Solitary Waves in Abelian Gauge Theories

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Received 4 October 2007
Communicated by Shair Ahmad

Abstract
Abelian gauge theories consist of a class of field equations which provide a model for the interaction between matter and electromagnetic fields. In this paper we analyze the existence of solitary waves for these theories. We assume that the lower order term $W$ is positive and we prove the existence of solitary waves if the coupling between matter and electromagnetic field is small. We point out that the positiveness assumption on $W$ implies that the energy is positive: this fact makes these theories more suitable to model physical phenomena.

2000 Mathematics Subject Classification. 47J30, 35J50, 81V10,

Key words. Abelian gauge theories, solitary waves, critical points

1 Introduction

Roughly speaking a solitary wave is a solution of a field equation whose energy travels as a localized packet and which preserves this localization in time.
In this paper we are interested in investigating the existence of solitary waves relative to Abelian gauge theories. These theories consist of a class of field equations that provide a model for the interaction of matter with the electromagnetic field.

This problem has already been treated in [1], [7], [8], [6], [14], [4], [9] and [20]. In those papers the lower order term $W$ has the form

$$W(u) = \frac{1}{2} u^2 - \frac{1}{p} |u|^p, \quad 2 < p < 6.$$ 

In this paper, we analyze the case

$$W(u) \geq 0. \quad (1.1)$$

This assumption implies that the energy density is positive; the interest of this feature will be discussed at the end of section 1.2 and in section 1.3.

**1.1 Abelian gauge theories**

Let $G$ be a subgroup of $U(N)$, the unitary group in $\mathbb{C}^N$, and denote by $\Lambda^k(\mathbb{R}^4, \mathfrak{g})$ the set of $k$-forms defined in $\mathbb{R}^4$ with values in the Lie algebra $\mathfrak{g}$ of the group $G$. A 1-form

$$\Gamma = \sum_{j=0}^{3} \Gamma_j dx^j \in \Lambda^1(\mathbb{R}^4, \mathfrak{g})$$

is called connection form. The operator

$$d\Gamma = \Lambda^k(\mathbb{R}^4, \mathfrak{g}) \to \Lambda^{k+1}(\mathbb{R}^4, \mathfrak{g})$$

defined by

$$d\Gamma = d + \Gamma = \sum_{j=0}^{3} \left( \frac{\partial}{\partial x^j} + \Gamma_j \right) dx^3$$

is called covariant differential and the operators

$$D_j = \frac{\partial}{\partial x^j} + \Gamma_j : \mathcal{C}^1(\mathbb{R}^4, \mathbb{C}^N) \to \mathcal{C}^0(\mathbb{R}^4, \mathbb{C}^N), \quad j = 0, ..., 3$$

are called covariant derivatives. The 2-form

$$F = d\Gamma \Gamma = \sum_{i,j=0}^{3} (\partial_i \Gamma_j + [\Gamma_i, \Gamma_j]) dx^i \wedge dx^j$$

is called curvature.

Now, we equip $\mathbb{R}^4$ with the Minkowski quadratic form given by

$$\langle v, v \rangle_M = -|v_0|^2 + \sum_{j=1}^{3} |v_j|^2,$$
where \( v = (v_0, v_1, v_2, v_3) \) is a 4-vector with components in \( \mathbb{R} \) or in \( \mathbb{C} \). The Minkowski quadratic form can be extended to the space of the differential forms \( \alpha \in \Lambda^k(\mathbb{R}^4, g) \), and it will be denoted by \( \langle \alpha, \alpha \rangle_M \). We set

\[
\mathcal{L}_0 = -\frac{1}{2} \langle d\Gamma \psi, d\Gamma \psi \rangle_M,
\]

where \( \psi \in \mathbb{C}^N \).

We set

\[
\mathcal{L}_1 = -\frac{1}{2q^2} \langle d\Gamma \Gamma, d\Gamma \Gamma \rangle_M,
\]

where \( q > 0 \) is a real parameter which controls the coupling of \( \mathcal{L}_1 \) with \( \mathcal{L}_0 \).

A gauge field, by definition (see e.g. [22], [15]), is a critical point of the action functional

\[
S = \int \mathcal{L} \, dx \, dt, \quad \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 - W(\psi),
\]

where \( W : \mathbb{C}^N \to \mathbb{R} \) is a function which is assumed to be \( G \)-invariant, namely

\[
W(g\psi) = W(\psi), \quad g \in G.
\]

We are interested in the Abelian gauge theory, namely in the case in which \( G = U(1) = S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \).

In this case, the \( \Gamma_j(x, t) \) are imaginary numbers and

\[
[\Gamma_i, \Gamma_j] = 0.
\]

Then

\[
\mathcal{L}_1 = -\frac{1}{2q^2} \langle d\Gamma \Gamma, d\Gamma \Gamma \rangle_M = -\frac{1}{2q^2} \langle d\Gamma, d\Gamma \rangle_M.
\]

If we set

\[
A^j = A_j = -\frac{1}{iq} \Gamma_j, \quad j = 1, 2, 3
\]

and

\[
\varphi = A^0 = -A_0 = \frac{1}{iq} \Gamma_0,
\]

it turns out that \( (A^0, A^1, A^2, A^3) \) is a real valued 4-vector field and \( A := (A^1, A^2, A^3) \) is its spatial component. Setting \( x^0 = t \), the covariant derivatives take the form

\[
D_t = \frac{\partial}{\partial t} + iq\varphi, \quad D_j = \frac{\partial}{\partial x^j} - iqA_j, \quad j = 1, 2, 3
\]

and, for \( q = 0 \), they reduce to the usual ones. Using the above notation, the Lagrangian density \( \mathcal{L}_0 \), can be written as follows

\[
\mathcal{L}_0 = \frac{1}{2} |D_t \psi|^2 - \frac{1}{2} |D_x \psi|^2
\]

and

\[
= \frac{1}{2} \left( \left| \left( \frac{\partial}{\partial t} + iq\varphi \right) \psi \right|^2 - |(\nabla - iqA) \psi|^2 \right)
\]
where \(D_x \psi = (D_1 \psi, D_2 \psi, D_3 \psi)\) and \(L_1\) takes the form

\[
L_1 = \frac{1}{2} \left| \frac{\partial A}{\partial t} + \nabla \phi \right|^2 - \frac{1}{2} |\nabla \times A|^2.
\]

Here \(\nabla \times\) and \(\nabla\) denote respectively the curl and the gradient operators.

Making the variation of \(S\) with respect to \(\psi, \phi\) and \(A\), we get the following system of equations

\[
D_t^2 \psi - D_x^2 \psi + W'(\psi) = 0 \quad (1.6)
\]

\[
\nabla \cdot \left( \frac{\partial A}{\partial t} + \nabla \phi \right) = q \left( \text{Im} \frac{1}{\psi} \frac{\partial \psi}{\partial t} + q \phi \right) |\psi|^2 \quad (1.7)
\]

\[
\nabla \times (\nabla \times A) + \frac{\partial}{\partial t} \left( \frac{\partial A}{\partial t} + \nabla \phi \right) = q \left( \text{Im} \frac{\nabla \psi}{\psi} - q A \right) |\psi|^2 \quad (1.8)
\]

where

\[
D_x^2 \psi = \sum_{j=1}^{3} D_j^2 \psi
\]

is the "equivariant Laplacian"; also we have assumed the notation

\[
W'(\psi) = \frac{\partial W}{\partial \psi_1} + i \frac{\partial W}{\partial \psi_2}, \quad \psi = \psi_1 + i \psi_2.
\]

Notice that (1.3) implies that \(W\) is a function of \(|\psi|\) and we have

\[
W(e^{i\theta} \psi) = W(\psi), \quad W'(e^{i\theta} \psi) = e^{i\theta} W'(\psi).
\]

In order to give a more meaningful form to the above equations, we will write \(\psi\) in polar form

\[
\psi(x,t) = u(x,t) e^{iS(x,t)}, \quad u \geq 0, \quad S \in \mathbb{R}/2\pi \mathbb{Z}.
\]

