WELL-POSEDNESS OF STRONGLY DISPERSIVE
TWO-DIMENSIONAL SURFACE WAVES BOUSSINESQ
SYSTEMS

FELIPE LINARES, DIDIER PILOD, AND JEAN-CLAUDE SAUT

Abstract. We consider in this paper the well-posedness for the Cauchy problem associated to two-dimensional dispersive systems of Boussinesq type which model weakly nonlinear long wave surface waves. We emphasize the case of the strongly dispersive ones with focus on the “KdV-KdV” system which possesses the strongest dispersive properties and which is a vector two-dimensional extension of the classical KdV equation.

1. Introduction

1.1. General Setting. In general, nonlinear dispersive equations and systems are not derived from first principles. They are obtained as asymptotic models (normal forms) when some small parameter tends to zero, of more general systems, under a suitable scaling. There are supposed to describe the dynamics under suitable scaling conditions. We will study in this article small amplitude, long wave models which have the general form

\begin{equation}
U_t + AU + \epsilon F(U, \nabla U) + \epsilon LU = 0.
\end{equation}

Here \( \epsilon \) is a “small” parameter which takes into account the nonlinear and dispersive effects, which are assumed of the same order. \( U = U(x, t) \) is a vector in \( \mathbb{R}^{n+1} \), \( n = 1, 2 \) and \( x \in \mathbb{R}^n, \ t > 0 \). The order zero part \( U_t + AU \) is just the linear wave equation, while the third and fourth terms in (1) represents the nonlinear and dispersive effects. The right-hand side of (1) should in fact be \( O(\epsilon^2) \).

A typical example is that of the \( abcd \) Boussinesq systems for long, small amplitude gravity surface water waves introduced in [6], [5],

\begin{equation}
\begin{cases}
\eta_t + \nabla \cdot \mathbf{v} + \epsilon [\nabla \cdot (\eta \mathbf{v}) + a \nabla \cdot \Delta \mathbf{v} - b \Delta \eta_t] = 0 \\
\mathbf{v}_t + \nabla \eta + \epsilon [\frac{1}{2} \nabla |\mathbf{v}|^2 + c \nabla \Delta \eta - d \Delta \mathbf{v}_t] = 0.
\end{cases}
\end{equation}

Here, \( \eta \) is the deviation of the free surface from its rest state and \( \mathbf{v} \) is an \( O(\epsilon^2) \) approximation of the horizontal velocity taken at a certain depth (see [6], [5]). The constants \( a, b, c, d \) are modeling parameters subject to the constraint \( a + b + c + d = \frac{1}{3} \). Those three degrees of freedom arise from the choice of the height at which the velocity is taken and from a double use of the BBM trick.

The small parameter \( \epsilon \) is defined by

\[ \epsilon = a/h \sim (h/\lambda)^2, \]

Date: April 11, 2011.
where $h$ denotes the mean depth of the fluid, $a$ a typical amplitude of the wave and $\lambda$ a typical horizontal wavelength.

Those systems are approximations of the full water wave system in the so-called Boussinesq regime (see [6], [5]). They degenerate into the KdV (or BBM) equation in the one-dimensional case, for waves traveling in one direction. They also appear as models for internal waves, in an appropriate regime (see [9], [11]). When surface tension effects are taken into account, the coefficient $c$ should be changed into $c - \tau$ where $\tau \geq 0$ is the surface tension parameter (see [14]). Throughout the paper we will consider only the case of purely gravity waves, that is $\tau = 0$.

Of course restrictions are to be imposed on $a, b, c, d$ in order that the linear part of (2) be well-posed. It was established in [7] that, when $n = 1$, all the linearly well-posed systems are locally nonlinearly well-posed. As for the two-dimensional case ($n = 2$), it has been proved in [15] that in the generic case where $b > 0$, $d > 0$ one has well-posedness on time scales of order $O(\frac{1}{\sqrt{\epsilon}})$ for data in the Sobolev space $H^1$. The local well-posedness in other cases (but not in the strongly dispersive “KdV-KdV” case) was proved in [13], but the question of the dependence of the time existence with respect to $\epsilon$ was not addressed there.

One of the goals of the present paper is to complete the local well-posedness theory for $a, b, c, d$ systems in two dimensions, with focus on the size of the lifespan of the solutions with respect to $\epsilon$.

An interesting fact (already noticed in [6]) from the PDE viewpoint is that, though the $a, b, c, d$ systems describe the same wave propagation phenomenon, their dispersive properties are quite different. We will precise the dispersion matrix in Section 2. After diagonalization, one is led to two-dimensional systems with strong dispersion (of KdV or Schrödinger type) or with weak dispersion (for instance of BBM type). One also obtains a system which can be viewed as a dispersive perturbation of the two-dimensional Saint-Venant (shallow-water) system, generalizing the one-dimensional system studied by Amick [2] and Schonbek [30].

On the other hand, the solutions of the Cauchy problem associated to (2) should exist on time scales of order $O(\frac{1}{\epsilon})$ in order to prove that the asymptotic system is a good approximation of the water wave system [1]. In fact, the solutions of (2) cease to be relevant as approximations of those of the original system on time scales larger than $O(\frac{1}{\epsilon^2})$.

It is worth noticing that it is unlikely that all the $abcd$ systems would have global solutions. In fact they are shown to be Hamiltonian (and thus possess a formally conserved energy) only when $b = d$ (see [6]). This can be used in the one-dimensional case to obtain the global well-posedness of the Cauchy problem for a few of the systems see [7] but this situation is exceptional and moreover does not apply to the two-dimensional case where the Hamiltonian does not control any Sobolev type norm.

This situation is in strong contrast with one-directional models such as the Korteweg-de Vries (KdV) or Benjamin-Bona-Mahony (BBM) equations or with quasi one-directional models such as the Kadomtsev-Petviashvili (KP) equation where global a priori bounds are available (possibly at low degree of regularity), which allows to prove the global well-posedness of the Cauchy problem.
As far as we know the problem of solving any of the systems \((2)\) on time intervals of order \(O(\frac{1}{\epsilon})\) has been completely open until very recently, except for some one-dimensional ones which have global solutions (under some restrictions on the initial data).

In [5] Bona, Colin and Lannes obtained an order \(O(\frac{1}{\epsilon})\) existence interval for a fully symmetric class of systems, in one and two dimensions. Those systems have a skew-adjoint linear part \((a = c)\) and are obtained from \((2)\) via the nonlinear change of variables \(\tilde{v} = v(1 + \frac{\epsilon}{2} \eta)\) which does not affect the linear part (modulo higher order terms in \(\epsilon\)) and which symmetrizes the nonlinear part. Neglecting the higher order terms in \(\epsilon\), one obtain skew-adjoint perturbations of symmetric quasilinear hyperbolic systems and the classical theory of this kind of systems provides the \(O(\frac{1}{\epsilon})\) existence time. This method does not use the dispersive part of the systems and of course does not solve the long time existence problem for the \(abcd\) systems.

In [27] an approach based on a Nash-Moser theorem is developed to prove well-posedness results for the 2D- Boussinesq systems \(^1\) on time intervals of order \(1/\epsilon\).

In [27] the generic case, that is when \(b > 0, d > 0, a < 0, c < 0\) and the \(BBM/BBM\) case, that is when \(b > 0, d > 0, a = c = 0\) are emphasized but the approach could very likely be applied to other cases. This method does not use the dispersive part of the systems and a high regularity level in required on the initial data, with loss of regularity on the solution.

The aim of the present paper is to prove the well-posedness for the most dispersive of the two-dimensional Boussinesq systems \(^2\) on time scales of order \(\epsilon^{-1/2}\), using the dispersive properties of the systems. This allows to consider relatively rough initial data. We will in particular obtain uniformly bounded in \(\epsilon\) solutions on time scales of order \(T/\sqrt{\epsilon}\) in suitable Sobolev spaces, achieving the rigorous justification of those models on the corresponding time scales (see [1]). Note however that we are not able in our functional setting to reach the optimal time scales \(O(\frac{1}{\epsilon})\).

The heart of the paper concerns the most dispersive Boussinesq system (the \(KdV-KdV\) system) where \(a = c = 1/6, b = d = 0\) which is an interesting two-dimensional extension of the Korteweg-de Vries equation \(^3\) for which the local well-posedness is not a simple matter. Our method deeply lies on dispersive estimates for the underlying linear problem. In particular we establish new Strichartz and maximal function estimates.

For the other cases (which are less dispersive), at least when \(d > 0\), the proofs are the extension to the two-dimensional case of those given by energy estimates in [7] for the one-dimensional case, keeping track of the \(\epsilon\)'s but we do not pursue this issue here. We will focus instead on the order 2 Boussinesq systems which can be written as systems of nonlinear nonlocal Schrödinger type equations coupled by the nonlinear terms. Some of those cases have been studied in [13] but the \(\epsilon\) dependence of the existence time interval was not addressed here and the difficulty linked to the case \(d = 0\) was underestimated.

1.2. Organization of the paper. The paper is organized as follows.

Section 2 recall briefly some useful facts on the Boussinesq systems.

---

\(^1\)This approach gives also similar results in the one dimensional case.

\(^2\)There are relatively few physically relevant systems of this form.

\(^3\) When \(d = 0\) it is unclear whether the local well-posedness can be obtained by elementary energy methods in the 2D case without imposing the irrotationality of the velocity.
In Section 3 we will consider the \( KdV-KdV \) Boussinesq system, and establish the well-posedness in the two-dimensional case, on time intervals of order \( \epsilon^{-1/2} \), after establishing various dispersive estimates for the solutions of the linear part.

In Section 4, we consider the well-posedness of the remaining *strongly dispersive* cases (Schrödinger type) to prove that there are also well-posed on time intervals of order \( \epsilon^{-1/2} \). There are essentially two cases, depending whether \( b \) or \( d \) vanishes.

1.3. Notations. For any positive numbers \( a \) and \( b \), the notation \( a \lesssim b \) means that there exists a positive constant \( c \) such that \( a \leq cb \). \( C \) or \( c \) will also denote various positive constants independent of \( \epsilon \).

We denote the horizontal variables by \( x \) when \( n = 1 \) and by \( x = (x_1, x_2) \) when \( n = 2 \). We will also denote by \( \cdot \) the euclidian scalar product of two vectors \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) of \( \mathbb{R}^2 \), which is to say \( x \cdot y = x_1 y_1 + x_2 y_2 \), and by \( |x| \) the euclidian norm of \( x = (x_1, x_2) \), i.e. \( |x| = \sqrt{x_1^2 + x_2^2} \).

We use the Fourier multiplier notation: \( f(D)u \) is defined as \( \mathcal{F}(f(D)u)(\xi) = f(\xi)\hat{u}(\xi) \), where \( \mathcal{F} \) and \( \hat{} \) stand for the Fourier transform, which is defined by

\[
\mathcal{F}(\phi)(\xi) = \hat{\phi}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \phi(x) e^{-ix \cdot \xi} \, dx,
\]

for any function \( \phi : \mathbb{R}^2 \to \mathbb{C} \).

For \( s \in \mathbb{R} \), we define the Bessel and Riesz potentials of order \( -s \) by \( \Lambda^s = (1 + |D|^2)^{s/2} \) and \( D^s = |D|^s \). Moreover, \( \sqrt{-\Delta} \) will denote the Fourier multiplier of symbol \( |\xi|^2 \). Observe that \( \sqrt{-\Delta} = D^1 \).

Let \( R_j \) be the Riesz transforms, defined via Fourier transform by \( R_j \phi = (-i \xi_j \hat{\phi})^\vee \), for \( j = 1, 2 \).

The divergence operator will be denoted by \( \nabla \cdot \) or \( \text{div} \).

We denote by \( \| \cdot \|_{L^p} \) \((1 \leq p \leq \infty)\) the standard norm of the Lebesgue spaces \( L^p(\mathbb{R}^n) \) \((n = 1, 2)\) and \( (\cdot, \cdot) \) the scalar product in \( L^2 \).

If \( v = (v_1, v_2)^T \in L^2(\mathbb{R}^2)^2 \), then we write \( \|v\|_{L^2} = (\|v_1\|_{L^2}^2 + \|v_2\|_{L^2}^2)^{1/2} \).

If \( v = (v_1, v_2)^T \in L^\infty(\mathbb{R}^2)^2 \), then we write \( \|v\|_{L^\infty} = \|v_1\|_{L^\infty} + \|v_2\|_{L^\infty} \).

