Topological quantum phase transitions in topological superconductors

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Abstract – In this paper we show that BF topological superconductors (insulators) exhibit phase transitions between different topologically ordered phases characterized by different ground-state degeneracy on manifold with non-trivial topology. These phase transitions are induced by the condensation of (or lack of) topological defects. We concentrate on the (2+1)-dimensional case where the BF model reduces to a mixed Chern-Simons term and we show that the superconducting phase has a ground-state degeneracy \(k\) and not \(k^2\). When the symmetry is \(U(1) \times U(1)\), namely when both gauge fields are compact, the mixed Chern-Simons model is not equivalent to the sum of two Chern-Simons terms with opposite chirality (even if naively diagonalizable) since the \(U(1)\) symmetry requires an ultraviolet regularization that makes the diagonalization impossible. We analyze this aspect using a lattice regularization, where the gauge fields become angular variables. In addition, we will show that the phase in which both gauge fields are compact is not allowed dynamically.

Quantum phase transitions describe changes in the entanglement pattern of the complex-valued quantum ground-state wave function. The universality classes of these macroscopic quantum ground states define the corresponding quantum orders [1]. When there is a gap in the spectrum, the quantum ordered ground state is called topologically ordered [2]; its remarkable hallmark is a ground-state degeneracy depending only on the topology of the underlying space.

The best-known example of topological order is given by Laughlin’s quantum incompressible fluids [3] describing the ground states responsible for the quantum Hall effect [4].

In [5] we proposed a superconductivity mechanism which is based on a topologically ordered ground state rather than on the usual Landau mechanism of spontaneous symmetry breaking. Topologically ordered superconductors have a long-distance hydrodynamic action which can be entirely formulated in terms of generalized compact gauge fields, the dominant term being the topological BF action.

In this paper we show that BF models exhibit phase transitions between different topologically ordered phases. Different topological order are characterized by different ground-state degeneracy. We will point out that these phase transitions are induced by the condensation of (or lack of) topological defects, which are present due to the compactness of gauge fields in the model.

We will address here the (2+1)-dimensional case where the BF model reduces to a mixed Chern-Simons term. We point out that, when the symmetry is \(U(1) \times U(1)\), namely when the two gauge fields are compact, this model is not equivalent to the sum of two Chern-Simons terms with opposite chirality (even if the model is naively diagonalizable) since the \(U(1)\) symmetry requires an ultraviolet regularization. The ultraviolet regularization is typically achieved if one obtains the \(U(1)\) group from spontaneous symmetry breaking of a non-Abelian group or formulates the gauge theory on a lattice; in the latter situation, the gauge fields are angular variables. In both regularization methods the ultraviolet cutoff actually prevents the naive diagonalization of the mixed

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Chern-Simons term, in one case because of the non-quadratic character of the action and in the other because of the angular character of both gauge fields. The diagonalization is, instead, permitted when only one of the gauge field is compact or when both are non-compact, i.e. for gauge groups \(U(1) \times R, R \times U(1)\) or \(R \times R\). As we have shown in [6] and confirmed in the continuum formulation [7] these are actually the only cases that can occur due to the condensation of topological defects, the \(U(1) \times U(1)\) case being dynamically excluded. The three corresponding phases are superconducting, superinsulator and metallic phases.

BF theories are topological theories that can be defined on manifolds \(M_{d+1}\) of any dimension (here \(d\) is the number of spatial dimensions) and play a crucial role in models of two-dimensional gravity [8]. In [6] we have shown that the BF term also plays a crucial role in the determination of the phase diagram accessible to Josephson junction arrays.

The BF term [9] is the wedge product of a \(p\)-form \(B\) and the curvature \(dA\) of a \((d-p)\) form \(A\):

\[
S_{BF} = \frac{k}{2\pi} \int_{M_{d+1}} B \wedge dA_{d-p},
\]

where \(k\) is a dimensionless coupling constant. This action has a generalized Abelian gauge symmetry under the transformation

\[
B \to B + \eta, \quad A \to A + \xi,
\]

where \(\eta\) and \(\xi\) are a closed \(p\) and a closed \((d-p)\) form, respectively.

