AN ALGEBRAIC GEOMETRIC FOUNDATION FOR A
CLASSIFICATION OF SUPERINTEGRABLE SYSTEMS IN
ARBITRARY DIMENSION

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Abstract. Second order superintegrable systems in dimensions two and three are essentially classified, but current methods become unmanageable in higher dimensions because the system of non-linear partial differential equations they rely on grows too fast with the dimension. In this work we prove that the classification space of non-degenerate second order superintegrable systems is naturally endowed with the structure of a projective variety with a linear isometry action. Hence the classification is governed by algebraic equations. For constant curvature manifolds we provide a single, simple and explicit equation for a variety of superintegrable Hamiltonians that contains all known non-degenerate as well as some degenerate second order superintegrable systems. We establish the foundation for a complete classification of second order superintegrable systems in arbitrary dimension, derived from the geometry of the classification space, with many potential applications to related structures such as quadratic symmetry algebras and special functions.

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1. Introduction

Symmetries are an essential tool in the study of Hamiltonian systems and superintegrable systems are the most symmetric of these. Their study has a long history due to the attractive possibility of determining almost all important features using algebraic methods alone. Special functions appear naturally as a manifestation of a superintegrable system’s symmetry and, for many authors, are the main motivation for their study.

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1.1. **Special functions and superintegrable systems.** Since the appearance of the first tables of chords [Ptory], special functions have been ubiquitous in science and technology. Their fundamental role necessitates not only explicit formulae or the tabulation of a function’s values, but also a thorough documentation of its properties and interrelations. Traditionally, this has been done in the form of handbooks, most notably the Bateman Manuscript Project [Bat53, Bat54], filling five thick volumes, and the “Abramowitz and Stegun” Handbook of Mathematical Functions [AS], with more than 40,000 citations one of the most cited works in the literature [BCLO11]. In the dawn of the era of digitisation, the use of sophisticated symbolic computation engines has overcome the limitations of books and manual calculations, and handbooks have been replaced by extensive online databases. The most complete resources today are the Mathematical Functions Site [WMT], which comprises at present more than 300,000 formulae and is steadily growing, and the NIST Digital Library of Mathematical Functions [DLMF], the online version of the above mentioned Handbook of Mathematical Functions.

Yet, special functions have always been organised in an ad hoc manner and all handbooks and databases are mere compilations. Meanwhile, the quest for a unified theory of special functions has continued since the nineteenth century – a theory that would explain and systematically organise, for a reasonably wide class of special functions, their properties, interrelations, symmetry principles and other related structures behind the façade of seemingly endless formulae in rows.

The present work lays the foundation for subsequent research leading to such a theory – a “Periodic Table of Special Functions” comprising a wide variety of special functions derived from a sequence of projective $G$-varieties whose dimensions and geometric invariants resemble the role of the atomic and quantum numbers in the Periodic Table of Elements. Inasmuch as the Schrödinger Equation provides the basis for a systematic mathematical description of chemical elements and their properties, we establish a single, simple equation defining these varieties, which provides the basis for a systematic algebraic geometric description of special functions and their properties.

A theory aiming to classify special functions may naturally start from some rich source of such functions. This is where superintegrable systems come into play: Besides the hypergeometric differential equation, they are a particularly prolific source of special functions. Notably, it has been shown that superintegrable systems give rise to hypergeometric orthogonal polynomials [KMP07, KMP13], to Painlevé transcendants [Mar19, Gra04], to Jacobi-Dunkl polynomials [GIVZ13] and to the recently discovered exceptional polynomials [PTV12, HMPZ18]. *The present work establishes the foundations for a complete classification of second order superintegrable systems.*

1.2. **What are superintegrable systems?** The prototypical example of a superintegrable system is the Kepler-Coulomb system of planetary motion around a central celestial body and its quantum counterpart, the hydrogen atom. By the nature of the equations of motion, the movement of the planet is completely determined by its position and momentum given at any fixed point in time. More abstractly, we would say this movement defines a curve in the six-dimensional space of position and momentum called *phase-space.*

In a conservative central force field, the energy and the angular momentum vector are conserved under the temporal evolution of the system. The Kepler system has
the remarkable property that it possesses an additional conserved quantity: the Laplace-Runge-Lenz vector, pointing from the force centre towards the perihelion of the planetary orbit. Together, these form seven scalar constants of motion. Each of them defines a function on phase-space and confines the trajectory of the system to a level set of this function. Since phase space is six-dimensional, only five out of them can be (functionally) independent. Indeed, there are two scalar identities among them.

With the Kepler system in mind, a (maximally) superintegrable system is defined as an \(n\)-dimensional Hamiltonian system possessing the maximal number of functionally independent constants of motion, which is \(2n - 1\). The superintegrable system is called quadratic if the constants of motion can be chosen quadratic in momenta.

### 1.3. Classification of superintegrable systems – State-of-the-art.

To date, the most well-developed results on the classification and structure concern second order (conformally) superintegrable systems on conformally flat spaces in dimensions two and three. In particular, such results exist for the Euclidean plane [Tem04, KKM07b], Koenigs and Darboux spaces [KKW02, KKMW03], general 2D spaces [DY06, KKM05a, KKM05b], degenerate 2D superintegrable systems [KKMP09], 2D quantum superintegrable systems [BDK93, KKM06b, DT07], 3D flat space [KKM07c], 3D conformally flat spaces [KKM05c, KKM06a, KKM07a, KKM07c, CK14] and for quantum superintegrable systems on 3D conformally flat spaces [KKM06b]. For an overview see [Win04], [MPW13] or the comprehensive monograph [KKM18]. Most of the systems were known prior to their classification and have been constructed under the additional assumption of separability or multiseparability [KMP00a, KMP00b, KK02].

Above dimension three, only sporadic families of second-order superintegrable systems are known, such as the harmonic oscillator, a generalisation of the Kepler system [PP90, BH09], the Smorodinsky-Winternitz system I [FMS+65], also referred to as the generic system, and the Smorodinsky-Winternitz system II [KKMP07], which can be interpreted as an anisotropic oscillator model [RTW08].\(^1\) From these, further \(n\)-dimensional families can be obtained through Bôcher contractions [Bôc94, KKMP07] or coupling constant metamorphosis (aka Stäckel transforms) [Pos10].

To summarise, complete classification results are only known in dimensions two and three. Despite the substantial use of computer algebra, an extension of the classification to higher dimensions is out of the scope of current methods and therefore one of the most challenging problems in the theory of superintegrability. The main reason for this is that the number and the complexity of the partial differential equations used in current approaches grows way too fast with the dimension. In this work we shall overcome this hindrance and outline a new approach to the classification of second order superintegrable systems in arbitrary dimension.

### 1.4. What does a “classification” actually mean?

Before one begins to classify superintegrable systems, one should first clarify what is actually meant by the word “classification”. In its simplest meaning, it stands for an explicit list of all objects

\(^1\) We limit ourselves to second order systems here. If one includes higher order superintegrable systems, additional families are known, such as the Calogero-Moser system [Woj83], the Toda lattice [ADS06].
under consideration or, more formally, a bijection with some explicitly given set—called the classification space. Usually, however, this set carries much more structure. In the present case of superintegrable systems, for instance, the classification space can be endowed with at least three natural structures:

**Topology:** As solutions to a system of partial differential equations, the classification space inherits a natural topology.

**Group action:** The definition of superintegrability is invariant under isometries. We therefore have a well-defined action of the isometry group on the classification space.

**Equivalence relation:** Apart from equivalence under isometries, there is a second well-known transformation for superintegrable systems, called Stäckel equivalence [BKM86] or coupling constant metamorphosis [HGDR84].

So instead of a bare set, the classification space for superintegrable systems is at least a topological $G$-space. This suggests that a classification of superintegrable systems for a given (pseudo-)Riemannian manifold should be considered as an isomorphism in the category of topological $G$-sets, namely between the classification space and some explicitly given topological $G$-space.

More generally, one should first fix a category in which to consider the classification problem for superintegrable systems. A solution then consists of the following:

(i) A proof that the classification space is an object of this category.

(ii) An explicit object in this category.

(iii) An isomorphism between the classification space and this object.

Here we will provide the foundations for a classification of second order superintegrable systems in the category of projective $G$-varieties.

1.5. **First result: The classification space is a variety.** We prove that the kinetic part of the constants of motion completely determines a non-degenerate second order superintegrable system. Since the kinetic part is given by a Killing tensor, a non-degenerate superintegrable system on an $n$-dimensional manifold defines a $(2n - 1)$-dimensional subspace in the finite dimensional space $K(M)$ of Killing tensors. We define a classification space that in addition to all irreducible non-degenerate second order superintegrable systems also comprises part of the degenerate systems. Our classification space can naturally be identified with a subset in the Grassmannian $G_{2n-1} (K(M))$. We then prove that this subset is actually a subvariety.

Classical theory has always dealt with partial differential equations to solve the classification problem for superintegrable systems. Our result now shows that these equations are, at base, purely algebraic equations which come disguised as partial differential equations in an intricate manner. This also indicates that classical techniques are inadequate: Instead of solving partial differential equations, one should try to understand the geometry of the classification space using powerful algebraic-geometric methods, as has been noticed in the review paper [MPW13]:

"The possibility of using methods of algebraic geometry to classify superintegrable systems is very promising and suggests a method to extend the analysis in arbitrary dimension as well as a way to understand the geometry underpinning superintegrable systems."
Despite the fact that experts in the field agree that an algebraic-geometric approach is a promising route to a classification of superintegrable systems in arbitrary dimension, such a route has never been outlined concretely. *The subject of the present work is to provide exactly this.*

1.6. **State-of-the-art, revised.** In the light of the aforesaid, it should also be mentioned that the explicit question about the nature of the classification space has never been raised in the literature. All currently known classification results for superintegrable systems consist in writing down lists of normal forms under isometries. In other words, they study the quotient of the classification space under isometries. Although never proven in general, this quotient turns out to be finite in all known cases. While passing to the quotient is convenient, as it yields finite lists of simple normal forms, it destroys most information about the geometry of the classification space. The latter is only studied implicitly by considering limits of superintegrable systems in the form of orbit degenerations and so called Böcher contractions [KJS16a, KJS16b, RKWMS17, RWMS17], induced by İnönü-Wigner-contractions of the isometry group. Hence the currently known classification of second order superintegrable systems can be considered a classification in the category of sets, i.e. in the most elementary category. *The results of the present work entail that the classification problem for superintegrable systems should be considered in the category of projective $G$-varieties. In this category, the classification problem remains unsolved, even in dimensions two and three.*

1.7. **Desiderata.** In the present paper, we propose to approach the classification of superintegrable systems by studying the geometry of the classification space. Abstractly proving that the classification space is endowed with the structure of a variety is, however, insufficient, as it does not provide us with explicit and manageable algebraic equations. Particularly, for a viable approach we desire certain properties to hold. Ideally, the equations should be:

**Explicit:** The equations should be written down explicitly.

**Concise:** There should not be too many equations, and they should be simple.

**Generic:** The equations should have the same form in any dimension, except for dimension dependant constants.

**Tensorial:** The equations should be tensorial, making them independent of coordinate changes on the base manifold.

**Equivariant:** The equations should be explicitly equivariant under isometries.

**Natural:** The equations should naturally arise from the definition of superintegrability and not, e.g., be derived a posteriori from a known classification.

