ON HYPERCYCLICITY AND LINEAR CHAOS
IN A NONCLASSICAL SEQUENCE SPACE AND BEYOND

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Abstract. We analyze the hypercyclicity, chaoticity, and spectral structure of (bounded and unbounded) weighted backward shifts in a nonclassical sequence space, which the space $l_1$ of summable sequences is both isometrically isomorphic to and continuously and densely embedded into.

Based on the weighted backward shifts, we further construct new bounded and unbounded linear hypercyclic and chaotic operators both in the nonclassical sequence space and the classical space $l_1$, including those that are hypercyclic but not chaotic.

1. Introduction

We analyze the hypercyclicity, chaoticity, and spectral structure of the weighted backward shifts, bounded

$$X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto A_w x := w(x_{k+1})_{k \in \mathbb{N}} \in X \quad (w \in \mathbb{F})$$

($\mathbb{F} := \mathbb{R}$ or $\mathbb{F} := \mathbb{C}$), as well as unbounded

$$A_w x := (w^k x_{k+1})_{k \in \mathbb{N}} \quad (w \in \mathbb{F}, \ |w| > 1)$$

with maximal domain

$$D(A_w) := \left\{ x := (x_k)_{k \in \mathbb{N}} \in X \left| (w^k x_{k+1})_{k \in \mathbb{N}} \in X \right\} ,$$

(for the first mention in the classical setting, see [15, 19]) in a nonclassical sequence space $X$ introduced in [8] (see also [10]), which the space $l_1$ of summable sequences is both isometrically isomorphic to and continuously and densely embedded into.

Based on the weighted backward shifts, we further construct new bounded and unbounded linear hypercyclic and chaotic operators both in the nonclassical sequence space $X$ and the classical space $l_1$, including those that are hypercyclic but not chaotic.

2. Preliminaries

The subsequent preliminaries are essential for our discourse.

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2.1. Certain Facts on Classical Sequence Spaces.

**Definition 2.1 (Schauder Basis).**
For a Banach space \((X, \| \cdot \|)\) over \(\mathbb{F}\), a subset \(\{e_n\}_{n \in \mathbb{N}} \subseteq X\) is called a Schauder basis if
\[
\forall x \in X \ \exists! (c_k)_{k \in \mathbb{N}} \in \mathbb{F}^\mathbb{N} : \ x = \sum_{k=1}^{\infty} c_k e_k.
\]
The series is called the *Schauder expansion* of \(x\) and the numbers \(c_k \in \mathbb{F}, k \in \mathbb{N}\), are called the coordinates of \(x\) relative to \(\{e_n\}_{n \in \mathbb{N}}\) (see, e.g., [14, 16, 18]).

Classical examples of such spaces are the spaces
\[
l_p := \{ x := (x_k)_{k \in \mathbb{N}} \in \mathbb{F}^\mathbb{N} \mid \sum_{k=1}^{\infty} |x_k|^p < \infty \}
\]
of \(p\)-summable sequences \((1 \leq p < \infty)\) and the space
\[
c_0 := \{ x := (x_k)_{k \in \mathbb{N}} \in \mathbb{F}^\mathbb{N} \mid \lim_{k \to \infty} x_k = 0 \}
\]
of vanishing sequences.

For these spaces, the set \(\{e_n := (\delta_{nk})_{k \in \mathbb{N}}\}_{n \in \mathbb{N}}\), where
\[
\delta_{nk} := \begin{cases} 
1 & \text{if } k = n, \\
0 & \text{if } k \neq n,
\end{cases}
\]
is the Kronecker delta, is the standard Schauder basis and
\[
\forall x := (x_k)_{k \in \mathbb{N}} \in X : \ x = \sum_{k=1}^{\infty} x_k e_k.
\]
\((X := l_p \ (1 \leq p < \infty)\) or \(X := c_0)\).

For the space
\[
c := \{ x := (x_k)_{k \in \mathbb{N}} \in \mathbb{F}^\mathbb{N} \mid \exists \lim_{k \to \infty} x_k \in \mathbb{F} \}
\]
of convergent sequences, the standard Schauder basis is \(\{e_n := (\delta_{nk})_{k \in \mathbb{N}}\}_{n \in \mathbb{N}}\), where
\[
e_0 := (1,1,1,\ldots) \quad \text{and} \quad e_n := (\delta_{nk})_{k \in \mathbb{N}}, \ n \in \mathbb{N},
\]
and
\[
\forall x := (x_k)_{k \in \mathbb{N}} \in c : \ x = \sum_{k=0}^{\infty} c_k e_k,
\]
where
\[
c_0 = \lim_{m \to \infty} x_m, \ c_k = x_k - \lim_{m \to \infty} x_m, \ k \in \mathbb{N}.
\]
The space
\[
l_\infty := \{ x := (x_k)_{k \in \mathbb{N}} \in \mathbb{F}^\mathbb{N} \mid \sup_{k \in \mathbb{N}} |x_k| < \infty \}
\]
of bounded sequences has no Schauder basis since it is not separable.

See, e.g. [14, 16, 18].
Remark 2.1. Observe that $c$ is a proper subspace of $l_\infty$ and $c_0$ is a proper subspace (a hyperplane) of $c$, i.e.,

$$c_0 \subset c \subset l_\infty,$$

(see, e.g., [14, 16, 17]).

Henceforth, make use of the following

Theorem 2.1 (General Characterization of Convergence [18, Theorem 1]).

Let $(X, \| \cdot \|)$ be a Banach space with a Schauder basis $\{e_n\}_{n \in \mathbb{N}}$ and corresponding coordinate functionals $c_n(\cdot)$, $n \in \mathbb{N}$.

For a sequence $(x_n)_{n \in \mathbb{N}}$ and a vector $x$ in $X$,

$$x_n \to x, \; n \to \infty,$$

iff

(1) $\forall k \in \mathbb{N}: \; c_k(x_n) \to c_k(x), \; n \to \infty$, and

(2) $\forall \varepsilon > 0 \; \exists K_0 \in \mathbb{N} \; \forall K \geq K_0 \; \forall n \in \mathbb{N}: \; \left\| \sum_{k=K+1}^{\infty} c_k(x_n)e_k \right\| < \varepsilon$.

Remark 2.2. In the chain of the proper inclusions

$$c_0 \subset l_p \subset l_q \subset c_0,$$

where $1 \leq p < q < \infty$, (see, e.g., [14, 16]),

$$l_p \hookrightarrow l_q \hookrightarrow c_0$$

are continuous and dense embeddings.

In particular, the continuity of embeddings

$$l_p \hookrightarrow l_q \; \; (1 \leq p < q < \infty)$$

is implied by the Characterization of Convergence in $l_p$ ($1 \leq p < \infty$) (see, e.g., [16, Proposition 2.16], [14, Proposition 2.17]), which, in its turn, is a particular case of the prior general characterization [18].

2.2. Spectrum.

The spectrum $\sigma(A)$ of a closed linear operator $A$ in a complex Banach space $X$ is the union of the following pairwise disjoint sets:

$$\sigma_p(A) := \{ \lambda \in \mathbb{C} \mid A - \lambda I \text{ is not injective, i.e., } \lambda \text{ is an eigenvalue of } A \},$$

$$\sigma_c(A) := \{ \lambda \in \mathbb{C} \mid A - \lambda I \text{ is injective, not surjective, and } R(A - \lambda I) = X \},$$

$$\sigma_r(A) := \{ \lambda \in \mathbb{C} \mid A - \lambda I \text{ is injective and } R(A - \lambda I) \neq X \}$$

($R(\cdot)$ is the range of an operator and $\overline{\cdot}$ is the closure of a set), called the point, continuous, and residual spectrum of $A$, respectively (see, e.g., [6, 14]).
2.3. Hypercyclicity and Linear Chaos.

**Definition 2.2** (Hypercyclic and Chaotic Linear Operators).
For a (bounded or unbounded) linear operator $A$ in a (real or complex) Banach space $X$, a nonzero vector $x \in C^\infty(A) := \bigcap_{n=0}^\infty D(A^n)$ (where $D(\cdot)$ is the domain of an operator, $A^0 := I$, $I$ is the identity operator on $X$) is called hypercyclic if its orbit under $A$

$$\text{orb}(x, A) := \{ A^n x \}_{n \in \mathbb{Z}_+}$$

is dense in $X$, i.e.,

$$\text{orb}(x, A) = X.$$

Linear operators possessing hypercyclic vectors are said to be hypercyclic.

If there exist an $N \in \mathbb{N}$ and a vector $x \in D(A^N)$ with $A^N x = x$, such a vector is called a periodic point for the operator $A$ of period $N$. If $x \neq 0$, we say that $N$ is a period for $A$. Hypercyclic linear operators with a dense in $X$ set $\text{Per}(A)$ of periodic points, i.e.,

$$\text{Per}(A) = X,$$

are said to be chaotic.

See [2,5,9].

**Examples 2.1.**

1. On the space $X := l_p$ ($1 \leq p < \infty$) or $X := c_0$, the classical Rolewicz weighted backward shifts

$$X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto A_w x := w (x_{k+1})_{k \in \mathbb{N}} \in X,$$

where $w \in \mathbb{F}$ with $|w| > 1$ are chaotic [9,19].

2. On the nonclassical sequence space

$$X := \left\{ (x_k)_{k \in \mathbb{N}} \in \mathbb{F}^\mathbb{N} \mid \sum_{k=1}^\infty \left| \frac{x_{k+1}}{k+1} - \frac{x_k}{k} \right| < \infty \text{ and } \lim_{k \to \infty} \frac{x_k}{k} = 0 \right\},$$

equipped with the norm

$$\| x \| := \sum_{k=1}^\infty \left| \frac{x_{k+1}}{k+1} - \frac{x_k}{k} \right|,$$

which takes the center stage in the subsequent discourse (see Section 3), the backward shift

$$X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto Ax := (x_{k+1})_{k \in \mathbb{N}} \in X$$

is hypercyclic but not densely periodic, and hence, not chaotic [8] (see also [10, Exercise 4.1.3]).
3. On an infinite-dimensional separable Banach space \((X, \| \cdot \|)\), the identity operator \(I\) is densely periodic but \textit{not} hypercyclic, and hence, \textit{not} chaotic.

**Remarks 2.1.**

- In the prior definition of hypercyclicity, the underlying space is necessarily \textit{infinite-dimensional} and \textit{separable} (see, e.g., [10]).
- For a hypercyclic linear operator \(A\), the set \(HC(A)\) of its hypercyclic vectors is necessarily dense in \(X\), and hence, the more so, is the subspace \(C^\infty(A) \supseteq HC(A)\).
- Observe that \(\text{Per}(A) = \bigcup_{N=1}^{\infty} \text{Per}_N(A)\), where \(\text{Per}_N(A) = \ker(A^N - I), ~ N \in \mathbb{N}\) is the \textit{subspace} of \(N\)-periodic points of \(A\).
- As immediately follows from the inclusions \(HC(A^n) \subseteq HC(A), ~ \text{Per}(A^n) \subseteq \text{Per}(A), \, n \in \mathbb{N}\), if, for a linear operator \(A\) in an infinite-dimensional separable Banach space \(X\) and some \(n \geq 2\), the operator \(A^n\) is hypercyclic or chaotic, then \(A\) is also hypercyclic or chaotic, respectively.

Prior to [2,3], the notions of linear hypercyclicity and chaos had been studied exclusively for \textit{continuous} linear operators on Fréchet spaces, in particular for \textit{bounded} linear operators on Banach spaces (for a comprehensive survey, see [1,10]).

The following extension of Kitai’s criterion for bounded linear operators (see [7,11]) is a useful shortcut for establishing hypercyclicity for (bounded or unbounded) linear operators without explicitly furnishing a hypercyclic vector as in [19].

**Theorem 2.2** (Sufficient Condition for Hypercyclicity [2, Theorem 2.1]).

Let \(X\) be a (real or complex) infinite-dimensional separable Banach space and \(A\) be a densely defined linear operator in \(X\) such that each power \(A^n, ~ n \in \mathbb{N}\), is a closed operator. If there exists a set

\[ Y \subseteq C^\infty(A) := \bigcap_{n=1}^{\infty} D(A^n) \]

dense in \(X\) and a mapping \(B : Y \to Y\) such that

\[ (1) \forall x \in Y : ABx = x \text{ and} \]
\[ (2) \forall x \in Y : A^n x , B^n x \to 0, ~ n \to \infty, \]

then the operator \(A\) is hypercyclic.

The subsequent newly established sufficient condition for linear chaos [12], obtained via strengthening one of the hypotheses of the prior sufficient condition for hypercyclicity, serves as a shortcut for establishing chaoticity for (bounded or unbounded)
linear operators without explicitly furnishing both a hypercyclic vector and a dense set of periodic points and is fundamental for our discourse.

