3D transformation for rotating frames and temporal behavior of spin in rotating electromagnetic fields

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The Dirac equation is invariant under rotations with a constant frequency and invariable cylindrical radius. 3D transformation for rotating frames is found with help of this invariance. Exact localized solutions of the Dirac equation in the field of a traveling circularly polarized electromagnetic wave and constant magnetic field exist and possess unusual properties. Temporal changes of spin pertaining to the solutions are studied.

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INTRODUCTION

Usually the transformation to the rotating coordinate system is postulated without a strong basis like the relativity principle for the Lorentz transformation. As a rule it is non-Galilean transformation. It means that time is the same in both frames. A characteristic example is in \[1\].

In the paper the 3D transformation is deduced using an invariance of Dirac’s equation under rotation with a constant frequency.

We describe properties of exact localized solutions of Dirac equation in electromagnetic field. We consider temporary evolution of spin by using spatial averaging.

Invariance of Dirac’s equation under rotation with a constant frequency and invariable cylindrical coordinates \(x, y, z, \tau\) corresponds to the rotating frame.

The Dirac equation

The Dirac equation is invariant under rotations with a constant frequency and invariable cylindrical radius. An argument in favor such an "invariance by constant cylindrical radius" can be found in general relativity. Consider Murder’s space \[3\], that is the solution of Einstein’s equation with the cylindrical symmetry with interval

\[ ds^2 = Ar^a + bdr^2 + r^2d\varphi^2 + r^b dz^2 + Cr^a dt^2, \]  

where \(A, C, a, b\) are constants. After equating \(a = b = 2\) and a normalization of \(r\) and \(t\) the metric can be reduced to the form

\[ ds^2 = (1 + \frac{1}{Lr})[dr^2 + l^2 d\varphi^2 + dz^2 - c^2 dt^2], \]

where \(l\) and \(L\) are constants with the dimension of length. Obviously, the interval \(\varphi, z, ic\tau\) is invariant under any orthogonal transformation of coordinates \(l\varphi, z, ic\tau\) and invariable \(r\).

3D TRANSFORMATION TO ROTATING FRAMES

General form

Assume that the general form of 3D transformation for cylindrical coordinates \(\varphi, z, t\), is

\[ \tilde{\varphi} = a_{11}\varphi + a_{12}z + a_{13}t, \]

\[ \tilde{z} = a_{21}\varphi + a_{22}z + a_{23}t, \]

\[ \tilde{t} = a_{31}\varphi + a_{32}z + a_{33}t, \]

The matrix \(a_{nk}\) is a function of the cylindrical radius \(r\) and the frequency of rotation \(\Omega\). The cylindrical radius is considered as an invariable parameter: \(\tilde{r} = r\). The tilde corresponds to the rotating frame.

We use the dimensionless units

\[ t \rightarrow \frac{ct}{\lambda}, \quad (x, y, z) \rightarrow \frac{(x, y, z)}{\lambda} \quad \lambda = \frac{\hbar}{mc}, \]

\(\lambda\) is the Compton wavelength.

The transformation must obey the sacral condition of the speed of light constancy. That is: if \(V \equiv z/t = 1\), then \(\tilde{V} = \tilde{z}/\tilde{t} = 1\) for any \(\omega \equiv \varphi/t\). This gives two connections between coefficients \(a_{nk}\)

\[ a_{21} = a_{31}, \quad a_{22} + a_{23} = a_{32} + a_{33}. \]  

Below we determine all the coefficients with help of invariance of Dirac’s equation and using conditions \[5\]. Together with the 3D transformation we define the transformation of spinor to the rotation frame.

\(A, C, a, b\) are constants with the dimension of length. Obviously, the interval \(\varphi, z, ic\tau\) is invariant under any orthogonal transformation of coordinates \(l\varphi, z, ic\tau\) and invariable \(r\).
Invariance

It is well known that 3D transformation for the Cartesian coordinates can be composed by rotations around two or three axes. An similar composition can be used for coordinates \( r \varphi, z, t \). Consider rotation of the frame around the \( z \)-axis with the frequency \( \Omega \). The frequency in the normalized units \( \Omega \to \Omega \lambda/c \). The Galilean transformation for this rotation is

\[
\varphi' = \varphi - \Omega t, \quad t' = t
\]  
(9)

