ON THE BOUCKSOM-ZARISKI DECOMPOSITION FOR IRREDUCIBLE SYMPLECTIC VARIETIES AND BOUNDED NEGATIVITY

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Abstract. Zariski decomposition plays an important role in the theory of algebraic surfaces due to many applications. For irreducible symplectic manifolds Boucksom provided a characterization of his divisorial Zariski decomposition in terms of the Beauville-Bogomolov-Fujiki quadratic form. Different variants of singular holomorphic symplectic varieties have been extensively studied in recent years. In this note we first show that the “Boucksom-Zariski” decomposition holds for effective divisors in the largest possible framework of varieties with symplectic singularities. On the other hand in the case of projective surfaces, it was recently shown that there is a strict relation between the boundedness of coefficients of Zariski decompositions of pseudoeffective integral divisors and the bounded negativity conjecture. In the present note, we show that an analogous phenomenon can be observed in the case of projective irreducible symplectic manifolds. We furthermore prove an effective analog of the bounded negativity conjecture in the smooth case. Combining these results we obtain information on the denominators of “Boucksom-Zariski” decompositions for holomorphic symplectic manifolds. From such a bound we easily deduce a result of effective birationality for big line bundles on projective holomorphic symplectic manifolds, answering to a question asked by F. Charles.

1. Introduction

Zariski decomposition is a fundamental tool in the theory of linear series on algebraic surfaces. One possible generalization in higher dimension is the divisorial Zariski decomposition (see [14] in the projective case and [15] in the Kähler setting). When $X$ is an irreducible symplectic manifold, by replacing the intersection form with the Beauville-Bogomolov-Fujiki quadratic form $q_X$ on $H^2(X, \mathbb{C})$, Boucksom [15, Theorem 4.3 part (i), Proposition 4.4, Theorem 4.8 and Corollary 4.11] gave a characterization of his divisorial Zariski decomposition in terms of the quadratic form $q_X$ (see Definition 3.5). We will refer to it as a Boucksom-Zariski decomposition. Several notions of singular irreducible symplectic varieties have received much attention in recent years, for several reasons. Let us simply mention here that the minimal model program naturally leads to singular minimal models. “Singular” irreducible symplectic varieties are studied in the theory of orbifolds or V-manifolds. They are also studied as moduli spaces of sheaves on $K3$ or abelian surfaces (see [48] and references to prior works therein). The period map and the moduli theory

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of “singular” irreducible symplectic varieties are extensively studied in [3, 4]. Finally “singular” irreducible symplectic varieties appear as building blocks of mildly singular projective varieties with trivial canonical class (see [29, 20, 24]). See the recent survey [49] for all the different definitions and results. As the Boucksom-Zariski decomposition proved to be a very useful tool in the theory of smooth irreducible symplectic varieties it is natural to ask whether it holds for some classes of singular irreducible symplectic varieties. In this article we show that the (Boucksom characterization of the divisorial) Zariski decomposition actually holds in the largest possible framework, namely it holds on any variety \( X \) with symplectic singularities (see Definition 2.1). In the smooth case Boucksom actually proves it for pseudo-effective \( \mathbb{R} \)-classes, while here we restrict ourselves to effective \( \mathbb{Q} \)-divisors. To state our result recall that, following Kirchner [35], one can define a quadratic form \( q_{X, \sigma} \) on \( H^2(X, \mathbb{C}) \), where \( \sigma \) is a reflexive 2-form on \( X \) which is symplectic on \( X_{\text{reg}} \). When \( X \) is a primitive symplectic variety (in the sense of Definition 2.2) and \( \sigma \) is a normalized symplectic form (see Definition 2.5) one shows that \( q_{X, \sigma} \) is independent of \( \sigma \) and is called the Beauville-Bogomolov-Fujiki quadratic form (and denoted \( q_X \)). To deal with the non-\( \mathbb{Q} \)-factorial case we extend the definition of \( q_{X, \sigma} \) to Weil divisors (cf. Definition 2.9). We refer the reader to Section 2.4 for further details.

**Theorem 1.1.** If \( X \) is a compact Kähler space with symplectic singularities and \( \sigma \in H^0(X, \Omega_X^2) \) a reflexive 2-form which is symplectic on \( X_{\text{reg}} \), then all \( \mathbb{Q} \)-Weil effective divisors on \( X \) have a unique \( q_{X, \sigma} \)-Zariski decomposition, i.e. \( D \) can be written in a unique way as

\[
D = P(D) + N(D),
\]

where \( P(D) \) and \( N(D) \) are \( \mathbb{Q} \)-effective Weil divisors satisfying the following:

1) \( P(D) \) is \( q_{X, \sigma} \)-nef, i.e. \( q_{X, \sigma}(P(D), E) \geq 0 \) for all effective Cartier divisors;
2) \( N(D) \) is \( q_{X, \sigma} \)-exceptional, i.e. the Gram matrix \( (q_X(N_i, N_j))_{i,j} \) of the irreducible components of the support of \( N(D) \) is negative definite;
3) \( q_{X, \sigma}(P(D), N(D)) = 0 \);
4) for all integer \( k \geq 0 \) such that \( kP(D) \) and \( kD \) are integral, then the natural map

\[
H^0(X, \mathcal{O}_X(kP(D))) \twoheadrightarrow H^0(X, \mathcal{O}_X(kD))
\]

is a surjection.

In particular, if \( X \) is a primitive symplectic variety, any \( \mathbb{Q} \)-Cartier effective divisor has a unique \( q_X \)-Zariski decomposition with respect to the Beauville-Bogomolov-Fujiki quadratic form \( q_X \).

The sheaves \( \mathcal{O}_X(kP(D)) \) and \( \mathcal{O}_X(kD) \) above are the rank one reflexive sheaves associated to the Weil divisors \( kP(D) \) and \( kD \) (see Section 2.3 for further details on Weil divisors and reflexive sheaves).

Our proof goes along the lines of Bauer’s approach [6] to the Zariski decomposition for surfaces. As observed in [7], in order to have a Zariski-type decomposition on a compact Kähler space it is sufficient to have a quadratic form on the second cohomology group (or on the divisor class group) that behaves like an intersection product, i.e., the intersection of two distinct prime divisors is always non-negative. It is hence enough to prove that on any Kähler variety \( X \) with symplectic singularities...
the quadratic form $q_{X,\sigma}$ introduced above behaves like an intersection product, see Theorem 3.11. Even if one wants to deal only with $\mathbb{Q}$-Cartier divisors, without further hypotheses these may a priori have non $\mathbb{Q}$-Cartier irreducible components. Our approach to the Boucksom-Zariski decomposition requires to evaluate the quadratic form on those components. This is a technical difficulty that we can circumvent but it renders the proof more involuted.

In the case of algebraic surfaces, as well as in the case of irreducible symplectic manifolds, the geometric significance of Zariski decompositions lies in the fact that, given a pseudo-effective integral divisor $D$ with Zariski decomposition $D = P + N$, one has for every sufficiently divisible integer $m > 1$ the equality

$$H^0(X, \mathcal{O}_X(mD)) = H^0(X, \mathcal{O}_X(mP)),$$

i.e. all sections in $H^0(X, \mathcal{O}_X(mD))$ come from global sections of the “positive” part $\mathcal{O}_X(mP)$. The term “sufficiently divisible” here means that one needs to pass to a multiple $mD$ that clears denominators in $P$ for the statement to hold. In general, we do not know how to find numbers $m$ for a given surface $X$ and any integral pseudoeffective divisors. In [8], the main result tells us that the question about the possible values of $m$ is strictly related to the bounded negativity conjecture, which is another open problem in the theory of linear series conjecturing the boundedness of negative self-intersections of irreducible divisors on any surface. Let us illustrate this phenomenon using the well-known case of smooth projective $K3$ surfaces. By the adjunction formula, for any irreducible and reduced curve $C$ one has that $C^2 \geq -2$, and in order to clear denominators in the Zariski decomposition for any pseudoeffective divisors $D$ one can take $m = 2^{\rho - 1}$, where $\rho$ denotes the Picard number.

