Running spectral index and mode-mode correlation of inflationary perturbations from off-equilibrium effects

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We study the off-equilibrium effects of inflaton on the dynamics of primordial perturbations in the $O(N)$ model. A self-consistent off-equilibrium formalism is employed to investigate the evolution of the inflationary background field and its fluctuations with the back-reaction effects. We find two observable remains left behind the off-equilibrium processes: the running spectral index of primordial density perturbations and the correlations between perturbation modes in phase space, which would serve as the imprints to probe the epoch of inflation, even beyond.

In the inflationary scenario, the primordial perturbations of the universe originate from vacuum fluctuations of the scalar field(s), the inflaton $\phi$, driving the inflation. If the dynamics of the fluctuations is approximated by single massless free field during the inflation, the power spectrum of curvature perturbations for the mode $k$, $P_R(k)$ is roughly $k$-independent [1]. However the $k$-dependence, or the running of the spectral index, revealed by the recently released Wilkinson Microwave Anisotropy Probe (WMAP) data can be as large as $dn(k)/d\ln k = (d/d\ln k)^2\ln P_R(k) \simeq -0.055$ to $-0.077$ [2]. Hence, the origin of the primordial perturbations cannot be solely accommodated with the quantum fluctuations of single free field. In a slow-roll inflation, one has $d/d\ln k = (1/H)d/dt = -(1/8\pi G)(V'/(\sqrt{V})) (d/d\phi)$, and thus the derivative of the power spectrum $(d/d\ln k)^2\ln P_R(k)$ will no longer be negligible for the second order ($n = 2$) as long as the third or higher derivatives of the inflaton potential becomes substantial. To fit in with the running of the spectral index, models beyond single scalar field with quadratic potential $V(\phi)$ have been proposed accordingly [3]. In this context, the running index of the perturbation power spectrum is considered as an essential feature due to the interactions or the self-interactions of the inflaton(s).

It should be pointed out here that in deriving the derivative of the power spectrum above, the effective potential $V(\phi)$ is involved. However, this approach may be problematic, when in particular, the one-loop effective potential becomes complex where the background field $\phi$ is in the region of $V''(\phi) < 0$ [4]. The imaginary part of the effective potential would inevitably lead to a dynamically unstable state [5]. This so-called “spinodal instability” will allow long wavelength fluctuations to grow non-perturbatively. Therefore, the primordial perturbations must be dealt consistently with a different method to account for the amplification of vacuum fluctuations in the presence of spinodal instabilities. This motivates us to study the self-interaction effects of the inflaton on the dynamics of primordial perturbations within a context of the self-consistent off-equilibrium formalism. We will use the $O(N)$ model of inflation as an example. The $O(N)$ model with spontaneously broken symmetry has been extensively used in modelling the quantum off-equilibrium processes in the early universe, as well as the chiral phase transition in relativistic heavy ion collision [6, 7, 8, 9]. Here, we will focus on searching for the possible observable imprints caused by the off-equilibrium evolution of inflationary primordial perturbations [10].

Consider the dynamics of an inflation driven by a field $\Phi$ of the $O(N)$ vector model with spontaneous symmetry breaking. The action is defined by

$$S = \int d^4x\sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \cdot \partial_\nu \Phi - V(\Phi \cdot \Phi) \right],$$

(1)

where $V(\Phi \cdot \Phi)$ is a self-interaction potential given by

$$V(\Phi \cdot \Phi) = \frac{\lambda}{8N} \left( \Phi \cdot \Phi - \frac{2Nm^2}{\lambda} \right)^2.$$

(2)

As an inflaton, the $N$ components of the field generally are represented as $\Phi = (\sigma, \vec{\pi})$, where $\vec{\pi}$ represent $N-1$ scalar fields. The cosmic inflation is characterized by the state in which the component $\sigma$ has a spatially homogeneous expectation, i.e. $\sigma$ can be decomposed into a background field plus fluctuations around the background field as

$$\sigma(x,t) = \sqrt{N}\phi(t) + \chi(x,t)$$

(3)

with the expectation value $\langle \sigma(x,t) \rangle = \sqrt{N}\phi(t)$ and thus, $\langle \chi(x,t) \rangle = 0$. During the inflationary epoch, the background space-time can be described by a spatially flat Friedmann-Robertson-Walker metric,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)\delta_{ij}dx^i dx^j,$$

(4)
where the scale factor \( a = \exp(Ht) \) with the expansion rate \( H = \sqrt{8\pi G \rho/3} \) determined by the mean of energy density of the inflaton field, \( \rho \).

