Parallel Batch-Dynamic $k$-Clique Counting

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Abstract

In this paper, we study new batch-dynamic algorithms for $k$-clique counting, which are dynamic algorithms where the updates are batches of edge insertions and deletions. We study this problem in the parallel setting, where the goal is to obtain algorithms with low (poly-logarithmic) depth. Our first result is a new parallel batch-dynamic triangle counting algorithm with $O(\Delta \sqrt{\Delta + m})$ amortized work and $O(\log^*(\Delta + m))$ depth with high probability (w.h.p.), and $O(\Delta + m)$ space for a batch of $\Delta$ edge insertions or deletions. Our second result is a simple parallel batch-dynamic $k$-clique counting algorithm that uses a newly developed parallel $k$-clique counting algorithm to bootstrap itself, by enumerating smaller cliques, and intersecting them with the batch. Instantiating this idea gives a simple batch-dynamic algorithm running in $O(\Delta(m + \Delta)2^{k-4})$ expected work and $O(\log^k n)$ depth w.h.p., all in $O(m + \Delta)$ space. Our third result is an algebraic algorithm based on parallel fast matrix multiplication. Assuming that a parallel fast matrix multiplication algorithm exists with parallel matrix multiplication constant $\omega_p$, the same algorithm solves dynamic $k$-clique counting with $O\left(\min\left(\Delta m^{\frac{2k-1}{3m\omega_p+1}}, (\Delta + m)^{\frac{2k+1}{3m\omega_p+1}}\right)\right)$ amortized work, $O(\log(\Delta + m))$ depth, and $O\left((\Delta + m)^{\frac{2k+1}{3m\omega_p+1}}\right)$ space.
1 Introduction

Subgraph counting problems are fundamental graph analysis tools, with numerous applications in network classification in domains including social network analysis and bioinformatics. A particularly important type of subgraph for these applications is the triangle, or 3-clique—three vertices which are all mutually connected [New03]. Counting the number of triangles is a basic and fundamental task that is used in numerous social and network science measurements [Gra77, WS98, NG04].

In this paper, we study the triangle counting problem and its generalization to higher cliques from the perspective of dynamic algorithms. A $k$-clique consists of $k$ vertices and all $\binom{k}{2}$ possible edges among them (for applications of $k$-cliques, see, e.g., [HR05]). As many real-world graphs change rapidly in real-time, it is crucial to design dynamic algorithms that efficiently maintain $k$-cliques upon updates, since the cost of re-computation from scratch can be prohibitive. Furthermore, due to the fact that dynamic updates can occur at a rapid rate in practice, it is increasingly important to design batch-dynamic algorithms which can take arbitrarily large batches of updates (edge insertions or deletions) as their input. Finally, since the batches, and corresponding update complexity can be large, it is also desirable to use parallelism to speed-up maintenance and design algorithms that map to modern shared-memory architectures.

Due to the broad applicability of $k$-clique counting in practice and the fact that $k$-clique counting is a fundamental theoretical problem of its own right, there has been a large body of prior work on the problem. Theoretically, the fastest static algorithm for arbitrary graphs uses fast matrix multiplication, and counts $3\ell$ cliques in $O(n^k\omega)$ time where $\omega$ is the matrix multiplication exponent [NP85]. Similar effort has also been devoted to efficient combinatorial algorithms. Chiba and Nishizeki [CN85] show how to compute $k$-cliques in $O(\alpha^{k-2}m)$ work, where $m$ is the number of edges in the graph and $\alpha$ is the arboricity of the graph. This algorithm was recently parallelized by Danisch et al. [DBS18] (although not in polylogarithmic depth). Worst-case optimal join algorithms can perform $k$-clique counting in $O(m^{k/2})$ work as a special case [NPRR18, ALT+17]. Alon, Yuster, and Zwick [AYZ97] design an algorithm for triangle counting in the sequential model, based on fast matrix multiplication. Eisenbrand and Grandoni [EG04] then extend this result to $k$-clique counting based on fast matrix multiplication. Vassilevska designs a space-efficient combinatorial algorithm for $k$-clique counting [Vas09]. Finocchi et al. give clique counting algorithms for MapReduce [FFF15]. Jain and Seshadri provide probabilistic algorithms for estimating clique counts [JS17].

The $k$-clique problem is also a classical problem in parametrized-complexity, and is known to be $W[1]$-complete [DF95]. In general graphs, Chen et al. show that if $k$-clique admits a $o(n^k)$ time algorithm, many classic NP-complete problems admit sub-exponential time algorithms [CHKX04].

The problem of maintaining $k$-cliques under dynamic updates began more recently. Eppstein et al. [ES09, EGST12] design sequential dynamic algorithms for maintaining size-3 subgraphs in $O(h)$ amortized time and $O(mh)$ space and size-4 subgraphs in $O(h^2)$ amortized time and $O(mh^2)$ space, where $h$ is the $h$-index of the graph ($h = O(\sqrt{m})$). Ammar et al. extend the worst-case optimal join algorithms to the parallel and dynamic setting [AMSJ18]. However, their update time is not better than the static worst-case optimal join algorithm. Recently, Kara et al. [KNN+19] present a sequential dynamic algorithm for maintaining triangles in $O(\sqrt{m})$ amortized time and $O(m)$ space. Dvorak and Tuma [DT13] present a dynamic algorithm that maintains $k$-cliques as a special case in $O(\alpha^{k-2}\log n)$ amortized time and $O(\alpha^{k-2}m)$ space by using low-outdegree orientations for graphs with arboricity $\alpha$. To the best of our knowledge, no prior work has designed algorithms for the problem that support batch updates, or dynamic algorithms with non-trivial update bounds that are also highly parallel (polylogarithmic depth).

Other Related Work. There has been significant amount of work on practical parallel algorithms for the case of static 3-clique counting, also known as triangle counting (e.g., [SV11, AKM13, PC13, PSKP14, ST15], among many others). Due to the importance of the problem, there is even an annual competition for parallel triangle counting solutions [Gra]. Ediger et al. [EJRB10] and Makkar et al. [MBG17] design parallel algorithms for dynamic triangle counting, but in the worst case their updates take linear work. Practical static
counting algorithms for the special cases of $k = 4$ and $k = 5$ have also been developed [HD14, ESBD16, PSV17, ANR+17, DAH17].

Theoretically-efficient parallel dynamic algorithms have been designed for a variety of other graph problems, including minimum spanning tree [KPR18, FL94, DF94], Euler trees [TDB19], connectivity [STTW18, AABD19, FL94], tree contraction [RT94, AAW17], and depth-first search [Kha17]. Very recently, parallel dynamic algorithms were also designed for the Massively Parallel Computation (MPC) setting [ILMP19, DDK+20].

Lastly, dynamic algorithms have had been studied in distributed models of computation under the framework of self-stabilization [Sch93]. In this setting, the system undergoes various changes, for example topology changes, and must quickly converge to a stable state. Most of the existing work in this setting focuses on a single change per round [CHHK16, BCH19, AOSS19], although algorithms studying multiple changes per round have been considered very recently [BKM19, CHDK+19]. Understanding how these algorithms relate to parallel batch-dynamic algorithms is an interesting question for future work.

Summary of Our Contributions. In this paper, we design parallel algorithms in the batch-dynamic setting, where the algorithm receives a batch of $\Delta \geq 1$ edge updates which are not internally ordered (and can therefore be processed in parallel). Our focus is on parallel batch-dynamic algorithms that admit strong theoretical bounds on their work and have polylogarithmic depth with high probability. Note that although our work bounds may be amortized, our depth will be polylogarithmic with high probability, leading to efficient RNC algorithms. As a special case of our results, we obtain algorithms for parallelizing single updates ($\Delta = 1$). We first design a parallel batch-dynamic triangle counting algorithm based on the sequential algorithm of Kara et al. [KNN+19]. For triangle counting, we obtain an algorithm that takes $O(\Delta \sqrt{\Delta + m})$ amortized work and $O(\log^*(\Delta + m))$ depth w.h.p. assuming a fetch-and-add instruction that runs in $O(1)$ work and depth, running in $O(\Delta + m)$ space.

We then give a simple (and potentially practically interesting) batch-dynamic $k$-clique listing algorithm, based on enumerating smaller cliques and intersecting them with edges in the batch. The algorithm runs in $O(\Delta (m + \Delta)^{\alpha^{k-4}})$ work and has polylogarithmic depth, using $O(m + \Delta)$ space.

Lastly, we present a new parallel batch-dynamic algorithm based on fast matrix multiplication. Using the best currently known parallel matrix multiplication [Wil12, LG14], our algorithm dynamically maintains the number of $k$-cliques in $O\left(\min\left(\Delta m^{0.469k-0.235}, (\Delta + m)^{0.469k+0.469}\right)\right)$ amortized work w.h.p. per batch of $\Delta$ updates where $m$ is defined as the maximum number of edges in the graph before and after all updates in the batch are applied. Our approach is based on the algorithm of [AYZ97, EG04, NP85], and maintains triples of $k/3$-cliques that together form $k$-cliques. The depth is $O(\log(\Delta + m))$ w.h.p. and space is $O\left((\Delta + m)^{0.469k+0.469}\right)$. Our results also imply an amortized time bound of $O\left(m^{0.469k-0.235}\right)$ per update for dense graphs in the sequential setting. Of potential independent interest, we present the first proof of logarithmic depth in the parallelization of any tensor-based fast matrix multiplication algorithms.

1.1 Preliminaries

Given an undirected graph $G = (V, E)$ with $n$ vertices and $m$ edges, and an integer $k$, a $k$-clique is defined as a set of $k$ vertices $v_0, \ldots, v_k$ such that for all $i \neq j$, $(v_i, v_j) \in E$. The $k$-clique count is the total number of $k$-cliques in the graph. The dynamic $k$-clique problem maintains the number of $k$-cliques in the graph upon edge insertions and deletions, given individually or in a batch. The arboricity $\alpha$ of a graph is the minimum number of forests that the edges can be partitioned into and its value is between $\Omega(1)$ and $O\left(\sqrt{m}\right)$ [CN85].

In this paper, we analyze algorithms in the work-depth model, where the work of an algorithm is defined to be the total number of operations done, and the depth is defined to be the longest sequential dependence

\footnote{We use “with high probability” (w.h.p.) to mean with probability at least $1 - 1/n^c$ for any constant $c > 0.$}
in the computation (or the computation time given an infinite number of processors) [Ja92]. Our algorithms can run in the nested-parallel model or the PRAM model. We use the concurrent-read concurrent-write (CRCW) model, where reads and writes to a memory location can happen concurrently. We assume either that concurrent writes are resolved arbitrarily, or are reduced together (i.e., fetch-and-add PRAM).

We use the following primitives throughout the paper. Approximate compaction takes a set of $m$ objects in the range $[1, n]$ and allocates them unique IDs in the range $[1, O(m)]$. The primitive is useful for filtering (i.e. removing) out a set of obsolete elements from an array of size $n$, and mapping the remaining $m$ live elements to a sparse array of size $O(m)$. Approximate compaction can be implemented in $O(n)$ work and $O(\log^* n)$ depth [GMV91]. We also use a parallel hash table which supports $n$ operations (insertions, deletions) in $O(n)$ work and $O(\log^* n)$ depth with high probability, and $n$ lookup operations in $O(n)$ work and $O(1)$ depth [GMV91].

Our algorithms in this paper make use of the widely used fetch-and-add instruction. A fetch-and-add instruction takes a memory location and atomically increments the value stored at the location. In this paper we assume that the fetch-and-add instruction can be implemented in $O(1)$ work and depth. Existing simulation results show that the CRCW PRAM augmented with a fetch-and-add instruction can be simulated work-efficiently at the cost of a space increase proportional to the number of fetch-and-adds done, and a multiplicative $O(\log n)$ factor increase in the depth [MV91].

2 Technical Overview

In this section, we present a high-level technical overview of our approach in this paper.

Parallel Batch-Dynamic Triangle Counting

Our parallel batch-dynamic triangle counting algorithm is based on a recently proposed sequential dynamic algorithm due to Kara et al. [KNN+19]. They describe their algorithm in the database setting, in the context of dynamically maintaining the result of a database join. We provide a self-contained description of their sequential algorithm in Appendix A.

High-Level Approach. The basic idea of the algorithm from [KNN+19] is to partition the vertex set using degree-based thresholding. Roughly, they specify a threshold $t = \Theta(\sqrt{m})$, and classify all vertices with degree less than $t$ to be low-degree, and all vertices with degree larger than $t$ to be high-degree. This thresholding technique is widely used in the design of fast static triangle-counting and $k$-clique counting algorithms, (e.g., [NP85, AYZ97]). Observe that if we insert an edge $(u, v)$ incident to a low-degree vertex, $u$, we can enumerate all $w$ in $N(u)$ in $O(\sqrt{m})$ expected time and check if $(u, v, w)$ forms a triangle (checking if the $(v, w)$ edge is present in $G$ can be done by storing all edges in a hash table). In this way, edge updates incident to low-degree vertices are handled relatively simply. The more interesting case is how to handle edge updates between high-degree vertices. The main problem is that a single edge insertion $(u, v)$ between two high-degree vertices can cause up to $O(n)$ triangles to appear in $G$, and enumerating all of these would require $O(n)$ work—potentially much more than $O(\sqrt{m})$. Therefore, the algorithm maintains an auxiliary data structure, $\mathcal{T}$, over wedges (2-paths). $\mathcal{T}$ stores for every pair of high-degree vertices $(v, w)$, the number of low-degree vertices $u$ that are connected to both $v$ and $w$ (i.e., $(u, v)$ and $(u, w)$ are both in $E$). Given this structure, the number of triangles formed by the insertion of the edge $(v, w)$ going between two high-degree vertices can be found in $O(1)$ time by checking the count for $(v, w)$ in $\mathcal{T}$. Updates to $\mathcal{T}$ can be handled in $O(\sqrt{m})$ time, since $\mathcal{T}$ need only be updated when a low-degree vertex inserts/deletes a neighbor, and the number of entries in $\mathcal{T}$ that are affected is at most $t = O(\sqrt{m})$. Some additional care needs to be taken when specifying the threshold $t$ to handle re-classifying vertices (going from low-degree to high-degree, or
vice versa), and also to handle rebuilding the data structures, which leads to a bound of $O(\sqrt{m})$ amortized work per update for the algorithm.

**Incorporating Batching and Parallelism.** The input to the parallel batch-dynamic algorithm is a batch containing (possibly) a mix of edge insertions and deletions (vertex insertions and deletions can be handled by inserting or deleting its incident edges). For simplicity, and without any loss in our asymptotic bounds, our algorithm handles insertions and deletions separately. The algorithm first removes all *nullifying* updates, which are updates that have no effect after applying the entire batch (i.e., an insertion which is subsequently deleted within the same batch, an insertion of an edge that already exists or a deletion of an edge that doesn’t exist). This can easily be done within the bounds using basic parallel primitives. The algorithm then updates tables representing the adjacency information of both low-degree and high-degree vertices in parallel. To obtain strong parallel bounds, we represent these sets using parallel hash tables. For each insertion (deletion), we then determine the number of new triangles that are created (deleted). Since a given triangle could incorporate multiple edges within the same batch of insertions (deletions), our algorithm must carefully ensure that the triangle is counted only once, assigning each new inserted (deleted) triangle uniquely to one of the updates forming it. We then update the overall triangle count with the number of distinct triangles inserted (deleted) into the graph by the current batch of insertions (deletions). The remaining work of the algorithm cleans up mutable state such as marking of edges contained in the current update in the hash tables, and also migrating vertices between low-degree and high-degree states.

**Worst-Case Optimality.** We note that the Kara et al. algorithm which is the basis for our parallel batch-dynamic triangle counting algorithm is conditionally optimal under the Online Matrix-Vector Multiplication (OMv) conjecture [HKNS15, KNN+19]. The same result in the sequential setting implies that our parallel work-bounds, which are work-efficient with respect to the Kara et al. algorithm [KNN+19], are conditionally optimal. It is an interesting question whether our depth bounds are conditionally optimal on the CRCW PRAM.

**Dynamic $k$-Clique Counting via Fast Static Parallel Algorithms**

Next, we present a very simple, and potentially practical algorithm for dynamically maintaining the number of $k$-cliques based on statically enumerating smaller cliques in the graph, and intersecting the enumerated cliques with the edge updates in the input batch. The algorithm is space-efficient, and is asymptotically more efficient than other methods for sparse graphs. Our algorithm is based on a recent and concurrent work proposing a work-efficient parallel algorithm for counting $k$-cliques in work $O(m\alpha^{k-2})$ and polylogarithmic depth [SDS20]. Using this algorithm, we show that updating the $k$-clique count for a batch of $\Delta$ updates can be done in $O(\Delta(m + \Delta)\alpha^{k-4})$ work, and polylogarithmic depth, using $O(m + \Delta)$ space by using the static algorithm to (i) enumerate all $(k-2)$-cliques, and (ii) checking whether each $(k-2)$-clique forms a $k$-clique with an edge in the batch. This algorithm strictly outperforms re-computation using the new static parallel algorithm for $\Delta < \alpha^2$.

