Explicit Formulas for General Euler Type Sums

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Abstract We define a new kind of classical digamma function, and establish its some fundamental identities. Then we apply the formulas obtained, and extend tools developed by Flajolet and Salvy to study more general Euler type sums. The main results of Flajolet and Salvy’s paper [11] are the immediate corollaries of main results in this paper. Furthermore, we provide some parameterized extensions of Ramanujan-type identities that involve hyperbolic series.

Keywords: Euler type sum; Polygamma function; Zeta value; Contour integration; Residue theorem; Ramanujan-type identity.

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1 Introduction and Notations

Investigation of Euler sums has a long history. The origin of the study of linear Euler sums

\[ S_{p,q} := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}, \quad p \geq 1, q \geq 2 \]

which correspond to the \( d = 1 \) case of (1.2), goes back to the correspondence of Euler with Goldbach in 1742–1743; see Berndt [5, p. 253] for a discussion. Euler elaborated a method to show that the linear sums \( S_{p,q} \) can be evaluated in terms of zeta values in the following cases: \( p = 1, p = q, p + q \) odd, \( p + q \) even but with the pair \((p,q)\) being restricted to the set \{(2, 4), (4, 2)\}. For more details on linear Euler sums, the readers are referred to [4, 8, 11]. Here

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$H_n^{(p)}$ stands for the $p$-th generalized harmonic number, which is defined by

$$H_n^{(p)} := \sum_{k=1}^{n} \frac{1}{k^p} \quad \text{and} \quad H_0^{(p)} := 0.$$  

If $p > 1$, the generalized harmonic number $H_n^{(p)}$ converges to the (Riemann) zeta value $\zeta(p)$:

$$\lim_{n \to \infty} H_n^{(p)} = \zeta(p).$$

When $p = 1$, $H_n^{(1)} \equiv H_k$ is the classical harmonic number. A twin sibling of the harmonic number is called alternating harmonic number defined by

$$\bar{H}_n^{(p)} := \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^p} \quad \text{and} \quad \bar{H}_0^{(p)} := 0,$$

which was introduced in [11]. When taking the limit $n \to \infty$ in above, we get the so-called the alternating Riemann zeta value

$$\bar{\zeta}(p) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^p}, \quad (p \geq 1).$$

Flajolet and Salvy [11] introduced and studied the following a kind of special Dirichlet series that involve harmonic numbers

$$S_{p,q} := \sum_{n=1}^{\infty} \frac{H_n^{(p_1)} H_n^{(p_2)} \cdots H_n^{(p_r)}}{n^q},$$

which is called the generalized (non-alternating) Euler sums, if $r > 1$, then it is called nonlinear Euler sums. Here $p := (p_1, p_2, \ldots, p_r)$ ($r, p_i \in \mathbb{N}, i = 1, 2, \ldots, r$) with $p_1 \leq p_2 \leq \ldots \leq p_r$ and $q \geq 2$. The quantity $w := p_1 + \cdots + p_r + q$ is called the weight and the quantity $r$ is called the degree (order). As usual, repeated summands in partitions are indicated by powers, so that for instance

$$S_{1^22^34, q} = S_{112224, q} = \sum_{n=1}^{\infty} \frac{H_n^{(2)} H_n^{(3)} H_n^{(4)}}{n^q}.$$  

They considered the contour integration involving classical digamma function and used the residue computations to establish more explicit reductions of generalized Euler sums to Euler sums with lower degree. In particular, they proved the famous theorem: a nonlinear Euler sum $S_{p_1p_2\cdots p_r,q}$ reduces to a combination of sums of lower orders whenever the weight $p_1 + p_2 + \cdots + p_r + q$ and the order $r$ are of the same parity. The study of nonlinear Euler sums have attracted a lot of research in the area in the last three decades. Some related results may be seen in the works of [14, 20–22, 24] and the references therein.

In addition, Flajolet and Salvy [11] also introduced and studied the three alternating linear Euler sums,

$$S_{\bar{p},q} := \sum_{n=1}^{\infty} \frac{\bar{H}_n^{(p)}}{n^q}, \quad S_{\bar{p},q} := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} (-1)^{n-1}, \quad S_{\bar{p},q} := \sum_{n=1}^{\infty} \frac{\bar{H}_n^{(p)}}{n^q} (-1)^{n-1}. \quad (1.2)$$
Moreover, they shown that the alternating linear Euler sums \( S_{p,q}, S_{p,q} \) and \( S_{p,q} \) can be evaluated in terms of polynomials of zeta values. For convenience, in (1.3), if replace \( "H_n^{(p)}" \) by \( "\bar{H}_n^{(p)}" \) in the numerator of the summand, we put a “bar” on the top of \( p_j \). In particular, we put a bar on top of \( q \) if there is a sign \( (-1)^{n-1} \) appearing in the denominator on the right. For example,

\[
S_{p_1p_2p_3p_4,q} := \sum_{n=1}^{\infty} \frac{H_n^{(p_1)}\bar{H}_n^{(p_2)}H_n^{(p_3)}\bar{H}_n^{(p_4)}}{n^q} \quad \text{and} \quad S_{p_1p_2p_3p_4,q} := \sum_{n=1}^{\infty} \frac{\bar{H}_n^{(p_1)}\bar{H}_n^{(p_2)}H_n^{(p_3)}\bar{H}_n^{(p_4)}}{n^q} (-1)^{n-1}.
\]

The sums of types above (one of more the \( p_j \) or \( q \) barred) are called the alternating Euler sums. There are many other researches on alternating Euler sums. For example, Zhao [25] gave the explicit evaluations of all 89 alternating Euler sums with weight \( \leq 5 \). The author and Wang [22] have developed the Maple package to evaluate the non-alternating Euler sums of weight \( 2 \leq w \leq 16 \) and the alternating Euler sums of weight \( 1 \leq w \leq 6 \).

In this paper, we define a new kind of classical digamma function. Then, we consider the contour integration involving new digamma function to obtain some new identities of Euler type sums.

Next, we give three definitions. Let \( A := \{a_k\}, -\infty < k < \infty \) be a sequence of complex numbers with \( a_k = o(k^\alpha) \ (\alpha < 1) \) if \( k \to \pm\infty \). For convenience, we let \( A_1 \) and \( A_2 \) to denote the constant sequence \( \{1^k\} \) and alternating sequence \( \{(-1)^k\} \), respectively.

**Definition 1.1** With \( A \) defined above, we define the parametric digamma function \( \Psi (-s; A) \) by

\[
\Psi (-s; A) := \frac{a_0}{s} + \sum_{k=1}^{\infty} \left( \frac{a_k}{k} - \frac{a_k}{k - s} \right).
\]

(1.3)

If \( a_n = 1 \), then the parametric digamma function \( \Psi (-s; A) \) reduces the classical digamma function \( \psi (-s) + \gamma \) which is defined as the logarithmic derivative of the well know gamma function

\[
\psi (s) := \frac{d}{ds} (\ln \Gamma (s)) = \frac{\Gamma' (s)}{\Gamma (s)} = -\gamma - \frac{1}{s} + \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k + s} \right).
\]

and it satisfies the complement formula

\[
\psi (s) - \psi (-s) = \frac{1}{s} - \pi \cot (\pi s),
\]

as well as an expansion at \( s = 0 \) that involves the zeta values:

\[
\psi (s) + \gamma = -\frac{1}{s} + \zeta (2) s - \zeta (3) s^2 + \cdots.
\]

From the definition of Riemann zeta function and Hurwitz zeta function, we know that

\[
\psi^{(n)} (1) = (-1)^{n+1} n! \zeta (n+1) \quad \text{and} \quad \psi^{(n)} (z) = (-1)^{n+1} n! \zeta (n+1; z).
\]

The Riemann zeta function and Hurwitz zeta function are defined by

\[
\zeta (s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re (s) > 1,
\]

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and
\[ \zeta(s; \alpha + 1) := \sum_{n=1}^{\infty} \frac{1}{(n + \alpha)} \] , \quad (\Re(s) > 1, \alpha \notin \mathbb{N}^- := \{-1, -2, \ldots\}).

The evaluation of the polygamma function \( \psi^{(n)}(p/q) \) at rational values of the argument can be explicitly done via a formula as given by Kölbig [13], or Andrews, Askey and Roy in terms of the polylogarithmic or other special functions. Some specific values are listed in the books [1].

**Definition 1.2** Define the cotangent function with sequence \( A \) by
\[
\pi \cot (\pi s; A) = -\frac{a_0}{s} + \Psi (-s; A) - \Psi (s; A)
\]
\[
= \frac{a_0}{s} - 2s \sum_{k=1}^{\infty} \frac{a_k}{k^2 - s^2}.
\] (1.4)

It is clear that if letting \( A = A_1 \) or \( A_2 \) in (1.4), respectively, then it become
\[
\cot (\pi s; A_1) = \cot (\pi s) , \quad \cot (\pi s; A_2) = \csc (\pi s) .
\]

We now provide notations that will be used throughout this paper.

