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ON THE MAXIMALITY OF THE TRIANGULAR SUBGROUP

by Jean-Philippe FURTER & Pierre-Marie POLONI (*)

Abstract. — We prove that the subgroup of triangular automorphisms of the complex affine $n$-space is maximal among all solvable subgroups of $\text{Aut}(\mathbb{A}^n_C)$ for every $n$. In particular, it is a Borel subgroup of $\text{Aut}(\mathbb{A}^n_C)$, when the latter is viewed as an ind-group. In dimension two, we prove that the triangular subgroup is a maximal closed subgroup and that nevertheless, it is not maximal among all subgroups of $\text{Aut}(\mathbb{A}^2_C)$. Given an automorphism $f$ of $\mathbb{A}^2_C$, we study the question whether the group generated by $f$ and the triangular subgroup is equal to the whole group $\text{Aut}(\mathbb{A}^2_C)$.

Résumé. — Nous montrons que le sous-groupe des automorphismes triangulaires est un sous-groupe résoluble maximal de $\text{Aut}(\mathbb{A}^n_C)$ pour tout $n$. Il forme ainsi un sous-groupe de Borel du ind-groupe $\text{Aut}(\mathbb{A}^n_C)$. En dimension deux, nous montrons que le sous-groupe triangulaire est un sous-groupe fermé maximal mais qu’il n’est néanmoins pas maximal parmi tous les sous-groupes de $\text{Aut}(\mathbb{A}^2_C)$. Un automorphisme $f$ de $\mathbb{A}^2_C$ étant donné, nous étudions la question suivante : le sous-groupe engendré par $f$ et par les automorphismes triangulaires est-il égal au groupe $\text{Aut}(\mathbb{A}^2_C)$ tout entier ?

1. Introduction

The main purpose of this paper is to study the Jonquières subgroup $\mathcal{B}_n$ of the group $\text{Aut}(\mathbb{A}^n_C)$ of polynomial automorphisms of the complex affine $n$-space, i.e. its subgroup of triangular automorphisms. We will settle the titular question by providing three different answers, depending on to which properties the maximality condition is referring to.

Keywords: Polynomial automorphisms, triangular automorphisms, ind-groups.

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**Theorem 1.1.**

(1) For every $n \geq 2$, the subgroup $B_n$ is maximal among all solvable subgroups of $\text{Aut}(\mathbb{A}_n^n)$.  

(2) The subgroup $B_2$ is maximal among the closed subgroups of $\text{Aut}(\mathbb{A}^2_C)$.  

(3) The subgroup $B_2$ is not maximal among all subgroups of $\text{Aut}(\mathbb{A}^2_C)$.

Recall that $\text{Aut}(\mathbb{A}^n_C)$ is naturally an ind-group, i.e. an infinite dimensional algebraic group. It is thus equipped with the usual ind-topology (see Section 2 for the definitions). In particular, since $B_n$ is a closed connected solvable subgroup of $\text{Aut}(\mathbb{A}_n^n)$, the first statement of Theorem 1.1 can be interpreted as follows:

**Corollary 1.2.** — The group $B_n$ is a Borel subgroup of $\text{Aut}(\mathbb{A}_n^n)$.  

This generalizes a remark of Berest, Eshmatov and Eshmatov [4] stating that triangular automorphisms of $\mathbb{A}^2_C$, of Jacobian determinant 1, form a Borel subgroup (i.e. a maximal connected solvable subgroup) of the group $\text{SAut}(\mathbb{A}^2_C)$ of polynomial automorphisms of $\mathbb{A}^2_C$ of Jacobian determinant 1. Actually, the proofs in [4] also imply Corollary 1.2 in the case $n = 2$. Nevertheless, since they are based on results of Lamy [15], which use the Jung–van der Kulk–Nagata structure theorem for $\text{Aut}(\mathbb{A}^2_C)$, these arguments are specific to the dimension 2 and cannot be generalized to higher dimensions.

The Jonquières subgroup of $\text{Aut}(\mathbb{A}_n^n)$ is thus a good analogue of the subgroup of invertible upper triangular matrices, which is a Borel subgroup of the classical linear algebraic group $\text{GL}_n(\mathbb{C})$. Moreover, Berest, Eshmatov and Eshmatov strengthen this analogy when $n = 2$ by proving that $B_2$ is, up to conjugacy, the only Borel subgroup of $\text{Aut}(\mathbb{A}^2_C)$. On the other hand, it is well known that there exist, if $n \geq 3$, algebraic additive group actions on $\mathbb{A}^n_C$ that cannot be triangularized [1, 21]. Therefore, we ask the following problem.

**Problem 1.3.** — Show that Borel subgroups of $\text{Aut}(\mathbb{A}_n^n)$ are not all conjugate ($n \geq 3$).

This problem turns out to be closely related to the question of the boundedness of the derived length of solvable subgroups of $\text{Aut}(\mathbb{A}_n^n)$. We give such a bound when $n = 2$. More precisely, the maximal derived length of a solvable subgroup of $\text{Aut}(\mathbb{A}_2^2)$ is equal to 5 (see Proposition 3.14). As a consequence, we prove that the group $\text{Aut}_z(\mathbb{A}^2_C)$ of automorphisms of $\mathbb{A}^3$ fixing the last coordinate admits non-conjugate Borel subgroups (see...
Corollary 3.22). Note that such a phenomenon has already been pointed out in [4].

The paper is organized as follows. Section 1 is the present introduction. In Section 2, we recall the definitions of ind-varieties and ind-groups given by Shafarevich and explain how the automorphism group of the affine n-space may be endowed with the structure of an ind-group.

In Section 3, we prove the first two statements of Theorem 1.1 and discuss the question, whether the ind-group Aut(\(\mathbb{A}^n_k\)) does admit non-conjugate Borel subgroups. We then study the group of all automorphisms of \(\mathbb{A}^3_k\) fixing the last variable, proving that it admits non-conjugate Borel subgroups. In the last part of Section 3, we give examples of maximal closed subgroups of Aut(\(\mathbb{A}^n_k\)).

Finally, we consider Aut(\(\mathbb{A}^2_k\)) as an “abstract” group in Section 4. We show that triangular automorphisms do not form a maximal subgroup of Aut(\(\mathbb{A}^2_k\)). More precisely, after defining the affine length of an automorphism in Definition 4.1, we prove the following statement:

**Theorem 1.4.** — For any field \(k\), the two following assertions hold.

1. If the affine length of an automorphism \(f \in \text{Aut}(\mathbb{A}^2_k)\) is at least 1 (i.e. \(f\) is not triangular) and at most 4, then the group generated by \(B_2\) and \(f\) satisfies

\[
\langle B_2, f \rangle = \text{Aut}(\mathbb{A}^2_k).
\]

2. There exists an automorphism \(f \in \text{Aut}(\mathbb{A}^2_k)\) of affine length 5 such that the group \(\langle B_2, f \rangle\) is strictly included into \(\text{Aut}(\mathbb{A}^2_k)\).

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## 2. Preliminaries: the ind-group of polynomial automorphisms

In [24, 25], Shafarevich introduced the notions of ind-varieties and ind-groups, and explained how to endow the group of polynomial automorphisms of the affine n-space with the structure of an ind-group. Since these two papers are well-known to contain several inaccuracies, we now recall the definitions from Shafarevich and describe the ind-group structure of the automorphism group of the affine n-space.

For simplicity, we assume in this section that \(k\) is an algebraically closed field.
2.1. Ind-varieties and ind-groups

We first define the category of infinite dimensional algebraic varieties (ind-varieties for short).

**Definition 2.1** (Shafarevich [24]).

1. An ind-variety $V$ (over $k$) is a set together with an ascending filtration $V_{\leq 0} \subseteq V_{\leq 1} \subseteq V_{\leq 2} \subseteq \cdots \subseteq V$ such that the following holds:
   
   (a) $V = \bigcup_d V_{\leq d}$.
   
   (b) Each $V_{\leq d}$ has the structure of an algebraic variety (over $k$).
   
   (c) Each $V_{\leq d}$ is Zariski closed in $V_{\leq d+1}$.

2. A morphism of ind-varieties (or ind-morphism) is a map $\varphi: V \to W$ between two ind-varieties $V = \bigcup_d V_{\leq d}$ and $W = \bigcup_d W_{\leq d}$ such that there exists, for every $d$, an $e$ for which $\varphi(V_{\leq d}) \subseteq W_{\leq e}$ and such that the induced map $V_{\leq d} \to W_{\leq e}$ is a morphism of varieties (over $k$).

In particular, every ind-variety $V$ is naturally equipped with the so-called ind-topology in which a subset $S \subseteq V$ is closed if and only if every subset $S_{\leq d} := S \cap V_{\leq d}$ is Zariski-closed in $V_{\leq d}$.

We remark that the product $V \times W$ of two ind-varieties $V = \bigcup_d V_{\leq d}$ and $W = \bigcup_d W_{\leq d}$ has the structure of an ind-variety for the filtration $V \times W = \bigcup_d V_{\leq d} \times W_{\leq d}$.

**Definition 2.2.** — An ind-group is a group $G$ which is an ind-variety such that the multiplication $G \times G \to G$ and inversion $G \to G$ maps are morphisms of ind-varieties.

If $G$ is an abstract group, we denote by $D(G) = D^1(G)$ its (first) derived subgroup. It is the subgroup generated by all commutators $[g,h] := ghg^{-1}h^{-1}$, $g,h \in G$. The $n$-th derived subgroup of $G$ is then defined inductively by $D^n(G) = D^1(D^{n-1}(G))$ for $n \geq 1$, where by definition $D^0(G) = G$. A group $G$ is called solvable if $D^n(G) = \{1\}$ for some integer $n \geq 0$. Furthermore, the smallest such integer $n$ is called the derived length of $G$.

For later use, we state (and prove) the following results which are well-known for algebraic groups and which extend straightforwardly to ind-groups.

**Lemma 2.3.** — Let $H$ be a subgroup of an ind-group $G$. Then, the following assertions hold.

1. The closure $\overline{H}$ of $H$ is again a subgroup of $G$. 
(2) We have $D(H) \subseteq D(H)$.

(3) If $H$ is solvable, then $\overline{H}$ is solvable too.

Proof.

(1). The proof for algebraic groups given in [11, Proposition 7.4A, p. 54] directly applies to ind-groups. This proof being very short, we give it here. Inversion being a homeomorphism, we get $(H)^{-1} = H^{-1} = H$. Similarly, left translation by an element $x$ of $H$ being a homeomorphism, we get $xH = xH = H$, i.e. $H \subseteq HH$. In turn, right translation by an element $x$ of $H$ being a homeomorphism, we get $Hx = Hx \subseteq H$ and $H \subseteq Hx$. This says that $H$ is a subgroup.

