AUTOMATA FINITENESS CRITERION IN TERMS OF VAN DER PUT SERIES OF AUTOMATA FUNCTIONS

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To Professor Igor Volovich on the occasion of his 65-th birthday

Abstract. In the paper we develop the \( p \)-adic theory of discrete automata. Every automaton \( A \) (transducer) whose input/output alphabets consist of \( p \) symbols can be associated to a continuous (in fact, 1-Lipschitz) map from \( p \)-adic integers to \( p \) integers, the automaton function \( f_A \). The \( p \)-adic theory (in particular, the \( p \)-adic ergodic theory) turned out to be very efficient in a study of properties of automata expressed via properties of automata functions. In the paper we prove a criterion for finiteness of the number of states of automaton in terms of van der Put series of the automaton function. The criterion displays connections between \( p \)-adic analysis and the theory of automata sequences.

1. Introduction

The \( p \)-adic numbers, which appeared more than a century ago in Kurt Hensel’s works as a pure mathematical construction, see e.g. [7], at the end of XX century were recognized as a base for adequate descriptions of physical, biological, cognitive and information processing phenomena. The pioneer papers in these studies were works of Vladimirov and Volovich [12], [13], [15] followed by monograph [14]. Although the papers (and the monograph) are focused on application of the \( p \)-adic theory to mathematical physics, the impact of these works was much wider than physical models only: Inspired by these works, many scientists started applying \( p \)-adic methods to their own areas of research. Now the \( p \)-adic theory, and wider, ultrametric analysis and ultrametric dynamics, is a rapidly developing area that finds applications to various sciences (physics, biology, genetics, cognitive sciences, information sciences, computer science, cryptology, numerical methods, etc.). On the contemporary state-of-the-art, the interested reader is referred to the monograph [2] and references therein. The current paper concerns application of \( p \)-adic methods to automata theory, the both mathematical and applied science which has numerous applications in engineering sciences, linguistics, computer science, etc. The paper displays tight connections between the \( p \)-adic theory and the theory of automata sequences. Note also that automata may be regarded as a mathematical model of the ‘causality law’, [18]: so the present paper may have some relations to physics as well.

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By the definition, the (non-initial) automaton is 5-tuple $\mathfrak{A} = \langle I, S, O, S, O \rangle$ where $I$ is a finite set, the input alphabet; $O$ is a finite set, the output alphabet; $S$ is a non-empty (possibly, infinite) set of states; $S: I \times S \rightarrow S$ is a state transition function; $O: I \times S \rightarrow O$ is an output function. The automaton where both input alphabet $I$ and output alphabet $O$ are non-empty is called the transducer, see e.g. [1]; the automaton where the input alphabet is empty whereas the output alphabet is not empty is called the generator. The initial automaton $\mathfrak{A}(s_0) = \langle I, S, O, S, O, s_0 \rangle$ is an automaton $\mathfrak{A}$ where one state $s_0 \in S$ is fixed; it is called the initial state. We stress that the definition of the initial automaton $\mathfrak{A}(s_0)$ is nearly the same as the one of Mealy automaton (see e.g. [2]) with the only important difference: the set of states $S$ of $\mathfrak{A}(s_0)$ is not necessarily finite.

Given a non-empty alphabet $A$, its elements are called symbols, or letters. By the definition, a word of length $n$ over alphabet $A$ is a finite sequence (stretching from right to left) $\alpha_{n-1} \cdots \alpha_1 \alpha_0$, where $\alpha_{n-1}, \ldots, \alpha_1, \alpha_0 \in A$. The empty word is a sequence of length 0, that is, the one that contains no symbols. Hereinafter the length of the word $w$ is denoted via $|w|$. Given a word $w = \alpha_{n-1} \cdots \alpha_1 \alpha_0$, any word $v = \alpha_{k-1} \cdots \alpha_1 \alpha_0$, $k \leq n$, is called a prefix of the word $w$; whereas any word $u = \alpha_{n-1} \cdots \alpha_i \alpha_i$, $0 \leq i \leq n - 1$ is called a suffix of the word $w$. Given words $a = \alpha_{n-1} \cdots \alpha_1 \alpha_0$ and $b = \beta_{k-1} \cdots \beta_1 \beta_0$, the concatenation $a \circ b$ is the following word (of length $n + k$):

$$a \circ b = \alpha_{n-1} \cdots \alpha_1 \alpha_0 \beta_{k-1} \cdots \beta_1 \beta_0.$$

Given an input word $w = \chi_{n-1} \cdots \chi_1 \chi_0$ over the alphabet $I$, an initial transducer $\mathfrak{A}(s_0) = \langle I, S, O, S, O, s_0 \rangle$ transforms $w$ to output word $w' = \xi_{n-1} \cdots \xi_1 \xi_0$ over the output alphabet $O$ as follows (cf. Figure 1): Initially the transducer $\mathfrak{A}(s_0)$ is at the state $s_0$; accepting the input symbol $\chi_0 \in I$, the transducer outputs the symbol $\xi_0 = O(\chi_0, s_0) \in O$ and reaches the state $s_1 = S(\chi_0, s_0) \in S$; then the transducer accepts the next input symbol $\chi_1 \in I$, reaches the state $s_2 = S(\chi_1, s_1) \in S$, outputs $\xi_1 = O(\chi_1, s_1) \in O$, and the routine repeats. Throughout the paper, ‘automaton’ mostly stands for ‘initial automaton’; we make corresponding remarks if not. Further in the paper we mostly consider transducers. Furthermore, throughout the paper we consider only reachable transducers; that is, we assume that all the states of an

