Randomized Rounding Revisited with Applications

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Abstract

We develop new techniques for rounding packing integer programs using iterative randomized rounding. It is based on a novel application of multidimensional Brownian motion in $\mathbb{R}^n$. Let $\tilde{x} \in [0,1]^n$ be a fractional feasible solution of a packing constraint $Ax \leq 1$, $A \in \{0,1\}^{m \times n}$ that maximizes a linear objective function. The independent randomized rounding method of Raghavan-Thompson rounds each variable $x_i$ to 1 with probability $\tilde{x}_i$ and 0 otherwise. The expected value of the rounded objective function matches the fractional optimum and no constraint is violated by more than $O\left(\frac{\log m}{\log \log m}\right)$. In contrast, our algorithm iteratively transforms $\tilde{x}$ to $\hat{x} \in \{0,1\}^n$ using a random walk, such that the expected values of $\hat{x}_i$'s are consistent with the Raghavan-Thompson rounding. In addition, it gives us intermediate values $x'$ which can then be used to bias the rounding towards a superior solution.

The reduced dependencies between the constraints of the sparser system can be exploited using Lovasz Local Lemma. Using the Moser-Tardos’ constructive version, $x'$ converges to $\hat{x}$ in polynomial time to a distribution over the unit hypercube $H_n = \{0,1\}^n$ such that the expected value of any linear objective function over $H_n$ equals the value at $\tilde{x}$.

For $m$ randomly chosen packing constraints in $n$ variables, with $k$ variables in each inequality, the constraints are satisfied within $O\left(\frac{\log m}{\log \log (mk \log m/n)}\right)$ with high probability where $p$ is the ratio between the maximum and minimum coefficients of the linear objective function. For example, when $m, k = \sqrt{n}$ and $p = \text{polylog}(n)$, this yields $O\left(\log \log n / \log \log \log n\right)$ error for polylogarithmic weighted objective functions that significantly improves the $O\left(\frac{\log m}{\log \log m}\right)$ error incurred by the classical randomized rounding method of Raghavan and Thompson [RT87].

Further, we explore trade-offs between approximation factors and error, and present applications to well-known problems like circuit-switching, maximum independent set of rectangles and hypergraph $b$-matching. Our methods apply to the weighted instances of the problems and are likely to lead to better insights for even dependent rounding.

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Many combinatorial optimization problems can be modeled using a weighted packing integer program. Maximize $\sum_{i=1}^{m} c_i \cdot x_i$ subject to $\sum_{i \in S_j} x_i \leq 1$ for all $j \leq m$. We can assume that $\max_i c_i = 1$. Although the above formulation appears somewhat restrictive having a fixed right hand side of each inequality, the methods that we develop extend to more general parameters. The constraints are expressed as $C_j : V_j \cdot x \leq 1$ where $V_j$ is a 0-1 incidence vector corresponding to set $S_j \subset \{1, 2 \ldots n\}$. We will also use $x_i$ to denote the $i$-th coordinate of a vector $x$.

Since this version is also NP-hard as it captures many intractable independent set problems, a common strategy is to solve the Linear Program (LP) corresponding to the relaxation $0 \leq x_i \leq 1$. Then use the LP optimum $OPT$ to obtain a good approximation of the optimal integral solution. Suppose the optimum is achieved by the vector $x' = [x'_1, x'_2 \ldots x'_n]$. In the conventional randomized rounding [RT87], we round each $x_i$ independently in the following manner.

\[
\hat{x}_i = \{1 \text{ with probability } x'_i \text{ and 0 otherwise}\}.
\]

Since $E[\hat{x}_i] = x'_i$, it follows that $E[\sum_i c_i \hat{x}_i] = \sum_i c_i \cdot x'_i$ and using Chernoff bounds, we can show that for all $j$, w.h.p. $\sum_{i \in S_j} \hat{x}_i \leq \left(\frac{\log m}{\log \log m}\right).$

The primary motivation for improved randomized rounding is improved approximation of integer linear packing problems. Historically, the original randomized rounding technique was proposed for obtaining better approximation of multicommodity flows [RT87]. Srinivasan [Sri95] presents an extensive survey of many sophisticated variations of randomized rounding techniques and applications to approximation algorithms. It was established much later [CGKT07] that the Raghavan-Thompson bound cannot be improved as a consequence of the following result.

**Lemma 1.1 (CGKT07)** There exists a constant $\delta > 0$ such that the integrality gap of the multicommodity flow relaxation problem is $\Omega(1/c' \cdot n^{\frac{\delta}{c' \cdot n^{1+\delta}}})$ for any congestion $c'$, $1 \leq c' \leq (\delta \log n)/\log \log n$ where $n$ is the number of vertices in the graph and the integrality gap for $c'$ is with respect to congestion 1.

Therefore no rounding algorithm can simultaneously achieve error $c' = o(\log n/\log \log n)$, and an objective function value of $\Omega(OPT)$ for all polynomial size input ($m \leq n^{O(1)}$). Note that this does not preclude improvement of the multicommodity flow problem approximation by alternate formulation - for this, there are alternate hardness bounds given by [CGKT07].

Subsequent to the work of Raghavan-Thompson, Srinivasan [Sri95], Srinivasan [Sri96] Baveja and Srinivasan [BS00], Kolliopoulos and Stein [KS98] obtained results that focussed on circumventing the above bottleneck by making use of some special properties of the constraint matrices like column-restricted matrices. In particular, the focus shifted to designing approximation algorithms that are feasible (unlike the basic independent rounding of RT) by bounding dependencies between inequalities and using Lovasz Local Lemma (LLL) or even more sophisticated correlation inequalities like FKG [Sri96]. Leighton, Rao and Srinivasan [LRS98] made very clever use of LLL to achieve nearly optimal results in graphs with short flow paths. The notion of short flow paths was developed further in the context of the unsplittable multicommodity flow problem with an explicit objective function in many papers including [KR96] [BS00], [KS06], [CCGK07]. Intuitively, short flow paths reduce dependencies between conflicting routing paths and makes the algorithms more efficient in terms of assigning paths without exceeding the maximum allowed congestion (for edge-disjoint paths it is 1).

In this paper, we attempt to provide a uniform framework behind many prior uses of independent rounding by recasting the process as a Brownian walk in high dimension and analyzing its convergence. For readers familiar with this technique, it may appear to be an overkill since the final outcome of Brownian motion and independent rounding are identical (we shall formalize this in the next section). However, we will demonstrate that this framework offers a cleaner exposition and simpler explanations of many of the previous clever, yet ad-hoc techniques. We have presented a detailed characterization of the time-dependent behavior of the random variables associated with
each inequality executing Brownian motion which was not studied before to the best of our know-
edge. It could potentially offer new tools for analysis of more complicated rounding methods where
the random variables may not be independent.

We demonstrate two distinct applications of our new framework by addressing the rounding
problem for the average case of an input family that we will define more precisely.
(i) A simple approximation algorithm for weighted objective function.
(ii) An improvement of the Raghavan-Thompson rounding error for a restricted class of weighted
objective function.

As opposed to the one-shot rounding of \cite{RT87}, also referred as independent rounding, our
rounding is iterative and based on successive refinements of the LP solution that has a lower variance
per step, to yield a sparser set of constraints. This reduces dependencies between inequalities and
makes the use of techniques like Lovasz Local Lemma (LLL) more effective.

1.1 Main results and some applications

Theorem 1.1 Given an \(m \times n\) 0-1 matrix \(A\), such that \(A \cdot x' \leq b\) for \(x' \in [0, 1]\), and an objective
function \(\sum_i c_i x_i\), in randomized polynomial time \(x'\) can be transformed into \(y'\) which are \([0, 1]\) valued
random variables such that for all \((\log n - \log \log m + \log b) \geq p \geq 0\)
(i) All constraints have \(\leq \frac{M}{p}\) non-zero variables from \(y'\) and
(ii) For all \(i\), \(\sum_i A_i \cdot y' \leq O\left(\sqrt{\frac{mB \log m}{n}}\right)\) where \(A_i\) is the \(i\)-th row of \(A\).
(iii) \(\mathbb{E}[c_i y'_i] = \sum_i c_i x_i\)

In other words we have a sparse system of equivalent constraints with the objective functional value
unchanged in the expectation.

