Research Article

Adolf R. Mirotin*

On the description of multidimensional normal Hausdorff operators on Lebesgue spaces

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Abstract: Hausdorff operators originated from some classical summation methods. Now this is an active research field. In the present article, a spectral representation for multidimensional normal Hausdorff operator is given. We show that normal Hausdorff operator in $L^2(\mathbb{R}^n)$ is unitary equivalent to the operator of multiplication by some matrix-valued function (its matrix symbol) in the space $L^2(\mathbb{R}^n; \mathbb{C}^{2n})$. Several corollaries that show that properties of a Hausdorff operator are closely related to the properties of its symbol are considered. In particular, the norm and the spectrum of such operators are described in terms of the symbol.

Keywords: Hausdorff operator, symbol of an operator, normal operator, spectrum, compact operator, spectral representation

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1 Introduction

The notion of a Hausdorff operator with respect to a positive measure on the unit interval was introduced by Hardy [8, Chapter XI] as a continuous analogue of the Hausdorff summability methods for series. This class of operators contains some important examples, such as Hardy operator, the Cesàro operator and its q-calculus version, and there adjoints. As mentioned in [22] the Riemann–Liouville fractional integral and the Hardy–Littlewood–Polya operator can also be reduced to the Hausdorff operator, and as was noted in [4] in the one-dimensional case, the Hausdorff operator is closely related to a Calderón–Zygmund convolution operator, too.

The modern theory of Hausdorff operators begins with the paper by Liflyand and Moricz [15]. Now this is an active research field. The survey articles [5, 13] contain main results on Hausdorff operators and bibliography up to 2014. For more recent results, see, e.g., [4, 14, 17–19]. The last paper is devoted to generalizations of Hausdorff operators to locally compact groups. In this paper, we accept the following special case of the definition from [18].

Definition 1. Let $(\Omega, \mu)$ be some $\sigma$-compact topological space endowed with a positive regular Borel measure $\mu$, $K$ a locally integrable function on $\Omega$, and $(A(u))_{u \in \Omega}$ a $\mu$-measurable family of $n \times n$-matrices that are nonsingular for $\mu$-almost every $u$ with $K(u) \neq 0$. We define the Hausdorff operator with the kernel $K$ by $(x \in \mathbb{R}^n$ is a column vector)

$$(\mathcal{H}f)(x) = (\mathcal{H}_{K,A}f)(x) = \int_{\Omega} K(u)f(A(u)x) \, d\mu(u).$$

*Corresponding author: Adolf R. Mirotin, Department of Mathematics and Programming Technologies, F. Skorina Gomel State University, Sovietskaya, 104, 246019, Gomel, Belarus, e-mail: amirotin@yandex.ru. http://orcid.org/0000-0001-7340-4522
To our knowledge, all known results on Hausdorff operators refer to the boundedness of such operators in various settings only (exceptions are the papers [1, 21] in which some spectra were calculated for the one-dimensional case). In particular, multidimensional normal Hausdorff operators have not been studied. Our main goal is to obtain a spectral representation for such operators. As is known, an explicit diagonalization of a normal operator can be obtained only in a few cases. In this paper, using the $n$-dimensional Mellin transform, we show that a normal Hausdorff operator in $L^2(\mathbb{R}^n)$ with self-adjoint $A(u)$ is unitary equivalent to the operator of multiplication by some matrix-valued function (its matrix symbol) in the space $L^2(\mathbb{R}^n; C^{2n})$. This is an analogue of the spectral theorem for the class of operators under consideration. This allows us to find the norm and to study the spectrum of such operators. The cases of positive definite and negative definite $A(u)$ have been examined. We give also for the case of normal operators a negative answer to the problem of compactness of (nontrivial) Hausdorff operators posed by Liflyand [12] (see also [13]). Several other corollaries are considered that show that properties of a Hausdorff operator are closely related to the properties of its symbol. Several examples are worked out, as well. The results were announced in [20]. It should be noted that the case of $L^p$ spaces is more complicated for the lack of the Plancherel theorem; cf. [16, 17].

In the following, we assume that all the conditions of Definition 1 are fulfilled.

2 Notation and preliminaries

Let $U_j (j = 1; \ldots ; 2^n)$ be some fixed enumeration of the family of all open hyperoctants in $\mathbb{R}^n$. For every pair $(i; j)$ of indices, there is a unique $\varepsilon(i; j) \in [-1, 1]^n$ such that

$$\varepsilon(i; j) U_i := \{(\varepsilon(i; j)_1 x_1; \ldots ; \varepsilon(i; j)_n x_n) : x \in U_i\} = U_j.$$

It is clear that $\varepsilon(i; j) U_j = U_i$ and $\varepsilon(i; j) U_i \cap U_i = \emptyset$ as $l \neq j$. We will assume that $A(u)$ form a commuting family. Then, as is well known, there are an orthogonal $n \times n$-matrix $C$ and a family of diagonal nonsingular real matrices $A'(u) = \text{diag}(a_1(u); \ldots ; a_n(u))$ such that $A'(u) = C^{-1} A(u) C$ for $u \in \Omega$. Then $a(u) := (a_1(u); \ldots ; a_n(u))$ is the family of all eigenvalues (with their multiplicities) of the matrix $A(u)$. We put

$$\Omega_{ij} := \{u \in \Omega : \text{sgn}(a_1(u)); \ldots ; \text{sgn}(a_n(u))) = \varepsilon(i; j)\}.$$

If $|\det A(u)|^{-\frac{1}{2}} K(u) \in L^1(\Omega)$, we put also

$$\varphi_{ij}(s) := \int_{\Omega_{ij}} K(u) |a(u)|^{-\frac{1}{2} - is} d\mu(u).$$

(Above, we assume that $|a(u)|^{-\frac{1}{2} - is} := \prod_{i=1}^n |a_i(u)|^{-\frac{1}{2} - is_i}$, where $|a_i(u)|^{-\frac{1}{2} - is_i} := \exp((-\frac{1}{2} - is_i) \log |a_i(u)|))$.

