Streaming with Oracle: New Streaming Algorithms for Edit Distance and LCS

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Abstract

The edit distance (ED) and longest common subsequence (LCS) are two fundamental problems which quantify how similar two strings are to one another. In this paper, we consider these problems in the streaming model where one string is available via oracle queries and the other string comes as a stream of characters. Our main contribution is a constant factor approximation algorithm in this setting for ED with memory $O(n^\delta)$ for any $\delta > 0$. In addition to this, we present an upper bound of $\tilde{O}(\sqrt{n})$ on the memory needed to approximate ED or LCS within a factor $1 + o(1)$ in our setting. All our algorithms run in a single pass.

For approximating ED within a constant factor, we discover yet another application of triangle inequality, this time in the context of streaming algorithms. Triangle inequality has been previously used to obtain subquadratic time approximation algorithms for ED. Our technique is novel and elegantly utilizes triangle inequality to save memory at the expense of an exponential increase in the runtime.

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1 Introduction

We consider edit distance (ED) and longest common subsequence (LCS) which are classic problems measuring the similarity between two strings. Edit distance is defined on two strings $s$ and $\bar{s}$ and seeks the smallest number of character insertions, character deletions, and character substitutions to transform $s$ into $\bar{s}$. While in edit distance the goal is to make a transformation, longest common subsequence asks for the largest string that appears as a subsequence in both $s$ and $\bar{s}$.

Edit distance and longest common subsequence have applications in various contexts, such as computational biology, text processing, compiler optimization, data analysis, image analysis, among others. As a result, both problems have been subject to a plethora of studies since 1950 (e.g. see e.g. \cite{17, 18, 2, 12, 11, 10, 42, 15, 27, 29, 41, 49, 28, 36, 46, 20, 5, 23, 55, 21, 54, 43, 34, 32, 50, 3, 22, 39, 7, 9, 8, 6, 13, 33, 25, 45, 56, 37, 4, 19, 38, 23}).

Both of the problems are often used to measure the similarity of large strings. For example, a human genome consists of almost three billion base pairs that are modeled as a string for similarity testing. Classic algorithms for the problems require quadratic runtime as well as linear memory to find a solution. Unfortunately, none of these bounds seem practical for real-world applications. Therefore, recent work on ED and LCS focus on obtaining fast algorithms \cite{53, 38, 18, 11, 10, 7, 8, 52, 51, 44, 23} as well as solutions with small memory \cite{40, 18, 24, 35}.

The streaming setting is an increasingly popular framework to model memory constraints. In this setting, the input arrives as a stream data while only sublinear memory is available to the algorithm. The goal is to design an algorithm that solves/approximates the solution by reading the input in a few rounds\footnote{Typically only a single round is allowed.}. While several works have studied ED and LCS in the streaming model (see Section 1.1 for a detailed discussion), positive results are known only for the low-distance regime \cite{48, 58, 16, 24}. For ED and LCS in particular, one clear shortcoming of the streaming model is that local access to the input may be too little in return for a a global alignment between the two strings. In other words, for two strings $s$ and $\bar{s}$, it seems to be overly optimistic to wish to find out which interval of $\bar{s}$ corresponds to a small fragment of $s$ (which is available to the algorithm) provided only local access to characters of $\bar{s}$. In addition to this, strong lower bounds are given for the streaming variant of LCS \cite{48, 58}.

To alleviate this problem, we introduce a streaming variant of the problems that does not suffer from the above issue. In our model, one of the strings ($\bar{s}$) is available via oracle queries and the other string ($s$) comes as in the classic streaming setting. In other words, while characters of $s$ come as a stream of data, at each point in time, we can access every element $i$ of $s$ ($s[i]$) via a value query to the oracle. Similar to the streaming model, the memory available to the algorithm is sublinear in our setting.

We present a constant factor approximation algorithm for ED that uses only $O(n^\delta)$ memory for any constant $\delta > 0$. In addition to this, we show that with memory $\tilde{O}(\sqrt{n})$ one can approximate both ED and LCS within a factor $1 + o(1)$. All our algorithms run in a single round. Moreover, our algorithm for LCS is tight due to a lower bound given in \cite{31}.

LIS and distance to monotonicity (DTM) are special cases of LCS and ED that are also studied in the streaming model \cite{35}. In these two problems, one of the strings is a permutation of numbers in $[n]$ and the second string is the sorted permutation $(1, 2, \ldots, n)$. In our setting, oracle queries are no longer needed for these special cases; $\bar{s}[i]$ is always equal to $i$. Therefore, for these problems, our setting is the same as the classic streaming setting and therefore our algorithms can be seen as a generalization of previous work on streaming LIS and distance to monotonicity.
1.1 Related work

Quadratic time solutions for ED and LCS have been known for many decades [47]. Recently, it has been shown that a truly subquadratic time solution for either ED or LCS refutes Strong Exponential Time Hypothesis (SETH), a conjecture widely believed in the community (see [12, 2, 22]). Therefore, much attention is given to approximation algorithms for the two problems. For edit distance, a series of works [46], [14], [15], and [11] improve the approximation factor culminating in the seminal work of Andoni, Krauthgamer, and Onak [10] that finally obtains a polylogarithmic approximation factor in near-linear time. More recently constant factor approximation algorithms with truly subquadratic runtimes are obtained for edit distance (a question which was open for a few decades): first a quantum algorithm [18], then a classic solution [23], and finally for far strings, near linear time solutions are also given [51, 44]. LCS has also received tremendous attention in recent years [38, 52, 53, 1, 4, 26]. Only trivial solutions were known for LCS until very recently: a 2 approximate solution when the alphabet is 0/1 and an \( O(\sqrt{n}) \) approximate solution for general alphabets in linear time. Both these bounds are recently improved by Hajiaghayi et al. [38] and Rubinstein and Song [52] (see also a recent approximation algorithms given by Rubinstein et al. [53]).

Streaming algorithms for edit distance have been limited to the case that the distance between the two strings is bounded by a parameter \( k \) which is substantially smaller than \( n \). A parameterized streaming algorithm that makes one-pass over its input \( s \) and \( \bar{s} \) with space \( O(k^6) \) (which can be as large as the input size) and running time \( O(n + k^6) \) [24] (STOC’16) is presented recently as well.

1.2 Preliminaries

For a string \( s \), we use \( s[i] \) to denote the \( i \)th character in \( s \). We use \( s[i,j] \) to denote the substring of \( s \) from the \( i \)th character to \( j \)th character. We also use \( s[i,j] \) to denote the substring of \( s \) from the \( i \)th character to \( (j - 1) \)th character (\( s[i,i] \) is an empty string). We use a similar notation for \( \bar{s} \).

Given two strings \( s \) and \( \bar{s} \), the longest common subsequence (LCS) of \( s \) and \( \bar{s} \) is a string \( t \) with the maximum length such that \( t \) is a subsequence of both \( s \) and \( \bar{s} \). In other words, \( t \) can be obtained from both \( s \) and \( \bar{s} \) by removing some of the characters. We use \( \text{lcs}(s, \bar{s}) \) to denote the length of the LCS of two strings \( s \) and \( \bar{s} \). The edit distance (ED) between two strings \( s \) and \( \bar{s} \), denoted by \( \text{ed}(s, \bar{s}) \), is the minimum number of character insertions, deletions, and substitutions needed to transform one string to the other string.

