On $L^2$-Harmonic Forms of Complete Almost Kähler Manifold

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Abstract
In this article, we study the $L^2$-harmonic forms on the complete $2n$-dimensional almost Kähler manifold $X$. We observe that the $L^2$-harmonic forms can decomposition into Lefschetz powers of primitive forms. Therefore we can extend vanishing theorems of $d$(bounded) (resp. $d$(sublinear)) Kähler manifold proved by Gromov (resp. Cao-Xavier, Jost-Zuo) to almost Kählerian case, that is, the spaces of all harmonic $(p, q)$-forms on $X$ vanishing unless $p + q = n$. We also give a lower bound on the spectra of the Laplace operator to sharpen the Lefschetz vanishing theorem on $d$(bounded) case.

Keywords Symplectic hyperbolic (parabolic) · $L^2$-Harmonic forms · Vanishing theorem

Mathematics Subject Classification 53D05 · 58A10 · 58A12

1 Introduction

A differential form $\alpha$ in a Riemannian manifold $(X, g)$ is called bounded with respect to the metric $g$ if the $L^\infty$-norm of $\alpha$ is finite, namely,

$$\|\alpha\|_{L^\infty} = \sup_{x \in X} |\alpha(x)| < \infty.$$ 

By definition, a $k$-form $\alpha$ is said to be $d$(bounded) if $\alpha = d\beta$, where $\beta$ is a bounded $(k - 1)$-form. It is obvious that if $X$ is compact, then every exact form is $d$(bounded). However, when $X$ is not compact, there exist smooth differential forms which are exact but not $d$(bounded). For instance, on $\mathbb{R}^n$, $\alpha = dx^1 \wedge \cdots \wedge dx^n$ is exact, but it is not $d$(bounded) [4,10]. Let’s recall some concepts introduced in [2,13,17].
**Definition 1.1** A differential form $\alpha$ on a complete non-compact Riemannian manifold $(X, g)$ is called $d$-(sublinear) if there exist a differential form $\beta$ and a number $c > 0$ such that $\alpha = d\beta$ and

$$\|\beta(x)\|_{L_\infty} \leq c(1 + \rho_g(x, x_0)),$$

where $\rho_g(x, x_0)$ stands for the Riemannian distance between $x$ and a base point $x_0$ with respect to $g$.

Let $(X, g)$ be a Riemannian manifold and $\pi : (\tilde{X}, \tilde{g}) \to (X, g)$ be the universal covering with $\tilde{g} = \pi^*g$. A form $\alpha$ on $X$ is called $\tilde{d}$-(bounded) (resp. $\tilde{d}$-(sublinear)) if $\pi^*\alpha$ is a $d$-(bounded) (resp. $d$-(sublinear)) form on $(\tilde{X}, \tilde{g})$. In geometry, various notions of hyperbolicity have been introduced, and the typical examples are manifolds with negative curvature in suitable sense [4]. The starting point for the present investigation is Gromov’s notion of Kähler hyperbolicity [10]. Extending Gromov’s terminology, Cao-Xavier [2] and Jost-Zuo [17] proposed the Kähler parabolicity.

Let $(X^{2n}, \omega)$ be a closed symplectic manifold. Let $J$ be an $\omega$-compatible almost complex structure, i.e., $J^2 = -I$, $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$, and $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ is a Riemannian metric on $X$. The triple $(\omega, J, g)$ is called an almost Kähler structure on $X$. Notice that any one of the pairs $(\omega, J)$, $(J, g)$ or $(g, \omega)$ determines the other two. An almost-Kähler structure $(\omega, J, g)$ is Kähler if and only if $J$ is integrable. For the symplectic case, inspired by Kähler geometry, Tan–Wang–Zhou [18,22] gave the definition of a symplectic hyperbolic (resp. parabolic) manifold.

**Definition 1.2** A closed almost Kähler manifold $(X, \omega)$ is called symplectic hyperbolic (resp. parabolic) if the lift $\tilde{\omega}$ of $\omega$ to the universal covering $(\tilde{X}, \tilde{\omega}) \to (X, \omega)$ is $d$-(bounded) (resp. $d$-(sublinear)) on $(\tilde{X}, \tilde{\omega})$.

**Example 1.3** (1) Let $(X^{2n}, \omega)$ be a closed symplectic manifold. If $[\omega]$ is aspherical and $\pi_1(X)$ is hyperbolic then, $\omega$ is hyperbolic. In particular, if $X^{2n}$ admits a Riemannian metric of negative sectional curvature then, $\omega$ is hyperbolic, see [18, Corollary 1.13].

(2) Let $X^{2n}$ be a closed manifold of non-positive sectional curvature. If $X^{2n}$ is homeomorphic to a symplectic manifold, then $X^{2n}$ is symplectically parabolic [2,17].

Tan–Wang–Zhou proved that if $(X^{2n}, \omega)$ is a closed symplectic parabolic manifold which satisfies the Hard Lefschetz Condition (HLC), then the spaces of $L^2$-harmonic forms $\mathcal{H}_{k}(X)$ on the universal space $\tilde{X}$ are zero unless $k = n$ [15,22]. The Hard Lefschetz Condition is necessary in Tan–Wang–Zhou’s theorem. Hind-Tomassini [12] constructed a $d$-(bounded) complete almost Kähler manifold $X$ satisfying $\mathcal{H}_{k}(X) \neq \{0\}$ by using methods of contact geometry.

In [10], Gromov developed $L^2$-Hodge theory for Kähler manifolds, proving an $L^2$-Hodge decomposition Theorem for $L^2$-forms. As a consequence, for a complete and $d$-(bounded) Kähler manifold $X$, denoting by $\mathcal{H}_{k}^{k}$, respectively $\mathcal{H}_{k}^{p,q}$, the space of $\Delta_d$-harmonic $L^2$-forms of degree $k$, respectively $\Delta_d = 2\Delta_\partial$-harmonic $L^2$-forms of bi-degree $(p, q)$, he showed that $\mathcal{H}_{k}^{k} \cong \bigoplus_{p+q=k} \mathcal{H}_{(p,q)}^{p,q}$; furthermore, denoting by $n = \dim \tilde{X}$, that $\mathcal{H}_{k}^{k} = \{0\}$, for all $k \neq n$ and hence $\mathcal{H}_{1}^{1} = \{0\}$, for all $(p, q)$. Springer
such that \( p + q \neq n \). Gromov [10] also gave a lower bound on the spectra of the Laplace operator \( \Delta_d := dd^* + d^*d \) on \( L^2 \)-forms \( \Omega^{p,q}(X) \) for \( p + q \neq n \) to sharpen the Lefschetz vanishing theorem. The main purpose in this article is to extend the Gromov’s results to almost Kählerian case.

**Theorem 1.4** (=Theorem 3.7 and 3.15) Let \((X, J, \omega)\) be a complete \(2n\)-dimensional almost Kähler manifold with a \(d\)(sublinear) symplectic form \(\omega\). Then

\[
\mathcal{H}^{p,q}_{(2),J}(X) = \{0\}
\]

unless \( k := p + q = n \), where

\[
\mathcal{H}^{p,q}_{(2),J}(X) := \{ \alpha \in \Omega^{p,q}_{(2),J}(X) : \Delta_d \alpha = 0 \}.
\]

In particular, if \(\omega\) is \(d\)(bounded), i.e., there exists a bounded 1-form \(\theta\) such that \(\omega = d\theta\), then any \(\alpha \in \Omega^{p,q}_J \cap \Omega^0_0\) on \(X\) of degree \(k := p + q \neq n\) satisfies the inequality

\[
c_n,k \|\theta\|_{L^\infty}^2 \|\alpha\|_{L^2(X)}^2 \leq \|d\alpha\|_{L^2(X)}^2 + \|d^*\alpha\|_{L^2(X)}^2,
\]

where \(c_{n,k} > 0\) is a constant which depends only on \(n, k\).