Evidently, all functions $\varphi_{ij}$ belong to the algebra $C_0(\mathbb{R}^n)$ of bounded and continuous functions on $\mathbb{R}^n$ and $\varphi_{ij} \equiv \varphi_{ji}$.

Definition 2. Let $|a(u)|^{-\frac{1}{2}} K(u) \in L^1(\Omega)$. We define the matrix symbol of a Hausdorff operator $\mathcal{H}_{K,A}$ by

$$\Phi = (\varphi_{ij})_{i,j=1}^{2^n}.$$

So $\Phi$ is a symmetric element of the matrix algebra $\text{Mat}_{2^n}(C_0(\mathbb{R}^n))$.

The symbol was first introduced in [17] for the case of positive definite $A(u)$ (in the simplest one-dimensional case, the symbol was in fact considered also in [4, Theorem 2.1]). As we shall see, properties of a Hausdorff operator are closely related to the properties of its matrix symbol.

Remark 1. Of course, $\Phi$ depends on the enumeration of the family of all hyperoctants in $\mathbb{R}^n$ we choose.

Lemma 1 ([18], cf. [8, (11.18.4)], [2]). Let $|\det A(u)|^{-\frac{1}{2}} K(u) \in L^1(\Omega)$. Then the operator $\mathcal{H}_{K,A}$ is bounded in $L^2(\mathbb{R}^n)$, and

$$\|\mathcal{H}_{K,A}\| \leq \int_{\Omega} |K(u)| |\det A(u)|^{-\frac{1}{2}} d\mu(u).$$

This estimate is sharp (see [17, Theorem 1]).
3 The main result

Theorem 1. Let \( A(u) \) be a commuting family of nonsingular self-adjoint \( n \times n \)-matrices, and

\[
|\det A(u)|^{-\frac{1}{2}} K(u) \in L^1(\Omega).
\]

Then the Hausdorff operator \( \mathcal{H}_{K,A} \) in \( L^2(\mathbb{R}^n) \) with matrix symbol \( \Phi \) is normal and unitary equivalent to the operator \( M_0 \) of multiplication by the normal matrix \( \Phi \) in the space \( L^2(\mathbb{R}^n; \mathbb{C}^2) \) of \( \mathbb{C}^2 \)-valued functions. In particular, the spectrum of \( \mathcal{H}_{K,A} \) equals the spectrum \( \sigma(\Phi) \) of \( \Phi \) in the matrix algebra \( \text{Mat}_2(\mathcal{C}_k(\mathbb{R}^n)) \); in other words,

\[
\sigma(\mathcal{H}_{K,A}) = \{ \lambda \in \mathbb{C} : \inf_{\omega \in \mathbb{R}^n} |\det(\lambda - \Phi(s))| = 0 \}.
\]

The point spectrum \( \sigma_p(\mathcal{H}_{K,A}) \) of \( \mathcal{H}_{K,A} \) consists of such complex numbers \( \lambda \) for which the closed set

\[
E(\lambda) := \{ s \in \mathbb{R}^n : \det(\lambda - \Phi(s)) = 0 \}
\]

has positive Lebesgue measure. The residual spectrum of \( \mathcal{H}_{K,A} \) is empty.

Proof. It is known (see [2]) that, under the conditions of Lemma 1, the adjoint for the Hausdorff operator in \( L^2(\mathbb{R}^n) \) has the form

\[
(\mathcal{H}^* f)(x) = (\mathcal{H}^*_{K,A} f)(x) = \int_{\Omega} K(v) |\det A(v)|^{-\frac{1}{2}} f(A(v)^{-1} x) \, d\mu(v)
\]

(Thus, the adjoint for a Hausdorff operator is also Hausdorff.) Since \( A(u) \) form a commuting family, the normality of \( \mathcal{H}_{K,A} \) follows from the equalities

\[
(\mathcal{H}^* \mathcal{H} f)(x) = \int_{\Omega} \int_{\Omega} K(u) \overline{K(v)} |\det A(v)|^{-\frac{1}{2}} f(A(v)^{-1} A(u)x) \, d\mu(v) \, d\mu(u),
\]

\[
(\mathcal{H} \mathcal{H}^* f)(x) = \int_{\Omega} \int_{\Omega} K(u) \overline{K(v)} |\det A(v)|^{-\frac{1}{2}} f(A(v)A(u)^{-1} x) \, d\mu(u) \, d\mu(v),
\]

and the Fubini theorem.