(2). Fix an element $y$ of $H$. The map $\phi: G \rightarrow G$, $x \mapsto [x,y] = xyx^{-1}y^{-1}$ being an ind-morphism, it is in particular continuous. Since $H$ is obviously contained in $\phi^{-1}(D(H))$, we get $H \subseteq \phi^{-1}(D(H))$. Consequently, we have proven that

$$\forall x \in H, \forall y \in H, [x,y] \in D(H).$$

In turn (and analogously), for each fixed element $x$ of $\overline{H}$, the map $\psi: G \rightarrow G$, $y \mapsto [x,y]$ is continuous. Since $H$ is included into $\psi^{-1}(D(H))$, we get $\overline{H} \subseteq \psi^{-1}(D(H))$ and thus

$$\forall x, y \in \overline{H}, [x,y] \in D(H).$$

This implies the desired inclusion.

(3). If $H$ is solvable, it admits a sequence of subgroups such that

$$H = H_0 \supseteq H_1 \supseteq \cdots \supseteq H_n = \{1\} \text{ and } D(H_i) \subseteq H_{i+1} \text{ for each } i.$$ 

This yields $\overline{H} = \overline{H_0} \supseteq \overline{H_1} \supseteq \cdots \supseteq \overline{H_n} = \{1\}$ and by (2) we get $D(\overline{H_i}) \subseteq D(H_i) \subseteq \overline{H_{i+1}}$ for each $i$. □

2.2. Automorphisms of the affine $n$-space

As usual, given an endomorphism $f \in \text{End}(\mathbb{A}^n_k)$, we denote by $f^*$ the corresponding endomorphism of the algebra of regular functions $\mathcal{O}(\mathbb{A}^n_k) = k[x_1, \ldots, x_n]$. Note that every endomorphism $f \in \text{End}(\mathbb{A}^n_k)$ is uniquely determined by the polynomials $f_i = f^*(x_i), 1 \leq i \leq n$.

In the sequel, we identify the set $\mathcal{E}_n(k) := \text{End}(\mathbb{A}^n_k)$ with $(k[x_1, \ldots, x_n]^n)$. We thus simply denote by $f = (f_1, \ldots, f_n)$ the element of $\mathcal{E}_n(k)$ whose corresponding endomorphism $f^*$ is given by

$$f^*: \mathcal{O}(\mathbb{A}^n_k) \rightarrow \mathcal{O}(\mathbb{A}^n_k), \quad P(x_1, \ldots, x_n) \mapsto P \circ f = P(f_1, \ldots, f_n).$$
The composition \( g \circ f \) of two endomorphisms \( f = (f_1, \ldots, f_n) \) and \( g = (g_1, \ldots, g_n) \) is equal to
\[
g \circ f = (g_1(f_1, \ldots, f_n), \ldots, g_n(f_1, \ldots, f_n)).
\]

Note that for each nonnegative integer \( d \), the following set is naturally an affine space (and therefore an algebraic variety!)
\[
\mathbf{k}[x_1, \ldots, x_n]_{\leq d} := \{ P \in \mathbf{k}[x_1, \ldots, x_n], \ \text{deg} \ P \leq d \}.
\]
If \( f = (f_1, \ldots, f_n) \in \mathcal{E}_n(\mathbf{k}) \), we set \( \text{deg} f := \max_i \{ \text{deg} f_i \} \) and define
\[
\mathcal{E}_n(\mathbf{k})_{\leq d} := \{ f \in \mathcal{E}_n(\mathbf{k}), \ \text{deg} f \leq d \}.
\]
The equality \( \mathcal{E}_n(\mathbf{k})_{\leq d} = (\mathbf{k}[x_1, \ldots, x_n]_{\leq d})^n \) shows that \( \mathcal{E}_n(\mathbf{k})_{\leq d} \) is naturally an affine space. Moreover, the filtration \( \mathcal{E}_n(\mathbf{k}) = \bigcup_d \mathcal{E}_n(\mathbf{k})_{\leq d} \) defines a structure of ind-variety on \( \mathcal{E}_n(\mathbf{k}) \).

We denote by \( \mathcal{G}_n(\mathbf{k}) = \text{Aut}(\mathbb{A}^n_{\mathbf{k}}) \) the automorphism group of \( \mathbb{A}^n_{\mathbf{k}} \). The next result allows us to endow \( \mathcal{G}_n(\mathbf{k}) \) with the structure of an ind-variety.

**Lemma 2.4.** Denote by \( \mathcal{C}_n(\mathbf{k}) \), resp. \( \mathcal{J}_n(\mathbf{k}) \), the set of elements \( f \) in \( \mathcal{E}_n(\mathbf{k}) \) whose Jacobian determinant \( \text{Jac}(f) \) is a constant, resp. a nonzero constant. Then, the following assertions hold:

1. The set \( \mathcal{C}_n(\mathbf{k}) \) is closed in \( \mathcal{E}_n(\mathbf{k}) \).
2. The set \( \mathcal{J}_n(\mathbf{k}) \) is open in \( \mathcal{C}_n(\mathbf{k}) \).
3. The set \( \mathcal{G}_n(\mathbf{k}) \) is closed in \( \mathcal{J}_n(\mathbf{k}) \).

**Proof.**

1. Since \( \text{deg}(\text{Jac}(f)) \leq n(\text{deg}(f) - 1) \), the map \( \text{Jac}: \mathcal{E}_n(\mathbf{k}) \to \mathbf{k}[x_1, \ldots, x_n] \) is an ind-morphism. By definition, \( \mathcal{C}_n(\mathbf{k}) \) is the preimage of the set \( \mathbf{k} \) which is closed in \( \mathbf{k}[x_1, \ldots, x_n] \).

2. The Jacobian morphism induces a morphism \( \varphi: \mathcal{C}_n(\mathbf{k}) \to \mathbf{k}, \ f \mapsto \text{Jac}(f) \). By definition, \( \mathcal{J}_n(\mathbf{k}) \) is the preimage of the set \( \mathbf{k}^* \) which is open in \( \mathbf{k} \).

3. Set \( \mathcal{J}_{n,0} := \{ f \in \mathcal{J}_n(\mathbf{k}), \ f(0) = 0 \} \). Every element \( f \in \mathcal{J}_{n,0} \) admits a formal inverse for the composition (see e.g. [7, Theorem 1.1.2]), i.e. a formal power series \( g = \sum_{d \geq 1} g_d \), where each \( g_d = (g_{d,1}, \ldots, g_{d,n}) \) is a \( d \)-homogeneous element of \( \mathcal{E}_n(\mathbf{k}) \), meaning that \( g_{d,1}, \ldots, g_{d,n} \) are \( d \)-homogeneous polynomials in \( \mathbf{k}[x_1, \ldots, x_n] \) such that
\[
f \circ g = g \circ f = (x_1, \ldots, x_n) \quad \text{(as formal power series)}.
\]
Furthermore, for each \( d \), the map \( \psi_d: \mathcal{J}_{n,0} \to \mathcal{E}_n(\mathbf{k}) \) sending \( f \) onto \( g_d \) is a morphism because each coefficient of every component of \( g_d \) can be expressed as a polynomial in the coefficients of the components of \( f \) and in
the inverse \((\text{Jac } f)^{-1}\) of the polynomial \(\text{Jac } f\). Recall furthermore (see [2, Theorem 1.5]) that every automorphism \(f \in G_n(k)\) satisfies
\[
(2.1) \quad \deg(f^{-1}) \leq (\deg f)^{n-1}.
\]
Therefore, an element \(f \in J_n(k) \leq d\) is an automorphism if and if \(\tilde{f} := f - f(0)\) is an automorphism. This amounts to saying that \(f\) is an automorphism if and only if \(\psi_e(\tilde{f}) = 0\) for all integers \(e > d^{n-1}\). These conditions being closed, we have proven that \(G_n(k) \leq d\) is closed in \(J_n(k) \leq d\) for each \(d\), i.e. that \(G_n(k)\) is closed in \(J_n(k)\). Note that when the field \(k\) has characteristic zero, the Jacobian conjecture (see for example [2, 7]) asserts that the equality \(G_n(k) = J_n(k)\) actually holds.  
\[\square\]

Since the multiplication \(G_n(k) \times G_n(k) \rightarrow G_n(k)\) and inversion \(G_n(k) \rightarrow G_n(k)\) maps are morphisms (for the inversion, this again relies on the fundamental inequality \((2.1)\)), we obtain that \(G_n(k)\) is an ind-group.

3. Borel subgroups

Throughout this section, we work over the field \(k = \mathbb{C}\) of complex numbers.

Note that the affine subgroup
\[
\mathcal{A}_n = \{f = (f_1, \ldots, f_n) \in G_n(\mathbb{C}) \mid \deg(f_i) = 1 \text{ for all } i = 1 \ldots n\}
\]
and the Jonquières (or triangular) subgroup
\[
B_n = \{f = (f_1, \ldots, f_n) \in G_n(\mathbb{C}) \mid \forall i, f_i \in \mathbb{C}[x_i, \ldots, x_n]\}
= \{f \in G_n(\mathbb{C}) \mid \forall i, f_i = a_i x_i + p_i, a_i \in \mathbb{C}^*, p_i \in \mathbb{C}[x_{i+1}, \ldots, x_n]\}
\]
are both closed in \(G_n(\mathbb{C})\).

It is well known that the group \(G_n(\mathbb{C})\) is connected (see e.g. [25, proof of Lemma 4], [13, Proposition 2] or [22, Theorem 6]). The same is true for \(B_n\).

**Lemma 3.1.** — The groups \(G_n(\mathbb{C}) = \text{Aut}(\mathbb{A}^n_\mathbb{C})\) and \(B_n\) are connected.

**Proof.** — We say that a variety \(V\) is curve-connected if for all points \(x, y \in V\), there exists a morphism \(\varphi: C \rightarrow V\), where \(C\) is a connected curve (not necessarily irreducible) such that \(x\) and \(y\) both belong to the image of \(\varphi\). The same definition applies to ind-varieties.

We prove that \(G_n(\mathbb{C})\) and \(B_n\) are curve-connected. Let \(f\) be an element in \(G_n(\mathbb{C})\). We first consider the morphism \(\alpha: \mathbb{A}^n_\mathbb{C} \rightarrow G_n(\mathbb{C})\) defined by
\[
\alpha(t) = f - tf(0, \ldots, 0)
\]
which is contained in $B_n$ if $f$ is triangular. Note that $\alpha(0) = f$ and that
the automorphism $\tilde{f} := \alpha(1)$ fixes the origin of $\mathbb{A}^n_k$.

Therefore the morphism $\beta: A^1_{\mathbb{C}} \setminus \{0\} \to G_n(\mathbb{C})$, $t \mapsto (t^{-1} \cdot \text{id}_{A^1_{\mathbb{C}}}) \circ f \circ (t \cdot \text{id}_{A^1_{\mathbb{C}}})$
extends to a morphism $\beta: A^1_{\mathbb{C}} \to G_n(\mathbb{C})$ (with values in $B_n$ if $f$, thus $\tilde{f}$, is
triangular) such that $\beta(1) = \tilde{f}$ and such that $\beta(0)$ is a linear map, namely
the linear part of $\tilde{f}$. This concludes the proof since $\text{GL}_n(\mathbb{C})$ (resp. the set
of all invertible upper triangular matrices) is curve-connected.

