On sofic groups

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Abstract. Answering some queries of Weiss, we prove that the free product and amenable extensions of sofic groups are sofic as well, and give an example of a finitely generated sofic group that is not residually amenable.

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1 Introduction

Sofic groups (originally: initially subamenable groups) were introduced by Gromov [2]. They can be viewed as a common generalizations of amenable and residually finite groups. Our interest in sofic groups arouse when in [1] we proved Kaplansky’s direct finiteness conjecture for sofic groups. First of all let us recall the definition of soficity. For a finite set $A$ let $\text{Map}(A)$ denote the monoid of self-maps of $A$ acting on the right. We use multiplicative notation for the action, that is we write $a \cdot f$ for $f(a)$ and multiplication in $\text{Map}(A)$ works as usual: $a \cdot fg = (a \cdot f) \cdot g = g(f(a))$. Let $\epsilon \in (0, 1)$ be a real number, then we say that two elements $e, f \in \text{Map}(A)$ are $\epsilon$-similar, or $e \sim_{\epsilon} f$, if the number of points $a \in A$ with $a \cdot e \neq a \cdot f$ is at most $\epsilon |A|$. We say that these $e, f$ are $(1 - \epsilon)$-different, or $e \not\sim_{1-\epsilon} f$, if they are not $(1 - \epsilon)$-similar.

Definition 1.1 Let $G$ be a group, $\epsilon \in (0, 1)$ a real number and $F \subseteq G$ a finite subset. An $(F, \epsilon)$-quasi-action of $G$ on a finite set $A$ is a function $\phi : G \to \text{Map}(A)$ with the following properties:

(a) For any two elements $e, f \in F$ the map $\phi(ef)$ is $\epsilon$-similar to $\phi(e)\phi(f)$.

(b) $\phi(1)$ is $\epsilon$-similar to the identity map of $A$.

(c) For each $e \in F \setminus \{1\}$ the map $\phi(e)$ is $(1 - \epsilon)$-different from the identity map of $A$.

Definition 1.2 The group $G$ is sofic if for each number $\epsilon \in (0, 1)$ and any finite subset $F \subseteq G$ there exists an $(F, \epsilon)$-quasi-action of $G$.

Remark 1.1

1. If a countable group is sofic then there exists a countable sequence $(A_n, \phi_n)$ of quasi-actions such that for each finite set $F \subseteq G$ and each $\epsilon \in (0, 1)$ the $(A_n, \phi_n)$ are $(F, \epsilon)$-quasi-actions for all sufficiently large index $n$.

2. It is enough to define the function $\phi$ of an $(F, \epsilon)$-quasi-action on the subset $\{1\} \cup F \cdot F$, then one can choose an arbitrary extension to the whole of $G$.

The obvious examples for sofic groups are the residually amenable groups. The goal of this paper is to answer some queries from the survey of Weiss [5]. Namely, we prove that the class of sofic groups is closed under free products and extensions by amenable groups and construct an example of a finitely generated sofic group which is not residually amenable.

2 The class of sofic groups

Lemma 2.1 If a group $G$ is sofic then for each finite subset $F \subseteq G$ and for each $\epsilon \in (0, 1)$ there is an $(F, \epsilon)$-quasi-action $\phi$ of $G$ on a finite set $A$ satisfying the following extra conditions:
(b') $\phi(1)$ is the identity map of $A$, for each $1 \neq g \in G$ the map $\phi(g)$ is a fixpoint free bijection and $\phi(g^{-1}) = \phi(g)^{-1}$.

(c') For different elements $e, f \in F \cup \{1\}$ the map $\phi(e)$ is $(1 - \epsilon)$-different from $\phi(f)$.

**Proof:** Let $F^{-1}$ denote the set of inverses of the elements of $F$, and let $\bar{F} = \left( F \cup F^{-1} \cup \{1\} \right) \cdot \left( F \cup F^{-1} \cup \{1\} \right)$ denote the collection of products of all pairs from the set $\left( F \cup F^{-1} \cup \{1\} \right)$. Let us choose an $\left( \bar{F}, \epsilon/10 \right)$-quasi-action $(A, \phi)$ and then we define a new quasi-action $\psi$ on the set $A' = A \coprod A$ (the disjoint union of two copies of $A$) as follows: Since $\psi : X \to X' = X \coprod X$ is a functor, so $\text{Map}(A)$ has a natural action on $A' = A \coprod A$, acting the same way on both copies of $A$. Using this action we get from another $\left( \bar{F}, \epsilon/10 \right)$-quasi-action $\phi' = \phi \coprod \phi$ on the set $A'$. We use this trick only to make sure that important subsets have even number of elements. Let $\psi(1)$ be the identity map on $A'$. For each element $g$ outside of $\bar{F}$ we define $\psi(g)$ to be an arbitrary fixpoint free involution on $A'$. Here we need that $A'$ has an even number of elements. This way condition (b') holds for these elements, and the other conditions are certainly unaffected.