Let the orthogonal \( n \times n \)-matrix \( C \) and a family of diagonal real matrices \( A'(u) = \text{diag}(a_1(u); \ldots; a_n(u)) \) be such that \( A'(u) = C^{-1} A(u) C \) for \( u \in \Omega \) as in the Section 2. Then all functions \( a_j(u) \) are \( \mu \)-measurable and real, and \( \det A(u) = a_1(u) \ldots a_n(u) \neq 0 \) for \( \mu \)-almost all \( u \). Consider the operator \( \check{C}f(x) := f(Cx) \) in \( L^2(\mathbb{R}^n) \). Since

\[
\mathcal{H}_{K,A} = \check{C}^{-1} \mathcal{H}_{K,A'} \check{C}
\]

and \( \check{C} \) is unitary, the operator \( \mathcal{H}_{K,A} \) is unitary equivalent to \( \mathcal{H}_{K,A'} \).

For every pair \((i, j)\) of indices, consider the following operator in \( L^2(U_j)\):

\[
(H_{ij} f)(x) = \int_{\Omega_{ij}} K(u) f(A'_{ij}(u)x) \, d\mu(u).
\]

Then \( H_{ij} \) maps \( L^2(U_j) \) into \( L^2(U_j) \) (because \( f(A'_{ij}(u)x) = 0 \) for \( f \in L^2(U_j) \) and \( x \not\in U_j \)). Moreover, if \( f \in L^2(\mathbb{R}^n) \) and \( f_i := f1_{U_i} \) (1\(_E\) denotes the indicator of a subset \( E \subset \mathbb{R}^n \)), then, for a.e. \( x \in \mathbb{R}^n \),

\[
(\mathcal{H}_{K,A'} f)(x) = \sum_{j=1}^{2^n} \sum_{i=1}^{2^n} (H_{ij} f_i)(x). \tag{3.1}
\]

Indeed, for every \( x \in \mathbb{R}^n \) that does not belong to any coordinate hyperplane, there is a unique index \( j_0 \) such that \( x \in U_{j_0} \). Then \( A'(u)x \in U_i \) if and only if \( u \in \Omega_{ij} \). Therefore,

\[
(\mathcal{H}_{K,A'} f)(x) = \sum_{i=1}^{2^n} \int_{\Omega} K(u) f_i(A'(u)x) \, d\mu(u) = \sum_{i=1}^{2^n} \int_{\Omega_{ij}} K(u) f_i(A'(u)x) \, d\mu(u). \tag{3.2}
\]

On the other hand, if \( x \not\in U_j \), then \( A'(u)x \not\in U_i \) for all \( i \) and \( u \in \Omega_{ij} \), and therefore

\[
\int_{\Omega_{ij}} K(u) f_i(A'(u)x) \, d\mu(u) = 0.
\]
Thus, for \( x \in U_{j_0} \) and \( f \in L^2(\mathbb{R}^n) \), we have in view of (3.2) that

\[
\sum_{j=1}^{2^n} \sum_{i=1}^{2^n} (H_{ij}f)(x) = \sum_{j=1}^{2^n} \sum_{i=1}^{2^n} \int K(u)f_i(A'(u)x) \, d\mu(u)
\]

\[= \sum_{i=1}^{2^n} \int K(u)f_i(A'(u)x) \, d\mu(u) = (\mathcal{H}_{K,A}f)(x).
\]

Now, since \( \sum_{i=1}^{2^n} H_{ij}f_i \in L^2(U_j) \) for all \( j \), formula (3.1) can be rewritten as

\[
\mathcal{H}_{K,A}f = \bigoplus_{j=1}^{2^n} \sum_{i=1}^{2^n} H_{ij}f_i.
\]

(3.3)

In turn, if we identify \( L^2(\mathbb{R}^n) \) with the orthogonal sum \( L^2(U_1) \oplus \cdots \oplus L^2(U_{2^n}) \), the last formula can be rewritten in the following way:

\[
\mathcal{H}_{K,A}f = (H_{ij_1}f_1; \ldots; f_{2^n})^T
\]

\((B^T \) denotes the transposed to the matrix \( B \)). So we get the following block operator matrix representation for \( \mathcal{H}_{K,A}^T \):

\[
\mathcal{H}_{K,A}^T = (H_{ij})_{ij}.
\]

Consider the modified \( n \)-dimensional Mellin transform for the \( n \)-hyperoctant \( U_i \) (i = 1, ..., 2^n):

\[
(\mathcal{M}_i f)(s) := \frac{1}{(2\pi)^{n/2}} \int_{U_i} |x|^{-\frac{n}{2} + is} f(x) \, dx, \ s \in \mathbb{R}^n.
\]

