Thermalization time bounds for Pauli stabilizer Hamiltonians

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\[ t_{\text{mix}} \leq \mathcal{O}(N^2 e^{2\beta \varepsilon}) \]
Overview

- Motivation & previous results
- Mixing and thermalization
- The spectral gap bound
- Proof sketch
Thermalization in Kitaev’s 2D model

- Spectral gap bound for the 2D toric code and 1D Ising

R. Alicki, M. Fannes, M. Horodecki J. Phys. A: Math. Theor. 42 (2009) 065303
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Spectral gap bound:

\[ \lambda \geq \frac{1}{3} e^{-8\beta J} \]
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Spectral gap bound:

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Implies mixing time bound:
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Figure 2. Partitioning Kitaev’s lattice.

The Hamiltonian of the model is

$$H_{\text{Kit}} = - \sum_s J X_s - \sum_p J Z_{p,J > 0}.$$  (82)

Similarly to the Ising model, the ground states are totally unfrustrated: all $X_s$ and $Z_p$ have expectation 1. This is actually not sufficient to fully determine the state of all spins as the star and plaquette observables are not independent: because of the periodic boundary conditions they satisfy

$$\prod s X_s = 1$$

and

$$\prod p Z_p = 1.$$  (83)

As a consequence, two topological qubit freedoms are left which may be used for encoding. The Hamiltonian (82) chooses non-ergodically by multiplying the individual star and plaquette observables by positive but otherwise arbitrary coefficients, this will not change the set of ground states. Here too, it is natural to consider the commutant of such a generic Hamiltonian which consists of a product of two qubit algebras and

$$A_{XZ}.$$  This is seen quite explicitly by introducing, similarly to (42), observables for two encoded qubits

$$X_1 = \prod_{j \in c_1} \sigma_x^j,$$

$$X_2 = \prod_{j \in c_2} \sigma_x^j,$$

$$Z_1 = \prod_{j \in d_1} \sigma_z^j,$$

$$Z_2 = \prod_{j \in d_2} \sigma_z^j.$$  (84)

Here, $c_1, d_1, c_2$ and $d_2$ are the loops shown in figure 1. Unlike for the Ising ring, all qubit observables are very delocalized.

Let us divide the set of spins into four disjoint subsets, see figure 2: the snake, the comb, spin 1 and spin 2. Spin 1 is located at the crossing of $X_1$ and $Z_1$, i.e. $c_1$ and $d_1$, and similarly for spin 2. Note that the qubit $X_1$ has been modified a little, so that it closely follows the snake.
The energy barrier

\[ E_B \sim f(N) \]
The energy barrier

- Arrhenius law
  \[ t_{mem} \sim e^{\beta E_B} \]

Phenomenological law of the lifetime

\[ E_B \sim f(N) \]

Bravyi, Sergey, and Barbara Terhal, J. Phys. 11 (2009) 043029

Olivier Landon-Cardinal, David Poulin Phys. Rev. Lett. 110, 090502 (2013)
The energy barrier

- **Arrhenius law**
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  Bravyi, Sergey, and Barbara Terhal. J. Phys. 11 (2009) 043029

  Olivier Landon-Cardinal, David Poulin. Phys. Rev. Lett. 110, 090502 (2013)

- **Question:**
  Can we prove a connection between the energy barrier and thermalization?
Stabilizer Hamiltonians

A set of commuting Pauli matrices

\[ G = \{g_1, \ldots, g_M\} \quad [g_i, g_j] = 0 \]

Example: Toric Code

\[ A_v = Z_1 Z_2 Z_3 Z_4 \]
\[ B_p = X_1 X_2 X_3 X_4 \]

Kitaev, A.Y. (2003). Fault-tolerant quantum computation by anyons. Annals of Physics, 303(1), 2–30.
Stabilizer Hamiltonians

A set of commuting Pauli matrices

\[ \mathcal{G} = \{ g_1, \ldots, g_M \} \quad [g_i, g_j] = 0 \]