**Dynamic $k$-Clique via Fast Matrix Multiplication**

We then present a parallel batch-dynamic $k$-clique counting algorithm using parallel fast matrix multiplication (MM). Our algorithm is inspired by the static triangle counting algorithm of Alon, Yuster, and Zwick (AYZ) [AYZ97] and the static $k$-clique counting algorithm of [EG04] that uses MM-based triangle counting. We present a new dynamic algorithm that obtains better bounds than the simple algorithm based on static lower-clique enumeration above (and also presented in Section 4) for larger values of $k$. Specifically, assuming a parallel matrix multiplication exponent of $\omega_p$, our algorithm handles batches of $\Delta$ edge insertions/deletions using $O\left(\min\left(\frac{(2k-3)\omega_p}{3(1+\omega_p)}, (m + \Delta)^\frac{2k\omega_p}{3(1+\omega_p)}\right)\right)$ work and $O(\log m)$ depth w.h.p., in
We determine the specific value for \( t \) when a \( 2\)-\( \ell \)-vertices with degrees in the range \( \ell \) vertices, because there is an upper bound on the maximum number of such vertices in the graph, we update vertices that creates or destroys a triangle that contains a low-degree vertex in \( G \). Thus, a triangle in \( G' \) represents a \( k \)-clique in \( G \). Specifically, there exist exactly \( \binom{k}{k/3} \binom{2k/3}{k/3} \) different triangles in \( G' \) for each clique in \( G \).

Given a batch of edge insertions and deletions to \( G \), we create a set of edge insertions and deletions to \( G' \). An edge is inserted in \( G' \) when a new \( 2k/3 \)-clique is created in \( G \) and an edge is deleted in \( G' \) when a \( 2k/3 \)-clique is destroyed in \( G \). Suppose, for now, that we have a dynamic algorithm for processing the edge insertions/deletions into \( G' \). Counting the number of triangles in \( G' \) after processing all edge insertions/deletions and dividing by \( \binom{k}{k/3} \binom{2k/3}{k} \) provides us with the exact number of cliques in \( G \).

There are a number of challenges that we must deal with when formulating our dynamic triangle counting algorithm for counting the triangles in \( G' \):

1. We cannot simply count all the triangles in \( G' \) after inserting/deleting the new edges as this does not perform better than a trivial static algorithm.

2. Any trivial dynamization of the AYZ algorithm will not be able to detect all new triangles in \( G' \). Specifically, because the AYZ algorithm counts all triangles containing a low-degree vertex separately from all triangles containing only high-degree vertices, if an edge update only occurs between high-degree vertices, a trivial dynamization of the algorithm will not be able to detect any triangle that the two high-degree endpoints make with low-degree vertices.

To solve the first challenge, we dynamically count low-degree and high-degree vertices in different ways. Let \( \ell = k/3 \) and \( M = 2m + 1 \). For some value of \( 0 < t < 1 \), we define low-degree vertices to be vertices that have degree less than \( M^{\ell t}/2 \) and high-degree vertices to have degree greater than \( 3M^{\ell t}/2 \). Vertices with degrees in the range \([M^{\ell t}/2, 3M^{\ell t}/2]\) can be classified as either low-degree or high-degree. We determine the specific value for \( t \) in Lemma 5.12. We perform rebalancing of the data structures as needed as they handle more updates. For low-degree vertices, we only count the triangles that include at least one newly inserted/deleted edge, at least one of whose endpoints is low-degree. This means that we do not need to count any pre-existing triangles that contain at least one low-degree vertex. For the high-degree vertices, because there is an upper bound on the maximum number of such vertices in the graph, we update an adjacency matrix \( A \) containing edges only between high-degree vertices. At the end of all of the edge updates, computing \( A^3 \) gives us a count of all of the triangles that contain three high-degree vertices.

This procedure immediately then leads to our second challenge. To solve this second challenge, we make the observation (proven in Lemma 5.3) that if there exists an edge update between two high-degree vertices that creates or destroys a triangle that contains a low-degree vertex in \( G' \), then there must exist at least one new edge insertion/deletion that creates or destroys a triangle representing the same clique to that low-degree vertex in the same batch of updates to \( G' \). Thus, we can use one of those edge insertions/deletions to determine the new clique that was created and, through this method, find all triangles containing at least one low-degree vertex and at least one new edge update. Some care must be observed in implementing this procedure in order to not increase the runtime or space usage; such details can be found in Section 5.2.

Incorporating Batching and Parallelism When dealing with a batch of updates containing both edge insertions and deletions, we must be careful when vertices switch from being high-degree to being low-
degree and vice versa. If we intersperse the edge insertions with the edge deletions, there is the possibility that a vertex switches between low and high-degree multiple times in a single batch. Thus, we batch all edge deletions together and perform these updates first before handling the edge insertions. After processing the batch of edge deletions, we must subsequently move any high-degree vertices that become low-degree to their correct data structures. After dealing with the edge insertions, we must similarly move any low-degree vertices that become high-degree to the correct data structures. Finally, for triangles that contain more than one edge update, we must account for potential double counting by different updates happening in parallel. Such challenges are described and dealt with in Section 5.2 and Algorithm 5.

3 Parallel Batch-Dynamic Triangle Counting

In this section, we present our parallel batch-dynamic triangle counting algorithm, which is based on the \(O(m)\) space and \(O(\sqrt{m})\) amortized update, sequential, dynamic algorithm of Kara et al. [KNN+19]. Theorem 3.1 summarizes the guarantees of our parallel batch-dynamic triangle counting algorithm.

**Theorem 3.1.** There exists a parallel batch-dynamic triangle counting algorithm that requires \(O(\Delta(\sqrt{\Delta + m}))\) amortized work and \(O(\log^* (\Delta + m))\) depth with high probability, and \(O(\Delta + m)\) space for a batch of \(\Delta\) edge updates, using fetch-and-add.

We provide a detailed description of the fully dynamic sequential algorithm of [KNN+19] in Appendix A for reference, and a brief high-level overview of that algorithm in this section.

3.1 Sequential Algorithm Overview

Given a graph \(G = (V, E)\) with \(n = |V|\) vertices and \(m = |E|\) edges, let \(M = 2m + 1\), \(t_1 = \sqrt{M}/2\), and \(t_2 = 3\sqrt{M}/2\). We classify a vertex as low-degree if its degree is at most \(t_1\) and high-degree if its degree is at least \(t_2\). Vertices with degree in between \(t_1\) and \(t_2\) can be classified either way.

**Data Structures.** The algorithm partitions the edges into four edge-stores \(\mathcal{HH}, \mathcal{HL}, \mathcal{LH},\) and \(\mathcal{LL}\) based on a degree-based partitioning of the vertices. \(\mathcal{HH}\) stores all of the edges \((u, v)\) where both \(u\) and \(v\) are high-degree. \(\mathcal{HL}\) stores edges \((u, v)\), where \(u\) is high-degree and \(v\) is low-degree. \(\mathcal{LH}\) stores the edges \((u, v)\) where \(u\) is low-degree and \(v\) is high-degree. Finally, \(\mathcal{LL}\) stores edges where both \(u\) and \(v\) are low-degree.

The algorithm also maintains a wedge-store \(\mathcal{T}\) (a wedge is a triple of distinct vertices \((x, y, z)\) where both \((x, y), (y, z)\) are edges in \(E\)). For each pair of high-degree vertices \(u\) and \(v\), the wedge-store \(\mathcal{T}\) stores the number of wedges \((u, w, v)\), where \(w\) is a low-degree vertex. \(\mathcal{T}\) has the property that given an edge insertion (resp. deletion) \((u, v)\) where both \(u\) and \(v\) are high-degree vertices, it returns the number of wedges \((u, w, v)\), where \(w\) is low-degree, that \(u\) and \(v\) are part of in \(O(1)\) expected time. \(\mathcal{T}\) is implemented via a hash table indexed by pairs of high-degree vertices that stores the number of wedges for each pair.

**Initialization.** Given a graph with \(m\) edges, the algorithm first initializes the triangle count \(C\) using a static triangle counting algorithm in \(O(m^{3/2})\) work and \(O(m)\) space [Lat08]. The \(\mathcal{HH}, \mathcal{HL}, \mathcal{LH},\) and \(\mathcal{LL}\) tables are created by scanning all edges in the input graph and inserting them into the appropriate hash tables. \(\mathcal{T}\) can be initialized by iterating over edges \((u, w)\) in \(\mathcal{HL}\) and for each \(w\), iterating over all edges \((w, v)\) in \(\mathcal{LH}\) to find pairs of high-degree vertices \(u\) and \(v\), and then incrementing \(\mathcal{T}(u, v)\).

**The Kara et al. Algorithm [KNN+19].** Given an edge insertion \((u, v)\) (deletions are handled similarly, and for simplicity assume the edge does not already exist in \(G\)), the update algorithm must identify all tuples \((u, w, v)\) where \((u, w)\) and \((v, w)\) already exist in \(G\), since such triples correspond to new triangles formed by the edge insertion. The algorithm proceeds by considering how a triangle’s edges can reside in the data structures. For example, if all of \(u, v,\) and \(w\) are high-degree, then the algorithm will enumerate these triangles by checking \(\mathcal{HH}\) and finding all neighbors \(w\) of \(u\) that are also high-degree (there are at most
We consider batches of $\Delta$ edge insertions and/or deletions. Let $\text{insert}(u, v)$ represent the update corresponding to inserting an edge between vertices $u$ and $v$, and $\text{delete}(u, v)$ represent deleting the edge between $u$ and $v$. We first preprocess the batch to account for updates that nullify each other. For example, an $\text{insert}(u, v)$ update followed chronologically by a $\text{delete}(u, v)$ update nullify each other because the $(u, v)$ edge that is inserted is immediately deleted, resulting in no change to the graph. To process the batch consisting of nullifying updates, we claim that the only update that is not nullifying for any pair of vertices is the chronologically last update in the batch for that edge. Since all updates contain a timestamp, to account for nullifying updates we first find all updates on the same edge by hashing the updates by the edge, and then nullifying updates nullify each other. For example, an $\text{insert}(u, v)$ update followed chronologically by a $\text{delete}(u, v)$ update nullify each other because the $(u, v)$ edge that is inserted is immediately deleted, resulting in no change to the graph. To process the batch consisting of nullifying updates, we claim that the only update that is not nullifying for any pair of vertices is the chronologically last update in the batch for that edge. Since all updates contain a timestamp, to account for nullifying updates we first find all updates on the same edge by hashing the updates by the edge, and then nullifying updates nullify each other.

3.2 Parallel Batch-Dynamic Update Algorithm

We consider batches of $\Delta$ edge insertions and/or deletions. Let $\text{insert}(u, v)$ represent the update corresponding to inserting an edge between vertices $u$ and $v$, and $\text{delete}(u, v)$ represent deleting the edge between $u$ and $v$. We first preprocess the batch to account for updates that nullify each other. For example, an $\text{insert}(u, v)$ update followed chronologically by a $\text{delete}(u, v)$ update nullify each other because the $(u, v)$ edge that is inserted is immediately deleted, resulting in no change to the graph. To process the batch consisting of nullifying updates, we claim that the only update that is not nullifying for any pair of vertices is the chronologically last update in the batch for that edge. Since all updates contain a timestamp, to account for nullifying updates we first find all updates on the same edge by hashing the updates by the edge, and then nullifying updates nullify each other.

Before we go into the details of our parallel batch-dynamic triangle finding algorithm, we first describe some challenges that must be solved in using Kara et al. [KNN+19] for the parallel batch-dynamic setting.

Challenges. Because Kara et al. [KNN+19] only considers one update at a time in their algorithm, they do not deal with cases where a set of two or more updates creates a new triangle. Since, in our setting, we must account for batches of multiple updates, we encounter the following set of challenges:

1. We must be able to efficiently find new triangles that are created via two or more edge insertions.
2. We must be able to handle insertions and deletions simultaneously meaning that a triangle with one inserted edge and one deleted edge should not be counted as a new triangle.
3. We must account for over-counting of triangles due to multiple updates occurring simultaneously.

For the rest of this section, we assume that $\Delta \leq m$, as otherwise we can re-initialize our data structure using the static parallel triangle-counting algorithm [ST15] to get the count in $O(\Delta^{3/2})$ work, $O(\log^* \Delta)$ depth, and $O(\Delta)$ space (assuming fetch-and-add), which is within the bounds of Theorem 3.1.

Parallel Initialization. Given a graph with $m$ edges, we initialize the triangle count $C$ using a static parallel triangle counting algorithm in $O(m^{3/2})$ work, $O(\log^* m)$ depth, and $O(m)$ space [ST15], using fetch-and-add. We initialize $\mathcal{H}, \mathcal{L}$, and $\mathcal{CH}$ by scanning the edges in parallel and inserting them into the appropriate parallel hash tables. We initialize $\mathcal{D}$ by scanning the vertices. Both steps take $O(m)$ work and $O(\log^* m)$ depth w.h.p. $\mathcal{T}$ can be initialized by iterating over edges $(u, w)$ in $\mathcal{H}$ in parallel and for each $w$.

\footnote{The hashing-based version of the algorithm given in [ST15] can be modified to obtain the stated bounds if it does not do ranking and when using the $O(\log^* n)$ depth w.h.p. parallel hash table and uses fetch-and-add.}
Algorithm Overview. We first remove updates in the batch that either insert edges already in the graph or incrementing \( T(u, v) \). The number of entries in \( \mathcal{HL} \) is \( O(m) \) and each \( w \) has \( O(\sqrt{m}) \) neighbors in \( \mathcal{HL} \), giving a total of \( O(m^{3/2}) \) work and \( O(\log^* m) \) depth w.h.p. for the hash table insertions. The amortized work per edge update is \( O(\sqrt{m}) \).

Data Structure Modifications. We now describe additional information that is stored in \( \mathcal{HH}, \mathcal{HL}, \mathcal{LH}, \mathcal{LL} \), and \( T \), which is used by the batch-dynamic update algorithm:

1. Every edge stored in \( \mathcal{HH}, \mathcal{HL}, \mathcal{LH}, \text{ and } \mathcal{LL} \) stores an associated state, indicating whether it is an old edge, a new insertion or a new deletion, which correspond to the values of 0, 1, and 2, respectively.
2. \( T(u, v) \) stores a tuple with 5 values instead of a single value for each index \((u, v)\). Specifically, a 5-tuple entry of \( T(u, v) = (t^{(u,v)}_1, t^{(u,v)}_2, t^{(u,v)}_3, t^{(u,v)}_4, t^{(u,v)}_5) \) represents the following:
   - \( t^{(u,v)}_1 \) represents the number of wedges with endpoints \( u \) and \( v \) that include only old edges.
   - \( t^{(u,v)}_2 \) and \( t^{(u,v)}_3 \) represent the number of wedges with endpoints \( u \) and \( v \) containing one or two newly inserted edges, respectively.
   - \( t^{(u,v)}_4 \) and \( t^{(u,v)}_5 \) represent the number of wedges with endpoints \( u \) and \( v \) containing one or two newly deleted edges, respectively. In other words, they are wedges which do not exist anymore due to one or two edge deletions.

Algorithm Overview. We first remove updates in the batch that either insert edges already in the graph or delete edges not in the graph by using approximate compaction to filter. Next, we update the tables \( \mathcal{HH}, \mathcal{HL}, \mathcal{LH}, \text{ and } \mathcal{LL} \) with the new edge insertions. Recall that we must update the tables with both \((u, v)\) and \((v, u)\) (and similarly when we update these tables with edge deletions). We also mark these edges as newly inserted. Next, we update \( \mathcal{D} \) with the new degrees of all vertices due to edge insertions. Since now the degrees of some vertices have increased, for low-degree vertices whose degree exceeds \( t_2 \) in parallel, we promote them into high-degree vertices, which involves updating the tables \( \mathcal{HH}, \mathcal{HL}, \mathcal{LH}, \mathcal{LL} \), and \( T \). Next, we update the tables \( \mathcal{HH}, \mathcal{HL}, \mathcal{LH}, \text{ and } \mathcal{LL} \) with new edge deletions, and mark these edges as newly deleted. We then call the procedures \text{update_table_insertions} and \text{update_table_deletions} which updates the wedge-table \( T \) based on all new insertions, and similarly for deletions. At this point, our auxiliary data structures contain both new triangles formed by edge insertions, and triangles deleted due to edge deletions.

For each update in the batch, we then determine the number of new triangles that are created by counting different types of triangles that the edge appears in (based on the number of other updates forming the triangle). We then aggregate these per-update counts to update the overall triangle count.

Now that the count is updated, the remaining steps of the algorithm handle unmarking the edges and restoring the data structures so that they can be used by the next batch. We unmark all newly inserted edges from the tables, and delete all edges marked as deletes in this batch. Finally, we handle updating \( T \), the wedge-table for all insertions and deletions of edges incident to low-degree vertices. The last steps in our algorithm are to update the degrees in response to the newly inserted edges and the now truly deleted edges. Then, since the degrees of some high-degree vertices may drop below \( t_1 \) (and vice versa), we convert them to low-degree vertices and update the tables \( \mathcal{HH}, \mathcal{HL}, \mathcal{LH}, \mathcal{LL} \), and \( T \) (and vice versa). Finally, if the number of edges in the graph becomes less than \( M/4 \) or greater than \( M \) we reset the values of \( M, t_1 \), and \( t_2 \), and re-initialize all of the data structures.

