Occupants in manifolds

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Abstract. Let $K$ be a subset of a smooth manifold $M$. In some cases, functor calculus methods lead to a homotopical formula for $M \setminus K$ in terms of the spaces $M \setminus S$, where $S$ runs through the finite subsets of $K$.

1. Occupants in a submanifold

1.1. Formulation of the problem. Imagine a smooth manifold $M$ and a compact smooth submanifold $L$, both with empty boundary, of dimensions $m$ and $\ell$ respectively. We look for a homotopical description of $M \setminus L$ in terms of the spaces $M \setminus S$, where $S$ runs through the finite subsets of $L$. The finite subsets $S$ of $L$ could be regarded as finite sets of occupants.

For one of the more geometric formulations of the problem, choose a Riemannian metric on $L$. Instead of working with finite subsets $S$ of $L$, we work with thickenings of finite subsets of $L$ and we pay attention to inclusions of one such thickening in another. More precisely we work with pairs $(S, \rho)$ where $S$ is a finite subset of $L$ and $\rho$ is a function from $S$ to the positive real numbers subject to two conditions.

- For each $s \in S$, the exponential map $\exp_s$ at $s$ is defined and regular on the (compact) disk of radius $\rho(s)$ about the origin in $T_s L$.
- The images in $L$ of these disks under the exponential maps $\exp_s$ are pairwise disjoint.

For a pair $(S, \rho)$ satisfying the two conditions, let $V_L(S, \rho) \subset L$ be the union of the open balls of radius $\rho(s)$ about points $s \in S$. Then $V_L(S, \rho)$ is diffeomorphic to $\mathbb{R}^\ell \times S$. The inclusion of $M \setminus V_L(S, \rho)$ in $M \setminus S$ is a homotopy equivalence.

Let $C_k(L)$ be the space of unordered configurations of $k$ points in $L$. For fixed $k \geq 0$, the pairs $(S, \rho)$ that satisfy the two conditions and the additional condition $|S| = k$ form a space $C_k^{fat}(L)$, an open subspace of the total space of some $k$-dimensional vector bundle on $C_k(L)$. The projection $C_k^{fat}(L) \to C_k(L)$ is a homotopy equivalence. Form the topological disjoint union of the spaces $C_k^{fat}(L)$ for all $k \geq 0$ and view that as a (topological) poset $\mathcal{P}(L)$ where

$$(S, \rho) \leq (T, \sigma)$$

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means simply that $V_L(S, \rho) \subset V_L(T, \sigma)$. The poset $P(L)$ can also be viewed as a category. A contravariant functor $\Phi$ from $P(L)$ to spaces is defined by

\[(1.1.1) \quad \Phi(S, \rho) = M \setminus V_L(S, \rho).\]

There is a map

\[(1.1.2) \quad M \setminus L \to \text{holim } \Phi,\]

determined by the inclusions $M \setminus L \to \Phi(S, \rho)$ for $(S, \rho) \in P(L)$. With a view to occupants and the problem of finding unoccupied places, we ask:

- is the map $(1.1.2)$ a weak homotopy equivalence?

There is a more numerical variant. For $j \geq 0$ let $P_j(L)$ be the subspace and full topological sub-poset of $P(L)$ consisting of all $(S, \rho)$ in $P(L)$ that satisfy $|S| \leq j$. There is a map

\[(1.1.3) \quad M \setminus L \to \text{holim } \Phi|_{P_j(L)}\]

determined by the inclusions $M \setminus L \to \Phi(S, \rho)$. We ask:

- is the map $(1.1.3)$ highly connected?

In reading these questions, keep in mind that $\Phi$ has some continuity properties. This affects the meaning or definition of the homotopy inverse limits. Here is a quick definition using the fact that $\Phi$ is a subfunctor of a constant functor. More details can be found in definition 1.1.2 below. Let $NP(L)$ be the nerve of $P(L)$, a simplicial space. So $N_rP(L)$ is the space of order-reversing maps $u$ from $[r] = \{0, 1, \ldots, r\}$ to $P(L)$. Order-reversing means that $u(0) \geq u(1) \geq \cdots \geq u(r)$ in $P(L)$. Now $\text{holim } \Phi$ in $(1.1.2)$ can be described as a subspace of the space of all maps from the geometric realization $|NP(L)|$ to $M$, with the compact-open topology. A map $f: |NP(L)| \to M$ belongs to that subspace if and only if for every $r$ and $u \in N_rP(L)$ with characteristic map $c_u: \Delta^r \to |NP(L)|$, the composition $fc_u$ lands in $\Phi(u(r)) \subset M$. This description gives a rather good idea what $\text{holim } \Phi$ is: the space of all homotopy coherent ways to choose a place in $M$ when some places in $L$ are occupied. — The homotopy limit in $(1.1.3)$ can be defined analogously.

**Theorem 1.1.1.** If $m - \ell \geq 3$, then the map in $(1.1.2)$ is a weak homotopy equivalence and the map in $(1.1.3)$ is $(1 + (j+1)(m-\ell-2))$-connected.

The proof of theorem 1.1.1 is given at the end of this section. It is based on a reduction to standard theorems in manifold calculus as found in [11] and [8]. (See also remark 1.3.5 below.) The main idea is this: apply manifold calculus to the contravariant functor $F$ defined by $F(V) = M \setminus V$ for open subsets $V$ of $L$. Then the left-hand side of $(1.1.2)$ is $F(L)$ and the right-hand side is very reminiscent of $(T_\infty F)(L)$, where $T_\infty F$ is the “Taylor series” of $F$. Therefore the work consists mainly in showing that $F$ is analytic. (But the definition $F(V) := M \setminus V$ should be regarded as provisional.)

We begin with a slightly more systematic description or definition of the homotopy inverse limits in $(1.1.2)$ and $(1.1.3)$. The idea is to use the trusted formula of Bousfield-Kan [3] while paying attention to topologies where appropriate. Let $\Gamma_r(\Phi)$ be the space of (continuous) sections, with the compact-open topology, of the fiber bundle $E_L(\Phi) \to N_rP(L)$ such that the fiber over a point $u \in N_rP(L)$ is $\Phi(u(r))$. So if $u$ is given by a string $(S_0, \rho_0) \geq \cdots \geq (S_r, \rho_r)$ in $P(L)$, then the fiber over $u$ is $\Phi(S_r, \rho_r) = M \setminus V_L(S_r, \rho_r)$.
DEFINITION 1.1.2. The homotopy limit in (1.1.2) is \( \text{Tot} ([r] \mapsto \Gamma_r(\Phi)) \).

REMARK 1.1.3. The cosimplicial space \([r] \mapsto \Gamma_r(\Phi)\) is Reedy fibrant. This means that for each \( r \geq 0 \) the matching map

\[
\Gamma_r(\Phi) \longrightarrow \text{lim}_{[r] \to [q]} \Gamma_q(\Phi)
\]
determined by the non-identity (co)degeneracy operators is a fibration. (Here \( q < r \) and \([r] \to [q]\) stands for a monotone surjection; see e.g. [5] for more details.) It is a desirable property to have because a map \( X \to Y \) between Reedy fibrant cosimplicial spaces which is a degreewise weak equivalence induces a weak equivalence \( \text{Tot}(X) \to \text{Tot}(Y) \). Sketch of a proof showing that \([r] \mapsto \Gamma_r(\Phi)\) is Reedy fibrant:

since \( E^r_x(\Phi) \to N_r\mathcal{P}(L) \) is a fiber bundle it is enough to note that the simplicial space \([r] \mapsto N_r\mathcal{P}(L)\) is Reedy cofibrant. It is Reedy cofibrant because, for every \( r \geq 0 \), the latching map

\[
\text{colim}_{[r] \to [q]} N_q\mathcal{P}(L) \longrightarrow N_r\mathcal{P}(L)
\]
(where \( q < r \) etc.) is the inclusion of one ENR, Euclidean neighborhood retract, in another ENR as a closed subspace. Such an inclusion is a cofibration [9 ch.III,Thm.3.2].

1.2. Discrete variants. Write \( \delta\mathcal{P}(L) \) for the discrete variant of \( \mathcal{P}(L) \). So \( \delta\mathcal{P}(L) \) is a discrete poset and there is a map of posets \( \delta\mathcal{P}(L) \to \mathcal{P}(L) \) which is bijective and full. That is, \((S, \rho) \leq (T, \sigma)\) has the same meaning in \( \delta\mathcal{P}(L) \) and in \( \mathcal{P}(L) \). That map from \( \delta\mathcal{P}(L) \) to \( \mathcal{P}(L) \) induces a map of homotopy inverse limits:

\[
\text{holim}_{(S, \rho) \in \mathcal{P}(L)} \Phi(S, \rho) \longrightarrow \text{holim}_{(S, \rho) \in \delta\mathcal{P}(L)} \Phi(S, \rho)
\]

Similarly, restricting cardinalities of configurations we have a comparison map

\[
\text{holim}_{(S, \rho) \in \mathcal{P}_j(L)} \Phi(S, \rho) \longrightarrow \text{holim}_{(S, \rho) \in \delta\mathcal{P}_j(L)} \Phi(S, \rho)
\]

LEMMA 1.2.1. The maps (1.2.1) and (1.2.2) are weak equivalences.

PROOF. This proof is rather long but not very new, since it mainly recycles some ideas from [11]. We prove that the map (1.2.1) is a weak equivalence; the other statement has a similar proof. — The first step is to replace the topological poset \( \mathcal{P}(L) \) by a simplicial poset \([t] \mapsto P_t\) and the functor \( \Phi \) by a simplicial functor \([t] \mapsto \Phi_t \). Therefore let \( P_t \) be the set of continuous maps from \( \Delta^t \) to the underlying space of \( \mathcal{P}(L) \). For \( \sigma, \tau \in P_t \) we write \( \sigma \leq \tau \) to mean that \( \sigma(x) \leq \tau(x) \) for all \( x \in \Delta^t \). Let \( \Phi_t \) be the functor from \( P_t \) to spaces taking \( \sigma: \Delta^t \to \mathcal{P}(L) \) to the space of maps \( f: \Delta^t \to M \) which satisfy \( f(x) \notin V_L(\sigma(x)) \) for all \( x \in \Delta^t \).