In order to take account of the growth of fluctuations due to spinodal instabilities as we will see later, we will employ the method of the Hartree factorization, which approximates the potential \( V \) with an effective quadratic potential while keeping \( N \) finite. Then the equation of motion of the background field \( \phi \) can be directly obtained from the Hartree-factorized Lagrangian by means of the tadpole condition \( \langle \chi(x,t) \rangle = 0 \) given by

\[
\ddot{\phi}(t) + 3H \dot{\phi}(t) + \left[ M^2_{\phi}(t) - \lambda \phi^2(t) \right] \phi(t) = 0 \quad (5)
\]

We then decompose \( \chi(t) \) and \( \pi(t) \) in the Fourier basis. In the Heisenberg picture, one has

\[
\chi(x,t) = \int \frac{d^3k}{8\pi^3} \left[ b_k \chi(k,t) + b_k^\dagger \chi^*(k,-t) \right] e^{i\mathbf{k} \cdot \mathbf{x}},
\]

\[
\pi_i(x,t) = \int \frac{d^3k}{8\pi^3} \left[ a_{ik} \pi_{i,k}(t) + a_{ik}^\dagger \pi_{i,-k}^*(t) \right] e^{i\mathbf{k} \cdot \mathbf{x}},
\]

where \( a_{ik}, b_k \) and \( a_{ik}^\dagger, b_k^\dagger \) are the creation and annihilation operators which obey the commutation relations. The equations of the mode functions \( f_{\chi,k}(t) \) and \( f_{\pi,k}(t) \) can be found from the Heisenberg field equations as follows:

\[
\left[ \frac{d^2}{dt^2} + 3H \frac{d}{dt} + \frac{k^2}{a^2} + M^2_{\chi}(t) \right] f_{\chi,k}(t) = 0,
\]

\[
\left[ \frac{d^2}{dt^2} + 3H \frac{d}{dt} + \frac{k^2}{a^2} + M^2_{\pi}(t) \right] f_{\pi,k}(t) = 0 \quad (7)
\]

The time dependent mass terms are given by

\[
M^2_{\chi}(t) = -m^2 + \frac{3\lambda}{2} \phi^2(t) + \frac{3\lambda}{2N} \langle \chi^2(t) \rangle
\]

\[
+ \frac{\lambda}{2} \left( 1 - \frac{1}{N} \right) \langle \psi^2(t) \rangle + \xi R,
\]

\[
M^2_{\pi}(t) = -m^2 + \frac{\lambda}{2} \phi^2(t) + \frac{\lambda}{2N} \langle \chi^2(t) \rangle
\]

\[
+ \frac{\lambda}{2} \left( 1 + \frac{1}{N} \right) \langle \psi^2(t) \rangle + \xi R, \quad (8)
\]

where \( \langle \psi^2 \rangle \) is defined by \( \langle \psi^2 \rangle = (N-1) \langle \psi^2 \rangle \). The initial conditions of the mode functions can be specified as effective free massive scalar fields in an expanding universe:

\[
f_{\chi,k}(0) = \frac{1}{\sqrt{2[k^2 + M^2_{\chi}(0)]}}, \quad \dot{f}_{\chi,k}(0) = -i \sqrt{\frac{k^2 + M^2_{\chi}(0)}{2}}; \]

\[
f_{\pi,k}(0) = \frac{1}{\sqrt{2[k^2 + M^2_{\pi}(0)]}}, \quad \dot{f}_{\pi,k}(0) = -i \sqrt{\frac{k^2 + M^2_{\pi}(0)}{2}},
\]

where we have set \( a(0) = 0 \). The values of \( M^2_{\chi}(0) \) and \( M^2_{\pi}(0) \) depends on the details of the onset of the inflation.

