EFFECTIVE MACROSCOPIC DYNAMICS OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS IN PERFORATED DOMAINS

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ABSTRACT. An effective macroscopic model for a stochastic microscopic system is derived. The original microscopic system is modeled by a stochastic partial differential equation defined on a domain perforated with small holes or heterogeneities. The homogenized effective model is still a stochastic partial differential equation but defined on a unified domain without holes. The solutions of the microscopic model is shown to converge to those of the effective macroscopic model in probability distribution, as the size of holes diminishes to zero. Moreover, the long time effectiveness of the macroscopic system in the sense of convergence in probability distribution, and the effectiveness of the macroscopic system in the sense of convergence in energy are also proved.

1. INTRODUCTION

In recent years there has been explosive growth of activities in multiscale modeling of complex phenomena in many areas including material science, climate dynamics, chemistry and biology [15, 31]. Stochastic partial differential equations (SPDEs or stochastic PDEs) — evolutionary equations containing noises — arise naturally as mathematical models of multiscale systems under random influences. In fact the need to include stochastic effects in mathematical modelling of realistic physical behavior has become widely recognized in, for example, condensed matter physics, climate and geophysical sciences, and materials sciences. But implementing this idea poses some challenges both in theory and for computation [17, 33].

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This paper is devoted to the effective macroscopic dynamics of microscopic systems modeled by parabolic SPDEs in perforated media which exhibit small-scale heterogeneities. One example of such microscopic systems of interest is composite materials with microscopic heterogeneities under the impact of external random fluctuations.

The heterogeneity scale is taken to be much smaller than the macroscopic scale, which is equivalent, here, to assuming that the heterogeneities are evenly distributed. From a mathematical point of view, one can assume that microscopic heterogeneities (holes) are periodically placed in the media. This periodicity can be represented by a small positive parameter $\epsilon$ (i.e., the period). In fact we work on the space-time cylinder $D_\epsilon \times (0, T)$, with $T > 0$, and $D_\epsilon$ being the spatial domain obtained by removing a number $N_\epsilon$ of holes, of size $\epsilon$, periodically distributed, from a fixed domain $D$. When taking $\epsilon \to 0$, the holes covering $D$ are smaller and smaller and their numbers goes to $\infty$. This signifies that the heterogeneities are finer and finer.

There are lots of work on the homogenization problem for the deterministic systems defined in such perforated domain or other heterogeneous media, see for example [6, 24, 25, 28, 30] for heat transfer in a composite material, [6, 8, 11] for the wave propagation in a composite material and [21, 23] for the fluid flow in a porous media. For an introduction see [9, 18, 27].

Recently there are also works on homogenization in the random context; see [19, 22, 26] for general random coefficients, and [5, 35, 36] for randomly perforated domains. And also see a survey book about the homogenization results in a random context [18]. A basic assumption in these texts is the ergodic hypotheses on the coefficients for the passing of the limit of $\epsilon \to 0$. Note that the microscopic models in these works are partial differential equations with random coefficients, so-called random partial differential equations (random PDEs), instead of stochastic PDEs.

In the present paper, the microscopic model is a SPDE defined in a perforated domain. Homogenization techniques are employed to derive an effective, simplified, macroscopic model. Homogenization is a formal mathematic procedure for deriving macroscopic models from microscopic systems. It has been applied to a variety of problems including composite materials modelling, porous media and climate modelling; see [9, 10, 18, 27]. Homogenization provides effective macroscopic behavior of the system with microscopical heterogeneities for which direct numerical simulations are usually too expensive.

We consider a spatially extended system where stochastic effects are taken into account in the model equation, defined on a deterministic domain but perforated with small scale holes. Specifically, we study a class of stochastic
partial differential equations driven by white noise on a perforated domain in the following form

\[ du_\epsilon(t) = (A_\epsilon u_\epsilon + F_\epsilon(x, t))dt + G_\epsilon(x, t)dW(t), \quad 0 < t < T, \epsilon > 0, \]

which will be described in detail in the next section. For the general theory of SPDEs we refer to [12]. The goal here is to derive the homogenized equation (effective equation), which is still a stochastic partial differential equation, for [11] by the homogenization techniques in the sense of probability.

Homogenization theory has been developed for deterministic systems, and compactness discussion for the solutions \( \{u_\epsilon\} \) in some function space is a key step in the homogenized approach [9]. However, due to the appearance of the stochastic term in the above microscopic system considered in this paper, such compact result does not hold for this stochastic system. Fortunately the compactness in the sense of probability, that is, the tightness of the distributions for \( \{u_\epsilon\} \), still holds. So one appropriate way is to homogenize the stochastic system in the sense of probability. The goal in this paper is to derive an effective macroscopic equation for the above microscopic system, by homogenization in the sense of probability. It is shown that the solution \( u_\epsilon \) of the microscopic or heterogeneous system converges to that of the macroscopic or homogenized system as \( \epsilon \downarrow 0 \) in probability. That also implies the distributions of \( \{u_\epsilon\} \) weakly converge, in some appropriate space, to the distribution of a stochastic process which solves the macroscopic effective equation. Moreover, the long time effectivity of the homogenized macroscopic system is demonstrated, that is, the solution \( u_\epsilon(t) \) is shown to converge to the stationary solution of the homogenized equation as \( t \to \infty \) and \( \epsilon \downarrow 0 \) in the sense of probability distribution. Furthermore, the effectivity of the macroscopic system in the sense of convergence in energy is also shown.

In our approach, one difficulty is that the spatial domain is changing as \( \epsilon \to 0 \). To overcome this we use the extension operator introduced in [8] and introduce a new probability space depending on a parameter in which the solution is uniformly bounded. One novelty here is that the original microscopic model is a stochastic PDE, instead of a random PDE as studied by others, e.g., [19, 26, 7].

This paper is organized as follows. The problem formulation is stated in §2. Section 3 is devoted to basic properties of the microscopic system. The effective macroscopic equation is derived in §4. The long time effectivity of the homogenized macroscopic system is considered in §6. Finally, the effectivity of the macroscopic system in the sense of convergence in energy is shown in §5.
Moreover, in the Appendix we present the explicit expression of the homogenization matrix.

2. Problem formulation

Let $D$ be an open bounded set in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial D$ and $\epsilon > 0$ is a small parameter. Let $Y = [0, l_1) \times [0, l_2) \times \cdots \times [0, l_n)$ be a representative (cubic) cell in $\mathbb{R}^n$ and $S$ an open subset of $Y$ with smooth boundary $\partial S$, such that $\overline{S} \subset Y$. Write $l = (l_1, l_2, \ldots, l_n)$. Define $\epsilon S = \{\epsilon y : y \in S\}$. Denote by $S_{\epsilon, k}$ the translated image of $\epsilon S$ by $kl$, $k \in \mathbb{Z}^n$, $kl = (k_1l_1, k_2l_2, \ldots, k_nl_n)$. And let $S_\epsilon$ be the set of all the holes contained in $D$ and $D_\epsilon = D \setminus S_\epsilon$. Then $D_\epsilon$ is a periodically perforated domain with holes of the same size as period $\epsilon$. We assume that the holes do not intersect with the boundary $\partial D$, which implies that $\partial D_\epsilon = \partial D \cup \partial S_\epsilon$. See Fig. 1 for the case $n = 2$. This assumption is for avoiding technicalities and the results of our paper will remain valid without this assumption; see [1].

In the sequel we use the notations

$$Y^* = Y \setminus \overline{S}, \quad \vartheta = \frac{|Y^*|}{|Y|}$$

with $|Y|$ and $|Y^*|$ the Lebesgue measure of $Y$ and $Y^*$ respectively. And denote by $\tilde{v}$ the zero extension to the whole $D$ for any function defined on $D_\epsilon$:

$$\tilde{v} = \begin{cases} v & \text{on } D_\epsilon, \\ 0 & \text{on } S_\epsilon. \end{cases}$$

Fig. 1: Geometric setup in $\mathbb{R}^2$
Now for $T > 0$ fixed final time, we consider the following Itô type nonautonomous stochastic partial differential equation defined on the perforated domain $D_\epsilon$ in $\mathbb{R}^n$:

$$
du_\epsilon(x,t) = \left(\text{div}(A_\epsilon(x)\nabla u_\epsilon(x,t)) + f_\epsilon(x,t)\right)dt + g_\epsilon(t)dW(t) \quad (2.1)
$$

in $D_\epsilon \times (0,T)$,

$$
u = 0 \quad \text{on} \quad \partial D \times (0,T),
$$

$$
\frac{\partial u_\epsilon}{\partial n_\epsilon} = 0 \quad \text{on} \quad \partial S_\epsilon \times (0,T),
$$

$$
\left.u_\epsilon(0) = u_0\right|_{D_\epsilon}, \quad (2.4)
$$

where the matrix $A_\epsilon$ is

$$
A_\epsilon = \left(a_{ij}\left(\frac{x}{\epsilon}\right)\right)_{ij}
$$

and

$$
\frac{\partial \cdot}{\partial n_\epsilon} = \sum_{ij} a_{ij}\left(\frac{x}{\epsilon}\right) \frac{\partial \cdot}{\partial x_j} n_i
$$

with $n$ the exterior unit normal vector on the boundary $\partial D_\epsilon$.