Recall that the subgroup of upper triangular matrices in $\text{GL}_n(\mathbb{C})$ is solv-
able and has derived length $\lceil \log_2(n) \rceil + 1$, where $\lceil x \rceil$ denotes the smallest
integer greater than or equal to the real number $x$ (see e.g. [26, p. 16]). In
contrast, we have the following result.

**Lemma 3.2.** — The group $B_n$ is solvable of derived length $n + 1$.

**Proof.** — For each integer $k \in \{0, \ldots, n\}$, denote by $U_k$ the subgroup of
$B_n$ whose elements are of the form $f = (f_1, \ldots, f_n)$ where $f_i = x_i$ for all
$i > k$ and $f_i = x_i + p_i$ with $p_i \in \mathbb{C}[x_{i+1}, \ldots, x_n]$ for all $i \leq k$. We will prove
$D(B_n) = U_n$ and $D^1(U_n) = U_{n-j}$ for all $j \in \{0, \ldots, n\}$.

For this, we consider the dilatation $d(j, \lambda_j)$ and the elementary auto-
morphism $e(j, q_j)$ which are defined for every integer $j \in \{1, \ldots, n\}$, every
nonzero constant $\lambda_j \in \mathbb{C}^*$ and every polynomial $q_j \in \mathbb{C}[x_{j+1}, \ldots, x_n]$ by

$$d(j, \lambda_j) = (g_1, \ldots, g_n) \quad \text{and} \quad e(j, q_j) = (h_1, \ldots, h_n),$$

where $g_j = \lambda_j x_j$, $h_j = x_j + q_j$ and $g_i = h_i = x_i$ for $i \neq j$. Note that an
element $f \in U_k$ as above is equal to

$$f = e(k, p_k) \circ \cdots \circ e(2, p_2) \circ e(1, p_1).$$

In particular, this tells us that $U_k$ is generated by the elements $e(j, q_j)$,
$j \leq k$, $q_j \in \mathbb{C}[x_{j+1}, \ldots, x_n]$.

The inclusion $D(B_n) \subseteq U_n$ is straightforward and left to the reader. The
converse inclusion $U_n \subseteq D(B_n)$ follows from the equality

$$[e(j, q_j), d(j, \lambda_j)] = e(j, (1 - \lambda_j)q_j).$$

Finally, we prove $D^1(U_n) = U_{n-j}$ by proving that the equality $D(U_{k+1}) = U_k$
holds for all $k \in \{0, \ldots, n-1\}$. The inclusion $D(U_{k+1}) \subseteq U_k$ is straight-
forward and left to the reader. To prove the converse inclusion, let us intro-
duce the map $\Delta_i: \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n], q \mapsto q(x_1, \ldots, x_n) - q(x_1 - 1, x_{i+1}, \ldots, x_n)$. Note that $\Delta_i$ is surjective and that

$$[e(j, q_j), e(j + 1, 1)] = e(j, \Delta_{j+1}(q_j))$$

for all $j \in \{1, \ldots, n - 1\}$ and all $q_j \in \mathbb{C}[x_{j+1}, \ldots, x_n]$. This implies $U_k \subseteq D(U_{k+1})$ and concludes the proof. □
### 3.1. Triangular automorphisms form a Borel subgroup.

In this section, we prove the first two statements of Theorem 1.1 from the introduction. For this, we need the following result.

**Proposition 3.3.** — Let \( n \geq 2 \) be an integer. If a closed subgroup of \( \text{Aut}(\mathbb{A}^n_\mathbb{C}) \) strictly contains \( B_n \), then it also contains at least one linear automorphism that is not triangular.

**Proof.** — Let \( H \) be a closed subgroup of \( \text{Aut}(\mathbb{A}^n_\mathbb{C}) \) strictly containing \( B_n \). We first prove that \( H \) contains an automorphism whose linear part is not triangular. Let \( f = (f_1, \ldots, f_n) \) be an element in \( H \setminus B_n \). Then, there exists at least one component \( f_i \) of \( f \) that depends on an indeterminate \( x_j \) with \( j < i \), i.e. such that \( \frac{\partial f_i}{\partial x_j}(c) \neq 0 \). Now, choose \( c = (c_1, \ldots, c_n) \in \mathbb{A}^n_\mathbb{C} \) such that \( \frac{\partial f_i}{\partial x_j}(c) \neq 0 \) and consider the translation \( t_c := (x_1 + c_1, \ldots, x_n + c_n) \in B_n \).

Since \( f_i(x + c) = f_i(c) + \sum_k \frac{\partial f_i}{\partial x_k}(c)x_k + (\text{terms of higher order}) \), the linear part \( l \) of \( f \circ t_c \) is not triangular because it corresponds to the (non-triangular) invertible matrix \( \left( \frac{\partial f_i}{\partial x_k}(c) \right)_{ik} \). Composing on the left hand side by another translation \( t' \), we obtain an element \( g := t' \circ f \circ t \in H \) which fixes the origin of \( \mathbb{A}^n_\mathbb{C} \) and whose linear part is again \( l \).

For every \( \varepsilon \in \mathbb{C}^* \), set \( h_\varepsilon := (\varepsilon x_1, \ldots, \varepsilon x_n) \in B_n \). We can finally conclude by noting that \[ \lim_{\varepsilon \to 0} h_\varepsilon^{-1} \circ g \circ h_\varepsilon = l \in H, \] where the limit means that the ind-morphism \( \varphi : \mathbb{C}^* \to \text{Aut}(\mathbb{A}^n_\mathbb{C}), \varepsilon \mapsto h_\varepsilon^{-1} \circ g \circ h_\varepsilon \) extends to a morphism \( \psi : \mathbb{C} \to \text{Aut}(\mathbb{A}^n_\mathbb{C}) \) such that \( \psi(0) = l \). Since we have \( \psi(\varepsilon) \in H \) for each \( \varepsilon \in \mathbb{C}^* \), it is clear that \( \psi(0) \) must also belong to \( H \). Indeed, note that the set \( \{ \varepsilon \in \mathbb{C}, \psi(\varepsilon) \in H \} \) is Zariski-closed in \( \mathbb{C} \). \( \square \)

**Proposition 3.4.** — Let \( n \geq 2 \) be an integer. Then, the Jonquières group \( B_n \) is maximal among all solvable subgroups of \( \text{Aut}(\mathbb{A}^n_\mathbb{C}) \).

**Proof.** — Suppose by contradiction that there exists a solvable subgroup \( H \) of \( \text{Aut}(\mathbb{A}^n_\mathbb{C}) \) that strictly contains \( B_n \). Up to replacing \( H \) by its closure \( \overline{H} \) (see Lemma 2.3), we may assume that \( H \) is closed. By Proposition 3.3, the group \( H \cap \mathbb{A}_n \) strictly contains \( B_n \cap \mathbb{A}_n \). But since \( B_n \cap \mathbb{A}_n \) is a Borel subgroup of \( \mathbb{A}_n \), this prove that \( H \cap \mathbb{A}_n \) is not solvable, thus that \( H \) itself is not solvable. Notice that we have used the fact that every Borel subgroup of \( \mathbb{A}_n \) is not solvable.
a connected linear algebraic group is a maximal solvable subgroup. Indeed, every parabolic subgroup (i.e. a subgroup containing a Borel subgroup) of a connected linear algebraic group is necessarily closed and connected. See e.g. [11, Corollary B of Theorem (23.1), p. 143]. □

In dimension two, we establish another maximality property of the triangular subgroup which is actually stronger than the above one (see Remark 3.7 below).

**Proposition 3.5.** — The Jonquières group $B_2$ is maximal among the closed subgroups of $\text{Aut}(\mathbb{A}^2_C)$.

**Proof.** — Let $H$ be a closed subgroup of $\text{Aut}(\mathbb{A}^2_C)$ strictly containing $B_2$. By Proposition 3.3 above, $H$ contains a linear automorphism which is not triangular. This implies that $H$ contains all linear automorphisms, hence $A_2$, and it is therefore equal to $\text{Aut}(\mathbb{A}^2_C)$. Recall indeed that the subgroup $B_2 = B_2 \cap \text{GL}_2(\mathbb{C})$ of invertible upper triangular matrices is a maximal subgroup of $\text{GL}_2(\mathbb{C})$, since the Bruhat decomposition expresses $\text{GL}_2(\mathbb{C})$ as the disjoint union of two double cosets of $B_2$, which are namely $B_2$ and $B_2 \circ f \circ B_2$, where $f$ is any element of $\text{GL}_2(\mathbb{C}) \setminus B_2$. □

**Remark 3.6.** — Proposition 3.5 can not be generalized to higher dimension, since, if $n \geq 3$, then $B_n$ is strictly contained into the (closed) subgroup of automorphisms of the form $f = (f_1, \ldots, f_n)$ such that $f_n = a_n x_n + b_n$ for some $a_n, b_n \in \mathbb{C}$ with $a_n \neq 0$.

**Remark 3.7.** — Proposition 3.5 implies Proposition 3.4 for $n = 2$. Indeed, suppose that $B_2$ is strictly included into some solvable subgroup $H$ of $\text{Aut}(\mathbb{A}^2_C)$. Up to replacing $H$ by $\overline{H}$ (see Lemma 2.3), we may further assume that $H$ is closed. By Proposition 3.5, we would thus get that $H = \text{Aut}(\mathbb{A}^2_C)$. But this is a contradiction because the group $\text{Aut}(\mathbb{A}^2_C)$ is obviously not solvable, since it contains the linear group $\text{GL}(2, \mathbb{C})$ which is not solvable.

By Proposition 3.4, we can say that the triangular group $B_n$ is a Borel subgroup of $\text{Aut}(\mathbb{A}^n_C)$. This was already observed, in the case $n = 2$ only, by Berest, Eshmatov and Eshmatov in the nice paper [4] in which they obtained the following strong results. (In [4], these results are stated for the group $\text{SAut}(\mathbb{A}^2_C)$ of polynomial automorphisms of $\mathbb{A}^2_C$ of Jacobian determinant 1, but all the proofs remain valid for $\text{Aut}(\mathbb{A}^2_C)$.)

**Theorem 3.8 ([4]).**

(1) All Borel subgroups of $\text{Aut}(\mathbb{A}^2_C)$ are conjugate to $B_2$.

(2) Every connected solvable subgroup of $\text{Aut}(\mathbb{A}^2_C)$ is conjugate to a subgroup of $B_2$. 

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Recall that there exist, for every \( n \geq 3 \), connected solvable subgroups of \( \text{Aut}(\mathbb{A}^n \mathbb{C}) \) that are not conjugate to subgroups of \( \mathcal{B}_n \) [1, 21]. Hence, the second statement of the above theorem does not hold for \( \text{Aut}(\mathbb{A}^n \mathbb{C}) \), \( n \geq 3 \). Similarly, we believe that not all Borel subgroups of \( \text{Aut}(\mathbb{A}^n \mathbb{C}) \) are conjugate to \( \mathcal{B}_n \) if \( n \geq 3 \). This would be clearly the case, if we knew that the following question has a positive answer.

**Question 3.9.** — Is every connected solvable subgroup of \( \text{Aut}(\mathbb{A}^n \mathbb{C}) \), \( n \geq 3 \), contained into a maximal connected solvable subgroup?

