Optical Physics of Imaging and Interferometric Phased Arrays

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ABSTRACT

Microwave, submillimetre-wave, and far-infrared phased arrays are of considerable importance for astronomy. We consider the behaviour imaging phased arrays and interferometric phased arrays from a functional perspective. It is shown that the average powers, field correlations, power fluctuations, and correlations between power fluctuations at the output ports of an imaging or interferometric phased array can be found once the synthesised reception patterns are known. The reception patterns do not have to be orthogonal or even linearly independent. It is shown that the operation of phased arrays is intimately related to the mathematical theory of frames, and that the theory of frames can be used to determine the degree to which any class of intensity or field distribution can be reconstructed unambiguously from the complex amplitudes of the travelling waves at the output ports. The theory can be used to set up a likelihood function that can, through Fisher information, be used to determine the degree to which a phased array can be used to recover the parameters of a parameterised source. For example, it would be possible to explore the way in which a system, perhaps interferometric, might observe two widely separated regions of the sky simultaneously.

Keywords: Phased arrays, Imaging arrays, Interferometry, Frames, Partially coherent optics

1. INTRODUCTION

There is considerable interest in developing phased arrays for radio astronomy. Projects include the Square Kilometer Array (SKA), the Low Frequency Array (LOFAR), the Electronic Multibeam Radio Astronomy Concept (EMBRACE), and the Karoo Array Telescope (KAT).\textsuperscript{1-3} All of these projects are aimed constructing phased arrays for microwave astronomy, but as technological capability improves, phased arrays will eventually be constructed for submillimetre-wave and far-infrared astronomy.\textsuperscript{4, 5}

Two types of phased array are of interest: (i) imaging phased arrays, where an array of coherent receivers is connected to a beam-forming network such that synthesised beams can be created and swept across the sky; (ii) interferometric phased arrays, where the individual antennas of an aperture synthesis interferometer are equipped with phased arrays such that fringes are formed within the synthesised beams. In this way it is possible to extend the field of view, to observe completely different regions of the sky simultaneously, to steer the field of view electronically, and to observe spatial frequencies that are not available to an interferometer because the baselines cannot be made smaller than the diameters of the individual antennas.

It is important to recognise that the synthesised beams of a phased array need not be orthogonal, and may even be linearly dependent. Non-orthogonality may be built into a system intentionally as a way of increasing the fidelity with which an image can be reconstructed, or it may arise inadvertently as a consequence of RF coupling and post-processing cross-talk. In some situations, say in the case of interacting planar antennas, it may not even be clear how to distinguish one basis antenna from another, even before the beam-forming network has been connected.

In this paper, we show that the only information that is needed to determine the average powers, the correlations between the complex travelling wave amplitudes, the fluctuations in power, and the correlations...
between the fluctuations in power at the output ports of a phased array, or between the output ports of phased arrays on different antennas, is the synthesised beams. It is not necessary to know anything about the internal construction of the arrays or the beam forming networks. Beam patterns may be taken from electromagnetic simulations or experimental data. In the case of interferometric phased arrays, the arrays on the individual antennas do not have to be the same.

The ability to assess the behaviour of a system simply from the synthesised beam patterns separates the process of choosing the best beams for a given application from the process of understanding how to realise the beams in practice. It also suggests important techniques for simulating phased arrays, and for analysing experimental data.

2. BASIC PRINCIPLES

In practice, an imaging phased array comprises a sequence of optical components, an array of single-mode antennas, and an electrical beam-forming network such that each output port corresponds to a synthesised reception pattern on the input reference surface, usually the sky. In some cases, the synthesised reception patterns may be static and designed to give optimum sampling on a given class of object, whereas in other cases, the beam-forming network may be controlled electrically to generate a set of synthesised beams that can be swept across the field of view. In the case of microwave astronomy, the optical system would be a telescope, the single-mode antennas would be the horns or planar antennas of an array of HEMT amplifiers or SIS mixers, and the beam-forming network would be a system of microwave or digital electronics.

Our analysis is based on the generic system shown in Fig. 1. \( A \) denotes the input reference surface, \( B \) the output ports of the horns, and \( C \) the output ports of the beam-forming network. We shall assume that an array of \( M \) horns is connected to a beam-forming network having \( P \) output ports. Each of the \( P \) ports is thus associated with a reception pattern on the input reference surface. For simplicity, we shall assume paraxial optics throughput. When a pseudomonochromatic field, \( x(r) \), is incident on the system, a set of travelling waves will appear at \( B \): we shall denote their complex amplitudes by \( \{ y_m : m \in 1, \cdots, M \} \). We shall use the notion of complex analytic signals throughout, which for most practical purposes means that one can integrate the final result over some bandwidth to calculate general behaviour. Likewise, a set of travelling waves will appear at \( C \): we shall denote their complex amplitudes by \( \{ z_p : p \in 1, \cdots, P \} \). When \( M \) and \( P \) are finite, the complex amplitudes can be assembled into column vectors \( y \in \mathbb{C}^M \) and \( z \in \mathbb{C}^P \), respectively.

In what follows, it will sometimes be beneficial to represent the primary variables as abstract vectors. Because the incoming field, \( x(r) \), is square integrable over the input reference surface, \( A \), it can be represented by a vector \( |x \rangle \) in Hilbert space \( \mathbb{H} \). The input surface may extend to infinity, or it may be bounded by an aperture, and therefore of finite extent. Regions having different shapes and sizes correspond to different Hilbert spaces. \( y \) and \( z \) can also be represented by abstract vectors, \( |y \rangle \in \ell^2 \) and \( |z \rangle \in \ell^2 \) respectively, where \( \ell^2 \) is the space of square-summable sequences. These definitions lead to two operators, one of which, \( \hat{H} : \mathbb{H} \to \ell^2 \), maps the
incoming optical field onto the outputs of the horns, and the other \( \Phi : \ell^2 \rightarrow \ell^2 \) maps the outputs of the horns onto the outputs of the beam-forming network. These individual operators can be combined into a single composite operator \( T = \Phi H : \mathbb{H} \rightarrow \ell^2 \), which describes the system as a whole.

It can be shown, Appendix A, using only the concepts of inner product, operators, and adjoints in Hilbert space, that the complex travelling-wave amplitude appearing at port \( p \), when a field, \( x(r) \), is incident on a system is given by

\[
z_p = \int_A t_p^*(r) \cdot x(r) \, d^2r, \tag{1}
\]

where \( t_p(r) \) is the functional form of the \( p \)’th synthesised reception pattern. \( A \) corresponds to the surface and region over which the Hilbert space is defined. In expressions such as (1) we shall show the complex conjugate explicitly, even though some notation includes it in the dot product, as an inner product, implicitly. The reason for the formality in stating, and indeed deriving (1), is that (1) can be shown to be true even when the beam patterns are not orthogonal.