Remark For \(p = \log n - \log \log m + \log B\), there are \(\log m\) variables in each inequality summing
upto \(O(1)\). This specific result has been observed before by using a clever application of independent
rounding in the following manner. Round each \(x'_i < \frac{1}{\log m}\) to \(\frac{1}{\log m}\) with probability \(x'_i \log m\). Then
\(\mathbb{E}[x_i] = x'_i\) and no inequality has more than \(\log m\) variables with high probability.

Using the above sparsification, we obtain a number of interesting results arguably in simpler
ways than known previously.

Theorem 1.2 Let \(A_k^{m \times n}\) denote the family of \(m \times n\) matrices where each row (independently) has
\(k\) ones in randomly chosen columns from \(\{1, 2, \ldots, n\}\) and 0 elsewhere\(^{1}\). Let \(A \in A_k^{m \times n}\), then for
any point \(x' = (x'_1, x'_2, \ldots, x'_n)\), \(0 \leq x'_i \leq 1\) such that \(A \cdot x' \leq e^m\), where \(e^m\) is a vector of \(m\) 1’s, and
\(OPT = \sum_i c_i \cdot x'_i\) where \(1 = \max_i\{c_i\}\) and \(\forall i\ c_i \geq \frac{1}{p}\). Then, \(x'\) can be rounded to \(\hat{x} \in \{0, 1\}^n\) such
that the following holds with probability \(\geq 1 - \frac{1}{m}\)

\[
\|A\hat{x}\|_\infty \leq O\left(\frac{\log (mkp \log m/n) + \log \log m}{\log \log (mkp \log m/n + \log m)}\right) \quad \text{and} \quad (1)
\]

\[
\sum_i c_i \hat{x}_i \geq \Omega(OPT) \quad (2)
\]

Moreover such an \(\hat{x}\) can be computed in randomized polynomial time.

Remark (i) The result relates the rounding error to the average number of ones in a column, i.e.,
\(\frac{mk}{n}\) and characterizes tradeoffs between the parameters \(m, n, k\). For example, for \(k = \sqrt{n}\), we can
bound the error to \(O(\log \log n)\) for \(m \leq \sqrt{n} \cdot \text{poly}(n)\) for \(p \leq \frac{\log \text{poly}(n)}{\log \log n}\). This is a significant
improvement over the \(O(\log n / \log \log n)\) bound of the independent rounding.

\(^{1}\) Alternately we can set every entry to be 1 with probability \(k/n\) independently
(ii) For \( k = \log n \), the bound is indeed tight. A proof is given in the appendix (\[Vis\]).

(iii) For the unweighted case, the result holds for arbitrary distribution of the \( k \) 1’s in each row.

**Theorem 1.3** Let \( A \in \mathcal{A}_k^{m \times n} \), such that \( x' \in [0,1]^n \) maximizes \( \sum_i c_i x' \) and satisfies \( A \cdot x \leq b \cdot e^m \) with \( \text{OPT} = \sum_c c x'_c \). If \( 1 = \max_c c \geq 1/p \), \( p \geq 1 \) and \( m \leq \frac{n \log m}{k} \), then \( x' \) can be rounded to \( \hat{x} \in \{0,1\}^n \) in polynomial time such that the following holds with high probability for \( b \geq 1 \)

\[
\|Ax\|_\infty \leq b \quad \text{and} \quad \sum_i c_i \hat{x}_i \geq \Omega(OPT/(\max(p \log m, \log^2(m))^{1/b}))
\]

We observe a trade-off between the weights and the approximation factor \((p \log m)^{1/b}\). For \( b = O(\frac{\log \log m}{\log \log \log m}) \), we can obtain \( O(1) \) approximation for \( c_i \geq \frac{1}{\log n} \) for any constant \( C \geq 1 \).

The random matrices provide a natural framework for combinatorial results related to random hypergraphs. We sketch one such application to \( b \)-matching of \( k \)-regular hypergraphs that yields an approximation \( k^{1/b} \) for \( b \geq 2 \) using the result of Theorem 1.3 and extends a result of Srinivasan \[Sri96\]. This implies that for \( b = \Omega(\log \log n / \log \log \log n) \), we can obtain a \( b \)-matching of size \( \Omega(OPT) \) for most hypergraphs which matches the best possible size given by the fractional optimum. It is known that \( k \)-uniform \( b \)-matching problem cannot be approximated better than \( \frac{1}{\log k} \) for \( b \leq k / \log k \) unless \( P = NP \) \[ OFS11, HSS06\].

1.2 An overview and related work

Our algorithm can be best characterized as a randomized iterated rounding where we begin from a fractional feasible (specifically optimal) solution \( x' \) and iteratively converge to a good integral solution. Our algorithm has two distinct stages - random walk stage (more precisely, Brownian walk) and subsequently in the second stage invokes the Moser-Tardos iterative scheme for constructive Lovasz Local Lemma (LLL). In the first stage we effectively slow down the RT rounding process. Our approach is intuitive - starting from \( x' \), for each variable (dimension), we will roughly increment (actually a normal Gaussian increment) \( x_i \) by \( \pm \gamma \) for a suitably chosen \( 1 > \gamma > 0 \) where the sign (direction) is a random variable. In each iteration, the values of \( x_i \)'s are modified and we continue this process for each \( x_i \) until it is in the range \([0, \delta] \cup [1-\delta, 1] \) for an appropriate \( 1 > \delta > 0 \) such that \( \delta > \gamma \). At this point we fix the variable and we terminate when all inequalities have less than some predetermined value \( u \) of unfixed variables. This stage has some similarities with the method of Lovette and Meka \[LMT12\] but our analysis requires completely different techniques. The crux of the method called partial coloring lemma is a rounding strategy of an arbitrary \( x \in [-1,1]^n \) vector within the discrepancy polytope defined by the constraints starting with \( x = (0,0,\ldots,0) \). Their method can be mapped to \( \{0,1\} \) rounding as well, that was observed by Rothvoss \[Rot13\]. Compared to the setting of the discrepancy rounding, we are dealing with smaller error margins and the variable and polytope constraints are not widely separated in terms of distances from the starting point. Moreover, one needs to also account for the deviation of the objective function which is not required for Spencer’s discrepancy result.

2 A Rounding procedure using random walks

The simple random walk based algorithm outlined in the introduction doesn’t take into account any of the constraints \( V_j \cdot x \leq 1 \) and therefore likely to violate them after some random walk steps.

Set each variable to 1 with probability \( 1/k \), and for each violated constraint that has more than \( q = \log((mk)/n) \) ones, zero all its variables (or zero enough of its variables, chosen arbitrarily, so that it has only \( q \) ones). At most \( n/2 \) of the variables are heavy in the sense that they appear in more than \( 2mk/n \) constraints. For a light variable, even if it comes up 1, the probability that it is a member of a violated constraint is small (say, below 1/2), and hence the light variables (using linearity of expectation) guarantee a value of \( \Omega(n/k) \). This proof sketch was given by an anonymous reviewer of an earlier version.
Algorithm Iterative Randomized Rounding

Input : \( x'_i \) \( 1 \leq i \leq n \) satisfying \( Ax \leq b \cdot e^m \). \( C^0 \): set of constraints. 
0 < \( \gamma < \delta < 1 \) - the exact values are discussed in the analysis. 
Output : \( \hat{x}_i \in \{0,1\} \)

Initialize all variables as un-fixed and set \( X_0 = [x'_1, x'_2, \ldots, x'_n] \).

