The rise of fully turbulent flow

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Over a century of research into the origin of turbulence in wall-bounded shear flows has resulted in a puzzling picture in which turbulence appears in a variety of different states competing with laminar background flow\(^1\)–\(^6\). At moderate flow speeds, turbulence is confined to localized patches; it is only at higher speeds that the entire flow becomes turbulent. The origin of the different states encountered during this transition, the front dynamics of the turbulent regions and the transformation to full turbulence have yet to be explained. By combining experiments, theory and computer simulations, here we uncover a bifurcation scenario that explains the transformation to fully turbulent pipe flow and describe the front dynamics of the different states encountered in the process. Key to resolving this problem is the interpretation of the flow as a bistable system with nonlinear propagation (advection) of turbulent fronts. These findings bridge the gap between our understanding of the onset of turbulence\(^7\) and fully turbulent flows\(^8,9\).

The sudden appearance of localized turbulent patches in an otherwise quiescent flow was first observed by Osborne Reynolds for pipe flow\(^1\) and has since been found to be the starting point of turbulence in most shear flows\(^2,3,10\)–\(^15\). In this regime of localized turbulence it is impossible to maintain turbulence over extended regions as it automatically\(^16,17\) reduces to discrete patches, each of approximately the same size. Such patches are called puffs in the context of pipe flow (see Fig. 1a). Puffs can decay, or else split and thereby multiply. For same size. Such patches are called puffs in the context of pipe flow (see Fig. 1a). Puffs can decay, or else split and thereby multiply.

process outweighs decay, resulting in sustained disordered motion\(^7\). Although sustained, this turbulence appears only as discrete puffs surrounded by laminar flow (Fig. 1a), and larger clusters of turbulence cannot form\(^7,18\).

At flow rates larger than those sustaining the regime of localized turbulent patches, the situation is fundamentally different: once triggered, turbulence aggressively expands and eliminates all laminar motion (Fig. 1b). The flow is then fully turbulent and only in this state do wall-bounded shear flows have characteristic mean properties such as the Blasius or Prandtl–von Karman friction laws.\(^7\) This rise of fully turbulent flow has remained unexplained, despite the fact that this transformation occurs in virtually all shear flows and generally dominates the dynamics at sufficiently large Reynolds numbers. A classic diagnostic for the formation of turbulence\(^2,3,7,19,20\) is the propagation speed of the upstream and downstream fronts of a turbulent patch. We carried out such measurements for pipe and square-duct flow (Fig. 1c), focusing on the regime where turbulence first begins to expand. In both experiments, fluid enters the conduit through a smoothly contracting inlet, which ensures that, without external perturbations, flows are laminar over the Reynolds number range shown in Fig. 1. Turbulence is triggered 120d from the inlet (where d is the pipe diameter; see Methods) by a short-duration, localized perturbation. A pressure sensor at the outlet determines the subsequent arrival of first the downstream and then the upstream turbulent–laminar front. Speeds are averaged over many realizations for each R, corresponding to a total travel distance of typically 5 \times 10^4d. As an independent verification, speeds in pipe flow were determined from direct numerical simulations in pipes of length 180d, with averaging over typically 20 runs.

In both pipe and square-duct flows, initially the speeds of the downstream fronts are indistinguishable from the upstream ones, signalling localized turbulence. For R \(\lesssim 2,250\) in pipe flow and R \(\lesssim 2,030\) in duct flow, the downstream speed increases with R; these values mark the point where turbulence begins to aggressively invade the surrounding fluid. With further increases in R, the downstream front speeds exhibit complex changes of curvature as a function of R. The spreading of turbulence shows neither a square-root scaling nor an exponent associated with a percolation-type process, as proposed in earlier studies\(^2,3,21\); the speed of the downstream spreading exhibits far more complex behaviour than these theories imply.

In a previous theoretical approach\(^22\)–\(^24\), puffs in pipe flow were categorized as localized excitations, analogous to action potentials in axons, from which the numerous features of puff turbulence were captured. However, in that model, the transition leading to an expanding state is first-order (discontinuous), which does not reflect the observed continuous behaviour at the onset of fully turbulent flow (Fig. 1c). Moreover, this model did not include nonlinear advection, a feature intrinsic to fluid dynamics. We have devised an extended model incorporating an advective nonlinearity that enables us to fully capture the sequence encountered in the transformation to fully turbulent flow. The model is

\[
q_t + (u - \zeta)q_x = f(q, u) + Dq_{xx}, \quad u_t + uu_x = cg(q, u)
\]

where

\[
f(q, u) = q(r - u + 2 - (r + 0.1)(q - 1)^2),
\]

\[
g(q, u) = 2 - u + 2g(1 - u)
\]

and the subscripts denote partial derivatives. The variables q and u depend only on the streamwise coordinate x and time t. q denotes the turbulence level within the flow, which is physically representative of a cross-sectional integration of the turbulent fluctuations. u represents the centreline velocity of the fluid and plays two important roles: it accounts for nonlinear advection in the streamwise direction and captures the physical state of the shear profile, with u = 2 corresponding to parabolic flow and u < 2 to plug flow. The functions f(q, u) and g(q, u) describe, with minimal nonlinearities, the known interplay between turbulence (the excited state) and the shear profile\(^2,3,18,22\). An explicit derivation of these functions from the Navier–Stokes equations has yet to be achieved. The parameter r models the Reynolds number, \(\zeta\) accounts for the fact that turbulence is advected more slowly than the centreline velocity, D controls the coupling strength of the turbulent patches to the laminar flow (via diffusion) and \(\epsilon\) sets the timescale ratio between the fast excitation of q and the slow recovery of u following relaminarization; see Methods for details.