A monotone map \( \alpha: [t] \to [u] \) determines a map of posets \( \alpha^*: P_u \to P_t \) and a natural transformation from \( \Phi_u \) to \( \Phi_t \circ \alpha^* \). This is already a fairly complicated situation and it means that \([t] \mapsto \text{holim} \Phi_t \) does not qualify as a functor, covariant or contravariant. Instead we need to look at the following:

\[
([t] \mapsto [u]) \mapsto \text{holim} (\Phi_t \circ \alpha^*).
\]
This is a covariant functor from the twisted arrow category \( \text{tw}(\Delta) \) of \( \Delta \) to the category of spaces. (The objects of the twisted arrow category of a category \( D \) are morphisms \( f: c \to d \) in \( D \), and a morphism from \( f: c \to d \) to \( g: u \to v \) is a pair of morphisms \( h: c \to u \) and \( k: v \to d \) such that \( f = kgh \).) The map (1.2.1) can be written in the form of a restriction \( \text{holim} \Phi \to \text{holim} \Phi_0 \) and as such it is a composition of two maps:

\[
(1.2.3) \quad \text{holim} \Phi \to \text{holim} \text{holim} (\Phi_t \circ \alpha^*) \to \text{holim} \Phi_0.
\]

We are going to show that both of these maps are weak equivalences. For the one on the left it is more of a routine task. We can write

\[
\text{holim} \alpha: [t] \to [u] \to \text{holim} \text{holim} (\Phi_t \circ \alpha^*) = \text{holim} \text{holim} \text{Tot} \left( \{ r \mapsto \prod_{\sigma_0 \geq \cdots \geq \sigma_r} \Phi_t(\sigma_r \circ \alpha^*) \} \right)
\]

where

\[
\text{holim} \alpha: [t] \to [u] \to \prod_{\sigma_0 \geq \cdots \geq \sigma_r} \Phi_t(\sigma_r \circ \alpha^*)
\]

is the space of lifts as in the following diagram:

\[
(1.2.4) \quad E_r^t(\Phi) \quad \xrightarrow{\text{holim}} \quad \text{holim} (\Delta^t \times N_rP_u) \quad \xrightarrow{\text{eval.}} \quad N_rP(L)
\]

But the map in the lower row of diagram (1.2.4) is a weak equivalence, so that the induced map from the section space \( \Gamma_r(\Phi) \) of \( E_r^t(\Phi) \to N_rP(L) \) to

\[
\text{holim} \alpha: [t] \to [u] \to \prod_{\sigma_0 \geq \cdots \geq \sigma_r} \Phi_t(\sigma_r \circ \alpha^*)
\]

is a weak equivalence for every \( r \geq 0 \). It follows (by application of Tot) that the first arrow in (1.2.3) is a weak equivalence.

For the other arrow in (1.2.3) it suffices to show that the functor

\[
(\alpha: [t] \to [u]) \mapsto \text{holim} (\Phi_t \circ \alpha^*)
\]

on \( \text{tw}(\Delta) \) takes all morphisms to weak equivalences. This reduces easily to the statement that the prolongation map \( \text{holim} \Phi_0 \to \text{holim} \Phi_0 \circ \alpha^* \) is a weak equivalence, where \( \alpha: [0] \to [u] \) is any map and \( \alpha^*: P_u \to P_0 \) is the induced map of posets, given by evaluation at a vertex \( w \) of \( \Delta^u \). Now [5, thm 6.12] can be applied. Then it only remains to show that \( \alpha^*: P_u \to P_0 \) is homotopically terminal, i.e., that for every element \( z \) of \( P_0 \) the over category \( z/\alpha^* \) has a contractible classifying space.

Here we can make good use of a fact from fibration theory: a Serre microfibration with contractible fibers is a Serre fibration [12, Lemma 2.2], hence a weak equivalence. Write \( z = (S, \rho) \) where precision is necessary, and for elements of \( P_u \) write \( ((T^x, \sigma^x))_x \) and the like, on the understanding that \( x \) runs in \( \Delta^u \). The category \( z/\alpha^* \) is again a poset. Indeed it is the sub-poset of \( P_u \) consisting of the elements
consisting of the pairs \((T^x, \sigma^x)\) where \(S, \rho \leq (T^w, \sigma^w)\); remember that \(w\) is a specified vertex of \(\Delta^u\). Let \(K\) be the contractible space of continuous maps \(g: \Delta^u \to \text{emb}(S, L)\) such that \(g(w)\) is the inclusion \(S \to L\). We write \(g(x, s)\) instead of \(g(x)(s)\) for \(x \in \Delta^u\) and \(s \in S\). Let \(W \subset |N(z/\alpha^s)| \times K\) be the open subspace consisting of pairs \((y, g)\) where \(y\) belongs to a cell of \(|N(z/\alpha^s)|\) corresponding to a nondegenerate simplex \([(T_0^x, \sigma_0^x)]_x \times [(T_1^x, \sigma_1^x)]_x \times \cdots \times [(T_r^x, \sigma_r^x)]_x \in N_r(z/\alpha^s)\) and \(g\) satisfies \(g(x, s) \in V_L(T_0^x, \sigma_0^x)\) for all \(x \in \Delta^u\) and \(s \in S\). (The word cell is used here as in [4]: distinct cells in a CW-space are disjoint and each is homeomorphic to a euclidean space.) The projections from \(W\) to \(|N(z/\alpha^s)|\) and \(K\) are both Serre microfibrations, since \(W\) is open in the product. The fibers of the projection from \(W\) to \(|N(z/\alpha^s)|\) are contractible by inspection, so that \(|N(z/\alpha^s)| \simeq W\). Next we want to show that the fibers of the projection from \(W\) to \(K\) are contractible. To that end we introduce

\[ W' \subset |N(z/\alpha^s)| \times K, \]

consisting of the pairs \((y, g)\) where \(y\) belongs to a cell of \(|N(z/\alpha^s)|\) corresponding to a nondegenerate simplex \([(T_0^x, \sigma_0^x)]_x \times [(T_1^x, \sigma_1^x)]_x \times \cdots \times [(T_r^x, \sigma_r^x)]_x \in N_r(z/\alpha^s)\) and \(g\) satisfies \(g(x, s) \in V_L(T_0^x, \sigma_0^x)\) for all \(x \in \Delta^u\) and \(s \in S\). The fiber of the projection \(W \to K\) over \(g \in K\) contains as a deformation retract the fiber of the projection \(W' \to K\) over the same \(g\). The fiber of \(W' \to K\) over \(g\) can be identified with the classifying space of a sub-poset \(H_g\) of \(z/\alpha^s\). It is easy to show that the classifying space of \(H_g\) is contractible by producing a homotopically initial functor from the poset of the negative integers to \(H_g\). Therefore the fibers of \(W' \to K\) are contractible and the fibers of \(W \to K\) are contractible. The conclusion from this argument with Serre microfibrations is that \(|N(z/\alpha^s)| \simeq W \simeq K \simeq \ast\). \(\square\)

For the next lemma, let \(U \subset L\) be an open subset. Then we have \(P(U) \subset P(L)\) and \(\delta P(U) \subset \delta P(L)\). If \(U\) is diffeomorphic to \(\mathbb{R}^t \times T\) for some finite set \(T\), which we now want to assume, then we can specify a full sub-poset \(\delta P(U)^\times\) of \(\delta P(U)\) as follows. An element \((S, \rho)\) of \(\delta P(U)\) belongs to \(\delta P(U)^\times\) if and only if every connected component of \(U\) contains exactly one element of \(S\).

**Lemma 1.2.2.** The classifying space of \(\delta P(U)^\times\) is contractible.

**Proof.** Let \(U(1), \ldots, U(k)\) be the connected components of \(U\). Then we have

\[
\delta P(U)^\times \simeq \prod_{s=1}^k \delta P_{U(s)}^\times.
\]

Therefore it suffices to consider the case where \(U\) is connected, and that means \(U\) is diffeomorphic to \(\mathbb{R}^t\). Of course \(U\) still comes with a Riemannian metric which may not be flat. We shall make use of the preferred CW-structure on the classifying space \(|N(\delta P(U)^\times)|\) of \(\delta P(U)^\times\). Let

\[ V \subset U \times |N(\delta P(U)^\times)| \]

be the subspace (with the subspace topology) consisting of all pairs \((x, z)\) with \(x \in U\) and \(z \in |N(\delta P(U)^\times)|\), where the unique cell containing \(z\) corresponds to a
nondegenerate simplex \((S_0, \rho_0) > (S_1, \rho_1) > \cdots > (S_k, \rho_k)\) in \(N_k\delta\mathcal{P}(U)^\times\), and \(x\) belongs to \(V_L(S_0, \rho_0)\). Now we have the projections
\[
U \leftarrow V \rightarrow |N(\delta\mathcal{P}(U)^\times)|.
\]

It suffices to show that both are weak equivalences, and for that it suffices to show that both are Serre microfibrations with contractible fibers \([12] \text{ Lemma 2.2}\). It is clear that both are Serre microfibrations because both are obtained by restricting a Serre fibration (namely, projection from a product to one of the factors) to an open subset of the source, alias total space. By construction, if \(z \in |N(\delta\mathcal{P}(U)^\times)|\) is in a cell corresponding to a nondegenerate simplex
\[
(S_0, \rho_0) > (S_1, \rho_1) > \cdots > (S_k, \rho_k),
\]
then the fiber of \(V \to |N(\delta\mathcal{P}(U)^\times)|\) over \(z\) is identified with \(V_L(S_0, \rho_0)\), which is indeed contractible. The fiber over \(x \in U\) of the projection from \(V\) to \(U\) is identified with the classifying space of a full sub-poset \(\mathcal{H}_x\) of \(\delta\mathcal{P}(U)^\times\). The sub-poset \(\mathcal{H}_x\) consists of all \((S, \rho)\) in \(\delta\mathcal{P}(U)^\times\) with the property that \(x\) is contained in the open ball of radius \(\rho\) about the singleton \(S\). It is easy to see that the classifying space of \(\mathcal{H}_x\) is contractible by producing a homotopically initial functor from the poset of the negative integers to \(\mathcal{H}_x\).

1.3. Using manifold calculus. Let \(\mathcal{O}(L)\) be the (discrete) poset of open subsets of \(L\). In view of lemmas \([1.2.1] \text{ and } [1.2.2]\), the following plan for a proof of theorem \([1.1.1]\) looks promising. There is a contravariant functor
\[
(1.3.1) \quad V \mapsto M \setminus V
\]
from \(\mathcal{O}(L)\) to spaces. Manifold calculus as in \([11]\) and \([8]\) was created to help in understanding such functors. In particular, if a contravariant functor \(F\) from \(\mathcal{O}(L)\) to spaces has some reasonable properties such as isotopy invariance and satisfies some approximate excision conditions, then manifold calculus has a formula
\[
(1.3.2) \quad F(V) := \operatorname{holim}_{C \subseteq V} M \setminus C
\]
(Here \(\mathcal{O}k(L) \subset \mathcal{O}(L)\) is the full sub-poset whose elements are the open subsets of \(L\) which are abstractly diffeomorphic to \(\mathbb{R}^t \times S\) for some set \(S\) with \(|S| \leq k\). Therefore \(\bigcup_k \mathcal{O}k(L) \subset \mathcal{O}(L)\) is the full sub-poset whose elements are the open subsets of \(L\) which are abstractly diffeomorphic to \(\mathbb{R}^t \times S\) for some finite set \(S\).) Using lemmas \([1.2.1] \text{ and } [1.2.2]\) we should be able to work from there to arrive at theorem \([1.1.1]\).

