Phase Structure
of the $O(n)$ Model on a Random Lattice for $n > 2$

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Abstract

We show that coarse graining arguments invented for the analysis of multi-spin systems on a randomly triangulated surface apply also to the $O(n)$ model on a random lattice. These arguments imply that if the model has a critical point with diverging string susceptibility, then either $\gamma = +\frac{1}{2}$ or there exists a dual critical point with negative string susceptibility exponent, $\tilde{\gamma}$, related to $\gamma$ by $\gamma = \frac{5}{\tilde{\gamma} - 1}$. Exploiting the exact solution of the $O(n)$ model on a random lattice we show that both situations are realized for $n > 2$ and that the possible dual pairs of string susceptibility exponents are given by $(\tilde{\gamma}, \gamma) = \left(-\frac{1}{m}, \frac{1}{m+1}\right)$, $m = 2, 3, \ldots$. We also show that at the critical points with positive string susceptibility exponent the average number of loops on the surface diverges while the average length of a single loop stays finite.
1 Introduction

Two-dimensional gravity coupled to conformal matter fields of central charge $c \leq 1$ is a reasonably well understood subject both in the continuum approach [1, 2, 3] and in the framework of dynamical triangulations, see [4] for reviews. The case $c > 1$ remains, however, to a large extent ununderstood. In the continuum this is reflected e.g. by the fact that critical exponents become non-real (see, however, [5]) whereas in the discrete approaches the naive regularized models are generally not exactly solvable and most results rely upon numerical simulations [6]. In [7] a kind of dual relationship between a class of discretized $c > 1$ theories, or rather models with diverging susceptibility, e.g. multiple Ising spin models, and corresponding $c < 1$ models was proposed on the basis of universality considerations. For unitary models this leads to useful restrictions on the values of critical exponents [7].

It is well known that the $O(n)$ model on a random triangulation [8] for suitable rational values of the parameter $\nu \in ]0, 1[$, related to $n$ by $n = 2 \cos(\nu \pi)$, represents (sectors of) two-dimensional gravity coupled to minimal conformal field theories of central charge $c < 1$ [8, 9, 10, 11]. Recently the exact solution of the model, valid for all values of $n$, has been found [12, 13]. Although it is not clear that the model for $n > 2$ can reasonably be considered as representing $c > 1$ conformal field theories coupled to two-dimensional gravity it is clearly of interest to investigate its properties beyond the threshold $n = 2$ where critical points with diverging susceptibility are expected to exist. The purpose of this note is to point out that the $O(n)$ model (at least for integer $n$) falls within the class of models to which the arguments in [7] apply and to exploit the exact solution in [12, 13] to evaluate the dual pairs of susceptibility exponents. We find that for $n > 2$ the model is either in a branched polymer phase or that the pairs of exponents are given by $\left(-\frac{1}{m}, \frac{1}{m+1}\right)$, where $m = 2, 3, \ldots$. This result in turn corroborates the universality assumption of [7] and thereby indirectly the existence of a continuum limit at the points with diverging susceptibility. We show that at these points the average number of loops on the surfaces diverges while the average length of a single loop stays finite as suggested in an early paper by I. Kostov [14]. The investigation of the continuum limit is, however, beyond the scope of this paper.

We start in section 2 by introducing the $O(n)$ model on a random lattice and present in section 3 a version of the coarse graining argument of [7] valid for this model. In section 4 we recall a few important characteristics of the exact solution of the model and in section 5, using the exact solution, we determine the possible values of the string susceptibility exponent for $n > 2$. Section 6 is devoted to the calculation of the average number of loops on the surfaces and finally section 7 contains some concluding remarks.
2 The $O(n)$ model on a random triangulation

In matrix model language the partition function of the $O(n)$ model on a random triangulation is defined as (see [8])

$$Z(g) = \int_{\mathbb{R}^{N \times N}} dM \prod_{i=1}^{n} dA_i \exp \left\{-N \text{ tr} \left( \frac{1}{2}M^2 + V(M) + \left(g_0M + \frac{1}{2}\right) \sum_{i=1}^{n} A_i^2 \right) \right\} \quad (2.1)$$

where $M$ and $A_1, \ldots, A_n$ are hermitian $N \times N$ matrices and

$$V(M) = \sum_{j=3}^{N} \frac{g_j}{j} M^j \quad (2.2)$$

is a polynomial potential depending on the coupling constants $g_3, g_4, \ldots$ and we use the collective notation $g = (g_0, g_3, g_4, \ldots)$. By expanding the non-quadratic part of the exponent in powers of $g_0, g_3, g_4, \ldots$ we obtain an interpretation of $Z(g)$ as the partition function of a gas of $n$ species of self-avoiding and non-intersecting loops living on a random triangulation (see [8]). Specifically, after a suitable normalization of the measures $dM$ and $dA_i$ we have

$$F(g) = \frac{1}{N^2} \log Z(g) = \sum_{h=0}^{\infty} N^{-2h} F_h(g) \quad (2.3)$$

where

$$F_h(g) = \sum_{\tau \in T_h} \sum_{L_1, \ldots, L_n} C_\tau^{-1}(L_1, \ldots, L_n) \prod_{j \geq 3} (-g_j)^{N_j(\tau)} \left( \begin{array}{c} g_0 \\ g_3 \end{array} \right)^{|L_1| + \ldots + |L_n|} \quad (2.4)$$

Here $T_h$ denotes the set of two-dimensional closed, connected complexes of genus $h$ obtained by gluing together $N_j(\tau)$, $j \geq 3$, $j$-gons along pairs of links and $L_i$, $i = 1, \ldots, n$, is any collection of loops $\omega_1^i, \ldots, \omega_{m_i}^i$ on the dual complex whose links are dual to links shared by pairs of triangles in $\tau$, i.e. the links in the loops may be considered as connecting centres of neighbouring triangles in $\tau$. By $|L_i|$ we denote the total number of links in $\omega_1^i, \ldots, \omega_{m_i}^i$, i.e. their total length, and all loops in $L_1, \ldots, L_n$ are required to be self-avoiding and pairwise disjoint. Finally, $C_\tau(L_1, \ldots, L_n)$ denotes the order of the automorphism group of $\tau$ with the configuration of loops $L_1, \ldots, L_n$.