The degeneracy of the ground state of the BF theory on a manifold with non-trivial topology was determined in [10]. Consider the model (1) with \(k = \frac{e^2}{2\pi}\) on a manifold \(M_d \times R_1\), with \(M_d\) compact, path-connected, orientable \(d\)-dimensional manifold without boundaries. The degeneracy of the ground state is expressed in terms of the intersection matrix \(M_{mn}\) [11] with \(m, n = 1, \ldots, N_p\) and \(N_p\) the rank of the matrix, between \(p\)-cycles and \((d-p)\)-cycles. \(N_p\) corresponds to the number of generators of the two homology groups \(H_p(M_d)\) and \(H_{d-p}(M_d)\) and is essentially the number of non-trivial cycles on the manifold \(M_d\). The degeneracy of the ground state is given by \(|k_1 k_2 M|^{N_p}\), where \(M\) is the integer-valued determinant of the linking matrix. In our case \(p = (d-1)\) and the degeneracy reduces to

\[
|k_1 k_2 M|^{N_{d-1}}.
\]

In this paper we consider the special case of \((2+1)\) dimensions \((d = 2)\). In this case also \(B\) becomes a 1-form and, correspondingly the BF term reduces to a mixed Chern-Simons term with action

\[
S_{CS} = \frac{k}{2\pi} \int_{M_{2+1}} A_1 \wedge dB_1.
\]

In this case the degeneracy on a manifold of genus \(g\) is \((k_1 \times k_2)^{2g}\) and thus \((k_1 \times k_2)^2\) on a torus.

In the application to superconductivity, the conserved current \(j_1 = \ast dB_1\) represents the charge fluctuations, while the current \(j_1 = \ast A_1\) describes the conserved fluctuations of vortices. As a consequence, the form \(B_1\) must be considered as a pseudo-vector, while \(A_1\) is a vector, as usual. The BF coupling is thus \(P\)- and \(T\)-invariant.

The low-energy effective theory of the superconductor can be entirely expressed in terms of the generalized gauge fields \(A_1\) and \(B_1\). The dominant term at long distances is the BF term; the next terms in the derivative expansion of the effective theory are the kinetic terms for the two gauge fields (for simplicity of presentation we shall assume relativistic invariance), giving

\[
S_{TM} = \int_{M_{2+1}} \left(-\frac{1}{2e^2} dA_1 \wedge \ast dA_1 + \frac{k}{2\pi} A_1 \wedge dB_1 + \frac{1}{2g^2} dB_1 \wedge \ast dB_1\right),
\]

where \(e^2\) and \(g^2\) are coupling constants of dimension \(m\). Note that in \((2+1)\) dimensions the action 4 is invariant under the duality transformation \(e \leftrightarrow g, A_1 \leftrightarrow B_1\). This action, including its non-Abelian generalization with kinetic terms was first considered in [12].

Naively one could diagonalize [4] by a transformation \(A = \frac{1}{2}(a + b), B = (a - b)\), giving

\[
S_{BF}(d = 2) = \frac{k}{4\pi} \int a \wedge da - \frac{k}{4\pi} \int b \wedge db.
\]

The result is a doubled Chern-Simons model for gauge fields of opposite chirality. It is the simplest example of the class of \(P\)- and \(T\)-invariant topological phases of strongly correlated \((2+1)\)-dimensional electron systems considered in [13].

The compactness of the gauge fields allows for the presence of topological defects, both electric and magnetic. The electric (magnetic) topological defects couple to the form \(A_1\) \((B_1)\) and are string-like objects described by a singular closed 1-form \(Q_1\) \((M_d)\). These forms represent the singular parts of the field strengths \(dA_1\) and \(dB_1\), allowed by the compactness of the gauge symmetries. The condensation of (or lack of) topological defects will lead to different topological phases characterized by a different ground-state degeneracy on manifold with non-trivial topology. To show this we will use an ultraviolet (lattice) regularization.

