The Acrobot

The acrobot usually refers to a 2-link underactuated robot with one end fixed at some point in the coordinate frame. It looks similar to a 2-link robot arm, but with only one actuator at the elbow. This feature makes it an underactuated robot. Underactuated robot is a family of robots which draws particular interests among engineers. They are hard to control since they are usually highly non-linear. The acrobot is one of the simplest examples in this family, as its equations of motion is obtainable by manual derivation.

In this note, we focus on the classical swing up task for acrobot. The task aims at bring the acrobot from the bottom position to the up right position. We present solutions for this task using an optimization based method with manually derived system equations. We studied two kind of acrobot: the classical 2-link acrobot and an extended version with three links. The computer programming environment we used in the experiments is MATLAB, with a open-sourced MATLAB library called OptimTraj.

2 Theoretical Preliminaries

2.1 State Space Model of Dynamical Systems

People have been using differential equations to describe nature since the establishment of morden science. These mathematical objects equating variables and their derivatives have numerous application in many different fields. Some of the most celebrated examples including heat equation describing heat diffusion in rods, equations describing the growth of population and our favorite equations describing the motion of rigid bodies.

The state space model of dynamical system is a set of first-order ordinary differential equation(ODE for short) describing the behaviour of a dynamical system. Using a set of generalized coordinates, we can reduce high-order differential equations into first-order.
\[ \dot{x} = f(x(t), u(t), t), \]

is a typical state space model. \( x \) denotes the state space variables, \( \dot{x} \) denotes their first-order derivatives, \( u \) is the control input. Control input isn’t a necessity for every system. A passive system is a system with no control input. A special form of ODE we will use in is note is ODEs with a mass matrix.

\[ M\dot{x} = f(x(t), u(t), t) \]

The mass matrix \( M \) raise when the function \( f \) no longer directly describe the derivatives of state space variables, but a weighted sum of the them. This form of ODEs are common results of motion of rigid bodies, where we directly acquire the mass matrix. We are going to use this form in the numerical solution of passive systems. However, we need to recover the standard form by calculating the inverse of mass matrix.

\[ \dot{x} = M^{-1}f(x(t), u(t), t) \]

The state space model of a dynamical is mostly non-linear, which means that the function \( f(x, u) \) is usually non-linear. The classical tools analyzing the behaviour of a non-linear systems is Lyapunov theory. In this note, we want to emphasize the trajectory optimization method for control, instead of the theoretical properties of the objective robot. So we will leave theoretical analysis of the system to study in the future.

### 2.2 The General Optimal Trajectory Design Problem

Trajectory optimization is a class of optimal control methods. We are going to give up solving optimal feedback controller, and instead using optimization tool to find a good solution for a single initial condition. The dynamics of the system is

\[ \dot{x} = f(x(t), u(t), t), \]

and initial and final conditions can be defined within some prescribed lower and upper bounds

\[ \psi_{i,l} \leq \psi_i(x(t_i), u(t_i), t_i) \leq \psi_{i,u}, \]

\[ \psi_{f,l} \leq \psi_f(x(t_f), u(t_f), t_f) \leq \psi_{f,u}. \]

In addition, we can impose simple bounds on the state variables

\[ x_l \leq x(t) \leq x_u \]

and on control variables

\[ u_l \leq u(t) \leq u_u. \]

The basic problem is to determine the control vector \( u(t) \) to minimize the performance index
\[ J = \phi(x(t_f), t_f), \]

where \( \phi \) is the pre-defined cost function. To solve the optimal trajectory design problem, we form it into a non-linear programming (NLP) problem. The general NLP problem requires finding the \( n \) vectors to solve

\[ \min_x F(x), \]

subject to \( m \) constraints

\[ c_L \leq c(x) \leq c_U, \]

and bounds

\[ x_L \leq x \leq x_U, \]

where \( F(x) \) is a scalar function, and \( c(x) \) are non-linear constraints.

