A note on mixture representations for the Linnik and Mittag-Leffler distributions and their applications

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Abstract: We present some product representations for random variables with the Linnik, Mittag-Leffler and Weibull distributions and establish the relationship between the mixing distributions in these representations. The main result is the representation of the Linnik distribution as a normal scale mixture with the Mittag-Leffler mixing distribution. As a corollary, we obtain the known representation of the Linnik distribution as a scale mixture of Laplace distributions. Another corollary of the main representation is the theorem establishing that the distributions of random sums of independent identically distributed random variables with finite variances converge to the Linnik distribution under an appropriate normalization if and only if the distribution of the random number of summands under the same normalization converges to the Mittag-Leffler distribution.

1 Introduction

Usually the Mittag-Leffler and Linnik distributions are mentioned in the literature together as examples of geometric stable distributions. Since these distributions are very often pointed at as weak limits for geometric random sums, there might have emerged a prejudice that the scheme of geometric summation is the only asymptotic setting within which these distributions can be limiting for sums of independent and identically distributed random variables. This prejudice is accompanied by the suspicion that non-trivial \((\delta < 1, \alpha < 2)\) Mittag-Leffler and Linnik laws can be limiting only for sums in which the summands have infinite variances.

Another obvious reason for which the Mittag-Leffler and Linnik distributions are often brought together in the literature is the formal similarity of the Laplace transform of the former and the Fourier–Stieltjes transform of the latter.

The main aim of the present paper is to study the analytic and asymptotic relations between these two laws. We will show that actually the link between these two laws is much more interesting. Namely, it turns out that the Linnik distribution with parameter \(\alpha\) is a scale mixture of the normal distributions with the mixing distribution being the Mittag-Leffler law with parameter \(\delta = \alpha/2\). As a corollary of this result we obtain a theorem establishing that the distributions of random sums of independent identically distributed random variables with finite variances converge to the Linnik distribution under an appropriate normalization if and only if the distribution of the random number of summands under the same normalization converges to the Mittag-Leffler distribution.

Our main tools are mixture representations for the Linnik, Mittag-Leffler and Weibull distributions. Mixture representations for the Linnik and Mittag-Leffler laws were the objects of investigation in [3, 5, 6, 16, 23, 19, 20]. Some of the results of these papers will be used in what follows and will be presented as lemmas below. As well, the proofs of our results are based on some new mixture representations for the Weibull distributions.

However, we are unaware of the results concerning the possibility of representation of the Linnik distribution as a scale mixture of normals. Perhaps, the paper [16] is the closest to this conclusion and exposes the representability of the Linnik law as a scale mixture of Laplace distributions with the mixing distribution written out explicitly.

We develop the results of [16] and obtain our main result which is the representation of the Linnik distribution as a normal scale mixture with the Mittag-Leffler mixing distribution, thus finding a tight and clear analytical link between the Linnik and Mittag-Leffler distributions. As a corollary, we have the representation of the Linnik distribution as a scale mixture of Laplace distributions obtained in [16]. Our proof of this result together with the explicit formula for the mixing density obtained in [16] lead to a by-product corollary of this representation which

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is the explicit formula for the distribution density of the ratio of two independent positive strictly stable random variables. Another consequence of the main representation is the theorem establishing that the distributions of random sums of independent identically distributed random variables with finite variances converge to the Linnik distribution under an appropriate normalization if and only if the distribution of the random number of summands under the same normalization converges to the Mittag-Leffler distribution. On the one hand, this theorem offers a new «asymptotic» link between the Linnik and Mittag-Leffler distributions and, on the other hand, it shows that the Linnik law can be limiting for the distribution of sums of independent identically distributed random variables with finite variances thus dispelling the suspicion mentioned above.

The paper is organized as follows. Section 2 presents the definitions and basic properties of the Linnik and Mittag-Leffler distributions. Section 3 contains all the definitions and auxiliary results. The proofs of our main results are purposely indirect and essentially rely on some new mixture properties of the Weibull distribution also presented in Section 3. In Section 4 we prove the representability of the Linnik distribution as the scale mixture of normal laws with the Mittag-Leffler mixing distribution and as the scale mixture of the Laplace laws with the mixing distribution being that of the ratio of two independent random variables with the same strictly stable distribution concentrated on the nonnegative halfline. We use this representation together with the result of [16] to obtain a by-product corollary which is the explicit representation of the distribution density of the ratio of two independent positive strictly stable random variables. In Section 5 we prove and discuss the theorem establishing that the distributions of random sums of independent identically distributed random variables with finite variances converge to the Linnik distribution under an appropriate normalization if and only if the distribution of the random number of summands under the same normalization converges to the Mittag-Leffler distribution.

2 The Mittag-Leffler and Linnik distributions

2.1 The Mittag-Leffler distributions

The Mittag-Leffler probability distribution is the distribution of a nonnegative random variable \( M_\delta \) whose Laplace transform is

\[
\psi_\delta(s) \equiv E e^{-sM_\delta} = \frac{1}{1 + \lambda s^\delta}, \quad s \geq 0,
\]

where \( \lambda > 0 \), \( 0 < \delta \leq 1 \). For simplicity, in what follows we will consider the standard scale case and assume that \( \lambda = 1 \).