**Theorem 2.3** (Sufficient Condition for Linear Chaos [12, Theorem 3.2]).
Let \((X, \| \cdot \|)\) be a (real or complex) infinite-dimensional separable Banach space and \(A\) be a densely defined linear operator in \(X\) such that each power \(A^n, n \in \mathbb{N}\), is a closed operator. If there exists a set
\[
Y \subseteq C^\infty(A) := \bigcap_{n=1}^\infty D(A^n)
\]
dense in \(X\) and a mapping \(B: Y \to Y\) such that
1. \(\forall x \in Y : ABx = x\) and
2. \(\forall x \in Y \ \exists \alpha = \alpha(x) \in (0, 1), \ \exists c = c(x, \alpha) > 0 \ \forall n \in \mathbb{N} :\)
\[
\max (\|A^n x\|, \|B^n x\|) \leq c \alpha^n,
\]
then the operator \(A\) is chaotic.

For applications, see [13].

We also need the following statements.

**Corollary 2.1** (Chaoticity of Powers [12, Corollary 4.3]).
For a chaotic linear operator \(A\) in a (real or complex) infinite-dimensional separable Banach space subject to the Sufficient Condition for Linear Chaos (Theorem 2.3), each power \(A^n, n \in \mathbb{N}\), is chaotic.

**Theorem 2.4** (Kitai [10, Theorem 5.6]).
For a bounded hypercyclic operator \(A\) on a complex infinite-dimensional separable Banach space, every connected component of its spectrum \(\sigma(A)\) meets the unit circle.

**Proposition 2.1** ([10, Proposition 5.7]).
For a bounded hypercyclic operator \(A\) on a complex infinite-dimensional separable Banach space, its spectrum has no isolated points and its point spectrum \(\sigma_p(A)\) contains infinitely many roots of unity.

### 3. Nonclassical Sequence Space

#### 3.1. Space and Its Elements.

For us, underlying is the following nonclassical infinite-dimensional sequence space
\[
X := \left\{ (x_k)_{k \in \mathbb{N}} \in \mathbb{F}^\mathbb{N} \left| \sum_{k=1}^\infty \left| x_{k+1} - \frac{x_k}{k} \right| < \infty \text{ and } \lim_{k \to \infty} \frac{x_k}{k} = 0 \right. \right\}
\]
introduced in [8], which is separable and Banach relative to the norm
\[
X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto \|x\| := \sum_{k=1}^\infty \left| \frac{x_{k+1}}{k+1} - \frac{x_k}{k} \right|
\]
(see also [10, Exercise 4.1.3]) and referred to as \((X, \| \cdot \|)\) henceforth.
Remark 3.1. Thus, a sequence \((x_k)_{k \in \mathbb{N}} \in \mathbb{F}^\mathbb{N}\) is in the space \(X\) iff it satisfies the two conditions:

\[
\left( \frac{x_{k+1}}{k+1} - \frac{x_k}{k} \right)_{k \in \mathbb{N}} \in l_1 \quad \text{and} \quad \left( \frac{x_k}{k} \right)_{k \in \mathbb{N}} \in c_0,
\]

the latter condition being equivalent to

\[x_k = o(k), \quad k \to \infty.\]

The following examples illustrate the independence of the above conditions.

Examples 3.1.

1. \((1, 1, 1, \ldots) \in X\) since

\[
\left( \frac{1}{k+1} - \frac{1}{k} \right)_{k \in \mathbb{N}} = \left( -\frac{1}{k(k+1)} \right)_{k \in \mathbb{N}} \quad \text{and} \quad \left( \frac{1}{k} \right)_{k \in \mathbb{N}} \in c_0.
\]

2. \(\left( \frac{(-1)^k}{\ln(k+1)} \right)_{k \in \mathbb{N}} \notin X\) since

\[
\left( \frac{(-1)^{k+1}}{k} - \frac{(-1)^k}{k+1} \right)_{k \in \mathbb{N}} = \left( \frac{(-1)^{k+1}}{(k+2)\ln(k+1) + k} \right)_{k \in \mathbb{N}} \notin l_1
\]

although \(\left( \frac{(-1)^k}{k \ln(k+1)} \right)_{k \in \mathbb{N}} \in c_0.\)

3. \((k)_{k \in \mathbb{N}} \notin X\) since

\[
\left( \frac{k+1}{k+1} - \frac{k}{k} \right)_{k \in \mathbb{N}} = (0, 0, 0, \ldots) \in l_1 \quad \text{but} \quad \left( \frac{k}{k} \right)_{k \in \mathbb{N}} = (1, 1, 1, \ldots) \notin c_0.
\]

4. \((k^2)_{k \in \mathbb{N}} \notin X\) since

\[
\left( \frac{(k+1)^2}{k+1} - \frac{k^2}{k} \right)_{k \in \mathbb{N}} = (1, 1, 1, \ldots) \notin l_1 \quad \text{and} \quad \left( \frac{k^2}{k} \right)_{k \in \mathbb{N}} = (k)_{k \in \mathbb{N}} \notin c_0.
\]

The following example relates the well-known class of power sequences to the space \(X\).

Example 3.2 (Power Sequences).

For \(p \in \mathbb{R}\),

\[
(k^p)_{k \in \mathbb{N}} \in X \iff p < 1.
\]

In particular,

\((1, 1, 1, \ldots), \left( \sqrt[k]{k} \right)_{k \in \mathbb{N}} \in X\) and \((k)_{k \in \mathbb{N}}, \ (k^2)_{k \in \mathbb{N}} \notin X.\)

Indeed,

\[
\left( \frac{k^p}{k} \right)_{k \in \mathbb{N}} = (k^{p-1})_{k \in \mathbb{N}} \in c_0 \iff p - 1 < 0 \iff p < 1.
\]
Further, since, for \( p - 1 < 0 \),
\[
\sum_{k=1}^{\infty} \left| \frac{(k+1)^p}{k+1} - \frac{k^p}{k} \right| = \sum_{k=1}^{\infty} \left| (k+1)^{p-1} - k^{p-1} \right| = \sum_{k=1}^{\infty} (k^{p-1} - (k+1)^{p-1})
\]
\[
= \lim_{n \to \infty} \sum_{k=1}^{n} (k^{p-1} - (k+1)^{p-1}) = 1 - \lim_{n \to \infty} (n+1)^{p-1} = 1,
\]
we infer that \( (k^p)_{k \in \mathbb{N}} \in X \).

In particular,
\[
\forall p < 1 : \| (k^p)_{k \in \mathbb{N}} \| = 1.
\]

**Remark 3.2.** As follows from Examples 3.1 and Example 3.2 (see also Remark 2.1),
\[
l_\infty \supset c \supset c_0 \ni \left( \frac{(-1)^k}{\ln(k+1)} \right)_{k \in \mathbb{N}} \notin X \ni \left( \sqrt{n} \right)_{k \in \mathbb{N}} \notin l_\infty.
\]
Hence,
\[
c_0 \not\subseteq X \not\subseteq l_\infty.
\]

Thus, we conclude that the space \( X \) is not a part of the hierarchy of the classical spaces \( c_0, c, \text{ and } l_\infty \).

### 3.2. Isometric Isomorphisms.

Let us look at the following classical space
\[
bv_0 := \left\{ (x_k)_{k \in \mathbb{N}} \in c_0 \left| \sum_{k=1}^{\infty} |x_{k+1} - x_k| < \infty \right. \right\},
\]
of vanishing sequences of bounded variation with the norm
\[
bv_0 \ni x := (x_k)_{k \in \mathbb{N}} \mapsto \|x\|_0 := \sum_{k=1}^{\infty} |x_{k+1} - x_k|.
\]
(see, e.g., [6]).

**Proposition 3.1** (Isometric Isomorphisms Proposition).

1. The space \((X, \| \cdot \|)\) is isometrically isomorphic to the space \(bv_0\) under the mapping
\[
X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto J_1 x := \left( \frac{x_k}{k} \right)_{k \in \mathbb{N}} \in bv_0
\]
with the inverse
\[
bv_0 \ni y := (y_k)_{k \in \mathbb{N}} \mapsto J_1^{-1} y := (ky_k)_{k \in \mathbb{N}} \in X.
\]

2. The space \(bv_0\) is isometrically isomorphic to the space \(l_1\) under the mapping
\[
bv_0 \ni y := (y_k)_{k \in \mathbb{N}} \mapsto J_2 y := (y_{k+1} - y_k)_{k \in \mathbb{N}} \in l_1,
\]
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with the inverse

\[ l_1 \ni z := (z_k)_{k \in \mathbb{N}} \mapsto J_2^{-1} z := \left( \sum_{j=0}^{k-1} z_j \right)_{k \in \mathbb{N}} \in \mathcal{B}v_0, \]

where

\[ z_0 := -\sum_{k=1}^{\infty} z_k. \]

3. The space \((X, \| \cdot \|)\) is isometrically isomorphic to the space \(l_1\) under the mapping \(J := J_2J_1\), i.e.,

\[ X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto Jx := \left( \frac{x_{k+1}}{k+1} - \frac{x_k}{k} \right)_{k \in \mathbb{N}} \in l_1 \]

with the inverse \(J^{-1} = J_1^{-1}J_2^{-1}\), i.e.,

\[ l_1 \ni z := (z_k)_{k \in \mathbb{N}} \mapsto J^{-1} z := \left( \sum_{j=0}^{k-1} z_j \right)_{k \in \mathbb{N}} \in X, \]

where

\[ z_0 := -\sum_{k=1}^{\infty} z_k. \]

Proof.

1. As is easily seen, \(J_1 : X \to \mathcal{B}v_0\) is an injective linear operator which is a bijection between \(X\) and \(\mathcal{B}v_0\) since

\[ \forall y := (y_k)_{k \in \mathbb{N}} \in \mathcal{B}v_0 : x := (ky_k)_{k \in \mathbb{N}} \in X \text{ and } J_1 x = \left( \frac{ky_k}{k} \right)_{k \in \mathbb{N}} = y. \]

Since further

\[ \forall x := (x_k)_{k \in \mathbb{N}} \in X : \|J_1 x\|_0 = \left\| \left( \frac{x_k}{k} \right)_{k \in \mathbb{N}} \right\|_0 \]

\[ = \sum_{k=1}^{\infty} \left| \frac{x_{k+1}}{k+1} - \frac{x_k}{k} \right| = \|x\|, \]

we infer that \(J_2\) is an isometric isomorphism between the spaces \(X\) and \(\mathcal{B}v_0\) with the inverse

\[ \mathcal{B}v_0 \ni y := (y_k)_{k \in \mathbb{N}} \mapsto J_1^{-1} y := (ky_k)_{k \in \mathbb{N}} \in X. \]

2. As is easily seen, \(J_2 \colon \mathcal{B}v_0 \to l_1\) is a linear operator.

Suppose that for an arbitrary \(y := (y_k)_{k \in \mathbb{N}} \in \mathcal{B}v_0,\)

\[ J_2 y = (y_{k+1} - y_k)_{k \in \mathbb{N}} = (0, 0, 0 \ldots). \]

Then, inductively,

\[ \forall k \in \mathbb{N} : y_k = y_1, \]

i.e., \(y\) is a constant sequence.
Thus, in view of the fact that
\[ y_k \to 0, \quad k \to \infty, \]
implies that \( y = (0, 0, 0 \ldots) \).

Therefore, the linear operator \( J_2 \) is \textit{injective} (see, e.g., [14,16]).

For an arbitrary \( z := (z_k)_{k \in \mathbb{N}} \in l_1 \), let
\[
\sum_{k=1}^{\infty} z_k \in \mathbb{F}.
\]

Then
\[
y := \left( \sum_{j=0}^{k-1} z_j \right)_{k \in \mathbb{N}} \in bv_0
\]
since
\[
\lim_{k \to \infty} y_k = \lim_{k \to \infty} \sum_{j=0}^{k-1} z_j = z_0 + \sum_{j=1}^{\infty} z_j = 0,
\]
and
\[
\sum_{k=1}^{\infty} |y_{k+1} - y_k| = \sum_{k=1}^{\infty} \left| \sum_{j=0}^{k} z_j - \sum_{j=0}^{k-1} z_j \right| = \sum_{k=1}^{\infty} |z_k| < \infty.
\]

Moreover,
\[
J_2 y = \left( \sum_{j=0}^{k} z_j - \sum_{j=0}^{k-1} z_j \right)_{k \in \mathbb{N}} = (z_k)_{k \in \mathbb{N}} = z.
\]

Hence, the operator \( J_2 \) is a \textit{bijection} between \( bv_0 \) and \( l_1 \).