It is natural to ask whether we can generalize the above considerations to the case of higher dimensional projective varieties. First of all, we have several variations on the classical Zariski decomposition, for instance the Cutkosky-Kawamata-Moriwaki-Zariski decomposition, but this decomposition, as it was shown by Cutkosky [18], cannot exists in general. On the other hand, there is no natural and meaningful generalization of the bounded negativity conjecture in general. In [39], the authors constructed an example of a sequence of divisors $D_k$ on smooth projective 3-fold $Y$ such that $D_k^3 \to -\infty$.

However, as recalled before, we do have the Boucksom-Zariski decomposition on irreducible symplectic manifolds.

In the case of projective irreducible symplectic manifolds, similarly to the case of $K3$ surfaces, thanks to the presence and properties of the Beauville-Bogomolov quadratic form we are able to prove an effective version of the analogue of the bounded negativity conjecture. More precisely in Proposition 4.9 we prove that for a projective irreducible symplectic manifold $X$ the Beauville-Bogomolov self-intersection of any irreducible divisor is bounded from below by $4 \text{Card}(A_X)$, where $\text{Card}(A_X)$ is the cardinality of the (finite) discriminant group $A_X := H^2(X, \mathbb{Z})^\vee / H^2(X, \mathbb{Z})$ of the intersection lattice. The proof relies on two known results of Druel and Markman that we recall in Propositions 4.6, 4.7. These results will potentially also...
hold and give an analogous consequence in the singular case. For the boundedness of denominators in the Boucksom-Zariski decomposition we follow the lines of [8] to conclude with an effective bound on denominators:

**Theorem 1.2.** Let $X$ be a smooth projective irreducible symplectic variety of Picard number $\rho(X)$. The denominators of the coefficients of the negative and positive parts of the Boucksom-Zariski decompositions of all pseudoeffective Cartier divisors are bounded by $(4 \text{ Card}(A_X))^{\rho(X)-1!}$.

For the proof see Corollary 4.11. Note that $\rho(X) \leq h^{1,1}(X)$ hence $(4 \text{ Card}(A_X))^{h^{1,1}(X)-1!}$ gives a bound that is uniform for the whole family of deformations of $X$.

From Theorem 1.2 we easily deduce a result of effective birationality which is interesting on its own.

**Corollary 1.3.** Let $X$ be a smooth projective irreducible symplectic variety of dimension $2n$ and $L \in \text{Pic}(X)$ a big line bundle on it. Then for all

$$m \geq \frac{1}{2} (2n+2)(2n+3)(4 \text{ Card}(A_X))^{\rho(X)-1!} \quad (1)$$

the map associated to the linear system $|mL|$ is birational onto its image.

Again, by replacing $\rho(X)$ with $h^{1,1}(X)$ in equation (1), we obtain an effective bound that holds for the whole family of deformations of $X$. In particular, the corollary above answers affirmatively and effectively to a strong version of a question asked by Charles in the Introduction of [17]. It is important to notice that even without answering to this question Charles is able to obtain a birational boundedness result for smooth projective irreducible symplectic varieties, see [17, Theorem 1.2]. Such result follows from Corollary 1.3 by standard arguments, see Corollary 5.1 in Section 5. After the completion of the paper we were informed by C. Birkar that, for any $d > 0$, he proved the existence of a bounded number $m$ giving the birationality of $|mL|$, where $L$ is a big integral divisor on any Calabi-Yau variety of fixed dimension $d$ with klt singularities (see [11, Corollary 1.4]). If on the one hand his result is obviously much more general than ours, on the other hand the integer $m$ in his theorem is not explicit, while, in the case of smooth irreducible symplectic varieties, Corollary 1.3 provides such an explicit bound.

At the moment it is not yet clear whether the bounded negativity conjecture may be true in the most general setting of singular irreducible symplectic varieties. However, having now established Zariski decomposition in this setting in Theorem 1.1 we can follow the lines of [8] to prove in Theorem 4.2 that the bounded negativity conjecture is equivalent to the boundedness of Zariski denominators also for (smooth or singular) irreducible symplectic varieties. Indeed, we observe that the arguments of [8] are, up to slight adaptations, also valid in this context.

2. Preliminaries.

We will use the word “manifold” to stress the smoothness, while we use the words “variety” or “space” when smoothness is not required.
2.1. Irreducible symplectic varieties: the smooth case. An irreducible symplectic manifold is a compact Kähler manifold $X$ which is simply connected and carries a holomorphic symplectic form $\sigma$ such that $H^0(X, \Omega^2_X) = \mathbb{C} \cdot \sigma$. For a general introduction of the subject, we refer to [30].

Let $X$ be an irreducible symplectic manifold of dimension $2n \geq 2$. Let $\sigma \in H^0(X, \Omega^2_X)$ such that $\int_X \sigma^n \bar{\sigma}^n = 1$. Then, following [9], the second cohomology group $H^2(X, \mathbb{C})$ is endowed with a quadratic form $q_X$ defined as follows

$$q_X(a) := \frac{n}{2} \int_X (\sigma \bar{\sigma})^{n-1} a^2 + (1 - n) \left( \int_X \sigma^n \bar{\sigma}^{n-1} a \right) \cdot \left( \int_X \sigma^{n-1} \bar{\sigma}^n a \right), \quad a \in H^2(X, \mathbb{C}),$$

which is non-degenerate and, up to a positive multiple, is induced by an integral nondivisible quadratic form on $H^2(X, \mathbb{Z})$ of signature $(3, b_2(X) - 3)$. The form $q_X$ is called the Beauville-Bogomolov-Fujiki quadratic form of $X$. By Fujiki [23], there exists a positive rational number $c_X$ (the Fujiki constant of $X$) such that

$$c_X \cdot q_X^n(\alpha) = \int_X \alpha^{2m}, \quad \forall \alpha \in H^2(X, \mathbb{Z}).$$

2.2. Irreducible symplectic varieties: the singular case. Let $X$ be a normal complex variety. Recall that, for any $p \geq 1$, the sheaf $\Omega^p_X$ of reflexive holomorphic $p$-forms on $X$ is $\iota^* \Omega^p_{X_{reg}}$, where $\iota : X_{reg} \hookrightarrow X$ is the inclusion of the regular locus of $X$. It can be alternatively (and equivalently) defined by the double dual $\Omega^p_X = (\Omega^p_X)^{**}$. Recall the following definition which is due to Beauville [10].

**Definition 2.1.** Let $X$ be a normal Kähler variety.

i) A symplectic form on $X$ is a closed reflexive 2-form $\sigma$ on $X$ which is non-degenerate at each point of $X_{reg}$.

ii) If $\sigma$ is a symplectic form on $X$, the variety $X$ has symplectic singularities if for one (hence for every) resolution $f : \tilde{X} \to X$ of the singularities of $X$, the holomorphic symplectic form $\sigma_{reg} := \sigma|_{X_{reg}}$ extends to a holomorphic 2-form on $\tilde{X}$.

**Definition 2.2.** A primitive symplectic variety is a normal compact Kähler variety $X$ such that $h^1(X, \mathcal{O}_X) = 0$ and $H^0(X, \Omega^2_X)$ is generated by a holomorphic symplectic form $\sigma$ such that $X$ has symplectic singularities.

For a normal variety $X$ such that $X_{reg}$ has a symplectic form $\sigma$, having canonical singularities is in fact equivalent to Beauville’s condition above that the pullback of $\sigma$ to a resolution of $X$ extends as a regular 2-form. By [21] and [34, Corollary 1.8] this is also equivalent to $X$ having rational singularities. For the definition and basic properties of Kähler forms on possibly singular complex spaces we refer the reader to e.g. [4, Section 2].

