THE MILLENNIUM ARECIBO 21 CENTIMETER ABSORPTION-LINE SURVEY. IV. STATISTICS OF MAGNETIC FIELD, COLUMN DENSITY, AND TURBULENCE

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ABSTRACT

We discuss observations of the magnetic field, column density, and turbulence in the cold neutral medium (CNM). The observed quantities are only indirectly related to the intrinsic astronomical ones. We relate the observed and intrinsic quantities by relating their univariate and bivariate probability distribution functions (pdf’s). We find that observations of the line-of-sight component of a magnetic field do not constrain the pdf of the total field \( B_{\text{tot}} \) very well but do constrain the median value of \( B_{\text{los}} \). In the CNM, we find a well-defined median magnetic field \( 6.0 \pm 1.8 \) \( \mu \)G. The CNM magnetic field dominates thermal motions. Turbulence and magnetism are in approximate equipartition. We find that the probability distribution of column density \( N_\perp (\text{H}) \) in the sheets closely follows \( N_\perp (\text{H})^{-1} \) over a range of 2 orders of magnitude, \( 0.026 \leq N_\perp (\text{H}) \leq 2.6 \times 10^{20} \) \( \text{cm}^{-2} \). The bivariate distributions are not well enough determined to constrain structural models of CNM sheets.

Subject headings: ISM: magnetic fields — ISM: structure — radio lines: ISM — turbulence

1. INTRODUCTION

Beginning in 1999 February we used the Arecibo telescope\(^1\) to begin a series of Zeeman-splitting measurements of the 21 cm line in absorption against continuum radio sources. Heiles & Troland (2004, hereafter Paper III, and references therein) discussed technical details of the observing technique and data reduction and presented observational results, including magnetic field measurements. As described in Paper III, our analysis of the data identified Gaussian components in the \( \text{H}_i \) absorption spectra. We assume that each component samples a single, isothermal, sheetlike region of the cold neutral medium (CNM). For 69 such components, we are able to estimate (1) the line-of-sight magnetic field strength \( B_{\text{los}} \), often subject to significant error, (2) the line-of-sight column density \( N_\perp (\text{H})_{\text{los}} \), and (3) the contribution of turbulence \( V_{\text{turb, los}} \) to the line-of-sight velocity dispersion. Systematic instrumental errors are small, and the uncertainties are Gaussian distributed. Therefore, our survey has yielded a statistically well-defined ensemble of observed values for each of these three CNM quantities.

From these data, we seek to answer several astrophysically significant questions. For example, what are the probability distribution functions (pdf’s) of magnetic field strengths, column densities, and turbulent energies in the CNM? Are any of these quantities statistically related to each other? If so, can we determine whether the magnetic field lies preferentially in the planes of CNM sheets or perpendicular to them? What is the ratio of magnetic to turbulent energy in the CNM? As we will see below, the data provide answers to some, but not all, of these questions.

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We observe quantities that are only indirectly related to the intrinsic astronomical ones. For the magnetic field, we observe the line-of-sight component \( B_{\text{los}} \), not the total field \( B_{\text{tot}} \). For the column density \( N (\text{H}) \), we must account for the fact that the CNM is in sheets, not spheres; our observed column densities \( N_{\text{los}} \) are always larger than the intrinsic column density normal to the sheet \( N_\perp \). To properly interpret our results and to address the questions listed above, we must consider how the intrinsic and observed quantities are related in a statistical sample. These statistical transformations, and the results of applying them to our data, are the focus of the current paper.

We treat both univariate distributions and bivariate distributions. The univariate distributions are more interesting because we obtain definitive results; in contrast, the possible correlations that could be revealed by bivariate distributions are obscured by noise and inadequate numbers of data. Accordingly, the reader who is interested in astrophysical results can concentrate on the sections dealing with univariate distributions of magnetic field, column density, and turbulent velocity. We recommend beginning with § 2, which introduces the notation and concepts of intrinsic and observed quantities. We develop the theoretical relationships between observed and intrinsic univariate distributions in § 3 and use these to obtain the actual intrinsic univariate distributions in § 3.1. Finally, we discuss astrophysical implications in §§ 7, 8, 9, and 10.

2. SOME IMPORTANT WORDS ON NOTATION

We need to introduce important distinguishing nomenclature because of the anisotropic nature of magnetic field related quantities. First, we make the crucial distinction between line-of-sight (los) quantities, such as \( B_{\text{los}} \), and the intrinsic astronomical ones such as \( B_{\text{tot}} \). We often refer to the latter with one of the shorter terms “intrinsic” or “astronomical.” The important point is to distinguish between observed and intrinsic quantities.
The line-of-sight component is the quantity to which the telescope responds, and we designate these with the subscript $\text{los}$. There is an additional complication for the magnetic field, which is the presence of significant instrumental noise. Thus, we must distinguish between the line-of-sight component of the field, designated by $B_{\text{los}}$, and the actual measured value, designated by $B_{\text{obs}}$. We define

$$B_{\text{los}} \equiv \text{the line-of-sight component of } B_{\text{tot}},$$ (1a)$$
$$B_{\text{obs}} \equiv \text{the observed field strength.}$$ (1b)

The essential difference between $B_{\text{los}}$ and $B_{\text{obs}}$ is

$$B_{\text{obs}} = B_{\text{los}} + \delta B_{\text{noise}},$$ (1c)

where $\delta B_{\text{noise}}$ is the uncertainty contributed by random measurement noise.

Noise is small enough to neglect for column density and velocity, so the observed quantities are essentially identical to the los ones; we use the subscript los for the observed quantities, with the implicit assumption that their uncertainties from noise are negligible. For column density, we always refer to $N$ at H I, so we often write $N$ instead of $N(H\text{ I})$. We usually consider sheets, for which the column density perpendicular to the face is $N(H\text{ I})_{\perp}$, and for which the apparent column density along the line of sight is

$$N_{\text{los}} = N(H\text{ I})_{\text{los}} = N(H\text{ I})_{\text{los}} \cos \theta,$$ (2)

where $\theta$ is the angle between the sheet’s normal vector and the line of sight. Similarly, for velocity line widths, we use the symbol $V_{\text{turb,los}}$ to indicate the line-of-sight component of the turbulent velocity.

Finally, the pdf’s of intrinsic quantities, such as $B_{\text{tot}}$, differ from those of the line-of-sight quantities. We always use the symbol $\phi$ for the pdf’s of intrinsic quantities and the symbol $\psi$ for line-of-sight (los) or the observed ones. Unless otherwise specified, units are always as follows: magnetic field $B$ in $\mu$G, column density $N$ in H I atoms $10^{20}$ cm$^{-2}$, and velocity $V$ in km s$^{-1}$.

3. THEORETICAL CONVERSIONS OF INTRINSIC ASTRONOMICAL PROBABILITY DENSITY FUNCTIONS TO OBSERVED ONES

We begin with some light theory by considering elementary transformations of magnetic field and column density in statistical distributions. First we consider how the intrinsic pdf of $B_{\text{tot}}$, defined as $\phi(B_{\text{tot}})$, converts to the observed histogram or pdf of observed $B_{\text{los}}$, defined as $\psi(B_{\text{los}})$, under the assumption that fields are randomly oriented with respect to the observer. Note this important distinction: $B_{\text{tot}}$ is the total field strength; $B_{\text{los}}$ is the observed line-of-sight component, which is always smaller.

Next we incorporate one of the fundamental results of Heiles & Troland (2003, hereafter Paper II), namely, that the CNM components are thin sheets. We define $N_\perp = N(H\text{ I})_{\perp}$ as the H I column density perpendicular to the sheet; the observed quantity is $N_{\text{los}} = N(H\text{ I})_{\text{los}}$, which is always larger. As with the magnetic field, we consider how the pdf $\phi(N_\perp)$ converts to the observed histogram or pdf $\psi(N_{\text{los}})$, again under the assumption that the sheets are randomly oriented with respect to the observer. We also consider the statistical transformation of the distribution of the actual nonthermal velocity dispersion $V_{\text{turb}}$ to that of the observed one $V_{\text{turb,los}}$ under the assumption that turbulence is perpendicular to the magnetic field.

We then tackle the bivariate distributions. First we consider how the bivariate distribution $\phi(B_{\text{tot}}, N_\perp)$ converts to the observed one $\psi(B_{\text{los}}, N_{\text{los}})$. We then assume two extreme models, one with the fields always perpendicular to the sheets and one with fields parallel to the sheets, and assuming random orientations derive the observed bivariate distributions $\psi(B_{\text{los}}, N_{\text{los}})$ for the two cases. We illustrate and discuss the result by considering delta-function distributions of $B_{\text{tot}}$ and $N_\perp$, and we also apply the transformation of the observed $\psi(B_{\text{los}}, N_{\text{los}})$ to its intrinsic counterpart $\phi(B_{\text{tot}}, N_\perp)$. Finally, we examine the bivariate distributions involving the pairs $(V_{\text{turb,los}}, N_{\text{los}})$ and $(V_{\text{turb,los}}, B_{\text{los}})$, which produces little in the way of useful results.