The standard notation \( H^s(\mathbb{R}^n) \), or simply \( H^s \) if the underlying domain is clear from the context, is used for the \( L^2 \)-based Sobolev spaces; their norm is written \( \| \cdot \|_{H^s} \).

Finally, if \( u = u(x, t) \) is a function defined in \( \mathbb{R}^2 \times [0, T] \), respectively in \( \mathbb{R}^2 \times \mathbb{R} \) and \( 1 \leq p, q \leq \infty \), we define the mixed space-time spaces \( L^p_t L^q_x \) and \( L^q_x L^p_t \), respectively, by the norms

\[
\|u\|_{L^p_t L^q_x} = \left( \int_0^T \|u(\cdot, t)\|_{L^q_x}^p \, dt \right)^{1/p}, \quad \text{resp.} \quad \|u\|_{L^q_x L^p_t} = \left( \int_\mathbb{R} \|u(\cdot, t)\|_{L^p_t}^q \, dt \right)^{1/q},
\]

and \( L^p_t L^q_x \), respectively \( L^q_x L^p_t \), by the norms

\[
\|u\|_{L^p_t \dot{L}^q_x} = \left( \int_{\mathbb{R}^d} \|u(x, \cdot)\|_{L^q_x}^p \, dx \right)^{1/p}, \quad \text{resp.} \quad \|u\|_{\dot{L}^q_x L^p_t} = \left( \int_{\mathbb{R}^d} \|u(x, \cdot)\|_{L^p_t}^q \, dx \right)^{1/q}.
\]

2. Classification of Boussinesq Systems

The Boussinesq systems can be conveniently classified according to the linearization at the null solutions which display their dispersive properties \cite{1}. More precisely, the dispersion matrix writes in Fourier variables,
\[
\hat{A}(\xi_1, \xi_2) = i \begin{pmatrix}
0 & \frac{\xi_2(1-c|\xi|^2)}{1+ed|\xi|^2} & \frac{\xi_2(1-c|\xi|^2)}{1+ed|\xi|^2} \\
\frac{\xi_1(1-\epsilon a|\xi|^2)}{1+ed|\xi|^2} & 0 & 0 \\
\frac{\xi_1(1-\epsilon c|\xi|^2)}{1+ed|\xi|^2} & 0 & 0
\end{pmatrix}.
\]

The corresponding non zero eigenvalues are
\[
\lambda_{\pm} = \pm i |\xi| \left( \frac{(1 - \epsilon a|\xi|^2)(1 - \epsilon c|\xi|^2)}{(1 + \epsilon d|\xi|^2)(1 + \epsilon b|\xi|^2)} \right)^{\frac{1}{2}}.
\]

Recall [6] that the well-posedness of the linearized Boussinesq system requires that \( b \geq 0, \ d \geq 0 \) and \( a \leq 0, \ c \leq 0 \), (or \( a = c \)).

The order of \( \lambda_{\pm} \) determine the strength of the dispersion in the Boussinesq systems, the more dispersive one corresponding to \( b = d = 0 \), the so called KdV-KdV case which will be studied in details in the next section.

By “weakly dispersive” Boussinesq systems, we mean the case where \( b > 0 \) and \( d > 0 \), so that \( \lambda_{\pm} \) have order 1, 0 or \(-1\).

This situation has been studied in [15] where it is established for instance in the generic case with \( a < 0 \) and \( c < 0 \) that the Cauchy problem is well-posed in \( C([0,T_0]; H^1(\mathbb{R}^2))^2 \) where \( T_0 \) is of order \( O\left(\lambda^{\beta}\right) \) for any \( \beta < \frac{1}{2} \).

As already mentioned, existence on time intervals of order \( O\left(\lambda^{\beta}\right) \) has been established for weakly dispersive Boussinesq systems when the initial data are smooth (and with a loss of derivatives) in [27].

3. The KdV-KdV Boussinesq system

We are interested in the following dispersive system

\[
\begin{align*}
\partial_t \eta + \text{div} \, v + \epsilon \text{div} (\eta v) + cd \text{div} \Delta v &= 0, \\
\partial_t v + \nabla \eta + \epsilon \frac{1}{2} \nabla (|v|^2) + \epsilon \nabla \Delta \eta &= 0,
\end{align*}
\]

which corresponds to the case where \( b = d = 0 \) and \( a = c = \frac{1}{6} \). We have scaled \( a \) and \( c \) to the value 1. This system might not be the best one as modeling and numerical purposes are concerned (in particular the cubic dispersion terms induce serious numerical difficulties). Nevertheless [3] is a mathematically interesting system since it can be written as a nonlocal two-dimensional version of a KdV-type system. Previous results concerned the one-dimensional version:

\[
\begin{align*}
\eta_t + v_x + \epsilon (\eta v)_x + \epsilon v_{xxx} &= 0, \\
v_t + \eta_x + \frac{\epsilon}{2} (v^2)_x + \epsilon \eta_{xxx} &= 0,
\end{align*}
\]

Actually, as noticed in [7], the change of variable \( \eta = u + w, \ v = u - w \) reduces [4] to the following system:

\[
\begin{align*}
u_t + u_x + \epsilon u_{xxx} + \epsilon [2uw_x + (uw)_x] &= 0, \\
w_t - w_x - \epsilon w_{xxx} + \epsilon [-2wv_x + (uw)_x] &= 0,
\end{align*}
\]

which is a system of KdV type with uncoupled (diagonal) linear part. Thus (see [7]) the Cauchy problem is easily seen to be locally well-posed for initial data in \( H^s(\mathbb{R}) \times H^s(\mathbb{R}) \), \( s > \frac{3}{4} \) by the results in [20], [21]. On the other hand, as noticed in
Appendix A in a slightly different context, a minor modification of Bourgain’s method as used in [23] allows to solve the Cauchy problem for [3] for data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s > -\frac{1}{2}$. We refer to [8] for details. It is worth noticing that in [8] the question of the dependence of the existence time with respect to $\epsilon$ is not considered.

Coming back to the two-dimensional system, we will establish that the Cauchy problem is locally well-posed for data in $H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)^2$ where $s > \frac{3}{2}$.

**Theorem 1.** Let $s > \frac{3}{2}$ and $0 < \epsilon \leq 1$ be fixed. Then for any $(\eta_0, v_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)^2$ with curl $v_0 = 0$, there exist a positive time $T = T(\|\eta_0, v_0\|_{H^s(\mathbb{R}^2)^2})$, a space $Y_T^s$, such that

$$
Y_T^s \hookrightarrow C([0, T_s] ; H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)^2),
$$

and a unique solution $(\eta, v)$ to (3) in $Y_T^s$, satisfying $(\eta, v)|_{t=0} = (\eta_0, v_0)$, where $T_s = T \epsilon^{-\frac{1}{2}}$.

Moreover, for any $T' \in (0, T_s)$, there exists a neighborhood $\Omega^s$ of $(\eta_0, v_0)$ in $H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)^2$ such that the flow map associated to (3) is smooth from $\Omega^s$ into $Y_{T'}^s$.

**Remark 1.** Our proof applies as well with minor modifications to Boussinesq systems with $b = d = 0$ and $a < 0$, $c < 0$. While this case is excluded for systems modeling purely gravity waves, it may happen for capillary-gravity waves when the surface tension parameter $\tau$ is large enough.

**Remark 2.** It transpires from the proof of Theorem 1 that the solution is uniformly bounded (with respect to $\epsilon$) in the corresponding spaces on a time interval $[0, \frac{T}{\epsilon^{\frac{1}{2}}}].$

The strategy is first to diagonalize the linear part of (3). We thus reduce the original system to a **nonlocal** one. In particular, the nonlinear part involves order zero pseudo-differential operators (in fact Riesz transforms). We then solve the underlying Duhamel integral formulation by a fixed point argument in a ball of a Banach space constructed from the various dispersive estimates satisfied by the linear part. Note that here, we are able to compute the dependence on $\epsilon$ of the constants appearing in the linear estimates. This allows us to obtain an existence result in a time interval $[0, T]$ depending on $\epsilon$, in our case $T \sim \epsilon^{-\frac{1}{2}}$.

We will thus proceed as follows. We first derive the Hamiltonian formulation of (3). We then perform the diagonalization and state the dispersive estimates which are to be used in the fixed point argument. The maximal function and Strichartz estimates seem to be new. Finally, we solve the new system (10) which is equivalent to (3), assuming curl $v_0 = 0$.

3.1. **Hamiltonian structure.** The system (3) can be rewritten on the form

$$
\partial_t u + A_\epsilon u + \epsilon N(u) = 0,
$$

where

$$
\begin{pmatrix}
\eta \\
v_1 \\
v_2
\end{pmatrix}, 
A_\epsilon =
\begin{pmatrix}
0 & (1 + \epsilon \Delta)\partial_{x_1} & (1 + \epsilon \Delta)\partial_{x_2} \\
(1 + \epsilon \Delta)\partial_{x_1} & 0 & 0 \\
(1 + \epsilon \Delta)\partial_{x_2} & 0 & 0
\end{pmatrix},
$$

$$
\begin{pmatrix}
\eta \\
v_1 \\
v_2
\end{pmatrix}.
$$
Therefore, it follows that
\[ H \]
where we used the fact that \( J \)
\[ J = \begin{pmatrix} 0 & \partial_{x_1} & \partial_{x_2} \\ \partial_{x_1} & 0 & 0 \\ \partial_{x_2} & 0 & 0 \end{pmatrix}. \]

We will denote by \((\cdot, \cdot)\) the scalar product on \( L^2(\mathbb{R}^2; \mathbb{R}^3) \), i.e.
\[ (u, \tilde{u}) = \int_{\mathbb{R}^2} (\eta \tilde{u} + v_1 \tilde{v}_1 + v_2 \tilde{v}_2) \, dx_1 \, dx_2 \]
and by \( J \) the skew adjoint matrix
\[ J = \begin{pmatrix} 0 & \partial_{x_1} & \partial_{x_2} \\ \partial_{x_1} & 0 & 0 \\ \partial_{x_2} & 0 & 0 \end{pmatrix}. \]

Then, the system (3) is equivalent to
\[ \partial_t u = -J \begin{pmatrix} (1 + \epsilon \Delta) \eta + \frac{\epsilon}{2} |v|^2 \\ (1 + \epsilon \Delta) v_1 + \epsilon \eta v_1 \\ (1 + \epsilon \Delta) v_2 + \epsilon \eta v_2 \end{pmatrix} = J(\text{grad } H_\epsilon)(u), \]
where \( H_\epsilon(u) \) is the functional given by
\[ H_\epsilon(u) = \frac{1}{2} \int_{\mathbb{R}^2} (\epsilon |\nabla \eta|^2 + \epsilon |\nabla v|^2 - \eta^2 - |v|^2 - \epsilon \eta |v|^2) \, dx_1 \, dx_2. \]
Therefore, it follows that \( H_\epsilon \) is a conserved quantity by the flow of (3), since
\[ \frac{d}{dt} H_\epsilon(u) = H'_\epsilon(u) \partial_t u = (\langle \text{grad } H_\epsilon(u), \partial_t u \rangle, \langle \text{grad } H_\epsilon(u), J(\text{grad } H_\epsilon(u)) \rangle) = 0, \]
where we used the fact that \( J \) is skew adjoint.

