Gravitational waves from compact binary systems

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Abstract. Inspiralling compact binaries are among the plausible sources of detectable gravitational radiation. Here we compute the gravitational wave polarizations in the extreme mass ratio limit and give their formal expressions with the inclusion of higher order corrections beyond the standard quadrupole term. We discuss the effects caused by the rotation of the central, massive body.

1. Introduction

Direct detection of gravitational radiation is expected by the gravitational wave observatories, i.e. LIGO [1], VIRGO [2], GEO600 [3], TAMA [4] and LISA [5] in the near future. There are three types of gravitational wave sources relevant for detection. The stochastic background is the sum of signals from uncorrelated sources, burst signals are produced during energetic, short duration events and periodic signals are emitted by, for example, oscillating, precessing neutron stars and compact binary systems. Compact binaries are among the most relevant gravitational wave sources since they generate well defined chirp signals. Depending on the complexity of the system these signals, however, can be characterized by many parameters. The orbit of the binary slowly decays under radiation reaction and the frequency and amplitude of the emitted signal increase over time. During data analysis the method of matched filtering is employed [6], whose effectiveness requires a precise knowledge of the emitted gravitational waves.

As a first approximation we are considering compact binaries in the test particle limit. Operating at low frequencies the inspiral of stellar mass compact objects into supermassive black holes is one of the most important sources for the planned Laser Interferometer Space Antenna. The motion of the binary system is described in the Lense-Thirring approach [7], i.e. with the geodesic motion around a spinning body. To investigate the effects of the rotation in the lowest approximation, up to 1.5 post-Newtonian (PN) order, we focus on terms linear in the spin of the central, massive body. We give the explicit form of the vectors describing the relative position of the binary and the detector which are necessary to express the polarization states \( h_+ \) and \( h_\times \). Having in hand the description of the classical motion with standard methods we calculate the analytic expressions of \( h_+ \) and \( h_\times \) of the emitted gravitational waves for eccentric orbits including higher order corrections beyond the quadruple term. The description of the method is completed by giving the explicit contributions to the gravitational wave signal which belong to different PN orders, polarizations and spin effects. The spinless and circular orbit limits of our calculations are in agreement with existing results.
2. The motion of the binary system

The classical motion of the binary is described by a test particle with mass \( m \) orbiting around a massive, \( M \gg m \), rotating body. The mass ratio \( m/M \) is negligible and the Lagrangian of the system is

\[
L = \frac{\mu \dot{r}^2}{2} + \frac{G \mu M}{r} + \frac{2G\mu}{c^2 r^3} \mathbf{S} \cdot (\dot{\mathbf{r}} \times \mathbf{r}) ,
\]

where \( \mu = mM/(m+M) \approx m \) is the reduced mass of the system and \( \mathbf{S} \) denotes the spin vector of the central mass. The equations of motion of a test particle in the field of a uniformly rotating rigid body were investigated by Lense and Thirring [7]. The motion of the particle is described by the Lagrangian (1).

When the two bodies have comparable masses the dynamics is determined by the equations of motion and the spin precession equations [8]. Since the Lense-Thirring dynamics describes geodesic motion in our case the spin vector is constant, which also follows from an order of magnitude estimate of the precession equations. It states that the change of the spin is

\[
\dot{\mathbf{S}} \sim \frac{m}{M} \epsilon
\]

which can be neglected in our approximation.

To describe the dynamics of the orbiting bodies we use the results of [9], where the complete radial and angular dynamics are given in the Lense-Thirring approximation. Moreover, an appropriate parametrization of the orbit is developed [10] for the integration of the dynamics. Following [9] we chose comoving coordinates and perform Euler rotations,

\[
r = R_z(\hat{\Psi}) R_z(\iota) R_z(\Psi) \mathbf{r}_0 ,
\]

to place the system in a general orientation. This orientation is determined by the condition that in this invariant system the \( z \) axis is aligned with the constant spin vector. In the comoving system the components of the relative separation, velocity and spin vectors are \( \mathbf{r} = (r, 0, 0) \), \( \mathbf{v} = (v_\parallel = \dot{r}, v_\perp, 0) \) and \( \mathbf{S} = S(\sin \psi \sin \iota, \cos \psi \sin \iota, \cos \iota) \), respectively. \( \iota \) is the angle between the orbital angular momentum \( \mathbf{L} \) and the spin and \( \Phi \) and \( \Psi \) denotes the orientation of the separation vector and the \( x \) axis of the invariant system with respect to the node line. Using these angles the coordinates of the separation vector is expressed as