\textbf{Repeat} for iterations \( i = 1, 2 \ldots \)

1. Generate a random vector \( R^t = U_i \) where \( U_i \) is a multidimensional gaussian r.v. restricted to the un-fixed variables.
2. \( X_{i+1} = X_i + \gamma \cdot R^t \).
3. Fix a variable if it is less than \( \delta \) or greater than \( 1 - \delta \). If all variables are fixed then exit.
   \{ * multiple variables may get fixed in a single iteration. * \}
4. Update the set of constraints \( C^i \) that contains at least one unabsorbed variable.

until stopping condition \( \mathcal{S} (\max_j \{ \text{number of un-fixed variables in } C_j \}) \leq \log n \) *

Run Moser-Tardos \([MT10]\) algorithm on \( C^T \) on the un-fixed variables of \( X_T \) according given in Figure 2

Round the fixed variables to 0 or 1 whichever is closer and return this vector denoted by \( \hat{x} \).

Figure 1: An iterative randomized rounding algorithm based on random walks

However, the probability that an \( x_i \) reaches 1 before it reaches 0 is equal to the ratio \( \frac{x'_i / \gamma}{1 - x'_i + x'_i / \gamma} = x'_i \). This is a consequence of a more general property of martingales known as Doob’s optional stopping theorem (\([Fel68]\)).

**Theorem 2.1** Let \((\Omega, \Sigma, P)\) be a probability space and \( \{F_i\} \) be a filtration of \( \Omega \), and \( X = \{X_i\} \) a martingale with respect to \( \{F_i\} \). Let \( T \) be a stopping time such that \( \forall \omega \in \Omega, \forall i, |X_i(\omega)| < K \) for some positive integer \( K \), and \( T \) is almost surely bounded. Then \( \mathbb{E}[X_T] = \mathbb{E}[X_0] \).

In the above application, the stopping times are \( X_T = \{-b, a\} \) whichever is earlier. So \( -b \cdot \Pr[X = -b] + a \cdot \Pr[X = a] = 0 \cdot \Pr[X = 0] \). Since \( \Pr[X = -b] + \Pr[X = a] = 1 \) from the stopping criteria, \( \Pr[X = a] = \frac{b}{b + a} \). In our context, we start from \( X = x'_i \) and \( b = \frac{x'_i}{\gamma} \) and \( a = \frac{1 - x'_i}{\gamma} \), so the probability that \( \hat{x}_i = 1 \) equals \( \frac{x'_i / \gamma}{1 - x'_i + x'_i / \gamma} = x'_i \). So the distributions of the variables being absorbed at 0 or 1 are identical to the independent rounding. However, the random walk process has many intermediate steps that do not have corresponding mappings in the \( 2^n \) possible configurations of the independent rounding. Whether it makes this framework more powerful compared to one-step independent rounding is a difficult question but we feel that it could give us a superior understanding of the process of independent rounding.

In the algorithm presented in Figure 1 instead of the simple Bernoulli random walk step, we use a normal Gaussian random walk on each coordinate. We run the basic algorithm for \( T \) iterations such that the number of un-fixed variables in each constraint of \( C^T \) (the active set of constraints after \( T \) iterations) is bounded by \( \log n \). The value of \( T \), more specifically \( \mathbb{E}[T] \), will be determined during the course of our analysis.

Now we run the algorithm of \([MT10]\) (c.f. section 4.1) on the constraints in \( C^T \) which are projections of the original constraints on the un-fixed variables after \( T \) steps.

### 2.1 The framework and some notations

In the rounding algorithm, \( X_t \) denotes the random walk vector after \( t \) steps. Let \( U^n \) denote a \( d \)-dimensional Gaussian random variable and let \( U_t \) denote the projection of \( U^n \) on the current
subspace corresponding to the unfixed variables in the \( t \)th iteration. The normal distribution is denoted by \( \mathcal{N}(\mu, \sigma^2) \) where \( \mu \) is the mean and \( \sigma^2 \) is the variance. Unless otherwise stated, we will refer to the standard normal distribution where \( \mu = 0 \) and \( \sigma^2 = 1 \).

The parameters \( \gamma, \delta \) are chosen to ensure that the walk stays within the feasible region. It suffices to have \( \gamma \leq \frac{\delta}{\log n} \) from the pdf of the normal distribution if we are executing \( O\left(\frac{1}{\delta^2} \right) \) steps (see [LM12] for a rigorous proof). The value of \( \delta \) will be determined according to the approximation factor and the error bound that we have in a specific application. In our analysis, the the focus will be on variables absorbed at 0 that will be rounded down. The decrease in the objective value can be at most \( n\delta \) (recall the maximum weight of the coefficients is 1). If we choose \( \delta \) such that \( n\delta \) is \( o(OPT) \) it will suffice. For the main theorems, choosing \( \delta \leq \frac{1}{\text{polylog} n} \) will work.

After \( t \) iterations, the \( j \)-th constraint is denoted by \( C_j^t := V_j, X_t \leq 1 + \beta_t \) that has many of variables fixed. We shall denote the set of unfixed variables in iteration \( t \) by \( U(t) \subset \{1, 2 \ldots n\} \) and the corresponding vector by \( X_t^U \) where the the fixed variables are set to 0. The constraint vector \( V_j^t = V_j \cap U(t) \), where only the coefficients in \( V_j^t \) corresponding to \( U(t) \) can be non-zero. Then
\[
| < V_j^t, X_t - X_{t+1}^t > | = | < \frac{V_j^t}{||V_j^t||_2}, X_t^t - X_{t+1}^t > | \cdot ||V_j^t||_2 \text{ is the change in the value of } C_j \text{ in iteration } t \text{ as the remaining variables do not change. Note that while } X^t \text{ changes in every step, } V_j^t \text{ changes only when some variable is absorbed.}
\]

For convenience of the analysis, we will club successive iterations into phases, where within a phase \( p, \beta_p \) remains unchanged. Equivalently, \( \beta_p - \beta_{p-1} \) reflects the cumulative effect of a number of random walk steps within the phase \( p \) referred to as the accumulated error or simply error. The phase \( p \) corresponds to the \( L_2 \) norm of any constraint \( ||V_j^p||_2 \) that is bounded by \( \sqrt{n/2p} \). Intuitively, with additional random walk steps, we are more likely to violate the original constraints and \( \beta_p \) is a measure of the violation. In terms of the above notation, it is obvious that \( ||V_j^{p+1}||_2 \leq ||V_j^p||_2 \). We will use the index \( p \) (respectively \( t \)) to indicate reference to phases (resp. iterations).

Wlog, we assume that all constraints have at least 2 variables and at most \( n - 1 \) variables.

The successive Brownian motion steps (defined by multidimensional normal Gaussian) form a martingale sequence. The following result forms the crux of our analysis - see [Ban10, LM12] for more details and proof.

**Observation 2.1** In iteration \( t \), let \( Y_t = < V_j, U_t \cdot \gamma > \), that is the measure of the change in \( < V_j, x > \) in the step \( t \). Then \( Y_t \) is a Gaussian random variable with mean \( 0 \) and variance \( \gamma \cdot ||V_j||^2_2 \).

Note that the scaled random variable \( Y_t' = \frac{Y_t}{\gamma ||V_j||_2} \) is a Gaussian with mean \( 0 \) and variance \( \leq 1 \). When \( Y_t' \) correspond to standard Gaussian, then

**Lemma 2.1** For any \( \beta > 0 \), \( \Pr[|Y_1' + Y_2' \ldots Y_j'| > \beta] \leq 2 \exp(-\beta^2/2T) \).

The behavior of random walks starting from an arbitrary initial position and subsequently absorbed at 0 can be obtained from the gambler’s ruin problem where the underlying martingale is the Brownian motion. There exists a wealth of literature on Brownian motion [Fel68, Ros06], but the specific form in which we invoke them for analyzing our algorithm is stated below. A proof is presented in the appendix.