The origin of the term *Mittag-Leffler distribution* is due to that the probability density corresponding to Laplace transform (1) has the form

\[
f_\delta^{ML}(x) = \frac{1}{x^{1+\delta}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{\delta n}}{\Gamma(\delta n + 1)} = -\frac{d}{dx} E_\delta(-x^\delta), \quad x \geq 0,
\]

where \( E_\delta(z) \) is the Mittag-Leffler function with index \( \delta \) that is defined as the power series

\[
E_\delta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\delta n + 1)}, \quad \delta > 0, \quad z \in \mathbb{Z}.
\]

Here \( \Gamma(s) \) is Euler’s gamma-function,

\[
\Gamma(s) = \int_0^{\infty} z^{s-1} e^{-z} dz, \quad s > 0.
\]

The distribution function corresponding to density (2) will be denoted \( F_\delta^{ML}(x) \).

With \( \delta = 1 \), the Mittag-Leffler distribution turns into the standard exponential distribution, that is, \( F_1^{ML}(x) = [1 - e^{-x}] 1(x \geq 0), \ x \in \mathbb{R} \) (here and in what follows the symbol \( 1(C) \) denotes the indicator function of a set \( C \)). But with \( \delta < 1 \) the Mittag-Leffler distribution density has the heavy power-type tail: from the well-known asymptotic properties of the Mittag-Leffler function it can be deduced that

\[
f_\delta^{ML}(x) \sim \frac{\sin(\delta \pi) \Gamma(\delta + 1)}{\pi x^{\delta + 1}}
\]

as \( x \to \infty \), see, e. g., [11].
It is well-known that the Mittag-Leffler distribution is stable with respect to geometric summation (or geometrically stable). This means that if \(X_1, X_2, \ldots\) are independent random variables and \(N_p\) is the random variable independent of \(X_1, X_2, \ldots\) and having the geometric distribution

\[
\mathbb{P}(N_p = n) = p(1-p)^{n-1}, \quad n = 1, 2, \ldots, \quad p \in (0, 1),
\]

then for each \(p \in (0, 1)\) there exists a constant \(a_p > 0\) such that \(a_p(X_1 + \ldots + X_{N_p}) \Rightarrow M_p\) as \(p \to 0\), see, e. g., [2] or [13] (the symbol \(\Rightarrow\) hereinafter denotes convergence in distribution). Moreover, as far ago as in 1965 it was shown by I. Kovalenko [18] that the distributions with Laplace transforms (1) are the only possible limit laws for the distributions of appropriately normalized geometric sums of the form \(a_p(X_1 + \ldots + X_{N_p}) \Rightarrow 0\), where \(X_1, X_2, \ldots\) are independent identically distributed nonnegative random variables and \(N_p\) is the random variable with geometric distribution (3) independent of the sequence \(X_1, X_2, \ldots\) for each \(p \in (0, 1)\). The proofs of this result were reproduced in [7, 8] and [9]. In these books the class of distributions with Laplace transforms (1) was not identified as the class of Mittag-Leffler distributions but was called class \(K\) after I. Kovalenko.

Twenty five years later this limit property of the Mittag-Leffler distributions was re-discovered by A. Pillai in [26, 27] who proposed the term Mittag-Leffler distribution for the distribution with Laplace transform (1). Perhaps, since the works [18, 7, 8] were not easily available to probabilists, the term Mittag-Leffler distribution became conventional. The Mittag-Leffler distributions are of serious theoretical interest in the problems related to thinned (or rarefied) homogeneous flows of events such as renewal processes or anomalous diffusion or relaxation phenomena, see [29, 10] and the references therein.

### 2.2 The Linnik distributions

In 1953 Yu. V. Linnik [22] introduced the class of symmetric probability distributions defined by the characteristic functions

\[
\varphi_{\alpha}(t) = \frac{1}{1 + |t|^\alpha}, \quad t \in \mathbb{R}, \tag{4}
\]

where \(\alpha \in (0, 2]\). Later the distributions of this class were called Linnik distributions [17] or \(\alpha\)-Laplace distributions [25]. In this paper we will keep to the first term that has become conventional. With \(\alpha = 2\), the Linnik distribution becomes the Laplace distribution corresponding to the density

\[
f^\Lambda(x) = \frac{2}{\pi}e^{-|x|}, \quad x \in \mathbb{R}. \tag{5}
\]

A random variable with Laplace density (5) and its distribution function will be denoted \(\Lambda\) and \(F^\Lambda(x)\), respectively.

The Linnik distributions possess many interesting analytic properties such as unimodality [21] and infinite divisibility [3], existence of an infinite peak of the density for \(\alpha < 1\) [3], etc. However, perhaps, most often Linnik distributions are recalled as examples of geometric stable distributions.

A random variable with the Linnik distribution with parameter \(\alpha\) will be denoted \(L_\alpha\). Its distribution function and density will be denoted \(F^L_\alpha\) and \(f^L_\alpha\), respectively. As this is so, from (4) and (5) it follows that \(F^L_2(x) \equiv F^\Lambda(x), \quad x \in \mathbb{R}\).