**Definition 1.3** For nonnegative integers \( j \geq 1 \) and \( n \), we define
\[
D^{(A)}(j) := \sum_{k=1}^{\infty} \frac{a_k}{k^j} , \quad D^{(A)}(1) := 0 ,
\]
\[
E^{(A)}_n(j) := \sum_{k=1}^{n} \frac{a_{n-k}}{k^j} , \quad E^{(A)}_0(j) := 0 ,
\]
\[
\tilde{E}^{(A)}_n(j) := \sum_{k=1}^{n} \frac{a_{k-n-1}}{k^j} , \quad \tilde{E}^{(A)}_0(j) := 0 ,
\]
\[
F^{(A)}_n(j) = \begin{cases} 
\sum_{k=1}^{\infty} \frac{a_{k+n} - a_k}{k} , & j = 1 , \\
\sum_{k=1}^{\infty} \frac{a_{k+n}}{k^j} , & j > 1 , 
\end{cases}
\]
\[
\tilde{F}^{(A)}_n(j) = \begin{cases} 
\sum_{k=1}^{\infty} \frac{a_{k-n} - a_k}{k} , & j = 1 , \\
\sum_{k=1}^{\infty} \frac{a_{k-n}}{k^j} , & j > 1 , 
\end{cases}
\]
\[
G^{(A)}_n(j) := E^{(A)}_n(j) - E^{(A)}_{n-1}(j) - \frac{a_0}{n^j} , \quad G^{(A)}_0(j) := 0 ,
\]
\[
L^{(A)}_n(j) := F^{(A)}_n(j) + (-1)^j \tilde{F}^{(A)}_n(j) ,
\]
\[
M^{(A)}_n(j) := E^{(A)}_n(j) + (-1)^j F^{(A)}_n(j) ,
\]
\[
\tilde{M}^{(A)}_n(j) := \tilde{F}^{(A)}_n(j) - \tilde{E}^{(A)}_{n-1}(j) , \quad n \geq 1 ,
\]
\[
R^{(A)}_n(j) := G^{(A)}_n(j) + (-1)^j L^{(A)}_n(j) .
\]
It is clear that \( M_n^{(A)}(j) + \bar{M}_n^{(A)}(j) = R_n^{(A)}(j) + \frac{a_0}{n^j} \), and if \( A = A_1 \) and \( A_2 \), then

\[
\begin{align*}
M_n^{(A_1)}(j) &= H_n^{(j)} + (-1)^j \zeta(j), \\
\bar{M}_n^{(A_1)}(j) &= \zeta(j) - H_{n-1}^{(j)}, \\
R_n^{(A_1)}(j) &= (1 + (-1)^j) \zeta(j),
\end{align*}
\]

\[
\begin{align*}
M_n^{(A_2)}(j) &= (-1)^{n-1} \bar{H}_n^{(j)} + (-1)^j \left\{ \begin{array}{ll}
(1 - (-1)^n) \log(2), & j = 1, \\
(-1)^{n-1} \bar{\zeta}(j), & j > 1,
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\bar{M}_n^{(A_2)}(j) &= (-1)^n \bar{H}_{n-1}^{(j)} + \left\{ \begin{array}{ll}
(1 - (-1)^n) \log(2), & j = 1, \\
(-1)^{n-1} \bar{\zeta}(j), & j > 1,
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
R_n^{(A_2)}(j) &= (-1)^{n-1} (1 + (-1)^j) \bar{\zeta}(j).
\end{align*}
\]

In order to distinguish, we let \( A^{(l)} := \{a_k^{(l)}\} \), \(-\infty < k < \infty \) (\( l \) is any positive integer) be any sequences of complex numbers with \( a_k^{(l)} = o(k^\alpha) \) (\( \alpha < 1 \)) if \( k \to \pm \infty \). In above three definitions, the sequence \( A \) can be replaced by sequence \( A^{(l)} \).

The purposes of this paper are to establish some explicit relations of Euler type sums involving finite sums \( M_n^{(A^{(1)})}(j), \bar{M}_n^{(A^{(2)})}(j) \) and \( R_n^{(A^{(3)})}(j) \) by using the contour integration that involve the parametric digamma function \( \Psi(-s; A^{(j)}) \) and residue computations. Then applying the formulas obtained and letting \( A^{(j)} = A_1 \) or \( A_2 \), we can obtain many evaluations for Euler sums and hyperbolic series.

The remainder of this paper is organized as follows.

In the second section we establish several formulas of \( \Psi(-s; A) \) and \( \cot(\pi s; A) \). In the third and fourth sections we use contour integration and residue computations with the help of formulas of \( \Psi(-s; A) \) and \( \cot(\pi s; A) \) established in the second section to obtain some closed form representations of linear and quadratic Euler type sums. In the fifth section we evaluate infinite series involving \( \cot(\pi s; A) \) by using residue computations. Moreover, we provide some Ramanujan-type identities that involve hyperbolic series.

2 Several Identities Among Parametric Digamma Function

In [11], Flajolet and Salvy used residue computations on large circular contour and specific functions to obtain more independent relations for Euler sums. These functions are of the form \( \xi(s)r(s) \), where \( r(s) := 1/s^q \) and \( \xi(s) \) is a product of cotangent (or cosecant) and polygamma function. Hence, they gave the following equivalent formulas of cotangent, cosecant and polygamma function at the poles of \( \xi(s)r(s) \),

\[
\begin{align*}
\pi \cot(\pi s) &\xrightarrow{s \to n} \frac{1}{s-n} - 2 \sum_{k=1}^{\infty} \zeta(2k)(s-n)^{2k-1}, \\
\frac{\pi}{\sin(\pi s)} &\xrightarrow{s \to n} (-1)^n \left( \frac{1}{s-n} + 2 \sum_{k=1}^{\infty} \zeta(2k)(s-n)^{2k-1} \right), \\
\psi(-s) + \gamma &\xrightarrow{s \to n} \frac{1}{s-n} + H_n + \sum_{k=1}^{\infty} \left( (-1)^k \zeta_n(k+1) - \zeta(k+1) \right)(s-n)^k, \quad n \geq 0 \\
\psi(-s) + \gamma &\xrightarrow{s \to -n} H_{n-1} + \sum_{k=1}^{\infty} \left( \zeta_{n-1}(k+1) - \zeta(k+1) \right)(s+n)^k, \quad n > 0 \\
\frac{\psi^{(p-1)}(-s)}{(p-1)!} &\xrightarrow{s \to n} \frac{1}{(s-n)^p} \left( 1 + (-1)^p \sum_{i \geq p} \binom{i-1}{p-1} \left( \zeta(i) + (-1)^i \zeta_n(i) \right)(s-n)^i \right), \quad n \geq 0, \quad p > 1
\end{align*}
\]
\[
\frac{\psi(p-1)(-s)}{(p-1)!}_{s \to -n} = (-1)^p \sum_{i \geq 0} \binom{p-1+i}{p-1} (\zeta(p+i) - \zeta_{n-1}(p+i)) (s+n)^i, \ n > 0, \ p > 1.
\]

In below, we also consider the \( \xi(s) \psi(s) \) (only replace polygamma \( \psi(p-1)(-s) \) by parametric polygamma \( \Psi(p-1)(-s; A) \)) to establish some independent relations for Euler type sums. Thus, we need to obtain the Laurent expansions for parametric polygamma \( \Psi(p-1)(-s; A) \) about \( s = n \) (\( n \) is a any integer).

In this section, we will establish the explicit formulas of parametric polygamma function \( \Psi(p-1)(-s; A) \) in terms of infinite series that involve sums \( M_n^{(A)}(j) \) and \( \bar{M}^{(A)}(j) \). The results in this section are basic tools that will be used throughout this paper.

**Theorem 2.1** Let \( p \geq 1 \) and \( n \) be nonnegative integers, if \( s \in (n-1, n+1) \setminus \{n\} \), then

\[
\frac{\Psi(p-1)(-s; A)}{(p-1)!} = \frac{1}{(s-n)^p} \left\{ a_n - \sum_{j=1}^{\infty} (-1)^j \binom{j+p-2}{p-1} M_n^{(A)}(j+p-1)(s-n)^{j+p-1} \right\}. \quad (2.1)
\]

**Proof.** From the definition of \( \Psi(-s; A) \), if \( s \in (n-1, n+1) \setminus \{n\} \), then it can be rewritten in the form

\[
\Psi(-s; A) = \frac{a_0}{s} + \frac{a_n}{n} - \frac{a_n}{n-s} + \sum_{k=1}^{n-1} \left( \frac{a_k}{k} - \frac{a_k}{k-s} \right) + \sum_{k=n+1}^{\infty} \left( \frac{a_k}{k} - \frac{a_k}{k-s} \right).
\]

Using the elementary identity

\[
(1-x)^{-1} = \sum_{k=1}^{\infty} x^{k-1}, \quad x \in (-1,1)
\]

we find that

\[
\Psi(-s; A) = \frac{a_n}{s-n} - \sum_{j=1}^{\infty} \left\{ (-1)^j E_n^{(A)}(j) + E_n^{(A)}(j) \right\} (s-n)^{j-1}.
\]

Differentiating (2.2) \( p-1 \) times with respect to \( s \), we complete the proof of (2.1). \( \square \)

When \( n = 0 \) in (2.1), then for \( s \in (-1,1) \setminus \{0\} \), we have

\[
\frac{\Psi(p-1)(-s; A)}{(p-1)!} = \frac{a_0}{s^p} + (-1)^p \sum_{j=1}^{\infty} \binom{j+p-2}{j-1} D^{(A)}(j+p-1)s^{j-1}. \quad (2.3)
\]