One can show that a smooth primitive symplectic variety $X$ is simply connected (see e.g. [50, Lemma 15]), hence such $X$ is a primitive symplectic manifold. In order to put Definition 2.2 into perspective, recall first the following.
**Definition 2.3.** An irreducible symplectic variety is a normal compact Kähler variety $X$ with canonical singularities and such that for all quasi-étale morphisms $f : X' \to X$ the reflexive pull-back $f^! \sigma$ of the symplectic form $\sigma$ on $X$ generates the exterior algebra of reflexive forms on $X'$.

Irreducible symplectic varieties appear in the Beauville-Bogomolov decomposition for minimal models with trivial canonical class, obtained thanks to contributions of several groups of people [29, 20, 24, 25, 16, 2]. In the smooth case being primitive symplectic or irreducible symplectic is equivalent by [9, Proposition 3], while in the singular case an irreducible symplectic variety is primitive symplectic, see e.g. [25, Proposition 6.9], but it is a more restrictive notion. Indeed, the Kummer singular surface $A/\mathbb{Z}/2$ is primitive symplectic but has a quasi-étale cover by $A$, hence it is not irreducible symplectic. Moduli spaces of sheaves on a projective $K3$ surface (or the Albanese fiber of moduli spaces of sheaves on an abelian surface) are shown to be irreducible symplectic varieties by Perego-Rapagnetta in [48].

We now recall the work of Namikawa, Kirchner, Schwald and Bakker-Lehn on the Beauville-Bogomolov-Fujiki form in the singular setting. We start by recalling the following definition which make sense in general, but will be mainly relevant for symplectic varieties. We will only use it once outside this framework, in the proof of Theorem 3.11 where we will consider it on a resolution of a variety with symplectic singularities.

**Definition 2.4.** ([35, Definition 3.2.1]) Let $X$ be a $2n$ dimensional, complex compact variety and $w \in H^2(X, \mathbb{C})$ a nonzero class. Then, for all $a \in H^2(X, \mathbb{C})$ we define a quadratic form $q_{X,w}$ on $X$ as follows

$$q_{X,w}(a) := \frac{n}{2} \int_X (w \bar{w})^{n-1} a^2 + (1 - n) \left( \int_X w^n \bar{w}^{n-1} a \right) \cdot \left( \int_X w^{n-1} \bar{w}^n a \right).$$

In what follows whenever $X$ has symplectic singularities, in Kirchner’s definition, we will take $w$ equal to the symplectic form $\sigma$ on $X$. When $X$ is a primitive symplectic variety, Namikawa also defines a quadratic form on $X$ by pulling everything back to the resolution of singularities of $X$. Kirchner’s and Namikawa’s approaches are equivalent, as checked in [35, Proposition 3.2.15] (see also [50]). As noticed in [4, Section 5.1] the hypothesis of the projectivity of $X$ that appears in [50] is unnecessary.

To obtain a uniquely determined quadratic form $q_X$ on a primitive symplectic variety $X$, it is convenient to normalize the symplectic form $\sigma$. Let $I(\sigma) := \int_X (\sigma \bar{\sigma})^n$ and consider $s := I(\sigma)^{-1/2n} \cdot \sigma$. Then $I(s) = 1$ and one verifies that, once the normalization is performed, the resulting quadratic form $q_{X,s}$ does not depend on the choice of the symplectic form (see [50, Lemma 22] and [4, Lemma 5.3]).

**Definition 2.5.** Let $X$ be a primitive symplectic variety. The Beauville-Bogomolov-Fujiki form of $X$ is defined as $q_X := q_{X,s}$ for any $\sigma \in H^0(X; \Omega^2_X)$ with $I(\sigma) = 1$.

As noticed in [4, Lemma 5.5] there exists a real multiple of $q_X$ which takes integral values on integral classes. We finally collect in a single statement several observations and results due to Schwald and Bakker-Lehn which we will use.
Theorem 2.6.  
(1) \cite[Lemma 25]{50} Let $f : Y \to X$ be a proper birational morphism from a normal, complex compact variety $Y$ to a variety $X$ with symplectic singularities. Then $q_{X,Y}(v) = q_{X,f^*}(f^*v)$, for all $v, w \in H^2(X, \mathbb{C})$.

(2) \cite[Theorem 2, item (1)]{50} and \cite[Proposition 5.15]{4} Let $X$ be an irreducible symplectic variety of dimension $2n$. There exists a positive number $c_X \in \mathbb{R}_{>0}$ such that for all $a \in H^2(X, \mathbb{C})$ the following holds

$$c_X \cdot q_X(a)^n = \int_X a^{2n}.$$ 

(3) \cite[Theorem 2, item (2)]{50} and \cite[Lemma 5.3]{4} Let $X$ be a primitive symplectic variety. The restriction of $q_X$ to $H^2(X, \mathbb{R})$ is a non-degenerate real quadratic form of signature $(3, b_2(X) - 3)$.

In the singular IHS setting, the second cohomology carries a pure Hodge structure (see e.g. \cite[Corollary 3.5]{4}), therefore we can define the positive cone $\mathcal{P}$ as the connected component of \{ $\alpha \in H^{1,1}(X, \mathbb{R})$ : $q(\alpha) > 0$ \} containing a Kähler class, and the movable cone $\mathcal{M}(X)$ is defined as the closure of the cone generated by classes of effective Cartier divisors $L$ such that the base locus of the linear series $|L|$ has codimension at least 2. Moreover, we denote by $\mathcal{E}$ the pseudoeffective cone (which is closed and convex). In the case of irreducible symplectic manifolds, one can show that the dual of the pseudoeffective cone $\mathcal{E}^*$ coincides with the movable cone $\mathcal{M}(X)$.

2.3. Weil divisors and reflexive sheaves. In this subsection we collect some known facts on Weil divisors and rank one reflexive sheaves. We refer the reader to \cite{51} for the proofs, further details and references.

By a Weil divisor on a normal variety $X$, we mean a formal sum of integral codimension 1 subschemes (prime divisors). As in the smooth case, each prime divisor $D$ corresponds to some discrete valuation $v_D$ of the fraction field of $X$, because by the normality of $X$ the stalk at the generic point of a prime divisor is still a regular ring. Therefore to any non-zero rational function $f \in K(X)$ on $X$ we associate the principal divisor $\text{div}(f)$ as in the regular case: $\text{div}(f) = \sum_D v_D(f)D$, where the sum runs through all the prime Weil divisors $D$ on $X$. Likewise, we say that two Weil divisors $D_1$ and $D_2$ are linearly equivalent if $D_1 - D_2$ is principal.

If $\mathcal{F}$ is a coherent sheaf on a variety $X$ its dual is $\mathcal{F}^\vee := \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. There is a natural map from $\mathcal{F}$ to the double-dual $(\mathcal{F}^\vee)^\vee$ and the coherent sheaf is called reflexive if this map is an isomorphism.

Given a (Weil) divisor $D$ on a normal variety $X$, we consider the coherent sheaf $\mathcal{O}_X(D)$ defined as follows:

$$\Gamma(V, \mathcal{O}_X(D)) = \{ f \in K(X) : \text{div}(f)|_V + D|_V \geq 0 \}$$

for any open subset $V \subset X$. If $D$ is a prime divisor, then $\mathcal{O}_X(-D) = \mathcal{I}_D$ and furthermore, if $D$ is any divisor, then $\mathcal{O}_X(D)$ is a reflexive sheaf, cf. \cite[Proposition 3.4]{51}.

Proposition 2.7. Let $X$ be a normal variety.

(1) Any reflexive rank 1 sheaf $\mathcal{F}$ is of the form $\mathcal{O}_X(D)$ for some (uniquely determined) divisor $D$. 

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(2) Two Weil divisors $D_1$ and $D_2$ are linearly equivalent if and only if $O_X(D_1) = O_X(D_2)$.

(3) To every non-zero global section $s \in H^0(X, \mathcal{F})$ of a reflexive rank 1 sheaf $\mathcal{F}$, we can associate an effective divisor $D$ on $X$.

(4) Let $\mathcal{F}$ be a reflexive rank 1 sheaf. For every effective Weil divisor $D$ such that $O_X(D) = \mathcal{F}$, there is a section $s \in H^0(X, \mathcal{F})$ such that $s$ corresponds to $D$.