There is a small problem with this plan. The functor \([1.3.1]\) does not have all the good properties required such as isotopy invariance. (Example: take \(L\) to be \(S^2 = \mathbb{R}^2 \cup \infty\) and take \(M\) to be \(S^m = \mathbb{R}^m \cup \infty\) for some \(m \geq 2\). Let \(V_1 \subset \mathbb{R}^2 \subset L\) be the union of the open rectangles \([2^{-1-i}, 2^{-i}] \times [0, 1]\) for \(i = 0, 1, 2, \ldots\) and let \(V_0 \subset \mathbb{R}^2 \subset L\) be the union of the open squares \([2^{-1-i}, 2^{-i}] \times [0, 2^{-1-i}]\) for \(i = 0, 1, 2, \ldots\). The inclusion \(V_0 \to V_1\) is an isotopy equivalence, but the induced homomorphism \(\pi_{m-1}(M \setminus V_1) \to \pi_{m-1}(M \setminus V_0)\) is not surjective.) But that is easy to fix. We rectify \([1.3.1]\) by setting
\[
(1.3.2) \quad F(V) := \operatorname{holim}_{C \subseteq V} M \setminus C
\]
for $V \in \mathcal{O}(L)$, where $C$ runs through all compact subsets of $V$. Note that $F(L)$ has a forgetful map to $M \setminus L$ which is a weak equivalence; and that map has a preferred section which is therefore also a weak equivalence. Now we need to show that $F$ has reasonable properties such as isotopy invariance, and that it satisfies some approximate excision conditions.

**Lemma 1.3.1.** The functor $F$ of (1.3.2) is good. That is to say:
- if $V_0 \subset V_1$ are open subets of $L$ such that the inclusion $V_0 \to V_1$ is abstractly isotopic to a diffeomorphism, then the map $F(V_1) \to F(V_0)$ induced by the inclusion is a weak homotopy equivalence;
- if $W \in \mathcal{O}(L)$ is a union of open subsets $W_i$ where $i = 0, 1, 2, 3, \ldots$ and $W_i \subset W_{i+1}$, then the map from $F(W)$ to $\holim_i F(W_i)$ determined by the inclusions $W_i \to W$ is a weak equivalence.

**Proof.** The second of the two properties claimed is obvious from the definition of $F$. By contrast the first property is not easy to establish. Choose a sequence $(C_i)_{i \geq 0}$ of compact subsets of $V_1$ such that $C_i \subset C_{i+1}$ for all $i \geq 0$ and every compact subset of $V_1$ is contained in one of the $C_i$. Then the projection from $F(V_1)$ to the sequential homotopy limit

$$\holim_i M \setminus C_i$$

is a weak equivalence. Choose a smooth isotopy $(e_t: V_0 \to V_1)_{t \in [0,1]}$ such that $e_0: V_0 \to V_1$ is the inclusion and $e_1: V_0 \to V_1$ is a diffeomorphism. (See remark [1.3.4]) Let $C_{t,i} := e_t(e_1^{-1}(C_i))$ and note that $C_i$ has just been renamed $C_{1,i}$. Now the projection from $F(V_0)$ to

$$\holim_i M \setminus C_{0,i}$$

is also a weak equivalence. Let $Y_i$ be the space of continuous maps $w: [0,1] \to M$ such that $w(t) \notin C_{t,i}$ for all $t \in [0,1]$. By a straightforward application of Thom’s isotopy extension theorem, the maps

$$M \setminus C_{0,1} \leftarrow Y_i \rightarrow M \setminus C_{1,i}$$

given by evaluation, $w \mapsto w(0)$ and $w \mapsto w(1)$, are homotopy equivalences. Therefore in the resulting diagram of sequential homotopy limits

$$\holim_i M \setminus C_{0,i} \leftarrow \holim_i Y_i \rightarrow \holim_i M \setminus C_{1,i}$$

the two arrows are also weak equivalences. Summing up: we have a diagram

$$F(V_0) \cong \holim_i M \setminus C_{0,i} \cong \holim_i Y_i \cong \holim_i M \setminus C_{1,i} \cong F(V_1)$$

showing that $F(V_0) \cong F(V_1)$, abstractly. But it is not yet clear that this weak equivalence is homotopic to the map $F(V_1) \to F(V_0)$ induced by the inclusion of $V_0$ in $V_1$. To show that it is, choose a monotone injective function $\psi: \mathbb{N} \to \mathbb{N}$ such that $C_{1,\psi(t)}$ contains $C_{t,i}$ for all $t \in [0,1]$. This is possible by assumption on the sets $C_i = C_{1,i}$. The (weak) homotopy class of the map $F(V_1) \to F(V_0)$ induced by the inclusion $V_0 \to V_1$ can be described by the diagram

$$\begin{array}{ccc}
F(V_0) & \cong & F(V_1) \\
\holim_i M \setminus C_{0,i} & \xrightarrow{w\mapsto w(1)} & \holim_i Y_{\psi(t)} & \xrightarrow{w\mapsto w(1)} & \holim_i M \setminus C_{1,\psi(t)}
\end{array}$$
The other map or homotopy class of maps $F(V_1) \to F(V_0)$, which we know to be a weak equivalence, can be and has been described similarly, but using the formula $w \mapsto w(0)$ on the left-hand horizontal arrow instead of $w \mapsto w(1)$. This makes it clear that the two homotopy classes are the same, since we can maneuver between $w \mapsto w(0)$ and $w \mapsto w(1)$ by writing $w \mapsto w(t)$, where $t \in [0, 1]$. □

**Proof of theorem 1.1.1** We start by showing that the functor $F$ of (1.3.2) is analytic and by giving some excision estimates for it. Since we know already that $F$ is good, it suffices to look into the following situation. Let $V \in \mathcal{O}(L)$ and let $A_1, A_2, \ldots, A_{k+1}$ be pairwise disjoint closed subsets of $V$. We may assume that $V$ is the interior of a smooth compact codimension zero submanifold of $L$, and that the $A_i$ are also compact smooth codimension zero submanifolds, and that $A_i$ has a handle decomposition with handles of index $\leq q_i$ only. For a subset $R$ of $\{1, 2, \ldots, k+1\}$ let $A_R = \bigcup_{i \in R} A_i$. The commutative cube of spaces

$$R \mapsto F(V \setminus A_R)$$

(1.3.3)

determines a map from $F(V) = F(V \setminus A_0)$ to

$$\operatorname{holim}_{\emptyset \neq R \subseteq \{1, 2, \ldots, k+1\}} F(V \setminus A_R).$$

We need an estimate for the connectivity of that map, in terms of the dimensions $m$ and $\ell$, the number $k$ and and the numbers $q_i$ for $i = 1, \ldots, k+1$. This is easy for the following reason. Because of the special assumptions on $V$ and the $A_i$, the canonical map

$$M \setminus (V \setminus A_R) \to F(V \setminus A_R)$$

(from the definition of $F$) is a homotopy equivalence. Therefore the cube (1.3.3) can be replaced by

$$R \mapsto G(R) := (M \setminus V) \cup A_R.$$  

(1.3.4)

That cube is evidently a strongly cocartesian cube in Goodwillie’s terminology [7]. The map $G(\emptyset) \to G(\{i\})$ is evidently $(q_i - 1 + m - \ell)$-connected. Therefore the excision estimates of [6] and [7] Thm.2.3 for such cubes apply here. Plugging these estimates into [8] Defn. 2.1 we deduce that $F$ is $(m - 2)$-analytic with excess 1 for manifold calculus purposes. More to the point, the comparison map

$$F(L) \to \operatorname{holim}_{U \in \bigcup_k \mathcal{O}(L)} F(U)$$

is a weak equivalence and the comparison map

$$F(L) \to \operatorname{holim}_{U \in \mathcal{O}(L)} F(U)$$

is $(1 + (k + 1)(m - \ell - 2))$-connected by [8] Thm. 2.3. — Now we need to relate $\bigcup_k \mathcal{O}(L)$ to $\mathcal{P}(L)$. There is a full monomorphism of posets

$$\mathcal{P}(L) \to \bigcup_k \mathcal{O}(L).$$
which takes \((S, \rho)\) to \(V_L(S, \rho)\). This leads to a commutative diagram

\[
\begin{array}{ccc}
M \smallsetminus L & \longrightarrow & \mathrm{holim}_{(S, \rho) \in \mathcal{P}(L)} M \smallsetminus V_L(S, \rho) \\
\cong & \searrow & \cong \\
F(L) & \longrightarrow & \mathrm{holim}_{(S, \rho) \in \mathcal{P}(L)} F(V_L(S, \rho)) \\
\cong & \searrow & \cong \\
F(L) & \cong & \mathrm{holim}_{U \in \bigcup \mathcal{O}_k(L)} F(U)
\end{array}
\]

(1.3.5)

Since we want to know that the top horizontal arrow is a weak equivalence, we ought to show that the lower right-hand vertical arrow, call it \(g\), is a weak equivalence. The map

\[
\mathrm{holim}_{(S, \rho) \in \mathcal{P}(L)} F(V_L(S, \rho))
\]

\[
h \downarrow
\]

\[
\mathrm{holim}_{U \in \bigcup \mathcal{O}_k(L)} \left( \mathrm{holim}_{(S, \rho) \in \mathcal{P}(U)^*} F(V_L(S, \rho)) \right)
\]

(self-explanatory) has the property that \(hg\) is a weak equivalence, by lemma 1.2.2. But it is also clear that \(h\) has a weak left homotopy inverse. Therefore \(g\) is a weak equivalence. This completes the proof of the first statement in theorem 1.1.1. The proof of the other statement can be completed similarly, by focusing on \(\mathcal{O}_j(L)\) instead of \(\bigcup \mathcal{O}_k(L)\). \(\square\)

**Remark 1.3.2.** The above proof of theorem 1.1.1 might suggest that the \(k\)-th Taylor approximation of the functor \(F\) in the sense of manifold calculus can be obtained by post-composing \(F\) with the \(k\)-th Taylor approximation of the identity functor from spaces to spaces, in the sense of Goodwillie’s homotopy functor calculus. Surprisingly, this is false. (It is also easy to see that it is false in the case \(k = 1\).) A partial explanation is as follows. If \(G\) is a functor from spaces to spaces which is polynomial of degree \(\leq k\) in the sense of homotopy functor calculus, then \(GF\) is polynomial of degree \(\leq k\) in the manifold calculus sense. This is due to the similarity of the definitions of polynomial functor in the two functor calculuses, and to a property of \(F\) which was emphasized in the proof above. But if \(G\) is a functor from spaces to spaces which is homogeneous of degree \(k\) in the sense of homotopy functor calculus, then \(GF\) need not be homogeneous of degree \(k\) in the sense of manifold calculus. This is due to obvious differences in the classification of homogeneous functors in the two functor calculuses.