To elucidate the core of the transition from localized to expanding excitations, and to identify the different states occurring in the process, we carry out a standard asymptotic analysis\(^25,20\) in the limit of sharp laminar–turbulent fronts (\(\epsilon \to 0\)). Three distinct turbulent structures are predicted: a localized state (Fig. 2a), an asymmetric expanding state
Pipe axis (RESEARCH)

Equilibrium is (upper equilibrium stream front occurs at spreading rate is modest (Fig. 2b, e); the fronts themselves are not very front lags the upstream front, giving rise to a growing turbulent region bistable. Here, fully turbulent flow begins to arise. The downstream most intersection of the nullclines in Fig. 2e, f) and the system is now independent of how turbulence is triggered. c, Speeds of turbulent–laminar fronts as a function of Reynolds number for pipe flow and duct flow. Error bars give 95% confidence interval for average values (symbols). Speeds are nondimensionalized by the mean streamwise velocity \( U \). A speed difference between the upstream and downstream fronts corresponds to expanding turbulence. The arrows at the top of the left panel indicate the Reynolds numbers at which the simulations in a and b were performed.

(Fig. 2b) and a symmetric expanding state (Fig. 2c). The essence of each state is seen in the local phase plane (Fig. 2d–f). Equilibrium points are located at the intersections of the \( q \) and \( u \) nullclines (curves where time derivatives of \( u \) and \( q \) are zero). For low values of \( r \) (Fig. 2d) the only equilibrium is \( (u = 2, q = 0) \), corresponding to parabolic laminar flow. Nevertheless, the system can be excited locally; when perturbed, the state jumps to the upper branch \( q^+ \). This forms the upstream laminar-to-turbulent front. On the upper branch, \( u < 0 \) (where the overdot indicates differentiation with respect to time) and \( u \) decreases to a point where turbulence is not maintained and the system jumps back to \( q = 0 \), forming the downstream front. The downstream front follows the upstream one at a fixed distance, thus creating a localized excitation: a puff in pipe flow analogous to an action potential in excitatory media.

For larger values of \( r \), a second stable equilibrium appears (uppermost intersection of the nullclines in Fig. 2e, f) and the system is now bistable. Here, fully turbulent flow begins to arise. The downstream front lags the upstream front, giving rise to a growing turbulent region between the fronts. Initially the expansion is asymmetric and the spreading rate is modest (Fig. 2b, e); the fronts themselves are not very different in appearance from those of the localized state. The downstream front occurs at \( u < 2 \) and is formed by a drop directly from the upper equilibrium \( q = q^+ \) to \( q = 0 \) (Fig. 2e). We refer to this as the ‘weak front state’. For larger \( r \) the weak front becomes unstable, giving rise to the final state, a much more rapidly expanding ‘strong front state’ (Fig. 2c, f). The strong downstream front occurs at \( u = 2 \) and is the mirror image of the upstream front. As seen in Fig. 2c, the value of \( q \) increases above \( q^+ \) just before the drop to \( q = 0 \) at the downstream front. The downstream speed is opposite to the upstream speed with respect to what we term the ‘neutral speed’.

Before comparing the model to the experimental data, we discuss features of the front-speed scaling that are intrinsic to this model. Figure 2g shows front speeds of the three states. (From the asymptotic analysis presented in the Methods, the front speeds explicitly scale as \( \sqrt{D} \); the results in Fig. 2 are for \( D = 0.13 \).) Starting at low \( r \), excitations are strictly localized and their speed monotonically decreases with \( r \) (red curve in Fig. 2g). Expanding turbulence is first encountered when this curve intersects the weak-front curve (green in Fig. 2g). The turbulent state (upper fixed point in Fig. 2e, f) bifurcates at lower \( r \), but initially the downstream speed is smaller than the upstream one, resulting in a contraction back to a localized excitation. Thus onset of bistability and the expansion do not coincide, masking the transition and resulting in a non-standard front speed scaling (in contrast to the case without nonlinear advection shown in Extended Data Fig. 1a). The strong front (blue in Fig. 2g) is stable at slightly higher \( r \) (solid portion of the curve) and is perfectly symmetric to the downstream front (red in Fig. 2g) about the neutral speed. In the asymptotic limit \( c \rightarrow 0 \), weak and strong fronts co-exist over a range of \( r \), but for finite \( c \) the front speed continuously varies from a weak to increasingly strong front (solid black curve in Fig. 2g). During this adjustment the front speed exhibits two curvature changes. This, together with the eventual approach to the upper branch of the parabola, is a distinct signature of the scenario described by this model.

Using the theoretical model as a guide, we combine the measured front speeds from pipe and duct flow and compare them directly with theory (Fig. 3a). Initially, at lower values of \( R \), turbulent excitations are localized (as illustrated for duct flow in Fig. 3b and pipe flow in Fig. 3c) and the front speed data from both flows agree very well with the parabolic scaling predicted by the model asymptotics (solid red curve in Fig. 3a). At \( R \approx 2,250 \) in pipe flow and \( R \approx 2,050 \) in duct flow, expansion begins with the formation of the weak downstream front (illustrated for duct and pipe flow in Fig. 3c, f, respectively). Although upstream fronts of both data sets continue to follow the simple asymptotic form, the weak downstream fronts do not display the same scaling. Nevertheless, with appropriate choices of the parameters \( \zeta \) and \( \epsilon \), the model precisely captures the two curvature changes (solid black curves in Fig. 3a) encountered as each flow continuously adjusts from the weak front (green dashed line in Fig. 3a) to the strong front (blue dashed line in Fig. 3a), corresponding to the emergence of the final strong front state (Fig. 3d, g). As the downstream front...
approaches the scaling given by the strong-front asymptotics, its speed forms a parabola with the upstream front speed, a feature overlooked in previous studies.

