MATRIX FACTORIZATIONS AND DOUBLE LINE IN $\mathfrak{sl}_n$ QUANTUM LINK INVARIANT

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Dedicated to Professor Akihiro Tsuchiya on the occasion of his retirement

Abstract. This article gives matrix factorizations for the trivalent diagrams and double line appearing in $\mathfrak{sl}_n$ quantum link invariant. These matrix factorizations reconstruct Khovanov-Rozansky homology. And we show that the Euler characteristic of the matrix factorization for a double loop equals the quantum dimension of the representation $\wedge^2 V$ of $U_q(\mathfrak{sl}_n)$ in Section 3.2.

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1. Introduction

L. Kauffman introduced a graphical link invariant which is the normalized Jones polynomial [4][5]. It is well-known that the polynomial is derived from the fundamental representation of the quantum group $U_q(\mathfrak{sl}_2)$. Further, G. Kuperberg constructed a graphical link invariant associated with the fundamental representation of the quantum group $U_q(\mathfrak{sl}_3)$ [10]. H. Murakami, T. Ohtsuki and S. Yamada introduced a graphical regular link invariant for the fundamental representation of the quantum group $U_q(\mathfrak{sl}_n)$ [11]. In general, we can also obtain a graphical link invariant for a given quantum group $U_q(\mathfrak{g})$ and the fundamental representation. These invariants are collectively called $\mathfrak{g}$ quantum link invariants.

M. Khovanov constructed a categorification of $\mathfrak{sl}_2$ quantum link invariant [6]. A categorification generally means the replacement of a set with a category by corresponding an element to an object. The morphism of the category is properly chosen to carry theory well-done. For a categorification, there is an inverse operation called a decategorification which is the replacement of a category with a set. The decategorification of equivalent objects in the category is a same element in the set.

The Khovanov’s theory is a beautiful example of the categorification; this is the replacement of Jones polynomial $J(L)$, which is a map from the set of links to “$\mathbb{Z}[g, q^{-1}]$”, by a map $C_K$ from the set of links to “the homotopy category of the bounded complex of graded $\mathbb{Z}$-modules”. The bounded complex $C_K(L)$ and the $\mathbb{Z} \oplus \mathbb{Z}$-graded homology groups $KH^{i,j}(L)$ associated with $C_K(L)$ also become link invariants under the Reidemeister
moves. The decategorification is a $\mathbb{Z}$-graded Euler characteristic $\chi_{KH}$ with the normalized Jones polynomial $J(L)$:

$$\chi_{KH} \left( \bigoplus_{i,j \in \mathbb{Z}} KH^{i,j}(L) \right) := \sum_{i,j \in \mathbb{Z}} (-1)^i \dim_{\mathbb{Q}}(KH^{i,j}(L) \otimes \mathbb{Q}) q^j = J(L).$$

Recently, M. Khovanov and L. Rozansky introduced a categorification of $\mathfrak{sl}_n$ quantum link invariant using the homotopy category of the bounded complex of matrix factorizations [7][8]. Since a resolution of a link diagram consists of a combination of the two local diagrams (See FIGURE 2), a matrix factorization for the resolution is defined by a tensor product of some matrix factorizations for the two local diagrams. Then, the categorification of $\mathfrak{sl}_n$ quantum link invariant is constructed as the map $C_{KR}$ from the set of links to the homotopy category of the bounded complex of matrix factorizations for some resolutions.

The bounded complex $C_{KR}$ and the $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$-graded homology groups $KRH^{i,j,k}(L)$ associated with $C_{KR}(L)$ also become link invariants under the Reidemeister moves. The Euler characteristic $\chi_{KRH}$ is defined by

$$\chi_{KRH} \left( \bigoplus_{i,j,k \in \mathbb{Z}, \mathbb{Z}_2} KRH^{i,j,k}(L) \right) := \sum_{i,j,k \in \mathbb{Z}, \mathbb{Z}_2} (-1)^i \dim_{\mathbb{Q}}(KRH^{i,j,k}(L)) q^j.$$

This equals $\mathfrak{sl}_n$ quantum link invariant $\langle L \rangle_n$ for the link $L$.

![Figure 1. $\mathfrak{sl}_n$ quantum link invariant](image-url)
In this paper, we define matrix factorizations for the more general local diagrams and the double line. And we show that these matrix factorizations have some suitable properties; Proposition 3.5 and corollary 3.6 in Section 3.2.

Proposition 3.5 claims that the inner marking made by gluing these local diagrams can be removed in the homotopy category of a matrix factorization. For example, the matrix factorization for the diagram is obtained as a tensor product of two matrix factorizations for and .

M. Khovanov and L. Rozansky showed that the Euler characteristic of a matrix factorization for the single loop equals to the quantum dimension of the fundamental representation of . Corollary 3.6 claims that the Euler characteristic of a matrix factorization for the double loop also equals to the quantum dimension of of .

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2. Category of matrix factorization

2.1. Matrix factorization. We describe a category of a matrix factorization. M. Khovanov and L. Rozansky first imported this algebraic object into link theory [7] [8].

Let be a polynomial ring over and let be free -modules permitted infinite rank. is a matrix factorization with a potential such that and .

For a polynomial ring and a polynomial , let be a category of a matrix factorization whose object is a matrix factorization and whose morphism between and consists of a pair of -module morphisms such that .

For , we define the tensor product by

\[
\mathbf{M} \boxtimes \mathbf{N} := \begin{pmatrix}
(M_0 \otimes N_0) & (M_1 \otimes N_0) \\
(M_1 \otimes N_1) & (M_0 \otimes N_1)
\end{pmatrix}
\begin{pmatrix}
d_{M_0} & d_{M_1} \\
d_{N_0} & d_{N_1}
\end{pmatrix}
\begin{pmatrix}
d_{M_0} \cdot d_{N_0} \\
d_{M_1} \cdot d_{N_1}
\end{pmatrix},
\]

Figure 2. Two type resolutions for crossing
where
\[ \begin{pmatrix} d_{M_0} & -d_{N_1} \\ d_{N_0} & d_{M_1} \end{pmatrix} \text{ and } \begin{pmatrix} d_{M_1} & d_{N_1} \\ -d_{N_0} & d_{M_0} \end{pmatrix} \]
simply denote
\[ \begin{pmatrix} d_{M_0} \otimes \text{Id}_{N_0} & -\text{Id}_{M_1} \otimes d_{N_1} \\ \text{Id}_{M_0} \otimes d_{N_0} & d_{M_1} \otimes \text{Id}_{N_1} \end{pmatrix} \text{ and } \begin{pmatrix} d_{M_1} \otimes \text{Id}_{N_0} & \text{Id}_{M_0} \otimes d_{N_1} \\ -\text{Id}_{M_1} \otimes d_{N_0} & d_{M_0} \otimes \text{Id}_{N_1} \end{pmatrix}. \]

**Remark 2.1.** We consider \( R \otimes R \) as a tensor product of polynomial rings \( R \) and \( R \) over \( R \cap R \). We also consider \( M \otimes M' \) as a tensor product of an \( R \)-module \( M \) and an \( R \)-module \( M' \) over \( R \cap R \).