The natural strategy to attack the above question would be to apply Zorn’s lemma, as we do in the proof of the following general proposition.

**Proposition 3.10.** — Let \( G \) be a group endowed with a topology. Suppose that there exists an integer \( c > 0 \) such that every solvable subgroup of \( G \) is of derived length at most \( c \). Then, every solvable (resp. connected solvable) subgroup of \( G \) is contained into a maximal solvable (resp. maximal connected solvable) subgroup.

**Proof.** — Let \( H \) be a solvable (resp. connected solvable) subgroup of \( G \). Denote by \( \mathcal{F} \) the set of solvable (resp. connected solvable) subgroups of \( G \) that contain \( H \). Our hypothesis, on the existence of the bound \( c \), implies that the poset \( (\mathcal{F}, \subseteq) \) is inductive. Indeed, if \( (H_i)_{i \in I} \) is a chain in \( \mathcal{F} \), i.e. a totally ordered family of \( \mathcal{F} \), then the group \( \bigcup_i H_i \) is solvable, because we have that
\[
D^j \left( \bigcup_i H_i \right) = \bigcup_i D^j (H_i)
\]
for each integer \( j \geq 0 \). Moreover, if all \( H_i \) are connected, then so is their union. Thus, \( \mathcal{F} \) is inductive and we can conclude by Zorn’s lemma. \( \square \)

**Remark 3.11.** — Proposition 3.10 does not require any compatibility conditions between the group structure and the topology on \( G \). Let us moreover recall that an algebraic group (and all the more an ind-group) is in general not a topological group.

We are now left with another concrete question.

**Definition 3.12.** — Let \( G \) be a group. We set
\[
\psi(G) := \sup \{ l(H) \mid H \text{ is a solvable subgroup of } G \} \in \mathbb{N} \cup \{ +\infty \},
\]
where \( l(H) \) denotes the derived length of \( H \).

**Question 3.13.** — Is \( \psi(\text{Aut}(\mathbb{A}^n \mathbb{C})) \) finite?
Recall that $\psi(\text{GL}(n, \mathbb{C}))$ is finite. This classical result has been first established in 1937 by Zassenhaus [27, Satz 7] (see also [16]). More recently, Martelo and Ribón have proved in [17] that $\psi((\mathcal{O}_{\text{ana}}(\mathbb{C}^n), 0)) < +\infty$, where $(\mathcal{O}_{\text{ana}}(\mathbb{C}^n), 0)$ denotes the group of germs of analytic diffeomorphisms defined in a neighbourhood of the origin of $\mathbb{C}^n$.

Our next result answers Question 3.13 in the case $n = 2$.

**Proposition 3.14.** — We have $\psi(\text{Aut}(\mathbb{A}^2_2)) = 5$.

**Proof.** — The proof relies on a precise description of all subgroups of $\text{Aut}(\mathbb{A}^2_2)$, due to Lamy, that we will recall below. Using this description, the equality $\psi(\text{Aut}(\mathbb{A}^2_2)) = 5$ directly follows from the equality $\psi(A_2) = 5$ that we will establish in the next section (see Proposition 3.16). The description of all subgroups of $\text{Aut}(\mathbb{A}^2_2)$ given by Lamy uses the amalgamated structure of this group, generally known as the theorem of Jung, van der Kulk and Nagata: The group $\text{Aut}(\mathbb{A}^2_2)$ is the amalgamated product of its subgroups $A_2$ and $B_2$ over their intersection

$$\text{Aut}(\mathbb{A}^2_2) = A_2 \ast_{A_2 \cap B_2} B_2.$$ 

In the discussion below, we will use the Bass–Serre tree associated to this amalgamated structure. We refer the reader to [23] for details on Bass–Serre trees in full generality and to [15] for details on the particular tree associated to the above amalgamated structure. That latter tree is the tree whose vertices are the left cosets $g \circ A_2$ and $h \circ B_2$, $g, h \in \text{Aut}(\mathbb{A}^2_2)$. Two vertices $g \circ A_2$ and $h \circ B_2$ are related by an edge if and only if there exists an element $k \in \text{Aut}(\mathbb{A}^2_2)$ such that $g \circ A_2 = k \circ A_2$ and $h \circ B_2 = k \circ B_2$, i.e. if and only if $g^{-1} \circ h \in A_2 \circ B_2$. The group $\text{Aut}(\mathbb{A}^2_2)$ acts on the Bass–Serre tree by left translation: For all $g, h \in \text{Aut}(\mathbb{A}^2_2)$, we set $g \cdot \left(h \circ A_2\right) = (g \circ h) \circ A_2$ and $g \cdot \left(h \circ B_2\right) = (g \circ h) \circ B_2$. Each element of $\text{Aut}(\mathbb{A}^2_2)$ satisfies one property of the following alternative:

1. It is triangularizable, i.e. conjugate to an element of $B_2$. This is the case where the automorphism fixes at least one point on the Bass–Serre tree.

2. It is a Hénon automorphism, i.e. it is conjugate to an element of the form

$$g = a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k,$$

where $k \geq 1$, each $a_i$ belongs to $A_2 \setminus B_2$ and each $b_i$ belongs to $B_2 \setminus A_2$. This is the case where the automorphism acts without fixed points, but preserves a (unique) geodesic of the Bass–Serre tree on which it acts as a translation of length $2k$. 


Furthermore, according to [15, Theorem 2.4], every subgroup \( H \) of \( \text{Aut}(A_2^2) \) satisfies one and only one of the following assertions:

1. It is conjugate to a subgroup of \( A_2 \) or of \( B_2 \).
2. Every element of \( H \) is triangularizable and \( H \) is not conjugate to a subgroup of \( A_2 \) or of \( B_2 \). In that case, \( H \) is Abelian.
3. The group \( H \) contains some Hénon automorphisms (i.e. non triangularizable automorphisms) and all those have the same geodesic on the Bass–Serre tree. The group \( H \) is then solvable.
4. The group \( H \) contains two Hénon automorphisms having different geodesics. Then, \( H \) contains a free group with two generators.

Let \( H \) be now a solvable subgroup of \( \text{Aut}(A_2^2) \). If we are in case (1), then we may assume that \( H \) is a subgroup of \( A_2 \) or of \( B_2 \). Since \( \psi(A_2) = 5 \) and \( \psi(B_2) = 3 \) (the group \( B_2 \) being solvable of derived length 3), this settles this case. In case (2), \( H \) is Abelian hence of derived length at most 1. In case (3), there exists a geodesic \( \Gamma \) which is globally fixed by every element of \( H \). Therefore, we may assume without restriction that

\[
H = \{ f \in \text{Aut}(A_2^2), \; f(\Gamma) = \Gamma \}.
\]

Note that \( D^2(H) \) is included into the group \( K \) that fixes pointwise the geodesic \( \Gamma \). Up to conjugation, we may assume that \( \Gamma \) contains the vertex \( B_2 \), i.e. that \( K \) is included into \( B_2 \). By [15, Proposition 3.3], each element of \( \text{Aut}(A_2^2) \) fixing an unbounded set of the Bass–Serre tree has finite order. If \( f, g \in K \), their commutator is of the form \( (x + p(y), y + c) \). This latter automorphism being of finite order, it must be equal to the identity, showing that \( K \) is Abelian. Therefore, we get \( D^3(H) = \{1\} \).

Finally, we cannot be in case (4), because a free group with two generators is not solvable.

From Propositions 3.10 and 3.14, we get at once the following result, which also follows from Theorem 3.8 above.

**Corollary 3.15.** — *Every solvable connected subgroup of \( \text{Aut}(A_2^2) \) is contained into a Borel subgroup.*

### 3.2. Proof of the equality \( \psi(A_2) = 5 \).

Recall that Newman [20] has computed the exact value \( \psi(\text{GL}(n, \mathbb{C})) \) for all \( n \). It turns out that \( \psi(\text{GL}(n, \mathbb{C})) \) is equivalent to \( 5 \log_9(n) \) as \( n \) goes to infinity (see [26, Theorem 3.10]). Let us give a few particular values for
\[ \psi(\text{GL}(n, \mathbb{C})) \text{ taken from [20].} \]

| \(n\) |
|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 18 | 26 | 34 | 66 | 74 |
| 1 | 4 | 5 | 6 | 7 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |

We now consider the affine group \(A_n\). On the one hand, observe that \(A_n\) is isomorphic to a subgroup of \(\text{GL}(n+1, \mathbb{C})\). Hence, \(\psi(A_n) \leq \psi(\text{GL}(n+1, \mathbb{C}))\). On the other hand, we have the short exact sequence

\[ 1 \to C^n \to A_n \xrightarrow{L} \text{GL}_n(\mathbb{C}) \to 1, \]

where \(L: A_n \to \text{GL}(n, \mathbb{C})\) is the natural morphism sending an affine transformation to its linear part. Thus, if \(H\) is a solvable subgroup of \(A_n\), we have a short exact sequence

\[ 1 \to H \cap (C^n) \to H \xrightarrow{L} L(H) \to 1. \]

Since \(L(H)\) is solvable of derived length at most \(\psi(\text{GL}_n(\mathbb{C}))\) and since \(H \cap (C^n)\) is Abelian, this implies that \(l(H) \leq \psi(\text{GL}_n(\mathbb{C})) + 1\). Therefore, we have proved the general formula

\[ \psi(\text{GL}_n(\mathbb{C})) \leq \psi(A_n) \leq \min\{\psi(\text{GL}(n, \mathbb{C})) + 1, \psi(\text{GL}(n + 1, \mathbb{C}))\}. \]

For \(n = 2\), this yields \(\psi(A_2) = 4\) or 5. We shall now prove that \(A_2\) contains solvable subgroups of derived length 5 (see Lemma 3.19 below), hence the following desired result.

**Proposition 3.16.** — The maximal derived length of a solvable subgroup of the affine group \(A_2\) is 5, i.e. we have \(\psi(A_2) = 5\).

As explained above, it still remains to provide an example of a solvable subgroup of \(A_2\) of derived length 5. In that purpose, recall that the group \(\text{PSL}(2, \mathbb{C})\) contains a subgroup isomorphic to the symmetric group \(S_4\) and that all such subgroups are conjugate (see for example [3]).

**Definition 3.17.** — The binary octahedral group \(2O\) is the pre-image of the symmetric group \(S_4\) by the \((2 : 1)\)-cover \(\text{SL}(2, \mathbb{C}) \to \text{PSL}(2, \mathbb{C})\).

The following result is also well-known.