**Streaming model.** Throughout this paper, we assume that the input of the algorithm consists of two strings \( s \) and \( \bar{s} \). We assume for simplicity and without loss of generality that the two strings have equal length \( n \). We call the string \( s \) the **offline** string and assume the algorithm has a random access to the characters of \( s \) by making a query on an oracle. In our model, one of the strings comes as a stream of characters and the other one is available via oracle queries. We refer to the first one by \( s \) and the second one by \( \bar{s} \).

| problem | approximation factor | memory | reference |
|---------|----------------------|--------|-----------|
| ED      | \( 2^{1/6} \)        | \( O(n^6) \) | Theorem 2.4 |
| ED      | \( 1 + \epsilon \)   | \( O(\sqrt{n}) \) | Theorem 4.1 |
| LCS     | \( 1 + \epsilon \)   | \( O(\sqrt{n}) \) | Theorem 3.4 |
| LIS     | \( 1 + \epsilon \)   | \( O(\sqrt{n}) \) | 35 |
| DTM     | \( 1 + \epsilon \)   | \( O(\sqrt{n}) \) | 35 |
| DTM     | 4                    | \( O(\log^2 n) \) | 35 |
| DTM     | 2                    | \( O(\log^2 n) \) | 30 |

Table 1: The results of this paper along with previous work are summarized in this table.
1.3 Our Technique: Triangle Inequality

As mentioned earlier, our main result is an algorithm with memory $O(n^\delta)$ for any $\delta > 0$ that approximates edit distance within a constant factor in our model. When the available memory is limited, a typical approach to approximating edit distance is to break each of the strings into smaller pieces and find a solution in which each piece of a string is entirely transformed into another piece of the other string. Such solutions have been referred to as “window-compatible solutions” [18] or “matching between candidate intervals” in previous work [39] (a similar technique is also used in [23] to obtain a constant-factor approximate solution for ED). One should construct the pieces in a way that there always exists such a solution whose approximation factor is bounded. Previous work give several constructions with small approximation factors [18, 23, 39, 53].

Let us refer to these pieces as windows and to such solutions as window-compatible solutions. It is not hard to see that if the edit distance between every pair of windows is available, then one can find an optimal window-compatible solution without any knowledge of the strings. That is, just knowing the distances between the windows suffices to find the optimal window-compatible solution. On the other hand, computing the edit distance between each pair of windows requires memory proportional to the window sizes. Therefore, a convenient way to design a memory-efficient algorithm (in certain settings such as MPC) is to give a construction for the windows in which the maximum window size is small and that it guarantees the existence of an almost optimal window-compatible solution.

The problem becomes more challenging in our streaming setting as the online string ($s$) is only available in a single pass. Therefore, when the characters of a window of $s$ are stored in the memory, we have to use that information immediately to compute the edit distance of that particular window with all windows of the offline string. If the maximum window size is $\ell$, then we need memory $\Omega(\ell)$ for that purpose. Moreover, the number of windows for such a construction should be at least $\Omega(n/\ell)$, otherwise some parts of the strings are not included in any window and such a construction cannot guarantee any approximation factor. Thus, one needs to keep track of $O(n/\ell)$ values for each window of the online string, determining its distance from the windows of the offline string. Roughly speaking, this suggests that this approach can only take us as far as obtaining a solution with memory $O(\sqrt{n})$. We more formally show in Section 4 that this technique leads to a solution with approximation factor $1 + o(1)$ and memory $O(\sqrt{n})$.

Triangle inequality is the key to improving the memory of the algorithm. The key idea is summarized in the following: consider a window $w$ of the online string for which we would like to store its distance from all windows of the offline string. Instead of directly storing these values, we find a substring $[\ell, r]$ of the offline string whose edit distance is the smallest to $w$. Let the distance be $d$. We only keep 3 integer numbers $\ell$, $r$, $d$ for this window. Surprisingly, these 3 numbers suffice to recover a 3-approximate solution for the edit distance of $w$ from any substring of the offline strings (including all the windows) without even knowing $w$! More precisely, whenever the distance of $w$ from an interval $\tilde{s}[\ell', r']$ of the offline string is desired, we approximate $\text{ed}(w, \tilde{s}[\ell', r'])$ by $d + \text{ed}(\tilde{s}[\ell, r], \tilde{s}[\ell', r'])$. It is not hard to see by triangle inequality that $d + \text{ed}(\tilde{s}[\ell, r], \tilde{s}[\ell', r'])$ is at least as large and at most $3$ times larger than the actual distance between $\tilde{s}[\ell', r']$ and $w$. Moreover, both substrings $\tilde{s}[\ell', r']$ and $\tilde{s}[\ell, r]$ are available via oracle queries since they both belong to the offline string. Finally, when two windows of the offline string are available via oracle queries, we show that their edit distance can be computed with polylogarithmic memory.

To improve the memory of the algorithm down to $O(n^\delta)$ for any $\delta > 0$, we recursively apply the above idea to make the window sizes smaller in every recursion. This comes at the expense of a multiplicative factor of 3 in the approximation for each level of recursion. Finding the optimal window-compatible solution for our setting is also cumbersome due to memory constraints. Instead of determining that with dynamic programming, we use a brute force. This takes a significant hit
on the runtime of the algorithm while keeping the memory small. More details about this algorithm is given in Section 2.

Theorem 1.1. Given an offline and online strings of length $n$ and any constant $\delta > 0$, there exists a streaming algorithm that finds a $O(2^{1/\delta})$ approximation of the edit distance using $\tilde{O}(n^\delta/\delta)$ memory.

## 2 Constant Approximation for Edit Distance

One of our main results is a streaming algorithm that given any constant $\delta > 0$ finds a constant approximation of the edit distance using $\tilde{O}(n^\delta)$ memory. Instead of directly approximating the edit distance, we define the following subproblem and we show that how solving this subproblem also gives us a good approximation of ED. In this subproblem, given an offline string $\bar{s}$ and an online string $s$, the goal is to find a substring of $\bar{s}$ with the minimum edit distance from $s$.

**Closest Substring**

**Input:** An offline string $\bar{s}$ and an online string $s$.

**Result:** Indices $l, r$ and $ed(\bar{s}[l, r], s)$ such that $ed(\bar{s}[l, r], s) \leq ed(\bar{s}[i, j], s)$ for every $1 \leq i \leq j \leq n$.

We first show that how solving the closest substring problem can gives us a good approximation of the edit distance. Let $\bar{s}[l, r]$ be the substring of $\bar{s}$ with the minimum edit distance from $s$. By definition edit distance meets the triangle inequality$^2$. By the triangle inequality we have

$$ed(\bar{s}, s) \leq ed(\bar{s}, \bar{s}[l, r]) + ed(\bar{s}[l, r], s).$$

We also have,

$$ed(\bar{s}, \bar{s}[l, r]) + ed(\bar{s}[l, r], s) \\ \leq ed(\bar{s}, s) + ed(\bar{s}[l, r], s) + ed(\bar{s}[l, r], s) \quad \text{By the triangle inequality.} \\ \leq 3ed(\bar{s}, s) \quad \text{Since } \bar{s}[l, r] \text{ has the minimum ED from } s. \quad (2)$$

It follows from (1) and (2) that $ed(\bar{s}, \bar{s}[l, r]) + ed(\bar{s}[l, r], s)$ gives us a $3$-approximation for the edit distance of $\bar{s}$ and $s$. Therefore, if we design a streaming algorithm that finds $\bar{s}[l, r]$ and its edit distance from $s$, then we can estimate the edit distance of $s$ and $\bar{s}$ by computing $ed(\bar{s}, \bar{s}[l, r]) + ed(\bar{s}[l, r], s)$. The following theorem shows that the edit distance of any two substrings of the offline string can be computed using a very small memory $O(\log^2 n)$.

**Theorem 2.1.** For given two strings $s$ and $\bar{s}$ of length $n$, suppose we have a random access to the characters of $s$ and $\bar{s}$ using a oracle. Then $\text{lcs}(s, \bar{s})$ and $ed(s, \bar{s})$ can be computed using $O(\log^2 n)$ memory.

Therefore, by finding the substring that has the minimum edit distance from $s$, we can approximate the edit distance. We design a streaming algorithm with the memory of $n^\delta$ to approximate this subproblem. Given an online string, we divide the online string into $n^{1-\delta}$ blocks of size $n^\delta$. Our algorithm (formally as Algorithm 1), then recursively finds substrings of $\bar{s}$ that have the minimum edit distance from each of these blocks. Note that for each block we can store the result of solving the closest substring problem in $O(\log n)$ (We can store only three numbers which are the start and the end of the interval and the edit distance from the online string). Therefore, by the end of all recursive calls the memory that our algorithm has been used is $\tilde{O}(n^\delta)$.