The synthesised reception patterns are central to what follows because, according to (1), the complex travelling wave appearing at port \( p \) is given by calculating the inner product, over the input reference surface, between the synthesised reception pattern \( t_p(r) \) and the incoming field. It would be naive to assume, however, that when a system is illuminated by a field having the form \( t_p(r) \), a travelling wave only appears at \( p \). In the case of phased arrays, the synthesised reception patterns do not have to be orthogonal, and can even be linearly dependent. Thus, although the output at a given port is given by the inner product between a field and a reception pattern, as for ordinary antennas, one cannot assume that there is a one-to-one mapping between the antenna patterns and the ports.

For example, in the case of Fig. 1, the beam patterns of the horns, \( h_m(r) \), are orthogonal, and the outputs of the horns, \( y_m \), are given by

\[
y_m = \int_A h_m^*(r) \cdot x(r) \, d^2r, \tag{2}
\]

but the beam-forming network is described by a linear operator \( \Phi \), and therefore

\[
z_p = \sum_m \phi_{pm} y_m. \tag{3}
\]

Substituting (3) in (2) we find

\[
z_p = \int_A \sum_m \phi_{pm} h_m^*(r) \cdot x(r) \, d^2r, \tag{4}
\]

which can be cast into the form of (1) by defining

\[
t_p^*(r) = \sum_m \phi_{pm} h_m^*(r). \tag{5}
\]

As expected, the synthesised reception patterns are merely weighted linear combinations of the horn patterns. The orthogonality of the synthesised reception patterns can now be tested through

\[
\int_A t_p^*(r) \cdot t_{p'}(r) \, d^2r = \sum_m \phi_{pm} \phi_{p'm}^*, \tag{6}
\]

where (5) has been used, together with the orthonormality of the horn patterns.

In the case where the numbers of horns and ports are finite, (6) takes the form of a matrix equation:

\[
\int_A t_p^*(r) \cdot t_{p'}(r) \, d^2r = \Phi \Phi^\dagger. \tag{7}
\]

Because \( \Phi \) is mapping between \( \mathbb{C}^M \) and \( \mathbb{C}^P \), \( \Phi \) is under complete if \( P > M \), and \( \Phi \Phi^\dagger \) is singular; contrariwise, \( \Phi \) is over complete if \( P < M \), and \( \Phi \Phi^\dagger \) is not singular. In both cases, except trivially when certain ports are
not connected, the synthesised reception patterns are not orthogonal, because $\Phi \Phi^\dagger \neq I_P$ is not diagonal. In the case where $\Phi$ is unitary, $\Phi \Phi^\dagger = I_P$, where $I_P$ is the identity operator of dimension $P$, the synthesised reception patterns are orthogonal. Butler beam forming networks are used in practice to realise this situation.

In summary, the complex travelling-wave amplitudes appearing at the output ports of a phased array are found by calculating the inner products of the incoming field with respect to a set of synthesised reception patterns, but the synthesised reception patterns do not have to be orthogonal. Even if a system is designed to have orthogonal beams, practical issues relating to coupling and cross talk will cause the beam patterns to be non-orthogonal at some level. One would, therefore, like to derive an analysis procedure based on the beam patterns alone, where it is not necessary to know anything about the internal construction of the array. For our purposes, we shall assume that the behaviour of all phased arrays is described by (1) regardless of whether it is known how the arrays are constructed or not.

In many cases we are interested in using phased arrays to image incoherent or partially coherent fields—in the context of astronomy, although the field on the sky is usually incoherent, the input reference plane, as far as the phased array is concerned, may be internal to the optics of the telescope. To this end, it is convenient to introduce correlation dyadics. We shall define the correlation dyadic of the incident field according to

$$\overline{X}(r', r) = \langle x(r)x^*(r') \rangle,$$

where $\langle \rangle$ denotes the ensemble average, and $x(r)$ is interpreted as a complex analytic signal. The tensor $\overline{X}(r', r)$ contains complete information about the correlation between the fields at any two points and in any two polarisations. Once the correlation dyadic is known, all classical measures of coherence follow.

The correlation between the travelling wave amplitudes at any two ports can be written $\langle z_p z^*_p' \rangle$, or in matrix form

$$Z = zz^\dagger,$$

where $Z \in \mathbb{C}^{P \times P}$ is a correlation matrix. The matrix elements of $Z$ can be found by using (1):

$$Z_{pp'} = \int_A \int_A t^*_p(r) \cdot \overline{X}(r', r) \cdot t_{p'}(r') \, d^2r \, d^2r'.$$

Now illuminate the system with an unpolarised, spatially fully incoherent source

$$\overline{X}(r', r) = \overline{I} \delta(r - r'),$$

where $\overline{I}$ is the dyadic identity operator. Substituting (11) in (10), we find

$$Z_{pp'} = \int_A t^*_p(r) \cdot t_{p'}(r) \, d^2r,$$

which shows that, because the synthesised reception patterns are generally not orthogonal, the travelling waves at the output ports are correlated. Ultimately, it is these correlations that prevent one from extracting more and more information from a source, using a finite number of horns, by synthesising more and more beams.

3. FRAMES

In what follows, we shall need to make use of the mathematical theory of frames. Suppose for the moment we have some general monochromatic field $|x\rangle$, and that we determine the inner products with respect to a set of basis vectors $T = \{ |t_p\rangle, \ p \in 1, \cdots, P \}$: $z_p = \langle t_p | x \rangle$. $P$ can extend to infinity, and we do not make any assumptions about the orthonormality or linear independence of $T$. Under what circumstances can the original vector $|x\rangle$, which represents a continuous function, be recovered unambiguously from a discrete set of complex coefficients, possibly countable, and how can this be achieved? In the context of phased arrays, we are essentially asking under what circumstances can the form of an incident field be recovered from the outputs of an array, when the synthesised beams are possibly non-orthogonal and linearly dependent.
Evaluate the square moduli of the inner products between \( T \) and any general vector \(|x|\), and sum the results. If there are two constants \( A \) and \( B \) such that \( 0 < A < \infty \) and \( 0 < B < \infty \), and

\[
A \| x \|^2 \leq \| T|x| \|^2 \leq B \| x \|^2,
\]

which can also be written

\[
A \| x \|^2 \leq \sum_p |\langle t_p|x|H \rangle|^2 \leq B \| x \|^2,
\]

\( \forall |x| \in \mathbb{H} \), then the basis set \( T \) is called a frame with respect to \( \mathbb{H} \). Notice the use of strict inequalities in the allowable values of \( A \) and \( B \). In the case where \( A \approx B \), the frame is called a ‘tight frame’ because the inner products for all \(|x| \in \mathbb{H} \) lie within some small range, and the dynamic range needed for inversion is small. When the original basis is orthonormal, the frame bounds, \( A \) and \( B \), are equal, as can be appreciated by inserting \(|x| = |t_{p'}|\) in (14). If the frame is over complete, but normalised, \( A \) is a measure of the redundancy in the frame.