In [5] we have shown that the zero-temperature partition function of a \((2+1)\)-dimensional BF theory may be written on the lattice as

\[
Z = \sum_{\{Q_0\}} \int D\alpha \exp(-S),
\]

\[
S = \int dt \sum_{\nu} \left(-\frac{ik}{2\pi} \alpha_{\nu} K_{\mu\nu} B_{\nu} + ikA_0 Q_0 + ikB_0 M_0\right),
\]

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where $A_\mu (B_\mu)$ is a (pseudo) vector field, $K_{\mu
u}$ is the lattice Chern-Simons term [14], defined by $K_{00} = 0$, $K_{0i} = -\epsilon_i d_j$, $K_{ij} = -S_i \epsilon_j d_0$, in terms of forward (backward) shift and difference operators $S_i$ ($\tilde{S}_i$) and $d_i$ ($\tilde{d}_i$). Its conjugate $\tilde{K}_{\mu
u}$ is defined by $\tilde{K}_{00} = 0$, $\tilde{K}_{0i} = -\tilde{S}_i \epsilon_j \tilde{d}_0$, $\tilde{K}_{ij} = -\tilde{S}_i \epsilon_j \tilde{d}_0$. The two Chern-Simons kernels $K_{\mu
u}$ and $\tilde{K}_{\mu\nu}$ are interchanged upon integration (summation) by parts on the lattice. The topological excitations are described by the integer-valued fields $Q_0$ and $M_0$ and represent unit charges and vortices rendering the gauge field components $A_0$ and $B_0$ integers via the Poisson summation formula; their fluctuations determine the phase diagram [5]. The lattice spacing $l$ is assumed $l = 1$.

On the lattice the fields $A_\mu$ and $B_\mu$ are angular variables [15] defined on the interval $[-\pi, \pi]$. If we write $A = \frac{1}{2} (a + b)$ and $B = (a - b)$, it is now clear that the ultraviolet regularization, required by the presence of topological defects, make impossible to diagonalize naively the mixed Chern-Simons term.

In the phase in which electric and magnetic topological defects condense the partition function requires a formal sum also over the forms $Q_1$ and $M_1$:

$$Z = \int \mathcal{D}A \mathcal{D}B \mathcal{D}Q \mathcal{D}M \exp i \frac{k_1}{2\pi} \times \int_{M_{2+1}} \left( A_1 \wedge dA_1 + A_1 \wedge *Q_1 + B_1 \wedge *M_1 \right).$$

(7)

Topological excitations may be absorbed into the compact gauge field $A_\mu^c$, $B_\mu^c$. This situation is described by the doubled Chern-Simons with both gauge fields compact described by the action [3], with a ground-state degeneracy $(k_1 \times k_2)^2$ on the torus. If only one of the two topological defects condenses we have two dual phases [5]: when the magnetic excitations are dilute and the charge excitations condense rendering the system a superconductor: vortex confinement amounts here to the Meissner effect. If the magnetic excitations condense while the charged ones become dilute the system exhibits insulating behavior due to vortex superconductivity accompanied by a charge Meissner effect. Since the two phases are dual we will discuss only the charge condensation effect.

First of all let us notice that, associated with the confinement of vortices, there is a breakdown of the original $U(1)$ matter symmetry under the transformations $A_1 \to A_1 + d\lambda$. To see this let us consider the effect of such a transformation on the partition function (7) with an electric condensate. Upon integration by parts, the exponential of the action acquires a multiplicative factor

$$\exp i \frac{k_1}{2\pi k_2} \left( \int_{M_{2+1}} \lambda \wedge *Q_1 - \int_{M_{2+1}} \lambda \wedge *Q_1 \right).$$

(8)