In figure 1 we show a surface contributing to $F_1(g)$.

Although in general $\tau$ does not consist of triangles alone we shall refer to $\tau$ as a triangulation. In the following we shall be interested in the spherical limit $N \to \infty$. 

3
3 Coarse graining

Given a polygonal loop $\gamma_0$ we let $T(\gamma_0)$ be the set of triangulations of planar topology bounded by $\gamma_0$ and define the one-loop function $W_{\gamma_0}(g)$ by

$$W_{\gamma_0}(g) = \sum_{\tau \in T(\gamma_0)} \sum L_1, \ldots, L_n \prod_{j \geq 3} (-g_j)^{N_j(\tau)} \frac{g_0}{g_3}^{(L_1 + \ldots + L_n)}$$

(3.1)

where $L_1, \ldots, L_n$ are as above. Due to the relation

$$\sum_{j \geq 3} jN_j(\tau) = 2L(\tau) - |\gamma_0|,$$

(3.2)

where $L(\tau)$ denotes the number of links in $\tau \in T(\gamma_0)$, it follows that if $\gamma_0$ has length two, which we henceforth assume, then $W_{\gamma_0}(g) \equiv W(g)$ is obtained by applying a suitable linear combination of the derivatives $g_j \frac{\partial}{\partial g_j}$, $j = 0, 3, \ldots$ to $F_0(g)$. Applying a further differentiation yields the susceptibility

$$\chi_{00}(g) = \sum_{\tau \in T(\gamma_0, \gamma')} \sum L_1, \ldots, L_n \prod_{j \geq 3} (-g_j)^{N_j(\tau)} \frac{g_0}{g_3}^{(L_1 + \ldots + L_n)}$$

(3.3)

where $T(\gamma_0, \gamma')$ denotes the set of triangulations of cylindrical topology bounded by two loops $\gamma_0$ and $\gamma'$, both of length two.

It is well known that the sums (3.1) and (3.3) are convergent for $|g_j|, j = 0, 3, 4, \ldots$, small enough and hence that $W(g)$ and $\chi_{00}(g)$ are analytic in a domain $\mathcal{A}$ of coupling constants. We shall be interested in the singular behaviour of these quantities at the boundary $\partial \mathcal{A}$ of the domain of analyticity, in the following referred to as the critical surface. Assuming that the singular (or critical) behaviour reflects properties of an underlying continuum surface theory we expect the detailed structure at the scale of the lattice cutoff of the discrete surfaces (triangulations) that are summed over to be unimportant. In particular, we may restrict the ensembles $T(\gamma_0)$ and $T(\gamma_0, \gamma')$ to the corresponding ensembles $\tilde{T}(\gamma_0)$ and $\tilde{T}(\gamma_0, \gamma')$ consisting of surfaces which do not
contain any loops of length two (except for the boundary loops). In this way we define a one-loop function $\bar{W}$ and a susceptibility $\bar{\chi}_{00}$ in a domain of analyticity $\tilde{A}$ and we expect by universality that there is a one-to-one mapping between $\partial\mathcal{A}$ and $\partial\tilde{\mathcal{A}}$ by which corresponding points represent identical critical behaviour.

In order to exploit this, let us first express $W$ and $\chi_{00}$ in terms of $\tilde{W}$ and $\tilde{\chi}_{00}$. This is done simply by performing a partial summation in (3.1) and (3.3) over outgrowths on $\tau$ bounded by loops of length two as follows. Given a loop $\gamma$ of length two in $\tau \in T(\gamma_0)$ there are two cases to consider:

1. There is no loop in $L_1 \cup \ldots \cup L_n$ which crosses $\gamma$.
2. A loop $\omega_j$ crosses $\gamma$, and since $\omega_j$ is closed it actually crosses both links in $\gamma$.

By cutting $\tau$ along $\gamma$ the triangulation $\tau$ decomposes into a triangulation $\tau_0' \in T(\gamma_0, \gamma)$ and an outgrowth $\tau_1 \in T(\gamma)$. In case 2 a segment of $\omega_j$ is contained in $\tau_0'$ and the remaining part in $\tau_1$. Both segments emerge from one link in $\gamma$ and end at the other one. Thus when summing over all outgrowths $\tau_1$ we obtain in case 2 a contribution $(1 + W_1(g))$ where $W_1(g)$ is given by the right hand side of equation (3.1) with the sum restricted to triangulations $\tau \in T(\gamma)$ whose boundary polygons are triangles with collections of self-avoiding and non-intersecting curves on the dual triangulation, all of which are closed except one which connects the two boundary links of $\gamma$ and which has half-links at its ends, see figure 2.

By gluing the two links in the boundary component $\gamma$ in $\tau_0'$ together we obtain a triangulation $\tau_0 \in T(\gamma_0)$ and the segment of $\omega_j$ in $\tau_0'$ turns into a closed loop. It follows that summation over maximal outgrowths allows us to write (see also [7])

$$W(g) = \sum_{\tau \in T(\gamma_0)} \sum_{L_1, \ldots, L_n} \prod_{j \geq 3} (-g_j)^{N_j(\tau)} \left( \frac{g_0}{g_3} \right)^{|L_1|+\ldots+|L_n|} \times$$

$$\left(1 + W(g)\right)^{L(\tau) - (|L_1|+\ldots+|L_n|) - 1} \left(1 + W_1(g)\right)^{|L_1|+\ldots+|L_n|}, \quad (3.4)$$

Figure 2: A surface contributing to $W_1(g)$
which as a consequence of (3.2) can be rewritten as

$$W(g) = \tilde{W}(\tilde{g}),$$

(3.5)

with

$$\tilde{g}_j = \left(1 + W(g)\right)^{j/2} g_j, \quad j \geq 3,$$

(3.6)

$$\tilde{g}_0 = \left(1 + W_1(g)\right) \left(1 + W(g)\right)^{1/2} g_0.$$  

(3.7)

