Conservation laws for a $q$-deformed nonrelativistic particle

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Abstract

We derive $q$-versions of Green’s theorem from the Leibniz rules of partial derivatives for the $q$-deformed Euclidean space. Using these results and the Schrödinger equations for a $q$-deformed nonrelativistic particle, we derive continuity equations for the probability density, the energy density, and the momentum density of a $q$-deformed nonrelativistic particle.

1 Introduction

As is well-known, the divergences that plagued QED from the beginning were eliminated by the concept of renormalization. On the other hand, various attempts to overcome the problem with divergences in QED by introducing a fundamental length have not yet been successful [1]. Heisenberg’s idea of a “lattice-world”, for example, led to a breakdown of continuous rotational and translational symmetries [2,3].

However, some considerations within the framework of a future theory of quantum gravity suggest that space-time reveals a discrete structure at small distances [4]. For example, the attempt to increase the accuracy of a position measurement more and more should disturb the background metric more and more [5]. Such a fundamental uncertainty in space could require a space-time algebra generated by non-commuting coordinates. Realistic noncommutative space-time algebras can be obtained, for example, by $q$-deformation [6,7].

The $q$-deformed Euclidean space has symmetries we can interpret as $q$-analogs of continuous rotational and translational symmetries. Due to this property, there are conservation laws for energy and momentum of a nonrelativistic particle living in $q$-deformed Euclidean space. In what follows, we are briefly describing how we derive these conservation laws in this article.

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First of all, we summarize some results and characteristic features of our approach [cf. Chap. 2]. The commutation relations for the coordinates of $q$-deformed Euclidean space satisfy the so-called Poincaré-Birkhoff-Witt property. Due to this fact, we can associate the noncommutative algebra of $q$-deformed Euclidean space with a commutative coordinate algebra by using the star-product formalism [8]. The star-product formalism enables us to construct a $q$-deformed version of mathematical analysis [9, 10]. As was shown in Ref. [11], the time evolution operator of a quantum system in $q$-deformed Euclidean space is of the same form as in the undeformed case. These findings enable us to write down Schrödinger equations for a $q$-deformed nonrelativistic particle (cf. Ref. [12] and Chap. 3).

In Chap. 4 of this article, we derive $q$-versions of Green’s theorem. We achieve this by applying the Leibniz rules for $q$-deformed partial derivatives. From the $q$-versions of Green’s theorem and the Schrödinger equations for a $q$-deformed particle, we derive continuity equations for the probability density, the energy density, and the momentum density of a $q$-deformed, nonrelativistic particle. Our reasonings also include the case where a $q$-deformed particle interacts with an electromagnetic field. In this respect, we will show in Chap. 8 that our $q$-deformed continuity equations are invariant under gauge transformations. Finally, we calculate Heisenberg’s equation for observables of a $q$-deformed nonrelativistic particle interacting with an electromagnetic field. In doing so, we obtain evolution equations that are in agreement with our $q$-deformed continuity equations.

2 Preliminaries

2.1 Star-products

The three-dimensional $q$-deformed Euclidean space $\mathbb{R}^3_q$ has the generators $X^+$, $X^3$, and $X^-$, subject to the following commutation relations [13]:

\[
\begin{align*}
X^3X^+ &= q^2X^+X^3, \\
X^3X^- &= q^{-2}X^-X^3, \\
X^-X^+ &= X^+X^- + (q - q^{-1})X^3X^3.
\end{align*}
\]

We can extend the algebra of $\mathbb{R}^3_q$ by a time element $X^0$, which commutes with the generators $X^+$, $X^3$, and $X^-$ [11]:

\[
X^0X^A = X^AX^0, \quad A \in \{+, 3, -\}.
\]

In the following, we refer to the algebra spanned by the generators $X^i$ with $i \in \{0, +, 3, -\}$ as $\mathbb{R}^{3,4}_q$.

There is a $q$-analog of the three-dimensional Euclidean metric $g^{AB}$ with its inverse $g_{AB}$ [13] (rows and columns are arranged in the order $+, 3, -$):

\[
g_{AB} = g^{AB} = \begin{pmatrix}
0 & 0 & -q \\
0 & 1 & 0 \\
-q^{-1} & 0 & 0
\end{pmatrix}.
\]
We can use the $q$-deformed metric to raise and lower indices:

$$X_A = g_{AB} X^B, \quad X^A = g^{AB} X_B.$$  \hfill (4)

The algebra $\mathbb{R}^{3,t}_q$ has a semilinear, involutive, and anti-multiplicative mapping, which we call *quantum space conjugation*. If we indicate conjugate elements of a quantum space by a bar $\bar{}$, we can write the properties of quantum space conjugation as follows ($\alpha, \beta \in \mathbb{C}$ and $u, v \in \mathbb{R}^{3,t}_q$):

$$\bar{\alpha u + \beta v} = \bar{\alpha} \bar{u} + \bar{\beta} \bar{v}, \quad \bar{u} = u, \quad \bar{uv} = v \bar{u}.$$ \hfill (5)

The conjugation for $\mathbb{R}^{3,t}_q$ is compatible with the commutation relations in Eq. (1) and Eq. (2) if the following applies \cite{11}:

$$X^A = X_A = g_{AB} X^B, \quad X^0 = X_0.$$ \hfill (6)

We can only prove a physical theory if it predicts measurement results. The problem, however, is: How can we associate the elements of the noncommutative space $\mathbb{R}^{3,t}_q$ with real numbers? One solution to this problem is to introduce a vector space isomorphism between the noncommutative algebra $\mathbb{R}^{3,t}_q$ and a corresponding commutative coordinate algebra $\mathbb{C}[x^+, x^3, x^-, t]$.

We recall that the normal-ordered monomials in the generators $X^i$ form a basis of the algebra $\mathbb{R}^{3,t}_q$, i.e. we can write each element $F \in \mathbb{R}^{3,t}_q$ uniquely as a finite or infinite linear combination of monomials with a given normal ordering (*Poincaré-Birkhoff-Witt property*):

$$F = \sum_{n_+, \ldots, n_0} a_{n_+, \ldots, n_0} (X^+)^{n_+} (X^3)^{n_3} (X^-)^{n_-} (X^0)^{n_0}, \quad a_{n_+, \ldots, n_0} \in \mathbb{C}. \hfill (7)$$

Since the monomials $(x^+)^{n_+} (x^3)^{n_3} (x^-)^{n_-} t^{n_0}$ with $n_+, \ldots, n_0 \in \mathbb{N}_0$ form a basis of the commutative algebra $\mathbb{C}[x^+, x^3, x^-, t]$, we can define a vector space isomorphism \(W : \mathbb{C}[x^+, x^3, x^-, t] \to \mathbb{R}^{3,t}_q\) with

$$W ((x^+)^{n_+} (x^3)^{n_3} (x^-)^{n_-} t^{n_0}) = (X^+)^{n_+} (X^3)^{n_3} (X^-)^{n_-} (X^0)^{n_0}.$$ \hfill (9)

In general, we have

$$\mathbb{C}[x^+, x^3, x^-, t] \ni f \mapsto F \in \mathbb{R}^{3,t}_q,$$ \hfill (10)

where

$$f = \sum_{n_+, \ldots, n_0} a_{n_+, \ldots, n_0} (x^+)^{n_+} (x^3)^{n_3} (x^-)^{n_-} t^{n_0},$$

$$F = \sum_{n_+, \ldots, n_0} a_{n_+, \ldots, n_0} (X^+)^{n_+} (X^3)^{n_3} (X^-)^{n_-} (X^0)^{n_0}. \hfill (11)$$

\(^1\)A bar over a complex number indicates complex conjugation.
The vector space isomorphism $\mathcal{W}$ is nothing else but the Moyal-Weyl mapping, which gives an operator $F$ to a complex valued function $f$.

We can extend this vector space isomorphism to an algebra isomorphism if we introduce a new product on the commutative coordinate algebra. This so-called star-product symbolized by $\star$ satisfies the following homomorphism condition:

$$\mathcal{W}(f \star g) = \mathcal{W}(f) \cdot \mathcal{W}(g).$$  \hspace{1cm} (12)

Since the Moyal-Weyl mapping is invertible, we can write the star-product as follows:

$$f \star g = \mathcal{W}^{-1}(\mathcal{W}(f) \cdot \mathcal{W}(g)).$$  \hspace{1cm} (13)

To get explicit formulas for calculating star-products, we first have to write a noncommutative product of two normal-ordered monomials as a linear combination of normal-ordered monomials again (see Ref. [17] for details):

$$(X^+)^{n_+} \cdots (X^0)^{n_0} \cdot (X^+)^{m_+} \cdots (X^0)^{m_0} = \sum_{k=0}^{\infty} B_{n,m}^k (X^+)^{k_+} \cdots (X^0)^{k_0}. \hspace{1cm} (14)$$

We achieve this by using the commutation relations for the noncommutative coordinates [cf. Eq. (1)]. From the concrete form of the expansion in Eq. (14), we can finally read off a formula to calculate the star-product of two power series in commutative space-time coordinates ($\lambda = q - q^{-1}$):

$$f(x, t) \star g(x, t) = \sum_{k=0}^{\infty} \lambda^k \left( \frac{x^3}{[[k]]_q!} q^{2(\hat{n}_+ \hat{n}_0 + \hat{n}_- \hat{n}_0')} D^{k}_{q^4,x} f(x, t) |_{x' \to x} \right). \hspace{1cm} (15)$$

The expression above depends on the operators

$$\hat{n}_A = x^A \frac{\partial}{\partial x^A}$$  \hspace{1cm} (16)

and the so-called Jackson derivatives [15]:

$$D^{k}_{q^4,x} f = \frac{f(q^k x) - f(x)}{q^k x - x}. \hspace{1cm} (17)$$

Moreover, the $q$-numbers are given by

$$[[a]]_q = \frac{1 - q^a}{1 - q}, \hspace{1cm} (18)$$

and the $q$-factorials are defined in complete analogy to the undeformed case:

$$[[n]]_q! = [[1]]_q[[2]]_q \cdots [[n - 1]]_q[[n]]_q, \hspace{1cm} [[0]]_q! = 1. \hspace{1cm} (19)$$

The algebra isomorphism $\mathcal{W}^{-1}$ also enables us to carry over the conjugation for the quantum space algebra $\mathbb{R}^{3,t}_q$ to the commutative coordinate algebra\footnote{The argument $x$ indicates a dependence on the spatial coordinates $x^+, x^0, and x^-$.}.
\( \mathbb{C}[x^+, x^3, x^-, t] \). In other words, the mapping \( W^{-1} \) is a \(*\)-algebra homomorphism:

\[
W(f) = \overline{W(f)} \quad \iff \quad \overline{f} = W^{-1}(\overline{W(f)}).
\] (20)

This relationship implies the following property for the star-product:

\[
\overline{f \star g} = g \star \overline{f}.
\] (21)

With \( \overline{f} \), we designate the power series obtained from \( f \) by quantum space conjugation. It follows from Eq. (6) and Eq. (20) that \( \overline{f} \) takes the following form (if \( \overline{a}_{n+} n_3, n_-, n_0 \) stands for the complex conjugate of \( a_{n+} n_3, n_-, n_0 \)) [10, 11]:

\[
\overline{f}(x, t) = \sum_{n} \overline{a}_{n+} n_3, n_-, n_0 (x^-)^n (x^3)^n (x^+) n_--t n_0
\]
\[
= \sum_{n} (x^-)^n (x^3)^n (x^+) n_--t n_0
\]
\[
= \overline{f}(x, t).
\] (22)

### 2.2 Partial derivatives and integrals

There are partial derivatives for \( q \)-deformed space-time coordinates [19, 20]. These partial derivatives again form a quantum space with the same algebraic structure as that of the \( q \)-deformed space-time coordinates. Thus, the \( q \)-deformed partial derivatives \( \partial_i \) satisfy the same commutation relations as the covariant coordinate generators \( X_i \):

\[
\partial_0 \partial_+ = \partial_+ \partial_0, \quad \partial_0 \partial_- = \partial_- \partial_0, \quad \partial_0 \partial_3 = \partial_3 \partial_0,
\]
\[
\partial_+ \partial_3 = q^2 \partial_3 \partial_+, \quad \partial_3 \partial_- = q^2 \partial_- \partial_3,
\]
\[
\partial_- \partial_+ - \partial_+ \partial_- = \lambda \partial_3 \partial_3.
\] (23)

The commutation relations above are invariant under conjugation if the derivatives show the following conjugation properties\(^3\)

\[
\overline{\partial_A} = -\partial^A = -g^{AB} \partial_B, \quad \overline{\partial_0} = -\partial^0 = -\partial_0.
\] (24)

There are two ways of commuting \( q \)-deformed partial derivatives with \( q \)-deformed space-time coordinates. One is given by the following \( q \)-deformed Leibniz rules [11, 19, 20]:

\[
\partial_B X^A = \delta_B^A + q^4 R^{AC} BD X^D \partial_C,
\]
\[
\partial_A X^0 = X^0 \partial_A,
\]
\[
\partial_0 X^A = X^A \partial_0,
\]
\[
\partial_0 X^0 = 1 + X^0 \partial_0.
\] (25)

\(^3\)The indices of partial derivatives are raised and lowered in the same way as those of coordinates [see Eq. (4) in Chap. 2.1].
Note that $\hat{R}^{AC}_{BD}$ denotes the vector representation of the R-matrix for the three-dimensional $q$-deformed Euclidean space.