3.2. Linear estimates.

3.2.1. Diagonalization. We transform here (3) into an equivalent system with a diagonal linear part.

First, we observe that the Fourier transform of \( A \) is given by
\[ \hat{A}_\epsilon(\xi_1, \xi_2) = \begin{pmatrix} 0 & i(1 - \epsilon |\xi|^2) \xi_1 & i(1 - \epsilon |\xi|^2) \xi_2 \\ i(1 - \epsilon |\xi|^2) \xi_1 & 0 & 0 \\ i(1 - \epsilon |\xi|^2) \xi_2 & 0 & 0 \end{pmatrix}. \]

We compute the characteristic polynomial of \( \hat{A}_\epsilon(\xi_1, \xi_2) \),
\[ \chi_{\hat{A}_\epsilon(\xi_1, \xi_2)}(\lambda) = -\lambda(\lambda^2 + (1 - \epsilon |\xi|^2)^2 |\xi|^2), \]
so that its eigenvalues are given by
\[ \lambda_0 = 0, \quad \lambda_1 = i(1 - \epsilon |\xi|^2) |\xi|, \quad \text{and} \quad \lambda_2 = -i(1 - \epsilon |\xi|^2) |\xi|, \]
with associated eigenvectors
\[ E_0 = \begin{pmatrix} \frac{\xi_1}{|\xi|} \\ \frac{\xi_2}{|\xi|} \\ -\frac{\xi_1}{|\xi|} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 1 \\ \frac{\xi_1}{|\xi|} \\ \frac{\xi_2}{|\xi|} \end{pmatrix}, \quad \text{and} \quad E_2 = \begin{pmatrix} -1 \\ \frac{\xi_1}{|\xi|} \\ \frac{\xi_2}{|\xi|} \end{pmatrix}. \]
Then, if we denote by \( \hat{P} \) the matrix of \((E_0,E_1,E_2)\) in the canonical basis, we have that
\[
\hat{P} = \begin{pmatrix}
0 & 1 & -1 \\
\frac{\epsilon}{\xi} & \frac{\epsilon}{\xi} & \frac{\epsilon}{\xi} \\
0 & 0 & 0
\end{pmatrix}, \quad \text{and} \quad \hat{P}^{-1} = \frac{1}{2} \begin{pmatrix}
0 & 2iR_2 & -2iR_1 \\
iR_1 & iR_2 & iR_2 \\
iR_1 & iR_2 & iR_2
\end{pmatrix}.
\]

Therefore, the linear part of \((\ref{equation})\) is equivalent to \(\partial_t \mathbf{w} + D \mathbf{w} = 0\), where
\[
\mathbf{w} = \begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix} = P^{-1} \mathbf{u}, \quad P^{-1} = \frac{1}{2} \begin{pmatrix}
0 & 2iR_2 & -2iR_1 \\
iR_1 & iR_2 & iR_2 \\
iR_1 & iR_2 & iR_2
\end{pmatrix},
\]
and
\[
D = \begin{pmatrix}
0 & 0 & 0 \\
0 & i(1+\epsilon \Delta)\sqrt{-\Delta} & 0 \\
0 & 0 & -i(1+\epsilon \Delta)\sqrt{-\Delta}
\end{pmatrix}.
\]

Next, we turn to the nonlinear part of \((\ref{equation})\). \(\mathcal{N}(\mathbf{u})\) is given as function of \(\mathbf{w}\) by
\[
\begin{pmatrix}
\partial_{x_1} \left( ((w_1 - w_2)(R_2w_0 + R_1(w_1 + w_2))) + \partial_{x_2} \left( (w_1 - w_2)(-R_1w_0 + R_2(w_1 + w_2)) \right) \right) \\
\frac{1}{\xi} \partial_{x_1} \left( (R_2w_0 + R_1w_1 + R_1w_2)^2 + (-R_1w_0 + R_2w_1 + R_2w_2)^2 \right) \\
\frac{1}{\xi} \partial_{x_2} \left( (R_2w_0 + R_1w_1 + R_1w_2)^2 + (-R_1w_0 + R_2w_1 + R_2w_2)^2 \right)
\end{pmatrix}.
\]

Then, deduce, using the identities \( R_2 \partial_{x_1} = R_1 \partial_{x_2} \) and \( R_1 \partial_{x_1} + R_2 \partial_{x_2} = \sqrt{-\Delta} \), that
\[
P^{-1} \mathcal{N}(\mathbf{u}) = \frac{1}{2} \begin{pmatrix}
0 \\
I + II \\
-1 + II
\end{pmatrix} =: \tilde{\mathcal{N}}(\mathbf{w}),
\]
where
\[
I = I(w_1, w_2) = (w_1 - w_2)\sqrt{-\Delta}(w_1 + w_2) + \partial_{x_1}(w_1 - w_2)R_1(w_1 + w_2) + \partial_{x_2}(w_1 - w_2)R_2(w_1 + w_2),
\]
and
\[
II = II(w_1, w_2) := i\sqrt{-\Delta} \left( (R_1(w_1 + w_2))^2 + (R_2(w_1 + w_2))^2 \right).
\]

Summarizing we have that \((\ref{equation})\) is equivalent to
\[
\begin{cases}
\partial_t w_0 + i(1+\epsilon \Delta)\sqrt{-\Delta}w_1 + (I + II)(w_1, w_2) = 0 \\
\partial_t w_2 - i(1+\epsilon \Delta)\sqrt{-\Delta}w_2 + (-I + II)(w_1, w_2) = 0
\end{cases},
\]
where \(I(w_1, w_2)\) and \(II(w_1, w_2)\) are defined in \((\ref{eq:I})\) and \((\ref{eq:II})\).

**Remark 3.** Note that we use in our analysis that \(w_0 = 0\). Indeed, the equation on \(w_0\) is \(\partial_t w_0 = 0\). Moreover,
\[
w_0 = 0 \iff R_2v_1 = R_1v_2 \iff \text{curl } \mathbf{v} = 0.
\]

We observe that this condition is physically relevant. The Boussinesq systems derived from the water waves equations, where the fluid is supposed to be irrotational and \(\mathbf{v}\) is an \(O(\epsilon^2)\) approximation of the horizontal velocity at a certain depth which is a gradient. Note also that since the equation for \(\mathbf{v}\) writes \(\partial_t \mathbf{v} = \nabla F\), the condition \(\text{curl } \mathbf{v} = 0\) is preserved by the evolution.
Theorem 3. Without loss of generality, we can assume that
\[ \begin{aligned}
  \text{Proof.} \\
  \end{aligned} \]

where the implicit constant does not depend on \( \epsilon \).

\[ \begin{aligned}
  \text{The maximal function estimate.} \\
  \end{aligned} \]

3.2.3. Dispersive smoothing estimates. We will use the results in \([19]\) (see also the general results in \([25]\)) to deduce the following local smoothing estimates.

Let \( \{Q_\alpha\}_{\alpha \in \mathbb{Z}^2} \) denote a family of nonoverlapping cubes of unit size such that \( \mathbb{R}^2 = \bigcup_{\alpha \in \mathbb{Z}^2} Q_\alpha \).

**Theorem 2.** Let \( T > 0 \) and \( \epsilon > 0 \). Then, it holds that
\[ \begin{aligned}
  \sup_{\alpha} \left( \int_{Q_\alpha} \int_0^T \left| D_x^\epsilon \left( U_\epsilon^\pm(t) u_0(x) \right) \right|^2 dt dx \right)^{\frac{1}{2}} \lesssim \epsilon^{-\frac{1}{2}} \| u_0 \|_{L^2},
\end{aligned} \]
where the implicit constant does not depend on \( \epsilon \) and \( T \).

**Proof.** Without loss of generality, we can assume that \( Q_\alpha = Q := \{ x : |x| < 1 \} \). We fix a function \( \eta \) in \( C_c^\infty(\mathbb{R}) \) such that \( 0 \leq \eta \leq 1 \), \( \eta \equiv 1 \) on \( [-1,1] \) and \( \text{supp} \eta \subset [0,2] \). Let us define \( \eta_\epsilon(x) = \epsilon^2 \eta(\epsilon^2 x) \). Then \( P_{\leq \epsilon^{-\frac{1}{2}}} \), respectively \( P_{>\epsilon^{-\frac{1}{2}}} \) denote the operators defined by
\[ \begin{aligned}
  P_{\leq \epsilon^{-\frac{1}{2}}} u_0 &= \mathcal{F}^{-1}(\eta_\epsilon(|\cdot|) \hat{u}_0) \\
  P_{>\epsilon^{-\frac{1}{2}}} &= 1 - P_{\leq \epsilon^{-\frac{1}{2}}},
\end{aligned} \]

Observe that in the support of \( 1 - \eta_\epsilon(|\cdot|) \), we have that \( |\nabla \varphi_\epsilon(\xi)| = 3\epsilon |\xi|^2 - 1 > 0 \). Then it follows by using Theorem 4.1, formula (4.2), p 54-55 in \([19]\) that
\[ \begin{aligned}
  \| D_x^\epsilon \left( U_\epsilon^\pm(t) u_0(x) \right) \|_{L^2_{Q_\alpha \times [0,T]}} \lesssim \left( \int_{|\xi| > \epsilon^{-\frac{1}{2}}} \frac{|\xi|^2}{3\epsilon |\xi|^2 - 1} |\hat{u}_0(\xi)|^2 d\xi \right)^{\frac{1}{2}} \lesssim \epsilon^{-\frac{1}{2}} \| u_0 \|_{L^2},
\end{aligned} \]
which concludes the proof of Theorem 2.

3.2.3. The maximal function estimate. We will prove our maximal function estimate in the \( n \)-dimensional case. In other words, we will consider the unitary group \( U_\epsilon^\pm(t) = e^{\pm it\varphi_\epsilon(D)} \), where \( \varphi_\epsilon(\xi) = \varphi_\epsilon(|\xi|) = \epsilon |\xi|^3 - |\xi| \) and \( \xi \in \mathbb{R}^n \), for \( n \geq 2 \). Let \( \{Q_\alpha\}_{\alpha \in \mathbb{Z}^n} \) denote the mesh of dyadic cubes of unit size. The main result of this subsection reads as follows.

**Theorem 3.** With the above notation, for any \( s > \frac{3n}{4} \), \( \epsilon > 0 \) and \( T > 0 \) satisfying \( \epsilon T \leq 1 \), it holds that
\[ \begin{aligned}
  \left( \sum_{\alpha \in \mathbb{Z}^n} \sup_{|t| \leq T} \sup_{x \in Q_\alpha} |U_\epsilon^\pm(t) u_0(x)|^2 \right)^{\frac{1}{2}} \lesssim (1 + T^{\frac{n}{2} - \frac{1}{2}}) \| u_0 \|_{L^s},
\end{aligned} \]
where the implicit constant does not depend on \( \epsilon \) and \( T \).
We will only treat the case of $U_+^e$, since the case of $U_-^e$ is similar. The proof of Theorem 3 is based on the next lemma.

**Lemma 1.** For $k \in \mathbb{Z}_+$, let $\psi_k \in C_0^\infty([2^{k-1}, 2^{k+1}])$ be such that $0 \leq \psi_k \leq 1$. Then for $ct \in (0, 2]$,
\begin{equation}
\left| \int_{\mathbb{R}^n} e^{i(x \cdot \xi)} \psi_k(|\xi|) d\xi \right| \leq c H_k(|x|),
\end{equation}
where $H_k$ is decreasing and satisfies
\begin{equation}
\int_{\mathbb{R}^n} H_k(|x|) dx \lesssim (1 + |t|^{\frac{n}{2} - \frac{k}{2}})^{\frac{k}{2}} c^n,
\end{equation}
and $H_k(r) \leq c^{\frac{k}{2}}$ for $r \in (0, 10)$. Also, a similar result holds for $\psi \in C_0^\infty([-10, 10])$ with $2^{\frac{k}{2}}$ replaced by $c$. Observe that the implicit constant does not depend on $\epsilon$ and $t$, but may depend on the dimension $n$.

The following estimate, essentially proved in Proposition 2.6 of [20], will be useful.

**Proposition 1.** For $k \in \mathbb{Z}_+$, $ct \in (0, 2]$, $r \in \mathbb{R}$ and $\psi_k$ as in Lemma 1, define
\begin{equation}
I_k(t, r) = \int_0^{+\infty} e^{i(cts^3 + s \alpha)} \psi_k(s) ds.
\end{equation}
Then
\begin{equation}
|I_k(t, r)| \leq F_k(r) = \left\{ \begin{array}{ll}
\frac{2^k}{c} & \text{for } |r| \leq 1 \\
\frac{2^k}{c} r^{-\frac{1}{2}} & \text{for } 1 \leq |r| \leq c 2^k \\
\frac{1}{c^2} r^{-N} & \text{for } |r| > c 2^k
\end{array} \right.
\end{equation}
for any $N \in \mathbb{Z}_+$.

Moreover, it is known (see [32] for example) that the Fourier transform of a radial function $f(|x|) = f(s)$ is still radial and is given by
\begin{equation}
\hat{f}(r) = \hat{f}(|\xi|) = r^{-\frac{\alpha d}{2}} \int_0^{+\infty} f(s) J_{\frac{\alpha d}{2}}(rs) s^{\frac{d}{2}} ds,
\end{equation}
where $J_m$ is the Bessel function, defined by
\begin{equation}
J_m(r) = \frac{(r/2)^m}{\Gamma(m + 1/2) \pi^{1/2}} \int_{-1}^{1} e^{irs} (1 - s^2)^{m-\frac{1}{2}} ds, \quad \text{for } m > -\frac{1}{2}.
\end{equation}

Next, we list some properties of the Bessel functions (see [22], [22], [16] and the references therein).

