Note on one inequality and its application in intuitionistic fuzzy sets theory. Part 1

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Abstract: The inequality $\mu^{\frac{1}{2}} + \nu^{\frac{1}{2}} \leq \frac{1}{2}$ is introduced and proved, where $\mu$ and $\nu$ are real numbers, for which $\mu, \nu \in [0,1]$ and $\mu + \nu \leq 1$. The same inequality is valid for $\mu = \mu_A(x)$, $\nu = \nu_A(x)$, where $\mu_A$ and $\nu_A$ are the membership and the non-membership functions of an arbitrary intuitionistic fuzzy set $A$ over a fixed universe $E$ and $x \in E$. Also, a generalization of the above inequality for arbitrary $n \geq 2$ is proposed and proved.

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1 Introduction

The Intuitionistic Fuzzy Sets (IFSs) are introduced by K. Atanassov in [1,2] as follows. Let $E$ be a universal set, $\mu_A, \nu_A : E \rightarrow I := [0,1]$ be mappings and for each $x \in E$: $\mu_A(x) + \nu_A(x) \leq 1$. (1)

Then the set

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$$

is called an IFS.

Mappings $\mu_A$ and $\nu_A$ are called membership and non-membership functions for the element $x \in E$ to the set $A \subseteq E$.

When for each $x \in E$:

$$\mu_A(x) + \nu_A(x) = 1,$$

(2)

the set $A$ is transformed to the ordinary fuzzy (Zadeh’s) set [4].
2 Main results

The main result of the paper is the following Theorem 1.

**Theorem 1.** Let \( \mu, \nu \in I \) be real numbers satisfying inequality

\[
\mu + \nu \leq 1. \tag{3}
\]

Then the inequality

\[
\mu^{\frac{1}{\nu}} + \nu^{\frac{1}{\mu}} \leq \frac{1}{2} \tag{4}
\]

holds, where the equality is possible if and only if \( \mu = \nu = \frac{1}{2} \).

Before giving the proof of Theorem 1, we need the following lemma.

**Lemma.** Let the function \( f \) be given by

\[
f(x) = (1 - x)^{\frac{1}{x}} := e^{\frac{\ln(1-x)}{x}}. \tag{5}
\]

Then function \( f \) is strictly concave on interval \((0, 1)\) and also strictly decreasing on the same interval. Also, \( f(1) = 0 \) and if we define \( f(0) := \lim_{x \to 0^+} f(x) \), then \( f(0) = \frac{1}{e} \).

**Proof.** Using (5) we obtain:

\[
\left( \frac{d}{dx} \right) f(x) = f(x) \cdot \frac{\ln(1-x)}{x} \tag{6}
\]

and

\[
\left( \frac{d}{dx} \right)^2 f(x) = f(x) \left( \frac{\ln(1-x)}{x} \right)^2 + \left( \frac{d}{dx} \right)^2 \left( \frac{\ln(1-x)}{x} \right). \tag{7}
\]

For \( x \in (0, 1) \) we have

\[
\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.
\]

Hence,

\[
\frac{\ln(1-x)}{x} = -\sum_{n=0}^{\infty} \frac{x^n}{n+1}. \tag{8}
\]

The series \( \sum_{n=0}^{\infty} \frac{x^n}{n+1} \) converges uniformly on each compact set \([\delta_1, \delta_2]\), where \( 0 < \delta_1 < \delta_2 < 1 \). This follows from Weierstrass criterion for uniform convergence of series (see [3]), since

\[
\sum_{n=0}^{\infty} \frac{x^n}{n+1} \leq \sum_{n=0}^{\infty} \frac{\delta_2^n}{n+1},
\]

and the series \( \sum_{n=0}^{\infty} \frac{\delta_2^n}{n+1} \) converges (from D’Alembert criterion for convergence of series (see [3]). Therefore, (8) yields

\[
\frac{d}{dx} \left( \frac{\ln(1-x)}{x} \right) = -\sum_{n=0}^{\infty} \frac{(n+1)x^n}{n+2}. \tag{9}
\]
Since \( f(x) > 0 \), formulas (6) and (9) imply that \( f \) is strictly decreasing on \((0, 1)\).