From the viewpoint of contemporary physics time in the rotating and resting frame should be different. Eq. (9) is not invariant by such a change of coordinates \( \varphi, t \). The desirable change is a somewhat modified Lorentz transformations

\[
\varphi' = \frac{-\varphi - \Omega t}{\sqrt{1 - r^2 \Omega^2}}, \quad t' = \frac{-r^2 \Omega \varphi + t}{\sqrt{1 - r^2 \Omega^2}}
\]  
(10)

Eq. (10) is invariant under this transformation. This may be shown by multiplying Eq. (10) by the operator

\[
P_\Phi = \exp\left(\frac{1}{2} \alpha_2 \Phi\right)
\]  
(11)

and the change of spinor \( \Psi' = \tilde{P}_\Phi \Psi \), where \( \tilde{P}_\Phi = \beta P_\Phi \beta = \alpha_1 P_\Phi \alpha_1 = \exp(-\frac{1}{2} \alpha_2 \Phi) \),

\[
cosh \Phi = \frac{1}{\sqrt{1 - r^2 \Omega^2}}, \quad \sinh \Phi = \frac{r \Omega}{\sqrt{1 - r^2 \Omega^2}}
\]  
(12)

The second rotation is in the plane \( (r \varphi, z) \). Eq. (10) is invariant under the transformation

\[
r \tilde{\varphi} = r \varphi' \cos \Phi_1 - z \sin \Phi_1, \quad z' = \varphi' r \sin \Phi_1 + z \cos \Phi_1,
\]  
(13)

where \( \Phi_1 \) is a vague angle depending on \( r \) and \( \Omega \). The wave function changes as \( \Psi'' = P_{\Phi_1} \Psi' \), where

\[
P_{\Phi_1} = \exp\left(\frac{1}{2} \alpha_2 \alpha_3 \Phi_1\right) = \beta \exp\left(\frac{1}{2} \alpha_2 \alpha_3 \Phi_1\right) \beta
\]  
(15)

The dependence \( \Phi_1(r, \Omega) \) is defined from the conditions [5].

The third rotation could be the Lorentz transformation for coordinates \( z' \) and \( t' \)

\[
\tilde{z} = z' \cosh \Phi_2 - t' \sinh \Phi_2, \quad \tilde{t} = -z' \sinh \Phi_2 + t' \cosh \Phi_2,
\]  
(16)

where the angle \( \Phi_2 \) is defined by relations

\[
cosh \Phi_2 = \frac{1}{\sqrt{1 - v^2}}, \quad \sinh \Phi_2 = \frac{r \Omega}{\sqrt{1 - r^2 \Omega^2}}
\]  
(18)

\( v \) is the arbitrary normalized velocity \( v \to v/c \). However, for simplicity, we suppose that \( v = 0, \Phi_2 = 0 \).

With help of conditions [5] define the angle \( \Phi_1 \)

\[
\sin \Phi_1 = -r \Omega, \quad \cos \Phi_1 = \sqrt{1 - r^2 \Omega^2}.
\]

Emphasize, for this definition, only one condition from [5] is sufficient, the second is fulfilled automatically.

Using Eqs. (10), (13) we express \( r \tilde{\varphi}, \tilde{z}, \tilde{t} \) as functions of \( r \varphi, z, t \). The transformation of coordinates takes the form

\[
r \tilde{\varphi} = \varphi + z \Omega - \Omega t
\]  
(19)

\[
\tilde{z} = \frac{-r^2 \Omega \varphi}{\sqrt{1 - r^2 \Omega^2}} + z \sqrt{1 - r^2 \Omega^2} + \frac{r^2 \Omega^2 t}{\sqrt{1 - r^2 \Omega^2}},
\]  
(20)

\[
\tilde{t} = \frac{-r^2 \Omega \varphi}{\sqrt{1 - r^2 \Omega^2}} + \frac{t}{\sqrt{1 - r^2 \Omega^2}}
\]  
(21)

The determinant of this transformation \( |a_{kn}| = 1 \). It can be straightforwardly shown that two invariants exist

\[
r^2 \tilde{\varphi}^2 + \tilde{z}^2 - \tilde{t}^2 = r^2 \varphi^2 + z^2 - t^2
\]  
(22)

\[
\frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} = \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}
\]  
(23)

Note, such an invariance exists by including in the 3D transformation the third rotation at angle \( \Phi_2 \).

