Non-trivial Extension of Starobinsky Inflation

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Abstract

We consider the Starobinsky inflation coupling to a non-trivial (non-canonic) field. We work in Einstein-frame, in this frame, the gravitational part of the action is equivalent to Hilbert-Einstein action plus a scalar field called scalaron. We investigate a model with a heavy scalaron trapped in the effective potential minimum, where its fluctuations are negligible. Although the DBI field governs the inflation, the boost factor and other quantities are different from the standard DBI model through implicit dependence on Scalaron. For appropriate parameters, this model is consistent with the Planck results.

keywords: Early universe; Inflation; F(R) theory; Starobinsky model; Dirac-Born-Infeld(DBI) inflation;
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1 Introduction

Inflation theory is proposed to solve fundamental problems of standard cosmology\cite{1, 2, 3}. It also explains the origin of the primordial fluctuations. Although observational data supports the inflation theory in general, there is no fundamental theory that can describe the nature of this theory. In the simplest model, the inflaton field which is responsible for inflating the universe rolls down in an almost flat potential (slow-roll regime). Observational data indicate that in single field models the monomial potentials are disfavored including the famous potential $m^2\phi^2$, other models with more intricate potentials specially with exponential tails and also brane inflation provide good fits to data \cite{4, 5}. On the other hand, in recent years F(R) theories, which are an extension of Hilbert-Einstein’s action, have attracted
attention. These theories are another approach to explain the acceleration periods of our universe. Maybe the simplest and most famous model of \( F(R) \) theory is \( R + cR^2 \). Being within the Planck 68\% confidence level constraints, arouse enthusiasm for this model\[5\]. In fact, this model is proposed many years ago by Starobinsky \[7, 8\] as a model for inflation. The \( F(R) \) action is written in the Jordan frame. To do the calculation, we transform to Einstein frame in which the action is equivalent to a scalar field plus Hilbert-Einstein action \[9\]; this dual scalar field called scalaron can take the role of inflaton with exponential potential.

Inspired by string theory and high-energy physics there is motivation to have more than one field. A multi-field model has more phenomenology than a single field model. The simplest extension of the Starobinsky model is considering an extra canonical scalar field in \( R + cR^2 \) gravity, it is shown that this model is also a robust one\[10, 11\]. On the other hand, it is shown that inflation can be derived from non-canonical fields, this kind of model are investigated in k-inflation\[14\] and \( P(X^{ij}, \phi) \) \[18, 17\] context. DBI model of inflation\[12\], which is first considered in the context of brane-inflation, is one of these models\[13\]. Apart from its theoretical origin, its square root feature causes several novelties. Under some constraints, the brane inflation and DBI model are consistent with observational data \[6\].

This work aims is to investigate DBI inflation in the context of \( R^2 \) gravity. We consider a heavy sclaron, therefore after a while, the DBI field governs the inflation. Transforming to Einstein frame and making field redefinition causes coupling between the DBI field and dual field, this coupling modifies the dynamics of the DBI field and hence affects the cosmological parameters.

This paper is organized as follows; in section \( 2 \) the setup of the model is described. In section \( 3 \) the background solution is considered. The field perturbations are investigated in section \( 4 \) we also do some numerical analysis in \( 4.1 \). We conclude in section \( 5 \).

2 The Setup

In principle a generic \( f(R) \) model is given by the below action,

\[
S = \frac{1}{2} \int d^4x \sqrt{-g} f(R)
\]

This model is connected to scalar-tensor theory via Legendre transformation as,

\[
S = \frac{1}{2} \int d^4x \sqrt{-g} \left( f(\phi) + f'(\phi)(R - \phi) \right).
\]
We define $\Omega^2 \equiv f'(\phi)$ and rewrite the above action as,

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2} \Omega^2 R - V(\phi) \right),$$  

(3)

with $V(\phi) \equiv \frac{1}{2} (\phi f'(\phi) - f(\phi))$.

It is also possible to add a matter sector. We are interested in matter with non-canonic kinetic term,

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2} \Omega^2 R - V(\phi) \right) + \int d^4x \sqrt{-g} P(X, \chi^I)$$  

(4)

where $X = -\frac{1}{2} g_{\mu\nu} \partial_\mu \chi \partial_\nu \chi$ and $P$ is function of $X$ and $\chi$.

It is feasible to go to Einstein-frame under a conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, we define a new field as $\Omega^2 = f'(\phi) = e^{2\alpha \psi}$. First we consider the gravitational part which is equivalent to the below action,

$$S_G' = \int d^4x \sqrt{-\tilde{g}} \left( \frac{\tilde{R}}{2\kappa} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - \tilde{V}(\psi) \right),$$

where $\alpha = \frac{\kappa}{\sqrt{6}}$. The matter part also transforms as

$$S_M' = \int d^4x \sqrt{-\tilde{g}} e^{-4\alpha \psi} P(X, \chi^I)$$  

(5)

where $X = -\frac{1}{2} \tilde{g}_{\mu\nu} \partial_\mu \chi \partial_\nu \chi$. In Einstein-frame there are two fields $\psi$ which comes from the correction of Einstein gravity and $\chi$ which is the matter field. Both of these fields influence on inflation. The conformal transformation causes $\chi$ to be coupled with $\psi$.

