Local and Nonlocal Optimal Control in the Source

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Abstract. The analysis of an optimal control problem of nonlocal type is analyzed. The obtained results are applied to the study of the corresponding local optimal control problem. The state equations are governed by p-laplacian elliptic operators, of local and nonlocal type, and the costs belong to a wide class of integral functionals. The nonlocal problem is formulated by means of a convolution of the states with a kernel. This kernel depends on a parameter, called horizon which, is responsible for the nonlocality of the equation. The input function is the source of the elliptic equation. The existence of nonlocal controls is obtained and a $G$-convergence result is employed in this task. The limit of the solutions of the nonlocal optimal control problem, when the horizon tends to zero, is analyzed and compared to the solutions of the underlying local optimal control problem.

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1. Introduction

In many branches of science, the nonlocal models have shown a high proficiency to study phenomena. They have been one of the main alternatives to reformulate different types of problems in Applied Mathematics. The usage of these models has been notable in fields, such as kinetic equations, phase transitions, diffusion models and other themes of continuum mechanics [9, 15, 17, 28, 38, 39, 42, 48, 50]. There are several ways to introduce nonlocality when we try to model some classical problems. Among others works we must highlight [7, 26, 27, 32, 49] in the nonlocal framework and [16, 20, 29, 33, 34, 40, 43, 47] from the point of view of the fractional analysis. In a general context, the main idea to built a nonlocal model, basically relies on considering derivatives of nonlocal type, or of fractional derivatives, instead of the
classical ones. This new way to measure the variability, somehow, allows to introduce and modulate long-range interactions.

In our specific setting, of optimal control problems governed by partial differential equations, instead of considering differential equations, we shall present a nonlocal model built through integral equations. These integrals are, somehow, the convolution of the states with a specific family of kernels. This family is parametrized by a number called horizon, which is responsible for the degree of the nonlocal interaction. The optimization problem is driven by the nonlocal \( p \)-laplacian as state equation, and Dirichlet boundary conditions are imposed. The control is the right-hand side forcing function, the source, and the cost to minimize belongs to a fairly general class of integral functional.

The purpose of the present article is the analysis of this type of nonlocal optimal control problem, the existence of solutions and their asymptotic behavior when the nonlocality, the horizon, tends to zero. Since in the limit we recover the formulation of certain classical control problems, some meaningful conclusions about approximation or existence of classical solutions are obtained as well. Consequently, two different problems will be addressed in the article, the nonlocal model and the classical or local counterpart. To go into the details, we first specify the ambient space we work on, and then, we shall formulate these two optimal control problems.

\subsection{Hypotheses}

Specifically, the framework in which we shall work can be described as follows. The domain is \( \Omega \subset \mathbb{R}^N \), an open and bounded set with Lipschitz boundary. We define its extension \( \Omega_{\delta} = \Omega \cup_{p \in \partial \Omega} B(p, \delta) \), where \( B(x, r) \) is the notation of an open ball centered at \( x \in \mathbb{R}^N \) and radius \( r > 0 \) and \( \delta \) is a positive number.

About the right term of the elliptic equations, the function \( f \), called the source, we assume \( f \in L^{p'}(\Omega) \), where \( p' = \frac{p}{p-1} \) and \( p > 1 \). Concerning the kernels \( (k_\delta)_{\delta > 0} \), we assume that it is a sequence of nonnegative radial functions such that for any \( \delta \):

\[
\text{supp} \ k_\delta \subset B(0, \delta)
\]  

and

\[
\frac{1}{C_N} \int_{B(0, \delta)} k_\delta(|z|) \, dz = 1
\]

where \( C_N = \frac{1}{\text{meas}(S^{N-1})} \int_{S^{N-1}} |\omega \cdot e|^p \, d\sigma^{N-1}(\omega) \), \( \sigma^{N-1} \) stands for the \( N-1 \) dimensional Haussdorff measure on the unit sphere \( S^{N-1} \) and \( e \) is any unitary vector in \( \mathbb{R}^N \). In addition, the kernels satisfy the uniform estimate

\[
k_\delta(|z|) \geq \frac{c_0}{|z|^{N+(s-1)p}}
\]

where \( c_0 > 0 \) and \( s \in (0, 1) \) are given constants.

The natural frame in which we shall work is the nonlocal energy space:

\[
X = \{ u \in L^p(\Omega_\delta) : B(u, u) < \infty \}
\]
where $B$ is the operator defined in $X \times X$ by means of the formula:

$$
B(u,v) = \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta(|x'| - |x|) \frac{|u(x') - u(x)|}{|x'|^p} |u(x') - u(x)|^{p-2} \frac{(v(x') - v(x))}{|x'|^p} \, dx'dx.
$$

(1.4)

We also define the constrained energy space as

$$
X_0 = \{ u \in X : u = 0 \text{ in } \Omega_\delta \setminus \Omega \}
$$

It is well-known that for any given $\delta > 0$ the space $X = X(\delta)$ is a Banach space with the norm:

$$
\|u\|_X = \|u\|_{L^p(\Omega_\delta)} + (B(u,u))^{1/p}.
$$

The dual of $X$ will be denoted by $X'$ and can be endowed with the norm defined by

$$
\|g\|_{X'} = \sup \{ \langle g, w \rangle_{X' \times X} : w \in X, \|w\|_X = 1 \}.
$$

Analogous definitions apply to the space $X_0 = X_0(\delta)$.

There is another functional space that we will use in the formulation of our problem and that is susceptible to be used as a set of controls. It is the space of diffusion coefficients, that is

$$
\mathcal{H} = \{ h : \Omega_\delta \rightarrow \mathbb{R} \mid h(x) \in [h_{\text{min}}, h_{\text{max}}] \text{ a.e. } x \in \Omega, h = 0 \text{ in } \Omega_\delta \setminus \Omega \},
$$

where $h_{\text{min}}$ and $h_{\text{max}}$ are positive constants such that $0 < h_{\text{min}} < h_{\text{max}}$.

1.2. Formulation of the Problems

1.2.1. Nonlocal Optimal Control in the Source. The nonlocal optimal control in the source is an optimal control problem denoted by $\mathcal{P}^\delta$ whose formulation is as follows: the problem, for each $\delta > 0$ fixed, consists on finding $g \in L^p(\Omega)$ such that minimizes the functional:

$$
I_\delta(g,u) = \int_{\Omega} F(x,u(x),g(x)) \, dx + \gamma B_{h_0}(u,u),
$$

(1.5)

where $u \in X$ is the solution of the nonlocal boundary problem $\mathcal{P}^\delta$

$$
\begin{cases}
B_h(u,w) = \int_{\Omega} g(x) w(x) \, dx, & \text{for any } w \in X_0, \\
u = u_0 & \text{in } \Omega_\delta \setminus \Omega,
\end{cases}
$$

(1.6)

where

$$
B_h(u,w) = \int_{\Omega_\delta} \int_{\Omega_\delta} H(x',x) k_\delta(|x'| - |x|) \frac{|u(x') - u(x)|}{|x'|^p} |u(x') - u(x)|^{p-2} \frac{(v(x') - v(x))}{|x'|^p} \, dx'dx
$$