Suppose that \((X^{2n}, \omega)\) is a complete Kähler manifold. Let \(\alpha_k\) be a \(k\)-form in \(X^{2n}\). We denote \(\alpha_k := \sum_{p+q=k} \alpha_{p,q}\), where \(\alpha_{p,q} \in \Omega^{p,q}(X)\). We have

\[
\langle \Delta_d \alpha_k, \alpha_k \rangle_{L^2(X)} = \sum_{p+q=k} \langle \Delta_d \alpha_{p,q}, \alpha_{p,q} \rangle_{L^2(X)} \Rightarrow \mathcal{H}^k_{(2)}(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{(2),J}(X).
\]

Following Theorem 1.4, we get some well-known results proved by Gromov [10], Cao-Xavier [2] and Jost-Zuo [17].

**Corollary 1.5** (=Corollary 3.9 and 3.16) Let \((X^{2n}, \omega)\) be a complete Kähler manifold with a \(d\)(sublinear) Kähler form \(\omega\). Then

\[
\mathcal{H}^k_{(2)}(X) = \{0\}
\]

unless \(k = n\). In particular, if \(\omega\) is \(d\)(bounded), i.e., there exists a bounded 1-form \(\theta\) such that \(\omega = d\theta\), then any \(\alpha \in \Omega^k_0\) on \(X\) of degree \(k \neq n\) satisfies the inequality

\[
c_{n,k} \|\theta\|_{L^\infty}^2 \|\alpha\|_{L^2(X)}^2 \leq \|d\alpha\|_{L^2(X)}^2 + \|d^*\alpha\|_{L^2(X)}^2,
\]

where \(c_{n,k} > 0\) is a constant which depends only on \(n, k\).

## 2 L²-Hodge Theory

We recall some basic on \(L^2\) harmonic forms [3,8]. Let \(X\) be a smooth manifold of dimension \(n\), let \(\Omega^k(X)\) and \(\Omega^k_0(X)\) denote the smooth \(k\)-forms on \(X\) and the smooth \(k\)-forms with compact support on \(X\), respectively. We assume now that \(X\) is endowed
with a Riemannian metric $g$. Let $(\cdot, \cdot)$ denote the pointwise inner product on $\Omega^k(X)$ given by $g$. The global inner product is defined

$$\langle \alpha, \beta \rangle = \int_X (\alpha, \beta) d\text{vol},$$

where $d\text{vol}$ is the Riemannian volume form of metric $g$.

We also write $|\alpha|^2 = (\alpha, \alpha)$, $\|\alpha\|^2 = \int_X |\alpha|^2 d\text{vol}$, and let

$$\Omega^k_{(2)}(X) = \{ \alpha \in \Omega^k(X) : \|\alpha\|^2 < \infty \}.$$

Denote by $(A^k_{(2)}(X), d)$ the sub-complex of $(\Omega^k(X), d)$ formed by differential forms $\alpha$ such that both $\alpha$ and $d\alpha$ are in $L^2$. Then the reduced $L^2$-cohomology group of degree $k$ of $X$ is defined as

$$H^k_{(2)}(X) = A^k_{(2)}(X) \cap \ker d / (d\Omega^{k-1}_{(2)}(X)).$$

We recall the following

**Lemma 2.1** [10, Lemma 1.1 A] Let $(X, g)$ be a complete Riemannian manifold of dimension $n$ and let $\eta$ be an $L^1$-form on $X$ of degree $n - 1$, that is

$$\int_X |\eta| < \infty.$$ 

Assume that also the differential $d\eta$ is also $L^1$. Then

$$\int_X d\eta = 0.$$ 

Let $d^*$ denote the adjoint operator of the differential operator $d$ with respect to $g$. The Laplacian operator is given by $\Delta_d = dd^* + d^*d : \Omega^k(X) \to \Omega^k(X)$. A $k$-form $\alpha \in \Omega^k_{(2)}(X)$ is called $L^2$-harmonic form if $\Delta_d \alpha = 0$. It is well known that $\alpha$ is $L^2$-harmonic if only if $d\alpha = 0$ and $d^*\alpha = 0$. We denote by

$$\mathcal{H}^k_{(2)}(X) = \{ \alpha \in \Omega^k_{(2)}(X) : \Delta_d \alpha = 0 \}$$

the space of $L^2$-harmonic $k$-forms on $X$. We have the Hodge–de Rham–Kodaira orthogonal decomposition of $\Omega^k_{(2)}(X)$

$$\Omega^k_{(2)}(X) = \mathcal{H}^k_{(2)}(X) \oplus \overline{d(\Omega^{k-1}(X))} \oplus d^*\overline{(\Omega^{k+1}(X))},$$

where $\overline{d(\Omega^{k-1}(X))}$ and $d^*\overline{(\Omega^{k+1}(X))}$ are closure of $d(\Omega^{k-1}(X))$ and $d^*(\Omega^{k+1}(X))$ with respect to $L^2$-norm respectively (see [3] and [10, 1.1.C.]). We have the following
Lemma 2.2 [12, Lemma 2.2] Let \((X, g)\) be a complete Riemannian manifold of dimension \(n\) and let \(\alpha \in \Omega^2_2(X)\). Denote by
\[
\alpha = \alpha_H + \lambda + \mu
\]
the Hodge decomposition of \(\alpha\), where \(\alpha_H \in \mathcal{H}^{k}(X)\), \(\lambda \in \overline{d\Omega^{k-1}(X)}\), \(\mu \in \overline{d^*\Omega^{k+1}(X)}\). Then

1. \(d\lambda = 0\),
2. If \(d\alpha = 0\), then \(\mu = 0\).

3 Vanishing Theorems

As we derive estimates in this section, there will be many constants which appear. Sometimes we will take care to bound the size of these constants, but we will also use the following notation whenever the value of the constants are unimportant. We write \(\alpha \lesssim \beta\) to mean that \(\alpha \leq C \beta\) for some positive constant \(C\) independent of certain parameters on which \(\alpha\) and \(\beta\) depend. The parameters on which \(C\) is independent will be clear or specified at each occurrence. We also use \(\beta \lesssim \alpha\) and \(\alpha \approx \beta\) analogously.

3.1 \(L^2\)-Harmonic Forms of Bi-degree

We review and point out certain special structures of differential forms on symplectic manifolds. Let \((X, \omega)\) be a closed symplectic manifold of dimension \(2n\). Using the symplectic form \(\omega = \sum \frac{1}{2} \omega_{ij} dx^i \wedge dx^j\), the Lefschetz operator \(L : \Omega^k \rightarrow \Omega^{k+1}\) and the dual Lefschetz operator \(\Lambda : \Omega^k \rightarrow \Omega^{k-2}\) are defined acting on a \(k\)-form \(\alpha_k\) by
\[
L(\alpha_k) = \omega \wedge \alpha_k, \quad \Lambda(\alpha_k) = \frac{1}{2} (\omega^{-1})^{ij} i_{\partial_i} i_{\partial_j} \alpha_k.
\]

Definition 3.1 [23, Definition 2.1] A differential \(k\)-from \(B_k\) with \(k \leq n\) is calledprimitive, i.e., \(B_k \in P^k(X)\), if it satisfies the two equivalent conditions: (i) \(\Lambda B_k = 0\); (ii) \(L^{n-k+1} B_k = 0\).