**Lemma 2.2.** (1) For \( \overline{M} \in \text{Ob}(\text{MF}_{R,\omega}) \) and \( \overline{N} \in \text{Ob}(\text{MF}_{R,\omega'}) \), there is an isomorphism in \( \text{MF}_{R \otimes R, \omega + \omega'} \)
\( \overline{M} \otimes \overline{N} \cong \overline{N} \otimes \overline{M} \).
(2) For \( \overline{L} \in \text{Ob}(\text{MF}_{R,\omega}) \), \( \overline{M} \in \text{Ob}(\text{MF}_{R,\omega'}) \) and \( \overline{N} \in \text{Ob}(\text{MF}_{R,\omega'}) \), there is an isomorphism in \( \text{MF}_{R \otimes R \otimes R, \omega + \omega' + \omega''} \)
\( (\overline{L} \otimes \overline{M}) \otimes \overline{N} \cong \overline{L} \otimes (\overline{M} \otimes \overline{N}) \).

**Proof.** (1) For \( R \)-modules \( M \) and \( N \), we define \( T : M \otimes N \rightarrow N \otimes M \) by \( T(m \otimes n) := n \otimes m \). \( \overline{T} = \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix} \) is a morphism from \( \overline{M} \otimes \overline{N} \) to \( \overline{N} \otimes \overline{M} \) and also a morphism from \( \overline{N} \otimes \overline{M} \) to \( \overline{M} \otimes \overline{N} \). Since \( \overline{T}^2 = \text{Id} \), \( \overline{T} \) gives isomorphic between these matrix factorizations.
(2) By definition, we have
\[ d_{(\overline{L} \otimes \overline{M}) \otimes \overline{N})} = \begin{pmatrix} d_{L_0} & -d_{M_1} & -d_{N_1} & 0 \\ d_{M_0} & d_{L_1} & 0 & -d_{N_1} \\ d_{N_0} & 0 & d_{M_1} & d_{L_0} \\ 0 & d_{N_0} & -d_{M_0} & d_{L_0} \end{pmatrix}, \]
d\[ d_{(\overline{L} \otimes \overline{M} \otimes \overline{N})} = \begin{pmatrix} d_{L_0} & 0 & -d_{M_1} & -d_{N_1} \\ 0 & d_{L_0} & d_{N_0} & -d_{M_1} \\ d_{M_0} & -d_{N_1} & d_{L_1} & 0 \\ d_{N_0} & d_{M_1} & 0 & d_{L_1} \end{pmatrix}. \]
Thus it is obvious that \( \overline{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \) : \( (\overline{L} \otimes \overline{M}) \otimes \overline{N} \rightarrow \overline{L} \otimes (\overline{M} \otimes \overline{N}) \) is an isomorphism.

**Lemma 2.3.** The matrix factorization \( \overline{M} \xrightarrow{R \otimes R} \overline{M} \) is the unit object in \( \text{MF}_{R,\omega} \). That is, for any matrix factorization \( \overline{M} \in \text{MF}_{R,\omega} \),
\( \overline{M} \otimes (\overline{M} \otimes R) \) is isomorphic to \( \overline{M} \).

**Proof.** By definition, we have
\[ \overline{M} \otimes (\overline{M} \otimes R) = \begin{pmatrix} 0 & d_{M_0} & 0 \\ 0 & d_{M_1} & 0 \\ M_0 \otimes R & M_1 \otimes R & M_0 \otimes R \end{pmatrix} \]
\[ \simeq \overline{M}. \]
\( \square \)
The translation functor \( (1) \) changes the matrix factorization \( \overline{M} = (M_0, M_1) \) into
\[
\overline{M} \langle 1 \rangle = \left( M_1 \xrightarrow{-d_{M_1}} M_0 \xrightarrow{-d_{M_0}} M_1 \right).
\]
The functor \( (2) = (\langle 1 \rangle)^2 \) is the identity functor.

**Lemma 2.4.** For \( \overline{M} \in \text{Ob}(\text{MF}_{R,\omega}) \) and \( \overline{N} \in \text{Ob}(\text{MF}_{R,\omega}) \), there is an isomorphism in \( \text{MF}_{R \otimes R, \omega + \omega} \)
\[
(\overline{M} \boxtimes \overline{N}) \langle 1 \rangle = (\overline{M} \langle 1 \rangle) \boxtimes \overline{N} \simeq \overline{M} \boxtimes (\overline{N} \langle 1 \rangle).
\]

**Proof.** We directly find that \((\overline{M} \boxtimes \overline{N}) \langle 1 \rangle\) equals \((\overline{M} \langle 1 \rangle) \boxtimes \overline{N}\) by definition. The second equivalence is correct by Lemma 2.4 (1) and the first equality. \(\square\)

The morphism \( \overline{f} : \overline{M} \rightarrow \overline{N} \in \text{Mor}(\text{MF}_{R,\omega}) \) is null-homotopic if morphisms \( h_0 : M_0 \rightarrow N_1 \) and \( h_1 : M_1 \rightarrow N_0 \) exist such that \( f_0 = h_1 d_{M_0} + d_{N_1} h_0 \) and \( f_1 = h_0 d_{M_1} + d_{N_0} h_1 \). And \( \overline{f}, \overline{g} : \overline{M} \rightarrow \overline{N} \in \text{Mor}(\text{MF}_{R,\omega}) \) are homotopic if \( \overline{f} - \overline{g} \) is null-homotopic.

Let \( \text{HMF}_{R,\omega} \) be the quotient category of \( \text{MF}_{R,\omega} \) which has the same objects to \( \text{Ob}(\text{MF}_{R,\omega}) \) and has morphisms of \( \text{Mor}(\text{MF}_{R,\omega}) \) modulo null-homotopic. A matrix factorization in \( \text{MF}_{R,\omega} \) is called contractible if it is isomorphic to the zero matrix factorization
\[
\left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \right)
\]
in \( \text{HMF}_{R,\omega} \).

**Example 2.5.** Let \( R_0, R_1 \) be a ring \( R \) and \( \omega \in R \).
\[
\left( \begin{array}{ccc}
R_1 & \omega & \rightarrow & R_0 & \rightarrow & 1 & \rightarrow & R_1 \\
\end{array} \right)
\]
and
\[
\left( \begin{array}{ccc}
R_1 & \rightarrow & 1 & \rightarrow & R_0 & \rightarrow & \omega & \rightarrow & R_1 \\
\end{array} \right)
\]
are contractible.

