Pocket resonances in low-energy antineutrons reactions with nuclei

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Upon investigating whether the variation of the antineutron-nucleus annihilation cross-sections at very low energies satisfy Bethe-Landau’s power law of $\sigma_{\text{ann}}(p) \propto 1/p^\alpha$ behavior as a function of the antineutron momentum $p$, we uncover unexpected regular oscillatory structures in the low antineutron energy region from 0.001 to 10 MeV, with small amplitudes and narrow periodicity in the logarithm of the antineutron energies, for large-$A$ nuclei such as Pb and Ag. Subsequent semiclassical analyses of the $S$ matrices reveal that these oscillations are pocket resonances that arise from quasi-bound states inside the pocket and the interference between the waves reflecting inside the optical potential pockets with those from beyond the potential barriers, implicit in the nuclear Ramsauer effect. They are the continuation of bound states in the continuum. Experimental observations of these pocket resonances will provide vital information on the properties of the optical model potentials and the nature of the antineutron annihilation process.
I. INTRODUCTION

Matter-antimatter asymmetry in the Universe is one of the greatest mysteries of modern physics [1]. To unravel this mystery, there has been a great deal of experimental and theoretical investigations on matter-antimatter interactions. So far, most of the obtained information centers around antiproton-nucleus ($\bar{p}A$) reactions and structures [2–14]. The corresponding information on antineutron-nucleus ($\bar{n}A$) reaction [15], on the other hand, remains comparatively sparse and limited with the most recent work from the OBELIX’s collaboration [16, 17]. Nevertheless, the $\bar{n}A$ annihilation is essential in the process of quantifying signals from $n \rightarrow \bar{n}$ oscillations in matter [18–21], and the significant the connection between the $\bar{n}A$ interaction-potential and the $n \rightarrow \bar{n}$ oscillations rates have also been examined theoretically [22–24].

It has been generally expected that in the $s$-wave limit the $\bar{n}A$ and $\bar{p}A$ annihilation cross-sections are to obey Bethe-Landau’s power-law, $\sigma_{\text{ann}}(\bar{n}A, \bar{p}A) \propto 1/p^\alpha$ as a function of the antineutron momentum $p$, with $\alpha = 1$ for $\bar{n}A$ and $\alpha = 2$ for $\bar{p}A$ [25]. But recent experimental data (see Fig. 5 of Ref. [3]) revealed that the annihilation cross sections of $\bar{n}$, and $\bar{p}$, on C nuclei at low energies, appear to follow a similar trend with only minor differences. Much curious about the puzzle, we recently introduced a new momentum-dependent optical potential to investigate the behavior of the annihilation cross-sections for the $\bar{n}$, and the $\bar{p}$, on C, Al, Fe, Cu, Ag, Sn, and Pb nuclei in the momentum range 50–500 MeV/c [26], via the ECIS code [27]. The calculated results were concordant with the OBELIX’s annihilation cross-section data [16, 17]. However, between 40 and 100 MeV/c, the optical model calculations indicated that $\alpha \approx 1/2$ for $\bar{n}A$ and $\alpha \approx 1.5$ for $\bar{p}A$, leading us to conclude our low-energy annihilation reaction in question was yet to reach the $s$-wave limit.

In the course of investigating the energy dependence of the cross-sections at even lower energies, we uncover, to our surprise, unexpected regular oscillatory structures with small amplitudes and narrow periods (in the logarithm of the energy) in the annihilation and elastic cross-sections, in the region from $0.001 \text{ MeV}$ to $10 \text{ MeV}$, as shown in Fig. 1. Such oscillations are absent for small nuclei and gain in strength as the nuclear radius increases. Its amplitude is larger for the elastic scattering than for the annihilation process. The dependency on the size of the nucleus reveals that it occurs when many partial-wave $L$ values are involved in its interference as many more partial waves can

![Energy variation of the (a) $\bar{n}A$ annihilation and (b) $\bar{n}A$ elastic cross sections obtained from the optical model calculations for different nuclei. The semiclassical results for $\bar{n}$Pb reactions are shown as the dotted-curve.](image-url)
be accommodated when the size of the nucleus increase. Such behavior, undoubtedly, contradicts Bethe-Landau’s power-law. It also reminiscences of wave-interference.

