TRIVIAL $L(F_1)$-VALUED MOMENT SERIES OF THE GENERATING OPERATOR OF $L(F_2)$

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Abstract. In this paper, we will consider the generating operator $x = a + b + a^{-1} + b^{-1}$ of the free group factor $L(F_2)$, where $F_2 = \langle a, b \rangle$ is the free group with two generators $a$ and $b$. Let $F_1 = \langle h \rangle$ be a free group with one generator $h = aba^{-1}b^{-1}$ which is group isomorphic to the integers $\mathbb{Z}$. Then we can construct the free group factor $L(F_1)$ and the conditional expectation $E : L(F_2) \to L(F_1)$, defined by $E \left( \sum_{g \in F_2} \alpha_g g \right) = \sum_{k \in F_1} \alpha_k k$, for all $\sum_{g \in F_2} \alpha_g g$ in $L(F_2)$. Then $(L(F_2), E)$ is the $W^*$-probability space with amalgamation over $L(F_1)$. In this paper, we will compute the trivial $L(F_1)$-valued moment series of the generating operator $a + b + a^{-1} + b^{-1}$ of $L(F_2)$, over $L(F_1)$. This computation is the good example for studying the operator-valued distribution, since the operator-valued moment series of random variables contain algebraic and combinatorial free probability information about operator-valued distribution.

From mid 1980’s, Free Probability Theory has been developed. Here, the classical concept of Independence in Probability theory is replaced by a noncommutative analogue called Freeness (See [9]). There are two approaches to study Free Probability Theory. One of them is the original analytic approach of Voiculescu and the other one is the combinatorial approach of Speicher and Nica (See [1], [2] and [3]). Speicher defined the free cumulants which are the main objects in Combinatorial approach of Free Probability Theory. The free cumulants of random variables are gotten from the free moments of random variables via Möbius inversion. But in this paper, we will concentrate only on computing the free moments of random variables. And he and Nica developed free probability theory by using Combinatorics and Lattice theory on collections of noncrossing partitions (See [3]). Also, Speicher considered the operator-valued free probability theory, which is also defined and observed originally by Voiculescu (See [1]). In this paper, we will observe the important example of such operator-valued free probability.

Let $F_N$ be a free group with $N$-generators and let $L(F_N)$ be the free group factor defined by

$L(F_N) = \mathcal{C}[F_N]^\text{op}.$

In this paper, by using so-called the recurrence diagram found in [13] and [14], we will compute the trivial $L(F_1)$-valued moment series of the generating operator $G$ of the free group factor $L(F_2)$, defined by

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\[ G = a + b + a^{-1} + b^{-1} \in L(F_2), \]

over the free group factor \( L(F_1) \), where \( F_2 = \langle a, b \rangle \) and \( F_1 = \langle aba^{-1}b^{-1} \rangle \).

Throughout this paper, we will fix \( a \) and \( b \) as the generators of the free group \( F_2 \) and we will also fix \( h = aba^{-1}b^{-1} \) as the generator of the free group \( F_1 \) which is group isomorphic to the integers \( \mathbb{Z} \). Let \( x \) be an operator in \( L(F_2) \). Then there exists the Fourier expansion of \( x \),

\[ x = \sum_{g \in F_2} \alpha_g u_g, \text{ with } \alpha_g \in \mathbb{C}, \text{ for all } g \in F_2. \]

We can regard all \( g \in F_2 \) as unitaries \( u_g \) in \( L(F_2) \). For the convenience, we will denote these unitaries \( u_g \) just by \( g \). With this notation, it is easy to check that

\[ g^* = u_g^* = u_g^{-1} = u_{g^{-1}} = g^{-1} \text{ in } L(F_2), \]

where \( g^{-1} \) is the group inverse of \( g \) in \( F_2 \). We can define the conditional expectation \( E : L(F_2) \to L(F_1) \) by

\[ E \left( \sum_{g \in F_2} \alpha_g g \right) = \sum_{k \in F_1} \alpha_k k. \]