**Lemma 3.18.** — The derived length of the binary octahedral group \(G = 2O\) is 4.

**Proof.** Using the short exact sequence

\[ 0 \to \{\pm I\} \to G \xrightarrow{\pi} S_4 \to 0, \]

we get \(\pi(D^2G) = D^2(\pi(G)) = D^2(S_4) = V_4\), where \(V_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2\) is the Klein group. One could also easily check that \(\pi^{-1}(V_4)\) is isomorphic to
the quaternion group \( Q_8 \). The equality \( \pi(D^2G) = V_4 \) is then sufficient for showing that \( D^2G = \pi^{-1}(V_4) \). Indeed, if \( D^2G \) was a strict subgroup of \( \pi^{-1}(V_4) \simeq Q_8 \), it would be cyclic, hence \( \pi(D^2G) = V_4 \) would be cyclic too. A contradiction. Since \( D^2G \simeq Q_8 \) has derived length 2, this shows us that the derived length of \( G \) is \( 2 + 2 = 4 \). □

**Lemma 3.19.** — Consider the pre-image \( L^{-1}(G) \simeq G \times \mathbb{C}^2 \) of the binary octahedral group \( G := 2O \subseteq \text{SL}(2, \mathbb{C}) \) by the natural morphism \( L: \mathcal{A}_2 \to \text{GL}(2, \mathbb{C}) \) sending an affine transformation onto its linear part. Then, the derived length of \( L^{-1}(G) \) is equal to 5.

**Proof.** — By Lemma 3.18, the derived length of \( G \) is 4. The short exact sequence

\[ 1 \to \mathbb{C}^2 \to G \times \mathbb{C}^2 \to G \to 1 \]

implies that the derived length of \( G \times \mathbb{C}^2 \) is at most \( 4 + 1 = 5 \). Moreover, the strictly decreasing sequence \( G = D^0(G) > D^1(G) > D^2(G) > D^3(G) > D^4(G) = 1 \) shows that the group \( D^2(G) \) is non-Abelian and in particular non-cyclic. By Lemma 3.20 below, we thus have \( D^i(G \times \mathbb{C}^2) = D^i(G) \times \mathbb{C}^2 \) for every \( i \leq 3 \). But since \( D^3(G) \) is non-trivial, the group \( D^3(G \times \mathbb{C}^2) = D^3(G) \times \mathbb{C}^2 \) strictly contains the subgroup \( (\mathbb{C}^2, +) \) of translations and cannot be Abelian, because the group \( \mathbb{C}^2 \) is its own centralizer in \( \mathcal{A}_2 \). Finally, we get \( D^4(G \times \mathbb{C}^2) \neq 1 \), proving that the derived length of \( G \times \mathbb{C}^2 \) is indeed 5. □

**Lemma 3.20.** — Let \( H \) be a finite non-cyclic subgroup of \( \text{GL}(2, \mathbb{C}) \). Then the derived subgroup of \( L^{-1}(H) = H \times \mathbb{C}^2 \subseteq \mathcal{A}_2 \) is the group \( D(H) \times \mathbb{C}^2 \).

**Proof.** — Set \( K := D(H \times \mathbb{C}^2) \cap \mathbb{C}^2 \). Note that \( K \) contains the commutator \([\text{id} + v, h] \) for all \( v \in \mathbb{C}^2 \), \( h \in H \), i.e. it contains all elements \( h \cdot v - v \). It is enough to show that these vectors generate \( \mathbb{C}^2 \). Indeed, it would then imply that there exist \( h_1, v_1, h_2, v_2 \) such that the vectors \( h_1 \cdot v_1 - v_1 \) and \( h_2 \cdot v_2 - v_2 \) are linearly independent. But then, \( K \) would also contain the vectors \( h_1 \cdot (\lambda_1 v_1) - (\lambda_1 v_1) + h_2 \cdot (\lambda_2 v_2) - \lambda_2 v_2 \) for any \( \lambda_1, \lambda_2 \in \mathbb{C} \), proving that \( K = \mathbb{C}^2 \). Therefore, let us assume by contradiction that there exists a non-zero vector \( w \in \mathbb{C}^2 \) such that \( h \cdot v - v \) is a multiple of \( w \) for all \( h \in H \), \( v \in \mathbb{C}^2 \). Take \( w' \in \mathbb{C}^2 \) such that \((w, w')\) is a basis of \( \mathbb{C}^2 \). In this basis, any element of \( H \) admits a matrix of the form

\[
\begin{pmatrix}
a & b \\
0 & 1
\end{pmatrix}.
\]
Therefore, by the theory of representations of finite group, we may assume, up to conjugation, that each element of \( H \) admits a matrix of the form
\[
\begin{pmatrix}
a & 0 \\
0 & 1
\end{pmatrix}.
\]
This would imply that \( H \) is isomorphic to a finite subgroup of \( \mathbb{C}^* \), hence that it is cyclic. A contradiction. \( \square \)

### 3.3. An ind-group with nonconjugate Borel subgroups.

In this section, we consider the subgroup \( \text{Aut}_z(\mathbb{A}_3^3) \) of \( \text{Aut}(\mathbb{A}_3^3) \) consisting of all automorphisms \( f = (f_1, f_2, z) \) fixing the last coordinate of \( \mathbb{A}_3^3 = \text{Spec}(\mathbb{C}[x, y, z]) \). Since it is clearly a closed subgroup, it is also an ind-group. Note that \( \text{Aut}_z(\mathbb{A}_3^3) \) is naturally isomorphic to a subgroup of \( \text{Aut}(\mathbb{A}_2^2(z)) \). In its turn, the field \( \mathbb{C}(z) \) can be embedded into the field \( \mathbb{C} \), so that the group \( \text{Aut}(\mathbb{A}_2^2(z)) \) is isomorphic to a subgroup of \( \text{Aut}(\mathbb{A}_2^2) \). Therefore, by Proposition 3.14, we get
\[
\psi\left( \text{Aut}_z(\mathbb{A}_3^3) \right) \leq \psi\left( \text{Aut}(\mathbb{A}_2^2(z)) \right) \leq \psi\left( \text{Aut}(\mathbb{A}_2^2) \right) = 5.
\]
Recall moreover that \( \text{Aut}_z(\mathbb{A}_3^3) \) contains nontriangularizable additive group actions [1]. Let us briefly describe the example given by Bass. Consider the following locally nilpotent derivation of \( \mathbb{C}[x, y, z] \):
\[
\Delta = -2y\partial_x + z\partial_y.
\]
Then, the derivation \((xz + y^2)\Delta\) is again locally nilpotent. We associate it with the morphism
\[
(\mathbb{C}, +) \to \text{Aut}_\mathbb{C}(\mathbb{C}[x, y, z]), \quad t \mapsto \exp(t(xz + y^2)\Delta).
\]
The automorphism of \( \mathbb{A}_3^3 \) corresponding to \( \exp(t(xz + y^2)\Delta) \) is given by
\[
f_t := (x - 2ty(xz + y^2) - t^2z(xz + y^2)^2, y + tz(xz + y^2), z) \in \text{Aut}(\mathbb{A}_3^3).
\]
For \( t = 1 \), we get the famous Nagata automorphism. Note that the fixed point set of the corresponding \((\mathbb{C}, +)\)-action on \( \mathbb{A}_3^3 \) is the hypersurface \( \{xz + y^2 = 0\} \) which has an isolated singularity at the origin. On the other hand, the fixed point set of a triangular \((\mathbb{C}, +)\)-action on \( \mathbb{A}_3^3 \)
\[
\quad t \mapsto g_t = \exp(t(a(y, z)\partial_x + b(z)\partial_y)) \in \text{Aut}(\mathbb{A}_3^3)
\]
is the set \( \{a(y, z) = b(z) = 0\} \), which is isomorphic to a cylinder \( \mathbb{A}_1^1 \times Z \) for some variety \( Z \). This implies that the \((\mathbb{C}, +)\)-action \( t \mapsto f_t \) is not triangularizable.
By Proposition 3.10, it follows that $\text{Aut}_z(A_3^3_C)$ contains Borel subgroups that are not conjugate to a subgroup of the group

$$B_z = \{(f_1, f_2, z) \in \text{Aut}(A_3^3_C) \mid f_1 \in \mathbb{C}[x, y, z], f_2 \in \mathbb{C}[y, z]\}$$

of triangular automorphisms of $\text{Aut}_z(A_3^3_C)$.

**Proposition 3.21.** — The group $B_z$ is a Borel subgroup of $\text{Aut}_z(A_3^3_C)$.

**Proof.** — With the same proof as for Lemma 3.1, we obtain easily that $B_z$ is connected. It is also solvable, since it can be seen as a subgroup of the Jonquières subgroup of $\text{Aut}(A_3^2_C(z))$, which is solvable.

Now, we simply follow the proof of Proposition 3.3. Let $H \subset \text{Aut}_z(A_3^3_C)$ be a closed subgroup strictly containing $B_z$ and take an element $f$ in $H \setminus B_z$, i.e. an element $f = (f_1, f_2, z)$ with $f_2 \in \mathbb{C}[x, y, z] \setminus \mathbb{C}[y, z]$. Arguing as before, we can find suitable translations $t_c = (x + c_1, y + c_2, z)$ and $t_{c'} = (x + c'_1, y + c'_2, z)$ such that the automorphism $g = t_c \circ f \circ t_{c'}$ fixes the point $(0, 0, 0)$ and is of the form $g = (g_1, g_2, z)$ with $g_2 = xc(z) + yd(z) + h(x, y, z)$ for some $c(z), d(z) \in \mathbb{C}[z], c(z) \neq 0$, and some polynomial $h(x, y, z)$ belonging to the ideal $(x^2, xy, y^2)$ of $\mathbb{C}[x, y, z]$.

Conjugating this $g$ by the automorphism $(tx, ty, z) \in H$, $t \neq 0$, and taking the limit when $t$ goes to 0, we obtain an element of the form $(a(z)x + b(z)y, c(z)x + d(z)y, z)$ with $c(z) \neq 0$ in $H$. By Lemma 3.23 below, this implies that the group $H$ is not solvable. □

**Corollary 3.22.** — The ind-group $\text{Aut}_z(A_3^3_C)$ contains non-conjugate Borel subgroups.

In the course of the proof of Proposition 3.21, we have used the following lemma that we prove now.

**Lemma 3.23.** — The subgroup $B_2(\mathbb{C}[z])$ of upper triangular matrices of $\text{GL}_2(\mathbb{C}[z])$ is a maximal solvable subgroup.

**Proof.** — For every $\alpha \in \mathbb{C}$, denote by $\text{ev}_\alpha : \text{GL}_2(\mathbb{C}[z]) \to \text{GL}_2(\mathbb{C})$ the evaluation map that associates to an element $M(z) \in \text{GL}_2(\mathbb{C}[z])$ the constant matrix $M(\alpha)$ obtained by replacing $z$ by $\alpha$. Let $H$ be a subgroup of $\text{GL}_2(\mathbb{C}[z])$ strictly containing the group $B_2(\mathbb{C}[z])$. By definition, $H$ contains a non-triangular matrix, i.e. a matrix of the form

$$M = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}, \text{ with } c \neq 0.$$

Choose a complex number $\alpha$ such that $c(\alpha) \neq 0$. Then, the group $\text{ev}_\alpha(H)$ contains the upper triangular constant matrices $B_2(\mathbb{C})$ and a non-triangular matrix. Therefore, $\text{ev}_\alpha(H) = \text{GL}_2(\mathbb{C})$ and $H$ is not solvable. □
Remark 3.24. — By Nagao’s theorem (see [18] or e.g. [23, Chapter II, no. 1.6]), we have an amalgamated product structure
\[ \text{GL}_2(\mathbb{C}[z]) = \text{GL}_2(\mathbb{C}) \ast_{B_2(\mathbb{C})} B_2(\mathbb{C}[z]). \]
However, contrarily to the case of Aut(A^2), the group B_2(\mathbb{C}[z]) is not a maximal closed subgroup. Indeed, for every complex number α, this group is strictly included into the group ev_α^{-1}(B_2(\mathbb{C})).

