THE BAIRE CLASSIFICATION OF STRONGLY SEPARATELY CONTINUOUS FUNCTIONS ON $\ell_\infty$

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Abstract. We prove that for any $\alpha \in [0, \omega_1)$ there exists a strongly separately continuous function $f : \ell_\infty \to [0, 1]$ such that $f$ belongs to the $(\alpha + 1)’$th /$(\alpha + 2)’$th/ Baire class and does not belong to the $\alpha'$th Baire class if $\alpha$ is finite /infinite/.

1. Introduction

The notion of real-valued strongly separately continuous function defined on $\mathbb{R}^n$ was introduced and studied by Dzagnidze in his paper [2]. He proved that the class of all strongly separately continuous real-valued functions on $\mathbb{R}^n$ coincides with the class of all continuous functions. Later, Cincura, Salat and Visnyai [1] considered strongly separately continuous functions defined on the Hilbert space $\ell_2$ of sequences $x = (x_n)_{n=1}^{\infty}$ of real numbers such that $\sum_{n=1}^{\infty} x_n^2 < +\infty$ and showed that there are essential differences between some properties of strongly separately continuous functions defined on $\ell_2$ and the corresponding properties of functions on $\mathbb{R}^n$. In particular, they noticed that there exists a strongly separately continuous function $f : \ell_2 \to \mathbb{R}$ which does not belong to the first Baire class. Extending these results, Visnya [8] constructed a strongly separately continuous function $f : \ell_2 \to \mathbb{R}$ of the third Baire class which is not quasi-continuous at every point of $\ell_2$. It was shown recently in [6] that for every $2 \leq \alpha < \omega$ there exists a strongly separately continuous function $f : \ell_p \to \mathbb{R}$ which belongs to the $\alpha'$th Baire class and does not belong to the $\beta'$th Baire class on $\ell_p$ for $\beta < \alpha$, where $p \in [1, +\infty)$.

The aim of this paper is to generalize results from [6] to the case of $p = +\infty$. We develop arguments from [3] and prove that for any $\alpha \in [0, \omega_1)$ there exists a strongly separately continuous function $f : \ell_\infty \to [0, 1]$ such that $f$ belongs to the $(\alpha + 1)’$th /$(\alpha + 2)’$th/ Baire class and does not belong to the $\alpha'$th Baire class if $\alpha$ is finite /infinite/.

2. Definitions and notations

Let $\ell_\infty$ be the Banach space of all bounded sequences of reals with the norm

$$\|x\|_\infty = \sup_{k \in \omega} |x_k|$$

for all $x = (x_k)_{k \in \omega} \in \ell_\infty$. For $x, y \in \ell_\infty$ we denote $d_\infty(x, y) = \|x - y\|_\infty$. If $x \in \ell_\infty$ and $\delta > 0$, then

$$B_\infty(x, \delta) = \{y \in \ell_\infty : \|x - y\|_\infty < \delta\}.$$

Definition 2.1. Let $x^0 = (x^0_k)_{k \in \omega} \in \ell_\infty$ and $(Y, | \cdot |)$ be a metric space. A function $f : \ell_\infty \to Y$ is said to be strongly separately continuous at $x^0$ with respect to

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to the $k$-th variable if
\[ \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x = (x_k)_{k \in \omega} \in B_\infty(x^0, \delta) \]
\[ |f(x_1, \ldots, x_k, \ldots) - f(x_1, \ldots, x_{k-1}, x^0_k, x_{k+1}, \ldots)| < \varepsilon. \] (1)

If $f$ is strongly separately continuous at $x^0$ with respect to each variable, then $f$ is said to be strongly separately continuous at $x^0$. Moreover, $f$ is strongly separately continuous on $\ell_\infty$ if it is strongly separately continuous at each point of $\ell_\infty$.

Strongly separately continuous functions we will also call ssc functions for short.

