Resonance phenomena in the annular array of underdamped Josephson junctions

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Appearance and origin of resonance phenomena have been studied in the annular system of underdamped Josephson junctions. If no fluxon is trapped in the system, dynamics is governed by the motion of fluxon-antifluxon pairs, while if trapped fluxons are present, they can move solely but also simultaneously with the pairs. Locking between the rotating excitations (fluxons and antifluxons) and the Josephson frequency leads to the appearance of zero field steps in the current-voltage characteristics, which can further exhibit branching due to resonance between the rotating excitations and plasma oscillations in their tail. The number of zero field steps and their branching are strongly determined by the total number of excitations present in the system. High resolution analysis further reveals not only some interesting properties of zero field steps, but also that the current-voltage characteristics is determined not only by the number, but also by the type of excitations, i.e., whether the dynamics is governed only by the motion of fluxon-antifluxon pairs or the trapped fluxons, or they move simultaneously in the system.

I. INTRODUCTION

Resonance phenomena in Josephson junction (JJ) systems have been for years an active research topic for science and technology. From the fundamental point of view the JJs are an excellent devices for the studies of nonlinear dynamics of discrete systems as they represent an experimental realization of the Frenkel-Kontorova model (discrete sine-Gordon model1,2). At the same time from the point of their applications, JJs are promising devices for development of various fields from generation and detection of electromagnetic radiations in very low terahertz range3, development of quantum information technologies4,5,6,7, superconducting metamaterials4,8,9, to the fields as far as biology10.

The systems described by the sine-Gordon equations can exhibit three types of dynamical solutions: small-amplitude waves, solitons and breathers11,12. In the JJ systems, the soliton solution has a real physical meaning since it corresponds to a quantum of flux trapped into a junction also called fluxon or Josephson vortex13. The idea that fluxon behaves as a particle-like solitary wave, which can be manipulated and controlled, motivated creation of a new logic circuit by using Josephson fluxon as elementary bits of information14,15,16. In the creation of a new logic elements, particularly important are the long Josephson junctions described by continuous sine-Gordon equation, and the Josephson junctions parallel array by its discreet counterpart i.e. Frenkel-Kontorova model17. The very good overview of the most prominent fluxon dynamics results in both natural and man-made systems can be found in Ref. 1.

Dynamics of long Josephson junctions have been subjects of numerous theoretical and experimental studies18-24. However, in long JJs, motion of fluxon strongly depends on the geometry and boundaries of the junctions, which act as a mirror reflecting fluxon into antifluxon (this process can be viewed as a fluxon-antifluxon collision). Thus, the current-voltage (I-V) characteristics of long JJs could be very complicated which makes studies of fluxon dynamics very challenging. These problems led to the creation of annular Josephson junction as well as annular systems of Josephson junctions with a ring-shape geometry25. Annular junctions provide an undisturbed and very tunable fluxon motion, which today makes them an ideal system for the studies of fluxon dynamics26,27

One of the interesting properties of JJs systems is the appearance of resonant steps in the current-voltage characteristics in the absence of any external radiation28-33. Though the first experimental indication about their existence have been earlier34, the name zero filed steps (ZFS) for the observed resonant structures and the first theoretical explanation in terms of vortex motion inside the long junction were introduced by Fulton and Dyens in their pioneering work on long Josephson junction35. The always present need for an undisturbed fluxon motion without collisions further motivated studies of resonance phenomena in annular systems36-38, where the appearance of ZFSs can be interpreted in terms of circulating motion of fluxons and antifluxons.

