Labeling on Line Digraphs

R. Thamizharasi and R. Rajeswari
Sathyabama University, Chennai - 600119, Tamil Nadu, India; thamizhvisu@gmail.com, rajeswarivel@yahoo.in

Abstract

Objectives: In this paper, we prove that the line digraphs of Cayley digraphs admit super vertex (a,d) antimagic labeling, product anti magic and vertex magic total labeling. Also we have shown the labeling formula to get the same. Findings: If a Cayley digraph admits the above three labelings, then the line digraph of Cayley digraph also admits the same.

Keywords: Antimagic, Labelings, Line Digraphs, Magic

1. Introduction

Graph labeling can be described as placing integers on the vertices and or edges of a graph subject to certain conditions. The first labeling was introduced by Rosa in 1967. In early 1980’s Bloom and Hsu defined labeling for directed graphs. Hartfield and Ringel introduced antimagic graphs in 1990. Baca et al introduced the notion of (a,d) vertex antimagic total labeling in 2000. Further (a,d) antimagic labelings discussed for many graphs such as Cayley digraphs, twin graphs and so on. Magic labelings were introduced sedlack in1963. Figueroa-Centeno, Ichishma and Muntanerbate have introduced multiplicative analogs of magic and anti-magic labeling. Thirusangu et al introduced super vertex (a,d) antimagic and vertex magic total labeling for some classes of Cayley digraphs. Thirusangu et al discussed the product antimagic labeling for Cayley digraphs with two generators and also they discussed various labelings for the competition graph of Cayley digraphs. Thamizharasi and Rajeswari discussed different labelings on the line digraph of Cayley digraphs.

The Cayley digraphs serve as excellent models for interconnection networks. Line digraphs are also used as models of Fault tolerant networks. Graphs actually used in networks are labeled so here we present the super vertex (a,d) antimagic, total vertex magic and product antimagic labeling of the line digraph of Cayley digraphs. The below results would be very useful to study the network addressing problems.

2. Preliminaries

In this section let us see some basic definitions which would be useful to our further investigations

2.1 Definition

Let G be a finite group. S is a generating subset of G. The Cayley digraph Cay (G;S) is the digraph whose vertices are the elements of G, and there is an arc from g to gs whenever g ∈ G and s ∈ S. If S=S⁻¹ then there is an arc from g to gs if and only if there is an arc from gs to g.

2.2 Definition

The line Graph L(G) of a graph G is the Graph defined by V(LG)) = E(G) and {e₁,e₂} ∈ E(L(G)) if e₁,e₂ are incident to a common vertex in G. If {x,y} ∈ E(G) we will denote the corresponding vertex of L(G) by [x y]. We will denote the vertex set by V (L (G)) and the edge set by E (L (G)).

3. Labelings on Line Digraph of Cayley Digraph

3.1 Super Vertex (a,d) Antimagic Labeling for the Line Digraph of Cayley Digraph

In this section we prove super vertex (a,d) antimagic labeling for the line digraph of the Cayley digraph and
present an example to illustrate super vertex \((a, d)\) antimagic labeling.

### 3.1.1 Definition

A digraph \(G=(V, E)\) is said to admit super vertex \((a, d)\) antimagic labeling if there exist a function \(f\) from \(V \cup E\) onto the set \(\{1, 2, \ldots, p+q\}\) such that \(f(v) = 1, 2, \ldots, p\) and for any vertex \(v\), sum of the label of vertex \(v\) and the labels of its outgoing edges are distinct and the set of all such distinct elements corresponds to \(V\) of \(G\) is equal to \(\{a, a+d, a+2d, \ldots, a+(q-1)d\}\), where \(a\) and \(d\) are any two positive integers.

### 3.1.2 Theorem

The line digraph of the Cayley digraph admits super vertex \((a, d)\) antimagic labeling.

