Trust but verify: The case for astrophysical black holes

Scott A. Hughes
Department of Physics and MIT Kavli Institute,
77 Massachusetts Avenue, Cambridge, MA 02139

This article is based on a pair of lectures given at the 2005 SLAC Summer Institute. Our goal is to motivate why most physicists and astrophysicists accept the hypothesis that the most massive, compact objects seen in many astrophysical systems are described by the black hole solutions of general relativity. We describe the nature of the most important black hole solutions, the Schwarzschild and the Kerr solutions. We discuss gravitational collapse and stability in order to motivate why such objects are the most likely outcome of realistic astrophysical collapse processes. Finally, we discuss some of the observations which — so far at least — are totally consistent with this viewpoint, and describe planned tests and observations which have the potential to falsify the black hole hypothesis, or sharpen still further the consistency of data with theory.

I. BACKGROUND

Black holes are among the most fascinating and counterintuitive objects predicted by modern physical theory. Their counterintuitive nature comes not from obtuse features of gravitation physics, but rather from the extreme limit of well-understood and well-observed features — the bending of light and the redshifting of clocks by gravity. This limit is inarguably strange, and drives us to predictions that may reasonably be considered bothersome. Given the lack of direct observational or experimental data about gravity in the relevant regime, a certain skepticism about the black hole hypothesis is perhaps reasonable. Indeed, one can understand Eddington’s somewhat plaintive cry “I think there should be a law of Nature to prevent a star from behaving in this absurd way!” [1].

To the best of our knowledge, there is no such law of Nature. We expect that in any object which is sufficiently massive, gravity can overwhelm all opposing forces, causing that object to collapse into a black hole — a region of spacetime in which gravity is so intense that not even light can escape. Indeed, astronomical observations have now provided us with a large number of candidate black holes — dark objects which are so massive and compact that the hypothesis that they are the black holes of general relativity (GR) is quite plausible. These objects are now believed to be quite common — extremely massive black hole candidates (several $10^5$ solar masses to $\sim 10^9$ solar masses) reside in the cores of almost all galaxies, including our own; stellar mass candidates (several to several tens of solar masses) are seen in orbit with stars in our galaxy; and evidence suggests that “intermediate” mass black holes may fill the gap between these limits. Not only is there no law preventing stars from “behaving in this absurd way”, Nature appears to revel in such absurdities.

It must be emphasized that, from the purely observational perspective, at present these objects are black hole candidates. We have not unequivocally established that any of these objects exhibits all of the properties that GR predicts for “real” black holes. GR is consistent with Newton’s theory of gravity in the appropriate limit. This means that, for example, orbits of black holes are totally consistent with Newtonian gravity except for strong field (small radius) orbits. Establishing whether these objects are the black holes of GR requires observations that probe close to the black hole itself. Because black holes are so compact, this is a challenging proposition. Consider the putative black hole at the core of our galaxy. If it is a black hole as described by GR, its radius should be about $10^7$ kilometers. It is located roughly 7500 parsecs from the earth. The black hole thus subtends an angle $\delta \theta \sim 10^7 \text{ km}/(7.5 \times 10^3 \text{ pc}) \sim 0.01$ milliarcseconds on the sky! Resolving such small scale phenomena is extraordinarily difficult.

Given that no observations have unequivocally established that black hole candidates are GR’s black holes, it is not surprising that alternate hypotheses have been advanced to describe these candidates. Since GR is a classical theory and Nature is fundamentally quantum, one might expect something other than classical GR to describe black hole candidates. Such a modification certainly seems necessary in order to, for example, resolve black hole information paradoxes [2].
The standard view is that, though modifications to classical GR are needed, their impact is most likely negligible for astrophysical black holes. Black holes describe gravitational effects far beyond what we normally encounter, and so appear to be “extreme objects”. Within the context of general relativity, however, they are not so extreme. In GR, the “fundamental” gravitational field is not manifested as acceleration; instead, it is the spacetime curvature, which is manifested as tidal fields. Though black holes have enormous gravitational acceleration, their tides can be quite gentle. In particular, astrophysical black hole tides are far smaller than the level at which one would expect quantum corrections to be important.

Regardless of whether one accepts this standard view of modifications to GR theory or whether one postulates more drastic modifications, one needs some kind of “gold standard” against which to compare observations. The view we take here is that, given the number of possible ways in which one can modify classical GR, it is currently most reasonable to treat the classical black hole solution as our gold standard. If observations show that this is inadequate, then we would have an experimentally motivated direction to pursue modifications to general relativity and to black holes.

The goal of these lectures is to summarize the case for black holes within classical GR. We first discuss, in Sec. II, the simplest, exact black hole solutions of general relativity. We illustrate that they have a completely reasonable, non-singular nature (at least in their exteriors!), but that uncovering this nature can be a bit subtle owing to some pathologies in the coordinates that are commonly used to describe them. Changing to coordinates which eliminate these pathologies, we show that light can never escape once inside a certain radius (the “black”), and thus any object which falls inside is doomed to fall until diverging tidal forces tear it apart (the “hole”).

The solution we use to build this description corresponds to a black hole that exists for all time. One might worry that the features we discovered using it are a contrivance of that artificial, eternal solution. Can we really create a region of spacetime with these features beginning with well-behaved “normal” matter? As shown in Sec. III we can. A well-known solution, developed by Oppenheimer and his student Snyder in 1939 [3] shows analytically that pressureless, spherical matter quickly collapses from a smooth, well-behaved initial distribution to form a black hole with exactly the properties of the exact solution that we initially discussed. Numerical calculations show that including pressure and eliminating spherical symmetry leads to the same result.

These results depend on the solution having no angular momentum. Roy Kerr lifted this requirement when he discovered the solution which now bears his name [4]. Spinning black holes are the generic case that we expect astrophysically; as discussed in Secs. IV and V classical general relativity predicts the Kerr black hole to be the unique and stable endpoint of gravitational collapse. We discuss the perturbative techniques which were first used to advance this result, and discuss numerical calculations which show that it holds generically.

The Kerr black hole is thus the object which general relativity predicts describes the many black hole candidates which we observe today. What remains is to test this prediction. In Sec. VI we briefly describe some of the observations being developed today which can probe into the strong field of these objects and confront the Kerr black hole hypothesis with data.

Throughout this article, we assume that the reader has basic familiarity with general relativity; in particular, the notion of geodesics and the Einstein field equations will play an important role. Carroll’s SLAC Summer Institute lectures covered this material; the reader is referred to those lectures for GR background. The presentation of Carroll’s textbook [5] and the classic textbook by Misner, Thorne, and Wheeler (MTW) [6] influenced the development of these lectures quite a bit. Readers interested in a very detailed discussion of black hole physics would be well-served with Poisson’s textbook [7].

In most of the article, we use “relativist’s units” in which \( G = 1 = c \). These units nicely highlight the fact that general relativity has no intrinsic scale; as we shall see, all important lengthscales and timescales are set by and become proportional to black hole mass. We shall sometimes restore \( Gs \) and \( cs \) to make certain quantities more readable. Useful conversion factors for discussing astrophysical black holes come from expressing the solar mass \((M_\odot)\) in these units:

\[
M_\odot \rightarrow \frac{GM_\odot}{c^2} = 1.47663 \text{km} \simeq 1.5 \text{km}
\]
\[ \frac{GM_\odot}{c^3} = 4.92549 \times 10^{-6} \text{ sec } \simeq 5 \mu \text{sec} \ . \]

II. THE SCHWARZSCHILD BLACK HOLE

The simplest black hole solution is described by the Schwarzschild metric, discovered by Karl Schwarzschild in 1916 \[ \text{[8]} \]. In what are now known as “Schwarzschild coordinates”, this metric takes the form

\[ ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \, . \]  

By building the Einstein tensor from this metric, it is a simple matter to show that the Schwarzschild solution is an exact solution to the Einstein field equations \( G_{ab} = 8\pi T_{ab} \) for \( T_{ab} = 0 \) — i.e., it’s a vacuum solution.

Since it satisfies Einstein’s equations, we can be satisfied that this metric is mathematically consistent. Is it however a physically meaningful solution? Is this metric one that we might use to describe real objects in the universe, or is it too idealized to be worthwhile? To answer this, we first examine several limiting regimes of this spacetime. First consider \( r \gg M \):

\[ ds^2 \approx -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 + \frac{2M}{r}\right)dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \, . \]

Examining slow motion geodesics (spatial velocity \( |dx^i/dt| \ll c \)) in this spacetime. The equation of motion which we find describes the coordinate motion of bodies reduces, in this limit, to

\[ \frac{d^2 x^i}{dt^2} = -\frac{M x^i}{r^3} \, . \]  

In other words, the \( r \gg M \) limit of the Schwarzschild metric reproduces Newtonian gravity! Clearly, the Schwarzschild metric represents the external “gravitational field” of a monopolar mass \( M \) in general relativity \[ \text{[5]} \].