Then we have the \( W^\ast \)-probability space \( (L(F_2), E) \) with amalgamation over \( L(F_1) \). Let \( G \) be the generating operator \( a + b + a^{-1} + b^{-1} \) in \( L(F_2) \). It is easy to see that the first, second and third trivial \( L(F_1) \)-moments of \( G \) vanish, i.e.,

\[ E(G^k) = 0_{L(F_1)}, \text{ for } k = 1, 2, 3, \]

since \( G^k \) does not contain the \( h^n \)-term, for \( k = 1, 2, 3 \) and for \( n \in \mathbb{Z} \), where \( h = aba^{-1}b^{-1} \) and \( h^{-1} = bab^{-1}a^{-1} \). However, fourth trivial \( L(F_1) \)-moment \( E(G^4) \) of \( G \) contains the \( h \)-term and the \( h^{-1} \)-term. So, finding the trivial \( L(F_1) \)-moments of \( G \) is to find the \( h^k \)-terms of \( G^n \), for all \( k \in \mathbb{Z} \) and \( n \in \mathbb{N} \).

The following recurrence diagram will play a key role to find such trivial \( L(F_1) \)-valued moment series of the generating operator \( G \) of \( L(F_2) \);
In this paper, we obtain good applications about the above recurrence diagram. We will re-compute the moment series of the generating operators of the free group factor $L(F_N)$, for all $N \in \mathbb{N}$, by using the above recurrence diagram. This would be the one application of this recurrence diagram (See Chapter 1). When $N = 2$, we can apply this recurrence diagram to compute the trivial $L(F_1)$-valued moment series of the generating operator of $L(F_2)$ (See Chapter 2). Remark that to study (scalar-valued or operator-valued) moment series of elements in an operator algebra is to study (scalar-valued or operator-valued) free distributions of elements in that operator algebra. So, the computations in this paper about generating operators contain the free probability information about free distribution of those generating operators. And the free probability information is determined by the above recurrence diagram.

In Chapter 1, we will re-compute the (scalar-valued) moment series of the generating operator of $L(F_N)$, by using the recurrence diagram found in [13] and [14]. The moment series of the generating operator of $L(F_N)$ is already known, but here we will compute it again, by using the above recurrence diagram. In Chapter 2, by using the recurrence diagram when $N = 2$, we will compute the trivial $L(F_1)$-valued moment series of the generating operator $G = a + b + a^{-1} + b^{-1}$ of $L(F_2)$. Remark that the moment series in Chapter 1 is a scalar-valued moment series and the trivial $L(F_1)$-valued moment series in Chapter 2 is an operator-valued ($L(F_1)$-valued) moment series.
1. Moment Series of the Generating Operator of $L(F_N)$

Let $A$ be a von Neumann algebra and let $\tau : A \to \mathbb{C}$ be the normalized faithful trace. Then we call the algebraic pair $(A, \tau)$, the $W^*$-probability space and we call elements in $(A, \tau)$, random variables. Define the collection $\Theta_s$, consists of all formal series without the constant terms in noncommutative indeterminants $z_1, ..., z_s$ ($s \in \mathbb{N}$). Then we can regard the moment series of random variables as elements of $\Theta_s$.

**Definition 1.1.** Let $(A, \tau)$ be a $W^*$-probability space with its normalized faithful trace $\tau$ and let $a \in (A, \tau)$ be a random variable. The moment series of $a$ is defined by the formal series in $\Theta_1$,

$$M_a(z) = \sum_{n=1}^{\infty} \tau(a^n) z^n .$$

The coefficients $\tau(a^n)$ are called the $n$-th moments of $a$, for all $n \in \mathbb{N}$.

Let $H$ be a group and let $L(H)$ be a group von Neumann algebra. i.e,

$$L(H) = \overline{\mathbb{C}[H]}^\sigma .$$

Precisely, we can regard $L(H)$ as a weak-closure of the group algebra generated by $H$ and hence

$$L(H) = \overline{\{ \sum_{g \in H} t_g g : g \in H \}}^\sigma .$$

It is well known that $L(H)$ is a factor if and only if the given group $H$ is icc. (Since the free group $F_N$ with $N$-generators is icc, the von Neumann group algebra $L(F_N)$ is a factor and it is called the free group factor.)