### 3 Basic notation and auxiliary results

Most results presented below actually concern special mixture representations for probability distributions. However, without any loss of generality, for the sake of visuality and compactness of formulations and proofs we will formulate the results in terms of the corresponding random variables assuming that all the random variables mentioned in what follows are defined on the same probability space \((\Omega, \mathfrak{A}, \mathbb{P})\).

The random variable with the standard normal distribution function \(\Phi(x)\) will be denoted \(X\),

\[
\mathbb{P}(X < x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}dz, \quad x \in \mathbb{R}.
\]

Let \(\Psi(x), \quad x \in \mathbb{R}\), be the distribution function of the maximum of the standard Wiener process on the unit interval, \(\Psi(x) = 2\Phi\left(\max\{0, x\}\right) - 1, \quad x \in \mathbb{R}\). It is easy to see that \(\Psi(x) = \mathbb{P}(|X| < x)\). Therefore, sometimes \(\Psi(x)\) is said to determine the half-normal or folded normal distribution.

Throughout the paper the symbol \(\equiv\) will denote the coincidence of distributions.
The distribution function and density of the strictly stable distribution with the characteristic exponent $\alpha$ and shape parameter $\theta$ defined by the characteristic function
\[
f_{\alpha,\theta}(t) = \exp\left\{-|t|^{\alpha}\exp\left\{-\frac{1}{2}i\theta \text{sgn}(t)\right\}\right\}, \quad t \in \mathbb{R},
\]
with $0 < \alpha \leq 2$, $|\theta| \leq \min\{1, \frac{2}{\alpha} - 1\}$, will be denoted by $G_{\alpha,\theta}(x)$ and $g_{\alpha,\theta}(x)$, respectively (see, e.g., [30]). Any random variable with the distribution function $G_{\alpha,\theta}(x)$ will be denoted $S_{\alpha,\theta}$.

From (6) it follows that the characteristic function of a symmetric ($\theta = 0$) strictly stable distribution has the form
\[
f_{\alpha,0}(t) = e^{-|t|^\alpha}, \quad t \in \mathbb{R}.
\]
(7)

From (7) it is easy to see that $S_{2,0} \overset{d}{=} \sqrt{2}X$.

**Lemma 1.** Let $\alpha \in (0, 2]$, $\alpha' \in (0, 1]$. Then
\[
S_{\alpha',0} \overset{d}{=} S_{\alpha,0}^{1/\alpha'}
\]
where the random variables on the right-hand side are independent.

**Proof.** See, e.g., theorem 3.3.1 in [30].

**Corollary 1.** A symmetric strictly stable distribution with the characteristic exponent $\alpha$ is a scale mixture of normal laws in which the mixing distribution is the one-sided strictly stable law ($\theta = 1$) with the characteristic exponent $\alpha/2$:
\[
G_{\alpha,0}(x) = \int_{0}^{\infty} \Phi(x/\sqrt{2z})dG_{\alpha/2,1}(z), \quad x \in \mathbb{R}.
\]

In terms of random variables the statement of corollary 1 can be written as
\[
S_{\alpha,0} \overset{d}{=} X \sqrt{2S_{\alpha/2,1}}
\]
(8)

with the random variables on the right-hand side being independent.

Let $\gamma > 0$. The distribution of the random variable $W_{\gamma}$:
\[
P(W_{\gamma} < x) = [1 - e^{-x/\gamma}]1(x \geq 0), \quad x \in \mathbb{R},
\]
is called the *Weibull distribution* with shape parameter $\gamma$. It is obvious that $W_{1}$ is the random variable with the standard exponential distribution: $P(W_{1} < x) = [1 - e^{-x}]1(x \geq 0)$. The Weibull distribution with $\gamma = 2$, that is, $P(W_{2} < x) = [1 - e^{-x^2}]1(x \geq 0)$ is called the Rayleigh distribution.

It is easy to see that if $\gamma > 0$ and $\gamma' > 0$, then $P(W_{\gamma'}^{1/\gamma} \geq x) = P(W_{\gamma'} \geq x'^{\gamma}) = e^{-x'^{\gamma'}} = P(W_{\gamma'} \geq x), \quad x \geq 0$, that is, for any $\gamma > 0$ and $\gamma' > 0$
\[
W_{\gamma'} \overset{d}{=} W_{\gamma}^{1/\gamma'.}
\]
(9)

In particular, for any $\gamma > 0$ we have $W_{\gamma} \overset{d}{=} W_{1}^{1/\gamma}$.

It can be shown that each Weibull distribution with parameter $\gamma \in (0, 1]$ is a mixed exponential distribution. In order to prove this we first make sure that each Weibull distribution with parameter $\gamma \in (0, 2]$ is a scale mixture of the Rayleigh distributions.

For $\alpha \in (0, 1]$ denote $V_{\alpha} = S_{\alpha,1}^{-1}$, where $S_{\alpha,1}$ is a random variable with one-sided strictly strictly stable density $g_{\alpha,1}(x)$.