For each element $e \in \bar{F} \setminus \{1\}$ we define $\psi(e)$ and $\psi(e^{-1})$ together. Let $A_e \subseteq A$ denote the difference of the fixpoint sets of $\phi(e)\phi(e^{-1})$ and $\phi(e)$. In other words, $A_e$ is the largest subset of $A$ such that $\phi(e)\phi(e^{-1})$ is the identity map on $A_e$, but $\phi(e)$ has no fixpoint in it. We define $A_{e^{-1}}$ similarly (with $e^{-1}$ in place of $e$), it is clear that $A_{e^{-1}} = A_e \cdot \phi(e)$.

Now $\phi(e)$ and $\phi(e^{-1})$ are inverse bijections between $A_e$ and $A_{e^{-1}}$. Our new $\psi(e)$ will be equal to $\phi'(e)$ on $A'_e = A_e \coprod A_e$ and we extend it to the whole of $A'$ in two steps. First we extend to $(A_{e^{-1}} \cup A_e)'$ via an arbitrary bijection $(A_{e^{-1}} \setminus A_e)' \to (A_e \setminus A_{e^{-1}})'$. Then extend it further via any fixpoint free involution on the complement $(A \setminus (A_{e^{-1}} \cup A_e))'$. (One can do it since the complement has an even number of elements.) This $\psi(e)$ is a fixpoint free bijection. Then we define $\psi(e^{-1})$ to be its inverse. (Note that our construction is symmetric in $e$ and $e^{-1}$.) In case $e$ is an involution we find that $\phi(e)$ is an involution on $A_e$, hence our $\psi(e)$ is also an involution. Hence this $\psi$ satisfies condition (b') and therefore also condition (b) of Definition 1.1.

Let’s pick an element $e \in \bar{F} \setminus \{1\}$. We know that $\phi(e^{-1})\phi(e)$ is $\epsilon/5$-similar to $\phi(1)$, hence $\epsilon/5$-similar to the identity map. The fixpoint set of $\phi(e)$ has at most $\epsilon/10 |A|$ elements. So the size of $A_e$ is at least $(1 - \frac{3}{10}\epsilon)|A|$, therefore $\psi(e)$ and $\phi'(e)$ are $\frac{3}{10}\epsilon$-similar. The same is true for $e = 1$ as well. Now let $e, f$ be elements of $F \cup F^{-1}$. Then $\psi(e)$ resp. $\psi(f)$ are $\frac{3}{10}\epsilon$-similar to $\phi'(e)$ resp. $\phi'(f)$. Since $\psi(e)$ is a bijection, we find that $\psi(e)\psi(f)$ is $\frac{3}{10}\epsilon$-similar to $\psi(e)\phi'(f)$. Hence

$$
\psi(e)\psi(f) \sim_{\frac{3}{10}\epsilon} \psi(e)\phi'(f) \sim_{\frac{3}{10}\epsilon} \phi'(e)\phi'(f) \sim_{\epsilon/10} \phi'(ef) \sim_{\frac{3}{10}\epsilon} \psi(ef).
$$

Putting it all together we find that $\psi(e)\psi(f)$ is $\frac{7}{10}\epsilon$-similar to $\phi'(ef)$ and $\epsilon$-similar to $\psi(ef)$. Thus $\psi$ satisfies condition (a) of Definition 1.1.

Now let $e, f$ be different elements of $F$. We have shown above that $\psi(e)\psi(f^{-1})$ is $\frac{7}{10}\epsilon$-similar to $\phi'(ef^{-1})$ and the latter is $(1 - \epsilon/10)$-different from the identity map. Hence $\psi(e)\psi(f^{-1})$ is $(1 - \frac{8}{10}\epsilon)$-different from the identity. Since $\psi(f^{-1}) = \psi(f)^{-1}$, we see that
Proof: We start with the proof of Theorem 1. Let \( \{G_i\}_{i \in I} \) be sofic groups, let \( G = \prod_{i \in I} G_i \) and \( F \subseteq G \) a finite subset, and fix a number \( \epsilon \in (0,1) \). Then there exists a finite subset \( J \subseteq I \) such that the natural projection \( \pi_J : G \to G_J = \prod_{j \in J} G_j \) is injective on the set \( F \cup \{1\} \). Then each \((\pi_J(F), \epsilon)\)-quasi-action of \( G_J \) is also an \((F, \epsilon)\)-quasi-action of \( G \), hence it is enough to prove soficity for finite direct products. So we assume that the index set is \( I = \{1, 2, \ldots, n\} \). Let \( F_i \) denote the image of the projection of \( F \) into the factor \( G_i \), and choose some \((F_i, \epsilon)\)-quasi-actions \( \phi_i : G_i \to \text{Map}(A_i) \). We define the finite set \( A = \prod_{i \in I} A_i \) and the quasi-action \( \phi : G \to \text{Map}(A) \) via the formula

\[
\left( a_1, a_2, \ldots, a_n \right) \cdot \phi(g) = \left( a_1 \cdot \phi(g_1), a_2 \cdot \phi(g_2), \ldots, a_n \cdot \phi(g_n) \right)
\]

