Optimal Simulation of Quantum Measurements via the Likelihood POVMs

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Abstract

We provide a new and simplified proof of Winter’s measurement compression [1] via likelihood POVMs. Secondly, we provide an alternate proof of the central tool at the heart of this theorem - the Quantum covering lemma. Our proof does not rely on the Ahlswede-Winter’s operator Chernoff bound [2] and is applicable even when the random operators are pairwise independent. We leverage these results to design structured POVMs and prove their optimality in regards to communication rates.

I. INTRODUCTION

Exploiting the unique behaviour of sub-atomic particles can enable us transform information processing. Measurements play a central role in extracting information and reading out the result of computations performed on a quantum computing device. The design and analysis of quantum measurements thus plays a central role in quantum information processing. In this article, we address the question of how much information is contained in the outcome of a Positive Operator Valued Measurement (POVM).

The outcome of a quantum measurement is random and one of the challenges in addressing the above question is to precisely characterize and quantify that component of the randomness that is intrinsic to the quantum state. This question, being of fundamental importance, has been addressed in a long series of works [3]–[6]. In 2004, Winter adopted an information theoretic viewpoint and derived an elegant formulation. His definitive solution to this problem stands as one among the important findings in quantum information theory.

Our work is aimed at answering analogous questions involving multiple and distributed measurements. One question of interest is to study how much information is contained in successive POVMs that are jointly measurable. A second question of interest is to characterize the information contained in the outcomes of distributed measurements performed on entangled states. In this work, we present some of the foundational tools required to answer these questions. Our first contribution involves a new treatment of Winter’s findings. This includes a new proof using likelihood POVMs which are much simpler to describe in comparison to those studied in [1], [7] including more recent studies [8]–[10]. While the likelihood POVMs are indeed the most intuitive to design, its performance analysis for the problem at hand has remained elusive. Indeed, even while [7] provides an excellent exposition of [1], the simulation POVM and its study is identical and involved. Through a carefully chosen sequence of steps, highlighted in Sec. II-B, II-C we demonstrate the simplicity of likelihood POVMs also facilitates a simplified proof.

As a next step, we revisit the fundamental tool - quantum covering lemma (QCL) [11, Ch. 17]. Known proofs of QCL are based on the Ahlswede and Winter’s [2] operator Chernoff bound (OCB). The OCB requires that the random operators be mutually independent. Its use for the problem at hand precludes the simulation POVM to have any additional structure. For example, relying on the OCB simulation precludes proving optimality of a simulation POVM with a 'algebraic closure structure’ (Sec. IV). This is because, if one picks a random POVM with an algebraic structure, its operators are not mutually independent. Our second contribution is a new proof (Lem. [1] Sec. III-E) of the QCL that does not rely on the OCB and the underlying concentration only requires pairwise independence. Building on this we design - as an application of all our findings - structured likelihood POVMs (Sec. IV) that simulate POVMs with optimal communication costs (Thm. 2).

Given the technical nature of our work on a studied problem, this article is heavy on details. Yet, the simplicity of the proofs have enabled us illustrate the key new steps through Sec. III-B, III-C and also leverage elegant, useful operator properties such as the Löwner Heinz theorem [12] and the operator monotonicity of the square root function that will hopefully find other applications in quantum information theory.

II. PRELIMINARIES AND PROBLEM DESCRIPTION

A. Notation

We supplement standard quantum information theory notation with the following. \( \alpha^* \in \mathbb{C} \) denotes complex conjugate of \( \alpha \in \mathbb{C} \). For a positive integer \( n \), we let \( [n] \triangleq \{1, \ldots, n\} \), \( [n] \) \( \triangleq \{0\} \cup [n] \). All Hilbert spaces are assumed to be finite dimensional. \( \mathcal{L}(\mathcal{H}), \mathcal{R}(\mathcal{H}), \mathcal{P}(\mathcal{H}), \mathcal{D}(\mathcal{H}) \) denote the collection of linear, Hermitian, positive and density operators acting on Hilbert space \( \mathcal{H} \) respectively. \( T^\dagger \in \mathcal{L}(\mathcal{H}) \) denotes the adjoint of \( T \in \mathcal{L}(\mathcal{H}) \).

POVMs will play a central role in this article. \( \mathcal{M}(\mathcal{H}, \mathcal{Y}) \) denotes the set of all POVMs acting on \( \mathcal{H} \) with outcomes in \( \mathcal{Y} \). An element of \( \mathcal{M}(\mathcal{H}, \mathcal{Y}) \) is therefore a collection of \( |\mathcal{Y}| \) positive operators\(^1\) acting on \( \mathcal{H} \) that sum to \( I_\mathcal{H} \) - the identity operator on \( \mathcal{H} \). Often times, we denote a POVM \( \lambda = \{ \lambda_y \in \mathcal{P}(\mathcal{H}) : y \in \mathcal{Y} \} \in \mathcal{M}(\mathcal{H}, \mathcal{Y}) \) by adding the outcome set as a

\(^1\)not necessarily distinct
subscript, as in $\lambda_Y = \lambda$. For POVM $\lambda = \{\lambda_y : y \in \mathcal{Y}\}$, $\lambda^{\otimes n} \triangleq \{\lambda_{y^n} \triangleq \lambda_{y_1} \otimes \cdots \otimes \lambda_{y_n} : y^n \in \mathcal{Y}^n\}$. To reduce clutter, we let $\lambda^n = \lambda_Y^n = \lambda^{\otimes n} \in \mathcal{M}(H^{\otimes n}_A, Y^n)$ denote the same latter POVM.

Associated with a POVM $\lambda = \{\lambda_y : y \in \mathcal{Y}\}$ is a Hilbert space $\mathcal{H}_Y \triangleq \text{span}\{|y\rangle : y \in \mathcal{Y}\}$ with $\langle y|y\rangle = \delta_{yy}$ and the CPTP map $\delta^\lambda : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H}_Y)$, defined as $\delta^\lambda(s) = \sum_{y \in \mathcal{Y}} \text{tr}(s)\langle y|y\rangle$. For a stochastic matrix $(p_{y|w}(w)|y\rangle : (w, y) \in \mathcal{W} \times \mathcal{Y})$, we let $\delta^\lambda_{p|w} : \mathcal{L}(\mathcal{H}_Y) \rightarrow \mathcal{L}(\mathcal{H}_Y)$ denote the CPTP map $\delta^\lambda_{p|w}(a) \triangleq \sum_{(w,y) \in \mathcal{W} \times \mathcal{Y}} p_{y|w}(y|w)\langle y|w\rangle a |w\rangle |y\rangle$. The composition of CPTP maps $\mathcal{L}(\mathcal{H}_A) \xrightarrow{\delta^\lambda_{\mathcal{H}_B}} \mathcal{L}(\mathcal{H}_B) \xrightarrow{\delta^\lambda_{\mathcal{H}_C}} \mathcal{L}(\mathcal{H}_C)$ is denoted $\delta^\lambda \circ \delta^\lambda_{\mathcal{H}_B}$.

A quantum ensemble $(\rho_w, p_{w|w}, \mathcal{W})$ comprises of (i) a finite set $\mathcal{W}$, (ii) a collection of density operators $\rho_w \in \mathcal{D}(\mathcal{H}) : w \in \mathcal{W}$ and (iii) a probability mass function (PMF) $p_w$ on $\mathcal{W}$ such that $\sum_{w \in \mathcal{W}} p_{w|w}(w)\rho_w = \rho$. We also refer to a quantum ensemble $(\rho_w, p_{w|w}, \mathcal{W})$ as an ensemble $(\rho_w, p_{w|w}, \mathcal{W})$ or an ensemble $(\rho_w, p_{w|w}, \mathcal{W})$ with average $\sum_{w \in \mathcal{W}} p_{w|w}(w)\rho_w$. For an ensemble $(\rho_w, p_{w|w}, \mathcal{W})$, $\chi(\rho_w, p_{w|w}, \mathcal{W}) \triangleq S(\sum_{w \in \mathcal{W}} p_{w|w}(w)\rho_w) - \sum_{w \in \mathcal{W}} p_{w|w}(w)S(\rho_w)$ denotes Holevo information. SCD, WHP abbreviate spectral decomposition and with high probability.

We let $\{|b_1\}, \cdots, |b_d\rangle \subseteq \mathcal{H}_A$ denote an arbitrary but fixed orthonormal basis and $\mathcal{H}_X = \mathcal{H}_A$. For any $|x\rangle = \sum_{i=1}^d \alpha_i |b_i\rangle \in \mathcal{H}_A$, we let $|x\rangle = \sum_{i=1}^d \alpha_i^* |b_i\rangle \in \mathcal{H}_A$ denote the complex conjugation with respect to orthonormal basis $\{|b_1\}, \cdots, |b_d\rangle\}$. For any $\sigma \in \mathcal{D}(\mathcal{H}_A)$, with a spectral decomposition $\sigma = \sum_{\gamma} \gamma |u_i\rangle \langle u_i|$, we let $|\phi_{\sigma}\rangle \triangleq \sum_{\gamma} \sqrt{\gamma} |u_i\rangle \otimes |u_i\rangle \in \mathcal{H}_X \otimes \mathcal{H}_A \otimes^n$ denote a purification of $\sigma$, henceforward referred to as the canonical purification $\mathcal{P}$. 

### B. Problem Description

Let Hilbert space $\mathcal{H}_A$ have dimension $d_A$. Let $\rho \in \mathcal{D}(\mathcal{H}_A)$ model the behaviour of a given sub-atomic particle and $\lambda \triangleq \lambda_Y \triangleq \{\lambda_y \in \mathcal{P}(\mathcal{H}) : y \in \mathcal{Y}\}$ denote a given POVM. We follow [1], [7] in modelling the following question. Suppose a measurement modeled via POVM $\lambda_Y$ is performed on the particle $\rho \in \mathcal{D}(\mathcal{H}_A)$, how much ‘intrinsic’ information does the outcome contain? The statistical nature of the outcome leads us to a Shannon-theoretic modeling. If one performs $n$ (independent) measurements $\lambda^{\otimes n}$ on $n$ identically prepared particles $\rho^{\otimes n}$, how many bits are necessary to compress the $n$ random outcomes?

Since a measurement outcome is random, what fraction of the inherent randomness is ‘intrinsic’ to the particle $\rho$, and what fraction is ‘extrinsic’, or unrelated to $\rho$? To quantify this, we design an ‘alternate $n$-letter measurement’ - a simulated POVM - that is supplemented with an external source of common randomness of $C$ bits/letter (Fig. 1) available at both terminals. These $C$ bits of common randomness are statistically independent of the particle and are available to (i) design this simulated POVM and (ii) postprocess its outcome to simulate the outcome of the original POVM $\lambda_Y$ on $\rho$. We require the outcome of the simulated POVM - both the post measurement particle and the observed outcome $Y^n \in \mathcal{Y}^n$ - to be statistically indistinguishable from that of the original POVM $\lambda_Y$. Enforcing this, we aim to quantify the minimum rate $R$ bits/letter that enables Bob reconstruct the classical outcome. Characterizing all possible $(R, C)$ pairs enables us quantify the trade-off between intrinsic information and extrinsic randomness contained in the outcome of POVM $\lambda_Y$.

