ON THE CARDINALITY
OF EXTREMALLY DISCONNECTED GROUPS
WITH LINEAR TOPOLOGY

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Abstract. A group topology is said to be linear if open subgroups form a base of neighborhoods of the identity element. It is proved that the existence of a nondiscrete extremally disconnected group of Ulam nonmeasurable cardinality with linear topology implies that of a nondiscrete extremally disconnected group of cardinality at most $2^\omega$ with linear topology.

A topological space $X$ is said to be extremally disconnected if the closure of any open set in $X$ is open. Extremally disconnected spaces have the following obvious properties:

(1) Any two disjoint open sets in an extremally disconnected space have disjoint closures.
(2) Any open subset of an extremally disconnected space is extremally disconnected.
(3) The image of an extremally disconnected space under an open continuous map is extremally disconnected (this follows from the observation that, for any open continuous map $f: X \to Y$ and any $A \subset Y$, we have $\overline{A} = f(f^{-1}(A))$).

Yet another, not so obvious, property is the following theorem of Isbell [1]:

(4) A nondiscrete extremally disconnected space of Ulam nonmeasurable cardinality cannot be a $P$-space, i.e., must contain a nonopen $G_\delta$ set.

(Recall that a cardinal $\kappa$ is Ulam measurable if it carries a $\sigma$-complete (i.e., closed under countable intersections) nonprincipal ultrafilter on $\kappa$. The least Ulam measurable cardinal is measurable, so the nonexistence of Ulam measurable cardinals is consistent with ZFC, while the consistency of their existence cannot be derived from ZFC.)

The problem of the existence in ZFC of a nondiscrete extremally disconnected group [2] is still unsolved. However, it is known that the nonexistence of a countable nondiscrete extremally disconnected group is consistent with ZFC [3]. In this note, we prove that if there exists an extremally disconnected group of Ulam nonmeasurable cardinality which contains a countable family of open subgroups with nonopen intersection, then there

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exists a Boolean nondiscrete extremally disconnected group of cardinality at most $2^{\omega}$.

In what follows we use Malykhin’s theorem that any extremally disconnected group contains an open Boolean subgroup [4]. Recall that a Boolean group is a group in which all nonidentity elements are of order 2. It is easy to see that all such groups are Abelian. Any Boolean group $B$ can be treated as a vector space over the two-element field $\mathbb{F}_2$ (and hence all Boolean groups of the same cardinality $\kappa$ are isomorphic to each other and to the direct sum $\bigoplus \mathbb{Z}_2$ of $\kappa$ copies of the two-element group $\mathbb{Z}_2$). If $E$ is a basis of $B$, then $B$ is the set $[E]^\omega$ of all finite subsets of $E$, and the group operation of $B$ is symmetric difference: $g + h = g \triangle h$ for $g,h \in B = [E]^\omega$. The zero element of $B$ is the empty set. When considering the group operation (addition) on subsets of $B$ represented as $[E]^\omega$, we use the standard notation $+$, while for the addition of elements, we often use the symmetric difference symbol $\triangle$. Thus, for $X,Y \subset B = [E]^\omega$, we use the standard notation $+$, while $X \triangle Y = \{x \triangle y : x \in X, y \in Y\}$.

**Theorem.** Let $G$ be a Boolean extremally disconnected group with zero element 0 containing a countable family of open subgroups such that $\bigcap H_n = \{0\}$. Then $G$ has an open subgroup which admits a continuous isomorphism onto a subgroup of the countable Cartesian product of discrete countable Boolean groups with the product topology. In particular, any such group $G$ contains an open subgroup of cardinality at most $2^\omega$.

**Lemma 1.** If a topological group $G$ with identity element 1 contains open subgroups $G = H_0 \supset H_1 \supset H_2 \supset \ldots$ such that $\cap H_n = \{1\}$, then $G$ admits a continuous isomorphism onto a subgroup of the Cartesian product $\prod D_n$ of discrete groups $D_n \cong G/H_n$ with the product topology.

