Research Article

A New Inequality for Frames in Hilbert Spaces

Zhong-Qi Xiang

College of Mathematics and Computer Science, Shangrao Normal University, Shangrao, Jiangxi 334001, China

Correspondence should be addressed to Zhong-Qi Xiang; lxsy20110927@163.com

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We obtain a new inequality for frames in Hilbert spaces associated with a scalar and a bounded linear operator induced by two Bessel sequences. It turns out that the corresponding results due to Balan et al. and Găvruţa can be deduced from our result.

1. Introduction

A frame for a Hilbert space firstly emerged in the work on nonharmonic Fourier series owing to Duffin and Schaeffer [1], which has made great contributions to various fields because of its nice properties; the reader can examine the papers [2–12] for background and details of frames.

Balan et al. in [13] showed us a surprising inequality when they further investigated the Parseval frame identity derived from their study on efficient algorithms for signal reconstruction, which was then extended to general frames and alternate dual frames by Găvruţa [14]. In this paper, we establish a new inequality for frames in Hilbert spaces, where a scalar and a bounded linear operator with respect to two Bessel sequences are involved, and it is shown that our result can lead to the corresponding results of Balan et al. and Găvruţa.

The notations $\mathcal{H}$, $\text{Id}_\mathcal{H}$, and $\mathcal{J}$ are reserved, respectively, for a complex Hilbert space, the identity operator on $\mathcal{H}$, and an index set which is finite or countable. The algebra of all bounded linear operators on $\mathcal{H}$ is designated as $\mathcal{B}(\mathcal{H})$.

One says that a family $\{f_j\}_{j \in \mathcal{J}}$ of vectors in $\mathcal{H}$ is a frame, if there are two positive constants $C, D > 0$ satisfying

$$\|f\|^2 \leq \sum_{j \in \mathcal{J}} |\langle f, f_j \rangle|^2 \leq D \|f\|^2, \quad \forall f \in \mathcal{H}. \quad (1)$$

The frame $\{f_j\}_{j \in \mathcal{J}}$ is said to be Parseval if $C = D = 1$. If $\{f_j\}_{j \in \mathcal{J}}$ satisfies the inequality to the right in (1), we call that $\{f_j\}_{j \in \mathcal{J}}$ is a Bessel sequence for $\mathcal{H}$.

For a given frame $\mathcal{F} = \{f_j\}_{j \in \mathcal{J}}$, the frame operator $S_{\mathcal{F}}$, a positive, self-adjoint, and invertible operator on $\mathcal{H}$, is defined by

$$S_{\mathcal{F}} : \mathcal{H} \rightarrow \mathcal{H},$$

$$S_{\mathcal{F}} f = \sum_{j \in \mathcal{J}} \langle f, f_j \rangle f_j, \quad \forall f \in \mathcal{H}, \quad (2)$$

from which we see that

$$f = \sum_{j \in \mathcal{J}} \langle f, f_j \rangle S_{\mathcal{F}}^{-1} f_j = \sum_{j \in \mathcal{J}} \langle f, S_{\mathcal{F}}^{-1} f_j \rangle f_j, \quad \forall f \in \mathcal{H}, \quad (3)$$

where the involved frame $\{\tilde{f}_j = S_{\mathcal{F}}^{-1} f_j\}_{j \in \mathcal{J}}$ is said to be the canonical dual of $\{f_j\}_{j \in \mathcal{J}}$.

For any $\mathcal{I} \subset \mathcal{J}$, denote $\mathcal{I}^c = \mathcal{J} \setminus \mathcal{I}$. A positive, bounded linear, and self-adjoint operator induced by $\mathcal{I}$ and the frame $\mathcal{F} = \{f_j\}_{j \in \mathcal{I}}$ is given below

$$S_{\mathcal{F}}^\mathcal{I} : \mathcal{H} \rightarrow \mathcal{H},$$

$$S_{\mathcal{F}}^\mathcal{I} f = \sum_{j \in \mathcal{I}} \langle f, f_j \rangle f_j, \quad \forall f \in \mathcal{H}. \quad (4)$$

Suppose that $\mathcal{F} = \{f_j\}_{j \in \mathcal{J}}$ and $\mathcal{G} = \{g_j\}_{j \in \mathcal{J}}$ are two Bessel sequences for $\mathcal{H}$. An application of the Cauchy-Schwartz inequality can show that the operator