Since a POVM obfuscates phase information, the above stated requirement is specified in terms of demanding that the combined operators of the reference and outcome post measurement of both the original measurement $\lambda_Y^{\otimes n}$ and the simulated one are statistically ‘close’. Adopting trace distance to quantify ‘closeness’ we are led to the following.

**Defn. 1.** Suppose $\rho \in \mathcal{D}(\mathcal{H}_A)$, $\mathcal{H}_X = \mathcal{H}_A$ and $\lambda \in \mathcal{M}(\mathcal{H}_A^{\otimes n}, Y^n)$ is a POVM. A sequence $\Xi^{(n)} \in \mathcal{M}(\mathcal{H}_A^{\otimes n}, Y^n) : n \geq 1$ of POVMs simulates $\lambda$ on $\rho$ if for all $\eta > 0$, $\exists N_\eta \in \mathbb{N}$ such that for all $n \geq N_\eta$, we have $||\alpha_o - \alpha_{sp}||_1 \leq \eta$, where

$$
\alpha_o \triangleq (i^{\otimes n}_X \otimes \delta^{\otimes \lambda_Y})(|\phi_{\rho^{\otimes n}}\rangle \langle \phi_{\rho^{\otimes n}}|) \quad \text{and} \quad \alpha_{sp} = (i^{\otimes n}_X \otimes \delta^{\Xi^{(n)}})(|\phi_{\rho^{\otimes n}}\rangle \langle \phi_{\rho^{\otimes n}}|).
$$

$$
|| (i^{\otimes n}_X \otimes \delta^{\otimes \lambda_Y})(|\phi_{\rho^{\otimes n}}\rangle \langle \phi_{\rho^{\otimes n}}|) - (i^{\otimes n}_X \otimes \delta^{\Xi^{(n)}})(|\phi_{\rho^{\otimes n}}\rangle \langle \phi_{\rho^{\otimes n}}|) ||_1 \leq \eta.
$$

![Fig. 1. Illustrates original and simulated POVMs. The components in the two blue ellipses must be statistically indistinguishable.](image-url)
Remark 1. Though the purification of a state is not unique, we have chosen the canonical purification $|\phi_\rho\rangle$ in our definition above. In Appendix $\text{[R]}$ we prove the choice of the purification does not matter.

Since the simulated POVM can utilize independent randomness at both terminals, we let (i) $\rho^\otimes_n = \frac{1}{n!} \sum_{k \in [K]} \rho^\otimes_k$ model its input state, (ii) design the simulated measurement to be of the form $\theta = (\delta_{k,m} \otimes |k\rangle\langle k| : (k, m) \in [K] \times [M]) \in \mathcal{M}(\mathcal{H}_{A^K} \otimes [K] \times [M])$, where $\mathcal{H}_{A^K} = \mathcal{H}_{X^K} \otimes \mathcal{H}_K$, $\mathcal{H}_K = \text{span} \{ |k\rangle : k \in [K] \}$ and $\langle k|k\rangle = \delta_{kk}$. Essentially, the simulated POVM $\theta$ observes the common randomness $k$ and chooses to perform the POVM $\{ \delta_{k,m} : m \in [M] \} \in \mathcal{M}(\mathcal{H}_{X^K} \otimes [M])$ and hands over the nature provided common randomness $k$ and the POVM outcome $m$ to the Dec. Denoting $\mathcal{N}(\mathcal{H}_{A^K}, [K] \times [M]) \subseteq \mathcal{N}(\mathcal{H}_{A^K}, [K])$ as the set of POVMs of the above form, we define the quantity of interest.

Defn. 2. The communication cost of a simulation POVM $\theta \in \mathcal{N}(\mathcal{H}_{A^K}, [K])$ is $(\frac{\log K}{n}, \frac{\log M}{n})$, POVM $\lambda_Y$ on $\rho$ can be simulated at a cost $(C, R)$ if $R > C$ and there exists a $\theta \in \mathcal{N}(\mathcal{H}_{A^K}, [K])$ and a POVM $\Delta_{Y^n} \in \mathcal{N}(\mathcal{H}_X \otimes \mathcal{H}_M, \mathcal{Y}^n)$ such that $C + \frac{\log K}{n} \leq C + \eta$, $R + \frac{\log M}{n} \leq R + \eta$ and $||\alpha_\rho - \alpha_\lambda||_1 \leq \eta$, where

\begin{align}
\alpha_\rho &= (i_X^n \otimes \mathcal{E}^\lambda_n)(|\phi_\rho\rangle \langle \phi_\rho|), \quad \alpha_\lambda = \mathcal{E}_\lambda(\rho^\otimes_n), \\
\mathcal{E}_\lambda &= (i_X^n \otimes \mathcal{E}^\lambda_n) \circ (i_X^n \otimes i_{\mathcal{H}_K} \otimes \mathcal{E}^\delta_n) \circ (i_X^n \otimes i_{\mathcal{H}_K} \otimes \mathcal{E}^\theta_n),
\end{align}

$\mathcal{H}_X = \mathcal{H}_A$, $i_X^n$ abbreviates the identity map $i_X^n$ on $\mathcal{L}(\mathcal{H}_X^\otimes_n)$.

Remark 2. In Defn. 2, $\alpha_\rho$ is the outcome of the original POVM and hence takes only $\rho^\otimes_n$ as input. Note that the purification of $\rho^\otimes_n$ contains a component in the reference space corresponding to common randomness $K$. This justifies the presence of $i_{\mathcal{H}_K}$ in $\mathcal{E}_\lambda(\cdot)$. Finally, we wish to only guarantee faithful simulation of the particle post measurement. This explains the $\mathcal{E}_\lambda(\cdot)$ in the final CPTP map of $\mathcal{E}_\lambda(\cdot)$.

Remark 3. In the interest of brevity, we have only provided a heuristic explanation for the above formulation. In Appendix $\text{[P]}$ we argue this formulation involving purification more directly. The interested reader will find the contents therein similar to the discussion provided in $\text{[7]}$.

C. Communication Cost of Simulating $\lambda_Y$ on $\rho$

Winter $\text{[1]}$ derived an elegant computable characterization for the communication cost of simulating $\lambda_Y$ on $\rho$ for the case when both terminals wish to possess outcome of the simulated POVM. We state below Wilde et. al.’s $\text{[7]}$ generalization for the case when only Bob wishes to possess the simulated outcome.

Defn. 3. For $\rho \in \mathcal{D}(\mathcal{H}_A)$ and $\lambda_Y \in \mathcal{M}(\mathcal{H}_A, \mathcal{Y})$, let $C(\rho, \lambda_Y)$ be a collection of triples $(W, \mu_W, p_{Y|W})$ wherein (i) $W$ is a finite set, (ii) $\mu_W \in \mathcal{M}(\mathcal{H}_A, W)$ is a POVM, and (iii) $(p_{Y|W}(w) : (w, y) \in W \times Y)$ is a stochastic matrix such that

\begin{equation}
(i_X \otimes \mathcal{E}^\rho_{Y|W}) \circ (i_X \otimes \mathcal{E}^\gamma_{Y^i})(|\phi_\rho\rangle \langle \phi_\rho|) = (i_X \otimes \mathcal{E}^\lambda_{Y})(|\phi_\rho\rangle \langle \phi_\rho|),
\end{equation}

For $(W, \mu_W, p_{Y|W}) \in C(\rho, \lambda_Y)$, let $p_W(w) \triangleq \mathcal{E}(\rho_W)$,

\begin{equation}
\beta_w \triangleq \sqrt{p_W(w)} \mathcal{E}(\rho_W), \quad \gamma_w \triangleq \sum_{y \in Y} p_{Y|W}(w|y) |y\rangle |y\rangle,
\end{equation}

Let $\mathcal{R}(W, \mu_W, p_{Y|W}) \triangleq \chi(\gamma_w, p_W : W)$ and $\mathcal{D}(W, \mu_W, p_{Y|W}) \triangleq \chi(\beta_w, p_W : W)$.

Remark 4. For $(W, \mu_W, p_{Y|W}) \in C(\rho, \lambda_Y)$, we note $\mathcal{E}(\rho_W) \circ (i_X \otimes \mathcal{E}^\rho_{Y|W}) \circ (i_X \otimes \mathcal{E}^\gamma_{Y^i})(|\phi_\rho\rangle \langle \phi_\rho|) = \sum_{w \in W} p_W(w) \gamma_w$.

\begin{equation}
(i_X \otimes \mathcal{E}^\rho_{Y|W})(|\phi_\rho\rangle \langle \phi_\rho|) = \sum_{w \in W} \sqrt{p_W(w)} \mathcal{E}(\rho_w), \quad \mathcal{E}(\rho_w), \quad \gamma_w = \sum_{w \in W} p_W(w) \gamma_w,
\end{equation}

\begin{equation}
\mathcal{E}(\rho_w), \quad \gamma_w = \sum_{w \in W} p_W(w) \gamma_w.
\end{equation}

Theorem 1. POVM $\lambda_Y$ on $\rho$ can be simulated at a cost $(C, R)$ iff there exists $(W, \mu_W, p_{Y|W}) \in C(\rho, \lambda_Y)$ for which $R > \mathcal{R}(W, \mu_W, p_{Y|W})$ and $R + C > \mathcal{D}(W, \mu_W, p_{Y|W})$.

III. PROOF VIA UNSTRUCTURED LIKELIHOOD POVMs

A. Proof Setup : Notations, Definitions and Likelihood POVM

Our proof will involve specifying an $(n, C, R, \theta, \Delta)$ simulation protocol, analyzing $||\alpha_\rho - \alpha_\lambda||_1$ where $\alpha_\rho, \alpha_\lambda$ is as defined in $\text{[61], [62]}$ respectively, and thereby characterizing conditions under which $||\alpha_\rho - \alpha_\lambda||_1$ shrinks exponentially in $n$. Towards that end, we begin by introducing notations and conventions used throughout the proof. Let $\omega \triangleq \rho^\otimes n$. Choose $(W, \mu_W, p_{Y|W}) \in$
Furthermore, in order to bound \( \mu \), we derive an upper bound on the RHS of (13).

