Roughness of geodesics in Liouville quantum gravity

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Abstract: The metric associated with the Liouville quantum gravity (LQG) surface has been constructed through a series of recent works and several properties of its associated geodesics have been studied. In the current article we confirm the folklore conjecture that the Euclidean Hausdorff dimension of LQG geodesics is strictly greater than 1 for all values of the so-called Liouville first passage percolation (LFPP) parameter $\xi$. We deduce this from a general criterion due to Aizenman and Burchard [AB99] which in our case amounts to near-geometric bounds on the probabilities of certain crossing events for LQG geodesics in the number of crossings. We obtain such bounds using the axiomatic characterization of the LQG metric after proving a special regularity property for the Gaussian free field (GFF). We also prove an analogous result for the LFPP geodesics.

Keywords and phrases: Liouville quantum gravity (LQG), Liouville first passage percolation (LFPP), Gaussian free field (GFF), Random metrics, Random curves, Hausdorff dimension.

1. Introduction

1.1. Background and motivation

Liouville quantum gravity (LQG) is a one-parameter family of random fractal surfaces that was first studied by physicists in the 1980s as a class of canonical models of random two-dimensional Riemannian manifolds [Pol81, Dav88, DK89]. We refer the reader to [Gwy20, BP21] for a mathematical introduction to this topic.

Formally speaking, a $\gamma$-Liouville quantum gravity ($\gamma$-LQG) surface is a random “Riemannian manifold” with Riemannian metric tensor $e^{\gamma h} ds^2$ where $h$ is a variant of the Gaussian free field (GFF) on some domain $U \subset \mathbb{C}$ and $\gamma \in (0, 2]$ is the underlying parameter. This is of course not well-defined as the GFF is a random Schwartz distribution (see, e.g., [She07, BP21, PW20] for a comprehensive introduction to the GFF). Rigorous mathematical investigation into this surface as a random metric measure space was set off with the construction of the associated volume measure in [DS11] which is a special instance of the Gaussian multiplicative chaos (GMC) developed by Kahane in 1985 [Kah85] (see [RV14] for a current introduction to this subject).

On the metric side, Miller and Sheffield constructed the $\sqrt{8/3}$-LQG metric through a series of papers [MS20, MS21a, MS21b] where they established a deep link with the Brownian map.
For a general $\gamma \in (0, 2)$ (the so-called subcritical phase), the metric was recently constructed as a culmination of several works due to Ding, Dubédat, Dunlap, Falconet, Gwynne, Miller, Pfeffer and Sun [DDDF20, DFG+20, GM21a, GM21b]. Their construction starts by producing candidate distance functions which are obtained as subsequential limits [DDDF20] of a family of random metrics known as the Liouville first passage percolation (LFPP). The limiting metric is then shown to be unique in two stages. Firstly, every possible subsequential limit is shown to be a measurable function of the GFF $h$ satisfying a list of axioms motivated by the natural properties and scaling behavior of the LFPP metrics [DFG+20]. Subsequently these axioms are shown to uniquely characterize a random metric [GM21a, GM21b] on the plane. See the ICM article by Ding, Dubédat and Gwynne [DDG21] for a detailed overview of the construction of LQG using LFPP.

Associated with the LFPP metric is a parameter $\xi = \xi(\gamma)$ which gives a reparametrization of the LQG metric although its explicit dependence on $\gamma$ is currently unknown (see [DG20a, DDG21]). There is yet another parametrization of LQG which is perhaps more popular in the physics community, namely the matter central charge $c_M$. The subcritical phase (i.e., $\gamma \in (0, 2)$) corresponds to $c_M \in (-\infty, 1)$ whereas the critical ($\gamma = 2$) and the supercritical phases ($\gamma$ complex, $|\gamma| = 2$) correspond to $c_M = 1$ and $c_M \in (1, 25]$ respectively. See [DDG21] and [DG21, Section 1] to get a better understanding of the interplay between these different parameters. Of these three phases, the supercritical phase is the most mysterious, not least because $\gamma$ is complex. However, one can still assign a LFPP parameter $\xi > 0$ to such $\gamma$ [DG20b] and consider subsequential limits of LFPP metrics like before. This program was carried out in a series of works by Ding, Gwynne and Pfeffer [DG20b, Pfe21, DG21] and hence the LQG metric is now defined for all values of $\xi \in (0, \infty)$.

In parallel to these works, many properties of the LQG metric and its associated geodesics (which are known to exist, see Section 2 below) have been studied and discovered in recent times; see [DDG21] for an overview. Of particular relevance to the current article is the paper [GPS22] where it was proved (see Theorem 1.8), amongst several other results, that there is a deterministic constant $\Delta_{\geo} = \Delta_{\geo}(\gamma) > 0$ for any $\gamma \in (0, 2)$ (equivalently $\xi \in (0, \xi_{\text{crit}} = \xi(2))$) such that a.s. the Euclidean Hausdorff dimension of every $\gamma$-LQG geodesic started from 0 is equal to $\Delta_{\geo}$. Let us note at this point that the dimension of a geodesic is always equal to 1 w.r.t. the metric for which it is a geodesic, i.e., the LQG metric in this case. Not much is known about the precise value of $\Delta_{\geo}$ except for an upper bound given in [GP19a, Corollary 1.10].

As to the lower bound on the dimension, it is expected “...that it is strictly greater than 1” where the quoted text is from Ding, Dubédat and Gwynne’s ICM paper [DDG21] (see Section 3.3). A closely related question was addressed in a work by Ding and Zhang [DZ19] where they showed under a positivity assumption on the distance exponent for a variant of LFPP that the Euclidean length of any LFPP-geodesic joining two macroscopically separated points diverges, with high probability, as a (negative) power of $\varepsilon$ as $\varepsilon \to 0$ where $\varepsilon$ is the “regularization” parameter for LFPP (see, e.g., [DDG21] and Section 1.3 below for a precise definition). But this property of the exponent is currently known to hold only for very small values of $\xi$ from [DG19] and for $\xi \in (0.266, \ldots, 1/\sqrt{2})$ from [GP19a, Theorem 2.3]. In the present article we confirm this conjecture for the LQG geodesics corresponding to any value of $\xi \in (0, \infty)$. We also extend the result of Ding and Zhang to all values of $\xi$ for a variant of LFPP.
1.2. Lower bound on the dimension of LQG geodesics

Based on the discussion in the previous subsection, we know that the LQG metric is a measurable function of the underlying field $h$ which we will refer to as $D_h$ in the sequel while keeping its dependence on the LFPP parameter $\xi$ implicit. Whenever we speak of “a $D$-geodesic” for some metric $D$ in this paper, we mean a geodesic associated with $D$ connecting two points $z, w \in \mathbb{C}$ (see Section 2 below for a definition). We can now state our main result on LQG geodesics.

**Theorem 1.1.** Let $h$ be the whole-plane GFF and $D_h$ be the associated $\xi$-LQG metric. Then for each $\xi > 0$, with probability 1, the Euclidean Hausdorff dimension of any $D_h$-geodesic is strictly greater than 1.

**Remark 1.2** (Other variants of GFF). Although we stated Theorem 1.1 for the whole-plane GFF only, the result also holds for zero boundary GFF on proper subdomains of $\mathbb{C}$ by local absolute continuity. See Section 2 below.

Broadly speaking, the identifying characteristic of a rough random curve is its propensity to deviate from a straight line segment in multiple scales. This can be manifested, for example, in the rapid decay of simultaneous crossing probabilities of thin rectangles (or annuli) by the curve. Aizenman and Burchard showed in their celebrated work [AB99] that a geometric decay of such crossing probabilities in the number of rectangles (see [AB99, display (1.15)]) is sufficient for the (Euclidean) Hausdorff dimension to be greater than 1. Unfortunately, we can not prove this condition for LQG geodesics in full generality and it is not clear to us whether this is even true. However, it turns out that the proof of the lower bound on Hausdorff dimension involves some specific configurations of rectangles (or annuli) for which we can establish a near-geometric decay. To this end let us introduce:

**Definition 1.3.** A collection of disks $\{A_j\}_{j=1}^n$ is said to be $(\lambda, \nu)$-balanced for some $\lambda, \nu > 1$ if the following three conditions are satisfied.

**C1** For any $j \neq j'$, $d(A_j, A_{j'}) \geq \max(\text{diam}(A_j), \text{diam}(A_{j'}))$ where $d(\cdot, \cdot)$ denotes the Euclidean distance.

**C2** There exists a sequence of non-negative integers $k_1 < k_2 < \cdots < k_m$ and $L_0 > 0$ such that each $\text{diam}(A_j) \in [L_0 \lambda^{-k_j}/100, 100L_0 \lambda^{-k_j}] := I_i$ for some $i \in [1, m]$.

**C3** $|\{j : \text{diam}(A_j) \in I_i\}| \in \left[\nu/100, 100\nu\right]$ for all $i \in [1, m]$.

As we will see below, such families of disks arise naturally when we subdivide a path into smaller crossings in a certain hierarchical fashion. We just need a few more definitions before we can resume our discussion on Theorem 1.1. If $A$ is the (open) Euclidean disk $B(z, r)$ centered at $z$ with radius $r$, we abbreviate the (Euclidean) annulus $A(z, r/2, r) := B(z, r) \backslash B(z, r/2)$ as $A_\circ$. We say that a path $P$ crosses $A$, if it has a segment that is contained in $A$, has its both endpoints on $\partial A$ and intersects the circle $\partial B(z, r/2)$.

We can now state the main ingredient of our proof of Theorem 1.1.