Furthermore, the predicted oscillations appear somewhat different from that of neutron-induced total reaction cross sections for Pb, Cd, and Ho nuclei, which arises from the Ramsauer’s resonances [28–30]. They are also unlike those present in the high-energy 
$^{12}\text{C} + ^{12}\text{C}$ fusion, which is due to successive addition of contributions from even values of partial waves to the identical-particle fusion cross-section, with increasing energies [31–33].

To understand the nature of these oscillations, we shall use a semiclassical $S$ matrix method as it is capable of providing a clear intuitive picture to guide our understanding of the reaction process. Since the oscillations show up more clearly in the $\bar{n}$Pb reaction, we shall focus on this system. We wish to show that the cross-section oscillations are physical and constitute pocket resonances. Such resonance phenomena are general features of potential scattering in atomic, molecular, and heavy-ion collisions. A thorough analysis of the reaction process is therefore needed to understand the origin of such resonances.

II. SEMICLASSICAL ANALYSES OF THE ANNIHILATION OSCILLATIONS AND ELASTIC OSCILLATIONS

Semiclassical descriptions of quantum effects in a potential scattering problem have been discussed in great details by Ford and Wheeler [34] using the Wentzel-Kramers-Brillouin (WKB) approximation, and later consolidated by other pioneering works [35–39]. However, we shall follow a collection of WKB works for complex-potential scattering by Brink, Takigawa, Lee, Marty, and Ohkubo [37–39] as they deemed proper for the present analyses.

![FIG. 2: Real part of the $\bar{n}$Pb interaction potential Re($V_{\text{eff}}(r, E)$) at a sample energy $E = 5.3$ MeV, where only the potential curves for even orbital angular momentum $L$ are displayed. The $r_{1,2,3}$ denote three turning points for Re($V_{\text{eff}}(r, E)$).](image)

We adapt the same momentum-dependent optical potential used in our earlier work [26] in the present semiclassical theory. We consider the concept of the interference between a barrier wave reflected at the potential barrier and modified by the tunneling effect, and an internal wave which penetrates the barrier, into the potential pocket, and reemerges through the barrier out to $r \to \infty$.

According to [37], implicitly for a given $E$ and a partial wave $L$, the total phase function $\delta = \delta(E, L)$ is related to the WKB phase function $\delta_{\text{WKB}}$, the action phase angle $\delta_{ij}$ between the turning points $r_i$ and $r_j$, tunneling phase $\phi$, and
tunneling phase angle $\xi$ and tunneling coefficient $w$ by

$$\delta = \delta_{WKB} - \phi + \tan^{-1}\{w(\xi) \tan(\delta_{32} - \phi)\},$$

$$\delta_{WKB} = \frac{\pi}{2}(L + \frac{1}{2}) - kr_1 + \sqrt{\frac{2\mu}{\hbar^2}} \int_{r_1}^{\infty} (K - k)dr,$$

$$\delta_{ij} = \sqrt{\frac{2\mu}{\hbar^2}} \int_{r_i}^{r_j} Kdr,$$

$$\phi = \frac{1}{2} \left( \xi \ln(\xi/e) - \arg(\frac{1}{2} + i\xi) \right),$$

$$\xi = -\frac{i}{\pi} \delta_{21},$$

$$w(\xi) = \frac{\sqrt{(1 + e^{-2\pi\xi}) - 1}}{\sqrt{1 + e^{-2\pi\xi}} + 1},$$

where $K = (E - V_{\text{eff}}(r,E))^{1/2}$, $k = E^{1/2}$, $E$ is the center-of-mass energy and $\mu$ is the reduced mass of the collision-pair. The turning points $r_{1,2,3}$ are the roots of $E - V_{\text{eff}}(r,E) = 0$, as displayed in Fig. 2. In our case, $V_{\text{eff}}(r,E)$ is the Langer’s modified interaction potential,

$$V_{\text{eff}}(r,E) = \frac{-(V(E) + iW(E))}{1 + \exp((r - r_o)/a_o)} + \frac{\hbar^2(L + 1/2)^2}{2\mu r^2},$$

which is complex and the values for the parameters $r_o$, $a_o$ and volume terms $V$ and $W$ are listed in Table 1, 2 and 3 in [26]. To keep the analyses simple, we consider only the nuclear volume terms in the present semiclassical theory.

