. (A-2). Here we have used the relation $V_2(0,0) V_2^\dagger(0,0) = V_2(t,\tau)$.

We first apply Eq. (A-2) to factorize $U_D(t,0) = T \exp\{-i \int_0^t d\tau H_D(\tau)\}$ out of the system-reservoir evolution operator $U_S(t,0) = T \exp\{-i \int_0^t d\tau \{H_S(\tau) + H_D(\tau)\}\}$, where $H_S(\tau)$ and $H_D(\tau)$ are given in Eqs. (1) and (2) of the main text, respectively. In this way one has

$$U_S(t,0) = T \exp\{-i \int_0^t d\tau U_D(t,\tau)H_S(\tau)U_D^\dagger(t,\tau)\} T \exp\{-i \int_0^t d\tau H_D(\tau)\}.$$  \hspace{1cm} (A-4)

The cavity mode operator $\hat{a}$ in $H_S(\tau)$ is transformed to

$$U_D(t,\tau)\hat{a} U_D^\dagger(t,\tau) = e^{-\frac{1}{2} (t-\tau)} \hat{a} + \hat{n}_c(\tau,\tau) \equiv \hat{A}(t,\tau)$$  \hspace{1cm} (A-5)

in $U_D(t,\tau)H_S(\tau)U_D^\dagger(t,\tau)$, and the mechanical mode operator is transformed to

$$U_D(t,\tau)\hat{b} U_D^\dagger(t,\tau) = e^{-\frac{1}{2} \gamma (t-\tau)} \hat{b} + \hat{n}_m(\tau,\tau) \equiv \hat{B}(t,\tau),$$  \hspace{1cm} (A-6)

where $\hat{n}_c(\tau,\tau) = \sqrt{\kappa} \int_\tau^t d\tau' e^{-\kappa (\tau'-\tau)/2} \xi_c(\tau')$ and $\hat{n}_m(\tau,\tau) = \sqrt{\gamma m} \int_\tau^t d\tau' e^{-\gamma (\tau'-\tau)/2} \xi_m(\tau')$. The transformed operators satisfy the equal-time commutation relation $[\hat{A}(t,\tau), \hat{A}^\dagger(t,\tau)] = [\hat{B}(t,\tau), \hat{B}^\dagger(t,\tau)] = 1$. Then the Hamiltonian in the first time-ordered exponential of (A-4) becomes

$$U_D(t,\tau)H_S(\tau)U_D^\dagger(t,\tau) = (i E(t) \hat{A}^\dagger(t,\tau) e^{i \Delta_0 t} - i E^*(t) \hat{A}(t,\tau) e^{-i \Delta_0 t}) - g \hat{K}_m(t,\tau) \hat{A}^\dagger \hat{A}(t,\tau),$$  \hspace{1cm} (A-7)

where

$$\hat{K}_m(t,\tau) = \cos(\omega_m \tau) (\hat{B}_m(t,\tau) + \hat{B}_m^\dagger(t,\tau)) + \sin(\omega_m \tau) (-i \hat{B}_m(t,\tau) + i \hat{B}_m^\dagger(t,\tau)).$$
By using (A-1) we factorize out the drive Hamiltonian in (A-7) as follows:

\[
T \exp \{-i \int_0^t d\tau U_D(t, \tau) H_S(\tau) U_D^\dagger(t, \tau) \}
= T \exp \{-i \int_0^t d\tau (iE(t)\hat{A}^\dagger(t, \tau)e^{i\Delta_0 t} - iE^*(t)\hat{A}(t, \tau)e^{-i\Delta_0 t}) \}
T \exp \{ig \int_0^t d\tau U_E^\dagger(\tau, 0) \hat{K}_m(t, \tau)\hat{A}^\dagger(\tau, \tau) U_E(\tau, 0) \}
= T \exp \{-i \int_0^t d\tau (iE(t)\hat{A}^\dagger(t, \tau)e^{i\Delta_0 t} - iE^*(t)\hat{A}(t, \tau)e^{-i\Delta_0 t}) \}
\times T \exp \{ig \int_0^t d\tau \hat{K}_m(t, \tau)(\hat{A}^\dagger(t, \tau) + D^*(\tau))(\hat{A}(t, \tau) + D(\tau)) \},
\]
(A-8)

where \(U_E(\tau, 0) = T \exp \{\int_0^\tau dt' E(t')e^{i\Delta_0 t'} \hat{A}^\dagger(t, t') - H.c. \} \). In (A-8) the effect of \(U_E(\tau, 0) \) on the cavity operator \(\hat{A}(t, \tau) \) is the displacement