The susceptibility may be treated similarly. Cutting along loops which decompose $\tau \in T(\gamma_0, \gamma)$ into two cylindrical triangulations yields a geometric series for the $2 \times 2$ matrix $\chi_{ij}(g)$, $0 \leq i, j \leq 1$, of susceptibilities defined as in (3.3) but with boundary conditions imposed at $\gamma_0$ and $\gamma$ as follows: For $\chi_{01}(g) = \chi_{10}(g)$ the summation in (3.3) is subject to the constraint that the boundary polygons in $\tau$ at, say, $\gamma_0$ are triangles and there is a polygonal curve in $L_1 \cup \ldots \cup L_n$ connecting the two links in $\gamma_0$, i.e. $L_1, \ldots, L_n$ are as in the definition of $W_1(g)$; $\chi_{11}(g)$ is given by (3.3) with $\tau$ restricted so that its four boundary polygons are all triangles and in $L_1, \ldots, L_n$ there are two polygonal curves connecting two pairs of boundary links. We then have

$$\chi(g) = \tilde{\chi}(\tilde{g}) \left(1 - \tilde{\chi}(\tilde{g})\right)^{-1}$$

(3.8)

where 1 is the $2 \times 2$ unit matrix. Equations (3.3) and (3.8) are the desired expressions for $W(g)$ and $\chi(g)$ in terms of the corresponding quantities in the coarse grained tilde-model. At this point we can adopt the arguments of [7]. For the sake of completeness we shall briefly recapitulate them. First, let us note that the strongest divergence of the susceptibilities $\chi_{ij}$ will always be contained in $\chi_{00}$. The susceptibilities $\chi_{01}, \chi_{10}$ and $\chi_{11}$ can be less divergent than $\chi_{00}$, namely if the number of links in $L_1 \cup \ldots \cup L_n$ on the average occupies a vanishing fraction of the total number of links in $\tau$, but the opposite situation can not occur. We consider now a critical point $\bar{g} \in \partial A$ at which $\chi_{00}(g)$ diverges with an associated critical exponent $\gamma > 0$ such that

$$\chi_{00}(\bar{g}) \sim |\bar{g} - g|^\gamma$$

(3.9)

as $g$ approaches $\bar{g}$ transversally to $\partial A$. As noted previously, $\chi_{00}(g)$ is obtained as a derivative of $W(g)$ (cf. page 4). Since $W(g)$ is finite as a consequence of (3.5), (3.6) and (3.7) it follows that

$$W(g) \sim W(\bar{g}) + cst \cdot |g - \bar{g}|^{1-\gamma}. $$

(3.10)

\footnote{For (3.5) to hold we actually need to distinguish the links in each boundary two-loop. This convention, which we note does not influence the singular behaviour of the quantities under consideration, is the reason for the absence of symmetry factors in (3.1), (3.3) and (3.4).}
By diagonalising the susceptibility matrices in (3.8) it is seen that the divergence of \( \chi(g) \) is a consequence of the largest of the eigenvalues of \( \tilde{\chi}(\tilde{g}) \) tending to 1 as \( g \to g_c \). Thus assuming that this approach is given by a susceptibility exponent \( \tilde{\gamma} \leq 0 \) such that

\[
\tilde{\chi}((\tilde{g})) \sim \tilde{\chi}_c + C \cdot |\tilde{g} - \tilde{g}(g_c)|^{-\tilde{\gamma}}
\]

(3.11)

where \( 1 - \tilde{\chi}_c \) is singular we obtain from (3.6)–(3.10) that

\[
|\tilde{g} - \tilde{g}(g_c)|^{-\gamma} \sim |\tilde{g} - \tilde{g}(g_c)|^{\tilde{\gamma}(1-\gamma)}
\]

(3.12)
i.e.,

\[
\gamma = \frac{-\tilde{\gamma}}{1-\tilde{\gamma}}.
\]

(3.13)

Alternatively, it may happen that \( \tilde{g}(g_c) \in \tilde{A} \) i.e. that \( \tilde{\chi} \) is analytic at \( \tilde{g}(g_c) \). This case corresponds to \( \tilde{\gamma} = -1 \) in (3.11), i.e.

\[
\gamma = \frac{1}{2}.
\]

(3.14)

Returning to the universality assumption discussed previously we have thus obtained that if there exists a critical point \( g_c \) with susceptibility exponent \( \gamma > 0 \) then either \( \gamma = \frac{1}{2} \) or there exists a corresponding critical point with critical exponent \( \tilde{\gamma} \) related to \( \gamma \) by (3.13). In section 5 we show by exploiting the exact solution of the model that both cases (3.13) and (3.14) are actually realized for \( n > 2 \), and that by suitable choices of the potential, \( \tilde{\gamma} \) may assume the values \( -\frac{1}{m} \), \( m = 2, 3, \ldots \), with corresponding values of \( \gamma = \frac{1}{m+1} \).

4 The exact solution

For the following considerations it is convenient to write the partition function of the model in the form

\[
Z = e^{N^2 F} = \int_{N \times N} dM \prod_{i=1}^{n} dA_i \exp \left\{ -N \text{tr} \left( V(M) + \sum_{i=1}^{n} MA_i^2 \right) \right\}
\]

(4.1)

where \( V(M) \) is an arbitrary polynomial potential, \( V(M) = \sum_{j=1}^{m} g_j / j M^j \). From (4.1) the form of (2.1) is easily recovered by a linear shift of \( M \). Expressed in terms of the eigenvalues, \( \{\lambda_i\} \), of the matrix \( M \) the integral (4.1) reads

\[
Z \propto \int \prod_{i=1}^{N} d\lambda_i \exp \left\{ -N \sum_{j} V(\lambda_j) \right\} \prod_{j<k}(\lambda_j - \lambda_k)^2 \prod_{i,l}(\lambda_i + \lambda_l)^{-n/2}.
\]

(4.2)

In the large \( N \) limit the eigenvalue configuration is determined by the saddle point of the integral above and the eigenvalues are confined to a compact region of the real
axis \[15\]. In the following we consider the situation where the eigenvalues live on only one interval \([a, b]\) on the positive real axis. We shall make use of the exact solution of the model found in this situation in \[12, 13\].