Figure 2 gives the real part of the potential $\Re(V_{\text{eff}}(r,E))$ as a function of internuclear distance $r$ and different $L$ at a sample energy $E = 5.3$ MeV in the $\bar{n}$Pb interaction. With the nuclear potential that has a negative imaginary potential $W$, the turning points $r_{1,3}$ are below the real axis of the complex-plane and $r_2$ lies above, and the $V_{\text{eff}}(r,E)$ for a given $E$ must be an analytic function of $r$. The integrands in the action integrals are multiple-valued involving branch cuts. The integration paths in the complex plane need to be chosen so that the action angles $\delta_{ij}$ and $\xi$ are positive, as discussed in [37].

Upon finding the oscillating behavior from the optical model calculation which is rather unusual behavior, we have carried out the above semiclassical calculations for comparison to analyze the underlying origin of the oscillations. We find that the semiclassical calculation using the above formulation not only reproduces the result of the optical model but fits the oscillatory structures reasonably well in energy region $0.1 - 10$ MeV as shown in Fig. 1. At much lower energies and $L = 0$, the semiclassical cross-section is expected to deviate from that of the optical model because the de Broglie wavelengths at such low energies are significantly larger than the width of the nuclear potential.

As the semiclassical calculations contains an explicit contributions from physical processes which can illuminate the interplay between partial waves of various origins, our task is to carry out a detailed study of the partial wave analysis to pinpoint the precise origins of such oscillations. Accordingly, we focus on the interference between the barrier waves and the internal waves and split the total $S$ matrix element into the barrier wave contribution $S_B$, and the internal wave contribution $S_I$:

$$S = e^{2i\delta} = S_B + S_I,$$

$$S_B = \frac{e^{2i\delta_{WKB}}}{N},$$

$$S_I = \left[ \frac{e^{2i\delta_{WKB}}}{N} \right] e^{-2\pi\xi} \left[ \frac{e^{2i\delta_{32}}}{N} \right] \left[ 1 + e^{2i\delta_{32}} \right] + \frac{\left( e^{2i\delta_{32}} \right)^2}{N} + \frac{\left( e^{2i\delta_{32}} \right)^3}{N} + \cdots$$

where $N = \sqrt{2\pi} \exp(-2\pi\xi + i\xi \ln(\xi/e)) / \Gamma(\frac{1}{2} + i\xi)$ is the barrier-penetration factor. The factor $1/(1 + e^{2i\delta_{32}}/N)$ in the $S_I$ term is from Padé approximation, where $e^{2i\delta_{32}}/N$ is the amplitude of penetration-weighted internal-wave. Thus, the factor $e^{2i\delta_{32}}/(N + e^{2i\delta_{32}})$ essentially describes the sum of amplitudes of multiple reflections of internal-wave inside the potential pocket [37, 38]. And knowing these $S$ matrices, the annihilation cross section: $\sigma_{\text{ann}} = \frac{\pi}{k^2} \sum_{L}(2L + 1)(1 - |S|^2)$ and elastic cross section: $\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_{L}(2L + 1)|1 - S|^2$ can be readily evaluated.
Partial wave analysis of the annihilation oscillations in terms of the interference of the barrier and the internal waves gives

\[ \sigma_{\text{ann}} = \sigma_{\text{ann}}^{\text{BI}} + \sigma_{\text{ann}}^{\text{int}} = \sum_{L=0}^{\infty} (\sigma_{\text{ann}}^{\text{BI}}(L) + \sigma_{\text{ann}}^{\text{int}}(L)), \tag{11} \]

\[ \sigma_{\text{ann}}^{\text{BI}}(L) = \frac{\pi}{k^2} (2L + 1)(1 - |S_B|^2 - |S_I|^2), \tag{12} \]

\[ \sigma_{\text{ann}}^{\text{int}}(L) = \frac{\pi}{k^2} (2L + 1)(-2\text{Re}(S_B S_I^*)), \tag{13} \]

where \(\sigma_{\text{ann}}^{\text{int}}(L)\) constitutes the interference between the internal and barrier waves, and \(\sigma_{\text{ann}}^{\text{BI}}(L)\) is the remaining non-interference term.