**Remark 1.3.3.** How useful, interesting or faithful is the map (1.1.2) when the codimension \(m - \ell\) is less than 3? Here is a codimension 2 case which is not encouraging. Let \(M = S^3\) and let \(L\) be a knot in \(S^3\), your favorite knot, but not
the unknot. Let \( Z_\infty \) be the Bousfield-Kan \( \mathbb{Z} \)-completion functor from spaces to spaces. It comes with a natural transformation \( e : \text{id} \to Z_\infty \). For simply connected spaces \( X \), the natural map \( e : X \to Z_\infty X \) is a weak homotopy equivalence; this is applicable when \( X \) is \( \Phi(S, \rho) \) for some \((S, \rho) \in P(L)\). But for \( X = M \setminus L \) the natural map \( X \to Z_\infty X \) is not a weak equivalence because \( Z_\infty (M \setminus L) \simeq S^1 \). (Instead it is a well-known map \( M \setminus L \to S^1 \) which induces an isomorphism in ordinary integer homology.) We obtain a commutative diagram

\[
\begin{array}{ccc}
M \setminus L & \xrightarrow{(1.1.2)} & \text{holim} \Phi \\
\downarrow e & & \downarrow \simeq e \\
Z_\infty (M \setminus L) & \xrightarrow{} & \text{holim} (Z_\infty \circ \Phi)
\end{array}
\]

It follows that the map (1.1.2), top horizontal arrow in the diagram, factors up to homotopy through the notorious map \( M \setminus L \to S^1 \). That seems to make (1.1.2) tragically un-faithful, in this codimension 2 example.

**Remark 1.3.4.** In articles on manifold calculus, the meaning of *isotopy equivalence* is sometimes ambiguous. According to one definition, call it (a), a smooth codimension zero embedding \( e : U \to V \) (of smooth manifolds with empty boundary) is an isotopy equivalence if and only if there exists an embedding \( f : V \to U \) such that \( ef \) and \( fe \) are smoothly isotopic to the respective identity maps. According to another definition, call it (b), the embedding \( e : U \to V \) is an isotopy equivalence if and only if it is isotopic (as a smooth embedding) to a diffeomorphism from \( U \) to \( V \). I do not know whether definitions (a) and (b) are equivalent. Fortunately it is easy to see that, if a functor from \( O(L) \) to spaces takes isotopy equivalences as in definition (b) to weak equivalences, then it takes isotopy equivalences as in definition (a) to weak equivalences.

**Remark 1.3.5.** Two slightly different views exist on what manifold calculus is about. In the older view laid out in [11] and [8], manifold calculus is about (some) contravariant functors from \( O(L) \) to spaces, where \( L \) is a fixed background manifold. In a more modern view, described for example in [2], though it was also heralded in [1], manifold calculus is about contravariant functors from a certain category \( \text{Man}_\ell \) of *all* smooth \( \ell \)-manifolds to spaces (for some \( \ell \)). The morphisms in \( \text{Man}_\ell \) are smooth embeddings between \( \ell \)-manifolds. More precisely, the morphisms from \( L_0 \) to \( L_1 \) are organized into a space (or simplicial set), composition of morphisms is continuous (or is a simplicial map) etc., which means that \( \text{Man}_\ell \) is *enriched* over the category of spaces (or simplicial sets). Similarly the category of spaces is enriched over spaces (or simplicial sets), and the contravariant functors from \( \text{Man}_\ell \) to spaces that we consider in manifold calculus should respect the enrichments.

A functor like \( \text{emb}(\cdot, W) \) for a fixed smooth manifold \( W \) lives comfortably in both settings: the placeholder – can be interpreted as an open subset of the fixed manifold \( L \), or as an object of \( \text{Man}_\ell \). By contrast the functor \( F \) of (1.3.2) which we have used in proving theorem 1.1.1 seems to belong to the older setting; \( F(V) \) makes sense only for open subsets \( V \) of \( L \).

Does this mean that the modern reformulation of manifold calculus as in [2] has thrown out the baby with the bathwater? Nothing could be further from the truth. I believe that most of the old manifold calculus can be subsumed in the new one as the branch concerned with contravariant functors \( G \) from \( \text{Man}_\ell \) to spaces
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(preserving enrichment) which come equipped with a natural transformation $\gamma$ to a representable functor

$$\text{mor}_{\text{Man}_L}(-,L) = \text{emb}(-,L)$$

for fixed $L$ in $\text{Man}_L$. Such a pair $(G,\gamma)$ gives rise to a contravariant functor $G_\gamma$ from $\mathcal{O}(L)$ to spaces by

$$G_\gamma(V) := \text{hofiber}[\gamma:G(V) \to \text{emb}(V,L)]$$

for $V \in \mathcal{O}(L)$, where the homotopy fiber is taken over the base point of $\text{emb}(V,L)$. The construction $(G,\gamma) \to G_\gamma$ should be seen as a transform, i.e., it is often reversible. In particular most of the contravariant functors from $\mathcal{O}(L)$ to spaces that we encounter in the old manifold calculus are weakly equivalent to $G_\gamma$ for some $G$ and $\gamma:G \to \text{emb}(-,L)$. Exercise: confirm this for the functor $F$ of (1.3.2). — In any case, the proposed subsuming of the old manifold calculus in the new one has not yet been carried out. That is why it could not be used here.

2. Occupants in the interior of a manifold

2.1. Formulation of the problem. Let $M$ be a smooth compact manifold with boundary. We look for a homotopical description of $\partial M$ in terms of the spaces $M \smallsetminus S$, where $S$ runs through the finite subsets of $M \smallsetminus \partial M$. To make that more precise, choose a Riemannian metric on $M$. Then $M \smallsetminus \partial M$ also has a Riemannian metric and the topological poset $\mathcal{P}(M \smallsetminus \partial M)$ is defined as in section 1. Thus, elements of $\mathcal{P}(M \smallsetminus \partial M)$ are pairs $(S,\rho)$ where $S$ is a finite subset of $M \smallsetminus \partial M$ and $\rho$ is a function from $S$ to the positive reals such that, for each $s \in S$, the exponential map $\exp_s:T_s(M \smallsetminus \partial M) \to M \smallsetminus \partial M$ is defined and regular on the disk of radius $\rho(s)$ about the origin, and the images of these disks in $M \smallsetminus \partial M$ are pairwise disjoint. For $(S,\rho) \in \mathcal{P}(M \smallsetminus \partial M)$ let $V(S,\rho) \subset M \smallsetminus \partial M$ be the union of the open balls of radius $\rho(s)$ about points $s \in S$. Then $V(S,\rho)$ is homeomorphic to $\mathbb{R}^m \times S$. A contravariant functor $\Psi$ from $\mathcal{P}(M \smallsetminus \partial M)$ to spaces is defined by

$$(2.1.1) \quad \Psi(S,\rho) = M \smallsetminus V(S,\rho).$$

There are maps

$$(2.1.2) \quad \partial M \longrightarrow \text{holim} \ \Psi,$$

$$(2.1.3) \quad \partial M \longrightarrow \text{holim} \ \Psi|_{\mathcal{P}_j(M \smallsetminus \partial M)}$$

induced by the inclusions $\partial M \to M \smallsetminus V(S,\rho)$. The precise definition of the homotopy limits follows the pattern of definition 1.1.2.

THEOREM 2.1.1. Suppose that $M$ is the total space of a smooth disk bundle $p:M \to L$ of fiber dimension $c$ on a smooth closed manifold $L$. If $c \geq 3$, then the map (2.1.2) is a weak equivalence and the map (2.1.3) is $(1 + (j + 1)(c - 2))$-connected.

The expression disk bundle means just that: a smooth fiber bundle whose fibers are diffeomorphic to disks (of a fixed dimension). It is not necessary to assume that $p$ is the disk bundle associated with a smooth vector bundle on $L$. 
2.2. The tube lemma. Theorem 2.1.1 will be proved by a reduction to theorem 1.1.1. The main idea for the reduction is in lemma 2.2.1 below. The lemma uses the notation of theorem 2.1.1 but we can allow a disk bundle \( p: M \to L \) of any fiber dimension \( \geq 0 \). We choose a Riemannian metric on \( L \). Then in addition to the topological poset \( \mathcal{P}(M \setminus \partial M) \) there is the topological poset \( \mathcal{P}(L) \), and there are some interactions between the two which will be explored. For \( (S, \rho) \in \mathcal{P}(M \setminus \partial M) \) the open set \( V(S, \rho) \subset M \setminus \partial M \) was defined just above. To be more consistent with section 1 we ought to write \( V_{M \setminus \partial M}(S, \rho) \), but that would be cumbersome. For \( (S, \rho) \in \mathcal{P}(L) \) we still write \( V_L(S, \rho) \subset L \) in the style of section 1.

**Lemma 2.2.1.** The map

\[
\text{hocolim}_{(S, \rho) \in \delta \mathcal{P}(L)} C_k(p^{-1}(V_L(S, \rho)) \setminus \partial M) \to C_k(M \setminus \partial M)
\]
determined by the inclusions \( C_k(p^{-1}(V_L(S, \rho)) \setminus \partial M) \to C_k(M \setminus \partial M) \) is a weak equivalence.

**Proof.** As in the proof of lemma 1.2.2, it suffices to show that (2.2.1) is a Serre microfibration with contractible fibers. Contractibility of the fibers is straightforward. The fiber over a configuration \( T \in C_k(M \setminus \partial M) \) is identified with the classifying space of the sub-poset \( \mathcal{H}_T \) of \( \delta \mathcal{P}(L) \) consisting of all \( (S, \rho) \) such that \( p(T) \subset V_L(S, \rho) \). The poset \( \mathcal{H}_T \) is the target of a homotopy initial functor from the poset of the negative integers.

