Heat kernel expansion and induced action
for the matrix model Dirac operator

Daniel N. Blaschke∗, Harold Steinacker†, Michael Wohlgenannt‡

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∗∗†‡ University of Vienna, Faculty of Physics
Boltzmanngasse 5, A-1090 Vienna (Austria)
† Vienna University of Technology, Institute for Theoretical Physics
Wiedner Hauptstrasse 8-10, A-1040 Vienna (Austria)

Abstract

We compute the quantum effective action induced by integrating out fermions in Yang-Mills matrix models on a 4-dimensional background, expanded in powers of a gauge-invariant UV cutoff. The resulting action is recast into the form of generalized matrix models, manifestly preserving the $SO(D)$ symmetry of the bare action. This provides non-commutative (NC) analogs of the Seeley-de Witt coefficients for the emergent gravity which arises on NC branes, such as curvature terms. From the gauge theory point of view, this provides strong evidence that the non-commutative $\mathcal{N} = 4$ SYM has a hidden $SO(10)$ symmetry even at the quantum level, which is spontaneously broken by the space-time background. The geometrical view proves to be very powerful, and allows to predict non-trivial loop computations in the gauge theory.
1 Introduction

The aim of this work is to study the quantum effective action induced by integrating out fermions in Yang-Mills-type matrix models, using heat-kernel techniques.

The matrix models under consideration were introduced originally in the context of string theory, where they are viewed as non-perturbative definitions of superstrings on flat $R^{10}$. On the other hand, one can also consider non-commutative brane solutions (or generic brane configurations) in these matrix models. It is well-known that this leads to non-commutative gauge...
theory on the branes \[1\]. Upon closer examination, it turns out that the $U(1)$ sector of this
gauge theory can be understood in terms of geometry, and encodes an effective gravity theory
on the brane \[2, 4\]. This gives a novel and direct gauge/gravity relation specific to the non-
commutative setting. However, to understand the dynamics of this emergent gravity, quantum
effects must be taken into account. The reason is that quantization on non-trivial backgrounds
leads to induced gravitational actions \[5\]. On a semi-classical level, this can be understood in
terms of Seeley-de Witt coefficients.

In order to properly understand the physical content of these models, one must study their
quantization at the level of matrix models, taking their non-commutative nature into account.
As an important step in this program, we study in this paper the quantum effective actions
due to fermions, and show that it can be recast in the form of generalized matrix models\[4\].
This is the language which is appropriate to understand the geometric aspects of these models.
Moreover, it turns out to provide a powerful and predictive new tool for the description of non-
commutative gauge theory, notably for the maximally supersymmetric $\mathcal{N} = 4$ gauge theory.

We thus consider fermions coupled to a generic non-commutative background in the matrix
model. The matrices $X^a$ can be considered as perturbations around the Groenewold-Moyal
quantum plane $\mathbb{R}^4_\theta$. Hence, a scale of non-commutativity $\Lambda_{NC}$ is introduced, as will be explained
in more detail in subsequent sections. We will then compute the quantum effective action
induced by integrating out fermions. To this end we make a heat kernel expansion for $D^2$. For
this expansion (more generally for the quantum effective action) to make sense, it is essential
to consider a UV cutoff $\Lambda$ which satisfies the bound

$$ p^2 \Lambda^2 \ll \Lambda_{NC}^4 $$

(1.1)

for any momentum scale $p$ in the background. This condition \[11\] guarantees that the UV/IR
mixing is “mild”, so that the geometrical interpretation in terms of emergent gravity is expected to apply \[11\]. Only in that case there is indeed a meaningful expansion of the fermionic
effective action. In contrast, the induced action appears to be pathological in the limit $\Lambda \rightarrow \infty$.
The physical motivation for such a “low” cutoff comes from supersymmetric matrix models
such as the IKKT model \[6\] i.e. $\mathcal{N} = 4$ SYM, where such a cutoff could be provided by the
SUSY breaking scale. Only such supersymmetric models are expected to be well-behaved upon
quantization.

Using this setup, we compute the heat kernel expansion of the Dirac operator with a generic
4-dimensional NC background, using a perturbative (Duhamel) expansion and a covariant cut-
of. This allows to systematically obtain all terms in the induced action with any given operator
dimension. Using the language of non-commutative gauge theory, we compute in this way the
complete induced action including all terms of operator dimension 6 or less. Gauge invariance
is recovered upon collecting the appropriate contributions. This results in an effective action
for a non-commutative $U(1)$ gauge theory, incorporating UV/IR mixing in a controlled way. In
a second, crucial step, we show that this action can be recast in the form of an effective general-
ized matrix model. The resulting action has a manifest $SO(D)$ symmetry, which is completely
hidden in the gauge theory language. This is a remarkable and non-trivial result, which is very
natural from the geometric point of view of emergent gravity.

It is interesting to compare this with the results of \[12\], where the asymptotic expansion
of a similar operator on the non-commutative torus (note: the “infinite-dimensional” one, not

\[3\]While the quantization of Yang-Mills matrix models has been studied before e.g. in \[1, 6–10\], the results
available so far are not very explicit, and not in the form of generalized matrix models.
the fuzzy torus) was studied. The result was found to depend crucially on number-theoretical properties of the non-commutativity parameter, and for a certain class of $\theta$ an asymptotic expansion of the heat-kernel was found to be quite similar to the commutative case. However in these previous papers [12, 13], the condition (1.1) is not satisfied, so that their results are not related with ours. In contrast, the results of the present paper are robust and independent of any specific properties of $\theta$. They are expected to apply also to the case of NC tori, assuming a similar cutoff.

The geometrical point of view of the matrix model and the $SO(D)$ symmetry turn out to be very powerful and predictive tools for the quantization. Indeed the complete “vacuum energy” term in the effective matrix model can be inferred from a 2-line computation combined with geometrical insights. This completely predicts a series of highly non-trivial loop results in the gauge theory picture, in complete agreement with our computations. The $SO(D)$ symmetry contains both space-time rotations as well as internal rotations, but — most remarkably — mixes the space-time and internal sector. More specifically, it relates the gauge fields with the scalar fields. It is somewhat related to extended supersymmetry, i.e. $\mathcal{N} = 4$ SUSY for $D = 10$, since the internal symmetry $SO(6)$ coincides with the $SO(6)$ R-symmetry of $\mathcal{N} = 4$ SUSY. However, the $SO(D)$ symmetry goes beyond SUSY, and leads to additional restrictions on the effective action. This is manifest in the present computation of the heat kernel associated to the Dirac operator, which amounts to a one-loop computation involving only fermions. It underlines the fact that NC gauge theory is richer than commutative gauge theory. We expect that this symmetry also extends to the non-Abelian $SU(N)$ sector, which is less affected by UV/IR mixing [14] and should reduce to the commutative gauge theory in a suitable limit. It is thus natural to suspect some relation to the recent results on hidden symmetries of SUSY gauge theory [15–18], although the specific relation is not clear at present.

Finally we should point out that although the induced action due to fermions is of course only one piece of the complete effective action, it is closely related to Connes spectral action [19], which is most naturally computed using the heat kernel expansion. Our results thus provide the leading contributions to this spectral action for the matrix model Dirac operator. The missing bosonic integral can be viewed as an integral over the backgrounds, thus providing a measure for the integration over geometries.

This paper is organized as follows. We first recall in Section 2 the basic definition of the Dirac operator and the geometrical interpretation of branes in matrix models. The essential steps and the novel aspects of the heat kernel expansion and the effective action are explained in Section 3. The detailed and lengthy computation of the effective gauge theory action is given in Sections 4 and 5; a reader not interested in the details can jump to the main results which are (5.7), (5.8) and (5.11). In Section 6 this effective gauge theory is rewritten as an effective matrix model, culminating in (7.1) which is the main result of this paper.

## 2 The fermionic action

Our starting point is the matrix model fermion action [4, 6, 20, 21] in Euclidean space

$$S_\Psi = (2\pi)^2 \text{Tr} \Psi^\dagger \slashed{D} \Psi = (2\pi)^2 \text{Tr} \Psi^\dagger \gamma_a [X^a, \Psi],$$

where $X^a, a = 1, 2, \ldots, D$ are Hermitian matrices, and

$$\slashed{D} \Psi := \gamma_a [X^a, \Psi].$$
Latin indices are pulled up/down with the flat embedding metric \( g_{ab} = \delta_{ab} \) of \( \mathbb{R}^D \), and the \( \gamma \)-matrices form the usual Clifford algebra \( [\gamma_a, \gamma_b]_+ = 2\delta_{ab} \). The matrices can be interpreted as operators on a separable Hilbert space \( \mathcal{H} \). This action is invariant under the following symmetries:

\[
\begin{align*}
X^a &\to U^{-1}X^aU, & \Psi &\to U^{-1}\Psi U, & U &\in U(\mathcal{H}), & \text{gauge invariance,} \\
X^a &\to \Lambda(g)^{ab}_b X^b, & \Psi_\alpha &\to \bar{\pi}(g)^{\alpha}_{\beta}\Psi_\beta, & g &\in \widetilde{SO}(D), & \text{rotational symmetry,} \\
X^a &\to X^a + c^a 1, & c^a &\in \mathbb{R}, & & \text{translational symmetry,}
\end{align*}
\]

where the tilde indicates the corresponding spin group. The induced effective action \( \Gamma[X] \) is defined as

\[
e^{-\Gamma[X]} = \int d\Psi d\bar{\Psi} e^{-S_\Psi} = (\text{const.}) \exp \left( \frac{1}{2} \text{Tr} \log (D^2) \right),
\]

\[
D^2 \Psi = \gamma_\alpha \gamma_\beta [X^a, [X^b, \Psi]],
\]

(2.4)

As a background, we will consider matrices \( X^a \) which define 4-dimensional NC spaces (branes) embedded in \( \mathbb{R}^D \). The simplest example is the Groenewold-Moyal quantum plane \( \mathbb{R}^4_\theta \), defined by \( X^a = (\bar{X}^\mu, 0) \) where the \( \bar{X}^\mu \) satisfy

\[
[\bar{X}^\mu, \bar{X}^\nu] = i \bar{\theta}^{\mu\nu} = i \Lambda^{-2}_{\text{NC}} \left( \begin{array}{cccc}
0 & \alpha & 0 & 0 \\
-\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & \pm \alpha^{-1} \\
0 & 0 & \mp \alpha^{-1} & 0
\end{array} \right),
\]

(2.5)

we can assume this standard form of \( \bar{\theta}^{\mu\nu} \) using a \( SO(4) \) transformation if necessary. More generally, we assume that the matrices can be split as

\[
X^a = (X^\mu, \phi^i(X^\mu))
\]

(2.6)

where \( X^\mu, \mu = 1, \ldots, 4 \) are considered as independent quantized coordinate “functions” satisfying generic commutation relations \( [X^\mu, X^\nu] = i \theta^{\mu\nu}(X) \), while the \( \phi^i(X^\mu) \) are “smooth” functions of these coordinates. As explained in \([4]\), there are two different interpretations of such a background. First, the action (2.1) can be viewed as describing fermions propagating on a (generally curved) brane \( \mathcal{M}^4 \subset \mathbb{R}^D \), with effective metric

\[
G^{\mu\nu} = \Lambda^{-2}_{\text{NC}} \theta^{\mu\nu} \delta_{\mu\nu},
\]

(2.7)

in the semi-classical limit where \( X^a \sim x^a \) and \( \theta^{\mu\nu}(X) \sim \theta^{\mu\nu}(x) \). Note that the fixed background metric \( g_{ab} = \delta_{ab} \) of \( \mathbb{R}^D \) defines a scale, and all subsequent quantities are measured with respect to this scale. In that sense, \( X^a \) has dimension length, and \( \theta^{\mu\nu} \) encodes the non-commutativity scale

\[
\Lambda^{-2}_{\text{NC}} = \det \theta^{-1}_{\mu\nu},
\]

(2.8)

which may depend on \( x \); note also \( \det g_{\mu\nu} = \det G^{\mu\nu} \). Second, (2.1) can be viewed as describing fermions on \( \mathbb{R}^4_\theta \) coupled to non-commutative gauge fields and scalars, which arise through (3.7). Hence the flat resp. free case corresponds to the Groenewold-Moyal quantum plane \( \mathbb{R}^4_0 \). To proceed, we need to regularize the divergent functional determinant using a gauge-invariant

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\( ^2 \)Other regularizations might be easier to work with, however we choose a covariant cutoff in order to ensure that both gauge invariance as well as the \( SO(D) \) symmetry are preserved. In general, we will be cavalier about precise mathematical definitions, aiming at physically meaningful and finite results. In particular we will not worry about the “trivial” divergence associated to the infinite volume of \( \mathbb{R}^4 \), which poses no problem in the Duhamel expansion.
cutoff as follows:
\[
\frac{1}{2} \text{Tr}\left( \log D^2 \right) \rightarrow -\frac{1}{2} \text{Tr} \int_0^\infty \frac{d\alpha}{\alpha} e^{-\alpha D^2} e^{-\frac{1}{\alpha L}} =: \Gamma_L[X]. \quad (2.9)
\]

Here \( L \) is a cutoff of dimension ‘length’, which essentially sets a lower limit \( \alpha > \frac{1}{L} \) for the \( \alpha \) integral. Although \( X^a \) and \( D \) have dimension ‘length’ in the present setting, this will amount to a UV cutoff
\[
\Lambda := \Lambda_{NC}^2 L \quad (2.10)
\]
of dimension \((\text{length})^{-1}\) in a NC background. This defines an effective (generalized) matrix model \( \Gamma_L[X] \) in \( X^a \), which depends on the cutoff \( L \) and satisfies the scaling relation
\[
\Gamma_{cL}[cX] = \Gamma_L[X]. \quad (2.11)
\]

It corresponds to the induced action in NC gauge theory, resp. to the induced gravitational action in emergent gravity. Our goal is to compute this \( \Gamma_L[X] \) explicitly, using a systematic approximation.

For completeness, we recall that the fermionic action (2.1) together with the bosonic action
\[
S_{YM} = -(2\pi)^2 \text{Tr} \left( [X^a, X^b] [X_a, X_b] \right) \quad (2.12)
\]
defines the class of Yang-Mills matrix models. In particular, the IKKT or IIB model [6] is obtained for \( D = 10 \) imposing a Majorana-Weyl condition on the fermions, and admits a maximal supersymmetry.

### 3 Strategy of the heat kernel expansion

Before diving into the computations, we first explain the setup and the essential ideas behind the complicated details. We will take advantage of the two complementary points of view of the model, 1) as NC gauge theory and 2) as matrix model (for emergent gravity). Gauge invariance is essential for both points of view. The second makes also the global \( SO(D) \) symmetry manifest, which is hidden in the gauge theory point of view. We will use the gauge theory point of view for the explicit (perturbative) computations, and then recast the result in the matrix model language.

The form (2.9) of the induced action suggests to consider the associated heat kernel expansion
\[
\text{Tr} e^{-\alpha D^2} = \sum_n \frac{\alpha^n}{n!} \Gamma^{(n)}[X]. \quad (3.1)
\]
The \( \Gamma^{(n)}[X] \) are by construction gauge-invariant, and at least formally they are also invariant under \( SO(D) \). In the commutative case, the analog of (3.1) involves a sum over positive \( n \) only, and the Seeley-de Witt coefficients \( \Gamma^{(n)} \) turn out to be integrals of gauge-invariant densities over \( \mathbb{R}^4 \); cf. [22, 23]. This yields an effective action organized as
\[
\Gamma_\Lambda \sim \Lambda^4 \sum_{n \geq 0} \int d^4 x \sqrt{g} O \left( \frac{p_\Lambda}{\Lambda} \right)^n, \quad (3.2)
\]
where \( O\left( \frac{L}{\Lambda}^n \right) \) stands for some Lagrangian density involving \( n \) powers of momentum (due to background curvature, field strength, etc.), starting with the vacuum energy \( \Lambda^4 \int d^4x \sqrt{g} \).

In the NC case, things are much more subtle due to UV/IR mixing. In a perturbative expansion of NC gauge theory, one finds additional terms such as \( e^{-p^2\Lambda_{\text{NC}}^4/\alpha} \) in (3.1) originating from non-planar diagrams. These lead to a sum over arbitrary \( n \in \mathbb{Z} \) in (3.1). It is then not clear in general how to extract a meaningful limit or asymptotic series for \( \Lambda \to \infty \). This is the infamous UV/IR mixing problem (for a review see [24, 25] and references therein), which leads to various strange or pathological phenomena.\(^3\)

In the framework of emergent gravity [4], this pathological UV/IR mixing is turned into a desirable feature, by restricting oneself to well-behaved models with sufficient supersymmetry, such as the \( \mathcal{N} = 4 \) Super-Yang Mills model resp. the IKKT model [6]. That model is expected (and to some extent verified) to be UV finite just like its commutative counterpart [26], and hence free of UV/IR mixing. If the supersymmetry is (spontaneously or softly) broken at some scale \( \Lambda \), then a mild form of UV/IR mixing arises from non-planar diagrams below this scale. This can be understood semi-classically in terms of induced gravity [11], and this is what we want to compute in the present paper. We therefore consider the case of a finite cutoff \( \Lambda \) resp. \( L \), as expected in the full model due to SUSY breaking. We can then compute the induced action \( \Gamma_L[\mathcal{X}] \) (2.9) by studying the heat kernel not in the limit \( \Lambda \to \infty \) but for finite \( \Lambda \), such that the external momenta \( p \) (due to curvature, field strength, etc.) satisfy the condition

\[
\epsilon_L(p) := \frac{p^2\Lambda^2}{\Lambda_{\text{NC}}^4} = \frac{p^2L^2}{\Lambda} \ll 1.
\]

This characterizes the “semi-classical” low-energy regime.\(^4\) It certainly includes the regime of interest for gravity, since both \( \Lambda \) and \( \Lambda_{\text{NC}} \) are assumed to be physical high-energy scales, presumably related to the Planck scale. Under this assumption, we can expand the UV/IR mixing terms such as

\[
e^{-p^2\Lambda_{\text{NC}}^4/\alpha} = \sum_{m \geq 0} \frac{1}{m!} (-p^2\Lambda_{\text{NC}}^4/\alpha)^m \approx \sum_{m \geq 0} a_m \epsilon_L(p)^m
\]

replacing \( \Lambda^2 \geq \alpha^{-1} \) by \( \Lambda^2 \) as justified by the cutoff in (2.9). Thus the heat-kernel expansion becomes an expansion of the form

\[
\Gamma_L \sim \Lambda^4 \sum_{n,l,k \geq 0} \int d^4x O\left( \epsilon_L(p)^n \left( \frac{p^2}{\Lambda_{\text{NC}}^2} \right)^l \left( \frac{p^2}{\Lambda^2} \right)^k \right)
\]

in powers of three small parameters \( \epsilon_L(p) \), \( (p\theta q) \sim \frac{p^2}{\Lambda_{\text{NC}}^2} \) and \( \frac{p^2}{\Lambda^2} \), which makes sense provided (3.3) holds. This condition is very important, and our results no longer make sense in the limit \( \Lambda \to \infty \) for fixed \( \Lambda_{\text{NC}} \) as considered in [12], because then \( \epsilon_L(p) \) diverges. A related observation on the appropriate definition of the heat kernel expansion on fuzzy spaces was made in [27].

A remark on the symmetries is in order. By construction, the expansion of \( \Gamma_L \) preserves gauge invariance at each order in \( L^n \) resp. \( \Lambda^n \) as well as the SO(\( D \)) symmetry, at least formally.\(^5\)

\(^3\)For example, the heat kernel on the non-commutative torus was found to depend on number-theoretical properties of \( \theta \) [12].

\(^4\)Meaning that the phases in the loop integrals are less than 1, i.e. UV/IR mixing is weak and within the semi-classical regime [11].
Although the background $R^4_θ$ of course breaks both symmetries, these symmetries are still realized in a non-linear way on the fluctuations, and should therefore be respected in the quantum effective action. While this is quite evident for the gauge symmetry, the argument is not strictly valid for $SO(D)$, since rotating the $X^μ$ into the scalar fields leads to unbounded $δφ^i$ which might not be admissible. Nevertheless, it turns out that the effective actions obtained in this way do indeed respect this $SO(D)$ symmetry, as one would hope.

Finally, one may wonder about the relation with the commutative case. The commutative limit should be approximated by imposing $Λ ≪ ΛNC$, (hence $ΛNC L ≪ 1$) in addition to (3.3); in particular, we cannot send $ΛNC → ∞$ for fixed $L$. Finally, non-Abelian gauge fields and scalar fields are naturally obtained within the same model by replacing the “single-brane” background by $n$ coinciding branes. However, then the following analysis needs to be refined, which should be worked out elsewhere.

3.1 Perturbative expansion in NC gauge theory.

Our aim is to compute the leading (divergent) terms in the momentum expansion of $Γ_L[X]$ (3.5). For this purpose, we take the point of view of NC gauge theory on $R^4_θ$, and consider small fluctuations around $R^4_θ$. We accordingly split the matrices into background plus fluctuations,

$$X^a = (\bar{X}^μ) + (-\bar{θ}^μ A^μϕ^i),$$

and treat $A^μ = A^μ(\bar{X})$ and $ϕ^i = ϕ^i(\bar{X})$ as gauge fields resp. scalar fields on $R^4_θ$. In order to recover the standard dimensional assignment of quantum field theory, we define

$$ϕ^i := Λ^2 NC φ^i$$

which has dimension $\text{dim} ϕ = L^{-1}$. The corresponding field-theoretic normalization of the Dirac operator is given by $Λ^2 NC \bar{D}$. We can now perform a perturbative expansion of

$$\frac{1}{2} \text{Tr} \left( \log B^2 - \log B^2_0 \right) \to -\frac{1}{2} \text{Tr} \left( \int_0^∞ \frac{dα}{α} \left( e^{-αB^2} - e^{-αB^2_0} \right) e^{-\frac{1}{αL^2}} \right) \sum_{k>0} O(V^k)$$

$$= Λ^4 \sum_{n≥0} \int d^4x O \left( \frac{(p, A, ϕ)^n}{(Λ, ΛNC)^n} \right),$$

where

$$\bar{D}_0 \Psi := γ_a [\bar{X}^a, Ψ], \quad B^2 ψ = \bar{D}^2_0 ψ + V.$$ (3.10)

Each term $O(V^k)$ contributes one or two fields, given by the Duhamel expansion as explained in Appendix[A]. However, each of these terms contains arbitrarily high powers of momenta due to UV/IR mixing, and their appropriate organization is not obvious a priori. Moreover, gauge invariance is not respected by this expansion. The most physical organization is according to their engineering dimension i.e. according to powers of fields and momenta, as indicated in

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5Here and in most of the following, the scales $ΛNC$ and $Λ$ will be defined by the Moyal-Weyl background. Otherwise we will write $Λ(x)$ etc.
This makes sense due to the IR condition \((3.3)\), involving three small parameters \(\epsilon_L(p)\), \((p\theta q) \sim \frac{p^2}{\Lambda_{NC}^2}\). It leads to an effective action of the type

\[
\Gamma_L = \sum_{n,k,l \geq 0} \Lambda^{2n-2k} \Lambda_{NC}^{-4n-2l} \int d^4 x \, \Gamma^{(n,k,l)}[A_\mu, \varphi^i] \tag{3.11}
\]
on \(\mathbb{R}^4\), where \(\Gamma^{(n,k,l)}[A_\mu, \varphi^i]\) has engineering dimension \(r = 4 + 2k + 2l + 2n\). Only finitely many terms in the Duhamel expansion contribute to each given \(r\). However, recovering gauge invariance requires a non-trivial re-organization of this expansion, since e.g. a commutator of fields such as \([A_\mu, A_\nu]\) can be rewritten in momentum space as an expansion in powers of \((p\theta q) \sim \frac{p^2}{\Lambda_{NC}^2}\). This can be done iteratively, and one obtains

\[
\Gamma_L = \sum_{m>0;n} \Lambda^n \Lambda_{NC}^{-m} \int d^4 x \, \Gamma^{(n,m)}[A_\mu, \varphi^i], \tag{3.12}
\]

where each \(\Gamma^{(n,m)}[A_\mu, \varphi^i]\) is manifestly gauge-invariant, interpreted as an effective higher-order NC gauge theory. Note that the \(\mathcal{O}(V^k)\) contribution in the perturbative expansion \((3.9)\) contains at least \(k\) powers of fields \(A\) resp. \(\varphi\), so that any given term in \((3.12)\) is completely determined at some finite order of \(k\). These steps are carried out in Section 5 where the first terms in this expansion are found to be \((5.1)\), \((5.8)\), and \((5.11)\).