Then \( \mathcal{M}_i \) is a unitary operator acting from \( L^2(U_i) \) to \( L^2(\mathbb{R}^n) \). This can be easily obtained from the Plancherel theorem for the Fourier transform by using an exponential change of variables (see [3]). Moreover, if we assume that \( |y|^{-\frac{1}{2}} f(y) \in L^1(U_i) \), then making use of Fubini’s theorem and integrating by substitution \( x = A'(u)^{-1}y = (\frac{y_1}{a_1(u)}; \ldots; \frac{y_n}{a_n(u)}) \) yield the following (\( s \in \mathbb{R}^n \)):

\[
(\mathcal{M}_i H_{ij} f)(s) = \frac{1}{(2\pi)^{n/2}} \int_{U_i} |x|^{-\frac{n}{2} + is} \int_{\Omega_{ij}} K(u)f(A'(u)x) \, d\mu(u)
\]

\[= \int_{U_i} K(u) \, d\mu(u) \int_{\Omega_{ij}} |x|^{-\frac{n}{2} + is} f(A'(u)x) \, dx
\]

\[= \int_{U_i} K(u) |a(u)|^{-\frac{n}{2} + is} \, d\mu(u) \frac{1}{(2\pi)^{n/2}} \int_{U_i} |y|^{-\frac{n}{2} + is} f(y) \, dy = \varphi_{ij}(s)(\mathcal{M}_i f)(s).
\]

By continuity, we get for all \( f \in L^2(U_j) \) that \( \mathcal{M}_i H_{ij} f = \varphi_{ij} \mathcal{M}_i f \). So \( H_{ij} = \mathcal{M}_i^{-1}M_{\varphi_{ij}} \mathcal{M}_j \) (\( M_{\varphi_{ij}} \) denotes the operator in \( L^2(\mathbb{R}^n) \) of multiplication by \( \varphi_{ij} \)), and therefore \( \mathcal{H}_{K,A}^T = (\mathcal{M}_i^{-1}M_{\varphi_{ij}} \mathcal{M}_j)_i \).

Let \( U := \text{diag}(U_1; \ldots; U_{2^n}) \). If we identify \( L^2(\mathbb{R}^n) \) with \( L^2(U_1) \times \cdots \times L^2(U_{2^n}) \), then \( U \) is a unitary operator between \( L^2(\mathbb{R}^n) \) and \( L^2(\mathbb{R}^n, C^{2^n}) \), and \( \mathcal{H}_{K,A} = U^{-1}M_{\Phi} U \). This proves the first statement of the theorem.

To compute the spectrum, let \( \lambda \in \mathbb{C} \). The operator \( \lambda - \mathcal{H}_{K,A} \) is unitary equivalent to the operator \( M_{\lambda - \Phi} \), in \( L^2(\mathbb{R}^n, C^{2^n}) \). The last operator is invertible if and only if the matrix \( \lambda - \Phi(s) \) is invertible (i.e. \( \det(\lambda - \Phi(s)) \neq 0 \) for all \( s \in \mathbb{R}^n \)) and \( (\lambda - \Phi(s))^{-1} \) acts in \( L^2(\mathbb{R}^n, C^{2^n}) \). This condition is fulfilled if and only if

\[
(\lambda - \Phi)^{-1}(0; \ldots; 0; f; 0; \ldots; 0)^T \quad (f \text{ is in the } j \text{-th column})
\]

belongs to \( L^2(\mathbb{R}^n, C^{2^n}) \) for all \( f \in L^2(\mathbb{R}^n) \) and \( j = 1, \ldots, 2^n \). Let \( (\lambda - \Phi(s))^{-1} = (a_{ij}(\lambda; s))_{ij} \). Then we have

\[
a_{ij}(\lambda; \cdot) \in C(\mathbb{R}^n) \quad \text{and} \quad (\lambda - \Phi)^{-1}(0; \ldots; 0; f; 0; \ldots; 0)^T = (a_{ij}(\lambda; s)f)^{2^n}_{\text{ij}}.
\]

So \( \lambda \) is a regular point for \( \mathcal{H}_{K,A} \) if and only if the matrix \( \lambda - \Phi(s) \) is invertible and \( a_{ij}(\lambda; \cdot) \in C_b(\mathbb{R}^n) \) for every pair \((i; j)\) of indices. This means that the matrix \( \lambda - \Phi \) is invertible in the algebra \( \text{Mat}_{2^n}(C_b(\mathbb{R}^n)) \) (i.e. \( \lambda \not\in \sigma(\Phi) \)). But it is known that the last condition is equivalent to the fact that \( \det(\lambda - \Phi) \) is invertible in \( C_b(\mathbb{R}^n) \) (see, e.g., [9, Proposition VII.3.7, p. 353]), i.e. \( \inf_{x \in \mathbb{R}^n} |\det(\lambda - \Phi(s))| > 0 \).
Now let \( \lambda \in \sigma_p(M_{\Phi}) \), and let \( f \in L^2(\mathbb{R}^n, \mathbb{C}^{2^m}) \) be a corresponding eigenvalue. Then \((\lambda - \Phi(s))f(s) = 0 \) for a.e. \( s \in \mathbb{R}^n \). It follows that \( \det(\lambda - \Phi(s)) = 0 \) for a.e. \( s \in S \), where the set \( S := \{ s \in \mathbb{R}^n : f(s) \neq 0 \} \) has a positive Lebesgue measure. So \( \text{mes}(E(\lambda)) > 0 \).