In Fig. 3(a), we use Eq. (11) to separate the total annihilation cross section \(\sigma_{\text{ann}}\) into the non-interference part, \(\sigma_{\text{ann}}^{\text{BI}}\), and the interference part, \(\sigma_{\text{ann}}^{\text{int}}\). The non-interference \(\sigma_{\text{ann}}^{\text{BI}}\) produces the familiar \(1/p\)-like dependence with decreasing energy without any oscillations. The interference part \(\sigma_{\text{ann}}^{\text{int}}\), oscillates, giving rise to a total sum \(\sigma_{\text{ann}}\) that oscillates on the smooth \(1/p\) background in Fig. 3(a).

Fig. 3(b) shows the break-up of the total annihilation cross section \(\sigma_{\text{ann}}\) into different partial waves contributions, \(\sigma_{\text{ann}}(L)\). Except for the \(L = 0\) partial-wave curve that resembles the \(1/p\)-like behavior, the rest, on the other hand, display a double peak with the largest peak at the lower energy end. In the low-energy region in Fig. 3(c) and Fig. 3(d), the \(\sigma_{\text{ann}}^{\text{BI}}(L)\) given by Eq. (12) produces the broad peak of the cross-section, whereas the interference term \(\sigma_{\text{ann}}^{\text{int}}(L)\) given by Eq. (13) produces the finer oscillations. For a particular partial wave \(L\), the sum of \(\sigma_{\text{ann}}^{\text{BI}}(L)\) and \(\sigma_{\text{ann}}^{\text{int}}(L)\) resulted in the observed double peaks in \(\sigma_{\text{ann}}(L)\). As \(E\) increases, these double-peak structures gradually turn to a single-peak where the contribution from higher \(L\) becomes dominant. This is because, at high energies, for a given high partial-wave, \(\sigma_{\text{ann}}^{\text{BI}}(L)\) with broad maximum overtakes the less relevant, finer oscillatory \(\sigma_{\text{ann}}^{\text{int}}(L)\).

Focusing on the non-interference part from different partial waves \(L\), Fig. 3(c) shows a plot of \(\sigma_{\text{ann}}^{\text{BI}}(L)\) given by Eq. (12). As expected, the \(L = 0\) curve faithfully follows the \(1/p\) dependence which seems to imply the difference
\(|S_I|^2 - |S_B|^2\) is small in comparison to the first term of 1. In contrast, the \(L > 0\) family, in general, displays curves with broad maximum characteristic akin to an optical diffraction pattern which we qualitatively interpret this as the Ramsauer interference in terms of the “path-length” difference between \(|S_I|^2\) and \(|S_B|^2\) in Eq.(12); the interference between the part of the \(\bar{\eta}\) wave passing through the nucleus with the part of the wave which has gone around the nucleus, as pointed out in Peterson\[29\]. Note that \(\sigma_{\text{ann}}^\text{BI}(L)\) contributes the most to the broad maximum feature in \(\sigma_{\text{ann}}(L)\) of Fig.3(b).

Now, focusing on the interference part from different partial waves \(L\), Fig. 3(d) indicates that the oscillations in the annihilation reaction are attributed to the barrier wave interfering with the internal wave. Inspection of Eq.(9), (10) and (13) reveal that the oscillations are governed by \([-2\text{Re}(S_B S_I^*)]\) in which \(S_B S_I^*\) can be factorized into

\[
S_B S_I^* = \left[ \frac{e^{2i\delta_{\text{WKB}}}}{N} \right] \left[ \frac{e^{2i(\delta_{\text{WKB}} + \delta_{32})}}{N} \right]^* \left[ \frac{e^{2i\delta_{32}} / N}{1 + e^{2i\delta_{32}} / N} \right]^*
\]

