APPROXIMATE SOLUTIONS OF VECTOR FIELDS AND AN APPLICATION TO DENJOY-CARLEMAN REGULARITY OF SOLUTIONS OF A NONLINEAR PDE

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ABSTRACT. In this paper we study microlocal regularity of a $C^2$ solution $u$ of the equation

$$u_t = f(x,t,u,u_x),$$

where $f(x,t,u,u_x)$ is ultradifferentiable in the variables $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and holomorphic in the variables $(\zeta_0, \zeta) \in \mathbb{C} \times \mathbb{C}^N$. We proved that if $c^M$ is a regular Denjoy-Carleman class (including the quasianalytic case) then:

$$\text{WF}_{c^M}(u) \subset \text{Char}(L^u),$$

where $\text{WF}_{c^M}(u)$ is the Denjoy-Carleman wave-front set of $u$ and $\text{Char}(L^u)$ is the characteristic set of the linearized operator $L^u$:

$$L^u = \frac{\partial}{\partial t} - \sum_{j=1}^N \frac{\partial f}{\partial \zeta_j}(x,t,u,u_x) \frac{\partial}{\partial x_j}.$$

1. Introduction

Let $\Omega \subset \mathbb{R}^N \times \mathbb{R}$ and $\Omega'' \subset \mathbb{C} \times \mathbb{C}^N$ be open sets and let $f \in C^\infty(\Omega' \times \Omega'')$ be holomorphic with respect to the variables $(\zeta_0, \zeta) \in \mathbb{C} \times \mathbb{C}^N$. Suppose that $u \in C^2(\Omega')$ is a solution of the nonlinear equation:

$$u_t = f(x,t,u,u_x),$$

and consider the linearized operator:

$$L^u = \frac{\partial}{\partial t} - \sum_{j=1}^N \frac{\partial f}{\partial \zeta_j}(x,t,u,u_x) \frac{\partial}{\partial x_j}.$$

Many authors have studied the relation between the microlocal regularity of $u$ and the characteristic set of the linearized operator $L^u$ for different assumptions on the regularity of the function $f$ in the variables $(x,t)$. In [10] F. Treves and N. Hanges proved that if $f$ is real-analytic in $(x,t)$ then the real-analytic wave front set of $u$ is contained in the characteristic set of $L^u$. The $C^\infty$ version of this result is a consequence of a result proved by J. Y. Chemin in [7], a different proof of it being obtained by C. H. Asano, in [3], by adapting Hanges-Treves’ techniques. Later on, R. F. Barostichi and G. Petronilho proved in [4] that if $f$ is Gevrey in $(x,t)$ then the same result is valid for the Gevrey wave front set. Finally, Z. Adwan and G. Hoepfner proved in [1] analogous results for strongly non-quasianalytic Denjoy-Carleman classes. The main difference between Asano’s and Treves-Hanges’ proofs is the availability of Cauchy-Kowalevski in the analytic setting while in the $C^\infty$ case the proof relies on approximate solutions of vector fields and almost-analytic extensions. The main difficulty in the Gevrey and in the strongly non-quasianalytic case is to find a suitable approximate solution that belongs to the class under consideration.

In this work we deal with the same problem as in [10], [3], [4] and [1], but in the case of regular Denjoy-Carleman classes. The only extra hypothesis that we make is that the space of the real-analytic functions is properly contained in the Denjoy-Carleman class under consideration. This includes the quasi-analytic case, and in that case, we gain a difficulty: the absence of non-trivial flat functions. This is an obstruction for the technique that Asano, Barostichi-Petronilho and Adwan-Hoepfner used in their proofs.

Loosely speaking if $u_0$ is a function ($C^\infty$, Denjoy-Carleman, Gevrey) in an open set $\Omega$, and $L$ is a vector field in $\Omega \times [-1,1)$, a function $u$ on $\Omega \times [-1,1]$ is an approximate solution of $L$ with initial datum

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we adapt Dyn'kin’s proof for the case of a vector field of the form \( m \) defined in \([9]\). and then applying the same argument of Hanges-Treves in \([10]\) we prove the desired regularity result. Carleman classes following \([9]\), in Section 3 we prove the theorem about approximate solutions, Theorem Hanges-Treves result for general regular Denjoy-Carleman classes. characterization of the Denjoy-Carleman wave-front set given by the FBI-transform, we can prove the With this in our hands and other results concerning general Denjoy-Carleman functions, such as the given a \( C \)-function on an open set \( \Omega \subset \mathbb{R}^N \) there exists a suitable almost-analytic extension of \( u \) in the complex space, i.e. there exists a function \( U \in \mathcal{C}^\infty(\mathbb{C}^N) \) such that \( U(x) = u(x), \forall x \in \Omega \), and

\[
\left| \frac{\partial U}{\partial z_j}(z) \right| \leq \frac{C^{k+1}M_k}{k!} |\text{Im } z|^k, \quad k \in \mathbb{Z}_+, \quad j = 1, 2, \ldots, N.
\]

Stated differently, \( U \) is an \((\mathcal{M}, |\text{Im } z|)\)-approximate solution for the complex \( \{\partial/\partial z_j\}_{j=1}^N \). In this paper we adapt Dyn’kin’s proof for the case of a vector field of the form

\[
L = \frac{\partial}{\partial t} + \sum_{j=1}^N a_j(x, t) \frac{\partial}{\partial x_j}.
\]

With this in our hands and other results concerning general Denjoy-Carleman functions, such as the characterization of the Denjoy-Carleman wave-front set given by the FBI-transform, we can prove the Hanges-Treves result for general regular Denjoy-Carleman classes.

We organize the paper as follows: in Section 2 we state and prove some results about regular Denjoy-Carleman classes following \([9]\), in Section 3 we prove the theorem about approximate solutions, Theorem 3.6, and finally in the Section 4 we use Theorem 3.6 to prove the main result of this paper, Theorem 4.3, and then applying the same argument of Hanges-Treves in \([10]\) we prove the desired regularity result.

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2. Denjoy-Carleman classes

In this section we recall the definitions and some properties of the regular Denjoy-Carleman classes as defined in \([9]\).

Let \( \mathcal{M} = (M_k)_{k=0}^\infty \) be a sequence of positive real numbers. We say that \( \mathcal{M} \) is regular if the sequence \((m_k)_{k=0}^\infty \), where \( m_k = M_k/k! \), has the following properties:

a) \( m_0 = m_1 = 1; \)
b) \( m_k^2 \leq m_{k-1}m_{k+1}, \quad k \geq 1; \)
c) \( \sup (m_{k+1}/m_k)_{k=1}^{1/k} < \infty; \)
d) \( \lim_{k \to \infty} m_k^{1/k} = \infty. \)

The conditions a) and b) imply that the sequence \( m_k \) is increasing; condition c) gives us a constant \( c > 0 \) such that \( m_{k+1} \leq c^k m_k \), for all \( k = 0, 1, 2, \ldots; \) condition b) is often called strong log-convexity. If \( \Omega \subset \mathbb{R}^N \) is an open set, the space \( \mathcal{C}^\infty(\Omega) \) of ultradifferentiable functions associated to the regular sequence \( \mathcal{M} \) is the space of all \( \mathcal{C}^\infty \)-functions \( f \) such that for every compact \( K \subset \Omega \) there is a positive constant \( A \) for which the following inequality holds:

\[
\sup_{x \in K} |D^\alpha f(x)| \leq A^{[\alpha]+1} M_\alpha, \quad \forall \alpha \in \mathbb{Z}_+^N.
\]
Thus, setting $M_k = k!^s$, $s > 1$, one obtain the Gevrey classes $G^s$. As in [8] we define the FBI transform of a compactly supported distribution $u$ by
\[ \mathcal{F}[u](x, \xi) = u_y \left( e^{i(x-y) \cdot \xi - |\xi|^2/(4y)} \right). \]

In [11] it is proved that a compactly supported distribution $u$ belongs to $C^M$ if and only if for every compact $K$ there is a positive constant $A$ such that:
\[ |\mathcal{F}[u](x, \xi)| \leq \frac{A^{k+1}M_k}{|\xi|^k}, \quad k \in \mathbb{Z}_+, \; x \in K, \; \xi \in \mathbb{R}^N. \]

This last inequality can be used to microlocalize the notion of $C^M$ regularity, thus we can define the Denjoy-Carleman wave-front set of a distribution $u$ at a point $x$, denoted by $\text{WF}_M(u)_x$, as the complementary set of the $C^M$-regular directions. Now we will recall some functions defined in [9] that play a crucial role in the proof of the approximate solution result.