$^2 ed(s_1, s_3) \leq ed(s_1, s_2) + ed(s_2, s_3)$ for any strings $s_1, s_2, s_3$. 

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Then, in order to find the solution of the closest substring problem using these partial solutions, our algorithm considers all different substrings $\bar{s}[l, r]$ of $\bar{s}$. It then considers all different mappings between the blocks of the online string and substrings of $\bar{s}[l, r]$. Then, for any mapping it estimates the edit distance between a block of the online string and its mapped substring of $\bar{s}[l, r]$, using the solution of the closest substring problem that we have found for each block.

We first show that finding any approximation of the closest substring problem, can yield us an approximation for the edit distance. We first define an approximate version of the closest substring problem as follows.

**Algorithm 1:** Algorithm APPROXIMATE-CLOSEST-SUBSTR for approximating ED.

**Data:** An offline string $\bar{s}$ of length $n$, a stream of characters of the online string $s$, and a parameter $\delta > 0$.

1. if $|s| \leq n\delta$ then
2. Store all characters of $s$ in the memory.
3. Find a substring of $\bar{s}$ that has the minimum edit distance from $s$. Let $\bar{s}[l, r]$ be this substring and $d$ be its edit distance.
4. return $l$, $r$ and $d$.
5. else
6. $\xi \leftarrow n\delta$.
7. Divide $s$ into $\xi$ blocks $s^*_1, s^*_2, \ldots, s^*_\xi$ of size $|s|/\xi$.
8. for $i \in [\xi]$ do
9. Recursively find the closest substring of $\bar{s}$ from $s^*_i$. Let $l_i, r_i$ be the start and the end of this substring respectively, and $d_i$ be the approximate edit distance of this substring from $s^*_i$.
10. $\text{min}_\text{dist} \leftarrow \infty$.
11. for $1 \leq p_0 \leq p_1 \leq \ldots \leq p_\xi \leq n + 1$ do
12. $\text{dist} = \sum_{i=1}^{\xi} d_i + \text{ed}(\bar{s}[p_{i-1}, p_i], \bar{s}[l_i, r_i])$.
13. if $\text{dist} < \text{min}_\text{dist}$ then
14. $\text{min}_\text{dist} \leftarrow \text{dist}$.
15. $l \leftarrow p_0$.
16. $r \leftarrow p_\xi - 1$.
17. return $l$, $r$ and $\text{min}_\text{dist}$.

**Definition 2.2.** Given an offline string $\bar{s}$ and online string $s$, we say that the substring $\bar{s}[l, r]$ along with the edit distance $d$ is an $\alpha$-approximation for the closest substring problem if for any substring $\bar{s}[l^*, r^*]$ we have

$$\text{ed}(\bar{s}[l, r], s) \leq d \leq \alpha \cdot \text{ed}(\bar{s}[l^*, r^*], s).$$

(3)

In the following claim we show that we can use any $\alpha$-approximation of the closest substring problem to get a $O(\alpha)$-approximation for the edit distance.

**Claim 2.3.** Let $\bar{s}[l, r]$ be the $\alpha$-approximation solution of the closest substring problem and let $d$ be the approximate edit distance between $\bar{s}[l, r]$ and $s$. Then for any substring $\bar{s}[l^*, r^*],$

$$d + \text{ed}(\bar{s}[l, r], \bar{s}[l^*, r^*])$$

gives $(2\alpha + 1)$-approximation for the edit distance between $\bar{s}[l^*, r^*]$ and $s$. (2)
Proof. First we show that \((d + \text{ed}(\bar{s}[l], \bar{s}[l^*, r^*]))\) is not less than the edit distance between \(\bar{s}[l^*, r^*]\) and \(s\).

\[
d + \text{ed}(\bar{s}[l], \bar{s}[l^*, r^*]) \geq \text{ed}(\bar{s}[l], \bar{s}[l^*, s]) + \text{ed}(\bar{s}[l], \bar{s}[l^*, r^*])
\]

By \((3)\).

\[
\geq \text{ed}(\bar{s}[l^*, r^*], s) .
\]

By the triangle inequality.

We now show that the value of \((d + \text{ed}(\bar{s}[l], \bar{s}[l^*, r^*]))\) is at most \((2\alpha + 1)\) \(\text{ed}(s, \bar{s}[l^*, r^*])\). Thus it gives us a \((2\alpha + 1)\)-approximation of the edit distance. We have

\[
d + \text{ed}(\bar{s}[l], \bar{s}[l^*, r^*]) \leq d + \text{ed}(s, \bar{s}[l^*, r^*]) + \text{ed}(s, \bar{s}[l^*, r^*])
\]

By the triangle inequality.

\[
\leq d + \alpha \cdot \text{ed}(s, \bar{s}[l^*, r^*]) + \text{ed}(s, \bar{s}[l^*, r^*])
\]

By \((3)\).

\[
= d + (\alpha + 1) \text{ed}(s, \bar{s}[l^*, r^*])
\]

\[
\leq \alpha \cdot \text{ed}(s, \bar{s}[l^*, r^*]) + (\alpha + 1) \text{ed}(s, \bar{s}[l^*, r^*])
\]

By \((3)\).

\[
= (2\alpha + 1) \text{ed}(s, \bar{s}[l^*, r^*]) ,
\]

which completes the proof of the claim. \(\square\)

Based on these observations, we now design an algorithm that finds a constant approximation of the edit distance using \(n^\delta\) memory for any \(\delta > 0\). The algorithm first divides the online string into \(n^\delta\) blocks with the equal length. Therefore, the length of each block is \(n^{1-\delta}\). It then finds the solution of the closest substring problem recursively. By Claim 2.3, we can use the approximate solution of the closest substring problem for each block, to find its edit distance from every other substring of the offline string. The algorithm uses these approximations to approximate the edit distance between the entire online string and any substring of the offline string.

Note that by each recursive call the length of the online string will get smaller by a multiplicative factor of \(n^{-\delta}\). Therefore, when the depth of the recursive calls becomes \(1/\delta\), the length of the remaining online string is bounded by \(O(n^\delta)\) and we can store all of this block in the memory and find the exact solution of the closest substring problem. The following theorem shows that the approximation ratio of our algorithm is \(O(2^{1/\delta})\).

**Theorem 2.4.** Given an offline string \(\bar{s}\), an online string \(s\) and any constant \(\delta > 0\), let \(n\) be the length of the offline string and \(n^\gamma\) be the length of the online string where \(\gamma > 0\). Then, Algorithm 1 finds a \(O(2^{\gamma/\delta})\) approximation for the closest substring problem.

**Proof.** We use induction on the length of the online string to prove the theorem. In specific, using induction on \(\gamma\) we show that the approximation ratio of the algorithm on a online string with the length of at most \(n^\gamma\) is bounded by \(2^{(\gamma-\delta)/\delta}\). Therefore, if the length of the online string is at most \(n^\delta\), then the algorithm stores all of the characters of the online string and find the exact solution. In other words, for \(\gamma \leq \delta\), the algorithm finds the exact solution and the induction clearly holds.

Otherwise, we can assume the length of the online string is \(n^\gamma\) where \(\gamma > \delta\). In that case the algorithm divides the online string into \(n^\delta\) blocks of equal length. For the simplicity of the presentation, we assume that the length of the online string is divisible by \(n^\delta\). Therefore, the algorithm divides \(s\) into \(n^\delta\) blocks \(s_1^\delta, s_2^\delta, \ldots, s_{n^\delta}^\delta\) each with the length of \(n^{\gamma-\delta}\). The algorithm then recursively finds the closest substring problem for each of these blocks. For the block \(s_i^\delta\), let \(\bar{s}[l_i, r_i]\) be the substring returned by the algorithm and let \(d_i\) be its approximate edit distance from \(s_i^\delta\).