**Algebraic:** The equations should be polynomial.

**Low-degree:** The algebraic equations should have a low polynomial degree.

**Solvable:** It should be possible to solve the equations in any dimension, at best without resorting to the (excessive) use of computer algebra.

Note that the equations used in the existing literature to classify superintegrable systems do not satisfy most of these conditions. *Somewhat surprisingly, however,*
it turns out that all of them can be satisfied, as we are going to show in the present work.\textsuperscript{2}

1.8. Second result: Explicit algebraic equations. We give explicit algebraic equations for the variety of those superintegrable Hamiltonians for which all necessary integrability conditions are generically satisfied. For constant curvature manifolds this variety comprises all known non-degenerate as well as some of the degenerate second order superintegrable Hamiltonians. We show that it is isomorphic to the variety of cubic forms $\Psi_{ijk}x^ix^jx^k$ on $\mathbb{R}^n$ satisfying the simple algebraic equation
\begin{equation}
\Psi^{a}_{i|j|k} = -9\kappa g_{i|j|k},
\end{equation}
where $\kappa$ is the (constant) sectional curvature and the brackets denote antisymmetrisation in $j$ and $k$. Conjecturally, the whole classification space fibres over this variety.

We show that every superintegrable system in the classification space gives rise to a torsion-free affine connection, which is flat exactly if the above equation holds. The origin of this structure must be sought in the fact that one can develop a conformally invariant notion of superintegrability, for which Stäckel transforms are nothing but conformal equivalences. This suggests that in the corresponding conformal geometry the Bertrand-Darboux condition gives rise to several tractor bundles equipped with connections parameterised by superintegrable systems. We emphasise that classical superintegrability theory, although dealing with conformally superintegrable systems on conformally flat manifolds, has never regarded superintegrability from this geometric perspective.

A reformulation of superintegrability in terms of projective or conformal geometry is out of the scope of the present publication, as well as a comprehensive solution of the above equation, a description of the geometry of the corresponding variety, a derived complete classification of second order superintegrable systems on constant curvature manifolds and of related structures such as quadratic symmetry algebras and hypergeometric orthogonal polynomials. This program will be carried out in future publications and is based on the results in this article.

1.9. What may we expect? The algebraic geometric approach employed here to the classification of superintegrable systems is inspired by a similar approach to the classification of separable systems developed by the first author, together with Alexander P. Veselov, which has culminated in a remarkable isomorphism between the classification space of separable systems (in normal form) on an $n$-dimensional sphere and the real Deligne-Mumford-Knudsen moduli space of stable genus zero curves with $n + 2$ marked points $[SV15]$:

\[\mathcal{M}_{0,n+2}(\mathbb{R})\]

Separable and superintegrable systems are closely related, suggesting that we may deal with a renowned variety with prominent geometry in the case under consideration here as well.

Most known superintegrable systems are multiseparable, meaning that they contain different separable systems. This might even be true for all known superintegrable systems in a broader sense of multiseparability, allowing for degenerations

\textsuperscript{2}In dimension two our equations have a slightly different form, but our methods apply as well. The flat case is treated in [KS19] and a general formulation will be subject to a forthcoming paper.
with multiplicities. We therefore expect the classification space for superintegrable systems to be related to symmetric products of Deligne-Mumford moduli spaces.

The geometry of the moduli spaces $\mathcal{M}_{0,n}$ has revealed an operad structure on the classification spaces of separable systems on spheres which provides a simple explicit construction of separable systems on spheres, avoiding intricate limit procedures [SV15]. We expect similar structures for superintegrable systems.

Both classification approaches – to separable and superintegrable systems – are contrasted in more detail in Table 1.

1.10. Perspectives. Our proposed approach will provide – in the truest sense of the word – a variety of explicit superintegrable systems, i.e. Hamiltonian systems that can be solved exactly by algebraic means. Apart from this immediate result, the actual potential of our approach lies in the fact that it transfers the classification problem for superintegrable systems from the domain of analysis to that of algebraic geometry, representation theory and geometric invariant theory, making it accessible to a whole new range of powerful methods. This will lead to a series of generalised and induced classifications of many structures related to superintegrable systems, such as:

- degenerate superintegrable systems
- conformally superintegrable systems and superintegrable systems on conformal manifolds
- multiseparable superintegrable systems
- quantum superintegrable systems
- quadratic symmetry algebras and their representations
- special functions arising from superintegrable systems

Let us give an instructive example. It has been observed that second order superintegrable systems in dimension two are in correspondence to hypergeometric...
orthogonal polynomials [KMP07, KMP13]. This correspondence can probably be formulated properly as an isomorphism in the category of oriented graphs, with one graph being given by the Askey scheme [Ask85, AW85] and the other by a graph whose vertices represent superintegrable systems and whose edges represent Bôcher contractions. Such a correspondence will be generalised by our approach into an isomorphism of $G$-varieties, where the corresponding graph is recovered as the partial order of orbits and orbit closure inclusions. It is interesting to note in this context that a structure of a glued manifold with corners has been revealed on the Askey scheme by analysing the limits of hypergeometric orthogonal polynomials [Koo09] and that any variety naturally carries such a structure as well. We expect our approach to lead to a proper definition of higher-dimensional hypergeometric polynomials and to higher dimensional generalisations of the Askey-Wilson scheme. Indeed, the generic superintegrable system on the 3-sphere can be related to 2-variable Wilson polynomials [KMP11], and interbasis expansions for the isotropic 3D harmonic oscillator are linked with bivariate Krawtchouk polynomials [GVZ14]. Also, it has been shown that the only free degenerate quadratic algebras that can be constructed in phase space are those that arise from superintegrability [ERMS17].

Note that classically, hypergeometric polynomials have always been studied in families, each parametrised by a few complex parameters. What we propose here is a paradigm shift: Rather than regarding hypergeometric polynomials as many families, each parametrised by a parameter in a subset of $\mathbb{C}^k$, we propose to describe them as a single family, parametrised by a parameter in a projective variety.

Structure of the paper. After briefly reviewing theory, terminology and notation in Section 2, we introduce the pivotal object of our approach in Section 3: A valence three tensor field encoding all information about a superintegrable system, called the structure tensor. Sections 4 and 5, are devoted to the integrability conditions that superintegrability imposes on this tensor. Our first main statement is proven in Section 6, namely that the classification space forms an algebraic variety. In Section 7 we derive explicit algebraic equations for this variety on constant curvature spaces. Finally, in Section 8 the known $n$-dimensional families of superintegrable systems on constant curvature spaces are reviewed from the point of view developed in this paper.

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2. Preliminaries

2.1. Superintegrable systems. An $n$-dimensional Hamiltonian system is a dynamical system characterised by a Hamiltonian function $H(p, q)$ on the phase space of positions $q = (q_1, \ldots, q_n)$ and momenta $p = (p_1, \ldots, p_n)$. Its temporal evolution is governed by the equations of motion

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}. $$

A function $F(p, q)$ on the phase space is called a constant of motion or first integral, if it is constant under this evolution, i.e. if

$$\dot{F} = \frac{\partial F}{\partial q} \dot{q} + \frac{\partial F}{\partial p} \dot{p} = \frac{\partial F}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial H}{\partial q} = 0$$

or

$$\{F, H\} = 0,$$

where

$$\{F, G\} = \sum_{i=1}^{n} \left( \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i} \frac{\partial F}{\partial p_i} \right)$$

is the canonical Poisson bracket. Such a constant of motion restricts the trajectory of the system to a hypersurface in phase space. If the system possesses the maximal number of $2n - 1$ functionally independent constants of motion $F^{(0)}, \ldots, F^{(2n-2)}$, then its trajectory in phase space is the (unparametrised) curve given as the intersection of the hypersurfaces $F^{(\alpha)}(p, q) = c^{(\alpha)}$, where the constants $c^{(\alpha)}$ are determined by the initial conditions. In this case one can solve the equations of motion exactly and in a purely algebraic way, without having to solve explicitly any differential equation.

**Definition 2.1.** (i) A maximally superintegrable system is a Hamiltonian system together with a space generated by $2n - 1$ functionally independent constants of motion $F^{(\alpha)}$,

$$\{F^{(\alpha)}, H\} = 0, \quad \alpha = 0, 1, \ldots, 2n - 2, \quad (2.1a)$$

one of which is the Hamiltonian itself:

$$F^{(0)} = H. \quad (2.1b)$$

(ii) A constant of motion is second order if it is of the form

$$F^{(\alpha)} = K^{(\alpha)} + V^{(\alpha)}, \quad (2.2a)$$

where

$$K^{(\alpha)}(p, q) = \sum_{i=1}^{n} K_{ij}^{(\alpha)}(q)p^i p^j \quad (2.2b)$$

is quadratic in momenta and $V^{(\alpha)} = V^{(\alpha)}(q)$ is a function depending only on positions.

(iii) A (maximally) superintegrable system is second order if its constants of motion $F^{(\alpha)}$ can be chosen to be second order

$$H = g + V, \quad (2.3a)$$
where
\[ g(p, q) = \sum_{i=1}^{n} g_{ij}(q) p^i p^j \] (2.3b)

is given by the Riemannian metric \( g_{ij}(q) \) on the underlying manifold.

(iv) We call \( V \) a superintegrable potential if the Hamiltonian (2.3) defines a superintegrable system.

In this article we will be concerned exclusively with second order maximally superintegrable systems and thus omit the terms “second order” and “maximally” without further mentioning.

2.2. Bertrand-Darboux condition. The condition (2.1a) for (2.2) and (2.3) splits into two parts, which are cubic respectively linear in the momenta \( p \):

\[
\begin{align*}
\{K^{(\alpha)}, g\} &= 0 \quad (2.4a) \\
\{K^{(\alpha)}, V\} + \{V^{(\alpha)}, g\} &= 0 \quad (2.4b)
\end{align*}
\]

Definition 2.2. A (second order) Killing tensor is a symmetric tensor field on a Riemannian manifold satisfying the Killing equation

\[ \{K, g\} = 0 \]

or, in components,

\[ K_{ij,k} + K_{jk,i} + K_{ki,j} = 0, \quad (2.5) \]

where the comma denotes covariant derivatives.

Example 2.3. The metric \( g \) is trivially a Killing tensor, since it is covariantly constant.

The metric \( g \) allows us to identify symmetric forms and endomorphisms. Interpreting a Killing tensor in this way, as an endomorphism on 1-forms, Equation (2.4b) can be written in the form

\[ dV^{(\alpha)} = K^{(\alpha)} dV, \]

and shows that, once the Killing tensors \( K^{(\alpha)} \) are known, the potentials \( V^{(\alpha)} \) can be recovered from \( V = V^{(0)} \), up to an irrelevant constant, provided the integrability conditions

\[ d(K^{(\alpha)} dV) = 0 \quad (2.6) \]

are satisfied. This eliminates the potentials \( V^{(\alpha)} \) for \( \alpha \neq 0 \) from our equations.

2.3. Young projectors. We will make extensive use of Young projectors, mainly to make tensor symmetries explicit and to simplify lengthy tensor expressions. Since here is not the place for a comprehensive introduction to the representation theory of symmetric and linear groups, we refer to the literature on this subject, e.g. [Ful97, FH00] and content ourselves with providing only those examples appearing in the present work.

