On hypergraph Lagrangians

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Abstract

It is conjectured by Frankl and Füredi that the $r$-uniform hypergraph with $m$ edges formed by taking the first $m$ sets in the colex ordering of $\mathbb{N}^r$ has the largest Lagrangian of all $r$-uniform hypergraphs with $m$ edges in [1]. Motzkin and Straus’ theorem confirms this conjecture when $r = 2$. For $r = 3$, it is shown by Talbot in [15] that this conjecture is true when $m$ is in certain ranges. In this paper, we explore the connection between the clique number and Lagrangians for $r$-uniform hypergraphs. As an implication of this connection, we prove that the $r$-uniform hypergraph with $m$ edges formed by taking the first $m$ sets in the colex ordering of $\mathbb{N}^r$ has the largest Lagrangian of all $r$-uniform graphs with $t$ vertices and $m$ edges satisfying $(\binom{r}{r-1} - 1) \leq m \leq (\binom{r}{r-1} + (\binom{r-2}{r-3}) - ([2r-6] \times 2^{r-1} + 2^{r-3} + (r-4)(2r-7)-1)/((\binom{r}{r-2}) - 1)$ for $r \geq 4$.

Key Words: Cliques of hypergraphs; Colex ordering; Lagrangians of hypergraphs; Optimization.

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1 Introduction

For a set $V$ and a positive integer $r$, let $V^{(r)}$ be the family of all $r$-subsets of $V$. An $r$-uniform hypergraph or $r$-graph $G$ consists of a set $V(G)$ of vertices and a set $E(G) \subseteq V(G)^{(r)}$ of edges. When $r = 2$, an $r$-graph is a simple graph. When $r \geq 3$, an $r$-graph is often called a hypergraph. An edge $e = \{a_1, a_2, \ldots, a_r\}$ will be simply denoted by $a_1a_2 \ldots a_r$. Let $K_i^{(r)}$ denote the complete $r$-graph on $t$ vertices, that is the $r$-graph on $t$ vertices containing all possible edges. A complete $r$-graph on $t$ vertices is also called a clique with order $t$. A clique is said to be maximum if it has maximum cardinality. Let $\mathbb{N}$ be the set of all positive integers. For an integer $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, 2, 3, \ldots, n\}$. Let $[n]^{(r)}$ represent the complete $r$-graph on the vertex set $[n]$.

For an $r$-graph $G = (V, E)$, denote the $(r-1)$-neighborhood of a vertex $i \in V$ by $E_i = \{A \in V^{(r-1)} : A \cup \{i\} \in E\}$. Similarly, denote the $(r-2)$-neighborhood of a pair of vertices $i, j \in V$ by $E_{ij} = \{B \in V^{(r-2)} : B \cup \{i, j\} \in E\}$. Denote the complements of $E_i$ by $E_i^c = \{A \in V^{(r-1)} : A \cup \{i\} \not\in E\}$. Also, denote the complements of $E_{ij}$ by $E_{ij}^c = \{B \in V^{(r-2)} : B \cup \{i, j\} \not\in V^{(r)} \setminus E\}$ and $E_{i \cup j} = E_i \cap E_j^c$.

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**Theorem 1.2**\(^1\) If \(G = ([n], E(G))\) and a vector \(\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n\), define

\[
\lambda(G, \vec{x}) = \sum_{i_1, i_2, \ldots, i_r \in E(G)} x_{i_1} x_{i_2} \ldots x_{i_r}.
\]

Let \(S = \{\vec{x} = (x_1, x_2, \ldots, x_n) : \sum_{i=1}^n x_i = 1, x_i \geq 0\text{ for } i = 1, 2, \ldots, n\}\). The Lagrangian\(^2\) of \(G\), denoted by \(\lambda(G)\), is the maximum of the above homogeneous function over the standard simplex \(S\). Precisely,

\[
\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in S\}.
\]

The value \(x_i\) is called the weight of the vertex \(i\). A vector \(\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) is called a feasible weighting for \(G\) if \(\vec{x} \in S\). A vector \(\vec{y} \in S\) is called an optimal weighting for \(G\) if \(\lambda(G, \vec{y}) = \lambda(G)\).

The following fact is easily implied by the definition of the Lagrangian.

**Fact 1.1** Let \(G_1, G_2\) be \(r\)-uniform graphs and \(G_1 \subseteq G_2\). Then \(\lambda(G_1) \leq \lambda(G_2)\).

In \([8]\), Motzkin and Straus established a remarkable connection between the clique number and the Lagrangian of a graph.

**Theorem 1.2** \([8]\) If \(G\) is a 2-graph in which a maximum clique has order \(t\) then \(\lambda(G) = \lambda(K_t^{(2)}) = \frac{1}{2}(1 - \frac{1}{t})\).