### 2.2. \( \mathbb{Z} \)-graded matrix factorization

Let \( R \) be replaced with a \( \mathbb{Z} \)-graded polynomial ring over \( Q \) whose each parameter has a \( \mathbb{Z} \)-grading and let \( M_0, M_1 \) be also replaced with free \( \mathbb{Z} \)-graded \( R \)-modules. The category \( \text{MF}_{R,\omega}^{gr} \) is the category of a \( \mathbb{Z} \)-graded matrix factorization whose object is the same object to \( \text{Ob}(\text{MF}_{R,\omega}) \) except having a \( \mathbb{Z} \)-grading and whose morphism consists of a morphism with preserving a \( \mathbb{Z} \)-grading shift \( \{n\} \) turns the matrix factorization \( \overline{M} \) into
\[
\left( \begin{array}{ccc}
M_0 \{n\} & \xrightarrow{d_{M_0}} & M_1 \{n\} & \xrightarrow{d_{M_1}} & M_0 \{n\} \\
\end{array} \right).
\]

**Lemma 2.6.** For \( \overline{M} \in \text{Ob}(\text{MF}_{R,\omega}^{gr}) \) and \( \overline{N} \in \text{Ob}(\text{MF}_{R,\omega}^{gr}) \), there is an equality in \( \text{MF}_{R \otimes R, \omega + \omega}^{gr} \)
\[
(\overline{M} \boxtimes \overline{N}) \{n\} = (\overline{M} \{n\}) \boxtimes \overline{N} = \overline{M} \boxtimes (\overline{N} \{n\}).
\]

**Proof.** We find that these objects are really identical by definition. \(\square\)

For the \( \mathbb{Z} \)-graded matrix factorization \( \overline{M} \) in \( \text{MF}_{R,\omega}^{gr} \), we can consider \( \mathbb{Z} \oplus \mathbb{Z}_2 \)-graded homology group as follows;
\[
\text{H}(\overline{M}) = \bigoplus_{j \in \mathbb{Z}, k \in \mathbb{Z}_2} \text{H}^{j,k}(\overline{M})
\]
whose $k$ is a complex grading of the matrix factorization $\overline{M}$, i.e. $i = 0$ or 1, and $j$ is a $\mathbb{Z}$-grading induced by the $\mathbb{Z}$-graded modules of the matrix factorization $\overline{M}$. The Euler characteristic $\chi$ is defined by

$$\chi(H(M)) = \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}_2} \dim_{\mathbb{Q}} H^j \overline{M} q^j.$$ 

2.3. Koszul matrix factorization. Let $R$ be a $\mathbb{Z}$-graded polynomial ring over $\mathbb{Q}$ and let $\deg_{\mathbb{Z}}(a)$ denote a $\mathbb{Z}$-grading of the polynomial $a \in R$. For polynomials $a, b \in R$ and a $\mathbb{Z}$-graded $R$-module $M$, we define the matrix factorization $K(a; b)_M$ with the potential $ab$ by

$$K(a; b)_M := \left( \begin{array}{ccc} M & a & M \left( \frac{1}{2}(\deg_{\mathbb{Z}}(b) - \deg_{\mathbb{Z}}(a)) \right) & b & M \end{array} \right).$$

Remark 2.7. Let $R$ be a polynomial ring over $\mathbb{Q}$ and let $R_y$ be the polynomial ring $R[y]$. For polynomials $a$ and $b$ in $R$, $K(a; b)_{R_y}$ is a matrix factorization of $R_y$-modules with rank 1 as an object in $\text{MF}^{gr}_{R_y, ab}$. And we can consider that $K(a; b)_{R_y}$ is a matrix factorization of $R$-modules with infinite rank as an object in $\text{MF}^{gr}_{R, ab}$.

Lemma 2.8.

$$K(a; b)_M \langle 1 \rangle = K(-b; -a)_M \left\{ \frac{1}{2}(\deg_{\mathbb{Z}} b - \deg_{\mathbb{Z}} a) \right\}.$$

Proof. By definition, we have

$$K(a; b)_M \langle 1 \rangle = \left( \begin{array}{ccc} M & a & M \left( \frac{1}{2}(\deg_{\mathbb{Z}} b - \deg_{\mathbb{Z}} a) \right) & b & M \end{array} \right) \langle 1 \rangle$$

$$= \left( \begin{array}{ccc} M \left\{ \frac{1}{2}(\deg_{\mathbb{Z}} b - \deg_{\mathbb{Z}} a) \right\} & -b & M \end{array} \right) - a \left( \begin{array}{ccc} M \left\{ \frac{1}{2}(\deg_{\mathbb{Z}} b - \deg_{\mathbb{Z}} a) \right\} \end{array} \right)$$

$$= \left( \begin{array}{ccc} M & -b \end{array} \right) - a \left( \begin{array}{ccc} M \left\{ \frac{1}{2}(\deg_{\mathbb{Z}} a - \deg_{\mathbb{Z}} b) \right\} \end{array} \right) - a \left( \begin{array}{ccc} M \left\{ \frac{1}{2}(\deg_{\mathbb{Z}} b - \deg_{\mathbb{Z}} a) \right\} \end{array} \right)$$

$$= K(-b; -a)_M \left\{ \frac{1}{2}(\deg_{\mathbb{Z}} b - \deg_{\mathbb{Z}} a) \right\}.$$ 

□

In general, for the sequences $a = \langle a_1, a_2, \ldots, a_k \rangle$ and $b = \langle b_1, b_2, \ldots, b_k \rangle$ of polynomials in $R$, we define the matrix factorization $K(a; b)_M$ with the potential $\sum_{i=1}^k a_i b_i$ by

$$K(a; b)_M := \bigotimes_{i=1}^k K(a_i; b_i)_M.$$

This matrix factorization is called a Koszul matrix factorization.

Lemma 2.9. Let $c$ be a non-zero element in $\mathbb{Q}$.

There is an isomorphism in $\text{MF}^{gr}_{R, ab}$

$$K(a; b)_M \simeq K(ca; c^{-1}b)_M.$$

Proof. $\overline{f} = (1, c) : K(a; b)_M \to K(ca; c^{-1}b)_M$ and $\overline{g} = (1, c^{-1}) : K(ca; c^{-1}b)_M \to K(a; b)_M$ satisfy that $\overline{f} \overline{g} = \overline{1}$ and $\overline{g} \overline{f} = \overline{1}$. Thus, these matrix factorizations are isomorphic.

□

Theorem 2.10. [Khovanov-Rozansky, Theorem 2.2.] We put $R = \mathbb{Q}[x_1, x_2, \ldots, x_n]$ and $R_y = R[y]$. Let $\langle a_1, a_2, \ldots, a_k \rangle$ and $\langle b_1, b_2, \ldots, b_k \rangle$ be sequences of polynomials $a_j, b_j \in R_y$.

We assume that $K(a; b)_{R_y}$ is a Koszul matrix factorization with the potential $\omega \in R$. That is to say, we can see that this matrix factorization is an object in $\text{HMF}^{gr}_{R, \omega}$, which consists of infinite rank $R$-modules $R_y$. 

□
Furthermore, we assume that $b_i = cy^n + p$ for some $i$, where $c$ is a non-zero element in $\mathbb{Q}$ and $p$ is the polynomial in $R_y$ whose degree for $y$ is less than $n$. Then, there is an isomorphism in $\text{HMF}_{R,\omega}^{gr}$:

$$K(a; b)_{R_y} \simeq K(\hat{a}; \hat{b})_{R_y}/(b_i),$$

where $\hat{a}$ and $\hat{b}$ are associated with $a$ and $b$ removing the $i$-th polynomial.