By the induction hypothesis we have that the approximation ratio of the solution for each block is bounded by

\[
2^{(\gamma-\delta)/\delta} - 1 = 2^{\gamma/\delta} - 1 .
\]
Let \( \bar{s}[l^*, r^*] \) be an arbitrary substring of \( \bar{s} \). Consider the optimal mapping between \( s_i^* \)'s blocks and \( \bar{s}[l^*, r^*] \). Let assume that in the optimal mapping, the block \( s_i^* \) is mapped to \( \bar{s}[p_{i-1}^*, p_i^*] \) where we have

\[
l^* = p_0^* \leq p_1^* \leq \cdots \leq p_n^* = r^* + 1.
\]

Then, we have

\[
\text{ed}(s, \bar{s}[l^*, r^*]) = \sum_{i=1}^{n^\delta} \text{ed} \left( s_i^*, \bar{s}[p_{i-1}^*, p_i^*] \right). \tag{4}
\]

Recall that for each block \( s_i^* \), the substring \( \bar{s}[l_i, r_i] \) and the distance \( d_i \) is a \((2^{\gamma/\delta} - 1)\) approximation of the closest substring problem for this block. Therefore by Claim 2.3 we have

\[
d_i + \text{ed} \left( \bar{s}[l_i, r_i], \bar{s}[p_{i-1}^*, p_i^*] \right) \leq (2^{\gamma/\delta} + 1) \text{ed} \left( s_i^*, \bar{s}[p_{i-1}^*, p_i^*] \right). \tag{5}
\]

For each substring \( \bar{s}[l^*, r^*] \), Algorithm 1 tries all different mappings between \( s_i^* \) blocks and this substring. Note that in order to iterate over all different mappings, we can iterate over the variables \( p_0, p_1, \cdots, p_{n^\delta} \) such that

\[
l^* = p_0 \leq p_1 \leq \cdots \leq p_{n^\delta} = r^* + 1,
\]

and these variables can be stored in a memory of \( \tilde{O}(n^\delta) \). For each different mapping the algorithm estimates the edit distance of each block and the mapped substring using Claim 2.3. We claim that for each substring \( \bar{s}[l^*, r^*] \), the algorithm finds \((2^{\gamma/\delta}+1) - 1\)-approximation of the edit distance between this substring and the online string. To show that consider the optimal mapping \( p_0^*, p_1^*, \cdots, p_{n^\delta}^* \), then the distance that algorithm estimates is at most

\[
\sum_{i=0}^{n^\delta} d_i + \text{ed} \left( \bar{s}[l_i, r_i], \bar{s}[p_{i-1}^*, p_i^*] \right)
\leq \sum_{i=0}^{n^\delta} \left(2^{\gamma/\delta} + 1\right) \text{ed} \left( s_i^*, \bar{s}[p_{i-1}^*, p_i^*] \right) \quad \text{By (5).}
\]

\[
= \left(2^{\gamma/\delta} + 1\right) \sum_{i=0}^{n^\delta} \text{ed} \left( s_i^*, \bar{s}[p_{i-1}^*, p_i^*] \right)
= \left(2^{\gamma/\delta} + 1\right) \text{ed}(s, \bar{s}[l^*, r^*]). \quad \text{By (4).}
\]

Therefore for each substring \( \bar{s}[l^*, r^*] \), the algorithm finds \((2^{\gamma/\delta}+1) - 1\) approximation of its edit distance from \( s \). Thus, the algorithm finds \((2^{\gamma/\delta}+1) - 1\) approximation of the closest substring problem. This completes the induction and proves the theorem.

\[\square\]

**Theorem 1.1.** Given an offline and online strings of length \( n \) and any constant \( \delta > 0 \), there exists a streaming algorithm that finds a \( O(2^{1/\delta}) \) approximation of the edit distance using \( \tilde{O}(n^{\delta}/\delta) \) memory.

**Proof.** By Theorem 2.4, Algorithm 1 finds a \( O(2^{1/\delta}) \) approximation of the closest substring problem. Recall that by Theorem 2.1, we can find the edit distance of any two substrings of \( \bar{s} \) in a very small memory. Therefore by Claim 2.3, we can find a \( O(2^{1/\delta}) \) approximation of the edit distance between \( s \) and \( \bar{s} \).

\[8\]
Claim 3.2. the following holds. For any \( \text{LCSPosition}_{l,r} \) the function from the substrings of \( s \) it immediately derives from the definition of the function.

Proof. It immediately derives from the definition of the function. \( \square \)

3 \((1 - \epsilon)\)-Approximation of LCS

In this section, we design a streaming algorithm for finds \((1 - \epsilon)\) approximation of the LCS using \( \tilde{O}(\sqrt{n}/\epsilon) \) memory. Let LCSPosition be the function defined as below.

\[
\text{LCSPosition}_{l,r} \quad \text{Input:} \quad \text{A position } p \text{ in } \bar{s} \text{ and a non-negative integer } k. \\
\text{Result:} \quad \text{The smallest position } q \text{ such that } \text{LCS}(\bar{s}[p,q], s[l,r]) \geq k. \text{ If no such } q \text{ exists, the output is } \infty.
\]

For a position \( p \) in \( \bar{s} \), a substring \( s[l,r] \) of \( s \), and a non-negative integer \( k \), we use \( \text{LCSPosition}_{l,r}(p, k) \) to denote the result of the mentioned function which is the smallest position \( q \) such that LCS of \( \bar{s}[p,q] \) and \( s[l,r] \) is at least \( k \). We also define \( \text{LCSPosition}_{l,r}(p, 0) \) to be \( p - 1 \). Note that the LCS of two strings \( \bar{s} \) and \( s \) is equal to the largest \( k \) such that \( \text{LCSPosition}_{1,n}(1, k) < \infty \). Therefore, instead of solving the LCS problem, we can solve the \( \text{LCSPosition}_{1,n}(1, k) < \infty \) problem and report the largest \( k \) such that \( \text{LCSPosition}_{1,n}(1, k) < \infty \). We start designing our algorithm, by observing some properties of the function LCSPosition.

Observation 3.1. Function \( \text{LCSPosition}_{l,r} \) is non-decreasing on \( p \) and \( k \). In other words, for every numbers \( p_1 \leq p_2 \) and \( k_1 \leq k_2 \), we have

\[
\text{LCSPosition}_{l,r}(p_1, k_1) \leq \text{LCSPosition}_{l,r}(p_2, k_2).
\]

Proof. It immediately derives from the definition of the function. \( \square \)

Another observation about the function \( \text{LCSPosition}_{l,r} \) is about how we can find the value of the function from the substrings of \( s[l,r] \). Consider the function \( \text{LCSPosition}_{l,r}(p, k) \), and let \( s[l,m] \) and \( s[m+1,r] \) be an arbitrary division of the substring \( s[l,r] \) into two substrings. We claim that the following holds.

Claim 3.2. For any \( k \geq 0 \), the following holds.

\[
\text{LCSPosition}_{l,r}(p, k) = \min_{k_1, k_2 \geq 0, k_1 + k_2 = k, \text{LCSPosition}_{m+1,r}(\text{LCSPosition}_{l,m}(p, k_1) + 1, k_2) < \infty} \text{LCSPosition}_{m+1,r}(\text{LCSPosition}_{l,m}(p, k_1) + 1, k_2). \tag{6}
\]

Proof. For any \( k_1, k_2 \geq 0 \) such that \( k = k_1 + k_2 \) and \( \text{LCSPosition}_{l,m}(p, k_1) < \infty \), the value of \( \text{LCSPosition}_{m+1,r}(\text{LCSPosition}_{l,m}(p, k_1) + 1, k_2) \) indicates the ending of a common subsequence of size \( k \) such that exactly \( k_1 \) characters from \( s[l,m] \) are in this common subsequence and \( k_2 \) characters from \( s[m+1,r] \) are in this subsequence. Therefore, we always have

\[
\text{LCSPosition}_{l,r}(p, k) \leq \min_{k_1, k_2 \geq 0, k_1 + k_2 = k, \text{LCSPosition}_{m+1,r}(\text{LCSPosition}_{l,m}(p, k_1) + 1, k_2) < \infty} \text{LCSPosition}_{m+1,r}(\text{LCSPosition}_{l,m}(p, k_1) + 1, k_2).
\]
In order to complete the proof of the claim, we show that there always exists $k_1$ and $k_2$ such that \( \text{LCSPosition}_{m+1,r}(\text{LCSPosition}_{l,m}(p, k_1) + 1, k_2) \leq \text{LCSPosition}_{l,r}(p, k) \).