It is obviously an \((F, n\epsilon)\)-quasi-action of \( G \). This shows that \( G \) is a sofic group.

Quasi-actions of a group \( G \) are also quasi-actions for its subgroups, hence if \( G \) is sofic then all subgroups are sofic as well. Inverse limit of sofic groups is by definition a subgroup of their product, and therefore it is also sofic.

Now let \( \{G_i\}_{i \in I} \) be a directed system of sofic groups, let \( G = \lim_{i \in I} G_i \), let \( F \subseteq G \) be a finite subset, and fix a number \( \epsilon \in (0,1) \). Then there is an index \( i \in I \) and a finite subset \( F_i \subseteq G_i \) such that the natural homomorphism \( \sigma_i : G_i \to G \) is a bijection \( F_i \to F \). Let \( G' \) denote the image of \( G_i \) in \( G \) and choose a coset representing system \( s : G' \to G_i \) (i.e. it has the property that \( \sigma_i(s(g)) = g \) for all \( g \in G' \)). Let \( \phi : G_i \to \text{Map}(A) \) be an \((F_i, \epsilon)\)-quasi-action of \( G_i \). Then we define \( \Phi : G \to \text{Map}(A) \) via the formula

\[
a \cdot \phi(g) = \begin{cases} 
a \cdot \phi(s(g)) & \text{when } g \in G', \ a \in A \\
 a & \text{when } g \in G \setminus G', \ a \in A
\end{cases}
\]

It is clearly an \((F, \epsilon)\)-quasi-action of \( G \). Hence \( G \) is also sofic. We finished proving Theorem 1.

Now we turn to the proof of Definition 2.1 of the theorem.

Definition 2.1 Let \( G \) and \( H \) be groups. Then each element \( g \neq 1 \) of the free product \( G \ast H \) has a unique shortest decomposition of the form \( g = g_1 h_1 \ldots g_k h_k \) where \( g_i \in G, \ h_i \in H \). For \( g = 1 \) we define the shortest decomposition to be \( g = 1 \cdot 1 \). One can easily see that a decomposition \( g = g_1 h_1 \ldots g_k h_k \) is shortest if and only if none of the factors are 1 except possibly \( g_1 \) or \( h_k \).
It is enough to prove soficity for the free product of two groups. So let $G, H$ be sofic groups. Let $F_G$ and $F_H$ be finite subsets of $G$ and $H$ respectively, $\epsilon \in (0, 1)$ and fix a positive integer $N$. We pick an $(F_G, \epsilon)$-quasi-action $\phi$ on a finite set $A$ and an $(F_H, \epsilon)$-quasi-action $\psi$ on a finite set $B$, both satisfying the conditions of Lemma 2.1. Let $F \subseteq G \ast H$ be the subset consisting of all elements with shortest decomposition of the form $g_1 h_1 \ldots g_k h_k$ where $g_i \in F_G, h_i \in F_H$ and $k \leq N$. Our goal is to construct an $(F, \epsilon)$-quasi-action of the free product $G \ast H$. This is enough for proving 2. First we define the incidence graph of two partitions. Then we construct our quasi-action in two steps.

**Definition 2.2** Let $\alpha$ and $\beta$ be partitions of a finite set $C$. The incidence graph of $\alpha$ and $\beta$ is a bipartite graph, whose two sets of vertices consist of the classes of $\alpha$ and the classes of $\beta$, and the edges are the elements of $C$, each element $c \in C$ connects its $\alpha$-class with its $\beta$-class.

**Step 1.** We construct a set $C$ with two partitions $\alpha$ and $\beta$ on it with the following properties: each $\alpha$-class has $|A|$ elements, each $\beta$-class has $|B|$ elements, an $\alpha$-class can meet a $\beta$-class in at most one element, and in the incidence graph of $\alpha$ and $\beta$ each circle is longer than $2N$.