In view of the term \( \sqrt{1 - r^2 \Omega^2} \) an upper boundary exists for the radius \( r^2 \Omega^2 \leq 1 \). In the non-normalized units this inequality is equivalent to

\[
\frac{2 \pi r}{c T} \leq 1,
\]  
(24)

where \( T \) is the period corresponding to the frequency \( \Omega \). This obviously means that the speed of points on a circle of radius \( r \) cannot be greater than the speed of light.

It is noteworthy that in the rotation (10) can not arise a new constant, because then in the inequality (24) the limit speed would differ from the speed of light.

Time interval in the rotating and resting frame should be measured at the same angle and longitudinal coordinate \( z \) in corresponding frame. The dependence of time intervals \( \Delta \tilde{t} \) and \( \Delta t \) is

\[
\Delta \tilde{t} = \Delta t \frac{\sqrt{1 - \nu^2} \sqrt{1 - r^2 \Omega^2}}{\sqrt{1 - \nu^2 r^2 \Omega^2}}, \quad \text{and}
\]  
(25)

\[
\Delta \tilde{t} = \Delta t \frac{(1 - \nu^2 \Omega^2)}{\sqrt{1 - \nu^2 \Omega^2}}
\]  
(26)

for \( \Delta \varphi = \Delta \tilde{z} = 0 \) and \( \Delta \varphi = \Delta \varphi = 0 \) respectively. For the fullness the intervals include rectilinear move with velocity \( v \), that is, the rotation (10).

The operator of the tree dimensional rotations is the product

\[
P = P_{\Phi_1} P_\Phi.
\]  
(27)
Define operator $\hat{P}$

$$\hat{P} = \beta P\beta = \exp\left(\frac{1}{2}a_2a_3\Phi_1 - \frac{1}{2}a_2\Phi\right). \quad (28)$$

Multiplication by $\hat{P}$ transforms spinor to the rotating frame $\hat{P}\Psi = \tilde{\Psi}$.

**Frequency and velocity in the rotating frame**

In accordance with the found transformation, consider the change of frequency and velocity by transition to the rotating frame

$$\tilde\omega = \frac{\omega + v\Omega - \Omega}{-r^2\Omega + 1}\sqrt{1 - r^2\Omega^2}, \quad (29)$$

$$\tilde{v} = \frac{-r^2\Omega + v(1 - r^2\Omega^2) + r^2\Omega^2}{-r^2\Omega + 1} \quad (30)$$

At $r^2\Omega^2 \to 1$ the frequency $\tilde{\omega}$ tends to zero and the velocity $\tilde{v}$ along the $\tilde{z}$-axis tends to speed of light. At $r^2\Omega^2 \to 0$, $\tilde{\omega} \to \omega - v\Omega - \Omega, \tilde{v} \to v$

**THE DIRAC EQUATION IN ELECTROMAGNETIC FIELD**

In this section we study temporal evolution of average spin in rotating magnetic or electromagnetic fields. This section consist of two parts. In the first part, for comparison, we present non-relativistic case based on the Pauli equation in rotating magnetic field. The second is devoted to the Dirac equation in the field of circularly polarized electromagnet wave.

**Averaging. Nonrelativistic case**

Consider the Pauli equation in the normalized units $i\hbar$

$$i\partial t\Psi = \frac{1}{2}(\mathbf{p} - \mathbf{A})^2\Psi - \frac{g}{2}(\mathbf{\sigma}\mathbf{H})\Psi, \quad (31)$$

in magnetic field consisting of a constant and circularly polarized part, $g$ is the g-factor. The field is described by the potential $\mathbf{A}$

$$A_x = -H_0z/2, \quad A_y = H_0z/2, \quad A_z = H[-x\sin\Omega t + y\cos\Omega t], \quad (32)$$

$H_z$ is the constant component of the magnetic field along the $z$-axis, $H$ is the amplitude of the circularly polarized component. In the normalized units the potential and magnetic field are

$$\mathbf{A} \rightarrow \frac{eA}{\hbar}, \quad \frac{eH}{\hbar} \rightarrow \mathbf{H}, \quad (34)$$

The charge, for definiteness, is assumed to be negative $e = -|e|$.

In some cases when the equation for the operator of spin can be separated from the Pauli equation, the temporal dependence of the spin can be obtained [3]. However the averaging is more general approach.

In experiment, usually, a linearly oscillating magnetic field is used. Such a field is a combination of two circularly polarized fields with opposite rotation. Fermion itself chooses the convenient polarization. The second polarization gains a weak dependence of spin on time, which can be neglected.

