Truncation of Einstein equations through Gravitational Foliation

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Abstract

In previous works we suggested to consider a \((3 + 1)D\) quantum gravitational field as an “evolution” of a \((2+1)D\) renormalized quantum gravitational field along the direction of the gravitational force. The starting point of the suggestion is derivation of a unique hypersurface which looks effectively like \((2 + 1)D\) from the point of view of Einstein equations in \((3 + 1)D\). In this paper we derive such unique hypersurfaces for different kinds of static spherical metrics. We find that these hypersurfaces exist whenever all the components of the gravitational force field vanish on the hypersurface. We discuss the implication of this result and the necessary further work.

1 Introduction

The conventional attempts to quantize the gravitational theory by means of Hamiltonian and ADM formalism lead to a non renormalizable theory and to the problem of time. Rather than giving up the powerful Hamiltonian formalism when general relativity theories are concerned, we suggested \[1\] to use this formalism differently. We suggested to consider the symmetry breaking caused by a gravitational force field, and to use the force field direction as an independent parameter through which states evolve. This

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mean that instead of singling out the direction of a time vector field in
the Hamiltonian formalism, we single out the direction of the gravitational
force.\footnote{We understand that the term “gravitational force field” is confusing since in gravity
we rarely talk about the “gravitational force”. Here the term “gravitational force field” is
used simply to refer to the acceleration field of a family of observers which do not change
their spatial coordinates in a given coordinate system.}

Though this suggestion is supported by several works \cite{2, 3, 4, 5, 6, 7},
which will be reviewed in the next section, its effect on causality is unclear.
To see this note that the direction of the foliation, which is along the grav-
itational force field gives non-causal brackets since the foliation is actually
space-like directed and not time-like directed. Recently we proved \cite{8} that
under some conditions, one can develop a causal quantum theory using space-
like directed foliation and it turns out that these conditions are useful from
the quantum gravity point of view. However, these conditions are useful only
if one can find, for a given metric background, a kind of unique holographic
hypersurface which looks effectively as a \((2+1)D\) from the point of view of
Einstein equations in \((3+1)D\).

The purpose of this paper is to find specific examples for such a unique
hypersurface in static spherically symmetric metrics. To begin with we ob-
tain the gravitational force field direction by considering the acceleration
vector field of static observers in different kinds of spherically symmetric
metrics. Next, by using the ADM formalism, we foliate spacetime along this
direction and find the conditions needed for hypersurfaces to appear effect-
ively as \((2+1)D\) from the point of view of the Einstein equations. We
found that these conditions are fulfilled whenever all the components of the
acceleration vector field vanish on the hypersurface.

Before discussing the implications of this result, let us first see why
this unique hypersurface is expected to be helpful for constructing a causal
\((3+1)D\) quantum gravity theory, even though its construction involves "evolu-
tion" along a spatial direction. As was shown in \cite{8}, Poisson brackets be-
come non-causal when foliating along a space like directed vector field. Thus
the only way to obtain the required causal classical field brackets for this
kind of foliation is by finding the fields’ commutation relations in some other way. In [8] they were obtained from the given theory on the hypersurface and taking them to their classical limit. Moreover, since the hypersurface looks effectively a (2+1)D from the point of view of the Einstein equations, a renormalized quantum gravity theory can indeed be constructed on it. This construction leads to a causal quantum gravitational theory in (3+1)D, even though it "evolves" along a spatial direction.

Now we examine the advantages of this construction and the significance of our result.

The advantage of constructing a gravitational theory using our unique hyperspace is obvious. The unique hyperspace enables derivation of quantum gravitational fields which are already renormalized on the hypersurface. Whether the evolution of these specific fields "out from the hypersurface" along the gravitational force direction preserves their property of "being renormalizable" remains to be seen. But the fact that the structure is based on a renormalized quantum gravity looks promising and may lead the way to construction of a (3+1)D renormalized quantum theory.

Moreover, this contraction leads to a very interesting and important result. It relates the acceleration of static observers in a given coordinate system to the non-renormalizability of the quantum gravitational theory. To see this connection note that our findings suggest that when all the components of the acceleration vector field vanish, construction of a renormalized gravitational theory is possible. In other words, this relates the non-renormalizability property of the gravitational theory to the existence of acceleration in a curved spacetime. This relationship is predicted in [2]. We expand on this subject in our conclusions.

Note that usually the Hamiltonian method is typically used to achieve a background-independent nonperturbative quantization. In our analysis, one uses the Hamiltonian method in order to limit the (3+1)D gravitational theory to an affective (2+1)D on a specific fixed background. Whereas the quantization of the affective (2+1)D gravitational theory can be done in various methods, we expect that in order to obtain a quantized (3+1)D gravitational theory, i.e. an extension to the forth direction, one should
consider fluctuations around the background. Thus although we use the Hamiltonian method, our analysis will not be a background-independent nonperturbative quantization.

The rest of the paper is organized as follows: Section 2 defines the gravitational foliation force and discuss its implications. In section 2.1 we define and perform the gravitational foliation. In subsection 2.2 we discuss quantum gravity properties supporting foliation along the gravitational force. In subsection 2.3 we deal with the expected ambiguity regarding causality and its solution in the context of the gravitational theory. In section 2.4 we derive the conditions for hypersurfaces that appear effectively as \((2 + 1)D\) from the point of view of the Einstein equations. In Section 3 we find these hypersurfaces in different static spherical metrics. In section 3.1 we consider the extremal black hole and show that although its horizon fulfills all the necessary conditions, it can not be a proper candidate for the hypersurface to appear effectively as a \((2 + 1)D\) from the point of view of the Einstein equation. In section 3.2 we take a toy model and derive the condition in order for this hypersurface to exist. In section 3.3 we find the condition for this case for a general static spherical symmetric metric, and show that it is fulfilled whenever all the components of the acceleration vector field vanish on the hypersurface. In section 4 we discuss the implications of our findings as well as possible future work. Section 5 is a summary. Finally, in the appendix, we derive the Einstein equation when singling out a space like vector field direction instead of time like vector field.

2 Quantum gravity and gravitational foliation

Obtaining a gravitational theory from microscopic objects or quantum fields is an extremely important and challenging aim. Attempts to describe gravity using microscopic objects, such as strings, loops or triangles, have not yet led us to the desired Einstein equations. Attempts to treat the gravitational metric as simply another quantum field are problematic. Use of the Hamiltonian formalism in order to quantize the gravitational theory leads to a non renormalizable theory since it involves spin-2 massless fields and this kind of
theory is not renormalizable in more than $2 + 1$ dimensions.

Instead of giving up the powerful Hamiltonian formalism where general relativity theories are concerned, on the one hand, and in attempt to use the renormalized gravitational theories on the $2 + 1$ dimensions, on the other hand, we suggested [1] a different use of the Hamiltonian formalism. In our approach, we consider the symmetry breaking caused by the gravitational force field for static observers in a given coordinate system and use the direction of that field as the direction through which states “evolve.”