Let us now study this metric in this coordinate systems more carefully. First, consider a “slice” of the spacetime with constant \( r \) and constant \( t \):

\[ ds^2 = r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \, . \]  

This is the metric that is used to measure the distance between points on the surface of a sphere; in other words, this is a spherically symmetric spacetime, with \( \theta \) and \( \phi \) chosen to be the “usual” spherical angles that cover the sphere. More interestingly, this construction helps us to understand the meaning of the coordinate \( r \): Eq. \[ \text{(6)} \] shows us that the surface area of this sphere is just \( 4\pi r^2 \) — just as our intuition would lead us to expect.

Now, allow \( r \) to vary:

\[ ds^2 = \frac{dr^2}{1 - 2M/r} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \, . \]

The distance between two spatial points \((r_1, \theta, \phi)\) and \((r_2, \theta, \phi)\) is not just \( \Delta r = r_2 - r_1 \) as our intuition would have led us to believe! Instead we must integrate the line element,

\[ \text{Distance} = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - 2M/r}} = \frac{r_2 - 2M}{\sqrt{1 - 2M/r_2}} - \frac{r_1 - 2M}{\sqrt{1 - 2M/r_1}} + 2M \ln \left[ \frac{r_2}{r_1} \left(1 + \sqrt{1 - 2M/r_2^2} \right) \right] \, . \]  

The coordinate \( r \) is an “areal” radius — it labels spherical surfaces of area \( 4\pi r^2 \), but it does not label proper distance in a simple way.

\[ \text{[8]} \]

---

1. This limiting behavior is used to derive the Schwarzschild metric from first principles. Beginning with the most general static, spherically symmetric line element, \( ds^2 = e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + R(r)^2(d\theta^2 + \sin^2 \theta \, d\phi^2) \), one enforces the vacuum Einstein equations and finds \( R(r) = r, \Lambda(r) = -\Phi(r) \). Correspondence with the Newtonian limit leads to \( e^{2\Phi(r)} = 1 - 2M/r \).
Finally, we need to understand the meaning of the timelike coordinate \( t \). When we are extremely far from the mass \( M \), the line element (3) reduces to that of flat spacetime:

\[
ds^2 \approx -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) .
\]

The \( t \) coordinate is clearly time as measured by observers in this “asymptotically flat” region. This is the meaning of \( t \): It is the label that very distant observers use to measure the passing of time. This is a very convenient time label for many purposes, but is not so good for many others. For example, it is a superb coordinate for describing features that would be measured by distant observers watching processes near a black hole. It is an absolutely lousy coordinate for describing what occurs to an object that falls into a black hole. Many misunderstandings about the nature of black holes are fundamentally due to the nature of the Schwarzschild time coordinate.

Having examined and explained the nature of the coordinates with which we have presented the Schwarzschild solution, let us now begin to understand the spacetime itself. It is clear that there are some oddities we should worry about — some metric components diverge or go to zero at \( r = 2M \); all metric components show funny behavior as \( r \to 0 \). As we shall see, the first behavior reflects a coordinate singularity — the spacetime is perfectly well behaved at \( r = 2M \), but the coordinates are not. The second pathology by contrast reflects a real singularity — a divergence in tidal forces, reflecting a breakdown in the classical solution.

A. Birkhoff’s Theorem

In deciding how we should regard the Schwarzschild solution, perhaps the most important result to bear in mind is Birkhoff’s Theorem:

The exterior spacetime of all spherical gravitating bodies is described by the Schwarzschild metric.

The Schwarzschild metric describes a portion of the spacetime of any spherically symmetric body. Consider for example a spherical star of radius \( R_\ast \). Birkhoff’s theorem then guarantees that Eq. (3) describes the star’s spacetime for \( r > R_\ast \). (Obviously, some other metric must describe the spacetime in the star’s interior, \( r < R_\ast \), since the interior contains matter.)

In fact, Birkhoff’s theorem is even more powerful than this: It applies even if the spacetime is time dependent, so long as the time dependence is spherical in nature. Thus, the exterior spacetime of a star that undergoes radial pulsations likewise is described by the Schwarzschild metric. This tells that spherical oscillations cannot affect the star’s exterior gravitational field.\(^2\)

Birkhoff’s Theorem is in fact quite simple to prove. Carroll \([5]\) gives a very careful proof; here, we outline a simpler proof, based on that given in MTW \([6]\). We begin by writing down the general line element for a time dependent, spherical spacetime:

\[
ds^2 = -a(r, t)^2 dt^2 - 2a(r, t)b(r, t) \, dr \, dt + d(r, t)^2 dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) .
\]

\[\text{[Without loss of generality, we have chosen the coefficient of the angular sector to be } r^2 \text{ rather than some function } R(r)^2. \text{ By doing so, we are choosing our coordinate } r \text{ to be an areal radius.]} \]

We can immediately simplify this by changing our time coordinate: define

\[
e^{\Phi(r,t')} dt' = a(r, t) \, dt + b(r, t) \, dr .
\]

Insert this into (10), define \( e^{2\Lambda(r,t')} = b^2(r, t) + d^2(r, t) \), and drop the primes on \( t \) in the result. We find

\[
ds^2 = -e^{2\Phi(r,t)} dt^2 + e^{2\Lambda(r,t)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) .
\]

\[\text{[In retrospect, this makes a lot of sense: The only way to causally affect the exterior field is for information about changes to the star’s gravity to propagate radiatively. Monopolar oscillations cannot radiate in general relativity, just as they don’t radiate in electrodynamics.]} \]

\[\text{L006 4}\]
We now enforce the vacuum Einstein equations: Compute the Einstein tensor \( G_{ab} \) and equate it to 0. (We see here why Birkhoff applies to a gravitating body’s exterior; on the interior, we would set the right hand side to something appropriate to the body’s interior matter distribution.) We find several differential equations that constrain the functions \( \Phi(r, t) \) and \( \Lambda(r, t) \). For example, the \( r, t \) component of the Einstein tensor yields [cf. MTW, Eq. (32.3b)]

\[
\frac{2e^{-(\Phi+\Lambda)}}{r} \frac{\partial \Lambda}{\partial t} = 0 \quad \rightarrow \quad \frac{\partial \Lambda}{\partial t} = 0.
\] (13)

\( \Lambda \) is a function of \( r \) only — it has no time dependence. When this is taken into account, the components of the Einstein tensor that determine \( \Lambda \) are identical to the static case, and we find

\[
e^{2\Lambda} = \frac{1}{1 - 2M/r},
\] (14)

just as before. The remaining Einstein tensor components constrain the function \( \Phi(r, t) \); enforcing the vacuum Einstein equation gives us

\[
e^{2\Phi} = (1 - 2M/r) e^{2f(t)},
\] (15)

where \( f(t) \) is an arbitrary function of time (arising as an integration constant). It can be eliminated by changing time variables once again: \( dt' = e^{f(t)} dt \). The Schwarzschild metric is the final result of these manipulations.

### B. The event horizon

Recall that our goal in this section is to understand how relevant the Schwarzschild solution is as a description of physics, as opposed to as a simple mathematical solution of the Einstein field equations. Birkhoff’s Theorem tells us that, at the very least, this solution describes the exterior of at least some objects we might expect to encounter in the universe. However, what about for extremely compact objects? We raised concerns about oddness at the radius \( r = 2M \). Might there exist objects whose surfaces satisfy \( r \leq 2M \), and if so, how do we understand the nature of the spacetime in that region?

Newtonian intuition [9] suggests that we should in fact be concerned about the character of spacetime in this region. The escape speed from a spherical object of mass \( M \) and radius \( R \) is found by requiring that the kinetic energy per unit mass equal the work per unit mass it takes to move from \( r = R \) to infinity:

\[
\frac{1}{2} v_{esc}^2 = \frac{GM}{R}.
\] (16)

What is the radius at which the Newtonian escape speed is the speed of light? Plugging in \( v_{esc} = c \) and solving for \( R \), we find

\[
R = \frac{2GM}{c^2} = 2M.
\] (17)

This is precisely the radius at which the Schwarzschild metric behaves oddly! Though this calculation abuses quite a few physical concepts, it properly illustrates the fact that at this radius the impact of gravity on radiation is likely to be extreme.