3 DBI field Dynamics in Starobinsky Model

To be more clear, we choose a DBI field as the non-canonic field in matter sector,

$$S = \int d^4x \sqrt{-g} \left( \frac{\tilde{R}}{2\kappa} + \frac{\mu}{2} R^2 \right) + \int d^4x \sqrt{-g} \left( \frac{1}{f(\chi)} \left( 1 - \sqrt{1 + f(\chi) g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi} \right) - U(\chi) \right),$$

(6)

where $\kappa = 8\pi G = M_{pl}^{-2}$ with $M_{pl}$ is the reduced Planck mass. In the following, we will work in natural units in which $\kappa = 1$. The coupling
parameter, \( \mu \) with units \([\text{mass}]^{-2}\) is assumed to satisfy the condition \( \mu \gg \kappa \); in the following we set \( \mu = 10^9/M_{pl}^2 \) (in natural units). In the DBI part, \( f(\chi) \approx \frac{1}{\chi} \) is the warp factor of DBI field and \( U(\chi) \) is its potential. Originally, this model proposed in the context of \( D3-\bar{D}3 \) brane-inflation in a warped throat. We assume \( D3 \)-brane starts inside the throat, so the effective potential takes the simple form as\[^{19}\],

\[
U(\chi) = \frac{1}{2} m^2 \chi^2 + V_0 \left( 1 - \frac{v V_0}{4 \pi^2 \chi^4} \right) \tag{7}
\]

\( V_0 \) is the effective cosmological constant and depend on the warp factor of the \( \bar{D}3 \) branes position, factor \( v \) depends on the properties of the warped throat, we choose \( v = 27/16\[^{19}\].

The total action in Einstein-frame is given by,

\[
S' = \int d^4x \sqrt{-\tilde{g}} \left( \frac{\tilde{R}}{2\kappa} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - e^{-4\alpha \psi} \left( e^{2\alpha \psi} - 1 \right)^2 \right) + \int d^4x \sqrt{-\tilde{g}} e^{-4\alpha \psi} \left( \frac{1}{f(\chi)} \left( 1 - \sqrt{1 + f(\chi) e^{2\alpha \psi} \tilde{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi} \right) - U(\chi) \right). \tag{8}
\]

For the sake of simplicity we define the potential of \( \psi \) as \( w(\psi) \equiv \frac{1}{8\kappa^2 \mu} e^{-4\alpha \psi} \left( e^{2\alpha \psi} - 1 \right)^2 \) and its mass as \( m^2_\psi = \frac{1}{6\kappa \mu} \). The equations of motion for \( \psi \) and \( \chi \) in flat FRW space-time are as follows,

\[
\ddot{\psi} + 3H \dot{\psi} + w,\psi = -\alpha e^{-4\alpha \psi} T_{DBI}^\psi, \tag{9}
\]

\[
\ddot{\chi} + 3H \gamma^{-2} \chi + e^{-2\alpha \psi} \frac{f,\chi}{2 \gamma^2} \left( 1 + 2\gamma^{-1} - 3\gamma^{-2} + e^{-2\alpha \psi} \gamma^{-3} U,\chi \right) = \alpha \dot{\psi} \dot{\chi} \left( 3\gamma^{-2} - 1 \right). \tag{10}
\]

Where \( \gamma = 1/\sqrt{1 - e^{2\alpha \psi} f,\chi^2} \) is the modified boost factor of DBI field, the presence of \( e^{2\alpha \psi} \) under the square root affects the dynamics of \( \chi \). \( T_{DBI}^\psi \equiv [f^{-1}(\chi) \left( 4 - 3\gamma^{-1} - 4U(\chi) \right)] \) is the trace of energy-momentum tensor of DBI part. \( (),\psi \) and \( (),\chi \) denote derivative with respect to the fields \( \psi \) and \( \chi \) respectively. Einstein’s field equations in flat FRW background are given as below,

\[
3H^2 = \frac{1}{2} \dot{\psi}^2 + e^{-4\alpha \psi} \left( e^{2\alpha \psi} - 1 \right)^2 \frac{1}{8\kappa^2 \mu} + \rho_{DBI} \tag{11}
\]

\[
-2\dot{H} = \dot{\psi}^2 + e^{-2\alpha \psi} \chi^2 \gamma, \tag{12}
\]
where $\rho_{DBI} = e^{-4\alpha\psi}[f^{-1}(\gamma - 1) + U(\chi)]$.

