The maximum agreement subtree problem

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Abstract

In this paper we investigate an extremal problem on binary phylogenetic trees. Given two such trees $T_1$ and $T_2$, both with leaf-set $\{1, 2, \ldots, n\}$, we are interested in the size of the largest subset $S \subseteq \{1, 2, \ldots, n\}$ of leaves in a common subtree of $T_1$ and $T_2$. We show that any two binary phylogenetic trees have a common subtree on $\Omega(\sqrt{\log n})$ leaves, thus improving on the previously known bound of $\Omega(\log \log n)$ due to M. Steel and L. Szekely. To achieve this improved bound, we first consider two special cases of the problem: when one of the trees is balanced or a caterpillar, we show that the largest common subtree has $\Omega(\log n)$ leaves. We then handle the general case by proving and applying a Ramsey-type result: that every binary tree contains either a large balanced subtree or a large caterpillar. We also show that there are constants $c, \alpha > 0$ such that, when both trees are balanced, they have a common subtree on $cn^\alpha$ leaves. We conjecture that it is possible to take $\alpha = 1/2$ in the unrooted case, and both $c = 1$ and $\alpha = 1/2$ in the rooted case.

1 Preliminaries

All trees considered in this paper are binary. Although we mainly talk about rooted trees, we introduce the problem in terms of unrooted trees to be consistent with earlier papers on the subject.

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1.1 Unrooted phylogenetic trees

A phylogenetic tree is a binary, unrooted tree in which the leaves are labelled bijectively with elements from a finite set. All internal vertices of a phylogenetic tree have degree 3. For such a tree \( T \), the set of vertices is denoted by \( V(T) \), the set of edges by \( E(T) \), and the set of leaves by \( L(T) \).

In phylogenetics, it is common to consider isomorphism between trees in a more restricted sense. We say that trees \( T_1 \) and \( T_2 \) are isomorphic (and write \( T_1 \cong T_2 \)) if there is a bijection \( \varphi : V(T_1) \to V(T_2) \) such that

(i) \( \{\varphi(u), \varphi(v)\} \in E(T_2) \iff \{u, v\} \in E(T_1) \),

(ii) \( \varphi(i) = i \) for all leaves \( i \in L(T_1) \).

Observe that, while this notion of isomorphism also works for non-binary trees, there can be no isomorphism between trees that have distinct leaf-sets.

For a subset \( X \subseteq L(T) \), define the restriction of \( T \) to \( X \) to be the phylogenetic tree \( T|X \) with leaf-set \( L(T|X) = X \), and the property that there exists an isomorphism (in the sense of the previous paragraph) from a subdivision of \( T|X \) to the unique minimal connected subgraph of \( T \) containing \( X \). We loosely call the tree \( T|X \) a subtree of \( T \). Given trees \( T_1 \) and \( T_2 \), if \( X \) is a subset of \( L(T_1) \cap L(T_2) \) of maximum cardinality with the property that \( T_1|X \cong T_2|X \), we say that \( T_1|X \) (or \( T_2|X \)) is a maximum agreement subtree of \( T_1 \) and \( T_2 \). We also define the parameter \( \text{mast}\{T_1, T_2\} \) as

\[
\text{mast}\{T_1, T_2\} := \max \{|X| : X \subseteq L(T_1) \cap L(T_2), T_1|X \cong T_2|X\}.
\]

Figure 1 illustrates trees \( T_1 \) and \( T_2 \) and one of their maximum agreement subtrees.

![Figure 1](image)

Now let

\[
\text{mast}(n) := \min \{\text{mast}\{T_1, T_2\} : L(T_1) = L(T_2) = \{1, 2, \ldots, n\}\}.
\]

It was shown by [Kubicka, Kubicki, and McMorris] in [1] that

\[
c_1 (\log \log n)^{1/2} \leq \text{mast}(n) \leq c_2 \log n,
\]

(1)

1Logarithms in this paper are always taken with base 2.
for some positive constants $c_1$ and $c_2$. To see that the upper bound in (1) is tight, consider the case when $T_1$ is a caterpillar with $n$ leaves and $T_2$ is a balanced tree of height $\log n$ (to be defined in Section 2): any common subtree must be a caterpillar, and there is no caterpillar of length more than $2 \log n$ in $T_2$. The lower bound in (1) was improved by Steel and Székely [2] who showed that $\text{mast}(n) \geq c \log \log n$ for a positive constant $c$. In fact, in a remark following Theorem 1 in their paper, they mention that a more explicit bound of $\frac{1}{3} \log \log(n - 1)$ may be derived, and suggest that a much stronger lower bound of $c \log n$ might hold, for some positive constant $c$.

Problem 1. Is there a constant $c > 0$ such that any two phylogenetic trees $T_1$ and $T_2$ with $L(T_1) = L(T_2) = \{1, 2, \ldots, n\}$ have a maximum agreement subtree on at least $c \log n$ leaves?

One of the goals of this paper is to further improve the lower bound in (1). In particular, in Theorem 14 we show that any two phylogenetic trees $T_1$ and $T_2$ with leaf-set $\{1, 2, \ldots, n\}$ have an agreement subtree on $\Omega(\sqrt{\log n})$ leaves.

1.2 Rooted phylogenetic trees

Next, we develop some terminology for rooted phylogenetic trees. In a rooted phylogenetic tree $T$, for $|L(T)| > 1$, all internal nodes have degree 3 except the root, which has degree 2. If $|L(T)| = 1$, then $T$ has exactly one vertex which is both its only leaf and root. Let us denote the root of a tree $T$ by $\rho(T)$. For a vertex $u \in V(T)$, we denote by $T^u$ the subtree of $T$ rooted at $u$ containing all descendants of $u$ in $T$. We denote by $\ell(u)$ and $r(u)$, respectively, the left and the right children of an internal vertex $u$ of a rooted tree.

For a rooted tree $T$ and vertices $x$ and $y$ in $V(T)$, we define $x \land y$ to be the most recent common ancestor of $x$ and $y$. Since $\land$ is associative and commutative, we may define the most recent common ancestor of a set $X = \{x_1, x_2, \ldots, x_k\} \subseteq V(T)$ to be

$$\land X := x_1 \land x_2 \land \cdots \land x_k.$$  

Given rooted trees $S$ and $T$, we say that $S$ is a subtree of $T$ (and write $S \preceq T$) if there exists an injective map $f : V(S) \to V(T)$ satisfying

1. $f(x) = x$ for all $x \in L(S)$

2. $f(x \land y) = f(x) \land f(y)$ for all $x, y \in V(S)$

We say that rooted phylogenetic trees $T_1$ and $T_2$ are isomorphic if $T_1 \preceq T_2$ and $T_2 \preceq T_1$.

Now we define the restriction of $T$ to the set of leaves $X$ as the unique binary, rooted phylogenetic tree $T|X$ having leaf-set $X$ and satisfying $T|X \preceq T$. The rooted maximum agreement subtree and mast$\{\cdot\}$ are defined as in the unrooted case.

Proposition 2 allows us to recursively construct agreement subtrees (but not necessarily maximum agreement subtrees) of rooted trees.
Proposition 2. Let $T_1$ and $T_2$ be rooted, binary trees, with roots $u := \rho(T_1)$ and $v := \rho(T_2)$, respectively. If $S_\ell \preceq T_1(\ell(u))$, $T_2(\ell(v))$ and $S_r \preceq T_1(r(u))$, $T_2(r(v))$, then the tree $S$ with left and right subtrees $S_\ell$ and $S_r$, respectively, is such that $S \preceq T_1$, $T_2$.