3.1. Conversion of the Univariate Distributions

3.1.1. Conversion of the Intrinsic $\phi(B_{\text{tot}})$ to the Observed $\psi(B_{\text{los}})$

We first consider the simple case in which all clouds have the same $B_{\text{tot}}$, which is randomly oriented with respect to the observer. The line-of-sight component $B_{\text{los}}$ is

$$B_{\text{los}} = B_{\text{tot}} \cos \theta,$$ (3)

where $\theta$ is the angle between the field direction and the line of sight; $\theta$ can run from 0 to $\pi$, but the intervals from 0 to $\pi/2$ and from $\pi/2$ to $\pi$ are identical except for a change of sign in $B_{\text{los}}$. It is simpler and no less general to consider the smaller interval $\theta$ from 0 to $\pi/2$ so that we can ignore the slight complications of the sign change. In this case, the pdf of $\theta$ is the familiar

$$\phi_{\theta}(\theta) = \sin \theta,$$ (4)

and we wish to know the pdf of $B_{\text{los}}$, which is given by (see Trumpler & Weaver [1953] for a discussion of these conversions)

$$\psi(B_{\text{los}}) = \int \phi_{\theta}(\theta) \left[ \frac{d\phi(B_{\text{los}})}{dB_{\text{los}}} \right] d\theta,$$ (5)

gives which provides

$$\psi(B_{\text{los}}) = \begin{cases} 
1, & 0 \leq B_{\text{los}} \leq B_{\text{tot}}, \\
0, & \text{otherwise}.
\end{cases}$$ (6)

In other words, $B_{\text{los}}$ is uniformly distributed between the maximum possible extremes 0 and $B_{\text{tot}}$ (actually $\pm B_{\text{tot}}$). This leads to the well-known results that in a large statistical sample for which a constant $B_{\text{tot}}$ is viewed at random angles, both the median and the mean observed field strengths are half the total field strength and $B_{\text{los}}^2 = (B_{\text{tot}}^2)/3$. More generally, observed fields are always smaller than the actual total fields, and with significant probability they range all the way down to zero.

Now suppose $B_{\text{tot}}$ has an arbitrary pdf $\phi(B_{\text{tot}})$. The univariate pdf $\phi(\theta)$ becomes the bivariate pdf $\phi(B_{\text{tot}}, \theta)$, and we assume $B_{\text{tot}}$ is independent of the observer’s location so that $\phi(B_{\text{tot}}, \theta) = \phi_{B_{\text{tot}}}(B_{\text{tot}})\phi_{\theta}(\theta)$. Note that we introduce subscripts on the different $\phi$’s to distinguish them, instead of designating them with different Greek letters. To obtain $\psi(B_{\text{los}})$ we again follow the standard techniques; it is easy to integrate over $\theta$ and obtain

$$\psi(B_{\text{los}}) = \int_{|B_{\text{los}}| > B_{\text{los, min}}} \left( \frac{\phi(B_{\text{tot}})}{B_{\text{tot}}} \right) dB_{\text{tot}},$$ (7)

where the symbol $[B_{\text{los}} > B_{\text{tot}, \min}]$ means the larger of the two quantities. The presence of $B_{\text{tot}}$ in the denominator means that smaller ranges of $B_{\text{los}}$ are emphasized. This is commensurate with equation (6)'s uniform pdf for a single field value. We note that this can be regarded as an integral equation for $\phi(B_{\text{los}})$, and it is straightforward to invert.

Figure 1 illustrates the solution of equation (7) for four functional forms of $\phi(B_{\text{los}})$ plotted against $|B| / |B_{1/2}|$, where the subscript $1/2$ denotes the median value. These forms include the following:

1. a Kronecker delta function (DELTA FCN), $\phi_{\text{los}}(B_{\text{tot}}) = \delta(B_{\text{tot}} - B_{\text{tot}, \min})$, yielding a flat function $\psi$ (as discussed immediately above; see eq. [6]);
2. a flat distribution (FLAT FCN) between $0 \leq |B_{\text{tot}}| \leq B_0$, yielding $\psi \propto \ln (B_0 / B_{\text{los}})$;
3. a weighted Gaussian (EXP FCN),

$$
\phi(B_{\text{tot}}) = \frac{1}{\sqrt{2\pi B_0^2}} e^{-\left(\frac{B_{\text{los}}}{2B_0}\right)^2},
$$

yielding $\psi$, a Gaussian with dispersion $B_0$;
4. a Gaussian (GAUSS FCN) with dispersion $B_0$, yielding $\psi \propto E_1 \left( B_{\text{los}}^2 / 2B_0^2 \right)$, where $E_1$ is the exponential integral of order 1.

All four $\phi(B_{\text{los}})$ are plotted with respect to $B_{\text{tot}}/B_{\text{tot},1/2}$, so the medians of all lie at unity on the x-axis. However, the means differ. Similarly, the medians and means of the associated $\psi(B_{\text{los}})$ differ from each other. These relationships between median and mean are summarized in Table 1. The medians and means for $\psi(B_{\text{los}})$ are all about half those for $\langle B_{\text{tot}} \rangle$, which is a direct result of the weighting by $B_{\text{tot}}^{-1}$ in equation (7).

Figure 1 is disappointing from the observer’s standpoint, because the observed distributions $\psi(B_{\text{los}})$ do not differ very much. These differences become smaller (inconsequential, in fact) when one includes measurement noise, as we discuss in detail for our data in $\S$ 4.1. Unfortunately, given the inevitable errors in any observation that is sensitive to $B_{\text{los}}$, it seems practically impossible to distinguish among different functional forms for $\phi(B_{\text{tot}})$. Nevertheless, the average value of $B_{\text{los}}$ is close to half the average value of $B_{\text{tot}}$ for a wide range of intrinsic pdf’s of the latter; this also applies to the medians, but less accurately. Therefore, this rule of thumb may be used to estimate the median or average $B_{\text{tot}}$ from an ensemble of measurements of $B_{\text{los}}$.

### 3.1.2. Conversion of the Intrinsic $\psi(N_{\perp})$ to the Observed $\psi(N_{\text{los}})$ for Sheets

Here we assume the CNM is distributed in sheets having H I column density $N_{\perp}$ in the direction perpendicular to the sheet. If the normal vector to the sheet is oriented at angle $\theta$ with respect to the line of sight, we have

$$
N_{\text{los}} = \frac{N_{\perp}}{\cos \theta}.
$$

To find $\psi(N_{\text{los}})$ we follow the same procedures as in $\S$ 3.1.1. For a single value of $N_{\perp}$ we obtain

$$
\psi(N_{\text{los}}) = \begin{cases} 
\frac{N_{\perp}}{N_{\text{los}}} & N_{\text{los}} \geq N_{\perp}, \\
0 & \text{otherwise}. 
\end{cases}
$$

For a single $N_{\perp}, N_{\text{los}}$ has a long tail extending to infinity. The median value of $N_{\text{los}}$ is $N_{\text{los},1/2} = 2N_{\perp}$, reflecting the increased observed column for tilted sheets. The mean value of $N_{\text{los}}$ ($\langle N_{\text{los}} \rangle$) is not defined, because the integral diverges logarithmically; of course, this does not occur in the real world, where sheets do not extend to infinity. For example, if all sheets have an aspect ratio of 5:1, then $\langle N_{\text{los}} \rangle = 1.6N_{\perp}$ and $N_{\text{los},1/2} = 1.7N_{\perp}$.

For an arbitrary pdf $\phi(N_{\perp})$ and infinite slabs, we obtain

$$
\psi(N_{\text{los}}) = \frac{1}{N_{\text{los}}^{\text{hi}}} \int_{N_{\text{los},\text{min}}}^{N_{\perp}} N_{\perp} \phi(N_{\perp}) dN_{\perp}.
$$

As above, this can be regarded as an integral equation for $\phi(N_{\perp})$, and it is almost as straightforward to invert.

### 3.1.3. Conversion of the Intrinsic $\psi(V_{\text{turb}})$ to the Observed $\psi(V_{\text{turb,los}})$ for Turbulence Perpendicular to $B_{\text{tot}}$

The observed velocity width comes from two sources, thermal and nonthermal. We can separate these because we have

| Table 1 | Medians and Means of B_{tot} and B_{los} for Representative pdf’s |
|---------|---------------------------------------------------------------|
| $\phi(B_{\text{tot}})$ | $B_{\text{tot},1/2}$ | $\langle B_{\text{los}} \rangle$ | $B_{\text{tot},1/2}$ | $\langle B_{\text{los}} \rangle$ |
| DELTA FCN | 1.00 | 1.00 | 0.50 | 0.50 |
| FLAT FCN | 1.00 | 1.00 | 0.40 | 0.52 |
| GAUSS FCN | 1.00 | 1.18 | 0.38 | 0.59 |
| EXP FCN | 1.00 | 1.04 | 0.44 | 0.51 |

Fig. 1.—Top: Intrinsic $\phi(B_{\text{los}})$ for four representative functional forms. Bottom: Their line-of-sight counterparts $\psi(B_{\text{los}})$. The vertical scales are arbitrary.
independent measurements of the kinetic temperature. Thus, we can derive the line-of-sight nonthermal (“turbulent”) line width and its associated energy density. How this relates to the total turbulent energy density depends on whether the turbulence is one-dimensional, two-dimensional, or three-dimensional.

Here we assume that this turbulence is two-dimensional, i.e., we assume that it is restricted to motions perpendicular to the mean magnetic field. We make this assumption because we show below that the typical turbulent Mach number is equal to 3.7. For velocities that are parallel to the magnetic field, such turbulence would produce strong shocks that would damp very rapidly. However, velocities that are perpendicular to the magnetic field can be as high as the Alfvén velocity without producing shocks; this is the basis for considering turbulent motions perpendicular to the mean field. We caution, however, that numerical simulations of magnetohydrodynamical turbulence find that magnetic fields do not ameliorate turbulent dissipation (Mac Low et al. 1998), so our assumption might not have any basis in physical reality; in this case the turbulence is isotropic (three-dimensional) and the observed distribution \( \psi(V_{\text{turb,los}}) \) is equal to the intrinsic one \( \phi(V_{\text{turb}}) \).