This approach is supported by several examples which relate the gravitational foliation to different aspects of quantum gravity, on the one hand, but is expected to be problematic since the direction of any force field is spacelike, and not time-like and thus leads to ambiguity regarding causality.

We begin by foliating along the gravitational force direction and use this in order to rewrite the Einstein equation in the ADM formalism. We then provide examples which relates this approach to different aspects in quantum gravity. Next we deal with the causality issue and its possible solution. We conclude this section by deriving the necessary condition for the hypersurface to appear effectively as $(2 + 1)D$ from the point of view of the Einstein equations.

### 2.1 Gravitational foliation

The first step is to define the gravitational force field direction. For a given metric in a given coordinate system we calculate the gravitational force direction for static observers. The 4-velocity vector field of the static observer is $u^a = (\sqrt{-g^{00}}, 0, 0, 0)$ and her 4-acceleration vector field is given by $a^a = u^i \nabla_i u^a$. Thus the direction of the gravitational force field is given by $n^a = a^a / a$ where $a$ is the magnitude of the acceleration $a = \sqrt{a^a a_i}$. Note that $n^a$ is a space like vector field.

Next we use the standard foliation of spacetime with respect to some spacelike hypersurfaces whose directions are $n^a$. The lapse function $N$ and shift vector $W_a$ satisfy $r_a = N n_a + W_a$ where $r^a \nabla_a r = 1$ and $r$ is constant on $\Sigma_r$. The $\Sigma_r$ hyper-surface metric $h_{ab}$ is given by $g_{ab} = h_{ab} + n_a n_b$. The
extrinsic curvature tensor of the hyper-surfaces is given by $K_{ab} = -\frac{1}{2} \mathcal{L}_n h_{ab}$ where $\mathcal{L}_n$ is the Lie derivative along $n^a$. Instead of the (3 + 1)D Einstein equations

$$R_{ab}^{(4)} = 8\pi \left( T_{ab} - \frac{1}{2} T g_{ab} \right)$$

one finds (see appendix) a kind of (2 + 1)D Einstein equations:

$$R_{ab}^{(3)} - K K_{ab} + 2 K_a K_b + N^{-1} (\mathcal{L}_r K_{ab} - D_a D_b N) = 8\pi \left( S_{ab} - \frac{1}{2} (S + P) h_{ab} \right)$$

and the two constraints:

$$R^{(3)} - K^2 + K_{ab} K^{ab} = -16\pi P,$$

$$D_a K - D_b K^b_a = 8\pi F_a.$$

where $D_a$ represent the 2+1 covariant derivatives, $S_{ab} = h_{ac} h_{ad} T^{cd}$, $P = n_c n_d T^{cd}$ and $F_a = -h_{ac} n_b T^{cb}$. We will see that this foliation is relevant for several different areas in physics which deal with the expected quantum gravity properties.

### 2.2 Background supporting foliation along the gravitational force

We now provide some examples from our own work as well as that of others, showing that gravitational foliation can be useful for aspects of quantum gravity. The first example involves the surface density of space time degrees of freedom (DoF). These are expected to be observed by an accelerating observer in curved spacetime. This DoF surface density was first derived by Padmanabhan [9] for a static spacetime using thermodynamic considerations. We found that this density can also be constructed from specific canonical conjugate pairs as long as they are derived in a unique way [2]. These canonical conjugate pairs must be obtained by foliating spacetime with respect to the direction of the gravitational vector force field. Note that this aspect
reinforces the importance of singling out a very unique spatial direction: the
direction of a gravitational force.

The second example involves string theory excitation. It was found that
some specific kind of singularity is obtained by string theory excitations of
a D1D5 black hole [10]. We found [3] that these singularities can also be
explained using the uncertainty principle, as long as the variables in the
uncertainty principle are obtained in a unique way: they must be canonical
conjugate pairs which are obtained by singling out the radial direction. The
radial direction can be regarded as the direction of a gravitational force
for observers that are static with respect to this coordinate system. Thus,
these singularities, which according to string theory are expected in quantum
gravity theories, are derived by the uncertainty principle only when singling
out the gravitational force direction.

The third example involves “holographic quantization” which uses spatial
foliation in order to quantize the gravitational fields for different backgrounds
in Einstein theory. This is carried out by singling out one of the spatial
directions in a flat background, and also singling out the radial direction
for a Schwarzschild metric [4]. Moreover, other works [5] even suggest that
the holographic quantization causes the (3+1)D Einstein gravity to become
effectively reduced to (2+1)D after solving the Lagrangian analogues of the
Hamiltonian and momentum constraints.

The fourth example involves the developing of a quantum black hole wave
packet [6]. In this case, the gravitational foliation is used in order to obtain
a quantum Schwarzschild black hole, at the mini super spacetime level, by a
wave packet composed of plane wave eigenstates.

The fifth and final example involves the Wheeler-De Witt metric prob-
ability wave equation. Recently, in [7], foliation in the radial direction was
used to obtain the Wheeler-De Witt equation on the apparent horizon hyper-
surface of the Schwarzschild de Sitter black hole. By solving this equation,
the authors found that a quantized Schwarzschild de Sitter black hole has a
nonzero value for the mass in its ground state. This property of quantum
black holes leads to stable black hole remnants.

Whereas our current approach relies on these examples, which relate
gravitational foliation to different aspects of quantum gravity, it also strongly relies on the fact that it is possible to quantize a \((2+1)D\) gravitational theory. Though a \((2+1)D\) gravitational theory is believed to be a toy model for quantum gravity, we suggest that a \((3+1)D\) theory may be regarded as a continuation along the gravitational force field direction of a quantized \((2+1)D\) gravitational theory. Given the fact that a renormalized \((2+1)D\) quantum gravity theory can be obtained, this construction leads to a \((3+1)D\) quantum gravity originated from a renormalized theory. Whether or not this construction leads to an effectively renormalized gravitational theory on the \((3+1)D\) remains to be seen. However, we argue that even if this way of construction does not lead to a renormalized quantum gravitational field theory in the \((3+1)D\) but leads to the correct Einstein field equations, this construction can be considered as a proof that our inability to renormalized the \((3+1)D\) theory is related directly to the acceleration relative to a given coordinate system. We expand this subject in section 4.

Our suggestion leads to a \((3+1)D\) gravitational theory which “evolves” along the gravitational force direction, i.e. an evolution along a space like directed vector field. This leads to the causality vagueness.