Since \( S \), where \((k,m) \in [K] \times [M] \), is a POVM leading to much complexity. We therefore illustrate the key steps below, which is also revealing. For \( \alpha \) above and \( \alpha_s \) in (62) are the result of applying the same CPTP map - \( \mathcal{E}_{\text{im}} \) - on two different states provides us with the right clue. Indeed, we have

\[
\|\alpha_s - \alpha\|_1 \leq \|\sigma_{\alpha} - \sigma_{\alpha_s}\|_1 + \|\alpha - \alpha_s\|_1
\]

from the Triangular inequality and the fact that CPTP maps shrink the trace distance \([11, \text{Eq. 9.69}]\). Through the rest of the proof, we analyze each of the terms in the RHS of (12). In Sec. III-C we analyze the second term where we leverage an elegant bound that relates the distance between states and their canonical purifications. In Sec. III-D we evaluate the RHS of (11) and thereby knock off the inconvenient outer normalizing factors \( S_k^{\frac{1}{2}} \) in \( \theta_{k,m} \) defined in (7).

## B. Key Steps Outlining the Proof

While it is easy to conjecture the above simulation POVMs, the non-commutativity of quantum operations has obfuscated its analysis for the task at hand, leading studies \([1, 7]\) including the more recent ones \([8–10]\) to adopt an alternate simulation POVM leading to much complexity. We therefore illustrate the key steps below, which is also revealing. For \( \alpha \in [K] \), let

\[
T_a = \frac{S_a}{M} \quad \text{have SCD } T_a = \sum_{i=1}^d \nu_{i a} |x_{i a} \rangle \langle x_{i a} |, \quad \text{i.e., } |x_{i a} \rangle \langle x_{i a} | = \delta_{i a} \quad \text{for } a \in [K] \quad \text{and } \sigma_{\alpha a} = \frac{1}{K} \sum_{a \in [K]} T_a \otimes |a\rangle \langle a|.
\]

In order to bound \( \|\alpha_0 - \alpha_s\|_1 \) we define

\[
\alpha \triangleq \mathcal{E}_{\text{sm}}(|\phi_{\sigma_{\alpha a}^n} \rangle \langle \phi_{\sigma_{\alpha a}^n} |), \quad \text{where } |\phi_{\sigma_{\alpha a}^n} \rangle \triangleq \sum_{i=1}^d \sum_{a \in [K]} \frac{1}{M} \sqrt{\nu_{i a}} |x_{i a} a\rangle \otimes |x_{i a} a\rangle.
\]

is the canonical purification of \( \sigma_{\alpha a}^n \). Recognizing that \( \alpha \) above and \( \alpha_s \) in (62) are the result of applying the same CPTP map - \( \mathcal{E}_{\text{im}} \) - on two different states provides us with the right clue. Indeed, we have

\[
\|\alpha_0 - \alpha_s\|_1 \leq \|\alpha_0 - \alpha_s\|_1 + \|\alpha - \alpha_s\|_1
\]

\[
\leq \|\alpha_0 - \alpha\|_1 + \|\sigma_{\alpha a} - \sigma_{\alpha a}^n\|_1
\]

from the Triangular inequality and the fact that CPTP maps shrink the trace distance \([11, \text{Eq. 9.69}]\). Through the rest of the proof, we analyze each of the terms in the RHS of (12). In Sec. III-C we analyze the second term where we leverage an elegant bound that relates the distance between states and their canonical purifications. In Sec. III-D we evaluate the RHS of (11) and thereby knock off the inconvenient outer normalizing factors \( S_k^{\frac{1}{2}} \) in \( \theta_{k,m} \) defined in (7).

## C. Relating distance between states and their canonical purifications

Our approach to deriving an upper bound on the second term in (12) is by relating the latter to \( \|\sigma_{\alpha a} - \sigma_{\alpha a}^n\|_1 \). However, note that tracing over components decreases the trace distance, and hence

\[
\|\sigma_{\alpha a} - \sigma_{\alpha a}^n\|_1 \leq \|\phi_{\sigma_{\alpha a}^n} \rangle \langle \phi_{\sigma_{\alpha a}^n} | - |\phi_{\rho_{\alpha a}^n} \rangle \langle \phi_{\rho_{\alpha a}^n} | \|_1.
\]

However, we need an inequality in the reverse. The choice of the canonical purification \([11, \text{Pg. 166}]\) enables us to suitably reverse the above inequality. Specifically, by leveraging the relationship between fidelity and trace distance \([11, \text{Thm. 9.3.1}]\) and the specific form of the canonical purification, we have

\[
\|\phi_{\sigma_{\alpha a}^n} \rangle \langle \phi_{\sigma_{\alpha a}^n} | - |\phi_{\rho_{\alpha a}^n} \rangle \langle \phi_{\rho_{\alpha a}^n} | \|_1 \leq 4 \sqrt{\|\sigma_{\alpha a} - \rho_{\alpha a}^n\|_1}.
\]

A proof of (13) can be found in \([1, \text{App. A, Lem. 14}]\) and the same is elaborated in Lemma 5 of Appendix A. In Sec. III-E we derive an upper bound on the RHS of (13).
D. Characterizing $\alpha = \mathcal{E}_m(\ket{\phi_{\sigma A_n K}}\bra{\phi_{\sigma A_n K}})$ and $\alpha_o$

In this section, we focus on the first term in (12) and evaluate $\alpha_o, \alpha$. In characterizing $\alpha$, we ought to evolve $\ket{\phi_{\sigma A_n K}}\bra{\phi_{\sigma A_n K}}$ through three CPTP maps that define $i_{\Delta}^{\alpha}$. From definition of $\delta_0^{\alpha}$, we have $(i_{\alpha}^{\alpha} \otimes i_{\mathcal{H}_K} \otimes \delta_0^{\alpha})(\ket{\phi_{\sigma A_n K}}\bra{\phi_{\sigma A_n K}}) =$

$$
\sum_{t=1}^{d_\alpha} \sum_{a=1}^{d_\alpha} \sum_{a=1}^{d_{\alpha}} \sum_{b=1}^{d_{\alpha}} K^{-1} \sqrt{\nu_{\alpha} \nu_{\beta}} |x_{ta} a \rangle \langle x_{vb} b | \otimes \\
\text{tr}(\theta_{k,m} x_{ta}(x_{vb}) \text{tr}(|k \rangle \langle k| a \rangle b)) |k m \rangle \langle k m |
$$

$$
= \sum_{t=1}^{d_\alpha} \sum_{a=1}^{d_\alpha} \sum_{a=1}^{d_{\alpha}} \sum_{b=1}^{d_{\alpha}} K^{-1} \sqrt{\nu_{\alpha} \nu_{\beta}} |x_{ta} a \rangle \langle x_{vb} b | \otimes \\
\langle x_{vk} \theta_{k,m} x_{tk} |k m \rangle \langle k m |
$$

$$
= \sum_{t=1}^{d_\alpha} \sum_{a=1}^{d_\alpha} \sum_{a=1}^{d_{\alpha}} \sum_{b=1}^{d_{\alpha}} K^{-1} |x_{ta} a \rangle \langle x_{vb} b | \otimes \\
\langle x_{vk} \sqrt{T_{\alpha}} \theta_{k,m} \sqrt{T_{\alpha}} x_{tk} |k m \rangle \langle k m |
$$

$$
= \sum_{t=1}^{d_\alpha} \sum_{a=1}^{d_\alpha} \sum_{a=1}^{d_{\alpha}} \sum_{b=1}^{d_{\alpha}} K^{-1} |x_{ta} a \rangle \langle x_{vb} b | \otimes \\
\langle x_{vk} \sqrt{T_{\alpha}} \theta_{k,m} \sqrt{T_{\alpha}} x_{tk} |k m \rangle \langle k m |
$$

$$
= \sum_{k \in [K]} \sum_{m \in [M]} \sum_{t=1}^{d_\alpha} \sum_{a=1}^{d_\alpha} |x_{ta} a \rangle \langle x_{vb} b | \otimes \\
\langle x_{vk} \sqrt{T_{\alpha}} \theta_{k,m} \sqrt{T_{\alpha}} x_{tk} |k m \rangle \langle k m |
$$

where (15) follows from shifting scalars $\sqrt{\nu_{\alpha}}, \sqrt{\nu_{\beta}}$, (16) follows from spectral decomposition in (10), (17) follows from Part 4 of Lemma 3, (18) follows from the relationship between $T_k, S_k$ in (10) and $[M] = [M] \cup \{0\}$, (19) follows from definition of $\theta_{m,k}$ and $T_k$ in (7) and the fact stated in (8). Next, we evolve the state in RHS of (19) through the CPTP map $(i_{X}^{\alpha} \otimes i_{\mathcal{H}_K} \otimes \delta_{0}^{\alpha})(\cdot)$ to yield the state

$$
\frac{1}{KM} \sum_{k \in [K]} \sum_{m \in [M]} \sum_{y^n} \beta_{Y^n|W}(y^n | c(k, m)) \frac{\sqrt{\nu_{\alpha}} \nu_{m, \sqrt{\nu_{\beta}}}}{\text{tr}(\omega_{\mu_{k,m}})} \otimes |k y^n \rangle \langle k y^n |
$$

where the above follows from defn. (9). Finally, evolving above state through CPTP map $(i_{X}^{\alpha} \otimes \text{tr}_{K} \otimes \delta_{0}^{\alpha})(\cdot)$ yields

$$
\alpha = \frac{1}{KM} \sum_{(k, m, y^n) \in [K] \times [M] \times Y^n} \beta_{Y^n|W}(y^n | c(k, m)) \frac{\sqrt{\nu_{\alpha}} \nu_{m, \sqrt{\nu_{\beta}}}}{\text{tr}(\omega_{\mu_{k,m}})} \otimes |y^n \rangle \langle y^n |
$$

An identical sequence of steps yields

$$
\alpha_0 = \sum_{y^n \in Y^n} \left( \sqrt{\omega} \lambda_{y^n} \sqrt{\omega} \right)^t \otimes |y^n \rangle \langle y^n |
$$

\text{This is a standard computation and can be verified in [7] Proof of Lem. 4] }
E. A new proof of the quantum covering lemma

Substituting (20), (21) and (13) in the RHS of (12), we have

$$||\alpha_o - \alpha||_1 \leq ||\alpha_o - \alpha||_1 + 4^4||\sigma_{A^K} - \rho_K^{\otimes n}||_1.$$  

(22)

Collating definitions of $S_k$ from (7), Substituting $T_k$ into $\sigma_{A^K}$ in (10), recalling definition of $\rho_K^{\otimes n}$ from Lemma 8, recognizing the block diagonal structure of $\rho_K^{\otimes n}$ and $\sigma_{A^K}$, we have

$$||\sigma_{A^K} - \rho_K^{\otimes n}||_1 = \frac{1}{K} \sum_{k=1}^{K} ||\rho_K^{\otimes n} - \frac{1}{M} \sum_{m=1}^{M} \sqrt{\omega} \mu_{k,m} \sqrt{\omega} ||_1$$

$$= \frac{1}{K} \sum_{k=1}^{K} ||\rho_K^{\otimes n} - \frac{1}{M} \sum_{m=1}^{M} \beta_{c(k,m)}||_1.$$  

(23)

from the definition of $\beta_w$ in (4) and $\beta_{c(k,m)} \triangleq \beta_{c(k,m)} \otimes \cdots \otimes \beta_{c(k,m)}$. Analogously defining $\gamma_{c(k,m)} \triangleq \gamma_{c(k,m)} \otimes \cdots \otimes \gamma_{c(k,m)}$, recognizing $\alpha = \frac{1}{KM} \sum_{k=1}^{K} \sum_{m=1}^{M} \gamma_{c(k,m)}$ from (20) and $\alpha_0 = (\sum_{w \in W} p_W(w) \gamma_w)^{\otimes n}$ from (6), (21), we have

$$||\alpha_o - \alpha||_1 = ||\gamma^{\otimes n} - \frac{1}{KM} \sum_{k,m} \gamma_{c(k,m)}||_1.$$  

(24)

where we have let $\gamma = \sum_{w \in W} p_W(w) \gamma_w$. Noting $\rho = \sum_{w} p_W(w) \beta_{w}$, one recognizes similarity in the RHSs (23) and (24). Indeed, they are instances of the following QCL [11, Sec. 17.4], for which we provide a new proof. The reader is also referred to [10] for another proof of QCL.