Assuming a constant $\lambda$, we see that the only values for which the partition function remains invariant are

$$\lambda = 2\pi n \frac{k_2}{k_1}, \quad n = 1 \ldots k_1;$$

(9)

this shows that the global symmetry is broken from $U(1)$ to $Z_{k_1}$. In this phase all charges different from $nk_1e$ are screened and the Dirac quantization condition imposes that diluted magnetic vortices are quantized in units $\frac{2\pi}{k_1} \epsilon_i$. Note that this is not the usual Landau mechanism of spontaneous symmetry breaking. Indeed, there is no local order parameter and the order is characterized rather by the expectation value of a non-local, topological operator such as the ’t Hooft loop [5]. The important point is that, in this phase, the ground-state degeneracy on the torus is not $k^2$ but rather only $k$, as we will now show.

By rewriting the charge topological excitations as the curl of an integer-valued axial field $\varphi_\mu$: $Q_0 = \epsilon_i d_i \varphi_j$, the partition function becomes:

$$Z_{\text{LE}} = \sum_{\{\varphi_i\}} \int \mathcal{D}A_\mu \int \mathcal{D}B_\mu \exp (-S_{\text{LE}}),$$

$$S_{\text{LE}} = -i \frac{k_1}{2\pi} \int dt \sum_i A_0 K_{0i} (B_i + \varphi_i) + A_i K_{i0} B_0 + A_i K_{ij} B_j.$$  

(10)

From (10) one sees that the gauge field components $B_i$ are angular variables due to their invariance under time-independent integer shifts. Such shifts do not affect the last term in the action, which contains a time derivative, and may be reabsorbed in the topological excitations $\varphi_i$, leaving also the first term of the action invariant. Topological excitations may be absorbed only into the compact gauge $B^c_\mu$ and one may define

$$\sum_{\{\varphi_i\}} \int \mathcal{D}A_\mu \int \mathcal{D}B_\mu \exp (-S_{\text{LE}}(A_1, B_i, \varphi_i)) = \int \mathcal{D}A_\mu \int \mathcal{D}B^c_\mu \exp (-S_{\text{LE}}(A_1, B^c_i)).$$

(11)

The canonical quantization of the low energy effective action $S_{\text{LE}}(A_1, B^c_i)$ is carried out by imposing the Gauss law constraints associated to the two Lagrange multipliers $A_0$ and $B_0$, $K_{0i} B^c_i = 0$ and $K_{0i} A_i = 0$ and then enforcing the usual Weyl gauge condition $A_0 = 0$, $B^c_0 = 0$: the canonical commutation relations read $[A_i(x), B^c_j(x + i)] = - \frac{2\pi}{k_1} \epsilon_i j$, where $i$ denotes a unit lattice vector in direction $i$.

Since now only one of the two gauge fields is compact, we can introduce two pertinently normalized chiral gauge fields defined by $A^L_\mu \equiv (\frac{1}{2} A_1 + B^c_\mu)$ and $A^R_\mu \equiv (\frac{1}{2} A_1 - B^c_\mu)$. Following [16], the non-trivial windings around closed loops can be encoded in the two pairs of global variables:

$$q_L = \sum_i A^L_1 (l, 0), \quad p_L = - \frac{k_1}{2\pi} \sum_i A^L_2 (0, l),$$

$$q_R = \sum_i A^R_1 (l, 0), \quad p_R = - \frac{k_1}{2\pi} \sum_i A^R_2 (0, l),$$

(12)

where the sums run over a period $P_i$ ($l = 1, 2$) of the torus. The commutators between $A_i$ and $B_j$ imply that

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The generators of large gauge transformations can thus be written as

\[
U_1(n, m) = \exp \left[ 2\pi i (np_L + mp_R) \right],
\]

\[
U_2(t, l) = \exp \left[ -ik(tq_L + lp_R) \right],
\]

and they satisfy the algebra:

\[
U_1(n, m)U_2(t, l) = U_2(t, l)U_1(n, m) \times \exp \left[ i2\pi k(nt + ml) \right].
\]