Suppose that the magnetic field is oriented at angle \( \theta \) with respect to the line of sight, as in \S 3.1.1. Suppose that turbulent motions are perpendicular to the field lines and isotropic in the azimuthal directions around the field line, and along one direction perpendicular to \( B \) they have width \( V_{\text{turb}} \); this makes the full turbulent width \( 2^{1/2}V_{\text{turb}} \). Then the line-of-sight width is

\[
V_{\text{turb,los}} = V_{\text{turb}} \sin \theta.
\]

To find \( \psi(V_{\text{turb,los}}) \) we again follow the same procedures as in \S 3.1.1 and obtain

\[
\psi(V_{\text{turb,los}}) = \begin{cases} 
\frac{V_{\text{turb,los}}}{V_{\text{turb}}} \left( \frac{V_{\text{turb}}}{2} - V_{\text{turb,los}}^2 \right)^{-1/2}, & V_{\text{turb,los}} < V_{\text{turb}}, \\
0, & \text{otherwise}.
\end{cases}
\]

Given a value for \( V_{\text{turb}} \), \( \psi(V_{\text{turb,los}}) \rightarrow \infty \) as \( V_{\text{turb,los}} \rightarrow V_{\text{turb}} \), but the cumulative distribution is well defined with

\[
\text{cum}(V_{\text{turb,los}}) = 1 - \left[ 1 - \left( \frac{V_{\text{turb,los}}}{V_{\text{turb}}} \right)^2 \right]^{1/2},
\]

which gives the median \( V_{\text{turb,los}}\overline{1/2} = 0.87V_{\text{turb}} \) median and mean \( \langle V_{\text{turb,los}} \rangle = 0.79V_{\text{turb}} \). These high values reflect the large fraction of sheets tilted to the line of sight, for which \( V_{\text{turb,los}} \) is large.

For an arbitrary pdf \( \phi(V_{\text{turb}}) \) we obtain

\[
\psi(V_{\text{turb,los}}) = \int_{-\infty}^{\infty} \frac{V_{\text{turb,los}}}{V_{\text{turb}}} \left( \frac{V_{\text{turb}}}{2} - V_{\text{turb,los}}^2 \right)^{-1/2} \phi(V_{\text{turb}}) \, dV_{\text{turb}},
\]

In contrast to the two cases above, this integral equation is not straightforward to invert. It is a Volterra equation of the first kind, and following the identical example in Trumpler & Weaver (1953), it can be rewritten as an Abel integral equation. The analytic solution is

\[
\phi(V_{\text{turb}}) = -\frac{2V_{\text{turb}}^2}{\pi} \frac{d}{dV_{\text{turb}}} \int_{V_{\text{turb}}}^{\infty} \frac{V_{\text{turb,los}}}{V_{\text{turb}}} \left( \frac{V_{\text{turb}}}{2} - V_{\text{turb,los}}^2 \right)^{-1/2} \psi(V_{\text{turb,los}}) \, dV_{\text{turb,los}}.
\]

### 3.2. Conversion of the Intrinsic Bivariate Distribution

\( \phi(B_{\text{tot}}, N_{\perp}) \) to the Observed \( \psi(B_{\text{los}}, N_{\text{los}}) \)

We have measured both \( B_{\text{los}} \) and \( N_{\text{los}} \) and wish to know the bivariate distribution \( \psi(B_{\text{los}}, N_{\text{los}}) \). We proceed by first assuming that \( B_{\text{tot}}, N_{\perp} \), and of course \( \theta \) are all uncorrelated. To proceed we consider two different models.

#### 3.2.1. Case of \( B_{\text{tot}} \) Perpendicular to the Sheet

We refer to the case in which \( B_{\text{tot}} \) is perpendicular to the sheet as the perpendicular model. The only angle involved is \( \theta \), so the original pdf is the trivariate \( \phi(B_{\text{tot}}, N_{\parallel}, \theta) \). This case is simplified because only \( \cos \theta \) is involved, which makes \( B_{\text{los}} \propto 1/N_{\text{los}} \). Converting the original distribution \( \phi(B_{\text{tot}}, N_{\perp}, \theta) \) to the one involving the observed parameters yields

\[
\psi(B_{\text{los}}, N_{\text{los}}; N_{\perp}) = \frac{1}{N_{\text{los}}} \phi_{\text{tot}} \left( \frac{B_{\text{los}} N_{\text{los}}}{N_{\perp}} \right) \phi_{N_{\perp}}(N_{\perp}).
\]

Here we have chosen to eliminate \( B_{\text{tot}} \) and express the result in terms of \( N_{\perp} \); we could have gone the other way. To obtain \( \psi \) in terms of only the observed quantities, we need to integrate over \( N_{\perp} \), but we cannot do this without knowing \( B_{\text{tot}} \). Later we use the one obtained from observations.

For now we consider the illustrative case for which all \( B_{\text{tot}} \) are identical. In this case \( \phi_{\text{tot}} = \delta(B_{\text{los}} - B_{\text{los,0}}) \) and \( \phi_{N_{\perp}}(N_{\perp}) = \delta(N_{\perp} - N_{\perp,0}) \), where \( \delta \) is the Kronecker delta function. This is a trivial case, because all observed points fall on the line

\[
B_{\text{los}} = B_{\text{los,0}} \frac{N_{\perp,0}}{N_{\text{los}}},
\]

which is shown in the top panel of Figure 2.

#### 3.2.2. Case of \( B_{\text{tot}} \) Parallel to the Sheet

The case in which \( B_{\text{tot}} \) is parallel to the sheet is more complicated, because \( B_{\text{los}} \) depends on two angles. These are \( \theta \) and \( \phi \), which is the tilt of the sheet with respect to the observer, and \( \Phi \), which is the azimuthal angle of the field within the sheet. We have \( B_{\text{los}} = B_{\text{los}} \sin \theta \sin \Phi \). This means the pdf \( \phi \) is quadivariate. Again we assume everything is uncorrelated and eliminate \( \theta \) and \( \Phi \). Obtaining the observed distribution is somewhat cumbersome but yields the surprisingly straightforward result

\[
\psi(B_{\text{los}}, N_{\text{los}}, B_{\text{los}}; N_{\perp}) = \frac{N_{\perp}}{\pi N_{\text{los}}} \left[ (B_{\text{los}} N_{\text{los}})^2 \right. \\
- (B_{\text{los}} N_{\perp})^2 \left. - (B_{\text{los}} N_{\text{los}})^2 \right] ^{-1/2} \phi_{\text{los}}(B_{\text{los}}) \phi_{N_{\perp}}(N_{\perp}).
\]
Fig. 2.—Theoretical observed joint pdf's \( \psi(B_{\text{los}}, N_{\text{los}}) \) for the illustrative case of delta-function distributions for \( B_{\text{los}} \) and \( N_{\text{los}} \). The top panel shows the pdf for \( B_{\text{los}} \) perpendicular to the sheets; it degenerates into a single line. The bottom panel is for \( B_{\text{los}} \) parallel to the sheets; contours are spaced by factors of 2 with arbitrary scaling, and the dashed line shows the median \( B_{\text{los}} \) vs. \( N_{\text{los}} \).
Later we use observed distributions, but for now we again consider the illustrative case of delta functions for \( B_{\text{los}} \) and \( N_\perp \). Integrating over \( B_{\text{los}} \) and \( N_\perp \) yields

\[
\psi(B_{\text{los}}, N_\perp) = \frac{N_\perp}{\pi N_\text{los}} \left( \frac{B_{\text{los}}}{N_\text{los}} \right)^2 - \left( \frac{B_{\text{los}}}{N_\text{los}} \right)^2 - \left( \frac{B_{\text{los}}}{N_\text{los}} \right)^{-1/2},
\]

which is illustrated in the bottom panel of Figure 2.

3.2.3. Discussion of Figure 2

Figure 2 exhibits the joint pdf’s for the two sheet models (\( B_{\text{tot}} \) perpendicular and parallel to the sheets, i.e., the “perpendicular” and “parallel” models). The median observed column density \( N_{\text{los},1/2} \) is twice the assumed \( N_\perp \), and the median observed magnetic field \( B_{\text{los},1/2} \) is half the assumed \( B_{\text{los}} \); these univariate medians are indicated by squares in the two panels. The significance of these squares is that half the observed \( B_{\text{los}} \), and half the observed \( N_{\text{los}} \), are smaller and half larger. Finally, the dashed line in the bottom panel exhibits the median \( B_{\text{los},1/2} \) versus \( N_{\text{los}} \); we calculate this by extracting the conditional pdf \( \psi(B_{\text{los}}|N_{\text{los}}) \) versus \( N_{\text{los}} \) and calculating the median from its cumulative distribution, thus obtaining \( B_{\text{los},1/2} \) versus \( N_{\text{los}} \).

Figure 2 illustrates a crucial observational signature at large \( N_{\text{los}} \) that distinguishes between the two sheet models. More specifically, for the perpendicular model, large \( N_{\text{los}} \) goes with small \( B_{\text{los}} \), and vice versa for the parallel model. For the perpendicular model, all of the data points having \( N_{\text{los}} \) above its univariate median (indicated by the square) have \( B_{\text{los}} \) below its univariate median. In contrast, for the parallel model most (66%) of the data points have \( B_{\text{los}} \) above its univariate median: as \( N_{\text{los}} \) gets large, the conditional pdf \( \psi(B_{\text{los}}|N_{\text{los}}) \rightarrow (N_\perp \sqrt{\pi N_{\text{los}}} B_{\text{los}}^{2} B_{\text{los}}^{-1/2}) \), which produces the median \( B_{\text{los},1/2} \approx 0.7 B_{\text{los}} \).

3.3. Conversion of the Intrinsic to Observed Bivariate Distributions Involving Turbulent Velocity

In this section we derive and display the observed bivariate distributions involving \( \psi(B_{\text{los}}, V_{\text{turb}, \text{los}}) \) and \( \psi(N_{\text{los}}, V_{\text{turb}, \text{los}}) \), under the assumption that the turbulent velocity is perpendicular to the magnetic field.