**Lemma 2.** It holds that
\begin{equation}
J_m(r) = O(r^m), \quad r \to 0
\end{equation}
\begin{equation}
J_m(r) = \sum_{j=0}^{N} \alpha_{m,j} r^{-\left(j + \frac{1}{2}\right)} + e^{ir} \sum_{j=0}^{N} \alpha_{m,j} r^{-\left(j + \frac{1}{2}\right)} + O(r^{-\left(N + \frac{1}{2}\right)}), \quad r \to +\infty,
\end{equation}
for any $N \in \mathbb{Z}_+$, and
\begin{equation}
r^{-\frac{\alpha d}{2}} J_{\frac{\alpha d}{2}}(r) = c_n \mathcal{R}(e^{i \alpha h(r)}),
\end{equation}
where $\mathcal{R}$ is the real part of a complex function.
where \( h \) is a smooth function satisfying

\[
|\partial_x^k h(r)| \leq c_k (1 + r)^{-\frac{n-1}{2}-k},
\]

for any \( k \in \mathbb{Z}_+ \).

**Proof of Lemma 1** By using (21), it follows that

\[
I_k(t, r) := \int_{\mathbb{R}^n} e^{i(\varphi_x(\xi) + x \cdot \xi)} \psi_k(|\xi|) d\xi = r^{-\frac{n-2}{2}} \int_0^{+\infty} e^{it(s^2 - s)} \psi_k(s) J_{\frac{n-2}{2}}(rs) s^{\frac{n}{2}} ds,
\]

where \( r = |x| \in [0, +\infty) \) and \( c \in [0, 2] \).

When \( 0 \leq |r - t| \leq 1 \) or \( 0 \leq r \leq 1 \), it follows from (24) (or (25)–(26)) that

\[
|I_k(t, r)| \lesssim \int_0^{+\infty} \psi_k(s) s^{n-1} ds = c2^k.
\]

In the case \( r > 1 \) and \( |r - t| > 1 \), we substitute \( J_{\frac{n-2}{2}} \) by the right-hand side of (24) in (27) and evaluate successively each term of the sum and the remainder. Here we consider only the most difficult case, when \( \alpha_{m,j} = 0 \). Then, the \( j^{th} \) term has the form

\[
I_{k,j}(t, r) := r^{-\frac{n-2}{2}} \int_0^{+\infty} e^{it(s^3 - s) + sr} \psi_k(s)(rs)^{-\frac{j}{2}} s^\frac{n}{2} ds,
\]

so that

\[
|I_{k,j}(t, r)| \leq r^{-\frac{n-2}{2}} \left( 2^k r^{-\frac{j}{2}} \right) 2^k \left| \int_0^{+\infty} e^{it(s^3 + s(r-t))} \tilde{\psi}_k(s) ds \right|,
\]

where \( \tilde{\psi}_k \) is another function satisfying \( \tilde{\psi}_k \in C^\infty([2^k-1, 2^k+1]) \) and \( 0 \leq \tilde{\psi}_k \leq 1 \).

Therefore, we deduce from Proposition 1 that

\[
|I_{k,j}(t, r)| \lesssim \left\{ \begin{array}{ll}
2^k (\frac{j}{2} - j) r^{-\frac{j}{2}} \left( \frac{j}{2} + j \right) |r - t|^{-\frac{j}{2}} & \text{for } |r - t| \in [1, 2^k] \\
2^k (\frac{j}{2} - j) r^{-\frac{j}{2}} \left( \frac{j}{2} + j \right) |r - t|^{-m} & \text{for } |r - t| > 2^k,
\end{array} \right.
\]

for any \( m \in \mathbb{Z}_+ \). Next, we fix \( N = N(n) > \frac{n-1}{2} \) and bound the remainder

\[
R_k(t, r) := r^{-\frac{n-2}{2}} \int_0^{+\infty} e^{it\varphi_x(s) + sr} \psi_k(s)(rs)^{-\frac{j}{2}} s^\frac{n}{2} ds
\]

as follows

\[
|R_k(t, r)| \lesssim 2^k (\frac{j}{2} - N - \frac{j}{2}) r^{-\frac{j}{2} + N + \frac{j}{2}} \lesssim r^{-m},
\]

with \( m > n \).

Therefore, if we define

\[
H_k(|x|) = \left\{ \begin{array}{ll}
2^k n \sum_{j=0}^{N} 2^{k(\frac{j}{2} - j)} |x|^{-\frac{j}{2}} & \text{for } |x| \leq 1 \text{ or } ||x| - t| \leq 1 \\
2^k (\frac{j}{2} - j) |x| - t|^{-m} & \text{for } 1 \leq ||x| - t| \leq 2^k \\
2^k (\frac{j}{2} - j) |x| - t|^{-m} & \text{for } ||x| - t| > 2^k,
\end{array} \right.
\]

and

\[
H_k(|x|) = H_k(|x|) + \frac{2^k n}{(1 + |x|)^m}
\]
it follows from (27)–(30) that $|I_k(t, |x|)| \leq cH_k(|x|)$ and a simple computation leads to

$$
\int_{\mathbb{R}^n} H_k(|x|) dx \leq c2^{kn} + c2^{kn} \int_{1 \leq |r-t| \leq c2^k} r^{\frac{n}{2} - \frac{k}{2}} |r-t|^{-\frac{k}{2}} dr
$$

$$
\leq c2^{\frac{3kn}{4}} (1 + |t|^{\frac{n}{2} - \frac{k}{2}}),
$$

which concludes the proof of Lemma 1. \(\square\)

Finally, at this point, the proof of Theorem 3 follows closely the argument of Kenig, Ponce and Vega in the case of the Schrödinger equation in Theorem 3.2 of [22]. Therefore, we will omit it.

3.2.4. Strichartz estimates. Strichartz estimates for unitary groups of the form $e^{it\phi(D)}$ in $\mathbb{R}^n$, $n \geq 2$, were derived in the case where $\phi$ is an elliptic polynomial by Kenig, Ponce and Vega [19] and in the case where $\phi$ is a general polynomial in $\mathbb{R}^2$ by Ben-Artzi, Koch and Saut [3]. When the phase function $\phi$ is a radial (non-homogeneous) function and its derivative does not vanish, some techniques were recently developed by Cho and Ozawa [12] and Guo, Peng and Wang [16].

In the sequel, we will use the techniques developed in [16], based on the ones used in [19] and on the representation of the Fourier transform of a radial function in terms of the Bessel function (see formula (21)), to prove Strichartz estimates associated to the unitary groups $U^\pm_\epsilon$ defined in (12). However, in our case, we do not need to perform a dyadic decomposition in frequencies.

**Theorem 4.** Let $0 < \epsilon \leq 1$, $T > 0$ and $0 \leq \alpha < \frac{1}{2}$. Then, it holds that

$$
\|D^\alpha x U^\pm_\epsilon u_0\|_{L^q_T L^\infty_x} \lesssim \epsilon^{-\kappa_\alpha} \|u_0\|_{L^2},
$$

for all $u_0 \in L^2(\mathbb{R}^2)$, where the implicit constant is independent of $\epsilon$ and $T$, $q_\alpha$ is the root of the polynomial

$$
3q^2 - 2(7 - 2\alpha)q + 12 = 0,
$$

satisfying $q_\alpha > 2$ and $\kappa_\alpha = \frac{1}{2} + \frac{\alpha}{4} - \frac{1}{4q_\alpha}$.

**Remark 4.** When $\alpha = 0$, then $q = \frac{7+\sqrt{13}}{2} = \frac{7}{2} +$ and $\kappa = \frac{1}{2} - \frac{1}{q_\alpha} = \frac{3}{2} +$. On the other hand, we have that $\lim_{\alpha \to \frac{1}{2}} q_\alpha = 2$ and $\lim_{\alpha \to \frac{1}{2}} \kappa_\alpha = \frac{5}{8}$.

For sake of simplicity, we will fix $U_\epsilon = U^+_\epsilon$ in the rest of this subsection. First, we derive the following decay estimate for the solution to the linear problem [12].

**Proposition 2.** Let $0 < \epsilon \leq 1$ and $0 \leq \beta \leq 1$. Then, it holds that

$$
\|D^\beta x U_\epsilon(t) u_0\|_{L^\infty_x} \lesssim k_{\beta, \epsilon}(t) \|u_0\|_{L^1},
$$

for all $t \in \mathbb{R}$, where $k_{\beta, \epsilon}$ is given by

$$
k_{\beta, \epsilon}(t) = \begin{cases} 
(\epsilon t)^{-\frac{\beta}{1-\epsilon}} & \text{if } t \leq \theta \epsilon^{\frac{1}{2}} \\
\epsilon^{-\frac{\beta}{4} - \frac{\beta}{2} t^{\frac{1}{2}}} & \text{if } t \geq \theta \epsilon^{\frac{1}{2}},
\end{cases}
$$

and $\theta$ is any positive constant independent of $\epsilon$.

The proof of Proposition 2 is based on formula (21), Lemma 2 and Van der Corput’s lemma:
Lemma 3. Suppose that $f$ is a real valued $C^2$-function defined in $[a, b]$ such that $|f''(\xi)| > 1$ for any $\xi \in [a, b]$. Then
\[
\left| \int_a^b e^{i\lambda f(\xi)} \psi(\xi) d\xi \right| \lesssim |\lambda|^{-\frac{1}{2}} (\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}),
\]
where the implicit constant does not depend on $a$ and $b$.

Proof of Proposition 2. First observe that
\[
\|D_x^2 U(t)u_0\|_{L_x^\infty} \lesssim \|(|\beta| e^{-it\psi})\|_{L_x^\infty} \|u_0\|_{L^1}.
\]
On the other hand, formulas (21) and (25) imply that
\[
\left( |\beta| e^{-it\psi} \right)^{\vee} (x) = \int_0^{+\infty} s^\beta e^{it(s^3 - s)} J_0(rs) s ds
\]
where $r = |x|$. For sake of simplicity, we will assume that $t \geq 0$ and only deal with $II_\beta$ since $I_\beta$ can be handled by similar techniques. We change variables $u = (et)^{\frac{1}{2}} s$ and deduce that
\[
II_\beta(r) = (et)^{-\frac{2+\beta}{3}} A_\beta \left( \frac{r}{(et)^{\frac{3}{2}}} \right),
\]
where
\[
A_\beta(r) = \int_0^{+\infty} e^{ij r(u) h(ru)} u^{\beta + 1} du,
\]
the phase function $f_r(u)$ is given by
\[
f_r(u) = u^3 - (r + \alpha) u, \quad \text{and} \quad \alpha = t^2 e^{-\frac{r}{3}}.
\]
Therefore
\[
\sup_{r \geq 0} |A_\beta(r)| \lesssim \max\{1, \alpha^{\frac{2}{3} + \frac{1}{3}}\} = \max\{1, t^{\frac{2}{3} + \frac{1}{3}} e^{-\frac{r}{3} - \frac{1}{3}}\}, \quad \forall \beta \in [0, 1],
\]
would imply formula (32).

To prove (37), we introduce the smooth real-values functions $(\psi_1, \psi_2) \in C^\infty_0 \times C^\infty$ such that $0 \leq \psi_1, \quad \psi_2 \leq 1, \quad \psi_1(u) + \psi_2(u) = 1,$
\[
supp \psi_1 \subset \{ u : |3u^2 - (r + \alpha)| \leq \frac{r + \alpha}{2} \}
\]
and
\[
\psi_2 = 0 \quad \text{in} \quad \{ u : |3u^2 - (r + \alpha)| \leq \frac{r + \alpha}{3} \}.
\]
It follows that
\[
|A_\beta(r)| \leq |A_\beta^1(r)| + |A_\beta^2(r)|,
\]
where
\[
A_\beta^j(r) = \int_0^{+\infty} e^{ij f_r(u) h(ru)} u^{\beta + 1} \psi_j(u) du, \quad j \in \{1, 2\}.
\]

First, we deal with $A_\beta^2$. Observe that we can restrict the oscillatory integral in the range $u \in [1, +\infty]$, since otherwise when $u \in [0, 1]$, the estimate is trivial.
Moreover, we deduce from the triangle inequality that in the support of \( \psi_2 \), the derivative of the phase function satisfies \( |f'_r(u)| = |3u^2 - (r + \alpha)| > \frac{1}{3}(u^2 + (r + \alpha)). \n\)

Then, we obtain integrating by parts that
\[
J = i \int_1^{+\infty} f_r(u) \frac{d}{du} \left( \frac{h(ru)u^{\beta+1}\psi_2(u)}{f'_r(u)} \right) du.
\]

Therefore, it follows from (20) that
\[
|J|^2 \lesssim \int_1^{+\infty} \left( \frac{ru^{\beta+1}}{(u^2 + (r + \alpha))(1 + ru)^{\frac{1}{2}}} + \frac{u^{\beta+2}}{(u^2 + (r + \alpha))^2(1 + ru)^{\frac{1}{2}}} \right) du \lesssim 1.
\]

Note that the implicit constant does not depend on \( r, \epsilon \) or \( t \).