We solve the equations of motion, (9) and (10) together with Einstein’s field equations, (11) and (12) to arrive at the evolution of the fields which are plotted in FIG.1. In order to satisfy the constraint on the maximum length of the throat[25], we choose the initial value of $\chi$ equals 1.5 (which is less than the initial value of $\psi$). At the end of inflation $\chi$ decreases to small value( from brane-inflation view points, branes and anti-branes annihilate near the bottom of the throat). We define the mass ratio parameter as $\beta = \frac{m_\psi}{m_\chi}$. These figures show that when $\beta$ becomes much larger than one, the scalaron traps in a minimum and the energy density of DBI field overcomes, thus DBI field governs the dynamics. The effect of scalaron is hidden in the boost factor, $\psi$ provides enough e-folds and keep the boost factor around 1, which allows us to use slow-roll approximation and also assume that the DBI field is potential dominated.

3.1 Background

When the scalaron is heavier than DBI field, it rolls down in the effective potential to goes to its minimum, where it is trapped. The effective potential
relays on both fields. It is written as follows,

\[ U_{\text{eff}}(\psi) = \frac{1}{8\kappa^2 \mu} e^{-4\alpha \psi} (e^{2\alpha \psi} - 1)^2 - \frac{1}{4} e^{-4\alpha \psi} T_{DBI}^\phi, \]  

the extremum value at \( \psi_{\text{min}} \) satisfies,

\[ \left[ \frac{\alpha}{2\kappa^2 \mu} e^{-4\alpha \psi} (e^{2\alpha \psi} - 1) + \alpha e^{-4\alpha \psi} T_{DBI}^\phi \right] |_{\psi_{\text{min}}} = 0, \]

solving the above equation gives,

\[ \psi_{\text{min}} = \frac{1}{2\alpha} \ln \left( 1 - 2\kappa^2 \mu T_{DBI}^\phi \right), \]  

the condition for having a minimum \( \left( \frac{d^2 U_{\text{eff}}}{d\psi^2} |_{\psi_{\text{min}}} > 0 \right) \) is always satisfied because we have,

\[ \frac{d^2 U_{\text{eff}}}{d\psi^2} |_{\psi_{\text{min}}} = \frac{\alpha^2}{\kappa^2 \mu} e^{-2\alpha \psi_{\text{min}}} > 0. \]  

We assume that the fields are potential dominated i.e. \( T_{DBI}^\phi \simeq -4U(\chi) \) and \( e^{2\alpha \psi_{\text{min}}} \approx 1 + 8\kappa^2 \mu U(\chi) \), the Friedmann equations can be approximated as,

\[ 3H^2 \simeq \frac{1}{8\kappa^2 \mu} e^{-4\alpha \psi} [(e^{2\alpha \psi} - 1)^2 + 8\kappa^2 \mu U(\chi)] |_{\psi_{\text{min}}}, \]  

\[ \simeq e^{-2\alpha \psi} U(\chi) \]  

\[ -2\dot{H} \simeq e^{-2\alpha \psi} \dot{\chi}^2, \]

from now on, we dropped the index min. As mentioned before, when \( \psi \) is trapped in its minimum the dynamics is controlled by \( \chi \). Comparing with usual DBI in the general relativity context, shows that the effect of \( \psi \) or equivalently \( R^2 \) term appears in \( e^{-2\alpha \psi} \) factor. To arrive to above equations, we assumed \( \dot{\psi}^2 \ll e^{-2\alpha \psi} \dot{\chi}^2 \) in [12], this assumption is equivalent to,

\[ \gamma \gg \frac{4\kappa^2}{9\alpha^2 \beta^2} \frac{\chi^2}{1 + \frac{2\kappa}{3\beta} \chi^2} \]  

(17)

this condition is satisfied when \( \beta \gg 1 \). Differentiating (14) with respect to time gives the change of the minimum of \( \psi \) as \( \chi \) evolves,

\[ \dot{\psi} = \frac{4\kappa^2 \mu}{\alpha} e^{-2\alpha \psi} U_\chi \dot{\chi}. \]  

(18)
The coupling between the two fields make Hubble friction term and potential terms dominate in the DBI equation of motion \(10\), then we have

\[
\dot{\chi} \simeq -e^{-2\alpha\psi} \frac{U_x}{3H\gamma}
\]  

(19)

We also define the slow-roll parameters as usual,

\[
\epsilon \equiv -\frac{\dot{H}}{H^2} = 3 \frac{\dot{\psi}^2 + \gamma e^{-2\alpha\psi} \dot{\chi}^2}{2\rho_{DBI} + \frac{1}{2} \dot{\psi}^2 + w(\psi)},
\]  

(20)

Using \(16\) we arrive at,

\[
\epsilon \approx \frac{3}{2} \frac{\gamma \dot{\chi}^2}{U(\chi)}
\]  