**Proof.** We can replace $b_i$ with $y^n + p$ using Lemma 2.9. Then, we repeat proof by M.Khovanov and L.Rozansky in [9]. By definition and the above assumption, we have

$$K(a; b)_{R_y} = K(\hat{a}; \hat{b})_{R_y} \boxtimes K(a_i; y^n + p)_{R_y}.$$ 

The Koszul matrix factorization $K(\hat{a}; \hat{b})_{R_y}$ is described as:

$$(R_y^{\hat{a}} \xrightarrow{D_0} R_y^{\hat{b}} \xrightarrow{D_1} R_y^{\hat{c}}),$$

where $D_0, D_1 \in M_r(R_y)$ and $r = 2^{k-2}$. Thus, we obtain

$$K(a; b)_{R_y} = \begin{pmatrix}
R_y^{a_i} & \oplus & R_y^{b_i} \\
D_0' & \oplus & D_1'
\end{pmatrix},$$

where $D_0' = \begin{pmatrix} D_0 & (-y^n - p) \text{Id}_{R_y^{a_i}} \\
a_i \text{Id}_{R_y^{a_i}} & D_1
\end{pmatrix}, \quad D_1' = \begin{pmatrix} D_1 & (y^n + p) \text{Id}_{R_y^{b_i}} \\
-a_i \text{Id}_{R_y^{b_i}} & D_0
\end{pmatrix}.$

The ring $R_y$ is split into the direct sum as an $R$-module as follows;

$$R_y \simeq R_{<n} \oplus R_{\geq n},$$

where $R_{<n} = \bigoplus_{i=0}^{n-1} R \cdot y^i$ and $R_{\geq n} = \bigoplus_{i=0}^{\infty} R \cdot y^i (y^n + p)$.

The $R$-module morphism $R_y \xrightarrow{y^n + p} R_y$ induces the $R$-module isomorphism

$$f_{iso} : R_y \xrightarrow{y^n + p} R_{\geq n}.$$

Moreover, there are the natural $R$-module injection

$$f_{inj} : R_{<n} \rightarrow R_y,$$

and the natural $R$-module projections

$$f_{proj_{<n}} : R_y \rightarrow R_{<n}$$

and

$$f_{proj_{\geq n}} : R_y \rightarrow R_{\geq n}.$$

The $R$-module $R_y^r$ is also split into the direct sum

$$R_y^r \simeq R_{<n}^r \oplus R_{\geq n}^r.$$ 

Then, there are the $R$-module isomorphism

$$F_{iso} : R_y^r \xrightarrow{(y^n + p) \text{Id}_{R_y^r}} R_{\geq n}^r,$$

the $R$-module injection

$$F_{inj} : R_{<n}^r \rightarrow R_y^r.$$
and the $R$-module projections
\[
F_{\text{proj} < n} : R_y^r \rightarrow R_{< n}^r
\]
and
\[
F_{\text{proj} \geq n} : R_y^r \rightarrow R_{\geq n}^r.
\]

It is easy to find that the $R$-module morphisms
\[
\phi_0 = \begin{pmatrix}
R_{< n}^r \\
\oplus \\
R_{\geq n}^r \\
\oplus \\
R_y^r
\end{pmatrix}
\rightarrow
\begin{pmatrix}
R_y^r \\
\oplus \\
R_{< n}^r \\
\oplus \\
R_{\geq n}^r \\
\oplus \\
R_y^r
\end{pmatrix}
\]
\[
\phi_1 = \begin{pmatrix}
R_{< n}^r \\
\oplus \\
R_{\geq n}^r \\
\oplus \\
R_y^r
\end{pmatrix}
\rightarrow
\begin{pmatrix}
R_y^r \\
\oplus \\
R_{< n}^r \\
\oplus \\
R_{\geq n}^r \\
\oplus \\
R_y^r
\end{pmatrix}
\]
are $R$-module isomorphisms. Let $\overline{M}$ be the matrix factorization
\[
\overline{M} = \begin{pmatrix}
R_{< n}^r \\
\oplus \\
R_{\geq n}^r \\
\oplus \\
R_y^r
\end{pmatrix}
\rightarrow
\begin{pmatrix}
R_{< n}^r \\
\oplus \\
R_{\geq n}^r \\
\oplus \\
R_y^r
\end{pmatrix}
\rightarrow
\begin{pmatrix}
R_{< n}^r \\
\oplus \\
R_{\geq n}^r \\
\oplus \\
R_y^r
\end{pmatrix}
\]

Then, $\overline{\phi} = (\phi_0, \phi_1)$ is an isomorphism from $\overline{M}$ to $K(a; b)_{R_y}$ in $\text{MF}_{R_{\omega}}^{gr}$.

Since $\overline{\phi}^{-1}$ consists of
\[
\phi_0^{-1} = \begin{pmatrix}
F_{\text{proj} < n} \\
\oplus \\
\phi_1^{-1} D_0 \phi_0 \\
\oplus \\
R_y^r
\end{pmatrix}
\]
and $\phi_1^{-1} = \begin{pmatrix}
F_{\text{proj} < n} \\
\oplus \\
\phi_0^{-1} D_1 \phi_1 \\
\oplus \\
R_y^r
\end{pmatrix}$,
the morphisms $\phi_0^{-1} D_0 \phi_0$ and $\phi_0^{-1} D_1 \phi_1$ are described by
\[
\phi_0^{-1} D_0 \phi_0 = \begin{pmatrix}
F_{\text{proj} < n} D_0 F_{\text{inj}} \\
\oplus \\
\omega F_{i_0}^{-1} 0 \\
\oplus \\
R_y^r
\end{pmatrix}
\]
and $\phi_0^{-1} D_1 \phi_1 = \begin{pmatrix}
F_{\text{proj} < n} D_1 F_{\text{inj}} \\
\oplus \\
\omega F_{i_0}^{-1} 0 \\
\oplus \\
R_y^r
\end{pmatrix}$.

The matrix factorization obtained by restricting $R_{< n}^r \oplus R_{\geq n}^r \oplus R_y^r$ of $\overline{M}$ to $R_{< n}^r \oplus R_y^r$
\[
\begin{pmatrix}
R_{< n}^r \\
\oplus \\
R_y^r
\end{pmatrix}
\rightarrow
\begin{pmatrix}
R_{\geq n}^r \\
\oplus \\
R_y^r
\end{pmatrix}
\rightarrow
\begin{pmatrix}
R_{\geq n}^r \\
\oplus \\
R_y^r
\end{pmatrix}
\]
is contractible in $\text{MF}_{R_{\omega}}^{gr}$. Thus, $\overline{M}$ is isomorphic to the quotient matrix factorization
\[
\begin{pmatrix}
R_{< n}^r \\
\oplus \\
R_y^r
\end{pmatrix}
\rightarrow
\begin{pmatrix}
R_{< n}^r \\
\oplus \\
R_y^r
\end{pmatrix}
\rightarrow
\begin{pmatrix}
R_{< n}^r \\
\oplus \\
R_y^r
\end{pmatrix}
\]

in $\text{HMFR}^R_{R_\infty}$. By the choice of a basis of $R_\infty$ as an $R$-module, it is easy to find that this quotient matrix factorization equals $K(\pi_1; \pi_2)_{R_\infty/(b_i)}$. □

3. Matrix factorization for trivalent diagrams and double line

3.1. Khovanov Rozansky homology. We briefly recall Khovanov-Rozansky link homology theory, only the definition of matrix factorizations for planar diagrams and some properties of the matrix factorizations. See Section 6 in [7] for further details.

In [7], M. Khovanov and L. Rozansky defined a categorification of planar graphs as a Koszul matrix factorization.