### 2.3 Direct Collocation

To solve the general optimal trajectory design problem with NLP solver, we need to form a scalar cost function \( J \) over trajectories, and convert dynamic constraint into inequalities. According to method taken in this step, trajectory optimization methods are often divided into two classes: indirect method and direct method. In the direct setting, we will first discretize the state variable \( x \) and control input \( u \), then formulate an optimization over the trajectory. In this note, we use the direct method.

The idea of direct transcribing is to discretize time into intervals (usually equal intervals) as

\[ t_i, t_2, t_3, \ldots, t_f. \]

The states and controls can be discretized over the trajectory by defining \( x_k = x(t_k) \) and \( u_k = u(t_k) \). The discretized states and controls are used as decision variables of NLP problem, which is

\[ y = [x_1, u_1, \ldots, x_N, u_N]. \]

For the dynamic constraints (the equations of motion), we can form an equality constraint on every interval

\[ x_{k+1} - x_k - hf_k = 0, \]

where \( h \) is the step size of discretization (usually fixed), and \( f_k = f(x_k, u_k, t_k) \). We can use the technique of constructing a defect variable for each of these equality constraint, as

\[ \zeta_k = x_{k+1} - x_k - hf_k. \]

As a result of the transcription, the optimal control constraints can be replaced by a set of inequalities of decision variables
\[ c_L \leq c(y) \leq c_U \]

where
\[ c(y) = [\zeta_1, \zeta_2, \ldots, \zeta_{N-1}, \psi_1, \psi_N], \]

and
\[ c_L = [0, 0, \ldots, 0, \psi_{i,l}, \psi_{f,l}], \]
\[ c_U = [0, 0, \ldots, 0, \psi_{i,u}, \psi_{f,u}]. \]

Collocation is an important method used to transcribe differential dynamic constraints into a set of algebraic constraints, in order to form an NLP problem. The basic idea is to choose a polynomial up to a certain degree with a number of points with boundary values matched control and state value at knots, and to enforce the polynomials to satisfy the equations of motion at the collocation points (intermediate points between knots).

3 Mathematical Notation

There will be plenty of equations in this note, so it’s good to clarify mathematical notation before jumping into the equations. Our acrobot lives in a physical world with a gravitational acceleration \( g \). It has two massless link, and two point mass. A figure depicting our acrobot look as the follow.

In this figure, the two line segments in blue represent the massless links. We will use \( l_1 \) to represent the length of the first link (the upper one in this figure), and use \( l_2 \) for the length of the second link (the lower one in this figure). The three small circles in different colors (red, green and blue) are the ends of links. The red point is the fixed end of the first link; the green point is a revolut joint connecting two links with a mass \( m_1 \); the blue point is a blue point is the end of the second link with a mass \( m_2 \). The angle of the first link is represent by \( q_1 \), and the second angle in represent by \( q_2 \). Since there are plenty of trigonometry
functions in the equations of motion, we use a short hand for sinesoid functions.
We use $c_\theta$ to denote function $\cos(\theta)$, and use $s_\theta$ to denote function $\sin(\theta)$. We
use $\dot{}q$ to denote the first-order time derivative of $q$, and use $\ddot{}q$ to denote the
second time derivative of $q$. $\tau$ denotes torques, from gravity or control input.

4 Acrobot Swing Up

4.1 Equations of Motion

The acrobot is a two-link pendulum, while only the second joint has an actuator.
Thus, it is an underactuated robot.

We will first derive the dynamics model of the double pendulum, and sim-
ulate a passive one in MATLAB. We are going to use the method of Lagrange
to derive the equations of motion.