Now, define the canonical trace $\tau : L(H) \to \mathbb{C}$ by

$$\tau \left( \sum_{g \in H} t_g g \right) = t_{e_H}, \text{ for all } \sum_{g \in H} t_g g \in L(H),$$

where $e_H$ is the identity of the group $H$. It is easy to check that the trace $\tau$ is normalized and faithful. So, the algebraic pair $(L(H), \tau)$ is a $W^*$-probability space. Assume that the group $H$ has its generators $\{g_j : j \in I\}$. We say that the operator $G = \sum_{j \in I} g_j + \sum_{j \in I} g_j^{-1}$.
the generating operator of $L(H)$. For instance, if we have a free group $F_N = \langle g_1, ..., g_N \rangle$, then the generating operator of the free group factor $L(F_N)$ is

$$g_1 + ... + g_N + g_1^{-1} + ... + g_N^{-1}.$$ 

Rest of this chapter, we will consider the moment series and the R-transform of the generating operator $G$ of $L(F_N)$.

From now, fix $n \in \mathbb{N}$. And we will denote free group factor $L(F_N)$ by $A$, i.e

$$A = \left\{ \sum_{g \in F_N} t_g g : t_g \in \mathbb{C} \right\}^w.$$ 

Recall that there is the canonical trace $\tau : A \to \mathbb{C}$ defined by

$$\tau \left( \sum_{g \in F_N} t_g g \right) = t_e,$$

where $e \in F_N$ is the identity of $F_N$ and hence $e \in L(F_N)$ is the unity $1_{L(F_N)}$. The algebraic pair $(L(F_N), \tau)$ is a $W^*$-probability space. Let $G$ be the generating operator of $L(F_N)$, i.e

$$G = g_1 + ... + g_N + g_1^{-1} + ... + g_N^{-1},$$

where $F_N = \langle g_1, ..., g_N \rangle$. It is well-known that if we denote the sum of all words with length $n$ in $\{g_1, g_1^{-1}, ..., g_N, g_N^{-1}\}$ by

$$X_n = \sum_{|w|=n} w \in A,$$

then

(1.1) \quad X_1X_1 = X_2 + 2N \cdot e \quad (n = 1)

and

(1.2) \quad X_1X_n = X_{n+1} + (2N - 1)X_{n-1} \quad (n \geq 2)

(See [15]). In our case, we can regard our generating operator $G$ as $X_1$ in $A$, by the very definition of $G$.

By using the relation (1.1) and (1.2), we can express $G^n$ in terms of $X_i$’s; For example, $G = X_1$,

$$G^2 = X_1X_1 = X_2 + 2N \cdot e,$$
Then we have the following recurrence relations;

\[ G^3 = X_1 \cdot X_2^2 = X_1 (X_2 + (2N)e) = X_1 X_2 + (2N)X_1 \]
\[ = X_3 + (2N - 1)X_1 + (2N)X_1 = X_3 + ((2N - 1) + 2N)X_1, \]

\[ G^4 = X_4 + ((2N - 1) + (2N - 1) + 2N) X_2 + (2N) ((2N - 1) + (2N)) e, \]

\[ G^5 = X_5 + ((2N - 1) + (2N - 1) + (2N - 1) + 2N) X_3 \]
\[ + ((2N - 1) ((2N - 1) + (2N - 1) + (2N)) + (2N) ((2N - 1) + (2N))) X_1, \]

\[ G^6 = X_6 + ((2N - 1) + (2N - 1) + (2N - 1) + (2N - 1) + 2N) X_4 \]
\[ + \{(2N - 1) ((2N - 1) + (2N - 1) + (2N - 1) + (2N)) \}
\[ + (2N - 1) ((2N - 1) + (2N - 1) + (2N)) \]
\[ + (2N - 1) ((2N - 1) + (2N - 1) + (2N))) \}
\[ + (2N) ((2N - 1) ((2N - 1) + (2N - 1) + (2N))) + (2N)((2N - 1) + (2N))) e, \]

etc.