Since further
\[
\forall y := (y_k)_{k \in \mathbb{N}} \in bv_0 : \|J_2 y\|_1 = \| (y_{k+1} - y_k)_{k \in \mathbb{N}} \|_1
\]
\[
= \sum_{k=1}^{\infty} |y_{k+1} - y_k| = \| y \|_0,
\]
where \( \| \cdot \|_1 \) is the norm of \( l_1 \), we infer that \( J_2 \) is an \textit{isometric isomorphism}
between the spaces \( bv_0 \) and \( l_1 \) with the inverse
\[
l_1 \ni z := (z_k)_{k \in \mathbb{N}} \mapsto J_2^{-1} z := \left( \sum_{j=0}^{k-1} z_j \right)_{k \in \mathbb{N}} \in bv_0,
\]
where
\[
z_0 := - \sum_{k=1}^{\infty} z_k.
\]
3. By parts 1 and 2, it is clear that \( J := J_2 J_1 \) is an isometric isomorphism between \( X \) and \( l_1 \), and hence,
\[
\forall x := (x_k)_{k \in \mathbb{N}} \in X : J x = J_2 J_1 x = J_2 \left( \frac{x_k}{k} \right)_{k \in \mathbb{N}} = \left( \frac{x_{k+1}}{k+1} - \frac{x_k}{k} \right)_{k \in \mathbb{N}} \in l_1.
\]

Moreover, \( J^{-1} := J_2^{-1} J_1^{-1} \) is the inverse of \( J \), and thus,
\[
\forall z := (z_k)_{k \in \mathbb{N}} \in l_1 : J^{-1} z = J_1^{-1} J_2^{-1} z = J_1^{-1} \left( \sum_{j=0}^{k-1} z_j \right)_{k \in \mathbb{N}} = \left( k \sum_{j=0}^{k-1} z_j \right)_{k \in \mathbb{N}} \in X,
\]
where
\[
z_0 := - \sum_{k=1}^{\infty} z_k.
\]

\[\square\]

Remarks 3.1.

- The fact that the space \( bv_0 \) is isometrically isomorphic to \( l_1 \) is stated as [6, Exercise IV.13.11] (see also [4]).

- Since \( l_1 \) is an infinite dimensional separable Banach space, as follows from the prior proposition, so are the spaces \( bv_0 \) and \( (X, \| \cdot \|) \).

- Since the subspace \( c_00 \) of eventually zero sequences is dense in the space \( l_1 \) and \( J_2^{-1}(c_00) = J_1^{-1}(c_00) = c_00 \), by the prior proposition, \( c_00 \) is also dense in the spaces \( bv_0 \) and \( (X, \| \cdot \|) \).

3.3. Continuous and Dense Embeddings.

The latter remark is consistent with the following proposition further relating the space \( (X, \| \cdot \|) \) to the spaces \( l_1 \) and \( bv_0 \).

**Proposition 3.2** (Continuous and Dense Embeddings).

The chain of proper inclusions
\[
c_00 \subset l_1 \subset bv_0 \subset X.
\]
holds, where the embeddings
\[
l_1 \hookrightarrow bv_0 \quad \text{and} \quad bv_0 \hookrightarrow X
\]
amore continuous and dense with
\[
\|z\|_0 \leq 2\|z\|_1, \quad z \in l_1, \quad \text{and} \quad \|y\| \leq \|y\|_0, \quad y \in bv_0.
\]
Furthermore, the embedding \( l_1 \hookrightarrow X \) is a continuous and dense embedding with
\[
\|z\| \leq \|z\|_1, \quad z \in l_1.
\]
Proof. First, it is clear that $c_0 \subset l_1$ is a proper inclusion. Moreover,

\[ \forall z := (z_k)_{k \in \mathbb{N}} \in l_1 : z \in c_0 \]

and

\[ (3.4) \quad \sum_{k=1}^{\infty} |z_{k+1} - z_k| \leq \sum_{k=1}^{\infty} |z_{k+1}| + \sum_{k=1}^{\infty} |z_k| \leq 2\|z\|_1 < \infty. \]

Hence, the inclusion $l_1 \subseteq bv_0$ holds.

Moreover, $bv_0 \ni \left(\frac{1}{k}\right)_{k \in \mathbb{N}} \notin l_1,$
since $\left(\frac{1}{k}\right)_{k \in \mathbb{N}} \in c_0$ and

\[ \sum_{k=1}^{\infty} \frac{1}{k} = \infty \quad \text{while} \quad \sum_{k=1}^{\infty} \left| \frac{1}{k+1} - \frac{1}{k} \right| = \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 < \infty. \]

Therefore, the inclusion $l_1 \subset bv_0$ is proper.

Due to estimate (3.4),

\[ \|z\|_0 \leq 2\|z\|_1, \quad z \in l_1, \]

which makes the embedding $l_1 \hookrightarrow bv_0$ continuous.

Consider an arbitrary $y := (y_k)_{k \in \mathbb{N}} \in bv_0$.

In view of $bv_0 \subseteq c_0$,

\[ \left(\frac{y_k}{k}\right)_{k \in \mathbb{N}} \in c_0. \]

Furthermore, since $y \in bv_0$,

\[ \left( r_k := \sum_{j=k}^{\infty} |y_{j+1} - y_j| \right)_{k \in \mathbb{N}} \in c_0(\mathbb{R}_+). \]

Hence, the following is clear

\[ (3.5) \quad \forall k \in \mathbb{N} : |y_{k+1} - y_k| = r_k - r_{k+1} \]

and

\[ (3.6) \quad 0 \leq |y_k| = \sum_{j=k}^{\infty} (|y_j| - |y_{j+1}|) \leq \sum_{j=k}^{\infty} |y_{j+1} - y_j| = r_k. \]

In view of $r_1 = \|y\|_0$, using (3.5) and (3.6), for an arbitrary $n \in \mathbb{N}$, we deduce

\[ \sum_{k=1}^{n} \left| \frac{y_{k+1}}{k+1} - \frac{y_k}{k} \right| = \sum_{k=1}^{\infty} \left| \frac{y_{k+1}}{k+1} - \frac{y_k}{k+1} + \frac{y_k}{k+1} - \frac{y_k}{k} \right| \]

\[ = \sum_{k=1}^{n} \left| \frac{y_{k+1} - y_k}{k+1} - \frac{y_k}{k(k+1)} \right|. \]
\[
\leq \sum_{k=1}^{n} \left( \frac{|y_{k+1} - y_k|}{k+1} + \frac{|y_k|}{k(k+1)} \right) = \sum_{k=1}^{n} \left( \frac{r_k - r_{k+1}}{k+1} + \frac{|y_k|}{k(k+1)} \right)
\]

\[
\leq \sum_{k=1}^{n} \left( \frac{r_k - r_{k+1}}{k+1} + \frac{r_k}{k(k+1)} \right)
\]

\[
= \sum_{k=1}^{n} \left( \frac{r_k}{k} - \frac{r_{k+1}}{k+1} + \frac{r_k}{k} - \frac{r_{k+1}}{k+1} \right)
\]

\[
= \sum_{k=1}^{n} \left( \frac{r_k}{k} - \frac{r_{k+1}}{k+1} \right) = r_1 - \frac{r_{n+1}}{n+1} = \|y\|_0 - \frac{r_{n+1}}{n+1} \leq \|y\|_0.
\]

Hence, passing to the limit as \( n \to \infty \),

\[
\|y\| = \sum_{k=1}^{\infty} \left| \frac{y_{k+1}}{k+1} - \frac{y_k}{k} \right| \leq \|y\|_0.
\]

Thus, the inclusion \( bv_0 \subseteq X \) holds and

\[
\|y\| \leq \|y\|_0, \quad y \in bv_0,
\]

so that the embedding \( bv_0 \hookrightarrow X \) is continuous.

Moreover, since

\[
X \ni (1, 1, 1, \ldots) \notin c_0 \supseteq bv_0
\]

(see Examples 3.1), the inclusion \( bv_0 \subset X \) is proper.

Thus, chain of proper inclusions (3.1) holds, the embeddings

\[
l_1 \hookrightarrow bv_0 \quad \text{and} \quad bv_0 \hookrightarrow X
\]

being continuous with estimates (3.2) in place.

Since \( \{e_n := (\delta_{nk})_{k \in \mathbb{N}}\}_{n \in \mathbb{N}} \) is a Schauder basis for \( l_1 \) (see Section 2.1), \( bv_0 \), and \((X, \| \cdot \|)\) (see Section 3.4) we infer that

\[
c_{00} = \text{span} \left( \{e_n := (\delta_{nk})_{k \in \mathbb{N}}\}_{n \in \mathbb{N}} \right)
\]

is a dense subspace in \( l_1, bv_0 \), and \((X, \| \cdot \|)\).

Thus, the continuous embeddings

\[
l_1 \hookrightarrow bv_0 \quad \text{and} \quad bv_0 \hookrightarrow X
\]

are also dense.

The continuity and denseness of the embeddings

\[
l_1 \hookrightarrow bv_0 \quad \text{and} \quad bv_0 \hookrightarrow X
\]

instantly implies, the continuity and denseness for the embedding

\[
l_1 \hookrightarrow X,
\]

in which, by estimates (3.2),

\[
\|z\| \leq \|z\|_0 \leq 2\|z\|_1, \quad z \in l_1.
\]
Let us show that the latter estimate can be refined to estimate (3.3). Indeed,
\[
\forall z := (z_k)_{k \in \mathbb{N}} \in l_1, \forall n \in \mathbb{N}, n \geq 2 : \sum_{k=1}^{n} \left| \frac{z_{k+1}}{k+1} - \frac{z_k}{k} \right| \leq \sum_{k=1}^{n} \left| \frac{z_{k+1}}{k+1} \right| + \sum_{k=1}^{n} \left| \frac{z_k}{k} \right|
\]
\[
= |z_1| + \frac{|z_{n+1}|}{n+1} + 2 \sum_{k=2}^{n} \left| \frac{z_k}{k} \right|
\]
\[
\leq |z_1| + |z_{n+1}| + 2 \sum_{k=2}^{n} \left| \frac{z_k}{2} \right|
\]
\[
= |z_1| + |z_{n+1}| + \sum_{k=2}^{n} |z_k| = \sum_{k=1}^{n+1} |z_k|.
\]
Hence,
\[
\|z\| := \sum_{k=1}^{\infty} \left| \frac{z_{k+1}}{k+1} - \frac{z_k}{k} \right| = \lim_{n \to \infty} \sum_{k=1}^{n} \left| \frac{z_{k+1}}{k+1} - \frac{z_k}{k} \right| \leq \lim_{n \to \infty} \sum_{k=1}^{n+1} |z_k| = \sum_{k=1}^{\infty} |z_k| = \|z\|_1.
\]

**Example 3.3** (Exponential Sequences).

For \(\lambda \in \mathbb{C}\),
\[
(\lambda^k)_{k \in \mathbb{N}} \in X \iff |\lambda| < 1 \quad \text{or} \quad \lambda = 1.
\]
In particular,
\[
(1, 1, 1, \ldots), (2^{-k})_{k \in \mathbb{N}} \in X \quad \text{and} \quad (i^k)_{k \in \mathbb{N}}, ((-1)^k)_{k \in \mathbb{N}}, (2^k)_{k \in \mathbb{N}} \notin X.
\]
\((i \text{ is the imaginary unit})\).

Indeed, for \(\lambda = 1\),
\[
(\lambda^k)_{k \in \mathbb{N}} = (1, 1, 1, \ldots)_{k \in \mathbb{N}} \in X
\]
(see Examples 3.1).
For \(|\lambda| < 1\), since
\[
(\lambda^k)_{k \in \mathbb{N}} \in l_1,
\]
by the prior proposition, \((\lambda^k)_{k \in \mathbb{N}} \in X\).
For \(|\lambda| > 1\), since
\[
\left(\frac{\lambda^k}{k}\right) \notin c_0,
\]
we infer that \((\lambda^k)_{k \in \mathbb{N}} \notin X\).
For \(\lambda = -1\),
\[
\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k+1} - \frac{(-1)^k}{k} \right| = \sum_{k=1}^{\infty} \left| (-1)^{k+1} \left( \frac{1}{k+1} + \frac{1}{k} \right) \right| = \sum_{k=1}^{\infty} \left( \frac{1}{k+1} + \frac{1}{k} \right) = \infty,
\]
so that \((\lambda^k)_{k \in \mathbb{N}} = ((-1)^k)_{k \in \mathbb{N}} \notin X\).
Lastly, for \(|\lambda| = 1\) and \(\lambda \notin \{-1, 1\}\),
Hence, by the Isometric Isomorphisms Proposition

\[ \exists \theta \in (-\pi, \pi) \setminus \{0, \pi\} : \lambda = e^{i\theta} = \cos \theta + i \sin \theta. \]

Hence,

\[
\forall k \in \mathbb{N} : \left| \frac{e^{i\theta(k+1)}}{k+1} - \frac{e^{i\theta k}}{k} \right| = \left| \frac{e^{i\theta k}}{k} \right| \left| \frac{e^{i\theta}}{k+1} - \frac{1}{k} \right| = \left| \frac{\cos \theta + i \sin \theta}{k+1} - \frac{1}{k} \right| = \left| \frac{\cos \theta}{k+1} + \frac{i \sin \theta}{k+1} \right| \geq \frac{|\sin \theta|}{k+1}.
\]

Moreover, since \( \theta \in (-\pi, \pi) \setminus \{0, \pi\} \) so that \( |\sin \theta| \neq 0 \), by the Comparison Test,

\[
\sum_{k=1}^{\infty} \left| \frac{e^{i\theta(k+1)}}{k+1} - \frac{e^{i\theta k}}{k} \right| = \infty.
\]

Therefore, \( (\lambda^k)_{k \in \mathbb{N}} \notin X \).