\approx \frac{1}{2} e^{-2\alpha\psi} \left(\frac{U_x}{U}\right)^2,
\]  

similar to previous results the only difference with usual DBI model is \(e^{-2\alpha\psi}\) factor. As usual we have \(\ddot{a}/a = H^2 (1 - \epsilon)\). Differentiate \(21\) with respect to time, we arrive at the rate of change of this slow-roll parameter,

\[
\frac{\dot{\epsilon}}{2H\epsilon} \simeq \frac{1}{2} s - \delta + 2 \left(1 + 12\kappa^2 \mu U(\chi)\right) \epsilon
\]  

(22)

with

\[
s = -\frac{\dot{\gamma}}{H\gamma} \quad \text{and} \quad \delta = \frac{1}{\gamma} \frac{U_{\chi\chi}}{U}.
\]  

(23)

where "s" measures the rate of change of the sound speed and \(\delta\) is equivalent to \(\eta\) parameter. Note that both of these parameters has implicit dependence on \(\psi\) through \(e^{-2\alpha\psi}\) factor in \(\gamma\). From a mathematical point of view, our model is equivalent to a scalar-tensor theory\[15, 16\].

4 Perturbations evolution and cosmological parameters

First we consider the evolution of linear perturbation of this model. We perturb the action \(8\) in standard way by decomposition of the fields \(\psi\) and \(\chi\) into a homogeneous and perturbed part,

\[
\psi (t, x) = \psi (t) + \delta \psi (t, x) \quad \chi (t, x) = \chi (t) + \delta \chi (t, x).
\]  

(24)
The field perturbations are of linear order. We shall work in Fourier space in which the spatial derivative, \(\partial\), can be replaced by \(-ik\). Assume that the anisotropic stress is absent, in longitudinal gauge, the scalar perturbation of the flat FRW metric is expressed as below

\[
ds^2 = -(1 + 2\Phi)\, dt^2 + a^2(t)\left(1 - 2\Phi\right)\delta_{ij}dx^i dx^j. \tag{25}\]

The equations of field perturbation are as follows

\[
\begin{align*}
\ddot{\psi} + 3H\dot{\psi} - 4\Phi \dot{\psi} + \alpha \delta \chi \dot{\chi} e^{-2\alpha \psi} (3\gamma - \gamma^3) \\
+ \delta \psi \left(\frac{k^2}{a^2} + w_{\psi\psi} - 4\alpha^2 e^{-4\alpha \psi} [f^{-1}(4-3\gamma - 1) - 4U(\chi)] + \alpha^2 e^{-2\alpha \psi} (3\gamma - \gamma^3) \dot{\chi}^2 \right) \\
+ \delta \chi (-4\alpha e^{-4\alpha \psi} u_{\chi \chi} - \alpha e^{-4\alpha \psi} \frac{f_{\chi}}{f} (8 - 3\gamma - 1 - 6\gamma + \gamma^3)) \\
+ 2\Phi (w_{\psi} + \alpha e^{-4\alpha \psi} [f^{-1}(4-3\gamma - 1) - 4U(\chi)] + \alpha e^{-2\alpha \psi} (3\gamma - \gamma^3) \dot{\chi}^2) = 0,
\end{align*}
\]

and

\[
\begin{align*}
\delta \dot{\chi} + (3H + 3\dot{\gamma} - 2\alpha \psi) \delta \chi - \dot{\Phi} \chi (1 + 3\gamma - 2) + \alpha \delta \psi \chi (1 - 3\gamma - 2) \\
+ \delta \chi \left\{ \gamma^{-2} \frac{k^2}{a^2} + \gamma^{-3} u_{\chi \chi} e^{-2\alpha \psi} + \frac{f_{\chi}}{f} \frac{\dot{\chi}}{\gamma} - \frac{1}{2} U_{\chi} f_{\chi} \dot{\chi}^{-1} \chi^2 \right\} \\
+ \frac{1}{2} e^{-2\alpha \psi} (1 + \gamma)^{-2} \dot{\gamma}^{-2} [\gamma \left(\frac{f_{\chi}}{f}\right)_{,\chi} + \left(\frac{f_{\chi}}{f}\right)_{,\chi} (1 + \gamma)^{-1} \gamma^2] \\
- \alpha \delta \psi (\gamma^{-1} (1 + \gamma^{-2}) U_{,\chi} e^{-2\alpha \psi} + \frac{f_{\chi}}{f^2} \gamma^{-1} (1 - \gamma^{-1})^2 e^{-2\alpha \psi} - 2\dot{\gamma}^2) \\
+ \Phi (e^{-2\alpha \psi} \gamma^{-1} (1 + \gamma^{-2}) u_{,\chi} - 2\dot{\gamma}^2 + \frac{f_{\chi}}{f^2} e^{-2\alpha \psi} \gamma^{-1} (1 - \gamma^{-1})^2) = 0.
\end{align*}
\]