**Definition 2.1.** For each $r > 0$ we define:
\[ h_1(r) = \inf_{k \in \mathbb{Z}_+} m_k r^{k-1}, \]
\[ h(r) = \inf_{k \in \mathbb{Z}_+} m_k r^k. \]

**Remark 2.2.** Note that for $r \geq 1$, we have $h_1(r) = 1$.

**Proposition 2.3.** Let $n \in \mathbb{Z}_+$. There are constants $C_1, C_2, Q_1, Q_2 > 0$ such that
\[ \frac{1}{r^n} h_1(r) \leq C_1 h_1(Q_1 r), \quad \forall r > 0, \]
\[ \frac{1}{r^n} h(r) \leq C_2 h(Q_2 r), \quad \forall r > 0. \]

**Proof.** Let $k \geq n$.
\[ \frac{1}{r^n} m_k r^{k-1} \leq e^{k-1} m_{k-1} r^{k-n-1} \leq e^{(k-1)+(k-2)+\cdots+(k-n)} m_{k-n} r^{k-n-1} = C_1 m_{k-n} (Q_1 r)^{k-n-1}, \]
where the constants $C_1$ and $Q_1$ only depend on $n$. Then:
\[ \frac{1}{r^n} h(r) = \frac{1}{r^n} \inf_{k \in \mathbb{Z}_+} m_k r^{k-1} \leq \inf_{k \geq n} m_k r^{k-n-1} \leq C_1 \inf_{k-n \geq 0} m_{k-n} (Q_1 r)^{k-n-1} = C_1 h(Q_1 r). \]

The proof for the function $h_1$ is analogous. \qed

**Definition 2.4.** For $r > 0$ we define:
\[ N(r) = \min \{ n : h_1(r) = m_n r^{n-1} \}. \]

**Proposition 2.5.** There exists a subsequence $(m_{n_k})_{k=1}^\infty$ such that:
\[ N \left( \frac{m_{n_k}}{m_{n_{k+1}}} \right) = n_k. \]

**Proof.** We first assume $m_n^2 < m_{n-1} m_{n+1}$ for all $n \in \mathbb{Z}_+ \setminus \{0\}$, then for each such $n$ we shall prove:
\[ N(r) = n, \quad \frac{m_n}{m_{n+1}} \leq r < \frac{m_{n-1}}{m_n}. \]
Let $k < n$ be a non-negative integer, we have:
\[ m_k^{r^{k-1}} = \frac{m_k}{m_{k+1}} \frac{m_{k+1}}{m_{k+2}} \cdots \frac{m_{n-1}}{m_n} m_n r^{n-1} > m_n r^{n-1}, \]
by our assumption on $(m_n)_{n=0}^\infty$. Thus $h_1(r) \leq m_n r^{n-1} < m_r r^{k-1}$, and $N(r) \geq n$. On the other hand, for each non-negative integer $j > n$ we have:
\[ m_n r^{n-1} = \frac{m_n}{m_{n+1}} \frac{m_{n+1}}{m_{n+2}} \cdots \frac{m_{j-1}}{m_j} m_j r^{j-1} \leq m_j r^{j-1}. \]
Therefore $N(r) = n$. In particular, we have $N(m_n/m_{n+1}) = n$. In the general case one has to take the least subsequence of $(m_n)$ which is strictly log-convex.

\[ \square \]

**Corollary 2.6.** The function $N$ is a decreasing step function such that $N(r) = 0$ for every $r \geq 1$ and $\lim_{r \to 0} N(r) = \infty$.

**Lemma 2.7.** Let $r > 0$. If $n \leq k \leq N(r)$, then:
\[ m_k r^k \leq m_n r^n. \]

**Proof.** Let $n \leq k \leq N(r)$. Condition b) implies that:
\[ m_k^{N(r)-n} \leq m_n^{N(r)-k} m_k^{k-n} m_n^{n-k} \]
Thus:
\[ (m_k r^k)^{N(r)-n} \leq m_n^{N(r)-k} m_k^{N(r)-n-k} \leq m_n^{N(r)-k} m_n^{n-k} \leq m_n^{N(r)-n} m_n^{n-k} \leq (m_n r^n)^{N(r)-n}. \]

\[ \square \]

### 3. Approximate solutions for vector fields

We shall denote the coordinates on $\mathbb{R}^N \times \mathbb{R}$ and on $\mathbb{C}^M$ by $(x, t) = (x_1, \ldots, x_N, t)$ and $\zeta = (\zeta_1, \ldots, \zeta_M)$, respectively. For this section, we fix $\Omega'$, an open neighborhood of the origin in $\mathbb{R}^N$, and $\Omega''$, an open set in $\mathbb{C}^M$. Let
\[ L = \frac{\partial}{\partial t} + \sum_{i=1}^N a_i(x, t, \zeta) \frac{\partial}{\partial x_i} + \sum_{j=1}^M b_j(x, t, \zeta) \frac{\partial}{\partial \zeta_j}, \]
be a vector field in $\Omega' \times \mathbb{R} \times \Omega''$ where $a_i, b_j$ are holomorphic in the variable $\zeta$ and of class $\mathcal{C}^1$ in $(x, t)$.

**Definition 3.1.** Let $u_0 \in \mathcal{C}^1(\Omega' \times \Omega'')$ be given. An $(\mathcal{M}, t)$-approximate solution of $L$ on $\Omega' \times \mathbb{R} \times \Omega''$ with initial datum $u_0$ is a function $u \in \mathcal{C}^1(\Omega' \times \mathbb{R} \times \Omega'')$ with the following properties:

1. For $(x, \zeta) \in \Omega' \times \Omega''$ we have $u(x, 0, \zeta) = u_0(x, \zeta);
2. For every compact set $K \subset \Omega' \times \Omega''$ there are constants $A, \gamma, \delta > 0$ such that:
\[ \sup_{(x, \zeta) \in K} |Lu(x, t, \zeta)| \leq Ah(\gamma |t|), \quad 0 < |t| \leq \delta. \]

Condition (2) in the definition above is equivalent to: for every compact set $K \subset \Omega' \times \Omega''$ there are positive constants $A, \gamma, \delta$ such that
\[ \sup_{(x, \zeta) \in K} |Lu(x, t, \zeta)| \leq A^{k+1} \frac{M_k}{k!} (|t|)^k, \quad 0 < |t| \leq \delta. \]

In this section, we shall prove that there exists an $(\mathcal{M}, t)$-approximate solution $u$ of $L$ for every $u_0 \in \mathcal{C}^M(\Omega' \times \Omega'')$ as initial datum when the coefficients of $L$ are functions of class $\mathcal{C}^M$ in $(x, t)$. Let $\mathcal{A}$ be the subspace of $\mathcal{C}^\infty(\Omega' \times \Omega'')$ consisting of all functions that are holomorphic with respect to $\zeta$ and of class $\mathcal{C}^M$ in the variable $x$. First we shall assume that $a_i, b_j \in \mathcal{A}$ for the vector field (3.1) (thus the coefficients of $L$ do not depend on $t$, the general case follows from this particular one) and denote by $\mathcal{A}[|t|]$ the
space of formal power series in the variable \( t \) with coefficients on \( \mathcal{A} \). Then the vector field (3.1) is an endomorphism of \( \mathcal{A}[[t]] \). Let \( f \in \mathcal{A} \) be given and let \( u^f(x, \zeta, t) = \sum_{k=0}^{\infty} u_k(x, \zeta) t^k \) be a formal solution of the problem:

\[
\begin{aligned}
L u^f(x, \zeta, t) &= 0, \\
u^f(x, \zeta, 0) &= f(x, \zeta).
\end{aligned}
\]

In fact, we have:

\[
u_0(x, \zeta) = f(x, \zeta), \quad u_k(x, \zeta) = -\frac{1}{k} \left\{ \sum_{i=1}^{N} a_i(x, \zeta) \frac{\partial u_{k-1}}{\partial x_i}(x, \zeta) + \sum_{j=1}^{M} b_j(x, \zeta) \frac{\partial u_{k-1}}{\partial \zeta_j}(x, \zeta) \right\},
\]

for each \( k \in \mathbb{Z}_+ \setminus \{0\} \) and each \((x, \zeta) \in \Omega' \times \Omega''\).