This result would be perfectly satisfactory from the point of view of NC gauge theory, since each of the \(\Gamma^{(n,m)}\) respects gauge invariance as well as the global \(SO(D-4)\) symmetry. However, the effective action appears to violate \(SO(4)\) invariance due to terms such as \(\theta^{\mu\nu} F_{\mu\nu}\), not to mention the original \(SO(D)\) symmetry of the matrix model. These will be recovered in the next, non-trivial step, suggested by the gravity point of view.

### 3.2 Re-assembling the effective matrix model.

In the last step, we collect the gauge-invariant actions \(\Gamma^{(n,m)}[A_\mu, \varphi^i]\) on \(\mathbb{R}^4\) and translate the result into the form of a generalized matrix model,

\[
\Gamma_L = \sum_{n,m} \Lambda^n \Lambda_{NC}^{-m} \int d^4 x \, \Gamma^{(n,m)}[A_\mu, \varphi^i] = \text{Tr} \mathcal{L}(X^a), \tag{3.13}
\]

which manifestly respects gauge invariance as well as the \(SO(D)\) symmetry. This is a highly non-trivial (and at this point non-systematic) step, which will be carried out explicitly in Section 6.

We obtain the following effective matrix model action \((7.1)\)

\[
\Gamma_L = -\frac{1}{4} \text{Tr} \frac{L^4}{\sqrt{\frac{1}{2} H^2 - H^{ab} H_{ab} + \frac{c_1}{L^2} [X^c, H^{ab}][X_c, H_{ab}] + \frac{c_2}{L^4} H^{cd} [X^c, \Theta^{ab}][X_d, \Theta_{ab}] + \ldots}} \tag{3.14}
\]

which constitutes the main result of this paper. This action has a manifest global \(SO(D)\) symmetry, which is hidden in the gauge theory point of view. The dependence on the specific background \(R^4\) has disappeared (except for its 4-dimensional nature), which allows to translate the result into the geometric language of emergent gravity where the \(X^a\) are interpreted as embedding of \(M^4 \subset \mathbb{R}^D\). The higher-order terms then correspond e.g. to curvature terms as
shown in [28, 29]. The ellipses in Eqn. (3.14) stand for terms which contain more derivatives resp. commutators, corresponding to higher-order curvature terms, etc.

The same type of effective action should be expected from the bosonic part of the matrix model, integrating out the $X^a$ at one loop or beyond. We plan to report on this elsewhere. In this way, the quantization of NC gauge theory allows to obtain the quantization of (emergent) gravity.

In general this resulting expansion depends on the background, in particular it is different e.g. for the non-Abelian modes. Thus it is not “the complete” effective matrix model, and there should be some universal form which applies to all backgrounds. Note also that the dependence on $\Lambda$ characterizes some kind of “matrix” renormalization group, which should be studied in detail.

4 Details of the heat kernel expansion

4.1 Square of the Dirac operator

We choose coordinates such that

$$g_{\mu\nu} = \partial_\mu x^a \partial_\nu x^b g_{ab} = \text{diag}(1,1,1,1),$$

and using a suitable $SO(4)$ rotation we can assume that $\bar{\theta}^{\mu\nu}$ has the standard form (2.5). According to (2.7), the effective metric on $\mathbb{R}^4$ is

$$\bar{G}^{\mu\nu} = \Lambda_{NC}^4 \bar{\theta}^{\mu\mu} \bar{\theta}^{\nu\nu} g_{\mu\nu}.$$ (4.1)

We consider the square of the Dirac operator on $\mathbb{R}^4$, and treat $A_\mu$ as well as $\phi^i$ as perturbations. This gives

$$\bar{D}^2 \Psi = \gamma_\alpha \gamma_\beta [X^a, [X^b, \Psi]] = (\delta_{ab} - 2i\Sigma_{ab})[X^a, [X^b, \Psi]]$$

where we define

$$\Sigma_{ab} = \frac{i}{4} [\gamma_\alpha, \gamma_\beta], \quad \Theta^{ab} = -i[X^a, X^b].$$ (4.2)

Furthermore, we split the square of the Dirac operator according to

$$\bar{D}^2 \Psi = (H_0 + V) \Psi,$$

$$H_0 \Psi := \delta_{\mu\nu}[\bar{X}^\mu, [\bar{X}^\nu, \Psi]] = -\Lambda_{NC}^{-4} G^{\mu\nu} \partial_\mu \partial_\nu \Psi$$

Using

$$X^\mu = \bar{X}^\mu + A_\mu = \bar{X}^\mu - \bar{\theta}^{\mu\nu} A_\nu,$$ (4.5)

one has

$$[X^a, [X_a, \Psi]] = \bar{\square} \Psi + \delta_{\mu\nu}([\bar{X}^\mu, [A_\nu, \Psi]] + [A_\mu, [\bar{X}^\nu, \Psi]] + [A^\mu, [A^\nu, \Psi]]) + \delta_{ij}([\phi^i, [\phi^j, \Psi]])$$

$$= \bar{\square} \Psi - \frac{i\bar{G}^{\mu\nu}}{\Lambda_{NC}^4} (2[A_\mu, \partial_\nu \Psi] + [\partial_\mu A_\nu, \Psi] + i[A_\mu, [A_\nu, \Psi]] + \Lambda_{NC}^{-4} [\phi^i, [\phi^j, \Psi]])$$

$$2\Sigma_{ab}[X^a, [X^b, \Psi]] = \Sigma_{ab} \left( [X^a, [X^b, \Psi]] - [X^b, [X^a, \Psi]] \right)$$

$$= i\Sigma_{ab}[\Theta^{ab}, \Psi].$$ (4.6)
Note that the scalar fields have been rescaled above according to \( \varphi_i = \Lambda_{NC}^2 \phi_i \). The components of \( \Theta^{ab} \) are given by

\[
[X^\mu, X^\nu] = i \bar{\Theta}^{\mu\nu} + i \bar{\Theta}^{\mu\rho} \partial_\rho A^\nu - i \bar{\Theta}^{\nu\rho} \partial_\rho A^\mu + [A^\mu, A^\nu]
\]

\[
= i \bar{\Theta}^{\mu\nu} + i \mathcal{F}^{\mu\nu},
\]

\[
[X^\mu, \phi^i] = i \bar{\Theta}^{\mu\nu} D_\nu \phi^i,
\]

where

\[
\mathcal{F}^{\mu\nu} = -\bar{\Theta}^{\mu\rho} \bar{\Theta}^{\nu\sigma} F_{\rho\sigma}
\]

\[
= -\bar{\Theta}^{\mu\rho} \bar{\Theta}^{\nu\sigma} (\partial_\rho A_\sigma - \partial_\sigma A_\rho + i[A_\rho, A_\sigma]),
\]

\[
D_\alpha \phi = \partial_\alpha \phi + i[A_\alpha, \phi].
\]

Then the above gives

\[
\Lambda_{NC}^4 V \Psi = -i \bar{G}^{\mu\nu} \left( 2[A_\mu, \partial_\nu \Psi] + [\partial_\mu A_\nu, \Psi] + i[A_\mu, [A_\nu, \Psi]] \right) + \delta_{ij} \{ \varphi^i, [\varphi^j, \Psi] \}
\]

\[
+ \Lambda_{NC}^4 \Sigma_{ab} \{ \Theta^{ab}, \Psi \}
\]

\[
= -i \bar{G}^{\mu\nu} \left( 2[A_\mu, \partial_\nu \Psi] + [\partial_\mu A_\nu, \Psi] + i[A_\mu, [A_\nu, \Psi]] \right) + \delta_{ij} \{ \varphi^i, [\varphi^j, \Psi] \}
\]

\[
+ \Lambda_{NC}^4 \left( \Sigma_{\mu\nu} \mathcal{F}^{\mu\nu}, \Psi \right) + 2 \Sigma_{\mu\nu} \bar{\Theta}^{\mu\rho}[\partial_\nu \phi^\rho + i[A_\nu, \phi^\rho], \Psi] - i \Sigma_{ij} \{ [\phi^i, \phi^j], \Psi \}. \quad (4.9)
\]

Note that \( V \) contains both linear and quadratic terms in the fluctuations \( \varphi^i \) resp. \( A^\mu \). All these formulas are exact on \( \mathbb{R}^4_\theta \).

### 4.2 Setup for the trace computations

First, we must specify the Hilbert space under consideration. The algebra \( \mathcal{A} \) of (non-commutative) functions on \( \mathbb{R}^4_\theta \) can be identified with the Heisenberg algebra, and therefore is represented as (infinite-dimensional) matrix algebra \( \mathcal{A} \subset \text{End}(\mathcal{H}) \) acting on a separable Hilbert space \( \mathcal{H} \). We suppress the spinor indices for simplicity in this discussion. As usual \( \mathcal{A} \) (resp. a suitable subspace of \( \mathcal{A} \)) can be equipped with an inner product structure

\[
\langle \Psi_1, \Psi_2 \rangle = \text{Tr}_\mathcal{H} \Psi_1^\dagger \Psi_2 = \Lambda_{NC}^4 \int \frac{d^4 x}{(2\pi)^2} \sqrt{g} \Psi_1^\dagger \Psi_2
\]

(4.10)

for \( \Psi_i \in \mathcal{A} \cong \text{End}(\mathcal{H}) \), considered as a (pre-)Hilbert space of wave-functions on \( \mathbb{R}^4_\theta \). Here the integral over \( x \) is understood as integral over \( \mathbb{R}^4_\theta \). We will always assume coordinates \( x^\mu \) with \( g_{\mu\nu} = \delta_{\mu\nu} \) and often drop the \( \sqrt{g} \). Then \( \partial_i \in \text{End}(\mathcal{A}) \) resp. \( \partial_i^2 \) are Hermitian operators on \( \mathcal{A} \),

\[
\text{Tr}_\mathcal{H} \Psi_1^\dagger \partial_i^2 \Psi_2 = \Lambda_{NC}^4 \int \frac{d^4 x}{(2\pi)^2} \sqrt{g} \Psi_1^\dagger \partial_i^2 \Psi_2 = \Lambda_{NC}^4 \int \frac{d^4 x}{(2\pi)^2} \sqrt{g} (\partial_i^2 \Psi_1)^\dagger \Psi_2.
\]

(4.11)

\( \mathcal{A} \) can be identified via Fourier transform with square-integrable functions on \( \mathbb{R}^4 \), so that \( \mathcal{A} \cong L^2(\mathbb{R}^4_\theta) \cong L^2(\mathbb{R}^4) \). This identification (the Weyl quantization map) is defined by mapping plane waves \( e^{ipx^\mu} \) to the “generalized eigenfunctions”

\[
|p \rangle = e^{ipx^\mu} \in \tilde{\mathcal{A}} \supset \mathcal{A}
\]

(4.12)

\*We do not consider functional-analytic details here. One way to make this more rigorous would be to use the fuzzy torus, which is compact while having a very similar non-commutative structure.
of NC plane waves, which satisfy \( \hat{P}_\mu(p) = ip_\mu(p) \) for \( \hat{P}_\mu = -i\partial^{-1}_\mu[X^\nu,.] \); note that \([\hat{P}_\mu, \hat{P}_\nu] = 0\). We can compute their inner product formally

\[
\langle q|p \rangle = \text{Tr}(|p\rangle\langle q|) = \text{Tr}_\mathcal{H}(e^{-ip_\mu X^\mu} e^{ip_\nu X^\nu}) = (2\pi \Lambda_{NC}^2)^2 \delta^4(p - q). \tag{4.13}
\]

(Note that \( \text{dim} \mathcal{H} = \text{Tr}_\mathcal{H}1 = \text{Tr}(|0\rangle\langle 0|) = (2\pi \Lambda_{NC}^2)^2 \delta^4(0) \) so that \( \delta^4(0) \sim \text{Vol} \) corresponds to a divergent volume factor). Hence the trace \( \text{Tr} \) over operators on \( \mathcal{A} \) (not to be confused with \( \text{Tr}_\mathcal{H} \)) can be computed using the following relations:

\[
\text{Tr} \mathcal{O} = \int \frac{d^4p}{(2\pi \Lambda_{NC}^2)^2} \langle p|\mathcal{O}|p \rangle, \tag{4.14a}
\]

\[
1 = \int \frac{d^4p}{(2\pi \Lambda_{NC}^2)^2} |p\rangle\langle p|. \tag{4.14b}
\]

Notice the presence of the NC scale. Analogous formulas can be justified rigorously on (compact) fuzzy spaces such as \( S_N^2 \) or \( T^4_\theta \). In particular, a field \( \Psi \in \mathcal{A} \) is conveniently written in momentum basis as

\[
|\Psi\rangle = \int \frac{d^4p}{(2\pi \Lambda_{NC}^2)^2} |p\rangle \langle p|\Psi\rangle = \int \frac{d^4p}{(2\pi \Lambda_{NC}^2)^2} \psi(p) e^{ip_\mu X^\mu},
\]

\[
\psi(p) = \langle p|\Psi\rangle = \text{Tr}_\mathcal{H}(e^{-ip_\mu X^\mu}\Psi) = \Lambda_{NC}^4 \int \frac{d^4x}{(2\pi)^2} \psi(x) e^{-ip_\mu x^\mu},
\]

\[
\psi(x) = \langle x|\Psi\rangle = \int \frac{d^4p}{(2\pi \Lambda_{NC}^2)^2} e^{ip_\mu x^\mu} \psi(p) = \int \frac{d^4p}{(2\pi \Lambda_{NC}^2)^2} \text{Tr}_\mathcal{H}(e^{-ip_\mu (x^\mu - \bar{X}^\mu)}\Psi). \tag{4.15}
\]

For example, \( |p\rangle \) corresponds to \( \psi(p') = (2\pi \Lambda_{NC}^2)^2 \delta^{(4)}(p' - p) \). Similarly, we consider the Fourier representation of the external fields

\[
\phi^i = \int \frac{d^4p}{(2\pi \Lambda_{NC}^2)^2} \phi^i(p) e^{ip_\mu X^\mu},
\]

\[
A_\mu = \int \frac{d^4p}{(2\pi \Lambda_{NC}^2)^2} A_\mu(p) e^{ip_\mu X^\mu}. \tag{4.16}
\]

We can now start with the formula \( (A.4) \) of Appendix \( A \) which expresses the effective action as a trace of certain operators acting on the spinor field on \( \mathbb{R}^4_\theta \). In subsequent computations we will need:

\[
H_0|p\rangle = \Lambda_{NC}^{-4} p \cdot p |p\rangle, \tag{4.17}
\]

where

\[
p \cdot p := \bar{\mathcal{G}}^{\mu\nu} p_\mu p_\nu, \tag{4.18}
\]

and

\[
e^{ik \bar{X}}_e e^{it \bar{X}} = e^{-\frac{i}{2} k \partial_t} e^{i(k + l) \bar{X}}, \tag{4.19}
\]

on \( \mathbb{R}^4_\theta \). It therefore follows that

\[
[e^{ik \bar{X}} e^{it \bar{X}}, e^{ik' \bar{X}} e^{it' \bar{X}}] = -2i \sin \left( \frac{k \partial t}{2} \right) e^{i(k + l) \bar{X}},
\]

\[
e^{ip \bar{X}} e^{ik \bar{X}} [e^{ik' \bar{X}} e^{it \bar{X}}, e^{ik' \bar{X}} e^{it' \bar{X}}] = -2i e^{\frac{i}{2} (k + l) \partial_p} \sin \left( \frac{k \partial t}{2} \right) e^{i(k + l + p) \bar{X}},
\]

\[
[[e^{ik \bar{X}} e^{it \bar{X}}, e^{ip \bar{X}} e^{ik' \bar{X}} e^{it' \bar{X}}] = (-2i)^2 \sin \left( \frac{k \partial t}{2} \right) \sin \left( \frac{(k + l) \partial p}{2} \right) e^{i(k + l + p) \bar{X}}. \tag{4.20}
\]
and similarly, for anti-commutators \(-2i\sin()\) is replaced by \(2\cos()\) in the above formulas. Furthermore, we have the following spinorial trace

\[
\text{tr}(\Sigma_{ab} \Sigma_{cd}) = \frac{\text{tr} \mathbb{1}}{4} (g_{ac}g_{bd} - g_{ad}g_{bc}) .
\]  

(4.21)

Having collected all basic ingredients, we may proceed with the explicit computations presented in the next section. The reader not interested in the details may jump to section 5 where the effective gauge theory action is presented.

### 4.3 Order by order computations

To start the computation, consider first the general matrix element

\[
\langle \Psi'_\beta | V | \Psi_\alpha \rangle = \int \frac{d^4x}{(2\pi)^2} \Psi'_\beta \left( [\varphi^i, [\varphi_i, \Psi_\alpha]] + \Lambda^4_{\text{NC}} \Sigma_{ab} [\Theta^{ab}, \Psi_\alpha] \right.
\]

\[
- iG^{\mu\nu} (2[A_\mu, \partial_\nu \Psi_\alpha] + [\partial_\mu A_\nu, \Psi_\alpha] + i[A_\mu, [A_\nu, \Psi_\alpha]]) \bigg)
\]

\[
= \Lambda^4_{\text{NC}} \int \frac{d^4p}{(2\pi \Lambda^2_{\text{NC}})^2} \int \frac{d^4q}{(2\pi \Lambda^2_{\text{NC}})^2} \psi_\beta^\dagger(p) \left( 2iG^{\mu\nu} (p + q) A_\mu(p - q) \sin \left( \frac{q_\mu q_\nu}{2} \right) \right.
\]

\[
- 4G^{\mu\nu} \int \frac{d^4l}{(2\pi \Lambda^2_{\text{NC}})^2} A_\mu(p - q - l) A_\nu(l) \sin \left( \frac{q_\mu l_\nu}{2} \right) \sin \left( \frac{l_\mu q_\nu}{2} \right) \left( \frac{l_\mu q_\nu}{2} \right)
\]

\[
- 4\delta_{ij} \int \frac{d^4l}{(2\pi \Lambda^2_{\text{NC}})^2} \varphi^i(p - q - l) \varphi^j(l) \sin \left( \frac{q_\mu l_\nu}{2} \right) \sin \left( \frac{l_\mu q_\nu}{2} \right) \left( \frac{l_\mu q_\nu}{2} \right)
\]

\[
+ 2i\Lambda^4_{\text{NC}} \Sigma_{ab} \Theta^{ab}(p - q) \sin \left( \frac{q_\mu q_\nu}{2} \right) \psi_\alpha(q) ,
\]  

(4.22)

where

\[
\Theta^{ab}(x) = \int \frac{d^4k}{(2\pi \Lambda^2_{\text{NC}})^2} \Theta^{ab}(k) e^{ik_\mu x^\mu} .
\]  

(4.23)

We note that this interaction \(V\) vanishes when any of the external fields \(A\) or \(\varphi\) has zero momentum. This is clear because they arise only from commutators, which vanish in the commutative case. Hence all the non-trivial results are due to non-commutative effects resp. UV/IR mixing.

**First order.** From (4.22) it follows that

\[
\text{Tr} \left( V e^{-\alpha H_0} \right) = \int \frac{d^4p}{(2\pi \Lambda^2_{\text{NC}})^2} \text{tr} V_{\mu\nu} e^{-\tilde{\alpha} p_\mu p_\nu}
\]

\[
= \frac{4\text{tr} \mathbb{1}}{\Lambda^4_{\text{NC}}} \int \frac{d^4l}{(2\pi \Lambda^2_{\text{NC}})^2} \left( \tilde{G}^{\mu\nu} A_\mu(-l) A_\nu(l) + \varphi^i(-l) \varphi_i(l) \right) \int \frac{d^4p}{(2\pi \Lambda^2_{\text{NC}})^2} \sin \left( \frac{l_\mu q_\nu}{2} \right) \left( \frac{l_\mu q_\nu}{2} \right) e^{-\tilde{\alpha} p_\mu p_\nu}
\]

\[
= \frac{\text{tr} \mathbb{1}}{2} \frac{1}{\Lambda^8_{\text{NC}}} \int \frac{d^4l}{(2\pi \Lambda^2_{\text{NC}})^2} \sqrt{7} \left( \tilde{G}^{\mu\nu} A_\mu(-l) A_\nu(l) + \varphi^i(-l) \varphi_i(l) \right) \frac{1}{\tilde{\alpha}^2} \left( 1 - e^{-\frac{l_\mu l_\nu}{4\tilde{\alpha}}} \right) ,
\]  

(4.24)

where we define

\[
\tilde{\alpha} = \Lambda^4_{\text{NC}} \alpha ,
\]

\[
\tilde{p}_\mu := \tilde{G}_{\mu\nu} p^\nu := \tilde{G}_{\mu\nu} \theta^{\mu\nu} p_\rho ,
\]

\[
\Lambda^2 = \Lambda^4_{\text{NC}} L^2 ,
\]

(4.25)
Keeping only the leading non-trivial term $\Lambda_2^2$ given some finite cutoff $\Lambda$. Hence we can expand which involves the famous “effective cutoff” the embedding metric $g$

Notice that this is already a “non-planar” (UV/IR mixing) contribution, as it involves

where

The point is now that gravity is an infrared phenomenon, so that we are interested in the regime

This gives the following contribution to the effective action

This is the essential difference to the previous work \cite{12, 13}. This is consistent with the semi-classical result in \cite{21}, but the present derivation is exact to all orders in $\theta$. Higher-order terms in the expansion \cite{4, 30} will contribute in particular to the curvature action.

