On minimal ideal triangulations of cusped hyperbolic 3–manifolds

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Abstract Previous work of the authors studies minimal triangulations of closed 3–manifolds using a characterisation of low degree edges, embedded layered solid torus subcomplexes and 1–dimensional $\mathbb{Z}_2$–cohomology. The underlying blueprint is now used in the study of minimal ideal triangulations. As an application, it is shown that the monodromy ideal triangulations of the hyperbolic once-punctured torus bundles are minimal.

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1 Introduction

Thurston observed that ideal triangulations are a useful tool to study the geometry and topology of cusped hyperbolic 3–manifolds. Geometric triangulations are useful to study geometric properties of a manifold. Minimal triangulations, i.e. topological ideal triangulations using the least number of ideal 3–simplices, are used in census enumeration, and as a platform to study the topology of the manifold using normal surface theory. In this paper we establish basic facts about minimal ideal triangulations of cusped hyperbolic 3–manifolds in analogy with our previous work for closed 3–manifolds [13, 16, 17].

Throughout this paper, by a cusped hyperbolic 3–manifold we mean an orientable non-compact 3–manifold $M$ that admits a complete hyperbolic structure of finite volume. Such a manifold is known to be the interior of a compact, irreducible, $\partial$–irreducible, atoroidal, anannular 3–manifold $\overline{M}$ with non-empty boundary a finite union of tori. If $\overline{M}$ has exactly $n$ boundary components, we say that $M$ is $n$–cusped.

Let $c(M)$ be the minimal number of ideal tetrahedra in a (topological) ideal triangulation of the cusped hyperbolic 3–manifold $M$. In this paper, we use the topology of $M$ to study the anatomy of a minimal ideal triangulation, and with this derive lower bounds on the complexity, and apply it to describe some infinite families of minimal ideal triangulations. Before stating our results, we describe previous work on minimal ideal triangulations using the geometry of the manifold.

1.1 Minimal ideal triangulations via geometry

The canonical cell decompositions of cusped hyperbolic 3–manifolds due to Epstein and Penner [4] are generically ideal triangulations. It follows from work of Guéritaud and Schleimer [9] on Dehn surgery on the Whitehead link complement that the canonical cell decompositions of twist knots are ideal triangulations. However, these are not minimal in general. We thank Jessica Purcell for pointing us to these examples.

If four ideal tetrahedra in a geometric ideal triangulation form a non-convex ideal octahedron, then there are 4–4 moves on this triangulation that result in non-geometric ideal triangulations. Neil Hoffman verified that in the current censuses of cusped hyperbolic 3–manifolds up to 6 tetrahedra, there is always at least one minimal triangulation that is geometric, but that non-geometric minimal ideal triangulations appear through the presence
of $4$–$4$ moves as described above. In particular, it is currently not known whether there is always at least one minimal ideal triangulation that is geometric.

Denote $v_3$ the volume of a regular ideal hyperbolic tetrahedron; this is approximately $1.0149$. An argument due to Thurston \cite{28} shows that

$$c(M) \geq \frac{\text{Vol}(M)}{v_3}.$$  \hspace{1cm} (1.1)

In particular, any geometric triangulation only involving regular ideal hyperbolic tetrahedra is minimal. The figure eight knot complement has such a triangulation. By taking finite sheeted coverings this leads to infinitely many minimal triangulations that are geometric and for which the above inequality is an equality. A census of such manifolds decomposed into at most 25 ideal regular ideal hyperbolic tetrahedra was given by Fominykh, Garoufalidis, Goerner, Tarkaev and Vesnin \cite{6}.

In general the gap in inequality (1.1) can be arbitrarily large. To see this neither requires explicit minimal triangulations nor computation of volumes: Thurston’s theory of hyperbolic Dehn surgery implies that for any $n \geq 1$ there are sequences of $n$–cusped hyperbolic 3–manifolds with bounded volume and whose complexities are an unbounded sequence.

An equivalent approach to complexity is via Matveev’s theory of special spines. From this point of view, Petronio and Vesnin \cite{23} give a lower bound on complexity using a volume estimate. Ishikawa and Nemoto \cite{11} combined this with an upper bound, on the complexity of 2–bridge links and determined the complexity of an infinite family of 2–bridge links. These ideal triangulations were described by Sakuma and Weeks \cite{26}. Akiyoshi, Sakuma, Wada and Yamashita have announced a proof that these triangulations are canonical \cite{1}. A proof of this fact for the fibered 2–bridge links was given by Sakata \cite{25}.

1.2 Minimal ideal triangulations via topology

In this paper, we are first concerned with the anatomy of a minimal ideal triangulation. In particular, in §2 we characterise edges of low degree, and analyse subcomplexes that we call maximal layered $\partial$–punctured solid tori. These are solid tori with a point missing on the boundary, and imbued with a special ideal triangulation. These subcomplexes were exhibited by Guéritaud and Schleimer \cite{9} in canonical decompositions of cusped hyperbolic 3–manifolds satisfying certain genericity hypotheses, and are thus a natural part of the structure of an ideal triangulation to analyse from a geometrical viewpoint. Our methods show that they also have an important topological role to play. In §3 we describe normal surfaces representing $\mathbb{Z}_2$–homology classes in ideal triangulations. We use this in §4 to prove our main result Theorem 1, which is analogous to the main result of \cite{17}. Before we can state it, we need to introduce some extra definitions.

Let $M$ be a cusped hyperbolic 3–manifold, and let $S$ be an embedded closed surface representing a given $c \in H_2(M; \mathbb{Z}_2)$. We wish to emphasise that we do not work with $H_2(M, \partial M; \mathbb{Z}_2)$. An analogue of Thurston’s norm \cite{29} is defined in \cite{17} as follows. If $S$ is connected, let $\chi_-(S) = \max\{0, -\chi(S)\}$, and otherwise let

$$\chi_-(S) = \sum_{S_i \subset S} \max\{0, -\chi(S_i)\},$$

where the sum is taken over all connected components of $S$. Note that $S_i$ is not necessarily orientable. Define:

$$\| c \| = \min\{\chi_-(S) \mid [S] = c\}.$$

The surface $S$ representing a class $c \in H_2(M; \mathbb{Z}_2)$ is said to be $\mathbb{Z}_2$–taut if no component of $S$ is a sphere or torus and $\chi(S) = -\| c \|$. Note that $M$ contains no projective plane or Klein bottle. As in \cite{29}, one observes that every component of a $\mathbb{Z}_2$–taut surface is non-separating and geometrically incompressible.
Theorem 1  Let $\mathcal{T}$ be an ideal triangulation of the cusped hyperbolic 3–manifold $M$. If $H \leq H_2(M, \mathbb{Z}_2)$ is a rank 2 subgroup, then

$$|\mathcal{T}| \geq \sum_{0 \neq c \in H} ||c||.$$

Moreover, in the case of equality the triangulation is minimal and each canonical normal representative of a non-zero element in $H \leq H_2(M, \mathbb{Z}_2)$ is taut and meets each tetrahedron in a quadrilateral disc.

As an application, we show in §5 that the monodromy ideal triangulation of a hyperbolic once-punctured torus bundle is its unique minimal triangulation. This triangulation is described by Floyd and Hatcher [5] and attributed to Thurston. Lackenby [19] showed that the monodromy ideal triangulation of each hyperbolic once-punctured torus bundle is canonical, and hence also geometric. Such a triangulation may contain edges of arbitrarily high degree, hence arbitrarily many ideal hyperbolic tetrahedra of arbitrarily small volume. Thus, our family contains explicit minimal triangulations where the gap between complexity and volume can be arbitrarily large.

The monodromy ideal triangulation $\mathcal{T}$ of a hyperbolic once-punctured torus bundle $M$ with the property that $H_2(M, \mathbb{Z})$ contains a Kleinian 4–group $H$ satisfies the equality

$$|\mathcal{T}| = \sum_{0 \neq c \in H} ||c||.$$

There are other once-cusped hyperbolic 3–manifolds, for which this equality is achieved. These are not fibred, and are described in §5.4. It remains an open problem to classify all triangulations for which the lower bound in Theorem 1 is achieved. Such a classification is given in [13] for our lower bound on the complexity for closed atoroidal 3–manifolds.

1.3 Other classes of hyperbolic 3–manifolds

We remark that for closed hyperbolic 3–manifolds, currently no infinite families of minimal triangulations are known. In the case of compact manifolds with non-empty boundary, lower bounds on complexity are known through work of [2] and [14], where it is independently shown that they are attained by $S_g \times [0, 1]$, where $S_g$ is a closed orientable surface of genus $g \geq 1$.

There are infinite families of minimal triangulations of manifolds with a totally geodesic boundary component of genus $g \geq 2$ and some cusp due to Frigerio, Martelli and Petronio [7]. These triangulations can be thought of as subdivisions into partially truncated tetrahedra; or alternatively as ideal triangulations with one ideal vertex having link a surface of genus $g$ and all others having link a torus. We remark that our methods and results do not generalise to this setting of higher genus vertex links. The main reason is that there is less control over the anatomy of these triangulations (cf. Remark 6).

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2 The anatomy of a minimal ideal triangulation

2.1 Triangulations and pseudo-manifolds

Let $\Delta$ be a finite union of pairwise disjoint Euclidean 3–simplices with the standard simplicial structure. Every $k$–simplex $\tau$ in $\Delta$ is contained in a unique 3–simplex $\sigma_{\tau}$. A 2–simplex in $\Delta$ is termed a face.

Let $\Phi$ be a family of affine isomorphisms pairing the faces in $\Delta$, with the properties that $\varphi \in \Phi$ if and only if $\varphi^{-1} \in \Phi$, and every face is the domain of a unique element of $\Phi$. The elements of $\Phi$ are termed face pairings.

Consider the quotient space $\hat{M} = \Delta/\Phi$ with the quotient topology, and denote the quotient map $\hat{\tau} : \Delta \to \hat{M}$.

We make the additional assumption on $\Phi$ that for every $k$–simplex $\tau$ in $\Delta$ the restriction of $\hat{\tau}$ to the interior of $\tau$ is injective. Then the set of non-manifold points of $\hat{M}$ is contained in the 0–skeleton, and in this case $\hat{M}$ is called a closed 3–dimensional pseudo-manifold. (See Seifert-Threfall [27].) We also always assume that $\hat{M}$ is connected. The triple $\mathcal{T} = (\Delta, \Phi, p)$ is called a (singular) triangulation of $\hat{M}$.

Let $\hat{M}^{(k)} = \{ p(\tau^k) \mid \tau^k \subseteq \Delta \}$ denote the set of images of the $k$–simplices of $\Delta$ under the projection map. Then the elements of $\hat{M}^{(k)}$ are precisely the equivalence classes of $k$–simplices in $\Delta$. The elements of $\hat{M}^{(0)}$ are termed the vertices of $\hat{M}$ and the elements of $\hat{M}^{(1)}$ are the edges. The triangulation is a $k$–vertex triangulation if $\hat{M}^{(0)}$ has size $k$.

2.2 Edge paths and edge loops

To each edge $e$ in $\hat{M}^{(1)}$ we associate a continuous path $\gamma_e : [0, 1] \to \hat{M}$ with endpoints in $\hat{M}^{(0)}$ via the composition of maps $[0, 1] \to \Delta \to \hat{M}$, where the first map is an affine map onto a 1–simplex in the equivalence class and the second is the quotient map. We call $\gamma_e$ an edge path. An edge path is termed an edge loop if its ends coincide. For instance, in a 1–vertex triangulation every edge path is an edge loop.

We often abbreviate “edge path” or “edge loop” to “edge”, and we denote the edge path $\gamma_e$ with reversed orientation by $\gamma_e$. The notions of interest in this paper are independent of the parametrisation.

2.3 Ideal triangulations

Let $M = \hat{M} \setminus \hat{M}^{(0)}$. Then $M$ is a topologically finite, non-compact 3–manifold, and the pseudo-manifold $\hat{M}$ is referred to as the end-compactification of $M$, the elements of $\hat{M}^{(0)}$ as the ideal vertices of $M$ and the elements of $\hat{M}^{(1)}$ as the ideal edges of $M$. We also refer to the triangulation $\mathcal{T} = (\hat{\Delta}, \hat{\Phi}, p)$ of $\hat{M}$ as an ideal (singular) triangulation of $M$.

The adjective singular is usually omitted: we do not need to distinguish between simplicial and singular triangulations. If $\hat{M}$ is a closed 3–manifold, we may write $M = \hat{M}$, and hope this does not cause any confusion. We suppress the notation $\mathcal{T} = (\hat{\Delta}, \hat{\Phi}, p)$ and simply say that $\hat{M}$ is triangulated or $M = \hat{M} \setminus \hat{M}^{(0)}$ is ideally triangulated.