The series (9) converges uniformly on each compact set \([\delta_1, \delta_2]\), where \(0 < \delta_1 < \delta_2 < 1\). Therefore,

\[
\left( \frac{d}{dx} \right)^2 \left( \frac{\ln(1 - x)}{x} \right) = -\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{n+3}x^n
\]  

and the series (10) converges (from D’Alembert criterion). Then, to prove that \( f \) is strictly concave, we need to prove that

\[
\left( \frac{d}{dx} \right)^2 f(x) < 0
\]

for \( x \in (0, 1) \). Because of (7), (9) and (10), inequality (11) is equivalent to the inequality

\[
\left( \sum_{n=0}^{\infty} \frac{(n+1)}{n+2}x^n \right)^2 < \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{n+3}x^n. \tag{12}
\]

The left-hand side of inequality (12) must be understood as a Cauchy–Mertens multiplication of the series \( \sum_{n=0}^{\infty} \frac{(n+1)}{n+2}x^n \) by itself, i.e.,

\[
\left( \sum_{n=0}^{\infty} \frac{(n+1)}{n+2}x^n \right)^2 = \left( \sum_{n=0}^{\infty} \frac{(n+1)}{n+2}x^n \right) \cdot \left( \sum_{n=0}^{\infty} \frac{(n+1)}{n+2}x^n \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{k+1}{k+2} \frac{n-k+1}{n-k+2} \right) x^n. \tag{13}
\]

From (13), inequality (12) is equivalent to

\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{k+1}{k+2} \frac{n-k+1}{n-k+2} \right) x^n < \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{n+3}x^n. \tag{14}
\]

To prove (14), it is enough to prove that the inequality

\[
\sum_{k=0}^{n} \frac{k+1}{k+2} \frac{n-k+1}{n-k+2} < \frac{(n+1)(n+2)}{n+3} \tag{15}
\]

holds. For this aim, we use the equality

\[
\frac{1}{n+4} \left( \frac{1}{k+2} + \frac{1}{n-k+2} \right) = \frac{1}{(k+2)(n-k+2)}. \tag{16}
\]

Because of (16), we may rewrite (15) in the form

\[
\sum_{k=0}^{n} \frac{1}{n+4} \left( \frac{1}{k+2} + \frac{1}{n-k+2} \right) \cdot (k+1)(n-k+1) < \frac{(n+1)(n+2)}{n+3}. \tag{17}
\]

But (17) is equivalent to

\[
\sum_{k=0}^{n} \frac{1}{n+4} \frac{k+1}{k+2} (n-k+1) + \sum_{k=0}^{n} \frac{1}{n+4} (k+1) \frac{n-k+1}{n-k+2} < \frac{(n+1)(n+2)}{n+3}. \tag{18}
\]
Since $\frac{k + 1}{k + 2} < 1$ and $\frac{n - k + 1}{n - k + 2} < 1$, (18) will be proved if we know that the inequality

$$\frac{1}{n + 4} \cdot \left( \sum_{k=0}^{n} (n - k + 1) + \sum_{k=0}^{n} (k + 1) \right) < \frac{(n + 1)(n + 2)}{n + 3},$$

holds. But after a simple calculation, the left-hand side of (19) equals to $\frac{(n + 1)(n + 2)}{n + 4}$. This proves (19) since

$$\frac{(n + 1)(n + 2)}{n + 4} < \frac{(n + 1)(n + 2)}{n + 3}.$$ 

Then the above mentioned function $f$ is strictly concave on interval $(0, 1)$. Also,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1 - x)^{\frac{1}{x}} = \lim_{y \to \infty} \left( 1 - \frac{1}{y} \right)^y = \frac{1}{e}$$

and the Lemma is proved. \(\square\)

**Corollary 1.** If $x \in (0, 1)$ and $x \neq \frac{1}{2}$, then

$$f(x) + f(1 - x) < 2f \left( \frac{1}{2} \right) = 2 \cdot \frac{1}{4} = \frac{1}{2}. \quad (20)$$

When $x = \frac{1}{2}$, (20) is an equality.

*Proof.* Since $f$ is strictly concave on $(0, 1)$, then

$$\frac{f(x) + f(1 - x)}{2} < f \left( \frac{x + (1 - x)}{2} \right) = f \left( \frac{1}{2} \right)$$

and (20) holds. \(\square\)

**Corollary 2.** If $\mu, \nu \in (0, 1)$ and $\mu + \nu = 1$, then (4) holds.

*Proof.* If $\mu = \nu = \frac{1}{2}$, then, obviously, (4) is an equality.
Let $\mu \neq \nu$. We put $\mu = 1 - x$, $\nu = x$. Hence, $x \in (0, 1)$. Since

$$f(x) + f(1 - x) = (1 - x)^{\frac{1}{x}} + x^{\frac{1}{1-x}},$$

then (20) yields exactly (4). \(\square\)

We must note that Corollary 2 means that for fuzzy sets (4) is always true for $\mu, \nu \in I$, since

$$f(0) = \frac{1}{e} < \frac{1}{2}$$

and $f(1) = 0$.