So, we can find a recurrence relation to get \( G^n \) \((n \in \mathbb{N})\) with respect to \( X_k \)'s \((k \leq n)\). Inductively, \( G^{2k-1} \) and \( G^{2k} \) have their representations in terms of \( X_j \)'s as follows:

\[ G^{2k-1} = X_1^{2k-1} = X_{2k-1} + q_{2k-3}^{2k-1}X_{2k-3} + q_{2k-5}^{2k-1}X_{2k-5} + \ldots + q_3^{2k-1}X_3 + q_1^{2k-1}X_1 \]

and

\[ G^{2k} = X_1^{2k} = X_{2k} + p_{2k-2}^{2k}X_{2k-2} + p_{2k-4}^{2k}X_{2k-4} + \ldots + p_2^{2k}X_2 + p_0^{2k}e, \]

where \( k \geq 2 \). Also, we have the following recurrence relation;

**Proposition 1.1.** Let's fix \( k \in \mathbb{N} \setminus \{1\} \). Let \( q_i^{2k-1} \) and \( p_j^{2k} \) \((i = 1, 3, 5, \ldots, 2k - 1, \ldots \)
and \( j = 0, 2, 4, \ldots, 2k, \ldots \)) be given as before. If \( p_0^2 = 2N \) and \( q_1^3 = (2N - 1) + (2N)^2 \),
then we have the following recurrence relations:

\[ 1) \text{ Let } G^{2k-1} = X_{2k-1} + q_{2k-3}^{2k-1}X_{2k-3} + \ldots + q_3^{2k-1}X_3 + q_1^{2k-1}X_1. \]

Then

\[ G^{2k} = X_{2k} + ((2N - 1) + q_{2k-3}^{2k-1}) X_{2k-2} + ((2N - 1)q_{2k-3}^{2k-1} + q_{2k-5}^{2k-1}) X_{2k-4} \]
\[ + ((2N - 1)q_{2k-5}^{2k-1} + q_{2k-7}^{2k-1}) X_{2k-6} + \ldots + ((2N - 1)q_3^{2k-1} + q_1^{2k-1}) X_2 + (2N)q_1^{2k-1} e. \]

i.e,
Example 1.1. Suppose that $N = 2$. and let $p_0^2 = 4$ and $q_1^3 = 3 + p_0^2 = 3 + 4 = 7.

Put

$$G^8 = X_8 + p_0^8 X_6 + p_4^8 X_4 + p_2^8 X_4 + p_0^8 e.$$  

Then, by the previous proposition, we have that

$$p_0^8 = 3 + q_5^7, \quad p_4^8 = 3q_5^7 + q_3^7, \quad p_2^8 = 3q_3^7 + q_1^7 \quad \text{and} \quad p_0^8 = 4q_1^7.$$  

Similarly, by the previous proposition,

$$q_5^7 = 3 + p_4^6, \quad q_3^7 = 3p_4^6 + p_2^6 \quad \text{and} \quad q_1^7 = 3p_2^6 + p_0^6,$$

$$p_4^6 = 3 + q_3^5, \quad p_2^6 = 3q_3^5 + q_1^5 \quad \text{and} \quad p_0^6 = 4q_1^5,$$

$$q_3^5 = 3 + p_2^4 \quad \text{and} \quad q_1^5 = 3p_2^4 + p_2^4,$$

$$p_2^4 = 3 + q_1^3 \quad \text{and} \quad p_0^4 = 4q_1^3.$$
and 

\[ q_1^3 = 3 + p_0^2 = 7. \]

Therefore, combining all information,

\[ G^8 = X_8 + 22X_6 + 202X_4 + 744X_2 + 1316e. \]

We have the following diagram with arrows which mean that

\[ \checkmark \checkmark : (2N - 1) + [\text{former term}] \]
\[ \checkmark \downarrow : (2N - 1) \cdot [\text{former term}] \]
\[ \checkmark : \cdot + [\text{former term}] \]

and

\[ \downarrow \downarrow : (2N) \cdot [\text{former term}]. \]

\[ \begin{align*}
  p_0^2 &= 2N \\
  q_1^3 &= (2N - 1) + 2N \\
  \vdots & \quad \vdots \\
  p_6^8 & \quad p_4^8 & \quad p_2^8 & \quad p_0^8
\end{align*} \]