3.4. Convergence.

The set \( \{e_n := (\delta_{nk})_{k \in \mathbb{N}} \}_{n \in \mathbb{N}} \), where \( \delta_{nk} \) is the Kronecker delta, is a Schauder basis for the space \( l_1 \) (see Preliminaries).

Hence, by the Isometric Isomorphisms Proposition (Proposition 3.1),

\[
J^{-1}\left\{\{e_n\}_{n \in \mathbb{N}}\right\} = \left\{J^{-1}e_n\right\}_{n \in \mathbb{N}}
\]

is a Schauder basis for \( X \) \((J : X \rightarrow l_1 \) is the corresponding isometric isomorphism), where

\[
\forall n \in \mathbb{N} : J^{-1}e_n := k \left( -\sum_{j=1}^{\infty} \delta_{nj} + \sum_{j=1}^{k-1} \delta_{nj} \right)_{k \in \mathbb{N}} = \left( k \left( -\sum_{j=k}^{\infty} \delta_{nj} \right) \right)_{k \in \mathbb{N}} = (-1, -2, \ldots, -n, 0, 0, \ldots) = -\sum_{k=1}^{n} ke_k.
\]

Furthermore, the set \( \{e_n := (\delta_{nk})_{k \in \mathbb{N}}\}_{n \in \mathbb{N}} \) is also a Schauder basis for \( X \) and

\[
\forall x := (x_k)_{k \in \mathbb{N}} \in X : x = \sum_{k=1}^{\infty} x_ke_k
\]

since

\[
\forall x := (x_k)_{k \in \mathbb{N}} \in X : \left\| x - \sum_{k=1}^{n} x_k e_k \right\| = \left\| \sum_{k=n+1}^{\infty} x_k e_k \right\| = \frac{\left| x_{n+1} \right|}{n+1} + \sum_{k=n+1}^{\infty} \left| \frac{x_{k+1}}{k+1} - \frac{x_k}{k} \right| \rightarrow 0, \quad n \rightarrow \infty.
\]

Remark 3.3. Similarly, by the Isometric Isomorphisms Proposition (Proposition 3.1),

\[
J_2^{-1}\left\{\{e_n\}_{n \in \mathbb{N}}\right\} = \left\{ -\sum_{k=1}^{n} e_k \right\}_{n \in \mathbb{N}}
\]
is a Schauder basis for the space $bv_0$ ($J_2 : bv_0 \to l_1$ is the corresponding isometric isomorphism) as well as the set $\{e_n := (\delta_{nk})_{k \in \mathbb{N}}\}_{n \in \mathbb{N}}$.

Based on the above, we obtain the following

**Corollary 3.1 (Characterization of Convergence in $(X, \| \cdot \|)$).**

A sequence $\left(x^{(n)} := \left(x_k^{(n)}\right)_{k \in \mathbb{N}}\right)_{n \in \mathbb{N}}$ converges to $x := (x_k)_{k \in \mathbb{N}}$ in the space $(X, \| \cdot \|)$, i.e.,

$$x^{(n)} \to x, \ n \to \infty,$$

iff

1. $\forall k \in \mathbb{N} : \ x_k^{(n)} \to x_k, \ n \to \infty$, and
2. $\forall \varepsilon > 0 \ \exists K \in \mathbb{N} \ \forall n \in \mathbb{N} : \ \sum_{k=K+1}^{\infty} \left|\frac{x_{k+1}^{(n)}}{k+1} - \frac{x_k^{(n)}}{k}\right| < \varepsilon.$

**Proof.** Since the set $\{e_n := (\delta_{nk})_{k \in \mathbb{N}}\}_{n \in \mathbb{N}}$ is a Schauder basis for the space $(X, \| \cdot \|)$ with the coordinate functionals

$$X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto c_k(x) = x_k, \ k \in \mathbb{N},$$

(see (3.7)), by the **General Characterization of Convergence Proposition** (Theorem 2.1), a sequence $\left(x^{(n)} := \left(x_k^{(n)}\right)_{k \in \mathbb{N}}\right)_{n \in \mathbb{N}}$ converges to $x := (x_k)_{k \in \mathbb{N}}$ in the space $(X, \| \cdot \|)$, i.e.,

$$x^{(n)} \to x, \ n \to \infty,$$

iff

1. $\forall k \in \mathbb{N} : \ c_k \left(x^{(n)}\right) = x_k^{(n)} \to c_k(x) = x_k, \ n \to \infty$ and
2. $\forall \varepsilon > 0 \ \exists K_0 \in \mathbb{N} \ \forall K \geq K_0 \ \forall n \in \mathbb{N} : \ \left\|\sum_{k=K+1}^{\infty} x_k^{(n)} c_k\right\| = \left|\frac{x_{K+1}^{(n)}}{K+1}\right| + \sum_{k=K+1}^{\infty} \left|\frac{x_{k+1}^{(n)}}{k+1} - \frac{x_k^{(n)}}{k}\right| < \varepsilon$

(see (3.8)).

Thus, condition (1) is equivalent to condition (1').

It is clear that condition (2') implies seemingly weaker condition (2).

For the reverse implication, by condition (2), suppose that

$$\forall \varepsilon > 0 \ \exists K_0 \in \mathbb{N} \ \forall n \in \mathbb{N} : \sum_{k=K_0+1}^{\infty} \left|\frac{x_{k+1}^{(n)}}{k+1} - \frac{x_k^{(n)}}{k}\right| < \frac{\varepsilon}{2}.$$
Then, since \( x^{(n)} \in X, n \in \mathbb{N} \), we further have:

\[
\forall K \geq K_0 \forall n \in \mathbb{N} : \ \frac{\varepsilon}{2} > \sum_{k=K_0+1}^{\infty} \left| \frac{x^{(n)}_{k+1}}{k+1} - \frac{x^{(n)}_k}{k} \right| \geq \sum_{k=K+1}^{\infty} \left| \frac{x^{(n)}_{k+1}}{k+1} - \frac{x^{(n)}_k}{k} \right| \\
\geq \sum_{k=K+1}^{\infty} \left( \left| \frac{x^{(n)}_k}{k} \right| - \left| \frac{x^{(n)}_{k+1}}{k+1} \right| \right) \\
= \frac{|x^{(n)}_{K+1}|}{K+1} - \lim_{m \to \infty} \frac{|x^{(n)}_{m+1}|}{m+1} = \frac{|x^{(n)}_{K+1}|}{K+1},
\]

and hence,

\[
\frac{|x^{(n)}_{K+1}|}{K+1} + \sum_{k=K+1}^{\infty} \left| \frac{x^{(n)}_{k+1}}{k+1} - \frac{x^{(n)}_k}{k} \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Therefore, condition \((2)'\) is implied by condition \((2)\) so that both conditions are equivalent. \(\square\)

**Remark 3.4.** Condition \((2)\) can be equivalently restated as follows:

\[
\sup_{n \in \mathbb{N}} \sum_{k=K+1}^{\infty} \left| \frac{x^{(n)}_{k+1}}{k+1} - \frac{x^{(n)}_k}{k} \right| \to 0, \ K \to \infty.
\]

4. **Bounded Weighted Backward Shifts**

**Lemma 4.1 (Norm Identities).**

*On the space \((X, \| \cdot \|)\), the backward shift*

\[
X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto Ax := (x_{k+1})_{k \in \mathbb{N}} \in X
\]

*is a bounded linear operator subject to the following norm identities:*

\[
\| A^n \| = n + 1, \ n \in \mathbb{N}.
\]

*In particular, \(\| A \| = 2\).*

**Proof.** Let

\[
bv_0 \ni y := (y_k)_{k \in \mathbb{N}} \mapsto \hat{A} y := (y_{k+1})_{k \in \mathbb{N}} \in bv_0.
\]

The fact that the mapping \(\hat{A}\) is well defined and linear on \(bv_0\) is obvious.

The linear operator \(\hat{A}\) is also bounded since

\[
\forall y := (y_k)_{k \in \mathbb{N}} : \sum_{k=1}^{\infty} |y_{k+2} - y_{k+1}| = \sum_{k=2}^{\infty} |y_{k+1} - y_k| \leq \sum_{k=1}^{\infty} |y_{k+1} - y_k| = \| y \|_0.
\]

In particular, we infer

\[
\| \hat{A} \| \leq 1.
\]

Let \(x := (x_k)_{k \in \mathbb{N}} \in X\) be arbitrary.
By the Isometric Isomorphisms Proposition (Proposition 3.1),

\[(4.4) \quad \exists y := (y_k)_{k \in \mathbb{N}} \in bv_0 : x = J_1^{-1} y := (ky_k)_{k \in \mathbb{N}} \]

\((J_1 : X \to bv_0)\) is the corresponding isometric isomorphism).

Hence, for any \(n \in \mathbb{N}\),

\[(4.5) \quad (x_{k+n})_{k \in \mathbb{N}} \quad \text{by (4.4);} \]

\[= ((k + n)y_{k+n})_{k \in \mathbb{N}} = (ky_{k+n})_{k \in \mathbb{N}} + n(y_{k+n})_{k \in \mathbb{N}} \]

by the Isometric Isomorphisms Proposition (Proposition 3.1) and (4.2);

\[= J_1^{-1} \hat{A}^n y + n \hat{A}^n y \quad \text{since } n \hat{A}^n y \in bv_0 \subset X \]

by the Continuous and Dense Embeddings Proposition (Proposition 3.2) and (4.4);

\[= J_1^{-1} \hat{A}^n J_1 x + n \hat{A}^n J_1 x \in X.\]

Hence, the operator \(A\) is well defined on \(X\) by (4.1) and is, obviously, linear.

Let \(x \in X\) with \(\|x\| = 1\) be arbitrary.

For any \(n \in \mathbb{N}\), by (4.5),

\[\|A^n x\| = \left\| J_1^{-1} \hat{A}^n J_1 x + n \hat{A}^n J_1 x \right\| \leq \left\| J_1^{-1} \hat{A}^n J_1 x \right\| + n \left\| \hat{A}^n J_1 x \right\| \]

by the Continuous and Dense Embeddings Proposition (Proposition 3.2), estimates (3.2);

\[\leq \left\| J_1^{-1} \hat{A}^n J_1 x \right\| + n \left\| \hat{A}^n J_1 x \right\|_0 \]

by the Isometric Isomorphisms Proposition (Proposition 3.1)

\[= \left\| \hat{A}^n J_1 x \right\|_0 + n \left\| \hat{A}^n J_1 x \right\|_0 = (n + 1) \left\| \hat{A}^n J_1 x \right\|_0 \leq (n + 1) \left\| \hat{A} \right\|_0^n \left\| J_1 x \right\|_0 \]

by estimate (4.3);

\[\leq (n + 1) \left\| J_1 x \right\|_0 \]

by the Isometric Isomorphisms Proposition (Proposition 3.1)

\[= (n + 1) \|x\| = n + 1.\]

Therefore, \(A\) is a bounded linear operator on \((X, \| \cdot \|)\) and

\[\|A^n\| \leq n + 1, \ n \in \mathbb{N}.\]

Let \(n \in \mathbb{N}\) be arbitrary and

\[x_k^{(n)} := \begin{cases} k, & k \leq n + 1, \\ 0, & k > n + 1, \end{cases} \quad k \in \mathbb{N}.\]

Then, in view of the Continuous and Dense Embeddings Proposition (Proposition 3.2),

\[x^{(n)} := (x_k^{(n)})_{k \in \mathbb{N}} \in c_0 \subset X.\]
Since 
\[ \| x^{(n)} \| = \sum_{k=1}^{n} \left| \frac{k+1}{k} - \frac{k}{k} \right| + \left| 0 - \frac{n+1}{n+1} \right| = 1 \]
and
\[ \| A^n x^{(n)} \| = \left\| (x^{(n)}_{k+n})_{k \in \mathbb{N}} \right\| = \left| 0 - \frac{n+1}{1} \right| = n + 1, \]
we infer that 
\[ n + 1 \leq \| A^n \|, \; n \in \mathbb{N}. \]
Thus, we conclude that
\[ \| A^n \| = n + 1, \; n \in \mathbb{N}. \]

\[ \Box \]

**Remarks 4.1.**

- By the prior lemma, we conclude that the space \((X, \| \cdot \|)\) is **shift invariant**.
- As follows from the proof of the prior lemma, for the backward shift \(X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto Ax := (x_{k+1})_{k \in \mathbb{N}} \in X\) on the space \((X, \| \cdot \|)\), the following representations
  \[ \forall n \in \mathbb{N} : A^n = J_1^{-1} \hat{A}^n J_1 + n \hat{A}^n J_1 \]
hold.