If a basis set constitutes a frame, then it can be shown, through (13) alone, that \( \hat{T} \) is injective, one-to-one, but not surjective, onto: \( T \) maps \( \mathbb{H} \) onto a subspace of \( \ell^2 \), or when \( P \) is finite, a subspace of \( \mathbb{C}^P \). Consequently, \( T \) has a left inverse, \( T^{-1} \), such that \(|x| = T^{-1}T|x| : \forall x \in \mathbb{H} \). \( T^{-1} \) maps the image of \( T \), Im[\( T \)], back onto \( \mathbb{H} \), and maps the null complement of Im[\( T \)], Im[\( T \)]+, onto the zero vector in \( \mathbb{H} \). The inverse operator is given by

\[
\hat{T}^{-1} = \left( \hat{T}^T \hat{T} \right)^{-1} \hat{T}^T = \hat{S}^{-1} \hat{T}^T.
\]

\( \hat{S} \) is bijective such that \( \hat{S}^{-1} \) exists.

It can also be shown that \( T^{-1} \) satisfies the frame condition

\[
\frac{1}{B} \| z \|^2 \leq \| T^{-1}z \|^2 \leq \frac{1}{A} \| z \|^2.
\]

Thus, the more tightly bound the frame, the more tightly bound the inverse, and the more stable the reconstruction process.

The reconstruction of the original field through \( \hat{T}^{-1} \) can be best implemented by the introduction of dual vectors. The dual vectors \( |\hat{t}_p| \) of any given frame \( T \), with respect to Hilbert space \( \mathbb{H} \), can be derived through

\[
|\hat{t}_p| = \hat{S}^{-1}|t_p|.
\]

The dual basis set, which we shall call \( \hat{T} = \{ |\hat{t}_p|, p = 1, \cdots, P \} \), has the same degree of completeness as the original frame, \( T \), and therefore it too constitutes a frame with respect to \( \mathbb{H} \). Indeed, two representations of any general \(|x|\) are possible:

\[
|x| = \sum_p \langle t_p|x|H \rangle |\hat{t}_p| \quad (18)
\]

\[
|x| = \sum_p \langle \hat{t}_p|x\rangle |t_p|.
\]

If one calculates a set of coefficients by taking the inner products with a frame, then one inverts the process by reconstructing the field using the dual vectors. Alternatively, if one calculates the inner products with respect to the dual vectors, then one inverts the process by reconstructing the field using the frame. In the case where the basis vectors are perfectly complete with respect to \( \mathbb{H} \), but not necessarily orthogonal, the basis is called a Riesz basis, and the basis set \( T \) and dual set \( \hat{T} \) are then biorthogonal: \( \langle t_p|t_{p'} \rangle = \delta_{pp'} : \forall p, p' \in 1, \cdots, P \).

In the case where the basis vectors do not constitute a frame, that is to say they are under complete with respect to \( \mathbb{H} \), then one can go through the same procedure as before, but now, when an attempt is made to reconstruct the original field vector, by using the duals,

\[
|x'| = \sum_p \langle t_p|x\rangle |\hat{t}_p|, \quad (19)
\]
the reconstructed vector $|x'\rangle$ cannot, for all vectors in $\mathbb{H}$, be the same as the original vector $|x\rangle$. It can be shown, straightforwardly, that the error vector $|x\rangle - |x'\rangle$ is orthogonal to the basis vectors; in other words, the solution is as close as possible within the degrees of freedom available. $|x'\rangle$ is the orthogonal projection of $|x\rangle$ onto $\mathbb{S}$, the subspace spanned by the under complete set of basis vectors. In the context of functions, the reconstructed field is a least-square fit to the original field. The same conclusion is reached if the inner products are taken with the dual vectors of an incomplete basis, and the field reconstructed using the original vectors.

The relevance to phased arrays is clear, one can measure the complex outputs of a phased array, and if the reception patterns constitute a frame with respect to the Hilbert space defined by the shape, extent, and illumination of the input reference surface, then the continuous incoming field can be reconstructed completely from the discrete set of outputs. If the reception patterns do not constitute a frame, reconstruction leads to the least square fit that is consistent with the degrees of freedom to which the phased array is sensitive. If the field is a spatially fully incoherent source, the number of degrees of freedom in the field is infinite, even if the field only extends over a finite region, and an infinite number of horns is needed to realise a frame. In reality, however, all optical fields only contain a finite number of degrees of freedom, and therefore frames are, at least in principle, possible.

4. MATRIX REPRESENTATIONS

The theory of frames is intimately related to the operation of phased arrays. Suppose, for example, that we wish to describe the behaviour of a phased array by means of a scattering matrix that relates, for any incoming field, the $P$ reception-pattern coupling coefficients to the $P$ output ports. One such representation is simply the $P \times P$ identity matrix, $I_P$, because the outputs are given by the coupling coefficients in any case, but such a representation does not correctly represent the throughput of the system if the number of horns is less than the number of ports, because there are fewer degrees of freedom in the calculated coefficients than the number of coefficients. The identity matrix does not contain any information about the physics of the array, which becomes apparent when one comes to consider internally generated noise.

A better approach is as follows. Using the concept of frames, we can generate a set of coupling coefficients by representing the field in terms of the duals of the reception patterns. From (1)

$$z_p = \int_\mathcal{A} t_p^*(r) \cdot \sum_{p'} z_{p'} t_{p'}(r) \, d^2r = \sum_{p'} R_{pp'} z_{p'}'. \tag{20}$$

where

$$z_{p'}' = \int_\mathcal{A} t_{p'}^*(r) \cdot x(r) \, d^2r. \tag{21}$$

Alternatively, according to (18), we may represent the field in terms of the reception patterns themselves:

$$z_p = \int_\mathcal{A} t_p^*(r) \cdot \sum_{p'} z'_{p'} t_{p'}(r) \, d^2r = \sum_{p'} R'_{pp'} z'_{p'}. \tag{22}$$

where

$$z'_{p'} = \int_\mathcal{A} t_{p'}^*(r) \cdot x(r) \, d^2r. \tag{23}$$

In (20) and (22), $R_{pp'}$ and $R'_{pp'}$ are both $P \times P$ scattering matrices, which are equally good at describing the behaviour of the array. Unlike the identity matrix, however, they can only transmit the same number of degrees of freedom as the array itself regardless of whether the beam patterns constitute a frame with respect to the incoming field or not. They also lead to the appropriate correlations for internally generated noise, as will be shown later.