By conjugation, we can obtain the Leibniz rules for another differential calculus from the identities in Eq. (25). Introducing $\hat{\partial}_A = q^6 \partial_A$ and $\hat{\partial}_0 = \partial_0$, we can write the Leibniz rules of this second differential calculus in the following form:

$$\hat{\partial}_B X^A = \delta^A_B + q^{-4}(\hat{R}^{-1})^{AC}_{BD} X^D \hat{\partial}_C,$$

$$\hat{\partial}_A X^0 = X^0 \hat{\partial}_A,$$

$$\hat{\partial}_0 X^A = X^A \hat{\partial}_0,$$

$$\hat{\partial}_0 X^0 = 1 + X^0 \hat{\partial}_0. \tag{26}$$

Using the Leibniz rules in Eq. (25) or Eq. (26), we can calculate how partial derivatives act on normal-ordered monomials of noncommutative coordinates. We can carry over these actions to commutative coordinate monomials with the help of the Moyal-Weyl mapping:

$$\partial^i \triangleright (x^+)^n (x^-)^m x^n + t^n = W^{-1}(\partial^i \triangleright (X^+)^n (X^-)^m (X^-)^n - (X^0)^n). \tag{27}$$

Since the Moyal-Weyl mapping is linear, we can apply the action above to space-time functions that can be written as a power series:

$$\partial^i \triangleright f(x,t) = W^{-1}(\partial^i \triangleright W(f(x,t))). \tag{28}$$

If we use the ordering given in Eq. (9) of the previous chapter, the Leibniz rules in Eq. (25) will lead to the following operator representations:

$$\partial_+ \triangleright f(x,t) = D_{q^4,x^+} f(x,t),$$

$$\partial_3 \triangleright f(x,t) = D_{q^2,3} f(q^2 x^+, x^3, x^-, t),$$

$$\partial_- \triangleright f(x,t) = D_{q^4,x^-} f(x^+, q^2 x^3, x^-, t) + \lambda x^+ D_{q^2,3} f(x,t). \tag{29}$$

The derivative $\partial_0$, however, is represented on the commutative space-time algebra by an ordinary partial derivative:

$$\partial_0 \triangleright f(x,t) = \partial_t \triangleright f(x,t) = \frac{\partial f(x,t)}{\partial t}. \tag{30}$$

Using the Leibniz rules in Eq. (26), we get operator representations for the partial derivatives $\hat{\partial}^i$. The Leibniz rules in Eq. (25) and Eq. (26) are transformed into each other by the following substitutions:

$$q \rightarrow q^{-1}, \quad X^- \rightarrow X^+, \quad X^+ \rightarrow X^-,$$

$$\partial^+ \rightarrow \hat{\partial}^-, \quad \partial^- \rightarrow \hat{\partial}^+, \quad \partial^3 \rightarrow \hat{\partial}^3, \quad \partial^0 \rightarrow \hat{\partial}^0. \tag{31}$$

For this reason, we obtain the operator representations of the partial derivatives $\hat{\partial}_A$ from those of the partial derivatives $\partial_A$ [cf. Eq. (25)] if we replace $q$ by $q^{-1}$.
and exchange the indices + and −:

\[
\hat{\partial}_- \triangleright f(x,t) = D_{q^{-1},x^-} f(x,t), \\
\hat{\partial}_3 \triangleright f(x,t) = D_{q^{-2},x^3} f(q^{-2}x^-,x^3,x^+,t), \\
\hat{\partial}_+ \triangleright f(x,t) = D_{q^{-1},x^+} f(x^-,q^{-2}x^3,x^+,t) - \lambda x^- D_{q^{-2},x^3}^2 f(x,t).
\]  

(32)

Once again, \( \hat{\partial}_0 \) is represented on the commutative space-time algebra by an ordinary partial derivative:

\[
\hat{\partial}_0 \triangleright f(x,t) = \partial_t \triangleright f(x,t) = \frac{\partial f(x,t)}{\partial t}.
\]  

(33)

Due to the substitutions given in Eq. (31), the actions in Eqs. (32) and (33) refer to normal-ordered monomials different from those in Eq. (9) of the previous chapter:

\[
\hat{\mathcal{W}} \left( t^{n_0} (x^+)^{n_1} (x^3)^{n_3} (x^-)^{n_4} \right) = (X^0)^{n_0} (X^-)^{n_4} (X^3)^{n_3} (X^+)^{n_1}.
\]  

(34)

We should not forget that we can also commute \( q \)-deformed partial derivatives from the right side of a normal-ordered monomial to the left side by using the Leibniz rules. This way, we get so-called right representations of partial derivatives, for which we write \( f \triangleright \hat{\partial}^i \) or \( f \triangleright \hat{\partial}^i \): Note that the operation of conjugation transforms left actions of partial derivatives into right actions and vice versa [21]:

\[
\partial^i \triangleright f = - \bar{f} \triangleright \partial^i, \\
\hat{\partial}^i \triangleright f = - f \triangleright \hat{\partial}^i.
\]  

(35)

In general, the operator representations in Eqs. (29) and (32) consist of two terms, which we call \( \partial^A_{\text{cla}} \) and \( \partial^A_{\text{cor}} \):

\[
\partial^A \triangleright F = (\partial^A_{\text{cla}} + \partial^A_{\text{cor}}) \triangleright F.
\]  

(36)

In the undeformed limit \( q \rightarrow 1 \), \( \partial^A_{\text{cla}} \) becomes an ordinary partial derivative, and \( \partial^A_{\text{cor}} \) disappears. We get a solution to the difference equation \( \partial^A \triangleright F = f \) with given \( f \) by using the following formula [22]:

\[
F = (\partial^A)^{-1} \triangleright f = (\partial^A_{\text{cla}} + \partial^A_{\text{cor}})^{-1} \triangleright f \\
= \sum_{k=0}^{\infty} \left[ -(\partial^A_{\text{cla}})^{-1} \partial^A_{\text{cor}} \right]^k (\partial^A_{\text{cla}})^{-1} \triangleright f.
\]  

(37)

Applying the above formula to the operator representations in Eq. (29), we get

\[
(\partial_+)^{-1} \triangleright f(x,t) = D_{q^{-1},x^+}^{-1} f(x,t), \\
(\partial_3)^{-1} \triangleright f(x,t) = D_{q^{-2},x^3}^{-1} f(q^{-2}x^-,x^3,x^+,t),
\]  

(38)
and

\[(\partial_-)^{-1} \triangleright f(x, t) = \sum_{k=0}^{\infty} q^{2k(k+1)} \left(-\lambda x^+ D_{q^4, x^+}^{-1} D_{q^2, x^3}^2\right)^k D_{q^4, x^-}^{-1} f(x^+, q^{-2k+1} x^3, x^-, t). \]  

(39)

Note that \(D_{q,x}^{-1}\) stands for a Jackson integral with \(x\) being the variable of integration [23]. The explicit form of this Jackson integral depends on its limits of integration and the value for the deformation parameter \(q\). If \(x > 0\) and \(q > 1\), for example, the following applies:

\[
\int_0^x d_q z f(z) = (q - 1) x \sum_{j=1}^{\infty} q^{-j} f(q^{-j} x). 
\]

(40)

Finally, the integral for the time coordinate is an ordinary integral since \(\partial_0\) acts on the commutative space-time algebra like an ordinary partial derivative [cf. Eq. (30)]:

\[
(\partial_0)^{-1} \triangleright f(x, t) = \int dt \ f(x, t). 
\]

(41)

The above considerations also apply to the partial derivatives with a hat. However, we can obtain the representations of \(\hat{\partial}_i\) from those of the derivatives \(\partial_i\) if we replace \(q\) with \(q^{-1}\) and exchange the indices + and −. Applying these substitutions to the expressions in Eqs. (38) and (39), we immediately get the corresponding results for the partial derivatives \(\hat{\partial}_i\).

By successively applying the integral operators given in Eqs. (38) and (39), we can explain an integration over all space [10, 22]:

\[
\int_{-\infty}^{+\infty} d_q^3 x f(x^+, x^3, x^-) = (\partial_-)^{-1} f(x^-) - (\partial_3)^{-1} f(x^+) + (\partial_+)^{-1} f(x). 
\]

(42)

On the right-hand side of the above relation, the different integral operators can be simplified to Jackson integrals [22, 24]:

\[
\int_{-\infty}^{+\infty} d_q^3 x f(x) = D_{q^4, x^-}^{-1} f(x^-) - D_{q^4, x^3}^{-1} f(x^3) + D_{q^4, x^+}^{-1} f(x^+). 
\]

(43)

Note that the Jackson integrals in the formula above refer to a smaller \(q\)-lattice. Using such a smaller \(q\)-lattice ensures that our integral over all space is a scalar with trivial braiding properties [25].

The \(q\)-integral over all space shows some significant features [10]. In this respect, \(q\)-deformed versions of Stokes’ theorem apply:

\[
\int_{-\infty}^{+\infty} d_q^3 x \hat{\partial}^A \triangleright f = \int_{-\infty}^{+\infty} d_q^3 x f \hat{\partial}^A = 0, \quad \int_{-\infty}^{+\infty} d_q^3 x \hat{\partial}^A \triangleright f = \int_{-\infty}^{+\infty} d_q^3 x f \hat{\partial}^A = 0. 
\]

(44)
The \( q \)-deformed Stokes’ theorem also implies rules for integration by parts:

\[
\int_{-\infty}^{+\infty} d^3 q \, f \otimes (\partial^A \triangleright g) = \int_{-\infty}^{+\infty} d^3 q \, (f \triangleleft \partial^A) \otimes g,
\]

\[
\int_{-\infty}^{+\infty} d^3 q \, f \otimes (\partial^A \downarrow g) = \int_{-\infty}^{+\infty} d^3 q \, (f \downarrow \partial^A) \otimes g.
\]  

\( (45) \)

Finally, we mention that the \( q \)-integral over all space behaves as follows under quantum space conjugation:

\[
\int_{-\infty}^{+\infty} d^3 q \, f = \int_{-\infty}^{+\infty} d^3 q \, \bar{f}.
\]  

\( (46) \)

2.3 Hopf structures and L-matrices

The three-dimensional \( q \)-deformed Euclidean space \( \mathbb{R}^3_q \) is a three-dimensional representation of the Drinfeld-Jimbo algebra \( \mathcal{U}_q(\mathfrak{su}_2) \). The latter is a deformation of the universal enveloping algebra of the Lie algebra \( \mathfrak{su}_2 \) \( [26] \). Accordingly, the algebra \( \mathcal{U}_q(\mathfrak{su}_2) \) has three generators \( T^+, T^-, \) and \( T^3 \) which satisfy the following relations \( [27] \):

\[
q^{-1}T^+T^- - qT^-T^+ = T^3,
\]

\[
q^2T^3T^- - q^{-2}T^+T^3 = (q + q^{-1})T^+,
\]

\[
q^2T^-T^3 - q^{-2}T^3T^- = (q + q^{-1})T^-.
\]

\( (47) \)

The algebra of the \( q \)-deformed partial derivatives \( \partial^A, A \in \{+, 3, -\} \), together with \( \mathcal{U}_q(\mathfrak{su}_2) \) form the cross-product algebra \( \mathbb{R}^3_q \times \mathcal{U}_q(\mathfrak{su}_2) \) \( [28, 29] \). We know that the algebra \( \mathbb{R}^3_q \times \mathcal{U}_q(\mathfrak{su}_2) \) is a Hopf algebra \( [30] \). Accordingly, the \( q \)-deformed partial derivatives as elements of \( \mathbb{R}^3_q \times \mathcal{U}_q(\mathfrak{su}_2) \) have a co-product, an antipode, and a co-unit.

