Bell’s Experiment in Quantum Mechanics and Classical Physics

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Abstract. Both the quantum mechanical and classical Bell’s experiment are within the focus of this paper. The fact that one measures different probabilities in both experiments is traced back to the superposition of two orthogonal but non-entangled substates in the quantum mechanical case. This superposition results in an interference term that can be splitted into two additional states representing a sink and a source of probabilities in the classical event space related to Bell’s experiment. As a consequence, a statistical operator can be related to the quantum mechanical Bell’s experiment that contains already negative quasi probabilities, as usually known from quantum optics in conjunction with the Glauber-Sudarshan equation. It is proven that the existence of such negative quasi probabilities are neither a sufficient nor a necessary condition for entanglement. The equivalence of using an interaction picture in a fixed basis or of employing a change of basis to describe Bell’s experiment is demonstrated afterwards. The discussion at the end of this paper regarding the application of the complementarity principle to the quantum mechanical Bell’s experiment is supported by very recent double slit experiments performed with polarization entangled photons.
1. Introduction

Even if existing since nearly 100 years there are still controversial discussions regarding the epistemological consequences of quantum mechanics. We are obviously able to describe the behaviour of objects on the atomic and subatomic level in a quite formal mathematical way rather than to align it with our experience from classical physics. In the younger days of quantum mechanics those contradictions have been discussed on a pure philosophical level. But since the beginning of the 1980’s there exist several experimental results (and the number of corresponding experiments is growing continuously even in our days) which seem to confirm the correctness of the strange behaviour of quantum objects. Two experiments are within the focus of these discussions. These are the famous double slit experiment, and Bell’s experiment. The latter is usually considered to be an essential indication of the non-local character of quantum mechanics, and, strongly related to this, as an evidence of the existence of “entangled states”. It has its roots in the basic discussion regarding the completeness of quantum mechanics. This discussion was initiated by Einstein, Podolsky and Rosen (EPR) on the one side, and by Bohr on the other side in two famous papers published in 1935 (see [1], [2]). In the paper of EPR quantum mechanics was accused of being incomplete, and, therefore, that one has to look for hidden parameters to replace it by a complete theory. In his answer, Bohr defended his position of understanding quantum mechanics as a complete theory and his insistence on the principle of complementarity. But again, this discussion was a pure philosophical ones until the famous paper of Bell [3]. He derived therein an inequality (now called Bell’s inequality) that allows for an experimental proof of the non-local character of quantum mechanics as well as the existence of entangled states. But it took again more than one decade until the first experiments with polarization entangled photons provided us with an indication that Bell’s inequality can indeed be violated in quantum mechanics. These experiments have been performed by A. Aspect and co-workers at the beginning of the 1980’s [4]. The existence of entangled states is the most essential difference between quantum mechanics and classical physics, according to Schrödinger. He wrote in [5]:

\[ When \text{two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the system separate again, then they can no longer be described in the same way as before viz. by endowing each of them with a representative of its own. I would not call that one but rather that characteristic trait of Quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives have become entangled.}\]

Today, Schrödinger’s position as well as the assumption that entanglement is responsible for the violation of Bell’s inequality in quantum mechanics is well accepted among most of the physicists. However, it will be demonstrated in what follows that it is not primarily the entanglement but the superposition of 2 non-entangled and orthogonal substates which do not belong to orthogonal subspaces that results in a violation of Bell’s inequality. I.e., it is a similar reason we know already from the double slit experiment. Beside the well-known description of Bell’s experiment in terms of a basis transformation it is shown moreover that these 2 substates are considered to be the result of 2 additional but local and stochastically independent interactions. This avoids the assumption of any ”spooky action at a distance”. In this context it is quite interesting to see that we are able to relate a statistical operator already to both the quantum mechanical and classical Bell’s experiment.
The necessity of taking interference terms into account results in negative weights of the statistical operator related to the quantum mechanical Bell's experiment. Such negative weights are known so far only in quantum optics as negative "quasi probabilities". They are proven afterwards to represent neither a sufficient nor a necessary condition for entanglement. The statistical operator that belongs to a fixed parameter configuration of Bell's experiment can be simply extended to a "Bell's ensemble", i.e. to an incoherent mixture of different experimental configurations, as usually known from quantum mechanics. Some consequences especially with respect to the complementarity principle are finally discussed.

2. The quantum mechanical Bell's experiment

2.1. The quantum mechanical Bell's box

To commemorate the well-known results let's start with a quite phenomenological description of the quantum mechanical Bell's experiment by introducing a "quantum mechanical Bell's box" (QBB). In its first level of configuration our QBB consists of a box with 3 compartments (see Fig. 1). In the center compartment we have placed a source that emits 2 polarization entangled photons (horizontally (h)- and vertically (v)-polarized) into opposite directions once we push the button. But we don't know the state of polarization of the photon emitted in a certain direction. I.e., we don’t know if we have the combination (h,v) or (v,h) with respect to the polarization of both photons in a single event. The first term in the brackets is related to the photon on the left hand side, and the second term is related to the photon on the right hand side. For the time being, the other 2 compartments on the left- and right hand side remain empty. 2 additional lamps $L_A$ and $L_B$ are mounted at the ends of the box. Each lamp is equipped with a detector that switches the lamp on if a h-polarized photon is detected. The lamp remains switched off otherwise. After performing a multitude of experiments with this first level setup of the QBB we realize that there are just two possible events (2 non-local measurement values). These are "lamp $L_A$ on, lamp $L_B$ off: $(y_A, n_B)$", and "lamp $L_A$ off, lamp $L_B$ on: $(n_A, y_B)$". This defines