**Notation** From now, we will call the above diagram the recurrence diagram for \( N \). \( \Box \)

For example, when \( N = 2 \), we can compute \( p_4^6 \), as follows:

\[ p_0^2 = 4, \quad q_1^3 = 7, \]
\[ p_2^4 = 3 + 7 = 10, \quad p_4^3 = 28, \]
\[ q_3^5 = 3 + 10 = 13, \quad p_1^5 = 3 \cdot 10 + 28 = 58. \]

and hence \( p_4^6 = 3 + 13 = 16. \)

Recall that Nica and Speicher defined the even random variable in a \(*\)-probability space. Let \((B, \tau_0)\) be a \(*\)-probability space, where \( \tau_0 : B \to \mathbb{C} \) is a linear functional satisfying that \( \tau_0(b^*) = \overline{\tau_0(b)} \), for all \( b \in B \), and let \( b \in (B, \tau_0) \) be a random
variable. We say that the random variable \( b \in (B, \tau_0) \) is even if it is self-adjoint and it satisfies the following moment relation:

\[ \tau_0 (b^n) = 0, \text{ whenever } n \text{ is odd.} \]

By the recurrence diagram for \( N \), we can get that

**Theorem 1.2.** Let \( G \in (A, \tau) \) be the generating operator. Then the moment series of \( G \) is

\[
\tau (G^n) = \begin{cases} 
0 & \text{if } n \text{ is odd} \\
 p_n^0 & \text{if } n \text{ is even,}
\end{cases}
\]

for all \( n \in \mathbb{N} \).

**Proof.** Assume that \( n \) is odd. Then

\[ G^n = X_n + q_n^{n-2} X_{n-2} + \ldots + q_3^n X_3 + q_1^n X_1. \]

So, \( G^n \) does not contain the \( e \)-terms. Therefore,

\[ \tau (G^n) = \tau (X_n + q_n^{n-2} X_{n-2} + \ldots + q_3^n X_3 + q_1^n X_1) = 0. \]

Assume that \( n \) is even. Then

\[ G^n = X_n + p_n^{n-2} X_{n-2} + \ldots + p_2^n X_2 + p_0^n e. \]

So, we have that

\[ \tau (G^n) = \tau (X_n + p_n^{n-2} X_{n-2} + \ldots + p_2^n X_2 + p_0^n e) = p_0^n. \]

Remark that the \( n \)-th moments of the generating operator in \((A, \tau)\) is totally depending on the recurrence diagram for \( N \).

**Corollary 1.3.** Let \( G \in (A, \tau) \) be the generating operator. Then \( G \) is even in \((A, \tau)\). \( \square \)

**Corollary 1.4.** Let \( G \in (A, \tau) \) be the generating operator. Then the operator \( G \) has its moment series,

\[ M_G(z) = \sum_{n=1}^{\infty} p_0^{2n} z^{2n} \in \Theta_1. \]

\( \square \)
2. Trivial $L(F_1)$-valued Moment Series of the Generating Operator of $L(F_2)$

Let $M_0 \subset M$ be von Neumann algebras with $1_{M_0} = 1_M$ and let $\varphi : M \to M_0$ be the conditional expectation satisfying that

\[ \varphi(m_0) = m_0, \text{ for all } m_0 \in M, \]

\[ \varphi(m_0 mm'_0) = m_0 \cdot \varphi(m) \cdot m'_0, \]

for all $m_0, m'_0 \in M_0$, $m \in M$, and

\[ \varphi(m^*) = \varphi(m)^*, \text{ for all } m \in M. \]

Then the algebraic pair $(M, \varphi)$ is a $W^*$-probability space over $M_0$. If $m \in (M, \varphi)$, then we will call $m$ a $M_0$-valued random variable.