**Theorem 4.1 (Bounded Weighted Backward Shifts).**

Let \(w \in \mathbb{F}\), the bounded linear weighted backward shift operator
\[ X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto A_w x := w (x_{k+1})_{k \in \mathbb{N}} \in X \]
on the space \((X, \| \cdot \|)\) is

1. nonhypercyclic for \(|w| < 1\),
2. hypercyclic but not chaotic for \(|w| = 1\), and
3. chaotic along with every power \(A^n_w, \; n \in \mathbb{N}\), for \(|w| > 1\).

Provided the space \((X, \| \cdot \|)\) is complex (i.e., \(\mathbb{F} = \mathbb{C}\)),
\[ \sigma (A_w) = \{ \lambda \in \mathbb{C} \mid |\lambda| \leq |w| \} \]
with
\[ \sigma_p(A_w) = \{ \lambda \in \mathbb{C} \mid |\lambda| < |w| \} \cup \{ w \} \quad \text{and} \quad \sigma_c(A_w) = \{ \lambda \in \mathbb{C} \mid |\lambda| = |w| \} \setminus \{ w \}. \]

**Proof.** The fact that \(A_w = wA\), where
\[ X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto Ax := (x_{k+1})_{k \in \mathbb{N}} \in X \]
is the unweighted backward shift, is a **bounded linear operator** on \(X\) for arbitrary \(w \in \mathbb{F}\) follows from the **Norm Identities Lemma** (Lemma 4.1) by which
\[ \forall n \in \mathbb{N} : \| A^n_w \| = \| w^n A^n \| = |w|^n \| A^n \| = |w|^n (n+1). \]

With this in mind, for the following below consider \(w \in \mathbb{F}\).
(1) Let $|w| < 1$.

By identities (4.6), we infer that

$$
\forall x \in X : 0 \leq \lim_{n \to \infty} \|A^n_w x\| \leq \lim_{n \to \infty} |w|^n (n+1) \|x\| = 0.
$$

Therefore, $A_w$ is not hypercyclic.

(2) Let $|w| = 1$.

In view of $A_w$ being a bounded linear operator on $X$, for every $n \in \mathbb{N}$, $A^n_w$ is a bounded linear operator on $X$, and hence, by Characterization of Closedness for Bounded Linear Operators (see [14, 16]), is also a closed linear operator.

Since $w \neq 0$, let

$$
c_0 \ni x := (x_k)_{k \in \mathbb{N}} \mapsto B_w x := w^{-1} (x_{k-1})_{k \in \mathbb{N}},
$$

with $x_0 := 0$ and the linearity being obvious.

Inductively,

$$
\forall x := (x_k)_{k \in \mathbb{N}} \in c_0 \forall n \in \mathbb{N} : B^n_w x = w^{-n} (x_{k-n})_{k \in \mathbb{N}} \in c_0 \subset X
$$

$(x_{k-n} := 0, k = 1, \ldots, n)$.

Furthermore, for arbitrary $x := (x_k)_{k \in \mathbb{N}} \in c_0 \subset l_1 \subset X$, we have:

$$
\forall n \in \mathbb{N} : \left(\frac{x_{k-n}}{k}\right)_{k \in \mathbb{N}} \in c_0 \subset l_1 \subset X
$$

and

$$
\|B^n_w x\| = \|w^{-n} (x_{k-n})_{k \in \mathbb{N}}\| = |w|^{-n} \|(x_{k-n})_{k \in \mathbb{N}}\|
$$

since $|w| = 1$;

$$
= \left\|\left(\frac{x_{k-n}}{k}\right)_{k \in \mathbb{N}}\right\|_0
$$

by the Isometric Isomorphisms Proposition (Proposition 3.1);

$$
= 2 \left\|\left(\frac{x_{k-n}}{k}\right)_{k \in \mathbb{N}}\right\|_1 = 2 \left\|\left(\frac{x_k}{k+n}\right)_{k \in \mathbb{N}}\right\|_1 \leq \frac{2}{n+1} \|x\|_1 \to 0, n \to \infty.
$$

Considering $x \in c_0$, it is clear that

$$
A^n_w x \to 0, n \to \infty.
$$

Therefore,

$$
\forall x \in c_0 : A^n_w x, B^n_w x \to 0, n \to \infty.
$$

i.e., the second condition of the Sufficient Condition for Hypercyclicity (Theorem 2.2) is met.
On the other hand, the first condition is clearly met, i.e.,
\[ \forall x := (x_k)_{k \in \mathbb{N}} \in c_{00} : A_w B_w x = w ((B_w x)_{k+1})_{k \in \mathbb{N}} = (x_k)_{k \in \mathbb{N}} := x. \]

Furthermore, the following is clear:
\[ c_{00} \subseteq C^\infty (A) := \bigcap_{n=1}^\infty D(A^n) \subseteq X. \]

Therefore, since \( c_{00} \) is dense in the space \((X, \| \cdot \|)\) (see Remarks 3.1 and Proposition 3.2), by the Sufficient Condition for Hypercyclicity (Theorem 2.2), \( A_w \) is hypercyclic.

In [8], it is shown that the only periodic points of \( A_w \) are constant sequences.

Here, we generalize the proof and obtain the following:

\[ \text{(4.7)} \quad \text{Per}(A_w) \subseteq \text{span}\{(1, 1, 1, \ldots )\}. \]

Indeed, by way of contradiction, suppose
\[ \exists x := (x_k)_{k \in \mathbb{N}} \in \text{Per}(A_w) \setminus \text{span}\{(1, 1, 1, \ldots )\}. \]

Then,
\[ \exists N \in \mathbb{N} : x = A^N_w x = w^N (x_{k+N})_{k \in \mathbb{N}} \]
and
\[ \exists j \in \mathbb{N} : x_j \neq x_{j+1}. \]

Therefore,
\[ \forall k \in \mathbb{N} : w^{kN} x_{j+kN} = x_j \neq x_{j+1} = w^{kN} x_{j+1+kN} \]
and hence, in view of \( |w| = 1 \),
\[ \infty > \|x\| \geq \sum_{k=1}^\infty \left| \frac{x_{j+kN}}{j+1+kN} - \frac{x_{j+kN}}{j+kN} \right| = \sum_{k=1}^\infty \left| \frac{x_{j+1}}{j+1+kN} - \frac{x_j}{j+kN} \right|. \]

However, since \( x_j \neq x_{j+1} \),
\[ \lim_{k \to \infty} \left| \frac{x_{j+1} - x_j}{j+1+kN} - \frac{x_j}{(j+kN)(j+1+kN)} \right| = 1. \]
so that, by the *Comparison Test*,
\[
\sum_{k=1}^{\infty} \left| \frac{x_{j+1} - x_j}{j + 1 + kN} - \frac{x_j}{(j + kN)(j + 1 + kN)} \right| = \infty.
\]

The obtained contradiction implies inclusion (4.7), which, in view of the fact that the one-dimensional subspace span \{1, 1, 1, \ldots\} of constant sequences is nowhere dense in the infinite-dimensional space \((X, \| \cdot \|)\) (see, e.g., [14, 16]), implies that
\[
\overline{\text{Per}(A_w)} \neq X.
\]

Whence, we conclude that the operator \(A_w\) is *not* chaotic.

(3) Let \(|w| > 1\).

Similar to the case of \(|w| = 1\), it is clear that \(A_w^n\) is a closed linear operator for any \(n \in \mathbb{N}\).

Further, since \(w \neq 0\), let
\[
epsilon \exists x := (x_k)_{k \in \mathbb{N}} \rightarrow B_w x := w^{-1} (x_{k-1})_{k \in \mathbb{N}} \subseteq c_0 \subset X
\]
with \(x_0 := 0\).

Inductively,
\[
\forall x := (x_k)_{k \in \mathbb{N}} \subseteq c_0 \forall n \in \mathbb{N} : \quad B_w^n x = w^{-n} (x_{k-n})_{k \in \mathbb{N}} \subseteq c_0 \subset X
\]
\((x_{k-n} := 0, k = 1, \ldots, n)\).

Hence, for arbitrary \(x := (x_k)_{k \in \mathbb{N}} \subseteq c_0 \subset l_1 \subset X \) and \(n \in \mathbb{N}\),
\[
\|B_w^n x\| = \|w^{-n} (x_{k-n})_{k \in \mathbb{N}}\| = |w|^{-n} \| (x_{k-n})_{k \in \mathbb{N}}\|
\]
by the *Continuous and Dense Embeddings Proposition* (Proposition 3.2),

\[
\leq |w|^{-n} \| (x_{k-n})_{k \in \mathbb{N}}\|_1 = |w|^{-n} \|x\|_1.
\]

In view of \(\|x\|_1 < \infty\) being fixed, \((A_w^n x)_{n \in \mathbb{N}}\) eventually zero, and \(|w| > 1\),
\[
\forall x \in c_0 \exists \alpha(x) \in ([|w|^{-1}, 1) \subset (0, 1), \quad \exists c = c(x, \alpha) > \|x\|_1 \geq 0 \forall n \in \mathbb{N} : \quad \max (\|A_w^n x\|, \|B_w^n x\|) \leq \max (\|A_w^n x\|, |w|^{-n} \|x\|_1) \leq c \alpha^n.
\]
i.e., the second condition of the *Sufficient Condition for Linear Chaos* (Theorem 2.3) is met.

Furthermore,
\[
\forall x := (x_k)_{k \in \mathbb{N}} \subseteq c_0 : \quad A_w B_w x = w ((B_w x)_{k+1})_{k \in \mathbb{N}} = (x_k)_{k \in \mathbb{N}} := x.
\]
i.e., the first condition of the *Sufficient Condition for Linear Chaos* (Theorem 2.3) is also met.

Lastly, the space \(c_0 \subset X\) is a dense subspace satisfying
\[
c_0 \subseteq C^\infty (A_w) := \bigcap_{n=1}^{\infty} D(A_w^n) \subseteq X.
\]
Hence, by the **Sufficient Condition for Linear Chaos** (Theorem 2.3) and the **Chaoticity of Powers Corollary** (Corollary 2.1), \( A_w \) is a chaotic linear operator as well as its every power \( A_n^w \) \((n \in \mathbb{N})\).

Let us now analyze the spectral structure of \( A \).

For this purpose, suppose that the space \((X, \| \cdot \|)\) is complex.

Then, for \( \lambda \in \mathbb{C} \setminus \{0\} \), by the **Exponential Sequences Example** (Example 3.3),
\[
A - \lambda I \text{ is not injective } \iff \exists x := (x_k)_{k \in \mathbb{N}} \in X \setminus \{(0,0,0,\ldots)\} : (A - \lambda I)x = 0
\]
\[
\iff \exists x := (x_k)_{k \in \mathbb{N}} \in X \setminus \{(0,0,0,\ldots)\} : Ax = \lambda x
\]
\[
\iff \exists x := (x_k)_{k \in \mathbb{N}} \in X \setminus \{(0,0,0,\ldots)\} : x_{k+1} = \lambda x_k
\]
\[
\iff \exists x_1 \in \mathbb{C} : x_1 (\lambda^{k-1})_{k \in \mathbb{N}} \in X \setminus \{(0,0,0,\ldots)\}
\]
\[
\iff (\lambda^{k-1})_{k \in \mathbb{N}} \in X \setminus \{(0,0,0,\ldots)\}
\]
\[
\iff (\lambda^k)_{k \in \mathbb{N}} = \lambda (\lambda^{k-1})_{k \in \mathbb{N}} \in X \setminus \{(0,0,0,\ldots)\}
\]
\[
\iff 0 < |\lambda| < 1 \text{ or } \lambda = 1.
\]

Furthermore, for the case of \( \lambda = 0 \), it is clear that \( A \) is *not* injective since for \( e_1 := (\delta_{1k})_{k \in \mathbb{N}} \in X \setminus \{(0,0,0,\ldots)\} \),
\[
Ae_1 = (0,0,0,\ldots).
\]

Hence,
\[
\sigma_p(A) = \{ \lambda \in \mathbb{C} \mid |\lambda| < 1 \} \cup \{1\}.
\]

In view of \( A \) being a closed linear operator, \( \sigma(A) \subseteq \mathbb{C} \) is a closed subset (see, e.g., [14]) so that
\[
\{ \lambda \in \mathbb{C} \mid |\lambda| \leq 1 \} = \overline{\sigma_p(A)} \subseteq \sigma(A).
\]

Furthermore, since \( A \) is a bounded linear operator on the complex Banach space \((X, \| \cdot \|)\), by **Gelfand’s Spectral Radius Theorem** (see, e.g., [14]), considering the **Norm Identities Lemma** (Lemma 4.1),
\[
\max_{\lambda \in \sigma(A)} |\lambda| = \lim_{n \to \infty} ||A^n||^{\frac{1}{n}} = \lim_{n \to \infty} (n + 1)^{\frac{1}{n}} = \lim_{n \to \infty} e^{\frac{\ln(n+1)}{n}} = e^0 = 1.
\]

Therefore, in view of (4.9) and (4.10), the following holds:
\[
\sigma(A) = \{ \lambda \in \mathbb{C} \mid |\lambda| \leq 1 \}.
\]

Since the operator \( A \) is hypercyclic, by [12, Proposition 4.1]
\[
\sigma_r(A) = \emptyset.
\]

Therefore, since \( \sigma_p(A), \sigma_c(A), \) and \( \sigma_r(A) \) partition \( \sigma(A) \),
\[
\sigma_c(A) = \{ \lambda \in \mathbb{C} \mid |\lambda| = 1 \} \setminus \{1\}.
\]

Hence, provided the space \((X, \| \cdot \|)\) is complex, since \( \sigma_p(A) = \sigma(A) = \{0\} \) is clear, by (4.8), (4.11), and (4.12),
\( \forall w \in \mathbb{C} : \sigma(A_w) = \{ \lambda \in \mathbb{C} \mid |\lambda| \leq |w| \} \)

with

\( \sigma_p(A_w) = \{ \lambda \in \mathbb{C} \mid |\lambda| < |w| \} \cup \{ w \} \)

and

\( \sigma_c(A_w) = \{ \lambda \in \mathbb{C} \mid |\lambda| = |w| \} \setminus \{ w \} \).

