Self-organization and the Maximum Empower Principle in the Framework of max-plus Algebra

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Abstract

Self-organization is a process where order of a whole system arises out of local interactions between small components of a system.

Emergy, defined as the amount of (solar) energy used to make a product or a service, is becoming an important ecological indicator. To explain observed self-organization of systems by emergy the Maximum Empower Principle (MEP) was proposed initially without a mathematical formulation.

Emergy analysis is based on four rules called emergy algebra. Most of emergy computations in steady state are in fact approximate results, which rely on linear algebra. In such a context, a mathematical formulation of the MEP has been proposed by Giannantoni (2002).

In 2012 Le Corre and the second author of this paper have proposed a rigorous mathematical framework for emergy analysis. They established that the exact computation of emergy is based on the

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so-called max-plus algebra and seven coherent axioms that replace the emergy algebra. In this paper the MEP in steady state is formalized in the context of the max-plus algebra and graph theory. The main concepts of the paper are (a) a particular graph called “emergy graph”, (b) the notion of compatible paths of the emergy graph, and (c) sets of compatible paths, which are called “emergy states”. The main results of the paper are as follows:

(1) Emergy is mathematically expressed as a maximum over all possible emergy states.

(2) The maximum is always reached by an emergy state.

(3) Only prevail emergy states for which the maximum is reached.

Keywords: Emergy algebra, Formal language, Graph, Max-plus algebra, Sustainability.

1. Introduction

Self-organization, or spontaneous order principle, states that any living or non-living disordered system evolves towards an “equilibrium state” or coherent state, also called attractor. Self-organization is observed e.g. in physical, biological, social, mathematical systems/models, economics, information theory and informatics.

Optimization (i.e. minimization or maximization) appears to be a natural way of thinking of the human being. So, it seems natural to try to explain self-organization phenomena by optimization approach. For example, the least action principle is the basis of the Lagrangian mechanics; the extremization of statistical entropy is the basis of statistical mechanics and information theory; the free energy principle (minimization) has also been successfully applied to explain, for example, Bayesian inference, thermodynamics equilibrium, optimal behavior in neural organization; the minimization of some
Dirichlet functional corresponding to the dissipative power of an electrical network explains Kirchhov’s current law (e.g. Baez and Fong, 2016).

All these approaches are based on a functional that has to be minimized or maximized (e.g. Fath et al., 2001; and references therein in the context of ecology).

It has been considered since a long time that ecological systems, social and economic systems are energy driven systems (e.g. Podolinsky, 1880; Boltzmann, 1886). In this context the emergy is defined as the available energy (or exergy) of one kind used up directly or indirectly to make a service or product (Odum, 1996). Emergy evaluation is thus based on energy system theory. Self-organization of energy systems is based on the maximum power principle, which has been proposed by, for example, Lotka (1922a, 1922b), Odum and Pinkerton (1955), and Odum (1996). This principle states that “system designs develop and prevail that maximize power intake, energy transformation, and those uses that reinforce production and efficiency” (Odum, 1995).

But the major drawback of such approach is that different kinds of energy have different abilities to do work. Odum introduced the notion of transformity, which is defined as the emergy required to generate a unit of the available energy in another form. Thus, each type of energy has a transformity. The reference for transformity being the one of the sun, which is equal to one solar equivalent Joule per Joule (denoted 1 sej/J). The empower is defined as the emergy per time. Odum expanded the maximum power principle to the maximum empower principle: “All self-organizing systems tend to maximize their empower, and those systems that maximize empower prevail”.

Emergy evaluation of steady state is based on four rules, called emergy algebra, and are stated as follows (e.g. Brown and Herendeen, 1996):
R1: When only one product is obtained from a process (i.e. a process with only one output), all source-emergy is assigned to it.

Concerning processes with more than one output we have two cases. In the split case, a pathway of the emergy system is divided into several branches of the same kind, e.g. as in hydraulic systems. Then we have the following rule:

R2: When a flow (of emergy) splits the total emergy splits accordingly, based on the exergy flowing through each pathway.

In the co-product case, the flow in each branch is of different kinds, e.g. as in combined heat and power plants (described in e.g. Horlock, 1996). We have the following rule:

R3: When two or more co-products are generated in a process, the total source-emergy is assigned to each of them.

Finally, a fourth rule describes how emergy is assigned within a system of interconnected processes

R4: Emergy cannot be counted twice within a system.

R4.1: Emergy in feedbacks cannot be double counted.

R4.2: Co-products, when reunited, cannot be summed. Only the emergy of the largest co-product flow is accounted for (see also Odum, 1996, p.51 Fig. 3.7).

To respect the track summing method introduced in Tennenbaum (1988) the rule R4.1 has been introduced. This method involves some kinds of paths of emergy graphs known as simple paths and the class of terminal non-feedback cycle paths (e.g. Whipple, 1999). The observation is that from these rules it is not trivial to establish the MEP. The idempotent max operator only appears in the rule R4.2, but despite of this rule most of the emergy analysis are based on linear algebra which do not take into account neither the notion of path (i.e. energy memory), as pointed out e.g. by Sciubba (2010), nor the
max operator. This leads to absurd results such as negative transformities (e.g. Patterson, 2014). However, a major attempt to mathematically formalize MEP in steady state based on linear algebra is due to Giannantoni (2002).

The implied rule of splits, which states that we have to sum when splits of emergy are reunited, and the rule R4.2 lead to consider the emergy computation as a max-plus linear problem (e.g. Baccelli et al., 1992). Based on this remark, a rigorous framework for emergy computation has been proposed by Le Corre and Truffet (2012). The whole emergy analysis is a path oriented method based on idempotent algebra and graph theory. The emergy computation part of this method assumes that the emergy paths (see Definition 1) are given. The emergy computation relies on seven coherent axioms that are the reinterpretations of the four emergy rules of the emergy algebra.

In this paper, it is proved that the emergy analysis method of Le Corre and Truffet (2012) can be seen as a maximization principle. This principle is based on emergy state, which is a set of particular paths (see Definition 5), and on the emergy associated with the emergy state. In this context, the maximization principle is expressed as follows: “only prevail emergy states for which the maximum of emergy is reached”.