Now consider the situation were an optical system is placed in front of the phased array described by (22). We wish to describe the behaviour of the optical system itself in terms of a scattering matrix. Moreover, we wish to use the synthesised reception patterns as the basis set on the output side of the optical system, which we shall now call $\mathcal{T}_2 = \{ |t_{2,p} \rangle, \ p \in \{1, \cdots, P^2\} \}$, and some other basis set on the input side of the optical system, which
we shall call $T_1 = \{ |t_{1,p1}\}, p1 \in \{1, \cdots, P1\}$. $T_2$ does not have to be a frame with respect to all possible field distributions that can appear at the output, say $\mathbb{H}_2$, because we are only interested in those fields to which the array can couple. $T_1$ does, however, have to be a frame with respect to all possible field distributions that can appear at the input, because we are not sure how incoming fields will scatter. If we choose to use the synthesised reception patterns as the basis for the input reference surface, we must supplement the set with the complement of $T_2$ relative to $\mathbb{H}_1$. Indeed, through this process we can define a virtual array whose beams are $T_2 \cap [T_2]^{-\perp}$. We shall not develop this idea here.

The behaviour of the optical system can be described by

$$x_2(r_2) = \int_{S_1} \mathcal{N}(r_2, r_1) \cdot x(r_1) d^2 r_1,$$  \hspace{1cm} (24)

where, $x_2(r_2)$ and $x_1(r_1)$ are the fields on the input and output sides, respectively, and again, paraxial optics is assumed. Now we can use the dual frame of $T_1$, $\overline{T}_1$ say, to generate a set of expansion coefficients on the input side:

$$\bar{a}_{p1} = \int_{S_1} \breve{t}_{1,p1}^*(r_1) \cdot x_1(r_1) d^2 r_1.$$  \hspace{1cm} (25)

and then (24) becomes

$$x_2(r_2) = \int_{S_1} \mathcal{N}(r_2, r_1) \cdot \sum_{p1} \bar{a}_{p1} \breve{t}_{1,p1}(r_1) d^2 r_1.$$  \hspace{1cm} (26)

We can also express the output field in terms of a set of coefficients

$$\bar{b}_{p2} = \int_{S_2} \breve{t}_{2,p2}^*(r_2) \cdot x_2(r_2) d^2 r_2.$$  \hspace{1cm} (27)

Substituting (26) in (27) we find

$$\bar{b}_{p2} = \sum_{p1} M_{p2p1} \bar{a}_{p1},$$  \hspace{1cm} (28)

where the matrix elements are given by

$$M_{p2p1} = \int_{S_1} \int_{S_2} \breve{t}_{2,p2}^*(r_2) \cdot \mathcal{N}(r_2, r_1) \cdot \breve{t}_{1,p1}(r_1) d^2 r_1 d^2 r_2.$$  \hspace{1cm} (29)

(29) is an operator, which is a matrix for finite dimensional spaces, that maps the field coefficients on the input side onto the field coefficients on the output side. The operator describes the process of reconstructing the field in the space domain, scattering in the space domain, and then projecting the scattered field onto the output basis set.

If we assume finite dimensionality for all surfaces, and that the output frame of one optical component is used as the input frame of the next optical component, then we can cascade a number of components, $I$, according to

$$M = \prod_{I} M^i,$$  \hspace{1cm} (30)

where the $\{ M^i : i \in \{1 \cdots, I\} \}$ are the scattering matrices of the individual components. The last component, $M^I$, could be the phased array itself, $R'$ in (22), giving a description of the system as a whole.

Earlier, we showed that it is possible to describe the behaviour of a phased array in terms of the duals of the reception patterns, rather than the reception patterns themselves. Equally, we can use either frames or dual frames on the input and output reference surfaces of an optical component to generate a variety of scattering matrices, each of which describes the behaviour of the component equally well. Moreover, we can choose whether to use frames or dual frames, or a mixture, in the definition of the correlation dyadics, thereby generating a variety of equally good ways of describing correlations. When representing the process of scattering a partially coherent field through an optical component, the correlation dyadics should be chosen to match the bases used for the scattering matrices themselves.
5. IMAGING PHASED ARRAYS

5.1. Imaging field distributions

It is not possible to construct a phased array that forms a frame with respect to any undefined complex function, even over a finite-sized region, because an infinite number of individual horns would be needed. In reality, however, optical fields have finite dimensionality, and frames become feasible. Often, a phased array will be placed on the back of an optical system, and the role of the phased array is to collect as much of the information that appears at the output of the optical system as possible. We now consider whether the outputs of a given phased array form a frame with respect to any field that can pass through a preceding optical system.

We have shown previously\(^\text{11}\) that the behaviour of paraxial optical systems is best described using the Hilbert-Schmidt decomposition of the operator that projects the field at the input reference surface onto the output reference surface: \(\overline{\mathbf{N}}(\mathbf{r}_2, \mathbf{r}_1)\) in (24). A Hilbert-Schmidt decomposition is needed because optical systems generally map fields between different Hilbert spaces, and therefore eigenfunctions are not suitable for describing behaviour.

Thus, the dyadic Green’s function in (24) becomes

\[
\overline{\mathbf{N}}(\mathbf{r}_2, \mathbf{r}_1) = \sum_i \sigma_i \mathbf{u}_i(\mathbf{r}_2) \mathbf{v}_i^*(\mathbf{r}_1). \tag{31}
\]

After substituting (31) into (24) it becomes clear that the process of scattering a field through an optical system consists of projecting the incoming field onto the input eigenfields \(\mathbf{v}_i(\mathbf{r}_1)\), scaling by the singular values \(\sigma_i\), and reconstructing the outgoing field through the outgoing eigenfields \(\mathbf{u}_i(\mathbf{r}_2)\). It is also clear, and an intrinsic feature of the Hilbert Schmidt decomposition, that the field, possibly partially coherent, at the output reference surface has only a limited number of degrees of freedom. In the context of (31), the Hilbert Schmidt decomposition has only a finite number of singular values that are significantly different from zero.

What we require is for the synthesised reception patterns of our phased array to create a frame with respect to the vector space spanned by the \(\mathbf{u}_i(\mathbf{r}_2)\) having singular values significantly different from zero: say Hilbert subspace \(\mathcal{S}\). In this case the frame is finite, and could, in principle at least, be realised by a finite number of horns. How do we determine whether the synthesised reception patterns constitute a frame with respect to \(\mathcal{S}\)?