We first present a simplified version of our algorithm without the full implementation details for the purposes of intuition. Then, in Section 3.3, we present the full algorithm. The following \text{COUNT-TRIANGLE} procedure takes as input a batch of \( \Delta \) updates \( \mathcal{B} \) and returns the count of the updated number of triangles in the graph (assuming the initialization process has already been run on the input graph and all associated data structures are up-to-date).
Algorithm 1. Batch-Dynamic-Triangle-Counting
1: function COUNT-TRIANGLES(B)
2:       parfor insert\((u, v) \in B\) do
3:           Update and label edges \((u, v)\) and \((v, u)\) in HH, HL, LH, and LL as inserted edges.
4:       parfor delete\((u, v) \in B\) do
5:           Update and label edges \((u, v)\) and \((v, u)\) in HH, HL, LH, and LL as deleted edges.
6:       parfor insert\((u, v) \in B\) or delete\((u, v) \in B\) do
7:           Update \(T\) with \((u, v)\). \(T\) records the number of wedges that have 0, 1, or 2 edge updates.
8:       parfor insert\((u, v) \in B\) or delete\((u, v) \in B\) do
9:           Count the number of new triangles and deleted triangles incident to edge \((u, v)\), and account for duplicates.
10:      Rebalance data structures if necessary.

The above algorithm is presented only for understanding the intuition behind our batch-dynamic triangle counting algorithm. The full implementation is provided in the following Section 3.3.

3.3 Parallel Batch-Dynamic Triangle Counting Detailed Algorithm

We now provide the full details of our parallel batch-dynamic triangle counting algorithm. Recall that the update procedure for a set of \(\Delta \leq m\) non-nullifying updates is as follows (the subroutines used in the following steps are described afterward):

Algorithm 2. Parallel Batch-Dynamic Triangle Counting

1. Remove updates in the batch that either insert edges already in the graph or delete edges not in the graph using approximate compaction [GMV91].
2. Update the tables HH, HL, LH, and LL with the new edge insertions using insert\((u, v)\). Recall that we must update the tables with both \((u, v)\) and \((v, u)\). To mark these edges as newly inserted, we run mark_inserted_edges\((B)\) on the batch of updates \(B\).
3. Update the tables HH, HL, LH and/or LL with new edge deletions delete\((u, v)\). Recall that we must update the tables with both \((u, v)\) and \((v, u)\). To mark these edges as newly deleted, we run mark_deleted_edges\((B)\) on the batch of updates \(B\).
4. Call update_table_insertions\((B)\) for the set \(B\) of all edge insertions insert\((u, v)\) in the batch where either \(u\) or \(w\) is a low-degree vertex and the other is a high-degree vertex.
5. Call update_table_deletions\((B)\) for the set \(B\) of all edge deletions delete\((u, v)\) in the batch where either \(u\) or \(w\) is a low-degree vertex and the other is a high-degree vertex.
6. For each update in the batch, determine the number of new triangles that are created by counting 6 values. We count the values using a 6-tuple, \((c_1, c_2, c_3, c_4, c_5, c_6)\) which represents the contributions of an updated-edge in a triangle, based on the number of other updates contained in that triangle:
   a. For each edge insertion update insert\((u, v)\) resulting in a triangle containing only one newly inserted edge (and no deleted edges), increment \(c_1\) by count_triangles\((1, 0, insert(u, v))\).
   b. For each edge insertion update insert\((u, v)\) resulting in a triangle containing two newly inserted edges (and no deleted edges), increment \(c_2\) by count_triangles\((2, 0, insert(u, v))\).
   c. For each edge insertion update insert\((u, v)\) resulting in a triangle containing three newly inserted edges, increment \(c_3\) by count_triangles\((3, 0, insert(u, v))\).
   d. For each edge deletion update delete\((u, v)\) resulting in a deleted triangle with one newly
deleted edge, increment $c_4$ by $\text{count_triangles}(0, 1, \text{delete}(u, v))$.

(e) For each edge deletion update $\text{delete}(u, v)$ resulting in a deleted triangle with two newly deleted edges, increment $c_5$ by $\text{count_triangles}(0, 2, \text{delete}(u, v))$.

(f) For each edge deletion update $\text{delete}(u, v)$ resulting in a deleted triangle with three newly deleted edges, increment $c_6$ by $\text{count_triangles}(0, 3, \text{delete}(u, v))$.

Let $C$ be the previous count of the number of triangles in the graph. We update $C$ to be $C + c_1 + (1/2)c_2 + (1/3)c_3 - c_4 - (1/2)c_5 - (1/3)c_6$, which is the new count of the number of triangles in the graph.

(7) Scan through the edge updates again in the batch. For each edge update, if the value stored in the tables $\mathcal{HH}$, $\mathcal{HL}$, $\mathcal{LH}$, and/or $\mathcal{LL}$ is a 2, then remove this edge from the tables. If instead, the stored value is 1, change the value to 0. For all edge updates where both vertices are high-degree or both vertices are low-degree, we are done. For each update on edge $(u, w)$, if either $u$ or $w$ is low-degree (assume without loss of generality that $w$ is) and the other is high-degree, we look for all high-degree neighbors $v$ of $w$ and update $T(u, v)$ by summing all the first, second, and third values of the tuple and subtracting the fourth and fifth values.

(8) Update $\mathcal{D}$ with the new degrees of all vertices due to edge updates.

(9) Perform all minor rebalancing due to updates for all vertices $v$ that exceed $t_2$ in degree or fall under $t_1$ in degree in parallel by calling the procedure $\text{minor_rebalance}(v)$ for all such $v$. This type of minor rebalance updates the structure by making a formerly low-degree vertex high-degree (and vice versa) by updating all associated data structures.

(10) Perform major rebalancing if necessary (i.e., the total number of edges in the graph is less than $M/4$ or greater than $M$). Major rebalancing is done by simply re-initializing the data structure.

**Procedure** $\text{mark_inserted_edges}(B)$. We scan through each of the $\text{insert}(u, v)$ updates in $B$ and mark $(u, v)$ and $(v, u)$ as newly inserted edges in $\mathcal{HH}$, $\mathcal{HL}$, $\mathcal{LH}$, and/or $\mathcal{LL}$ by storing a value of 1 associated with the edge.

**Procedure** $\text{mark_deleted_edges}(B)$. Because we removed all nullifying updates before $B$ is passed into the procedure, none of the deletion updates in $B$ should delete newly inserted edges. For all edge deletions $\text{delete}(u, v)$, we change the values stored under $(u, v)$ and $(v, u)$ from 0 to 2 in the tables $\mathcal{HH}$, $\mathcal{HL}$, $\mathcal{LH}$, and $\mathcal{LL}$.

**Procedure** $\text{count_triangles}(i, d, \text{update})$. This procedure returns the number of triangles containing the update $\text{insert}(u, v)$ or $\text{delete}(u, v)$ and exactly $i$ newly inserted edges or exactly $d$ newly deleted edges (the update itself counts as one newly inserted edge or one newly deleted edge). If at least one of $u$ or $v$ is low-degree, we search in the tables, $\mathcal{HL}$, $\mathcal{LH}$, and $\mathcal{LL}$ for new triangles and the number of marked edges per triangle: edges marked as 1 for insertion updates and edges marked as 2 for deletion updates. If both $u$ and $v$ are high-degree, we first look through all high-degree vertices using $\mathcal{HH}$ to see if any form a triangle with both high-degree endpoints $u$ and $v$ of the update. This allows us to find all newly updated triangles containing only high-degree vertices. Then, we look at the $j$'th value stored in the tuple given by $T(u, v)$, i.e., $t_j^{(u,v)}$, to determine the count of triangles containing $u$ and $v$ and one low-degree vertex if we are given an update where:

- Return the first value $t_1^{(u,v)}$ if either $i = 1$ or $d = 1$.
- Return the second value $t_2^{(u,v)}$ if $i = 2$.
- Return the third value $t_3^{(u,v)}$ if $i = 3$.
- Return the fourth value $t_4^{(u,v)}$ if $d = 2$.
- Return the fifth value $t_5^{(u,v)}$ if $d = 3$.

Note that we ignore all triangles that include more than one insertion update and more than one deletion.
update.

Procedure minor_rebalance(u). This procedure performs a minor rebalance when either the degree of u decreases below \( t_1 \) or increases above \( t_2 \). We move all edges in \( \mathcal{H} \mathcal{H} \) and \( \mathcal{H} \mathcal{L} \) to \( \mathcal{L} \mathcal{H} \) and \( \mathcal{L} \mathcal{L} \) and vice versa.

Procedure update_table_insertions(\( B \)). For each \( (u, w) \in B \), assume without loss of generality that \( w \) is the low-degree vertex and do the following. We first find all of \( w \)'s neighbors, \( v \), in \( \mathcal{L} \mathcal{H} \) in parallel. Then, we determine for each neighbor \( v \) if \( (w, v) \) is new (marked as 1). If the edge \( (w, v) \) is not new, then increment the second value stored in the tuple with index \( \mathcal{T}(u, v) \). If \( (w, v) \) is newly inserted, then increment the third value stored in \( \mathcal{T}(u, v) \). The first, fourth, and fifth values stored in \( \mathcal{T}(u, v) \) do not change in this step. Intuitively, the first, second, and third values will tell us later on whether a newly created triangle will have one, two, or three, respectively, new edge insertions (resp. deletions) from the batch update. We ignore triangles which contain a mix of edge insertion updates and edge deletion updates.

Procedure update_table_deletions(\( B \)). For each \( (u, w) \in B \), assume without loss of generality that \( w \) is the low-degree vertex and do the following. We first find all of \( w \)'s neighbors, \( v \), in \( \mathcal{L} \mathcal{H} \) in parallel. Then, we determine for each neighbor \( v \) if \( (w, v) \) is a newly deleted edge (marked as 2). If \( (w, v) \) is not a newly deleted edge, increment the fourth value in the tuple stored in \( \mathcal{T}(u, v) \). Otherwise, if \( (w, v) \) is a newly deleted edge, increment the fifth value of \( \mathcal{T}(u, v) \). The first, second, and third values in \( \mathcal{T}(u, v) \) do not change in this step. Intuitively, the first, fourth, and fifth values tell us later on whether a newly deleted triangle will have one, two, or three, respectively, new edge deletions from the batch update.

### 3.4 Analysis

We prove the correctness of our algorithm in the following theorem. The proof is based on accounting for the contributions of an edge to each triangle it participates in based on the number of other updated edges found in the triangle.

**Theorem 3.2.** Our parallel batch-dynamic algorithm maintains the number of triangles in the graph.

**Proof.** All triangles containing at least one low-degree vertex can be found either in \( \mathcal{T} \) or by searching through \( \mathcal{L} \mathcal{H} \) and \( \mathcal{L} \mathcal{L} \). All triangles containing all high-degree vertices can be found by searching \( \mathcal{H} \mathcal{H} \). Suppose that an edge update insert(\( u, v \)) (resp. delete(\( u, v \))) is part of \( I(u,v) \) (resp. \( D(u,v) \)) triangles. We need to add or subtract from the total count of triangles \( I(u,v) \) or \( D(u,v) \), respectively. However, some of the triangles will be counted twice or three times if they contain more than one edge update. By dividing each triangle count by the number of updated edges they contain, each triangle is counted exactly once for the total count \( C \). \( \square \)

**Overall Bound.** We now prove that our parallel batch-dynamic algorithm runs in \( O(\Delta\sqrt{\Delta + m}) \) work, \( O(\log^*(\Delta + m)) \) depth, and uses \( O(\Delta + m) \) space. Henceforth, we assume that our algorithm uses the fetch-and-add instruction (see Section 1.1). Removing nullifying updates takes \( O(\Delta) \) total work, \( O(\log^* \Delta) \) depth w.h.p., and \( O(\Delta) \) space for hashing and the find-maximum procedure outlined in Section 3.1. In step (1), we perform table lookups for the updates into \( D \) and once in \( \mathcal{H} \mathcal{H}, \mathcal{H} \mathcal{L}, \mathcal{L} \mathcal{H}, \) or \( \mathcal{L} \mathcal{L} \), followed by approximate compaction to filter. The hash table lookups take \( O(\Delta) \) work and \( O(\log^* m) \) depth with high probability and \( O(m) \) space. Approximate compaction [GMV91] takes \( O(\Delta) \) work, \( O(\log^* \Delta) \) depth, and \( O(\Delta) \) space. Steps (2), (3), and (8) perform hash table insertions and updates on the batch of \( O(\Delta) \) edges, which takes \( O(\Delta) \) amortized work and \( O(\log^* m) \) depth with high probability.

The next lemma shows that updating the tables based on the edges in the update (steps (4) and (5)) can be done in \( O(\Delta\sqrt{m}) \) work and \( O(\log^* m) \) depth w.h.p., and \( O(m) \) space.

**Lemma 3.3.** update_table_insertions(\( B \)) and update_table_deletions(\( B \)) on a batch \( B \) of size \( \Delta \) takes \( O(\Delta\sqrt{m + \Delta}) \) work and \( O(\log^*(\Delta + m)) \) depth w.h.p., and \( O(\Delta + m) \) space.
Proof. For each \( w \), we find all of its high-degree neighbors in \( \mathcal{LH} \) and perform the increment or decrement in the corresponding entry in \( \mathcal{T} \) in parallel (at this point, the vertices are still classified based on their original degrees). The total number of new neighbors gained across all vertices is \( O(\Delta) \) since there are \( \Delta \) updates. Therefore, across all updates, this takes \( O(\Delta \sqrt{m} + \Delta) \) work and \( O(\log^*(\Delta + m)) \) depth w.h.p. due to hash table lookup and updates. Then, for all high-degree neighbors found, we perform the increments or decrements in corresponding entries in \( \mathcal{T} \) in parallel, taking the same bounds. All vertices can be processed in parallel, giving a total of \( O(\Delta \sqrt{m} + \Delta) \) work and \( O(\log^*(\Delta + m)) \) depth w.h.p.

The next lemma bounds the complexity of updating the triangle count in step (6).

Lemma 3.4. Up\ding{122}ting the triangle count takes \( O(\Delta \sqrt{m} + \Delta) \) work and \( O(\log^*(\Delta + m)) \) depth w.h.p., and \( O(\Delta + m) \) space.

Proof. We initialize \( c_1, \ldots, c_6 \) to 0. For each edge update in \( \mathcal{B} \) where both endpoints are high-degree, we perform lookups in \( \mathcal{T} \) and \( \mathcal{HH} \) for the relevant values in parallel and increment the appropriate \( c_i \). Finding all triangles containing the edge update and containing only high-degree vertices takes \( O(\Delta \sqrt{m}) \) work and \( O(\log^*(\Delta + m)) \) depth w.h.p. Performing lookups in \( \mathcal{T} \) takes \( O(\Delta) \) work and \( O(\log^*(\Delta + m)) \) depth w.h.p. For each update containing at least one endpoint with low-degree, we perform lookups in the tables \( \mathcal{HL}, \mathcal{CH}, \) and \( \mathcal{LL} \) to find all triangles containing the update and increment the appropriate \( c_i \). This takes \( O(\Delta \sqrt{m} + \Delta) \) work and \( O(\log^*(\Delta + m)) \) depth w.h.p. Incrementing all \( c_i \)'s for all newly updated triangles takes \( O(\Delta) \) work and \( O(1) \) depth. We then apply the equation in step (6) to update \( C \). This takes \( O(1) \) work and \( O(1) \) depth.

The following lemma bounds the cost for minor rebalancing, where a low-degree vertex becomes high-degree or vice versa (step (9)).

Lemma 3.5. Minor rebalancing for edge updates takes \( O(\Delta \sqrt{m}) \) amortized work and \( O(\log^*(\Delta + m)) \) depth w.h.p., and \( O(\Delta + m) \) space.

Proof. We describe the case of edge insertions, and the case for edge deletions is similar. Using approximate compaction to perform the filtering, we first find the set \( S \) of low-degree vertices exceeding \( t_2 \) in degree. This step takes \( O(\Delta) \) work and \( O(\log^*\Delta) \) depth. For vertices in \( S \), we then delete the edges from their old hash tables and move the edges to their new hash tables. The work for each vertex is proportional to its current degree, giving a total work of \( O(\sum_{v \in S} \deg(v)) = O(\Delta \sqrt{m} + \Delta) \) w.h.p. since the original degree of low-degree vertices is \( O(\sqrt{m}) \) and each edge in the batch could have caused at most 2 such vertices to have their degree increase by 1 (the w.h.p. is for parallel hash table operations).

In addition to moving the edges into new hash tables, we also have to update \( \mathcal{T} \) with new pairs of vertices that became high-degree and delete pairs of vertices that are no longer both high-degree.

To update these tables, we need to find all new pairs of high-degree vertices. There are at most \( O(\Delta \sqrt{m} + \Delta) \) such new pairs, which can be found by filtering neighbors using approximate compaction [GMV91] of vertices in \( S \) in \( O(\Delta \sqrt{m} + \Delta) \) work and \( O(\log^*(\Delta + m)) \) depth w.h.p. For each pair, we check all neighbors of the endpoints that just became high-degree and increment the entry \( \mathcal{T}(u, v) \) for each low-degree neighbor \( w \) found that have edges \((u, w)\) and \((w, v)\). Low-degree neighbors have degree \( O(\sqrt{\Delta + m}) \), and so the total work is \( O(\Delta(\Delta + m)) \) and depth is \( O(\log^*(\Delta + m)) \) w.h.p using fetch-and-add. There must have been \( \Omega(\sqrt{m}) \) updates on a vertex before minor rebalancing is triggered, and so the amortized work per update is \( O(\Delta \sqrt{m}) \) and the depth is \( O(\log^* m) \) w.h.p. The space for filtering is \( O(m + \Delta) \).

