EXOTIC SPHERES AND THE TOPOLOGY OF SYMPLECTOMORPHISM GROUPS

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ABSTRACT. We show that, for certain families $\phi_s$ of diffeomorphisms of high-dimensional spheres, the commutator of the Dehn twist along the zero-section of $T^*S^n$ with the family of pullbacks $\phi_s^*$ gives a noncontractible family of compactly-supported symplectomorphisms. In particular, we find examples: where the Dehn twist along a parametrised Lagrangian sphere depends up to Hamiltonian isotopy on its parametrisation; where the symplectomorphism group is not simply-connected, and where the symplectomorphism group does not have the homotopy-type of a finite CW-complex. We show that these phenomena persist for Dehn twists along the standard matching spheres of the $A_n$-Milnor fibre. The nontriviality is detected by considering the action of symplectomorphisms on the space of parametrised Lagrangian submanifolds. We find related examples of symplectic mapping classes for $T^*(S^n \times S^1)$ and of an exotic symplectic structure on $T^*(S^n \times S^1)$ standard at infinity.

1. INTRODUCTION

1.1. Dehn twists. Arnol’d [5] first noticed that the monodromy of the Lefschetz fibration $(z_0, \ldots, z_{n+1}) \rightarrow \sum_{i=0}^{n} z_i^2$ around the unit circle is naturally realised by a compactly-supported symplectomorphism $\tau$ of the fibre $T^*S^n$ called the model Dehn twist. Since any Lagrangian sphere admits a Weinstein neighbourhood symplectomorphic to a neighbourhood of the zero-section $S^n \subset T^*S^n$, the model Dehn twist can be implanted into the symplectomorphism group of any symplectic manifold containing a Lagrangian sphere. Symplectomorphisms obtained this way are called Dehn twists. Dehn twists have since been studied extensively by Seidel [31, 32, 33, 34] and for exact symplectic manifolds they provide one of the few known sources of symplectomorphisms which are nontrivial in the symplectic mapping class group (the group of components of the symplectomorphism group). Indeed, in some low-dimensional cases [15, 18, 26, 31, 39] their isotopy classes are known to generate the symplectic mapping class group.

1.2. Dependence on parametrisation. The Weinstein neighbourhood theorem allows us to embed an open subset of $T^*S^n$ symplectically as a tubular neighbourhood of a Lagrangian sphere but this requires us to choose a metric and a parametrisation of the sphere. The choice of metric is contractible, so the resulting Dehn twist does not depend on the metric up to isotopy. This leaves us with a choice of parametrisation (“framing” in the language of [33]) of the Lagrangian sphere. It was asked in [32, Remark 5.1] whether the Hamiltonian isotopy class of the Dehn
twist depends on this choice. We will prove (Corollary 1.10) that the Hamiltonian
isotopy class does indeed depend on the parametrisation in many cases.

In high dimensions there are many known exotic diffeomorphisms of spheres
which one could use to change the parametrisation. If \( \phi: S^n \to S^n \) is a diffeo-
morphism, denote by \( \phi^*: T^* S^n \to T^* S^n \) the (not compactly-supported) symplec-
tomorphism of the cotangent bundle which pulls one-forms back along \( \phi \).

**Definition 1.1.** If \( \ell: S^n \to X \) is a Lagrangian embedding, we will denote by \( \tau_\ell \)
the associated Dehn twist. Note that \( \tau_{\ell \circ \phi} \) is isotopic to \( \phi^* \tau_\ell (\phi^{-1})^* \). If we write \( \tau \) without further decoration, we mean the model Dehn twist of \( T^* S^n \) and if we write \( \tau_\phi \) we mean \( \phi^* \tau (\phi^{-1})^* \).

**Remark 1.2.** Note that \( \phi^* \tau_\ell (\phi^{-1})^* \) has compact support even though \( \phi^* \) does not.

**Remark 1.3.** We refer to Section 8 for some results concerning the dependence of
the smooth isotopy class on the choice of parametrisation; in particular we show
that \( \tau \) and \( \tau_\phi \) are smoothly isotopic when \( n \equiv 3 \) \( \text{mod} \) \( 4 \) (Theorem D).

### 1.3. Recollections on exotic spheres

We write \( \text{Diff}(M) \) for the group of orientation-preserving diffeomorphisms of an orientable manifold \( M \) and \( \text{Diff}(M, \partial) \) for the
subgroup of diffeomorphisms fixing the boundary pointwise. There is a fibration

\[
\text{Diff}(D^{n+1}, \partial) \to \text{Diff}(D^n) \to \text{Diff}(S^n)
\]

over the identity component of \( \text{Diff}(S^n) \) (the second map is restriction to the boundary)
which yields a long exact sequence

\[
\cdots \to \pi_{k+1}(\text{Diff}_0(S^n)) \xrightarrow{\partial_{k+1}} \pi_k(\text{Diff}(D^{n+1}, \partial)) \to \pi_k(\text{Diff}(D^n)) \to \pi_k(\text{Diff}_0(S^n)) \to \cdots
\]

By Smale’s h-cobordism theorem [27] together with Cerf’s pseudo-isotopy theorem [9] we also have an isomorphism \( \partial_0: \pi_0(\text{Diff}(S^n)) \to \Theta_{n+1} \), where \( \Theta_{n+1} \)
is the group of smooth homotopy \((n+1)\)-spheres under connected sum and \( \partial_0 \) is the
map which sends a mapping class \( \phi \) to the homotopy sphere obtained by gluing
two \((n+1)\)-discs along their common boundary using \( \phi \).

There is an embedding \( \iota: \text{Diff}(D^{n+1}, \partial) \to \text{Diff}(S^{n+1}) \) and we can form the composition \( \iota \circ \partial_k: \pi_k(\text{Diff}(S^n)) \to \pi_{k-1}(\text{Diff}(S^{n+1})) \). Using this we obtain maps

\[
\lambda_{n+k}^{n+k}: \pi_k(\text{Diff}(S^n)) \to \Theta_{n+k+1}.
\]

The images of these maps give a filtration of \( \Theta_N \) by subgroups; this is called the Gromoll filtration [16]. There are many results on
the nontriviality of the Gromoll filtration, for example [8 Theorem 7.4], [3, 10].

The image of \( \lambda_{n+k}^{n+k} \) is usually written \( \Gamma_{n+k+1}^{n+k+1} \subset \Theta_{n+k+1} \).

**Remark 1.4.** The \( k = 1 \) part of the Gromoll filtration is completely understood.
Cerf’s pseudo-isotopy theorem [9] states that \( \pi_0(\text{Diff}(D^N)) = 0 \) for \( N \geq 6 \). Since any diffeomorphism of \( S^N \) can be modified by an isotopy to fix a ball pointwise,
\( \pi_0(\text{Diff}(D^N, \partial)) = \pi_0(\text{Diff}(S^N)) \). The exact sequence (11) and Cerf’s theorem therefore imply that \( \lambda_{1}^{1} \) is surjective.

1 More generally there are maps \( \lambda_{n,j}^{n-j}: \pi_j(\text{Diff}(D^{n-j}, \partial)) \to \pi_{j-1}(\text{Diff}(D^{n-j+1}, \partial)) \).
Definition 1.5. Given \( \alpha \in \pi_k(\text{Diff}(S^n)) \), we will write \( S^{n+k}_\alpha \) for the exotic sphere \( \lambda^{n+k}_{k,k}(\alpha) \).

Remark 1.6. We can also understand \( \lambda^{n+k}_{k,k} \) as follows. Given a family \( \phi_s \in \text{Diff}(S^n) \) of diffeomorphisms parametrised by \( s \in S^k \), representing a homotopy class \( \alpha \in \pi_k(\text{Diff}(S^n)) \) one obtains the homotopy \( (n+k+1) \)-sphere \( S^{n+k+1}_\alpha \) by gluing \( S^k \times D^{n+1} \) to \( D^{k+1} \times S^n \) along their common boundary \( S^k \times S^n \) using the diffeomorphism \((x, s) \mapsto (\phi_s(x), s)\).

Suppose that \( \phi_s = \text{id} \) for all \( s \) in a small neighbourhood of \( 0 \in S^k \) and that there is some open ball \( U \subset S^n \) for which \( \phi_s|_U = \text{id}|_U \) for all \( s \in S^k \); pick \( x_0 \in U \). Then the diffeomorphism \((x, s) \mapsto (\phi_s(x), s)\) descends to a diffeomorphism

\[
S^k \times S^n / (\{0\} \times S^n \vee S^k \times \{x_0\}) = S^{n+k} \to S^{n+k}
\]

(the suspension of a representative of \( \alpha \)) and we can equivalently obtain \( S^{n+k}_\alpha \) by gluing two copies of \( D^{n+k+1} \) along this suspension.

Finally, we recall that there is a natural subgroup \( bP_{N+1} \subset \Theta_N \) comprising the homotopy spheres which bound parallelisable \((N+1)\)-manifolds. There is an exact sequence [24]

\[
0 \to bP_{N+1} \to \Theta_N \to \text{coker}(J_N) \to C_N \to 0
\]

where \( J_N : \pi_N(O) \to \pi_N^S \) is the \( J \)-homomorphism, \( \pi_N^S \) is the \( N \)th stable homotopy group of the spheres and \( C_N \) is either \( \mathbb{Z}/2 \) or \( 1 \) according to whether there exists, respectively does not exist, an \( N \)-dimensional framed manifold of nonzero Kervaire invariant.

1.4. \( A_m \)-Milnor fibre.

Definition 1.7. We will denote by \( A^n_m \) the (complex) \( n \)-dimensional \( A_m \)-Milnor fibre

\[
A^n_m := \left\{ \prod_{j=0}^m (x_0 - j) + x_1^2 + \cdots + x_n^2 = 0 \right\} \subset \mathbb{C}^{n+1}.
\]

The space \( A^n_1 \) is symplectomorphic to \( T^*S^n \).

The Milnor fibre \( A^n_m \) admits a Lefschetz fibration

\[
\pi^n_m : A^n_m \to \mathbb{C},
\]

\[
(x_0, \ldots, x_n) \mapsto x_0,
\]

with \( m + 1 \) critical points \( x_0 \in \{0, \ldots, m\} \). Its general fibre is symplectomorphic to \( A^{n-1}_1 \). Over any embedded arc \( \gamma : [0, 1] \to \mathbb{C} \) with \( \gamma^{-1}(\{0, \ldots, m\}) = \{0, 1\} \) (matching path) there is an embedded Lagrangian \( n \)-sphere (a matching sphere)

\[
L_\gamma \subset \gamma^{-1}(\{0, 1\}) \subset A^n_m.
\]
1.5. **The set \( \mathcal{L}(N) \).** Fix a cotangent fibre \( \Lambda \subset T^* S^N \) and let \( \mathcal{L}(N) \subset \Theta_N \) denote those homotopy spheres which admit a Lagrangian embedding into \( T^* S^N \), with the additional requirement that the embedding intersects \( \Lambda \) transversely in exactly one point. It is readily checked that \( \mathcal{L}(N) \) is a sub-monoid with identity given by the standard sphere and hence, because of the finiteness of \( \Theta_N \), it is moreover a subgroup (however, these properties will not be needed).

**Remark 1.8.**
1. If the nearby Lagrangian conjecture is true, it would follow, *a fortiori* that \( \mathcal{L}(N) \) is the trivial group.
2. It is a priori not clear if every exact Lagrangian embedding in \( T^* M \) can be made to intersect a given fibre in exactly one point after a Hamiltonian isotopy. Of course, this would also follow from the nearby Lagrangian conjecture.

1.6. **Main results.** By combining the results of Abouzaid [1], Ekholm-Smith [12], [13], and Abouzaid-Kragh [2], we will prove the following statement.

**Theorem A** ([1, 2, 12, 13]). For \( N \geq 6 \), we have the inclusion \( \mathcal{L}(N) \subset bP_{N+1} \).

**Remark 1.9.** Note that if \( N = 2\ell \), \( bP_{N+1} = \{1\} \), so in these dimensions the only condition is that the mapping class is nontrivial. If \( N = 6, 7 \) then \( \Theta_N = bP_{N+1} \) so the theorem is empty.

Our other main results follow from this by using Lagrangian suspension (see Section 3) to turn questions about symplectomorphisms into questions about Lagrangian submanifolds. The first theorem we prove is Theorem B which says, roughly, that it is difficult to change the parametrisation of a Lagrangian \( S^n \) inside its cotangent bundle by a Hamiltonian isotopy. We call this phenomenon *rigidity of parametrisation*.