**Proof of Theorem 1.** From Corollary 1, we have that Theorem 1 is valid for $\mu + \nu = 1$.
Let $\mu, \nu \in (0, 1)$ and $\mu + \nu < 1$. Also, let $\alpha = 1 - \mu$. Therefore, $\alpha \in (0, 1)$. Then we have

$$\mu^{\frac{1}{\alpha}} + \alpha^{\frac{1}{\mu}} \leq \frac{1}{2}. \quad (21)$$
But, also, we have
\[ 0 < \nu < \alpha < 1 \]  \hspace{1cm} (22)
and
\[ 1 < \frac{1}{\alpha} < \frac{1}{\nu}. \]  \hspace{1cm} (23)
From \( 0 < \mu < 1 \) and (23) we obtain
\[ \mu^{\frac{1}{\nu}} < \mu^{\frac{1}{\alpha}}. \]  \hspace{1cm} (24)
From (22) we obtain
\[ \nu^{\frac{1}{\mu}} < \alpha^{\frac{1}{\mu}}. \]  \hspace{1cm} (25)
Now, (24) and (25) yield
\[ \mu^{\frac{1}{\nu}} + \nu^{\frac{1}{\mu}} < \mu^{\frac{1}{\mu}} + \alpha^{\frac{1}{\mu}}. \]  \hspace{1cm} (26)
From (21) and (26) inequality (4) holds immediately. Thus, Theorem 1 holds for the case \( \mu, \nu \in (0, 1) \).

It remains only to consider the following cases:
1. \( \mu = 0 \),
2. \( \mu = 1 \).

If Case 1 holds, we consider the subcases:
1.1. If \( \nu = 0 \), then \( \mu^{\frac{1}{\nu}} \) and \( \nu^{\frac{1}{\mu}} \) takes the form \( 0^{\frac{1}{\nu}} \), which we may consider (after putting \( \mu = x \)) as
\[ \lim_{x \to 0^+} x^{\frac{1}{\nu}} = \lim_{y \to \infty} \left( \frac{1}{y} \right)^y = 0. \]
Therefore, (4) is obviously true.
1.2. If \( \nu \neq 0 \) is true, then \( \mu^{\frac{1}{\nu}} = 0^{\frac{1}{\nu}} = 0 \).
Also, when \( 0 < \nu < 1 \), \( \nu^{\frac{1}{\mu}} = \nu^{+\infty} = 0 \).
If \( \nu = 1 \), then \( \nu^{\frac{1}{\mu}} \) takes the form \( 1^{+\infty} \), which we understand (after putting \( \mu = x \)) as
\[ \lim_{x \to 0^+} (1 - x)^{\frac{1}{\nu}} = \lim_{y \to \infty} \left( 1 - \frac{1}{y} \right)^y = \frac{1}{e}. \]
Therefore, (4) is true, since \( \frac{1}{e} < \frac{1}{2} \).
Let Case 2 hold. Then \( \nu = 0 \), because of the conditions \( 0 \leq \nu \) and \( \mu + \nu = 1 \). Therefore,
\[ \nu^{\frac{1}{\mu}} = 0^1 = 0. \]
Also, we have that \( \mu^{\frac{1}{\nu}} \) takes the form \( 1^{+\infty} \), which (after putting \( \nu = x \)) we consider as
\[ \lim_{x \to 0^+} (1 - x)^{\frac{1}{\nu}} = \lim_{y \to \infty} \left( 1 - \frac{1}{y} \right)^y = \frac{1}{e}. \]
Therefore, (4) is again true and Theorem 1 is proved. \( \square \)
Finally, we will give an unexpected form of Theorem 1 for the case of fuzzy sets.

**Theorem 2.** Let $\mu, \nu \in (0, 1)$ and $\mu + \nu = 1$. Then the inequality

$$\mu^{1+\mu^2+\cdots} + \nu^{1+\nu^2+\cdots} \leq \frac{1}{2}$$

(27)

holds and the equality is possible if and only if $\mu = \nu = \frac{1}{2}$.

**Proof.** We use (4) that is proved in Theorem 1. But since $\mu + \nu = 1$, we may rewrite (4) in the form

$$\mu \frac{1}{1-\mu} + \nu \frac{1}{1-\nu} \leq \frac{1}{2}.$$  

(28)

Since $\mu, \nu \in (0, 1)$, we have

$$\frac{1}{1-\mu} = 1 + \mu + \mu^2 + \ldots, \frac{1}{1-\nu} = 1 + \nu + \nu^2 + \ldots.$$  

(29)

Then, from (28) and (29) inequality (27) holds.

Let us now look at one generalization of (4) for arbitrary $n \geq 2$, proving the following theorem.