The following result is implicit in Matveev [21] and shows that every topologically finite 3–manifold arises in this way from a closed pseudo-manifold:

**Proposition 2** (Topologically finite manifolds have ideal triangulations) If $M$ is the interior of a compact 3–manifold $\overline{M}$ with non-empty boundary, then $M$ admits an ideal triangulation. The ideal vertices of the ideal triangulation are in one-to-one correspondence with the boundary components of $\overline{M}$.

**Proof** The statement is implied by the following results in [21]. Theorem 1.1.13 due to Casler asserts that $\overline{M}$ possesses a special spine $\Sigma$; it has the property that $\overline{M}$ is homeomorphic to a regular neighbourhood of $\Sigma$ in $M$. Theorem 1.1.26 implies that $\Sigma$ is dual to an ideal triangulation of $M$ with the property that the ideal vertices are in one-to-one correspondence with the boundary components of $\overline{M}$. □
2.4 Properties of minimal triangulations

An ideal triangulation of a topologically finite 3–manifold $M$ is minimal if it uses the smallest number of simplices in an ideal triangulation of $M$. The number of ideal simplices in a minimal ideal triangulation of $M$ is denoted $c(M)$ and termed the complexity of $M$.

We assume that the reader is familiar with normal surface theory. We refer to [30] for a thorough introduction, but remark that we only need to study closed normal surfaces in ideal triangulations.

An ideal triangulation of $M$ is $0$–efficient if no normal surface is a 2–sphere. An ideal triangulation of $M$ is $\partial$–efficient if it has the property that any closed normal surface that is isotopic to a vertex linking surface is normally isotopic to that vertex-linking surface. A normal isotopy is an isotopy of $M$ that preserves each simplex of each dimension.

**Theorem 3** Suppose $M$ is the interior of a compact, irreducible, $\partial$–irreducible anannular 3–manifold. Then each minimal ideal triangulation of $M$ is $0$–efficient and $\partial$–efficient.

**Proof** It is shown in [15] Corollary 7.3 that each minimal triangulation of $M$ is $0$–efficient. It follows from the proof of [12] Theorem 4.7 that each minimal triangulation of $M$ is also $\partial$–efficient. \[\square\]

We now give a definition of what we mean by an ideal edge to be homotopic or isotopic into the boundary. The key is that intermediate paths in the homotopy are not allowed to pass through the ideal vertices, except for their endpoints. The reader is reminded than an ideal edge of $M$ is an edge of $\hat{M}$. An expanded discussion of the following material can be found in [10].

A path homotopy (resp. isotopy) $H: [0,1] \times [0,1] \to \hat{M}$ between two edge paths is admissible if $H((0,1) \times [0,1]) \subset M$, where the first factor parametrises the paths. The edge path $\gamma$ is admissibly null-homotopic (resp. null-isotopic) if there is a path homotopy (resp. isotopy) $H: [0,1] \times [0,1] \to \hat{M}$ with $H(x,0) = \gamma(x)$, $H(x,1) = \gamma(0)$ for all $x \in [0,1]$ and $H((0,1) \times [0,1]) \subset M$.

We say that an ideal edge $e$ of $M$ is homotopic (resp. isotopic) into the boundary if it is not admissibly null-homotopic (resp. null-isotopic) in $\hat{M}$.

**Corollary 4** Suppose $M$ is the interior of a compact, irreducible, $\partial$–irreducible anannular 3–manifold other than a 3–ball, and $\mathcal{T}$ is a minimal ideal triangulation of $M$. Then no ideal edge is isotopic into the boundary. In particular, no edge in $\hat{M}$ bounds an embedded disc in $\hat{M}$.

**Proof** Suppose the edge $e$ is isotopic into the boundary. Choose a compact core $M^c$ of $M$ with the property that each boundary component of $M^c$ is a vertex linking surface. There exists an embedded disc $D$ in $M^c$ whose boundary consists of $e \cap M^c$ and a subarc on a boundary component $B$ of $M^c$. The boundary of a regular neighbourhood of $B \cup D$ has two components; one is a vertex linking surface normally isotopic to $B$ and the other is a surface $S$ isotopic to $B$ and disjoint from $e$. Since $S$ is incompressible and not a 2–sphere and $M$ is irreducible, it follows that there is a normal surface isotopic but not normally isotopic to $B$. Since the triangulation is $\partial$–efficient according to Theorem 3 it follows that $M$ is homeomorphic with $S \times (0,1)$. However, this contradicts our topological hypotheses on $M$. \[\square\]

Note that the above result does not apply to the unique minimal ideal triangulation $\mathcal{T}$ of the 3–ball. This is the 1–tetrahedron, 1–vertex triangulation of $S^3$ with the vertex removed. The triangulation of $S^3$ has precisely two edges; one is a trefoil knot in $S^3$ and the other the unknot in $S^3$. The former gives an ideal edge in $\mathcal{T}$ that

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1i.e. a homotopy (resp. isotopy) keeping endpoints fixed
is only homotopic into the boundary, whilst the latter is isotopic into the boundary. The above proofs breaks down for the 3–ball because the surface \( S \) in the proof shrinks to a sphere contained in a tetrahedron.

We now specialise to the class of manifolds of interest, namely cusped hyperbolic 3–manifolds, where more specific results can be shown about their minimal triangulations. In order to limit the cases to consider, we not only assume that all vertex links are of Euler characteristic zero, but also that the manifolds are orientable.

2.5 Faces in minimal ideal triangulations of cusped hyperbolic 3–manifolds

![Types of faces. The Möbius, 3–fold and dunce faces only have one vertex due to the edge identifications. A cone face may have one or two vertices and a triangle face may have up to 3 vertices.](image)

An ideal triangle in an ideal triangulation of a cusped hyperbolic 3–manifold \( M \) may have some of its edges identified. There are eight possible types of triangular faces in the triangulation of the end-compactification \( \hat{M} \), depending on how vertices or edges are identified (see Figure 1 in [15]). Ignoring possible identifications of vertices as in [20], this reduces to five types in \( M \) based on edge identifications only (see Figure [1]). We name the faces in \( M \) according to their topology in \( \hat{M} \). A face with no edge-identifications is termed a triangle face; it is a 2–simplex, possibly with some or all of its vertices identified. If a pair of edges is identified, the face is either a cone (possibly with the tip identified with the vertex on the boundary) and called a cone face or it is a Möbius band and called a Möbius face. In a Möbius face, we distinguish the boundary edge and the core edge. If all three edges are identified, the face is either a 3–fold or a dunce hat, and called a 3–fold face or dunce face respectively.

For minimal ideal triangulations, we can prove the following statement.

**Theorem 5** Let \( \mathcal{T} \) be a minimal ideal triangulation of a cusped hyperbolic 3–manifold \( M \). Then there is no 3–fold face and there is no dunce face.

**Proof** Choose a compact core \( M^c \) of \( M \) with the property that each boundary component of \( M^c \) is a vertex linking surface. Let \( t \) be a triangle of \( \mathcal{T} \).

**Case 1:** \( t \) is a 3–fold face. Refer to Figure [2] Let \( e \) be the edge of \( t \), \( N(e) \) a small regular neighbourhood of \( e \) in \( M \) and \( N^c(e) = N(e) \cap M^c \). The frontier of \( N^c(e) \) in \( M^c \) is an annulus; we call this the exchange annulus around \( e \). Let \( N(t) \) be a regular neighbourhood of \( t \) in \( M \). Then the boundary of \( N^c(t) = N(t) \cap M^c \) has a natural cell decomposition into two hexagons corresponding to the two sides of \( t \) and three quadrilaterals that are subsurfaces of the exchange annulus around \( e \). The edges of the hexagons through the interior of \( M^c \) are termed the long edges. Seen with respect to \( (t \cap M^c) \setminus N(e) \) the long edges of the hexagons have a natural labelling 1,2,3 in cyclic order. See Figure [2] on the right for a drawing of \( N^c(t) \), in particular the two hexagons and their long edges.

Give \( t \) a transverse orientation. This allows us to refer to one of the hexagons as the top hexagon and to the other as the bottom hexagon. The intersection \( t \cap N^c(e) \) is a tripod times \([0,1]\). Figure [2] on the left depicts a cross-section of the tripod times \([0,1]\) at some \( x \in (0,1) \). The transverse orientation of \( t \) is in the same direction
at the legs of the tripod (either all clockwise or all anti-clockwise around the vertex of the tripod). It follows that the quadrilaterals join each long edge of the top hexagon to a long edge of the bottom hexagon. Moreover, on the labels this induces a 3–cycle. Thus, modulo orientation, the identification must be as shown in Figure 2 (both on the left and on the right).

After collapsing the sections of the exchange annulus to the long edges of the hexagons, we obtain a connected, orientable, bounded surface $S$ decomposed into two hexagons, nine edges and six vertices – and thus of Euler characteristic $-1$. As can be deduced from the vertex labels, the surface has three boundary components and thus must be a 3–punctured sphere.

![Figure 2: Left: Cross-section through the neighbourhood $N^c(t)$ of an ideal 3–fold $t$ near its boundary edge $e$. The cross-section through the top and bottom hexagon of the right picture appear as green and purple lines respectively. Their endpoints mark the cross-section through the long edges of the respective hexagons. Labels coincide with the ones on the right. Right: Truncated boundary $N^c(t)$ of regular neighbourhood of the 3–fold face $t$ (drawn in the centre in black). Its boundary surface is connected, of Euler characteristic $-1$ with three boundary components (each containing two edges, as indicated by the edge labels) and thus a 3–punctured sphere. The dotted blue lines with labels represent the intersection of $t$ with the boundary torus $T_0$ – a triple arc between nodes 1 and 2, or the $\Theta$–graph.](image)

Because $t$ is a 3–fold, the intersection of $t$ with a boundary torus $T_0 \subset \partial M^c$ consists of two nodes and three (normal) arcs, all running between the two nodes, also known as the $\Theta$–graph, see the dotted arcs in Figure 2. Since $S$ has three boundary components meeting $T_0$ along the boundary of a neighbourhood of $t \cap T_0$, the complement of the embedding of the $\Theta$–graph into $T_0$ must have three boundary components.

If no two arcs form a separating (i.e., inessential) curve on $T_0$, then the $\Theta$–graph forms a spine of $T_0$. In particular, its complement is a disc with a single boundary component, a contradiction. Hence, two arcs must bound a disc in $T_0$. The remaining arc now either runs parallel to the first two, or completes each of the other two to an essential curve on $T_0$. In both cases the complement has three boundary components. Both embeddings are shown in Figure 3. Since the homotopy type of the $\Theta$–graph $t \cap T_0$ coincides with the one of $\mathcal{S}$, we can think of $\mathcal{S}$ as running parallel to a regular neighbourhood $N(t \cap T_0)$ of the $\Theta$–graph in both embeddings. Uniting $\mathcal{S}$ with the complement of $N(t \cap T_0)$ in $T_0$ thus produces a torus $T_0 \setminus N(t \cap T_0) \cup \mathcal{S}$

boundary parallel to $T_0$ and disjoint from $e$ (which is itself incident to $T_0$). This is impossible since we assume that $\mathcal{S}$ is minimal and thus boundary efficient by Lemma 3.
Figure 3: Two embeddings of the intersection of the 3–fold face $t$ with torus boundary component $T_0$. Left: trivial embedding contained in a disc of $T_0$; in this case the boundary parallel torus is made up of a once-punctured torus and two discs that are subsurfaces of $T_0$ and the 3–punctured sphere $\mathcal{S}$. Right: two parallel essential loops; in this case the boundary parallel torus is made of a disc and an annulus that are subsurfaces of $T_0$ and the 3–punctured sphere $\mathcal{S}$.

Case 2: $t$ is a dunce face. Refer to Figure 4. Let $e$ be the edge of $t$. We use the notation and set-up as in the previous case. The intersection of $t$ with $N_c(e)$ is again a tripod times an interval (see Figure 4 on the left for a cross-section). However, now the transverse orientations do not all agree: two are in one direction and the last is in the opposite direction. In particular, this implies that each of the hexagons has two of its long edges connected by a quadrilateral on the exchange annulus, and exactly one long edge of the top hexagon is joined to a long edge of the bottom hexagon (and these edges do not share the same label). It follows that, up to symmetry, there is only a single way to identify the long edges of the hexagons, shown in Figure 4 (both on the left and on the right).