However, there are two ways of choosing the Hopf structure of the \( q \)-deformed partial derivatives. It is so because the two different co-products of the \( q \)-deformed partial derivatives are related to the two versions of Leibniz rules given in Eq. \( (25) \) or Eq. \( (26) \) of the last subchapter. For a better insight, we note that you can generalize these Leibniz rules by introducing so-called L-matrices \( L_\partial \) and \( \hat{L}_\partial \) \( (u \in \mathbb{R}^3_q) \):

\[
\partial^A u = (\partial^A(1) \triangleright u) \partial^A(2) = \partial^A \triangleright u + ((L_\partial)^A_B \triangleright u) \partial^B,
\]

\[
\hat{\partial}^A u = (\hat{\partial}^A(1) \triangleright u) \hat{\partial}^A(2) = \hat{\partial}^A \triangleright u + ((\hat{L}_\partial)^A_B \triangleright u) \hat{\partial}^B.
\]  

\( (48) \)

You can see from the above identities that the two L-matrices determine the two co-products \( ^3 \) of the \( q \)-deformed partial derivatives \( [31] \):

\[
\partial^A(1) \otimes \partial^A(2) = \partial^A \otimes 1 + (L_\partial)^A_B \otimes \partial^B,
\]

\[
\hat{\partial}^A(1) \otimes \hat{\partial}^A(2) = \hat{\partial}^A \otimes 1 + (\hat{L}_\partial)^A_B \otimes \hat{\partial}^B.
\]

\( (49) \)

\( ^3 \)We write the co-product in the so-called Sweedler notation, i. e. \( \Delta(a) = a(1) \otimes a(2) \).
The entries of the two L-matrices consist of generators of the Hopf algebra $U_q(\text{su}_2)$ and powers of a scaling operator $\Lambda$ [also see Eq. (52)]. For this reason, the L-matrices can act on any element of $\mathbb{R}^3_q$. In this respect, we say that an element of $\mathbb{R}^3_q$ has trivial braiding if the L-matrices act on it as follows:

$$
(L_\partial)^A_B \triangleright u = \delta^A_B u,
(\bar{L}_\partial)^A_B \triangleright u = \delta^A_B u.
$$

(50)

In Ref. [21] and Ref. [32], we have written down the co-products of the partial derivatives $\partial^A$ or $\hat{\partial}^A$, $A \in \{+, 3, -\}$, explicitly. By taking into account Eq. (49), you can read off the entries of the L-matrices $L_\partial$ and $\bar{L}_\partial$ from these co-products. In doing so, you find, for example:

$$
(L_\partial)^{−−} = \Lambda^{−1/2} \tau^{−1/2} \text{ and } (\bar{L}_\partial)^{++} = \Lambda^{−1/2} \tau^{−1/2}.
$$

(51)

The scaling operator $\Lambda$ acts on the spatial coordinates or the corresponding partial derivatives as follows:

$$
\Lambda \triangleright X^A = q^4 X^A, \quad \Lambda \triangleright \partial^A = q^{-4} \partial^A.
$$

(52)

These actions imply the commutation relations

$$
\Lambda X^A = q^4 X^A \Lambda, \quad \Lambda \partial^A = q^{-4} \partial^A \Lambda
$$

if we take into account the Hopf structure of $\Lambda$ [31]:

$$
\Delta(\Lambda) = \Lambda \otimes \Lambda, \quad S(\Lambda) = \Lambda^{-1}, \quad \varepsilon(\Lambda) = 1.
$$

(54)

The Hopf structure of the partial derivatives includes a co-product as well as an antipode and a co-unit. Regarding the co-unit of the partial derivatives, the following holds [31]:

$$
\varepsilon(\partial^A) = 0.
$$

We can obtain the antipodes of the partial derivatives from their co-products using the following Hopf algebra axioms:

$$
a_{(1)} \cdot S(a_{(2)}) = \varepsilon(a) = S(a_{(1)}) \cdot a_{(2)}.
$$

(55)

Due to this axiom, we have:

$$
S(\partial^A) = - S(L_\partial)^A_B \partial^B, \quad \bar{S}(\hat{\partial}^A) = - S^{-1}(\bar{L}_\partial)^A_B \hat{\partial}^B.
$$

(56)

This way, for example, we get the following expressions for the antipodes of the partial derivatives $\partial^−$ and $\hat{\partial}^+$ (also see Ref. [21]):

$$
S(\partial^−) = - \Lambda^{-1/2} \tau^{1/2} \partial^−, \quad \bar{S}(\hat{\partial}^+) = - \Lambda^{1/2} \tau^{1/2} \hat{\partial}^+.
$$

(57)

Instead of $T^3$, one often uses $\tau = 1 - \lambda T^3$. 

10
The antipodes of partial derivatives allow us to write the left actions of partial derivatives as right actions, and vice versa. Concretely, we have

\[
\partial^A \triangleright f = f \triangleleft S(\partial^A) = -(f \triangleleft S(\mathcal{L}_\partial)^A_B) \triangleleft \partial^B \\
= -(\mathcal{L}_\partial)^A_B \triangleright f, \\
\hat{\partial}^A \triangleright f = f \triangleright S(\hat{\partial}^A) = -(f \triangleright S(\hat{\mathcal{L}}_\partial)^A_B) \triangleright \hat{\partial}^B \\
= -((\hat{\mathcal{L}}_\partial)^A_B \triangleright f) \triangleright \hat{\partial}^B,
\]

and

\[
f \triangleleft \hat{\partial}^A = S^{-1}(\hat{\partial}^A) \triangleright f = -\hat{\partial}^B \triangleright (S^{-1}(\hat{\mathcal{L}}_\partial)^A_B \triangleright f) \\
= -(\hat{\mathcal{L}}_\partial)^A_B \triangleright (f \triangleleft (\hat{\mathcal{L}}_\partial)^A_B), \\
f \triangleright \partial^A = \mathcal{S}^{-1}(\partial^A) \triangleright f = -\partial^B \triangleright (S^{-1}(\mathcal{L}_\partial)^A_B \triangleright f) \\
= -\partial^B \triangleright (f \triangleleft (\mathcal{L}_\partial)^A_B). \tag{58}
\]

3 Schrödinger equations for a \(q\)-deformed non-relativistic particle

In Ref. [12], we have chosen the following expression as Hamilton operator for a free nonrelativistic particle with mass \(m\):

\[
H_0 = -(2m)^{-1}g_{AB} \partial^A \partial^B = -(2m)^{-1}\partial^A \partial_A. \tag{60}
\]

This choice ensures that \(H_0\) behaves like a scalar under the action of the Hopf algebra \(\mathcal{U}_q(su_2)\). Moreover, \(H_0\) has trivial braiding if we require:

\[
\Lambda m = q^{-8} m \Lambda.
\]

Due to its definition, the Hamilton operator \(H_0\) is also a central element of the algebra of \(q\)-deformed partial derivatives:

\[
[H_0, \partial^A] = 0, \quad A \in \{+, 3, -\}. \tag{61}
\]

The conjugation properties of the partial derivatives imply that \(H_0\) is invariant under conjugation [cf. Eq. (24) of Chap. 2.2]:

\[
\overline{H_0} = H_0. \tag{62}
\]

We can add a potential \(V(x)\) to the free Hamilton operator:

\[
H = H_0 + V(x). \tag{63}
\]

Under the action of the Hopf algebra \(\mathcal{U}_q(su_2)\), \(V(x)\) should behave like a scalar. We also require \(V(x)\) to have trivial braiding and to be central in the algebra of position space:

\[
V(x) \triangleleft f(x) = f(x) \triangleleft V(x). \tag{64}
\]
Moreover, $V(x)$ has to be invariant under conjugation:

$$\overline{V(x)} = V(x).$$  \hfill (65)

If we deal with a charged particle moving in the presence of a magnetic field, we start from the Hamilton operator

$$H = -(2m)^{-1} D^C D_C,$$  \hfill (66)

which depends on the covariant derivatives

$$D_C = \partial_C - ie A^C(x, t).$$  \hfill (67)

$A^C(x, t)$ denotes the component of the vector potential and $e$ is the charge of the particle. To ensure that $H$ has well-defined braiding properties, the L-matrices have to act on the component $A^C(x, t)$ in the same way as on the partial derivative $\partial_C$. Since $H$ has to be invariant under quantum space conjugation, we also require

$$A^D(x, t) = A^D(x, t).$$  \hfill (68)

We can use the Hamilton operators given in Eq. (63) or Eq. (66) to write down Schrödinger equations [11]:

$$i \partial_t \triangleright \psi_R(x, t) = H \triangleright \psi_R(x, t), \quad \psi_L(x, t) \triangleright \partial_t i = \psi_L(x, t) \triangleright H,$$

$$i \partial_t \triangleright \psi_R^*(x, t) = H \triangleright \psi_R^*(x, t), \quad \psi_L^*(x, t) \triangleright \partial_t i = \psi_L^*(x, t) \triangleleft H. \hfill (69)$$

As was shown in Ref. [11], the time evolution operator for the quantum space $\mathbb{R}^3_q$ is of the same form as in the undeformed case. For this reason, we get solutions to our $q$-deformed Schrödinger equations by applying the operators $\exp(-itH)$ and $\exp(itH)$ to wave functions at time $t = 0$:

$$\psi_R(x, t) = \exp(-itH) \triangleright \psi_R(x, 0), \quad \psi_L(x, t) = \psi_L(x, 0) \triangleright \exp(iHt),$$

$$\psi_R^*(x, t) = \exp(-itH) \triangleright \psi_R^*(x, 0), \quad \psi_L^*(x, t) = \psi_L^*(x, 0) \triangleleft \exp(iHt). \hfill (70)$$

Last but not least, we require that the solutions to the Schrödinger equations in Eq. (69) behave as follows under quantum space conjugation:

$$\overline{\psi_L(x, t)} = \psi_R(x, t), \quad \overline{\psi_L^*(x, t)} = \psi_R^*(x, t). \hfill (71)$$

This condition ensures that quantum space conjugation transforms the Schrödinger equations on the left side of Eq. (69) into the Schrödinger equations on the right side of Eq. (69) and vice versa.

## 4 q-Versions of Green’s theorem

We need $q$-versions of Green’s theorem to derive conservation laws. We will show in this chapter how to get these $q$-deformed analogs from the Leibniz rules for $q$-deformed partial derivatives and the properties of L-matrices$^6$.