**Lemma 1.** Suppose $p_X(\cdot)$ is a PMF on a finite set $\mathcal{X}$, $s_x \in D(\mathcal{H}) : x \in \mathcal{X}$ and $s = \sum_x p_X(x) s_x \in D(\mathcal{H})$. Suppose the $2^{nR}$ elements of $A = (X^n(1), \cdots, X^n(2^{nR}))$ are identically distributed according to $P(X^n(i) = x^n) = p_X^n(x^n) \forall x^n \in \mathcal{X}^n$, $\forall i \in \{2^{nR}\}$, and pairwise independent, then

$$\mathbb{E}_P \{||s^{\otimes n} - (s - A)||_1\} \leq \exp \left\{ -\frac{n}{2} (R - \chi(s_x; p_X : \mathcal{X})) \right\}$$  

(25)

where $s(A) \triangleq \frac{1}{2^n} \sum_{m=1}^{2^n} s_X^n(m)$. In particular, there exists a map $c : [2^{nR_0}] \times [2^{nR_1}] \to \mathcal{X}^n$ such that

$$\frac{1}{2^{nR_0}} \sum_{k=1}^{2^{nR_0}} ||s^{\otimes n} - \frac{1}{2^{nR_1}} \sum_{m=1}^{2^{nR_1}} s_{c(k,m)}||_1 \leq 2^{-\frac{nr}{2}}.$$  

(26)

if $R_1 > \chi(s_x; p_X : \mathcal{X}) + 2\eta$.

**Remark 5.** Lem. 1 yields an achievability of Thm. 1 if one chooses (i) $R_0 = \log K + R_1 = \log M + R_1$ to bound (23) and (ii) $R_0 = 0$ and $R_1 = \frac{\log K M}{n}$ to bound (24).

**Proof of Lem. 1** In the interest of brevity, we only elaborate on the key new steps and how it differs from earlier proofs [11, Sec. 17.4]. Let

$$s_x = \sum_y \gamma_{y|x} f_{y|x} : x \in \mathcal{X}, s = \sum_y p_Y(y) f_y$$

be SCDS, $\pi_x^n \triangleq \pi_{x^n,p_X \gamma_{Y|X}} \mathbf{1}_{x^n \in T_k(p_X)}$ be a conditional typical projector of $s_x^n$ and $\pi^n \triangleq \pi_{p_X^n}$ the (unconditional) typical projector of $s$. For $a = (x^n(m) : m \in \{2^{nR}\})$, let

$$s(a) \triangleq \sum_{m=1}^{2^{nR}} s_{x^n(m)}, w(a) \triangleq \sum_{m=1}^{2^{nR}} \pi^n_x s_{x^n(m)} \pi^n_{x^n(m)} s_{x^n(m)} \pi^n$$

$$w \triangleq \sum_{x^n} p_X^n(x^n) w(a) \pi^n_{x^n}, s_x^{n, \pi}, \pi^n$$ and note $w = \mathbb{E}_P \{w(A)\}$.  

(27)

Let $s(A), w(A)$ be corresponding random quantities. The quantity of interest

$$\mathbb{E}_P \{||s(A) - s^{\otimes n}||_1\} \leq T_1 + T_2 + T_3$$  

(28)

$T_1 = \mathbb{E}_P \{||s(A) - w(A)||_1\}, T_2 = \mathbb{E}_P \{||w(A) - w||_1\}, T_3 = \mathbb{E}_P \{||w - s^{\otimes n}||_1\}$. $T_1, T_3$ are handled in a straightforward sequence of arguments leveraging (i) properties of conditional and unconditional typicality projectors, (ii) gentle operator

\footnote{Note that $\pi_x^n = 0$ if $x^n$ is not typical}
lemma, (iii) and the bound $||AB||_1 \leq ||A||_1||B||_\infty$. These steps can be verified at [11] Sec. 17.4.3. We address the crucial term $T_2$ that yields the rate bound and this is where we differ from earlier proofs. While the proof in [11] Sec. 17.4] relies on the Ahlswede-Winter operator Chernoff bound [2], our proof takes a clue from Cuff’s [13] Lem. 19 ‘root-variance’ technique. Letting $v(A) \triangleq w(A) - \mathbb{E}_P[w(A)]$, we have

$$T_2 = \mathbb{E}_P[||v(A)||_1] = \mathbb{E}_P[\text{tr}\{\sqrt{(v(A))^tv(A)}\}]$$

(29)

$$= \text{tr}[\mathbb{E}_P[\sqrt{(v(A))^tv(A)}]] \leq \sqrt{\mathbb{E}_P[(v(A))^tv(A)]}$$

(30)

where (29) follows from definition of trace, (30) from linearity of trace and the operator concavity [12, Thm. 2.6] of the square root function (a consequence of the Löwner-Heinz theorem [12, Thm. 2.6]). Substituting $w(A), \mathbb{E}_P[w(A)]$, suppressing $\eta$, expanding out the expectation and following a sequence of standard steps, yields

$$\mathbb{E}_P[(v(A))^tv(A)] \leq \sum_{x^n} p_X(x^n) p_{\pi_{x^n}} s_{x^n} \pi_{s_{x^n}} \pi_{s_{x^n}} p_{x^n} \sum_n \frac{p_X(x^n)p_X(\hat{x}^n)\pi_{x^n} s_{x^n} \pi_{x^n} s_{x^n} \pi_{x^n}}{2^nR}$$

$$- 2^nR \sum_{x^n, \hat{x}^n, \hat{x}^n \neq x^n} \frac{p_X(x^n)p_X(\hat{x}^n)\pi_{x^n} s_{x^n} \pi_{x^n} s_{x^n} \pi_{x^n}}{2^nR}$$

$$\leq \sum_{x^n} \frac{p_X(x^n)p_{\pi_{x^n}} s_{x^n} \pi_{s_{x^n}} \pi_{s_{x^n}} \pi_{x^n} s_{x^n} \pi_{x^n}}{2^nR}$$

(31)

$$= \sum_{x^n \in \mathcal{T}_u(p_X)} \frac{p_X(x^n)p_{\pi_{x^n}} s_{x^n} \pi_{s_{x^n}} \pi_{s_{x^n}} \pi_{x^n} s_{x^n} \pi_{x^n}}{2^nR}$$

(32)

$$\leq \sum_{x^n \in \mathcal{T}_u(p_X)} \frac{p_X(x^n)p_{\pi_{x^n}} s_{x^n} \pi_{s_{x^n}} \pi_{s_{x^n}} \pi_{x^n} s_{x^n} \pi_{x^n}}{2^n(R + \sum_a p_X(a)S(a))}$$

(33)

$$\leq \sum_{x^n \in \mathcal{T}_u(p_X)} \frac{p_X(x^n)p_{\pi_{x^n}} s_{x^n} \pi_{s_{x^n}} \pi_{s_{x^n}} \pi_{x^n} s_{x^n} \pi_{x^n}}{2^n(R + \sum_a p_X(a)S(a))}$$

(34)

$$\leq \pi 2^{-n(R + \sum_a p_X(a)S(a)) - \frac{1}{2}(\sum_a p_X(a)S(a))}$$

(35)

where (31) follows from commutativity of $s_{x^n}$ and its conditional typicality projector $\pi_{x^n}$ and $\pi \leq R^{\otimes n}$, (32) from earlier definition of conditional typicality projector being 0 on non-typical elements, (33) from [11] Prop. 15.2.6, (34) follows from commutativity of $s_{x^n}$ and its conditional frequency projector $p_{x^n}$ and $\pi_{x^n} \leq R^{\otimes n}$, expanding the sum to the entire set and (35) from [11] Prop. 15.1.3. From operator monotonicity of square root function [12, Thm. 2.6], we can substitute RHS of (35) under the square root in (30) as an upper bound. Following through (29) - (35), we have

$$T_2 \leq \text{tr}\left\{\sqrt{2^n(R + \sum_a p_X(a)S(a)) + S(\sum_a p_X(a)s(a))}\right\}$$

$$= 2^nS(\sum_a p_X(a)s(a))$$

(36)

where (36) follows from the bound $\text{tr}(\sqrt{\pi}) = \text{tr}(\pi) \leq 2^nS(\sum_a p_X(a)s(a))$ on the trace of the unconditional projector [11] Prop. 15.1.2, thus completing the proof.

IV. SIMULATION VIA ALGEBRAICALLY CLOSED POVMs

We briefly revisit the simulated POVM in Sec. III. Alice possesses POVM operators $\tilde{\theta} \triangleq \{\theta_{k,m} : 1 \leq K, 0 \leq m \leq M\}$ and Bob has a corresponding table $\tilde{c} \triangleq \{w^n(k, m) : 1 \leq K, 0 \leq m \leq M\}$. On observing (common) random bits $k^*$, Alice performs POVM $\{\theta_{k^*, m} : 0 \leq m \leq M\}$. Bob chooses $w^n(k^*, m^*)$, where $m^*$ is the POVM outcome, and evolves this through the stochastic matrix $p_{w^n}(\cdot | w^n(k^*, m^*))$. The question we ask in this section is whether the table $\tilde{c}$ and the corresponding POVMs $\tilde{\theta}$ be endowed with certain algebraic structure? Specifically, suppose $\mathcal{V}$ is a finite field or a group, can the table $\tilde{c}$ be chosen to be a coset of a linear code?