First we choose a finite group $V$ generated by $A \times B$ such that all relations among the generators are longer than $2N$. The existence of such finite groups follows from the fact that free groups are residually finite. In formulas we shall use the notation $(\overline{a, b})$ for generators of $V$. Let $C = A \times B \times V$. Our goal is to show that $F \subseteq G \ast H$ is a sofic group. In this case their intersection is the single element $(a, b, v)$. Suppose now that the classes

$$A[b, v] = \left\{ (a, b, v) \mid a \in A \right\} \quad \text{for each } b \in B, \ v \in V$$

We define $\beta$ classes as the subsets of the form

$$B[a, w] = \left\{ (a, b, w \cdot (\overline{a, b})) \mid b \in B \right\} \quad \text{for each } a \in A, \ w \in V$$

Two classes $A_{b,v}$ and $B_{a,w}$ can meet only if $v = w \cdot (\overline{a, b})$, and in this case their intersection is the single element $(a, b, v)$. Suppose that

$$A[b_1, v_1], \ B[a_2, v_2], \ A[b_3, v_3], \ B[a_4, v_4], \ldots A[b_{2k-1}, v_{2k-1}], \ B[a_{2k}, v_{2k}]$$

form a circle with minimal length in the incidence graph of $\alpha$ and $\beta$. Our goal is to show that $k > N$. Assume indirectly that $k \leq N$. To simplify notation, we shall use indexes modulo $2k$, hence $b_{2k+1} = b_1$ and $v_{2k+1} = v_1$. The above criterion for meeting classes now reads:

$$v_{2i-1} = v_{2i} \cdot (\overline{a_{2i}, b_{2i-1}}), \quad v_{2i+1} = v_{2i} \cdot (\overline{a_{2i}, b_{2i+1}}) \quad \text{for } 1 \leq i \leq k$$

This means that $v_1, v_2, v_3 \ldots v_{2k+1} = v_1$ is a returning path in the Cayley graph of $V$. Since this graph have no loops of length $2k$, this path must return along itself. This implies in turn that there is a turning point, i.e. $v_{j+2} = v_j$ for some index $j$. If $j$ is odd then we find that

$$v_j = v_{j+1} \cdot (\overline{a_{j+1}, b_j}) = v_{j+2} \cdot (\overline{a_{j+1}, b_{j+2}})^{-1} \cdot (\overline{a_{j+1}, b_j})$$
$$v_j = v_j \cdot (a_{j+1}, b_{j+2})^{-1} \cdot (a_{j+1}, b_j)$$

hence
$$(a_{j+1}, b_j) = (a_{j+1}, b_{j+2})$$

so
$$b_j = b_{j+2}$$

But then $$A[b_j, v_j] = A[b_{j+2}, v_{j+2}]$$, the original circle can be shortened:

$$A[b_1, v_1], B[a_2, v_2], \ldots A[b_j, v_j], B[a_{j+3}, v_{j+3}], \ldots B[a_{2k}, v_{2k}]$$

This contradicts the minimality of the length 2k. For even $$j$$ the same argument gives us $$B[a_j, v_j] = B[a_{j+2}, v_{j+2}]$$, which is again a contradiction. In both cases we run into contradiction, hence $$k \leq N$$ is impossible. This proves our claims about $$\alpha$$, $$\beta$$ and $$C$$.

**Step 2.** Using the two partitions we construct an $$(F, \epsilon)$$-quasi-action of the free product $$G \ast H$$ on our $$C$$.

First we build an $$(F_G, \epsilon)$$-quasi-action $$\phi' : G \rightarrow \mathcal{M}ap(C)$$. Since the $$\alpha$$-classes have $$|A|$$ elements, we make an identification $$C \approx A \times (C/\alpha)$$ so that the $$\alpha$$-classes are the subsets of the form $$A \times \{t\}$$. We define $$\phi'$$ to act on the first coordinate:

$$(a, x) \cdot \phi'(g) = \left( a \cdot \phi(g), x \right) \quad g \in G, \ (a, x) \in C \approx A \times (C/\alpha).$$

Similarly we define the $$(F_H, \epsilon)$$-quasi-action $$\psi' : H \rightarrow \mathcal{M}ap(C)$$ via the identification $$C \approx B \times (C/\beta)$$.

Now let $$g \in G \ast H$$ be any element with shortest decomposition $$g = g_1 h_1 \ldots g_k h_k$$ (where $$g_i \in G$$ and $$h_i \in H$$ as in Definition 2.1). We define

$$\Phi(g_1 h_1 \ldots g_k h_k) = \phi'(g_1) \psi'(h_1) \ldots \phi'(g_k) \psi'(h_k)$$

Clearly $$\Phi(1)$$ is the identity map on $$C$$.