\[
U_E^\dagger(\tau, 0)\hat{A}(t, \tau)U_E(\tau, 0) = \hat{A}(t, \tau) + e^{-\frac{\pi}{2}(t-\tau)} \int_0^\tau dt' E(t')e^{i\Delta_0 t'}e^{-\frac{\pi}{2}(t-t')} + \int_0^\tau dt' \Gamma_c(t', \tau)E(t')e^{i\Delta_0 t'}
\equiv \hat{A}(t, \tau) + D(\tau),
\]
(A-9)

where

\[
\Gamma_c(t', \tau) = [\hat{n}_c(t', t') - \hat{n}_c(t, t')] = e^{-\kappa(t-\tau)/2} - e^{-\kappa(t-\tau)/2}e^{-\kappa(t-t')/2}.
\]

The next step is to factorize the second time-ordered exponential in (A-8) as follows:

\[
T \exp \{ig \int_0^t d\tau \hat{K}_m(t, \tau)(\hat{A}^\dagger(t, \tau) + D^*(\tau))(\hat{A}(t, \tau) + D(\tau)) \}
= T \exp \{igU_K(t, \tau)(\hat{K}_m(t, \tau)(\hat{A}^\dagger(t, \tau)D(\tau) + \hat{A}(t, \tau)D^*(\tau) + |D(\tau)|^2)U_K^\dagger(t, \tau) \}
\times T \exp \{ig \int_0^t d\tau \hat{K}_m(t, \tau)\hat{A}(t, \tau) \},
\]
(A-10)

where \(U_K(t, \tau) = T \exp \{ig \int_\tau^t dt' \hat{K}_m(t, t')\hat{A}^\dagger(t, t') \}. \) The operation \(U_K(t, \tau) \) in the first time-ordered exponential of the above equation is to displace the mechanical operator

\[
U_K(t, \tau)\hat{K}_m(t, \tau)U_K^\dagger(t, \tau) = \hat{K}_m(t, \tau) - 2g \int_\tau^t d\tau' e^{-\gamma_m(\tau-\tau')/2} \sin\omega_m(\tau - \tau')\hat{A}^\dagger(t, \tau')\hat{A}(t, \tau')
\equiv \hat{K}_m(t, \tau) - g\hat{N}(t, \tau),
\]
(A-11)

and add a non-Abelian phase factor to the operator \(\hat{A}(t, \tau) \):

\[
U_K(t, \tau)\hat{A}(t, \tau)U_K^\dagger(t, \tau) = \left\{ T \exp \{-ig \int_\tau^t dt' e^{-\kappa(t-\tau')/2}U_K(t, t')\hat{K}_m(t, t')U_K^\dagger(t, t') \} \right\} \hat{A}(t, \tau) \equiv \hat{P}(\tau)\hat{A}(t, \tau),
\]
(A-12)

giving the effective optomechanical coupling Hamiltonian

\[
H_{OM}(\tau) = -g(\hat{K}_m(t, \tau) - g\hat{N}(t, \tau))(D^*(\tau)\hat{P}(\tau)\hat{A}(t, \tau) + D(\tau)\hat{A}^\dagger(t, \tau)\hat{P}^\dagger(\tau) + |D(\tau)|^2)
\]
(A-13)

for the first time-ordered exponential in (A-10), which is defined as \(U_{OM}(t, 0) = T \exp \{-i \int_0^t d\tau H_{OM}(\tau) \}. \) Now we have exactly factorized the joint evolution operator as

\[
U_S(t, 0) = U_E(t, 0)U_{OM}(t, 0)U_K(t, 0)U_D(t, 0).
\]
(A-14)
B. Expectation Value of System Operators

We apply the factorization of the joint evolution operator in (A-14) to find the expectation value of a system operator \( \hat{O} \):

\[
\text{Tr}_S \{ \hat{O} \rho(t) \} = \text{Tr}_S \{ \hat{O} \text{Tr}_R \{ U_E(t,0)U_{OM}(t,0)U_K(t,0)U_D(t,0)\rho(0)R(0)U_R(0)U_{OM}^\dagger(t,0)U_K^\dagger(t,0)U_D^\dagger(t,0)U_E^\dagger(t,0) \} \}
\]

(8)

The action \( U_K(t,0)U_D(t,0)\rho(0)R(0)U_R(0)U_K^\dagger(t,0)U_D^\dagger(t,0) \) is on the the product of the initial system state

\[
\rho(0) = |0\rangle_c\langle 0| \otimes \sum_{n=0}^{\infty} \frac{n^n}{(1+n_m)^{n+1}} |n\rangle_m \langle n| \equiv |0\rangle_c\otimes \rho_m
\]

and the associate reservoir state \( R(0) \) in thermal equilibrium with \( \rho(0) \), where \( n_m \) is the thermal phonon number at the temperature \( T \).