![Figure 1. Initial transducer, schematically](image-url)
initial transducer $\mathfrak{A}(s_0)$ are reachable from $s_0$: Given $s \in S$, there exists input word
$w$ over alphabet $I$ such that after the word $w$ has been feeded to the automaton
$\mathfrak{A}(s_0)$, the automaton reaches the state $s$. A reachable transducer is called finite if
its set $S$ of states is finite, and is called infinite if otherwise.

To the initial automaton $\mathfrak{A}(s_0)$ we put into a correspondence a family $I(\mathfrak{A})$ of all
subautomata $\mathfrak{A}(s) = (I, S, O, \delta, O, s), s \in S$, where $S = \delta(s) \subset S$ is the set of all
states that are reachable from the state $s$ and $S, O$ are respective restrictions of the
state transition and output functions $S, O$ on $I \times S$.

Hereinafter in the paper the word ‘automaton’ stands for an initial transducer
whose input and output alphabet consists of $p$ symbols. We mostly assume that $p$ is
a prime although many of further results (for instance, the following Theorem 2.2)
are true without this restriction and so we identify input/output symbols with the
$p$-element field $F_p = \{0, 1, \ldots, p-1\}$. Thus, for every $n = 1, 2, 3, \ldots$ the automaton
$\mathfrak{A}(s_0) = (F_p, S, F_p, S, O, s_0)$ maps $n$-letter words over $F_p$ to $n$-letter words over $F_p$
according to the procedure described above, cf. Figure 1.

We identify $n$-letter words over $F_p$ with non-negative integers in a natural way:
Given an $n$-letter word $w = \chi_{n-1}\chi_{n-2} \cdots \chi_0$ (i.e., $\chi_i \in F_p$ for $i = 0, 1, 2, \ldots, n-1$),
we consider $w$ as a base-$p$ expansion of the number $\chi_0 + \chi_1 \cdot p + \cdots + \chi_{n-1} \cdot p^{n-1}$. In
turn, the latter number can be considered as an element of the residue ring $Z/p^nZ$
modulo $p^n$. Thus, to every automaton $\mathfrak{A}$ there corresponds a map $f_{n, \mathfrak{A}}$ from $Z/p^nZ$
to $Z/p^nZ$, for every $n = 1, 2, 3, \ldots$.

Note that when necessary we may also identify $n$-letter words over $F_p$ with
elements of $F_p^n$, the $n$-th Cartesian power of $F_p$; so further we use these one-to-one
correspondences between $n$-letter words and residues modulo $p^n$ (as well as between
the words and elements from $F_p^n$) without extra comments.

In a similar manner, every automaton $\mathfrak{A} = \mathfrak{A}(s_0)$ defines a map $f_{n, \mathfrak{A}}$ from $Z_p$ to
$Z_p$: Given an infinite word $w = \ldots \chi_{n-1}\chi_{n-2} \cdots \chi_0$ (that is, an infinite sequence)
over $F_p$ we consider a $p$-adic integer whose $p$-adic canonical expansion is $z = z(w) =
\chi_0 + \chi_1 \cdot p + \cdots + \chi_{n-1} \cdot p^{n-1} + \cdots$; so, by the definition, for every $z \in Z_p$ we put

\begin{equation}
\delta_i(f_{n, \mathfrak{A}}(z)) = O(\delta_i(z), s_i) \quad (i = 0, 1, 2, \ldots),
\end{equation}

where $s_i = S(\delta_{i-1}(z), s_{i-1})$, $i = 1, 2, \ldots$, and $\delta_i(z)$ is the $i$-th $p$-adic digit of
$z$; that is, the $i$-th term coefficient in the $p$-adic canonical representation of $z$:
$\delta_i(z) = \chi_i \in F_p$, $i = 0, 1, 2, \ldots$. The so defined map $f_{n, \mathfrak{A}}$ is called the automaton
function (or, the automaton map) of the automaton $\mathfrak{A}$.

The point is that the class of all automata functions that correspond to automata
with $p$-letter input/output alphabets coincides with the class of all maps from
$Z_p$ to $Z_p$ that satisfy $p$-adic Lipschitz condition with a constant 1 (1-Lipschitz
maps, for brevity), see further Theorem 2.2. The claim of the theorem is not a
completely new result: Actually the claim can be derived from a more general result
on asynchronous automata $[6]$, Proposition 3.7; also, in a special case $p = 2$ the
claim was proved in $[10]$. Nonetheless, as we consider only synchronous automata,
however, for arbitrary $p$, further in Section 2 we present a direct $p$-adic proof of
Theorem 2.2. Thus, to study automata functions one can apply various techniques
of $p$-adic analysis.