To understand what happens in GR, we need to examine “stuff” moving around in this region[3]. Consider dropping a particle from some starting radius \( r_0 \) on a purely radial trajectory (no angular momentum). By integrating the geodesic equation, we find \( r(t) \), the position of the particle as a function of time measured by distant observers, as

---

[3] Note added Spring 2024: when this document was originally written, I apparently made an elementary error integrating the equations of motion to find these solutions presented in Eqs. (18) and (19). Reposting nearly 20 years later on the off-chance that someone might stumble on these formulas.
The radial position of an infalling particle as a function of two different measures of time: \( t \), the time measured by distant observers’ clocks; and “proper time” \( \tau \), time that is measured by observers who ride with the particle. The particle reaches \( r = 0 \) in finite proper time, but only asymptotically approaches \( r = 2M \) according to our distant observers.

\[
\frac{t}{2M} = \ln \left[ \sqrt{\frac{r(r_0 - 2M)}{2M(r_0 - r)}} + 1 \right] + \sqrt{\frac{r(r_0 - r)}{(2M)^2}} - 1 + \left( \frac{r_0}{2M} + 2 \right) \sqrt{\frac{r_0}{2M}} - 1 \left[ \frac{\pi}{2} - \arctan \left( \sqrt{\frac{r}{r_0} - r} \right) \right],
\]

and

\[
\frac{\tau}{2M} = \left( \frac{r_0}{2M} \right)^{3/2} \arctan \left( \sqrt{\frac{r_0 - r}{r}} \right) + \frac{rr_0(r_0 - r)}{\sqrt{(2M)^3}}.
\]

Expanding near \( r = 2M + \delta r \), the behavior according to distant observers takes the asymptotic form

\[
\delta r = C(r_0)e^{-t/2M}.
\]

The quantity \( C(r_0) \) we introduced here gather together several \( r_0 \) dependent factors appearing in Eq. (18) into the constant \( C(r_0) \). These two solutions for \( r \) as a function of “time” are shown in Fig. 1.

The two observers have radically different views. The observer who rides along with the infall finds that it takes finite time pass through \( r = 2M \). By contrast, distant observers using \( t \) as their time label never actually see the particle fall through \( r = 2M \) — the particle only asymptotically approaches this radius as \( t \to \infty \). There appears to be a severe inconsistency here.

Before explaining the source of this apparent inconsistency, it’s worth examining what, if anything, the infalling body experiences at \( r = 2M \). As already mentioned, the fundamental gravitational “force” in general relativity is subsumed by the notion of tides that a body experiences. These tides are given by tensors that describe spacetime
curvature; the most important is the Riemann curvature tensor. As measured by the freely falling particle, the Riemann tensor’s components are given by

\[ R_{\alpha\beta\gamma\delta} = \frac{AM}{r^3}, \quad (21) \]

where \( A = \pm 2, \pm 1, 0 \) depending on specific index values. Notice that there is nothing special about the curvature at \( r = 2M \) — the small body passes through the radius, feeling continually increasing tidal forces, without any particular notification that \( r = 2M \) has come and gone. The curvature diverges as \( r \to 0 \) — that represents a true, infinite tidal force singularity (at least classically).

The disagreement between the “infalling view” and the “distant observer view” thus remains mysterious. To clarify this mystery somewhat, consider radial “null geodesics”, the trajectories followed by radiation:

\[ ds^2 = 0 \quad \rightarrow \quad dt = \pm \frac{dr}{1 - 2M/r} \]

\[ \equiv \pm dr^* \quad (23) \]

where \( r^* = r + 2M \ln(r/2M - 1) \). It takes an extremely long time, according to distant observers, for light to leak out from near \( 2M \). The delay becomes infinite if the light starts at \( r = 2M \) — photons released at that radius never reach distant observers. Consider next the energy associated with radiation moving on this geodesic. To do so, we use the rule that the measured energy of a photon with 4-momentum \( p \) is given by

\[ E = -p \cdot u, \quad (24) \]

where \( u \) is the 4-velocity of the observer who makes the measurement. For a static observer at radius \( r \),

\[ u^a = \left[ (1 - 2M/r)^{-1/2}, 0, 0, 0 \right]. \quad (25) \]

We imagine that a photon is emitted at radius \( r \) and observed very far away:

\[ \frac{E_{\text{obs}}}{E_{\text{emit}}} = \sqrt{1 - \frac{2M}{r}}. \quad (26) \]

Radiation emitted near \( r = 2M \) is highly redshifted, with \( r = 2M \) corresponding to a surface of infinite redshift\(^4\).

Both the “infinite time delay” and the “infinite redshift” tell us that our time coordinate is broken in the strong field of the black hole. The surface \( r = 2M \) corresponds to a singularity in Schwarzschild coordinates — as seen by distant observers, clocks slow to a halt as they approach this surface. From far away, we never actually see our test particle cross this surface; no paradox is involved, however, since the photons we might use to observe this behavior are infinitely redshifted away, and infinitely delayed in reaching us.

C. Other coordinate systems

The punchline of the preceding section is that Schwarzschild coordinates are not a good way to describe the kinematics of objects that pass through the event horizon. Fortunately, we can transform to different coordinates which do a much better job of illustrating aspects of motion in the strong field of the Schwarzschild spacetime. For example “Eddington-Finkelstein coordinates” ([5], §5.6; [6], Box 31.2) are very well adapted to describing what would be observed by someone falling into a black hole; the Painlevé-Gullstrand coordinate system ([7], §5.1.4) give a very nice description of the spacetime using coordinates adapted to freely falling particles. Here, we will focus almost entirely on Kruskal-Szekeres (K-S) coordinates — a (rather nonintuitive) relabeling of time and radius that clarifies which regions of spacetime are in causal contact with one another.

\[ ^4 \text{Recall that redshift } z \text{ is defined via } E_{\text{obs}}/E_{\text{emit}} = 1/(1 + z). \]
II THE SCHWARZSCHILD BLACK HOLE

FIG. 2. The Schwarzschild spacetime in Kruskal-Szekeres coordinates, and some example radial null geodesics. The coordinate $u$ is horizontal, $v$ is vertical. The dotted lines represent the surface $r = 2M$; the heavy black lines are the “point” $r = 0$. The smaller 45° lines represent null geodesics, or radiation. Notice that light directed radially outward never crosses $r = 2M$. Notice also that any radiative trajectory that at any time crosses $r = 2M$ must hit the $r = 0$ singularity at some point.

1. Kruskal-Szekeres

K-S coordinates $u$ and $v$ are related to Schwarzschild coordinates $r$ and $t$ via

$$u = \sqrt{1 - r/2M} e^{r/4M} \sinh(t/4M) \quad (27)$$

$$v = \sqrt{1 - r/2M} e^{r/4M} \cosh(t/4M) \quad r < 2M \; ;$$

$$u = \sqrt{r/2M - 1} e^{r/4M} \cosh(t/4M) \quad (29)$$

$$v = \sqrt{r/2M - 1} e^{r/4M} \sinh(t/4M) \quad r > 2M \; ;$$

the opposite signs for $u$ and $v$ are also valid. Notice that

$$u^2 - v^2 = (r/2M - 1)e^{r/2M} \; .$$

(31)

This means that the surface $r = 2M$ is represented by the lines $u = \pm v$, and the point $r = 0$ is represented by the hyperbolae $v = \pm \sqrt{1 + u^2}$; these locations are plotted in Fig. 2. The right-hand quadrant of this figure represents $r > 2M$; the top quadrant represents $r < 2M$. For discussion of the meaning of the left-hand and bottom quadrants, see [3], §5.7; [4], §31.5; or [7], §5.1. The seemingly obtuse definitions of $u$ and $v$ in fact follow in a natural way from Eddington-Finkelstein coordinates, as discussed in the afore-referenced texts.

In terms of $u$ and $v$, the metric (3) becomes

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} (-dv^2 + du^2) + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \; .$$

(32)
The virtue of these coordinates is made clear by considering radial ($d\theta = d\phi = 0$) null geodesics ($ds = 0$), describing light or other radiation:

$$ds^2 = 0 \quad \rightarrow \quad du = \pm dv .$$

(33)

*Radiation travels on 45° lines in K-S coordinates.* A few radiative trajectories are illustrated in Fig. 2; rays emitted from the same point in opposite directions form a light cone with opening angle 90°.

Consider next radial *timelike* trajectories, which are followed by material objects and observers. For these trajectories, we can put $ds = d\tau$, the proper time measured along the trajectory itself. We then find

$$-1 = \frac{32M^3}{r} e^{-r/2M} \left[ -\left( \frac{dv}{d\tau} \right)^2 + \left( \frac{du}{d\tau} \right)^2 \right].$$

(34)

Rearranging slightly, we have

$$\frac{r/32M^3}{(dv/d\tau)^2} e^{r/2M} = \left[ 1 - \left( \frac{du}{dv} \right)^2 \right].$$

(35)

Since the left-hand side of this equation is positive, we must have

$$\left| \frac{du}{dv} \right| < 1 .$$

(36)

In other words, material objects move such that their angle relative to the $v$ axis is $< 45°$. A timelike trajectory from a particular point must live inside that point’s lightcone.