**Proposition 1.4.** Let $h$ be as in the statement of Theorem 1.1. Then for any $\lambda_0 > 1$ and $\xi > 0$, there exist $\rho = \rho(\lambda_0, \xi) \in (0, 1)$ and a positive absolute constant $C$ such that the following holds. For any collection of disks $\{A_j\}_{j=1}^n$ that is $(\lambda, \nu)$-balanced for some $\lambda \geq \lambda_0$ and $\nu > 1$, one has

$$\mathbb{P}[\text{a } D_h\text{-geodesic crosses } A_{j, \circ} \text{ for all } j] \leq C \rho^{n \sqrt{\nu}}. \quad (1.1)$$
Theorem 1.1 follows from Proposition 1.4 by a similar argument as in [AB99] with some adjustments along the way. For sake of completeness, we present the essential parts of the argument below. Let us start by recalling the notion of straight runs from [AB99]. For a \( \lambda > 1 \) which we fix at the outset, a path \( P \) in \( \mathbb{R}^2 \) is said to exhibit a straight run at scale \( L \) if it traverses some rectangle of length \( L \) and cross-sectional diameter \( (9/\sqrt{\lambda})L \) in the “length” direction, joining the centers of the corresponding sides. Two straight runs are nested if one of the defining rectangles contains the other.

The straight runs of \( P \) are \((\lambda, k_0)\)-sparse if \( P \) does not exhibit any nested collection of straight runs on a sequence of scales \( L_{k_1} > \cdots > L_{km} \) with \( L_k := L_0 \lambda^{-k} \) and \( m \geq \frac{1}{2} \max\{k_m, k_0\} = \frac{1}{2} k_m \).

We have the following deterministic result from \cite{AB99}.

**Theorem 1.5.** [\cite{AB99}, Theorem 5.1] If the straight runs of a given path \( P \) are \((\lambda, k_0)\)-sparse for some \( k_0 > 0 \), then the Hausdorff dimension of \( P \) is at least \( s \) with \( s \) given by \( \lambda^s = \sqrt{p(p+1)} \) and \( p \) an integer strictly smaller than \( \lambda \).

We can now finish the proof.

**Proof of Theorem 1.1.** It suffices to show that there exist \( p < \infty \) such that for every \( \lambda > p \) and \( K \subset \subset \mathbb{R}^2 \),

\[
\mathbb{P}[\text{straight runs of } D_h\text{-geodesics are } (\lambda, k_0)\text{-sparse in } K] \geq 1 - C_K 2^{-k_0} \tag{1.2}
\]

where \( C_K \) depends only \( K \) and \( L_0 = 1 \). Indeed, using Borel-Cantelli lemma we can conclude from (1.2) that there exists, almost surely, \( k_0 = k_0(\omega) < \infty \) such that all the \( D_h\)-geodesics in \( K \) are \((\lambda, k_0)\)-sparse. Taking \( \lambda \) close enough to \( p \), we then get from Theorem 1.5 that the Hausdorff dimension of any \( D_h\)-geodesic in \( K \) is strictly greater than 1. Theorem 1.1 now follows by letting \( K \uparrow \mathbb{R}^2 \).

Let us now return to the proof of (1.2). To this end we first compute the probability that there is a nested sequence of straight runs at scales \( L_{k_1}, \ldots, L_{km} \). If a path crosses a rectangle \( R \) of length \( L \) and width \( (9/\sqrt{\lambda})L \) in the long direction, then it also crosses a rectangle \( R' \) of width \( (10/\sqrt{\lambda})L \) and length \( L/2 \) centered at a line segment joining discretized points in \( L' \mathbb{Z}^d \), provided that \( L' \leq L/\lambda \) is picked in a suitable way. Furthermore if \( L = L_k \) and \( R \) contains a smaller rectangle with length \( L_{k'} \) and the same aspect ratio for some \( k' > k \), then \( R' \) can be chosen so that it contains the smaller rectangle as well. Therefore the number of possible locations of the \( m \) nested rectangles obtained in this manner from the straight runs at scales \( L_{k_1}, \ldots, L_{km} \) is bounded by

\[
C_K \lambda^{4k_1} \lambda^{4(k_2-1)} \cdots \lambda^{4(k_m-k_{m-1})} \leq C_K \lambda^{4k_m}. \tag{1.3}
\]

Let us now fix a sequence \( R_i, i = 1, \ldots, m \) of nested rectangles of length \( L_{k_i}/2 \) and width \( (10/\sqrt{\lambda})L_{k_i} \). In the sequel we will call a collection \( S \) of subsets of the plane as well-separated if the distance between any \( S \in S \) and the rest is at least as large as the diameter of \( S \) (recall the first condition in Definition 1.3). Now split each of the rectangles \( R_i \) to get \( \sqrt{\lambda}/40 \) shorter rectangles of aspect ratio 2. Since \( R_{i+1} \) intersects at most two of the shorter rectangles obtained by subdividing \( R_i \), the number of rectangles in a maximal well-separated collection is at least \( m(\sqrt{\lambda}/80 - 2) \). Let us call these new rectangles \( \{R'_j\}_{j=1}^p \). Now observe that any path that crosses \( R'_j \) also crosses \( A_{j,0} \) (cf. (1.1)) where \( A_j \) is the disk centered at the line joining the midpoints of the shorter sides of \( R'_j \). Hence the probability of a geodesic crossing all the \( R_i \)'s is bounded above by the probability of crossing \( A_{j,0} \) for all \( j \in [1, n] \). It also follows from the definition of
$R'_j$ and $A_j$ that the family of disks $\{A_j\}_{j=1}^n$ is $(\sqrt{\lambda}, \sqrt{\lambda})$-balanced for any $\lambda$ satisfying $\sqrt{\lambda} > 800$. Therefore using Proposition 1.4 we get for any such $\lambda$

$$P[R_1, \ldots, R_m \text{ are crossed by a } D_h\text{-geodesic}] \leq C \rho \frac{m\lambda^{1/4}}{\log \lambda}$$

where $\rho = \rho(\xi) \in (0,1)$. Combined with (1.3) this yields that for all $\lambda$ larger than some fixed number and $k_m$ satisfying $m \geq k_m/2$,

$$P \left[ \text{there exists a nested sequence of staright runs of} \right.$$

$$\left. \text{a } D_h\text{-geodesic at scales } L_{k_1}, \ldots, L_{k_m} \text{ inside } K \right] \leq C_K e^{(C \log \lambda + \log \rho \lambda^{1/4}/\log \lambda)m}$$

where $C$ is an absolute constant. Now choosing $p > (800)^2$ large enough so that

$$C \log \lambda + \log \rho \lambda^{1/4}/\log \lambda < -4,$$

for all $\lambda > p$ and summing over all the sequences $k_1 < \cdots < k_m$ satisfying $m \leq k_m \leq 2m$, we get

$$P \left[ \text{there exists a nested sequence of staright runs of a } D_h\text{-geodesic} \right.$$

$$\left. \text{at (log-)scales } k_1, \ldots, k_m \text{ inside } K \text{ with } m \leq k_m \leq 2m \right] \leq C_K 4^m e^{-4m} \leq C_K 4^{-m}.$$

Finally, summing over $m \geq \frac{k_m}{2}$ yields (1.2).

\[ \square \]

1.3. Lower bound on the length of LFPP geodesics

Let us first give a definition of the Liouville first passage percolation (LFPP) that is standard in recent literature. Soon we will give another definition which is equivalent to the former one in the limiting sense (i.e., they both converge to the LQG metric) and is more convenient to work with for our purpose. In the remainder of this section, $h$ is a whole-plane GFF with the additive constant chosen so that its average over the unit circle is 0. The starting point in all definitions of LFPP is a family of continuous functions which approximate $h$. For $\varepsilon > 0$, consider a mollified version of $h$ by

$$h^*_\varepsilon(z) := (h \ast p_{\varepsilon^2/2})(z) = \int_C h(w) p_{\varepsilon^2/2}(z - w) \, dw, \quad \forall z \in \mathbb{C},$$

where $p_{\varepsilon}(z) := \frac{1}{2\pi\varepsilon} \exp \left(-\frac{|z|^2}{2\varepsilon}\right)$ is the heat kernel and the integration is in the sense of distributional pairing.

Now we define the Liouville first passage percolation (LFPP) with parameter $\xi$ as the family of random metrics $\{D^\varepsilon_h\}_{\varepsilon > 0}$ defined by

$$D^\varepsilon_h(z,w) := \inf_{P : z \to w} \int_0^1 e^{\varepsilon h^*_\varepsilon(P(t))} |P'(t)| \, dt, \quad \forall z,w \in \mathbb{C}$$

(1.4)

where the infimum is over all piecewise $C^1$-paths $P : [0,1] \to \mathbb{C}$ from $z$ to $w$. To get a non-trivial limit of the metrics $D^\varepsilon_h$ in some suitable topology, one needs to re-normalize them. The standard, although somewhat arbitrary, choice for the normalizing factor is

$$a_\varepsilon := \text{median of } \inf \left\{ \int_0^1 e^{\varepsilon h^*_\varepsilon(P(t))} |P'(t)| \, dt : P \text{ is a left-right crossing of } [0,1]^2 \right\}.$$  

(1.5)
The tightness of the rescaled metrics \( \{a_{\epsilon}^{-1}D_h^\epsilon\}_{\epsilon>0} \) was established in [DDDF20] and [DG20b] for subcritical and general \( \xi \) respectively albeit in different topologies. See [DDG21] for more details.