The Stabilizer Group \( \mathcal{S} = \langle \mathcal{G} \rangle \)

Logical operators \( \mathcal{C}(\mathcal{S}) \backslash \mathcal{S} \)

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Stabilizer Hamiltonian

\[ H = -J \sum_k g_k \]

Example: Toric Code

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Open system dynamics

- Lindblad master equation

\[ \dot{\rho} = \mathcal{L}(\rho) = -i[H, \rho] + \sum_{k} L_k \rho L_k^\dagger - \frac{1}{2} \{ L_k^\dagger L_k, \rho \} + \]
Open system dynamics

• Lindblad master equation

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\[ T_t(f) = \exp(\mathcal{L}t)(f) \]
Open system dynamics

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- With a unique fixed point

\[ \mathcal{L}(\sigma) = 0 \quad \sigma > 0 \]
Thermal noise model & Weak coupling limit

\[ H_I = \lambda \sum_{\alpha} S_{\alpha} \otimes B_{\alpha} \]

\[ \rho_R \propto e^{-\beta H_B} \]
Thermal noise model & Weak coupling limit

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The evolution:

\[ \rho_S(t + \Delta t) = \text{tr}_R \left[ e^{-iH\Delta t} (\rho(t) \otimes \rho_R) e^{iH\Delta t} \right] \]
Thermal noise model & Weak coupling limit

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Weak coupling limit & Markovian approximation:

\[ \partial_t \rho = \mathcal{L}(\rho) \]

Davies, E. B. (1974). Markovian master equations. Communications in Mathematical Physics, 39(2), 91–110.
The Davies generator

\[ \mathcal{L}_\beta(\rho) = \sum_{\alpha, \omega} h^\alpha(\omega) \left( S_\alpha(\omega) \rho S_\alpha^\dagger(\omega) - \frac{1}{2} \{ S_\alpha^\dagger(\omega) S_\alpha(\omega), \rho \} \right) \]
The Davies generator

\[ \mathcal{L}_\beta(\rho) = \sum_{\alpha, \omega} h^\alpha(\omega) \left( S_{\alpha}(\omega) \rho S_{\alpha}^\dagger(\omega) - \frac{1}{2} \{ S_{\alpha}^\dagger(\omega) S_{\alpha}(\omega), \rho \} \right) \]

For a single thermal bath:

KMS conditions*:

\[ h^\alpha(-\omega) = e^{-\beta \omega} h^\alpha(\omega) \]

\[ \sigma S_{\alpha}(\omega) = e^{\beta \omega} S_{\alpha}(\omega) \sigma \]

\[ \sigma \propto e^{-\beta H_S} \]

Ensures detail balance with:

Gibbs state as steady state

* Kubo, R. (1957). Statistical-Mechanical Theory of Irreversible Processes. I. General Theory and Simple Applications to Magnetic and Conduction Problems. Journal of the Physical Society of Japan, 12(6), 570–586.

Martin, P., & Schwinger, J. (1959). Theory of Many-Particle Systems. I. Physical Review, 115(6), 1342–1373.
Davies generator for Pauli stabilizers

- Lindblad operators

\[ e^{iHt} S_\alpha e^{-iHt} = \sum_\omega S_\alpha(\omega) e^{i\omega t} \]
Davies generator for Pauli stabilizers

- Lindblad operators

\[ e^{iHt} S_\alpha e^{-iHt} = \sum_\omega S_\alpha(\omega) e^{i\omega t} \]

- Syndrome projectors

\[ S_\alpha(\omega) = \sum_{\omega = \epsilon_a - \epsilon_a \alpha} \sigma_i^\alpha P(a) \]
Davies generator for Pauli stabilizers

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  \[ e^{iHt} S_\alpha e^{-iHt} = \sum_\omega S_\alpha(\omega) e^{i\omega t} \]

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  \[ S_\alpha(\omega) = \sum_{\omega=\epsilon_a-\epsilon_\alpha} \sigma_i^\alpha P(a) \]