Optimal compression requires that each outcome $w^n(k, m)$ is an equally likely POVM outcome, forcing the entries of the table to be $p_{w^n} -$typical, where, we recall $p_{w^n}(w) = \text{tr}(\rho_{w^n})$. Requiring table entries to be algebraically closed forces us to choose entries $w^n(k, m)$ that are not $p_{w^n} -$typical. Non-$p_{w^n} -$typical outcomes are extremely rare lending the simulation protocol sub-optimal (in terms of communication rates).
Does this imply that the table cannot have algebraic properties? There is one way to get around this obstacle. Can we add redundant operators and corresponding table entries in a controlled manner to guarantee algebraic closure, yet not suffer on the communication and common randomness rate? The idea is to enlarge the table into a third dimension. Let \( \hat{c} : [K] \times [M] \times [B] \to \mathcal{W}^n \) be a map,

\[
\hat{S}_{k} \triangleq \sum_{(b,m) \in \mathcal{B}[K] \times [M]} \sqrt{\omega_{k,m,b} \omega_{k}^{-\frac{1}{2}}} \hat{\theta}_{k,m,b} \triangleq \frac{\hat{S}_{k}^{-\frac{1}{2}} \sqrt{\omega_{k,m,b} \omega_{k}^{-\frac{1}{2}}}}{\text{tr}(\omega_{k,m,b})},
\]

and \( \hat{\theta} \triangleq \{ \hat{\theta}_{k,m,b} \mid |k|/k : 1 \leq k \leq K, 1 \leq m \leq M, 1 \leq b \leq B \} \) be a POVM. On observing random bits \( k^* \), Alice performs POVM \( \hat{\theta}_{k^*} \) and simulates the POVM outcome. Since the POVM outcome is \( \hat{p}_{w} \)-typical, then Bob can evolve \( u^n(k^*, m^*, b^*(k^*, m^*)) \) through the stochastic matrix \( p^0_{w|w}(|w^n(k^*, m^*, b^*(k^*, m^*))|) \) and simulate the POVM outcome.

This provides us with the clue. For simplicity, let us assume \( W = F_2 \) to be the binary field. In order to prove achievability in Lem. 1, we picked entries of the table \( c \) independently and randomly with distribution \( p^0_{w} \). Instead, suppose we let table \( \hat{C} \) be of size \( 2^{2r+\beta} \times |\hat{K}| \) whose entries are picked uniformly independently from \( F_2 = \{0, 1\} \), then its range is a random linear code with uniform \( pairwise independent \) codewords. Common randomness specifies \( c \) bits. For each choice of these \( c \) bits, we build a POVM with \( 2^r \times \beta \) operators. Only the \( r \) \((r + \beta)\) outcome bits is communicated to Bob. Having been provided \( c + r \) bits, Bob looks for a unique collection of \( \beta \) bits for which the corresponding entry in \( \hat{C} \) is \( \hat{p}_{w} \)-typical.

Since the entries of \( \hat{C} \) are uniformly distributed, the expected number of \( \hat{p}_{w} \)-typical codewords in any collection of \( 2^\beta \) entries is

\[
\frac{2^\beta [\hat{p}_{w}]}{2^{2r}} = 2^{-n(1 - H(p_{w}) - \beta)}. \tag{37}
\]

Therefore, so long as \( \beta < 1 - H(p_{w}) \), it is natural to expect that Bob will find just one \( \hat{p}_{w} \)-typical entry whose index agrees with the \( c + r \) bits he has been provided. This suggests that, if we can enlarge our table by a factor not greater than \( 2^{n(1 - H(p_{w}))} \) and prove a QCL analogous to Lem. 1 but with entries of the table \( A \) uniformly chosen from \( F_2^n \), instead of \( p_{w} \), then we can perform POVM simulation with an ‘algebraically closed POVM’. This is indeed true. A proof of Lem. 2 is similar to proof of Lem. 1 and is provided in an enlarged version of this preprint. We follow this up with a final statement on the existence of structured POVMs for simulation. A proof is provided in an enlarged version of this preprint.

**Lemma 2.** Suppose \( p_X(\cdot) \) is a PMF on a finite field \( X = \mathcal{F}_q \) of size \( q \), \( s_x \in \mathcal{D}(\mathcal{H}_X) : x \in X \) and \( s = \sum_x p_X(x) s_x \in \mathcal{D}(\mathcal{H}_X) \). Suppose the \( q^{nR} \) elements of \( A = \{ X^n(1), \ldots, X^n(q^{nR}) \} \) are uniformly distributed and pairwise independent, then

\[
\mathbb{E}_X \left[ ||s \otimes n - s(A)||_1 \right] \leq \exp \left\{ -\frac{n(1 - H(p_{X}) - \log q + H(p_{X}))}{2} \right\} \text{ where } s(A) \triangleq \sum_{m=1}^{q^{nR}} q^n p_X(X^n(m)) s_{X^n(m)}. \tag{38}
\]

In particular, \( \exists \ a \map c : [q^{nR}] \times [q^{nR}] \to X^n \) whose range is a coset such that

\[
|q^{nR} | \sum_{k=1}^{q^{nR}} ||s \otimes n - \frac{1}{q^{nR}} \sum_{m=1}^{q^{nR}} q^n p_X(c(k,m)) s_{c(k,m)}||_1 \leq 2^{-n\eta}
\]

if \( R > \chi(s_x : X) + \log q - H(p_X) + 2\eta \).

**Theorem 2.** Let \( \rho \in \mathcal{D}(\mathcal{H}_A), \lambda_Y \in \mathcal{M}(\mathcal{H}_A, \mathcal{Y}) \) and \( (\mathcal{W} = \mathcal{F}_q, \mu_W, p_{W|W}) \in C(\rho, \lambda_Y) \), where \( W = \mathcal{F}_q \) is a finite field with \( q \) elements. Suppose \( c+r+\beta > \mathcal{R}(\mathcal{W} = \mathcal{F}_q, \mu_W, p_{W|W}) + \log q - H(p_{W}) \) and \( r+\beta > H(p_{W} = \mathcal{W} = \mathcal{F}_q, \mu_W, p_{W|W}) + \log q - H(p_{W}) \), where \( p_{W}(w) = \text{tr}(\rho_{W}w) : w \in W \), then there exists \( \exists \ c : [q^{nR}] \times [q^{nr}] \times [q^{n\beta}] \to W^n \) whose range is a coset for which the POVM \( \hat{\theta} \triangleq \{ \hat{\theta}_{k,m,b} \mid |k|/k : 1 \leq k \leq n^c, 1 \leq m \leq n^r, 1 \leq b \leq n^{\beta} \} \) defined through (37) simulates POVM \( \lambda_Y \) on \( \rho \) with communication cost \( \mathcal{R}(\mathcal{W} : W, \mu, p_{W|W}) \).

Remark 6. To prove achievability of Thm. 2 using Lem. 2 one needs to pick a random linear code with uniform and IID generator matrix indices for \( \beta \) is just smaller than but within \( \eta \) of \( \log q - H(p_{W}) \). The rest of the steps are identical to Proof of Thm. 1.

**APPENDIX A**

**POVM Simulation with Respect to Purifications**

We recall notation that is adopted throughout the article. We let \( \{ b_1, \ldots, b_d \} \subseteq H_A \) denote an arbitrary but fixed orthonormal basis and \( H_X = H_A \). For any \( |x| = \sum_{i=1}^d \alpha_i |b_i \rangle \in H_A \), we let \( |x| = \sum_{i=1}^d \alpha_i |b_i \rangle \in H_A \) denote the complex conjugation with respect to orthonormal basis \( \{ |b_1 \rangle, \ldots, |b_d \rangle \} \). Similarly, for \( A \in \mathcal{R}(\mathcal{H}) \) with a spectral decomposition \( A = \sum_{i=1}^d \alpha_i |a_i \rangle \langle a_i | \), we let \( A^* \triangleq A^t \triangleq \sum_{i=1}^d \alpha_i |a_i \rangle \langle a_i | \). We collate simple facts regarding these in the following.

\footnote{This notation can be justified by the fact that the matrices of \( A \) and \( A^* \) in the orthonormal basis \( \{|b_1 \rangle, \ldots, |b_d \rangle\} \subseteq H_A \) are complex conjugates or transposes of each other.}
Lemma 3. 1) If \( \{ |u_1\rangle, \ldots, |u_d\rangle \} \subseteq \mathcal{H}_A \) is an orthonormal basis, then \( \{ |\pi_1\rangle, \ldots, |\pi_d\rangle \} \subseteq \mathcal{H}_A \) is an orthonormal basis.

2) For any \( |x\rangle, |y\rangle \in \mathcal{H}_A \), we have \( \langle x|y \rangle = \langle y|x \rangle \).

3) Suppose \( \sigma \in D(\mathcal{H}_A) \) has a spectral decomposition \( \sigma = \sum_{i=1}^{d} \gamma_i |u_i\rangle \langle u_i| \), then \( \phi_{\sigma} \triangleq \sum_{i=1}^{d} \sqrt{\gamma_i} |\pi_i\rangle \otimes |u_i\rangle \in \mathcal{H}_X \otimes \mathcal{H}_A \) is a purification of \( \sigma \). Analogously, \( \phi_{\sigma^{\otimes n}} = \sum_{i=1}^{d} \sqrt{\gamma_i} |\pi_i\rangle \otimes \cdots \otimes |\pi_i\rangle \otimes |u_i\rangle \cdots \otimes |u_i\rangle \) is a purification of \( \sigma^{\otimes n} \), and

4) Suppose \( A, C \in \mathcal{R}(\mathcal{H}_A) \) and \( \{ |u_1\rangle, \ldots, |u_d\rangle \} \subseteq \mathcal{H}_A \) is an orthonormal basis, then \( \langle u_j|ACA|u_i\rangle = \langle \pi_i|(ACA)^*|\pi_j\rangle \) for all \( i, j \in [d] \) and hence

\[
\sum_{i=1}^{d} \langle u_j|ACA|u_i\rangle \langle \pi_i|\pi_j\rangle = (ACA)^* = (ACA)^t.
\]

5) For any \( A, C \in \mathcal{R}(\mathcal{H}) \), we have \( (A - C)^* = A^* - C^* \) and \( \| A \|_1 = \| A^* \|_1 = \| A \|_1 \).

Proof. For \( i \in [d] \), let \( |u_i\rangle = \sum_{k=1}^{d} \alpha_{ki} |b_k\rangle \) and \( |\pi_i\rangle = \sum_{k=1}^{d} \alpha_{ki}^* |b_k\rangle \). Since \( \{ |u_1\rangle, \ldots, |u_d\rangle \} \subseteq \mathcal{H}_A \) is an orthonormal basis, we have \( \delta_{ji} = \langle u_j|u_i\rangle = \sum_{k=1}^{d} \alpha_{ki} \alpha_{jk} \). We therefore have \( \langle \pi_j|\pi_i\rangle = \sum_{k=1}^{d} \alpha_{ki}^* \alpha_{kj} = \delta_{ji} \).

Suppose \( |x\rangle = \sum_{i=1}^{d} \alpha_i |b_i\rangle \) and \( |y\rangle = \sum_{i=1}^{d} \beta_i |b_i\rangle \), then \( \langle x| = \sum_{i=1}^{d} \alpha_i \phi^*_i = \sum_{i=1}^{d} \beta_i \gamma_i \) and \( \langle y| = \sum_{i=1}^{d} \beta_i \phi_i \). Therefore, \( \langle x|y \rangle = \sum_{i=1}^{d} \alpha_i \beta_i = \langle x|y \rangle = \langle y|x \rangle \).