**Proof:**
Assume the Cayley digraph \(\text{Cay}(G, S)\) with \(n(G) = p\) and \(n(S) = m\). The present Cayley digraph has \(p\) vertices and \(m\) generators. We always know that the line digraph of the above regular digraph is also regular. That is every vertex of the line digraph has \(m\) incoming and \(m\) outgoing arcs. If the Cayley digraph of a group contains \(p\) vertices and \(q\) edges, then the corresponding line digraph contains \(q\) vertices and \(mq\) arcs. Let us denote the vertex set of \(L(\text{Cay}(G, S))\) as \(V = \{v_1, v_2, v_3, \ldots, v_n\}\) where \(n=q\) and denote the edge set of \(L(\text{Cay}(G, S))\) as \(E = \{e_{11}, e_{12}, \ldots, e_{1m}, e_{21}, e_{22}, \ldots, e_{2m}, \ldots, e_{n1}, \ldots, e_{nm}\}\).

To prove \(L(\text{Cay}(G, S))\) admits super vertex \((a, d)\) antimagic labeling, we have to show that for any vertex \(v\), the sum of the label of vertex \(v\) and the labels of outgoing arcs are distinct and the set of all such distinct elements corresponds to \(V\) is equal to \(\{a, a+d, a+2d, \ldots, a+(k-1)d\}\) where \(a\) and \(d\) are any two positive integers.

We prove this theorem in two cases accordingly \(m\) is even and odd.

**Case (i): when \(m\) is even**

Define \(f: V \rightarrow \{1, 2, \ldots, n\}\) as \(f(v) = i\) for \(1 \leq i \leq n\)

And \(g_{\text{sm}}(v): E \rightarrow \{n+1, n+2, \ldots, mn\}\) as

\[
g_{\text{sm}}(v) = \{ (j + 1)n + 1 - i \quad \text{for } j = 1, 3, \ldots, m - 1 \} \quad \text{for } j = 2, 4, \ldots, m
\]

Then the sum \(S = i + 2n+1 - i + 2n+ i + 4n+1 - i + 4n+ i + \ldots + mn+1 - i + mn+i\)

\[= i + (m/2)(1)+2[2n+4n+\ldots+4n]/2\]

Moreover for any two integers \(i, j\) such that \(i \neq j, \quad f(v) \neq f(v)\) and the sum of the labels are also distinct. Also for any integer \(i\), \(f(v_{i+1}) - f(v_i) = 1 = d\) (say). The initial value of the label is \(a = 1+(m+4nm+4mn)/2\) which proves that the vertex sums form an arithmetic progression \(\{a, a+d, a+2d, \ldots, a+(k-1)d\}\).

Hence the line graph of Cayley digraph with even number of generators admits super vertex \((a, d)\)- antimagic labeling.

**Case (i): when \(m\) is odd**

Define \(f: V \rightarrow \{1, 2, \ldots, n\}\) as \(f(v) = i\) for \(1 \leq i \leq n\)

And \(g_{\text{sm}}(v): E \rightarrow \{n+1, n+2, \ldots, mn\}\) as

\[
g_{\text{sm}}(v) = \{ (j + 1)n + 1 - i \quad \text{for } j = 1, 3, \ldots, m - 2 \} \quad \text{for } j = 2, 4, \ldots, m
\]

Then the sum \(S = i + 2n+1 - i + 2n+ i + 4n+1 - i + 4n+ i + \ldots + (m-1)n+1 - i + (m-1)n+i + mn+i\)

\[= i + (m(n-1)/2)(1)+2[2n+4n+\ldots+(m-1)n]+mn\]

\[= 2i+(m+2nm^2-1)/2\]

Moreover for any two integers \(i, j\) such that \(i \neq j, \quad f(v) \neq f(v)\) and the sum of the labels are also distinct. Also for any integer \(i\), \(f(v_{i+1}) - f(v_i) = 2 = d\) (say). The starting value of the label is \(a = 2+(m+4nm^2+4mn)/2\). From which we know the vertex sums form an arithmetic progression \(\{a, a+d, a+2d, \ldots, a+(k-1)d\}\). Hence the line digraph of Cayley digraph with odd number of generators admits super vertex \((a, d)\)- antimagic labeling.

Therefore the line graph of Cayley digraph admits super vertex \((a, d)\)- antimagic labeling.

### 3.1.3 Proposition

If any Cayley digraph admits super vertex \((a, d)\) antimagic labeling then its corresponding line digraph also admits super vertex \((a, d)\) antimagic labeling.

### 3.1.4 Proposition

Every regular directed graph admits super vertex \((a, d)\) antimagic labeling.