Suppose that \(|\mathbf{x}_2\rangle\) is some general vector in the Hilbert space \(\mathbb{H}_2\) at the output reference surface of an optical system. \(\mathcal{S}\) corresponds to that subspace of \(\mathbb{H}_2\) spanned by the output eigenfields having singular values greater than some threshold value, say \(\sigma_{\text{min}} > \epsilon\). In other words, \(\mathcal{S}\) contains, for all practical purposes, any information that could have been transmitted through the optical system. The set of output eigenfields having singular values greater than \(\epsilon\) is \(\{\mathbf{u}_i: i \in 1, \ldots, I\}\), where \(I < \infty\), because the throughput of the optical system is finite. Now suppose that we have some other set of vectors \(|\mathbf{t}_p\rangle: 1, \ldots, P\}, and wish to determine whether \(|\mathbf{t}_p\rangle: 1, \ldots, P\} constitutes a frame with respect to \(\mathcal{S}\). That is to say, if we determine the complex coupling coefficients between the \(|\mathbf{t}_p\rangle\) and any vector in \(\mathcal{S}\), can we recover the vector in \(\mathcal{S}\) without ambiguity?

If \(|\mathbf{x}_2\rangle\) is some general vector in \(\mathcal{S}\), then the frame condition reads

\[
A \parallel \mathbf{x}_2 \parallel^2 \leq \sum_p |\langle \mathbf{t}_p | \mathbf{x}_2 \rangle|^2 \leq B \parallel \mathbf{x}_2 \parallel^2 \quad \forall |\mathbf{x}_2\rangle \in \mathcal{S}, \tag{32}
\]

or, assuming that \(|\mathbf{x}_2\rangle\) has been normalised

\[
A \leq \sum_p |\langle \mathbf{t}_p | \mathbf{x}_2 \rangle|^2 \leq B \quad \forall |\mathbf{x}_2\rangle \in \mathcal{S}. \tag{33}
\]

For a given set of vectors \(|\mathbf{t}_p(\mathbf{r})\rangle\), the inner products can be written explicitly, such that (33) takes the form

\[
A \leq \sum_p \left| \int_A \mathbf{t}_p^*(\mathbf{r}) \cdot \mathbf{x}_2(\mathbf{r}) \, d^2\mathbf{r} \right|^2 \leq B \quad \forall \mathbf{x}_2(\mathbf{r}) \in \mathcal{S}. \tag{34}
\]
We can, however, describe $x_2(r)$ completely in terms of the output eigenfields

$$x_2(r) = \sum_i a_i u_i(r). \tag{35}$$

Substituting (35) into (34) gives

$$A \leq \sum_p \left| \sum_i E_{pi} a_i \right|^2 \leq B \quad \forall a \in \mathbb{C}^I, \tag{36}$$

where

$$E_{pi} = \int_A t_p^*(r) \cdot u_i(r) d^2r. \tag{37}$$

Expanding (36), we get

$$A \leq \sum_{ii'} a_i^* a_i R_{ii'} \leq B \quad \forall a \in \mathbb{C}^I, \tag{38}$$

where

$$R_{ii'} = \sum_p E_{ii'}^p E_{pi}. \tag{39}$$

Or, because the number of basis functions $I$ is finite, $R$ can be written as $R = E^\dagger E$, where the elements of $E$ correspond to the overlap integrals between the output eigenfields and the synthesised reception patterns: (37). Although, the final relationship expresses a mapping of a finite dimensional space onto itself, the mapping passes through a space having infinite dimensions and therefore the integral in (37) should be evaluated analytically if at all possible. The frame condition (38) then becomes

$$A \leq a^\dagger R a \leq B = A \leq a^\dagger E^\dagger E a \leq B \quad \forall a \in \mathbb{C}^I. \tag{40}$$

In order to establish whether $\{t_p : 1, \cdots, P\}$ constitutes a frame with respect to the output eigenfields having non-zero singular values, we need to determine the limits $A$ and $B$ by rotating $a$ throughout $\mathbb{C}^I$. Another way of thinking about the same problem is that we have some general $a$, and we wish to determine whether it always be described in terms of the vector space spanned by the set of vectors $z_p$, corresponding to the set of all possible measurements, given the mapping $E$.

The operator $R = E^\dagger E$ is Hermitian, and can be diagonalised:

$$R = W \Lambda W^\dagger. \tag{41}$$

The frame condition then becomes

$$A \leq a^\dagger W \Lambda W^\dagger a \leq B \quad \forall a \in \mathbb{C}^I. \tag{42}$$

The middle term of (42) takes on its maximum value when the vector $a$ corresponds to the eigenvector of $W$ having the largest eigenvalue: remembering that $a$ must have unit length and therefore can only be rotated. If $W$ is degenerate in the largest eigenvalue, there is a range of vectors that lead to a maximum, but the outcome is still that $B = \lambda_{\text{max}}$. Likewise, (42) takes on its minimum value when the vector $a$ corresponds the the eigenvector having the smallest singular value, $A = \lambda_{\text{min}}$. If the smallest singular value is zero, $R$ is singular, and $\{t_p : 1, \cdots, P\}$ does not constitute a frame with respect to $S$.

The operator $R = E^\dagger E$ simply maps the eigenfield coefficients of the optical system onto the output ports of the array and then back again onto the eigenfield coefficients. If the set of basis vectors $T$ do not span all possible vectors in $S$, either because there are too few of them, or because they do not span the same space, information is lost when the frame coefficients are calculated, and the frame is incomplete. It is not possible, therefore, to recover complete information about the output field of the optical system from the outputs of the phased array. In this case, recovering $a$ with the dual vectors, and then reconstructing the field using the eigenfields, will give
the best least square approximation to the field. In reality, because of the presence of noise a Bayesian method would probably be used to reconstruct the field.

For infinite-dimensional frames, \( P \to \infty \), we can use the same procedure, but now we must calculate the eigenvalues of the matrix \( R \), where

\[
R_{i'i} = \int_A \int_A u_i^*(r) \cdot \overline{T}(r', r) \cdot u_i(r') \, d^2r \, d^2r',
\]

(43)

where

\[
\overline{T}(r', r) = \sum_p t_p(r) t_p^*(r'),
\]

(44)

and the sum over \( p \) extends to infinity. Again, these integrals should be evaluated analytically. Clearly, in the case where the frame is complete and orthonormal \( \sum_p t_p(r) t_p^*(r') = \mathbb{I} \delta(r-r') \), giving \( R = \mathbb{I} \), supporting the validity of the result.

We now have a measure of how effectively a phased array can image a complex field; it is easy to show that when a phased array forms a frame with respect to a fully coherent field at a surface, then it is also possible to recover completely the spatial correlations of a partially coherent field at the same surface: essentially because the natural modes of the partially coherent field lie within the same Hilbert subspace \( \mathcal{S} \).