**Theorem B.** Let \( \alpha \in \pi_k(\text{Diff}(S^n)) \) be a class such that \( S^{n+k+1} \alpha / \mathcal{L}(n + k + 1) \) and let \( \psi_s, s \in I^k \), be a family of diffeomorphisms representing \( \alpha \), satisfying \( \psi_s = \text{id} \) for \( s \in \partial(I^k) \). There is no family \( \phi_{(s,t)} \) of compactly-supported Hamiltonian symplectomorphisms parametrised by \( (s,t) \in I^k \times I^k \) satisfying:

- \( \phi_{(s,t)} = \text{id} \) for \( (s,t) \in \partial(I^k) \times I^k \times \{1\} \);
- \( \phi_{(s,1)} \) fixes the zero-section \( S^n \) setwise and \( \phi_{(s,1)}|_{\Lambda} = \psi_s \).

This can be used to deduce dependence of a Dehn twist on its parametrisation when the reparametrisation has order > 2 in the quotient \( \Theta_{n+1}/bP_{n+2} \) of the mapping class group (Corollary 5.5).

The same idea, but applied to the parametrisation of a cotangent fibre, allows us to prove:

**Theorem C.** Suppose that \( \alpha \in \pi_k(\text{Diff}(S^n)) \) satisfies the property that \( S^{n+k+1} \alpha / \mathcal{L}(n + k + 1) \), and let \( \phi_s : S^n \rightarrow S^n, s \in S^k \) be a family of diffeomorphisms (based at the identity) representing the class \( \alpha \). Let \( A_{n+1} \) denote the (complex) \( n \)-dimensional \( A_{m+1} \)-Milnor fibre and...
let $\gamma$ be a matching path. Suppose that $\ell: S^n \to A^n_{m}$ is a Lagrangian embedding whose image is $L_{\gamma}$. Then

$$s \mapsto \tau_{\ell}^{-1}\tau_{\ell\circ\phi}$$

is a nontrivial element in $\pi_k(\text{Symp}^c(A^n_{m}))$.

In Section 8, we investigate the smooth isotopy properties of reparametrised Dehn twists and prove:

**Theorem D.** Let $\tau$ denote the model Dehn twist on $T^*S^{4\ell+3}$, $\ell \geq 1$, and let $\phi$ be a diffeomorphism of $S^{4\ell+3}$. Then $\tau^{-1}\tau_{\phi}$ is isotopic to the identity by a compactly supported smooth isotopy, and hence $\tau$ and $\tau_{\phi}$ are smoothly isotopic.

Further results include:

**Theorem E.** Given a nontrivial loop $\alpha \in \pi_1(\text{Diff}(S^n))$ with the property that $S^{n+2}_n \notin L(n+2)$, there is a symplectomorphism $\Psi \in \text{Symp}^c(T^*(S^n \times S^1))$ which is not isotopic to the identity through compactly supported symplectomorphisms.

**Theorem F.** If $n \equiv 3 \mod 4$ and $L(n+1) \subset \Theta(n+1)$, there is a symplectic form on $T^*(S^n \times S^1)$ which is standard at infinity, homotopic to the standard form through nondegenerate two-forms standard at infinity, but not symplectomorphic relative infinity to the standard form.

1.7. Corollaries of Theorems A and C

**Corollary 1.10.** Suppose that $n \geq 5$. If $\phi \in \text{Diff}^+(S^n)$ is a diffeomorphism representing a mapping class which is nontrivial in $\Theta_{n+1}/bP_{n+2}$, then the symplectomorphism $\tau^{-1}\tau_{\phi}$ is not isotopic to the identity through compactly-supported Hamiltonian diffeomorphisms. In particular, $\tau$ and $\tau_{\phi}$ are not isotopic through compactly-supported Hamiltonian diffeomorphisms.

**Remark 1.11.** A result of Seidel shows that the Dehn twist has infinite order inside $\pi_0(\text{Symp}^c(A^n_{m}))$, see e.g. [32, Lemma 5.7] for a proof. The same proof also shows that the non-trivial elements in $\pi_0(\text{Symp}^c(A^n_{m}))$ produced by Theorem C are not Hamiltonian isotopic to any non-zero power of a (reparametrised) Dehn twist. The reason is that these elements act trivially on the grading of the Lagrangian sphere $\ell$ (viewed as a graded Lagrangian submanifold) while, whenever $n > 1$, any non-zero power of the Dehn twist acts non-trivially on this grading.

**Corollary 1.12.** Let $n \geq 5$ and suppose that $\phi_\alpha$ is a loop of diffeomorphisms of $S^n$ representing a class $\alpha \in \pi_1(\text{Diff}(S^n))$ such that $S^{n+2}_n$ is nontrivial in $\Theta_{n+2}/bP_{n+3}$. Then the loop $\tau^{-1}\tau_{\phi_\alpha}$ is nontrivial in $\pi_1(\text{Symp}^c(T^*S^n))$.

**Remark 1.13.** Surjectivity of $\lambda_{n+1}^1$ implies that there are many examples to which Corollary 1.12 applies. As far as we know, these are the first examples of Stein manifolds for which the fundamental group of the group of compactly-supported symplectomorphisms is known to be nontrivial.
The existence of nontrivial elements in $\pi_k(\text{Symp}(T^*S^n))$ follows similarly whenever we have an element of $\Gamma_{n+k+1} \cap (\Theta_{n+k+1} \setminus bP_{n+k+2})$. The next corollary was brought to our attention by Ivan Smith and Oscar Randall-Williams. It is a (relatively) low-dimensional example where we can say something concrete; similar arguments have been used in [4] and elsewhere.

Corollary 1.14. The group $\text{Ham}^c(T^*S^{10})$ does not have the homotopy-type of a finite CW complex.

Proof. By the table in [4] Appendix A.3, $\Gamma_{13}^{13} = \Theta_{13} = \mathbb{Z}/3$ (and $bP_{14} = 0$). Therefore there is an element $\alpha \in \pi_2(\text{Diff}(S^{10}))$ with $S_{13}^{13} \not\in bP_{14}$ and hence $S_{13}^{13} \not\in \mathcal{L}(13)$. By Theorem C, $\pi_2(\text{Ham}^c(T^*S^{10}))$ is then nontrivial and since $\text{Ham}^c(T^*S^n)$ is a path-connected H-space, [7] Theorem 6.11 implies that it cannot have the homotopy-type of a finite CW-complex. □

1.8. Relation to Seidel’s work. This paper is related to, but orthogonal to, Seidel’s paper [35] where he constructs symplectomorphisms of cotangent bundles of exotic spheres analogous to iterated Dehn twists (although, it is currently unknown if these constructions can be made to have compact support). In particular, Corollary 1.10 answers a question posed in that paper. The idea we use, which amounts to detecting nontrivial symplectic mapping classes by looking for a change in the parametrisation of a cotangent fibre, was inspired by Keating’s argument in Section 5 of that paper.

1.9. Outline of contents by section.

2. We review the literature on Arnold’s nearby Lagrangian conjecture and deduce Theorem A.
3. We generalise the suspension construction for Lagrangian submanifolds and Hamiltonian paths to $k$-parameter families.
4. We construct an open embedding of an open subset of $A^m_n \times T^*I^{k+1}$ into $T^*S^n \times S^{k+1}$, which will be used extensively in the subsequent proofs.
5. We prove Theorem B on rigidity of parametrisation.
6. We prove Theorem C.
7. We find some nontrivial symplectic mapping classes of $T^*(S^n \times S^1)$ (Theorem D).
8. We study the question of when the new elements of $\pi_k(\text{Symp}(T^*S^n))$ are nullhomotopic in $\pi_k(\text{Diff}(T^*S^n))$ (Theorem E).
9. We use the same methods to find an exotic symplectic structure on $T^*(S^n \times S^1)$ standard at infinity (Theorem F).
10. We give some relations in the symplectic mapping class group satisfied by the mapping classes of $\tau_\phi$.
11. We give some open questions which our methods leave unanswered.
1.10. **Acknowledgements.** The authors would like to thank Ivan Smith and Oscar Randall-Williams for helpful discussions, in particular for pointing out the Gromoll filtration and Corollary 1.14.

2. **Proof of Theorem A**

2.1. **Ingredients.** Before showing how Theorem A follows from the results of [1, 2, 12, 13], we will recall the main ingredients. The first result in this direction was the following, even stronger, result by Abouzaid.

**Theorem 2.1.** Suppose \( N = 4\ell + 1 \geq 8 \). If \( L \subset T^*S^N \) is a Lagrangian homotopy sphere, then \( L \in bP_{N+1} \).

Ekholm and Smith [12, 13] later proved similar restrictions for Lagrangian immersions in \( \mathbb{C}^N \) with one transverse double-point. We will only need the following, slightly weaker, formulation of their theorem. First, recall the definition of the stable Lagrangian Gauss map in Section 2.3 below, whose homotopy class is determined by an element in \( \pi_N(U/O) \) for any Lagrangian immersion of an \( N \)-dimensional homotopy sphere into \( \mathbb{C}^N \). Furthermore, to a generic Lagrangian immersion in \( \mathbb{C}^N \) with vanishing Maslov class, one can associate an integer to each double point via the so-called Conley-Zehnder index. We will call this integer the Legendrian contact homology grading of the double-point (see [13] for more details).

Recall that the Whitney immersion is the Lagrangian immersion

\[
\begin{align*}
w : S^N & \hookrightarrow \mathbb{C}^N, \\
(x, y) & \mapsto (1 + iy)x, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R},
\end{align*}
\]

having exactly one transverse double-point.

**Theorem 2.2.** Suppose that \( N \geq 8 \). If \( K \hookrightarrow \mathbb{C}^N \) is an exact Lagrangian immersion of a homotopy sphere with only one transverse double-point for which

- the Legendrian contact homology grading of the double-point of \( K \) coincides with that of the Whitney immersion (i.e. it is two), and
- the stable Lagrangian Gauss map of \( K \) coincides with that of the Whitney immersion (i.e. it is null homotopic),

then \( K \in bP_{N+1} \).

2.2. **The proof.** In order to relate the previous result to our setting, we need the following lemmas.

**Lemma 2.3.** Suppose \( L \subset T^*S^N \) intersects a cotangent fibre \( \Lambda \) once transversely. Let \( \pi : T^*S^N \to S^N \) be the bundle projection. There exists an open ball \( U \subset S^N \) containing \( \{x\} = \pi(\Lambda) \) and a Lagrangian embedding \( L' \subset T^*S^N \) which is Hamiltonian isotopic to \( L \) and satisfies

\[
L' \cap \pi^{-1}(U) = U \subset S^N \subset T^*S^N.
\]
Proof. Since the Lagrangian intersects a cotangent fibre $\Lambda$ transversely at one point there is a neighbourhood $U'$ of the point $x = \pi(\Lambda)$ over which $L \cap \pi^{-1}(U')$ agrees with the graph of an exact differential $df : U' \to \mathbb{R}$. Now we define a family of Lagrangian embeddings by replacing $L \cap \pi^{-1}(U')$ with the graph of $df_t$, $t \in [0, 1]$, where $f_t$ is a family of functions such that $f_0 = f$ and $f_1$ agrees with $f$ outside of a compact set $K \subset U'$ and equals zero on a subset $U \subset K$. A standard argument shows that this path of Lagrangian embeddings can be realised by a Hamiltonian isotopy.

By Weinstein’s neighbourhood theorem there is an open neighbourhood $W \subset T^*S^N$ of the zero-section and an immersion $\tilde{w} : W \to C^N$ with $\tilde{w}|_{S^N} = w$ coinciding with the Whitney immersion. We can assume that $\tilde{w}^{-1}(w(S^N))$ consists of the zero-section and a pair of cotangent fibres; moreover by reparametrising the embedding we can assume these cotangent fibres are $T^*_xS^N$ and $T^*_yS^N$ for points $x$ and $y$ contained in some given open ball $U$.

**Lemma 2.4.** If $L \in \mathcal{L}(N)$ then $L$ admits a Lagrangian immersion into $C^N$ with precisely one transverse double point. Moreover, the Maslov grading of this double point agrees with that of the double point for the Whitney immersion.