Consider an optimal solution of the function \( \text{LCSPosition}_{l,r}(p, k) \), and let suppose that \( q = \text{LCSPosition}_{l,r}(p, k) \). In that solution there is a common subsequence of size \( k \) between the characters in \( s[l, r] \) and \( \bar{s}[p, q] \). Let suppose that in that solution character \( s[a_i] \) is matched to \( \bar{s}[b_i] \) for each \( 1 \leq i \leq k \). W.l.o.g., we can assume

\[
l \leq a_1 < a_2 < \cdots < a_k \leq r.
\]

It also implies that

\[
p \leq b_1 < b_2 < \cdots < b_k = q.
\]

We consider two different cases. The first case is when all indices \( a_i \) are larger than \( m \). In this case \( k \) characters from \( s[m+1, r] \) are matched to \( \bar{s}[p, q] \). Therefore, \( \text{LCSPosition}_{m+1,r}(p, k) \leq q \). By setting \( k_1 = 0 \) and \( k_2 = k \), we get

\[
\text{LCSPosition}_{m+1,r}(\text{LCSPosition}_{l,m}(p, k_1) + 1, k_2) = \text{LCSPosition}_{m+1,r}(p - 1 + 1, k_2) = \text{LCSPosition}_{m+1,r}(p, k) \leq q.
\]

The other case is when for at least one \( a_i \), we have \( a_i \leq m \). Let assume that \( k_1 \) the largest number such that \( a_{k_1} \) is at most \( m \). In this solution \( k_1 \) characters from \( s[l, m] \) are matched to the characters in \( \bar{s}[p, b_{k_1}] \). Let \( q_1 \) the result of \( \text{LCSPosition}_{l,m}(p, k_1) \), then we have

\[
q_1 \leq b_{k_1}.
\]

We also know that there are \( k_2 = k - k_1 \) characters from \( s[m+1, r] \) that are matched to the characters in \( \bar{s}[b_{k_1}+1, q] \). Therefore we have

\[
\text{LCSPosition}_{m+1,r}(b_{k_1} + 1, \ k_2) \leq q.
\]

Thus,

\[
\text{LCSPosition}_{m+1,r}(\text{LCSPosition}_{l,m}(p, k_1) + 1, k_2) = \text{LCSPosition}_{m+1,r}(q_1 + 1, k_2) \leq \text{LCSPosition}_{m+1,r}(b_{k_1} + 1, k_2) \leq q
\]

By (7).

\[
= \text{LCSPosition}_{l,r}(p, k),
\]

which proves the claim. \( \square \)

Algorithm 2 first divides the online string into \( \sqrt{n} \) blocks of equal sizes. We assume w.l.o.g., that length of the strings is divisible by \( \sqrt{n} \). Otherwise we can always pad offline and online strings with different characters that are not in \( \Sigma \) such that their new length get divisible by \( \sqrt{n} \). The algorithm divides \( s \) into \( \sqrt{n} \) blocks \( s_1^*, s_2^*, \ldots, s_{\sqrt{n}}^* \) blocks of size \( \sqrt{n} \) where \( s_i^* \) is the substring \( s[(i - 1)\sqrt{n} + 1, i\sqrt{n}] \). Given an \( \epsilon > 0 \), the algorithm keeps an array \( D \) of the size \( \lceil \log_{1+\epsilon^*} n \rceil \) where \( D[k] \) is an estimation of \( \text{LCSPosition}_{1,l}(1, [(1 + \epsilon^*)k]) \) in the subsequence of the online string that has arrived so far in the stream. Specifically, after arrival of the block \( s_i^* \) in the stream, the algorithm keeps an estimation of \( \text{LCSPosition}_{1,l}(1, [(1 + \epsilon^*)k]) \) in \( D[k] \). First we show that how
Algorithm 2: Algorithm APPROXIMATE-LCS for approximation the LCS.

**Data:** An offline string \( s \) of length \( n \), a stream of characters of the online string \( s \), and an \( \epsilon^* > 0 \).

1. Divide \( s \) into \( \sqrt{n} \) blocks \( s_1^*, s_2^*, \ldots, s_{\sqrt{n}}^* \) of size \( \sqrt{n} \).
2. \( D \leftarrow \) an array of size \( \lfloor \log_{1+\epsilon^*} n \rfloor \) initially containing \( \infty \) in all cells.
3. for \( i \in [\sqrt{n}] \) do
4.   \( T \leftarrow \) an array of size \( \lfloor \log_{1+\epsilon^*} n \rfloor \) initially containing \( \infty \) in all cells.
5.   for \( 0 \leq k \leq \lfloor \log_{1+\epsilon^*} n \rfloor \) do
6.     \( T[k] \leftarrow \text{LCSPosition}_{(i-1)\sqrt{n}+1,i\sqrt{n}} (1, \lfloor (1+\epsilon^*)^k \rfloor) \).
7.   for \( 0 \leq k_1 \leq k \) do
8.     if \( D[k_1] < \infty \) then
9.       \( D[k] \leftarrow \text{LCSPosition}_{(i-1)\sqrt{n}+1,i\sqrt{n}} (D[k_1]+1, \lfloor (1+\epsilon^*)^k \rfloor - \lfloor (1+\epsilon^*)^{k_1} \rfloor) \) using any offline algorithm. \( q \) be this result.
10.    \( T[k] \leftarrow \min \{ T[k], q \} \).
11.  \( D \leftarrow T \).
12. return The largest value \( \lfloor (1+\epsilon^*)^k \rfloor \) such that \( D[k] < \infty \). If no such \( k \) exists, return 0.

The algorithm can update the array \( D \) upon arrival of a new block, and after that we demonstrate the approximation guarantee of our method.

Let assume that we have an array \( D \) in which \( D[k] \) is an approximation of \( \text{LCSPosition}_{(i-1)\sqrt{n}+1,i\sqrt{n}} (1, \lfloor (1+\epsilon^*)^k \rfloor) \) for different values of \( 0 \leq k \leq \lfloor \log_{1+\epsilon^*} n \rfloor \). Upon arrival of a new block \( s_i^* \), the algorithm has to update the array \( D \). Suppose that we want to find \( \text{LCSPosition}_{1,i\sqrt{n}} (1, \lfloor (1+\epsilon^*)^k \rfloor) \). According to Claim 3.2, there are integers \( k_1^* \) and \( k_2^* \) such that \( k_1^* + k_2^* = \lfloor (1+\epsilon^*)^k \rfloor \) and

\[
\text{LCSPosition}_{1,i\sqrt{n}} (1, \lfloor (1+\epsilon^*)^k \rfloor) = \text{LCSPosition}_{(i-1)\sqrt{n}+1,i\sqrt{n}} (\text{LCSPosition}_{1,(i-1)\sqrt{n}} (k_1^*)+1, k_2^*). \tag{9}
\]

