STRUCTURE FORMATION IN INFLATIONARY COSMOLOGIES

Andrew R. Liddle

Astronomy Centre, University of Sussex, Falmer, Brighton BN1 9QH, U. K.

Abstract

A brief account is given of large-scale structure modelling based on the assumption that the initial perturbations arise from inflation. A recap is made of the implications of inflation for large-scale structure; under the widely applicable slow-roll paradigm inflation adds precisely two extra parameters to the normal scenarios, which can be taken to be the tilt of the density perturbation spectrum and the amplitude of gravitational waves. Some comments are made about the COBE normalization. A short description is given of an analysis combining cosmic microwave background anisotropy data and large-scale structure data to constrain cosmological parameters, and the case of cold dark matter models with a cosmological constant is used as a specific illustration.

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1 Introduction

In the medium term, the cosmic microwave background (CMB) shall undoubtedly prove an extremely powerful tool in constraining cosmological parameters. Already, the COBE observations of large-angle anisotropies provide the most accurate and unambiguous constraint on the spectra of perturbations in the universe. However, at the present time the best route to constraining cosmological parameters is not through the CMB alone, but from the combination of microwave data with a large number of measures of the power spectrum from large-scale structure observations.

In recent work, my collaborators and I have sought to test a wide parameter space of large-scale structure models, using linear and quasi-linear theory. We have written three papers on this topic, covering cold dark matter (CDM) models in open universes\(^1\) and in flat universes with a cosmological constant\(^3\), and the case of a critical density universe\(^2\) where we also allow a fraction of the dark matter to be hot. The key ingredient of our work is to take the inflationary hypothesis seriously, and to take advantage of the extra parameters that slow-roll inflation lends to large-scale structure modelling.

There isn’t space here to give the full details of this work, so instead I’ll concentrate on a couple of aspects and illustrate the outcome by showing some results from our investigation of CDM models in flat universes\(^3\).

2 Inflationary parameters

The simplest approximation for the production of perturbations from inflation, that it gives a scale-invariant density perturbation spectrum and nothing else, is woefully inadequate to describe the output of most inflation models. It is necessary to make a better approximation, which acknowledges that as well as producing density perturbations inflation will produce a spectrum of gravitational waves. Based on the slow-roll approximation, one finds that inflation predicts power-law spectra of both density perturbations and gravitational waves. This gives a total of four parameters, two amplitudes and two spectral indices. This turns out to be an excellent approximation for almost all known inflationary models, and may well be all that is ever needed.

Interestingly, these four parameters are not all independent. The reason is that we have extracted two continuous functions, the spectra, from a single input function, the inflaton potential. The only way you can get two continuous functions from one is if they are related, and at this level of approximation it turns out that one of the four parameters is redundant. This is interesting, because it offers the possibility of a consistency check on the inflationary hypothesis which is \textit{independent} of the choice of inflationary potential.

The three inflationary parameters for large-scale structure are

1. The amplitude of density perturbations, \(\delta_H\), defined below.

2. The spectral index of the density perturbations \(n\).

3. The relative contribution of gravitational waves to large-angle CMB anisotropies, \(R \equiv \frac{C_\ell^{GW}}{C_\ell^{DP}}\), where \(\ell\) could for instance be taken to be 10, the COBE pivot scale.

The equation describing the redundancy is

\[ R \simeq -2\pi n_{GW}, \]  
\[ (1) \]
where $n_{GW}$ is the spectral index of the gravitational waves. Although a very distinctive signature of inflation, it seems very unlikely that one will ever be able to carry out this test since $n_{GW}$ is almost certainly impossible to measure. By contrast, one should be extremely optimistic about measuring the non-degenerate parameters $n$ and $R$.

One sometimes sees the relation $R \approx 2\pi (1 - n)$ quoted as an inflationary signature. This is in fact the prediction of a specific model, the power-law inflation model (which always gives $n < 1$), and is not general enough to describe generic slow-roll inflation, for which $n$ and $R$ enter independently at the same order in the slow-roll expansion. It is quite a useful relation nevertheless, because for $n < 1$ it gives about the largest amount of gravitational waves one finds in any inflation model, and so a good range to consider is $R$ going from 0 up to this value. However, for purposes such as fitting CMB anisotropies, there seems little purpose in adopting such a relation, because simply by including $n$ and $R$ independently one is looking at the generic slow-roll inflation situation. Since one is anyway fitting for a large number of cosmological parameters, the addition of one more can hardly degrade the fit at all and has the enormous benefit of generality.