In showing that (2.2.1) is a Serre microfibration it is very important to understand that although (2.2.1) and the projection to \( |N\delta \mathcal{P}(L)| \) together determine an injective continuous map

\[
\text{hocolim}_{(S, \rho) \in \delta \mathcal{P}(L)} C_k(p^{-1}(V_L(S, \rho)) \setminus \partial M) \to |N\delta \mathcal{P}(L)| \times C_k(M \setminus \partial M),
\]

that injective continuous map is not an embedding (homeomorphism onto the image). Here is a lengthy example to illustrate the phenomenon and the advantages that it has for us. Suppose for simplicity that \( p: M \to L \) is an identity map, i.e., disk bundle with fiber dimension 0. Take two elements \((S, \rho)\) and \((T, \sigma)\) of \( \delta \mathcal{P}(L) \) such that \((S, \rho) > (T, \sigma)\). The inequality \((S, \rho) > (T, \sigma)\) determines a nondegenerate 1-simplex in \( N(\delta \mathcal{P}(L)) \), and an injective map \( \Delta^1 \to |N\delta \mathcal{P}(L)| \). View that as a path \( w: [0, 1] \to N(\delta \mathcal{P}(L)) \), beginning at \((T, \sigma)\) and ending at \((S, \rho)\). Suppose that \( w \) has a lift \( \tilde{w} \) to a path in hocolim \( C_k(V_L(S, \rho)) \). It is clear that the composition

\[
[0, 1] \xrightarrow{\tilde{w}} \text{hocolim}_{(S, \rho) \in \delta \mathcal{P}(L)} C_k(V_L(S, \rho)) \xrightarrow{\text{proj}} C_k(L)
\]

has the form \( t \mapsto R_t \) where the configuration \( R_t \) is contained in \( V(T, \sigma) \) if \( 0 \leq t < 1 \) and in \( V(S, \rho) \) when \( t = 1 \). But more careful reasoning shows that \( R_t \) must be contained in \( V(T, \sigma) \), too. This is fortunate for us because it implies at once that a sufficiently small homotopy of the composition (2.2.3) can be lifted to a homotopy of \( \tilde{w} \) itself, as the Serre microfibration condition wants to have it. — Now we return to our business, which is to establish the Serre microfibration condition for the map (2.2.1). Let \( Z \) be a compact subset of the source in (2.2.1) and let

\[
h: Z \times [0, 1] \to C_k(M \setminus \partial M)
\]

be a homotopy such that the map \( z \mapsto h(z, 0) \) agrees with the projection from \( Z \) to \( C_k(M \setminus \partial M) \). The image of \( Z \) in \( N\delta \mathcal{P}(L) \) is contained in a finite union of cells,
say $E_1, E_2, \ldots, E_q$. Let $X_1, \ldots, X_q$ be the closed subspaces of the source in (2.2.1) which are the preimages of the closures of $E_1, \ldots, E_q$ in $|N\delta\mathcal{P}(L)|$, respectively. Let $Z_i = Z \cap X_i$ for $i = 1, \ldots, q$. For sufficiently small $\varepsilon_i > 0$, the restriction of $h$ to $Z_i \times [0, \varepsilon_i]$ admits a unique lift to the source in (2.2.1), with the prescribed initial values, such that the composition

$$
Z_i \times [0, \varepsilon_i] \xrightarrow{\text{lift}} \text{hocolim}_{(S, \rho)} C_k(p^{-1}(V_L(S, \rho)) \setminus \partial M) \longrightarrow N\delta\mathcal{P}(L)
$$

is a stationary homotopy. (The example/digression above was meant to illustrate that slightly counter-intuitive claim.) These partial lifts are compatible by construction and so define a continuous lift

$$
Z \times [0, \varepsilon] \rightarrow \text{hocolim}_{(S, \rho)} C_k(p^{-1}(V_L(S, \rho)) \setminus \partial M)
$$

of $h$, where $\varepsilon$ is the minimum of the $\varepsilon_i$.

**Corollary 2.2.2.** The map

$$
\text{hocolim}_{(S, \rho) \in \delta\mathcal{P}(L)} N_0\mathcal{P}(p^{-1}(V_L(S, \rho)) \setminus \partial M) \longrightarrow N_0\mathcal{P}(M \setminus \partial M)
$$

determined by the inclusions $p^{-1}(V_L(S, \rho)) \setminus \partial M \longrightarrow M \setminus \partial M$ is a weak equivalence.

**Proof.** This is obtained from lemma 2.2.1 essentially by taking the disjoint union over all $k \geq 0$, noting that the hocolim respects disjoint unions. Perhaps it should be clarified that $\mathcal{P}(U)$, for an open subset $U$ of $M \setminus \partial M$, is defined or can be defined as the full topological sub-poset of $\mathcal{P}(M \setminus \partial M)$ consisting of all elements $(S, \rho)$ such that $V(S, \rho)$ is contained in $U$ and has compact closure in $U$. □

**Corollary 2.2.3.** For every $r \geq 0$ the map

$$
\text{hocolim}_{(S, \rho) \in \delta\mathcal{P}(L)} N_r\mathcal{P}(p^{-1}(V_L(S, \rho)) \setminus \partial M) \longrightarrow N_r\mathcal{P}(M \setminus \partial M)
$$

determined by the inclusions $p^{-1}(V_L(S, \rho)) \setminus \partial M \longrightarrow M \setminus \partial M$ is a weak equivalence.

**Proof.** This is obtained from the previous corollary by noting that for open $U$ in $M \setminus \partial M$ there is a homotopy pullback square

$$
\begin{array}{ccc}
N_r\mathcal{P}(U) & \longrightarrow & N_r\mathcal{P}(M \setminus \partial M) \\
\downarrow & & \downarrow \\
N_0\mathcal{P}(U) & \longrightarrow & N_0\mathcal{P}(M \setminus \partial M)
\end{array}
$$

where the horizontal arrows are inclusions and the vertical arrows are given by the ultimate target operator, also known as 0-th vertex operator. □
**Proof of theorem 2.1.1, first part.** There is a commutative square

\[
\begin{array}{ccc}
\text{holim} \quad \Psi(T, \sigma) & \longrightarrow & \text{holim} \quad \Psi(T, \sigma) \\
(T, \sigma) \in \mathcal{P}(M \setminus \partial M) & & (S, \rho) \in \delta \mathcal{P}(L)
\end{array}
\]

By corollary 2.2.3, the top horizontal arrow, given by specialization, is a weak equivalence. By theorem 1.1.1 and lemma 1.2.1, the lower horizontal arrow is a weak equivalence. We want to know that the left-hand vertical arrow is a weak equivalence. So it suffices to show that the right-hand vertical arrow is a weak equivalence.

For that it suffices to show that for fixed \((S, \rho) \in \delta \mathcal{P}(L)\) the map

\[
\text{holim} \quad \Psi(T, \sigma) \\
(T, \sigma) \in \mathcal{P}(M \setminus \partial M) \\
\ \\
\text{cls. of } p(K(T, \sigma)) \subseteq V_L(S, \rho)
\]

induced by the inclusion of \(\partial M \cup (M \setminus p^{-1}(V_L(S, \rho)))\) in the various \(\Psi(T, \sigma)\) is a weak equivalence. The target can also be written as

\[
\text{holim} \quad \Psi(T, \sigma) \\
(T, \sigma) \in \mathcal{P}(U) \\
\text{cls. of } p(K(T, \sigma)) \subseteq V_L(S, \rho)
\]

where \(U\) is the open subset \(p^{-1}(V_L(S, \rho)) \setminus \partial M\) of \(M \setminus \partial M\). We now make a few alterations to that expression, which turn out to be weak equivalences under \(\partial M \cup (M \setminus p^{-1}(V_L(S, \rho)))\).

1. Replace \(\mathcal{P}(U)\) by \(\delta \mathcal{P}(U)\).
2. Let \(F\) be the contravariant functor from \(\mathcal{O}(U)\) to spaces taking \(W \in \mathcal{O}(U)\) to \(\text{holim}_C(M \setminus C)\), where \(C\) runs through the compact subsets of \(W\). Replace \(\Psi(T, \sigma)\) by \(F(V(T, \sigma))\).
3. After implementing (1) and (2), replace \(\delta \mathcal{P}(U)\) by \(\bigcup_k \mathcal{O}_k(U)\) and replace \(F(V(T, \sigma))\) for \((T, \sigma) \in \delta \mathcal{P}(U)\) by \(F(W)\) for \(W \in \bigcup_k \mathcal{O}_k(U)\).

Alterations (1) and (3) can be justified by arguments which we have seen in section 1. Alteration (2) is justified because there is a comparison map from \(\Psi(T, \sigma)\) to \(F(V(T, \sigma))\) which is a weak equivalence. In this way, expression (2.2.4) turns into

\[
\text{holim} \quad F(W) \\
W \in \bigcup_k \mathcal{O}_k(U)
\]

But the poset \(\bigcup_k \mathcal{O}_k(U)\) has a maximal element, which is \(U\) itself. Therefore expression (2.2.5) can be replaced by \(F(U)\). It is easy to see that the reference map from \(\partial M \cup (M \setminus p^{-1}(V_L(S, \rho)))\) to \(F(U)\) is a weak equivalence. □
Proof of theorem 2.1.1, second part. Fix an integer \( j > 0 \) as in (2.1.3). We need a modification of lemma 2.2.1. Let \( k \) be another integer such that \( j \geq k \geq 0 \). The modification states that the projection map

\[
\text{holim}_{(S,\rho) \in \delta P_j(L)} C_k(p^{-1}(V_L(S,\rho)) \setminus \partial M) \to C_k(M \setminus \partial M)
\]

is a weak equivalence. The proof is exactly like the proof of lemma 2.2.1 itself: the map is again a Serre microfibration with contractible fibers. Note in passing that we need \( j \geq k \) for the contractibility of the fibers. — There is a commutative square

By a modification of corollary 2.2.3 which comes from the modification of lemma 2.2.1 just formulated, the top horizontal arrow is a weak equivalence. By theorem 1.1.1 and lemma 1.2.1, the lower horizontal arrow is \((1 + (j + 1)(c - 2))\)-connected. We want to know that the left-hand vertical arrow is \((1 + (j + 1)(c - 2))\)-connected. So it suffices to show that the right-hand vertical arrow is a weak equivalence. This can be verified as in the proof of the first half of theorem 2.1.1.

3. Gates

This section generalizes the previous two. Consequently it has two slightly different themes.

3.1. Submanifold case. For the first theme, imagine a smooth manifold \( M \) with boundary and a neat smooth compact submanifold \( L \), so that \( \partial L \subset \partial M \). We look for a homotopical description of \( M \setminus L \) in terms of the spaces \( M \setminus S \), where \( S \) runs through the finite subsets of \( L \setminus \partial L \). In the case where \( \partial L \) and \( \partial M \) are empty, this is exactly the situation of section 1. Also, in the case where \( \partial L \) is empty but \( \partial M \) is nonempty, it is almost exactly the situation of section 1 because in such a case it makes no substantial difference if we delete \( \partial M \) from \( M \).

For a more precise formulation we extend the definition of \( P(L) \) given in section 1 so that \( L \) is allowed to have a nonempty boundary. Choose a Riemannian metric on \( L \). The elements of \( P(L) \) are going to be pairs \((S,\rho)\) where \( S \) is a finite subset of \( L \setminus \partial L \) and \( \rho \) is a function from \( S \cup \partial L \) to the positive reals, locally constant on \( \partial L \) and subject to a few more conditions.