3.3.1. Conversion of the Intrinsic \( \phi(B_{\text{los}}, V_{\text{turb}}) \) to the Observed \( \psi(B_{\text{los}}, V_{\text{turb}, \text{los}}) \)

The joint pdf \( \psi(B_{\text{los}}, V_{\text{turb}, \text{los}}) \) is the same for both the parallel and perpendicular models, and indeed does not depend on whether there are sheets or not, because \( V_{\text{turb}} \) is perpendicular to \( B_{\text{los}} \) so that there is a unique relationship \( B_{\text{los}}/B_{\text{turb}} = [1 - (V_{\text{turb}, \text{los}}/V_{\text{turb}})^{2}]^{1/2} \). As before, we begin with trivariate distributions involving \( \theta, B_{\text{los}} \), and \( V_{\text{turb}} \). We eliminate \( \theta \) to obtain

\[
\psi(B_{\text{los}}, V_{\text{turb}, \text{los}}) = \int_{V_{\text{turb}}}^{\infty} \frac{V_{\text{turb}, \text{los}}}{V_{\text{turb}}} \left( \frac{1 - V_{\text{turb}, \text{los}}^2}{V_{\text{turb}}^2} \right)^{-1/2} \times \phi_{B_{\text{los}}}(B_{\text{los}} \left( \frac{1 - V_{\text{turb}, \text{los}}^2}{V_{\text{turb}}^2} \right)^{-1/2}) \times \phi_{V_{\text{turb}}}(V_{\text{turb}}) \, dV_{\text{turb}}.
\]

The top panel of Figure 3 displays this joint pdf for delta-function distributions of \( B_{\text{los}} \) and \( V_{\text{turb}} \). Because we assume that the \( B_{\text{los}} \) and \( V_{\text{turb}} \) are perpendicular, strong fields go with small velocities and vice versa. There are no contours; they all collapse into a line, because for the delta-function distributions, there is a one-to-one relationship between \( B_{\text{los}} \) and \( V_{\text{turb}, \text{los}} \).

3.3.2. Conversion of the Intrinsic \( \phi(N_{\perp}, V_{\text{turb}}) \) to the Observed \( \psi(N_{\text{los}}, V_{\text{turb}, \text{los}}) \) for the Perpendicular Model

In contrast to the above case, the relation between \( \phi(N_{\perp}, V_{\text{turb}}) \) and \( \psi(N_{\text{los}}, V_{\text{turb}, \text{los}}) \) does depend on the model. For the model with \( B_{\text{tot}} \) perpendicular to the sheet, we have \( N_{\text{los}} = N_\perp/\cos \theta \) and \( V_{\text{turb}, \text{los}} = V_{\text{turb}} \sin \theta \). We obtain

\[
\psi(N_{\text{los}}, V_{\text{turb}, \text{los}}) = \int_{N_{\text{los}}}^{\infty} \left( \frac{N_\perp}{N_{\text{los}}} \left( 1 - \frac{N_\perp^2}{N_{\text{los}}^2} \right)^{-1/2} \times \phi_{V_{\text{turb}}}(V_{\text{turb}, \text{los}} \left( 1 - \frac{N_\perp^2}{N_{\text{los}}^2} \right)^{-1/2}) \right) \times \phi_{N_{\perp}}(N_\perp) \, dN_\perp.
\]

The middle panel of Figure 3 displays this joint pdf for delta-function distributions of \( N_{\perp} \) and \( V_{\text{turb}} \). There are no contours; they all collapse into a line, because for the delta-function distributions, there is a one-to-one relationship between \( B_{\text{los}} \) and \( V_{\text{turb}, \text{los}} \). Because we assume that the \( B_{\text{los}} \) and \( V_{\text{turb}} \) are perpendicular, high \( H \beta \) columns go with small velocities and vice versa.

3.3.3. Conversion of the Intrinsic \( \phi(N_{\perp}, V_{\text{turb}}) \) to the Observed \( \phi(N_{\text{los}}, V_{\text{turb}, \text{los}}) \) for the Parallel Model

The parallel model is more complicated, as it was in § 3.2.2, because \( \Phi \) enters explicitly: because \( V_{\text{turb}} \) is assumed to be perpendicular to \( B_{\text{los}} \), \( V_{\text{turb}, \text{los}} = V_{\text{turb}} [1 - \sin \theta \sin \Phi]^{1/2} \). The final expression equivalent to equation (22) is quite complicated, so we deal with it numerically. The bottom panel of Figure 3 displays this joint pdf for delta-function distributions of \( N_{\perp} \) and \( V_{\text{turb}} \), the contours come from a Monte Carlo calculation.

3.3.4. Summary

The complicated nature of the univariate distribution \( \psi(V_{\text{turb}, \text{los}}) \) in equation (15) means that the bivariate distributions that involve \( V_{\text{turb}, \text{los}} \) are even more complicated and preclude closed-form solutions when we use the observationally derived pdf’s. However, we can easily present the bivariate results for delta-function distributions of \( B_{\text{los}} \), \( V_{\text{turb}} \), and \( N_{\perp} \), shown in Figure 3. Two, \( \psi(B_{\text{los}}, V_{\text{turb}, \text{los}}) \) and \( \psi(N_{\text{los}}, V_{\text{turb}, \text{los}}) \), degenerate into lines because the observed variables depend only on \( \theta \).

4. DERIVATION OF INTRINSIC ASTRONOMICAL UNIVARIATE pdf’s FROM OBSERVED HISTOGRAMS

4.1. Derivation of the Intrinsic \( \phi(B_{\text{los}}) \) from the Histogram of Observed \( B_{\text{obs}} \)

The top panel of Figure 4 exhibits the histogram of measured field strengths \( B_{\text{obs}} \). It contains 69 measurements, of which only 12 have the measurement error \( \delta B_{\text{obs}, m} < 2.5 B_{\text{los}, m} \). This histogram is not symmetric: it has 42 instances of \( B_{\text{obs}} > 0 \) and 27 instances of \( B_{\text{obs}} < 0 \). For a randomly distributed angle between the line-of-sight and the field directions, one expects the numbers to be equal; the probability that we obtain this imbalance, or worse, is given by integrating the binomial (coin tossing) pdf and is equal to 0.042. This is fairly low but is hardly low enough.
Fig. 3.—Bivariate distributions for the model having turbulent velocities perpendicular to the magnetic field, discussed in § 3.3. These distributions assume delta-function distributions of the astronomical parameters $V_{\text{turb}}, B_{\text{tot}},$ and $N_{\perp}$. Top: The locus of contours of the bivariate distribution $\psi(B_{\text{los}}, V_{\text{turb},\text{los}})$ (observed quantities). Middle and bottom: Contours of $\psi(N_{\text{los}}, V_{\text{turb},\text{los}})$. Squares show the univariate medians.
to rule out the random distribution. However, the small probability might indicate that the selection of sources in the Arecibo sky does, in fact, involve nonrandomness.

We proceed on the assumption that the distribution of angles is, indeed, random. This allows us to compare the histogram of the absolute value of measured field $|B_{\text{tot}, m}|$ with the theoretical expectation for various assumed intrinsic pdf's of $B_{\text{tot}}$. The bottom panel of Figure 4 exhibits this distribution, but it is symmetrized so that every entry for a positive $B_{\text{obs}}$ is matched by one with negative $B_{\text{obs}}$. The smooth curve is the best fit for the Gaussian pdf

$$\psi(B_{\text{obs}}) = \frac{1}{\sqrt{2\pi B_{\text{obs}, 0}^2}} e^{-\frac{(B_{\text{obs}} - B_{\text{obs}, 0})^2}{2B_{\text{obs}, 0}^2}}$$  \hspace{0.5cm} (23a)
with
\[ B_{\text{obs},0} = 5.2 \pm 1.3 \, \mu G. \] (23b)

To find this best fit and error for \( B_{\text{obs},0} \), we numerically sampled a range of trial \( B_{\text{obs},0} \) values and for each trial value compared the calculated cumulative distribution with that of the data by performing the Kolmogorov-Smirnov (K-S) test. This provides sets of the K-S statistic \( D \) and its associated probability \( P_{\text{KS}}(B_{\text{obs},0}) \) that the assumed distribution matches the observed one (see Press et al. 1997). We determined the best fit for \( B_{\text{obs},0} \) by choosing the one that maximizes \( P_{\text{KS}} \), which is \( P_{\text{KS}} = 1.0 \) (meaning that a Gaussian is an excellent fit), and we defined its uncertainty as being where the \( P_{\text{KS}} \) falls to 32% of its peak value (thus mimicking the definition of the 1 \( \sigma \) error for Gaussian statistics).

Of course, the 5.2 \( \pm 1.3 \) \( \mu G \) of equation (23b) represents the pdf of \( B_{\text{obs}} \). This is not the same as the pdf of \( B_{\text{los}} \) because of observational noise, which is very significant. Therefore, the dispersion of \( B_{\text{los}} \) is considerably smaller. Of course, it is \( B_{\text{los}} \), not \( B_{\text{obs}} \), which is the quantity of interest, so we need to statistically account for the observational noise. This is not straightforward, because the 69 measurement errors are all different. If they had all been identical, then we could have used Gaussian statistics and convolutions to derive the true dispersion of \( B_{\text{los}} \) from the measured \( B_{\text{obs},0} \). Because the errors are not identical, we instead use a Monte Carlo analysis and employ the actual measurement uncertainties instead of their rms.

4.1.1. Monte Carlo Method

Here we use the observed values of \( B_{\text{obs}} \), together with the uncertainties \( \delta B_{\text{noise}} \), in a Monte Carlo simulation. We begin with four different possibilities for the functional form of \( \phi(B_{\text{los}}) \); each of these is characterized by the median total field strength \( B_{\text{tot}} \), and we calculate 46 uniformly spaced field possibilities ranging from 1 to 10 \( \mu G \).