### 3.1.5 Example

Consider the group \(S_3 = \langle v_1=(1)(2)(3), v_2=(1 23)\rangle\) with generating set \(\{v_3=(12)(3), v_4=(1 23), v_5=(13)(2), v_6=(132)\}\). The line digraph of the Cayley digraph for the group \(S_3\) and its super vertex \((a, d)\) anti magic labeling is shown figure 1.
Figure 1. Super vertex (a,d) antimagic labeling of the line digraph of Cayley digraph for the group $S_3$.

3.2 Product Antimagic Labeling for the Line digraph of Cayley digraph

In this section we will prove that the Line digraph of the Cayley digraphs admit the product antimagic labeling and illustrated by an example.

3.2.1 Definition

A digraph $G=(V, E)$ of size $q$ is said to admit product antimagic if there exist a function $f: E(G) \rightarrow \{1, 2, \ldots, q\}$ such that the product of the labels of the outgoing edges of every vertex are distinct.

3.2.2 Theorem

The line digraph of the Cayley digraph with generating set $(a, b)$ admits product antimagic labeling

Proof:

If the Cayley digraph of a group contains $p$ vertices and $q$ edges, then the corresponding line digraph contains $q$ vertices and $mq$ arcs. Let us denote the vertex set of $L(Cay(G,S))$ as $V=\{v_1, v_2, v_3, \ldots, v_n\}$ where $n=q$ and denote the edge set of $L(Cay(G,S))$, as $E = \{e_{i_1}, e_{i_2}, \ldots, e_{i_m}, e_{21}, e_{22}, \ldots, e_{2m}, \ldots, e_{n1}, \ldots, e_{nm}\}$.

To prove $L(Cay(G,S))$ admits product antimagic labeling we have to show that there is a labeling $f$ from $E$ onto $\{1, 2, 3, \ldots, mn\}$ such that the product of the labels of the outgoing arcs of each vertex are distinct. Consider an arbitrary vertex $v_i \in V$ of the line graph of Cayley digraph $L\{Cay(G,S)\}$.

Now define $f: V \rightarrow \{1, 2, \ldots, n\}$ such that $f(v_i) = i$ for $1 \leq i \leq n$ and $f(v_j):=\{j-1\}$ if $j$ is odd

$$g_{ij}(v_i) = \begin{cases} (j-1) + i & \text{if } j \text{ is odd} \\ jn - i & \text{if } j \text{ is even} \end{cases}$$

Then the product $P(v_i) = f_{i_1}(e_i) \cdot f_{i_2}(e_i) \cdot \ldots \cdot f_{i_m}(e_i) = (i)(2n+1-i) \ldots ((m-1)n + i))$ if $m$ is odd

This gives the distinct value for each vertex.

Hence the line digraph of the Cayley digraphs Cay $(G,S)$ admits product antimagic labeling.

3.2.3 Proposition

Every regular graph is product antimagic.

3.2.4 Example

Consider the group $Z_6 = \{0, 1, 2, 3, 4, 5\}$ with generating set $\{2, 3\}$. The line digraph of the Cayley digraph for the group $Z_6$ and its product antimagic labeling is shown in figure 2.

Figure 2: Product antimagic labeling of the line graph of Cayley digraph for $Z_6$.

3.3. Vertex-Magic Total labeling for the Line digraph of Cayley digraph

In this section we prove vertex magic total labeling for the line digraph of the Cayley digraph with odd number of generators and present an example.

3.3.1 Definition

A $(p, q)$ digraph $G$ is said to admit vertex magic total labeling if there exist a function $f: V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ such that for every vertex $v$, the sum of the label of vertex $v$ and the labels of the outgoing arcs of $v$ is constant for all $v \in V$. 

Define $g_v(u): E \rightarrow \{n+1, n+2, \ldots, mn\}$ such that $g_{ij}(v_i) = \begin{cases} (j-1)n + i & \text{if } j \text{ is odd} \\ jn - i & \text{if } j \text{ is even} \end{cases}$

Then the product $P(v_i) = f_{i_1}(e_i) \cdot f_{i_2}(e_i) \cdot \ldots \cdot f_{i_m}(e_i) = (i)(2n+1-i) \ldots ((m-1)n + i)$ if $m$ is odd

This gives the distinct value for each vertex.