- For each \( s \in S \), the exponential map \( \exp_s \) at \( s \) is defined and regular on the disk of radius \( \rho(s) \) about the origin in \( T_s L \).
- The (boundary-normal) exponential map is defined and regular on the set of all tangent vectors \( v \in T_z L \) where \( z \in \partial L \), where the vector \( v \) is inward perpendicular to \( T_z \partial L \) and \( |v| \leq \rho(z) \).
- The images in $L$ of these disks and the image of this band under the 
exponential map(s) are pairwise disjoint.

For a pair $(S, \rho)$ satisfying these conditions, let $V_L(S, \rho) \subset L$ be the union of 
the open balls of radius $\rho(s)$ about elements $s \in S$ and the open collar on $\partial L$ 
determined the normal distance function $\rho|_{\partial L}$. Then $V_L(S, \rho)$ is diffeomorphic to 
$(\mathbb{R}^l \times S) \sqcup [0,1] \times \partial L$ and the inclusion of $M \setminus V_L(S, \rho) \subset M \setminus S$ is a homotopy 
equivalence. The partial order on $\mathcal{P}(L)$ is defined so that $(S_0, \rho_0) \leq (S_1, \rho_1)$ if and 
only if $V_L(S_0, \rho_0) \subset V_L(S_1, \rho_1)$. In this partial order the boundary $\partial L$ acts like a 
gate which allows occupants to leave.

The poset $\mathcal{P}(L)$ can also be viewed as a category. A contravariant functor $\Phi$ from 
$\mathcal{P}(L)$ to spaces is defined by

\begin{equation}
\Phi(S, \rho) = M \setminus V_L(S, \rho).
\end{equation}

There is a map

\begin{equation}
M \setminus L \longrightarrow \text{holim} \Phi,
\end{equation}
determined by the inclusions $M \setminus L \rightarrow \Phi(S, \rho)$ for $(S, \rho) \in \mathcal{P}(L)$. Also, let $\mathcal{P}_j(L)$ 
be the subspace and full topological sub-poset of $\mathcal{P}(L)$ consisting of all $(S, \rho)$ in 
$\mathcal{P}(L)$ that satisfy $|S| \leq j$. Then again there is a map

\begin{equation}
M \setminus L \longrightarrow \text{holim} \Phi|_{\mathcal{P}_j(L)}
\end{equation}
determined by the inclusions $M \setminus L \rightarrow \Phi(S, \rho)$.

**Theorem 3.1.1.** If $m - \ell \geq 3$, then the map \[3.1.2\] is a weak homotopy equiv-
ance and the map \[3.1.3\] is $(1 + (j + 1)(m - \ell - 2))$-connected.

The proof of is very similar to the proof of theorem \[1.1.1\] and the details are therefore omitted.

### 3.2. Absolute case.

Our second topic is a generalization of theorem \[2.1.1\] to a situation with more complicated boundary conditions. Let $M$ be a compact 
smooth manifold with boundary and corners in the boundary. In particular $\partial M$ 
is the union of two codimension zero smooth submanifolds $\partial_0 M$ and $\partial_1 M$ that intersect in the corner set 
$\partial_0 M \cap \partial_1 M = \partial(\partial_0 M) = \partial(\partial_1 M)$.

We look for a homotopical description of $\partial_1 M$ in terms of the spaces $M \setminus S$, where 
$S$ runs through the finite subsets of $M \setminus \partial M$.

Now $M \setminus \partial_1 M$ is a smooth manifold with boundary $\partial_0 M \setminus \partial_1 M$; both $M \setminus \partial_1 M$ 
and its boundary can be noncompact. Again we choose a Riemannian metric on 
all of $M$. For simplicity we require it to be a product metric in a neighborhood of 
$\partial_1 M$, i.e., the product of a Riemannian metric on $\partial_1 M$ and a Riemannian metric on $[0, \varepsilon]$ for some $\varepsilon > 0$. Then we can define a topological poset $\mathcal{P}(M \setminus \partial_1 M)$ 
roughly as in section \[2\] The elements are pairs $(S, \rho)$ where $S$ is a finite subset of 
$M \setminus \partial M = (M \setminus \partial_1 M) \setminus \partial(M \setminus \partial_1 M)$ and $\rho$ is a function from $S \sqcup (\partial_0 M \setminus \partial_1 M)$ to 
the positive reals which is locally constant on $\partial_0 M \setminus \partial_1 M$ and subject to the usual 
conditions. The usual conditions are, briefly stated: regularity of the exponential 
maps on the tangential disks and the tangential closed band defined by $\rho$, and 
pairwise disjointness of their images in $M \setminus \partial_1 M$. The union of the open balls of 
radius $\rho(s)$ about elements $s \in S$ and of the (half)-open band determined by the
normal distance function $\rho$ on $\partial_0 M$ is denoted by $V(S, \rho)$. Therefore $V(S, \rho)$ is an open subset of $M \setminus \partial_1 M$ diffeomorphic to $(\mathbb{R}^n \times S) \cup (\partial_0 M \setminus \partial_1 M) \times [0,1]$.

A contravariant functor $\Psi$ from $\mathcal{P}(M \setminus \partial_1 M)$ to spaces is defined by

$$\Psi(S, \rho) = M \setminus V(S, \rho).$$

There are maps

$$\partial_1 M \rightarrow \text{holim } \Psi,$$

$$\partial_1 M \rightarrow \text{holim } \Psi|_{\mathcal{P}(M \setminus \partial_1 M)}$$

induced by the inclusions $\partial_1 M \rightarrow M \setminus V(S, \rho)$.

**Theorem 3.2.1.** Suppose that $M \setminus \partial_1 M$ has a neat compact smooth submanifold $L$ of codimension $c \geq 3$ making $M$ into a smooth thickening of $L \cup \partial_0 M$; see definition (3.2.2) below. Then the map (3.2.2) is a weak equivalence and the map (3.2.3) is $(1 + (j + 1)(c - 2))$-connected.

**Definition 3.2.2.** On the meaning of smooth thickening in theorem (3.2.1) the main points are that the inclusion of $\partial_0 M \cup L$ in $M$ is a homotopy equivalence and that the inclusion of $\partial_1 M$ in $M \setminus L$ is a homotopy equivalence.

In detail, we start with a smooth compact manifold $L$, another smooth compact manifold $A$ (think $A = \partial_0 M$) and a smooth embedding $u: \partial L \rightarrow A$ which avoids the boundary of $A$. Then we can speak of $L \cup A$, the pushout of $L \leftarrow \partial L \rightarrow A$. Let $Q \subset L$ be a closed collar, so that $\partial L \subset Q$ and there is a diffeomorphism $Q \rightarrow \partial L \times [0,1]$ extending the map $x \mapsto (x,0)$ on $\partial L$. Let $L_1$ be the closure of $L \setminus Q$ in $L$. To make a smooth thickening of $L \cup A$ we need in addition

- a smooth disk bundle $E \rightarrow L_1$ whose total space has dimension $\dim(A) + 1$;
- an identification of $E|\partial L_1 \rightarrow \partial L_1 \cong \partial L$ with the normal disk bundle of the embedding $u: \partial L \rightarrow A$.

Then the pushout $T$ of $A \times [0,1] \leftarrow E|\partial L_1 \rightarrow E$ is defined, on the understanding that the left-hand arrow embeds $E|\partial L_1$ into $A \times \{1\}$. Now $T$ is a compact manifold with boundary, smooth with corners. The corner set has three disjoint parts: $\partial A \times \{0\}$, $\partial A \times \{1\}$ and $\partial(E|\partial L_1)$. Think of $\partial T$ as the union of $\partial_0 T := A \times \{0\}$ and $\partial_1 T$, the closure of $\partial T \setminus \partial_0 T$ in $\partial T$. The parts of the corner set not accounted for by $\partial_0 T \setminus \partial_1 T$ should be subjected to smoothing. There is a copy of $L = Q \cup L_1$ contained in $T$. And of course there is also a copy of $A$ contained in $T$, in the shape of $A \times \{0\}$. Any smooth manifold with corners which is diffeomorphic to such a $T$ relative to $L \cup A = L \cup \partial_0 T$ can be called a smooth thickening of $L \cup A$ or of $L \cup \partial_0 T$.

**3.3. Something like engulfing.** We turn to the proof of theorem (3.2.1). This is broken up into remarks, definitions, lemmas and even a corollary.

**Remark 3.3.1.** The second part of theorem (3.2.1) (the high connectivity statement) implies the first part (the weak equivalence statement). This is easy to see if we use the definition of holim $\Psi$ as a subspace of the space of maps from $|NP(M \setminus \partial_1 M)|$ to $M$. The inclusion of $|NP_j(M \setminus \partial_1 M)|$ in $|NP_k(M \setminus \partial_1 M)|$, for $k > j$, is a cofibration. It follows that the projection

$$\text{holim } \Psi|_{\mathcal{P}(M \setminus \partial_1 M)} \rightarrow \text{holim } \Psi|_{\mathcal{P}(M \setminus \partial_1 M)}$$
is a weak equivalence. The homotopy groups \( \pi_r \) of the right-hand side can be calculated as inverse limits

\[
\lim_j \pi_r (\text{holim } P_j(M \times_{\partial_1} M) )
\]

is a weak equivalence. The homotopy groups \( \pi_r \) of the right-hand side can be calculated as inverse limits

\[
\lim_j \pi_r (\text{holim } P_j(M \times_{\partial_1} M) )
\]

The higher derived inverse limit \( \text{lim}^1 \) does not contribute to this calculation because of the Mittag-Leffler criterion [10] ch.7.App.]. The criterion is applicable here because we are assuming the second part of theorem 3.2.1.

\textbf{Remark 3.3.2.} Since the validity of theorem 3.2.1 does not depend on the Riemannian metric which we select for \( M \), we can choose a Riemannian metric with very convenient properties. We shall assume that it is a product metric near \( \partial M \) as well, i.e., a neighborhood of \( \partial_0 M \) is isomorphic as a Riemannian manifold to a product \( \partial_0 M \times [0, \varepsilon] \) where the interval \([0, \varepsilon]\) has the standard Riemannian metric, and the isomorphism takes \( \partial_0 M \subset M \) to \( \partial_0 M \times \{0\} \subset \partial_0 M \times [0, \varepsilon] \). (The product structure near \( \partial_0 M \) is automatically compatible with the product structure near \( \partial_1 M \) which we assumed earlier, so that \( \partial \partial_0 M = \partial_0 M \cap \partial_1 M \) has a neighborhood in \( M \) which is isomorphic to \( \partial \partial_0 M \times [0, \varepsilon] \times [0, \varepsilon] \) as a Riemannian manifold.) As a result we have a standard compact collar for \( \partial_0 M \) (of width \( \varepsilon \)). We shall also assume that the intersection of \( L \) with that collar has the form \( \partial L \times [0, \varepsilon] \) in the collar coordinates.