**Lemma 2.2** Consider a random walk starting from position \( a \) in an interval of length \( a+b \) with absorbing barriers at both end-points. Then the expected number of steps for the walk to get absorbed (at any of the ends) is \( a \cdot b \). Moreover, the probability of the random walk being absorbed at 0 is \( \frac{b}{a+b} - 1/k \) after \( k \cdot a \cdot b \) steps for any \( k > 1 \).

Remark By choosing \( b > k \cdot a \), the probability can be made arbitrarily close to 1 for \( k \gg 1 \) - conversely, the probability of non-absorption at 0 is \( O\left(\frac{1}{k}\right) \) after \( k^2a^2 \) steps.

\( ^3 \) A constraint with \( n \) variables has a trivial solution where any one variable can be set to 1 and a constraint with one variable is redundant.
3 Brownian walk analysis

We will divide the analysis into two components - First we will compute the rate at which the variables are absorbed at 0. Second, we compute the increments of all variables during each iteration that causes an inequality to be violated. This causes the right hand side of any inequality to behave as a martingale and we will refer to it as the error. Note that the increase in error is related to the number of unabsorbed variables as they execute random walk. Once all variables are absorbed, then the error doesn’t change. For subsequent applications, we will derive the bounds for starting position scaled by $S \geq 1$, i.e., from $\frac{x}{S}$. Readers who are familiar with [LM12] may note that our analysis focuses on variables associated with each constraint as opposed to the global number of variables.

**Lemma 3.1** Let $x' \in \mathbb{R}^n$ be a feasible solution to $A \cdot x \leq B \cdot \bar{1}^m$, $A \in \{0,1\}^{m \times n}$, and $\bar{x} = \frac{x'}{S}$, $S \geq 1$ be chosen as the starting point for brownian walk. Then, after $T_p = \frac{2p^2}{n \gamma^2}$ steps, with probability $\geq 1 - \frac{1}{m^{\Omega(1)}}$, the number of unabsorbed variables in constraint $i$ is $O \left( \frac{nB}{S^{5/2}} \right)$ for $2p \leq \frac{nB}{S (\log(m))}$.

First, we can use Lemma 2.2 to get a bound on the probability of absorption.

**Claim 1** For $b \geq ra$, after $O \left( \frac{r^2 a^2}{\gamma^2} \right)$ Brownian motion steps, the probability of non-absorption at 0 is $\leq O \left( \frac{1}{r} \right)$.

The above claim is trivially true for $r \leq 1$. In Lemma 2.2, use $k = r, b = ra/\gamma$ that gives absorption probability $\leq O \left( \frac{1}{r} \right)$ after $r \cdot a/\gamma \cdot ar/\gamma = \frac{r^2 a^2}{\gamma^2}$ steps.

**Proof:** (of Lemma 3.1) Choosing $T_p = \frac{2p^2}{n \gamma^2} = \frac{\left( \frac{x_i}{S} \right)^2 \left( \frac{nB}{S} \right)^2}{\gamma^2}$, we can apply Claim 1 with $r = \frac{2p}{n x_i}$ to obtain that the probability that $x_i$ is not absorbed at 0 is $\leq \frac{n x_i}{S \gamma^2}$. Let $V_{i}^p x \leq B$ be the $i^{th}$ constraint after $p$ phases. If $u_i^p = \| ||V_i^p||^2 \|$ denotes the number of unabsorbed variables in constraint $i$ after $p$ phases then $E(u_i^p) = \sum_{j=1}^{n} \frac{n x_i}{S \gamma^2} \cdot A_{i,j} \leq \frac{nB}{S^{5/2}}$ since $\sum_{j=1}^{n} A_{i,j} x_j \leq \frac{B}{S}$.

Note that the variables are independently executing Brownian walks. For $\frac{nB}{S^{5/2}} \geq \log(m)$ or equivalently, $p \leq \log(n) - \log(\log(m)) + \log(B) - \log(S)$, we can apply Chernoff bound to claim that $u_i^p$ is $O \left( \frac{nB}{S^{5/2}} \right)$ with probability $\geq 1 - \frac{1}{m}$.

**Corollary 3.1.1** Since $E(u_i^p) = E(||V_i^p||^2_2)$ for $V_i^p \in \{0,1\}^n$, we can bound $||V_i^p||^2_2$ by $\frac{nB}{S^{5/2}}$ with high probability for $p \leq \log(n) - \log(\log(m) \log(n)) + \log(B) - \log(S)$.

**Remark** (i) For $A_{i,j} \in [0,1]$, the above proof on $E(u_i^p)$ doesn’t hold directly but the bound on $||V_i^p||^2_2$ is still valid.

(ii) If we proceed upto $p^* = \log(n) - \log(\log(m)) + \log(B) - \log(S) - \log(c)$, all constraints have at most $O(\log m)$ unfixed coordinates.

3.1 Error Bound

From the previous result, for all $j$, $\| V_j^p \|_2^2 \leq O \left( \frac{nB}{S^{5/2}} \right)$ with high probability. To bound the error consider a fixed constraint $C_j : V_j, x \leq B$, we denote the error accumulated in the $p^{th}$ phase by $\delta_p$, so that $\sum_{q=1}^{p} \delta_q = \beta_p$.

\[\text{For } m \text{ polynomial in } n \text{ we need not distinguish between } \log m \text{ and } \log n\]

\[\text{Applying a union bound over all constraints, we satisfy the conditions of the lemma for each of the phases and constraints with high probability}\]
Lemma 3.2  For all constraints $C_j$, the error $\delta_p \leq \sqrt{\frac{2B\log m}{S \cdot n}} \cdot \frac{2^p}{2^p - 2^q - 2^q}$.

Thus the total error $\beta_p$ up to $\log n - \log(\log m) + \log(B) - \log(S)$ is bounded by $c' \frac{B}{S}$ for some constant $c'$.

Proof: Consider $\Pr(< \gamma(\sum_{i=T_{p-1}+1}^{T_{p+1}} U_i), V_j > | \geq \delta_p) = \Pr(|| \sum_{i=T_{p-1}+1}^{T_{p+1}} U_i, V_j \cdot \frac{V_j}{||V_j||} > | \geq \frac{\delta_p}{\gamma||V_j||})$.

Now $< U_i, \frac{V_j}{||V_j||} \sim N(0, \sigma^2)$ where $\sigma^2 \leq 1$. From Lemma 2.1, it follows that the above is bounded by $\exp(-\frac{\delta_p^2}{\gamma||V_j||^2(T_p - T_{p-1})})$. It will hold simultaneously for all the $m$ constraints in the $p^{th}$ phase, if $\delta_p$ satisfies:

$$\frac{(\delta_p)^2}{\gamma||V_j||^2(T_p - T_{p-1})} \geq \Omega(\log(m))$$

i.e. $\delta_p \geq \gamma||V_j||\sqrt{2 \cdot \log m \cdot (T_p - T_{p-1})}$

From Corollary 3.1.1 $||V_j^{p-1}||_2 \leq \frac{nB}{S \cdot 2^p - 1}$ and $T_p = \frac{2^p \gamma p}{n^2 \gamma^2}$, it suffices to choose

$$\delta_p \geq \gamma \sqrt{\frac{nB}{S \cdot 2^p - 1}} \cdot \sqrt{2 \log m} \cdot \sqrt{\frac{2^p \cdot 2^p}{n^2 \gamma^2}} = 2 \sqrt{B \log m} \frac{2^p}{S \cdot n}$$

The above bound for $\delta_p$ holds with high probability when $p \leq \log n - \log(\log m) + \log(B) - \log(S)$ since only in this situation can we bound effective value of $V_j$.

Thus the total error up to $\log n - \log(\log m) + \log(B) - \log(S)$ is bounded by

$$\sum_{p=1}^{\log n - \log(\log m) + \log(B) - \log(S)} 2 \sqrt{B \log m} \frac{2^p}{S \cdot n} \cdot \frac{2^p}{2^p - 2^q - 2^q} \leq 2 \sqrt{\frac{B \log m}{S \cdot n}} \cdot O(\frac{\log n - \log(\log m) + \log(B) - \log(S)}{2}) \leq 2 \sqrt{\frac{B \log m}{S \cdot n}} \cdot O(\frac{\log(m)}{\log(m)}) = O(\frac{B}{S})$$

Therefore, the total error in every constraint is bounded by $< V_j, x^p > = < V_j, \hat{x} + \gamma \sum_{i=1}^{T_p} U_i > = O(\frac{B}{S})$.