We assume random orientation with respect to the line of sight. We assume the observational errors \( \delta B_{\text{noise}} \) to be Gaussian distributed. For each test possibility, we perform the set of 69 observations many times (20,000 trial runs). Each time, for each of the 69 measurements, we randomly generate a value of \( B_{\text{tot}} \) according to the assumed \( \phi(B_{\text{los}}) \), orient it randomly with respect to the line of sight, derive its observed value \( B_{\text{los}} \), generate an uncertainty \( \delta B_{\text{los}} \) from a Gaussian pdf whose dispersion is equal to the associated value of \( \delta B_{\text{noise}} \), and record the resulting value of the observed \( B_{\text{obs}} = B_{\text{los}} + \delta B_{\text{los}} \), which includes measurement errors. For each trial run, all four possibilities for the functional form use the same values of \( B_{\text{tot}} \), orientation, and \( \delta B_{\text{los}} \). We make histograms of the absolute values \( B_{\text{obs}} \) and use the K-S test to compare to the cumulative histogram of our data.

Figure 5 displays the K-S fits to the four functional forms. As anticipated in § 3.1.1, all four K-S fits are reasonably good, showing that the data cannot distinguish among these functional forms. However, medians for all four functional forms are quite similar, so the data do select the median with a reasonably small uncertainty.

To collapse the results of four functional fits into a single number for the median total field, we simply average the fields for the four peaks to obtain 6.0 \( \mu G \). To obtain the uncertainty, we use the maximum spread of the 32% points, which are 4.2 and 7.8 \( \mu G \). These combine to make our derived total median field strength
\[ |B_{\text{tot},1/2}| = 6.0 \pm 1.8 \, \mu G. \] (24)

4.1.2. A Single Uncertainty \( \delta B_{\text{noise}} \) for Purposes of Convolution

In § 4.1.1 we derived the median value \( B_{\text{tot},1/2} = 6.0 \, \mu G \). Under the assumption that the distribution of observed values without noise is Gaussian, the pdf of \( B_{\text{los}} \) is the weighted exponential of equation (8). In this equation, the median \( B_{\text{tot},1/2} = 6.0 \, \mu G \) corresponds to \( B_{0} = 3.9 \, \mu G \).

This value of \( B_{0} = 3.9 \, \mu G \) is really \( B_{\text{los},0} \), i.e., it refers to what we would see without noise; in contrast, with noise we found \( B_{\text{obs}} = 5.2 \, \mu G \). Thus, for this particular pdf for \( B_{\text{tot}} \), our ensemble of \( B_{\text{obs}} \) is equivalent to convolving the ensemble of actual values \( B_{\text{los}} \) with a single Gaussian of dispersion
\[ \delta B_{\text{noise},\text{engl}} = (5.2^{2} - 3.9^{2})^{1/2} = 3.4 \, \mu G. \] (25)

Below, in our interpretation of bivariate histograms of our results, we convolve the Monte Carlo–derived values of \( B_{\text{los}} \) (which have no noise) with a Gaussian of this dispersion to obtain simulations of the measured histograms involving \( B_{\text{obs}} \) (which includes noise). This is a quick, approximate way to illustrate how measurement errors affect the histograms of observed values.

4.2. Derivation of the Intrinsic \( \phi(N_{\perp}) \) from the Histogram of Observed \( N_{\perp} \)

We first distinguish between two distinct sets of data for \( N_{\perp} \) and \( V_{\perp} \). One is the entire sample of CNM components from Paper II. The second is the sample we have been discussing, namely that comprising the 69 statistically interesting measurements of \( B_{\text{obs}} \); these are drawn from Paper III, restricted to those having small uncertainties \( \delta B_{\text{los}} \). This smaller sample has a bias eliminating small column densities, which are irrevocably associated with large uncertainties in \( \delta B_{\text{los}} \).

We should not use the smaller data set of 69 samples for discussing the statistics of column density and velocity, because it is biased. Instead, we use the larger data set, but with some restrictions. We include all components from Paper II that satisfy the restrictions that \( |b| \geq 10^{\circ} \), \( V_{\text{turb,los}} > 0 \), and \( V_{\text{turb,los}} < 5.25 \, \text{km s}^{-1} \). The first restriction helps to eliminate blending of CNM components; the second eliminates five components for which the errors happen to yield \( T_{\text{K, max}} > T_{\text{K}} \); and the third eliminates three outliers to the analytic approximation below. All this leaves us with 138 for the sample from Paper II. In the ensuing discussion, we specify which data set we are using.
In § 4.2.2 we will assume that the sheets are oriented randomly and derive the intrinsic $\phi(N_\perp)$ from the histogram of observed $N_{\text{los}}$. Before doing this, however, we consider whether the distribution is, in fact, random. In particular, we might expect the sheets to lie parallel to the Galactic plane, in which case we would expect $N_{\text{los}} \propto 1/\sin|b|$, so we discuss this possibility first.

Figure 6 exhibits the latitude dependence of $N_{\text{los}}$. Looking at the points, one sees some higher values near smaller $|b|$, leading one to suspect that this might be a statistical trend. To test this suspicion we examine the latitude dependence of binned medians. The error bars on the medians marked with asterisks were calculated using the absolute values of the residuals, i.e., the error is $\Sigma|N_{\text{los}} - N_{\text{los,1/2}}|/[M^{1/2}(M - 1)]$, where $N_{\text{los,1/2}}$ is the median and $M$ is the number of points in the bin. The dashed line connects the medians of $N_{\text{los}}$ for 10° wide bins in $|b|$.

If $N_{\text{los}} \propto 1/\sin|b|$, then these medians should follow the same dependence; we provide the dotted line, which is $N_{\text{los}} = 1/\sin|b|$, for a visual comparison. Visually, there is absolutely no tendency for the medians to follow $1/\sin|b|$. The presence of data points with high $N_{\text{los}}$ at small $|b|$ might be real or it might be a statistical artifact. If it is real, it might mean that sheets are in fact aligned with the Galactic plane to some degree and/or have their own intrinsic distribution of $N_\perp$.

There is more to this relationship than random statistics, namely our nonuniform and nonrandom sampling of the sky. Of course, we are restricted to Arecibo’s sky, which is a 38° wide swath in declination. In the vicinity of right ascension 04h30m, it passes through the Taurus/Perseus region at $l \sim 180°$, a large spread in latitude and a small range in longitude. Large column densities persist here up to $|b| \sim 50°$, which is an anomaly in the overall latitude distribution of the neutral interstellar medium (ISM). Thus, in Taurus and Perseus, if we were to plot the total observed column density toward any direction (as opposed to $N_{\text{los}}$, which is the observed column density in one CNM Gaussian component) versus $|b|$, it would not behave anything like $1/\sin|b|$. On the contrary, in other longitude ranges the situation differs. For example, in the vicinity of $l = 240°$, the ISM column density is anomalously small. Thus, plotting quantities versus $|b|$ also plots them versus a biased distribution in $l$. Accordingly, given the nonuniform sampling of the sky, it is very difficult to establish a latitude dependence.

From the above, we conclude that we cannot extract a latitude dependence of $N_{\text{los}}$ from the data. Accordingly, we proceed under the simplest assumption, namely that the sheets are randomly oriented with respect to the observer’s line of sight.

4.2.2. Derivation of $\phi(N_\perp)$ Assuming the Sheets Are Randomly Oriented

The top panel of Figure 7 exhibits the histograms of observed column densities $N_{\text{los}}$ together with a curve that, over most of the region, goes roughly as $N_{\text{los}}^{-1}$ (see eq. [27] below). The solid histogram is for the full Paper II sample of 138 data points. Eyeballing the figure shows that this is a reasonably good fit. Using equation (11), we find that if $\psi(N_{\text{los}}) \propto N_{\text{los}}^{-2}$, then $\phi(N_\perp)$ has the same dependence, namely $\phi(N_\perp) \propto N_\perp^{-1}$. Unfortunately, its integral diverges so it is not a valid pdf. Rather, we must impose lower and upper limits on this pdf.

To proceed, we assume

$$\phi(N_\perp) = \begin{cases} \kappa, & N_{\perp, \text{min}} \leq N_\perp \leq N_{\perp, \text{max}}, \\ 0, & \text{otherwise}, \end{cases}$$

(26)

where $\kappa = \left[ \ln \left( N_{\perp, \text{max}}/N_{\perp, \text{min}} \right) \right]^{-1}$, and derive the corresponding $\psi(N_{\text{los}})$ from equation (11). This yields

$$\psi(N_{\text{los}}) = \begin{cases} 0, & N_{\text{los}} < N_{\perp, \text{min}}, \\ \kappa \frac{N_{\text{los}} - N_{\perp, \text{min}}}{N_{\text{los}}^2}, & N_{\perp, \text{min}} \leq N_{\text{los}} \leq N_{\perp, \text{max}}, \\ \kappa \frac{N_{\perp, \text{max}} - N_{\perp, \text{min}}}{N_{\text{los}}^2}, & N_{\text{los}} > N_{\perp, \text{max}}. \end{cases}$$

(27)

We then numerically cover a grid of trial values for $(N_{\perp, \text{min}}, N_{\perp, \text{max}})$ and for each combination derive the corresponding $\psi(N_{\text{los}})$. From that, we calculate the cumulative distribution and compare with the observed one by performing the K-S test, from which we obtain the bivariate probability $P_{\text{KS}}(N_{\perp, \text{min}}, N_{\perp, \text{max}})$ that the assumed distribution matches the observed one. We determine the best-fit parameters by choosing the combination $(N_{\perp, \text{min}}, N_{\perp, \text{max}})$ that maximizes $P_{\text{KS}}$.

Figure 8 shows contour plots of $P_{\text{KS}}(N_{\perp, \text{min}}, N_{\perp, \text{max}})$. The top panel is for the 138 data points from Paper II, which is the larger and unbiased sample. The highest contour encloses a well-defined area that has two peaks with $P_{\text{KS}} \sim 0.89$ connected by a saddle. We do not regard the differences within the highest contour to be significant, and we adopt the point indicated by an asterisk as the best solution; here $P_{\text{KS}} = 0.86$. We adopt its uncertainty as defined by the contours where $P_{\text{KS}}$ drops to ~0.30, which is ~32% of the peak value 0.89. This yields

$$N_{\perp, \text{min}} = 0.026^{+0.019}_{-0.010} \times 10^{20} \text{ cm}^{-2},$$

(28a)

$$N_{\perp, \text{max}} = 2.6^{+1.9}_{-1.2} \times 10^{20} \text{ cm}^{-2}.$$  

(28b)

These two limits differ by 2 orders of magnitude! For most of the range of $N_{\perp}$ between these limits, $\phi(N_{\perp}) \propto N_{\perp}^{-1}$ to a very good approximation. In Figure 7 the smooth solid line in the top panel is equation (27) with these values of $(N_{\perp, \text{min}}, N_{\perp, \text{max}})$. The bottom panel shows the cumulative distributions, both for the data (diamonds) and for this derived pdf. To the eye the match looks excellent, and this is confirmed by the high value of the K-S probability $P_{\text{KS}} = 0.86$.  