The algorithm stores all of the characters of \( s_i^* \) in the memory. Therefore, for every \( p \) and \( k \) we can compute the function \( \text{LCSPosition}_{(i-1)\sqrt{n}+1,i\sqrt{n}} (p,k) \) in our streaming algorithm. In order to update the array \( D \), the algorithm iterates over all \( k_1^* \) such that \( k_1^* \) is a power of \( (1+\epsilon^*) \) and pick the one that minimizes the r.h.s. of (9). Specifically, let \( T \) an array of length \( \lfloor \log_{1+\epsilon^*} n \rfloor \) which represents the updated estimates. Initially for each \( k \) we set

\[
T[k] = \text{LCSPosition}_{(i-1)\sqrt{n}+1,i\sqrt{n}} (1, \lfloor (1+\epsilon^*)^k \rfloor),
\]

which represents the case that all characters in the optimal solution of \( \text{LCSPosition}_{1,i\sqrt{n}} (1, \lfloor (1+\epsilon^*)^k \rfloor) \) are from block \( s_i^* \), i.e., when \( k_1^* \) is zero in (9). Then the algorithm considers values of \( k_1^* \) such that \( k_1^* \) is a power of \( (1+\epsilon^*) \), i.e., we have \( k_1^* = \lfloor (1+\epsilon^*)^{k_1} \rfloor \) for some integer \( k_1 \). Recall that we have assumed that \( D[k] \) is an approximation \( \text{LCSPosition}_{1,(i-1)\sqrt{n}} (1, \lfloor (1+\epsilon^*)^k \rfloor) \). Therefore we can approximate the r.h.s. of (9) for \( k_1^* = \lfloor (1+\epsilon^*)^{k_1} \rfloor \) by computing

\[
\text{LCSPosition}_{(i-1)\sqrt{n}+1,i\sqrt{n}} (D[k_1]+1, k_2^*),
\]

where \( k_2^* = \lfloor (1+\epsilon^*)^k \rfloor - k_1^* \). In our algorithm we compute the value above for all different value of \( k_1^* \) and set the \( T[k] \) equal to minimum of these values. In other words, by the end of the arrival of
the block \( s^*_i \), we have

\[
T[k] = \min \left\{ \text{LCSPosition}_{(i-1)\sqrt{n}+1,i\sqrt{n}}(1, [(1+\epsilon^*)k]), \right. \\
\left. \text{min}_{k^*_1,k^*_2 \geq 0,k^*_1+k^*_2 = [(1+\epsilon^*)k]}, \quad \text{LCSPosition}_{(i-1)\sqrt{n}+1,i\sqrt{n}}(D[k_1] + 1, k^*_2) \right\}. 
\]

(10)

After computing the values in the array \( T \), we can replace values in the array \( D \) with the values in \( T \), and update the array \( D \).

In order to provide an approximation guarantee for our algorithm, we prove the following claim.

Claim 3.3. Let \( D_i \) be the array \( D \) after arrival of the block \( s^*_i \), then for each \( 1 \leq k^* \leq n \), there exists a \( 0 \leq k \leq \lfloor \log_{1+\epsilon^*} n \rfloor \) such that

\[
k^*(1-\epsilon^*)^i \leq \lfloor (1+\epsilon^*)^k \rfloor \leq k^*,
\]

and,

\[
D_i[k] \leq \text{LCSPosition}_{1,i\sqrt{n}}(1,k^*).
\]

Proof. We prove the claim by induction on \( i \) which represents the number of blocks that have arrived in the stream. For \( i = 1 \), the algorithm finds the exact solution of \( \text{LCSPosition}_{1,\sqrt{n}}(1, [(1+\epsilon^*)k]) \) for all \( 0 \leq k \leq \lfloor \log_{1+\epsilon^*} n \rfloor \). Consider an integer \( 1 \leq k^* \leq n \), then there exists some number with the form of \( \lfloor (1+\epsilon^*)^k \rfloor \) between \( k^*/(1+\epsilon^*) \) and \( k^* \). Let \( \lfloor (1+\epsilon^*)^k \rfloor \) be that number. Then we have,

\[
D_1[k] = \text{LCSPosition}_{1,\sqrt{n}}(1, [(1+\epsilon^*)^k]) \leq \text{LCSPosition}_{1,\sqrt{n}}(1,k^*).
\]

We also have

\[
k^*(1-\epsilon^*) \leq \frac{k^*}{1+\epsilon^*} \leq \lfloor (1+\epsilon^*)^k \rfloor \leq k^*,
\]

which proves the claim for \( i = 1 \).

Now consider an \( i > 1 \), and a \( 1 \leq k^* \leq n \). If \( \text{LCSPosition}_{1,i\sqrt{n}}(1,k^*) \) is \( \infty \), then the claim clearly holds. Otherwise we can assume \( \text{LCSPosition}_{1,i\sqrt{n}}(1,k^*) = q \) where \( q < \infty \). By (10) and the way our algorithm computes the array \( D_i \) we have

\[
D_i[k] = \min \left\{ \text{LCSPosition}_{(i-1)\sqrt{n}+1,i\sqrt{n}}(1, [(1+\epsilon^*)^k]), \right. \\
\left. \text{min}_{k^*_1,k^*_2 \geq 0,k^*_1+k^*_2 = [(1+\epsilon^*)^k]}, \quad \text{LCSPosition}_{(i-1)\sqrt{n}+1,i\sqrt{n}}(D_{i-1}[k_1] + 1, k^*_2) \right\}. 
\]

(11)

By Claim 3.2, there exists integers \( k^*_1, k^*_2 \geq 0 \) such that \( k^*_1 + k^*_2 = k^* \) and

\[
\text{LCSPosition}_{1,i\sqrt{n}}(1,k^*) = \text{LCSPosition}_{(i-1)\sqrt{n}+1,i\sqrt{n}}(\text{LCSPosition}_{1,(i-1)\sqrt{n}}(1,k^*_1) + 1, k^*_2).
\]

Let \( q_1 = \text{LCSPosition}_{1,(i-1)\sqrt{n}}(1,k^*_1) \), then we have

\[
\text{LCSPosition}_{1,i\sqrt{n}}(1,k^*) = \text{LCSPosition}_{(i-1)\sqrt{n}+1,i\sqrt{n}}(q_1 + 1, k^*_2).
\]

(12)

We consider two different cases on \( k^*_1 \).
• The first case is when $k_1^* = 0$. In that case we have $q_1 = 0$, and by (12) we have

$$\text{LCSPosition}_{1,i\sqrt{n}(1,k^*)} = \text{LCSPosition}_{(i-1)\sqrt{n+1},i\sqrt{n}(1,k^*)}.$$  

Then there exists some number with the form of $\lceil (1 + \varepsilon)^k \rceil$ between $k^*/(1 + \varepsilon)$ and $k^*$. Then $\lceil (1 + \varepsilon)^k \rceil$ is also between $k^*(1 - \varepsilon)$ and $k^*$. Moreover, by (11) we have

$$D_i[k] \leq \text{LCSPosition}_{(i-1)\sqrt{n+1},i\sqrt{n}(1,\lceil (1 + \varepsilon)^k \rceil)}$$

$$\leq \text{LCSPosition}_{(i-1)\sqrt{n+1},i\sqrt{n}(1,k^*)}$$

$$= \text{LCSPosition}_{1,i\sqrt{n}(1,k^*)}, \quad \text{By (13)}.$$ 

which proves the claim for this case.

• The other case is when $k_1^* > 0$. In this case, in the optimal solution $k_1^*$ characters from $s[1,(i-1)\sqrt{n}]$ are matched to the characters in $\hat{s}[1,q_1]$. By the induction hypothesis, we know there exists some $k_1$ such that

$$k_1^*(1 - \varepsilon)^i \leq \lceil (1 + \varepsilon)^{k_1} \rceil \leq k_1^*,$$  

and

$$D_i-1[k] \leq \text{LCSPosition}_{1,(i-1)\sqrt{n}(1,k_1^*)} = q_1.$$  

Let $k' = \lceil (1 + \varepsilon)^{k_1} \rceil + k_2^*$. Then, we have

$$k' \geq k^*(1 - \varepsilon)^i.$$  

By (14).