2.3 The causality ambiguity and its solution

In our approach, it is necessary to single out the direction of the gravitational force vector field instead of the direction of time like vector field. Since the direction of any force field is space-like, and not time-like this suggestion leads to a lack of clarity regarding the basic concepts of relativistic quantum field theories: causality, probability, conservation, unitarity and more. In order to overcome these anticipated difficulties, we investigated the outcome of such foliation on relativistic free scalar fields. We found that, under some conditions, one can derive a causally quantum theory using non-Cauchy foliation. However, it seems that the main problem is contraction of the causal fields brackets on the non Cauchy hypersurface, since the usual Poisson brackets are not useful when using non-Cauchy foliation.

At this point it is not clear how to obtain such "Poisson-like" brackets
using the conventional mathematical definition, and we need to derive them in some other way. Therefore we propose to derive the quantum commutation relations between the fields on the unique non-Cauchy hypersurface and then use these to obtain the classical brackets.

Though in general this restriction seems problematic, from the quantum gravity point of view it turns out to be promising. The main reason is that in some cases we do know how to quantize a $(2 + 1)D$ gravitational theory. Thus for a given metric background in a $(3 + 1)D$ gravitational theory, we can look for unique hypersurfaces that appear effectively as $(2 + 1)D$ from the point of view of the Einstein equations. If we manage to do so, we can construct a renormalized quantum gravity on this unique $(2 + 1)D$ hypersurface and obtain the commutation relations of the quantum gravitational fields on the hypersurface. In this way, we can easily deduce the causal classical brackets of the fields without using the Poisson brackets. Thus, as was found in the scalar case, one can derive the Hamilton-like equations along the hypersurface direction and use these classical causal brackets in order to obtain the causal quantum gravitational theory in $(3 + 1)D$. We expand this subject in section 4.

This construction of quantum gravity in $(3+1)D$ relies on known $(2+1)D$ renormalized gravitational theory on a unique hypersurface. In the next subsection, we define the unique hypersurface that truncates the Einstein equations and find the condition for it to generate an effectively $(2 + 1)D$ Einstein equations.

\footnote{Note that a $(2 + 1)$-dimensional gravity is tricky to quantize: for example a partition function for 3D pure gravity with a negative cosmological constant was derived in \cite{12} and \cite{13} and exhibit several problematic features, including a negative density of states in certain regimes.}
2.4 Truncation of Einstein equations: the condition for effectively (2 + 1)D Einstein equations on the hypersurface

In the first subsection we found that the gravitational foliation gives a kind of (2 + 1)D Einstein equations:

\[ R_{ab}^{(3)} - K K_{ab} + 2 K_{ai} K_i^b + N^{-1} (\mathcal{L}_r K_{ab} - D_a D_b N) = 8 \pi \left( S_{ab} - \frac{1}{2} (S + P) h_{ab} \right), \]

and the two constraints:

\[ R^{(3)} - K^2 + K_{ab} K^{ab} = -16 \pi P, \]
\[ D_a K - D_b K^b_a = 8 \pi F_a. \]

Thus if we able to find a unique hypersurface \( r = r_0 \) so that:

\[ B_{ab} \equiv K K_{ab} - 2 K_{ai} K_i^b - N^{-1} (\mathcal{L}_r K_{ab} - D_a D_b N) = 0 \quad (5) \]

then we have:

\[ R_{ab}^{(3)} = 8 \pi \left( S_{ab} - \frac{1}{2} (S + P) h_{ab} \right), \quad (6) \]

which is almost the Einstein equations in 2+1 dimension, when \( P \) serves as a cosmological constant. However, note that the Einstein equations in (2+1)D are \( R_{ab}^{(3)} = 8 \pi (S_{ab} - S h_{ab}) \) and thus we have only obtained an “Einstein-like equation.” Only if \( S = P \) do we get exactly the expected Einstein equation in (2 + 1)D. Thus, for example, we do not expect the conservation of the energy momentum \( S_{ab} \) to hold on this hypersurface. Moreover, one has to consider the fact that in (2+1)D and (3+1)D gravitational constant and even the energy-momentum tensor have different units.

This unique hypersurface is interesting. Although we do not know how to obtain a renormalized quantum gravity theory in 3+1 dimensions, a renormalized quantum theory in 2+1 dimensions can nevertheless be obtained [11]. Thus, when the (3+1)D Einstein equations reduce to (2 + 1)D Einstein-like equations on some hypersurface \( r = r_0 \), we can quantize the gravitational fields on the hypersurface \( r = r_0 \) at least with respect to this foliation. In
the next section we find this kind of unique hypersurface for different static spherical metrics.

3 Spherical symmetry examples

In this section we deal with different examples of truncation hypersurfaces. All of them assume static spherically symmetric metrics. Note that although one may expect that it is useful to first tackle a pure gravity system, we will see that our analysis becomes easier when dealing with a gravity-matter system.

3.1 First example: extremal black hole

We start with the metric of an extremal black hole and show that although its horizon fulfills all the necessary conditions, this metric can not be a proper candidate for the hypersurface to appear effectively as a \((2+1)D\) from the point of view of the Einstein equation.

In this case

\[
\begin{align*}
  ds^2 &= - \left( 1 - \frac{M}{r} \right)^2 dt^2 + \left( 1 - \frac{M}{r} \right)^{-2} dr^2 + r^2 d\Omega^2.
\end{align*}
\]

The 4-velocity and 4-acceleration vector fields for a static observer in this metric are

\[
\begin{align*}
  u^a &= \left( \left( 1 - \frac{M}{r} \right)^{-1}, 0, 0, 0 \right) \\
  a^a &= \left( 0, \frac{M}{r^2(1 - \frac{M}{r})}, 0, 0 \right),
\end{align*}
\]

\[\text{In general, given an arbitrary vector field on a spacetime, there will not exist hypersurfaces normal to the vector field. For a family of surfaces normal to } a^a \text{ to exist, Frobenius' theorem requires that the vector field } a^a \text{ must be hypersurface orthogonal. In special cases with a high degree of symmetry (like the static, spherically symmetric cases), the symmetries can ensure that } a^a \text{ is indeed hypersurface orthogonal and therefore the foliation exists.}\]
The direction of acceleration is

\[ n^a = (0, (1 - \frac{M}{r}), 0, 0). \]

The induced metric becomes

\[
h_{ab} = \begin{pmatrix} (1 - \frac{M}{r})^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}
\]

and the extrinsic curvature

\[
K_{ab} = \begin{pmatrix} -\frac{M(r-M)^2}{r^4} & \times & 0 & 0 \\ \times & \times & \times & \times \\ 0 & \times & (r-M) & \times \\ 0 & \times & 0 & (r-M) \sin^2 \theta \end{pmatrix}
\]