We make the following assumptions on the coefficients:

1. $a_{ij} \in L^\infty(\mathbb{R}^n)$, $i, j = 1, \ldots, n$;
2. $\sum_{i,j} a_{ij}\xi_i\xi_j \geq \alpha \sum_{i=1}^n \xi_i^2$ for $\xi \in \mathbb{R}^n$ and $\alpha$ a positive constant;
3. $a_{ij}$ are $Y$-periodic.

Furthermore we assume that

$$
f_\epsilon \in L^2(D_\epsilon \times [0,T]) \quad (2.5)
$$

and for $0 \leq t \leq T$, $g_\epsilon(t)$ is a linear operator from $\ell^2$ to $L^2(D_\epsilon)$ defined as

$$
g_\epsilon(t)k = \sum_{i=1}^\infty g_i^\epsilon(x,t)k_i, \quad k = (k_1, k_2, \cdots) \in \ell^2
$$

where $g_i^\epsilon(x,t) \in L^2(D_\epsilon \times [0,T])$, $i = 1, 2, \cdots$, are measurable functions with

$$
\sum_{i=1}^\infty |g_i^\epsilon(x,t)|^2_{L^2(D_\epsilon)} < C_T, \quad t \in [0,T] \quad (2.6)
$$

for some positive constant $C_T$ independent of $\epsilon$. In (2.1), $W(t) = (W_1(t), W_2(t), \cdots)$ is a Wiener process in $\ell^2$ with covariance operator $Q = Id_{\ell^2}$ and $\{W_i(t) : i = 1, 2, \cdots\}$ are mutually independent real valued standard Wiener processes on a
complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a canonical filtration \((\mathcal{F}_t)_{t \geq 0}\). Then
\[
|g_\varepsilon(t)|^2_{L^2} = \sum_{i=1}^{\infty} |g_i^\varepsilon(x,t)|^2_{L^2(D_\lambda)} < C_T, \quad t \in [0, T]. \tag{2.7}
\]
Here \(L^2_{\mathcal{Q}}\) is the space of Hilbert-Schmit operators \([12, 16]\). Denote by \(\mathbb{E}\) the expectation operator with respect to \(\mathbb{P}\).

The following compactness result \([20]\) will be used in our approach. Let \(\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z}\) be three reflective Banach spaces and \(\mathcal{X} \subset \mathcal{Y}\) with compact and dense embedding. Define Banach space
\[
G = \{ v : v \in L^2(0,T; \mathcal{X}), \frac{dv}{dt} \in L^2(0,T; \mathcal{Z}) \}
\]
with norm
\[
|v|_G^2 = \int_0^T |v(s)|^2_{\mathcal{X}} ds + \int_0^T \left| \frac{dv}{ds} (s) \right|^2_{\mathcal{Z}} ds, \quad v \in G.
\]

**Lemma 2.1.** If \(B\) is bounded in \(G\), then it is precompact in \(L^2(0,T; \mathcal{Y})\).

Let \(\mathcal{S}\) be a Banach space and \(\mathcal{S}'\) be the strong dual space of \(\mathcal{S}\). We recall the definitions and some properties of weak convergence and weak\(^*\) convergence \([34]\).

**Definition 2.2.** A sequence \(\{s_n\}\) in \(\mathcal{S}\) is said to converge weakly to \(s \in \mathcal{S}\) if \(\forall s' \in \mathcal{S}'\),
\[
\lim_{n \to \infty} (s', s_n)_{\mathcal{S}' \mathcal{S}} = (s', s)_{\mathcal{S}' \mathcal{S}}
\]
which is written as \(s_n \rightharpoonup s\) weakly in \(\mathcal{S}\). Note that \((s', s)\) denotes the value of the continuous linear functional \(s'\) at the point \(s\).

**Lemma 2.3.** (Eberlein-Shmulyan) Assume that \(\mathcal{S}\) is reflexive and let \(\{s_n\}\) be a bounded sequence in \(\mathcal{S}\). Then there exists a subsequence \(\{s_{n_k}\}\) and \(s \in \mathcal{S}\) such that \(s_{n_k} \rightharpoonup s\) weakly in \(\mathcal{S}\) as \(k \to \infty\). If all the weak convergent subsequence of \(\{s_n\}\) has the same limit \(s\), then the whole sequence \(\{s_n\}\) weakly converges to \(s\).

**Definition 2.4.** A sequence \(\{s'_n\}\) in \(\mathcal{S}'\) is said to converge weakly\(^*\) to \(s' \in \mathcal{S}'\) if \(\forall s \in \mathcal{S}\),
\[
\lim_{n \to \infty} (s'_n, s)_{\mathcal{S}' \mathcal{S}} = (s', s)_{\mathcal{S}' \mathcal{S}}
\]
which is written as \(s'_n \rightharpoonup s'\) weakly\(^*\) in \(\mathcal{S}'\).

**Lemma 2.5.** Assume that the dual space \(\mathcal{S}'\) is reflexive and let \(\{s'_n\}\) be a bounded sequence in \(\mathcal{S}'\). Then there exists a subsequence \(\{s'_{n_k}\}\) and \(s' \in \mathcal{S}'\)
such that $s'_n \rightharpoonup s'$ weakly* in $S'$ as $k \to \infty$. If all the weakly* convergent subsequence of $\{s'_n\}$ has the same limit $s'$, then the whole sequence $\{s'_n\}$ weakly* converges to $s'$.

We also use the following definition of the weak convergence of the Borel probability measures on $S$, for more we refer to [14].

**Definition 2.6.** Let $\{\mu_\epsilon\}_\epsilon$ be a family of Borel probability measures on the Banach space $S$. We say $\mu_\epsilon$ weakly converges to a Borel measure $\mu$ on $S$ if

$$\int_S h d\mu_\epsilon \to \int_S h d\mu, \quad \text{as} \quad \epsilon \downarrow 0,$$

for any $h \in C_b(S)$, the space of bounded continuous functions on $S$.

In the following, for a fixed $T > 0$, we always denote by $C_T$ a constant independent of $\epsilon$.

### 3. Basic properties of the microscopic model

In this section we will present some estimates of the solutions of microscopic model (2.1), useful for the tightness result of the distributions of solution processes in some appropriate space.

Let $H = L^2(D)$ and $H_\epsilon = L^2(D_\epsilon)$. Define the following space

$$V_\epsilon = \{ u \in H^1(D_\epsilon), u|_{\partial D} = 0 \}$$

provided with the norm

$$|v|_{V_\epsilon} = |\nabla A_\epsilon v|_{\oplus_n H_\epsilon} = \left| \left( \sum_{j=1}^n a_{ij} \left( \frac{x}{\epsilon} \right) \frac{\partial v}{\partial x_j} \right)_{i=1}^n \right|_{\oplus_n H_\epsilon}.$$

This norm is equivalent to the usual $H^1(D_\epsilon)$-norm, with an embedding constant independent of $\epsilon$, due to the assumptions on $a_{ij}$ in the last section. Here $\oplus_n$ denotes the direct sum of the Hilbert spaces with usual direct sum norm. Let

$$\mathcal{D}(A_\epsilon) = \{ v \in V_\epsilon : \text{div}(A_\epsilon \nabla v) \in H_\epsilon \text{ and } \frac{\partial v}{\partial v_{A_\epsilon}}|_{\partial S_\epsilon} = 0 \}$$

and define operator $A_\epsilon v = \text{div}(A_\epsilon \nabla v)$ for $v \in \mathcal{D}(A_\epsilon)$. Then system (2.1)-(2.4) can be written as the following abstract stochastic evolutionary equation

$$du_\epsilon = (A_\epsilon u_\epsilon + f_\epsilon) dt + g_\epsilon dW, \quad u_\epsilon(0) = u_0^\epsilon. \quad (3.1)$$

By the assumptions on $a_{ij}$, operator $A_\epsilon$ generates a strongly continuous semigroup $S_\epsilon(t)$ on $H_\epsilon$. Solution of (3.1) can then be written in the mild sense

$$u_\epsilon(t) = S_\epsilon(t) u_0^\epsilon + \int_0^t S_\epsilon(t-s) f_\epsilon(s) ds + \int_0^t S_\epsilon(t-s) g_\epsilon(s) dW(s) \quad (3.2)$$
And the variational formulation is
\[(du_ε(t), v)_{H_ε^{-1}, V_ε} = \left( - \int_{D_ε} A_ε(x) \nabla u_ε(x, t) \nabla v(x) dx + \int_{D_ε} f_ε(x, t)v(x)dx \right) dt + \int_{D_ε} g_ε(x, t)v(x)dW(t), \text{ in } \mathcal{D}'(0, T), \ v \in V_ε, \quad (3.3)\]

with \( u_ε(0, x) = u_ε^0(x) \).