Weak fronts move more slowly than the bulk advection velocity of turbulence; once the downstream front speed exceeds the bulk advection velocity, the front switches to a strong front. At that point, a turbulent patch invades (nearly) fully recovered laminar flow at the downstream front, in much the same way that turbulence first begins to expand (Fig. 3c, f). Fronts fluctuate, especially the downstream front, and it is common for the system to sometimes exhibit a strong and sometimes a weak downstream front. The bifurcation scenario predicted by the model is only recovered in average quantities. Likewise, turbulence for $2,250 \leq R \leq 3,000$ is not always uniform, but commonly contains intermittent laminar patches.

The simplicity of the model permits investigation of new phenomena associated with fully turbulent flow. In the model the creation of extended turbulent regions hinges on the upper intersection of the $q$ and $u$ nullclines, and by manipulating $u$ this fixed point can be destroyed (see Methods). Likewise for pipe flow an analogous profile manipulation leads to a reverse transition. As demonstrated in the Methods, fully turbulent flow is eliminated and only localized excitations remain, offering a very simple and robust way to control turbulence and to reduce frictional drag.

Although much progress has been made in our understanding of how turbulence in wall-bound flows is formed from unstable invariant solutions at moderate $R$, little to no progress has been made in connecting this transitional regime to studies of high-$R$ turbulence. Explaining the origin of the fully turbulent state is a decisive step towards connecting these regimes and paves the way for a bottom-up approach to turbulence.
The rise of fully turbulent flow. a. Front speeds as a function of Reynolds number for pipe and duct flow. Points are experimental results from Fig. 1c. Red, blue and dark green curves are front speeds in the asymptotic limit of sharp fronts in $q$ (as in Fig. 2g). The only model parameter used to fit these curves is $D = 0.13$. Black curves are the downstream front speed at finite front width ($\epsilon = 0.2$, $\zeta = 0.79$ for pipe flow and $\epsilon = 0.11$, $\zeta = 0.56$ for duct flow). The distinct weak and strong asymptotic branches (dashed) form the skeleton for the formation of fully turbulent flow, while at finite front width the model captures the complex behaviour of front speeds as a smooth switching between the asymptotic branches. b–d, Cross-stream velocity fluctuations $v^\prime/U$ for the three front states in a square duct: localized puff ($R = 1,700$), expanding turbulence with a weak downstream front ($R = 3,300$) and the strong front state ($R = 3,000$), which exhibits the characteristic energy overshoot at the downstream edge$^{15}$ (the arrows to a indicate the Reynolds number to which b–d correspond). e–g, Space–time plots from simulations of pipe flow at $R = 2,000$, $R = 2,800$ and $R = 4,500$, respectively (as indicated by the arrows to a). $\sqrt{u^2 + v^2}$ is plotted in the reference frame moving at the neutral speed. White lines indicate front speeds from the model converted to physical units. At $R = 2,000$, turbulence is localized with equal upstream and downstream front speeds. At $R = 4,500$, turbulence expands with a strong downstream front and the long-time flow is fully turbulent. The upstream and downstream fronts have the same character (compare with the symmetric overshoot in Fig. 2d) and the spreading is symmetric in the neutral reference frame. At $R = 2,800$, the downstream front moves at a speed between the weak and strong branches and exhibits some characteristics of both fronts as it fluctuates. This, as well as the intermittent laminar patches appearing within the turbulent flow, is typical of turbulence as fully turbulent flow first arises.

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Author Contributions V.M., G.L. and B.H. designed and performed the experiments and analysed the experimental results. B.S. and M.A. designed and performed the computer simulations of the Navier–Stokes equations. B.S., M.A. and B.H. analysed the numerical results. B.S. generated the corresponding visualizations. D.B. performed the theoretical analysis. D.B., B.S., V.M., G.L. and M.A. and B.H. wrote the paper.

Author Information Reprints and permissions information is available at www.nature.com/reprints. The authors declare no competing financial interests. Readers are welcome to comment on the online version of the paper. Correspondence and requests for materials should be addressed to D.B. (D.Barclay@warwick.ac.uk, for theoretical aspects) and B.H. (bhof@ist.ac.at, for experimental aspects).
METHODS

Speed measurements. Speeds of laminar–turbulent fronts were measured in experiments and highly resolved computer simulations. In both cases, long observation times were necessary to average out stochastic fluctuations that, although intrinsic to turbulence, may disguise the underlying transition scenario. All measured speeds are nondimensionalized by the mean streamwise velocity \( U \). Times are reported in units of \( d/U \) for pipe flow and \( h/U \) for duct flow, where \( d \) is the pipe diameter and \( h \) is the duct width. The corresponding Reynolds numbers for the two flows are \( R = Ud/\nu \) and \( R = Uh/\nu \), where \( \nu \) is the kinematic viscosity.