We will make use of the Weil relation for primitive \(k\)-forms \(B_k\), see [23] Equation (2.19):
\[
\ast \frac{1}{r!} L^r B_k = (-1)^{\frac{k(k+1)}{2}} \frac{1}{(n-k-r)!} L^{n-k-r} \mathcal{J}(B_k), \quad (3.1)
\]
where
\[
\mathcal{J} = \sum_{p,q} (\sqrt{-1})^{p-q} \Pi^{p,q}
\]
projects a $k$-form onto its $(p, q)$ parts time the multiplicative factor $(\sqrt{-1})^{p-q}$.

Recall the symplectic star operator, $\ast_s : \Omega^k(X) \rightarrow \Omega^{2n-k}(X)$ defined by

$$\alpha \wedge \ast_s \beta = (\omega)^{-1}(\alpha, \beta) d\text{vol}$$

$$= \frac{1}{k!} (\omega^{-1})^{i_1j_1} (\omega^{-1})^{i_2j_2} \cdots (\omega^{-1})^{i_kj_k} \alpha_{i_1i_2\cdots i_k} \beta_{j_1j_2\cdots j_k} \frac{\omega^n}{n!},$$

for any two $k$-forms $\alpha, \beta \in \Omega^k$. This definition is in direct analogy with the Riemannian Hodge star operator where here $\omega^{-1}$ has replaced $g^{-1}$. Notice, however, that $\ast_s$ as defined in (3.2) does not give a positive-definite local inner product, as $\alpha \wedge \ast_s \alpha$ is $k$-symmetric. Thus, for instance, $\alpha_k \wedge \ast_s \alpha_k$ for $k$ odd [23]. The symplectic star operator permits us to consider $\Lambda$ and $d\Lambda$ as the symplectic adjoints of $L$ and $d$, respectively. Specifically, we have the relations [20, 24] $\Lambda = \ast_s L \ast_s$, and [1]

$$d\Lambda := [d, \Lambda] = (-1)^{k+1} \ast_s d \ast_s,$$

acting on $\alpha_k \in \Omega^k$. Thus we easily find that $d\Lambda$ squares to zero, that is, $d\Lambda d\Lambda = - \ast_s d^2 \ast_s = 0$.

Let $(X, \omega, g)$ be a closed almost Kähler manifold. We can used the metric $g$ to define the Hodge star operator. The dual Lefschetz operator $\Lambda$ is then just the adjoint of $L$, $\Lambda = (-1)^k \ast L \ast$. The $d\Lambda$ operator is related via the Hodge star operator defined with respect to the compatible metric $g$ by the relation, see [23, Lemma 2.9],

$$d\Lambda = (-1)^{k+1} \ast J^{-1} d \ast J^{-1} = - \ast J^{-1} d J \ast.$$ 

Let $X$ be a $2n$-dimensional manifold (without boundary) and $J$ be a smooth almost-complex structure on $X$. There is a natural action of $J$ on the space $\Omega^k(X, \mathbb{C}) := \Omega^k(X) \otimes \mathbb{C}$, which induces a topological type decomoposition

$$\Omega^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}_J(X, \mathbb{C}),$$

where $\Omega^{p,q}_J(X, \mathbb{C})$ denotes the space of complex forms of type $(p, q)$ with respect to $J$ [11]. If $k$ is even, $J$ also acts on $\Omega^k(X)$ as an involution. The space $\Omega^k(X)$ of real smooth differential $k$-forms has a type decomposition:

$$\Omega^k(X) = \bigoplus_{p+q=k} \Omega^{p,q}_J(X),$$

where

$$\Omega^{p,q}_J(X) = \left\{ \alpha \in \Omega^{p,q}_J(X, \mathbb{C}) \oplus \Omega^{q,p}_J(X, \mathbb{C}) : \alpha = \bar{\alpha} \right\}.$$
We denote by
\[ H^{p,q}_{(2);J}(X) := \left\{ \alpha \in \Omega^{p,q}_{(2);J}(X) : \Delta_d \alpha = 0 \right\} \]
the space of $L^2$-harmonic forms of bi-degree $(p, q)$. Here
\[ \Omega^{p,q}_{(2);J}(X) := \{ \alpha \in \Omega^{p,q}_{J}(X) : \| \alpha \|_{L^2(X)} < \infty \}. \]

In a closed symplectic manifold $(X, \omega)$, Tseng-Yau considered the symplectic cohomology group $H^k_{d+d^\Lambda}$ which are just the symplectic version of well-known cohomologies in complex geometry already studied by Kodaira-Spencer [19]. They also defined a four-order differential operator as follows
\[ \Delta_{d+d^\Lambda} = dd^\Lambda (dd^\Lambda)^* + \lambda (d^* d + d^* d^\Lambda). \]
where $d^\Lambda = ([d, \Lambda])^* = [L, d^*] = *d^* (\text{see [23, Equation (2.25)])}$. In [23], the authors proved that there exists a Lefschetz decomposition for symplectic cohomology group $H^k_{d+d^\Lambda}$.

In a complete symplectic manifold $(X, \omega)$, we only consider a two-order differential operator as follows
\[ D_{d+d^\Lambda} = d^* d + d^\Lambda d^\Lambda. \]

We denote by
\[ \mathcal{H}^{p,q}_{(2);d+d^\Lambda} := \{ \alpha \in \Omega^{p,q}_{(2);d+d^\Lambda} : D_{d+d^\Lambda} \alpha = 0 \} \]
the space of $L^2 d + d^\Lambda$-harmonic $(p, q)$-forms. We also denote by
\[ P \mathcal{H}^{p,q}_{(2);d+d^\Lambda} := \{ \alpha \in P^{p,q}_{(2);J} : D_{d+d^\Lambda} \alpha = 0 \} \]
the space of $L^2 d + d^\Lambda$-harmonic $(p, q)$-forms, where $P^{p,q}_{(2);J} := \ker \Lambda \cap \Omega^{p,q}_{(2);J}$ is the space of the primitive $L^2 (p, q)$-forms.

**Lemma 3.2** For any $\alpha_{p,q} \in \Omega^{p,q}_J$, we have the identity
\[ \|d^\Lambda \alpha_{p,q}\|^2 = \|d^* \alpha_{p,q}\|^2. \]
In particular,
\[ \mathcal{H}^{p,q}_{(2);J}(X) = \mathcal{H}^{p,q}_{(2);d+d^\Lambda}(X). \]

**Proof** Noting that $J^2 = (-1)^k$ acting on a $k$-form. We then have
\[ d \ast J^{-1} \alpha_{p,q} = d \ast (-1)^k J \alpha_{p,q} = (-1)^k (\overline{\square-1})^{p-q} d \ast \alpha_{p,q}. \]
Therefore,
\[\|d^\Lambda \alpha_{p,q}\|^2 = \|\mathcal{J}^{-1} d \ast \mathcal{J}^{-1} \alpha_{p,q}\|^2 = \|d \ast \mathcal{J}^{-1} \alpha_{p,q}\|^2 = \|d \ast \alpha_{p,q}\|^2 = \|d^\Lambda \alpha_{p,q}\|^2.\]

Therefore, \(d^\Lambda \alpha_{p,q} = 0\) if only if \(d^* \alpha_{p,q} = 0\). Suppose that \(\alpha_{p,q} \in H^{p,q}_{(2)}(\mathbb{R}^n)\); \(\mathcal{J}(X)\).

