Superintegrability on $sl(2)$-coalgebra spaces

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We review a recently introduced set of $N$-dimensional quasi-maximally superintegrable Hamiltonian systems describing geodesic motions, that can be used to generate “dynamically” a large family of curved spaces. From an algebraic viewpoint, such spaces are obtained through kinetic energy Hamiltonians defined on either the $sl(2)$ Poisson coalgebra or a quantum deformation of it. Certain potentials on these spaces and endowed with the same underlying coalgebra symmetry have been also introduced in such a way that the superintegrability properties of the full system are preserved. Several new $N = 2$ examples of this construction are explicitly given, and specific Hamiltonians leading to spaces of non-constant curvature are emphasized.

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I. INTRODUCTION

Two infinite families of $N$-dimensional (ND) quasi-maximally superintegrable Hamiltonians endowed with a set of $(2N-3)$ integrals of the motion have been recently introduced in [1–5]. In the first family, the superintegrability properties of all these Hamiltonians are shown to be a consequence of a hidden $sl(2)$ Poisson coalgebra symmetry [1]. The second family is just a $q$-deformation of the former (see [2–5] and references therein), and the deformed coalgebra symmetry is given by $sl_q(2) (q = e^z)$, the Poisson analogue of the non-standard quantum deformation of $sl(2)$ [6]. As a concrete application of these general results, some of these Hamiltonians can be shown to generate superintegrable geodesic motions on certain curved manifolds (see [1, 3, 5]). In this contribution, we briefly review this approach and we provide new 2D explicit examples of such $sl(2)$-coalgebra spaces.

II. $sl(2)$-COALGEBRA SPACES

We recall that an ND completely integrable Hamiltonian $H^{(N)}$ is called maximally superintegrable (MS) if there exists a set of $(2N-2)$ globally defined functionally independent constants of the motion that Poisson-commute with $H^{(N)}$. Among them, at least, two different subsets of $(N-1)$ constants in involution can be found. In the same way, a system will be called quasi-maximally superintegrable (QMS) if there are $(2N-3)$ independent integrals with the abovementioned properties.

Let us now consider the $sl(2)$ Poisson coalgebra generated by the following Lie–Poisson brackets and comultiplication map:

\[
\{ J_3, J_+ \} = 2J_+ , \quad \{ J_3, J_- \} = -2J_- , \quad \{ J_-, J_+ \} = 4J_3 ,
\]

\[
\Delta (J_l) = J_l \otimes 1 + 1 \otimes J_l , \quad l = +, - , 3 .
\]

(1)

The Casimir function for $sl(2)$ reads

\[
C = J_- J_+ - J_3^2 .
\]

(3)

The following result holds [1]:

**Theorem 1.** Let $\{ q, p \} = \{ (q_1, \ldots, q_N), (p_1, \ldots, p_N) \}$ be $N$ pairs of canonical variables. The ND Hamiltonian

\[
H^{(N)} = \mathcal{H} \left( J_-, J_+, J_3 \right) ,
\]

(4)

with $\mathcal{H}$ any smooth function and

\[
J_- = \sum_{i=1}^{N} q_i^2 \equiv q^2 , \quad J_+ = \sum_{i=1}^{N} \left( p_i^2 + \frac{b_i}{q_i^2} \right) \equiv p^2 + \sum_{i=1}^{N} \frac{b_i}{q_i^2} , \quad J_3 = \sum_{i=1}^{N} q_ip_i \equiv q \cdot p ,
\]

(5)
where \( b_i \) are arbitrary real parameters, is a QMS system. The \((2N - 3)\) functionally independent and “universal” integrals of the motion are explicitly given by

\[
C^{(m)} = \sum_{1 \leq i < j} \left\{ (q_i p_j - q_j p_i)^2 + \left( b_i \frac{q_i^2}{q_j^2} + b_j \frac{q_j^2}{q_i^2} \right) \right\} + \sum_{i=1}^{m} b_i,
\]

\[
C_{(m)} = \sum_{N-m+1 \leq i < j} \left\{ (q_i p_j - q_j p_i)^2 + \left( b_i \frac{q_i^2}{q_j^2} + b_j \frac{q_j^2}{q_i^2} \right) \right\} + \sum_{i=N-m+1}^{N} b_i,
\]

where \( m = 2, \ldots, N \) and \( C^{(N)} = C_{(N)} \). Moreover, the sets of \( N \) functions \( \{H^{(N)}, C^{(m)}\} \) and \( \{H^{(N)}, C_{(m)}\} \) \((m = 2, \ldots, N)\) are in involution.