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{bell_experiment.png}
\caption{Bell's box with only a primary quantum mechanical source}
\end{figure}
Figure 2. Bell’s box with the primary quantum mechanical source of Fig. 1 but now with 2 additional, local stochastic interaction modules (polarization filters)

our classical event space. Let’s further assume that the primary stochastic source acts in such a way that the probability is 1/2 for both measurement pairs. If we relate the eigenvector

$$|\varphi_1 > = (1, 0)$$

(1)

to the measurement value "lamp A/B on (y_{A/B})”, and the eigenvector

$$|\varphi_2 > = (0, 1)$$

(2)

to the measurement value "lamp A/B off (n_{A/B})” then we are able to characterize this first level QBB by the probability state vector

$$|\Phi^{(0)}_{QBB} > = \frac{1}{\sqrt{2}} (|\varphi_1, \varphi_2 > - |\varphi_2, \varphi_1 >)$$

(3)

in the direct product space. Calculating the scalar product

$$< \Phi^{(0)}_{QBB} | \Phi^{(0)}_{QBB} >$$

(4)

provides as with the probabilities 1/2 for each pair (y_{A}, n_{B}) and (n_{A}, y_{B}) observed before in the experiments.

In a second level of configuration we insert 2 additional stochastic interaction modules into the so far empty compartments of the QBB. These modules are nothing but 2 polarization filters the photon on each side is interacting with. The position of the polarization filters can be continuously varied between [0, π] with corresponding rotating switches $D_\alpha$ and $D_\beta$. The local interactions become in this way functions of the local parameters $\alpha$ and $\beta$. Finally, the detectors of the lamps are substituted by new detectors which act in such a way that the lamps are switched on if a photon is detected, independent of its polarization. The lamps remain switched off otherwise. This second level configuration of our Bell’s experiment is depicted in Fig. 2. It is now related to a 4-dim. event space defined by the 4 possible pairs (y_{A}, y_{B}), (y_{A}, n_{B}), (n_{A}, y_{B}), and (n_{A}, n_{B}). It is an essential advantage of Bell’s experiment that it can be exclusively described in this 4-dim. space since there exists a condition to prove entanglement in a straightforward way. Performing a large number
of experiments for different sets of the local parameters $\alpha$ and $\beta$ results in the following probability state in the 4-dim. direct product space that characterizes this second level QBB:

$$\left| \Phi_{QBB} \right> = c_{11} \cdot \left| \varphi_1, \varphi_1 \right> + c_{12} \cdot \left| \varphi_2, \varphi_1 \right> + c_{21} \cdot \left| \varphi_1, \varphi_2 \right> + c_{22} \cdot \left| \varphi_2, \varphi_2 \right>, \quad (5)$$

with the $\alpha$ and $\beta$ dependent probability amplitudes

$$c_{11} = c_{22} = \frac{1}{\sqrt{2}} \sin(\alpha - \beta) \quad (6)$$

$$c_{21} = c_{12} = \frac{1}{\sqrt{2}} \cos(\alpha - \beta). \quad (7)$$

The additional eigenvectors $\left| \varphi_1, \varphi_1 \right>$ and $\left| \varphi_2, \varphi_2 \right>$ are related to the 2 additional measurement pairs $(y_A, y_B)$ and $(n_A, n_B)$. If $\alpha = \beta$ is chosen, then we end up with (3) as a special case. This probability state is normalized to unity, i.e.,

$$< \Phi_{QBB} | \Phi_{QBB} > = 1 \quad (8)$$

holds. By the way, the same normalization holds obviously for the probability state vector (3). It must again be emphasized that (5) with probability amplitudes (6) and (7) is considered to be the result of the measured probabilities

- probability $w(y, y)/w(n, n)$ that both lamps are switched on/switched off:

  $$w(y, y) = w(n, n) = \frac{1}{2} \sin^2(\alpha - \beta) \quad (9)$$

- probability $w(y, n)/w(n, y)$ that just one lamp is switched on and the other lamp remains switched off:

  $$w(n, y) = w(y, n) = \frac{1}{2} \cos^2(\alpha - \beta) \quad (10)$$

These probabilities have been indeed observed in real experiments and can therefore accepted to be a fact. I would also like to point out that in the next subsection, when deriving the probability amplitudes (6) and (7) by use of a T-matrix, we will end up with relation $c_{21} = -c_{12}$ instead of relation $c_{21} = c_{12}$ in (7). But in this subsection we started from the phenomenological point of view that we have first measured the always positive probabilities. The related probability amplitudes are obtained afterwards by simply taking the square root of these probabilities. But this sign is of no importance when calculating the probabilities related to each single event from the scalar product of (5) with itself.

Regarding Bell’s experiment the most general probability state is obviously given by

$$\left| \Phi \right> = \sum_{i,k=1}^{2} c_{ik} \cdot \left| \varphi_i, \varphi_k \right> \quad (11)$$

with

$$\sum_{i,k=1}^{2} c_{ik}^2 = 1. \quad (12)$$
On the other hand, we have the 2 general substates

\[ |\Phi_l > = \sum_{i=1}^{2} c_i \cdot |\varphi_i > \]  
(13)

\[ |\Phi_r > = \sum_{i=1}^{2} \tilde{c}_i \cdot |\varphi_i > \]  
(14)

of the 2-dim. subspaces related to the events on the left- and right hand side of Bell’s experiment. Their real valued probability amplitudes are also normalized to unity,

\[ \sum_{i=1}^{2} c_i^2 = 1 \]  
(15)

\[ \sum_{i=1}^{2} \tilde{c}_i^2 = 1 \]  
(16)