and is depended upon a combination of complex phases \(\delta_{21}, \delta_{32}\) and \(\delta_{\text{WKB}}\), which are functions of \(E\) and \(L\). To isolate the precise origin of the oscillations, we must consider Fig.4 that shows the energy dependence of the real part of the pocket and the barrier factors as separate elements of the whole Eq.(14) term for various partial waves. For incident energy greater than zero and below the barrier, in the \(r_2 < r < r_1\) tunneling region, we find from our numerical calculations that \(\delta_{21}\) are predominantly imaginary, resulting \(\exp(2i\delta_{21}) \approx \exp(-2\pi \xi)\). We also find that the contribution from the difference between \(\delta_{\text{WKB}}\) and \(\bar{\delta}_{\text{WKB}}\) of the \(r > r_1\) outer most region are predominantly imaginary as well, rendering \(\exp(2i(\delta_{\text{WKB}} - \bar{\delta}_{\text{WKB}})) \approx \exp(-2\eta)\). We recast Eq.(13) into

\[
\sigma_{\text{ann}}^\text{int}(L) \sim \frac{\pi}{k_2^2} (2L + 1) |S_B||S_I|e^{-2(\pi \xi + \eta)}[-2\cos(2\delta_{32})].
\]

We find consequently that the the maxima of the annihilation cross sections \(\sigma_{\text{ann}}^\text{int}(L)\) with \(L = 1\) in Fig. 3(d) as “annihilation resonances” are located at the minima of the real part of the pocket factor of Eq. (14) in Fig. 4(a) and

\[\text{FIG. 4: Energy dependence of (a) the real part of the pocket factor, (b) the real part of the barrier factor, (c) Re}(S_B S_I^*)\text{ of Eq.(14) and (d) Re}(\delta_{32})\text{ for various partial waves.} \]
in \( \text{Re}(S_B S_I^*) \) in Fig. 4(c), as marked by the vertical dashed lines in Fig. 4(c) and 4(d). Though the barrier factor of Fig. 4(b) is seen to increase with increasing energy, it reduces the oscillation-amplitude of the pocket factor of Fig. 4(a) without varying the final positions of maximum and minimum of the oscillations as shown in Fig. 4(c). The minimum of the pocket factor, according to Eq. (15), must occur whenever \( \cos(2\delta_{32}) = -1 \) (or \( 2\delta_{32} = (2n + 1)\pi \)) and the minima are indeed located at \( \delta_{32} \approx 7.5\pi, 6.5\pi \) and \( 5.5\pi \), respectively, as indicated by horizontal lines in Fig. 4(d). The quantity \( \delta_{32} \) varies by one \( \pi \) unit between resonances with \( \partial\delta_{32}/\partial n \approx \pi \). We further examine and find that wherever the horizontal lines cross the \( \delta_{32} \) curves of Fig. 4(d), every cross-point matches every single minimum of \( \text{Re}\{S_B S_I^*\} \) in Fig. 4(c) and a maximum \( \sigma_{\text{ann}}(L) \) in Fig. 3(d). This means that the condition for an annihilation resonance at a cross section maximum, at the energy \( E \), is

\[
\delta_{32}(E, L) \approx (n_L + \frac{1}{2})\pi, \tag{16}
\]

corresponding to the quantization condition for a quasi-bound state inside the pocket, where \( n_L \) is the number of bound and quasi-bound states below \( E \) for the angular momentum \( L \). As \( \delta_{32}(E, L) \) is the action integral for a path within the confining potential \( V_{\text{eff}}(r, E) \), the “pocket resonance” is in fact a continuation of the bound states of the potential into the continuum. While the locations of these pocket resonances for the present problem are specific to the optical potential in question, the general condition at which the annihilation resonance occurs suggests that the condition of Eq. (16) is quite general in nature.

FIG. 5: Semiclassical elastic cross sections for \( ^{\text{n}}\text{Pb} \) as a function of energy. (a) is from Eq.(17), (b) is from the sum of Eq.(18) and Eq.(19), (c) is from Eq.(18) and (d) is from Eq.(19).