A partition of a positive integer \( n \) is a decomposition of \( n \) into a sum of ordered positive integers:

\[ n = \lambda_1 + \lambda_2 + \cdots + \lambda_r \quad \lambda_i \in \mathbb{N} \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0. \]
A Young frame is a visualisation of a partition by consecutive, left-aligned rows of square boxes, such as

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[ ] [ ] [ ]
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for \( 9 = 4 + 2 + 2 + 1 \).

Young frames are used to label irreducible representations of the permutation group \( S_n \) and the induced Weyl representations of \( \text{GL}(n) \). A Young tableau is a Young frame filled with distinct objects, in our case tensor index names. Young tableaux are used to define explicit projectors onto irreducible representations. Let us illustrate this with a couple of examples used in this article.

A Young tableau consisting of a single row is used to denote complete symmetrisation, as in

\[
S_{ijk} = S_{ijk} + S_{ikj} + S_{kij} + S_{jki} + S_{kji}.
\]

Similarly, a single column Young tableau denotes complete antisymmetrisation,

\[
A_{ijk} = A_{ijk} - A_{ikj} + A_{kij} - A_{kji} + A_{jki} - A_{jik}.
\]

A general Young tableau denotes the composition of its row symmetrisers and column antisymmetrisers. By convention, we apply antisymmetrisers first. Operators of this type are (scalar multiples of) projectors, called Young projectors. The Young projectors used most here are hook symmetrisers, composed of a single row and a single column. For instance,

\[
T_{ijk} = T_{ijk} - T_{ikj} + T_{jik} = T_{ijk} - T_{ikj} + T_{jik} - T_{jki}.
\]

If we want to apply the symmetrisers first, we can use the adjoint operator. For example

\[
T_{ijk} = T_{ijk} + T_{jik} - T_{ikj} - T_{kij}.
\]

Next, it is easy to see that tensors of the form

\[
R_{ijkl} = T_{ijkl} - T_{i j k l} T_{ijkl}
\]

are algebraic curvature tensors, i.e. satisfy

(i) antisymmetry: \( R_{ijkl} = -R_{ijkl} \),

(ii) pair symmetry: \( \hat{R}_{klij} = R_{ijkl} \),

(iii) the Bianchi identity: \( R_{ijkl} + R_{iklj} + R_{ljk i} = 0 \).

We will use a subscript "\( o \)" to indicate a projector onto the completely trace-free part. For example,

\[
W_{ijkl} = \frac{1}{12} R_{ijkl} - R_{ijkl}
\]

is the Weyl part in the well known Ricci decomposition

\[
R_{ijkl} = W_{ijkl} + \frac{1}{4(n-1)} R_{iklj} + \frac{1}{8n(n-1)} R_{ijkl} - \frac{1}{n} g_{ijkl},
\]

where

\[
R_{iklj} = \frac{1}{2} R_{iklj} - R_{ijkl}.
\]
is the trace-free part of the Ricci tensor and $R$ the scalar curvature.

We will also use Young tableaux to denote symmetrisations in a subset of a tensor’s indices, such as in

$$\begin{bmatrix} j \\ k \end{bmatrix} T_{ijk} = T_{ijk} - T_{ikj}.$$ 

### 3. The Structure Tensor of a Superintegrable System

For the following definition recall that the Riemannian metric $g$ on the base manifold provides an isomorphism between bilinear forms and endomorphisms on the tangent space.

**Definition 3.1.** We call a superintegrable system *irreducible* if the Killing tensors in this system, regarded as endomorphisms, form an irreducible set, i.e. do not admit a non-trivial invariant subspace.

For simplicity, we will – here and in what follows – denote covariant derivatives with a comma.

**Proposition 3.2.** Every irreducible superintegrable system admits a tensor $T_{ijk}$, symmetric and trace-free in its first two indices,

$$T_{ijk} = T_{ijk} \quad g^{ij}T_{ijk} = 0,$$  

such that the corresponding superintegrable potential satisfies

$$V_{ij} = T_{ij}^m V_{,m} + \frac{1}{n} g_{ij} \Delta V.$$  

Moreover, at each point $x$ the tensor components $T_{ijk}(x)$ are rational expressions in the components of the Killing tensors and their derivatives at $x$, $K^{(\alpha)}_{ij}(x)$ respectively.

Equations similar to (3.2) appear in [KKM05c], in local coordinates and for the special case of dimension three.

**Proof.** In components, the Bertrand-Darboux condition (2.6) for a Killing tensor $K = K^{(\alpha)}$ in a superintegrable system reads

$$\begin{bmatrix} i \\ j \end{bmatrix} \left( K^{\alpha}_{,i} V_{,jm} + K^{m}_{,i,j} V_{,m} \right) = 0.$$  

Observe that the first term in the sum can be written as a commutator $[K, V'']$ of endomorphisms, where $V''$ denotes the Hessian of $V$. Let us consider (3.3) for all Killing tensors $K^{(\alpha)}$ as a linear system for $V''$. The kernel of its coefficient matrix consists of all symmetric endomorphisms commuting with all Killing tensors. Since the latter form an irreducible set by assumption, the kernel consists of multiples of the identity and the linear system can be solved for $V''$ up to a multiple of the identity. Note that the identity corresponds to the metric under the identification of endomorphisms and bilinear forms. This establishes Equation (3.2), where we have, without loss of generality, assumed $T_{ijk}$ to be trace free in $(i, j)$.

In what follows we consider a class of superintegrable systems that is even more general than irreducible superintegrable systems. We refer to this class by the name of the author of a series of papers on the projective differential geometry of surfaces, which inspired our methods [Wil07, Wil09].
Definition 3.3. We call a superintegrable system Wilczynski, if its potential satisfies (3.2) for some (not necessarily unique) tensor $T_{ijk}$, which will be called its structure tensor.

Example 3.4. The harmonic oscillator on flat space is the (trivial) Wilczynski system with vanishing structure tensor.

4. Superintegrable potentials

4.1. Prolongation of a superintegrable potential. Equation (3.2) expresses the derivative of $\nabla V$ linearly in $\nabla V$ and $\Delta V$, where the coefficients are determined by the structure tensor. The following Proposition shows that this equation can be extended by a second one to a system expressing the derivatives of $\nabla V$ and $\Delta V$ both linearly in $\nabla V$ and $\Delta V$, with the coefficients determined by the structure tensor. Extensions of such type are called prolongation.

Proposition 4.1. A Wilczynski superintegrable system satisfies

\begin{align}
V_{ij} &= T^{m}_{ij}V_{,m} + \frac{1}{n} q_{ij} \Delta V, \\
\frac{n-1}{n} (\Delta V)_{,k} &= q_{k}^{m} V_{,m} + \frac{1}{n} t_{k} \Delta V,
\end{align}

with

\begin{align}
t_{j} &:= T^{i}_{ij} \\
q_{j}^{m} &:= Q^{im}_{ij},
\end{align}

where

\begin{align}
Q_{ijk}^{m} &:= T^{m}_{ij}V_{,k} + T^{m}_{ij}T^{l}_{lk} - R^{m}_{ijk}.
\end{align}

Proof. Equation (4.1a) is a copy of Equation (3.2). Substituting it into its covariant derivative, we obtain

\begin{align}
V_{ijk} &= T^{m}_{ij}V_{,m} + T^{m}_{ij}V_{,mk} + \frac{1}{n} q_{ij} (\Delta V)_{,k} \\
&= (T^{m}_{ij}V_{,m} + T^{m}_{ij}T^{l}_{lk}V_{,m} + \frac{1}{n} (T_{ijk}\Delta V + g_{ij}(\Delta V)_{,k}).
\end{align}

Antisymmetrisation in $(j,k)$ and application of the Ricci identity yields

\begin{align}
R^{m}_{ijk}V_{,m} = \frac{1}{k} \left[(T^{m}_{ij}V_{,m} + T^{m}_{ij}T^{l}_{lk})V_{,m} + \frac{1}{n} (T_{ijk}\Delta V + g_{ij}(\Delta V)_{,k})\right].
\end{align}

Solving for the last term on the right hand side, we get

\begin{align}
\frac{1}{n} \frac{1}{k} g_{ij}(\Delta V)_{,k} = -\frac{1}{k} (Q_{ijk}^{m}V_{,m} + \frac{1}{n} T_{ijk}\Delta V).
\end{align}

The contraction of this equation in $(i,j)$ now yields (4.1b), since $T_{ijk}$ and $Q_{ijk}^{m}$ are trace-free in $(i,j)$ by definition. □

The System (4.1) can be used to express all higher derivatives of $\nabla V$ and $\Delta V$ linearly in $\nabla V$ and $\Delta V$. In particular, all higher derivatives of $V$ in a fixed point are determined by the values of $\nabla V$ and $\Delta V$ in that point. So if $V$ is analytic, this determines $V$ locally up to a constant. That is, the initial partial differential equation (3.2) has a finite dimensional space of solutions. Systems of this type are called overdetermined of finite type.

It is known that the Killing equation also extends to an overdetermined system of finite type, since all third covariant derivatives of a Killing tensor can be expressed
as linear combinations of its derivatives up to order two, with the Riemannian curvature tensor as coefficients (see Section 5).

**Corollary 4.2.** The potential \( V \) of an irreducible superintegrable system on a Riemannian manifold with analytic metric is analytic and determined locally by the values of \( V, \nabla V \) and \( \Delta V \) in a non-singular point of \( V \).

**Proof.** We use the fact that solutions of overdetermined systems of finite type with analytic coefficients are analytic [KM16, Kur57]. In particular, Killing tensors on a Riemannian manifold with an analytic metric are analytic. For an irreducible superintegrable system the components of the structure tensor are rational in the components of the Killing tensors and their derivatives and hence also analytic. By Proposition 4.1 the superintegrable potential \( V \) of an irreducible superintegrable system is the solution of an overdetermined system of finite type, with the components of the structure tensor as coefficients. This implies that \( V \) is analytic. The rest follows from Proposition 4.1. \( \square \)

Note that the values of \( V, \nabla V \) and \( \Delta V \) at a given point are \( n + 2 \) scalars. This motivates the following generalisation of the notion of non-degeneracy commonly employed in dimensions two and three [KKPM01].

**Definition 4.3.** We call a Wilczynski system *non-degenerate*, if \( (3.2) \) admits an \( (n + 2) \)-dimensional space of solutions \( V \).

### 4.2. Integrability conditions for a superintegrable potential

Non-degeneracy is just the condition that assures that the integrability conditions of the System \((4.1)\) are satisfied generically, i.e. independently of the potential. This will eliminate the potential \( V \), leaving equations involving only the structure tensor, respectively the Killing tensors of the superintegrable system.