- The Lindblad operators are local! (when the code is)
Davies generator for Pauli stabilizers

- **Lindblad operators**
  \[ e^{iHt} S_\alpha e^{-iHt} = \sum_\omega S_\alpha(\omega)e^{i\omega t} \]

- **Syndrome projectors**
  \[ S_\alpha(\omega) = \sum_{\omega=\epsilon_\alpha-\epsilon_\alpha\alpha} \sigma_\alpha P(a) \]

- **The Lindblad operators are local! (when the code is)**
Convergence to the fixed point $\sigma$

- For a unique fixed point:

\[
 t > t_{\text{mix}}(\epsilon) \quad \Rightarrow \quad \| e^{\mathcal{L}t} (\rho_0) - \sigma \|_{tr} \leq \epsilon
\]
Convergence to the fixed point $\sigma$

- For a unique fixed point:

$$ t > t_{\text{mix}}(\epsilon) \implies \| e^{\mathcal{L}t}(\rho_0) - \sigma \|_{tr} \leq \epsilon $$

- Exponential convergence

$$ \| e^{t\mathcal{L}}(\rho_0) - \sigma \|_{tr} \leq A e^{-Bt} $$
Convergence to the fixed point $\sigma$

- For a unique fixed point:

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- Exponential convergence

$$\| e^{t\mathcal{L}}(\rho_0) - \sigma \|_{tr} \leq \sqrt{\| \sigma^{-1} \|} e^{-\lambda t}$$

Temme, K., et al. "The $\chi^2$-divergence and mixing times of quantum Markov processes." *Journal of Mathematical Physics* 51.12 (2010): 122201.
Convergence to the fixed point $\sigma$

- For a unique fixed point:

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$$ \|e^{t\mathcal{L}}(\rho_0) - \sigma\|_{tr} \leq \sqrt{\|\sigma^{-1}\|} e^{-\lambda t} $$

Temme, K., et al. "The $\chi^2$-divergence and mixing times of quantum Markov processes." *Journal of Mathematical Physics* 51.12 (2010): 122201.

- A thermal $\sigma$ implies the bound

$$ \|\sigma^{-1}\| \sim e^{c\beta N} \implies t_{\text{mix}} \sim O(\beta N \lambda^{-1}) $$
Theorem 14 For any commuting Pauli Hamiltonian $H$, eqn. (1), the spectral gap $\lambda$ of the Davies generator $\mathcal{L}_\beta$, c.f. eqn (15), with weight one Pauli couplings $W_1$ is bounded by

$$\lambda \geq \frac{h^*}{4\eta^*} \exp(-2\beta \bar{\epsilon}),$$

(81)
Spectral gap bound

**Theorem 14** For any commuting Pauli Hamiltonian $H$, eqn. (1), the spectral gap $\lambda$ of the Davies generator $\mathcal{L}_\beta$, c.f. eqn (15), with weight one Pauli couplings $W_1$ is bounded by

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(81)

The constants are:

The largest Pauli path: $\eta^* = \mathcal{O}(N)$

smallest transition rate: $h^* \geq c_0 e^{-\beta \Delta}$

generalized energy barrier : $\bar{\epsilon}$
Generalized energy barrier

Paths on the Pauli Group \( \gamma_0, \gamma_1, \ldots, \gamma_t \in \mathcal{P} \)

\[
\gamma_0 = I
\]
Generalized energy barrier

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Generalized energy barrier

Paths on the Pauli Group \( \gamma_0, \gamma_1, \ldots, \gamma_t \in \mathcal{P} \)

\[ \gamma_t = \eta \]
Generalized energy barrier

Paths on the Pauli Group

\[ \gamma_0, \gamma_1, \ldots, \gamma_t \in \mathcal{P} \]

Reduced set of generators

\[ \mathcal{G}_\eta = \left\{ g \in \mathcal{G} \mid [g, \eta] = 0 \right\} \]
Generalized energy barrier