---

Fig. 6.—Diamonds are data points, $N_{\text{los}}$ (in $10^{20}$ cm$^{-2}$) vs. $|b|$. Each asterisk represents the median of $N_{\text{los}}$ within the 10° wide bin centered on the asterisk; the asterisks are connected by the dashed line. The dotted line is $N_{\text{los}} = 1/\sin|b|$ and is meant only to guide the eye. Two points, located at $(|b|, N(\text{H i})) = (28, 12)$ and (19, 13), are excluded from the figure (but not the medians) to save space.
Figure 7 shows the distributions for the current magnetically selected sample of 69 data points from Paper III. The top panel shows the histogram with a dotted line. The dotted histogram differs significantly from the solid one, because it does not have the large increase for small $N_{\text{los}}$. The magnetic selection excludes data points whose uncertainty $\delta B_{\text{los}}$ exceeds an upper limit. This restriction biases the column densities, because it is impossible to obtain small uncertainties on $B_{\text{los}}$ for small $N_{\text{los}}$. Consequently, the pdf of $N_{\text{los}}$ is cut off at small values. The bottom panel of Figure 8 shows contour plots of $P_{\text{KS}}(N_{\perp \min}, N_{\perp \max})$ for this set of 69 data points. Clearly, the contours peak at a different location. The bias in the magnetically selected points is reflected in the larger value of $N_{\perp \min}$, 0.090 for the magnetically selected sample versus 0.026 for the...
distribution is $\psi(V_{\text{turb,los}}) \propto x^2 e^{-x}$, i.e., $n = 2$ in the previous paragraph, with $V_{\text{turb,los,0}} = 0.45 \text{ km s}^{-1}$. This line width is dispersion.

5. DERIVATION OF PREDICTED BIVARIATE OBSERVED DISTRIBUTIONS AND COMPARISON WITH DATA

In this section we use the previous determinations of the pdf of intrinsic quantities $\phi(B_{\text{tot}})$, $\phi(N_\perp)$, and $\phi(V_{\text{turb}})$ to determine the expected bivariate distributions of observed pairs $\psi(B_{\text{obs}}, N_{\text{los}})$, $\psi(V_{\text{turb,los}}, B_{\text{obs}})$, and $\psi(V_{\text{turb,los}}, N_{\text{los}})$. If the observations were good enough, this would enable us to use the correlations among observed quantities to infer astrophysical model parameters. Unfortunately, we will find that we cannot.

5.1. $\phi(B_{\text{tot}}, N_{\perp})$ and $\psi(B_{\text{obs}}, N_{\text{los}})$

Here we employ the observationally derived univariate distributions for $B_{\text{tot}}$ and $N_\perp$ to predict the bivariate distribution of the observed quantities $\psi(B_{\text{los}}, N_{\text{los}})$ for the two cases of $B_{\text{tot}}$ perpendicular and parallel to the sheets. However, this single step does not provide a firm basis for comparison with the data, because the observational uncertainties on $B_{\text{los}}$ are so large.

As we saw in § 4.1.1, the noise precludes our deriving even the univariate $\phi(B_{\text{tot}})$, so for discussion purposes we adopt the EXP FCN of equation (8) with $B_0 = 3.9 \mu G$ (which provides the required median value 6.0 $\mu G$; § 4.1.1). Even with this, however, there is an additional step: we must predict the bivariate distribution of the measured $B_{\text{los}}$ (which includes observational error), i.e., we must obtain $\psi(B_{\text{los}}, N_{\text{los}})$ instead of $\psi(B_{\text{los}}, N_{\text{los}})$. We accomplish this by convolving the conditional distribution $\phi(B_{\text{los}}|N_{\text{los}})$ with a Gaussian having the rms measurement dispersion $3.4 \mu G$ (§ 4.1.2) and doing this as a function of $N_{\text{los}}$.

5.1.1. Case of $B_{\text{tot}}$ Perpendicular to the Sheet

Here we use equation (17) to predict the observed $\psi(B_{\text{los}}, N_{\text{los}})$ for the perpendicular case. As explained above, for $\phi(B_{\text{tot}})$ we use the EXP FCN of equation (8) with $B_0 = 3.9 \mu G$, and $\phi(N_{\perp})$ comes from § 4.2. The analytic solution is somewhat cumbersome and yields

$$
\psi(B_{\text{los}}, N_{\text{los}}) = \begin{cases} 
\frac{\kappa \sqrt{2}}{\pi B_0 N_{\text{los}}} e^{-\left[\frac{\sigma_{\text{los}}^2}{2 B_0^2} + \left(2 N_{\text{los}}^2 \sigma_{\text{los}}^2 \right)\right]}, & N_{\text{los}} \leq N_{\perp,\text{max}}, \\
\frac{\kappa \sqrt{2}}{\pi B_0 N_{\text{los}}} e^{-\left[\frac{\sigma_{\text{los}}^2}{2 B_0^2} + \left(2 N_{\text{los}}^2 \sigma_{\text{los}}^2 \right)\right]}, & N_{\text{los}} > N_{\perp,\text{max}}.
\end{cases}
$$

(29)

This produces contours in the $(B_{\text{los}}, N_{\text{los}})$ plane; the top panel of Figure 10 shows the results. The version including measurement errors is in the middle panel.

The general trend in the top panel is clear from the discussion of § 2: large measured column densities $N_{\text{los}}$ produce small measured fields $B_{\text{los}}$, because the field lines are perpendicular to the sheets. Unfortunately, including the observational uncertainties, as we do in the middle panel, obscures this trend. We defer further discussion to § 6.

5.1.2. Case of $B_{\text{tot}}$ Parallel to the Sheet

Here we proceed as in § 5.1.1, but we use equation (19) instead of equation (17) to predict the observed $\psi(B_{\text{los}}, N_{\text{los}})$ for
the parallel case. The math is complicated, so we proceed by calculating \( \psi(B_{los}, N_{los}) \) using a Monte Carlo simulation. Figure 11 shows the results. The version including measurement errors is in the middle panel.

The general trend in the top panel is clear from the discussion of § 2: for \( N_{los} \gg N_{los, max} \), we see sheets more nearly edge-on, for which \( B_{los} \) is usually large and the conditional pdf \( \psi(B_{los}|N_{los}) \) becomes independent of \( N_{los} \). We defer further discussion to § 6.

5.2. \( \phi(V_{turb}, B_{tot}) \) and \( \psi(V_{turb, los}, B_{los}) \)

As noted in § 3.3.1, the bivariate distribution does not depend upon any model regarding the geometrical shape of the \( \text{H} \) clouds. The top panel of Figure 12 shows contours of \( \psi(V_{turb, los}, B_{los}) \) from a Monte Carlo simulation using the observationally derived \( \phi(V_{turb}, B_{tot}) \); these \( B_{los} \) values do not include measurement errors. The middle panel shows \( \psi(V_{turb, los}, B_{obs}) \), and the bottom panel
Fig. 10.—Top: Contours from eq. (29) of $\psi(B_{\text{los}}, N_{\text{los}})$ for the perpendicular case derived using the observationally derived $\phi(B_{\text{tot}}, N)$. Units are $\mu G$ and $10^{20} \text{ cm}^{-2}$.

Middle: Contours of $\psi(B_{\text{obs}}, N_{\text{los}})$, which includes measurement errors on $B_{\text{los}}$. The square marks the univariate medians.

Bottom: The data.
Fig. 11.—Top: Contours of $\phi(B_{\text{los}}, N_{\text{los}})$ from a Monte Carlo simulation for the parallel case derived using the observationally derived $\phi(B_{\text{tot}}, N_{\mid})$. Units are $\mu G$ and $10^{20} \text{ cm}^{-2}$. Middle: Contours of $\psi(B_{\text{obs}}, N_{\text{los}})$, which includes measurement errors on $B_{\text{obs}}$. The square marks the univariate medians. Bottom: The 69 data points from Paper III.
Fig. 12.—Top: Contours of $\psi(V_{\text{turb,los}}, B_{\text{los}})$ from a Monte Carlo simulation using the observationally derived $\phi(V_{\text{turb}}, B_{\text{tot}})$. Units are km s$^{-1}$ and $\mu$G. Middle: $\psi(V_{\text{turb,los}}, B_{\text{obs}})$, which includes measurement errors on $B_{\text{los}}$. Bottom: The data.
Fig. 13.—Contours of $\psi(V_{\text{turb,los}}, N_{\text{los}})$ from a Monte Carlo simulation using the observationally derived $\phi(V_{\text{turb}})$ and $\phi(B_{\text{tot}})$. Units are km s$^{-1}$ and $10^{20}$ cm$^{-2}$. Top and middle: Perpendicular and parallel sheet models, respectively. Bottom: The data.
shows the data. Contours are spaced by a factor of 2. Eyeballing the figure reveals that the fit to these 69 data points is not bad, which makes it tempting to conclude that the turbulence is, in fact, perpendicular to the magnetic field. Unfortunately, eyeballing also shows that the distribution of points does not differ much from one in which \( B_{\text{los}} \) and \( V_{\text{turb,los}} \) are uncorrelated, meaning that such a conclusion is unwarranted.