The paper is organized as follows: we recall the main mathematical tools borrowed from Le Corre and Truffet (2012) in Section 2. Using these tools, we provide in Section 3 a mathematical formulation of Odum’s Maximum Empower Principle, which is the main result of the paper. This mathematical formulation is illustrated by a numerical example in Section 4. Finally, in Section 5, we explain how this could lead to future works.
2. Emergy evaluation reminder

We present the basic concepts and axioms introduced by Le Corre and Truffet (2012). They are necessary to establish our main results (see Section 3).

2.1. Emergy graph

The way by which emergy circulates in a multicomponent system is modelled by an oriented graph, which is called emergy graph (Le Corre and Truffet, 2012). Formally, it is the following 10-tuple:

\[
G = (L, L^s, L^i, L^o, F, \Delta, \text{id}, \cup, \setminus, \emptyset)
\]  

(1)

where \( L, L^s, L^i, L^o, F, \Delta, \text{id}, \cup, \setminus, \emptyset \) are defined hereafter. The set \( L \), where \( L \subseteq \mathbb{N} \) and \( \mathbb{N} \) is the set of natural numbers, represents the set of nodes of \( G \). We have \( L = L^s \cup L^i \cup L^o \), where \( (L^s; L^i; L^o) \) is a partition, since an emergy graph contains three types of nodes: the set of emergy sources \( L^s \), the set of intermediate nodes \( L^i \) and the set of output nodes \( L^o \). The modelling of emergy circulation is based on particular words (which are called paths in the sequel) over alphabet \( F \), where \( F = L \times L \), i.e. \( F = \{ [u;v] \mid u \in L, v \in L \} \): a path in the emergy graph is represented by a word over \( F \) (see Definition 1 below).

The reader must be aware that, in our case, a letter is of the form \([u;v]\) with \( u \in L \) and \( v \in L \), and thus will be called an arc in the sequel. Le Corre and Truffet (2012) used this approach because it allowed to compute relevant paths for emergy analysis by borrowing some results of formal language theory. It is applied to the formal language \( F \), which is the particular idempotent semiring

\[
F = (F_0^*, \cup, \bullet, 0, 1),
\]  

(2)
where

1. \( F_0^* = F^* \cup \{0\} \).

2. \( F^* \) is the set of all words of finite length constructed over the alphabet \( F \). Notice that \( F^* \) is the free monoid (for a very formal definition see e.g. Bourbaki, 2006, A.I, pp. 77-79). It is defined as the following disjoint union (which coincides with the union of sets \( \cup \)):

\[
F^* = \bigcup_{n \geq 0} F^n,
\]

where \( F^n \) denotes the set of all words which contain exactly \( n \) letters, for \( n \geq 1 \). We have \( F^0 = \{0\} \), which means that \( 1 \) is the empty word.

3. \( \cup \) is the union of two words, which can be identified with the union if a word \( m \) is identified with the set \( \{m\} \). It means that \( F_0^* \) is identified with the set of all parts of \( F_0^* \), which is denoted by \( 2^{F_0^*} \).

4. \( \circ \) is the concatenation of two words, which is defined as follows:

\[
\circ : F^*_0 \times F^*_0 \rightarrow F^*_0 \quad \quad (m, m') \mapsto m \circ m'
\]

The word \( m \circ m' \) is the new word obtained by joining the letters of \( m \) and the letters of \( m' \) end-to-end. When there is no ambiguity, the concatenated word \( m \circ m' \) will be denoted by \( mm' \). Notice that \( 1 \) is the neutral element for concatenation.

5. \( 0 \) is the neutral element for \( \cup \) and is absorbing for \( \circ \), i.e. \( \forall m \in F_0^* \), \( 0 \circ m = m \circ 0 = 0 \). Notice that \( 0 \) is the empty set.
The set $A$, where $A \subseteq F$, denotes the set of arcs of the emergy graph $G$. It satisfies

$$A \cap L_s \times L_s = A \cap L_o \times L_o = \emptyset.$$  \hspace{1cm} (6)

Every pair of arcs of an emergy graph must satisfy one of the four binary symmetric relations $id$, $\perp$, $\parallel$, and $\emptyset$, which are defined as follows:

- For all $a, a' \in A$: $a \emptyset a'$ means that there is no relation between arcs $a$ and $a'$.
- For all $a, a' \in A$: $a id a'$ means that $a = a'$ (identity relation over $A$).
- For all $l, l', l'' \in L$: $[l; l'] \parallel [l; l'']$ means that there is a co-product at node $l$. Node $l$ is called a co-product.
- For all $l, l', l_1, l_2 \in L$: $[l; l_1] \perp [l'; l_2]$ means that if $l = l'$ there is a split of emergy at node $l$, else $l$ and $l'$ are emergy sources. In the case $l = l'$, node $l$ is called a split.

The relations $\emptyset$, $id$, $\perp$ and $\parallel$ satisfy 7 axioms (Le Corre and Truffet, 2012, Section 3.1). These axioms are mainly used to prove that the algorithm in Le Corre and Truffet (2012, Section 4) begins and terminates. In this paper, only the last axiom is of importance:

(A1). By convention, each source of the emergy graph is connected to only one node of the emergy graph, i.e. $\forall l \in L^s, |[l; L \setminus L^s] \cap A| = 1$. 

8
Example 1.

Let us consider the emergy graph of Figure 1, as illustrated by Li et al. (2010, Figures 8 and 9). The drawing conventions for the emergy graph are given at the bottom: a source is represented by symbol A, an intermediate node on the emergy graph is represented by symbol B, and an output node is represented by symbol C. Splits and co-products are represented by symbols D and E respectively.

There are splits at nodes 3, 5, 6, 7 and 10, and a co-product at node 4. The set of sources is $L^s = \{1, 2\}$, the set of internal nodes is $L^i = \{3, 4, 5, 6, 7, 8, 9, 10\}$ and the set of the output nodes is $L^o$.
=\{11,12,13,14\}. Because 1 and 2 are sources we have: \([1;3] \perp [2;10]\). Because 3, 5, 6, 7 and 10 are splits we have: \([3;4] \perp [3;5], [6;8] \perp [6;9], [7;9] \perp [7;10]\) and \([10;4] \perp [10;11]\). Because of the co-product at node 4 we have: \([4;6] // [4;7]\).