**Proof.** By Lemma 2.4 there is a Lagrangian embedding $f : L \to T^*S^N$ which coincides with the zero-section over some open set $U \subset S^N$. Rescaling using the Liouville flow we can assume that the image of $L$ is contained in the domain of $\tilde{w}$ and so we can form the Lagrangian immersion $\tilde{w} \circ f$. By construction this has precisely one transverse double point, whose grading moreover agrees with that of the double point for the Whitney immersion (to see this, it suffices to consider a path lying in $U$ joining the preimages $x$ and $y$ of the double point).

**Proof of Theorem 2.1**  
**Case** $N = 4 \ell + 1$: This follows *a fortiori* from Theorem 2.1.

**Case** $N = 2 \ell, 8 \ell - 1$: This follows directly from Theorem 2.2 since the stable Lagrangian Gauss map always is null-homotopic in these dimensions (see [13]).

**Case** $N = 8 \ell + 3$: The result follows from Theorem 2.2 together with Proposition 2.5 below.

### 2.3. Recollections on the Lagrangian Gauss map

A Lagrangian immersion has an associated Lagrangian frame map, which is an $O(N)$-equivariant map to the principal bundle of $U(N)$-frames in the target manifold. By considering the corresponding section in the induced $U(N)/O(N)$-bundle, one obtains the Lagrangian Gauss map induced by the immersion. In the case of a Lagrangian immersion into $C^N$, the Lagrangian Gauss map can naturally be considered as a mapping values in $U(N)/O(N)$. There is an associated map to $U/O$, where $U = \lim_{k \to \infty} U(k)$ and $O = \lim_{k \to \infty} O(k)$, which is the so-called stable Lagrangian Gauss map.

It is well-known that stable Lagrangian Gauss map for the Whitney immersion is contractible. Moreover, the following (even stronger) statement holds for its frame map: In the trivialisation of $T^*S^N \oplus \mathbb{R}$ induced by the canonical inclusion $T^*S^N \subset TR^{N+1}$, the $U(N+1)$-frame $Dw \oplus id : T^*S^N \oplus \mathbb{R} \to C^{N+1}$ is null-homotopic as a map $S^N \to U(N+1)$.
To see this claim, we argue as follows. The isotropic immersion \((w,0)\) inside the stabilisation \(\mathbb{C}^N \times \mathbb{C}\) is regular homotopic to the standard embedding \(S^N \subset \mathbb{R}^{N+1} \subset \mathbb{C}^{N+1}\) into the real part, where this homotopy moreover can be taken to be through isotropic immersions. For instance, one can simply take a linear interpolation. We let \(w_t: S^N \to \mathbb{C}^{N+1}, t \in [0,1]\), denote this regular homotopy. The symplectic normal bundle of \(w_0\) is obviously trivial, and hence the same is true for each \(w_t\). Consider the frame of the symplectic normal bundle which is given by the constant frame \((0, \ldots, 0, 1) \in \mathbb{C}^{N+1}\) for \(t = 0\) and which is given by the outward-normal in the real-part for \(t = 1\). This frame extends to a frame of the symplectic normal bundle for all \(t \in [0,1]\). Stabilising the frame map \(Dw_t\) by adjoining this frame, and using the canonical trivialisation, we obtain the sought null-homotopy of the stabilised Lagrangian tangent-frame \(S^N \to U(N + 1)\) of the Whitney immersion.

Recall that Weinstein’s Lagrangian neighbourhood theorem provides a symplectic immersion \(\tilde{w}: D^*S^N \to \mathbb{C}^N\) which restricts to the Whitney immersion \(w\) along the zero-section. Any Lagrangian immersion \(f: L \hookrightarrow T^*S^N\) hence gives rise to a Lagrangian immersion into \(\mathbb{C}^N\) via the composition \(\tilde{w} \circ f\) (after a suitable rescaling). By the above properties of the Lagrangian frame map of the Whitney immersion, the homotopy class of the stable Lagrangian Gauss map of \(\tilde{w} \circ f\) can be computed as follows. Choose an immersion of \(S^N\) into the real part \(\mathbb{R}^{N+M} \subset \mathbb{C}^{N+M}\). The pull-back of \(TC^{N+M}\) along this immersion can be canonically identified with \((\mathbb{C} \otimes \nu) \oplus T(T^*S^N)\), where \(\nu\) denotes the normal bundle of the immersion inside \(\mathbb{R}^{N+M}\). By adjoining the real-part of \(\mathbb{C} \otimes \nu\) to the Lagrangian tangent planes of the immersion \(f\), we obtain a family of Lagrangian planes in \(\mathbb{C}^{N+M}\) parametrised by \(L\).

Abouzaid and Kragh [2] have shown that there are restrictions on the homotopy class of the stable Lagrangian Gauss map for an exact Lagrangian embedding inside \(T^*S^N\). We will only need the following special case.

**Proposition 2.5.** [2 Proposition 2.2] Let \(f: L \hookrightarrow T^*S^N\) be an exact Lagrangian embedding of a homotopy sphere, where \(N = 8\ell + 3\). The Lagrangian immersion \(\tilde{w} \circ f: L \to \mathbb{C}^N\) has a null-homotopic stable Lagrangian Gauss map.

**Proof.** For any \(T \in \pi_N(U/O)\), Abouzaid and Kragh [2, Section 2] construct an associated map \(BJ \circ T_{\mathcal{L}_0}: \mathcal{L}L \to BH\), where:

- \(\mathcal{L}L\) denotes the free loop space of \(L\),
- \(H\) denotes the stable group of self-homotopy equivalences of spheres,
- the homotopy class of \(T_{\mathcal{L}_0}\) is determined by the homotopy class of \(T\).

Observe that \(\pi_N(U/O) = \mathbb{Z}/2\) when \(N = 8\ell + 3\). In this case, the map \(BJ \circ T_{\mathcal{L}_0}\) induced by the non-trivial element of \(\pi_N(U/O)\) is not null-homotopic, as is seen by studying the last row of [2 Table 1] for the appropriate dimension. On the other hand, in the case when \(T\) is the stable Lagrangian Gauss map of an exact Lagrangian homotopy sphere in \(T^*S^N\), [2 Proposition 2.2] shows that \(BJ \circ T_{\mathcal{L}_0}\) is null-homotopic. Therefore if \(T\) is the stable Lagrangian Gauss map of \(\tilde{w} \circ f\) it must be equal to zero in \(\pi_N(U/O) = \mathbb{Z}/2\).

\(\square\)
3. Lagrangian suspension

The main tool we use is the Lagrangian suspension construction [6] which allows us to turn a Hamiltonian isotopy $\phi^t$ for which $\phi^1$ preserves a Lagrangian submanifold $L \subset X$ into a Lagrangian embedding $L \times [0, 1] \to X \times T^*[0, 1]$. We need a generalisation of this construction to $k$-parameter families $\psi_q$, $q \in I^k$ where $\psi_q$ is the identity for $q \in \partial(I^k) \setminus (I^{k-1} \times \{1\})$ and $\psi_q$ preserves $L$ for all $q \in \partial(I^k)$. We call the resulting $L \times I^k \subset X \times T^*I^k$ the Lagrangian suspension of $L$ along $\psi_q$. The case $k = 0$ is the usual Lagrangian suspension.

In the following we let $I = [0, 1]$ denote the unit interval and identify $T^*[0, 1]$ with $\mathbb{R} \times [0, 1]$. We will only need the case when $(X, \omega)$ is a simply-connected symplectic manifold. We can therefore omit the requirements of being Hamiltonian in the following section.

**Proposition 3.1.** Suppose we have a family $\psi_q$ of symplectomorphisms, parametrised by $q \in I^k$, for which there is a nonempty open subset $U \subset X$ such that $\psi_q|_U = \text{id}$ for all $q$. Then there exists a symplectomorphism

$$\Psi: (X \times T^*I^k, \omega \oplus d\theta_{I^k}) \to (X \times T^*I^k, \omega \oplus d\theta_{I^k}),$$

$$(x, q, p) \mapsto (\psi_q(x), q, p - \mathbf{H}(x, q)), $$

for a smooth function

$$\mathbf{H}: X \times I^k \to \mathbb{R}^k$$

which may be taken to vanish on $U \times I^k$. Moreover, if $\psi_q(L) = L$ for a Lagrangian submanifold $L \subset (X, \omega)$ and all $q \in A$ for a subset $A \subset I^k$, it follows that $\mathbf{H}$ is constant along each subset $L \times \{q\}$ with $q \in A$.

**Proof.** The smooth map

$$(x, q, p) \mapsto (\psi_q(x), q, p)$$

is unfortunately not symplectic, since the pull-back of $\omega \oplus d\theta_{I^k}$ is of the form

$$\omega \oplus d\theta_{I^k} - \eta,$$

$$\eta := \sum_{i=1}^k dq_i \wedge \alpha_i(x, q) + \sum_{i, j=1}^k f_{i,j}(x, q) dq_i \wedge dq_j.$$

Here $\alpha_i$ is a $k$-parameter family of one-forms on $X$ and $f_{i,j}: X \times I^k \to \mathbb{R}$ are functions satisfying $f_{i,j} = -f_{j,i}$. The fact that both the forms $\omega \oplus d\theta_{I^k} - \eta$ and $\omega \oplus d\theta_{I^k}$ are symplectic, and hence that $\eta$ is closed, implies that $d_x \alpha_i = 0$. Consequently,

$$\alpha_i = d_x H_i(x, q)$$

for some smooth function

$$\mathbf{H} = (H_1, \ldots, H_k): X \times I^k \to \mathbb{R}^k$$

uniquely determined by the requirement that it vanishes on $U \times I^k$. Here we have used the fact that $\eta$, and hence $\alpha_i$, vanishes on $U \times I^k$.

Closedness moreover implies that

$$d_x(f_{i,j} - f_{j,i}) = \partial_{q_i} \alpha_i(x, q) - \partial_{q_j} \alpha_j(x, q) = \partial_{q_j} d_x H_i - \partial_{q_i} d_x H_j.$$

Changing the order of the partial derivations, we obtain that

$$f_{i,j} - f_{j,i} = \partial_{q_j} H_i - \partial_{q_i} H_j + F_{i,j}(q),$$

where $F_{i,j}(q)$ is a smooth function of $q$. This shows that $\Psi$ is a symplectomorphism.
where \( F_{i,j}(q) \) vanishes since \( \eta_i \) and hence \( f_{i,j} \), vanishes on \( U \times I^k \). It is now easy to check that the modified diffeomorphism \( \Psi \) preserves the symplectic form.

Suppose that \( L \subset (X, \omega) \) is a Lagrangian submanifold preserved set-wise by \( \psi_{q_0} \).
The Lagrangian condition implies that \( \eta \) vanishes along \( L \times \{q_0\} \) and, hence, so does the one-forms \( \alpha_i \) when pulled back to \( L \times \{q_0\} \subset X \times \{q_0\} \). It follows that \( H \) is constant along \( L \times \{q_0\} \).

**Remark 3.2.** Note that one can replace \( H \) by \( H(x, q) + F(q) \) for an arbitrary function \( F(q) \) and one still gets a symplectomorphism. If \( \psi_q(L) = L \) so that \( H \) is constant equal to \( C(q) \) along \( L \times \{q\} \), we can choose \( F(q) = -C(q) \) so that \( \Psi(L, q, 0) = (L, q, 0) \).

**Definition 3.3** (Lagrangian suspension). Let \( \iota : L \to X \) be a Lagrangian embedding and \( \psi_q \) be a family of symplectomorphisms parametrised by \( q \in I^k \). Suppose that \( \psi_q(L) = L \) for all \( q \) in a neighbourhood of \( \partial I^k \). Then the Lagrangian

\[
\Psi(L) \subset X \times T^* I^k
\]

is called the **Lagrangian suspension**, \( \Sigma L \), of \( L \) along \( \psi_q \). By Remark 3.2 one can ensure that \( \Psi(L) \) agrees with \( L \times I^k \) in a neighbourhood of the boundary.

**Remark 3.4.** We will further assume that \( \psi_q = \text{id} \) for \( q \in \partial I^k \setminus \{I^{k-1} \times \{1\}\} \) and that the map \( I^{k-1} \times \{1\} \to \text{Diff}(L) \) which sends \( q \) to \( \psi_q|_{L} \) represents a given homotopy class \( \alpha \in \pi_{k-1}(\text{Diff}(L)) \). The suspension \( \Sigma L \) agrees with \( L \times I^k \) near the boundary, but we get a new parametrisation of \( L \times \{(q_1, \ldots, q_{k-1}, 1)\} \) by sending \( x \in L \) to \( \psi(q_1, \ldots, q_{k-1}, 1)(x) \). As \( (q_1, \ldots, q_{k-1}) \) varies, this new parametrisation traces out the class \( \alpha \) in \( \text{Diff}(L) \).