We note that the $p$-adic approach (and wider the non-Archimedean one) has
already been successfully applied to automata theory. Seemingly the paper $[8]$ is
the first one where the $p$-adic techniques is applied to study automata functions;
the paper deals with linearity conditions of automata maps. For application of
the non-Archimedean methods to automata and formal languages see expository
paper [10] and references therein; for applications to automata and group theory
see [6, 5]. In [16, 18, 17] the 2-adic methods are used to study binary automata
(the ones whose input/output alphabet is \(F_2\)), in particular, to obtain the finiteness
criterion for these automata. In monograph [2] the \(p\)-adic ergodic theory is studied
(see numerous references therein) aiming at applications to computer science and
cryptography (in particular, to automata theory, to pseudorandom number gener-
ation and to stream cipher design) as well as to applications in other areas like
quantum theory, cognitive sciences and genetics.

The central result of the paper is the finiteness criterion for automata in terms
van der Put series of automata function, Theorem 4.2. Compared to the finiteness
criterion from [17], our’s criterion

- is more general (\(p\) is arbitrary prime, and not only \(p = 2\) as in [17]);
- is more convenient, as the criterion from [17] is in terms of values of func-
tions \(\delta_i(f_A)\), \(i = 0, 1, 2, \ldots\), which are difficult (or even impossible) to
obtain from standard \(p\)-adic representations of functions; whereas our’s
directly uses one of these standard representations, the van der Put series.

The paper is organized as follows:

- In Section 2 we show that the class of all automata functions of au-
tomata with \(p\)-letter input/output alphabets coincides with the class of
all 1-Lipschitz functions defined on \(\mathbb{Z}_p\) and valuated in \(\mathbb{F}_p\).
- In Section 3 we remind basic facts about van der Put series; for insta-
ce, the criterion if a function is 1-Lipschitz, in terms of its van der Put series.
- In Section 4 we prove main result of the paper, the automaton finiteness
criterion in terms of van der Put series of the automaton function.
- We conclude in Section 5.

2. Automata functions = 1-Lipschitz functions

In the section, we prove that automata functions constitutes the class of all
1-Lipschitz functions. We first remind a characterization of 1-Lipschitz functions
\(f: \mathbb{Z}_p \to \mathbb{Z}_p\) via properties of coordinate functions; the latter are functions
\(\delta_i(f(x))\) defined on \(\mathbb{Z}_p\) and valuated in \(\mathbb{F}_p\).

Proposition 2.1 (cf. [2 Proposition 3.35]). A function \(f: \mathbb{Z}_p \to \mathbb{Z}_p\) is 1-Lipschitz
if and only if for every \(i = 1, 2, \ldots\) the \(i\)-th coordinate function \(\delta_i(f(x))\) does not depend on \(\delta_{i+k}(x)\), for all \(k = 1, 2, \ldots\).

Theorem 2.2. The automaton function \(f_{\mathfrak{A}(s_0)}: \mathbb{Z}_p \to \mathbb{Z}_p\) of the automaton \(\mathfrak{A}(s_0) = \langle \mathbb{F}_p, S, \mathbb{F}_p, O, s_0 \rangle\) is 1-Lipschitz. Conversely, for every 1-Lipschitz function \(f: \mathbb{Z}_p \to \mathbb{Z}_p\) there exists an automaton \(\mathfrak{A}(s_0) = \langle \mathbb{F}_p, S, \mathbb{F}_p, O, s_0 \rangle\) such that \(f = f_{\mathfrak{A}(s_0)}\).

Proof. As \(s_i = S(\delta_{i-1}(z), s_{i-1})\) for every \(i = 1, 2, \ldots\), the \(i\)-th output symbol
\(\xi_i = \delta_i(f_{\mathfrak{A}}(z))\) depends only on input symbols \(\chi_0, \chi_1, \ldots, \chi_i\); that is
\[
\delta_i(f_{\mathfrak{A}}(z)) = \psi_i(\delta_0(z), \delta_1(z), \ldots, \delta_i(z))
\]
for all $i = 0, 1, 2, \ldots$ and for suitable maps $\psi_i : \mathbb{F}_p^{i+1} \rightarrow \mathbb{F}_p$. That is, $f = f_{\mathfrak{A}(s_0)} : \mathbb{Z}_p \rightarrow Z_p$ is of the form

$$
(2.2) \quad f : x = \sum_{i=0}^{\infty} \chi_i p^i \mapsto f(x) = \sum_{i=0}^{\infty} \psi_i(\chi_0, \ldots, \chi_i)p^i.
$$

By Proposition 2.1 this means that the function $f_{\mathfrak{A}(s_0)}$ is 1-Lipschitz.

Conversely, let $f : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ be a 1-Lipschitz map; the by Proposition 2.1 $f$ may be represented in the form (2.2) for suitable maps $\psi_i : \mathbb{F}_p^{i+1} \rightarrow \mathbb{F}_p$. We now construct an automaton $\mathfrak{A}(s_0) = (F, S, F_p, S, O, s_0)$ such that $f_{\mathfrak{A}(s_0)} = f$.