**Definition 2.1.** Let $(M, \varphi)$ be a $W^*$-probability space over $M_0$ and let $m \in (M, \varphi)$ be a $M_0$-valued random variable. Define the $n$-th $M_0$-valued moment of $m$ by

\[ E((m_1 m)(m_2 m) \ldots (m_n m)), \]

for all $n \in \mathbb{N}$, where $m_1, \ldots, m_n \in M_0$ are arbitrary. When $m_1 = \ldots = m_n = 1_{M_0}$, for all $n \in \mathbb{N}$, we say that the $M_0$-valued moment of $m$ is trivial, i.e., the $n$-th trivial $M_0$-valued moments of $m \in (M, \varphi)$ are $E(m^n)$, for all $n \in \mathbb{N}$. We will say that the $M_0$-valued formal series

\[ M^t_m(z) = \sum_{n=1}^\infty E(m^n) z^n \in M_0[[z]] \]

is the trivial $M_0$-valued moment series of $m \in (M, \varphi)$, where $z$ is the indeterminant. (Here $M_0[[z]]$ is the formal-series-ring in its indeterminant $z$)

In this chapter, by using the recurrence diagram for $N = 2$, we will compute the trivial $L(F_1)$-valued moment series of the given generating operator

\[ G = a + b + a^{-1} + b^{-1} \]

of the free group factor $L(F_2)$, where $F_2 = \langle a, b \rangle$ and $F_1 = \langle h \rangle$ are free groups, where
\[ h = aba^{-1}b^{-1}. \]

First, we will define the conditional expectation \( E : L(F_2) \to L(F_1) \) by

\[
E \left( \sum_{g \in F_2} \alpha g \right) = \sum_{k \in F_1} \alpha_k k,
\]

for all \( \sum_{g \in F_2} \alpha g \in (L(F_2), E) \).

Then we can construct the \( W^* \)-probability space \((L(F_2), E)\) over its \( W^* \)-subalgebra \( L(F_1) \). Notice that to find the conditional expectational value of the \( L(F_1) \)-valued random variable \( x \in (L(F_2), E) \) is to find the \( h^k \)-terms of the \( L(F_1) \)-valued random variables \( x \), for \( k \in \mathbb{Z} \), where \( h = aba^{-1}b^{-1} \) and \( h^{-1} = bab^{-1}a^{-1} \).

First, let us provide the recurrence diagram for \( N = 2 \):

\[
\begin{align*}
p_0^2 &= 4 \\
\downarrow & \quad q_3^3 = 7 \\
p_2^4 & \quad p_0^4 \\
q_1^5 & \quad q_1^5 \\
p_6^8 & \quad p_4^8 \\
p_8^8 & \quad p_2^8 \\
p_6^8 & \quad p_0^8
\end{align*}
\]

where

\[
\begin{align*}
\checkmark \checkmark & : 3 + \text{[former term]} \\
\downarrow & : 3 \cdot \text{[former term]} \\
\checkmark & : \cdot + \text{[former term]}
\end{align*}
\]

and

\[
\downarrow \downarrow \downarrow : 4 \cdot \text{[former term]}.
\]

By the above recurrence diagram for \( N = 2 \), we have that if

\[
p_0^2 = 4 \quad \text{and} \quad q_1^3 = 3 + p_0^2 = 7,
\]
and if we put $X_n = \sum_{|w|=n} w$, as the sum of all words with length $n$ in $\{a, b, a^{-1}, b^{-1}\}$, for $n \in \mathbb{N}$, then we have the following recurrence relations (1) and (2):

(1) If

$$G^{2k-1} = X_{2k-1} + q_{2k-3}^{2k-1} X_{2k-3} + \ldots + q_3^{2k-1} X_3 + q_1^{2k-1} X_1.$$ 

then

$$G^{2k} = X_{2k} + (3 + q_{2k-3}^{2k-1}) X_{2k-2} + (3q_{2k-3}^{2k-1} + q_{2k-5}^{2k-1}) X_{2k-4}$$

$$+ (3q_{2k-5}^{2k-1} + q_{2k-7}^{2k-1}) X_{2k-6} + \ldots + (3q_3^{2k-1} + q_1^{2k-1}) X_2 + 4q_1^{2k-1} e.$$ 

where

$$p_{2k-2}^{2k} = 3 + q_{2k-3}^{2k-1}, \quad p_{2k-4}^{2k} = 3q_{2k-3}^{2k-1} + q_{2k-5}^{2k-1},$$

$$\ldots, \quad p_2^{2k} = 3q_3^{2k-1} + q_1^{2k-1} \quad \text{and} \quad p_0^{2k} = 4q_1^{2k-1},$$

by the recurrence diagram.