We note that for \( k \) an integer the cocycle is always trivial. Performing the transformation in the integer \( n, m, l, t \):

\[
n = x + y; \quad m = x - y; \quad t = z + s; \quad l = z - s,
\]

one can write

\[
U_1(n, m) = U_1(x)U_1(y) = \exp \left[ 2\pi ix(p_L + p_R) \right] \exp \left[ 2\pi iy(p_L - p_R) \right],
\]

\[
U_2(t, l) = U_2(s)U_2(z) = \exp \left[ ikz(q_L + qr) \right] \exp \left[ ikz(qL - qr) \right];
\]

\[
U_1(x) \text{ and } U_1(y), \quad U_2(s) \text{ and } U_2(z) \text{ are two sets of commuting operators corresponding to the form } A^L \text{ and } \tilde{A}^R.
\]

Although one may define the winding numbers separately for the two chiral sectors, there is only one generator of large gauge transformations per homology cycle. This is because only the \( U(1)_{axial} \) subgroup is compact; thus, large gauge transformations in the left sector, \( A^L \), with \( \lambda(x_i + P_i) = \lambda(x_i) + 2\pi m_i \) and \( m_i \in \mathbb{Z} \) for \( i = 1, 2 \) must be combined with the corresponding large gauge transformations \( \tilde{A}^R \), with \( \chi(x_i + P_i) = \chi(x_i) - 2\pi m_i \) in the right sector. From [16] one sees that this implies \( x = z = 0 \) making one of the generators in [15] act trivially on the physical states. This already hints to the fact that the Hilbert space is only \( k_1 \times k_2 \) on the torus. In the following we shall provide an explicit derivation of this result.

The generators of combined large gauge transformations \( q_L \rightarrow q_L + 2\pi, \quad q_R \rightarrow q_R - 2\pi \) and \( p_L \rightarrow p_L - k, \quad p_R \rightarrow p_R + k \) are

\[
U_1(1) = U_1 = U^L_1 U^R_1 = \exp \left[ 2\pi i (p_L - p_R) \right],
\]

\[
U_2(1) = U_2 = U^L_2 U^R_2 = \exp \left[ ik (q_L - qr) \right],
\]

and satisfy the algebra \( U_1U_2 = U_2U_1 \exp[2\pi ik] \).

One is now ready to compute the ground-state degeneracy for a generic rational value of \( k, i.e. k = k_1/k_2 \). On the torus, the wave function splits into a part depending on the local variables \( x \) and a part depending on the global variables defined in (12); the degeneracy of the ground state is entirely determined by this second part [17]. In the Schrödinger representation, the global variables \( q \) are realized as generalized coordinates and the \( p \) are the corresponding momenta; thus, the wave function is a function of the \( q \)'s only and must carry a representation of the algebra of large gauge transformations (LGT). The representations of the algebra of LGTs are specified by two angles \( \theta_1 \) and \( \theta_2 \), which are entirely determined by the windings around the 2 homology cycles of the torus; correspondingly, the ground-state wave function must also be labeled by these two angles, \( \psi = \psi_{\theta_1, \theta_2} \).

Assuming \( U_1\psi_{\theta_1, \theta_2} = \exp[i\theta_1]\psi_{\theta_1, \theta_2} \) one has

\[
U_1U_2 \psi_{\theta_1, \theta_2} = U_2U_1 \exp \left[ 2\pi i k \right] \psi_{\theta_1, \theta_2} = \exp \left[ 2\pi i k \right] \exp \left[ i\theta_1 \right] U_2 \psi_{\theta_2},
\]

and for \( l = k_2 \) one obtains the same eigenvalue since \( \exp[2\pi i(k_1/k_2)k_2] = 1 \); this leads to a first set of \( k_2 \)-degenerate independent states \( U_1^l \psi_{\theta_1, \theta_2}, \quad l = 1, \ldots, k_2 \). For \( l = k_2 \), one has \( U_2^l \psi_{\theta_1, \theta_2} = \exp[i\theta_2]\psi_{\theta_1, \theta_2} \).