Since the function \( h_\epsilon^* \) is continuous, the geodesics associated with \( D_h^\epsilon \) are (locally) rectifiable and hence their Hausdorff dimension is 1. Therefore we can instead look at the (Euclidean) lengths of geodesics and ask whether they diverge as \( \epsilon \to 0 \) with high probability. Ding and Zhang [DZ19] proved a power law divergence for the length when the LFPP distance is defined using the discrete Gaussian free field (DGFF) under the assumption that, with high probability, maximum LFPP distance (appropriately scaled) between any points in a compact set is at most \( \epsilon^c \) for some \( c>0 \). As already mentioned in Section 1.1, this condition is currently known to hold only for very small values of \( \xi \) [DG19] and \( \xi \in (0.266\ldots,1/\sqrt{2}) \) [GP19a]. Using similar ideas as involved in the proof of Theorem 1.1, we can deduce the power law lower bound on the length of LFPP geodesics for all values of \( \xi \). We state this result for a slightly different choice of mollification as described below.

In several situations like in ours, it turns out to be more convenient to work with mollifications that depend locally on \( h \) unlike \( h_\epsilon^*(\cdot) \) above. To this end, let us consider

\[
\hat{h}_\epsilon^*(z) := \int_C \psi(\epsilon^{-1/2}(z-w)) h(w) p_{\epsilon z/2}(z-w) \, dw, \quad \forall z \in \mathbb{C},
\]

where \( \psi : \mathbb{C} \to [0,1] \) is a deterministic, smooth, radially symmetric bump function supported in \( B(0,1) \) that is identically equal to 1 on \( B(1/2,0) \). We can define a LFPP metric \( \hat{D}_h^\epsilon \) similarly as \( D_h^\epsilon \) with \( \hat{h}_\epsilon^* \) playing the role of \( h_\epsilon^* \) (recall \( (1.4) \)). It was proved in [DFG+20, Lemma 2.1] that \( \text{a.s. } \lim_{\epsilon \to 0} \frac{\hat{D}_h^\epsilon(z,w;U)}{\hat{h}_\epsilon(z,w;U)} = 1 \) uniformly over all \( z,w \in U \) \( (z \neq w) \) for each bounded open set \( U \subset \mathbb{C} \) where \( d(\cdot,\cdot;U) \) is the internal metric of \( d \) on \( U \) (see Definition 2.1 below). Consequently, the families of metrics \( \{a_{\epsilon}^{-1}D_h^\epsilon\}_{\epsilon>0} \) and \( \{a_{\epsilon}^{-1}\hat{D}_h^\epsilon\}_{\epsilon>0} \) have the same (weak) subsequential limits [Pfe21, Lemma 2.15]. We now state our result on the (Euclidean) lengths of \( \hat{D}_h^\epsilon \)-geodesics. Below and in rest of the article, two points \( z,w \in \mathbb{C} \) are said to be \( \kappa \)-separated if their Euclidean distance is at least \( \kappa \).

**Theorem 1.6.** For each \( \xi \in (0,\infty) \), there exists \( \alpha = \alpha(\xi)>0 \) such that for every \( \kappa \in (0,1) \) and \( K \subset \mathbb{R}^2 \) compact,

\[
\lim_{\epsilon \to 0} \mathbb{P}[\text{the length of any } \hat{D}_h^\epsilon \text{ geodesic connecting two any } \kappa \text{-separated points in } K \geq \epsilon^{-\alpha}] = 1.
\]

**Remark 1.7** ([Relationship between Theorem 1.1 and Theorem 1.6]). The results given by Theorem 1.1 and Theorem 1.6 are independent a priori since neither implies the other even if one assumes that the LFPP geodesics converge in Hausdorff distance to the LQG geodesics. Although the ideas behind their proofs are similar, the arguments involved vary significantly in their details.

**Remark 1.8** ([Quantitative lower bounds on the dimension and the length exponent]). It is possible to extract quantitative lower bounds on the (Euclidean) Hausdorff dimension and the length exponent for LQG and LFPP geodesics respectively from our proofs. In fact, it can be shown that the lower bound on the Hausdorff dimension is asymptotic to \( 1 + e^{-C\xi^{-2}} \) as \( \xi \to 0 \) along with an analogous lower bound for the length exponent \( \alpha \). However, we do not expect these bounds to be optimal.
We now briefly describe the organization of this article. In Section 2 we review the axiomatic characterization as well as some basic properties of the LQG metric which we will need in our proof. Section 3 is devoted to proving Proposition 1.4 which comprises several intermediate lemmas including a certain regularity estimate for the harmonic extensions of GFF inside a collection of disks that is \((\lambda, \nu)\)-balanced (Lemma 3.3). Finally, in Section 4 we give the proof of Theorem 1.6.

Our convention regarding constants is the following. Throughout, \(c, c', C, C', \ldots\) denote positive constants that may change from place to place. Numbered constants are defined the first time they appear and remain fixed thereafter. Unless mentioned otherwise, all the constants are assumed to be absolute. Their dependence on other parameters, if any, will always be made explicit.

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2. Definition and some properties of the LQG metric

In this section, we review the definition and some basic properties of the \(\xi\)-LQG metric. Let us start by recalling some basic notions from metric geometry.

Definition 2.1. Let \((X, d)\) be a metric space, with \(d\) allowed to take on infinite values.

- For a path \(P : [a, b] \mapsto X\), the \(d\)-length of \(P\) is defined by
  \[
  \text{len}(P; d) := \sup_{T} \sum_{i=1}^{\#T} d(P(t_i), P(t_{i-1}))
  \]
  where the supremum is over all partitions \(T : a = t_0 < \cdots < t_{\#T} = b\) of \([a, b]\). Note that the \(d\)-length of a path may be infinite.

- We say that \((X, d)\) is a length space if for each \(x, y \in X\) and each \(\varepsilon > 0\), there exists a path of \(d\)-length at most \(d(x, y) + \varepsilon\) from \(x\) to \(y\). If \(d(x, y) < \infty\) a path from \(x\) to \(y\) of \(d\)-length exactly \(d(x, y)\) is called a geodesic.

- For \(Y \subset X\), the internal metric of \(d\) on \(Y\) is defined by
  \[
  d(x, y; Y) := \inf_{P \subset Y} \text{len}(P; d), \forall x, y \in Y
  \]
  where the infimum is over all paths \(P\) in \(Y\) from \(x\) to \(y\). Note that \(d(\cdot, \cdot; Y)\) is a metric on \(Y\), except that it is allowed to take infinite values.

- If \(X \subset \mathbb{C}\) we say that \(d\) is a lower semicontinuous metric if the function \((x, y) \mapsto d(x, y)\) is lower semicontinuous w.r.t. the Euclidean topology. We equip the set of lower semicontinuous metrics on \(X\) with the following topology (the so-called Beer topology [Bee82]) on lower semicontinuous functions on \(X \times X\) and the associated Borel \(\sigma\)-algebra. A sequence of functions \(\{f_n\}_{n \in \mathbb{N}}\) converges in this topology to another function \(f\) if and only if
(i) Whenever \((z_n, w_n) \in X \times X\) with \((z_n, w_n) \to (z, w)\), we have \(f(z, w) \leq \liminf_{n \to \infty} f_n(z_n, w_n)\).

(ii) For each \((z, w) \in X \times X\), there exists a sequence \((z_n, w_n) \to (z, w)\) such that \(f_n(z_n, w_n) \to f(z, w)\).

We would like to emphasize at this point for the sake of clarity that all the paths we consider in the sequel are Euclidean paths, i.e., paths in the Euclidean topology.

We now define the LQG metric with parameter \(\xi > 0\). The following axiomatic characterization of the LQG metric is taken from [DG21] which covers all the phases of LQG. Also see the earlier works [MQ20, GM21b] for closely related formulations. The definition involves an additional parameter \(Q = Q(\xi) > 0\) whose exact functional dependence on \(\xi\) is currently unknown. We refer the reader to [DG21, Section 1.2] and [DDG21, Section 2.3.1] for a detailed discussion on the relationship of this parameter with the coupling constant \(\gamma\) and the matter central charge \(c_M\).

**Definition 2.2.** Let \(\mathcal{D}'\) be the space of distributions (generalized functions) on \(\mathbb{C}\) equipped with the usual weak topology. For \(\xi > 0\), an LQG metric with parameter \(\xi\) is a measurable function \(h \mapsto D_h\) from to the space of lower semicontinuous metrics on \(\mathbb{C}\) with the following properties\(^1\).

Let \(h\) be a GFF plus a continuous function on \(\mathbb{C}\): i.e., \(h\) is a random distribution on \(\mathbb{C}\) which can be coupled with a random continuous function \(f\) in such a way that \(h - f\) has the law of the whole-plane GFF. Then the associated metric \(D_h = D_h^{(\xi)}\) satisfies the following axioms.

I **Length space.** Almost surely, \((\mathbb{C}, D_h)\) is a length space.

II **Locality.** Let \(U \subset \mathbb{C}\) be a deterministic open set. The \(D_h\)-internal metric \(D_h(\cdot, \cdot; U)\) is a.s. given by a measurable function of \(h|_U\) (see, e.g., [DG21, Section 2.2] for a precise meaning of this).

III **Weyl scaling.** For a continuous function \(f: \mathbb{C} \to \mathbb{R}\), define

\[
(e^{\xi f} \cdot D_h)(z, w) := \inf_{P: z \to w} \int_0^{\text{len}(P; D_h)} e^{\xi f(P(t))} dt, \quad \forall z, w \in \mathbb{C}, \tag{2.1}
\]

where the infimum is over all \(D_h\)-rectifiable paths from \(z\) to \(w\) in \(\mathbb{C}\) parametrized by \(D_h\)-length (we use the convention that \(\inf \emptyset = \infty\)). Then a.s. \(e^{\xi f} \cdot D_h = D_{h+f}\) for every continuous function \(f: \mathbb{C} \to \mathbb{R}\).