The Motzkin-Straus result provides solutions to the optimization problem of a class of homogeneous multilinear functions over the standard simplex of the Euclidean space. The Motzkin-Straus result and its extension were also successfully employed in optimization to provide heuristics for the maximum clique problem (see \([11, 2, 8, 5, 11]\)). It is interesting to explore whether similar results holds for hypergraphs. The obvious generalization of Motzkin and Straus’ result to hypergraphs is false because there are many examples of hypergraphs that do not achieve their Lagrangian on any proper subhypergraph.

Lagrangians of hypergraphs has been proved to be a useful tool in hypergraph extremal problems. Applications of Lagrangian method can be found in \([4, 5, 7, 9, 14]\). In most applications, an upper bound is needed. Frankl and Füredi \([4]\) asked the following question. Given \(r \geq 3\) and \(m \in \mathbb{N}\) how large can the Lagrangian of an \(r\)-graph with \(m\) edges be? For distinct \(A, B \in \mathbb{N}^{(r)}\) we say that \(A\) is less than \(B\) in the colex ordering if \(\max(A \Delta B) \in B\), where \(A \Delta B = (A \setminus B) \cup (B \setminus A)\). For example, the first \(\binom{t}{r}\) \(r\)-tuples in the colex ordering of \(\mathbb{N}^{(r)}\) are the edges of \([t]^{(r)}\). The following conjecture of Frankl and Füredi (if it is true) proposes a solution to the question mentioned above.

**Conjecture 1.3** \([4]\) The \(r\)-graph with \(m\) edges formed by taking the first \(m\) sets in the colex ordering of \(\mathbb{N}^{(r)}\) has the largest Lagrangian of all \(r\)-graphs with \(m\) edges. In particular, the \(r\)-graph with \(\binom{t}{r}\) edges and the largest Lagrangian is \([t]^{(r)}\).

This conjecture is true when \(r = 2\) by Theorem 1.2. For the case \(r = 3\), Talbot in \([15]\) proved the following.

**Theorem 1.4** \([7, 9]\) Let \(m\) and \(t\) be integers satisfying \(\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2} - (t - 1)\). Then Conjecture 1.3 is true for \(r = 3\) and this value of \(m\).

\(^1\)Let us note that this use of the name Lagrangian is at odds with the tradition. Indeed, names as Laplacian, Hessian, Gramian, Grassmanian, etc., usually denote a structured object like matrix, operator, or manifold, and not just a single number.

\(^2\)From now on we use the notation \(\lambda(G)\) for the Lagrangian of \(G\).
Recently, in [17], using some different approaches, Conjecture 1.3 is confirmed for \( r = 3 \) when the value of \( m \) satisfying \( (t-1) \leq m \leq (t-1) + \binom{t-2}{2} - \frac{1}{2}(t-1) \).

Although the obvious generalization of Motzkin and Straus’ result to hypergraphs is false as mentioned earlier, we attempt to explore the relationship between the Lagrangian of a hypergraph and the size of its maximum cliques for hypergraphs when the number of edges is in certain ranges. In [12], it is conjectured that the following Motzkin and Straus type results are true for hypergraphs.

**Conjecture 1.5** [12] Let \( t, m, \) and \( r \geq 3 \) be positive integers satisfying \( (t-1) \leq m \leq (t-1) + \binom{t-2}{2} \). Let \( G \) be an \( r \)-graph with \( m \) edges and \( G \) contain a clique of order \( t-1 \). Then \( \lambda(G) = \lambda([t-1]^{(r)}) \).

**Conjecture 1.6** [12] Let \( t, m, \) and \( r \geq 3 \) be positive integers satisfying \( (t-1) \leq m \leq (t-1) + \binom{t-2}{2} \). Let \( G \) be an \( r \)-graph with \( m \) edges without containing a clique of order \( t-1 \). Then \( \lambda(G) < \lambda([t-1]^{(r)}) \).

Note that the upper bound \( (t-1) + \binom{t-2}{2} \) in Conjecture 1.5 is the best possible (see [12]). Conjecture 1.5 is confirmed when \( r = 3 \) in [12]. Let \( C_r,m \) denote the \( r \)-graph with \( m \) edges formed by taking the first \( m \) sets in the colex ordering of \( \mathbb{N}^{(r)} \). The following result was given in [15].

**Lemma 1.7** [15] For any integers \( m, t, \) and \( r \) satisfying \( (t-1) \leq m \leq (t-1) + \binom{t-2}{2} \), we have \( \lambda(C_r,m) = \lambda([t-1]^{(r)}) \).

In [11], the following result is obtained for \( r \)-graphs.

**Theorem 1.8** [11] Let \( t, m \) and \( r \) be positive integers satisfying \( (t-1) \leq m \leq (t-1) + \binom{t-2}{2} - (2r^{r-3} - 1)\binom{t-2}{2} \). Let \( G \) be an \( r \)-graph with \( t \) vertices and \( m \) edges and contain a clique of order \( t-1 \). Then \( \lambda(G) = \lambda([t-1]^{(r)}) \).

In [15], the following result is also proved, which is the evidence for Conjecture 1.3 for \( r \)-graphs \( G \) on exactly \( t \) vertices.