### 4. Open symplectic embeddings in \( T^* S^n \)

In the proofs of the main results we will continually need to appeal to the existence of certain open symplectic embeddings of open subsets of \( A^n_m \times T^* I^{k+1} \) into \( T^* S^{n+k+1} \). In this section we will establish the result we need and give two alternative proofs.

Consider the Lefschetz fibration

\[
\pi^n_m : A^n_m \to \mathbb{C}
\]

from Section 3.4 with \( m + 1 \) critical points at \( 0, 1, \ldots, m \). If we write \( \gamma_i : [0, 1] \to \mathbb{C}, i = 1, \ldots, m \), for the arc \( \gamma_i(t) = i + t - 1 \) (considered as a matching path for \( \pi^n_m \)). For brevity we will write \( L_i \) for the matching sphere \( L_{\gamma_i} \subset A^n_m \).

**Proposition 4.1.** For all \( k, n \geq 0 \) there is an open symplectic embedding of an open neighbourhood \( W \) of \( \bigcup_{i=1}^m L_i \times I^{k+1} \subset A^n_m \times T^* I^{k+1} \) into \( T^* S^{n+k+1} \) such that:

- \( L_1 \times I^{k+1} \) is sent to a subset of the zero-section \( S^{n+k+1} \);
- the image of the embedding is disjoint from particular a cotangent fibre \( \Lambda \subset T^* S^{n+k+1} \).
The proposition follows from the case \( k = 0 \) because there is an open symplectic embedding of \( T^*S^{n+1} \times T^*I^k \) into \( T^*S^{n+k+1} \). Take the left-inverse of the pullback along the open inclusion \( S^{n+1} \times I^k \to S^{n+k+1} \) of a tubular neighbourhood of a sphere \( S^{n+1} \subset S^{n+k+1} \). Note that the image of this embedding is disjoint from \( \Lambda = T^*_xS^{n+k+1} \) for any \( x \) not contained in the tubular neighbourhood of \( S^{n+1} \).

There are two ways to prove the \( k = 0 \) case. We include both for completeness.

**First proof of Proposition 4.1.** The left-inverse of the pullback along the open inclusion \( S^n \times I \subset S^{n+1} \) of a tubular neighbourhood of an equator \( S^n \subset S^{n+1} \) gives an open embedding \( T^*S^n \times T^*I \to T^*S^{n+1} \) whose image is disjoint from \( \Lambda = T^*_xS^{n+1} \) for some \( x \in S^{n+1} \setminus (S^n \times I) \). Our open embedding will have its image inside \( T^*(S^n \times I) \), so it will automatically satisfy the second criterion in the statement of the proposition.

Let \( L'_1, \ldots, L'_m \subset T^*S^n \) be exact Lagrangian submanifolds obtained as generic perturbations of the zero-section in \( T^*S^n \), where \( L'_1 = S^n \subset T^*S^n \), and where \( L'_i \cap L'_j \) moreover intersect transversely (and hence in at least two points) whenever \( i \neq j \). The exact Lagrangian cylinders

\[
L'_i \times I \subset T^*S^n \times T^*I
\]

intersect in a collection of paths. We claim that, after perturbing each \( L'_i \times I \), we can resolve any given subset of paths of intersections (while keeping the other paths intact). Moreover, this can be done by a Hamiltonian perturbation supported in some arbitrarily small neighbourhood of the intersection locus.

To see this, we argue in the local symplectic model of a component of the above intersection locus, which is given by

\[
(\Re(B^{2n}) \cup \Im(B^{2n})) \times I \subset B^{2n} \times T^*I.
\]

Translating the sheet \( \Re(B^{2n}) \) in the \( p \)-direction of the \( T^*I \)-factor completely resolves this intersection. Since this is a Hamiltonian diffeomorphism, we can use a cut-off function to produce a Hamiltonian whose flow performs this operation locally.

For a suitable choice of perturbation \( C_i \) of \( L'_i \times I \) as above, ensuring as before that \( C'_i = L'_i = S^n \subset T^*S^n \) is the zero-section, we can remove all intersections except \( (L'_i \cap L'_{i+1}) \times I, i = 1, \ldots, m \). The produced union \( \bigcup_{i=1}^m C_i \subset T^*(S^n \times I) \) of Lagrangian cylinders has a neighbourhood in \( T^*S^{n+1} \) symplectomorphic to a neighbourhood \( W \) of

\[
\bigcup_{i=1}^m L_i \times I \subset A^n_m \times T^*I,
\]

as follows by Weinstein’s Lagrangian neighbourhood theorem. We may moreover assume that \( C_i \) is identified with \( L_i \times I \) under this symplectomorphism and, in particular, that the zero-section is identified with \( L_1 \times I \subset A^n_m \times T^*I \).

**Second proof of Proposition 4.1.** The manifold \( T^*S^{n+1} \) admits a sequence of Lefschetz fibrations \( p_m, m = 1, 2, 3 \ldots \), where:

- \( p_1 = s_1^{n+1}: A_1^{n+1} = T^*S^{n+1} \to \mathbb{C} \) is the standard Lefschetz fibration with two singular fibres, whose general fibre is \( A_1^n \).
• $p_m$ is obtained from $p_1$ by stabilising $m - 1$ times: the smooth fibre is $A^n_m$ and there are $m + 1$ singular fibres living over the critical points $0, 1, \ldots, m$.

The vanishing cycles associated to the $m + 1$ critical points of the fibration $p_m$ are $L_1, L_1, L_2, \ldots, L_{m-1}, L_m \subset A^n_m$. Moreover $\gamma_1$ defines a matching path for $\pi^{n+1}_m$ whose matching sphere is the zero-section in $T^* S^{n+1}$. See Figure 1.

By symplectic parallel transport, we can try to trivialise $p_m$ over a compact subset $D^* I = \{ x + iy \in \mathbb{C} : x \in [\epsilon, 1 - \epsilon], |y| \leq 1 \} \subset \mathbb{C}$. Since the fibres are noncompact then, a priori, we are only able to achieve this over some compact neighbourhood $W'$ of the skeleton $\bigcup_{i=1}^n L_i \subset A^n_m$. This gives us a symplectic embedding of $W = W' \times D^* I$ into $T^* S^{n+1}$ and by construction $L_1 \times I$ is identified with a subset of the zero-section (the matching sphere for $\gamma_1$).

There is a cotangent fibre of $T^* S^{n+1}$ which arises as the Lefschetz thimble for $p_m$ living over the ray $(-\infty, 0]$, and which hence is disjoint from $p_1^{-1}(D^* I)$.

5. RIGIDITY OF PARAMETRISATION

Our proof of Theorem C essentially detects nontrivial families of symplectomorphisms by their action on parametrised Lagrangian embeddings. In this section we explain a related phenomenon we call rigidity of parametrisation, where two different parametrised Lagrangian submanifolds with the same geometric image are not Hamiltonian isotopic. This will also give a simple proof that the Dehn twist depends on its parametrisation in certain cases.

Rigidity of parametrisation is equivalent to putting restrictions on the Lagrangian monodromy group, that is the group of self-diffeomorphisms of a Lagrangian arising as time-one maps of Hamiltonian flows whose time-one map preserves the Lagrangian; there are several papers proving restrictions on the homology action of the Lagrangian monodromy group [22, 28, 40]. We point out several results which go beyond the homology action.
5.1. Local rigidity of parametrisation. Here “local” means “inside a Weinstein neighbourhood”.

**Theorem** [B] Let \( \alpha \in \pi_k(Diff(S^n)) \) be a class such that \( S^{n+k+1}_0 \not\in \mathcal{L}(n+k+1) \) and let \( \psi_s, s \in I^k \), be a family of diffeomorphisms representing \( \alpha \), satisfying \( \psi_s = \text{id} \) for \( s \in \partial(I^k) \). There is no family \( \phi_{(s,t)} \) of compactly-supported Hamiltonian symplectomorphisms parametrised by \((s,t) \in I^{k+1} \) satisfying:

- \( \phi_{(s,t)} = \text{id} \) for \((s,t) \in \partial(I^{k+1}) \setminus I^k \times \{1\} \);
- \( \phi_{(s,1)} \) fixes the zero-section setwise and \( \phi_{(s,1)}|_{\mathcal{L}} = \psi_s \).

**Proof.** If we take the suspension of \( S^n \times I^{k+1} \) along \( \phi_{(s,t)} \) we obtain a Lagrangian \( S^n \times I^{k+1} \) in \( T^*S^n \times T^*I^{k+1} \) which agrees with \( S^n \times I^{k+1} \) near the boundary.

Proposition 4.1 tells us there is a neighbourhood \( S^n \times I^{k+1} \subset U \subset C \) which is disjoint from a given cotangent fibre \( \Lambda \). Using the inward Liouville flow of \( T^*S^n \times T^*I^{k+1} \) we can ensure that the Lagrangian suspension \( \Sigma \) lies inside \( U \). We can therefore replace \( \psi \) by \( \psi S^n \times I^{k+1} \cap U \subset S^{n+k+1} \) with \( \psi \). By construction, the result is a Lagrangian \( S^{n+k+1} \) in \( T^*S^{n+k+1} \) which intersects \( \Lambda \) at the single point \( S^{n+k+1} \cap \Lambda \) transversely. Therefore \( S^{n+k+1} \in \mathcal{L}(n+k+1) \). \( \Box \)

The proof in the case \( k = 0 \) is somewhat easier to visualise and gives as a corollary:

**Corollary 5.1.** Let \( i: S^n \to \mathcal{T}^*S^n \) be the embedding of the zero section into its cotangent bundle and let \( \phi \) be a diffeomorphism such that \( S^{n+1}_0 \not\in \mathcal{L}(n+1) \). The two Lagrangian embeddings \( i \) and \( i \circ \phi \) (whose images coincide) are not isotopic through compactly-supported Hamiltonian symplectomorphisms of \( T^*S^n \).

**Remark 5.2.** Indeed, a completely general theorem on local rigidity of parametrisation in cotangent bundles would follow by the same argument from the nearby Lagrangian conjecture.

**Remark 5.3.** Recall that the map \( \pi_1(Diff(S^n)) \to \pi_0(Diff(S^{n+1})) \) is surjective by Cerf’s theorem. In particular there exist loops of diffeomorphisms of \( S^n \) which do not arise as the restriction of a loop of compactly-supported Hamiltonian diffeomorphisms preserving \( S^n \) setwise and extending as a disc of compactly-supported Hamiltonian diffeomorphisms of \( T^*S^n \). More \( k \)-parameter examples arise from elements in \( \Gamma^{n+k+1}_{n+k+1} \cap (\Theta_{n+k+1} \setminus bP_{n+k+2}) \).

5.2. A simple proof that Dehn twists depend on parametrisation. We use local rigidity of parametrisation to prove a simple case of Theorem [C] for \( k = 0 \) which requires the mapping class of \( \phi \) to have order \( > 2 \) in \( \Theta_{n+1}/bP_{n+2} \). We require the following lemma of Seidel.

**Lemma 5.4** ([35 Lemma 2.2]). Let \( \phi \in Diff^+(S^n) \) and let \( a \) be an element of \( O(n+1) \setminus SO(n+1) \) (for example the antipodal map when \( n \) is even). Then

\[ a \phi a \simeq \phi^{-1}. \]
Corollary 5.5 (Dependence of Dehn twist on parametrisation: simple case). Suppose \( n \geq 6 \) is even. Let \( \phi \) be a diffeomorphism of \( S^n \) which has order \( m > 2 \) in the quotient \( \Theta_{n+1}/bP_{n+2} \) of the mapping class group. Let \( \phi^*: T^*S^n \to T^*S^n \) denote the symplectomorphism (not compactly-supported) given by pulling back along \( \phi \) and let \( \tau: T^*S^n \to T^*S^n \) denote the model Dehn twist. Then the compactly-supported symplectomorphism \( \tau_\phi \) is not isotopic to \( \tau \) through symplectomorphisms of compact support.