Next we prove that if $$1 \neq g \in F$$ then $$\Phi(g)$$ has no fixpoints. Our proof is indirect. Assume that some $$c_0 \in C$$ is a fixpoint of $$\Phi(g)$$. Then we define a sequence of elements of $$C$$ via induction:

$$c_{2i+1} = c_{2i} \cdot \phi'(g_{i+1}) \quad \text{and} \quad c_{2i+2} = c_{2i+1} \cdot \psi'(h_{i+1})$$

for all $$0 \leq i \leq k-1$$. Now let $$A_i$$ be the $$\alpha$$-class of $$c_{2i}$$ and $$B_i$$ denote the $$\beta$$-class of $$c_{2i+1}$$. Since $$\phi'(g_{i+1})$$ respects the $$\alpha$$-classes we see that $$c_{2i+1}$$ also belongs to $$A_i$$, hence $$c_{2i+1}$$ connects $$A_i$$ and $$B_i$$ in the incidence graph of $$\alpha$$ and $$\beta$$. Similarly, $$c_{2i+2}$$ connects $$B_i$$ with $$A_{i+1}$$ in this incidence graph. Hence we found a path $$A_0, B_0 \ldots A_{k-1}, B_{k-1}, A_k$$ in the incidence graph.

It is clear from the definition of $$\Phi$$ that

$$c_{2k} = c_0 \cdot \phi'(g_1) \psi'(h_1) \ldots \phi'(g_k) \psi'(h_k) = c_0 \cdot \Phi(g) = c_0,$$

hence $$A_{2k} = A_0$$, our path returns to $$A_0$$. But $$g \in F$$, so $$k \leq N$$. Since the graph cannot not contain such a short circle, our path must turn back at some point. Let’s look at the first turning point: either we find $$A_i = A_{i+1}$$ or we have $$B_i = B_{i+1}$$ for some index $$0 \leq i \leq k-2,$
or we have $k = 1$. (Indeed, if $A_{k-1} = A_k$ is a turning point and $k > 1$, then $A_1 \ldots A_{k-1}$ is a shorter returning path, so there is another turning point before.) In the first case

$$\{c_{2i+1}\} = A_i \cap B_i = A_{i+1} \cap B_i = \{c_{2i+2}\} = \{c_{2i+1} : \psi'(h_{i+1})\}.$$

This is impossible since $\psi'(h_{i+1})$ have no fixpoint for $0 \leq i \leq k - 2$. Similarly, $B_i = B_{i+1}$ would imply that $c_{2i+2} = c_{2i+3}$ is a fixed point of $\phi'(g_{i+2})$, which is again impossible. In the third case we have $k = 1$, and $c_1 = c_2 = c_0$ is a fixed point of both $g_1$ and $h_1$, hence both $g_1 = h_1 = 1$. This is also impossible, because $g_1 h_1 = g \neq 1$. Either way we have got a contradiction, so $\Phi(g)$ must be fixpoint free.

Now we verify that our $\Phi$ is an $(F, \varepsilon)$-quasi-action. $\Phi(1)$ is the identity map, hence condition (b) of Definition 14.1 is satisfied. Moreover, if $g \in F$ is different from 1 then $\Phi(g)$ is fixpoint free, hence condition (c) holds as well. We need to show (a). Let $g, g' \in F$ with their shortest decompositions $g = g_1 h_1 \ldots g_k h_k$ and $g' = g'_1 h'_1 \ldots g'_l h'_l$. We distinguish three cases according to $h_k$ and $g'_1$:

**First**, if $h_k \neq 1$ and $g'_1 \neq 1$ then there is no cancellation and the shortest decomposition of $gg'$ is $g_1 h_1 \ldots h_{k-1} g'_1 \ldots g'_l h'_l$. Then by definition $\Phi(gg') = \Phi(g) \Phi(g')$.

**Second**, if $h_k = 1$ and $g'_1 = 1$ then the shortest decomposition of $gg'$ is shorter, it is $g_1 h_1 \ldots g_k h'_1 \ldots g'_l h'_l$. Then both $\psi'(h_k)$ and $\phi'(g'_1)$ are the identity map, so again $\Phi(gg') = \Phi(g) \Phi(g')$.

**Third**, if only one of $h_k$ and $g'_1$ is 1 then in the product of the shortest decompositions of $g$ and $g'$ one factor is 1 (so must be omitted), some cancellations may happen (e.g. if $g'_1 = 1$ then $h_k$ and $h'_1$ are neighbors and may cancel each other), and finally we must collapse the last remaining factor of $g$ and the first remaining factor of $g'$ into one (e.g. if $h_k = 1$ and nothing cancels then $g_k$ and $g'_1$ become neighbors, and both are from the same group $G$). The shortest decomposition of $gg'$ has the form