(5) Two non-zero global sections $s_1, s_2 \in H^0(X, \mathcal{F})$ of a reflexive rank 1 sheaf $\mathcal{F}$ determine the same divisor if and only if there is a unit $u \in H^0(X, O_X)$ such that $s_1 = us_2$.

For the proofs see [51, Propositions 3.7, 3.11 and 3.12]. We also have the following.

**Proposition 2.8** ([51], Theorem 2.8 and Proposition 3.13). Let $X$ be a normal variety. Let $D_1$ and $D_2$ be two Weil divisors on $X$. Then we have the following facts:

1. $O_X(D_1 + D_2) = (O_X(D_1) \otimes O_X(D_2))^\vee$.
2. $O_X(-D_1) = O_X(D_1)^\vee$.
3. $O_X(D_1 - D_2) = O_X(D_1) \otimes O_X(-D_2)$.

Thus one can turn the set of (isomorphism classes of) rank 1 reflexive sheaves into a group as follows. To add two sheaves, simply tensor them together and then double-dualize. To invert a rank 1 reflexive sheaf, simply dualize. The sheaf $O_X$ is the identity. This group is clearly isomorphic to the divisor class group by the previous results.

Thanks to the previous propositions it makes sense to talk about the complete linear system $|D|$ associated to a Weil divisor $D$ intended as the set of all Weil divisors $D'$ linearly equivalent to $D$ or, equivalently, to $\mathbb{P}H^0(X, O_X(D))$. The base ideal $b(|D|)$ associated to $|D|$ is the image of the map

$$H^0(X, O_X(D)) \otimes O_X(D)^\vee \cong H^0(X, O_X(D)) \otimes O_X(-D) \to O_X$$

induced by the evaluation morphism. The base locus $\text{Bs}(|D|)$ is the closed locus cut out by the base ideal $b(|D|)$.

### 2.4. The quadratic form on Weil divisors

Even if one wants to deal only with $\mathbb{Q}$-Cartier divisors, without further hypotheses these may a priori have non $\mathbb{Q}$-Cartier irreducible components. The approach to the Bunction-Zariski decomposition that we will present in the next section will require to evaluate the quadratic form defined in Definition 2.4 on those components. A way out via a minimal $\mathbb{Q}$-factorialization is possible in the projective case thanks to the MMP (see Remark 3.13 for the details), but $\mathbb{Q}$-factorializations are not available yet in the non-projective one. Therefore in order to deal with the most general framework (non projective, non $\mathbb{Q}$-factorial varieties) we extend the definition of the quadratic form to Weil divisors.

Let $X$ be a normal compact Kähler variety. Let $\Sigma_0 \subset X$ be the closed sublocus of points where the singularities of $X$ are worse than ADE (also known as the dissident locus of $X$). Through the rest of the paper we will consider the complement open subset

$$\iota : U := X \setminus \Sigma_0 \to X.$$
Notice that $U$ is $\mathbb{Q}$-factorial as the image of a divisorial extremal contraction is. Moreover
\[
\text{codim}_X(\Sigma_0) \geq 4 \tag{2}
\]
by \cite{[46], Proposition 1.6}.

Let $D$ a Weil divisor on $X$ and $D_U$ its restriction to $U$. By the $\mathbb{Q}$-factoriality of $U$ it makes sense to consider the cohomology class of $D_U$ inside $H^2(U, \mathbb{C})$ (which by abuse of notation we will again denote by $D_U$). Let $\pi : Y \to X$ be any resolution of singularities.

**Definition 2.9.** Let $X$ be a $2n$ dimensional, compact Kähler variety with symplectic singularities and $w \in H^2(X, \mathbb{C})$ the class of a symplectic form. Let $U \subset X$ be the open subset defined above. For any Weil divisor $D$ on $X$ we define $q_{X,w}^\text{Weil}(D)$ as follows:
\[
q_{X,w}^\text{Weil}(D) := q_Y,\pi^*w((\iota_{\pi^{-1}(U)})_* \circ (\pi|_{\pi^{-1}(U)})^*(D_U))
\]
where $\pi : Y \to X$ is any resolution of singularities and $q_Y,\pi^*w$ is as in Definition 2.4.

**Proposition 2.10.** The definition above does not depend on the choice of the resolution of singularities.

**Proof.** Let $\pi_j : Y_j \to X$, $j = 1, 2$, be two resolutions of singularities. Consider a smooth birational model $\tilde{Y}$ dominating both $Y_1$ and $Y_2$ and sitting in the following commutative diagram:

\[
\begin{array}{ccc}
Y_1 & \xrightarrow{\nu} & Y_2 \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
X & \xrightarrow{\nu} & \tilde{Y}
\end{array}
\tag{3}
\]

Let $w \in H^2(X, \mathbb{C})$ be the class of a symplectic form. Then for any Weil divisor $D$ on $X$ and $j = 1, 2$ we have
\[
q_{Y_j,\pi_j^*w}((\iota_{\pi_j^{-1}(U)})_* \circ (\pi_j|_{\pi_j^{-1}(U)})^*(D_U)) = q_{\tilde{Y},\tilde{\pi}_j^*w}((\iota_{\tilde{\pi}_j^{-1}(U)})_* \circ (\tilde{\pi}_j|_{\tilde{\pi}_j^{-1}(U)})^*(D_U))
\]
\[
= q_{\tilde{Y},\nu^*w}((\iota_{\nu^{-1}(U)})_* \circ (\nu|_{\nu^{-1}(U)})^*(D_U))
\]
where the first equality follows from Theorem 2.6 item (1), and the second from the commutativity of the diagram (3). \hfill \square

**Remark 2.11.** It is important to notice that if $X$ is a compact Kähler variety with symplectic singularities and $D$ is a Cartier divisor then the two Definitions 2.9 and 2.4 do coincide. To see this, let $\pi : Y \to X$ be a resolution of singularities and $\sigma \in H^2(X, \mathbb{C})$ be the class of a symplectic form. We first remark that
\[
\pi^*D = (\iota_{\pi^{-1}(U)})_*((\pi|_{\pi^{-1}(U)})^*(D_U)) + \sum a_i E_i,
\tag{4}
\]
where the $E_i$’s are $\pi$-exceptional divisors whose image is supported on $\Sigma_0$. On the other hand we have

$$q_{Y, \pi^* \sigma}(E_i) = q_{Y, \pi^* \sigma}((\pi|_{\pi^{-1}U})^*(D_U)), E_i) = 0$$

(5)

because $(\pi^* \sigma)|_{E_i}$ has rank at most $2n - 4$ by [32, Proof of Lemma 2.7] (see also [22, Theorem 3.2 and Remark 3.1]) and therefore $(\pi^* \sigma)|_{E_i}$ is zero. Thus, from (4) and (5) we deduce

$$q_{X, \sigma}(D) = q_{Y, \pi^* \sigma}(\pi^* D) = q_{Y, \pi^* \sigma}((\pi|_{\pi^{-1}U})^*(D_U))) = =: q_{X, \sigma}^{\text{Weil}}(D)$$

where the first equality follows again from Schwald’s result Theorem 2.6, item (1).

From now on we will therefore drop the “Weil” exponent in the notation introduced in Definition 2.9 and write $q_{X, \sigma}$ (or simply $q_X$ when a choice of $\sigma$ such that $\int_X (\sigma \bar{\sigma})^n = 1$ is made).

3. BOUCKSOM-ZARISKI DECOMPOSITIONS ON PRIMITIVE SYMPLECTIC VARIETIES

Then we prove that the generalization of the Beauville-Bogomolov-Fujiki form to primitive symplectic varieties behaves like an intersection product in the sense below and therefore by [7] it allows to obtain a Boucksom-Zariski decomposition. A technical difficulty comes from fact that the varieties need not be $\mathbb{Q}$-factorial.

We start by recalling some definitions.