The Lie derivative along \( r^a \) of the extrinsic curvature is

\[
\mathcal{L}_r K_{ab} = \begin{pmatrix} 2 \frac{M^2}{r^2} \left(1 - \frac{M}{r}\right)^2 \left(1 - \frac{2M}{r}\right) & \times & 0 & 0 \\ \times & \times & \times & \times \\ 0 & \times & \frac{M}{r} \left(1 - \frac{M}{r}\right) & \times \\ 0 & \times & 0 & \frac{M}{r} \left(1 - \frac{M}{r}\right) \sin^2 \theta \end{pmatrix}
\]

and since \( N = (1 - \frac{M}{r})^{-1} \) and \( W_a = 0 \) we obtain

\[
D_0 D_0 N = \frac{M^2}{r^4} \left(1 - \frac{M}{r}\right)
\]
\[
D_2 D_2 N = \frac{M}{r^2}
\]
\[
D_3 D_3 N = \frac{M}{r^2} \sin^2 \theta.
\]

Finally, we have everything we need in order to calculate the truncation tensor \( B_{ab} \) defined in eq. \( 13 \). It turns out that all its components vanish on the horizon: \( r = M \), and thus we conclude that the term in Einstein equations that depends on the extrinsic curvature vanishes on the horizon:
\( r = M \) and eq. 5 holds. This means that from the point of view of Einstein equations, the hypersurface denoted by \( r = M \) does not "feel" the radial direction and we may expect that effectively the Einstein equations can be described as leaving on \((2 + 1)D\). However, note that in this case the time-time component of the induced metric becomes \( h_{00} = 0 \) and thus this surface is not a good candidate for Einstein equations leaving on \((2 + 1)D\).

### 3.2 Second example: toy model

It turns out that although the extremal black hole is not a good candidate for our suggestion, one can find examples of gravitational foliation that lead to an effectively \((2 + 1)D\) theory from the point of view of the Einstein equations. For example we consider the metric

\[
ds^2 = - \left(1 - \frac{A^2}{r^2} + \frac{B^3}{r^3}\right) dt^2 + \left(1 - \frac{A^2}{r^2} + \frac{B^3}{r^3}\right)^{-1} dr^2 + r^2 d\Omega^2
\]

In this case the 4-velocity and 4-acceleration vector fields for a static observer in this metric are

\[
u^a = \left(\left(1 - \frac{A^2}{r^2} + \frac{B^3}{r^3}\right)^{-1/2}, 0, 0, 0\right).
\]

\[
a^a = \left(0, \frac{2Ar - 3B^2}{2r^4}, 0, 0\right),
\]

Note that all the components to the accelerating vector field vanish on \( r = 3B^2/2A \).

Foliating spacetime along the direction of the acceleration vector field, gives the induced metric

\[
h_{ab} = \begin{pmatrix}
1 - \frac{A^2}{r^2} + \frac{B^3}{r^3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}
\]
and the extrinsic curvature of hypersurfaces directed along the acceleration vector field is

\[ K_{ab} = \sqrt{(3B^2 - 2Ar)^2(B^2 - Ar + r^3)} \begin{pmatrix} -\frac{1}{2}r^{-11/2} & \times & 0 & 0 \\ \times & \times & \times & \times \\ 0 & \times & r^{-3} & 0 \\ 0 & \times & 0 & r^{-3}\sin^2\theta \end{pmatrix}. \]

The Lie derivative along \( r^a \) of the extrinsic curvature vanishes everywhere except the following components:

\[ \mathcal{L}_r K_{00} = \sqrt{\frac{(3B^2 - 2Ar)^2}{r^3(B^2 - Ar + r^3)}} \frac{-33B^4 - 24B^2r(-2A + r^2) + 4Ar(-4A + 3r^2)}{8r^5} \]
\[ \mathcal{L}_r K_{22} = \sqrt{\frac{(3B^2 - 2Ar)^2}{r^3(B^2 - Ar + r^3)}} \frac{(B^2 - 2r^3)}{4r^3} \]
\[ \mathcal{L}_r K_{33} = \sqrt{\frac{(3B^2 - 2Ar)^2}{r^3(B^2 - Ar + r^3)}} \frac{(B^2 - 2r^3)}{4r^3} \sin^2\theta \]

and since 
\[ N = \left(1 - \frac{A^2}{r^2} + \frac{B^3}{r^3}\right)^{-1/2} \]
we obtain

\[ D_0 D_0 N = \frac{(3B^2 - 2Ar)^2}{2r^2\sqrt{B^2 - Ar + r^3}} \]
\[ D_2 D_2 N = \frac{3B^2 - 2Ar}{2r^2\sqrt{B^2 - Ar + r^3}} \]
\[ D_3 D_3 N = \frac{3B^2 - 2Ar}{2r^2\sqrt{B^2 - Ar + r^3}} \sin^2\theta \]

Finally, we have everything we need in order to calculate the truncation tensor \( B_{ab} \) defined in eq (5). As one might expect, eq. (5) holds when the component of the acceleration vector field vanishes, as in the extremal black hole case. Since \( a^a = \left(0, \frac{2Ar - 3B^2}{2r^4}, 0, 0\right) \) we find that the acceleration vector field vanishes on \( r = \frac{3B^2}{2A} \). Note that since for general \( A \) and \( B \) the term \( f(r)_{r=1/2} = \left(1 - \frac{4A^3}{27B^2}\right) \) does not vanish on \( r = \frac{3B^2}{2A} \), the hypersurface denoted by \( r = \frac{3B^2}{2A} \) is not a horizon.

It is interesting to note that in this example the energy momentum tensor
$S^a_b$ and momentum $P$ on the hyper-surface $r = \frac{3B^2}{2A}$ are

$$S^a_b = \begin{pmatrix} -\frac{16A^5}{243B^8} & 0 & 0 \\ 0 & \frac{16A^5}{81B^8} & 0 \\ 0 & 0 & \frac{16A^5}{81B^8} \end{pmatrix}, \quad P = -\frac{16A^5}{243B^8}. \quad (7)$$

and thus it does not represent an AdS universe.