Finally, to close these equations self-consistently, the terms \( \langle \chi^2(t) \rangle \) and \( \langle \psi^2(t) \rangle \) in the mass-squared [Eq. (5)] can be determined by the mode functions which involve the divergences associated with the loop integrals. To get the self-consistent renormalized quantities defined as

\[
\langle \chi^2 \rangle_R(t) = \int_{t-r}^{t} dt' f_{\chi,k}(t')^2 - \frac{1}{8\pi^2} \frac{\Lambda^2}{a^2},
\]

\[
\langle \psi^2 \rangle_R(t) = \int_{t-r}^{t} dt' f_{\pi,k}(t')^2 - \frac{1}{8\pi^2} \frac{\Lambda^2}{a^2}, \quad (9)
\]

one has to absorb the divergences into renormalization of the bare mass and the bare coupling constant. However, for a weak coupling \( \lambda < 10^{-14} \) in a typical inflation model, the logarithmic divergences can be neglected.

Figure 1 plots a self-consistent solution of the inflationary background field \( \phi(t) \) and the fluctuations \( \langle \psi^2 \rangle (t) \), \( \langle \chi^2 \rangle (t) \), in which the parameters are taken to be \( h = H/m = 2.0, \lambda = 10^{-14}, N = 4, \) and \( M^2_{\chi}(0) = M^2_{\pi}(0) \approx m^2 > 0 \). In fact, the value we choose for \( H/m \) above can be determined by the energy density of the inflaton field. During the inflation, the energy density of the inflaton is dominated by that of the potential energy that leads to \( (H/m)^2 \approx N m^2 / M^2_{\chi} \lambda \approx O(1) \) for \( N \approx O(1), \lambda \approx 10^{-14} \), and the inflaton mass \( m \approx 10^{12} \) GeV. We see that \( \phi(t) \approx 0 \) and \( \langle \psi^2 \rangle (t) \approx \langle \chi^2 \rangle (t) \approx 0 \) during the first stage of the inflation as \( t < 40 m^{-1} \) in the spinodal regime where the initially positive \( M^2_{\chi} \) and \( M^2_{\pi} \) turn negative. The spinodal instabilities become important eventually, and leads to the significant growth of both \( \langle \psi^2 \rangle \) and \( \langle \chi^2 \rangle \) starting at \( t = 50 m^{-1} \). Then, the spinodal condition is only weakly satisfied over a span of \( 70 m^{-1} \leq t \leq 80 m^{-1} \) where the background \( \phi \) starts the rapid falling into the valley of the inflaton potential. Finally, the process of spinodal instability terminates at time \( t_c \) given by \( M^2_{\pi}(t_c) \approx 0 \). It renders \( t_c \approx (70-80) m^{-1} \). The background inflaton \( \phi \) sets in the minimum of a broken symmetry phase at the time \( t_c \) where \( \phi \approx \sqrt{2/\lambda m} \), which signifies the end of inflation. Therefore, the number of total inflationary e-foldings is \( N_c \approx Ht = 2mt \approx 160 \). It is evident that most of energy density of the background of the \( \phi \) field is transferred to the

![FIG. 1: The evolution of \( \sqrt{2m\langle \psi^2 \rangle} \), \( \frac{1}{2m\langle \chi^2 \rangle} \), and \( \frac{1}{2m\langle \chi^2 \rangle} \) respectively vs. \( t \) (in units of \( m^{-1} \)) for initial conditions \( \phi(0) \approx 0 \), \( \phi(0) = 0 \), and the parameters \( h = \frac{H}{m} = 2.0, \lambda = 10^{-14} \) in the case of \( N = 4 \).](image-url)
fluctuations ($\psi^2$) via the production of $\pi$ modes during the inflation. Since $M^2_{\text{pl}}(t_*) \approx 0$, these $\pi$ modes are in fact the massless Goldstone modes of the broken symmetry phase. The Goldstone theorem is fulfilled dynamically. Although the Hubble parameter $H$ is treated here as a constant, the energy density of the inflaton evolves dynamically. In fact, the time dependent $H$ can be determined by the evolution of the inflaton field $\phi(t)$ as the effects of quantum fluctuations ($\approx 10^{-5}$) on the dynamics of the Hubble parameter is negligible. The approximation of the constant $H$ can be argued to be valid when the $\phi(t)$ reaches the maximum value at $t = 70m^{-1}$. A truly self-consistent approach has to involve the correct dynamics of the Hubble parameter through the semiclassical Einstein equations as in Ref. [7]. However, the qualitative features of the above solutions are shown to remain the same.