It is convenient to introduce gauge-invariant quantity, so-called Sasaki-Mokhanuv variables\[15, 20,\]

\[
Q_\psi \equiv \delta \psi + \frac{\dot{\psi}}{H}, \quad Q_\chi \equiv \delta \chi + \frac{\dot{\chi}}{H}, \tag{26}\]

which are the scalar field perturbations in the flat gauge. In terms of these new variables the equations form a closed system,

\[
\begin{align*}
\ddot{Q}_\psi + 3H \dot{Q}_\psi + B_\psi \dot{Q}_\chi + \left(\frac{k^2}{a^2} + C_{\psi \psi}\right) Q_\psi + C_{\psi \chi} Q_\chi &= 0, \tag{27} \\
\ddot{Q}_\chi + \left(3H - 2\alpha \psi + 3\dot{\gamma} \right) \dot{Q}_\chi + B_\chi Q_\psi \left(\frac{k^2}{a^2 \gamma^2} + C_{\chi \chi}\right) Q_\chi + C_{\chi \psi} Q_\psi &= 0. \tag{28}
\end{align*}
\]
with the coefficients as

\[ B_\chi = -\alpha \left( \frac{3}{\gamma^2} - 1 \right) \dot{\chi} - \frac{\psi \dot{\chi}}{2H} (1 - \frac{1}{\gamma^2}), \]

\[ B_\psi = -e^{-2\alpha_0 \psi} \beta B_\chi, \]

\[ C_{\psi \psi} = -\alpha \frac{\psi}{H} e^{-4\alpha_0 \psi} f^{-1}(\frac{3}{\gamma} + 1) (1 - \gamma)^3 - \alpha^2 e^{-4\alpha_0 \psi} f^{-1}(16 - 8\gamma - \frac{9}{\gamma} + \gamma^3) + 3\dot{\psi}^2 \]

\[- \gamma^3 (1 + \frac{1}{\gamma^2}) e^{-2\alpha_0 \psi} \frac{\dot{\psi}^2 \dot{\chi}^2}{4H^2} - \frac{\dot{\psi}^4}{2H^2} + \alpha e^{-4\alpha_0 \psi} \frac{2\dot{\psi}}{H} \frac{1}{2\kappa e} (1 - e^{2\alpha_0 \psi}) - 4U(\chi) \]

\[ + \alpha^2 e^{-4\alpha_0 \psi} \left( \frac{1}{\kappa^2 \mu} (2 - e^{2\alpha_0 \psi}) + 16 U(\chi) \right), \]

\[ C_{\psi \chi} = \frac{e^{-4\alpha_0 \psi} \dot{\psi}}{4H} \frac{\dot{f}_x}{f^2} (1 - \frac{1}{\gamma})^2 - \frac{(\dot{f}_x f)}{f} + \frac{e^{-2\alpha_0 \psi} \dot{\chi} \dot{\gamma}}{\gamma} \dot{\chi} \]

\[- \frac{1}{2\gamma} f_x \dot{\gamma} U_\chi + \frac{1}{2} e^{-2\alpha_0 \psi} (1 - \frac{1}{\gamma})^2 \frac{1}{\gamma} \left( \frac{\dot{f}_x f}{f^2} \right)_\chi + (1 + \frac{1}{\gamma}) f^{-1} (\frac{\dot{f}_x f}{f} t)_\chi \]

\[ + \frac{3}{2} e^{-2\alpha_0 \psi} \frac{\dot{\chi}^2 \gamma (1 + \frac{1}{\gamma})^2}{\gamma} - \frac{e^{-4\alpha_0 \psi} \dot{\chi}^2 \gamma}{4H^2} - e^{-2\alpha_0 \psi} (1 + \frac{1}{\gamma}) \frac{\dot{\chi} \dot{\psi} \dot{\chi}}{2H^2} \]

\[ + e^{-4\alpha_0 \psi} \frac{\dot{\chi}}{H} (1 + \frac{1}{\gamma^2}) U_\chi + \frac{1}{\gamma^2} e^{-2\alpha_0 \psi} U_\chi, \]

\[ C_{\chi \psi} = (-2e^{-2\alpha_0} + \frac{\dot{\psi}}{H}) \frac{1}{2} e^{-2\alpha_0 \psi} \frac{\dot{f}_x f}{f^2} \gamma (1 - \frac{1}{\gamma})^2 - \frac{\gamma \dot{\psi} \dot{\chi}}{2H^2} \]

\[ + 2\alpha e^{-4\alpha_0 \psi} \frac{\dot{\chi}}{H} f^{-1} (1 - \frac{1}{\gamma})^2 \]

\[ \left( 3\frac{\dot{\psi} \dot{\chi}}{2H^2} - \frac{2\alpha}{\gamma} e^{-2\alpha_0 \psi} U_\chi + \frac{\dot{\psi} e^{-2\alpha_0 \psi} U_\chi}{\gamma H} + \alpha e^{-4\alpha_0 \psi} \frac{1}{2\kappa^2 \mu} (e^{2\alpha_0 \psi} - 1) - 4U(\chi) \right) \frac{\dot{\chi}}{H}. \]