The proof of this result is based on the fact that, for any choice of the function \( H \), the Hamiltonian \( H^{(N)} \) has an \( sl(2) \) Poisson coalgebra symmetry (see [7, 8]). Notice that for arbitrary \( N \) there is a single constant of the motion left in order to assure maximal superintegrability. In case that an additional integral does exist, the latter does not come from the coalgebra symmetry and has to be found directly. Finally, note that in the case \( N = 2 \) quasi-maximal superintegrability is just equivalent to integrability (the coalgebra symmetry provides only \((2N - 3) = 1\) integral of the motion). However, these coalgebra systems can, by construction, be generalized to arbitrary dimension.

Let us now give some explicit examples of the QMS spaces coming from this construction as particular geodesic motion Hamiltonians.

**A. \( sl(2) \)-coalgebra spaces with constant curvature**

Let us consider the symplectic realization (5) with \( b_i = 0, \forall i \). The kinetic energy \( T \) of a particle on the \( ND \) Euclidean space \( E^N \) can be directly interpreted as the generator \( J_+ \). Therefore, the \( E^N \) space can be thought of as the manifold with geodesic motions given by

\[
\mathcal{H} = T = \frac{1}{2} J_+ = \frac{1}{2} p^2.
\]

Moreover, the kinetic energy on \( ND \) Riemannian spaces with constant curvature \( \kappa \) can be expressed in Hamiltonian form as a function of the \( sl(2) \) generators in two different ways (see [1] for a detailed geometrical interpretation of this result):

\[
\mathcal{H}^P = T^P = \frac{1}{2} (1 + \kappa J_-)^2 J_+ = \frac{1}{2} (1 + \kappa q^2)^2 p^2,
\]

\[
\mathcal{H}^B = T^B = \frac{1}{2} (1 + \kappa J_- \cdot (J_+ + \kappa J_3^2) = \frac{1}{2}(1 + \kappa q^2) (p^2 + \kappa (q \cdot p)^2).
\]

The first one \( \mathcal{H}^P \) is just the kinetic energy for a free particle on the spherical \( S^N \) \((\kappa > 0)\) and hyperbolic \( H^N \) \((\kappa < 0)\) spaces in terms of Poincaré coordinates \( q \) (coming from a stereographic projection in \( \mathbb{R}^{N+1} \)) and their associated canonical momenta \( p \). The second one \( \mathcal{H}^B \) corresponds to Beltrami coordinates and momenta (central projection). In the framework...
here presented, both Hamiltonians can immediately be interpreted as deformations (in terms of the curvature parameter \(\kappa\)) of the flat Euclidean motion given by \(\kappa = 0\). By construction, and for any dimension, both geodesic motions are QMS ones since they Poisson-commute with the integrals (6).

**B. \(sl(2)\)-coalgebra spaces with non-constant curvature**

Note that, in principle, any homogeneous quadratic function of the canonical momenta can provide an admissible geodesic motion. In particular, we can consider the \(sl(2)\) Hamiltonian (with \(b_i = 0, \forall i\))

\[
H = T = \frac{1}{2} f(J_-) J_+ = \frac{1}{2} f(q^2) p^2, \tag{9}
\]

with \(f\) an arbitrary smooth function, and we can derive from it the corresponding kinetic energy Lagrangian. Hence the QMS geodesic motion is defined on a Riemannian manifold whose metric is given by

\[
ds^2 = \frac{2}{f(q^2)} dq^2. \tag{10}
\]