Figure 4: Left: Cross-section through the neighbourhood $N_c(t)$ of an ideal dunce hat $t$ near its boundary edge $e$. Labels coincide with the ones on the right. Right: Truncated boundary $N_c(t)$ of regular neighbourhood of the dunce hat $t$ (drawn in the centre in black). The boundary surface is connected, of Euler characteristic $-1$ with three boundary components (one containing four edges and two containing a single edge, as indicated by the edge labels) and thus a 3–punctured sphere. The dotted lines with vertex labels represent the intersection of $t$ with the boundary torus $T_0$, two disjoint loops and a connecting arc.

The two hexagons again form a properly embedded 3–punctured sphere $\mathcal{S} \subset M^c$. The dunce hat $t$ itself meets $\partial M^c$ in two nodes and three normal arcs: two loops, one at each node, and one arc connecting the two nodes – also known as the barbell graph. Up to the action of the mapping class group, there are four possibilities of embedding the barbell graph into a torus boundary component $T_0 \subset \partial M^c$, producing three
boundary components in a regular neighbourhood of the graph: One where both loops are essential in $T_0$ and thus necessarily parallel; one where one loop is inessential and the other one is an arbitrary essential curve in $T_0$; and two where both loops are inessential. See Figure 5 for a picture of all four cases. Since, again, the homotopy type of $t \cap T_0$ coincides with the one of $\mathcal{S}$, we can think of $\mathcal{S}$ as running parallel to $N(t \cap \partial T_0)$. Again, uniting $\mathcal{S}$ with the complement of $N(t \cap T_0)$ produces a torus boundary parallel to $T_0$ and disjoint from $e$. Thus, Theorem 3 produces a contradiction.

Figure 5: Four embeddings of the intersection of the dunce hat $t$ with torus boundary component $T_0$. Top left: two inessential loops, not nested. Bottom left: two inessential loops, nested. Top right: two parallel essential loops. Bottom right: one essential loop and one inessential loop.

Remark 6 The preconditions of Theorem 5 are necessary: the minimal triangulation of the Gieseking manifold has dunce faces, and there exists a minimal 2-tetrahedron triangulation of a 3–manifold with two Klein-bottle cusps, in which one of the four faces is a 3–fold face.

Moreover, Frigerio, Martelli and Petronio describe classes of orientable hyperbolic manifolds $\mathcal{M}_{g,k}$ with $k$ torus cusps and one end of type an orientable surface of genus $g$ [7]. They show, that these manifolds admit minimal $(g + k)$–tetrahedra ideal triangulations if and only if $g > k$, or $g = k$ and $g$ even, see [7] Proposition 1.4. Such triangulations must necessarily have $3k$ cone faces, and $(2g - k)$ dunce- or 3–fold faces, see [7] Lemma 2.1. Hence, for $g$ large enough, there exist minimal ideal triangulations of orientable hyperbolic 3–manifolds with all but an arbitrarily small portion of faces being dunce- or 3–fold-faces.

2.6 Low degree edges in minimal ideal triangulations

We now show that there are no edges of degree one or two in a minimal triangulation of a cusped hyperbolic 3–manifold, and that edges of degree three are contained in special subcomplexes. These subcomplexes are layered solid tori with the vertex removed. We term this a layered $\partial$–punctured solid torus.

Lemma 7 If $\mathcal{T}$ is a minimal ideal triangulation of a cusped hyperbolic 3–manifold $M$, then $\mathcal{T}$ has no edge of degree one.

Proof Suppose $e_1$ is an edge of $\mathcal{T}$ of degree 1. Let $\tilde{\Lambda}_1$ be the single tetrahedron containing $e_1$. Then $\tilde{\Lambda}_1 = e_1 \ast e$ is the join of $e_1$ and the edge $e$ opposite $e_1$. The edge $e$ of $\mathcal{T}$ bounds a disc. This contradicts Corollary 4. $\square$

Lemma 8 If $\mathcal{T}$ is a minimal ideal triangulation of a cusped hyperbolic 3–manifold $M$, then $\mathcal{T}$ has no edge of degree two.
Proof The figure-eight knot complement $m004^2$ and its sister $m003$ are the only cusped hyperbolic 3–manifold having an ideal triangulation with two ideal tetrahedra. Neither of these triangulations has an edge of degree two. Hence, we may assume $\mathcal{T}$ has at least three tetrahedra and that the preimage of an edge $e$ of degree two consists of two edges $e'$ and $e''$ in two distinct tetrahedra, $\tilde{\Delta}'$ and $\tilde{\Delta}''$, respectively (note that an edge of degree two cannot be contained in a single tetrahedron).

We fix the following notation. The vertices of $\tilde{\Delta}'$ are denoted by $A', B', C'$, and $D'$ with $e' = A'D'$. The vertices of $\tilde{\Delta}''$ are denoted by $A'', B'', C''$, and $D''$ with $e'' = A''D''$. The face identifications are $(A'B'D') \leftrightarrow (A''B''D'')$. We include the names of manifolds from the cusped hyperbolic manifold census of SnapPea.

If this is possible, then there is a triangulation of $M$ having fewer tetrahedra than $\mathcal{T}$, contradicting that $\mathcal{T}$ is minimal. It follows that under the assumption of $\mathcal{T}$ minimal and $e$ an edge of degree two, there must be an obstruction to such a re-identification. We consider the possible obstructions and show that each leads to a contradiction, establishing that there are no edges of degree two.

**Obstruction 1.** The edges $B'C'$ and $B''C''$ are identified, preventing the collapse, see Figure 7. In this situation we have two subcases:

1. Edges are identified with opposite orientation (i.e., $B'C'$ is identified with $C''B''$). If this is the case, then in the truncated 3–manifold $M^c$ of $M$ with an open neighbourhood of the vertices removed, we have a properly embedded Möbius band $\mathcal{N} \subset M^c$. If $\partial \mathcal{N}$ is an inessential curve on the vertex linking torus $T_0 \subset \partial M^c$. $M$ admits an embedded projective plane and we have a connected summand of $\mathbb{RP}^3$ in $M$, contradiction to $M$ being hyperbolic. Hence, assume that $\partial \mathcal{N}$ is an essential curve on $T_0$. From this situation we can construct a Seifert fibred space properly embedded in $M$ which is impossible since our manifold is hyperbolic. To see this, note that the boundary of a small regular neighbourhood of $T_0 \cup \mathcal{N}$ is a torus. Since our manifold is hyperbolic this torus must be inessential, and thus it must bound a solid torus outside $T_0 \cup \mathcal{N}$. Hence the Seifert structure on $T_0 \cup \mathcal{N}$ extends to a Seifert fibering over this solid torus with at most two exceptional fibres, one at the centre line of the Möbius band $\mathcal{N}$.

2. Edges are identified with coinciding orientation (i.e., $B'C'$ is identified with $B''C''$). In this case we have a properly embedded annulus $\mathcal{A} \subset M^c$. If one boundary component of $\mathcal{A}$ is an essential curve, the other one is, too, and both are on the same torus boundary component $T_0 \subset \partial M^c$, since otherwise $T_0$ is compressible (note that $\mathcal{A}$ must be compressible since $M$ is hyperbolic). In this case the annulus $\mathcal{A}$ runs parallel to $T_0$ forcing either $AB$ and $AC$ or $BD$ and $CD$ to be contained in a neighbourhood of $T_0$. This gives a contradiction to the triangulation being minimal and thus boundary efficient due to Lemma 3. If the boundary components are inessential, colour the bounding discs in the respective torus.

\[\text{Figure 6: Edge of degree two.}\]

Typically, in such a case of an edge of degree two, one can discard the two tetrahedra $\tilde{\Delta}'$ and $\tilde{\Delta}''$ and make new identifications $(A'B'C') \leftrightarrow (A''B''C'')$ and $(B'C'D') \leftrightarrow (B''C''D'')$.

We include the names of manifolds from the cusped hyperbolic manifold census of SnapPea.
boundary components. More precisely, colour the disc near vertex $B$ purple, and the one near vertex $C$ green. W.l.o.g., edges $AB$ and $AC$ (truncated to run inside $M^c$) run inside the solid torus bounded by $A$. By construction, the endpoint near $B$ is purple and the endpoint near $C$ is green. Moreover, both endpoints near $A$ must either both be purple or both be green (note that $A$ must be compressible). It follows that one of the edges $AB$ and $AC$ has endpoints with equal colours. It thus follows that this edge must be in the neighbourhood of the respective torus boundary components of $M^c$, see Figure 8. Again, we use Lemma 3 to produce a contradiction to the fact that the triangulation is minimal.

It follows that Obstruction 1. cannot occur.

**Figure 7:** Obstruction 1 (1). Edge $B'C'$ identified with edge $C'B''$ forming a punctured $\mathbb{R}P^2$, or Möbius strip $\mathcal{N}$ (left). Obstruction 1 (2). Edge $B'C'$ identified with edge $B''C''$ forming an annulus $A$ (right).

**Figure 8:** Obstruction 1 (2). Both boundary components of $A$ are inessential $\partial M^c$.

**Obstruction 2.** Faces from the front are identified with faces of the back (cf. Figure 6). As a result of such a face identification, the two edges $B'C'$ and $B''C''$ may become identified. If this the case we fall back onto the cases dealt with in Obstruction 1. Again, we have two subcases:

1. Face $(A'B'C')$ is identified with the face $(B'C'D')$ or $(A''B''C'')$ is identified with the face $(B'C'D')$. These situations are symmetric. As explained above we may assume that $B'C'$ and $B''C''$ are not identified. Suppose we crush the two tetrahedra so that $(A'B'C')$ is identified with $(A''B''C'')$ and $(B'C'D')$ is identified with $(B''C''D'')$. Since the interiors of $B'C'$ and $B''C''$ were initially disjoint, the preimage of a point under this operation is either a point or an interval. In particular this crushing map is cell-like and hence preserves the homeomorphism type of the manifold $M$. This gives a new triangulation of $M$ with fewer tetrahedra which contradicts our assumption that the triangulation $\mathcal{T}$ is minimal.
(2) The face \((A'B'C')\) is identified with the face \((A''B''C'')\) or the face \((B'C'D')\) is identified with the face \((B''C''D'')\). Again, these situations are symmetric. However, if face \((A'B'C')\) is glued to face \((A''B''C'')\) without a twist, \(B'C'\) is glued to \(B''C''\) and we are done (alternatively, if \((A'B'C') \leftrightarrow (A''B''C'')\), then the vertex at \(A' = A''\) is a manifold point and \(T\) is not an ideal triangulation). If the two faces are identified with a twist, all edges of the two triangles become identified. In particular \(B'C'\) and \(B''C''\) are identified and, again, we are done.

Since we have that \(T\) contains at least three tetrahedra, and that all possibilities of having an edge of degree two lead to a contradiction, there are no edges of degree two.

Lemma 9  Let \(T\) be a minimal ideal triangulation of a cusped hyperbolic 3–manifold \(M\). Suppose LST is a layered \(\partial\)–punctured solid torus and there is a combinatorial map \(\varphi\) : LST \(\rightarrow M\). Then \(\varphi\) is an embedding.

Proof  We need to show that no identifications of the boundary faces of LST are possible in \(T\). If the two triangles of \(\partial\)LST are identified, then \(M\) is a punctured lens space, contrary to the assumption that \(M\) is hyperbolic.

It follows from Theorem 5 that not all three edges of \(\partial\)LST can be identified. There are three choices of how two edges of \(\partial\)LST can become identified. In all cases, the image of LST is a twice punctured lens space. It follows again that \(M\) must contain a punctured lens space. In particular, \(M\) is not irreducible, a contradiction.

Lemma 10  Let \(T\) be a minimal ideal triangulation of a cusped hyperbolic 3–manifold \(M\) and \(e\) be an ideal edge of degree three. Then \(T\) contains, as an embedded subcomplex, the two tetrahedron layered \(\partial\)–punctured solid torus \(\text{LST}^*\{1,3,4\}\). Moreover, \(T\) contains at least three ideal tetrahedra and each ideal edge of degree three is contained in such a subcomplex.

Proof  We may and will work with the pseudo-manifold \(\hat{M}\). Let \(e \in T\) be an edge of degree 3. If \(e\) is incident with three distinct tetrahedra, then a 3-2–move produces a triangulation with strictly less tetrahedra, contradicting minimality of the triangulation. If \(e\) is incident with exactly one tetrahedron, it cannot be of degree 3. Hence, \(e\) is incident with exactly two tetrahedra. As in \([16]\), one concludes that there exists only one 2–tetrahedra complex with an internal degree 3 edge – the layered solid torus LST\((1,3,4)\). It follows from Lemma 9 that no identifications between the faces of this layered solid torus are possible, and hence the triangulation contains at least three tetrahedra.