For the well-posedness of system (3.1) we have the following result.

**Theorem 3.1.** *(Global well-posedness of microscopic model)* Assume (2.7) and (2.7) hold. Let \( u_ε^0 \) be a \( (\mathcal{F}_0, \mathcal{B}(H_ε)) \)-measurable random variable. Then system (3.1) has a unique mild solution \( u \in L^2(Ω, C(0, T; H_ε)∩L^2(0, T; V_ε)) \), which is also a weak solution in the following sense

\[(u_ε(t), v)_{H_ε} = (u_ε^0, v)_{H_ε} + \int_0^t (A_ε u_ε(s), v)_{H_ε} ds + \int_0^t (f_ε, v)_{H_ε} ds + \int_0^t (g_ε, dw, v)_{H_ε}, \quad (3.4)\]

for \( t \in [0, T) \) and \( v \in V_ε \). Moreover, if \( u_ε^0 \) is independent of \( W(t) \) with \( E|u_ε^0|^2_{H_ε} < ∞ \), then

\[E|u_ε(t)|^2_{H_ε} + E \int_0^t |u_ε(s)|^2_{V_ε} ds \leq E|u_ε^0|^2_{H_ε} + C_T, \text{ for } t \in [0, T], \quad (3.5)\]

\[E \int_0^t |\dot{u}_ε(s)|^2_{H_ε^{-1}} ds \leq C_T(E|u_ε^0|^2_{H_ε} + 1), \text{ for } t \in [0, T]. \quad (3.6)\]

If further assume that

\[|∇_{A_ε} g_ε(t)|^2_{L^2} = \sum_{i=1}^{∞} |∇_{A_ε} g^i_ε(t)|^2_{H_ε} \leq C_T, \text{ for } t \in [0, T] \quad (3.7)\]

with \( u_ε^0 \in V_ε \) and \( E|u_ε^0|^2_{V_ε} < ∞ \), then

\[E|u_ε(t)|^2_{V_ε} + E \int_0^t |A_ε u_ε(s)|^2_{H_ε} ds \leq E|u_ε^0|^2_{V_ε} + C_T, \text{ for } t \in [0, T]. \quad (3.8)\]

Moreover, system (3.1) is well-posed on \([0, ∞)\) if

\[f_ε \in L^2(0, ∞; H_ε), \ g_ε \in L^2(0, ∞; L^2_ε). \quad (3.9)\]

**Proof.** By the assumption (2.7), we have

\[||g_ε(t)||_{L^2_ε}^2 = \sum_{i=1}^{∞} |g^i_ε(t, x)|^2_{H_ε} < ∞. \]
Then the classical result of [12] yields the local existence of \( u_\varepsilon \). And applying the stochastic Fubini theorem, it is easy to verify the local mild solution is also a weak solution.

Now we give the following a priori estimates which yields the existence of weak solution on \([0, T]\) provide (2.5) and (2.7) hold.

Applying Itô formula to \(|u_\varepsilon|^2\), we derive

\[
d|u_\varepsilon(t)|_{H_\varepsilon}^2 - 2\langle A_\varepsilon u_\varepsilon, u_\varepsilon \rangle_{H_\varepsilon} dt = 2\langle f_\varepsilon, u_\varepsilon \rangle_{H_\varepsilon} dt + 2\langle g_\varepsilon dW, u_\varepsilon \rangle_{H_\varepsilon} + |g_\varepsilon|^2_{L^2_\varepsilon} dt. \tag{3.10}
\]

By the assumption on \( a_{ij} \), we see that

\[-\langle A_\varepsilon u_\varepsilon, u_\varepsilon \rangle_{H_\varepsilon} \geq \lambda |u_\varepsilon|_{H_\varepsilon}^2 \]

for some constant \( \lambda > 0 \) independent of \( \varepsilon \). Then integrating (3.10) with respect to \( t \) yields

\[
|u_\varepsilon(t)|_{H_\varepsilon}^2 + \int_0^t |u_\varepsilon|^2_{V_\varepsilon} ds \
\leq |u_\varepsilon(0)|_{H_\varepsilon}^2 + \lambda^{-1} |f_\varepsilon|^2_{L^2(0,T;H_\varepsilon)} + \int_0^t |g_\varepsilon|^2_{L^2_\varepsilon} ds + \int_0^t |g_\varepsilon|^2_{L^2_\varepsilon} ds.
\]

Taking expectation on both sides of the above inequality, we derive (3.5).

In a similar way, application of Itô formula to \(|u_\varepsilon|^2_{V_\varepsilon} = |\nabla A_\varepsilon u_\varepsilon|^2_{H_\varepsilon} \) results in the relation

\[
d|u_\varepsilon(t)|_{V_\varepsilon}^2 + 2\langle A_\varepsilon u_\varepsilon, A_\varepsilon u_\varepsilon \rangle_{H_\varepsilon} dt \
= -2\langle f_\varepsilon, A_\varepsilon u_\varepsilon \rangle_{H_\varepsilon} dt - 2\langle g_\varepsilon dW, A_\varepsilon u_\varepsilon \rangle_{H_\varepsilon} + |\nabla A_\varepsilon g_\varepsilon|^2_{L^2_\varepsilon} dt. \tag{3.11}
\]

Integrating both sides of (3.11) and by the Cauchy-Schwarz inequality, it is easily to have

\[
|u_\varepsilon(t)|_{V_\varepsilon}^2 + \int_0^t |A_\varepsilon u_\varepsilon|^2_{H_\varepsilon} ds \
\leq |u_\varepsilon(0)|_{V_\varepsilon}^2 + |f_\varepsilon|^2_{L^2(0,T;H_\varepsilon)} - 2 \int_0^t |\nabla A_\varepsilon u_\varepsilon|^2_{V_\varepsilon} ds + \int_0^t |\nabla A_\varepsilon g_\varepsilon|^2_{L^2_\varepsilon} ds,
\]

Then taking the expectation, we derive (3.8). By (3.3) and the property of the stochastic integral we easily have (3.6).

Thus, by the above estimates, the solution can be extended to \([0, \infty)\) if (3.9) hold. The proof is complete. \( \square \)

We recall a probability concept. Let \( z \) be a random variable taking values in a Banach space \( S \), namely, \( z : \Omega \rightarrow z \). Denote by \( \mathcal{L}(z) \) the distribution (or law) of \( z \). In fact, \( \mathcal{L}(z) \) is a Borel probability measure on \( S \) defined as [12]

\[
\mathcal{L}(z)(A) = \mathbb{P}\{\omega : z(\omega) \in A\},
\]
for every event (i.e., a Borel set) $A$ in the Borel $\sigma$-algebra $\mathcal{B}(S)$, which is the smallest $\sigma$-algebra containing all open balls in $S$.

As stated in §1, for the SPDE (2.1) we aim at deriving an effective equation in the sense of probability. A solution $u_\epsilon$ may be regarded as a random variable taking values in $L^2(0,T;H_\epsilon)$. So for a solution $u_\epsilon$ of (2.1)-(2.4) defined on $[0,T]$, we focus on the behavior of distribution of $u_\epsilon$ in $L^2(0,T;H_\epsilon)$ as $\epsilon \to 0$. For this purpose, the tightness [14] of distributions is needed. Note that the function space changes with $\epsilon$, which is a difficulty for obtaining the tightness of distributions. Thus we will treat $\{L^2(u_\epsilon(D))\}_{\epsilon > 0}$ as a family of distributions on $L^2(0,T;H)$ by extending $u_\epsilon$ to the whole domain $D$. Recall that the distribution (or law ) of $u_\epsilon$ is defined as:

$$L^2(u_\epsilon)(A) = \mathbb{P}\{\omega : u_\epsilon(\cdot,\cdot,\omega) \in A\}$$

for Borel set $A$ in $L^2(0,T;H_\epsilon)$. First we define an extension operator $P_\epsilon$ in the following lemmas.

In the following we denote by $\mathbb{L}(\mathcal{X},\mathcal{Y})$ the space of bounded linear operator from Banach space $\mathcal{X}$ to Banach space $\mathcal{Y}$.

**Lemma 3.2.** There exists a bounded linear operator

$$\hat{Q} \in \mathbb{L}(H^k(Y^*)^*, H^k(Y)), \quad k = 0, 1,$$

such that

$$|\nabla \hat{Q}v|_{\oplus_n L^2(Y)} \leq C|\nabla v|_{\oplus_n L^2(Y^*)}, \quad v \in H^1(Y^*)$$

for some constant $C > 0$.

For the proof of Lemma 3.2 see [8].

We define an extension operator $P_\epsilon$ in terms of the above bounded linear operator $\hat{Q}$ in the following lemma.