Similar to single-field perturbation analysis in canonical and DBI models we introduce two auxiliary fields as

\[ u_\psi = a Q_\psi, \quad u_\chi = a e^{-\alpha_0 \psi} c_s^{-3/2} Q_\chi. \]
The equations of motion in terms of conformal time can be rewritten in a more symmetric form,

\[
\begin{align*}
\ddot{u}_\psi &- Bu'_\chi + \left[k^2 + a^2 C_{\psi\psi} - \frac{r'_\psi}{r_\psi}\right] u_\psi + \left[\frac{r_\psi}{r_\chi} a^2 C_{\psi\chi} + B \frac{r'_\chi}{r_\psi}\right] u_\chi = 0 \\
\ddot{u}_\chi &+ Bu'_\psi + \left[k^2 c_s^2 + a^2 C_{\chi\chi} - \frac{r'_\chi}{r_\chi}\right] u_\chi + \left[\frac{r_\chi}{r_\psi} a^2 C_{\psi\chi} - B \frac{r'_\psi}{r_\psi}\right] u_\psi = 0
\end{align*}
\]

(30) (31)

where \( ()' \) denotes the derivative with respect to conformal time and we define \( c_s = 1/\gamma, \ r_\chi = ae^{-\alpha \psi} \gamma^{3/2}, \ r_\psi = a \) and \( B = r_\chi B_\chi \). The co-moving curvature perturbation, can be express in terms of gauge invariant variables \( Q_\psi \) and \( Q_\chi \) in a simple form\[15],

\[
\mathcal{R} = \frac{H}{-2H}[\dot{\psi}Q_\psi + e^{-2\alpha \psi} \gamma \dot{\chi} Q_\chi]
\]

(32)

The evolution of perturbations for a trapped scalaron:

When scalaron \( \psi \) traps in the minimum of the effective potential, the contribution of \( Q_\psi \) in curvature perturbation can be ignored and the system of equation is treated as a single field DBI model with modified boost factor, numerical analysis supports this approximation \[21]. In this case, the dynamics are governed by \( \chi \)

Figure 2: The blue and red curves depict the contribution of scalaron and DBI fields in curvature perturbation (32) respectively after \( \psi \) trapped in the minimum of effective potential.

\[1\] In our numerical code we got some help from numerical code mTransport\[24\]
Therefore the perturbation equations (30 and 31) are estimated as follows,

\[ u''_\chi + \left[ k^2 c_s^2 + a^2 C_{\chi\chi} - \frac{r''_\chi}{r_\chi}\right] u_\chi \simeq 0. \]  

(33)

insertion of (19) into (23), gives \( C_{\chi\chi} \) and the derivative of \( r_\chi \) in terms of slow-roll parameters (up to the first order) as

\[ a^2 C_{\chi\chi} \simeq 3\mathcal{H}^2[\delta - s - 2\epsilon + 8\kappa^2 \mu U \epsilon (1 - \gamma^{-2})], \]  

(34)

and

\[ \frac{r''_\chi}{r_\chi} \simeq \mathcal{H}^2 \left( 2 - \frac{2}{9} s - (1 - 24\kappa^2 \mu U) \epsilon \right) \]  

(35)

where \( \mathcal{H} = a'/a \) and \( \mathcal{H}' = \mathcal{H}^2 (1 - \epsilon) \). The background variable \( z \) is defined as usual,

\[ z \equiv \frac{a\gamma \sqrt{\rho + p}}{H} = a\gamma \sqrt{2\epsilon} \]  

(36)

where we used the fact \(-2\dot{H} = \rho + p\). Combination of (34) and (35) gives,

\[ a^2 C_{\chi\chi} - \frac{r''_\chi}{r_\chi} \simeq -\frac{z''}{z} + 24\kappa^2 \mu U \epsilon \mathcal{H}^2 (1 + c_s^2). \]  

(37)

The first term is almost the same as single field k-inflation[14], in which \( u_\chi \) is constant for small \( k \); the second term is a small correction of order \( \epsilon \) which is proportional to \((1 + c_s^2)\). At lowest order, we ignore the second term;

\[ u''_\chi + \left( k^2 c_s^2 - \mathcal{H}^2 [2 - \frac{3}{2} s - 3\delta + (5 + 72\kappa^2 \mu U) \epsilon] \right) u_\chi \simeq 0. \]  

(38)

Ignoring the perturbation of \( \psi \) in the co-moving curvature perturbation (32) gives,

\[ \mathcal{R} \simeq \frac{e^{-\alpha \psi} \gamma^{1/2}}{\sqrt{2\epsilon}} Q_\chi = \frac{u_\chi}{z}. \]  

(39)

The power spectrum is as

\[ P_{\mathcal{R}} \simeq \frac{k^3}{2\pi^2} \left| \frac{u_\chi}{z} \right|^2. \]  