We now finish showing Theorem 3.1. Lemma 3.2 shows that our algorithm maintains the correct count of triangles. Lemmas 3.3, 3.4, and 3.5 show that the cost of minor rebalancing, updating tables to reflect the batch, and updating the triangle counts all run in \( O(\Delta \sqrt{m} + \Delta) \) work and \( O(\log^*(\Delta + m)) \) depth w.h.p., and \( O(\Delta + m) \) space.
Step (7) can be done in \(O(\sqrt{m})\) work and \(O(\log^* m)\) depth as follows. We scan through the batch \(B\) in parallel and update the hash tables \(H_H, H_L, L_H,\) and \(L_L\) in \(O(\Delta)\) work and \(O(\log^*(\Delta + m))\) depth w.h.p. For all updates in \(B\) containing one high-degree vertex and one low-degree vertex, we update the table \(T\) in parallel by scanning the neighbors in \(L_H\) of the low-degree vertex. This step takes \(O(\sqrt{m} + \Delta)\) work and \(O(\log^*(\Delta + m))\) depth w.h.p. Major rebalancing (step (10)) takes \(O((\Delta + m)^{3/2})\) work and \(O(\log^*(\Delta + m))\) depth by re-initializing the data structures. The rebalancing happens every \(\Omega(m)\) updates, and so the amortized work per update is \(O(\sqrt{m} + \Delta)\) and depth is \(O(\log^*(\Delta + m))\) w.h.p.

Therefore, our update algorithm takes \(O(\Delta/\sqrt{m} + \Delta)\) amortized work and \(O(\log^*(\Delta + m))\) depth w.h.p., and \(O(\Delta + m)\) space overall using fetch-and-add as stated in Theorem 3.1.

### 4 Dynamic \(k\)-Clique Counting via Fast Static Parallel Algorithms

In this section, we present a very simple algorithm for dynamically maintaining the number of \(k\)-cliques for \(k > 3\) based on statically enumerating a number of smaller cliques in the graph, and intersecting the enumerated cliques with the edge updates in the input batch. Importantly, the algorithm is space-efficient, and only relies on simple primitives such as clique enumeration of cliques of size smaller than \(k\), for which there are highly efficient algorithms both in theory and practice.

**Fast Static Parallel \(k\)-Clique Enumeration.** The main tool used by algorithm is the following theorem, which is presented in concurrent and independent work [SDS20]:

**Theorem 4.1** (Theorem 4.2 of [SDS20]). There is a parallel algorithm that given a graph \(G\) can enumerate all \(k\)-cliques in \(G\) in \(O(\max k - 2)\) expected work and \(O(\log^{k - 2} n)\) depth w.h.p., using \(O(m)\) space.

Theorem 4.1 is proven by modifying the Chiba-Nishizeki (CN) algorithm in the parallel setting, and combining the CN algorithm with parallel low-outdegree orientation algorithms [BE10, GP11].

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**Algorithm 3. Dynamic \(k\)-Clique Counting**

1. **function** \(k\)-Clique-Count\((G = (V, E), B)\)
2. Remove nullifying updates from \(B\).
3. **if** \(\Delta \geq m\) **then**
   4. Rerun the static \(k\)-clique counting algorithm.
5. **else**
6. Insert all updates that are edge insertions in \(B\) in \(G\).
7. Let \(H\) be a static parallel hash table representing \(B\).
8. **parfor** \(e = \{u, v\} \in B\) **do**
   9. Enumerate all \((k - 2)\)-cliques in \(G\) in parallel using the Algorithm from Theorem 4.1.
10. **for** each enumerated \((k - 2)\)-clique, \(C\) **do**
11. **if** \(C\) forms a newly inserted or newly deleted \(k\)-clique with \(e^a\) **then**
12. **if** \(e = (u, v)\) is the lexicographically-first edge in \(C\) in the batch \(b\) **then**
13. Atomically update the \(k\)-clique count for \(C \cup \{u, v\}\).
14. Delete all updates that are edge deletions in \(B\) in \(G\).

\(^a\)A newly inserted \(k\)-clique is one where at least one edge in the clique is an edge insertion in \(B\), and all other clique edges are not deleted in \(B\). Similarly a newly deleted \(k\)-clique is one where at least one edge in the clique is an edge deletion in \(B\) and all other edges in the clique are not edge insertions in \(B\). We use \(H\) to check whether edges are new insertions or deletions in \(B\).

\(^b\)An edge \(e = (u, v)\) is the lexicographically first edge in the batch in a clique \(C\) if, \(\forall e' = (u', v') \in C\) s.t. \((u', v') \in B\), \(e\) is lexicographically smaller than \(e'\). Note that we are working over an undirected graph without self-loops. By convention, when discussing lexicographic comparison, we have that for any \(e = (u, v)\) that \(u < v\); in other words, where the order in
A Dynamic $k$-Clique Counting Algorithm. Given Theorem 4.1, one approach to maintain the number of $k$-cliques in $G$ upon receiving a batch of insertions or deletions $B$ is to, for each edge $e$ in the batch simply enumerate all $(k-2)$-cliques, check whether $e$ forms a $k$-clique with any of these $(k-2)$-cliques, and update the clique counts based on the newly discovered (or deleted) cliques.

Algorithm 3 presents a formalized version of this idea. The algorithm first removes all nullifying updates from $B$. It then checks whether the batch is large ($\Delta \geq m$), and if so simply recomputes the overall $k$-clique count by re-running the static enumeration algorithm. Otherwise, the algorithm inserts the edge insertions in the batch into $G$, and stores them in a static parallel hash table $H$ which maps each edge in the batch to a value indicating whether the edge is an insertion or deletion in $B$.

Then, in parallel, for each edge $e = (u, v)$ in the batch, it enumerates all $(k-2)$-cliques in the graph. For each $k-2$ clique, $C$, the algorithm checks whether this clique forms a newly inserted or newly deleted $k$-clique with $e$. A newly inserted $k$-clique is one where at least one edge is an edge insertion in $B$ and all other edges are not deleted in $B$. Similarly a newly deleted $k$-clique is one where at least one edge is an edge deletion in $B$ and all other edges are not edge insertions in $B$. This step is done by querying the static parallel hash table $H$ for each edge in the clique to check whether it is an insertion or deletion in $B$. Cliques consisting of a mix of edge insertions and deletions are cliques that are not previously present before the batch, and will not be present after the batch, and are thus ignored.

For a newly inserted or newly deleted clique, the algorithm then checks whether $e$ is the lexicographically-first edge in the batch inside of this clique formed by $C \cup \{u, v\}$ (otherwise, a different edge update from the batch will find and handle the processing of this clique). Checking whether $e$ is the lexicographically-first edge in a clique $C$ is done by querying the static parallel hash table $H$. For each clique where $e$ is the lexicographically-first edge in the batch in the clique, we either atomically increment, or decrement the count, based on whether this clique is newly inserted or newly deleted. After the clique count has been updated, the algorithm updates $G$ by performing the edge deletions from $B$.

We note that we could just as well enumerate all of the $(k-2)$-cliques a single time, and then for each $(k-2)$-clique we discover, check whether it forms a $k$-clique with each edge in the batch. A practical optimization of this idea may store edges in a batch incident to their corresponding endpoints, and so vertices in the discovered $(k-2)$-clique would only need to check updates incident to the vertices in this clique. We remark that the asymptotic complexity of both ideas—joining cliques with edges, instead of edges with cliques, and pruning edges from the batch to consider—do not change the asymptotic complexity of our result in the worst case.

Correctness and Bounds. If a $k$-clique in the graph is not incident to any edges in the batch, then its count is unaffected (since we only perform modifications to the count for cliques containing edges in $B$). For cliques incident to edges in $B$, we consider two cases. If the clique $C$ is deleted after applying $B$, observe that by decomposing $C$ into a $(k-2)$-clique and the lexicographically-first marked edge $e$ in $C$, $C$ will be found and counted by $e$. The argument that a newly inserted clique, $C$, will be found is similar. Lastly, cliques consisting of both edge insertions and deletions in $B$ will be correctly ignored by the check in Line 11. This argument proves the following theorem:

Theorem 4.2. Algorithm 3 correctly maintains the number of $k$-cliques in the graph.

Theorem 4.3. Given a collection of $\Delta$ updates, there is a dynamic $k$-clique counting algorithm that updates the $k$-clique counts running in $O(\Delta(m + \Delta)\alpha^{k-4})$ expected work and $O(\log^{k-2} n)$ depth w.h.p., using $O(m + \Delta)$ space.

Proof. We analyze Algorithm 3. First, updating the graph, assuming the edges incident to each vertex are represented sparsely using a parallel hash table requires $O(\Delta)$ work in expectation and $O(\log^* n)$ depth w.h.p.
If $\Delta \geq m$, the algorithm calls the static $k$-clique counting algorithm which costs $O((m + \Delta)\alpha^{k-2})$ expected work. Since $m = O(\Delta)$ and $\alpha^2 = O(m + \Delta)$, the work of calling the static algorithm is upper-bounded by $O(\Delta(m + \Delta)\alpha^{k-4})$ as required. Finally, the depth bound is $O(\log^{k-2} n)$ as required.

Otherwise, $\Delta < m$. Then, the algorithm first inserts and marks the batch into the graph (in the worst case, no updates are nullifying). It also stores the edges in the batch in a parallel hash table. Creating the parallel hash table costs $O(\Delta)$ work and $O(\log^* n)$ depth w.h.p., which are both subsumed by the overall work and depth for the relevant setting of $k > 2$. For each update, we list all $(k - 2)$-cliques using the algorithm from Theorem 4.1. This step can be done using $O((m + \Delta)\alpha^{k-4})$ expected work and $O(\log^{k-4} n)$ depth w.h.p. by Theorem 4.1. If the $(k - 2)$-clique $C$ forms a $k$-clique with $e$, the cost of checking whether the clique is newly inserted or newly deleted using $H$ costs $O(k)$ work, which is a constant, and $O(1)$ depth. The cost of checking whether $e$ is the lexicographically first edge in $B$ is identical. Multiplying the cost of enumeration by the number of edges in the batch completes the proof.

We note that it is an interesting open question whether our dependence on $m$ could be entirely removed from the update bound. Existing work has provided efficient sequential dynamic algorithms maintaining the $k$-clique count in $\tilde{O}(\alpha^{k-2})$ work per update using dynamic low-outdegree orientations [DT13]. It would be interesting to understand whether such an algorithm can be parallelized in the parallel batch-dynamic setting, and achieve a similar work bound, which would allow the dynamic algorithm to match the work of static, parallel recomputation up to logarithmic factors.

5 Dynamic $k$-Clique via Fast Matrix Multiplication

In this section, we present a parallel batch-dynamic algorithm for counting $k$-cliques based on fast matrix multiplication. Using parallel matrix multiplication (discussed in Section 5.6), we achieve a better work bound (in terms of $m$) for large values of $k$ than our bound of $O(\Delta(m + \Delta)\alpha^{k-4})$ obtained from the simple algorithm presented in Section 4. To the best of our knowledge, our algorithm (when made sequential) also achieves the best runtime for any sequential dynamic $k$-clique counting algorithm on dense graphs for large $k$ when using the best currently known matrix multiplication algorithm [Wil12, LG14]. For larger values of $k$, our MM based algorithm achieves $o(m^{k/2 - 1})$ amortized time compared to the arboricity-based algorithm of [DT13] that dynamically counts cliques in $\tilde{O}(\alpha^{k-2})$ amortized time where $\alpha$ is the arboricity of the graph (or $\tilde{O}(m^{k/2 - 1})$ amortized time when $\alpha = \Omega(\sqrt{m})$) or the trivial $O(m^{k/2 - 1})$ algorithm of choosing all $k/2 - 1$ combinations of edges containing neighbors of the incident vertices of the inserted edge.

Our dynamic algorithm modifies the algorithm of [AYZ97] for counting triangles based on fast matrix multiplication and combines it with a dynamic version of the static $k$-clique counting algorithm of [EG04] to count the number of $k$-cliques under edge updates in batches of size $\Delta$. Sections 5.1–5.4 prove the following theorem for the case when $k \bmod 3 = 0$. Section 5.5 describes the changes needed for the case when $k \bmod 3 \neq 0$.

**Theorem 5.1.** There exists a parallel batch-dynamic algorithm for counting the number of $k$-cliques, where $k \bmod 3 = 0$, that takes $O\left(\min\left(\Delta m \frac{2k - 3\omega_p}{\omega_p + \omega_p + 1}, (m + \Delta) \frac{2k - 3\omega_p}{\omega_p + \omega_p + 1}\right)\right)$ amortized work and $O(\log(m + \Delta))$ depth w.h.p., in $O\left((m + \Delta)\frac{2k - 3\omega_p}{\omega_p + \omega_p + 1}\right)$ space, given a parallel matrix multiplication algorithm with exponent $\omega_p$.

Using the best currently known matrix multiplication algorithms with exponent $\omega_p = 2.373$, we obtain the following work and space bounds.
Corollary 5.2. There exists a parallel batch-dynamic algorithm for counting the number of $k$-cliques, where $k \mod 3 = 0$, which takes $O\left(\min(\Delta m^{0.469k-0.704}, (m + \Delta)^{0.469k})\right)$ work and $O(\log(m+\Delta))$ depth w.h.p., in $O((m + \Delta)^{0.469k})$ space by Corollary 5.19.

Specifically, when amortized over the total number of edge updates $\Delta$, we obtain an amortized work bound of $O(m^{0.469k-0.704})$ per edge update which is asymptotically better than the combinatorial bound of $O((m^{k/2-1})$ per update for larger values of $k$.

Observe that our update algorithm only needs to handle batches of size $0 < \Delta \leq m^{\omega_p/(1+\omega_p)}$. For batches which have size $\Delta > m^{\omega_p/(1+\omega_p)}$, we can reinitialize our data structures in $O((m + \Delta)^{0.469k})$ work ($O(m^{0.469k-0.704})$ amortized work per update in the batch), $O(\log \Delta)$ depth, and $O((m + \Delta)^{0.469k})$ space using our initialization algorithm described in Lemma 5.5 and the fast parallel matrix multiplication of Corollary 5.19, which is faster than using the update algorithm (in general, we can use any fast matrix multiplication algorithm that has low depth, but the cutoff for when to reinitialize would be different). The analysis of the reinitialization procedure (similar to the static case presented by Alon, Yuster, and Zwick [AYZ97]) is provided in Section 5.4. Thus, in the following sections, we only describe our dynamic update procedures for batches of size $0 < \Delta \leq m^{\omega_p/(1+\omega_p)}$.

5.1 Parallel Batch-Dynamic $k$-Clique Algorithm

In what follows, we assume that $k \mod 3 = 0$ (please refer to Section 5.5 for $k \mod 3 \neq 0$). We use a batch-dynamic triangle counting algorithm as a subroutine for our batch-dynamic $k$-clique algorithm. Our algorithm for maintaining triangles is a batch-dynamic version of the triangle counting algorithm by Alon, Yuster, and Zwick (AYZ) [AYZ97]. However, our dynamic algorithm cannot directly be used for the case of $k = 3$ (and only applies for cases $k > 3$) due to the following challenge which we resolve in Section 5.2. Furthermore, our analysis also assumes $k > 6$ for greater simplicity and since for smaller $k$, our algorithm from Section 4 is also faster.

Adapting the Static Algorithm. We face a major challenge when adapting the algorithm of Alon, Yuster, and Zwick [AYZ97] for our setting as well as for the sequential setting. Because the AYZ algorithm is meant to count cliques in the static setting, it is fine to consider two different types of triangles and count the triangles of each type separately. The two different types of triangles considered are triangles which contain at least one low-degree vertex and triangles which contain only high-degree vertices. In the static case, we can find all low-degree vertices, but in the dynamic case, we cannot afford to look at all low-degree vertices. If we only look at low-degree vertices incident to edge updates, then the following case may occur: an edge update between two high-degree nodes forms a new triangle incident to a low-degree node. In such a case, only looking at the vertices adjacent to this edge update will not find this triangle. We resolve this issue for $k > 3$ via Lemma 5.3 in Section 5.2.

Definitions and Data Structures. Given a graph $G$, we construct an auxiliary graph $G'$ consisting of vertices where each vertex represents a clique of size $\ell = k/3$ in $G$. An edge $(u, v)$ between two vertices in $G'$ exists if and only if the cliques represented by $u$ and $v$ form a clique of size $2\ell$ in $G$. Our algorithm maintains a dynamic total triangle count $C$ on $G'$. Let $M = 2m + 1$ and let a low-degree vertex in $G'$ be a vertex with degree less than $M^{t\ell/2}$ (for some $0 < t < 1$ to be determined later) and a high-degree vertex in $G'$ be a vertex with degree greater than $3M^{t\ell/2}$. The vertices with degree in the range $[M^{t\ell/2}, 3M^{t\ell/2}]$ can be classified as either low-degree or high-degree. In addition to the total triangle count, we maintain a count, $C_L$, of all triangles involving a low-degree vertex. Using the algorithm of AYZ [AYZ97], we assume we have a two-level hash table, $L$, representing the neighbors of low-degree vertices in $G'$ (a table mapping a low-degree vertex to another hash table containing its incident edges). We also maintain the adjacency

\footnote{We use a hash table $Q$ that stores each vertex in $G'$ as an index to a set of vertices in $G$ and also stores each set of vertices composing an $\ell$-clique in $G$ (lexicographically sort the vertices and turn into a string) as an index to a vertex in $G'$.}
matrix $A$ of high-degree vertices in $G'$ used in AYZ as a two-level hash table for easy insertion and deletion of additional high-degree vertices. Finally, we maintain another hash table $D$ which dynamically maintains the degrees of the vertices.