**Definition 2.2.** A subset $A \subseteq X$ of a Cartesian product $X = \prod_{k=1}^{\infty} X_k$ of sets $X_1, X_2, \ldots$ is called $S$-open [4], if
\[ \sigma_1(a) = \{(x_k)_{k=1}^{\infty} \in X : \{|k : x_k \neq a_k| \leq 1\} \subseteq A \]
for all $a = (a_k)_{k=1}^{\infty} \in A$.

If $x \in \ell_\infty$ and $N \subseteq \omega$, then we put
\[ \pi_N(x) = (x_k)_{k \in N}. \]

In the case $N = \{n\}$, we write $\pi_n(x)$ instead of $\pi_{\{n\}}(x)$.

3. **Main result**

Define a function $(\alpha)^*$ as the following
\[ (\alpha)^* = \begin{cases} \alpha, & \alpha \in [0, \omega), \\ \alpha + 1, & \alpha \in [\omega, \omega_1). \end{cases} \] (2)

**Theorem 3.1.** For any $\alpha \in [0, \omega_1]$ there exists a strongly separately continuous function $f : \ell_\infty \to [0, 1]$ which belongs to the $(\alpha + 1)^*$ th Baire class and does not belong to the $\alpha$ th Baire class on $\ell_\infty$.

**Proof.** We define inductively transfinite sequences $(A_\alpha)_{1 \leq \alpha < \omega_1}$ and $(B_\alpha)_{1 \leq \alpha < \omega_1}$ of subsets of $\ell_\infty$ in the following way. Put
\[ A_1 = \{(x_n)_{n=1}^{\infty} \in \ell_\infty : \exists m \forall n \geq m \ x_n = 0\} \quad \text{and} \quad B_1 = \ell_\infty \setminus A_1. \]

Let $(T_n : n \in \omega)$ be a partition of $\omega$ onto infinite sets $T_n = \{t_{n0}, t_{n1}, \ldots\}$, where $(t_{nm})_{m \in \omega}$ is a strictly increasing sequence of numbers $t_{nm} \in \omega$. We put
\[ \ell_{\infty}^n = \{(x_{t_{nm}}) \in \ell_\infty : t_{nm} \in T_n \ \forall m \in \omega\}. \]

For every $n \in \omega$ we denote by $A_n^1 / B_n^1$ the copy of the set $A_1 / B_1$, which is contained in the space $\ell_{\infty}^n$. Assume that for some $\alpha > 1$ we have already defined sequences $(A_\beta)_{1 \leq \beta < \alpha}$ and $(B_\beta)_{1 \leq \beta < \alpha}$ (and their copies $(A^a_n)_{1 \leq \beta < \alpha}$ and $(B^a_n)_{1 \leq \beta < \alpha}$ in $\ell_{\infty}^n$) of subsets of $\ell_\infty$. Now we put
\[ A_\alpha = \begin{cases} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \pi_{T_n}^{-1}(B_{\beta_n}^a), & \alpha = \beta + 1, \\ \bigcup_{n=1}^{\infty} \pi_{T_n}^{-1}(A_{\beta_n}^a), & \alpha = \sup \beta_n, \end{cases} \]
and
\[ B_\alpha = \ell_\infty \setminus A_\alpha. \]

**Claim 1.** For every $\alpha \in [1, \omega_1]$ the following statements are true:

1. the sets $A_\alpha$ and $B_\alpha$ are $S$-open in $\ell_\infty$;
2. for any $y = (y_n)_{n=1}^{\infty} \in \ell_\infty$ with $y_n \neq 0$ for all $n \in \omega$ we have
\[ x = (x_n)_{n \in \omega} \in A_\alpha \iff z = (x_n \cdot y_n)_{n \in \omega} \in A_\alpha. \]
Moreover, \( C \) of sets \( \subseteq C \) for all \( n \geq m \). Since \( \pi_{\tau_\beta}(y) \in \sigma_1(\pi_{\tau_\beta}(x)) \) and \( B_\beta \) is \( S \)-open, \( \pi_{\tau_\beta}(y) \in B_\beta \). Therefore, \( y \in A_\beta \). We argue similarly in the case where \( \alpha \) is a limit ordinal.