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i,$$

where $L$ denotes the Lagrangian, $q_i$ denotes the generalized coordinates and $Q_i$ denotes the force applied.

$$L = T - U$$

$$T = \frac{1}{2}(m_1 + m_2)l_1^2(q_\dot{}_1)^2 + \frac{1}{2}m_2l_2^2(q_\dot{}_1 + q_\dot{}_2)^2 + m_2l_1l_2q_\dot{}_1(q_\dot{}_1 + q_\dot{}_2)c_2$$

$$U = -(m_1 + m_2)glc_1 - m_2gls_{c_1+2}$$

Plugging the Lagrangian into the Lagrange Equations, we obtain the equa-
tions of motion.

Then we can reform the equations of motion into a characteristic form called
the Manipulator Equations, which takes the form:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} = \tau_g(q) + Bu,$$

where $q$ is the state vector, $I$ is the inertia matrix, $C$ captures Coriolis forces,
and $\tau_g$ is the gravity vector. The matrix $B$ maps control inputs $u$ into generalized
force. In our first task to simulate the passive double pendulum, we don’t have
to worry about $B$ or $u$. The other matrix take the following form:

$$M(q) = \begin{bmatrix}
(m_1 + m_2)l_1^2 + m_2l_2^2 + 2m_2l_1l_2c_2 & m_2l_2^2 + m_2l_1l_2c_2 \\
 m_2l_2^2 + m_2l_1l_2c_2 & m_2l_2^2
\end{bmatrix}$$

$$C(q, \dot{q}) = \begin{bmatrix}
0 & -m_2l_1l_2(2\dot{q}_1 + \dot{q}_2)s_2 \\
 m_2l_1l_2\dot{q}_1s_2 & 0
\end{bmatrix}$$

$$\tau_g(q) = -g \begin{bmatrix}
(m_1 + m_2)l_1s_1 + m_2l_2s_{1+2} \\
m_2l_2s_{1+2}
\end{bmatrix}$$
4.2 Dynamics Model of 2-link Acrobot

Alright, we now have a horrific looking second-order differential equation. The next necessary step is to reform the Manipulator Equations into first-order differential equations, so that we can form a state space model. The state space variable we are going to use is \( q = [q_1, q_2, \dot{q}_1, \dot{q}_2] \). The first-order differential equations we wish to have takes the form:

\[
\text{Mass}(q) \ddot{q} = f(q)
\]

We use the matrix I and C to construct the Mass matrix:

\[
\text{Mass}(q) = \begin{bmatrix}
I_{2 \times 2} & 0_{2 \times 2} \\
C(q) & M(q)
\end{bmatrix},
\]

which is a 4-by-4 matrix. Then we can reformulate the Manipulator Equation into a set of 4 differential equations:

\[
\begin{bmatrix}
I_{2 \times 2} & 0_{2 \times 2} \\
C(q) & M(q)
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix}
= 
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
-g(\text{m}_1 + \text{m}_2)l_1s_1 + \text{m}_2l_2s_1 + 2 \text{m}_2l_2s_1 + 2 \\
\text{m}_2l_2s_1 + 2
\end{bmatrix}
\]

Finally, we have a set of first-order differential equations, which we can throw into some numerical integration solver. Here we will use the built-in solver from MATLAB. The following two figures are frames from the simulation of a passive acrobot swing from right to left.

4.3 Trajectory Optimization

Alright we have a passive double pendulum swinging back and forth, but that is not actually very exciting. The exciting part of this note is to demonstrate how to swing up the acrobot from the bottom using a control technique called trajectory optimization. The software library we are going to use is an open-sourced trajectory optimization library called OptimTraj.

Now, we have to reformulate the equations of motion into the cannonical form, so our plan is:

\[
\dot{q} = \text{Mass}(q)^{-1}f(q, u),
\]
which is a non-linear dynamical system. Then, we can construct an optimization problem using this model, which we can throw into our trajectory optimization library. The following figure plots the trajectories of the two joints (red and green joints).

The values of these two actuated joints start from $[0, 0]$ and end at $[\pi, 0]$. Then, let’s animate the swing up sequence of the 2-link acrobot. Here are three frames from the animation.