(1.7)

$$
H(x',x) = \frac{h(x') + h(x)}{2}, \quad h \in \mathcal{H}, \text{ and } u_0 \text{ is a given function}. \text{ The above nonlocal boundary condition, } u = u_0 \text{ in } \Omega_\delta \setminus \Omega, \text{ must be interpreted in the sense of traces. Indeed, to make sense, } u_0 \text{ must belong to the space } \tilde{X}_0 =
$$
\{ w_{|\Omega_s \setminus \Omega} : w \in X \}. This space is well defined independently of the parameter \( s \in (0, 1) \) we choose in (1.2). It is easy to check that a norm for this space is the one defined as

\[ \| v \|_{X_0} = \inf \left\{ \| w \|_{L^p(\Omega_s)} + (B(w, w))^{1/p} : w \in X \text{ such that } w_{|\Omega_s \setminus \Omega} = v \right\}. \]

The integrand \( F \) that appear in the cost we want to optimize is under the format:

\[
F(x, u(x), g(x)) = G(x, u(x)) + \beta |g(x)|^{p'}, \tag{1.8}
\]

where \( \beta \) and \( \gamma \) are given positive constants, \( h_0 \in \mathcal{H} \) is also given, and \( G : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) assumed to be a measurable positive function such that \( G(x, \cdot) \) is uniformly Lipschitz continuous, that is, for any \( x \in \Omega \) and any \( (u, v) \in \mathbb{R}^2 \) there exists a positive constant \( L \) such that \( |G(x, u) - G(x, v)| \leq L |u - v| \).

We formulate the nonlocal optimal control problem as

\[
\min_{(g, u) \in \mathcal{A}_\delta} I_{\delta}(g, u), \tag{1.9}
\]

where

\[ \mathcal{A}_\delta \doteq \left\{ (f, v) \in L^{p'}(\Omega) \times X : v \text{ solves (1.6) with } g = f \right\}. \]

Recall that under the above circumstances, the nonlocal participating states in (1.6), can be viewed as elements of the convex space

\[ u_0 + X_0 \doteq \{ v \in X : v = u_0 + w \text{ where } w \in X_0 \}. \]

**1.2.2. The Local Optimal Control in the Source.** The corresponding local optimal control is a problem denoted by \((P^{loc})\) whose goal is to find \( g \in L^{p'}(\Omega) \) such that minimizes the functional:

\[
I(g, u) = \int_{\Omega} F(x, u(x), g(x)) \, dx + \gamma b_h (u, u), \tag{1.10}
\]

where \( u \in W^{1,p}(\Omega) \) is the solution of the local boundary problem \((P^{loc})\):

\[
\begin{cases}
  b_h(u, w) = \int_{\Omega} g(x) w(x) \, dx, & \text{for any } w \in W^{1,p}_0(\Omega) \\
  u = u_0 & \text{in } \partial\Omega,
\end{cases} \tag{1.11}
\]

\( b(\cdot, \cdot) \) is the operator defined in \( W^{1,p}(\Omega) \times W^{1,p}(\Omega) \) by means of

\[
 b_h(u, v) \doteq \int_{\Omega} h(x) |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) \, dx,
\]

\( h_0 \in \mathcal{H} \) and \( u_0 \) is a given function from the trace fractional Sobolev space \( W^{1-1/p, p} (\partial\Omega) \).

The statement of the local optimal control problem is

\[
\min_{(g, u) \in \mathcal{A}^{loc}} I(g, u), \tag{1.12}
\]

where

\[ \mathcal{A}^{loc} \doteq \left\{ (f, v) \in L^{p'}(\Omega) \times W^{1,p}(\Omega) : v \text{ solves (1.11) with } g = f \right\}. \]
As usual, if we identify \( u_0 \) with a function \( V_0 \in W^{1,p}(\Omega) \) whose trace is \( u_0 \), and at the same time, \( V_0 \) is denoted by \( u_0 \) too, then, as usual, the competing states are those that form space:

\[
u_0 + W^{1,p}_0(\Omega) = \{ v \in W^{1,p}(\Omega) : v = u_0 + w \text{ where } w \in W^{1,p}_0(\Omega) \}.
\]

The analysis of this type of problems is a subject that has been studied extensively in previous works \([2, 23, 24, 27, 30]\). As far as the author knows, the first work dealing with nonlocal optimal control problems is \([22]\). A series of articles containing a different type of controls have appeared in the last years. Some good samples are \([8, 12, 22, 23]\). About the analysis of \( G \)-convergence or \( \Gamma \)-convergence the reader can consult \([11, 22, 36, 37, 45, 52]\). We can find some theoretical advances about the explicit computation of the limit problem. In this sense we must underline among others \([10, 13, 26, 36, 51]\). It is worth noting the influence that this type of problem has received from a list of seminal works coming from the field of the analysis and characterisation of Sobolev Spaces. See for instance \([6, 7, 14, 25, 26, 35, 45]\). In what concerns the numerical analysis of nonlocal problems see \([21]\) and references therein.

We must say that to a great extent, the work \([22]\) has served as inspiration for the present article. Nonetheless, we must emphasize the techniques we use here, in some aspects, substantially differ from the ones employed there. One of the features of our development is the usage of a principle of minimum energy to characterize the \( G \)-convergence of the state equation (see \([31, \text{ Chapter 5, p. 162}]\) for a detailed study in a concrete linear case). Recall that since we are dealing with the exponent \( p > 1 \), the linearity for the \( p \)-laplacian disappears, and consequently, the classical Lax–Milgram Theorem no longer applies. Besides, in \([22]\) this linearity, and the specificity of the type of cost functionals, jointly with the necessary conditions of optimality are the key points for the achievement of existence of optimal controls. By contrast, in our context, the proof of existence, both for the state equation and the optimal control problem, is obtained using the Direct Method and the result of \( G \)-convergence. After, we prove convergence of the nonlocal state equation and the nonlocal optimal control problem to the local ones. Even though these achievements could be significant, since the present analysis can be applied to a rather general class of cost functionals, the results obtained for the particular case \( p = 2 \) are not less attractive. The reason is that, for such a case, the non-local model can approximate classical problems including the squared gradient within the cost functional. Though we have not examined any numerical method for the approximation of solutions yet, some techniques derived from a maximum principle (see \([18]\) for the local case) could be explored to implement a descent method for the case \( p = 2 \) (see \([3]\)).

1.3. Results and Organization

The purpose of this manuscript is twofold: there is a first part of the paper devoted to studying the existence of nonlocal optimal designs. This goal is achieved for a cost functional class whose format may include the non-linear term of the non-local operator. See Theorem 3 in Sect. 3. The proof of this
theorem is basically, based on a previous result of $G$-convergence (Theorem 2 in Sect. 3). The aim of the second part is the convergence of the nonlocal problem toward the local optimal design one. In a first stage we prove convergence of the state equation to the classical $p$-laplacian when the horizon $\delta \to 0$ (Theorem 4 in Sect. 4). Then, we face the study of convergence for the optimal design problem. The main result is Theorem 5, in Sect. 5. A case of particular interest, the one that we assume $p = 2$, is analyzed. The type of cost functional for which we study the convergence, includes the nonlocal gradient, and consequently, the local counterpart optimal problem we approximate contains the square of the gradient (Theorem 6 in Sect. 5). To facilitate the reading of the article, some specific preliminary results are previously explained in Sect. 2. Some compactness and basic inequalities are commented, and the proof of existence of solution for the nonlocal state problem is analyzed (Theorem 1).

2. Preliminary Results and Well-Posedness of the State Equation

2.1. Preliminaries

Here we review some technical tools.