**Proposition 4.4.** The structure tensor of a non-degenerate superintegrable system satisfies the following equations, which are the integrability conditions for the prolongation \((4.1)\) of a superintegrable potential:

\[
\begin{align*}
\frac{j}{k} \left( T_{ijk} + \frac{1}{n-1}g_{ij}t_k \right) &= 0 \quad (4.4a) \\
\frac{j}{k} \left( Q_{ijkl} + \frac{1}{n-1}g_{ij}q_{kl} \right) &= 0 \quad (4.4b) \\
\frac{k}{l} \left( q_{n,j} + T_{ml} q_{k}^{m} + \frac{1}{n-1}t_{k}q^{n} \right) &= 0. \quad (4.4c)
\end{align*}
\]

**Proof.** The system \((4.1)\) allows us to write all higher derivatives of \( \nabla V \) and \( \Delta V \) as linear combinations of \( \nabla V \) and \( \Delta V \). Necessary and sufficient integrability conditions are then obtained by applying this procedure to the left hand sides of the Ricci identities

\[
\begin{align*}
\frac{j}{k} V_{ijk} &= R_{ijk}^{m} V_{m} \\
\frac{k}{l} (\Delta V)_{,kl} &= 0.
\end{align*}
\]

This results in

\[
\begin{align*}
\frac{j}{k} \left( Q_{ijk}^{m} + \frac{1}{n-1}g_{ij}q_{k}^{m} \right) V_{m} + \frac{1}{n} \frac{j}{k} \left( T_{ijk} + \frac{1}{n-1}g_{ij}t_{k} \right) \Delta V &= 0
\end{align*}
\]
and, respectively,

\[ \frac{k}{l} \left( q^n_k + T_{ml}^n q^n_k + \frac{1}{n-1} t_k q^n_l \right) V_n + \frac{1}{n} \frac{k}{l} \left( t_{k,l} + q_{kl} \right) \Delta V = 0. \]

For a non-degenerate superintegrable potential the coefficients of \( \Delta V \) and \( \nabla V \) must vanish. In addition to the stated integrability conditions, this yields the condition

\[ \frac{k}{l} \left( t_{k,l} + q_{kl} \right) = 0. \] (4.5)

The latter is redundant, however, as it can be obtained from (4.4b) via a contraction over \((i, l)\).

In the remainder of this section we cast the above integrability conditions for a superintegrable potential into the following simpler form.

**Proposition 4.5.** The integrability conditions (4.4) for a superintegrable potential are equivalent to the algebraic conditions

\[ \begin{align*}
J^j_i & \ T_{ijk} = 0 \\
\frac{1}{8} \ T_{ijkl}^a T_{ijkl} &= W_{ijkl}
\end{align*} \] (4.6a)

and the differential condition

\[ \frac{k}{l} \left( T_{ijk,l} + \frac{2}{n-2} g_{ik} Z_{jl} \right) = 0 \] (4.6c)

with

\[ Z_{ij} := T_{i}^{ab} T_{j}^{ab} - (n-2) (T_{ij}^{a} T_{a}^{i} + T_{ij}^{i} T_{i}^{j}) - R_{ij}, \] (4.6d)

where \( T_{ijk} \) and \( T_{ij} \) are the trace-free part and the rescaled non-vanishing trace of the structure tensor, given in (4.8).

**Proof.** Proposition 4.5 follows from Propositions 4.6, 4.8 and 4.10 below. \( \square \)

4.2.1. The 1st integrability condition. We can solve Equation (4.4a) right away, because it is linear and does not involve derivatives.

**Proposition 4.6.** The first integrability condition for a superintegrable potential, Equation (4.4a), can be written in the form (4.6a) and is equivalent to the following decomposition of the structure tensor:

\[ T_{ijk} = \hat{T}_{ijk} + \bar{T}^{ij} \left( \bar{T}_{ijk} - \frac{1}{n} T_{ij} T_{ik} \right), \] (4.7)

where

\[ \begin{align*}
\hat{T}_{ijk} &= \frac{1}{6} \ T_{ijk} \\
\bar{T}_{ij} &= \frac{n}{(n+2)(n-1)} T_{ij}
\end{align*} \] (4.8a)

are the trace-free part and the rescaled non-vanishing trace of the structure tensor. Note that both are uniquely determined by \( T_{ijk} \) and vice versa.
Proof. First note that (4.4a) can be written in the form (4.6a) by the definition of the Young projector.

The structure tensor $T_{ijk}$ is symmetric in $(i,j)$. According to the Littlewood-Richardson rule its symmetry class is therefore

$$\begin{array}{c}
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This means that $T_{ijk}$ can be decomposed into a trace-free and totally symmetric part, a trace-free part of hook symmetry, and two independent traces. By (4.6a) the trace-free hook symmetric part vanishes. This implies that the trace-free part $\tilde{T}_{ijk}$ of the structure tensor is totally symmetric. The two remaining trace terms are of the form

$$\begin{array}{c}
\begin{array}{c}
j_1 \\
k_1
\end{array}\begin{array}{c}
j_2 \\
k_2
\end{array}\begin{array}{c}
j_3 \\
k_3
\end{array}
g_{ij}\rho_k
\end{array}
\begin{array}{c}
j_1 \\
k_1
\end{array}\begin{array}{c}
j_2 \\
k_2
\end{array}\begin{array}{c}
j_3 \\
k_3
\end{array}
g_{ij}\sigma_k
$$

and their traces $\rho_k$ respectively $\sigma_k$ can be determined from (3.1) and (4.2a).

\[\Box\]

Corollary 4.7.

(i) The tensor $q_{ij}$ is symmetric: $q_{ji} = q_{ij}$.

(ii) The tensor $t_i$ is the derivative of a function $t$, i.e. $t_i = t_{ji}$, and similarly for $\bar{t}_i$.

Proof. The first statement follows from substituting (4.7) into the definition (4.2b) of $q_{ij}$. The second then follows from (4.5).

4.2.2. The 2nd integrability condition.

Proposition 4.8. Assume the structure tensor of a superintegrable system satisfies the first integrability condition (4.4a). Then the second integrability condition (4.4b) is equivalent to (4.6b) and (4.6c).

Proof. Equation (4.6b) follows directly by symmetrising (4.4b) in $(i,l)$. Antisymmetrising instead, we obtain

$$\begin{array}{c}
\begin{array}{c}
j_1 \\
k_1
\end{array}\begin{array}{c}
j_2 \\
k_2
\end{array}\begin{array}{c}
j_3 \\
k_3
\end{array}
\begin{array}{c}
j_4 \\
k_4
\end{array}\begin{array}{c}
j_5 \\
k_5
\end{array}\begin{array}{c}
j_6 \\
k_6
\end{array}
T_{ijk,l} + \frac{2}{n-2}g_{ij}Z_{jl} = 0
\end{array}$$

with $Z_{ij}$ given by (4.6d). The symmetriser can be written as

$$\begin{array}{c}
\begin{array}{c}
j_1 \\
k_1
\end{array}\begin{array}{c}
j_2 \\
k_2
\end{array}\begin{array}{c}
j_3 \\
k_3
\end{array}
\begin{array}{c}
j_4 \\
k_4
\end{array}\begin{array}{c}
j_5 \\
k_5
\end{array}\begin{array}{c}
j_6 \\
k_6
\end{array}
\frac{1}{2} \begin{array}{c}
j_1 \\
k_1
\end{array}\begin{array}{c}
j_2 \\
k_2
\end{array}\begin{array}{c}
j_3 \\
k_3
\end{array}
\frac{1}{2} \begin{array}{c}
j_4 \\
k_4
\end{array}\begin{array}{c}
j_5 \\
k_5
\end{array}\begin{array}{c}
j_6 \\
k_6
\end{array}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
j_1 \\
k_1
\end{array}\begin{array}{c}
j_2 \\
k_2
\end{array}\begin{array}{c}
j_3 \\
k_3
\end{array}
\frac{1}{96} \begin{array}{c}
j_1 \\
k_1
\end{array}\begin{array}{c}
j_2 \\
k_2
\end{array}\begin{array}{c}
j_3 \\
k_3
\end{array}
\frac{1}{96} \begin{array}{c}
j_4 \\
k_4
\end{array}\begin{array}{c}
j_5 \\
k_5
\end{array}\begin{array}{c}
j_6 \\
k_6
\end{array}
\end{array}$$

making explicit the projectors in the Littlewood-Richardson rule

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\end{array}$$

The last component vanishes, because

$$\begin{array}{c}
\begin{array}{c}
j_1 \\
k_1
\end{array}\begin{array}{c}
j_2 \\
k_2
\end{array}\begin{array}{c}
j_3 \\
k_3
\end{array}
\begin{array}{c}
j_4 \\
k_4
\end{array}\begin{array}{c}
j_5 \\
k_5
\end{array}\begin{array}{c}
j_6 \\
k_6
\end{array}
\left(T_{ijk,l} + \frac{2}{n-2}g_{ij}Z_{jl}\right)
= 2 \begin{array}{c}
j_1 \\
k_1
\end{array}\begin{array}{c}
j_2 \\
k_2
\end{array}\begin{array}{c}
j_3 \\
k_3
\end{array}
\left(-g_{ij}t_{kl} + \frac{1}{n-2}(g_{ik}Z_{jl} + g_{jk}Z_{il})\right)
= 0,
\end{array}$$

where we have used (4.4a). The equivalence of (4.6c) and (4.9) now follows from the fact that $PP^*P$ is proportional to $P$ for any Young projector.\[\Box\]
Proposition 4.9. The integrability condition (4.6c) implies

\[ R_{ilm}^a T_{a jk} + \frac{2}{n-2} g_{ik} Z_{jl,m} = 0. \]  

(4.10)

Proof. Differentiation and antisymmetrisation of (4.9) yields

\[ (T_{ijk,lm} + \frac{2}{n-2} g_{ik} Z_{jl,m} = 0. \]

The statement now follows from applying the Ricci identity to the first term and using the Bianchi identity. □

4.2.3. The 3rd integrability condition.

Proposition 4.10. The third of the integrability conditions (4.4) is redundant.

Proof. A straightforward computation confirms that, up to trace terms, Equation (4.4c) is a linear combination of the contraction of (4.6c) and (4.10), if we take into account the Ricci identity for \( t_k \). Similarly, the trace of (4.4c) is a linear combination of (4.6c), contracted with \( \tilde{T}_{ijk} \), again taking into account the Ricci identity for \( t_k \). This proves the claim. □

5. Superintegrable Killing tensors

5.1. Prolongation of a superintegrable Killing tensor. For arbitrary second order Killing tensors \( K_{ij} \) it is well known that all higher covariant derivatives are determined by the derivatives up to second order. More precisely, an explicit but complicated expression for \( K_{ij,klm} \) can be given which is linear in \( K_{ij} \) and \( K_{ij,k} \), the coefficients being linear in the Riemannian curvature tensor and its derivative [Wol98]. Symbolically,

\[ \nabla^3 K = (\nabla R) \otimes K + R \otimes (\nabla K), \]  

(5.1)

where “\( \otimes \)” is a placeholder for some complicated bilinear operation. This defines the standard prolongation of the Killing equation.

The following proposition shows that a Killing tensor which arises from a superintegrable system satisfies another, much simpler prolongation, namely that all its covariant derivatives are already determined by the Killing tensor itself. Symbolically:

\[ \nabla K = T \otimes K. \]

This generalizes, to arbitrary dimension, equations found in [KKM05c] for dimension three.

Proposition 5.1. A Killing tensor in a non-degenerate superintegrable system with structure tensor \( T_{ijk} \) satisfies

\[ K_{ij,k} = \frac{1}{3} \sum_{j,k} T^a_{ji} K_{ak}. \]

(5.2)

Proof. Substituting (3.2) into (3.3) gives

\[ \frac{1}{k} (K_{j,k}^a - T_{j,k}^a K_{k}^{,b}) V_a = 0. \]
From the definition of non-degeneracy it then follows that
\[
\frac{j}{k} K_{ij,k} = \frac{j}{k} T_{ji}^a K_{ak}.
\] (5.3)

On the other hand, the Killing equation (2.5) implies that
\[
K_{ij,k} = \frac{1}{3} \frac{j}{k} \frac{i}{l} K_{ij,k}.
\]

Combining the last two equations proves (5.2).