**Proposition 3.2.** For each compact set \( K \subset \Omega' \times \Omega'' \) there exists \( C = C_K > 0 \) such that:

\[
sup_{(x, \zeta) \in K} \left| \frac{\partial^\alpha_x \partial^\beta_\zeta u_k(x, \zeta)}{k!} \right| \leq \frac{C^{1+|\alpha|+|\beta|+k}M^{|\alpha|+k|\beta|}}{k!},
\]

for all \( \alpha \in \mathbb{Z}_+^N \) and \( \beta \in \mathbb{Z}_+^M \).

For a proof of the above proposition, see Lemma 4.1 in [4] and Lemma 18 in [1], where the Gevrey case and the strongly non-quasi-analytic case, respectively, are proved; the proofs also hold in our case for they are based only on the log-convexity property. We save the the symbol \( C \) for the constant in Proposition 3.2.

**Definition 3.3.** For \( n \in \mathbb{Z}_+ \) define \( T^n : \mathcal{A}[[t]] \rightarrow \mathcal{A}[[t]] \) by

\[
T^n \left[ \sum_{k=0}^{\infty} s_k(x, \zeta) t^k \right] = \sum_{k=0}^{n} s_k(x, \zeta) t^k,
\]

where \( \sum_{k=0}^{\infty} s_k(x, \zeta) t^k \in \mathcal{A}[[t]] \).

**Proposition 3.4.** For each compact set \( K \subset \Omega' \times \Omega'' \) there exists \( B = B_K > 0 \) such that:

\[
sup_{(x, \zeta) \in K} |L(T^n u^f)(x, \zeta, t)| \leq B^{n+1} n_m |t|^n.
\]

**Proof.** We have the following identity of formal power series:

\[
L \left( T^n u^f \right)(x, \zeta, t) = L \left( \sum_{k=0}^{n} u_k(x, \zeta) t^k \right)
= L \left( u^f(x, \zeta, t) - \sum_{k=n+1}^{\infty} u_k(x, \zeta) t^k \right)
= -L \left( \sum_{k=n+1}^{\infty} u_k(x, \zeta) t^k \right)
= [(n+1)u_{n+1}(x, \zeta)] t^n + Q(x, \zeta, t),
\]

where \( Q(x, \zeta, t) \in t^{n+1} \mathcal{A}[[t]] \). But since the left-hand side of the previous equation is a polynomial in the variable \( t \) of degree \( n \), we have that \( L \left( T^n u^f \right)(x, \zeta, t) = [(n+1)u_{n+1}(x, \zeta)] t^n \). Now the result follows from Proposition 3.2 combined with property (c) of the regular Denjoy-Carleman classes definition.

Now we can use the technique presented in [9] to define an \((\mathcal{A}, t)\)-approximate solution \( u \) for the vector field (3.1) with initial datum \( f \in \mathcal{A} \). Let \( \varepsilon > 0 \) be given and let \( \psi \in C_\infty^\infty(D_\varepsilon(0)) \) be a cutoff function such that \( \psi \geq 0, \psi(z) = \psi(|z|) \) for all \( z \), and

\[
\int_C \psi(z) dz = \frac{2}{i}.
\]
Fix $U \in \Omega'$ a neighborhood of the origin and $V \in \Omega''$ an open set. Now define for $x \in U$, $\zeta \in V$ and $|t| > 0$ 

$$u(x, t, \zeta) = \frac{i}{2\pi^2} \int_{C} \psi \left( \frac{z - t}{|t|} \right) N((1 + \epsilon)|z|) \sum_{k=0}^{n} u_k(x, \zeta) z^k dz \wedge d\bar{z}.$$ 

The function under the integral sign is measurable since $N(r)$ is a step function, so $u$ is well defined. Differentiating under the integral sign we conclude that $u$ is holomorphic in $\zeta$. Because of the choice of $\psi$ we have that $\lim_{r \rightarrow 0} u(x, t, \zeta) = u_0(x, \zeta) = f(x, \zeta)$. So we can set $u(x, 0, \zeta) = f(x, \zeta)$. In view of the symmetry of $\psi$, we have:

$$\frac{i}{2\pi^2} \int_{C} \psi \left( \frac{z - t}{|t|} \right) P(z) dz \wedge d\bar{z} = P(t),$$

for every polynomial $P(z)$, in fact:

$$\frac{i}{2\pi^2} \int_{C} \psi \left( \frac{z - t}{|t|} \right) P(z) dz \wedge d\bar{z} = \frac{i}{2} \int_{C} \psi(w) P(|t| w + t) dw \wedge d\bar{w} = P(t) + \frac{i}{2} \int_{C} \psi(w) Q(t, |t| w) dw \wedge d\bar{w},$$

where $Q(t, z)$ is a polynomial such that $Q(t, 0) = 0$, hence

$$\int_{C} \psi(w) Q(t, |t| w) dw \wedge d\bar{w} = 0.$$ 

Therefore we have

$$Lu(x, t, \zeta) = L \left[ \sum_{k=0}^{n} u_k(x, \zeta) t^k + \frac{i}{2\pi^2} \int_{C} \psi \left( \frac{z - t}{|t|} \right) N((1 + \epsilon)|z|) \sum_{k=n+1}^{\infty} u_k(x, \zeta) z^k dz \wedge d\bar{z} \right]$$

$$= L \left( T^n u^* \right)(x, \zeta, t) + \frac{i}{2} \int_{C} L \left[ \frac{1}{i^2} \psi \left( \frac{z - t}{|t|} \right) N((1 + \epsilon)|z|) \sum_{k=n+1}^{\infty} u_k(x, \zeta) z^k \right] dz \wedge d\bar{z}$$

$$= L \left( T^n u^* \right)(x, \zeta, t) + \frac{i}{2} \int_{C} L \left[ \frac{1}{i^2} \psi \left( \frac{z - t}{|t|} \right) N((1 + \epsilon)|z|) \sum_{k=n+1}^{\infty} u_k(x, \zeta) z^k dz \wedge d\bar{z} + C_{1} \frac{1}{|t|^4} \right]$$