In the \(N = 2\) case we can easily compute the corresponding Gaussian curvature \(K\) of the space; namely

\[
K = \frac{1}{f(q^2)} \left\{ -q^2 f'(q^2) + f(q^2) \left[ f'(q^2) + q^2 f''(q^2) \right] \right\}, \tag{11}
\]

where \(f'\) and \(f''\) are the derivatives with respect to the variable \(q^2 = (q_1^2 + q_2^2)\). Therefore, we have obtained an infinite family of \(N = 2\) spaces with, in general, non-constant curvature depending on the “radial” coordinate \(q^2\) (see [9–11] for the study of 2D and 3D superintegrable systems on spaces with non-constant curvature). Obviously, the constant curvature spaces given in terms of Poincaré coordinates (8) are just particular cases of this construction with \(f(J_-) = (1 + \kappa J_-)^2/2\), that is, \(K = \kappa\).

Another remarkable \(sl(2)\)-coalgebra space, contained in (9), is obtained by setting

\[
H = \mathcal{T} = \frac{1}{\alpha + J_-} = \frac{1}{\alpha + \frac{p^2}{q^2}}, \tag{12}
\]

The corresponding space is just the so-called Darboux space of type III [9], whose \(N = 2\) non-constant Gaussian curvature reads

\[
K = -\frac{\alpha}{(\alpha + q_1^2 + q_2^2)^3}. \tag{13}
\]

A MS (intrinsic) Smorodinsky–Winternitz system [12, 13] on the ND generalization of this Darboux space has been recently obtained in [14].

We stress that the Hamiltonian (9) does not exhaust all the possibilities for free motion, since \(J^2_3 = (q \cdot p)^2\) is also a quadratic homogeneous function in the momenta. Therefore, we could consider more complicated kinetic energy terms including \(J^2_3\). A particular choice of
this type would be the one given by the Beltrami kinetic energy presented in (8), for which the metric of the space reads
\[ ds^2 = \frac{(1 + \kappa \mathbf{q}^2) d\mathbf{q}^2 - \kappa (\mathbf{q} \cdot d\mathbf{q})^2}{(1 + \kappa \mathbf{q}^2)^2}, \] (14)
and all its sectional curvatures are constant and equal to \( \kappa \). Another choice is given by the Hamiltonian
\[ \mathcal{H} = \frac{1}{2} J_+ + \alpha J_3^2, \] (15)
where \( \alpha \) is a real parameter. For \( N = 2 \) this gives rise to a space with a metric
\[ ds^2 = \frac{2}{1 + 2 \alpha (q_1^2 + q_2^2)} \left\{ (1 + 2 \alpha q_1^2) dq_1^2 + (1 + 2 \alpha q_2^2) dq_2^2 - 4 \alpha q_1 q_2 dq_1 dq_2 \right\}, \] (16)
also endowed with a constant Gaussian curvature \( K = -\alpha \). In contrast to the latter case, if one considers the free Hamiltonian
\[ \mathcal{H} = \frac{1}{2} J_+ + \alpha J_3^2 = \frac{1}{2} \mathbf{p}^2 + \alpha \mathbf{q}^2 (\mathbf{q} \cdot \mathbf{p})^2, \] (17)
one finds a space whose metric for \( N = 2 \) reads
\[ ds^2 = \frac{2(1 + 2 \alpha q_1^2 (q_1^2 + q_2^2))}{1 + 2 \alpha (q_1^2 + q_2^2)^2} dq_1^2 + \frac{2(1 + 2 \alpha q_2^2 (q_1^2 + q_2^2))}{1 + 2 \alpha (q_1^2 + q_2^2)^2} dq_2^2 - 8 \alpha \frac{q_1 q_2 (q_1^2 + q_2^2)}{1 + 2 \alpha (q_1^2 + q_2^2)^2} dq_1 dq_2, \] (18)
and whose nonconstant Gaussian curvature is found to be
\[ K = -2 \alpha (q_1^2 + q_2^2). \] (19)
In any case, we stress that the geometric interpretation of the canonical variables \((\mathbf{q}, \mathbf{p})\) can be completely different for each \( sl(2) \)-coalgebra space.