To prove the converse, let \( \lambda \in \mathbb{C} \) and \( \text{mes}(E(\lambda)) > 0 \). Consider a multifunction \( \Gamma(s) := \ker(\lambda - M_{\Phi(s)}) \setminus \{0\} \) on \( E(\lambda) \) taking values in the set of all subsets of \( \mathbb{C}^{2^m} \). Then \( \Gamma(s) \neq \emptyset \) for all \( s \in E(\lambda) \). Moreover, since the map \((s, x) \mapsto (\lambda - \Phi(s))x \) is continuous on \( E(\lambda) \times \mathbb{C}^{2^m} \), the graph
\[
G_{\Gamma} := \{(s, x) \in E(\lambda) \times (\mathbb{C}^{2^m} \setminus \{0\}) : (\lambda - \Phi(s))x = 0 \}
\]
of \( \Gamma \) is a Borel subset of \( E(\lambda) \times \mathbb{C}^{2^m} \) (the disjoint union \( G_{\Gamma} \cup (E(\lambda) \times \{0\}) \) is closed). By the measurable selection theorem (see, e.g., [11]), there is a measurable selection \( \xi : E(\lambda) \to \mathbb{C}^{2^m} \), \( \xi(s) \in \ker(\lambda - M_{\Phi(s)}) \setminus \{0\} \). Let \( \chi_C \) be the indicator of a compact subset \( C \subset E(\lambda) \) of positive Lebesgue measure. Then the function \( f(s) := \left( \frac{\xi(s)}{\|\xi(s)\|_2} \right) \chi_C(s) \) belongs to \( L^2(\mathbb{R}^n, \mathbb{C}^{2^m}) \) and is an eigenvalue of \( M_{\Phi} \) which corresponds to \( \lambda \). Finally, as is well known, a normal operator has empty residual spectrum. This completes the proof of the theorem. \( \square \)

### 4 Corollaries and examples

In the following corollaries, we assume that the assumptions and conclusions of Theorem 1 are fulfilled.

**Corollary 1.** The operator \( \mathcal{H}_{K,A} \) is invertible if and only if inf \( \{ \det \Phi(s) : s \in \mathbb{R}^n \} > 0 \). In this case, its inverse is unitary equivalent to the operator \( M_{\Phi^{-1}} \) in \( L^2(\mathbb{R}^n, \mathbb{C}^{2^m}) \).

**Corollary 2.** Let \( \mathcal{H}_{K_1,A} \) and \( \mathcal{H}_{K_2,B} \) be two Hausdorff operators with the same measure space \( (\Omega, \mu) \) such that \( (A(u); B(v)) \) is a commuting family of self-adjoint \( n \times n \)-matrices that are nonsingular for \( \mu \)-almost \( u \) and \( v \) respectively, and \( |\det A(u)|^2 K_1(u), |\det B(v)|^2 K_2(v) \in L^1(\Omega) \). Then the product \( \mathcal{H}_{K_1,A} \mathcal{H}_{K_2,B} \) is unitary equivalent to the operator \( M_{\Phi_1 \Phi_2} \) in \( L^2(\mathbb{R}^n, \mathbb{C}^{2^m}) \). \( (\Phi_2 \Phi_1) \) denotes the matrix symbol of \( \mathcal{H}_{K_1,B} \).

**Proof.** First note that the orthogonal matrix \( C \) exists such that both \( A'(u) = C^{-1} A(u) C \) and \( B'(v) = C^{-1} B(v) C \) are diagonal. Then the proof of Theorem 1 shows that \( \forall \mathcal{H}_{K_1,A} \mathcal{H}_{K_2,B} V^{-1} = M_{\Phi_1 \Phi_2} \) and \( \forall \mathcal{H}_{K_2,B} V^{-1} = M_{\Phi_2} \) for some unitary operator \( V \) from \( L^2(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n, \mathbb{C}^{2^m}) \) which depends only on \( n \) and \( C \), and the result follows. \( \square \)

**Corollary 3.** We have
\[
\|\mathcal{H}_{K,A}\| = \max|\lambda| : \inf_{s \in \mathbb{R}^n} |\det(\lambda - \Phi(s))| = 0 = \sup_{s \in \mathbb{R}^n} \|\Phi(s)\|,
\]
where \( \|\Phi(s)\| \) stands for the norm of the operator in \( \mathbb{C}^{2^m} \) of multiplication by the matrix \( \Phi(s) \).

**Proof.** The first equality follows from Theorem 1 and the normality of \( \mathcal{H}_{K,A} \) (the norm of this operator is equal to its spectral radius), and the second one follows from Theorem 1 and the equality \( \|M_{\Phi}\| = \sup_{s \in \mathbb{R}^n} \|\Phi(s)\| \) (see [23, Theorem 4.1.1] for a more general result). \( \square \)

**Corollary 4.** The matrix symbol of the adjoint operator \( \mathcal{H}_{K,A}^\ast \) is the adjoint matrix \( \Phi^\ast = (\Phi_{ij})^\ast \).

**Proof.** The adjoint for a Hausdorff operator \( \mathcal{H}_{K,A} \) is also Hausdorff [2]. By Theorem 1, this adjoint is unitary equivalent to the adjoint for the operator \( M_{\Phi} \), i.e. to \( M_{\Phi}^\ast \). \( \square \)

**Corollary 5.** The Hausdorff operator \( \mathcal{H}_{K,A} \) is self-adjoint (positive, unitary) if and only if the matrix \( \Phi(s) \) is self-adjoint (respectively, positive definite, unitary) for all \( s \in \mathbb{R}^n \).