Let $F_p$ be a set of all non-empty words over the alphabet $F_p$. We consider these words as base-$p$ expansions of numbers from $\mathbb{N} = \{1, 2, 3, \ldots\}$ and enumerate all these words by integers $1, 2, 3, \ldots$ in lexicographical order in accordance with the natural order on $\mathbb{F}_p$: $0 < 1 < 2 < \cdots < p - 1$. This way we establish a one-to-one correspondence between the words $w \in F_p$ and integers $i \in \mathbb{N}$: $w \leftrightarrow \nu(w)$, $i \leftrightarrow \omega(i)$ ($\nu(w) \in \mathbb{N}$, $\omega(i) \in F_p$). Note that $\nu(\omega(i)) = i$, $\omega(\nu(w)) = w$ for all $i \in \mathbb{N}$ and all non-empty words from $w \in F_p^*$. Assume that $\omega(0)$ is empty word.

Now put $S = S_0 = \{0, 1, 2, 3, \ldots\}$, the set of all states of the automaton $\mathfrak{A}$ under construction, and take the initial state $s_0 = 0$. The state transition function $S$ is defined as follows:

$$
(2.3) \quad S(r, i) = \nu(r \circ \omega(i)),
$$

where $i = 0, 1, 2, \ldots$ and $r \in F_p$. That is, $S(r, i)$ is the number of the word $r \circ \omega(i)$ that is a concatenation of the word $\omega(i)$ (the word that has number $i$), the prefix, with the single-letter word $r$, the suffix.

Now consider a one-to-one map $\theta_i(\chi_{n-1} \cdots \chi_1 \chi_0) = (\chi_0, \chi_1, \ldots, \chi_{n-i})$ from the $n$-letter words onto $F_p^n$ and define the output function of the automaton $\mathfrak{A}$ as follows:

$$
(2.4) \quad O(r, i) = \psi_{\nu(\omega(i))}(\theta_{\omega(i)}+1(r \circ \omega(i))),
$$

where $i = 0, 1, 2, \ldots$ and $r \in F_p$.

As both $f$ and $f_{\mathfrak{A}(s_0)}$ are 1-Lipschitz, thus continuous with respect to $p$-adic metric, and as $\mathbb{N}_0$ is dense in $\mathbb{Z}_p$, to prove that $f = f_{\mathfrak{A}(s_0)}$ is suffices to show that

$$
(2.5) \quad f_{\mathfrak{A}(s_0)}(\tilde{w}) \equiv f(\tilde{w}) \pmod{p^{\lfloor |w|/p^i \rfloor}}
$$

for all finite non-empty words $w \in F_p^*$, where $\tilde{w} \in \mathbb{N}_0$ stands for an integer whose base-$p$ expansion is $w$. We prove that (2.5) holds for all $w \in F_p^*$ once $|w| = n > 0$ by induction on $n$.

If $n = 1$ then $\tilde{w} \in F_p$; so once $w$ is feeded to $\mathfrak{A}$, the automaton reaches the state $S(w, 0) = \nu(w)$ (cf. (2.3)) and outputs $O(w, 0) = \psi_0(\theta_1(w)) = f(\tilde{w}) \pmod{p}$ (cf. (2.4)), see (2.2). Thus, (2.5) holds in this case.

Now assume that (2.5) holds for all $w \in F_p^*$ such that $|w| = n < k$ and prove that (2.5) holds also when $|w| = n = k$. Represent $w = r \circ v$, where $r \in F_p$ and $|v| = n - 1$. By the induction hypothesis, after the word $v$ has been feeded to $\mathfrak{A}$, the automaton reaches the state $\nu(v)$ and outputs the word $v_1$ of length $n - 1$ such that $\tilde{v}_1 \equiv f(\tilde{v}) \pmod{p^{n-1}}$. Next, being feeded by the letter $r$, the automaton (which is in the state $\nu(v)$ now) outputs the letter $O(r, \nu(v)) = \psi_{\nu(\nu(v)))}(\theta_{\nu(\nu(v)))+1(r \circ \omega(\nu(v)))}) = \psi_{\nu(\nu(v)))}(\theta_{\nu(\nu(v)))+1(r \circ \omega(\nu(v)))})$. This means that once feeded by $w$, the automaton $\mathfrak{A}(s_0)$ outputs the word $v_2 = (\psi_{\nu(\nu(v)))}(\theta_{\nu(\nu(v)))+1(r \circ \omega(\nu(v)))}) \circ v_1$. However, $\tilde{v}_2 \equiv f(\tilde{w}) \pmod{p^n}$, cf. (2.2).
Note 2.3. From the proof of Theorem 2.2 it is clear that the mapping
\[ f_{n, \mathfrak{A}(s_0)} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \]
is just a reduction modulo \( p^n \) of the automaton function \( f_{\mathfrak{A}(s_0)} : f_{n, \mathfrak{A}(s_0)} = f_{\mathfrak{A}(s_0)} \mod p^n \) for all \( n = 1, 2, 3, \ldots \).