**Lemma 3.3.** There exists an extension operator

$$P_\epsilon \in \mathbb{L}(L^2(0,T;H^k(D_\epsilon)), L^2(0,T;H^k(D))), \quad k = 0, 1,$$

such that for any $v \in H^k(D_\epsilon)$

1. $P_\epsilon v = v$ on $D_\epsilon \times (0,T)$
2. $|P_\epsilon v|_{L^2(0,T;H)} \leq C_T|v|_{L^2(0,T;H_\epsilon)}$
3. $|\nabla A_\epsilon(P_\epsilon v)|_{L^2(0,T;\oplus_n L^2(D_\epsilon))} \leq C_T|\nabla A_\epsilon v|_{L^2(0,T;\oplus_n L^2(D))}$

where $C_T$ is a constant independent of $\epsilon$.

**Proof.** For $\varphi \in H^k(D_\epsilon)$, then

$$\varphi_\epsilon(y) = \frac{1}{\epsilon} \varphi(\epsilon y)$$
belongs to \( H^k(Y^*_l) \) with \( Y^*_l \) the translation of \( Y^* \) for some \( l \in \mathbb{R}^n \). Define

\[
\hat{Q}_\varepsilon \varphi(x) = \varepsilon \hat{Q} \varphi(x) = \frac{\varphi(x)}{\varepsilon}.
\]

Now for \( \varphi \in L^2(0, T; H^k(D_\varepsilon)) \), we define

\[
(P_\varepsilon \varphi)(x, t) = \left[ \hat{Q}_\varepsilon \varphi(t, \cdot) \right] \left( \frac{x}{\varepsilon} \right) = \varepsilon \left[ \hat{Q} \varphi(t, \cdot) \right] \left( \frac{x}{\varepsilon} \right).
\]

It is known \[8\] that the operator \( P_\varepsilon = 0 \), of \( P_\varepsilon \) estimates (3.5) and (3.8) are easily derived due to the property of the operator \( H_\varepsilon \) we define.

**Remark 3.4.** In Lemma 2.1 of \[8\], the operator \( P_\varepsilon \) defined in \( L(L^\infty(0, T; H^k(D_\varepsilon)), L^2(0, T; H^k(D_\varepsilon))) \), \( k = 0, 1 \), coincides with the operator defined in Lemma 3.3 above.

**Remark 3.5.** The estimates in Theorem 3.1 for \( u_\varepsilon \) also hold for \( P_\varepsilon u_\varepsilon \). In fact estimates (3.3) and (3.8) are easily derived due to the property of the operator of \( P_\varepsilon \). Since the operator \( P_\varepsilon \) is defined on \( L^2(0, T; H^k(D_\varepsilon)) \), \( k = 0, 1 \), for (3.4) we define

\[
P_\varepsilon \dot{u}_\varepsilon = A_\varepsilon P_\varepsilon u_\varepsilon + \dot{f}_\varepsilon + \dot{g}_\varepsilon \dot{W}, \text{ on } D \times (0, T).
\]

By the property of \( P_\varepsilon \) and the estimates of \( u_\varepsilon \), it is easy to see that

\[
P_\varepsilon \dot{u}_\varepsilon = (P_\varepsilon \dot{u}_\varepsilon), \text{ in } D_\varepsilon \times (0, T)
\]

and

\[
\mathbb{E}|P_\varepsilon \dot{u}_\varepsilon|_{L^2(0, T; H^{-1})} \leq \mathbb{E}|\dot{u}_\varepsilon|_{L^2(0, T; H^{-1})}.
\]

### 4. Effective macroscopic model

We now derive the effective macroscopic model for the original model (2.1). Let \( u_\varepsilon \in L^2(0, T; H_\varepsilon) \) be the solution of system (2.1)-(2.4). Then by the estimates in Theorem 3.1, Remark 3.5 and the Chebyshev inequality \[12, 14\], it is clear that for any \( \delta > 0 \) there is a bounded set \( K_\delta \subset G \) with spaces \( \mathcal{X}, \mathcal{Y} \) and \( \mathcal{Z} \) in Lemma 2.1 (and in the paragraph immediately before it) are replaced by \( H_0^1(D) \), \( H \) and \( H^{-1}(D) \) respectively, such that

\[
P\{ P_\varepsilon u_\varepsilon \in K_\delta \} \geq 1 - \delta.
\]

Thus \( K_\delta \) is compact in \( L^2(0, T; H) \) by Lemma 2.1. Then \( L(P_\varepsilon u_\varepsilon) \) is tight in \( L^2(0, T; H) \). The Prokhorov Theorem and the Skorohod embedding theorem \((12)\) assure that for any sequence \( \{ \varepsilon_j \} \) with \( \varepsilon_j \to 0 \) as \( j \to \infty \), there exists a subsequence \( \{ \varepsilon_{j(k)} \} \), random variables \( \{ \hat{u}_{\varepsilon_{j(k)}} \} \subset L^2(0, T; H_{\varepsilon_{j(k)}}) \) and \( u \in L^2(0, T; H) \) defined on a new probability space \( (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}) \), such that

\[
\mathcal{L}(P_{\varepsilon_{j(k)}} \hat{u}_{\varepsilon_{j(k)}}) = \mathcal{L}(P_{\varepsilon_{j(k)}} u_{\varepsilon_{j(k)}})
\]
and
\[ P_{\epsilon j(k)} \hat{u}_{\epsilon j(k)} \to u \text{ in } L^2(0, T; H) \text{ as } k \to \infty, \]
for almost all \( \omega \in \hat{\Omega} \). Moreover \( P_{\epsilon j(k)} \hat{u}_{\epsilon j(k)} \) solves system (2.1)-(2.4) with \( W \) replaced by Wiener process \( \hat{W}_k \) defined on probability space \( (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}) \) with same distribution as \( W \). The limit \( u \) is unique; see [4], p.333. In the following, we will determine the limiting equation (homogenized effective equation) that \( u \) satisfies and the limiting equation is independent of \( \epsilon \). After this is done we see that \( L(u_\epsilon) \) weakly converges to \( L(u) \) as \( \epsilon \downarrow 0 \).

We always assume the following
\[ \tilde{f}_\epsilon \rightharpoonup f, \text{ weakly in } L^2(0, T; H) \] (4.1)
and
\[ \tilde{g}^i_\epsilon \rightharpoonup g^i, \text{ weakly in } L^2(0, T; H). \] (4.2)
Define a new probability space \( (\Omega_\delta, \mathcal{F}_\delta, \mathbb{P}_\delta) \) as
\[ \Omega_\delta = \{ \omega \in \Omega : u_\epsilon(\omega) \in K_\delta \}, \]
\[ \mathcal{F}_\delta = \{ F \cap \Omega_\delta : F \in \mathcal{F} \} \]
and
\[ \mathbb{P}_\delta(F) = \frac{\mathbb{P}(F \cap \Omega_\delta)}{\mathbb{P}(\Omega_\delta)}, \text{ for } F \in \mathcal{F}_\delta. \]
Denote by \( \mathbb{E}_\delta \) the expectation operator with respect to \( \mathbb{P}_\delta \).

Now we restrict the system on the probability space \( (\Omega_\delta, \mathcal{F}_\delta, \mathbb{P}_\delta) \). In the following discussion we aim at obtaining \( L^2(\Omega_\delta) \) convergence for any \( \delta > 0 \) which means the convergence in probability [3, 14].

From the estimates (3.5), (3.6), Remark 3.5 and the compact injection \( G \subset L^2(0, T; H) \), there exists a subsequence of \( u_\epsilon \) in \( K_\delta \), still denoted by \( u_\epsilon \), such that for a fixed \( \omega \in \Omega_\delta \)
\[ P_\epsilon u_\epsilon \rightharpoonup u \text{ weakly* in } L^\infty(0, T; H) \] (4.3)
\[ P_\epsilon u_\epsilon \rightharpoonup u \text{ weakly in } L^2(0, T; H^1) \] (4.4)
\[ P_\epsilon u_\epsilon \rightharpoonup u \text{ strongly in } L^2(0, T; H) \] (4.5)
\[ P_\epsilon \dot{u}_\epsilon \rightharpoonup \dot{u} \text{ weakly in } L^2(0, T; H^{-1}). \] (4.6)

Define
\[ \xi_\epsilon = \left( \sum_{j=1}^n a_{ij} \left( \frac{x}{\epsilon} \right) \frac{\partial u_\epsilon}{\partial x_j} \right) = A_\epsilon \nabla u_\epsilon \]
which satisfies

$$- \text{div} \xi_{\epsilon} = f_{\epsilon} + g_{\epsilon} \dot{W} - \dot{\bar{u}}_{\epsilon} \text{ in } D_{\epsilon} \times (0, T) \quad (4.7)$$

$$\xi_{\epsilon} \cdot n = 0 \text{ on } \partial D_{\epsilon} \times (0, T). \quad (4.8)$$

By the hypothesis of \( a_{ij} \) and the fact that \((\tilde{u}_{\epsilon})_{\epsilon}\) being bounded in \( L^2(0, T; H_0^1) \), we have

$$\tilde{\xi}_{\epsilon} \rightharpoonup \xi \text{ weakly in } L^2(0, T; \oplus_n H) \quad (4.9)$$

We make use of Tartar’s method of oscillating test functions to determine the limiting equation \[9\].