$^6$A one-dimensional $q$-version of Green’s theorem was given in Ref. [33].
The Leibniz rules in Eq. (48) of Chap. 2.2 imply the following rule for left actions of \(q\)-deformed partial derivatives:

\[
\psi \triangleleft \partial^A \triangleright \phi = \partial^B \triangleright \left[ \psi \triangleleft (L_\partial)^A_B \triangleright \phi \right] + \psi \triangleleft \partial^A \triangleright \phi.
\] (72)

We can prove the identity above in the following way:

\[
\partial^B \triangleright \left[ \psi \triangleleft (L_\partial)^A_B \triangleright \phi \right] = \partial^B \triangleright \left[ S^{-1}(L_\partial)^A_B \triangleright \psi \triangleright \phi \right] = -\psi \triangleleft \partial^A \triangleright \phi + \delta^A_B \psi \triangleright \partial^C \triangleright \phi
\] (73)

In the first step, we have used the fact that the entries \((L_\partial)^A_B\) are elements of a Hopf algebra (also see Chap. 2.3). Thus, we can write the right action as a left-action by using the inverse antipode of this Hopf algebra. The second step results from Eq. (48) of Chap. 2.2. The third step and the fourth step are a consequence of the Hopf algebra axiom

\[
a(2) \cdot S^{-1}a(1) = \varepsilon(a)
\] (74)

and the following properties of L-matrices:

\[
\Delta(L_\partial)^A_B = (L_\partial)^A_C \otimes (L_\partial)^C_B, \quad \varepsilon(L_\partial)^A_B = \delta^A_B.
\] (75)

Similar arguments lead to

\[
\psi \triangleleft \partial^A \triangleright \phi = \left[ \psi \triangleleft (L_\partial)^A_B \triangleright \phi \right] \triangleleft (2) = \left[ \psi \triangleleft (L_\partial)^A_B \triangleright \phi \right] \triangleleft \partial^B + \psi \triangleleft \partial^A \triangleright \phi.
\] (76)

We can now derive a \(q\)-version of Green’s theorem. Due to Eq. (72), we have

\[
\psi \triangleleft \partial^A \triangleright \partial^A \triangleright \phi = \psi \triangleleft \partial^A \triangleright \partial^A \triangleright \phi - \partial^A \triangleright \left[ \psi \triangleleft (L_\partial)^B_C \triangleright \partial^B \triangleright \phi \right] g_{AB}
\] (77)

and

\[
\psi \triangleleft \partial^A \triangleright \partial^A \triangleright \phi = \psi \triangleleft \partial^A \triangleright \partial^A \triangleright \phi + \partial^B \triangleright \left[ \psi \triangleleft (L_\partial)^A_C \triangleright \partial^B \triangleright \phi \right] g_{AB}.
\] (78)

Combining Eq. (77) and Eq. (78), we obtain:

\[
\psi \triangleleft \partial^A \triangleright \partial^A \triangleright \phi - \psi \triangleleft \partial^A \triangleright \partial^A \triangleright \phi =
\]

\[
-\partial^C \triangleright \left[ \psi \triangleleft (L_\partial)^A_C \triangleright \partial^B \triangleright \phi \right] g_{AB}
\]

and

\[
\psi \triangleleft \partial^A \triangleright \partial^A \triangleright \phi = \psi \triangleleft \partial^A \triangleright \partial^A \triangleright \phi + \partial^A \triangleright \left[ \psi \triangleleft (L_\partial)^A_C \triangleright \partial^B \triangleright \phi \right] g_{AB}.
\] (79)

The last step of the above calculation results from the following identities:

\[
g_{AB} \partial^A (L_\partial)^B_C = g_{AB} (L_\partial)^D_C [\partial^A \triangleleft (L_\partial)^B_D] = g_{AB} (L_\partial)^D_C q^A R^{AB}_{DE} \partial^E = q^{-2} q^A g_{DE} (L_\partial)^D_C \partial^E = g_{AB} q^{-2} (L_\partial)^A_C \partial^B g_{AB}.
\] (80)
The first identity of Eq. (80) follows from the co-product of the L-matrices \[\text{cf. Eq. (75)}\] and the commutation relations of the Hopf algebra \(\mathbb{R}_q^3 \rtimes \mathcal{U}_q(\mathfrak{su}_2)\):

\[
\partial \cdot u = u_{(2)} \cdot (\partial \lrcorner u_{(1)}).
\]

(81)

The second identity of Eq. (80) holds because the vector representation of the R-matrix determines the action of \((L_\phi)^A_B\) on a vector. The penultimate identity of Eq. (80) follows from the projector decomposition of the R-matrix of the \(q\)-deformed Euclidean space \[\text{i.e.}\]

\[
\hat{R} = P_S + q^{-6}P_T - q^{-4}P_A,
\]

(82)

if we take into account:

\[
g_{AB}(P_S)^{AB}_{CD} = g_{AB}(P_A)^{AB}_{CD} = 0, \quad g_{AB}(P_T)^{AB}_{CD} = g_{CD}.
\]

(83)

By similar reasonings using Eq. (76), we can also derive the following identity:

\[
\psi \lrcorner D^C D_C \lrcorner \phi - \psi \lrcorner D^C D_C \lrcorner \phi =
\]

\[
= g_{BC} [\psi \lrcorner (L_\phi)^B_C \lrcorner \phi + q^2 \psi \lrcorner (L_\phi)^C_B \lrcorner \phi] \lrcorner \delta^D
\]

(84)

If there is a vector potential, we have to substitute the partial derivatives \(\partial^C\) by the operators \(D^C\) \[\text{cf. Eq. (67) of Chap. 3}\]. Instead of Eq. (79) and Eq. (84), we have the identities

\[
\psi \lrcorner D^C D_C \lrcorner \phi - \psi \lrcorner D^C D_C \lrcorner \phi =
\]

\[
= - \partial^F \lrcorner [q^{-2} \psi \lrcorner (L_\phi)^B \lrcorner D^C \lrcorner \phi + \psi \lrcorner (L_\phi)^C_B \lrcorner D^C \lrcorner \phi] g_{BC}
\]

\[
= g_{BC} [\psi \lrcorner D^B \lrcorner (L_\phi)^C_F \lrcorner \phi + q^2 \psi \lrcorner D^B \lrcorner (L_\phi)^C_F \lrcorner \phi] \lrcorner \delta^F
\]

(85)

with

\[
D^C \lrcorner \phi = \partial^C \lrcorner \phi - i\epsilon A^C \lrcorner \phi,
\]

\[
\psi \lrcorner D^C = \psi \lrcorner \partial^C - i\psi \lrcorner A^C \epsilon.
\]

(86)

We can derive these identities in the same way as the \(q\)-versions of Green’s theorem if we use the formula

\[
\psi \lrcorner D^B \lrcorner \phi - \psi \lrcorner D^B \lrcorner \phi = \partial^C \lrcorner \left[\psi \lrcorner (L_\phi)^B_C \lrcorner \phi\right]
\]

\[
= - \left[\psi \lrcorner (L_\phi)^B_C \lrcorner \phi\right] \lrcorner \delta^C
\]

(87)

and recall that the rules in Eq. (48) of Chap. 2.3 also hold for the operators \(D^C\).

\[7\text{The projector } P_A \text{ is a } q\text{-analog of an antisymmetrizer, the projector } P_S \text{ is the } q\text{-deformed trace-free symmetrizer, and } P_T \text{ is the } q\text{-deformed trace-projector.}\]
We require that the solutions to the free $q$-deformed Schrödinger equations given in Eq. (69) of Chap. 3 are subject to the following normalization condition:

$$1 = \frac{1}{2} \int d^3x \left( \psi^*_L(x, t) \otimes \psi_R(x, t) + \psi^*_L(x, t) \otimes \psi^*_R(x, t) \right).$$

(88)

This condition confirms that the probability density for a nonrelativistic particle depends on the following expressions:

$$\rho(x, t) = \psi^*_L(x, t) \otimes \psi_R(x, t),$$

$$\rho^*(x, t) = \psi^*_L(x, t) \otimes \psi^*_R(x, t).$$

(89)

Due to Eq. (71) of Chap. 3, the expressions above transform into each other by conjugation:

$$\rho(x, t) = \rho^*(x, t).$$

(90)

Since the wave functions stay normalized as they evolve in time, $\rho(x, t)$ and $\rho^*(x, t)$ must satisfy continuity equations. In the following, we will derive these continuity equations. We first consider a nonrelativistic particle in an external force field with a scalar potential $V(x)$. In this case, we use the Hamilton operator $H$ given by Eq. (63) of Chap. 3. We calculate the time derivative of $\rho(x, t)$ by taking into account the Schrödinger equations in Eq. (69) of Chap. 3:

$$\partial_t \triangleright \rho(x, t) = \partial_t \triangleright \psi^*_L(x, t) \otimes \psi_R(x, t) + \psi^*_L(x, t) \otimes \partial_t \triangleright \psi_R(x, t)$$

$$= i\left( \psi^*_L \triangleright H \otimes \psi_R - \psi^*_L \otimes H \triangleright \psi_R \right)$$

$$= i\left( \psi^*_L \triangleright H_0 \otimes \psi_R - \psi^*_L \otimes H_0 \triangleright \psi_R \right)$$

$$= -i\left( \psi^*_L \triangleright \partial^A \partial_A (2m)^{-1} \otimes \psi_R - \psi^*_L \otimes (2m)^{-1} \partial^A \partial_A \triangleright \psi_R \right).$$

(91)

Note that the contributions due to $V(x)$ cancel each other out. Applying one of the $q$-versions of Green’s theorem [cf. Eq. (79) of Chap. 4] to the last expression in Eq. (91), we obtain a continuity equation for the probability density $\rho(x, t)$, i.e.

$$\partial_t \triangleright \rho(x, t) + \partial^A \triangleright j_A(x, t) = 0$$

(92)

with the probability current

$$j_A(x, t) = -\frac{i}{2m} q^{-2} \psi^*_L(x, t) \triangleright (L_0)^B A \partial_B \otimes \psi_R(x, t)$$

$$- \frac{i}{2m} \psi^*_L(x, t) \triangleright (L_0)^B A \otimes \partial_B \triangleright \psi_R(x, t).$$

(93)

The continuity equation for the probability density $\rho^*(x, t)$ can be derived by similar considerations as above or by conjugating Eq. (92). This way, we get

$$\rho^*(x, t) \triangleright \partial_t + (j^*)^A(x, t) \triangleright \partial_A = 0$$

(94)

The considerations of the present chapter and the following ones are similar to those in Ref. [34].
with the new probability current

\[
(j^*)^A(x,t) = -\frac{i}{2m} q^{-2} g_{BC}^L \psi_L(x,t) \otimes \partial^B(\mathcal{L}_\partial)^C_F \tilde{\psi}_R^*(x,t) g^{FA}
- \frac{i}{2m} g_{BC}^L \psi_L(x,t) \tilde{\otimes} \partial^B (\mathcal{L}_\partial)^C_F \tilde{\psi}_R^*(x,t) g^{FA}.
\] (95)

In analogy to Eq. (90), it holds:

\[
j_A(x,t) = (j^*)^A(x,t).
\] (96)

Next, we derive the continuity equation for the probability density of a charged particle in a magnetic field. We calculate the time derivative of the probability density by taking into account the Hamilton operator in Eq. (66) of Chap. 3 and the identity in Eq. (85) of Chap. 4:

\[
\partial_t \tilde{\rho}(x,t) = -i\psi^*_L \Leftrightarrow D_C^A \otimes \psi_R + i\psi^*_L \otimes (2m)^{-1} D_C^A \otimes \psi_R
- i^{-1}(2m)^{-1} \partial^F \Leftrightarrow [g^{-2} \psi^*_L \Leftrightarrow (\mathcal{L}_\partial)^B_F D_C^A \otimes \psi_R] g_{BC}
- i^{-1}(2m)^{-1} \partial^F \Leftrightarrow [\psi^*_L \Leftrightarrow (\mathcal{L}_\partial)^B_F \otimes D_C^A \otimes \psi_R] g_{BC}.
\] (97)