\[
x = r (\cos \Phi \cos \Psi - \cos \iota \sin \Phi \sin \Psi) , \\
y = r (\sin \Phi \cos \Psi + \cos \iota \cos \Phi \sin \Psi) , \\
z = r \sin \iota \sin \Psi
\]

and the field equations take the following form

\[
\dot{\Psi} = \frac{L}{mr^2} , \quad \dot{\Phi} = \frac{2S}{r^3} ,
\]

\[
r^2 = \frac{2E}{m} + \frac{2M}{r} - \frac{L^2}{m^2 r^2} - \frac{4LzS}{mr^3} .
\]

It should be noted that the equations for the angles \( \Psi \) and \( \Phi \) are decoupled unlike in the case of spherical coordinates. \( \Psi \) has the Newtonian value, while \( \Phi \) describes the rotation around the spin vector \( \mathbf{S} \).

We determine the components of the orthonormal triad \( (\mathbf{N}, \mathbf{p}, \mathbf{q}) \) which vectors are used to express the projections of the polarization states \( h_+ \) and \( h_\times \). \( \mathbf{N} \) denotes the direction of the line
of sight, \( \mathbf{p} \) is chosen to be perpendicular to the Newtonian angular momentum and \( \mathbf{q} = \mathbf{N} \times \mathbf{p} \). The vectors \( \mathbf{N} \) and \( \mathbf{S} \) are constants and we set the \( y \) axis of the invariant system so that \( N_y \) vanishes. In this case \( \mathbf{N} = (\sin \gamma, 0, \cos \gamma) \), where \( \gamma \) is the angle between \( \mathbf{N} \) and \( \mathbf{S} \). In the comoving system the components of \( \mathbf{N} \) are

\[
N = \begin{pmatrix} 
\cos \Psi \cos \Phi \sin \gamma - \sin \Psi \cos \Phi \sin \gamma + \sin \Psi \sin \iota \cos \gamma \\
-\sin \Psi \cos \Phi \sin \gamma - \cos \Psi \cos \Phi \sin \gamma + \cos \Psi \sin \iota \cos \gamma \\
\sin \iota \sin \Phi \sin \gamma + \cos \iota \cos \gamma
\end{pmatrix} .
\tag{7}
\]

The unit vector \( \mathbf{p} \) is perpendicular to \( \mathbf{N} \) and the Newtonian angular momentum. These conditions determine the form of \( \mathbf{p} \) in the comoving system which is

\[
\mathbf{p} = \frac{1}{N} \begin{pmatrix} 
\sin \Psi \cos \Phi \sin \gamma + \cos \Psi \cos \Phi \sin \gamma - \cos \Psi \sin \iota \cos \gamma \\
\cos \Psi \cos \Phi \sin \gamma - \sin \Psi \cos \Phi \sin \gamma + \sin \Psi \sin \iota \cos \gamma \\
\sin \iota \sin \Phi \sin \gamma + \cos \iota \cos \gamma
\end{pmatrix} ,
\tag{8}
\]

where

\[
N = \sqrt{N_x^2 + N_y^2} = \sqrt{1 - N_x^2} = \sqrt{1 - (\sin \iota \sin \Phi \sin \gamma + \cos \iota \cos \gamma)^2} .
\tag{9}
\]

We will use these vectors and the comoving system to express the polarization states.

**3. Polarization states**

The signal of a laser interferometric gravitational wave detector can be decomposed into two polarization states,

\[
h(t) = F_+(\alpha, \beta, \xi)h_+(t) + F_\times(\alpha, \beta, \xi)h_\times(t) ,
\tag{10}
\]

where the beam-pattern functions \( F_+ \) and \( F_\times \) depend on the direction of the source, \( \alpha \) and \( \beta \), and the polarization angle \( \xi \) describing the relative orientation of the detector and the source,

\[
F_+ = \frac{1}{2}(1 + \cos \alpha^2) \cos 2\beta \cos 2\xi + \cos \alpha \sin 2\beta \sin 2\xi ,
\tag{11}
\]

\[
F_\times = -\frac{1}{2}(1 + \cos \alpha^2) \cos 2\beta \sin 2\xi + \cos \alpha \sin 2\beta \cos 2\xi .
\tag{12}
\]