Pipe experiments. Experiments were carried out in a pipe with a diameter \( d = 10 \) mm (\( \pm 0.01 \) mm) and a length of 1,500 mm. The 15-m-long pipe was assembled on a straight aluminium base and made of precision bore glass tubes with lengths of 1–1.2 m. Customized connectors made from perspex allowed an accurate fit of the pipe segments. A specially made pipe inlet consisting of several meshes and a smooth convergence from a 100-mm-wide section to the 10-mm pipe was used to avoid inlet disturbances and eddie formation (see ref. 17 for details). In this way, the water flow could be held laminar for \( R > 8,000 \).

The laminar flow was left to develop its parabolic velocity profile over a length of 200d. At this downstream location, the flow was perturbed by an impulsive jet of water injected (for 10 ms) through a 1-mm hole in the pipe wall. The perturbed flow was left to develop into a turbulent patch over the next 250d and at this location (and from the inlet), a pressure sensor recorded the arrival of the upstream and downstream laminar–turbulent interfaces. A second sensor was located a further 1,000d downstream (50d upstream of the pipe exit), once again determining the arrival of the interfaces so that the average interface speed over the intermediate stretch of 1,000d was measured. At each Reynolds number, the measurement of the interface velocity was repeated 10 times.

The flow was gravity driven from a reservoir at a fixed height above the pipe exit. Because the turbulent fraction in the pipe is increasing over the course of a measurement, the overall drag in the pipe also increases (turbulent flow has a higher skin friction than does laminar flow). This unavoidably leads to a decrease in the flow rate (and hence \( R \)) during a measurement. To minimize this effect, a large reservoir height was chosen; in this case 23 m above the pipe exit. A precision valve positioned directly in front of the pipe inlet was used to adjust the flowrate and hence to select \( R \). For the Reynolds-number regime investigated here (\( R < 6,000 \)), the total pressure drop across the pipe is much smaller than the 23-m water head, and most of the pressure drop occurs across the valve. The increase in drag caused by the expansion of turbulence is only a small fraction (\(<0.5\%\) of the overall pressure drop) and hence, even at the highest Reynolds numbers, investigated flow rates were constant to within \(<0.5\%\) throughout the measurement.

Duct experiments. Experiments were carried out in a square duct with width \( h = 5 \) mm and a length of 1,200h (61m). The duct was made of eight perspex sections precisely machined to an accuracy of \( \pm 0.01 \) mm. They were assembled and mounted straight together on an aluminium frame. A well-designed entrance reservoir height was chosen; in this case 23 m above the pipe exit. A precision valve as the working fluid. Analogous to the pipe experiment, a precision valve was positioned directly in front of the pipe inlet was used to adjust the flowrate and hence to select \( R \). For the Reynolds-number regime investigated here (\( R < 6,000 \)), the total pressure drop across the pipe is much smaller than the 23-m water head, and most of the pressure drop occurs across the valve. The increase in drag caused by the expansion of turbulence is only a small fraction (\(<0.5\%\) of the overall pressure drop) and hence, even at the highest Reynolds numbers, investigated flow rates were constant to within \(<0.5\%\) throughout the measurement.

Numerical simulations. We consider the motion of incompressible fluid driven through a circular pipe with a fixed mass flux. Normalizing lengths with the diameter \( d \) and velocities with the mean velocity \( U \), the Navier–Stokes equations read

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla p + \frac{1}{R} \nabla \mathbf{u}, \\
\nabla \cdot \mathbf{u} &= 0,
\end{align*}
\]

where \( \mathbf{u} \) is the velocity of the fluid and \( p \) is the pressure. These equations were solved in cylindrical coordinates \((r, \theta, z, t)\) using a code developed by A. P. Willis who uses a spectral finite-difference method with no-slip boundary conditions at the pipe wall, \( \mathbf{u}(r, \theta, z, t) = 0 \) and periodicity in the axial direction. The pressure term was eliminated from the equations by using a toroidal–poloidal potential formulation of the velocity field, in which the velocity is represented by toroidal \( \psi \) and poloidal potentials \( \phi \), such that \( \mathbf{u} = \nabla \times (\psi \mathbf{\hat{r}}) + \nabla \times \phi \mathbf{\hat{\phi}} \).

After projecting the curl and double curl of the Navier–Stokes equations onto the \( x \) axis, a set of equations for the potentials \( \psi \) and \( \phi \) is obtained. A difficulty, due to the coupled boundary conditions on the potentials, is solved with an influence-matrix method. In the radial direction, spatial discretization is performed using a finite-difference method with a 9-point stencil. Assuming periodicity in azimuthal and axial directions, the potentials are expanded in Fourier modes

\[
A(r, \theta, x, t) = \sum_{k=-M}^{M} \sum_{m=-m}^{m} \hat{A}_{k,m}(r, t)e^{ikz + imm}
\]

where \( z \) and \( m \) are the wavenumbers of the modes in the axial and azimuthal directions respectively, \( 2\pi/\alpha \) fixes the pipe length \( L_p \), and \( \hat{A}_{k,m} \) is the complex Fourier coefficient of mode \((k, m)\). The time-dependent equations are integrated in time using a second-order predictor–corrector scheme with a dynamic timestep size, which is controlled using information from a Crank–Nicolson corrector step. The nonlinear term is evaluated using a pseudo-spectral technique with the de-aliasing \( \alpha \)-rule. Using the expansion in equation (3), the resultant linear differential equations for the potentials \( \psi \) and \( \phi \) decouple for each \((k, m)\) mode. This linear system is solved using LU decompositions of the resultant banded matrices; see ref. 31 for more details of the formulation and solution.

