Heron Angles, Heron Triangles, and Heron Parallelograms

Walter Wyss

Abstract
Heron angle: both its sine and cosine are rational
Heron triangle: all its sides and area are rational
Heron Parallelogram: all its sides, diagonals and area are rational
We give one-to-one (bijective) parametrizations for all three concepts.

1 Introduction
Special angles, triangles, rectangles, and parallelograms have been studied for ages by Mathematician and Scientists (Pythagoras, Heron, Diophantus, Brahmagupta to mention a few). Parameter representations, some in integers, have been found in all cases [1]. However, there was a lack in retrieving the parameters. Here we give parameter representations that are bijective (one-to-one).

2 Bijective parameter representation of a relation

Theorem 1
let $a, x, y, z$ be real numbers. Given $a$, the relation

$$x^2 + 2axy + y^2 = z^2$$ (1)

has the following bijective parameter representation, with parameters $\lambda, \sigma$

$$x = \frac{\lambda}{1 + a\sigma}(1 - \sigma^2)$$ (2)
$$y = 2\lambda\sigma$$ (3)
$$z = \frac{\lambda}{1 + a\sigma}(1 + 2a\sigma + \sigma^2)$$ (4)

cursively $2\lambda = x + z$, $\sigma = \frac{y}{x + z}$ (5)
Proof. The relation (1) also reads
\[ x^2 + (y + ax)^2 = z^2 + (ax)^2 \] \hspace{1cm} (6)

According to [2] this relation has the following bijective parameter representation with parameters \( s_1, s_2, \lambda_2 \)

\[
\begin{align*}
  y + ax &= s_1 + \lambda_2 s_2 \\
  ax &= s_1 - \lambda_2 s_2 \\
  x &= s_2 - \lambda_2 s_1 \\
  z &= s_2 + \lambda_2 s_1
\end{align*}
\] \hspace{1cm} (7)

Conversely

\[
\begin{align*}
  2s_1 &= y + 2ax \\
  2s_2 &= x + z \\
  2\lambda_2 s_2 &= y
\end{align*}
\] \hspace{1cm} (8)

Now

\[ ax = a_1(s_2 - \lambda_2 s_1) = s_1 - \lambda_2 s_2 \] \hspace{1cm} (9)

resulting in

\[ s_1 = \frac{a + \lambda_2}{1 + a\lambda_2} s_2 \] \hspace{1cm} (10)

Then

\[
\begin{align*}
  x &= s_2 - \lambda_2 \frac{a + \lambda_2}{1 + a\lambda_2} s_2 \\
  x &= \frac{s_2}{1 + a\lambda_2}(1 - \frac{\lambda_2}{2})
\end{align*}
\] \hspace{1cm} (11)

\[
\begin{align*}
  z &= s_2 + \lambda_2 \frac{a + \lambda_2}{1 + a\lambda_2} s_2 \\
  z &= \frac{s_2}{1 + a\lambda_2}(1 + 2a\lambda_2 + \lambda_2^2)
\end{align*}
\] \hspace{1cm} (12)

and

\[ y = 2\lambda_2 s_2 \] \hspace{1cm} (13)

Finally let \( s_2 = \lambda, \lambda_2 = \sigma \)
Corollary 1

The relation, given $\varepsilon$,

$$m^2 + \varepsilon mn + n^2 = 1 \quad (17)$$

has the following bijective parameter representation, with parameter $\sigma$

$$m = \frac{1 - \sigma^2}{1 + \varepsilon \sigma + \sigma^2} \quad (18)$$
$$n = \frac{\sigma(2 + \varepsilon \sigma)}{1 + \varepsilon \sigma + \sigma^2} \quad (19)$$

Conversely

$$\sigma = \frac{n}{1 + m} \quad (20)$$

Proof. In (2, 3, 4, 5) let $x = m$, $y = n$, $z = 1$, and $2a = \varepsilon$

3 The relation

$$\lambda^2 = \frac{m}{n} \frac{1 - n^2}{1 + m^2} \quad (21)$$

We have two one-parameter families of fundamental solutions given by

(a) Type I:

$$\lambda = \frac{m}{n} \quad (22)$$

Then

$$m(1 - m^2) = n(1 - n^2)$$
$$m - n = m^3 - n^3$$

or

$$m^2 + mn + n^2 = 1 \quad (23)$$

This relation has already been considered by Diophantus of Alexandria [3].