**Lemma 2.** For any $\gamma \in (0, 2]$ we have
\[
W_{\gamma} \overset{d}{=} W_{2} \sqrt{V_{\gamma/2}},
\]
(10)

where the random variables on the right-hand side of (10) are independent.

**Proof.** Write relation (8) with $\alpha$ replaced by $\gamma$ in terms of characteristic functions with the account of (7):
\[
e^{-|t|^\gamma} = \int_{0}^{\infty} \exp\{-t^2z\}g_{\gamma/2,1}(z)dz, \quad t \in \mathbb{R}.
\]
(11)

Formally letting $|t| = x$ in (11), where $x \geq 0$ is an arbitrary nonnegative number, we obtain
\[
P(W_{\gamma} > x) = e^{-x/\gamma} = \int_{0}^{\infty} \exp\{-x^2z\}g_{\gamma/2,1}(z)dz.
\]
(12)
At the same time it is obvious that if $W_2$ and $S_{\gamma/2,1}$ are independent, then
\[
P(W_2 \sqrt{V_{\gamma/2}} > x) = P(W_2 > x \sqrt{S_{\gamma/2,1}}) = \int_0^\infty \exp[-x^2z]g_{\gamma/2,1}(z)dz.
\] (13)

Since the right-hand sides of (12) and (13) coincide identically in $x \geq 0$, the left-hand sides of these relations coincide as well. The lemma is proved.

**Lemma 3.** For any $\gamma \in (0, 1]$, the Weibull distribution with parameter $\gamma$ is a mixed exponential distribution:
\[
W_\gamma \overset{d}{=} W_1 V_\gamma,
\] (14)
where the random variables on the right-hand side of (14) are independent.

**Proof.** It is easy to see that $P(W_1^{1/\gamma} \geq x) = P(W_1 \geq x^\gamma) = e^{-x^\gamma} = P(W_\gamma \geq x)$, $x \geq 0$, that is,
\[
W_\gamma \overset{d}{=} W_1^{1/\gamma}.
\] (15)

From (15) it follows that $W_2 \overset{d}{=} \sqrt{W_1}$. Therefore, from lemma 2 it follows that for $\gamma \in (0, 2]$ we have
\[
W_\gamma \overset{d}{=} W_2 \sqrt{V_{\gamma/2}} \overset{d}{=} \sqrt{W_1 V_{\gamma/2}}
\]
or, with the account of (15),
\[
W_{\gamma/2} \overset{d}{=} W_2^{\gamma} \overset{d}{=} W_1 V_{\gamma/2}.
\]
Re-denoting $\gamma/2 \mapsto \gamma \in (0, 1]$, we obtain the desired assertion.

In [3] the following statement was proved. Here its formulation is extended with the account of (9).

**Lemma 4** [3]. For any $\alpha \in (0, 2]$, the Linnik distribution with parameter $\alpha$ is a scale mixture of a symmetric stable distribution with the Weibull mixing distribution with parameter $\alpha/2$, that is,
\[
L_\alpha \overset{d}{=} S_{\alpha,0} W_\alpha \overset{d}{=} S_{\alpha,0} \sqrt{W_{\alpha/2}},
\]
where the random variables on the right-hand side are independent.

**Lemma 5.** For any $\delta \in (0, 1]$, the Mittag-Leffler distribution with parameter $\delta$ is a scale mixture of a one-sided stable distribution with the Weibull mixing distribution with parameter $\delta/2$, that is,
\[
M_\delta \overset{d}{=} S_{\delta,1} W_\delta \overset{d}{=} S_{\delta,1} \sqrt{W_{\delta/2}},
\]
where the random variables on the right-hand side are independent.

**Proof.** This statement has already become folklore. For the purpose of convenience we give its elementary proof without any claims for priority. Let $S_{\delta,1}$ be a positive strictly stable random variable. As is known, its Laplace transform is $\psi(s) = E e^{-sS_{\delta,1}} = e^{-sz^\delta}$, $s \geq 0$. Then with the account of (9) by the Fubini theorem the Laplace transform of the product $S_{\delta,1} W_\delta$ is
\[
E \exp\{-sS_{\delta,1} W_\delta\} = E \exp\{-sS_{\delta,1} W_1^{1/\delta}\} = EE\{\exp\{-sS_{\delta,1} W_1^{1/\delta}\}|W_1\} = \int_0^\infty e^{-(sz^{1/\delta})^\delta} e^{-z}dz =
\[
= \int_0^\infty e^{-z(s^{1/\delta}+1)}dz = \frac{1}{1+s^\delta} = E e^{-sM_\delta}, \quad s \geq 0.
\]
The lemma is proved.

Let $\rho \in (0, 1)$. In [19] it was demonstrated that the function
\[
f_\rho^K(x) = \frac{\sin(\pi\rho)}{\pi\rho^2 x^2 + 2x\cos(\pi\rho) + 1}, \quad x \in (0, \infty),
\] (16)
is a probability density on $(0, \infty)$. Let $K_\rho$ be a random variable with density (16).