5 A 3-link Acrobot

5.1 Equations of Motion

Let’s derive the equations of motion for a 3-link acrobot in 2-dimensional space. Here we are again using the method of Lagrange. Let’s get start with the Lagrangian, which comes from the kinetic and potential energy of the system.

$$T = \frac{1}{2}(m_1 + m_2 + m_3)l_1^2 \dot{q}_1^2 + \frac{1}{2}(m_2 + m_3)l_2^2 (\dot{q}_1 + \dot{q}_2)^2 + \frac{1}{2}m_3l_3^2 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2 + (m_2 + m_3)l_1l_2 \dot{q}_1 (\dot{q}_1 + \dot{q}_2)c_2 + m_3l_1l_3 \dot{q}_1 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)c_{2+3} + m_3l_2l_3 (\dot{q}_1 + \dot{q}_2)(\dot{q}_1 + \dot{q}_2 + \dot{q}_3)c_3$$

$$U = -(m_1 + m_2 + m_3)l_1gc_1 - (m_2 + m_3)l_2gc_{1+2} - m_3l_3gc_{1+2+3}$$

$$L = T - U$$

Next, let’s recall our favorite Lagrange Equations.
\[ \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i \]

Plugging the Lagrangian into these equations yields the equations of motion.

\[ \tau_1 = (m_1 + m_2 + m_3)\ddot{q}_1 + (m_2 + m_3)\ddot{q}_2 + m_3\ddot{q}_3 + (m_2 + m_3)l_1l_2(\ddot{q}_1 + \ddot{q}_2)c_2 - (m_2 + m_3)l_1l_2(\ddot{q}_1 + \ddot{q}_2)q_2s_2 + m_3l_1l_3(\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3)c_{2+3} - m_3l_1l_3(\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3)(q_2 + q_3)s_{2+3} + m_3l_2l_3(\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3)c - m_3l_2l_3(\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3)q_1s_3 + (m_1 + m_2 + m_3)l_1g_s_1 + (m_2 + m_3)l_2g_s_{1+2} + m_3l_3g_{s_{1+2}} \\
\tau_2 = (m_2 + m_3)\ddot{q}_2 + m_3\ddot{q}_3 + (m_2 + m_3)l_1l_2q_2c - (m_2 + m_3)l_1l_2q_2s_2 + m_3l_1l_3q_1c_{2+3} - m_3l_1l_3q_1(q_2 + q_3)s_{2+3} + m_3l_2l_3(\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3)c_3 - m_3l_2l_3(\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3)q_{3s_3} + (m_2 + m_3)l_1l_2q_1(q_1 + \ddot{q}_2)s_2 + m_3l_1l_3(q_1 + \ddot{q}_2)(q_1 + \ddot{q}_2 + \ddot{q}_3)s_2 + (m_2 + m_3)l_2g_s_{1+2} + m_3l_3g_{s_{1+2}} \\
\tau_3 = m_3\ddot{q}_3 + m_3l_1l_3q_1c_3 - m_3l_1l_3q_1(q_2 + q_3)s_{2+3} + m_3l_2l_3(q_1 + \ddot{q}_2)c_3 - m_3l_2l_3(q_1 + \ddot{q}_2)q_{3s_3} + m_3l_1l_3q_1(q_1 + \ddot{q}_2 + \ddot{q}_3)s_{2+3} + m_3l_2l_3(q_1 + \ddot{q}_2)(q_1 + \ddot{q}_2 + \ddot{q}_3)s_3 + m_3l_3g_{s_{1+2}} \\
\]

5.2 Dynamics Model of 2-link Acrobat

These equations may look horrific for normal people (include me). However, we can put them into a nice looking matrix form, which is what we have to tell the computer. The Manipulator Equations of our 3-link acrobot takes the form:

\[ M(q)\ddot{q} + C(q, \dot{q})\dot{q} = \tau(q) + Bu \]

We have to again reformulate the problem into a first-order differential equation by constructing the mass matrix. This time, the mass matrix is a 6 × 6 matrix, since our state variable has become 6-dimensional.