We now examine the elastic oscillations with

\[
\sigma_{\text{el}} = \sigma_{\text{el}}^{\text{BI}} + \sigma_{\text{el}}^{\text{int}} = \sum_{L=0}^{\infty} (\sigma_{\text{el}}^{\text{BI}}(L) + \sigma_{\text{el}}^{\text{int}}(L)), \tag{17}
\]

\[
\sigma_{\text{el}}^{\text{BI}}(L) = \frac{\pi}{k^2}(2L + 1)(1 + |S_I|^2 + |S_B|^2), \tag{18}
\]

\[
\sigma_{\text{el}}^{\text{int}}(L) = \frac{\pi}{k^2}(2L + 1)(2\text{Re}\{S_B^* S_I - S_I^* S_B\}). \tag{19}
\]
The quantity $\sigma_{el}^{HI}$ and its partial wave decomposition in Fig. 5(a) and 5(c) show the all $\sigma_{el}^{HI}(L)$ are monotonically decreasing functions of the energy. There is a significant cancellation of $\sigma_{el}^{HI}$ by $\sigma_{int}^{HI}$ where $\sigma_{int}^{HI}$ has an oscillating behavior, resulting in an oscillating total elastic cross section $\sigma_{el}$ as a function of energy as shown in Fig. 5(a).

Fig. 5(b) shows the break-up of the total elastic $\sigma_{el}$ into different partial waves contributions $\sigma_{el}(L)$. Here we notice only the $L = 0$ partial wave has three, narrow, sinusoidal oscillations with decreasing amplitudes. Those of $L > 0$, however, depict only one single maximum with similar shape. The first, broad maximum seen in $\sigma_{el}$ near 0.05 MeV seems to correspond to the first maximum of $L = 0$ curve at the same energy. The subsequent, second maximum of $\sigma_{el}$ that is near 0.2 MeV, corresponds to the contributions from the $L = 1$ curve and the second maximum of $L = 0$ curve. Beyond the second maximum, each consecutive maximum now appears to associate with a specific partial wave of $L = 2, 3, 4,$ and so forth.

In Fig. 5(c) we show the curves for various partial-wave non-interference cross-sections $\sigma_{el}^{HI}(L)$ from Eq. (18). They all obey Bethe-Landau’s power law in proper order as $L$ increases and unlike the diffraction pattern depicted earlier in the case of annihilation. Such behavior can be understood from Eq. (18) and qualitatively interpreted as there is “no” path-length difference between $|S_I|^2$ and $|S_B|^2$.

Focusing on the interference part of the contributions to different partial waves, Fig. 5(d) exhibits the rapid oscillation functions of $\sigma_{el}(L)$ as a function of $E$ for different partial waves. The oscillation is similar to the wave-interference in the annihilation cross section. Although Eq. (19) is more complicated than Eq. (13) for the annihilation, we have found from our numerical calculations that the $|−2\text{Re}(S_I)]$ term dominates in the evaluation of $\sigma_{int}(L)$ over the energy range. Furthermore, the maximum of $\sigma_{el}(L)$ as an “elastic resonance” for various partial waves is located at the same energy as the minimum of $\text{Re}(S_I)$. The phase of $S_I$, accordingly to Eq. (10), is given predominantly by $(\delta_{WKB} + \delta_{32})$. Numerical calculations of $(\delta_{WKB} + \delta_{32})$ indicate that the difference of the phases between elastic resonances is $\pi$, i.e., $\partial(\delta_{WKB} + \delta_{32})/\partial n \approx \pi$. Figures 5(d) and 5(e) show that the condition for the elastic resonances at the energy $E$ arising from the pocket is

$$
(\delta_{WKB} + \delta_{32})(E, L) \approx (n_L + \frac{1}{2})\pi,
$$

at which $\cos[2(\delta_{WKB} + \delta_{32})(E, L)] = -1$, substantiating the quantization condition for the elastic resonance similar to that for the annihilation resonance in Eq. (16), but with the addition of $\delta_{WKB}$ to $\delta_{32}$. And because of this additional $\delta_{WKB}$, the location of the elastic resonance energy $E$ is slightly shifted from the annihilation resonance energy for the same $n$ and $L$.