So (1.2) takes the following form

\[
S(u, S, \phi, A) = \int \int \left[ \frac{1}{2} \left( \frac{\partial u}{\partial t} \right)^2 - \frac{1}{2} |\nabla u|^2 - W(u) \right] dx dt +
\]

\[
+ \frac{1}{2} \int \int \left[ \left( \frac{\partial S}{\partial t} + q \phi \right)^2 - |\nabla S - q A|^2 \right] u^2 dx dt
\]

\[
+ \frac{1}{2} \int \int \left( \left| \frac{\partial A}{\partial t} + \nabla \phi \right|^2 - |\nabla \times A|^2 \right) dx dt
\]

and the equations (1.6,1.7,1.8) take the form:

\[
\Box u + W'(u) + \left[ |\nabla S - q A|^2 - \left( \frac{\partial S}{\partial t} + q \phi \right)^2 \right] u = 0 \quad (1.9)
\]
\[ \frac{\partial}{\partial t} \left[ \left( \frac{\partial S}{\partial t} + q\phi \right) u^2 \right] - \nabla \cdot \left[ (\nabla S - qA) u^2 \right] = 0 \] (1.10)

\[ \nabla \cdot \left( \frac{\partial A}{\partial t} + \nabla \phi \right) = q \left( \frac{\partial S}{\partial t} + q\phi \right) u^2 \] (1.11)

\[ \nabla \times (\nabla \times A) + \frac{\partial}{\partial t} \left( \frac{\partial A}{\partial t} + \nabla \phi \right) = q (\nabla S - qA) u^2. \] (1.12)

As we will see in the next section, these equations provide a model for Electrodynamics.

### 1.2 The Maxwell equations

In order to show the relation of the above equations with the Maxwell equations and to get a model for Electrodynamics, we make the following change of variables:

\[ E = - \left( \frac{\partial A}{\partial t} + \nabla \phi \right) \] (1.13)

\[ H = \nabla \times A \] (1.14)

\[ \rho = - \left( \frac{\partial S}{\partial t} + q\phi \right) qu^2 \] (1.15)

\[ j = (\nabla S - qA) qu^2. \] (1.16)

So (1.11) and (1.12) are the second couple of the Maxwell equations with respect to a matter distribution whose charge and current density are respectively \( \rho \) and \( j \):

\[ \nabla \cdot E = \rho \] (1.17)

\[ \nabla \times H - \frac{\partial E}{\partial t} = j. \] (1.18)

(1.13) and (1.14) give rise to the first couple of the Maxwell equation:

\[ \nabla \times E + \frac{\partial H}{\partial t} = 0 \] (1.19)

\[ \nabla \cdot H = 0. \] (1.20)

Equation (1.9) can be written as follows

\[ \Box u + W'(u) + \frac{j^2 - \rho^2}{q^2 u^3} = 0 \] (1.21)

and finally Equation (1.10) is the charge continuity equation

\[ \frac{\partial}{\partial t} \rho + \nabla \cdot j = 0. \] (1.22)
Notice that equation (1.22) can be deduced by (1.17) and (1.18). Thus, equations (1.17,..,1.21) are equivalent to equations (1.9,...,1.12). Concluding, an Abelian gauge theory, via equations (1.17,..,1.21), provides a model of interaction of the matter field $\psi$ with the electromagnetic field $(E, H)$.

The gauge group is given by $G \cong C^\infty(\mathbb{R}^4)$ with the additive structure and it acts on the variables $\psi, \varphi, A$ as follows

\[
T_\chi \psi = \psi e^{i\chi}; \\
T_\chi \varphi \rightarrow \varphi - \frac{\partial \chi}{\partial t}; \\
T_\chi A = A + \nabla \chi;
\]

with $\chi \in C^\infty(\mathbb{R}^4)$. Equations (1.9,...,1.12) are gauge invariant by the way they have been constructed. However, this fact, in eq. (1.17,..,1.21), can be checked directly since the variables $u, \rho, j, E, H$ are themselves gauge invariant.

In order to show the relevance of assumption (1.1), we will compute the formula of the energy.

**Theorem 1.1** If $(u, S, E, H)$ is a solution of the field equations (1.9,...1.12), its energy is given by

\[
E(u, S, E, H) = \int \left[ \frac{1}{2} \left( \frac{\partial u}{\partial t} \right)^2 + \frac{1}{2} |\nabla u|^2 + W(u) + \frac{\rho^2 + j^2}{2q^2 u^2} + \frac{E^2 + H^2}{2} \right] dx.
\]

**Proof.** We recall the well known expression for the energy density (see e.g. [13]):

\[
\frac{\partial L}{\partial \left( \frac{\partial u}{\partial t} \right)} \cdot \frac{\partial u}{\partial t} + \frac{\partial L}{\partial \left( \frac{\partial S}{\partial t} \right)} \cdot \frac{\partial S}{\partial t} + \frac{\partial L}{\partial \left( \frac{\partial \varphi}{\partial t} \right)} \cdot \frac{\partial \varphi}{\partial t} + \frac{\partial L}{\partial \left( \frac{\partial A}{\partial t} \right)} \cdot \frac{\partial A}{\partial t} - L.
\]

Now we will compute each term. We have:

\[
\frac{\partial L}{\partial \left( \frac{\partial u}{\partial t} \right)} \cdot \frac{\partial u}{\partial t} = \left( \frac{\partial u}{\partial t} \right)^2
\]

\[
\frac{\partial L}{\partial \left( \frac{\partial S}{\partial t} \right)} \cdot \frac{\partial S}{\partial t} = \left( \frac{\partial S}{\partial t} + q \varphi \right) \frac{\partial S}{\partial t} u^2
\]

\[
= \left( \frac{\partial S}{\partial t} + q \varphi \right) \frac{\partial S}{\partial t} u^2 + \left( \frac{\partial S}{\partial t} + q \varphi \right) q \varphi u^2 - \left( \frac{\partial S}{\partial t} + q \varphi \right) q \varphi u^2
\]

\[
= \left( \frac{\partial S}{\partial t} + q \varphi \right)^2 u^2 - \left( \frac{\partial S}{\partial t} + q \varphi \right) q \varphi u^2
\]

\[
= \frac{\rho^2}{q^2 u^2} + \rho \varphi.
\]
In the following we write \( \int g(x)\,dx \) instead of \( \int_{\mathbb{R}^3} g(x)\,dx \). By the Gauss equation (1.17), multiplying by \( \varphi \) and integrating, we get

\[- \int \mathbf{E} \nabla \varphi \, dx = \int \rho \varphi \, dx.\]

Thus, replacing this expression in the above formula, we get

\[
\int \left( \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial S}{\partial t} \right)} \cdot \frac{\partial S}{\partial t} \right) \, dx = \int \left( \rho^2 - \frac{\rho^2 - J^2}{q^2 u^2} \mathbf{E} \cdot \nabla \varphi \right) \, dx. \tag{1.24}
\]

Also we have

\[
\frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \varphi}{\partial t} \right)} \frac{\partial \varphi}{\partial t} = 0 \tag{1.25}
\]

and

\[
\frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \mathbf{A}}{\partial t} \right)} \frac{\partial \mathbf{A}}{\partial t} = \left( \frac{\partial \mathbf{A}}{\partial t} + \nabla \varphi \right) \cdot \frac{\partial \mathbf{A}}{\partial t} = - \mathbf{E} \cdot \frac{\partial \mathbf{A}}{\partial t} \tag{1.26}
\]

Moreover, using the notation (1.13,...,1.16), we have that

\[
\mathcal{L} = \frac{1}{2} \left( \frac{\partial u}{\partial t} \right)^2 - \frac{1}{2} |\nabla u|^2 - W(u) + \frac{\rho^2 - J^2}{2q^2 u^2} + \frac{\mathbf{E}^2 - \mathbf{H}^2}{2}.
\]