Proof. Let $f_\ell$ and $f_r$ be maps that realize $S_\ell \preceq T_1(\ell(u))$ and $S_r \preceq T_1(r(u))$, respectively. Then $S \preceq T_1$ is realized by defining a map $f : V(S) \to V(T_1)$ so that

\[
f(x) := \begin{cases} f_\ell(x) & \text{if } x \in V(S_\ell) \\ f_r(x) & \text{if } x \in V(S_r) \\ \rho(T_1) & \text{if } x = \rho(S) \end{cases}
\]

A similarly constructed map shows that $S \preceq T_2$. \hfill \square

We denote the tree $S$ constructed from $S_\ell$ and $S_r$ as in the above lemma by $S_\ell \circ S_r$.

To prove the results in this paper, we first obtain agreement subtrees of rooted trees (constructed by rooting the given unrooted trees suitably), and then agreement subtrees of unrooted trees by ignoring the roots.

2 When one of the trees is balanced

A rooted, phylogenetic tree is balanced if all leaves are at the same distance from the root. For unrooted trees, the definition is analogous. We first define the center of a graph $G$ as the set of vertices $u \in V(G)$ for which

\[
\max\{d_G(u, v) : v \in V(G)\}
\]

is minimum, where $d_G(u, v)$ is the length of a shortest path from $u$ to $v$ in $G$. The center of a tree contains either a single vertex or two adjacent vertices. For example, in Figure 1, $T_1$ has center $\{u\}$, while $T_2$ has center $\{x, y\}$. We say that a phylogenetic tree is balanced if all leaves are at the same distance from the center of the tree. For example, the tree $T_1$ in Figure 1 is balanced. Note that a rooted balanced tree has $2^m$ leaves for some $m \geq 0$, while an unrooted balanced tree has either $2^m$ or $3 \cdot 2^m$ leaves for some $m \geq 0$. We refer to Section 2.2 for more explanation.

In this section, we solve Problem 1 when one of the trees is balanced and binary.

2.1 The rooted case

We first consider the case of rooted trees, thus assuming that one tree is a rooted, balanced tree of height $m$ and another is a general rooted, binary tree. In Theorem 8 we prove a result for the unrooted case by appropriately rooting the trees and applying Lemma 3.

In what follows, for nodes $x \in V(T_1)$ and $y \in V(T_2)$, let $t_{xy}$ be the number of elements in the set $L(T_1^x) \cap L(T_2^y)$. 

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Lemma 3. Suppose $T_1$ is a rooted, balanced, binary tree on a leaf-set of cardinality $2^m$, and $T_2$ is an arbitrary rooted, binary tree on $t > 0$ leaves with $L(T_2) \subseteq L(T_1)$. Then for all $\delta \in (0, 1/2)$, the two trees have a maximum agreement subtree that has at least

$$\frac{m \log(1 - \delta) + \log t}{1 - \log \delta}$$

leaves.

Proof. Let $g(m, t)$ be the minimum value of $\max \{T_1, T_2 \}$ (over all choices of $T_1$ and $T_2$), where $T_1$ and $T_2$ are as in the statement of the lemma. Observe that $g(m, t)$ is a monotonically non-decreasing function of $t$. We show by induction on $m$ that

$$g(m, t) \geq \frac{m \log(1 - \delta) + \log t}{1 - \log \delta}. \quad (2)$$

Base case: If $m = 0$, then $g(m, t) = 1$ and the right-hand-side of (2) is 0. So we may assume that $m > 0$.

Induction step: Let $u := \rho(T_1)$ and $v := \rho(T_2)$. Observe that

$$t = t_{uv} = t_{\ell(u)\ell(v)} + t_{r(u)r(v)} + t_{\ell(u)r(v)} + t_{r(u)\ell(v)}. \quad (3)$$

Without loss of generality, assume that $t_{\ell(u)\ell(v)} + t_{r(u)r(v)} \geq t_{\ell(u)r(v)} + t_{r(u)\ell(v)}$, and $t_{r(u)r(v)} \geq t_{\ell(u)\ell(v)}$. Therefore, by (3), we have

$$t_{r(u)r(v)} \geq \lceil t/4 \rceil. \quad (4)$$

Case 1: $t_{\ell(u)\ell(v)} > 0$.

In this case, we take a maximum agreement subtree $S_\ell$ of $T_1^{(\ell(u))}$ and $T_2^{(\ell(v))}$, and a maximum agreement subtree $S_r$ of $T_1^{(r(u))}$ and $T_2^{(r(v))}$. We then construct $S = S_\ell \circ S_r$, which by Proposition 2 is an agreement subtree of $T_1$ and $T_2$. Now

$$g(m, t) \geq |L(S)| = |L(S_\ell)| + |L(S_r)| \geq 1 + g(m - 1, t_{r(u)r(v)}).$$

By (4), we have

$$g(m, t) \geq 1 + g(m - 1, \lceil t/4 \rceil).$$

Therefore, applying the induction hypothesis, we obtain

$$g(m, t) \geq 1 + \frac{(m - 1) \log(1 - \delta) + \log(t/4)}{1 - \log \delta}$$

$$= \frac{m \log(1 - \delta) + \log t}{1 - \log \delta} + \frac{-1 - \log(1 - \delta) - \log \delta}{1 - \log \delta}$$

$$> \frac{m \log(1 - \delta) + \log t}{1 - \log \delta}.$$
Case 2: \( t_{ℓ(u)ℓ(v)} = 0 \).

From now on, we assume that \( t_{r(u)ℓ(v)} \) is non-zero; otherwise, together with the assumption of this case, it would contradict the hypothesis that \( L(T_2) \subseteq L(T_1) \).

Subcase 2.1: \( t_{ℓ(u)r(v)} + t_{r(u)ℓ(v)} \geq δt \) and \( t_{ℓ(u)r(v)} > 0 \). Also, without loss of generality, assume that \( t_{r(u)ℓ(v)} \geq t_{ℓ(u)r(v)} \), so that \( t_{r(u)ℓ(v)} \geq δt/2 \).

In this case, we construct \( S := S_ℓ \circ S_r \), where \( S_ℓ \) is a rooted maximum agreement subtree of \( T_1^{(ℓ(u))} \) and \( T_2^{(r(v))} \), and \( S_r \) is a rooted maximum agreement subtree of \( T_1^{(r(u))} \) and \( T_2^{(ℓ(v))} \). Therefore,

\[
g(m, t) \geq |L(S_ℓ)| + |L(S_r)| \\
\geq 1 + g(m - 1, [δt/2]) \\
\geq 1 + \frac{(m - 1) \log (1 - δ) + \log (δt/2)}{1 - \log δ} \\
\geq \frac{m \log (1 - δ) + \log t}{1 - \log δ} + \frac{- \log (1 - δ)}{1 - \log δ} \\
> \frac{m \log (1 - δ) + \log t}{1 - \log δ}.
\]

Subcase 2.2: \( t_{ℓ(u)r(v)} + t_{r(u)ℓ(v)} < δt \). In this case, \( t_{r(u)r(v)} = t - t_{ℓ(u)ℓ(v)} - t_{ℓ(u)r(v)} - t_{r(u)ℓ(v)} > (1 - δ)t \).

Let \( S \) be a rooted maximum agreement subtree of \( T_1^{(r(u))} \) and \( T_2^{(r(v))} \). We have

\[
g(m, t) \geq |L(S)| \\
= g(m - 1, [(1 - δ)t]) \\
\geq \frac{(m - 1) \log (1 - δ) + \log (1 - δ)t}{1 - \log δ} \\
= \frac{m \log (1 - δ) + \log t}{1 - \log δ}.
\]

Subcase 2.3: \( t_{ℓ(u)r(v)} = 0 \).