(1) The embedding

$$X_0 \subset L^p (\Omega)$$

is compact. To check that we first notice $X_0 \subset W^{s,p} (\Omega_\delta)$, and since the elements of $X_0$ vanish in $\Omega_\delta \setminus \Omega$, then extension by zero outside $\Omega_\delta$ gives rise to elements of $W^{s,p} (\mathbb{R}^N)$ (see [25, Lemma 5.1]). Then

$$X_0 \subset \tilde{W}^{s,p}_0 (\Omega) \doteq \{ f \in W^{s,p} (\mathbb{R}^N) : f = 0 \text{ in } \mathbb{R}^N \setminus \Omega \}.$$

Besides, since $\Omega$ has Lipschitz boundary, $W^{s,p}_0 (\Omega) = \tilde{W}^{s,p}_0 (\Omega)$. Under the previous circumstances, we are in position to state the following Poincaré’s inequality: there exists a constant $c = c (N, s, p, \Omega) > 0$ such that for any $w \in X_0$

$$c \| w \|^p_{L^p (\Omega)} \leq \int_\Omega \int_\Omega \left| \frac{w(x') - w(x)}{|x' - x|^{N+sp}} \right|^p dx' dx$$

(2.1)

(see [1,25]). By paying attention to the hypotheses on the kernel (1.2), and using (2.1) we conclude there is a positive constant $C$ such that the nonlocal Poincaré inequality:

$$C \| w \|^p_{L^p (\Omega_\delta)} \leq B_h (w, w).$$

(2.2)

holds for any $w \in X_0$.

We consider now a sequence $(w_j)_j \in X_0$ uniformly bounded in $X_0$, that is, there is a constant $C$ such that for every $j$

$$B_h (w_j, w_j) \leq C.$$

(2.3)

By (2.2) $(w_j)_j$ is uniformly bounded $L^p (\Omega_\delta)$ which, jointly with (1.2) guarantees $(w_j)_j$ is uniformly bounded in $W^{s,p}_0 (\Omega_\delta)$. We employ now
the compact embedding $W_0^{s,p} (\Omega_\delta) \subset L^p (\Omega_\delta)$ (see [25, Th. 7.1]) to ensure the existence of a subsequence from $(w_j)_j$, still denoted by $(w_j)_j$, such that $w_j \to w$ strongly in $L^p (\Omega_\delta)$, for some $w \in X_0$. 

(2) If we take a sequence $(w_j)_j$ from $u_0 + X_0$ such that $B_h (w_j, w_j) \leq C$, then $w_j - u_0 \in X_0$. Moreover, since

$$B_h (w_j - u_0, w_j - u_0) \leq c \left( B_h (w_j, w_j) + B_h (u_0, u_0) \right)$$

for a certain constant $c > 0$, then

$$B_h (w_j - u_0, w_j - u_0) \leq C$$

If we apply now the nonlocal Poincaré inequality (2.2), we have

$$C \|w_j - u_0\|_{L^p}^p \leq B_h (w_j - u_0, w_j - u_0), \quad (2.4)$$

whereby we state the sequence $(w_j)_j$ is uniformly bounded in $L^p (\Omega_\delta)$, and therefore, there exists a function $w \in L^p (\Omega)$ such that, for a subsequence of $(w_j)_j$, still denoted by $(w_j)_j$, $w_j \to u$ strongly in $L^p (\Omega_\delta)$. Moreover, since $u_0 + X_0$ is a closed set, $w \in u_0 + X_0$.

(3) Let $(g_\delta, u_\delta)_\delta$ be a sequence of pairs such that the uniform estimate $B_h (u_\delta, u_\delta) \leq C$, is fulfilled (where $C$ is a positive constant). Then, from $(u_\delta)_\delta$ we can extract a subsequence, labelled also by $u_\delta$, such that $u_\delta \to u$ strongly in $L^p (\Omega)$ and $u \in W^{1,p} (\Omega)$ (see [44, Th. 1.2]). Furthermore, the following inequality is fulfilled

$$\lim_{\delta \to 0} B_h (u_\delta, u_\delta) \geq \int_{\Omega} h (x) |\nabla u (x)|^p \, dx \quad (2.5)$$

(see [5,41,44]). Besides, it is also well-known that if $u_\delta = u \in W^{1,p} (\Omega)$, then the above limit is

$$\lim_{\delta \to 0} B_h (u, u) = \int_{\Omega} h (x) |\nabla u (x)|^p \, dx \quad (2.6)$$

(see [14, Corollary 1] and [4, Th. 8]).

2.2. The State Equation

For the well-posedness of the nonlocal control problem $(P_\delta)$ it is imperative to prove existence and uniqueness for the nonlocal boundary problem $(P^\delta)$. A remarkable fact that will be employed for this goal is the characterization of (1.1) by means of a Dirichlet principle. For the proof we just need to adapt (because we have to include the nonlocal boundary condition $u_0$) the lines given in [13].

Throughout this section, $u_0 \in \tilde{X}_0$, $\delta > 0$ and $g \in L^{p'} (\Omega)$ are assumed to be fixed. We seek a solution to the problem (1.6) and as we have commented, the crucial point in this searching, is the inherent relation of the nonlocal boundary problem with the following minimization problem:

$$\min_{w \in u_0 + X_0} \mathcal{J} (w) \quad (2.7)$$
where
\[ J(w) = \frac{1}{p} B_h(w, w) - \int_\Omega g(x) w(x) \, dx. \]

**Lemma 1.** There exists a solution \( u \in u_0 + X_0 \), to the minimization problem (2.7).

**Proof.** First of all, we check \( J \) is bounded from below. Let \( w \) be any function from \( X \) such that \( w - u_0 \in X_0 \). Using the nonlocal Poincaré inequality (2.4) there is a constant \( c > 0 \) such that
\[ c \| w - u_0 \|^p_{L^p} \leq B_h(w, w) + B_h(u_0, u_0), \]
whence we have
\[ \| w \|^p_{L^p} - \| u_0 \|^p_{L^p} \leq \| w - u_0 \|^p_{L^p} \leq \left( \frac{B_h(w, w) + B_h(u_0, u_0)}{c} \right)^{1/p}. \tag{2.8} \]

If we apply now the Hölder’s inequality and Young’s inequality, we get
\[
J(w) \geq \frac{1}{p} B_h(w, w) - \| g \|_{L^{p'}} \| w \|_{L^p} \\
\geq \frac{1}{p} B_h(w, w) - \| g \|_{L^{p'}} \left( \frac{1}{c^{1/p}} \right) (B_h(w, w) + B_h(u_0, u_0))^{1/p} - \| g \|_{L^{p'}} \| u_0 \|_{L^p} \\
\geq \frac{1}{p} B_h(w, w) - \frac{1}{p} (B_h(w, w) + B_h(u_0, u_0)) - \frac{1}{p} \left( \frac{\| g \|_{L^{p'}}}{c^{1/p}} \right)^{p'} - \| g \|_{L^{p'}} \| u_0 \|_{L^p} \\
= -\frac{1}{p} B_h(u_0, u_0) - \frac{1}{p} \left( \frac{\| g \|_{L^{p'}}}{c^{1/p}} \right)^{p'} - \| g \|_{L^{p'}} \| u_0 \|_{L^p}.
\]