Note that

$$\int_0^T \int_D \tilde{\xi}_{\epsilon} \cdot \nabla v \phi \, dx \, dt = \int_0^T \int_D \tilde{f}_{\epsilon} v \phi \, dx \, dt + \sum_{i=1}^{\infty} \int_0^T \int_D \tilde{g}_i v dx \phi dW_i(t) + \int_0^T \int_D P_{\epsilon} u_{\epsilon} \chi_{D_{\epsilon}} \phi dx \, dt \quad (4.10)$$

for all \( v \in H_0^1(D) \) and \( \phi \in \mathcal{D}(0, T) \). We pass to the limit in \( (4.10) \) as \( \epsilon \to 0 \).

Due to the fact

$$P_{\epsilon} u_{\epsilon} \to u \text{ strongly in } L^2(0, T; H) \quad (4.11)$$

$$\chi_{D_{\epsilon}} \to \vartheta \text{ weakly* in } L^\infty(D) \quad (4.12)$$

and the estimate

$$E \left| \sum_{i=1}^{\infty} \int_0^T \int_D \tilde{g}_i v dx \phi dW_i(t) \right|^2 \leq \sum_{i=1}^{\infty} \left| \tilde{g}_i \right|^2_{L^2(0, T; H)} \left| v \phi \right|^2_{L^2(0, T; H)};$$

by the assumption \( (4.2) \), we see that

$$\sum_{i=1}^{\infty} \int_0^T \int_D \tilde{g}_i v dx \phi dW_i(t) \to \sum_{i=1}^{\infty} \int_0^T \int_D g_i v dx \phi dW_i(t), \text{ in } L^2(\Omega).$$

Thus letting \( \epsilon \to 0 \) in \( (4.10) \) and since \( L^2(\Omega_\delta) \) is a subspace of \( L^2(\Omega) \) one finds that in \( L^2(\Omega_\delta) \)

$$\int_0^T \int_D \xi \cdot \nabla v \phi \, dx \, dt = \int_0^T \int_D f v \phi \, dx \, dt + \sum_{i=1}^{\infty} \int_0^T \int_D g_i v dx \phi dW_i(t) + \int_0^T \int_D \vartheta u \phi dx \, dt. \quad (4.13)$$

Hence

$$- \text{div} \xi(x, t) = f(x, t) + g(x, t) \dot{W} - \vartheta \dot{u} \text{ in } D \times (0, T). \quad (4.14)$$
In the following we identify the limit $\xi$. We follow the approach of deterministic case for the elliptic problem with homogeneous Neumann boundary condition [9].

For any $\lambda \in \mathbb{R}^n$, let $w_\lambda$ be the solution of

\[- \sum_{j=1}^n \frac{\partial}{\partial y_j} \left( \sum_{i=1}^n a_{ij}(y) \frac{\partial w_\lambda}{\partial y_i} \right) = 0 \quad \text{in } Y^* \tag{4.15}\]

\[w_\lambda - \lambda \cdot y \quad Y - \text{periodic} \tag{4.16}\]

\[\frac{\partial w_\lambda}{\partial \nu_A} = 0 \quad \text{on } \partial S \tag{4.17}\]

and define

\[w_\epsilon^\xi = \epsilon (\hat{Q} w_\lambda) \left( \frac{x}{\epsilon} \right) \]

where $\hat{Q}$ is in Lemma 3.2. Then we have [9],

\[w_\epsilon^\xi \rightharpoonup \lambda \cdot x \quad \text{weakly in } H^1(D), \tag{4.18}\]

\[\nabla w_\lambda \rightharpoonup \lambda \quad \text{weakly in } \bigoplus_n L^2(D). \tag{4.19}\]

Now we define

\[(\eta^\lambda_j(y))_j = \left( \sum_{i=1}^n a_{ji}(y) \frac{\partial w_\lambda(y)}{\partial y_i} \right), \quad y \in Y^* \]

and $(\eta^\lambda_j)(x) = (\eta^\lambda_j(x/\epsilon))_j = A^t \nabla w_\lambda^\epsilon$. Then

\[- \text{div } \tilde{\eta}^\lambda = 0 \quad \text{in } D \tag{4.20}\]

and due to (4.18) and (4.19)

\[\tilde{\eta}^\lambda \rightharpoonup \mathcal{M}_Y(\eta^\lambda) \quad \text{weakly in } L^2(D). \tag{4.21}\]

It is easy to see that $\mathcal{M}_Y(\eta^\lambda) = B^t \lambda$ with $B^t = (\beta_{ji})$ a constant matrix which is determined in the appendix.

Using test function $\varphi w_\lambda^\epsilon$ with $\varphi \in \mathcal{D}(0,T)$, $v \in \mathcal{D}(D)$ in (4.10) and multiplying both sides of (4.20) with $\varphi v_{\epsilon t}$, one has

\[\int_D \int_0^T \tilde{\xi} \cdot \nabla v \varphi w_\lambda^\epsilon dx dt + \int_0^T \int_D \xi \cdot \nabla w_\lambda^\epsilon v \varphi dx dt \]

\[\quad - \int_0^T \int_D \tilde{\eta}^\lambda \cdot \nabla v \varphi P_{\epsilon t} u_{\epsilon t} dx dt - \int_0^T \int_D \tilde{\eta}^\lambda \cdot \nabla (P_{\epsilon t} u_{\epsilon t}) v \varphi dx dt = \]

\[\int_0^T \int_D \tilde{f} w_\lambda^\epsilon dx dt + \sum_{i=1}^\infty \int_0^T \int_D g_i^i w_\lambda^\epsilon dW_i(t) + \int_0^T \int_D P_{\epsilon t} u_{\chi_D} \varphi w_\lambda^\epsilon dx dt. \]
Then by the definition of $\xi$, $\eta^\lambda$, and the assumptions (4.1), (4.2), using the convergence (4.9), (4.11), (4.12), (4.18), (4.19), and (4.21), we have in $L^2(\Omega)$

$$
\int_0^T \int_D \xi \cdot \nabla \varphi \lambda \cdot x dx dt - \int_0^T \int_D B^t \lambda \cdot \nabla \varphi u dx dt
$$

$$
= \int_0^T \int_D f \varphi \lambda \cdot x dx dt + \sum_{i=1}^{\infty} \int_0^T \int_D g^i \lambda \cdot x dx \varphi dW_i(t) + \int_0^T \int_D \vartheta u \varphi \lambda \cdot x dx dt.
$$

That is

$$
\int_0^T \int_D \xi \cdot (v \lambda \cdot x) \varphi dx dt - \int_0^T \int_D \xi \cdot \lambda \varphi dx dt - \int_0^T \int_D B^t \lambda \cdot \nabla \varphi u dx dt
$$

$$
= \int_0^T \int_D f \varphi \lambda \cdot x dx dt + \sum_{i=1}^{\infty} \int_0^T \int_D g^i \lambda \cdot x dx \varphi dW_i(t) + \int_0^T \int_D \vartheta u \varphi \lambda \cdot x dx dt.
$$

Then by using (4.13) with the test function replaced by $v \lambda \cdot x \varphi$ one has

$$
\int_0^T \int_D \xi \cdot \lambda \varphi dx dt = \int_0^T \int_D B^t \lambda \cdot \nabla \varphi v dx dt
$$

which yields

$$
\xi \cdot \lambda = B^t \lambda \cdot \nabla u = B \nabla u \cdot \lambda.
$$

Then

$$
\xi = B \nabla u
$$

since $\lambda$ is arbitrary. Then $u$ satisfies the following equation

$$
\partial_t u = (\text{div}(B \nabla u) + f) dt + g dW(t). \quad (4.22)
$$

Suppose

$$
\bar{u}_e^0 \rightharpoonup u^0, \text{ weakly in } H. \quad (4.23)
$$

We now determine the initial value by suitable test-functions. In fact, taking $v \in \mathcal{D}(D)$ and $\varphi \in \mathcal{D}([0, T])$ with $\varphi(T) = 0$ we have

$$
\int_0^T \int_D \xi \varphi dx dt = \int_0^T \int_D \bar{f}_e \varphi dx dt + \sum_{i=1}^{\infty} \int_0^T \int_D \bar{g}_i^e v dx \varphi dW_i(t) - \int_0^T \int_D \bar{u}_e \varphi dx dt + \int_D \bar{u}_e^0 \varphi(0) v dx.
$$
Now let $\epsilon \to 0$, noticing that
\[
\int_0^T \int_D \tilde{u}_\epsilon \psi dx dt = \int_0^T \int_D \chi_D, P_\epsilon \tilde{u}_\epsilon \psi dx dt \to \int_0^T \int_D \vartheta u \psi dx dt = \\
- \int_0^T \int_D \vartheta u \psi dx dt + \int_D \vartheta u(0) \varphi(0) vdx
\]
by (4.14), we have
\[
u(0) = \frac{u^0}{\vartheta}.
\]

Here one should notice that the above result is in the sense of $L^2(\Omega_\delta)$. Then the above analysis yields the following results
\[
\lim_{\epsilon \to 0} E_\delta |P_\epsilon u_\epsilon - u|^2_{L^2(0,T;H)} = 0 \tag{4.24}
\]
and
\[
\lim_{\epsilon \to 0} E_\delta \int_0^T \int_D (A_\epsilon P_\epsilon u_\epsilon - B\nabla u) \psi dx dt = 0 \tag{4.25}
\]
for any $v \in D(D)$ and $\varphi \in D([0,T])$.