In this paper, we will examine the underdamped dynamics of an annular array of Josephson junctions (AAJJ) particularly focusing on the origin and appear-
ance of various resonance phenomena in the absence external radiation. In contrast to the previous studies of annular Josephson junctions, which were mainly focused on the zero field steps in the case of usually one trapped fluxon in a small range of currents and voltages, here, we will examine the resonance phenomena of the AJJJ in various cases (without trapped fluxons, when dynamics is characterized by the motion of fluxon-antifluxon pairs, as well as in the case with trapped fluxons) in a wide range of currents and voltages in order to get full picture of dynamical behavior. Our results show that dynamics of the AJJJ strongly depends not only on the number but also on the type of excitations, i.e., whether there are only trapped fluxons or the fluxon-antifluxon pairs in the system, or the trapped fluxons circulate simultaneously with fluxon-antifluxon pairs.

The paper is organized as follows. The model is introduced in Sec. II, while the simulation results are presented in Sec. III-VII. Influence of the discreteness of the system on the appearance of ZFS is examined in Sec. III, while their branching is analyzed in Sec. IV. The correlation between the current-voltage characteristics in the case without and in the case with trapped fluxons is presented in Sec. V. The appearance of the effect, which we call a pulsating fluxon, on the current-voltage characteristics is shown in Sec. VI, while the influence of the type of excitations on the system dynamics is examined in Sec. VII. Finally, Sec. VIII concludes the paper.

II. MODEL

We consider an annular parallel array of $N$ underdamped Josephson junctions.\cite{23,24} The total length of a chain is $L = Na$, where $a$ is a distance between the neighboring junctions. In order to derive dynamical equations for a description of such discrete system, we shall start from the equations for the continuous annular JJ. In fact, the annular JJ is actually a long JJ with a periodic boundary conditions, and so, we will start from the equations for a long JJ. The Josephson junction is considered to be long or short if its length is longer or shorter than the Josephson penetration depth $\lambda_J = \sqrt{\hbar S/(2e\mu_0 DL_s)}$, respectively. Here $S$ is the surface area of superconducting layer, $e$ is electron charge, $\mu_0$ is magnetic constant, $D = 2\lambda_L + d$ is effective magnetic thickness, $\lambda_L$ is London penetration depth, and $d$ is the thickness of insulating layers.

According to the resistively and capacitively shunted junction (RCSJ) model\cite{37,38} the total current through the junction is a sum of the Josephson supercurrent, a quasiparticle (resistive normal) current, and a displacement (capacitive) current:

$$I = I_s + I_{qp} + I_d,$$  

where $\varphi$ and $V$ are the phase difference and voltage across the junction, while $R$ and $C$ are resistance and capacitance of the JJ, respectively. The voltage $V$ is given by the Josephson relation:

$$V = \frac{\hbar}{2e} \frac{d\varphi}{dt} = \frac{\hbar}{2e} \omega_J,$$  

where $\omega_J$ is the Josephson frequency. In the case of long JJ, the surface current $I_{sf}$ along the superconducting layer given as

$$I_{sf} = L_3 \lambda_J^2 \frac{d^2 \varphi}{dx^2},$$

should also be taken into account.

Using the Eq. (1), (2) and (3), the total or bias current $I$ through the junction can be written as

$$\frac{h}{2e} C \frac{d^2 \varphi}{dt^2} - I_c \lambda_J^2 \frac{d^2 \varphi}{dx^2} + I_c \sin \varphi + \frac{h}{2e} R \frac{d\varphi}{dt} = I.$$  

In the normalized form, the equation (4) for the phase difference in JJ can be simplified as

$$\frac{d^2 \varphi}{dt^2} - \alpha \frac{d\varphi}{dt} + \sin \varphi + \alpha \varphi = I,$$  

where the time is normalized with respect to the inverse plasma frequency $\omega_p^{-1}$, $\omega_p = \sqrt{2\pi I_c/(\Phi_0 C)}$, the coordinate is normalized with respect to $\lambda_J$, and the bias current $I$ with respect to the critical current $I_c$. The dissipation parameter is given as $\alpha = \sqrt{\Phi_0/(2\pi L_s R^2 C)}$ where $L_0$, $C$ and $R$ are the inductance, capacitance and the differential resistance of a single cell, respectively, and $\Phi_0 = \frac{\hbar}{2e}$ is the flux quantum.\cite{29} The equation (5) represents the well known perturbed sine-Gordon equation. The boundary conditions for the long JJ described by the Eq. (5) have the form $\varphi / \partial x |_{x=0} = \varphi / \partial x |_{x=L} = H_{ext}$, which for the annular case are periodic, i.e. $\varphi(x = 0) = \varphi(x = L) + 2\pi M$, where $H_{ext}$ is external magnetic field and $M$ represents number of trapped fluxons inside the system.

