Post-Newtonian effects of planetary gravity on the perihelion shift

Kei Yamada, Hideki Asada
Faculty of Science and Technology, Hirosaki University, Hirosaki 036-8561, Japan

ABSTRACT
We consider a coplanar system comprised of a massive central body (a star), a less massive secondary (a planet) on a circular orbit, and a test particle on a bound orbit exterior to that of the secondary. The gravitational pull exerted on the test particle by the secondary acts as a small perturbation, wherefore the trajectory of the particle can be described as an ellipse of a precessing perihelion. While the apsidal motion is defined overwhelmingly by the Newtonian portion of the secondary’s gravity, the post-Newtonian portion, too, brings its tiny input. We explore whether this input may be of any astrophysical relevance in the next few decades. We demonstrate that the overall post-Newtonian input of the secondary’s gravity can be split into two parts. One can be expressed via the orbital angular momentum of the secondary, another via its orbital radius. Despite some moderately large numerical factors showing up in the expressions for these two parts, the resulting post-Newtonian contributions from the secondary’s gravity into the apsidal motion of the test particle turn out to be small enough to be neglected in the near-future measurements.

Key words: celestial mechanics – gravitation – planets and satellites: general – methods: analytical

1 INTRODUCTION
The perihelion shift of Mercury gave an experimental evidence for the theory of general relativity a century ago, e.g. (Will 2006). This relativistic effect is caused by the solar gravity, which is $10^{-8}$ smaller than the Newtonian ones. Substantial improvements including Cassini mission have been brought into the ephemerides, e.g. (Pitjeva 2008, 2010), where the post-Newtonian effects in the solar system have been considered numerically in a combined manner, e.g. (Maindl and Dvorak 1994; Simon et al. 1994).

We should note that the post-Newtonian effects of planetary gravity on a light-like trajectory are of astrophysical relevance. Two decades ago, the bending angle of light by a planetary mass was observed by VLBI (Treuhaft and Lowe 1991). It is interesting to investigate the post-Newtonian effects of planetary gravity on a particle motion.

We may insist, before doing explicit calculations, that even the post-Newtonian effects of the jovian gravity should be $10^{-3}$ smaller than the solar ones and thus they should be negligible. This argument seems reasonable, unless a large numerical factor comes in. For instance, $10^{-3}$ multiplied by say $6\pi^2$ is nearly five percents. It is not the best idea to use numerical simulations for the purpose of discussing the smaller post-Newtonian effects of planetary gravity separately from other major effects.

The main purpose of this paper is to clarify whether a large numerical factor is involved in the post-Newtonian effects of planetary gravity on the perihelion shift. For this purpose, we propose a model calculation that is easily accessible.

This paper is organized as follows. In section II, analytical calculations are carried out to study the post-Newtonian effects of planetary gravity on the perihelion shift. The effects are evaluated for the solar planets in section III. Section IV is devoted to the conclusion. We provide some calculations in Appendices.

Latin indices run from 1 to 3 and Greek ones run from 0 to 3. Einstein summation convention is used for both of the indices. We take the units of $G = c = 1$.

2 POST-NEWTONIAN EFFECTS OF PLANETARY GRAVITY

2.1 Effective metric by an orbiting planet
In the post-Newtonian approximation, the line element for $N$ masses is expressed in an inertial frame as
where \((x^\mu) = (t, r, \theta, \phi)\) for \(\mu = 0, 1, 2, 3\), the position of each mass \(m_A\) is denoted as \(R_A\), the relative vectors and distances are defined as \(v_A \equiv \nabla - R_A\), \(R_{AB} \equiv R_A - R_B\), \(r_A \equiv |r_A|\), \(R_{AB} \equiv |R_{AB}|\) and \(r \equiv |r|\) and the velocity is \(v_A\).