3.4. Maximal closed subgroups

In this section, we mainly focus on the following question.

Question 3.25. — What are the maximal closed subgroups of Aut(A^n_C)?

First of all, it is easy to observe that, since the action of Aut(A^n_C) on A^n_C is infinite transitive, i.e. m-transitive for all integers m \geq 1, the stabilizers of a finite number of points are examples of maximal closed subgroups.

Proposition 3.26. — For every finite subset \( \Delta \) of A^n_C, n \geq 2, the group \( \text{Stab}(\Delta) = \{ f \in \text{Aut}(A^n_C), \ f(\Delta) = \Delta \} \) is a maximal subgroup of Aut(A^n_C). Furthermore, it is closed.

Proof. — Let \( \Delta = \{a_1, \ldots, a_k\} \) be a finite subset of A^n_C. Let \( f \in \text{Aut}(A^n_C) \setminus \text{Stab}(\Delta) \). We will prove that \( \langle \text{Stab}(\Delta), f \rangle = \text{Aut}(A^n_C) \), where \( \langle \text{Stab}(\Delta), f \rangle \) denotes the subgroup of Aut(A^n_C) that is generated by Stab(\Delta) and \( f \). We will use repetitively the well-known fact that Aut(A^n_C) acts 2k-transitively on A^n_C.

We first observe that \( \langle \text{Stab}(\Delta), f \rangle \) contains an element \( g \) such that \( g(\Delta) \cap \Delta = \emptyset \). To see this, denote by \( m := |\Delta \cap f(\Delta)| \) the cadinality of the set \( \Delta \cap f(\Delta) \). Up to composing it by an element of Stab(\Delta), we can suppose that \( f \) fixes the points \( a_1, \ldots, a_m \) and maps \( a_{m+1}, \ldots, a_k \) outside \( \Delta \). If \( m \geq 1 \), then we consider an element \( \alpha \in \text{Stab}(\Delta) \) that maps the point \( a_m \) onto \( a_{m+1} \) and sends all points \( f(a_{m+1}), \ldots, f(a_k) \) outside the set \( f^{-1}(\Delta) \). Remark that \( g = f \circ \alpha \circ f \) is an element of \langle \text{Stab}(\Delta), f \rangle with \( |\Delta \cap g(\Delta)| < m \). By descending induction on \( m \), we can further suppose that \( |\Delta \cap g(\Delta)| = 0 \) as desired.

Now, consider any \( \varphi \in \text{Aut}(A^n_C) \). Let us prove that \( \varphi \) belongs to the subgroup \( \langle \text{Stab}(\Delta), g \rangle \). Take an element \( \beta \in \text{Stab}(\Delta) \) such that \( \beta(\varphi(\Delta)) \cap g^{-1}(\Delta) = \emptyset \). Then, \( g(\beta(\varphi(\Delta)) \cap \Delta = \emptyset \) and we can find an element \( \gamma \in \text{Stab}(\Delta) \) such that \( \gamma(\beta(\varphi(\Delta)) = f(\Delta) \). As desired.
Stab(Δ) such that \((\gamma \circ g \circ \beta \circ \varphi)(a_i) = g(a_i)\) for all \(i\). We have \(\varphi = \beta^{-1} \circ g^{-1} \circ \gamma^{-1} \circ g \circ \delta \in \langle \text{Stab}(\Delta), g \rangle\), where \(\delta := g^{-1}(\gamma \circ g \circ \beta \circ \varphi)\) is an element of \(\text{Stab}(\Delta)\), proving that \(\langle \text{Stab}(\Delta), g \rangle\) is equal to the whole group \(\text{Aut}(\mathbb{A}_n)\). Therefore, the group \(\text{Stab}(\Delta)\) is actually maximal in \(\text{Aut}(\mathbb{A}_n^C)\). Finally, note that for each point \(a \in \mathbb{A}_n^C\) the evaluation map \(\text{ev}_a : \text{Aut}(\mathbb{A}_n^C) \to \mathbb{A}_n^C, f \mapsto f(a)\), is an ind-morphism. Since \(\Delta\) is a closed subset of \(\mathbb{A}_n^C\) the equality

\[
\text{Stab}(\Delta) = \bigcap_i (\text{ev}_{a_i})^{-1}(\Delta)
\]

implies that \(\text{Stab}(\Delta)\) is closed in \(\text{Aut}(\mathbb{A}_n^C)\). □

Besides the above examples and the triangular subgroup \(\mathcal{B}_2\), the only other maximal closed subgroup of \(\text{Aut}(\mathbb{A}_2^C)\) that we are aware of is the affine subgroup \(\mathcal{A}_2\). The fact that \(\mathcal{A}_2\) is maximal among all closed subgroups of \(\text{Aut}(\mathbb{A}_2^C)\) is a particular case of the following recent result of Edo [5]. (We recall that the so-called tame subgroup of \(\text{Aut}(\mathbb{A}_2^C)\) is its subgroup generated by \(\mathcal{A}_n\) and \(\mathcal{B}_2\).)

**Theorem 3.27 ([5]).** — If a closed subgroup of \(\text{Aut}(\mathbb{A}_n^C), n \geq 2\), contains strictly the affine subgroup \(\mathcal{A}_n\), then it also contains the whole tame subgroup, hence its closure. In particular, for \(n = 2\), the affine group \(\mathcal{A}_2\) is maximal among the closed subgroups of \(\text{Aut}(\mathbb{A}_2^C)\).

**Remark 3.28.** — Note that Theorem 3.27 does not allow us to settle the question of the (non) maximality of \(\mathcal{A}_n\) among the closed subgroups of \(\text{Aut}(\mathbb{A}_n^C)\) when \(n \geq 3\). Indeed, on the one hand, it was recently shown that, in dimension 3, the tame subgroup is not closed (see [6]). But, on the other hand, it is still unknown whether it is dense in \(\text{Aut}(\mathbb{A}_2^C)\) or not. For \(n \geq 4\), the three questions, whether the tame subgroup is closed, whether it is dense, or even whether it is a strict subgroup of \(\text{Aut}(\mathbb{A}_n^C)\), are all open.

Let us finally remark that the affine group \(\mathcal{A}_2\) is not a maximal among all abstract subgroups of \(\text{Aut}(\mathbb{A}_2^C)\). Indeed, using the amalgamated structure

\[
\text{Aut}(\mathbb{A}_2^C) = \mathcal{A}_2 \ast_{\mathcal{A}_2 \cap \mathcal{B}_2} \mathcal{B}_2
\]

and following [8], we can define the multidegree (or polydegree) of any automorphism \(f \in \text{Aut}(\mathbb{A}_2^C)\) in the following way. If \(f\) admits an expression

\[
f = a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k \circ a_{k+1},
\]

where each \(a_i\) belongs to \(\mathcal{A}_2\), each \(b_i\) belongs to \(\mathcal{B}_2\) and \(a_i \notin \mathcal{B}_2\) for \(2 \leq i \leq k\), \(b_i \notin \mathcal{A}_2\) for \(1 \leq i \leq k\), the multidegree of \(f\) is defined as the finite sequence (possibly empty) of integers at least equal to 2:

\[
m\text{deg}(f) = (\deg b_1, \deg b_2, \ldots, \deg b_k).
\]
Then, the subgroup $M_r := \langle A_2, (B_2)_{\leq r} \rangle \subseteq \text{Aut}(\mathbb{A}^2_\mathbb{C})$ coincides with the set of automorphisms whose multidegree is of the form $(d_1, \ldots, d_k)$ for some $k$ with $d_1, \ldots, d_k \leq r$. We thus have a strictly increasing sequence of subgroups
\[ \mathcal{A}_2 = M_1 < M_2 < \cdots < M_d < \cdots, \]
showing in particular that $\mathcal{A}_2$ is not a maximal abstract subgroup.

4. Non-maximality of the Jonquières subgroup in dimension 2

Throughout this section, we work over an arbitrary ground field $k$.

Recall that by the famous Jung–van der Kulk–Nagata theorem [12, 14, 19], the group $\text{Aut}(\mathbb{A}^2_k)$, of algebraic automorphisms of the affine plane, is the amalgamated free product of its affine subgroup
\[ A = \{(ax + by + c, a'x + b'y + c') \in \text{Aut}(\mathbb{A}^2_k) \mid a, b, c, a', b', c' \in k\} \]
and its Jonquières subgroup
\[ B := \{(ax + p(y), b'y + c') \in \text{Aut}(\mathbb{A}^2_k) \mid a, b', c' \in k, p(y) \in k[y]\} \]
above their intersection. Therefore, every element $f \in \text{Aut}(\mathbb{A}^2_k)$ admits a reduced expression as a product of the form
\[ f = t_1 \circ a_1 \circ t_2 \circ \cdots \circ a_n \circ t_{n+1}, \]
where $a_1, \ldots, a_n$ belong to $A \setminus A \cap B$, and $t_1, \ldots, t_{n+1}$ belong to $B$ with $t_2, \ldots, t_n \notin A \cap B$.

**Definition 4.1.** — The number $n$ of affine non-triangular automorphisms appearing in such an expression for $f$ is unique. We call it the affine length of $f$ and denote it by $\ell_A(f)$.

**Remark 4.2.** — Instead of counting affine elements to define the length of an automorphism of $\mathbb{A}^2$, one can of course also consider the Jonquières elements and define the triangular length $\ell_B(f)$ of every $f \in \text{Aut}(\mathbb{A}^2_k)$. Actually, this is the triangular length, that one usually uses in the literature. Let us in particular recall that this length map $\ell_B : \text{Aut}(\mathbb{A}^2_\mathbb{C}) \to \mathbb{N}$ is lower semicontinuous [9], when considering $\text{Aut}(\mathbb{A}^2_\mathbb{C})$ as an ind-group. Since
\[ \ell_A(f) = \max_{b_1, b_2 \in B} \ell_B(b_1 \circ f \circ b_2) - 1 \]
for every $f \in \text{Aut}(\mathbb{A}^2_k)$ and since the supremum of arbitrarily many lower semicontinuous maps is lower semicontinuous, we infer that $\ell_A$ has also this property.
Proposition 4.3. — The affine length map \( \ell_A : \text{Aut}(\mathbb{A}_n^2) \to \mathbb{N} \) is lower semicontinuous.

The next result shows that the Jonquières subgroup is not a maximal subgroup of \( \text{Aut}(\mathbb{A}_n^2) \).