Let $k$ an integer such that $\lceil (1 + \varepsilon)^k \rceil$ is between $k'/ (1 + \varepsilon)$ and $k'$. We show that $D_i[k]$ satisfies the claim. From the previous equation, we have

$$\lceil (1 + \varepsilon)^k \rceil \geq \frac{k'}{1 + \varepsilon} \geq \frac{1}{1 + \varepsilon} \cdot k^*(1 - \varepsilon)^i$$

$$\geq k^*(1 - \varepsilon)^i.$$ 

Let $k_2 = \lceil (1 + \varepsilon)^k \rceil - \lceil (1 + \varepsilon)^{k_1} \rceil$. Then we have,

$$k_2 = \lceil (1 + \varepsilon)^{k_1} \rceil - \lceil (1 + \varepsilon)^{k_1} \rceil$$

$$\leq k' - \lceil (1 + \varepsilon)^{k_1} \rceil$$

Since $\lceil (1 + \varepsilon)^k \rceil \leq k'$. 

$$= k_2^*.$$ 

Since $k' = \lceil (1 + \varepsilon)^{k_1} \rceil + k_2^*$.  

(16)

By (11), we have

$$D_i[k] \leq \text{LCSPosition}_{(i-1)\sqrt{n+1},i\sqrt{n}(D_i-1[k_1] + 1, k_2)}$$

$$\leq \text{LCSPosition}_{(i-1)\sqrt{n+1},i\sqrt{n}(q_1 + 1, k_2)} \quad \text{By (15).}$$

$$\leq \text{LCSPosition}_{(i-1)\sqrt{n+1},i\sqrt{n}(q_1 + 1, k_2^*)} \quad \text{By (16).}$$

$$= \text{LCSPosition}_{1,i\sqrt{n}(1,k^*)}. \quad \text{By (12).}$$

This proves the second case and completes the induction and proves the claim.
Theorem 3.4. For any \( \epsilon^* > 0 \), Algorithm 2 finds a \((1 - \epsilon^*)\sqrt{n}\) approximation of the LCS between \( \bar{s} \) and \( s \) using \( \tilde{O}(\sqrt{n} + \log_{1+\epsilon^*} n) \) memory.

Proof. Let OPT be the size of LCS between \( \bar{s} \) and \( s \). Then OPT is the largest \( k \) such that \( \text{LCSPosition}_{1,n}(1,k) < \infty \). By Claim 3.3, in the final array \( D \) computed by the algorithm there exists an integer \( k \) such that

\[
\lceil (1 + \epsilon^*)^k \rceil \geq \text{OPT}(1 - \epsilon^*)\sqrt{n},
\]

and

\[
D[k] \leq \text{LCSPosition}_{1,n}(1,\text{OPT}) < \infty.
\]

Therefore, the answer returned by the algorithm is at least \( \lceil (1 + \epsilon^*)^k \rceil \) and it gives us a \((1 - \epsilon^*)\sqrt{n}\) approximation.

To show the memory bound of Algorithm 2, observe that the algorithm needs a memory of \( \tilde{O}(\sqrt{n}) \) to store each block \( s^*_i \) and compute the LCS between a substring of this block and a substring of the offline string (using Theorem 2.1). Also, the algorithm keeps an array \( D \) and \( T \) of size \( \log_{1+\epsilon^*} n \). Therefore, the memory of the algorithm is bounded by \( \tilde{O}(\sqrt{n} + \log_{1+\epsilon^*} n) \).

\( \square \)

Theorem 3.5. There exists a streaming algorithm that finds \((1 - \epsilon)\) approximation of the LCS between \( \bar{s} \) and \( s \) using \( \tilde{O}(\sqrt{n}/\epsilon) \) memory.

Proof. By setting \( \epsilon^* = \epsilon/\sqrt{n} \), Theorem 3.4 immediately gives us an algorithm with the approximation ratio of

\[
(1 - \epsilon^*)\sqrt{n} \geq 1 - \epsilon^* \cdot \sqrt{n} = 1 - \epsilon.
\]

Also, the memory of this algorithm is bounded by

\[
\tilde{O}(\sqrt{n} + \log_{1+\epsilon^*} n) = \tilde{O}(\sqrt{n}/\epsilon).
\]

\( \square \)

4 (1 + \epsilon)-Approximation of ED

In this section, we design a streaming algorithm that finds \((1 + \epsilon)\) approximation of the edit distance for an arbitrary \( \epsilon > 0 \). The memory of our algorithm is \( \tilde{O}(\sqrt{n}/\epsilon) \). Our algorithm is inspired by the algorithm of [38] for approximating the edit distance in the Massively Parallel Computation (MPC) model.

Suppose that we are given a distance \( d \), and we want to verify whether the edit distance of \( \bar{s} \) and \( s \) is close to \( d \) or not. If we can solve this subproblem, we can also find an approximation of the edit distance between \( \bar{s} \) and \( s \). In order to do that, we can run the algorithm for different values of \( d \) in \( \{1, [(1+\epsilon)], [(1+\epsilon)^2], \cdots \} \) and return the minimum \( d \) that our algorithm thinks is close to the edit distance between \( \bar{s} \) and \( s \). The number of guesses for \( d \) is also bounded by \( O(\log_{1+\epsilon}(n)) \) and we can run the algorithm for all different guesses of \( d \) in parallel and return the best answer. Thus, our goal in the rest of the section is to design a streaming algorithm that given an approximate size of the edit distance, verifies whether a solution with that size exists.
Similar to our algorithm for LCS, we divide the online string into $\sqrt{n}$ blocks of size $\sqrt{n}$. We can always assume that $n$ is a perfect square and $\sqrt{n}$ is an integer. The reason is that, if $n$ is not a perfect square, we can pad both online and offline strings with the same character which is not in $\Sigma$ and make sure that their new length is a perfect square. We can always access the padded characters using an oracle. Also, this work does not change the edit distance. Therefore we can assume that $n$ is a perfect square. Our algorithm divides the online string into $\sqrt{n}$ blocks $s^*_1, s^*_2, \ldots, s^*_\sqrt{n}$ where $s^*_i$ is the substring $s[(i-1)\sqrt{n} + 1, i\sqrt{n}]$ of the online string. Let assume that in the optimal solution of the edit distance, block $s^*_1$ is mapped to the substring $\bar{s}[l_1, r_1)$. For each block $s^*_i$, our algorithm finds a set of candidate intervals for the mapping of this block. Roughly speaking, we show that our candidate set always contains an interval which is very close to $[l_i, r_i)$. We then show that using these intervals we can get a good approximation of the edit distance.

**Finding Candidate Intervals.** Consider a block $s^*_i$ of the online string. Let us suppose that in the optimal solution, it is mapped to the substring $\bar{s}[l_i, r_i)$. We can always assume that $l_i = r_{i-1}$ for $i > 1$, $l_1 = 1$ and $r_{\sqrt{n}} = n + 1$. Then we have

$$\text{ed}(s, \bar{s}) = \sqrt{n} \sum_{i=1}^{\sqrt{n}} \text{ed}(s^*_i, \bar{s}[l_i, r_i)).$$

Our goal is to find a set of candidate intervals for $s^*_i$ such that at least one of these intervals is very close to $[l_i, r_i)$. In order to design our algorithm, we first explore some properties of the interval $[l_i, r_i)$. We use $\alpha_i = (i-1)\sqrt{n} + 1$, and $\beta_i = i\sqrt{n}$ to denote the starting and the ending of the block $s^*_i$ respectively. Therefore, we have $s^*_i = s[\alpha_i, \beta_i]$. Recall that we have assumed that we are given a bound $d$ on the size of the edit distance. Therefore, we should have

$$|r_i - 1 - \beta_i| \leq d.$$

It follows that $r_i \in [\beta_i + 1 - d, \beta_i + 1 + d]$. Let $\kappa = \lfloor d \cdot \epsilon / \sqrt{n} \rfloor$. The algorithm considers all intervals such that their ending points are in $[\beta_i + 1 - 2d, \beta_i + 1 + 2d]$ and the ending points are divisible by $\kappa$ (see Figure 1). We call these intervals, *candidate intervals* and we call their endings *candidate endings*. We also consider all intervals ending in 1, i.e. intervals $[l, 1)$, as candidate intervals if $1 \in [\beta_i + 1 - 2d, \beta_i + 1 + 2d]$.