C. QMS potentials

The underlying coalgebra symmetry of this construction is also helpful in order to define QMS potentials \( \mathcal{V} \) on \( sl(2) \)-coalgebra spaces. This can be achieved, in general, by adding some suitable functions depending on \( J_- \) to the kinetic energy term and by considering arbitrary centrifugal terms that come from symplectic realizations of the \( J_+ \) generator with constants \( b_i \)'s that are different from zero. Such a Hamiltonian would be given by
\[ \mathcal{H} = \mathcal{T}(J_+, J_-, J_3) + \mathcal{V}(J_-). \] (20)
In the constant curvature case, the Hamiltonians that we would obtain in this way are the curved counterpart of the Euclidean systems for different values of the sectional curvature \( \kappa \), that lead to QMS potentials on the spaces \( S^N (\kappa > 0), H^N (\kappa < 0), \) and \( E^N (\kappa = 0) \). A detailed description for such potentials can be found in [1]. The same scheme can be applied to the non-constant curvature spaces given in Section II.B. In all the cases, the fact that the Hamiltonian (20) is defined on the “abstract” \( sl(2) \)-coalgebra generators ensures the existence of the set of \((2N - 3)\) “universal” integrals given in theorem 1, whatever the functions \( \mathcal{T} \) and \( \mathcal{V} \) be.
III. $\text{sl}_z(2)$-COALGEBRA SPACES

We recall that the non-standard $\text{sl}_z(2)$ Poisson coalgebra is given by the following deformed Poisson brackets and coproduct [8]:

$$\{J_3, J_+\} = 2J_+ \cosh zJ_-, \quad \{J_3, J_-\} = -2 \frac{\sinh zJ_-}{z}, \quad \{J_-, J_+\} = 4J_3, \quad (21)$$

$$\Delta_z (J_-) = J_- \otimes 1 + 1, \quad \Delta_z (J_l) = J_l \otimes e^{zJ_-} + e^{-zJ_-} \otimes J_l, \quad l = +, 3. \quad (22)$$

The Casimir function for $\text{sl}_z(2)$ reads

$$C_z = \frac{\sinh zJ_-}{z} J_+ - J_3^2. \quad (23)$$

The construction presented in the previous Section can be generalized to the case of this quantum deformation of $\text{sl}(2)$, and the associated spaces will be, in general, of non-constant curvature. Explicitly, we have the following general result [2, 8]:

**Theorem 2.** Let $\{q, p\} = \{(q_1, \ldots, q_N), (p_1, \ldots, p_N)\}$ be $N$ pairs of canonical variables. The $ND$ Hamiltonian

$$H_z^{(N)} = \mathcal{H}_z \left( J^{(N)}_-, J^{(N)}_+, J^{(N)}_3 \right), \quad (24)$$

where $\mathcal{H}_z$ is any smooth function and

$$J^{(N)}_- = \sum_{i=1}^{N} q_i^2 \equiv q^2, \quad J^{(N)}_3 = \sum_{i=1}^{N} \frac{\sinh zq_i^2}{zq_i^2} q_i p_i e^{zK^{(N)}_i(q^2)} \equiv (q \cdot p)_z,$$

$$J^{(N)}_+ = \sum_{i=1}^{N} \left( \frac{\sinh zq_i^2}{zq_i^2} p_i^2 + \frac{zb_i}{\sinh zq_i^2} \right) e^{zK^{(N)}_i(q^2)} \equiv \tilde{p}^2_z, \quad (25)$$

where

$$K^{(N)}_i(q^2) = - \sum_{k=1}^{i-1} q_k^2 + \sum_{l=i+1}^{N} q_l^2, \quad (26)$$

is QMS for any choice of the function $\mathcal{H}$ and for arbitrary real parameters $b_i$.

We remark that the explicit expressions for the $(2N - 3)$ functionally independent and “universal” integrals of the motion can be found in [2].