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Lemma 1. Suppose that \( U, V, L \in B(\mathcal{H}) \) and that \( U + V = L \). Then for each \( \lambda \in [0, 1] \) we have

\[
U^*U + \lambda (V^*L + L^*V) = U^*V + (1 - \lambda) (U^*L + L^*U) + (2\lambda - 1) L^*L \geq (2\lambda - \lambda^2) L^*L.
\]

Proof. A direct calculation gives

\[
U^*U + \lambda (V^*L + L^*V) = U^*U + \lambda ((L^* - U^*) (L - U)) + (2\lambda - 1) L^*L
\]

\[
= U^*U + \lambda (L^*L - U^*U + L^*L - L^*L) + (2\lambda - 1) L^*L
\]

we arrive at the relation stated in the lemma. \( \square \)

Corollary 2. Suppose that \( U, V \in B(\mathcal{H}) \) and that \( U + V = \text{Id}_{\mathcal{H}} \). Then for every \( \lambda \in [0, 1] \) we have

\[
U^*U + \lambda (V^* + V) = V^*V + (1 - \lambda) (U^* + U)
\]

\[
+ (2\lambda - 1) \text{Id}_{\mathcal{H}} \geq (2\lambda - \lambda^2) \text{Id}_{\mathcal{H}}.
\]
Therefore,
\[
\|Uf\|^2 = \|Vf\|^2 + 2(1-\lambda) \text{Re} \langle S_{\mathcal{F}}^* Uf, f \rangle \\
+ (2\lambda - 1) \text{Re} \langle S_{\mathcal{F}}^* f, S_{\mathcal{F}} f \rangle \\
- 2\lambda \text{Re} \langle S_{\mathcal{F}}^* Vf, f \rangle
\]
\[
= \|Vf\|^2 + 2 \text{Re} \langle S_{\mathcal{F}}^* Uf, f \rangle \\
- 2\lambda \text{Re} \langle S_{\mathcal{F}}^* f, S_{\mathcal{F}} f \rangle
\]
\[
= \|Vf\|^2 + 2 \text{Re} \langle S_{\mathcal{F}}^* Uf, f \rangle \\
- \text{Re} \langle S_{\mathcal{F}}^* (U + V) f, f \rangle
\]
\[
= \|Vf\|^2 + \text{Re} \langle S_{\mathcal{F}}^* Uf, f \rangle - \text{Re} \langle S_{\mathcal{F}}^* Vf, f \rangle .
\]
It follows that
\[
\left\| \sum_{j \in J} \langle f, g_j - h_j \rangle f_j \right\|^2 + \text{Re} \sum_{j \in J} \langle f, h_j \rangle \langle f_j, S_{\mathcal{F}} f \rangle \\
= \|Uf\|^2 + \text{Re} \langle S_{\mathcal{F}}^* Vf, f \rangle \\
= \|Vf\|^2 + \text{Re} \langle S_{\mathcal{F}}^* Uf, f \rangle \\
= \left\| \sum_{j \in J} \langle f, h_j \rangle f_j \right\|^2 \\
+ \text{Re} \sum_{j \in J} \langle f, g_j - h_j \rangle \langle f_j, S_{\mathcal{F}} f \rangle .
\]
We now prove the inequality in (10). Again by Lemma 1,
\[
\|Uf\|^2 + 2\lambda \text{Re} \langle S_{\mathcal{F}}^* Vf, f \rangle \\
\geq (2\lambda - \lambda^2) \langle S_{\mathcal{F}}^* S_{\mathcal{F}} f, f \rangle
\]
for every $f \in \mathcal{F}$. Hence,
\[
\|Uf\|^2 \geq (2\lambda - \lambda^2) \langle S_{\mathcal{F}}^* S_{\mathcal{F}} f, f \rangle \\
- 2\lambda \text{Re} \langle S_{\mathcal{F}}^* Vf, f \rangle \\
= (2\lambda - \lambda^2) \langle S_{\mathcal{F}}^* f, S_{\mathcal{F}} f \rangle \\
- 2\lambda \text{Re} \langle Vf, S_{\mathcal{F}} f \rangle \\
= (2\lambda - \lambda^2) \text{Re} \langle (U + V) f, S_{\mathcal{F}} f \rangle \\
- 2\lambda \text{Re} \langle Vf, S_{\mathcal{F}} f \rangle \\
= (2\lambda - \lambda^2) \text{Re} \langle Uf, S_{\mathcal{F}} f \rangle \\
- \lambda^2 \text{Re} \langle Vf, S_{\mathcal{F}} f \rangle .
\]
from which we conclude that
\[
\left\| \sum_{j \in J} \langle f, g_j - h_j \rangle f_j \right\|^2 + \text{Re} \sum_{j \in J} \langle f, h_j \rangle \langle f_j, S_{\mathcal{F}} f \rangle \\
= \|Uf\|^2 + \text{Re} \langle Vf, S_{\mathcal{F}} f \rangle \\
\geq (2\lambda - \lambda^2) \text{Re} \langle Uf, S_{\mathcal{F}} f \rangle \\
+ (1 - \lambda^2) \text{Re} \langle Vf, S_{\mathcal{F}} f \rangle
\]
\[
= (2\lambda - \lambda^2) \text{Re} \sum_{j \in J} \langle f, g_j - h_j \rangle \langle f_j, S_{\mathcal{F}} f \rangle \\
+ (1 - \lambda^2) \text{Re} \sum_{j \in J} \langle f, h_j \rangle \langle f_j, S_{\mathcal{F}} f \rangle .
\]
The proof of (11) is similar to the proof of (10); we leave the details to the reader. \qed