**Theorem 1.9** [15] For any \( r \geq 4 \) there exist constants \( \gamma_r \) and \( \kappa_0(r) \) such that if \( m \) satisfies

\[
\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1} - \gamma_r(t-1)^{r-2},
\]

with \( t \geq \kappa_0(r) \), let \( G \) be an \( r \)-graph on \( t \) vertices with \( m \) edges, then \( \lambda(G) \leq \lambda([t-1]^{(r)}) \).

The main result in this paper is Theorem 1.10 which is an accompany result of Theorem 1.8.

**Theorem 1.10** Let \( m, t, \) and \( r \geq 4 \) be integers satisfying \( (t-1) \leq m \leq (t-1) + \binom{t-2}{2} - (2r-6) \times \binom{t-2}{2} \). Let \( G \) be an \( r \)-graph with \( t \) vertices and \( m \) edges without containing a clique of order \( t-1 \). Then \( \lambda(G) < \lambda([t-1]^{(r)}) \).

Theorem 1.10 and Theorem 1.8 give a Motzkin-Straus result for some \( r \)-graph. Combining Theorems 1.8 and 1.10 we have the following result immediately.

**Corollary 1.11** Let \( m, t, \) and \( r \geq 4 \) be integers satisfying \( (t-1) \leq m \leq (t-1) + \binom{t-2}{2} - (2r-6) \times \binom{t-2}{2} \). Let \( G \) be an \( r \)-graph with \( t \) vertices and \( m \) edges. Then \( \lambda(G) \leq \lambda([t-1]^{(r)}) \).
Note that \((t^{r-1}) \leq m \leq \binom{t}{r-1} + \binom{t-2}{r-1} - (2r - 6) \times 2^{r-1} + 2^{r-3} + (r - 4)(2r - 7) - 1\) implies the number of vertices \(t\) should be sufficiently large such that \(\binom{t-2}{r-1} \geq (2r - 6) \times 2^{r-1} + 2^{r-3} + (r - 4)(2r - 7) - 1\) in Theorem \ref{thm:main} and Corollary \ref{cor:main}

Theorem \ref{thm:main} and Corollary \ref{cor:main} provide evidence for both Conjecture \ref{conj:main} and Conjecture \ref{conj:main2} respectively. The contribution of Corollary \ref{cor:main} is that the method developed in the proof of Theorem \ref{thm:main} is simpler and different from that in Theorem \ref{thm:main2} in some ways. The upper bound in Corollary \ref{cor:main} for the number of edges \(m\) is more explicit and an improvement comparing to the bound in Theorem \ref{thm:main2}

The proof of Theorem \ref{thm:main} will be given in Section \ref{sec:proof-main}. Further remarks and conclusions are in Section \ref{sec:conclusion}.

## 2 Proof of Theorem \ref{thm:main}

We will impose one additional condition on any optimal weighting \(\bar{x} = (x_1, x_2, \ldots, x_n)\) for an \(r\)-graph \(G\):

\[
\{|i : x_i > 0\}| \text{ minimal, i.e. if } \bar{y} \text{ is a feasible weighting for } G \text{ satisfying } |\{i : y_i > 0\}| < |\{i : x_i > 0\}|, \text{ then } \lambda(G, \bar{y}) < \lambda(G). \tag{1}
\]

When the theory of Lagrange multipliers is applied to find the optimum of \(\lambda(G, \bar{x})\), subject to \(\sum_{i=1}^n x_i = 1\), notice that \(\lambda(E_i, \bar{x})\) corresponds to the partial derivative of \(\lambda(G, \bar{x})\) with respect to \(x_i\). The following lemma gives some necessary conditions of an optimal weighting for \(G\).

**Lemma 2.1** \cite{0} Let \(G = (V, E)\) be an \(r\)-graph on the vertex set \([n]\) and \(\bar{x} = (x_1, x_2, \ldots, x_n)\) be an optimal weighting for \(G\) with \(k\) (\(\leq n\)) non-zero weights \(x_1, x_2, \ldots, x_k\) satisfying condition \(\{1\}\). Then for every \(\{i, j\} \in [k]^{(2)}\), (a) \(\lambda(E_i, \bar{x}) = \lambda(E_j, \bar{x}) = r \lambda(G)\), (b) there is an edge in \(E\) containing both \(i\) and \(j\).

**Definition 2.1** An \(r\)-graph \(G = (V, E)\) on the vertex set \([n]\) is left-compressed if \(j_1, j_2, \ldots, j_r \in E\) implies \(i_1i_2\ldots i_r \in E\) whenever \(i_k \leq j_k, 1 \leq k \leq r\). Equivalently, an \(r\)-graph \(G = (V, E)\) on the vertex set \([n]\) is left-compressed if \(E_{i,j} = \emptyset\) for any \(1 \leq i < j \leq n\).