### 5.2 Algorithm Overview

Our algorithm proceeds as follows. Each edge in an update in the batch (edges in $G$) can either create at most $O(m^{k/3-1})$ new $(2k/3)$-cliques or disrupt $O(m^{k/3-1})$ existing $(2k/3)$-cliques in $G$. We treat each of these newly created or destroyed cliques as an edge insertion or deletion in $G'$. Since we preprocess the updates to $G$ such that there are no duplicate or nullifying updates, a destroyed clique cannot be created again or vice versa. This means that the set of updates to $G'$ will also contain no nullifying updates.

Importantly, the AYZ algorithm does not take into account edge insertions and deletions between two high-degree vertices that create or destroy triangles containing at least one low-degree vertex. Thus, we must prove the following lemma for any edge insertion/deletion in $G$ that results in an edge insertion in $G'$ between two high-degree vertices which creates or destroys a triangle containing a low-degree vertex. This lemma is crucial for our algorithm, since it ensures that a triangle formed by two high-degree vertices and a low-degree vertex will be discovered by enumerating all triangles formed or deleted by an edge update incident to the low-degree vertex, and its current edges. Furthermore, this lemma is the reason why our algorithm does not work for $k = 3$ cliques.

**Lemma 5.3.** Given a graph $G = (V, E)$, the corresponding $G' = (V', E')$, and for $k > 3$, suppose an edge insertion (resp. deletion) between two high-degree vertices in $G'$ creates a new triangle, $(u_H, w_H, x_L)$, in $G'$ which contains a low-degree vertex $x_L$. Let $R(y)$ denote the set of vertices in $V$ represented by a vertex $y \in V'$. Then, there exists a new edge insertion (resp. deletion) in $G'$ that is incident to $x_L$ and creates a new triangle $(u', w', x_L)$ such that $R(u') \cup R(w') = R(u_H) \cup R(w_H)$.

**Proof.** We prove this lemma for edge insertions in $G$. The proof can be easily modified to account for the case of edge deletions in $G$. Suppose an edge insertion $(y, z)$ in $G$ leads to an edge insertion in $G'$ between the two high-degree vertices $u_H$ and $w_H$ that creates the new triangle $(u_H, w_H, x_L)$. The creation of the new triangle signifies that a new clique was created in $G$ consisting of vertices $R(u_H) \cup R(w_H) \cup R(x_L)$. Then, the edge insertion $(y, z)$ created a new $2k/3$-clique in $G$ consisting of the vertices in $R(u_H) \cup R(w_H)$. Since the edge $(y, z)$ between $y, z \in V$ did not exist previously but now exists, $(2k/3-2)$ new cliques were created using the set of vertices in $R(u_H) \cup R(w_H)$. Each of these new cliques corresponds to a new vertex in $G'$. Suppose $u'$ is one such new vertex representing vertex set $R(u') \subseteq R(u_H) \cup R(w_H)$ and $u'$ represents vertex set $R(u') = (R(u_H) \cup R(w_H)) \setminus R(u')$. Then, new edges are inserted between $u'$ and $w'$ and between $u'$ and $x_L$ (the edge $(u', x_L)$ might be a newly inserted edge or it is already present in the graph) since all triangles representing the clique of vertices $(u_H, w_H, x_L)$ must be present in $G'$. Thus, the new triangle $(u', w', x_L)$ is created in $G'$.

We now describe our dynamic clique counting algorithm that combines the AYZ algorithm [AYZ97] with the clique counting algorithm of [EG04]. Given the batch of edge insertions/deletions into $G$, we first compute the duplicate and nullifying updates and remove them. Then, for a set of insertions/deletions into $G'$, we form two batches, one containing the edge insertions and one containing the edge deletions. Given the batch of updates to $G'$, we now formulate a dynamic version of the AYZ algorithm [AYZ97] on the updates to $G'$. For the batch of updates, we first look at the updates pertaining to the low-degree vertices. For every update $(u, v)$ that contains at least one low-degree vertex (without loss of generality, let $v$ be a low-degree vertex), we search all of $v$’s $O(3M^{2/3}/2)$ neighbors and check whether a triangle is formed (resp.

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Note that this is fine for the static case but not for the dynamic case.
deleted). For each triangle formed (resp. deleted), we update the total triangle count of the graph \( G' \). For high-degree vertices, we update our adjacency matrix \( A \) containing vertices with high-degree. To compute the triangles containing high-degree vertices, we need only compute \( A^3 \) (the diagonal will then provide us with the triangle counts). Lastly, one clique results in many different copies of triangles. We must obtain the correct clique count by dividing the number of triangles by the number of ways we can partition the vertices in a \( k \)-clique into triples of subcliques of size \( k/3 \). We provide a simplified version of our algorithm first, and then a detailed implementation in Section 5.3.

**Algorithm 4. Parallel-\( k \)-Clique Algorithm**

1: function COUNT-CLIIQUES(\( B \))
2: Update graph \( G' \) with \( B \) by inserting new \( \ell \)- and \( 2\ell \)-cliques.
3: Find batch of insertions into \( G', B'_I \), and batch of deletions, \( B'_D \).
4: Determine the final degrees of every vertex in \( G' \) after performing updates \( B'_I \) and \( B'_D \).
5: parfor insert \((u, v)\) \in \( B'_I \), delete \((u, v)\) \in \( B'_D \) do
6: if either \( u \) or \( v \) is low-degree: \( d(u) \leq \delta \) or \( d(v) \leq \delta \) then
7: Enumerate all triangles containing \((u, v)\). Let this set be \( T \).
8: By Lemma 5.3, find all possible triangles representing the same triangle \( t \in T \).
9: Correct for duplicate counting of triangles.
10: else
11: Update \( A \) (adjacency list for high-degree vertices).
12: Compute \( A^3 \). The diagonal provides the triangle counts for all triangles containing only high-degree vertices.
13: Sum the counts of all triangles.
14: Correct for duplicate counting of cliques.

\(^a\)Some care must be taken to ensure that rebalancing does not incur too much work. The details of how to deal with rebalancing are given in the full implementation, Algorithm 5.

### 5.3 Detailed Parallel Batch-Dynamic Matrix Multiplication Based Algorithm

The analysis we perform in Section 5.4 on the efficiency of our algorithm is with respect to the detailed implementation. We provide the detailed description and implementation of our algorithm below:

**Algorithm 5. Parallel Batch-Dynamic Matrix Multiplication Based \( \ell \)-Clique Counting Algorithm**

1. Given a batch \( B \) of non-nullifying edge updates, first update the graph \( G' \). If the update is an insertion, insert \((u, v)\), add all new \( \ell \)-cliques created by it into \( G' \). If the update is a deletion, delete \((u, v)\), mark all \( \ell \)-cliques destroyed by it in \( G' \). For each update, insert \((u, v)\) or delete \((u, v)\), determine all \( 2\ell \)-cliques that include it. This will determine the set of edge insertions/deletions into \( G' \). Let all edge updates that destroy \( 2\ell \)-cliques be a batch \( B'_D \) of edge deletions in \( G' \). Then, let all \( 2\ell \)-cliques formed by edge updates be a batch of edge insertions \( B'_I \) into \( G' \). Note that edge insertions in the batch could be edges for newly created vertices; for each such newly created vertex, we also add the vertex into \( G' \) and its associated data structures.
2. Determine the final degree of each vertex after all insertions in \( B'_I \) and all deletions in \( B'_D \). (We do not perform the updates yet—only compute the final degrees.) For all vertices, \( X \), which become low-degree after the set of all updates (and were originally high-degree), we create a batch of updates \( B'_{I,L} \) consisting of old edges (not update edges) that are adjacent to vertices in \( X \) and were not deleted by the batches of updates. For all vertices, \( Y \), which become high-degree after the set
of updates (and were originally low-degree), we create a batch of updates \( B'_{D,H} \) consisting of old edges adjacent to vertices in \( Y \) that were not deleted after the batches of updates.

(3) Let the edges in \( B'_D \cup B'_{D,H} \) be the batch of edge deletions to \( G' \). For each of the edges in \( B'_D \cup B'_{D,H} \), we first count the number of triangles it is a part of that contain at least one low-degree vertex. We call this the set of deleted triangles. Let this number of deleted triangles be \( T_D \) (initially set \( T_D = 0 \)).

(a) To count the number of triangles that contain at least one low-degree vertex, we first check for each edge whether one of its endpoints is low-degree. Let this set of edge deletions be \( D'_L \subseteq B'_D \cup B'_{D,H} \).

(b) For every edge \((u', v') \in D'_L\), without loss of generality let \( u' \) be the lexicographically first low-degree vertex. For every edge \((u', w')\) incident to \( u' \), check whether \((u', v')\) forms a triangle with \((v', w')\).

(c) For every \((u', v', w')\) triangle deleted (where \((u', v', w')\) is sorted lexicographically), call \( t \leftarrow \text{count}_\text{updated}_\text{low_degree_triangles}((u', v', w'), (u', v')) \), and atomically update \( T_D \leftarrow T_D + t \).

(4) Update \( C_L \leftarrow C_L - T_D \).

(5) Update the data structures using the batches of edges insertions and deletions, \( B'_D \) and \( B'_I \):

(a) Using \( B'_D \), delete the relevant edges in \( L \) (containing neighbors of low-degree vertices) and then change the relevant values in \( A \) to 0. We also update \( D \) with the new degrees of the vertices for which an adjacent edge was deleted.

(b) For the batch of edge insertions into \( G', B'_I \), we first insert the relevant edges into \( L \). Then, we change the relevant entries in \( A \) from 0 to 1. Finally, we update \( D \) with the new degrees of the vertices following the edge insertions.

(c) Remove all vertices which are no longer high-degree (i.e. their degree is now less than \( M^{\ell t}/2 \)) from \( A \). Create entries in \( L \) for all edges adjacent to each vertex that was removed from \( A \).

(d) Remove the edges of all vertices which are no longer low-degree (i.e. their degree is now greater than \( 3M^{\ell t}/2 \)) from \( L \) and create new entries in \( A \) with the new high-degree vertices. Set the relevant entries in \( A \) corresponding to edges adjacent to the new high-degree vertices to 1.

(6) Let the edges in \( B'_I \cup B'_{I,L} \) be the batch of edge insertions to \( G' \). For each of the edges in \( B'_I \cup B'_{I,L} \), we first count the number of triangles it is a part of that contain at least one low-degree vertex. We call this the set of inserted triangles. Let this value be \( T_I \) (\( T_I = 0 \) initially).

(a) To count the number of triangles that contain at least one low-degree vertex, we first check for each edge whether one of its endpoints is low-degree. Let this set of edge insertions be \( I'_L \subseteq B'_I \cup B'_{I,L} \).

(b) For every edge \((u', v') \in I'_L\), without loss of generality let \( u' \) be the lexicographically first low-degree vertex. For every edge \((u', w')\) of \( u' \), check whether \((u', v')\) forms a triangle with \((u', w')\).

(c) For every newly inserted triangle \((u', v', w')\) (where \((u', v', w')\) is sorted lexicographically), call \( t = \text{count}_\text{updated}_\text{low_degree_triangles}((u', v', w'), (u', v')) \), and atomically update \( T_I \leftarrow T_I + t \).
(7) Update $C_L \leftarrow C_L + T_I$.
(8) We perform parallel matrix multiplication after all entries in $A$ have been modified to calculate $S = A^3$. Then, $C_H = \frac{1}{2} \sum_{i \in n} S_{i,i}$.
(9) Update $C \leftarrow C_L + C_H$.
(10) Compute the number of $k$-cliques by dividing $C$ by $\binom{k}{(k/3)} \binom{2k/3}{k/3}$.
(11) If $m$ falls outside the range $[M/4, M]$, then reinitialize the degree thresholds and data structures.

\begin{quote}
\textsuperscript{a}Recall that we can always remove nullifying edge updates as given in Section 3.2.
\textsuperscript{b}We check in our hash table $Q$ whether each newly created (deleted) $\ell$-clique is already represented (non-existent) in the graph $G'$. If not, we insert the new clique and/or remove an old clique from $Q$.
\textsuperscript{c}The batch of updates $B'_{I,L}$ is used to rebalance the data structures when vertices need to be removed from $A$ after becoming low-degree. Because the edges adjacent to these vertices need to be inserted into the structures maintaining low-degree vertices, $B'_{I,L}$, then, can be thought of as a set of edge insertions to update low-degree data structures. Similarly, vertices which become high-degree need to be deleted from low-degree structures, and hence, $B'_{D,H}$ can be thought of as a set of edge deletions from low-degree structures.
\textsuperscript{d}The specific lexicographical order for the vertices in $G'$ is fixed but can be arbitrary.
\end{quote}

Algorithm 5 uses a subroutine $\text{count\_updated\_low\_degree\_triangles}((u', v', w'), (u', v'))$ (defined below) to find all triangles containing a low-degree vertex but only contains an edge update between two high-degree vertices. By Lemma 5.3, if such an edge update occurs, then another edge update to a low-degree vertex must occur such that the two triangles represent the same clique in $G$. Thus, it is sufficient to find all $k$-cliques represented by triangles containing at least one low-degree vertex by looking at edge updates to low-degree vertices in $G'$. To prevent parallel double counting such triangles, we use a sorting procedure such that whichever triangle is first in the sort updates the total count of triangles. In more detail, the procedure below ensures that we count each newly inserted or deleted triangle exactly once, which guarantees that our count contains no duplicate triangles.

\begin{algorithm}
\caption{\text{count\_updated\_low\_degree\_triangles}((u', v', w'), (u', v'))}
(1) Let $u', v', w' \in V'$ represent the sets of vertices $U', X', W' \subseteq V$, respectively.
(2) Enumerate all possible triangles that represent the clique containing vertices $U' \cup X' \cup W'$.
(3) Sort the vertices of each triangle lexicographically to obtain tuples of vertices representing the triangles. Let $\text{ID}(u', v')$ be the ID of edge $(u', v')$.
(4) For each enumerated tuple $(x', y', z')$, create a label containing the tuple representing the triangle concatenated with all labels (sorted lexicographically) of edges that are updates in the triangle. Thus, each label can have 4 to 6 entries consisting of the three vertices of a triangle tuple and at most 3 edge labels. For example, suppose that $(x', y')$ is the only edge that is an updated edge in triangle $(x', y', z')$. Then, the label representing this triangle is $(x', y', z', \text{ID}(x', y'))$ where the ID of the edge is given by $\text{ID}(x', y')$. The IDs of all deleted or inserted edges are appended to the end of the label in the order $\text{ID}(x', y'), \text{ID}(y', z'), \text{ID}(z', x')$.
(5) Sort all labels lexicographically.
(6) Without loss of generality, let $L = (x', y', z', \text{ID}(x', y'))$ be the lexicographically-first of these triangle labels which contains at least one edge deletion (resp. edge insertion) of an edge that is incident to at least one low-degree vertex.
(7) If $(u', v', w')$ corresponds to the lexicographically-first label $L$ and $\text{ID}(u', v')$ is the first edge ID in the label that contains a low-degree vertex, then $(u', v')$ performs the following steps:
\begin{itemize}
\item[(a)] Count the number of unique triangles (using the labels, one can count the unique triangles) containing at least one edge deletion (resp. insertion) and at least one low-degree vertex as $T_D$ (resp. $T_I$). We count using the generated labels for the triangles enumerated in step (2) of this procedure.
\end{itemize}
\end{algorithm}
5.4 Analysis

In Theorem 5.4, we prove that the procedure correctly returns the exact number of $k$-cliques in $G$. The proof is similar to AYZ except that each $\ell$-clique can appear multiple times in $G'$ so we need to normalize by the constant stated in step (10) of Algorithm 5.

**Theorem 5.4.** Algorithm 5 correctly computes the exact number of cliques in a graph $G = (V, E)$ when $k \mod 3 = 0$.

**Proof.** We first show that all triangles in $G'$ represent a $k$-clique in $G$. A vertex exists in $G'$ if and only if it is a $(k/3)$-clique in $G$. Similarly, an edge exists in $G'$ if and only if it connects two vertices in $G'$ that form a $(2k/3)$-clique in $G$. Thus, a triangle connects 3 pairs of 3 distinct $(k/3)$-cliques. This implies that each pair represents a complete subgraph, which necessarily means by the pigeonhole principle that the triangle represents a $k$-clique. Now we show that for each unique $k$-clique in $G$, there exist exactly $\binom{k}{k/3} \binom{2k/3}{k/3}$ triangles representing it in $G'$. For each $k$-clique in $G$, there are $\binom{k}{k/3}$ distinct $(k/3)$-subcliques. Each of these subcliques is represented by a vertex in $G'$. Each distinct triple of subcliques will be a triangle in $G'$. There are $\binom{k}{k/3}$ ways to choose the first subclique, $\binom{2k/3}{k/3}$ ways to choose the second subclique, and $\binom{k/3}{k/3}$ ways to choose the third subclique in the triple. Thus, the total number of duplicate triangles is $\binom{k}{k/3} \binom{2k/3}{k/3}$.