Let us briefly recall the main characteristics of this solution. It proved convenient to parametrize the solution in terms of two sets of basis functions, \(\{G_a^{(k)}(p), G_b^{(k)}(p)\}\) and \(\{\tilde{G}_a^{(k)}(p), \tilde{G}_b^{(k)}(p)\}\), \(k \in \{0, 1, \ldots\}\). The \(G\)-functions are defined by certain requirements on their analyticity structure and their asymptotic behaviour. The \(\tilde{G}\)-functions are obtained from the \(G\)-functions by the replacement \(\nu \rightarrow (1 - \nu)\). Furthermore, functions with subscript \(b\) are obtained from functions with subscript \(a\) by the interchange \(a \leftrightarrow b\). Explicit expressions for all basis functions in terms of \(\theta\)-functions can be written down. In the following we shall not need the explicit expressions for the \(G\) and \(\tilde{G}\)-functions, only the fact that they fulfill certain recursion relations. For instance it holds that

\[
\begin{align*}
\frac{\partial G_a^{(k)}(p)}{\partial a} &= \lambda_a^{(k)} G_a^{(k+1)}(p), \\
\rho a G_a^{(k)}(p) &= \tilde{G}_a^{(k-1)}(p) + s_a^{(k)} \tilde{G}_a^{(k)}(p)
\end{align*}
\]

where \(\{\lambda_a^{(k)}\}\) and \(\{s_a^{(k)}\}\) are some (finite) constants. These constants can be expressed entirely in terms of the endpoints of the support of the eigenvalue distribution, \(a\) and \(b\), and two additional parameters \(e\) and \(\alpha\). The parameters \(e\) and \(\alpha\) are given by

\[
\begin{align*}
e &= a \text{ sn} \left( i (1 - \nu) K', k \right) \\
\alpha &= b \left\{ Z \left( i (1 - \nu) K', k \right) + i (1 - \nu) \frac{\pi}{2K} \right\}, \\
k &= \frac{a}{b}
\end{align*}
\]

where \(Z\) is the Jacobian zeta function and \(K\) and \(K'\) are the complete elliptic integrals of the first kind. The quantity \(\alpha\) is essentially a first derivative of \(e\),

\[
(b^2 - a^2) \rho_a = \frac{\sqrt{\tilde{e}}}{e} \alpha + b^2 - e^2
\]

where

\[
\rho_a = \frac{a^2 \partial e^2}{e^2 \partial a^2}, \quad \sqrt{\tilde{e}} = \sqrt{(e^2 - a^2)(e^2 - b^2)}.
\]

Obviously, recursion relations similar to (4.3) and (4.4) hold in the case where \(a\) is replaced by \(b\). Furthermore similar recursion relations hold when untilded functions and constants are replaced by their tilded analogues. In this connection it is useful to note that

\[
\tilde{\rho}_a = 1 - \rho_a = \rho_b, \quad \tilde{e} = -\frac{ab}{e}, \quad \sqrt{\tilde{e}} = \frac{\sqrt{e}}{\tilde{e}}.
\]
analysis we shall need the following results:

\[ \lambda_a^{(0)} = -\frac{1}{2} i \tan \left( \frac{\nu \pi}{2} \right) \frac{e^2 e^2 - a^2}{a^2 e \sqrt{e}} \rho_a, \]  

(4.10)

\[ s_a^{(1)} = \frac{1}{2} \left( 1 - \frac{\rho_a}{\lambda_a^{(0)}} \right), \quad s_a^{(k+1)} = s_a^{(k)} \frac{\lambda_a^{(k)}}{\lambda_a^{(k)}}. \]  

(4.11)

The quantities mentioned so far depend on the matrix model potential only implicitly, namely via \( a \) and \( b \). All explicit dependence on the potential \( V(M) \) is described via a set of moment variables defined by

\[ M_k = \oint_C \frac{d\omega}{2\pi i} V'(\omega) \tilde{G}_a^{(k)}(\omega), \quad J_k = M_k(a \leftrightarrow b) \]  

(4.12)

where \( C \) is a curve which encircles the interval \([a, b]\). In particular, the equations which determine \( a \) and \( b \) can conveniently be expressed in terms of moment variables. They read

\[ M_0 = \oint_C \frac{d\omega}{2\pi i} V'(\omega) \tilde{G}^{(0)}(\omega) = 0, \]  

(4.13)

\[ M_{-1} = \oint_C \frac{d\omega}{2\pi i} V'(\omega) \omega G^{(0)}(\omega) = 2 - n \]  

(4.14)

where \( G^{(0)}(p) \equiv G_a^{(0)}(p) = G_b^{(0)}(p) \).

5 Critical behaviour for \( n > 2 \)

We now turn to discussing the critical properties of the model. For that purpose we introduce an overall coupling constant in front of our potential, i.e. we replace \( V(M) \) by \( V(M)/T \) where \( T \) is to be thought of as the cosmological constant. Then we define the string susceptibility \( U(T) \) by

\[ U(T) = \frac{d^2}{dT^2} \left( T^2 F_0 \right) \]  

(5.1)

where \( F_0 \) is the genus zero contribution to the free energy. The leading singularity of \( U(T) \) is the same as that of the susceptibility \( \chi_{00}(q) \), introduced in the previous section. In geometrical terms \( U(T) \) expresses the average number of links minus the average area for surfaces with one boundary component, i.e.

\[ U(T) \sim \langle L(\tau) - A(\tau) \rangle_{\tau \in T_0(\gamma)} \]  