### 2.2. Max-plus algebra as a tool for emergy evaluation

Since double-counting must be avoided (see Rule 4.1, Introduction) emergy evaluation is based on considering particular paths in the emergy graph.

**Definition 1.**

- **Path.** A path \(\pi\) is an element of the set \(\mathbb{F}_0^*\) which has the form \(\pi = 0\), or \(\pi = 1\), or \(\pi = [l_1;l_2]\) or \(\forall k > 3, \pi = [l_1;l_2][l_3;l_4] \ldots [l_{k-2};l_{k-1}] [l_{k-1};l_k]\), with \(l_j \in \mathbb{L}\), for \(1 \leq j \leq k\). The length \(l_g(\pi)\) of a path \(\pi\) is equal to \(-\infty\) if \(\pi = 0\), and is equal to 0 if \(\pi = 1\); otherwise the length of \(\pi\) is equal to the number of arcs \([l_i;l_{i+1}]\) which compose the path.

- **Simple path.** A simple path is a path \(\pi = [l_1;l_2] \ldots [l_{k-1};l_k]\), such that \(l_j \neq l_j'\) for \(1 \leq j < j' \leq k\).

- **Subpath.** A subpath \(\pi\) of a path \(\pi'\) is a path such that \(\pi' = \pi_1\pi\pi_2\), where \(\pi_1\) and \(\pi_2\) are two (maybe empty) paths.

- **Emergy path.** An emery path is a path \(\pi = [l_1;l_2] \ldots [l_{k-1};l_k]\), such that \([l_1;l_2] \ldots [l_{k-2};l_{k-1}]\) is a simple path, \(l_1 \in \mathbb{F}_0^*\) and \(\forall 2 \leq j \leq k, l_j \in \mathbb{L}\setminus\mathbb{L}_s\).
Example 1 (continued).

Path $[3;4][4;7][7;10][10;4][4:6]$ is not a simple path since node 4 is repeated. Path $[1;3][3;5][5;7][7;10][10;4][4:6][6;8]$ is an emergy path since it starts from a source and it is a simple path.

From now on, it is assumed that the set of all emergy paths is given. It is denoted by $\mathcal{E}$. There exist several algorithms for computing this set: see for example Tennenbaum (1988), Marvuglia et al. (2011 and 2013), or Le Corre and Truffet (2012). At this step, it is necessary for emergy analysis to introduce the following auxiliary functions and the axioms they satisfy.

Let us recall that $\mathbb{L}$ denotes the set of all nodes of the emergy graph $G$ defined by (1). We define the following functions. Let $\mathbb{R}_+$ be the set of nonnegative reals. The emergy function $\theta : \mathbb{L} \rightarrow \mathbb{R}_+$ such that $\theta(l) = 0$ for $l \notin \mathbb{L}_s$. Thus, for $l \in \mathbb{L}_s$, $\theta(l)$ is the value of the source $l$. The weight function $\omega : \mathcal{F} \rightarrow \mathbb{R}_+$ which satisfies the following axioms:

- (ω.0). $\omega(\perp) = 1$.
- (ω.1). $\omega([l;l'])$ corresponds to the fraction of emergy (which is assumed to be given in this paper) circulating on $[l;l']$ if $[l;l']$ is an arc of the emergy graph; otherwise $\omega([l;l']) = 0$. 

\[ \omega([l;l']) \]
Definition 2. (Auxiliary function $\varphi$, Le Corre and Truffet (2012))

Let us define the set function $\varphi: \mathbb{F}^2 \to \mathbb{R}^+$ which allows us to compute the emergy flowing on every arc of the emergy graph. Denoting concatenation of a word $m$ with a set of words $\mathbb{U}$ by $m\mathbb{U}$, where $m\mathbb{U} \overset{\text{def}}{=} \{mu \mid u \in \mathbb{U}\}$, the function $\varphi$ satisfies the following axioms:

- (ϕ.0). $\varphi(\emptyset)=1$, $\varphi(Ø)=0=\varphi(Ø)$.
- (ϕ.1). $\forall m \in \mathbb{F}^2, \varphi(m) = \varphi(\{m\})$.
- (ϕ.2). $\varphi([l; l']) = \begin{cases} \omega([l; l']) & \text{if } l, l' \notin \mathbb{L}, \\ \theta(l)\omega([l; l']) & \text{if } l \in \mathbb{L} \text{ and } l' \notin \mathbb{L}, \\ 0 & \text{otherwise.} \end{cases}$
- (ϕ.3). $\forall m \in \mathbb{F}^2, \forall \mathbb{U} \in 2^\mathbb{F}^2, \varphi(m\mathbb{U})=\varphi(m)\varphi(\mathbb{U})$
- (ϕ.4). Let us consider the situation where the quantity flows have a common upstream flow $m$ and are reunited at arc $[l; l']$ after a split ($\perp$) or a co-product ($//$). That is $\forall m \in \mathbb{F}^2, \forall a_1, ..., a_k \in \mathbb{A}$ s.t. $a_1 \perp a_2 ..., \perp a_k$ with $\perp \in \{\perp, //\}$, $\forall \mathbb{U}_1, ..., \mathbb{U}_k \in 2^\mathbb{F}^2$:
  - (ϕ.4.1). If the arcs $a_i$ are linked by the relation $\perp$ then the total quantity $\varphi(\bigcup_{1 \leq i \leq k} ma_i \mathbb{U}_i)$ flowing on arc $[l; l']$ is equal to the sum of the quantities flowing on arc $[l; l']$ of the system as if there was only one arc $a_i$ after the upstream flow $m$, $\varphi(ma_i \mathbb{U}_i)$, for $1 \leq i \leq k$, when reunited, i.e.:
    \[
    \varphi(\bigcup_{i=1}^k ma_i \mathbb{U}_i) = \sum_{1 \leq i \leq k} \varphi(ma_i \mathbb{U}_i) \text{ if } \perp = \perp. \tag{7}
    \]
(ϕ.4.2). If the arcs $a_i$ are linked by the relation $\parallel$, then the total quantity flowing on arc $[l; l']$, $\phi(\bigcup_{l \in \varepsilon} ma_i \cup_l)$ is equal to the maximum of the quantities flowing on arc $[l; l']$ of the system as if there was only one arc $a_i$, after the upstream flow $m$, $\phi(ma \cup_l)$, for $1 \leq i \leq k$, when reunited, i.e.:

$$\phi(\bigcup_{l \in \varepsilon} ma_i \cup_l) = \max_{l \in \varepsilon} \phi(ma \cup_l) \text{ if } \varepsilon = \varepsilon \parallel.$$  

Hence, function $\phi$ is max-plus linear (e.g. Baccelli et al., 1992).