5.3. \( \phi(V_{\text{turb}},N_{\parallel}) \) and \( \psi(V_{\text{turb,los}},N_{\text{los}}) \)

Here, as above with the relationship between \( B_{\text{los}} \) and \( N_{\text{los}} \), the predicted bivariate distribution depends on whether the magnetic field is perpendicular or parallel to the sheet. The top panel of Figure 13 shows contours of \( \psi(V_{\text{turb,los}},B_{\text{los}}) \) for the perpendicular model; the middle panel shows the parallel model. The contours are derived from Monte Carlo simulations using the observationally derived \( \phi(V_{\text{turb}},N_{\parallel}) \), assuming that two quantities are independent. The bottom panel shows the data.

Unfortunately, the difference between the contours in the top and middle panels is not large and it does not take a quantitative analysis to tell that the data cannot distinguish between the two models. To the eye, the data look in reasonable agreement with both models.

6. WHICH SHEET MODEL FITS BETTER?

The key to distinguishing between the two sheet models is the bivariate distributions, because the univariate distributions do not depend on whether the perpendicular or parallel model reigns. Of the three bivariate distributions, two are relevant.

Figure 13 displays \( \psi(V_{\text{turb,los}},N_{\text{los}}) \). Unfortunately, the differences between the contours in the top panel (perpendicular model) and middle panel (parallel model) are not large. We can make a quantitative comparison of these bivariate distributions with the data by using the two-dimensional generalization of the K-S test described by Press et al. (1997), which provides \( P_{\text{KS}} \) in a similar fashion as does the usual one-dimensional K-S test. The values are \( P_{\text{KS}} = 0.032 \) and 0.016 for the perpendicular and parallel cases, respectively. These values are both small but not so small as to rule out agreement of the data with the model. They differ by a factor of 2, but this is nowhere near enough of a difference to distinguish between the two cases. Moreover, our basic assumption in deriving these distributions is the anisotropy of the turbulent velocity \( V_{\text{turb}} \), which might not be correct. Unfortunately, this bivariate distribution cannot distinguish between the two models.

The bivariate pair \( \psi(N_{\text{los}},B_{\text{los}}) \) is more directly related to the two models, because there is no additional assumption involving anisotropy of the turbulence. Figures 10 and 11 exhibit the pair for the perpendicular and parallel cases, respectively. After including the measurement errors (middle panels), the two bivariate distributions look similar, as do the runs of the median \( B_{\text{obs,1/2}} \) versus \( N_{\text{los}} \). Neither represents the data very well. We can be quantitative and perform the two-dimensional K-S test, as above; the perpendicular and parallel cases yield \( P_{\text{KS}} = 0.022 \) and 0.024, respectively. Again, the values are small but not so small as to rule out the models; and again, the test cannot distinguish between the two cases.

One can argue that the single reliably detected point at \( (N_{\text{los}},B_{\text{los}}) \sim (7.8 \, \mu G) \) is more consistent with the parallel model, for which the two quantities tend to be correlated. This particular Gaussian component of 3C 142.1 (Paper III) has among the largest values for both quantities, which means that for the perpendicular model we need either a \textit{very} thick sheet and/or a \textit{very} high intrinsic field strength. But we cannot extend this argument to other sources, whose values are more representative.

A confident choice of a model requires more and better data. Unfortunately, this is not likely given current instrumentation, because we have already used \( \sim 800 \) hr of Arecibo time for this project.

7. ASTROPHYSICAL DISCUSSION: MAGNETIC FIELDS

7.1. Observational Issues

As discussed in § 3.1.1, we observe the line-of-sight component of a magnetic field \( B_{\text{los}} \), which is always less than the actual magnetic field \( B_{\text{tot}} \). Figure 1 and its associated discussion shows that, in practice, we cannot determine the intrinsic distribution \( \phi(B_{\text{tot}}) \) from observations of \( B_{\text{los}} \). Fortunately, the median is well determined, with \( [B_{\text{tot,1/2}}] = 6.0 \pm 1.8 \) (§ 4.1.1).

Figure 1 and Table 1 show that the median line-of-sight field strength \( B_{\text{los,1/2}} \) is less than half that of the total field \( B_{\text{tot,1/2}} \), with most of the histograms of \( \psi([B_{\text{los}}]) \) peaking at zero. This functional behavior of \( B_{\text{los}} \), which results purely from geometry, is responsible for the large number of nondetections in our sample. Moreover, it explains particular cases in which \( B_{\text{los}} \) is small. A spectacular example is the Local arm field seen against Cas A, \( B_{\text{los}} = -0.3 \pm 0.6 \mu G \). This surprisingly small result is perfectly consistent with statistical expectation. Of course, we cannot rule out the field actually is really small in any particular case like this, but one needs additional data to draw such a conclusion!

Now consider the large set of magnetic fields observed in 21 cm line \textit{emission} in morphologically obvious structures, reviewed by Heiles & Crutcher (2005). The term “morphologically obvious” means filaments or edge-on sheets. Edge-on sheets should be edge-on shocks in which the field is parallel to the observationally derived \( B_{\text{los}} \). Unfortunately, this is not likely given current instrumentation, as discussed in § 3.2.3, as the line of sight becomes parallel to the sheet (i.e., for a morphologically obvious sheet) the median \( B_{\text{tot,1/2}} \sim 0.71B_{\text{tot}} \). For these structures, measured fields are strong, ranging from \( \sim 5 \) to \( \sim 10 \mu G \). This is not inconsistent with a uniform \( B_{\text{tot}} \sim 10 \mu G \), which is almost a factor of 2 above the median CNM field strength. This suggests that shocks enhance the field strength but not by large factors.

Finally, compare the CNM median field \( B_{\text{tot,1/2}} \) with other estimates of field strength. Beck (2003) reviews the most recent estimate of field strength derived from synchrotron emission, minimum energy arguments, measured cosmic-ray flux, and polarization. He finds the regular component to be \( \sim 4 \mu G \) and the total component to be \( \sim 6 \mu G \). Pulsars give a much smaller value for the regular component, but they provide an underestimate if field and electrons are uncorrelated, as is likely (Beck et al. 2003); nevertheless, they give about \( 5 \mu G \) for the total component (review by Heiles 1996).

The difference between the regular and total components is the fluctuating component, whose turbulent spectrum covers a wide range of scales ranging up to at least tens of parsecs (e.g., the North Polar Spur). Our CNM structure sizes are typically on the order of tenths of a parsec (Paper III), smaller than much of the magnetic field’s turbulence scale range. For this reason we think that it is more appropriate to compare the CNM field strengths with the total component given by Beck (2003), not the regular one. Our CNM median of \( \sim 6.0 \mu G \) is close to the local Galactic total component of \( \sim 6 \mu G \).

7.2. Astrophysical Issues

We find that \( B_{\text{los}} \) in the CNM is comparable to the field strength in other ISM components. This is at first surprising, because the volume density \( n(H) \) in the CNM greatly exceeds that in all other interstellar structures except molecular clouds.
Flux freezing applies almost rigorously in the diffuse gas, even in the H I, and as the interstellar H I changes from CNM to warm neutral medium (WNM) and back again, whether by thermal instability or dynamical processes, the transition must occur under the constraints imposed by flux freezing. Under the usual flux-freezing ideas, magnetic field strength should increase with volume density. If this increase would actually occur, then we would expect higher field strengths in the CNM than in other diffuse gas phases, because the ISM should exhibit approximate thermal pressure equality among the phases. This evidently does not happen. In fact, this absence of field strength increase for small n(H I) is well known from past studies (e.g., § 3.4 of Crutcher et al. 2003), so this is hardly news.

The field is strong enough to dominate the gas pressure and therefore the dynamics. With a CNM median field Btot ∼ 6.0 μG, which also applies elsewhere in the interstellar volume, the magnetic pressure is Pmag/k ∼ 10,400 cm⁻³ K. This dominates the CNM pressure PCNM ∼ 3000 cm⁻³ (Jenkins & Tripp 2001; Wolfire et al. 2003). When the field dominates the pressure, it is much less affected by the gas pressure or thermodynamic state.

CNM structures are magnetically subcritical. When gravitation is important, the distinction between magnetically sub- and supercritical clouds occurs at Btot/N⊥ ∼ 0.38 (Nakano & Nakamura 1978). In subcritical clouds, the magnetic field dominates gravity and prevents collapse. Our CNM sheets have N⊥ ≤ 2.6 and the median Btot,1/2 ∼ 6.0 μG, yielding a minimum ratio of ∼2.3, far above the supercritical upper limit. Gravity is far from important in these clouds, but if several clouds were to coalesce into a gravitationally important one, then magnetic forces would prevent gravitational collapse, unless the field were destroyed in the process, e.g., by the annihilation of oppositely directed fields during coalescence of individual clouds. In this sense, the field is strong and must dominate the act of star formation in denser clouds that form from less dense interstellar gas.

8. ASTROPHYSICAL DISCUSSION: COLUMN DENSITIES

It is well accepted that interstellar H I often lies in sheets. Paper II showed this convincingly for the CNM structures. In the present paper we note that the observed histogram of observed column density Nlos falls monotonically with behavior close to N⁻¹. We then derive the pdf of the intrinsic column density for the sheets, N⊥ and find that it follows equation (27) between two limits, which are rather well defined. Equation (27) behaves much like a N⁻¹ distribution. Figure 7 convincingly shows that this is a good description of the observations.

Suppose, first, that the CNM results from shocks. We have in mind the McKee & Ostriker (1977) model, in which a supernova shock adiabatically compresses and heats the ambient gas; as the gas cools, it does so under roughly constant pressure so its density increases, it cools faster, and so becomes the CNM. As the shocked gas cools, it slows. For low-velocity CNM, which is primarily what we observe, the swept-up column density depends on the energy injected and the ambient density. This dependence is complicated (Cioffi et al. 1988), so we do not attempt a detailed discussion. However, the swept-up column densities are not incomparable with our upper limit N⊥,max = 2.6 × 10²⁰ cm⁻² in equations (27), (28a), and (28b). Thus, the ~N⁻¹ dependence in equation (27) could be a reflection of the statistical distribution of the relevant function of injected energy and ambient density.