We assign a parameter $x_i$ on the end point of a single line. And each $\mathbb{Z}$-grading of parameters $x_i$ is 2. This $\mathbb{Z}$-grading induces a $\mathbb{Z}$-grading of a matrix factorization in $\text{MF}^R_{R_\infty}$ and $\text{HMFR}^R_{R_\infty}$. The function $f(s_1, s_2)$ is obtained by expanding the power sum $x^{n+1} + y^{n+1}$ with the elementary symmetric polynomials $x + y$ and $xy$, i.e. the function $f(s_1, s_2)$ satisfies that

$$f(x + y, xy) = x^{n+1} + y^{n+1}.$$}

We define matrix factorizations for the planar diagrams

$$C\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_n = K(\pi_{12}; x_1 - x_2)_{\mathbb{Q}[x_1, x_2]},$$

$$C\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_n = K(\begin{pmatrix} u_{1, 2}, v_{1, 2} \\ u_{3, 4}, v_{3, 4} \end{pmatrix}; \begin{pmatrix} x_1 + x_2 - x_3 - x_4 \\ x_1 x_2 - x_3 x_4 \end{pmatrix})_{\mathbb{Q}[x_1, x_2, x_3, x_4]} \cdot (-1),$$

where $\pi_{12}(x_1 - x_2) = x_1^{n+1} - x_2^{n+1}$, $u_{1, 2} = f(x_2, x_3, x_4) - f(x_2, x_1, x_4)$ and $v_{3, 4} = f(x_1, x_3, x_4) - f(x_2, x_3, x_4)$.

Furthermore, we construct a matrix factorization for the more complex planar diagrams produced by combinatorially joining the diagrams and . Now, suppose the planar diagrams and , which can match at $x$ and $x'$ with keeping the orientation. For the matrix factorizations

$$C\begin{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \end{pmatrix}_n \in \text{Ob}(\text{MF}^R_{R_\infty + x^{n+1}}) \quad \text{and} \quad C\begin{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \end{pmatrix}_n \in \text{Ob}(\text{MF}^R_{R_\infty - x^{n+1}}),$$

we define the matrix factorization

$$C\begin{pmatrix} \begin{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \end{pmatrix} & \begin{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \end{pmatrix} \end{pmatrix}_n \in \text{Ob}(\text{MF}^R_{R_\infty \otimes R_\infty}) \quad \text{by} \quad C\begin{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \end{pmatrix}_n \otimes C\begin{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \end{pmatrix}_n |_{x = x'}.$$
Proposition 3.2. \(1\) There is an isomorphism in \(\text{HMF}^\text{gr}_{Q[x] \otimes R, x^{n+1} - \omega}\), where \(R\) is a polynomial ring generated by boundary parameters except \(x\),

\[
C\left(\begin{array}{c}
\text{Diagram 1}
\end{array}\right)_n \cong C\left(\begin{array}{c}
\text{Diagram 2}
\end{array}\right)_n.
\]

\(2\) There is an isomorphism in \(\text{HMF}^\text{gr}_{Q[x] \otimes R, \omega - x^{n+1}}\), where \(R\) is a polynomial ring generated by boundary parameters except \(x\),

\[
C\left(\begin{array}{c}
\text{Diagram 3}
\end{array}\right)_n \cong C\left(\begin{array}{c}
\text{Diagram 4}
\end{array}\right)_n.
\]

\(3\) There is an isomorphism in \(\text{HMF}^\text{gr}_{Q, 0}\)

\[
C\left(\begin{array}{c}
\text{Diagram 5}
\end{array}\right)_n = K(\pi_{12}; x_1 - x_2)_{Q[x_1, x_2]} \bigg|_{x_1 = x_2} \cong \left(0 \to Q[x]/(x^n) \{1 - n\} \to 0\right).
\]

\(4\) There is an isomorphism in \(\text{HMF}^\text{gr}_{Q[x_1, x_2, x_5, x_6, x_1^{n+1} + x_2^{n+1} - x_5^{n+1} - x_6^{n+1}}\)

\[
C\left(\begin{array}{c}
\text{Diagram 6}
\end{array}\right)_n \cong C\left(\begin{array}{c}
\text{Diagram 7}
\end{array}\right)_n \{-1\} \oplus C\left(\begin{array}{c}
\text{Diagram 8}
\end{array}\right)_n \{1\}.
\]

\(5\) There is an isomorphism in \(\text{HMF}^\text{gr}_{Q[x_1, x_2], x_1^{n+1} - x_2^{n+1}}\)

\[
C\left(\begin{array}{c}
\text{Diagram 9}
\end{array}\right)_n \cong \bigoplus_{i=0}^{n-2} C\left(\begin{array}{c}
\text{Diagram 10}
\end{array}\right)_n \{2 - n + 2i\} \{1\}.
\]

\(6\) There is an isomorphism in \(\text{HMF}^\text{gr}_{Q[x_1, x_2, x_3, x_4], x_1^{n+1} - x_2^{n+1} + x_3^{n+1} - x_4^{n+1}}\)

\[
C\left(\begin{array}{c}
\text{Diagram 11}
\end{array}\right)_n \cong C\left(\begin{array}{c}
\text{Diagram 12}
\end{array}\right)_n \oplus \bigoplus_{i=0}^{n-3} C\left(\begin{array}{c}
\text{Diagram 13}
\end{array}\right)_n \{3 - n + 2i\}.
\]

\(7\) There is an isomorphism in \(\text{HMF}^\text{gr}_{Q[x_1, x_2, x_3, x_4, x_5, x_6, x_1^{n+1} + x_2^{n+1} + x_3^{n+1} - x_4^{n+1} - x_5^{n+1} - x_6^{n+1}}\)

\[
C\left(\begin{array}{c}
\text{Diagram 14}
\end{array}\right)_n \oplus C\left(\begin{array}{c}
\text{Diagram 15}
\end{array}\right)_n \cong C\left(\begin{array}{c}
\text{Diagram 16}
\end{array}\right)_n \oplus C\left(\begin{array}{c}
\text{Diagram 17}
\end{array}\right)_n.
\]

\[\text{Proof.}\] See [7].
The matrix factorization for the single loop \( \bigcirc \) is defined as the following matrix factorization:

\[
C\left( \bigcirc \right) = \left( 0 \to \mathbb{Q}[x]/(x^n) \to \{1-n\} \to 0 \right).
\]

Since the potential of the matrix factorization \( C\left( \bigcirc \right)_n \) is 0, we can take the homology group \( H \) of this matrix factorization.

**Corollary 3.3.** The Euler characteristic of the homology \( H(C(\bigcirc)_n) \) equals the value of sl\(_n\) quantum link invariant for the single loop:

\[
\chi(H(C(\bigcirc)_n)) = [n].
\]

**Proof.** See [7]. □

The map \( C \) from a planar diagram to a matrix factorization extends the map \( C \) from the projection diagram associated with a link to a complex of matrix factorizations.

\[
C\left( \begin{array}{c}
\text{projection diagram} \\
\text{single line}
\end{array} \right) := \left( \begin{array}{c}
C\left( \begin{array}{c}
\text{end point of a single line} \\
\text{and a double line}
\end{array} \right) \{n\} (1) \\
C\left( \begin{array}{c}
\text{end point of a single line}
\end{array} \right) \{n-1\} (1)
\end{array} \right) \to 0
\]

\[
C\left( \begin{array}{c}
\text{projection diagram} \\
\text{double line}
\end{array} \right) := \left( \begin{array}{c}
\text{end point of a single line}
\end{array} \right) \{0\} (1) \to C\left( \begin{array}{c}
\text{end point of a double line}
\end{array} \right) \{0\} (1)
\]

**Theorem 3.4.** [Khovanov-Rozansky, 7] The map \( C \) is an oriented link invariant in the homotopy category of a complex of matrix factorizations.