**Corollary 4.** Suppose that $\{f_j\}_{j \in J}$ is a frame for $\mathcal{F}$ with frame operator $S_{\mathcal{F}}$ and that $f_j = S_{\mathcal{F}}^{-1/2} f_j$ for any $j \in J$. Then for all $\lambda \in [0, 1]$, for any $l \in J$ and any $f \in \mathcal{F}$, we have
\[
\left\| \sum_{j \in J} \langle f, f_j \rangle (g_j - h_j) \right\|^2 = \left\| \sum_{j \in J} \langle f, f_j \rangle S_{\mathcal{F}}^{-1/2} f_j \right\|^2
\]
\[
\geq \sum_{j \in J} \left| \langle f, f_j \rangle \right|^2 + \sum_{j \in J} \left| \langle S_{\mathcal{F}}^* f, S_{\mathcal{F}}^* f_j \rangle \right|^2
\]
\[
\geq (2\lambda - \lambda^2) \sum_{j \in J} \left| \langle f, f_j \rangle \right|^2 \\
+ (1 - \lambda^2) \sum_{j \in J} \left| \langle f, f_j \rangle \right|^2 .
\]
Proof. Setting $g_j = S_{\mathcal{F}}^{-1/2} f_j$ for each $j \in J$, then $S_{\mathcal{F}} = S_{\mathcal{F}}^{1/2}$. Taking
\[
h_j = \begin{cases} 0, & j \in \emptyset, \\
g_j, & j \in \mathcal{F}, \end{cases}
\]
then $\{g_j\}_{j \in J}$ and $\{h_j\}_{j \in J}$ are both Bessel sequences for $\mathcal{F}$. For any $f \in \mathcal{F}$ we have
\[
\left\| \sum_{j \in J} \langle f, f_j \rangle (g_j - h_j) \right\|^2 = \left\| \sum_{j \in J} \langle f, f_j \rangle S_{\mathcal{F}}^{-1/2} f_j \right\|^2
\]
\[
= \left\| S_{\mathcal{F}}^{-1/2} S_{\mathcal{F}} f \right\|^2 = \left\| S_{\mathcal{F}}^{1/2} f, S_{\mathcal{F}}^{-1/2} S_{\mathcal{F}} f \right\|^2
\]
\[
= \left\| S_{\mathcal{F}}^{-1/2} S_{\mathcal{F}}^* f, S_{\mathcal{F}}^{-1/2} S_{\mathcal{F}}^* f \right\|^2
\]
\[
= \left\| S_{\mathcal{F}}^{-1/2} S_{\mathcal{F}}^* f, S_{\mathcal{F}}^{-1/2} S_{\mathcal{F}}^* f \right\|^2
\]
\[
= \sum_{j \in J} \left( S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^* f, f_j \right) \left( S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^* f, f_j \right)
\]
\[
= \sum_{j \in J} \left| \left( S_{\mathcal{F}}^{-1} S_{\mathcal{F}}^* f, f_j \right) \right|^2 \\
= \sum_{j \in J} \left| \left( S_{\mathcal{F}}^{-1} f, S_{\mathcal{F}}^* f_j \right) \right|^2 \\
= \sum_{j \in J} \left| \left( S_{\mathcal{F}}^* f, S_{\mathcal{F}} f_j \right) \right|^2 .
\]
A similar discussion yields
\[
\left\| \sum_{j \in J} \langle f, f_j \rangle h_j \right\|^2 = \sum_{j \in J} \left| \langle S^F \bar{f}, f_j \rangle \right|^2. \tag{22}
\]