Let \( S \) be a rooted maximum agreement subtree of \( T_1^{(r(u))} \) and \( T_2^{(r(v))} \). Therefore,

\[
g(m, t) \geq |L(S)| \\
= g(m - 1, t) \\
\geq \frac{(m - 1) \log (1 - δ) + \log t}{1 - \log δ} \\
= \frac{m \log (1 - δ) + \log t}{1 - \log δ} + \frac{- \log (1 - δ)}{1 - \log δ} \\
> \frac{m \log (1 - δ) + \log t}{1 - \log δ}.
\]
With this we complete all subcases of the induction step. Therefore (2) holds and the lemma is proved. \qed

An immediate consequence of the above lemma is the following corollary.

**Corollary 4.** For $\delta \in (0, 1/2)$, set
\[
\alpha := \frac{1 + \log (1 - \delta)}{1 - \log \delta}.
\] (5)

If $T_1$ is a rooted, balanced, binary tree on a leaf-set of cardinality $2^m$, and $T_2$ is an arbitrary rooted, binary tree on $2^m$ leaves such that $L(T_2) = L(T_1)$, then $T_1$ and $T_2$ have a maximum agreement subtree on at least $\alpha m$ leaves.

**Remark 5.** We found numerically that the maximum value of $\alpha$ is approximately 0.2055 obtained when $\delta$ is approximately 0.1705.

**Remark 6.** An algorithm to construct an agreement tree (though not necessarily a maximum agreement subtree) is implicit in the proof of Lemma 3. Observe that in each of Case 1 and Subcase 2.1, we may take the agreement subtree $S_\ell$ to be a tree with a single leaf (e.g. any leaf from $L(T_1^{(l(u))}) \cap L(T_2^{(l(v))})$) in Case 1 and any leaf from $L(T_1^{(l(u))}) \cap L(T_2^{(r(v))})$ in Subcase 2.1). Such a choice gives us an agreement subtree that is a caterpillar of length $\alpha m$ in Corollary 4.

We explicitly describe a recursive algorithm, which we call $\text{Match1}$. The algorithm takes as input two rooted, binary trees, the first one being balanced, and returns a set of leaves in a common subtree that is a caterpillar. As in Lemma 3, algorithm $\text{Match1}$ also depends on a parameter $\delta \in (0, 1/2)$.

**Algorithm Match1($T_1^{(u)}$, $T_2^{(v)}$)**

1: if $|L(T_1^{(u)})| = 1$ or $|L(T_2^{(v)})| = 1$ then return $L(T_1^{(u)}) \cap L(T_2^{(v)})$.

2: if necessary, interchange left and right subtrees in $T_1^{(u)}$ and/or $T_2^{(v)}$ so that $t_{l(u)r(v)} + t_{r(u)l(v)} \leq t_{l(u)l(v)} + t_{r(u)r(v)}$ and $t_{l(u)l(v)} \leq t_{r(u)r(v)}$.

3: if $(t_{l(u)l(v)} > 0)$ then
   (a) select any leaf $z$ from $L(T_1^{(l(u))}) \cap L(T_2^{(l(v))})$,
   (b) return \{ $z$ \} $\cup \text{Match1}(T_1^{(r(u))}, T_2^{(r(v))})$.

4: if $t_{r(u)l(v)} = 0$ then return $\text{Match1}(T_1^{(u)}, T_2^{(v)})$.

5: if $t_{l(u)r(v)} = 0$ then return $\text{Match1}(T_1^{(r(u))}, T_2^{(v)})$.

6: if $(t_{l(u)r(v)} + t_{r(u)l(v)} \geq \delta t_{uv})$ then
   (a) if necessary, interchange the left and the right subtrees of both $T_1^{(u)}$ and $T_2^{(v)}$ so that $t_{l(u)r(v)} \leq t_{r(u)l(v)}$,
(b) select any leaf $z$ from $L(T_1^{(\ell(u))}) \cap L(T_1^{(r(v))})$,
(c) return $\{z\} \cup \text{Match1}(T_1^{(r(u))}, T_2^{(r(v))})$.

7: return $\text{Match1}(T_1^{(r(u))}, T_2^{(r(v))})$.

Steel and Warnow [3] devised an efficient polynomial time algorithm for finding the maximum agreement subtree of two given binary trees. Our algorithm is not optimal, but it is easy to analyze; moreover, it is easy to guarantee a lower bound on the returned value. We show how the value of $\alpha$ in Lemma 3 may also be obtained by analyzing the algorithm.

Alternative proof of Corollary 4. Set $x := \rho(T_1)$ and $y := \rho(T_2)$. We now analyze the execution of the a call to $\text{Match1}(T_x, T_y)$.

First observe that, each time an instance of algorithm $\text{Match1}$ is being executed, it calls itself recursively only once in that instance. That happens, say $k$ times, each time going deeper in the recursion levels, until a base case is reached in line 1. Thus, we may define two sequences of nodes $x = x_0, x_1, x_2, \ldots, x_k$ and $y = y_0, y_1, y_2, \ldots, y_k$ that correspond to the roots of the trees passed as arguments in each triggered call. That is, $\text{Match1}(T_1^{x_0}, T_2^{y_0})$ calls $\text{Match1}(T_1^{x_1}, T_2^{y_1})$, which in turn calls $\text{Match1}(T_1^{x_2}, T_2^{y_2})$, and so on. Note that the nodes in each sequence need not be distinct. For example, if $\text{Match1}$ is called from line 4, then $x_{i+1} = x_i$; and similarly, if $\text{Match1}$ is called from line 5 then $y_{i+1} = y_i$.

As a shorthand notation, define

$$t_i := t_{x_iy_i}.$$ 

Now suppose $\text{Match1}(T_1^n, T_2^n)$ is being called with $u = x_i$ and $v = y_i$. Using our notation, we have $t_i = t_{uv}$. For each possibility of calling $\text{Match1}$ recursively, we obtain a lower bound for $t_{i+1}$ in terms of $t_i$.

After executing line 2, we have $t_{r(u)r(v)} \geq t_{uv}/4$. Hence, if the recursive call in line 3b is triggered, we have

$$t_{i+1} \geq t_i/4.$$  \hspace{1cm} (6)

If the recursive call in line 4 is triggered, it is because $t_{\ell(u)\ell(v)} = 0$ and $t_{r(u)\ell(v)} = 0$, which implies $t_{ur(v)} = t_{uv}$. Hence, in that case, we must have

$$t_{i+1} = t_i.$$  \hspace{1cm} (7)

Similarly, if the recursive call in line 5 is triggered, we have $t_{r(u)v} = t_{uv}$, which also implies (7). The conditions in lines 6 and 6a imply that $t_{r(u)\ell(v)} \geq \delta t_{uv}/2$. Hence, if the recursive call in line 6c is made, we must have

$$t_{i+1} \geq \delta t_i/2.$$  \hspace{1cm} (8)
In line 7, since $t_{\ell(v)} + t_{r(v)} < \delta t_{uv}$ and $t_{\ell(v)} = 0$, we have $t_{r(v)} > (1 - \delta)t_{uv}$. Hence, if $\text{Match1}$ is called from line 7, then we have

$$t_{i+1} > (1 - \delta)t_i.$$ (9)

Now suppose that, during the entire execution of the recursive algorithm, line 3b is executed $a$ times, line 4 is executed $b$ times, line 5 is executed $c$ times, line 6c is executed $d$ times, and line 7 is executed $e$ times. Since a new leaf $z$ is returned each time one of the lines 3b or 6c is executed, the set returned by the outermost call $\text{Match1}(T_x^1, T_y^2)$ (when the execution halts) contains precisely $a + d + 1$ leaves. Therefore, it is enough to show the following.