**Proof.** See [7]. □

3.2. Definition of matrix factorization for trivalent diagrams and double line. We extend the map \( C' \) for a planar diagram with two kinds of the boundaries, which consist of end points of a single line and a double line. We define matrix factorizations for the essential planar diagrams \( \searrow, \nearrow \), and \( \triangle \) as follows.

After this, we assume that \( n \geq 3 \). We assign the parameter \( x_i \) on the end point of a single line and the pair of parameters \((y_j, z_j)\) on the end point of a double line. And each \( \mathbb{Z} \)-grading of \( x_i \) and \( y_j \) also equals 2, but the \( \mathbb{Z} \)-grading of \( z_i \) equals 4:

\[
\deg (x_i) = 2, \quad \deg (y_i) = 2 \quad \text{and} \quad \deg (z_i) = 4.
\]

Define a matrix factorization for the first diagram, double line, by

\[
C\left( \begin{array}{c}
y_1, z_1 \\
y_2, z_2
\end{array} \right) = K\left( \begin{array}{c}
f(y_1, z_1) - f(y_2, z_1) \\
f(y_2, z_2) - f(y_2, z_1)
\end{array} \right) \left( \begin{array}{c}
y_1 - y_2 \\
z_1 - z_2
\end{array} \right) \mathbb{Q}[y_1, y_2, z_1, z_2]
\]

in \( \text{MF}^{gr}_{\mathbb{Q}[y_1, y_2, z_1, z_2], f(y_1, z_1) - f(y_2, z_2)} \).

Define a matrix factorization for the second trivalent diagram by

\[
C\left( \begin{array}{c}
y_3, z_3 \\
x_1, x_2
\end{array} \right) = K\left( \begin{array}{c}
f(y_3, z_3) - f(x_1 + x_2, z_3) \\
f(y_3, z_3) - f(x_1 + x_2, z_3)
\end{array} \right) \left( \begin{array}{c}
y_3 - x_1 - x_2 \\
z_3 - x_1 - x_2
\end{array} \right) \mathbb{Q}[x_1, x_2, y_3, z_3]
\]
in $\text{MF}^{gr}_{\mathbb{Q}[x_1,x_2,y_3,z_3], f(y_3,z_3)-z_3^{n+1}-x_3^{n+1}}$.

Define a matrix factorization for the third trivalent diagram by

$$C\left(\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}\right) = K\left(\begin{array}{cc}
\frac{f(x_1+x_2,x_1x_2)-f(y_3,x_1x_2)}{x_1+x_2-y_3} & \frac{y_3-x_1x_2}{x_1x_2-z_3}
\end{array}\right) \in \text{Ob}(\text{MF}^{gr}_{\mathbb{Q}[x_1,x_2,y_3,z_3], f(y_3,z_3)-z_3^{n+1}-x_3^{n+1}}).$$

The basic potential of a double line is the polynomial $f(y,z)$ obtained by expanding the power sum with the elementary symmetric polynomials. This polynomial $f(y,z)$ is a non-homogeneous polynomial, but it has the homogeneous $\mathbb{Z}$-grading $2n+2$.

Now, we consider two planar diagrams, which can match at end points of oriented double lines with keeping the orientation. For the matrix factorizations $C\left(\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}\right) \in \text{Ob}(\text{MF}^{gr}_{\mathbb{Q}[x_1,x_2,y_3,z_3], f(y_3,z_3)})$ and $C\left(\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}\right) \in \text{Ob}(\text{MF}^{gr}_{\mathbb{Q}[x_1,x_2,y_3,z_3], f(y_3,z_3)})$, the matrix factorization $C\left(\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}\right) \in \text{Ob}(\text{MF}^{gr}_{\mathbb{Q}[x_1,x_2,y_3,z_3], f(y_3,z_3)})$ is defined by the tensor product

$$C\left(\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}\right) \otimes C\left(\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}\right) \in \text{Ob}(\text{MF}^{gr}_{\mathbb{Q}[x_1,x_2,y_3,z_3], f(y_3,z_3)})$$

For a polynomial $f \in \mathbb{Q}[x_1,x_2,\ldots,x_k]$, the Jacobi algebra $J_f$ is defined as the quotient ring

$$\mathbb{Q}[x_1,x_2,\ldots,x_k]/\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_k} \rangle,$$

where $\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_k} \rangle$ is the ideal of $\mathbb{Q}[x_1,x_2,\ldots,x_k]$ generated by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_k}$.

**Proposition 3.5.**

1. There is an isomorphism in $\text{HMF}^{gr}_{\mathbb{Q}[y,z] \otimes R, f(y,z)-\omega}$, where $R$ is a polynomial ring generated by boundary parameters except $y$ and $z$,

$$C\left(\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}\right) \simeq C\left(\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}\right).$$

2. There is an isomorphism in $\text{HMF}^{gr}_{\mathbb{Q}[y,z] \otimes R, f(y,z)}$, where $R$ is a polynomial ring generated by boundary parameters except $y$ and $z$,

$$C\left(\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}\right) \simeq C\left(\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}\right).$$

3. There is an isomorphism in $\text{HMF}^{gr}_{\mathbb{Q}, 0}$

$$C\left(\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}\right) \simeq K\left(\begin{array}{cc}
\frac{f(y_1,z_1)-f(y_2,z_2)}{y_1-z_2} & \frac{y_1-z_2}{z_1-z_2}
\end{array}\right) \mid (y_1,z_1)=(y_2,z_2).$$

4. There is an isomorphism in $\text{HMF}^{gr}_{\mathbb{Q}[y_1,y_2,z_1,z_2], f(y_1,z_1)-f(y_2,z_2)}$

$$C\left(\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}\right) \simeq C\left(\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}\right) \otimes C\left(\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}\right) \{1\}.$$
(5) There is an isomorphism in $\text{HMF}^{gr}_{\mathbb{Q}[x_1, x_2, x_3, x_4, x_1^{n+1} + x_2^{n+1} - x_3^{n+1} - x_4^{n+1}]}$

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
x_1 & x_2 \\
\ar@{-}[rr] & & \ar@{-}[rr] \ar@{.}[rr] & & \ar@{-}[rr] \ar@{.}[rr] & & (y_5, z_5) \\
\}
\end{array}
\end{array}
\] 

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
x_1 & x_2 \\
\ar@{-}[rr] & & \ar@{-}[rr] \ar@{.}[rr] & & \ar@{-}[rr] \ar@{.}[rr] & & n \\
\}
\end{array}
\end{array}
\]

Proof. (1) The matrix factorization $C\left(\begin{array}{c} (y, z), \ \\
\end{array}\right)_n$ is described by

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
M_0 & M_1 & M_0 \\
\}
\end{array}
\end{array}
\] 

where $M_i$ is an $R \otimes \mathbb{Q}[x', y']$-module. Then, we have

\[
C\left(\begin{array}{c} (y, z), \ \\
\end{array}\right)_n = \left(\begin{array}{c} M_0 & M_1 & M_0 \\
\end{array}\right) \otimes \mathbb{K} \left(\begin{array}{c} f(y, z) - f(y', z) \ \\
\end{array}\right)_n \left(\begin{array}{c} y - y' \ \\
\end{array}\right)_n \mathbb{Q}[y, y', z, z].
\]