To prove the existence of solution, we take a minimizing sequence \((u_j) \subset u_0 + X_0\) so that
\[ m = \lim_{j \to \infty} \left( \frac{1}{p} B_h(u_j, u_j) - \int_\Omega g(x) u_j(x) \, dx \right) \tag{2.9} \]
where \( m \) is the infimum \( \inf_{w \in u_0 + X_0} J(w) \). From this convergence, we ensure that there is a constant \( C > 0 \) such that
\[ \frac{1}{p} B_h(u_j, u_j) - \int_\Omega g(x) u_j(x) \, dx \leq C \]
for any \( j \). Thus we get
\[ 0 \leq B_h(u_j, u_j) \leq \left( 1 + \left| \int_\Omega g(x) u_j(x) \, dx \right| \right) \]
\[ \leq C \left( 1 + \| g \|_{L^2(\Omega)} \| u_j \|_{L^2(\Omega)} \right) \]
with \( C > 0 \). Again, the nonlocal Poincaré inequality gives
\[ c \| u_j - u_0 \|^p_{L^p} \leq B_h(u_j, u_j) + B_h(u_0, u_0) \leq C \left( 1 + \| g \|_{L^{p'}(\Omega)} \| u_j \|_{L^{p'}(\Omega)} \right) + B_h(u_0, u_0) \]
and, therefore
\[ \| u_j \|^p_{L^p} \leq C \left( 1 + \| g \|_{L^{p'}(\Omega)} \| u_j \|_{L^{p'}(\Omega)} + \| u_0 \|^p_{L^p} \right) \]
From the two above inequalities we deduce the sequences $B_h(u_j, u_j)$ and $\|u_j\|_{L^p}$ are uniformly bounded. By virtue of the compactness embedding $X_0 \subset L^p$, we know there is a subsequence of $(u_j)$, which will be denoted also by $(u_j)$, strongly convergent in $L^p(\Omega)$ to some $u \in u_0 + X_0$.

We retake (2.9) and use the lower semicontinuity in $L^p$ of the operator $J$ to write

$$m = \lim_j \frac{1}{p} B_h(u_j, u_j) - \lim_j \int_\Omega g(x) u_j(x) \, dx$$

$$\geq \frac{1}{p} B_h(u, u) - \int_\Omega g(x) u(x) \, dx$$

$$= J(u).$$

From this inequality, we conclude that $u$ is a minimizer. □

**Lemma 2.** $u$ is a solution of the minimization principle (2.7) if, and only if, $u$ solves the problem (1.6).

The proof is standard. Assume $u$ solves (1.6). We have only note that if we take any $v \in u_0 + X_0$ then $w = u - v \in X_0$ and $B_h(u, w) = \int_\Omega g(x) w(x) \, dx$, that is

$$B_h(u, u) = B_h(u, v) + \int_\Omega g(u - v) \, dx.$$

By applying Young’s inequality to the first term of the right part in the above equality, we get

$$B_h(u, u) \leq \frac{1}{p} B_h(v, v) + \frac{1}{p} B_h(u, u) + \int_\Omega g(u - v) \, dx,$$

and thus

$$\frac{1}{p} B_h(u, u) - \int_\Omega gudx \leq \frac{1}{p} B_h(v, v) - \int_\Omega gudx$$

which is equivalent to write $J(u) \leq J(v)$.

And reciprocally, if $u$ is a minimizer of $J$ on $u_0 + X_0$ then we can take the admissible function $w = u + t\xi$, where $\xi$ is any element form $X_0$. Since the function $j(t) = J(u + t\xi)$ attains a minimum at $t = 0$, then $j'(0) = 0$ and this equality can be easily rewritten as $B_h(u, \xi) = \int_\Omega g(x) \xi(x) \, dx$.

**Theorem 1.** There exists a unique solution to the nonlocal boundary problem $(P^\delta)$ given in (1.6).

All that remains is to prove the uniqueness. The proof is automatic due to the convexity of $J$: indeed, if $u$ and $v$ are minimizers, then

$$m = \min_{w \in u_0 + X_0} J(w) = J(u) = J(v).$$

Besides, for any $\alpha \in (0, 1)$, the function $\alpha u + (1 - \alpha) v$ is admissible for minimization principle. Then, thanks to the strict convexity of $J$, we deduce that

$$m \leq J(\alpha u + (1 - \alpha) v) < \alpha J(u) + (1 - \alpha) J(v) = m$$

which is a contradiction.
Remark 1. The result remains valid if we assume \( g \) to be in the space \( X_0' \) and the proof follows along the same lines from above.

The existence and uniqueness of solution for the local state equation \((P^{loc})\) is a basic issue. Even we have to adapt some details, [19] is a reference we can follow in order to carry out this task. For clarity in the exposition of the text, we recall the proof of the uniqueness here. If we assume that \( u \) and \( v \) are two different solutions to the state equation (1.11), then \( b_h (u, w) = b(v, w) \) for any \( w \in W_0^{1,p} \). This is equivalent to say that for any \( w \in W_1^{1,p} \):

\[
\int_{\Omega} h(x) \left( |\nabla u(x)|^{p-2} \nabla u(x) \nabla w(x) - |\nabla v(x)|^{p-2} \nabla v(x) \nabla w(x) \right) \, dx = 0.
\]

(2.10)

By taking \( w = u - v \), we obtain

\[
\int_{\Omega} h(x) \left( |\nabla u(x)|^{p-2} \nabla u(x) - |\nabla v(x)|^{p-2} \nabla v(x) \right) \nabla (u - v)(x) \, dx = 0.
\]

At this point, we take into account the next elementary inequality: if \( 1 < p < \infty \), then there exist two positive constants \( C = C(p) \) and \( c = c(p) \) such that for every \( a, b \in \mathbb{R}^N \)

\[
c \{ |a| + |b| \}^{p-2} |a - b|^2 \leq \left( |a|^{p-2} a - |b|^{p-2} b \right) \cdot (a - b)
\]

\[
\leq C \{ |a| + |b| \}^{p-2} |a - b|^2. \tag{2.11}
\]

Finally, by applying (2.11) in (2.10) the uniqueness follows (see [19, Prop.17.3 and Th. 17.1]).

3. \( G \)-Convergence for the State Equation and Existence of Nonlocal Optimal Controls

Let \((g_j)_j\) be a minimizing sequences of controls for the problem \((P^\delta)\) and let \((u_j)_j\) be the corresponding sequence of states. As in the end, the sequences we are going to work with, are minimizing sequences, we shall assume that there is a constant \( C > 0 \) such that for any \( j \):

\[
\int_{\Omega} |g_j(x)|^{p'} \, dx < C.
\]

Hence, we can extract a subsequence weakly convergent in \( L^{p'}(\Omega) \) to some \( g \in L^{p'}(\Omega) \). We also know the following variational equality for any \( v \in X_0 \):

\[
B_h (u_j, v) = \int_{\Omega} g_j(x) \, v(x) \, dx
\]

In particular

\[
B_h (u_j, u_j - u_0) = \int_{\Omega} g_j(x) \, (u_j(x) - u_0(x)) \, dx.
\]

Holder’s inequality and the linearity of \( B_h (w, \cdot) \), for any \( w \in X \), lead us the estimate

\[
B_h (u_j, u_j) \leq \|g_j\|_{L^{p'}} \left( \|u_j\|_{L^p} + \|u_0\|_{L^p} \right) + B_h (u_j, u_0).
\]
If we take into account (2.8) and make use of the Young’s inequality, we deduce

\[
B_h(u_j, u_j) \leq \|g_j\|_{L^{p'}} \left( \left( \frac{B_h(u_j, u_j) + B_h(u_0, u_0)}{c} \right)^{1/p} + 2\|u_0\|_{L^p} \right)
\]

and thereby

\[
\left( 1 - \frac{1}{p'} \right) B_h(u_j, u_j) \leq C + D (B_h(u_j, u_j))^{1/p}
\]

for some positive constants C and D. The above inequality implies \(B_h(u_j, u_j)\) is uniformly bounded and by (2.8) \(\|u_j\|_{L^p}\) too. If at this point we use point 2 from Sect. 2.1, we can state the strong convergence in \(L^p\), at least for a subsequence of \((u_j)_j\), to some function \(u^* \in u_0 + X_0\). Let \(u\) be the state associated to \(g\). We pose whether the identity \(u = u^*\) is true or not:

**Theorem 2.** (\(G\)-convergence) Under the above circumstances

\[
\lim_{j \to \infty} \min_{w \in u_0 + X_0} \left\{ \frac{1}{p} B_h(w, w) - \int_{\Omega} g_j(x) w(x) \, dx \right\}
\]

\[
= \min_{w \in u_0 + X_0} \left\{ \frac{1}{p} B_h(w, w) - \int_{\Omega} g(x) w(x) \, dx \right\}
\]

and \(u = u^*\).