\( \square \)

**Remarks 4.2.**

- For \( w \in \mathbb{F} \) and \( |w| < 1 \), when the space \((X, \| \cdot \|)\) is complex, by (4.13),
  \[ \sigma(A_w) = \sigma(wA) = w \sigma(A) = \{ \lambda \in \mathbb{C} \mid |\lambda| \leq |w| < 1 \}. \]
  Hence,
  \[ \sigma(A_w) \cap \{ \lambda \in \mathbb{C} \mid |\lambda| = 1 \} = \emptyset, \]
  so that, by Theorem 2.4 (see [10, Theorem 5.6]), \( A_w \) is not hypercyclic.

- For \( w \in \mathbb{F} \) and \( |w| = 1 \), the fact that \( A_w \) is not chaotic coincides with \( B_w \) not satisfying the second condition of the Sufficient Condition for Linear Chaos (Theorem 2.3).
  Indeed, for \( e_1 := (1, 0, 0, 0, \ldots) \in c_0 \),
  \[ \| B^n_w e_1 \| = |w|^n \| B^n_1 e_1 \| = \| B^n_1 e_1 \| \]
  \[ = \left| \frac{1}{n+1} - \frac{0}{n} \right| + \left| \frac{0}{n+2} - \frac{1}{n+1} \right| = \frac{2}{n+1}. \]
  However,
  \[ \forall \alpha \in (0, 1) \ \forall c > 0 : \lim_{n \to \infty} \frac{\| B^n_w e_1 \|}{c \alpha^n} = \frac{1}{c} \lim_{n \to \infty} \frac{2}{(n+1)\alpha^n} = \infty. \]

- For \( w \in \mathbb{F} \) and \( |w| = 1 \),
  \[ \text{Per}(A_w) \subseteq \text{span}\{(1, 1, 1, \ldots)\}. \]
  From this, the following holds:
  \[ \text{Per}(A_w) = \text{span}\{(1, 1, 1, \ldots)\} \iff \exists N \in \mathbb{N} : w^N = 1 \]
  and
  \[ \text{Per}(A_w) = \emptyset \iff \forall N \in \mathbb{N} : w^N \neq 1. \]

- For \( w \in \mathbb{F} \) with \( |w| = 1 \), when the space \((X, \| \cdot \|)\) is complex, by (4.14),
  \[ \sigma_p(A_w) \cap \{ \lambda \in \mathbb{C} \mid |\lambda| = 1 \} = \{ w \}. \]
  Hence, \( \sigma_p(A_w) \) doesn’t contain infinitely many roots of unity so that, by Proposition 2.1 (see [10, Proposition 5.7]), \( A_w \) is not chaotic.
5. Unbounded Weighted Backward Shifts

Lemma 5.1. Let \( w \in \mathbb{F} \) and \(|w| > 1\). Then, for the weighted backward shift operator

\[
A_w x := (w^k x_{k+1})_{k \in \mathbb{N}}
\]

in the space \((X, \| \cdot \|)\) with maximal domain

\[
D(A_w) := \left\{ x := (x_k)_{k \in \mathbb{N}} \in X \left| (w^k x_{k+1})_{k \in \mathbb{N}} \in X \right. \right\},
\]

each power

\[
A_n^w x = \left( \prod_{j=k}^{k+n-1} w^j \right) x_{k+n}, \quad n \in \mathbb{N},
\]

with domain

\[
D(A_n^w) = \left\{ x := (x_k)_{k \in \mathbb{N}} \in X \left| \left( \prod_{j=k}^{k+n-1} w^j \right) x_{k+n} \right. \in X \right\},
\]

is a densely defined unbounded closed linear operator and the subspace

\[
C^\infty(A_w) := \bigcap_{n=1}^{\infty} D(A_n^w)
\]

is dense in \((X, \| \cdot \|)\).

Proof. Let \( w \in \mathbb{F} \) with \(|w| > 1\) be arbitrary.

Similarly to [17, Lemma 3.1],

\[
\forall n \in \mathbb{N} : A_n^w x = \left( \prod_{j=k}^{k+n-1} w^j \right) x_{k+n},
\]

with domain

\[
D(A_n^w) = \left\{ x := (x_k)_{k \in \mathbb{N}} \in X \left| \left( \prod_{j=k}^{k+n-1} w^j \right) x_{k+n} \right. \in X \right\},
\]

where

\[
c_{00} \subseteq D(A_{n+1}^w) \subseteq D(A_n^w).
\]

Since \( c_{00} \subset X \) is a dense subspace (see Remarks 3.1 and Proposition 3.2), it also follows verbatim [17, Lemma 3.1], that each power \( A_n^w \) (\( n \in \mathbb{N} \)) is densely defined with

\[
C^\infty(A_w) := \bigcap_{n=1}^{\infty} D(A_n^w)
\]

being dense in the space \((X, \| \cdot \|)\).

Let \( x_m := (\delta_{mk})_{k \in \mathbb{N}} \in c_{00} \subset X, \ m \in \mathbb{N} \).

If \( m \geq 2 \), then

\[
\| \frac{m}{2} e_m \| = \frac{m}{2} \| e_m \| = \frac{m}{2} \left( \left| \frac{1}{m} \right| - \left| \frac{0}{m-1} \right| + \left| \frac{0}{m+1} - \frac{1}{m} \right| \right) = \frac{m}{2} \frac{2}{m} = 1.
\]
For an arbitrary \( n \in \mathbb{N} \), in view of \(|w| > 1\),
\[
\forall m \geq 2 : \left\| A^n_w e^{m+n} \right\| = \frac{m+n}{2} \left\| A^n_w e^m \right\|
= \frac{m+n}{2} \left\| \left( \prod_{j=k}^{k+n-1} w^j \right) \delta_{(m+n)(k+n)} \right\|
= \frac{m+n}{2} \left( \prod_{j=m}^{m+n-1} \left| \frac{0}{m-1} \right| + \left| \frac{m+n-1}{m} \right| \right)
= \frac{m+n}{m} \prod_{j=m}^{m+n-1} \left| w^j \right| \geq \prod_{j=m}^{m+n-1} \left| w^j \right| \geq |w|^{mn} \to \infty, \ m \to \infty.
\]

Therefore, in view of (5.1), \( A^n_w \) is unbounded.

Let \( n \in \mathbb{N} \) and \( \left( x^{(m)} := \left( x_k^{(m)} \right)_{k \in \mathbb{N}} \right)_{m \in \mathbb{N}} \) be a sequence in \( D(A^n_w) \) such that
\[
x^{(m)} \to x := (x_k)_{m \in \mathbb{N}} \in X, \ m \to \infty,
\]
and
\[
A^n_w x^{(m)} = \left[ \prod_{j=k}^{k+n-1} w^j \right] x_k^{(m)} \to y := (y_k)_{k \in \mathbb{N}} \in X, \ m \to \infty.
\]

Then, by the Characterization of Convergence in \((X, \| \cdot \|)\) Corollary (Corollary 3.1), for any \( k \in \mathbb{N} \),
\[
x_k^{(m)} \to x_k, \ m \to \infty,
\]
and
\[
\left[ \prod_{j=k}^{k+n-1} w^j \right] x_k^{(m)} \to y_k, \ m \to \infty.
\]

Proceeding as in [17, Lemma 3.1], we infer that, for each \( k \in \mathbb{N} \),
\[
\left[ \prod_{j=k}^{k+n-1} w^j \right] x_{k+n} = y_k,
\]
which means that
\[
\left( \left[ \prod_{j=k}^{k+n-1} w^j \right] x_{k+n} \right)_{k \in \mathbb{N}} = y \in X.
\]

Whence, we conclude that \( x \in D(A^n_w) \) and \( y = A^n_w x \), which, by the Sequential Characterization of Closed Linear Operators (see, e.g., [14, 16]), implies the operator \( A^n_w \) is closed. \( \square \)
Theorem 5.1 (Unbounded Weighted Backward Shifts).

For an arbitrary $w \in F$ with $|w| > 1$, the unbounded linear weighted backward shift operator

$$A_w x := \left( w^k x_{k+1} \right)_{k \in \mathbb{N}}$$

in the space $(X, \| \cdot \|)$ with maximal domain

$$D(A_w) := \left\{ x := (x_k)_{k \in \mathbb{N}} \in X \mid \left( w^k x_{k+1} \right)_{k \in \mathbb{N}} \in X \right\}$$

is chaotic as well as its every power $A^n_w$, $n \in \mathbb{N}$.

Furthermore, each $\lambda \in F$ is an eigenvalue for $A_w$ of geometric multiplicity 1, i.e.,

$$\dim \ker(A_w - \lambda I) = 1.$$ 

In particular, provided the space $(X, \| \cdot \|)$ is complex,

$$\sigma_p(A_w) = \mathbb{C}.$$ 

Proof. Let $w \in F$ with $|w| > 1$ be arbitrary.

Consider the following mapping:

$$c_00 \ni x := (x_k)_{k \in \mathbb{N}} \mapsto B_w x := \left( w^{-(k-1)} x_{k-1} \right)_{k \in \mathbb{N}} \in c_00 \quad (x_0 := 0).$$

For which, it is clear that $B_w$ is a well-defined linear operator such that

$$(5.2) \quad A_w B_w x = x, \quad x \in Y.$$ 

i.e., the first condition of the Sufficient Condition for Linear Chaos (Theorem 2.3) is met.

Notice

$$X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto B^2_w x = \left( w^{-(k-1)} w^{-(k-2)} x_{k-2} \right)_{k \in \mathbb{N}} \quad (x_{k-2} := 0, \quad k = 1, 2)$$

and

$$X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto B^3_w x = \left( w^{-(k-1)} w^{-(k-2)} w^{-(k-3)} x_{k-3} \right)_{k \in \mathbb{N}} \quad (x_{k-3} := 0, \quad k = 1, 2, 3).$$

Inductively, for any $n \in \mathbb{N}$,

$$X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto B^n_w x = \left( \prod_{j=1}^{n} w^{-(k-j)} x_{k-n} \right)_{k \in \mathbb{N}} \quad (x_{k-n} := 0, \quad k = 1, \ldots, n).$$

Hence, for arbitrary $x := (x_k)_{k \in \mathbb{N}} \in c_00 \subset l_1 \subset X$ and $n \in \mathbb{N},$

$$\| B^n_w x \| = \left\| \left( \prod_{j=1}^{n} w^{-(k-j)} x_{k-n} \right)_{k \in \mathbb{N}} \right\|$$

by the Continuous and Dense Embeddings Proposition (Proposition 3.2),

estimate (3.3);
\[
\leq \left\| \left( \prod_{j=1}^{n} w^{-(k-j)} \right) x_{k-n} \right\|_{1} = \left\| \left( \prod_{j=1}^{n} w^{-(k+n-j)} \right) x_{k} \right\|_{1},
\]

since \(|w| > 1\);

\[
\leq \left\| \left( \prod_{j=1}^{n} w^{-1} \right) x_{k} \right\|_{1} = |w|^{-n} \|x\|_{1}.
\]

In view of \(|x|_{1} < \infty\) being fixed, \((A_{w}^{n} x)_{n \in \mathbb{N}}\) eventually zero, and \(|w| > 1\),

\[
\forall x \in c_{00} \exists \alpha(x) \in (|w|^{-1}, 1) \subset (0, 1), \exists c = c(x, \alpha) > \|x\|_{1} \geq 0 \forall n \in \mathbb{N}:
\]

\[
\max (\|A_{w}^{n} x\|, \|B_{w}^{n} x\|) \leq \max (\|A_{w}^{n} x\|, |w|^{-n} \|x\|_{1}) \leq c \alpha^{n}.
\]

i.e. the second condition of the \textit{Sufficient Condition for Linear Chaos} (Theorem 2.3) is also met.