Definition 3.1. Let $q$ be a quadratic form on a vector space $V$ and $B$ a basis of $V$. We say that $q$ is an intersection product with respect to $B$ if for any $D, D' \in B$ such that $D \neq D'$ we have

$$q(D, D') \geq 0.$$

(6)

Definition 3.2. Let $X$ be a compact complex space endowed with a quadratic form $q_X$ on $H^2(X, \mathbb{C})$ (respectively on the Weil divisor class group $\text{Cl}(X)$). A reduced and irreducible effective divisor $D \subset X$ is called a prime divisor. An effective $\mathbb{Q}$-Cartier ($\mathbb{Q}$-Weil) divisor $E = \sum a_i E_i$ is called $q_X$-exceptional if the Gram matrix $(q_X(E_i, E_j))_{i,j}$ of the irreducible components of the support of $E$ is negative definite. Furthermore, we say that a $\mathbb{Q}$-Cartier ($\mathbb{Q}$-Weil) divisor $D$ is $q_X$-nef if

$$q_X(D, E) \geq 0$$

for every effective $\mathbb{Q}$-Cartier ($\mathbb{Q}$-Weil) divisor $E$ (or equivalently for every prime Cartier ($\mathbb{Q}$-Weil) divisor $E$).

Remark 3.3. If $X$ is a complex variety with no further hypotheses we shall apply the definition of intersection product to the Weil divisor class group $\text{Cl}(X)$. If $X$ is $\mathbb{Q}$-factorial we will rather apply it to the Neron-Severi space $V = \text{NS}_R(X)$ and $B$ any basis consisting of classes of prime divisors. In particular, we will say that a quadratic form $q$ on $H^2(X, \mathbb{C})$ is an intersection product if restricted to $\text{NS}_R(X)$ it is an intersection product with respect to any basis $B$ consisting of prime divisors, or in other words (6) is satisfied for any two distinct prime divisors $D$ and $D'$. 
Remark 3.4. If $X$ is an irreducible symplectic manifold and $q_X$ the Beauville-Bogomolov-Fujiki form we know that closed cone of $q_X$-nef classes coincides with the closure of the birational Kähler cone (see e.g. [41, Proposition 5.6]).

Definition 3.5. Let $X$ be a compact Kähler space endowed with a quadratic form $q_X$ on $\text{Cl}(X)$. Let $D$ be a $\mathbb{Q}$-Weil effective divisor on $X$. A rational $q_X$-Zariski decomposition for $D$ is a decomposition

$$D = P(D) + N(D),$$

where $P(D)$ and $N(D)$ are $\mathbb{Q}$-Weil effective divisors satisfying the following:

1) $P(D)$ is $q_X$-nef;
2) $N(D)$ is $q_X$-exceptional or trivial;
3) $q_X(P(D), N(D)) = 0$;

The divisors $P(D)$ and $N(D)$ are called respectively the positive and the negative part of $D$.

Notice that even if the divisor $D$ is Cartier we do not require the positive and the negative parts of $D$ to be ($\mathbb{Q}$-)Cartier.

Let us formulate the following theorem.

Theorem 3.6. Let $X$ be a compact Kähler space endowed with a quadratic form $q_X$ on $\text{Cl}(X)$ which is an intersection product (in the sense of Definition 3.1). Then all $\mathbb{Q}$-Weil effective divisors on $X$ have a unique rational $q_X$-Zariski decomposition.

Proof. The theorem follows from [7, Theorem 3.3] by taking $V$ to be the (infinite dimensional) $\mathbb{Q}$-vector space generated by $\mathbb{Q}$-Weil prime divisors. □

Notice that by [15, Proposition 4.2, item (ii)] the Beauville-Bogomolov-Fujiki quadratic form is an intersection product on smooth irreducible symplectic varieties.

Remark 3.7. If $X$ is $\mathbb{Q}$-factorial then of course $P(D)$ and $N(D)$ are $\mathbb{Q}$-Cartier. In the general case if every component of the divisor $D$ is $\mathbb{Q}$-Cartier, then the proof above shows that $P(D)$ and $N(D)$ are $\mathbb{Q}$-Cartier. Nevertheless if we only assume that the divisor $D$ is $\mathbb{Q}$-Cartier we do not know even in the IHS case whether $P(D)$ and $N(D)$ must automatically be $\mathbb{Q}$-Cartier or not.

Proposition 3.8. The unique decomposition from Theorem 3.6 satisfies the following additional condition: for all integer $k \geq 0$ such that $kP(D)$ and $kD$ are integral, then the natural map

$$H^0(X, \mathcal{O}_X(kP(D))) \to H^0(X, \mathcal{O}_X(kD))$$

is an isomorphism, where $\mathcal{O}_X(kP(D))$ and $\mathcal{O}_X(kD)$ are the rank one reflexive sheaves associated to $kP(D)$ and $kD$, see Section 2.3.

Proof. Let $k$ be a positive integer such that $kD$ and $kP(D)$ are integral and consider $M \in |kD|$. Then $M \sim k(P(D) + N(D))$ and by item 3) there exists an index $i$ such that

$$q_X(M, N_i) = kq_X(N(D), N_i) < 0.$$
Therefore, since \( q_X \) is an intersection product, we have that

\[ N_i \subset \text{Supp}(M). \tag{7} \]

By letting \( M \) move in the linear system we conclude that for some \( i \)
\[ N_i \subset \text{Bs}(kD), \]
where \( \text{Bs}(\cdot) \) denotes the base locus.

We thus have
\[ H^0(X, \mathcal{O}_X(kD - N_i)) = H^0(X, \mathcal{O}_X(kD) \otimes \mathcal{I}_{N_i}) \rightarrow H^0(X, \mathcal{O}_X(kD)), \]
where the first equality follows from Proposition 2.8 item (3). Consider now the decomposition \( M - N_i = kP(D) + (kN(D) - N_i) \). It is clearly a \( q_X \)-Zariski decomposition. We can hence repeat the above argument for \( M_1 = kD - N_i \). Iterating this process one obtains a sequence of divisors \( M_j \) the last of which being \( kP(D) \) and a sequence of surjections whose composition is the desired surjection
\[ H^0(X, \mathcal{O}_X(kP(D))) \rightarrow H^0(X, \mathcal{O}_X(kD)). \]

\[ \square \]

Notice that our proof of Theorem 3.6 above leads to a slightly stronger statement on Zariski decomposition.

**Corollary 3.9.** The decomposition \( D = P(D) + N(D) \) provided by Theorem 3.6 for an effective \( \mathbb{Q} \)-Weil divisor \( D \) satisfies \( \text{Supp}(P(D)) \cup \text{Supp}(N(D)) = \text{Supp}(D) \).

**Proof.** The inclusion \( \text{Supp}(P(D)) \subset \text{Supp}(D) \) is given by the natural map of sections, while the inclusion \( \text{Supp}(N(D)) \subset \text{Supp}(D) \) is given by (7).

\[ \square \]

**Remark 3.10.** In fact, in Theorem 3.6 the Zariski decomposition can be performed for \( q_X \) being an intersection product in different ambient spaces \( V \) and bases \( \mathcal{B} \) (cf. Definition 3.1). We can consider \( D \) to be either an effective \( \mathbb{Q} \)-divisor or a linear equivalence class of such divisors or their numerical class. For each such \( D \) one can choose a basis consisting of classes of prime divisors in which the class of \( D \) is presented with positive coefficients and perform a decomposition with respect to the chosen basis. By the uniqueness of Zariski decomposition as seen in Theorem 3.6 all these decompositions are compatible. More precisely, if \( [D] \) is a numerical class (or linear equivalence class) of an effective \( \mathbb{Q} \)-divisor \( D \) then there exists a unique decomposition \( [D] = [P] + [N] \) into numerical classes (or linear equivalence classes) of effective (with respect to our chosen basis) \( \mathbb{Q} \)-divisors and it satisfies \( [P] = [P(D)], [N] = [N(D)] \).

Recall that if a bilinear form satisfies condition (6) on distinct prime divisors we say that it is an intersection product.