Now, we are able to recall the definition of the emergy measure flowing on an arc of an emergy graph.

**Definition 3. (Emergy evaluation, Le Corre and Truffet (2012))**

Let us consider the emergy graph $G$ defined by (1), i.e. $G = (\mathbb{L}, \mathbb{L}^s, \mathbb{L}^i, \mathbb{L}^o, F, \Lambda, \text{id}, \bot, \parallel, \emptyset)$. The emergy flowing on arc $[l;l']$ with $l,l' \in \mathbb{L}$, $\text{Em}([l;l'])$, is defined by:

$$\text{Em}([l;l']) = \phi(\varepsilon([l;l'])),$$

where $\varepsilon([l;l'])$ denotes the subset of the elements of the set $\varepsilon$ ending by $[l;l']$, and $\phi$ satisfies axioms (ϕ.0-ϕ.4).

**Example 1 (continued).**

Emergy paths ending by $[9;13]$ (i.e. paths of $\varepsilon([9;13])$) are enumerated in Table 1: there are 6 emergy paths in $\varepsilon([9;13])$. 

13
| Path | Description               |
|------|--------------------------|
| \(\pi_1\) | \([1;3] [3;4] [4;6] [6;9] [9;13]\) |
| \(\pi_2\) | \([1;3] [3;4] [4;7] [7;9] [9;13]\) |
| \(\pi_3\) | \([1;3] [3;5] [5;7] [7;9] [9;13]\) |
| \(\pi_4\) | \([1;3] [3;5] [5;7] [7;10] [10;4][4;6][6;9][9;13]\) |
| \(\pi_5\) | \([2;10] [10;4] [4;6] [6;9] [9;13]\) |
| \(\pi_6\) | \([2;10] [10;4] [4;7] [7;9] [9;13]\) |

**Table 1.** Paths of \(\varepsilon([9;13])\), i.e. emergy paths ending by arc \([9;13]\).

### 3. A mathematical formulation of Odum’s Maximum Empower Principle

From now on we shall use the following notation:

**Notation 1.**

Let \(X\) be a set of simple paths ending by arc \([l;l']\), and let \(l \in \mathbb{L}\). We denote by \(X_l\) the set of subpaths of paths of \(X\) that start by \(l\) and end by arc \([l;l']\). Formally, it is defined by \(X_l = \{\pi : \exists \pi', \pi' \in X, \pi \text{ starts by } l\}\).

#### 3.1. Properties of function \(\varphi\)

**Proposition 1** (Superposition principle).

If \(X\) is a subset of \(\varepsilon([l;l'])\) then
\[ \varphi(X) = \sum_{s \in \text{sel.}'} \varphi(X_s). \quad (10) \]

**Proof.** Recall that, by Axiom (A1), a source \( s \) is connected to only one node, say \( \text{succ}(s) \), so we have

\[ X = \bigcup_{s \in \text{sel.}'} X_s, \] with \( X_s = \{ [s; \text{succ}(s)] \pi : \pi \in X_{\text{succ}(s)} \} \). By definition of sources, arcs \([s; \text{succ}(s)]\) are linked by the relation \( \perp \), hence Axiom (\( \phi.4.1 \)) applies and we get

\[ \varphi(\bigcup_{s \in \text{sel.}'} [s; \text{succ}(s)] \pi : \pi \in X_{\text{succ}(s)}) = \sum_{s \in \text{sel.}'} \varphi([s; \text{succ}(s)] \pi : \pi \in X_{\text{succ}(s)}), \]

i.e. \( \varphi(X) = \sum_{s \in \text{sel.}'} \varphi(X_s) \).

**Proposition 2 (Monotonicity).**

Let \( A \) and \( B \) be two subsets of \( \varepsilon([l; l']) \). If \( A \subseteq B \), then

\[ \varphi(A) \leq \varphi(B). \quad (11) \]

**Proof.** First, we prove that \( \varphi(A_s) \leq \varphi(B_s) \) for \( s \in \mathbb{L}^S \). Let us consider a source \( s \) of \( \mathbb{L}^S \) and let \( m_s \) be the sequence of arcs that are common to the paths of \( A_s \), i.e. \( A_s = \bigcup_{1 \leq i \leq k} m_s A_i \) where \( l_i \) is the end of arc \( a_i \) for \( 1 \leq i \leq k \). Since \( A \subseteq B \), we also have \( A_s \subseteq B_s \), so we just have to prove that \( \varphi(A_s) \leq \varphi(B_s) \).

Hence, if \( A_s = B_s \) we get \( \varphi(A_s) = \varphi(B_s) \). Else, there exists at least one path, say \( \pi \), of \( B_s \) that does not belong to \( A_s \). Let \( m_{A_s\cup\{\pi\}} \) be the sequence of arcs that are common to the paths of \( A_s \cup \{\pi\} \), i.e.

\[ A_s \cup \{\pi\} = \bigcup_{1 \leq i \leq k'} m_{A_s \cup \{\pi\}} A_i \cup \{m_{A_s \cup \{\pi\}} \pi'\} \] where \( l_i' \) is the end of arc \( a_i' \) for \( 1 \leq i \leq k' \), and \( \pi = m_{A_s \cup \{\pi\}} \pi' \).