From 1), we have \( \langle \phi_{\sigma} \phi_{\sigma}^* \rangle = \sum_{i=1}^{d} \sum_{j=1}^{d} \sqrt{\gamma_i} \sqrt{\gamma_j} \langle \pi_i|\pi_j\rangle \langle u_j|u_i\rangle = \sum_{i=1}^{d} \sum_{j=1}^{d} \sqrt{\gamma_i \gamma_j} \delta_{ji} = \sum_{i=1}^{d} \gamma_i \delta_{ji} = 1 \) and \( \text{tr}_{\mathcal{H}_A}(\langle \phi_{\sigma} \phi_{\sigma}^* \rangle) = \sum_{i=1}^{d} \sum_{j=1}^{d} \sqrt{\gamma_i} \sqrt{\gamma_j} \langle \pi_i|\pi_j\rangle \langle u_j|u_i\rangle = \sum_{i=1}^{d} \sum_{j=1}^{d} \sqrt{\gamma_i \gamma_j} \delta_{ji} \sum_{j=1}^{d} \gamma_j = 1 \) and \( \mathcal{H} \subseteq \mathcal{H}_A \).

The next statement in regards to \( \sigma^{\otimes n} \) can be verified analogously.

With regard to the fourth assertion, clearly \( A, C \in \mathcal{R}(\mathcal{H}) \) implies \( ACA \in \mathcal{R}(\mathcal{H}) \). Let \( ACA = \sum_{i=1}^{d} \chi_i |x_i\rangle |x_i\rangle \) with \( \chi_i \in \mathbb{R} \) for \( i \in [d] \). We have \( (ACA)^* = (ACA)^t = \sum_{i=1}^{d} \chi_i |\pi_i\rangle |\pi_i\rangle \) and \( \nu_{st} = \langle u_s|ACA|u_t\rangle \). Substituting spectral decomposition of \( ACA \) and using the third assertion, we have

\[
\nu_{st} = \langle u_s|ACA|u_t\rangle = \sum_{i=1}^{d} \chi_i \langle u_s|x_i\rangle \langle x_i|u_t\rangle = \sum_{i=1}^{d} \chi_i \langle \pi_i|\pi_s\rangle \langle \pi_t|\pi_i\rangle = \sum_{i=1}^{d} \chi_i \langle \pi_i|\pi_i\rangle = \langle \pi_i|(ACA)^*|\pi_s\rangle = \langle \pi_i|(ACA)^t|\pi_s\rangle.
\]

From the first assertion, equality of the second and the last terms in (39), we have

\[
(ACA)^* = (ACA)^t = \sum_{i=1}^{d} \langle \pi_i|(ACA)^*|\pi_j\rangle |\pi_i\rangle |\pi_j\rangle = \sum_{i=1}^{d} \langle \pi_i|(ACA)^t|\pi_j\rangle |\pi_i\rangle |\pi_j\rangle = \sum_{i=1}^{d} \langle u_j|ACA|u_i\rangle |\pi_i\rangle |\pi_j\rangle
\]

which proves our claimed assertion.

Suppose \( A, C \in \mathcal{R}(\mathcal{H}) \), \( A = \sum_{i=1}^{d} \chi_i |x_i\rangle |x_i\rangle \) and \( C = \sum_{i=1}^{d} \epsilon_i |y_i\rangle |y_i\rangle \) with \( \chi_i, \epsilon_i \in \mathbb{R} \) for \( i \in [d] \). Let \( A - C \in \mathcal{R}(\mathcal{H}) \) have spectral decomposition \( A - C = \sum_{i=1}^{d} \mu_i |w_i\rangle |w_i\rangle \) with \( \mu_i \in \mathbb{R} \) for \( i \in [d] \). We also observe

\[
\nu_i = \langle u_i|A - C|w_i\rangle = \langle u_i|A|w_i\rangle - \langle u_i|C|w_i\rangle = \sum_{i=1}^{d} \chi_i \langle w_i|x_i\rangle \langle x_i|u_i\rangle - \sum_{i=1}^{d} \epsilon_i \langle w_i|y_i\rangle \langle y_i|w_i\rangle = \sum_{i=1}^{d} \chi_i \langle \pi_i|\pi_i\rangle \langle \pi_i|\pi_i\rangle - \sum_{i=1}^{d} \epsilon_i \langle \pi_i|\pi_i\rangle \langle \pi_i|\pi_i\rangle = \langle \pi_i|A^s|\pi_i\rangle - \langle \pi_i|C^s|\pi_i\rangle.
\]

We now have \( (A - C)^* = \sum_{i=1}^{d} \mu_i |w_i\rangle |w_i\rangle = \sum_{i=1}^{d} \langle \pi_i|A^s|\pi_i\rangle |\pi_i\rangle |\pi_i\rangle - \langle \pi_i|C^s|\pi_i\rangle |\pi_i\rangle |\pi_i\rangle = A^* - C^* \). In regards to the assertion relating norms, we have \( A^* = A^t = \sum_{i=1}^{d} |\pi_i|\langle \pi_i|\pi_i\rangle |\pi_i\rangle \langle \pi_i|\pi_i\rangle \) from the spectral decomposition of \( A \), and \( U_X \triangleq \sum_{i=1}^{d} |\pi_i\rangle \langle x_i| \) is unitary. Since trace norm is invariant under multiplication by isometries [11], Prop. 9.1.4, we have

\[
\| A^* \|_1 = \| U_X^t A^t U_X \|_1 = \left\| \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{l=1}^{d} \langle x_j|\pi_i\rangle \chi_i \langle \pi_i|\pi_l\rangle \langle \pi_l|\pi_i\rangle \langle \pi_i|x_i\rangle \right\|_1 = \left\| \sum_{i=1}^{d} \chi_i |x_i\rangle \langle x_i| \right\|_1 = \| A \|_1.
\]

The following lemma enables us relate simulation of POVMs as stated in Defn. [4] to simulation of POVMs with respect to purifications. To begin with, we recall our notation. Associated with any POVM \( \lambda \in \mathcal{P}(\mathcal{H}) \) is a Hilbert space \( \mathcal{H}_Y \triangleq \text{span}\{ |y\rangle : y \in \mathcal{Y} \} \) with \( \langle y|y \rangle = \delta_{yy} \) and the CPTP map \( \delta^\lambda : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_Y) \), defined as \( \delta^\lambda(s) = \sum_{y \in \mathcal{Y}} \text{tr}(s \lambda_y) |y\rangle \langle y| \).

\[\square\]
Lemma 4. Suppose $\rho \in \mathcal{D}(\mathcal{H}_A)$, $\lambda_3, \theta_3 \in \mathcal{M}(\mathcal{H}_A, \mathcal{Y})$ are POVMs and $\eta > 0$. Then $\eta$-simulates $\lambda_3$ on $\rho$ if

$$\|(i_X \otimes \mathcal{E}^\lambda)(|\phi_\rho\rangle\langle\phi_\rho|) - (i_X \otimes \mathcal{E}^\theta)(|\phi_\rho\rangle\langle\phi_\rho|)\|_1 \leq \eta. \tag{40}$$

Proof. Our proof is identical to that in [7] and is provided here only for completeness. Let $(\sigma_k, p_K : K)$ be an ensemble with average $\rho$. Let $\rho = \sum_{i=1}^d \gamma_i |u_i\rangle\langle u_i|$ be a spectral decomposition and we have $|\phi_\rho\rangle = \sum_{i=1}^d \sqrt{\gamma_i} |u_i\rangle \otimes |u_i\rangle \in \mathcal{H}_X \otimes \mathcal{H}_A$. We let $S_k \equiv \rho^{-\frac{1}{2}} p_K(k) \sigma_k \rho^{-\frac{1}{2}}$ and we note $S_k \in \mathcal{P}(\mathcal{H}_A)$ and $A_y = A_y^+ - A_y^- \in \mathcal{P}(\mathcal{H})$. We are led to

$$\sum_{k \in \mathcal{K}} \sum_{y \in \mathcal{Y}} \text{tr}(\lambda_y p_K(k) \sigma_k) - \text{tr}(\theta_y p_K(k) \sigma_k) = \sum_{k \in \mathcal{K}} \sum_{y \in \mathcal{Y}} \text{tr}(\rho \lambda_y \rho \frac{1}{2} \rho \frac{1}{2} p_K(k) \sigma_k) - \text{tr}(\rho \theta_y \rho \frac{1}{2} \rho \frac{1}{2} p_K(k) \sigma_k) \geq 0 \tag{41}$$

where the equalities in (41), (42) follows from cyclicality, linearity of trace operator, the inequality in (43) follows from the triangular inequality, the last equality in (43) follows from the fact that $A, B \in \mathcal{P}(\mathcal{H})$ implies $\text{tr}(AB) \geq 0$. We have that $\sum_{k \in \mathcal{K}} S_k = I_{\mathcal{H}_A}$, the second equality in (44) follows from (14) Eqn. 1.8 and 1.15, and finally the last equality in (44) follows from the fact that the trace norm of a block diagonal matrix is the sum of the trace norm of each of the blocks. We shall now evaluate the LHS in (40) and prove that the same is equal to the RHS of (44). Towards the end, we have

$$\|(i_X \otimes \mathcal{E}^\lambda)(|\phi_\rho\rangle\langle\phi_\rho|) - (i_X \otimes \mathcal{E}^\theta)(|\phi_\rho\rangle\langle\phi_\rho|)\|_1 = \sum_{y \in \mathcal{Y}} \sum_{i=1}^d \sum_{j=1}^d \sqrt{\gamma_i} \gamma_j \langle u_j | \lambda_y u_i \rangle |\pi_i\rangle |\pi_j\rangle \otimes |u_i\rangle |u_j\rangle = \sum_{y \in \mathcal{Y}} \sum_{i=1}^d \sum_{j=1}^d \langle u_j | \sqrt{\rho} \lambda_y \sqrt{\rho} |u_i\rangle |\pi_i\rangle |\pi_j\rangle \otimes |y\rangle |y\rangle = \sum_{y \in \mathcal{Y}} \langle \sqrt{\rho} \lambda_y \sqrt{\rho} |y\rangle |y\rangle - \langle \sqrt{\rho} \theta_y \sqrt{\rho} |y\rangle |y\rangle \tag{44}$$

where the last equality follows the fourth assertion in Lemma 4. Substituting this, (40) implies

$$\eta \geq \|(i_X \otimes \mathcal{E}^\lambda)(|\phi_\rho\rangle\langle\phi_\rho|) - (i_X \otimes \mathcal{E}^\theta)(|\phi_\rho\rangle\langle\phi_\rho|)\|_1 = \sum_{y \in \mathcal{Y}} \sum_{i=1}^d \sum_{j=1}^d \langle \sqrt{\rho} \lambda_y \sqrt{\rho} |y\rangle |y\rangle - \langle \sqrt{\rho} \theta_y \sqrt{\rho} |y\rangle |y\rangle \tag{45}$$

and

$$\eta \geq \sum_{k \in \mathcal{K}} \sum_{y \in \mathcal{Y}} \langle \text{tr}(\lambda_y p_K(k) \sigma_k) - \text{tr}(\theta_y p_K(k) \sigma_k) |y\rangle |y\rangle \tag{46}$$

where we have leveraged the fifth assertion in Lemma 3 and (41) - (44) in asserting the last inequality.}

For any $\sigma \in \mathcal{D}(\mathcal{H}_A)$, with a spectral decomposition $\sigma = \sum_{i=1}^d \gamma_i |u_i\rangle\langle u_i|$, we let $|\phi_\sigma\rangle \equiv \sum_{i=1}^d \sqrt{\gamma_i} |u_i\rangle \otimes |\pi_i\rangle$ denote a purification of $\sigma$. Analogously, we let $|\phi_{\sigma \odot \mathcal{E}}\rangle = \sum_{i=1}^d \sum_{j=1}^d \sqrt{\gamma_i \gamma_j} |u_i\rangle \otimes |\pi_i\rangle \otimes |u_j\rangle \otimes |\pi_j\rangle$. Finally, we prove that that if two quantum states $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ are close, then their canonical purifications are not too far apart. Our proof mimics the proof steps of [1] App. 14 and is provided only for self-containment.