The probability state (11) can be resolved into the direct product of the substates (13) and (14) if the real valued amplitude functions \( c_{ik} \) are given by

\[ c_{ik} = c_i \cdot \tilde{c}_k , \]  
(17)

and if condition

\[ c_{11} \cdot c_{22} = c_{12} \cdot c_{21} \]  
(18)

holds. The probability amplitudes of (13) and (14) can then be recalculated from the amplitudes of (11) according to

\[ c_i^2 = \sum_{k=1}^{2} c_{ik}^2 \quad ; \quad i = 1, 2 \]  
(19)

\[ \tilde{c}_i^2 = \sum_{k=1}^{2} \tilde{c}_{ki}^2 \quad ; \quad i = 1, 2 \]  
(20)

However, if condition (18) is violated, we are unable to resolve (11) into the direct product of (13) and (14), and (11) is called “entangled”. Regarding probability state (11) with amplitudes (6) and (7) condition (18) holds only if

\[ \sin^2(\alpha - \beta) = \cos^2(\alpha - \beta) , \]  
(21)

i.e., if we have \( \alpha - \beta = \pi/4 \) for the corresponding local interaction parameters. Only then we are able to resolve this probability state into the direct product of the two 2-dim. subspaces. For all other parameter configurations \( \alpha \) and \( \beta \) this state becomes entangled.

Once we know the probabilities of a certain experimental configuration (i.e., for a certain choice of the local parameters \( \alpha \) and \( \beta \)) we are able to calculate the so-called ”correlation function” \( K(\alpha, \beta) \) according to

\[ K(\alpha, \beta) = w(y, y) + w(n, n) - w(y, n) - w(n, y) = - \cos 2(\alpha - \beta) . \]  
(22)

This function is the essential quantity in Bell’s inequality. Looking into the relevant literature (see [6], for example) we find

\[ |K(\alpha, \beta) - K(\alpha, \beta')| + |K(\alpha', \beta) + K(\alpha', \beta')| - 2 \leq 0 . \]  
(23)

It should be mentioned that there exist different expressions of Bell’s inequality. [23] is the original expression derived by Bell. Now we are able to verify its correctness for
the QBB experiment. For this we consider the probabilities of the following 4 different experimental configurations:

\[(\alpha, \beta) = \left(0, \frac{\pi}{8}\right)\]  \hspace{1cm} (24)

\[(\alpha, \beta') = \left(0, \frac{3\pi}{8}\right)\]  \hspace{1cm} (25)

\[(\alpha', \beta) = \left(\frac{\pi}{4}, \frac{\pi}{8}\right)\]  \hspace{1cm} (26)

\[(\alpha', \beta') = \left(\frac{\pi}{4}, \frac{3\pi}{8}\right)\]  \hspace{1cm} (27)

Surprisingly, Bell’s inequality (23) is violated. A closer look onto the probabilities which belong to the different configurations reveals that configurations (24), (26), and (27) result into the same probabilities, and, therefore, into identical correlation functions. This can also be inferred from the corresponding probability states. These are given by

- \(\alpha = 0, \beta = \frac{\pi}{8}\):

\[
\left|\Phi_{QBB}\right> = \frac{1}{\sqrt{2}} \sin(-\frac{\pi}{8}) \cdot |\varphi_1, \varphi_1> +
\frac{1}{\sqrt{2}} \cos(-\frac{\pi}{8}) \cdot |\varphi_1, \varphi_2>
\]

- \(\alpha = 0, \beta = \frac{3\pi}{8}\):

\[
\left|\Phi_{QBB}\right> = \frac{1}{\sqrt{2}} \sin(-\frac{3\pi}{8}) \cdot |\varphi_1, \varphi_1> +
\frac{1}{\sqrt{2}} \cos(-\frac{3\pi}{8}) \cdot |\varphi_1, \varphi_2>
\]

- \(\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{8}\):

\[
\left|\Phi_{QBB}\right> = \frac{1}{\sqrt{2}} \sin(-\frac{\pi}{8}) \cdot |\varphi_1, \varphi_1> +
\frac{1}{\sqrt{2}} \cos(-\frac{\pi}{8}) \cdot |\varphi_1, \varphi_2>
\]

- \(\alpha = \frac{\pi}{4}, \beta = \frac{3\pi}{8}\):

\[
\left|\Phi_{QBB}\right> = \frac{1}{\sqrt{2}} \sin(-\frac{3\pi}{8}) \cdot |\varphi_1, \varphi_1> +
\frac{1}{\sqrt{2}} \cos(-\frac{3\pi}{8}) \cdot |\varphi_2, \varphi_2>
\]

Thus we have in fact only 2 different experimental configurations. From (28) and (29) we can see moreover that the probabilities related to the single events are only interchanged.
2.2. Description of Bell’s experiment in terms of local interactions

Fig. 3 shows the general interaction scheme that holds for Bell’s experiment (second level of configuration). $h$ represents a horizontally polarized photon emitted from the primary source into a certain direction. Correspondingly, $v$ represents a vertically polarized photon emitted by the same source and at the same time into the opposite direction. The local events measured on each side are $n_{A/B}$ ("lamp A/B off") and $y_{A/B}$ ("lamp A/B on") of the local events measured in the experiment after the additional local interactions took place. The corresponding probability amplitudes are given in the square brackets.