Second order. The second order in the heat kernel expansion yields

\begin{align}
\text{Tr} \left( V e^{-\tilde{t}H_0} V e^{-(\alpha-l)H_0} \right) &= \int \frac{d^4p}{(2\pi \Lambda_{NC}^2)^2} \int \frac{d^4q}{(2\pi \Lambda_{NC}^2)^2} \text{tr} \left( V_{p,q} e^{-i\tilde{t}q} V_{q,p} e^{-(\alpha-l)p} \right)
\end{align}

\footnote{This is the essential difference to the previous work \cite{12, 13}.}
\[
\begin{align*}
\frac{16\alpha_f}{\Lambda_{\text{NC}}^8} & \int \frac{d^4p\,d^4q}{(2\pi\Lambda_{\text{NC}}^2)^4} e^{-\nu q \cdot (\alpha - \bar{\nu})p} \left( \bar{G}^{\mu\nu} G_{\rho\sigma} \frac{(p+q)^\nu(p+q)^\sigma}{4} A_\mu(p-q) A_\rho(q-p) \sin^2\left(\frac{q\bar{\theta}p}{2}\right) \right) \\
+ \frac{1}{4} \Lambda_{\text{NC}}^8 (g_{ac}g_{bd} - g_{ad}g_{bc}) \Theta^{ab}(p-q) \Theta^{cd}(q-p) \sin^2\left(\frac{q\bar{\theta}p}{2}\right) \\
+ \frac{i}{2} \int \frac{d^4l}{(2\pi\Lambda_{\text{NC}}^2)^2} \left[ iG^{\mu\nu} G_{\rho\sigma} \frac{(p+q)^\nu}{2} A_\mu(p-q) A_\rho(q-p-l) A_\eta(l) \sin\left(\frac{q\bar{\theta}l}{2}\right) \sin\left(\frac{q\bar{\theta}p}{2}\right) \sin\left(\frac{(l+q)\bar{\theta}p}{2}\right) \right] \\
+ \frac{i}{2} \int \frac{d^4k}{(2\pi\Lambda_{\text{NC}}^2)^2} \left[ \sin\left(\frac{q\bar{\theta}l}{2}\right) \sin\left(\frac{q\bar{\theta}k}{2}\right) \sin\left(\frac{(l+q)\bar{\theta}p}{2}\right) \sin\left(\frac{(k+q)\bar{\theta}q}{2}\right) \right] \\
\times \left( G^{\mu\nu} G_{\rho\sigma} A_\mu(p-q-l) A_\rho(q-p-k) A_\eta(l) A_\sigma(k) + \phi^i(p-q-l) \phi^j(l) \phi^j(q-p-k) \phi^j(k) \right) \\
+ \delta_{ij} G^{\mu\nu} A_\mu(p-q-l) A_\nu(l) \phi^i(q-p-k) \phi^j(k) \sin\left(\frac{q\bar{\theta}l}{2}\right) \sin\left(\frac{q\bar{\theta}k}{2}\right) \\
\times \sin\left(\frac{(l+q)\bar{\theta}p}{2}\right) \sin\left(\frac{(p+k)\bar{\theta}q}{2}\right) + \left\{ q \leftrightarrow p \quad l \leftrightarrow k \right\} \right) \right), \quad (4.32)
\end{align*}
\]

involving two, three and four field contributions. We will ultimately do the computations of this section up to third order, and hence neglect the four field contributions — a sample expression is nonetheless given in Appendix [13]. In order to compute at least one of the momentum integrals, we need to make clever variable substitutions where the fields are independent of one of the new integration variables. For example, for the terms which only involve integrals over \(p\) and \(q\), the substitution \(P = p + q\) and \(Q = p - q\) is favourable and leads to integrals of the type

\[
\int \frac{d^4P}{(2\pi\Lambda_{\text{NC}}^2)^2} \sin^2\left(\frac{Q\bar{\theta}P}{4}\right) e^{-\frac{1}{4}(Q-P)\cdot(Q-P) - \frac{1}{4}(\bar{Q} - \bar{\nu})(P+Q)\cdot(P+Q)}
\]

\[
= \frac{2\sqrt{G}}{\Lambda_{\text{NC}}^4 \alpha^2} \left( e^{\frac{Q\bar{\phi}}{4\alpha}} - 1 \right) e^{-\frac{4\nu\Phi Q}{4\alpha}}, \quad (4.33a)
\]

\[
\int \frac{d^4P}{(2\pi\Lambda_{\text{NC}}^2)^2} P_\nu P_\sigma \sin^2\left(\frac{Q\bar{\theta}P}{4}\right) e^{-\frac{1}{4}(Q-P)\cdot(Q-P) - \frac{1}{4}(\bar{Q} - \bar{\nu})(P+Q)\cdot(P+Q)}
\]

\[
= \frac{2\sqrt{G}}{\Lambda_{\text{NC}}^4 \alpha^2} e^{-\frac{4\nu\Phi Q}{4\alpha}} \left( Q_\nu \bar{Q}_\sigma + (Q_\nu Q_\sigma (\bar{\nu} - 2\nu)^2 + 2\alpha G_{\nu\sigma}) \left( e^{\frac{Q\bar{\phi}}{4\alpha}} - 1 \right) \right), \quad (4.33b)
\]

15
\[
\int \frac{d^4 P}{(2\pi \Lambda_{NC}^2)^2} P_\nu \sin \left( \frac{Q \theta P}{4} \right) \sin \left( \frac{\theta(Q + P)}{4} \right) e^{-\frac{i}{4}(Q - P)\cdot(Q - P) - \frac{i}{4}(\alpha - i)(P + Q)\cdot(P + Q)}
= \frac{\sqrt{G}}{2\Lambda_{NC}^4 \alpha^3} \exp \left( -\frac{2 \tilde{t} \cdot \tilde{Q} + \tilde{t}^2 + \tilde{Q} \cdot \tilde{Q} + 2i(2\tilde{t} + \tilde{\alpha})(l \theta Q) + 4Q \cdot Q\tilde{t}(\tilde{\alpha} - \tilde{t})}{4\tilde{\alpha}} \right) \times \\
\times \left( \tilde{I}_\nu \left( e^{\frac{2i \tilde{Q} \cdot \tilde{\alpha} \cdot \tilde{Q}}{4\tilde{\alpha}}} - 1 \right) \left( e^{\frac{2 \tilde{t}(l \theta Q)}{\alpha}} + e^{i(l \theta Q)} \right) \right. \\
+ \left. e^{\frac{2i \tilde{Q} \cdot \tilde{t} + 4\tilde{t}(l \theta Q)}{4\tilde{\alpha}}} \left( \tilde{Q}_\nu \left( 1 + e^{i(l \theta Q)} \right) - iQ_\nu(\tilde{\alpha} - 2\tilde{t}) \left( e^{\frac{i(t \theta Q)}{4\tilde{\alpha}}} - 1 \right) \left(-1 + e^{i(l \theta Q)} \right) \right) \\
+ i \left( Q_\nu(\tilde{\alpha} - 2\tilde{t}) \left( e^{\frac{2i \tilde{Q} \cdot \tilde{\alpha} \cdot \tilde{Q}}{4\tilde{\alpha}}} - 1 \right) \left( e^{i(l \theta Q)} - e^{\frac{2 \tilde{t}(l \theta Q)}{\alpha}} \right) + i\tilde{Q}_\nu \left( e^{\frac{2 \tilde{t}(l \theta Q)}{\alpha}} + e^{i(l \theta Q)} \right) \right) \right),
\]

for the two and three field contributions. In order to make sense of the UV/IR mixing terms, we consider \( \Lambda \) as a finite cutoff, and assume that \( \epsilon_L(p) = p^2 \Lambda^2 / \Lambda_{NC}^4 \ll 1 \). We can then expand the “UV/IR mixing terms” e.g. as \( e^{\tilde{p}^\nu / \alpha} = 1 + \sum_{\alpha = \Lambda_{NC}}^{2n} \), which amounts to an expansion in the NC parameter \( \theta \) after performing the loop integral.

Adopting the expansion in \( \epsilon_L(p) \) as justified above, we only need

\[
\int_0^\infty \! d\tilde{t} \! d\tilde{Q} \frac{1}{\Lambda_{NC}^4} \left( \Lambda^2 \tilde{Q} \cdot \tilde{Q} + \frac{Q \cdot Q\tilde{Q} \cdot \tilde{Q}}{6} \left( \ln \left( \frac{Q \cdot Q}{\Lambda^2} \right) + 2\gamma_E - \frac{8}{3} \right) \right) + \mathcal{O}(Q^6 / \Lambda^6), \quad (4.34a)
\]

\[
\int_0^\infty \! d\tilde{t} \! d\tilde{Q} \frac{1}{\Lambda_{NC}^4} \left( \frac{Q \cdot Q}{8\tilde{Q} \cdot \tilde{Q}} \left( 2\tilde{t} \cdot \tilde{Q} - \tilde{Q} \cdot \tilde{Q} \right) \left( 2\tilde{t} \cdot (\tilde{t} + \tilde{Q}) + \tilde{Q} \cdot \tilde{Q} \right) \sin \left( \frac{l \theta Q}{2} \right) \right) \right. \\
\left. + \frac{\Lambda^4}{4Q \cdot Q} \left( Q \cdot Q \left( \tilde{Q} \cdot \tilde{Q} \left( 2\tilde{Q}_\nu \sin^2 \left( \frac{l \theta Q}{4} \right) + \tilde{I}_\nu + \tilde{I} \tilde{Q}_\nu \right) \right) + \tilde{Q}_\nu(2\tilde{t} + \tilde{Q}) \tilde{Q} \left( l \theta Q - 2 \sin \left( \frac{l \theta Q}{2} \right) \right) + 2\tilde{t} \cdot \tilde{Q}_\nu \cdot \tilde{Q} \left( \tilde{I}_\nu + \tilde{Q}_\nu \right) \right) \right. \\
\left. + \Lambda^2 \tilde{Q}_\nu \left( \cos \left( \frac{l \theta Q}{2} \right) - 1 \right) \right. \\
\left. + \frac{1}{9} Q \cdot Q\tilde{Q}_\nu \sin^2 \left( \frac{l \theta Q}{4} \right) \left( 3\ln \left( \frac{\Lambda^2}{Q \cdot Q} \right) - 6\gamma_E + 8 \right) \right) + \mathcal{O}(Q^6 / \Lambda^6), \quad (4.34c)
\]

Note that every power of \( Q \) is suppressed by either \( \frac{1}{\Lambda} \) or \( \frac{1}{\Lambda_{NC}} \), along possibly with factors \( \frac{1}{\Lambda_{NC}} \) which we assume to be finite.
Collecting all 2-field contributions in (4.32) we hence find with (4.33a), (4.33b), (4.34a) and (4.34b) and $d^4pd^4q = \frac{1}{16} d^4 P d^4 Q$:

$$\int_0^\infty \int_0^\alpha dt \int_0^\alpha dq Tr \left( V e^{-tH_0} V e^{-(\alpha - t)H_0} \right) e^{-\frac{1}{\alpha L^2}} \left| \text{2-fields} \right.$$  

\[ \approx \frac{\text{tr} \sqrt{G}}{2\Lambda_{NC}^4} \int \frac{d^4 Q d^4 l}{(2\pi \Lambda_{NC}^2)^4} i \tilde{G}^{\mu \nu} \left\{ \tilde{G}^{\nu \sigma} A_\mu(Q)A_\rho(-Q) \left( \frac{1}{2} \tilde{G}_{\nu \sigma} - Q \cdot \tilde{Q} - \Lambda^2 \right) \right. 
- \frac{\Lambda^6}{2} \tilde{Q} \cdot \tilde{Q} \left( \tilde{Q}_\mu \tilde{Q}_\nu + \frac{1}{4} \tilde{Q} \cdot \tilde{Q} \tilde{G}_{\nu \sigma} \right) - \frac{\Lambda^2}{12} (\tilde{G}_{\nu \sigma} Q \cdot Q - Q_\nu Q_\sigma) \tilde{Q} \cdot \tilde{Q} - \frac{\Lambda^2}{6} Q \cdot Q \tilde{Q}_\nu \tilde{Q}_\sigma 
+ \frac{\Lambda^6}{2} \Theta^{ab}(Q) \Theta_{ab}(-Q) \left( \Lambda^2 \tilde{Q} \cdot \tilde{Q} + \frac{Q_\nu Q_\sigma}{6} \tilde{Q} \cdot \tilde{Q} \left( \ln \left( \frac{Q Q}{\Lambda^2} \right) + 2 \gamma_E - \frac{8}{3} \right) \right) + O \left( \frac{Q^6}{\Lambda^6} \right) \right\} \tag{4.35} \]

Similarly, for the 3-field contributions in (4.32) using (4.33c) and (4.34c) we have:

$$\int_0^\infty \int_0^\alpha dt \int_0^\alpha d\alpha dq Tr \left( V e^{-tH_0} V e^{-(\alpha - t)H_0} \right) e^{-\frac{1}{\alpha L^2}} \left| \text{3-fields} \right.$$  

\[ \approx \frac{\text{tr} \sqrt{G}}{2\Lambda_{NC}^4} \int \frac{d^4 Q d^4 l}{(2\pi \Lambda_{NC}^2)^4} i \tilde{G}^{\mu \nu} \left\{ \tilde{G}^{\nu \sigma} A_\mu(Q)A_\rho(-Q - l)A_\sigma(l) + A_\mu(Q) \phi^i(-Q - l) \phi_i(l) \right\} 
\times \left( \frac{\Lambda^6 Q_\nu}{8Q \cdot Q} \left( 2 \tilde{Q} \cdot \tilde{Q} + \tilde{Q} \cdot \tilde{Q} \right) \left( 2 \tilde{Q} \cdot \tilde{Q} + \tilde{Q} \cdot \tilde{Q} \right) \sin \left( \frac{\angle Q}{2} \right) \right. 
+ \frac{\Lambda^4}{4Q \cdot Q} \left( Q \cdot Q \left( 2 \tilde{Q}_\nu \sin^2 \left( \frac{\angle Q}{2} \right) + \tilde{Q} \cdot \tilde{Q} \right) + \tilde{Q} \cdot \tilde{Q} Q \cdot Q(\tilde{Q} + \tilde{Q}_\nu) 
+ Q_\nu(2 \tilde{Q} + \tilde{Q} \cdot \tilde{Q} \cdot \tilde{Q} \left( \cos \left( \frac{\angle Q}{2} \right) - 1 \right) 
+ \frac{1}{9} Q \cdot Q \tilde{Q}_\nu \sin^2 \left( \frac{\angle Q}{2} \right) \left( 3 \ln \left( \frac{\Lambda^2}{Q^2} \right) - 6 \gamma_E + 8 \right) \right) \left. \right) + O \left( \frac{Q^6}{\Lambda^6} \right) \tag{4.36} \]

**Third order.** The third order in the heat kernel expansion yields for the three field contributions:

$$\text{Tr} \left( V e^{-rH_0} V e^{-(t-r)H_0} V e^{-(\alpha - t)H_0} \right) \left| \text{3fields} \right.$$  

\[ = \int \frac{d^4 p}{(2\pi \Lambda_{NC}^2)^2} \int \frac{d^4 q}{(2\pi \Lambda_{NC}^2)^2} \int \frac{d^4 l}{(2\pi \Lambda_{NC}^2)^2} \text{tr} \left( V_{p,q} e^{-\frac{r}{2} q q V_{q,l} e^{-(\alpha - t)H_0} V_{l,p} e^{-(\alpha - t)H_0} p p} \right) \left| \text{3fields} \right. 
\]
\[ A \]

In order to be able to perform these integrals explicitly, we need to find an appropriate 3-dimensional coordinate transformation so that all fields become independent of one of these variables. One possibility would be the following:

\[
\begin{pmatrix}
  P \\
  Q \\
  L
\end{pmatrix} = \begin{pmatrix}
  1 & 0 & 1 \\
  1 & -1 & 0 \\
  0 & 1 & -1
\end{pmatrix} \begin{pmatrix}
  p \\
  q \\
  l
\end{pmatrix}.
\] (4.38)

Then e.g. \( A_\mu(p-q)A_\rho(q-l)A_\tau(l-p) = A_\mu(Q)A_\rho(L)A_\tau(-Q-L) \) is independent of \( P \), allowing explicit integration over that variable. We hence find

\[
\text{Tr} \left( V e^{-rH_0} V e^{-(t-r)H_0} V e^{-(\alpha-t)H_0} \right) \bigg|_{3\text{fields}}
\]

\[
= -\frac{16\text{tr}1}{\Lambda_{\text{NC}}^2} \int \frac{d^4P d^4Q d^4L}{(2\pi \Lambda_{\text{NC}}^2)^6} e^{-\frac{P}{4}(P-Q+L)(P-Q+L)} e^{-\frac{Q}{4}(Q-L)(Q-L)} e^{-\frac{L}{4}(L-P)(L-P)} e^{-rP-Q-(t-r)P-(\alpha-t)L} P \times \]

\[
\times \left( \frac{\tilde{\alpha}}{\alpha} \right)^2 \left( P \right)^\sigma \left( Q \right)_{\tau} \left( L \right)_\rho \left( P \right)_\nu A_\mu(P)A_\rho(Q)A_\tau(L)A_\sigma(P-Q-L) \sin \left( \frac{(P+L)\theta Q}{4} \right) \times \]

\[
\times \sin \left( \frac{(P-Q)\theta L}{4} \right) \sin \left( \frac{(Q-L)\theta P}{4} \right) \left( P + L \right)_\nu A_\mu(Q) \Theta^{ab}(L) \Theta_{ab}(-Q-L) + \Theta^{ab}(Q) \Theta_{ab}(L) P_\nu A_\mu(-Q-L) \right).
\] (4.39)

Integration over \( P \) yields a rather lengthy expression, and in order to continue with the parameter integrals (once more regularized by a cutoff \( \Lambda \)), we consider the following approximations after the \( P \) integration:

1. We replace all phases such as \( e^{iQ\theta L} \) by 1, which amounts to dropping higher-order terms in the \( \theta \)-expanded action resp. commutators in the NC action.

2. Since the divergent contributions are due to the parameter region \( t < r < \alpha \approx 0 \), we keep only the leading terms in the exponent of type \( \propto \frac{1}{\theta} \) (i.e. dropping contributions such as \( rt/\alpha \), which would correspond to higher-order terms in the action).

3. Finally, an expansion of type \( e^{\frac{Q}{\alpha}} \approx \left( 1 + \frac{Q}{\alpha} \right) \) is made, as explained in Section 3.
Therefore, we get the following leading order contributions

\[
\int_0^\infty d\tilde{\alpha} \int_0^\tilde{\alpha} d\tilde{t} \int d^4P e^{-\frac{\tilde{\alpha}}{\Lambda^4}(P-Q+L)\cdot(P-Q+L)} e^{-\frac{\tilde{\alpha}}{\Lambda^4}(P-Q-L)\cdot(P-Q-L)} e^{-\frac{\tilde{\alpha}}{\Lambda^4}(P+Q+L)\cdot(P+Q+L)} \times \\
\times e^{-\frac{\tilde{\alpha}}{\Lambda^4}(P+P_{\text{shift}})\cdot(t-t')H_0} \sin\left(\frac{(P+P_{\text{shift}})\cdot\Gamma}{4}\right) \sin\left(\frac{(P-Q)\cdot\nu\Gamma}{4}\right) \sin\left(\frac{(Q+L)\cdot\nu\Gamma}{4}\right)
\]

\[
\approx -\frac{\pi^2\Lambda^4}{3} \left(\tilde{Q} \cdot \tilde{Q} \left(\tilde{L}_c G_{\nu\sigma} + \tilde{L}_v G_{\sigma\nu} + \tilde{L}_\sigma G_{\nu\nu}\right) + \tilde{L} \cdot \tilde{L} \left(\tilde{Q}_c \tilde{G}_{\nu\sigma} + \tilde{Q}_v \tilde{G}_{\sigma\nu} + \tilde{G}_{\nu\nu}\right)\right) + 2\tilde{L} \cdot \tilde{Q} \left(G_{\nu\sigma}(\tilde{L} + \tilde{Q})_\sigma + \tilde{G}_{\sigma\nu}(\tilde{L} + \tilde{Q})_\nu + \tilde{G}_{\nu\nu}(\tilde{L} + \tilde{Q})_\nu\right) + 2\tilde{Q}_\sigma \tilde{L}_\nu(\tilde{L} + \tilde{Q})_\nu + 2\tilde{L}_\nu \tilde{Q}_\nu(\tilde{L} + \tilde{Q})_\nu + 2\tilde{L}_\sigma \tilde{Q}_\nu(\tilde{L}_\nu + \tilde{Q}_\nu)\bigg), \quad (4.40a)
\]

\[
\int_0^\infty d\alpha \int_0^\alpha dt \int dr \int d^4P e^{-\frac{\alpha}{\Lambda^4}(P-Q+L)\cdot(P-Q+L)} e^{-\frac{\alpha}{\Lambda^4}(P-Q-L)\cdot(P-Q-L)} e^{-\frac{\alpha}{\Lambda^4}(P+Q+L)\cdot(P+Q+L)} \times \\
\times e^{-\frac{\alpha}{\Lambda^4}(P+P_{\text{shift}})\cdot(t-t')H_0} \sin\left(\frac{(P+P_{\text{shift}})\cdot\Gamma}{4}\right) \sin\left(\frac{(P-Q)\cdot\nu\Gamma}{4}\right) \sin\left(\frac{(Q+L)\cdot\nu\Gamma}{4}\right)
\]

\[
\approx -\frac{1}{6} \pi^2\Lambda^4 \left(\tilde{L}_c \tilde{Q} \cdot \tilde{Q} + \tilde{L} \cdot \tilde{L} \tilde{Q}_\nu + 2 \left(\tilde{L}_\nu + \tilde{Q}_\nu\right) \tilde{L} \cdot \tilde{Q}\right). \quad (4.40b)
\]

Putting everything together, (4.40a) and (4.40b) finally lead to the 3-field contribution

\[
\int_0^\infty d\alpha \int_0^\alpha dt \int d^4P e^{-\frac{\alpha}{\Lambda^4}(P-Q+L)\cdot(P-Q+L)} e^{-\frac{\alpha}{\Lambda^4}(P-Q-L)\cdot(P-Q-L)} e^{-\frac{\alpha}{\Lambda^4}(P+Q+L)\cdot(P+Q+L)} \times \\
\times e^{-\frac{\alpha}{\Lambda^4}(P+P_{\text{shift}})\cdot(t-t')H_0} \sin\left(\frac{(P+P_{\text{shift}})\cdot\Gamma}{4}\right) \sin\left(\frac{(P-Q)\cdot\nu\Gamma}{4}\right) \sin\left(\frac{(Q+L)\cdot\nu\Gamma}{4}\right)
\]

\[
\approx -\frac{1}{6} \pi^2\Lambda^4 \left(\tilde{L}_c \tilde{Q} \cdot \tilde{Q} + \tilde{L} \cdot \tilde{L} \tilde{Q}_\nu + 2 \left(\tilde{L}_\nu + \tilde{Q}_\nu\right) \tilde{L} \cdot \tilde{Q}\right). \quad (4.41)
\]

\[
\begin{align*}
\text{Putting everything together, (4.40a) and (4.40b) finally lead to the 3-field contribution}
\int_0^\infty d\alpha \int_0^\alpha dt \int d^4P e^{-\frac{\alpha}{\Lambda^4}(P-Q+L)\cdot(P-Q+L)} e^{-\frac{\alpha}{\Lambda^4}(P-Q-L)\cdot(P-Q-L)} e^{-\frac{\alpha}{\Lambda^4}(P+Q+L)\cdot(P+Q+L)} \times \\
\times e^{-\frac{\alpha}{\Lambda^4}(P+P_{\text{shift}})\cdot(t-t')H_0} \sin\left(\frac{(P+P_{\text{shift}})\cdot\Gamma}{4}\right) \sin\left(\frac{(P-Q)\cdot\nu\Gamma}{4}\right) \sin\left(\frac{(Q+L)\cdot\nu\Gamma}{4}\right)
\end{align*}
\]

\[
\approx -\frac{1}{6} \pi^2\Lambda^4 \left(\tilde{L}_c \tilde{Q} \cdot \tilde{Q} + \tilde{L} \cdot \tilde{L} \tilde{Q}_\nu + 2 \left(\tilde{L}_\nu + \tilde{Q}_\nu\right) \tilde{L} \cdot \tilde{Q}\right). \quad (4.41)
\]

\[
\begin{align*}
\text{Putting everything together, (4.40a) and (4.40b) finally lead to the 3-field contribution}
\int_0^\infty d\alpha \int_0^\alpha dt \int d^4P e^{-\frac{\alpha}{\Lambda^4}(P-Q+L)\cdot(P-Q+L)} e^{-\frac{\alpha}{\Lambda^4}(P-Q-L)\cdot(P-Q-L)} e^{-\frac{\alpha}{\Lambda^4}(P+Q+L)\cdot(P+Q+L)} \times \\
\times e^{-\frac{\alpha}{\Lambda^4}(P+P_{\text{shift}})\cdot(t-t')H_0} \sin\left(\frac{(P+P_{\text{shift}})\cdot\Gamma}{4}\right) \sin\left(\frac{(P-Q)\cdot\nu\Gamma}{4}\right) \sin\left(\frac{(Q+L)\cdot\nu\Gamma}{4}\right)
\end{align*}
\]

\[
\approx -\frac{1}{6} \pi^2\Lambda^4 \left(\tilde{L}_c \tilde{Q} \cdot \tilde{Q} + \tilde{L} \cdot \tilde{L} \tilde{Q}_\nu + 2 \left(\tilde{L}_\nu + \tilde{Q}_\nu\right) \tilde{L} \cdot \tilde{Q}\right). \quad (4.41)
\]

Note that the replacement \(\alpha = \Lambda^4\tilde{\alpha}\) etc. provides a factor \(\Lambda_{NC}^2\).