\[ Mass(q) = \begin{bmatrix} I_{3 \times 3} & 0_{3 \times 3} \\ C(q)_{3 \times 3} & M(q)_{3 \times 3} \end{bmatrix} \]

As expected, the mass matrix is going to be a little bit complicated, so we decide to present the matrix element-by-element. \( M_{ij} \) denotes elements from matrix \( M(q) \), while \( C_{ij} \) denotes elements from matrix \( C(q, \dot{q}) \).

\[ M_{11} = (m_1 + m_2 + m_3)l_1^2 + (m_2 + m_3)l_2^2 + m_3l_3^2 + 2(m_2 + m_3)l_1l_2c_2 + 2m_3l_1l_3c_{2+3} + 2m_3l_2l_3c_{3} \]

\[ M_{12} = (m_2 + m_3)l_1^2 + m_3l_3^2 + (m_2 + m_3)l_1l_2c_2 + m_3l_1l_3c_{2+3} + 2m_3l_2l_3c_3 \]

\[ M_{13} = m_3l_3^2 + m_3l_1l_3c_{2+3} + m_3l_2l_3c_3 \]

\[ M_{21} = (m_2 + m_3)l_1^2 + m_3l_3^2 + (m_2 + m_3)l_1l_2c_2 + m_3l_1l_3c_{2+3} + 2m_3l_2l_3c_3 \]

\[ M_{22} = (m_2 + m_3)l_2^2 + m_3l_3^2 + 2m_3l_2l_3c_3 \]

\[ M_{23} = m_3l_3^2 + m_3l_2l_3c_3 \]

\[ M_{31} = m_3l_3^2 + m_3l_1l_3c_{2+3} + m_3l_2l_3c_3 \]
\[ M_{32} = m_3 l_3^2 + m_3 l_2 l_3 c_3 \]
\[ M_{33} = m_3 l_3^2 \]
\[ C_{11} = 0 \]
\[ C_{12} = -(m_2 + m_3) l_1 l_2 (2 \dot{q}_1 + \dot{q}_2) s_2 - m_3 l_1 l_3 (2 \dot{q}_1 + \dot{q}_2 + \dot{q}_3) s_{2+3} \]
\[ C_{13} = -m_3 l_1 l_3 (2 \dot{q}_1 + \dot{q}_2 + \dot{q}_3) s_{2+3} - m_3 l_2 l_3 (2 \dot{q}_1 + \dot{q}_2 + \dot{q}_3) s_3 \]
\[ C_{21} = -(m_2 + m_3) l_1 l_2 \dot{q}_2 s_2 - m_3 l_1 l_3 (\dot{q}_2 + \dot{q}_3) s_{2+3} + (m_2 + m_3) l_1 l_2 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) s_3 \]
\[ C_{22} = m_3 l_1 l_3 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) s_3 \]
\[ C_{23} = -m_3 l_2 l_3 (2 \dot{q}_1 + \dot{q}_2 + \dot{q}_3) s_3 \]
\[ C_{31} = -m_3 l_1 l_3 (\dot{q}_2 + \dot{q}_3) s_{2+3} + m_3 l_1 l_3 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) s_{2+3} + m_3 l_2 l_3 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) s_3 \]
\[ C_{32} = m_3 l_2 l_3 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) s_3 \]
\[ C_{33} = -m_3 l_2 l_3 (\dot{q}_1 + \dot{q}_2) s_3 \]