If one has a particularly complex inflationary potential, one might need more parameters to describe its influence. This can be achieved by a technique known as the slow-roll expansion. Up to a point, having to introduce extra parameters is actually a good thing, because if you feel you need them to describe the data, that means that potentially there is more information, in the form of extra parameters, to be learnt about inflation from the observations. Since there are a lot of parameters anyway, a couple more shouldn’t particularly degrade the fit. Equally though, it would be possible to have too much of a good thing and I can’t imagine that anyone would be pleased if it was thought that inflation might add say twenty new parameters. Fortunately that will not be the case.

### 3 Cosmological parameters

When one considers short-scale information, such as large-scale structure data, the theoretical prediction depends on the whole range of cosmological parameters, including

1. The Hubble parameter $h$.
2. The total matter density $\Omega_0$.
3. The cosmological constant $\Lambda$.
4. The baryon density $\Omega_B$.
5. The hot dark matter (HDM) density $\Omega_{\nu}$. (It is always assumed there’ll be some CDM).
6. The effective number (at the photon temperature) of massless species $g_*$.
7. The redshift of reionization $z_R$.

Further complexity can be introduced in many ways, for example by allowing the dark matter to decay, by allowing the HDM to be composed of more than one particle type or by allowing the hot component to violate the usual mass–density relation.

All of these parameters have effects on various kinds of observations, and coupled with the three input parameters from inflation give a fairly large number of parameters to be dealt with. The sort of data a new generation CMB anisotropy satellite might produce could quite conceivably just simultaneously fit all, or a sizeable subset, of them, but present data are a long way off and it is usual to work within some simplifying assumptions. Our analyses have been unusual in that they attempt to retain all the inflationary parameters, as well as a reasonable number of the cosmological ones.

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1. Though it is possible to have models, such as double inflation, that have features so drastic that they can’t be described within this perturbative framework at all.
Although the list of cosmological parameters looks intimidating, combining all the options seems very unattractive. Also, various scalings can be utilized, since many changes have an effect very similar to changing the Hubble parameter (the main effect of which is to shift the epoch of matter–radiation equality), which allows many options to be considered at once. The epoch of reionization is only important if one considers intermediate-scale CMB anisotropies.

4 The COBE normalization

More than anything else, this section is a request to those carrying out fittings to the COBE data to quote the best-fit amplitude of the matter power spectrum, as well as radiation anisotropy quantities such as $Q_{rms}$. In general it is far from trivial to get from one to the other. It doesn’t really matter in what form this is done, but I’d like to put in a word in favour of a quantity we call $\delta_H$.

When the spectrum of density perturbations $\mathcal{P}$ is defined so that the variance is

$$\sigma^2(R) = \int_0^\infty W^2(kR) \mathcal{P}(k) \frac{dk}{k},$$

(2)

where $W(kR)$ is whatever window function does the smoothing, then the spectrum can be broken up into pieces as

$$\mathcal{P}(k) = \left(\frac{k}{aH}\right)^4 \delta_H^2(k) T^2(k, t) \frac{g^2(\Omega)}{g^2(\Omega_0)}.$$  

(3)

The final term is the usual growth suppression factor, which can be applied at any epoch provided the $\Omega$ at that time is used. This term vanishes if $\Omega = 1$. The transfer function is normalized to unity on large scales. In general it may be time dependent, for example if there is an HDM component, but in a CDM universe it is time independent. The very first term carries the remaining time dependence, that of a critical density CDM universe.

Finally then, $\delta_H$ is the initial spectrum of perturbations. It is time-independent, and amounts (when multiplied by the growth suppression term) to a proper definition of what is meant by the perturbation at horizon crossing. In the special case $n = 1$ it is also $k$-independent. One can specify a fit to the matter power spectrum by giving its value at $k = a_0H_0$, the present Hubble radius. For the four-year COBE data, one finds for CDM models with a cosmological constant$^3$ that

$$\delta_H(a_0H_0) = 1.94 \times 10^{-5} \Omega_0^{-0.785 - 0.05\ln\Omega_0} \exp[f(n)]$$

(4)

where

$$f(n) = -0.95(n - 1) - 0.169(n - 1)^2 \quad \text{No gravitational waves}$$

(5)

$$= 1.00(n - 1) + 1.97(n - 1)^2 \quad \text{Power-law inflation}$$

(6)

A similar fitting function can also be found for the open universe case. The joy of this formula is that it says everything about the COBE normalization in a single fitting formula. To a good approximation, this fit is independent of $h$, $\Omega_B$ and the nature of the dark matter.