5 Effective NC gauge theory action

Now we can recast the above results into an effective gauge theory action, organized in terms of engineering dimension. As discussed in Section 3, this arises systematically due to the expansion
in three small parameters $\epsilon_L(p), (p\theta q) \sim \frac{\Lambda^2}{\Lambda_{NC}^2}$ and $\frac{\Lambda^2}{\Lambda_{NC}^2}$, imposing the IR condition \[3.3\]. We will systematically compute all terms of operator dimension $\leq 6$.

It is important to note that since $V$ is either linear or quadratic in the fields $(A, \varphi)$, there is only a finite number of terms in the perturbation expansion which can produce gauge theory terms involving $n$ fields (i.e. of order $V^n$ up to $V^{2n}$). Therefore these are completely determined by the perturbative expansion.

We will first consider the quadratic terms in $A_\mu$, resp. $\phi^i$, which arise from the first and second order terms in $V$. They may contain arbitrarily high powers of momenta resp. derivatives. Due to translational invariance \[2.3\], the leading term in this expansion is quadratic in momenta and quadratic in the fields, of type $\Lambda^4 \int O(p^2(A, \varphi)^2)$. This is the usual vacuum energy contribution in quantum field theory, which diverges as $\Lambda^4$. In the present context, we will denote it as “potential” term since it governs the vacuum structure of the NC brane solution in the flat case, in particular $\theta^{\mu\nu}$ and the dilaton. Gauge invariance then requires the presence of certain cubic terms in the fields, which will be verified in detail \[8\].

Next, we will analyze the dimension 6 operators with structure $\Lambda^2 \int O(p^4(A, \varphi)^2)$ in a similar way, leading to curvature-type terms. Again, gauge invariance will be verified up to the cubic terms in the fields. However, there will also be terms proportional $\Lambda^6 \int O((A, \varphi)^2 p^4)$ due to UV/IR mixing resp. factors of $\epsilon_L(p)$, which also correspond to curvature contributions. Higher-order terms of dimension 8 and higher will not be analyzed further in this paper.

To make contact with the commutative case, one should consider the case $\frac{\Lambda}{\Lambda_{NC}} \ll 1$. Then UV/IR mixing terms would give an expansion in this small parameter.

### 5.1 $\Lambda^4$ potential terms

**Two field contributions.** We have contributions from the first-order term \[4.31\] as well as from the second-order term \[4.35\], where both the $AA$ terms as well as the $\Theta\Theta$ terms contain $D\phi D\phi$. The complete action with engineering dimension 4 is as expected proportional to $\Lambda^4$, given by

$$
\Gamma^{(2)}_{\Lambda^4}((A, \varphi)^2, p^2) = \frac{\text{tr} \mathbb{1}}{16} \frac{\Lambda^4}{\Lambda_{NC}^4} \int \frac{d^4l}{(2\pi\Lambda_{NC}^2)^2} \sqrt{g} \left( l^2 \varphi^i(-l)\varphi_i(l) - 2\Lambda^2 \Theta^{\mu\nu} l_\nu A_\mu(l)\Theta^{\rho\sigma} l_\sigma A_\rho(-l) \right) 
$$

$$
= \frac{\text{tr} \mathbb{1}}{16} \frac{\Lambda^4}{\Lambda_{NC}^4} \int \frac{d^4x}{(2\pi)^2} \sqrt{g} \left( g^{\alpha\beta} \partial_\alpha \varphi^i \partial_\beta \varphi_i - \frac{1}{2} \Lambda^4 \Theta^{\mu\nu} F_{\nu\rho} \Theta^{\rho\sigma} F_{\sigma\rho} + \mathcal{O}(A^3) \right)
$$

\[5.1\]

using \[4.26\]. The result essentially gauge-invariant up to $\mathcal{O}(A^3)$ and $\mathcal{O}(A^4)$ terms, which will be recovered from higher-order terms. This is consistent with previous results \[20\], where the fermionic one-loop action was computed on $R_{\theta}^4$. Note that there is no renormalization of the bare Yang-Mills action \[21.12\].

\[8\]The quartic terms are not verified here in order to keep the paper within reasonable bounds.
Three field contributions. The dimension 6 contributions from $O(V^2)$ due to (4.36) proportional to $\Lambda^4$ are given by

$$
\Gamma^{(2)}((A, \varphi)^3, p^3) = -i\frac{\Lambda^4\text{tr}\Pi \sqrt{G}}{16\Lambda_{\text{NC}}^8} \int \frac{d^4q d^4l}{(2\pi)^4} \left\{ \left( q^2 \tilde{\varphi}^\mu + l^2 \tilde{\varphi}^\mu + 2ql(\tilde{l} \tilde{\varphi}^\mu + \tilde{\varphi}^\mu) \right) \times \left( \tilde{G}^{\rho\sigma} A_\rho(q) A_\sigma(q-l) A_\sigma(l) + A_\mu(q) \varphi^\rho(-q-l) \varphi_\rho(l) \right) + O(p^4 A^3) \right\}
$$

(5.2)

using (4.26) and dropping higher-order terms arising e.g. from $(l^2 q^2)$. The contributions from $O(V^3)$ due to (4.41) are

$$
\Gamma^{(3)}((A, \varphi)^3, p^3) = \frac{i\Lambda^4\text{tr}\Pi \sqrt{G}}{16\Lambda_{\text{NC}}^8} \int \frac{d^4q d^4l}{(2\pi)^4} A_\nu(q) A_\sigma(l) A_\nu(l) \times \left( \tilde{G}^{\rho\sigma} (q^2 \tilde{\varphi}^\rho + l^2 \tilde{\varphi}^\rho + 2ql(\tilde{l} \tilde{\varphi}^\rho + \tilde{\varphi}^\rho)) \right)
$$

(5.2)

(using $q \rightarrow -l - q$ at some point). This cancels precisely the term involving $G^{\mu\nu} A_\mu A_\nu$ in (5.2), and the combined 3-field contribution up to dimension 6 operators proportional to $\Lambda^4$ is

$$
\Gamma^{(4)}((A, \varphi)^3, p^3) = \frac{\Lambda^4\text{tr}\Pi \sqrt{G}}{16\Lambda_{\text{NC}}^8} \int \frac{d^4q d^4l}{(2\pi)^4} \left( A_\mu(q) \varphi^\rho(-q-l) \varphi_\rho(l) \left( 2ql(\tilde{l} \tilde{\varphi}^\mu + \tilde{\varphi}^\mu) \right) \right)
$$

$$
= - \frac{\Lambda^4\text{tr}\Pi \sqrt{G}}{16\Lambda_{\text{NC}}^8} \int \frac{d^4q d^4l}{(2\pi)^4} \left( A_\mu(q) \varphi^\rho(-q-l) \varphi_\rho(l) \left( 2ql(\tilde{l} \tilde{\varphi}^\mu + \tilde{\varphi}^\mu) \right) \right)
$$

$$
= \frac{\Lambda^4\text{tr}\Pi \sqrt{G}}{16\Lambda_{\text{NC}}^8} \int \frac{d^4q d^4l}{(2\pi)^4} \left( A_\mu(q) \varphi^\rho(-q-l) \varphi_\rho(l) \left( 2ql(\tilde{l} \tilde{\varphi}^\mu + \tilde{\varphi}^\mu) \right) \right)
$$

Now the AAA terms can be simplified using

$$
-2(\tilde{G}^{\rho\sigma} A_\rho A_\sigma) \tilde{\varphi}^\mu A_\mu A_\nu A_\nu \varphi\varphi_i \varphi_i \varphi_i = (\tilde{G}^{\rho\sigma} F_{\rho\sigma}) \tilde{\varphi}^\mu \left( F_{\nu\nu} + \partial_\nu A_\nu \right) \tilde{\varphi}^\mu \varphi_i \varphi_i
$$

$$
= (\tilde{G}^{\rho\sigma} F_{\rho\sigma}) \tilde{\varphi}^\mu \left( \frac{1}{2} F_{\nu\nu} \tilde{\varphi}^\mu \varphi_i \varphi_i - i[A_\nu, A_\nu] \varphi_i \varphi_i \right)
$$

(5.4)

up to quartic terms, replacing the commutator with a Poisson bracket to leading order in $\theta$. Here we used the identity

$$
F_{\nu\nu} \tilde{\varphi}^\mu \varphi_i \varphi_i = \frac{1}{2} F_{\nu\nu} \tilde{\varphi}^\mu \varphi_i \varphi_i
$$

(5.5)

(up to cubic terms), which can be seen by renaming $\nu \leftrightarrow \nu'$ and $\mu \leftrightarrow \nu$. Similarly, the $A \varphi\varphi$ terms can be simplified using partial integration as follows

$$
\int \frac{d^4q}{(2\pi)^4} \tilde{G}^{\rho\sigma} g^{\alpha\beta} (2\partial_\alpha A_\mu \partial_\nu \varphi^i \partial_\beta \varphi_\rho - \partial_\alpha \partial_\beta A_\mu \varphi^i \partial_\nu \varphi_\rho - \partial_\nu A_\mu \varphi^i \partial_\alpha \partial_\beta \varphi_\rho)
$$

$$
= \int \frac{d^4q}{(2\pi)^4} \tilde{G}^{\rho\sigma} g^{\alpha\beta} (2\partial_\alpha A_\mu \partial_\nu \varphi^i \partial_\beta \varphi_\rho - \partial_\alpha \partial_\beta A_\mu \varphi^i \partial_\nu \varphi_\rho - \partial_\nu A_\mu \varphi^i \partial_\alpha \partial_\beta \varphi_\rho)
$$

= \int \frac{d^4q}{(2\pi)^4} \tilde{G}^{\rho\sigma} g^{\alpha\beta} (2F_{\alpha\mu} + \partial_\alpha A_\mu) \partial_\nu \varphi^i \partial_\beta \varphi_\rho + \frac{1}{2} F_{\nu\mu} \partial_\beta \varphi^i \partial_\alpha \varphi_\rho) + h.o.
$$

(5.6)
Thus we get

$$\Gamma_{A^2}(A, \varphi, p^4) = -\frac{\Lambda^4 \text{tr} \sqrt{g}}{16 \Lambda_{NC}^4} \int \frac{d^4 x}{(2\pi)^2} \left( g^{\alpha \beta} (2 \bar{\theta}^{\mu \nu} F_{\alpha \mu} \partial_{\mu} \varphi^i \partial_{\nu} \varphi_i - 2i [A_\alpha, \varphi^i] \theta \partial_{\nu} \varphi_i \\
+ \frac{1}{2} (\bar{\theta}^{\mu \nu} F_{\mu \nu}) \partial_{\beta} \varphi^i \partial_{\alpha} \varphi_i + \Lambda_{NC}^4 (\bar{\theta}^{\sigma \tau} F_{\sigma \tau}) \partial^{\alpha} (\frac{1}{2} F_{\alpha \nu} \bar{\theta}^{\nu \mu} F_{\mu \epsilon} - i [A_\epsilon, A_\mu]) \right).$$

The commutator terms provide precisely the missing cubic terms for the gauge-invariant completion of (5.1), so that the complete induced potential including all terms with dimension up to 6 is given by

$$\Gamma_{A^2}(A, \varphi, p^{4-6}) = \frac{\text{tr} \Lambda^4}{16 \Lambda_{NC}^4} \int \frac{d^4 x}{(2\pi)^2} \sqrt{g} \left( g^{\alpha \beta} D_\alpha \varphi^i D_\beta \varphi_i \\
- \frac{1}{2} \Lambda_{NC}^4 (\bar{\theta}^{\mu \nu} F_{\mu \nu} \bar{\theta}^{\rho \sigma} F_{\rho \sigma} + (\bar{\theta}^{\sigma \tau} F_{\sigma \tau})(\bar{\theta} F \theta)) \\
- 2 \bar{\theta}^{\mu \nu} F_{\mu \rho} g^{\alpha \beta} \partial_{\nu} \varphi^i \partial_{\alpha} \varphi_i + \frac{1}{2} (\bar{\theta}^{\mu \nu} F_{\mu \nu}) g^{\alpha \beta} \partial_{\beta} \varphi^i \partial_{\alpha} \varphi_i + \text{h.o.} \right), \quad (5.7)$$

which is manifestly gauge invariant. Remarkably, this result will be precisely recovered from the simple matrix model effective action (6.7). This is a strong confirmation of the $SO(D)$ symmetry, demonstrating the power of the matrix model point of view.

### 5.2 $O(\Lambda^2)$ curvature terms

Consider now the dimension 6 terms proportional to $\Lambda^2$. They must be gauge-invariant by themselves. The terms quadratic in the fields are

$$\Gamma_{A^2}(A, \varphi^2, p^4) = \frac{1}{4} \text{tr} \frac{\Lambda^2}{16 \Lambda_{NC}^4} \int \frac{d^4 q}{(2\pi)^2} \sqrt{g} \left( 6 \Lambda_{NC}^8 \Theta_{ab}(q) \Theta_{ab}(-q) \bar{q} \cdot \bar{q} \\
- (G^{\nu \sigma} \cdot q - q_\nu q_\sigma) \bar{q} \cdot \bar{q} G^{\mu \nu} G^{\rho \sigma} A_\mu(q) A_\rho(-q) - 2q_\nu q_\sigma G^{\mu \nu} G^{\rho \sigma} A_\mu(q) A_\rho(-q) \right)$$

$$= \frac{1}{4} \text{tr} \frac{\Lambda^2}{16 \Lambda_{NC}^4} \int \frac{d^4 x}{(2\pi)^2} \sqrt{g} \left( - 6 \Lambda_{NC}^8 \Theta_{ab} \bar{\square}_g \Theta_{ab} \\
+ \frac{1}{2} \Lambda_{NC}^4 F_{\mu \nu} \bar{\square}_g F_{\mu \nu} \bar{G}^{\mu \nu} + \frac{1}{2} \Lambda_{NC}^4 (\bar{\theta}^{\mu \nu} F_{\mu \nu}) \bar{G}^{\rho \sigma} G^{\rho \sigma} + \text{O}(A^3) \right)$$

$$= \frac{1}{4} \text{tr} \frac{\Lambda^2}{16 \Lambda_{NC}^4} \int \frac{d^4 x}{(2\pi)^2} \sqrt{g} \left( \frac{11}{2} \Lambda_{NC}^4 F_{\mu \nu} \bar{\square}_g F_{\sigma \tau} G^{\rho \sigma} G^{\mu \nu} - 12 \Lambda_{NC}^8 \bar{\square}_g \bar{\square}_g \varphi \cdot \varphi_i \\
+ \frac{1}{2} \Lambda_{NC}^4 (\bar{\theta}^{\mu \nu} F_{\mu \nu}) \bar{G}^{\rho \sigma} G^{\rho \sigma} + \text{O}(A^3) \right). \quad (5.8)$$

This is indeed gauge-invariant up to $O(A^3)$ terms, which should be recovered later. Note that there are different Laplacians (6.18) in this expression such as $\bar{\square}_g$, corresponding to the matrix operators but for the Groenewold-Moyal background. They contain powers of $\Lambda_{NC}$. This will facilitate the comparison with the matrix model expressions.
Three field contributions. The 3-field contributions proportional to $\Lambda^2$ from $\mathcal{O}(V^2)$ start at $\mathcal{O}(p^6)$ i.e. dimension 8, which is not considered here. The contributions from $\mathcal{O}(V^3)$ due to (4.36) are given by
\[
\Gamma^{(3)}((A, \varphi)^3, p^3) = \frac{i}{12} \int \frac{d^4q}{(2\pi)^4} \frac{\Lambda^2}{16} \bar{G}^{\mu\nu} \left(\mathcal{L}_\nu \bar{Q} \cdot \dot{Q} + \mathcal{L} \cdot \mathcal{Q} \nu + 2 \left(\mathcal{L}_\nu + \mathcal{Q}_\nu\right) \bar{L} \cdot \dot{Q}\right) \times \\
\left\{A_\mu(Q)\Theta^{ab}(L)\Theta_{ab}(-Q-L) + \Theta^{\mu\nu}(Q)A_\mu(L)\Theta_{ab}(-Q-L) + \Theta^{\mu\nu}(Q)\Theta_{ab}(L)A_\mu(-Q-L)\right\}
\]
\[
\Gamma^{(3)}((A, \varphi)^3, p^3) = \frac{1}{4} \Lambda^2 \frac{A^4}{16 \Lambda^2_{NC}} \int \frac{d^4x}{(2\pi)^4} \bar{\theta}^{\mu\nu} g^{\alpha\beta} \left(\partial_\alpha \partial_\beta A_\mu(\varphi^\beta \Theta^{ab} \Theta_{ab} + \partial_\mu A_\mu \partial_\alpha \theta^{ab} \Theta_{ab} - 2\partial_\alpha A_\mu \partial_\beta \Theta^{ab} \partial_\beta \Theta_{ab}\right)
\]
\[
\Gamma^{(3)}((A, \varphi)^3, p^3) = \frac{1}{4} \Lambda^2 \frac{A^4}{16 \Lambda^2_{NC}} \int \frac{d^4x}{(2\pi)^4} \bar{\theta}^{\mu\nu} g^{\alpha\beta} \left(2(\mathcal{L}_\alpha + \partial_\alpha A_\mu) \partial_\nu \Theta^{ab} \partial_\beta \Theta_{ab} + \frac{1}{2} F_{\mu\nu} \partial_\beta \Theta^{ab} \partial_\alpha \Theta_{ab} + \text{h.o.}\right)
\]
\[
\Gamma^{(3)}((A, \varphi)^3, p^3) \sim \frac{1}{64} \Lambda^2 \frac{A^4}{16 \Lambda^2_{NC}} \int \frac{d^4x}{(2\pi)^4} \bar{\theta}^{\mu\nu} g^{\alpha\beta} \left(2i[A_\alpha, \Theta^{\beta\gamma}] \partial_\nu \Theta_{ab} - 2\bar{\theta}^{\mu\nu} F_{\alpha \mu} \partial_\nu \Theta^{ab} \partial_\beta \Theta_{ab} - \frac{1}{2} \bar{\theta}^{\mu\nu} F_{\gamma \nu} \partial_\beta \Theta^{ab} \partial_\alpha \Theta_{ab}\right)
\]
\[
(5.9)
\]
using (5.6). The middle term is clearly part of a covariant derivative $D_\alpha \Theta^{ab} D_\beta \Theta_{ab}$. However these are also dimension 8 operators which we will not consider any further in this paper.

5.3 $\mathcal{O}(\Lambda^6)$ curvature terms

Finally consider the dimension 6 terms proportional to $\Lambda^6$, quadratic in the fields. There are also contributions from the $\mathcal{O}(V)$ terms (1.27), due to
\[
\Lambda^2 - \Lambda^2_{\text{eff}}(l) = \Lambda^2 - \frac{\Lambda^2}{1 + \frac{1}{4} \Lambda^2_{NC} l^2} = \frac{1}{4} \Lambda^4 l^2 - \frac{1}{16} \Lambda^6 l^4 + \ldots
\]
\[
(5.10)
\]
This gives
\[
\Gamma_{\Lambda^6((A, \varphi)^2, p^4)} = \frac{1}{16} \Lambda^2 \frac{A^4}{16 \Lambda^2_{NC}} \int \frac{d^4l}{(2\pi)^2} \sqrt{g} \left[-\frac{1}{4} (\mathcal{L}_\nu)^2 \varphi^i(-l) \varphi_i(l) + \Lambda^4 \frac{A_{\mu}(q) A_{\mu}(-q) q^2 \bar{\nu}_{\alpha} \bar{\nu}_{\beta}\right]
\]
\[
= \frac{1}{16} \Lambda^2 \frac{A^4}{16 \Lambda^2_{NC}} \int \frac{d^4x}{(2\pi)^4} \sqrt{g} \left[-\frac{1}{4} \Lambda^4 \Lambda^4 \Box_g \varphi_i(\square_g \varphi_i) + \frac{1}{4} \Lambda^4 \Lambda^4 \Box_g (\bar{\varphi}^{\mu\nu} F_{\mu\nu} \boxdot_g (\bar{\varphi}^{\mu\nu} F_{\mu\nu}) + \mathcal{O}(A^3)\right]
\]
\[
(5.11)
\]
Again this is indeed gauge-invariant, which constitutes a non-trivial check of our 1-loop computations. The 3-field contributions at $\mathcal{O}(\Lambda^6)$ due to (4.36) as well as the missing terms for the gauge-invariant completion of $\mathcal{F}$ or $\square_g$ have dimension 8 or higher (upon expanding $[A, A]$), which we do not consider here.