We fix \( y = (y_n)_{n=1}^\infty \in \ell_\infty \) such that \( y_n \neq 0 \) for all \( n \in \mathbb{N} \). The statement is true for \( \alpha = 1 \), since \( A_1 = \sigma(0) \). Assume that for some \( \alpha < \omega_1 \) the property is valid for all \( \beta < \alpha \). Let \( \alpha = \beta + 1 \) for some \( \beta \). The inductive assumption implies that

\[
x \in A_\alpha \iff \exists m \in \mathbb{N} \ \forall n \geq m \ \pi_{\tau_\alpha}(x) \in B_\beta^n
\]

\[
z \in A_\alpha \iff \exists m \in \mathbb{N} \ \forall n \geq m \ \pi_{\tau_\alpha}(z) \in B_\beta^n
\]

We argue similarly in the case of limit \( \alpha \).

Consider the equivalent metric

\[
d(x, y) = \min\{d_\infty(x, y), 1\}
\]

on the space \( \ell_\infty \).

**Claim 2.** For every \( \alpha \in [1, \omega_1) \) the following condition holds:

\( (*) \) for every set \( C \subseteq (\ell_\infty, d) \) of the additive/multiplicative class \( \alpha \) there exists a contracting mapping \( f : (\ell_\infty, d) \to (\ell_\infty, d) \) with the Lipschitz constant \( L = \frac{1}{2} \) such that

\[
C = f^{-1}(A_\alpha) \quad / C = f^{-1}(B_\alpha)/,
\]

\[
|\pi_n(f(x))| < 1 \quad \forall x \in \ell_\infty \ \forall n \in \omega.
\]

**Proof of Claim 2.** We will argue by the induction on \( \alpha \). Let \( C \) be an arbitrary \( F_\gamma \)-subset of \( (\ell_\infty, d) \). Then there exists an increasing sequence \( (C_n)_{n \in \omega} \) of closed subsets of \( (\ell_\infty, d) \) such that \( C = \bigcup_{n \in \omega} C_n \). Consider a map \( f : \ell_\infty \to \ell_\infty \), defined by the rule

\[
f(x) = (\frac{1}{2}d(x, C_1), \ldots, \frac{1}{2}d(x, C_n), \ldots)
\]

for all \( x \in \ell_\infty \).

We show that \( C = f^{-1}(A_1) \). Take \( x \in C \) and choose \( m \in \omega \) such that \( x \in C_n \) for all \( n \geq m \). Then \( d(x, C_n) = 0 \) and \( \pi_n(f(x)) = 0 \) for all \( n \geq m \). Hence, \( x \) belongs to the right-hand side of the equality. Now we prove the inverse inclusion. Let \( x \in f^{-1}(A_1) \). Then there exists \( m \in \omega \) such that \( \pi_n(f(x)) = 0 \) for all \( n \geq m \). Consequently, \( d(x, C_n) = 0 \) for all \( n \geq m \). Since \( C_n \) is closed, \( x \in C_n \) for all \( n \geq m \). Therefore, \( x \in \bigcup_{n \in \omega} C_n = C \).

Since

\[
d(f(x), f(y)) \leq d_\infty(f(x), f(y)) = \sup_{n \in \omega} \frac{1}{2}d(x, C_n) - \frac{1}{2}d(y, C_n) \leq \frac{1}{2}d(x, y)
\]

for all \( x, y \in \ell_\infty \), the mapping \( f \) is contracting with the Lipschitz constant \( L = \frac{1}{2} \).

Moreover,

\[
|\pi_n(f(x))| = \frac{1}{2}d(x, C_n) < 1
\]

for every \( n \in \omega \).