We conclude by proving that our algorithm finds the exact number of triangles in $G'$. All triangles containing edge updates where at least one of its endpoints is low-degree can be found by searching all of the neighbors of the low-degree vertex. All such neighbors will be in $\mathcal{L}$, thus, searching through the entries in $\mathcal{L}$ is enough to find all triangles containing at least one low-degree vertex and an edge update to a low-degree vertex. By Lemma 5.3, all triangles with a low-degree vertex, containing a single edge update between high-degree vertices can be found via the count_new_low_degree_triangles procedure. The same logic handles vertices that change status from high-degree to low-degree, since we treat edges incident to these vertices as new edge insertions. Finally, the procedure ensures that no duplicate triangles are added to the update triangle count because the lexicographically first triangle counts all possible triangles representing the same clique (and no others increment the count). Table $A$ is used to compute (via transitive closure) the number of triangles that contain no low-degree vertices. Thus, by computing $A^3$, we find the remaining triangles which only contain high-degree vertices. Finally, dividing by the total number of different triangles that are created per unique clique gives us the precise count of the number of $k$-cliques in $G$. \qed

**Cost.** We now analyze the work, depth, and space of the dynamic algorithm. Our analysis assumes that $m^{\omega_p/(1+\omega_p)} = O(m^{\ell t})$ so that the $O(m^{\ell t})$ terms in our analysis are only affected by a constant factor for our batch size of $\Delta \leq m^{\omega_p/(1+\omega_p)}$. This is true for $k > 6$ because $t \geq 1/3$ and $\ell \geq 3\omega_p/(1+\omega_p)$. For small $\ell$ we use the combinatorial algorithm from Section 4, which is also faster.

First, we compute the work and depth bound of performing preprocessing on an initial graph $G = (V, E)$ with $m$ edges. We can also apply this preprocessing directly without running the update algorithm whenever we receive a batch of size $\Delta > m^{\omega_p/(1+\omega_p)}$.

For preprocessing, we use a different threshold $m^{\ell/2}$ for low-degree and high-degree vertices. Searching for all the triangles containing at least one low-degree vertex takes $O\left(m^{(1+\ell/2)^t}\right)$ work by a similar calcula-
tion as in Lemma 5.9 and searching for triangles containing all high-degree vertices takes $O\left(m^{(1-t')\ell} \omega_p\right)$ work by Lemma 5.10. Thus, the optimal value $t'$ is when $m^{(1+t')\ell} = m^{(1-t')\ell} \omega_p$, which gives $t' = \frac{\omega_p-1}{\omega_p+1}$ as in [AYZ97].

**Lemma 5.5.** Preprocessing the graph $G = (V, E)$ with $m$ edges into $G'$, creating the data structures $\mathcal{L}$, $A$, and $D$, and counting the number of $k$-cliques takes $O\left(\frac{2k\omega_p}{m^{1+\omega_p}}\right)$ work and $O(\log m)$ depth w.h.p., and $O\left(\frac{2k\omega_p}{m^{1+\omega_p}}\right)$ space assuming a parallel matrix multiplication algorithm with coefficient $\omega_p$. Using the fastest parallel matrix multiplication currently known ([LG14], Corollary 5.19), preprocessing takes $O\left(m^{0.469k}\right)$ work and $O(\log m)$ depth w.h.p., and $O(m^{0.469k})$ space.

**Proof.** The graph $G'$ has size $O(m^\ell)$ by Lemma 5.6. We can find all $\ell$-cliques using $O(m^{\ell/2})$ work and $O(1)$ depth and all $2\ell$-cliques using $O(m^\ell)$ work and $O(1)$ depth. Initializing the data structures $\mathcal{L}$ and $D$ with $O(m^\ell)$ entries requires insertions into two parallel hash tables. This takes $O(m^\ell)$ work and $O(\log^* m)$ depth w.h.p., and $O(m^\ell)$ space. There are $O\left(\frac{4\ell}{m^{1+\omega_p}}\right)$ high-degree vertices which means that initializing $A$, the adjacency matrix, requires creating a 2-level hash table with $O\left(\frac{4\ell}{m^{1+\omega_p}}\right)$ entries. This takes $O\left(\frac{4\ell}{m^{1+\omega_p}}\right)$ work and $O(\log^* m)$ depth w.h.p., and $O\left(\frac{4\ell}{m^{1+\omega_p}}\right)$ space. Computing $A^3$ requires $O\left(\frac{2\omega_p}{m^{1+\omega_p}}\right)$ work, $O(\log m)$ depth, and $O\left(\frac{2\omega_p}{m^{1+\omega_p}}\right)$ space. Finally, counting all the triangles with at least one low-degree vertex requires $O\left(\frac{2\omega_p}{m^{1+\omega_p}}\right)$ work and $O(1)$ depth (by performing $O\left(m^{(1+t')\ell}\right)$ lookups in $\mathcal{L}$). By Corollary 5.19, $\omega_p = 2.373$, and since $\ell = k/3$, preprocessing takes $O\left(m^{0.469k}\right)$ work, $O(\log m)$ depth, and $O(m^{0.469k})$ space.

Next, we analyze the update procedure of our dynamic algorithm. To start, we bound the number of vertices and edges in $G'$ (representing the number of $\ell$ and $2\ell$ cliques in $G$, respectively) in terms of $m$ (the number of edges in $G$) below.

**Lemma 5.6 ([CN85]).** Given a graph $G = (V, E)$ with $m$ edges, the number of $k$-cliques that $G$ can have is bounded by $O(m^{k/2})$.

**Lemma 5.7.** $G'$ uses $O(m^\ell)$ space.

**Proof.** Each vertex in $G'$ represents an $\ell$-clique. By Lemma 5.6, $G'$ has $O(m^\ell)$ vertices and thus $O(m^\ell)$ edges.

Before we compute the number of triangles in $G'$, we must update $G'$ and the data structures associated with $G'$ with our batch of updates.

**Lemma 5.8.** Updating $G'$ and the associated data structures $\mathcal{L}$ and $A$ after a batch of $\Delta$ edge updates in $G$ takes $O(\Delta m^{\ell-1} + \Delta m^{(2-2t)\ell-1})$ amortized work and $O(\log^* m)$ depth w.h.p., and $O\left(m^\ell + m^{(2-2t)\ell}\right)$ space.

**Proof.** In step (1) we first add and/or delete vertices in $G'$. Since each vertex in $G'$ represents a different clique of size $\ell$, one edge update in $G$ can result in $O(m^{(\ell/2)-1})$ new vertices (or vertex deletions) since given two vertices (the endpoints of the edge update) that must be in the $\ell$-clique, we only need to look
for all \((\ell - 2)\)-cliques in \(G\). For a batch of size \(\Delta\), the total number of vertices added or deleted in \(G'\) is 
\[ O(\Delta m^{(\ell/2)−1}) \]

In steps (5)a and (5)b, updating the data structures \(L, A,\) and \(D\) by insertions/deletions into parallel hash tables requires \(O(\Delta m^{\ell−1})\) amortized work and \(O(\log^* m)\) depth w.h.p.

Recall that the number of edges in \(G'\) is determined by the total number of \(2\ell\)-cliques in \(G\). One edge update can affect at most \(O(m^{\ell−1})\) \(2\ell\)-cliques in \(G\), thus, given a \(\Delta\)-batch of edge updates in \(G\), there will be \(O(\Delta m^{\ell−1})\) edge updates in \(G'\), separated into a deletion batch \(B'_{D}\) and an insertion batch \(B'_{I}\).

We now analyze the cost for steps (5)c and (5)d. Adding/removing a row and column from \(A\) takes \(O(m^{(1−t)\ell})\) amortized work. Since there are \(O(m^{\ell−1})\) edge updates in \(G'\) per update in \(G\), the total work for resizing is \(O(m^{(2−t)\ell−1})\) per edge update in \(G\). The work for adding/removing a vertex from \(L\) is \(O(m^{\ell−1})\), and since there are \(O(m^{\ell−1})\) edge updates per update in \(G\), the total work is \(O(m^{(1+t)\ell−1})\) per update in \(G\). We must have \(\Omega(m^{\ell})\) updates in \(G'\) before a vertex changes statuses (becomes high-degree if it originally was low-degree and vice versa) and needs to update \(A\) and \(L\). Therefore, we can charge the work of updating \(A\) and \(L\) against \(O(m^{\ell})\) updates in \(G'\). Thus, the amortized work for updating \(A\) and \(L\) given a batch of \(\Delta\) updates in \(G\) is \(O(\Delta m^{(2−2t)\ell−1} + m^{\ell−1})\) for steps (1) and (5). The depth is \(O(\log^* m)\) w.h.p. due to hash table operations.

The data structures \(L, D,\) and \(A\) use a combined \(O(m^{\ell} + m^{(2−2t)\ell})\) space because there are \(O(m^{\ell})\) edges in the graph and \(A\) contains \(O(m^{(2−2t)\ell})\) entries.

By Lemma 5.8, step (2) takes \(O(\Delta m^{\ell−1})\) amortized work to determine the final degrees and \(O(\Delta m^{\ell−1} + \Delta m^{(2−2t)\ell−1})\) amortized work to compute \(B'_{I, L}\) and \(B'_{D, H}\). In total, step (2) takes \(O(\Delta m^{\ell−1} + \Delta m^{(2−2t)\ell−1})\) amortized work, \(O(\log m)\) depth (dominated by computing the final degrees), and \(O(m^{\ell} + m^{(2−2t)\ell})\) space by Lemma 5.8. Steps (4), (7), (9), and (10) of the algorithm take \(O(1)\) work. The following lemmas bound the cost for the remaining steps.

Lemma 5.9 below bounds the cost for steps (3) and (6). The proof is based on counting the number of new edge updates necessary in \(G'\).

**Lemma 5.9.** Computing all new \(k\)-cliques represented by triangles that contain at least one low-degree vertex in \(G'\) takes \(O(\Delta m^{(t+1)\ell−1})\) work and \(O(\log^* m)\) depth w.h.p., and \(O(m^{\ell})\) space.

**Proof:** We first bound the work necessary to perform steps (3) and (6) for new edge insertions and deletions. Given one edge update in \(G\), there can be at most \(O(m^{\ell−1})\) edge updates necessary in \(G'\) by Lemma 5.6. For each of these edge updates, we consider whether each edge update in \(G'\) contains a low-degree vertex. By Lemmas 5.3 and 5.4, to find all updated triangles containing at least one low-degree vertex, it is only necessary to consider edge updates to low-degree vertices. For every edge update to a low-degree vertex, we search the neighbors of that low-degree vertex to see if new triangles are formed/destroyed. Since each low-degree vertex has degree \(\deg\), we search the neighbors of that low-degree vertex to see if new triangles are formed/destroyed. Since each low-degree vertex has degree \(\deg\), we search the neighbors of that low-degree vertex to see if new triangles are formed/destroyed.

The proof is based on counting the number of new edge updates necessary in \(G'\).
must have degree $O(m^\ell)$ at the time of rebalancing. Thus, the total work performed for these updates is $O(Xm^{2\ell})$. However, in order for a rebalancing on a vertex to happen, there must be $\Omega(m^\ell)$ updates. Thus, if $X$ vertices are rebalanced, then there must be $\Omega(Xm^\ell)$ updates. Hence, we can charge the work of rebalancing to the $\Omega(Xm^\ell)$ updates to obtain $O(m^\ell)$ amortized work per update in $G'$. Then, we obtain $O(\Delta m^{(t+1)\ell-1})$ amortized work for a $\Delta$ batch updates to $G$. Rebalancing requires $O(\log^* m)$ depth w.h.p. due to hash table operations and $O(m^\ell)$ space (the total number of edges in the graph).}

Lemma 5.10 bounds the cost for step (8) by using the matrix multiplication bounds for the adjacency matrix containing high-degree vertices.

**Lemma 5.10.** Computing $A^3$ using parallel matrix multiplication takes $O(m^{(1-t)\ell}\omega_p)$ work, where $\omega_p$ is the parallel matrix multiplication constant, $O(\log m)$ depth, and $O(m^{\omega_p(1-t)\ell})$ space, assuming that there exists a parallel matrix multiplication algorithm with coefficient $\omega_p$ and using $O(\log n)$ depth and $O(n^{\omega_p})$ space given $n \times n$ matrices.

**Proof.** There are $O(m^{(1-t)\ell})$ high-degree vertices because each high-degree vertex has degree $\Omega(m^\ell)$ and there are $O(m^\ell)$ edges in $G'$. Since the table $A$ is an adjacency matrix on the high-degree vertices, by Corollary 5.19, parallel matrix multiplication can be done in $O(m^{(1-t)\ell}\omega_p)$ work.

Lemma 5.11 bounds the cost for step (11). The proof is based on amortizing the cost for reconstruction over $\Omega(m)$ updates.

**Lemma 5.11.** Step (11) requires $O(\Delta m^{(2-2t)\ell-1} + m^{\ell-1})$ amortized work and $O(\log^* m)$ depth w.h.p., and $O(m^{(2-2t)\ell} + m^\ell)$ space.

**Proof.** We reconstruct $A$ from scratch, which has one entry for every pair of high-degree vertices, which takes $O(m^{2(1-t)\ell}) = O(m^{(2-2t)\ell})$ work and space. However, this is amortized against $\Omega(m)$ updates, and so the amortized work is $O(m^{(2-2t)\ell-1})$ per update. The work and space for creating $L$ can be bounded by $O(m^\ell)$, the number of edges in $G'$. Amortized against $\Omega(m)$ updates gives $O(m^{\ell-1})$ work per update. The depth is $O(\log^* m)$ w.h.p. using parallel hash table operations.

Given these costs, we can now compute the optimal value of $t$ in terms of $\omega_p$ that minimizes the work. Note that here we compute for $t$ assuming $\Delta = 1$ because to adaptively change our threshold requires too much work in terms of rebalancing the data structures. However, if we have a fixed batch size, $\Delta$, we can further optimize our threshold $t$ to take into account the fixed batch size.

**Lemma 5.12.** $t = \frac{3-k+k\omega_p}{k+k\omega_p}$ gives us an optimal work bound assuming $\Delta = 1$.

**Proof.** From Lemmas 5.8, 5.9, 5.10, and 5.11, we have that the work is $O(\Delta m^{(t+1)\ell-1} + m^{(1-t-k)\omega_p})$ w.h.p. (the $O(\Delta m^{(2-2t)\ell-1})$ term is dominated by the $O(\Delta m^{(1-t)\ell-1})$ term since $\omega_p \geq 2$ implies $t \geq 1/3$). Assuming $\Delta = 1$, balancing the two sides of the equation yields:

$$m^{(1-t-k)\omega_p} = m^{(t+1)\ell-1}.$$  

Solving for $t$ gives

$$t = \frac{3-k+k\omega_p}{k+k\omega_p}.$$

Plugging in our value for $t$ from Lemma 5.12, we prove Theorem 5.1 and Corollary 5.2 for the cost of our algorithm when $0 < m \leq m^{\omega_p/(1+\omega_p)}$.  

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5.5 Accounting for \( k \mod 3 \neq 0 \)

We now modify the algorithm above to account for all values \( k \) following the algorithm presented in [EG04]. This requires several changes to how we construct our graph \( G' \) from a graph \( G = (V, E) \), resulting in changes to our data structures which we detail below. We recall the notation \( R(x) \) for vertex \( x \in G' \) to denote the vertices in \( G \) that \( x \) represents.

5.5.1 Construction of \( G' \)

For \( k \mod 3 \neq 0 \), the fundamental problem we face in this case in constructing the graph \( G' \) is that triangles in the graph \( G' \) representing cliques of size \( \lceil \frac{k}{3} \rceil \) no longer create \( k \)-cliques. In fact, they now create \((k - 1)\)-cliques or \((k - 2)\)-cliques for \( k \mod 3 = 1 \) and \( k \mod 3 = 2 \), respectively. We modify the creation of \( G' \) in the two following ways to account for this issue:

\( k \mod 3 = 1 \): In this case, we create two sets of vertices. One set, \( A \), of vertices represents all \( \left( \frac{k-1}{3} \right) \)-cliques in the graph \( G \). Edges exist between \( v_1, v_2 \in A \) if and only if the vertices, \( R(v_1) \) and \( R(v_2) \), in the \( \left( \frac{k-1}{3} \right) \)-cliques represented by \( v_1 \) and \( v_2 \) form a \( 2 \left( \frac{k-1}{3} \right) \)-clique and there are no duplicate vertices, i.e., \( R(v_1) \cap R(v_2) = \emptyset \). We create a second set of vertices \( B \) which contains vertices which represent cliques of size \( \frac{k+2}{3} \). Edges exist between \( v \in A \) and \( w \in B \) if and only if \( R(v) \) and \( R(w) \) form a \( \left( \frac{2k+1}{3} \right) \)-clique and \( R(v) \cap R(w) = \emptyset \).

\( k \mod 3 = 2 \): In this case, we still create two sets of vertices but \( A \) instead represents \( \left( \frac{k+1}{3} \right) \)-cliques in the graph \( G \). Edges exist between \( v_1, v_2 \in A \) if and only if \( R(v_1) \cup R(v_2) \) form a \( \left( \frac{2(k+1)}{3} \right) \)-clique and \( R(v_1) \cap R(v_2) = \emptyset \). We create a second set of vertices \( B \) which contains vertices which represent cliques of size \( \frac{k-2}{3} \). Edges exist between \( v \in A \) and \( w \in B \) if and only if \( R(v) \) and \( R(w) \) form a \( \left( \frac{2k-1}{3} \right) \)-clique and \( R(v) \cap R(w) = \emptyset \).

We first prove the properties the new graph \( G' \) has, namely the number of vertices it contains as well as the number of edges in the graph.