The terms in front of \(dt^2\) in the R.H.S. of Eq. (1) make a contribution as shown in the Appendix A. It is caused by the orbital angular momentum of both the central object and the secondary one. Under the above approximations, however, the orbital angular momentum of the central object is much smaller than that of the secondary one, so that it can be neglected (See Appendix B). This effect is distinct from the Lense-Thirring effect that comes from a single spinning object (Thirring and Lense [1918]).

We consider three masses \(m_A\) (\(A = 1, 2, 3\)). See Figure 1 for the configuration of the system and our notation. For simplicity, approximations are made for the system:

(i) \(m_1 \gg m_2 \gg m_3\),
(ii) \(\ell << a\) (\(R_0 \ll r\) at the third body position),
(iii) on the same orbital plane (coplanar),
(iv) a circular motion of the secondary mass, where the orbital radius of the secondary object is denoted as \(\ell\) and the semi-major axis of the third mass orbit is denoted as \(a\). The approximation (i) implies that the third body can be treated as a test particle in the spacetime produced by the two-body system of \(m_1\) and \(m_2\). The approximation (ii) suggests that the orbital period of the third body is much longer than that of the secondary one according to Kepler’s third law. The approximation (iv) means that the primary mass is also moving on circular orbit around the common center of mass, because \(m_3\) is a test mass.

The post-Newtonian center of mass may affect the multipole expansion. Fortunately, the correction by the post-Newtonian center of mass vanishes for the circular motion (Lincoln and Will [1990]).

The slowly-moving body feels the gravitational field that is averaged over fast-changing fluctuations caused by the moving primary and secondary objects. For discussing the motion of the third body, therefore, it is convenient to use the time averaging \(< \cdot >\equiv P^{-1} \int dt\) for the orbital period \(P\). For a circular motion, this averaging is the same as the angular one \(< \cdot >\equiv (2\pi)^{-1} \int d\theta\). This simplifies the following calculations. The averaging is denoted simply as \(< \cdot >\).

Under the approximations (i)-(iv) with the averaging, the averaged metric becomes

\[
< ds^2 > = \left[ -1 + \frac{2m_{tot}}{r} \left( 1 - \frac{m_2^2}{2r^3} \right) + \frac{1}{2} \frac{m_2^2}{r^3} - \frac{2m_2^2}{r^3} - \frac{m_{tot}m_2^2}{r^3} \right] dt^2 + \left[ 1 + \frac{2m_{tot}}{r} + \frac{1}{2} \frac{m_2^2}{r^3} \right] \delta_{ij} dx^i dx^j. \tag{2}
\]

The calculations for deriving this are given in the Appendix C, where our estimate shows that the post-Newtonian effects stemming from the circular motion of the primary can be neglected.

Let us introduce the polar coordinates as \(ds^2 = r^2 (d\theta^2 + \sin^2 \theta d\phi^2) = \delta_{ij} dx^i dx^j\). For a radial coordinate \(R\) to be the circumference radius, we introduce the isotropic coordinates as

\[
R^2 \equiv \left( 1 + \frac{2m_{tot}}{r} + \frac{1}{2} \frac{m_2^2}{r^3} \right) r^2. \tag{3}
\]

By using this transformation, we reach the effective metric acting on the third mass as

\[
< ds^2 > = \left( -1 + \frac{2m_{tot}}{R} \left( \frac{m_2^2}{2R^3} + \frac{m_{tot}m_2^2}{R^4} - \frac{m_{tot}m_2^2}{\ell^3 R} \right) \right) dt^2 + \left( 1 + \frac{2m_{tot}}{R} + \frac{3m_2^2}{2R^3} \right) dR^2 + R^2 d\phi^2
\]

\[
= \left[ -1 + \frac{r_s}{R} \left( 1 - \frac{Q}{R^3} \right) + \frac{Q}{R^3} + \frac{r_s Q}{R^4} \right] dt^2 + \left( 1 + \frac{r_s}{R} + \frac{3Q}{R^3} \right) dR^2 + R^2 d\phi^2, \tag{4}
\]

where we have used \(\theta = \pi/2 = \text{const.}\) for the coplanar configuration. Here, we defined the effective Schwarzschild radius as \(r_s = 2m_{tot}\) and the effective moment induced by the secondary body as

\[
Q \equiv \frac{m_2^2}{2}. \tag{5}
\]

It follows that the line element given by Eq. (1) agrees with the Schwarzschild metric on the equatorial plane if \(Q = 0\).