Initial conditions were prepared at \( R = 2,000 \) in 133d and 180d pipes for simulations at \( R > 2,000 \). At low \( R = 2,000 \), puff-splitting is extremely unlikely and puffs remain approximately constant in length (about 20d) as they travel downstream along the pipe. Hence, simulations at \( R = 1,910, 1,920, 2,000 \) were carried out in a shorter 24m = 75d pipe, with initial conditions prepared at \( R = 1,950 \). The lengths of the pipes and numerical resolutions used at each Reynolds number are listed in Extended Data Table 1.

The fronts were detected by setting an appropriate cut-off. Here, the local intensity was computed as

\[
\int (u_r^2 + u_t^2)d\mathbf{r}d\theta
\]

and a cut-off of \( 5 \times 10^{-4} \) was chosen for all the simulations to determine the position of the laminar–turbulent fronts. We tested different cut-off values and found that the front speed was insensitive to the chosen value.

The expansion speed of the downstream front was found to accelerate substantially during the initial stages of the simulation. To obtain the asymptotic value of the speed, we determined the length of the turbulent region \( L_t \) beyond which the speed statistics become length-independent. We found that for \( R < 4,000 \), \( L_t > 60d \) was sufficient, whereas for \( R = 4,000 \), \( L_t > 100d \) was required. This is the reason why very long pipes were used, as reported in Extended Data Table 1. At each \( R \), the speed was determined by computing \( (x_{\text{max}} - x_0)/(t_{\text{end}} - t_0) \) for each run and then averaging over a total of 20 runs. The initial time \( t_0 \) corresponds to the time at which the turbulent region has reached the length \( L_t \).

Model details. The model is a two-component system of advection-reaction-diffusion equations

\[
\begin{align*}
\frac{\partial q}{\partial t} &+ (u_r - q)|u_r| \frac{\partial q}{\partial x} = f(q, u) + D \frac{\partial^2 q}{\partial x^2}, \\
\frac{\partial u}{\partial t} &+ \frac{\partial (u_r q)}{\partial x} = g(q, u),
\end{align*}
\]

where \( q \) represents the level of turbulent fluctuations and \( u \) is the velocity at the centreline. The nonlinear reaction functions \( f(q, u) \) and \( g(q, u) \) are

\[
\begin{align*}
\frac{df}{du} &= q(\frac{q}{u} - u + 2 - (r + 0.1)(q - 1)), \\
\frac{dg}{du} &= 2u - 2q + 2q(1 - u)
\end{align*}
\]

where the parameter \( r \) corresponds to the model Reynolds number.

The model and the role of the fitting parameters \((D, I, \alpha, \epsilon)\) is most easily understood by first considering the equations in the absence of spatial derivatives. In this case, the model reduces to the ordinary differential equations (ODEs)

\[
\dot{q} = f(q, u), \quad \dot{u} = g(q, u),
\]

where the overdot indicates differentiation with respect to time. These ODEs are the core of the model as they describe the interaction between the turbulent fluctuations \( q \) and axial velocity \( u \) locally in space. The functional forms are designed to qualitatively capture the well-established physics of this interaction with minimal nonlinearities. (In a previous approach, the variable \( u \) corresponded to the axial velocity of pipe flow in the frame of reference moving at the mean or bulk velocity \( U \); here, \( u \) corresponds to velocity in the lab frame so that \( u \approx 2 \) for laminar flow.)

The nullclines for the ODEs are \( f(q, u) = 0 \) and \( g(q, u) = 0 \). For all parameter values, these nullclines intersect at the fixed point \((u = 2, q = 0)\) corresponding to laminar, Hagen–Poiseuille flow. \( \epsilon \) sets the ratio of the timescale of \( u \) relative to \( q \).
(Previously\textsuperscript{22}, two parameters $c_1$ and $c_2$ appeared in the model; here we have simplified the model to a single timescale ratio $c$, where $c_1 = c$ and $c_2 = 2c$.)

Now consider the full model equations. In addition to the local terms given by $f(q, u)$ and $g(q, u)$, the model has first and second spatial derivatives. The first-derivative terms account for nonlinear advection in the streamwise direction. For the $u$ equation, we use the advective nonlinearity that follows directly from the Navier–Stokes equations. The parameter $\zeta$ accounts for diminished advection of $q$ in comparison with the centreline velocity $u$. The streamwise velocity is maximal on the centreline and the turbulent field is not advected at this speed. We simulated turbulent flow in short pipes ($L = 12d$) and verified that turbulent structures are advected considerably more slowly than is the centreline velocity. This effect leads to complex processes in the pipe cross-section. We include in the model the simplest term that can describe the diminished advection. (Previously\textsuperscript{22}, the model contained only linear advection; the fixed difference in the advection of the $q$ and $u$ fields was expressed by an additional first-derivative term on the right-hand side of the $u$ equation, which effectively corresponded to $\zeta = 1$ in the current model.) We describe the importance of the parameter $\zeta$ after we derive expressions for front speeds in the model.

The diffusive term in equation (4) accounts for the processes by which a region of turbulent flow couples to, and thereby excites, adjacent laminar flow. The physical fields was expressed by an additional first-derivative term on the right-hand side of equation (4). The asymptotic analysis follows very closely that of refs 26, 27. Extended Data Figure 2 illustrates solutions to the boundary value problem in equation (8) in the case of a downstream front. For a fixed value of $r$, the eigenvalue $\eta$ and solution $q$ depend on $u_0$ as do the boundary conditions in equation (7).

Extended Data Figure 1 shows model front speeds as a function of model Reynolds number (Extended Data Fig. 1b is the same as Fig. 2g); speeds are from equations (10) and (11).