Bearing these features of motion in K-S coordinates in mind, re-examine Fig. 2. Notice that nothing which is inside $r = 2M$ can ever come out! Consider a radial light ray emitted just inside the $r = 2M$ line. It either goes at a 45° angle smack into the $r = 0$ singularity; or, it skims slightly inside the $r = 2M$ line. This second light ray can never cross $r = 2M$ — that line and the light ray are parallel. Instead, after some finite time, it will eventually smash into the $r = 0$ singularity — even an “outward” directed light ray eventually hits the singularity if it originates at $r < 2M$. Since timelike trajectories must move at an angle $< 45°$ with respect to the $v$ axis, material objects and observers inside $r = 2M$ are likewise guaranteed to smash into the singularity.

By contrast, consider radial light rays emitted outside $r = 2M$. The ray directed toward the event horizon clearly hits the singularity; however, the outward directed ray always remains outside $r = 2M$. As we’ll show in the next section, it travels arbitrarily far away — “escaping” the region of the black hole.

The punchline is that all physical trajectories which begin inside or cross $r = 2M$ must eventually hit the singularity at $r = 0$ — even the trajectories of radiation. Anything that goes inside this region can never cross back into the exterior region. For this reason, the spherical surface $r = 2M$ is called the “event horizon”: Events which occur inside this horizon cannot ever impact events on the outside. Spacetime interior to the horizon is causally disconnected from the rest of the universe.

2. *Penrose diagram*

Although it is somewhat beyond the original scope of these lectures, it is hard to resist including the following brief discussion. With one further coordinate transformation, we can map infinitely distant events in the spacetime to a finite coordinate label: put

$$v + u = \tan \left[ (\psi + \xi)/2 \right] ,$$

$$v - u = \tan \left[ (\psi - \xi)/2 \right] .$$

(37)

(38)

The $u, v$ coordinate representation of Schwarzschild remaps to the $\xi$ (horizontal axis) and $\psi$ (vertical axis) representation shown in Fig. 3. This way of showing Schwarzschild is called a “Penrose diagram”. The heavy black horizontal
FIG. 3. Penrose diagram of the Schwarzschild spacetime. The heavy black lines are the classical singularity at \( r = 0 \); the diagonal dashed lines show the event horizon at \( r = 2M \). The region labeled “I” represents the exterior spacetime \((r > 2M)\); that labeled “II” is the interior \((r < 2M)\).

The line is the \( r = 0 \) singularity; the 45° dashed lines are the event horizon, \( r = 2M \); the 45° thin solid lines represent infinitely distant parts of the spacetime. Indeed, this diagram makes clear that there are several different varieties of “infinity”: Spacelike trajectories in the infinite limit \((r \to \infty, t \to \text{finite})\) all accumulate at the single point \( \psi = 0, \xi = \pi \) — the lower right-most point on the thin solid line. Timelike trajectories \((t \to \infty, r \to \text{finite})\) accumulate at \( \psi = \pi/2, \xi = \pi/2 \) — the upper left-most point on this line. Null trajectories \((t \to \infty, r \to \infty)\) reach various points on that line. See [5–7] for further discussion.

The line element \( ds^2 \) in the coordinates \( \psi, \xi \) is given by

\[
ds^2 = \frac{32M^3}{r} e^{-r/2M} \left( \frac{-d\psi^2 + d\xi^2}{4 \cos^2 \left(\frac{\psi + \xi}{2}\right) \cos^2 \left(\frac{\psi - \xi}{2}\right)} \right) + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \, .
\]

Just as in K-S coordinates, radiation travels on 45° lines: the condition for null trajectories, \( ds = 0 \), yields \( d\xi = \pm d\psi \). By examining light trajectories in the Penrose diagram, we see even more clearly that any signal emitted at \( r < 2M \) (region II of Fig. 3) will forever be trapped inside. Outward directed signals in region I, by contrast, eventually reach “infinity”. Two observers in region I can easily send signals to one another; they can also send signals to a friend in region II. However, that friend cannot send a signal back out: region II cannot send any signal to region I.

D. Summary of Schwarzschild spacetime

We wrap up this section by summarizing what this brief examination of the features of the Schwarzschild spacetime revealed:

- The Schwarzschild spacetime represents, in general relativity, a monopolar “gravitational field”. One can regard it is the GR analog of the Coulomb point charge electric field.
- This solution describes the exterior spacetime of any spherically symmetric body, even one that is time dependent (as long as the time dependence preserves spherical symmetry).
- The spacetime contains an event horizon: A spherical surface at \( r = 2M \) that causally disconnects all events at \( r < 2M \) from those at \( r > 2M \). Things can go into the horizon (from \( r > 2M \) to \( r < 2M \)), but they cannot get out; once inside, all causal trajectories (timelike or null) take us inexorably into the classical singularity at \( r = 0 \).
- From the perspective of distant observers, dynamics near the horizon appears weird — such observers never actually see anything cross it and fall in. This is largely a consequence of extreme redshifting — clocks slow
down so much, relative to distant observers, that physical processes appear to come to a stop. At any rate, this weirdness is hidden from distant observers due to the extreme redshifting and deflection of the photons that they would use to observe these processes.

For all these reasons, the Schwarzschild solution is known as a black hole: A region of spacetime that is completely dark and whose interior is completely cut off from the rest of the universe.

III. DYNAMICS: CREATING A SCHWARZSCHILD BLACK HOLE

The Schwarzschild solution that we have discussed so far is eternal — it has existed for all time. Far more interesting are objects that are created through astrophysical processes. One may then ask whether this seemingly special solution can actually be created from “reasonable” initial conditions. In other words, can we make a Schwarzschild black hole by starting out with something made from “normal” matter?

In an idealized but illustrative calculation, Oppenheimer and Snyder [3] showed in 1939 that a black hole in fact does form in the collapse of ordinary matter. They considered a “star” constructed out of a pressureless “dustball”. By Birkhoff’s Theorem, the entire exterior of this dustball is given by the Schwarzschild metric, Eq. (3). Due to the self-gravity of this “star”, it immediately begins to collapse. Each mass element of the pressureless star follows a geodesic trajectory toward the star’s center; as the collapse proceeds, the star’s density increases and more of the spacetime is described by the Schwarzschild metric. Eventually, the surface passes through $r = 2M$. At this point, the Schwarzschild exterior includes an event horizon: A black hole has formed. Meanwhile, the matter which formerly constituted the star continues collapsing to ever smaller radii. In short order, all of the original matter reaches $r = 0$ and is compressed (classically!) into a singularity. This evolution is illustrated in Fig. 4. Notice that the event horizon first appears at $t \approx 6M$, and rapidly expands outwards.

The particularly attractive feature of the Oppenheimer-Snyder construction is that it is completely analytic. The metric which describes the interior of a spherical dustball is

$$\text{ds}^2 = -d\tau^2 + a^2(\tau) \left[ d\chi^2 + \sin^2 \chi \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right].$$

(40)

This is the Friedmann-Robertson-Walker (FRW) metric; it is more typically used to represent the large-scale spacetime of a homogeneous, isotropic universe. The radial coordinate inside the star is $\chi$; it runs from the center ($\chi = 0$) to the star’s surface ($\chi = \chi_s$, to be determined later). The function $a(\tau)$ is the “scale factor”; it provides an overall scale to $\chi$. The coordinate $\tau$ is proper time measured by a matter element as it evolves with the dustball.

Enforcing Einstein’s equations and applying appropriate initial conditions (namely that each mass element is initially at rest, so that the dustball collapses rather than expands) yields the following parametric solutions for the proper time $\tau$ and the scale factor $a$:

$$a = \frac{a(0)}{2} \left( 1 + \cos \eta \right),$$

(41)

$$\tau = \frac{a(0)}{2} \left( \eta + \sin \eta \right).$$

(42)

Collapse begins when the parameter $\eta = 0$; when $\eta = \pi$, all of the matter has fallen into the singularity.

By Birkhoff’s Theorem, the exterior spacetime is given by the Schwarzschild metric. The two descriptions must match smoothly at the star’s surface. Matter on the star’s surface can thus be described either using the FRW coordinate description of Eqs. (41) and (42), or by integrating the geodesic equation in Schwarzschild coordinates.

---

5 Since all of the matter is squashed into a point of zero size, this classical singularity must be modified in a a complete, quantum description. However, since all the singular nastiness is hidden behind an event horizon where it is causally disconnected from us, we need not worry about it (at least for astrophysical black holes).
FIG. 4. Oppenheimer-Snyder collapse of a "star" with an initial radius $R_{\text{surf}} = 5M$. The surface of the star passes through $r = 2M$ at a time of roughly $11M$; after that time, all of the star’s matter is behind the event horizon. The matter continues collapsing, eventually crushing into a zero-size singularity at roughly $t = 12.5M$. The horizon actually first came into existence just before $t = 6M$ and rapidly expanded outward.