We have \( \varphi(A_s \cup \{\pi\}) = \varphi(\bigcup_{1 \leq i \leq k'} m_{A_s \cup \{\pi\}} A_i \cup \{m_{A_s \cup \{\pi\}} \pi'\}) \). Now, notice that, since \( m_{A_s \cup \{\pi\}} \) is a subpath of
there exists some path \( \pi' \) (which may be the empty path) such that \( m_A = m_{A \cup \{\pi\}} \pi' \): we get

\[
\varphi(A \cup \{\pi\}) = \varphi(\bigcup_{i \in s'} m_{A \cup \{\pi\}} \pi'' a_i A_i) \cup \{m_{A \cup \{\pi\}} \pi''\}.
\]

Thus, we have two cases: if the last node of \( m_{A \cup \{\pi\}} \) is a split we apply Axiom (\( \varphi.4.1 \)) and get \( \varphi(A \cup \{\pi\}) = \sum_{1 \leq i \leq s'} \varphi(m_{A \cup \{\pi\}} \pi'' a_i A_i) + \varphi(m_{A \cup \{\pi\}} \pi') \); else we apply Axiom (\( \varphi.4.2 \)) and get \( \varphi(A \cup \{\pi\}) = \max(\varphi(m_{A \cup \{\pi\}} \pi'' a_i A_i), \varphi(m_{A \cup \{\pi\}} \pi')) \). In the first case we have \( \varphi(A \cup \{\pi\}) = \varphi(A) + \varphi(\pi) \) whereas in the second one we have \( \varphi(A \cup \{\pi\}) = \max(\varphi(A), \varphi(\pi)) \).

Since \( \varphi \) is a nonnegative function, we have \( \varphi(A) \leq \varphi(A \cup \{\pi\}) \) in both cases. Hence, if \( \pi_1, \ldots, \pi_{k'} \) denote the paths of \( B_s \) that do not belong to \( A_s \), we can apply this reasoning successively to these paths, which implies \( \varphi(A) \leq \varphi(A \cup \{\pi_1\}) \leq \varphi(A \cup \{\pi_1\} \cup \{\pi_2\}) \leq \cdots \leq \varphi(A \cup \{1 \leq i \leq s' \}) = \varphi(B) \).

Finally, by (10) we have \( \varphi(A) = \sum_{s \in 1, l} \varphi(A_s) \) and \( \varphi(B) = \sum_{s \in 1, l} \varphi(B_s) \), that is \( \varphi(A) \leq \varphi(B) \).

**Proposition 4 (Decomposition).**

If \( \pi \) is an emergy path of \( \varepsilon([l'; l]) \) then

\[
\varphi(\pi) = \theta(s) \prod_{1 \leq i \leq k} \omega([l_i, l_{i+1}]) \text{ for } \pi = [s; \ell_1][\ell_1; \ell_2] \cdots [\ell_k; \ell_{k+1}].
\]  

(12)

**Proof.** Let \( \pi = [s; \ell_1][\ell_1; \ell_2] \cdots [\ell_k; \ell_{k+1}] \). By repeated application of Axiom (\( \varphi.3 \)) we get

\[
\varphi(\pi) = \varphi([s; \ell_1]) \varphi([\ell_1; \ell_2]) \cdots \varphi([\ell_k; \ell_{k+1}]), \text{ i.e. } \varphi(\pi) = \theta(s) \omega([s; \ell_1]) \omega([\ell_1; \ell_2]) \cdots \omega([\ell_k; \ell_{k+1}]) \text{ by Axioms (\( \varphi.2 \)) and (A1).}.
\]
3.2. Main results

Recall (see Definition 1) that $\epsilon([l;l'])$ is the set of all emergy paths ending by arc $[l;l']$. We define the compatibility relation $\hat{+}$ on $\epsilon([l;l'])$ as the following binary relation.

**Definition 4. (Compatible paths)**

Two paths $\pi$ and $\pi'$, where $\pi=[l_1;l_2][l_2;l_3] \cdots [l;l']$ and $\pi'=[l'_1;l'_2][l'_2;l'_3] \cdots [l;l']$, are compatible (which is denoted by $\pi \hat{+} \pi'$) if and only if one of the following cases occurs:

1. $\pi = \pi'$,

2. $[l_1;l_2] \perp [l'_1;l'_2]$,

3. $\exists k$, with $k > l$, such that $\forall 1 \leq i \leq k, l_i = l'_i$ and $[l_k;l_{k+1}] \perp [l'_k;l'_{k+1}]$, i.e. paths $\pi$ and $\pi'$ coincide till a split node $l_k$ and then divide.

**Example 1 (continued).**

The compatibility relation between paths of $\epsilon([9;13])$ is given in Table 2.

| $\hat{+}$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ |
|-----------|--------|--------|--------|--------|--------|--------|
| $\pi_1$   | T      | F      | T      | T      | T      | T      |
| $\pi_2$   | F      | T      | T      | T      | T      |
| $\pi_3$   | T      | T      | T      | T      | T      |
| $\pi_4$   | T      | T      | T      | T      | T      |
| $\pi_5$   | T      | T      | T      | T      | F      |
| $\pi_6$   | T      | T      | T      | F      | T      |

17
Table 2. Compatibility relation between paths of $\varepsilon([9;13])$ (‘F’ means False, ‘T’ means True).

An emergy state, relatively to an arc $[l;l']$, is then defined as the following subset of $\varepsilon([l;l'])$.

**Definition 5. (Emergy state)**

An emergy state $\hat{\varepsilon}$, relatively to an arc $[l;l']$, is a set of pairwise compatible paths, i.e. such that $\hat{\varepsilon} \subseteq \varepsilon([l;l'])$ and $\forall \pi \in \hat{\varepsilon}, \forall \pi' \in \hat{\varepsilon}, \pi \hat{+} \pi'$. The set of all emergy states, relatively to an arc $[l;l']$, is denoted by $\hat{E}([l;l'])$. Note that $\hat{E}([l;l'])$ is a part of $2^{\varepsilon([l;l'])}$.