Thus the solution obtained satisfies the constraints $A \cdot x \leq \frac{B}{S}$.  

The above method can be extended to obtain an alternate proof the $O(\log m/\log \log m)$ error bound of Raghavan-Thompson that we have omitted from this version.

Based on the above results, we summarize as follows.

Corollary 3.2.1  In the first stage of Algorithm Iterative Randomized Rounding, we run the algorithm for $T = \frac{B^2}{S^2 \log^2 m \gamma^2}$ steps. Then with high probability

(i) all constraints have $\leq \log m$ unfixed variables and

(ii) the total error in any constraint is bounded by $c' \frac{B}{S}$ for some constant $c'$.

Using multiple copies of a variable, results similar to this section have been used before that are ad hoc to specific applications [CC09, Sri96]. However, it requires two stages of rounding - once choosing exactly one copy followed by a phase of independent rounding. In comparison, our technique and analysis are more general.

4 Applications using LLL

At the end of brownian walk (that is after $\frac{B^2}{S^2 \log^2 m \gamma^2}$ steps), we have at most $\log(m)$ unconverged variables occurring (with non zero coefficients) in each equation. Now we need to bound the error in
independantly rounding the unconverged Now we need to bound the error in independantly rounding the unconverged variables. As per the notations setup in section \( V^p_i \) represents the coefficient vector of \( i^{th} \) constraint after \( p^\text{th} \) phas.

Define \( \hat{A} \) as a matrix having rows as \( V^p_i \).

Now if the unconverged variables are rounded from \( \hat{x} \) to \( \hat{x} \), the change in the value of RHS is \( A \cdot (\hat{x} - \hat{x}) = A \cdot (\hat{x} - \hat{x}) \). (Since only unconverged variables change).

Hence to calculate additional error due to independant rounding we only need to consider the "sparsified" matrix \( \hat{A} \).

Before we prove the main results, we consider the simpler case of columns with bounded number of ones - Suppose the matrix \( A^{m\times n} \) has no more than \( \rho \) 1’s in any column. Let \( OPT \) be the optimal fractional objective value for the weighted objective function \( \sum_i c_i \cdot x_i \). Consider a fixed constraint \( C_r \), that contains \( m \) 1’s after the Brownian motion and let \( j_1(r), j_2(r), \ldots \) denote the (at most) \( \log n \) columns that contain 1. We say that a constraint \( C_y \) is dependent on \( C_r \) if they share at least one column where the value is 1. So, the dependency of any single constraint can be bound by \( \rho \log m \) (Since there are \( \log m \) 1’s per row in the sparsified matrix \( \hat{A} \)). If we use the independent rounding to round the fractional solution \( x \), the probability that the value of a constraint exceeds \( t \) is bounded \( \frac{1}{t} \) from Chernoff bounds. Let \( E_i \) denote the event that \( C_i \) exceeds \( t \) when we use randomized rounding. We are interested to know the probability of the event \( \bigcap_{1 \leq i \leq m} E_i \) since this implies the event that all the inequalities are less than \( t \). This is tailor-made for Lovasz Local Lemma (LLL). We also want to guarantee a large value of the objective function. For example, setting all variables equal to zero would guarantee feasibility but also return an objective function value 0. Thus we define an additional event \( A_{m+1} \) corresponding to the objective function value less than \( (1-\epsilon) \cdot OPT \) for some suitable \( 1 > \epsilon > 0 \). Since \( A_{m+1} \) is a function of all the variables, it has dependencies with all other \( A_i \ where \ i = 1 \ldots m \), therefore we have to use the generalized version of LLL in this case.

**Theorem 4.1 (Lovasz Local Lemma [PL75])** Let \( A_i, 1 \leq i \leq N \) be events such that \( \Pr[A_i] = p \) and each event is dependent on at most \( d \) other events. Then if \( ep(d+1) < 1 \), then 
\[
\Pr[\bigcap_{1 \leq i \leq m} A_i] > 0.
\]

Alternately, in a more general (asymmetric) case, where the dependencies are described by a graph \( ((1, 2, \ldots, N), E) \) where an edge between \( i, j \) denotes dependency between \( A_i, A_j \) and \( y_i \) are real numbers such that \( \Pr(A_i) \leq y_i \cdot \prod_{(i,j) \in E} (1 - y_j) \) then \( \Pr[\bigcap_{i=1}^N \hat{A}_i] \geq \prod_{i=1}^N (1 - y_i) \).

Moreover, such an event can be computed in randomized polynomial time using an algorithm of Moser and Tardos [MT10].

If we choose \( t \) such that \( e \cdot 2^{-t} \cdot (d+1) \leq 1 \) or equivalently \( t = \log d \), then we can apply the previous theorem to obtain a rounding that satisfies an error bound of \( O(\log \rho + \log \log m) \). The error \( t \) can be improved to \( O\left(\frac{\log d}{\log \log d}\right) \) by using a tighter version of the Chernoff bound (Equation [LS]).

We define \( A_{m+1} \) as the event where the objective value is less than \( (1-\epsilon)OPT \). From Chernoff-Hoeffding bounds, we know that \( \Pr(A_{m+1}) \leq \exp(-\epsilon^2 OPT/2) \). We define \( y_i \) for \( i = 1, 2, \ldots m \) as before, corresponding to the probability of exceeding \( t = \log d / \log d \).

By choosing \( y_i = 1/(\alpha d) \) \( i \leq m \) and \( y_{m+1} = \frac{1}{2} \), for some suitable scaling factor \( \alpha \geq e \), we must
Algorithm LLL based Iterative Randomized Rounding

Input: \( x'_i, 1 \leq i \leq n, t \) (error parameter)
Output: \( \hat{x}_i \in \{0, 1\} \)

Do independent rounding on all the variables having values \( x'_i \) to \( \hat{x}_i \).
Compute the value of each constraint \( C_i \) as \( < V_i, \hat{x} > \)

While any inequality exceeds \( t \) or objective value is \( < OPT/2 \)
1. Pick an arbitrary constraint \( C_j \) that exceeds \( t \) and perform independent rounding on all the variables in \( V_j \).
2. Update the value of the constraints whose variables have changed

Return the rounded vector \( \hat{x} \).

Figure 2: An iterative randomized rounding algorithm based on Moser-Tardos

satisfy the following inequalities

\[
\text{Pr}(A_i) \leq 1/(\alpha d)(1 - 1/(\alpha d))^d \cdot \frac{1}{2} \quad i = 1, 2 \ldots m \tag{7}
\]
\[
\text{Pr}(A_{m+1}) \leq \frac{1}{2}(1 - 1/(\alpha d))^m \leq \frac{1}{2} \cdot \exp(-m/(\alpha d)) \tag{8}
\]

The first condition is easily satisfied when \( \text{Pr}(A_i) = \Omega(1/\alpha d) \).
To satisfy the second inequality, we can choose \( \alpha \) so that \( OPT \geq \Omega(\frac{m}{\alpha d}) \). This implies that \( \exp(-m/(\alpha d)) \geq \exp(-\epsilon^2 OPT/2) \), so condition (8) is satisfied for LLL to be applicable.