Now we are in the position to give the homogenized effective equation in the following theorem.

**Theorem 4.1.** *(Effective macroscopic model)* For any $T > 0$, assume that (4.1), (4.2) and (4.23) hold. Let $u_\epsilon$ be the solution of (2.1)-(2.4). Then the distribution $L(P_\epsilon u_\epsilon)$ converges weakly to $\mu$ in the space of probability measures on $L^2(0,T;H)$ as $\epsilon \downarrow 0$, with $\mu$ being the distribution of $u$, which is the solution of the following homogenized effective equation
\[
\vartheta du = (\text{div}(B\nabla u) + f) dt + gdW(t) \text{ in } D \times (0,T), \tag{4.26}
\]
\[
u = 0 \text{ on } \partial D \times (0,T), \tag{4.27}
\]
\[
u(x,0) = \frac{u^0}{\vartheta} \text{ in } D, \tag{4.28}
\]
where $B = (\beta_{ij})$ is determined by (7.4) in Appendix at the end of this paper.

**Remark 4.2.** This theorem implies that the macroscopic model (4.26) is an effective approximation for the microscopic model (2.1), on any finite time interval $0 < t < T$, in the sense of probability distribution. In other words, if we intend to numerically simulate the microscopic model up to finite time, we could use the macroscopic model as an approximation when $\epsilon$ is sufficiently small.
Remark 4.3. Due to the appearance of the stochastic integral term (see (4.10)), this theorem on weak convergence of probability measures does not follow directly from the deterministic homogenization results and the mild formulation (3.2).

Remark 4.4. The stochastic PDE (4.26) is defined on the homogenized domain $D$. By the analysis in [12], for any fixed $T > 0$, the macroscopic system (4.26)-(4.28) is well-posed, as long as $f \in L^2(0, T; H)$ and $g \in L^2(0, T; L^2_2)$.

Proof. Noticing the arbitrariness of $\delta$, this is direct result of the analysis of the first part in this section by the Skorohod theorem and the $L^2(\Omega_\delta)$ convergence of $P_\epsilon u_\epsilon$ on $(\Omega_\delta, \mathcal{F}_\delta, \mathbb{P}_\delta)$.

We finish this section by the following remark.

Remark 4.5. Note that there are several papers on effective dynamics for partial differential equations with random coefficients (so called random PDEs; not stochastic PDEs); see [19, 26, 32] and reference therein. In [19, 26], a random partial differential equation is obtained as the homogenized effective equation for a random system with fast or small scales on both time or spatial variable. And the distribution of solution of heterogeneous system converges weakly to that of homogenized equation. However in [32], the effective equation is obtained as an averaged deterministic equation for a random system with small scale just on time. And the fluctuation of the solution of the random equation around the solution of the averaged equation converges to a generalized Ornstein-Uhlenbeck process in distribution. In the present paper, the original microscopic model is a stochastic PDE (i.e., PDE with white noise) and the effective macroscopic equation is still a stochastic partial differential equation.

5. Long time effectivity of the macroscopic model

In this section we consider the long time effectivity of the homogenized system (4.26) in the autonomous case. It is proved in section 4 that for fixed $T > 0$ the macroscopic behavior of the microscopic system (2.1)-(2.4) can be approximated by the macroscopic model (4.26) in the sense of probability distribution. In fact we can show the long time approximation. More specifically, we now prove that in the sense of distribution, all solutions of (2.1)-(2.4) converge to the unique stationary solution of (4.26) as $T \to \infty$ and $\epsilon \to 0$, under the assumption that $f_\epsilon \in H_\epsilon$ and $g_i^\epsilon \in V_\epsilon$ are independent of time $t$ and

$$\sum_{i=1}^{\infty} |\nabla A_i g_i^\epsilon(x)|_{H_\epsilon}^2 < C^*.$$ (5.1)

Here $C^*$ is a positive constant independent of $\epsilon$. 
By the above assumptions, the property of \(a_{ij}\) and \(\beta_{ij}\), a standard argument (see [13], Section 6) yields that the system (3.1) and (4.26) have unique stationary solutions \(u^{*}\) and \(u^{*}\) for \(t > 0\). We denote by \(\mu^{*}\) and \(\mu^{*}\) the distributions of \(P_{\epsilon}u^{*}\) and \(u^{*}\) in the space \(H\), respectively. Then if \(E|u^{0}|^{2} < \infty\) and \(E|u^{0}|^{2} < \infty\),

\[
\left| \int_{H} h d\mu_{\epsilon}(t) - \int_{H} h d\mu^{*}_{\epsilon} \right| \leq C(u^{0})e^{-\gamma t}, \quad t > 0, \tag{5.2}
\]

\[
\left| \int_{H} h d\mu(t) - \int_{H} h d\mu^{*} \right| \leq C(u^{0})e^{-\gamma t}, \quad t > 0 \tag{5.3}
\]

for some constant \(\gamma > 0\) and any \(h : H \to \mathbb{R}^{1}\) with sup \(|h| \leq 1\) and Lip\((h) \leq 1\). Here \(\mu_{\epsilon}(t) = \mathcal{L}(P_{\epsilon}u_{\epsilon}(t, u^{0})\)), \(\mu(t) = \mathcal{L}(u(t, u^{0}))\), and \(C(u^{0})\) and \(C(u^{0})\) are positive constants depending only on the initial value \(u^{0}\) and \(u^{0}\) respectively. The above convergence also yields that \(\mu_{\epsilon}(t)\) and \(\mu(t)\) weakly converges to \(\mu^{*}\) and \(\mu^{*}\) respectively, as \(t \to \infty\).

We will give some additional a priori estimates which is uniform with respect to \(\epsilon\) to ensure the tightness of the stationary distributions. For Banach space \(U\) and \(p > 1\), we define \(W^{1,p}(0, T; U)\) as the space of functions \(h \in L^{p}(0, T; U)\) such that

\[
|h|_{W^{1,p}(0, T; U)} = |h|_{L^{p}(0, T; U)} + \left| \frac{dh}{dt} \right|_{L^{p}(0, T; U)} < \infty.
\]

And for any \(\alpha \in (0, 1)\), define \(W^{\alpha,p}(0, T; U)\) as the space of function \(h \in L^{p}(0, T; U)\) such that

\[
|h|_{W^{\alpha,p}(0, T; U)} = |h|_{L^{p}(0, T; U)} + \int_{0}^{T} \int_{0}^{T} \frac{|h(t) - h(s)|^{p}}{|t - s|^{1+\alpha p}} dsdt < \infty.
\]

For \(p \in (0, 1)\), we denote by \(C^{\alpha}(0, T; U)\) the space of functions \(h : [0, T] \to \mathcal{X}\) that are Hölder continuous with exponent \(\rho\).

In the following part of this section we always assume that \(f_{\epsilon}\) and \(g^{i}_{\epsilon}\) are independent of time \(t\) with (5.1) hold. And for \(T > 0\) denote by \(u^{*}_{\epsilon,T}\) (respectively, \(u^{*}_{\epsilon}\)) the distribution of stationary process \(P_{\epsilon}u^{*}_{\epsilon}(\cdot)\) (respectively, \(u^{*}(\cdot)\)) in the space \(L^{2}(0, T; H^{1})\). Then we have the following result.

**Lemma 5.1.** For any \(T > 0\) the family \(u^{*}_{\epsilon,T}\) is tight in the space \(L^{2}(0, T; H^{2-\epsilon})\) with \(\epsilon > 0\).