Then, by (1.23,...,1.26) and the above expression for \( \mathcal{L} \) we get

\[
\mathcal{E}(u, S, \varphi, \mathbf{A}) = \int \left( \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial S}{\partial t} \right)} \cdot \frac{\partial S}{\partial t} + \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial S}{\partial t} \right)} \cdot \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \mathbf{A}}{\partial t} \right)} \cdot \frac{\partial \mathbf{A}}{\partial t} - \mathcal{L} \right) \, dx
\]

\[
= \int \left( \left( \frac{\partial u}{\partial t} \right)^2 + \frac{\rho^2}{q^2 u^2} - \mathbf{E} \cdot \nabla \varphi - \mathbf{E} \cdot \frac{\partial \mathbf{A}}{\partial t} - \mathcal{L} \right) \, dx
\]

\[
= \int \left( \left( \frac{\partial u}{\partial t} \right)^2 + \frac{\rho^2}{q^2 u^2} + \mathbf{E}^2 \right) \, dx
\]

\[
- \int \left( \frac{1}{2} \left( \frac{\partial u}{\partial t} \right)^2 - \frac{1}{2} |\nabla u|^2 - W(u) + \frac{\rho^2 - J^2}{2q^2 u^2} + \frac{\mathbf{E}^2 - \mathbf{H}^2}{2} \right) \, dx
\]

\[
= \int \left( \frac{1}{2} \left( \frac{\partial u}{\partial t} \right)^2 + \frac{1}{2} |\nabla u|^2 + W(u) + \frac{\rho^2 + J^2}{2q^2 u^2} + \frac{\mathbf{E}^2 + \mathbf{H}^2}{2} \right) \, dx.
\]

By the above theorem, the condition (1.1) implies that the energy density is positive and this fact makes these equations more suitable to model physical phenomena.
1.3 Solitary waves and solitons in the nonlinear wave equation

By solitary wave we mean a solution of a field equation whose energy travels as a localized packet; by soliton, we mean a solitary wave which exhibits some strong form of stability. This is a rather weak definition of soliton but probably it is the most commonly used. Solitons have a particle-like behavior and they occur in many questions of mathematical physics, such as classical and quantum field theory, nonlinear optics, fluid mechanics, plasma physics (see e.g. [10], [12], [17], [21]).

In order to prove the existence of solitons, first it is necessary to prove the existence of solitary waves and then to prove their stability.

We start our discussion of solitary waves considering eq. (1.9,...,1.12) in the case in which the covariant derivative is replaced by the usual one. In this case the matter decouples with the Maxwell equations, equation (1.6) becomes the nonlinear wave equation

\[ \frac{\partial^2 \psi}{\partial t^2} - \Delta \psi + W'(\psi) = 0 \]  

(1.27)

and the Maxwell equations become homogeneous.

The easiest way to produce solitary waves of (1.27) consists in solving the static equation

\[ -\Delta u + W'(u) = 0 \]  

(1.28)

and setting

\[ \psi_v(t,x) = \psi_v(t,x_1,x_2,x_3) = u \left( \frac{x_1 - vt}{\sqrt{1-v^2}}, x_2, x_3 \right); \]  

(1.29)

\( \psi_v(t,x) \) is a solution of eq. (1.27) which represents a bump which travels in the \( x_1 \)-direction with speed \( v \).

In [16] and [19], it has been proved that eq. (1.28) has nontrivial solutions provided that \( W \) has the following form:

\[ W(u) = \frac{1}{2} m^2 u^2 - \frac{1}{p} u^p, \quad m > 0, \quad 2 < p < 6. \]  

(1.30)

Moreover Shatah [18] found a condition which guarantees the ”orbital stability” of the solitary waves of eq. (1.27); if \( W \) is given by (1.30), this condition becomes \( 2 < p < \frac{10}{3} \) (see e.g. [2] or [3]).

However, it would be interesting to assume \( W \geq 0 \); in fact the energy of a solution of equation (1.27) is given by

\[ E(\psi) = \int \left[ \frac{1}{2} \left( \frac{\partial \psi}{\partial t} \right)^2 + \frac{1}{2} |\nabla \psi|^2 + W(\psi) \right] dx. \]

In this case, the positivity of the energy, not only is an important request for the physical models related to this equation, but it provides good \textit{a priori} estimates for the solutions of the relative Cauchy problem. These estimates allow to prove the existence and well-posedness results under very general assumptions on \( W \).
Unfortunately Derrick [11], in a very well known paper, has proved that request (1.1) implies that equation (1.28) has only the trivial solution. His proof is based on the following equality (which in a different form was found also by Pohozaev [16]; for details see also [3]). The Derrick-Pohozaev identity states that for any finite energy solution $u$ of eq.(1.28) we have

$$\frac{1}{6} \int |\nabla u|^2 \, dx + \int W(u) \, dx = 0. \quad (1.31)$$

Clearly (1.31) and (1.1) imply that $u \equiv 0$.

However, we can try to prove the existence of solitons of eq. (1.27) (with assumption (1.1)) exploiting the possible existence of standing waves, since this fact is not prevented by equation (1.31). A standing wave is a finite energy solution of (1.27) having the following form

$$\psi_0(t, x) = u(x) e^{-i\omega_0 t}, \quad u \geq 0, \text{ real.} \quad (1.32)$$

Substituting (1.32) in eq.(1.27), we get

$$-\Delta u + W'(u) = \omega_0^2 u. \quad (1.33)$$

The Lagrangian of eq. (1.27) is given by

$$\mathcal{L} = \frac{1}{2} \left( \frac{\partial \psi}{\partial t} \right)^2 - \frac{1}{2} |\nabla \psi|^2 - W(\psi).$$

It is easy to check that this Lagrangian is invariant for the Lorentz group. Thus given a solution $\psi(t, x)$ of (1.27), we can obtain an other solution $\psi_1(t, x)$ just making a Lorentz transformation on it. Namely, if we take the velocity $v = (v, 0, 0)$, $|v| < 1$, and set

$$t' = \gamma (t - vx_1), \quad x'_1 = \gamma (x_1 - vt), \quad x'_2 = x_2, \quad x'_3 = x_3 \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1 - v^2}},$$

it turns out that

$$\psi_v(t, x) = \psi(t', x')$$

is a solution of (1.27).

In particular, given a standing wave $\psi(t, x) = u(x) e^{-i\omega_0 t}$, the function $\psi_v(t, x) := \psi(t', x')$ is a solitary wave which travels with velocity $v$. Thus, if $u(x) = u(x_1, x_2, x_3)$ is any solution of Eq. (1.33), then

$$\psi_v(t, x_1, x_2, x_3) = u (\gamma (x_1 - vt), x_2, x_3) e^{i(k \cdot x - \omega t)} \quad (1.34)$$

is a solution of Eq. (1.27) provided that

$$\omega = \gamma \omega_0 \quad \text{and} \quad k = \gamma \omega_0 v. \quad (1.35)$$

Notice that (1.29) is a particular case of (1.34) when $\omega_0 = 0$.

We have the following result:

**Theorem 1.2** Assume that
• (i) $W(u) \geq 0$, 
• (ii) $W(0) = W''(0) = 0$; $W''(0) = m_0 > 0$, 
• (iii) there exists $u_0 \in \mathbb{R}^+$, $W(u_0) < \frac{1}{2} m_0^2 u_0^2$.

Then eq. (1.27) has finite energy solitary waves of the form $\psi_0(t, x) = u(x)e^{-i\omega_0 t}$ for every frequency $\omega_0 \in (m_1, m_0)$ where

$$m_1 = \inf \left\{ m > 0 : \exists u \in \mathbb{R}^+, W(u) - \frac{1}{2} m^2 u^2 < 0 \right\}.$$

Notice that by (iii), $m_1 < m_0$, then the interval $(m_1, m_0)$ is not empty.

**Proof.** By the previous discussion, it is sufficient to show that equation (1.33) has a solution $u$ with finite energy. The solutions of finite energy of (1.33) are the critical points in the Sobolev space $H^1(\mathbb{R}^3)$ of the reduced action functional:

$$J(u) = \frac{1}{2} \int |\nabla u|^2 \, dx + \int G(u) \, dx, \quad G(u) = W(u) - \frac{1}{2} \omega_0^2 u^2.$$

By a theorem of Berestycki and Lions [5], the existence of nontrivial critical points of $J$ is guaranteed by the following assumptions on $G$:

• $G(0) = G'(0) = 0$
• $G''(0) > 0$
• $\limsup_{s \to \infty} \frac{G(s)}{s} \geq 0$
• $\exists u_0 \in \mathbb{R}^+ : G(u_0) < 0$.