**Lemma 6** [19]. Let $0 < \delta < \delta' \leq 1$ and $\rho = \delta/\delta' < 1$. Then
\[
M_\delta \overset{d}{=} M_{\delta'} K_\rho^{1/\delta}
\]
where the random variables on the right-hand side are independent.

With $\delta' = 1$ we have

**Corollary 2** [19]. Let $0 < \delta < 1$. Then the Mittag-Leffler distribution with parameter $\delta$ is mixed exponential:

$$M_\delta \overset{d}{=} K_\delta^{1/\delta}W_1$$

where the random variables on the right-hand side are independent.

Let $0 < \alpha < \alpha' \leq 2$. In [16] it was shown that the function

$$f_{\alpha,\alpha'}^Q(x) = \frac{\alpha' \sin(\pi\alpha/\alpha')x^{\alpha-1}}{\pi[1 + x^{2\alpha} + 2x^\alpha \cos(\pi\alpha/\alpha')]}$$

$x > 0$,

is a probability density on $(0, \infty)$. Let $Q_{\alpha,\alpha'}$ be a random variable whose probability density is $f_{\alpha,\alpha'}^Q(x)$.

**Lemma 7** [16]. Let $0 < \alpha < \alpha' \leq 2$. Then

$$L_\alpha \overset{d}{=} L_{\alpha',Q_{\alpha,\alpha'}}$$

where the random variables on the right-hand side are independent.

With $\alpha' = 2$ we have

**Corollary 3** [16]. Let $0 < \alpha < 2$. Then the Linnik distribution with parameter $\alpha$ is a scale mixture of Laplace distributions corresponding to density (5):

$$L_\alpha \overset{d}{=} \Lambda Q_{\alpha,2}$$

where the random variables on the right-hand side are independent.

For the sake of completeness, we will demonstrate that the Weibull distributions possess the same property as the Linnik and Mittag-Leffler distributions presented in lemmas 6 and 7: any distribution of the corresponding class can be represented as a scale mixture of a distribution from the same class with larger parameter.

Relation (14) implies the following statement generalizing lemmas 2 and 3 and stating that the Weibull distribution with an arbitrary positive shape parameter $\gamma$ is a scale mixture of a Weibull distribution with an arbitrary positive shape parameter $\gamma' > \gamma$.

**Lemma 8**. Let $\gamma' > \gamma > 0$ be arbitrary numbers. Then

$$W_\gamma \overset{d}{=} W_{\gamma'} \cdot V_\alpha^{1/\gamma'}$$

where $\alpha = \gamma/\gamma' \in (0, 1)$ and the random variables on the right-hand side are independent.

**Proof.** In lemma 3 we showed that a Weibull distribution with parameter $\alpha \in (0, 1]$ is a mixed exponential distribution. Indeed, from (14) it follows that

$$e^{-x^\alpha} = P(W_\alpha > x) = P(W_1 > S_{\alpha,1}x) = \int_0^{\infty} e^{-z^\alpha} g_{\alpha,1}(z)dz, \quad x \geq 0.$$

Therefore, for any $\gamma' > \gamma > 0$, denoting $\alpha = \gamma/\gamma'$ (as this is so, $\alpha \in (0, 1)$), for any $x \in \mathbb{R}$ we obtain

$$P(W_\gamma > x) = e^{-x^\gamma} = e^{-x^{\gamma'}} = P(W_\alpha > x^{\gamma'}) = P(W_1 > S_{\alpha,1}x^{\gamma'}) =$$

$$= \int_0^{\infty} e^{-z^{\gamma'}} g_{\alpha,1}(z)dz = \int_0^{\infty} P(W_{\gamma'} > z^{1/\gamma'}) g_{\alpha,1}(z)dz = P(W_{\gamma'} \cdot V_\alpha^{1/\gamma'} > x),$$

The lemma is proved.

It should be noted that if $0 < \gamma < \gamma' < 2$, then the assertion of lemma 8 directly follows from theorem 3.3.1 of [30] due to the formal coincidence of the characteristic function of a strictly stable law and the complementary Weibull distribution function (see the proof of lemma 2).
4 Representation of the Linnik distribution as a scale mixture of normal or Laplace distributions and related results

In all the products of random variables mentioned below the multipliers are assumed independent.

The following statement presents the main result of this paper.

**Theorem 1.** Let $\alpha \in (0, 2]$, $\alpha' \in (0, 1]$. Then

$$L_{\alpha\alpha'} \overset{d}{=} S_{\alpha, 0} M_{\alpha'}^{1/\alpha}. \tag{18}$$

**Proof.** From lemma 4 we have

$$L_{\alpha\alpha'} \overset{d}{=} S_{\alpha\alpha'} M_{1/\alpha}. \tag{19}$$

Continuing (18) with the account of lemma 1, we obtain

$$L_{\alpha\alpha'} \overset{d}{=} S_{\alpha, 0} S_{\alpha'}^{1/\alpha} \sqrt{W_{\alpha\alpha'}/2}. \tag{20}$$

From (9) and lemma 5 it follows that

$$S_{\alpha', 1} \sqrt{W_{\alpha\alpha'}/2} \overset{d}{=} S_{\alpha', 1}^{1/\alpha} W_{\alpha\alpha'}/\alpha \overset{d}{=} M_{\alpha'/2}. \tag{21}$$

The theorem is proved.