We summarize our discussion as follows

Lemma 4.1 For \( A \in \{0, 1\}^{m \times n} \) with a maximum of \( \rho \) 1’s in each column, we can round the optimum solution \( x^* \) of the linear program \( \max_x \sum_i c_i x_i \) s.t. \( Ax \leq 1 \), \( 0 \leq x_i \leq 1 \) to \( \hat{x} \in \{0, 1\}^n \) such that

\[
||A\hat{x}||_\infty \leq \max \left( \frac{\log \rho + \log \log m}{\log \log(\rho \log m)}, \frac{\log \left( \frac{m}{OPT} \right)}{\log \left( \frac{m}{OPT} \right)} \right) \quad \text{and} \quad \sum_i c_i \hat{x}_i \geq (1 - \epsilon)OPT
\]

Proof: The error bound follows from the previous discussion by setting \( y_i = \Omega(\frac{1}{\alpha d}) \) and using the stronger form of Chernoff bound in Equation 18. Note that Equation 3 can be satisfied for LLL to be applicable.

\( \frac{m}{OPT} \cdot \log(\frac{m}{OPT}) \) by choosing an appropriately large \( \alpha \).
The above is satisfied for \( \alpha \geq \frac{m}{OPT} \cdot \log(\frac{m}{OPT}) \), i.e. if \( \alpha d = \frac{m}{OPT} \).
In this case the discrepancy is \( \frac{m}{OPT} \cdot \log(\frac{m}{OPT}) \).
Hence the result follows. \( \square \)

The objective function still follows the martingale property since the variables starting the random walk at \( x'_i \) have probability \( x'_i \) of being absorbed at 1 which is identical to the independent rounding that we use in the second phase for Moser-Tardos algorithm. Although, the random walk is short-cut by a single step independent rounding, the distribution for absorption at 0/1 remains unchanged. To see this, consider the last time a variable \( x_i \) is rounded by the Moser-Tardos algorithm - the probability is \( x'_i \) since it is done independently every time.

4.1 Random matrices

Let \( A_{k}^{m \times n} \) denote the family of \( m \times n \) 0-1 matrix with exactly \( k \) 1s in each of the \( m \) rows chosen uniformly at random. Clearly \( x'_i = 1/k \) is a feasible solution with objective function value \( n/k \).
After rounding $x'$ to $\hat{x}$, we want to achieve an objective value $\Omega(n/k)$ 1s in the rounded vector. In addition, $A \cdot x' \leq t \cdot e^m$ for error guarantee $t$.

To compute the dependency $d$ for $C_r$, we observe that another constraint $C_i$ will contain a 1 in $j_1(r)$ with probability $\frac{k}{n}$, i.e., if it had one of the $k$ randomly chosen 1’s in that column, which is $\frac{k}{n}$. Since all the rows were chosen independently, the expected number of rows among $m$ rows that have a 1 in column $j_1(r)$ can be bounded by $\frac{mk}{n}$ and by $O(\max(\frac{mk}{n}, \log m))$ with probability greater than $1 - \frac{1}{m}$. Since this holds for all the positions, $j_1(r)$, by the union bound, and using $\text{max}\{a, b\} \leq (a + b)$, we can claim the following

**Claim 2** The total number of constraints that are correlated to $C_r$ can be bounded by $O(\frac{mk \log m}{n} + \log^2 m)$ with high probability.

In our context, for $k = \log m$, $d = O(\frac{m \log^2 m}{n} + \log m^2)$. We can verify this by exhaustively computing the dependence - if it exceeds this then our algorithm is deemed to have failed which is bounded by inverse polynomial probability. If we choose $t$ such that $e \cdot 2^{-t} \cdot (d + 1) \leq 1$ or equivalently $t = \Omega(\log(\frac{m \log^2 m}{n} + \log^2 m))$, then we can apply the previous theorem to obtain a rounding that satisfies an error bound of $O(\log(\frac{m \log^2 m}{n} + \log^2 m))$.

For $m$ bounded by $n \log^{O(1)} m$ this is $O(\log m)$ which is substantially better than the Raghavan-Thompson bound. However for $m \geq n^{1+\epsilon}$ for any constant $\epsilon > 0$, it is no better.

**Proof:** (Completing the Proof of Theorem 1.2) Now we formally define our bad events $E_i$ here. \forall 1 \leq i \leq m, define $E_i \equiv (A^{\theta}_i x > \delta).$ (where we choose error to be $\delta$). Define $E_{m+1} \equiv (c^T x < (1 - \epsilon)OPT)$. We have $Pr(E_i) < \frac{\delta^2}{1 + \epsilon^{1+\epsilon}}$, and $Pr(E_{m+1}) < e^{-2^{\log^{OPT}}}$. We choose $y_i = \frac{1}{\alpha \cdot d}$ for some $\alpha \geq 1$. and choose $y_{m+1} = \frac{1}{2}$. To apply LLL we need, $Pr(E_i) < \frac{\alpha \cdot d}{\log(\alpha \cdot d)}$, $(1 - \frac{1}{2})$ and $Pr(E_{m+1}) < \frac{1}{2}(1 - \frac{\alpha \cdot d}{m})$. To satisfy these equations we get error, $\delta = \frac{\log(\alpha \cdot d)}{\log(\log(\alpha \cdot d))} + \frac{1}{2}$.

Also we get $e^{-2^{\log^{OPT}}} < e^{\frac{m}{n^2}}$. The second equation is satisfied for $OPT > \frac{m}{n^2}$ or $\alpha \cdot d > \frac{m}{OPT}$.

Combining the 2 results we have error $\delta$ in constraints as $\text{max} \left\{ \frac{\log(d)}{\log(\log(d))}, \frac{\log(\frac{OPT}{m})}{\log(\log(\frac{OPT}{m}))} \right\}$ which is $\text{max} \left\{ \frac{\log(m \log^2 m)}{\log(\log(m \log^2 m))}, \frac{\log(m \log^2 m)}{\log(\log(m \log^2 m))} \right\}$. When $\epsilon_1 \geq \frac{1}{2}$, we have $OPT \geq \frac{n}{k \cdot p}$. Substituting the value of OPT in the equation and using the fact that $\frac{mk}{n} = O(\log(m))$, we prove Theorem 1.2 The expected running time for the second phase is $O(\frac{m^2}{n^2} \cdot \log(n))$ in our case, due to application of Moser Tardos. Here the error bound may fail for the additional reason that the dependence may exceed $\frac{mk \log \log(n)}{n}$ after the Brownian motion phase. This can happen with probability $\frac{1}{m^2}$. This part is related the input distribution of $A^t_{mk \times n}$ and repeating the algorithm may not work. \qed

**Remark** A direct application of LLL (without running the Brownian motion) would increase the dependence to $O(\frac{mk^2 \log n}{n} + \log m \log n)$. In order to maintain the same asymptotic error bound, the number of rows in the matrix, $m$, could be significantly less. For example, if $m, k = n^{1/2}$, then the difference would be $O(\log n/ \log \log n)$ versus $O(\log \log n/ \log \log \log n)$.

The proof of Theorem 1.3 is along similar lines after appropriate scaling and is given in the appendix.

## 5 Applications of our rounding results

In this section we briefly sketch the applications of our rounding theorems. Although these do not significantly improve prior results, our parameterization could simplify further applications.
5.1 Application to Switching circuits

Consider an \( n \)-input butterfly network (with \( n \log n \) total nodes) with \( (s_i, t_i) \ i \leq n \log n \) source destination pairs where each input/output node has \( \log n \) sources and \( \log n \) destinations. For any arbitrary instance of routing, we can do a two phase routing with a random intermediate destination, say, we choose a random intermediate destination \( r_i \) for the source-destination pair \( (s_i, t_i) \). We want to route a maximum number of pairs subject to some edge capacity constraints.

If \( n = 2^L \), then the the expected congestion \( k \) of an edge for a random permutation is \( \log n \cdot \frac{2^L - 1}{2^L} = \log n \) and moreover it can be bounded by \( c \log n \) with high probability for some constant \( c \). So there exists a fractional solution with flow value \( \frac{1}{\log n} \) for each of the \( n \log n \) paths with objective function value \( n \).