**Theorem 2.2** Let \( p \) and \( n \) be positive integers, if \( s \in (-n-1, -n+1) \), then

\[
\frac{\Psi(p-1)(-s; A)}{(p-1)!} = (-1)^p \sum_{j=1}^{\infty} \binom{j+p-2}{p-1} \bar{M}_n^{(A)}(j+p-1)(s+n)^{j-1}. \quad (2.4)
\]
Proof. The proof is similar to the previous proof. For \( s \in (-n - 1, -n + 1) \), by a straightforward calculation we see that
\[
\Psi (-s; A) = \sum_{k=1}^{n-1} \frac{a_k}{k} + \sum_{k=1}^{n-1} \left( \frac{a_{k-n}}{k-s-n} - \frac{a_k}{k} \right) + \sum_{k=1}^{\infty} \left( \frac{a_k}{k} - \frac{a_{k-n}}{k-s-n} \right)
\]
\[
= \sum_{k=1}^{n-1} \frac{a_k}{k} + \sum_{k=1}^{n-1} \left( \frac{a_{k-n}}{k} \frac{1}{1 - \frac{s + n}{k}} - \frac{a_k}{k} \right) + \sum_{k=1}^{\infty} \left( \frac{a_k}{k} - \frac{a_{k-n}}{k} \frac{1}{1 - \frac{s + n}{k}} \right).
\]
Formally expand the summands on the right side into geometric series to deduce that
\[
\Psi (-s; A) = \sum_{j=1}^{\infty} \left\{ F_n(j) - F_n(A)(j) \right\} (s + n)^{j-1}
\]
\[
= - \sum_{j=1}^{\infty} M_n(j)(s - n)^{j-1}. \tag{2.5}
\]
If differentiating (2.5) \( p - 1 \) times with respect to \( s \), we obtain (2.4) to complete the proof. \( \square \)

**Theorem 2.3** With \( \cot(\pi s; A) \) defined above, if \( s \in (n - 1, n + 1) \) \( \backslash \{n\} \) \( \{n\} \) \( \mathbb{N} \), then
\[
\pi \cot(\pi s; A) = \frac{a_{\lfloor n \rfloor}}{s - n} - \sum_{j=1}^{\infty} (-\sigma_n)^j R^{(A)}_n(j)(s - n)^{j-1}, \tag{2.6}
\]
where \( \sigma_n \) is defined by the symbol of \( n \), namely,
\[
\sigma_n := \begin{cases} 1, & n \geq 0 \\ -1, & n < 0. \end{cases}
\]
Proof. Return to (1.4), use (2.1) and (2.4) to arrive at, if \( s \in (n - 1, n + 1) \) \( \backslash \{n\} \),
\[
\pi \cot(\pi s; A) = \frac{a_{\lfloor n \rfloor}}{s - n} - \sum_{j=1}^{\infty} \left\{ (-1)^j G^{(A)}_n(j + 1) + L^{(A)}_n(j + 1) \right\} (s - n)^{j-1},
\]
if \( s \in (-n - 1, -n + 1) \) \( \backslash \{-n\} \)
\[
\pi \cot(\pi s; A) = \frac{a_{\lfloor n \rfloor}}{s + n} - \sum_{j=1}^{\infty} \left\{ G^{(A)}_n(j + 1) + (-1)^j L^{(A)}_n(j + 1) \right\} (s + n)^{j-1},
\]
Thus, summing these two contributions yields the desired evaluation. \( \square \)

**Corollary 2.4** ([11]) Let \( n \) be a nonnegative integer, then the following formulas hold:
\[
\tilde{\psi} (-s) = (-1)^n \left\{ \frac{1}{n - s} + \sum_{k=0}^{\infty} \left( (-1)^k H^{(k+1)}_n - \tilde{\zeta}(k + 1) \right) (s - n)^k \right\}, \quad s \in (n - 1, n + 1) \backslash \{n\},
\]
\[
\tilde{\psi} (-s) = (-1)^n \sum_{k=0}^{\infty} \left( H^{(k+1)}_{n-1} - \tilde{\zeta}(k + 1) \right) (s + n)^k, \quad s \in (-n - 1, -n + 1), \quad n > 0,
\]
where \( \tilde{\psi}(s) \) denotes the modified digamma function, which is defined by
\[
\tilde{\psi} (s) := \sum_{k=0}^{\infty} \frac{(-1)^k}{s + k} = \frac{1}{2} \psi \left( \frac{s + 1}{2} \right) - \frac{1}{2} \psi \left( \frac{s}{2} \right), \quad (s \in \mathbb{C}).
\]
Proof. Corollary 2.4 follows immediately from (2.2) and (2.5) with \( A = A_2 \). \( \square \)
3 Linear and Quadratic Euler Type Sums

We first state a lemma that will subsequently be used in our proofs of main results.

Flajolet and Salvy [11] defined a kernel function $\xi(s)$ by the two requirements: 1. $\xi(s)$ is meromorphic in the whole complex plane. 2. $\xi(s)$ satisfies $\xi(s) = o(s)$ over an infinite collection of circles $|s| = \rho_k$ with $\rho_k \to \infty$. Applying these two conditions of kernel function $\xi(s)$, they shown the following residue theorem.

**Lemma 3.1** ([11]) Let $\xi(s)$ be a kernel function and let $r(s)$ be a rational function which is $O(s^{-2})$ at infinity. Then

$$
\sum_{\alpha \in O} \text{Res}[r(s)\xi(s), s = \alpha] + \sum_{\beta \in S} \text{Res}[r(s)\xi(s), s = \beta] = 0.
$$

where $S$ is the set of poles of $r(s)$ and $O$ is the set of poles of $\xi(s)$ that are not poles $r(s)$. Here $\text{Res}[r(s), s = \alpha]$ denotes the residue of $r(s)$ at $s = \alpha$.

Flajolet and Salvy proved every linear sum $S_{p,q}$ whose weight $p + q$ is odd is expressible as a polynomial in zeta values by applying the kernel function

$$
\frac{1}{2} \pi \cot(\pi s) \frac{\psi(p-1)(-s)}{(p-1)!}
$$

to the base function $r(s) = s^{-q}$. Elaborating on Euler’s work, Nielsen [16] also proved this result by a method based on partial fraction expansions. Let $B := \{b_k\}, -\infty < k < \infty$ be a sequence of complex numbers with $b_k = o(k^\beta)$ ($\beta < 1$) if $k \to \pm\infty$. Replacing $\cot(\pi s)\psi(p-1)(-s)$ by $\cot(\pi s; A)\Psi(p-1)(-s; B)$, we can get the following theorem.

**Theorem 3.2** For positive integers $p$ and $q > 1$, 

$$
(-1)^{p+q} \sum_{n=1}^{\infty} \frac{M^{(B)}_n(p)}{n^q}a_n + \sum_{n=1}^{\infty} \frac{M^{(B)}_n(p)}{n^q}a_n
$$

$$
= (-1)^p \sum_{j=1}^{p} \left( \begin{array}{c} p + q - j - 1 \\ q - 1 \end{array} \right) \sum_{n=1}^{\infty} \frac{R^{(A)}_n(j)}{n^{p+q-j}}b_n
$$

$$
+ (-1)^{p+q} \sum_{j=1}^{[q/2]} \left( \begin{array}{c} p + q - 2j - 1 \\ p - 1 \end{array} \right) D^{(A)}(2j)D^{(B)}(p + q - 2j)
$$

$$
+ b_0(1 + (-1)^{p+q})D^{(A)}(p + q) - (-1)^pa_0\left( \begin{array}{c} p + q - 1 \\ q \end{array} \right)D^{(B)}(p + q)
$$

$$
- (-1)^p\left( \begin{array}{c} p + q - 1 \\ p \end{array} \right) D^{(AB)}(p + q),
$$

(3.2)

where $D^{(AB)}(q)$ is defined by

$$
D^{(AB)}(q) := \sum_{n=1}^{\infty} \frac{a_n b_n}{n^q}.
$$
Proof. In the context of this paper, the theorem results from applying the kernel function

\[ \pi \cot (\pi s; A) \frac{\Psi^{(p-1)} (-s; B)}{(p-1)!} \]

to the base function \( r(s) = s^{-q} \). Namely, we need to compute the residue of the function

\[ f_1(s; A, B) := \pi \cot (\pi s; A) \frac{\Psi^{(p-1)} (-s; B)}{(p-1)! s^q}. \]

The only singularities are poles at the integers. At a negative integer \(-n\) the pole is simple and the residue is

\[ \text{Res} [f_1(s; A, B), s = -n] = (-1)^{p+q} \frac{M_n^{(B)}(p)}{n^q} a_n. \]

At a positive integer \( n \), the pole has order \( p+1 \) and the residue is

\[ \text{Res}[f_1(s; A, B), s = n] = (-1)^p \left( \frac{p+q-1}{p} \right) a_n b_n \frac{M_n^{(B)}(p)}{n^{p+q}} + \left( \frac{p+q-1}{q-1} \right) R_n^{(A)}(j). \]

Finally the residue of the pole of order \( p+q+1 \) at 0 is found to be

\[ \text{Res}[f_1(s; A, B), s = 0] = (-1)^p \left( \frac{p+q-1}{q} \right) a_0 D^{(B)}(p+q) - b_0 (1 + (-1)^{p+q}) D^{(A)}(p+q) \]

\[ - (-1)^p 2 \sum_{j=1}^{[q/2]} \left( \frac{p+q-2j-1}{p-1} \right) D^{(A)}(2j) D^{(B)}(p+q-2j). \]

Summing these three contributions yields the statement of the theorem. □

Putting \( p = q = 2, A = A_1 \) and \( b_n = 1/2^{[n]} \), a simple example is as follows:

\[ \sum_{n=1}^{\infty} \frac{1}{n^{2+2n}} \sum_{k=1}^{n} \frac{2^k}{k^2} = \frac{51}{16} \zeta(4) - 3 \text{Li}_4 \left( \frac{1}{2} \right) - \frac{3}{4} \zeta(2) \ln^2(2) - \frac{1}{8} \ln^4(2). \]

Here \( \text{Li}_s(x) \) denotes the polylogarithm function which is defined by

\[ \text{Li}_s(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^s} \quad (\Re(s) \geq 1, \ x \in [-1, 1]). \]

If setting \( A, B \in \{A_1, A_2\} \) in Theorem 3.2, then we deduce these well-known results of (alternating) linear Euler sums.