(40)
For solving eq. (38) we follow the approach in [16] and references therein, define a new time variable,
\[ y \equiv \frac{c_s k}{aH} = \frac{c_s k}{H} \]  
with this definition at sound horizon crossing we have \( y = 1 \). The derivatives of \( u_\chi \) can express in terms of slow-roll parameters,
\[ u'_\chi = -c_s k (1 - \epsilon - s) \frac{du_\chi}{d\tau} \]
and
\[ u''_\chi = \mathcal{H}^2 (1 - \epsilon - s)^2 y^2 \frac{du^2_\chi}{dy^2} - s (1 - \epsilon - s) y \frac{du_\chi}{dy} \]
with \( \mathcal{H}' = \mathcal{H}^2 (1 - \epsilon) \). Substituting in (38) gives,
\[ y^2 \frac{d^2 u_\chi}{dy^2} + (1 - 2p) y \frac{du_\chi}{dy} + (l^2 y^2 + p^2 - \nu^2) u_\chi = 0, \]
with
\[ p = \frac{1}{2} (1 + s), \]  
\[ l = (1 - \epsilon - s)^{-1}, \]  
\[ \nu = \frac{3}{2} + s - \delta + 3\epsilon (1 + 8\kappa^2 \mu U). \]

The solution of (42) is of the form \( u_\chi = y^p J_\nu (ly) \) where \( J_\nu \) is a Bessel function of order \( \nu \). Instead of Bessel function we write the solution in terms of Hankel functions which are more appropriate for our purpose; in the short wavelength limit \( y \gg 1 \) the solution is given by positive frequency mode \( \frac{1}{\sqrt{2c_s k}} e^{-i c_s k \tau} \) where \( \tau \) is conformal time. Only \( H^{(1)}_{\nu}(ly) \) can satisfy this initial condition, the solution is
\[ u_\chi(y) = \frac{1}{2} \sqrt{\frac{\pi}{c_s k}} \sqrt{\frac{y}{1 - \epsilon - s}} H^{(1)}_{\nu} \left( \frac{y}{1 - \epsilon - s} \right). \]  
In the long-wavelength limit \( y \ll 1 \), \( H^{(1)}_{\nu}(ly) \sim \sqrt{\frac{2}{\pi e^{-i\pi/2}}} 2^{\nu - \frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} y^{-\nu} \), and the solution is
\[ |u_\chi| \sim 2^{\nu - \frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} (1 - \epsilon - s)^{\nu - \frac{1}{2}} \frac{y^{\nu - \frac{1}{2}}}{\sqrt{2c_s k}}, \]  

(46)
Replacing in (40) we arrive at
\[ P_{R}^{1/2} \simeq \left( \frac{V(\nu)}{\pi} \right) \frac{H}{\sqrt{c_s} \epsilon} y^{3-\nu}, \]  
(47)

with \( V \equiv 2^{\nu-3}(1 - \epsilon - s)^{\nu-\frac{1}{2}} \Gamma(\nu)/\Gamma(\frac{3}{2}) \).

It can be shown that \( \frac{d}{dy} \left( \frac{H}{\sqrt{c_s} \epsilon} y^{3-\nu} \right) \simeq 0 \), which insures us that the power spectrum is independent of \( y \) and can be evaluated at any preferred \( y \) value \([16, 21]\), hence the sound crossing formalism is applicable.

Using this gives the spectral index (up to first order in slow-roll parameters) as,
\[ n_s - 1 = 3 - 2\nu \]
\[ = -2s + 2\delta - 6\epsilon(1 + 8\kappa^2 \mu U). \]  
(48)

Replacing the slow-roll parameters we arrive at
\[ n_s - 1 = 2 \frac{\dot{\gamma}}{H \gamma} - 8 \frac{1}{\gamma \chi^2}. \]  
(49)

Since at the end of inflation only DBI field drives the inflation and we ignore the perturbation of scalaron, it is reasonable to assume that the results obtained in the DBI inflation apply to this model; for example, the tensor-to-scalar ratio must be \( r \simeq 16\epsilon c_s \) or \( f_{NL}^{DBI} \simeq -0.3 (c_s^{-2} - 1) \). The effect of \( R^2 \) gravity on the DBI field keeps the sound speed close to one (see Fig\( 3 \)) i.e. keeps \( f_{NL}^{DBI} \) very small.

Figure 3: Sound speed versus number of e-folds. Parameters value are chosen as \( \lambda = 2 \times 10^{12}, \ V_0 = 10^{-12} \) and \( \beta = 55 \).
4.1 Numerical Analysis

In this section, we check the compatibility of our model with Planck2018 data. Our analysis shows that the amount of inflation depends on $\psi$ value, the initial value of $\psi$ is picked in such a way to obtain enough e-folds. Motivated by brane inflation, we choose the initial value of $\chi$ around $10^{25}$ (through this work we choose 1.5). There is also another parameter in the $R^2$ part of the action, there is no observational strong limit on it i.e. $\mu$, we select $\mu \sim 10^9$ (in natural units), therefore the mass of $\psi$ is fixed.