By Lemma 5.1, the \textit{Sufficient Condition for Linear Chaos} (Theorem 2.3), and the \textit{Chaoticity of Powers Corollary} (Corollary 2.1), we conclude that the operator \(A_{w}\) is chaotic as well as its every power \(A_{w}^{n}\ (n \in \mathbb{N})\).

What follows almost verbatim mimics the reasoning of [15, Theorem 3.1] (also [17, Theorem 3.2]).

For arbitrary \(\lambda \in \mathbb{F} (\mathbb{F} := \mathbb{R} \text{ or } \mathbb{F} := \mathbb{C})\) and \(x := (x_{k})_{\mathbb{N}} \in D(A_{w})\), the equation

\[
A_{w} x = \lambda x
\]

is equivalent to

\[
(w^{k} x_{k+1})_{k \in \mathbb{N}} = \lambda (x_{k})_{k \in \mathbb{N}},
\]

i.e.,

\[
w^{k} x_{k+1} = \lambda x_{k}, \ k \in \mathbb{N}
\]

Whence, we recursively infer that

\[
x_{k} = \left( \prod_{j=1}^{k-1} \frac{\lambda}{w^{k-j}} \right) x_{1} = \frac{\lambda^{k-1}}{w^{\sum_{j=1}^{k-1} (k-j)}} x_{1} = \frac{\lambda^{k-1}}{w^{\frac{k(k-1)}{2}}} x_{1} = \left( \frac{\lambda}{w^{\frac{k}{2}}} \right)^{k-1} x_{1}, \ k \in \mathbb{N},
\]

where for \(\lambda = 0, 0^{0} := 1\).

Considering that \(|w| > 1\), for all sufficiently large \(k \in \mathbb{N}\), we have:

\[
\left| \frac{\lambda}{w^{\frac{k}{2}}} \right|^{k-1} = \left( \frac{|\lambda|}{|w|^{\frac{k}{2}}} \right)^{k-1} \leq \left( \frac{1}{2} \right)^{k-1},
\]

which implies, by the \textit{Comparison Test} and the \textit{Continuous and Dense Embeddings Proposition} (Proposition 3.2),

\[
y := (y_{k})_{k \in \mathbb{N}} := \left( \left( \frac{\lambda}{w^{\frac{k}{2}}} \right)^{k-1} \right)_{k \in \mathbb{N}} \in l_{1} \subset X.
\]
Further, since
\[ w^k y_{k+1} = w^k \frac{\lambda^k}{\sum_{j=1}^{\infty} (k+1-j)} = \frac{\lambda^k}{w^{\sum_{j=2}^{\infty} (k+1-j)}} = \frac{\lambda^k}{w^{k+1}} x_1 = \left( \frac{\lambda}{w^{k+1}} \right)^k, \quad k \in \mathbb{N}, \]
we similarly conclude that
\[ (w^k y_{k+1})_{k \in \mathbb{N}} \in X, \]
and hence,
\[ y \in D(A_w) \setminus \{0\}. \]
Thus, we have shown that, for any \( \lambda \in \mathbb{F} \),
\[ \ker(A_w - \lambda I) = \text{span} \{ \{ y \} \} \subseteq D(A_w), \]
and hence,
\[ \dim \ker(A_w - \lambda I) = 1, \]
which completes the proof. \( \Box \)

6. More Hypercyclicity and Linear Chaos

Here, we discuss how the known chaos generates new chaos via the conjugacy relative the isometric isomorphism between the spaces \((X, \| \cdot \|)\) and \(l_1\) (see Section 3.2).

**Theorem 6.1** (More Bounded Linear Chaos in \(X\)).
For \(w \in \mathbb{F}\), the bounded linear operator
\[ X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto \hat{A}_w x := w \left( \frac{k}{k + 1} x_{k+1} \right)_{k \in \mathbb{N}} \in X \]
on the space \((X, \| \cdot \|)\) is chaotic as well as its every power\( \hat{A}_w^n x = w^n \left( \frac{k}{k + n} x_{k+n} \right)_{k \in \mathbb{N}} \in X, \quad n \in \mathbb{N} \)
when \(|w| > 1\) and nonhypercyclic otherwise when \(|w| \leq 1\).
Provided the space \((X, \| \cdot \|)\) is complex
\[ \sigma \left( \hat{A}_w \right) = \{ \lambda \in \mathbb{C} \mid |\lambda| \leq |w| \}\]
with
\[ \sigma_p \left( \hat{A}_w \right) = \{ \lambda \in \mathbb{C} \mid |\lambda| < |w| \} \quad \text{and} \quad \sigma_c \left( \hat{A}_w \right) = \{ \lambda \in \mathbb{C} \mid |\lambda| = |w| \}. \]

**Proof.** Let \(w \in \mathbb{F}\) be arbitrary.
The weighed backward shift
\[ l_1 \ni z := (z_k)_{k \in \mathbb{N}} \mapsto A_w z := w(z_{k+1})_{k \in \mathbb{N}} \in l_1 \]
is a bounded linear operator which is chaotic \([19]\) as well as its every power \( \hat{A}_w^n \) \((n \in \mathbb{N})\) when \(|w| > 1\), being subject to the Sufficient Condition for Linear Chaos (Theorem 2.3) (see \([12, \text{Examples 3.1}]\)), and is nonhypercyclic when \(|w| \leq 1\) since \(\|A_w\| = |w| \leq 1\) (see, e.g., \([14]\)).
By the *Isometric Isomorphisms Proposition* (Proposition 3.1), the mapping
\[
(6.1) \quad X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto Jx := \left(\frac{x_{k+1}}{k+1} - \frac{x_k}{k}\right)_{k \in \mathbb{N}} \in l_1
\]
is an isometric isomorphism between \((X, \| \cdot \|)\) and \(l_1\) with the inverse
\[
(6.2) \quad l_1 \ni z := (z_k)_{k \in \mathbb{N}} \mapsto J^{-1}z := \left(\sum_{j=0}^{k-1} z_j\right)_{k \in \mathbb{N}} \in X,
\]
where
\[
z_0 := -\sum_{k=1}^{\infty} z_k.
\]

On the space \((X, \| \cdot \|)\), consider the linear operator
\[
\hat{A}_w := J^{-1}A_w J
\]
naturally emerging from the commutative diagram
\[
\begin{array}{ccc}
  l_1 & \xrightarrow{A_w} & l_1 \\
  J & \downarrow & J \\
  X & \xrightarrow{\hat{A}_w} & X
\end{array}
\]

Since
\[
\hat{A}_w^n := J^{-1}A_w^n J, \quad n \in \mathbb{N},
\]
where
\[
l_1 \ni z := (z_k)_{k \in \mathbb{N}} \mapsto A_w^n z := w^n(z_{k+n})_{k \in \mathbb{N}} \in l_1,
\]
for any \(x := (x_k)_{k \in \mathbb{N}} \in X\),
\[
\hat{A}_w^n x = J^{-1}A_w^n Jx = J^{-1}A_w^n \left(\frac{x_{k+1}}{k+1} - \frac{x_k}{k}\right)_{k \in \mathbb{N}} = J^{-1}w^n \left(\frac{x_{k+1+n}}{k+1+n} - \frac{x_{k+n}}{k+n}\right)_{k \in \mathbb{N}}
\]
\[
= w^n \left(\frac{1}{k+n} \sum_{j=0}^{k-1} z_j\right)_{k \in \mathbb{N}} = w^n \left(\frac{k}{k+n} x_{k+n}\right)_{k \in \mathbb{N}}
\]
since, for
\[
z_k := \frac{x_{k+1+n}}{k+1+n} - \frac{x_{k+n}}{k+n}, \quad k \in \mathbb{N},
\]
\[
z_0 := -\sum_{k=1}^{\infty} z_k = -\sum_{k=1}^{\infty} \left(\frac{x_{k+1+n}}{k+1+n} - \frac{x_{k+n}}{k+n}\right) = \frac{x_{1+n}}{1+n},
\]
and hence,
\[
\sum_{j=0}^{k-1} z_j = z_0 + \sum_{j=1}^{k-1} \left(\frac{x_{j+1+n}}{j+1+n} - \frac{x_{j+n}}{j+n}\right) = \frac{x_{1+n}}{1+n} + \frac{x_{k+n}}{k+n} - \frac{x_{1+n}}{1+n} = \frac{x_{k+n}}{k+n}, \quad k \in \mathbb{N}.
\]

Further, since \(J : X \rightarrow l_1\) is an isometric isomorphism, the operator \(\hat{A}_w^n (n \in \mathbb{N})\) inherits the boundedness and chaotic/hypercyclic properties of \(A_w\) as well as its spectral structure.
Hence, $\hat{A}_w$ is chaotic as well as its every power $\hat{A}_w^n$ ($n \in \mathbb{N}$) when $|w| > 1$ and is nonhypercyclic when $|w| \leq 1$.

Provided the underlying space is complex, the spectral part of the theorem follows from the fact that

$$\sigma(A_w) = \{ \lambda \in \mathbb{C} \, | \, |\lambda| \leq |w| \}$$

with

$$\sigma_p(A_w) = \{ \lambda \in \mathbb{C} \, | \, |\lambda| < |w| \} \quad \text{and} \quad \sigma_c(A_w) = \{ \lambda \in \mathbb{C} \, | \, |\lambda| = |w| \}$$

(see, e.g., [14]).

**Theorem 6.2** (More Unbounded Linear Chaos in $X$).

For arbitrary $w \in F$ with $|w| > 1$, the linear operator

$$\hat{A}_w := \left( k \sum_{j=k}^{\infty} w^j \left( \frac{x_{j+1} - x_{j+2}}{j+1 - j+2} \right) \right)_{k \in \mathbb{N}}$$

in the space $(X, \| \cdot \|)$ with domain

$$D(\hat{A}_w) := \left\{ x := (x_k)_{k \in \mathbb{N}} \in X \, \bigg| \, \left( w^k \left( \frac{x_{k+2}}{k+2} - \frac{x_{k+1}}{k+1} \right) \right)_{k \in \mathbb{N}} \in l_1 \right\}$$

is unbounded and chaotic as well as its every power

$$\hat{A}_w^n := \left( k \sum_{j=k}^{\infty} \prod_{m=j}^{j+n-1} w^m \left( \frac{x_{j+n} - x_{j+n+1}}{j+n - j+n+1} \right) \right)_{k \in \mathbb{N}}$$

in the space $(X, \| \cdot \|)$ with domain

$$D(\hat{A}_w^n) := \left\{ x := (x_k)_{k \in \mathbb{N}} \in X \, \bigg| \, \left( \prod_{m=j}^{k+n-1} w^m \left( \frac{x_{k+n+1} - x_{k+n}}{k+1 - k+n} \right) \right)_{k \in \mathbb{N}} \in l_1 \right\}.$$

Furthermore, each $\lambda \in F$ is an eigenvalue for $\hat{A}_w$ of geometric multiplicity 1, i.e.,

$$\dim \ker(\hat{A}_w - \lambda I) = 1.$$

In particular, provided the space $(X, \| \cdot \|)$ is complex,

$$\sigma_p(\hat{A}_w) = \mathbb{C}.$$

**Proof.** Let $w \in F$ with $|w| > 1$ be arbitrary.

As is known, the unbounded weighted backward shift

$$D(A_w) \ni z := (z_k)_{k \in \mathbb{N}} \mapsto A_w z := (w^k z_{k+1})_{k \in \mathbb{N}} \in l_1$$

in $l_1$ with domain

$$D(A_w) := \left\{ z := (z_k)_{k \in \mathbb{N}} \in l_1 \bigg| \left( w^k z_{k+1} \right)_{k \in \mathbb{N}} \in l_1 \right\}.$$

is a chaotic linear operator [15, Theorem 3.1] along with every power $A_w^n$ ($n \in \mathbb{N}$) [12, Examples 3.1].
Based the Isometric Isomorphisms Proposition (Proposition 3.1), for the isometric isomorphism $J : X \to l_1$ subject to (6.1) and (6.2), in the space $(X, \| \cdot \|)$, consider the linear operator

$$\hat{A}_w := J^{-1} A_w J$$

as is given by the following commutative diagram:

$$\begin{align*}
l_1 \ni D(A_w) & \xrightarrow{A_w} l_1 \\
\uparrow & \quad \uparrow \\
X \ni D(\hat{A}_w) & \xrightarrow{\hat{A}_w} X
\end{align*}$$

where the domain of $\hat{A}_w$ is

$$D(\hat{A}_w) := J^{-1}(D(A_w)).$$

Since

$$\hat{A}_w^n = J^{-1} A_w^n J, \quad n \in \mathbb{N},$$

where

$$A_w^n z = \left( \prod_{j=k}^{k+n-1} w^j \right) z_{k+n}$$

with

$$D(A_w^n) = \left\{ z := (z_k)_{k \in \mathbb{N}} \in l_1 \mid \left( \prod_{j=k}^{k+n-1} w^j \right) z_{k+n} \in l_1 \right\}$$

(cf. [15,17]), in view of (6.1),

$$D(\hat{A}_w^n) = \left\{ x := (x_k)_{k \in \mathbb{N}} \in X \mid \left( \prod_{j=k}^{k+n-1} w^j \right) \left( \frac{x_{k+1}+n}{k+1+n} - \frac{x_{k+n}}{k+n} \right) \in l_1 \right\}.$$
where \( z_0 := -\sum_{k=1}^{\infty} \prod_{j=k}^{k+n-1} w^j \left( \frac{x_{k+1+n}}{k+1+n} - \frac{x_{k+n}}{k+n} \right) \).