Assume that for some \( \alpha < \omega_1 \) the condition \( (*) \) is valid for all \( \beta < \alpha \). Let \( C \subseteq (\ell_\infty, d) \) be any set of the \( \alpha \)th additive class. Take an increasing sequence of sets \( C_n \) such that \( C = \bigcup_{n \in \omega} C_n \), where every \( C_n \) belongs to the multiplicative class \( \beta \) if \( \alpha = \beta + 1 \), and in the case \( \alpha = \sup \beta_n \), we can assume that \( C_n \) belongs to the additive class \( \beta_n \) for every \( n \in \omega \). By the inductive assumption there exists a sequence \( (f_n)_{n \in \omega} \) of contracting maps \( f_n : (\ell_\infty, d) \to (\ell_\infty, d) \) with the Lipschitz
constant $L = \frac{1}{2}$ such that
\begin{align}
C_n &= \begin{cases} 
  f_n^{-1}(B_\beta), & \alpha = \beta + 1, \\
  f_n^{-1}(A_{\beta_n}), & \alpha = \sup \beta_n, 
\end{cases} \\
|\pi_m(f_n(x))| < 1 & \forall x \in \ell_\infty \forall n, m \in \omega.
\end{align}

(5) (6)

For every $k \in \omega$ we choose a unique pair $(n(k), m(k)) \in \omega^2$ such that
\begin{align}
k = t_{n(k)m(k)} \in T_{n(k)}. 
\end{align}

(3)

For all $x \in \ell_\infty$ and $n, m \in \omega$ we put $f_{nm}(x) = \pi_m(f_n(x))$ and consider a map $f : \ell_\infty \to \ell_\infty$, defined by the rule
\begin{align}
f(x) = (\frac{1}{2} f_{n(1)m(1)}(x), \ldots, \frac{1}{2} f_{n(k)m(k)}(x), \ldots)
\end{align}

(4) for all $x \in \ell_\infty$. The inequalities
\begin{align}
|f_{nm}(x) - f_{nm}(y)| &= |\pi_m(f_n(x)) - \pi_m(f_n(y))| \\
&\leq \sup_{m \in \omega} |\pi_m(f_n(x)) - \pi_m(f_n(y))| = d_\infty(f_n(x), f_n(y))
\end{align}

(5)

and
\begin{align}
|f_{nm}(x) - f_{nm}(y)| \leq 2
\end{align}

(6)

imply that
\begin{align}
\frac{1}{2} |f_{nm}(x) - f_{nm}(y)| \leq d(f_n(x), f_n(y)) \leq \frac{1}{2} d(x, y)
\end{align}

(7)

for all $x, y \in \ell_\infty$ and $n, m \in \omega$. Then
\begin{align}
d(f(x), f(y)) \leq d_\infty(f(x), f(y)) = \\
= \sup_{k \in \omega} |\frac{1}{2} (f_{n(k)m(k)}(x) - f_{n(k)m(k)}(y))| \leq \frac{1}{2} d(x, y)
\end{align}

(8)

for all $x, y \in \ell_\infty$. Therefore, $f : (\ell_\infty, d) \to (\ell_\infty, d)$ is a Lipschitz map with the constant $L = \frac{1}{2}$.

It remains to show that $C = f^{-1}(A_\alpha)$. Assume that $\alpha = \beta + 1$ (we argue similarly if $\alpha$ is limit). Let us observe that $x \in C$ if and only if there exists $m \in \omega$ such that $f_n(x) \in B_\beta$ for all $n \geq m$. Since
\begin{align}
\pi_{T_n}(f(x)) = \left( \frac{1}{2} \pi_k(f_n(x)) \right)_{k \in T_n},
\end{align}

we have
\begin{align}
f_n(x) \in B_\beta \iff \pi_{T_n}(f(x)) \in B_\beta^\omega.
\end{align}

(9)

by statement (2) of Claim 1. Therefore, $C = f^{-1}(A_\alpha)$. \hfill \Box

CLAIM 3. For every $\alpha \in [1, \omega_1)$ the set $A_\alpha$ belongs to the additive class $\alpha$ and does not belong to the multiplicative class $\alpha$ in $\ell_\infty$.