Another possibility is that the CNM arises from kinematical and thermal processes in the turbulent interstellar medium. We have in mind structures like those seen in numerical simulations of interstellar turbulence such as Vázquez-Semadeni et al. (2000; see references in Paper II). Consider the “Triad region” discussed extensively in § 8.2 of Paper II and in Heiles & Crutcher (2005). It has a line-of-sight extent of ∼0.05 pc and a plane-of-sky extent of ∼20 pc, for an aspect ratio of ∼200. It also has a typical turbulent velocity of ∼1 km s⁻¹, which makes the line-of-sight crossing time ∼5 × 10⁴ yr. This is very short—interstellar kinematical evolution over human history! If this sheet were the result of a slowed shock, it seems remarkable that, in the presence of ISM density fluctuations, the distance over which the swept-up column density is accumulated would allow the sheet to appear so coherent and additionally that it would retain its coherence to be observable. The alternative, we suppose, is that CNM structures are transient and that we can map as extensive only those that currently appear to be coherent.

This N⁻¹ behavior does not seem to be a capricious result. Rather, it is a challenge to the theorists to reproduce it. Deciding between the above two possibilities, or others, is a matter of the explanation reproducing the N⁻¹ behavior.

9. ASTROPHYSICAL DISCUSSION: TURBULENT VELOCITIES

In § 4.3 we modeled the turbulent velocities as being anisotropic, i.e., perpendicular to the magnetic field, and derived the intrinsic pdf ϕ(Vturb) from the observed one ψ(Vturb,los). Here Vturb is the one-dimensional component of turbulent velocity; for the two-dimensional case we modeled, the full component is 2¹⁄₂ Vturb. In § 5.2 we discussed the bivariate distribution ψ(Vturb,los, Blos). The data fit the model with anisotropic turbulence quite well, but the data also fit no correlation quite well, so the results are inconclusive. The observations cannot distinguish between anisotropic and isotropic turbulent velocities.

We can, however, discuss topics such as the relative energy densities in turbulent motions and magnetism, that is, whether turbulent velocities are super-Alfvénic. Doing this requires some care in definitions of parameters: we measure line-of-sight turbulent velocity dispersions, and these must be converted to their two- or three-dimensional counterparts; and we must include He as a component of the ISM mass density. Finally, we can discuss results in terms of the conventional plasma parameter β, in terms of energy densities, or in terms of supersonic and super-Alfvénic.

First we define velocities and the Mach number. Let ΔVturb,1D be the one-dimensional turbulent velocity dispersion and ΔVturb,3D be the thermal velocity dispersion. For isotropic turbulence, the full turbulent velocity is ΔVturb = ΔVturb,3D = 3¹⁄₂ ΔVturb,1D. With this isotropic turbulence, the turbulent Mach number Mturb is

\[ M^{2}_{turb} = \frac{3\Delta V^{2}_{turb,1D}}{C^{2}_{s}}. \]  

(30)

Here Cs is the velocity of sound; the appropriate sound velocity is the isothermal one, because thermal equilibrium is reached quickly in the CNM, so

\[ C^{2}_{s} = \frac{nkT}{\rho}. \]  

(31a)

The volume density is

\[ \rho = (1 + 4f_{H})n_{H}m_{H}. \]  

(31b)
where \( f_{\text{He}} \) is the fractional abundance of He by number and \( m_{\text{H}} \) is the mass of the H atom; we adopt \( f_{\text{He}} = 0.1 \). Similarly, for the Alfvén velocity \( V_{\text{Alf}} \), we have

\[
V_{\text{Alf}}^2 = \frac{B_{\text{tot}}^2}{4\pi \rho}.
\] (32)

The information propagation velocity perpendicular to the field lines is equal to the Alfvén velocity \( V_{\text{Alf}} \). The mean square velocity perpendicular to the field lines is twice the line-of-sight value. Consequently, we define the Alfvénic turbulent Mach number \( M_{\text{Alf}, \text{turb}} \) as

\[
M_{\text{Alf}, \text{turb}}^2 = \frac{2\Delta V_{\text{turb}, \text{1D}}^2}{V_{\text{Alf}}^2}.
\] (33)

If \( M_{\text{Alf}, \text{turb}} > 1 \), then shocks will develop; this is the super-Alfvénic case.

Next we define energy densities. The turbulent energy density is

\[
E_{\text{turb}} = \frac{\rho F \Delta V_{\text{turb}, \text{1D}}^2}{2}.
\] (34)

Here we include the quantity \( F \) to allow for anisotropic turbulence. If turbulent velocities are only perpendicular to \( B \), then \( F = 2 \); isotropic turbulence has \( F = 3 \) (the case assumed in Paper II). Of course, the magnetic energy density is

\[
E_{\text{mag}} = \frac{B_{\text{tot}}^2}{8\pi}.
\] (35)

The ratio is

\[
\frac{E_{\text{turb}}}{E_{\text{mag}}} = \frac{F \Delta V_{\text{turb}, \text{1D}}^2}{V_{\text{Alf}}^2} = \frac{F}{2} M_{\text{Alf}, \text{turb}}^2.
\] (36)

Finally, we define the conventional plasma parameter \( \beta_{\text{th}} \), which compares thermal and magnetic pressures:

\[
\beta_{\text{th}} = \frac{P_{\text{th}}}{P_{\text{mag}}} = \frac{2\Delta V_{\text{turb}, \text{1D}}^2}{2} \frac{1 + f_{\text{He}}}{1 + 4 f_{\text{He}}}. \quad (37)
\]

To compare turbulent and magnetic effects, we calculate the relevant ratios for the following adopted parameter values, which are close to the medians:

\[
T_{\text{CNM}} = 50 \text{ K}, \quad (38a)
\]

\[
\frac{P_{\text{CNM}}}{k} = 3000 \text{ cm}^{-3} \text{ K}, \quad (38b)
\]

\[
n(\text{H})_{\text{CNM}} = 54 \text{ cm}^{-3}, \quad (38c)
\]

\[
\Delta V_{\text{turb}, \text{1D}} = 1.2 \text{ km s}^{-1}, \quad (38d)
\]

\[
B_{\text{tot}} = 6.0 \text{ } \mu \text{G}. \quad (38e)
\]

Here \( T \) is a typical CNM temperature from Paper II; \( P_{\text{CNM}} \) is from Jenkins & Tripp (2001) and Wolfire et al. (2003). The value for \( \Delta V_{\text{turb}, \text{1D}} = 1.2 \text{ km s}^{-1} \) is the median from \( \S \) 4.3. The value for \( B_{\text{tot}} \) is the median from \( \S \) 4.1.1.

These values provide

\[
M_{\text{turb}} = 3.7, \quad (39a)
\]

\[
\beta_{\text{th}} = 0.29, \quad (39b)
\]

\[
M_{\text{Alf}, \text{turb}}^2 = 1.3, \quad (39c)
\]

\[
V_{\text{Alf}} = 1.5 \text{ km s}^{-1}. \quad (39d)
\]

If the field were small, then turbulence would be isotropic with \( F = 3 \). However, it is not so small, but neither is it so large that we could say for certain that \( F = 2 \). The limits \( 2 < F < 3 \) correspond to \( 1.3 < E_{\text{turb}}/E_{\text{mag}} < 1.9 \). These values should be regarded as representative. Not all CNM clouds have the median values, so these parameters have a considerable spread.

One interesting question is whether any CNM clouds have high \( \beta_{\text{th}} \), i.e., whether any CNM clouds have negligible magnetic field. We cannot answer this, because \( \beta_{\text{th}} \propto P/\sqrt{B_{\text{tot}}^2} \) and because neither \( P \) nor \( B_{\text{tot}}^2 \) is attainable for an individual cloud. Moreover, the functional form of \( B_{\text{tot}} \) is ill determined: we cannot even distinguish between all values being the same (a delta-function distribution) and a Gaussian (which has a significant population of very high field strengths); see \( \S \) 4.1.1.

10. FINAL COMMENTS: TURBULENT AND MAGNETIC ENERGY EQUIPARTITION

Our numbers indicate that magnetism and turbulence are in approximate equipartition. The approximate equipartition suggests that turbulence and magnetism are intimately related by mutual exchange of energy. In the absence of turbulence magnetic energies do not dissipate, because the magnetic field cannot decay on short timescales. But with turbulence, the field may be able to decay rapidly (Heitsch & Zweibel 2003; Lazarian & Vishniac 1999; Zweibel 2002). Moreover, numerical simulations suggest that supersonic turbulence also dissipates rapidly, even when the field is strong (Mac Low et al. 1998). However, it is not obvious to us that these dissipative processes should lead to the observed equipartition between turbulence and magnetism.

We suspect the answer lies in Hennebelle & Peralta’s (1999) result and M. Mac Low’s (2004, private communication) observation that the CNM components result from the transient nature of turbulent flow: the CNM occupies regions where densities are high, produced by converging flows, and the density rise is limited by pressure forces. These limiting pressures are magnetic because the gas has small \( \beta_{\text{th}} \), meaning that thermal pressure is negligible and the dynamical equality makes the magnetic pressure comparable to the converging ram pressure and leads to apparent equipartition.

The equipartition looks like a steady state equilibrium, but it is really a snapshot of time-varying density fields, and our immediate observational view is a statistical result over a large sample. In other words, our current observational snapshot shows an ensemble at a given time. Against this we compare the numerical simulations, which are stationary in the sense that they have been allowed to run long enough that the statistical properties become time independent. Such simulations are also ergodic, with statistical properties over time being equivalent to those over space. With this view, the ISM dynamically evolves through turbulence and its properties are governed by statistical equilibrium of energy inputs and dissipation. These matters are discussed at length in the excellent review by Mac Low & Klessen (2004).
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