It is easy to check that for every frequency $\omega_0 \in (m_1, m_0)$, the above assumptions are satisfied.

### 1.4 Solitary waves in Abelian gauge theories

Now let us consider the problem of the existence of solitary waves for an Abelian gauge theory. The Lagrangian $L$ is invariant for the following representation of the Lorentz group:

$$\psi_v(t, x) = \psi(t', x')$$
$$\phi_v(t, x) = \gamma [\phi(t', x') + v \cdot A(t', x')]$$
$$A_v(t, x) = \gamma [A(t', x') + \phi(t', x')v].$$

Thus, similar to the case of eq. (1.27), in order to produce solitary waves, it is sufficient to find stationary solutions of eq. (1.6), (1.7), (1.8) and to make a Lorentz transform. In particular we shall look for solutions of (1.6), (1.7), (1.8) of the type

$$\psi(t, x) = u(x)e^{-i\omega t}, \quad u \geq 0, \omega \text{ real}, A = 0, \phi = \phi(x).$$

(1.37)
We shall assume that $W$ is a $C^2$ function satisfying the following assumptions:

$W_1$) $W \geq 0$, $W(0) = W'(0) = 0$

$W_2$) $W''(0) = m_0^2 > 0$

$W_3$) There exist $m_1$, $c > 0$ with $m_1 < m_0$ s.t.

$$W(s) \leq \frac{1}{2} m_1^2 s^2 + c \text{ for all } s \in \mathbb{R}.$$ 

$W_4$) for all $s \in \mathbb{R}$

$$0 \leq \frac{1}{2} W'(s)s \leq W(s).$$

$W_5$) There exist constants $c_1, c_2 > 0$ and $p < 4$ such that for all $s$

$$|W''(s)| \leq c_1 |s|^p + c_2.$$

The main result of this paper is the following theorem:

**Theorem 1.3** Assume that $W$ satisfies $W_1, \ldots, W_5$. Then there exists $q_* > 0$ such that for any $q < q_*$ equations (1.6), (1.7), (1.8) possess a (non trivial) finite energy solution of the type (1.37).

## 2 Existence of solitary waves in Abelian Gauge Theories

The remaining part of this paper is devoted to the proof of a theorem of which theorem 1.3 is an immediate consequence.

### 2.1 Statement of the main theorem

In this section we introduce some technical preliminaries and state the main theorem. We look for solutions of (1.6), (1.7), (1.8) of type (1.37). With this ansatz, equations (1.10) and (1.12) are identically satisfied, while (1.9) and (1.11) become

$$-\Delta u - (q\phi - \omega)^2 u + W'(u) = 0 \quad (2.38)$$

$$\Delta \phi = q(q\phi - \omega) u^2. \quad (2.39)$$

We shall set

$$\Phi = \frac{\phi}{\omega}.$$ 

Equations (2.38), (2.39) become

$$-\Delta u - \omega^2 (q\Phi - 1)^2 u + W'(u) = 0 \quad (2.40)$$

$$-\Delta \Phi + q^2 u^2 \Phi = qu^2. \quad (2.41)$$
Let $H^1 = H^1(\mathbb{R}^3)$ denote the usual Sobolev space with norm
\[ \|u\|_{H^1} = \left( \int (|\nabla u|^2 + u^2) \, dx \right)^{\frac{1}{2}} \]
and $D$ denote the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the inner product
\[ (v \mid w)_D = \int (\nabla v \mid \nabla w) \, dx. \tag{2.42} \]

The request that $\psi(t, x) = u(x)e^{-i\omega t}$ and $\phi$ (or $\Phi$) possess finite energy is satisfied if and only if $u \in H^1$ and $\phi$ (or $\Phi$) $\in D$.

Now we can state the following theorem

**Theorem 2.1 (Main theorem)** Assume that $W$ satisfies $W_1, \ldots, W_5$. Then there exists $q_* > 0$ such that for any $q < q_*$, there exist $\omega \neq 0$, $\omega^2 < m_0^2$, and non trivial solutions $u \in H^1$, $\Phi \in D$ of (2.40) and (2.41).

### 2.2 The variational framework

Consider the functional
\[ F_\omega(u, \Phi) = J(u) - \omega^2 A(u, \Phi) \tag{2.43} \]
where
\[ J(u) = \frac{1}{2} \int |\nabla u|^2 \, dx + \int W(u) \, dx \tag{2.44} \]
and
\[ A(u, \Phi) = \frac{1}{2} \int |\nabla \Phi|^2 \, dx + \frac{1}{2} \int u^2(1 - q\Phi)^2 \, dx. \tag{2.45} \]

Standard arguments show that $F_\omega$ is $C^1$ on $H^1(\mathbb{R}^3) \times D$ and its critical points $(u, \Phi)$ are weak solutions of (2.40) and (2.41); so equations (2.40) and (2.41) can be written as follows
\[ \frac{\partial F_\omega}{\partial u}(u, \Phi) = 0 \tag{2.46} \]
\[ \frac{\partial F_\omega}{\partial \Phi}(u, \Phi) = 0 \text{ or } \frac{\partial A(u, \Phi)}{\partial \Phi} = 0 \tag{2.47} \]
where $\frac{\partial F_\omega}{\partial u}(u, \Phi)$, $\frac{\partial F_\omega}{\partial \Phi}(u, \Phi)$, $\frac{\partial A(u, \Phi)}{\partial \Phi}$ denote the partial derivatives of $F_\omega$ and $A$ at $(u, \Phi) \in H^1(\mathbb{R}^3) \times D$, namely for any $v \in H^1(\mathbb{R}^3)$ and $w \in D$
\[ \frac{\partial F_\omega}{\partial u}(u, \Phi) [v] = \int ((\nabla u \mid \nabla v) + W'(u) v) \, dx - \omega^2 \int (1 - q\Phi)^2 uv \, dx \tag{2.48} \]
\[ \frac{\partial F_\omega}{\partial \Phi}(u, \Phi) [w] = -\omega^2 \frac{\partial A(u, \Phi)}{\partial \Phi}(u, \Phi) [w] = -\omega^2 \int ((\nabla \Phi \mid \nabla w) + (q\Phi - 1) u^2 w) \, dx \tag{2.49} \]
Now we reduce the study of (2.43) to the study of a functional of the only variable $u$. Following [1], it can be shown that, for any $u \in H^1(\mathbb{R}^3)$, there exists a unique solution $\Phi \in D$ of (2.41). So we can define the map

$$u \in H^1(\mathbb{R}^3) \rightarrow \Phi(u) \in D$$ (2.50)

where $\Phi(u)$ is the unique solution of (2.41).

Now we set

$$I_\omega(u) = F_\omega(u, \Phi(u)) = J(u) - \omega^2 A(u, \Phi(u)), \quad u \in H^1(\mathbb{R}^3) \quad (2.51)$$

where $\Phi(u)$ is defined in (2.50). The functional $I_\omega$ is $C^1$. Moreover, by (2.41), we easily get

$$\int |\nabla \Phi(u)|^2 \, dx = \int (qu^2 \Phi(u) - q^2 u^2 \Phi(u)^2) \, dx.$$ (2.52)

Inserting (2.52) in (2.45), we get

$$A(u, \Phi(u)) = \frac{1}{2} \int u^2 (1 - q \Phi(u)) \, dx.$$ (2.53)

So by (2.51) and (2.53) we can write

$$I_\omega(u) = F_\omega(u, \Phi(u)) = J(u) - \frac{\omega^2}{2} \int u^2 (1 - q \Phi(u)) \, dx.$$ (2.54)

The following proposition holds

**Proposition 2.1** Let $(u, \Phi) \in H^1(\mathbb{R}^3) \times D$. Then the following statements are equivalent:

a) $(u, \Phi)$ is a critical point of $F_\omega$

b) $u$ is a critical point of $I_\omega$ and $\Phi = \Phi(u)$ solves (2.47)