IV **Affine coordinate change.** There is a specific choice of \(Q = Q(\xi) > 0\) such that for each fixed deterministic \(r > 0\) and \(z \in \mathbb{C}\), a.s.

\[
D_h(ru + z, rv + z) = D_{h(r^{-1}z)} + Q\log r(u, v), \quad \forall u, v \in \mathbb{C}. \tag{2.2}
\]

V **Finiteness.** Let \(U \subset \mathbb{C}\) be a deterministic, open, connected set and let \(K_1, K_2 \subset U\) be disjoint, deterministic, compact, connected sets which are not singletons. Almost surely, \(D_h(K_1, K_2; U) < \infty\).

The following theorem [DG21] (see also [GM21b]) asserts that the LQG metric as defined in Definition 2.2 exists and is unique.

---

\(^1\)The definition of \(D\) on any zero measure subset of \(\mathcal{D}'\) w.r.t. the law of any whole-plane GFF plus a continuous function is inconsequential to us.
Theorem 2.3. For each $\xi > 0$ there exists an LQG metric $D$ with parameter $\xi$ satisfying the axioms of Definition 2.2. This metric is unique in the following sense. If $D$ and $\bar{D}$ are two LQG metrics with parameter $\xi$, then there is a deterministic constant $C > 0$ such that a.s. $\bar{D}_h = CD_h$ whenever $h$ is a whole-plane GFF plus a continuous function.

In view of Theorem 2.3, we can refer to the unique metric satisfying Definition 2.2 as the LQG metric (with parameter $\xi$). To be precise, the metric is unique only up to a global deterministic multiplicative constant. When referring to the LQG metric, we fix the constant in some arbitrary way. For example, we could require that the median distance between the left and right sides of $r_0$, $1$ is $2$ when $h$ is a whole-plane GFF normalized so that its average over the unit circle is zero. It is clear that the geodesics underlying the metric, whenever they exist, remain invariant with respect to the choice of this constant.

Please note that the metric $D_h$ given by Theorem 2.3 implicitly depends on our particular choice for the normalization of $h$. Indeed, it follows from the Weyl scaling (Axiom III) that the metric corresponding to any other choice of normalization (say, we set the average of $h$ to be $0$ on a different circle) is related to $D_h$ by a random positive prefactor.

The LQG metric associated with variants of GFF on other domains (like the zero-boundary GFF on a proper subdomain of $\mathbb{C}$) can be constructed from $D_h$ via restriction and/or local absolute continuity in view of locality (Axiom II) and the Weyl scaling; see [GM21b, Remark 1.5].

Of special interest are two types of $D_h$-distances which we now introduce:

Definition 2.4. For an annulus $A \subset \mathbb{C}$, we define $D_h$ (across $A$) to be the $D_h$-distance between the inner and outer boundaries of $A$. We define $D_h$ (around $A$) to be the infimum of the $D_h$-lengths of paths in $A$ which separate the inner and outer boundaries of $A$.

These two types of (random) distances and the events comparing them play a crucial role in the study of LQG metric (see [DDG21] and the references therein) as they are going to do in our work as well. Notice that these distances are determined by the internal metric of $D_h$ on $A$. It is a.s. the case that $D_h$ (across $A$) and $D_h$ (around $A$) are finite and positive (see [Pfe21] for more explicit tail bounds and also the discussion following [DG21, Lemma 2.6]).

Finally, we come to the question of existence of geodesics. In the supercritical case $\xi > \xi_{\text{crit}}$, the limiting metric in Theorem 2.3 does not induce the Euclidean topology on $\mathbb{C}$. Rather, there exists an uncountable, Euclidean-dense set of singular points $z \in \mathbb{C}$ such that $D_h(z, w) = \infty$ for all $w \in \mathbb{C}\{z\}$ [DG21, (1.8)]. However, for each fixed $z \in \mathbb{C}$, a.s. $z$ is a non-singular point and hence the set of singular points has zero Lebesgue measure [DG21]. On the subspace of non-singular points, we have the following result from [Pfe21, Proposition 1.12].

Proposition 2.5 ([Pfe21]). Almost surely, the metric $D_h$ is complete and finite-valued on $\mathbb{C}\{\text{singular points}\}$. Moreover, every pair of points in $\mathbb{C}\{\text{singular points}\}$ can be joined by a $D_h$-geodesic.

Furthermore, the geodesics connecting two given points (which exist almost surely) is also a.s. unique. In fact, we have the following slightly more general statement from [DG21, Lemma 2.7].

Lemma 2.6 ([DG21]). Let $K_1, K_2 \subset \mathbb{C}$ be deterministic disjoint Euclidean-compact sets. Almost surely, there is a unique $D_h$-geodesic from $K_1$ to $K_2$. 
3. Simultaneous crossings of annuli

In this section we will prove Proposition 1.4. Let us start with the simplest scenario where there is only one disk, i.e., \( n = 1 \) in which case Proposition 1.4 is implied by the following result.

**Lemma 3.1.** For any \( z \in \mathbb{C} \) and \( r > 0 \), let \( F_{z,r} := \{ a \text{ a } D_h \text{-geodesic crosses } A(z, r/2, r) \} \). Then there exists \( c'(\xi) \in (0, 1) \) such that

\[
\mathbb{P}[a \text{ a } D_h \text{-geodesic crosses } A(z, r/2, r)] \leq c'(\xi).
\]

**Proof.** The argument presented below is similar to the one given in the proof of [MQ20, Lemma 4.3] and will be used as a reference for our later arguments. We will bound \( \mathbb{P}[F_{z,r}] \) by invoking a new event \( G_{z,r} \) which involves a comparison between two special types of \( D_h \)-distances. To this end let us define

\[
G_{z,r} := \{ D_h(\text{around } A(z, 3r/4, 7r/8)) < D_h(\text{across } A(z, r/2, 5r/8)) \}.
\]

and observe that \( G_{z,r} \subset F_{z,r} \). Indeed, on the event \( G_{z,r} \), we can reroute any path \( P \) crossing \( A(z, r/2, r) \) through a path separating \( A(z, 3r/4, 7r/8) \) such that the resulting path \( P' \) has (strictly) smaller \( D_h \)-length. In particular no geodesic can cross \( A(z, r/2, r) \) on the event \( G_{z,r} \).

As we now explain, probability of the event \( G_{z,r} \) is independent of \( z \) and \( r \), i.e., \( \mathbb{P}[G_{z,r}] = \mathbb{P}[G_{0,1}] \). To see this notice that due to the Weyl scaling (Axiom III) adding a (random) constant to \( h \) only changes the metric \( D_h \) by a multiplicative constant and consequently the event \( G_{z,r} \) is a.s. determined by \( h - h_r(z) \) where \( h_r(z) \) is the average of \( h \) over the circle \( \partial B(z, r) \) (see [DS11, Section 3.1] for an introduction to the circle average processes). Since the whole-plane GFF satisfies, for any \( z \in \mathbb{C} \) and \( r > 0 \),

\[
h(r \cdot + z) - h_r(z) \overset{\text{law}}{=} h - h_t(0)
\]

(see, e.g. [DDG21, Section 2.2.3]), we can immediately deduce \( \mathbb{P}[G_{z,r}] = \mathbb{P}[G_{0,1}] \).

Hence it suffices to show that \( \mathbb{P}[G_{0,1}] > 0 \). To this end let us recall from the previous section that both \( D_h(\text{around } A(3/4, 7/8)) \) and \( D_h(\text{across } A(1/2, 5/8)) \) are finite, positive random variables and hence there exists \( C = C(\xi) \geq 1 \) such that

\[
\mathbb{P}[D_h(\text{around } A(3/4, 7/8)) < CD_h(\text{across } A(1/2, 5/8))] > \frac{1}{2}
\]

where \( A(r, R) := A(0, r, R) \). Now consider a non-negative, radially symmetric (bump) function \( \phi \in C^\infty_c(\mathbb{C}) \) supported in \( B(0, 3/4) \) which is equal to 1 on \( B(0, 5/8) \). As \( (\mathbb{C}, D_h) \) is a length space (Axiom I), \( D_h(\text{across } A) \) is determined by the internal metric of \( D_h \) on \( A \) and so is \( D_h(\text{around } A) \) by definition. Hence, with \( C \) as in the last display, we get for the metric \( D_{h+\xi^{-1}\log C\phi} \),

\[
\mathbb{P}[D_{h+\xi^{-1}\log C\phi}(\text{around } A(3/4, 7/8)) < D_{h+\xi^{-1}\log C\phi}(\text{across } A(1/2, 5/8))] > \frac{1}{2}
\]

Indeed, by the Weyl scaling, the \( D_{h+\xi^{-1}\log C\phi} \)-internal metric is \( e^{\xi^{-1}\log C} = C \) times the \( D_h \)-internal metric inside \( A(1/2, 5/8) \) whereas it is same as the \( D_h \)-internal metric in \( A(3/4, 7/8) \) which yields the above bound in view of (3.3). However, the laws of \( h \) and \( h + \phi \) are mutually absolutely continuous (see, e.g., [MS16] for a proof) and consequently there exists \( c'(\xi) < 1 \) such that

\[
\mathbb{P}[G_{0,1}] = \mathbb{P}[G_{z,r}] \geq 1 - c'(\xi)
\]

which gives us \( \mathbb{P}[F_{z,r}] \leq c'(\xi) \), i.e., (3.1). 

\( \Box \)
Now we come to main part of the proof where we have to deal with multiple disks. To this end let us consider the collection of disks \( \{A_j\}_{j=1}^n \equiv \{B(z_j, r_j)\}_{j=1}^n \) that is \((\lambda, \nu)\)-balanced for some \( \lambda \geq \lambda_0 > 1 \) and \( \nu > 1 \). In order to obtain a near-geometric decay as in (1.1), we will produce two collections of events \( \{H_{z,j,r,j}\}_{j=1}^n \) and \( \{\tilde{F}_{z,j,r,j}\}_{j=1}^n \) satisfying the following properties for all \( j \in [1, n] \).