Lemma 5.2. Any Killing tensor satisfies the following identity
\[
\frac{i}{l} \frac{j}{k} (K_{ij,kl} + \frac{j}{k} R_{ijk}^a K_{ai}) = 0.
\] (5.4)

Proof. Using the identities
\[
\frac{j}{k} K_{ij,k} = \frac{1}{2} \frac{j}{k} K_{jk,i} \quad \frac{j}{k} R_{ji,k}^a = -\frac{1}{2} \frac{j}{k} R_{ijk}^a,
\]

which follow from a symmetrisation of the Killing equation respectively an anti-symmetrisation of the Bianchi identity, we have
\[
\frac{i}{l} \frac{j}{k} K_{ij,kl} = \frac{i}{l} \frac{j}{k} \left( K_{ij,kl} + \frac{k}{l} K_{ij,kl} \right)
= \frac{i}{l} \frac{j}{k} \left( \frac{1}{2} K_{il,jk} + \frac{i}{j} R_{ikl}^a K_{aj} \right)
= \frac{i}{l} \frac{j}{k} \left( \frac{1}{2} R_{ijk}^a K_{ai} + R_{ikl}^a K_{aj} + R_{jkl}^a K_{ai} \right)
= \frac{i}{l} \frac{j}{k} \left( R_{ikl}^a K_{aj} - R_{ijk}^a K_{ai} \right)
\]

Proposition 5.3. For an irreducible non-degenerate superintegrable system the values of the Killing tensors and of the structure tensor at a given point uniquely determine the values of the covariant derivative of the structure tensor at this point.

Proof. Substituting the derivative of (5.3),
\[
\frac{j}{k} K_{ij,kl} = \frac{j}{k} \left( T_{ji,l}^a K_{ak} + T_{ji}^a K_{ak,l} \right),
\]

together with (5.2) into (5.4) yields
\[
\frac{i}{l} \frac{j}{k} \left( T_{ji,l}^a K_{ak} + \frac{1}{3} T_{ji}^a \frac{k}{l} T_{ka}^b K_{bl} + \frac{j}{l} R_{ijk}^a K_{ai} \right) = 0.
\] (5.5)

Now suppose we have two structure tensors with the same values in a fixed point \(x_0\). Denote their difference by \(\delta T_{ij,k}\). Then the difference of the two copies of the above equation at \(x_0\), obtained for each of the structure tensors, is
\[
\frac{i}{l} \frac{j}{k} \delta T_{ji,l}^a (x_0) K_{ak}(x_0) = 0.
\]
This equation is satisfied by all Killing tensors in the superintegrable system. Note that, for fixed $i$ and $l$, the left hand side is a commutator between $\delta T^a_{ji,l}(x_0)$ and $K^a_k$. Hence, if the system is irreducible, we have

$$\delta T^a_{ji,l}(x_0) = g^a_j \Lambda_{il}$$

for some symmetric tensor $\Lambda_{il}$. Contracting $a$ and $j$ shows that $\Lambda_{il} = 0$ and hence

$$\delta T^a_{ji,l}(x_0) = 0.$$ 

On the other hand, by (4.9),

$$\delta T^a_{ji,l}(x_0) = 0,$$

implying $\delta T^a_{ji,l}(x_0) = 0$. This shows that both structure tensors must have the same derivatives at $x_0$. □

The following property of superintegrable systems shows that for a local classification it is sufficient to consider only the algebraic restrictions the Killing tensors and the structure tensor of a superintegrable system have to satisfy in a single point.

**Corollary 5.4.** For an irreducible non-degenerate superintegrable system the values of the Killing tensors and of the structure tensor at a given point uniquely determine the superintegrable system locally.

**Proof.** Substituting (5.2) into its own derivative yields

$$K_{ij,kl} - K_{kl,ij} = \frac{1}{2} \delta T^a_{ji,l} K_{ak} + \frac{1}{3} \delta T^b_{ji} T^a_{kl} K_{al}.$$ 

Together with (5.2) and Proposition 5.1, this implies that the values of the Killing tensors and of the structure tensor at a fixed point also determine the first and second derivatives of the Killing tensors at this point. As explained at the beginning of this section, this determines the Killing tensors and hence the structure tensor locally. The superintegrable potential is then determined locally by (3.2), completing the superintegrable system. □

### 5.2. Integrability conditions for a superintegrable Killing tensor

The integrability conditions for the standard prolongation of a Killing tensor, Equation (5.1), are

$$K_{ij,kl} - K_{kl,ij} = \frac{1}{2} \delta T^a_{ji,l} K_{ak} + \frac{1}{3} \delta T^b_{ji} T^a_{kl} K_{al} + \frac{1}{2} \delta T^a_{ji,l} K_{ak} - \frac{1}{2} \delta T^b_{ji} T^a_{kl} K_{al}.$$ 

and a very lengthy expression of the form

$$(\nabla^2 R + R \otimes R) \otimes K + (\nabla R) \otimes (\nabla K) + R \otimes (\nabla^2 K) = 0,$$

involving (even in low dimension) several hundreds of terms [Wol98], see also [GL19]. In contrast, the following proposition shows that the integrability conditions for the prolongation of a superintegrable Killing tensor, Equation (5.2), are much simpler.

**Proposition 5.5.** The integrability conditions for the prolongation (5.2) of a superintegrable Killing tensor read

$$K_{ij,kl} - K_{kl,ij} = \frac{1}{2} \delta T^a_{ji,l} K_{ak} + \frac{1}{3} \delta T^b_{ji} T^a_{kl} K_{al} - \frac{1}{2} \delta T^a_{ji,l} K_{ak} - \frac{1}{2} \delta T^b_{ji} T^a_{kl} K_{al} = 0,$$

(5.6)
where
\[
P_{ijk}^{ab} := \frac{1}{6} \left[ a \begin{pmatrix} b & j \end{pmatrix} \kappa \right] g^{a}_{k} T^{b}_{ji}.
\]

Proof. Equation (5.2) can be used to express all higher derivatives of the Killing tensor linearly in the Killing tensor itself. Explicitly, writing (5.2) as
\[
K_{ij,k} = P_{ijk}^{ab} K_{ab} (5.7)
\]
and substituting it back into its own derivative, yields
\[
K_{ij,kl} = P_{ijk}^{ab,l} K_{ab} + P_{ijk}^{cd} K_{cd,l} = \left( P_{ijk}^{ab,l} + P_{ijk}^{cd} P_{cdl}^{\ ab} \right) K_{ab}.
\]
This expression must satisfy the Ricci identity
\[
\frac{\hat{R}}{\hat{k}} K_{ij,kl} = \frac{\hat{i} \hat{j}}{\hat{k} \hat{l}} R^{\ a}_{\ ikl} K_{aj},
\]
which is the integrability condition for the existence of a local solution to (5.2). □

The following definition plays the same role for superintegrable Killing tensors as that of non-degeneracy for superintegrable potentials: It assures that the integrability conditions (5.6) are generically satisfied, that is independently of the Killing tensors.

Definition 5.6. We call a superintegrable system abundant if Equation (5.2) has \( n(n+1)/2 \) linearly independent solutions.

Lemma 5.7. A superintegrable system on a Riemannian manifold with analytic metric is abundant if and only if it satisfies the generic integrability conditions for a superintegrable Killing tensor:
\[
\begin{vmatrix} n \times n \end{vmatrix} \left( P_{ijk}^{mn,l} + P_{ijk}^{pq} P_{pq}^{\ mn} - \frac{1}{2} \left[ i \ j \ k \right] g^{n}_{i} R^{\ a}_{\ jkl} \right) = 0 \quad (5.8)
\]

Proof. Equation (5.2) can be used to express all higher derivatives of a superintegrable Killing tensor linearly in the Killing tensor. As we have seen in the proof of Corollary 4.2, Killing tensors on Riemannian manifolds with analytic metric are analytic. Therefore, a superintegrable Killing tensor is locally determined by its values \( K_{ij}(x_0) \) in a fixed point \( x_0 \), provided that the integrability conditions (5.6) hold. Equation (5.8) expresses the fact that they hold generically, i.e. independently of the Killing tensor in question. This is exactly the case if they do not put any constraint on the \( n(n+1)/2 \) initial values \( K_{ij}(x_0) \). □

Remark 5.8. In dimension two, every superintegrable system is trivially abundant, since \( 2n - 1 = n(n+1)/2 \) for \( n = 2 \). For \( n = 3 \) we have \( 2n - 1 = 5 \) and \( n(n+1)/2 = 6 \). The so-called “5 ⇒ 6 Lemma” states that every non-degenerate second order maximally superintegrable system on a conformally flat manifold of dimension three is abundant [KKM05c].

Theorem 5.9. The generic integrability conditions (5.8) for superintegrable Killing tensors are equivalent to the following polynomial expressions for the derivatives of
the structure tensor (respectively, its trace and tracefree component),
\[ T_{ijk,l} = \hat{T}_{ijk,l} + \frac{i}{j} \left( \tilde{t}_{ij} g_{jk} - \frac{1}{n} g_{ij} \tilde{t}_{kl} \right) \]  
(5.9a)
\[ \tilde{T}_{ijk,l} = \frac{1}{18} i j k \left[ \hat{T}_{ij} a T_{kla} + \hat{T}_{ijk} \hat{t}_{l} + 3 \hat{T}_{ijl} \hat{t}_{k} + \left( \frac{4}{n-2} \hat{T}_{i} ab \hat{t}_{jab} - 3 \hat{T}_{ij} a \hat{t}_{a} \right) g_{kl} \right] \]  
(5.9b)
\[ \tilde{t}_{k,l} = \frac{1}{3} \left( - \frac{2}{n-2} \hat{T}_{k} ab \hat{t}_{lab} + 3 \hat{T}_{kl} a \hat{t}_{a} + 4 \hat{t}_{k} \hat{t}_{l} \right) + \frac{1}{n} g_{kl} \left( \frac{3n+2}{6(n+2)(n-1)} T_{abc} T_{abc} - \frac{n-2}{6} \tilde{R}_{ia} \tilde{R}_{a} + \frac{3}{2(n-1)} R \right) \]  
(5.9c)
together with the polynomial equations
\[ \frac{1}{8} i j k l T_{ik} T_{ajl} = W_{ijkl} = 0 \]  
(5.10a)
\[ -\frac{1}{4} \hat{R}_{ij} = \hat{R}_{ij} . \]  
(5.10b)
Here “\( \circ \)” denotes the trace free part, \( W_{ijkl} \) and \( R_{ij} \) are the Weyl respectively the Ricci tensor of the Riemannian manifold and \( Z_{ij} \) is defined in (4.6d).

Proof. The generic integrability conditions (5.8) are linear in the derivatives of the structure tensor. They can be solved for these derivatives, which yields (5.9). Substituting (5.9) back into (5.8) yields (5.10).

\( \square \)
Corollary 5.10. Abundant superintegrable systems can only exist on conformally flat manifolds.

We now change our point of view, and consider (5.9) as a system of partial differential equations for a tensor \( T_{ijk} \), and ask for necessary and sufficient conditions such a tensor has to satisfy in a single point, in order to be a structure tensor of a superintegrable system.