Paths on the Pauli Group \[ \gamma_0, \gamma_1, \ldots, \gamma_t \in \mathcal{P} \]

Reduced set of generators \[ G_\eta = \left\{ g \in \mathcal{G} \mid [g, \eta] = 0 \right\} \]

Energy barrier of the Pauli \[ \bar{\epsilon}(\eta) = \max_t \left\{ 2 \# \left\{ g_k \in G_\eta \mid \{g_k, \gamma_t\} = 0 \right\} \right\} \]
Generalized energy barrier

Paths on the Pauli Group
\[ \gamma_0, \gamma_1, \ldots, \gamma_t \in \mathcal{P} \]

Reduced set of generators
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The generalized energy barrier
\[ \bar{\epsilon} = \min_{\{\gamma\}} \max_\eta \bar{\epsilon}(\eta) \]
Generalized energy barrier

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The generalized energy barrier
\[ \bar{\epsilon} = \min_{\{\gamma\}} \max_{\eta} \bar{\epsilon}(\eta) \]

Example: 2D Toric Code
\[ \lambda \geq \frac{1}{8N} e^{-\beta 6J} \]
3D Toric Code

Consider the toric code on an $L \times L \times L$ lattice

$$H = -J \sum_v A_v - J \sum_p B_p$$
3D Toric Code

Consider the toric code on an $L \times L \times L$ lattice

$\bar{\epsilon} \sim 2JL$
3D Toric Code

Consider the toric code on an $L \times L \times L$ lattice

\[ \bar{\epsilon} \sim 2JL \]

leads to a bound \[ \lambda \geq \mathcal{O}(L^{-3}e^{-4J\beta L}) \]
3D Toric Code

Consider the toric code on an $L \times L \times L$ lattice

$$\bar{\epsilon} \sim 2JL$$

leads to a bound

$$\lambda \geq O(L^{-3} e^{-4J\beta L})$$

High temperature bound

$$\kappa(\beta^*) \leq 1$$

$$\lambda \geq \frac{1 - \kappa(\beta)}{\log(2)}$$
Discussion of the bound

- Relationship to Arrhenius law

\[ t_{\text{mem}} \sim e^{\beta E_B} \quad t_{\text{mix}} = O(\beta N^2 e^{2 \beta \bar{e}}) \]
Discussion of the bound

- Relationship to Arrhenius law

\[ t_{\text{mem}} \sim e^{\beta E_B} \]

\[ t_{\text{mix}} = \mathcal{O}(\beta N^2 e^{2\beta \bar{e}}) \]

- It would be nicer to have a bound that includes "entropic contributions"
Discussion of the bound

- Relationship to Arrhenius law
  \[ t_{mem} \sim e^{\beta E_B} \quad t_{mix} = \mathcal{O}(\beta N^2 e^{2\beta \bar{\epsilon}}) \]

- It would be nicer to have a bound that includes “entropic contributions”

- Can we get rid of the 1/N factor?
  \[ \lambda \geq \frac{h^*}{4N} e^{-2\beta \bar{\epsilon}} \]
Proof sketch

• The Poincare Inequality
• Matrix pencils and the PI
• The canonical paths bound
• The spectral gap and the energy barrier
The Poincare Inequality

\[ \lambda \text{Var}_\sigma(f, f) \leq \mathcal{E}(f, f) \]
The Poincare Inequality

\[
\lambda \left( \text{tr} \left[ \sigma f^\dagger f \right] - \text{tr} \left[ \sigma f \right]^2 \right) \leq -\text{tr} \left[ \sigma f^\dagger \mathcal{L}(f) \right]
\]
The Poincare Inequality

\[ \lambda \left( \text{tr} \left[ \sigma f^\dagger f \right] - \text{tr} \left[ \sigma f \right]^2 \right) \leq -\text{tr} \left[ \sigma f^\dagger \mathcal{L}(f) \right] \]