Next we turn to \( A_3^3 \). In the support of \( \psi_1 \), we have \( u \sim (r + \alpha)^{\frac{1}{2}} \), so that \( |f''_r(u)| = 6u \gtrsim (r + \alpha)^{\frac{3}{2}} \). Thus, it follows from Van der Corput’s lemma that
\[
|A_3^3(r)| \lesssim \int_1^{+\infty} \left( \frac{r}{(u + (r + \alpha))(1 + ru)^{\frac{1}{2}}} + \frac{1}{(u^2 + (r + \alpha)^2(1 + ru)^{\frac{1}{2}}} \right) du \lesssim \max(1, \alpha^{\frac{3}{2}} + \epsilon^2).
\]

Finally, we deduce formula (37) combining (38)–(40), which concludes the proof of Proposition 2.

We are now in position to give a proof of Theorem 4.

**Proof of Theorem 4.** Fix \( 0 \leq \alpha < \frac{1}{2}, \beta = 2\alpha \in [0, 1), \kappa = \kappa_\alpha, q = q_\alpha \) and \( q' \) its conjugate exponent, i.e. \( \frac{1}{q} + \frac{1}{q'} = 1 \). We first observe by using a P. Tomas’ duality argument (see for example (26)) that estimate (31) is equivalent to
\[
\| \int_{-\infty}^{+\infty} D_x^\beta U_\epsilon(t - t') g(\cdot, t') dt' \|_{L^2 L^\infty_x} \lesssim \epsilon^{-2\alpha} \| g \|_{L^q_x L^\infty_t}^q,
\]
for all \( g \in L^q(R; L^1(R^2)) \).

Next we prove estimate (41). It follows from Minkowski’s inequality and estimate (32) that
\[
\| \int_{-\infty}^{+\infty} D_x^\beta U_\epsilon(t - t') g(\cdot, t') dt' \|_{L^\infty_x} \lesssim k_{\beta, \epsilon} \ast \| g(\cdot) \|_{L^1_t(t)} := J_{\beta, \epsilon}(t),
\]
where \( k_{\beta, \epsilon} \) is defined in (24). We will denote \( \varphi(t) = \| g(\cdot, t) \|_{L^1_t} \). Then we divide the kernel \( k_{\beta, \epsilon} \) in two parts, \( k_{\beta, \epsilon}^0 = k_{\beta, \epsilon} + k_{\beta, \epsilon}^\infty \), where
\[
k_{\beta, \epsilon}^0(t) = (et)^{-\frac{\beta-1}{2}} \chi_{(|t| \leq \theta \epsilon^{-\frac{1}{2}})} \quad \text{and} \quad k_{\beta, \epsilon}^\infty(t) = e^{-\frac{\beta}{2} t} t^{\frac{\beta}{2}} \chi_{(|t| \geq \theta \epsilon^{-\frac{1}{2}})},
\]
so that
\[
J_{\beta, \epsilon}(t) = J_{\beta, \epsilon}^0(t) + J_{\beta, \epsilon}^\infty(t),
\]
where \( J_{\beta, \epsilon}^0 \), respectively \( J_{\beta, \epsilon}^\infty \), is the convolution operator associated to the kernel \( k_{\beta, \epsilon}^0 \), respectively \( k_{\beta, \epsilon}^\infty \).
To estimate $J_{\beta,\epsilon}^0$, we observe that

$$
\int_{\mathbb{R}} k_{\beta,\epsilon}^0(t) \, dt = 2 \int_0^{1/\epsilon^2} (\epsilon t)^{-2+\beta} \, dt = c \epsilon^{1+\beta} \theta^{1-\beta},
$$

since $0 \leq \beta < 1$. Therefore, it follows from Theorem 2 in Chapter III of [11] that

$$
|J_{\beta,\epsilon}^0(t)| \lesssim c \epsilon^{1+\beta} \theta^{1-\beta} \mathcal{M}(\varphi(t)),
$$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal function. On the other hand, Young’s theorem implies that

$$
|\varphi|_q \leq \frac{1}{2} \theta^{-\frac{1}{q}} \|\varphi\|_{L^q},
$$

since $q > 2$. Observe that $1 + \frac{\beta}{2} - \frac{1}{2q} > \frac{1+\beta}{2}$. Thus, we deduce gathering (43)–(46) that

$$
|J_{\beta,\epsilon}^0(t)| \lesssim \epsilon^{-1+\frac{\beta}{2} - \frac{1}{3q}} \theta^{-\frac{1}{q}} \|\varphi\|_{L^q} \mathcal{M}(\varphi(t)).
$$

Now, we choose $\theta = \theta(t)$ to minimize the term on the right-hand side of (46), which is to say

$$
\theta(t) \lesssim \frac{1}{\epsilon} \theta^{-\frac{1}{q}} \|\varphi\|_{L^q} \mathcal{M}(\varphi(t))^{-1}.
$$

This implies together with (46) that

$$
|J_{\beta,\epsilon}^0(t)| \lesssim \epsilon^{-1+\frac{\beta}{2} - \frac{1}{3q}} \|\varphi\|_{L^q} \mathcal{M}(\varphi(t))^{-1+\gamma},
$$

where $\gamma = \frac{(1-\beta)q}{5q - 2\beta q - 6}$. Then it follows that

$$
\|J_{\beta,\epsilon}\|_{L^q} \lesssim \epsilon^{-1+\frac{\beta}{2} - \frac{1}{3q}} \|\varphi\|_{L^q} \mathcal{M}(\varphi(t))^{-1} \|\varphi\|_{L^q}^{-\gamma} \|\varphi\|_{L^q(1-\gamma)}^{-\gamma}.
$$

Moreover, observe that

$$(1 - \gamma)q = q' \iff 3q^2 - 2(7 - \beta)q + 12 = 0.$$

Finally, we conclude from the fact the maximal function is continuous in $L^q$ that

$$
\|J_{\beta,\epsilon}\|_{L^q} \lesssim \epsilon^{-1+\frac{\beta}{2} - \frac{1}{3q}} \|\varphi\|_{L^q},
$$

which concludes the proof of Theorem 4. \hfill \Box

3.3. The nonlinear Cauchy problem. The objective of this subsection is to prove the following result.

**Theorem 5.** Let $s > \frac{3}{2}$ and $0 < \epsilon \leq 1$ be fixed. Then for any $(w_0^0, w_0^2) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}^2)$, there exist a positive time $T = T(||(w_0^0, w_0^2)||_{H^s \times H^s})$, a space $X_{T_*}^s$ such that

$$
X_{T_*}^s \hookrightarrow C([0, T_*] ; H^s(\mathbb{R}) \times H^s(\mathbb{R}^2)),
$$

and a unique solution $(w_1, w_2)$ to (11) in $X_{T_*}^s$ satisfying $(w_1, w_2)_{t=0} = (w_0^0, w_0^2)$, where $T_* = T_\epsilon^{-\frac{1}{q'}}$.

Moreover, for any $T' \in (0, T_\epsilon)$, there exists a neighborhood $\Omega^s$ of $(w_0^0, w_0^2)$ in $H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ such that the flow map associated to (11) is smooth from $\Omega^s$ into $X_{T'}^s$.

First, we list some well-known properties of the Riesz transforms.
Proposition 3. It holds that
\begin{equation}
\| R_j f \|_{L^p} \leq C \| f \|_{L^p},
\end{equation}
for all $1 < p < \infty$ and $j \in \{1, 2\}$, and
\begin{equation}
D^1_x = R_1 \partial_{x_1} + R_2 \partial_{x_2}.
\end{equation}

The following Leibniz’ rule for fractional derivative, derived by Kenig, Ponce and Vega in Theorem A.12 of [21], will also be needed.

Lemma 4. Let $0 < \gamma < 1$ and $1 < p < \infty$. Then
\begin{equation}
\| D_x^\alpha (fg) - f D_x^\alpha g - g D_x^\alpha f \|_{L^p} \lesssim \|g\|_{L^\infty} \| D_x^\alpha f \|_{L^p},
\end{equation}
for all $f, g : \mathbb{R}^n \to \mathbb{C}$.

Proof of Theorem 3. We will treat only the most difficult case when $\frac{3}{2} < s < 2$ and we define $\gamma = s - 1 \in (\frac{1}{2}, 1)$. The integral system associated to (10) with initial data $(w^0_1, w^0_2)$ can be written as
\begin{equation}
\begin{cases}
   w_1 = F^+(w_1, w_2) := U_1^+(t) w_1^0 + \int_0^t U_1^+ (t - t') (I + II)(w_1, w_2)(t') dt' \\
   w_2 = F^-(w_1, w_2) := U_2^-(t) w_2^0 + \int_0^t U_2^- (t - t') (-I + II)(w_1, w_2)(t') dt'
\end{cases}
\end{equation}
where the nonlinearities $I$ and $II$ appearing on the right-hand side of (51) are defined in (35) and (39).

Next, for $0 < T \lesssim e^{-\frac{1}{4}}$, we fix $\alpha = 1 - \gamma \in (0, \frac{1}{2})$, $q_\alpha$ and $\kappa_\alpha$ defined in Theorem 4 and consider the following semi-norms
\begin{align*}
\chi_T^2(f) &= \sup_{0 \leq t \leq T} \| f(t) \|_{H^\alpha}, \\
\chi_T^3(f) &= \mathcal{E}^+ \sum_{|\beta| \leq 1} \| \partial_\beta f \|_{L^2_T L^\infty_x} + \mathcal{E}^+ \| D^1_x f \|_{L^2_T L^\infty_x} + \mathcal{E}^+ \sum_{l=1, 2} \sum_{|\beta| \leq 1} \| \partial_\beta R_l f \|_{L^2_T L^\infty_x} \\
&+ \mathcal{E}^+ \sum_{l=1, 2} \| D^1_x R_l f \|_{L^2_T L^\infty_x}, \\
\chi_T^4(f) &= \mathcal{E}^\alpha \sum_{|\beta| = 1} \| D^1_x \partial_\beta f \|_{L^q_{0,T} L^\infty_x} + \mathcal{E}^\alpha \sum_{l=1, 2} \sum_{|\beta| = 1} \| D^1_x \partial_\beta R_l f \|_{L^q_{0,T} L^\infty_x}, \\
\chi_T^5(f) &= \mathcal{E}^\alpha \sum_{|\beta| = 1} \sup_{\alpha \in \mathbb{Z}^2} \| P > e^{-\frac{1}{2}} \partial_\beta D^1_x f \|_{L^2_{Q_\alpha \times [0, T]}} \\
&+ \mathcal{E}^\alpha \sum_{l=1, 2} \sum_{|\beta| = 1} \sup_{\alpha \in \mathbb{Z}^2} \| P > e^{-\frac{1}{2}} \partial_\beta D^1_x R_l f \|_{L^2_{Q_\alpha \times [0, T]}}, \\
\chi_T^6(f) &= \mathcal{E}^\alpha \left( \sum_{\alpha \in \mathbb{Z}^2} \| f \|^2_{L^2_{Q_\alpha \times [0, T]}} \right)^{\frac{1}{2}} + \mathcal{E}^\alpha \sum_{l=1, 2} \left( \sum_{\alpha \in \mathbb{Z}^2} \| R_l f \|^2_{L^2_{Q_\alpha \times [0, T]}} \right)^{\frac{1}{2}},
\end{align*}
where $\{Q_\alpha\}_{\alpha \in \mathbb{Z}^2}$ denotes a family of nonoverlapping cubes of unit size such that $\mathbb{R}^2 = \cup_{\alpha \in \mathbb{Z}^2} Q_\alpha$, as in the precedent subsection. We also define the Banach space $X_T^3$ by
\begin{equation}
X_T^3 = \{(w_1, w_2) \in C([0, T]; H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)) \mid \| (w_1, w_2) \|_{X_T^3} < \infty \}.
\end{equation}
where
\begin{equation}
\| (w_1, w_2) \|_{X_T^3} = \sum_{j=1}^4 \chi_T^j(w_1) + \sum_{j=1}^4 \chi_T^j(w_2).
\end{equation}
We will show that, for adequate $T$ and $a$, the map $(\mathcal{F}^+, \mathcal{F}^-)$ is a contraction in a closed ball $X_T^2(a)$ in $X_T^2$ of radius $a > 0$ and centered at the origin.