The modulation by a circularly polarized magnetic field assumes matching the polarization of the magnetic field and oscillation spin.

For averaging it is important to assume that solutions are continuous and square integrable over all the cross-section.

First of all translate spinor to a rotating frame. In this frame the energy connected with the magnetic moment $g(\mathbf{\sigma}\mathbf{H})/2$ does not depend on time. As it is known rotation the coordinate system by an angle $\varphi$ corresponds to rotation of spinor by the angle $\varphi/2$. Rotation the coordinate system by a frequency $\Omega$, or by angle $\Omega t$ corresponds rotation of spinor by the frequency $\Omega/2$. The transformation is realized by multiplication Eq. (31) by $\exp(i\sigma_3\Omega t/2)$ and the change of spinor

$$\tilde{\Psi} = \exp\left(i\sigma_3\Omega t/2\right)\Psi. \quad (35)$$

For the Pauli equation, as a non-relativistic equation, the Galilean transformation coordinates may be used $\tilde{t} = t, \tilde{z} = z, \tilde{\varphi} = \varphi - \Omega t, \tilde{x} = r\cos\tilde{\varphi}, \tilde{y} = r\sin\tilde{\varphi}$. However the same results can be obtained as with as without the coordinate transformation.

Multiply the equation by $\bar{\Psi}\sigma_k$, integrate over all cross-section and subtract the complex conjugated. We obtain the system of equations for components of the average spin

$$\frac{\partial}{\partial \tilde{t}}\tilde{s}_1 = (\Omega + gH_z)\tilde{s}_2, \quad (36)$$

$$\frac{\partial}{\partial \tilde{t}}\tilde{s}_2 = gH\tilde{s}_3 - (\hbar\Omega + gH_z)\tilde{s}_1, \quad (37)$$

$$\frac{\partial}{\partial \tilde{t}}\tilde{s}_3 = -gH\tilde{s}_2, \quad (38)$$

where $\tilde{s}_k = \int \bar{\Psi}\sigma_k\Psi ds$ is the average spin component with the wave function in the rotating frame, the integration is over all the cross-section. From this system one follows that the sum $\tilde{s}_1^2 + \tilde{s}_2^2 + \tilde{s}_3^2$ do not depend on time and, without loss generality, it can be normalized to 1.

A solution of this system easily can be found. The required input and output conditions are: spin oscillations should have a maximum amplitude; the input and output
values of the amplitude should be maximum with differ sign. Using that we obtain
\[ \hat{s}_1 = 0, \quad \hat{s}_2 = \sin \omega t, \quad \hat{s}_3 = \cos \omega t, \quad \omega = gH, \quad (39) \]
provided that the condition of the magnetic resonance
\[ \Omega + gH = 0 \quad (40) \]
holds.

With help of the connection between the wave function in the rotating and lab frame \cite{35} the spin components with the wave function in the lab frame \( s_k = \int \Psi^* \sigma_k \Psi ds \) also can be found
\[ s_1 = -\sin \omega t \sin \Omega t, \quad s_2 = \sin \omega t \cos \Omega t, \quad s_3 = \cos \omega t, \]
This result is obtained without the concretization of solutions.

In \cite{3} on basis of exact solutions of the Pauli equation in the magnetic field \cite{92, 93}, it was shown that the principle of Pauli should be modified as follows: in such a field, pairs of non-stationary states exist with the same wave function but with energy different by \( \pm \frac{1}{2} gH \). The sum and difference of the wave functions corresponds to two states with spins, oscillating as sine and cosine at the frequency \( \omega = gH \).

**Relativistic case**

In this section unusual properties of exact solutions of the Dirac equation are described.

Consider Dirac’s equation
\[ \{-i \frac{\partial}{\partial t} - i \alpha \frac{\partial}{\partial \mathbf{x}} - \alpha \mathbf{A} + \beta\} \Psi = 0. \quad (41) \]
in the electromagnetic field with the potential
\[ A_x = -\frac{1}{2} H_y + \frac{1}{\Omega} H \cos(\Omega t - \Omega z), \quad (42) \]
\[ A_y = \frac{1}{2} H_x + \frac{1}{\Omega} H \sin(\Omega t - \Omega z). \quad (43) \]
This potential describes a traveling circularly polarized electromagnetic wave propagating along constant magnetic field. The normalized coordinates \cite{4} as well as the normalized potential are used in the equation \cite{11}. In the normalized dimensionless units the propagation constant equals \( \Omega \).