### 3.3 Third example: general static spherical metric

We can now investigate the conditions that are required in order that a hypersurface on $(3 + 1)D$ spherical static universe can be regarded as a $(2 + 1)D$ gravitational theory. In order to do so, we consider a general static spherically symmetric metric

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\Omega^2.$$ 

In this case the 4-velocity and 4-acceleration vector fields for a static observer are

$$u^a = \left( f^{-1/2}(r), 0, 0, 0 \right),$$

$$a^a = \left( 0, \frac{1}{2}f', 0, 0 \right).$$

and so $N = f^{-1/2}(r)$, $W_a = 0$ and the direction of the acceleration is always radial. Thus foliating spacetime along the direction of the acceleration vector field gives the induced metric

$$h_{ab} = \begin{pmatrix} f(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

and the extrinsic curvature of hypersurfaces directed along the acceleration vector field is
\[
K_{ab} = \sqrt{f(r)} \begin{pmatrix}
-\frac{1}{2} f' & \times & 0 & 0 \\
\times & \times & \times & \times \\
0 & \times & r & 0 \\
0 & \times & 0 & r \sin^2 \theta
\end{pmatrix}.
\]

The Lie derivative along \( r^a \) of the extrinsic curvature vanishes everywhere except the following components:

\[
\mathcal{L}_r K_{00} = -\frac{1}{8} \frac{f'}{\sqrt{f}} (f'^2 + 2ff'')
\]
\[
\mathcal{L}_r K_{22} = \frac{1}{4} \frac{f'}{\sqrt{f}} (rf' + 2f)
\]
\[
\mathcal{L}_r K_{33} = \frac{1}{4} \frac{f'}{\sqrt{f}} (rf' + 2f) \sin^2 \theta
\]

and since \( N = f^{-1/2}(r) \) we get

\[
D_0 D_0 N = \frac{1}{4} \frac{f'^2}{\sqrt{f}}
\]
\[
D_2 D_2 N = -\frac{1}{2} \frac{f'}{\sqrt{f}}
\]
\[
D_3 D_3 N = -\frac{1}{2} \frac{f'}{\sqrt{f}} \sin^2 \theta
\]

Thus the truncation tensor gives

\[
B_{ab} = f' \begin{pmatrix}
\frac{1}{8} f'^2 + \frac{1}{4} f'' f - \frac{1}{2} f & \times & 0 & 0 \\
\times & \times & \times & \times \\
0 & \times & r - \frac{1}{2} f - \frac{1}{4} rf' & 0 \\
0 & \times & 0 & (r - \frac{1}{2} f - \frac{1}{4} rf') \sin^2 \theta
\end{pmatrix}.
\]

As we see, the conditions of equation (5) hold on a hypersurface \( r = r_0 \) if \( f'(r_0) = 0 \) or if \( \left( \frac{1}{8} f'^2 + \frac{1}{4} f'' f - \frac{1}{2} f \right)_{r=r_0} = \left( r - \frac{1}{2} f - \frac{1}{4} rf' \right)_{r=r_0} = 0 \). Note that when the condition \( f'(r_0) = 0 \) holds, then the acceleration vector field \( a^a = (0, \frac{1}{2} f', 0, 0) \) vanishes on \( r = r_0 \) and we see that the hypersurface does not have to be a horizon.

The term \( K^2 - K_{ab} K^{ab} \) from the first constraint (3) equals to

\[
K^2 - K_{ab} K^{ab} = \frac{2}{r^2} (f + rf')
\].
As expected from the second example, this term does not vanish when equation (5) holds.

It is interesting to note that just as in eq. (7), in this example the energy momentum tensor $S^a_b$ and momentum $P$ on the hyper-surface $r = r_0$ are

$$
S^a_b = \begin{pmatrix}
\frac{1}{r^2} (f + rf' - 1) & 0 & 0 \\
0 & \frac{1}{2} f' + \frac{1}{2} f'' & 0 \\
0 & 0 & \frac{1}{2} f' + \frac{1}{2} f''
\end{pmatrix},
\quad P = \frac{1}{r^2} (f + rf' - 1).
$$

and thus it does not represent an AdS universe. Moreover, note that at least for a static spherically symmetric metrics our analysis becomes easier whenever one deals with a gravity-matter system. Since for a static spherically symmetric metrics one can get a pure gravity system only when $\frac{1}{r^2} (f + rf' - 1) = \frac{1}{2} f' + \frac{1}{2} f'' = 0$. \footnote{We do not believe that the consistent with the usual energy conditions in general relativity, like the null energy condition, the weak energy condition, are relevant in this surface.}

4 Further work

In this paper, we proved that under some conditions there exists a unique hypersurface that causes the Einstein equations to look like $(2 + 1)D$. But in order to construct a renormalized quantum gravity this is not sufficient.

To start with, note that in this approach the "evolution" is not along "time" but along the spatial coordinate. This means that all the dynamics must be encoded on our unique hypersurface in advance. In other words, the hypersurface must be holographic. The holographic principle is not new \cite{14, 15}. According to this principle, the number of degrees of freedom fundamentally scales like the area of surfaces and not the enclosed volume as one would expect from local field theory. In order to extend this idea to any spacetime, the Bousso bound \cite{16} was formulated. Next by defining two kinds of holographic screens (future and past), Bousso and Engelhardt \cite{17, 18} proved an extended new area law. Recently, in \cite{19} we related the entropy of any holographic screen to the phase space entropy derived in \cite{2}.

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and by using the new area law we identified uniquely the foliation direction needed for any given holographic screen. Using this explicit foliation, the next step should be deriving the truncation of Einstein equations for the holographic screens defined by Bousso and Engelhardt.

Next, it is probably necessary to derive a Hamilton-like ADM equation for the evolution along the direction of the gravitational force. Except the constraint, this is expected to look like

\[ L_R \Pi_{ab} = - \left\{ \tilde{H}, h_{ab} \right\}, \quad L_R h_{ab} = \left\{ \tilde{H}, \Pi_{ab} \right\} \]

where

\[ \Pi_{ab} := \frac{\partial L}{\partial L_R h_{ab}} = \sqrt{-h} \left( K h^{ab} - K^{ab} \right) \]

\[ \tilde{H} = \Pi_{ab} L_R h_{ab} - L \]

\[ = \sqrt{-h} \left[ N \left( -^{(3)}R + h^{-1} \Pi_{ij} \tilde{\Pi}^{ij} - \frac{1}{2} h^{-1} \Pi^2 \right) - 2 W_i D_j \left( h^{-1/2} \Pi^{ij} \right) \right] \]

Note that the brackets (8) are not trivial, and must be identified correctly. This happens because when using spatially directed foliation, instead of the usual time like foliation, one can no longer use the Poisson brackets. Thus it is necessary to derive a kind of "Poisson-like" brackets between the classical fields. In that case we would be able to obtain the relations

\[ \{ h_{ij}, h_{ab} \}_{r=r_0}, \{ h_{ij}, \Pi_{ab} \}_{r=r_0}, \{ \Pi_{ij}, \Pi_{ab} \}_{r=r_0}. \]

Our construction is helpful in order to derive the first one: \( \{ h_{ij}, h_{ab} \}_{r=r_0} \). These brackets are known whenever one constructs a quantum gravity theory on the unique \((2 + 1)D\) hypersurface. Thus the next step upon quantizing \((2 + 1)D\) should be a derivation of the quantum gravitational theory on the unique hypersurfaces specified above. In order to do that note first that in \((2 + 1)D\) and \((3 + 1)D\) both the gravitational constant and the energy-
momentum tensor have different units.