**Remark 2.2** (a) In Lemma 2.1 part (a) implies that \(x_j \lambda(E_{i,j}, \bar{x}) + \lambda(E_{i\setminus j}, \bar{x}) = x_i \lambda(E_{i,j}, \bar{x}) + \lambda(E_{j\setminus i}, \bar{x})\). In particular, if \(G\) is left-compressed, then \((x_i - x_j) \lambda(E_{i,j}, \bar{x}) = \lambda(E_{i\setminus j}, \bar{x})\) for any \(i, j\) satisfying \(1 \leq i < j \leq k\) since \(E_{j\setminus i} = \emptyset\).

(b) If \(G\) is left-compressed, then for any \(i, j\) satisfying \(1 \leq i < j \leq k\),

\[
x_i - x_j = \frac{\lambda(E_{i\setminus j}, \bar{x})}{\lambda(E_{i,j}, \bar{x})}. \tag{2}
\]

holds. If \(G\) is left-compressed and \(E_{i,j} = \emptyset\) for \(i, j\) satisfying \(1 \leq i < j \leq k\), then \(x_i = x_j\).

(c) By \cite{3}, if \(G\) is left-compressed, then an optimal weighting \(\bar{x} = (x_1, x_2, \ldots, x_n)\) for \(G\) must satisfy \(x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\).

Denote \(\lambda^*_r(m, t) = \max\{\lambda(G) : G \text{ is an } r\text{-graph with } t \text{ vertices and } m\text{ edges}\}\). The following Lemma is proved in \cite{13}.

**Lemma 2.3** \cite{13} There exists a left-compressed \(r\)-graph \(G\) with \(t\) vertices and \(m\) edges such that \(\lambda(G) = \lambda^*_r(m, t)\).
Remark 2.4 Since the only left-compressed r-graph with t vertices and m = \( \binom{t}{r} \) edges is \( |t|^r \). Hence by Lemma 2.3 and Fact 1.1, we have \( \lambda'_{m,t} \leq \lambda(|t|^r) \).

Denote \( \lambda'_{m,t-1,t} = \max\{\lambda(G) : G \text{ is an } r\text{-graph with } t \text{ vertices and } m \text{ edges not containing a clique of order } t - 1\} \). The following lemma implies that we only need to consider left-compressed r-graphs \( G \) when we prove Theorem 1.10.

Lemma 2.5 Let \( m \) and \( t \) be integers satisfying \( \binom{t-1}{r} \leq m \leq \binom{t-2}{r-1} - [(2r - 6) \times 2^{r-1} + 2^{r-3} + (r - 4)(2r - 7) - 1](\binom{t-3}{2} - 1) \). There exists a left-compressed r-graph \( G \) on vertex set \([t]\) with \( m \) edges without containing \([t-1]^r\) such that \( \lambda(G) = \lambda'_{m,t-1,t} \).

In the proof of Lemma 2.5 we need to define some partial order relation. An \( r \)-tuple \( i_1 i_2 \cdots i_r \) is called a descendant of an \( r \)-tuple \( j_1 j_2 \cdots j_r \), if \( i_s \leq j_s \) for each \( 1 \leq s \leq r \), and \( i_1 + i_2 + \cdots + i_r < j_1 + j_2 + \cdots + j_r \). In this case, the \( r \)-tuple \( j_1 j_2 \cdots j_r \) is called an ancestor of \( i_1 i_2 \cdots i_r \). The \( r \)-tuple \( i_1 i_2 \cdots i_r \) is called a direct descendant of \( j_1 j_2 \cdots j_r \) if \( i_1 i_2 \cdots i_r \) is a descendant of \( j_1 j_2 \cdots j_r \) and \( j_1 + j_2 + \cdots + j_r = i_1 + i_2 + \cdots + i_r + 1 \). We say that \( i_1 i_2 \cdots i_r \) has lower hierarchy than \( j_1 j_2 \cdots j_r \) if \( i_1 i_2 \cdots i_r \) is a descendant of \( j_1 j_2 \cdots j_r \). This is a partial order on the set of all \( r \)-tuples.

Proof of Lemma 2.5. Let \( G \) be an \( r \)-graph with \( t \) vertices and \( m \) edges without containing a clique of order \( t - 1 \) such that \( \lambda(G) = \lambda'_{m,t-1,t} \). We call \( G \) an extremal \( r \)-graph for \( m, t - 1 \) and \( t \). Let \( \vec{x} = (x_1, x_2, \ldots, x_t) \) be an optimal weighting of \( G \). We can assume that \( x_i \geq x_j \) when \( i < j \) since otherwise we can just relabel the vertices of \( G \) and obtain another extremal \( r \)-graph for \( m, t - 1 \) and \( t \) with an optimal weighting \( \vec{x} = (x_1, x_2, \ldots, x_t) \) satisfying \( x_i \geq x_j \) when \( i < j \). Next we obtain a new \( r \)-graph \( H \) from \( G \) by performing the following:

1. If \( (t-r) \cdots (t-1) \in E(G) \), then there is at least one \( r \)-tuple in \([t-1]^r \setminus E(G)\), we replacing \( (t-r) \cdots (t-1) \) by this \( r \)-tuple;
2. If an edge in \( G \) has a descendant other than \( (t-r) \cdots (t-1) \) that is not in \( E(G) \), then replace this edge by a descendant other than \( (t-r) \cdots (t-1) \) with the lowest hierarchy. Repeat this until there is no such an edge.