Corollary 3.3 ([11]) For an odd weight \( m = p + q \ (q \geq 2) \), the four linear Euler sums are
reducible to zeta values,

\[ S_{p,q} := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} \]

\[ = \frac{1}{2} \zeta(m) + \frac{1}{2} \frac{(-1)^p}{\zeta(p) \zeta(q)} \]

\[ + (-1)^p \sum_{k=0}^{[p/2]} \binom{m-2k-1}{q-1} \zeta(2k) \zeta(m-2k) \]

\[ + (-1)^p \sum_{k=0}^{[q/2]} \binom{m-2k-1}{p-1} \zeta(2k) \zeta(m-2k), \]

\[ S_{\bar{p},\bar{q}} := \sum_{n=1}^{\infty} \frac{\bar{H}_n^{(p)}}{n^q} (-1)^{n-1} \]

\[ = \frac{1}{2} \tilde{\zeta}(m) + \frac{1}{2} \frac{(-1)^p}{\tilde{\zeta}(p) \tilde{\zeta}(q)} \]

\[ - (-1)^p \sum_{k=0}^{[p/2]} \binom{m-2k-1}{q-1} \tilde{\zeta}(2k) \tilde{\zeta}(m-2k) \]

\[ - (-1)^p \sum_{k=0}^{[q/2]} \binom{m-2k-1}{p-1} \tilde{\zeta}(2k) \tilde{\zeta}(m-2k), \]

\[ S_{\bar{p},\bar{q}} := \sum_{n=1}^{\infty} \frac{\bar{H}_n^{(p)}}{n^q} (-1)^{n-1} \]

\[ = \frac{1}{2} \tilde{\zeta}(m) + \frac{1}{2} \frac{(-1)^p}{\tilde{\zeta}(p) \tilde{\zeta}(q)} \]

\[ - (-1)^p \sum_{k=0}^{[p/2]} \binom{m-2k-1}{q-1} \tilde{\zeta}(2k) \tilde{\zeta}(m-2k) \]

\[ + (-1)^p \sum_{k=0}^{[q/2]} \binom{m-2k-1}{p-1} \tilde{\zeta}(2k) \tilde{\zeta}(m-2k), \]

where \( \zeta(1) \) should be interpreted as 0 wherever it occurs, and \( \zeta(0) = -1/2, \tilde{\zeta}(0) = 1/2. \)

Obviously, when \( A, B \in \{A_1, A_2\} \) in Theorem 3.2, we see that for even weights, four modified
forms of the identity hold, but without any (alternating) linear Euler sum occurring. This gives back well-known nonlinear relations between (alternating) zeta values at even arguments. Flajolet and Salvy [11] applied the kernels \((\psi^{(j)}(-s))^2\) to \(s^q\) to yield some further relations of even weights. Please see their article for further reference. Next, in a same way, we establish a ‘duality’ sum formula of Euler type sums.

**Theorem 3.4** For positive integers \(m\) and \(p\),

\[
(-1)^m \sum_{i+j=m-1, i,j \geq 0} \binom{p+i-1}{i} \binom{q+j-1}{j} \sum_{n=1}^{\infty} \frac{M_n^{(B)}(p+i)}{n^{q+j}} a_n \\
+ (-1)^p \sum_{i+j=p-1, i,j \geq 0} \binom{m+i-1}{i} \binom{q+j-1}{j} \sum_{n=1}^{\infty} \frac{M_n^{(A)}(m+i)}{n^{q+j}} b_n \\
= (-1)^{p+m-1} \binom{p+q+m-2}{q-1} D^{(AB)}(p+q+m-1) \\
+ a_0(-1)^p \binom{p+q+m-2}{p-1} D^{(B)}(p+q+m-1) \\
+ b_0(-1)^m \binom{p+q+m-2}{m-1} D^{(A)}(p+q+m-1) \\
+ (-1)^{m+p} \sum_{j_1+j_2=q+1, j_1,j_2 \geq 1} \binom{j_1+m-2}{j_1-1} \binom{j_2+p-2}{j_2-1} D^{(A)}(j_1+m-1) D^{(B)}(j_2+p-1). \quad (3.7)
\]

**Proof.** Consider

\[
f_2(s; A, B) := \frac{\Psi^{(m-1)}(-s; A)\Psi^{(p-1)}(-s; B)}{(m-1)!(p-1)!s^q},
\]

which has poles of order \(p + m\) at \(s = n\) \((n \in \mathbb{N})\). With the help of Theorem 2.1, we compute the residues

\[
\text{Res}[f_2(s; A, B), s = n] = (-1)^{p+m-1} \binom{p+q+m-2}{q-1} a_n b_n \\
- (-1)^m \sum_{i+j=m-1, i,j \geq 0} \binom{p+i-1}{i} \binom{q+j-1}{j} \frac{M_n^{(B)}(p+i)}{n^{q+j}} a_n \\
- (-1)^p \sum_{i+j=p-1, i,j \geq 0} \binom{m+i-1}{i} \binom{q+j-1}{j} \frac{M_n^{(A)}(m+i)}{n^{q+j}} b_n.
\]

Clearly, \(f_2(s; A, B)\) also has a pole of order \(p + q + m\) at \(s = 0\). Using (2.3), we find that

\[
\text{Res}[f_2(s; A, B), s = 0] \\
= a_0(-1)^p \binom{p+q+m-2}{p-1} D^{(B)}(p+q+m-1) \\
+ b_0(-1)^m \binom{p+q+m-2}{m-1} D^{(A)}(p+q+m-1)
\]
\[ (+1)^{m+p} \sum_{j_1+j_2=q+1, j_1,j_2 \geq 1} \left( \frac{j_1+m-2}{j_1-1} \right) \left( \frac{j_2+p-2}{j_2-1} \right) D(A)(j_1+m-1)D(B)(j_2+p-1). \]

Summing these two contributions, we thus immediately deduce (3.7) to complete the proof. \( \square \)

Hence, setting \( A, B \in \{A_1, A_2\} \) in Theorem 3.4 yields many linear relations between (alternating) linear Euler sums and polynomials in zeta values. Some illustrate examples please see [8,9,11].

**Corollary 3.5** For integer \( q > 1 \),

\[
\frac{3}{2} (S_{1^2,q} - S_{2,q}) = (q + 1) S_{1,q+1} - \sum_{j_1+j_2=q-1, j_1,j_2 \geq 1} S_{1,j_1+1} \zeta(j_2+1),
\]

\[
S_{1^3,q} - 3S_{1^2,q} = qS_{1^2,q+1} - \sum_{j_1+j_2=q-1, j_1,j_2 \geq 1} S_{1,j_1+1}S_{1,j_2+1}.
\]

**Proof.** Corollary 3.5 follows immediately from Theorem 3.4 by setting \( m = p = 1 \), \((a_k,b_k) = (H_k,1)\) and \((a_k,b_k) = (H_k,H_k)\).

Further, by applying the kernels

\[
\pi \cot(\pi s) (s-1)\psi^{-1}(-s) \psi^{-1}(-s)
\]

\[
(m-1)! (p-1)!
\]

to \( s^{-q} \), Flajolet and Salvy gave the explicit formulas of quadratic Euler sums via linear Euler sums and zeta values, see [11, Theorem 4.2]. Now, we evaluate more general relation for quadratic Euler type sums in the same manner as in the above. Let \( C := \{c_k\}, -\infty < k < \infty \) be a sequence of complex numbers with \( c_k = o(k^\lambda) \) \( (\lambda < 1) \) if \( k \rightarrow \pm\infty \), and let

\[
D^{(ABC)}(q) := \sum_{n=1}^{\infty} \frac{a_n b_n c_n}{n^q}.
\]

**Theorem 3.6** Let \( m, p \) and \( q > 1 \) be positive integers with \( A, B \) and \( C \) defined above, we have

\[
(-1)^{p+q+m} \sum_{j=1}^{\infty} \frac{M_n^{(B)}(m)M_n^{(C)}(p)}{n^q} a_n + \sum_{n=1}^{\infty} \frac{M_n^{(B)}(m)M_n^{(C)}(p)}{n^q} a_n
\]

\[
+ (-1)^{p+q+m} \binom{p+q+m-1}{q-1} D^{(ABC)}(p+q+m)
\]

\[
+ (-1)^m \sum_{j=1}^{\infty} \binom{j+p-2}{p-1} \binom{m+q-1}{q-1} \sum_{n=1}^{\infty} \frac{M_n^{(C)}(j+p-1)}{n^{m+q-j+1}} a_n b_n
\]

\[
+ (-1)^p \sum_{j=1}^{\infty} \binom{j+m-2}{m-1} \binom{p+q-1}{q-1} \sum_{n=1}^{\infty} \frac{M_n^{(B)}(j+m-1)}{n^{p+q-j+1}} a_n c_n
\]

\[
- (-1)^{p+q} \sum_{j=1}^{\infty} \binom{p+q+m-j-1}{q-1} \sum_{n=1}^{\infty} \frac{F_n^{(A)}(j)}{n^{p+q+m-j+1}} b_n c_n
\]