Proposition 4.4. — Let \( p \in k[y] \) be a polynomial that fulfils the following property:

\[(WG) \forall \alpha, \beta, \gamma \in k, \deg[p(y) - \alpha p(\beta y + \gamma)] \leq 1 \implies \alpha = \beta = 1 \text{ and } \gamma = 0,\]

and consider the following elements of \( \text{Aut}(\mathbb{A}_n^2) \):

\[\sigma = (y, x), \quad t = (-x + p(y), y), \quad f = (\sigma \circ t)^2 \circ \sigma \circ (t \circ \sigma)^2.\]

Then, the subgroup generated by \( B \) and \( f \) is a strict subgroup of \( \text{Aut}(\mathbb{A}_n^2) \), i.e. \( \langle B, f \rangle \neq \text{Aut}(\mathbb{A}_n^2) \).

Remark 4.5. — Polynomials satisfying the above property \( (WG) \) are called weakly general in [10], where a stronger notion of a general polynomial is also given (see [10, Definition 15, p. 585]). In particular, by [10, Example 65, p. 608], the polynomial \( q = y^5 + y^4 \) is weakly general if \( k \) is a field of characteristic zero.

Moreover, the polynomial \( q = y^{2p} - y^{2p-1} \) is weakly general if \( \text{char}(k) = p > 0 \). This follows directly from the fact that the coefficients of \( y^{2p}, y^{2p-1} \) and \( y^{2p-2} \) in the polynomial \( q(y) - \alpha q(\beta y + \gamma) \) are equal to \( 1 - \alpha \beta^{2p}, 1 - \alpha \beta^{2p-1} \), and \(-\alpha \beta^{2p-2} \gamma, \) respectively.

Proof of Proposition 4.4. — Remark that \( \sigma \) and \( t \), hence \( f \), are involutions. Therefore, every element \( g \in \langle B, f \rangle \) can be written as

\[g = b_1 \circ f \circ b_2 \circ f \circ \cdots \circ b_k \circ f \circ b_{k+1},\]

where the elements \( b_i \) belong to \( B \) and where we can assume without restriction that \( b_2, \ldots, b_k \) are different from the identity (otherwise, the expression for \( g \) could be shortened using that \( f^2 = \text{id} \)).

In order to prove the proposition, it is enough to show that no element \( g \) as above is of affine-length equal to 1. Note that \( \ell_A(g) = 0 \) if \( k = 0 \) and that \( \ell_A(g) = \ell_A(f) = 5 \) if \( k = 1 \). It remains to consider the case where \( k \geq 2 \).
For this, let us define four subgroups $B_0, \ldots, B_3$ of $B$ by

$B_0 = B,$

$B_1 = A \cap B = \{(ax + by + c, b'y + c') \mid a, b, c, b', c' \in k, a, b' \neq 0\},$

$B_2 = (A \cap B) \cap [\sigma \circ (A \cap B) \circ \sigma]
\quad = \{(ax + c, b'y + c') \mid a, c, b', c' \in k, a, b' \neq 0\},$

$B_3 = \{(x, y + c') \mid c' \in k\}.$

Note that $B = B_0 \supseteq B_1 \supseteq B_2 \supseteq B_3$. We will now give a reduced expression of $u_i := (t \circ \sigma)^2 \circ b_i \circ (\sigma \circ t)^2$ for each $i \in \{2, \ldots, k\}$. We do it by considering successively the four following cases:

1. $b_i \in B_0 \setminus B_1$; 2. $b_i \in B_1 \setminus B_2$; 3. $b_i \in B_2 \setminus B_3$; 4. $b_i \in B_3 \setminus \{id\}$.

Case 1. — $b_i \in B_0 \setminus B_1$.

Since $b_i \in B \setminus A$, the element $u_i$ admits the following reduced expression

$$u_i = (t \circ \sigma)^2 \circ b_i \circ (\sigma \circ t)^2.$$

Case 2. — $b_i \in B_1 \setminus B_2$.

Since $\tilde{b}_i := \sigma \circ b_i \circ \sigma \in A \setminus B$, the element $u_i$ has the following reduced expression

$$u_i = t \circ \sigma \circ t \circ \tilde{b}_i \circ t \circ \sigma \circ t.$$

Case 3. — $b_i \in B_2 \setminus B_3$.

Let us check that $\overline{b}_i := t \circ \sigma \circ b_i \circ \sigma \circ t \in B \setminus A$. We are in the case where $b_i = (ax + c, b'y + c')$ with $(a, c, b') \neq (1, 0, 1)$. A direct calculation gives that

$$\overline{b}_i = (b'x + p(ay + c) - b'p(y) - c', ay + c).$$

By the assumption made on $p$, we have that $\deg[p(ay + c) - b'p(y)] \geq 2$, hence that $\overline{b}_i \in B \setminus A$. Therefore $u_i$ admits the following reduced expression

$$u_i = t \circ \sigma \circ \overline{b}_i \circ \sigma \circ t.$$

Case 4. — $b_i \in B_3 \setminus \{id\}$.

Let us check that $\tilde{b}_i := (t \circ \sigma)^2 \circ b_i \circ (\sigma \circ t)^2 \in B \setminus A$. We are in the case where $\tilde{b}_i = (x, y + c')$ with $c' \in \mathbb{C}^*$. Using the computation in case 3 with $(a, c, b') = (1, 0, 1)$, we then obtain that

$$\tilde{b}_i = t \circ \sigma \circ (x - c', y) \circ \sigma \circ t = t \circ (x, y - c') \circ t
\quad = (x + p(y - c') - p(y), y - c') \in B \setminus A.$$

Therefore, the element $u_i$ has the following reduced expression

$$u_i = \tilde{b}_i.$$
Finally we obtain a reduced expression for an element \( g \in \langle B, f \rangle \) from the above study of cases, since we can express
\[
g = b_1 \circ f \circ b_2 \circ f \circ \cdots \circ b_k \circ f \circ b_{k+1}
= b_1 \circ (\sigma \circ t)^2 \circ \sigma \circ u_2 \circ \sigma \circ \cdots \circ \sigma \circ u_k \circ \sigma \circ (t \circ \sigma)^2 \circ b_{k+1}.
\]
In particular, observe that \( \ell_A(g) \geq 6 \) if \( k \geq 2 \). This concludes the proof. \( \square \)

Note that the element \( f \) such that \( \langle B, f \rangle \neq \text{Aut}(A^2_k) \), that we constructed in Proposition 4.4, is of affine-length \( \ell_A(f) = 5 \). Our next result shows that 5 is precisely the minimal length for elements \( f \in \text{Aut}(A^2_k) \setminus B \) with that property.

**Proposition 4.6.** — Suppose that \( f \in \text{Aut}(A^2_k) \) is an automorphism of affine length \( \ell \) with \( 1 \leq \ell \leq 4 \). Then, the subgroup generated by \( B \) and \( f \) is equal to the whole group \( \text{Aut}(A^2_k) \), i.e. \( \langle B, f \rangle = \text{Aut}(A^2_k) \).

In order to prove the above proposition, it is useful to remark that we can impose extra conditions on the elements \( t_1, \ldots, t_{n+1}, a_1, \ldots, a_n \) appearing in a reduced expression \( (*) \) of an automorphism \( f \in \text{Aut}(A^2_k) \). We do it in Proposition 4.10 below. First, we need to introduce some notations.

**Notation 4.7.** — In the sequel, we will denote, as in the proof of Proposition 4.4, by \( \sigma \) the involution
\( \sigma = (y, x) \in \text{Aut}(A^2_k) \)
and by \( B_2 \) the subgroup
\( B_2 = \{ (ax + c, b'y + c') \in \text{Aut}(A^2_k) \mid a, c, b', c' \in k \} \subset A \cap B. \)
Moreover, we denote by \( I \) the subset
\( I = \{ (-x + p(y), y) \in \text{Aut}(A^2_k) \mid p(y) \in k[y], \deg p(y) \geq 2 \} \subset B \setminus A \cap B. \)
Note that the elements of \( I \) are all involutions.

**Lemma 4.8.** — The followings hold:

1. \( B_2 \circ \sigma = \sigma \circ B_2. \)
2. \( B \setminus A \cap B = I \circ B_2 = B_2 \circ I = B_2 \circ I \circ B_2. \)
3. \( A \setminus A \cap B \subset (A \cap B) \circ \sigma \circ (A \cap B). \)

**Remark 4.9.** — In particular, Assertion (3) implies that the group generated by \( \sigma \) and all triangular automorphisms is equal to the whole \( \text{Aut}(A^2_k) \), i.e. \( \langle B, \sigma \rangle = \text{Aut}(A^2_k) \).
Proof. — The first assertion is an easy consequence of the following equalities:

\[(ax + c, b'y + c') \circ \sigma = (ay + c, b'x + c') = \sigma \circ (b'x + c', ay + c)\].

Let us now prove the second assertion. It is easy to check that \(I \circ B_2 = B_2 \circ I \subset B \setminus A \cap B\). On the other hand, let \(f = (ax + p(y), b'y + c')\) be an element of \(B \setminus A \cap B\). Then \(f\) belongs to \(I \circ B_2\), since we can write

\[f = (−x + p\left(\frac{y - c'}{1 - x}ight), y) = (−ax, b'y + c').\]

It remains to prove the last assertion. For this, it suffices to write, given an element \(f = (ax + by + c, a'x + b'y + c')\) of \(A \setminus A \cap B\) with \(a' \neq 0\), that

\[f = (ax + by + c, a'x + b'y + c') = (x + \frac{a}{a'}y + c, y + c') \circ \sigma \circ \left(a'x + b'y, \frac{ba' - ab'}{a'}y\right).\]  

\[\square\]

**Proposition 4.10.** — Let \(f \in \text{Aut}(\mathbb{A}^2_k)\) be an automorphism of affine length \(\ell = n + 1\) with \(n \geq 0\). Then there exist triangular automorphisms \(\tau_1, \tau_2 \in B\) and triangular involutions \(i_1, \ldots, i_n \in I\) such that

\[(**): \quad f = \tau_1 \circ \sigma \circ i_1 \circ \sigma \circ \cdots \circ \sigma \circ i_n \circ \sigma \circ \tau_2.\]

In particular, the inverse of \(f\) is given by

\[f^{-1} = \tau_2^{-1} \circ \sigma \circ i_n \circ \sigma \circ \cdots \circ \sigma \circ i_1 \circ \sigma \circ \tau_1^{-1}.\]

Proof. — Let \(f\) be an automorphism of affine length \(\ell = n + 1\). By definition,

\[f = t_1 \circ a_1 \circ t_2 \circ \cdots \circ a_n \circ t_{n+1},\]

for some \(a_1, \ldots, a_n \in A \setminus A \cap B, t_1, t_{n+1} \in B\) and \(t_2, \ldots, t_n \in B \setminus A \cap B\). Using Assertion (3) of Lemma 4.8, we may replace every \(a_i\) by \(\sigma\). The proposition then follows from Assertions (1) and (2) of Lemma 4.8. \(\square\)

We can now proceed to the proof of Proposition 4.6.

**Proof of Proposition 4.6.**

Case \(\ell = 1\). — Let \(f \in B\) with \(\ell_A(f) = 1\). By Proposition 4.10, we can write \(f = \tau_1 \circ \sigma \circ \tau_2\) for some \(\tau_1, \tau_2 \in B\). Thus, \(\langle B, f \rangle = \langle B, \sigma \rangle = \text{Aut}(\mathbb{A}^2_k)\) follows from Remark 4.9.