The neutral speed in the model is $c = 2 - \zeta$. This follows immediately from equation (10) where one can see that the upstream speed (minus sign) and strong downstream speed (plus sign) are symmetric with respect to $2 - \zeta$. This is the advection speed of turbulence in the absence of front dynamics due to transitions between laminar and turbulent flow. Without the parameter $\zeta$, the neutral speed would be the maximum centreline velocity. This is neither consistent with the observed neutral speed, nor is it reasonable that turbulent structures would be advected at the maximum speed found in the flow.

Extended Data Figure 1a shows front speeds without the inclusion of advection terms (first derivatives in $x$) in the model equations. Without these terms the front speeds become

$$c = \pm \sqrt{D} q(2, r)$$

for the strong downstream front and all upstream fronts, and

$$c = \sqrt{D} u(\eta, r)$$

for the weak downstream front. The transition to expanding turbulence is discontinuous. Including linear advection (as was done previously\textsuperscript{22}) will result in an overall shift in all front speeds, and can affect the asymptotic stability of branches, but will not change the discontinuous nature of the transition.

This highlights the role of nonlinear advection in the bifurcation scenario: without the physical effect of nonlinear advection, the weak front branch has a distinct critical point and the transition to expanding turbulence is first-order (discontinuous).

In Fig. 2a–c, solutions $q(x)$ are obtained from the full model equations (4) with $c = 0.002$, which is sufficiently small that these solutions are visually close approximations to the $c \to 0$ limit. Figure 2d–f shows the nullclines for the cases shown in Fig. 2a–2c; however, the trajectories in the phase portraits are sketches (with the fronts coloured for clarity): even at this small $c$, the jumps between the branches of $q$ are not completely vertical in the plane phase.

A further calculation determines the stability of the asymptotic branches (D.B., manuscript in preparation). The result is that the weak downstream front is stable in the asymptotic limit ($c \to 0$) if $c < u_0 = u_{\text{in}},$ whereas the strong downstream front is stable in the asymptotic limit if $c > u_0 = \frac{2}{5}$. These criteria determine the stable portions of the branches (plotted as solid) in Fig. 2g.

There are many documented exact coherent structures in pipe flow. Most of these are spatially extended, in the form of travelling waves\textsuperscript{28–30,34}, but spatially localized states have also been found\textsuperscript{22,32,36}. The model captures these states in a minimal way. The fixed points $q^*$ (one stable and one unstable) arising as the model transitions to bistability can be viewed as upper and lower branches of spatially extended travelling-wave solutions. The cubic nonlinearity in $f(q, u)$ is the minimum requirement for this separation into upper and lower branch states. The model also has localized states (puffs) and, importantly, unstable small-amplitude localized solutions (not discussed here; see refs 22, 25, 26) corresponding to edge states, both in the puff regime and in the fully turbulent regime.

Finally, we comment on what takes place at the critical point where the system first becomes bistable. As with all material in this section, the discussion follows closely refs 25, 26. Extended Data Figure 2 illustrates solutions to the boundary value problem in equation (8) in the case of a downstream front. For a fixed value of $r$, the eigenvalue $\eta$ and solution $q$ depend on $u_0$, as do the boundary conditions in equation (7).

Downstream fronts are heteroclinic connections from $q^* \to q^*$, where ‘time’ in the phase plane corresponds to space (in the reference frame co-moving at the front speed). The phase plane is two-dimensional, coordinates $q$ and $q^*$, because equation (8) is the second-order ODE. As illustrated in Extended Data Fig. 2c, for generic $u_0$, both $q^*$ and $q^*$ are hyperbolic fixed points (saddles) in the phase plane and a heteroclinic connection exists only for a unique value of $s$. This determines $s$ as a function of $u_0$ as shown by the bold curve in Extended Data Fig. 2a. However, when $u_0$ is such that $q^* = q^*$ (at the nose of the $q$ nullcline), the upper fixed point ($q = q^* = q^*$) is no longer hyperbolic and there exist infinitely many heteroclinic connections from $q^* = q^*$ to $q^*$, and hence infinitely many possible values of $s$. These appear as the thin line in Extended Data Fig. 2a.

As the parameter $r$ varies (as in Fig. 2), the nullclines vary. The critical point is where the upper-branch steady state occurs at the limit point of the $q$ nullcline, that is the upper fixed point is at $q^* = q^*$. For $r$ smaller than this value, the downstream front speed can take any of an infinite range of values because the downstream front occurs at $u_0 = u_{\text{in}}$. For a puff, this infinite range of possible values is the mechanism that allows the speed of the downstream front to select the same value as for the upstream front. As a result, puffs remain localized while travelling along the pipe. However, for $r$ larger than the critical value, the upper-branch fixed point
no longer permits downstream fronts to occur at \( u_0 = u_e \), as seen in Fig. 2e. This restricts the possible values of \( s \) and, hence, the possible speeds of the downstream front to the bold portion of the branch illustrated in Extended Data Fig. 2a. Hence, as \( r \) passes through the critical point there is an abrupt change in the allowed values of the downstream front speeds, from an infinite to a finite range. Without non-linear advection, the abrupt change is manifested as a discontinuous change in the speed of the downstream front. With nonlinear advection, there is still a discontinuous change to the allowed values of \( s \), but the speeds are smaller than those of the upstream front, so the discontinuity in allowed solutions is masked.

**Combining pipe and duct data.** To combine data from pipe and duct flow into a single plot, it is necessary to determine specific Reynolds numbers and speeds from measured data (see Fig. 3), which will then be used to align the data from the two flows. Although the procedure is informed from the model analysis, it requires only measured data and the same procedure could be applied to data from other shear flows.