**Proof.** This follows from Corollaries 1 and 4.

**Example 1** (Discrete Hausdorff operators; see also Example 3 below). Let \( \Omega = \mathbb{Z}^+ \), and let \( \mu \) be a counting measure. Then Definition 1 takes the form \((f \in L^2(\mathbb{R}^n))\)
\[
(f_{K,A})\langle x \rangle = \sum_{k=0}^{\infty} K(k)f(A(k)x).
\]
Assume that \( \sum_{k=0}^{\infty} |K(k)| \left| \det A(k) \right|^{-\frac{1}{2}} < \infty \). Then \( H_{K,A} \) is bounded on \( L^2(\mathbb{R}^n) \), and

\[
\varphi_{ij}(s) = \sum_{k \in \Omega_i} K(k) |\det A(k)|^{-\frac{1}{2}} |a(k)|^{-is},
\]

where \( s = (s_i) \in \mathbb{R}^n \) and the principal values of the exponential functions are considered. Since this series converges on \( \mathbb{R}^n \) absolutely and uniformly, \( \varphi_{ij} \) is uniformly almost periodic. So the matrix symbol of \( H_{K,A} \) is a uniformly almost periodic matrix-valued function.

Assume, in addition, that \( A(k) = A^k \), where the matrix \( A \) is self-adjoint, but not positive definite and (\( \lambda_1, \ldots, \lambda_n \)) are all eigenvalues of \( A \) (with their multiplicities). Let \( \beta := (\text{sgn}(\lambda_1); \ldots; \text{sgn}(\lambda_n)) \). Then we have \( \beta \in \{-1, 1\}^n \) and

\[
(\text{sgn}(a_1(k)); \ldots; \text{sgn}(a_n(k))) = (\text{sgn}\lambda^k_1; \ldots; \text{sgn}\lambda^k_n) = \begin{cases} (1, \ldots, 1) & \text{if } k \in 2\mathbb{Z}_+, \\ \beta & \text{if } k \in 2\mathbb{Z}_+ + 1. \end{cases}
\]

Let us enumerate \( n \)-hyperoctants \( U_j \) in such a way that \( U_{2^{n-1}+i} = \beta U_i \) (the coordinate-wise multiplication) for \( i = 1, \ldots, 2^{n-1} \). Then \( \Omega_i = 2\mathbb{Z}_{-}, \Omega_i = 2\mathbb{Z} + 1 \) if \( |j - i| = 2^{n-1} \), and \( \Omega_i = 0 \) otherwise. It follows that

\[
\varphi_{ij}(s) = \varphi_+(s) := \sum_{m=0}^{\infty} \frac{K(2m)|a(2m)|^{-\frac{1}{2}}}{| \det(A) |^{\frac{1}{2}} s} \prod_{j=1}^{n} |\lambda_j|^{2^{n-1}m s_j},
\]

Analogously, if \( |j - i| = 2^{n-1} \),

\[
\varphi_{ij}(s) = \varphi_-(s) := \sum_{m=0}^{\infty} \frac{K(2m+1)|a(2m+1)|^{-\frac{1}{2}}}{| \det(A) |^{\frac{1}{2}} s} \prod_{j=1}^{n} |\lambda_j|^{2^{n-1}m s_j},
\]

and \( \varphi_{ij} = 0 \) otherwise. So the matrix symbol is the block matrix

\[
\Phi = \begin{pmatrix} \varphi_+ I_{2^{n-1}} & \varphi_- I_{2^{n-1}} \\ \varphi_- I_{2^{n-1}} & \varphi_+ I_{2^{n-1}} \end{pmatrix},
\]

where \( I_{2^{n-1}} \) denotes the identity matrix of order \( 2^{n-1} \). Then, for every \( \lambda \in \mathbb{C}, \)

\[
\lambda - \Phi = \begin{pmatrix} (\lambda - \varphi_+) I_{2^{n-1}} & - \varphi_- I_{2^{n-1}} \\ - \varphi_+ I_{2^{n-1}} & (\lambda - \varphi_-) I_{2^{n-1}} \end{pmatrix},
\]

and therefore, by the formula of Schur (see, e.g., [7, p. 46]),

\[
\det(\lambda - \Phi) = \det((\lambda - \varphi_+)^2 I_{2^{n-1}} - \varphi_-^2 I_{2^{n-1}}) = ((\lambda - \varphi_+)(\lambda - \varphi_-))^2 - (\lambda - \varphi_+) \varphi_-((\lambda - \varphi_-))^2 = 0,
\]

where \( \varphi := \varphi_+ + \varphi_- \), \( \varphi^* := \varphi_+ - \varphi_- \). Theorem 1 implies (we use the boundedness of \( \varphi, \varphi^* \))

\[
\sigma(\mathcal{H}_{K,A}) = \{ \lambda \in \mathbb{C} : \inf_{x \in \mathbb{R}^n} |\lambda - \varphi(s)(\lambda - \varphi^*(s))| = 0 \} = c(\varphi(\mathbb{R}^n) \cup \varphi^*(\mathbb{R}^n)).
\]

In view of the normality of \( \mathcal{H}_{K,A} \), this implies \( \mathcal{H}_{K,A} = \max\{\sup|\varphi|, \sup|\varphi^*|\} \).