For classical Markov processes

- Sampling the Permanent: M. Jerrum, A. Sinclair. "Approximating the permanent." SIAM journal on computing 18.6 (1989): 1149-1178.
- Powerful because it can lead to a geometric interpretation

Cheeger’s bound

Canonical paths
The Poincare Inequality

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Challenges in the quantum setting

- We are missing a general geometric picture

Cheeger’s bound

Canonical paths
Poincare and a Matrix pencil \( \lambda^{-1} = \tau \)

Equivalent formulation for \( \lambda \text{Var}_\sigma(f, f) \leq \mathcal{E}(f, f) \)

minimize \( \tau \quad \text{subject to} \quad \tau \hat{\mathcal{E}} - \hat{\mathcal{V}} \geq 0 \)

where \( \mathcal{E}(f, f) = (f|\hat{\mathcal{E}}|f) \quad \text{and} \quad \text{Var}_\sigma(f, f) = (f|\hat{\mathcal{V}}|f) \)
Poincare and a Matrix pencil \( \lambda^{-1} = \tau \)

Equivalent formulation for \( \lambda \text{Var}_\sigma(f, f) \leq \mathcal{E}(f, f) \)

\[
\begin{align*}
\text{minimize } \tau & \quad \text{subject to } \quad \tau \hat{\mathcal{E}} - \hat{\mathcal{V}} \geq 0 \\
\text{where } \quad \mathcal{E}(f, f) = (f|\hat{\mathcal{E}}|f) \quad \text{and} \quad \text{Var}_\sigma(f, f) = (f|\hat{\mathcal{V}}|f)
\end{align*}
\]

Lemma: Let \( \hat{\mathcal{E}} = AA^\dagger \) and \( \hat{\mathcal{V}} = BB^\dagger \)

\[
\tau = \min \|W\|^2 \quad \text{subject to } \quad AW = B
\]

Boman, Erik G., and Bruce Hendrickson. "Support theory for preconditioning." SIAM Journal on Matrix Analysis and Applications 25.3 (2003): 694-717.
Suitable matrix factorization

Some intuition from $\beta \to 0$

$$\mathcal{E}(f, f) \quad \xrightarrow{\beta \to 0} \quad \mathcal{L}(f) \sim \sum_{i: \alpha_i} (\sigma_i^{\alpha_i} f \sigma_i^{\alpha_i} - f)$$
Suitable matrix factorization

Some intuition from $\beta \to 0$

$\mathcal{E}(f, f) \quad \rightarrow \quad \mathcal{L}(f) \sim \sum_{i: \alpha_i} (\sigma_i^{\alpha_i} f \sigma_i^{\alpha_i} - f)$

$\text{Var}(f, f) \quad \rightarrow \quad \mathcal{V}(f) \sim \frac{1}{4N} \sum_{\gamma} (\sigma_1^{\gamma_1} \ldots \sigma_N^{\gamma_N} f \sigma_1^{\gamma_1} \ldots \sigma_N^{\gamma_N} - f)$
Suitable matrix factorization

Some intuition from $\beta \to 0$

\[ \mathcal{E}(f, f) \rightarrow \mathcal{L}(f) \sim \sum_{i: \alpha_i} (\sigma^\alpha_i f \sigma^\alpha_i - f) \]

\[ \text{Var}(f, f) \rightarrow \mathcal{V}(f) \sim \frac{1}{4N} \sum_{\gamma} (\sigma_1^\gamma \ldots \sigma_N^\gamma f \sigma_1^\gamma \ldots \sigma_N^\gamma - f) \]

Choosing a decomposition in terms of

\[ (\sigma_1^x f \sigma_1^x - f) + (\sigma_2^z f \sigma_2^z - f) + (\sigma_3^x f \sigma_3^x - f) \sim (\sigma_1^x \sigma_2^z \sigma_3^x f \sigma_1^x \sigma_2^z \sigma_3^x - f) \]
Suitable matrix factorization