We first look at \( U_D(t,0)\chi(0)U_D^\dagger(t,0) \), where \( \chi(0) = \rho(0)R(0) \) and

\[
U_D(t,0) = T \exp \left\{ \int_0^t d\tau \left( \sqrt{\gamma_m} \hat{b} \xi_m(\tau) - \sqrt{\gamma_m} \hat{b}^\dagger \xi_m^*(\tau) \right) \right\} \left( \int_0^t d\tau \left( \sqrt{\gamma_m} \hat{b} \xi_m(\tau) - \sqrt{\gamma_m} \hat{b}^\dagger \xi_m^*(\tau) \right) \right) \chi(0) \right\}
\]

(B-3)

for the product state \( |0\rangle_C \) of the cavity vacuum and its associate vacuum reservoir. If the action of the first operator involving mechanical mode and mechanical reservoir coupling changes the joint initial state \( \chi(0) \), the system state

\[
\hat{\rho}(t) = \text{Tr}_R \{ Te^{\int_0^t d\tau \left( \sqrt{\gamma_m} \hat{b} \xi_m(\tau) - \sqrt{\gamma_m} \hat{b}^\dagger \xi_m^*(\tau) \right) \} \chi(0) Te^{-\int_0^t d\tau \left( \sqrt{\gamma_m} \hat{b} \xi_m(\tau) - \sqrt{\gamma_m} \hat{b}^\dagger \xi_m^*(\tau) \right) } \}
\]

(B-4)

evolved under such coupling will be different from \( \rho(0) \). The system quantum state \( \hat{\rho}(t) \) is the solution to the master equation

in Lindblad form [42]. The initial state for the above master equation is \( \hat{\rho}(0) = \rho_m \), and \( n_{th} \) is the thermal quantum number of the reservoir. Here we assume the possible non-equilibrium between system and reservoir, so that \( n_{th} \) could be different from \( n_m \) (in the main text we only consider the situation of thermal equilibrium). This master equation can be exactly solved by the super-operator technique [3] as

\[
\hat{\rho}(t) = \sum_{n=0}^{\infty} \frac{\gamma(n_{th} + n_m - n_{th}) c^{(n_{th}/2)}}{(1 + n_{th} + n_m - n_{th})^{n+1}} |n\rangle_m \langle n|
\]

(B-5)

If the system and reservoir is in thermal equilibrium, i.e. \( n_{th} = n_m \), the above state will be \( \rho_m \) constantly with time. Under this condition, therefore, the operation \( U_D(t,0) \) keeps the joint initial state \( \chi(0) \) invariant. Moreover, similar to (B-3), one has \( U_K(t,0)\chi(0)U_K^\dagger(t,0) = \chi(0) \). Thus the system operator expectation value in (B-1) will reduce to the form in (7) of the main text.

C. Calculation of Entanglement Measured by Logarithmic Negativity

The entanglement of bipartite Gaussian states is quantified via the correlation matrix

\[
\hat{V} = \left( \begin{array}{cc} \hat{A} & \hat{C} \\ \hat{C}^T & \hat{B} \end{array} \right).
\]

(C-1)

with the elements \( \hat{V}_{ij}(t) = 0.5 \langle \hat{u}_i \hat{u}_j + \hat{u}_j \hat{u}_i \rangle - \langle \hat{u}_i \hat{u}_j \rangle \), where \( \hat{u} = (\hat{x}_c(t), \hat{\rho}_c(t), \hat{x}_m(t), \hat{\rho}_m(t))^T \). The logarithmic negativity as a measure for the entanglement is given as [43]

\[
E_N = \max[0, -\ln 2\eta^-],
\]

(C-2)
where

$$\eta^- = \frac{1}{\sqrt{2}} \sqrt{\Sigma - \sqrt{\Sigma^2 - \det V}}$$

(C-3)

and

$$\Sigma = \det \hat{A} + \det \hat{B} - 2\det \hat{C}.$$  

(C-4)