The 2 local stochastic interactions on each side and the resulting local probability amplitudes (the sine and cosine functions in the square brackets in Fig 3) may be obtained by use of local and unitary T-matrices which are identical with the matrix of rotation,

$$T_{\alpha/\beta} = D_{\alpha/\beta} = \begin{pmatrix} \cos \alpha/\beta & -\sin \alpha/\beta \\ \sin \alpha/\beta & \cos \alpha/\beta \end{pmatrix}.$$ (34)
The amplitudes $c_1$, and $c_2$, of the local states after the interaction are then the result of relation

$$
\begin{pmatrix}
c_1' \\
c_2'
\end{pmatrix} = T_{\alpha/\beta} \cdot 
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}
$$

(35)

with $c_1$ and $c_2$ being the amplitudes of the local states before the interaction. This description is identical with the description of the classical interaction of a linearly polarized plane wave with a polarization filter rotated by an angle of $\alpha$ (or $\beta$) against the plane of linear polarization. Such $T$-matrices are the decisive elements in electromagnetic wave scattering as well as in quantum scattering theory (see [7, 8], for example). After the interaction, from (35) and the respective $T$-matrices we get therefore the following local probability states of the 4 2-dim. subspaces:

- local observation point $A$ and $h$-polarization:

$$
|\phi(A, h)\rangle = \frac{1}{\sqrt{2}} \cdot (\cos \alpha \cdot |\varphi_1\rangle + \sin \alpha \cdot |\varphi_2\rangle)
$$

(36)

- local observation point $A$ and $v$-polarization:

$$
|\phi(A, v)\rangle = -\frac{1}{\sqrt{2}} \cdot (-\sin \alpha \cdot |\varphi_1\rangle + \cos \alpha \cdot |\varphi_2\rangle)
$$

(37)

- local observation point $B$ and $h$-polarization:

$$
|\phi(B, h)\rangle = \cos \beta \cdot |\varphi_1\rangle + \sin \beta \cdot |\varphi_2\rangle
$$

(38)

- local observation point $B$ and $v$-polarization:

$$
|\phi(B, v)\rangle = -\sin \beta \cdot |\varphi_1\rangle + \cos \beta \cdot |\varphi_2\rangle
$$

(39)

Please, note that we have assigned the probability amplitudes of the primary source to the local probability states on the left hand side. In the next step we pass on from these local substates to the direct product states which belong to the event pairs from both sides. This procedure is accomplished separately for the upper and lower part of Fig. 3. That’s because these parts belong to different experimental situations. Moreover, performing the direct product requires that the local interactions on each side are stochastically independent. In doing so, we get the following 2 probability states:

$$
|\Phi_1\rangle = |\phi(A, h)\rangle \otimes |\phi(B, v)\rangle =
\frac{1}{\sqrt{2}} \cdot \left[ -\cos \alpha \cdot \sin \beta \cdot |\varphi_1, \varphi_1\rangle + \cos \alpha \cdot \cos \beta \cdot |\varphi_1, \varphi_2\rangle - 
\sin \alpha \cdot \sin \beta \cdot |\varphi_2, \varphi_1\rangle + \sin \alpha \cdot \cos \beta \cdot |\varphi_2, \varphi_2\rangle \right]
$$

(40)

(this state belongs to the upper part of Fig. 3), and

$$
|\Phi_2\rangle = |\phi(A, v)\rangle \otimes |\phi(B, h)\rangle =
\frac{1}{\sqrt{2}} \cdot \left[ \sin \alpha \cdot \cos \beta \cdot |\varphi_1, \varphi_1\rangle + \sin \alpha \cdot \sin \beta \cdot |\varphi_1, \varphi_2\rangle - 
\cos \alpha \cdot \cos \beta \cdot |\varphi_2, \varphi_1\rangle - \cos \alpha \cdot \sin \beta \cdot |\varphi_2, \varphi_2\rangle \right]
$$

(41)

(this state belongs to the lower part of Fig. 3). These 2 probability states are obviously not entangled! And, although they are orthogonal, they do not belong to orthogonal subspaces any longer! Now it is quite easy to see that the superposition of these both non-entangled substates results indeed into state (5) with amplitudes (6) and (7) (except that now $c_{21} = -c_{12}$ holds, as already mentioned).
Introducing the normalized probability states
\[ |\tilde{\Phi}_1 \rangle := -\cos \alpha \cdot \sin \beta \cdot |\varphi_1, \varphi_1 \rangle + \cos \alpha \cdot \cos \beta \cdot |\varphi_1, \varphi_2 \rangle - \sin \alpha \cdot \sin \beta \cdot |\varphi_2, \varphi_1 \rangle + \sin \alpha \cdot \cos \beta \cdot |\varphi_2, \varphi_2 \rangle \] (42)
\[ |\tilde{\Phi}_2 \rangle := \sin \alpha \cdot \cos \beta \cdot |\varphi_1, \varphi_1 \rangle + \sin \alpha \cdot \sin \beta \cdot |\varphi_1, \varphi_2 \rangle - \cos \alpha \cdot \cos \beta \cdot |\varphi_2, \varphi_1 \rangle - \cos \alpha \cdot \sin \beta \cdot |\varphi_2, \varphi_2 \rangle \] (43)
\[ |\tilde{\Phi}_3 \rangle := \frac{1}{\sqrt{2}} \left[ |\varphi_1, \varphi_2 \rangle + |\varphi_2, \varphi_1 \rangle \right] \] (44)
\[ |\tilde{\Phi}_4 \rangle := \frac{1}{\sqrt{2}} \left[ |\varphi_1, \varphi_1 \rangle + |\varphi_2, \varphi_2 \rangle \right] \] (45)
will allow us to relate a statistical operator to the quantum mechanical Bell’s experiment. This operator reads
\[ \hat{\rho}^{(QBB)} := \sum_{i=1}^{4} p_i \cdot |\tilde{\Phi}_i \rangle < \tilde{\Phi}_i \rangle \] (46)
with the weights given by
\[ p_1 = p_2 = \frac{1}{2} \] (47)
\[ p_3 = -p_4 = 2 \cdot c(\alpha, \beta) \] (48)
and
\[ c(\alpha, \beta) = \sin \alpha \cdot \sin \beta \cdot \cos \alpha \cdot \cos \beta . \] (49)