**Theorem 3.** Let $n \geq 2$ be an arbitrary integer and $x_i \in (0, 1), i = 1, 2, \ldots, n$ be real numbers, such that

$$\sum_{i=1}^{n} x_i = 1.$$  

(30)

Then the inequality

$$\left(1 - x_1\right)^{\frac{1}{n_1}} + \left(1 - x_2\right)^{\frac{1}{n_2}} + \cdots + \left(1 - x_n\right)^{\frac{1}{n_n}} \leq \left(1 - \frac{1}{n}\right)^n$$  

(31)

holds and the equality is possible if and only if $x_1 = x_2 = \cdots = x_n = \frac{1}{n}$.

**Proof.** Since $f(x) = (1 - x)^{\frac{1}{n}}$ is concave on $(0, 1)$, then for arbitrary $x_i \in (0, 1), i = 1, 2, \ldots, n$ and from (30), we have

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \leq f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) = f\left(\frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)^n$$

and the equality holds if and only if $x_1 = x_2 = \cdots = x_n = \frac{1}{n}$.

If we rewrite (31) in the form

$$\left(1 - x_1\right)^{\frac{1}{n_1}} + \left(1 - x_2\right)^{\frac{1}{n_2}} + \cdots + \left(1 - x_n\right)^{\frac{1}{n_n}} \leq n \left(1 - \frac{1}{n}\right)^n,$$  

(32)

then, for $n = 2$, putting $x_1 = \nu, x_2 = \mu$, we obtain exactly (4). So, (32) is a generalization of (4) for arbitrary $n \geq 2$.

Since the sequence $\{(1 - \frac{1}{n})^n\}_{n=1}^{\infty}$ is strictly increasing and $\lim_{n\to\infty} (1 - \frac{1}{n})^n = \frac{1}{e}$, then as a corollary of Theorem 3, we obtain the inequality

$$\frac{\left(1 - x_1\right)^{\frac{1}{n_1}} + \left(1 - x_2\right)^{\frac{1}{n_2}} + \cdots + \left(1 - x_n\right)^{\frac{1}{n_n}}}{n} < \frac{1}{e}.$$  

(33)

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But now, we observe that if \( x_i \in (0, 1) \), the (33) holds, i.e., we do not need the condition \( x_1 + x_2 + \cdots + x_n = 1 \).

Indeed, since \( f(x) = (1 - x)^\frac{1}{n} \) is strictly decreasing on \((0, 1)\) (see the Lemma), then

\[
(1 - x_i)^\frac{1}{n} < \lim_{x \to 0^+} f(x) = \frac{1}{e}.
\]

Therefore, we have

\[
\frac{(1 - x_1)^\frac{1}{n} + (1 - x_2)^\frac{1}{n} + \cdots + (1 - x_n)^\frac{1}{n}}{n} < \frac{\frac{1}{e} + \frac{1}{e} + \cdots + \frac{1}{e}}{n} = \frac{1}{e}
\]

and (33) is proved. \( \square \)

Finally, we must mention that if \( A \) is a fixed IFS over a universe \( E \), then we can construct the following two new sets

\[
B = \{ \langle x, \mu_A(x)^\frac{1}{\nu_A(x)^\pi_A(x)} \rangle \mid x \in E \}
\]

and

\[
C = \{ \langle x, \mu_A(x)^\frac{1}{\nu_A(x)^\pi_A(x)} + \nu_A(x)^\frac{1}{\pi_A(x)^\mu_A(x)} \rangle \mid x \in E \}.
\]

These sets are IFSs, because for each \( x \in E \): \( \mu_A(x)^\frac{1}{\nu_A(x)^\pi_A(x)} \), \( \nu_A(x)^\frac{1}{\pi_A(x)^\mu_A(x)} \) \( \in [0, 1] \) and

\[
\mu_A(x)^\frac{1}{\nu_A(x)^\pi_A(x)} + \nu_A(x)^\frac{1}{\pi_A(x)^\mu_A(x)} \leq \mu_A(x) + \nu_A(x) \leq 1;
\]

and \( \mu_A(x)^\frac{1}{\nu_A(x)^\pi_A(x)} + \nu_A(x)^\frac{1}{\pi_A(x)^\mu_A(x)} \), \( \pi_A(x)^\frac{1}{\mu_A(x)^\nu_A(x)} \) \( \in [0, 1] \) and

\[
\mu_A(x)^\frac{1}{\nu_A(x)^\pi_A(x)} + \nu_A(x)^\frac{1}{\pi_A(x)^\mu_A(x)} + \pi_A(x)^\frac{1}{\mu_A(x)^\nu_A(x)} \leq \mu_A(x) + \nu_A(x) + \pi_A(x) = 1.
\]

### 3 Conclusion

In the second part of the present paper, we will represent a new inequality which one may deduce with the help of (4) and the well-known Young’s inequality for product. This new inequality also allows IFS interpretation.

### References

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