$$gg' = g_1 h_1 \ldots g_{k-t} g'_1 h'_1 \ldots g'_l h'_l$$

or

$$gg' = g_1 h_1 \ldots g_{k-t} g'_1 h'_1 \ldots g'_l h'_l$$

where $t$ is the number of cancellations, possibly 0. When we multiply together the formulas for $\Phi(g)$ and $\Phi(g')$ the same cancellations will occur, so we must compare

$$\Phi(g) \Phi(g') = \phi'(g_1) \ldots \psi'(h_{k-t-1}) \phi'(g_{k-t}) \phi'(g'_{1+t}) \psi'(h'_{1+t}) \ldots \psi(h'_l)$$

with

$$\Phi(gg') = \phi'(g_1) \ldots \psi'(h_{k-t-1}) \phi'(g_{k-t} g'_{1+t}) \psi'(h'_{1+t}) \ldots \psi(h'_l)$$

or we compare

$$\Phi(gg') = \phi'(g_1) \ldots \psi'(h_{k-t-1}) \phi'(g_{k-t} h'_1) \phi'(g'_{2+t}) \ldots \psi(h'_l)$$

with

$$\Phi(g) \Phi(g') = \phi'(g_1) \ldots \phi'(g_{k-t}) \psi'(h_{k-t}) \psi'(h'_{1+t}) \phi'(g'_{2+t}) \ldots \psi(h'_l)$$

7
In any case the factor \( \phi'(g_{k-t}g'_{1+t}) \) is \( \epsilon \)-similar to \( \phi'(g_{k-t})\phi'(g'_{1+t}) \), the factor \( \psi'(h_{k-t}h'_{1+t}) \) is \( \epsilon \)-similar to \( \psi'(h_{k-t})\psi'(h'_{1+t}) \), and all other factors \( \phi'(g_{t}) \), \( \psi'(h_{t}) \) are bijections, hence \( \Phi(gg') \) is \( \epsilon \)-similar to the product \( \Phi(g)\Phi(g') \).

Therefore condition (a) of Definition 1.1 holds, our \( \Phi \) is an \((F,\epsilon)\)-quasi-action. This finish the proof of the fact that the free products of sofic groups are sofic groups as well.

Now we prove 3 of the theorem. Let \( N \vartriangleleft G \) be groups such that \( N \) is sofic and \( G/N \) is amenable, and choose a finite set \( F \subseteq G \) and a number \( \epsilon \in (0,1) \). Our goal is to build an \((F,3\epsilon)\)-quasi-action of \( G \).

For elements \( g \in G \) let \( \overline{g} \in G/N \) denote their image. Let \( \sigma : G/N \rightarrow G \) be a section of the natural homomorphism \( G \rightarrow G/N \), i.e. it has the property that \( \overline{\sigma(h)} = h \) for each \( h \in G/N \), or equivalently, \( g\sigma(\overline{g})^{-1} \in N \) for all \( g \in G \). By Folner’s theorem we can choose a nonempty finite subset \( \overline{A} \subseteq G/N \) with the property that \( |\overline{A}\overline{g} \setminus \overline{A}| \leq \epsilon |\overline{A}| \) for all \( g \in F \). Let \( A = \sigma(\overline{A}) \). Moreover, let \( H = N \cap (A \cdot F \cdot A^{-1}) \), and choose an \((H,\epsilon)\)-quasi-action \( \psi \) of the group \( N \) on a finite set \( B \).

We define the map \( \Phi : G \rightarrow \text{Map}(B \times A) \) as follows:

\[
(b, a) \cdot \Phi(g) = \begin{cases} 
(b \cdot \psi(a\sigma(\overline{ag})^{-1}), \sigma(\overline{ag})) & \text{when } \overline{ag} \in \overline{A} \\
(b, a) & \text{otherwise}
\end{cases}
\]

It is well defined since \( ag \sigma(\overline{ag})^{-1} \in N \). We shall prove that \( \Phi \) is an \((F,3\epsilon)\)-quasi-action. Clearly \( \psi(a\sigma(\overline{a}))^{-1} = \psi(1) \) is \( \epsilon \)-similar to the identity map of \( B \), hence \( \Phi(1) \) is \( \epsilon \)-similar to the identity map of \( B \times A \), so (b) of Definition 1.1 holds.

Next we check condition (c). Let \( e \in F \setminus \{1\} \), and assume first that \( e \notin N \). Then \( \overline{e} \neq 1 \), hence \( a \neq \sigma(\overline{ae}) \) for all \( a \in G \). Therefore

\[
(b, a) \cdot \Phi(e) = (b \cdot \psi(\ldots), \sigma(\overline{ae})) \neq (b, a) \quad \text{whenever } \overline{ae} \in \overline{A}
\]

Hence \( (b, a) \) is a fixpoint of \( \Phi(e) \) only if \( \overline{ae} \notin \overline{A} \). By assumption the number of such \( \overline{e} \) is at most \( \epsilon |A| \), and each \( \overline{a} \) gives \( |B| \) fixpoints. So the total number of fixpoints is at most \( \epsilon |A| \cdot |B| \). Hence in this case \( \Phi(e) \) is \( (1-\epsilon)\)-different from the identity.