From the last expression, we can read off the probability current for a charged particle in a magnetic field:

\[
j_A(x,t) = -\frac{i}{2m} q^{-2} \psi^*_L(x,t) \Leftrightarrow (\mathcal{L}_\partial)^B_A D_B \otimes \psi_R
- \frac{i}{2m} \psi^*_L(x,t) \Leftrightarrow (\mathcal{L}_\partial)^B_A \otimes D_B \otimes \psi_R.
\] (98)

We can also obtain the expression above from that given in Eq. (93) if we replace \(\partial^C\) by \(D_C\). Using the explicit form of \(D_C\) [cf. Eq. (67) of Chap. 3], the probability current of a charged particle in a magnetic field can be written as follows:

\[
j_A = -\frac{iq^{-2}}{2m} q^{-2} \psi^*_L \Leftrightarrow (\mathcal{L}_\partial)^B_A \partial_B \otimes \psi_R
- \frac{i}{2m} \psi^*_L \Leftrightarrow (\mathcal{L}_\partial)^B_A \otimes \partial_B \otimes \psi_R
+ \frac{e}{2m} (q^{-2} + 1) \psi^*_L \Leftrightarrow (\mathcal{L}_\partial)^B_A \otimes A_B \otimes \psi_R.
\] (99)

There is another expression for the probability current of a particle in a magnetic field. In analogy to Eq. (96), we can get this expression by conjugation:

\[
(j^*)^A(x,t) = -\frac{i}{2m} q^{-2} g_{BC}^L \psi_L(x,t) \otimes \partial^B(\mathcal{L}_\partial)^C_F \tilde{\psi}_R^*(x,t) g^{FA}
- \frac{1}{2m} g_{BC}^L \psi_L(x,t) \tilde{\otimes} \partial^B (\mathcal{L}_\partial)^C_F \tilde{\psi}_R^*(x,t) g^{FA}
+ \frac{e}{2m} (q^{-2} + 1) g_{BC}^L \psi_L \otimes A_B \otimes (\mathcal{L}_\partial)^C_F \tilde{\psi}_R g^{FA}.
\] (100)
By integrating the above continuity equations, we can show that the wave functions stay normalized as they evolve in time. The following calculation using Eq. (92) shall serve as an example:

\[ \partial_t \psi^*_L(x, t) \otimes \psi_R(x, t) = \partial_t \int d^3q \rho(x, t) = -\int d^3q \partial \cdot j_A(x, t) = 0. \quad (101) \]

The last step of the above calculation follows from the fact that the action of the derivative \( \partial_A \) leads to surface terms vanishing at infinity [cf. Eq. (44) of Chap. 2.2].

6 Conservation of momentum

We introduce the following expressions for momentum density:

\[ i^A = \frac{1}{2i} \left( \psi^*_L \otimes \partial_A \cdot \psi_R + \psi^*_R \otimes \partial_A \cdot \psi_L \right), \]

\[ i^*_A = \frac{1}{2i} \left( \psi_L \otimes \partial_A \bar{\psi}_R^* + \psi_R \otimes \partial_A \bar{\psi}_L^* \right). \quad (102) \]

If we integrate the above expressions over all space and take into account Eq. (45) of Chap. 2.2 we get expectation values for the momentum components of a particle:

\[ \langle p^A \rangle = \int d^3q x i^A = \int d^3q x \psi^*_L \otimes i^{-1} \partial_A \cdot \psi_R, \]

\[ \langle p^*_A \rangle = \int d^3q x i^*_A = \int d^3q x \psi_L \otimes i^{-1} \partial_A \bar{\psi}_R. \quad (103) \]

Using Eq. (71) in Chap. 3 together with Eqs. (35), (45), and (46) in Chap. 2.2 we can show that quantum space conjugation transforms the first expression in Eq. (102) or Eq. (103) into the second one and vice versa:

\[ i^A = i^*_A, \quad \langle p^A \rangle = \langle p^*_A \rangle. \quad (104) \]

Next, we calculate the time derivative of the momentum density \( i^A \). To this end, we need the Schrödinger equations given in Eq. (69) of Chap. 3

\[ \partial_t i^A = \frac{1}{2} \left( \psi^*_L \cdot H \otimes \partial_A \cdot \psi_R - \psi^*_R \otimes \partial_A \cdot (H \cdot \psi_R) \right) \]

\[ + \frac{1}{2} \left( (\psi^*_L \cdot H) \cdot \partial_A \otimes \psi_R^* - \psi^*_L \cdot \partial_A \otimes H \cdot \psi_R \right). \quad (105) \]

With the explicit form of the Hamilton operator \( H \) [cf. Eq. (63) of Chap. 3], we
get the following expression for time derivative $i^A$:

$$
\partial_t \circ i^A = -\frac{1}{4m} \left( \psi_L^* \circ (\partial^B \partial_B \circ \partial^A \circ \partial R) - \psi_L^* \circ (\partial^A \circ \partial^B \partial_B \circ \partial R) \right)
+ \frac{1}{2} \left( \psi_L^* \circ (V \circ \partial^A \circ \partial R) - \psi_L^* \circ (V \circ \partial^A \circ \partial R) \right)
+ \frac{1}{2} \left( (\psi_L^* \circ V) \circ (\partial^A \circ \partial R) - \psi_L^* \circ (V \circ \partial^A \circ \partial R) \right).
$$

(106)

Since $V$ is invariant under the action of $U_q(su_2)$ as well as that of $\Lambda$, we have:

$$
(L_\partial)^A_B \circ V = V \circ (L_\partial)^A_B = \varepsilon((L_\partial)^A_B) V = \delta^A_B V.
$$

(107)

For this reason, the Leibniz rules of $q$-deformed partial derivatives imply [see Eq. (48) of Chap. 2.2]:

$$
\psi_L^* \circ (\partial^A \circ \partial R) \circ (V \circ \partial^A \circ \partial R) = \psi_L^* \circ (\partial^A \circ V) \circ \partial R + \psi_L^* \circ V \circ (\partial^A \circ \partial R).
$$

(108)

With these identities, we can combine the last two expressions on the left-hand side of Eq. (106) to a force density:

$$
f^A = -\frac{1}{2} \left( \psi_L^* \circ (\partial^A \circ V) \circ \partial R - \psi_L^* \circ (V \circ \partial^A \circ \partial R) \right).
$$

(109)

Moreover, we can write the first two expressions on the left-hand side of Eq. (106) as divergence. To achieve this, we apply Eq. (79) of Chap. 4 with

$$
\psi = \psi_L^*, \quad \phi = \partial^A \circ \partial R.
$$

(110)

or with

$$
\psi = \psi_L^* \circ \partial^A, \quad \phi = \partial R.
$$

(111)

This way, we get the continuity equation

$$
\partial_t \circ i_A = -\partial^B \circ T_{BA} + f_A
$$

(112)

with the following stress tensor:

$$
T_{BA} = -\frac{1}{4m} q^{-2} \psi_L^* \circ (L_\partial)^C_B \circ \partial_C \circ \partial_A \circ \partial R
- \frac{1}{4m} \psi_L^* \circ (L_\partial)^C_B \circ \partial_C \circ \partial A \circ \partial R
- \frac{1}{4m} q^{-2} \psi_L^* \circ (L_\partial)^C_B \circ \partial_C \circ \partial R
- \frac{1}{4m} \psi_L^* \circ (L_\partial)^C_B \circ \partial_C \circ \partial R.
$$

(113)
We can obtain the continuity equation for $i^*_A$ by conjugating Eq. (112). This way, it holds
\[(i^*)^A \bar{\partial}_t = -(T^*)^{AB} \bar{\partial}_B - (f^*)^A\] (114)
with
\[f_A = (f^*)^A, \quad T_{BA} = (T^*)^{AB}.\] (115)
Explicitly, we have
\[(f^*)^A = -\frac{1}{2}(\psi_L \circ (\partial^A \triangleright V) \circ \psi_R^* - \psi_L \circ (V \triangleleft \partial^A) \circ \psi_R^*)\] (116)
and
\[(T^*)^{AB} = -\frac{1}{4m} q^{-2} g_{CF} \psi_L \circ \partial^A \circ (\bar{\mathcal{L}}_\psi)^F_E \triangleright \psi_R^* g^{EB} \]
\[-\frac{1}{4m} q^{-2} g_{CF} \psi_L \circ \partial^A \circ (\bar{\mathcal{L}}_\psi)^F_E \triangleright \psi_R^* g^{EB} \]
\[-\frac{1}{4m} q^{-2} g_{CF} \psi_L \circ \partial^A \circ (\bar{\mathcal{L}}_\psi)^F_E \triangleright \psi_R^* g^{EB} \]
\[-\frac{1}{4m} g_{CF} \psi_L \circ \partial^C \circ (\bar{\mathcal{L}}_\psi)^F_E \triangleright \psi_R^* g^{EB}.\] (117)

If we integrate both sides of Eq. (112) over all space, we obtain Newton’s second law as part of the Ehrenfest theorem [also see Eq. (172) of Chap. 9]:
\[\bar{\partial}_t \triangleright \langle p^A \rangle = \int d^3 q \bar{\partial}_t \triangleright i^A = \int d^3 q f^A = -\langle \partial^A \triangleright V \rangle.\] (118)
In the last step, we have used the identity
\[f^A = \psi_L \circ (\partial^A \triangleright V) \circ \psi_R,\] (119)
which follows from Eq. (109) by taking into account the following calculation [also see Eq. (59) of Chap. 2.3 and Eq. (107)]:
\[V \triangleleft \partial^A = -\partial^F \triangleright (V \triangleleft (\mathcal{L}_\psi)^A_E) = -\delta^A_F \partial^F \triangleright V = -\partial^A \triangleright V.\] (120)
Next, let us consider a charged particle moving in a magnetic field. To this end, we replace the partial derivatives $\partial^C$ by the operators $D^C$ [cf. Eq. (67) of Chap. 8] and obtain the following expressions from Eq. (102):
\[i^C = \frac{1}{2i} \left( \psi_L^\dagger \circ D^C \triangleright \psi_R + \psi_L^\dagger \triangleleft D^C \circ \psi_R \right)\]
\[= \frac{1}{2i} \left( \psi_L^\dagger \circ \partial^C \triangleright \psi_R + \psi_L^\dagger \triangleleft \partial^C \circ \psi_R \right) - \psi_L^\dagger \circ e A^C \circ \psi_R,\] (121)
\[i^*_C = \frac{1}{2i} \left( \psi_L \circ D^C \triangleright \psi_R^* + \psi_L \triangleleft D^C \circ \psi_R^* \right)\]
\[= \frac{1}{2i} \left( \psi_L \circ \partial^C \triangleright \psi_R^* + \psi_L \triangleleft \partial^C \circ \psi_R^* \right) - \psi_L \circ e A^C \circ \psi_R^*.\] (122)
Accordingly, the Hamilton operator takes on the following form:

\[ H = -(2m)^{-1} D^C D_R + V. \]  

(123)

Again, we use the Schrödinger equations from Eq. (69) of Chap. 3 to calculate the time derivative of the momentum density \( i^C \) given in Eq. (121):

\[
\partial_t i^C = \frac{1}{2} \left( \psi^*_L \triangleleft H \triangleright D^C \triangleright \psi_R - \psi^*_L \triangleright D^C \triangleright (H \triangleright \psi_R) \right) \\
+ \frac{1}{2} \left( (\psi^*_L \triangleleft H) \triangleright D^C \triangleright \psi_R - \psi^*_L \triangleright D^C \triangleright H \triangleright \psi_R \right) \\
- \psi^*_L \triangleleft e \partial_t A^C \triangleright \psi_R.
\]  

(124)