Proof. Both \( \tau \) and \( \phi^* \) preserve the zero-section; since \( \tau \) acts by the antipodal map on the zero-section, by Lemma 5.4, the effect of \( \tau^{-1} \phi^* \tau(\phi^{-1})^* \) on the embedding \( \iota: S^n \to T^*S^n \) of the zero-section is to change the parametrisation by \( \phi^{-2} \), which is nontrivial in \( \Theta_{n+1}/bP_{n+2} \). By Theorem A, \( S_{n+1, [0,2]}^n \notin L(n+1) \). The resulting embedding \( \iota \circ \phi^{-2} \) is not Hamiltonian isotopic to \( \iota \) by Corollary 5.1 hence \( \tau^{-1} \phi^* \tau(\phi^{-1})^* \) is not Hamiltonian isotopic to the identity, which proves the claim. \( \square \)

Remark 5.6. The lowest-dimensional examples of diffeomorphism satisfying the hypotheses of the theorem occur for \( n = 12, 22, \ldots \), in which dimensions

\[
\Theta_{n+1}/bP_{n+2} \subset \text{coker}(J_{n+1})
\]

has an element of order 3.

5.3. Global rigidity of parametrisation for Lagrangian surfaces. The following result is not directly related to the main thrust of the paper, but we felt it desirable to point it out. The mapping class group of a surface of genus \( g > 1 \) is very rich. A theorem of Thurston tells us that any mapping class \( \phi \) of a compact surface of genus \( g > 1 \) falls into one of three disjoint classes:

- **Periodic**: \( \phi \) has finite order and arises as an isometry of some hyperbolic metric on the surface;
- **Reducible**: \( \phi \) preserves an isotopy class of simple closed curves on the surface;
- **Pseudo-Anosov**: \( \phi \) preserves a pair of transversely measured singular foliations and acts with stretch-factor \( \lambda > 1 \) on one of the measures and \( 1/\lambda < 1 \) on the other. By Thurston’s hyperbolisation theorem for mapping tori [38], we can also characterise pseudo-Anosov mapping classes as those whose mapping tori admit hyperbolic metrics.

This third class is “generic”: the result of a (suitably-defined) random walk of length \( N \) on the Cayley graph of the mapping class group is pseudo-Anosov with probability \( p_N \) where \( p_N \to 1 \) as \( N \to \infty \) [30, 27]. The following result is a global rigidity of parametrisation theorem, in the sense that the ambient manifold could be anything, not just the cotangent bundle of \( L \).

Proposition 5.7 (Global rigidity of parametrisation for Lagrangian surfaces). Let \( L \subset X \) be a compact orientable Lagrangian submanifold in a symplectic 4-manifold and suppose that \( \phi_1 \) is a Hamiltonian isotopy of \( X \). If \( \phi_1(L) = L \) then \( \phi_1|_L: L \to L \) does not represent a pseudo-Anosov mapping class.

Proof. The Lagrangian suspension of \( L \) along \( \phi_1 \) is a Lagrangian submanifold of \( X \times T^*[0, 1] \). If we identify \( X \times T^*_0[0, 1] \) with \( X \times T^*_1[0, 1] \) then we get a Lagrangian
mapping torus in \( X \times T^* S^1 \) and if \( \phi_1|_L \) is pseudo-Anosov then this mapping torus is a hyperbolic 3-manifold. Since the Lagrangian is compact, there exists an \( r < \infty \) for which the mapping torus is contained in \( X \times D_r^* S^1 \) where \( D_r^* S^1 \) denotes the radius \( r \) disc subbundle of \( T^* S^1 \). This disc subbundle embeds symplectically into \( S^2 \) with a large multiple of the standard area form. Therefore we obtain a hyperbolic Lagrangian submanifold of the symplectic 6-manifold \( X \times S^2 \). This 6-manifold is symplectically uniruled, but a theorem of Eliashberg and Viterbo [14, Theorem 1.7.5] rules out the existence of orientable hyperbolic Lagrangian submanifolds in uniruled symplectic manifolds of dimension 6 or more. □

6. Proof of Theorem

We first prove a lemma describing how the conjugated Dehn twist acts on a (parametrised) cotangent fibre. We prove Theorem in the case \( k = 0 \), that is when there is a single nontrivial mapping class of \( S_n \) giving rise to a nontrivial symplectic mapping class of \( A_n^\circ \); this case is conceptually simpler and illustrates the main idea. We then explain how the proof generalises to \( k > 0 \).

6.1. Action on a cotangent fibre.

Lemma 6.1. Let \( \iota_\mathbb{R} : \mathbb{R}^n \to T^* S^n \) be a parametrisation of the cotangent fibre \( T_x S^n \) and let \( \phi_s, s \in I^k \), be a \( k \)-parameter family of diffeomorphism of \( S^n \) supported in a ball \( U \subset S^n \setminus \{x\} \). When \( k \geq 1 \) we require this family to be trivial over \( \partial I^k \). It follows that there is a compactly-supported family \( \psi_s \) of symplectomorphisms which is trivial in \( \pi_k(\text{Symp}(T^* S^n)) \) such that

- the symplectomorphism \( C_s = \psi_s \tau^{-1} \phi_s^* \tau(\phi_s^{-1})^* \) preserves \( \iota_\mathbb{R}(\mathbb{R}^n) \), and

- the diffeomorphism \( \iota_\mathbb{R}^{-1} \circ \iota_\mathbb{R} \circ \iota_\mathbb{R} : \mathbb{R}^n \to \mathbb{R}^n \) is a family of compactly-supported diffeomorphism which, when extended to \( S^n = \mathbb{R}^n \cup \{\infty\} \), is isotopic to the family \( \phi_s \).

Proof. We prove the statement when \( k = 0 \); the general case is similar. Let \( L_1 \) denote \( \iota_\mathbb{R}(\mathbb{R}^n) \) and \( L_2 \) denote the zero-section.

First, the domain of \( (\phi^{-1})^* \) is disjoint from \( L_1 \), so \( (\phi^{-1})^* \) fixes \( L_1 \) pointwise.

Second, we apply the Dehn twist \( \tau \). After a Hamiltonian isotopy \( \psi' \) we can assume that \( \psi'((\tau(L_1)) \cap \text{supp}(\phi^*) = U \subset S^n \) is contained in the zero-section. See Figure

Third, \( \phi^* \) fixes \( \psi'((\tau(L_1)) \) setwise because \( \psi'((\tau(L_1)) \cap \text{supp}(\phi^*) \) is contained in the zero-section, which is obviously fixed setwise by \( \phi^* \).

Fourth, \( \tau^{-1}(\psi') \) sends

\[
\phi^*(\psi'((\tau(L_1))) = \psi'((\tau(L_1)))
\]

to \( L_1 \). Thus we see that

\[
C = \tau^{-1}(\psi')^{-1} \phi^* \psi'((\phi^{-1})^*\tau^{-1}
\]
EXOTIC SPHERES AND THE TOPOLOGY OF SYMPLECTOMORPHISM GROUPS

\[ W \subset T^*S^n \]

\[ L_1 \cap W = T^*_xS^n \cap W \]

\[ L_1 = L_2 = S^n \]

**Figure 2.** The cotangent bundle \( T^*S^n \), where \( L_1 \) is the cotangent fibre \( T^*_xS^n \) for a point \( x \) in the complement of \( \text{supp} \phi \) and \( L_2 \) is the zero-section. After an isotopy \( \psi' \) we can assume that \( \psi'(\tau(L_1)) \cap \text{supp}(\phi^*) = U \subset S^n \) is contained in the zero-section. It follows that \( C(L_1) = L_1 \) setwise, where \( C = \tau^{-1}(\psi')^{-1} \phi^* \psi' \tau(\phi^{-1})^* = \psi \tau^{-1} \phi^* \tau(\phi^{-1})^* \) for some Hamiltonian symplectomorphism \( \psi \).

fixes \( L_1 \) setwise. Since \( \text{Ham} \) is a normal subgroup of \( \text{Symp} \), we can move \( \psi' \) and \( (\psi')^{-1} \) to the left and concatenate them into a single compactly-supported Hamiltonian isotopy \( \psi \), so

\[ C = \psi \tau^{-1} \phi^* \tau(\phi^{-1})^*. \]

Post-composing \( C \circ \iota_x \) with \( \iota_x^{-1} \) gives a diffeomorphism of \( \mathbb{R}^n \). At the third stage, the parametrisation changed by \( \phi \), so \( \iota_x^{-1} \circ C \circ \iota_x \) is isotopic (as a diffeomorphism of \( S^n = \mathbb{R}^n \cup \{\infty\} \)) to \( \phi \).

6.2. Case \( k = 0 \).

**Proof of Theorem C when \( k = 0 \).** Without loss of generality, we work with the \( A_m \)-Milnor fibre with \( m \geq 2 \); if the theorem holds for \( m \geq 2 \) then in particular \( \tau_1 \) and \( \tau_{0\phi} \) are not isotopic inside \( A^n_0 \) or they would be isotopic in a Weinstein neighbourhood. We will also assume that \( \ell(S^n) = L_2 \); the matching spheres are all conjugate under the group of compactly-supported symplectomorphisms.

Suppose that \( C' = \tau_{0\phi}^{-1} \circ \tau_{0\phi} \) is Hamiltonian isotopic to the identity. We will study the effect of \( \tau_{0\phi}^{-1} \circ \tau_{0\phi} \) on parametrisations of the sphere \( L_1 \) to derive a contradiction. Pick a parametrisation \( \kappa \) of \( L_1 \).

Consider a Weinstein neighbourhood \( W \) of \( \ell \) where \( L_1 \cap W \) coincides with the cotangent fibre at \( x \in L_1 \cap L_2 \). In this Weinstein neighbourhood, \( C' \) is given by the
commutator $\tau^{-1} \phi^* \tau (\phi^{-1})^*$. We may realise the mapping class of $\phi$ by a diffeomorphism of $L_2$ supported away from $x$. By Lemma 6.1 we can find a Hamiltonian isotopic symplectomorphism $C = \psi \circ C'$ which preserves $L_1$ setwise and changes its parametrisation to $\kappa \circ \phi$.

If $C'$ (and hence $C$) is Hamiltonian isotopic to the identity in $A_n^m$ then we can take the Lagrangian suspension of $L_1$ under this Hamiltonian isotopy to give a Lagrangian $S^{n+1}$ in $\mathbb{A}^n \times T^* I$, agreeing with $L_1 \times I$ near the boundary.

Proposition 4.1 tells us there is an open symplectic embedding $e : W \to T^* S^{n+1}$. This embedding has the properties that $L_1 \times I$ is identified with a subset of $S^{n+1}$ and the image is disjoint from a given cotangent fibre $\Lambda$. Using a suitable inward Liouville flow of $A_n^m \times T^* I$ preserving the product structure, we can ensure that the Lagrangian suspension $\Sigma$ lies inside $W$. We can therefore replace $e((S^n \times I) \cap U) \subset S^{n+1}$ with $e(\Sigma)$. See Figure 3 for an illustration. By construction, the result is a Lagrangian $S^{n+1}_\alpha$ in $T^* S^{n+1}$ which intersects $\Lambda$ at the single point $S^{n+1} \cap \Lambda$ transversely. Therefore $S^{n+1}_\alpha \in \mathcal{L}(n+1)$.

6.3. Case $k > 0$.

Lemma 6.2. Let $\alpha \in \pi_k(\text{Diff}(S^n))$ be a homotopy class represented by a family $\phi_s : S^n \to S^n$ of diffeomorphisms depending on a parameter $s \in I_k$ such that for $s \in \partial(I^k)$, $\phi_s = \text{id}$. Let $U \subset S^n$ be an open ball. There exists a homotopy class $\alpha' \in \pi_k(\text{Diff}(S^n))$ represented by a family $\psi_s$ of diffeomorphisms such that

- $\psi_s = \text{id}$ for $s \in \partial(I^k)$,
- $\psi_s|_U = \text{id}_U$ for all $s \in I^k$,
- $\psi_s$ is homotopic rel $\partial(I^k)$ to $\phi_s \circ r_s$ for some family of rotations $r_s \in SO(n+1)$.
Proof. Let $P \to S^n$ denote the principal $GL^+(n, \mathbf{R})$-frame bundle of $S^n$. Fix a point $x \in S^n$ and a frame $p \in P_x$. There is an evaluation map $ev : \text{Diff}(S^n) \to P$ which sends $\phi$ to $\phi_x p$. The space $P$ is homotopy equivalent to $SO(n + 1)$ and the composite map $F : SO(n + 1) \subset \text{Diff}(S^n) \to P \to SO(n + 1)$ is a homotopy equivalence. The pushforward of $ev_x \alpha \in \pi_k(P)$ is therefore equal to $F_* \beta$ for some $\beta \in \pi_k(SO(n + 1))$. Let $r^{-1}_s$ be a family of rotations representing the class $\beta$, then

$s \mapsto (\phi_s \circ r^{-1}_s) p$

is nullhomotopic and $\psi_s : = \phi_s \circ r_s$ represents the class $\alpha' = \alpha \cdot \beta^{-1} \in \pi_k(\text{Diff}(S^n))$.