Theorem 1 also implies

\[
\sigma_p(\mathcal{H}_{K,A}) = \{ \lambda \in \mathbb{C} : \text{mes}(\varphi^{-1}(\{\lambda\})) > 0 \} \cup \{ \lambda \in \mathbb{C} : \text{mes}(\varphi^*-1(\{\lambda\})) > 0 \}
\]

As was mentioned above, the problem of compactness of nontrivial Hausdorff operators was posed in [12]. In our case, the answer to this question is negative (the case of positive definite matrices was considered in [17, 18]).

**Corollary 6.** The Hausdorff operator \( \mathcal{H}_{K,A} \) is noncompact provided it is nonzero.

**Proof.** Let \( \mathcal{H}_{K,A} \) be compact in \( L^2(\mathbb{R}^n) \) and nonzero. We shall use notation and formulas from the proof of Theorem 1. There is \( H_{ij} \) that is nonzero, too. Moreover, \( H_{ij} \) is compact because it is equal to \( P_1 \mathcal{H}_{K,A} \) by (3.3) \( (P_1 \) denotes the orthogonal projection of \( L^2(\mathbb{R}^n) \) onto \( L^2(U_j) \)). It follows that the operator \( M_1 H_{ij} M_1^{-1} = M_{\varphi_{ij}} \) is nonzero and compact in \( L^2(\mathbb{R}^n) \), as well. A contradiction. \qed
Corollary 7 ([17]). Let the matrices $A(u)$ be positive definite. Then the operator $\mathcal{H}_{K,A}$ is unitary equivalent to the operator of coordinate-wise multiplication by a function $\varphi \in C_b(\mathbb{R}^n)$ (the scalar symbol) in the space $L^2(\mathbb{R}^n, \mathbb{C}^2^n)$. In particular,

(i) the spectrum, the point spectrum and the continuous spectrum of $\mathcal{H}_{K,A}$ are equal to the spectrum (i.e. to the closure of the range of $\varphi$), to the point spectrum and to the continuous spectrum of the operator $M_\varphi$ of multiplication by $\varphi \in L^2(\mathbb{R}^n)$ respectively, and the residual spectrum of $\mathcal{H}_{K,A}$ is empty;

(ii) $\|\mathcal{H}_{K,A}\| = \sup |\varphi|$.

Proof. Indeed, if all the matrices $A(u)$ are positive definite, then \[
\langle \varphi(a_1(u)), \ldots, \varphi(a_n(u)) \rangle = (1; \ldots; 1).
\]
It follows that $\Omega_{ii} = \Omega$ and $\Omega_{ij} = 0$ for $i \neq j$. Therefore, $\varphi_{ii} = \varphi$, where \[
\varphi(s) := \int_\Omega K(u)|a(u)|^{-\frac{1}{2}} \text{d}u(u)
\]
and $\varphi_{ij} = 0$ for $i \neq j$. So $\Phi = \text{diag}(\varphi; \ldots; \varphi)$, and the corollary follows from Theorem 1 and Corollary 3. \hfill \Box

Example 2. Consider the following Cesàro operator in $L^2(\mathbb{R}^n)$ (see, e.g., [10]):

\[
\left(\mathcal{C}_{a,n}\right)(x) = a \frac{1}{0} \left(1 - u\right)^{a-1} \frac{f(x)}{u} \text{d}u(a > 0; x \in \mathbb{R}^n).
\]

This is a bounded Hausdorff operator, where $\Omega = [0; 1]$ is endowed with the Lebesgue measure, $K(u) = \frac{1-u^{a-1}}{u}$, and $A(u) = \text{diag}(\frac{1}{2}; \ldots; \frac{1}{2})$ ($u \in (0; 1)$) is a positive definite matrix. Its scalar symbol is $(s = (s_j) \in \mathbb{R}^n)$ \[
\varphi(s) = a \frac{1}{0} \left(1 - u\right)^{a-1} u^{-\frac{a}{2}} \text{d}u = \frac{\Gamma(a + 1) \Gamma\left(\frac{n}{2} + i \sum_j s_j\right)}{\Gamma\left(a + \frac{n}{2} + i \sum_j s_j\right)}.
\]

Since the modulus of the function \[
\gamma(t) := \frac{\Gamma(a + 1) \Gamma\left(\frac{n}{2} + it\right)}{\Gamma\left(a + \frac{n}{2} + it\right)}, \quad t \in \mathbb{R},
\]
attains its maximum at $t = 0$ (this follows, e.g., from [24, Section 12.13, Example 1]), by Corollary 7, we get \[
||\mathcal{C}_{a,n}|| = \sup_s |\varphi(s)| = \gamma(0) = \frac{\Gamma(a + 1) \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(a + \frac{n}{2}\right)}.
\]
Moreover, Corollary 7 implies that the spectrum of $\mathcal{C}_{a,n}$ is a curve given by the range of $\gamma(t)$, $t \in \mathbb{R} \cup \{\infty\}$.

Corollary 8. Let the matrices $A(u)$ be negative definite. Then the matrix symbol of the operator $\mathcal{H}_{K,A}$ for some enumeration of $n$-hyperoctants is the block matrix \[
\Phi = \begin{pmatrix} O & \varphi I_{2^{n-1}} \\ \varphi I_{2^{n-1}} & O \end{pmatrix},
\]
where $\varphi$ is given by formula (4.1). Moreover, $\sigma(\mathcal{H}_{K,A}) = \text{cl}(\mathcal{H}(\mathbb{R}^n) \cup \varphi(\mathbb{R}^n))$. In particular, $||\mathcal{H}_{K,A}|| = \sup |\varphi|$.