**Proof.** Assume \(m_j\) and \(m\) denote the minimum values from the left and right, respectively. We prove \(\lim_j m_j \leq m\):

\[
\lim_j m_j = \lim_j \left( \frac{1}{p} B_h(u_j, u_j) - \int_{\Omega} g_j(x) u_j(x) \, dx \right)
\]

\[
\leq \lim_j \left( \frac{1}{p} B_h(u, u) - \int_{\Omega} g_j(x) u(x) \, dx \right)
\]

\[
= \frac{1}{p} B_h(u, u) - \int_{\Omega} g(x) u(x) \, dx
\]

\[
= \min_{w \in u_0 + X_0} \left\{ \frac{1}{p} B_h(w, w) - \int_{\Omega} g(x) w(x) \, dx \right\}.
\]

We check \(\lim_j m_j \geq m\) : we know \(u_j \to u^*\) strongly in \(L^p\), \(g_j \rightharpoonup g\) weakly in \(L^{p'}\) and, therefore

\[
\lim_j \int_{\Omega} g_j(x) u_j(x) \, dx = \int_{\Omega} g(x) u^*(x) \, dx.
\]

We apply these convergences to analyze the limit of the energy functional:

\[
\lim_j m_j = \lim_j \left( \frac{1}{p} B_h(u_j, u_j) - \int_{\Omega} g_j(x) u_j(x) \, dx \right)
\]

\[
= \frac{1}{p} \lim_j B_h(u_j, u_j) - \int_{\Omega} g(x) u^*(x) \, dx
\]
\[
\geq \frac{1}{p} B_h (u^*, u^*) - \int_{\Omega} g (x) u^* (x) \, dx
\]
\[
\geq \frac{1}{p} B_h (u, u) - \int_{\Omega} g (x) u (x) \, dx
\]
\[
= \min_{w \in u_0 + X_0} \left\{ \frac{1}{2} B_h (w, w) - \int_{\Omega} g (x) w (x) \, dx \right\}
\]
where the first inequality is due to the lower semicontinuity of the operator \( B_h (\cdot, \cdot) \) with respect to the weak convergence in \( L^p \). We have proved \( \lim_{j \rightarrow \infty} m_j = m \). Also, from the above chain of inequalities it is obvious to see that both \( u \) and \( u^* \) are solutions to the problem (2.7), then according to Theorem 1 \( u = u^* \).

**Corollary 1.** The following convergences hold
\[
\lim_{j \rightarrow \infty} B_h (u_j, u_j) = B_h (u, u),
\]
and
\[
\lim_{j \rightarrow \infty} B_h (u_j - u, u_j - u) = 0.
\]

**Proof.** (3.1) follows from the proof of the above theorem and can be rewritten as this convergence of norms:
\[
\lim_{j \rightarrow \infty} \int_{\Omega_\delta} \int_{\Omega_\delta} H (x', x) k_\delta (|x' - x|) \frac{|u_j (x') - u_j (x)|^p}{|x' - x|^p} \, dx' \, dx = \int_{\Omega_\delta} \int_{\Omega_\delta} H (x', x) k_\delta (|x' - x|) \frac{|u (x') - u (x)|^p}{|x' - x|^p} \, dx' \, dx.
\]
But this convergence is equivalent to say that the norm in \( L^p (\Omega_\delta \times \Omega_\delta) \) of the sequence
\[
\Psi_j (x', x) = H^{1/p} (x', x) k_\delta^{1/p} (|x' - x|) \frac{(u_j (x') - u_j (x))}{|x' - x|}
\]
converges to the norm of the function
\[
\Psi (x', x) = H^{1/p} (x', x) k_\delta^{1/p} (|x' - x|) \frac{(u (x') - u (x))}{|x' - x|}.
\]
Since, additionally, up to a subsequence, \( (\Psi_j) \) converges pointwise a.e. \( (x', x) \in \Omega_\delta \times \Omega_\delta \) to \( \Psi \), then \( \Psi_j \) strongly converges to \( \Psi (x', x) \) in \( L^p (\Omega_\delta \times \Omega_\delta) \) (see [46, P. 78]) and (3.2) has been proved.

**Remark 2.** The convergence (3.2), together with the strong convergence of \( (u_j) \) in \( L^p (\Omega_\delta) \) is equivalent to the strong convergence in \( X \).

We also realize that for any \( h_0 \in H \), we have
\[
\lim_{j \rightarrow \infty} B_h (u_j - u, u_j - u) = \lim_{j \rightarrow \infty} B_{h_0 h_0} (u_j - u, u_j - u)
\]
\[
\geq \frac{h_{\min}}{h_{\max}} \lim_{j \rightarrow \infty} B_{h_0} (u_j - u, u_j - u).
\]
The checking of that follows from the expression we have for the operator $B_h(\cdot,\cdot)$: for any $w \in X$

$$B_h(w,w) = \int_{\Omega} h(x) \int_{B(x,\delta)} k_\delta(|x'-x|) \frac{|w(x') - w(x)|^p}{|x'-x|^p} dx' dx.$$ 

Consequently, from (3.2) we deduce $\lim_{j \to \infty} B_{h_0}(u_j - u, u_j - u) = 0$, for any $h_0 \in \mathcal{H}$ and thereby

$$\lim_{j \to \infty} B_{h_0}(u_j, u_j) = B_{h_0}(u, u). \quad (3.3)$$

The convergences of the states that we have just described above, are still valid if we consider a sequence of sources $(g_j)_j$ uniformly bounded in the dual space $X'_0$. Since $X_0$ is reflexive, $X'_0$ too, and we can ensure the sequence $(u_j)_j$ is weakly convergent, up to a subsequence, to an operator $g \in X'_0$. Let $(u_j)_j$ and $u$ the underlying states of $(g_j)_j$ and $g$, respectively. Then, thanks to the precedent analysis, we know the sequence $(u_j)_j$, the states associated to the controls $(g_j)_j$, converges weakly to $u$ in $L^p \supset X$, where $u$ is the stated associated to $g$. Take now any element $L \in X'_0$. Under these circumstances there exists a function $u_L \in u_0 + X_0$ such that $B_h(u_L, w) = \langle L, w \rangle_{X'_0 \times X}$ for any $w \in X_0$. Then, we easily deduce

$$\lim_{j \to \infty} \langle L, u_j - u_0 \rangle_{X'_0 \times X_0} = \langle L, u - u_0 \rangle_{X'_0 \times X_0}$$

The explanation of that relies on the strong convergence achieved in the above corollary: indeed, Hölder’s inequality and (3.2) straightforwardly provide

$$\lim_{j \to \infty} \left| \langle L, u_j - u_0 \rangle_{X'_0 \times X_0} - \langle L, u - u_0 \rangle_{X'_0 \times X_0} \right| = \lim_{j \to \infty} |B_h(u_L, u_j - u)|$$

$$\leq \lim_{j \to \infty} B_h^{1/p'}(u_L, u_L) B_h^{1/p}(u_j - u, u_j - u) = 0.$$

The analysis performed explicitly confirm the fact that the sequence $(u_j)_j$ is weakly convergent to $u$ in $X_0$, or in other words, the sequence of problems:

$$\min_{w \in u_0 + X_0} \left\{ \frac{1}{p} B_h(w, w) - \int_{\Omega} g_j(x) w(x) dx \right\}$$

$G$-converges to the problem:

$$\min_{w \in u_0 + X_0} \left\{ \frac{1}{p} B_h(w, w) - \int_{\Omega} g(x) w(x) dx \right\}$$

(see the abstract energy criterion established in [31, Chapter 5, p. 162]).