(5.2)

where \( A(\tau) = \sum_{j \geq 3} N_j(\tau) \). We note that in the vicinity of a singular point the quantities \( \langle L(\tau) \rangle, \langle A(\tau) \rangle \) and \( U(T) \) obey the same scaling behaviour.
What we will aim at calculating is, in case the model has a critical point at $T = T_c$, the value of the critical index $\gamma$

$$U(T) \sim (T_c - T)^{-\gamma}.$$ \hfill (5.3)

In reference [13] the following expression for $dU(T)/dT$ was derived

$$\frac{dU(T)}{dT} = \left(1 - \frac{n}{2}\right) \frac{1}{b^2 - a^2} \left\{ \frac{b^2 - \bar{c}^2}{\bar{c}^2} \frac{da^2}{dT} - \frac{a^2 - \bar{c}^2}{\bar{c}^2} \frac{db^2}{dT} \right\}. \hfill (5.4)$$

This expression is universal in the sense that it does not contain any direct reference to the matrix model potential. We can make the expression (5.4) even more explicit by deriving closed expressions for $da^2/dT$ and $db^2/dT$. To do so we first differentiate the boundary equation (4.13) with respect to $T$. Using the relation (4.3) and the definition of the moment variables (4.12) we get

$$\tilde{\lambda}_a^{(0)} M_1 \frac{da^2}{dT} + \tilde{\lambda}_b^{(0)} J_1 \frac{db^2}{dT} = 0. \hfill (5.5)$$

Next we differentiate the boundary equation (4.14) with respect to $T$. This gives

$$\frac{1}{2} (1 - \rho_a) M_1 \frac{da^2}{dT} + \frac{1}{2} (1 - \rho_b) J_1 \frac{db^2}{dT} = 2 - n \hfill (5.6)$$

where on the way we have made use of the relations (4.3), (4.4) and (4.11). Combining (5.5) and (5.6) and using (4.10) we get

$$\frac{da^2}{dT} = 2(2 - n) \frac{(b^2 - \bar{c}^2)}{(b^2 - a^2)\bar{c}^2} \frac{a^2}{(1 - \rho_a)M_1}, \hfill (5.7)$$

and similarly $db^2/dT$ equals the right hand side of (5.7) with $a$ and $b$ interchanged. Inserting the expressions for $da^2/dT$ and $db^2/dT$ into (5.4) we get

$$\frac{dU(T)}{dT} = (2 - n)^2 \frac{1}{(b^2 - a^2)^2} \left\{ \frac{a^2(b^2 - \bar{c}^2)^2}{\bar{c}^4} \frac{1}{(1 - \rho_a)M_1} - \frac{b^2(a^2 - \bar{c}^2)^2}{\bar{c}^4} \frac{1}{\rho_a J_1} \right\}. \hfill (5.8)$$

We stress that the expression (5.8) is valid for any value of $n$ and any potential $V(M)$. In formula (5.8) it is easy to spot singular points of the model. From now on we shall concentrate our analysis on the case $n > 2$. Accordingly we set $\nu = i\bar{\nu}$ with $\bar{\nu}$ real. We hence have

$$n = 2 \cosh(\bar{\nu}\pi) \hfill (5.9)$$

Obviously a singularity appears if $\bar{c}$ becomes equal to zero. This happens whenever $k = \frac{4}{6}$ takes one of the values for which

$$\bar{\nu} K' = 2mK, \quad m = 1, 2, \ldots \hfill (5.10)$$
As argued in reference [13] the model only has meaning until the first of these singularities is reached i.e. for $k > k_c$ where $k_c$ is given by

$$\frac{K'}{K}(k_c) = \frac{2}{\nu}. \quad (5.11)$$

We note that the critical value of $k$ corresponds to a finite (positive) value of $a$. In particular, this means that for $n > 2$ the model makes no sense at the point $a = 0$ which would be the naive analytical continuation of the critical point for $n \in [-2, 2]$ to $n > 2$. In addition to the singularity which occurs for $\tilde{e} = 0$ we might expect to encounter a singularity if $1 - \rho_a = 0$ or $\rho_a = 0$. However, as shown in reference [13] the first possibility can be realized only for $k < k_c$ and the second one does not imply any kind of non analytical behaviour. Let us also note, that the parameter $\tilde{e}$ can not diverge when $n > 2$. It is possible for $\rho_a$ to diverge but only at the points $\tilde{e} = 0$. Finally the model can become singular if one of the moments $M_1$ (or $J_1$) acquires a zero of some order. It follows immediately from the analysis of [13] that at the simplest possible of these points, $k = k_c$, $M_1^c \neq 0$, $J_1^c \neq 0$, the model is in a branched polymer phase, i.e.

$$\gamma = \frac{1}{2}, \quad k = k_c, \quad M_1^c \neq 0, \quad J_1^c \neq 0. \quad (5.12)$$

We shall now argue that the remaining critical points for $n > 2$ can be classified in the following way

- $M_1^c = M_2^c = \ldots = M_{m-1}^c = 0$, $M_m^c \neq 0$, $k > k_c$
  \[ \gamma = -\frac{1}{m}. \]

- $M_1^c = M_2^c = \ldots = M_{m-1}^c = 0$, $M_m^c \neq 0$, $k = k_c$
  \[ \gamma = +\frac{1}{m+1}. \]

In other words we have exactly the situation predicted by the coarse graining argument: By fine tuning the potential we can reach critical points with positive values of the string susceptibility exponent and these points have dual partners where the string susceptibility exponent is negative. It appears that the possible values for the dual pairs of exponents are $(\tilde{\gamma}, \gamma) = \left(-\frac{1}{m}, \frac{1}{m+1}\right)$, $m = 2, 3, \ldots$ and that the critical points at which $\gamma = \frac{1}{m+1}$, $m = 2, 3, \ldots$ are located where the critical surface corresponding to a $c < 1$ theory $(\tilde{\gamma} = -\frac{1}{m})$ intersects the critical surface corresponding to branched polymer behaviour.