Then following Lemma 3.3, we get \(d^\Lambda \alpha_{p,q} = 0\) and \(d \alpha_{p,q} = 0\). Hence \(d^* \alpha_{p,q} = 0\), i.e., \(\alpha_{p,q} \in H^{p,q}_{(2)}(\mathbb{R}^n)\).

\(\square\)

We follow the method of Gromov’s [10] to choose a sequence of cutoff functions \(\{f_\varepsilon\}\) satisfying the following conditions:

(i) \(f_\varepsilon\) is smooth and takes values in the interval \([0, 1]\), furthermore, \(f_\varepsilon\) has compact support.

(ii) The subsets \(f_\varepsilon^{-1} \subset X\), i.e., of the points \(x \in X\) where \(f_\varepsilon(x) = 1\) exhaust \(X\) as \(\varepsilon \to 0\).

(iii) The differential of \(f_\varepsilon\) everywhere bounded by \(\varepsilon\),

\[\|df_\varepsilon\|_{L^\infty} = \sup_{x \in X} |df_\varepsilon| \leq \varepsilon.\]

Thus one obtains another useful

**Lemma 3.3** If an \(L^2(p, q)\)-form \(\alpha\) is \(\mathcal{D}_{d+d^\Lambda}\)-harmonic form, then \(d \alpha = 0, d^\Lambda \alpha = 0\).

**Proof** We want to justify the integral identity

\[ \langle \mathcal{D}_{d+d^\Lambda} \alpha, \alpha \rangle = \langle d \alpha, d \alpha \rangle + \langle d^\Lambda \alpha, d^\Lambda \alpha \rangle \]

If \(d \alpha\) and \(d^\Lambda \alpha\) are \(L^2\) (i.e., square integrable on \(X\)), then this follows by Lemma 2.1. To handle the general case we cutoff \(\alpha\) and obtain by a simple computation

\[
0 = \langle \mathcal{D}_{d+d^\Lambda} \alpha, f_\varepsilon^2 \alpha \rangle \\
= \langle d \alpha, d(f_\varepsilon^2 \alpha) \rangle + \langle d^\Lambda \alpha, d^\Lambda (f_\varepsilon^2 \alpha) \rangle \\
= \langle d \alpha, f_\varepsilon^2 d \alpha \rangle + \langle d \alpha, 2 f_\varepsilon d f_\varepsilon \wedge \alpha \rangle \\
+ \langle d^\Lambda \alpha, f_\varepsilon^2 d^\Lambda \alpha \rangle + \langle d^\Lambda \alpha, 2 f_\varepsilon d f_\varepsilon \wedge (\Lambda \alpha) \rangle - \langle d^\Lambda \alpha, \Lambda(2 f_\varepsilon d f_\varepsilon \wedge \alpha) \rangle \\
= I_1(\varepsilon) + I_2(\varepsilon),
\]

where

\[
|I_1(\varepsilon)| = \langle d \alpha, f_\varepsilon^2 d \alpha \rangle + \langle d^\Lambda \alpha, f_\varepsilon^2 d^\Lambda \alpha \rangle \\
= \int_X f_\varepsilon^2 (|d \alpha|^2 + |d^\Lambda \alpha|^2)
\]

and

\[
|I_2(\varepsilon)| \leq |\langle d \alpha, 2 f_\varepsilon d f_\varepsilon \wedge \alpha \rangle| + |\langle d^\Lambda \alpha, 2 f_\varepsilon d f_\varepsilon \wedge (\Lambda \alpha) \rangle| + |\langle d^\Lambda \alpha, \Lambda(2 f_\varepsilon d f_\varepsilon \wedge \alpha) \rangle| \\
\lesssim \int_X |df_\varepsilon| \cdot |f_\varepsilon| \cdot |\alpha| (|d \alpha| + |d^\Lambda \alpha|).
\]

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Then we choose $f_{\varepsilon}$ such that $|df_{\varepsilon}|^2 < \varepsilon f_{\varepsilon}$ on $X$ and estimate $I_2$ by Schwartz inequality. Then

$$|I_2(\varepsilon)| \lesssim \varepsilon \|f_{\varepsilon} \alpha\|_{L^2(X)} \left( \int_X f_{\varepsilon}^2 (|d\alpha|^2 + |d^\Lambda \alpha|^2) \right)^{\frac{1}{2}},$$

and hence $|I_1| \to 0$ for $\varepsilon \to 0$.

**Proposition 3.4** Let $(X^{2n}, J, \omega)$ be a complete almost Kähler manifold. We then have a decomposition for the space of the $L^2$-harmonic forms of bi-degree $(p, q)$:

$$
\mathcal{H}^{p,q}_{(2); J}(X) = \bigoplus_{r \geq \max\{0,k-n\}} L^r P^{p-r,q-r}(X),
$$

where $k := p + q$.

**Proof** Let $\alpha_k$ be a $L^2$ $\mathcal{D}_{d+d^\Lambda}$-harmonic form of bi-degree $(p, q)$ on $X$, $k := p + q$. Following primitive decomposition formula, see [16, Proposition 1.2.30] or [7, Charp VI. (5.15)], we can denote

$$
\alpha_k = \sum_{r \geq \max\{0,k-n\}} L^r \beta_{k-2r},
$$

where $\beta_{k-2r} \in P^{k-2r}$. Following [23, Lemma 2.3, 2.10], we have

$$
[D_{d+d^\Lambda}, L] = [d^* d + d^\Lambda * d^\Lambda, L] = [d^*, L] d + d^\Lambda * [d^\Lambda, L] = 0.
$$

Therefore,

$$
0 = D_{d+d^\Lambda} \alpha_k = \sum_{r \geq \max\{0,k-n\}} L^r (D_{d+d^\Lambda} \beta_{k-2r}). \tag{3.4}
$$

Noting that the operator $D_{d+d^\Lambda}$ communicates with $\Lambda$, see [23, Lemma 3.7],

$$
[D_{d+d^\Lambda}, \Lambda] = [d^* d + d^\Lambda * d^\Lambda, \Lambda] = d^*[d, \Lambda] + [d^\Lambda, \Lambda] d^\Lambda = 0.
$$

Then $\Lambda D_{d+d^\Lambda} \beta_{k-2r} = 0$, i.e., $D_{d+d^\Lambda} \beta_{k-2r} \in P^{k-2r}$. Using the fact ([16, Proposition 1.2.30])

$$
\Omega^k(X) = \bigoplus_{r \geq \max\{0,k-n\}} L^r P^{k-2r}(X).
$$

Hence following (3.4), we then have

$$
D_{d+d^\Lambda} \beta_{k-2r} = 0.
$$
It implies that
\[ \mathcal{H}_{(2),d+d^A}^{p,q}(X) = \bigoplus_{r \geq \max\{0, k-n\}} L^r P \mathcal{H}_{(2),d+d^A}^{p-r,q-r}. \]

Noting that
\[ P \mathcal{H}_{(2),d+d^A}^{p-r,q-r} = P \mathcal{H}_{(2);J}^{p-r,q-r}. \]

Following Weil formula,
\[ \ast L^r \beta_{k-2r} = (-1)^{\frac{(k-2r)(k-2r+1)}{2}} (\sqrt{-1})^{p-q} L^{n-k+r} \beta_{k-2r}. \]

Therefore, \( L^r \beta_{k-2r} \in \mathcal{H}_{(2)}^{p,q}(X) \). We get
\[ \mathcal{H}_{(2);d+d^A}^{p,q} = \bigoplus_{r \geq \max\{0, k-n\}} L^r P \mathcal{H}_{(2),d+d^A}^{p-r,q-r} \subset \mathcal{H}_{(2);J}^{p,q}(X). \]

Hence the conclusion follows from Lemma 3.2.

\[ \square \]

**Remark 3.5** In [5,6], the authors extended the Kähler identities to the non-integrable setting. In fact, Proposition 3.4 is the generalized Hard Lefchetz Duality of the space of \((p, q)\)-harmonic forms on compact almost Kähler manifolds, see [5, Theorem 5.1].