**Claim 7.** $a + d > \alpha m$.

To prove the claim consider the sequence $t_0, t_1, \ldots, t_k$. For each $i$ in $\{0, 1, \ldots, k - 1\}$, we know that $t_{i+1}$ satisfies one of (6), (7), (8) or (9). Therefore, we have

$$t_k \geq t_0 \left( \frac{1}{4} \right)^a \left( \frac{\delta}{2} \right)^d (1 - \delta)^e.$$ (10)

Now observe that $t_0 = 2^m$. Also, since the $k$-th recursive call is a base case, we have $t_k = 1$. Moreover, because $\delta < 1/2$, we have $1/4^a > (\delta/2)^a$. Using these observations and the fact that $e \leq m$, equation (10) yields

$$1 > 2^m \left( \frac{\delta}{2} \right)^{a+d} (1 - \delta)^m.$$ 

Solving for $a + d$, we obtain

$$a + d > \left( \frac{1 + \log(1 - \delta)}{1 - \log \delta} \right)m = \alpha m.$$

The claim is proved, and the corollary follows. \hfill $\square$

### 2.2 The unrooted case

We now consider the case when one of the trees is unrooted and balanced. We define two classes of unrooted, balanced trees. The center of a tree may be either a single vertex or a pair of adjacent vertices. Let $m \geq 0$ be an integer. When all leaves of a tree are at distance $m$ from the center and the center is a single vertex, we say that the tree is in class $C_m$. When all leaves are at distance $m$ from the center and the center is a pair of adjacent vertices, we say the phylogenetic tree is in class $B_{m+1}$. By construction, trees in the class $C_m$ have $3 \times 2^{m-1}$ leaves, and trees in the class $B_m$ have $2^m$ leaves.

**Theorem 8.** If $T_1$ is a balanced phylogenetic tree on $n$ leaves, and $T_2$ is an arbitrary phylogenetic tree on the same leaf-set, then they have an agreement subtree on at least $\alpha \log \frac{2n}{3}$ leaves, where $\alpha$ is the constant defined in (5).
Proof. If $T_1$ is in class $B_m$, for some $m$, then $n = 2^m$. Let $\{x, y\}$ be its central edge. We add a new vertex $z$, and replace the edge $\{x, y\}$ by edges $\{x, z\}$ and $\{y, z\}$, and root the tree at $z$. For $T_2$, we add a new vertex $w$, replace an arbitrary edge $\{u, v\}$ by edges $\{u, w\}$ and $\{v, w\}$, and root $T_2$ at $w$. Notice that, for any rooted agreement subtree of $T_1^z$ and $T_2^w$, we may ignore the root and obtain an unrooted agreement subtree of $T_1$ and $T_2$. Applying Lemma 3 to $T_1^z$ and $T_2^w$ gives a lower bound of $\alpha m$ on the size of the maximum agreement subtree of $T_1$ and $T_2$. The desired bound follows.

If $T_1$ is in class $C_m$, for some $m$, then $n = 3 \times 2^{m-1}$. Let $z$ be the center of $T_1$. Let $X$ be the set of leaves in two of the three branches rooted at $z$. Note that $T_1|X$ is in class $B_m$ and $T_2|X$ is an arbitrary phylogenetic tree. Proceeding as in the above paragraph, we obtain a lower bound of $\alpha m$ on the size of the maximum agreement subtree of $T_1|X$ and $T_2|X$. Hence, $T_1$ and $T_2$ have an agreement subtree on at least $\alpha \log \frac{n}{3}$ leaves.

The following proposition for the case when one of the trees is “almost balanced” is proved with little extra effort.

**Proposition 9.** For every $k > 0$, there is a constant $\alpha_k > 0$ such that, if $T_1$ and $T_2$ are binary trees on the same leaf-set of cardinality $n$, and $T_1$ has radius at most $k \log n - 1$, then they have a maximum agreement subtree on at least $\alpha_k \log n$ leaves.

**Proof.** In tree $T_1$, we subdivide the central edge (if it has a central edge) or an edge adjacent to the center (if its center is a single vertex), and root the tree at the newly inserted vertex of degree 2. (We have bounded the radius of $T_1$ by $k \log n - 1$ and not $k \log n$ only to allow the possibility that when we root $T_1$, its radius may increase by 1.) We root $T_2$ by subdividing an arbitrarily chosen edge. We then construct a rooted, balanced, binary tree $T'_1$ of height $k \log n$ that contains $T_1$ as a subtree (in the sense that $T_1 \preceq T'_1$). Now by Lemma 3 we assert that $T'_1$ and $T_2$ (hence also $T_1$ and $T_2$) have an agreement subtree on at least

\[
\frac{k \log n \log(1 - \delta) + \log n}{1 - \log \delta} = \left(1 + k \log(1 - \delta)\right) \log n
\]

leaves. We select $\delta$ sufficiently small to satisfy $1 + k \log(1 - \delta) > 0$, and set

\[
\alpha_k := \frac{1 + k \log(1 - \delta)}{1 - \log \delta}.
\]

Then there is an agreement subtree on at least $\alpha_k \log n$ leaves. Indeed, the above value of $\alpha_k$ may also be obtained by (re)analyzing algorithm Match 1 as in the alternative proof of Corollary 4. \qed

### 3 General binary trees

Our approach to general binary trees is based on the following intuition: every binary tree has large diameter or contains (as a restriction) a balanced subtree of large height.
For $0 \leq k \leq h$, let $f(h, k)$ be the maximum number of leaves a rooted tree of height at most $h$ can have so that no restriction of the tree is a balanced, binary tree of height more than $k$.

**Lemma 10.** If $h = k$ or $k = 0$, then $f(h, k) = 2^k$. If $0 < k < h$, then

$$f(h, k) = \sum_{i=0}^{k} \binom{h - i - 1}{k - i} 2^i.$$  \hspace{1cm} (11)

**Proof.** We claim the following recurrence for $f(h, k)$:

$$f(h, k) = \begin{cases} 2^k & \text{if } h = k \text{ or } k = 0, \\ f(h - 1, k) + f(h - 1, k - 1) & \text{if } 0 < k < h. \end{cases}$$  \hspace{1cm} (11)

This is proved as follows. If $h = k$ or $k = 0$, then we have $f(h, k) = 2^k$, the extremal tree being the rooted, balanced tree of height $k$.

Now suppose that $h > k > 0$. We first prove that $f(h, k) \leq f(h - 1, k) + f(h - 1, k - 1)$. Let $T$ be a binary tree of height at most $h$ with more than $f(h - 1, k) + f(h - 1, k - 1)$ leaves. Suppose $T$ has $x$ leaves in the left subtree and $y$ leaves in the right subtree. Without loss of generality assume $y \leq x$. If $x > f(h - 1, k)$, then the left subtree of $T$ would have a restriction to a balanced, binary tree of height $k + 1$. Therefore, we may assume that $x \leq f(h - 1, k)$, which implies $f(h - 1, k - 1) < y \leq x$. It follows that both the left and the right subtrees have restrictions to balanced trees of height $k$, and that $T$ has a restriction to a balanced tree of height $k + 1$, which is a contradiction.

Next we show that $f(h, k) \geq f(h - 1, k) + f(h - 1, k - 1)$. Consider the tree $T(h, k)$ defined as follows: if $h = k$ or $k = 0$, then $T(h, k)$ is a balanced, binary tree of height $k$; otherwise, its left subtree is an extremal tree for parameters $h - 1$ and $k$, and its right subtree is an extremal tree for parameters $h - 1$ and $k - 1$. Thus $T(h, k)$ has precisely $f(h - 1, k) + f(h - 1, k - 1)$ leaves, and does not contain a restriction that is a balanced tree of height more than $k$.

Thus we have $f(h, k) = f(h - 1, k) + f(h - 1, k - 1)$ for $0 < k < h$, and the tree $T(h, k)$ constructed above is an example of an extremal tree for parameters $h$ and $k$. In fact the above arguments, together with induction on $h + k$, show that $T(h, k)$ is the unique such tree. We skip the details.