By simple computations one can show that

$$\left| L \left[ \frac{1}{i^2} \psi \left( \frac{z - t}{|t|} \right) \right] \right| \leq C_1 \frac{1}{|t|^4},$$

for some positive constant $C_1$. Since $0 \leq \psi \leq 1/(\pi \epsilon^2)$, we have:

$$|Lu(x, t, \zeta)| \leq \left| L \left( T^n u^* \right)(x, \zeta, t) \right| + \frac{1}{2} \int_{C} \left| \frac{1}{i^2} \psi \left( \frac{z - t}{|t|} \right) N((1 + \epsilon)|z|) \right| \sum_{k=n+1}^{\infty} \left| u_k(x, \zeta) \right| |z|^k dz \wedge d\bar{z}$$

$$+ \frac{1}{2} \int_{C} \left| \frac{1}{i^2} \psi \left( \frac{z - t}{|t|} \right) \right| N((1 + \epsilon)|z|) \sum_{k=n+1}^{\infty} |Lu_k(x, \zeta)| |z|^k dz \wedge d\bar{z}$$

$$\leq \left| L \left( T^n u^* \right)(x, \zeta, t) \right| + \frac{C_1}{2|t|^4} \int_{|z - t| \leq |t| \epsilon} \sum_{k=n+1}^{\infty} \left| u_k(x, \zeta) \right| |z|^k dz \wedge d\bar{z}$$

$$+ \frac{1}{2\pi \epsilon^2} \int_{|z - t| \leq |t| \epsilon} \sum_{k=n+1}^{\infty} \left| Lu_k(x, \zeta) \right| |z|^k dz \wedge d\bar{z},$$
Now we fix \( n = N ((1 + \varepsilon)^2 C |t|) - 1 \). Note that \( n \) must be positive, so from now on we shall assume \(|t| \leq 1/(1 + \varepsilon)^2 C = \delta\). Applying Lemma 2.7 we can estimate:

\[
M_k \left( (1 + \varepsilon) C|z| \right)^k \leq \frac{M_{n+1}}{(n + 1)!} (1 + \varepsilon)^{n+1},
\]

for \( n < k \leq N((1 + \varepsilon) C|z|) \). Therefore, by Proposition 3.2 and using \(|z - t| \leq \varepsilon |t|\), we have:

\[
\sum_{k=n+1}^{N((1+\varepsilon)C|z|)} |u_k(x, \zeta)||z|^k \leq \sum_{k=n+1}^{N((1+\varepsilon)C|z|)} C \frac{M_k}{k!} ((1 + \varepsilon) C|z|)^k \frac{1}{(1 + \varepsilon)^k}
\]

\[
\leq C_2 M_{n+1} C^{n+1}(1 + \varepsilon)^{2(n+1)}|t|^{n+1}
\]

\[
\leq C_2 (1 + \varepsilon)^2 |t| h_1 ((1 + \varepsilon)^2 C|t|),
\]

where this last equality follows from our choice of \( n \). Analogously, we have:

\[
\sum_{k=n+1}^{N((1+\varepsilon)C|z|)} |L u_k(x, \zeta)||z|^k \leq C_3 \sum_{k=n+1}^{N((1+\varepsilon)C|z|)} C^2 \frac{M_{k+1}}{(k + 1)!} ((1 + \varepsilon) C|z|)^k \frac{1}{(1 + \varepsilon)^k}
\]

\[
+ C_4 \sum_{k=n+1}^{N((1+\varepsilon)C|z|)} C \frac{M_k}{k!} ((1 + \varepsilon) C|z|)^k \frac{1}{(1 + \varepsilon)^k}
\]

\[
\leq C_5 M_{n+1} (1 + \varepsilon)^{2(n+1)} C^n |t|^{n+1}
\]

\[
\leq C_5 (1 + \varepsilon)^2 |t| h_1 ((1 + \varepsilon)^2 C|t|).
\]

By Proposition 3.4, we can also estimate the remaining term:

\[
|Lu(x, t, \zeta)| \leq C_6 h(Q|t|), \quad (x, \zeta) \in U \times V, \quad 0 < |t| \leq \delta.
\]

We claim that \( u \) is a \( C^\infty \)-function. We just have to check if \( u \) is of class \( C^\infty \) at \( \{(x,0,\zeta)\} \). For \( n > 0 \) we have:

\[
\frac{1}{|t|^n} \left| \partial_x^n \partial_\zeta^0 u(x, t, \zeta) - \sum_{k=0}^n \partial_x^k \partial_\zeta^0 u_k(x, \zeta) t^k \right| \leq \frac{1}{2|t|^n} \int_C \left| \frac{1}{|z - t|} \right| \left| \sum_{k=n+1}^{N((1+\varepsilon)C|z|)} \left| \partial_x^k \partial_\zeta^0 u_k(x, \zeta) \right| |z|^k |d\zeta \right|
\]

\[
= \frac{1}{2|t|^n} \int_C \left| \psi(w) \right| \left| \sum_{k=n+1}^{N((1+\varepsilon)C|t|+|w|)} \left| \partial_x^k \partial_\zeta^0 u_k(x, \zeta) \right| |t|^{n+1} \right| |dw| \longrightarrow 0,
\]

when \( t \to 0 \). We proved the following theorem:

**Theorem 3.5.** Let \( \Omega = \Omega' \times \mathbb{R} \times \Omega' \subset \mathbb{R}^N \times \mathbb{R} \times C^M \) be an open set, where \( \Omega' \subset \mathbb{R}^N \) is an open neighborhood of the origin. Let:

\[
L = \frac{\partial}{\partial t} + \sum_{i=1}^{N} a_i(x, \zeta) \frac{\partial}{\partial x_i} + \sum_{j=1}^{M} b_j(x, \zeta) \frac{\partial}{\partial \zeta_j},
\]

be a vector field defined on \( \Omega \), where \( a_i,b_j \in C^\infty(\Omega' \times \Omega') \) are functions of class \( C^M \) with respect to \( x \) and holomorphic in the variable \( \zeta \). Let \( f \in C^\infty(\Omega' \times \Omega') \) be a function of class \( C^M \) with respect to \( x \) and holomorphic in the variable \( \zeta \). Then for every open neighborhood of the origin \( U \subset \Omega' \) and every open set
\( V \in \Omega'' \), there are a \( C^\infty \)-function \( u = u(x,t,\zeta) \) defined on \( U \times \mathbb{R} \times V \) and holomorphic in \( \zeta \) and constants \( A, Q, \delta > 0 \) such that:
\[
\begin{align*}
|Lu(x,t,\zeta)| &\leq Ah(Q|t|), & (x,t,\zeta) &\in U \times (-\delta, \delta) \times V, \\
u(x,0,\zeta) &= f(x,\zeta), & (x,\zeta) &\in U \times V.
\end{align*}
\]
i.e., the function \( u \) is an \((M,t)\)-approximate solution of \( L \) on \( U \times \mathbb{R} \times V \) with initial datum \( f \).

In Theorem 3.5 we assumed that the coefficients of \( L \) do not depend on \( t \), however one can obtain the general case from it:

**Theorem 3.6.** Let \( \Omega = \Omega' \times I \times \Omega'' \subset \mathbb{R}^N \times \mathbb{R} \times \mathbb{C}^M \), where \( \Omega' \subset \mathbb{R}^N \) is an open neighborhood of the origin and \( \Omega'' \subset \mathbb{C}^M \) is an open set. Let
\[
L = \frac{\partial}{\partial t} + \sum_{i=1}^{N} a_i(x,t,\zeta) \frac{\partial}{\partial x_i} + \sum_{j=1}^{M} b_j(x,t,\zeta) \frac{\partial}{\partial \zeta_j},
\]
be a vector field defined on \( \Omega' \), where \( a_i, b_j \) are functions of class \( \mathbb{C}^M \) with respect to the variables \((x,t)\) and holomorphic in the variable \( \zeta \). Let \( f \in C^\infty(\Omega' \times \Omega'') \) holomorphic in \( \zeta \) and \( \mathbb{C}^M \) in \( x \). Then for every open neighborhood of the origin \( U \subset \Omega' \) and every open neighborhood of the origin \( V \subset \Omega'' \), there are a \( C^\infty \)-function \( u = u(x,t,\zeta) \) defined on \( U \times \mathbb{R} \times V \) and holomorphic in \( \zeta \) and constants \( A, Q, \delta > 0 \) such that:
\[
\begin{align*}
|Lu(x,t,\zeta)| &\leq Ah(Q|t|), & (x,t,\zeta) &\in U \times (-\delta, \delta) \times V, \\
u(x,0,\zeta) &= f(x,\zeta), & (x,\zeta) &\in U \times V.
\end{align*}
\]
i.e., the function \( u \) is an \((M,t)\)-approximate solution of \( L \) on \( U \times \mathbb{R} \times V \) with initial datum \( f \).