Let us explicitly write the 2-particle symplectic realization of $\text{sl}_z(2)$ (25):

$$J^{(2)}_- = q_1^2 + q_2^2, \quad J^{(2)}_3 = \frac{\sinh zq_1^2}{zq_1^2} e^{zq_2^2} q_1 p_1 + \frac{\sinh zq_2^2}{zq_2^2} e^{-zq_1^2} q_2 p_2,$$

$$J^{(2)}_+ = \frac{\sinh zq_1^2}{zq_1^2} e^{zq_2^2} p_1 + \frac{\sinh zq_2^2}{zq_2^2} e^{-zq_1^2} p_2 + \frac{zb_1}{\sinh zq_1^2} e^{zq_2^2} + \frac{zb_2}{\sinh zq_2^2} e^{-zq_1^2}. \quad (27)$$

In this case there is a single constant of the motion:

$$C_z^{(2)} = \frac{\sinh zJ^{(2)}_+}{z} J^{(2)}_+ - \left( J^{(2)}_3 \right)^2. \quad (28)$$
Again we are dealing with free motion, thus we will take the symplectic realization with \( b_1 = b_2 = 0 \) in order to avoid centrifugal terms. In a parallel way to (9), we can consider an infinite family of integrable (and quadratic in the momenta) free \( N = 2 \) motions with \( sl_z(2) \)-coalgebra symmetry through Hamiltonians of the type

\[
H_z^{(2)} = \frac{1}{2} J_+^{(2)} f(zJ_-^{(2)}),
\]

where \( f \) is an arbitrary smooth function such that \( \lim_{z \to 0} f(zJ^{(2)}_z) = 1 \), that is, \( \lim_{z \to 0} H_z^{(2)} = \frac{1}{2}(p_1^2 + p_2^2) \). We shall explore in the sequel some specific choices for \( f \), and we shall analyse the spaces generated by them.

A. An \( sl_z(2) \)-coalgebra space with non-constant curvature

Of course, the simplest choice will be just to set \( f \equiv 1 \) [3]:

\[
H^I_z = \frac{1}{2} J_+^{(2)} = \frac{1}{2} \left( \frac{\sinh zq_1^2}{zq_1^2} e^{zq_1^2} p_1^2 + \frac{\sinh zq_2^2}{zq_2^2} e^{-zq_2^2} p_2^2 \right).
\]

Hence the kinetic energy \( T^I_z(q_i, \dot{q}_i) \) coming from \( H^I_z \) is

\[
T^I_z(q_i, \dot{q}_i) = \frac{1}{2} \left( \frac{zq_1^2}{\sinh zq_1^2} e^{-zq_1^2} \dot{q}_1^2 + \frac{zq_2^2}{\sinh zq_2^2} e^{zq_2^2} \dot{q}_2^2 \right),
\]

and defines a geodesic flow on a 2D Riemannian space with signature \( \text{diag}(+,+) \) and metric given by:

\[
ds^2 = \frac{2zq_1^2}{\sinh zq_1^2} e^{-zq_1^2} dq_1^2 + \frac{2zq_2^2}{\sinh zq_2^2} e^{zq_2^2} dq_2^2.
\]

The Gaussian curvature \( K \) for this space turns out to be nonconstant and negative:

\[
K(q_1, q_2; z) = -z \sinh \left( z(q_1^2 + q_2^2) \right) = -z \sinh \left( z q^2 \right).
\]

Thus, the underlying 2D space is of hyperbolic type and endowed with a “radial” symmetry.

We consider the following transformation that includes a new parameter \( \lambda_2 \neq 0 \):

\[
cosh(\lambda_1 \rho) = \exp \left\{ z(q_1^2 + q_2^2) \right\}, \quad \sin^2(\lambda_2 \theta) = \frac{\exp \left\{ 2zq_1^2 \right\} - 1}{\exp \left\{ 2z(q_1^2 + q_2^2) \right\} - 1},
\]

where \( z = \lambda_1^2 \) and both \( \lambda_1, \lambda_2 \) can take either a real or a pure imaginary value. Note that the zero-deformation limit \( z \to 0 \) is in fact the flat contraction \( K \to 0 \). Under this limit

\[
\rho \to 2(q_1^2 + q_2^2), \quad \sin^2(\lambda_2 \theta) \to \frac{q_1^2}{q_1^2 + q_2^2}.
\]