Now the solution to the recurrence relation is obtained by expanding it until all terms are expressed as $f(i, i) = 2^i$ for some $i > 0$ or $f(j, 0) = 1$ for some $j > 0$. 

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Figure 2: An illustration for the recurrence in (11)

In Figure 2 above, each directed path from the point \((h, k)\) to \((i, i)\) contributes the term \(f(i, i) = 2^i\), and there are \(\binom{h-i-1}{k-i}\) such paths; similarly every path from \((h, k)\) to \((j, 0)\) contributes \(f(j, 0) = 1\), and there are \(\binom{k+j-1}{j}\) such paths. Hence, for \(0 < k < h\), we have

\[
f(h, k) = \sum_{i=1}^{k} \binom{h-i-1}{k-i} 2^i + \sum_{j=0}^{h-k-1} \binom{k+j-1}{j}
\]

Since the second sum is \(\binom{h-1}{k}\), the desired bound follows. \(\square\)

**Corollary 11.** For \(1 \leq k \leq h\), we have \(f(h, k) \leq (2h)^k\).

**Proof.** We apply Lemma 10. If \(k = 1\), then \(f(h, k) = h + 1 \leq (2h)^1\). If \(1 < k < h\), then we have

\[
f(h, k) = \sum_{i=0}^{k} \binom{h-i-1}{k-i} 2^i < \binom{h}{k} \sum_{i=0}^{k} 2^i < (2h)^k.
\]

If \(k = h\), then \(f(h, k) = 2^k \leq (2h)^k\). \(\square\)

Define \(\phi(n, a) = \frac{(\log n)^a}{2}\) and \(\psi(n, b) = \frac{(\log n)^b}{\log \log n}\).

**Corollary 12.** Given any \(a, b \in (0, 1)\) such that \(a + b = 1\), every tree with \(n > 2\) leaves contains either a path of length at least \((\log n)^{\psi(n,b)}\) or a balanced subtree of height at least \(\phi(n, a)\).

**Proof.** Let \(a, b \in (0, 1)\) such that \(a + b = 1\). Let \(k \leq \phi(n, a)\) and \(h \leq (\log n)^{\psi(n,b)}\). We have

\[
k \log(2h) = k + k \log h \leq \phi(n, a) + \phi(n, a)\psi(n, b) \log \log n = \frac{(\log n)^a}{2} + \frac{\log n}{2} < \log n.
\]

Hence, by Corollary 11, we conclude that \(f(h, k) < n\). Now the corollary follows. \(\square\)
Proposition 13. If $T_1$ and $T_2$ are binary trees on the same leaf-set of cardinality $n$, and $T_1$ is a caterpillar, then they have a maximum agreement subtree on at least $\frac{1}{3} \log n$ leaves.

Proof. The proof of this fact goes along the lines of the proof of Theorem 1 in Steel and Székely [2]. We sketch it here. We embed $T_1$ and $T_2$ in the plane so that the leaves of $T_1$ are on one side of the longest path in $T_1$. Without loss of generality, suppose that the leaves of $T_1$ appear in the order $1, 2, \ldots, n$. The embedding of $T_2$ imposes a circular order on its leaves. We cut this circular order arbitrarily to get a linear order $i_1, i_2, \ldots, i_n$. Next we find the longest monotone subsequence of $i_1, i_2, \ldots, i_n$; it has length at least $\sqrt{n}$ by the Erdős-Szekeres Theorem [4]. Let $X$ be the set of leaves in this subsequence. We restrict $T_1$ and $T_2$ to $X$ obtaining $T_1|_X$ and $T_2|_X$. Notice that $T_1|_X$ is still a caterpillar. We further restrict both trees to $Y \subseteq X$ so that $T_2|_Y$ is a caterpillar with a maximum number of leaves. Thus $|Y|$ is at least $\log n$ (the extremal case being when $T_2|X$ is balanced). Now both $T_1|Y$ and $T_2|Y$ are caterpillars (see Figure 3 below). Now let $y_1, y_2, \ldots, y_k$ be the elements of $Y$ in the order they appear in the embedding of $T_1$.

In Figure 3, we can see that there are three maximal agreement caterpillars, namely, caterpillars with leaf-sets $\{y_1, y_2, \ldots, y_i\}$, $\{y_i, y_{i+1}, \ldots, y_j\}$ and $\{y_j, y_{j+1}, \ldots, y_k\}$. One of them must have length at least $(k + 2)/3 \geq \frac{1}{3} \log n$.

Theorem 14. If $T_1$ and $T_2$ are binary trees on the same leaf-set of cardinality $n > 2$, then they have a maximum agreement subtree having at least $\frac{a}{2} \sqrt{\log n} + \alpha \log \frac{2}{3}$ leaves.

Proof. Applying Corollary 12 (with $a = b = 1/2$), one of the trees must contain a balanced subtree of height at least $\phi(n, 1/2)$ or a path of length at least $(\log n)^{\psi(n, 1/2)}$. Suppose that one of the trees contains a balanced subtree of height at least $\phi(n, 1/2)$. Let $A$ be the leaf-set of such a balanced subtree. Therefore, after restricting the other tree to $A$, we can claim by Theorem 8 that $T_1$ and $T_2$ have a common subtree on at least $a\phi(n, 1/2) + \alpha \log \frac{2}{3}$ leaves from $A$.

If such a balanced subtree does not exist in either of the two trees, then there is a path (and hence a caterpillar) of length at least $(\log n)^{\psi(n, 1/2)}$ in one of the trees. We restrict
both trees to the set of leaves in this caterpillar. Therefore, by Proposition 13, there must be a common subtree on at least $\frac{1}{3}\psi(n, 1/2) \log \log n = \frac{1}{3}\sqrt{\log n}$ leaves. Taking the maximum value of $\alpha$ as in Remark 5, this is a quantity larger than the desired bound. □

4 When both trees are balanced

We now investigate the size of a maximum agreement subtree of two balanced, binary trees. In this case, in Theorem 18 we obtain a much better bound than that of Theorem 8.

Lemma 15. Suppose $T_1$ and $T_2$ are rooted, balanced, binary trees of height $m_1$ and $m_2$, respectively. Suppose that $|L(T_1) \cap L(T_2)| = t > 0$. Then for all $\delta \in (0, \frac{1}{4})$, the two trees have a rooted maximum agreement subtree on at least $2^{g(m_1, m_2, t)}$ leaves, where

$$g(m_1, m_2, t) := \frac{(m_1 + m_2) \log(1 - 3\delta) + \log t}{\log (1 - 3\delta) - \log \delta}.$$ 

Proof. Let $M(m_1, m_2, t)$ be the minimum value of $\text{mast}\{T_1, T_2\}$ (over all choices of $T_1$ and $T_2$), where $T_1$ and $T_2$ are as in the statement of the lemma. Observe that $M(m_1, m_2, t)$ is a monotonically non-decreasing function of $t$. We show the result by induction on $m_1 + m_2$.

Base case: When $m_1 + m_2 \in \{0, 1\}$, at least one of the trees has a single vertex (which is its leaf and root), hence $\text{mast}\{T_1, T_2\} = 1$. Also, since $t = 1$, we have $g(m_1, m_2, t) \leq 0$, and the claim is true. So we assume below that $m_1 \geq 1$ and $m_2 \geq 1$.

Induction step: Let $u := \rho(T_1)$ and $v := \rho(T_2)$. As in Theorem 8, we have $t = t_{uv} = t_{\ell(u)\ell(v)} + t_{r(u)r(v)} + t_{r(u)\ell(v)} + t_{r(u)r(v)}$ and we assume, without loss of generality, that $t_{\ell(u)\ell(v)} + t_{r(u)r(v)} \geq t_{\ell(u)r(v)} + t_{r(u)\ell(v)}$ and $t_{r(u)r(v)} \geq t_{\ell(u)\ell(v)}$ to obtain

$$t_{r(u)r(v)} \geq \lceil t/4 \rceil. \quad (13)$$

Case 1: $t_{\ell(u)\ell(v)} \geq \delta t$.