### 3.2 Symplectic Parabolic

Let now \((X^{2n}, J)\) be a almost Kähler manifold and \(g\) be a Hermitian metric. Then according to Cao-Xavier [2] and Jost-Zuo [17], if \(J\) is integrable and \(\omega\) is \(d\) (sublinear), then \(\mathcal{H}_{(2)}^{p,q}(X) = \{0\}\), unless \(k := p + q = n\). In this section we will see that the same conclusions hold in the category of almost Kähler manifolds.

**Proposition 3.6** Let \((X^{2n}, J, \omega)\) be a complete symplectic manifold with a \(d\) (sublinear) symplectic form \(\omega\). Then any \(\alpha \in P \mathcal{H}_{(2);J}^{p,q}(X)\) of degree \(k := p + q < n\) vanishes.

**Proof** For any \(\alpha \in P^k, k := p + q < n\), following Weil formula (3.1), it implies that
\[ \ast \alpha = C(n, p, q) \alpha \wedge \omega^{n-k}, \]

where \(C(n, p, q) = \sqrt{-1}^{p-q} (-1)^{\frac{k(k+1)}{2}} \frac{1}{(n-k)!}\). By hypothesis, there exists a 1-form \(\theta\) with \(\omega = d\theta\) and
\[ \| \theta(x) \|_{L_\infty} \leq c (1 + \rho(x, x_0)), \]

where \(c\) is an absolute constant. In what follows we assume that the distance function \(\rho(x, x_0)\) is smooth for \(x \neq x_0\). The general case follows easily by an approximation.
argument. We observe that

\[ *\alpha = d\eta, \]

where

\[ \eta = C(n, p, q)(\theta \wedge \alpha \wedge \omega^{n-k-1}) \]

Let \( h : \mathbb{R} \to \mathbb{R} \) be smooth, \( 0 \leq h \leq 1 \),

\[ h(t) = \begin{cases} 
1, & t \leq 0 \\
0, & t \geq 1
\end{cases} \]

and consider the compactly supported function

\[ f_j(x) = h(\rho(x_0, x) - j), \]

where \( j \) is a positive integer.

Noticing that \( f_j * \alpha \) has compact support, one has

\[
\langle *\alpha, f_j * \alpha \rangle_{L^2(X)} = \langle d\eta, f_j * \alpha \rangle_{L^2(X)} = \langle \eta, d^* (f_j * \alpha) \rangle_{L^2(X)} = \langle \theta \wedge \alpha \wedge \omega^{n-k-1}, d^* (f_j * \alpha) \rangle_{L^2(X)} = \langle \theta \wedge \alpha \wedge \omega^{n-k-1}, * (df_j \wedge \alpha) \rangle_{L^2(X)}. \tag{3.5}
\]

Since \( 0 \leq f_j \leq 1 \) and \( \lim_{j \to \infty} f_j(x)(*\alpha)(x) = *\alpha(x) \), it follows from the dominated convergence theorem that

\[
\lim_{j \to \infty} \langle *\alpha, f_j * \alpha \rangle_{L^2(X)} = \| \alpha \|^2_{L^2(X)}. \tag{3.6}
\]

Since \( \omega \) is bounded, \( \text{supp}(df_j) \subset B_{j+1} \setminus B_j \) and \( \| \theta(x) \|_{L^\infty} = O(\rho(x_0, x)) \), one obtains that

\[
\langle \theta \wedge \alpha \wedge \omega^{n-k-1}, * (df_j \wedge \alpha) \rangle_{L^2(X)} \leq (j + 1)C \int_{B_{j+1} \setminus B_j} |\alpha(x)|^2 dx, \tag{3.7}
\]

where \( C \) is a constant independent of \( j \).

We claim that there exists a subsequence \( \{j_i\}_{i \geq 1} \) such that

\[
\lim_{i \to \infty} (j_i + 1) \int_{B_{j_i+1} \setminus B_{j_i}} |\alpha(x)|^2 dx = 0. \tag{3.8}
\]
If not, there exists a positive constant $a$ such that

$$\lim_{j \to \infty} (j + 1) \int_{B_{j+1} \setminus B_j} |\alpha(x)|^2 dx \geq a > 0.$$ 

This inequality implies

$$\int_X |\alpha(x)|^2 dx = \sum_{j=0}^{\infty} \int_{B_{j+1} \setminus B_j} |\alpha(x)|^2 dx \geq a \sum_{j=0}^{\infty} \frac{1}{j + 1} = +\infty$$

which is a contradiction to the assumption $\int_X |\alpha(x)|^2 dx < \infty$. Hence, there exists a subsequence $\{j_i\}_{i \geq 1}$ for which (3.8) holds. Using (3.7) and (3.8), one obtains

$$\lim_{i \to \infty} (\theta \wedge \alpha \wedge \omega^{n-k-1}, *(df_{j_i} \wedge \alpha))_{L^2(X)} = 0 \quad (3.9)$$

It now follows from (3.5), (3.6) and (3.9) that $\|\alpha\|_{L^2(X)} = 0$, i.e, $\alpha = 0$. \qed

Following the $L^2$-decomposition in Proposition 3.4, we then have

**Theorem 3.7** Let $(X^{2n}, J, \omega)$ be a complete almost Kähler manifold with a $d$-sublinear symplectic form $\omega$. Then

$$\mathcal{H}^{p,q}_{(2)}(X) = [0]$$

unless $k := p + q = n$.

**Proof** The conclusion follows from Propositions 3.4 and 3.6. \qed

Suppose that $J$ is integrable, i.e, $(X^{2n}, J, \omega)$ is a complete Kähler manifold. We have a $L^2$-decomposition for the space of the $L^2$-harmonic $k$-forms.