**P1** \( F_{z,j,r,j} \subset \tilde{F}_{z,j,r,j} \cup H_{z,j,r,j} \).

**P2** \( \tilde{F}_{z,j,r,j} \) is independent of the \( \sigma \)-algebra generated by \( \{H_{z,j,r,j}\}_{i=1}^n, \{\tilde{F}_{z,i,r,i}\}_{i \neq j} \).

**P3** There exists \( c'(\xi) \in (0, 1) \) such that \( P(\tilde{F}_{z,j,r,j}) \leq c'(\xi) \).

Assuming that two families of events exist satisfying **P1–P3**, we can immediately write with \( S := \{j \in [1, n] : H_{z,j,r,j} \text{ occurs}\} \subset [1, n] \),

\[
P\left[ \bigcap_{j=1}^n F_{z,j,r,j} \right] \leq P\left[ \bigcap_{j \in S} \tilde{F}_{z,j,r,j} \right] \leq \mathbb{E}[c'(\xi)^{|S|}] \tag{3.4}
\]

where in the second step we also used the fact that the random set \( S \) is determined by the events \( \{H_{z,j,r,j}\}_{j=1}^n \). This leads to the bound

\[
P\left[ \bigcap_{j=1}^n F_{z,j,r,j} \right] \leq c'(\xi)^K + P[|S| < K] \tag{3.5}
\]

for any \( K \in [1, n] \) which yields Proposition 1.4 provided we also have

\[
P[|S| < cn] \leq Ce^{-c(\lambda_0) \frac{m\sqrt{n}}{\log n}} \tag{3.6}
\]

(recall \( m \) and also that \( n \geq m\nu/100 \) from Definition 1.3).

Having laid out the basic strategy, we now proceed to defining the events \( \{\tilde{F}_{z,j,r,j}\}_{j=1}^n \) and \( \{H_{z,j,r,j}\}_{j=1}^n \) to which end we will use the **Markov decomposition** of \( h \) as stated below (see, e.g., [She07, PW20, BP21] and [She16, Section 3.2] for a brief review on the whole-plane GFF in particular).

For any \( z \in \mathbb{C} \) and \( r > 0 \), \( h \) can be decomposed as

\[
h = \tilde{h}_{z,r} + h_{z,r}
\]

where \( \tilde{h}_{z,r} \) is a GFF on \( B(z, r) \) with zero boundary condition and \( h_{z,r} \) is a distribution modulo absolute constant that is **harmonic** on \( B(z, r) \) and agrees with \( h \) in \( \mathbb{C} \setminus B(z, r) \). Furthermore, \( \tilde{h}_{z,r} \) is independent of \( \mathcal{F}_{z,r} \) — the \( \sigma \)-algebra generated by \( h|_{\mathbb{C} \setminus B(z, r)} \) (see, e.g., [DG21, (2.2)]) for a precise definition). Note that \( \tilde{h}_{z,r} \) is measurable w.r.t. \( \mathcal{F}_{z,r} \).

At this point we would like to draw the reader’s attention to the fact that \( h \) is treated in this decomposition as a distribution modulo additive constant, i.e., without any particular choice of normalization. In order to use the aforementioned independence between \( \mathcal{F}_{z,r} \) and \( \tilde{h}_{z,r} \) after normalizing \( h \) so that \( h_s(w) = 0 \) for some \( s > 0 \) and \( w \in \mathbb{C} \) (recall that \( s = 1 \) and \( w = 0 \) in our preexisting choice), we would need \( B(z, r) \) to be disjoint from \( \partial B(w, s) \). However, changing the normalization to \( h_s(w) = 0 \) only amounts to subtracting the (random) constant \( h_s(w) \) from...
$h$ and hence, by the Weyl scaling, changes $D_h$ by the (random) factor $e^{-\xi h_0(w)}$. This does not affect the relative distances nor the geodesics and hence the events $F_{z,r}$ and $G_{z,r}$ are invariant w.r.t. the choice of the circle on which we normalize $h$. Therefore, in the rest of the paper we assume that $\partial B(w, s)$ is disjoint from the union of $B(z_j, r_j)$; $j \in [1, n]$ so that we can use the Markov property inside each $B(z_j, r_j)$ with $h$ normalized in this manner.

Now for some $M > 0$ whose precise value would be chosen in Lemma 3.3 below, let us define the events $\tilde{F}_{z,r} = \tilde{F}_{z,r,M}$ and $H_{z,r} = H_{z,r;M}$ as follows:

$$
\tilde{F}_{z,r} := \{ D_{\tilde{h}_{z,r}} (\text{around } A(z, 3r/4, 7r/8)) \geq e^{-\xi M} D_{\tilde{h}_{z,r}} (\text{across } A(z, r/2, 5r/8)) \}, \quad H_{z,r} := \{ \sup_{u,v \in B(z, 7r/8)} | \tilde{h}_{z,r}(u) - \tilde{h}_{z,r}(v) | \geq M \}
$$

(3.7)

where $D_{\tilde{h}_{z,r}}$ is the LQG metric on $B(z, r)$ associated with $\tilde{h}_{z,r}$ (recall the discussion from Section 2). We then have:

**Lemma 3.2.** Consider a collection of disjoint disks $\{ B(z_j, r_j) \}_{j=1}^n$. Then the families of events $\{ \tilde{F}_{z_j, r_j} \}_{j=1}^n$ and $\{ H_{z_j, r_j} \}_{j=1}^n$ defined as in (3.7) satisfy properties P1–P3 with $c'$ depending only on $\xi$ and $M$.

**Proof.** We verify each of the properties P1–P3 below for the collections $\{ \tilde{F}_{z_j, r_j} \}_{j=1}^n$ and $\{ H_{z_j, r_j} \}_{j=1}^n$.

**Property P1.** It immediately follows from (3.7) and the Weyl scaling that $\tilde{F}_{z,r} \cap H_{z,r} \subset F_{z,r}$ (recall the argument previously given for $G_{z,r} \subset F_{z,r}$ in the proof of Lemma 3.1).

**Property P2.** Since $\tilde{F}_{z,r}$ is determined by $\tilde{h}_{z,r}$ and the disks $B(z_j, r_j)$’s are disjoint, we get that the events $\{ H_{z_j, r_j} \}_{j=1}^n, \{ \tilde{F}_{z_j, r_j} \}_{j \neq j}$ are measurable relative to $\tilde{F}_{z_j, r_j}$. The Markov property then gives us the required independence between $\tilde{F}_{z_j, r_j}$ and $(\{ H_{z_i, r_i} \}_{i=1}^n, \{ \tilde{F}_{z_i, r_i} \}_{i \neq j})$ for all $j \in [1, n]$.

**Property P3.** For the upper bound on the probability of $\tilde{F}_{z,r}$, first notice that due to the scale and translation invariance of $h$ (recall (3.2)) as well as the Weyl scaling we have $\mathbb{P}[\tilde{F}_{z,r}] = \mathbb{P}[\tilde{F}_{0,1}]$. It is known from [MQ20, Lemma 4.1] that the laws of $h_{| B(0, 7/8)} - h_1(0)$ and $\tilde{h}_{0,1}|_{B(0, 7/8)}$ are mutually absolutely continuous and hence both $D_{\tilde{h}_{0,1}} (\text{around } A(3/4, 7/8))$ and $D_{\tilde{h}_{0,1}} (\text{across } A(1/2, 5/8))$ are finite and positive random variables (note that they are both determined by the internal metric of $D_{\tilde{h}_{0,1}}$ on $B(0, 7/8)$). Given this fact, we can deduce an upper bound on $\mathbb{P}[\tilde{F}_{0,1}]$ (and hence $\mathbb{P}[\tilde{F}_{z,r}]$) depending only on $\xi$ and $M$ by the exact same argument as that for the upper bound on $\mathbb{P}[G_{0,1}]$. □

Finally, it remains to check whether (3.6) also holds for these events.

**Lemma 3.3.** Suppose $\{ B(z_j, r_j) \}_{j=1}^n$ is $(\lambda, \nu)$-balanced for some $\lambda \geq \lambda_0 > 1$ and $\nu > 1$ and the events $\{ H_{z_j, r_j} \}_{j=1}^n$ are defined by (3.7). Then there exists an absolute constant $M > 0$ such that with $S$ defined as in (3.4), we have

$$
\mathbb{P}[|S| \leq n/2] \leq Ce^{-c(\lambda_0)\frac{m^{\nu}}{\log \nu}}.
$$

(3.8)

Combined with (3.5) and Lemma 3.2, this finishes the proof of Proposition 1.4. The proof of Lemma 3.3 will be given in the next subsection. □
3.1. Proof of Lemma 3.3

Notice that Lemma 3.3 is essentially a statement about the regularity of harmonic extensions. In the case when the sets $A_j$’s concentric annuli, such estimates have been used many times in the LQG literature; see, e.g., [MQ20, Proposition 4.3]. However, our situation is quite different from theirs and necessitates a different approach. One key ingredient in the proof is the following variance estimate.

**Lemma 3.4.** Let $z \in \mathbb{C}$, $r > 0$ and $s \in (0, 1)$. Then there exists $C = C(s) > 0$ such that for any two points $u, v \in B(z, sr)$,

$$
\mathbb{E}[(\hat{h}_{z,r}(u) - \hat{h}_{z,r}(v))^2] \leq C \left( \frac{|u - v|}{r} \right)^2.
$$

(3.9)

Estimates of this flavor are prevalent in the literature; see, e.g., [BDZ16, Lemma 3.10] for the analogous result in the context of a discrete GFF on a box and also [PW20, Section 4.1] which gives a weaker bound for the zero-boundary GFF on a proper subdomain of $\mathbb{C}$. Using it we can now finish the proof.