By (13), we also have $t_{r(u)r(v)} \geq \delta t$. In this case, we take a maximum agreement subtree $S_\ell$ of $T_1^{(\ell(u))}$ and $T_2^{(\ell(v))}$, and a maximum agreement subtree $S_r$ of $T_1^{(r(u))}$ and $T_2^{(r(v))}$. We then construct $S = S_\ell \circ S_r$, which by Proposition 2 is an agreement subtree of $T_1$ and $T_2$. Therefore, we have

$$M(m_1, m_2, t) \geq 2M(m_1 - 1, m_2 - 1, \lceil \delta t \rceil) \geq 2 \times 2^{g(m_1 - 1, m_2 - 1, \lceil \delta t \rceil)} \geq 2^{1 + g(m_1 - 1, m_2 - 1, \lceil \delta t \rceil)} \geq 2^{g(m_1, m_2, t)}.$$
where the last step follows from
\[
1 + g(m_1 - 1, m_2 - 1, \lceil \delta t \rceil) = 1 + \left( \frac{(m_1 - 1 + m_2 - 1) \log(1 - 3\delta) + \log(\delta t)}{\log(1 - 3\delta) - \log \delta} \right)
\]
\[
= \frac{(m_1 + m_2) \log(1 - 3\delta) + \log t - \log(1 - 3\delta)}{\log(1 - 3\delta) - \log \delta}
\]
\[
> g(m_1, m_2, t),
\]
where the last inequality requires that \( \delta \in (0, \frac{1}{4}) \).

**Case 2:** \( t_{\ell(u)r(v)} \geq \delta t \) and \( t_{r(u)\ell(v)} \geq \delta t \).

The calculation in this case is identical to that of Case 1, so we omit it.

**Case 3:** \( t_{\ell(u)r(v)} < \delta t \) and \( t_{r(u)\ell(v)} < \delta t \).

Since Case 1 has been examined, we assume that \( t_{\ell(u)\ell(v)} < \delta t \), which implies \( t_{r(u)r(v)} > (1-3\delta)t \). Since \( \max\{T_1, T_2\} \) must be at least \( \max\{T_{11^{(r(u))}}, T_{21^{(r(v))}}\} \), we have
\[
M(m_1, m_2, t) \geq M(m_1 - 1, m_2 - 1, \lceil (1-3\delta)t \rceil).
\]
Now the result follows from the assumption that \( \delta \in (0, \frac{1}{4}) \) and the following:
\[
g(m_1 - 1, m_2 - 1, \lceil (1-3\delta)t \rceil) \geq \frac{(m_1 - 1 + m_2 - 1) \log(1 - 3\delta) + \log(1 - 3\delta)t}{\log(1 - 3\delta) - \log \delta}
\]
\[
= \frac{(m_1 + m_2) \log(1 - 3\delta) + \log t - \log(1 - 3\delta)}{\log(1 - 3\delta) - \log \delta}
\]
\[
> g(m_1, m_2, t).
\]

**Case 4:** \( t_{\ell(u)r(v)} < \delta t \) and \( t_{r(u)\ell(v)} \geq \delta t \).

Since Case 1 has been examined, we assume that \( t_{\ell(u)\ell(v)} < \delta t \), which implies \( t_{r(u)r(v)} + t_{r(u)\ell(v)} > (1 - 2\delta)t \). In this case, since \( \max\{T_1, T_2\} \) must be at least \( \max\{T_{11^{(r(u))}}, T_{21^{(r(v))}}\} \), we can write
\[
M(m_1, m_2, t) \geq M(m_1 - 1, m_2, \lceil (1-2\delta)t \rceil).
\]
Now the result follows from the assumption that \( \delta \in (0, \frac{1}{4}) \) and the following:
\[
g(m_1 - 1, m_2, \lceil (1-2\delta)t \rceil) \geq \frac{(m_1 - 1 + m_2) \log(1 - 3\delta) + \log(1 - 2\delta)t}{\log(1 - 3\delta) - \log \delta}
\]
\[
= \frac{(m_1 + m_2) \log(1 - 3\delta) + \log t + \log(1 - 2\delta) - \log(1 - 3\delta)}{\log(1 - 3\delta) - \log \delta}
\]
\[
> g(m_1, m_2, t).
\]

**Case 5:** \( t_{\ell(u)r(v)} \geq \delta t \) and \( t_{r(u)\ell(v)} < \delta t \).

The analysis of this case is similar to Case 4, except that we have the inequality
\[
M(m_1, m_2, t) \geq M(m_1, m_2 - 1, \lceil (1-2\delta)t \rceil). \quad \blacksquare
\]
Corollary 16. Let

\[ \delta \in \left(0, \frac{1}{3} - \frac{1}{3\sqrt{2}}\right) \quad \text{and} \quad \beta := \left(\frac{1 + 2 \log(1 - 3\delta)}{\log(1 - 3\delta) - \log \delta}\right). \]

If \( T_1 \) and \( T_2 \) are rooted, balanced, binary trees on the same leaf-set of cardinality \( 2^m \), then \( T_1 \) and \( T_2 \) have a maximum agreement subtree on at least \( 2^{\beta m} \) leaves.

Proof. We set \( m_1 = m_2 = m \) and \( t = 2^m \) in Lemma 15. Moreover, we now require \( \delta \) to be less that \( \left(\frac{1}{3} - \frac{1}{3\sqrt{2}}\right) \) (which is less that 1/4) so as to ensure that \( \beta \) is positive.

As in Section 2, we present algorithm Match2 that closely follows the recursions in the roof of Lemma 15. It takes as input two rooted, balanced, binary trees, and returns a set of leaves in a common subtree. Algorithm Match2 depends on a real positive \( \delta \), which we require to be sufficiently small for the algorithm to give a desired bound on the size of a common subtree. The algorithm is somewhat greedy and suboptimal. The analysis of the performance of Match2 makes Lemma 15 much more transparent, giving an alternative proof of Corollary 16. We then apply the corollary to prove the main results of this section for unrooted, balanced (or “almost balanced”) trees.
Algorithm Match2\( (T_1^u, T_2^v) \)

1: if \(|L(T_1^u)| = 1\) or \(|L(T_1^v)| = 1\) then
   return \(L(T_1^u) \cap L(T_1^v)\).

2: if necessary, interchange left and right subtrees in \(T_1^u\) and/or \(T_2^v\) so that
   \(t_{\ell(u)}r(v) + t_{r(u)}r(v) \leq t_{\ell(u)}r(v) + t_{r(u)}r(v)\) and \(t_{\ell(u)}r(v) \leq t_{r(u)}r(v)\).

3: if \((t_{\ell(u)}r(v) \geq \delta_{uw}\) and \(t_{r(u)}r(v) \geq \delta_{uw}\) then
   return \(\text{Match2}(T_1^{(\ell(u))}, T_2^{(\ell(v))}) \cup \text{Match2}(T_1^{(r(u))}, T_2^{(r(v))})\).

4: if \((t_{\ell(u)}r(v) \geq \delta_{uw}\) and \(t_{r(u)}r(v) \geq \delta_{uw}\) then
   return \(\text{Match2}(T_1^{(\ell(u))}, T_2^{(r(v))}) \cup \text{Match2}(T_1^{(r(u))}, T_2^{(\ell(v))})\).

5: if \((t_{\ell(u)}r(v) < \delta_{uw}\) and \(t_{r(u)}r(v) < \delta_{uw}\) then
   return \(\text{Match2}(T_1^{(r(u))}, T_2^{(r(v))})\).