(2) If

$$G^{2k} = X_{2k} + p_{2k-2}^{2k} X_{2k-2} + \ldots + p_2^{2k} X_2 + p_0^{2k} e.$$ 

Then

$$G^{2k+1} = X_{2k+1} + (3 + p_{2k-2}^{2k}) X_{2k-1} + (3p_{2k-2}^{2k} + p_{2k-4}^{2k}) X_{2k-3}$$

$$+ (3p_{2k-4}^{2k} + p_{2k-6}^{2k}) X_{2k-5} + \ldots + (3p_4^{2k} + p_2^{2k}) X_3 + (3p_2^{2k} + p_0^{2k}) X_1.$$ 

where

$$q_{2k-1}^{2k+1} = 3 + p_{2k-2}^{2k}, \quad q_{2k-3}^{2k+1} = 3p_{2k-2}^{2k} + p_{2k-4}^{2k},$$

$$\ldots, \quad q_3^{2k+1} = 3p_4^{2k} + p_2^{2k} \quad \text{and} \quad q_1^{2k+1} = 3p_2^{2k} + p_0^{2k},$$

by the recurrence diagram.

Note that $h$ and $h^{-1}$ are words with their length 4. Therefore, $X_{4k}$ contains $h^k$-terms and $h^{-k}$-terms, for all $k \in \mathbb{N} \cup \{0\}$. Thus we can compute the trivial $L(F_1)$-valued moments of the operator $G$ as follows;
Theorem 2.1. Fix $k \in \mathbb{N}$ and Let $G \in (L(F_2), E)$ be the generating operator of $L(F_2)$. Then

1. $E(G^k) = 0_{L(F_1)}$, if $k$ is odd.

2. $E(G^{4k}) = (h^k + h^{-k}) + \sum_{j=1}^{k-1} p_{4k-4j}^{4k} (h^{k-j} + h^{-(k-j)}) + p_0^{4k} h^0$,

   where $p_0^2 = 28$.

3. If $4 \nmid 2k$, in the sense that $2k$ is not a multiple by 4, then

   $E(G^{2k}) = \sum_{j=1}^{k-1} p_{2k-2j}^{2k} (h^{\frac{k-j}{2}} - 2j + h^{-(\frac{k-j}{2})-2j}) + p_0^{2k} h^0$,

   where $p_0^2 = 4$.

Proof. (1) Suppose that $k$ is odd. Then $G^k$ does not have the words with length $4p$, for some $p \in \mathbb{N}$, by the recurrence diagram for $N = 2$, since $G^k$ does not have the $X_{4n}$-terms, for $n \in \mathbb{N}$, $4n < k$. This shows that there's no $h^n$-terms and $h^{-n}$-terms in $G^k$, where $n$ is previously given such that $2n < k$. Therefore, all odd trivial $L(F_1)$-valued moments of $G$ vanish.

(2) By the straightforward computation using the recurrence diagram, we have that

$$
E(G^{4k})
= E(X_{4k} + p_{4k-2}^{4k} X_{4k-2} + p_{4k-4}^{4k} X_{4k-4} + \ldots + p_4^{4k} X_4 + p_2^{4k} X_2 + p_0^{4k} h^0)
$$

(2.2)

$$
= E(X_{4k}) + p_{4k-2}^{4k} E(X_{4k-2}) + p_{4k-4}^{4k} E(X_{4k-4}) + \ldots + p_4^{4k} E(X_4) + p_2^{4k} E(X_2) + p_0^{4k} h^0.
$$

Since $h^p$ and $h^{-p}$ terms are in $X_{4p}$, for any $p \in \mathbb{N} \cup \{0\}$, the formula (2.2) is

(2.3)

$$
E(X_{4k}) + p_{4k-2}^{4k} E(X_{4k-2}) + \ldots + p_4^{4k} E(X_4) + p_0^{4k} h^0
= (h^k + h^{-k}) + p_{4k-4}^{4k} (h^{k-1} + h^{-(k-1)}) + \ldots + p_4^{4k} (h + h^{-1}) + p_0^{4k} h^0.
$$