**Proof.** Consider the vector field \( \tilde{L} \) in \( \Omega \times \mathbb{R} \) defined by
\[
\tilde{L} = \partial_s + L,
\]
and consider the function \( \tilde{f}(x,t,\zeta) = f(x,\zeta) \). Let \( U \subset \Omega', \ V \subset \Omega'' \) both neighborhoods of the origin and \( r > 0 \) such that \((r,-r) \subset I \). By Theorem 3.5 there exists a function \( \tilde{u} \in C^\infty(U \times (-r,r) \times R \times V) \) and constants \( A, Q, \delta > 0 \) such that \( U \times (-\delta, \delta) \times V \subset \Omega \times \mathbb{R} \), \( \tilde{u}(x,t,0,\zeta) = \tilde{f}(x,t,\zeta) \), for every \((x,t,\zeta) \in U \times (-r,r) \times V \), and
\[
|\tilde{L}\tilde{u}(x,t,s,\zeta)| \leq Ah(Q|s|), \quad (x,t,s,\zeta) \in U \times (-r,r) \times (-\delta, \delta) \times V.
\]
We shall assume \( \delta < r \). Set \( F(x,t,\zeta) = \tilde{u}(x,t,t,\zeta) \) for \( x \in U \), \( \zeta \in V \) and \( |t| < \delta \). We have
\[
Lu(x,t,\zeta) = L(\tilde{u}(x,t,t,\zeta))
\]
\[
= \tilde{L}\tilde{u}(x,t,t,\zeta).
\]
Therefore the desired estimate follows from (3.2). \( \square \)

4. Nonlinear PDEs

The following lemmas are the \( \mathbb{C}^M \)-counterparts of results found in [3]. We shall denote the coordinates in \( \mathbb{R}^{N+1} = \mathbb{R}^N \times \mathbb{R} \) by \((x,t) = (x_1, \ldots, x_N, t)\).

**Lemma 4.1.** Let \( \Omega \subset \mathbb{R}^{N+1} \) be an open neighborhood of the origin. Let
\[
L = \frac{\partial}{\partial t} + \sum_{j=1}^{N} a_j(x,t) \frac{\partial}{\partial x_j},
\]
be a vector field in \( \Omega \) with coefficients in \( C^1(\Omega) \). For each \( 1 \leq j \leq N \), suppose that there exists \( Z_j \in C^1(\Omega) \) an \((M,t)\)-approximate solution of \( L \) with initial condition \( Z_j(x,0) = x_j \). Then there exists a vector field
\[
L_1 = \frac{\partial}{\partial t} + \sum_{j=1}^{N} b_j(x,t) \frac{\partial}{\partial x_j},
\]
defined on an open neighborhood of the origin \( \Omega_1 \subset \Omega \) and with coefficients in \( C^1(\Omega_1) \) such that:
(1) For each $1 \leq j \leq N$ we have:

$$L_1(Z_j) = 0,$$

$$a_j(x,0) = b_j(x,0);$$

(2) Every $(\mathcal{M},t)$-approximate solution of $L$ is an $(\mathcal{M},t)$-approximate solution of $L_1$.

For a proof of Lemma 4.1 see Section 2 of [3], pp. 3010-3011.

**Lemma 4.2.** Let $\Omega \subset \mathbb{R}^{N+1}$ be an open neighborhood of the origin. Let

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{N} a_j(x,t) \frac{\partial}{\partial x_j},$$

be a vector field in $\Omega$ where $a_j \in C^1(\Omega)$, $1 \leq j \leq N$. Suppose that there exists $Z_j \in C^1(\Omega)$ an $(\mathcal{M},t)$-approximate solution of $L$ with initial condition $Z_j(x,0) = x_j$. Let $\xi_0 \in \mathbb{R}^N \setminus \{0\}$ be such that $\text{Im} a(0) \cdot \xi_0 < 0$. Then there exists an open cone $\Gamma \subset \mathbb{R}^N \setminus \{0\}$, an open neighborhood of the origin $U \subset \mathbb{R}^N$, a cutoff function $\chi \in C_c^\infty(\mathbb{R}^N)$, with $\chi = 1$ on $U$, and constants $A > 0$ such that $\xi_0 \in \Gamma$ and

$$|F[\chi \Psi_0](x,\xi)| \leq \frac{A^{k+1}M_k}{|\xi|^k}, \quad (x,\xi) \in V \times \Gamma, \quad k \in \mathbb{Z}_+,$$

where $V \subset U$ is an open neighborhood of the origin, and $\Psi_0(x) = \Psi(x,0)$ is the trace of any $(\mathcal{M},t)$-approximate solution of $L$.

**Proof.** Let $\xi_0 \in \mathbb{R}^N$ be such that $\text{Im} a(0) \cdot \xi_0 < 0$. We apply Lemma 4.1 and obtain an open neighborhood of the origin $\Omega_1 \subset \Omega$ and a vector field $L_1$ on $\Omega_1$. We have $d(HdZ) = (L_1H)dt \wedge dZ$, for every function $H \in C^1(\Omega_1)$. Let $B \subset \mathbb{R}^N$ be an open ball around the origin and $I \subset \mathbb{R}$ an open interval around zero such that $B \times I \cap \Omega_1$. Choose a cutoff function $\chi \in C_c^\infty(B)$ such that $\chi \equiv 1$ on a neighborhood $U \subset B$ of the origin and $0 \leq \chi \leq 1$. Thus, if we fix an approximate solution $\Psi$ of $L$ and choose $H = H_{y,\xi}$ by the formula:\footnote{where $y,\xi$ are parameters.}

$$H(x,t) = e^{i\xi \cdot (y-Z(x,t)) - \xi \cdot (y-Z(x,t))^2} \chi(x)\Psi(x,t), \quad (x,t) \in \Omega_1,$$

then, for each positive $\lambda \in I$ we can apply Stokes’ theorem and get:

$$\int_{B \times [0,\lambda]} (L_1H)dt \wedge dZ = \int_{B \times [0,\lambda]} \chi(x)e^{i\xi \cdot (y-Z(x,t)) - \xi \cdot (y-Z(x,t))^2} L_1[\Psi(x,t)] dt \wedge dZ$$
$$+ \int_{B \times [0,\lambda]} e^{i\xi \cdot (y-Z(x,t)) - \xi \cdot (y-Z(x,t))^2} \Psi(x,t)L_1[\chi(x)] dt \wedge dZ$$
$$= \int_{\partial(B \times [0,\lambda])} HdZ$$
$$= \int_{x \in B} e^{i\xi \cdot (y-Z(x,\lambda)) - \xi \cdot (y-Z(x,\lambda))^2} \chi(x)\Psi(x,\lambda)d_xZ(x,\lambda)$$
$$- \int_{x \in B} e^{i\xi \cdot (y-Z(x,0)) - \xi \cdot (y-Z(x,0))^2} \chi(x)\Psi(x,0)d_xZ(x,0),$$

thus:

$$|F[\chi \Psi_0](y,\xi)| \leq \int_{B \times [0,\lambda]} \chi(x) \left| e^{i\xi \cdot (y-Z(x,t)) - \xi \cdot (y-Z(x,t))^2} L_1[\Psi(x,t)] \right| dt \wedge dZ$$
$$+ \int_{B \times [0,\lambda]} \left| e^{i\xi \cdot (y-Z(x,t)) - \xi \cdot (y-Z(x,t))^2} \Psi(x,t)L_1[\chi(x)] \right| dt \wedge dZ$$
$$+ \int_{x \in B} \left| e^{i\xi \cdot (y-Z(x,\lambda)) - \xi \cdot (y-Z(x,\lambda))^2} \chi(x)\Psi(x,\lambda) \right| d_xZ(x,\lambda).$$
Let \( Q(x, t, y, \xi) = i\xi \cdot (y - Z(x, t)) - |\xi|^2(y - Z(x, t))^2 \), then as in [3] there exists an open cone \( \Gamma \subset \mathbb{R}^N \), with \( \xi_0 \in \Gamma \), an open neighborhood of the origin \( V \subset \mathbb{R}^N \) and constants \( C_0, \delta > 0 \) such that
\[
\text{Re} \ Q(x, t, y, \xi) \leq -C_0|\xi|^2/2,
\]
for all \( x \in B, \xi \in \Gamma, y \in V \) and \( 0 < t < \delta \). Taking \( \delta \in I \) and \( V \subset U \), we can estimate:
\[
|\xi|^k \chi(x) \left| e^{i\xi \cdot (y - Z(x, t))} - |\xi|^2(y - Z(x, t))^2 \right| L_1 |\Psi(x, t)| \leq \int_{B \times [0, \delta]} |\xi|^k e^{-C_0|\xi|^2/2} C^k \frac{M_{k-1}}{(k-1)!} t^{k-1} \sup_{(x, t) \in B \times [0, \delta]} |\det Z_x(x, t)| \, dt \, dZ
\]
\[
= C^k m(B) \sup_{(x, t) \in B \times [0, \delta]} |\det Z_x(x, t)| \frac{M_{k-1}}{(k-1)!} \int_0^\delta e^{-C_0|\xi|^2/2} t^{k-1} \, dt
\]
\[
\leq C^k m(B) \sup_{(x, t) \in B \times [0, \delta]} |\det Z_x(x, t)| \frac{M_{k-1}}{(k-1)!} \int_0^\infty e^{-C_0|\xi|^2/2} t^{k-1} \, dt
\]
\[
= C^k m(B) \sup_{(x, t) \in B \times [0, \delta]} |\det Z_x(x, t)| \frac{M_{k-1}}{(k-1)!} \left( \frac{2C_0}{C_0} \right)^k
\]
\[
\leq \left( \frac{2C_0}{C_0} \right)^k m(B) \sup_{(x, t) \in B \times [0, \delta]} |\det Z_x(x, t)| M_k, \quad \xi \in \Gamma, y \in V.
\]
As in [3], the remaining terms in (4.2) have exponential decay in some conic neighborhood of the origin.

\( \square \)

**Theorem 4.3.** Let \( \Omega = \Omega' \times I \subset \mathbb{R}^N \times \mathbb{R} \) be an open neighborhood of the origin and let \( \Omega'' \subset \mathbb{C}^{N+1} \) be an open set. Let \( u \in C^2(\Omega) \) be a solution of the nonlinear PDE:
\[
u_t = f(x, t, u, u_x),
\]
where \( f(x, t, \zeta_0, \zeta) \) is a function of class \( C^M \) with respect to \((x, t) \in \Omega \) and holomorphic with respect to \((\zeta_0, \zeta) \in \Omega'' \). Let \( L^u \) be the linearized operator:
\[
L^u = \frac{\partial}{\partial t} - \sum_{j=1}^N \frac{\partial f}{\partial \zeta_j}(x, t, \zeta_0, \zeta) \frac{\partial}{\partial x_j}.
\]
Then for each open set \( U \subset \Omega' \) there exist \( C^1 \)-functions \( Z_j(x, t) \) and \( \Psi(x, t) \) that are \((\mathcal{M}, t)\)-approximate solutions of \( L^u \) on \( U \times \mathbb{R} \) with initial data \( x_j \) and \( u_0 = u(\cdot, 0) \), respectively, \( j = 1, \ldots, N \).

**Proof.** In this proof we follow closely the proof of Theorem 4.1 of [3]. Consider the vector field
\[
\mathcal{L} = \frac{\partial}{\partial t} - \sum_{j=1}^N \frac{\partial f}{\partial \zeta_j}(x, t, \zeta_0, \zeta) \frac{\partial}{\partial x_j},
\]
and the functions
\[
h_0(x, t, \zeta_0, \zeta) = f(x, t, \zeta_0, \zeta) - \sum_{j=1}^N \zeta_j \frac{\partial f}{\partial \zeta_j}(x, t, \zeta_0, \zeta)
\]
\[
h_i(x, t, \zeta_0, \zeta) = \frac{\partial f}{\partial x_i}(x, t, \zeta_0, \zeta) + \zeta_i \frac{\partial f}{\partial \zeta_0}(x, t, \zeta_0, \zeta), \quad i = 1, \ldots, N.
\]
This functions \( h_j \) satisfies \( h(x, u(x, t)) = L^u w(x, t) \), where \( w(x, t) = (u(x, t), u_x(x, t)) \). We can introduce now the holomorphic Hamiltonian
\[
H = \mathcal{L} + h_0 \frac{\partial}{\partial \zeta_0} + \sum_{j=1}^N h_j \frac{\partial}{\partial \zeta_j}.
\]
So it follows as in [10] that for every \( \Phi(x, t, \zeta_0, \zeta) \) a \( C^\infty \)-function,
\[
L^u \Phi = (H \Phi)^w,
\]
and $\mathcal{L}^w = L^w$, with the notation $\Phi^w(x, t) = \Phi(x, t, w(x, t))$. Let $U \Subset \Omega'$ be an open neighborhood of the origin and let $V \Subset \Omega''$ be an open neighborhood of $w(0, 0) = (u(0, 0), u_x(0, 0))$ such that $w(x, t) \in V$ for all $(x, t) \in U$. Applying Theorem 3.6 there exist functions $Z_j(x, t, \zeta_0, \zeta)$, $\Xi_k(x, t, \zeta_0, \zeta)$, $j = 1, \ldots, N$ and $k = 0, 1, \ldots, N$, $C^\infty$ in $(x, t)$ and holomorphic in $(\zeta_0, \zeta)$, $(\mathcal{M}, t)$-approximate solutions of $H\Phi = 0$ on $U \times \mathbb{R} \times V$ with initial conditions $Z_j(x, 0, \zeta_0, \zeta) = x_j$, for $j = 1, \ldots, N$ and $\Xi_k(x, 0, \zeta_0, \zeta) = \zeta_k$, for $k = 0, 1, \ldots, N$. So there are constants $C_1, \rho, \delta > 0$ such that

$$
\begin{align*}
|HZ_j(x, t, \zeta_0, \zeta)| &\leq C_1 h(\rho |t|), & \forall j = 1, \ldots, N, \\
|H\Xi_k(x, t, \zeta_0, \zeta)| &\leq C_1 h(\rho |t|), & \forall k = 0, 1, \ldots, N,
\end{align*}
$$

for $(x, \zeta_0, \zeta) \in U \times V$ and $|t| \leq \delta$. The identity (4.5) implies that $Z_j^w(x, t)$ is an $(\mathcal{M}, t)$-approximate solution of $\mathcal{L}^w$ with initial condition $Z_j^w(x, 0) = x_j$, for $j = 1, \ldots, N$. So it only remains to find an approximate solution of $\mathcal{L}^w$ with initial condition $u_0$. Let $\tilde{Z}(z, \tau, t, \zeta_0, \zeta)$ and $\tilde{\Xi}(z, \tau, t, \zeta_0, \zeta)$ be $\mathcal{M}$-almost holomorphic extensions of $Z(x, t, \zeta_0, \zeta)$ and $\Xi(x, t, \zeta_0, \zeta)$ on $U \times \mathbb{R} \times V$, see [9]. Note that $\tilde{Z}(z, \tau, t, \zeta_0, \zeta)$ and $\tilde{\Xi}(z, \tau, t, \zeta_0, \zeta)$ are both holomorphic in $(\zeta_0, \zeta)$. Then there are positive constants $C_2, \gamma$ such that, shrinking $\delta$ if necessary,