We vary the mass ratio parameter, $\beta$, to obtain the spectral index and the tensor-to-scalar ratio. There is also two other parameters in the DBI part of the action, $\lambda$ and $V_0$. We check different values of these parameters. First we consider different value of $V_0$, figure 4 shows the tensor-to-scalar ratio versus spectral index, by increasing the mass ratio the spectral index also increases but the tensor-to-scalar ratio remains almost constant. The tensor-to-scalar ratio value is very small (in comparison with Planck upper limit 0.064). To be more clear we plot spectral index (5) and the tensor-to-scalar ratio (6) with respect to mass ratio. ($\beta$).

Figure 4: The tensor to scalar ratio versus the spectral index is depicted, we choose $\lambda = 2 \times 10^{12}$. The colored regions are 68% and 95% confidence level of TT,TE,EE+lowE+lensing Planck2018 data.
Figure 5: We plot $n_s$ versus beta (the mass ratio), the narrow grey band shows the planck limit. We choose $\lambda = 2 \times 10^{12}$.

Figure 6: The tensor to scalar ratio is shown versus beta (the mass ratio). We choose $\lambda = 2 \times 10^{12}$. Our results are much smaller than the Planck limit $r < 0.064$.

From the above figures, we conclude that very small and very large values of $V_0$ are not compatible with observations. For small $V_0$, the DBI potential is almost $\frac{1}{2} m^2 \phi^2$, which is not compatible with the Planck data. On the opposite side, for large values of $V_0$, the DBI potential is dominated by the second term. It seems that to get good results we need both parts of the potential, therefore we choose an intermediate value, i.e. $V_0 = 10^{-12}$.

To find out the effect of the other parameter, $\lambda$ we again plot $r$ with respect to $n_s$ by varying the mass ratio ($\beta$) for different value of $\lambda$ (7). We also plot $r$ (8) and $n_s$ (9) with respect to $\beta$ separately.
Figure 7: The tensor to scalar ratio versus the spectral index is depicted, we choose $V_0 = 10^{-12}$. The colored regions are 68% and 95% confidence level of TT,TE,EE+lowE+lensing Planck2018 data.

Figure 8: We plot $n_s$ versus beta (the mass ratio), we choose $V_0 = 10^{-12}$. The narrow grey band shows the planck limit.
Figure 9: The tensor to scalar ratio is shown versus beta (the mass ratio), we choose $V_0 = 10^{-12}$. These plots are for different value of constant part of DBI potential. Our results are much smaller than the Planck limit $r < 0.064$.

These figures indicate that, only intermediate values, around $10^{12}$ to $10^{13}$ gives compatible results, therefore for a closer look, we plot $r$ and $n_s$ for $\lambda$ in this range.

Figure 10: We plot the spectral index by varying the $\lambda$ parameter. The gray area is allowed value by Planck2018. As before $V_0 = 10^{-12}$
Our analysis shows that it is possible to get the spectral index and the tensor to scalar ratio in the Planck range for the appropriate choice of parameters and initial conditions.

Our numerical analysis shows that regardless of the mass ratio, the sound speed is near one (see FIG 3). As before we set the initial values of $\chi = 1.5$ and $\psi = 5.3$. According to our analysis, the main results are not sensitive to initial conditions.

5 Conclusion

In this work, we studied the effect of the existence of a DBI field in the Starobinsky inflation i.e. $R + cR^2$ gravity. When the mass of the DBI field is greater than or equal to the mass of scalaron, the scalaron dominates. But when the DBI field is lighter, after a while the scalaron traps in its minimum and DBI takes the main role. We consider the latter case i.e. heavy scalaron, then it is possible to ignore the fluctuation of scalaron which is trapped in its minimum. Before trapping of $\psi$, the DBI field is almost constant, from the brane inflation point of view, it means that the branes move very slowly. After $\psi$ traps in its minimum, the DBI field begins to decrease, the branes get closer together. Although the DBI field drives inflation, the boost factor and other quantities have implicit dependence on $\psi$. In this model, the boost factor is smaller than the single DBI model due to the existence of $e^{2\alpha \psi}$ in the square root. Hence the level of non-Gaussianity decreases. Since
before trapping the scalaron contribution to the energy density is much
greater than the contribution of DBI, Hubble parameter and consequently
the maximum number of e-folds, have a strong dependence on scalaron;
but due to heaviness of scalaron, its perturbations are suppressed; only the
perturbations of DBI field contribute to curvature perturbation, which has
been checked numerically.

It seems that with appropriate initial conditions, we get 50-60 e-folds
at the end of inflation and this model can be compatible with the Planck
constraints on the spectral index and the tensor to scalar ratio. (see figures
4 and 7).

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