**Proof.** Clearly we have

b) $\Leftrightarrow \frac{\partial F_\omega}{\partial u}(u, \Phi) + \frac{\partial F_\omega}{\partial \Phi}(u, \Phi) \Phi'(u) = 0$ and $\Phi = \Phi(u)$ solves (2.47) $\Leftrightarrow \frac{\partial F_\omega}{\partial u}(u, \Phi) = 0$ and $\frac{\partial F_\omega}{\partial \Phi}(u, \Phi) = 0$ $\Leftrightarrow$ a). $\blacksquare$

Thus we are reduced to finding the critical points of

$$I_\omega(u) = J(u) - \frac{\omega^2}{2} \int u^2 (1 - q \Phi(u)) \, dx, \quad u \in H^1(\mathbb{R}^3).$$ (2.55)

The functional $I_\omega$ presents a lack of compactness due to its invariance under the group transformations $u(x) \rightarrow u(x + a)$ $(a \in \mathbb{R}^3)$. To overcome this difficulty we restrict ourselves to radial functions $u = u(r), \quad r = |x|$. More precisely we shall consider the functional $I_\omega$ on the subspace of the radially symmetric functions

$$H^1_r = \{ u \in H^1(\mathbb{R}^3) : u = u(r), \quad r = |x| \}.$$(2.56)
We recall (see [19] or [5]) that, for $6 > p > 2$, $H^1_r$ is compactly embedded into $L^p_r$, where $L^p_r = \{ u \in L^p(\mathbb{R}^3) : u \text{ radially symmetric} \}$.

Now $H^1_r$ is a natural constraint for $I_\omega$, namely the following lemma holds.

**Lemma 2.1** Any critical point $u \in H^1_r$ of $I_\omega|_{H^1_r}$ is also a critical point of $I_\omega$.

**Proof.** Consider the $O(3)$ group action $T_g$ on $H^1(\mathbb{R}^3)$ defined by

$$T_g u(x) = u(g(x)).$$

Clearly $H^1_r$ is the set of the fixed points for this action, namely

$$H^1_r = \{ u \in H^1(\mathbb{R}^3) : T_g u = u \text{ for all } g \in O(3) \}.$$

Then the conclusion can be achieved by usual arguments (see [19]), if we show that $I_\omega$ is invariant under the $T_g$ action, namely if

$$I_\omega(T_g u) = I_\omega(u) \quad (2.57)$$

Now, for $u \in H^1(\mathbb{R}^3)$, $\Phi(u)$ solves the equation

$$-\Delta \Phi + q^2 u^2 \Phi = qu^2.$$

Then, if $g \in O(3)$, we have

$$T_g(-\Delta \Phi(u) + q^2 u^2 \Phi(u)) = qT_g(u^2)$$

and

$$-\Delta(T_g \Phi(u)) + q^2(T_g u)^2(T_g \Phi(u)) = q(T_g u)^2.$$

This equality and the definition of $\Phi$ imply that

$$T_g \Phi(u) = \Phi(T_g u). \quad (2.58)$$

Therefore, using (2.58), we easily deduce (2.57).

We want to find non trivial $u$, $\Phi$, $\omega$, such that (2.40) and (2.41) are solved. Then, by lemma 2.1 and proposition 2.1, we have to find for some $\omega \neq 0$ a non trivial critical point $u \in H^1_r$ of $I_\omega|_{H^1_r}$. In this case $u$, $\Phi(u)$ will be non trivial weak solutions of (2.40) and (2.41). To do this we set, for $\sigma > 0$

$$V_\sigma = \left\{ u \in H^1_r : \frac{1}{2} \int u^2(1 - q\Phi(u))dx = \sigma^2 \right\}.$$

We shall look for critical points (in particular minimizers) of the functional $J$ defined in (2.44) on $V_\sigma$ and $\omega^2$ will be obtained as Lagrange multiplier. Then in order to prove theorem 2.1, it will be enough to prove the following proposition.
Proposition 2.2 Assume that $W$ satisfies the assumptions $W_1, \ldots, W_5$. Then there exists $q_*>0$ such that for any $q<q_*$ there exists $\sigma>0$ s.t. the functional $J$ has a minimizer on the manifold

$$V_{q} = \left\{ u \in H^1_r : \frac{1}{2} \int u^2(1-q\Phi(u))dx = \sigma^2 \right\}.$$ 

Moreover, for the corresponding Lagrange multiplier $\omega^2$ we have

$$0 < \omega^2 < m_0^2.$$ 

2.3 Preliminary lemmas

In this section we shall prove some technical lemmas.

Lemma 2.2 The functional

$$\Lambda : H^1 \to \mathbb{R}^3, \Lambda(u) = \frac{1}{2} \int u^2(1-q\Phi(u))dx$$ 

is $C^1$, and for any $u \in H^1$,

$$\Lambda'(u) = u(1-q\Phi(u)). \quad (2.59)$$

Proof. Standard arguments show that $\Lambda$ is $C^1$. Now we prove (2.59). By (2.53) and (2.45) we have

$$\Lambda(u) = A(u, \Phi(u)) = \frac{1}{2} \int |\nabla \Phi(u)|^2 dx + \frac{1}{2} \int u^2(1-q\Phi(u))^2 dx.$$ 

Then

$$\Lambda'(u) = \frac{\partial A}{\partial u}(u, \Phi(u)) + \frac{\partial A}{\partial \Phi}(u, \Phi(u))\Phi'(u). \quad (2.60)$$

Since $\Phi(u)$ solves (2.41), we have

$$\frac{\partial A}{\partial \Phi}(u, \Phi(u)) = 0.$$ 

Then (2.60) gives

$$\Lambda'(u) = \frac{\partial A}{\partial u}(u, \Phi(u)) = u(1-q\Phi(u)).$$ 

■
The following lemma holds

**Lemma 2.3** Let \( u \in H^1 \) and \( \Phi(u) \in D \) be the solution of (2.41). Then

\[
0 \leq \Phi(u) \leq \frac{1}{q}. \tag{2.61}
\]

**Proof.** Arguing by contradiction, we assume that there exists an open subset \( \Omega \subset \mathbb{R}^3 \) such that

\[
\Phi(u) > \frac{1}{q} \text{ in } \Omega. \tag{2.62}
\]

Then, since \( \Phi(u) \) solves (2.41), we have

\[
-\Delta(\Phi(u) - \frac{1}{q}) + q^2u^2(\Phi(u) - \frac{1}{q}) = -\Delta \Phi(u) + q^2u^2\Phi(u) - qu^2 = 0.
\]

So \( v = \Phi(u) - \frac{1}{q} \) satisfies

\[
-\Delta v + q^2u^2v = 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega.
\]

Then \( v = 0 \), contradicting (2.62). An analogous argument shows that \( \Phi(u) \geq 0 \). □

**Lemma 2.4** For any \( \sigma > 0 \), the set

\[
V_\sigma = \left\{ u \in H^1_r : \frac{1}{2} \int u^2(1 - q\Phi(u))dx = \sigma^2 \right\}
\]

is not empty and it is a one codimensional manifold.

**Proof.** Let \( \sigma > 0 \) and first prove that

\[
V_\sigma = \left\{ u \in H^1_r : \frac{1}{2} \int u^2(1 - q\Phi(u))dx = \sigma^2 \right\} \neq \emptyset.
\]

Fix \( u \in H^1_r, u \neq 0 \) a.e. in \( \mathbb{R}^3 \), and set for \( \lambda > 0 \)

\[
u_\lambda(x) = \lambda u(\lambda x), \quad \Phi_\lambda(u)(x) = \Phi(u)(\lambda x).
\]

We have

\[
\Phi_\lambda(u)(x) = \Phi(u)(\lambda x). \tag{2.63}
\]

In fact

\[
-\Delta \Phi_\lambda(u)(x) + q^2u^2_\lambda(x)\Phi_\lambda(u)(x) = \\
= \lambda^2 \left( -\Delta \Phi(u)(\lambda x) + q^2u^2(\lambda x)(\Phi(u)(\lambda x)) \right) \tag{2.64}
\]
and, since $\Phi(u)$ satisfies (2.41), we have
\[-\Delta \Phi(u)(\lambda x) + q^2 u^2(\lambda x)\Phi(u)(\lambda x) = qu^2(\lambda x). \quad (2.65)\]
From (2.64) and (2.65) we get
\[-\Delta \Phi_\lambda(u)(x) + q^2 u^2_\lambda(x)\Phi_\lambda(u)(x) = \lambda^2 qu^2(\lambda x) = qu^2_\lambda(x) \]
and this implies (2.63).