We remark that an analysis of edges of degree four and five can be carried out following \([16]\), but this is not needed for the main applications of this paper and hence omitted.

2.7 Layered punctured solid tori in ideal triangulations

The argument for closed 3–manifolds in \([17]\) hinged on an understanding of the intersections of maximal layered solid tori. We now carry such an analysis out in the case of cusped hyperbolic 3–manifolds. This requires a different approach due to the fact that the combinatorics of minimal ideal triangulations is less restricted than the combinatorics of minimal and 0–efficient (material) triangulations.

An ideal triangulation of a \(\partial\)–punctured solid torus is understood to have the ideal vertex at the puncture on the boundary. Deleting the vertex from a layered solid torus gives a layered \(\partial\)–punctured solid torus. As in \([17]\) we focus on subcomplexes in triangulations that are maximal layered \(\partial\)–punctured solid tori – here, maximal implies that such a subcomplex is not properly contained in a subcomplex that is also combinatorially equivalent with a layered \(\partial\)–punctured solid torus.
Lemma 11  Let \( \mathcal{T} \) be a minimal ideal triangulation of a cusped hyperbolic 3–manifold \( M \). Then the intersection of any two maximal layered \( \partial \)–punctured solid tori is either empty, an ideal vertex, or an ideal edge of \( \mathcal{T} \).

Proof  Let \( \text{LST}_a, \text{LST}_b \subset \mathcal{T} \) be two maximal \( \partial \)–layered solid tori. If their boundaries \( \partial \text{LST}_a \) and \( \partial \text{LST}_b \) meet in two faces, \( \mathcal{T} \) must be homeomorphic to a punctured lens space. A contradiction. If \( \text{LST}_a \) and \( \text{LST}_b \) share a common tetrahedron, remove the tetrahedra in the intersection from one of them and observe, that they must meet in two faces, \( \mathcal{T} \) must be homeomorphic to a punctured lens space. A contradiction. If \( \text{LST}_a \) and \( \text{LST}_b \) share a common tetrahedron, remove the tetrahedra in the intersection from one of them and observe, that they now meet in two faces. Again, a contradiction. Hence, we can assume that two maximal \( \partial \)–layered solid tori meet along a single face, three edges, or two edges.

1. Assume that \( \partial \text{LST}_a \) and \( \partial \text{LST}_b \) meet along exactly two edges. It follows, that they must be identified along a spine of each \( \partial \text{LST}_a \) and \( \partial \text{LST}_b \), referred to as their common spine. Let \( \mathcal{D}_a \) and \( \mathcal{D}_b \) be the discs obtained by taking the interior of \( \partial \text{LST}_a \) and \( \partial \text{LST}_b \) after truncating their vertex and removing their common spine. \( \mathcal{D}_a \) and \( \mathcal{D}_b \) can be glued together along sections of the exchange annuli in a neighbourhood of the common spine yielding a 4–punctured sphere \( \mathcal{T} \) properly embedded in \( M' \). By construction, the four boundary components of \( \mathcal{T} \) in one of the torus boundary components \( T_0 \subset \partial M' \) bound a regular neighbourhood of the intersection of the maximal layered solid tori \( \text{LST}_a \) and \( \text{LST}_b \) with \( T_0 \) – a regular neighbourhood of two discs pinched along four points (the four points being the intersection of the common spine with \( T_0 \)). Since this regular neighbourhood is connected, of Euler characteristic \( -2 \), and with four boundary components, it must be a 4–punctured sphere contained in \( T_0 \). There are two cases: this 4–punctured sphere is either a thrice-punctured disc subsurface of \( T_0 \) or a twice punctured annulus. First suppose \( (\text{LST}_a \cup \text{LST}_b) \cap T_0 \) is contained in a disc on \( T_0 \) and hence three of its boundary components bound discs in the complement of the intersection. Pasting these three discs of \( T_0 \) into \( \mathcal{T} \) and pushing them slightly off \( T_0 \) yields a disc properly embedded in \( M' \). This disc can be extended to a torus parallel to boundary component \( T_0 \) of \( M' \) and disjoint from two edges of \( \mathcal{T} \) (which are both incident to \( T_0 \)). A contradiction to the assumption that \( \mathcal{T} \) is boundary efficient (cf. Lemma 3). Hence suppose \( (\text{LST}_a \cup \text{LST}_b) \cap T_0 \) is contained in an annulus on \( T_0 \) and hence two of its boundary components bound discs in the complement of the intersection. Pasting these two discs of \( T_0 \) into \( \mathcal{T} \) and pushing them slightly off \( T_0 \) yields an annulus properly embedded in \( M' \) and disjoint from two edges of \( \mathcal{T} \) (which are both incident to \( T_0 \)). This annulus together with the annulus in \( T_0 \) complementary to \( (\text{LST}_a \cup \text{LST}_b) \cap T_0 \) extends to a boundary parallel torus since otherwise \( M \) would contain a non-trivial Seifert fibred space. Again, this boundary parallel torus is disjoint from two edges of \( \mathcal{T} \), a contradiction to the assumption that \( \mathcal{T} \) is boundary efficient (cf. Lemma 3).

2. Assume that \( \partial \text{LST}_a \) and \( \partial \text{LST}_b \) meet along exactly three edges. Following an analogous procedure as in the previous case, we arrive at two 3–punctures spheres \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) (instead of one 4–punctured sphere \( \mathcal{T} \)), both properly embedded in \( M' \), both meeting the same boundary component \( T_0 \subset \partial M' \). Again, it follows that a regular neighbourhood of \( (\text{LST}_a \cup \text{LST}_b) \cap T_0 \) is connected, of Euler characteristic \( -4 \) with six boundary components and thus a 6–punctured sphere. A surgery argument as in the previous case again yields a torus parallel to boundary component \( T_0 \) and disjoint from three edges of the triangulation which are all incident to \( T_0 \) – and we conclude as before by using Lemma 3.

3. Assume that \( \partial \text{LST}_a \) and \( \partial \text{LST}_b \) meet along a single face. Again, we argue as in the previous cases. Here, we end up with one 3–punctured sphere \( \mathcal{T} \) (the other one being filled in as a result of the face identification). A regular neighbourhood of \( (\text{LST}_a \cup \text{LST}_b) \cap T_0 \) now is connected, of Euler characteristic \( -1 \), has three boundary components, and thus is contained in a disc or an annulus on \( T_0 \). We again obtain a disc or an annulus that can be extended to a torus running parallel to \( T_0 \) which is disjoint of three edges of the triangulation incident to \( T_0 \) and we use Lemma 3 once again to finish the proof.

This completes the proof of the lemma. \( \square \)
3 Normal surfaces representing $\mathbb{Z}_2$–homology classes

Normal surfaces in 1–vertex triangulations of closed 3–manifolds that are dual to non-trivial first cohomology classes were used in $[16, 17]$ in order to obtain lower bounds on the complexity of 3–manifolds. This section develops a similar theory for ideal triangulations. In this section, $\hat{M}$ is a compact, irreducible 3–manifold with non-empty boundary a finite union of tori, and $M$ is its interior.

We consider surfaces representing elements of $H_2(\hat{M}; \mathbb{Z}_2)$. We wish to emphasise that we do not work with $H_2(\hat{M}, \partial \hat{M}; \mathbb{Z}_2)$. We recall the following definitions from the introduction. If $S$ is connected, let $\chi(S) = \max\{0, -\chi(S_i)\}$, and otherwise let

$$\chi(S) = \sum_{S_i \subset S} \max\{0, -\chi(S_i)\},$$

where the sum is taken over all connected components of $S$. Note that $S_i$ is not necessarily orientable. Define:

$$|| e || = \min\{\chi(S) \mid [S] = c\}.$$

Then the surface $S$ representing the class $c \in H_2(M; \mathbb{Z}_2)$ is $\mathbb{Z}_2$–taut if no component of $S$ is a sphere, real projective plane, Klein bottle or torus and $\chi(S) = -|| e ||$. As in $[29]$, one observes that every component of a $\mathbb{Z}_2$–taut surface is non-separating and geometrically incompressible.

3.1 Rank-1 colouring of edges and canonical surface

We have $M = \hat{M} \setminus \hat{M}^{(0)}$, and thus Lefschetz duality gives a natural isomorphism

$$D_M: H_2(M; \mathbb{Z}_2) \cong H^1(\hat{M}, \hat{M}^{(0)}; \mathbb{Z}_2).$$

An element $\varphi \in H^1(\hat{M}, \hat{M}^{(0)}; \mathbb{Z}_2)$ can be viewed as an assignment of values 0 or 1 to each edge in the triangulation such that in each triangle the sum of all values at its edges equals zero modulo two. We say that the edge $e$ is $\varphi$–even if $\varphi(e) = 0$ and $\varphi$–odd if $\varphi(e) = 1$. It follows that a tetrahedron falls into one of the following categories, which are illustrated in Figure 9.

Type $\Delta_0$: A pair of opposite edges are $\varphi$–even, all others are $\varphi$–odd.

Type $\Delta_1$: The three edges incident to a vertex are $\varphi$–odd, all others are $\varphi$–even.

Type $\Delta_0$: All edges are $\varphi$–even.

If $\varphi$ is non-trivial, one obtains a unique normal surface, $S_\varphi = S_{\varphi(\mathcal{T})}$, with respect to $\mathcal{T}$ by introducing a single vertex on each $\varphi$–odd edge. This surface is disjoint from the tetrahedra of type $\Delta_0$; it meets each tetrahedron of type $\Delta_1$ in a single triangle meeting all $\varphi$–odd edges; and each tetrahedron of type $\Delta_0$ in a single quadrilateral dual to the $\varphi$–even edges. Moreover, $S_\varphi$ is Lefschetz dual to $\varphi$ and is termed the canonical (normal) surface dual to $\varphi$. Given $0 \neq c \in H_2(M; \mathbb{Z}_2)$ we also write $S_c = S_{D_M(c)}$ and call $S_c$ the canonical (normal) representative of $c$.

Lemma 12 Let $M$ be a cusped hyperbolic 3–manifold with ideal triangulation $\mathcal{T}$ and $0 \neq c \in H_2(M; \mathbb{Z}_2)$. Then $|| c || \leq -\chi(S_c)$.
Proof It follows from the construction and definitions that $[S_e] = c$. It suffices to show that no component of $S_\varphi$ is a sphere or a projective plane. Since $M$ is irreducible and not $\mathbb{R}P^3$, no component can be a projective plane. Since each component of $S_\varphi$ meets some edge, and it meets each edge in at most one point, no component can be a sphere.

3.2 Rank-2 colouring of edges

Given the subgroup $H = \langle \varphi_1, \varphi_2 \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ of $H^1(\hat{M}, \hat{M}^{(0)}; \mathbb{Z}_2)$, we now introduce a refinement of the above colouring. Since $\varphi_1 + \varphi_2 = \varphi_3$, there are four types of edges:

edge $e$ is called $H$–even or 0–even if $\varphi_i[e] = 0$ for each $i \in \{1, 2, 3\}$; and

edge $e$ is called $i$–even if $\varphi_i[e] = 0$ for a unique $i \in \{1, 2, 3\}$.

Let $\{i, j, k\} = \{1, 2, 3\}$. The normal corners of the normal surface $S_\varphi(\mathcal{T})$ are precisely on the $j$–even and $k$–even edges. Edge $e$ is $\varphi_i$–even if it is $i$–even or 0–even. It follows that a face of a tetrahedron either has all of its edges 0–even; or it has two 1–even (or 2–even or 3–even respectively) and one 0–even edge; or it has one 1–even, one 2–even and one 3–even edge. Whence an oriented tetrahedron falls into one of the following categories:

Type $\Delta_{qtt}$: One edge is 0–even, the opposite edge is $i$–even, and one vertex of the latter is incident with two $j$–even edges, and the other with two $k$–even edges, where $\{i, j, k\} = \{1, 2, 3\}$. (There are six distinct colourings of such a tetrahedron, in the following referred to as sub-types.)

Type $\Delta_{qq}$: A pair of opposite edges are 0–even, all others are $i$–even for a unique $i \in \{1, 2, 3\}$. (There are hence three distinct sub-types.)

Type $\Delta_{tt}$: The three edges incident to a vertex are $i$–even for a fixed $i \in \{1, 2, 3\}$, and all others are 0–even. (There are hence three distinct sub-types.)

Type $\Delta_0$: All edges are 0–even.

Type $\Delta_{qqq}$: Each vertex is incident to an $i$–even edge for each $i \in \{1, 2, 3\}$. (In particular, no edge is 0–even, opposite edges are of the same type, and there are two distinct sub-types of tetrahedra.)