**Theorem 3.11.** Let \( X \) be a \( 2n \)-dimensional compact Kähler space with symplectic singularities and \( \sigma \in H^0(\Omega^2_X) \) a reflexive 2-form which is symplectic on \( X_{\text{reg}} \). Then the quadratic form \( q_{X,\sigma} \) defined in Definition 2.9 is an intersection product on \( \text{Cl}(X) \).
Proof. Let \( \pi : \tilde{X} \to X \) be a resolution of singularities and \( \tilde{\sigma} \) the holomorphic 2-form on \( \tilde{X} \) extending \( \sigma \). By [34, Theorem 1.4] we have that
\[
\tilde{\sigma} = \pi^* \sigma.
\] (8)

We first show that condition (6) holds for \( q_{\tilde{X}, \tilde{\sigma}} \) on \( \tilde{X} \) and then deduce from this that it also holds on \( X \).

To prove that (6) holds for \( q_{\tilde{X}, \tilde{\sigma}} \) we argue as in [15]. Firstly, we notice that if \( D \) and \( D' \) are two distinct effective divisors on \( \tilde{X} \) then the (numerical equivalence) class \( \{ D \cdot D' \} \) contains the closed positive \( (2, 2) \)-current given by the integration along the effective intersection cycle \( D \cap D' \).

Moreover the form \( (\tilde{\sigma} \tilde{\sigma})^{n-1} \) is a smooth positive form (of bidimension \( (2, 2) \)).

Then from Definition 2.4 we immediately deduce that
\[
q_{\tilde{X}, \tilde{\sigma}}(D, D') = \frac{n}{2} \int_{D \cap D'} (\tilde{\sigma} \tilde{\sigma})^{n-1}
\]
and the latter is \( \geq 0 \) by the above.

Since condition (6) holds for \( q_{\tilde{X}, \tilde{\sigma}} \) on \( \tilde{X} \) the \( q_{\tilde{X}, \tilde{\sigma}} \)-Zariski decomposition holds for effective divisors on \( \tilde{X} \) by Theorem 3.6.

To deduce that condition (6) holds for \( q_{X, \sigma} \) on \( X \) we argue as follows. As is Section 2.4 we will consider the open subset \( U \) effective divisors on \( \tilde{X} \) Let \( U \) their restrictions to \( \tilde{X} \). Then from Definition 2.4 we immediately deduce that
\[
(q_{\tilde{X}, \tilde{\sigma}}(D, D')) = \frac{n}{2} \int_{D \cap D'} (\tilde{\sigma} \tilde{\sigma})^{n-1}
\]
and the latter is \( \geq 0 \) by the above.

Since condition (6) holds for \( q_{\tilde{X}, \tilde{\sigma}} \) on \( \tilde{X} \) the \( q_{\tilde{X}, \tilde{\sigma}} \)-Zariski decomposition holds for effective divisors on \( \tilde{X} \) by Theorem 3.6.

To deduce that condition (6) holds for \( q_{X, \sigma} \) on \( X \) we argue as follows. As is Section 2.4 we will consider the open subset \( U = X \setminus \Sigma_0 \) where \( \Sigma_0 \) is the dissident locus. Let \( D \) and \( D' \) be two distinct prime and effective \( \mathbb{Q} \)-Weil divisors on \( X \) and \( D_U, D'_U \) their restrictions to \( U \). To render the notation less heavy we will denote by \( \pi^*_U D \) the divisor
\[
(t_{\pi^{-1}(U)})^* (\pi|_{\pi^{-1}(U)})^*(D_U)
\]
on \( \tilde{X} \) (and similarly for \( \pi^*_U D' \)). Consider the \( q_{\tilde{X}, \tilde{\sigma}} \)-Zariski decomposition on \( \tilde{X} \)
\[
\pi^*_U D' = P(\pi^*_U D') + N(\pi^*_U D')
\]
with \( P' := P(\pi^*_U D') \) and \( N' := N(\pi^*_U D') = \sum c_i N'_i \) satisfying all the items of Definition 3.5. Then
\[
q_{X, \sigma}(D, D') = q_{\tilde{X}, \tilde{\sigma}}(\pi^*_U D, \pi^*_U D') = q_{\tilde{X}, \tilde{\sigma}}(\pi^*_U D, P') + \sum c_i q_{\tilde{X}, \tilde{\sigma}}(\pi^*_U D, N'_i),
\]
where the first equality follows from Theorem 2.6 item (1) and equation (8). Notice that
\[
q_{\tilde{X}, \tilde{\sigma}}(\pi^*_U D, P') \geq 0
\]
because \( \pi^*_U D \) is effective and, by Theorem 3.6 the positive part \( P' \) is \( q_{\tilde{X}, \tilde{\sigma}} \)-nef.

For a component \( N'_i \) which is not a \( \pi \)-exceptional divisor we have that
\[
q_{\tilde{X}, \tilde{\sigma}}(\pi^*_U D, N'_i) \geq 0,
\]
because \( D \) and \( D' \) are distinct and we have already proved that (6) holds on \( \tilde{X} \).

Now if a component \( N'_i \) is a \( \pi \)-exceptional divisor we have two cases. Either \( \pi(N'_i) \) is contained in \( \Sigma_0 \) and then \( q_{\tilde{X}, \tilde{\sigma}}(\pi^*_U D, N'_i) = 0 \) because \( \tilde{\sigma}|_{N'_i} = 0 \) as observed in
Remark 2.11, equation (5). If $\pi(N'_i)$ is not contained in $\Sigma_0$, then $q_{\tilde{X},\tilde{\sigma}}(\pi^*_U D, N'_i) = 0$ by the push-pull formula

$$q_{\tilde{X},\tilde{\sigma}}(\pi^*_U D, N'_i) = \int_{\tilde{X}} (\tilde{\sigma}\tilde{\tilde{\sigma}})^{n-1}[\pi^*_U D][N'_i] = \int_{\tilde{X}} \pi^*(\sigma\tilde{\sigma})^{n-1}[\pi^*_U D][N'_i] = \int_{X} (\sigma\tilde{\sigma})^{n-1}[(\omega)_* D_U][\pi_* N'_i]$$

where the last equality holds because over $U$ we have $\pi^*_U D = \pi^* D_U$ and outside of it the forms $(\tilde{\sigma}\tilde{\tilde{\sigma}})^{n-1}$ and $(\sigma\tilde{\sigma})^{n-1}$ are identically zero by (5). Then the last integral is zero as $N'_i$ is contracted by $\pi$. Therefore

$$q_{X,\sigma}(D, D') \geq 0$$

and we are done. \qed

Proof of Theorem 1.1. It follows immediately from the combination of Theorems 3.6 and 3.11.

Remark 3.12. It is important to notice that our proof of the existence of the Boucksom-Zariski decomposition in the singular case is very different from the one in the smooth case due to Boucksom. Indeed, as mentioned in the introduction, Boucksom first shows the existence of a divisorial Zariski decomposition on a compact complex manifold $X$ (and, contrary to Bauer’s approach, this is done by defining directly the negative part, by attaching asymptotically defined multiplicities to components of the stable base-locus of $D$) and then characterizes it when $X$ is a smooth irreducible symplectic variety in terms of the Beauville-Bogomolov-Fujiki form. If one wants to follow the same path in the singular setting the first thing to do is to extend to this framework the divisorial Zariski decomposition. This is done in [13, Section 4] in the projective case. In the non-projective case things seem to be more subtle, though experts believe it should work the same (for the 3-dimensional case see e.g. [28, Section 4.B]). Still, even if we had a divisorial Zariski decomposition in the singular case its characterization in terms of the Beauville-Bogomolov-Fujiki form uses several results (see [15, Section 4]) for smooth irreducible symplectic varieties not available yet in the singular case.