**Proof.** Since \(u^{*}\) is stationary, by (3.8), we see that

\[
E|u^{*}_{\epsilon}|^{2}_{L^{2}(0, T; H^{2})} < C_{T}. \tag{5.4}
\]
Now represent $u^*_\epsilon$ in the form

$$u^*_\epsilon(t) = u^*_\epsilon(0) + \int_0^t A_\epsilon u^*_\epsilon(s) ds + \int_0^t f_\epsilon(x) ds + \int_0^t g_\epsilon(x) dW(s).$$

Also by the stationarity of $u^*_\epsilon$ and (3.8) we obtain

$$\mathbb{E} \left| \int_0^t A_\epsilon P_\epsilon u^*_\epsilon(s) ds + \int_0^t f_\epsilon(x) ds \right|^2_{W^{1,2}(0,T; H)} \leq C_T.$$  \hspace{1cm} (5.5)

Let $M_\epsilon(s, t) = \int_s^t \tilde{g}_\epsilon(x) dW(s)$. By Lemma 7.2 of [12] and Hölder inequality, we derive that

$$\mathbb{E}|M_\epsilon(s, t)|_{V^*_\epsilon}^4 \leq c \left( \int_s^t |\nabla A_\epsilon \tilde{g}_\epsilon(x)|_{L^2}^2 d\tau \right)^2 \leq K(t-s) \int_s^t |\nabla A_\epsilon \tilde{g}_\epsilon(x)|_{L^2}^4 d\tau \leq KC^*(t-s)^2$$

for $t \in [s, T]$, where $K$ is a positive constant independent of $\epsilon$, $s$ and $t$. Then

$$\mathbb{E} \int_0^T |M_\epsilon(0,t)|_{V^*_\epsilon}^4 dt \leq C_T$$  \hspace{1cm} (5.6)

and

$$\mathbb{E} \int_0^T \int_0^T \frac{|M_\epsilon(0,t) - M_\epsilon(0,s)|_{V^*_\epsilon}^4}{|t-s|^{1+4\alpha}} dsdt \leq C_T.$$  \hspace{1cm} (5.7)

Combining (5.4)-(5.7), and the compact embedding of

$L^2(0, T; H^2) \cap W^{1,2}(0, T; H) \subset L^2(0, T; H^{2-\iota})$

and

$L^2(0, T; H^2) \cap W^{\alpha,2}(0, T; H^1) \subset L^2(0, T; H^{2-\iota})$

we obtain the tightness of $u^*_\epsilon$. This completes the proof. \hspace{1cm} \Box

The above lemma directly yields the following result

**Corollary 5.2.** The family $\{\mu^*_\epsilon\}$ is tight in the space $H^1$.

By Lemma 5.1 for any fixed $T > 0$, the Skorohod embedding theorem asserts that for any sequence $\{\epsilon_n\}_n$ with $\epsilon_n \to 0$ as $n \to \infty$, there is subsequence $\{\epsilon_n(k)\}_k$, a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and random variables $\tilde{u}^*_\epsilon(\epsilon_n(k)) \in L^2(0, T; V_\epsilon)$, $\tilde{u}^* \in L^2(0, T; H^1)$ such that

$$\mathcal{L}(P_\epsilon \tilde{u}^*_\epsilon(\epsilon_n(k))) = u^*_\epsilon(\epsilon_n(k), T), \quad \mathcal{L}(\tilde{u}^*) = u^*_T$$

and

$$\tilde{u}^*_\epsilon(\epsilon_n(k)) \to \tilde{u}^* \text{ in } L^2(0, T; H^1) \text{ as } k \to \infty.$$  

Moreover $\tilde{u}^*_\epsilon(\epsilon_n(k))$ (respectively, $\tilde{u}^*$) is the unique stationary solution of equation (3.1) (respectively, (4.26)) with $W$ replaced by $W_k$ (respectively, $\mathbb{W}$). Here $W_k$
and \( W \) are some Wiener processes defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \) with same distribution as \( W \). Then by the analysis of Section 4 and the uniqueness of the invariant measure, we have
\[
u_{\epsilon,T} \to \nu_T, \quad \text{as} \quad \epsilon \to 0
\]
for any \( T > 0 \).

To show the long time effectivity, let \( u_{\epsilon}(t), t \geq 0 \), be a weak solution of system (2.1)-(2.4) and define \( u_{\epsilon}^*(\cdot) = u_{\epsilon}(t+\cdot) \) which is in the space \( L^{2}_{\text{loc}}(\mathbb{R}^+; \mathcal{V}_{\epsilon}) \) by Theorem 3.1. Then by (5.2)
\[
\mathcal{L}(P\epsilon u_{\epsilon}^*(\cdot)) \to \mathcal{L}(P\epsilon u^*(\cdot)), \quad t \to \infty
\]
in the space of probability measures on \( L^{2}_{\text{loc}}(\mathbb{R}^+; \mathcal{H}^1) \). Having the above analysis we draw the following result which implies the long time effectivity of the homogenized effective equation (4.26).

**Theorem 5.3.** *(Long time effectivity of macroscopic model)*

Assume that \( f_{\epsilon} \in \mathcal{H}_{\epsilon} \) and \( g_{\epsilon}^i \in \mathcal{V}_{\epsilon} \) are independent of time \( t \) with (5.1) hold, and further assume that (4.1) and (4.2) hold in \( \mathcal{H} \). Denote by \( u_{\epsilon}(t), t \geq 0 \), the solution of (2.1)-(2.4) and \( u^* \) the unique stationary solution of (4.26). Then
\[
\lim_{\epsilon \to 0} \lim_{t \to \infty} \mathcal{L}(P\epsilon u_{\epsilon}^*(\cdot)) = \mathcal{L}(u^*(\cdot)), \quad (5.8)
\]
where the limits are understood in the sense of weak convergence of Borel probability measures in the space \( L^{2}_{\text{loc}}(\mathbb{R}^+; \mathcal{H}^1) \). That is, the solution of (2.1)-(2.4) converges to the stationary solution of (4.26) in probability distribution as \( t \to \infty \) and \( \epsilon \to 0 \).

**Remark 5.4.** This theorem implies that the macroscopic model (4.26) is an effective approximation for the microscopic model (2.1), on very long time scale. In other words, if we intend to numerically simulate the long time behavior of the microscopic model, we could just simulate the macroscopic model as an approximation when \( \epsilon \) is sufficiently small.

6. **Effectivity in energy convergence**

In the last two sections, we have considered finite time and long time effectivity of the macroscopic model (4.26), in the sense of convergence in probability distribution. In this section we focus on the finite time effectivity of the macroscopic model (4.26), but in the sense of convergence in energy. Namely, we show that the solution of the microscopic model (2.1) or (3.1), converges to the solution of the macroscopic model (4.26), in an energy norm.
Let $u_\varepsilon$ be a weak solution of (3.1) and $u$ be a weak solution of (4.26). We introduce the following energy functionals:

$$E^\varepsilon(u_\varepsilon)(t) = \frac{1}{2} E|\tilde{u}_\varepsilon|^2_H + E \int_0^t \int_D \chi_{\varepsilon} A_\varepsilon \nabla (P_\varepsilon u_\varepsilon(x, \tau)) \nabla (P_\varepsilon u_\varepsilon(x, \tau)) \, dx \, d\tau$$

(6.1)

and

$$E^0(u)(t) = \frac{1}{2} E|u|^2_H + E \int_0^t \int_D B \nabla u(x, \tau) \nabla u(x, \tau) \, dx \, d\tau.$$  

(6.2)

By the Itô formula, it is clear that

$$E^\varepsilon(u_\varepsilon)(t) = \frac{1}{2} E|\tilde{u}_\varepsilon|^2_H + E \int_0^t \int_D \tilde{f}_\varepsilon(x, \tau) \tilde{u}_\varepsilon(x, \tau) \, dx \, d\tau + \frac{1}{2} E \int_0^t \tilde{g}_\varepsilon(x, \tau) \, dx \, d\tau$$

and

$$E^0(u)(t) = \frac{1}{2} E|u|^2_H + E \int_0^t \int_D f(x, \tau) u(x, \tau) \, dx \, d\tau + \frac{1}{2} E \int_0^t |g(x, \tau)|^2_{L_2^Q} \, d\tau.$$  

Then we have the following result on effectivity of the macroscopic model in the sense of convergence in energy.

**Theorem 6.1. (Effectivity in energy convergence)**

Assume that (4.1) and (4.2) hold. If

$$\tilde{u}_\varepsilon^0 \to u^0, \text{ strongly in } H, \text{ as } \varepsilon \to 0,$$

then

$$E^\varepsilon(u_\varepsilon) \to E^0(u) \text{ in } C([0, T]), \text{ as } \varepsilon \to 0.$$

**Proof.** By the analysis of section 4 for any $\delta > 0$, $u_\varepsilon \to u$ strongly in $L^2(0, T; H)$ on $\Omega_\delta$, then by the arbitrariness of $\delta$, it is easy to see that

$$E \int_0^t \int_D \tilde{f}_\varepsilon(x, \tau) \tilde{u}_\varepsilon(x, \tau) \, dx \, d\tau \to E \int_0^t \int_D f(x, \tau) u(x, \tau) \, dx \, d\tau, \text{ for } t \in [0, T].$$

Then by $\tilde{g}_\varepsilon \to g$ weakly in $L^2(0, t; L^2_Q)$, we have

$$E^\varepsilon(u_\varepsilon)(t) \to E^0(u)(t) \text{ for any } t \in [0, T].$$

(6.3)

We now only need to show that $\{E^\varepsilon(u_\varepsilon)(t)\}_\varepsilon$ is equicontinuous, as then the Ascoli-Arzela’s theorem [13] will imply the result in the theorem.