Thus \( \rho \) can be interpreted as a radial coordinate and \( \theta \) is a either circular (\( \lambda_2 \) real) or hyperbolic angle (\( \lambda_2 \) imaginary). Notice that in the latter case, say \( \lambda_2 = i \), the coordinate
$q_1$ is imaginary and can be written as $q_1 = i\tilde{q}_1$ where $\tilde{q}_1$ is a real coordinate; then $\rho \to 2(q_2^2 - \tilde{q}_1^2)$ which corresponds to a relativistic radial distance. Therefore the introduction of the additional parameter $\lambda_2$ will allow us to obtain Lorentzian metrics.

In this new coordinates, the metric (32) reads

$$ds_I^2 = \frac{1}{\cosh(\lambda_1 \rho)} \left( d\rho^2 + \frac{\lambda_2^2}{\lambda_1^2} \frac{\sinh^2(\lambda_1 \rho)}{\cosh(\lambda_1 \rho)} d\theta^2 \right) = \frac{1}{\cosh(\lambda_1 \rho)} ds_0^2. \quad (36)$$

where $ds_0^2$ is just the metric of the 2D Cayley–Klein spaces in terms of geodesic polar coordinates [15, 16] provided that we identify $z = \lambda_1^2 \equiv -\kappa_1$ and $\lambda_2^2 \equiv \kappa_2$; hence $\lambda_2$ determines the signature of the metric. The Gaussian curvature turns out to be

$$K(\rho) = -\frac{1}{2} \frac{\lambda_1^2}{\lambda_2^2} \frac{\sinh^2(\lambda_1 \rho)}{\cosh(\lambda_1 \rho)}. \quad (37)$$

In this way we find the following spaces:

- When $\lambda_2$ is real, we get a 2D deformed sphere $S_z^2 (z < 0)$, and a deformed hyperbolic or Lobachewski space $H_z^2 (z > 0)$.

- When $\lambda_2$ is imaginary, we obtain a deformation of the (1+1)D anti-de Sitter spacetime $AdS_z^{1+1} (z < 0)$ and of the de Sitter one $dS_z^{1+1} (z > 0)$.

- In the nondeformed case $z \to 0$, we recover the Euclidean space $E_2$ ($\lambda_2$ real) and Minkowskian spacetime $M_1^{1+1}$ ($\lambda_2$ imaginary).

In the new variables the kinetic energy (31) is transformed into

$$T_I z (\rho, \theta; \dot{\rho}, \dot{\theta}) = \frac{1}{2 \cosh(\lambda_1 \rho)} \left( \dot{\rho}^2 + \frac{\lambda_2^2}{\lambda_1^2} \frac{\sinh^2(\lambda_1 \rho)}{\cosh(\lambda_1 \rho)} \dot{\theta}^2 \right), \quad (38)$$

and the free motion Hamiltonian (30) is written as

$$\tilde{H}_z^I = \frac{1}{2} \cosh(\lambda_1 \rho) \left( p_\rho^2 + \frac{\lambda_1^2}{\lambda_2^2} \frac{\sinh^2(\lambda_1 \rho)}{\cosh(\lambda_1 \rho)} p_\theta^2 \right), \quad (39)$$

where $\tilde{H}_z^I = 2\tilde{H}_z^I$. The unique constant of the motion $C_z^{(2)}$ (28) is simply given by

$$\tilde{C}_z = p_\theta^2, \quad (40)$$

provided that $\tilde{C}_z = 4\lambda_2^2 C_z^{(2)}$, and through the usual radial-symmetry reduction we find

$$\tilde{H}_z^I = \frac{1}{2} \cosh(\lambda_1 \rho) p_\rho^2 + \frac{\lambda_1^2 \cosh(\lambda_1 \rho)}{2\lambda_2^2 \sinh^2(\lambda_1 \rho)} \tilde{C}_z. \quad (41)$$
B. \( sl_z(2) \)-coalgebra spaces with constant curvature