**Example 1 (continued).**

The set $\{\pi_1, \pi_5, \pi_6\}$ is an emergy state relatively to $[9;13]$ since the three paths are pairwise compatible (see Table 2). On the contrary, the set $\{\pi_1, \pi_2, \pi_6\}$ is not an emergy state relatively to $[9;13]$ because paths $\pi_1$ and $\pi_2$ divide at node 4 which is a co-product node.

Note that there are at most $2^6$ elements in $\hat{E}([9;13])$ since $\varepsilon([9;13])$ contains 6 emergy paths.

In order to establish the main results of the paper (Theorem 1 and 2) we need the following algorithm, which removes irrelevant paths, relatively to the compatibility relation $\hat{+}$, of a given set of paths.
Algorithm 1. 

Function CPATHS:

Input: a subset $A$ of $\varepsilon_j([l;′l'])$ where $[l;′l'] \in A$ and $j \in \mathbb{L}$;

Output: a subset of $A$;

If $j=l$ then

return $\{[l;′l']\}$;

Else

Let $a_1,\ldots, a_k$ be the arcs starting by $j$;

Let $l_1,\ldots, l_k$ the last nodes of arcs $a_1,\ldots, a_k$, respectively;

If $j$ is a split or a source then

return $\bigcup_{1 \leq i \leq k} a_i$CPATHS($A_{l_i}$); /* keep all paths */

Else

Let $i'$ be such that $\phi(a_i$CPATHS($A_{l_i}$)) = max $\phi(a_i$CPATHS($A_{l_i}$))

return $a_i$CPATHS($A_{l_i}$); /* keep a path that maximizes value $\phi(a_i$CPATHS($A_{l_i}$)) */
Proposition 3.

If \( A \) is a subset of \( \varepsilon_j([l;l']) \), with \( j \in \mathbb{L} \setminus \mathbb{L}^o \), then Algorithm 1 applied to \( A \) returns a set \( \text{CPATHS}(A) \) which verifies the following properties:

\[
\varphi(\text{CPATHS}(A)) = \sum_{\pi \in \text{CPATHS}(A)} \varphi(\pi), \quad (13.1)
\]

\[
\varphi(A) = \varphi(\text{CPATHS}(A)), \quad (13.2)
\]

\[\forall \pi, \pi' \in \text{CPATHS}(A), \pi \neq \pi'. \quad (13.3)\]

Proof. We give a proof by induction on the maximum length of a path of \( A \). If \( \max_{\pi \in A} \lg(\pi) = 1 \) then \( j = l \) and \( \text{CPATHS}(A) \) returns \( \{[l;l']\} \) : since \( A = \{[l;l']\} \) properties (13.1)-(13.3) are true. So assume that \( \text{CPATHS}(A) \) returns a set that verifies (13.1)-(13.3) when \( 1 \leq \max_{\pi \in A} \lg(\pi) \leq n \), and consider a set \( A \) such that \( \max_{\pi \in A} \lg(\pi) = n + 1 \). We have two cases, according to the type of node \( j \):

- If \( j \) is a split or a source, a call to \( \text{CPATHS}(A) \) returns the set \( \bigcup_{1 \leq i \leq k} a_i \text{CPATHS}(A_i) \). Then, Axiom (\( \varphi.4.1 \)) applies (taking \( m = l \) and \( U_i = A_i \)) and we get

\[
\varphi(\text{CPATHS}(A)) = \sum_{1 \leq i \leq k} \varphi(a_i \text{CPATHS}(A_i))
\]

and \( \varphi(A) = \sum_{1 \leq i \leq k} \varphi(a_i A_i) \). By Axiom (\( \varphi.3 \)) we get

\[
\varphi(\text{CPATHS}(A)) = \sum_{1 \leq i \leq k} \varphi(a_i) \varphi(\text{CPATHS}(A_i))
\]

and \( \varphi(A) = \sum_{1 \leq i \leq k} \varphi(a_i) \varphi(A_i) \). The hypothesis of induction applied to the sets \( A_i \) implies, on the one hand, that

\[
\varphi(A) = \sum_{1 \leq i \leq k} \varphi(a_i) \varphi(\text{CPATHS}(A_i)),
\]

which shows that (13.2) is true, and on the other the hand that

\[
\varphi(\text{CPATHS}(A)) = \sum_{1 \leq i \leq k} \varphi(a_i) \sum_{\pi \in \text{CPATHS}(A_i)} \varphi(\pi).
\]

Hence, by Axiom (\( \varphi.3 \)), we have
\( \phi(\text{PATHS}(A)) = \sum_{C_{1 \leq k}} \sum_{\pi \in \text{PATHS}(A_{i})} \phi(a, \pi), \) i.e. \( \phi(\text{PATHS}(A)) = \sum_{\pi \in \text{PATHS}(A)} \phi(\pi), \) which implies that (13.1) is true.

Finally, paths of \( a_{i} \text{PATHS}(A_{i}) \) satisfy (13.3) because paths of \( \text{PATHS}(A_{i}) \) do so by hypothesis of induction. Since \( j \) is a split or a source, (13.3) is also true for paths of \( \bigcup_{1 \leq k} a_{i} \text{PATHS}(A_{i}). \)

- If \( j \) is a co-product, the reasoning is almost the same. We have \( \phi(\text{PATHS}(A)) = \phi(a_{i} \text{PATHS}(A_{i})) = \max_{1 \leq k} \phi(a_{i} \text{PATHS}(A_{i})) \) by definition of \( i' \) in Algorithm 1. Since Axiom (φ.4.2) applies, we also have \( \phi(A) = \max_{1 \leq k} \phi(a_{i}A_{i}) \) (taking \( m=I_{i} \) and \( U_{i} = A_{i}. \).)

Applying Axiom (φ.3), we get \( \phi(\text{PATHS}(A)) = \max_{1 \leq k} \phi(a_{i}) \phi(\text{PATHS}(A_{i})) \) and \( \phi(A) = \max_{1 \leq k} \phi(a_{i}) \phi(A_{i}) \) respectively. The hypothesis of induction implies then \( \phi(A) = \max_{1 \leq k} \phi(a_{i}) \phi(\text{PATHS}(A_{i})), \) which proves that (13.2) is true.