The proofs for affine length \(\ell = 2, 3, 4\) will be based on explicit computations. In particular, it will be useful to observe that all \(i = (−x + p(y), y) \in I\).
satisfy that
\begin{align}
(4.1) & \quad i \circ (x + 1, y) \circ i = (x - 1, y), \\
(4.2) & \quad \sigma \circ i \circ (x + 1, y) \circ i \circ \sigma = (x, y - 1)
\end{align}

and
\begin{align}
(4.3) & \quad i \circ (x, y - 1) \circ i \circ (-x, y + 1) = (-x + (p(y) - p(y + 1)), y).
\end{align}

Case \( \ell = 2 \). — Let \( f \in B \) with \( \ell_A(f) = 2 \). By Proposition 4.10, we can suppose that \( f = \sigma \circ i \circ \sigma \) for some involution \( i = (-x + p(y), y) \in I \). Consider the elements \( b_1 = \sigma \circ (x, y - 1) \circ \sigma \) and \( b_2 = \sigma \circ (-x, y + 1) \circ \sigma \) of \( B_2 \). Since

\[ f \circ b_1 \circ f \circ b_2 = \sigma \circ i \circ (x, y - 1) \circ i \circ (-x, y + 1) \circ \sigma, \]

it follows from Equality (4.3) above that the automorphism \( \sigma \circ (-x + (p(y) - p(y + 1)), y) \circ \sigma \) belongs to \( \langle B, f \rangle \). By induction, we thus obtain an element in \( \langle B, f \rangle \) of the form \( \sigma \circ (-x + q(y), y) \circ \sigma \) with \( \deg(q) = 1 \). This element is in fact an element of \( A \setminus A \cap B \) and has therefore affine length 1. This implies that \( \langle B, f \rangle = \text{Aut}(A^2_k) \).

Case \( \ell = 3 \). — Let \( f \in B \) with \( \ell_A(f) = 3 \). By Proposition 4.10, we can suppose that \( f = \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma \) for some \( i_1 = (-x + p_1(y), y), i_2 = (-x + p_2(y), y) \in I \). We first use Equality (4.2), which implies that
\begin{align}
(4.4) & \quad \sigma \circ i_2 \circ \sigma \circ b \circ \sigma \circ i_2 \circ \sigma = (x, y - 1),
\end{align}

where \( b \) denotes the element \( b = \sigma \circ (x + 1, y) \circ \sigma \in B_2 \). Hence, denoting by \( b' \) the element \( b' = \sigma \circ (-x, y + 1) \circ \sigma \) in \( B_2 \) and using Equalities (4.3) and (4.4), we obtain that

\[ f \circ b \circ f^{-1} \circ b' = \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma \circ b \circ \sigma \circ i_2 \circ \sigma \circ i_1 \circ \sigma \circ b' \]
\[ = \sigma \circ i_1 \circ (x, y - 1) \circ i_1 \circ \sigma \circ b' \]
\[ = \sigma \circ i_1 \circ (x, y - 1) \circ i_1 \circ (-x, y + 1) \circ \sigma \]
\[ = \sigma \circ (-x + (p_1(y) - p_1(y + 1)), y) \circ \sigma \]

is an element of affine length 2 (or 1 in the case where \( \deg(p_1) = 2 \)), which belongs to \( \langle B, f \rangle \). Consequently, \( \langle B, f \rangle = \text{Aut}(A^2_k) \).

Case \( \ell = 4 \). — Let \( f \in B \) with \( \ell_A(f) = 4 \). By Proposition 4.10, we can suppose that \( f = \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma \circ i_3 \circ \sigma \) for some \( i_j = (-x + p_j(y), y) \in I \),
\( j = 1, 2, 3 \). Letting \( b = \sigma \circ (x + 1, y) \circ \sigma \) as above, one get that
\[
f \circ b \circ f^{-1} = \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma \circ i_3 \circ \sigma \circ b \circ \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma \circ i_1 \circ \sigma \\
= \sigma \circ i_1 \circ \sigma \circ i_2 \circ (x, y - 1) \circ i_2 \circ \sigma \circ i_1 \circ \sigma \\
= \sigma \circ i_1 \circ \sigma \circ i_2 \circ (x, y - 1) \circ i_2 \circ (-x, y + 1) \\
\circ (-x, y - 1) \circ \sigma \circ i_1 \circ \sigma \\
= \sigma \circ i_1 \circ \sigma \circ i' \circ (-x, y - 1) \circ \sigma \circ i_1 \circ \sigma \\
= \sigma \circ i_1 \circ \sigma \circ i_2 \circ \sigma \circ (x - 1, -y) \circ i_1 \circ \sigma \\
= \sigma \circ i_1 \circ \sigma \circ i'_2 \circ \sigma \circ i'_1 \circ (x + 1, -y) \circ \sigma \\
= \sigma \circ i_1 \circ \sigma \circ i' \circ \sigma \circ i'_2 \circ \sigma \circ i'_1 \circ \sigma \circ (-x, y + 1),
\]
where \( i'_2 = (-x + p'_2(y), y) \) and \( i'_1 = (-x + p'_1(y), y) \) for the polynomials
\( p'_2(y) = p_2(y) - p_2(y + 1) \) and \( p'_1(y) = p_1(-y) \), respectively. In particular,
\( \langle B, f \rangle \) contains the element \( \sigma \circ i_1 \circ \sigma \circ i'_2 \circ \sigma \circ i'_1 \circ \sigma \circ (x, y) \circ \sigma \circ i_1 \circ \sigma \circ i'_2 \circ \sigma \circ i'_1 \circ \sigma \circ (-x, y + 1) \),
\( \sigma \circ i_1 \circ \sigma \circ i'_2 \circ \sigma \circ i'_1 \circ \sigma \circ (-x, y + 1) \),
where \( i'_2 = (-x + p'_2(y), y) \) and \( i'_1 = (-x + p'_1(y), y) \) for the polynomials
\( p'_2(y) = p_2(y) - p_2(y + 1) \) and \( p'_1(y) = p_1(-y) \), respectively. Since \( \deg(p'_2) = \deg(p_2) - 1 \),
we obtain by induction an element in \( (B, f) \) of the form \( \sigma \circ i_1 \circ \sigma \circ i'_2 \circ \sigma \circ i'_1 \circ \sigma \circ \sigma \circ (x, y + 1) \),
with \( \tilde{\sigma}_2 = (-x + \tilde{p}_2(y), y) \) and \( \deg(\tilde{p}_2) = 1 \). Since \( \sigma \circ \tilde{\sigma}_2 \circ \sigma \) is an element of
\( A \setminus A \cap B \), the above \( \sigma \circ i_1 \circ \sigma \circ i'_2 \circ \sigma \circ i'_1 \circ \sigma \circ (x, y) \circ \sigma \circ i_1 \circ \sigma \circ i'_2 \circ \sigma \circ i'_1 \circ \sigma \circ (-x, y + 1) \),
and the proposition follows. \( \square \)

To conclude, let us emphasize that, as pointed to us by S. Lamy, our results concerning the non-maximality of \( B \) are related to those of [10] about
the existence of normal subgroups for the group \( \text{SAut}(\mathbb{A}_2^\mathbb{C}) \) of automorphisms of the complex affine plane whose Jacobian determinant is equal to 1. Indeed, the subgroup \( \langle B, f \rangle \), generated by \( B \) and a given automorphism
\( f \), is contained into the subgroup \( B \circ \langle f \rangle_N = \{ h \circ g | h \in B, g \in \langle f \rangle_N \} \),
where \( \langle f \rangle_N \) denotes the normal subgroup of \( \text{Aut}(\mathbb{A}_2^\mathbb{C}) \) that is generated
by \( f \).

Combined with Proposition 4.6, the above observation gives us a short
proof of the following result.

**Theorem 4.11** ([10, Theorem 1]). — If \( f \in \text{SAut}(\mathbb{A}_2^\mathbb{C}) \) is of affine length
at most 4 and \( f \not= \text{id} \), then the normal subgroup \( \langle f \rangle_N \) generated by \( f \) in
\( \text{SAut}(\mathbb{A}_2^\mathbb{C}) \) is equal to the whole group \( \text{SAut}(\mathbb{A}_2^\mathbb{C}) \).

**Proof.** — The case where \( f \) is a triangular automorphism being easy to
treat (see [10, Lemma 30, p. 590]), suppose that \( f \in \text{SAut}(\mathbb{A}_2^\mathbb{C}) \) is of affine length
at most 4 and at least 1. By Proposition 4.6, we have \( \langle B, f \rangle = \text{Aut}(\mathbb{A}_2^\mathbb{C}) \).
Since the group \( B \circ \langle f \rangle_N \) contains \( B \) and \( f \), we get \( B \circ \langle f \rangle_N = \text{Aut}(\mathbb{A}_2^\mathbb{C}) \). In particular, the element \( (-y, x) \) can be written as \( (-y, x) = b \circ g \)
for some \( b \in B \) and \( g \in \langle f \rangle_N \). Consequently, \( \langle f \rangle_N \) contains the element
\( g = b^{-1} \circ (-y, x) \) which is of affine length 1.
Remark that the Jacobian determinant of $b$ is equal to 1. Therefore, we can write $b^{-1} = (ax + P(y), a^{-1}y + c)$ for some $a \in \mathbb{C}^*$, $c \in \mathbb{C}$ and $P(y) \in \mathbb{C}[y]$. Thus, $g$ is given by

$$g = (-ay + P(x), a^{-1}x + c).$$

Next, we consider the translation $\tau = (x + 1, y)$ and compute the commutator $[\tau, g] = \tau \circ g \circ \tau^{-1} \circ g^{-1}$, which is an element of $\langle f \rangle_N$. Since

$$[\tau, g] = (x + 1, y) \circ (-ay + P(x), a^{-1}x + c) \circ (x + 1, y)$$

$$\circ (ay - ac, -a^{-1}x + a^{-1}P(ay - ac))$$

$$= (x - P(ay - ac) + P(ay - ac - 1) + 1, y - a^{-1})$$

is a triangular automorphism different from the identity, the theorem follows directly from [10, Lemma 30, p. 590]. □

On the other hand, we can retrieve the fact that the Jonquières subgroup is not a maximal subgroup of $\text{Aut}(\mathbb{A}^2 \mathbb{C})$ as a corollary of [10, Theorem 2]. Indeed, the latter produces elements $f \in \text{SAut}(\mathbb{A}^2 \mathbb{C})$ of affine length $\ell_A(f) = 7$ such that $\langle f \rangle_N \neq \text{SAut}(\mathbb{A}^2 \mathbb{C})$. In particular, by [10, Theorem 1] above, the identity is the only automorphism of affine length smaller than or equal to 4 contained in $\langle f \rangle_N$. Therefore, since $\langle B, f \rangle \subset B \circ \langle f \rangle_N$, the subgroup $\langle B, f \rangle$ does not contain any non-triangular automorphism of affine length $\leq 4$. Consequently, $\langle B, f \rangle$ is a strict subgroup of $\text{Aut}(\mathbb{A}^2 \mathbb{C})$.

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