This $\Lambda^6$ contribution is expected to arise from $\mathcal{O}(X^{14})$ terms in the matrix model such as $\Box_g X^a \Box_g X^a$, which however will not be worked out in this paper. Note that this term is comparable with the $\Lambda^2$ contribution if $\Lambda \approx \Lambda_{NC}$, but is negligible if $\Lambda \ll \Lambda_{NC}$. It is also worth pointing out that this contribution may not be obtained in a semi-classical analysis along the lines of [20, 21], because it involves a higher-order contribution in the UV/IR mixing term (5.10).
5.4 Free contribution

The induced action resp. vacuum energy for the free case (i.e. the constant term in (3.9)) is given by

\[ \Gamma_L[X] = -\frac{1}{2} \text{Tr} \int_0^\infty \frac{d\alpha}{\alpha} e^{-\alpha \beta \frac{1}{\alpha x^2}} = -\frac{1}{2} \text{tr} \int_0^\infty \frac{d\bar{\alpha}}{\bar{\alpha}} \int \frac{d^4 p}{(2\pi)^2} e^{-\bar{\alpha} p^2 - \frac{1}{\alpha x^2}} \int \frac{d^4 x}{(2\pi)^2} \sqrt{g} \]

\[ = -\frac{\Lambda^4 \text{tr} \frac{1}{8}}{(2\pi)^2} \int \frac{d^4 x}{\sqrt{g}}. \tag{5.12} \]

Remarkably, this short computation — along with general geometrical considerations — suffices to predict all of the above loop computations (and beyond) for the potential, as explained below.

6 Effective matrix model action

In this section, we will determine an effective matrix model action which reproduces the induced gauge theory action obtained above. The scaling law (2.11) combined with gauge invariance suggests to consider action functionals of the form

\[ \Gamma_L[X] = \text{Tr} \mathcal{L} \left( \frac{X^a}{L} \right). \tag{6.1} \]

Since we need to reproduce terms which diverge as \( \sim \Lambda^n = L^n \Lambda_{\text{NC}}^n \), for \( n > 0 \), it is clearly not sufficient to assume that \( \mathcal{L}(X) \) is a polynomial or a power series in \( X^a \). For example, the effective matrix model action corresponding to the vacuum energy contributions (5.7) must be homogeneous functions of degree \(-4\) in \( X^a \). This lack of “analyticity” in \( X^a \) should not be surprising, since the loop computation is based on the assumption of a 4-dimensional background, described by a deformation of \( \mathbb{R}^4 \). On such 4-dimensional backgrounds, one can generalize the polynomial functions to a certain class of holomorphic functions in \( X^a \) which are analytic near such backgrounds. This is indeed what happens, which implies the existence of singularities at other locations in the moduli space of Hermitian matrices. This should of course be expected, since the same matrix model accommodates spaces of different dimensions, which surely lead to very different quantum effects.

Gauge invariance strongly restricts the possible form of the action. We restrict ourselves to single-trace terms, which is sufficient for the present context without non-Abelian gauge fields. Furthermore, the \( SO(D) \) symmetry of the bare matrix model and the \( SO(D) \)-invariant regularization (2.9) strongly suggest that the effective action should also be manifestly invariant under \( SO(D) \). However this is a non-trivial statement, which might be spoiled by anomalies due to the infinite-dimensional nature of the matrix model. Nevertheless, we will indeed find an effective matrix model action with manifest \( SO(D) \) symmetry, which reproduces all of the loop results as far as we can verify them. This is a highly non-trivial result, which predicts infinitely many higher-order terms in the induced action of non-commutative gauge theory. All terms of dimension 6 are verified in detail, providing very strong support for the \( SO(D) \) symmetry and the effective action given below.

It should be pointed out that since the effective matrix model action holds for generic deformations of \( \mathbb{R}^4 \), it provides a background-independent action for gravity (as well as non-Abelian gauge theory) for 4-dimensional non-commutative branes \( \mathcal{M}_C^4 \subset \mathbb{R}^{10} \). This is a key issue of emergent gravity.
6.1 Effective potential $V(X)$

Due to translational invariance \(^{(2.3)}\), the effective action $\Gamma_L = \text{Tr} \mathcal{L}(X)$ can be written in terms of commutators, and organized in terms of the order of commutators. Commutators correspond to derivative operators for the gauge fields. They may arise as simple commutators $[X^a, X^b]$, or multiple commutators such as $[X^a, [X^b, X^c]]$ corresponding to higher derivatives. The leading term $\mathcal{L}(X) = V(X) + \text{h.o.}$ in such a “momentum expansion” of the effective matrix action will have only simple commutators, resp. first-order derivatives of the gauge theory language. We will denote such simple-commutator terms as effective potential $V(X) \equiv V([X, X])$. From the emergent gravity point of view, it depends only on the tensor fields $g_{\mu\nu}(x)$ and $\theta^{\mu\nu}(x)$ rather than their derivatives (notably curvature). Hence $V(X)$ will govern the vacuum in the flat case, justifying the name potential. It corresponds to the vacuum energy, cf. \([28]\) for a related discussion.

Taking into account (or assuming) also the $SO(D)$ symmetry, it follows that $V(X)$ can be written in terms of contractions of $\Theta^{ab}$ with $g_{ab} = \delta_{ab}$. Since different orderings of products of $\Theta^{ab}$ differ by commutators, it is enough to consider only contractions of neighbouring indices, in the cyclic sense. This means that $V(X)$ can be written in terms of products of\(^{[\Theta]}\)

$$ J^a_b := i\Theta^{ae}g_{eb} = [X^a, X_b], \quad \text{tr} J \equiv J^a_a = 0, \quad (6.2) $$

where $\text{tr}$ will denote the trace over the $SO(D)$ indices. Since $\Theta^{ab}$ is anti-symmetric, $V(X)$ can be written in terms of products of traces of even powers of $J$.

By analyzing the possible effective potential terms in the semi-classical limit, we can further narrow down the possible form of $V(X)$ by recalling the (semi-classical) characteristic equation which holds for generic 4-dimensional branes $\mathcal{M}^4 \subset \mathbb{R}^D$ \([29, 30]\):

$$ (J^1)^a_b - \frac{1}{2}(\text{tr} J^2) (J^2)^a_b \sim -\Lambda_{\text{NC}}^{-8}(x)(\mathcal{P}T)\mathcal{J}^a_b, \quad J^5 - \frac{1}{2}(\text{tr} J^2) J^3 \sim -\Lambda_{\text{NC}}^{-8}(x) J, \quad (6.3) $$

where the semi-classical object $\mathcal{P}^{ab} := g^{\mu\nu}\partial_\mu x^a \partial_\nu x^b$ is the projector on the tangential bundle (cf. \((2.3)\)), and $\Lambda_{\text{NC}}^{-4}(x) = \sqrt{[\theta^{\mu\nu}(x)]}$ denotes the scale defined by the full NC structure $\theta^{-1}_{\mu\nu}(x) = \partial^{-1}_{\mu\nu} + F_{\mu\nu}$. This follows from the fact that $J$ defines a 4-dimensional anti-symmetric tensor field in the semi-classical limit. Furthermore,

$$ \text{tr} J^2 \sim \Lambda_{\text{NC}}^{-4}(x) G^{\mu\nu}(x)g_{\mu\nu}(x) \equiv \Lambda_{\text{NC}}^{-4}(Gg). \quad (6.4) $$

Due to these relations, any expression

$$ \text{tr} J^{2n}, \quad n \geq 6, \quad (6.5) $$

can be reduced semi-classically\(^{10}\) to a function of $\text{tr} J^4$ and $\text{tr} J^2$. Therefore the most general (single-trace) form of the effective potential compatible with the scaling has the form $V(X) = V\left(\frac{L^4}{\text{tr} J^2}, \frac{\text{tr} J^4}{(\text{tr} J^2)^2}\right)$, or equivalently

$$ V(X) = V\left(-\frac{L^4}{\text{tr} J^2} + \frac{\text{tr} J^2}{\text{tr} J^4} \right) \sim V\left(\frac{L^4}{\Lambda_{\text{NC}}^{-4}(x) (Gg)}, \frac{4}{(Gg)^2}\right). \quad (6.6) $$

\(^{10}\)Note that $J$ defines an almost-complex structure under certain natural conditions \([2\theta]\).

\(^{10}\)In the fully NC case, this argument strictly speaking applies only up to higher-order corrections in $\theta^{\mu\nu}$. Nevertheless, it appears to work.
Note that both arguments of $V(z_1, z_2)$ are Hermitian matrices in the adjoint of $U(\infty)$, which are invertible on 4-dimensional NC branes with $\Lambda_{\text{NC}} \neq 0$. Therefore the above $V(X)$ is an admissible candidate for the effective potential, and $\text{Tr} V(X)$ is well-defined and gauge invariant provided $V$ is analytic (and real-valued) if both $z_1$ and $z_2$ are on the positive real line.

It only remains to determine the function $V(z_1, z_2)$. Remarkably, this can be determined already from the vacuum energy in the free case (5.12), which is proportional to $\Lambda^4$ (in agreement with (5.7)) and does not depend on $(Gg)$ (which measures the deviation of $\theta^{\mu\nu}$ from the (anti-)selfdual case). It follows that $V \sim z_1/\sqrt{z_2}$, i.e. $\Gamma_L[X] = \text{Tr} V(X) + \text{h.o.}$

$$\text{Tr} V(X) = -\frac{1}{4} \text{Tr} \left( \frac{L^4}{\sqrt{-\text{tr} J^4 + \frac{1}{2} (\text{tr} J^2)^2}} \right) \sim -\frac{1}{8} \frac{1}{(2\pi)^2} \int d^4x \Lambda^4(x) \sqrt{g}. \quad (6.7)$$

The normalization factor is obtained from (5.12), identifying the rhs with the semi-classical vacuum energy. “h.o.” stands for higher-order commutator terms, i.e. curvature contributions etc. which will be considered below. As a quick check observe that $\sqrt{g} \sim \sqrt{\bar{g}}(1 + \frac{1}{2} g \partial \phi \partial \phi + \ldots)$ for a non-trivial background, in agreement with (5.7). We therefore conjecture that the potential term in the induced effective action for generic 4-dimensional backgrounds is given by (6.7). We will verify below that this reproduces precisely the above loop computations up to the order computed here. This demonstrates the power of the geometric view of the matrix model.

Note that the bare matrix model (2.12) could be viewed as a potential with $V(z_1, z_2) = \frac{1}{z_1}$. However that term would be proportional to $\Lambda^{-4}$ (rather than log $\Lambda$ as one might suspect), which is highly suppressed and not considered here.

It is remarkable that the vacuum energy (6.7) has a very non-trivial structure from the matrix model point of view. This should have very interesting physical consequences in particular for the cosmological constant problem. However a more complete picture, notably including the contributions from the bosonic sector, is required before its physical consequences can be addressed.

An analogous expression should work in other dimensions, which should be studied elsewhere. In the presence of non-Abelian gauge fields, the action also will need to be generalized.

In order to verify (6.7), we simply have to include fluctuations to the matrices $X^a$ around $R^4_{\theta}$, expand it to any desired order, and compare the result with the induced gauge theory action computed above. This will be done in detail below. However a partial comparison can be made easily using the geometrical point of view.

**Semi-classical analysis** The gauge sector of the above terms can be obtained quickly by setting $\partial_\mu \phi^i = 0$, resp. more generally by going to normal embedding coordinates [4]. Then (6.3) gives for the semi-classical limit

$$\text{tr} J^4 - \frac{1}{2} (\text{tr} J^2)^2 \sim -\Lambda_{\text{NC}}^8(x) \text{tr} P_T = -4 (\text{Pfaff}(\theta(x)))^2 = -\frac{1}{16} \left( \varepsilon_{\mu\nu\alpha\beta} \theta^{\mu\nu}(x) \theta^{\alpha\beta}(x) \right)^2 \quad (6.8)$$

where $\theta^{-1}_{\mu\nu}(x) = \theta^{-1}_{\mu\nu} + F_{\mu\nu}(x)$ is the full symplectic structure, and $P_T$ is the projector on the tangential bundle. The Pfaffian of an anti-symmetric $4 \times 4$ matrix is defined as

$$\text{Pfaff}(F_{\mu\nu}) = \frac{1}{8} \varepsilon^{\mu\nu\rho\eta} F_{\mu\nu} F_{\rho\eta} \quad (6.9)$$

[11] Recall that all higher-order commutators vanish on $R^4_{\theta}$.
and coincides with $\pm \sqrt{|F|}$. This yields
\[
\sqrt{-\text{tr} J^4 + \frac{1}{2}(\text{tr} J^2)^2} \sim \frac{1}{4} \varepsilon^{\mu \nu \alpha \beta} \theta_{\mu \nu} \theta_{\alpha \beta} = \frac{1}{4} \det \bar{\theta} \varepsilon^{\mu \nu \alpha \beta} (\bar{\theta}^{-1}_{\mu \nu} + F_{\mu \nu})(\bar{\theta}^{-1}_{\alpha \beta} + F_{\alpha \beta})
\]
\[= 2\Lambda_{\text{NC}}^{-4} \left(1 - \frac{1}{2} \bar{\theta}^\mu \theta_{\rho \nu} + \Lambda_{\text{NC}}^{-4} \text{Pfaff}(F)\right) \quad (6.10)
\]
which is actually correct to all orders in $F$. Here we use
\[
\frac{1}{8} \varepsilon^{\mu \nu \alpha \beta} \bar{\theta}^{-1}_{\mu \nu} = -\frac{1}{4} \text{Pfaff} \bar{\theta}^{-1} \bar{\theta}^{\alpha \beta},
\]
which can be seen e.g. using the standard form (2.5) for $\bar{\theta}$. Using (6.26a) this gives
\[
\sqrt{-\text{tr} J^4 + \frac{1}{2}(\text{tr} J^2)^2} = \frac{1}{2} \Lambda_{\text{NC}}^{-4} \left(1 + \frac{1}{2} \bar{\theta}^\mu \theta_{\rho \nu} + \frac{1}{4} (\bar{\theta} F)^2 + \frac{1}{8} (F^\mu_\nu \theta_{\rho \sigma})^3 - \bar{\theta}^\mu \theta_{\rho \nu} \Lambda_{\text{NC}}^{-4} \text{Pfaff}(F)\right.
\]
\[- \Lambda_{\text{NC}}^{-4} \text{Pfaff}(F) + \mathcal{O}(F^4)\biggr) = \Lambda_{\text{NC}}^{-4} \left(1 + \frac{1}{2} \bar{\theta}^\mu \theta_{\rho \nu} + \frac{1}{4} (\bar{\theta} F)^2 + \frac{1}{4} \bar{\theta}^\mu \theta_{\rho \nu} (F \bar{\theta} F) + \mathcal{O}(F^4)\right), \quad (6.12)
\]
which is in agreement with the pure gauge sector of (5.7).

### 6.2 Building blocks: the matrix tensors $H^{ab}$

Now consider the general expansion of the effective matrix model action for fluctuating matrices. Besides $\Theta^{ab} = -i[X^a, X^b]$, the following “matrix tensors” [29] play an important role
\[
H^{ab} = \frac{1}{2} [[X^a, X^c], [X^b, X^c]]_+, \\
H = H^{ab} g_{ab} = [X_c, X^d][X^c, X_d] = -\text{tr} J^2,
\]
where we use coordinates such that $g_{ab} = \delta_{ab}$. On the Groenewold-Moyal vacuum $\mathbb{R}^{4}_{\bar{\theta}} \subset \mathbb{R}^{D}$, they reduce to $H^{ab} \rightarrow \bar{H}^{ab}$ where
\[
\bar{H}^{\mu \nu} = -\Lambda_{\text{NC}}^{-4} G^{\mu \nu}, \quad \bar{H}^{\mu i} = \bar{H}^{ij} = 0, \\
\bar{H} = \bar{H}^{ab} g_{ab} = -\Lambda_{\text{NC}}^{-4} G^{\mu \nu} g_{\mu \nu}.
\]
Using the following characteristic relation for 4-dimensional Moyal space [29]
\[
(G g \bar{G})^{\mu \nu} = -\frac{1}{2} \Lambda_{\text{NC}}^{-4} \bar{H} G^{\mu \nu} - g^{\mu \nu} = \frac{1}{2} (G g) \bar{G}^{\mu \nu} - g^{\mu \nu}, \quad (6.15)
\]
(which follows from (6.3)) as well as
\[
\Theta^{\mu \nu} = \bar{\theta}^{\mu \nu} + F^{\mu \nu} = -\bar{\theta}^{\mu \rho} \bar{\theta}^{\nu \sigma} (\bar{\theta}^{-1}_{\rho \sigma} + F_{\rho \sigma}), \\
\Theta^{\mu i} = \bar{\theta}^{\mu \nu} D_\nu \phi^i,
\]
this gives
\[
(H^{ab} - \frac{1}{2} \bar{H} g^{ab})[, X_a, \Phi][X_b, \Psi] = (\Lambda_{\text{NC}}^{-4} G^{\mu \nu} + \frac{1}{2} \bar{H} g^{\mu \nu}) g_{\mu \nu'} g_{\nu \sigma'} \bar{\theta}^{\mu \rho} \bar{\theta}^{\nu \gamma} \partial_\rho \Phi \partial_\gamma \Psi \\
= -\Lambda_{\text{NC}}^{-8} g^{\rho \eta} \partial_\rho \Phi \partial_\eta \Psi. \quad (6.17)
\]
We hence define
\[ \Box_g \Phi := [X_a, (H^a - \frac{1}{2} H g^{ab}) [X_b, \Phi]], \]
\[ \Box_g \Phi := [\tilde{X}_a, (\tilde{H}^a - \frac{1}{2} \tilde{H} g^{ab}) [\tilde{X}_b, \Phi]] = -\Lambda_{NC}^{-8} (g^{a\alpha} \partial_{\rho} \partial_{\eta} \Phi), \]
\[ \Box \Phi := [X^a, [X^b, \Phi]] g_{ab}, \]
\[ \Box \Phi := [\tilde{X}^a, [\tilde{X}^b, \Phi]] g_{ab} = -\Lambda_{NC}^{-4} G^{a\beta} \partial_{\rho} \partial_{\eta} \Phi. \] (6.18)

(It was shown in [29] that they reduce to the appropriate Laplace operators in the semi-classical limit for general 4-dimensional branes.) In particular, this yields
\[ (H^a - \frac{1}{2} \tilde{H} g^{ab}) [X_a, e^{ipx}] [\tilde{X}_b, e^{iqx}] = \Lambda_{NC}^{-8} g^{a\alpha} p_{\rho} q_{\eta} e^{ipx} e^{iqx}. \] (6.19)

Including fluctuations around \( R^4 \subset \mathbb{R}^D \) as in (3.7), we have
\[ H^{\mu\nu} = \frac{1}{2} [\Theta^{\alpha \beta}, \Theta^{\nu \beta} + g_{cd} = \frac{1}{2} [\Theta^{\alpha \beta} + F^{\alpha \beta}, \theta^{\nu \beta} + F^{\nu \beta} + g_{a\beta} - \frac{1}{2} \theta^{\mu \alpha} \hat{\theta}^{\nu \beta} [D_{\alpha \beta}, D_{\mu \nu}], + g_{ij} \]
\[ = -\frac{\hat{G}^{\mu \nu}}{\Lambda_{NC}^{4}} - \frac{\hat{\theta}^{\mu \nu}}{\Lambda_{NC}^{4}} \left( \Theta^{\alpha \beta} \hat{\theta}^{-1} F_{\mu \nu}^{\alpha \beta} + \theta^{-1} F_{\mu \nu}^{\alpha \beta} + \frac{1}{2} [F_{\mu \nu}^{\alpha \beta}, F_{\alpha \beta}^{\nu \beta} + 1] + \frac{1}{2} D_{\alpha \beta}^{\mu \nu}, D_{\mu \nu} \right), \]
\[ H^{ii} = \frac{1}{2} [\Theta^{\alpha \beta}, \Theta^{i \beta} + g_{ij} = \frac{1}{2} [\Theta^{\alpha \beta}, \Theta^{i \beta} + \frac{1}{2} \Theta^{\alpha \beta} + g_{a\beta} \]
\[ = \frac{1}{2} \Lambda_{NC}^{-8} \hat{\theta}^{\alpha \beta} \left( \left( [D_{\mu \nu}^{\alpha \beta}, i [\varphi^{\alpha}, \varphi^{\beta}]] + g_{ij} \right) - \left( \left( \left( [\varphi^{\alpha}, \varphi^{\beta}] + g_{kl} - \frac{1}{2} \Theta^{\alpha \beta} + \frac{1}{2} \Theta^{i \beta} + g_{a\beta} \right) \right) \right) \]
\[ = \frac{1}{2} \Lambda_{NC}^{-8} \hat{G}^{\alpha \beta} [D_{\alpha \beta}, D_{\mu \nu}], + g_{ij} \]
\[ = -\Lambda_{NC}^{-4} g^{\mu \nu} g_{ij} - \Lambda_{NC}^{-8} \left( -2 \Lambda_{NC}^{4} \hat{\theta}^{\alpha \beta} F_{\mu \nu} + \frac{\hat{G}^{\mu \nu} \hat{G} \alpha \beta} F_{\mu \nu} + 2 \hat{G}^{\mu \nu} D_{\mu \nu} \varphi^{i} D_{\nu \varphi^{i}}, g_{ij} \right) \]
\[ + \Lambda_{NC}^{-8} \left( [\varphi^{i}, \varphi^{i}][\varphi^{j}, \varphi^{j}], g_{kl} g_{ij} \right) \]. (6.20)

where we defined
\[ D_{\alpha \beta} = \partial_{\alpha \beta} + i [A_\alpha, \phi], \]
\[ \hat{\theta}^{\mu \nu} = (\hat{G} \theta)^{\mu \nu}, \quad \text{and} \quad \hat{\theta}^{\mu \nu} = (\hat{G} g \theta)^{\mu \nu} = -\hat{\theta}^{\mu \nu}. \] (6.21)

Note that the expressions (6.20) are exact.