After some further isotopy, we can assume that $p$ is fixed by each element in the family $\psi_s$ and that a small open ball centred at $x$ is fixed. \hfill \Box

Remark 6.3. The classes $\alpha$ and $\alpha'$ differ by the class $\beta$ in $\pi_k(\text{Diff}(S^n))$, but the rotations $r_s$ extend to the disc $D^{n+1}$. This means that $S^{n+k+1}_\alpha$ and $S^{n+k+1}_{\alpha'}$ are diffeomorphic.

Remark 6.4. The model Dehn twist commutes with pullback along isometries because it is defined in terms of the geodesic flow for the round metric. Therefore if $r$ is a rotation and $\ell : S^n \to X$ is a Lagrangian embedding then $\tau_0 \circ r = r \circ \tau_0$.

Corollary 6.5. In the proof of Theorem \textcircled{C}, we can assume without loss of generality that $\alpha$ is represented by a family of diffeomorphisms $\phi_s$ which are the identity on some ball $U \subset S^n$.

Proof of Theorem \textcircled{C} when $k > 0$. Suppose that $\phi_s, s = (s_1, \ldots, s_k) \in I^k$, is a $k$-parameter family of diffeomorphisms of $S^n$ such that $\phi_s = \text{id}$ for $s$ in a neighbourhood of $\partial(I^k)$. This gives rise to a $k$-parameter family of symplectomorphisms

$C' : = \tau^{-1}_s \tau_0 \circ \phi_s, A^m_n \to A^m_n$

where $\ell : S^n \to A^m_n$ is a Lagrangian embedding whose image is a matching cycle $L_\gamma \subset A^m_n$. As for the case $k = 0$ we can assume without loss of generality that $m \geq 2$ and that the matching cycle is associated to the matching path $[1, 2] \subset C$. Furthermore, as in the case $k = 0$, the family $C' \phi$ can be seen to be Hamiltonian isotopic to a family $C_s = \psi_s \circ C'_s$ which maps the matching sphere associated to the matching path $[0, 1] \subset C$ to itself.

We will assume that $s \mapsto C'_s$, and hence $s \mapsto C_s$, is nullhomotopic in $\text{Symp}^+(T^* A^m_n)$ and derive a contradiction. Suppose $C_{(s,t)}$ is a nullhomotopy of $C_s$ satisfying $C_{(s,t)} = \text{id}$ for all $(s,t) \in \partial(I^{k+1}) \setminus (I^k \times \{1\})$, and $C_{(s,1)} = C_s$.

By Proposition \textcircled{4.1}, we have an open symplectic embedding $e : W \to T^* S^{n+k+1}$ where $W$ is a Weinstein neighbourhood of $\bigcup_{i=1}^n L_i \times I^{k+1} \subset A^m_n \times T^* I^{k+1}$. Taking the Lagrangian suspension of $L_1 \subset A^m_n$ along $C_{(s,t)}$ and pushing inwards along a suitable Liouville flow on $A^m_n \times T^* I^{k+1}$ preserving the product structure, we obtain a Lagrangian embedding $L_1 \times I^{k+1} \to W$ which agrees with the standard embedding near the boundary. Replacing $e(L_1 \times I^{k+1})$ with this Lagrangian suspension gives us a Lagrangian submanifold of $T^* S^{n+k+1}$ diffeomorphic to $S^{n+k+1}_\alpha$. This intersects a cotangent fibre once transversely, by the second property of the embedding $e$ from Proposition \textcircled{4.1}, so $S^{n+k+1}_\alpha \in \mathcal{L}(n + k + 1)$. \hfill \Box
7. SYMPLECTOMORPHISMS OF $T^*(S^n \times S^1)$

We now describe how a one-parameter family of compactly-supported symplectomorphisms $\tau^{-1} \tau_0$, of $T^* S^n$ can induce a compactly supported symplectomorphism $\Psi \in \text{Symp}^c(T^*(S^n \times S^1))$ that is not isotopic to the identity through compactly supported symplectomorphisms.

We can write $\tau^{-1} \tau_0 = \phi^t_H$, where $H_t: T^* S^n \to \mathbb{R}$ is a time-dependent Hamiltonian with compact support. The suspension

$\Phi_H: T^* S^n \times T^* I \to T^* S^n \times T^* I$,

$(x, q, p) \mapsto (\phi^t_H(x), q, p - H_q(\phi^t_H(x)))$,

is not compactly-supported, but descends along the projection

$T^* S^n \times T^* I \to T^* S^n \times I \times \mathbb{R}/\mathbb{Z}$,

$(x, q, p) \mapsto (x, q, [p])$,

to a compactly supported symplectomorphism $\Psi \in \text{Symp}^c(T^*(S^n \times I \times \mathbb{R}/\mathbb{Z})$, where we have endowed $I \times \mathbb{R}/\mathbb{Z} \subset T^* S^1$ with the standard symplectic form.

**Theorem** If the loop $\phi_t$ represents a class $\alpha \in \pi_1(\text{Diff}(S^n))$ such that $S_0^{n+2} \notin \mathcal{L}(n+2)$ then $\Psi \in \text{Symp}^c(T^*(S^n \times S^1))$ is not isotopic to the identity through compactly supported symplectomorphisms.

**Proof.** The proof is similar to the proof of Theorem (k = 1 case). If $\Psi$ were isotopic to the identity in some compact region (say contained in $T^* S^n \times I \times S^1$) then we could lift to the universal cover $T^* S^n \times I \times \mathbb{R}$ and get an isotopy of $\Phi_H$ with the identity fixing a neighbourhood of the boundary. After a Hamiltonian modification, using Lemma 6.1 the suspension of $T^* S^n \times I$ under this is a Lagrangian $e: \mathbb{R}^n \times I^2 \hookrightarrow T^* S^n \times (T^* I)^2$ which agrees with $T^* S^n \times I^2$ outside a compact subset and in a neighbourhood of $T^* S^n \times \partial(I^2)$.

There is an embedding $\iota: D^* S^n \to A_2^*$ of a disc subbundle into $A_2^*$ such that $\iota(S^n) = L_2$ and $\iota(D^* S^n) = V \subset L_1$. Note that

$L' = \iota(e(\mathbb{R}^n \times I^2) \cap (D^* S^n \times T^*(I^2)))$

agrees with $V \times I^2$ near the boundary.

Then use Proposition 4.1 embed an open neighbourhood of $(L_1 \cup L_2) \times I^2 \subset A_2^n \times T^*(I^2)$ into $T^* S^{n+2}$ so that $L_1 \times I^2$ is mapped into a subset of $S^{n+2}$. Excising $V \times I^2 \subset S^{n+2}$ and replacing it with $L'$ gives a Lagrangian $S_0^{n+2} \subset T^* S^{n+2}$ which intersects a cotangent fibre once transversely (for example a cotangent fibre based at a point outside the image of $S^n \times I^2 \subset S^{n+2}$). Thus $S_0^{n+2} \in \mathcal{L}(n+2)$. \hfill $\square$

8. SOME RESULTS IN THE SMOOTH CATEGORY

Fix a point $* \in S^n$. Let $\phi$ be a diffeomorphism of $S^n$ equal to the identity in a neighbourhood $U \ni *$ and homotopic rel $U$ (but not necessarily isotopic) to the identity. Let $\iota: S^n \to T^* S^n$ be the inclusion of the zero-section and let $\iota_*: TS^n \to \iota^* T(T^* S^n)$ be the derivative of $\iota$ considered as a morphism of bundles over $S^n$. Since $\iota$ is a Lagrangian embedding, $\iota_*$ is a Lagrangian bundle map (the image of
each fibre is a Lagrangian subspace). Since \( \phi \) is homotopic to the identity, we have an isomorphism of symplectic vector bundles \( h: \phi^*T^*(S^n) \to T^*(T^*S^n) \), equal to the identity over \( U \).

**Proposition 8.1.**

1. The map \( h \circ (i \circ \phi)_* \) is homotopic (rel \( U \)) through Lagrangian bundle morphisms to \( i_* \).

2. Let \( \phi_t \in \text{Diff}(S^n) \) be a loop based at the identity, which moreover is assumed to be homotopic through loops of continuous maps to the constant loop. The loop of maps \( \{ h_t \circ (i \circ \phi)_* \} \) is homotopic through Lagrangian bundle morphisms to the constant loop \( \{ i_* \} \).

**Proof.** (1): Since the symplectic group acts transitively on Lagrangian frames, the two maps differ by a symplectic change of coordinates, in other words a map \( \Phi: S^n \times I \to T^*(S^n) \). The homotopy class of \( \eta \) is the difference class which is the obstruction to finding a homotopy through Lagrangian bundle morphisms. If \( n \) is even then \([\eta] = 0 \) since \( \pi_n(U(n)) = 0 \). If \( n \) is odd then \( \pi_n(U(n)) \cong \mathbb{Z} \), so \([\eta]\) is either zero or has infinite order.

If we repeat the same construction with the diffeomorphisms \( \phi^k, k = 0, 1, 2, 3, \ldots \), then we obtain the difference classes \( k[\eta] \). Since mapping classes of spheres have finite order \( (n \geq 5) \) we see that \( k[\eta] = 0 \) for some \( k \), hence \([\eta] = 0 \).

(2): Consider the diffeomorphism

\[
\Phi: S^n \times I \to S^n \times I,
\]

\[
(x, t) \mapsto (\phi_t(x), t),
\]

as well as the bundle-isomorphism \( H: \Phi^*T^*(S^n \times I) \to T^*(S^n \times I) \), constructed analogously to the bundle-morphism \( h \) above. Observe that there is a natural map which takes path of Lagrangian frames over \( S^n \subset T^*S^n \) to an isotropic frame over \( S^n \times I \subset T^*(S^n \times I) \). By adjoining the tangent-vector \( \partial_\nu \), we obtain a Lagrangian frame over \( S^n \times I \). Furthermore, since the inclusion \( U(n) \subset U(n+1) \) is \((2n - 1)\)-connected, this map is a homotopy equivalence.

In other words, it suffices to show that the corresponding bundle map \( H \circ (i \circ \Phi)_* \) is homotopic to \( i_* \) relative \( S^2 \times \{0,1\} \). Since the group

\[
\pi_0(\text{Diff}(S^n \times I, S^n \times \partial I)) = \pi_0(\text{Diff}(S^{n+1}))
\]

is finite, the proof is the same as in the case (1). \( \Box \)

**Theorem.** Let \( \tau \) denote the model Dehn twist on \( T^*S^{4\ell+3}, \ell \geq 1 \), and let \( \phi \) be a diffeomorphism of \( S^{4\ell+3} \). Then \( \tau^{-1}\tau_\phi \) is isotopic to the identity by a compactly supported smooth isotopy, and hence \( \tau \) and \( \tau_\phi \) are smoothly isotopic.

We begin with some preparatory lemmas. If \( W \subset T^*S^n \) is a neighbourhood of the zero-section and \( \nu: W \to X \) is an embedding of \( W \) then there is a preferred isotopy class of diffeomorphisms of \( X \) acting as a model Dehn twist on \( \nu(W) \) and the identity elsewhere. If \( \iota: S^n \to X \) is an embedding and \( f: T^*S^n \to \nu_\iota \) is an identification of its normal bundle \( \nu_\iota \), with \( T^*S^n \) then the tubular neighbourhood theorem gives a preferred isotopy class of embeddings \( W \to X \) and hence a preferred isotopy class of Dehn twists. If \( \iota: S^n \to X \) is a totally real embedding into
an almost complex manifold then $J$ defines a preferred identification $T^*S^n \to \nu_1$, and hence we can associate a smooth Dehn twist canonically up to smooth isotopy to any totally real embedding.