We remind that the reduction map modulo \( p^n \) is the map \( \text{mod} p^n : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z} \) such that \( x \mod p^n = \sum_{i=0}^{n-1} \delta_i(x)p^i \); that is, the map \( \text{mod} p^n \) just deletes all terms starting with the \( n \)-th one in the canonical \( p \)-adic expansion of \( x \). Note that \( \text{mod} p^n \) is a continuous ring epimorphism of \( \mathbb{Z}_p \) onto the residue ring \( \mathbb{Z}/p^n\mathbb{Z} \). Thus, given a 1-Lipschitz map \( f : \mathbb{Z}_p \to \mathbb{Z}_p \), the reduction modulo \( p^n \) \( f \mod p^n : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \) such that \( (f \mod p^n)(x) = f(x) \mod p^n \) is well defined by Proposition 2.1; that is, the so defined map \( f \mod p^n \) does not depend on the choice of representatives in co-sets with respect to the ideal \( p^n\mathbb{Z}_p \).

Further, given a 1-Lipschitz function \( f : \mathbb{Z}_p \to \mathbb{Z}_p \) via \( \mathfrak{A} \) we denote an initial transducer \( \langle F, \mathcal{E}, F_p, S, O, s_0 \rangle \) whose automaton function is \( f \); that is, \( f_{\mathfrak{A}} = f \). Note that the automaton \( \mathfrak{A} \) is not unique: There are many automata that has the same automaton function. However, this non-uniqueness will not cause misunderstanding since in the paper we are mostly interested with automata functions rather than with ‘internal structure’ (e.g., with state sets, state transition and output functions, etc.) of automata themselves.

3. Van der Put series of 1-Lipschitz functions

Now we remind definition and some properties of van der Put series, see e.g. \cite{9, 11} for details. Given a continuous function \( f : \mathbb{Z}_p \to \mathbb{Z}_p \), there exists a unique sequence \( B_0, B_1, B_2, \ldots \) of \( p \)-adic integers such that

\[ f(x) = \sum_{m=0}^{\infty} B_m \chi(m, x) \]

for all \( x \in \mathbb{Z}_p \), where

\[ \chi(m, x) = \begin{cases} 1, & \text{if } |x - m|_p \leq p^{-n} \\ 0, & \text{otherwise} \end{cases} \]

and \( n = 1 \) if \( m = 0 \); \( n \) is uniquely defined by the inequality \( p^{n-1} \leq m \leq p^n - 1 \) otherwise. The right side series in (3.6) is called the van der Put series of the function \( f \). Note that the sequence \( B_0, B_1, \ldots, B_m, \ldots \) of van der Put coefficients of the function \( f \) tends \( p \)-adically to 0 as \( m \to \infty \), and the series converges uniformly on \( \mathbb{Z}_p \). Vice versa, if a sequence \( B_0, B_1, \ldots, B_m, \ldots \) of \( p \)-adic integers tends \( p \)-adically to 0 as \( m \to \infty \), then the series in the right part of (3.6) converges uniformly on \( \mathbb{Z}_p \) and thus define a continuous function \( f : \mathbb{Z}_p \to \mathbb{Z}_p \).

The number \( n \) in the definition of \( \chi(m, x) \) has a very natural meaning; it is just the number of digits in a base-\( p \) expansion of \( m \in \mathbb{N}_0 \):

\[ [\log_p m] = (\text{the number of digits in a base-}p \text{ expansion for } m) - 1; \]

therefore \( n = [\log_p m] + 1 \) for all \( m \in \mathbb{N}_0 \) (that is why we assume \( [\log_p 0] = 0 \)).

Note that coefficients \( B_m \) are related to the values of the function \( f \) in the following way: Let \( m = m_0 + \ldots + m_{n-2}p^{n-2} + m_{n-1}p^{n-1} \) be a base-\( p \) expansion
for $m$, i.e., $m_j \in \{0, \ldots, p - 1\}$, $j = 0, 1, \ldots, n - 1$ and $m_{n-1} \neq 0$, then

$$B_m = \begin{cases} f(m) - f(m - m_{n-1}p^{n-1}), & \text{if } m \geq p; \\ f(m), & \text{if otherwise.} \end{cases}$$

(3.7)

It worth noticing also that $\chi(m, x)$ is merely a characteristic function of the ball $B_{p^{-\lfloor \log_p m \rfloor} - 1}(m) = m + p^{\lfloor \log_p m \rfloor - 1}Z_p$ of radius $p^{-\lfloor \log_p m \rfloor - 1}$ centered at $m \in \mathbb{N}_0$:

$$\chi(m, x) = \begin{cases} 1, & \text{if } x \equiv m \pmod{p^{\lfloor \log_p m \rfloor + 1}}; \\ 0, & \text{if otherwise} \end{cases}$$

(3.8)

**Theorem 3.1** (Anashin-Khrennikov-Yurova, [3]). The function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is 1-Lipschitz if and only if it can be represented as

$$f(x) = \sum_{m=0}^{\infty} b_mp^{\lfloor \log_p m \rfloor} \chi(m, x),$$

where $b_m \in \mathbb{Z}_p$ for $m = 0, 1, 2, \ldots$

**4. Main theorem**

We first remind some notions and facts from the theory of automata sequences following [1].