Extended Data Figure 3a, c shows data from pipe and duct flow, respectively, plotted with the upstream speeds reflected about the neutral speed, labelled \( C_0 \). The value of \( C_0 \) is determined to be that for which reflected upstream data coincides with the downstream data at sufficiently large Reynolds number. Extended Data Figure 3b, d shows the same data, but with model speeds (determined subsequently) also plotted as a visual aid. In the case of pipe flow, it is possible to determine \( C_0 \) to better that 2% accuracy. For duct flow, we estimate that the downstream front speed has not quite reached the reflected upstream speed at the highest Reynolds number accessible to present experiments. Nevertheless, \( C_0 \) is quite well determined. From the same plots, the value of the Reynolds number \( R_0 \) at which the upstream front obtains the neutral speed \( C_0 \) is easily determined.

We also determine the Reynolds number \( R_1 \) from the data, where the downstream weak front first deviates from the downstream front. This can in principle be determined solely from the data, but using model fits to the weak branch gives further confidence in the determined values. \( C_1 \) is the front speed at \( R_1 \).

Once the values \( (R_0, C_0) \) and \( (R_1, C_1) \) have been found for each flow, the data is collated by plotting each data set such that these two points collapse, as seen in Extended Data Fig. 4. This is equivalent to simply choosing the origin and scaling the axes for the two flows. The upstream and strong-downstream fronts each coincide, whereas the weak-front branch does not.

**Determining model parameters.** There are three model parameters, \( D, \xi \) and \( \epsilon \), to be determined to quantitatively relate the model speeds to the measured data for each flow.

The generic model cannot be expected to predict the flow-specific values \( R_0, R_1, C_0 \) and \( C_1 \), and, moreover, there is nothing universal about these values. Instead, given these flow-specific values, the model should capture the form of the various branches seen in the combined data of Extended Data Fig. 4. When fitting model parameters, it is useful to plot the combined data in terms of the reduced Reynolds number and reduced speed

\[
\frac{R - R_0}{R_1 - R_0} = \frac{1}{2} \frac{C - C_0}{C_0 - C_1}
\]

which requires only relabelling of the axes in Extended Data Fig. 4 to shift the neutral speed \( C_0 \) to zero and scale the onset of the weak front to the point \( (R_1, C_1) = (1, -1/2) \). As will become apparent, the reason for including 1/2 in the reduced speed is so that model speeds are typically about half those of the reduced speeds from the experimental data.

We first consider the value of the parameter \( D \). We select \( D \) so as to fix a simple relationship between model and measured quantities for both flows. Specifically, in Extended Data Fig. 5a, we plot the combined pipe and duct data together with the asymptotic results from the model for different values of \( D \). The model results are plotted directly in terms of the model Reynolds number \( r \) and the model speed shifted by the model neutral speed, \( c - c_0 \). For \( D = 0.13 \) the upstream front and strong branches match the combined experimental data extremely well. Note that the strong and weak asymptotic curves in Extended Data Fig. 5a are independent of the other two model parameters, \( \epsilon \) and \( \zeta \).

Using only one parameter, \( D \), and fixing its value to 0.13, the model not only fits the upstream and strong-downstream front speeds very well for both flows, but a simple relationship between model and experimental data are fixed, namely

\[
r = \frac{R - R_0}{R_1 - R_0} \frac{1}{2} \frac{C - C_0}{C_0 - C_1}
\]

Given the flow-specific values \( R_0, R_1, C_0 \) and \( C_1 \), equation (13) is inverted to obtain the Reynolds number \( R \) and speed \( C \) from the model values \( r \) and \( c \); this is how we map the model results to Reynolds number and speed in Fig. 3.

The remaining two model parameters dictate the behaviour of the downstream fronts as they transition from weakly expanding to strongly expanding. Here pipe and duct flows differ and so the values of the fitting parameters will necessarily be different for the two flows; see Extended Data Fig. 5b.

The value of \( \epsilon \) dictates how quickly the system jumps from the weak to the strong branch. Large values give smoother transitions while smaller values give more abrupt transitions. The value of \( \zeta \) dictates how long the system follows the weak branch before transitioning to the strong branch. Larger values, as for pipe flow, result in a delay in transition, whereas smaller values, as for the fit to duct flow, result in more immediate transition. We did not apply a formal procedure for determining \( \zeta \) and \( \epsilon \) for each of the flows. Rather they were determined simply by eye. In both cases it is quite easy to adjust \( \zeta \) and \( \epsilon \) so that the transition from weak to strong fronts follows the measured data.

**Control.** The model suggests that the fully turbulent state can be destabilized by removing the upper turbulent fixed point as depicted in Extended Data Fig. 6a. In the model, this is achieved by forcing the variable \( u \), which corresponds to the state of the shear profile. The reduction of \( u \) by forcing corresponds to a blunting of the shear profile.

To demonstrate that the fully turbulent state can indeed be destabilized by removing the turbulent fixed point, as suggested by the model, we performed a direct numerical simulation of pipe flow for \( R = 5,000 \). Initially the forcing is not applied and the flow is fully turbulent. Starting at time \( t = 175 dU \), a global body force is gradually switched on (fully applied by time \( t = 200 dU \), which blunts the velocity profile to a more plug-like form (the same forcing is used as in ref. 18). As can be seen, turbulent intensity subsequently decreases, and eventually the fully turbulent flow destabilizes and degenerates into localized turbulent patches, similar to the natural ones (puffs) at lower Reynolds number (below about 2,300) in the absence of any additional force.