If we consider the function \( f(zJ_z^{(2)}) = e^{zJ_z^{(2)}} \) we obtain a MS \( sl_z(2) \)-coalgebra Hamiltonian given by

\[
\mathcal{H}_z^{MS} = \frac{1}{2} J_z^{(2)} + e^{zJ_z^{(2)}} = \frac{1}{2} \left( \frac{\sinh zq_1^2}{zq_1^2} e^{-2zq_1^2} \dot{q}_1^2 + \frac{\sinh zq_2^2}{zq_2^2} e^{-2zq_2^2} \dot{q}_2^2 \right). \tag{42}
\]

Its maximal superintegrability comes from the existence of an additional (and functionally independent) constant of the motion given by \([8]\):

\[
\mathcal{I}_z = \frac{\sinh zq_1^2}{2zq_1^2} e^{zq_1^2} p_1^2. \tag{43}
\]

In this case the kinetic energy Lagrangian is given by

\[
\mathcal{T}_z^{MS}(q_i, \dot{q}_i) = \frac{1}{2} \left( \frac{zq_1^2}{\sinh zq_1^2} e^{-2zq_1^2} \dot{q}_1^2 + \frac{zq_2^2}{\sinh zq_2^2} e^{-2zq_2^2} \dot{q}_2^2 \right), \tag{44}
\]

whose associated metric reads

\[
ds_{MS}^2 = \frac{2zq_1^2}{\sinh zq_1^2} e^{-2zq_1^2} dq_1^2 + \frac{2zq_2^2}{\sinh zq_2^2} e^{-2zq_2^2} dq_2^2. \tag{45}
\]

Surprisingly enough, the computation of the Gaussian curvature \( K \) for \( ds_{MS}^2 \) gives that \( K = z \). Therefore, we are dealing with a space of constant curvature which is just the deformation parameter \( z \). In \([3]\) it was shown that a certain change of coordinates (of the type \( \lambda_2 \) and that includes the signature parameter \( \kappa_2 \)) transforms the metric \((45)\) into

\[
ds_{MS}^2 = dr^2 + \lambda_2^2 \frac{\sin^2(\lambda_1 r)}{\lambda_1^2} d\theta^2, \tag{46}
\]

which exactly coincides with the metric \( ds_{0}^2 \) of the Cayley–Klein spaces written in geodesic polar coordinates \((r, \theta)\) provided that now \( z = \lambda_1^2 \equiv \kappa_1 \) and \( \lambda_2^2 \equiv \kappa_2 \). Obviously, after this change the geodesic motion can be reduced to a “radial” 1D system:

\[
\tilde{H}_z^{MS} = \frac{1}{2} p_r^2 + \frac{\lambda_1^2}{2\lambda_2^2 \sin^2(\lambda_1 r)} \tilde{C}_z, \tag{47}
\]

where \( \tilde{H}_z^{MS} = 2\mathcal{H}_z^{MS} \) and \( \tilde{C}_z = p_\theta^2 \) is again the generalized momentum for the \( \theta \) coordinate.

It can also be checked that other choices for the Hamiltonian yield constant curvature spaces. In fact, let us consider the generic Hamiltonian \((29)\) depending on \( f \), whose 2D Gaussian curvature can be expressed in terms of the function \( f(x) \) as

\[
K(x) = z \left( f'(x) \cosh x + \left( f''(x) - f(x) - \frac{f'^2(x)}{f(x)} \right) \sinh x \right), \tag{48}
\]

where \( x \equiv zJ_z^{(2)} = z(q_1^2 + q_2^2) \). In general, we obtain \( sl_z(2) \)-coalgebra spaces with non-constant curvature. In order to characterize the constant curvature cases \([2]\), we define \( g := f'/f \) and we get the nonlinear differential equation

\[
K/z = f' \cosh x + \left( f'' - f - (f')^2/f \right) \sinh x = f \left( g \cosh x + (g' - 1) \sinh x \right). 
\]
If we now require $K$ to be a constant we obtain the equation

$$K' = 0 \equiv 2y \cosh x + y' \sinh x = 0,$$

where $y := 2g' + g^2 - 1$.