The Dirac’s equation \cite{11} has exact solutions localized in the cross section perpendicular to the propagation direction of the wave \cite{3}. The solutions in the lab frame can be presented as follows
\[ \Psi = \exp[-iEt + ipz - \frac{1}{2} \alpha_1 \alpha_2 (\Omega t - \Omega z) + D] \psi. \quad (44) \]

\[ D = -\frac{d}{2} \gamma^2 - id_2 \hat{x} + d_2 \hat{y}, \quad (45) \]
\[ d = -\frac{1}{2} H_z, \quad d_2 = \frac{\mathcal{E}h}{2(\mathcal{E} - \mathcal{E}_0)} \Omega. \quad (46) \]

In the normalized units \( d_2 \to d_2 \mathbf{A}, \quad d \to d \mathbf{A}^2 \quad E \to E/mc^2, \quad p \to p/mc \).

The solution in the rotating frame may be found by means of the Galilean transformation
\[ \varphi = \varphi - \Omega t + \Omega z, \quad \hat{x} = (\hat{x} + \Omega t), \quad \hat{z} = z, \quad (47) \]
\[ \hat{x} = r \cos \varphi, \quad \hat{y} = r \sin \varphi, \quad (48) \]
\[ \tilde{\Psi} = \exp\left(\frac{1}{2} \alpha_1 \alpha_2 (\Omega t - \Omega z)\right) \psi. \quad (49) \]

In this section we use the term ”rotating frame” bearing in mind ”rotating frame by the Galilean transformation”. Using the non-Galilean transformation in the given case complicates the issue but not leads to new results.

The spinor \( \psi \) depend of \( \hat{x}, \hat{y} \). A constant spinor describe the ground state, a spinor polynomial corresponds to excited states.

In the lab frame only non-stationary states are possible. In contrast to that in the rotating frame stationary states exist.

\( E \) obeys the characteristic equation
\[ \mathcal{E}(\mathcal{E} + 2p - \Omega) - 1 - \frac{\mathcal{E}h^2}{\mathcal{E} - \mathcal{E}_0} = 0, \quad (50) \]
\[ \mathcal{E}_0 = \frac{2d}{\Omega}, \quad \mathcal{E} = E - p, \quad h = \frac{1}{\Omega} H, \quad (51) \]

Eq. \cite{50} is algebraic equation of the third order. This is one unusual property of the solutions of the Dirac equation.

The spinor \( \psi \) corresponding to the ground state has shape
\[ \psi = N \left( \begin{array}{c} \frac{\hbar \mathcal{E}}{-\mathcal{E} + 1}\mathcal{E} - \mathcal{E}_0 \\ -\frac{\hbar \mathcal{E}}{-\mathcal{E} - 1}\mathcal{E} - \mathcal{E}_0 \end{array} \right). \quad (52) \]

\( N \) is the normalization constant, defining from the normalization integral \( \int \Psi^* \Psi ds = 1 \).

\[ N^2 [\hbar^2 \mathcal{E}^2 + (\mathcal{E}^2 + 1)(\mathcal{E} - \mathcal{E}_0)^2] \pi = \frac{1}{2} \exp\left(\frac{d^2}{d^2}\right) = 1 \quad (53) \]

We restrict ourselves the consideration of the ground state.

Spinor \cite{92} as well as spinors of excited states never can be presented in the form a small and large two-component spinors. The second unusual property is the spinors always corresponds only to a relativistic case.
The parameter $\mathcal{E}_0$ in the non-normalized units is defined as

$$\mathcal{E}_0 = -\frac{2\mu H_z}{h\Omega},$$  \hspace{1cm} (54)$$

where $\mu = e\hbar/(2mc)$ is the Bohr magneton. Equating $2/\mathcal{E}_0$ to $g$-factor turns the definition [54] in the classical condition of the magnetic resonance [40].

Evaluate the normalized parameter $h$. Typically, the amplitude of the magnetic field $H$ is much smaller than the constant magnetic field $|H_z|$.

$$h = \frac{1}{\Omega}H \ll -\frac{1}{\Omega}H_z = \mathcal{E}_0 \sim \frac{2}{g},$$  \hspace{1cm} (55)$$

If $g \sim 2$ then $h \ll 1$. After Eq. (60): The typical frequency of the magnetic resonance $\sim 100GHz$. The ratio of $\lambda$ to the Compton wavelength is of the order of $10^9$. Therefore $h$ is extremely small.