6: if \((t_{\ell(u)}r(v) < \delta_{uw}\) and \(t_{r(u)}r(v) \geq \delta_{uw}\) then
   return \(\text{Match2}(T_1^{(r(u))}, T_2^{(r(v))})\).

7: if \((t_{\ell(u)}r(v) \geq \delta_{uw}\) and \(t_{r(u)}r(v) < \delta_{uw}\) then
   return \(\text{Match2}(T_1^{(u)}, T_2^{(v)})\).

We now analyze the above algorithm to compute \(\beta\) in Corollary 16 more transparently.

Alternative proof of Corollary 16. We prove the result by analyzing \(\text{Match2}\). In the beginning, we call \(\text{Match2}(T_1^{(x)}, T_2^{(y)})\), where \(x := \rho(T_1)\) and \(y := \rho(T_2)\). Let \(T\) be the tree of recursive calls to \(\text{Match2}\) constructed as follows: the pair \((x, y)\) is the root of \(T\). If \(\text{Match2}(T_1^{(u)}, T_2^{(v)})\) is called during the execution of the algorithm, then \((u, v)\) is a vertex of \(T\). If \(\text{Match2}(T_1^{(u)}, T_2^{(v)})\) calls \(\text{Match2}(T_1^{(u')}, T_2^{(v')})\), then \((u', v')\) is a child of \((u, v)\). The leaf vertices of \(T\) correspond to the function calls that return in line 1. Observe that in line 1, a set containing a single new leaf is returned. By construction, the number of leaves in the common subtree returned by \(\text{Match2}\) is precisely the number of leaves of \(T\).

The ideas in this lemma are similar to those in the alternative proof of Corollary 1. We consider an arbitrary root-to-leaf path in \(T\) and we show that it branches at least \(\beta m\) times. We then conclude that \(T\) has at least \(2^{\beta m}\) leaves, thereby proving the theorem.

Now consider an arbitrary root-to-leaf path \((x_0, y_0), (x_1, y_1), \ldots, (x_k, y_k)\) in \(T\), where \((x_0, y_0) := (x, y)\). As a shorthand notation, define

\[t_i := t_{x_iy_i}\]

Now suppose \(\text{Match2}(T_1^{u_i}, T_2^{y_i})\) is being called with \(u = x_i\) and \(v = y_i\). Using our notation, we have \(t_i = t_{u,v}\). For each possibility of calling \(\text{Match2}\) recursively, we obtain a lower bound for \(t_{i+1}\) in terms of \(t_i\).
First observe that, as in the case of $\text{Match1}$, we relabel $\ell(u), r(u)$ and $\ell(v), r(v)$ so that we have $t_{\ell(u)\ell(v)} + t_{r(u)r(v)} \geq t_{uv}/2$ and $t_{r(u)r(v)} \geq t_{\ell(u)\ell(v)}$ (which implies $t_{r(u)r(v)} \geq t_{uv}/4$). Hence, for the choice of $\delta$, we have

$$t_{r(u)r(v)} \geq t_{uv}/4 \geq \delta t_{uv}. \quad (14)$$

If a recursive call in line 3 is triggered, then $(x_{i+1}, y_{i+1})$ is either $(\ell(u), \ell(v))$ or $(r(u), r(v))$. In both cases, under the conditions in line 3, we have

$$t_{i+1} \geq \delta t_i. \quad (15)$$

Similarly, if a recursive call in line 4 is triggered, then we also have $\delta t_{uv}$.

After lines 3 and 4, we must have $t_{\ell(u)\ell(v)} < \delta t_{uv}$ because $t_{\ell(u)\ell(v)} \leq t_{r(u)r(v)}$ and the condition in line 3 has failed.

If the recursive call in line 5 is triggered, then $(x_{i+1}, y_{i+1}) = (r(u), r(v))$, and

$$t_{i+1} \geq (1 - 3\delta)t_i, \quad (16)$$

because $t_{\ell(u)\ell(v)} < \delta t_{uv}$, $t_{\ell(u)r(v)} < \delta t_{uv}$ and $t_{r(u)r(v)} < \delta t_{uv}$.

If the recursive call in line 6 is triggered, then $(x_{i+1}, y_{i+1}) = (r(u), v)$, and

$$t_{i+1} \geq (1 - 2\delta)t_i, \quad (17)$$

because $t_{\ell(u)\ell(v)} < \delta t_{uv}$ and $t_{\ell(u)r(v)} < \delta t_{uv}$.

Similarly, if the recursive call in line 7 is triggered, then $(x_{i+1}, y_{i+1}) = (u, r(v))$, and $\delta t_{uv}$ holds because $t_{\ell(u)\ell(v)} < \delta t_{uv}$ and $t_{r(u)r(v)} < \delta t_{uv}$.

Along the chosen path in $\mathcal{T}$, suppose that line 3 is executed $a$ times, line 4 is executed $b$ times, line 5 is executed $c$ times, line 6 is executed $d$ times, and line 7 is executed $e$ times. In each of the recursive calls, the height of one of the trees decreases by 1. Therefore, $a + b + c + d + e \leq 2m$. Consequently, we have

$$t_{uv} \geq t_0 \delta^{a+b}(1 - 3\delta)^c(1 - 2\delta)^{d+e}$$

$$> 2m \delta^{a+b}(1 - 3\delta)^{c+d+e}$$

$$\geq 2m \delta^{a+b}(1 - 3\delta)^{2m-a-b}$$

$$= 2^m \left( \frac{\delta}{1 - 3\delta} \right)^{a+b} (1 - 3\delta)^{2m} \quad (18)$$

Since $(x_k, y_k)$ is a leaf of $\mathcal{T}$, we must have $t_k = 1$. Hence the right-hand-side of (18) must be less than 1, which implies, for the choice of $\beta$, that $a + b > 3m$. Now, a positive $\delta$ less than $\frac{1}{3} - \frac{1}{3\sqrt{2}}$ guarantees that $\beta$ is positive.

We have shown that each root-to-leaf path in $\mathcal{T}$ branches at least $a + b \geq 3m$ times, which further implies that there must be at least $2^{3m}$ leaves in $\mathcal{T}$. Hence $T_1$ and $T_2$ must have a common subtree on at least $2^{3m}$ leaves.
Remark 17. Observe that in the above analysis, we showed that each root-to-leaf path in $\mathcal{T}$ has length at least $\beta m$, which implies that it is possible to find an agreement subtree (not necessarily a maximum agreement subtree) with at least $2^{\beta m}$ leaves that is also balanced (and of height at least $\beta m$). The agreement subtree obtained by algorithm Match2 is illustrated in Figure 4. We may choose a single leaf from each subtree rooted at depth $\lceil \beta m \rceil$, and restrict the tree to chosen leaves to obtain a balanced agreement subtree with precisely $2^{\lceil \beta m \rceil}$ leaves.

![Figure 4](image)

One of the implications of this observation, which we state without proof, is that Lemma 15 together with algorithm Match2 may be used to obtain a lower bound of $2^{\gamma m}$ for an agreement subtree of more than 2 balanced binary trees of height $m$ for a sufficiently small positive $\gamma$. For example, we call algorithm Match2 for two rooted, balanced trees $T_1$ and $T_2$. The resulting agreement subtree contains a rooted, balanced agreement subtree, say $T_{12}$, of height at least $\beta m$. We then call algorithm Match2 for $T_{12}$ and $T_3$. The resulting agreement subtree contains a rooted, balanced agreement subtree $T_{123}$, and so on.

Theorem 18. There exists a constant $c > 0$ such that, if $T_1$ and $T_2$ are balanced, binary trees on the same leaf-set, both in $B_m$ or both in $C_m$, then they have a maximum agreement subtree on at least $2^{\beta m - c}$ leaves.