Assume next that \( e \in N \cap F \setminus \{1\} \). In this case \( \overline{e} = 1 \) hence

\[
(b, a) \cdot \Phi(e) = (b \cdot \psi(\overline{aea}^{-1}), a) \quad \text{for all } b \in B, \ a \in A
\]

Hence \( (b, a) \) is a fixpoint of \( \Phi(e) \) if and only if \( b \) is a fixpoint of \( \psi(\overline{aea}^{-1}) \). But \( a\overline{e}a^{-1} \in H \setminus \{1\} \), so for a fixed \( a \in A \) there are at most \( \epsilon |B| \) such fixpoints, hence altogether there are at most \( \epsilon |A| \cdot |B| \) fixpoints, and thus \( \Phi(e) \) is again \( \epsilon \)-different from the identity map. This proves condition (c) of Definition 1.1.

Finally let \( e, f \in F \) and suppose that for certain \( a \in A \) the elements \( \overline{ae} \) and \( \overline{af} \) are both in \( \overline{A} \). This assumption holds for all but \( 2\epsilon |A| \) values of \( a \). Then

\[
(b, a) \cdot \Phi(e)\Phi(f) = (b \cdot \psi(\overline{ae} \sigma(\overline{ae})^{-1}), \sigma(\overline{ae})) \cdot \Phi(f) =
\]
\[ = \left( b \cdot \psi(ae \sigma(ae)^{-1}) \cdot \psi(\sigma(ae)f \sigma(aef)^{-1}) \right) \]

and

\[ \left( b, a \right) \cdot \Phi(ef) = \left( b \cdot \psi(aef \sigma(ae)^{-1}) \right) \]

But our assumption imply that \( \sigma(ae), \sigma(aef) \in A \), hence \( aef^{-1} \) and \( \sigma(ae)f \sigma(aef)^{-1} \) are elements of \( H \). Since \( \psi \) is an \((H, \epsilon)\)-quasi-action, we find that with the exception of at most \( \epsilon|B| \) values of \( b \)

\[ b \cdot \psi(ae \sigma(ae)^{-1}) \cdot \psi(\sigma(ae)f \sigma(aef)^{-1}) = b \cdot \psi(aef \sigma(aef)^{-1}) \]

Putting it together, we have

\[ (b, a) \cdot \Phi(e) \Phi(f) = (b, a) \cdot \Phi(ef) \] except for \( \{ 2\epsilon|A| \} \) values of \( a \) \( \{ \epsilon|B| \} \) values of \( b \)

Therefore \( \Phi(e) \Phi(f) \) is \( 3\epsilon \)-similar to \( \Phi(ef) \), condition (a) of Definition 1.1 holds.

We proved all three conditions of Definition 1.1 hence \( \Phi \) is indeed an \((F, 3\epsilon)\)-quasi-action. Hence 3 is proven. This completes the proof of the theorem.

Theorem 2  Free product of locally residually amenable groups are sofic. In particular, free product of residually finite groups and amenable groups are sofic.

Proof:  The one element group is sofic. Hence residually finite groups are sofic by (c) of Theorem 1. Subgroups of direct products of amenable groups are sofic by (a) of Theorem 1. Residually amenable groups are subgroups of the product of their amenable factors, therefore they are sofic. Locally residually amenable groups are the direct limit of residually amenable groups, hence they are also sofic. Finally, free product of locally residually amenable groups are sofic by (b) of Theorem 1.

3  A finitely generated non-residually amenable sofic group

In [4], the authors constructed a finitely generated amenable LEF-group that was not residually finite. Modifying their construction we show the existence of a finitely generated sofic group that is not residually amenable. Let \( K \) be a hyperbolic, residually finite group with Kazhdan’s Property (T) [3]. Let \( P \) be the set of all permutations of \( K \) which move only finitely many elements. Consider the group \( Q \) generated by \( P \) and the left translations by the elements of \( K \). Note that \( Q \) is the semidirect product of \( K \) and the locally finite group \( P \).

Theorem 3  \( Q \) is a finitely generated sofic group, but it is not residually amenable.
**Proof:** Let $S$ be symmetric generating system for $K$. Denote by $T_S$ the set of transpositions in the form $(1,s)$, where $s \in S$.