**Proof.** Since $\bigcap H_n = \{1\}$, the family $\mathcal{X}$ of natural homomorphisms $h_n : G \to G/H_n$ separates points of $G$. Each topological quotient $G/H_n$ is discrete (because the subgroups $H_n$ are open), and each $h_n$ is continuous. Therefore, the diagonal map $\Delta \mathcal{X}$ is a continuous monomorphism. \qed

**Lemma 2.** Let $G$ be a Boolean topological group with zero 0 which admits a continuous monomorphism $\varphi$ to the Cartesian product $\prod_{i \in \omega} D_i$ of discrete Boolean groups $D_i$ with the product topology, and let $\pi_n : \prod_{i \in \omega} D_i \to D_n$, $n \in \omega$, denote the canonical projections. Suppose that, for any neighborhood $U$ of $0$, there exists an $n$ such that $\pi_n(\varphi(U))$ is uncountable. Then there are two disjoint open sets $W_0$ and $W_1$ in $G$ such that $0 \in W_0 \cap W_1$.

**Proof.** For convenience, we identify $G$ with a subgroup of $\prod_{i \in \omega} D_i$, i.e., assume $\varphi$ to be the identity monomorphism.

Treating the groups $D_i$ as vector spaces over $\mathbb{F}_2$, we choose a basis $E_i$ in each $D_i$, so that $D_i = [E_i]^\omega$. Given a $g \in D_i$, by $|g|$ we denote its cardinality as a finite subset of $E_i$. For convenience, we denote the zero element of $D_i$ (the empty subset of $E_i$) by $0_i$. 
For each $k \in \omega$ and each $\delta \in \{0,1\}$, we set

$$W_\delta^k = \{(g_0)_{n \in \omega} \in G \cap \prod_{i \in \omega} D_i : x_i = 0, \text{ for } i < k, x_k \neq 0,$$

the number of 2's in the prime factorization of $|x_k|$ has the parity of $\delta$.}

Note that $W_\delta^k$ is a union of sets of the form $\{0_0\} \times \cdots \times \{0_{k-1}\} \times \{g\} \times \prod_{i > k} D_i \cap G$, each of which is open in $G$, because $D_k$ is discrete. Hence all $W_\delta^k$ are open. Clearly, $W_0^k \cap W_1^m = \emptyset$ for any $k,m \in \omega$. Therefore, $W_0 = \bigcup_{k \in \omega} W_0^k$ and $W_1 = \bigcup_{k \in \omega} W_1^k$ are disjoint open sets in $G$. Let us show that $0 \in \overline{W_0} \cap \overline{W_1}$.

Take any neighborhood $U$ of $0$ in $G$. We must prove that $U \cap W_0 \neq \emptyset$ and $U \cap W_1 \neq \emptyset$. Let $V$ be a neighborhood of $0$ for which $8 \in U$, and let $k$ be the least nonnegative integer for which $\pi_k(V)$ is uncountable. We have $V \subset \prod_{n \in \omega} \pi_n(V)$. Since all $\pi_i(V)$, $i < k$, are at most countable and the number of $k$-tuples of elements of countable sets is countable, it follows that there exists a $k$-tuple $(g_0, \ldots, g_{k-1}) \in \prod_{i < k} \pi_i(V)$ and an uncountable set $V' \subset V$ such that, for any $(x_n)_{n \in \omega}, (y_n)_{n \in \omega} \in V'$, $x_i = y_i = g_i$ for all $i < k$ and $x_i \neq y_i$ for all $i \geq k$. Take any $g \in V'$ and let $W = \{g + x : x \in V' \setminus \{\}\}$. Then $W \subset V + V$, $\pi_k(W)$ is uncountable, $\pi_i(W) = \{0\}$ for all $i < k$, and $0 \notin \pi_k(W)$.

The $k$th coordinates of elements of $W$ are different elements of the Boolean group $D_k$, i.e., finite subsets of its basis $E_k$. Among these finite sets are uncountably many different sets of the same cardinality $m$. By the $\Delta$-system lemma, there exists a finite $R \subset E_k$ and an uncountable $F \subset \pi_k(W)$ such that, for any $F,G \in F$, we have $|F| = |G| = m$ and $F \cap G = R$. Note that all symmetric differences $F \triangle G, F,G \in F$, are pairwise disjoint; moreover, for any different $F,G \in F$, we have

$$F \triangle G \in \pi_k(2W) \subset \pi_k(4V) \quad \text{and} \quad |F \triangle G| = 2(m - |R|) > 0.$$