\[ |\tilde{\Phi}_1 \rangle \text{ and } |\tilde{\Phi}_2 \rangle \text{ as well as } |\tilde{\Phi}_3 \rangle \text{ and } |\tilde{\Phi}_4 \rangle \] are again orthogonal among each other. Moreover, \[ |\tilde{\Phi}_3 \rangle \text{ and } |\tilde{\Phi}_4 \rangle \] belong to orthogonal subspaces, in contrast to \[ |\tilde{\Phi}_1 \rangle \text{ and } |\tilde{\Phi}_2 \rangle \]. Please, note also the negative weight (negative quasi probability) \( p_4 ! \) The measured probabilities (9) and (10) are then the result of
\[ w(y, y) = < \varphi_1, \varphi_1 |\hat{\rho}^{(QBB)}| \varphi_1, \varphi_1 \rangle \] (50)
\[ w(n, n) = < \varphi_2, \varphi_2 |\hat{\rho}^{(QBB)}| \varphi_2, \varphi_2 \rangle \] (51)
\[ w(y, n) = < \varphi_2, \varphi_1 |\hat{\rho}^{(QBB)}| \varphi_2, \varphi_2 \rangle \] (52)
\[ w(n, y) = < \varphi_1, \varphi_2 |\hat{\rho}^{(QBB)}| \varphi_2, \varphi_1 \rangle \] (53)
as usually known from quantum mechanics. Operator (46) may be called the ”basic Bell’s operator” since it is related to a single experiment with a fixed parameter configuration \( \alpha \) and \( \beta \). On the other hand, if we have a mixture of \( N \) such experiments (for different parameter configurations \( (\alpha_k, \beta_k) \) with \( k = 1, \ldots, N \)) with the classical weights \( r_k \), \( \sum_{k=1}^{N} r_k = 1 \), then we have the following statistical operator of the mixture:
\[ \hat{R} = \sum_{k=1}^{N} r_k \cdot \hat{\rho}_k^{(QBB)} = \sum_{k=1}^{N} \sum_{i=1}^{4} r_k \cdot p_i^{(k)} \cdot |\tilde{\Phi}_i^{(k)} \rangle < \tilde{\Phi}_i^{(k)} \rangle . \] (54)
This is again well-known from quantum mechanics where similar operators are used to describe incoherent mixtures of pure quantum states.

Regarding the statistical operator (46) the question of the linear independence of the normalized probability states \( |\tilde{\Phi}_i \rangle \) is of some importance. If this happens, then we are able to represent any probability state of our 4-dim. event space by a linear combination of these state vectors. To prove the linear independence we have to consider Grams’ matrix
\[ G = < \tilde{\Phi}_i |\tilde{\Phi}_j > ; \quad i, j = 1, 2. \] (55)
Because of (42) - (45) this matrix is a symmetric one,

\[
G = \begin{pmatrix}
1 & 0 & g_1 & g_2 \\
0 & 1 & -g_1 & g_2 \\
g_1 & -g_1 & 1 & 0 \\
g_2 & g_2 & 0 & 1
\end{pmatrix},
\]

with elements \(g_1\) and \(g_2\) given by

\[
g_1 = \frac{1}{\sqrt{2}} \cdot \cos(\alpha + \beta)
\]

\[
g_2 = \frac{1}{\sqrt{2}} \cdot \sin(\alpha - \beta).
\]

Its determinant reads

\[
\det(G) = \frac{8}{\sqrt{2}} \cdot c(\alpha, \beta),
\]

with \(c(\alpha, \beta)\) according to (49). Thus we have to meet the condition

\[
c(\alpha, \beta) \neq 0
\]

to ensure the linear independence of the vectors (42) - (45). On the other hand, if

\[
c(\alpha, \beta) = 0,
\]

holds (this happens if we have \((\alpha = 0\) or \(\pi/2, \beta \neq 0\) or \((\alpha \neq 0, \beta = 0\) or \(\pi/2\)), both weights \(p_3\) and \(p_4\) are identical zero. In other words: The probabilities of the events represented by the state vectors \(|\tilde{\Phi}_3\rangle\) (a "source" of the sum of probabilities \(w(y, n) + w(n, y)\)) and \(|\tilde{\Phi}_4\rangle\) (a "sink" of the sum of probabilities \(w(y, y) + w(n, n)\)) of the 2 orthogonal subspaces will no longer be redistributed to the probabilities of the events represented by the state vectors \(|\Phi_1\rangle\) and \(|\Phi_2\rangle\).

The essential result of the above considerations is the fact that the interference term, that results from the superposition of the two non-entangled probability states (40) and (41), is responsible for the violation of Bell’s inequality. This interference term was traced back to additional local and stochastically independent interactions. Thus, there seems to be no need to assume any "spooky action at a distance" or any "hidden parameters" behind the probability states to understand the quantum mechanical Bell’s experiment. But the following aspect is also of some importance. As one can see from Eqs. (21) and (48)/(49), entanglement is not necessarily linked to the existence of negative quasi probabilities. However, such statements can frequently be found in the literature (see [9], for example). If we choose \((\alpha = \pi/8, \beta = 0)\) as the local interaction parameters, for example, then the quasi probabilities \(p_3\) and \(p_4\) are identical zero. But the corresponding probability state of the QBB is still entangled! The existence of negative quasi probabilities is therefore neither a sufficient nor a necessary condition of entanglement.