Proof. As in Theorem 8 we consider the two cases: the trees are either both in class $B_m$ or both in class $C_m$. When the trees are both in class $B_m$, the proof is analogous to the corresponding case in Theorem 8, except that it invokes Corollary 10 instead of Lemma 3.

When the trees are in class $C_m$, the analysis differs only slightly from that in Theorem 8. We delete one of the branches of $T_1$ rooted at the center, and root the resulting tree at the center (which now has degree 2). Let $X$ be the leaf-set of the pruned tree. We cannot simply take a restriction of $T_2$ to the leaf-set $X$ as in Theorem 8 since $T_2|X$ may not be a balanced tree. We instead delete one of the branches of $T_2$ rooted at its center, and root the pruned tree at its center. Let $Y$ be the leaf-set of the pruned tree. Now we can ensure that $|X \cap Y| \geq 2^{m+1}/3$ by appropriately choosing the branches of $T_1$ and $T_2$ to be deleted. We apply Match2 to the rooted trees $T_1|X$ and $T_2|Y$. The analysis of Match2 does not change, except that we now have a constant factor 2/3 on the right-hand-side.
of (18). Therefore, with $a, b, c, d, e$ defined as in the alternative proof of Corollary 16, we have

$$a + b > m \left( \frac{1 + 2 \log(1 - 3\delta) - (1/m) \log(3/2)}{\log(1 - 3\delta) - \log \delta} \right).$$

Taking $c = (\log 3 - 1)/(\log(1 - 3\delta) - \log \delta)$, there are at least $2^{2m - c}$ leaves in a common subtree.

In fact, we have a similar result when the two trees are “almost balanced”.

**Proposition 19.** For every $k > 0$, there is a constant $\beta_k > 0$ such that, if $T_1$ and $T_2$ are binary trees on a leaf-set of cardinality $n$, each of radius at most $k \log n$, then they have a maximum agreement subtree on at least $n^{\beta_k}$ leaves.

**Proof.** The proof is analogous to the alternative proof of Corollary 16: the only change is that now we have $a + b + c + d + e \leq 2k \log n$. We use a value of $\delta > 0$ such that

$$\beta_k := \frac{1 + 2k \log(1 - 3\delta)}{\log(1 - 3\delta) - \log \delta}$$

is positive, and we have $a + b \geq \beta_k \log n$.

Concerning the maximum agreement subtree problem for balanced trees, we believe in the following.

**Conjecture 20.** Any two balanced, rooted, binary trees of height $m$ have an agreement subtree on at least $2^{m/2}$ leaves.

We now describe an example of a pair of rooted, balanced, binary trees of height $2k$, for each $k > 0$, which we believe is an extremal example. Let $T_1$ and $T_2$ be balanced, binary trees of height $2k$, rooted at $u$ and $v$, respectively, and both drawn top-down. Let the leaves of $T_1$ be labelled $1, 2, \cdots, 2^{2k}$ from left to right. We label the leaves of $T_2$ from left to right according to the sequence $\text{swap}(1, 2, \cdots, 2^{2k})$, which we define recursively as follows:

1. If $S$ is a sequence of length 1, then

$$\text{swap}(S) = S.$$

2. If $S$ is a sequence of length $4^i$, with $i > 0$, written as $S := S_1 : S_2 : S_3 : S_4$ as a concatenation of 4 sequences of length $4^{i-1}$ each, then

$$\text{swap}(S) := \text{swap}(S_1) : \text{swap}(S_3) : \text{swap}(S_2) : \text{swap}(S_4).$$

**Proposition 21.** Trees $T_1$ and $T_2$ have no rooted agreement subtree with more than $2^k$ leaves.
Therefore, mast children of leaves. Trees Proposition 22. Two or three of the four sets have no agreement subtree with more than 3 × 2^{k-1} leaves.

Proof. We prove the result by induction on k. When k = 1, the trees have 4 leaves, with leaves of T_1 labelled 1, 2, 3, 4 from left to right, while the leaves of T_2 labelled 1, 3, 2, 4 from left to right. In this case, a rooted agreement subtree cannot have more than 2 leaves.

In the general case, the inductive argument goes as follows. Let u_1 and u_2 be the children of ℓ(u), and let u_3 and u_4 be the children of r(u). Similarly, in T_2, we label the grandchildren of v by v_1, v_2, v_3, v_4. By construction, we have \( L(T_1^{(u_1)}) = L(T_2^{(v_1)}) \), \( L(T_1^{(u_2)}) = L(T_2^{(v_2)}) \), \( L(T_1^{(u_3)}) = L(T_2^{(v_3)}) \), and \( L(T_1^{(u_4)}) = L(T_2^{(v_4)}) \). But a rooted agreement subtree cannot have leaves from more than two of the sets \( L(T_1^{(u_i)}), i \in \{1, 2, 3, 4\} \). Therefore, \( \text{mast}\{T_1, T_2\} \leq 2 \times \text{mast}\{T_1^{(u_1)}, T_2^{(v_1)}\} \leq 2 \times 2^{k-1} = 2^k \).

An analogous but slightly weaker statement holds for unrooted trees. We use the same labelling scheme as in the rooted case, but remove the roots, i.e., we delete the vertex u, and the edges \( \{u, ℓ(u)\} \) and \( \{u, r(u)\} \), and add an edge \( \{ℓ(u), r(u)\} \), and similarly make T_2 unrooted.

**Proposition 22.** Trees T_1 and T_2 have no agreement subtree with more than 3 × 2^{k-1} leaves.

Proof. We prove the result by induction on k. When k = 1, the trees have an agreement subtree on 3 leaves, but not 4. In the general case, as in the rooted case, L(T_1^{(u_1)}) = L(T_2^{(v_1)}), L(T_1^{(u_2)}) = L(T_2^{(v_2)}), L(T_1^{(u_3)}) = L(T_2^{(v_3)}), and L(T_1^{(u_4)}) = L(T_2^{(v_4)}). For \( i \in \{1, 2, 3, 4\} \), let \( A_i \) denote the set of leaves from \( L(T_1^{(u_i)}) \) that are in a maximum agreement subtree \( R \). But an agreement subtree cannot have leaves from all four sets \( L(T_1^{(u_i)}), i \in \{1, 2, 3, 4\} \). Therefore, \( A_i = \emptyset \) for some \( i \in \{1, 2, 3, 4\} \).

**Case 1:** Two or three of the four sets \( A_i \) are non-empty. Without loss of generality, let \( A_1 \) and \( A_2 \) (and possibly also \( A_3 \)) be non-empty. For \( i \in \{1, 2, 3\} \), let \( x_i \) be the most recent common ancestor of leaves in \( A_i \) in the rooted subtree \( T_1^{(u_i)} \). Now we observe that \( R|A_1 \) rooted at \( x_1 \) is a rooted agreement subtree of \( T_1^{(u_1)} \) and \( T_1^{(v_1)} \), and \( R|A_2 \) rooted at \( x_2 \) is a rooted agreement subtree of \( T_1^{(u_2)} \) and \( T_1^{(v_2)} \), and, if \( A_3 \) is non-empty, \( R|A_3 \) rooted at \( x_3 \) is a rooted agreement subtree of \( T_1^{(u_3)} \) and \( T_1^{(v_3)} \). Therefore, \( |L(R)| = |A_1 \cup A_2 \cup A_3| \leq 3 \times 2^{k-1} \) (by Proposition 21).

**Case 2:** A maximum agreement subtree \( R \) has leaves from only one of the four sets \( L(T_1^{(u_i)}), i \in \{1, 2, 3, 4\} \). Without loss of generality, let \( L(R) \subseteq L(T_1^{(u_1)}) \). In this case, by induction, \( \text{mast}\{T_1, T_2\} \leq 3 \times 2^{k-2} < 3 \times 2^{k-1} \).

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