### 2.2 Motion in the effective potential

Equation (1) gives the Lagrangian for the third body as

\[
L = - \left[ 1 - \frac{r_s}{R} \left( 1 - \frac{Q}{R^3} \right) - \frac{Q}{R^3} - \frac{r_s Q}{R^4} \right] \frac{1}{2} \left( 1 - \frac{Q}{R^3} \right) \frac{r_s Q}{R^4} \right] \dot{t}^2 + \left( 1 + \frac{r_s}{R} + \frac{3Q}{R^3} \right) \dot{R}^2 + R^2 \dot{\phi}^2, \tag{6}
\]

where the dot denotes the derivative with respect to the proper time of the test mass as \(d/d\tau\).

We get constants of motion for this Lagrangian as

\[
\varepsilon \equiv \left[ 1 - \frac{r_s}{R} \left( 1 - \frac{Q}{R^3} \right) - \frac{Q}{R^3} - \frac{r_s Q}{R^4} \right] \dot{t}, \tag{7}
\]

\[
j \equiv R^2 \dot{\phi}. \tag{8}
\]

Substituting these constants into Eq. (6) leads to the energy
integral as
\[
\left(1 + \frac{2Q}{R^3} \right) R^2 - \frac{r_s}{R} \left(1 - \frac{Q}{R^3}\right) - \frac{Q}{R^3} - \frac{r_s Q}{R^4} + \frac{j^2}{R^2} \left(1 - \frac{r_s}{R} - \frac{Q}{R^3}\right) = \varepsilon^2 - 1.
\] (9)

Note that this includes both the Newtonian effect with a quadrupole potential (expressed as \(-Q/R^3\) at the third term of Eq. (9)) and the Schwarzschild case at the first post-Newtonian order.

It is convenient to introduce
\[
u \equiv \frac{1}{R}.
\] (10)
Using \(dR/d\tau = (dR/d\varphi)(d\varphi/d\tau)\) and Eq. (5) for Eq. (9), we formally obtain
\[
\left(\frac{du}{d\varphi}\right)^2 = F(u),
\] (11)
where \(F(u)\) is a function of \(u\) with coefficients including \(r_s\) and \(Q\).

### 2.3 Relativistic perihelion shift at \(O(Q)\)

Let \(r_a\) and \(r_p\) denote the aphelion and perihelion in the Keplerian orbit of the third mass (the semi-major axis \(a\) and the eccentricity \(e\)). We define \(u_a = r_a^{-1}\) and \(u_p = r_p^{-1}\), which are related with the orbital elements in the Newtonian limit as
\[
u_a = \frac{1}{a(1+e)},
\] (12)
\[
u_p = \frac{1}{a(1-e)}.
\] (13)

The perihelion shift for \((r = r_p \rightarrow r_a \rightarrow r_p)\) is written as (Weinberg 1972)
\[
\Delta \varphi = 2 \int_{r_p}^{r_a} d\varphi - 2\pi, \quad \text{where the factor 2 comes from the fact that the averaged metric by Eq. (1)}
\]
is axisymmetric.

By integrating Eq. (11), the perihelion shift of the third body \((u_a \rightarrow u_p \rightarrow u_a)\) is expressed at \(O(Q)\) as
\[
\Delta \varphi_Q = \left(24 + \frac{171}{2} e^2 + O(e^4)\right) \frac{\pi Q}{a^3}.
\] (14)
Note that \(r \gg R_A\) leads to \(e \ll 1\) and hence the above expression is valid for small eccentricity \(e \ll 1\). Eq. (14) yields the perihelion shift rate as
\[
\dot{\varpi}_Q = \left(24 + \frac{171}{2} e^2 + O(e^4)\right) \frac{Q}{a^3} \frac{\pi}{P} = \left(6 + \frac{171}{8} e^2 + O(e^4)\right) \frac{m_2 e^2}{a^5 n},
\] (15)
where \(n = 2\pi P^{-1}\) is the mean motion of the third mass. The post-Newtonian correction to \(P\) could cause the 2PN correction to \(\dot{\varpi}\).