For simplicity, let us first consider a critical point corresponding to $M_1^c = 0$, $M_2^c \neq 0$, $k > k_c$ and let us approach the point by fixing the coupling constants of $V(M)$ at their critical values and letting $T$ approach $T_c = 1$. The boundary equations (4.13) and (4.14) tell us how $a$ and $b$ approach their critical values $[a_c, b_c]$ under this fine
tuning. Expanding equation (4.13) in powers of \((b^2 - b_c^2)\) and \((a^2 - a_c^2)\) and keeping only leading order terms we get

\[
\tilde{\lambda}_b^{(0)} J_1^c (b^2 - b_c^2) + \frac{1}{2} \tilde{\lambda}_a^{(0)} \tilde{\lambda}_a^{(1)} M_2^c (a^2 - a_c^2)^2 = 0 \tag{5.13}
\]

where we have made use of the relation (4.3) and exploited the fact that \(M_1^s = 0\) while \(M_2^c \neq 0\). This gives

\[
(b^2 - b_c^2) = - \frac{1}{2} \tilde{\lambda}_a^{(0)} \tilde{\lambda}_a^{(1)} M_2^c \frac{a^2 - a_c^2}{J_1^c} \tag{5.14}
\]

It is easy to see that \(J_1^c\) must necessarily be different from \(M_1^c\) and hence non-vanishing. Treating the second boundary equation (4.14) in a similar manner and taking into account the relations (4.3) and (4.4) one finds

\[
\lambda_b^{(0)} s_b^{(1)} J_1^c (b^2 - b_c^2) + \frac{1}{2} \lambda_a^{(0)} \lambda_a^{(1)} s_a^{(2)} M_2^c (a^2 - a_c^2)^2 = (2 - n) (T - T_c) \tag{5.15}
\]

Inserting the expression (5.14) for \((b^2 - b_c^2)\) in (5.15) and making use of (4.10) and (4.11) we finally get

\[
\left. \frac{1}{4} M_2^c \tilde{\lambda}_a^{(1)} \tilde{\rho}_a \left\{ \frac{a^2 - b^2}{\bar{c}^2 - b^2} \right\} \frac{\bar{e}^2}{a^2} \right|_{T = T_c} (a^2 - a_c^2)^2 = (n - 2) (T_c - T) \tag{5.16}
\]

Since per assumption \(\bar{e} \neq 0\) \((k > k_c)\) and \(|\bar{e}| < \infty\) as well as \(\bar{e}^2 \neq b^2\) \((n > 2)\) we have

\[
(a^2 - a_c^2)^2 \sim (T_c - T) \tag{5.17}
\]

Furthermore, expanding \(M_1\) around \(a = a_c\) and exploiting the fact that \(M_2 \neq 0\) we get

\[
M_1 \sim (a^2 - a_c^2) \tag{5.18}
\]

Finally it follows from (4.6) and (4.7) that \(\rho_a\) stays finite as \(a \to a_c\). In conclusion we find

\[
\frac{dU(T)}{dT} \sim (a^2 - a_c^2) \sim (T_c - T)^{-1/2} \tag{5.19}
\]

which implies that

\[
\gamma = - \frac{1}{2}. \tag{5.20}
\]

It is obvious how the argument goes in the general case. For \(M_1^c = M_2^c = \ldots = M_m^c = 0\), \(M_m^c \neq 0 \text{ and } k > k_c\) one gets

\[
M_1 \sim (a^2 - a_c^2)^{m-1}, \quad (a^2 - a_c^2)^m \sim (T_c - T) \tag{5.21}
\]

which gives

\[
\frac{dU(T)}{dT} \sim (T - T_c)^{-\frac{m-1}{m}}, \quad \text{i.e.} \quad \gamma = - \frac{1}{m} \tag{5.22}
\]
Now let us turn to considering the type of critical point where in addition \( \tilde{e} = 0 \). For simplicity, let us assume that otherwise we have the same situation as before, i.e. \( M_1^c = 0, M_2^c \neq 0 \). In that case, obviously the relation (5.14) between \( b^2 - b_c^2 \) and \( a^2 - a_c^2 \) remains the same. So does the relation (5.16) but we see that now the coefficient of \( (a^2 - a_c^2)^2 \) becomes equal to zero: From (4.6) and (4.7) (expressed in terms of tilded variables instead of untilded ones) we conclude that \( \tilde{\rho}_a \cdot \tilde{e} \) is finite as \( k \to k_c \). This follows from the fact that \( \tilde{\alpha} \) is finite for \( k = k_c \). Hence the total coefficient vanishes. This implies that the relation (5.17) is replaced by
\[
(a^2 - a_c^2)^3 \sim (T_c - T).
\]
(5.23)

(It is easy to convince oneself that under the given circumstances the coefficient of the term cubic in \( a^2 - a_c^2 \) in the expansion of (4.14) can not vanish.) For the scaling of \( M_1 \) we still have the same behaviour as in (5.18). However, in the present case, in addition \( \tilde{e} \) and \( \tilde{\rho}_a \) will scale. As mentioned above \( \rho_a \cdot \tilde{e} \) is finite as \( k \to k_c \). Now, expanding \( \tilde{e} \) around its critical value we get
\[
\tilde{e} = \frac{1}{2 a^2} \tilde{\rho}_a \bigg|_{k=k_c} \left( a^2 - a_c^2 \right) + \frac{1}{2 b^2} (1 - \tilde{\rho}_a) \bigg|_{k=k_c} \left( b^2 - b_c^2 \right)
\]
(5.24)

from which we conclude that
\[
\tilde{e} \sim (a^2 - a_c^2) \quad \text{as} \quad k \to k_c.
\]
(5.25)

In total we get from (5.8)
\[
\frac{dU(T)}{dT} \sim \frac{1}{(a^2 - a_c^2)^4} \sim (T_c - T)^{-4/3}
\]
(5.26)
or
\[
\gamma = \frac{1}{3}
\]
(5.27)