Then \( H \) satisfies the following properties:

1. The number of edges in \( H \) is the same as the number of edges in \( G \).
2. \( \lambda(G) = \lambda(G, \vec{x}) \leq \lambda(H, \vec{x}) \leq \lambda(H) \).
3. \( (t-r) \cdots (t-1) \notin E(H) \).
4. For any edge in \( E(H) \), all its descendants other than \( (t-r) \cdots (t-1) \) will be in \( E(H) \).

If \( H \) is not left-compressed, then there is an ancestor of \( (t-r) \cdots (t-1) \), say \( e \), such that \( e \in E(H) \). Hence \( (t-r) \cdots (t-2)t \) and all the descendants of \( (t-r) \cdots (t-2)t \) other than \( (t-r) \cdots (t-1) \) will be in \( E(H) \). Then

\[
m \geq \left( \frac{t-1}{r} \right) - 1 + \left( \frac{t-2}{r-1} \right) > \left( \frac{t-1}{r} \right) + \left( \frac{t-2}{r-1} \right) - [(2r-6)\times 2^{r-1}+2^{r-3}+(r-4)(2r-7)-1](\frac{t-2}{r-2})-1
\]
which is a contradiction. \(H\) does not contain \([t-1]^{(r)}\) since \(H\) does not contain \((t-r)\ldots(t-1)\). Clearly \(H\) is on vertex set \([t]\). So we complete the proof of Lemma 2.5

In the rest of the paper we assume that \(r \geq 4\) is an integer. In the following three lemmas, Lemma 2.6 implies the maximum weight of \(G\) should distribute `uniform’ on the \(t\) vertices if \(\lambda(G) \geq \lambda([t-1]^{(r)})\), and Lemma 2.8 implies \(G\) contains most of the first \(\binom{t-2r+6}{r}\) edges in colex ordering of \(N^{(r)}\) if \(\lambda(G) \geq \lambda([t-1]^{(r)})\), while Lemma 2.7 implies \(G\) also contains most of the first \(\binom{t-2r+6}{r-1}\) edges containing \(t-1\). Since \(G\) is left-compressed, \(G\) also contains most of the first \(\binom{t-2r+6}{r-1}\) edges containing vertex \(i\), where \(t-2r+7 \leq i \leq t-1\). So \(G\) contains most edges of \([t-1]^{(r)}\). Note that, in the proof of Lemma 2.6, whenever the lower bound of a product is greater than the upper bound, we take this to be the empty product.

*Lemma 2.6* (a) Let \(G\) be an \(r\)-graph on vertex set \([t]\). Let \(\bar{x} = (x_1, x_2, \ldots, x_t)\) be an optimal weighting for \(G\) satisfying \(x_1 \geq x_2 \geq \ldots \geq x_t \geq 0\). Then \(x_1 < x_{t-2r+3} + x_{t-2r+4}\) or

\[
\lambda(G) \leq \frac{(t-r)^{r-1}}{r!} \left( \frac{t-2}{t-r+2} \right) < \frac{1}{r!} \frac{t-1}{(t-1)^r} = \lambda([t-1]^{(r)}).
\]

(b) Let \(G\) be an \(r\)-graph on vertex set \([t]\). Let \(\bar{x} = (x_1, x_2, \ldots, x_t)\) be an optimal weighting for \(G\) satisfying \(x_1 \geq x_2 \geq \ldots \geq x_t \geq 0\). Then \(x_1 < 2(x_{t-2r+4} + x_{t-2r+5})\) or

\[
\lambda(G) \leq \frac{(t-r)^{r-1}}{r!} \left( \frac{t-2}{t-r+2} \right) < \frac{1}{r!} \frac{t-1}{(t-1)^r} = \lambda([t-1]^{(r)}).
\]

*Proof.* (a) If \(x_1 \geq x_{t-2r+3} + x_{t-2r+4}\), then \(tx_1 + x_2 + \ldots + x_{t-2r+2} \geq x_1 + x_2 + \ldots + x_{t-2r+4} + x_{t-3} + x_{t-2r+6} + x_{t-1} + x_1 = 1\). Recalling that \(x_1 \geq x_2 \geq \ldots \geq x_{t-2r+2}\), we have \(x_1 \geq \frac{1}{t-r+1}\). Using Lemma 2.3 we have \(\lambda(G) = \frac{1}{r} \lambda(E_1, x)\). Note that \(E_1\) is an \((r-1)\)-graph with \(t-1\) vertices and total weights at most \(1 - \frac{1}{t-r+1}\). Hence by Remark 2.4 we change the total weights 1 to \(1 - \frac{1}{t-r+1}\).