2.3. Description of Bell’s experiment in terms of a basis transformation

The quantum mechanical Bell’s experiment has been described so far in terms of local interactions accomplished by 2 additional polarization filters. However, in several presentations I was faced again and again with an incomprehension regarding this interaction point of view. Therefore, let’s discuss the quantum mechanical Bell’s
experiment again but from the point of view of the well accepted basis transformation to demonstrate the equivalence of both approaches.

We ask for the transformation matrix that transforms the primary probability state

$$|\Phi^{(0)}\rangle = \frac{1}{\sqrt{2}} \cdot |\varphi_1, \varphi_2 \rangle - |\varphi_2, \varphi_1 \rangle$$

(62)
of the QBB into the new probability state

$$|\Phi_{QBB}\rangle = c_{yy} \cdot |y_A, y_B \rangle + c_{yn} \cdot |y_A, n_B \rangle + c_{ny} \cdot |n_A, y_B \rangle + c_{nn} \cdot |n_A, n_B \rangle$$

(63)

with probability amplitudes

$$c_{yy} = c_{nn} = \frac{1}{\sqrt{2}} \cdot \sin(\alpha - \beta)$$

(64)

$$c_{yn} = -c_{ny} = \frac{1}{\sqrt{2}} \cdot \cos(\alpha - \beta).$$

(65)

The new but so far unknown eigenvectors $|y_{A/B}\rangle$ and $|n_{A/B}\rangle$ are again related to the possible local measurements "lamp A/B on" and "lamp A/B off". Now, let’s assume that these eigenvectors are the result of a rotation of the local coordinate system on the left- and right hand side,

$$
\begin{pmatrix}
|y_{A/B}\rangle \\
|n_{A/B}\rangle 
\end{pmatrix} = D_{\alpha/\beta} \cdot 
\begin{pmatrix}
|\varphi_1 \rangle \\
|\varphi_2 \rangle 
\end{pmatrix},
$$

(66)
caused by the 2 polarization filters. $D_{\alpha/\beta}$ therein is again the matrix (34) of rotation. Thus we get

$$
\begin{align*}
|y_A\rangle &= (\cos \alpha, -\sin \alpha) \\
|y_B\rangle &= (\cos \beta, -\sin \beta) \\
|n_A\rangle &= (\sin \alpha, \cos \alpha) \\
|n_B\rangle &= (\sin \beta, \cos \beta) .
\end{align*}
$$

(67) (68) (69) (70)

Next we introduce the T-matrix

$$
T = 
\begin{pmatrix}
<y_B, y_A|\varphi_1, \varphi_2 > & < y_B, y_A|\varphi_2, \varphi_1 > \\
<y_B, n_A|\varphi_1, \varphi_2 > & < n_B, y_A|\varphi_2, \varphi_1 > \\
<y_B, n_A|\varphi_1, \varphi_2 > & < n_B, n_A|\varphi_2, \varphi_1 > \\
< n_B, n_A|\varphi_1, \varphi_2 > & < n_B, n_A|\varphi_2, \varphi_1 > \\
\end{pmatrix}
= 
\begin{pmatrix}
-\cos \alpha \cdot \sin \beta & -\sin \alpha \cdot \cos \beta \\
\cos \alpha \cdot \cos \beta & -\sin \alpha \cdot \sin \beta \\
-\sin \alpha \cdot \sin \beta & \cos \alpha \cdot \cos \beta \\
\sin \alpha \cdot \cos \beta & \cos \alpha \cdot \sin \beta \\
\end{pmatrix}
$$

(71)

Then the new probability amplitudes (64) and (65) are calculated according to

$$
\begin{pmatrix}
c_{yy} \\
c_{yn} \\
c_{ny} \\
c_{nn}
\end{pmatrix} = T \cdot 
\begin{pmatrix}
1/\sqrt{2} \\
0 \\
0 \\
-1/\sqrt{2}
\end{pmatrix}
$$

(72)

from the primary probability amplitudes of (62). The T-matrix (71) is again a unitary matrix, i.e.,

$$T^tp \cdot T = E$$

(73)
holds with \( \mathbf{E} \) representing the \( 2 \times 2 \) unit matrix. This ensures the conservation of the total probability. However, it is a disadvantage of the basis transformation that the aspect of the superposition of 2 non-entangled substates is covered up. But a closer look onto (71) reveals that the probability amplitudes of these 2 substates are identical with the elements of the first and second column of (71). The fact that T-matrices can be used for both the description of a certain interaction and the description of an equivalent transformation of corresponding eigenvectors is considered in detail in [7].

3. A classical Bell’s experiment

In this section we will discuss the classical counterpart of the quantum mechanical Bell’s experiment. To start with let’s venture the following guess: Regarding the statistical operator (46) we expect that non-vanishing weights \( p_3 \) and \( p_4 \) are the essential aspect that makes the quantum mechanical Bell’s experiment differ from its classical counterpart. If this is true, then we would get the probabilities of a corresponding classical experiment by use of (50) - (53) but with \( \hat{\rho}^{(\text{QBB})} \) replaced by the classical statistical operator

\[
\hat{\rho}^{(\text{cl})} := \sum_{i=1}^{2} p_i \cdot |\tilde{\Phi}_i > < \tilde{\Phi}_i|.
\]  

(74)

This corresponds to the procedure that we first calculate the probabilities of each substate (40) and (41) separately, and if adding up these probabilities afterwards. In this way we would end up with the following classical probabilities:

- Probability \( w(y, y)/w(n, n) \) that both lamps are switched on/switched off:
  \[
w(y, y) = w(n, n) = \frac{1}{2} \cdot (\sin^2 \alpha \cdot \cos^2 \beta + \sin^2 \beta \cdot \cos^2 \alpha)
  \]  

(75)

- Probability \( w(y, n)/w(n, y) \) that just one lamp is switched on and the other lamp remains switched off:
  \[
w(n, y) = w(y, n) = \frac{1}{2} \cdot (\cos^2 \alpha \cdot \cos^2 \beta + \sin^2 \beta \cdot \sin^2 \alpha).
  \]  

(76)

It would be of some benefit to prove these probabilities in a corresponding experiment, as it was done in the quantum mechanical case by A. Aspect and co-workers. And there is indeed a quite simple marble experiment that can be used to prove the correctness of the probabilities (75) and (76). This experiment is described in what follows for the 2 parameter configurations (24) and (25), i.e. the only 2 configurations of the 4 different QBB experiments required to verify Bell’s inequality which result in different correlation functions. But the extension to other parameter configurations is straightforward.

A Box \( B_p \) with 1 white and 1 black marble represents the primary stochastic source. 2 additional boxes \( B_w \) and \( B_b \) are filled with 17 white and 3 black marbles (box \( B_w \)) and 17 black and 3 white marbles (box \( B_b \)). This corresponds approximately to the probabilities of 0.85/0.15 to draw a white or black marble out of the respective box. These 2 additional boxes represent the local interaction on the right hand side!

Now, if the parameter configuration (24) is chosen, the experiment runs as follows: We draw both marbles blindly out of box \( B_p \) and put one marble on the left hand side and the other marble on the right hand side on our desk. The colour of the marble on the left hand side is already the result of this side. To get the result of the right hand
side requires an additional step. If the primary marble on the right hand side is white, then we have to draw another marble out of box $B_w$. Its colour is the result of the right hand side. But if the primary marble on the right hand side is black, then we have to draw another marble out of box $B_b$. This colour will then be the result of the right hand side. We repeat this procedure until we are able to calculate the probabilities within a sufficient accuracy. Then we are able to calculate the correlation function $K(\alpha = 0, \beta = \pi/8)$.

If the parameter configuration (25) is chosen, the experiment runs as follows: The first step to get the result of the left hand side is as before. But, now, if the primary marble on the right hand side is white, then we have to draw another marble out of box $B_b$. Its colour is the result of the right hand side. On the other hand, if the primary marble on the right hand side is black, then we have to draw another marble out of box $B_w$. This colour will then be the result of the right hand side. We repeat this procedure until we are able to calculate the probabilities within a sufficient accuracy. Then we are able to calculate the correlation function $K(\alpha = 0, \beta = 3\pi/8)$.

We can proceed in a similar way if the local parameter $\alpha$ is not zero. The only thing we have to do is to fill 2 additional boxes on the left hand side with an appropriate number of black and white marbles to meet the probabilities of the local interaction on this side. At least 200 single experiments for each parameter configuration are needed to approach the probabilities (75) and (76) within a sufficient accuracy. Once we accept the classical probabilities (75) and (76) as an experimental fact, and if we compare these probabilities with the quantum mechanical probabilities (9) and (10) we are able to make the following statements:

- Performing the classical Bell’s experiment with the 4 different sets (24) - (27) of the local interaction parameters shows that a violation of Bell’s inequality (23) cannot be observed! Thus we may state that Bell’s inequality can be used to distinguish whether Bell’s experiment was performed with classical or quantum mechanical objects.
- The probabilities of the quantum mechanical and classical Bell’s experiment are identical if $(\alpha = 0, \beta \neq 0)$, or if $(\alpha \neq 0, \beta = 0)$. From this we may conclude that:
- The probabilities of the quantum mechanical Bell’s experiment with fixed parameters $\alpha_{QBB}$ and $\beta_{QBB}$ can be reproduced by a corresponding classical Bell’s experiment with fixed parameters $(\alpha_{cl} = 0, \beta_{cl} = \alpha_{QBB} - \beta_{QBB})$ or $(\beta_{cl} = 0, \alpha_{cl} = \alpha_{QBB} - \beta_{QBB})$. Thus, looking only at the probabilities of a certain experiment without having the information about the local interaction parameters will not allow us to distinguish if this experiment was performed with classical or quantum objects.
- Each local observer of a Bell’s experiment will always measure the probabilities $1/2$ for both local events “lamp on: $y_{A/B}$” or “lamp of: $n_{A/B}$”, independent of the local interaction parameters $\alpha$ and $\beta$, and independent of whether classical or quantum objects are used.
- Beside the violation of Bell’s inequality there exists another criterion to decide whether Bell’s experiment was performed with quantum mechanical or classical objects. In contrast to the 4 necessary experimental configurations required to test Bell’s inequality we can simply choose one experimental configuration with the same local interaction parameters, let’s say $\alpha = \beta = \pi/4$, for example. If the quantum mechanical Bell’s experiment was performed, then the probabilities for
the 4 events are given by \( w(y, y) = w(n, n) = 0 \) and \( w(y, n) = w(n, y) = 1/2 \). On the other hand, we get the probabilities \( w(y, y) = w(n, n) = w(y, n) = w(n, y) = 1/4 \) if the classical Bell’s experiment was performed.