**Lemma 5.13.** \( G' \) constructed as in Section 5.5.1 contains \( O \left( m^{\frac{k+2}{6}} \right) \) vertices and \( O \left( m^{\frac{2k+1}{6}} \right) \) edges if \( k \mod 3 = 1 \). \( G' \) contains \( O \left( m^{\frac{k+1}{6}} \right) \) vertices and \( O \left( m^{\frac{k+1}{3}} \right) \) edges if \( k \mod 3 = 2 \).

**Proof.** When \( k \mod 3 = 1 \), the number of vertices is upper bounded (asymptotically) by the number of \( \left( \frac{k+2}{3} \right) \)-cliques in the graph. By Lemma 5.6, the number of vertices is then bounded by \( O \left( m^{\frac{k+2}{6}} \right) \). The number of edges is bounded by the number of \( \left( \frac{2k+1}{3} \right) \)-cliques in the graph which is \( O \left( m^{\frac{2k+1}{6}} \right) \). Similarly, when \( k \mod 3 = 2 \), by Lemma 5.6, the number of vertices and edges are bounded by \( O \left( m^{\frac{k+1}{6}} \right) \) and \( O \left( m^{\frac{k+1}{3}} \right) \), respectively. \( \square \)

5.5.2 Data Structure and Algorithm Changes

The major data structure change is to redefine the high-degree and low-degree vertices in terms of the number of edges in the graph. This means that low-degree is defined as having a degree less than \( \frac{M'(\frac{2k+1}{6})}{2} \) and high-degree as greater than \( \frac{3M'(\frac{2k+1}{6})}{2} \) for the \( k \mod 3 = 1 \) case; similarly we define low-degree to be less than \( \frac{M'(\frac{k+1}{3})}{2} \) and high-degree to be greater than \( \frac{3M'(\frac{k+1}{3})}{2} \) for the \( k \mod 3 = 2 \) case.
Another key difference between this case and the case when \( k \) is divisible by 3 is that the number of duplicate cliques is different for these two cases. For the \( k \mod 3 = 1 \) case, each \( k \)-clique in \( G \) will be represented by \( \binom{k}{(k+2)/3} \binom{(2k-2)/3}{(k-1)/3} \) triangles found by the algorithm. For the \( k \mod 3 = 2 \) case, each \( k \)-clique in \( G \) will be represented by \( \binom{k}{(k-2)/3} \binom{(2k-2)/3}{(k+1)/3} \) triangles. Thus, at the end of our algorithm, we must divide the count of the triangles by their respective number of duplicates.

The rest of the algorithm remains the same as before, except that we solve for different values of \( t \) depending on the case. Since the proofs for obtaining the following results are nearly identical to the ones for \( k \mod 3 = 0 \), we do not restate the proofs and only give our results.

**Lemma 5.14.** For the case when \( k \mod 3 = 1 \), there exists \( O\left(m^{2k+1}\right) \) edges in the graph and solving for the optimal value of \( t \) (assuming \( \Delta = 1 \)) gives \( t = \frac{2k\omega_p - k + \omega_p + 5}{2k\omega_p + 2k + \omega_p + 1} \). For the case when \( k \mod 3 = 2 \), there exists \( O\left(m^k\right) \) edges in the graph and solving for the optimal value of \( t \) gives \( t = \frac{k\omega_p - k + \omega_p + 2}{k\omega_p + k + \omega_p + 1} \).

Using our values for \( t \), we can obtain our final theorem, **Theorem 5.15**, for the work and depth bounds for these two cases.

**Theorem 5.15.** Our fast matrix multiplication based \( k \)-clique algorithm takes 

\[
O\left(\min\left(\Delta m^{\frac{2(k-1)ωp}{3(ωp+1)}}, (\Delta + m)^{\frac{(2k+1)ωp}{3(ωp+1)}}\right)\right) \text{ work and } O(\log(m+Δ)) \text{ depth w.h.p., and } O\left(\min\left(\Delta m^{\frac{(2k-1)ωp}{3(ωp+1)}}, (\Delta + m)^{\frac{2(k+1)ωp}{3(ωp+1)}}\right)\right) \text{ work and } O(\log(m+Δ)) \text{ depth w.h.p., and } O\left(\left(\Delta + m\right)^{\frac{2(2k+1)ωp}{3(ωp+1)}}\right) \text{ space when } k \mod 3 = 1 \text{, and}
\]

\[
O\left(\min\left(\Delta m^{\frac{(2k-1)ωp}{3(ωp+1)}}, (\Delta + m)^{\frac{2(k+1)ωp}{3(ωp+1)}}\right)\right) \text{ work and } O(\log(m+Δ)) \text{ depth w.h.p., and } O\left(\left(\Delta + m\right)^{\frac{(2k+1)ωp}{3(ωp+1)}}\right) \text{ space when } k \mod 3 = 2.
\]

**Corollary 5.16.** Using Corollary 5.19 with \( ω_p = 2.373 \), we obtain a parallel fast matrix multiplication \( k \)-clique algorithm that takes \( O\left(\min\left(\Delta m^{0.469k-0.469}, (\Delta + m)^{0.469k+0.235}\right)\right) \) work and \( O(\log m) \) depth w.h.p., and \( O\left(\left(\Delta + m\right)^{0.469k+0.235}\right) \) space when \( k \mod 3 = 1 \), and \( O\left(\min\left(\Delta m^{0.469k-0.469}, (\Delta + m)^{0.469k+0.235}\right)\right) \) work and \( O(\log m) \) depth w.h.p., and \( O\left(\left(\Delta + m\right)^{0.469k+0.235}\right) \) space when \( k \mod 3 = 2 \).

### 5.6 Parallel Fast Matrix Multiplication

In this section, we show that tensor-based matrix multiplication algorithms (including Strassen’s algorithm) can be parallelized in \( O(\log n) \) depth and \( O(n^ω) \) work. Such techniques are used for algorithms that achieve the best currently known matrix multiplication exponents [Wil12, LG14]. We assume, as is common in models such as the arithmetic circuit model, that field operations can be performed in constant work. We refer readers interested in learning more about current techniques in fast matrix multiplication to [Blä13, Alm19].

Before we prove our main parallel result in this section, we first define the **matrix multiplication tensor** as used in previous literature.

**Definition 5.17 (Matrix Multiplication Tensor (see, e.g., [Alm19])).** For positive integers \( a, b, c \), the matrix multiplication tensor \( \langle a, b, c \rangle \) is a tensor over \( \{x_{ij}\}_{i∈[a], j∈[b]} \), \( \{y_{jk}\}_{j∈[b], k∈[c]} \), \( \{z_{ki}\}_{k∈[c], i∈[a]} \), where

\[
\langle a, b, c \rangle = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} x_{ij} y_{jk} z_{ki}.
\]

The matrix multiplication tensor can be seen as a generating function for \( A × B \) multiplication where the coefficients of the \( z_{ki} \) terms are exactly the \( (i, k) \) entries in the matrix product \( A × B \) where \( A = (x_{11}, \ldots, x_{1b}) \) and \( B = (y_{11}, \ldots, y_{1c}) \), and \( x_{a1}, \ldots, x_{ab} \) and \( (x_{b1}, \ldots, x_{bc}) \).
Current matrix multiplications algorithms use this fact to obtain the best known exponents. The proof of the following lemma closely follows the proof of Proposition 4.1 given in [Alm19].

**Lemma 5.18.** Let \( R((q, q, q)) \leq r \) (over a field \( \mathbb{F} \)) be the rank of the matrix multiplication tensor \( (q, q, q) \). Assuming that field operations take \( O(1) \) work, then, there exists a parallel matrix multiplication algorithm that performs \( A \times B \) matrix multiplication (where \( A, B \in \mathbb{F}^{n \times n} \)) over \( \mathbb{F} \) using \( O\left(\ell_{k, i}^{\log_q(r)}\right) \) work and \( O((\log r + \log q) \log_q(n)) \) depth using \( O\left(n^{\log_q(r)}\right) \) space.

**Proof.** By definition of rank, since \( R((q, q, q)) \leq r \),

\[
(q, q, q) = \sum_{\ell=1}^{r} \left( \sum_{i,j \in [q]} a_{ij \ell} x_{ij} \right) \left( \sum_{j,k \in [q]} b_{jkt} x_{jk} \right) \left( \sum_{k,i \in [q]} c_{kit} x_{ki} \right)
\]

for some coefficients \( a_{ij \ell}, b_{jkt}, c_{kit} \in \mathbb{F} \). Computing this matrix multiplication tensor requires at most \( O\left(q^2\right) \) field operations.

Using this information, we perform parallel matrix multiplication via the following recursive algorithm. We assume that \( n \) is a power of \( q \); otherwise, we can pad \( A \) and \( B \) with 0’s until such a condition is satisfied—this would increase the dimensions by at most a factor of \( q \).

Partition the padded matrices \( A \) and \( B \) into \( q \times q \) block matrices where each block has size \( n/q \times n/q \).

This algorithm performs, in parallel, the following linear combinations for each \( \ell \),

\[
A'_{\ell} = \sum_{i,j \in [q]} a_{ij \ell} A_{ij}
\]

\[
B'_{\ell} = \sum_{j,k \in [q]} b_{jkt} B_{jk}
\]

where \( A_{ij} \) and \( B_{jk} \) are the \( n/q \times n/q \) blocks in \( A \) and \( B \), respectively. Such operations require \( O(rq^2) \) operations to perform; however, all such multiplication operations can be done in parallel, and the summation of the results can be done in \( O(\log q) \) depth, resulting in \( O(\log q) \) depth.

Then, for each \( \ell \in [r] \), we compute \( C'_{\ell} = A'_{\ell} \times B'_{\ell} \) by performing parallel \( n/q \times n/q \) matrix multiplication recursively on \( A'_{\ell} \) and \( B'_{\ell} \) where the base case is \( q \times q \) matrix multiplication. All field operations in the same level of the recursion can be performed in parallel. There are \( O(\log_q(n)) \) levels of recursion. Each level of recursion computes a number of field operations in parallel in \( O(\log q) \) depth as in the top level.

Finally, after obtaining the results \( C'_{\ell} \) of the recursive calls, we compute

\[
C_{ki} = \sum_{\ell \in [r]} c_{k\ell i} C'_{\ell, ki}
\]

for all \( k, i \in [q] \) where \( C'_{\ell, ki} \) are the results we obtain from our recursive calls. The blocks \( C_{ki} \) for all \( k, i \in [q] \) are the results of our matrix multiplication \( A \times B \).

This final step can compute in parallel the blocks \( C_{ki} \) for all \( k, i \in [q] \) in \( O(\log r) \) depth (assuming that we have the results \( C'_{(\ell, ki)} \)) since the multiplication operations can be done in parallel and the summation of the elements in the resulting matrices can be done in \( O(\log r) \) depth.

Thus, the depth required for this algorithm is \( O((\log r + \log q) \log_q(n)) \).

To compute the work and space usage, we compute the total number of field operations performed, which is \( O(n^2) \) per level of the recursion. For each level of recursion, there are \( r \) calls per subproblem of the recursion. Since we assume that each field operation is \( O(1) \) work, this results in total work given by
\[ W(n) = r \cdot W(n/q) + O(n^2). \]

Solving the recurrence gives \( W(n) = O(n^{\log_q r}) \) work for the entire algorithm. The space usage is also \( O(n^{\log_q r}) \).

Using Lemma 5.18, we obtain the following parallel matrix multiplication bounds:

**Corollary 5.19.** There exists a parallel matrix multiplication algorithm based on [Wil12, LG14] that multiplies two \( n \times n \) matrices with \( O(n^{2.373}) \) work and \( O(\log n) \) depth, using \( O(n^{2.373}) \) space.

### 6 Conclusion

In this paper, we have given new dynamic algorithms for the \( k \)-clique problem. We study this fundamental problem in the recently proposed batch-dynamic setting, which is better suited for parallel hardware that is widely available today, and enables dynamic algorithms to scale to high-rate data streams. We have presented a work-efficient parallel batch-dynamic triangle counting algorithm. We also gave a simple, enumeration-based algorithm for maintaining the \( k \)-clique count. Finally, we have presented a novel parallel batch-dynamic \( k \)-clique counting algorithm based on fast matrix multiplication, which is asymptotically faster than existing dynamic approaches on dense graphs. We believe that our parallel batch-dynamic triangle counting algorithm may be practical, and view implementing and empirically evaluating this algorithm as interesting future work. Another interesting future direction would be to develop a parallel batch-dynamic low-outdegree orientation algorithm, and use this maintenance structure to obtain a work-efficient combinatorial \( k \)-clique counting algorithm that matches the bounds of the more sophisticated dynamic subgraph counting algorithm of Dvorak and Tuma [DT13]. Finally, it would be interesting to study the \( k \)-clique counting problem in the batch-dynamic setting in other parallel and distributed models of computation, such as a bandwidth-restricted LOCAL model [BCH19], or the Massively Parallel Computation model.

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### References

[AABD19] Umut A. Acar, Daniel Anderson, Guy E. Blelloch, and Laxman Dhulipala. Parallel batch-dynamic graph connectivity. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, pages 381–392, 2019.

[AAW17] Umut A. Acar, Vitaly Aksenov, and Sam Westrick. Brief announcement: Parallel dynamic tree contraction via self-adjusting computation. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, pages 275–277, 2017.

[AKM13] Shaikh Arifuzzaman, Maleq Khan, and Madhav Marathe. PATRIC: A parallel algorithm for counting triangles in massive networks. In *ACM Conference on Information and Knowledge Management (CIKM)*, pages 529–538, 2013.
[Alm19] Josh Alman. *Linear Algebraic Techniques in Algorithms and Complexity*. PhD thesis, Massachusetts Institute of Technology, 2019.

[ALT+17] Christopher R. Aberger, Andrew Lamb, Susan Tu, Andres Nötzli, Kunle Olukotun, and Christopher Ré. EmptyHeaded: A relational engine for graph processing. *ACM Trans. Database Syst.*, 42(4):20:1–20:44, 2017.

[AMSJ18] Khaled Ammar, Frank McSherry, Semih Salihoglu, and Manas Joglekar. Distributed evaluation of subgraph queries using worst-case optimal low-memory dataflows. *Proc. VLDB Endow.*, 11(6):691–704, February 2018.

[ANR+17] Nesreen K. Ahmed, Jennifer Neville, Ryan A. Rossi, Nick G. Duffield, and Theodore L. Willke. Graphlet decomposition: framework, algorithms, and applications. *Knowl. Inf. Syst.*, 50(3):689–722, 2017.

[AOSS19] Sepehr Assadi, Krzysztof Onak, Baruch Schieber, and Shay Solomon. Fully dynamic maximal independent set with sublinear in $n$ update time. In *ACM-SIAM Symposium on Discrete Algorithms*, pages 1919–1936, 2019.

[AYZ97] N. Alon, R. Yuster, and U. Zwick. Finding and counting given length cycles. *Algorithmica*, 17(3):209–223, Mar 1997.

[BCH19] Matthias Bonne and Keren Censor-Hillel. Distributed detection of cliques in dynamic networks. In *International Colloquium on Automata, Languages, and Programming (ICALP 2019)*, pages 132:1–132:15, 2019.

[BE10] Leonid Barenboim and Michael Elkin. Sublogarithmic distributed MIS algorithm for sparse graphs using Nash-Williams decomposition. *Distributed Computing*, 22(5):363–379, Aug 2010.

[BKM19] Philipp Bamberger, Fabian Kuhn, and Yannic Maus. Local distributed algorithms in highly dynamic networks. In *IEEE International Parallel and Distributed Processing Symposium (IPDPS)*, pages 33–42, 2019.

[Blä13] Markus Bläser. Fast matrix multiplication. *Theory of Computing. Graduate Surveys*, 5:1–60, 2013.

[CHDK+19] Keren Censor-Hillel, Neta Dafni, Victor I Kolobov, Ami Paz, and Gregory Schwartzman. Fast and simple deterministic algorithms for highly-dynamic networks. *arXiv preprint arXiv:1901.04008*, 2019.

[CHKH16] Keren Censor-Hillel, Elad Haramaty, and Zohar Karnin. Optimal dynamic distributed MIS. In *ACM Symposium on Principles of Distributed Computing*, pages 217–226, 2016.

[CHKX04] Jianer Chen, Xiuzei Huang, Iyad A Kanj, and Ge Xia. Linear FPT reductions and computational lower bounds. In *ACM Symposium on Theory of Computing*, pages 212–221, 2004.

[CN85] Norishige Chiba and Takao Nishizeki. Arboricity and subgraph listing algorithms. *SIAM J. Comput.*, 14(1):210–223, February 1985.

[DAH17] V. S. Dave, N. K. Ahmed, and M. Hasan. PE-CLoG: Counting edge-centric local graphlets. In *IEEE International Conference on Big Data*, pages 586–595, 2017.
Maximilien Danisch, Oana Balalau, and Mauro Sozio. Listing \(k\)-cliques in sparse real-world graphs. In *International Conference on World Wide Web (WWW)*, pages 589–598, 2018.

David Durfee, Laxman Dhulipala, Janardhan Kulkarni, Richard Peng, Saurabh Sawlani, and Xiaorui Sun. Parallel batch-dynamic graphs: Algorithms and lower bounds. In *ACM-SIAM Symposium on Discrete Algorithms*, 2020.

Sajal K. Das and Paolo Ferragina. An \(o(n)\) work EREW parallel algorithm for updating MST. In *Annual European Symposium on Algorithms (ESA)*, pages 331–342, 1994.