Doing so the latter way yields the following solutions for the radius of the star’s surface, $R_{\text{surf}}$, and the elapsed time as measured by clocks at the surface, $\tau_{\text{surf}}$:

$$R_{\text{surf}} = \frac{R_{\text{surf}}(0)}{2} (1 + \cos \eta),$$  \hspace{1cm} (43)

$$\tau_{\text{surf}} = \sqrt{\frac{R_{\text{surf}}(0)^3}{8M}} (\eta + \sin \eta).$$  \hspace{1cm} (44)

By computing the circumference of the star, $C = \int \sqrt{g_{\phi\phi}(\theta = \pi/2)} \, d\phi$, in the two descriptions, we find

$$R_{\text{surf}}(0) = a(0) \sin \chi_s.$$  \hspace{1cm} (45)

By requiring that proper time at the surface be the same in the two descriptions, we find

$$a(0) = \sqrt{\frac{R_{\text{surf}}(0)^3}{2M}}.$$  \hspace{1cm} (46)

Given an initial choice for the radius of the star, Eqs. (45) and (46) fix the values of $a(0)$ and $\chi_s$. From there, the complete spacetime can be constructed. This is how Fig. 4 was generated.

One might object to the relevance of this calculation — after all, pressureless matter isn’t very interesting! The point which this calculation makes manifestly clear is that, at least in this idealized case, one easily transitions from smooth, well-behaved initial data to the rather odd black hole Schwarzschild solution.

Although, to our knowledge at least, there exist no similarly analytic calculations of such collapse when for realistic matter (including pressure, among other physically important characteristics), such a calculation may be performed numerically; the first such calculation was published in 1966 [10]. Although the quantitative details are changed, the
qualitative collapse picture is identical to that given here: If the pressure is not sufficient to hold up the infalling matter, the star’s matter collapses until its surface passes through \( r = 2M \), leaving only an event horizon visible to external observers. Inside the star, the matter continues collapsing until it is squashed into a zero-size singularity.

Black holes are a generic end product of spherical gravitational collapse.

IV. THE KERR METRIC: LIFE IS NOT SPHERICAL

In the limit of perfect spherical symmetry, the black hole picture is now complete. We have an exact solution that contains event horizons, and we know that this solution can be produced from smooth, well-behaved matter by gravitational collapse. However, spherical symmetry is a highly unrealistic limit — we expect astrophysical objects to have angular momentum (Schwarzschild black holes are non-rotating), and to go through dynamical processes that are extremely non-symmetric. We cannot pretend that the perfectly symmetric collapse process which produces Schwarzschild black holes even approximates realistic astrophysical processes.

As we describe here and in the following section, these concerns are addressed by what is called the **Kerr black hole**. Discovered by Roy Kerr in 1963 \cite{4}, this metric describes a rotating black hole. It is symmetric about the rotation axis, but not spherically symmetric. Furthermore, it can be shown that it is the generic result of collapse to a black hole — as we shall discuss in the next section, once a black hole forms, it will shake and vibrate until its metric settles down to the Kerr solution.

Written in “Boyer-Lindquist” coordinates \cite{12}, the Kerr metric is

\[
ds^2 = - \left( 1 - \frac{2Mr}{\Sigma} \right) dt^2 - \frac{4aMr \sin^2 \theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left( r^2 + a^2 + \frac{2Mra^2 \sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\phi^2 ,
\]

where

\[
a = \frac{|\vec{S}|}{M} = \frac{G |\vec{S}|}{c M},
\]
\[
\Delta = r^2 - 2Mr + a^2 ,
\]
\[
\Sigma = r^2 + a^2 \cos^2 \theta .
\]

In these coordinates, it is clear that the metric reduces to Schwarzschild for \( a = 0 \). It is also clear that this solution is not spherical. If it were spherical, we could find angular coordinates \( \theta', \phi' \) which are independent of \( r \) and which satisfy \( g_{\theta'\phi'} = \sin^2 \theta' g_{\theta\phi} \). This cannot be done for Kerr except in the Schwarzschild limit, \( a = 0 \). The event horizon of the Kerr black hole is located at \( r_{\text{horiz}} = M + \sqrt{M^2 - a^2} \), the larger root of \( \Delta = 0 \). (The smaller root corresponds to a horizon-like surface inside the hole itself; this “inner horizon” plays no role in astrophysical black hole physics, so we won’t discuss it here.)

A particularly interesting feature of the Kerr spacetime is **frame dragging**: The fact that \( g_{\phi t} \propto a \) tells us that the black hole spin “connects” time and space via the hole’s rotation. Geodesics in this spacetime show that objects are dragged by the hole’s rotation — in essence, the gravitational field rotates, dragging everything nearby to co-rotate with it. Because of this, Kerr is not a static spacetime, like Schwarzschild; the adjective used to describe it is “stationary”. The spacetime does not vary with time, but frame dragging makes it impossible for static observers to exist near the black hole. In the vicinity of a Schwarzschild black hole, one can always arrange for an observer to sit in a static manner relative to distant observers. It may require an enormously powerful rocket to prevent that observer from falling through the horizon; but no issue of principle prevents one from using such a rocket. By contrast, close enough to the horizon of a Kerr black hole, all observers are forced to co-rotate relative to very distant observers, no matter how strongly they fight that tendency.

\[\text{Note that } g_{\phi\phi} = \sin^2 \theta g_{\theta\theta} + O(a^2)\text{; the metric is thus approximately spherically symmetric for slow rotation.}\]
Unfortunately, though perhaps not surprisingly, there is no simple collapse model akin to a rotating O-S collapse which demonstrates that Kerr black holes are produced by collapse. Fortunately, numerical calculations due in fact show this; see for example Ref. [11].

The Kerr black hole is clearly far more relevant to astrophysical discussions than the Schwarzschild solution — it rotates, and it is not spherically symmetric. One might wonder: Are we done? For example, are there additional black hole solutions with no symmetry? As we shall now discuss, the Kerr black hole is in fact the unique black hole solution within general relativity. Furthermore, it is stable: If something momentarily disturbs the hole so that the Kerr solution does not perfectly describe the spacetime, the hole will vibrate momentarily and then settle back down to the Kerr solution. Taken together, these statements of uniqueness and stability tell us that Kerr black holes are the ultimate outcome of gravitational collapse in general relativity.

V. UNIQUENESS AND STABILITY OF THE BLACK HOLE SOLUTION

The purpose of this section is to describe (without going into too much technical detail) why the Kerr solution provides the complete description of astrophysical black holes. The first part of this exercise is a description of the black hole uniqueness theorems. They establish that the Kerr black hole is the unique uncharged black hole solution within general relativity. The uniqueness theorems are known collectively as the "no-hair" theorem, meaning that a black hole has no distinguishing features (no hair), except for its mass and spin. The next part is a discussion of stability and Price’s theorem, which establish the mechanism by which Nature enforces the no-hair theorem.

Before going into this discussion, one might wonder: Should we consider charged black holes? For example, a non-spinning black hole with mass $M$ and charge $Q$ is described by the Reissner-Nordstrom metric [5–7],

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{dr^2}{1 - 2M/r + Q^2/r^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right);$$  \hspace{1cm} (51)

generalization to includes spin (the Kerr-Newman metric) is also known. Should we consider these black holes in our discussion?

At least on the rather long timescales relevant to measuring the properties of black hole candidates, the answer to this question is clearly no. Typical astrophysical black holes are embedded in environments that are rich in gas and plasma. Any net charge will be rapidly neutralized by the ambient plasma. The time scale for this to occur is very rapid, scaling roughly with the hole’s mass: $T_{\text{neut}} \sim 5 \mu\text{sec} \left(M/M_\odot\right)$. If charged black holes play any role in astrophysics, it will only be on very short timescales — far shorter than is relevant for observations which probe the nature of the black hole itself.

A. “Black holes have no hair”

The no-hair theorem tells us that (modulo the above discussion of charge), the only black hole solution is given by the Kerr metric. A corollary of this is that black holes are completely described by only two parameters, mass and spin. No matter how complicated the initial state, collapse to a black hole removes all “hair”, leaving a simple final state.

This is a remarkable statement. Suppose that we begin with two objects which are completely different — a massive star near the end of its life on one hand; a gigantic pile of elderly Winnebago campers on the other. We crush each object into a black hole such that each has the same mass $M$ and spin $|\vec{S}|$. The no-hair theorem tells us that there is no way to distinguish these objects. All information about their highly different initial state is lost by the time the event horizon forms. This remarkable feature of classical black holes is among their most bothersome; it is thus worth examining this aspect with some care.
1. Israel’s theorem

The first result which begins to establish the general no-hair result was a theorem that was proved by Werner Israel in a 1967 publication [13]. Stated formally, Israel’s theorem proves that

Any spacetime which contains an event horizon and is static must be spherically symmetric. By Birkhoff’s theorem it is therefore described by the Schwarzschild metric.