Hence the line digraph of the Cayley digraphs Cay $(G,S)$ admits product antimagic labeling.
3.3.2 Theorem

The line digraph of the Cayley digraph with odd number of generators admits vertex-magic total labeling.

Proof:

Assume the Cayley digraph Cay(G,S) with \( n(G) = p \) and \( n(S) = m \). The present Cayley digraph has \( p \) vertices and \( m \) generators. We always know that the line digraph of the above regular digraph is also regular. That is every vertex of the line digraph has \( m \) incoming and \( m \) outgoing arcs where \( m \) is odd.

If the Cayley digraph of a group contains \( p \) vertices and \( q \) edges, then the corresponding line digraph contains \( q \) vertices and \( mq \) arcs. Let us denote the vertex set of \( L(\text{Cay}(G,S)) \) as \( V = \{v_1, v_2, v_3, \ldots, v_n \} \) where \( n = q \) and denote the edge set of \( L(\text{Cay}(G,S)) \), as \( E = \{e_{i1}, e_{i2}, \ldots, e_{im}, e_{m+1}, e_{m+2}, \ldots, e_{2m}, e_{2m+1}, \ldots, e_{nm} \} \).

\[ \text{Figure 3. Vertex magic total labeling of the line graph of Cayley digraph for the group } \mathbb{Z}_8. \]

To prove \( L(\text{Cay}(G,S)) \) admits vertex-magic total labeling we have to show that for every vertex \( v_i \), the sum of the label of vertex \( v_i \) and the label of outgoing arcs is same for all vertices. To prove the line digraph of Cayley digraph admits magic labeling, we have to show that there exist a function \( f \) from \( VUE \) onto the set \( \{1, 2, \ldots, p+q\} \) such that \( f(v_i) \) is constant for all vertices \( v_i \). Let \( f(v_i) = i \) for \( 1 \leq i \leq n \)

And \( g_{jm}(v_i) = \{j+1, n+1-i \} \) for \( j \neq i \) and \( m \) is odd.

Then the sum \( S = f(v_i) + g_{s1}(v_i) + g_{s2}(v_i) + \cdots + g_{sm}(v_i) \) is constant for any \( m \) and \( n \).

Hence the line digraph of Cayley digraph with odd number of generators admits vertex-magic total labeling.

3.3.3 Proposition

Every odd regular digraph admits vertex-magic total labeling.

3.3.4 Proposition

The Cayley digraph admits the vertex magic total labeling if the line digraph of the corresponding Cayley digraph admits the same labeling.

3.3.5 Example

Consider the group \(( \mathbb{Z}_8 : 5,3,7) \) consisting of 8 elements and the generating set

\[ A = \{5, 3, 7\}. \]

The line graph of the Cayley digraph for the group \( \mathbb{Z}_8 \) and its vertex-magic total labeling is shown in figure 3.

To prove \( L(\text{Cay}(G,S)) \) admits vertex-magic total labeling we have to show that for every vertex \( v_i \), the sum of the label of vertex \( v_i \) and the label of outgoing arcs is same for all vertices. To prove the line digraph of Cayley digraph admits magic labeling, we have to show that there exist a function \( f \) from \( VUE \) onto the set \( \{1, 2, \ldots, p+q\} \) such that \( f(v_i) = i \) for \( 1 \leq i \leq n \)

Define \( f : V \rightarrow \{1, 2, \ldots, n\} \) as \( f(v_i) = i \) for \( 1 \leq i \leq n \)

And \( g_{jm}(v_i) = \{j+1, n+1-i \} \) for \( j \neq i \) and \( m \) is odd.

Then the sum \( S = f(v_i) + g_{s1}(v_i) + g_{s2}(v_i) + \cdots + g_{sm}(v_i) \) is constant for any \( m \) and \( n \).

Hence the line digraph of Cayley digraph with odd number of generators admits vertex-magic total labeling.

4. Conclusion

We proved that

1. The line digraph of Cayley digraph \( \text{Cay}(G,S) \) admits \((a,d)\) super vertex antimagic labeling and product antimagic labeling.
2. The line digraph of the Cayley digraph admits vertex total magic labeling if the Cayley digraph has the odd number of generators.
3. If a Cayley digraph admits the above three labelings, then the line digraph of Cayley digraph also admits the same.
5. References

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