Rod G. Downey and Michael R. Fellows. Fixed-parameter tractability and completeness I: Basic results. *SIAM Journal on Computing*, 24(4):873–921, 1995.

Zdeněk Dvořák and Vojtěch Tůma. A dynamic data structure for counting subgraphs in sparse graphs. In *Algorithms and Data Structures*, pages 304–315, 2013.

Friedrich Eisenbrand and Fabrizio Grandoni. On the complexity of fixed parameter clique and dominating set. *Theor. Comput. Sci.*, 326(1-3):57–67, October 2004.

David Eppstein, Michael T. Goodrich, Darren Strash, and Lowell Trott. Extended dynamic subgraph statistics using \(h\)-index parameterized data structures. *Theoretical Computer Science*, 447:44 – 52, 2012. Combinational Algorithms and Applications.

D. Ediger, K. Jiang, J. Riedy, and D. A. Bader. Massive streaming data analytics: A case study with clustering coefficients. In *IEEE International Symposium on Parallel Distributed Processing, Workshops and Phd Forum (IPDPSW)*, pages 1–8, 2010.

David Eppstein and Emma S. Spiro. The \(h\)-index of a graph and its application to dynamic subgraph statistics. In *Algorithms and Data Structures*, pages 278–289, 2009.

Ethan R. Elenberg, Karthikeyan Shanmugam, Michael Borokhovich, and Alexandros G. Dimakis. Distributed estimation of graph 4-profiles. In *International Conference on World Wide Web (WWW)*, pages 483–493, 2016.

Irene Finocchi, Marco Finocchi, and Emanuele G. Fusco. Clique counting in MapReduce: Algorithms and experiments. *J. Exp. Algorithmics*, 20:1.7:1–1.7:20, October 2015.

Paolo Ferragina and Fabrizio Luccio. Batch dynamic algorithms for two graph problems. In *Parallel Architectures and Languages Europe (PARLE)*, pages 713–724, 1994.

J. Gil, Y. Matias, and U. Vishkin. Towards a theory of nearly constant time parallel algorithms. In *IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 698–710, 1991.

Michael T. Goodrich and Paweł Pszona. External-memory network analysis algorithms for naturally sparse graphs. In *European Symposium on Algorithms*, ESA’11, page 664–676, 2011.

GraphChallenge. [http://graphchallenge.mit.edu/](http://graphchallenge.mit.edu/).

Mark S Granovetter. The strength of weak ties. In *Social networks*, pages 347–367. Elsevier, 1977.

Tomaz Hocevar and Janez Demsar. A combinatorial approach to graphlet counting. *Bioinformatics*, pages 559–65, 2014.
[HKNS15] Monika Henzinger, Sebastian Krinninger, Danupon Nanongkai, and Thatchaphol Saranurak. Unifying and strengthening hardness for dynamic problems via the online matrix-vector multiplication conjecture. In *ACM Symposium on Theory of Computing*, pages 21–30, 2015.

[HR05] Robert A. Hanneman and Mark Riddle. *Introduction to social network methods*. University of California, Riverside, 2005.

[ILMP19] Giuseppe F. Italiano, Silvio Lattanzi, Vahab S. Mirrokni, and Nikos Parotsidis. Dynamic algorithms for the massively parallel computation model. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, 2019.

[IR77] Alon Itai and Michael Rodeh. Finding a minimum circuit in a graph. In *Annual ACM Symposium on Theory of Computing (STOC)*, pages 1–10, 1977.

[Jaj92] J. Jaja. *Introduction to Parallel Algorithms*. Addison-Wesley Professional, 1992.

[JS17] Shweta Jain and C. Seshadhri. A fast and provable method for estimating clique counts using Turán’s theorem. In *International Conference on World Wide Web (WWW)*, pages 441–449, 2017.

[Kha17] Shahbaz Khan. Near optimal parallel algorithms for dynamic DFS in undirected graphs. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, pages 283–292, 2017.

[KNN+19] Ahmet Kara, Hung Q. Ngo, Milos Nikolic, Dan Olteanu, and Haozhe Zhang. Counting triangles under updates in worst-case optimal time. In *International Conference on Database Theory (ICDT)*, volume 127, pages 4:1–4:18, 2019.

[KPR18] Tsvi Kopelowitz, Ely Porat, and Yair Rosenmüller. Improved worst-case deterministic parallel dynamic minimum spanning forest. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, pages 333–341, 2018.

[Lat08] Matthieu Latapy. Main-memory triangle computations for very large (sparse (power-law)) graphs. *Theor. Comput. Sci.*, 2008.

[LG14] François Le Gall. Powers of tensors and fast matrix multiplication. In *International Symposium on Symbolic and Algebraic Computation (ISSAC)*, pages 296–303, 2014.

[MBG17] D. Makkar, D. A. Bader, and O. Green. Exact and parallel triangle counting in dynamic graphs. In *IEEE International Conference on High Performance Computing (HiPC)*, pages 2–12, Dec 2017.

[MV91] Yossi Matias and Uzi Vishkin. On parallel hashing and integer sorting. *Journal of Algorithms*, 12(4):573–606, 1991.

[New03] Mark EJ Newman. The structure and function of complex networks. *SIAM review*, 45(2):167–256, 2003.

[NG04] Mark EJ Newman and Michelle Girvan. Finding and evaluating community structure in networks. *Physical review E*, 69(2):026113, 2004.

[NP85] Jaroslav Nešetřil and Svatopluk Poljak. On the complexity of the subgraph problem. *Commentationes Mathematicae Universitatis Carolinae*, 026(2):415–419, 1985.
A Sequential Fully Dynamic Triangle Counting of [KNN+19]

Here, we present the sequential fully dynamic triangle counting algorithm of Kara et al. [KNN+19] that operates in $O(m)$ space, $O(\sqrt{m})$ amortized work per edge update, and $O(m^{3/2})$ work for preprocessing. This algorithm returns the exact count of the number of triangles in an undirected graph under both edge insertions and deletions (their algorithm is presented for directed 3-cycles in [KNN+19], but we simplify their algorithm for the undirected case). Kara et al. [KNN+19] prove the following theorem.
Theorem A.1 (Fully Dynamic Triangle Counting [KNN⁺19]). There exists a sequential algorithm to count the number of triangles in an undirected graph $G = (V, E)$ using $O(m^{3/2})$ preprocessing work that can handle an edge update in $O(\sqrt{m})$ amortized work and $O(m)$ space.

We now explain the fully dynamic triangle counting algorithm of [KNN⁺19] in greater detail.

Given a graph $G = (V, E)$ with $n = |V|$ vertices and $m = |E|$ edges, we initialize the following variables: $M = 2m + 1$, $t_1 = \sqrt{M}/2$, and $t_2 = 3\sqrt{M}/2$. We define a vertex to be low-degree if its degree is at most $t_1$ and high-degree if its degree is at least $t_2$. Vertices with degree in between $t_1$ and $t_2$ can be classified either way. Let $C$ be the current count of the number of triangles in the graph. We compute the initial count of the number of triangles in the input graph $G$ using a static triangle counting algorithm [IR77] in $O(m^{3/2})$ work and $O(m)$ space. Thus, we immediately have a preprocessing work of $O(m^{3/2})$.

We create four data structures $\mathcal{HH}$, $\mathcal{HL}$, $\mathcal{LL}$, and $\mathcal{LH}$. $\mathcal{HH}$ stores all of the edges $(u, v)$ where both $u$ and $v$ are high-degree, $\mathcal{HL}$ stores edges $(u, v)$, where $u$ is high-degree and $v$ is low-degree, $\mathcal{LL}$ stores the edges $(u, v)$ where $u$ is low-degree and $v$ is high-degree, and $\mathcal{LH}$ stores edges where both $u$ and $v$ are low-degree. With our data structures, the following operations are supported:

1. Given a vertex $v$, determine whether it is low-degree or high-degree in $O(1)$ work.
2. Given an edge $(u, v)$, check if it is in $\mathcal{HH}$, $\mathcal{HL}$, $\mathcal{LH}$, or $\mathcal{LL}$ in $O(1)$ work.
3. Given a vertex $v$, return all neighbors of $v$ in $\mathcal{HH}$, $\mathcal{HL}$, $\mathcal{LH}$, and $\mathcal{LL}$ in $O(\deg(v))$ work.
4. Given an edge $(v, w)$ to insert or delete, update $\mathcal{HH}$, $\mathcal{HL}$, $\mathcal{LH}$, or $\mathcal{LL}$ in $O(1)$ work.

We can implement $\mathcal{HH}$, $\mathcal{HL}$, $\mathcal{LH}$, and $\mathcal{LL}$ to support these operations by using a two-level hash table for each of these structures and an additional array $D$. $D$ is a dynamic hash table containing a key for each vertex that has non-zero degree and stores the degree of the vertex as the value. The data structures support insertions and deletions in $O(1)$ work. $D$ can be initialized in $O(m)$ work by scanning over all vertices and computing their degree. $\mathcal{HH}$, $\mathcal{HL}$, $\mathcal{LH}$, and $\mathcal{LL}$ can be initialized in $O(m)$ work by scanning over all edges and inserting them into the right table based on the degrees of their endpoints.

We maintain one additional data structure $T$ that counts the number of wedges $(u, w, v)$, where $u$ and $v$ are high-degree vertices and $w$ is a low-degree vertex. $T$ has the property that given an edge insertion or deletion $(u, v)$ where both $u$ and $v$ are high-degree vertices, it returns the number of such wedges $(u, w, v)$ where $w$ is low-degree that $u$ and $v$ are part of in $O(1)$ work. We can implement this via a hash table indexed by pairs of high-degree vertices that stores the number of wedges for each pair. $T$ can be initialized in $O(m^{3/2})$ work by iterating over all edges $(u, w)$ in $\mathcal{HL}$ and then for each, iterating over all edges $(w, v)$ in $\mathcal{LH}$ to determine whether $v$ is high-degree, and so then increment $T(u, v)$ by 1. There are $O(m)$ edges $(u, w)$ in $\mathcal{HL}$, and for each $w$ there are at most $O(\sqrt{m})$ edges $(w, v)$ in $\mathcal{LH}$ since $w$ is low-degree. Each lookup and increment takes $O(1)$ work, giving an overall work of $O(m^{3/2})$.

A.1 Update Procedure [KNN⁺19]

The procedure for handling single edge updates in the sequential setting given by [KNN⁺19] as follows:

For an edge insertion (resp. deletion) $(u, v)$, we first find the degree of $u$ and $v$ in $D$ and then look up the edge in their respective tables $\mathcal{HH}$, $\mathcal{HL}$, $\mathcal{LH}$, or $\mathcal{LL}$. If the edge already exists (resp. does not exist) in the table, nothing else is done. Otherwise, we need to find all tuples $(u, w, v)$ such that $(v, u)$ and $(u, w)$ already exist in the graph because for each such tuple, a new triangle will be formed (resp. an existing triangle will be deleted). We first update the triangle count, and then we update the data structures. For updating the triangle count $C$, there are 4 different cases for such tuples, and so we check each of the following cases:
1. \((u, w)\) is in \(\mathcal{H_H}\) and \((w, v)\) is in \(\mathcal{H_y}\) where \(y \in \{\mathcal{H}, \mathcal{L}\}\): We extract all high-degree neighbors of \(u\) in \(\mathcal{H_H}\). Given that the degree of all high-degree vertices is \(\Omega(\sqrt{m})\), there are at most \(O(\sqrt{m})\) such vertices. For each of these neighbors, we can check in \(O(1)\) work for each \(w\) whether \((w, v)\) exists in \(\mathcal{H_y}\). This takes \(O(\sqrt{m})\) work.

2. \((u, w)\) is in \(\mathcal{H_L}\) and \((w, v)\) is in \(\mathcal{H_y}\) where \(y \in \{\mathcal{H}, \mathcal{L}\}\): Since both \(u\) and \(v\) are high-degree in this case, we perform an \(O(1)\) work lookup in \(\mathcal{T}\) for the count of the number of wedges \((u, w, v)\) in this case.

3. \((u, w)\) is in \(\mathcal{L_H}\) and \((w, v)\) is in \(\mathcal{H_y}\) where \(y \in \{\mathcal{H}, \mathcal{L}\}\): Scan through the neighbors of \(u\) in \(\mathcal{L_H}\). For each neighbors of \(u\), check whether \((w, v)\) exists in \(\mathcal{H_y}\). This takes \(O(\sqrt{m})\) work since \(u\) has low-degree.

4. \((u, w)\) is in \(\mathcal{L_L}\) and \((w, v)\) is in \(\mathcal{L_y}\) where \(y \in \{\mathcal{L}, \mathcal{H}\}\): Again, scan through the neighbors of \(u\) in \(\mathcal{L_H}\). For each neighbors of \(u\), check whether \((w, v)\) exists in \(\mathcal{L_y}\). This takes \(O(\sqrt{m})\) work since \(u\) has low-degree.

After updating the triangle count, we proceed with updating the data structures with the edge insertion (resp. deletion).

We first update \(\mathcal{T}\) given an edge insertion (resp. deletion) \((u, v)\) as follows:

1. If \(u\) is high-degree and \(v\) is low-degree, then we find all of \(v\)'s neighbors in \(\mathcal{L_H}\) and for each such neighbor \(x\), we increment (resp. decrement) the entry \(\mathcal{T}(u, x)\) by 1. It takes \(O(\sqrt{m})\) work to perform this update since \(v\) is low-degree.

2. If \(u\) is low-degree and \(v\) is high-degree, then we scan through all vertices in \(\mathcal{H_L}\) and for each vertex \(x\) in \(\mathcal{H_L}\) that has \(u\) as a neighbor, we increment (resp. decrement) \(\mathcal{T}(x, v)\) by 1. This takes \(O(\sqrt{m})\) work since there are at most \(O(\sqrt{m})\) high-degree vertices.

In addition to the updates to \(\mathcal{T}\), we also insert (resp. delete) \((u, v)\) into \(\mathcal{H_H}, \mathcal{H_L}, \mathcal{L_H},\) and \(\mathcal{L_L}\) depending on the degrees of \(u\) and \(v\), and update \(\mathcal{D}\). For a given edge \((u, v)\) insertion (resp. deletion), we first determine whether \(u\) and \(v\) are low-degree or high-degree by looking in \(\mathcal{D}\) for \(u\) and \(v\) in \(O(1)\) work. \(\mathcal{H_H}, \mathcal{H_L}, \mathcal{L_H},\) and \(\mathcal{L_L}\) are constructed as hash tables keyed by first the first vertex in the edge tuple and then the second vertex in the edge tuple with pointers to second-level hash tables storing the neighbors of that particular vertex. If \(u\) is high-degree, then the edge is inserted (resp. deleted) into \(\mathcal{H_H}\) or \(\mathcal{H_L}\) (depending on whether \(v\) is low or high-degree) using \(u\) as the key and adding \(v\) to the second level hash table. Similarly, if \(u\) is low-degree, \((u, v)\) is inserted (resp. deleted) into \(\mathcal{L_H}\) or \(\mathcal{L_L}\). Furthermore, \((v, u)\) is also inserted into its respective table depending on whether \(v\) is low or high-degree. The entries for \(u\) and \(v\) in \(\mathcal{D}\) are then incremented (resp. decremented) in \(\mathcal{D}\). The updates to these data structures take \(O(1)\) work.

We also have to deal with the cases where the degree classification of vertices have changed or the number of edges has changed by too much that the values of \(M, t_1,\) and \(t_2\) need to be updated. This is described in the next section.

A.2 Rebalancing [KNN+19]

We now describe the rebalancing procedure given in [KNN+19] when a low-degree vertex becomes a high-degree vertex (or vice versa) and when too many updates have been applied (and all the data structures must be changed according to the new values of \(M, t_1,\) and \(t_2\)).
**Minor rebalancing**  This type of rebalancing occurs if a vertex which was previously high-degree has its degree fall below $t_1$ or if a vertex that was previously low-degree has its degree increase above $t_2$. In the first case, we move the vertex and all its edges from $\mathcal{HH}$ to $\mathcal{HL}$, and from $\mathcal{LH}$ to $\mathcal{LL}$. In the second case, we move the vertex and all its edges from $\mathcal{HL}$ to $\mathcal{HH}$, and from $\mathcal{LL}$ to $\mathcal{LH}$. Since our data structures support additions and deletions of an edge in $O(1)$ work, and since the degree of $v$ is $\Theta(\sqrt{m})$ at this point, we perform $\Theta(\sqrt{m})$ updates. We showed in Section A.1 that updates take $O(\sqrt{m})$ work so we take $O(m)$ work overall for a minor rebalancing. However, $\Omega(\sqrt{m})$ updates must have occurred on this vertex before we have to perform minor rebalancing since $t_2 - t_1 = \Theta(\sqrt{m})$, and so we can amortize this cost over the $\Omega(\sqrt{m})$ updates, resulting in $O(\sqrt{m})$ amortized work per update.

**Major rebalancing**  A major rebalancing occurs when $m$, the number of edges in the graph, falls outside the range $[M/4, M]$. We simply reinitialize the data structures as in the original algorithm. Major rebalancing can only occur after $\Omega(M)$ updates, and so we can afford to re-initialize our data structure and recompute the triangle count from scratch using an $O(m^{3/2})$ work triangle counting algorithm. The amortized work of major rebalancing over $\Omega(m)$ updates is then $O(\sqrt{m})$. 