Lemma 5.11. The integrability conditions for the system (5.9) consist of (5.10) and the following scalar equation,
\[ \frac{n+2}{9n} \left( T_{abc} T_{abc} - (n+2)(n-1) \tilde{R}_{a} \tilde{R}_{a} - 9 \hat{R} \right) \tilde{t}_{i} = -R_{i} + \frac{2(5n+2)}{3(n-2)} \hat{T}_{i} ab \hat{R}_{ab} - \frac{2(n^2 - 2n + 8)}{3(n-2)} \hat{R}_{ab} \hat{R}_{a} . \]  
(5.11)

Proof. The integrability condition for (5.9) is obtained by taking the covariant derivative of this equation using the product rule, replacing all derivatives on the right hand side using the Equations (5.9) itself and applying the Ricci identity
\[ \frac{1}{m} T_{ijk,lm} = R_{ilm}^{a} T_{ajk} + R_{jlm}^{a} T_{iak} + R_{klm}^{a} T_{ija} . \]

It is straightforward to check that this integrability condition is equivalent to (5.11) due to (5.10).

\( \square \)

Corollary 5.12. Consider (5.9) as a system of partial differential equations with initial conditions \( T_{ijk}(x_{0}) \) at a fixed point \( x_{0} \) satisfying
(i) the symmetries (3.1)
(ii) the linear condition (4.4a)
(iii) the non-linear conditions (5.10) and (5.11).

Then this system has a unique solution $T_{ijk}$ in some local neighbourhood of $x_0$ satisfying (i)-(iii) in this neighbourhood.

6. THE VARIETY OF SUPERINTEGRABLE SYSTEMS

In the previous sections we have shown that the structure tensor of an irreducible non-degenerate superintegrable system satisfies the integrability conditions (4.4) for a superintegrable potential as well as (5.6) for a superintegrable Killing tensor. Instead of non-degeneracy, we directly require both integrability conditions in the following definition, as this will also allow degenerate superintegrable systems in the classification space.

**Definition 6.1.** We define the **classification space of second order superintegrable systems** to be the set of irreducible second order superintegrable systems whose structure tensor satisfies the integrability conditions (4.4) for a superintegrable potential as well as (5.6) for a superintegrable Killing tensor.

In the definition above we deliberately ignore the fact that in the definition of a superintegrable system the constants of motion are required to be **functionally** and not only linearly independent. Apart from simplicity of the exposition, the reason is twofold: First, from a pragmatic point of view, we can classify superintegrable systems in this weaker sense and check the property a posteriori. This is also how classical superintegrability theory deals with functional independence. Second, the condition of functional independence can be formulated in a purely algebraic fashion and thus be incorporated into our algebraic geometric description at a subsequent stage. Indeed, functional independence of the integrals means that their differentials are linearly independent almost everywhere and linear dependence is characterised by the vanishing of minors of a matrix.

**Theorem 6.2.** The classification space of second order superintegrable systems on a Riemannian manifold $M$ has a natural structure of a projective variety, isomorphic to a subvariety in the Grassmannian $G_{2n-1}(K(M))$ of $(2n-1)$-dimensional subspaces in the space $K(M)$ of Killing tensors on $M$.

**Proof.** Consider a point $x$ in the base manifold $M$. The evaluation $K \mapsto K(x)$ of a tensor $K$ at $x$ and the covariant derivative $K \mapsto \nabla K$ are linear operations. Therefore, the components of $K$ and $\nabla K$ at $x$, $K_{ij}(x)$ and $K_{ij,k}(x)$, are linear functions on the space $K(M)$ of Killing tensors on $M$.

By the definition of the structure tensor $T$ of a non-degenerate superintegrable system, its components $T_{ijk}(x)$ at $x$ are rational functions in the components of the Killing tensors $K^{(\alpha)}$ and their derivatives $\nabla K^{(\alpha)}$ at $x$, $K_{ij}^{(\alpha)}(x)$ and $K_{ij,k}^{(\alpha)}(x)$, $\alpha = 0, \ldots, 2n-2$. Therefore, the components of the structure tensor are rational functions on the space $K(M)^{2n-1}$.

Similarly, the components $T_{i,j,k}(x)$ of the derivative $\nabla T$ of the structure tensor are rational functions in the components of the Killing tensors and their first and second derivatives at $x$. So they are rational functions on the space $K(M)^{2n-1}$ as well.
We have shown that a tensor field $T_{ijk}$ with the symmetries (3.1) is the structure tensor of a non-degenerate superintegrable system if and only if it satisfies the integrability conditions (4.4) and (5.6). Evaluated at $x$, these conditions are polynomial in the components of the structure tensor $T$ and its derivative $\nabla T$ at $x$. As shown above, the latter are rational in the components of $K^{(\alpha)}$, $\nabla K^{(\alpha)}$ and $\nabla^2 K^{(\alpha)}$ at $x$. Consequently, the symmetries (3.1) and the integrability conditions (4.4) and (5.6), evaluated at a point $x$, can be written as polynomial equations on $K(M)^{2n-1}$. A more detailed analysis of the argument shows that they are homogeneous.

To summarise, the necessary equations for an arbitrary tensor $T$ to define a superintegrable system in the classification space are equivalent to a set of homogeneous equations on the space $K(M)^{2n-1}$ for every point $x \in M$. These equations are sufficient if we require in addition that the Killing tensors $K^{(\alpha)}$ are linearly independent. This defines a projective subvariety in the Stiefel manifold $V_{2n-1}(K(M))$.

The Bertrand-Darboux equations (2.6) are invariant under linear changes of the basis $K^{(\alpha)}$. Therefore, the above subvariety is invariant under the action of the general linear group $GL_{2n-1}$ on $V_{2n-1}(K(M))$ and descends to a subvariety of the Grassmannian $G_{2n-1}(K(M)) \simeq V_{2n-1}(K(M)) \, GL_{2n-1}$.

Every point on this variety corresponds to a superintegrable system in the classification space and vice versa.

While the above Theorem is not constructive in the sense that it does not provide explicit algebraic equations defining the classification space as an algebraic variety, Proposition 5.3 says that such equations are given by a complete set of algebraic restrictions on the Killing tensors and the structure tensor of a superintegrable system and can be established on the tangent space $T_x M$ in a fixed point in place of the space of Killing tensors, $K(M)$. This is the reason we will be able to give explicit equations in the next section.

7. Constant curvature

7.1. Codazzi tensors of a superintegrable system. In this section we express the integrability conditions (4.6) on constant curvature manifolds in terms of Codazzi tensors and use their local form to rewrite them as a system of partial differential equations for two scalar functions – the structure functions of the superintegrable system.

Definition 7.1. A second order Codazzi tensor is a symmetric tensor $C_{ij}$ satisfying

$$\frac{\partial}{\partial k} C_{ij,k} = 0.$$  \hspace{2cm} (7.1)

Proposition 7.2. For every non-degenerate superintegrable system on a constant curvature manifold, there exists a function $\Phi$ such that the trace modification

$$C_{ij} = Z_{ij} + \Phi g_{ij}$$  \hspace{2cm} (7.2)

of the symmetric tensor (4.6d) is a Codazzi tensor.

3Superintegrable systems are here described geometrically, as subspaces in the space of Killing tensors. This is slightly different from the traditional viewpoint, which characterises superintegrable systems by representatives, i.e. specific Killing tensors that generate this subspace.
Proof. On a constant curvature manifold we have
\[ R_{ijkl} = \frac{R}{n(n-1)} (g_{ik}g_{jl} - g_{il}g_{jk}), \]
where \( R = R_{ab}^{ab} \) is the scalar curvature. In this case the curvature term in (4.10) vanishes due to the first integrability condition (4.4a),
\[ Z_{ik,l}g_{jm} = 0. \]
The trace of this equation over \((j, k)\) yields a conformal Codazzi equation for the tensor \( Z_{ij} \):
\[ Z_{il,m} = 0. \] (7.3)
Skew symmetrising the covariant derivative of this equation in all indices but \( i \) and using the Ricci identity results in
\[ R^{a \ell mn}Z_{ai} + \frac{1}{n-1}g_{il}Z_{m,a}^{a} = 0. \]
Taking the trace of this equation over \((i, n)\) now shows that the divergence of \( Z_{ij} \) is closed,
\[ Z_{m,a}^{a} = 0, \]
and hence the differential of some function, more precisely
\[ Z_{m,a}^{a} = (n-1)\Phi_{m}. \]
Under this condition, the Codazzi equation (7.1) for \( Z_{ij} \) reduces to the conformal Codazzi equation (7.3) for \( C_{ij} \), which we have already shown to be satisfied. \( \square \)

Lemma 7.3. [Fer81] On a constant curvature manifold every second order Codazzi tensor \( C_{ij} \) is locally of the form
\[ C_{ij} = C_{,ij} + \frac{R}{n(n-1)} C g_{ij}, \]
for some function \( C \), where \( R \) is the scalar curvature. The function \( C \) is unique up to a gauge transformation \( C \mapsto C + \delta C \) with
\[ \delta C_{,ij} = 0. \]
Conversely, every tensor of this form on a constant curvature manifold is a Codazzi tensor.

Definition 7.4. A third order Codazzi tensor is a totally symmetric tensor \( B_{ijk} \) that satisfies
\[ B_{ijk,l} = 0. \] (7.4)

Proposition 7.5. For every non-degenerate superintegrable system on a constant curvature manifold, the tensor
\[ B_{ijk} = T_{ijk} + \frac{1}{n-1}g_{ij}t_{,k} + \frac{1}{2(n-2)} g_{ij}C_{,k} \] (7.5)
is a Codazzi tensor.
Proof. First note that the tensor $B_{ijk}$ is indeed symmetric, due to the first integrability condition (4.4a). Using the definitions (7.2) and (7.5) of $C_{ij}$ respectively $B_{ijk}$ together with the fact that $g_{ik}g_{jl} = 0$, we check that the Codazzi equation for $B_{ijk}$ is equivalent to the second integrability condition (4.4b):

\[ B_{ijk,l} = \frac{1}{2} B_{ij} \left( T_{ijk,l} + \frac{2}{n-2} g_{ik} Z_{jl} \right) = 0. \]

\[ \Box \]

The proof of the following Lemma is analogous to that of Lemma 7.3.

Lemma 7.6. On a constant curvature manifold every third order Codazzi tensor $B_{ijk}$ is locally of the form

\[ B_{ijk} = \frac{1}{6} B_{ij} \left( T_{ijk} + \frac{4R}{n(n-1)} g_{ij} B \right), \]

for some function $B$. The function $B$ is unique up to a gauge transformation $B \mapsto B + \delta B$ with

\[ i j k \delta B_{ijk} = 0. \]

Conversely, every tensor of this form on a constant curvature manifold is a Codazzi tensor.

7.2. Structure functions of a superintegrable system.

Proposition 7.7. The structure tensor of a superintegrable system on a constant curvature manifold has the decomposition (4.7) with

\[ \hat{T}_{ijk} = \frac{1}{6} i j k B_{ijk}, \]

\[ \frac{n}{n-1} t = \Delta B + \frac{2(n+1)}{n(n-1)} R B - \frac{n+2}{n-2} C + \text{constant} \]

for two functions $B$ and $C$. The functions $B$ and $C$ are unique up to a gauge transformation

\[ B \mapsto B + \delta B \]

\[ C \mapsto C + \delta C \]

satisfying the compatibility condition

\[ \frac{n+2}{n-2} \delta C = \Delta \delta B + \frac{2(n+1)}{n(n-1)} R \delta B + \text{constant}. \]

In particular, we can choose simultaneously

\[ C(x_0) = 0 \quad \nabla C(x_0) = 0 \quad \Delta C(x_0) = 0 \]

\[ B(x_0) = 0 \quad \nabla B(x_0) = 0 \quad \nabla \nabla B(x_0) = 0 \]

in a fixed point $x_0$. 