According to (18), in the infinite-dimensional case,

\[
z_p = \int_A t_p^*(r) \cdot x(r) \, d^2r
\]

(45)

\[
x(r) = \sum_p z_p \tilde{t}_p(r),
\]

which describes the recovery of a coherent field. Incidentally, (45) also shows that \( \sum_p \tilde{t}_p(r) t_p^*(r) = \mathbb{I} \delta(r) \) for an over complete or perfectly complete frame. Forming the correlation matrix \( Z \) and the correlation dyadic \( \overline{X}(r', r) \), using (45), we get

\[
Z_{pp'} = \int_A \int_A t_{p'}^*(r) \cdot \overline{X}(r', r) \cdot t_{p'}(r') \, d^2r \, d^2r'
\]

(46)

\[
\overline{X}(r', r) = \sum_{pp'} Z_{pp'} \tilde{t}_p(r) \tilde{t}_{p'}^*(r'),
\]

which describes the recovery of the spatial correlations of a field from measurements of the cross correlations between the outputs of a phased array, using the dual beams. (46) confirms that the correlations of a field can also be recovered, if the reception patterns constitute a frame.

### 5.2. Imaging intensity distributions

The previous section describes a calculation that can be performed to find out whether the synthesised reception patterns of a phased array form a frame with respect to the output eigenfields of an optical system. This procedure must be used when one is interested in recovering phase information from the field. Often, however, in the case of simple imaging, one is only interested in being able to recover the intensity distribution of a fully incoherent source. In this case, certain of the beam patterns needed to form a frame may be created by scanning the array physically across the source. It seems, however, that different frames are needed depending on whether one is trying to preserve phase or whether one is just interested in measuring intensity: we should distinguish between ‘field frames’ and ‘intensity frames’.

To this end, assume that the source is fully incoherent and unpolarised, but that the intensity varies from position to position. The correlation dyadic of the source then becomes

\[
\overline{X}(r', r) = \overline{T}(r) \delta(r-r'),
\]

(47)
where \(I(r)\) is the intensity as a function of position. Substituting (47) into (10) gives

\[
Z_{pp'} = \int_A I(r)t_p^*(r) \cdot t_{p'}(r) \, d^2r. \tag{48}
\]

But say that we only measure the diagonal elements of \(Z\) through the use of total power detectors, then

\[
Z_{pp} = \int_A I(r)k_p(r) \, d^2r, \tag{49}
\]

where 
\( k_p(r) = t_p^*(r) \cdot t_p(r) \).

Thus, for an incoherent source, the output powers of the individual ports of a phased array are related to the intensity distribution of the source through a set of inner products with the functions \( \{k_p(r) : p \in 1, \cdots, P\} \).

If the goal is to reconstruct the intensity distribution of a source, one could ask whether the basis \( \{k_p(r) : p \in 1, \cdots, P\} \) forms a frame with respect to the Hilbert space defining the range of possible intensity distributions. There is a problem, however, because in assuming that the source field is spatially incoherent, we assumed that the intensity is a member of an infinite dimensional space. To answer the question as to whether the phased array is suitable for recovering intensity, we must define more clearly the vector space of intensity distributions that is of interest.

One possible approach is to describe the intensity distribution as a weighted linear combination of basis functions, \( \psi_n(r) \). These functions could, for example, be radial basis functions, wavelets, or delta functions at sample points. If chosen carefully, these functions need not correspond to a single region, but could represent a number of spatially separated regions that one wishes to image simultaneously. If we characterise the space of intensity distributions according to

\[
I(r) = \sum_n a_n \psi_n(r), \tag{50}
\]

then the powers recorded at the output of the phased array become

\[
Z_{pp} = \sum_n a_n F_{pn}, \tag{51}
\]

where

\[
F_{pn} = \int_A k_p(r)\psi_n(r) \, d^2r. \tag{52}
\]

The frame condition then reads

\[
A \leq a^T F^T Fa \leq B \quad \forall a \in \mathbb{C}^N, \tag{53}
\]

where \(A\) and \(B\), and hence the tightness of the frame, can be determined by finding the eigenvalues of \(F^T F\), or the SVD of \(F\). In the case where the basis functions correspond to sample points \(r_n\), we have \(\psi_n(r) = \delta(r - r_n)\) and \(F_{pn} = k_p(r_n)\). Clearly, the original intensity distribution can be found, to within the degrees of freedom \(N\), by using the dual vectors of \(k_p(r)\), namely \(\tilde{k}_p(r)\), defined in the space of \(\psi_n(r)\). Usually, however, for stochastic sources, and when noise is included, a Bayesian method would be used to recover images.

It is instructive to see how this form of analysis compares with, and is applicable to, multimode bolometric imaging arrays.\(^{12}\) It has been\(^{13}\) shown that the expectation value, \(E[P]\), of the output of essentially any multimode bolometric detector is given by

\[
E[P] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_A \int_A \overline{X}(r, r', \omega) \odot \overline{T}(r, r', \omega) \, d^2r \, d^2r' \, d\omega, \tag{54}
\]

where \(\overline{T}(r, r', \omega)\) is a tensor that characterises completely the physics of the detector, and can include any optical system and filters that precede the detector. \(\odot\) denotes the full tensor contraction to a single real variable, and
\( \omega \) indicates the frequency dependence of the tensors. If we now assume a completely unpolarised, incoherent source, as described by (47), then the output of the detector becomes

\[
E[P] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{\mathcal{A}} I(\mathbf{r}, \omega) k(\mathbf{r}, \omega) \, d^2 \mathbf{r} \, d\omega,
\]

(55)

where \( k(\mathbf{r}, \omega) \) is the sum of the diagonal elements of \( \overline{T}(\mathbf{r}, \mathbf{r}', \omega) \) evaluated at a single position. In other words, it gives the output of a multimode bolometric detector as a function of position. (55) has precisely the same form as (49), and therefore one can use, as before, the theory of frames to determine the degree to which an array of multimode bolometric detectors creates a frame with respect to a given class of intensity distributions. In fact, a multimode bolometric detector can be created from a phased array by measuring the power arriving at each output port, multiplying each of the measurements by some weighting factor, and then adding all of the results together. This comparison will become important when we come to consider interferometric phased arrays, and the fluctuations and correlations that appear at the outputs of phased arrays.

To finish this section, it is important to stress that the above analysis applies only when the source is fully spatially incoherent; it does not apply to recovering the intensity distribution of a field that is partially coherent; in that case, the results of the previous section should be used. Because the source must be fully incoherent, the analysis applies to primary sources, although the critical point is that the coherence length must be smaller than the interval over which the reception patterns change appreciably. Thus the analysis is appropriate for many practical situations, but is not applicable, for example, in the case of recovering the intensity in the focal plane of a low throughput optical system.

### 6. INTERFEROMETRIC PHASED ARRAYS

We now consider the behaviour of phased arrays in the context of interferometry. By ‘interferometric phased array’ we mean any interferometer where the individual elements are equipped with phased arrays for the purpose of creating a number of primary beams on the sky simultaneously. Central to the analysis is the observation that an interferometric phased array is essentially a bolometric interferometer,11 where the individual phased arrays are associated with a number of natural modes, which are equivalent to the natural modes of a multimode bolometer.