**Proof of Lemma 3.3.** Let $\Delta_{z,r}(u, v)$ denote the difference $\hat{h}_{z,r}(u) - \hat{h}_{z,r}(v)$. In view of the variance bound (3.9), it follows from the Fernique’s inequality [Fer75] (see also [BDZ16, Lemma 3.5]) that

$$
\mathbb{E} \left[ \sup_{u, v \in B(z, 7r/8)} \Delta_{z,r}(u, v) \right] \leq C.
$$

(3.10)

This implies in particular,

$$
\mu := \mathbb{E} \left[ \sup_{(u_j, v_j) \in B(z, 7r_j/8)} \sum_j \Delta_{z_j, r_j}(u_j, v_j) \right] = \mathbb{E} \left[ \sum_j \sup_{(u_j, v_j) \in B(z, 7r_j/8)} \Delta_{z_j, r_j}(u_j, v_j) \right] \leq C \log n \leq C' m \nu
$$

(3.11)

where in the final step we used the bound $n \leq 100 m \nu$ as a consequence of the definition of $(\lambda, \nu)$-balanced sets. Now suppose that we also have, for any choice of pairs of points $(u_j, v_j) \in B(z_j, 7r_j/8)$,

$$
v := \text{Var} \left[ \sum_j \Delta_{z_j, r_j}(u_j, v_j) \right] \leq C(\lambda_0) m \nu \log \nu.
$$

(3.12)

Then, using the Borell-Tsirelson inequality [Bor75, SC74] (see, e.g., [AT07, Theorem 2.1.1]) we get

$$
\mathbb{P} \left[ \sup_{(u_j, v_j) \in B(z_j, 7r_j/8)} \Delta_{z_j, r_j}(u_j, v_j) \geq \mu + k \right] \leq e^{-k^2/2v} \leq e^{-c(\lambda_0) \frac{k^2}{m \nu \log \nu}}
$$

(3.13)

for all $k \geq 0$. Setting $M = 4C$ where $C$ is from (3.11), we can deduce from this:

$$
\mathbb{P} \left[ |S| \leq n/2 \right] = \mathbb{P} \left[ \# \left\{ j : \sup_{(u_j, v_j) \in B(z_j, 7r_j/8)} \sum_j \Delta_{z_j, r_j}(u_j, v_j) \geq 4C \right\} > n/2 \right]
$$

\[ \leq \mathbb{P} \left[ \sum_j \sup_{(u_j, v_j) \in B(z_j, 7r_j/8)} \Delta_{z_j, r_j}(u_j, v_j) \geq 2\mu + Cn \right] \leq e^{-c(\lambda_0) \frac{k^2}{m \nu \log \nu}} \leq e^{-c(\lambda_0) \frac{m \nu}{\log \nu}}
$$

where in the last step used the lower bound $n \geq m \nu/100$ as implied by Definition 1.3.
Let us now verify the bound in (3.12) to which end we start by expanding

\[
\text{Var}\left[ \sum_j \Delta_{z_j,r_j}(u_j, v_j) \right]
\leq \sum_j \text{Var}\left[ \Delta_{z_j,r_j}(u_j, v_j) \right] + 2 \sum_{i=1}^m \sum_{j \in S_i} \sum_{\ell \geq i} \sum_{k \in S_k, k \neq j} |\text{Cov}\left[ \Delta_{z_j,r_j}(u_j, v_j), \Delta_{z_k,r_k}(u_k, v_k) \right]| \tag{3.14}
\]

where \( S_i := \{ j : 2r_j \in [L_0 \lambda^{-k_j}/100, 100 L_0 \lambda^{-k_j}] \} \) (recall Definition 1.3). It follows from (3.9) that

\[
\text{Var}\left[ \Delta_{z_j,r_j}(u_j, v_j) \right] \leq C. \tag{3.15}
\]

For the covariance terms first observe that due to the well-separatedness of \( B(z_j, r_j)'s \), we have \( 2d_{j,k} := |z_j - z_k| \geq 2(r_j \lor r_k) \) whenever \( j \neq k \) and hence the two disjoint disks \( B(z_j, d_{j,k}) \) and \( B(z_k, d_{j,k}) \) contain \( B(z_j, r_j) \) and \( B(z_k, r_k) \) respectively. Therefore we can write

\[
\Delta_{z_j,r_j}(u_j, v_j) = \tilde{h}_{z_j,d_{j,k}}(u_j) - \tilde{h}_{z_j,d_{j,k}}(v_j) = (\tilde{h}_{z_j,d_{j,k}}(u_j) - \tilde{h}_{z_j,d_{j,k}}(v_j)) + (\tilde{h}_{z_j,d_{j,k}}(u_j) - \tilde{h}_{z_j,d_{j,k}}(v_j))
\]

where \( \tilde{h}_{z_j,d_{j,k}}(w) := \frac{1}{\partial B(z_j, r_j)} \mathbb{P}_{z_j,r_j}(w, y) \sigma_{z_j,r_j}(dy) \) is the harmonic average of the field \( h_{z_j,d_{j,k}} \) at \( w \) w.r.t. the circle \( \partial B(z_j, r_j) \) with \( \mathbb{P}_{z_j,r_j}(\cdot) \) being the Poisson kernel for the disk \( B(z, r) \) and \( \sigma_{z,r}(\cdot) \) being the uniform distribution on the circle \( \partial B(z, r) \). Similar decomposition holds for \( \Delta_{z_k,r_k}(u_k, v_k) \) as well. These averages are well-defined just as the circle averages by the continuity of Poisson kernel. Now notice that the random variable \( \Delta_{z,r}(u, v) \) is a difference of two averages involving \( h \) and hence is well-defined without any normalization, see, e.g., [She16, Section 3.2]). Hence, by the independence between \( \tilde{h}_{z,r} \) and \( \mathcal{F}_{z,r} \), we get in view of this decomposition

\[
\text{Cov}\left[ \Delta_{z_j,r_j}(u_j, v_j), \Delta_{z_k,r_k}(u_k, v_k) \right] = \text{Cov}\left[ \tilde{h}_{z_j,d_{j,k}}(u_j) - \tilde{h}_{z_j,d_{j,k}}(v_j), \tilde{h}_{z_j,d_{j,k}}(u_k) - \tilde{h}_{z_j,d_{j,k}}(v_k) \right].
\]

Bounding this by the Cauchy-Schwarz inequality and subsequently the resulting variance terms by Lemma 3.4 yields us

\[
\left| \text{Cov}\left[ \Delta_{z_j,r_j}(u_j, v_j), \Delta_{z_k,r_k}(u_k, v_k) \right] \right| \leq C \frac{r_j r_k}{d_{j,k}^2} \tag{3.16}
\]

We now go back to (3.14) and consider, for some \( j \in S_i \) and \( \ell \geq i \), the sum of the (absolute) covariance terms

\[
\sum_{k \in S_k} |\text{Cov}\left[ \Delta_{z_j,r_j}(u_j, v_j), \Delta_{z_k,r_k}(u_k, v_k) \right]| \leq \sum_{t \in \mathbb{N}_{>0}} \sum_{k \in T_{i,j,\ell}} |\text{Cov}\left[ \Delta_{z_j,r_j}(u_j, v_j), \Delta_{z_k,r_k}(u_k, v_k) \right]|
\]

where \( T_{i,j,\ell} := \{ k \in S_k : A(z_j, r_j + (t - 1)L_0 \lambda^{-k_j}, r_j + tL_0 \lambda^{-k_j}) \cap B(z_k, r_k) \neq \emptyset \} \). Since the disks \( B(z_k, r_k)'s \) are disjoint; and \( j \in S_i \) and \( k \in S_k \); it follows that

\[
|T_{i,j,\ell}| \leq C \frac{(r_j + tL_0 \lambda^{-k_j})L_0 \lambda^{-k_j}}{L_0^2 \lambda^{-2k_j}} \leq C \frac{\lambda^{k_j-k_i}+t}{\lambda^{k_i-k_j}}. \tag{3.17}
\]

As to \( \text{Cov}\left[ \Delta_{z_j,r_j}(u_j, v_j), \Delta_{z_k,r_k}(u_k, v_k) \right] \), using the same bounds on \( r_j \) and \( r_k \), we get from (3.16)

\[
\max_{k \in T_{i,j,\ell}} |\text{Cov}\left[ \Delta_{z_j,r_j}(u_j, v_j), \Delta_{z_k,r_k}(u_k, v_k) \right]| \leq C \frac{\lambda^{k_j-k_i}}{(\lambda^{k_i-k_j}+t)^2},
\]
Since this bound is decreasing in $t$ and $|S_t| \leq 100 \nu$ by Definition 1.3, we can write in view of (3.17),

$$
\sum_{t \in \mathbb{N}_{>0}} \sum_{k \in T_{t,j}, t} \left| \text{Cov}[\Delta_{x_j, r_j}(u_j, v_j), \Delta_{z_k, r_k}(u_k, v_k)] \right| \leq C \lambda^{k_t - k_i} t \nu \leq C \lambda^{k_t - k_i} \frac{t \nu}{\lambda^{k_t - k_i}} \leq C \lambda^{k_t - k_i} \log \frac{\lambda^{k_t - k_i} + t \nu}{\lambda^{k_t - k_i}}
$$