### 3 SOLAR PLANETS

Eqs. (14) and (15) involve not huge but moderately large numbers such as \(24\pi\) in the coefficients. Therefore, we make an estimation of the corrections for the solar planets.

The Jupiter has the largest mass and orbital angular momentum among the solar planets. First, we choose the secondary object as the Jupiter and estimate the magnitude of the jovian effects on the Saturn. Table 1 shows the plane-

| planet | \(m_2\) [kg] | \(\ell\) [AU] | \(\varpi_L\) [mas/cy] | \(\varpi_Q\) [mas/cy] |
|---------|-------------|--------------|-----------------|-----------------|
| Mercury | \(3.30 \times 10^{20}\) | 0.387 | \(-6.09 \times 10^{-7}\) | \(7.50 \times 10^{-9}\) |
| Venus   | \(4.87 \times 10^{24}\) | 0.723 | \(-1.23 \times 10^{-5}\) | \(3.86 \times 10^{-7}\) |
| Earth   | \(5.98 \times 10^{24}\) | 1.00 | \(-1.77 \times 10^{-5}\) | \(9.06 \times 10^{-7}\) |
| Mars    | \(6.42 \times 10^{24}\) | 1.52 | \(-2.35 \times 10^{-6}\) | \(2.26 \times 10^{-7}\) |
| Jupiter | \(1.90 \times 10^{27}\) | 5.20 | \(-1.28 \times 10^{-2}\) | \(7.80 \times 10^{-3}\) |

The above corrections to the perihelion shift for thirty years amount to \(\sim 1.3 \times 10^{-2}[\text{mas/cy} \times 30[\text{yr}] \sim 4 \times 10^{-3}\text{mas}]\) for the Saturn and \(\sim 1.4 \times 10^{-3}\text{mas/cy} \times 30[\text{yr}] \sim 4 \times 10^{-4}\text{mas}\) for the Venus. Therefore, the post-Newtonian effects of the planets on the orbital motion are small enough to be negligible even for near future measurements.

### 4 CONCLUSION

The post-Newtonian effects of planetary gravity do influence the perihelion shift. Analytical calculations were carried out to show that the post-Newtonian effects of planetary gravity originate from the orbital angular momentum and the orbital radius of a planet. Thereby, we argued that the post-
Newtonian effects are small enough to be negligible in the solar system even for near future measurements, though a moderately large numerical factor is involved. A study beyond the present approximations is left as future work.

Finally, let us mention a gauge issue of the present result. The metric Eq. [1] is based on the standard PN co-ordinates. Namely, the gauge freedom is fixed. The gauge choice at the post-Newtonian order makes a difference of $10^{-8}$ at most in the solar system dynamics, e.g. [Brumberg 1991, Klioner 2003]. Therefore, we should note that our result (e.g. the numerical coefficients) is dependent on the coordinates that we used.

ACKNOWLEDGMENTS

We would be grateful to M. Efroimsky for his invaluable comments on the manuscript. We would like to thank E. V. Pitjeva, Y. Itoh and H. Arakida for useful comments and discussions.

APPENDIX A: EFFECTS BY THE PLANETARY ORBITAL ANGULAR MOMENTUM

Let $h_{ij}$ denote the coefficient of $dt dx^j$ in $ds^2$ expressed by Eq. (1). The field $h_{ij}$ is generated by massive bodies 1 and 2. In the weak field approximation, $h_{ij}$ is known to take the form as [Landau and Lifshitz 1962]

$$h_{ij} = \frac{2}{r^2} (\mathbf{n} \times \mathbf{L})_j,$$  \hfill (A1)

where $\mathbf{L}$ denotes the vector for the total orbital angular momentum that we used.