The annular system that we are considering here can be described by the discrete version of perturbed sine-Gordon equation, which is well known as the dissipative Frenkel-Kontorova model.\cite{27}

$$\frac{d^2 \varphi_i}{dt^2} - \frac{\varphi_{i+1} + 2\varphi_i + \varphi_{i-1}}{a^2} + \sin \varphi_i + \alpha \frac{d\varphi_i}{dt} = I,$$  

where $\varphi_i$ is the phase difference across the $i$-th junction. The coupling between the neighboring junctions is described by the constant $\frac{1}{a}$, where $a = \sqrt{2\pi L_0 I_c/\Phi_0}$ is the discreteness parameter, i.e., distance between two junctions normalized to the $\lambda_J$.

The Eq. (6) can be linearized around the solution consisting of traveling kink (fluxon) and a small linear wave (perturbation).\cite{24} The dispersion law for linearized waves is given as:

$$\omega_m = \sqrt{1 + \frac{4}{a^2} \sin^2 \left( \frac{\pi na}{L} \right)}.$$  

where \( m \) is an integer.

In order to calculate the I-V characteristic of the AAJJ we have used the Eq. (6) and the Josephson relation:

\[
V_i = \frac{d\varphi_i}{dt} = \omega_J,
\]

where \( V_i \) is the voltage of \( i \)th junction normalized to \( V_0 = \hbar \omega_p/2e \).

Our numerical simulations were performed for the periodic boundary conditions, which in discrete case have the form:

\[
\varphi_{N+1} = \varphi_1 + 2\pi M, \quad \varphi_0 = \varphi_N - 2\pi M,
\]

where the spatial points \( i = 0 \) and \( i = N+1 \) were assumed to be equivalent to \( i = N \) and \( i = 1 \). We have applied the well known procedure used in Ref. 40 and 41. The algorithm for calculating the I-V characteristic consists of several stages. First of all, for the given value of the bias current \( I \) the Eqs. (6) and (8) are solved numerically in the time interval \([0, T_{max}]\) using the fourth order Runge-Kutta method with the corresponding boundary conditions (9) and the initial conditions \( \varphi_i(0) = 0, V_i(0) = 0 \) at \( t = 0 \). As a result we obtained the time dependence of \( \varphi_i(t) \) and \( V_i(t) \) for the fixed value of the bias current. Then, using the expression:

\[
\langle V_i \rangle = \frac{1}{T_{max} - T_{min}} \int_{T_{min}}^{T_{max}} V_i(t) dt,
\]

where \( T_{min} \) is the time necessary for system to reach the steady state, we calculated the average voltage for each junction in the system. Further, the total average voltage \( V \) is obtained using the expression \( V = \sum \langle V_i \rangle / N \). In this way, for the given value of bias current \( I \) the corresponding voltage \( V \) was found. Next, we change the bias current for some value \( \Delta I \) and repeat the above procedure in order to obtain the next point. By repeating this procedure for every value of the current, the I-V characteristic is produced. We note that the solutions \( \varphi_i(T_{max}) \) and \( V_i(T_{max}) \) obtained for the time \( T_{max} \) at certain value of \( I \) are used as the initial condition for the calculation of next point at the value of bias current \( I + \Delta I \).