Using the numerical solutions, we compute the power spectrum of primordial perturbations. The power spectrum of primordial perturbations is described by $P(k) = \langle |\delta k|^2 \rangle$ where the mass density perturbations are determined by the gauge invariant quantity [12]

$$\delta_k = \frac{\delta \rho}{\rho + p} \bigg|_{k = aH}.$$  \hspace{1cm} (10)

The mass density fluctuation $\delta \rho$ originates from the field fluctuations $\pi_1$ and $\chi$ given by

$$\frac{\delta \rho}{N} = \left(1 - \frac{1}{N}\right) \left[\frac{1}{2} \langle \psi^2 \rangle + \frac{1}{2a^2} \langle (\nabla \psi)^2 \rangle - \frac{1}{2} a^2 \langle \nabla \chi \rangle^2 \right]$$

$$+ \frac{\lambda}{4} \langle \psi^2 \rangle + \frac{\lambda}{4N} \langle \chi^2 \rangle + \frac{3\lambda}{4} \langle \psi \rangle^2$$

$$+ \frac{\lambda}{4} \left(1 - \frac{1}{N}\right) \langle \psi^2 \rangle \langle \chi^2 \rangle + \frac{3\lambda}{8N} \langle \chi^2 \rangle^2 \bigg].$$  \hspace{1cm} (11)

The term summing up the energy density and the pressure, $\rho + p$, in Eq. (11) can be obtained by

$$\frac{\rho + p}{N} = \delta^2 + \left(1 - \frac{1}{N}\right) \left[\langle \psi^2 \rangle + \frac{1}{a^2} \langle (\nabla \psi)^2 \rangle \right]$$

$$+ \frac{1}{N} \left[\langle \chi^2 \rangle + \frac{1}{a^2} \langle (\nabla \chi)^2 \rangle \right].$$  \hspace{1cm} (12)

The spectral index $n(k) - 1 = dP(k)/d\ln k$ and its $k$-dependence $dn(k)/d\ln k$ are computed numerically as shown in Fig. 2. The value of $n(k)$ varies from unity at the larger scales as $t < 60m^{-1}$ to about $n = 1 + 4(\nu - 3/2) \approx 1.32$ at the smaller scale as $t > 80m^{-1}$. Apparently, there is a significant index-running $dn(k)/d\ln k \approx 0.015$ in the wavelength range corresponding to the horizon-crossing times during $60 < nt < 80$. The running of the spectral index is in fact due to the energy transfer from the inflationary background field to the fluctuations which certainly can not be obtained from the classical effective potential approach.

The correlation of fluctuations caused by the off-equilibrium process is generic, because the evolution of fluctuations is typically with respect to a time-dependent background. When perturbations of scale $k$ cross the horizon at a time $t$, the equal-time two point correlation function between $(x, t)$ and $(x', t)$ will yield the correlation between two space-scale modes $(x, k)$-$(x', k)$ via the mapping formula $k = aH/t$ in Eq. (11). The discrete wavelet transform (DWT) is designed to do such scale-space [x-k] decomposition [13].