We also have
\[
\text{Re} \sum_{j \in J} \langle f, S \bar{f} h_j \rangle \langle f, f_j \rangle = \sum_{j \in J} \left| \langle f, f_j \rangle \right|^2,
\]
\[
\text{Re} \sum_{j \in J} \langle f, S \bar{f} (g_j - h_j) \rangle \langle f, f_j \rangle = \sum_{j \in J} \left| \langle f, f_j \rangle \right|^2. \tag{23}
\]

Thus the result follows from Theorem 3. \qed

Let \( \{f_j\}_{j \in J} \) be a Parseval frame for \( \mathcal{H} \); then \( S_\mathcal{F} = \text{Id}_\mathcal{H} \).

Thus for any \( \mathcal{l} \subset J \),
\[
\sum_{j \in J} \left| \langle S^F \bar{f}, f_j \rangle \right|^2 = \sum_{j \in J} \left| \langle S^F \bar{f}, f_j \rangle \right|^2 = \|S^F \bar{f}\|^2 \tag{24}
\]
\[
= \left\| \sum_{j \in \mathcal{L}} \langle f, f_j \rangle f_j \right\|^2, \quad \forall f \in \mathcal{H}.
\]

Similarly we have
\[
\sum_{j \in J} \left| \langle S^F \bar{f}, f_j \rangle \right|^2 = \left\| \sum_{j \in \mathcal{L}} \langle f, f_j \rangle f_j \right\|^2. \tag{25}
\]

This together with Corollary 4 leads to a result as follows.

**Corollary 5.** Suppose that \( \{f_j\}_{j \in J} \) is a Parseval frame for \( \mathcal{H} \). Then for each \( \lambda \in [0, 1] \), for any \( \mathcal{l} \subset J \) and any \( f \in \mathcal{H} \), we have
\[
\sum_{j \in \mathcal{L}} \left| \langle f, f_j \rangle \right|^2 + \left\| \sum_{j \in \mathcal{L}} \langle f, f_j \rangle f_j \right\|^2 \geq \left( 2\lambda - \lambda^2 \right) \sum_{j \in J} \left| \langle f, f_j \rangle \right|^2 + \left( 1 - \lambda^2 \right) \sum_{j \in \mathcal{L}} \left| \langle f, f_j \rangle \right|^2. \tag{26}
\]

**Corollary 6.** Suppose that \( \{f_j\}_{j \in J}, \{g_j\}_{j \in J} \) is an alternate dual frame pair for \( \mathcal{H} \). Then for each \( \lambda \in [0, 1] \), for any \( \mathcal{l} \subset J \) and any \( f \in \mathcal{H} \), we have
\[
\left\| \sum_{j \in \mathcal{L}} \langle f, g_j \rangle f_j \right\|^2 + \text{Re} \sum_{j \in \mathcal{L}} \langle f, g_j \rangle \langle f, f_j \rangle \geq \left( 2\lambda - \lambda^2 \right) \sum_{j \in J} \left| \langle f, f_j \rangle \right|^2 + \left( 1 - \lambda^2 \right) \sum_{j \in \mathcal{L}} \left| \langle f, f_j \rangle \right|^2. \tag{27}
\]

**Proof.** Since \( \{g_j\}_{j \in J} \) is an alternate dual frame of \( \{f_j\}_{j \in J} \), \( S^F = \text{Id}_\mathcal{H} \). For any \( j \in J \), let
\[
h_j = \begin{cases} 0, & j \in \mathcal{l}, \\ g_j, & j \in \mathcal{L}. \end{cases} \tag{28}
\]

On the one hand we have
\[
\left\| \sum_{j \in \mathcal{L}} \langle f, g_j - h_j \rangle f_j \right\|^2 = \left\| \sum_{j \in \mathcal{L}} \langle f, g_j \rangle f_j \right\|^2, \tag{29}
\]
\[
\left\| \sum_{j \in \mathcal{L}} \langle f, h_j \rangle f_j \right\|^2 = \left\| \sum_{j \in \mathcal{L}} \langle f, g_j \rangle f_j \right\|^2.
\]

On the other hand we have
\[
\text{Re} \sum_{j \in J} \langle f, h_j \rangle \langle f, S \bar{f} f \rangle = \text{Re} \sum_{j \in \mathcal{L}} \langle f, g_j \rangle \langle f, f_j \rangle, \tag{30}
\]
\[
\text{Re} \sum_{j \in J} \langle f, g_j - h_j \rangle \langle f, S \bar{f} f \rangle = \Rightarrow \sum_{j \in \mathcal{L}} \langle f, g_j \rangle \langle f, f_j \rangle.
\]

By Theorem 3 the conclusion follows. \qed

**Remark 7.** Theorems 2.2 and 3.2 in [14] and Proposition 4.1 in [13] can be obtained when taking \( \lambda = 1/2 \), respectively, in Corollaries 4, 6, and 5.