Now set
\[\sigma^2 = \frac{1}{2} \int u^2_\lambda(1 - q\Phi(u_\lambda))dx\]
and first show that
\[\sigma^2_1 = \frac{1}{2} \int u^2(1 - q\Phi(u))dx > 0. \quad (2.66)\]
Arguing by contradiction, assume that
\[\int u^2(1 - q\Phi(u))dx = 0.\]
Then, by lemma 2.3 and since $u \neq 0$ a.e. in $\mathbb{R}^3$, we have
\[\Phi(u) = \frac{1}{q}\]
contradicting the fact that $\Phi(u) \in D \subset L^6$.

Now we have
\[\sigma^2 = \frac{1}{2} \int u^2_\lambda(1 - q\Phi(u_\lambda))dx = (by \ (2.63)) = \frac{1}{2} \int u^2_\lambda(1 - q\Phi_\lambda(u))dx = \]
\[= \frac{1}{2\lambda} \int u^2(1 - q\Phi(u))dx = \frac{\sigma^2_1}{\lambda}.\]
We conclude that for any $\sigma > 0$ we can chose $\lambda = \frac{\sigma^2_1}{\sigma^2}$ so that $\sigma^2_\lambda = \sigma^2$. This means that $u_\lambda \in V_\sigma$.

$V_\sigma$ is a one codimensional manifold. In fact, by lemma 2.2, $\Lambda$ is $C^1$ and we have
\[\Lambda'(u) = u(1 - q\Phi(u)).\]
Then, since $\Phi(u) \leq \frac{1}{q}$, we have
\[(\Lambda'(u) = 0) \iff (u(1 - q\Phi(u)) = 0) \iff (u^2(1 - q\Phi(u)) = 0) \iff (\int (u^2(1 - q\Phi(u)))dx = 0) \iff (u \notin V_\sigma).\]
So for any $u \in V_\sigma$ we have
\[\Lambda'(u) \neq 0.\]
Lemma 2.5 Assume that $W$ satisfies $W_3$. Then there exists $q_* > 0$ such that for $q < q_*$ there exists $\bar{u} \in H^1_1, \bar{u} \neq 0$ s.t.
\[
\frac{J(\bar{u})}{\frac{1}{2} \int \bar{u}^2 (1 - q\Phi(\bar{u})) dx} < m_0^2.
\]

Proof. Let us first consider the case $q = 0$ and prove that there exists $\bar{u} \in H^1_1, \bar{u} \neq 0$ s.t.
\[
\frac{J(\bar{u})}{\frac{1}{2} \int \bar{u}^2 dx} < m_0^2.
\] (2.67)

To this end, let $u$ be a non trivial, smooth and radial function with compact support $K$ and set for $\lambda > 0$
\[
u = \lambda u(\frac{x}{\lambda}), \sigma_\lambda^2 = \frac{1}{2} \int u^2 dx, K_\lambda = \lambda K.
\]

By assumption $W_3)$ we have
\[
\frac{J(u_\lambda)}{\sigma_\lambda^2} = \frac{1}{2} \int_{K_\lambda} |\nabla u_\lambda|^2 dx + \int_{K_\lambda} W(u_\lambda) dx \leq \frac{1}{2} \int_{K_\lambda} |\nabla u_\lambda|^2 dx + \int_{K_\lambda} \left( \frac{m_1^2}{2} u_\lambda^2 + c \right) dx
\]
\[= \frac{1}{2} \int_{K_\lambda} |\nabla u_\lambda|^2 dx + \int_{K_\lambda} \frac{m_1^2}{2} u_\lambda^2 dx + c \frac{\text{meas}(K_\lambda)}{\sigma_\lambda^2}.
\]

Then by easy computations we get
\[
\frac{J(u_\lambda)}{\sigma_\lambda^2} \leq c_1 \lambda^{-2} + m_1^2
\] (2.68)

where $c_1$ is a positive constant depending only on the fixed map $u$. Then, since $m_1^2 < m_0^2$, by (2.68), we get that (2.67) is verified if we take
\[
\bar{u} = u_\lambda, \text{ for } \lambda \text{ large.}
\]

Now we shall write $\Phi_q(\bar{u})$ instead of $\Phi(u)$ in order to emphasize the dependence on $q$. Clearly the conclusion will easily follow from (2.67) if we show that
\[
\frac{1}{2} \int \bar{u}^2 (1 - q\Phi_q(\bar{u})) dx \to \frac{1}{2} \int \bar{u}^2 dx \text{ for } q \to 0.
\] (2.69)

So it remains to prove (2.69). The map $\Phi_q(\bar{u})$ satisfies equation (2.41). Thus
\[
-\Delta \Phi_q(\bar{u}) + q^2 u^2 \Phi_q(\bar{u}) = q\bar{u}^2.
\]
So, multiplying by $\Phi_\ell(\bar{u})$ and integrating, we have
\[ \|\Phi_\ell(\bar{u})\|_D^2 + q^2 \int \bar{u}^2 \Phi_\ell(\bar{u})^2 dx = q \int \bar{u}^2 \Phi_\ell(\bar{u}) dx \leq \]
\[ \leq q \|\bar{u}\|_{L^{\frac{13}{4}}}^2 \|\Phi_\ell(\bar{u})\|_{L^6} \] (2.70)
and then
\[ \frac{\|\Phi_\ell(\bar{u})\|_D^2}{\|\Phi_\ell(\bar{u})\|_{L^6}} \leq q \|\bar{u}\|_{L^{\frac{13}{4}}}^2. \]
Thus, since $D$ is continuously embedded into $L^6$, we easily get
\[ \|\Phi_\ell(\bar{u})\|_D \leq c_2 q \|\bar{u}\|_{L^{\frac{13}{4}}}^2, \] (2.71)
where $c_2$ is a positive constant. Then, by (2.71) and by using again (2.70), we get
\[ q \int \bar{u}^2 \Phi_\ell(\bar{u}) dx \leq q \|\bar{u}\|_{L^{\frac{13}{4}}}^2 \|\Phi_\ell(\bar{u})\|_{L^6} \leq c_2 q^2 \|\bar{u}\|_{L^{\frac{13}{4}}}^4. \]
So we have
\[ q \int \bar{u}^2 \Phi_\ell(\bar{u}) dx \to 0 \text{ for } q \to 0 \]
and (2.69) is proved. ■

2.4 Proof of the main theorem

Now we are ready to prove proposition 2.2. Take $q < q_\ast$ and $\bar{u}$ as in lemma 2.5 and set
\[ \sigma^2 = \frac{1}{2} \int \bar{u}^2 (1 - q\Phi(\bar{u})) dx. \]
Consider
\[ V_\sigma = \left\{ u \in H^1_r : \frac{1}{2} \int u^2 (1 - q\Phi(u)) dx = \sigma^2 \right\}. \]
We shall prove that $J|_{V_\sigma}$ has a minimizer whose Lagrange multiplier $\omega^2$ satisfies
\[ 0 < \omega^2 < m_0^2. \]
Let \( \{u_n\} \subset V_\sigma \) be a minimizing sequence for $J|_{V_\sigma}$, i.e. \( \{u_n\} \subset H^1_r \) s.t.
\[ \frac{1}{2} \int u_n^2 (1 - q\Phi(u_n)) dx = \sigma^2 \]
and
\[ J(u_n) \to \inf J(V_\sigma). \]
By standard variational arguments we can assume that \( \{u_n\} \) is also a criticizing sequence, i.e.
\[ J'|_{V_\sigma} (u_n) \to 0. \] (2.72)
Then there exists a sequence of real numbers \( \{\lambda_n\} \) such that
\[
J'(u_n) - \lambda_n \Lambda'(u_n) = \varepsilon_n \to 0 \quad \text{in} \quad H^{-1}.
\] (2.73)

By (2.59) we can write
\[
J'(u_n) - \lambda_n u_n (1 - q \Phi_n) = \varepsilon_n \to 0 \quad \text{in} \quad H^{-1},
\] (2.74)

where we have set
\[ \Phi_n = \Phi(u_n). \]

Now the study of the minimizing sequence \( \{u_n\} \) is essentially divided in two steps:

- **First step**: \( \{u_n\} \) is bounded in \( H^1 \).
- **Second step**: \( \{u_n\} \) strongly converges (up to a subsequence) in \( H^1 \).