Proof. Let us enumerate $n$-hyperoctants $U_i$ in such a way that $U_{2^{n-1}+i} = -U_i$ for $i = 1, \ldots, 2^{n-1}$. Since \[
\langle \varphi(a_1(u)); \ldots; \varphi(a_n(u)) \rangle = (-1; \ldots; -1),
\]
it follows that $\Omega_{ij} = \Omega$ for $|j - i| = 2^{n-1}$, and $\Omega_{ij} = 0$ otherwise. Therefore, $\varphi_{ij} = \varphi$ for $|j - i| = 2^{n-1}$, and $\varphi_{ij} = 0$ otherwise. Thus, $\Phi$ is given by (4.2), and for $\lambda \in \mathbb{C}$, we have \[
\lambda - \Phi = \begin{pmatrix} M_{2^{n-1}} & -\varphi I_{2^{n-1}} \\ -\varphi I_{2^{n-1}} & M_{2^{n-1}} \end{pmatrix},
\]
where $\varphi$ is given by formula (4.1). Moreover, $\sigma(\mathcal{H}_{K,A}) = \text{cl}(\mathcal{H}(\mathbb{R}^n) \cup \varphi(\mathbb{R}^n))$. In particular, $||\mathcal{H}_{K,A}|| = \sup |\varphi|$.
Now, as in Example 1, the formula of Schur implies
\[ \det(\lambda - \Phi) = \det(\lambda^2 I_{2n-1} - \varphi(x)^2 I_{2n-1}) = (\lambda^2 - \varphi(x)^2)^{2n-1}, \]
and by Theorem 1, we get
\[ \sigma(J_{K, A}) = \{ \lambda \in \mathbb{C} : \inf_{s \in \mathbb{R}^n} |\lambda^2 - \varphi^2(s)| = 0 \} = \text{cl}(-\varphi(\mathbb{R}^n) \cup \varphi(\mathbb{R}^n)). \]

The value of the norm follows from this formula and normality of \( J_{K, A} \).

**Example 3.** Consider the q-calculus version of a Cesàro operator (see, e.g., [6] for the definition of the q-integral)
\[ (C_q f)(x) := \frac{1}{x} \int_0^x f(t) \, d_q t := (1 - q) \sum_{k=0}^{\infty} q^k f(q^k x). \]

Here \( f \in L^2(\mathbb{R}), x \in \mathbb{R}, \) and \( q \) is real, \( 0 < |q| < 1. \) This is a bounded discrete Hausdorff operator in the sense of Example 1, where \( n = 1, K(k) = (1 - q)q^k, A = q, a(k) = q^k. \) Two cases are possible.

(1) \( 0 < q < 1. \) In this case, one can apply Corollary 7. By formula (4.1), the scalar symbol is
\[ \varphi(s) = (1 - q) \sum_{k=0}^{\infty} (q^k)^{\frac{1}{1-q}} = \frac{1 - q}{1 - \sqrt{q} q^{1-s}}. \]

Now Corollary 7 implies
\[ \sigma(C_q) = \left\{ \frac{1 - q}{1 - \sqrt{q} z} : z \in \mathbb{C}, |z| = 1 \right\} = \{ \lambda \in \mathbb{C} : |\lambda - 1| = \sqrt{q} \}. \]

It follows that \( \|C_q\| = \sqrt{q}. \) Moreover, the operator \( C_q \) is invertible, and its inverse \( (C_q^{-1} g)(x) = \frac{g(x) - qg(qx)}{1-q} \) is unitary equivalent to the operator of coordinate-wise multiplication by a function \( \frac{1}{\sqrt{q}} \) in the space \( L^2(\mathbb{R}, \mathbb{C}^2). \)

(2) \( -1 < q < 0. \) In this case, one can apply Corollary 8. Again by formula (4.1), the scalar symbol is
\[ \varphi(s) = (1 - q) \sum_{k=0}^{\infty} q^k (-q^k)^{\frac{1}{1-q}} = (1 - q) \sum_{k=0}^{\infty} (-1)^k (-q^k)^{\frac{1}{1-q}} = \frac{1 - q}{1 + \sqrt{-q} (-q)^{-1}}. \]

Since \( \varphi(\mathbb{R}) = \{ \lambda \in \mathbb{C} : |\lambda - 1| = \sqrt{-q} \} \) as in case (1) above, Corollary 8 implies
\[ \sigma(C_q) = \{ \lambda \in \mathbb{C} : |\lambda \pm 1| = \sqrt{-q} \}. \]

It follows that \( \|C_q\| = \sqrt{-q}. \) The operator \( C_q \) is invertible, and its inverse \( (C_q^{-1} g)(x) = \frac{g(x) - qg(qx)}{1+q} \) is unitary equivalent to the operator \( M_{\varphi^{-1}} \) in the space \( L^2(\mathbb{R}, \mathbb{C}^2), \) where (see formula (4.2))
\[ \Phi^{-1} = \begin{pmatrix} 0 & \frac{1}{\varphi} \\ \frac{1}{\varphi} & 0 \end{pmatrix}. \]

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