Since the potential of this matrix factorization does not include the parameters $y'$ and $z'$, we can apply Theorem 2.11 to $y - y'$ and $z - z$. Hence, we obtain the following isomorphic

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
M_0 & M_1 & M_0 \\
\}
\end{array}
\end{array} \otimes \mathbb{K} \left(\begin{array}{c} f(y, z) - f(y', z) \ \\
\end{array}\right)_n \left(\begin{array}{c} y - y' \ \\
\end{array}\right)_n \mathbb{Q}[y, y', z, z] 
\]

\[
\approx \left(\begin{array}{c} M_0 / (y - y', z - z) & M_1 / (y - y', z - z) & M_0 / (y - y', z - z) \\
\end{array}\right) 
\]

\[
\approx C\left(\begin{array}{c} (y, z), \ \\
\end{array}\right)_n.
\]

(2) This proof is similar to (1)

(3) The polynomial $f(y_1, z_1)$ is explicitly described as

\[
f(y_1, z_1) = y_1^{n+1} + (n + 1) \sum_{1 \leq 2i \leq n+1} (-1)^i \binom{n - i}{i - 1} y_1^{n+1-2i} z_1^i.
\]

Then, we have

\[
\frac{\partial f(y_1, z_1)}{\partial y_1} = (n + 1)y_1^n + (n + 1) \sum_{1 \leq 2i \leq n} (-1)^i(n + 1 - 2i) \binom{n - i}{i - 1} y_1^{n-2i} z_1^i,
\]

\[
\frac{\partial f(y_1, z_1)}{\partial z_1} = (n + 1) \sum_{1 \leq 2i \leq n+1} (-1)^i \binom{n - i}{i - 1} y_1^{n+1-2i} z_1^{i-1}.
\]

In the case that $n$ is even:

The sequence $\left(\frac{\partial f(y_1, z_1)}{\partial y_1}, \frac{\partial f(y_1, z_1)}{\partial z_1}\right)$ can be described as

\[
\left( (n + 1)y_1^n + (-1)^{\frac{n}{2}} (n + 1)z_1^{\frac{n}{2}} + p(y_1, z_1), -(n + 1)y_1^{n-1} + q(y_1, z_1) \right),
\]

where the polynomial degree $p(y_1, z_1)$ for $y_1$ satisfies

\[
\deg_{y_1}(p(y_1, z_1)) < n - 1
\]
and the polynomial degree for $z_1$ satisfies
\[ \deg_{z_1}(p(y_1, z_1)) < \frac{n}{2} \]
and the polynomial $q(y_1, z_1)$ satisfies
\[ \deg_{y_1}(q(y_1, z_1)) < n - 2 \]
and
\[ \deg_{z_1}(q(y_1, z_1)) < \frac{n}{2}. \]

It is easy to find that this sequence is a regular sequence. By Lemma 2.3 and Lemma 2.8, we have
\[ C(\bigcirc_{y_1}(y_1, z_1))_n \simeq K \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} -(n + 1)y_1^n - (-1)^\frac{n}{2}(n + 1)z_1^\frac{n}{2} - p(y_1, z_1) \\ (n + 1)y_1^{n-1} - q(y_1, z_1) \end{pmatrix} \right)_{\mathbb{Q}[y_1, z_1]} \{1 - n\} \{3 - n\} \]
\[ \simeq K \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} -(n + 1)y_1^n - (-1)^\frac{n}{2}(n + 1)z_1^\frac{n}{2} - p(y_1, z_1) \\ (n + 1)y_1^{n-1} - q(y_1, z_1) \end{pmatrix} \right)_{\mathbb{Q}[y_1, z_1]} \{4 - 2n\}. \]

We apply Theorem 2.10 to the polynomial $(n + 1)y_1^{n-1} - q(y_1, z_1)$ for $y_1$. Then, we obtain
\[ K \left( 0; -(-1)^\frac{n}{2}(n + 1)z_1^\frac{n}{2} - y_1q(y_1, z_1) - p(y_1, z_1) \right)_{\mathbb{Q}[y_1, z_1]/\langle (n + 1)y_1^{n-1} - q(y_1, z_1) \rangle} \{4 - 2n\}. \]

Since we have $\mathbb{Q}[y_1, z_1]/\langle (n + 1)y_1^{n-1} - q(y_1, z_1) \rangle \simeq \mathbb{Q}[y_1]/\langle y_1^{n-1} \rangle \{z_1\}$, $\deg_{z_1}(y_1q(y_1, z_1) + p(y_1, z_1)) < \frac{n}{2}$ and the sequence $\langle (n + 1)y_1^{n-1} - q(y_1, z_1) \rangle$ is regular, we can apply Theorem 2.10 to the polynomial $(-1)^\frac{n}{2}(n + 1)z_1^\frac{n}{2} - y_1q(y_1, z_1) - p(y_1, z_1)$ for $z_1$. Hence, we obtain
\[ C(\bigcirc_{y_1}(y_1, z_1))_n \simeq J_{f(y_1, z_1)} \{4 - 2n\} \rightarrow 0 \rightarrow J_{f(y_1, z_1)} \{4 - 2n\}. \]

In the case that $n$ is odd:
The sequence $\left( \frac{\partial f(y_1, z_1)}{\partial y_1}, \frac{\partial f(y_1, z_1)}{\partial z_1} \right)$ can be described as
\[ ((n + 1)y_1^n + p(y_1, z_1), -(n + 1)y_1^{n-1} + (-1)^\frac{n+1}{2}(n + 1)z_1^\frac{n+1}{2} + q(y_1, z_1)), \]
where the polynomial degree $p(y_1, z_1)$ for $y_1$ satisfies
\[ \deg_{y_1}(p(y_1, z_1)) < n - 1 \]
and the polynomial degree for $z_1$ satisfies
\[ \deg_{z_1}(p(y_1, z_1)) < \frac{n + 1}{2} \]
and the polynomial $q(y_1, z_1)$ satisfies that
\[ \deg_{y_1}(q(y_1, z_1)) < n - 2 \]
and
\[ \deg_{z_1}(q(y_1, z_1)) < \frac{n - 1}{2}. \]
By Lemma 2.3 and Lemma 2.8 we have
\[ C\left(\bigodot_{n}(y_1, z_1)\right) \cong K \left( \begin{array}{c}
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\end{array} \right) \begin{pmatrix}
-(n+1)y_1^n - p(y_1, z_1) \\
(n+1)y_1^n - (n+1)z_1^{n-1} - q(y_1, z_1)
\end{pmatrix} \} \{1 - n\} \{3 - n\}.\]

Now, we have that \(\deg_{y_1}(p(y_1, z_1)) < n - 1\), \(\deg_{z_1}(q(y_1, z_1) - (n + 1)y_1^{n-1}) < \frac{n - 1}{2}\) and the sequence \((\frac{\partial f(y_1, z_1)}{\partial y_1}, \frac{\partial f(y_1, z_1)}{\partial z_1})\) is regular. Thus, after we apply Theorem 2.10 to the polynomial \(z_1 \rightarrow (n + 1)y_1^n - (n+1)z_1^{n-1} - q(y_1, z_1)\) for \(z_1\). We also obtain
\[ C\left(\bigodot_{n}(y_1, z_1)\right) \cong J_{f(y_1, z_1)}(4 - 2n) \}
\]