**Proof of the first step**: Clearly, since \( W \geq 0 \), \( \{\nabla u_n\} \) is bounded in \( L^2 \).

So, in order to get the first step, we have only to show that \( \{u_n\} \) is bounded in \( L^2 \). We begin to show that
\[ \{\Phi_n\} \text{ is bounded in } D. \] (2.75)

Since the \( \Phi_n \)'s solve (2.41) (with \( u = u_n \)), we get
\[
\int |\nabla \Phi_n|^2 \, dx = \int (qu_n^2 \Phi_n - q^2 u_n^2 \Phi_n^2) \, dx = q \int u_n^2 \Phi_n (1 - q \Phi_n) \, dx \leq (\text{by lemma 2.3}) \leq \int u_n^2 (1 - q \Phi_n) \, dx = 2\sigma^2.
\]

Then (2.75) is verified.

The maps \( \Phi_n \) are radially symmetric and the sequence \( \{\Phi_n\} \) is bounded in \( D \). Then, by virtue of a well known radial lemma [5], we have
\[
|\Phi_n(x)| \leq c_1 |x|^{-\frac{1}{2}} \|\Phi_n\|_D \leq c_2 |x|^{-\frac{1}{2}} \quad \text{for } |x| \geq 1
\]
where \( c_1, c_2 \) are positive constants independent on \( n \). Then there exists \( R > 0 \) large enough so that for all \( n \in \mathbb{N} \) and for \( |x| \geq R \)
\[
1 - q \Phi_n(x) \geq k > 0.
\]

So
\[
2\sigma^2 = \int u_n^2 (1 - q \Phi_n) \, dx \geq \int_{B_R^c} u_n^2 (1 - q \Phi_n) \, dx \geq k \int_{B_R^c} u_n^2 \, dx \quad (2.76)
\]
where

\[ B_R = \{ x \in \mathbb{R}^3 : |x| < R \}, \quad B_R^c = \mathbb{R}^3 - B_R. \]

On the other hand, since \( \{ u_n \} \) is bounded in \( D \subset L^6 \), we have that

\[ \{ u_n \} \text{ is bounded in } L^2(B_R). \] (2.77)

Finally by (2.76) and (2.77) we conclude that \( \{ u_n \} \) is bounded in \( L^2 \). Now we prove the second step.

**Proof of the second step.** First we show that, up to subsequence,

\[ \lambda_n \to \omega^2 \quad \text{where} \quad \omega^2 < m_0^2. \] (2.78)

By (2.74) we get

\[ \langle J'(u_n), u_n \rangle - \lambda_n \int u_n^2 (1 - q \Phi_n) dx = \langle J'(u_n), u_n \rangle - 2 \lambda_n \sigma^2 = \delta_n, \] (2.79)

where \( \delta_n = \langle \varepsilon_n, u_n \rangle \) and \( \langle \quad \rangle \) denotes the pairing between \( H^1 \) and its dual \( H^{-1} \). Since \( \{ u_n \} \) is bounded in \( H^1 \) and \( \varepsilon_n \to 0 \) in \( H^{-1} \), we have

\[ \delta_n \to 0. \] (2.80)

Then

\[
\lambda_n = \frac{1}{2\sigma^2} (\langle J'(u_n), u_n \rangle - \delta_n) = \frac{1}{\sigma^2} \left( \frac{1}{2} \int \left( |\nabla u_n|^2 + W'(u_n)u_n \right) dx \right) - \frac{\delta_n}{2\sigma^2} \leq \frac{J(u_n)}{\sigma^2} - \frac{\delta_n}{2\sigma^2}. \] (2.81)

On the other hand,

\[ \langle J'(u_n), u_n \rangle = \int \left( |\nabla u_n|^2 + W'(u_n)u_n \right) dx \geq 0. \] (2.82)

Then by (2.81) and (2.82) we easily get

\[ -\frac{\delta_n}{2\sigma^2} \leq \lambda_n \leq \frac{J(u_n)}{\sigma^2} - \frac{\delta_n}{2\sigma^2}. \] (2.83)

Then, up to a subsequence, we have

\[ \lambda_n \to \omega^2 \] (2.84)

where

\[ 0 \leq \omega^2 \leq \inf \left\{ \frac{J(u)}{\sigma^2} : u \in V_\sigma \right\}. \] (2.85)
Since $\bar{u}$ and $q$ are chosen as in lemma 2.5, we have

$$
\frac{J(\bar{u})}{\sigma^2} < m_0^2.
$$

(2.86)

From (2.84), (2.85) and (2.86) we clearly get (2.78).

Now we show that $\{u_n\}$ is a Cauchy sequence in $H^1$. Set

$$
w_{nm} = u_n - u_m.
$$

Then writing (2.74) for $u_n$ and $u_m$ and subtracting we get

$$
-\Delta w_{nm} + W'(u_n) - W'(u_m) - \lambda_n u_n + \lambda_m u_m + q\lambda_n u_n \Phi_n - q\lambda_m u_m \Phi_m = \varepsilon_{nm}
$$

(2.87)

where

$$
\varepsilon_{nm} = \varepsilon_n - \varepsilon_m.
$$

If we add and subtract in (2.87) the terms $\lambda_n u_m$, $q\lambda_n u_m \Phi_m$ and multiply, by the duality map, both the sides by $w_{nm}$, we get

$$
\int (|\nabla w_{nm}|^2 + (W''(\xi_{nm}) - \lambda_n) w_{nm}^2) w_{nm} dx + q(\lambda_n - \lambda_m) \int u_m \Phi_m w_{nm} dx + q\lambda_n \int (u_n \Phi_n - u_m \Phi_m) w_{nm} dx = (\varepsilon_{nm}, w_{nm}) = \delta_{nm}.
$$

(2.88)

Since $w_{nm}$ are bounded in $H^1$ and $\varepsilon_{nm} \to 0$ in $H^{-1}$, we get

$$
\delta_{nm} \to 0.
$$

(2.89)

Now

$$
\int (W'(u_n) - W'(u_m)) w_{nm} dx = \int (W''(\xi_{nm})) w_{nm}^2 dx
$$

(2.90)

where

$$
\xi_{nm} = tu_n + (1 - t)u_m, 0 \leq t \leq 1.
$$

(2.91)

Therefore, (2.88) can be written as follows:

$$
\int (|\nabla w_{nm}|^2 + (W''(\xi_{nm}) - \lambda_n) w_{nm}^2) dx + B_{nm}^1 + B_{nm}^2 + B_{nm}^3 = \delta_{nm}
$$

(2.92)

where

$$
B_{nm}^1 = \int (\lambda_n - \lambda_m) u_m w_{nm} dx,
$$

$$
B_{nm}^2 = q(\lambda_n - \lambda_m) \int u_m \Phi_m w_{nm} dx
$$

and

$$
B_{nm}^3 = q\lambda_n \int (u_n \Phi_n - u_m \Phi_m) w_{nm} dx.
$$
Now we show that
\[ B_{1}^{n} m \to 0, B_{2}^{n} m \to 0, B_{3}^{n} m \to 0. \] (2.93)

Since \( \{u_{n}\} \) is bounded in \( H^{1} \), the sequence \( \{ \int u_{n} w_{nm} \} \) is also bounded. Moreover \( \lambda_{n} - \lambda_{m} \to 0 \), and hence
\[ B_{1}^{n} m \to 0. \]

Analogously, in order to prove that \( B_{2}^{n} m \to 0 \), it will be enough to show that
\[ \left\{ \int u_{m} \Phi_{m} w_{nm} dx \right\} \] is bounded. (2.94)

This is true; in fact,
\[ \left| \int u_{m} \Phi_{m} w_{nm} dx \right| \leq \|u_{m}\|_{L^{\frac{12}{5}}} \|\Phi_{m}\|_{L^{6}} \|w_{nm}\|_{L^{\frac{12}{5}}}. \]
and we know that \( \{u_{m}\} \) is bounded in \( H^{1} \) (and therefore in \( L^{\frac{12}{5}} \)) and \( \{\Phi_{m}\} \) is bounded in \( D \) (and therefore in \( L^{6} \)).