According to (17), the relation (23) has the bijective parameter representation with parameter $\sigma$, as

$$m = \frac{1 - \sigma^2}{1 + \sigma + \sigma^2}, \quad n = \frac{\sigma(2 + \sigma)}{1 + \sigma + \sigma^2}, \quad \lambda = \frac{1 - \sigma^2}{\sigma(2 + \sigma)} \quad (24)$$

conversely

$$\sigma = \frac{n}{1 + m} \quad (25)$$
(b) Type II: 
\[ \lambda = mn \] (26)

Then
\[ mn^3(1 - m^2) = 1 - n^2 \]
\[ 1 + m^3n^3 = n^2(1 + mn) \]
or
\[ 1 - mn + m^2n^2 = n^2 \]

and then
\[ \left( \frac{1}{n} \right)^2 - m \left( \frac{1}{n} \right) + m^2 = 1 \] (27)

According to (17), the relation (27) has the bijective parameter representation with parameter \( \sigma \), as
\[
m = \frac{1 - \sigma^2}{1 - \sigma + \sigma^2}, \quad n = \frac{1 - \sigma + \sigma^2}{\sigma(2 - \sigma)}, \quad \lambda = \frac{1 - \sigma^2}{\sigma(2 - \sigma)} \] (28)

conversely
\[ \sigma = \frac{1}{n(1 + m)} \] (29)

Example

(a) \[ \sigma = \frac{1}{2}, \quad m = \frac{3}{7}, \quad n = \frac{5}{7}, \quad \lambda = \frac{3}{5} \] (30)

(b) \[ \sigma = \frac{2}{3}, \quad m = \frac{5}{7}, \quad n = \frac{7}{8}, \quad \lambda = \frac{5}{8} \] (31)

4 Heron angles

Definition 1
1. An angle \( \alpha \) is called a Heron angle if both \( \sin \alpha \) and \( \cos \alpha \) are rational
2. The generator of an angle \( \alpha \) is defined by
\[
m = m(\alpha) = \frac{\sin \alpha}{1 + \cos \alpha} = \tan \left( \frac{\alpha}{2} \right) \] (32)
Lemma 1
For $0 < \alpha < \pi$, $m(\alpha) > 0$ and is an increasing function of $\alpha$.

Proof. Since $\sin \alpha > 0$ and $-1 < \cos \alpha < 1$ we see from (32) that $m(\alpha) > 0$.

From
$$ \frac{dm(\alpha)}{d\alpha} = \frac{1}{2 \cos^2(\frac{\alpha}{2})} > 0 $$
we see that $m(\alpha)$ is increasing. \qed

Lemma 2
Given the generator $m = m(\alpha), \ 0 < \alpha < \pi$, we find
$$ \cos \alpha = \frac{1 - m^2}{1 + m^2}, \ \sin \alpha = \frac{2m}{1 + m^2} \quad (33) $$

Proof. From
$$ m^2 = \frac{1 - \cos \alpha}{1 + \cos \alpha} $$
we find
$$ \cos \alpha = \frac{1 - m^2}{1 + m^2} \quad \text{and then} \quad \sin \alpha = \frac{2m}{1 + m^2} $$
Observe that the generator of a Heron angle is rational and vice versa. Thus there is a one-to-one relationship between rational numbers and Heron angles. \qed

5 Heron triangles

For a triangle with sides $u_1, u_2, u_3$ and interior angles $\phi_1, \phi_2, \phi_3$, where $\phi_k$ is the angle opposite $u_k, k = 1, 2, 3$

We have
1. The law of sine
$$ \frac{u_1}{\sin \phi_1} = \frac{u_2}{\sin \phi_2} + \frac{u_3}{\sin \phi_3} \quad (34) $$
2. The law of cosine
$$ u_3^2 = u_1^2 + u_2^2 - 2u_1u_2 \cos \phi_3 \quad (35) $$
with similar relations involving the other sides and angles.
3. The area $A$ of the triangle is given by
$$ A = \frac{1}{2} u_1u_2 \sin \phi_3 \quad (36) $$
and similarly for the other sides and angles.
Definition 2

A triangle is called a Heron triangle if all its sides and area are rational. Observe that from (35, 36) all the interior angles are Heron angles. For a Heron triangle we find from (34) and $\phi_1 = \pi - (\phi_2 + \phi_3)$ the representation

$$u_1 = \sin(\phi_2 + \phi_3)w$$  \hspace{1cm} (37)
$$u_2 = \sin \phi_2 \cdot w$$  \hspace{1cm} (38)
$$u_3 = \sin \phi_3 \cdot w$$  \hspace{1cm} (39)

where $w$ is a rational scaling parameter.