Also for the critical points at which \( \tilde{e} = 0 \) it is obvious how the scaling will be in the general case \( M_1^c = M_2^c = \ldots = M_{m-1}^c = 0, M_m^c \neq 0 \). One finds
\[
M_1 \sim (a^2 - a_c^2)^{m-1}, \quad (a^2 - a_c^2)^{m+1} \sim (T_c - T)
\]
(5.28)

whereas the scaling of \( \tilde{e} \) and \( \tilde{\rho}_a \) is unchanged. For \( \frac{dU(T)}{dT} \) one hence has
\[
\frac{dU(T)}{dT} \sim (a^2 - a_c^2)^{-m-2} \sim (T_c - T)^{-1 - \frac{1}{m+1}}
\]
(5.29)

which implies
\[
\gamma = \frac{1}{m + 1}.
\]
(5.30)
6 The average number of loops

In this section we calculate a geometric observable which shows strikingly different behaviour at the critical points with respectively positive and negative values of $\gamma$. To begin with, let us note that the dependence on $n$ in sums like (2.4), (3.1) and (3.3) can easily be explicitly exposed by first summing over configurations of (uncoloured) loops and afterwards summing over colours. For instance for $F_0(g)$ we have

$$F_0(g) = \sum_{\tau \in T_0(\mathcal{L})} \sum_{\{L\}} C_\tau^{-1}(\mathcal{L}) \prod_{j \geq 3} (-g_j)^{N_j(\tau)} \left( \frac{g_0}{g_3} \right)^{|\mathcal{L}|} n^{N_\mathcal{L}} \tag{6.1}$$

where the notation is as in (2.4) except that $\{\mathcal{L}\}$ now refers to a given configuration of loops. By $N_\mathcal{L}$ we mean the total number of loops in $\{\mathcal{L}\}$ and by $|\mathcal{L}|$ the total number of links of the loops in $\{\mathcal{L}\}$. Since $F_0(g)$ is finite at all critical points of the model, calculating $\frac{dF_0(g)}{dn}$ will give us the average value of $N_\mathcal{L}$ up to a finite normalization constant. In particular, if the quantity $\frac{dF_0(g)}{dn}$ diverges it means that the average number of loops on the surfaces contributing to the sum (6.1) diverges. The same argument applies if $F_0(g)$ is replaced by $W_{\gamma_0}(g)$ or any other correlation function which itself is finite at the critical point. In the following we will use the notation of equation (4.1) and the quantity we aim at calculating is $\frac{d}{dn} \langle \frac{1}{N} \text{tr} M \rangle$. First we note that due to (4.2) we have

$$\frac{dF}{dn} = -\frac{1}{2} \frac{1}{Z} \frac{1}{N^2} \int d\lambda_j \exp \left\{ -N \sum_i V(\lambda_i) \right\} \prod_{i < j} (\lambda_i - \lambda_j)^2 \times \prod_{i,j} (\lambda_i + \lambda_j)^{-n/2} \sum_{i,j} \log(\lambda_i + \lambda_j)$$

$$= -\frac{1}{2} \left\langle \frac{1}{N^2} \sum_{i,j} \log(\lambda_i + \lambda_j) \right\rangle. \tag{6.2}$$

In the large $N$ limit we can write this as

$$\frac{dF}{dn} = -\frac{1}{2} \int \frac{d\omega_1}{2\pi i} \int \frac{d\omega_2}{2\pi i} W(\omega_1)W(\omega_2) \log(\omega_1 + \omega_2) \tag{6.3}$$

where we have gone through the steps of reference [13] of introducing a density of eigenvalues and a corresponding generating functional of expectation values $W(p) = \langle \frac{1}{N} \text{tr} \frac{1}{p-M} \rangle$. The contours in (6.3) encircle the cut of $W(p)$ (and not that of $W(-p)$) or equivalently the support of the eigenvalue distribution (and not its mirror image with respect to zero). Now let us act on both sides of (6.3) with the loop insertion operator $d/dV(p)$ defined by [16]

$$\frac{d}{dV(p)} = -\sum_{j=1}^{\infty} \frac{j}{p^{j+1}} \frac{d}{dg_j}. \tag{6.4}$$
This gives
\[
\frac{dW(p)}{dn} = - \oint \frac{d\omega}{2\pi i} \frac{d\omega_1}{2\pi i} \frac{d\omega_2}{2\pi i} W(\omega_2, p) W(\omega_1) \log(\omega_1 + \omega_2)
\]
(6.5)
where \( W(\omega, p) = \langle \text{tr} \frac{1}{\omega-M} \text{tr} \frac{1}{p-M} \rangle_{\text{conn}} \). In [12] a closed expression for \( W(\omega, p) \) has been obtained in the form
\[
W(\omega, p) = \frac{\partial}{\partial \omega} \{ H(\omega, p) \}
\]
(6.6)
with
\[
H(\omega, p) = \frac{1}{4 - n^2} \left\{ \frac{1}{p^2 - \omega^2} \left( ip\tilde{G}(p) \left[ i(w^2 - e^2)\tilde{G}(\omega) - \omega \sqrt{\frac{e}{e}} G(\omega) \right] ight. \right.
\]
\[
- \left. G(p) \left[ (w^2 - e^2)\omega G(\omega) + i\omega^2 \sqrt{\frac{e}{e}} \tilde{G}(\omega) \right] \right) \right. - n \frac{1}{p + \omega} - 2 \frac{1}{p - \omega} \right\}
\]
Hence we can write
\[
\frac{dW(p)}{dn} = - \oint \frac{d\omega}{2\pi i} W(-\omega) H(\omega, p).
\]
(6.7)
Taking the coefficients of the \( \frac{1}{p^4} \) terms in this equation gives
\[
\frac{d}{dn} \langle \frac{1}{N} \text{tr} M \rangle = - \frac{2i \sin \left( \frac{\omega}{2} \right)}{4 - n^2} \oint \frac{d\omega}{2\pi i} W(-\omega) \left[ i(w^2 - e^2)\tilde{G}(\omega) - \omega \sqrt{\frac{e}{e}} G(\omega) \right]
\]
(6.8)
Let us now consider the quantity \( \frac{d}{dn} \langle \frac{1}{N} \text{tr} M \rangle \) in the vicinity of a critical point with \( \gamma > 0 \). In order to determine the possible divergence of the integrand it is necessary to write down the various contributions in terms of \( \theta \)-functions exploiting the explicit expression for \( G(\omega) \) found in reference [13]. The factor \( e^2 \) diverges as \( \frac{1}{\tilde{e}} \) and the factor \( \sqrt{\frac{e}{e}} \) as \( \frac{1}{\tilde{e}} \) (cf. Eq. (4.9)). The function \( \tilde{G}(\omega) \) is finite in the scaling limit whereas the function \( G(\omega) \) contains a term which diverges as \( \frac{1}{\tilde{e}} \). A careful analysis shows that the leading singularities of the two terms in square brackets in (6.8) can cancel, so that their contribution to the integrand diverges not as \( \frac{1}{\tilde{e}} \) but as \( \frac{1}{\tilde{e}} \), i.e. as \( (a^2 - a_0^2)^{-1} \) (cf. section 3). The one-loop correlator, \( W(p) \), although parametrized in terms of the divergent function \( G(p) \), is finite as it should be. This can again be seen by exploiting the explicit expression for \( G(p) \) in terms of \( \theta \)-functions. In addition, at the critical points under consideration, the cut of \( W(-p) \) remains at a finite distance from the cut of \( W(p) \) and hence \( W(-p) \) does not give rise to non-analytical behaviour of the integral. In conclusion, we have
\[
\langle \mathcal{N}_L \rangle \sim \frac{1}{\tilde{e}} \sim (a^2 - a_0^2)^{-1}, \quad \gamma > 0.
\]
(6.9)
The existence of critical points for \( n > 2 \) with \( \langle \mathcal{N}_L \rangle \) diverging was conjectured in [14]. The average area of surfaces with one boundary component obeys the following scaling
\[ \langle A \rangle \sim (T_c - T)^{-\gamma} \sim (a^2 - a_2^2)^{-1}, \quad \gamma > 0. \quad (6.10) \]