**Lemma 3.8** [16] Let $(X^{2n}, \omega)$ be a complete Kähler manifold with a Kähler form $\omega$. If $\alpha \in \Omega^k_0(X)$, we denote $\alpha := \sum_{p+q=k} \alpha_{p,q}, \alpha_{p,q} \in \Omega^p_0 \Omega^q_0(X)$, then we have

$$\|d\alpha\|^2 + \|d^*\alpha\|^2 = \sum_{p+q=k} (\|d\alpha_{p,q}\|^2 + \|d^*\alpha_{p,q}\|^2).$$

In particular, we have a decomposition for the space of the $L^2$-harmonic $k$-forms:

$$\mathcal{H}^k_{(2)}(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{(2)}(X).$$
**Proof** We denote $\Delta_d = dd^* + d^*d$ and $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. We have an identity $\Delta_d = 2\Delta_{\bar{\partial}}$. For any $\alpha \in \Omega^k_0(X)$, we get

$$
\|d\alpha\|^2 + \|d^*\alpha\|^2 = \langle \Delta_d \alpha, \alpha \rangle_{L^2(X)} = 2\langle \Delta_{\bar{\partial}} \alpha, \alpha \rangle_{L^2(X)} = 2 \sum_{p,q} \langle \Delta_{\bar{\partial}} \alpha_{p,q}, \alpha_{p,q} \rangle_{L^2(X)} = \sum_{p,q} (\|d\alpha_{p,q}\|^2 + \|d^*\alpha_{p,q}\|^2).
$$

Here we use the fact that $\Delta_{\bar{\partial}} \alpha_{p,q}$ is a $(p,q)$-form. □

Following the $L^2$-decomposition in Lemma 3.8, we have

**Corollary 3.9** [2, 17] Let $(X^{2n}, \omega)$ be a complete Kähler manifold with a $d$(sublinear) Kähler form $\omega$. Then

$$
\mathcal{H}^k_{(2)}(X) = \{0\}
$$

unless $k = n$.

**Proof** The conclusion follows from Lemma 3.8 and Theorem 3.7. □

### 3.3 Symplectic Hyperbolic

In this section, we extend the idea of [14, Theorem 3.7] to the case of symplectic manifold with a $d$(bounded) symplectic form $\omega$.

**Proposition 3.10** Let $(X^{2n}, J, \omega)$ be a complete almost Kähler manifold with a $d$(bounded) symplectic form $\omega$, i.e., there exists a bounded 1-form $\theta$ such that $\omega = d\theta$. Then every $(p,q)$-form $\alpha \in P^{p,q}_J \cap \Omega^k_0 \subset P^k \cap \Omega^k_0$ on $X$ of degree $k := p + q < n$ satisfies the inequality

$$
c_{n,k} \|\theta\|^2_{L^\infty} \|\alpha\|^2_{L^2(X)} \leq \|d\alpha\|^2_{L^2(X)},
$$

where $c_{n,k} > 0$ is a constant which depends only on $n, k$.

**Proof** Inequality (3.10) makes sense, strictly speaking, if $d\alpha$ (as well as $\alpha$) is in $L^2$. The linear map $L^{n-k} : \Omega^k \to \Omega^{2n-k}$ for $k \leq n - 1$ is a bijective quasi-isometry on $P^{p,q}_J$ ($p + q = k$), thus any $\alpha \in P^{p,q}_J$ satisfies

$$
\alpha = C(n, k) \ast L^{n-k} \alpha = C(n, k) \ast (\alpha \wedge \omega^{n-k}),
$$
where $C(n, k) = \sqrt{-1} p^{-q} (-1)^{\frac{k(k+1)}{2}} \frac{1}{(n-k)!}$. We denote $\star \alpha = d \eta - \tilde{\alpha}$, for

$$\eta = C(n, k)(\theta \land \alpha \land \omega^{n-k-1}), \quad \tilde{\alpha} = C(n, k)(\theta \land d \alpha \land \omega^{n-k-1})$$

Observe that $\|\alpha\|_{L^2(X)} = \|\star \alpha\|_{L^2(X)}$ and $\|\eta\|_{L^2(X)} \lesssim \|\theta\|_{L^\infty} \|\alpha\|_{L^2(X)}$. Now, we have

$$\|\alpha\|_{L^2(X)}^2 = \langle \star \alpha, d \eta - \tilde{\alpha} \rangle_{L^2(X)}$$

$$\leq |\langle \star \alpha, d \eta \rangle_{L^2(X)}| + |\langle \star \alpha, \tilde{\alpha} \rangle_{L^2(X)}| := I_1 + I_2.$$  \hspace{1cm} (3.11)

The first term of the right hand:

$$I_1 = |\langle \star \alpha, d \eta \rangle_{L^2(X)}| = |\langle \star d \alpha, \eta \rangle_{L^2(X)}|$$

$$\leq \|d \alpha\|_{L^2(X)} \|\eta\|_{L^2(X)}$$  \hspace{1cm} (3.12)

$$\lesssim \|d \alpha\|_{L^2(X)} \|\theta\|_{L^\infty} \|\alpha\|_{L^2(X)}.$$

The second term of right hand:

$$I_2 = |\langle \alpha, \tilde{\alpha} \rangle_{L^2(X)}| \leq \|\alpha\|_{L^2(X)} \|\tilde{\alpha}\|_{L^2(X)}$$

$$\lesssim \|\alpha\|_{L^2(X)} \|\theta\|_{L^\infty} \|d \alpha\|_{L^2(X)}.$$  \hspace{1cm} (3.13)

Substituting (3.12) and (3.13) into (3.11), it follows that

$$\|\alpha\|_{L^2(X)}^2 \lesssim \|\alpha\|_{L^2(X)} \|\theta\|_{L^\infty} \|d \alpha\|_{L^2(X)}.$$  \hspace{1cm} (3.14)

Therefore, we obtain the inequality (3.10). \hfill \Box

**Lemma 3.11** If $B_k \in P^k$, $(k \leq n)$, then for any $0 \leq j \leq i \leq (n-k)$, we have

$$\langle L^j B_k, L^i A_k \rangle_{L^2(X)} = \frac{(n-k-i+j)!}{(n-k-i)!(i-j)!} \langle L^{i-j} B_k, L^i A_k \rangle_{L^2(X)}, \quad \forall A_k \in \Omega^k.$$  \hspace{1cm} (3.14)

**Proof** If $\alpha \in \Omega^k$, there is a formula [16] Corollary 1.2.28:

$$[L^i, \Lambda] \alpha = i(k-n+i-1)L^{i-1} \alpha.$$  \hspace{1cm} (£ Springer
Therefore, we have

\[
\langle L^i B_k, L^j A_k \rangle_{L^2(X)} = \langle [\Lambda, L^i] B_k, L^{i-1} A_k \rangle_{L^2(X)}
\]

\[
= i(n + 1 - (k + i)) \langle L^{i-1} B_k, L^{i-1} A_k \rangle_{L^2(X)}
\]

\[
= i(n + 1 - (k + i)) \langle [\Lambda, L^{i-1}] B_k, B_k \rangle_{L^2(X)}
\]

\[
= i(n + 1 - (k + i)) (i - 1)(n + 1 - (k + i - 1)) \langle L^{i-2} B_k, L^{i-2} A_k \rangle_{L^2(X)}
\]

\[= \cdots \]

\[
= \frac{(n - k + 1)!}{(n - k - i)! (i - j)!} \langle L^{i-j} B_k, L^{i-j} A_k \rangle_{L^2(X)}
\]

We complete this proof. \( \square \)

Lefschetz decomposing \( dB_k \), we can formally write

\[
d B_k = B^0_{k+1} + L B^1_{k+1} + \cdots + \frac{1}{r!} L^r B_{k+1-2r} + \cdots.
\]

But in fact the differential operators acting on primitive forms have special properties.

**Lemma 3.12** [23, Lemma 2.4] Let \( B_k \in P^k \) with \( k \leq n \). The differential operators \((d, d^\Lambda)\) acting on \( B_k \) take the following forms:

(i) If \( k < n \), then \( dB_k = B^0_{k+1} + L B^1_{k-1} \);

(ii) If \( k = n \), then \( dB_k = L B^1_{k-1} \);

(iii) \( d^\Lambda B_k = -(n - k + 1) B^1_{k-1} \).