$U_{E}(t, 0)$ in (B-1) does not contribute to the correlation matrix elements. Given the quadratic Hamiltonian $H_{OM}$ in (6) of the main text, the operation $U_{OM}$ transforms the vector $(\hat{x}_c(t), \hat{\phi}_c(t), \hat{x}_m(t), \hat{\phi}_m(t))^T$ in terms of the following linear differential equation:

$$\frac{d}{dt} \begin{pmatrix} \hat{x}_c \\ \hat{\phi}_c \\ \hat{x}_m \\ \hat{\phi}_m \end{pmatrix} = \begin{pmatrix} 0 & 0 & l_3(t, \tau) & l_4(t, \tau) \\ 0 & 0 & l_1(t, \tau) & l_2(t, \tau) \\ -l_2(t, \tau) & l_4(t, \tau) & 0 & 0 \\ l_1(t, \tau) & -l_3(t, \tau) & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_c \\ \hat{\phi}_c \\ \hat{x}_m \\ \hat{\phi}_m \end{pmatrix} + \begin{pmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \\ \hat{f}_4 \end{pmatrix} = \frac{d}{dt} \hat{v} = \hat{M}(t, \tau) \hat{v} + \hat{f}(t, \tau),$$

(C-5)

where

$$l_1(t, \tau) = ge^{-\kappa(t-\tau)/2-\gamma_1(t-\tau)/2}(D(\tau) + D^*(\tau)) \cos(\omega_m \tau),$$

$$l_2(t, \tau) = ge^{-\kappa(t-\tau)/2-\gamma_1(t-\tau)/2}(D(\tau) + D^*(\tau)) \sin(\omega_m \tau),$$

$$l_3(t, \tau) = i ge^{-\kappa(t-\tau)/2-\gamma_1(t-\tau)/2}(D(\tau) - D^*(\tau)) \cos(\omega_m \tau),$$

$$l_4(t, \tau) = i ge^{-\kappa(t-\tau)/2-\gamma_1(t-\tau)/2}(D(\tau) - D^*(\tau)) \sin(\omega_m \tau),$$

(C-6)

and

$$\hat{f}_1(t, \tau) = \frac{i}{\sqrt{2}} ge^{-\kappa(t-\tau)/2}(D(\tau) - D^*(\tau)) \{ \cos(\omega_m \tau) (\hat{n}_m(t, \tau) + \hat{n}_m^\dagger(t, \tau)) - \sin(\omega_m \tau) (i \hat{n}_m(t, \tau) - i \hat{n}_m^\dagger(t, \tau)) \} ,$$

$$\hat{f}_2(t, \tau) = \frac{1}{\sqrt{2}} ge^{-\kappa(t-\tau)/2}(D(\tau) + D^*(\tau)) \{ \cos(\omega_m \tau) (\hat{n}_m(t, \tau) + \hat{n}_m^\dagger(t, \tau)) - \sin(\omega_m \tau) (i \hat{n}_m(t, \tau) - i \hat{n}_m^\dagger(t, \tau)) \} ,$$

$$\hat{f}_3(t, \tau) = -g (\hat{n}_c(t, \tau) D^*(\tau) + \hat{n}_c^\dagger(t, \tau) D(\tau) + |D(\tau)|^2) e^{-\gamma_1(t-\tau)/2} \sin(\omega_m \tau) ,$$

$$\hat{f}_4(t, \tau) = g (\hat{n}_c(t, \tau) D^*(\tau) + \hat{n}_c^\dagger(t, \tau) D(\tau) + |D(\tau)|^2) e^{-\gamma_1(t-\tau)/2} \cos(\omega_m \tau).$$

(C-7)

In the above the terms containing $\hat{n}_c$, $\hat{n}_m$ and their conjugates contribute to the correlation matrix (C-4), and the pure drive terms proportional to $|D(\tau)|^2$ do not contribute to $\hat{V}$, but they affect the system mean motion $\langle \hat{v}(t) \rangle$. The solution to (C-5) is

$$\hat{v}(t) = T e^{\int_0^t d\tau \hat{M}(t, \tau)} \hat{v}(0) + T e^{\int_0^t d\tau \hat{M}(t, \tau)} \int_0^t d\tau (T e^{\int_0^\tau d\tau' \hat{M}(t, \tau')} )^{-1} \hat{f}(t, \tau).$$

(C-8)