An infinite sequence $a = (a_i)_{i=0}^{\infty}$ over a finite alphabet $A$, $\#A = L < \infty$, is called $p$-automatic if there exists a finite transducer $\Sigma = (\mathcal{P}, S, A, \delta, s_0)$ such that for all $n = 0, 1, 2, \ldots$, if $\Sigma$ is fed by the word $\ldots \chi_{k-1} \chi_{k-2} \cdots \chi_0$ which is a base-$p$ expansion of $n = \chi_0 + \chi_1 p + \cdots + \chi_k p^k$, $\chi_k \neq 0$ if $n \neq 0$, then the $k$-th output symbol of $\Sigma$ is $a_n$; or, in other words, such that $\delta^A_k(f_\Sigma(n)) = a_n$ for all $n \in \mathbb{N}_0$, where $k = \lfloor \log_p n \rfloor$ and $\delta^A_k(r)$ stands for the $k$-th digit in the base-$L$ expansion of $r$.

The $p$-kernel of the sequence $a$ is the set $\ker_p(a)$ of all subsequences $(a_{jt^m+1})_{i=0}^{\infty}$, $m = 0, 1, 2, \ldots$, $0 \leq t < p^m$.

**Theorem 4.1** (Automaticity criterion, cf. [1] Theorem 6.6.2). Let $p \geq 2$; then the sequence $a$ is $p$-automatic if and only if its $p$-kernel is finite.

Now we are able to state the main result of the paper:

**Theorem 4.2** (Automata finiteness criterion). Given a 1-Lipschitz function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ represented by van der Put series (3.9),

$$f(x) = \sum_{m=0}^{\infty} b_mp^{\lfloor \log_p m \rfloor} \chi(m, x),$$

the function $f$ is the automaton function of a finite automaton if and only if the following conditions hold simultaneously:

(i) all coefficients $b_m$, $m = 0, 1, 2, \ldots$, constitute a finite subset $B_f \subset \mathbb{Q} \cap \mathbb{Z}_p$,

(ii) the $p$-kernel of the sequence $(b_m)_{m=0}^{\infty}$ is finite.

**Note 4.3.** Condition (ii) of the theorem is equivalent to the condition that the sequence $(b_m)_{m=0}^{\infty}$ is $p$-automatic, cf. Theorem 4.1.
Now we are going to present equivalent statement of Theorem 4.2 in terms of formal power series.

Given a $q$-element field $\mathbb{F}_q$, denote via $\mathbb{F}_q[[X]]$ the ring of formal power series in variable $X$ over $\mathbb{F}_q$:

$$\mathbb{F}_q[[X]] = \left\{ \sum_{i=0}^{\infty} a_i X^i : a_i \in \mathbb{F}_q \right\};$$

denote via $\mathbb{F}_q((X))$ the ring of formal Laurent series over $\mathbb{F}_q$:

$$\mathbb{F}_q((X)) = \left\{ \sum_{i=-\infty}^{\infty} a_i X^i : n_0 \in \mathbb{N}, a_i \in \mathbb{F}_q \right\}.$$

Denote via $\mathbb{F}_q(X)$ the field of (univariate) rational functions over $\mathbb{F}_q$:

$$\mathbb{F}_q(X) = \left\{ \frac{u(X)}{v(X)} : u(X), v(X) \in \mathbb{F}_p[X], u(X) \neq 0 \right\},$$

where $\mathbb{F}_p[X]$ is the ring of polynomials in variable $X$ over $\mathbb{F}_q$. As the field $\mathbb{F}_q((X))$ contains a subfield $\mathbb{F}_q(X)$, it is possible to define algebraicity over $\mathbb{F}_q(X)$: A formal Laurent series $F(X) = \sum_{i=-n_0}^{\infty} a_i X^i$ is algebraic over $\mathbb{F}_p(X)$ if and only if there exist $d \in \mathbb{N}$ and polynomials $u_0(X), \ldots, u_d(X) \in \mathbb{F}_p[X]$, not all zero, such that in the field $\mathbb{F}_q((X))$ the following identity holds:

$$u_0(X) + u_1(X) \cdot F(X) + \cdots + u_d(X) \cdot (F(X))^d = 0.$$

Now we remind Christol’s theorem, \cite[Theorem 12.2.5]{Christol}:

**Theorem 4.4** (Christol). Let $p$ be a prime, and let $\mathbf{a} = (a_i)_{i=0}^{\infty}$ be infinite sequence over a finite non-empty alphabet $\mathcal{A}$. The sequence $\mathbf{a}$ is $p$-automatic if and only if there exists an integer $\ell \in \mathbb{N}$ and an injection $\tau : \mathcal{A} \to \mathbb{F}_p$ such that the formal power series $\sum_{i=0}^{\infty} \tau(a_i) X^i$ is algebraic over $\mathbb{F}_p(X)$.

By Christol’s theorem, we now may replace condition (ii) from the statement of Theorem 4.2 by equivalent one, thus getting an equivalent finiteness criterion:

**Theorem 4.5** (Automata finiteness criterion, equivalent). Given a 1-Lipschitz function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ represented by van der Put series \cite{vdPut}, the function $f$ is the automaton function of a finite automaton if and only if the following conditions hold simultaneously:

(i) all coefficients $b_m$, $m = 0, 1, 2, \ldots$, constitute a finite subset $B_f \subset \mathbb{Q} \cap \mathbb{Z}_p$, and

(ii) under a suitable injection $\tau : B_f \to \mathbb{F}_p$, the formal power series

$$\sum_{m=0}^{\infty} \tau(b_m) X^m$$

over $\mathbb{F}_p$ is algebraic over $\mathbb{F}_p(X)$.