Very cool! Now we are ready to put everything into the computer. In MATLAB, again we construct the following first-order differential equations:

\[
\begin{bmatrix}
I_{3\times3} & 0_{3\times3} \\
C(q)_{3\times3} & M(q)_{3\times3}
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_3 \\
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3
\end{bmatrix}
= \begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_3 \\
\tau_{g1} + u_1 \\
\tau_{g2} + u_2 \\
\tau_{g3} + u_3
\end{bmatrix},
\]

where \( \tau_{gi} \) are the gravity terms, while \( u_i \) are the control input. Here we have three input, which means the system is fully actuated. Since the acrobot is underactuated, we will take away \( u_1 \) when we solve the acrobot problem. The gravity terms are:

\[ \tau_{g1} = -g [(m_1 + m_2 + m_3) l_1 s_1 + (m_2 + m_3) l_2 s_{1+2} + m_3 l_3 s_{1+2+3}] \]
\[ \tau_{g2} = -g [(m_2 + m_3) l_2 s_{1+2} + m_3 l_3 s_{1+2+3}] \]
\[ \tau_{g3} = -g m_3 l_3 s_{1+2+3} \]

Exciting news! All the mathematic is done and now let’s talk about how to do simulation on a computer.

The first thing we want to work out on the computer is to check whether we have the correct dynamics model. To check whether our model is correct, an intuitive way is simulate the passive system. We can give the system an initial state, such as \([ \frac{\pi}{3} \ 0 \ 0 \ 0 \ 0 \ 0 \ ]\), and watch it evolves over time. If there
are any bugs in our equations, chances are the system will looked unreal. I have
to clarify that this method doesn’t gauruntee the model is correct, but I choose
to believe my eyeballs and commonsense (because I’m an engineer, typically lazy
person). Since our model is barely a set of differential equations, we can use
numerical integration tools to solve it approximatly. Our plan here is using the
built-in solver from MATLAB. The dynamics model of acrobot fits right in one
of the standard form support by MATLAB, which called “ODE with time and
state dependent mass matrix”. In order to tell MATLAB what we intended to
do, we have to specify options for the ode45 solver.

\[
\begin{align*}
  \text{options} &= \text{odeset}(\text{’Mass’, @}\text{mass}), \\
  \begin{bmatrix} t, y \end{bmatrix} &= \text{ode45}(@f, tspan, q0, \text{options}), \\
\end{align*}
\]

where the name/value pair (’Mass’, @}mass) tells ode45 that we have a mass
matrix in our equations and we have a function called ’mass’ calculating it. In
the second line of code, ’tspan’ stands for the time span we wish to integrate;
’q0’ is the initial state of the system; ’@f’ link ode45 to a function called ’f’,
which calculates the time derivative of the state variables.

The ode45 solver will give us the numerical solution of our system over time,
given our manually picked initial state. Then we can plot the solution and
convince ourselves we have the correct model, if we are lucky to be typo-free.
In the following images, the acrobot swing from right to left passively.

\[5.3 \text{ Trajectory Optimization}\]

Finally we are here, after all the ugly mathematics and boring simulation. We
have all we need to wirte a piece of code to solve the 3-link acrobot swing-
up problem. We have a dynamics model of the system, which we’ve convinced
ourselves is correct, and a library to perform trajectory optimization. Now, we
will construct an trajectory optimization using the dynamics model and throw
it to OptimTraj.

To use the OptimTraj library, we need to first construct a function which
computes the dynamics. Here we have to put the inverse of the mass matrix on
the right hand side, like

\[
\dot{q} = \text{Mass}(q)^{-1} f(q, u),
\]

where the inverse matrix is provable to exist. The second thing is to supply
a cost function, which we will take the dot product of the control effort. This
will leads to the minimization of control effort. Then we also need to specify
some constraint for the optimization problem, including initial state, final state,
upper and lower bounds of state variables and initial guesses. After specifying all these stuff, we finally can leave the hard work to computer. The following two figures plot the trajectories of two actuated joints (green and blue joints).

The values of these two actuated joints start from \([0, 0]\) and end at \([\pi, 0]\). Then, let’s animate the swing up sequence of the 3-link acrobot. Here are three frames from the animation.

6 Remarks

This note presents an experimental study of acrobot, which is an easygoing member in the family of underactuated robots. To make the result mentioned here concrete, stability and controllability analysis are expected in the future.

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