### 6.3 Expansion of the effective potential

Using the above results, we can expand the various terms in the effective potential \( V(X) \) in terms of gauge and scalar fields. The contributions quadratic in the fields \( (A, \varphi) \) are
\[ H^{ab} H_{ab} = H^{\mu \nu} H^{\nu \rho} g_{\mu \nu} + H^{ij} H_{ij} g_{\mu \nu} g_{ij} + 2 H^{\mu \nu} H^{\nu \rho} g_{\mu \nu} g_{\nu \rho} + 4 \Lambda_{NC}^{-12} G^{\alpha \beta} (\hat{G} g \theta)^{\mu \nu} F_{\mu \alpha} F_{\nu \beta} \]
\[ + 2 \Lambda_{NC}^{-8} \hat{\theta}^{\mu \nu} \hat{\theta}^{\mu \nu} F_{\mu \nu} F_{\nu \alpha} + 4 \Lambda_{NC}^{-8} F_{\mu \alpha} \left( \frac{1}{2} (\hat{G} \theta) - \hat{\theta} \right)^{\mu \alpha} \]
\[ + 4 \Lambda_{NC}^{-12} (\hat{G} g \theta)^{\alpha \beta} D_{\alpha} \varphi^{i} D_{\beta} \varphi^{j} g_{ij} + \text{h.o.} \).
\[ H^2 = \Lambda_{NC}^{-8} (G^\mu g_{\mu})^2 + 4 \Lambda_{NC}^{-12} (\bar{G}g) \tilde{G}^{\mu \nu} \bar{D}_\mu \varphi^i \bar{D}_\nu \varphi^j g_{ij} + 4 \Lambda_{NC}^{-8} (\bar{\theta}^{\mu \nu} F_{\mu \nu})^2 \\
- 4 \Lambda_{NC}^{-8} (\bar{G}g)(\bar{\theta}^{\mu \nu} F_{\mu \nu}) + 2 \Lambda_{NC}^{-12} (\bar{G}g) \tilde{G}^{\mu \nu} \tilde{G}^{\alpha \beta} F_{\mu \alpha} F_{\nu \beta} + \text{h.o.} \] (6.22)

The \( O(A\varphi) \) terms are found to be

\[ H^{ab} H_{ab} |_{A \varphi \varphi} = 8 \Lambda_{NC}^{-12} \bar{\theta}^{\rho \beta} F_{\beta \nu} G^{\mu \eta} D_\rho \varphi^i D_\eta \varphi_i, \]

\[ H^2_{A \varphi \varphi} = -8 \Lambda_{NC}^{-12} \Lambda_{NC}^4 \bar{\theta}^{\rho \beta} F_{\rho \nu} G^{\mu \nu} D_\rho \varphi^i D_\nu \varphi^j g_{ij}, \]

\[ (H^{ab} H_{ab} - \frac{1}{2} H^2) |_{A \varphi \varphi} = 8 \Lambda_{NC}^{-12} \left( \bar{\theta}^{\rho \beta} F_{\beta \nu} G^{\mu \eta} D_\rho \varphi^i D_\eta \varphi_i + \frac{1}{2} \bar{\theta}^{\alpha \beta} F_{\alpha \beta} G^{\mu \nu} D_\mu \varphi^i D_\nu \varphi^j D_\nu \varphi^i \right) \] (6.23)

noting that

\[ (\bar{G}g G g \theta)^{\mu \nu} = \frac{1}{2} (\bar{G}g) \bar{\theta}^{\mu \nu} - \bar{\theta}^{\mu \nu}. \] (6.24)

We can rewrite \( \bar{\theta}^{\mu \beta} \bar{\theta}^{\nu \alpha} F_{\nu \beta} F_{\mu \alpha} \) using the identities in Lemma [1] giving

\[ H^{ab} H_{ab} - \frac{1}{2} H^2 = \Lambda_{NC}^{-8} (\bar{G}g \bar{G} - \frac{1}{2} (\bar{G}g) \bar{G}) g^{\alpha \beta} g_{\alpha \beta} + 4 \Lambda_{NC}^{-12} \bar{G} G (\bar{G}g \bar{G} - \frac{1}{4} \bar{G}g \bar{G})^{\mu \nu} F_{\mu \alpha} F_{\nu \beta} \\
+ 4 \Lambda_{NC}^{-12} (\bar{G}g \bar{G} - \frac{1}{2} (\bar{G}g) \bar{G})^{\alpha \beta} g^{\mu \nu} F_{\mu \alpha} F_{\nu \beta} - 8 \Lambda_{NC}^{-8} \tilde{\theta}^{\alpha \beta} \tilde{\theta}^{\mu \nu} \tilde{G}^{\mu \nu} \tilde{G}^{\alpha \beta} + \text{h.o.} \]

Here \( \text{Pfaff}(F_{\mu \nu}) \) is a purely topological surface term i.e. a total derivative.

**Lemma 1** For anti-symmetric matrices \( F_{\mu \nu} \) and \( \bar{\theta}^{\mu \nu} \) defined as above, the following identities hold:

\[ \frac{1}{8} (F \bar{\theta})(F \bar{\theta}) - \frac{1}{4} (F \bar{\theta}F \bar{\theta}) = \text{Pfaff}(F_{\mu \nu}) \text{Pfaff}(\bar{\theta}^{\mu \nu}), \] (6.26a)

\[ \bar{G}^{\alpha \beta} (\bar{G}g \bar{G} - \frac{1}{4} (\bar{G}g) \bar{G})^{\mu \nu} F_{\mu \alpha} F_{\nu \beta} = \frac{1}{4} \Lambda_{NC}^4 \left( (\bar{\theta}^{\mu \nu} F_{\mu \nu})^2 - (\bar{\theta}^{\mu \nu} F_{\mu \nu})^2 \right), \] (6.26b)

\[ \bar{\theta}^{\mu \alpha} F_{\alpha \beta} G^{\beta \nu} - \bar{\theta}^{\alpha \beta} F_{\mu \alpha} g^{\mu \nu} \partial_\mu \partial_\nu = \frac{1}{2} (\bar{\theta}^{\alpha \beta} F_{\alpha \beta} g^{\mu \nu} - \bar{\theta}^{\alpha \beta} F_{\alpha \beta} G^{\mu \nu}) \partial_\mu \partial_\nu. \] (6.26c)

**Proof** See Appendix [1].

This gives

\[ \sqrt{-H^{ab} H_{ab} + \frac{1}{2} H^2} \]

\[ \sim \Lambda_{NC}^{-4} \left( 1 + \Lambda_{NC}^{-4} g^{\alpha \beta} D_\alpha \varphi^i D_\beta \varphi^j + \frac{1}{4} (\bar{\theta}^{\mu \nu} F_{\mu \nu})^2 - F_{\mu \alpha} \bar{\theta}^{\mu \alpha} + 2 \Lambda_{NC}^{-4} \text{Pfaff}(F_{\mu \nu}) \right) \]
\[-2\Lambda_{NC}^{-12} \left( \bar{\theta}^{\rho\beta} F_{\beta\nu} g^{\eta\mu} D_\eta \varphi^i D_\eta \varphi_i + \frac{1}{2} \bar{\theta}^{\alpha\beta} F_{\alpha\beta} g^{\mu\nu} D_\mu \varphi^i D_\nu \varphi_i \right) + \mathcal{O}(FFF) + \text{h.o.} \right)^{-1/2}

= \frac{1}{2} \left( \Lambda_{NC}^4 - \frac{1}{2} g^{\alpha\beta} D_\alpha \varphi^i D_\beta \varphi_i + \frac{1}{4} \Lambda_{NC}^4 (\bar{\theta}^{\mu\nu} F_{\mu\nu})^2 + \frac{1}{2} \Lambda_{NC}^4 F_{\mu\alpha} \bar{\theta}^{\mu\alpha} - \text{Pfaff}(F_{\mu\nu}) \right.

\left. + \Lambda_{NC}^8 \left( \bar{\theta}^{\rho\beta} F_{\beta\nu} g^{\eta\mu} D_\eta \varphi^i D_\eta \varphi_i - \frac{1}{4} \bar{\theta}^{\alpha\beta} F_{\alpha\beta} g^{\mu\nu} D_\mu \varphi^i D_\nu \varphi_i \right) + \mathcal{O}(FFF) + \text{h.o.} \right) \quad (6.27)

consistent with (6.12), from which we will take the \(\mathcal{O}(A^3)\) terms for simplicity. This yields

\[
\text{Tr} V(X) = -\frac{1}{4} \text{Tr} \frac{L^4}{\sqrt{\frac{1}{2} H^2 - H^{ab} H_{ab}}}
\]

\[
= -\frac{1}{8} \Lambda^4 \int \frac{d^4 x}{(2\pi)^2} \sqrt{g} \left( 1 - \frac{1}{2\Lambda_{NC}^4} g^{\alpha\beta} D_\alpha \varphi^i D_\beta \varphi_i + \frac{1}{4} (\bar{\theta}^{\mu\nu} F_{\mu\nu})^2 + \bar{\theta}^{\mu\nu} F_{\mu\nu} (F \bar{\theta} F \bar{\theta}) \right)
\]

\[
+ \Lambda_{NC}^{-12} \left( \bar{\theta}^{\rho\beta} F_{\beta\nu} g^{\eta\mu} D_\eta \varphi^i D_\eta \varphi_i - \frac{1}{4} \bar{\theta}^{\alpha\beta} F_{\alpha\beta} g^{\mu\nu} D_\mu \varphi^i D_\nu \varphi_i \right) + \text{h.o.} \right) \quad (6.28)

\]

dropping surface terms. This agrees precisely with the induced action (5.7). In particular, the \(SO(D)\) symmetry is indeed preserved.

Note that the effective matrix model contains much more information than what was put in. It predicts an infinite series of higher-order terms in \(F\) and \(\varphi\) proportional to \(\Lambda^4\). Since the above expression was uniquely determined by very simple arguments in Section 6.1, this represents a strong prediction of the matrix-model framework, based on the \(SO(D)\) symmetry and the scaling law (2.11) combined with a very basic one-line loop computation.

It is interesting that these induced “vacuum energy” terms are distinct from the bare matrix model (2.12). This means that the physics of vacuum energy is different from GR, which may be very relevant to the cosmological constant problem.

### 6.4 Curvature terms

Having understood the single-commutator matrix model terms, we now consider contributions which contain double commutators, more precisely those which can be written such that there are double commutators. It turns out that these lead to curvature terms from the gravity point of view, which usually diverge as \(\Lambda^2\). They correspond to gauge theory terms which have at least four derivatives such as \(\partial \varphi \partial \varphi \partial \varphi \partial \varphi\), of dimension \(\geq 6\). We first discuss the structure of such terms in the matrix model, and then compute their gauge theory content.

#### 6.4.1 On the structure of curvature terms

We first want to understand what kind of terms in the matrix model can be written in terms of two double commutators\(^{12}\). We consider only single-trace terms here, and restrict ourselves to the case of \(\mathcal{O}(X^6)\) and \(\mathcal{O}(X^{10})\) contributions. Eventually, a more systematic classification should be given.

\(^{12}\)It is clear from \(SO(D)\) invariance that there is no term which has only one double commutator \([X, [X, X]]\) except for \(D = 3\), which is not considered here.
While simple commutators $[X, X]$ correspond to constants $\partial^{\mu\nu}$ or first-order derivatives $F_{\mu\nu}$ and $\partial_{\mu}\phi^i$ (resp. $\partial^i\partial\phi$), double commutators such as $[X, [X, X]]$ or $[[X, X], [X, X]]$ correspond to terms with second-order derivatives. Note that all terms which contain $[V(X), V(X)]$ can be reduced to terms of the structure $V(X)[[X, X], [X, X]]$ (up to higher commutators), which using the Jacobi identity can be rewritten as

$$V(X)[[X, X], [X, X]] = -V(X)[[X, [X, X]], [X, X]] - V(X)[[X, X], [X, X]].$$

Here $V(X)$ is a potential term, i.e. involves only single commutators. Under the trace, these can be rewritten as $\text{Tr}[X, V(X)][X, [X, X]]$. It remains to characterize the most general term of the structure $\text{Tr}V_1(X)[X, V_2(X)][X, V_3(X)]$. Using again the Leibnitz rule, this can be reduced to

$$\text{Tr} L_{\text{curv}}[X] = \text{Tr} V(X)[X, [X, X]][X, [X, X]]$$

up to additional or higher-order commutators. Here the indices can be contracted in any possible way.

We note the following relation

$$\Theta^{de[a} \Theta^{bc'][g_{cc']} = [X^a, \Theta^{de} \Theta^{bc'} g_{cc']} - \Theta^{de[a} \Theta^{bc']g_{cc'}] + \text{h.o.},$$

which implies

$$[X_{a}, H_{ab}] = \frac{1}{2} \left( [\Box X_{c}, [X_{b}, X_{c}]] + \frac{1}{2} [X_{b}, H] \right)$$

$$\sim \Box X_{c} [X_{b}, X_{c}] + \frac{1}{4} [X_{b}, H] + \text{h.o.}$$

### $O(X^6)$ curvature terms.

It is easy to see\textsuperscript{[28]} that there are only two independent terms of order $O(X^6)$, given by

$$S_{6,A} = \text{Tr} \Box X^a \Box X_a,$$

$$S_{6,B} = \text{Tr} X^c, [X^a, X^b]][X_{c}, [X_{a}, X_{b}]].$$

Notice that there is no potential term at order $O(X^6)$. From the geometrical point of view they contain curvature contributions. Although they are not induced in the effective action, they will occur as part of the $O(X^{10})$ terms.

### $O(X^{10})$ curvature terms.

At order $X^{10}$, we have to characterize the most general term of structure $\text{Tr} J J[X, J][X, J]$. There are several cases:

- Assume that the indices of the $X$ are contracted as in $\text{Tr} J J[X^a, J][X_a, J]$. Now consider the possible contractions of the indices of the $J$'s. The indices of the $J$'s can form two disjoint loops, or a single loop.

Consider first the case of two $J$ loops. If the indices of the internal $J$ (resp. the external $J$) are contracted among themselves, this gives $\text{Tr} H[X^a, J][X_a, J]$ i.e. $\text{Tr} HS_{6,\text{curv}}$.

Otherwise, each $J$ loop can be written using (6.31) as $[X^a, H]$, so that the action is $\text{Tr} [X^a, H][X_a, H]$.

Now consider the case of a single $J$ loop. Using again (6.31), these can be reduced to the form $\text{Tr} J[X^a, J] J[X_a, J]$ and $\text{Tr} [X^a, J^2][X_a, J^2]$ up to higher-order terms. These turn out to be independent.
• Now assume that the indices of the $X$ are not contracted among themselves. Then consider the two triple commutators $[X, \Theta]$, which altogether have 6 indices. Since there are only two external $\Theta$, at least two of the indices of the two double commutators must be contracted among themselves. Therefore either two indices of the same $[X, [X, X]]$ are contracted among themselves which gives $\square X^a$, or otherwise they must be contracted as in $[X^a, \Theta_{cd}] [X_b, \Theta_{ae}] \Theta \Theta$. The first case will be discussed below. In the second case, the double commutator has 4 remaining indices $b, c, d, e$. It is antisymmetric in $(cd)$, and we can assume that it is symmetric in $(be)$ because the anti-symmetric component in $(be)$ reduces to $[X^a, \Theta_{cd}] [X_{ae}, \Theta_{cd}] \Theta \Theta$ using the Jacobi identity, which has been covered above. Hence there is no (independent) way to contract them with the 4 free indices of $\Theta \Theta$. Therefore two of these must be contracted with $g^{ab}$. This leads to $[X^a, \Theta_{cd}] [X_b, J^c_d] H^{bd} \sim [X^a, \Theta_{cd}] [X_{ce}, \Theta_{cd}] H^{bd}$ or to $[X^a, J^c_d] [X_b, \Theta_{ae}] H^{ce}$ (which is the same) or to $[X^a, \Theta_{cd}] \square X_a H^{cd} \sim 0$ (apart from the case $S_6 H$ which has been covered before). Hence the only new term is

\[ 2 [X_a, \Theta_{cd}] H^{db} [X_b, \Theta^{ac}] = - [X_d, \Theta_{ae}] H^{db} [X_b, \Theta^{ac}]. \tag{6.34} \]

It remains to classify the terms of the form $\square X^a [X_b, \Theta_{cd}] \Theta \Theta$. Using the Jacobi identity, we can assume that the index $a$ is contracted either with an external $\Theta$, or with $\Theta_{cd}$. In the first case we get $\Theta_{ae} \square X^a [X_b, \Theta_{cd}] \Theta$, which gives either $H_{ac} \square X^a \square X^c$ or

\[ \Theta_{ae} \square X^a [X_b, \Theta_{cd}] \Theta \sim \Theta_{ae} \square X^a [X_b, H^{ef}] = H_{fa} \square X^a [X_b, \Theta^{fb}] \]

\[ = \Theta_{ae} \square X^a (\square X_c [X^e, X^c] + \frac{1}{4} [X^e, H]) - iH_{af} \square X^a \square X^f \]

\[ = \frac{1}{4} \Theta_{ae} \square X^a [X^e, H]. \tag{6.35} \]

In the second case, we can assume (using the Jacobi identity) that the term has the form $\square X^a [X_b, \Theta_{ad}] \Theta \Theta$. This leads either to $\square X^a \square X_a H \in S_6 H$, or using $(6.32)$ to

\[ \square X^a [X_b, \Theta_{ad}] H^{bd} = - \square X^a [X_d, \Theta_{ba}] H^{bd} \]

\[ = - \square X^a [X_d, \Theta_{ba} H^{bd}] + \square X^a \Theta_{ba} [X_d, H^{bd}] \]

\[ = - \square X^a [X_d, (J^3)^d] - \frac{1}{4} \square X^a \Theta_{ba} [X^b, H] - \square X^a H_{ac} \square X^e. \tag{6.36} \]

Hence, we have the following complete list of $O(X^{10})$ curvature terms:

\[ S_{10, \text{curv}} = \text{Tr} \left( HS_6 + c_1 [X^a, H] [X_a, H] + c_2 \Theta_{ae} \square X^a [X^e, H] \right. \]

\[ + c_3 \text{tr} J [X^a, J] [X_a, J] + c_4 [X^a, H^{cd}] [X_a, H_{cd}] \]

\[ + c_5 [X_d, \Theta_{ac}] H^{bd} [X_b, \Theta^{ac}] + c_6 \square X^a [X_d, (J^3)^d] \right). \tag{6.37} \]

which turn out to be independent. The last four can be replaced by

\[ \text{Tr} \left( c_3 J [X^a, J] [X_a, J] + c_4 H^{cd} H_{cd} + c_5 \Theta_{ae} \square g \Theta^{ac} + c_6 \square X^a \square g X_a \right). \tag{6.38} \]

There are also boundary terms which vanish under the trace, which will be supplemented below.
6.4.2 Expansion of $\mathcal{O}(X^6)$ curvature terms

As a warm-up, consider the two independent $\mathcal{O}(X^6)$ terms to quadratic order

\[ S_{6,A} = \text{Tr} \Box X^a \Box X_a \]

\[ \sim \int \frac{d^4x}{(2\pi)^2} \sqrt{G} \left( \frac{1}{2} \Lambda_{NC}^{-4} F_{\mu\nu} F_{\alpha\beta} G^{\alpha\mu} G^{\beta\nu} + \Lambda_{NC}^{4} \Box \phi_i \Box \phi_i + \text{h.o.} \right), \]

\[ S_{6,B} = \text{Tr} [X^c, [X^a, X^b]] [X_c, [X_a, X_b]] \]

\[ \sim \int \frac{d^4x}{(2\pi)^2} \sqrt{G} \left( \Lambda_{NC}^{4} F_{\mu\nu} F_{\alpha\beta} G^{\alpha\mu} G^{\beta\nu} + 2 \Lambda_{NC}^{4} \Box \phi_i \Box \phi_i + \text{h.o.} \right). \] (6.39)

Notice that the “intrinsic combination”

\[ \mathcal{L}_{6, \text{cubic}} [X] = \Box X^a \Box X_a - \frac{1}{2} [X^c, [X^a, X^b]] [X_c, [X_a, X_b]] \] (6.40)

has no quadratic contributions, and turns out to give a tensorial term involving the Riemann tensor [28]. To obtain the above form, we need the following identities

**Lemma 2** For an Abelian field strength tensor $F_{\mu\nu}$ and $\tilde{\theta}^{\mu\nu}$ etc. defined as before, the following identities hold:

\[ g^{\mu\nu} \partial_\mu F_{\nu\gamma} \tilde{G}^{\rho\beta} \partial_\rho F_{\beta\mu} \tilde{G}^{\gamma\mu'} = -\frac{1}{2} (g^{\mu\nu} \partial_\nu \partial_\mu) F_{\rho\gamma} F_{\beta\mu'} \tilde{G}^{\rho\beta} \tilde{G}^{\gamma\mu'} \quad + \text{h.o.} \] (6.41a)

\[ \tilde{G}^{\mu\nu} \partial_\mu F_{\nu\gamma} \tilde{G}^{\rho\beta} \partial_\rho F_{\beta\mu} \gamma^{\mu'} = \frac{1}{2} (g^{\mu\nu} \partial_\nu \partial_\mu) F_{\rho\gamma} F_{\beta\mu'} \tilde{G}^{\rho\beta} \tilde{G}^{\gamma\mu'} - (\tilde{G}^{\mu\nu} \partial_\nu \partial_\mu) F_{\rho\gamma} F_{\beta\mu'} \gamma^{\rho\beta} \tilde{G}^{\gamma\mu'} + \text{h.o.} \] (6.41b)

where $\text{h.o.}$ denotes surface terms. The proof is given in Appendix C.2 based on the Bianchi identity.

For completeness, we give here the full expansion of these terms around $R_\theta^4$. Consider first

\[ \Box \phi^k = [X^\mu, [X_\mu, \phi^k]] + [\phi^i, [\phi_i, \phi^k]] \]

\[ = -\Lambda_{NC}^{-4} \tilde{G}^{\mu\nu} D_\mu D_\nu \phi^k + [\phi^i, [\phi_i, \phi^k]], \]

and similarly

\[ \Box X^\nu = [X_\mu, [X^\mu, X^\nu]] + [\phi_i, [\phi^i, X^\nu]] \]

\[ = \Lambda_{NC}^{-4} \tilde{G}^{\mu\nu} D_\mu F_{\alpha\beta} - i \tilde{G}^{\mu\nu} [\phi_i, D_\beta \phi^i]. \] (6.43)

This gives

\[ S_{6,A} = \text{Tr} \Box X^a \Box X_a \]

\[ = \int \frac{d^4x}{(2\pi)^2} \sqrt{G} \left( \Lambda_{NC}^{-8} \tilde{G}^{\beta\gamma} \tilde{G}^{\alpha\rho} \tilde{G}^{\alpha'\rho'} D_\rho F_{\alpha\beta} D_\rho F_{\alpha'\beta'} + \Lambda_{NC}^{-4} \tilde{G}^{\mu\nu} D_\mu D_\nu \phi^k \tilde{G}^{\mu'\nu'} D_{\mu'} D_{\nu'} \phi_k \\
- 2i \Lambda_{NC}^{-4} \tilde{G}^{\beta\gamma} \tilde{G}^{\alpha\rho} D_\rho F_{\alpha\beta} [\phi_i, D_{\beta'} \phi^j] - 2 \tilde{G}^{\mu\nu} D_\mu D_\nu \phi^k [\phi^i, [\phi_i, \phi_k]] \right) \]

\[ = \left( \tilde{G}^{\beta\gamma} [\phi_i, D_{\beta'} \phi^j] [\phi_j, D_{\beta'} \phi^j] + [\phi^i, [\phi_i, \phi^k]] [\phi^j, [\phi_j, \phi_k]] \right), \] (6.44)
and a similar computation leads to

\[ S_{6,B} = \text{Tr}[X^c, [X^a, X^b]] [X_c, [X_a, X_b]] \]

\[ = \int \frac{d^4x}{(2\pi)^2} \sqrt{G} \left( \Lambda_{NC}^{-8} G^{\alpha \rho} G^{\beta \nu} G^{\alpha \varsigma} D_{\sigma} F_{\mu \nu} D_{\gamma} F_{\alpha \beta} + 2 \Lambda_{NC}^{-4} G^{\alpha \lambda} G^{\beta \phi} D_{\sigma} D_{\tau} D_{\sigma} \phi^{i} D_{\tau} \phi^{i} \right) \]

(6.45)

It is remarkable that this contains no explicit \( \hat{\theta}^{\mu \nu} \).