**Theorem 3.** (Well posedness) There exists a solution $(g,u)$ to the control problem $(P_\delta)$ given at (1.9).
Proof. Let \((g_j, u_j)\) be a minimizing sequence. Then, up to subsequence, we know that \(g_j \rightharpoonup g\) weakly in \(L^{p'}\) and \(u_j \rightarrow u\) strongly in \(L^p\). In addition, by Theorem 2 the couple \((g, u)\) is admissible for the control problem, that is \((g, u) \in \mathcal{A}^\delta\). Factually, this couple is a minimizer of the problem. To check that we observe the infimum \(i\) of the minimization principle can be computed as

\[
i = \lim_j I(g_j, u_j) = \lim_{j \to \infty} \int_\Omega F(x, u_j(x), g_j(x)) \, dx.
\]

If we take into account the properties of \(G\) is straightforward to verify the that

\[
\lim_{j \to \infty} \int_\Omega G(x, u_j(x)) \, dx = \int_\Omega G(x, u(x)) \, dx \quad (3.4)
\]

In addition, using the Fatou’s Lemma and the convergence (3.3), it is automatic to check that

\[
i \geq \liminf_{j \to \infty} \int_\Omega F(x, u_j(x), g_j(x)) \, dx + \gamma B_{h_0}(u_j, u_j)
\]

\[
\geq \int_\Omega F(x, u(x), g(x)) \, dx + \gamma B_{h_0}(u, u)
\]

\[
= I(g, u).
\]

The above inequality implies that \((g, u)\) is a minimizer. \(\square\)

Remark 3. If \(p = 2\) and \(G(x, \cdot)\) is convex, then the solution of (1.9) is unique. The uniqueness is guaranteed because of to the strict convexity of the function \(t \rightarrow |t|^p\) and the linearity of the state equation: if there are two different solutions \((g, u)\) and \((f, v)\), then the stated associated with the source \(y_s(x) = sg(x) + (1-s)f(x)\) \((s \in (0,1))\) is \(u_s(x) = su(x) + (1-s)v(x)\). If we apply the above properties of convexity, and the one of the operator \(B_{h_0}\) as well, then we arrive at

\[
J_\delta(y_s, u_s) = \int_\Omega F(x, u_s(x), y_s(x)) \, dx
\]

\[
= \int_\Omega \left( G(x, u_s(x)) + \beta |y_s(x)|^{p'} \right) \, dx + \gamma B_{h_0}(u_s, u_s)
\]

\[
< s \int_\Omega \left( G(x, u(x)) + \beta |g(x)|^{p'} \right) \, dx
\]

\[
+ (1-s) \int_\Omega \left( G(x, v(x)) + \beta |f(x)|^{p'} \right) \, dx + \gamma s B_{h_0}(u, u) + \gamma (1-s) B_{h_0}(v, v)
\]

\[
= s J_\delta(g, u) + (1-s) J_\delta(f, v),
\]

which is contradictory, because both \((g, u)\) and \((f, v)\) are minimizers of \(J_\delta\).
4. Convergence of the State Equation If $\delta \to 0$

Assume the source $g \in L^{p'}(\Omega)$ and $u_0 \in W^{1-1/p,p}(\partial\Omega)$ are fixed functions. If, for each $\delta$, we consider the corresponding sequence of states $(u_\delta)_\delta \subset X_0$, then

$$B_h (u_\delta, v) = \int_{\Omega} g(x) v(x) \, dx$$

for any $v \in X_0$. Consequently, as in the previous sections, we easily prove $\|u_\delta\|_{L^p(\Omega)}$ and $B_h (u_\delta, u_\delta)$ are sequences uniformly bounded in $\delta$. Then, using part 3 from Sect. 2.1, these estimates imply the existence of a function $u^* \in u_0 + W^{1,p}_0(\Omega)$ and a subsequence of $u_\delta$ (still denoted $u_\delta$), such that $u_\delta \to u^*$ strongly in $L^p(\Omega)$. Now, we look for the state equation that should be satisfied by the pair $(g, u^*)$. The answer to this question is given in the following convergence result:

**Theorem 4.**

$$\lim_{\delta \to 0} \min_{w \in u_0 + X_0} \left\{ \frac{1}{p} B_h (w, w) \right\} \geq \int_{\Omega} h(x) |\nabla u^*(x)|^p \, dx - \int_{\Omega} g(x) u^*(x) \, dx$$

(4.1)

and $(g, u^*) \in A^{loc}$.

**Proof.** We define the local state $u$ that corresponds to the source $g$ by means of the local problem:

$$\min_{w \in u_0 + W^{1,p}_0(\Omega)} \left\{ \frac{1}{p} b_h (w, w) - \int_{\Omega} g(x) w(x) \, dx \right\}$$

If $u_\delta$ is the solution to the nonlocal state equation, then $u_\delta$ solves

$$\min_{w \in u_0 + X_0} \left\{ \frac{1}{p} B_h (w, w) - \int_{\Omega} g(x) w(x) \, dx \right\}$$

and from $(u_\delta)_\delta$ we can extract a subsequence strongly convergent to $u^* \in u_0 + W^{1,p}_0(\Omega)$ in $L^p$. By means of

$$\lim_{\delta \to 0} B_h (u_\delta, u_\delta) \geq \int_{\Omega} h(x) |\nabla u^*(x)|^p \, dx,$$

(see (2.5)) we are allowed to write

\[
\lim_{\delta \to 0} \min_{w \in u_0 + X_0} \left\{ \frac{1}{p} B_h (w, w) - \int_{\Omega} g(x) w(x) \, dx \right\} = \lim_{\delta \to 0} \left( \frac{1}{p} B_h (u_\delta, u_\delta) - \int_{\Omega} g(x) u_\delta(x) \, dx \right) \\
\geq \frac{1}{p} \int_{\Omega} h(x) |\nabla u^*(x)|^p \, dx - \int_{\Omega} g(x) u^*(x) \, dx \\
\geq \frac{1}{p} \int_{\Omega} h(x) |\nabla u(x)|^p \, dx - \int_{\Omega} g(x) u(x) \, dx
\]
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\[ = \min_{w \in u_0 + W^{1,-p}_0(\Omega)} \left\{ \frac{1}{p} \int_{\Omega} h(x) |\nabla w(x)|^p \, dx - \int_{\Omega} g(x) \, w(x) \, dx \right\}. \]