This immediately implies:

**Lemma 8.2.** If $\iota_t : S^n \to X$ is a path of totally real embeddings of $S^n$ into an almost complex manifold $X$ then the Dehn twists $\tau_0$ and $\tau_1$ (associated to the totally real embeddings $\omega_0$ and $\omega_1$) are compactly-supported isotopic.

It follows from work of Haefliger [20] that if $\phi$ is a diffeomorphism of $S^n$ supported in the complement of a ball $U$ then the embeddings $\iota$ and $\iota \circ \phi$ of the zero-section are isotopic rel $U$. We will write $\iota_t$ for this isotopy. Our goal is to replace this by a path of totally real embeddings $\iota'_t$, where $\iota'_0 = \iota$ and $\iota'_1 = \iota \circ \phi$.

**Lemma 8.3.** Suppose $n = 3 \mod 4$. The space of immersions $S^n \to T^*S^n$ which agree with the standard embedding over an open ball $U \subset S^n$ is simply connected.

**Proof.** By the Hirsch-Smale h-principle for immersions [21], [36], the space of immersions $F : S^n \to T^*S^n$ is weakly homotopy equivalent (w.h.e.) to the space of injective bundle morphisms $TS^n \to F^*T(T^*S^n)$. Since $T(T^*S^n) \cong S^n \times \mathbb{R}^{2n}$ this is w.h.e. to the space of injective bundle morphisms $TS^n \to S^n \times \mathbb{R}^{2n}$. All of this remains true when immersions and bundle morphisms are fixed over $U$.

Trivialise the bundle $TS^n$ over the disc $D = S^n \setminus U$. The space of injective bundle morphisms fixed over $U$ is then identified with the space of maps $D \to V_{2n,n}$ to the Stiefel manifold of $n$-frames in $\mathbb{R}^{2n}$ such that $\partial D$ goes to a fixed basepoint. The fundamental group of this space is $\pi_{n+1}(V_{2n,n})$. When $n = 3 \mod 4$, this group vanishes [29, Table (a)]. □

**Lemma 8.4.** Suppose $n = 4\ell + 3$, $\ell \geq 1$. Along Haefliger’s isotopy $\iota_t : S^n \to T^*S^n$, the path of tangent maps $\iota_{t\star} : TS^n \to T(T^*S^n)$ is homotopic (rel its endpoints) to a path of Lagrangian bundle morphisms.

**Proof.** By Proposition 8.1 $\iota_{t_0}$ and $\iota_{t_1}$ are indeed homotopic through Lagrangian bundle morphisms $\Lambda_t (\Lambda_0 = \iota_{t_0 \star}, \Lambda_1 = \iota_{t_1 \star})$.

By the h-principle for Lagrangian immersions [25], [17], the path of bundle morphisms $\Lambda_t$ arises (after a small perturbation) as $\iota_{t\star}$ for a path $\ell_t$ of Lagrangian immersions (fixed on the open set $U$).

By Lemma 8.3 the paths $\iota_t$ and $\ell_t$ are homotopic rel their endpoints in the space of immersions fixed over $U$. Hence $\iota_{t\star}$ is homotopic to $\ell_{t\star}$. □

**Proof of Theorem 19** By Lemma 8.4 and the h-principle for totally real embeddings [19], the path $\iota_t$ of embeddings is homotopic (through paths of embeddings) to a path $\iota'_t$ of totally real embeddings $S^n \to T^*S^n$. By Lemma 8.2 this implies that $\tau$ and $\tau_\phi$ are compactly-supported isotopic, and that $\tau^{-1}\tau_\phi$ is isotopic to the identity. □
Remark 8.5. By an analogous argument to Theorem 9.1 one can prove that the loop 
\[ \tau^{-1}\tau_{\phi_t} \] is trivial in \( \pi_1(\text{Diff}(S^n)) \) if \( n = 2 \mod 4 \), using Proposition 8.1 (2) and Paechter’s computation \( \pi_{n+2}(V_{2n,n}) = 0 \) for \( n = 2 \mod 4 \).

Remark 8.6. The problem in proving Theorem 9.1 in general is to show that the identifications of \( T^*S^n \) with the normal bundles to \( t \) and \( t \circ \phi \) match along the smooth isotopy \( t \) given by Haefliger. There are \( \pi_n(\text{SO}(n)) \) possible identifications. If the isotopy \( t \) gives the identification corresponding to \( \beta \in \pi_n(\text{SO}(n)) \) then \( \phi^k \) gives the identification \( k\beta \), so if \( \pi_n(\text{SO}(n)) \) is \( k \)-torsion and the mapping class of \( \phi \) has order not dividing \( k \) then the identifications must match and the Dehn twists must be smoothly isotopic.

If \( n = 0, 1, 4, 5 \mod 8 \) (\( n \geq 8 \)) then \( \pi_n(\text{SO}(n)) \) is 2-torsion [23] so \( \tau_{\phi_t} \) is always smoothly isotopic to \( \tau \). In this way we obtain symplectomorphisms which are trivial in \( \pi_n(\text{Diff}^c(T^*S^n)) \) but nontrivial in \( \pi_0(\text{Symp}^c(T^*S^n)) \) when, additionally, \( \Theta_{n+2}/bP_{n+2} \) contains elements of order \( > 2 \) (for example when \( n = 9, 12, 17, \ldots \)). Similarly if \( n = 0, 3, 4, 7 \mod 8 \) (\( n \geq 8 \)) then \( \pi_{n+1}(\text{SO}(n)) \) is 2-torsion [23] and we obtain loops of symplectomorphisms which are contractible in \( \text{Diff}^c(T^*S^n) \) but not \( \text{Symp}^c(T^*S^n) \) whenever \( \Theta_{n+2}/bP_{n+3} \) contains elements of order \( > 2 \) (for example when \( n = 8, 11, 16, \ldots \)).

9. An exotic symplectic structure on \( T^*(S^n \times S^1) \) which is standard at infinity

By Theorem 9.1 \( \tau^{-1}\tau_{\phi_t} \in \text{Symp}^c(T^*S^n) \) is isotopic to the identity by a smooth compactly supported isotopy in the case when \( n = 4\ell + 3 \). In fact, as Lemma 9.2 shows, there is a smooth isotopy

\[
\Phi_t : T^*S^n \rightarrow T^*S^n, \\
\Phi_t = \begin{cases} 
\text{id}_{T^*S^n}, & t \leq 0, \\
\phi^t, & t \geq 1,
\end{cases}
\]

and hence, the former isotopy can be taken to be

\[ \tau^{-1}\Phi_t \tau \Phi_t^{-1}. \]

We use this isotopy to construct the “suspension”

\[
\Psi : T^*S^n \times T^*S^1 \rightarrow T^*S^n \times T^*S^1, \\
(x, q, p) \rightarrow (\tau^{-1}\Phi_p \tau \Phi_p^{-1}(x), q, p).
\]

Observe that the two-form \( \Psi^*(d\theta_{S^n \times S^1}) - d\theta_{S^n \times S^1} \) is compactly supported, even though \( \Psi_{\phi} \) is not.

**Theorem** Suppose that \( n = 4\ell + 3, \ell \geq 1 \), and let \( \Psi_{\phi} \) be as above. If \( S^{n+1} \notin L(n+1) \), then there exists no compactly supported symplectomorphism

\[
\Psi : (T^*(S^n \times S^1),d\theta_{S^n \times S^1}) \rightarrow (T^*(S^n \times S^1),\Psi^*(d\theta_{S^n \times S^1})).
\]

Furthermore, \( \Psi^*(d\theta_{S^n \times S^1}) \) is homotopic to \( d\theta_{S^n \times S^1} \) by a compactly supported homotopy through non-degenerate two-forms.
Remark 9.1. In other words, the above theorem shows that the one-parameter version of the \( h \)-principle for symplectic forms fails. Namely, if the path of non-degenerate two-forms joining \( \Psi^*_{r,t}(d\theta_{S^n \times S^1}) \) and \( d\theta_{S^n \times S^1} \) could be deformed to a path of symplectic forms relative infinity, an application of Moser’s trick would provide a compactly supported symplectomorphism between the two symplectic structures.

Proof. The symplectomorphism \( \Psi_{\phi} \circ \Psi \) lifts to a symplectomorphism of the universal cover \((T^*(S^n \times \mathbb{R}), d\theta_{S^n \times \mathbb{R}})\). There is some \( N > 0 \), such that this symplectomorphism is of the form

\[
\Psi_{\phi} \circ \Psi|_{\{p \leq -N\}} = (\id_{T^*S^n}, \id_{T^*\mathbb{R}})|_{\{p \leq -N\}},
\]

\[
\Psi_{\phi} \circ \Psi|_{\{p \geq N\}} = (\tau^{-1} \circ \tau_{\phi}, \id_{T^*\mathbb{R}})|_{\{p \geq N\}}.
\]

For convenience, let \( I \) denote the interval \([-N, N]\). The proof is similar to the proof of Theorems 9.1 and 9.3 (\( k = 0 \) case). Using Lemma 6.1 we can modify \( \Psi_{\phi} \circ \Psi \) slightly so that its composition with the standard embedding \( T^*_n \mathbb{S}^n \times \mathbb{R} \) is a new Lagrangian embedding \( e: \mathbb{R}^n \times \mathbb{R} \to T^*(\mathbb{S}^n \times \mathbb{R}) \) which agrees with \( T^*_n \mathbb{S}^n \times \mathbb{R} \) outside a compact set.

There is an embedding \( \iota: D^* \mathbb{S}^n \to \mathbb{A}^2_\mathbb{R} \) of a disc subbundle into \( \mathbb{A}^2_\mathbb{R} \) such that \( \iota(S^n) = L_2 \) and \( \iota(D^*_n \mathbb{S}^n) = V \subset L_1 \). Note that

\[
L' = \iota(e(\mathbb{R}^n \times I) \cap (D^* \mathbb{S}^n \times T^*I))
\]

agrees with \( V \times I \) near the boundary.

Then use Proposition 4.1 embed an open neighbourhood of \((L_1 \cup L_2) \times I \subset \mathbb{A}^2_\mathbb{R} \times T^*I\) into \( T^*(\mathbb{S}^{n+1}) \) so that \( L_1 \times I \) is mapped into a subset of \( \mathbb{S}^{n+1} \). Excising \( V \times I \subset \mathbb{S}^{n+1} \) and replacing it with \( L' \) gives a Lagrangian \( \mathbb{S}^{n+1} \subset T^*(\mathbb{S}^{n+1}) \) which intersects a cotangent fibre once transversely (for example a cotangent fibre based at a point outside the image of \( \mathbb{S}^n \times I \subset \mathbb{S}^{n+1} \)). Thus \( \mathbb{S}^{n+1} \in \mathcal{L}(n+1) \).

To establish the existence of the sought homotopy, we argue as follows. First, note that

\[
\omega := \Phi_{p,t}^*(d\theta_{S^n \times \mathbb{S}^1}) = ((\Phi_{p,t} \Phi_{p}^{-1})^* \pi_{T^*S^n}^*)(d\theta_{S^n}) - dq \wedge dp + \beta(p) \wedge dp,
\]

where \( \pi_{T^*S^n}: T^*(\mathbb{S}^n \times S^1) \to T^*\mathbb{S}^n \) is the natural projection, and where \( \beta(p) \) is a family of one-forms on \( T^*\mathbb{S}^n \) satisfying \( \beta(p) = 0 \) whenever \( p \leq 0 \) or \( p \geq 1 \). The two-form

\[
\omega' := ((\Phi_{p,t} \Phi_{p}^{-1})^* \pi_{T^*S^n}^*)(d\theta_{S^n}) - dq \wedge dp
\]

is non-degenerate, and coincides with \( \omega \) outside of a compact set. It can be explicitly checked that a linear interpolation \( (1-s)\omega + s\omega' \) is a compactly supported homotopy through non-degenerate two-forms.