The solution for this equation yields

$$y = \frac{A}{\sinh^2 x},$$

where $A$ is a constant, and solving for $g$, we find for $F := f^{\frac{1}{2}}$ that

$$F'' = \frac{1}{4} \left( 1 + \frac{A}{\sinh^2 x} \right) F,$$

whose general solution is ($A := \lambda(\lambda - 1)$):

$$F = (\sinh x)^\lambda \left( C_1 (\sinh(x/2))^{(1-2\lambda)} + C_2 (\cosh(x/2))^{(1-2\lambda)} \right),$$

where $C_1$ and $C_2$ are two integration constants.

Now, if we impose that $\lim_{x \to 0} f = 1$ we obtain that only the cases with $A = 0$ are possible, that is, either $\lambda = 1$ or $\lambda = 0$. Hence the two elementary solutions are

$$\mathcal{H}_z = \frac{1}{2} J_z^{(2)} e^{\pm z J_z^{(2)}},$$

and the Gaussian curvature of their associated 2D spaces is $K = \pm z$.

C. Other $sl_z(2)$-coalgebra spaces

Many other possibilities for the definition of the free motion Hamiltonian in terms of the $sl_z(2)$-coalgebra generators are indeed possible. In general, such choices would lead to non-constant curvature spaces with QMS geodesic motions. For instance, we can consider the kinetic energy Hamiltonian given by

$$\mathcal{H} = \frac{1}{2} J_+ + \alpha J_3^2.$$

A straightforward calculation shows that, in the $N = 2$ case, the Gaussian curvature of the associated space is just

$$K = \frac{\alpha}{2} - \frac{3\alpha}{2} \cosh \left( 2 z (q_1^2 + q_2^2) \right) - z \sinh \left( z (q_1^2 + q_2^2) \right).$$

Obviously, the nondeformed $z \to 0$ limit of this Hamiltonian is just the one given by (15) and its limiting Gaussian curvature is $-\alpha$. Through this example we see again that the introduction of the quantum deformation leads to an “algebraic generation” of nonconstant curvature on the underlying space.
D. QMS potentials

As we have just commented, we can also consider more general ND QMS Hamiltonians based on \(sl_z(2)\) (25) by considering arbitrary \(b_i\)'s (contained in \(J_+\)) and by adding some functions depending on \(J_-\). In particular, we have considered the Hamiltonians (see [4] for the 2D construction):

\[
\mathcal{H}_z = \frac{1}{2} J_+ f(z J_-) + \mathcal{U}(z J_-),
\]

(52)

where the arbitrary smooth functions \(f\) and \(\mathcal{U}\) are such that

\[
\lim_{z \to 0} \mathcal{U}(z J_-) = \mathcal{V}(J_-), \quad \lim_{z \to 0} f(z J_-) = 1.
\]

(53)

This, in turn, means that

\[
\lim_{z \to 0} \mathcal{H}_z = \frac{1}{2} p^2 + \mathcal{V}(q^2) + \sum_{i=1}^{N} \frac{b_i}{2q_i^2},
\]

(54)

recovering the superposition of a central potential \(\mathcal{V}(J_-) \equiv \mathcal{V}(q^2)\) with \(N\) centrifugal terms on \(E^N\) [17].

To end with, we stress that within this construction the function \(f(z J_-)\) fixes the type of curved background, which is characterized by the metric \(ds^2/f(z q^2)\) (where \(ds^2\) is the non-constant curvature metric associated to \(\mathcal{H}_z = \frac{1}{2} J_+\)). Among this infinite family of spaces, the two special cases with \(f(z J_-) = e^{\pm z J_-}\) give rise to Riemannian spaces of constant sectional curvatures, all of them equal to \(\pm z\).

Particular choices of the potential function for different \(sl_z(2)\)-coalgebra spaces have been proposed and analysed from a geometrical viewpoint in [2, 4, 14]. Among them, the non-constant curvature analogues of the Smorodinsky–Winternitz and generalized Kepler–Coulomb potentials have been proposed, and many of the well-known results concerning both potentials for the constant curvature spaces [18–28] have been recovered from the coalgebra symmetry approach.

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