Some intuition from $\beta \rightarrow 0$

$E(f, f) \quad \rightarrow \quad L(f) \sim \sum_{i: \alpha_i} (\sigma_i^{\alpha_i} f \sigma_i^{\alpha_i} - f)$

$\text{Var}(f, f) \quad \rightarrow \quad V(f) \sim \frac{1}{4N} \sum_{\gamma} (\sigma_1^{\gamma_1} \ldots \sigma_N^{\gamma_N} f \sigma_1^{\gamma_1} \ldots \sigma_N^{\gamma_N} - f)$

Choosing a decomposition in terms of

$$(\sigma_1^x \sigma_2^z f \sigma_1^x \sigma_2^z - f) + (\sigma_3^x f \sigma_3^x - f) \sim (\sigma_1^x \sigma_2^z \sigma_3^x f \sigma_1^x \sigma_2^z \sigma_3^x - f)$$
Suitable matrix factorization

Some intuition from $\beta \to 0$

$\mathcal{E}(f, f) \quad \longrightarrow \quad \mathcal{L}(f) \sim \sum_{i: \alpha_i} (\sigma_i^{\alpha_i} f \sigma_i^{\alpha_i} - f)$

$\text{Var}(f, f) \quad \longrightarrow \quad \mathcal{V}(f) \sim \frac{1}{4N} \sum_{\gamma} (\sigma_1^{\gamma_1} \ldots \sigma_N^{\gamma_N} f \sigma_1^{\gamma_1} \ldots \sigma_N^{\gamma_N} - f)$

Choosing a decomposition in terms of

$$(\sigma_1^x \sigma_2^z f \sigma_1^x \sigma_2^z - f) + (\sigma_3^x f \sigma_3^x - f) \sim (\sigma_1^x \sigma_2^z \sigma_3^x f \sigma_1^x \sigma_2^z \sigma_3^x - f)$$

A generalization yields to the matrix triple $[A, B, W]$

$\|W\|^2$ can be bounded by suitable norm bounds
Canonical paths bound

- The norm bound on $\|W\|^2$ can be evaluated in the following picture.

Dressed Pauli paths:

$\hat{\eta}_a = [(a, 0), (a^{\alpha_1}, \alpha_1), \ldots, (a^\eta, \eta)]$
Canonical paths bound

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Dressed Pauli paths:

$\hat{\eta}_a = [(a, 0), (a^{\alpha_1}, \alpha_1), \ldots, (a^n, \eta)]$
 Canonical paths bound

- The norm bound on $\|W\|^2$ can be evaluated in the following picture

  Dressed Pauli paths:

  $\tilde{\eta}_a = [(a, 0), (a^{\alpha_1}, \alpha_1), \ldots, (a^{\eta}, \eta)]$
Canonical paths bound

- The norm bound on $\|W\|^2$ can be evaluated in the following picture.

Dressed Pauli paths:

$\hat{\eta}_a = \{(a, 0), (a^{\alpha_1}, \alpha_1), \ldots, (a^{\eta}, \eta)\}$

The matrix norm bound yields

$$\tau \leq \max_{\xi} \frac{4\eta^*}{2^N h(\omega^\alpha(b))\rho_b} \sum_{\hat{\eta}_a \in \Gamma(\xi)} \rho_a \rho_a^{\eta}$$
The spectral gap and the energy barrier

• The only challenge is the maximum in the definition of $\tau$
The spectral gap and the energy barrier

- The only challenge is the maximum in the definition of $\mathcal{T}$

Injective map (Jerrum & Sinclair)

$\Phi_\xi : \Gamma(\xi) \to \mathcal{P}_N$
The spectral gap and the energy barrier

- The only challenge is the maximum in the definition of $\mathcal{T}$

Injective map (Jerrum & Sinclair)

$$\Phi_\xi : \Gamma(\xi) \to \mathcal{P}_N \quad [\Phi_\xi(\hat{\eta}_a)]_k = \begin{cases} (0, 0)_k : k \leq \xi \\ \eta_k : k \geq \xi \end{cases}$$
The spectral gap and the energy barrier