In addition we choose an open tubular neighborhood \( U \) of \( L \) such that the closure \( \bar{U} \) of \( U \) in \( M \) is a smooth disk bundle over \( L \). Also, the intersection of \( U \) with the standard collar on \( \partial_0 M \) (of width \( \varepsilon \)) is required to have the form \( \partial U \times [0, \varepsilon] \) in the collar coordinates. Here \( \partial U \subset \partial_0 M \) is a tubular neighborhood of \( \partial L \) whose closure in \( \partial_0 M \) is a smooth disk bundle over \( \partial L \).

\textbf{Definition 3.3.3.} Let \( \mathcal{C}_1 \) be the full topological sub-poset of \( \mathcal{P}_j(M \setminus \partial M) \) consisting of the objects \((S, \rho)\) such that the locally constant function \( \rho_{|\partial_0 M} \) is \( \leq \varepsilon \) everywhere and the disks of radius \( \rho(s) \) about elements \( s \in S \) are all contained in \( U \), the specified tubular neighborhood of \( L \). Write \( \delta \mathcal{C}_1 \) for the discrete variant.

\textbf{Definition 3.3.4.} Let \( \mathcal{C}_0 \) be the full topological sub-poset of \( \mathcal{P}_j(M \setminus \partial M) \) consisting of the objects \((S, \rho)\) such that \( \rho \leq \varepsilon/3j \), and the closure of \( U(S, \rho) \) is contained in the union of \( U \) and the standard collar on \( \partial_0 M \) of width \( \varepsilon/3j \). Nota bene: for an object \((S, \rho)\) of \( \mathcal{C}_0 \) it can happen that \( S \) is not contained in \( U \). Write \( \delta \mathcal{C}_0 \) for the discrete variant.

\textbf{Lemma 3.3.5.} For every element \((S, \rho)\) of \( \delta \mathcal{C}_0 \) there is some element of \( \delta \mathcal{C}_1 \) which is \( \geq (S, \rho) \) in \( \mathcal{P}_j(M \setminus \partial_0 M) \). Indeed the sub-poset of \( \delta \mathcal{C}_1 \) consisting of the elements which are \( \geq (S, \rho) \) has a contractible classifying space.

\textbf{Proof.} It is enough to note that there is a real number \( \tau \), strictly between \( \varepsilon/3j \) and \( \varepsilon \), such that the parallel hypersurface to \( \partial_0 M \) in \( M \) at distance \( \tau \) from \( \partial_0 M \) has empty intersection with the closure of \( V(S, \rho) \). This is due to our assumption \( |S| \leq j \) and the smallness of the radii in the metric balls which are part of \( V(S, \rho) \). \( \square \)
Corollary 3.3.6. A commutative diagram of the following shape can be supplied,

\[
\begin{array}{ccc}
\text{holim } \Psi|_{\mathcal{P}_j(M \smallsetminus \partial_1 M)} & \longrightarrow & \text{holim } \Psi|_{\mathcal{E}_1} \\
\downarrow & & \downarrow \\
\text{holim } \Psi|_{\mathcal{E}_0} & \simeq & Y
\end{array}
\]

where the solid arrows are given by restriction.

Proof. Define \( Y \) to be

\[
\text{holim } \Psi|_{(S, \rho) \in \mathcal{E}_0} \quad \text{holim } \Psi(S, \rho).
\]

The two broken arrows are then fairly obvious. The horizontal broken arrow is a weak equivalence by lemma 3.3.3. □

Lemma 3.3.7. The map \( \text{holim } \Psi|_{\mathcal{P}_j(M \smallsetminus \partial_1 M)} \longrightarrow \text{holim } \Psi|_{\mathcal{E}_0} \) given by restriction is a weak equivalence. The map \( \text{holim } \Psi|_{\mathcal{E}_0} \longrightarrow \text{holim } \Psi|_{\mathcal{E}_0} \) given by restriction is also a weak equivalence.

Proof. For the first statement, observe that the inclusion map from \( \mathcal{N}_r \mathcal{E}_0 \) to \( \mathcal{N}_r \mathcal{P}_j(M \smallsetminus \partial_1 M) \) is a homotopy equivalence for every \( r \geq 0 \). The second statement can be proved like lemma 1.2.1. □

Lemma 3.3.8. The composition

\[
\partial_1 M \quad \text{holim } \Psi|_{\mathcal{P}_j(M \smallsetminus \partial_1 M)} \quad \longrightarrow \text{holim } \Psi|_{\mathcal{E}_1}
\]

is \( (1 + (j + 1)(c - 2)) \)-connected.

Proof. That composition is one arrow in a commutative square

\[
\begin{array}{ccc}
\partial_1 M & \longrightarrow & \text{holim } \Psi|_{\mathcal{E}_1} \\
\downarrow \subset & & \downarrow \subset \\
M \smallsetminus L & \longrightarrow & \text{holim } \Phi|_{\mathcal{E}_1}
\end{array}
\]

where, in the lower row, \( \mathcal{E}_1 \) is viewed as a sub-poset of \( \mathcal{P}_j(L) \) and \( \Phi \) is the functor of theorem 3.1.1. That is, \( \Phi(S, \rho) \) means \( M \smallsetminus V_L(S, \rho) \). Now the inclusion of \( \mathcal{N}_r \mathcal{E}_1 \) in \( \mathcal{N}_r \mathcal{P}_j(L) \) is clearly a weak equivalence, for every \( r \geq 0 \). Consequently there is no need to distinguish carefully between \( \text{holim } \Phi|_{\mathcal{E}_1} \) and \( \text{holim } \Phi \) of theorem 3.1.1. In particular, it follows that the lower horizontal arrow in the above square is a weak equivalence. The inclusion of \( \partial_1 M \) in \( M \smallsetminus L \) is also a homotopy equivalence (definition 3.2.2). The right-hand vertical arrow in the diagram is a weak equivalence, too, because for each element \((S, \rho) \in \mathcal{E}_1\), the inclusion of \( \Psi(S, \rho) = M \smallsetminus V(S, L) \) in \( \Phi(S, \rho) = M \smallsetminus V_L(S, \rho) \) is a homotopy equivalence. Since three of the arrows in the square are weak equivalences, the remaining one must be a weak equivalence. □
Conclusion of proof of theorem 3.2.1. By remark 3.3.1 we can concentrate on the high connectivity statement, for a fixed $j \geq 0$. The commutative square of corollary 3.3.6 can be enlarged to a commutative diagram

$$
\begin{array}{c}
\partial_1 M \\
\text{holim } \Psi|_{\mathcal{P}_j(M \smallsetminus \partial_1 M)} \\
\text{holim } \Psi|_{\delta \mathcal{C}} \\
\text{holim } \Psi|_{\delta \mathcal{C}} \simeq Y
\end{array}
\end{array}
$$

The weak equivalence label in the lower left part of the diagram is justified by lemma 3.3.7. By lemma 3.3.8, the diagonal arrow in the upper half of the diagram is highly connected. It follows that the map (3.2.3) is split injective on homotopy groups or homotopy sets in a certain range, because the middle horizontal arrow in the diagram provides a splitting. But the middle horizontal itself induces a split injection on all homotopy groups or homotopy sets, too, as the lower part of the diagram shows.

4. Occupants near the gate

Imagine a smooth compact manifold $M$ with corners in the boundary, so that $M = \partial_0 M \cup \partial_1 M$ as in theorem 3.2.1. (There is no need to assume here that $M$ is a thickening according to definition 3.2.2.) We might ask for a homotopical description of the corner set $\partial \partial_1 M = \partial \partial_0 M = \partial_0 M \cap \partial_1 M$ in terms of the spaces $M \smallsetminus S$, where $S$ runs through the finite subsets of $M \smallsetminus \partial M$ which are close to $\partial \partial_0 M$. The description proposed here is conjectural, but the associated definitions will be useful elsewhere, in a sequel to this paper.

4.1. A conjecture. Let $\mathcal{P}(M \smallsetminus \partial_1 M)$ be defined as in theorem 3.2.1. Form the twisted arrow poset $\text{tw}(\mathcal{P}(M \smallsetminus \partial_1 M))$. (This is a special case of the twisted arrow category construction which was mentioned in the proof of lemma 1.2.1.) There is a contravariant functor $\Theta$ from $\text{tw}(\mathcal{P}(M \smallsetminus \partial_1 M))$ to spaces given by

$$
\Theta((S, \rho) \leq (T, \sigma)) := (\text{closure in } M \text{ of the collar part of } V(T, \sigma) \smallsetminus V(S, \rho)).
$$

This comes with a natural inclusion map $\partial \partial_1 M \to \Theta((S, \rho) \leq (T, \sigma))$ which in turn induces a map

$$
(4.1.1) \quad \partial \partial_1 M \longrightarrow \text{holim } \Theta.
$$

Conjecture 4.1.1. The map (4.1.1) is a weak equivalence under some conditions on $\partial_0 M$. (These conditions on $\partial_0 M$ should be reminiscent of the conditions in theorem 2.1.1. They might say, for example, that $\partial_0 M$ is the total space of a smooth disk bundle with fibers of dimension $\geq 2$ on a closed manifold $L$.)

We now arrange the maps (4.1.1) and (3.2.2) in a commutative diagram. This will also motivate conjecture 4.1.1. There is a natural transformation

$$
\Theta \longrightarrow \Psi \circ F_s
$$
given by inclusion, where \( F_* : \hom(\mathcal{P}(M \setminus \partial_1 M)) \to \mathcal{P}(M \setminus \partial_1 M) \) is the functor source. In more detail, for an object \( (S, \rho) \leq (T, \sigma) \) of \( \hom(\mathcal{P}(M \setminus \partial_1 M)) \) the space \( \Theta((S, \rho) \leq (T, \sigma)) = (\text{closure in } M \text{ of the collar part of } V(T, \sigma) \setminus V(S, \rho)) \) is obviously contained in \( \Psi(S, \rho) = M \setminus V(S, \rho) \). Using this we get a commutative diagram

\[
\begin{array}{ccc}
\partial \partial_1 M & \xrightarrow{\text{inclusion}} & \partial_1 M \\
\downarrow \leftarrow & \downarrow \leftarrow & \downarrow \leftarrow \\
\text{holim } \Theta & \longrightarrow & \text{holim } \Psi \\
\end{array}
\]

(4.1.2)

\[ \leftarrow \rightarrow \]

LEMMA 4.1.2. The map \( \Theta \longrightarrow \text{holim } \Psi \circ F_* \longrightarrow \text{holim } \Psi \) is a weak equivalence.