(3.18)

where $t \nu$ is the smallest integer satisfying $C_1 \sum_{d=1}^{t \nu} (\lambda^{k_t - k_i} + t) \leq 100 \nu$ and hence we have

$$
\text{Cov}[\Delta_{x_j, r_j}(u_j, v_j), \Delta_{z_k, r_k}(u_k, v_k)] \leq C \nu \lambda^{-(k_t - k_i)}. \tag{3.19}
$$

Let us further analyze the bound in (3.18). It is clear from the definition of $t \nu$ that $C_1 \sum_{d=1}^{t \nu} (\lambda^{k_t - k_i} + t) \leq 200 \nu$ and hence $\lambda^{k_t - k_i} t \nu \leq C \nu$. Consequently, we obtain from (3.18):

$$
\sum_{t \in \mathbb{N}_{>0}} \sum_{k \in T_{t,j}, t} \left| \text{Cov}[\Delta_{x_j, r_j}(u_j, v_j), \Delta_{z_k, r_k}(u_k, v_k)] \right| \leq C \lambda^{k_t - k_i} \log(1 + C \nu \lambda^{-(k_t - k_i)}) \leq C \nu \lambda^{-(k_t - k_i)} \tag{3.20}
$$

where in the final step we bounded $\log(1 + x)$ by $\lambda$. Notice that this bound is good as soon as $\nu \lambda^{-(k_t - k_i)} \leq 1$, i.e., $\nu \leq \lambda^{2(k_t - k_i)}$ and we can bound the log term by $\log \nu \lambda^{(k_t - k_i)}$ otherwise. Using these bounds in different `regimes" based on the value of $k_t$, we get (we suppress the constant prefactor $C$ by using "$\leq"$ instead of "$\leq"" in all steps)

$$
\sum_{t \geq 1} \sum_{k \in T_{t}} \left| \text{Cov}[\Delta_{x_j, r_j}(u_j, v_j), \Delta_{z_k, r_k}(u_k, v_k)] \right| \leq \sum_{\nu \leq \lambda^{2(k_t - k_i)}} \lambda^{k_t - k_i} \log \nu \lambda^{(k_t - k_i)} + \nu \lambda^{-(k_t - k_i)} \leq C(\lambda_0) \sqrt{\nu} \log \nu \lambda^{(k_t - k_i)}
$$

(recall that $\lambda \geq \lambda_0 > 1$). Plugging this as well as (3.15) into (3.14), we can conclude (3.11) in view of the bound $|S_t| \leq \lfloor \nu, C \nu \rfloor$ as implied by Definition 1.3. ∎

Finally we give

**Proof of Lemma 3.4.** Due to the scale and translation invariance of $h$ (see (3.2)), it suffices to prove the estimate for $z = 0$ and $r = 1$ which will drop from all the notations below. Since $\hat{h}(\cdot)$ is harmonic in $B(0, 1)$, we can write in the notations we introduced in the previous proof (see the discussion after (3.15)):

$$
\hat{h}(u) - \hat{h}(v) = \int_{\partial B(0, 1)} h(y) (\mathcal{P}(u, y) - \mathcal{P}(v, y)) \sigma(dy).
$$

As already noted, this average is well-defined even without any normalization and hence we have the following expression from the defining properties of the whole-plane GFF (see, e.g. [She16, display (3.4)]):

$$
E[(\hat{h}(u) - \hat{h}(v))^2] = \int_{\partial B(0, 1) \times \partial B(0, 1)} (\mathcal{P}(u, x) - \mathcal{P}(v, x)) G(x, y) (\mathcal{P}(u, y) - \mathcal{P}(v, y)) \sigma(dx) \sigma(dy)
$$

where $G(x, y) = -\log |x - y|$. From this we can immediately deduce (3.9) owing to the continuity of the Poisson kernel on $B(0, 1) \times \partial B(0, 1)$ (see, e.g., [MP10, Theorem 3.44]). ∎
4. Length of LFPP geodesics

In this section we will prove Theorem 1.6. Like in the case of Theorem 1.1, the principal tool is the following analogue of Proposition 1.4.

**Proposition 1.4'**. For any \( \lambda_0 > 1 \) and \( \xi > 0 \), there exist \( \rho = \rho(\lambda_0, \xi), \varepsilon_0 = \varepsilon_0(\xi) \in (0,1) \) and a positive absolute constant \( C \) such that the following holds. For any collection of disks \( \{ A_j \}_{j=1}^n \) that is \( (\lambda, \nu) \)-balanced for some \( \lambda \geq \lambda_0 \) and \( \nu > 1 \) and has maximum and minimum radius in \([32\varepsilon^{1/2}, 1]\), one has

\[
P[ \text{a } \hat{D}_k^\text{f}-\text{geodesic crosses } A_{j,0} \text{ for all } j ] \leq C\rho^{\log c\varepsilon}
\]  

(4.1)

whenever \( \varepsilon \in (0, \varepsilon_0) \).

In order to deduce Theorem 1.6 from Proposition 1.4', we will need a “finite-length” analogue of Theorem 1.5 which we now state. Let us recall the notion of straight runs and \( (\lambda, k_0) \)-sparsity from Section 1.2. We call the straight runs of a path \( P \) as \( (\lambda, k_0) \)-sparse down to scale \( \delta \) if \( P \) does not exhibit any nested collection of straight runs on a sequence of scales \( L_{k_1} > \cdots > L_{k_m} \) with \( L_k := L_0 \lambda^{-k} \), \( L_{k_m} \geq \delta \) and \( m \geq \frac{1}{4} \max\{k_m, k_0\} = \frac{1}{4}k_m \).

**Lemma 4.2**. Let the distance between the endpoints of a given path \( P \) be at least \( L_0 \). Also suppose that its straight runs are \( (\lambda, k_0) \)-sparse down to scale \( \delta \) for some \( \delta \in (0, L_0) \) and \( k_0 > 0 \). Then there exist \( s = s(\lambda) > 0 \) and \( c = c(\lambda, L_0) > 0 \) such that the Euclidean length of \( P \) is at least \( c\delta^{-s} \).

**Proof**. Choose \( p \in [\lambda/2, \lambda] \) such that \( \beta := \sqrt{p(p+1)} \) is strictly larger than \( \lambda \). Now using the same algorithmic construction as given in the proof of [AB99, Lemma 5.2], we get a nested sequence \( \Gamma_0, \ldots, \Gamma_{k_{\max}} \) of collections of segments of \( P \) such that:

- each \( \Gamma_k \) is a collection of segments of diameter at least \( L_k \);
- in each generation (as defined by \( k \)), distinct segments are at distances at least \( \varepsilon L_k \) with \( \varepsilon = (\lambda/p) - 1 \);
- each segment of \( \Gamma_k (k > 1) \) is contained in one of the segments of \( \Gamma_{k-1} \), with the number of immediate descendants thus contained in a given element of \( \Gamma_{k-1} \) at least \( p \) and is at least \( \beta^k \) unless it exhibits a straight run at the scale \( L_k \)

(here \( k_{\max} := \max\{k : L_k \geq \delta\} \)). Now, for points \( x \in \cup_{\eta \in \Gamma_k} \eta \), define \( n_k(x) \) to be the number of immediate descendants of the set containing \( x \) within \( \Gamma_{k-1} \) [AB99, Lemma 5.4]. Then, using the identity [AB99, display (5.19)], we can write

\[
1 = \sum_{\eta \in \Gamma_{k_{\max}}} \prod_{j=1}^{k_{\max}} n_j(\eta)^{-1} \leq |\Gamma_{k_{\max}}| \cdot \left( \min_{\eta \in \Gamma_{k_{\max}}} \prod_{j=1}^{k_{\max}} n_j(\eta) \right)^{-1}
\]

where the number \( n_j(\eta) \) is the constant value that \( n_j(x) \) takes for \( x \in \eta \). However, since the straight runs of \( P \) are \( (\lambda, k_0) \)-sparse down to scale \( \delta \), it follows from the aforementioned items that \( \prod_{j=1}^{k_{\max}} n_j(\eta) \geq \beta^{k_{\max}} \) (see also the proof of Theorem 5.1 in [AB99]). Therefore, the previous display gives us \( |\Gamma_{k_{\max}}| \geq \beta^{k_{\max}} \). From this we can deduce,

\[
|P| \geq |\Gamma_{k_{\max}}| \cdot L_{k_{\max}} \geq \left( \frac{\beta}{\lambda} \right)^{k_{\max}} L_0 = c(\lambda, p, L_0)\delta^{-s}
\]

with \( s = c \log(\beta/\lambda)/\log(\lambda) > 0 \) for some suitable absolute constant \( c > 0 \) upon noting that \( L_{k_{\max}} < \delta \).
We can now finish the

**Proof of Theorem 1.6.** The proof follows in essentially the same fashion from Proposition 1.4′ and Lemma 4.2 as did the proof of Theorem 1.1 from Proposition 1.4 and Theorem 1.5 by controlling the probability of the sparsity of straight runs in LFPP geodesics down to scale $32\varepsilon^{-1/2}$.