Let now $q$ be the generator of $\phi_2$, $p$ the generator of $\phi_3$, i.e.

$$\sin \phi_2 = \frac{2q}{1 + q^2}, \quad \cos \phi_2 = \frac{1 - q^2}{1 + q^2}$$  \hspace{1cm} (40)
$$\sin \phi_3 = \frac{2p}{1 + p^2}, \quad \cos \phi_3 = \frac{1 - p^2}{1 + p^2}$$  \hspace{1cm} (41)

Then

$$\sin \phi_1 = \sin(\phi_2 + \phi_3) = \sin \phi_2 \cos \phi_3 + \cos \phi_2 \sin \phi_3$$
$$\sin \phi_1 = \frac{2q(1 - p^2)}{(1 + q^2)(1 + p^2)} + \frac{2p(1 - q^2)}{(1 + q^2)(1 + p^2)}$$
$$\sin \phi_1 = \frac{2(p + q)(1 - pq)}{(1 + q^2)(1 + p^2)}$$  \hspace{1cm} (42)

Introduce the new rational scaling parameter $v$ by

$$w = \frac{1}{2}(1 + p^2)(1 + q^2)v$$  \hspace{1cm} (43)

Then we have the representation, with $p > 0, q > 0, v > 0$ and $pq < 1$

$$u_1 = (p + q)(1 - pq)v$$  \hspace{1cm} (44)
$$u_2 = q(1 + p^2)v$$  \hspace{1cm} (45)
$$u_3 = p(1 + q^2)v$$  \hspace{1cm} (46)
$$A = pq(p + q)(1 - pq)v^2$$  \hspace{1cm} (47)

This representation has the three parameters $p, q, v$. We now can retrieve these parameters from $u_1, u_2, u_3, A$ as follows:
From
\[
\sin \phi_3 = \frac{4A}{2u_1u_2} \tag{48}
\]
\[
\cos \phi_3 = \frac{u_1^2 + u_2^2 - u_3^2}{2u_1u_2} \tag{49}
\]
\[
1 + \cos \phi_3 = \frac{(u_1 + u_2)^2 - u_3^2}{2u_1u_2} \tag{50}
\]
we find the generator
\[
p = \frac{\sin \phi_3}{1 + \cos \phi_3} = \frac{4A}{(u_1 + u_2)^2 - u_3^2} \tag{51}
\]
and similarly
\[
q = \frac{\sin \phi_2}{1 + \cos \phi_2} = \frac{4A}{(u_1 + u_3)^2 - u_2^2} \tag{52}
\]
The scaling factor \(v\) is then given by
\[
v = \frac{u_3}{p(1 + q^2)} \tag{53}
\]
This is our one-to-one relationship.

**Example**

\[
u_1 = 9, \quad u_2 = 10, \quad u_3 = 17, \quad A = 36
\]
\[
p = 2, \quad q = \frac{1}{4}, \quad v = 8
\]

**6 Heron parallelograms**

A parallelogram has sides \(u_1, u_2\) and diagonals \(u_3, u_4\).

**Definition 3**

(a) A parallelogram with its sides and diagonals being rational is called a rational parallelogram.

(b) A rational parallelogram with rational area is called a Heron parallelogram.

In [4] we found a bijective parameter representation for rational parallelograms.

Parameters \(u > 0, \ 0 < m < 1, \ 0 < n < 1\) and all rational
\[ u_1 = (1 - mn)u \] (54)
\[ u_2 = (m + n)u \] (55)
\[ u_3 = [1 + mn - (n - m)]u \] (56)
\[ u_4 = [1 + mn + (n - m)]u \] (57)

conversely
\[ 4u = u_4 + u_3 + 2u_1 \] (58)
\[ m = \frac{u_4 + u_3 - 2u_1}{u_4 - u_3 + 2u_2} \] (59)
\[ n = \frac{u_4 - u_3 + 2u_2}{u_4 + u_3 + 2u_1} \] (60)

Not let \( \phi \) be then angle between the sides \( u_1, u_2 \). According to the law of cosine we find
\[ \cos \phi = \frac{u_1^2 + u_2^2 - u_3^2}{2u_1u_2} = \frac{2(u_1^2 + u_2^2) - 2u_3^2}{4u_1u_2} \] (61)