### 6.4.3 Expansion of \( \mathcal{O}(X^{10}) \) terms

We need to systematically compute the field theory content of all the above \( \mathcal{O}(X^{10}) \) terms, which start with \( \mathcal{O}(p^4(\varphi, A)^2) \). The results are as follows (see Appendix C.3):

\[ \mathcal{L}_{10,A} = [X_f, H^{a b}][X_f, H_{a b}] \]

\[ = \Lambda_{NC}^{-8} \left( \bar{H} \left( \frac{1}{2} F_{\mu \nu} D_{\rho} F_{\rho \sigma} G^{\mu \nu} + \Lambda_{NC}^{4} \bar{\varphi}^{b} \bar{\varphi}^{k} \right) \right. \]

\[ - \left. \frac{1}{2} \left( 3 (\hat{\theta} F) \bar{\omega} (\hat{\theta} F) - (\hat{\theta} F) \bar{\omega} (\hat{\theta} F) \right) + 2 \Lambda_{NC}^{4} \bar{\varphi}^{b} \bar{\varphi}^{k} \right) + \text{h.o.,} \]

\[ \mathcal{L}_{10,B} = - (H_{c d} - \frac{1}{2} H g^{c d}) [X_c, \Theta^{a b}][X_d, \Theta_{a b}] \]

\[ = \Lambda_{NC}^{-8} \left( \bar{G}^{\beta \rho} \bar{G}^{\gamma \tau} F_{\rho \mu} g F_{\sigma \tau} + 2 \Lambda_{NC}^{4} \bar{\varphi}^{j} \bar{\varphi}^{i} \right) + \text{h.o.,} \]

\[ \mathcal{L}_{10,C} = \Box^{a} \Box^{g} X_{a} \]

\[ = \Lambda_{NC}^{-8} \left( \Box^{g} F_{\rho \gamma} F_{\beta \mu} G^{\rho \beta} G^{\gamma \mu} + \frac{1}{2} (\hat{\theta} F) \bar{\omega} (\hat{\theta} F) \right. \]

\[ - \left. \frac{1}{4} (\hat{\theta} F) \bar{\omega} (\hat{\theta} F) + \Lambda_{NC}^{4} \bar{\varphi}^{j} \bar{\varphi}^{i} \right) + \text{h.o.,} \]

\[ \mathcal{L}_{10,D} = \text{tr} J [X^a, J] J [X_a, J] = - \frac{1}{2} \Lambda_{NC}^{-8} (\hat{\theta} F) \bar{\omega} (\hat{\theta} F) + \text{h.o.,} \]

\[ \mathcal{L}_{10,E} = [X_f, H][X^f, H] = - 4 \Lambda_{NC}^{-8} (\hat{\theta} F) \bar{\omega} (\hat{\theta} F) + \text{h.o.,} \]

\[ \mathcal{L}_{10,F} = \Box^{a} \Theta_{a b} [X^b, H] = - \Lambda_{NC}^{-8} (\hat{\theta} F) \bar{\omega} (\hat{\theta} F) + \text{h.o.} \]

(6.46)

Recall that \( \Box \) and \( \Box^{g} \) have different powers of \( \Lambda_{NC} \), see (6.18). We also note the following contributions from the \( \mathcal{O}(X^{10}) \) “boundary terms”:

\[ \Box H^2 = - 4 \Lambda_{NC}^{-8} (\bar{G} g) \bar{\omega} (\hat{\theta}^{\mu \nu} F_{\mu \nu}) + \Box \text{(h.o.),} \]

\[ \Box (H^{a b} H_{a b} - \frac{1}{2} H^2) = 4 \Lambda_{NC}^{-8} \Box (\hat{\theta}^{\mu \nu} F_{\mu \nu}) + \Box \text{(h.o.),} \]

\[ \Box^{g} H = 2 \Lambda_{NC}^{-4} \Box^{g} (\hat{\theta}^{\mu \nu} F_{\mu \nu}) + \Box^{g} \text{(h.o.)}. \]

(6.47)

Remarkably, the following terms have no quadratic contributions:

\[ \mathcal{L}_{10,\text{cubic}[X]} = q_1 \left( \frac{1}{2} [X_f, H^{a b}][X^f, H_{a b}] + (H_{c d} - \frac{1}{2} H g^{c d}) [X_c, \Theta^{a b}][X_d, \Theta_{a b}] + \Box^{a} \Box^{g} X^{a} \right) \]

\[ + q_2 \left( 2 \text{tr} J [X^a, J] J [X_a, J] - \Box^{a} \Theta_{a b} [X^b, H] \right) \]

\[ + q_3 \left( [X_f, H][X^f, H] - 4 \Box^{a} \Theta_{a b} [X^b, H] \right). \]

(6.48)
It would be desirable to include these “cubic” terms into the above computations, which is however beyond the scope of this paper.

6.4.4 \( \mathcal{O}(X^{14}) \) term.

A complete analysis for the \( \mathcal{O}(X^{14}) \) curvature terms is beyond the scope of this paper. We only consider the following term which is clearly generated in \([5.11]\):

\[
\square_g X^a \square_g X^a =
\]

\[
= \Lambda_{NC}^{-20} \left( G^\alpha\beta' g_{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} F_{\beta'\alpha'} + \left( (Gg)^2 - 1 \right) \tilde{G}^\alpha\beta' \partial_{\alpha'} \partial_{\beta'} F_{\beta'\alpha'} \right) - 2 \tilde{G}^\alpha\beta' g_{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} F_{\beta'\alpha'} + \frac{\Lambda^4_{NC}}{4} (Gg\tilde{G})^{\mu\nu} \partial_\mu (\bar{\theta} F) \partial_\nu (\bar{\theta} F)
\]

\[
+ \Lambda^4_{NC} \partial_\nu (\bar{\theta} F) (g_{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} \tilde{F} + \tilde{G}^\beta_{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} F_{\beta'\alpha'}) + \square_g \phi^i \square_g \phi_i
\]  

\[\text{(6.49)}\]

using

\[
\square_g X^\mu = [X, [(H^{cb} - \frac{1}{2} H g^{cb}) [X_b, X^\mu]] = [X, [(H^{\gamma\beta} - \frac{1}{2} H g^{\gamma\beta}) [X_\beta, X^\mu]] + \mathcal{O}(\varphi, A)^2
\]

\[
= \Lambda_{NC}^{-8} g^{\beta\mu} \partial_\nu F_{\beta\nu} \tilde{F}^{\mu\nu} + \Lambda_{NC}^{-8} \tilde{g}^{\mu\nu} \partial_\nu F_{\mu\alpha} + \frac{1}{2} \Lambda_{NC}^{-8} \tilde{g}^{\mu\nu} \partial_\nu (\tilde{\varphi}^{\alpha\beta} F_{\alpha\beta}) + \text{h.o.},
\]  

\[\text{(6.50)}\]

and \( GgGgG = (\frac{1}{2}(Gg)^2 - 1)G - \frac{1}{4}(Gg)g \). Note that for \( G = g \), the YM-type terms cancel, as in \([5.11]\).

7 Effective matrix model including curvature contributions

We now look for a generalization of the matrix model action \([6.7]\) involving curvature terms as above, such that the dimension 6 operators in \([5.8]\) proportional to \( \Lambda^2 = L^2 \Lambda_{NC}^4 \) are reproduced. In order to guess the appropriate form, observe first that the curvature terms vanish on a flat background \( R^4_\varphi \). Assuming analyticity in the \( X^a \) near \( R^4_\varphi \), this suggests the following form for the effective matrix model:

\[
\Gamma_L [X] = -\frac{1}{4} \text{Tr} \left( \frac{L^4}{\sqrt{-\text{tr} J^4 + \frac{1}{2} (\text{tr} J^2)^2} + \frac{1}{L^2} \mathcal{L}_{10, \text{curv}} [X] + \ldots} \right).
\]  

\[\text{(7.1)}\]

This should be expanded on \( R^4_\varphi \) up to the required order in \( A_\mu, \varphi^i \) and compared with the induced gauge theory action. Recalling \([6.3]\) we denote

\[
4 \Lambda_{NC}^{-8} [X] := -\text{tr} J^4 + \frac{1}{2} (\text{tr} J^2)^2 \sim 4 \Lambda_{NC}^{-8} (x) = 4 \Lambda_{NC}^{-8} - 4 \Lambda_{NC}^{-8} \tilde{g}_{\mu\nu} F_{\mu\nu} + \ldots
\]  

\[\text{(7.2)}\]

Then

\[
\Gamma_L [X] = -\frac{1}{4} \text{Tr} L^4 \left( 4 \Lambda_{NC}^{-8} [X] + \frac{1}{L^2} \mathcal{L}_{10, \text{curv}} + \ldots \right)^{-1/2}
\]

\[
= -\frac{1}{8} \text{Tr} \left( L^4 L^4_{NC} [X] - \frac{1}{8} L^2 L^2_{NC} [X] \mathcal{L}_{10, \text{curv}} [X] + \ldots \right)
\]

\[
= -\frac{1}{8} \Lambda_{NC}^4 \int \text{d}^4 x \sqrt{g} \left( L^4 L^4_{NC} (x) - \frac{1}{8} L^2 L^2_{NC} (x) \mathcal{L}_{10, \text{curv}} (x) + \ldots \right).
\]  

\[\text{(7.3)}\]
Observe that the scaling with $\Lambda^2 = L^2\Lambda_{NC}^4$, necessarily corresponds to a term $\mathcal{L}_{10,\text{curv}}[X]$ of order $X^{10}$. As shown above, the most general order 10 term in the matrix model has the form

$$
\begin{align*}
\mathcal{L}_{10,\text{curv}} &= c_1 [X_f, H^{ab}][X^f, H_{ab}] + c_2 \Box X^a \Box g^a + c_3 [X^f, H][X_f, H] \\
&+ \mathcal{L}_{10,\text{boundary}} + \mathcal{L}_{10,\text{cubic}} + d H \mathcal{L}_{6,\text{A}} + H \mathcal{L}_{6,\text{cubic}},
\end{align*}
$$

(7.4)

where $\mathcal{L}_{10,\text{cubic}}$ are given in (6.43), and $\mathcal{L}_{10,\text{boundary}}$ is a combination of the total derivative terms listed in (6.47). To determine the coefficients, we use the expansion of this action to quadratic order in the fields as given in section 6.4.3 and match it with the induced gauge theory action proportional to $\Lambda^2$. Comparing with (7.8), we indeed obtain terms with the desired structure. Note that the total derivatives in $\mathcal{L}_{10,\text{boundary}}$ may lead to non-trivial quadratic contributions, due to the multiplication with $\Lambda_{NC}^{-12}(X)$. Comparing their contributions (6.47) with the induced action, we see that $\Box (H^{ab} H_{ab} - \frac{3}{2} H^2)$ leads to $(\theta F) \Box (\theta F)$ which does indeed occur, while the other boundary terms are not induced. Therefore we set

$$
\mathcal{L}_{10,\text{boundary}} = b \Box \Lambda_{NC}^{-8}[X] = -4b \Lambda_{NC}^{-8} \Box (\bar{\theta} \mu \nu F_{\mu \nu}) + \text{h.o.},
$$

(7.5)

so that

$$
\begin{align*}
\text{Tr} \Lambda_{NC}^{12}(X) \mathcal{L}_{10,\text{boundary}}(X) &= -4b \text{Tr} (\Lambda_{NC}^{12} + \frac{3}{2} \Lambda_{NC}^{16} (\bar{\theta} \mu \nu F_{\mu \nu}) + \ldots) \Lambda_{NC}^{-8} (\bar{\theta} \mu \nu F_{\mu \nu}) + \text{h.o.} \\
&= -6b \Lambda_{NC}^{8} \text{Tr} (\bar{\theta} F) \Box (\bar{\theta} F) + \text{h.o.}
\end{align*}
$$

(7.6)

Then the quadratic gauge theory action resulting from (7.3) is given by

$$
\Gamma_L[X] = \frac{1}{64} \Lambda^2 \frac{A_{NC}^4}{(2\pi)^2} \int d^4 x \sqrt{g} \left( (\bar{\theta} F) \Box (\bar{\theta} F) \left( \frac{1}{2} c_1 - \frac{1}{4} c_2 - 6b \right) + \Lambda_{NC}^{4} \Box \varphi^k \Box g \varphi_k (2c_1 + c_2) \\
+ c_2 g^{\rho \sigma} G_{\rho \sigma} F_{\mu \nu} \Box g F_{\mu \nu} + \left( \frac{3}{2} c_1 + \frac{1}{2} c_2 - 4c_3 \right) (\bar{\theta} F) \Box (\bar{\theta} F) \\
+ \left( \frac{c_1}{2} + d \right) \tilde{H} \left( \frac{1}{2} F_{\mu \nu} \Box F_{\rho \sigma} \tilde{G}^{\mu \sigma} \tilde{G}^{\nu \tau} + \Lambda_{NC}^{4} \Box \varphi^k \Box \varphi_k \right) \right),
$$

(7.7)

which must coincide with $\Gamma_{\Lambda^2}(A, \varphi^2, p^4)$ given in (5.8). This yields

$$
\begin{align*}
\frac{1}{2} c_1 - \frac{1}{4} c_2 - 6b &= \frac{1}{3} \text{tr} I, \\
2c_1 + c_2 &= -8 \text{tr} I, \\
c_2 &= -\frac{11}{3} \text{tr} I, \\
\frac{3}{2} c_1 + \frac{1}{2} c_2 - 4c_3 &= 0, \\
d &= -\frac{c_1}{2},
\end{align*}
$$

(7.8)

which has the particular solution

$$
\begin{align*}
c_1 &= \frac{13}{6} \text{tr} I, \\
c_2 &= -\frac{11}{3} \text{tr} I, \\
b &= -\frac{1}{12} \text{tr} I, \\
c_3 &= \frac{17}{48} \text{tr} I, \\
d &= \frac{13}{12} \text{tr} I,
\end{align*}
$$

(7.9)

plus an undetermined contribution of $\mathcal{L}_{10,\text{cubic}}$ and $H \mathcal{L}_{6,\text{cubic}}$.

We emphasize again the non-trivial analytic structure of the effective matrix model (7.1), which is not adequately described by a power series in $X^a$. This is to be expected, since the bare matrix model (2.12) describes a very rich spectrum of backgrounds with various dimensions.

We have thus successfully reproduced the induced gauge theory action (5.8) proportional to $\Lambda^2$, using the above generalized matrix model (7.1). Even though the quadratic terms are insufficient to fully determine the action and there are less non-trivial checks compared with the potential term, it is non-trivial that the correct terms are indeed reproduced. It would of course be desirable to develop more efficient methods to compute the effective generalized matrix model. A more detailed comparison and confirmation should eventually be given, allowing to determine also the contributions due to $\mathcal{L}_{\text{cubic}}$. These are in fact the most interesting terms form the point of view of emergent gravity, as we recall below.
**Geometric action.** Finally, the geometrical meaning of the action (7.11) is obtained by expanding it not around $\mathbb{R}^{4}$ but around a generic 4-dimensional NC brane $M^{4} \subset \mathbb{R}^{D}$ generated by $X^{\mu} = \tilde{X}^{\mu} + A^{\mu}$, cf. [3, 4]. Using (6.3) and the semi-classical relation

$$\text{Tr}L(X) \sim \int \frac{d^{4}x}{(2\pi)^{2}} \sqrt{G} \Lambda_{NC}(x)^{4} L(x), \quad (7.10)$$

which holds on generic 4-dimensional branes $M^{4}$, we can write this action in the semi-classical limit as

$$\Gamma_{L}[X] \sim -\frac{1}{4} \frac{1}{(2\pi)^{2}} \int d^{4}x \sqrt{G} \Lambda_{NC}^{4}(x) \left(4 \Lambda_{NC}^{-8}(x) + \frac{1}{L^{2}} L_{10, \text{curv}} + \ldots \right)^{-1/2}$$

$$= -\frac{1}{8} \frac{1}{(2\pi)^{2}} \int d^{4}x \sqrt{G} \left(\Lambda^{4}(x) - \frac{1}{8} \Lambda^{2}(x) \Lambda_{NC}^{12}(x) L_{10, \text{curv}} + \ldots \right). \quad (7.11)$$

Here we define the effective cutoff on a curved brane as

$$\Lambda(x) := L \Lambda_{NC}^{4}(x), \quad (7.12)$$

in analogy to Eqn. (2.10). Now we recall the results of [28, 29] for the special case $G_{\mu\nu} = g_{\mu\nu}$, which arises for (anti-)selfdual $\theta^{\mu\nu}$ (resp. an almost-Kähler structure) on $M^{4}$. Generalized to the present case, we note that $L_{6, \text{cubic}}$ and the first term in $L_{10, \text{cubic}}$ (6.48)

$$L_{10, \text{cubic}} \geq \frac{1}{2} \left[ X_{f}, H^{ab} \right] \left[ X_{f}, H_{ab} \right] + (H^{cd} - \frac{1}{2} H g^{cd}) \left[ X_{c}, \Theta^{ab} \right] \left[ X_{d}, \Theta_{ab} \right] + \Box X^{a} \Box g X^{a}$$

$$\sim \left( -\frac{1}{2} H^{ab} \Box H_{ab} - \Box X^{a} \Box g X^{a} \right) + (-\Theta^{ab} \Box g \Theta_{ab} + 2 \Box X^{a} \Box g X^{a}) \quad (7.13)$$

(upto boundary terms) incorporate the Riemannian curvature. Indeed the first term $\frac{1}{2} H^{ab} \Box H_{ab} + \Box X^{a} \Box g X^{a}$ has been shown in [28, 29] to give essentially the curvature scalar $R$, up to contributions from the dilaton [13] $\Lambda_{NC}^{-4} := \Lambda_{NC}^{4}(x)$ and boundary terms. Similarly,

$$-\Theta^{ab} \Box g \Theta_{ab} + 2 \Box X^{a} \Box g X^{a} \overset{G}{\approx} 2 \Lambda_{NC}^{-4} L_{6, \text{cubic}}$$

$$\sim \Lambda_{NC}^{-8} e^{-2\sigma} \theta^{\mu\rho} \theta^{\rho\sigma} R_{\mu\rho\sigma} - 4 \Lambda_{NC}^{4} e^{-\sigma} R \quad (+\text{dilaton} + \partial())$$

has only cubic or higher-order contributions in the gauge theory point of view. Extrapolating these results to the present case (which involves different scalar factors), we can anticipate

$$\text{Tr} \Lambda_{NC}^{12}[X] H L_{6, \text{cubic}} \sim -\int \frac{d^{4}x}{(2\pi)^{2}} \sqrt{g} \Lambda(X)^{2} \left( \frac{1}{2} \Lambda_{NC}^{-8} e^{-\sigma} \theta^{\mu\rho} \theta^{\rho\sigma} R_{\mu\rho\sigma} - 2 c' \partial^{\mu} \sigma \partial_{\mu} \sigma \right),$$

$$\text{Tr} \Lambda_{NC}^{12}[X] L_{10, \text{cubic}} \sim -\int \frac{d^{4}x}{(2\pi)^{2}} \sqrt{g} \Lambda(X)^{2} \left( R + \Lambda_{NC}^{-8} e^{-\sigma} \theta^{\mu\rho} \theta^{\rho\sigma} R_{\mu\rho\sigma} - 4 R \right) + c' \partial^{\mu} \sigma \partial_{\mu} \sigma, \quad (7.14)$$

for $G_{\mu\nu} = g_{\mu\nu}$, where the overall coefficients and the scalar field contributions $c, c'$ are not determined here. These are clearly the analogs of the classical Seeley-de Witt coefficients corresponding to induced gravity. However, there are also terms such as $	ext{Tr}L_{10,C} \sim \frac{1}{(2\pi)^{2}} \int d^{4}x \sqrt{g} X^{a} \Box g X^{a}$

---

Footnote 13: The name dilaton is chosen because it couples to the curvature. It has a specific geometrical meaning in terms of $\Lambda_{NC}(x)$ here.

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which correspond to extrinsic curvature on $\mathcal{M} \subset \mathbb{R}^D$. The structure is consistent with the semi-classical results obtained in [21], and a more precise result should be given once the remaining coefficients for $\mathcal{L}_{\text{cubic}}$ are known.

To summarize, we found that the induced action due to fermions induces not only intrinsic curvature terms on $\mathcal{M}^4$ including a generalized Einstein-Hilbert action with a dilaton, but also terms which correspond to extrinsic curvature of the embedding $\mathcal{M}^4 \subset \mathbb{R}^D$. These might be cancelled by the contributions from the bosonic fields, but some extrinsic terms are expected to survive, in particular because the $\mathcal{N} = 4$ SUSY must be broken at low energies. Such terms would definitely go beyond general relativity, preferring flat vacuum geometries. This might be very interesting e.g. in the context of cosmology, however their physical relevance remains to be clarified.

8 Remarks and conclusion

In this paper we have computed the effective action in the Yang-Mills type matrix model induced upon integrating out the fermions, using a heat-kernel expansion. There are two important new results: First, we point out that one should not simply take the naive UV limit, in contrast to (most of the) previous work on the heat-kernel expansion on NC spaces. Rather, a specific IR condition (1.1) on the external momentum scales and the cutoff should be imposed, cf. [11]. Only then a non-trivial and robust induced effective action is obtained. The reason is that the induced action is entirely due to non-commutativity (for the $U(1)$ sector under consideration here) resp. UV/IR mixing, which becomes ill-defined in the strict UV limit. Such a finite cutoff should be realized notably in maximally supersymmetric models (i.e. the IKKT model) and close relatives, via some SUSY breaking scale. This result makes perfect sense from the emergent gravity point of view, which is relevant for the IR.