$$
\begin{align*}
\left| \frac{\partial \tilde{Z}_j}{\partial \tau_j}(z, \tau, t, \zeta_0, \zeta) \right| &\leq C_2 h(\gamma |\text{Im} \, z_j|), & \forall \tau, j = 1, \ldots, N, \\
\left| \frac{\partial \tilde{\Xi}_k}{\partial \tau_j}(z, \tau, t, \zeta_0, \zeta) \right| &\leq C_2 h(\gamma |\text{Im} \, z_j|), & \forall \tau, j = 1, \ldots, N, k = 0, 1, \ldots, N,
\end{align*}
$$

for $(z, \zeta_0, \zeta) \in (U + iB_\delta(0)) \times V$ and $|t| < \delta$. Since

$$
\frac{\partial(\tilde{Z}, \tilde{\Xi})}{\partial(z, \zeta_0, \zeta)}(\tau, \tau, t, \zeta_0, \zeta) = \tilde{Z}
$$

is non-singular if $t = 0$ and $\text{Im} \, z = 0$, shrinking if necessary $U$, $V$ and $\delta$, one can use the implicit function theorem to solve

$$
\begin{align*}
\tilde{Z}(z, \tau, t, \zeta_0, \zeta) &= \tilde{z} \\
\tilde{\Xi}(z, \tau, t, \zeta_0, \zeta) &= \tilde{\zeta},
\end{align*}
$$

with respect to $(z, \zeta_0, \zeta)$ in $(U + iB_\delta(0)) \times V$. So there are two $C^\infty$ functions $P$ and $Q$ such that

$$
\begin{align*}
z &\in P(\tilde{z}, \tilde{\zeta}, t, \zeta_0, \zeta) \\
(\zeta_0, \zeta) &\in Q(\tilde{z}, \tilde{\zeta}, t, \zeta_0, \zeta),
\end{align*}
$$

with $P(0, 0, \zeta_0, \zeta) = 0$ and $Q(0, 0, u(0), u_x(0)) = (u(0), u_x(0))$. Combining this four equations we obtain

$$
\begin{align*}
\tilde{Z}(P(\tilde{z}, \tilde{\zeta}, t, \zeta_0, \zeta), t, Q(\tilde{z}, \tilde{\zeta}, t, \zeta_0, \zeta)) &= \tilde{z} \\
\tilde{\Xi}(P(\tilde{z}, \tilde{\zeta}, t, \zeta_0, \zeta), t, Q(\tilde{z}, \tilde{\zeta}, t, \zeta_0, \zeta)) &= \tilde{\zeta}.
\end{align*}
$$

Differentiating the system (4.7) with respect to $\tilde{z}$ we obtain

$$
\begin{align*}
\frac{\partial(\tilde{Z}, \tilde{\Xi})}{\partial(z, \zeta_0, \zeta)}(P(\tilde{z}, \tilde{\zeta}, t, \zeta_0, \zeta), t, Q(\tilde{z}, \tilde{\zeta}, t, \zeta_0, \zeta)) \frac{\partial(P, Q)}{\partial \tilde{z}}(\tilde{z}, \tilde{\zeta}, t, \zeta_0, \zeta) + \\
\frac{\partial(\tilde{Z}, \tilde{\Xi})}{\partial(z, \zeta_0, \zeta)}(P(\tilde{z}, \tilde{\zeta}, t, \zeta_0, \zeta), t, Q(\tilde{z}, \tilde{\zeta}, t, \zeta_0, \zeta)) \frac{\partial(P, Q)}{\partial \tilde{z}}(\tilde{z}, \tilde{\zeta}, t, \zeta_0, \zeta) = 0.
\end{align*}
$$

Let $A(z, s, \zeta_0, \zeta)$ be a generic entry of the matrix

$$
\frac{\partial(\tilde{Z}, \tilde{\Xi})}{\partial(z, \zeta_0, \zeta)}(z, \tau, t, \zeta_0, \zeta).
$$

From the estimates (4.6) and that $\tilde{Z}$ and $\tilde{\Xi}$ are holomorphic in $(\zeta_0, \zeta)$ follows that

$$
|A(z, t, \zeta_0, \zeta)| \leq C_3 h(\gamma |\text{Im} \, z|), \quad \forall (z, t, \zeta_0, \zeta) \in (U + iB_\delta(0)) \times (-\delta, \delta) \times V,
$$

where $C_3$, $\gamma$ are positive constants.
for some positive constant $C_3$. Since the (complex) matrix
\[
\frac{\partial(\tilde{Z}, \tilde{\xi})}{\partial(z, \zeta, \xi, \eta)}(z, \bar{z}, t, \bar{\zeta}, \bar{\xi})
\]
is invertible for $\operatorname{Im} z = 0$ and $t = 0$, it follows that (shrinking $U, V$ and $\delta$ if necessary)
\[
(4.8) \quad \left| \frac{\partial Q_0}{\partial \bar{z}_j}(\bar{z}, \bar{\zeta}, t, \bar{\eta}) \right| \leq C_4 h(\gamma |\operatorname{Im} P(\bar{z}, \bar{\zeta}, t, \bar{\eta})) \quad \forall j = 1, \ldots, N.
\]
for some positive constants $C_4$. Analogously, differentiating the system (4.6) with respect to $\bar{z}$ and reasoning as before we have
\[
(4.9) \quad \left| \frac{\partial Q_0}{\partial \bar{\zeta}_j}(\bar{z}, \bar{\zeta}, t, \bar{\eta}) \right| \leq C_5 h(\gamma |\operatorname{Im} P(\bar{z}, \bar{\zeta}, t, \bar{\eta})) \quad \forall j = 0, \ldots, N,
\]
for some positive constants $C_5$. Define the function $\Psi(z, \bar{z}, t, \bar{\zeta}, \bar{\eta})$ for $(z, \bar{z}, t, \bar{\zeta}, \bar{\eta}) \in (U + iB_d(0)) \times (-\delta, \delta) \times V$ by
\[
\Psi(z, \bar{z}, t, \bar{\zeta}, \bar{\eta}) = Q_0\left(\tilde{Z}(z, \bar{z}, t, \bar{\zeta}, \bar{\eta}), \tilde{Z}(z, \bar{z}, t, \bar{\zeta}, \bar{\eta}), 0, \tilde{\xi}(z, \bar{z}, t, \bar{\zeta}, \bar{\eta}), \tilde{\xi}(z, \bar{z}, t, \bar{\zeta}, \bar{\eta})\right).
\]
And by the definition of $\Psi$ we have
\[
\Psi^w(x, 0) = \Psi(x, 0, w(x, 0)) = \Psi(x, 0, v(x, 0), v_x(x, 0)) = Q_0(x, 0, v(x, 0), v_x(x, 0)) = v(x, 0).
\]
Note that $H$ has no derivatives on $\operatorname{Im} z$, so $H\tilde{Z}(x, t, \xi, \eta) = H\tilde{Z}(x, t, \xi, \eta)$ and the same happens for $H\tilde{\xi}$ at $\operatorname{Im} z = 0$. We have:
\[
H\tilde{\xi} = \sum_{j=1}^{N} \left( \frac{\partial Q_0}{\partial \bar{z}_j} H\tilde{z}_j + \frac{\partial Q_0}{\partial \bar{\zeta}_j} H\tilde{\zeta}_j \right) + \sum_{k=0}^{N} \left( \frac{\partial Q_0}{\partial \bar{\xi}_k} H\tilde{\xi}_k + \frac{\partial Q_0}{\partial \bar{\zeta}_k} H\tilde{\zeta}_k \right),
\]
and also
\[
P(x, 0, \xi, \eta, \xi, \eta) = P(\tilde{Z}(x, 0, \xi, \eta), \tilde{Z}(x, 0, \xi, \eta), 0, \tilde{\xi}(x, 0, \xi, \eta), \tilde{\xi}(x, 0, \xi, \eta)) = x,
\]
so
\[
\operatorname{Im} P(\operatorname{Re} \tilde{z}, 0, \tilde{\zeta}, \tilde{\xi}) = 0.
\]
By the mean value inequality,
\[
|\operatorname{Im} P(\tilde{z}, \tilde{\zeta}, 0, \tilde{\xi})| = |\operatorname{Im} P(\tilde{z}, \tilde{\zeta}, 0, \tilde{\xi}) - \operatorname{Im} P(\operatorname{Re} \tilde{z}, 0, \tilde{\xi})|
\leq C_6 |\operatorname{Im} \tilde{z}|,
\]
For some positive constant $C_6$. On the other hand, since
\[
\tilde{Z}(x, 0, \xi, \eta) = x,
\]
there is $C_7 > 0$ such that
\[
|\operatorname{Im} \tilde{Z}(x, t, \xi, \eta) | \leq C_7 |t|, \quad (x, t, \xi, \eta) \in U \times (-\delta, \delta) \times V.
\]
Combining this two estimates with (4.8) and (4.9), taking $C = \max C_j$ we obtain
\[
\left| \frac{\partial Q_0}{\partial \bar{z}_j}(\tilde{Z}(x, t, \xi, \eta), \tilde{Z}(x, t, \xi, \eta), 0, \tilde{\xi}(x, t, \xi, \eta)) \right| \leq C h(\gamma |t|), \quad (x, t, \xi, \eta) \in U \times (-\delta, \delta) \times V,
\]
and
\[
\frac{\partial \bar{Q}_j}{\partial \zeta_j}(\bar{Z}(x,t,\zeta_0,\zeta), \bar{Z}(x,t,\zeta_0,\zeta), 0, \bar{Z}(x,t,\zeta_0,\zeta)) \leq Ch(\gamma|t|), \quad (x,t,\zeta_0,\zeta) \in U \times (-\delta,\delta) \times V.
\]