From the parallelogram equation
\[ 2(u_1^2 + u_2^2) = u_3^2 + u_4^2 \] (62)
we get
\[ \cos \phi = \frac{u_1^2 - u_2^2}{4u_1u_2} = \frac{(u_4 - u_3)(u_4 + u_3)}{4u_1u_2} \]
\[ \cos \phi = \frac{(n - m)(1 + mn)}{(n + m)(1 - mn)} \] (63)

Now the area \( A \) of a parallelogram is given by
\[ A = u_1u_2 \sin \phi \] (64)

where
\[ \sin^2 \phi = 1 - \cos^2 \phi \]
\[ \sin^2 \phi = \frac{4mn(1 - m^2)(1 - n^2)}{(n + m)^2(1 - mn)^2} \]
\[ \sin^2 \phi = \frac{4n^2(1 - m^2)^2}{(n + m)^2(1 - mn)^2} \cdot \frac{m(1 - n^2)}{n(1 - m^2)} \] (65)

or according to \( 21 \) with
\[ x^2 = \frac{m(1 - n^2)}{n(1 - m^2)} \]
we have
\[
\sin \phi = \frac{2n(1 - m^2)}{(n + m)(1 - mn)} \lambda \tag{66}
\]

The generator \( p \) of the angle \( \phi \) is given by
\[
p = \frac{\sin \phi}{1 + \cos \phi}
\]

where
\[
1 + \cos \phi = \frac{(n + m)(1 - mn) + (n - m)(1 + mn)}{(n + m)(1 - mn)}
\]
\[
1 + \cos \phi = \frac{2n(1 - m^2)}{(n + m)(1 - mn)} \tag{67}
\]

Therefore
\[
p = \lambda \tag{68}
\]

The area is now given by
\[
A = 2\lambda n(1 - m^2)u^2 \tag{69}
\]

Now, a Heron parallelogram with sides \( u_1, u_2 \), diagonals \( u_3, u_4 \) and area \( A \) is parameterized by
\[
0 < m < 1, \quad 0 < n < 1, \quad u > 0, \quad \lambda > 0 \text{ with } \lambda^2 = m(1 - n^2) = n(1 - m^2) \tag{70}
\]

For a type I Heron parallelogram, we have the relation (22) and thus the bijective parameter \( \sigma \). From (24, 25, 70) we find
\[
0 < \sigma < 1 \tag{71}
\]

For a type II Heron parallelogram, we have the relation (26) and thus the bijective parameter \( \sigma \). From (28, 29, 70) we find
\[
\frac{1}{2} < \sigma < 1 \tag{72}
\]

Observe that Heron parallelograms cover the case of Heron triangles with a rational median.

Finally, we have the relations

Type I:
\[
\lambda = \frac{m}{n} = \frac{2u_2 + u_3 - u_4}{2u_2 - u_3 + u_4}
\]

Type II:
\[
\lambda = mn = \frac{u_3 + u_4 - 2u_1}{u_3 + u_4 + 2u_1} \tag{73}
\]
Example

Type I : From (30)

\[
m = \frac{3}{7}, \quad n = \frac{5}{7}, \quad \lambda = \frac{3}{5}, \quad u = \frac{49}{2}
\]
\[u_1 = 17, \quad u_2 = 28, \quad u_3 = 25, \quad u_4 = 39, \quad A = 420\] (74)

Type II : From (31)

\[
m = \frac{5}{7}, \quad n = \frac{7}{8}, \quad \lambda = \frac{5}{8}, \quad u = 56
\]
\[u_1 = 21, \quad u_2 = 89, \quad u_3 = 82, \quad u_4 = 100, \quad A = 1680\] (75)

References

[1] Hermann C.H. Schubert, Die Ganzzahligkeit in der Algebrauschen Geometrie (1905), Translated by Ralph H. Buchholz, Integrability in algebraic geometry (2005)

[2] Walter Wyss, Sum of Squares, Bijective Parameter Representation, https://arxiv.org/abs/1402.0102

[3] T.L. Heath, Diophantus of Alexandria, Cambridge 1910

[4] Walter Wyss, Perfect Parallelograms, American Math Monthly, 119 (6) (2012), p.513-515

Department of Physics, University of Colorado Boulder, Boulder, CO 80309
Walter.Wyss@Colorado.EDU