Suppose that a number of telescopes configured as an interferometer, and that each telescope is equipped with a phased array. We know that each port of the beam forming network is associated with a synthesised reception pattern on the sky, but equally, we recognise that, in general, these synthesised beams are not orthogonal. In this context, we have already described the mapping \( T : |x\rangle \mapsto |z\rangle \) as \( T : \mathbb{H} \rightarrow \ell^2 \). The phased array acts as a linear operator between two Hilbert spaces: one being the space of square integrable functions over the input reference surface, and one being the space of square summable complex sequences. For any real system, this operator must be Hilbert Schmidt as the amount of information that can be transmitted is finite. The integral operator can be written

\[
z_p = \int_{\mathcal{A}} \sum_i \sigma_i U_i(p) V_i^* (r) \cdot x(r) \, d^2 r,
\]

(56)

which is the equivalent of (31), allowing for the fact that the output is a discrete vector.

The operation of a phased array can therefore be regarded as first mapping the incoming field onto the input eigenfields, \( V_i(r) \), which are orthogonal, scaling by the singular values, \( \sigma_i \), and then reconstructing the complex travelling wave amplitudes at the output through the basis vectors \( U_i(p) \), which are also orthogonal. Those input eigenfields associated with non-zero singular values span the field distributions at the input to which the phased array is sensitive, and those output eigenfields associated with non-zero singular values span the vectors at the output to which the phased array can couple. Moreover, it can be shown the the input eigenfields associated with different telescopes are mutually orthogonal, and therefore the eigenfields of different telescopes can be combined to form a single large, orthonormal, composite basis set that can be used to propagate any partially coherent field through a complete interferometer. The input eigenfields actually describe those field distributions that can be traced to the output ports and then back again onto the sky unchanged in form; they are the eigenfunctions of complete round trips.
The analysis of an interferometric phased array proceeds as follows. Calculate the Hilbert-Schmidt decomposition of each telescope, and pick out those eigenfields having non-zero singular values above the threshold, $\epsilon$, of interest. Place phase slopes on the eigenfields in accordance with the baselines of the interferometer. This procedure has already been described in detail in the context of bolometric interferometers, and will not be repeated here.\textsuperscript{14, 15} The elements of the correlation matrix describing the correlations between the different output ports of the phased arrays on different telescopes then become

$$Z_{pp'} = \sum_i \sum_{i'} \sigma_i \sigma_{i'} U_i(p) U_{i'}^*(p') \int_\mathcal{A} \mathbf{V}_i^* (r) \cdot \mathbf{T}(r', r) \cdot \mathbf{V}_{i'} (r') \, d^2r \, d^2r', \quad (57)$$

where the sums over the eigenfields $i$ and $i'$ extend to all telescopes. In the case where the source is spatially incoherent and unpolarised, (57) becomes

$$Z_{pp'} = \sum_i \sum_{i'} \sigma_i \sigma_{i'} U_i(p) U_{i'}^*(p') \int_\mathcal{A} I(r) \mathbf{V}_i^* (r) \cdot \mathbf{V}_{i'} (r') \, d^2r. \quad (58)$$

$Z$ contains complete information about the correlations between the outputs of phased arrays on the same and different telescopes in terms of the intensity, $I(r)$, of the field on the sky, and can be used to produce simulated fringes.\textsuperscript{14, 15}

Given that the combined set of input eigenfields of all antennas spans completely the fields on the sky to which an interferometer is sensitive, it would also be straightforward to determine whether a given set of baselines and phased arrays comprise a frame with respect to some class of intensity distributions. In reality, the Fourier plane is rarely sampled completely, and in any case the calculation of the frame bounds would be computationally intensive.

7. NOISE

We have described a scheme for analysing the behaviour of phased arrays and interferometric phased arrays, and it would be desirable to include noise. Ideally, one should be able to model any internally generated noise by using the synthesised reception patterns alone. Also, we need to determine not just the noise power appearing at the outputs of an array, but the fluctuations and correlations in the fluctuations in the power arriving at the output ports. After all, it is the fluctuations that determine the sensitivity with which a measurement can be made.

If the receiver noise temperatures associated with the primary horns are known, and equal, then the procedure is straightforward. By definition, the noise temperature of a receiving channel is the temperature that a matched source would need to have in order to generate the same output power as a noiseless, but otherwise identical, system.

Using (20) in matrix form, the correlations between the outputs of a phased array are given by

$$Z = R [Z' + Z_N'] R^\dagger, \quad (59)$$

where $Z_N'$ is the correlation matrix of an equivalent set of noise sources at the input, one for each synthesised beam, which are incoherent with respect to the true source $Z'$. Strictly speaking, (59) assumes that all of the noise is in the same spatial modes as the signal, which of course need not be true. (59) can be extended easily to account for the more general case, but we shall not do so here.

To find $Z_N'$ we simply need to project a uniform background source having an intensity that is equal to the noise temperature onto the synthesised beams; using (21) we get

$$Z_{N,pp'} = \int_\mathcal{A} T_n \mathbf{t}_p^* (r) \cdot \mathbf{t}_{p'} (r) \, d^2r. \quad (60)$$

The diagonal terms of $Z_{N,pp'}$ give the noise temperatures that must be associated with each of the synthesised beams, and importantly, the off-diagonal terms give the correlations between them. If the beams are orthogonal, the noise sources are uncorrelated, and one returns to the original definition of noise temperature.
8. CORRELATIONS AND FLUCTUATIONS

The net outcome of the previous sections is the ability to calculate the correlations that appear at the output ports of a phased array, or the output ports of phased arrays on different telescopes, from knowledge of the synthesised, possibly non-orthogonal and linearly dependent, reception patterns. Once this information is known, many measurable quantities follow; including average powers, fluctuations in power, field correlations, and fluctuations in power correlations between different ports. Moreover, these matrices contain the Hanbury Brown-Twiss correlations associated with phased arrays. The expressions that follow have been derived previously, but will be reproduced here for completeness.

Rather than simply determining the output powers of individual ports, it is more general to consider detectors that measure the powers at a number of ports simultaneously according to some weighting vector. Characterise each weighted combination of detectors by diagonal matrix $W \in \mathbb{C}^{P \times P}$, where the diagonal elements are the factors that weight the sensitivities of the individual detectors that are connected to the phased array. Under these circumstances, and assuming radiation whose intrinsic reciprocal coherence time is much greater than the bandwidth of the system $\Delta \omega$, which is valid for radio astronomy systems, it can be shown that the expectation value of the power $E[P]$, recorded by a detector combination is given by

$$E[P] = \int \text{Tr} Z W \, d\omega.$$  \hspace{1cm} (61)

Likewise, assuming Gaussian statistics, the fluctuations in the output $C_{ss}$ and the correlations between the fluctuations of two outputs $C_{st}$ are given by

$$C_{ss} = \frac{1}{\tau} \int \text{Tr} Z W_s Z W_s^* \, d\omega$$

$$C_{st} = \frac{1}{\tau} \int \text{Tr} Z W_s Z W_t \, d\omega,$$

where all quantities are allowed to be a function of frequency. The only restriction on (62) is that the post-detection integration time $\tau >> 1/\Delta \omega$ where $Z(\omega) \to 0$ for frequencies outside of $\Delta \omega$, which is necessary for all astronomical instruments.