\[
\lambda(G) = \frac{1}{r} \lambda(E_1, x) \leq \frac{1}{r} \lambda \left( \frac{t-1}{t-r+1} \right) \left( \frac{t-1}{t-r+1} \right)^{r-1}
\]

\[
= \frac{1}{r!} \frac{(t-r)^{r-1}}{(t-r+1)^{r-2} (t-1)^{r-2}}.
\]

Next we prove

\[
\frac{1}{r!} \frac{(t-r)^{r-1}}{(t-r+1)^{r-2} (t-1)^{r-2}} < \frac{1}{r!} \frac{t-1}{(t-1)^r} = \lambda([t-1]^{(r)}).
\]

To show this, we only need to prove

\[
(t-r)^{r-2} (t-1) < (t-r+1)^{r-1}.
\]

If \(t = r, r+1\), [5] clearly holds. Assuming \(t \geq r+2\), we prove this inequality by induction. Now we suppose that [5] holds for some \(r \geq 4\), we will show it also holds for \(r+1\). Replacing \(t\) by \(t-1\) in [5]. We have

\[
[t-(r+1)]^{r-2} (t-2) < (t-r)^{r-1}.
\]
Multiplying $t - (r + 1)$ to the above inequality, we have

$$[t - (r + 1)]^{r-1}(t - 2) < (t - r)^{r-1}[t - (r + 1)].$$

Adding $[t - (r + 1)]^{r-1}$ to the above inequality, we obtain

$$[t - (r + 1)]^{r-1}(t - 1) < (t - r)^{r-1}[t - (r + 1)] + [t - (r + 1)]^{r-1} = (t - r)^r - (t - r)^{r-1} + [t - (r + 1)]^{r-1} < (t - r)^r.$$  \hfill (6)

Hence (5) also holds for $r + 1$ and the induction is complete.

(b) If $x_1 \geq 2(x_{t-2r+5} + x_{t-2r+6})$, then $x_1 + x_2 + \cdots + x_{t-2r+4} + (r - 2)\frac{\bar{r}}{r} \geq x_1 + x_2 + \cdots + x_{t-2r+4} + x_{t-3} + x_{t-2r+6} + x_{t-1} + x_t = 1$. Recalling that $x_1 \geq x_2 \geq \cdots \geq x_{t-2r+4}$ and $r \geq 4$, we have $x_1 \geq \frac{1}{t-2r+4} \geq \frac{1}{t-r+1}$. The rest of the proof is identical to that in part (a), we omit the computation details here.

**Lemma 2.7** Let $G$ be a left-compressed $r$-graph on the vertex set $[t]$ without containing $[t-1]^{(r)}$, then $|[t-2r+6]^{(r-1)} \setminus E_{t-1}| \leq 2^{r-1}|E_{t-1}|$ or $\lambda(G) < \lambda([t-1]^{(r)})$.

*Proof.* Let $\vec{x} = (x_1, x_2, \ldots, x_t)$ be an optimal weighting for $G$. Since $G$ is left-compressed, by Remark 2.11(a), $x_1 \geq x_2 \geq \cdots \geq x_t \geq 0$. If $x_t = 0$, then $\lambda(G) = \lambda(G, \vec{x}) \leq \lambda([t-1]^{(r)})$ since $G$ does not contain $[t-1]^{(r)}$. So we assume that $x_t > 0$.

Consider a new weighting for $G$, $\vec{y} = (y_1, y_2, \ldots, y_t)$ given by $y_i = x_i$ for $i \neq t - 1, t$, $y_{t-1} = x_{t-1} + x_t$ and $y_t = 0$. By Lemma 2.11(a), $\lambda(E_{t-1}, \vec{x}) = \lambda(E_t, \vec{x})$, so

$$\lambda(G, \vec{y}) - \lambda(G, \vec{x}) = x_t(\lambda(E_{t-1}, \vec{x}) - x_t \lambda(E_{t-1}(t)\setminus \vec{x})) - x_{t-1}\lambda(E_{t-1}(t)\setminus \vec{x}) = -x_t^2 \lambda(E_{t-1}(t)\setminus \vec{x}).$$

Assume that $|[t-2r+6]^{(r-1)} \setminus E_{t-1}| > 2^{r-1}|E_{t-1}|$. If $\lambda(G) < \lambda([t-1]^{(r)})$ we are done. Otherwise if $\lambda(G) \geq \lambda([t-1]^{(r)})$ we will show that there exists a set of edges $F \subset [t-1]^{(r)} \setminus E$ satisfying

$$\lambda(F, \vec{y}) > x_t^2 \lambda(E_{t-1}(t)\setminus \vec{x}).$$

(8)

Then using (7) and (8), the r-graph $G^* = ([t], E^*)$, where $E^* = E \cup F$, satisfies $\lambda(G^*, \vec{y}) = \lambda(G, \vec{y}) + \lambda(F, \vec{y}) > \lambda(G, \vec{x}) = \lambda(G)$. Since $\vec{y}$ has only $t - 1$ positive weights, then $\lambda(G^*, \vec{y}) \leq \lambda([t-1]^{(r)})$, and consequently, $\lambda(G) < \lambda([t-1]^{(r)})$. This is a contradiction.