Using the integral equation (54), Minkowski’s integral inequality, the linear estimates (13), (16) and (31) with $\alpha = 0$ and $\alpha = 1 - \gamma$, and the fact that the Riesz transforms are unitary operators in $H^s$, we deduce that

$$\sum_{j=1}^{4} \lambda_j^T (\mathcal{F}^+(w_1, w_2)) + \sum_{j=1}^{4} \lambda_j^T (\mathcal{F}^-(w_1, w_2))$$

$$\lesssim \|\langle w_1^0, w_2^0 \rangle\|_{H^s \times H^s} + \epsilon \int_0^T (\|I(w_1, w_2)(t)\|_{H_x^s} + \|II(w_1, w_2)(t)\|_{H_x^s}) \, dt.$$  (55)

According to formulas (8) and (9), the nonlinearities $I(w_1, w_2)$ and $II(w_1, w_2)$ contain terms of the three following types: $w_j D_x^1 w_k, \partial_x w_j R_l w_k$ and $D_x^1 (R_i w_j R_m w_k)$ for $j, k, l, m = 1, 2$. For sake of simplicity, we only will present the computations for a nonlinear term of the first kind, since the other ones can be handled by similar arguments. In other words, we have to estimate the term $\epsilon \int_0^T \|w_j D_x^1 w_k\|_{H_x^s} \, dt$ as a function of the norms defined in (33). By the definition of the Sobolev space $H^s(\mathbb{R}^2)$, we have that

$$\epsilon \int_0^T \|w_j D_x^1 w_k\|_{H_x^s} \, dt \lesssim \epsilon \int_0^T \|w_j D_x^1 w_k\|_{L_x^2} \, dt + \epsilon \int_0^T \|D_x^s (w_j D_x^1 w_k)\|_{L_x^2} \, dt.$$  (56)

First, we use Hölder’s inequality and the Sobolev embedding $H^s(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ to estimate the first term on the right-hand side of (56) as

$$\epsilon \int_0^T \|w_j D_x^1 w_k\|_{L_x^2} \, dt \leq \epsilon T \|w_j\|_{L^\infty_T L_x^\infty} \|D_x^1 w_k\|_{L^\infty_T L_x^2} \leq \epsilon T \lambda_1^T (w_j) \lambda_1^T (w_k).$$  (57)

To treat the second term appearing on the right-hand side of (56), we observe from (52) that $D_x^s = D_x^2 R_1 \partial_x + D_x^2 R_2 \partial_y$. Therefore, it follows from (51) that

$$\epsilon \int_0^T \|D_x^s (w_j D_x^1 w_k)\|_{L_x^2} \, dt \leq \epsilon \sum_{l=1,2} \int_0^T \|D_x^2 (\partial_x w_j D_x^1 w_k)\|_{L_x^2} \, dt + \epsilon \sum_{l=1,2} \int_0^T \|D_x^2 (w_j \partial_x D_x^1 w_k)\|_{L_x^2} \, dt.$$  (58)

Then, we deduce from the fractional Leibniz’ rule (53) that

$$\epsilon \int_0^T \|D_x^2 (\partial_x w_j D_x^1 w_k)\|_{L_x^2} \, dt \leq \epsilon \|D_x^1 w_k\|_{L^\infty_T L_x^2} \|D_x^{1+\gamma} w_k\|_{L^\infty_T L_x^2}$$

$$+ \epsilon T \|D_x^1 w_k\|_{L^\infty_T L_x^2} \|\partial_x D_x^2 w_j\|_{L^\infty_T L_x^2} \|\partial_x D_x^2 w_j\|_{L^\infty_T L_x^2} \leq \epsilon \|D_x^1 w_k\|_{L^\infty_T L_x^2} \|\partial_x D_x^2 w_j\|_{L^\infty_T L_x^2} + \lambda_1^T (w_j) \lambda_1^T (w_k).$$  (59)
On the other hand, by using again Leibniz' rule and Hölder's inequality, we observe that

$$
\epsilon \int_0^T \| D_x^\gamma (w_j \partial_x D_x^1 w_k) \|_{L^2_x} dt
\leq \epsilon \int_0^T \| D_x^\gamma w_j \|_{L^q_x} \| \partial_x D_x^1 w_k \|_{L^\infty_x} dt + \epsilon \int_0^T \| w_j \partial_x D_x^{1+\gamma} w_k \|_{L^2_x} dt.
$$

(60)

We deal with the first term on the right-hand side of (60). We deduce by using Hölder's inequality in time that

$$
\epsilon \int_0^T \| D_x^\gamma w_j \|_{L^q_x} \| \partial_x D_x^1 w_k \|_{L^\infty_x} dt
\leq \epsilon^{1-\kappa_\alpha} T^{\frac{1}{\kappa_\alpha}} \| D_x^\gamma w_j \|_{L^q_x} \| \partial_x D_x^1 w_k \|_{L^\infty_x}^{\kappa_\alpha} \| D_x^1 w_k \|_{L^q_x} \| D_x^{1+\gamma} w_k \|_{L^q_x}
\leq \epsilon^{1-\kappa_\alpha} T^{\frac{1}{\kappa_\alpha}} \lambda_T^{(\kappa_\alpha)(w_j)} \lambda_T^{(\kappa_\alpha)(w_k)},
$$

(61)

where $q_{\alpha}$ is the conjugate exponent to $q_\alpha$.

Finally, to estimate the second term on the right-hand side of (60), we observe that

$$
\epsilon \int_0^T \| w_j \partial_x D_x^{1+\gamma} w_k \|_{L^2_x} dt
\leq \epsilon \int_0^T \| w_j P_{\leq \epsilon^{-\frac{1}{2}}} \partial_x D_x^{1+\gamma} w_k \|_{L^2_x} dt + \int_0^T \| w_j P_{> \epsilon^{-\frac{1}{2}}} \partial_x D_x^{1+\gamma} w_k \|_{L^2_x} dt.
$$

(62)

On the one hand, we use the Sobolev embedding $H^s(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, since $s > 1$, to obtain that

$$
\epsilon \int_0^T \| w_j P_{\leq \epsilon^{-\frac{1}{2}}} \partial_x D_x^{1+\gamma} w_k \|_{L^2_x} dt \leq \epsilon T \| w_j \|_{L^\infty_{T,x}} \| P_{\leq \epsilon^{-\frac{1}{2}}} \partial_x D_x^{1+\gamma} w_k \|_{L^\infty_{T,x}}
\leq \epsilon T \| w_j \|_{L^\infty_{T,x}} H_T \epsilon^{-\frac{1}{2}} \| D_x^{1+\gamma} w_k \|_{L^\infty_{T,x}}
\leq \epsilon^{\frac{1}{2}} T \lambda_T^{1}(w_j) \lambda_T^{1}(w_k).
$$

(63)

On the other hand, we deduce from Hölder's inequality that

$$
\epsilon \int_0^T \| w_j P_{> \epsilon^{-\frac{1}{2}}} \partial_x D_x^{1+\gamma} w_k \|_{L^2_x} dt
\leq \epsilon T^{\frac{1}{2}} \left( \sum_{\alpha \in \mathbb{Z}^2} \int_{Q_\alpha \times [0,T]} \| w_j P_{> \epsilon^{-\frac{1}{2}}} \partial_x D_x^{1+\gamma} w_k \| dx dt \right)^{\frac{1}{2}}
\leq \epsilon T^{\frac{1}{2}} \left( \sum_{\alpha \in \mathbb{Z}^2} \| w_j \|_{L^2_{Q_\alpha \times [0,T]}}^2 \right)^{\frac{1}{2}} \sup_{\alpha \in \mathbb{Z}^2} \| P_{> \epsilon^{-\frac{1}{2}}} \partial_x D_x^{1+\gamma} w_k \|_{L^2_{Q_\alpha \times [0,T]}},
\leq \epsilon^{\frac{1}{4}} T^{\frac{1}{2}} \lambda_T^{1}(w_j) \lambda_T^{1}(w_k).
$$

(64)

Thus, it follows gathering (60) – (62) that

$$
\|(\mathcal{F}^+(w_1, w_2), \mathcal{F}^-(w_1, w_2))\|_{X^3_T} \leq c \|(w_1^0, w_2^0)\|_{H^2 \times H^2}
+ c \left( \epsilon^{\frac{1}{4}} T^{\frac{1}{2}} + \epsilon^{1-\kappa_\alpha} T^{\frac{1}{\kappa_\alpha}} + \epsilon^{\frac{1}{2}} T + \epsilon^{\frac{1}{4}} T^{\frac{1}{2}} \right) \|(w_1, w_2)\|_{X^3_T}^2,
$$

(65)
and similarly,
\[
\left\| (F^+(w_1, w_2), F^-(w_1, w_2)) - (F^+(\tilde{w}_1, \tilde{w}_2), F^-(\tilde{w}_1, \tilde{w}_2)) \right\|_{X^a_{T}} 
\leq c(\epsilon^{\frac{1}{3}} - T^{\frac{1}{5}} + \epsilon^{1-\kappa_s}T_{\frac{1}{3}}\kappa_s + \epsilon^{\frac{1}{3}}T + \epsilon^{\frac{1}{2}}T^{\frac{1}{2}})
\times (\|(w_1, w_2)\|_{X^a_{T}} + \|(\tilde{w}_1, \tilde{w}_2)\|_{X^a_{T}})(w_1, w_2) - (\tilde{w}_1, \tilde{w}_2)\|_{X^a_{T}}.
\]
Observe that
\[
\epsilon^{\frac{1}{3}}T^{\frac{1}{5}} + \epsilon^{1-\kappa_s}T_{\frac{1}{3}}\kappa_s + \epsilon^{\frac{1}{3}}T + \epsilon^{\frac{1}{2}}T^{\frac{1}{2}} \lesssim 1,
\]
since \( T \lesssim \epsilon^{-\frac{1}{3}} \). Therefore, we deduce by choosing
\[
a = 2c\|(w_1^0, w_2^0)\|_{H^2 \times H^2} \quad \text{and} \quad T \sim \epsilon^{-\frac{1}{3}}
\]
that \((F^+, F^-)\) defines a contraction in the space \( X^a_{T}(a) \), which concludes the proof of Proposition 5 by the Picard fixed point theorem. \( \square \)

4. Other strongly dispersive Boussinesq systems

Other two-dimensional Boussinesq systems can be classified according to the order of the eigenvalues of the dispersion matrix (see Introduction). As we already mentioned, when \( b, d \) \( > \) 0, the well-posedness on time scales of order \( 1/\sqrt{\epsilon} \) has been established in [15] for initial data in low order Sobolev spaces and on time interval of order \( 1/\epsilon \) in [27] for smooth initial data (with loss of derivatives). We thus restrict ourselves to the case where at least one of the coefficients \( b \) or \( d \) vanishes, emphasizing the strongest dispersive case corresponding to eigenvalues of order 2 (the so-called “Schrödinger” type systems) where \( a < 0 \) and \( c < 0 \).

The local well posedness for some of those systems has been established in the two-dimensional case by Cung The Anh [13], but without looking for the dependence of the existence time with respect to \( \epsilon \). Note however that the proof given in [13] for the case where \( d = 0 \) (Theorem 3.5) does not seem to be correct.

We will complete the analysis, in particular by checking the \( \epsilon \) dependence. It turns out that when \( d > 0 \), one can establish the local well-posedness of Boussinesq “Schrödinger” type systems on time intervals of order \( 1/\sqrt{\epsilon} \) by elementary energy methods, in relatively low order Sobolev spaces.

We first state a useful lemma.

Lemma 5. Let \( s \geq 3/2 \). Then there exists \( C > 0 \) such that for any \( \mathbf{v} \in H^{s+1}(\mathbb{R}^2) \) and \( \eta \in H^s(\mathbb{R}^2) \),
\[
|\langle \Lambda^s \nabla \cdot (\eta \mathbf{v}), \Lambda^s \eta \rangle | \leq C\|\mathbf{v}\|_{H^{s+1}}\|\eta\|_{H^s}^2.
\]

The proof of this lemma is based on the following commutator estimate obtained by Ponce in Lemma 2.3 of [28].