Hence, Schwarzschild black holes give the only static black hole solutions.

The proof of this theorem is rather far beyond the scope of these lectures; see [13] for details. A key element of the proof is noting that the event horizon is a “null surface” — it can be thought of as “generated” by null (radiative) geodesics. (This is demonstrated by Figs. 2 and 3, in which the event horizon clearly corresponds to a piece of a light cone.) The technical details of the proof demonstrate that this null surface must be spherically symmetric in static spacetimes if the spacetime is asymptotically flat (more precisely, that spatial slices are asymptotically Euclidean) and if the curvature is well-behaved over the spacetime (i.e., it is smooth and bounded, with no singularities).

2. Carter & Robinson theorem

Applying as it does only to static and spherically symmetric spacetimes, Israel’s theorem, though a landmark in understanding the generic properties of black hole spacetimes, is of somewhat limited applicability to astrophysics. Fortunately, this theorem pointed the way to its own generalization. Work by Carter [14] and Robinson [15] showed that Israel’s theorem could be generalized from the static case as follows:

The only stationary spacetimes which contain event horizons are described by the Kerr metric, with spin parameter \( a \leq M \).

The detailed proof of this statement is, again, far beyond the scope of these lectures; the interested reader is referred to Refs. [14, 15] and references therein for the details.

B. Enforcing “no hair”: Price’s theorem

It may seem intuitively clear that the Israel-Carter-Robinson theorems cannot possibly apply to, for example, newly born black holes that form during supernova explosions; such situations are so highly dynamical and asymmetric that there must be some “structure” (broadly speaking) that makes the spacetime more complicated than that of the Kerr metric. Fortunately, the no-hair theorem has a built in loophole: It requires the spacetime to be stationary.

In other words, the no-hair theorem requires that the spacetime itself be time independent. Any deviation from the pure black hole solution means that this newly born black hole cannot be stationary!

This loophole leads us to what has come to be known as Price’s theorem: Anything that can be radiated IS radiated. Deviations from the “perfect” stationary black hole solution drive the emission of gravitational radiation. Backreaction from the that radiation removes the deviation. After a short time, the Kerr solution remains. The statement “black holes have no hair” is more accurately given as “black holes rapidly go bald”.

The mechanism by which the “balding” proceeds is most easily explained for small deviations using black hole perturbation theory; our discussion here follows that of Rezzolla [16]. We begin by considering only perturbations to the Schwarzschild metric; generalization to Kerr is conceptually straightforward, though (as we outline briefly below) considerably more complicated calculationally. Take the spacetime to be that of a Schwarzschild black hole plus a small perturbation:

\[
g_{ab} = g_{ab}^{BH} + h_{ab}, \quad \|h_{ab}\| / \|g_{ab}^{BH}\| \ll 1. \tag{52}
\]
The notation $||A_{ab}||$ means “the typical magnitude of non-zero components of the tensor $A_{ab}$”. We now use this metric to generate the Einstein tensor; since the black hole is a vacuum solution, it satisfies $G_{ab} = 0$:

$$G_{ab} \left[ g_{ab}^{BH} + h_{ab} \right] \simeq G_{ab} \left[ g_{ab}^{BH} \right] + \delta G_{ab} \left[ h_{ab} \right]$$

$$= \delta G_{ab} \left[ h_{ab} \right].$$

The approximate equality on the first line follows from expanding the Einstein equations to first order in the perturbation. The equality on the second line follows from the fact that the background black hole metric is itself an exact vacuum solution. This final result tells us

$$\delta G_{ab} \left[ h_{ab} \right] = 0.$$  

This gives us a wave-like operator acting on the perturbation.

Since the Schwarzschild background is static and spherically symmetric, the metric perturbation can be expanded in Fourier modes and spherical harmonics. The “time-time” part of the perturbation metric acts as a scalar function, and so it is expanded using ordinary scalar spherical harmonics:

$$h_{00} = \sum_{lm} H_{0}^{lm}(t, r) Y_{lm}(\theta, \phi).$$

The “time-space” piece $h_{0i}$ can be regarded as a spatial vector, and is thus expanded using vector harmonics:

$$h_{0i} = \sum_{lm} \left\{ H_{1}^{lm}(t, r) \left[ Y_{lm}^{E}(\theta, \phi) \right]_{i} + H_{2}^{lm}(t, r) \left[ Y_{lm}^{B}(\theta, \phi) \right]_{i} \right\}.$$  

The functions $[Y_{lm}^{E,B}(\theta, \phi)]_{i}$ are electric- and magnetic-type vector spherical harmonics. These functions have opposite parity: The electric harmonics are even parity, so $[Y_{lm}^{E}]_{i} \rightarrow (-1)^[l](Y_{lm}^{E})_{i}$, for $\theta \rightarrow \pi - \theta, \phi \rightarrow \phi + \pi$ (just like the usual scalar spherical harmonic), while the odd parity magnetic harmonics obey $[Y_{lm}^{B}]_{i} \rightarrow (-1)^{l+1}(Y_{lm}^{B})_{i}$. The components of the vector harmonics are given by ordinary spherical harmonics and their derivatives; see [16] for discussion tailored to this application.

Finally, the “space-space” components of the metric perturbation constitute a spatial tensor and are thus expanded in tensor harmonics:

$$h_{ij} = \sum_{lm} \left\{ H_{3}^{lm}(t, r) \left[ Y_{lm}^{pol}(\theta, \phi) \right]_{ij} + H_{4}(t, r) \left[ Y_{lm}^{nx}(\theta, \phi) \right]_{ij} \right\}.$$  

The “polar” modes are even parity; “axial” modes are odd. The angular functions again are given in terms of ordinary spherical harmonics and their derivatives.

The metric perturbation is described in total by solving mode by mode, for a particular choice of parity. In each case, the gauge freedom of general relativity allows us to set several of the functions $H_{0} - H_{4}$ to zero; running the remaining non-zero functions through the wave-like equation $\delta G_{ab} = 0$ yields an equation of the form

$$\frac{\partial^2 Q}{\partial t^2} = \frac{\partial^2 Q}{\partial r^2} + V(r)Q = 0.$$  

The coordinate $r_* = r + 2M \ln(r/2M - 1)$; it asymptotes to the normal Schwarzschild $r$ coordinate at large radius, but places the event horizon in the limit $r_* \rightarrow -\infty$. This facilitates setting boundary conditions on the black hole. The function $Q$ is simply related to the non-zero $H$ functions; the potential $V$ depends on the parity of the mode. At this point, we commonly insert our Fourier decomposition so that the $H$ and $Q$ functions all acquire a time dependence $\propto e^{-i\omega t}$.

The results for odd parity are particularly simple. The coefficients of the even parity harmonics are set to zero:

$$H_{0} = H_{1} = H_{3} = 0.$$
The function \(Q\) relates to the remaining functions \(H_2\) and \(H_4\) via
\[
Q = \frac{H_4}{r} \left(1 - \frac{2M}{r}\right),
\]
(61)

\[
-i\omega H_2 = \frac{\partial}{\partial r^*} (r^* Q).
\]
(62)

The potential we find is
\[
V(r) = \left(1 - \frac{2M}{r}\right) \left(\frac{l(l + 1)}{r^2} - \frac{6M}{r^3}\right).
\]
(63)

The equation governing odd-parity modes using the potential (63) is known as the Regge-Wheeler equation [17]. A similar result, the Zerilli equation, governs even-parity modes [18].

To finally solve for these modes we must impose boundary conditions. Since nothing can come out of the black hole, we require that the mode be purely ingoing at the event horizon:
\[
Q \propto e^{-\imath \omega (t + r^*)} \quad r \to 2M.
\]
(64)

For similar reasons, we require that the mode be purely outgoing at large radius:
\[
Q \propto e^{-\imath \omega (t - r^*)} \quad r \to \infty.
\]
(65)

Imposing these conditions, it is not too difficult to solve for \(Q\) and thus construct the metric perturbation. The result of this exercise is that all modes exponentially decay, taking the form of damped sinusoids. The longest lived mode has frequency and decay time
\[
\omega \simeq \frac{0.37}{M} \simeq 7.5 \times 10^4 \text{sec}^{-1} (M_\odot/M),
\]
(66)

\[
\tau \simeq 1.72 M \simeq 8.5 \mu\text{sec} (M/M_\odot).
\]
(67)

This tells us that a distorted black hole settles down to the “perfect” black hole solution extremely rapidly! This is the mechanism by which the no-hair theorem is enforced: Any multipolar distortion to the Schwarzschild metric is quickly shaken off. On a timescale of microseconds for stellar mass black holes, and seconds to hours for the most massive black holes, the black hole relaxes to the exact Schwarzschild solution. It is to be emphasized that these are extremely short astrophysical timescales — any system that we observe for a long period of time will certainly have relaxed into this state.