Let \( A \) be an \( m \times t \) matrix where \( m \) is the number of edges and \( t = n \log n \) is the number of paths. The edges are denoted by \( e_1, e_2 \ldots e_m \) and the paths are denoted by \( \Pi_1, \Pi_2 \ldots \Pi_t \). Then \( A_{i,j} = 1 \) iff the flow \( \Pi_j \) passes through edge \( e_i \). The number of edges in a path is bounded by \( L = \log n \). The value of the flow through path \( \Pi_j \) (denoted by \( f_j \)) is the amount of flow and let \( \bar{f} \) be the vector denoting all the flows. Then \( A \cdot \bar{f} = \bar{e} \) where \( \bar{e} = (c_1, c_2 \ldots c_m) \) is the vector corresponding to the congestion in edges \( (e_1, e_2 \ldots e_m) \).

Since the congestion is bounded by \( c \log n \), \( A \in \mathcal{A}_{c \log n}^{m \times n \log n} \) and there exists a fractional solution \( \bar{f} = (1/c \log n, 1/c \log n \ldots 1/c \log n) \) with objective value \( \Omega(n) \). Using \( \rho = O(\log n) \) in Lemma 4.1 we can round it to a 0-1 solution that yields the following result.

**Theorem 5.1** In an \( n \) input BBN butterfly network having \( 2(n \log n) \) edges, we can route the \( \Omega(n) \) source-sink pairs with congestion \( O(\log \log n / \log \log \log n) \). From the capacity constraint, it follows that no node contains more than \( O(\log \log n) \) sources or destination.

Note that we can also handle weighted objective functions.

The above result matches the previous results of [MS99, CMM+98] which have the advantage of being online. Using \( B = \log \log n / \log \log \log n \) in Lemma 4.2 we can obtain an optimal bound. Note that, in this case we can match asymptotically the optimal fractional flow of \( n \log \log n / \log \log \log n \) with \( x_i = \frac{\log \log n}{c \log \log n \log \log n} \).

**Theorem 5.2** In an \( n \) input multi-butterfly network having \( n \log n \) edges, we can route \( \Omega(n \log \log n / \log \log \log n) \) flows with congestion \( O(\log \log n / \log \log n) \) where each node can be the source or sink of at most \( O(\log \log n / \log \log n) \) flows.

The above result marginally extends a similar result of Maggs and Sitaraman [MS99] where they can route \( (n/\log^{1/2} n) \) pairs in the online case. It remains an open problem to find a fast online implementation of our rounding algorithms.

5.2 Maximum independent set of rectangles

Consider a \( \sqrt{n} \times \sqrt{n} \) grid and a set of axis-parallel rectangles that are aligned with the grid points, i.e., the upper-left and the lower-bottom corners are incident on the grid. Each rectangle contains \( A \) grid points for some \( A \leq n \) but they can have any aspect ratio. Trivially, the different types of such rectangles can be at most \( A \) where \( A = a \cdot b \) for \( a = 1, 2 \ldots A \). Two such rectangles \( r_1 \) and \( r_2 \) are overlapping if \( r_1 \cap r_2 \) contains one or more grid points.

From the \( A \) possible rectangles whose upper-right corners are anchored at a specific grid point \( p \), the input consists of one such rectangle. To avoid messy calculations, we assume that the grid is actually embedded on a torus. Given a set \( S \) of \( n \) such rectangles, our goal is to select a large non-overlapping set of rectangles. Although this is a restricted version of the general MISR problem it still captures many applications.

\footnote{We can assume that all the grid-points that are flush with the sides are in the interior by slightly enlarging the rectangle}
For any given grid point $p$, let $S(p)$ denote the set of rectangles containing $p$. This can be bounded by $\sum_{a=1}^{A} a \times A/a = O(A^2)$. After solving the relevant packing linear program, the $n \times n$ point-rectangle matrix $A$ contains 1 in the $(i, j)$ position if the $i$-th grid point is incident on the $j$-th rectangle. From our previous observation, no row contains more than $A^2$ 1's.

By setting $k = A^2$, we can apply our rounding results to obtain a wide range of trade-offs for this problem. For example, if $A$ is polylog($n$), then using $\rho = A$ in Lemma 1.1, we can choose $\Omega(OPT)$ rectangles with no more than $O(\log n/\log \log \log n)$ overlapping rectangles on any clique, where $OPT$ is the fractional optimal solution of the corresponding packing problem.

Further, using a result of [LENO04], we can obtain $\Omega(OPT \log \log^2 n / \log^2 n)$ non-overlapping rectangles as well as a $\frac{\log \log^2 n}{\log \log^2 n}$ approximate solution in the weighted case.

### 5.3 $b$-matching in random hypergraphs

In a hypergraph $H$ on $n$ vertices has hyper-edges defined by subsets of vertices. We wish to choose the maximum set of hyper-edges such that no more than $b$ hyperedges are incident on any vertex. An integer linear program can be written easily for this problem with $n$ constraints and $m$ variables corresponding to each of the $m$ edges. Let $A$ be an $n \times m$ matrix where $A_{i,j} = 1$ if vertex $i$ is incident on the $j$-th hyperedge. Let $x_i = \{0, 1\}$ depending on if the $i$-th hyperedge is selected in the $b$-matching.

$$\text{Maximize } \sum_i x_i \text{ s.t. } Ax \leq b \cdot e^m \quad x_i \in \{0, 1\}$$

The relaxed LP can be solved and the solution may be rounded. The weighted version can be formulated similarly by associating weights $w_i$ for $x_i$.

If the $m$ edges of $H$ are random subsets of vertices, then we can represent the fractional solution as an $n \times m$ random matrix with each entry of the matrix set to 1 with probability $k/n$. That is, each hyperedge has $k$ vertices and chooses $k$ vertices randomly. So the expected number of edges incident on a vertex is $\frac{mk}{n}$. There exists a feasible fractional solution using $x_i, 1 \leq i \leq m = \frac{n}{mk}$ with objective value $OPT \geq n/k$.

For $b = \Omega(\log \log n/\log \log \log n)$, we can obtain a $b$ matching of size $\Omega(OPT)$ for most hypergraphs which matches the best possible size given by the fractional optimum. In general, we obtain an approximation $k^{1/b}$ for $b \geq 2$ using the result of Theorem 1.3 including the weighted version.

It is known that $k$-uniform $b$-matching problem cannot be approximated better than $\frac{1}{\log k}$ for $b \leq k/\log k$ unless $P = NP$ [OFS11] [HSS06]. So our result shows that the bound can be much better for many $k$-uniform hypergraphs, for example $k = \log n$ and $b \geq 2$. A closely related result for bounded column case was observed by Srinivasan [Sri96].

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A Appendix

Proof of Lemma 2.2
Proof: The proof of the gambler’s ruin problem actually uses a quadratic martingale \( B^2(t) - t \) where \( B(t) \) is the Brownian motion random variable. Using the optional stopping theorem on this, we obtain
\[
E[B^2(T) - T] = E[B^2(0) - 0] = a^2
\]
Since \( p = \frac{b}{a+b} \) is the probability that it is at 0, we obtain
\[
E[T] = a \cdot b.
\]
Now, consider the events leading to the failure of being absorbed at 0 - these correspond to the absorption of the random walk at either ends or the non-absorption at either ends.
\[
\Pr[Failure] = \Pr[Failure \cap Absorption] + \Pr[Failure \cap Non-\text{absorption}]
\]
\[
= \Pr[Failure|Absorption] \cdot \Pr[Absorption] + \Pr[Failure|Non-\text{absorption}] \cdot \Pr[Non-\text{absorption}]
\]
\[
\leq \Pr[Failure|Absorption] + \Pr[Non-\text{absorption}]
\]
From the preceding argument, the probability that the random walk is not absorbed after \( ka \cdot b \) steps is less than \( \frac{1}{k} \) using Markov’s inequality. From optional stopping criteria, the probability of absorption at 0 is \( \frac{b}{a+b} \). The probability that the random walk is not absorbed at 0 after \( kab \) steps is bounded by \( \frac{1}{1 - \frac{b}{a+b} + 1/k} \). Consequently, the probability of being absorbed at 0 is at least \( \frac{b}{a+b} - 1/k \).