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Extended Data Figure 1 | Speed of model fronts in the asymptotic limit of sharp fronts. a, b, Speeds as a function of model Reynolds number \( r \) both without (a) and with (b) advection. Although strong downstream fronts cannot exist and have no physical meaning below the formation of the upper branch fixed point, the expression for strong front speeds in equation (10) still gives the speed that such a strong downstream front would have; these speeds are shown dashed. The effect of nonlinear advection in b is to mask the nominal critical point for the onset of fully turbulent flow. The neutral speed is naturally displaced from the mean speed \( U = 1 \).
Extended Data Figure 2 | Front speeds at critical point. Sketch illustrating solutions to the boundary value problem in equation (8) for a downstream front near the critical point. a, Eigenvalue $s$ as a function of $u_f$. $u_c$ is the value of $u_f$ such that $q_1 = q_2$. For this value there are infinitely many possible eigenvalues $s$, indicated by the thin line. b, c, Phase planes $(q, q')$ showing solutions for the second order differential equation (8). Downstream fronts are heteroclinic connections from the upper fixed point $q^+$ to the lower fixed point $q^0$. When $u_f = u_c$ and hence $q^- = q^+$, the upper fixed point is not hyperbolic and there are infinitely many connections, each corresponding to a value of $s$. When $u_f > u_c$, $q^+$ is hyperbolic and there is a unique connection and hence a unique value of $s$. 
Extended Data Figure 3 | Determination of corresponding Reynolds numbers and speeds for pipe and duct flow. a–d, Speeds from pipe (a, b) and duct flow (c, d) are plotted, as in Fig. 1c, but additionally with the upstream front speeds reflected about the neutral speed $C_0$. a, c, Experimental and simulation data only; b, d, model fits to the experimental and simulation data. The determined values for $R_0$, $R_1$, $C_0$ and $C_1$ are: $R_0 = 1,920$, $C_0 = 1.06$, $R_1 = 2,250$ and $C_1 = 0.92$ for pipe flow, and $R_0 = 1,490$, $C_0 = 1.12$, $R_1 = 2,030$ and $C_1 = 0.90$ for duct flow.
Extended Data Figure 4 | Combining pipe and duct data. Pipe and duct flow are plotted together using different axes. The data are plotted so that the two points \((R_0, C_0)\) and \((R_1, C_1)\) align for each data set; for example, \((R_1, C_1) = (2,250, 0.92)\) for pipe flow is aligned with \((R_1, C_1) = (2,030, 0.90)\) for duct flow, bringing into alignment the onset of weak fronts.
Extended Data Figure 5 | Determination of model parameters for pipe and duct flow. a, Determination of $D$. Points are data from pipe and duct flow (as in Extended Data Fig. 4) here plotted in terms of reduced Reynolds number $(R - R_0)/(R_1 - R_0)$ and reduced speed $(C - C_0)/(2(C_0 - C_1))$. Dashed curves are asymptotic speed curves (as in Extended Data Fig. 1) plotted in terms of model Reynolds number $r$ and speed $c - c_0$. For $D = 0.13$ there is very good agreement between the data and the model. This choice of $D$ fixes the asymptotic branches (dashed curves). b, Determination of $\zeta$ and $\epsilon$. Pipe and duct flow are necessarily considered separately. In each case, downstream branches are shown for four values of $\epsilon$. Smaller values yield more abrupt transitions between weak and strong branches.
Extended Data Figure 6 | Illustration of control by removing the turbulent fixed point. 

**a**, Control concept illustrated in the model phase plane. Without forcing (that is, without control), there is an upper-branch fixed point (upper intersection of nullclines) corresponding to fully turbulent flow. Applying an additive forcing term to the \( u \) equation corresponds to forcing the shear profile and blunting its shape. This can remove the turbulent fixed point thus eliminating fully turbulent flow. 

**b**, Proof of concept in a direct numerical simulation of pipe flow at \( R = 5,000 \). Without forcing the flow is fully turbulent. A global body force is applied that blunts the velocity profile to a more plug-like form. Subsequently, only localized turbulent patches remain, reminiscent of those at much lower \( R \).
Extended Data Table 1 | The domain size and resolution for the simulations at all the Reynolds numbers we considered

| $R$  | $L_x(d)$ | $N$  | $K$  | $M$  | $R$  | $L_x(d)$ | $N$  | $K$  | $M$  |
|------|---------|------|------|------|------|---------|------|------|------|
| 1920 | 24$\pi$ | 48   | 640  | 32   | 3200 | 180     | 72   | 2560 | 54   |
| 2000 | 24$\pi$ | 48   | 768  | 40   | 3500 | 180     | 72   | 2560 | 54   |
| 2200 | 133     | 48   | 768  | 40   | 3750 | 133     | 72   | 2048 | 54   |
| 2300 | 133     | 64   | 1536 | 40   | 4000 | 133     | 72   | 2048 | 54   |
| 2400 | 133     | 64   | 2048 | 48   | 4500 | 180     | 80   | 3072 | 64   |
| 2800 | 180     | 72   | 2560 | 48   | 5000 | 180     | 80   | 3072 | 64   |
| 2600 | 133     | 64   | 2048 | 48   | 5500 | 180     | 96   | 3840 | 80   |

In physical space there are $3K$ and $3M$ grid points in axial and azimuthal directions, respectively. $N$ is the number of grid points across the pipe radius $d/2$. 