Finally, in order to prove that \( B_{3}^{n} m \to 0 \), it will be enough to show that
\[ \int (u_{n} \Phi_{n} - u_{m} \Phi_{m}) w_{nm} dx \to 0. \] (2.95)

Clearly
\[ \int (u_{n} \Phi_{n}) w_{nm} dx \leq \|w_{nm}\|_{L^{3}} \|u_{n} \Phi_{n}\|_{L^{\frac{12}{5}}} . \] (2.96)

Since \( H^{1}_{c} \) is compactly embedded into \( L^{3} \) (see [5] or [19]), and \( w_{nm} = u_{n} - u_{m} \) weakly converges to 0 in \( H^{1}_{c} \), we have
\[ w_{nm} = u_{n} - u_{m} \to 0 \text{ in } L^{3}. \] (2.97)

On the other hand,
\[ \int (u_{n} \Phi_{n}) \frac{1}{2} dx \leq \left( \int u_{n}^{2} dx \right)^{\frac{1}{2}} \left( \int \Phi_{n}^{2} dx \right)^{\frac{1}{2}} . \]

Then, since \( \{u_{n}\} \) is bounded in \( L^{2} \) and \( \{\Phi_{n}\} \) is bounded in \( L^{6} \), \( \|u_{n} \Phi_{n}\|_{L^{\frac{12}{5}}} \) is bounded. So (2.95) easily follows from (2.96) and (2.97).

We conclude that (2.93) holds and therefore (2.92) implies that
\[ \int \left( |\nabla w_{nm}|^{2} + (W''(\xi_{nm}) - \lambda_{n}) w_{nm}^{2} \right) dx = \eta_{nm} \] (2.98)
where \( \eta_{nm} \to 0. \)

Now, since \( \lambda_{n} \to \omega^{2} < m_{0}^{2} \) (see (2.78) and \( W''(0) = m_{0}^{2} \), there exist \( \theta, \delta > 0 \) such that
\[ W''(s) - \lambda_{n} > \theta \text{ for } n \text{ large and } |s| < \delta. \] (2.99)
By a well known radial lemma ([5] or [19]) and since the sequence \(\{\xi_{nm}\}\) is bounded in \(H^1_r\), there exists \(M > 0\) such that for all \(n, m\)
\[
|\xi_{nm}(x)| < \delta \text{ for } |x| > M.
\]

(2.100)

Then, if we set
\[
B_M = \{x \in \mathbb{R}^3 : |x| < M\}, \quad B_M^c = \mathbb{R}^3 - B_M,
\]
from (2.99) and (2.100) we easily get for \(n\) large
\[
\int_{B_M^c} (W''(\xi_{nm}) - \lambda_n) w_{nm}^2 \, dx \geq \int_{B_M^c} \theta w_{nm}^2 \, dx.
\]

(2.101)

Now we show that
\[
\int_{B_M} (W''(\xi_{nm}) - \lambda_n) w_{nm}^2 \, dx \to 0.
\]

(2.102)

The Sobolev space \(H^1(B_M)\) is compactly embedded into \(L^q(B_M)\) (\(2 \leq q < 6\)), and the sequence \(w_{nm} = u_n - u_m\) weakly converges to 0 in \(H^1(B_M)\). Then we have
\[
\|w_{nm}\|_{L^q(B_M)} \to 0 \text{ for } 2 \leq q < 6.
\]

(2.103)

By assumption \(W_5\) we easily get
\[
\int_{B_M} |W''(\xi_{nm}) w_{nm}^2| \, dx \leq c_1 \int_{B_M} |\xi_{nm}|^p w_{nm}^2 \, dx + c_2 \int_{B_M} w_{nm}^2 \, dx \leq
\]
\[
c_1 \|\xi_{nm}\|_{L^p(B_M)}^p \|w_{nm}\|_{L^2(B_M)}^2 + c_2 \|w_{nm}\|_{L^2(B_M)}^2,
\]
where
\[
\bar{q} = \frac{12}{6 - p} < 6.
\]

Since the sequence \(\left\{\|\xi_{nm}\|_{L^p(B_M)}^p\right\}\) is bounded, we deduce by (2.103) and (2.104) that
\[
\int_{B_M} |W''(\xi_{nm}) w_{nm}^2| \, dx \to 0.
\]

Then we clearly have (2.102). From (2.103), (2.101) and (2.102) we easily get
\[
\int (W''(\xi_{nm}) - \lambda_n) w_{nm}^2 \, dx \geq \int \theta w_{nm}^2 \, dx + \mu_{nm}, \text{ with } \mu_{nm} \to 0.
\]

(2.105)

From (2.98) and (2.105) we get
\[
\int \left(|\nabla w_{nm}|^2 + \theta w_{nm}^2\right) \, dx \leq \gamma_{nm}
\]
where \(\gamma_{nm} = \eta_{nm} - \mu_{nm} \to 0\). Then we conclude that \(\{u_n\}\) is a Cauchy sequence in \(H^1\) and therefore
\[
u_n \to u \text{ strongly in } H^1.
\]

(2.106)
Then, since $J$ and $\Lambda$ are $C^1$, we have
\[ J(u_n) \to J(u) = \inf J(V_\sigma) \]
and
\[ J'(u_n) \to J'(u), \quad \Lambda'(u_n) \to \Lambda'(u). \tag{2.107} \]
Now we prove that $u \in V_\sigma$, i.e.
\[ \frac{1}{2} \int u^2(1 - q\Phi(u)) dx = \sigma^2. \tag{2.108} \]
Since $\{\Phi_n\}$ is bounded in $D$, we have (up to a subsequence)
\[ \Phi_n \to \Phi \text{ weakly in } D. \tag{2.109} \]
First observe that
\[ \Phi = \Phi(u) \tag{2.110} \]
i.e. $\Phi$ solves the equation
\[ -\Delta \Phi + q^2 u^2 \Phi = qu^2. \tag{2.111} \]
In fact, since $\Phi_n = \Phi(u_n)$, we have
\[ -\Delta \Phi_n + q^2 u_n^2 \Phi_n = qu_n^2. \tag{2.112} \]
Then, by (2.106) and (2.109), standard calculations permit to take the limit in (2.112) and to get (2.111). So (2.110) holds.
Since $u_n \in V_\sigma$, we have
\[ \frac{1}{2} \int u_n^2(1 - q\Phi_n) dx = \sigma_n^2 \text{ for all } n. \]
Again, by (2.106) and (2.109), standard calculations permit to show that
\[ \frac{1}{2} \int u_n^2(1 - q\Phi_n) dx \to \frac{1}{2} \int u^2(1 - q\Phi(u)) dx = \frac{1}{2} \int u^2(1 - q\Phi(u)) dx. \]
So (2.108) holds and $u$ is a minimizer of $J$ on $V_\sigma$.
By (2.107), (2.73) and (2.78) $u$ solves the equation
\[ J'(u) - \omega^2 \Lambda'(u) = 0 \]
where $\omega^2 < m_0^2$.
Finally we show that $\omega^2 > 0$. In fact, arguing by contradiction, assume $\omega^2 = 0$. Then $u \in H^1$ is a non trivial solution of the equation
\[ J'(u) = 0 \]
i.e.
\[ -\Delta u + W'(u) = 0. \]
However, since $W \geq 0$, this equation has no non trivial solution in $H^1$ by the Derrick-Pohozaev identity (1.31).
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