Another interesting feature besides the elastic oscillations is the resonance energies shifting toward lower energies with an increasing nuclear radius or mass. This is because for any quantum scattering by an effective potential such as $\delta_{WKB}$, if the pocket is sufficiently deep, resonances will occur whenever a whole number of wavelengths can be fitted into the pocket at energies below the barrier. Consequently, as the nuclear radius increases, the wavelengths of the wave function inside the pocket must correspondingly increase, and the energy $E$ of the wave function must correspondingly decrease to maintain the resonant condition. Note that the opposite behavior is also possible where the resonance shifts toward higher energies with increasing nuclear radius. And such behavior was exemplified in the neutron total cross-section data for Cu, Cd, and Pb [20] and Ho [30] nuclei from 2 MeV to 125 MeV. The cause of the broad resonance and its shifts to higher energies have been explained by Peterson [29] and by Wong in [30] using the concept of Ramsauer’s interference [29].

As the incident energy further decreases below 0.03 MeV, we see the optical model predicted annihilation cross-sections for C, Ag, and Pb nuclei merge into one single curve, whatever their mass. Parametrizing these curves with $\sigma_{ann}^{nA}(p) \propto 1/p^{\alphaA}$ in this energy region (i.e., $1.0 \leq p \leq 2.0$ MeV/c), we found $\alphaA = \partial\ln(\sigma_{ann}^{nA})/\partial\ln(p)$ for C, Ag and Pb nuclei to be $\sim-1.0$, $\sim-0.9$, and $\sim-0.8$, respectively, which are more attuned to Bethe-Landau’s s-wave prediction of $\alphaA = 1$. On the elastic scattering at energies below 0.03 MeV, we notice the scattering becomes isotropic and energy-independent. Consequently, the “s-wave” cross-section is purely geometrical effect and a constant, which can be described by a black-nucleus model.

Lastly, we find from additional calculations that as the imaginary potential $W$ increases, the amplitude of the pocket resonance oscillation decreases. And we further note that the annihilation and elastic curves for $W = 0.028$ MeV are essentially identical to those of $W = 2.8$ MeV.

### III. CONCLUSIONS

The optical-model calculations show oscillatory structures in $nA$ annihilation and elastic cross-sections in the low $n$ energy range $0.001 - 10$ MeV. The cross-sections oscillate as a function of log($E$) with small amplitudes and narrow periods for massive nuclei. This unexpected behavior contradicts the generally expected Bethe-Landau’s power-law in the s-wave limit.
We, therefore, used a semiclassical S-matrix method to obtain further insight into the nature and the precise origin of the oscillations. We found two main contributions to the structures in the reaction cross sections: (1) For annihilation and elastic scattering, the oscillations are attributed to the interference between the internal and barrier waves, while their smooth, $1/p$, backgrounds are mainly due to the non-interference term. (2) Delve deeper into the interference term for both reactions, we identified the maxima of the cross-sections as resonances occurred within the potential pocket. The condition for an annihilation resonance is the quantization rule: $\delta_{32}(E, L) \approx (n_L + 1/2)\pi$, and for an elastic resonance: $(\delta_{\text{WKB}} + \delta_{32})(E, L) \approx (n_L + 1/2)\pi$ where $n_L$ is the number of bound and quasi-bound $L$-states in the pocket below $E$. So in conclusion, the existence of resonances is connected to the potential pocket and barrier that, of course, depends sensitively on the depth and the radial dimension of the optical potential and its magnitude of the absorptive potential. It is for this reason that the pockets may not exists for lighter nuclei for which these pocket resonances do not appear. From this aspect, the occurrence of the pocket resonances may be a sensitive indicator of the depth of the potential. Pocket resonances may also not exist if the imaginary potential is large. Experimental observations of these pocket resonances will provide valuable information on the properties of the optical model potentials and the nature of the $\bar{n}$-annihilation process for different nuclei. Though here we focused on $\bar{n}A$ reactions, similar work on the potential pocket analysis can also be extended to $^{12}C+^{12}C$ nuclei $[41]$ which is an important reactions in nuclear astrophysics.

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