- The only challenge is the maximum in the definition of $\mathcal{T}$

Injective map (Jerrum & Sinclair)

$$\Phi_\xi : \Gamma(\xi) \to \mathcal{P}_N \quad \Phi_\xi(\hat{\eta}_a)_k = \begin{cases} (0, 0)_k : k \leq \xi \\ \eta_k : k > \xi \end{cases}$$

Bounding $\mathcal{T}$

$$\epsilon_{b\eta \oplus \xi} + \epsilon_{b\xi} - \epsilon_b - \epsilon_{bn} \leq 2\bar{\epsilon}$$
The spectral gap and the energy barrier

\[ \Phi_\xi : \Gamma(\xi) \rightarrow \mathcal{P}_N \]

\[ [\Phi_\xi(\hat{\eta}_a)]_k = \begin{cases} 
(0, 0)_k : k \leq \xi \\
\eta_k : k > \xi \end{cases} \]

Injective map (Jerrum & Sinclair)

Bounding \( \mathcal{T} \)

\[ h^\alpha(\omega^\alpha(a)) \rho_a \rho_b^{\Phi_\xi(\hat{\eta}_b)} \geq h^* e^{-\beta 2\overline{\epsilon}} \rho_b \rho_b^n \]
The spectral gap and the energy barrier

- The only challenge is the maximum in the definition of $\mathcal{T}$

Injective map (Jerrum & Sinclair)

$$\Phi_{\xi} : \Gamma(\xi) \to \mathcal{P}_N \quad [\Phi_{\xi}(\hat{\eta}_a)]_k = \left\{ \begin{array}{ll} (0, 0)_k : k \leq \xi \\ \eta_k : k > \xi \end{array} \right.$$  

Bounding $\mathcal{T}$

$$h^\alpha (\omega^\alpha (a)) \rho_{a} \rho_{b} \Phi_{\xi}(\hat{\eta}_b) \geq h^* e^{-\beta 2\hat{\alpha}} \rho_{b} \rho_{b_{\eta}}$$

$$\tau_{\gamma_0} \leq 4 \frac{\eta^*}{h^*} e^{\beta 2\hat{\alpha}} \max_{\hat{\xi}} \sum_{\hat{\eta}_b \in \Gamma(\hat{\xi})} \frac{1}{2N} \rho_{b} \Phi_{\xi}(\hat{\eta}_b)$$
The spectral gap and the energy barrier

• The only challenge is the maximum in the definition of $\mathcal{T}$

Injective map (Jerrum & Sinclair)

$$
\Phi_\xi : \Gamma(\xi) \to \mathcal{P}_N \quad [\Phi_\xi(\hat{\eta}_a)]_k = \begin{cases} 
(0, 0)_k : k \leq \xi \\
\eta_k : k > \xi
\end{cases}
$$

Bounding $\mathcal{T}$

$$
\lambda^\alpha (\omega^\alpha (a)) \rho_a \rho_b \Phi_\xi (\hat{\eta}_b) \geq \lambda^* e^{-\beta 2\bar{c}} \rho_b \rho_b^\eta
$$

$$
\tau_{\gamma_0} \leq 4 \frac{\lambda^*}{\lambda^*} e^{\beta 2\bar{c}} \max_{\hat{\eta}} \sum_{\hat{\eta}_b \in \Gamma(\hat{\xi})} \frac{1}{2N} \rho_b \Phi_\xi (\hat{\eta}_b) \leq 1
$$
Conclusion and Open Questions

• It would be great if one could extend the results to more general quantum memory models.

• This only provides a converse to the lifetime of the classical memory. It would be great if one could find a converse for the quantum memory time.

• Is it possible to find a bound that also takes the “entropic” contributions into account?

• Can we get rid of the prefactor? $N^{-1}$