Generalization of this argument to Kerr black holes is nontrivial, but doable. Largely due to the lack of spherical symmetry, expanding the perturbed Einstein equations to find an equation governing the metric perturbation does not yield separable equations. It turns out, though, that by developing a perturbation formalism based on expanding curvature tensors rather than the metric one can develop separable equations that are not too difficult to solve [19]. Doing so, imposing similar boundary conditions, and solving the resultant equations, one finds exactly the same qualitative behavior. The influence of spin causes some quantitative differences (modes typically decay more slowly), but the general form of a decaying sinusoid holds independent of spin. Objects with event horizons rapidly settle down to the Kerr metric, radiating away all distortions that push them from the “perfect” black hole solution.

One might worry that, since these conclusions are based on perturbative analyses, they are not relevant for “large” distortions such as might be expected for newly born black holes in supernovae, or the end result of black hole collisions. Such worries are unfounded: Detailed numerical analysis (by direct integration of the Einstein field equations) confirms that this qualitative behavior holds even for massive distortions from the quiescent black hole solution [20]. One always finds that the distorted black hole oscillates massive and violently, generating a large burst of gravitational waves; but, those waves carry away the high-multipole spacetime distortion. A black hole described by the Kerr metric is always what remains after the oscillations have damped out.

Any black hole that forms in the universe should rapidly decay into a form that is exactly described by the Kerr metric. Quiescent black holes should be perfectly described by an exact solution of general relativity!
VI. TESTING THE BLACK HOLE HYPOTHESIS

The discussion of the previous several sections establishes why, on theoretical grounds, the black hole solutions of general relativity are considered to be relevant to astrophysics. Astrophysical observations have now established many large, compact objects that are obvious candidate black holes. The big question becomes: Are these black hole candidates in fact described by general relativity’s black hole solutions? Several lines of investigation are working to answer — or at least constrain — this question.

A. Mass and compactness

Many black hole candidates are observed as companions to normal stars in binary systems. If the companion is clearly compact and dark — observations establish that it is far smaller than a typical star (several to several tens of kilometers), and that it is not luminous — then it is a good candidate to be a black hole. However, it could be a neutron star — a fascinating and important object, but not the subject of these lectures.

The first check to distinguish these two possibilities is to “weigh” the dark companion. By studying orbits of binaries and applying Kepler’s laws, one can infer the mass of the dark companion (or at least constrain it to some range). If the object turns out to be greater than about 3 solar masses, our current wisdom is that the object is very likely to be a black hole — to the best of our knowledge, cold “normal” matter cannot hold itself up against gravity at masses \( \gtrsim 3 M_\odot \) or so.\(^7\) Observations establish that the large dark object at the core of the Milky Way has a mass of about \( 4 \times 10^6 M_\odot \)\(^ \[22\] \), and have established that about 20 candidate black holes have masses around \( 5 - 20 M_\odot \)\(^ \[23\] \).

This argument relies on the fact that “normal” matter cannot hold itself up against gravity at these high masses. One can, however, invoke “non-normal” matter — condensations of scalar fields or fermion balls, or ... This is in fact is what many of the posited alternatives to the black hole hypothesis do. To definitively check the black hole hypothesis against these alternatives requires deeper tests.

B. Event horizons

The defining property of black holes is their event horizon. Rather than a true surface, black holes have a “one-way membrane” through which stuff can go in but cannot come out. At least two tests seek to test the black hole nature of black hole candidates by observationally probing the existence of event horizons.

1. X-ray bursts

One test involves observations of x-ray binaries. Such binaries generically consist of a compact object in orbit with a “normal” star. Gas flows from the star onto the compact object. The gas is heated during this accretion process to the point that it generates x-rays.

In some of these sources, x-ray bursts are observed in addition to the quiescent x-ray luminosity. Bursts are now understood to arise from thermonuclear detonation of accreted material\(^ \[24,25\] \): Material accreted from the star builds up on the surface of the compact object and is compressed by the object’s gravity. After sufficient material accumulates, it undergoes unstable thermonuclear burning, which we observe as an x-ray burst.

A key element of this model is that the compact object must have a surface. Material cannot accumulate on an event horizon, and so no bursts can come from an x-ray binary whose compact object is a black hole.

\(^7\) The limit on the maximum mass of a neutron star is not terribly well understood, as it depends on the behavior of nuclear matter at extremely high density. Our current best understanding of the nuclear equation of state at the relevant densities suggests that the maximum allowed neutron star mass is roughly \( 2 - 3 M_\odot \).\(^ \[21\] \).
Observations show that bursts are only seen if the compact object’s mass is sufficiently low that it is probably a neutron star \[26\]. Bursts are never seen from sources whose mass are high enough that they are black hole candidates. This is totally consistent with the idea that black holes lack hard surfaces on which material can accumulate. See \[26\] for additional discussion of this idea and further details.

2. Imaging a black hole’s shadow

The black hole candidate which covers the largest fraction of our sky is the one which resides in the center of our galaxy. This object has a mass of about $4 \times 10^6 M_\odot$ \[22\]; if it is indeed a black hole, this corresponds to a horizon radius of about 6 – 12 million km. The galactic center is about 7000 parsecs away, so the horizon subtends an angle of about 0.01 milliarcseconds on the sky. This is not too far from what might be achievable in the near future using very long baseline radio interferometry. Indeed, recent observations have shown that this source must have a size smaller than about 1 astronomical unit \[27\]; the precision required for this measurement is within an order of magnitude or so of what is needed to probe the region near the event horizon.

What we really want to measure is the horizon’s shadow: the “hole” cast in background radiation due to the presence of the event horizon. Because of the bending and focusing of radiation in the strong gravitational field of a black hole, the shadow will be somewhat larger than the event horizon. For a Schwarzschild black hole, the shadow would be a perfect circle $3\sqrt{3}/2 \approx 2.6$ times larger than the event horizon. For rotating black holes (the generic case which we expect), the shadow is expected to be somewhat asymmetric due to frame dragging — the bending of light is not symmetric. Getting the resolution required to do such imaging will require interferometry on the scale of the Earth’s size, observing at sub-mm wavelengths; such a measurement is plausible in the near future. This idea is discussed in detail in Ref. \[28\].

C. Orbits

Finally, there is great potential in probing the spacetimes of black hole candidates by observing orbits about them (much as we examined “thought orbits” to probe the nature of the event horizon in Sec. II). If orbits can be measured precisely, then we can measure informative characteristics like the timescales associated with different motions. These can be be connected to a model of the spacetime, from which the spacetime’s character can be inferred. Such measurements have already begun by studying orbits around the black hole at the core of the Milky Way \[22\]. Those stars are in orbits which, strictly speaking, a relativist would consider weak field\[8\] but they are constrained well enough that they make it possible to determine the putative black hole’s mass with good accuracy.

In general, motion around a black hole is described by integrating the geodesic equation for the black hole’s spacetime. The orbits admit a notion of conserved energy $E$, of conserved “axial” angular momentum $L_z$, as well as a third integral $Q$ which is essentially the “rest” of the orbit’s angular momentum: $Q = |\vec{L}|^2 - L^2_z$ for non-spinning black holes; for spinning black holes, the interpretation is more complicated, though this intuitive picture of $Q$ is useful. Using these conserved constants, the geodesic equations can be rewritten in terms of oscillations in the $r$ and $\theta$ coordinates, while the orbits “whirls” with respect to $\phi$:

\[
\Sigma^2 \left( \frac{dr}{d\tau} \right)^2 = \left[ E(r^2 + a^2) - aL_z \right]^2 - \Delta \left[ r^2 + (L_z - aE)^2 + Q \right] \equiv R(r), \tag{68}
\]

\[
\Sigma^2 \left( \frac{d\theta}{d\tau} \right)^2 = Q - \cot^2 \theta L^2_z - a^2 \cos^2 \theta (1 - E^2) \equiv \Theta(\theta), \tag{69}
\]

\[
\Sigma \left( \frac{d\phi}{d\tau} \right) = \csc^2 \theta L_z + aE \left( \frac{r^2 + a^2}{\Delta} - 1 \right) - \frac{a^2 L^2_z}{\Delta} \equiv \Phi(r, \theta), \tag{70}
\]

\* It should be noted that one star moves at about 5000 km/sec through the periapsis of its orbit. Though formally weak field, these orbits are still rather extreme!
One feature of these orbits is the existence of “innermost stable orbits”. For example, careful analysis of the radial potential \( R(r) \) for \( a = 0 \) shows that no stable circular orbits can exist at \( r < 6M \) — small perturbations drive such orbits to plunge almost immediately into the hole’s event horizon. Another feature is that the familiar Keplerian orbital period, \( T = 2\pi p a^3/M \) (where \( a \) is orbital semi-major axis) is split into 3 distinct periods, corresponding to motions in \( r \), \( \theta \), and \( \phi \). These periods are shown in Fig. 5; notice that they asymptote to the Kepler result far from the black hole, but are strongly split in the strong field. The distinct character of these orbital timescales is a fingerprint of the black hole strong field.