In the formulation of DWT, there are two sets of spatially localized bases given by the scaling functions $[j, l, s]$, and the wavelet functions $[j, l, w]$ both are characterized by the indices $j$ and $l$ where $j = 0, 1, 2, \ldots$ stands for a scale from $k_j$ to $k_j + \Delta k_j$ in which $k_j = 2\pi 2^j/L$ and $\Delta k_j = 2\pi 2^{j+1}/L$. The index $l$ is $0, 1, 2, \ldots L(1+1)/2$ and the location of the spatial point within $L/2^j < x < L(1+1)/2^j$. These bases are complete, and they satisfy the orthogonal relations $\langle j, l | j', l \rangle_s = \delta_{j,j'}\delta_{l,l'}$, and $\langle j, l | j', l \rangle_w = \delta_{j,j'}\delta_{l,l'}$. Then, the two-point correlation functions of the density contrast $\delta$ can be rewritten in terms of the DWT bases as

$$\langle \hat{\delta}_{j, l}\hat{\delta}_{j', l'} \rangle = \int \frac{\delta(k, t)}{(2\pi)^3} \delta \langle k, t \rangle \delta \langle -k, t \rangle \hat{\psi}_{j, l}(k) \hat{\psi}^*_{j', l'}(k),$$

$$\langle \hat{\delta}_{j, l}\hat{\delta}_{j', l'} \rangle = \int \frac{\delta(k, t)}{(2\pi)^3} \delta \langle k, t \rangle \delta \langle -k, t \rangle \hat{\phi}_{j, l}(k) \hat{\phi}^*_{j', l'}(k),$$  \hspace{1cm} (13)

where $\hat{\psi}_{j, l}(k) = \langle k|j, l \rangle_s$, and $\hat{\phi}_{j, l}(k) = \langle k|j, l \rangle_w$ are the scaling functions and the wavelet functions in the $k$-representation. The time $t$ is taken to be $t_j$ specified by the relation $2\pi 2^j/L = k = aH$. Since $a = \exp(HT)$, and one has

$$t_j = \frac{1}{H} \left[j \ln 2 + \ln \left(\frac{2\pi}{LH}\right)\right].$$  \hspace{1cm} (14)

Thus, Eq. (13) can be used to determine the correlations between fluctuations at different spatial points $I$ and $I'$, both crossing out of the Hubble horizon at the same time $t_j$ during the inflationary epoch.
As a numerical example, the normalized two-point space-scale correlations of density perturbations defined as \( \langle \delta_j(1)\delta_j(l) \rangle \) under different initial conditions are plotted in Fig. 3 in which the parameters are taken to be \( H = 2m \) and \( L = 2m^2 \). From Eq. (14), one has \( t_j \approx (1.63 - 2.33)m^{-1} \) for \( j = 4 - 6 \) which corresponds to the number of e-foldings \( Ht_j \approx 3.3 - 4.7 \). Since the inflation under consideration lasts from \( t = 0 \) to about \( t = 80m^{-1} \), the correlations in Fig. 3 actually probe the inflation dynamics at the beginning of the inflation. With \( M^2(0) = M^2(0) \approx m^2 > 0 \), the space-scale correlation function approaches zero drastically as the distance \( r \) between the two modes increases; while the correlation deviates from zero and lasts for a long range for perturbations with \( M^2 = M^2 = 0 \) initially. It indicates that the evolution of the correlation depends upon the initial conditions of the inflation. From Eq. (14), the \( \lambda \)-dependence of the mode-mode correlation can probe the evolution \( t \)-dependence of the primordial perturbations. Thus, one is capable of exploring the physics of the very early universe by means of the mode-mode correlations in phase space.

In summary, we study the off-equilibrium effects on the primordial perturbations in the O(N) model. When the interaction or self-interaction of inflaton becomes important, one must consider the evolution of the inflationary background field and its fluctuations with the back-reaction effects using a self-consistent off-equilibrium formalism. We find two observable remains left behind the off-equilibrium processes which would serve as the imprints to probe the epoch of inflation. The first one is the running spectral index of primordial perturbations. We find that the running spectral index depends essentially on the rate of the energy transfer from the background field to the inflaton fluctuations. The second remain is the correlation between phase space modes of the density perturbations. Under the influence of the self-interaction, fluctuations created from the background field are no longer white noises. Moreover, since the evolution of the correlation depends upon the initial conditions of the inflation, the mode-mode correlation of density perturbations also provides a window to study the dynamics of the self-interaction as well as the onset of the inflation. Thus, we may expect that the non-trivial mode-mode correlation in the phase space is detectable via a DWT analysis on the CMB temperature map, or other observations on the large scale structure relevant to the density perturbations [14].

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