With Eq. (123), we can write the first expression on the left-hand side of the above equation as follows:

\[
\psi^*_L \triangleleft H \triangleright D^C \triangleright \psi_R - \psi^*_L \triangleright D^C \triangleright (H \triangleright \psi_R) = \\
\psi^*_L \triangleleft D^F D_F (2m)^{-1} \triangleright D^C \triangleright \psi_R + \psi^*_L \triangleright (2m)^{-1} D^C D^F D_F \triangleright \psi_R \\
+ \psi^*_L \triangleright V \triangleright D^C \triangleright \psi_R - \psi^*_L \triangleright D^C \triangleright (V \triangleright \psi_R) \\
= - \psi^*_L \triangleleft D^F D_F (2m)^{-1} \triangleright D^C \triangleright \psi_R + \psi^*_L \triangleright (2m)^{-1} D^F D_F D^C \triangleright \psi_R \\
+ \psi^*_L \triangleright f^C_L \triangleright \psi_R - \psi^*_L \triangleright (\partial^C \triangleright V) \triangleright \psi_R.
\]  

(125)

The last expression in Eq. (125) is a consequence of the commutation relations [also see Eq. (198) of App. A]

\[ [D^F D_F, D^C] = -2m f^C_L \]  

(126)

and

\[ [D^C, V] = [\partial^C, V] + [i e A^C, V] = \partial^C \triangleright V. \]  

(127)

The operator for the Lorentz force density takes on the following form [cf. Eq. (199) of App. A]:

\[ f^D_L = \frac{e}{2m} g^{DG} \varepsilon_{ACG} \left( D^C B^A - B^C D^A \right). \]  

(128)

There are terms in the last expression of Eq. (125) which we can write as divergence if we apply Eq. (85) of Chap. 4 with the following identifications:

\[ \psi = \psi^*_L (2m)^{-1}, \quad \phi = D^C \triangleright \psi_R. \]  

(129)

This way, we obtain:

\[
- \psi^*_L \triangleleft D^B D_B (2m)^{-1} \triangleright D^C \triangleright \psi_R + \psi^*_L \triangleright (2m)^{-1} D^B D_B D^C \triangleright \psi_R = \\
= \partial^F \triangleright [g^{-2} \psi^*_L \triangleleft (L_\partial)^B_F D_B (2m)^{-1} \triangleright D^C \triangleright \psi_R] \\
+ \partial^F \triangleright [\psi^*_L \triangleleft (L_\partial)^B_F \triangleright (2m)^{-1} D_B D^C \triangleright \psi_R].
\]  

(130)
Similar reasonings hold for the second expression on the right-hand side of Eq. (124):

$$
(\psi^*_L \triangleleft H) \triangleleft D^C \circ \psi_R - \psi^*_L \triangleleft D^C \circ H \triangleright \psi_R = \\
= \partial^F \triangleright \left[ q^{-2} \psi^*_L \triangleleft D^C (\mathcal{L}_\partial)^B_F D_B (2m)^{-1} \circ \psi_R \right] \\
+ \partial^F \triangleright \left[ \psi^*_L \triangleleft D^C (\mathcal{L}_\partial)^B_F (2m)^{-1} \circ D_B \triangleright \psi_R \right] \\
- \psi^*_L \triangleleft f_{\text{Lor}} \circ \psi_R + \psi^*_L \circ (V \triangleleft \partial^C) \circ \psi_R.
$$

(131)

Summarizing the results so far, we finally get

$$\partial_t \triangleright i_C = - \partial^F \triangleright T_{FC} + f_C$$

(132)

with

$$
T_{FC} = - \frac{1}{4m} q^{-2} \psi^*_L \triangleleft (\mathcal{L}_\partial)^B_F D_B \circ D_C \triangleright \psi_R \\
- \frac{1}{4m} \psi^*_L \triangleleft (\mathcal{L}_\partial)^B_F \circ D_B D_C \triangleright \psi_R \\
- \frac{1}{4m} q^{-2} \psi^*_L \triangleleft D_C (\mathcal{L}_\partial)^B_F D_B \circ \psi_R \\
- \frac{1}{4m} \psi^*_L \triangleleft D_C (\mathcal{L}_\partial)^B_F D_B \triangleright \psi_R
$$

(133)

and

$$f^C = - \frac{1}{2} \left( \psi^*_L \circ (\partial^C \triangleright V) \circ \psi_R - \psi^*_L \circ (V \triangleleft \partial^C) \circ \psi_R \right) \\
+ \frac{1}{2} \left( \psi^*_L \circ f_{\text{Lor}} \triangleright \psi_R - \psi^*_L \circ f_{\text{Lor}} \circ \psi_R \right) - \psi^*_L \circ (e \partial_t \triangleright A^C) \circ \psi_R.
$$

(134)

Note that we can directly obtain the new stress tensor in Eq. (133) from that in Eq. (113) by replacing the partial derivatives $\partial^C$ with the operators $D^C$. If we integrate the continuity equation (132) over all space and take into account Eq. (45) of Chap. 2.2, we get the following evolution equation [also see Eq. (185) of Chap. 2.1]:

$$\partial_t \langle p^C \rangle = - (\partial^C \triangleright V) + \langle f_{\text{Lor}}^C \rangle - (e \partial_t \triangleright A^C).
$$

(135)

By conjugating Eq. (132), we can again obtain a second continuity equation for momentum density [also see Eq. (114)]. The corresponding expression for the stress tensor again follows from that of Eq. (117) by replacing the partial derivatives $\partial^C$ with the operators $D^C$. For the operator of the Lorentz force density, we now have

$$
(f_{\text{Lor}}^*)_D = \frac{e}{2m} g_{DG} \varepsilon^{GCA} (D_A B^*_C - B^*_A D_C)
$$

(136)

with

$$
(B^*)^F = B^F = i \partial_C \triangleright A_D \varepsilon^{FDC} = i A_C \triangleright \partial_D \varepsilon^{FDC}.
$$

(137)
7 Conservation of energy

In this chapter, we derive continuity equations for the energy density of a nonrelativistic particle. Once again, we first consider a nonrelativistic particle in an external force field with a scalar potential \( V(x) \). In this case, the energy density takes on the following form:

\[
\mathcal{H} = -\psi_L^* \odot \partial^A \odot (2m)^{-1} \partial_A \rhd \psi_R + \psi_L^* \odot V \odot \psi_R. \tag{138}
\]

To calculate the time derivative of the energy density, we apply the Schrödinger equations given in Eq. (69) of Chap. [3]

\[
\partial_t \rhd \mathcal{H} = -i(\psi_L^* \rhd H) \odot \partial^A \odot (2m)^{-1} \partial_A \rhd \psi_R
\]

\[
+ i\psi_L^* \odot \partial^A \odot (2m)^{-1} \partial_A \rhd (H \rhd \psi_R)
\]

\[
- i\psi_L^* \odot H \odot V \odot \psi_R - i\psi_L^* \odot V \odot H \rhd \psi_R. \tag{139}
\]

Inserting the expression for the Hamilton operator [cf. Eq. (63) in Chap. [3], we obtain:

\[
\partial_t \rhd \mathcal{H} = i\psi_L^* \odot \partial^A \partial B \partial_B \odot (2m)^{-2} \partial_A \rhd \psi_R
\]

\[
- i\psi_L^* \odot \partial^A \odot (2m)^{-2} \partial^B \partial_B \partial_A \rhd \psi_R
\]

\[
- i(\psi_L^* \odot V) \odot \partial^A \odot (2m)^{-1} \partial_A \rhd \psi_R
\]

\[
+ i\psi_L^* \odot \partial^A \odot (2m)^{-1} \partial_A \rhd (V \odot \psi_R)
\]

\[
- i\psi_L^* \odot \partial^B \partial_B (2m)^{-1} \odot V \odot \psi_R
\]

\[
+ i\psi_L^* \odot V \odot (2m)^{-1} \partial^B \partial_B \rhd \psi_R. \tag{140}
\]

We can write the first two terms on the right-hand side of the above equation as divergence by using Eq. (79) in Chap. [4]

\[
\psi_L^* \odot \partial^A \partial B \partial_B \odot (2m)^{-2} \partial_A \rhd \psi_R - \psi_L^* \odot \partial^A \odot (2m)^{-2} \partial^B \partial_B \partial_A \rhd \psi_R =
\]

\[
- \partial^C \rhd \left[ q^{-2} \psi_L^* \odot \partial^A \odot (\mathcal{L}_\beta)^{B_C} \partial_B \odot (2m)^{-2} \partial_A \rhd \psi_R
\]

\[
+ \psi_L^* \odot \partial^A \odot (\mathcal{L}_\beta)^{B_C} \odot (2m)^{-2} \partial_B \partial_A \rhd \psi_R \right]. \tag{141}
\]

On the right-hand side of Eq. (141), we can also combine the third and last term as well as the fourth and penultimate term into a divergence. However, this requires rewriting the third term with the help of Eq. (72) from Chap. [4] as follows:

\[
- (\psi_L^* \odot V) \odot \partial^A \odot (2m)^{-1} \partial_A \rhd \psi_R =
\]

\[
= -\psi_L^* \odot V \odot (2m)^{-1} \partial^A \partial_A \rhd \psi_R
\]

\[
+ \partial^C \rhd \left[ (\psi_L^* \odot V) \odot (\mathcal{L}_\beta)^{A_C} \odot (2m)^{-1} \partial_A \rhd \psi_R \right]. \tag{142}
\]

We do the same with the fourth term [also see Eq. (80) of Chap. [4]]:

\[
\psi_L^* \odot \partial^A \odot (2m)^{-1} \partial_A \rhd (V \odot \psi_R) =
\]

\[
= \psi_L^* \odot \partial^A \partial_A (2m)^{-1} \odot V \odot \psi_R
\]

\[
+ \partial^C \rhd \left[ q^{-2} \psi_L^* \odot (\mathcal{L}_\beta)^{A_C} \partial_A (2m)^{-1} \odot V \odot \psi_R \right]. \tag{143}
\]
Inserting these results into Eq. (140), we finally get
\[ \partial_t \triangleright H + \partial C \triangleright S_C = 0 \] (144)
with the following current density:
\[
S_C = -\frac{i}{2m} q^{-2} \psi_L^* \triangleleft (L_\partial)^A_C \partial_A \triangleright V \triangleright \psi_R \\
- \frac{i}{2m} \left( \psi_L^* \triangleright V \right) \triangleleft (L_\partial)^A_J \triangleright \partial_A \triangleright \psi_R \\
+ \frac{i}{4m^2} q^{-2} \psi_L^* \triangleleft \partial^A (L_\partial)^B_C \partial_B \triangleright \partial_A \triangleright \psi_R \\
+ \frac{i}{4m^2} \psi_L^* \triangleleft \partial^A (L_\partial)^B_C \triangleright \partial_B \triangleright \partial_A \triangleright \psi_R. \] (145)

If we integrate Eq. (144) over all space, the divergence leads to a surface term at spatial infinity. However, this surface term will be zero since the wave functions vanish at spatial infinity. Accordingly, the total energy of the nonrelativistic particle is constant over time:
\[
\partial_t \triangleright \int d^3 q \rho(x,t) = -\int d^3 q \partial^C \triangleright S_C = 0. \] (146)

By conjugating Eqs. (138) and (144), we get another expression for the energy density and a corresponding continuity equation, i.e.
\[ H^\ast - \psi_L \triangleright \partial^A (2m)^{-1} \circ \partial_A \triangleright \psi_R^* \triangleright V \triangleright \psi_R^* \] (147)
and
\[
H^\ast \triangleright \partial_t + (S^\ast)^C \triangleright \partial_C = 0 \] (148)
with
\[
(S^\ast)^C = -\frac{i}{2m} q^{-2} \psi_L \triangleright V \triangleright g_{AB} \partial^A (L_\partial)^B_E \triangleright \psi_R^* \triangleright g_{EC} \\
- \frac{i}{2m} \psi_L \triangleright \partial^A \triangleright g_{AB} (L_\partial)^B_E \triangleright (V \triangleright \psi_R^*) \triangleright g_{EC} \\
+ \frac{i}{4m^2} q^{-2} \psi_L \triangleright \partial^A \triangleright g_{BD} (L_\partial)^B_E \triangleright \partial_A \triangleright \psi_R^* \triangleright g_{EC} \\
+ \frac{i}{4m^2} \psi_L \triangleright \partial^A \triangleright g_{BD} (L_\partial)^B_E \triangleright \partial_A \triangleright \psi_R^* \triangleright g_{EC}. \] (149)