Proof. Combining Proposition 7.5 and Lemma 7.6, we get

\[ T_{ijk} + \frac{1}{n-1} g_{ij,t} = \left[ i \mid j \mid k \right] \left( \frac{1}{6} B_{ijk} + \frac{2R}{3n(n-1)} g_{ij} B_{jk} - \frac{1}{2(n-2)} g_{ij} C_{jk} \right). \]

Taking the trace-free part on each side results in (7.6a). Contracting this equation in \( i \) and \( j \) yields

\[ \frac{n}{n-1} t_{,k} = \frac{1}{3} \left( B_{,a} a_{,k} + 2 B_{,ka} a \right) + \frac{4R}{3n(n-1)} B_{,k} - \frac{n+2}{n-2} C_{,k}. \]

By the Ricci identity,

\[ B_{,ka} - B_{,a} a_{,k} = B_{,ak} a - B_{,a} a_{,k} = R_{,ak} a B_{,b} = \frac{R}{n} B_{,k}, \]

so that

\[ \frac{n}{n-1} t_{,k} = B_{,a} a_{,k} + \frac{2(n+1)R}{n(n-1)} B_{,k} - \frac{n+2}{n-2} C_{,k}. \]

This is equivalent to (7.6b) and imposes the constraint (7.7c) on the gauge transforms (7.7a) and (7.7b).

A flat constant curvature manifold is locally isometric to Euclidean space, so that (7.7a) and (7.7b) can easily be integrated to give

\[ \delta C(\vec{r}) = 2(n-2) \left( \frac{1}{2} c_2 r^2 + c_1 \vec{r} + c_0 \right) \]

\[ \delta B(\vec{r}) = \frac{1}{2} b_4 r^4 + r^2 b_3 \vec{r} + \vec{r} A \vec{r} + \vec{b}_1 \vec{r} + b_0, \]

where the \( b_0, b_4, c_0, c_2 \) are scalar constants, \( \vec{b}_1, \vec{b}_3, \vec{c}_1 \) are vectorial constants and \( A \) is a constant symmetric matrix. Imposing (7.7c) with \( R = 0 \), we find \( b_4 = c_2 \) and \( \vec{b}_3 = \vec{c}_1 \). This shows that we can choose the gauge (7.8) in a single point.

A non-flat constant curvature manifold is conformally equivalent to a flat one. Noting that on a constant curvature manifold the operators (7.7c) are invariant under conformal transformations and that the Laplace operator \( \Delta \) changes by a multiple of the identity, the proof is similar to the flat case.

\[ \square \]

**Proposition 7.8.** On a manifold of constant curvature \( \kappa \), the integrability conditions for a superintegrable potential in the form (4.6) are equivalent to the following equations for the structure functions:

\[ 0 = \left[ \frac{i}{j} \frac{j}{k} \frac{k}{l} \right] B_{,ik} a B_{,jla} \]

\[ \hat{C}_{,ij} = B_{,i} a B_{,jab} + B_{,ij} a \left( C - 2(n-2)\kappa B - \Delta B \right)_{,a} \]

\[ - \frac{1}{n-2} \left( C - (n-2)\kappa B \right)_{,i} \left( C - (n-2)\kappa B \right)_{,j} \]

\[ \text{(7.9a)} \]

\[ \text{(7.9b)} \]

\[ \text{(7.9c)} \]

**Proof.** By Proposition 4.6, Equation (4.6a) is equivalent to the decomposition (4.7) of the structure tensor according to (7.6). Equation (4.6b) is equivalent to (7.9a) after substituting (7.6). Equation (4.6c) is equivalent to the Codazzi equation for \( B_{ijk} \). Equation (4.6d) is equivalent to (7.9c) after substituting (7.6) and (7.2). \( \square \)
Remark 7.9. Equation (7.9a) can alternatively be written with the (symmetric) Codazzi tensor $B_{ijk}$ instead of the covariant derivative $B_{ijk}^a$ of the structure function $B$:

$$0 = \begin{bmatrix} i & k \\ j & l \end{bmatrix}^* B_{ik}^a B_{jla}. $$

The reason is that $B_{ijk}$ is symmetric up to trace terms, so that the only non-trace terms in the contraction $B_{ik}^a B_{jla}$ do not have Riemann symmetry according to the Littlewood-Richardson rule.

$$\begin{array}{ccc}
\begin{bmatrix} i \\ j \end{bmatrix} \otimes \begin{bmatrix} k \\ l \end{bmatrix} & \cong & \begin{bmatrix} i \\ j \end{bmatrix} \oplus \begin{bmatrix} k \\ l \end{bmatrix}.
\end{array}$$

Remark 7.10. Equation (7.9c) can be extended by a second equation expressing the covariant derivative of the Laplacian $\Delta C$ polynomially in $\nabla C$, where the coefficients are polynomial in the derivatives of $B$. Together, they define a (non-linear) prolongation of (7.9c). This leads to higher order integrability conditions for the prolongation of a superintegrable Killing tensor on a constant curvature manifold and will be pursued in a forthcoming paper.

7.3. Structure connection of a superintegrable system. We now encode the information about a superintegrable system on a constant curvature manifold in a torsion-free affine connection and formulate the integrability conditions for the corresponding potential and the Killing tensors as flatness of this connection.

Definition 7.11. For a non-degenerate superintegrable system on a manifold of constant curvature we introduce a torsion-free affine connection $\hat{\nabla}$ by

$$g(\hat{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + A(X, Y, Z),$$

where the symmetric tensor $A_{ijk}$ is given by

$$A_{ijk} := -\frac{1}{3} B_{ijk} + \frac{1}{6(n-2)} \begin{bmatrix} i & j & k \end{bmatrix} g_{ij} C_{,k}.$$

We call the connection $\hat{\nabla}$ the structure connection of the superintegrable system and denote its curvature tensor by $\hat{R}_{ijkl}$.

We remark that the structure connection $\hat{\nabla}$ is in general not a metric connection.

Proposition 7.12. The curvature of the structure connection $\hat{\nabla}$ is

$$\hat{R}^i_{jkl} = R^i_{jkl} + \frac{1}{9} \begin{bmatrix} k \\ l \end{bmatrix} \left( B^i_{ka} B^a_{jkl} + \frac{C_{,a}}{n-2} \left( g^i_j B_{jak} - g_{jl} B^i_{ak} \right) + \frac{3}{n-2} \left( g^i_j C_{,jk} + g_{jl} C^i_{,k} \right) \right. $$$

$$+ \frac{1}{(n-2)^2} \left( g^i_k C_{,j} C_{,l} - g_{jk} C_{,i} C_{,l} + g^i_k g_{jl} C^a_{,a} C_{,a} \right) \right) \quad (7.10)$$

Proof. The formula follows from applying the definition of the curvature of an affine connection,

$$\hat{R}^i_{jkl} = R^i_{jkl} + \begin{bmatrix} k \\ l \end{bmatrix} \left( \nabla_k A^i_{jl} + A^i_{mk} A^m_{jl} \right), \quad (7.11)$$

taking into account the Codazzi equation (7.4) for $B_{ijk}$. \qed
Proposition 7.13. On a constant curvature manifold, the Equations (5.9) and (5.10) imply
\[ \hat{T}^{abc}\hat{T}_{abc} - (n-1)(n+2) \hat{T}^a_a = 9R. \]

Proof. In the case of constant curvature (5.10b) becomes \( \hat{Z}_{ij} = 0 \). Consider the system of equations
\[ \begin{align*}
\hat{Z}^{ab} \nabla_c \hat{Z}_{ab} &= 0, \\
\hat{T}^{abc} \nabla_c \hat{Z}_{ab} &= 0, \\
\hat{t}^a \nabla_b \hat{Z}_{ab} &= 0, \\
\hat{t}^a \hat{T}^{abc} &= 0, \\
\hat{t}^a \hat{t}^b &= 0
\end{align*} \]
(7.12)
where we replace \( \nabla_l T^{ijk} \) using (5.9). It is easily confirmed that the system (7.12) has the algebraic consequences
\[ \begin{align*}
(\hat{T}^{abc}\hat{T}_{abc} - (n-1)(n+2) \hat{T}^a_a - 9R)^{abc} &= 24n(n-1) \hat{R}^{ab}
\end{align*} \]
and
\[ \begin{align*}
(\hat{T}^{abc}\hat{T}_{abc} - (n-1)(n+2) \hat{T}^a_a - 9R)^{a^2} &= \frac{6n}{(n-2)(n+2)} \hat{R}^{ab}
\end{align*} \]
(7.14)
Since for constant sectional curvature spaces we have \( \hat{R}_{ij} = 0 \), (7.16c) holds. This proves the assertion since, in the case \( T_{ijk} = 0 \), Equation (5.9) already implies \( R = 0 \), entailing (7.16c). \( \square \)

Corollary 7.14. Consider a manifold of constant curvature and in addition require the hypotheses of Corollary 5.12. If \( t_i = 0 \) holds in a (possibly smaller) neighbourhood of \( q_0 \), then the structure tensor vanishes already entirely in this neighbourhood, \( T_{ijk} = 0 \), and the underlying manifold must be flat.

Proof. By contraction of (5.9), and inserting the hypothesis, we obtain
\[ \left( \hat{T}_{i}^{ab}\hat{T}_{jab} \right) = 0 \quad (7.13) \]
and
\[ R = -\frac{3n+2}{9(n+2)} \hat{T}^{abc}\hat{T}_{abc}, \quad (7.14) \]
which for \( R = 0 \) already implies \( \hat{T}_{ijk} = 0 \). Otherwise, we insert (7.13) and (7.14) into the integrability condition for \( \hat{T}_{ijk} \), finding
\[ g^{ij} \left( \frac{\hat{T}_{ijk,lm}}{\hat{T}} - \frac{1}{2} \hat{R}^{lk}_{i} \hat{T}_{jk} \right) = 0 \]
and then \( R\hat{T}_{jkm} = 0 \).

Assuming \( R \neq 0 \) this implies \( \hat{T}_{ijk} = 0 \) in the neighbourhood, but this would imply \( R = 0 \) through (7.14). We have thus confirmed the claim. \( \square \)

We now summarise our results for superintegrable systems on constant curvature manifolds.

Theorem 7.15. The following are equivalent for a non-degenerate superintegrable system on a constant curvature space:

(i) The system is abundant.
(ii) The structure tensor of the system satisfies the generic integrability conditions (5.8).

(iii) The structure tensor satisfies (5.9) and the trace-free part of the tensor (4.6d) vanishes identically.

(iv) The structure tensor satisfies (5.9) and the structure function $C$ vanishes identically, up to gauge transformations of the form (7.7c).

(v) The structure tensor satisfies (5.9) and the structure connection $\hat{\nabla}$ has Riemann symmetry.

(vi) The structure connection $\hat{\nabla}$ is flat.