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\[-(-1)^m \sum_{j_1+j_2 \leq m+1, j_1, j_2 \geq 1} \binom{m+q-j_1-j_2}{m+q-j_1-j_2} \binom{j_2+p-2}{j_2+1} b_n \sum_{n=1}^{\infty} R_n^{(A)}(j_1) M_n^{C}(j_2+p-1) n^{m+q-j_1-j_2+1} \]

\[-(-1)^p \sum_{j_1+j_2 \leq p+1, j_1, j_2 \geq 1} \binom{p+q-j_1-j_2}{m-1} \binom{j_2+m-2}{j_2+1} c_n \sum_{n=1}^{\infty} R_n^{(A)}(j_1) M_n^{B}(j_2+m-1) n^{p+q-j_1-j_2+1} \]

\[+ \text{Res}[f_3(s; A, B, C), s = 0] = 0, \quad (3.10)\]

where

\[
\text{Res}[f_3(s; A, B, C), s = 0] = (-1)^p a_0 b_0 \binom{p+q+m-1}{p+q+m}(D^C(p+q+m) \\
+ (-1)^m a_0 c_0 \binom{p+q+m-1}{m-1} D^B(p+q+m) \\
- b_0 c_0 (1 + (-1)^{p+q+m}) D^A(p+q+m) \\
- (-1)^p b_0 \sum_{j_1+j_2=m+q+1, j_1, j_2 \geq 1} \binom{j_2+p-2}{j_2+1} D^A(2j_1) D^C(j_2+p+1) \\
- (-1)^m c_0 \sum_{j_1+j_2=p+q+1, j_1, j_2 \geq 1} \binom{j_2+m-2}{j_2+1} D^A(2j_1) D^B(j_2+m-1) \\
+ (-1)^p a_0 \sum_{j_1+j_2=q+2, j_1, j_2 \geq 1} \binom{j_1+m-2}{j_1+1} \binom{j_2+p-2}{j_2+1} D^B(j_1+m-1) D^C(j_2+p+1) \\
- (-1)^m b_2 \sum_{j_1+j_2=j_3+q+2, j_1, j_2, j_3 \geq 1} \binom{j_2+m-2}{j_2+1} \binom{j_3+p-2}{j_3+1} \\
\times D^A(2j_1) D^B(j_2+m-1) D^C(j_3+p+1) \quad (3.11)\]

**Proof.** Consider

\[
f_3(s; A, B, C) := \frac{\pi \cot(\pi s; A) \Psi^{(m-1)}(-s; B) \Psi^{(p-1)}(-s; C)}{(m-1)! (p-1)! s^q},
\]

which has simple poles at \( s = -n \ (n \in \mathbb{N}) \), with residues

\[
\text{Res}[f_3(s; A, B, C), s = -n] = (-1)^{p+q+m} \frac{M_n^{B}(m) M_n^{C}(p) n^q}{n^q} a_n.
\]

Clearly, \( s = n \ (n \in \mathbb{N}) \) are the poles of order \( m+p+1 \) of \( f_3(s; A, B, C) \), using Theorems 2.1 and 2.3, we find that

\[
\text{Res}[f_3(s; A, B, C), s = n] = \frac{M_n^{B}(m) M_n^{C}(p) a_n}{n^q} + (-1)^{p+m} \frac{a_n b_1 c_n}{n^{p+q+m}}.
\]

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yields the desired description. For arbitrary degree. We begin with some basic notations. For positive integer sequences \( j := (j_1, j_2, \ldots, j_r) \) and \( p := (p_1, p_2, \ldots, p_r) \), we let

\[
|j| := j_1 + j_2 + \cdots + j_r, \quad p := p_1 + p_2 + \cdots + p_r,
\]

\[
C_r(j; p) := \left(\frac{j_1 + p_1 - 2}{p_1 - 1}\right) \left(\frac{j_2 + p_2 - 2}{p_2 - 1}\right) \cdots \left(\frac{j_r + p_r - 2}{p_r - 1}\right), \quad C_0(j; p) := 1.
\]

Moreover, \( f_3(s; A, B, C) \) also has a pole of order \( p + q + m + 1 \) at \( s = 0 \), using (2.3) and Theorem 2.3, we arrive at (3.11). Thus, the desired evaluation holds.

Hence, if setting \( A = B = C = A_1 \) in Theorem 3.6, then it becomes the Theorem 4.2 of Flajolet and Salvy [11]. Further, we obtain the following description.

**Corollary 3.7** If \( p + q + m \) is even, and \( q > 1, m, p \) are positive integers, then the (alternating) quadratic Euler sums

\[
\sum_{n=1}^{\infty} \frac{\tilde{H}_n^{(m)} H_{\nu}^{(p)}}{n^q}, \sum_{n=1}^{\infty} \frac{\tilde{H}_n^{(m)} \tilde{H}_n^{(p)}}{n^q}, \sum_{n=1}^{\infty} \frac{\tilde{H}_n^{(m)} \tilde{H}_n^{(p)}}{n^q},
\]

\[
\sum_{n=1}^{\infty} \frac{\tilde{H}_n^{(m)} H_{\nu}^{(p)}}{n^q}(-1)^{n-1}, \sum_{n=1}^{\infty} \frac{\tilde{H}_n^{(m)} \tilde{H}_n^{(p)}}{n^q}(-1)^{n-1}, \sum_{n=1}^{\infty} \frac{\tilde{H}_n^{(m)} \tilde{H}_n^{(p)}}{n^q}(-1)^{n-1}
\]

are reducible to (alternating) linear Euler sums.

**Proof.** Letting \( A, B, C \in \{A_1, A_2\} \) in Theorem 3.6 yields the desired description.

It is obvious that (alternating) linear Euler sums reduce to (alternating) zeta values in the case of an odd weight, while (alternating) quadratic Euler sums reduce to (alternating) linear Euler sums in the case of an even weight. Flajolet and Salvy also shown a result to the effect that such reductions of order are general, but not explicit formulas, for detail see [11, Theorem 5.3].

### 4 Formulas for General Euler Type Sums

As with previous work, in this section, we prove some explicit relations of Euler type sums for arbitrary degree. We begin with some basic notations. For positive integer sequences \( j := (j_1, j_2, \ldots, j_r) \) and \( p := (p_1, p_2, \ldots, p_r) \), we let

\[
|j| := j_1 + j_2 + \cdots + j_r, \quad p := p_1 + p_2 + \cdots + p_r,
\]

\[
C_r(j; p) := \left(\frac{j_1 + p_1 - 2}{p_1 - 1}\right) \left(\frac{j_2 + p_2 - 2}{p_2 - 1}\right) \cdots \left(\frac{j_r + p_r - 2}{p_r - 1}\right), \quad C_0(j; p) := 1.
\]
Let $S_r$ be the symmetric group of all the permutations on $r$ symbols. For a permutation $\sigma \in S_r$, we let

\[ |j_\sigma| := j_\sigma(1) + j_\sigma(2) + \cdots + j_\sigma(l), \quad |p_\sigma| := p_{\sigma(1)} + p_{\sigma(2)} + \cdots + p_{\sigma(l)}, \]

\[ C_l(j_\sigma; p_\sigma) := \left(\frac{j_\sigma(1) + p_{\sigma(1)} - 2}{p_{\sigma(1)} - 1}\right) \cdots \left(\frac{j_\sigma(l) + p_{\sigma(l)} - 2}{p_{\sigma(l)} - 1}\right), \quad C_0(j_\sigma; p_\sigma) := 1, \]

where $l = 1, 2, \ldots, r$ and $|j_\sigma|_0 = |p_\sigma|_0 := 0$.

We are now ready to state general evaluations in closed form for general Euler type sums.