The magnetic field in the array have been calculated by the expressions:

\[
B_1 = \frac{\varphi_1 - \varphi_N + 2\pi M}{a}, \quad B_i = \frac{\varphi_i - \varphi_{i-1}}{a}.
\]

During our analysis, the calculation of magnetic field time dependence in JJs was often necessary in order to understand the origin of the observed features in the I-V characteristics.

### III. ZERO FIELD STEPS

In the AAJJ that we consider, in the absence of any external radiation, rotating excitations (fluxons and antifluxons) are passing repeatedly through the junctions, which leads to resonance between the circulating excitations and Josephson frequency. The signature of this effects are the zero field steps in the I-V characteristics of the system. Depending on the system properties and the circulating excitations, these steps can exhibits various interesting properties.

#### A. Zero field steps in the systems near continuum

In nonlinear systems, which exhibit resonance phenomena, discreteness plays an important role. Let us then examine first how discreteness of the AAJJ affects the zero field steps. We will start from the case close to the continuum limit. If we have an AAJJ of the length \( L = Na = 10 \), then, the near continuum limit can be achieved by placing \( N = 100 \) junctions at the distance \( a = 0.1 \) along the circle. In Fig. 1 the current-voltage characteristic of the annular system with 100 Josephson junctions is presented for \( M = 0 \), and \( M = 1 \) in (a) and (b), respectively, while the result for the long Josephson junction is shown in (c) for comparison. The presented curves are produced in the following way: starting from zero, the current is increased until the system transfers to the high voltage state. Then, it is decreased to zero revealing the staircase structure, after that it is increased again till the system is in the high voltage state. From there, the current is decreased to the first step. From the
first step it is again increased to the high voltage state, and from there it is decreased to the next step. In that way, the current was swept several times up and down, and the procedure repeated for all the steps.

These steps in Fig. 1 represent the well known zero field steps that appear when the voltage $V$, i.e., Josephson frequency satisfies the resonant condition $\omega_J = \frac{2\pi nu}{L}$, where $u$ is a speed of moving fluxon (antifluxon). Here, $n$ is the total number of excitations in the system, which can be written as $n = n_f + n_{af} = 2n_p + M$ where $n_f$, $n_{af}$ and $n_p$ represent the number of fluxons, antifluxons and fluxon-antifluxons pairs (FAP), respectively.

The AAJJ is a topologically closed system, therefore the number of trapped fluxons and new fluxons can be created only in the form of fluxon-antifluxon pairs. In the case when $M = 0$ (there are no trapped fluxons), dynamics is characterized by the motion of fluxon-antifluxon pairs. Thus, in Fig. 1 (a) the system exhibits only even mode resonances, where $n$ is an even number, which corresponds to the number of fluxons and antifluxons created in pairs in the system. If we introduce one fluxon, in which case $M = 1$, we can see in Fig. 1 (b) that only the odd mode resonances appear, where $n$ corresponds to the sum the trapped fluxon and the number of fluxons and antifluxons, which come in pairs. On the other hand, the long JJ in Fig. 1 (c) is not topologically closed, and it can exhibit both even and odd mode resonances.

B. Zero field steps in discrete systems

In highly discrete systems the fluxon dynamics changes significantly. In order to discretize our system, we reduce the number of junctions to $N = 10$, while keeping the same total length $L = 10$ ($a = 1$) as in previous case. In Fig. 2 the I-V characteristics for the case $M = 0$, when there are no fluxons trapped in the system, is presented. The curves are produced following the same procedure as in Fig. 1. In this case, the resonances come from the motion of fluxon-antifluxon pairs, and the four ZFSs correspond to $n = 2, 4, 6$ and 8. If we compare this result with the one in Fig. 1 (a), we can see that in the highly discrete case the number of steps is significantly reduced. This reduction of resonances comes from the fact that in order for a resonance to appear there should be a certain balance between the number of rotating excitations and the junctions in the system, i.e., the existence of stable fluxon-antifluxon pairs requires that the system is sufficiently large to permit the pair dynamics.