Lemma 6. If \( s > 1 \) and \( 1 < p < \infty \), then
\[
\|\Lambda^s f\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1} g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}),
\]
where \( p_1, p_4 \in (1, +\infty) \) such that
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.
\]
order 2. Because of the constraint on \(a, b, c, d \in \mathbb{R}^4\) for \(\alpha > 1\),

\[\text{(73)}\]

Schrödinger type Boussinesq systems when \(4.1\).

\[\text{Therefore, estimate (67) is deduced gathering (69)–(73).}\]

\[\text{Moreover, we can rewrite the terms appearing on the right-hand side of (69) as}\]

\[\text{(74)}\]

\[\text{Those systems are called in [7] “Benjamin-Ono type systems” since the dispersion}\]

\[\text{matrix has, in the one-dimensional case and when } \epsilon = 1, \text{ eigenvalues equal to}\]

\[\pm((\frac{a}{4\epsilon})^2 + r(k)) \text{ where } r(k) = 0 \left(\frac{1}{k^2}\right) \text{ as } k \to \infty.\]

\[\text{In the two-dimensional case, the dispersion matrix in Fourier variables reads}\]

\[\text{The corresponding nonzero eigenvalues are}\]

\[\text{When surface tension is large enough so that } \tau > 1/3, \text{ we have the admissible system } a = b = d = 0, \ c = \tau - 1/3 \text{ but we will note consider it here.}\]
We have therefore class terminology “Schrödinger type” Boussinesq systems.

After diagonalization of the linear part, (74) can thus be written as a (nonlinearly coupled) system of two Schrödinger type nonlocal equations involving derivatives of the unknowns, and we might think of solving the Cauchy problem by applying the general results of [24]. We will refrain to do that here because an elementary energy method on the original formulation (74), which is the extension to the two-dimensional case of the corresponding one-dimensional result in [7], applies when \( d > 0 \), and when \( d = 0 \) for curlfree velocities.

**Theorem 6.** Let \( s \geq 3/2 \) and \( 0 < \epsilon \leq 1 \) be given.

(i) Assume that \( a = c \). Then, for every \( (\eta_0, v_0) \in H^s(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2)^2 \), there exist \( T = T(||\eta_0||_{H^s}, ||v_0||_{H^{s+1}}) \) and a unique solution

\[
(\eta, v) \in C([0, T]; H^s(\mathbb{R}^2)) \times C([0, T]; H^{s+1}(\mathbb{R}^2)^2) ,
\]

where \( T_\epsilon = T \epsilon^{-1/2} \), of (74) with initial data \( (\eta_0, v_0) \), which is uniformly bounded on \([0, T]\

(ii) Assume that \( a \neq c \). Then, for every \( (\eta_0, v_0) \in H^{s+1}(\mathbb{R}^2) \times H^{s+2}(\mathbb{R}^2)^2 \), there exist \( T = T(||\eta_0||_{H^{s+1}}, ||v_0||_{H^{s+2}}) \) and a unique solution

\[
(\eta, v) \in C([0, T_\epsilon]; H^{s+1}(\mathbb{R}^2)) \times C([0, T_\epsilon]; H^{s+2}(\mathbb{R}^2)^2) ,
\]

where \( T_\epsilon = T \epsilon^{-1/2} \), of (74) with initial data \( (\eta_0, v_0) \) which is uniformly bounded on \([0, T]\

**Proof.** (i) **Uniqueness in the class** \( C([0, T]; H^{3/2}(\mathbb{R}^2)) \times C([0, T]; H^{5/2}(\mathbb{R}^2)^2) \).

In what follows we will use freely the embedding \( H^s(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2) \) and thus \( H^{\frac{3}{2}}(\mathbb{R}^2) \hookrightarrow W^{1,4}(\mathbb{R}^2) \) (see [4]).

Plainly it suffices to consider \( \epsilon = 1 \). Let \( (\eta_1, v_1), (\eta_2, v_2) \) two solutions in the class \( C([0, T]; H^{3/2}(\mathbb{R}^2)) \times C([0, T]; H^{5/2}(\mathbb{R}^2)^2) \) and let \( N = \eta_1 - \eta_2, V = v_1 - v_2 \).

We have therefore

\[
(75) \quad \begin{cases} 
N_t + \nabla \cdot V + \epsilon (\nabla \cdot (N v_1 + \eta_2 V) + a \nabla \cdot \Delta V) = 0 \\
V_t + \nabla N + \epsilon (\frac{1}{2} \nabla \cdot (V \cdot (v_1 + v_2))) + c \nabla \Delta N - d \Delta V \end{cases} = 0,
\]

We take the \( L^2 \) scalar product of the first equation by \( |c| N \), of the second one by \( a V \) and add the resulting equations. The contributions of the cubic dispersive terms cancel out \(^5\) and we find

\[
(76) \quad \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} (|c| |N|^2 + |a| |V|^2 + d|\nabla V|^2) = \int_{\mathbb{R}^2} (a N \nabla \cdot V + c \nabla N \cdot V) - \int_{\mathbb{R}^2} \nabla \cdot (N v_1 + \eta_2 V) N - \frac{1}{2} \int_{\mathbb{R}^2} \nabla(V \cdot (v_1 + v_2)) \cdot V .
\]

One has successively

\(^5\) One term has to be interpreted as a \( H^{1/2} - H^{-1/2} \) duality.
(77) $\int_{\mathbb{R}^2} \left(|a|N\nabla \cdot V + |c|\nabla N \cdot V\right) = \int_{\mathbb{R}^2} \left(|a| - |c|\right)N\nabla \cdot V \leq C\|N\|_{L^2}\|\nabla V\|_{L^2}$.

Then

$$\int_{\mathbb{R}^2} \nabla \cdot (Nv_1 + \eta_2V)N = \int_{\mathbb{R}^2} (N^2\nabla \cdot v_1 + \eta_2N\nabla \cdot V + (\nabla N \cdot v_1)N + (\nabla \eta_2 \cdot V)N).$$

Observing that $\int_{\mathbb{R}^2}(\nabla N \cdot v_1)N = -\frac{1}{2} \int_{\mathbb{R}^2} N^2\nabla \cdot v_1$, the contribution of this term is upper bounded by

$$C(\|\nabla \cdot v_1\|_{L^\infty}\|N\|_{L^2}^2 + \|\eta_2\|_{L^\infty}\|N\|_{L^2}\|\nabla V\|_{L^2} + \|V\|_{L^4}\|\nabla \eta_2\|_{L^4}\|N\|_{L^2}) \leq C(\|N\|_{L^2}^2 + \|\nabla V\|_{L^2}^2).$$

Finally,

$$\int_{\mathbb{R}^2} \nabla(\nabla \cdot v_1 + v_2) \cdot V \leq C(\|v_1\|_{W^{1,\infty}} + \|v_2\|_{W^{1,\infty}})\|V\|_{H^1}^2.$$  

The result follows from (76) and Gronwall’s lemma.

(ii) Existence. We will focus only on the derivation of the energy estimates checking the dependence of the existence interval on $\epsilon$. The complete proof would result from a standard compactness argument implemented on a regularized version of the system (for instance by truncating the high frequencies in the differential operators). The strong continuity in time and the continuity of the flow would result from a standard application of the Bona-Smith trick.

Energy estimates in the case $a = c$. We apply the operator $\Lambda^* \eta$ to both equations of (73), multiply the first one by $|a|\Lambda^* \eta$, take the scalar product of the second one by $|a|\Lambda^* V$, integrate and sum. All the linear terms cancel out and it remains to estimate

$$I_1 = \epsilon \int_{\mathbb{R}^2} (\Lambda^* \nabla \cdot (\eta V)\Lambda^* \eta + \frac{1}{2} \Lambda^* \nabla |v|^2 \cdot \Lambda^* v) = \epsilon (I_2 + I_3).$$

Lemma [5] implies that

$$|I_2| \leq C\|v\|_{H^{s+1}}\|\eta\|_{H^s}^2.$$  

Finally, by the Kato-Ponce commutator lemma (18),

$$|I_3| \leq C\|v\|_{H^{s+1}}\|v\|_{H^s}^2.$$  

Combining those estimates leads to the differential inequality

$$\frac{d}{dt}[\|\eta\|_{H^s}^2 + \|v\|_{H^s}^2 + \epsilon\|v\|_{H^{s+1}}^2] \leq C\epsilon(\|v\|_{H^{s+1}}^2\|\eta\|_{H^s}^2 + \|v\|_{H^s}^2\|v\|_{H^{s+1}}^2).$$

If we define $Y(t) = \|\eta\|_{H^s}^2 + \|v\|_{H^s}^2 + \epsilon\|v\|_{H^{s+1}}^2$, then inequality (78) can be rewritten as

$$Y'(t) \leq C\epsilon^{1/2}Y(t)^{3/2},$$

which in turn implies an a priori bound on a time interval $[0, \frac{T}{\epsilon^{1/2}}]$ where $T = T(\|\eta_0\|_{H^s}, \|v_0\|_{H^{s+1}})$.  


Energy estimates in the case \( a \neq c \). We apply the operator \( \Lambda^s \) to both equations of (74), multiply the first one by \( \Lambda^s \eta ccc \Delta \Lambda^s \eta \), take the scalar product of the second one by \( \Lambda^s \mathbf{v} + ac \Delta \Lambda^s \mathbf{v} \), integrate and sum to obtain that

\[
\frac{1}{2} \frac{d}{dt} \left[ \| \eta \|_{H^s}^2 - c c \| \nabla \eta \|_{H^s}^2 + \| \mathbf{v} \|_{H^s}^2 + (d-a) \epsilon \| \nabla \mathbf{v} \|_{H^s}^2 - ade^2 \| \Delta \mathbf{v} \|_{H^s}^2 \right] = \int \nabla \cdot \Lambda^s \mathbf{v} \Lambda^s \eta + c \int \nabla \cdot \Lambda^s \mathbf{v} \Delta \Lambda^s \eta + \int \nabla \Lambda^s \eta \cdot \Lambda^s \mathbf{v} + ac \int \nabla \cdot \Delta \Lambda^s \mathbf{v} \Lambda^s \eta + ac \epsilon \int \nabla \cdot \Delta \Lambda^s \mathbf{v} \Delta \Lambda^s \eta + c \int \nabla \Delta \Lambda^s \eta \cdot \Lambda^s \mathbf{v} + \epsilon \int \Lambda^s \nabla \cdot (\eta \mathbf{v}) \Lambda^s \eta + \epsilon^2 c \int \Lambda^s \nabla \cdot (\eta \mathbf{v}) \Delta \Lambda^s \eta + \frac{\epsilon}{2} \int \Lambda^s \nabla (|\mathbf{v}|^2) \cdot \Lambda^s \mathbf{v} + \frac{\epsilon^2 a}{2} \int \Lambda^s \nabla (|\mathbf{v}|^2) \cdot \Delta \Lambda^s \mathbf{v}
\]

\[
:= \sum_{i=1}^{12} J_i.
\]

Now, observe integrating by parts that \( J_1 + J_3 = J_2 + J_7 = J_4 + J_5 = J_6 + J_8 = 0 \). Moreover, we deduce from Lemma 4 that

\[
|J_9| \leq \epsilon \| \mathbf{v} \|_{L^{2+1}} \| \eta \|_{L^s}^2,
\]

and

\[
|J_{10}| \leq \epsilon^2 \| \mathbf{v} \|_{L^{2+2}} \| \eta \|_{L^{s+1}}^2.
\]

On the other hand, it follows integrating by parts and using Lemma 3 that

\[
|J_{11}| \leq \epsilon \| \mathbf{v} \|_{L^{2+1}} \| \mathbf{v} \|_{L^s}^2,
\]

and

\[
|J_{12}| \leq \epsilon^2 \| \mathbf{v} \|_{L^{2+2}} \| \mathbf{v} \|_{L^{s+1}}^2.
\]

Finally, let define \( Y \) by

\[
Y(t) = \| \eta \|_{H^s}^2 - c c \| \nabla \eta \|_{H^s}^2 + \| \mathbf{v} \|_{H^s}^2 + (d-a) \epsilon \| \nabla \mathbf{v} \|_{H^s}^2 - ade^2 \| \Delta \mathbf{v} \|_{H^s}^2.
\]

Then it follows gathering (79)–(83) that

\[
Y'(t) \leq C \epsilon^{1/2} Y(t)^{3/2},
\]

which in turn implies an a uniform in \( \epsilon \) priori bound on a time interval \( [0, \frac{T}{\sqrt{\epsilon}}] \). \( \square \)

4.2. Schrödinger type Boussinesq systems when \( d = 0 \) with curl free velocity field. We thus consider the case \( a < 0, c < 0, b > 0, d = 0 \).

\[
\begin{cases}
\eta_t + \nabla \cdot \mathbf{v} + \epsilon \nabla \cdot (\eta \mathbf{v}) + a \nabla \cdot \Delta \mathbf{v} - b \Delta \eta = 0 \\
\mathbf{v}_t + \nabla \eta + \epsilon \frac{1}{2} \nabla |\mathbf{v}|^2 + c \nabla \Delta \eta = 0.
\end{cases}
\]

We will here restrict the velocity \( \mathbf{v} \) to irrotational motions, a situation which, as we already indicated is relevant since the variable \( \mathbf{v} \) in the Boussinesq systems is curlfree up to a \( O(\epsilon^2) \) term. Note also that the curlfree condition is preserved by the evolution of (84). When \( \mathbf{v} \) is curlfree, the term \( \frac{1}{2} \nabla |\mathbf{v}|^2 \) in the second equation of (84) writes as two transport equations namely \( (\mathbf{v} \cdot \nabla) v_1, \mathbf{v} \cdot \nabla v_2)^T \) where \( \mathbf{v} = (v_1, v_2)^T \). This will permit to perform energy estimates on \( \mathbf{v} \).
Theorem 7. Let \( s > 2 \) and \( 0 < \epsilon \leq 1 \) be given.