As a matter of fact, we can establish a more general inequality for alternate dual frames than that shown in Corollary 6.
Theorem 8. Suppose that \( \{ f_j \}_{j \in \mathcal{J}}, \{ g_j \}_{j \in \mathcal{J}} \) is an alternate dual frame pair for \( \mathcal{H} \). Then for every bounded sequence \( \omega_j \) for all \( \lambda \in [0, 1] \) and all \( f \in \mathcal{H} \), we have
\[
\text{Re} \sum_{j \in \mathcal{J}} \omega_j \langle f, g_j \rangle \langle f_j, f \rangle + \left\| \sum_{j \in \mathcal{J}} (1 - \omega_j) \langle f, g_j \rangle f_j \right\|^2 \\
\geq (2\lambda - \lambda^2) \text{Re} \sum_{j \in \mathcal{J}} (1 - \omega_j) \langle f, g_j \rangle \langle f_j, f \rangle \\
+ (1 - \lambda^2) \text{Re} \sum_{j \in \mathcal{J}} \omega_j \langle f, g_j \rangle \langle f_j, f \rangle.
\]

Proof. We define the operators \( F_\omega \) and \( F_{1 - \omega} \) by
\[
F_\omega f = \sum_{j \in \mathcal{J}} \omega_j \langle f, g_j \rangle f_j, \\
F_{1 - \omega} f = \sum_{j \in \mathcal{J}} (1 - \omega_j) \langle f, g_j \rangle f_j.
\]
Then both series converge unconditionally and \( F_\omega, F_{1 - \omega} \in B(\mathcal{H}) \). Since \( F_\omega + F_{1 - \omega} = \text{Id}_{\mathcal{H}} \), by Corollary 2 we obtain
\[
\langle F_\omega f, f \rangle + \lambda \langle F_{1 - \omega} f, f \rangle + \lambda \langle F_\omega f, f \rangle \\
\geq (2\lambda - \lambda^2) \left\| f \right\|^2
\]
for each \( f \in \mathcal{H} \). Hence
\[
\left\| F_{1 - \omega} f \right\|^2 + 2\lambda \text{Re} \langle F_\omega f, f \rangle \geq (2\lambda - \lambda^2) \langle f, f \rangle.
\]
Therefore,
\[
\left\| F_{1 - \omega} f \right\|^2 \geq (2\lambda - \lambda^2) \langle f, f \rangle - 2\lambda \text{Re} \langle F_\omega f, f \rangle \\
= (2\lambda - \lambda^2) \text{Re} \langle F_\omega + F_{1 - \omega} f, f \rangle \\
- 2\lambda \text{Re} \langle F_\omega f, f \rangle \\
= (2\lambda - \lambda^2) \text{Re} \langle F_{1 - \omega} f, f \rangle \\
- \lambda^2 \text{Re} \langle F_\omega f, f \rangle.
\]

It follows that
\[
\text{Re} \sum_{j \in \mathcal{J}} \omega_j \langle f, g_j \rangle \langle f_j, f \rangle + \left\| \sum_{j \in \mathcal{J}} (1 - \omega_j) \langle f, g_j \rangle f_j \right\|^2 \\
= \text{Re} \langle F_\omega f, f \rangle + \left\| F_{1 - \omega} f \right\|^2 \\
\geq (2\lambda - \lambda^2) \text{Re} \langle F_{1 - \omega} f, f \rangle \\
+ (1 - \lambda^2) \text{Re} \langle F_\omega f, f \rangle \\
= (2\lambda - \lambda^2) \text{Re} \sum_{j \in \mathcal{J}} (1 - \omega_j) \langle f, g_j \rangle \langle f_j, f \rangle \\
+ (1 - \lambda^2) \text{Re} \sum_{j \in \mathcal{J}} \omega_j \langle f, g_j \rangle \langle f_j, f \rangle.
\]

This completes the proof.

Remark 9. If we take \( \lambda = 1/2 \) in Theorem 8, then we can obtain Theorem 3.3 in [14].

Data Availability
No data were used to support this study.

Conflicts of Interest
The author declares that he has no conflicts of interest.

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