For each type, one sub-type is shown in Figure 10; the dashed grey edges correspond to 0–even edges and the normal discs in $S_\varphi$ have the same colour as the $i$–even edges. The remaining sub-types are obtained by permuting the colours.

Figure 10: Tetrahedra of type $\Delta_{qtt}$, $\Delta_{qq}$, $\Delta_{tt}$, $\Delta_0$ and $\Delta_{qqq}$. Dashed grey edges are 0–even.
The set-up and notation of the previous subsection is continued. Let

\[ n_{\text{qtt}}(\mathcal{T}) = \text{number of tetrahedra of type } \Delta_{\text{qtt}}, \]
\[ n_{\text{qq}}(\mathcal{T}) = \text{number of tetrahedra of type } \Delta_{\text{qq}}, \]
\[ n_{\text{tt}}(\mathcal{T}) = \text{number of tetrahedra of type } \Delta_{\text{tt}}, \]
\[ n_{\text{t}}(\mathcal{T}) = \text{number of tetrahedra of type } \Delta_{\text{t}}, \]
\[ n_{\text{q}}(\mathcal{T}) = \text{number of tetrahedra of type } \Delta_{\text{q}}, \]
\[ n_{\text{qqq}}(\mathcal{T}) = \text{number of tetrahedra of type } \Delta_{\text{qqq}}, \]
\[ e(\mathcal{T}) = \text{number of } 0\text{-even edges}, \]
\[ \tilde{e}(\mathcal{T}) = \text{number of preimages of } 0\text{-even edges in } \tilde{\Delta}. \]

The number of tetrahedra in \( \mathcal{T} \) is

\[ |\mathcal{T}| = n_{\text{qtt}}(\mathcal{T}) + n_{\text{qq}}(\mathcal{T}) + n_{\text{tt}}(\mathcal{T}) + n_{\text{t}}(\mathcal{T}) + n_{\text{q}}(\mathcal{T}) + n_{\text{qqq}}(\mathcal{T}). \]

For the remainder of this subsection, we write \( n_{\text{qtt}} = n_{\text{qtt}}(\mathcal{T}) \), etc.

Let \( K \) be the complex in \( M \) spanned by all \( 0\text{-even} \) ideal edges. Let \( N \) be a small regular neighbourhood of \( K \). Then \( \partial N \) is a normal surface; it meets each tetrahedron in the same number and types of normal discs as \( S_{\varphi_1} \cup S_{\varphi_2} \cup S_{\varphi_3} \) except for the tetrahedra of type \( \Delta_{\text{qqq}} \), which it meets in four distinct normal triangle types instead of three distinct normal quadrilateral types. The normal coordinate of \( \partial \) solution has both positive and negative coordinates.) Hence subtracting all quadrilateral coordinates; see, for instance, [18, Section], or [30, Section 2.8]. Each tetrahedral \( \Delta \) of type \( \varphi = \varphi_1 \), and let \( \mathcal{T} \) be the canonical surface \( i \), and let \( \epsilon_d \) denote the number of \( 0\text{-even} \) edges of degree \( d \). Then

\[ \chi(S_{\varphi_1}) + \chi(S_{\varphi_2}) + \chi(S_{\varphi_3}) + n_{\text{qqq}} = \chi(\partial N) = 2\chi(N) = 2\chi(K) = -2\epsilon + n_{\text{tt}} + 2n_{\text{q}}, \]

where the Euler characteristic of \( K \) is computed from the combinatorial data (note that the ideal vertex is not counted). In particular, \( \chi(S_{\varphi_1}) + \chi(S_{\varphi_2}) + \chi(S_{\varphi_3}) \) is even. Rearranging the above equality gives

\[ n_{\text{tt}} + 2n_{\text{q}} - n_{\text{qqq}} = 2\epsilon + \chi(S_{\varphi_1}) + \chi(S_{\varphi_2}) + \chi(S_{\varphi_3}), \]

and hence:

\[ \tilde{e} = n_{\text{qtt}} + 2n_{\text{qq}} + 3n_{\text{tt}} + 6n_{\text{q}} = 2|\mathcal{T}| - n_{\text{qtt}} + n_{\text{tt}} + 4n_{\text{q}} - 2n_{\text{qqq}} = 2|\mathcal{T}| - n_{\text{qtt}} - n_{\text{tt}} + 4\epsilon + 2(\chi(S_{\varphi_1}) + \chi(S_{\varphi_2}) + \chi(S_{\varphi_3})). \]

**Lemma 13** Let \( M \) be a cusped hyperbolic 3–manifold of finite volume, and suppose that \( \varphi_1, \varphi_2 \in H^1(\tilde{M}, \tilde{M}(0); \mathbb{Z}_2) \) are non–trivial classes with \( \varphi_1 + \varphi_2 = \varphi_3 \neq 0 \). Let \( \mathcal{T} \) be a minimal triangulation, \( S_{\varphi_1} \) be the canonical surface dual to \( \varphi_1 \), and let \( \epsilon_d \) denote the number of \( 0\text{-even} \) edges of degree \( d \). Then

\[ \epsilon_3 = n_{\text{qtt}} + n_{\text{tt}} - 2(|\mathcal{T}| + \chi(S_{\varphi_1}) + \chi(S_{\varphi_2}) + \chi(S_{\varphi_3})) + \sum_{d=5}^{\infty} (d-4)\epsilon_d. \]

**Proof** Lemmata [7] and [8] imply that the smallest degree of an edge in \( \mathcal{T} \) is three. One has \( \tilde{e} = \sum d\epsilon_d \) and \( \epsilon = \sum \epsilon_d \). Inserting this into Equation (3.2) yields the desired equality. \( \square \)
4 Quadrilateral surfaces and minimal ideal triangulations

In this section we prove the following result, first stated in the introduction. The essence of the proof is a counting argument taking into account even edges, compression discs for the canonical normal representatives, and types of tetrahedra. This is modelled on the blueprint for closed 3–manifolds [13, 16, 17, 22].

**Theorem** 1 Let $\mathcal{T}$ be an ideal triangulation of a cusped hyperbolic 3–manifold $M$ and let $H \leq H_2(M, \mathbb{Z}_2)$ be a rank 2 subgroup. Then

$$|\mathcal{T}| \geq \sum_{0 \neq c \in H} ||c||.$$  

Moreover, in the case of equality the triangulation is minimal, and each canonical normal representative of a non-zero element in $H \leq H_2(M, \mathbb{Z}_2)$ is taut and each meets each tetrahedron in a quadrilateral disc.

Note that the conclusion that each canonical normal representative meets each tetrahedron in a quadrilateral disc is equivalent to saying that all tetrahedra are of type $\Delta_{qqq}$.

**Proof** It suffices to work with a minimal ideal triangulation. For if we can show any such a triangulation satisfies the desired inequality, then clearly an arbitrary ideal one does also. Moreover, if any ideal triangulation admits equality, then it clearly must be minimal.

Let $\mathcal{T}$ be a minimal ideal triangulation of a cusped hyperbolic 3–manifold $M$. We make the additional hypothesis that amongst all minimal triangulations of $M$, $\mathcal{T}$ has the smallest number of tetrahedra of type $\Delta_0$.

Let $H < H_2(M, \mathbb{Z}_2)$ be a subgroup of rank 2, and let $S_i$, $i = 0, 1, 2$, be the canonical normal representatives of the non-zero classes $0 \neq \phi_i \in H$. Following the notation used in Section 3, the union of the normal surfaces $S_i$, $i = 0, 1, 2$, intersects the tetrahedra of $\mathcal{T}$ in $n_{tt}$ tetrahedra of type $\Delta_{tt}$, $n_{qt}$ tetrahedra of type $\Delta_{qt}$, and so on. For the sake of contradiction, assume

$$|\mathcal{T}| < \sum_{i=0}^{2} ||c_i||.$$  

Let $d_i$ be the maximal number of independent compression discs for $S_i$. We have $||c_i|| \leq -\chi(S_i)$ due to Lemma [12]. Moreover, in the presence of compression discs, we obtain the stronger bound

$$\sum_{i=0}^{2} ||c_i|| \leq -\sum_{i=0}^{2} (\chi(S_i) + 2d_i)$$

since compressions do not change the homology class and do not introduce components that are spheres or projective planes. To simplify the expression, and with a view towards the second part of the theorem, we write

$$\sum_{i=0}^{2} ||c_i|| = -D - \sum \chi(S_i),$$

where $D \geq 0$. Here, we take care of the possibility that the canonical surfaces are incompressible but not norm minimising, and we note that each independent compression disc contributes 2 to $D$.

Lemma [13] now implies the strict inequality

$$n_{tt} + n_{qt} + 2D + \sum_{d=5}^{\infty} (d - 4)e_d < e_3.$$  

(4.1)

Hence, in order to prove the first part of Theorem [1] it suffices to prove that the left hand side of inequality (4.1) must be at least as large as the right hand side.
The idea of the proof is to counter-balance each contribution to the right hand side with a contribution to the left hand side of (4.1) of equal or larger size. In the argument, we identify certain subcomplexes in the triangulation and determine their contributions to the left hand side and the right hand side of (4.1). A deficit in (4.1) is when the contribution to the left is bigger than to the right, a balance in (4.1) is when they are equal, and otherwise we have a gain in (4.1). Whilst the first task at hand is to show that there is no gain, we need to keep track of balances for the characterisation result.

The core tetrahedron of any layered $\partial$-punctured solid torus $\mathrm{LST}$ is a standard one-tetrahedron layered $\partial$-punctured solid torus $\mathrm{LST}^*(1,2,3)$. Whence this tetrahedron is either of type $\Delta_{qq}$ or $\Delta_{0}$ (this can be seen by enumerating all $0/1$-colourings of edges in $\mathrm{LST}^*(1,2,3)$). It follows that either all tetrahedra in $\mathrm{LST}$ are of type $\Delta_{qq}$ or of type $\Delta_{0}$. Accordingly we say that $\mathrm{LST}$ is of type $\Delta_{qq}$ or $\Delta_{0}$. A layered $\partial$-punctured solid torus $\mathrm{LST}$ of type $\Delta_{qq}$ has one $H$-even boundary edge and the other two boundary edges are $H$-odd.

**Claim 0:** All edges of degree 3 in $\mathcal{T}$ are $H$-even.

**Proof of Claim 0:** Due to Lemma 10 (1), every edge $e \in \mathcal{T}$ of degree three can be associated to a maximal layered $\partial$-punctured solid torus $\mathrm{LST}_e \subset \mathcal{T}$ containing $e$ in its interior. If $\mathrm{LST}_e$ is of type $\Delta_{0}$, then $e$ is $H$-even. If $\mathrm{LST}_e$ is of type $\Delta_{qq}$, then $e$ is obtained by layering a tetrahedron on the degree 2 edge of $\mathrm{LST}^*(1,2,3)$. This edge is again $H$-even. ☐

A layered $\partial$-punctured solid torus $\mathrm{LST}$ has exactly one boundary edge of degree 1 with respect to $\mathrm{LST}$. We call the other two boundary edges non-unital. If $\mathrm{LST} = \mathrm{LST}^*(1,2,3)$, then the non-unital edges have degrees 2 and 3 with respect to $\mathrm{LST}$, and otherwise the degree of one is at least 3 and of the other is at least 4.

**Claim 1:** Consider the subcomplex $X$ consisting of all $H$-even edges in the triangulation that are non-unital in the boundary of some maximal layered $\partial$-punctured solid torus of type $\Delta_{0}$, together with all maximal layered $\partial$-punctured solid tori of type $\Delta_{0}$, of degree three can be associated to a maximal layered $\partial$-punctured solid tori of $\Delta_{qq}$ together with all interior edges of all maximal layered $\partial$-punctured solid tori of type $\Delta_{qq}$ contained in this subcomplex is a deficit in (4.1).

**Proof of Claim 1:** Let $\mathrm{LST}$ be a layered $\partial$-punctured solid torus of type $\Delta_{0}$ with at least two tetrahedra. Let $g$ be the top-degree boundary edge, hence $d_{\mathrm{LST}}(g) \geq 4$. Assume that $m$ additional maximal layered solid tori in $X$ are incident with $g$. Since no two maximal layered $\partial$-punctured solid tori can meet in a face (Lemma 11), the degree of $g$ is at least $5 + 2m$ in $M$. Thus, the contribution of $g$ to the left hand side of (4.1) is $1 + 2m$. The contribution to the right hand side of the collection of the $\partial$-punctured solid tori meeting in $e$ is at most $m + 1$. It follows that we have a deficit from this collection unless $m = 0$, and $\mathrm{LST}$ contains an edge of degree 3 and all other interior edges of $\mathrm{LST}$ have degree 4. But this forces $d_{\mathrm{LST}}(g) = 5$ and so the degree of $g$ is at least $6 + 2m$ in $M$. Whence we obtain a deficit from the solid tori meeting in $g$.