Summing up we have
\[
|H \Psi(x,t,\zeta_0,\zeta)| \leq Ch(\gamma|t|), \quad (x,t,\zeta_0,\zeta) \in U \times (-\delta,\delta) \times V.
\]

So in view of equation (4.5) we have
\[
\left\{ \begin{array}{l}
|L^u \Psi(x,t)| \leq Ch(\gamma|t|), \quad (x,t) \in U \times (-\delta,\delta), \\
\Psi(x,0) = u(x,0), \quad x \in U.
\end{array} \right.
\]

Thus we have constructed an \((\mathcal{M},t)\)-approximate solution of \(L^u\) with initial condition \(u_0\).

\[\square\]

**Theorem 4.4.** Let \(\Omega = \Omega' \times I \subset \mathbb{R}^N \times \mathbb{R}\) be an open neighborhood of the origin. Let \(v \in C^2(\Omega)\) be a solution of the nonlinear PDE:
\[
v_t = g(x,v,v_x),
\]
where \(g(x,\zeta_0,\zeta)\) is a function of class \(C^M\) with respect to \(x \in \Omega'\) and holomorphic with respect to \((\zeta_0,\zeta) \in \mathbb{C} \times \mathbb{C}^N\). Then:
\[
WF_{\mathcal{M}}(v_0)|_0 \subset \{(0,\xi) \in \Omega' \times \mathbb{R}^N : \text{Im } b(0) \cdot \xi \geq 0\},
\]
where \(v_0 \in C^2(\Omega')\) is given by \(v_0(x) = v(x,0), x \in \Omega'\), and \(b(x) = \nabla_x g(x,v_0(x),v_{0x}(x))\).

**Proof.** Just apply Lemma 4.2 with Theorem 4.3.

\[\square\]

Applying a technique of Hanges-Treves presented in [10], we have the regularity theorem as a consequence of Theorem 4.4:

**Theorem 4.5.** Let \(\Omega = \Omega' \times I \times \Omega'' \subset \mathbb{R}^N \times \mathbb{R} \times \mathbb{C}^N\), where \(\Omega' \times \mathbb{R}\) is an open neighborhood of the origin and \(\Omega''\) is an open set. Let \(u \in C^2(\Omega)\) be a solution of the nonlinear PDE:
\[
u_t = f(x,t,u,u_x),
\]
where \(f(x,t,\zeta_0,\zeta)\) is a function of class \(C^M\) with respect to \((x,t) \in \Omega\) and holomorphic with respect to \((\zeta_0,\zeta) \in \mathbb{C} \times \mathbb{C}^N\). Then:
\[
WF_{\mathcal{M}}(u) \subset \text{Char}(L^u),
\]
where \(L^u\) is the linearized operator:
\[
L^u = \frac{\partial}{\partial t} - \sum_{j=1}^N \frac{\partial f}{\partial \zeta_j}(x,t,u,u_x) \frac{\partial}{\partial x_j}.
\]

For the convenience of the reader we present Hanges-Treves’ argument. We shall prove:
\[
WF_{\mathcal{M}}(u)|_0 \subset \text{Char}(L^u)|_0.
\]

The direction \((0;\xi,\tau) \in \Omega \times (\mathbb{R}^N \times \mathbb{R})\) belongs to \(\text{Char}(L^u)\) if and only if:
\[
\left\{ \begin{array}{l}
\tau = -\text{Re } a(0) \cdot \xi, \\
0 = \text{Im } a(0) \cdot \tau,
\end{array} \right.
\]
where \(a(x,t) = \nabla_x f(x,t,u(x,t),u_x(x,t))\). For each \(\theta \in [0,2\pi)\) one can see that \(v(x,t,s) = u(x,t)\) is a \(C^2\)-solution of the following nonlinear PDE:
\[
v_s = f^\theta(x,t,v,v_x,v_t),
\]
where \(f^\theta(x,t,\zeta_0,\zeta,\zeta_{N+1}) = e^{-i\theta}(\zeta_{N+1} - f(x,t,\zeta_0,\zeta))\) and we are setting the coordinates in \(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}\) as \((x,t,s) = (x_1,\ldots,x_N,t,s)\) and the coordinates in \(\mathbb{C} \times \mathbb{C}^N \times \mathbb{C}\) as \((\zeta_0,\zeta,\zeta_{N+1}) = (\zeta_0,\zeta_1,\ldots,\zeta_N,\zeta_{N+1})\).

The corresponding linearized operator is:
\[
L^\theta = \frac{\partial}{\partial s} - e^{-i\theta}L^u.
\]
The direction \((0; \xi, \tau, \sigma) \in (\Omega \times \mathbb{R}) \times (\mathbb{R}^N \times \mathbb{R} \times \mathbb{R})\) belongs to \(\text{Char}(L^\theta)\) if and only if:

\[
\begin{align*}
\sigma &= \left[(\cos \theta) \text{Re} a(0) - (\sin \theta) \text{Im} a(0)\right] \cdot \xi + (\cos \theta) \tau, \\
0 &= \left[(\cos \theta) \text{Im} a(0) + (\sin \theta) \text{Re} a(0)\right] \cdot \xi + (\sin \theta) \tau.
\end{align*}
\]  

(4.16)

One can notice that the validity of (4.14) is equivalent to the validity of the second equation on (4.16) for every \(\theta \in [0, 2\pi)\). Now, let \((0; \xi_0, \tau_0) \notin \text{Char}(L^u)\). There exists \(\theta \in [0, 2\pi)\) such that \(\left[(\cos \theta) \text{Im} a(p) + (\sin \theta) \text{Re} a(p)\right] \cdot \xi_0 + (\sin \theta) \tau_0 < 0\). By choosing among \(\theta, \theta + \pi\) and \(\theta - \pi\), one can suppose \(\left[(\cos \theta) \text{Im} a(p) + (\sin \theta) \text{Re} a(p)\right] \cdot \xi_0 + (\sin \theta) \tau_0 < 0\). Applying Theorem 4.4 to the solution \(v(x, t, s) = u(x, t)\) of (4.15) we conclude \((0; \xi_0, \tau_0) \notin \text{WF}_{\mathcal{M}}(v_0)\) if \(\text{WF}_{\mathcal{M}}(u)\).

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