5 CDM models with a cosmological constant

Our strategy has been to assemble observational data which can be interpreted in terms of linear and quasi-linear theory. This allows a rapid assessment of which regions in parameter space are
Figure 1: Varying $n$, but with no gravitational waves, for spatially flat CDM models. All constraints are plotted at 95 per cent confidence, and models are normalized to the 4 year COBE data. The constraints are galaxy clustering (solid), cluster abundance (dashed) and DLAS abundance (dot-dashed). Bulk flows and quasar abundance are less constraining. Contours of constant age are shown as dotted lines. The allowed region is shown with two different shadings, both highlighting the parameter space not excluded at more than 95 per cent confidence on any single piece of data. The lighter shading shows models where the optical galaxies have to be antibiased at $8h^{-1}$ Mpc. Finally, the unshaded region in the $n = 0.8$ plot which is allowed by all plotted data is excluded by Doppler/acoustic peak height.

allowed or disallowed. The observations considered are large-angle CMB anisotropies (COBE), intermediate angle CMB anisotropies, bulk motions, galaxy clustering and the abundances of galaxy clusters, quasars and damped Lyman alpha systems (DLAS). Extra considerations that can be brought into play are the age of the universe, suppression of intermediate angle anisotropies by reionization and the question of whether (optically identified) galaxies are permitted to be anti-biased.

Figure 1 shows the sort of constraints that can be applied to CDM models in spatially flat universes. These plots show tilt $n$, but not gravitational waves. There is a sizeable allowed region for each $n$ shown, so observations do not really constrain the inflationary parameters. I’ll just make some brief points about these plots.

The age problem is usually cited as motivation for going to cosmological constant models. However, within the region fitting large-scale structure observations, the low density models are actually younger than the high density ones. Of course, as $\Omega_0$ approaches one the required $h$ is alarmingly low, and one might feel inclined to introduce a hot component, which permits larger $h$ with critical density.

In the low density region, optical galaxies typically have to be anti-biased (and of course IRAS galaxies more so). This is thought unlikely, though it’s not clear exactly what observations this is supposed to be in conflict with. For $n = 0.8$, there is a region which is allowed by all data except intermediate-scale CMB anisotropies; in that region of parameter space there isn’t
really a Doppler/acoustic peak at all. This is a sign of things to come; intermediate CMB anisotropies have the potential to exclude swathes of parameter space in the future.

6 Conclusions

Cosmologists are beginning to take seriously the possibility that one can determine the whole range of cosmological parameters. Within that context, one appreciates that it is possible to also include information from inflation, and attempt to fit for inflationary parameters at the same time as the cosmological parameters. The most popular inflationary paradigm, the slow-roll approximation, only introduces two extra parameters ($n$ and $R$) that one didn’t have to consider anyway, and there is good reason to be optimistic that one can constrain these.

However, once one takes the extra inflationary input into account, it is clear that present observational data fall some way short of providing any telling constraints. We have found that it is possible to get an adequate fit to present data within almost any context. There are viable regions of parameter space for

- **CDM models**$^2$: Requires some or all of low $h$, high $\Omega_B$ or tilt to $n < 1$. Gravitational waves don’t help much, but they are not very strongly constrained. It is however very hard to fit the data for $h \geq 0.50$. Adding extra massless species or decaying dark matter will also work though we haven’t investigated them in detail ourselves.

- **CHDM models**$^2$: The same general picture as CDM models, but allows a higher value of $h$, at least up to 0.6, provided the amount of hot dark matter is chosen wisely.

- **Low density CDM**$^{1,3}$: Can be made to work either in the open case or in the flat case with a cosmological constant. Observationally, no strong preference between the open and flat cases.

This situation should not remain for long. We stand at a tantalizing time, where observations are just good enough to exclude the more extreme inflationary models. We can look forward in the near future to a time when inflationary and cosmological parameters are extremely well determined, at which point we can expect most inflation models to be ruled out. Or maybe even all!

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**References**

I regret there has been no space here to provide an adequate reference list. Full references can be found in the papers cited here; apologies to the vast number of people who are missing out!

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