When the number of rotating fluxons and antifluxons increases, the time interval between their two consecutive passages is getting reduced. Consequently they are constantly passing through the junctions with the frequency much higher than $\omega_J$ leaving no time for any resonance to appear. Further, we will present some of the interesting phenomena that we observed in our examination of zero field steps.

IV. BRANCING OF THE ZERO FIELD STEPS

If we perform the high resolution analysis of the zero field steps in Fig. 2, a complex structure is revealed. In Fig. 3 the enlarged regions of the I-V curves corresponding to the each of four zero field steps are presented in (a) to (d), respectively. The first two steps in Fig. 3 (a) and (b) exhibit branching. Namely, an excitations moving...
through the system excites small-amplitude linear waves in its tale, which are called Josephson plasma waves due to their plasma-type dispersion relation (see Eq. (7)). The resonance appears due to frequency locking between the moving excitation and the small-amplitude oscillations or plasma waves (created by its motion), which results in the appearance of a series of branches at the zero field steps in the I-V characteristics of the junction. While the ZFSs appear in both continuous and discrete systems, branching can be observed only in discrete ones [23,24]. When \( M = 0 \), the dynamics is characterized by the motion of FAPs, i.e., the equal number of fluxons and antifluxons are circling in the opposite direction to each other along the AAJJ. The branch of order \( n \) appears when two consecutive passing of excitations through a junction corresponds to the \( n \)th oscillation of the plasma (linear) wave. As we can see in Fig. 3 (a), for \( n = 2 \), where one pair, i.e., one fluxon and one antifluxon rotate, the ZFS besides the main branch, which corresponds to the frequency \( \omega_j = 1.107 \), has also additional branches, corresponding to the frequencies \( \omega_j = 0.985, 0.882, 0.823, 0.69 \), and 0.635. For two FAPs in the system in Fig. 3 (b), the position of the ZFS \( n = 4 \) corresponds to the frequency \( \omega_j = 2.185 \), with additional two branches corresponding to the frequencies \( \omega = 1.977 \) and 1.87. On the other hand, in Fig. 3 (c) and (d) no branching appears in the case of ZFSs with \( n = 6 \) (\( \omega_j = 3.234 \)) and \( n = 8 \) (\( \omega_j = 3.81 \)), respectively.

Let us now examine the three highest branches \( \omega_j = 1.107, 0.985, \) and 0.823 in Fig. 3 (a), and analyze the time dependence of the magnetic field \( B \) at the end of each branch for the values of the current \( I \) marked by arrows. The time dependence of magnetic field \( B \) at different branches of the \( n = 2 \) ZFS is presented in Fig. 4. For the lowest branch in Fig. 4 (a), we could see that each consecutive passage of fluxon (antifluxon) corresponds to the 5th plasma oscillation. At the next branch in Fig. 4 (b), as the speed and the frequency of moving fluxon increases, the time between two consecutive passages of excitations decreases, and the branch appears due to the resonance with the 4th plasma oscillation. In Fig. 4 (c), which corresponds to the third branch, the speed of fluxon and antifluxon increases further so that their motion locked with the third plasma oscillation.

Examination of the time dependence of the magnetic field can also give us an answer, why the branching is most prominent for \( n = 2 \) in Fig. 3 (a), and as the number of pairs, i.e., fluxons and antifluxons, increases to \( n = 4 \) in Fig. 3 (b), the number of branches decreases, and completely disappears for \( n = 6 \) and 8 in Fig. 3 (c) and (d), respectively. In Fig. 5 the time dependence of magnetic field \( B \) corresponding to zero field steps \( n = 2, 4, 6, \) and 8 in (a), (b), (c) and (d), respectively, measured at the value of current \( I \), which corresponds to the resonant point at the end of each step.

with their motion. As the number of fluxon-antifluxon pairs increases, the time between each passage is getting more and more reduced. Large number of fluxons and antifluxons will therefore leave no time for plasma oscillations between each of their passage as we can see in Fig. 5 (c) and (d), and consequently, there would be no resonance. Resonances between the fluxon and plasma waves have been studied previously in annular system of under-
damped Josephson junctions with one trapped fluxon, where the analysis was focused only on the region of the I-V characteristics, which corresponds to the first step, i.e., one circulating fluxon, and the examination of the voltage time dependence was performed. In our case, instead of voltage it is more appropriate to use magnetic field time dependence since in that case fluxon can be distinguished from antifluxon.