**Proof of Theorem 4.2.** Given a 1-Lipschitz function $f$, for $n \in \mathbb{N}_0$, $k \geq \left\lfloor \log_p n \right\rfloor + 1$ consider functions $f_{n,k} : \mathbb{Z}_p \to \mathbb{Z}_p$ defined as follows:

$$f_{n,k}(z) = \frac{1}{p^k} \left( f(n + p^k z) - (f(n) \mod p^k) \right); \quad z \in \mathbb{Z}_p.$$

The function $f$ is an automaton function of a finite automaton if and only if in the collection $\mathcal{F}$ of functions $f_{n,k}$ ($n \in \mathbb{N}_0, k \in \mathbb{N}, k \geq \left\lfloor \log_p n \right\rfloor + 1$) contains only finitely
many pairwise distinct functions: Note that $f_{n,k}$ is the automaton function that corresponds to the automaton $\mathfrak{A}(s(n_k)) = (\mathbb{F}_p, S, \mathbb{F}_p, S, O, s(n_k))$, where $s(n_k) \in S$ is the state the automaton $\mathfrak{A} = (\mathbb{F}_p, S, \mathbb{F}_p, S, O, s_0)$ reaches after it has been feeded with the input word $n_k$ (of length $p^k$) that corresponds to a base-$p$ expansion of $n$ (so the word $n_k$ may contain some leading zeros that correspond to higher order digits of the expansion).

Note that by Theorem 3.1: $b_{n+p^k s} = \frac{1}{p^k}(f(n + p^k s) - f(n)) = \frac{1}{p^k}(f(n + p^k s) - (f(n) \mod p^k)) - \frac{1}{p^k}(f(n) - (f(n) \mod p^k)) = f_{n,k}(s) - f_{n,k}(0)$ if $n \leq p^k - 1$ and $s \in \{1, 2, \ldots, p - 1\}$, cf. (3.7); so the finiteness of $\mathcal{F}$ implies that in the sequence $(b_m)_{m=0}^{\infty}$ there are only finitely many pairwise distinct terms. We proceed with this in mind.

Take $n \in \mathbb{N}_0$ and $k \geq \lfloor \log_p n \rfloor + 1$. By (3.9), the value $f(n + p^k z)$ can be represented as $f(n + p^k z) = A_{n,k}(z) + B_{n,k}(z)$, where

$$A_{n,k}(z) = \sum_{m=0}^{\lfloor \log_p m \rfloor} b_{m p^{\lfloor \log_p m \rfloor}} \chi(m, n + p^k z);$$

$$B_{n,k}(z) = \sum_{m=p^k}^{\infty} b_{m p^{\lfloor \log_p m \rfloor}} \chi(m, n + p^k z).$$

By (3.8), once $m \leq p^k - 1$, the equality $\chi(m, n + p^k z) = 0$ holds if and only if $m \not\equiv n \pmod{p^{\lfloor \log_p m \rfloor} + 1}$; and once $m \geq p^k$, the equality $\chi(m, n + p^k z) = 0$ holds if and only if $m \not\equiv n \pmod{p^{\lfloor \log_p m \rfloor} + 1}$ (note that $\lfloor \log_p m \rfloor + 1 \geq k + 1$ under conditions of the latter case); thus

$$A_{n,k}(z) = \sum_{m=0}^{p^k - 1} b_{m p^{\lfloor \log_p m \rfloor}} \chi(m, n);$$

$$B_{n,k}(z) = \sum_{t=1}^{\infty} b_{n+p^k t} p^{k+\lfloor \log_p t \rfloor} \chi(n + p^k t, n + p^k z) = \sum_{t=1}^{\infty} b_{n+p^k t} p^{k+\lfloor \log_p t \rfloor} \chi(t, z).$$

From here in particular it follows that $A_{n,k}(z)$ does not depend on $z$ and that $B_{n,k}(z) \equiv 0 \pmod{p^k}$ for all $z \in \mathbb{Z}_p$; consequently,

$$f_{n,k}(z) = \frac{1}{p^k} (A_{n,k}(0) + B_{n,k}(z) - (A_{n,k}(0) \mod p^k)) = C_{n,k} + D_{n,k}(z),$$

where

$$C_{n,k} = \frac{1}{p^k} (A_{n,k}(0) - (A_{n,k}(0) \mod p^k));$$

$$D_{n,k}(z) = \sum_{t=1}^{\infty} b_{n+p^k t} p^{\lfloor \log_p t \rfloor} \chi(t, z).$$

Note that from (4.14) it follows that $D_{n,k}(0) = 0$ (cf. (3.8)), so from (4.14) we deduce that $f_{n,k}(0) = C_{n,k}$.