The most general form of the wave function yielding a representation of LGT is

\[
\psi_{\theta_1, \theta_2} = \sum_{n \in \mathbb{Z}} \exp \left[ i(n + \theta_1/2\pi)q + ikql \right] \psi_{\theta_1, \theta_2}(n),
\]

where \( q = (q_L - q_R)/2 \) is the normalized axial combination of winding numbers. Using (19) one has

\[
U_1^l \psi_{\theta_1, \theta_2} = \sum_{n \in \mathbb{Z}} \exp \left[ i(n + \theta_1/2\pi)q + ikql \right] \psi_{\theta_1, \theta_2}(n),
\]

which reduces, for \( l = k_2 \), to

\[
\sum_{n \in \mathbb{Z}} \exp \left[ ik(n + \theta_1/2\pi + k_1) \right] \psi_{\theta_1, \theta_2}(n) = \exp \left[ i\theta_2 \right] \sum_{n \in \mathbb{Z}} \exp \left[ i(n + \theta_1/2\pi)q \right] \psi_{\theta_1, \theta_2}(n).
\]

Combining this with \( U_2^k \psi_{\theta_1, \theta_2} = \exp[i\theta_2]\psi_{\theta_1, \theta_2} \) enables to derive the quasi-periodicity condition \( \psi_{\theta_1, \theta_2}(n) = \exp[i\theta_2]\psi_{\theta_1, \theta_2}(n + k_1) \). For each \( l \) there are \( k_1 \) independent states; thus, the dimension of the Hilbert space is \( k_1 \times k_2 \) for the torus (generically \( (k_1 \times k_2)^g \) on genus \( g \) Riemann surfaces).

For the superconducting and its dual insulator phase we have a degeneracy that is only \( k_1 \times k_2 \), while when both gauge fields are compact the degeneracy is \( (k_1 \times k_2)^2 \). From the phase structure analysis of BF topological fluids done in [6,7] we see however that the phase in which both topological defects condense is not present, but, instead, we have either electric or magnetic condensation or both are dilute. Thus the quantum order of the superconducting/insulator state BF topological fluids is in the universality class defined by the low-energy effective gauge theory

\[
S = \frac{k}{4\pi} \int d^3 x \ A^L \epsilon^{\mu\nu\alpha} \partial_\nu A^L_\alpha - \frac{k}{4\pi} \int d^3 x \ A^R \epsilon^{\mu\nu\alpha} \partial_\nu A^R_\alpha.
\]

The action (22) involves two separate Chern-Simons terms of opposite chirality. Contrary to previously considered examples of “doubled Chern-Simons theories” [13],
however, only the axial (vector) subgroup $U(1)_{axial\ (vector)}$ of the total gauge group $G = U(1)_L \otimes U(1)_R$ is compact; the diagonal vector group is the non-compact group $R(L)$, reflecting the fact that the charge (vortex) coupled to it is no good quantum number due to the existence of a superconducting condensate. Thus, the corresponding topological order is halved: as a consequence BF topological fluids belongs to a universality class which is different than the one of conventional doubled Chern-Simons theories.

We analyzed so far only the $T = 0$ superconducting phase BF topological fluids; by means of duality one may derive the properties of the corresponding insulating phase. What about the quantum transition point? To this end one should observe that a compact $U(1)$ theory always involves a scale, determining the radius of the gauge group: thus, at the quantum critical point both gauge groups must decompactify and long-range quantum fluctuations are described by a continuum gauge theory with gauge group $R_{vector} \otimes R_{axial}$. As a consequence, both charges and vortices must deconfine at the critical point, implying that, there, the BF topological fluids are — as expected [6] — in a metallic phase; this behavior supports the recent re-proposed scenario [18] that quantum critical points are generically described by continuum gauge theories with deconfined degrees of freedom.

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