We prove the reverse inequality: it suffices the usage of the convergence given at (2.6) to realize that

\[
\lim_{\delta \to 0} \min_{w \in u_0 + X_0} \left\{ \frac{1}{p} B_h(w, w) - \int_{\Omega} g(x) \, w(x) \, dx \right\} = \lim_{\delta \to 0} \left( \frac{1}{p} B_h(u_\delta, u_\delta) - \int_{\Omega} g(x) \, u_\delta(x) \, dx \right) \leq \lim_{\delta \to 0} \left( \frac{1}{p} B_h(u, u) - \int_{\Omega} g(x) \, u(x) \, dx \right) = \frac{1}{p} b_h(u, u) - \int_{\Omega} g(x) \, u(x) \, dx = \min_{w \in u_0 + W^{1,-p}_0(\Omega)} \left\{ \frac{1}{p} \int_{\Omega} h(x) |\nabla w(x)|^p \, dx - \int_{\Omega} g(x) \, w(x) \, dx \right\}.
\]

The above two estimates amount to state two consequences: on the one side, these estimates clearly give the convergence result (4.1). On the other side, from the above discussion, it can be read that both \( u \) and \( u^* \) are solutions to the classical boundary problem (1.11), and thus, by the uniqueness proved for this problem, we deduce \( u = u^* \). \( \square \)

The proof we have just done provides the convergence of energies

\[ \lim_{\delta \to 0} B_h(u_\delta, u_\delta) = b_h(u, u). \] (4.2)

If we use (2.6) the above limit can be rewritten as follows:

\[
\lim_{\delta \to 0} \int_{\Omega_\delta} \int_{\Omega_\delta} \left( \frac{|u_\delta(x') - u_\delta(x)|^p}{|x' - x|^p} - \frac{|u(x') - u(x)|^p}{|x' - x|^p} \right) \, dx' \, dx = 0. \]

Moreover, for the particular case \( p = 2 \), we have strong convergence in \( X_0 \) in the sense that

\[ \lim_{\delta \to 0} B_h(u_\delta - u, u_\delta - u) = 0 \] (4.4)

(the proof is automatic). Furthermore, if \( p = 2 \) and \( h_0 \) is any function from \( \mathcal{H} \), then

\[ \lim_{\delta \to 0} B_{h_0}(u_\delta - u, u_\delta - u) = 0 \]

and by (2.6),

\[ \lim_{\delta \to 0} B_{h_0}(u_\delta, u_\delta) = \lim_{\delta \to 0} B_{h_0}(u, u) = b_{h_0}(u, u). \] (4.5)
5. Approximation to the Optimal Control Problem

We know that, for each \(\delta\), there exists at least a minimizer \((g_\delta, u_\delta)\) to the problem (1.9). Our purpose is the asymptotic analysis of this sequence of minimizers. We shall prove that the limit in \(\delta\) of the sequence \((g_\delta, u_\delta)\), is the pair \((g, u)\) derived at the previous section. We also prove \((g, u)\) solves the corresponding local optimal control problem \((\mathcal{P}^\text{loc})\) defined in (1.12). The tools we use are nothing more than those used in Theorem 4.

**Theorem 5.** Let \((g_\delta, u_\delta)\) be the sequence of solutions to the control problem (1.9). Then there exists a pair \((g, u)\in\mathcal{A}^\delta\) and a subsequence of indexes \(\delta\) for which the following conditions hold:

1. \(g_\delta \to g\) weakly in \(L^p'(\Omega)\), \(u_\delta \to u\) strongly in \(L^p(\Omega)\) as \(\delta \to 0\).
2. \[
\lim_{\delta \to 0} \min_{w \in u_0 + X_0} \left\{ \frac{1}{p} B_h (w, w) - \int_\Omega g_\delta (x) w(x) \, dx \right\} = \min_{w \in u_0 + W^{1,p}_0(\Omega)} \left\{ \frac{1}{p} b_h (w, w) - \int_\Omega g (x) w(x) \, dx \right\}
\]
   and \((g, u)\in\mathcal{A}^\text{loc}\).
3. \((g, u)\) is a solution to the local control problem (1.12).
4. \(g_\delta \to g\) strongly in \(L^p'(\Omega)\) and
   \[
   \lim_{\delta \to 0} I_{\delta} (g_\delta, u_\delta) = I (g, u).
   \]

**Proof.**

(1) Let \((g_\delta, u_\delta)\) any sequence of minimizers of the problem \(\mathcal{P}^\delta\). We can extract a subsequence from \((g_\delta)\), weakly convergent to some function \(g\) in \(L^p'\). Indeed, if we fix any \(f \in L^p'(\Omega)\), then the optimality of \((g_\delta, u_\delta)\) ensures \(I_{\delta} (g_\delta, u_\delta) \leq I_{\delta} (f, u_\delta^*)\) for any \(\delta\), where \((f, u_\delta^*) \in \mathcal{A}^\delta\), that is
   \[
   \int_\Omega \left( G (x, u_\delta^*) + \beta |g_\delta|^p' \right) \, dx \leq \int_\Omega \left( G (x, u_\delta^*) + \beta |f|^p' \right) \, dx
   \]
   Since for a subsequence of \(\delta'\)'s, \(u_\delta^* \to u^* \in u_0 + W^{1,p}_0(\Omega)\) strongly in \(L^p\) if \(\delta \to 0\), then the right term of the above inequality is uniformly bounded in \(\delta\), and consequently, \(\int_\Omega |g_\delta|^p' \, dx\) is uniformly bounded too. Also, from the associated states \((u_\delta)\) we extract a subsequence that converges, strongly in \(L^p\), to a function \(u^* \in u_0 + W^{1,p}_0(\Omega)\). See Sect. 2.1 part 3.

(2) We are going to see the state function \(u^*\) is the one that corresponds to the control \(g\):
   Let \(u\) be the underlying local state of \(g\). Then, on the one side (2.5) allows us to write
   \[
   \lim_{\delta \to 0} \min_{w \in u_0 + X_0} \left\{ \frac{1}{p} B_h (w, w) - \int_\Omega g_\delta (x) w(x) \, dx \right\} = \lim_{\delta \to 0} \left( \frac{1}{p} B_h (u_\delta, u_\delta) - \int_\Omega g_\delta (x) u_\delta (x) \, dx \right)
   \]
\[ \geq \left( \frac{1}{p} b_h(u^*, u^*) - \int_{\Omega} g(x) u^*(x) \, dx \right) \]

On the other side, it is clear that (2.6) allows us to write

\[ \lim_{\delta \to 0} \min_{w \in u_0 + X_0} \left\{ \frac{1}{p} B_h(w, w) - \int_{\Omega} g_\delta(x) w(x) \, dx \right\} \]

\[ \leq \lim_{\delta \to 0} \left( \frac{1}{p} B_h(u, u) - \int_{\Omega} g(x) u(x) \, dx \right) \]

\[ = \frac{1}{p} b_h(u, u) - \int_{\Omega} g(x) u(x) \, dx \]

\[ = \min_{w \in u_0 + W_0^{1,p}(\Omega)} \left\{ \frac{1}{p} b_h(w, w) - \int_{\Omega} g(x) w(x) \, dx \right\}. \]

From the above lines we infer that both \( u \) and \( u^* \) are solutions to the local state problem (1.11). Thereby, if we use uniqueness, which have been checked at the end of Sect. 2, we deduce that \( u = u^* \) and consequently, that \( (g, u) \in A^{loc} \). Furthermore, from the above chain of equalities we derive the following convergence of energies:

\[ \lim_{\delta \to 0} B_h(u_\delta, u_\delta) = b_h(u, u). \]

(3) Take any \( (f, v) \in A^{loc} \) and consider the sequence of solutions \( (f, v_\delta) \) of the nonlocal boundary problem \((P_\delta)\) with \( g = f \). Since \((f, v_\delta) \in A^\delta\) then

\[ I(f, v) = \lim_{\delta \to 0} I_\delta(f, v_\delta) \geq \lim_{\delta \to 0} I_\delta(g_\delta, u_\delta) \geq I(g, u). \]