Typically $\mathcal{E}$ is expanded in power series in $h^2$. However, for pairs of singular solutions $\mathcal{E}$ is expanded in power series in $h$

$$\mathcal{E}_{1,2} = \mathcal{E}_0 + h\mathcal{E}_{1,2} + h^2\mathcal{E}_2 + \ldots, \quad \mathcal{E}_{1,2} = \pm \frac{\mathcal{E}_0}{\sqrt{\mathcal{E}_0^2 + 1}}.$$  \hspace{1cm} (56)$$

In this expansion odd terms have positive and negative signs for one and other solution in the pair. For such pairs in the first approximation

$$d_2 \approx \pm \frac{\sqrt{\mathcal{E}_0^2 + 1}}{2}.$$  \hspace{1cm} (57)$$

The necessary condition for existence of the expansion is the equality of the momentum for both the states in the pair.

$$p = \frac{1}{\mathcal{E}_0} - \mathcal{E}_0 + \frac{1}{2}\Omega.$$  \hspace{1cm} (58)$$

With this momentum the energy also coincide but with accuracy $\sim h$

$$E = \frac{1}{2}\left(\frac{1}{\mathcal{E}_0} + \mathcal{E}_0\right) + \frac{1}{2}\Omega + \ldots$$  \hspace{1cm} (59)$$

The pairs could be correspond to the above Pauli pairs. However, the sum and difference of the wave functions produce states with the spin oscillations with a vanishingly small amplitude. This amplitude is proportional the factor $\exp(-2d^2_2/d)$. In the non-normalized units

$$\frac{2d^2_2}{d} = \frac{\mathcal{E}_0^2 + 1}{\mathcal{E}_0^2},$$  \hspace{1cm} (60)$$

where $\lambda$ is the wavelength corresponding to the frequency $\Omega$. The typical frequency of the magnetic resonance $\sim 100GHz$, the ratio the wavelength corresponding to this frequency and Compton wavelength is of the order of $10^9$. Therefore the factor is extremely small.

The average value of spin components in the general case are $s_n = -\frac{1}{2} \int \Psi^* \sigma_n \Psi ds$, where

$$\sigma_1 = -i\alpha_2 \alpha_3, \sigma_2 = -i\alpha_3 \alpha_1, \sigma_3 = -i\alpha_1 \alpha_2.$$  \hspace{1cm} (61)$$

The components have the form

$$s_1 = \mp \frac{1}{2} \frac{2h\mathcal{E}_0^2(\mathcal{E} - \mathcal{E}_0)}{h^2\mathcal{E}_0^2 + (\mathcal{E}^2 + 1)(\mathcal{E} - \mathcal{E}_0)^2} \cos(\Omega t - \Omega z),$$

$$s_2 = \mp \frac{1}{2} \frac{2h\mathcal{E}_0^2(\mathcal{E} - \mathcal{E}_0)}{h^2\mathcal{E}_0^2 + (\mathcal{E}^2 + 1)(\mathcal{E} - \mathcal{E}_0)^2} \sin(\Omega t - \Omega z),$$

$$s_3 = \frac{1}{2} \frac{h\mathcal{E}_0^2 - (\mathcal{E}^2 + 1)(\mathcal{E} - \mathcal{E}_0)^2}{h^2\mathcal{E}_0^2 + (\mathcal{E}^2 + 1)(\mathcal{E} - \mathcal{E}_0)^2}.$$

For the singular solutions $\mathcal{E} \approx \mathcal{E}_0 = 2/g$ spin $s_3 = 0$ and

$$s_1 = -\frac{1}{\sqrt{4 + g^2}} \cos(\Omega t - \Omega z),$$

$$s_2 = -\frac{1}{\sqrt{4 + g^2}} \sin(\Omega t - \Omega z).$$

The temporal behavior of spin is similar to the magnetic field. This is the third radical dissimilarity of the solutions.

**Conclusion**

We have deduced 3D transformation for rotating frames on the basis of the general assumption regarding its form and the requirement on invariance of Dirac’s equation under rotation with a constant frequency. The problem arises by modulation of a rotating magnetic or electromagnetic field.

We have investigated the unusual properties of exact solutions of Dirac’s equation in the field of traveling circularly polarized electromagnetic wave and a constant magnetic field. The temporal dependence of the average spin is similar to that of the modulating magnetic field and can be tested by means of polarization measurements of the spin. It would be interesting to see how $g$-factor in these measurements matches known data.

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