(4) By definition, we have
\[ C\left(\bigodot_{n}(x_3, y_4)\right) \cong K \left( \begin{array}{c}
\begin{pmatrix}
\frac{f(y_1, z_1)}{f(x_3 + x_4, z_1)} \\
\frac{f(x_3 + x_4, z_1)}{f(x_3 + x_4, x_4)} \\
\frac{f(x_3 + x_4, x_4)}{f(y_2, x_2 + x_4)} - \frac{y_2}{x_2 x_4 - z_2}
\end{pmatrix}
\end{array} \right) \begin{pmatrix}
y_1 - x_3 - x_4 \\
z_1 - x_3 x_4 \\
x_3 + x_4 - y_2 \\
x_3 x_4 - z_2
\end{pmatrix} \}_{Q[y_1, z_1, y_2, z_2, x_3, x_4]}.\]

Since the potential of this matrix factorization does not include the parameter \(x_3\) and \(x_4\), we can apply Theorem 2.10 to the polynomial \(y_1 - x_3 - x_4\) for \(x_3\). Thus, this matrix factorization is equivalent to the following matrix factorization
\[ K \left( \begin{array}{c}
\begin{pmatrix}
\frac{f(y_1, z_1)}{f(x_3 + x_4, z_1)} \\
\frac{f(x_3 + x_4, z_1)}{f(y_2, x_2 + x_4)} - \frac{y_2}{x_2 x_4 - z_2}
\end{pmatrix}
\end{array} \right) \begin{pmatrix}
z_1 + x_2^2 - y_1 x_4 \\
y_1 - y_2 \\
-x_2^2 + y_1 x_4 - z_2
\end{pmatrix} \}_{Q[y_1, z_1, y_2, z_2, x_3, x_4]/(y_1 - x_3 - x_4)}.\]

Since the quotient ring \(Q[y_1, z_1, y_2, z_2, x_3, x_4]/(y_1 - x_3 - x_4)\) is isomorphic to \(Q[y_1, z_1, y_2, z_2, x_3, x_4]\), we once apply Theorem 2.10 to the polynomial \(z_1 + x_2^2 - y_1 x_4\) for \(x_4\). Then, this matrix factorization is equivalent to the following matrix factorization
\[ K \left( \begin{array}{c}
\begin{pmatrix}
\frac{f(y_1, z_1)}{f(y_2, z_2)} \\
\frac{f(y_1, z_1)}{f(y_2, z_2)}
\end{pmatrix}
\end{array} \right) \begin{pmatrix}
y_1 - y_2 \\
z_1 - z_2
\end{pmatrix} \}_{Q[y_1, z_1, y_2, z_2, x_3, x_4]/(z_1 + x_2^2 - y_1 x_4)}.\]

The quotient ring \(Q[y_1, z_1, y_2, z_2, x_3, x_4]/(z_1 + x_2^2 - y_1 x_4)\) \(-1\) is isomorphic to \(Q[y_1, z_1, y_2, z_2, x_3, x_4]\) \{-1\} \oplus \(Q[y_1, z_1, y_2, z_2] x_4 \{-1\}\). Furthermore, this ring is equivalent to \(Q[y_1, z_1, y_2, z_2] \{-1\} \oplus Q[y_1, z_1, y_2, z_2] \{1\}\).
as a $\mathbb{Z}$-graded $\mathbb{Q}[y_1, z_1, y_2, z_2]$-module. Hence, we obtain the following matrix factorization

$$K\left(\begin{pmatrix}
\frac{f(y_2, z_2)}{y_1 - y_2} & \frac{f(y_2, z_2)}{z_1 - z_2} \\
\frac{y_1 - y_2}{y_1 - y_2} & \frac{z_1 - z_2}{z_1 - z_2}
\end{pmatrix} : \begin{pmatrix}
y_1 - y_2 \\
z_1 - z_2
\end{pmatrix}\right) \{1\}.$$

This is the right-hand side of the equivalent of Proposition (4).

(5) By definition, we have

$$\text{deg} y_5 = 10 YASUYOSHI YONEZAWA$$

Since the potential of the matrix factorization does not include the parameter $y_5$ and $z_5$, we apply Theorem 2.10 to the polynomials $y_5 - x_3 - x_4$ and $z_5 - x_3 x_4$. Then, it is easy to find that we obtain the right-hand side of the equivalent of Proposition (5).

A matrix factorization for the double loop $\bigcirc$ is defined as the above matrix factorization:

$$C\left(\bigcirc\right)_n = \left(J_{f(y, z)}\{4 - 2n\} \to 0 \to J_{f(y, z)}\{4 - 2n\}\right).$$

Since the potential of the matrix factorization $C\left(\bigcirc\right)_n$ is 0, we can take the homology group $H$ of this matrix factorization.

**Corollary 3.6.** The Euler characteristic of the homology $H(C\left(\bigcirc\right)_n)$ equals the value of $\mathcal{A}_n$ quantum link invariant for the double loop;

$$\chi(H(C\left(\bigcirc\right)_n)) = \frac{n(n - 1)}{2}.$$

**Proof.** By proof of the above Proposition 3.5 (3), we have

$$J_{f(y, z)} \simeq \left\{
\begin{array}{ll}
\mathbb{Q}[y, z]/\langle y^{n-1}, z^{n+1} \rangle & n : \text{even} \\
\mathbb{Q}[y, z]/\langle y^n, z^{n+1} \rangle & n : \text{odd}
\end{array}\right.$$

(i) $n : \text{even}$

Since $\deg_z(y_1) = 2$ and $\deg_z(z_1) = 4$, we have

$$\chi(H(C\left(\bigcirc\right)_n)) = \chi(J_{f(y, z)}\{4 - 2n\})$$

$$= \sum_{i=0}^{n-2} \sum_{j=0}^{n-2} q^{4-2n+2i+4j}$$

$$= \sum_{i=0}^{n-2} q^{2-n+2i} \sum_{j=0}^{n-2} q^{2-n+4j}$$

$$= (q^{2-n} + q^{4-n} + q^{6-n} + \ldots + q^{n-4} + q^{n-2})(q^{2-n} + q^{6-n} + q^{10-n} + \ldots + q^{n-6} + q^{2-n})$$

$$= \frac{n(n - 1)}{2}.$$
(ii) $n$ : odd

We have

$$\chi \left( H(C'(\bigotimes_n)) \right) = \chi \left( J_{f(y,z)} \{4 - 2n\} \right)$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{\frac{n-3}{2}} q^{4-2n+2i+4j}$$

$$= \sum_{i=0}^{n-1} q^{4-2i+4n-3+4j}$$

$$= \left( q^{1-n} + q^{3-n} + q^{5-n} + \ldots + q^{n-3} + q^{n-1} \right) \left( q^{3-n} + q^{7-n} + q^{11-n} + \ldots + q^{n-7} + q^{n-3} \right)$$

$$= \left[ \frac{n}{2} \right] \left[ \frac{n-1}{2} \right]$$.

□

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