Remark 3.13. If $X$ is a projective variety with symplectic singularities and $\sigma$ a symplectic form on it, there is an alternative and easier way out to deal with non-$\mathbb{Q}$-Cartier components of $\mathbb{Q}$-Cartier divisors. Indeed by [12, Lemma 10.2] there exists a small birational morphism $\varphi : Y \to X$ from a $\mathbb{Q}$-factorial projective variety $Y$ onto $X$. Then for any Weil divisor $D$ on $X$ we can define $q_{X,\sigma}(D)$ as $q_{Y,\varphi^*\sigma}(\varphi^{-1}D)$. By the smallness of $\varphi$, if $D$ is a $\mathbb{Q}$-Cartier divisor we have that $\varphi^{-1}(D) = \varphi^* D$ and therefore the previous definition is consistent with the old one thanks to Theorem 2.6, item (1). Unfortunately the analogous existence result in the non-projective case is not known, which is why we were forced to our more involuted approach presented in Section 2.4. Anyway, in the projective case, by Remark 2.11 the two approaches are equivalent. \qed
Now we would like to present an effective way to bound coefficients in Zariski decompositions for (pseudo)effective divisors on any projective irreducible symplectic variety $X$. By Theorem 1.1 any effective Cartier divisor $D$ on $X$ can be uniquely presented as

$$D = P + \sum_{i=1}^{k} a_i N_i, \quad a_i \in \mathbb{Q}_{>0}$$

with $P$ and the $N_i$’s as in Definition 3.5 (if $X$ is smooth this also holds for any pseudoeffective divisor). Notice that we have $k \leq \rho(X) - 1 \leq h^{1,1}(X) - 1$. Indeed, we have that $q_X(N_i) < 0$. If for some $i \neq j$ the prime divisors $N_i$ and $N_j$ were linearly equivalent, then

$$0 > q_X(N_i) = q_X(N_i, N_j) \geq 0$$

since $q_X$ is an intersection product. Then necessarily $k \leq \text{rk}_{NS}(X) - 1 = \rho(X) - 1$.

It is natural to ask whether there exists an integer $d(X) > 1$ such that for every effective integral divisor $D$ the denominators in the Zariski decomposition of $D$ are bounded from above by $d(X)$. If such a bound $d(X)$ exists, then we say that $X$ has bounded Zariski denominators.

From now on, to lighten the notation, we set $q = q_X$. Consider the following system of linear equations which is given by taking $q(D, N_j)$ for each $j \in \{1, ..., k\}$, namely

$$q(D, N_j) = q(P, N_j) + \sum_{i=1}^{k} a_i q(N_i, N_j) = \sum_{i=1}^{k} a_i q(N_i, N_j).$$

Rewriting the above line in a compact way, we obtain

$$(q(D, N_1), ..., q(D, N_k))^T = \left[ q(N_i, N_j) \right]_{i,j=1,...,k} \cdot (a_1, ..., a_k)^T. \quad (9)$$

**Lemma 4.1.** Notation as above. We have

$$a_i = \frac{\det S_i}{\det \left[ q(N_i, N_j) \right]_{k \times k}}$$

where $S_i$ is the $(k \times k)$ matrix obtained from $\left[ q(N_i, N_j) \right]_{k \times k}$ by replacing its $i$-th column with the vector $(q(D, N_1), ..., q(D, N_k))^T$.

**Proof.** Item (1) follows from (9) using Cramer’s rule. For item (2), notice that the $a_i$’s are positive since $q(D, N_i) < 0$ for each $i \in \{1, ..., k\}$ – otherwise $N_i$ would not be in the negative part of $D$, for instance by [13, Lemma 4.9]. \qed

By the above considerations, we obtained an upper bound on the denominators of the coefficients $a_i$ by the value $|\det[ q(N_i, N_j) ]_{k \times k} |$. This gives a relation between boundedness of denominators in the Boucksom-Zariski decompositions and...
the boundedness of the squares of prime exceptional divisors. A similar phenomenon was observed in the case of algebraic surfaces in \cite{[S]}. This observation might not be very surprising, mostly due to the fact that one expects that irreducible symplectic manifolds behave like surfaces for what concerns linear series once one replaces the intersection form with the Beauville-Bogomolov-Fujiki quadratic form. The precise statement is the following:

**Theorem 4.2.** Let $X$ be an irreducible symplectic variety of dimension $2n \geq 2$. Then the following two conditions are equivalent:

1) the self-intersection numbers with respect to Bogomolov-Beauville form $q$ of integral prime divisors are bounded from below;

2) the denominators of coefficients $a_i$ of the negative parts in the Boucksom-Zariski decompositions of (pseudo)effective Cartier divisors are globally bounded.

**Proof of Theorem 4.2.** For the proof one proceeds along the same lines as in \cite{[S]} replacing the intersection form on the surface with the Beauville-Bogomolov-Fujiki quadratic form. □

**Remark 4.3.** Let us come back to the introduction, if $X$ is a $K3$ surface, then by the adjunction formula the self-intersection numbers of prime curves are bounded by $-2$, and the denominators of coefficients in Zariski decompositions are bounded from above by $2^{e-1}!$, where $\rho$ is equal to the Picard number, so from now on we assume that $n > 1$.

**Remark 4.4.** If $X$ is smooth in item 2) of the theorem above we can take the divisors to be pseudoeffective, as we do have a Boucksom-Zariski decomposition in this case, thanks to Boucksom. This will be relevant in our Corollary 1.3.

**Remark 4.5.** In fact, the proof provides an effective way to find a bound in any of the two items of Theorem 4.2 knowing the bound in the other. More precisely, if the self-intersections are bounded by $b$ then the denominators are bounded by $|b|^\rho(X)-1!$. Conversely if the denominators are bounded by $d$ then the self-intersections are bounded by $d! \cdot d \cdot \text{Card}(A_{NS}(X))$ where $\text{Card}(A_{NS}(X))$ is the discriminant of the Neron–Severi Lattice of $X$.

In order to find a global bound for denominators it is then sufficient to show that bounded negativity holds in the context of irreducible symplectic manifolds. In dimension 2 it is well known by adjunction that the squares of irreducible curves are bounded from below by $-2$. In higher dimension we first need to recall results devoted to the divisibility of exceptional prime divisors and their geometry by Druel and Markman.

**Proposition 4.6** (\cite{[L]}, Proposition 1.4 and Remark 4.3). Let $X$ be a projective irreducible symplectic manifold and let $E$ be a prime exceptional divisor on it.

(i) Then there exists another irreducible symplectic manifold $X'$ birational to $X$ and a contraction $\pi': X' \to Y'$ to a normal variety whose exceptional locus is the strict transform $E'$ of $E$. 

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(ii) If \( l \) is a general primitive exceptional curve (i.e. a curve whose strict transform is a general fiber of \( E' \to \pi'(E') \)), then \( l \) is either a smooth \( \mathbb{P}^1 \) or the union of two such \( \mathbb{P}^1 \) meeting in a point and moreover \( E \cdot l = -2 \).

**Proposition 4.7** ([42], Corollary 3.6 part 1 and 3). Let \( X, E \) and \( l \) be as above. Let us identify \( H^2(X, \mathbb{Q})^\vee \) with \( H_2(X, \mathbb{Q}) \) and let \( E^\vee \) be the map given by \( q(E, \cdot) \) with the Beauville-Bogomolov-Fujiki form. Then:

(i) the class of \( l \) is \( -\frac{2E^\vee}{q(E)} \);

(ii) either \( E \) is primitive or \( E/2 \) is.

**Remark 4.8.** The two propositions above, with the exception of Proposition 4.7, item (i) have been proved for \( \mathbb{Q} \)-factorial and projective primitive symplectic varieties by Lehn-Mongardi-Pacienza in [38, Theorem 1.2, item (1)]. The proof of Proposition 4.7, item (i) is the objet of a work in progress of the same authors.

From these two results, we deduce the following.

**Proposition 4.9.** Let \( X \) be a projective irreducible symplectic variety and let \( E \) be a prime exceptional divisor on \( X \). Assume that the conclusions of Propositions 4.6 and 4.7 hold. Then \( |q(E)| \leq 4 \text{Card}(A_X) \), where \( A_X \) is the finite discriminant group \( H^2(X, \mathbb{Z})^\vee / H^2(X, \mathbb{Z}) \).