In order to derive the second \( \{ h_{ij}, \Pi_{ab} \}_{r=r_0} \) and the third \( \{ \Pi_{ij}, \Pi_{ab} \}_{r=r_0} \), a different approach should be taken. For example, one may consider the use of Peierls bracket \(^{(20)}\) which is a more covariant structure equivalent to the Poisson bracket but which can be built directly from advanced and retarded Green’s functions for the linearized equations of motion.

Moreover, the proof that one can derive the Hamilton-like equations by foliating spacetime along spatial direction were proven only for scalars \(^{(8)}\). It seems that this result can easily be extended for any vector fields if one ignores complication associated with gauge invariance and work directly with physical components. In this case, the action of each physical component will be the same as for a scalar field. For example, though in \((3 + 1)D\) the metric has 10 components 8 of them are non-physical, and each of the two remaining physical components has an effective action of a scalar field. Thus, quantizing a vector field by foliating spacetime along the spatial direction is also possible whenever the vector field components have causal commutation relation on the non-Cauchy hypersurface. However, this result is relevant only when one ignores complications associated with gauge invariance in the Minkowski frame. Thus, the next step should be considering the implications of the gauge invariance on the derivation of causal quantum theory when singling out a spatial direction. Moreover, in order to extend our intuitiveness for gravity, one should also examine the derivation of causal quantum theory in the Rindler metric (for scalars and vector fields) when singling out a unique spatial direction: the acceleration direction.

5 Summary and discussion

In this paper, we used gravitational foliation in order to find a few static spherical symmetric examples of hypersurfaces that truncate the Einstein equations. We began with a derivation of the Einstein equation in the ADM formalism when foliating along the gravitational force direction. In order to find this direction we considered the direction of the acceleration vector field of static observers relative to the given coordinate system. Then we derived
the conditions for hypersurfaces that appear effectively \((2 + 1)D\) from the point of view of the Einstein equations. Finally, we found these hypersurfaces in different static spherical metrics. We found that the conditions in this case are fulfilled whenever all the components of the acceleration vector field vanish on the hypersurface.

Now further work is necessary. The next step should be to construct a renormalized quantum gravity on this unique \((2 + 1)D\) hypersurface and to obtain the commutation relations of the quantum gravitational fields on the hypersurface. In this way, we could easily deduce the causal classical brackets of the fields on this unique \((2 + 1)D\) hypersurface, without using the classical Poisson brackets.

The advantage of constructing a gravitational theory using the unique hyperspace is obvious. The unique hyperspace enables derivation of quantum gravitational fields which are already renormalized on the hypersurface, i.e., to obtain \(\{h_{ij}, h_{ab}\}_{r=r_0}\). Whether the evolution of these specific fields "out from the hypersurface" along the gravitational force direction keeps their feature of "being renormalizable" remains to be seen. But the fact that the structure is based on a renormalized quantum gravity looks promising and may lead the way to construct a \((3 + 1)D\) renormalized quantum theory.

Note that this procedure may be related to a kind of holography. In our suggested formalism, the evolution of the gravitational fields along the acceleration direction is determined by \(\{h_{ij}, h_{ab}\}_{r=r_0}, \{h_{ij}, \Pi_{ab}\}_{r=r_0}, \{\Pi_{ij}, \Pi_{ab}\}_{r=r_0}\) which plays the role of the "initial" or surface condition on the non-Cauchy hyper-surface \(r = r_0\). This construction gives all the information which encoded on the hyper-surface and is needed to describe the evolution of the gravitational field along the acceleration direction. However, whether this construction gives all the information in the balk is remain to be seen.

Moreover, this construction leads to a very interesting result since it relates the non-renormalizablity of the quantum gravitational theory to acceleration in curved spacetime. To see this, note that our conditions on the hypersurface are fulfilled whenever all the components of the acceleration vector field of static observers vanish. Moreover, these conditions cause our hypersurface to look effectively \((2 + 1)D\) from the point of view of the Ein-
stein equation. This means that if we find, for a given metric, a coordinate system that leads to the vanishing of the acceleration components for all the hypersurfaces directed along the acceleration vector field of static observers, construction of a renormalized $(3 + 1)D$ gravitational theory could be obtained. This happens because in this case, all hypersurfaces look effectively like $(2 + 1)D$ from the point of view of the Einstein equations. Since accelerated observers must use the $(3 + 1)D$ Einstein equations, this renormalized quantization cannot be applied for accelerated observers. This may suggest that acceleration in curved spacetime and the non-renormalizability of the quantum gravitational theory are connected.

Not surprisingly, foliation along a vanishing acceleration vector field is impossible. To see this note that if indeed all the components of the acceleration vector field vanish everywhere, we deal with freely falling observers. In this case, the direction of the acceleration cannot be defined and our suggested foliation cannot be done. However, this makes us wonder whether we may expect that freely falling observers in a $(3+1)D$ can only use a $(2+1)D$ coordinate system in order to describe the universe. This case is very interesting because this suggests that at least freely falling observers may be able to obtain a renormalized quantum gravitational theory by choosing the right coordinate system. This idea is supported by [2, 9] which relates the extra gravitational degrees of freedom seen by static observers to their acceleration in generalized theories of gravity. Moreover, it was found that these extra gravitational degrees of freedom vanish whenever the observers move on a geodesic. This reinforces our motivation to investigate the possibility that freely falling observers “see” less DoF and thus can renormalize the gravitational theory. This suggestion needs further investigation. First: Are our findings relating the vanishing of the acceleration components on the hypersurface to effectively $(2 + 1)D$ Einstein equations relevant even for non-spherically symmetric systems? What is the physical meaning of a system of coordinates that is relevant only for freely falling observers? Even if a $(2+1)D$ system of coordinates that is relevant only for freely falling observers exists, how do we extend it for accelerated observers which must use $(3+1)D$?
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Appendix: The “spatial ADM” formalism and derivation of Einstein equations for spacial foliation

When foliating along a space like vector field, instead of a time like vector field, the Einstein field equations in the ADM formalism slightly change. In this appendix we recalculate the Einstein equations in the ADM formalism using a space like vector field, instead of time like vector field.

We begin by considering the standard foliation of spacetime with respect to some spacelike hypersurfaces whose directions are $n^a$. The lapse function $N$ and shift vector $W_a$ satisfy $r_a = Nn_a + W_a$ where $r^a \nabla_a r = 1$ and $r$ is constant on $\Sigma_r$(Thus $n_a = N \nabla_a r$). The $\Sigma_r$ hyper-surfaces metric $h_{ab}$ is given by $g_{ab} = h_{ab} + n_a n_b$. The extrinsic curvature tensor of the hyper-surfaces is given by $K_{ab} = -\frac{1}{2} L_n h_{ab}$ where $L_n$ is the Lie derivative along $n^a$.