Now, notice that \( \phi(\text{PATHS}(A)) = \phi(a_{i}) \phi(\text{PATHS}(A_{i})) \) by Axiom (φ.3). By the hypothesis of induction we get \( \phi(\text{PATHS}(A)) = \phi(a_{i}) \sum_{\pi \in \text{PATHS}(A_{i})} \phi(\pi). \) Hence, by Axiom (φ.3) again, we have

\[ \phi(\text{PATHS}(A)) = \sum_{\pi \in \text{PATHS}(A_{i})} \phi(a_{i} \pi), \] i.e. \( \phi(\text{PATHS}(A)) = \sum_{\pi \in \text{PATHS}(A)} \phi(\pi) : \) (13.1) is true.

At last, paths of \( a_{i} \text{PATHS}(A_{i}) \) satisfy (13.3) because paths of \( \text{PATHS}(A_{i}) \) do so (by hypothesis of induction and because \( j \) has now only one successor, which is node \( i' \)).
Corollary 1.

If $\hat{e}$ is an emergy state of $\hat{E}([l;l'])$, with $[l,l'] \in \mathcal{A}$, then

$$\varphi(\hat{e}) = \sum_{\pi \in \hat{e}} \varphi(\pi).$$  \hspace{1cm} (14)

Proof. Since $\hat{e} = \bigcup_{s \in [l;l']} \hat{e}_s$, we have $\varphi(\hat{e}) = \sum_{s \in [l;l']} \varphi(\hat{e}_s)$ by the superposition principle (see Proposition 1).

Since $\hat{e}_s$ is an emergy state, any pair of paths of $\hat{e}_s$ are compatible, i.e. no paths can divide at a co-product node (recall case 3 of Definition 4). Hence, Algorithm 1 applied with $A = \hat{e}_s$ cannot remove a path, so $\text{CPATHS}(\hat{e}_s) = \hat{e}_s$, and since we have $\varphi(\text{CPATHS}(\hat{e}_s)) = \sum_{\pi \in \text{CPATHS}(\hat{e}_s)} \varphi(\pi)$, by (13.1), we get

$$\varphi(\hat{e}_s) = \sum_{\pi \in \hat{e}_s} \varphi(\pi).$$  \hspace{1cm} (15)

Therefore, we obtain

$$\varphi(\hat{e}) = \sum_{s \in [l;l']} \sum_{\pi \in \hat{e}_s} \varphi(\pi) = \sum_{\pi \in \hat{e}} \varphi(\pi).$$

Definition 6. (Emergy attractor)

An emergy attractor $\hat{e}^{\text{att}}$, for an arc $[l;l']$, is an emergy state of $\hat{E}([l;l'])$ such that $\varphi(\hat{e}^{\text{att}}) = \text{Em}([l;l'])$.

Theorem 1.

There exists an emergy attractor for every arc $[l;l']$ of $\mathcal{A}$.
\textbf{Proof.} We have $\varepsilon([l; l']) = \bigcup_{s \in s'} \varepsilon_s([l; l'])$ and, by Proposition 3, we can associate the set $\text{CPATHS}(\varepsilon_s([l; l']))$ with each $\varepsilon_s([l; l'])$. Property (13.3) implies that paths of $\text{CPATHS}(\varepsilon_s([l; l']))$ are pairwise compatible, i.e. $\text{CPATHS}(\varepsilon_s([l; l']))$ is an emergy state. Noticing that every path of $\text{CPATHS}(\varepsilon_s([l; l']))$ is compatible with every path of $\text{CPATHS}(\varepsilon'_s([l; l']))$, for $s, s' \in L_s$, we deduce that the set $\bigcup_{s \in s'} \text{CPATHS}(\varepsilon_s([l; l']))$ is also an emergy state.

Recall now that by (9) we have $\text{Em}([l; l']) = \varphi(\varepsilon([l; l']))$. By (10) we have $\varphi(\varepsilon([l; l'])) = \sum_{s \in s'} \varphi(\varepsilon_s([l; l']))$, i.e. $\varphi(\varepsilon([l; l'])) = \sum_{s \in s'} \varphi(\text{CPATHS}(\varepsilon_s([l; l'])))$ by (13.2). By Axiom (φ.4.1) we have $\sum_{s \in s'} \varphi(\text{CPATHS}(\varepsilon_s([l; l']))) = \varphi(\bigcup_{s \in s'} \text{CPATHS}(\varepsilon_s([l; l'])))$. Therefore, $\varphi(\bigcup_{s \in s'} \text{CPATHS}(\varepsilon_s([l; l']))) = \text{Em}([l; l'])$ and we deduce that $\bigcup_{s \in s'} \text{CPATHS}(\varepsilon_s([l; l']))$ is an emergy attractor for $[l; l']$.

\textbf{Theorem 2. (Maximum Empower)}

An attractor $\hat{\varepsilon}^{att}$ for an arc $[l; l']$ of $\mathcal{A}$ satisfies

$$\varphi(\hat{\varepsilon}^{att}) = \max_{\hat{\varepsilon} \in \hat{E}([l; l'])} \varphi(\hat{\varepsilon}).$$

(15)
Proof. Let us consider an emergy state \( \hat{e} \) of \( \hat{E}([l'; l]) \). Since \( \hat{e} \subseteq e([l; l']) \) we have, by Proposition 2, \( \varphi(\hat{e}) \leq \varphi(e([l; l'])) \), i.e. \( \varphi(\hat{e}) \leq \text{Em}([l; l']) \) by (9). By Theorem 1 we know that there exists an attractor \( \hat{e}_{\text{att}} \) for arc \([l; l']\), so \( \varphi(\hat{e}_{\text{att}}) = \text{Em}([l; l']) \) and the result follows.