(i) Assume that \( a = c \). Then, for every \((\eta_0, v_0)\) \( \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)^2 \) with \( \text{curl} \ v_0 = 0 \) there exist \( T = T(||\eta_0||_{H^{s+1}}, ||v_0||_{H^s}) \) and a unique solution

\[
(\eta, v) \in C([0, T]; H^{s+1}(\mathbb{R}^2)) \times C([0, T]; H^s(\mathbb{R}^2)^2),
\]

where \( T_e = T_{e}^{-1/2}, \) of (74) with initial data \((\eta_0, v_0)\), which is uniformly bounded on \([0, T_e]\).

(ii) Assume that \( a \neq c \). Then, for every \((\eta_0, v_0)\) \( \in H^{s+2}(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2)^2 \) with \( \text{curl} \ v_0 = 0 \) there exist \( T = T(||\eta_0||_{H^{s+2}}, ||v_0||_{H^{s+1}}) \) and a unique solution

\[
(\eta, v) \in C([0, T_e]; H^{s+2}(\mathbb{R}^2)) \times C([0, T_e]; H^{s+1}(\mathbb{R}^2)^2),
\]

where \( T_e = T_{e}^{-1/2}, \) of (74) with initial data \((\eta_0, v_0)\) which is uniformly bounded on \([0, T_e]\).

The proof of Theorem 7 is very similar to that of Theorem 6 and we only sketch it. We will use the following estimate.

Lemma 7. Let \( s > 2 \) and \( v \in H^s(\mathbb{R}^2) \) such that \( \text{curl} \ v = 0 \). Then

\[
\left| \int_{\mathbb{R}^2} \Lambda^s \nabla |v|^2 \cdot \Lambda^s v \right| \leq C ||v||^3_{H^s}.
\]

Proof. Let \( v = (v_1, v_2)^T \). Then \( \partial_{x_1} v_2 = \partial_{x_2} v_1 \) and

\[
\frac{1}{2} \nabla |v|^2 = (v_1 \partial_{x_1} v_1 + v_2 \partial_{x_2} v_1, v_1 \partial_{x_1} v_2 + v_2 \partial_{x_2} v_2)^T.
\]

The estimate then follows immediately from integration by parts, the Kato-Ponce commutator lemma and the embedding \( H^{s-1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2) \).

Proof of Theorem 7. (i) Uniqueness. Again we consider \( \epsilon = 1 \). Let \((\eta_1, v_1), (\eta_2, v_2)\) two solutions and let \( N = \eta_1 - \eta_2, V = v_1 - v_2 \). We have therefore

\[
\begin{align*}
N_t + \Delta \cdot V + c|V| (N v_1 + \eta_2 V) + a \Delta v_1 + b \Delta N_t &= 0 \\
V_t + \Delta N + c \frac{1}{2} \nabla (V \cdot (v_1 + v_2)) + c \nabla \Delta N &= 0.
\end{align*}
\]

We multiply the first equation by \(-c N\), take scalar product of the second one by \(-a V\), integrate and sum. One obtains after integrations by parts

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \left[ |c| N^2 + |a||V|^2 + |c| b |\nabla N|^2 \right] \\
\leq C(||v_1||_{L^\infty}, ||\eta_2||_{L^\infty}, ||\nabla v_1||_{L^\infty}, ||\nabla v_2||_{L^\infty}) ||N||^2_{L^2} + ||\nabla N||^2_{L^2} + ||V||^2_{L^2},
\]

and the result follows from Gronwall’s lemma.

(ii) Existence. We just indicate how to obtain the energy estimates in the case \( a = c \). We apply the operator \( \Lambda^s \) to both equations of (74), multiply the first one by \( \Lambda^s \eta \), take the scalar product of the second one by \( \Lambda^s v \), integrate and sum to get using Kato-Ponce Lemma and Lemma 7

\[
\frac{d}{dt} ||\Lambda^s \eta||^2_{L^2} + ||\Lambda^s v||^2_{L^2} + c b ||\nabla \Lambda^s \eta||^2_{L^2}
\]

\[
\leq C \epsilon \left| \int_{\mathbb{R}^2} \Lambda^s \nabla \cdot (\eta V) \Lambda^s \eta + \Lambda^s \nabla |v|^2 \cdot \Lambda^s v \right| \\
\leq C (||\nabla v||_{L^\infty} ||\Lambda^s \eta||^2_{L^2} + ||v||^3_{H^s}),
\]

...
from which one obtain the expected a priori uniform estimate on a time interval \([0, T/\sqrt{\epsilon}]\).

When \(a \neq c\) one has to modify the proof as in Theorem 6. □

5. Final comments

To keep this paper short and to maintain a certain unity, we have refrain here to consider the other Boussinesq systems, which have eigenvalues \(\lambda_\pm\) of order 1 or less. For instance the cases \(a = c = b = 0, d = 1/3\) or \(a = c = d = 0, b = 1/3\) are version of Boussinesq original system. The former case has been studied in the one-dimensional case by Schonbek [30] and Amick [2] who obtained well-posedness by viewing the system as a (dispersive) perturbation of the Saint-Venant (shallow-water) hyperbolic system.

In both cases, no results seem to be available in two-dimensions.

Acknowledgements. The Authors were partially supported by the Brazilian-French program in mathematics. J.-C. S. acknowledges support from the project ANR-07-BLAN-0250 of the Agence Nationale de la Recherche. F. L. was partially supported by CNPq and FAPERJ/Brazil.

References

[1] B. Alvarez-Samaniego and D. Lannes, Large time existence for 3D water-waves and asymptotics, Inventiones Math., 171 (2008), 485–541.
[2] C.J. Amick, Regularity and uniqueness of solutions of the Boussinesq system of equations, J. Diff. Eq., 54 (1984), 231–247.
[3] M. Ben-Artzi, H. Koch and J.-C. Saut, Dispersion estimates for third order equations in two dimensions, Comm. Part. Diff. Eq. 28, no. 11–12 (2003), 1943–1974.
[4] J. Bergh and J. Lofstrom, Interpolation Spaces, an Introduction, Springer-Verlag Berlin, 1976.
[5] J. L. Bona, T. Colin and D. Lannes, Long-wave approximation for water waves, Arch. Ration. Mech. Anal., 178 (2005), 373–410.
[6] J. L. Bona, M. Chen and J.-C. Saut, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media I : Derivation and the linear theory, J. Nonlinear Sci., 12 (2002), 283–318.
[7] J. L. Bona, M. Chen and J.-C. Saut, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II : The nonlinear theory, Nonlinearity, 17 (2004), 925–952.
[8] J. L. Bona, Z. Grudic and H. Kalisich, A KdV-type Boussinesq system : from energy level to analytic spaces, Disc. Cont. Dyn. Syst., 26, no. 2 (2010), 1121–1139.
[9] J. L. Bona, D. Lannes, and J.-C. Saut, Asymptotic models for internal waves, J. Math. Pures Appl., 89 (2008), 538–566.
[10] J. V. Boussinesq, Théorie des ondes et des remous qui se propagent le long d’un canal rectangulaire horizontal en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond, J. Math. Pures Appl., 17 (1872), 55–108.
[11] W. Craig, P. Guyenne, and H. Kalisich, Hamiltonian long-wave expansions for free surfaces and interfaces, Comm. Pure. Appl. Math., 58 (2005), 1587–1641.
[12] K. Cho and T. Ozawa, On small amplitude solutions to the generalized Boussinesq equations, Disc. Cont. Dyn. Syst., 17 (2007), 691–711.
[13] Cung The Anh, On the Boussinesq/Full dispersion and Boussinesq/Boussinesq systems for internal waves, Nonlinear Analysis, 72, no. 1 (2010), 409–429.
[14] P. Daripa and R.K. Dash, A class of model equations for bi-directional propagation of capillary-gravity waves, Int. J. Eng. Sc., 41 (2003), 201–218.
[15] V. Dougalis, D. Mitsotakis and J.-C. Saut, On some Boussinesq systems in two space dimensions : theory and Numerical Analysis, ESAIM : Mathematical Modelling and Numerical Analysis, 41, no. 5, (2007), 825–854.
Z. Guo, L. Peng and B. Wang, *Decay estimates for a class of waves equations*, J. Funct. Anal., **254** (2008), 1642–1660.

P. Grisvard, *Quelques propriétés des espaces de Sobolev, utiles dans l’étude des équations de Navier-Stokes*, Exposé 4 in “Problèmes d’évolution non linéaires”, Séminaire de Nice 1974–1975.

T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math., **41** (1988), 891–907.

C. E. Kenig, G. Ponce and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J., **40** (1991), 33–69.

C. E. Kenig, G. Ponce and L. Vega, *Well-posedness of the initial-value problem for the Korteweg-de Vries equation*, J. Amer. Math. Soc., **4** (1991), 323–347.

C. E. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation*, Comm. Pure Appl. Math., **46** (1993), 527–620.

C. E. Kenig, G. Ponce and L. Vega, *Small solutions to nonlinear Schrödinger equations*, Ann. Inst. Henri Poincaré, **10** (1993), 255–288.

C. E. Kenig, G. Ponce and L. Vega, *A bilinear estimate with application to the KdV equation*, J. Amer. math. Soc., **9** (1996), 573–603.

C. E. Kenig, G. Ponce and L. Vega, *Smoothing effects and local existence theory for generalized nonlinear Schrödinger equations*, Inventiones Math., **134** (1998), 489–545.

H. Koch and J.-C. Saut, *Local smoothing and local solvability for third order dispersive equations*, SIAM J. Math. Anal., **38**, no. 5, (2007), 1528–1541.

F. Linares and G. Ponce, *Introduction to Nonlinear Dispersive Equations*, Universitext, Springer, New York, 2009.

M. Ming, J.-C. Saut and P. Zhang, *Long-time existence of solutions to Boussinesq systems*, submitted.

G. Ponce, *Smoothing properties of solutions to the Benjamin-Ono equation*, Lect. Note Pure Appl. Math., **122**, Dekker, New York, (1990), 667–679.

J.-C. Saut and N. Tzvetkov, *On a model for the oblique interaction of internal gravity waves*, Math. Model. Numer. Anal., **34** (2000), 501–523.

M.E. Schonbek, *Existence of solutions for the Boussinesq system of equations*, J. Diff. Eq., **42** (1981), 325–352.

E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J., 1970

E. M. Stein and G. Weiss, *Introduction to Fourier Analysis in Euclidian Spaces*, Princeton University Press, Princeton, N.J., 1971.

IMPA, Estrada Dona Castorina 110, Rio de Janeiro 22460-320, RJ Brasil

E-mail address: linares@impa.br

INSTITUTO DE MATEMÁTICA, UFRJ, CAIXA POSTAL 68530 CEP 21941-97, RIO DE JANEIRO, RJ BRASIL

E-mail address: didier@im.ufrj.br, pilod@impa.fr

LABORATOIRE DE MATHÉMATIQUES, UMR 8628., UNIVERSITÉ PARIS-SUD ET CNRS., 91405 ORSAY, FRANCE

E-mail address: jean-claude.saut@math.u-psud.fr