Finally, it remains to give the:

**Proof of Proposition 1.4′.** Let us consider the following redefinition of the events in (3.7) for some $M > 0$ which will be later set to an absolute constant:

$$
\hat{F}_{z,r} := \left\{ \hat{D}_{h_{z,r}}^\varepsilon (\text{around } A(z, 3r/4, 7r/8)) \geq e^{-\xi M} \hat{D}_{h_{z,r}}^\varepsilon (\text{across } A(z, r/2, 5r/8)) \right\}, \text{ and}
$$

$$
H_{z,r} := \left\{ \sup_{u,v \in B(z,15r/16)} |\hat{h}_{z,r}(u) - \hat{h}_{z,r}(v)| \geq M \right\}
$$

where $\hat{D}_{h_{z,r}}^\varepsilon$ is the LFPP metric constructed from $\hat{h}_{z,r}$ in the same way as $\hat{D}_h^\varepsilon$ was constructed from $h$ (recall (1.6)). We will follow this convention of denoting LFPP metrics in the sequel. Similarly we redefine the event $F_{z,r}$ from the statement of Lemma 3.1 (see also P1) to involve $\hat{D}_h^\varepsilon$-geodesics instead of $D_h$-geodesics. In view of (3.5), Lemma 3.2 and Lemma 3.3 (with $H_{z,r}$ redefined as above for which the proof given in the previous section works similarly for a possibly new choice of the absolute constant $M$), it suffices to verify the properties P1–P3 for $\{\hat{F}_{z,r,j}\}_{j=1}^n$ and $\{H_{z,r,j}\}_{j=1}^n$ which we do in the remainder of the proof. A crucial observation to make here — which will be used repeatedly in the sequel often without any explicit reference — is that the random variable $\hat{h}_z^*(z)$ is a radially symmetric integral of (and therefore is measurable relative to) $h|_{B(z,\varepsilon^{1/2})}$ for all $z \in \mathbb{C}$. This is the main reason why we chose to work with $\hat{h}_z^*$ instead of $h_z^*$.

**Property P1.** This follows from the same reasoning as given in the proof of Lemma 3.2 for the same property except that in place of Weyl scaling, we can directly use the definition of the metric as given by (1.4) (with $\hat{h}_z^*$ in place of $h_z^*$). For this last part of the argument, it is crucial that $F_{z,r}$ be measurable relative to $h|_{B(z,15r/16)}$ which is true if $r \geq 32\varepsilon^{-1/2}$.

**Property P2.** Again the same argument as in the proof of Lemma 3.2 works in view of the lower bound on the minimum radius of the disks.

**Property P3.** This part requires a delicate argument since, unlike in the case of Lemma 3.2, we do not have $\mathbb{P}[\hat{F}_{z,r}] = \mathbb{P}[\hat{F}_{0,1}]$. The way to get around this is to use the tightness of the (re-scaled) LFPP metrics [DG20b]. However, to the best of our knowledge, there is no “readily available” tightness results for $\alpha_\varepsilon^{-1} \hat{D}_{h_{z,r}}^\varepsilon (\text{across/around } A(z, \cdot, \cdot)) (\varepsilon \in (0,1))$ in particular (see Section 1.3 for $\alpha_\varepsilon$) and some work is necessary to transfer the tightness bounds from $\alpha_\varepsilon^{-1} \hat{D}_h^\varepsilon (\text{across/around } A)$ to these random variables. Consequently, in the course of our proof, we will switch between and compare events similar to $\hat{F}_{z,r}$ defined for several LFPP metrics and hence it would be convenient to have a generic notation for such events. To this end let us define, for any random, intrinsic metric $D$ defined on $B(z,r)$ and prefactor $C > 0$,

$$
\hat{F}_{z,r}(D, C) := \{ D(\text{around } A(z, 3r/4, 7r/8)) \geq C D(\text{across } A(z, r/2, 5r/8)) \}
$$

(cf. the definition of $\hat{F}_{z,r}$ in (4.2)).
Now suppose for the moment being that we also have,

\[ \mathbb{P}\left[ \tilde{F}_{z,r}\left( \tilde{D}^\xi_{h_{z,r}}, C_2(\xi) \right) \right] \leq \frac{1}{2} \]  

for all \( r \in (32\varepsilon^{1/2}, 1), \varepsilon \in (0, c(\xi)) \) and some \( C_2(\xi) > 1 \). In order to verify \( \textbf{P3} \), we need to obtain a non-trivial upper bound (depending only on \( \xi \)) for the probability of \( \tilde{F}_{z,r}\left( \tilde{D}^\xi_{h_{z,r}}, e^{-\xi M} \right) \). To this end, let us consider a non-negative, radially symmetric (bump) function \( \phi \in C^\infty_c(\mathcal{C}) \) supported in \( B(0, 23/32) \) which is identically equal to 1 on \( B(0, 21/32) \). Then using similar argument as that for the upper bound on \( \mathbb{P}[G_{0,1}] \) in the proof of Lemma 3.1 and the fact that \( \tilde{h}^*_\xi(z) \) is determined by \( h|_{B(z, e^{t/2})} \) for any distribution \( h \) (provided all the distributional pairings are well-defined), we get a constant \( C_3 = C_3(C_2, M) > 0 \) such that

\[ \tilde{F}_{z,r}\left( \tilde{D}^\xi_{h_{z,r}}, e^{-\xi M} \right) = \tilde{F}_{z,r}\left( \tilde{D}^\xi_{h_{z,r}+C_3 \phi(r^{-1}\cdot - z)}, C_2 \right) \]  

whenever \( r \geq 32\varepsilon^{1/2} \). Denoting by \( \mathbb{P} \) and \( \tilde{\mathbb{P}} \) the laws of \( \tilde{h}_{z,r} \) and \( \tilde{h}_{z,r}+C_3 \phi(r^{-1}\cdot - z) \) respectively, let us recall the following fact which is a consequence of Jensen’s inequality, see e.g. the discussion following (2.7) in [BDZ95] for a proof. For any event \( A \) with positive \( \tilde{\mathbb{P}} \)-probability, one has

\[ \mathbb{P}[A] \geq \tilde{\mathbb{P}}[A]e^{-(1/\mathbb{P}[\tilde{A}])H(\tilde{\mathbb{P}}|\mathbb{P})+1/\varepsilon). \]  

where \( H(\tilde{\mathbb{P}}|\mathbb{P}) := \tilde{\mathbb{E}}\left[ \log \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \right] \) is the relative entropy of \( \tilde{\mathbb{P}} \) with respect to \( \mathbb{P} \). The Radon-Nikodym derivative \( \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \), in this case, is given by the formula (see, e.g. [MS16, Proposition 3.4])

\[ \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left( (h, \phi_{z,r})\nabla - \frac{1}{2}(\phi_{z,r}, \phi_{z,r})\nabla \right) \]

where \( (f, g)\nabla := \int_C \nabla f \cdot \nabla g \, d^2z \) is the Dirichlet inner product and \( \phi_{z,r}(\cdot) := C_3 \phi(r^{-1}\cdot - z) \).

From this we can now compute the relative entropy in terms of \( (\phi_{z,r}, \phi_{z,r})\nabla \) as follows:

\[ H(\tilde{\mathbb{P}}|\mathbb{P}) = \mathbb{E}[(h, \phi_{z,r})\nabla - \frac{1}{2}(\phi_{z,r}, \phi_{z,r})\nabla] = \mathbb{E}[(h + \phi_{z,r}, \phi_{z,r})\nabla - \frac{1}{2}(\phi_{z,r}, \phi_{z,r})\nabla] = \frac{1}{2}(\phi_{z,r}, \phi_{z,r})\nabla \]

(this is a special case of the famous Cameron-Martin theorem, see, e.g. [Jan97, Theorem 14.1]). By standard change of variable one gets that \( (\phi_{z,r}, \phi_{z,r})\nabla = C_3(\phi, \phi)\nabla \) and hence applying (4.5) to the event \( \tilde{F}_{z,r}(\tilde{D}^\xi_{h}, C_2)^c \), we obtain from (4.3) and (4.4)

\[ \mathbb{P}\left[ \tilde{F}_{z,r}\left( \tilde{D}^\xi_{h_{z,r}}, e^{-\xi M} \right) \right] \leq c(\xi) < 1 \]

which yields \( \textbf{P3} \).

Let us now show (4.3) for which we again start with an intermediate statement. So suppose that

\[ \mathbb{P}\left[ \tilde{F}_{z,r}\left( \tilde{D}^\xi_{h}, C_4(\xi) \right) \right] \leq \frac{1}{4} \]  

for some \( C_4(\xi) > 0 \), all \( r \in (\varepsilon, 1) \) and \( \varepsilon \in (0, c(\xi)) \). From this, (4.3) immediately follows by controlling the fluctuation of the harmonic field \( h_{z,r} \) on \( B(z, 15r/16) \) using Lemma 3.4, (3.10) and the Borell-Tsierlson inequality (see (3.13)). So we focus on (4.6) in the remaining part of the proof.
Continuing in the same vein, we first note that
\[ P[F_{z,r}(D^r_h, C_5(\xi))] \leq \frac{1}{8} \] (4.7)
for some $C_5(\xi) > 0$ and all $r > \varepsilon$ as the the random variables
\[ a_\delta^{-1} D^\delta_h(\text{around } A(3/4, 7/8)) \text{ and } (a_\delta^{-1} D^\delta_h(\text{across } A(1/2, 5/8)))^{-1} \]
are tight for $\delta \in (0, 1)$ [DG20b, Proposition 4.1]. Let us explain the connection between these two in more detail. It follows from the definition of $D^r_h$ that
\[ D^r_h(rz, rw) = r D^{r/r}_h(z, w), \quad \forall z, w \in \mathbb{C} \]
([DFG+20, Lemma 2.6]). Combined with the scale and translation invariance of $h$ (3.2), this yields (4.7) in view of the above-mentioned tightness of $D^\delta_h$ over $\delta \in (0, 1)$ (see also [Pfe21, Definition 1.6 – Axiom V]).

Let us now finish the proof of (4.6) which follows immediately from (4.7) and the following (local) uniform comparison result proved in [DFG+20, Lemma 2.1]:

Almost surely, $\lim_{\delta \to 0} \frac{\Delta h(z,w;B(0,1))}{D^\delta_h(z,w;B(0,1))} = 1$ uniformly over all $z, w \in B(0,1) (z \neq w)$. \qed

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