As far as we know, the following result has never been explicitly presented in the literature in full detail although the property of the Linnik distribution to be a normal scale mixture is something almost obvious.

**Corollary 4.** For each $\alpha \in (0, 2]$, the Linnik distribution with parameter $\alpha$ is the scale mixture of zero-mean normal laws with mixing Mittag-Leffler distribution with twice less parameter $\alpha/2$:

$$L_{\alpha} \overset{d}{=} X \sqrt{2 M_{\alpha/2}},$$

where the random variables on the right-hand side are independent.

It should be noted that in this case representation (19) takes the form

$$L_{\alpha} \overset{d}{=} X \sqrt{2 S_{\alpha/2, 1} W_{\alpha/2}}. \tag{20}$$

From lemma 3 we have

$$W_{\alpha/2} \overset{d}{=} \frac{W_1}{S_{\alpha/2, 1}}. \tag{21}$$

Hence, from (20) it follows that

$$L_{\alpha} \overset{d}{=} X \sqrt{2 W_1 S_{\alpha/2, 1} S_{\alpha'/2, 1}}$$

where the independent random variables $S_{\alpha/2, 1}$ and $S'_{\alpha'/2, 1}$ have one and the same one-sided strictly stable distribution with characteristic exponent $\alpha/2$ and are independent of the exponentially distributed random variable $W_1$. It is well known that

$$X \sqrt{2 W_1} \overset{d}{=} \Lambda \tag{21}$$

(see, e. g., the example on p. 272 of [1]). Therefore we obtain one more mixture representation for the Linnik distribution.

**Theorem 2.** For each $\alpha \in (0, 2]$, the Linnik distribution with parameter $\alpha$ is the scale mixture of the Laplace laws corresponding to density (5) with mixing distribution being that of the ratio of two independent random variables having one and the same one-sided strictly stable distribution with characteristic exponent $\alpha/2$:

$$L_{\alpha} \overset{d}{=} \Lambda \sqrt{S_{\alpha/2, 1} S'_{\alpha/2, 1}},$$

where the random variables on the right-hand side are independent.
It is easy to see that scale mixtures of Laplace distribution (5) are identifiable, that is, if

\[ \Lambda Y \overset{d}{=} \Lambda Y' \]

where \( Y \) and \( Y' \) are nonnegative random variables independent of \( \Lambda \), then \( Y \overset{d}{=} Y' \). Indeed, with the account of (21), the last relation turns into

\[ X \sqrt{2W_1Y^2} \overset{d}{=} X \sqrt{2W_1(Y')^2}, \]

where the random the multipliers on both sides are independent. But, as is known, scale mixtures of zero-mean normals are identifiable (see [28]). Therefore, (22) implies that

\[ W_1Y^2 \overset{d}{=} W_1(Y')^2. \]  

The complementary mixed exponential distribution functions of the random variables related by (23) are the Laplace transforms of \( Y^2 \) and \( (Y')^2 \), respectively. Relation (23) means that these Laplace transforms identically coincide:

\[ \int_0^\infty e^{-sz}dP(Y^2 < z) \equiv \int_0^\infty e^{-sz}dP((Y')^2 < z), \quad s \geq 0. \]

Hence, the distributions of the random variables \( Y^2 \) and \( (Y')^2 \) coincide and hence, the distributions of \( Y \) and \( Y' \) coincide as well since these random variables were originally assumed nonnegative.

Comparing the statement of theorem 2 with the assertion of corollary 3 with the account of identifiability of scale mixtures of Laplace distributions (5) we arrive at the relation

\[ Q_{\alpha,2} \overset{d}{=} \frac{S_{\alpha/2,1}}{S'_{\alpha/2,1}} \]

The combination of (17) and (24) gives one more, possibly simpler, proof of the following by-product result concerning the properties of stable distributions obtained in [4]. This result offers an explicit representation for the density of the ratio of two independent stable random variables in terms of elementary functions although with the exception of one case, the Lévy distribution (\( \alpha = \frac{1}{2} \)), such representations for the densities of nonnegative stable random variables themselves do not exist.

**Corollary 5.** Let \( S_{\alpha,1} \) and \( S'_{\alpha,1} \) be two independent random variables having one and the same one-sided strictly stable distribution with characteristic exponent \( \alpha \in (0,1) \). Then \( S_{\alpha,1}/S'_{\alpha,1} \overset{d}{=} K_{\alpha}^{1/\alpha} \overset{d}{=} Q_{2\alpha,2}^{\alpha/2} \), that is, the probability density \( p_{\alpha}(x) \) of the ratio \( S_{\alpha,1}/S'_{\alpha,1} \) has the form

\[ p_{\alpha}(x) = f_{\alpha,1}^{Q} (x) = \frac{\sin(\pi \alpha)x^{\alpha-1}}{\pi [1 + x^{2\alpha} + 2x^\alpha \cos(\pi \alpha)]}, \quad x > 0. \]

The proof immediately follows from the observation that \( p_{\alpha}(x) = (2\sqrt{x})^{-1}f_{2\alpha,2}^{Q}(\sqrt{x}) = f_{\alpha,1}^{Q}(x) \).