We now construct the set of edges $F$. Let $C = [t-2r+6]^{(r-1)} \setminus E_{t-1}$. Then by the assumption, $|C| > 2^{r-1}|E_{t-1}|$ and $\lambda(C, \vec{x}) \geq 2^{r-1}|E_{t-1}||x_{t-3r+8} \cdots x_{t-2r+6}|$.

Let $F$ consist of those edges in $[t-1]^{(r)} \setminus E$ containing the vertex $t - 1$. Since $\lambda(G) \geq \lambda([t-1]^{(r)})$ then $x_{t-2r+3} > \frac{\bar{r}}{r}$ by Lemma 2.6(a) and $x_{t-2r+4} \geq x_{t-2r+5} > \frac{\bar{r}}{r}$ by Lemma 2.6(b). Hence

$$\lambda(F, \vec{y}) = (x_{t-1} + x_t)\lambda(C, \vec{x}) > 2x_t \cdot 2^{r-1}|E_{t-1}|\lambda(x_{t-3r+8} \cdots x_{t-2r+6}) \geq x_t^2 |E_{t-1}| |x_t|^2 \sum_{i_1, \ldots, i_r \in E_{t-1}} x_{i_1} \cdots x_{i_r} = x_t^2 \lambda(E_{t-1}(t)\setminus \vec{x}).$$

(9)

Hence $F$ satisfies (8). This proves Lemma 2.7. \hfill □
Lemma 2.8 Let $G$ be a left-compressed $r$-graph on the vertex set $[t]$ without containing $[t - 1]^{(r)}$, then $|[t - 2r + 6]^{(r)} \setminus E| \leq 2^{r-1}|E^{(t-1)}_t|$ or $\lambda(G) < \lambda([t - 1]^{(r)})$.

Proof. Let $\bar{x} = (x_1, x_2, \ldots, x_t)$ be an optimal weighting for $G$. Since $G$ is left-compressed, by Remark 2.2(a), $x_1 \geq x_2 \geq \cdots \geq x_t \geq 0$. If $x_t = 0$, then $\lambda(G) < \lambda([t - 1]^{(r)})$ since $G$ does not contain $[t - 1]^{(r)}$. So we assume that $x_t > 0$.

Consider a new weighting for $G$, $\bar{y} = (y_1, y_2, \ldots, y_t)$ given by $y_i = x_i$ for $i \neq t - 1$, $y_{t-1} = x_{t-1} + x_t$ and $y_t = 0$. By Lemma 2.1(a), $\lambda(E_{t-1}, \bar{x}) = \lambda(E_t, \bar{x})$, similar to (4), we have

$$\lambda(G, \bar{y}) - \lambda(G, \bar{x}) = -x_t^2 \lambda(E_{t-1}^{(t-1)} t, \bar{x}).$$

(10)

Assume that $|[t - 2r + 6]^{(r)} \setminus E| > 2^{r-1}|E^{(t-1)}_t|$. If $\lambda(G) < \lambda([t - 1]^{(r)})$ we are done. Otherwise if $\lambda(G) \geq \lambda([t - 1]^{(r)})$ we will show that there exists a set of edges $F \subset [t - 2r + 6]^{(t)} \setminus E$ satisfying

$$\lambda(F, \bar{y}) < \frac{\lambda(F, \bar{x})}{x_t^2} \lambda(E_{t-1}^{(t-1)} t, \bar{x}).$$

(11)

Then using (10) and (11), the $r$-graph $G^* = ([t], E^*)$, where $E^* = E \cup F$, satisfies $\lambda(G^*, \bar{y}) = \lambda(G, \bar{y}) + \lambda(F, \bar{y}) > \lambda(G, \bar{x}) = \lambda(G)$. Since $\bar{y}$ has only $t - 1$ positive weights, then $\lambda(G^*, \bar{y}) \leq \lambda([t - 1]^{(r)})$, and consequently, $\lambda(G) < \lambda([t - 1]^{(r)})$. This is a contradiction.

We now construct the set of edges $F$. Let $C = [t - 2r + 6]^{(r)} \setminus E$. Then by the assumption, $|C| > 2^{r-1} |E^{(t-1)}_t|$ and $\lambda(C, \bar{x}) \geq 2^{r-1} |E^{(t-1)}_t| x_t \geq 3r \cdots x_{t-2r+6}$.