For some primitive forms \( B^0, B^1 \in P^* \).

**Lemma 3.13** If \( \alpha \in \Omega^k_0 \), then

\[
\|d\alpha\|^2_{L^2(X)} + \|d^\Lambda \alpha\|^2_{L^2(X)} \simeq \sum_{r \geq 0} \|d \beta_{k-2r}\|^2_{L^2(X)}.
\]

**Proof** We denote \( \alpha_k = \sum_{r \geq 0} L^r \beta_{k-2r} \), where \( \beta_{k-2r} \in P^{k-2r} \). For convenience, we denote \( \beta_{k-2r} \equiv 0 \) for all \( r > k/2 \). By the operator \( D_{d+d^\Lambda} \) communicates with \( L \), we have

\[
\|d\alpha\|^2_{L^2(X)} + \|d^\Lambda \alpha\|^2_{L^2(X)} = \langle D_{d+d^\Lambda} \alpha_k, \alpha_k \rangle_{L^2(X)}
\]

\[
= \left( \sum_{r \geq 0} L^r D_{d+d^\Lambda} \beta_{k-2r}, \sum_{r \geq 0} L^r \beta_{k-2r} \right)_{L^2(X)}
\]

\[
= \sum_{p=q} \langle L^p D_{d+d^\Lambda} \beta_{k-2p}, L^q \beta_{k-2q} \rangle_{L^2(X)} + \sum_{p \neq q} \langle L^p D_{d+d^\Lambda} \beta_{k-2p}, L^q \beta_{k-2q} \rangle_{L^2(X)}
\]

\[
(3.15)
\]

\( \square \) Springer
The first term of the right hand in (3.15) satisfies
\[
\sum_{p=q} \langle L^p D_{d+\Lambda} \beta_{k-2p}, L^q \beta_{k-2q} \rangle_{L^2(X)} \approx \sum_{r \geq 0} \langle D_{d+\Lambda} \beta_{k-2r}, \beta_{k-2r} \rangle_{L^2(X)}
\]
\[
= \sum_{r \geq 0} \|d \beta_{k-2r}\|_{L^2(X)}^2 + \|d^\Lambda \beta_{k-2r}\|_{L^2(X)}^2
\]
\[
\approx \sum_{r \geq 0} \|d \beta_{k-2r}\|_{L^2(X)}^2.
\]

Nothing that the operator \( D_{d+\Lambda} \) communicates with \( \Lambda \), see [23, Lemma 3.7]. Then
\( D_{d+\Lambda} \beta_{k-2p} \in P^{k-2p} \). For any \( p \neq q \), following the Lemma 3.11, we observe that
\[
\langle L^p D_{d+\Lambda} \beta_{k-2p}, L^q \beta_{k-2q} \rangle_{L^2(X)} = 0.
\]

Therefore the second term of the right hand in (3.15) is zero.

Corollary 3.14 If \( \alpha \in \Omega_{j}^{p,q} \), \((p + q \leq n)\), we denote \( \alpha = \sum_{r \geq 0} \beta_{k-2r} \), where \( \beta_{k-2r} \in P_{j}^{p-r,q-r} \), then
\[
\|d \alpha\|_{L^2(X)}^2 + \|d^* \alpha\|_{L^2(X)}^2 \approx \sum_{r \geq 0} \|d \beta_{k-2r}\|_{L^2(X)}^2.
\]

In particular, we have a decomposition for the space of the \( L^2 \)-harmonic forms of bi-degree \((p, q)\):
\[
\mathcal{H}_{(2)}^{p,q}(X) = \bigoplus_{r \geq 0} L^r P \mathcal{H}_{(2)}^{p-r,q-r}(X).
\]

Proof The conclusions follow from Lemmas 3.2 and 3.13.

Now we can give a lower bound on the spectra of the Laplace operator \( \Delta_d := dd^* + d^*d \) on \( L^2 \)-forms \( \Omega_{j}^{p,q}(X) \) for \( p + q \neq n \) to sharpen the vanishing theorem in \( d \)(bounded) case.

Theorem 3.15 Let \((X^{2n}, J, \omega)\) be a complete almost Kähler manifold with a \( d \)(bounded) symplectic form \( \omega \), i.e., there exists a bounded 1-form \( \theta \) such that \( \omega = d\theta \). Then any \( \alpha \in \Omega_{j}^{p,q}(X) \cap \Omega_{0}^{k}(X) \) on \( X \) of degree \( k := p + q \neq n \) satisfies the inequality
\[
c_n, k \|\theta\|_{L^\infty}^2 \|\alpha\|_{L^2(X)}^2 \leq \|d \alpha\|_{L^2(X)}^2 + \|d^* \alpha\|_{L^2(X)}^2,
\]
where \( c_n, k > 0 \) is a constant which depends only on \( n, k \). In particular,
\[
\mathcal{H}_{(2)}^{p,q}(X) = \{0\}
\]
unless \( k := p + q = n \).

\( \Box \) Springer
Proof We only need consider $k < n$ case. The case $k > n$ follows by the Poincaré duality as the operator $*$ : $\Omega^k \rightarrow \Omega^{2n-k}$ commutes with $\Delta_d$ and is isometric for the $L^2$-norms. Now we denote $\alpha = \sum_{r \geq 0} L^r \beta_{k-2r}$, where $\beta_{k-2r} \in P^{j-r \cdot j-r}$. We then have

$$||\alpha||^2_{L^2(X)} = \sum_{r \geq 0} ||L^r \beta_{k-2r}||^2_{L^2(X)} \lesssim \sum_{r \geq 0} ||\beta_{k-2r}||^2_{L^2(X)}.$$ 

Following Proposition 3.10, it implies that

$$||\beta_{k-2r}||_{L^2(X)} \|\theta||^{-1}_{L^\infty} \lesssim ||d \beta_{k-2r}||_{L^2(X)}.$$ 

Following Corollary 3.14, it implies that

$$||\alpha||^2_{L^2(X)} \lesssim ||\theta||^2_{L^\infty} \sum_{r \geq 0} ||d \beta_{k-2r}||^2_{L^2(X)}$$

$$\lesssim ||\theta||^2_{L^\infty}(||d\alpha||^2_{L^2(X)} + ||d^*\alpha||^2_{L^2(X)}).$$

We complete the proof. \qed

We can obtain a well-known result proved by Gromov, see [10, 1.4. A. Theorem] or [21].

Corollary 3.16 Let $(X^{2n}, \omega)$ be a complete Kähler manifold with a $d$ (bounded) Kähler form $\omega$, i.e., there exists a bounded 1-form $\theta$ such that $\omega = d \theta$. Then any $\alpha \in \Omega^k_0(X)$ satisfies the inequality

$$c_k \|\theta\|^{-2}_{L^\infty} ||\alpha||^2_{L^2(X)} \leq ||d\alpha||^2_{L^2(X)} + ||d^*\alpha||^2_{L^2(X)},$$

where $c_k > 0$ is a constant which depends only on $n, k$. In particular,

$$\mathcal{H}^k_{(2)}(X) = \{0\},$$

unless $k = n$.