Let us somehow split $F$ into two disjoint subfamilies $F'$ and $F''$. For any different $F', G' \in F'$ and any different $F'', G'' \in F''$, we have

$$F' \triangle G' \triangle F'' \triangle G'' \in \pi_k(4W) \subset \pi_k(8V)$$

and

$$|F' \triangle G' \triangle F'' \triangle G''| = 4(m - |R|).$$

Let $l$ be the number of 2's in the prime factorization of $m - |R|$. For any pairwise different $x', y', x'', y'' \in W$ such that $\pi_k(x'), \pi_k(y') \in F'$ and $\pi_k(x''), \pi_k(y'') \in F''$, we have either $x' + y' + x'' + y'' \in W_0^k$ and $x' + y' \not\in W_1^k$ (if $l$ is even) or $x' + y' + x'' + y'' \in W_1^k$ and $x' + y' \not\in W_0^k$ (if $l$ is odd). In any case, $x' + y' \in 2W \subset 4V \subset 8V \subset U$ and $x' + y' + x'' + y'' \in 4W \subset 8V \subset U$. Thus, $U \cap W_\delta^k \neq \emptyset$ for $\delta = 0,1$.

We have shown that any neighborhood of $0$ intersects both sets $W_0$ and $W_1$. This means that $0 \in \overline{W_0} \cap \overline{W_1}$. □
Proof of the theorem. Clearly, we can assume the subgroups $H_n$ in the statement of the theorem to be decreasing. By Lemma 1, there exists a continuous monomorphism $G \to \prod_{n \in \omega} D_n$, where all $D_n$ are discrete Boolean groups, and by Lemma 2 and property (1) of extremally disconnected spaces, $G$ contains a neighborhood $U$ of 0 such that $|\pi_n(U)| \leq \omega$ for all $n \in \omega$. Clearly, the subgroup $\langle U \rangle$ generated by $U$ in $G$ is open, and $\phi(\langle U \rangle) \subset \prod_{n \in \omega} \langle \pi_n(U) \rangle$, where each $\langle \pi_n(U) \rangle$ is the (at most countable) subgroup of $D_n$ generated by $\pi_n(U)$.

One of the immediate consequences of this theorem is concerned with extremally disconnected groups with linear topology.

Definition. A topological group is said to have linear topology if its open subgroups form a base of neighborhoods of the identity element.

All (consistent) examples of extremally disconnected groups known to the author have linear topology. Moreover, when the free Boolean topological group of a space with one nonisolated point is extremally disconnected, its free group topology turns out to be linear.

Corollary. If there exists a nondiscrete extremally disconnected group of Ulam nonmeasurable cardinality with linear topology, then there exists a nondiscrete extremally disconnected Boolean group with linear topology admitting a continuous isomorphism to a subgroup of $\mathbb{Z}_2^\omega$ with the product topology.

Proof. Suppose that there exists a nondiscrete extremally disconnected group of Ulam nonmeasurable cardinality with linear topology. According to Mal’tsev’s theorem mentioned above it has an open Boolean subgroup $B$. By property (2) of extremally disconnected spaces $B$ is extremally disconnected; obviously, it is linear and nondiscrete and has Ulam nonmeasurable cardinality.

In view of property (4) of extremally disconnected spaces and the linearity of $B$, there exist open subgroups $J_n$, $n \in \omega$, of $B$ whose intersection is not open (but closed, since any open subgroup is closed). The topological quotient $G = B/\bigcap_{n \in \omega} J_n$ is nondiscrete, and it is extremally disconnected by property (3) (because any quotient homomorphism of topological groups is open). Clearly, $G$ has linear topology, and its subgroups $H_n = J_n/\bigcap_{n \in \omega} J_n$ are open and satisfy the condition $\bigcap_{n \in \omega} H_n = \{0\}$ (here 0 is the zero element of $G$). By the theorem $G$ has an open (and hence nondiscrete and extremally disconnected) subgroup $H$ for which there exists a continuous monomorphism $\varphi: H \to \prod_{n \in \omega} D_n$ from $H$ to the topological product of discrete countable Boolean groups $D_n$. Clearly, the induced topology of $H$ is linear. Each group $D_n$ is isomorphic to the countable subgroup $\bigoplus^\omega \mathbb{Z}_2$ of $\mathbb{Z}_2^\omega$; let $\psi_n$ denote the corresponding isomorphism. Since the discrete topology of $D_n$ is coarser than the topology induced by the product topology, it follows that the Cartesian product $\psi = \prod_{n \in \omega} \psi_n: \prod_{n \in \omega} D_n \to \bigoplus^\omega \mathbb{Z}_2 \subset \mathbb{Z}_2^\omega$ is a continuous monomorphism. Therefore, so is $\psi \circ \varphi: G \to \mathbb{Z}_2^\omega$. □
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