**Lemma 5.** For any two states $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, we have

$$\| |\phi_\rho\rangle\langle\phi_\rho| - |\phi_\sigma\rangle\langle\phi_\sigma| \|_1 \leq 2\sqrt{2} \sqrt{\|\rho - \sigma\|_1}. \tag{48}$$

**Proof.** We note that $\| |\phi_\rho\rangle\langle\phi_\rho| - |\phi_\sigma\rangle\langle\phi_\sigma| \|_1 = 2\sqrt{1 - |\langle \phi_\rho | \phi_\sigma \rangle|^2}$ [11] Eqn. 9.173]. We therefore compute $\langle \phi_\rho | \phi_\sigma \rangle$. Let $\rho = \sum_{i=1}^d \gamma_i |u_i\rangle\langle u_i| \otimes |\pi_i\rangle$ be spectral decompositions and $|\phi_\rho\rangle = \sum_{i=1}^d \sqrt{\gamma_i} |u_i\rangle \otimes |\pi_i\rangle \in \mathcal{H}_X \otimes \mathcal{H}_A$. Then

$$\langle \phi_\rho | \phi_\sigma \rangle = \sum_{i, j=1}^d \gamma_i \gamma_j \langle u_i | u_j \rangle \langle \pi_i | \pi_j \rangle = \sum_{i=1}^d \gamma_i = \text{tr}(\rho) \geq 0.$$
and \(|\phi_\alpha\rangle = \sum_{j=1}^d \sqrt{\mu_j} |\varpi_j\rangle \otimes |w_j\rangle \in \mathcal{H}_X \otimes \mathcal{H}_A\) be canonical purifications. We have

\[
\langle \phi_\rho | \phi_\sigma \rangle = \sum_{i=1}^d \sum_{j=1}^d \sqrt{\gamma_i} \sqrt{\mu_j} \langle \varpi_i | \varpi_j \rangle \langle u_i | u_j \rangle = \sum_{j=1}^d \sqrt{\mu_j} \langle w_j | u_j \rangle \sum_{i=1}^d \sqrt{\gamma_i} \langle u_i | u_j \rangle \langle w_j | w_j \rangle
\]

where (53) follows from trace norm being invariant under multiplication by isometries [11, Prop. 9.1.4] and the fifth assertion which follows from a standard sequence of equalities. The RHS of (52) is therefore \(\rho\) (ii) the last equality in (50) follows from the definition of \(|\cdot|\). The last equality follows from a standard sequence of equalities. The RHS of (52) is therefore

\[
\rho\]

\[
\| (\rho \otimes \sigma^\lambda) |\alpha_\rho \rangle \rangle - (\rho \otimes \sigma^\lambda) |\alpha_\sigma \rangle \rangle \|_1 = \| (\rho \otimes \sigma^\lambda) |\alpha_\rho \rangle \rangle - (\rho \otimes \sigma^\lambda) |\alpha_\sigma \rangle \rangle \|_1.
\]

**Appendix B**

**INvariance in the choice of the purification**

It is natural to question whether we can replace the canonical purification \(\phi_\rho\) in (49) with any purification of \(\rho\). This is indeed true and we provide a proof of this statement in the following.

**Lemma 6.** Suppose \(|\omega_\rho\rangle \in \mathcal{H}_X \otimes \mathcal{H}_A\) and \(|\alpha_\rho\rangle \in \mathcal{H}_Z \otimes \mathcal{H}_A\) are two purifications of \(\rho \in \mathcal{D} (\mathcal{H}_A)\) with \(\dim (\mathcal{H}_Z) \geq \dim (\mathcal{H}_X)\). Let \(\lambda = \{\lambda_y : y \in Y\}, \theta = \{\theta_y : y \in Y\} \in \mathcal{M} (\mathcal{H}_A, Y)\) be POVMs, then

\[
\| (\rho \otimes \sigma^\lambda) |\alpha_\rho \rangle \rangle - (\rho \otimes \sigma^\lambda) |\alpha_\sigma \rangle \rangle \|_1 = \| (\rho \otimes \sigma^\lambda) |\alpha_\rho \rangle \rangle - (\rho \otimes \sigma^\lambda) |\alpha_\sigma \rangle \rangle \|_1.
\]

**Proof.** Suppose \(\rho \in \mathcal{D} (\mathcal{H}_A)\) has a spectral decomposition \(\rho = \sum_{i=1}^d \gamma_i |u_i\rangle \langle u_i|\) and \(\phi_\rho = \sum_{i=1}^d \sqrt{\gamma_i} |\varpi_i\rangle \otimes |u_i\rangle\) be the canonical purification. We shall prove (52) the stated assertion by proving equality (52) with \(|\omega_\rho\rangle = |\phi_\rho\rangle\) and a generic purification \(|\alpha_\rho\rangle\). Since any two purifications of \(\rho\) are related via an isometry acting on the reference Hilbert space [11, Thm. 5.1.1], there exists an isometry \(V : \mathcal{H}_X \rightarrow \mathcal{H}_Z\) such that \(|\alpha_\rho\rangle = (V \otimes I_A) |\phi_\rho\rangle = \sum_{i=1}^d \sqrt{\gamma_i} |\varpi_i\rangle \otimes |u_i\rangle\). we therefore have

\[
(i \otimes \sigma^\lambda) |\alpha_\rho \rangle \rangle = (i \otimes \sigma^\lambda) \left( \sum_{y \in Y} \sum_{i=1}^d \sqrt{\gamma_i} \langle \varpi_i | \varpi_j \rangle \langle u_i | u_j \rangle \right) = \sum_{y \in Y} \sum_{i=1}^d \sqrt{\gamma_i} \langle \varpi_i | \varpi_j \rangle \langle u_i | u_j \rangle \langle V \varpi_i | V \varpi_j \rangle \langle V^\dagger \otimes \rho \rangle \langle u_i | u_j \rangle \langle y | y \rangle
\]

which follows from a standard sequence of equalities. The RHS of (53) is therefore

\[
\| (i \otimes \sigma^\lambda) |\alpha_\rho \rangle \rangle - (i \otimes \sigma^\lambda) |\alpha_\sigma \rangle \rangle \|_1 = \| (i \otimes \sigma^\lambda) \left( \sum_{y \in Y} \langle V \varpi_i | V \varpi_j \rangle \langle V^\dagger \otimes \rho \rangle \langle u_i | u_j \rangle \langle y | y \rangle - \sum_{y \in Y} \langle V \varpi_i | V \varpi_j \rangle \langle V^\dagger \otimes \rho \rangle \langle u_i | u_j \rangle \langle y | y \rangle \right) \|_1
\]

\[
= \sum_{y \in Y} \| V \left( \sqrt{\rho \lambda} \otimes \rho \right) \langle V^\dagger \otimes \rho \rangle \langle u_i | u_j \rangle \langle y | y \rangle \|_1
\]

where (53) follows from trace norm being invariant under multiplication by isometries [11, Prop. 9.1.4] and the fifth assertion of Lemma 3 while the last equality follows from the equalities in (45), (46).
APPENDIX C
CHARACTERIZATION OF $\alpha_5$ VIA CPTP MAP $\mathcal{E}_{\text{sim}}$: PROOF OF LEMMA 8

We shall evaluate $\mathcal{E}_{\text{sim}}(\phi_{\rho}^{\otimes n})$, where $\mathcal{E}_{\text{sim}} \triangleq (i_X^n \otimes \text{tr}_K \otimes i_Y^n) \circ (i_X^n \otimes i_{H_K} \otimes \mathcal{D}) \circ (i_X^n \otimes i_{H_K} \otimes \mathcal{D}')$ and prove the desired equivalence. Let $\rho = \sum_{i=1}^{d_A} \gamma_i |u_i \rangle \langle u_i|$ be a spectral decomposition and

$$\phi_{\rho}^{\otimes n} = \sum_{i_1, i_2, \ldots, i_n} \frac{\sqrt{\gamma_{i_1} \cdots \gamma_{i_n}}}{\sqrt{K}} |\pi_{i_1} \otimes \cdots \otimes \pi_{i_n} \rangle \otimes |\hat{k} \rangle \otimes |u_{i_1} \otimes \cdots \otimes u_{i_n} \rangle \otimes |\hat{k} \rangle$$ (55)

be the canonical purification of $\rho_K^\otimes = \frac{1}{K} \sum_{k=1}^{K} \rho^{\otimes n} \otimes |k \rangle \langle k|$. Since $\theta \triangleq \{ \theta_{k,m} \triangleq |k \rangle \langle k| : (k,m) \in [K] \times [M] \} \in \mathcal{M}(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_K, [K] \times [M])$, we have

$$\alpha_{\theta} \triangleq (i_X^n \otimes i_{H_K} \otimes \mathcal{D})(\phi_{\rho}^{\otimes n}) = \sum_{i_1, i_2, \ldots, i_n} \frac{\sqrt{\gamma_{i_1} \cdots \gamma_{i_n}}}{\sqrt{K}} |\pi_{i_1} \otimes \cdots \otimes \pi_{i_n} \rangle \otimes |\hat{k} \rangle \otimes |k \rangle \otimes |m \rangle$$

$$= \sum_{i_1, i_2, \ldots, i_n} \frac{\sqrt{\gamma_{i_1} \cdots \gamma_{i_n}}}{\sqrt{K}} |\pi_{i_1} \otimes \cdots \otimes \pi_{i_n} \rangle \otimes |\hat{k} \rangle \otimes |k \rangle \otimes |m \rangle$$

$$= \sum_{i_1, i_2, \ldots, i_n} \frac{\sqrt{\gamma_{i_1} \cdots \gamma_{i_n}}}{\sqrt{K}} |\pi_{i_1} \otimes \cdots \otimes \pi_{i_n} \rangle \otimes |\hat{k} \rangle \otimes |k \rangle \otimes |m \rangle$$

and hence (57) where we have used the above stated spectral decomposition for $\rho$ and $\omega$ in (56), the fact stated in Part 4 of Lemma 8 to arrive at (57), and the last equality follows from the fact that $\Delta_{y^n|k,m}, K$ are real valued scalars and the definition of $\mathcal{D} \circ \theta_e$ in (59) associated with the simulation protocol in question.