We see that the divergence of the area of surfaces with one boundary component is of the same order as the divergence of the average number of loops. In order for (6.9) and (6.10) to be consistent the average length of a single loop must be finite.

7 Discussion

For the case \( n \in [-2, 2] \) the phase space structure of the \( O(n) \) model on a random lattice has been understood since the late eighties [8, 9, 10, 11]. For this range of \( n \) values there exists a series of critical surfaces where the model exhibits the scaling behaviour of pure 2D gravity and its multi-critical versions, i.e. 2D gravity coupled to minimal conformal field theories of the type \( (2,2m - 1) \). At these points \( \gamma \) takes one of the values \( \gamma = -\frac{1}{m}, m = 2, 3, \ldots \). In addition, there exists a critical surface at which \( \gamma = -\frac{\nu}{1-\nu} \), \( \nu \) being related to \( n \) by \( n = 2 \cos(\nu \pi) \). When the critical surface corresponding to \( \gamma = -\frac{1}{m} \) intersects the critical surface corresponding to \( \gamma = -\frac{\nu}{1-\nu} \), new critical behaviour emerges and \( \gamma \) changes to

\[ \gamma = \frac{-2\nu}{m - \eta_{m+1} + 1 - \nu}; \quad \eta_{2k} = \nu, \quad \eta_{2k+1} = 1 - \nu. \quad (7.1) \]

For \( \nu \) rational the corresponding continuum theory describes (a sector of) a minimal conformal model coupled to 2D quantum gravity and by a suitable choice of \( \nu \) and of the potential of the model one may reach any minimal conformal model. We note that the string susceptibility is always finite when \( n \in [-2, 2] \).

For \( n > 2 \) the critical surfaces corresponding to \( \gamma = -\frac{1}{m}, m = 2, 3, \ldots \) are still present. However, in this case the singularity corresponding to \( \gamma = -\frac{\nu}{1-\nu} \) is prevented by the occurrence of a new type of singularity at which the value of \( \gamma \) is \( +\frac{1}{2} \). It is interesting to note that this value is independent of \( n \). In analogy with what was the case for \( n \in [-2, 2] \) when a critical surface corresponding to \( \gamma = -\frac{1}{m}, m = 2, 3, \ldots \) intersects the critical surface corresponding to \( \gamma = \frac{1}{2} \) a new type of singular behaviour is seen. Here the associated value of the string susceptibility exponent is \( \gamma = \frac{1}{m+1} \). As opposed to the case \( n \in [-2, 2] \) we have for \( n > 2 \) critical points at which the string susceptibility diverges. This divergence can be traced back to a divergence of the average number of loops, \( \langle N_L \rangle \), on surfaces with one boundary component.

Let us mention that special cases of the phase structure found here for the \( O(n) \) model for \( n > 2 \) has been observed in models constructed by adding to the usual one-matrix model interaction terms of the type \( (\text{tr} M^2)^2 \) or \( (\text{tr} M^4)^2 \) [17]. These models,
however, do not as the $O(n)$ model contain all the minimal conformal models coupled to gravity and do not have a regular lattice formulation. Hence their interpretation as models describing 2D gravity interacting with matter is less clear. It is not surprising, however, that these models belong to the same universality class as the $O(n)$ model on a random lattice for some range of $n$ values, since the integral (2.1) can be rewritten as

$$Z = \int dM \exp \left\{ -N \text{tr} V(M) + \frac{n}{2} \sum_{m+p \geq 1} \frac{g_0^{m+p} (m+p)!}{m+p} \text{tr} M^m \text{tr} M^p \right\}$$ (7.2)

We finally note that in analogy with the multicritical models with $c < 1$, the possible continuum limits associated with critical points at which $\gamma = \frac{1}{m}$, $m = 4, 5, \ldots$, are expected to be non-unitary, whereas for $\gamma = \frac{1}{3}$ unitarity should hold. Clearly, the detailed properties of these theories constitute an interesting issue.

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