In fact, given any $t \in [0, T]$, and $h > 0$ small enough, we have

$$|E^\varepsilon(u_\varepsilon)(t + h) - E^\varepsilon(u_\varepsilon)(t)|$$

$$\leq \left| E \int_t^{t + h} \int_D \tilde{f}_\varepsilon(x, \tau) \tilde{u}_\varepsilon(x, \tau) \, dx \, d\tau \right| + E \int_t^{t + h} |\tilde{g}_\varepsilon(x, \tau)|^2_{L_2^Q} \, d\tau$$

$$\leq E \left\{ |\tilde{f}_\varepsilon|_{L^2(0, T; H)} \int_t^{t + h} |\tilde{u}_\varepsilon(x, \tau)|^2_H \, dx \, d\tau \right\} + E \int_t^{t + h} |\tilde{g}_\varepsilon(x, \tau)|^2_{L_2^Q} \, d\tau.$$
Noting that $\tilde{u}_\epsilon \in L^2(0, T; H)$ a.s. and (2.7), we have

$$|\mathcal{E}'(u_\epsilon)(t + h) - \mathcal{E}'(u_\epsilon)(t)| \to 0, \text{ as } h \to 0,$$

uniformly on $\epsilon$, which means the equi-continuity of the family $\{\mathcal{E}'(u_\epsilon)\}_\epsilon$. This completes the proof. □

7. Appendix: The homogenized matrix

In this Appendix, we give the explicit expression of the homogenized matrix $B$; for more details see [9]. Let $\chi^i, i = 1, \cdots, n$ be the solutions of

$$- \sum_{l,k=1}^n \frac{\partial}{\partial y_l} \left( a_{kl} \frac{\partial (\chi^i - y_l)}{\partial y_k} \right) = 0 \text{ in } Y^* \tag{7.1}$$

$$\sum_{l,k=1}^n a_{kl} \frac{\partial (\chi^i - y_l)}{\partial y_k} n_l = 0 \text{ on } \partial S \tag{7.2}$$

$$\chi^i \text{ is } Y - \text{periodic.} \tag{7.3}$$

It is easy to calculate that $\chi^i = -w_{e_i} + e_i$ with $\{e_i\}_{i=1}^n$ the canonical basis of $\mathbb{R}^n$. Then

$$\beta_{ij} = \frac{1}{|Y|} \int_Y \sum_{k=1}^n a_{kj} \frac{\partial w_{e_i}}{\partial y_k} dy = \frac{1}{|Y|} \int_Y a_{ij} dy - \frac{1}{|Y|} \int_Y \sum_{k=1}^n a_{kj} \frac{\partial \chi^i}{\partial y_k} dy. \tag{7.4}$$

Moreover the operator $B = (\beta_{ij})$ satisfies the uniform ellipticity condition: there is a constant $b > 0$ such that

$$\sum_{i,j=1}^n \beta_{ij} \xi_i \xi_j \geq b \sum_{i=1}^n \xi_i^2, \text{ for } \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n.$$

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REFERENCES

[1] G. Allaire, M. Murat & A. Nandakumar, Appendix of ”Homogenization of the Neumann problem with nonisolated holes”, Asymptotic Anal. 7(2), (1993), 81-95.
[2] A. Bensoussan, J. L. Lions & G. Papanicolaou, Asymptotic Analysis for Periodic Structure, North-Holland, Amsterdam, New York, 1978.
[3] P. Billingsley, Weak Convergence of Probability Measures, John Wiley/Sons, New York, 1968.
[4] P. Billingsley, Probability and Measure, Third Edition, John Wiley/Sons, New York, 1995.
[5] M. Briane & L. Mazliak, Homogenization of two randomly weakly connected materials, *Portugaliae Mathematica* **55**, (1998), 187-207.
[6] S. Brahim-Otsmane, G. A. Francfort & F. Murat, Correctors for the homogenization of the wave and heat equations, *J. Math. Pures Appl.* **71**, (1998), 197-231.
[7] L. A. Caffarelli, P. Souganidis and L. Wang, Homogenization of fully nonlinear, uniformly elliptic and parabolic partial differential equations in stationary ergodic media. *Comm. Pure Appl. Math.* Vol. LLVIII (2005), 1-43.
[8] D. Cioranescu & P. Donato, Exact internal controllability in perforated domains, *J. Math. Pures Appl.* **68**, (1989), 185-213.
[9] D. Cioranescu & P. Donato, *An Introduction to Homogenization*, Oxford University Press, New York, 1999.
[10] A. Cherkaev & R. V. Kohn, *Topics in the Mathematical Modelling of Composite Materials*. Birkhaeuser, Boston, 1997.
[11] D. Cioranescu, P. Donato, F. Murat & E. Zuazua, Homogenization and correctors results for the wave equation in domains with small holes, *Ann. Scuola Norm. Sup. Pisa* **18**, (1991), 251-293.
[12] G. Da Prato & J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, 1992.
[13] G. Da Prato & J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press, 1996.
[14] R. M. Dudley, *Real Analysis and Probability*. Cambridge Univ. Press, 2002.
[15] W. E, X. Li & E. Vanden-Eijnden, Some recent progress in multiscale modeling, *Multiscale modelling and simulation*, Lect. Notes Comput. Sci. Eng., **39**, 3–21, Springer, Berlin, 2004.
[16] Z. Huang and J. Yan, *Introduction to Infinite Dimensional Stochastic Analysis*. Science Press/Kluwer Academic Pub., Beijing/New York, 1997.
[17] P. Imkeller and A. Monahan (Eds.), *Stochastic Climate Dynamics*, a Special Issue in the journal *Stochastics and Dynamics*, Vol. 2, No. 3, 2002.
[18] V. V. Jikov, S.M. Kozlov & O. A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, 1994.
[19] M. L. Kleptsyna & A. L. Piatnitski, Homogenization of a random non-stationary convection-diffusion problem, *Russian Math. Surveys* **57**, (2002), 729-751.
[20] J. L. Lions, *Quelques méthodes de résolution des problèmes non linéaires*, Dunod, Paris, 1969.
[21] P. L. Lions & N. Masmoudi, Homogenization of the Euler system in a 2D porous medium, *J. Math. Pures Appl.* **84** (2005), 1-20.
[22] G. D. Maso & L. Modica, Nonlinear stochastic homogenization and ergodic theory, *J. Rea. Ang. Math. B.* **368**, (1986), 27-42.
[23] A. Mikelić & L. Paloi, Homogenization of the inviscid incompressible fluid flow through a 2D porous medium, *Proc. Amer. Math. Soc.* **127**, (1999), 2019-2028.
[24] A. K. Nandakumaran & M. Rajesh, Homogenization of a parabolic equation in a perforated domain with Neumann boundary condition, *Proc. Indian Acad. Sci. (Math. Sci.)* **112**, (2002), 195-207.
[25] A. K. Nandakumaran & M. Rajesh, Homogenization of a parabolic equation in a perforated domain with Dirichlet boundary condition, *Proc. Indian Acad. Sci. (Math. Sci.)* **112**, (2002), 425-439.
[26] E. Pardoux & A. L. Piatnitski, Homogenization of a nonlinear random parabolic partial differential equation, *Stochastic Process Appl.* **104**, (2003), 1-27.

[27] E. Sanchez-Palencia, *Non Homogeneous Media and Vibration Theory*, Lecture Notes in Physics, 127, Springer-Verlag, Berlin, 1980.

[28] J. Souza & A. Kist, Homogenization and correctors results for a nonlinear reaction-diffusion equation in domains with small holes, *The 7th Workshop on Partial Differential Equations II, Mat. Contemp.* **23**, (2002), 161-183.

[29] C. Timofte, Homogenization results for parabolic problems with dynamical boundary conditions, *Romanian Rep. Phys.* **56**, (2004), 131-140.

[30] M. B. Taghite, K. Taous & G. Maurice, Heat equations in a perforated composite plate: Influence of a coating, *Int. J. Eng. Sci.* **40** (2002), 1611-1645.

[31] R. Temam & A. Miranville, *Mathematical modeling in continuum mechanics*, Second edition, Cambridge University Press, Cambridge, 2005

[32] H. Watanabe, Averaging and fluctuations for parabolic equations with rapidly oscillating random coefficients, *Prob. Theory & Related Fields* **77**, (1988), 359-378.

[33] E. Waymire and J. Duan (Eds.), *Probability and Partial Differential Equations in Modern Applied Mathematics*. IMA Volume 140, Springer-Verlag, New York, 2005.

[34] K. Yosida, *Functional Analysis*, 5th Ed., Springer-Verlag, Berlin, 1978.

[35] V. V. Zhikov, On homogenization in random perforated domains of general type, *Matem. Zametki* **53**, (1993), 41-58.

[36] V. V. Zhikov, On homogenization of nonlinear variational problems in perforated domains, *Russian J. Math. Phys.* **2**, (1994), 393-408.

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