**Definition 4.1** With $A^{(l)}$ defined above, we define

\[ f(s; A) := \frac{\Psi_{p-1}(-s; A^{(1)})\Psi_{p-2}(-s; A^{(2)}) \cdots \Psi_{p-1}(-s; A^{(r)})}{(p_1 - 1)! (p_2 - 1)! \cdots (p_r - 1)!}. \tag{4.1} \]

From Theorems 2.1 and 2.2, we find that if $s \in (n - 1, n + 1) \setminus \{n\}$, $(n \geq 0)$, then

\[ f(s; A) = \sum_{l=0}^r \sum_{\sigma \in S_r} \sum_{j_\sigma(1) \cdots j_\sigma(l) = 1} \infty (-1)^{|j_\sigma|} C_l(j_\sigma; p_\sigma) \prod_{i=1}^l M_n^{(A^{(i))})}(j_\sigma(i) + p_{\sigma(i)} - 1) \prod_{k=l+1}^r a_n^{(\sigma(k))} \times (s - n)^{|j_\sigma| + |p_\sigma| - l - |p|}, \tag{4.2} \]

If $s \in (-n - 1, -n + 1)$, $(n \geq 1)$, then

\[ f(s; A) = (-1)^{|p|} \sum_{j_1, \ldots, j_{r-1}} \infty C_r(j; p) \prod_{i=1}^r M_n^{(A^{(i))})}(j_i + p - 1)(n + s)^{|j| - r}. \tag{4.3} \]

**Theorem 4.1** Let $p_1, p_2, \ldots, p_r, q$ and $r$ be positive integers. For a permutation $\sigma \in S_r$, we have

\[ \sum_{l=0}^{r-1} \sum_{\sigma \in S_r} \sum_{|j_\sigma| \leq w_\sigma(p, l)} (-1)^{|p| - |p_\sigma| + 1} C_l(j_\sigma; p_\sigma) \left(\frac{w_\sigma(p, l + q - |j_\sigma| - 1)}{q - 1}\right) \times \sum_{n=1}^\infty \prod_{i=1}^l M_n^{(A^{(i))})}(j_\sigma(i) + p_{\sigma(i)} - 1) \prod_{k=l+1}^r a_n^{(\sigma(k))} \]

\[ + \sum_{l=1}^r \sum_{\sigma \in S_r} \sum_{|j_\sigma| = q + w_\sigma(p, l)} (-1)^{|p_\sigma|} C_l(j_\sigma; p_\sigma) \prod_{i=1}^l D_n^{(A^{(i))})}(j_\sigma(i) + p_{\sigma(i)} - 1) \prod_{k=l+1}^r a_0^{(\sigma(k))} \]

\[ = 0, \tag{4.4} \]

where $w_\sigma(p, l) := |p| - |p_\sigma| + l - 1$.

**Proof.** Applying the kernel $f(s; A)$ to the base function $r(s) = 1/s^q$, and using (4.2) and (4.3), we achieve the desired expansion after a rather tedious computation. \hfill \Box

If setting $r = 3, p_1 = p_2 = p_3 = 1$ and $a_k^{(1)} = a_k^{(2)} = a_k^{(3)} = 1$, $(-1)^k, H_k$ in Theorem 4.1, we get the following examples.
Example 4.1 For integer $q > 1$,

$$S_{1^2,q} - S_{2,q} = p S_{1,q+1} - \frac{(q - 2) (q + 3)}{6} \zeta (q + 2) + (2) \zeta (q) - \sum_{j_1 + j_2 = q, j_1, j_2 \geq 1} \zeta (j_1 + 1) \zeta (j_2 + 1)$$

$$+ \frac{1}{3} \sum_{j_1 + j_2 = q - 1, j_1, j_2 \geq 1} \zeta (j_1 + 1) \zeta (j_2 + 1) \zeta (j_3 + 1), \tag{4.5}$$

$$S_{1^2,q} = - q S_{1,q+1} - S_{2,q} - \frac{(q - 2) (q + 3)}{6} \zeta (q + 2) + q \ln (2) \left( \zeta (q + 1) + \zeta (q + 1) \right)$$

$$- 2 \ln^2 (2) \left( \zeta (q) + \zeta (q) \right) + 2 \ln (2) \left( S_{1,q} + S_{1,q} \right) - \frac{1}{2} \zeta (2) \zeta (q)$$

$$+ \frac{1}{3} \sum_{j_1 + j_2 = q - 1, j_1, j_2 \geq 1} \zeta (j_1 + 1) \zeta (j_2 + 1) \zeta (j_3 + 1)$$

$$+ \sum_{j_1 + j_2 = q, j_1, j_2 \geq 1} \zeta (j_1 + 1) \zeta (j_2 + 1), \tag{4.6}$$

$$\frac{q (q + 1)}{6} S_{1^3,q+2} - \frac{q}{2} \left( S_{1^3,q+1} - 3 S_{1^2,q+1} \right) - 2 \zeta (3) S_{1^2,q} - \zeta (2) S_{1^3,q} - \frac{5}{2} S_{1^2,q} + 2 S_{1^2,q}$$

$$+ \frac{1}{4} \left( S_{1^2,q} + 9 S_{1^2,q} \right) - \frac{1}{3} \sum_{j_1, j_1 + j_2 = q - 1, j_1, j_2 \geq 1} S_{1,j_1+1} S_{1,j_2+1} S_{1,j_3+1} = 0. \tag{4.7}$$

Theorem 4.2 Let $p_1, p_2, \ldots, p_r, q$ and $r$ be positive integers. For a permutation $\sigma \in S_r$, we have

$$(-1)^{|\mathbf{p}|+q} \prod_{n=1}^{\infty} \sum_{l=0}^{r-1} \sum_{\sigma \in S_r} \sum_{l \leq w_{\sigma}^{(i)} (\mathbf{p}, l)} \left( -1 \right)^{|\mathbf{p}| - |\mathbf{p}_{\sigma}|} C_l (j_{\sigma} ; \mathbf{p}_{\sigma}) \left( w_{\sigma}^{(i)} (\mathbf{p}, l) + q - \left| j_{\sigma} - l - 1 \right| \right)$$

$$\times \sum_{n=1}^{\infty} \prod_{i=1}^{l} M_n^{A (\sigma (i))} (j_{\sigma (i)} + p_{\sigma (i)} - 1) \prod_{k=l+1}^{r} a_{\sigma (k)}^{(i)} n_{w_{\sigma}^{(i)} (\mathbf{p}, l) + q - \left| j_{\sigma} \right| - l - 1}$$

$$- \sum_{l=0}^{r-1} \sum_{\sigma \in S_r} \sum_{l \leq w_{\sigma}^{(i)} (\mathbf{p}, l)} \left( -1 \right)^{|\mathbf{p}| - |\mathbf{p}_{\sigma}|} C_l (j_{\sigma} ; \mathbf{p}_{\sigma}) \left( w_{\sigma}^{(i)} (\mathbf{p}, l) + q - \left| j_{\sigma} - l - 1 \right| \right)$$

$$\times \sum_{n=1}^{\infty} \prod_{i=1}^{l} M_n^{A (\sigma (i))} (j_{\sigma (i)} + p_{\sigma (i)} - 1) \prod_{k=l+1}^{r} a_{\sigma (k)}^{(i)} n_{w_{\sigma}^{(i)} (\mathbf{p}, l) + q - \left| j_{\sigma} \right| - l - 1}$$

$$+ \sum_{l=1}^{r} \sum_{\sigma \in S_r} \sum_{l \leq w_{\sigma}^{(i)} (\mathbf{p}, l) + q} \left( -1 \right)^{|\mathbf{p}_{\sigma}|} C_l (j_{\sigma} ; \mathbf{p}_{\sigma})$$

$$\times \prod_{i=1}^{l} D (A (\sigma (i))) (j_{\sigma (i)} + p_{\sigma (i)} - 1) \prod_{k=l+1}^{r} a_{\sigma (k)}^{(i)}$$
\[-2 \sum_{l=0}^{r} \sum_{\sigma \in S_r} \sum_{|j_\sigma|+2t = w_\sigma'(p,l)+q} (-1)^{|p_\sigma|} C_l(j_\sigma; p_\sigma) \times D^{(A)}(2t) \prod_{i=1}^{l} D^{(A^{(\sigma(i))})}(j_{\sigma(i)} + p_{\sigma(i)} - 1) \prod_{k=l+1}^{r} a_0^{(\sigma(k))} \]

where \( w_\sigma'(p,l) := |p| - |p_\sigma| + l \).

**Proof.** Applying the kernel \( \pi \cot(\pi s; A) f(s; A) \) to the base function \( r(s) = 1/s^q \) with the help of identities (4.2) and (4.3), we arrive at the desired evaluation. \( \square \)

It is clear that Theorems 3.4 and 3.6 are immediate corollaries of Theorems 4.1 and 4.2 with \( r = 2, p_1 = m, p_2 = p \) and \( A^{(1)} = A, A^{(2)} = B \).

Further, letting all \( A^{(l)} \in \{A_1, A_2\} \) \( (l = 1, 2, \ldots, r) \), we can get the following corollary.

**Corollary 4.3** (alternating) Euler sum \( S_{e_{12}\ldots e_r \sigma} \) \( (e_j \in \{p_j, \bar{p}_j\}, g \in \{q, \bar{q}\}) \) reduces to a combination of sums of lower orders whenever the weight \( p_1 + p_2 + \cdots + p_r + q \) and the order \( r \) are of the same parity.

It should be emphasized that every (alternating) Euler sum of weight \( w \) and degree \( n \) is clearly a \( \mathbb{Q} \)-linear combination of (alternating) multiple zeta values of weight \( w \) and depth less than or equal to \( n + 1 \) (explicit formulas was given by Xu and Wang [22]). The multiple zeta values are defined by ([12, 23, 24])

\[
\zeta(k) \equiv \zeta(k_1, \ldots, k_n) := \sum_{m_1 > \cdots > m_n \geq 1} \prod_{j=1}^{n} m_j^{-|k_j|} \mathrm{sgn}(k_j)^{m_j},
\]

where for convergence \( |k_1| + \cdots + |k_j| > j \) for \( j = 1, 2, \ldots, n \), and

\[
\mathrm{sgn}(k_j) := \begin{cases} 
1 & \text{if } k_j > 0, \\
-1 & \text{if } k_j < 0.
\end{cases}
\]

Here, we call \( l(k) := n \) and \( |k| := \sum_{j=1}^{n} |k_j| \) the depth and the weight of multiple zeta values, respectively. Rational relations among multiple zeta values are known through weight (sum of the indices) 22 and tabulated in the Multiple Zeta Value Data Mine (henceforth MZVDM) [7]. Rational relations among alternating multiple zeta values are also tabulated in the MZVDM (through weight 12 [7]). With the help of results of [7], the author and Wang [22] have developed the Maple package to evaluate the non-alternating Euler sums of weight 2 \( \leq w \leq 16 \) and the alternating Euler sums of weight 1 \( \leq w \leq 6 \).