**Proof.** By the conclusion of Proposition 4.7, either \( E \) or \( E/2 \) is primitive. Let \( D \) be this primitive class. Let \( d \) be the divisibility of \( D \), that is the positive generator of the ideal \( q(D, H^2(X, \mathbb{Z})) \subset \mathbb{Z} \). Notice that \( d \) divides \( q(E) \). From the conclusion of Proposition 4.7, item (i), we have that \( -\frac{2q(E)}{q(E)} \) is an integer class, which means that \( q(E) \) divides \( 2q(E, \cdot) \). This last term generates the ideal \( 2d\mathbb{Z} \subset \mathbb{Z} \) if \( E \) is primitive and \( 4d\mathbb{Z} \subset \mathbb{Z} \) otherwise. Therefore, we have one of the following:

a) \( D = E \) and \( q(E) = kd, k = 1, 2 \).

b) \( 2D = E \) and \( q(E) = kd, k = 1, 2, 4 \).

Hence, we only need to bound \( d \). Let us consider the lattice \( L := H^2(X, \mathbb{Z}) \) (which is a topological invariant) and the natural inclusion \( L \hookrightarrow L^\vee \) given by sending an element \( t \in L \) to \( q(t, \cdot) \). This inclusion has finite index and \( L^\vee / L = A_X \) is a finite group. Since \( -\frac{2q(E)}{q(E)} \) is an integer class, by a) or b) above, the element \( \frac{2q(D, \cdot)}{kd} \) lies in \( L^\vee \) and it gives a class in \( A_X \) of order \( kd/2 \), as \( D \) is primitive. Therefore, \( d \) divides \( 2 \text{Card}(A_X) \) and our claim holds.

Notice that, if \( \text{dim}(X) \geq 4 \), the bound \( 4 \text{Card}(A_X) \) is sharp, as it is obtained in the case of the Hilbert-Chow exceptional divisor on the Hilbert scheme of points on a \( K3 \) surface. We can say the same about generalized Kummer varieties. However, when \( A_X \) is not cyclic, its order can be replaced by the highest order of its elements.

In the following table we summarize in the four known deformation classes what is \( A_X \), the order \( d \) of its highest order element and the highest absolute value of the square of a prime exceptional divisor:
| Deformation type         | $A_X$       | Order $d$ | Highest negative divisor square |
|-------------------------|-------------|-----------|----------------------------------|
| $K3^{[n]}$-type         | $\mathbb{Z}/(2n-2)\mathbb{Z}$ | $2n-2$    | $8n-8$                           |
| Kummer $n$ type         | $\mathbb{Z}/(2n+2)\mathbb{Z}$ | $2n+2$    | $8n+8$                           |
| O’Grady sixfolds        | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | $2$       | $8$                              |
| O’Grady tenfolds        | $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ | $3$       | $6$                              |

The above values can be found in [41, Section 9] for $K3^{[n]}$-type manifolds, in [52] for Kummer $n$ type, in [47] for O’Grady’s sixfolds and [40] for O’Grady’s tenfolds.

**Remark 4.10.** There has been an effort to prove a similar boundedness result for a wider class of divisors, called Wall divisors (see [43, Definition 1.2]). Prime exceptional divisors are automatically wall divisors (see e.g. [43, Lemma 1.4]), while an example of a wall divisor which is not prime exceptional is obtained by considering a divisor whose class is dual to the class of a line in a projective plane inside an IHS fourfold. A complete boundedness result for wall divisors (with no hypothesis on the IHS deformation class) was proven in [11]. However their result does not give an explicit bound, which is what we need and prove here to obtain an explicit bound for the denominators. The result is rather formulated for MBM classes, but these are exactly the classes of curves dual to wall divisors (see e.g. [36, Remark 2.4].

**Corollary 4.11.** Let $X$ be a projective irreducible symplectic variety of dimension $> 2$. Assume that the conclusions of Propositions 4.6 and 4.7 hold for $X$. Then the denominators of the coefficients of the positive and negative parts in the Boucksom-Zariski decompositions of all effective Cartier divisors are bounded by $(4 \text{Card}(A_X))^{\rho(X)-1}!$. In particular, on every $Y$ deformation equivalent to $X$ these coefficients are globally bounded by $(4 \text{Card}(A_X))^{h^{1,1}(X)-1}!$

**Proof.** The first part follows immediately from the proof of Theorem 1.2 and Proposition 4.9. For the second statement we use the fact that the lattices $(H^2(X,\mathbb{Z}), q)$ and $(H^2(Y,\mathbb{Z}), q)$ are isometric to each other by Ehresmann Lemma.

5. Applications to effective birationality

We collect in this section the applications to effective birationality of big and effective line bundles $L$ on projective holomorphic symplectic manifolds. If one could control the singularities of members of the linear system $|L|$, the existence of a uniform (although not explicit) bound would follow from [27, Theorem 1.3]. For the history of the problem of the effectivity of birational plurigenera maps (and more generally of the Iitaka fibration) in the framework of the MMP and a list of references we refer the interested reader to [27], [14] and the very recent [11].

**Proof of Corollary 1.3.** Let $L$ be a big line bundle on $X$. Consider the Boucksom-Zariski decomposition

$$aL = P + N,$$

where $P \in \mathcal{FE}_X$ is a Cartier divisor, $N$ a Cartier $q_X$-exceptional divisor and

$$a = (4 \text{Card}(A_X))^{\rho(X)-1}!$$

is the integer given by Theorem 1.2 and clearing the denominators in the Boucksom-Zariski decompositions. Since $L$ is big by hypothesis, the positive part $P$ is big. By
the work of Huybrechts \cite{Huybrechts} and Boucksom \cite{Boucksom} (see e.g. \cite{Huybrechts} Proposition 5.6) we have that $\mathcal{FE}_X$ coincides with the closure of the birational Kähler cone. Therefore there exists a smooth projective irreducible symplectic variety $X'$ and a birational map

$$\varphi : X \dashrightarrow X'$$

such that

$$P' := f_* P$$

is an integral, nef and big divisor on $X'$. Now we can apply Kollár’s extension \cite{Kollár} Theorem 5.9] to nef and big divisors of the Angehrn-Siu result to $P'$, to deduce that for all $m \geq (\dim X + 2)(\dim X + 3)/2$ the morphism associated to $|mP'|$ is injective on $X'$. As a by-product we then obtain that the linear system $|a m L|$ separates two generic points on $X$. □

**Corollary 5.1.** Let $n$ be a positive integer and $C$ be a positive constant. Then the family of all smooth projective irreducible symplectic varieties of dimension $2n$, of a fixed deformation type and endowed with a big line bundle of volume at most $C$, is birationally bounded.

**Proof.** By Corollary \cite{Corollary 1.3} and \cite{Lemma 2.2] it follows that any projective irreducible symplectic varieties of dimension $2n$, endowed with a big line bundle $L$ having volume $\leq C$ is birational to a subvariety of a projective space (which by projection we may assume to be $\mathbb{P}^{4n+1}$) of degree at most $m_0^{2n}C$, where

$$m_0 = \frac{1}{2}(2n + 2)(2n + 3)(4 \text{ Card}(A_X))^{h^{1,1}(X) - 1!}$$

is the integer given by \cite{11}. Such subvarieties are parametrized by the points of an algebraic variety (the Chow scheme) and the result follows. □

**Remark 5.2.** It is interesting to notice that the proof of Corollary \cite{Corollary 1.3}, hence of Corollary \cite{Corollary 5.1} goes through for possibly singular projective irreducible symplectic varieties $X$ as soon as the Boucksom-Zariski decomposition holds for pseff divisors, Propositions \cite{Propositions 4.6} and \cite{Propositions 4.7} hold for $X$ and $\mathcal{FE}_X$ lies inside the closure of the birational Kähler cone. The outcome is the same effective birationality result and birational boundedness for such varieties.

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