Proof Following Lemma 3.8 and Theorem 3.15, we have

$$||d\alpha||^2 + ||d^*\alpha||^2 = \sum_{p, q} (||d\alpha_{p, q}||^2 + ||d^*\alpha_{p, q}||^2)$$

$$\geq \sum_{p, q} c_{n, p, q} ||\theta||^{-2}_{L^\infty} ||\alpha_{p, q}||^2_{L^2(X)}$$

$$\geq \min_{p, q} c_{n, p, q} ||\theta||^{-2}_{L^\infty} \sum_{p, q} ||\alpha_{p, q}||^2_{L^2(X)}$$

$$\geq \min_{p, q} c_{n, p, q} ||\theta||^{-2}_{L^\infty} ||\alpha||^2_{L^2(X)},$$

Here we use that fact that $||\alpha||^2_{L^2(X)} = \sum_{p, q} ||\alpha_{p, q}||^2_{L^2(X)}$. \qed
3.4 $L^2$-Decomposition for Almost Kähler Manifolds

Let $(X, J, \omega)$ be a $2n$-dimensional almost Kähler manifold. Then $J$ acts as an involution on the space of smooth 2-forms $\Omega^2(X)$: given $\alpha \in \Omega^2(X)$, for every pair of vector fields $u, v$ on $X$

$$J\alpha(u, v) = \alpha(Ju, Jv).$$

Therefore the space $\Omega^2(X)$ splits as the direct sum of $\pm$-eigenspaces $\Omega^\pm$, i.e., $\Omega^2(X) = \Omega^+(X) \oplus \Omega^-(X)$. Let us denote by $Z^2_J(X)$ the space of closed 2-forms which are in $L^2$ and set

$$Z^\pm_J = Z^2_J(X) \cap \Omega^\pm_J.$$

Define

$$H^\pm_{J} := \{ a \in H^2_J(X) : \exists \alpha \in Z^\pm_J such that a = [\alpha] \}.$$

For closed almost complex 4-manifolds Drăghici-Li-Zhang showed in [9] that there is a direct sum decomposition

$$H^2_{dR}(X; \mathbb{R}) = H^+(X) \oplus H^-(X).$$

In [12], Hind-Tomassini generalized such a decomposition to the $L^2$ setting.

**Theorem 3.17** [12, Theorem 4.8] Let $(X, J, \omega)$ be a complete almost Kähler $4$-dimensional manifold. Then, we have the following decomposition

$$H^2_J(X) = H^+_J(X) \oplus H^-_J(X).$$

Given any $J$-anti-invariant form $\alpha$ on a $2n$-dimensional almost Hermitian manifold $X$, we have

$$*\alpha^- = \frac{1}{(n-2)!} \alpha^- \wedge \omega^{n-2}.$$

We then have

**Proposition 3.18** [12, Corollary 4.1] Closed anti-invariant forms are harmonic, that is, we have an inclusion $Z^-_J \hookrightarrow H^2_J$.

Following the idea in Proposition 3.6, we get

**Corollary 3.19** Let $(X^{2n}, J, \omega)$ be a complete $2n$-dimensional almost Kähler manifold with a $d$ (sublinear) symplectic form $\omega$, $(n \geq 3)$. Then

$$H^-_J(X) = \{0\}.$$
Proof We denote by $\alpha^-$ the harmonic anti-invariant form on $X$. Noticing that $\alpha^-$ could be a sum of terms of type $(2, 0)$ and $(0, 2)$, it implies that $\Lambda_\omega \alpha^- = 0$, i.e., $\alpha \in \mathcal{H}^2_{(2); J}(X) \cap \ker P$. Even though $\mathcal{H}^2_{(2); J}(X) \subset \mathcal{H}^{2,0}_{(2); J}(X) \oplus \mathcal{H}^{0,2}_{(2); J}(X)$ does not hold, for any harmonic anti-invariant form $\alpha^-$ we have the identity

$$
*\alpha^- = \frac{1}{(n-1)!} \alpha^- \wedge \omega^{n-2} = \frac{1}{(n-1)!} d(\alpha^- \wedge \theta \wedge \omega^{n-3}) = d\eta,
$$

where

$$
\eta = \frac{1}{(n-1)!} (\alpha^- \wedge \theta \wedge \omega^{n-3}).
$$

We follow the method in Proposition 3.6 to choose the compactly supported function $f_j(x) = h(\rho(x_0, x) - j)$, where $j$ is positive integer. Noticing that $f_j \ast \alpha$ has compact support, one has

$$
\langle *\alpha^-, f_j \ast \alpha^- \rangle_{L^2(X)} = \langle d\eta, f_j \ast \alpha^- \rangle_{L^2(X)} = \langle \theta \wedge \alpha^- \wedge \omega^{n-3}, *(d f_j \wedge \alpha^-) \rangle_{L^2(X)}.
$$

Using the idea in Proposition 3.6, we obtain that there exists a subsequence $\{j_i\}_{i \geq 1}$ such that

$$
\langle \theta \wedge \alpha^- \wedge \omega^{n-3}, *(d f_{j_i} \wedge \alpha^-) \rangle_{L^2(X)} \to 0, \text{ as } i \to \infty.
$$

Therefore, we have

$$
0 = \lim_{i \to \infty} \langle *\alpha^-, f_{j_i} \ast \alpha^- \rangle_{L^2(X)} = \|\alpha^-\|^2_{L^2(X)},
$$

i.e., $\alpha^- = 0$. We complete this proof.

Let $\alpha_j^+$ be a $J$-self-dual-invariant form in $X^{2n}$. Then we can denote $\alpha_j^+ = f \omega + \alpha_{j,0}^{1,1}$, where $f \in \Omega^0(X)$ and $\alpha_{j,0}^{1,1} \in P^{1,1}$.

**Corollary 3.20** Let $(X^{2n}, J, \omega)$ be a complete $2n$-dimensional almost Kähler manifold with a $d$-(bounded) symplectic form $\omega$, $(n \geq 3)$. If $\alpha_j^+ \in H^+_{(2); J}(X)$, then there is a positive constant $c_n = c(n)$ such that

$$
c_n \|\theta\|^2_{L^\infty} \|\alpha_j^+\|^2_{L^2(X)} \leq \|d f\|^2_{L^2(X)},
$$

Furthermore, if $\alpha_j^+$ also satisfies $d^\Lambda \alpha_j^+ = 0$, then $\alpha_j^+ = 0$. 

\[ \Box \] Springer
Proof} If $\alpha^+_0 \in H^+_2(\mathcal{J}(X))$, i.e., $d\alpha^+_0 = 0$. Therefore, we have

$$df \wedge \omega + d\alpha^{1,1}_0 = 0 \Rightarrow \|df\|^2 = (n-1)\|d\alpha^{1,1}_0\|^2.$$ 

Following the Proposition 3.10, we get

$$\|\theta\|_{L^2_\infty} \|f\|_{L^2_2(X)} \lesssim \|df\|_{L^2_2(X)}$$

and

$$\|\theta\|_{L^2_\infty} \|\alpha^{1,1}_0\|_{L^2_2(X)} \lesssim \|d\alpha^{1,1}_0\|_{L^2_2(X)}.$$ 

Therefore, we get

$$c_n \|\theta\|_{L^2_\infty} \|\alpha^+_0\|_{L^2_2(X)} \lesssim \|df\|_{L^2_2(X)}.$$ 

Noticing that $d\Lambda \alpha^+_0 = ndf$. Therefore, if $\alpha^+_0 \in \ker d\Lambda$, i.e. $df = 0$, then $\alpha^+_0 = 0$. 

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