It remains to consider the case of a one-tetrahedron maximal layered $\partial$-punctured solid torus $\mathrm{LST}$ of type $\Delta_{0}$ that is not incident with the top-degree boundary edge of a layered $\partial$-punctured solid torus of type $\Delta_{0}$ with at least two tetrahedra. The top-degree boundary edge $g$ of $\mathrm{LST}$ has degree 3. The same argument as above, with different constants and taking into account that $\mathrm{LST}$ does not contribute a degree 3 edge gives a minimal contribution to the left hand side of $2m$ and a maximal contribution to the right hand side of $m$. Thus, we obtain a deficit if $m > 0$. If $m = 0$ and $d(g) \geq 5$ we again obtain a deficit. Hence assume $m = 0$ and $d(g) = 4$. In this case, the neighbourhood of $g$ is modelled on $\mathrm{LST}^*(2,3,5)$, but due to the maximality of $\mathrm{LST}$ there must be an additional edge identification on the boundary of $\mathrm{LST}^*(2,3,5)$. This contradicts Lemma 9. ☐

It follows from Claim 1 that the only potential balance or gain arises from layered $\partial$-punctured solid tori of type $\Delta_{qq}$ that meet no layered $\partial$-punctured solid tori of type $\Delta_{0}$ in an edge. Concerning their contribution to the left hand side of (4.1) we now have the following observation.

**Claim 2:** Let $\mathrm{LST} \subset \mathcal{T}$ be a maximal layered $\partial$-punctured solid torus of type $\Delta_{qq}$. Then the $H$-even edges of $\mathrm{LST}$ contribute a gain to (4.1) only if the following four conditions are satisfied:
(1) there is an interior $H$–even edge $e \in \mathcal{T}$ of degree three

(2) all interior $H$–even edges distinct from $e$ are of degree four, and

(3) the unique $H$–even boundary edge $f$ is of degree one with respect to LST,

(4) the edge $f$ is not a non-unital boundary edge of a maximal layered $\partial$–punctured solid torus of type $\Delta_{\emptyset}$.

We call such a maximal layered $\partial$–punctured solid torus internally unbalanced.

**Proof of Claim 2:** It is clear that LST must contain a degree three edge $e \in \mathcal{T}$ in order to achieve a gain.

If LST contains an interior $H$–even edge of degree $\geq 5$, then this edge contributes $\geq 1$ to the left hand side of (4.1). Moreover, no interior edge of LST distinct from $e$ can be of degree less than four (see Lemma [10](1)). Hence unless all interior $H$–even edges distinct from $e$ are of degree four we obtain a balance or a deficit.

If the unique $H$–even boundary edge $f \in \partial$ LST is of degree $> 4$ with respect to LST, then it is of degree $\geq 3$ with respect to LST. Assume that $m$ additional maximal layered solid tori meet $f$ in its unique $H$–even boundary edge, then $f$ is of degree at least $4 + 2m$ and thus, the maximal layered solid tori around $f$ contribute $2m$ to the left hand side of (4.1) while contributing at most $(m + 1)$ to the right hand side. Moreover, if $m = 0$, then we claim that the degree of $f$ is at least five. Suppose it is not. Then the neighbourhood of $f$ is modelled on a layered $\partial$–punctured solid torus having one more tetrahedron than LST but extra identifications between the boundary edges. This is not possible due to maximality and Lemma [9] Hence no identifications are possible amongst the edges and so $d(f) \geq 5$.

The last statement follows from Claim 1. \hfill $\square$

The set of internally unbalanced maximal layered $\partial$–punctured solid tori can now be canonically partitioned, by grouping together those which meet along their unique $H$–even boundary edge. Fix one such set of $m$ internally unbalanced maximal layered $\partial$–punctured solid tori around a common $H$–even edge $f$. Its contribution to the right hand side of (4.1) is $m$. But $f$ must be of degree at least $2m$ by Lemma [11] and its contribution to the left hand side of (4.1) is $\geq 2m - 4$. Hence, there is a deficit if $m > 4$. We therefore consider the cases where $m \leq 4$.

If $m = 4$ and $d(f) > 8$, we also obtain a deficit.

Since the degree $d = \deg(f)$ of $f$ is bounded below by four, we obtain at most a balance if:

$$(m, d) \in \{(4, 8), (3, 7), (2, 6), (1, 5)\},$$

and we obtain at most a gain if:

$$(m, d) \in \{(3, 6), (2, 5), (2, 4), (1, 4)\}.$$

Moreover, the maximal potential gain is 2 in the case $(2, 4)$ and it is 1 in all other cases. Our strategy now is to visit each edge $f$ that gives a potential gain, show that it either achieves a deficit, or (even stronger) that it achieves a deficit for itself and all $H$–even edges in its star.

We call the edges in the above collections the balancing edges and the gaining edges, respectively.

Before taking a closer look at these cases, we first prove an auxiliary statement. For this and for the remainder of this proof, we denote the tetrahedra around $f$ by $\Delta_1, \ldots, \Delta_d$ (labelled in cyclic order such that $\Delta_i$ shares a triangle with $\Delta_{i+1}$; some tetrahedron may appear in this sequence multiple times). Moreover, without loss of generality and following Lemma [11], the internally unbalanced maximal layered $\partial$–punctured solid torus LST, incident to $f$ can be assumed to contain $\Delta_{2i-1}$, $1 \leq i \leq m$.

**Claim 3:** In all four cases $(m, d) \in \{(3, 6), (2, 5), (2, 4), (1, 4)\}$, the edge $f$ is surrounded by $d$ pairwise distinct tetrahedra.
Proof of Claim 3: Tetrahedra contained in distinct internally unbalanced maximal layered $\partial$--punctured solid tori must be pairwise distinct. Hence the tetrahedra $\Delta_{2i-1}$, $1 \leq i \leq m$, are pairwise distinct since $\Delta_{2i-1}$ is contained in $\text{LST}_i$.

A tetrahedron of type $\Delta_{qqt}$ cannot occur twice around $f$ because it has only one $H$--even edge and, of course, there can be no tetrahedron of type $\Delta_{qqq}$. Moreover, only $\Delta_3$ in case $(1,4)$ can be of type $\Delta_0$, and hence a tetrahedron of type $\Delta_0$ cannot occur more than once around $f$.

If $f$ occurs more than once in a tetrahedron $\Delta$ of type $\Delta_{qqt}$, then we must be in $(2,5)$ or $(1,4)$ since $f$ is contained in a face only having $H$--even edges. In case $(2,5)$ we have $\Delta = \Delta_4 = \Delta_5$; in case $(1,4)$ we either have $\Delta = \Delta_2 = \Delta_3$, or $\Delta = \Delta_3 = \Delta_4$, or $\Delta = \Delta_2 = \Delta_4$. Assume we have one of the three cases where $\Delta = \Delta_i = \Delta_{i+1}$. Then two faces of $\Delta$ are identified. However, there is only one way to identify two faces of a tetrahedron of type $\Delta_{tt}$ in an orientation preserving manner, and this creates an edge of degree one, which is not possible. Hence we are in case $(1,4)$ and $\Delta = \Delta_2 = \Delta_4$. Then $\Delta_3$ is necessarily of type $\Delta_0$. But $\Delta_0$ cannot share two distinct faces with $\Delta_2 = \Delta_4$ because a tetrahedron of type $\Delta_{tt}$ only has one triangle of $H$--even edges.

The remaining tetrahedron type $\Delta_{qq}$ can occur in all cases and in all places around $f$. To start suppose $\Delta_2$ is of type $\Delta_{qq}$; one of the $H$--even edges of $\Delta_2$ is $f$ and we denote the other $f'$. If $\Delta_2$ appears more than once around $f$, then $f = f'$. In particular the two faces of $\Delta_2$ containing $f'$ meet $\Delta_1$ in at least two edges and hence cannot be identified with a tetrahedron $\Delta_{2i-1}$, $2 \leq i \leq m$. This shows that in the cases $(3,6)$ and $(2,4)$ all tetrahedra must be pairwise distinct by the symmetry of the situation.

In the case $(2,5)$ this implies that $\Delta_2$ cannot equal $\Delta_4$. However, $\Delta_2$ also meets a face of $\Delta_3$ and by the same reasoning cannot equal $\Delta_5$. The only remaining possibility in this case is that $\Delta_4$ equals $\Delta_5$. Here again, $\Delta_5$ meets $\Delta_1$ in a face, and hence if $\Delta_4$ equals $\Delta_5$, then $\Delta_1$ and $\Delta_3$ share two edges, which is not possible.

It remains to consider the case $(1,4)$. First suppose that $\Delta_2$ and $\Delta_3$ are the same tetrahedron. There are two possibilities that are combinatorially equivalent. Taking all the face pairings into account, we conclude that $\Delta_2$ is a 1--tetrahedron $LST^\ast(1,2,3)$. But this meets $\Delta_1$, and hence $\text{LST}_1$, in three edges, a contradiction. See Figure 11 for an illustration of this case.

![Figure 11](image-url)

**Figure 11:** Identifying $\Delta_2$ and $\Delta_3$ in case $(1,4)$ yields a one-tetrahedron layered solid torus meeting $\text{LST}_1$ in three edges.

Hence suppose without loss of generality that $\Delta_2$ and $\Delta_4$ are the same tetrahedron. Analysing the two different gluings that arise, we see that $\Delta_2$ is layered on one of the two $H$--odd boundary edges of $\Delta_1$. But this either contradicts maximality of $\text{LST}_1$ or it contradicts Lemma 9, see Figure 12.

This completes the proof of the claim. $\square$

**The case** $m = 2$ **and** $d = 4$: Since $f$ is $H$--even of degree four, we have to find a contribution to the left hand side of (4.1) of 2 to obtain a balance and $> 2$ to obtain a deficit.

By Claim 3 and by the preliminary observations above, we can assume that all four tetrahedra surrounding $f$ are distinct and tetrahedra $\Delta_1$ and $\Delta_3$ are of both of type $\Delta_{qq}$. If the quadrilaterals in $\Delta_1$ and $\Delta_3$ are of distinct
Figure 12: Identifying $\Delta_2$ and $\Delta_4$ in case (1,4) yields a layering on the boundary of $LST_1$.

colours, $\Delta_2$ and $\Delta_4$ must both be of type $\Delta_{qtt}$. Hence, they contribute 2 to the left hand side of (4.1). Moreover, since these two tetrahedra then do not have any $H$–even edges other than $f$, they cannot contribute to any other set of maximal layered solid tori united along an $H$–even edge, and their contribution to (4.1) is global. In particular, there is at most a balance and not a gain. Now one of the surfaces, $S_i$, links the edge $f$ in an annulus. We can perform a compression of this annulus, using a disc transverse to $f$. This is indeed a compression disc since the resulting surface is still connected, due to its intersection with the $H$-odd boundary edges of the layered $\partial$–punctured solid tori. Whence in total, we have a deficit of 4. We also note that this compression disc is confined to a neighbourhood of $f$ and that the tetrahedra $\Delta_2$ and $\Delta_4$ have a unique $H$–even edge. Thus, the quadrilateral discs of $S_i$ affected by this compression allow no other compressions of this kind and we may associate the total deficit with $f$.

We can thus assume that $\Delta_1$ and $\Delta_3$ are coloured the same. This implies that $\Delta_2$ and $\Delta_4$ must both be of type $\Delta_{qq}$ with the same colours as well. In particular, the octahedron that is the boundary of the star of $f$ has eight $H$-odd edges, all of degree two inside the star of $f$, and a 4–cycle of $H$–even edges, all of degree 1 inside the star of $f$. In this case we can perform compressions of two of the surfaces as above. Whence in total we have a contribution of 8 to the left hand side and a contribution of 2 to the right hand side. We now allocate a contribution of 3 to $f$ so that in total we have a deficit of 1 associated with $f$. We then associate a contribution of 2.5 to each of the other two $H$–even edges of $\Delta_2$ and $\Delta_4$. Hence each of these (if it is in our collection of gaining or balancing edges) now has a deficit associated with it and is not visited again. In particular, any further compression discs we find in the star of an $H$–even edge is disjoint from the ones we just found.

This completes the case of gaining edges of type $(2,4)$ showing that they all contribute a deficit. We may thus assume that the maximal potential gain of an edge we have not yet considered is 1.