4. Bell’s experiment and complementarity

This final section is concerned with the following question: What would be the result of the quantum mechanical Bell’s experiment if performing noncontacting measurements to detect the polarization state of the photon pairs emitted by the primary source, and before the additional local interactions with the polarization filters on both sides take place? It is the same situation we are faced with in the quantum mechanical double slit experiment. There we may ask what happens with the characteristic frequency distribution in the far field if performing noncontacting measurements to get the "which slit"-information. Both questions are strongly related to the complementarity principle. According to the usual understanding of the complementarity principle, and regarding the double slit experiment we would expect that the characteristic interference pattern will be lost if getting the "which slit"-information by noncontacting measurements. But the very recent experiments with polarization entangled photons and a special laser mode performed at the University of Potsdam have shown that the characteristic double slit interference pattern can still be observed (see [10] [11]). The "which slit"-information was obtained by noncontacting coincidence measurements in these experiments. I.e., despite the fact that the "which slit"-information was obtained both substates (the states related to each of the slits) must be superposed to explain this behaviour. If this result is transferred to the quantum mechanical Bell’s experiment the situation becomes as follows:

Let’s assume that we are equipped with a noncontacting measurement setup to get the information about the state of polarization of the primary emitted photon pairs, and before the additional local interactions with the polarization filters on each side of Bell’s experiment take place. According to the usual understanding of the complementarity principle one may expect that the probabilities (75) and (76) would be measured, instead of the probabilities (9) and (10). That’s because we are then in exactly the same situation we are confronted with in the classical marble experiment. Knowing the state of polarization of a photon would allow us to predict the probability of the photon to pass a polarization filter in a certain orientation. Therefore, the superposition of both substates (10) and (11) should be excluded. Such a noncontacting measurement of the state of polarization of a photon pair would be possible if there exists a primary source that emits 3 or more polarization entangled photons at the same time. The additional entangled photons may be used for coincidence measurements, for example, as it was done in the Potsdam experiments. Unfortunately, no such source or experiment is described in the literature, so far. But if we transfer the result of the Potsdam double slit experiment to the quantum mechanical Bell’s experiment we may speculate that we would still measure the probabilities (9) and (10). However, there is a quite interesting indication that supports our speculation, and that holds for the Potsdam double slit experiment as well. If we would indeed measure the probabilities (75) and (76) if performing the noncontacting measurements to get the "which state of polarization" information, then probabilities would no longer represent objectively measurable quantities since depending on the state of information of the observer! Two observers, one equipped with a noncontacting measurement setup and the other without such an equipment,
would measure different probabilities in the same experiment. This is not even conceivable since the considered event space is a pure classical one. The result of a measurement should be factual and independent of any state of information of the observer. Otherwise, physics would lose its objective character since objective measurements are its backbone. Abandon the objectivity of measurements would have serious consequences not only for quantum mechanics but for general physics. Fortunately, the result of the Potsdam experiment, and the speculative result of Bell's experiment are (or would be) in agreement with the requirement of the objectivity of any but especially quantum mechanical measurements.

The above discussion raises the question if we have to distinguish carefully between "objective measurements" and "noncontacting measurements" (or, better, the noncontacting gain of information). The former should be factual and independent of any state of information of the observer. Contrary, the latter is based on a certain knowledge (or theory) gained by performing local or non-local but objective measurements before. This knowledge is used afterwards in future experiments. This seems to be the only way to perform "noncontacting measurements", in my opinion. And that’s what was exactly done in the Potsdam experiments. In contrast to "noncontacting measurements" the "objective measurements" are based essentially on the interaction of the object under consideration with the measurement device. This interaction has an uncontrollable impact on the state of the object itself, as well-known from quantum mechanics. Therefore, if the Potsdam experiments would not be performed with noncontacting but objective measurements on the signal photon to get the "which slit" information, then we may expect an impact on the characteristic interference pattern in the far field.

5. Conclusion

The most essential results of this paper may be condensed into the following 2 statements:

(i) Entanglement is not responsible for the violation of Bell’s inequality in the quantum mechanical Bell’s experiment. It is in fact the superposition of 2 orthogonal and non-entangled substates, as already known from the quantum mechanical double slit experiment. The two substates which must be superposed in the quantum mechanical case turned out to be the result of 2 additional, local, and stochastically independent interactions.

(ii) It is possible to introduce a statistical operator for both the quantum mechanical and classical Bell’s experiment. This operator contains already negative quasi probabilities in the quantum mechanical case. But these negative quasi probabilities are neither sufficient nor necessary for entanglement. They are rather an indication of the existence of interference terms.

But what makes the superposition of probability states in quantum mechanics staying in conflict with our everyday experience? It cannot be the superposition itself since this process is already known from classical field theories if there exist 2 or more fields at the same time in a certain region of space. To answer the question we must first note that the measurable events in Bell’s as well as in the quantum mechanical double slit experiment are typical particelike (the flashing of a pixel on a screen or a lamp, a click, etc.). All these events are local with respect to time and position and are usually attributed to the interaction of a particle with the measurement device.
The very notation "photon" provokes already the picture of a particle, irregardless of the abstract formalism behind. Looking at the superposition of the 2 substates in the quantum mechanical Bell’s experiment we have to note that both substates belong to events which do not take place at the same time. But they must be superposed, as though existing at the same time, to get the correct probabilities. The same holds for the quantum mechanical double slit experiment. That’s indeed strange and hard to understand. Could this problem possibly be solved by reconsidering our understanding of "simultaneity" on the atomic and subatomic level? That’s what happened already on a macroscopic scale in the context of special relativity. The point of view that "probability" should usually represent a quantity that is independent of time (comparable to the steady state situation of classical fields?) would be an argument against this idea. Moreover, in quantum mechanics probability is rather related to the behaviour of an ensemble than to that of a single particle. And if an ensemble is considered in the equilibrium state we have the equality of time- and ensemble average. However, this aspect of time seems to be worth to further considerations, to my mind.

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