V. THE CORRELATION BETWEEN THE STEPS FOR M = 0 AND M = 1

So far, we presented only the results obtained for M = 0, but we have also performed analysis for the systems with trapped fluxons $M \neq 0$. In Fig. 6 the current-voltage characteristics for the case $M = 0$ and $M = 1$ are presented. As in the case $M = 0$, for $M = 1$, in Fig. 6 (a) four zero field steps will also appear, but this time only for the odd values of $n$ ($n = 1, 3, 5, 7$), so that $n = 1$ corresponds to the one rotating fluxon, $n = 3$ to the one fluxon and one fluxon-antifluxon pair, etc. In the high voltage state there is no difference between the curves for $M = 0$ and $M = 1$, however in the region of ZFSs the curves intersect each other. The steps for even values of $n$ ($M = 0$) alternate with the steps odd values of $n$ ($M = 1$) coming in between each other.

Interestingly, the examination of the I-V characteristics of the AAJJ with one trapped fluxon ($M = 1$) could help us understand some of the features, which appear on the ZFSs, when $M = 0$. Namely, the high resolution analysis of the I-V characteristics for $M = 0$ reveals another interesting property. In Fig. 6 (b), we could see some fine structure (a small peak), that appears on the step $n = 4$, at $I = 0.23$. The origin of this effect could be understood from the comparative analysis of the I-V curves for both values of $M$. As we can see, the appearance of this defect on $n = 4$ step is at the value of current ($I = 0.23$), for which the I-V curve for $M = 1$ changes from $n = 5$ to $n = 3$ state. For given system parameters the ability of the annular Josephson junction to exhibit certain number of FAPs is determined by the current $I$. Thus, as the current is decreased from the high voltage state, at some value of $I$, the AAJJ transfers to the state $n = 8$ (4 fluxons and 4 antifluxons), and as we decrease $I$ further, we can see that at certain values of the current, the AAJJ can be in the state with $n = 7, 6, ..., 1$ excitations. In Fig. 6 (b), at the step $n = 4$ ($M = 0$), we have 2 fluxon-antifluxon pairs moving along the AAJJ, and when the current decreases to the value $I = 0.23$, the AAJJ is in the I-V region which corresponds to $n = 3$ state. However, it is impossible for AAJJ to transfer to that state due to conservation of $M$ ($n_f = n_{af}$ for $M = 0$). On the contrary, the two pairs will, due to their inertia, continue moving till the current decreases to the value, for which the system is reduced to $n = 2$ state with only one fluxon-antifluxon pair. Also in Fig. 6 (b), we could see that the curve for $M = 1$ is slightly shifted to the right comparing to the curve $M = 0$. This might be due to inertial effect, since for $M = 1$ the system contains one more fluxon in addition to fluxon-antifluxon pairs, and more fluxons will simply have more inertia. Thus, when the current is decreased, 3 fluxons and 2 antifluxons ($n = 5$) would keep moving longer than 2 fluxons and 2 antifluxons ($n = 4$).