At least two types of measurements have the promise to probe orbits in the strong field and to observe these distinct behaviors: x-ray measurements of accretion (the flow of gas onto the black hole in these potentials), and gravitational waves (arising from the motion of bodies orbiting in these potentials). A particularly interesting strong-field hallmark visible in some accreting systems is a fluorescence line associated with iron, the Fe K\( \alpha \) line. In the rest frame, this line is expected to be a sharp feature at an energy of 6.4 keV. The observed lines are highly broadened from 4 to 7 keV — both blueshifted and redshifted due to a combination of the gravitational redshift and the rapid whirling motion of gas near the black hole. With some assumptions about the nature and structure of the gaseous disk from the radiation originates, measurement of such lines makes it possible to estimate the spin of massive black holes \cite{29}; clean, high signal-to-noise data may make it possible to constrain the spacetime even more sharply, perhaps enabling a careful test of the black hole hypothesis.

Future gravitational wave observations likewise promise to enable careful studies of black hole spacetimes. Gravitational waves are generated by a source with a varying mass quadrupole moment; binary systems are a perfect example of a strong gravitational radiator. It is expected that galactic black holes will occasionally capture a smaller compact

\[ \Sigma \left( \frac{dt}{d\tau} \right) = E \left[ \frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta \right] + aL_z \left( 1 - \frac{r^2 + a^2}{\Delta} \right) \equiv T(r, \theta) ; \]  

(71)

\[ \Sigma = r^2 + a^2 \cos^2 \theta \quad \Delta = r^2 - 2Mr + a^2 . \]  

(72)
object ($\sim 10 - 100 M_\odot$ black hole or $\sim 1 M_\odot$ neutron star) \cite{30}. That compact object will generate gravitational waves with spectral content at all of the orbital frequencies, $\Omega_r$, $\Omega_\theta$, and $\Omega_\phi$, as well as harmonics of those frequencies. Indeed, those frequencies will slowly evolve as the small body loses orbital energy and angular momentum due to the waves’ backreaction; by tracking the evolution, a great deal of information can in principle be extracted. Future space-based gravitational-wave missions (see Larson’s contribution to this summer school) are expected to measure these waves well enough to precisely determine the masses and spins of black holes \cite{31}, and should be able to check whether the spacetime satisfies the stringent requirements of the black hole hypothesis \cite{32}.

VII. CONCLUSIONS

Unambiguous observational evidence for the existence of astrophysical black holes has not yet been established. The moral of classical general relativity is that black holes must be created in the right circumstances; and, they must be driven to a particularly simple, unique form. Chandraskehar \cite{33} provides a typically clear summary:

It is well known that the Kerr solution with two parameters provides the unique solution for stationary black holes that can occur in the astronomical universe. But a confirmation of the metric of the Kerr spacetime (or some aspect of it) cannot even be contemplated in the foreseeable future.

(Emphasis added.) Chandra's final sentence makes sense in the context of 1986, when this statement was made. Technology has advanced considerably since then — as observations continue to improve, the amount of wiggle room in theoretical phase space for alternatives to the black hole hypothesis grows progressively smaller and smaller. We already are gathering evidence allowing confirmation of aspects of black hole spacetimes; it is likely that definitive, unambiguous evidence showing — or disproving! — that the many black hole candidates we see today are general relativity’s black holes is now not too far in the future.

ACKNOWLEDGMENTS

I wish to thank the organizations of the 2005 SLAC Summer Institute for giving me the opportunity to assemble these lectures, and for their patience as I wore out deadlines completing this final writeup. My work on black hole physics is supported by NSF Grants PHY-0244424 and CAREER Grant PHY-0449884, and by NASA Grants NAG5-12906 and NNG05G105G. I also very gratefully acknowledge the support of MIT’s Class of 1956 Career Development Fund.

[1] I took this quote from K. S. Thorne, *Black Holes and Time Warps: Einstein’s Outrageous Legacy* (W. W. Norton & Company, 1994), p. 160; Thorne in turn took the quote from A. S. Eddington, Observatory 58, 37 (1935).
[2] See, for example, D. N. Page, New J. Phys. 7, 203 (2005) for a review.
[3] J. R. Oppenheimer and H. Snyder, Phys. Rev. 56, 455 (1939).
[4] R. P. Kerr, Phys. Rev. Lett. 11, 237 (1963).
[5] S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity* (Addison-Wesley, 2004).
[6] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman and Company, 1973).
[7] E. Poisson, *A Relativist’s Toolkit: The Mathematics of Black-Hole Mechanics* (Cambridge University Press, 2004).
[8] K. Schwarzschild, Sitzungsberichte der Deutschen Akademie der Wissenschaften zu Berlin, Klasse für Mathematik, Physik, und Technik, 189 (1916).
[9] J. Mitchell, Phil. Trans. R. Soc. Lond. 74, 35 (1796).
[10] M. W. May and R. H. White, Phys. Rev. 141, 1232 (1966).
[11] T. Nakamura, Prog. Theor. Phys. 65, 1876 (1981).
[12] R. H. Boyer and R. W. Lindquist, J. Math. Phys. 8, 265 (1967).
[13] W. Israel, Phys. Rev. 164, 1776 (1967).
[14] B. Carter, Phys. Rev. Lett. 26, 331 (1971).
[15] D. C. Robinson, Phys. Rev. Lett. 34, 905 (1975).
[16] L. Rezzolla, “Gravitational waves from perturbed black holes and relativistic stars”, gr-qc/0302025 also A. Nagar and L. Rezzolla, Class. Quantum Grav. 22, R167 (2005).
[17] T. Regge and J. A. Wheeler, Phys. Rev. 108, 1063 (1957).
[18] F. Zerilli, Phys. Rev. Lett. 24, 737 (1970).
[19] S. A. Teukolsky, Astrophys. J. 185, 635 (1973).
[20] Some particularly spectacular examples of highly distorted black holes are those created in the collision of two black holes. See, for example, M. Alcubierre, W. Benger, B. Brügmann, G. Lanfermann, L. Nerger, E. Seidel, and R. Takahashi, Phys. Rev. Lett. 87, 271103 (2001).
[21] V. Kalogera and G. Baym, Astrophys. J. 470, L61 (1996).
[22] R. Schödel et al., Nature 419, 694 (2002); A. M. Ghez et al., Astrophys. J. 586, L127 (2003); A. M. Ghez, S. Salim, S. D. Hornstein, A. Tanner, J. R. Lu, M. Morris, E. E. Becklin, and G. Duchêne, Astrophys. J. 62, 744 (2005).
[23] J. E. McClintock and R. A. Remillard, in Compact Stellar X-ray Sources, eds. W. H. G. Lewin and M. van der Klis; astro-ph/0306213.
[24] L. Bildsten, in The Many Faces of Neutron Stars, eds. A. Alpar, L. Buccheri, J. van Paradijs (Kluwer, 419).
[25] R. Narayan and J. S. Heyl, Astrophys. J. 599, 419 (2003).
[26] R. Narayan, New J. Phys. 7, 199 (2005); R. A. Remillard, D. Lin, R. L. Cooper, and R. Narayan, astroph-ph/0509758.
[27] Z.-Q. Shen, K. Y. Lo, M.-C. Liang, P. T. P. Ho, and J.-H. Zhao, Nature 483, 62 (2005).
[28] H. Falcke, F. Melia, and E. Agol, Astrophys. J. 528, L13 (2000).
[29] C. S. Reynolds and M. A. Nowak, Phys. Rep. 377, 389 (2003).
[30] J. Gair et al., Class. Quantum Grav. 21, S1595 (2004).
[31] L. Barack and C. Cutler, Phys. Rev. D 69, 082005 (2004).
[32] F. D. Ryan, Phys. Rev. D 52, 5707 (1995); F. D. Ryan, Phys. Rev. D 56, 1845 (1997); N. A. Collins and S. A. Hughes, Phys. Rev. D 69, 124002; K. Glampedakis and S. Babak, gr-qc/0510057.
[33] S. Chandrasekhar, quoted from the Karl Schwarzschild Lecture of 18 Sept 1986; taken from Truth and Beauty: Aesthetics and Motivation in Science (Chicago University Press, 1987).