(vii) The structure function $B_{ijk}$ satisfies

$$i \overset{k}{j} \overset{l}{k} \overset{i}{l} \left( B_{ij}^{a} B_{kla} + \frac{9R}{n(n-1)} g_{jjg_{kl}} \right) = 0$$

and the structure function $C$ vanishes up to gauge transformations of the form (7.7c).

(viii) The structure tensor of the system satisfies the following Equations

$$\overset{k}{j} \overset{l}{k} \overset{i}{l} \left( \hat{T}_{ik}^{a} \hat{T}_{jla} = 0 \right)$$

$$\left( \hat{T}_{i}^{ab} \hat{T}_{jab} - (n-2)(\hat{T}_{ij} \overset{i}{t}_{a} + \overset{j}{t}_{i}) \right) = 0$$

$$\hat{T}_{abc} \hat{T}_{abc} - (n-1)(n+2) \overset{i}{t}_{a} = 9R.$$  

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is by Lemma 5.7. The equivalence (ii) $\Leftrightarrow$ (iii) follows from Theorem 5.9 and Proposition 4.5 for constant curvature manifolds. The equivalence (iii) $\Leftrightarrow$ (iv) is a consequence of Proposition 7.2, noting that $\hat{C}_{ij} = 0$ is equivalent to $C = 0$ up to gauge transforms. For the equivalence (iv) $\Leftrightarrow$ (v) we decompose the curvature tensor $\hat{R}_{ijkl}$ in (7.11) into its symmetric and antisymmetric part with respect to the indices $(i, j)$. Due to the symmetry of $A_{ijk}$ the antisymmetric part of $\hat{R}_{ijkl}$ has Riemann symmetry. The symmetric part is proportional to

$$g_{ij} C_{kl}$$

and vanishes exactly if $\hat{C}_{kl} = 0$.

The equivalence (vi) $\Leftrightarrow$ (vii) follows from (7.10), noting that the vanishing of the non-Riemann symmetric part of the curvature of $\nabla$ implies $\hat{C}_{ij} = 0$ as above. For the equivalence (vii) $\Leftrightarrow$ (viii) we note that (7.16) is nothing but the Ricci decomposition of the Riemann tensor in (7.15) under the substitution (7.6), using (4.6d) and (7.2).

We now finish the proof by showing the equivalence (iii) $\Leftrightarrow$ (viii). If $\hat{Z}_{ij} = 0$, the equations (4.6b) and (4.6d) imply (7.16a) respectively (7.16b), which together with (5.9) imply (7.16c) by Proposition 7.13. This shows (iii) $\Rightarrow$ (viii). The converse follows from Lemma 5.11 using (4.6d).

**Corollary 7.16.** The set of abundant superintegrable Hamiltonians on a constant curvature manifold is naturally endowed with the structure of a projective variety, isomorphic to the variety of cubic forms $\Psi_{ijk} x^{i} x^{j} x^{k}$ on $\mathbb{R}^{n}$ satisfying

$$\overset{i}{j} \overset{k}{i} \left( \Psi_{ijk}^{a} \Psi_{kla} + \frac{9R}{n(n-1)} g_{ij} g_{kl} \right) = 0.$$  

(7.17)
Proof. By the above theorem, the structure function $B$ of an abundant superintegrable system satisfies (7.15) and the structure function $C$ can be chosen to be identically zero. This completely determines the structure tensor $T_{ijk}$, which satisfies the hypotheses of Corollary 5.12. Consequently, the structure tensor can be recovered from the values $\Psi_{ijk} = B_{ijk}(x_0)$ in a single point $x_0$. Abundant superintegrable systems are therefore in bijective correspondence to points in the variety given by (7.17). □

8. Examples

Families of non-degenerate second-order superintegrable systems that can be defined for arbitrary dimension, can be obtained by two major procedures: First, by contracting known arbitrary-dimensional families. In this case, the contraction has to be such that it generalises to any dimension. Second, we may “extend” a system known from dimensions 2 and 3, by extrapolating its arbitrary-dimensional form, and then checking its validity.

Following the second approach, we have three natural examples of n-dimensional superintegrable systems on constant curvature spaces:

8.1. The isotropic harmonic oscillator. The isotropic harmonic oscillator exists only on flat space and has the non-degenerate potential

$$V = \omega^2 \sum_i x_i^2 + \sum_i \alpha_i x_i + \beta.$$ 

The structure tensor vanishes completely, $T_{ijk} = 0$, and so this may be considered the trivial superintegrable system.

8.2. The “generic” system. The generic system, on flat space, was first described by Friš et al. [FMS+65] and is referred to by the alternative name Smorodinsky-Winternitz $I^4$. It is one of the most prominent families of known superintegrable systems and significant as, in dimensions 2 and 3, any other non-degenerate second-order superintegrable system can be obtained from it by (an iteration of) limits of parametrised coordinate changes [KMP13, CKP15]. The generic system is the uppermost parent in the graph of Figure 1 in [KS19]. Concretely, the generic system on flat n-dimensional space with $g_{ij} = \delta_{ij}$, is given by the $(n + 2)$-parameter potential (the parameters are $a_1, \ldots, a_n, \omega, \beta$; on the $n$-sphere analogous statements hold)

$$V = \sum_{i=1}^{n} \left( \frac{a_i}{x_i^2} + \omega x_i^2 \right) + \beta .$$

It is compatible with the $n(n+1)/2$ Killing tensors

$$P_i = p_i^2, \quad J_{ij} = (x^i p_j - x^j p_i)^2 .$$

Moreover, in low dimensions, the following labels have been used in the literature: In the 2-dimensional case, on flat space (resp. the $n$-sphere) it has been labelled as $(0, 11, 0)$ [KS19] or [E1] (on the 2-sphere: [S9]) [KKPM01]. The corresponding label in three dimensions is $[I]$ (resp., $[I']$) in [KKM07c, Cap14].
8.3. The anisotropic oscillator. In the flat case, this system is also referred to as Smorodinsky-Winternitz II system. The anisotropic harmonic oscillator can be viewed as a blending of the generic system and the isotropic harmonic oscillator. For instance, the potential

\[ V = \omega^2 \left( 4x_m^2 + \sum_{i \neq m} x_i^2 \right) + \left( \alpha_m x_m + \sum_{i \neq m} \frac{\alpha_i}{x_i^2} \right) + \beta \]

is a non-degenerate superintegrable potential. More generally, we can define the potential

\[ V = \omega^2 \left( 4 \sum \tilde{x}_m^2 + \tilde{x}_i^2 \right) + \left( \sum \alpha_m x_m + \sum \alpha_i \right) \]

where \( \sum \) denotes a sum over one subset of indices, and \( \sum \) the summation over the remaining ones.

8.4. A non-Wilczynski system. In Proposition 3.2, we have assumed that the Killing tensors \( K^{(\alpha)} \), compatible with the superintegrable potential \( V \), do not have a common eigenvector. Indeed, the presence of a common eigenvector can complicate the situation considerably, as the following example illustrates.

Example 8.1. The potential

\[ V = \sum_i x_i^2 \]  

(8.1)

is second-order maximally superintegrable and Wilczynski (with non-unique structure tensor). The \( \frac{n(n+1)}{2} \) integrals defining a superintegrable system are given, via (2.6), by the Killing tensors

\[ K^{(ii)} = p_i^2 \quad \text{and} \quad K^{(ij)} = (x^i p_j - x^j p_i)^2 \, . \]

For each of the Killing tensors \( K^{(\alpha)} = K^{(ij)} \) (1 \( \leq i, j \leq n \)), compatible with \( V \) via (2.6), the differential \( dV \) is an eigenvector, meaning \( dV \wedge K^{(\alpha)} dV = 0 \). We therefore have

\[ dK^{(\alpha)} df(V) = f''(V) dV \wedge K^{(\alpha)} dV + f'(V) dK^{(\alpha)} dV = 0 \, , \]

for any smooth function \( f \in C^\infty \). As a result, any such \( f \) gives rise to a new superintegrable system, with potential \( f(V) \), compatible with the same Killing tensors \( K^{(\alpha)} \) as \( V \). However, these new potentials are in general not Wilczynski potentials, except when \( f'' = 0 \), because

\[ \nabla^2 f(V) = \nabla df(V) = f''(V) dV \otimes dV + f'(V) \nabla^2 V \, . \]
Families of non-degenerate second-order superintegrable systems in arbitrary dimension $n \geq 3$

| Example                                      | potential                                                                 | $B$                                                                 | $-\frac{n+2}{3} \bar{t}$ |
|----------------------------------------------|---------------------------------------------------------------------------|----------------------------------------------------------------------|----------------------------|
| Euclidean geometry $g = \sum_i dx_i^2$        | $\omega^2 \sum_i x_i^2 + \sum_i \alpha_i x_i$                          | 0                                                                  | 0                          |
| Isotropic oscillator                         | $\sum_{i=1}^n \left( \frac{a_i}{x_i^2} + \omega x_i^2 \right)$         | $-\frac{3}{n-1} \sum_i (x_i^2 \ln(x_i))$                         | $\sum_k \ln(x_k)$          |
| “Generic system” / Smorodinsky-Winternitz I   | $\omega^2 \left( 4x_m^2 + \sum_{i \neq m} x_i^2 \right) + \left( \alpha_m x_m + \sum_{i \neq m} \frac{a_i}{x_i^2} \right)$ | $-\frac{3}{n-1} \sum_{i \neq m} (x_i^2 \ln(x_i))$ | $\sum_{i \neq m} \ln(x_i)$ |
| Smorodinsky-Winternitz II                     | $\omega^2 \left( \frac{x^2}{n} \frac{1}{n(n-1)} \right)$              | $\frac{3}{2} n(n-1) \sum_i \frac{x_i^2 \ln(x_i)}{x_i^2}$          | $\sum_{i \neq n} \ln(x_i)$ |
| Complex n-sphere $g = -n(n-1) \sum_{i=1}^n dx_i^2$ |                                                                       |                                                                     |                             |
| “Generic system”                             | $-\frac{x_n^2}{n(n-1)} \sum_{i=1}^n \left( \frac{a_i}{x_i^2} + \omega x_i^2 \right)$ | $\frac{3}{2} n(n-1) \sum_i \frac{x_i^2 \ln(x_i)}{x_i^2}$          | $\sum_{i \neq n} \ln(x_i)$ |
| (Reduced)                                     | $\left( \frac{x_n^2}{n(n-1)} \right) \left[ \frac{x_n^2}{4} + \sum_{i \neq n} x_i^2 \right] + \left( \alpha_n x_n + \sum_{i \neq n} \frac{a_i}{x_i^2} \right)$ | $\frac{3}{2} n(n-1) \ln(x_n)$                                   | $-(n+1) \ln(x_n)$          |
| Anisotropic oscillator                       | $-\frac{x_n^2}{n(n-1)} \left( \frac{x_n^2}{4} + \sum_{i \neq n} x_i^2 \right) + \left( \alpha_n x_n + \sum_{i \neq n} \frac{a_i}{x_i^2} \right)$ | $\frac{3}{2} n(n-1) \ln(x_n)$                                   | $-(n+1) \ln(x_n)$          |

Table 2. The structure function $C$ is not shown as it is given only by gauge terms, and thus can be chosen as $C = 0$. In the second column, the constant term in the potentials has been suppressed. The last column the function $\bar{t}$ is shown for convenience of the reader, but is redundant due to formula (7.6b).
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