As concerns the Mittag-Leffler distribution, from lemmas 3 and 5 we obtain the following statement analogous to theorem 2.

**Theorem 3.** For each \( \delta \in (0,1] \), the Mittag-Leffler distribution with parameter \( \delta \) is the mixed exponential distribution with mixing distribution being that of the ratio of two independent random variables having one and the same one-sided strictly stable distribution with characteristic exponent \( \delta \):

\[ M_{\delta} \overset{d}{=} W_1 \frac{S_{\delta,1}}{S'_{\delta,1}}, \]

where the random variables on the right-hand side are independent.

From theorem 3 and corollary 5 we obtain the following representation of the Mittag-Leffler distribution function \( F_{\delta}^{ML}(x) \):

\[ F_{\delta}^{ML}(x) = 1 - \frac{\sin(\pi \delta)}{\pi} \int_0^\infty \frac{z^{\delta-1}e^{-zx}}{1 + z^\delta + 2z^\delta \cos(\pi \delta)}, \quad x > 0. \]  

Representation for the Linnik distribution similar to (25) was obtained in [19]. Using lemmas 1, 4 and 5 it is possible to obtain more product representations for the Mittag-Leffler- and Linnik-distributed random variables and hence, more mixture representations for these distributions.
5 Convergence of the distributions of random sums to the Linnik distribution

Product representations for the random variables with the Linnik and Mittag-Leffler distributions obtained in the previous works were aimed at the construction of convenient algorithms for the computer generation of pseudo-random variables with these distributions. The mixture representation for the Linnik distribution as a scale mixture of normals obtained in corollary 4 opens the way for the construction in this section of a random-sum central limit theorem with the Linnik distribution as the limit law. Moreover, in this version of the random-sum central limit theorem the Mittag-Leffler distribution must be the limit law for the normalized number of summands.

Recall that the symbol \( \Rightarrow \) denotes the convergence in distribution.

Consider a sequence of independent identically distributed random variables \( X_1, X_2, \ldots \) defined on the probability space \( (\Omega, \mathcal{A}, P) \). Assume that \( E X_1 = 0, 0 < \sigma^2 = DX_1 < \infty \). For \( n \in \mathbb{N} \) denote \( S_n^* = X_1 + \ldots + X_n \). Assume that \( \sum_{j=1}^{\infty} = 0 \).

Recall that a random sequence \( Z_1, Z_2, \ldots \) is said to infinitely increase in probability \( (Z_n \xrightarrow{P} \infty) \), if \( P(Z_n \leq m) \to 0 \) as \( n \to \infty \) for any \( m \in (0, \infty) \).

The proof of the main result of this section is based on the following version of the random-sum central limit theorem.

**Lemma 10.** Assume that the random variables \( X_1, X_2, \ldots \) and \( Z_1, Z_2, \ldots \) satisfy the conditions specified above and, moreover, let \( Z_n \xrightarrow{P} \infty \) as \( n \to \infty \). A distribution function \( F(x) \) such that

\[
P\left( \frac{S_n^*}{\sigma \sqrt{n}} < x \right) \Rightarrow F(x)
\]

as \( n \to \infty \) exists if and only if there exists a distribution function \( H(x) \) satisfying the conditions

\[
H(0) = 0, \quad F(x) = \int_0^\infty \Phi\left( \frac{x}{\sqrt{y}} \right) dH(y), \quad x \in \mathbb{R},
\]

and \( P(Z_n < nx) \Rightarrow H(x) \) \( (n \to \infty) \).

**Proof.** This statement is a particular case of a result proved in [14], also see theorem 3.3.2 in [9].

The following theorem gives a criterion (that is, necessary and sufficient conditions) of the convergence of the distributions of random sums of independent identically distributed random variables with finite variances to the Linnik distribution.

**Theorem 4.** Let \( \alpha \in (0, 2] \). Assume that the random variables \( X_1, X_2, \ldots \) and \( Z_1, Z_2, \ldots \) satisfy the conditions specified above and, moreover, let \( Z_n \xrightarrow{P} \infty \) as \( n \to \infty \). Then the distributions of the normalized random sums \( S_n^* \) converge to the Linnik law with parameter \( \alpha \), that is,

\[
P\left( \frac{S_n^*}{\sigma \sqrt{n}} < x \right) \Rightarrow F^L_\alpha(x)
\]

as \( n \to \infty \), if and only if

\[
\frac{Z_n}{n} \Rightarrow 2M_{\alpha/2} \quad (n \to \infty).
\]