The solar and planetary orbital angular momenta [kg m$^2$s$^{-1}$] with respect to the solar system’s barycenter are $1.8 \times 10^{39}$ (Sun), $9.2 \times 10^{38}$ (Mercury), $1.8 \times 10^{39}$ (Venus), $2.7 \times 10^{39}$ (Earth), $3.5 \times 10^{39}$ (Mars), $1.9 \times 10^{43}$ (Jupiter), $7.8 \times 10^{42}$ (Saturn), $1.7 \times 10^{42}$ (Uranus) and $2.5 \times 10^{42}$ (Neptune), where the solar orbital angular momentum is caused predominantly by outer planets such as the Jupiter and Saturn. The Sun’s orbital angular momentum is much smaller than that of the Jupiter (and the Saturn).

The assumption (i) leads to

$$R_1 = -\frac{m_2}{m_1} \ell + O\left(\frac{m_2^2}{m_1^4}\ell\right),$$ \hfill (C4)

where $\ell \equiv R_{23}$. Hence, we obtain $m_1 R_1^2 + m_2 R_2^2 = m_2 \ell^2$ at the lowest order, so that Eq. (C3) can be rewritten as

$$\left\langle \sum_A \frac{M_A}{r_A} \right\rangle \approx \frac{m_{\text{tot}}}{r} \frac{1}{4} \frac{m_2 \ell^2}{m_1^2} + O\left(\frac{R_A}{r}\right)^4.$$ \hfill (C6)

Similarly, under the assumptions (i) and (ii), the velocity-dependent part is averaged as

$$\left\langle \sum_A \frac{m_A v_A^2}{r_A} \right\rangle = \frac{m_{\text{tot}} m_2}{4 \ell r},$$ \hfill (C7)

where for computing this post-Newtonian part it is sufficient to make substitutions as $v_1^2 = R_1^2 \Omega_1^2 \rightarrow 0$ and $v_2^2 = R_2^2 \Omega_2^2 \rightarrow m_{\text{tot}} \ell^2$. Here, the secondary’s angular velocity is denoted as $\Omega$. Finally, the second-order part in masses is expanded as

$$\sum_A \sum_{B \neq A} \frac{m_A m_B}{r_{AB} R_{AB}} = \frac{2 m_1 m_2}{r \ell},$$ \hfill (C8)
which is not affected by the averaging.

By combining the above results, therefore, the averaged metric is expressed by Eq. (2).

REFERENCES

Brumberg, V. A. 1991, *Essential relativistic celestial mechanics*, (Bristol, England: Adam Hilger)
Klioner, S. A. 2003, Astron. J., 125, 1580
Landau, L. D., Lifshitz, E. M. 1962, *The Classical Theory of Fields* (Pergamon, New York)
Lincoln, C. W., Will, C. M. 1990, Phys. Rev. D, 42, 1123
Maindl, T. I., Dvorak, R. 1994, A&A, 290, 335
Misner, C. W., Thorne, K. S., Wheeler, J. A. 1973, *Gravitation*, (Freeman, New York)

Pitjeva, E. V. 2008, *IAU Symposium*, 248, 20 (Cambridge University, Cambridge)
Pitjeva, E. V. 2010, *IAU Symposium*, 261, 170 (Cambridge University, Cambridge)
Pitjeva, E. V. 2011, private communications
Simon, J. L., et al. 1994, A&A, 282, 663
Thirring, H., Lense, J. 1918, Phys. Zeit. 19, 156
Treuhaft, R. N., Lowe, S. T., 1991, Astron. J., 102, 1879
Weinberg, S. 1972, *Gravitation and Cosmology*, (Wiley, New York)
Will, C. M. 2006, Living Rev. Relativity, 3, 4

This paper has been typeset from a T\TeX/\LaTeX file prepared by the author.