4. Numerical example

We show how Theorem 2 applies. Let us consider the emergy graph of Figure 1. There are 6 emergy paths that end by arc \([9;13]\) (see Table 1). By (12) we have:

\[
\varphi(\pi_1) = \theta(1) \cdot \omega([3;4]) \cdot \omega([4;6]) \cdot \omega([6;9]) \cdot \omega([9;13]) = 1000 \cdot \frac{5}{8} \cdot 1 \cdot \frac{1}{5} = 15.625 ,
\]

\[
\varphi(\pi_2) = \theta(1) \cdot \omega([3;4]) \cdot \omega([4;7]) \cdot \omega([7;9]) \cdot \omega([9;13]) = 1000 \cdot \frac{5}{8} \cdot 1 \cdot \frac{2}{3} = 416.667 ,
\]

\[
\varphi(\pi_3) = \theta(1) \cdot \omega([3;5]) \cdot \omega([5;7]) \cdot \omega([7;9]) \cdot \omega([9;13]) = 1000 \cdot \frac{3}{8} \cdot \frac{4}{5} \cdot \frac{2}{3} = 200 ,
\]

\[
\varphi(\pi_4) = \theta(1) \cdot \omega([3;5]) \cdot \omega([5;7]) \cdot \omega([7;10]) \cdot \omega([10;4]) \cdot \omega([4;6]) \cdot \omega([6;9]) \cdot \omega([9;13])
\]
\[
= 1000 \cdot \frac{3}{8} \cdot \frac{4}{5} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{5} \cdot 1 = 6.667 ,
\]

\[
\varphi(\pi_5) = \theta(2) \cdot \omega([10;4]) \cdot \omega([4;6]) \cdot \omega([6;9]) \cdot \omega([9;13]) = 500 \cdot \frac{1}{3} \cdot \frac{1}{5} \cdot 1 = 33.333 ,
\]

\[
\varphi(\pi_6) = \theta(2) \cdot \omega([10;4]) \cdot \omega([4;7]) \cdot \omega([7;9]) \cdot \omega([9;13]) = 500 \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot 1 = 111.111 .
\]

According to Theorem 2 we have to find an attractor for arc \([9;13]\), i.e. an emergy state \( \hat{e} \) of \( \hat{E}([9;13]) \) that maximizes \( \varphi(\hat{e}) \). In our case, \( \hat{E}([9;13]) \) contains at most \( 2^6 \) emergy states because there are 6 emergy paths. However, it is possible to avoid enumerating the 64 sets by noticing that paths \( \pi_3 \) and \( \pi_4 \) 

24
are compatible with all other paths (see Table 2). Therefore, an attractor \( \hat{\epsilon}^{\text{att}} \) is of the form
\[
\hat{\epsilon}^{\text{att}} = \{\pi_3, \pi_4\} \cup \hat{\epsilon}' \quad \text{with} \quad \hat{\epsilon}' \subseteq \{\pi_1, \pi_2, \pi_5, \pi_6\}.
\]
By Corollary 1, we have
\[
\phi(\{\pi_3, \pi_4\} \cup \hat{\epsilon}') = \phi(\pi_3) + \phi(\pi_4) + \phi(\hat{\epsilon}')
\]
so we get
\[
\phi(\hat{\epsilon}^{\text{att}}) = \max_{\hat{\epsilon}' \subseteq \{\pi_1, \pi_2, \pi_5, \pi_6\}} (\phi(\pi_3) + \phi(\pi_4) + \phi(\hat{\epsilon}'))
\]
Hence, finding \( \hat{\epsilon}^{\text{att}} \) reduces to finding \( \hat{\epsilon}' \) with
\[
\phi(\hat{\epsilon}') = \max_{\hat{\epsilon}' \subseteq \{\pi_1, \pi_2, \pi_5, \pi_6\}} \phi(\hat{\epsilon}).
\]
Now, let us notice that \( \pi_1 \) and \( \pi_2 \) (resp. \( \pi_5 \) and \( \pi_6 \)) are not compatible. Hence, we only have to consider 4 candidates for \( \hat{\epsilon}' \): \( \hat{\epsilon}_1 = \{\pi_1, \pi_3\}, \hat{\epsilon}_2 = \{\pi_1, \pi_6\}, \hat{\epsilon}_3 = \{\pi_2, \pi_3\} \) and \( \hat{\epsilon}_4 = \{\pi_2, \pi_6\} \).

By (14) we have:
\[
\phi(\hat{\epsilon}_1) = \phi(\pi_1) + \phi(\pi_3) = 15.625 + 33.333 = 48.958,
\]
\[
\phi(\hat{\epsilon}_2) = \phi(\pi_1) + \phi(\pi_6) = 15.625 + 111.111 = 126.736,
\]
\[
\phi(\hat{\epsilon}_3) = \phi(\pi_2) + \phi(\pi_3) = 416.667 + 33.333 = 450.000,
\]
\[
\phi(\hat{\epsilon}_4) = \phi(\pi_2) + \phi(\pi_6) = 416.667 + 111.111 = 527.778.
\]
Hence, the four candidates for \( \hat{\epsilon}^{\text{att}} \) are: \( \{\pi_1, \pi_3, \pi_4, \pi_5\}, \{\pi_1, \pi_3, \pi_4, \pi_6\}, \{\pi_2, \pi_3, \pi_4, \pi_5\} \) and \( \{\pi_2, \pi_3, \pi_4, \pi_6\} \)
.(see Figures 2,3,4 and 5 respectively).

Therefore, an attractor for arc [9;13] is obtained by \( \hat{\epsilon}^{\text{att}} = \{\pi_2, \pi_3, \pi_4, \pi_6\} \) and we have
\[
\text{Em}[9;13] = \phi(\pi_3) + \phi(\pi_4) + \phi(\hat{\epsilon}_4) = 206.667 + \phi(\hat{\epsilon}_4) = 734.445,
\]
which is the value obtained in (Le Corre and Truffet 2012).
Figure 2. The emergy paths of emergy state $\hat{\varepsilon}_1$. 
Figure 3. The emergy paths of emergy state $\hat{e}_2$. 
Figure 4. The emergy paths of emergy state $\hat{\epsilon}_3$. 
5. Conclusions

We have shown that the maximum empower principle is equivalent to finding an emergy state among many (see Theorem 2). Hence, emergy evaluation can also be seen as a combinatorial optimization problem. This gives the opportunity to tackle emergy analysis by using techniques from the rich literature on combinatorial optimization.
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