The second important result is that the effective action can be written as a generalized matrix model, with manifest $SO(D)$ symmetry. We obtain an explicit form for this matrix model, which is conjectured to capture the full contribution from simple-commutator (potential) terms, and the leading contribution for the double commutator (curvature) terms. In particular, the geometrical insights gained from the geometrical point of view of emergent gravity allow to correctly predict the potential part of this effective action, thus predicting a series of highly non-trivial loop computations. This is very remarkable from the gauge theory point of view. It suggests that the effective non-Abelian gauge theory action induced on coinciding branes can also be computed efficiently by taking advantage of the full $SO(D)$ symmetry in the matrix model. Thus generalized effective matrix models should provide a powerful new tool in the context of matrix models, gauge theory and gravity.

Finally, the effective $SO(D)$ symmetric matrix model can be translated into the geometrical action for emergent gravity. This is an essential step towards a physical understanding of the emergent gravity which arises on generic 4D branes in this model. Supplemented with the contributions induced by the bosonic modes, a systematic analysis of the resulting physics should become possible.

The results presented here clearly are incomplete, and a much more systematic study of the induced effective matrix model should be carried out. While the potential terms have passed highly non-trivial checks, the curvature terms were only marginally resp. incompletely determined, and should be determined and verified in a more complete computation. Many aspects require a more detailed understanding. For example, the analysis of the potential [6,7]
in the effective matrix model was based on the 4D relation (6.3). This relation was derived only in the semi-classical limit, and is not expected to hold in the fully NC case. A related point is that the matrix model also admits backgrounds with different dimensions, around which the effective action would look very different. Therefore, the effective matrix model (7.1) should be seen as “leading part” of some more complete, “universal” effective action which holds for any background. These are only some issues in an interesting new line of research.

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Appendix A: Perturbative expansion of the heat kernel

Consider the one-loop effective action in terms of the gauge field $A$ (cf [11], App. B):

$$
\Gamma = \frac{1}{2} \text{Tr} \left( \log \frac{1}{2} \Delta_A - \log \frac{1}{2} \Delta_0 \right) \equiv -\frac{1}{2} \text{Tr} \int_0^\infty \frac{d\alpha}{\alpha} \left( e^{-\alpha \frac{1}{2} \Delta_A} - e^{-\alpha \frac{1}{2} \Delta_0} \right) e^{-\frac{1}{2} \alpha L^2}
$$

where the small $\alpha$ divergence is regularized as in (2.9) using a UV cutoff $L$. To obtain the expansion in $\alpha$ we use the Duhamel formula (cf. [31]):

$$
\left( e^{-\alpha H} - e^{-\alpha H_0} \right) =
- \int_0^\alpha dt_1 e^{-t_1 H_0} V e^{-(\alpha-t_1)H_0} + \int_0^\alpha dt_1 \int_0^{t_1} dt_2 e^{-t_2 H_0} V e^{-(t_1-t_2)H_0} V e^{-(\alpha-t_1)H_0}
- \int_0^\alpha dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 e^{-t_3 H_0} V e^{-(t_2-t_3)H_0} V e^{-(t_1-t_2)H_0} V e^{-(\alpha-t_1)H_0}
+ \int_0^\alpha dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 e^{-t_4 H_0} V e^{-(t_3-t_4)H_0} V e^{-(t_2-t_3)H_0} V e^{-(t_1-t_2)H_0} V e^{-(\alpha-t_1)H_0} + \ldots
\tag{A.1}
$$

where $H = H_0 + V$. The second-order term can be written as

$$
\int_0^\infty \frac{d\alpha}{\alpha} \int_0^\alpha dt_1 \int_0^{t_1} dt_2 \text{Tr} \left( e^{-t_2 H_0} V e^{-(t_1-t_2)H_0} V e^{-(\alpha-t_1)H_0} \right) e^{-\frac{1}{2} \alpha L^2}
= \int_0^\infty \frac{d\alpha}{\alpha} \int_0^\alpha dt' \int_0^{t'} \text{Tr} \left( V e^{-t' H_0} V e^{-(\alpha-t')H_0} \right) e^{-\frac{1}{2} \alpha L^2}
= \int_0^\infty \frac{d\alpha}{\alpha} \int_0^\alpha dt'' \int_0^{t''} \text{Tr} \left( V e^{-t'' H_0} V e^{-(\alpha-t'')H_0} \right) e^{-\frac{1}{2} \alpha L^2},
\tag{A.2}
$$

where $t' = t_1 - t_2$ and $t'' = \alpha - t'$. Combining the two last lines we obtain

$$
\int_0^\infty \frac{d\alpha}{\alpha} \int_0^\alpha dt_1 \int_0^{t_1} dt_2 \text{Tr} \left( e^{-t_2 H_0} V e^{-(t_1-t_2)H_0} V e^{-(\alpha-t_1)H_0} \right) e^{-\frac{1}{2} \alpha L^2}
= \frac{1}{2} \int_0^\infty \frac{d\alpha}{\alpha} \int_0^\alpha dt'' \text{Tr} \left( V e^{-t'' H_0} V e^{-(\alpha-t'')H_0} \right) e^{-\frac{1}{2} \alpha L^2}.
\tag{A.3}
$$
Hence one finds
\[
\Gamma = \frac{1}{2} \int_0^\infty d\alpha \int_0^\infty t \Tr \left( V e^{-tH_0} e^{-\alpha L} \right) e^{-\frac{\alpha}{\Lambda^2}} \\
+ \frac{1}{4} \int_0^\infty d\alpha \int_0^\infty dt' \int_0^\alpha t^\prime \Tr \left( V e^{-t^\prime H_0} V e^{-t^\prime H_0} e^{-\alpha L} \right) e^{-\frac{\alpha}{\Lambda^2}} \\
- \frac{1}{2} \int_0^\infty d\alpha \int_0^\alpha dt' \int_0^\alpha t'' \Tr \left( V e^{-t'' H_0} V e^{-t'' H_0} V e^{-t'' H_0} e^{-\alpha L} \right) e^{-\frac{\alpha}{\Lambda^2}} \\
+ \ldots
\]  
(A.4)

While gauge invariance is typically not preserved by this expansion, it must be recovered upon collecting all contributions at any given order in $\Lambda$. It may also be convenient to introduce a test function $f$, cf. [23].

Appendix B: Heat kernel expansion: four field contributions

In order to compute the four field contributions, a heat kernel expansion up to fourth order is required. Nonetheless, we would like to state here a partial result coming from the second order in the expansion in order to demonstrate which type of terms are likely to appear. Following the lines of Section 4.3 the four field contributions of (4.32) compute to:

\[
\int_0^\infty d\alpha \int_0^\infty dt \Tr \left( V e^{-tH_0} e^{-\alpha L} \right) e^{-\frac{\alpha}{\Lambda^2}} \bigg|_{\text{4-fields}} \\
\approx \frac{\Tr \sqrt{\mathcal{G}}}{4\Lambda_{NC}^4} \int \frac{d^4Q}{(2\pi\Lambda_{NC}^2)^4} \left\{ C^{\mu\nu} G^{\rho\sigma} A_\mu (Q - l) A_\rho (-Q - k) A_\nu (l) A_\sigma (k) \\
+ \varphi^i (Q - l) \varphi^j (-Q - k) \varphi_i (l) \varphi_j (k) + \tilde{G}^{\mu\nu} A_\mu (Q - l) A_\nu (l) \varphi^i (-Q - k) \varphi_i (k) \right\} \times \\
\times \left( \Lambda^2 \left( \hat{\mathbf{l}} \cdot (\hat{\mathbf{l}} + \hat{\mathbf{Q}}) \cos \left( \frac{(k\theta Q)}{2} \right) - 1 \right) - \hat{\mathbf{Q}} \cdot \hat{\mathbf{Q}} \sin \left( \frac{(2k\theta)}{4} \right) \sin \left( \frac{(l\theta)}{4} \right) \\
- \left( 2 \hat{\mathbf{k}} \cdot (\hat{\mathbf{k}} + \hat{\mathbf{Q}}) + \hat{\mathbf{Q}} \cdot \hat{\mathbf{Q}} \right) \sin^2 \left( \frac{(l\theta)}{4} \right) \right) \\
+ 16 \sin^2 \left( \frac{(k\theta)}{4} \right) \sin^2 \left( \frac{(l\theta)}{4} \right) \left( \ln \left( \frac{\Lambda^2}{Q \cdot Q} \right) - 2 \gamma_E + 2 \right) \right) + \left( Q \to -Q \right) + O \left( \frac{Q^4}{\Lambda^4} \right) \\
\right) .
\]  
(B.1)

Appendix C: Supplemental computations for the effective matrix model action

C.1 Proof of Lemma [1]

Eqn. (6.26a) can be shown using the fact that $\hat{\theta}^{ij} \hat{\theta}^{kl} - \hat{\theta}^{il} \hat{\theta}^{kj} - \hat{\theta}^{ij} \hat{\theta}^{kl}$ is totally anti-symmetric (cf. [2]). We verify the tensor identity (6.26a) by elaborating both sides in a local basis where
\( \tilde{\theta}^{\mu\nu} \) has canonical form \( [25] \) and \( g_{\mu\nu} = \delta_{\mu\nu}. \) Then
\[
(Gg) = \text{diag}(\alpha^2, \alpha^2, \alpha^{-2}, \alpha^{-2}),
\]
\[
Gg - \frac{1}{4}(Gg)\delta = \frac{1}{2} \text{diag}(\epsilon, \epsilon, -\epsilon, -\epsilon), \quad \epsilon = \alpha^2 - \alpha^{-2}.
\]
(C.1)

Now the lhs of (6.26b) is
\[
(\text{lhs}) = \frac{\epsilon}{2}(\alpha^4 F_{12} F_{12} + \alpha^4 F_{21} F_{21}) - \frac{\epsilon}{2}(\alpha^{-4} F_{34} F_{34} + \alpha^{-4} F_{43} F_{43}) - \frac{\epsilon}{2}(F_{13} F_{13} - F_{31} F_{31})
\]
\[- \frac{\epsilon}{2}(F_{14} F_{14} - F_{41} F_{41}) - \frac{\epsilon}{2}(F_{23} F_{23} - F_{32} F_{32}) - \frac{\epsilon}{2}(F_{24} F_{24} - F_{42} F_{42})
\]
\[= \epsilon(\alpha^4 F_{12} F_{12} - \alpha^{-4} F_{34} F_{34}).
\]
(C.2)

This agrees with the rhs of (6.26b), since
\[
(\tilde{\theta} F)^2 = 4\Lambda_{NC}^{-4}(\alpha F_{12} \pm \alpha^{-1} F_{34})^2,
\]
\[
(\tilde{\theta} F)^2 = 4\Lambda_{NC}^{-4}(\alpha^3 F_{12} \pm \alpha^{-3} F_{34})^2,
\]
\[
(\tilde{\theta} F)^2 - (\tilde{\theta} F)^2 = -4\Lambda_{NC}^{-4}(\alpha F_{12} \pm \alpha^{-1} F_{34})^2 + 4\Lambda_{NC}^{-4}(\alpha^3 F_{12} \pm \alpha^{-3} F_{34})^2
\]
\[= 4\Lambda_{NC}^{-4}\epsilon(\alpha^4 F_{12} F_{12} - \alpha^{-4} F_{34} F_{34}).
\]
(C.3)

Now consider (6.26c). It is easy to see using (C.1) that the lhs vanishes (assuming again that \( \tilde{\theta}^{\mu\nu} \) has canonical form) if \( \mu \) and \( \nu \) live in different blocks \((1,2)\) resp. \((3,2)\). Furthermore, \( \theta^{\mu\nu} F_{\mu\nu} \) vanishes if \( \mu \neq \nu \) live in the same blocks \((1,2)\) resp. \((3,2)\). Hence it is sufficient to check the relation for the diagonal elements \( \mu = \nu \), which can be checked explicitly using e.g.
\[
(GF\tilde{\theta})^{11} = (GF\tilde{\theta})^{22} = (\alpha^5 - \alpha) F_{12},
\]
\[
(GF\tilde{\theta})^{11} = (GF\tilde{\theta})^{22} = \pm(\alpha^5 - \alpha^{-1}) F_{34},
\]
(C.4)

which agrees with the rhs of (6.26c).

\section*{C.2 Proof of Lemma 2}

Using the Bianchi identity and partial integration, we have
\[
\tilde{G}^{\mu\nu} \partial_\mu F_{\gamma\rho} g^{\rho\beta} \partial_\nu F_{\beta\mu'} H^{\gamma\mu'} = \tilde{G}^{\mu\nu} \partial_\rho F_{\gamma\nu} g^{\rho\beta} \partial_\mu F_{\beta\mu'} H^{\gamma\mu'} + \partial()
\]
\[- \tilde{G}^{\mu\nu} (\partial_{\gamma} F_{\rho\nu} + \partial_{\rho} F_{\gamma\nu}) g^{\rho\beta} \partial_\mu F_{\beta\mu'} H^{\gamma\mu'} + \partial()
\]
\[= (F_{\gamma\rho} g^{\rho\beta} (\tilde{G}^{\mu\nu} \partial_\nu \partial_\mu) F_{\beta\mu'} H^{\gamma\mu'} - \tilde{G}^{\mu\nu} \partial_\mu F_{\rho\nu} g^{\rho\beta} \partial_\gamma F_{\beta\mu'} H^{\gamma\mu'}) + \partial()
\]
\[= (F_{\gamma\rho} g^{\rho\beta} (\tilde{G}^{\mu\nu} \partial_\nu \partial_\mu) F_{\beta\mu'} H^{\gamma\mu'} - \tilde{G}^{\mu\nu} \partial_\mu F_{\nu\gamma} H^{\rho\beta} \partial_\rho F_{\beta\mu'} g^{\gamma\mu'}) + \partial()
\]
(C.5)

for any symmetric matrix \( H^{\mu\nu}. \) Hence we get the identities
\[
\left[G^{\mu\nu} \partial_\mu F_{\nu\gamma} (g^{\rho\beta} \partial_\rho F_{\beta\mu'} H^{\gamma\mu'}) + G^{\rho\beta} \partial_\rho F_{\beta\mu'} g^{\gamma\mu'} \right] = -(G^{\mu\nu} \partial_\mu F_{\nu\gamma}) F_{\beta\mu'} g^{\rho\beta} G^{\gamma\mu'} + \partial(),
\]
\[2 g^{\mu\nu} \partial_\mu F_{\nu\gamma} G^{\rho\beta} \partial_\rho F_{\beta\mu'} G^{\gamma\mu'} = -(g^{\mu\nu} \partial_\mu F_{\nu\gamma}) F_{\beta\mu'} G^{\rho\beta} G^{\gamma\mu'} + \partial().
\]
Combining the first two gives (6.41b).
In particular, Computation for Eqn. (6.46)

Using (6.20), we find

\[
[X_f, H^{ab}][X_f, H_{ab}] = \Lambda_{NC}^{-4} \left( \left[ [\varphi^i, H^{ab}], [\varphi_i, H_{ab}] - \bar{G}^{\mu \nu} D_\mu H^{ab} D_\nu H_{ab} \right) 

= \Lambda_{NC}^{-12} g_{\mu \rho} g_{\sigma \sigma'} [\varphi^i, G^{\mu \nu} \bar{G}^{\rho \sigma'} F_{\nu \alpha} + \mu \leftrightarrow \nu] [\varphi_i, \bar{G}^{\rho \sigma} \bar{G}^{\sigma \rho'} F_{\sigma \beta} + \rho \leftrightarrow \sigma] 

- \Lambda_{NC}^{-12} g_{\mu \rho} g_{\nu \sigma} G^{\alpha \beta} D_\alpha \left( G^{\mu \nu} \bar{G}^{\rho \sigma'} F_{\nu \alpha} + \mu \leftrightarrow \nu \right) D_\beta \left( G^{\rho \sigma} \bar{G}^{\sigma \rho'} F_{\sigma \beta} + \rho \leftrightarrow \sigma \right) 

+ 2 \Lambda_{NC}^{-16} (\bar{G} g G) \left( \left[ [\varphi^i, D_\psi \varphi^k], [\varphi_i, D_\tau \varphi_k] - \bar{G}^{\mu \nu} D_\mu D_\psi \varphi^k D_\nu D_\tau \varphi_k \right) + \text{h.o.} 

= 2 \Lambda_{NC}^{-12} \left( [\varphi^i, D_\psi \varphi^k], [\varphi_i, D_\tau \varphi_k] - \bar{G}^{\mu \nu} D_\mu D_\psi \varphi^k D_\nu D_\tau \varphi_k \right) \left( \Lambda_{NC}^{-4} (\bar{G} g G) \mu \rho \bar{G}^{\mu \sigma} + \bar{G}^{\rho \sigma} \bar{G}^{\sigma \rho} \right) + \text{h.o.} 

(6.6)

Thus

\[
\text{Tr}[X_f, H^{ab}][X_f, H_{ab}] = \frac{2}{\Lambda_{NC}} \int \frac{d^4 x}{(2\pi)^2} \sqrt{G} \left( \left[ [\varphi^i, H^{ab}], [\varphi_i, H_{ab}] - \bar{G}^{\mu \nu} D_\mu H^{ab} D_\nu H_{ab} \right) 

= \frac{2}{\Lambda_{NC}^{-8}} \int \frac{d^4 x}{(2\pi)^2} \sqrt{G} \left( \left[ [\varphi^i, H^{ab}], [\varphi_i, H_{ab}] - \bar{G}^{\mu \nu} D_\mu H^{ab} D_\nu H_{ab} \right) \left( \Lambda_{NC}^{4} (\bar{G} g G) \right) \left( \bar{G}^{\mu \nu} D_\mu D_\psi \varphi^k D_\nu D_\tau \varphi_k \right) + \text{h.o.} 

(6.7)

using (6.26a) and (6.26b) (multiplied by q \cdot q in momentum representation) where H \sim - (\bar{G} g).

Similarly, using (6.20) and \bar{G} \theta^{-1} G = \Lambda_{NC}^{4} \bar{\theta}, we find

\[
H^{cd}[X_c, J^a_b][X_d, J^b_a] = H^{cd}[X_c, [X^a, X^b], X_d, X^a] = H^{cd}[X_c, [X^a, X^b], X_d, X^a] = H^{cd}[X_c, [X^a, X^b], X_d, X^a] 

= H^{\mu \nu} [X_\psi, \Theta_{ab}] [X_\tau, \Theta_{ab}] + 2 H^{\mu \nu} [X_\mu, \Theta_{ab}] [\varphi_i, \Theta_{ab}] + H^{ij} [\varphi_i, \Theta_{ab}] [\varphi_j, \Theta_{ab}] 

= \Lambda_{NC}^{-16} (\bar{G} g G)^{\mu \nu} \bar{G}^{\sigma \rho} (\bar{G}^\tau D_\mu F_{\rho \sigma} D_\nu F_{\sigma \tau} + 2 D_\rho \varphi^i D_\nu D_\sigma \varphi_i) + \text{h.o.} 

(8.8)

In particular,

\[
(H^{cd} - \frac{1}{2} H g^{cd})[X_c, \Theta_{ab}][X_d, \Theta_{ab}] 

\sim \Lambda_{NC}^{-16} (\bar{G} g G)^{\mu \nu} \bar{G}^{\sigma \rho} (\bar{G}^\tau D_\mu F_{\rho \sigma} D_\nu F_{\sigma \tau} + 2 D_\rho \varphi^i D_\nu D_\sigma \varphi_i) + \text{h.o.} 

= -\Lambda_{NC}^{-16} g^{\mu \nu} \bar{G}^{\sigma \rho} (\bar{G}^\tau D_\mu F_{\rho \sigma} D_\nu F_{\sigma \tau} + 2 D_\rho \varphi^i D_\nu D_\sigma \varphi_i) + \text{h.o.} 

(8.9)

Thus

\[
\text{Tr}(H^{cd} - \frac{1}{2} H g^{cd})[X_c, \Theta_{ab}][X_d, \Theta_{ab}] 

= \Lambda_{NC}^{-12} \int \frac{d^4 x}{(2\pi)^2} \sqrt{G} \left( \left[ [\varphi^i, H^{ab}], [\varphi_i, H_{ab}] - \bar{G}^{\mu \nu} D_\mu H^{ab} D_\nu H_{ab} \right) 

= \Lambda_{NC}^{-12} \int \frac{d^4 x}{(2\pi)^2} \sqrt{G} \left( - \Lambda_{NC}^{-8} G^{\mu \nu} D_\mu D_\psi \varphi^k D_\nu D_\tau \varphi_k - 2 \Lambda_{NC}^{-12} \bar{G}^\tau D_\mu F_{\rho \sigma} D_\nu F_{\sigma \tau} \right) + \text{h.o.} 

(8.10)

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which is consistent with the $\mathcal{O}(X^6)$ results \([6.39]\).

Furthermore, a straightforward computation gives

$$
\Box X^a \Box g X^a = \Lambda_{NC}^{-16} \bar{G}^{\mu \nu} \partial_\mu F_{\nu \gamma} \left( g^{\rho \beta} \partial_\rho F_{\beta \mu'} \bar{G}^{\mu' \gamma} + (GgG)^{\gamma \beta} \bar{G}^{\eta \mu} \partial_\eta F_{\nu \beta} - \frac{\Lambda_{NC}^4}{2} \hat{\partial}^{\gamma \nu'} \partial_{\nu'} (\hat{\partial}^{\beta \gamma} F_{\beta \nu}) \right) \\
+ \Box g \phi^i \Box g \phi^i.
$$

(C.11)

Under the integral resp. trace this can be rewritten as

$$
\text{Tr} \Box X^a \Box g X^a = \text{Tr} \Lambda_{NC}^{-16} \bar{G}^{\mu \nu} \partial_\mu F_{\nu \gamma} \left( g^{\rho \beta} \partial_\rho F_{\beta \mu'} \bar{G}^{\mu' \gamma} + \bar{G}^{\rho \beta} \partial_\rho F_{\beta \mu'} (GgG)^{\gamma \mu'} \right) \\
+ \frac{1}{4} \Lambda_{NC}^{-12} \bar{G}^{\mu \nu} \partial_\nu (\hat{\partial} F) \partial_{\mu} (\hat{\partial} F) + \Box g \phi^i \Box g \phi^i,
$$

(C.12)

noting that the Bianchi identity gives

$$
\hat{\partial}^{\gamma \nu'} \partial_{\nu'} F_{\gamma \nu} = -\frac{1}{2} \partial_\nu (\hat{\partial} F) \partial_{\nu} (\hat{\partial} F).
$$

(C.13)

Now we can use the second relation \((6.41b)\) of Lemma 2 replacing $g$ by $(GgG)$. This gives after some (by now standard) manipulations

$$
\text{Tr} \Box X^a \Box g X^a = \text{Tr} \Lambda_{NC}^8 \left( \Box g F_{\rho \gamma} F_{\beta \mu'} \bar{G}^{\rho \beta} \bar{G}^{\mu' \gamma} + \frac{1}{2} (\hat{\partial} F) \Box (\hat{\partial} F) - \frac{1}{4} (\hat{\partial} F) \Box (\hat{\partial} F) + \Lambda_{NC}^4 \Box g \phi^i \Box g \phi^i \right).
$$

(C.14)

The other terms of \((6.46)\) can be computed along the same lines.

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