**Proof.** This statement is a direct consequence of corollary 4 and lemma 10 with \( H(x) = F^ML_{\alpha/2}(x/2) \).

The convergence of the distributions of the normalized indices \( Z_n/n \) to the Mittag-Leffler distribution \( F^ML_{\alpha} \) is the main condition in theorem 4. Now we will give two examples of the situation where this condition can hold. The first example is trivial and is based on the geometric stability of the Mittag-Leffler distribution. The second example relies on a useful general construction of nonnegative integer-valued random variables which, under an appropriate normalization, converge to a given nonnegative (not necessarily discrete) random variable, whatever the latter is.
EXAMPLE 1. Let $\delta \in (0, 1)$ be arbitrary. For every $n \in \mathbb{N}$ let $N_{1/n}$ be a random variable having the geometric distribution (3) with $\rho = \frac{1}{n}$ independent of the sequence $Y_1, Y_2, \ldots$ of independent identically distributed nonnegative random variables such that

$$n^{-1/\delta} \sum_{j=1}^{N_{1/n}} Y_j \Longrightarrow 2M_\delta$$

(26)
as $n \to \infty$. To provide (26), the distributions of the random variables $Y_1, Y_2, \ldots$ should belong to the domain of the normal attraction of the one-sided strictly stable law with characteristic exponent $\delta$. As $Z_n$ for each $n \in \mathbb{N}$ take

$$Z_n = \left[ n^{1-1/\delta} \sum_{j=1}^{N_{1/n}} Y_j \right],$$

where square brackets denote the integer part. Then

$$\frac{Z_n}{n} = n^{-1/\delta} \sum_{j=1}^{N_{1/n}} Y_j - \frac{1}{n} \left\{ n^{1-1/\delta} \sum_{j=1}^{N_{1/n}} Y_j \right\},$$

(27)
where curly braces denote the fractional part. Since the second term on the right-hand side of (27) obviously tends to zero in probability, from (26) it follows that $Z_n/n \Longrightarrow 2M_\delta$ as $n \to \infty$.

EXAMPLE 2. In the book [9] it was proposed to model the evolution of non-homogeneous chaotic stochastic processes, in particular, the dynamics of financial markets by compound doubly stochastic Poisson processes (compound Cox processes). This approach got further grounds and development, say, in [1, 15]. According to this approach the flow of informative events, each of which generates the next observation, is described by the stochastic point process $P(Y(t))$ where $P(t), t \geq 0$, is a homogeneous Poisson process with unit intensity and $Y(t), t \geq 0$, is a random process independent of $P(t)$ and possessing the properties: $Y(0) = 0$, $P(Y(t) < \infty) = 1$ for any $t > 0$, the trajectories $Y(t)$ are non-decreasing and right-continuous. The process $P(Y(t)), t \geq 0$, is called a doubly stochastic Poisson process (Cox process) [12].

Within this model, for each $t$ the distribution of the random variable $P(Y(t))$ is mixed Poisson. For vividness, consider the case where in the model under consideration the parameter $t$ is discrete: $Y(t) = Y(n) = Y_n, n \in \mathbb{N}$, where $\{Y_n\}_{n \geq 1}$ is an infinitely increasing sequence of nonnegative random variables such that $Y_{n+1}(\omega) \geq Y_n(\omega)$ for any $\omega \in \Omega, n \geq 1$. Here the asymptotics $n \to \infty$ may be interpreted as that the intensity of the flow of informative events is (infinitely) large.

Assume that the random variable $M_\delta$ is independent of the standard Poisson process $P(t), t \geq 0$. For each natural $n$ take $Y_n = 2nM_\delta$. Respectively, let $Z_n = P(Y_n) = P(2nM_\delta), n \geq 1$. It is obvious that the random variable $Z_n$ so defined has the mixed Poisson distribution

$$P(Z_n = k) = P(P(2nM_\delta) = k) = \int_0^\infty e^{-2nz} \frac{(2nz)^k}{k!} dF_\delta^{ML}(z) \quad k = 0, 1, \ldots$$

This random variable $Z_n$ can be interpreted as the number of events registered up to time $n$ in the Poisson process with the stochastic intensity distributed as $2M_\delta$.

Denote $A_n(z) = P(Z_n < 2nz), z \geq 0 (A_n(z) = 0$ for $z < 0)$. It is easy to see that $A_n(z) \Longrightarrow F_\delta^{ML}(z)$ as $n \to \infty$. Indeed, as is known, if $\Pi(x; \ell)$ is the Poisson distribution function with the parameter $\ell > 0$ and $E(x; c)$ is the distribution function with a single unit jump at the point $c \in \mathbb{R}$, then $\Pi(\ell x; \ell) \Longrightarrow E(x; 1)$ as $\ell \to \infty$.

Since for $x \in \mathbb{R}$

$$A_n(x) = \int_0^\infty \Pi(2nx; 2nz) dF_\delta^{ML}(z),$$

then by the Lebesgue dominated convergence theorem, as $n \to \infty$, we have

$$A_n(x) \Rightarrow \int_0^\infty E(x/z; 1) dF_\delta^{ML}(z) = \int_0^x dF_\delta^{ML}(z) = F_\delta^{ML}(x),$$

that is, the random variables $Z_n$ defined above satisfy the condition of theorem 4. Moreover, $Z_n \xrightarrow{P} \infty$ as $n \to \infty$ since $P(M_\delta = 0) = 0$.

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