We use this foliation and rewrite the $(3+1)$D Einstein equations

\begin{equation}
(4) R_{ab} = 8\pi \left( T_{ab} - \frac{1}{2} T g_{ab} \right) .
\end{equation}

in terms of the induced metric. The starting point of the calculation (see for example see [21]) is the Gauss relation which we derive here for non-Cauchy foliation from the Ricci identity on the $\Sigma_r$ hyper-surfaces:

\begin{equation}
D_a D_b v^c - D_b D_a v^c = (3) R_{mab} v^m
\end{equation}

where $v^m$ is a generic vector field tangent to $\Sigma_r$. Relating the $D$-derivative to the $\nabla$-derivative and using $\nabla_m h_{ab} = \nabla_m (g_{ab} - n_a n_b) = -\nabla_m n_a n_b - n_a \nabla_m n_b$.

\footnote{The intrinsic curvature $R_{ab}^{(3)}$ is then given by the 2+1 Christoffel symbols: $\Gamma^{h}_{ab} = \frac{1}{2} h^{kl} \left( \frac{\partial h_{kb}}{\partial x^a} + \frac{\partial h_{ak}}{\partial x^b} - \frac{\partial h_{ab}}{\partial x^k} \right)$ so that $R^{(3)}_{ab} = \frac{\partial \Gamma^{k}_{ab}}{\partial x^k} - \frac{\partial \Gamma^{k}_{ba}}{\partial x^k} + \Gamma^{k}_{ab} \Gamma^{l}_{kl} - \Gamma^{l}_{ak} \Gamma^{k}_{lb}$.}
\( h^n_m n_n = 0 \) and \( h^m_n h^n_l \nabla_m n_n = -K_{ab} \) one gets:

\[
D_a D_b v^c = h^m_a h^n_b h^c_l \nabla_m \left( h^n_l h^i_j \nabla_s v^i \right)
\]

\[
= h^m_a h^n_b h^c_l \left( -n^s \nabla_m n_n h^i_l \nabla_s v^i - h^n_l \nabla_m v^l + h^n_l h^c_l \nabla_m \nabla_s v^l \right)
\]

\[
= -h^m_a h^n_b h^c_l n^s \nabla_m n_n \nabla_s v^i + h^m_a h^n_b h^c_l \nabla_m \nabla_s n_l + h^m_a h^n_b h^c_l \nabla_m \nabla_s v^l
\]

\[
= K_{ab} h^c_l n^s \nabla_s v^i + K^c_a K^d_b v^i + h^m_a h^n_b h^c_l \nabla_m \nabla_s v^i
\]

Next, using the symmetry of the extrinsic curvature, one gets for eq. (10):

\[
D_a D_b v^c - D_b D_a v^c = (K^c_a K^d_b - K^c_b K^d_a) v^i + h^m_a h^n_b h^c_l \left( \nabla_m \nabla_s v^i - \nabla_s \nabla_m v^i \right)
\]

Using \( \nabla_m \nabla_s v^i - \nabla_s \nabla_m v^i = (4) R_{lms} v^i \), eq. (10) and eq. (11) we find:

\[
(K^c_a K^d_b - K^c_b K^d_a) v^i + h^m_a h^n_b h^c_l (4) R_{lms} v^i = ^{(3)} R_{kab} v^k
\]

and thus the Gauss relation for non-Cauchy foliation is

\[
h^m_a h^n_b h^c_l h^k_i R^i_{hms} = ^{(3)} R_{kab} - K^c_a K^d_b + K^d_c K^c_a.
\]

Note that this term is different from the Gauss relation for Cauchy foliation. Using a non-Cauchy foliation leads to different signs for the two last terms.

If we contract the Gauss relation on the indices \( c \) and \( a \) and use \( h_{ma} h^a_l = h^a_l = g_{ml} - n^m n_l \), we get:

\[
h^a_l h^b_h (4) R_{hs} - n^l h^a_l h^b_h (4) R_{lsm} n^m = ^{(3)} R_{ab} - K K_{ab} + K^i_a K_{ib}.
\]

The next step is to derive \( n^l h^a_l h^b_h (4) R_{lsm} n^m \) with is the Ricci identity applied to the vector \( n^a \), and projecting it twice onto \( \Sigma_t \) and once along \( n^a \):

\[
h_{am} n_i h^b_h (\nabla_n \nabla_i n^m - \nabla_i \nabla_n n^m) = h_{am} n_i h^b_h (4) R^m_{jmn} n^i
\]

In order to calculate this term we work out \( \nabla_a n^b \):

\[
K_{ab} = \frac{1}{2} \mathcal{L}_n h_{ab} =
\]
Thus \( N_{[a n_b]} = -K_{ab} - n_{[a} D_{b]} \ln N \)

Returning to eq. (13) we get

\[
\begin{align*}
K_{ab} & = \frac{1}{2} \left( \nabla_a n_b + \nabla_b n_a - n_a n^i \nabla_i n_b - n_b n^i \nabla_i n_a \right) \\
\text{Using } n_a = N_{\nabla} n & \text{ we find that } n^{i} \nabla_{i} n_a = -D_{a} \ln N
\end{align*}
\]

we find

\[
K_{ab} = \left[ \frac{1}{2} \left( \nabla_a n_b + \nabla_b n_a - n_a n^i \nabla_i n_b - n_b n^i \nabla_i n_a \right) \right] _{a b}.
\]

Note we have used \( K_{a}^{n} n^{i} = 0, n^{i} \nabla_{i} n = 0, n_{i} n^{i} = 1, n^{i} \nabla_{i} n_{a} = -D_{a} \ln N \)

and \( h_{a}^{n} n_{a} = 0 \) to get the third equality. Let us now show that the term \( h_{a}^{n} n^{i} \nabla_{i} K_{n}^{m} \) is related to \( \mathcal{L}_{r} K_{ab} \). Indeed

\[
\mathcal{L}_{r} K_{ab} = r^{i} \nabla_{i} K_{ab} + K_{i b} \nabla_{a} r^{i} + K_{a i} \nabla_{b} r^{i}.
\]

Using \( r_{a} = N n_{a} + W_{a} \) and \( n_{[a n_{b}]} = -K_{ab} - n_{[a} D_{b]} \ln N \) we get

\[
\mathcal{L}_{r} K_{ab} = N n^{i} \nabla_{i} K_{ab} - 2 N K_{i b} K_{a}^{i} - K_{i b} n_{a} D_{i} N - K_{a i} n_{b} D_{i} N.
\]