These expressions can be extended to describe the quantum mechanical behaviour of phased arrays, as has been done for bolometric interferometers. The bolometric interferometer model did not, however, include the Poisson limit for low photon occupancies. In this volume, we describe how one can add a single term to (62) to create a statistical mixture that includes the Poisson noise of photon counting. Once the additional term is included, it is possible to take into account the transition from fully Poisson to fully bunched behaviour, as the photon occupancies of the incoming modes increase, as one moves from infrared to submillimetre wavelengths. It is entirely possible, therefore, to modal the quantum-statistical behaviour of phased arrays at all wavelengths.

Expressions (61) and (62) offer a further possibility of considerable importance. Clearly, we have a numerical procedure for determining the expectation values and the covariances of the powers that arrive at the output ports of an imaging, or interferometric, phased array when a source is present. A discretised version of the model has already been published. The model takes into account noise, and can be extended easily to include quantum effects. Also, the beam patterns do not have to be orthogonal. (61) and (62) therefore make it straightforward to set up a likelihood function for the outputs that would be recorded when some class of source is observed. Obviously the likelihood function would contain the signal, its fluctuations, and any instrumental noise, including quantum effects, as well as the Hanbury Brown-Twiss correlations between pixels. The source may be as simple as a single incoherent Gaussian on the sky, or if one is trying to design a phased array that can observe two different regions of the sky simultaneously, it could be two highly separated Gaussians. Any other parameterised source distribution could be used; for example, Cauchy functions are often used in astronomy to parameterise Sunyaev-Zel’dovich emission from clusters of galaxies, and in (62), we used a Gauss-Schell source as a convenient way of parameterising general partially coherent fields.

On the basis of the likelihood functions, one could then derive numerically, the Fisher information matrix, from which the covariance matrix of the source parameter estimators could be found. We have already started
to apply this technique, in a completely different context, for understanding the design of bolometric imaging arrays: see Saklatvala, in this volume. In other words, one could determine the minimum errors, and the confidence contours, that could be achieved when determining the parameters of sources. Exploring how these errors change as the design of a phased array changes, say by packing more and more overlapping beams into a finite region, would be of considerable interest, and the result should be related, in some way, to the effectiveness with which the array forms a frame with respect to the incoming field distributions, or intensity distributions, of interest.

9. CONCLUSION

We have studied the functional behaviour of imaging phased arrays and interferometric phased arrays, and shown that their operation is closely related to the mathematical theory of frames. In order to calculate the behaviour of an imaging phased array, or an interferometric phased array, it is only necessary to know the synthesised reception patterns, which may be non-orthogonal and linearly dependent. It is not necessary to know anything about the internal construction of the array itself. As a consequence, data can be taken from experimental measurements or from electromagnetic simulations. The theory of frames allows one to assess, in a straightforward manner, whether the outputs of a phased array contain sufficient information to allow a field or intensity distribution to be reconstructed in an unambiguous way.

Our model also allows straightforward calculation of quantities such as the correlations in the fluctuations at the output ports of phased arrays. The theory of interferometric phased arrays is almost identical to the theory of multimode bolometric interferometers, and therefore, recently developed techniques for modelling bolometric interferometers can be applied to phased arrays also: including quantum statistics. The work opens up the important possibility of constructing likelihood functions that enable the covariance matrices of source-parameter estimators to be determined. Thus, for example, one could explore the possibility of enhancing source reconstruction by packing in more and more overlapping synthesised beams into a region, or widely separated regions, of finite size. Any enhancement of the accuracy with which source parameters can be recovered, will be related, and to some extent determined, by the degree to which the beam patterns of the array form a frame with respect to all possible incoming field distributions.

In a later paper, we shall use the concepts described here, and the numerical techniques reported previously, to simulate and assess the behaviour of interferometric phased arrays when different optical systems and beam-forming networks are used.

APPENDIX A. DERIVING AN EXPRESSION FOR THE TRAVELLING WAVES AT THE OUTPUTS OF A PHASED ARRAY

Suppose that some field, $|x\rangle$, is incident on a phased array, we can represent a measurement of the amplitude and phase of the travelling wave $z_p$ at $p$, relative to a normalised reference signal $z'_p$, by the inner product $\langle z'_p|z_p \rangle_{\ell^2} = z'_p^* z_p$, where $|z'_p\rangle$ is a vector corresponding to a measurement at port $p$ alone. For example, the measurement could be carried out by homodyne mixing the travelling wave $z_p$ with a reference oscillator $z'_p$ at $p$, and then low-frequency filtering the result. Introducing the linear operator $\hat{T}$, leads to a measurement of $\langle z'_p|T|x\rangle_{\ell^2}$, which by definition of the adjoint, can be written $\langle T^\dagger z'_p|x \rangle_{\mathbb{H}}$. In other words the inner product between $|x\rangle$ and the field distribution represented by $|T^\dagger z'_p\rangle$ gives the same result as the measurement, but now the inner product is evaluated at the input reference surface. We shall call $|T^\dagger z'_p\rangle$ the synthesised reception pattern of port $p$. The canonical inner product in $\mathbb{H}$ takes the form

$$\langle T^\dagger z'_p|x \rangle_{\mathbb{H}} = \int_{\mathcal{A}} z'_p^* t_p^*(\mathbf{r}) \cdot x(\mathbf{r}) \, d^2\mathbf{r},$$

(63)

where $t_p(\mathbf{r})$ is the functional form of the synthesised reception pattern, because the result must be equal to $z'_p^* z_p$, and therefore conjugate linear in $z'_p$. In (63), the integral over $\mathcal{A}$ corresponds to the input reference surface, and
extends over the region associated with Hilbert space $\mathbb{H}$. Finally, because (63) must be equal to $z_p^* z_p$, we have an expression that relates the complex amplitude of the travelling wave at $p$ to the incident field:

$$z_p = \int_A t_p^*(r) \cdot x(r) d^2r,$$

(64)

It is clear from (64) that the synthesised reception pattern is the complex conjugate of what would be measured in an experiment where a point source is swept over the input surface. The key point is that (64) is valid even when the beams are not orthogonal.

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