VI. PULSATING FLUXON

Let us now examine the step $n = 1$ for the AAJJ with one trapped fluxon $M = 1$. The high resolution plot of the I-V curve from the Fig. corresponding to the step $n = 1$ (1 fluxon state) is presented in Fig. 7. As in the previous case for $M = 0$ in Sec. IV, now for $M = 1$ in...
In order to further examine how the type of excitations affects the AAJJ, let us compare two cases: $M = 0$ and $M = 2$. At first one might expect that the I-V characteristics for $M = 0$ and $M = 2$ should be the same since in both cases ZFSs appear for $n = 2, 4, 6, \text{and} 8$, however, this is not the case. In Fig. 7 the I-V characteristics for the case $M = 0$ with no trapped fluxon, and the case $M = 2$ with two trapped fluxons is presented. As we can see in Fig. 7, the steps are shifted respect to each other. On the ZFS $n = 2$, for $M = 0$ we have one fluxon and one antifluxon, while for $M = 2$ we have two fluxons. On the next step $n = 4$, we will have 2 fluxons and 2 antifluxons for $M = 0$, while for $M = 2$ there will be 3 fluxons and one antifluxon, etc. So, though the total number of excitations $n$ is the same for both values of $M$, the numbers of fluxons and antifluxons are different. For $M = 0$ we always have the same number of fluxons and antifluxons $n_f = n_{af}$, while for $M = 2$ in addition to fluxon-antifluxon pairs we have 2 trapped fluxons, so that $n_f \neq n_{af}$ and this completely change the dynamics of the system.

VII. COMPARATIVE ANALYSIS OF THE I-V CHARACTERISTICS FOR $M = 0$ AND $M = 2$

In the high resolution plot of the step $n = 2$ presented in Fig. 7, we can also see that for $M = 2$ voltage goes to zero as $I$ goes to zero, while for $M = 0$ voltage goes to zero around $I = 0.14$. This comes from that fact that for $M = 0$, fluxon-antifluxon pair can exist only above some critical current. So, when $I$ becomes smaller than that value, there are no fluxon-antifluxon pairs, and consequently, $V$ drops to zero. On the other hand, if we have two trapped fluxons, they will be moving as long as $I \neq 0$, and as $I$ goes to zero, their speed and consequently the voltage will go to zero.

VIII. CONCLUSION

In this work, detail analysis of the resonance phenomena in the annular array of Josephson junctions have been presented. The observed effects and their physical origins can be summarized as follows: in the absence of external radiation the system exhibits zero field steps due to locking between the rotating excitations (flux-
ons and antifluxons) and the Josephson frequency; these steps can further exhibit branching due to locking between the rotating excitations and plasma oscillations in their tale. The ability of system to exhibit ZFSs and their branching is determined by the number of excitations in the system. The branching appears always on the lower steps, and as the number of fluxons and antifluxons increases, gradually disappears at the higher steps. Further examinations reveal not only some interesting properties of those steps, such as pulsating fluxon and correlation between the current-voltage characteristics for the cases without and with trapped fluxons, but also show that the dynamics of AAJJ is determined not only by the number, but also by the type of excitations, i.e., the current-voltage characteristics will be completely different depending on whether there are only fluxon-antifluxon pairs or the trapped are also present in the system. In other words, the system with one fluxon and one antifluxon, for example, does not behave the same as the system with two trapped fluxons, though the total number of excitations is the same.

Annular Josephson junctions posses an enormous potential for various technological applications. Superconducting digital technology is capable of achieving much higher energy efficiency than other technologies, and fluxon dynamics as well as resonance phenomena are in the core of some of the most advanced ideas in those fields. Another interesting application of annular Josephson junctions is in superconducting metamaterials with a number of unique properties, which are difficult to achieve in any other way. A generic element of such a material is a superconducting ring split by a Josephson junction, and one of the most recent theoretical and experimental studies have been dedicated to the resonant response of such metamaterials to the external signal in strongly nonlinear regimes. Regardless of the field in which the annular Josephson junctions have application, a good understanding of their dynamics is crucial. For that reason, the resonance phenomena that we presented require further experimental examination that we will present in the future.

FIG. 9: (Color online). (a) The I-V characteristics of the AAJJ for $N = 10$, $a = 1$, $\alpha = 0.1$, $M = 0$ in red and $M = 2$ in green. (b) The high resolution plot of the $n = 2$ step.

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