We record two examples to illustrate Theorem 4.2 and Corollary 4.3.

**Example 4.2** For integer \( p > 1 \),

\[
\left( (-1)^{p-1} - 1 \right) S_{13, \bar{p}} - 6 S_{12, \bar{p}} + 3 \ln(2) (1 + (-1)^p) S_{12, \bar{p}} + 6 \ln(2) S_{2, \bar{p}} + 3 \left( \zeta(2) - (1 + (-1)^p) \ln^2(2) \right) S_{1, \bar{p}} - 3 S_{3, \bar{p}} + (-1)^p \ln^3(2) \zeta(p) - 3p S_{12, \bar{p}+1} - 3p S_{2, \bar{p}+1}
\]
\[+ 6 p \ln(2) S_{1, p+1} - p \left(3 \ln^2(2) - \frac{1}{2} \zeta(2)\right) \zeta(p + 1) - 3 \left(p \frac{p + 1}{2} + (-1)^p\right) S_{1, p+2} + 3 \left(p \frac{p + 1}{2} + (-1)^p\right) \ln(2) \zeta(p + 2) + \left(\ln^3(2) - 3 \ln(2) \zeta(2) + \frac{9}{4} \zeta(3)\right) \zeta(p) + (-1)^p \zeta(p + 3) \frac{p(p + 1)(p + 2)}{6} \zeta(p + 3) + 3(-1)^p \left(S_{1, p+1} - 2 \ln(2) S_{1, p+1} + \ln^2(2) \zeta(p + 1)\right) + \text{Res} \left[g_1(s), s = 0\right] = 0,\]

where

\[
\text{Res} \left[g_1(s), s = 0\right] = -3 \zeta(p + 3) - 3 \sum_{j_1 + j_2 = p+1, j_1, j_2 \geq 0} \zeta(j_1 + 1) \zeta(j_2 + 1) - \sum_{j_1 + j_2 + j_3 = p, j_1, j_2, j_3 \geq 0} \zeta(j_1 + 1) \zeta(j_2 + 1) \zeta(j_3 + 1) + (1 - (-1)^p) \zeta(p + 3) + 6 \sum_{j_1 + 2j_2 = p+2, j_1 \geq 0, j_2 \geq 1} \zeta(j_1 + 1) \zeta(2j_2) + 6 \sum_{j_1 + j_2 + 2j_3 = p+1, j_1, j_2 \geq 0, j_3 \geq 1} \zeta(j_1 + 1) \zeta(j_2 + 1) \zeta(2j_3) + 2 \sum_{j_1 + j_2 + j_3 + j_4 = p, j_1, j_2, j_3, j_4 \geq 1} \zeta(j_1 + 1) \zeta(j_2 + 1) \zeta(j_3 + 1) \zeta(2j_4).\]

**Example 4.3** For integer \( p > 1, \)

\[
\left((-1)^p - 1\right) S_{1, p} + 6 S_{1, p+1} - 3 \ln(2) \left(1 + (-1)^p\right) S_{1, p+1} - 6 \ln(2) S_{2, p} + 3 \ln^2(2) + 2 \zeta(2) S_{1, p} + 3 S_{3, p} - (-1)^p \ln^3(2) \zeta(p) + 3p S_{1, p+1} + 3p S_{2, p+1} - 6p \ln(2) S_{1, p+1} + p \left(3 \ln^2(2) - \frac{5}{2} \zeta(2)\right) \zeta(p + 1) + 3 \left(p \frac{p + 1}{2} + (-1)^p\right) S_{1, p+2} - 3 \left(p \frac{p + 1}{2} + (-1)^p\right) \ln(2) \zeta(p + 2) - \left(\ln^3(2) + 6 \ln(2) \zeta(2) + \frac{9}{4} \zeta(3)\right) \zeta(p) + (-1)^p \zeta(p + 3) \frac{p(p + 1)(p + 2)}{6} \zeta(p + 3) + 3(-1)^p \left(S_{1, p+1} - 2 \ln(2) S_{1, p+1} + \ln^2(2) \zeta(p + 1)\right) + \text{Res} \left[g_2(s), s = 0\right] = 0,\]

where

\[
\text{Res} \left[g_2(s), s = 0\right] = -3 \zeta(p + 3) - 3 \sum_{j_1 + j_2 = p+1, j_1, j_2 \geq 0} \zeta(j_1 + 1) \zeta(j_2 + 1) - \sum_{j_1 + j_2 + j_3 = p, j_1, j_2, j_3 \geq 0} \zeta(j_1 + 1) \zeta(j_2 + 1) \zeta(j_3 + 1) - (1 - (-1)^p) \zeta(p + 3)\]
- 6 \sum_{j_1+2j_2=p+2, \ j_1, j_2 \geq 0, j_1, j_2 \geq 1} \zeta(j_1 + 1) \zeta(2j_2)
- 6 \sum_{j_1+j_2+2j_3=p+1, \ j_1, j_2, j_3 \geq 0, j_1, j_2, j_3 \geq 1} \zeta(j_1 + 1) \zeta(j_2 + 1) \zeta(2j_3)
- 2 \sum_{j_1+j_2+j_3+2j_4=p, \ j_1, j_2, j_3, j_4 \geq 0, j_1, j_2, j_3, j_4 \geq 1} \zeta(j_1 + 1) \zeta(j_2 + 1) \zeta(j_3 + 1) \zeta(2j_4).

5 Parameterized Extensions of Ramanujan-Type Identities

Infinite series involving the trigonometric and hyperbolic functions have attracted the attention of many authors. Ramanujan evaluate many infinite series of hyperbolic functions in his notebooks [17] and lost notebook [18]. Andrews and Berndt’s books [2, 3] contain many such results as well as numerous references. In this section, we provide some parameterized extensions of Ramanujan-type identities that involve hyperbolic series.

We define the hyperbolic cotangent function with sequence \( A \) by

\[
\text{coth} \,(s; A) := i \cot (is; A) .
\]

Theorem 5.1 Let \( k > 1 \) be a integer and \( \alpha, \beta \) be reals with \( \alpha, \beta \neq 0 \) with \( A, B \) defined above, then

\[
\begin{align*}
\alpha^{2k-1}\beta \pi \sum_{n=1}^{\infty} a_n \coth \left( \frac{n\beta \pi}{\alpha} ; B \right) &+ (-1)^k \beta^{2k-1}\pi \sum_{n=1}^{\infty} b_n \coth \left( \frac{n\alpha \pi}{\beta} ; A \right) \\
&= b_0 \alpha^{2k} D^{(A)}(2k) + (-1)^k a_0 \beta^{2k} D^{(B)}(2k) - 2 \sum_{j_1+j_2=k, \ j_1, j_2 \geq 1} (-1)^{j_2} \alpha^{2j_1} \beta^{2j_2} D^{(A)}(2j_1) D^{(B)}(2j_2).
\end{align*}
\]

Proof. To prove (5.2), we consider the function

\[
f(s; A, B) := \frac{\pi^2 \cot (\pi \alpha s; A) \coth (\pi \beta s; B)}{s^{2k-1}}.
\]

It is obvious that the function \( f(s; A, B) \) is meromorphic in the entire complex plane with a simple pole at \( s = 0 \) and simple poles \( s = \frac{n}{\alpha} \) and \( s = \frac{n}{\beta} i \) for each integer \( n \). Then, brief calculations show that

\[
\begin{align*}
\text{Res} \left[ f(s; A, B), s = \frac{n}{\alpha} \right] &= \alpha^{2k-2} \pi \frac{a_n \coth \left( \frac{n\beta \pi}{\alpha} ; B \right)}{n^{2k-1}}, \\
\text{Res} \left[ f(s; A, B), s = \frac{n}{\beta} i \right] &= (-1)^k \beta^{2k-2} \pi \frac{b_n \coth \left( \frac{n\alpha \pi}{\beta} ; A \right)}{n^{2k-1}}, \\
\text{Res} \left[ f(s; A, B), s = 0 \right] &= -2 b_0 \alpha^{2k-1} \beta^{2k-1} \pi D^{(A)}(2k) - 2 (-1)^k a_0 \beta^{2k-1} \pi D^{(B)}(2k) \\
&+ 4 \sum_{j_1+j_2=k, \ j_1, j_2 \geq 1} (-1)^{j_2} \alpha^{2j_1} \beta^{2j_2-1} D^{(A)}(2j_1) D^{(B)}(2j_2).
\end{align*}
\]
for each integer $n \ (n \neq 0)$. Hence, using (5.3)-(5.5) and the residue theorem, we reach the
desired result. □

An elementary calculation, the (5.2) can be rewritten in the form

$$
\alpha \beta k \sum_{n=1}^{\infty} \frac{a_n \coth (n\alpha; B)}{n^{2k-1}} + (-1)^k \alpha k \pi \sum_{n=1}^{\infty} \frac{b_n \coth (n\beta; A)}{n^{2k-1}} = b_0 \beta k D(A)(2k) + (-1)^k a_0 \alpha k D(B)(2k) - 2 \sum_{j_1, j_2 \geq 1} (-1)^{j_2} \alpha \beta j_1 D(A)(2j_1)D(B)(2j_2), \quad (5.6)
$$

where $\alpha$ and $\beta$ are real numbers with $\alpha \beta = \pi^2$.