(3) If $4 \nmid 2k$, then $k = 1, 3, 5, \ldots$. If $k = 1$, then we have that :

$$
E(G^2) = E(X_2 + 4h^0) = 4h^0.
$$
If \( k \neq 1 \) is odd, then
\[
E(G^{2k}) = E(X_{2k} + p^{2k}_{2k-2}X_{2k-2} + p^{2k}_{2k-4}X_{2k-4} + p^{2k}_{2k-6}X_{2k-6} + \ldots + p^{2k}_{4}X_{4} + p^{2k}_{2}X_{2} + p^{2k}_{0}h^0)
\]
\[
= E(X_{2k}) + p^{2k}_{2k-2}E(X_{2k-2}) + p^{2k}_{2k-4}E(X_{2k-4}) + p^{2k}_{2k-6}E(X_{2k-6}) + \ldots + p^{2k}_{4}E(X_{4}) + p^{2k}_{2}E(X_{2}) + p^{2k}_{0}h^0
\]
\[
= 0_B + p^{2k}_{2k-2} (h^{k-1} + h^{-(k-1)}) + 0_B + p^{2k}_{2k-6} (h^{k-3} + h^{-(k-3)}) + \ldots + p^{2k}_{4} (h + h^{-1}) + 0_B + p^{2k}_{0}h^0,
\]
since \( X_{2k-2}, X_{2k-6}, \ldots, X_{4} \) contain \( h^p \)-terms and \( h^{-p} \)-terms, for \( p \in \mathbb{N} \cup \{0\} \). \( \square \)

By the previous trivial \( L(F_1) \)-valued moments of the generating operator \( G \) of \( L(F_2) \), we have the following result:

**Corollary 2.2.** Let \((L(F_2), E)\) be the \( W^* \)-probability space over \( L(F_1) \) and let \( G \in (L(F_2), E) \) be the generating operator of \( L(F_2) \). Then the trivial \( L(F_1) \)-valued moment series of \( G \) is
\[
M_G(z) = \sum_{n=1}^{\infty} b_{2n} z^n \in L(F_1)[[z]],
\]
where
\[
b_{4n} = (h^n + h^{-n}) + \sum_{j=1}^{n-1} p^{4n-4j}_{4n-4j} (h^{n-j} + h^{-(n-j)}) + p^{4n}_{0}h^0
\]
and
\[
b_{2k} = \sum_{j=1}^{k-1} p^{2k}_{2k-2j-4j} (h^{k-1-2j} + h^{-(k-1-2j)}) + p^{2k}_{0}h^0,
\]
where \( 4 \nmid 2k \), for all \( n, k \in \mathbb{N} \). \( \square \)

**Remark 2.1.** Suppose we have the free group factor \( L(F_N) \), where \( N \in \mathbb{N} \). Then we can extend the above result for the general \( N \). i.e., we can take \( F_N = < g_1, ..., g_N > \) and \( F_1 = < h >, \) where
\[
k = g_1 \cdots g_N \cdot g_1^{-1} \cdots g_N^{-1}.
\]

By defining the canonical conditional expectation \( E : L(F_N) \to L(F_1) \), we can construct the \( W^* \)-probability space with amalgamation over \( L(F_1) \). Similar to the
case when \( N = 2 \), by using the recurrence diagram for \( N \), we can compute the trivial \( L(F_1) \)-valued moments of the generating operator

\[
G_N = g_1 + ... + g_N + g_1^{-1} + ... + g_N^{-1}.
\]

When \( N = 2 \), to find the trivial \( L(F_1) \)-valued moments of \( G_2 \) is to find the \( h^k \)-terms of \( G_2^2 \), for \( k \in \mathbb{Z} \), \( n \in \mathbb{N} \). But in the general case, to find the trivial \( L(F_1) \)-valued moments of \( G_N \) is to find the \( k^p \)-terms of \( G_N^2 \), for \( p \in \mathbb{Z} \), \( q \in \mathbb{N} \). So, we have to choose the \( X_{(2N)p} \)-terms, for all \( p \in \mathbb{N} \), containing \( k^{\pm p} \)-terms!

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