Thus, we have obtained the following criterion of the finiteness of the number of distinct functions in the collection $\mathcal{F}$: There are only finite number of pairwise
distinct functions $f_{n,k} \in \mathcal{F}$, $n \in \mathbb{N}_0$, $k \geq \lfloor \log_p n \rfloor + 1$, if and only if the following two conditions hold simultaneously:

1. There are only finitely many pairwise distinct constants $C_{n,k} \in \mathbb{Z}_p$, $n \in \mathbb{N}_0$, $k \geq \lfloor \log_p n \rfloor + 1$.
2. There are only finitely many pairwise distinct functions $D_{n,k} : \mathbb{Z}_p \to \mathbb{Z}_p$, $n \in \mathbb{N}_0$, $k \geq \lfloor \log_p n \rfloor + 1$.

However, since representation (4.16) is a (unique) van der Put expansion of the function $D_{n,k}$, condition 2 is equivalent to the condition that there are only finitely many pairwise distinct subsequences $(b_{n+p^k})_{n=0}^\infty$, where $k \geq \lfloor \log_p n \rfloor + 1$, $n \in \mathbb{N}_0$. In turn, the latter condition is equivalent to the condition that there are only finitely many pairwise distinct subsequences $(b_{n+p^k})_{n=0}^\infty$, $n \in \mathbb{N}_0$, $k \in \mathbb{N}$. Note that if the condition holds, there are only finitely many pairwise distinct terms $b_n$ in the sequence $(b_m)_{m=0}^\infty$.

Consider condition 1. Note that for $k \geq \lfloor \log_p n \rfloor + 1$ we have that

$$
\sum_{m=0}^{p^k-1} b_m p^{\lfloor \log_p m \rfloor} \chi(m,n) = \sum_{m=0}^{p^{\lfloor \log_p n \rfloor}+1-1} b_m p^{\lfloor \log_p m \rfloor} \chi(m,n)
$$

since $\chi(m,n) = 0$ once $\lfloor \log_p m \rfloor > \lfloor \log_p n \rfloor$, cf. (3.8). Thus, $A_{n,k}(z)$ does not depend on $k$ (and on $z$ as we have already shown); so denoting the right hand side in (4.17) via $A(n)$, we have that $C_{n,k} = p^{-k}(A(n) - ((A(n)) \mod p^k))$, cf. (4.15) and (4.12).

From here by (3.8) we get that $C_{n,k} = p^{-k}(b_n - (b_n \mod p^k))$ once $n$ is such that $\lfloor \log_p n \rfloor = 0$. Consequently, given $n \in \{0,1,\ldots,p-1\}$, the finiteness of the number of pairwise distinct $C_{n,k}$, where $k \geq \lfloor \log_p n \rfloor + 1 = 1$, is equivalent to the condition that the sequence $(\delta_i(b_n))_{i=0}^\infty$ is eventually periodic, that is, to the condition that $b_n \in \mathbb{Q} \cap \mathbb{Z}_p$. Using this as a base for induction on $\lfloor \log_p n \rfloor$, assuming that all $b_n \in \mathbb{Q} \cap \mathbb{Z}_p$ for $n$ such that $\lfloor \log_p n \rfloor < N$, we see that, given $n$ such that $\lfloor \log_p n \rfloor = N$, the finiteness of the number of pairwise distinct $C_{n,k} = p^{-k}(A(n) - (A(n)) \mod p^k))$ for $k \geq \lfloor \log_p n \rfloor + 1 = N + 1$ is equivalent to the condition that $A(n) \in \mathbb{Q} \cap \mathbb{Z}_p$.

However, in view of (3.8) from the definition of $A(n)$ it follows that

$$
A(n) = b_n p^{\lfloor \log_p n \rfloor} + \sum_{m=0}^{p^{\lfloor \log_p n \rfloor}-1} b_m p^{\lfloor \log_p m \rfloor} \chi(m,n).
$$

The left side sum is in $\mathbb{Q} \cap \mathbb{Z}_p$ by induction hypothesis; so $b_n p^{\lfloor \log_p n \rfloor} \in \mathbb{Q} \cap \mathbb{Z}_p$ and whence $b_n \in \mathbb{Q} \cap \mathbb{Z}_p$ since $b_n \in \mathbb{Z}_p$.

We finally have proved that conditions 1–2 hold simultaneously if and only if the following conditions hold simultaneously:

1’. All coefficients $b_n$, $m = 0, 1, 2, \ldots$, constitute a non-empty finite subset (denoted as $B_f$) in $\mathbb{Q} \cap \mathbb{Z}_p$.

2’. There are only finitely many pairwise distinct subsequences $(b_{n+p^k})_{n=0}^\infty$, $n \in \mathbb{N}_0$, $k \in \mathbb{N}$, $k \geq \lfloor \log_p n \rfloor + 1$; that is, the $p$-kernel of the sequence $(b_m)_{m=0}^\infty$ is finite.

This ends the proof.
5. Discussion

In the paper, we found finiteness conditions for an automaton in terms of van der Put series of the automaton function. Any automaton function of a transducer with $p$-letter input/output alphabets can be considered as a continuous (with respect to the $p$-adic distance) map from $p$-adic integers to $p$-adic integers. The van der Put series is a standard representation for continuous $p$-adic maps. The paper discloses relations between the $p$-adic theory and the theory of automata sequences.

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