To prove that we notice that the first equality of (5.3) is true, because according to Theorem 4 \( v_\delta \to v \in u_0 + W_0^{1,p}(\Omega) \) strongly in \( L^p \) and \((f, v) \in A^{loc}\). If we use now the latter strong convergence and we pay attention to the convergence (3.4), then thanks to (4.2) it is straightforward to deduce

\[ \lim_{\delta \to 0} I_\delta(f, v_\delta) = \lim_{\delta \to 0} \int_{\Omega} \left( G(x, v_\delta(x)) + \beta |f(x)|^{p'} \right) \, dx \]

\[ \leq \lim_{\delta \to 0} \int_{\Omega} \left( G(x, v_\delta(x)) + \beta |f(x)|^{p'} \right) \, dx \]

\[ = \int_{\Omega} \left( G(x, v(x)) + \beta |f(x)|^{p'} \right) \, dx \]

\[ = I(f, v). \]

The first inequality of (5.3) is due to the fact that \((g_\delta, u_\delta)\) is a sequence of minimizers for the cost \( I_\delta \). The second inequality of (5.3) holds, because (2.5) and Fatou’s Lemma yield

\[ \lim_{\delta \to 0} I_\delta(g_\delta, u_\delta) \geq \lim_{\delta \to 0} \inf \int_{\Omega} \left( G(x, u_\delta(x)) + \beta |g_\delta(x)|^{p'} \right) \, dx \]
\[
\geq \int_{\Omega} \left( G(x, u(x)) + \beta |g(x)|^{p'} \right) dx = I(g, u).
\]

(4) Again, the key strategy is to take the pair \((g, u^{\star}_\delta) \in A^\delta\), take into account \(I^\delta(g, u^{\star}_\delta) \leq I^\delta(g, u^\ast_\delta)\) and pass to the limit in this inequality, to conclude that \(\lim_{\delta} I^\delta(f, v_\delta) \leq I(g, u)\). Since the reverse inequality holds, then (5.2) is proved. By applying this inequality we straightforwardly derive \((\|u^\delta\|_{L^{p'}})_\delta\) converges to \(\|u\|_{L^{p'}}\), which serves to confirm the strong convergence of \((u^\delta)_\delta\) in \(L^{p'}\).

\[\square\]

5.1. Case \(p = 2\)

The thesis of Theorem 5 remains true when we put \(\gamma > 0\) and \(p = 2\). To prove this statement we previously define the concrete optimization problems we have to face: the nonlocal control problem \((P^\delta)\) read as

\[
\min_{(g, u) \in A^\delta} J^\delta(g, u)
\]

where

\[
J^\delta(g, u) = \int_{\Omega} F(x, u(x), g(x)) \, dx + \gamma B_{h_0}(u, u),
\]

\(B_{h_0}(\cdot, \cdot)\) is defined as in (1.7) with \(h = h_0\) and \(p = 2\). The admissibility set is

\[
A^\delta = \{(f, v) \in L^2(\Omega) \times X : v \text{ solves } (1.6) \text{ with } g = f\},
\]

where (1.6) has also to considered for the specific case \(p = 2\). It must be underlined that for each \(\delta\), there is a solution there is at least a solution \((g^\delta, u^\delta) \in L^2(\Omega) \times (u_0 + H^1_0(\Omega))\) to the problem (5.4).

The corresponding local control problem \((P^{\text{loc}})\) is stated as

\[
\min_{(g, u) \in A^{\text{loc}}} J(g, u)
\]

where

\[
J(g, u) = \int_{\Omega} F(x, u(x), g(x)) \, dx + \gamma \int_{\Omega} h_0(x) |\nabla u(x)|^2 \, dx,
\]

with

\[
A^{\text{loc}} = \{(f, v) \in L^2(\Omega) \times W^{1,2}(\Omega) : v \text{ solves } (1.11) \text{ with } g = f\},
\]

and (1.11) is assumed to be constrained to the case \(p = 2\).

Theorem 6. Let \((g^\delta, u^\delta)\) be a sequence of solutions to the problem (5.4). Then there exists a pair \((g, u) \in A^{\text{loc}}\) and a subsequence of indexes \(\delta\) for which the following conditions hold:

1. \(g^\delta \rightharpoonup g\) weakly in \(L^2(\Omega)\) and \(u^\delta \rightarrow u\) strongly in \(L^2(\Omega)\) if \(\delta \rightarrow 0\).
2. The identity (5.1) holds and \((g, u) \in A^{\text{loc}}\).
3. \((g, u)\) is a solution to the local control problem (5.5). If in addition \(G(x, \cdot)\) is assumed to be convex, then the solution \((g, u)\) is unique.
(4) \( g_\delta \to g \) strongly in \( L^2(\Omega) \) and
\[
\lim_{\delta \to 0} J(g_\delta, u_\delta) = J(g, u).
\]

**Proof.** The procedure carried out for the proof of Theorem 5 moves perfectly into this context. It only remains to verify part 3. More concretely, we only need to show
\[
J(f, v) \geq J(g, u) \quad \text{for any} \quad (f, v) \in A_{loc}.
\]

To check (5.6) we take the sequence of solutions \((f, v_\delta)\) of the nonlocal boundary problem \((P_\delta)\) with \(g = f\). Using the inequality
\[
\lim_{\delta \to 0} B_{h_0}(v, v) = \int_\Omega h_0(x)|\nabla v(x)|^2 \, dx
\]
(see (2.6)) and taking into account the limit (4.5) we realize that
\[
J(f, v) = \int_\Omega \left( G(x, v(x)) + \beta |f(x)|^2 \right) \, dx + \gamma \int_\Omega h_0(x)|\nabla v(x)|^2 \, dx
\]
\[
= \int_\Omega \left( G(x, v(x)) + \beta |f(x)|^2 \right) \, dx + \gamma \lim_{\delta \to 0} B_{h_0}(v, v)
\]
\[
= \lim_{\delta \to 0} \int_\Omega \left( G(x, v_\delta(x)) + \beta |f(x)|^2 \right) \, dx + \gamma \lim_{\delta \to 0} B_{h_0}(v_\delta, v_\delta)
\]
\[
= \lim_{\delta \to 0} J_\delta(f, v_\delta).
\]

By applying now the optimality of \((g_\delta, u_\delta)\) for \(J_\delta\), we clearly infer
\[
J(f, v) = \lim_{\delta \to 0} J_\delta(f, v_\delta) \geq \lim_{\delta \to 0} J_\delta(g_\delta, u_\delta).
\]

And finally, from (4.5) we know
\[
\lim_{\delta \to 0} B_{h_0}(u_\delta, u_\delta) = b_{h_0}(u, u),
\]
whence we obtain
\[
\lim_{\delta \to 0} J_\delta(g_\delta, u_\delta) = \lim_{\delta \to 0} \int_\Omega \left( G(x, u_\delta(x)) + \beta |g_\delta(x)|^2 \right) \, dx + \gamma \lim_{\delta \to 0} B_{h_0}(u_\delta, u_\delta)
\]
\[
\geq \int_\Omega \left( G(x, u(x)) + \beta |g(x)|^2 \right) \, dx + \gamma b_{h_0}(u, u)
\]
\[
= J(g, u).
\]

By linking the above chain of inequalities we have proved (5.6). Regarding uniqueness, it is sufficient to repeat the same argument of Remark 3. The remain of the details follows along the same lines of the proof of Theorem 5. \(\Box\)

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