PROOF. Recall that holim \( \text{holim } \Psi \) was defined as \( \text{Tot}(X) \) for a certain cosimplicial space \( X \). Namely, \( X_r \) is the space of sections of the fiber bundle on \( N_r \mathcal{P}(M \setminus \partial M) \) whose fiber over \( ((S_r, \rho_r) \geq \cdots \geq (S_0, \rho_0)) \) is \( M \setminus V(S_0, \rho_0) \). Let \( \beta: \Delta \to \Delta \) be the functor \( [n] \mapsto [2n + 1] \). More precisely, \( \Delta \) is the category of totally ordered nonempty finite sets and order-preserving maps, or the equivalent full subcategory with objects \([n]\) for \( n \geq 0 \), and \( \beta \) is the functor which takes a totally ordered nonempty finite set \( S \) to \( S \sqcup S^\text{op} \) (with the total ordering where \( a < b \) whenever \( a \in S \subseteq S \sqcup S^\text{op} \) and \( b \in S^\text{op} \subseteq S \sqcup S^\text{op} \)). The inclusions \( S \to S \sqcup S^\text{op} \) define a natural transformation \( e: \text{id} \to \beta \). The cosimplicial space \( X \circ \beta \) is Reedy fibrant, by the same argument which we used to show that \( X \) is Reedy fibrant. We need to show that the map \( e_* : \text{Tot}(X) \to \text{Tot}(X \circ \beta) \) is a weak equivalence. Since both \( X \) and \( X \circ \beta \) are Reedy fibrant, we can use the easier variant \( \text{Tots} \) of \( \text{Tot} \) where only the (co)face operators are used; but we continue to view \( X \) and \( X \circ \beta \) as cosimplicial spaces. In this setting a more general statement can be made: if \( Y \) is any cosimplicial space, then \( e_* : \text{Tots}(Y) \to \text{Tots}(Y \circ \beta) \) is a weak equivalence. To show this, we use standard resolution procedures and note that \( \text{Tots} \) preserves degreewise weak equivalences. Therefore we may assume that \( Y \) is a homotopy inverse limit of cosimplicial spaces having the form

\[
[r] \mapsto \text{map}(\hom_{\Delta}([r], [t]), Z)
\]

(4.1.3)

for some \([t] \in \Delta \) and a space \( Z \). Since \( \text{Tots} \) commutes with such homotopy inverse limits, it suffices to show that \( e_* : \text{Tots}(Y) \to \text{Tots}(Y \circ \beta) \) is a weak equivalence when \( Y \) has the form \( (4.1.3) \). In that case \( \text{Tots}(Y) \) is just the space of maps from the geometric realization of the semi-simplicial set

\[
[r] \mapsto \text{hom}_\Delta([r], [t])
\]

(4.1.4)

to \( Z \). Similarly \( \text{Tots}(Y \circ \beta) \) is the space of maps from the geometric realization of the semi-simplicial set

\[
[r] \mapsto \text{hom}_\Delta([2r + 1], [t])
\]

(4.1.5)

to \( Z \). Now it is enough to show that the geometric realizations of \((4.1.4)\) and \((4.1.5)\) are both contractible. For that we may pretend or observe that both \((4.1.4)\) and \((4.1.5)\) are actually simplicial sets and realize them as such. The result is in one case a standard geometric \( t \)-simplex. In the other case it is an edgewise subdivided \( t \)-simplex. \[\square\]
4.2. Evidence for the conjecture. One important aspect of diagram (4.1.2) which can be established here is that the left-hand column depends only on an arbitrarily small open neighborhood $U$ of $\partial_0 M$ in $M$. This does not require any special geometric assumptions on $M$. To make a precise statement, let $\partial_1 U = U \cap \partial_1 M$ and $\partial_0 U = \partial_0 M$ and define $\mathcal{P}(U \setminus \partial_1 U)$ by analogy with $\mathcal{P}(M \setminus \partial_1 M)$. Alternatively define it as a full topological sub-poset of $\mathcal{P}(M \setminus \partial_1 M)$, consisting of the objects $(S, \rho)$ for which the closure of $V(S, \rho)$ in $M$ is contained in $U$. Let $\Theta_U$ be the restriction of $\Theta$ to $\text{tw}(\mathcal{P}(U \setminus \partial_1 U))$.

**Proposition 4.2.1.** The projection map from $\text{holim} \Theta$ to $\text{holim} \Theta_U$ is a weak equivalence.

**Proof.** We introduce another topological poset $Q = Q(M \setminus \partial_1 M)$ which is closely related to $\text{tw}(\mathcal{P}) = \text{tw}(\mathcal{P}(M \setminus \partial_1 M))$. Intuitively, $Q$ is the quotient poset obtained from $\text{tw}(\mathcal{P})$ by forcing a morphism in $\text{tw}(\mathcal{P})$ to be an equality (in $Q$) if the functor $\Theta$ takes it to an identity map of spaces. More formally, an element of $Q$ is (or can be represented) by an element

$$((S, \rho) \leq (T, \sigma))$$

of $\text{tw}(\mathcal{P})$ where $T = \emptyset$, so that $V(T, \sigma)$ is nothing but a collar. For two such pairs

$$((S, \rho) \leq (T, \sigma)), \quad ((S', \rho') \leq (T', \sigma'))$$

where $T = T' = \emptyset$, we say that the first is $\leq$ the second if

$$\Theta((S', \rho') \leq (T', \sigma')) \subset \Theta((S, \rho) \leq (T, \sigma)).$$

There is a functor (continuous map of posets) $K$ from $\text{tw}(\mathcal{P})$ to $Q$ which takes

$$((S, \rho) \leq (T, \sigma))$$

in $\text{tw}(\mathcal{P})$ to $((S_0, \rho_0) \leq (T_0, \sigma_0))$ where $T_0 = \emptyset$ and $\sigma_0$ is the appropriate restriction of $\sigma$, while $S_0$ is the part of $S$ which is contained in the collar part of $V(T, \sigma)$ and $\rho_0$ is the appropriate restriction of $\rho$. By construction we have

$$\Theta = \Theta_1 \circ K$$

for a functor $\Theta_1$ from $Q$ to spaces. This gives us a map

$$\text{holim} \Theta_1 \longrightarrow \text{holim} \Theta_1 \circ K = \text{holim} \Theta.$$

Making analogous constructions with $U$ instead of $M$, we obtain an analogous map from $\text{holim} \Theta_{1,U}$ to $\text{holim} \Theta_U$ and a commutative diagram of spaces

(4.2.1)

The remainder of the proof consists in showing that three of the arrows in (4.2.1) are weak equivalences.

To deal with the easiest case first, let us look at the top horizontal arrow. It is a weak equivalence because the inclusion of the nerve of $Q_U = Q(U \setminus \partial_1 U)$ in the nerve of $Q = Q(M \setminus \partial_1 M)$ is a degreewise weak equivalence. Next, let us try the left-hand vertical arrow (the case of the right-hand vertical arrow is similar). Let $K_* \Theta$ be the right derived pushforward of $\Theta$ along $K$. (In general, for a functor $\kappa: \mathcal{A} \rightarrow \mathcal{B}$ of small categories and a contravariant functor $g$ from $\mathcal{A}$ to spaces, the
right derived pushforward $\kappa_*g$ is the contravariant functor from $\mathcal{B}$ to spaces defined by $b \mapsto \text{holim} \ (g \circ v_b)$ where $v_b$ is the forgetful functor from the comma category $(b \downarrow \kappa)$ to $\mathcal{A}$. The objects of $(b \downarrow \kappa)$ are pairs $(a, f: b \to \kappa(a))$, etc.) There is a standard comparison map

$$(4.2.2) \quad \text{holim} \Theta \to \text{holim} K_*\Theta.$$

In the setting of discrete categories, it is well known that this type of comparison map is a weak equivalence. (The general statement is that a standard comparison map $\text{holim} g \to \text{holim} \kappa_*g$ is a weak equivalence, assuming that $\kappa: \mathcal{A} \to \mathcal{B}$ is a functor of small discrete categories.) The fact that we are working with topological posets and continuous homotopy limits might make it unsafe to use that. But we need rather less: we only need to know that the map $(4.2.2)$ has a homotopy left inverse. This is very easy to show. Consequently it suffices to show that the composition

$$\text{holim} \Theta_1 \longrightarrow \text{holim} \Theta \longrightarrow \text{holim} K_*\Theta$$

is a weak equivalence. This is a map of homotopy limits induced by a natural transformation of functors on $Q$, from $\Theta_1$ to $K_*\Theta$. It suffices to show that this natural transformation is objectwise a weak equivalence. To that end fix an object in $Q$, say

$$z = \left((S, \rho) \leq (T, \sigma)\right)$$

where $T = \emptyset$. We need to form the comma category $(z \downarrow K)$. It is still a topological poset, and indeed, a full topological sub-poset $\mathcal{A}(z)$ of $\text{tw}(\mathcal{P}) = \text{tw}(\mathcal{P}(M \setminus \partial_1 M))$. The elements of $\mathcal{A}(z)$ are all

$$y = \left((S', \rho') \leq (T', \sigma')\right)$$

in $\text{tw}(\mathcal{P})$ such that $\Theta_1(z) \subset \Theta(y)$; here both $\Theta_1(z)$ and $\Theta(y)$ are subspaces of $M$. So our remaining task is to show that the canonical map

$$\Theta_1(z) \longrightarrow \text{holim} \Theta|_{\mathcal{A}(z)},$$

induced by the inclusions $\Theta_1(z) \to \Theta(y)$ for each $y \in \mathcal{A}(z)$, is a weak equivalence. This can easily be done using standard adjunction tricks. Let $\mathcal{B}(z) \subset \mathcal{A}(z)$ be the full topological sub-poset consisting of all $y$ such that $\Theta_1(z) = \Theta(y)$. The inclusion $\mathcal{B}(z) \to \mathcal{A}(z)$ has a right adjoint, that is, for every $y \in \mathcal{A}(z)$ the set $\{y' \in \mathcal{B}(z) \mid y \leq y'\}$ has an absolute minimum. It follows that the projection

$$\text{holim} \Theta|_{\mathcal{A}(z)} \longrightarrow \text{holim} \Theta|_{\mathcal{B}(z)}$$

is a weak equivalence. Here we note that $\Theta|_{\mathcal{B}(z)}$ is already a constant functor, so it suffices to show that $\mathcal{B}(z)$ has a contractible classifying space. Let $\mathcal{C}(z) \subset \mathcal{B}(z)$ be the full topological subset consisting of all $y = ((S', \rho') \leq (T', \sigma'))$ in $\mathcal{B}(z)$ such that the source $(S', \rho')$ agrees with $(S, \rho)$, the source of $z$. The inclusion $\mathcal{C}(z) \to \mathcal{B}(z)$ has a left adjoint, that is, for every $y \in \mathcal{B}(z)$ the set $\{y' \in \mathcal{C}(z) \mid y' \leq y\}$ has an absolute maximum. Therefore the map of classifying spaces induced by $\mathcal{C}(z) \hookrightarrow \mathcal{B}(z)$ is a weak equivalence. But $\mathcal{C}(z)$ has a minimal element, so the classifying space of $\mathcal{C}(z)$ is contractible. \qed
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