The case $m = 3$ and $d = 6$: Since $f$ is $H$–even of degree six, it contributes 2 to the left hand side of (4.1). Hence, we have to find an additional contribution of 1 to obtain a balance and $> 1$ to obtain a deficit.

Since $LST_1$, $LST_2$ and $LST_3$ are internally unbalanced, $\Delta_1$, $\Delta_3$ and $\Delta_5$ are all of type $\Delta_{qq}$. If not all of them are coloured the same, at least two of $\Delta_2$, $\Delta_4$ and $\Delta_6$ must be of type $\Delta_{qtt}$, contributing 2 to the left hand side of (4.1) and we are done (again, note that the two tetrahedra of type $\Delta_{qtt}$ do not have any other $H$–even edges, and their contribution to the star of $f$ is global). In fact, with a little more effort one can do better by observing that either there are three tetrahedra of type $\Delta_{qtt}$ or there is a compression disc for one of the surfaces.

Hence all tetrahedra $\Delta_i$, $1 \leq i \leq 6$, are of type $\Delta_{qq}$, all with the same colours. This implies that one can perform a compression of two of the surfaces $S_i$, $i \in \{0, 1, 2\}$, by cutting along the two tubes around the $H$–even edge $f$, and pasting two discs, each intersecting $f$ once (see Figure 13 for an illustration of this process). The compressed surfaces are still embedded. Moreover, they are still connected: there exists a path from the top intersection of the compressed surface with $f$ to the bottom intersection, by passing through the interior of
one of the maximal layered solid tori. In particular, the compressed surfaces still have the same number of
connected components, are still one-sided, and their components still have non-positive Euler characteristic
and Lemma 12 applies. Altogether, the possibility of such compressions yields a contribution of 8 to the term
\[ 2 \sum_{i=0}^{2} d_i \] of the left hand side of (4.1).

We distribute this contribution onto \( f \) and the three remaining \( H \)-even edges of \( \Delta_2, \Delta_4 \) and \( \Delta_6 \) equally. Thus, all of the four edges receive a sufficient contribution to give a deficit in (4.1), and are not visited again (except
that they may receive additional deficit from other edges). In particular, any other compression discs we find
do not intersect the tetrahedra in the star of \( f \).

This completes the case of gaining edges of type \((3, 6)\) showing that they all contribute a deficit.

**The case \( m = 2 \) and \( d = 5 \):** Since \( f \) is \( H \)-even of degree five, it contributes 1 to the left hand side of (4.1). Hence, we have to find an additional contribution of 1 to achieve a balance and \( > 1 \) to achieve a deficit.

Again, \( \Delta_1 \) and \( \Delta_3 \) can be assumed to be of type \( \Delta_{qq} \). If \( \Delta_1 \) and \( \Delta_3 \) are of distinct colours, \( \Delta_2 \) must be of type \( \Delta_{tt} \). If there is at least one other tetrahedron of type \( \Delta_{tt} \) incident with \( f \), then there is a contribution to the left hand side of (4.1) of 2 and we have a deficit. Otherwise there are two distinct tetrahedra of type \( \Delta_{tl} \) also incident with \( f \). We obtain a contribution of 2 to the left hand side of (4.1) from these tetrahedra which needs to be shared with the two other \( H \)-even edges, \( g \) and \( h \) of the triangle shared by \( \Delta_4 \) and \( \Delta_5 \). We can distribute a contribution of \( \frac{2}{3} \) from these tetrahedra to the (possibly not pairwise distinct edges) \( f, g \) and \( h \). This yields a total contribution of \( \frac{5}{3} > 1 \) to \( f \) and hence a deficit.

Hence, assume that \( \Delta_1 \) and \( \Delta_3 \) are equally coloured and thus \( \Delta_2 \) must be of type \( \Delta_{tt} \) with matching colours. If any of \( \Delta_4 \) or \( \Delta_5 \) is of type \( \Delta_{qt} \), the other one must be as well, we obtain a contribution of 2, and we have a deficit.

If one of \( \Delta_4 \) and \( \Delta_5 \) is of type \( \Delta_{qq} \), the other one must be as well. As before, this implies that one can perform a compression of two of the surfaces \( S_i, i \in \{0, 1, 2\} \), by cutting along the two tubes around the \( H \)-even edge \( f \).

Again, there is an additional contribution to the left hand side of 8, distributed equally over the four \( H \)-even edges of \( \Delta_2, \Delta_4 \) and \( \Delta_5 \) (see Figure 13). In particular, each of these edges has an associated deficit.

Hence, we can assume that both \( \Delta_4 \) and \( \Delta_5 \) are of type \( \Delta_{tt} \), both with the same colouring. We obtain a contribution of 2 to the left hand side of (4.1) which needs to be shared with the two other \( H \)-even edges, \( g \) and \( h \) of the triangle shared by \( \Delta_4 \) and \( \Delta_5 \). However, for this contribution to be insufficient globally, both \( g \) and \( h \) must be surrounded by some internally unbalanced maximal layered solid tori, with the tetrahedra
surrounding $g$ and $h$ and distinct from $\Delta_4$ and $\Delta_5$ all being of type $\Delta_{qq}$ and of matching colours. This implies that one can perform two compressions of two of the surfaces $S_i$, $i \in \{0,1,2\}$, by cutting along the three times two incomplete tubes around the edges $f$, $g$ and $h$. This process replaces the two parallel triangles inside $\Delta_4$ and $\Delta_5$ by pairs of parallel normal triangles of all three other types. See Figure 14 for an illustration of this process in the case of one surface (rather than two parallel ones).

**Figure 14:** Double compression along a triangle of $H$–even edges. Only one of the two parallel copies of surfaces is drawn.

This double compression of two surfaces contributes a total of 16 to the left hand side of (4.1). We distribute two to each of $f$, $g$ and $h$, as well as to the at most three other $H$–even edges in the neighbourhoods of $f$, $g$ and $h$. The extra contribution of at least 4 is a surplus and not needed. We therefore again have a deficit.

This completes the case of gaining edges of type $(2,5)$ showing that they all contribute a deficit.

**The case $m = 1$ and $d = 4$:** In this case, $f$ does not contribute to the left hand side of (4.1) and we have to find a total contribution of 1 for a balance and $> 1$ for a deficit.

Again, $\Delta_1$ can be assumed to be of type $\Delta_{qq}$. If any of $\Delta_2$, $\Delta_3$ and $\Delta_4$ is of type $\Delta_{tt}$ then we again have two cases. Either at least two of these are of type $\Delta_{tt}$ and we obtain a contribution of at least 2 to the left hand side of (4.1) and we are done. Or one of them is of type $\Delta_{qtt}$ and two are of type $\Delta_{tt}$. In this case we distribute the contribution of 2 to the tetrahedra of type $\Delta_{tt}$ evenly over their three $H$–even edges and obtain a total contribution $> 1$ for $f$. If all $\Delta_i$, $1 \leq i \leq 4$ are of type $\Delta_{qq}$, there exist compressions for two of the surfaces $S_i$, $i \in \{0,1,2\}$, with a total contribution of 8, which can be distributed equally over the four $H$–even edges contained in $\Delta_2$, $\Delta_3$ and $\Delta_4$. This again gives a deficit for all $H$–even edges involved.

There are two cases remaining (up to symmetry). In the first case we have (without loss of generality) $\Delta_2$ of type $\Delta_{qq}$ and $\Delta_3$ and $\Delta_4$ of type $\Delta_{tt}$. Here, we fall back onto the double compression case along a triangle of $H$–even edges, as done in the case $m = 2$, $d = 5$ (see Figure 14). This again gives a deficit for all $H$–even edges involved.

In the second case we have $\Delta_2$ and $\Delta_4$ of type $\Delta_{tt}$ and $\Delta_3$ of type $\Delta_{qq}$. In this case, we obtain another minimal triangulation of $M$ by performing a 4-4-move on $f$. This results in two tetrahedra of type $\Delta_{tt}$ and two of type $\Delta_{qq}$. But this contradicts our hypothesis that $\mathcal{T}$ has the smallest number of tetrahedra of type $\Delta_{qq}$. In particular, this case does not happen.
This completes the case of gaining edges of type $(1,4)$ showing that they all contribute a deficit.

To sum up, it is not possible to achieve a gain, and hence we have proved the first part of the theorem, establishing the basic lower bound on the number of tetrahedra.

To prove the second part, we make some additional observations regarding the balancing edges.

If the balancing edge $f$ received a contribution $> 0$ as a member of the star of a gaining edge, then it has an associated deficit. Hence assume that $f$ has not received such a contribution. If $f$ is incident with a tetrahedron of type $\Delta_{qtt}$ then there is a deficit. If $f$ is incident with a tetrahedron of type $\Delta_{tt}$, then this tetrahedron has not been counted yet since otherwise $f$ would have received a contribution. We can therefore allocate $1/3$ to each edge incident with this tetrahedron and hence $f$ has a deficit.

In particular, this shows that associated to the subcomplex that is the union of $X$, all internally unbalanced tori and all tetrahedra of type $\Delta_{qtt}$ or $\Delta_{tt}$ we have an overall deficit.

These observations regarding the balancing edges suffice to analyse the case of equality. Hence suppose $|\mathcal{T}| = \sum_{0 \neq c \in H} ||c||$ and so

$$n_{tt} + n_{qtt} + 2D + \sum_{d=5}^{\infty} (d-4)e_d = e_3.$$

The above analysis shows that this equality is only possible if there is no deficit since otherwise the right hand side is smaller than the left hand side. In particular, the subcomplex that is the union of $X$, all internally unbalanced tori and all tetrahedra of type $\Delta_{qtt}$ or $\Delta_{tt}$ must be empty. Whence $n_{tt} = n_{qtt} = 0$. From the possible face pairings between tetrahedra of the different types, this implies that either all tetrahedra in $\mathcal{T}$ are of type $\Delta_0$ (but then $H$ has rank 0); or all tetrahedra are of type $\Delta_{qq}$ (but then $H$ has rank 1); or all are of type $\Delta_{qqq}$. Hence we have the last case and each canonical surface is a quadrilateral surface. Since all tetrahedra are of type $\Delta_{qqq}$, there are no $H$–even edges. Whence $e_3 = 0$, which implies $D = 0$ and the three surfaces therefore are taut.

5 The monodromy ideal triangulations of once-punctured torus bundles

In this section we apply Theorem 1 to a nice class of examples known as the (canonical) monodromy ideal triangulations of once-punctured torus bundles over the circle. In particular, we show that these are minimal.

In the last subsection we describe an infinite family of triangulations of semi-bundles that we conjecture to be minimal.

5.1 Definition of the triangulation

Monodromy ideal triangulations were studied in Lackenby [19] and Gueritaud [8], and we refer the reader to these references for details.

The monodromy ideal triangulations all have the property that they are ideal triangulations for which all edges are of even degree and there is a single ideal vertex. As we are only interested in the case of hyperbolic 3-manifolds amongst once punctured torus bundles, we restrict to monodromy as a map $A \in SL(2,\mathbb{Z})$ having trace different from 0, ±1, ±2.

Each tetrahedron in the monodromy ideal triangulation is layered on a once punctured triangulated torus with two ideal triangles. In particular, two sets of opposite edges of each tetrahedron are identified in pairs so that the
top and bottom pairs of faces form such triangulated once punctured tori. Each tetrahedron induces a diagonal flip on the triangulation of the once punctured torus, and tracing the sequence of flips gives rise to a conjugate of \( \pm A \). Indeed, each flip corresponds to one of the standard transvection matrices

\[
R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},
\]

and hence one obtains a factorisation of a conjugate of \( \pm A \) in terms of \( L \) and \( R \).

A key property of these triangulations is that the normal quadrilateral discs are of two types—called horizontal and vertical. A horizontal quadrilateral has vertices on the two pairs of opposite edges which are glued together. A vertical quadrilateral has two vertices on such a pair of edges and two vertices on the two edges which are not glued together.

A crucial fact about normal surfaces in these triangulations is that the Euler characteristic is exactly the negative of the total number of horizontal quadrilaterals. In particular no normal triangle and no vertical quadrilateral contributes to the Euler characteristic.

We now state the main result of this section. It is proved in Sections 5.2 and 5.3.

**Theorem 14**  **Monodromy ideal triangulations of once-punctured torus bundles are minimal.**

### 5.2 Reduction using coverings

Monodromy ideal triangulations of once punctured torus bundles \( M \) have the nice property that any cyclic covering induced by the bundle structure is again a once punctured torus bundle and the triangulation lifts to the monodromy ideal triangulation. Throughout this section, we assume that the monodromy has trace different from \( 0, \pm 1, \pm 2 \). Hence there is a unique class of infinite order in \( H_1(M, \mathbb{Z}) \). The cyclic coverings we are using in this section come from the kernels of maps \( \pi_1(M) \rightarrow H_1(M, \mathbb{Z}) \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_k \).