Further, since \( J : X \rightarrow \ell_1 \) is an isometric isomorphism, the operator \( \hat{A}_w^n \ (n \in \mathbb{N}) \) inherits the unboundedness and chaoticity of \( A_w \) as well as its eigenvalues coupled with their geometric multiplicities.

Therefore, the operator \( \hat{A}_w \) is unbounded and chaotic as well as its every power \( \hat{A}_w^n \ (n \in \mathbb{N}) \).

Furthermore, in view of,

\[ \forall \lambda \in \mathbb{F} : \dim \ker(A_w - \lambda I) = 1 \]

(see [15, Theorem 3.1]), each \( \lambda \in \mathbb{F} \) is a simple eigenvalue for \( \hat{A}_w \). \( \square \)

**Theorem 6.3** (More Bounded Linear Chaos in \( \ell_1 \)).

For \( w \in \mathbb{F} \), the bounded linear operator

\[ l_1 \ni z := (z_k)_{k \in \mathbb{N}} \mapsto \hat{A}_w z := w \left( \frac{k+2}{k+1} z_{k+1} - \frac{1}{k(k+1)} \sum_{j=0}^{k} z_j \right) \in \ell_1 \]

with

\[ z_0 := -\sum_{k=1}^{\infty} z_k, \]

on the space \( \ell_1 \) is

1. non-hypercyclic for \( |w| < 1 \),
2. hypercyclic but not chaotic for \( |w| = 1 \), and
3. chaotic as well as its every power

\[ l_1 \ni z := (z_k)_{k \in \mathbb{N}} \mapsto \hat{A}_w^n z := w^n \left( \frac{k+1+n}{k+1} z_{k+n} - \frac{n}{k(k+1)} \sum_{j=0}^{k-1+n} z_j \right) \in \ell_1 \]

with

\[ z_0 := -\sum_{k=1}^{\infty} z_k, \]

for \( |w| > 1 \).

Provided the space \( \ell_1 \) is complex (i.e., \( \mathbb{F} = \mathbb{C} \)),

\[ \sigma(\hat{A}_w) = \{ \lambda \in \mathbb{C} | |\lambda| \leq |w| \} \]

with

\[ \sigma_p(\hat{A}_w) = \{ \lambda \in \mathbb{C} | |\lambda| < |w| \} \cup \{w\} \quad \text{and} \quad \sigma_c(\hat{A}_w) = \{ \lambda \in \mathbb{C} | |\lambda| = |w| \} \setminus \{w\}. \]

**Proof.** For \( w \in \mathbb{F} \), let

\[ X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto A_w x := w(x_{k+1})_{k \in \mathbb{N}} \in X. \]
By the Bounded Weighted Backward Shifts Theorem (Theorem 4.1), \( A_w \) is a bounded linear operator that is

(i) nonhypercyclic for \(|w| < 1\),

(ii) hypercyclic but not chaotic for \(|w| = 1\), and

(iii) chaotic along with every power \( A_w^n, n \in \mathbb{N} \), for \(|w| > 1\).

By the Isometric Isomorphisms Proposition (Proposition 3.1), in view of (6.1) and (6.2), on the space \( l_1 \), consider the linear operator

\[ \hat{A}_w := JA_w J^{-1}. \]

i.e., the following diagram commutes.

\[ \begin{array}{ccc}
  l_1 & \xrightarrow{\hat{A}_w} & l_1 \\
  \uparrow & & \uparrow \\
  X & \xrightarrow{A_w} & X \\
\end{array} \]

Since

\[ \hat{A}_w^n = JA_w^n J^{-1}, \quad n \in \mathbb{N}, \]

where

\[ X \ni x := (x_k)_{k \in \mathbb{N}} \mapsto A_w^n x = w^n (x_{k+n})_{k \in \mathbb{N}} \in X, \]

for any \( z := (z_k)_{k \in \mathbb{N}} \in l_1 \), in view of (6.1) and (6.2), the following holds:

\[ \hat{A}_w^n z = JA_w^n J^{-1} z = JA_w^n \left( k \sum_{j=0}^{k-1} z_j \right)_{k \in \mathbb{N}} = Jw^n \left( (k+n) \sum_{j=0}^{k-1+n} z_j \right)_{k \in \mathbb{N}} \]

\[ = w^n \left( \frac{k+1+n}{k+1} \frac{k+n}{k} \sum_{j=0}^{k+n} z_j \right)_{k \in \mathbb{N}} \]

\[ = w^n \left( \frac{k+1+n}{k+1} \sum_{j=0}^{k+n} z_j + \frac{k+1+n}{k+1} \frac{k-1+n}{k} \sum_{j=0}^{k-1+n} z_j - \frac{k+n}{k} \sum_{j=0}^{k-1+n} z_j \right)_{k \in \mathbb{N}} \]

\[ = w^n \left( \frac{k+1+n}{k+1} z_{k+n} - \frac{n}{k(k+1)} \sum_{j=0}^{k-1+n} z_j \right)_{k \in \mathbb{N}} \]

where

\[ z_0 := -\sum_{k=1}^{\infty} z_k. \]

Furthermore, since \( J \) is an isometric isomorphism, \( \hat{A}_w \) inherits the boundedness and chaotic/hypercyclic properties of \( A_w \) as well as its spectral structure.

Hence, \( \hat{A}_w \) is a bounded linear operator that is

1. nonhypercyclic for \(|w| < 1\),
2. hypercyclic but not chaotic for $|w| = 1$, and

3. chaotic along with every power $A^*_w$, $n \in \mathbb{N}$, for $|w| > 1$.

Provided the underlying space is complex, the spectral part of this theorem follows from the Bounded Weighted Backward Shifts Theorem (Theorem 4.1) where

$$\sigma(A_w) = \{ \lambda \in \mathbb{C} \mid |\lambda| \leq |w| \}$$

with

$$\sigma_p(A_w) = \{ \lambda \in \mathbb{C} \mid |\lambda| < |w| \} \cup \{ w \} \quad \text{and} \quad \sigma_c(A_w) = \{ \lambda \in \mathbb{C} \mid |\lambda| = |w| \} \setminus \{ w \}.$$  

\[\square\]

**Theorem 6.4** (More Unbounded Linear Chaos in $l_1$).

*For an arbitrary $w \in \mathbb{F}$ with $|w| > 1$, the linear operator

$$\hat{A}_w z := \left( w^{k+1} \frac{k+2}{k+1} \sum_{j=0}^{k+1} z_j - w^k \sum_{j=0}^k z_j \right)_{k \in \mathbb{N}}$$

with

$$z_0 := -\sum_{k=1}^{\infty} z_k,$$

and domain

$$D(\hat{A}_w) := \left\{ z := (z_k)_{k \in \mathbb{N}} \in l_1 \mid \left( w^k (k+1) \sum_{j=0}^k z_j \right)_{k \in \mathbb{N}} \in X \right\}$$

is unbounded and chaotic as well as its every power

$$\hat{A}_w^n z := \left( \prod_{m=k}^{k+n} w^m \right) \left( w^n \frac{k+1+n}{k+1} \sum_{j=0}^{k+n} z_j - \frac{k+n}{k} \sum_{j=0}^k z_j \right)_{k \in \mathbb{N}}, \quad n \in \mathbb{N},$$

with

$$z_0 := -\sum_{k=1}^{\infty} z_k,$$

and domain

$$D(\hat{A}_w^n) = \left\{ z := (z_k)_{k \in \mathbb{N}} \in l_1 \mid \left( \prod_{m=k}^{k+n-1} w^m \right) (k+n) \sum_{j=0}^{k-1+n} z_j \right\} \in \mathbb{X} \right\}.$$  

Furthermore, each $\lambda \in \mathbb{F}$ is an eigenvalue for $\hat{A}_w$ of geometric multiplicity 1, i.e.,

$$\dim \ker(\hat{A}_w - \lambda I) = 1.$$  

In particular, provided the space $(X, \| \cdot \|)$ is complex,

$$\sigma_p(\hat{A}_w) = \mathbb{C}.$$
Proof. For arbitrary \( w \in F \) with \( |w| > 1 \), let
\[
D(A_w) \ni x := (x_k)_{k \in \mathbb{N}} \mapsto A_w x := (w^k x_{k+1})_{k \in \mathbb{N}} \in X
\]
where
\[
D(A_w) := \left\{ x := (x_k)_{k \in \mathbb{N}} \in X \mid (w^k x_{k+1})_{k \in \mathbb{N}} \in X \right\}.
\]

By Lemma 5.1 and the Unbounded Weighted Backward Shifts Theorem (Theorem 5.1), \( A_w \) is unbounded and chaotic along with every power \( A^n_w \) \((n \in \mathbb{N})\).

By the Isometric Isomorphisms Proposition (Proposition 3.1), in view of (6.1) and (6.2), in the space \( l_1 \), consider the following linear operator
\[
\hat{A}_w := JA_w J^{-1}
\]
that emerges from the commutative diagram below
\[
\begin{array}{ccc}
l_1 \supseteq D(\hat{A}_w) & \xrightarrow{\hat{A}_w} & l_1 \\
\uparrow J & & \uparrow J \\
X \supseteq D(A_w) & \xrightarrow{A_w} & X
\end{array}
\]
where the domain of \( \hat{A}_w \) is
\[
D(\hat{A}_w) := J(D(A_w)).
\]

Since
\[
\hat{A}_w^n = JA^n_w J^{-1}, \quad n \in \mathbb{N},
\]
where, by Lemma 5.1,
\[
A^n_w x = \left( \prod_{j=k}^{k+n-1} w^j \right) x_{k+n},
\]
with domain
\[
D(A^n_w) = \left\{ x := (x_k)_{k \in \mathbb{N}} \in X \mid \left( \prod_{j=k}^{k+n-1} w^j \right) x_{k+n} \}_{k \in \mathbb{N}} \in X \right\},
\]
in view of (6.2),
\[
D(\hat{A}_w^n) = \left\{ z := (z_k)_{k \in \mathbb{N}} \in l_1 \mid \left( \prod_{m=k}^{k+n-1} w^m \right) (k + n) \sum_{j=0}^{k-1+n} z_j \}_{k \in \mathbb{N}} \in X \right\}
\]
with
\[
z_0 := - \sum_{k=1}^{\infty} z_k.
\]
Moreover, for arbitrary $z := (z_k)_{k \in \mathbb{N}} \in l_1$, in view of (6.1) and (6.2),

\[
\hat{A}_w^n z = J A_w^n J^{-1} z = J A_w^n \left( \sum_{j=0}^{k-1} z_j \right)_{k \in \mathbb{N}} = J \left( \prod_{m=k}^{k+n} w^m \right) \left( k+n \sum_{j=0}^{k-1+n} z_j \right)_{k \in \mathbb{N}}
\]

\[
= \left( \frac{(k+1+n) \left( \prod_{m=k+1}^{k+n} w^m \right) \sum_{j=0}^{k+n} z_j}{k+1} - \frac{(k+n) \left( \prod_{m=k}^{k+1+n} w^m \right) \sum_{j=0}^{k+1+n} z_j}{k} \right)_{k \in \mathbb{N}}
\]

where

\[
z_0 := - \sum_{k=1}^{\infty} z_k.
\]

Further, since $J : X \to l_1$ is an isometric isomorphism, the operator $\hat{A}_w^n$ ($n \in \mathbb{N}$) inherits the boundedness and chaoticity of $A_w$ as well as its eigenvalues coupled with their geometric multiplicities.

Therefore, $\hat{A}_w$ is unbounded and chaotic as well as every power $\hat{A}_w^n$ ($n \in \mathbb{N}$).

Furthermore, by the Unbounded Weighted Backward Shifts Theorem (Theorem 5.1), every $\lambda \in \mathbb{F}$ is an simple eigenvalue for $A_w$. \qed

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