It now remains to prove that \( \omega' \) is homotopic to \( d\theta_{S^n \times S^1} \) by a compactly supported homotopy through non-degenerate two-forms. We will show that the path

\[
(\tau^{-1} \Phi_{t} \tau_{p} \Phi_{t}^{-1})^*(d\theta_{S^n}), \quad t \in [0,1],
\]

of symplectic forms on \( T^*\mathbb{S}^n \), which are standard outside of a compact set, is homotopic through non-degenerate two-forms to \( d\theta_{S^n} \) relative endpoints \( t \in \{0,1\} \).
Write $\eta_t := \Phi_t^* (d\theta_{S^n})$. Lemma 9.2 produces a homotopy $\eta_{t,s}$ of paths relative endpoints of non-degenerate two-forms satisfying

$$
\eta_{t,0} = \eta_t,
\eta_{t,1} = d\theta_{S^n}.
$$

Using a smooth compactly supported bump-function $\rho: T^* S^n \to \mathbb{R}$ which is equal to one in some neighbourhood of the zero-section, we can construct a compactly supported homotopy

$$
\tilde{\eta}_{t,s} := \eta_{t,\rho s}, \quad \tilde{\eta}_{t,0} = \eta_t.
$$

This homotopy may thus be assumed to satisfy $\tilde{\eta}_{t,s} = \eta_{t,s}$ on any given compact neighbourhood, while $\tilde{\eta}_{t,s} = \eta_t$ holds outside of some bigger compact set. It follows that the non-degenerate two-forms $(\Phi_t^{-1})^* (\tau^* (\tilde{\eta}_{t,s}))$ all coincide with $d\theta_{S^n}$ outside of some compact set. Moreover, since $\tilde{\eta}_{t,1} = d\theta_{S^n}$ holds in some neighbourhood of the zero-section, we may assume that

$$
(\Phi_t^{-1})^* (\tau^* (\tilde{\eta}_{t,1})) = (\Phi_t^{-1})^* (\tilde{\eta}_{t,1})
$$

holds on all of $T^* S^n$.

We can thus construct our homotopy of the loop

$$
(\tau^{-1} \Phi_t \tau \Phi_t^{-1})^* (d\theta_{S^n}) = (\Phi_t^{-1})^* (\tau^* (\Phi_t^* (d\theta_{S^n}))) = (\Phi_t^{-1})^* (\tau^* (\eta_t))
$$

as follows. First, by the above, there is a compactly supported homotopy to the loop

$$
(\Phi_t^{-1})^* (\tau^* (\tilde{\eta}_{t,1})) = (\Phi_t^{-1})^* (\tilde{\eta}_{t,1}).
$$

Finally, $(\Phi_t^{-1})^* (\tilde{\eta}_{t,1})$ is homotopic to

$$
(\Phi_t^{-1})^* (\eta_{t,0}) = (\Phi_t^{-1})^* (\eta_t) = d\theta_{S^n}
$$

by a compactly supported homotopy through non-degenerate two forms.

**Lemma 9.2.** Suppose that a parametrisation $\ell: L \to T^* L$ of the zero-section is formally Lagrangian isotopic to the canonical inclusion of the zero-section. It follows that there exists a (non compactly supported) isotopy $\Phi_t: T^* S^n \to T^* S^n$ satisfying the property that $\Phi_0 = \ell^*$, while $\Phi_1 = \text{id}_{T^* L}$. The corresponding path of diffeomorphisms $\Phi_t: T^* L \to T^* L$ may furthermore be taken to satisfy the property that the path of non-degenerate two-forms $\eta_t := \Phi_t^* (d\theta_L)$ is homotopic (through non-degenerate two-forms) to $d\theta_L$ relative endpoints.

**Proof.** We start by constructing $\Phi_t$. By assumption, there exists a compactly supported smooth isotopy $F_t: T^* L \to T^* L$, $t \in [0, 1/2]$, such that $F_0 = \text{id}_L$, $F_{t/2} \circ \ell$ is the canonical inclusion, and where $F_{t/2} \circ \ell$ is the formally Lagrangian isotopy.

Consider the smooth isotopy

$$
\Phi_t := F_t \circ \ell^* : T^* L \to T^* L, \quad t \in [0, 1/2],
$$

where hence $\Phi_0 = \ell^*$. The fact that the isotopy is formally Lagrangian implies that we may assume that $\Phi_{1/2} = \text{id}_{T^* L}$ holds in some neighbourhood of the zero-section. Since the Alexander trick now can be used to show that $\Phi_{1/2}$ is smoothly...
isotopic to $\text{id}_{T^*L}$ by an isotopy supported in the complement of some neighbourhood of the zero section, we can extend the above isotopy $\Phi_t$ to an isotopy defined for $t \in [0, 1]$, and for which $\Phi_1 = \text{id}_{T^*L}$ holds on all of $T^*L$.

It remains to show that the loop $\eta_t := \Phi_t^*(d\theta_L)$ of symplectic forms satisfies the sought properties.

The fact that the isotopy is formally Lagrangian implies that there exists a path of bundle-automorphisms $A_t : TT^*L|_L \rightarrow TT^*L|_L$ covering the identity, for which

$$\Phi_t^*(d\theta_L) \circ A_t|_L = d\theta_L|_L,$$

and where $A_t$ moreover is homotopic to the identity relative endpoints. We will use $A_{t,s}$ to denote this homotopy, where

$$A_{t,0} = \text{id}_{TT^*L|_L},$$

$$A_{t,1} = A_t.$$

Choosing a metric on $L$, the induced Levi-Civita connection on the bundle $T^*L$ induces an identification $TT^*L \cong V \oplus H$, where $H$ and $V$ denotes the so-called horizontal and vertical sub-bundles, respectively. Let $\pi : T^*L \rightarrow L$ denote the canonical projection. Recall that there is a canonical identification of $V_x$ with $\pi_\ast(V_x)L$, as well as of $H_x$ with $\pi_\ast(H_x)L$, where the latter identification is induced by the tangent map $D\pi : TT^*L \rightarrow TL$.

One obtains a compatible almost complex structure $J$ satisfying $H = JV$ by defining $J(v)$ on a vertical vector $v \in V_x$ as follows (see [11, Section 5, Appendix (iii)]). Identify $v \in V_x \cong T^*\pi(x)L$ with a vector $v' \in T\pi(x)L$ using the metric and set $J(v')$ to be the horizontal lift of $v'$ to $H_x$. The vertical distribution $V$ is Lagrangian and hence the horizontal distribution is Lagrangian because $H = JV$.

Using $r_s : T^*L \rightarrow T^*L$ to denote the bundle morphism which rescales each fibre via scalar multiplication by $s$, there exists a smooth family of bundle-morphisms

$$
\begin{array}{ccc}
TT^*L & \xrightarrow{R_s} & TT^*L \\
\downarrow \pi & & \downarrow \pi \\
T^*L & \xrightarrow{r_s} & T^*L,
\end{array}
$$

covering $r_s$ and which is uniquely determined by the fact that $R_s$ preserves the sub-bundles $V$ and $H$ and, moreover, commutes with the above canonical identifications. It follows that the above bundle-morphism satisfies

$$d\theta_L \circ R_s = d\theta_L,$$

$$R_1 = \text{id}_{TT^*L},$$

$$R_0 : TT^*L \rightarrow TT^*L|_L.$$

The sought homotopy from the loop $\Phi_t^*(d\theta_L)$ to the constant loop $d\theta_L$ is now be obtained by concatenating the path

$$\Phi_t^*(d\theta_L) \circ R_{1-s}, \; s \in [0, 1],$$
with the path
\[ \Phi_s^t (d\theta_L) \circ A_{s,t} \circ R_0, \ s \in [0, 1]. \]

\[\Box\]

10. SOME RELATIONS IN THE GROUP OF EXOTICALLY CONJUGATED DEHN TWISTS

Here we derive some basic relations satisfied by the symplectic mapping classes
\[ \tau_\phi = \phi^* \tau (\phi^{-1})^*: T^* S^n \to T^* S^n \]
considered up to Hamiltonian isotopy.

**Proposition 10.1.** The generators of the subgroup
\[ \langle [\tau_\phi] \rangle_{\phi \in \text{Diff}(S^n)} \subset \pi_0(\text{Symp}_0(T^* S^n)) \]
satisfy the relations
(2) \[ [\tau_\phi]^2 [\tau_\psi] = [\tau_\psi]^2, \]
(3) \[ [\tau_\phi][\tau_\psi] = \begin{cases} 
[\tau_\psi][\tau_\phi], & n = 2k + 1, \\
[\tau_\phi][\tau_{\psi^{-1} \phi}], & n = 2k.
\end{cases} \]

**Proof.** (2): This follows from the fact that, after a Hamiltonian isotopy, the symplectomorphism
\[ (\tau_\phi)^2 = \phi^* \tau^2 (\phi^{-1})^* \]
is the identity on some neighbourhood of the zero-section, which moreover may be assumed to contain the support of \( \tau_\psi \).

(3): After a Hamiltonian isotopy, the model Dehn twist \( \tau \) can be made to coincide with \( a^* \) in a neighbourhood \( U \) of the zero-section, where \( a \in \text{Diff}(S^n) \) is the antipodal map.

First, we consider the case \( n = 2k + 1 \). Since \( a \) is isotopic to the identity for odd-dimensional spheres, after a compactly supported Hamiltonian isotopy, we may suppose that \( \tau \) is the identity in the neighbourhood \( U \) of the zero-section. The same is thus true for \( \tau_\psi = \psi^* \tau (\psi^{-1})^* \). Since \( \tau_\phi \) can be taken to have support in \( U \), the claim now follows.

We now consider the case \( n = 2k \). Since \( a \) is orientation-reversing, Lemma 5.4 implies that \( (\phi \psi^{-1})^* a (\phi \psi^{-1})^{-1} \) is isotopic to \( a \). After a compactly supported Hamiltonian isotopy, we may therefore assume that \( \tau_\psi \) coincides with
\[ \psi^* ((\phi \psi^{-1})^* a (\phi \psi^{-1}))^* (\psi^{-1})^* = \phi^* a^* (\psi^{-1} \phi \psi^{-1})^* \]
in the neighbourhood \( U \).
Since $\tau_\phi$ may be taken to have support in $U$, the claim now follows from the calculation

$$
\tau_\phi(\phi^*a^*(\psi^{-1}\phi\psi^{-1})^*) = \\
= (\phi^*\tau(\psi^{-1})^*)(\phi^*a^*(\psi^{-1}\phi\psi^{-1})^*) \\
= \phi^*\tau a^*(\psi^{-1}\phi\psi^{-1})^* \\
= \phi^*a^*(\psi^{-1}\phi\psi^{-1})^*(\psi^{-1}\phi\psi^{-1})^* \tau(\psi^{-1}\phi\psi^{-1})^* \\
= (\phi^*a^*(\psi^{-1}\phi\psi^{-1})^*)(\psi^{-1}\phi\psi^{-1})^* \tau_{\psi\phi^{-1}}
$$

where the third equality holds under the assumption that $\tau$ is defined using the round metric, for which $a$ is an isometry. \qed

11. Open Questions

There is a Lagrangian embedding $S^n \times S^1 \hookrightarrow \mathbb{C}^{n+1}$ which, using Weinstein’s Lagrangian neighbourhood theorem, induces a symplectic embedding $D^*(S^n \times S^1) \hookrightarrow \mathbb{C}^{n+1}$ of some co-disc bundle of appropriate radius.

**Question 11.1.** The compactly supported symplectomorphism $\Psi$ of $T^*(S^n \times S^1)$ defined in Section 7 induces a compactly supported symplectomorphism of $\mathbb{C}^{n+1}$ by using the above inclusion. Is this symplectomorphism isotopic to the identity through compactly supported symplectomorphisms? Since any Lagrangian can be disjoined from the support of $\Psi$ by translation, $\Psi$ cannot act nontrivially on the isotopy classes of parametrised Lagrangians, so our methods cannot detect nontriviality of $\Psi$ (this is true more generally in any subcritical Stein manifold where any compact subset can be displaced).

**Question 11.2.** Again, using the above symplectic inclusion, the symplectic structure on $T^*(S^n \times S^1)$ constructed in Section 9 when $n = 4\ell + 3$, $\ell \geq 1$, induces a symplectic structure on $\mathbb{C}^{n+1}$ which is standard outside of a compact set. Is this symplectic structure symplectomorphic to the standard one by a compactly supported diffeomorphism?

**Question 11.3.** Do the symplectomorphisms $\tau^{-1}\tau_\phi$ have infinite order in the symplectic mapping class group $\pi_0(\text{Symp}^+(T^*S^n))$? Since there are only finitely many diffeomorphism classes of homotopy sphere in each dimension, our methods cannot detect infinite order.

**Question 11.4.** Are the symplectomorphisms $\tau_\phi$ and $\tau$ of $T^*S^n$ always smoothly isotopic?

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