The independent polarizations \( h_+ \) and \( h_\times \) can be projected from the metric perturbation \( h^{ij}_{TT} \)

\[
h_+ = \frac{1}{2}(p_ip_j - q_iq_j)h^{ij}_{TT} , \quad h_\times = \frac{1}{2}(p_iq_j + q_ip_j)h^{ij}_{TT} .
\tag{13}
\]

In the post-Newtonian approximation \( h^{ij}_{TT} \) can be written as

\[
h^{ij}_{TT} = \frac{2\mu}{D} \left[ Q^{ij} + P^{0.5}Q^{ij} + PQ^{ij} + PQ^{ij}_{SO} + P^{1.5}Q^{ij} + P^{1.5}Q^{ij}_{SO} \right]_{TT}
\tag{14}
\]

up to 1.5 PN order, where \( D \) is the distance between the source and the observer. The explicit form of the different terms given by \cite{11, 12}. These terms are the quadrupole, higher order PN corrections and spin terms. They can be given as functions of the dynamical elements, namely \( \dot{v} \) and \( v \).

To evaluate the polarization states we substitute the components of the vectors \( \mathbf{N} \), \( \mathbf{p} \), \( \mathbf{q} \) and \( \mathbf{S} \) into the transverse-traceless tensor, given by Kidder \cite{11}. Similarly to \( h^{ij}_{TT} \) the polarization states can be decomposed as

\[
h_+ = \frac{2m}{D} \left[ h^N_+ + h^{0.5}_+ + h^1_+ + h^{1SO}_+ + h^{1.5}_+ + h^{1.5SO}_+ \right] .
\tag{15}
\]
For the sake of simplicity the components of the vectors are substituted formally with the result

\[
\begin{align*}
h_+^N &= \left( \dot{r}^2 - \frac{M}{r} \right) \left( p_x^2 - q_x^2 \right) + 2v_\perp \dot{r} (p_x p_y - q_x q_y) + v_\perp^2 \left( p_y^2 - q_y^2 \right), \\
h_+^{1\text{SO}} &= \frac{1}{r^2} \left[ (\mathbf{qS}) p_x + (\mathbf{pS}) q_x \right], \\
h_+^{1,5\text{SO}} &= \frac{2}{r^2} \left[ 3v_\perp S_z (p_x^2 - q_x^2) + \dot{r} [\mathbf{S} \times (p_x \mathbf{p} - q_x \mathbf{q})]_x - 2v_\perp [\mathbf{S} \times (p_x \mathbf{p} - q_x \mathbf{q})]_y \right], \\
h_x^N &= 2 \left( \frac{\dot{r}^2 - M}{r} \right) p_x q_x + 2v_\perp \dot{r} (p_x q_y + q_x p_y) + 2v_\perp^2 p_y q_y, \\
h_x^{1\text{SO}} &= \frac{1}{r^2} \left[ (\mathbf{qS}) q_x - (\mathbf{pS}) p_x \right], \\
h_x^{1,5\text{SO}} &= \frac{2}{r^2} \left[ 6v_\perp S_z p_x q_x + \dot{r} [\mathbf{S} \times (p_x \mathbf{q} + q_x \mathbf{p})]_x - 2v_\perp [\mathbf{S} \times (p_x \mathbf{q} + q_x \mathbf{p})]_y \right].
\end{align*}
\]

To collect our main results we have listed here the lowest order Newtonian terms and all the contributions which are linear in spin. For circular orbits the length of the separation vector is constant, \( \dot{r} = 0 \) and \( v^2 = v_\perp^2 \), and one can make the time dependence explicit. Our formulae for the polarization states are found to be in agreement with the results of [11].

4. Conclusions

We have presented a method for the calculation of the polarization states of gravitational waves emitted by spinning compact binaries. We have considered eccentric orbits and focused on the effects of the rotation of the central, massive body. The results are given in terms of the components of the separation, the velocity and spin vectors and the \((\mathbf{N}, \mathbf{p}, \mathbf{q})\) triad. These results can be extended to more general systems, i.e. binaries with comparable masses and spins [13].

For circular orbits we have integrated the relation between the true anomaly parameter [10] and time \( t \). In this case the lowest order, Newtonian expressions have their frequency twice the orbital frequency.

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