In the general situation the time-ordered exponentials in the solution (C-8) should be expanded to infinite series (Magnus expansion [8]) for numerical calculations. Given a cavity drive with its profile $|E(t)| \leq C$ ($C$ is a constant) such that the function $D(t)$ defined in (A-9) is bounded, the decay factor $e^{-(\kappa(t-\tau)/2)}$ dominates the behavior of the matrix $\hat{M}(t, \tau)$, so one has the approximate commutator $[\hat{M}(t, \tau_1), \hat{M}(t, \tau_2)] \approx 0$ in the concerned regimes in which $gE(t)$ is not very large. Then the time-ordered exponentials in the above solution can be replaced by the ordinary exponentials to have a closed form of the solution to the differential equation (C-5) as

$$\hat{v}(t) \approx e^{\int_0^t d\tau \hat{M}(t, \tau)} \hat{v}(0) + \int_0^t e^{\int_\tau^t d\tau' \hat{M}(t, \tau')} \hat{f}(t, \tau) d\tau$$

$$= \left( \cosh(\sqrt{m(t, 0)}) \hat{I} + \frac{\sinh(\sqrt{m(t, 0)})}{\sqrt{m(t, 0)}} \hat{K}(0) \right) \hat{v}(0)$$

$$+ \int_0^t d\tau \left( \cosh(\sqrt{m(t, \tau)}) \hat{I} + \frac{\sinh(\sqrt{m(t, \tau)})}{\sqrt{m(t, \tau)}} \hat{K}(t, \tau) \right) \hat{f}(t, \tau).$$

(C-9)
FIG. C-1: Entanglement evolution without quantum noise effect. (a) Blue detuned regime. The long dashed (red) curve is for $\Delta_0 = -0.5\omega_m$, the short dashed (blue) curve for $\Delta_0 = -\omega_m$, the thin solid (black) curve for $\Delta_0 = -1.5\omega_m$, and the thick solid (purple) curve for $\Delta_0 = -2\omega_m$. (b) Red detuned regime. The thin solid (blue) curve is for $\Delta_0 = 0.5\omega_m$, and the long dashed (purple) curve for $\Delta_0 = \omega_m$, the solid (red) curve for $\Delta_0 = 1.5\omega_m$, and the short dashed (black) curve for $\Delta_0 = 2\omega_m$. The system parameters are $g/\kappa = 10^{-6}$, $E/\kappa = 2 \times 10^6$, $\omega_m/\kappa = 2.5$, $\omega_m/\gamma_m = 10^7$, and $T = 0$. The plots in (a) and (b) correspond to those in Fig. 1(b) and Fig. 2(b) of the main text, respectively. Here the entanglement without the effect of quantum noises becomes steady with time, while that under quantum noise effect shown in the main text could be destroyed after a finite period of time.

Here we have defined $\hat{K}(t, \tau) = \int_0^t d\tau' \hat{M}(t, \tau')$, and the function $m(t, \tau)$ from the relation $\hat{K}^2(t, \tau) = m(t, \tau) \hat{I}$ is

$$m(t, \tau) = \frac{1}{4} \left| \int_0^t d\tau' \left( l_1(t, \tau') + il_2(t, \tau') - il_3(t, \tau') + l_4(t, \tau') \right) \right|^2 - \frac{1}{4} \left| \int_0^t d\tau' \left( l_1(t, \tau') - il_2(t, \tau') - il_3(t, \tau') - l_4(t, \tau') \right) \right|^2.$$  

(C-10)

With arbitrary system parameters, the first term in (C-8) from the initial value $\hat{v}(0)$ of system operators contributes to one part of the correlation matrix $\hat{V}_1(t)$, where the average in the calculation of the matrix elements is taken with respect to the initial system state $\rho(0)$. This reflects the reliance of the system quantum state at the time $t$ on this initial state. Meanwhile, the second term of noise driving leads to another part of the correlation matrix $\hat{V}_2(t)$, where the average is over the reservoir state $\rho(0)$. Summing up the two matrices gives the total correlation matrix $\hat{V}(t) = \hat{V}_1(t) + \hat{V}_2(t)$. For a comparison with the entanglement evolution found in the main text, we give an example of entanglement evolution solely determined by matrix $\hat{V}_1(t)$ in Fig. (C-1). In the absence of quantum noise effect, the entanglement measured by logarithmic negativity tends to stable value with time, and there does not exist the phenomenon of entanglement sudden death in Fig. 1 and 2 of the main text.

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