If setting $A, B \in \{ A_1, A_2 \}$, then we get the following well-known results.

**Corollary 5.2** If $k > 1$ is a positive integer number and $\alpha, \beta$ are real numbers such as $\alpha \beta = \pi^2$, then

$$
\alpha \beta k \sum_{n=1}^{\infty} \frac{\coth (n\alpha)}{n^{2k-1}} + (-1)^k \alpha k \beta \sum_{n=1}^{\infty} \frac{\coth (n\beta)}{n^{2k-1}} = \left( \beta^k + (-1)^k \alpha^k \right) \zeta (2k) - 2 \sum_{k_1, k_2 \geq 1} (-1)^{k_2} \alpha k_2 \beta k_1 \zeta (2k_1) \zeta (2k_2). \quad (5.7)
$$

**Corollary 5.3** If $k$ is a positive integer number and $\alpha, \beta$ are real numbers such as $\alpha \beta = \pi^2$, then

$$
\alpha \beta k \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k-1} \sinh (n\alpha)} + (-1)^k \alpha k \beta \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k-1} \sinh (n\beta)} = \left( \beta^k + (-1)^k \alpha^k \right) \tilde{\zeta} (2k) + 2 \sum_{k_1, k_2 \geq 1} (-1)^{k_2} \alpha k_2 \beta k_1 \tilde{\zeta} (2k_1) \tilde{\zeta} (2k_2). \quad (5.8)
$$

**Corollary 5.4** If $k$ is a positive integer number and $\alpha, \beta$ are real numbers such as $\alpha \beta = \pi^2$, then

$$
\alpha \beta k \sum_{n=1}^{\infty} \frac{1}{n^{2k-1} \sinh (n\alpha)} + (-1)^k \alpha k \beta \sum_{n=1}^{\infty} \frac{\coth (n\beta)}{n^{2k-1}} (-1)^n = \left( \beta^k - (-1)^k \left( 1 - 2^{1-2k} \right) \alpha^k \right) \zeta (2k) + 2 \sum_{k_1, k_2 \geq 1} (-1)^{k_2} \alpha k_2 \beta k_1 \zeta (2k_1) \tilde{\zeta} (2k_2). \quad (5.9)
$$

Applying the fact

$$
\coth x = 1 + \frac{2}{e^{2x} - 1},
$$

to (5.7), we obtain the Ramanujan’s formula for $\zeta (2k - 1) \ (k > 1)$

$$
(4\beta)^{-(k-1)} \left\{ \frac{1}{2} \zeta (2k - 1) + \sum_{n=1}^{\infty} \frac{1}{n^{2k-1} (e^{2n\alpha} - 1)} \right\}
$$

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\[ -(-4\alpha)^{-(k-1)} \left\{ \frac{1}{2} \zeta (2k - 1) + \sum_{n=1}^{\infty} \frac{1}{n^{2k-1}(e^{2n\beta} - 1)} \right\} \]

\[ = \sum_{j=0}^{k} (-1)^{j-1} \frac{B_{2j}B_{2k-2j}}{(2j)! (2k-2j)!} \alpha^j \beta^{k-j}. \]  

(5.10)

The Ramanujan’s formula for \(\zeta(2k-1)\) (5.10) appears as Entry 21(i) in Chapter 14 of Ramanujan’s second notebook [17]. It also appears in a formerly unpublished manuscript of Ramanujan that was published in its original handwritten form with his lost notebook [18]. The first published proof of (5.10) is due to Malurkar [15] in 1925-1926. This partial manuscript was initially examined in detail by the Berndt in [6], and by Andrews and Berndt in their fourth book on Ramanujan’s lost notebook [3]. There are many results in the literature generalizing or extending the result. The two most extensive papers in this direction are perhaps those by Bradley [10] and Sitaramachandrarao [19].

**Theorem 5.5** Let \(\alpha, \beta, \gamma\) be reals with \(\alpha, \beta, \gamma \neq 0\), we have

\[
\alpha^{2k}\beta\gamma \pi^2 \sum_{n=1}^{\infty} a_n \coth(\frac{\pi n}{\alpha}; B) \coth(\frac{\pi n}{\alpha}; C)
\]

\[ + (-1)^{k-1} \beta^{2k} \alpha \gamma \pi^2 \sum_{n=1}^{\infty} b_n \coth(\frac{\pi n}{\beta}; A) \cot(\frac{\pi n}{\beta}; C)
\]

\[ + (-1)^{k-1} \gamma^{2k} \alpha \beta \pi^2 \sum_{n=1}^{\infty} c_n \coth(\frac{\pi n}{\gamma}; A) \cot(\frac{\pi n}{\gamma}; B)
\]

\[- 2a_0(-1)^k \sum_{j_1+j_2=k+1, j_1,j_2 \geq 1} D(B)(2j_1)D(C)(2j_2) \beta^{2j_1} \gamma^{2j_2}
\]

\[ + 2b_0 \sum_{j_1+j_2=k+1, j_1,j_2 \geq 1} (-1)^{j_2} D(A)(2j_1)D(C)(2j_2) \alpha^{2j_1} \gamma^{2j_2}
\]

\[ + 2c_0 \sum_{j_1+j_2=k+1, j_1,j_2 \geq 1} (-1)^{j_2} D(A)(2j_1)D(B)(2j_2) \alpha^{2j_1} \beta^{2j_2}
\]

\[ - 4 \sum_{j_1+j_2+j_3=k+1, j_1,j_2,j_3 \geq 1} (-1)^{j_2+j_3} D(A)(2j_1)D(B)(2j_2)D(C)(2j_3) \alpha^{2j_1} \beta^{2j_2} \gamma^{2j_3}
\]

\[ + a_0b_0 \gamma^{2k+2} (-1)^k D(C)(2k+2) + a_0c_0 \beta^{2k+2} (-1)^k D(B)(2k+2)
\]

\[ - b_0c_0 \alpha^{2k+2} D(A)(2k+2) = 0. \]

(5.11)

**Proof.** Consider

\[ f(s; A, B, C) := \frac{\pi^3 \cot(\pi s; A) \coth(\pi s; B) \coth(\pi s; C)}{s^{2k}} \quad (s \in \mathbb{C} \text{ and } \frac{\gamma}{\beta} \notin \mathbb{Q}). \]

Then, proceeding in the same fashion as in the proof of Theorem 5.1, we obtain the desired formula. \(\square\)
Setting \( a_n = c_n = 1, b_n = (-1)^n \) and \( k = \alpha = 1, \beta = \sqrt{2}, \gamma = 2 \) in (5.11) yields

\[
\sum_{n=1}^{\infty} \frac{\coth(2\pi n)}{n^2 \sinh(\sqrt{2}\pi n)} + \sqrt{2} \sum_{n=1}^{\infty} \frac{\coth\left(\frac{\pi n}{\sqrt{2}}\right) \cot\left(\sqrt{2}\pi n\right)}{n^2} (-1)^n + 2 \sum_{n=1}^{\infty} \frac{\coth\left(\frac{\pi n}{\sqrt{2}}\right)}{n^2 \sin\left(\frac{\pi n}{\sqrt{2}}\right)} = \frac{97\sqrt{2}}{120} \zeta(2).
\]

In general, let \( A := \{a_n\} \) and \( B_l := \{b_n(l)\} \) (\( l = 1, 2, \ldots, m; m \in \mathbb{N} \)) be sequences of complex numbers (if \( n \to \infty \), then \( a_n = o(n^{s_1}) \) and \( b_n(l) = o(n^{t_1}) \) with \( s_1, t_1 < 1 \)), \( k \) be a positive integer and \( \alpha, \beta_l \) \( (l = 1, 2, \ldots, m; m \in \mathbb{N}) \) be reals with \( \alpha, \beta_l \neq 0 \) and \( \frac{\beta_l}{\beta_j} \notin \mathbb{Q} \) \( (l \neq j) \), define

\[
f(s; A, B_1, B_2, \cdots, B_m) := \frac{\pi^{m+1} \cot(\pi \alpha s; A) \prod_{l=1}^{m} \cot(\pi \beta_l s; B_l)}{s^{2k + |1-(-1)^m|/2}} \quad (s \in \mathbb{C}),
\]

by elementary residue calculations, we can get the following general conclusion:

\[
(-1)^{k+m/2-|1-(-1)^m|/4} \pi^m m \sum_{j=1}^{m} \beta_j^{2k-\frac{|3-(-1)^m|}{2}} \sum_{n=1}^{\infty} \frac{b_n^{(j)}(l) \coth\left(\frac{\pi \alpha n}{\beta_j}; A\right) \prod_{l=1, l \neq j}^{m} \cot\left(\frac{\pi \beta_l n}{\beta_j}; B_l\right)}{n^{2k-|1-(-1)^m|/2}}
\]

\[
+ \alpha^{2k-\frac{|3-(-1)^m|}{2}} \pi^m m \sum_{n=1}^{\infty} \frac{a_n \prod_{l=1}^{m} \cot\left(\frac{\pi \beta_l n}{\alpha}; B_l\right)}{n^{2k-|1-(-1)^m|/2}} + \frac{1}{2} \text{Res}[f(s; A, B_1, \cdots, B_m), s = 0] = 0.
\]

(5.12)

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