**Lemma 3.1** The set $W = S \cup T_S$ generate the group $Q$.

**Proof:** Consider the right Cayley graph $\Gamma$ of $K$ with respect to the generating system $S$. Note that any transposition in the form $(g, gs)$, $g \in K$, $s \in S$ can be written as $g \cdot (1, s) \cdot g^{-1}$. If $g, h \in K$ then pick a shortest path

$$g \rightarrow gs_{i_1} \rightarrow gs_{i_1}s_{i_2} \rightarrow \ldots \rightarrow gs_{i_1}s_{i_2}\ldots s_{i_n} = h$$

in the Cayley graph. Any transposition in the form $(gs_{i_1}s_{i_2}\ldots s_{i_k}s_{i_{k+1}})$ is generated by $W$ by our previous observation, hence the transposition $(g, h)$ is generated by $W$ as well. This finishes the proof of our lemma. 

**Lemma 3.2** The group $Q$ is sofic.

**Proof:** Let $F_n \subseteq Q$ be the finite set of elements in the form of $k\sigma$, where $k \in B_n(K) \subseteq \Gamma$ and $\sigma \in P$ moves only the elements of $B_n(K)$. Clearly, $F_n \cdot F_n \subseteq F_{2n}$ and $\cup_{n=1}^{\infty} F_n = Q$, hence in order to prove that $Q$ is sofic, it is enough to construct for any $n$ an injective map $\Psi_n$ from $F_{2n}$ to a finite group $H_n$ such that if $f, g \in F_n$, then $\Psi_n(f)\Psi_n(g) = \Psi_n(fg)$.

Let $N_n$ be a normal subgroup of $K$ of finite index such that $N_n \cap B_{10n}(K) = \{1\}$. Such normal subgroup must exist since $K$ is residually finite. Let $\tau : K \to K/N_n$ be the quotient homomorphism. The map $\Psi_n : F_{2n} \to H_n = Sym(N_n)$ is defined as follows: $\Psi_n(k\sigma) = \Psi_n(k) \cdot \Psi_n(\sigma)$, where $\Psi_n(k)$ is the left translation by $\tau(k)$ and $\Psi_n(\sigma)(\tau(a)) = \tau(\sigma(a))$ if $a \in B_{2n}(K)$, otherwise $\Psi_n(\sigma)(b) = b$. Obviously, $\Psi_n$ is injective.

Now let $s \in B_{5n}(K)$, $f = k_1\sigma_1 \in F_1$, $g = k_2\sigma_2 \in F_1$. Then,

$$\Psi_n(f)\Psi_n(g)(\tau(s)) = \Psi_n(k_1\sigma_1)\Psi_n(k_2)\tau(\sigma_2(s)) = \Psi_n(k_1)\tau(\sigma_1(k_2(\tau(\sigma_2(s)))) = \tau(k_1\sigma_1k_2\sigma_2(s))$$

On the other hand,

$$\Psi_n(k_1\sigma_1k_2\sigma_2)(\tau(s)) = \Psi_n(k_1k_2(k_2^{-1}\sigma_1k_2)\sigma_2(\tau(s)) = \tau(k_1k_2)\tau((k_2^{-1}\sigma_1k_2)\sigma_2)) = \tau(k_1\sigma_1k_2\sigma_2(s))$$

Hence $\Psi_n(f)\Psi_n(g)$ coincides with $\Psi_n(fg)$ on $\tau(B_{5n}(K))$. However, if $l \notin \tau(B_{5n}(K))$ then

$$\Psi_n(k_1\sigma_1)\Psi_n(k_2\sigma_2)(l) = \tau(k_1k_2)(l)$$

and

$$\Psi_n(k_1\sigma_1k_2\sigma_2)(l) = \tau(k_1k_2)(l)$$

Hence $\Psi_n(f)\Psi_n(g) = \Psi_n(fg)$. 

**Lemma 3.3** The group $Q$ is not residually amenable.
**Proof:** Let $A$ denote the simple subgroup of even permutations in $P$. If $Q$ were residually amenable then there exists a homomorphism $\phi : Q \to M$, where $M$ is amenable, $\phi(t) \neq 1, \phi(a) \neq 1$, $t$ is a non-torsion element in the subgroup $K$ and $a$ is an even permutation. Note that such $t$ must exist since $K$ is hyperbolic. By the Kazhdan’s property (T) the image of $K$ must be finite, since any amenable quotient of a Kazhdan group is finite. On the other hand, $\phi$ must be injective on $A$ since $A$ is simple and $\phi(a) \neq 1$. Clearly, $t^{-n}at^{-n} \neq a$ for any $n$. However, if $n$ is the rank of $\phi(K)$, $\phi(t^n) = 1$. Therefore, $\phi(t^{-n}at^{-n}) = \phi(a)$ in contradiction with the injectivity of $\phi$ on the subgroup $A$. This finishes the proof of our lemma and of Theorem 3. as well.

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