APPENDIX D
EQUIVALENCE OF FORMULATIONS

A. Alternate Problem Statement

In this appendix, we shall justify the formulation stated in Sec. I-B. Specifically, we begin with an alternate formulation in Defn. 4 and prove equivalence of the same.

Defn. 4. Let $\rho \in \mathcal{D}(\mathcal{H}_A)$, $\lambda_Y = \{ \lambda_y : y \in Y \} \in \mathcal{M}(\mathcal{H}_A, \mathcal{Y})$, $\theta_Y = \{ \theta_y : y \in Y \} \in \mathcal{M}(\mathcal{H}_A, \mathcal{Y})$ be POVMs and $\eta > 0$. We say $\eta$-simulates $\lambda_Y$ on $\rho$ if

$$\sum_{k \in K, y \in Y} |p_K(k) \text{tr}(\lambda_y \sigma_k) - p_K(k) \text{tr}(\theta_y \sigma_k)| \leq \eta$$

for every ensemble $(\sigma_k, p_K, K)$ with average $\rho$. 
In order to simulate a POVM \( \lambda_Y \) on \( \rho \), one can utilize common randomness that is independent of the quantum system and available at both encoder and decoder. A simulation protocol, defined in the following, formalizes the components involved that make up a POVM simulation protocol that can exploit \( C \) bits/measurement of common randomness to simulate a target POVM.

**Defn. 5.** A simulation protocol \((n, C, R, \theta, \Delta)\) for \( \lambda_Y \) consists of

1. index sets \([K], [M]\), where \( K \triangleq [2^nC] \), \( M \triangleq [2^nR] \),
2. a collection of \( K \) POVMs \( \theta_k = \{ \theta_{k,m} \in P(H_{\lambda,n}^\otimes) : m \in [M] \} \in \mathcal{M}(H_{A,n}^\otimes, [k] \times [M]) \) for \( k \in [K] \), and the associated POVM \( \theta \triangleq \{ \theta_{k,m} \otimes |k\rangle \langle k| : (k,m) \in [K] \times [M] \} \in \mathcal{M}(H_{A,n}^\otimes \otimes H, [K] \times [M]) \), where \( H_K = \text{span}\{|k\rangle : k \in K\} \)
3. a sequence of simulation protocols \( A(n,C,R,\theta,\Delta) \) : \( n \geq 1 \) that make up a POVM simulation protocol that can exploit \( C \) bits/measurement of common randomness to simulate a target POVM.

**Proof.** The notations \( Y \eta \) denote the closure of the set. In the following, we recall an equivalent formulation involving purifications.

For such a simulation protocol \((n, C, R, \theta, \Delta)\), let \( \theta_e = \{ \theta_{m,n} \in P(H_{\lambda,n}) : (k,m) \in [K] \times [M] \} \in \mathcal{M}(H_{A,n}^\otimes, [K] \times [M]) \) and

\[
\Delta \circ \theta_e \triangleq \frac{1}{K} \sum_{k \in [K]} \sum_{m \in [M]} \Delta_{y^n|k,m} \theta_{k,m} \in \mathcal{M}(H_{A,n}^\otimes, y^n) \in \mathcal{M}(H_{A,n}^\otimes, [K] \times [M]).
\]

A sequence \((n, C, R, \theta, \Delta) : n \geq 1 \) of simulation protocols is said to have a communication cost \((C, R)\).

The notations \( K \triangleq [2^nC] \), \( M \triangleq [2^nR] \) suppress the dependence of \( K, M \) on \( n \). This abuse of notation reduces clutter. Moreover, since we focus throughout on a generic \( n \), this abuse is inconsequential.

**Defn. 6.** A sequence of simulation protocols \((n, C, R, \theta, \Delta) : n \geq 1 \) simulates \( \lambda_Y \) on \( \rho \in D(H_A) \) if for every \( \eta > 0 \), there exists \( N(\eta) \in \mathbb{N} \) such that for all \( n \geq N(\eta) \), the POVM \( \Delta \circ \theta \eta \)-simulates \( \lambda_Y \) on \( \rho^\otimes n \). POVM \( \lambda_Y \) on \( \rho \) can be simulated at a (communication) cost \((C, R)\) if there exists a sequence of simulation protocols \((n, C, R, \theta, \Delta) : n \geq 1 \) that simulates \( \lambda_Y \) on \( \rho \in D(H_A) \).

In this article, our goal is to characterize the set of all communication costs that permit simulation of \( \lambda_Y \) on \( \rho \). Specifically, we are interested in characterizing the set

\[
\mathcal{S}(\rho, \lambda_Y) = \text{cl}\left\{ (C, R) \in [0, \infty) : \text{there exists a sequence of simulation protocols} (n, C, R, \theta, \Delta) : n \geq 1 \text{ that simulates} \lambda_Y \text{ on} \rho \in D(H_A) \right\},
\]

where \( \text{cl}(\cdot) \) denotes the closure of the set. In the following, we recall an equivalent formulation involving purifications.

**B. Equivalent Formulation via Purifications**

We provide a formulation of the above problem statement in terms of purifications of the quantum state in question. As we shall see in the subsequent lemma, these formulations are equivalent.

**Defn. 7.** Let \( \rho \in D(H_A), \lambda_Y = \{ \lambda_y : y \in Y \} \in \mathcal{M}(H_A, Y), \theta_Y = \{ \theta_y : y \in Y \} \in \mathcal{M}(H_A, Y) \) be POVMs and \( \eta > 0 \). We say \( \theta_Y \eta \)-simulates \( \lambda_Y \) on purifications of \( \rho \) if for any purification \( |\phi_\rho\rangle \in H_X \otimes H_A \) of \( \rho \in D(H_A) \), we have

\[
\| (i_X \otimes \mathcal{E}_\lambda)(|\phi_\rho\rangle\langle \phi_\rho|) - (i_X \otimes \mathcal{E}_\theta)(|\phi_\rho\rangle\langle \phi_\rho|) \|_1 \leq \eta.
\]

**Lemma 7.** Let \( \rho \in D(H_A), \lambda_Y = \{ \lambda_y : y \in Y \} \in \mathcal{M}(H_A, Y), \theta_Y = \{ \theta_y : y \in Y \} \in \mathcal{M}(H_A, Y) \) be POVMs and \( \eta > 0 \). \( \theta_Y \eta \)-simulates \( \lambda_Y \) on \( \rho \) if and only if \( \theta_Y \eta \)-simulates \( \lambda_Y \) on purifications of \( \rho \).

**Proof.** We provide a proof in this article only for completeness. Our proof consists of two steps. In the first step, provided in Lemma 5 and its proof, we prove that the trace distance on the LHS of (59) is invariant with the choice of purification. Proof of Lemma 5 leverages the fact that any two purifications are related via an isometry acting on the purifying system and the invariance of trace norm under multiplication by isometries. This enables us focus on any chosen purification. We thus focus on a chosen purification \( |\phi_\rho\rangle \), henceforth referred to as the canonical purification, that plays a central role throughout this article. This purification facilitates simple uncluttered calculations and upper bounding of associated trace distances. In Lemma 5 and its proof, we prove that if \( \theta_Y \eta \)-simulates \( \lambda_Y \) on purification \( |\phi_\rho\rangle \) of \( \rho \), then \( \theta_Y \eta \)-simulates \( \lambda_Y \) on \( \rho \).
From Lemma [7], we conclude that to prove \((C, R) \in \mathcal{M}(\rho, \lambda_Y)\), it suffices to identify for every \(\eta > 0\), a sequence \((n, C + \eta, R + \eta, \theta, \Delta) : n \geq 1\) of simulation protocols for which

\[ \|\alpha_0 - \alpha_n\|_1 \leq \exp\{-nf(\eta)\} \] where \(\alpha_0 \triangleq (i_{X_n} \otimes \mathcal{E}^{\lambda_n})(|\phi_{p_n\otimes n}\rangle\langle \phi_{p_n\otimes n}|)\), and \(\alpha_n \triangleq (i_{X_n} \otimes \mathcal{E}^{\Delta \otimes \theta})(|\phi_{p_n\otimes n}\rangle\langle \phi_{p_n\otimes n}|)\). \hspace{1cm} (61)

for some positive valued function \(f : (0, \infty) \to (0, \infty)\). In the following, we characterize \(\alpha_n\) via an alternate evolution and employ the same in our proof.

**Lemma 8.** Consider a simulation protocol \((n, C, R, \theta, \Delta)\) for \(\lambda_Y \in \mathcal{M}(\mathcal{H}_A, \mathcal{Y})\) and the associated components as in Defn. [5] Let \(\rho_K \triangleq \frac{1}{K} \sum_{k=1}^{\infty} \rho^{\otimes n} \otimes |k\rangle\langle k|\). Then \(\alpha_n\) defined in (61) is also characterized as

\[ \alpha_n = \mathcal{E}_{\text{var}}(|\phi_{p_n\otimes n}\rangle\langle \phi_{p_n\otimes n}|), \text{ where } \mathcal{E}_{\text{var}} \triangleq (i_{X}^{\otimes n} \otimes \text{tr}_{K} \otimes i_{Y}^{\otimes n}) \circ (i_{X} \otimes \text{tr}_{K} \otimes \mathcal{E}^{\Delta}) \circ (i_{X}^{\otimes n} \otimes \text{tr}_{K} \otimes \mathcal{E}^{\theta}). \hspace{1cm} (62) \]

If there exists, for every \(\eta > 0\), a sequence of simulation protocols \((n, C + \eta, R + \eta, \theta, \Delta) : n \geq 1\) and some positive valued function \(f : (0, \infty) \to (0, \infty)\) for which \(\|\alpha_0 - \alpha_n\|_1 \leq \exp\{-nf(\eta)\}\) where \(\alpha_0, \alpha_n\) as defined in (61), (62) respectively, then \((C, R) \in \mathcal{M}(\rho, \lambda_Y)\).

A proof of (62) is provided in Appendix [C]. The proof of the last statement follows from our earlier discussion and (61). Our proof will revolve around establishing the above stated sufficient conditions for a specifically designed simulation protocol.

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