With some modifications, the above considerations also apply to a charged particle moving in a magnetic field. To this end, we replace the partial derivatives \( \partial^C \) in Eq. (138) with the operators \( D^C \):
\[ H = -\psi_L^* \triangleleft D^C \triangleright (2m)^{-1} D_C \triangleright \psi_R + \psi_L^* \triangleright V \triangleright \psi_R. \] (150)
Taking the time derivative of the new energy density, we obtain:

\[
\begin{align*}
\partial_t \triangleright H &= i \psi_L^* \odot \epsilon \partial_t \triangleright A^C \odot (2m)^{-1} D_C \triangleright \psi_R \\
&+ i \psi_L^* \odot D^C(2m)^{-1} \odot \epsilon \partial_t \triangleright A_C \odot \psi_R \\
&- i (\psi_L^* \triangleright H) \odot D^C \odot (2m)^{-1} D_C \triangleright \psi_R \\
&+ i \psi_L^* \odot D^C(2m)^{-1} D_C \triangleright (H \triangleright \psi_R) \\
&+ i \psi_L^* \triangleleft H \odot V \odot \psi_R - i \psi_L^* \odot V \odot H \triangleright \psi_R. 
\end{align*}
\] (151)

The first two terms on the right-hand side of the above equation form a $q$-deformed power density. Once again, we can turn the other expressions into a divergence by using Eqs. (85) and (87) of Chap. 4. These calculations are very similar to the ones that led to Eq. (144). Thus we finally obtain

\[
\partial_t \triangleright H + \partial^C \triangleright S_C = \frac{i \epsilon}{2m} \psi_L^* \odot \partial_t \triangleright A^C \odot D_C \triangleright \psi_R \\
+ \frac{i \epsilon}{2m} \psi_L^* \odot D^C \odot \partial_t \triangleright A_C \odot \psi_R 
\] (152)

with

\[
S_C = - \frac{i}{2m} q^{-2} \psi_L^* \triangleright (\mathcal{L}_\partial)^A_C D_A \odot V \odot \psi_R \\
- \frac{i}{2m} (\psi_L^* \odot V) \triangleright (\mathcal{L}_\partial)^A_C \odot D_A \triangleright \psi_R. \\
+ \frac{i}{4m^2} q^{-2} \psi_L^* \triangleright D^A(\mathcal{L}_\partial)^B_C D_B \odot D_A \triangleright \psi_R \\
+ \frac{i}{4m^2} \psi_L^* \triangleright D^A(\mathcal{L}_\partial)^B_C \odot D_B D_A \triangleright \psi_R. 
\] (153)

Note that we can get the current density in Eq. (153) from the expression in Eq. (145) by replacing the partial derivatives $\partial^C$ with the operators $D^C$.

### 8 Gauge transformations

We will show that the continuity equation in Eq. (132) of Chap. 6 is invariant under the gauge transformations

\[
e^A C = e A^C + \partial^C \triangleright e \Phi, \quad \tilde{V} = V - \partial_0 \triangleright e \Phi, 
\] (154)

and

\[
\begin{align*}
\hat{\psi}_R(x, t) &= \exp(i e \Phi) \odot \psi_R(x, t), \\
\hat{\psi}_L^*(x, t) &= \psi_L^*(x, t) \odot \exp(-i e \Phi). 
\end{align*}
\] (155)

Note that $\Phi(x)$ is a central element of the algebra of position space [cf. Eq. (64) of Chap. 3]. In addition to this, we require that $\Phi(x)$ has trivial braiding.
First, we show that the momentum density in Eq. (121) of Chap. 6 is invariant under the above gauge transformations. To this end, we do the following calculation:

\[
i^{-1} \tilde{D}^C \triangleright \tilde{\psi}_R = i^{-1} \partial^C \triangleright \left( \exp(i e \Phi) \otimes \psi_R \right) - e A^C \otimes \exp(i e \Phi) \otimes \psi_R \\
- \left( \partial^C \triangleright e \Phi \right) \otimes \exp(i e \Phi) \otimes \psi_R \\
= (i^{-1} \partial^C \triangleright \exp(i e \Phi)) \otimes \psi_R + \exp(i e \Phi) \otimes i^{-1} \partial^C \triangleright \psi_R \\
- e A^C \otimes \exp(i e \Phi) \otimes \psi_R - (\partial^C \triangleright e \Phi) \otimes \exp(i e \Phi) \otimes \psi_R. \tag{156}
\]

The last step follows from the Leibniz rules of the \( q \)-deformed partial derivatives and the trivial braiding of \( \Phi(x) \). For the same reasons, we also have

\[
\partial^C \triangleright (e \Phi)^n = \sum_{j=0}^{n-1} (e \Phi)^j \otimes (\partial^C \triangleright e \Phi) \otimes (e \Phi)^{n-1-j} \\
= n (\partial^C \triangleright e \Phi) \otimes (e \Phi)^{n-1} \tag{157}
\]

with

\[
(e \Phi)^n = e \Phi \otimes \ldots \otimes e \Phi, \tag{158}
\]

From Eq. (157) follows:

\[
i^{-1} \partial^C \triangleright \exp(i e \Phi) = \sum_{n=0}^{\infty} \frac{i^n}{n!} i^{-1} \partial^C \triangleright (e \Phi)^n \\
= \sum_{n=1}^{\infty} \frac{i^{n-1}}{(n-1)!} (\partial^C \triangleright e \Phi) \otimes (e \Phi)^{n-1} \\
= (\partial^C \triangleright e \Phi) \otimes \sum_{n=0}^{\infty} \frac{i^n}{n!} (e \Phi)^n = (\partial^C \triangleright e \Phi) \otimes \exp(i e \Phi). \tag{159}
\]

If we insert this result into Eq. (156), we obtain:

\[
i^{-1} \tilde{D}^C \triangleright \tilde{\psi}_R = (i^{-1} \partial^C \triangleright \exp(i e \Phi)) \otimes \psi_R + \exp(i e \Phi) \otimes i^{-1} \partial^C \triangleright \psi_R \\
- e A^C \otimes \exp(i e \Phi) \otimes \psi_R - (\partial^C \triangleright e \Phi) \otimes \exp(i e \Phi) \otimes \psi_R \\
= \exp(i e \Phi) \otimes i^{-1} \partial^C \triangleright \psi_R - \exp(i e \Phi) \otimes e A^C \otimes \psi_R \\
= \exp(i e \Phi) \otimes i^{-1} \tilde{D}^C \triangleright \psi_R. \tag{160}
\]

In the same manner, we get the following identity:

\[
\tilde{\psi}_L \diamond \tilde{D}^C = \tilde{\psi}_L \diamond D^C \otimes \exp(-i e \Phi). \tag{161}
\]

Applying Eqs. (160) and (161), we find that the momentum density in Eq. (121)
of Chap. 6 is invariant under gauge transformations:

\[
\tilde{C} = \frac{1}{2i} \left( \tilde{\psi}_L^* \circ \tilde{D}C \triangleright \tilde{\psi}_R + \tilde{\psi}_L \circ \tilde{D}C \triangleright \tilde{\psi}_R \right)
\]

\[
= \frac{1}{2i} \psi_L^* \circ \exp(-ie\Phi) \circ \exp(ie\Phi) \circ D^C \triangleright \psi_R
\]

\[
+ \frac{1}{2i} \psi_L^* \circ D^C \circ \exp(-ie\Phi) \circ \exp(ie\Phi) \circ \psi_R
\]

\[
= \frac{1}{2i} \left( \psi_L^* \circ D^C \triangleright \psi_R + \psi_L^* \circ D^C \triangleright \psi_R \right) = i^C.
\]

(162)

Similar considerations show that the \(q\)-deformed stress tensor in Eq. (133) of Chap. 6 is also invariant under gauge transformations. That the \(q\)-deformed force density in Eq. (134) of Chap. 6 is gauge invariant can be proven in complete analogy to the undeformed case. This result arises from the fact that the \(q\)-deformed magnetic field does not change under the gauge transformations in Eq. (154):

\[
\tilde{B}_F = i\partial^C \circ \tilde{A}^D \varepsilon_{DCF} = i\partial^C \circ \tilde{A}^D \varepsilon_{DCF} - i\partial^C \partial^D \circ \Phi \varepsilon_{DCF} = B_F.
\]

(163)

Note that the last step in the above calculation is a consequence of the following identity [35]:

\[
\partial^C \partial^D \varepsilon_{DCF} = 0.
\]

9 Equations of motion

In Ref. [11], we have shown that the Heisenberg equation of motion also holds for quantum-mechanical systems of \(q\)-deformed Euclidean space. Using Heisenberg’s equation, we can calculate the time derivative of an observable of a \(q\)-deformed nonrelativistic particle. We will see that these results are in agreement with the continuity equations in Chap. 6.

For the momentum operator of a free \(q\)-deformed nonrelativistic particle, we get the following equation of motion in the Heisenberg picture:

\[
\partial_i \triangleright (P^A)_H = i[H_0, (P^A)_H]
\]

\[
= -i(2m)^{-1} \partial^C \partial_C i^{-1} \partial^A + i^{-1} \partial^A i(2m)^{-1} \partial^C \partial_C = 0.
\]

(164)

In the above calculation, we have used the fact that the free Hamilton operator \(H_0\) commutes with the \(q\)-deformed partial derivatives [also see Eq. (61) in Chap. 3]:

\[
(P_A)_H = \exp(itH_0) P^A \exp(-itH_0)
\]

\[
= \exp(itH_0) i^{-1} \partial^A \exp(-itH_0) = i^{-1} \partial^A = P^A.
\]

(165)

By Eq. (164), we learn that the \(q\)-deformed momentum of a free particle is time-independent.

26
Next, we calculate the time derivative of the position operator in the Heisenberg picture:

\[ \partial_t \triangleright (X^A)_H = i[H_0, (X^A)_H] \]
\[ = -i(2m)^{-1} \partial^C \partial_C (X^A)_H + i(X^A)_H (2m)^{-1} \partial^C \partial_C. \]  

To evaluate the commutator with the free Hamilton operator, we have used the following identity:

\[ g_{CD} \partial^C \partial^D X^A = (1 + q^{-2}) \partial^A + q^4 X^A g_{CD} \partial^C \partial^D. \]

To get the latter, we have applied the Leibniz rules in Eq. (25) of Chap. 2.

The time derivative of the position operator should be proportional to the momentum operator. This assumption requires that mass \( m \) is a scaling operator:

\[ mX^A = q^4 X^A m. \]

From Eqs. (167) and (168) follows:

\[ -i(2m)^{-1} \partial^C \partial_C (X^A)_H = -i(1 + q^{-2})(2m)^{-1} \partial^A - i(X^A)_H (2m)^{-1} \partial^C \partial_C. \]

Inserting this result into Eq. (166), we finally get:

\[ \partial_t \triangleright (X^A)_H = i[H_0, (X^A)_H] = \frac{[2]}{2m} q^{-2} P^A. \]

Next, we will study how a time-independent scalar potential \( V(x) \) changes the Heisenberg equation of motion for the momentum operator or the position operator. For this reason, we consider the following Hamilton operator:

\[ H = H_0 + V(x). \]

The Heisenberg equation of motion for the momentum operator then reads:

\[ \partial_t \triangleright (P^A)_H = i[H_H, (P^A)_H] = i[(H_0)_H, (P^A)_H] + i[V_H, (P^A)_H] \]
\[ = (V(x) \partial^A)_H - (\partial^A V(x))_H \]
\[ = (V(x) \partial^A)_H - ((\partial^A \triangleright V(x))(\partial^A)_H \]
\[ = (V(x) \partial^A)_H - (\partial^A \triangleright V(x))_H - (V(x) \partial^A)_H \]
\[ = - (\partial^A \triangleright V(x))_H. \]

The above calculation employs the fact that \( H_0 \) commutes with the momentum operator [also see Eq. (164)] and that \( V(x) \) has trivial braiding properties. Note that Eq. (172) yields a \( q \)-analog of Newton’s second law.