A.1 A Lower Bound on error

Lemma A.1 For \( A \in A_{\log n}^{m \times n} \) such that \( A \cdot \bar{x} \leq e^m \), we cannot simultaneously obtain \( \sum_i \hat{x}_i = \Omega(n/\log n) \) and \( ||\hat{x}||_{\infty} \leq o(\log \log \log n) \) for \( \hat{x}_i \in \{0, 1\} \).

Proof:

As mentioned before, it is easy to see that \( x_i = 1/k \) is a solution for \( \bar{x} \) with objective function value \( n/k \). After rounding \( \bar{x} \) we want to have at least \( \Omega(n/k) \) 1s in the rounded vector, say \( \bar{y} \). We must have \( A \cdot \bar{y} \leq t \cdot e \) for some rounding guarantee \( t \). Let \( b \) be a fixed vector with \( n/k \) 1s.

Let \( r \) be a row vector with \( k \) 1s in random location - this corresponds to a row of \( A \). The probability that \( < r, \bar{y} > \geq t \) is given by
\[
\frac{\binom{n/k}{t} \cdot \binom{n-n/k}{k-t}}{\binom{n}{k}}
\]
Using \( \binom{n}{k} \leq (\frac{ne}{k})^{k} \), the above expression is
\[
\geq \frac{\binom{n/k}{t} \cdot (\frac{2n/k}{k-t})^{k-t}}{\binom{n}{k}} \geq \frac{1}{t} \cdot (\frac{k-1}{k})^{k-t} \geq \frac{k^{k-t}}{(k-t)^{k-t}} e^{k-t} \geq \frac{1}{1-t/k} e^{k-t} \geq \frac{1}{1-t/k} e^{k-t} \text{for } k \gg t
\]
\[
= \frac{1}{e^{k-t} t^t}
\]

\(8\)The proof that it is a martingale can be found in standard books on stochastic process
So the probability that for $m$ independently chosen rows, the probability that $< r, y > < t$ is

$$p(m, k, t) \leq \left(1 - \frac{1}{e^{k/t^2} \cdot t^l}\right)^m \leq \left(1 - \frac{1}{e^{k \cdot t^l}}\right)^m = \left(1 - \frac{1}{e^{k \cdot t^l}}\right)^m$$

(15)

$$\leq \left(\frac{1}{e\cdot k^t}\right)^m$$

(16)

Since there are no more than $\left(\frac{n}{\log n}\right)$ choice of columns with $n/\log n$ 1s, the probability that all the dot products are less than $t$ is less than $\left(\frac{n}{\log n}\right) \cdot p(m, k, t) \leq p(m, k, t) \cdot \left(\frac{ne}{n/\log n}\right)^{n/\log n} \leq e^n \cdot p(m, k, t)$. If $e^n \cdot p(m, k, t) < 1$ then there must exist matrices in $A_k^{m \times n}$ that do not satisfy the error bound $t$. So,

$$1 > \left(\frac{1}{e}\right)^m \cdot e^n \Rightarrow (e)^m > e^n$$

which is equivalent to the condition

$$\frac{m}{n} > e^{k \cdot t^l} \text{ or } \log(m/n) > k \cdot t \cdot log t$$

(17)

For $k = \log n$ and $m = n \cdot \text{polylog}(n)$, this holds for some $t \geq \frac{\alpha \log \log n}{\log \log \log n}$, i.e., for some constant $\alpha > 0$, i.e., the error cannot be of $\left(\frac{\log \log n}{\log \log \log n}\right)$.

\[\square\]

### A.2 Proof of the general approximation bound: Proof of Theorem 1.3

If we are interested in maintaining feasibility of constraints, we can ensure it by sacrificing the value of the objective function according to some trade-offs. The same algorithm works, where we use the known method of damping the probabilities of rounding ([RT87][Sri95]).

Let $X_1, X_2, \ldots, X_u$ denote the unabsorbed random variables in a constraint after the Brownian walk phase and let $U = \sum_i X_i = \frac{B}{\beta}$ for some $\beta \geq 1$. Let $\hat{X}_i$ be the $\{0, 1\}$ random variables where $Pr[\hat{X}_i = 1] = X_i$ and so $E[\sum_i \hat{X}_i] = \sum_i X_i = \frac{B}{\beta}$. To apply LLL, we need $Pr[\sum_i \hat{X}_i > B] < \frac{1}{\beta}$.

Since $U = \sum_i \hat{X}_i$, then from Chernoff bounds, we know that

$$Pr[U \geq (1 + \Delta)E[U] \leq \left[\frac{e^{\Delta}}{(1 + \Delta)^{1+\Delta}}\right]E[U]$$

(18)

Using $E[U] = \frac{B}{\beta}$ and $\beta = 1 + \Delta$ we obtain $Pr[U \geq B] \leq \left[\frac{e^{\beta-1}}{(\beta)^{\beta}}\right] \leq \left(\frac{\beta}{\beta}\right)^B \leq \frac{1}{\alpha d}$ for some scale parameter $\alpha \geq 1$. It follows that

$$(\beta/e)^B > 1/\alpha d$$

(19)

Therefore $\beta = e(d\alpha)^{1/B}$. Since our matrices are 0-1, the minimum infeasibility happens at $B + 1$, so we can substitute $B + 1$ instead of $B$ in the previous calculations to obtain $\beta = e(d\alpha)^{1/B+1}$. Using $S \geq 1$ as the initial scaling for Brownian walk, we obtain the following result using Lemma 3.2.

**Lemma A.2** For $B \geq 1$, we can round the fractional solution to a feasible integral solution with objective function value $\Omega(OPT/S)$ for $S=\max((\frac{m}{OPT})^{1/17}, d^{1/4})$. If $B \in Z, B$ can be replaced by $B+1$ in above expression and proof

**Proof:** If we start from $\hat{z} = \hat{x}'$, the solution produced by brownian walk satisfies the constraints with RHS $\frac{B}{\beta}$ (as shown in Lemma 3.2).

Now we define events for LLL similar to the proof of Lemma 1.1.
∀1 ≤ i ≤ m define $E_i \equiv (A^T_i . x > B)$.
and define $E_{m+1} \equiv c^T . x < (1 - \epsilon).OPT$.
As shown above, ∀1 ≤ i ≤ m, $Pr(E_i) \leq (\frac{\epsilon}{\beta})^B$.
Also by chernoff’s bound $Pr(E_{m+1}) \leq e^{-\epsilon^2 OPT/2}$.
We apply LLL with weights $y_i = \frac{1}{\alpha d}$, for 1 ≤ i ≤ m
and $y_{m+1} = \frac{1}{2}$.
, for some appropriately defined $\alpha \geq 1$.
Now we need $(\frac{\epsilon}{\beta})^B \leq \frac{1}{\alpha d}(1 - \frac{1}{\alpha d}) \frac{d}{2}$
and $e^{-\epsilon^2 OPT/2m} \leq \frac{1}{2}(1 - \frac{1}{\alpha d})^m \leq \frac{1}{2} e^{-\frac{m}{2\alpha d}}$(From Equation (8))
The above is satisfied for $\frac{OPT}{\alpha d} \geq \frac{m}{\alpha d}$
Thus we need to choose $\frac{OPT}{\alpha d} \geq \frac{m}{\alpha d}$
OR $\beta = (\alpha d)^{1/B} \geq (\frac{m}{OPT})^{\frac{1}{n-1}}$.
Also $\alpha \geq 1$. For $B \in \mathbb{Z}$, it would have sufficed to use $B' = B + 1$ as opposed to choosing B which proves the stated lemma.

To prove Theorem 1.3, we can observe that for $c_i \geq \frac{1}{p}$, we have $OPT \geq \frac{m}{np}$. For $\frac{mk}{n} = O(\log(m))$, we get the scaling as $\max((\log(m))^{\frac{1}{n}}, (p \log(m))^\frac{1}{n})$, which proves Theorem 1.3.