Projecting on \( \Sigma_{r} \) by applying \( h_{i m n} \) on both side and using \( \mathcal{L}_{r} K_{ab} = h_{a}^{n} h_{b}^{m} \mathcal{L}_{r} K_{n m} \) we

\[
6 n^{i} \nabla_{i} n_{a} = n^{i} \nabla_{i} \left( N \nabla_{a} r \right) = n^{i} \nabla_{i} N \nabla_{a} r + N n^{i} \nabla_{i} \nabla_{a} r = N^{-1} n_{a} n^{i} \nabla_{i} N + N n^{i} \nabla_{i} \nabla_{a} r = N^{-1} n_{a} n^{i} \nabla_{i} N + N n^{i} \nabla_{a} \left( N^{-1} n_{i} \right) = N - 1 \left( n_{a} n^{i} \nabla_{i} N - \nabla_{a} N \right) = -h_{a i} \nabla^{i} \ln N = -D_{a} \ln N
\]
get
\[ \mathcal{L}_r K_{ab} = N h^n_a h^m_b n^i \nabla_i K_{nm} - 2NK_{ib}K^i_a. \]

Thus
\[ h_{am}n^i h_b^{(4)} P^m_{jn} n^j = N^{-1} \mathcal{L}_r K_{ab} - N^{-1} D_b D_a N + K_{ai} K^i_b \quad (14) \]

The left hand side of (14) is a term which appears in the contracted Gauss equation (12). Therefore, by combining the two equations, we get:
\[ h^{s} h^{(4)} R_{hs} = N - 1\mathcal{L}_r K_{ab} - N^{-1} D_b D_a N + (3) R_{ab} - KK_{ab} + 2K^i_a K_{ib} \quad (15) \]

Note that this term is different from the one we get using Cauchy foliation. Using non-Cauchy foliation leads to different sign for the first, and the last two last terms of the left hand side terms.

Finally, contracting the (3 + 1)D Einstein equation (9) with the induced metric we get for the non-Cauchy surface foliation :
\[ R_{ab}((3) - K K_{ab} + 2K_{ai} K^i_b + N^{-1} (\mathcal{L}_r K_{ab} - D_a D_b N) = 8\pi \left( S_{ab} - \frac{1}{2} (S - P) h_{ab} \right), \quad (16) \]

Where \( D_a \) represent the 2+1 covariant derivatives, \( S_{ab} = h_{ac} h_{ad} T^{cd} \), \( P = n^c n^d T^{cd} \) and \( F_a = h_{ac} n_b T^{cb} \).

Next we find the constraint relevant for the non-Cauchy surface foliation.

The first is obtained by projection of the Einstein equation along \( n^a \), i.e. the normal to the hypersurface \( \Sigma_r \). Contracting eq. (12) with \( h^{ab} \) and using \( h^{ab(4)} R_{ab} = (g^{ab} - n^a n^b) \quad (4) R_{ab} = (4) R - (4) R_{ab} n^a n^b \) and \( h^{ab} n^i h^s_a h^b_h^{(4)} R_{lsn} n^m = n^i h^{(4)} R_{lsn} n^m = n^i R_{lm} n^m - n^i n_h n^s(4) R_{lsn} n^m = n^i(4) R_{lm} n^m \) we get
\[ (4) R - 2(4) R_{ab} n^a n^b = (3) R - K^2 + K^{ij} K_{ij}. \]

Using \( T = g_{ab} T^{ab} = h_{ab} T^{ab} + n_a n_b T^{ab} = S + P \) we find \( (4) R = -8\pi (S + P) \quad (4) R_{ab} n^a n^b = 8\pi \left( P - \frac{1}{2} T \right) = 4\pi (P - S) \) and thus we obtain the first constraint:
\[ (3) R - K^2 + K^{ij} K_{ij} = -16\pi P. \]

Finally, in order to derive the second constraint we need to derive the relevant
Codazzi relation by applying the Ricci identity to $n^a$:

$$\nabla_m \nabla_s n^k - \nabla_s \nabla_m n^k = (4) R^k_{ims} n^i.$$  

Projecting this onto $\Sigma_r$, we get

$$h^m_a h^s_b h^c_k (4) R^k_{ims} n^i = h^m_a h^s_b h^c_k \left( \nabla_m \nabla_s n^k - \nabla_s \nabla_m n^k \right).$$  

Since $h^s_b h^c_k \nabla_s n^k = -K^c_b$, $h^{ab} = g^{ab} - n^a n^b$ and $h^s_b n^m n^k (4) R^k_{ims} n^i = 0$ we get after contracting the indices $a$ and $c$

$$h^s_b (4) R_{is} n^i = -D_i K^i_b + D_b K.$$  

Finally, using Einstein equation once onto $\Sigma_r$ and once along the normal $n^a$ we find:

$$D_a K - D_i K^i_a = 8\pi F_a.$$  

To conclude:

Where as foliating spacetime using time-like vector field $u^a$ leads to

$$(3) \tilde{R} + \tilde{K}^2 - \tilde{K}^{ij} \tilde{K}_{ij} = 16\pi E.$$  

$$(3) \tilde{R} + \tilde{K}^2 - \tilde{K}^{ij} \tilde{K}_{ij} = 16\pi E.$$  

$$(3) \tilde{R} + \tilde{K}^2 - \tilde{K}^{ij} \tilde{K}_{ij} = 16\pi E.$$  

$$(3) \tilde{R} + \tilde{K}^2 - \tilde{K}^{ij} \tilde{K}_{ij} = 16\pi E.$$  

where the lapse function $N$ and shift vector $U_a$ satisfy $t_a = Nu_a + U_a$, $t^a \nabla_a t = 1$ and $t$ is constant on $\tilde{\Sigma}_t$ and the hyper-surface metric $\gamma_{ab}$ is given by $g_{ab} = \gamma_{ab} - u_a u_b$. In this case the extrinsic curvature tensor of the hyper-surfaces is given by $\tilde{K}_{ab} = -\frac{1}{2} \mathcal{L}_u \gamma_{ab}$ and $\tilde{D}_a$ represent the 3 spatial covariant...
derivatives, \( \bar{S}_{ab} = \gamma_{ac} \gamma_{ad} T^{cd} \), \( E = u_c u_d T^{cd} \) and \( p_a = -\gamma_{ac} u_b T^{cb} \).

Foliating along a spacelike vector fields leads to somewhat different equations:

\[
R_{ab}^{(3)} - K K_{ab} + 2 K_{ai} K^i_b + N^{-1} (\mathcal{L}_r K_{ab} - D_a D_b N) = 8 \pi \left( S_{ab} - \frac{1}{2} (S - P) h_{ab} \right),
\]

\[ (3) R - K^2 + K^{ij} K_{ij} = -16 \pi P. \tag{20} \]

\[
D_a K - D_i K^i_a = 8 \pi f_a. \tag{21} \]

where \( D_a \) represent the 2+1 covariant derivatives, \( S_{ab} = h_{ac} h_{ad} T^{cd} \), \( P = n_c n_d T^{cd} \) and \( f_a = -h_{ac} n_b T^{cb} \).

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