\(^9\)If we use the partial derivatives \( \partial^A \), we must replace \( q \) by \( q^{-1} \) in the following formulas.
The Heisenberg equation of motion for the position operator remains unchanged compared to Eq. (170) since \( V(x) \) commutes with the position operator:

\[
\partial_t \triangleright (X^A)_H = i[H, (X^A)_H] = i[(H_0)_H, (X^A)_H] + i[V, (X^A)_H] = \frac{[2]}{2m} \langle P^A \rangle_H.
\]

Combining Eq. (172) with Eq. (173) gives:

\[
(\partial^2_t) \triangleright (X^A)_H = - (\partial^A \triangleright V)_H.
\]

We can also write down \( q \)-deformed versions of Hamilton’s equations. To this end, we do the calculations

\[
\partial^A_x \triangleright H = \partial^A_x \triangleright H_0 + \partial^A_x \triangleright V(x) = \partial^A_x \triangleright V
\]

and

\[
\partial^A_p \triangleright H = \partial^A_p \triangleright H_0 + \partial^A_p \triangleright V = \partial^A_p \triangleright g_{AB} P^A P^B (2m)^{-1} = [[2]]_q \cdot P^A (2m)^{-1}.
\]

Comparing these results with those of Eqs. (172) and (173), we get the following identities:

\[
\partial_t \triangleright (P^A)_H = - (\partial^A_x \triangleright H)_H, \quad \partial_t \triangleright (X^A)_H = (\partial^A_p \triangleright H)_H.
\]

We again consider the case where the Hamiltonian operator depends not only on a scalar potential \( V(x) \) but also on a vector potential \( A(x) \) [cf. Eq. (66) of Chap. 3, i.e.

\[
H = (2m)^{-1} \Pi^C \Pi_C + V(x)
\]

with

\[
\Pi^C = P^C - e A^C(x).
\]

We recall that the components \( e A^C \) have the same braiding properties as the components \( P^C \) of the momentum operator. Consequently, if the momentum operator is represented by the derivatives \( \partial^C \), the components \( e A^C \) and the coordinate generators \( X^D \) satisfy the following commutation relations [also see Eq. (25) of Chap. 2.2]

\[
e A^C X^D = (\hat{R}^{-1})^{CD}_{EF} X^E e A^F.
\]

---

10. The partial derivatives \( \partial^A_p \) act on momentum space in the same way as the partial derivatives \( \partial^A_x \) act on position space.

11. If the momentum operator is represented by the derivatives \( \partial^C \), we must replace \( \hat{R}^{-1} \) by \( \hat{R} \) and \( q \) by \( q^{-1} \) in the following identities [also see Eqs. (25) and (26) of Chap. 2.2].
The same considerations that led to the identities in Eq. (167) yield:

\[
g_{CD} e^A e^C e^D X = q^4 X^E g_{CD} e^A e^D,
\]
\[
g_{CD} e^C P^D X = -ie A^E + q^4 X^E g_{CD} e^C P^D,
\]
\[
g_{CD} e^C P^D e^A X = -i q^{-2} e A^E + q^4 X^E g_{CD} e^C P^D e^A.
\]

Together with Eq. (168), the identities above imply:

\[
[(2m)^{-1} e^A e^D, X^E] = 0,
\]
\[
[(2m)^{-1} e^A P^D, X^E] = -i (2m)^{-1} e A^E,
\]
\[
[(2m)^{-1} P^D e^A, X^E] = -i (2m)^{-1} q^{-2} e A^E.
\]

We can calculate the time derivative of the position operator in the Heisenberg picture by using the commutators above:

\[
\partial_t \triangleright (X^D)_H = i [H_H, (X^D)_H] = [(2m)^{-1} (P^D - e A^D) - e A^D]_H
\]
\[
= \frac{[2]^m}{2m} (\Pi^D)_H.
\]

Next, we calculate the time derivative of kinetic momentum. Using Eq. (198) in App. A and the identities

\[
[V, \Pi^D] = [V, P^D - e A^D] = -\partial^D \triangleright V, \quad \frac{\partial \Pi^D}{\partial t} = -e \frac{\partial A^D}{\partial t},
\]

we can do the following calculation:

\[
\partial_t \triangleright (\Pi^D)_H = i [H_H, (\Pi^D)_H] + \frac{\partial (\Pi^D)_H}{\partial t}
\]
\[
= i [(2m)^{-1} (\Pi^C)_H (\Pi_C)_H + i [V_H, (\Pi^D)_H] + \frac{\partial (\Pi^D)_H}{\partial t}
\]
\[
= (f^D_{Lor})_H - e \frac{\partial (A^D)_H}{\partial t} - (\partial^D \triangleright V)_H.
\]

Combining Eq. (183) with the result above, we get:

\[
\frac{2m}{[2]_{q^{-2}}} (\partial_t)^2 \triangleright (X^D)_H = (f^D_{Lor})_H - e (\partial_t \triangleright A^D)_H - (\partial^D \triangleright V)_H.
\]

By taking the expectation values of both sides, we finally obtain

\[
\frac{2m}{[2]_{q^{-2}}} (\partial_t)^2 \triangleright \int d^3 x \psi_L^*(x,0) \otimes (X^D)_H \triangleright \psi_R(x,0) =
\]
\[
= \int d^3 x \psi_L^*(x,0) \otimes (f^D_{Lor})_H \otimes \psi_R(x,0)
\]
\[
- \int d^3 x \psi_L^*(x,0) \otimes e (\partial_t \triangleright A^D)_H \otimes \psi_R(x,0)
\]
\[
- \int d^3 x \psi_L^*(x,0) \otimes (\partial^D \triangleright V)_H \otimes \psi_R(x,0),
\]

29
or in abbreviated form:
\[
\frac{2m}{[2]_{q^{-2}}} (\partial_t)^2 \triangleright (X^D) = (f^D_{\text{loc}}) \psi - e (\partial_t \triangleright A^D) \psi - (\partial^D \triangleright V) \psi. \tag{188}
\]

Finally, we consider the time derivative of the \(q\)-deformed angular momentum operator. We recall that \(H\) is a scalar as to the actions of the Hopf algebra \(U_q(su_2)\). For this reason, \(H\) commutes with all \(g \in U_q(su_2)\):
\[
g H = (g_{(1)} \triangleright H) g_{(2)} = \varepsilon(g_{(1)}) H g_{(2)} = H g. \tag{189}
\]

The components of \(q\)-deformed angular momentum are elements of \(U_q(su_2)\) (also see App. 13). Due to this fact, \(H\) commutes with the \(q\)-deformed angular momentum operator:
\[
\partial_t \triangleright (L^A)_H = i[H, (L^A)_H] = iHL^A - iL^A H
= iHL^A - iHL^A = 0. \tag{190}
\]

The above result implies that the expectation value of each component of \(q\)-deformed angular momentum is constant over time:
\[
\partial_t \triangleright \langle L^D \rangle = \partial_t \triangleright \int d^3 x \psi^*_L(x, 0) \triangleright (L^D)_H \triangleright \psi_R(x, 0)
= \int d^3 x \psi^*_L(x, 0) \triangleright (\partial_t \triangleright (L^D)_H) \triangleright \psi_R(x, 0) = 0. \tag{191}
\]

\section{A \(q\)-Analog of Lorentz force density}

The commutation relations between the components of \(q\)-deformed kinetic momentum depend on the epsilon tensor of three-dimensional \(q\)-deformed Euclidean space. Its non-vanishing components are [13,38,39]:
\[
\begin{align*}
\varepsilon^{-3+} &= \varepsilon_{+3-} = -q^{-2}, & \varepsilon^{3+} &= \varepsilon_{+3} = 1, \\
\varepsilon^{-+3} &= \varepsilon_{3+-} = 1, & \varepsilon^{++3} &= \varepsilon_{3} = -1, \\
\varepsilon^{33} &= \varepsilon_{333} = -\lambda.
\end{align*}
\tag{192}
\]

We can get the commutation relations for the components of kinetic momentum [see Eq. (179) in Chap. 9] by the following calculation:
\[
\Pi^C \Pi^D \varepsilon_{DCF} = (P^C - eA^C)(P^D - eA^D) \varepsilon_{DCF}
= P^C P^D \varepsilon_{DCF} + eA^C eA^D \varepsilon_{DCF} - P^C eA^D \varepsilon_{DCF} - eA^C P^D \varepsilon_{DCF}
= -i^{-1} \partial^C \triangleright eA^D \varepsilon_{DCF} - q^{-1} \varepsilon_{DCF} (R^{-1})^{CD} G_{GH} eA^G P^H - eA^C P^D \varepsilon_{DCF}
= eB_F + eA^C P^H \varepsilon_{HGF} - eA^C P^D \varepsilon_{DCF} = eB_F. \tag{193}
\]
In the third step, we used the following identities [13]:

\[ P^C P^D \epsilon_{DCF} = A^C A^D \epsilon_{DCF} = 0. \]  

Furthermore, we took into account that q-deformed partial derivatives give a representation of canonical momentum:

\[ P^C \epsilon A^D = i^{-1} \partial^C \triangleright \epsilon A^D + q^{-4} (\hat{R}^{-1})^{CD} G_H \epsilon A^G P^H. \]  

The penultimate step in Eq. (193) follows with (also cf. Ref. [13])

\[ \epsilon_{DCF} (\hat{R}^{-1})^{CD} G_H = -q^4 \epsilon_{HGF} \]  

and the expression for q-deformed magnetic field:

\[ B_F = i \partial^C \triangleright A^D \epsilon_{DCF} = -q^{-4} (\hat{R}^{-1})^{CD} G_H A^G \triangleright \partial^H \epsilon_{DCF} \]
\[ = i A^G \triangleright \partial^H \epsilon_{HGF} = g_{FG} B^G. \]  

Using the commutation relations in Eq. (193), we can verify the following identity by a direct calculation:

\[ [\Pi^C \Pi_D, \Pi^D] = -2im f^D_{Lor}. \]  

Note that the q-deformed Lorentz force density reads as follows:

\[ f^D_{Lor} = \frac{ie}{2m} g^{DCG} \epsilon_{ACG} (\Pi^C B^A - B^C \Pi^A). \]  

B \quad q\text{-Analog of angular momentum}

We can express the components of q-deformed orbital angular momentum by q-deformed partial derivatives and q-deformed position coordinates [13,35,40]:

\[ L_A = q^{-2} \Lambda^{1/2} X^C \hat{\partial}^D \epsilon_{DCA} = -q^2 \Lambda^{1/2} \hat{\partial}^C X^D \epsilon_{DCA}. \]  

Once again, \( \epsilon_{DCA} \) denotes the q-deformed epsilon tensor [cf. Eq. (192) in Chap. A] and \( \Lambda \) stands for a scaling operator subject to the relations in Eq. (53) of Chap. 2.3. Together with the element

\[ W = \Lambda^{-1/2} (1 + q^3 \lambda g_{AB} X^A \hat{\partial}^B) = \Lambda^{1/2} (1 + q^{-3} \lambda g_{AB} X^A \hat{\partial}^B), \]  

the three components of q-deformed angular momentum form the Hopf algebra \( \mathcal{U}_q(\mathfrak{su}_2) \). In this respect, the following relations hold [13,35]:

\[ L^A L^B \epsilon_{BAC} = W g_{CD} L^D. \]
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