Proof of Claim 3. If $\alpha = 1$, then
\begin{align}
A_1 = \bigcup_{n \in \omega} \{x \in \ell_\infty : |\{k \in \omega : x_k \neq 0\}| \leq n\}
\end{align}

(10)

is an $F_\sigma$-subset of $\ell_\infty$, since every set $\{x \in \ell_\infty : |\{k \in \omega : x_k \neq 0\}| \leq n\}$ is closed. Consequently, $B_1$ is $G_\delta$-subset of $\ell_\infty$. Suppose that for some $\alpha \geq 1$ the set $A_\beta / B_\beta$ belongs to the additive / multiplicative / class $\beta$ in $\ell_\infty$ for every $\beta < \alpha$. Since every projection $\pi_{T_n} : \ell_\infty \to \ell_\infty$ is continuous, the set $A_\alpha$ belongs to the additive class $\alpha$ in $\ell_\infty$ and the set $B_\alpha$ belongs to the multiplicative class $\alpha$ in $\ell_\infty$.

Fix $\alpha \in [1, \omega_1)$. In order to show that $A_\alpha$ does not belong to the $\alpha$'th multiplicative class we assume the contrary. Claim 2 implies that there exists a contraction
$f: (\ell_\infty,d) \to (\ell_\infty,d)$ such that $A_n = f^{-1}(B_n)$. By the Contraction Map Principle, there would be a fixed point for $f$, which implies a contradiction. \hfill \Box

Now we are ready to construct a function $f$ from the statement of the theorem. Let $\alpha \in [0,\omega_1)$ be fixed. If $\alpha = 0$, then we put $A = c$, where $c$ is the subspace of $\ell_\infty$ consisting of all convergent sequences of real numbers. If $\alpha > 0$, then previous steps imply the existence of an $\mathcal{S}$-open set $A \subseteq \ell_\infty$ such that $A$ belongs to the $(\alpha)$’th additive class and does not belong to the $(\alpha)$’th multiplicative class. In any case for every $x \in \ell_\infty$ we put

$$f(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

We prove that $f: \ell_\infty \to [0,1]$ is strongly separately continuous. Fix $\varepsilon > 0$, $k \in \omega$ and $x = (x_n)_{n \in \omega} \in \ell_\infty$. We put $\delta = 1$ and notice that for all $y \in B_\infty(x,\delta)$ we have

$$y = (y_1,\ldots,y_k,\ldots) \in A \iff z = (y_1,\ldots,y_{k-1},x_k,y_{k+1},\ldots) \in A,$$

since $A$ is $\mathcal{S}$-open. Therefore,

$$|f(y) - f(z)| = 0$$

for all $y \in B_\infty(x,\delta)$ and $z = (y_1,\ldots,y_{k-1},x_k,y_{k+1},\ldots)$. Hence, $f$ is strongly separately continuous at $x$ with respect to the $k$’th variable.

Notice that both $A$ and $X \setminus A$ are of the $(\alpha + 1)$’th additive class, that is, $A$ is ambiguous set of the $(\alpha+1)$’th class in $\ell_\infty$. It is well-known that the characteristic function of any ambiguous set of the class $\xi$ in any metric space belongs to the $\xi$’th Baire class [17,31] for any $\xi \in [1,\omega_1)$. Therefore, $f \in B_{(\alpha+1)}(\ell_\infty,[0,1])$.

If $\alpha = 0$, then $f$ is discontinuous exactly on $A$ and hence $f \notin B_0(\ell_\infty,[0,1])$.

In case $\alpha > 0$ we assume that $f \in B_\alpha(\ell_\infty,[0,1])$. Then $f$ belongs to the $(\alpha)$’th Borel class. Therefore, $A = f^{-1}(1)$ is the set of the $(\alpha)$’th multiplicative class in $\ell_\infty$, which contradicts to the choice of $A$. \hfill \Box

**Remark 3.2.** The existence of an ssc function $f: \ell_\infty \to [0,1]$ which is not Baire measurable was proved in [18]. The Baire classification of ssc functions defined on $\mathbb{R}^\omega$ was studied in [1].

Theorem 3.1 suggests the following question.

**Question 3.3.** Does there exist a strongly separately continuous function $f: \ell_\infty \to [0,1]$ such that $f \in B_{\omega+1} \setminus B_\omega$?

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