Let $F = C$. Since $\lambda(G) \geq \lambda([t - 1]^{(r)})$ then $x_{t-2r+3} \geq \frac{x_t^2}{t}$ by Lemma 2.6(a) and $x_{t-2r+4} \geq x_{t-2r+5} > \frac{x_t}{t}$ by Lemma 2.6(b). Hence

$$\lambda(F, \bar{y}) = \lambda(C, \bar{x}) > 2^{r-1} |E^{(t-1)}_t| x_t \geq 3r \cdots x_{t-2r+6} \geq x_t^2 |E^{(t-1)}_t| x_t \geq x_t^2 \sum_{i=1}^{t-1} x_1 \cdots x_{i-2} = x_t^2 \lambda(E^{(t-1)}_t, \bar{x}).$$

(12)

Hence $F$ satisfies (11). This proves Lemma 2.8.

Proof of Theorem 1.10

Now we are ready to prove Theorem 1.10

Recalling that $G$ is left-compressed, we have $|[t - 2r + 6]^{(r-1)} \setminus E_t| \leq 2^{r-1} |E^{(t-1)}_t|$ for $t - 2r + 7 \leq i \leq t - 1$. We also have $|[t - 2r + 6]^{(t)} \setminus E_t| \leq 2^{r-1} |E^{(t-1)}_t|$ by Lemma 2.8. Note that $|E^{(t-1)}_t| \leq \binom{t-2}{r-2} - 1$, then

$$|[t - 1]^{(r)} \setminus E| \geq |[t - 2r + 6]^{(r)} \setminus E| + \sum_{i=t-2r+7}^{t-1} |[t - 2r + 6]^{(r-1)} \setminus E_t|$$

$$\geq \binom{t-2r+6}{r} - 2^{r-1} |E^{(t-1)}_t| + (2r-7) \left( \binom{t-2r+6}{r-1} - (2r-7) \times 2^{r-1} |E^{(t-1)}_t| \right)$$

$$\geq \binom{t-2r+6}{r} + (2r-7) \left( \binom{t-2r+6}{r-1} - (2r-6) \times 2^{r-1} \binom{t-2}{r-2} - 1 \right).$$

(13)
Repeated using the equality \( \binom{m+1}{n} = \binom{m}{n} + \binom{m}{n-1} \) to the above inequality, we have

\[
|t-1|^{(r)} \cap E \geq \left( \frac{t-1}{r} \right) - |(2r-6) \times 2^{r-1} + (r-4)(2r-7)|\left( \frac{t-2}{r-2} \right) - 1.
\]

So

\[
0 < |t-1|^{(r)} \setminus E \leq |(2r-6) \times 2^{r-1} + (r-4)(2r-7)|\left( \frac{t-2}{r-2} \right) - 1.
\]

Since \( G \) does not contain \( [t-1]^{(r)} \). Let \( E^* = E \cup [t-1]^{(r)} \) and \( G^* = ([t], E^*) \). Denote the number of edges of \( G^* \) by \( m^* \), then \( \left( \frac{t-1}{r} \right) \leq m^* = \left( \frac{t-1}{r} \right) + \left( \frac{t-2}{r-1} \right) - 2^{r-3}\left( \frac{t-2}{r-2} \right) - 1 \). So \( \lambda(G^*) = \lambda([t-1]^{(r)}) \) by Theorem 1.1. Clearly, \( \lambda(G^*, \vec{x}) - \lambda(G, \vec{x}) > 0 \) since \( x_1 \geq x_2 \geq \cdots \geq x_t > 0 \) and \( |[t-1]^{(r)} \setminus E| > 0 \). Hence \( \lambda(G) = \lambda(G, \vec{x}) < \lambda(G^*, \vec{x}) \leq \lambda(G^*) = \lambda([t-1]^{(r)}) \). This completes the proof of Theorem 1.10.

3 Remarks and conclusions

We remark that, in the proof of Theorem 1.10, we see that \( \gamma_r = 2^{2r} \) and \( t \geq \kappa_0(r) \), where \( \kappa_0(r) \) is a sufficiently large integer such that \( \left( \frac{t-2}{r-1} \right) \geq \gamma_r(t-1)^{r-2} = 2^{2r}(t-1)^{r-2} \) for \( t \geq \kappa_0(r) \). In Corollary 1.11, we improve the upper bound of \( m \) from \( \left( \frac{t-1}{r} \right) + \left( \frac{t-2}{r-1} \right) - \gamma_r(t-1)^{r-2} \) to \( \left( \frac{t-1}{r} \right) + \left( \frac{t-2}{r-1} \right) - |(2r-6) \times 2^{r-2} + (r-4)(2r-7) - 1|\left( \frac{t-2}{r-2} \right) - 1 \). Correspondingly, we improve the condition on \( t \) from \( \left( \frac{t-2}{r-1} \right) \geq 2^{2r}(t-1)^{r-2} \) to \( \left( \frac{t-2}{r-1} \right) \geq |(2r-6) \times 2^{r-2} + (r-4)(2r-7) - 1|\left( \frac{t-2}{r-2} \right) - 1 \).

The method developed in the proof of Theorem 1.10 can also be used to deal with the case for \( r = 3 \) (see [17]). A natural question in the future study is how to prove similar results as Theorem 1.10 and Corollary 1.11 without the restriction of the number of vertices. This will be considered in the future work.

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