We claim that either \( H_1(M, \mathbb{Z}_2) \) has rank 3 or there is a 2– or 3–fold cyclic covering \( \tilde{M} \) of \( M \) with the property that \( H_1(\tilde{M}, \mathbb{Z}_2) \) has rank 3. The natural presentation of the fundamental group of \( M \) is of the form

\[
\langle t, a, b \mid t^{-1}at = \alpha(a), t^{-1}bt = \alpha(b) \rangle,
\]

where \( \alpha \) is the automorphism of the free group \( \langle a, b \rangle \) induced by the monodromy \( A \). It follows that the map induced by \( A \) on \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) is precisely its image \( \tilde{A} \in SL(2, 2) \cong Sym(3) \). Whence the rank of \( H_1(M, \mathbb{Z}_2) \) is 3 if \( \tilde{A} \) is the identity. Let \( k \in \{1, 2, 3\} \) be the order of \( \tilde{A} \). Since the monodromy of the \( k \)--fold cyclic covering of \( M \) corresponding to the kernel of the map \( \pi_1(M) \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_k \) gives rise to the map \( \tilde{A}^k \in SL(2, 2) \), it follows that we obtain rank 3 after passing to a \( k \)--fold cyclic covering.

It now suffices to show that a monodromy ideal triangulation is minimal for a once punctured torus bundle \( M \) with \( H_1(M, \mathbb{Z}_2) \) of rank 3. For if the rank is not three, we can pass to a 2– or 3–fold cyclic covering \( \tilde{M} \) with \( H_1(\tilde{M}, \mathbb{Z}_2) \) of rank 3 as in the previous paragraph. If we can prove that the lifted monodromy ideal triangulation of \( M \) is minimal then the monodromy ideal triangulation of \( M \) must also be minimal.

### 5.3 Closed one-sided incompressible surfaces

**Lemma 15**  **Let \( M \) be a once-punctured hyperbolic torus bundle and let \( \mathcal{T} \) be a monodromy ideal triangulation of \( M \). Furthermore, let \( S \subset \mathcal{T} \) be a normal surface of \( \mathcal{T} \) which is a taut least weight representative of an even torsion class of \( H_2(M, \mathbb{Z}) \). Then \( S \) meets each tetrahedron of \( \mathcal{T} \) in a single quadrilateral. Moreover, \( \chi(S) = -(\# \text{ horizontal quads}) \).**

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Proof Let \( S \subset \mathcal{F} \) be a normal surface of \( \mathcal{F} \) which is a taut representative of a fixed even torsion class \( c \in H_2(M, \mathbb{Z}) \). We assume that \( S \) is of least weight amongst all taut representatives. The monodromy ideal triangulation \( \mathcal{F} \) naturally endows \( M \) with a circular Morse type function: The layering defines a cyclic ordering of pairs of faces forming a once-punctured torus fibre. Let \( T \) be such a two-triangle fibre.

Consider the intersection \( S \cap T \). This is a family of embedded disjoint normal simple closed curves in the two-triangle structure on \( T \). A well-known elementary fact is that any such a curve is either non separating or vertex linking. Next, note that \( S \cap T \) cannot contain two parallel curves, since otherwise we can perform an annular compression using an innermost fibre 0 weight annulus \( A \) and find a new homologous normal surface of less weight or less genus or both.

To be more specific, consider \( N(A) \), where \( N(A) \) is a small regular neighbourhood of \( A \). Then \( S \cap N(A) \) contains two annuli and these are replaced by two annuli in \( \partial N(A) \) to form a new surface with the same weight and Euler characteristic as \( S \) and representing the same class \( c \). If this surface is disconnected, it is easy to see one component is one-sided and the other is two-sided, as \( c \) is a torsion class. Hence we can discard the two-sided component (it must bound in \( M \)) and the resulting surface still represents \( c \). As this new surface is not normal, it can be isotoped to a normal surface of less weight, giving a contradiction.

Hence, from now on, we can assume that each fibre meets \( S \) in one or two curves. Note that in the latter case exactly one curve links the ideal vertex in the fibre and the other curve is non separating. Denote \( \gamma \) the non separating curve in \( S \cap T \).

The curve \( \gamma \subset T \) is determined by how often it intersects the three edges of \( T \). Since \( \gamma \) is connected, these three intersection numbers are pairwise coprime. This can be seen by the fact that, given two intersection numbers, the third one must either be the sum or the non-negative difference of the first two.

A key observation is that \( S \) propagates across the tetrahedron \( \Delta \in \mathcal{F} \) layered on top of \( T \) in a very well-behaved way. In what follows, we denote the edge of \( \Delta \) opposite \( T \) by \( e \), and the new fibre opposite \( T \) (i.e., the two triangles of \( \Delta \) containing \( e \)) by \( T' \).

Assume that \( S \cap T \) contains a vertex linking curve \( \alpha \). \( \alpha \) intersects \( T \) in six normal arcs (one of each type). Four of them necessarily yield two normal triangles in \( \Delta \) and two normal arcs in \( T' \). For the remaining two arcs, three cases need to be considered.

1) They are in the boundary of a quadrilateral normal disc in \( \Delta \). In this case, the two normal triangles and the normal quadrilateral containing \( \alpha \) intersect both triangles of \( T' \) in two parallel normal arcs. Moreover, all other normal arcs of \( S \cap T' \) must be of the same type (otherwise \( S \) self-intersects inside \( \Delta \)). In particular, \( S \cap T' \) has parallel curves, which contradicts our assumption that \( S \) is least weight, using the annular compression argument from above. See Figure 15 on the left.

2) One of them yields a normal triangle \( t \), and the other one is in the boundary of a normal quadrilateral in \( \Delta \) together with a normal arc of \( \gamma \). In this case, \( S \cap T' \), again consists of one type of normal arc per triangle of \( T' \), with two exceptional normal arcs coming from \( t \). Let one triangle of \( T' \) contain \( k \) and the other one contain \( \ell \) copies of parallel normal arcs. Hence, one edge of \( T' \) intersects \( S \) once, the second one \( k + 1 = \ell \) times and the third \( k \) times, hence, \( \ell + 1 = k \), a contradiction. See Figure 15 on the right.

3) All of the six normal arcs of \( \alpha \) are in the boundaries of normal triangles in \( \Delta \). Since 1) and 2) are not possible, this must be the case for all tetrahedra of \( \mathcal{F} \) and \( S \) contains a boundary parallel torus. This contradicts the assumption that \( S \) is a least weight taut representative of \( c \).

Hence, for the rest of the proof we can assume that \( S \cap T \) is connected for each fibre \( T \). Consequently, it follows that \( S \cap T \) has only one or two normal arc types in each triangle of \( T \), since otherwise \( S \cap T \) contains a vertex linking component.

We claim that \( \Delta \) can only contain vertical normal discs of \( S \), unless \( \Delta \cap S \) consists of a single horizontal quad and \( S \) intersects two edges of \( T \) exactly once, and is disjoint from \( e \).
Figure 15: Left: The normal arcs of $\alpha$ are in two normal triangles and one normal quadrilateral. Right: The normal arcs of $\alpha$ are in three normal triangles and one normal quadrilateral.

More precisely, there are two cases we have to consider:

1. The intersection of $S$ with $T$ is disjoint from one of the edges of $T$. In this case the other two edges must intersect $S$ exactly once. If we layer on the edge disjoint from $S$ then there is a single horizontal quadrilateral in the tetrahedron, if we layer on one of the other edges, there is a single vertical quadrilateral. See Figure 16 on the right.

2. The surface $S$ intersects $T$ in two normal arc types per triangle. In this case all normal discs of $S$ in $\Delta$ must necessarily be vertical. To see this note that a horizontal normal quadrilateral can only have two normal arcs of $S \cap T$ in its boundary. Since by the assumption, at least one other normal disc must exist in $\Delta$, this must be a vertical normal triangle. Arguing as in case 1) and 2) above we can then follow that all three normal arc types must be present per triangle, a contradiction. See Figure 16 on the left.

It is now easy to verify that the Euler characteristic of $S$ is the negative of the number of horizontal quadrilateral discs, using the circular Morse type function induced by the layering. For each horizontal quadrilateral disc induces a simple saddle type singularity of Morse index one, whereas there are no critical points on vertical triangles or quadrilaterals.

Since $S$ is taut, no component can be a torus or Klein bottle. Hence there must be at least one horizontal quadrilateral. We can cut $M$ open along a corresponding fibre $T$ to form a product of the form $T \times I$ where $T$ is a once punctured torus. This product admits an induced ideal triangulation which is minimal layered.

There are two steps to complete the proof. The first is to follow the argument in [24] which shows that there is a unique isotopy class of incompressible surfaces in a lens space with even fundamental group. We want to apply a similar argument to an incompressible surface $S^*$ in a product of a once punctured torus and an interval.

Consider a product $X = T \times I$ where $T$ is a once punctured torus and there are two essential boundary curves $C, C'$ in $T \times \{0\}, T \times \{1\}$ respectively and the intersection number of $C, C'$ is even when projected to a copy of $T$. We claim there is a unique incompressible surface $S^*$ up to isotopy with $\partial S^* = C \cup C'$.
Pick an annulus $A$ in the product structure of $X$ with boundary curve on $T \times \{1\}$ parallel to $C'$. We can assume without loss of generality that $A$ is transverse to $S^*$ and the intersection $A \cap S^*$ contains only arcs which are essential on $S^*$ and have both ends on $T \times \{0\}$. An innermost such an arc on $A$ cuts off a bigon which can be used to perform a boundary compression of $S^*$ across $T \times \{0\}$. Exactly as in [24], this boundary compression must change the boundary curve $C$ of $\partial S^*$ to a new curve which has smaller intersection number with $C'$. It is now easy to show as in [24] that this process produces a unique family of bands attached to a collar about $C$ forming $S^*$ which is therefore unique up to isotopy.

The second step is to observe that the canonical quadrilateral surface in the induced ideal layered triangulation of the product is incompressible. The proof again follows [24]. The key idea is to use induction and the recursion formula for the genus of the incompressible surface. By removing the tetrahedra up to the first containing a horizontal quadrilateral, we obtain a new product with an induced minimal layered triangulation. The recursion formula for the change in boundary slope and the genus matches the formula in [24] and this shows that the canonical quadrilateral surface must be incompressible as it is a minimal genus cobordism surface connecting $C$ and $C'$. This completes the proof.

**Corollary 16** Let $M$ be a once-punctured hyperbolic torus bundle and let $\mathcal{T}$ be a monodromy ideal triangulation of $M$. Suppose that $H_2(M, \mathbb{Z})$ contains a Kleinian 4-group $H$. Then

$$|\mathcal{T}| = \sum_{0 \neq c \in H} ||c||,$$

and in particular $\mathcal{T}$ is a minimal triangulation.

**Proof** For each of the three non-trivial classes in $H$ we have least weight taut normal representatives. We know from the first part of Lemma 15 that each of them meets each tetrahedron of $\mathcal{T}$ in exactly one normal quadrilateral. Moreover, since $H$ is a Kleinian 4-group, the sum of all three representatives must have edge weight zero mod two. It follows that every tetrahedron contains all three quadrilateral types exactly once. In particular, in each tetrahedron exactly one of the three representatives has a single horizontal normal quadrilateral.

Hence, by the second part of Lemma 15 the negative of the sum of Euler characteristics of these three representatives must equal the number of tetrahedra of $\mathcal{T}$. It now follows from Theorem 1 that $\mathcal{T}$ is minimal.

Combining Corollary 16 with the covering argument provided in §5.2 proves Theorem 14.

### 5.4 Further examples

The orientable cusped hyperbolic census shipped with Regina [3] contains 9,596 minimal triangulations of 4,815 cusped hyperbolic 3-manifolds. There are exactly nine triangulations for which the lower bound in
Theorem 1 is attained. Eight of these are once-punctured torus bundles, but the remaining manifold, $s781$, is not. The minimal triangulation of $s781$ has isomorphism signature $\text{gLLMQb\text{}efffehhqxhqq}$ and six ideal tetrahedra. There are precisely three non-orientable fundamental surfaces. Each only contains quadrilateral discs and has Euler characteristic $-2$. Hence these are forced to be taut representatives of the three $2$–torsion classes in $H_1(M,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$. It follows that this example attains the lower bound in Theorem 1. This example naturally falls into an infinite family of triangulations, which will be described in a separate note.

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