![Figure 1: The locations of the ending points for potential intervals of a block are illustrated in this figure. Thick segments show the ending points.](image-url)
Algorithm 3: Algorithm APPROXIMATE-ED for approximating the edit distance.

**Data:** An offline string \( \bar{s} \) of length \( n \), an online string \( s \), a bound \( d \) for the edit distance, and an \( \epsilon > 0 \).

1. \( \kappa = \lfloor d \cdot \epsilon / \sqrt{n} \rfloor \).
2. \( D \leftarrow \) an empty function.
3. \( D[1] \leftarrow 0 \).
4. Divide \( s \) into \( \sqrt{n} \) blocks \( s_1^*, s_2^*, \ldots, s_{\sqrt{n}}^* \) of size \( \sqrt{n} \).
5. for each block \( s_i^* \) do
6. \( T \leftarrow \) an empty function.
7. \( \beta = i \sqrt{n} \).
8. for every integer \( r \) in \( [\beta + 1 - 2d, \beta + 1 + 2d] \) such that \( r \) is 1 or is divisible by \( \kappa \) do
9. \( r \) is at least 1 and at most \( n + 1 \) then
10. \( T[r] \leftarrow \infty \).
11. for each \( l \in D \) such that \( l \leq r \) do
12. \( T[r] \leftarrow \min \{ T[r], D[l] + \text{ed}(\bar{s}[l, r), s_i^*) \} \).
13. \( D \leftarrow T \).
14. \( \text{min_dist} \leftarrow \infty \).
15. for each \( r \in D \) do
16. \( \text{min_dist} = \min\{ \text{min_dist}, D[r] + (n - r + 1) \} \).
17. return \( \text{min_dist} \).

Our algorithm uses the dynamic programming to find the best mapping of the \( s_i^* \) blocks to their candidate intervals. Define the function \( D_i \) as follows. Let \( D_i[r] \) be the best mapping of the first \( i \) blocks to their candidate intervals such that block \( s_i^* \) is mapped to an interval ending in \( r \). Note that for all candidate intervals for the block \( s_i^* \), their ending points are either 1 or an integer in \( [\beta_i + 1 - 2d, \beta_i + 1 + 2d] \) that is divisible by \( \kappa \). Therefore the number of different end points for the candidate intervals is bounded by \( O(d/\kappa) = O(\sqrt{n}/\epsilon) \). Thus, function \( D_i \) only takes \( O(\sqrt{n}/\epsilon) \) values and we can store all values for this function in a memory of \( O(\sqrt{n}/\epsilon) \). We say that \( r \in D_i \), if the function \( D_i \) takes the value \( r \). In other words, \( r \) is an end point for at least one of the candidate intervals for \( s_i^* \). Consider an ending point \( r \in D_i \). Consider the optimal solution for \( D_i[r] \). Let assume in that solution the block \( s_i^* \) is mapped to an interval \([l, r)\) of the offline string. Then, the first \( i - 1 \) blocks are mapped to the substring \( \bar{s}[l, l) \). Also, \( s_{i-1}^* \) is mapped to an interval with the ending point of \( l \). By the definition of the \( D_i \) functions, \( D_{i-1}[l] \) denotes the best mapping for the first \( i - 1 \) blocks. Thus, the value of the function \( D_i[r] \) can be derived as follows.

\[
D_i[r] = \min_{i \in D_i-1} D_{i-1}[l] + \text{ed}(\bar{s}[l, r), s_i^*) .
\]

According to the equation above, we can find the value for function \( D_i \) by only using the values of function \( D_{i-1} \). As we mentioned earlier, we can store the values of functions \( D_i \) and \( D_{i-1} \) in a memory of \( O(\sqrt{n}/\epsilon) \).

Our algorithm (formally as Algorithm 3), divides the online string into \( \sqrt{n} \) blocks \( s_1^*, s_2^*, \ldots, s_{\sqrt{n}}^* \).

**Theorem 4.1.** Algorithm 3 uses \( \tilde{O}(\sqrt{n}/\epsilon) \) memory and finds \((1 + 5\epsilon)\) approximation of the edit distance between \( \bar{s} \) and \( s \).
\textbf{Proof.} Consider an optimal solution for \(ed(\bar{s}, s)\). Let \(OPT\) be the size of this solution, and \(d\) be the best guess of our algorithm for the edit distance between \(\bar{s}\) and \(s\). Then, we have

\[OPT \leq d \leq (1 + \epsilon)OPT.\]  

(18)

Suppose in the optimal solution, block \(s_i^*\) is mapped to the substring \(\bar{s}[l, r)\) of the offline string. Then we have

\[OPT = ed(\bar{s}, s) = \sum_{i=1}^{\sqrt{n}} ed(\bar{s}[l, r), s_i^*).\]  

(19)

We also have that \(l_1 = 1\) and \(l_i = r_{i-1}\) for \(i > 1\). Also, \(r_{n+1} = n + 1\). Let \(C\) be the set of all integers such that they can be a candidate ending point for one of \(s_i^*\) blocks. In other words,

\[C = \{1, \kappa, 2\kappa, \cdots, \lfloor n/\kappa \rfloor \kappa \}.\]

For each \(l_i\) (respectively, \(r_i\)), let \(l_i'\) (resp., \(r_i'\)) be the largest number in \(C\) that is at most \(l_i\) (resp., \(r_i\)). Then, for each \(l_i\), we have

\[l_i - \kappa < l_i' \leq l_i.\]  

(20)

Similarly, for each \(r_i\), we have

\[r_i - \kappa < r_i' \leq r_i.\]  

(21)

It follows that for each interval \([l_i', r_i']\) we have

\[ed(\bar{s}[l_i', r_i'), s_i^*) \leq ed(\bar{s}[l_i', r_i'), \bar{s}[l, r)) + ed(\bar{s}[l, r), s_i^*) \quad \text{By the triangle inequality}.\]

\[\leq 2\kappa + ed(\bar{s}[l, r), s_i^*) \quad \text{By (20) and (21)}.\]  

(22)

It follows from (22) that

\[
\sum_{i=1}^{\sqrt{n}} ed(\bar{s}[l_i', r_i'), s_i^*) \\
\leq 2\kappa \cdot \sqrt{n} + \sum_{i=1}^{\sqrt{n}} ed(\bar{s}[l, r), s_i^*) \\
\leq 2\epsilon \cdot d + \sum_{i=1}^{\sqrt{n}} ed(\bar{s}[l, r), s_i^*) \\
= 2\epsilon \cdot d \ + OPT \quad \text{Since } \kappa = \lfloor d \cdot \epsilon / \sqrt{n} \rfloor. \\
\leq OPT(1 + 2\epsilon(1 + \epsilon)) \quad \text{By (19)}. \\
\leq OPT(1 + 3\epsilon). \quad \text{By (18)}. \\
(23)
\]

Therefore, the size of the solution that maps each block \(s_i^*\) to \(\bar{s}[l_i', r_i']\) is at most \((1 + 3\epsilon)OPT\). We show that our algorithm almost finds this solution. We claim that each \(r_i'\) is a candidate endpoint for \(s_i^*\). Since \(r_i' \in C\), it is either 1 or it is divisible by \(\kappa\). To show that \(r_i'\) can be the end point of some candidate interval for \(s_i^*\), it is sufficient to show that \(r_i'\) is in \([\beta_i + 1 - 2d, \beta_i + 1 + 2d]\) where \(\beta_i = i\sqrt{n}\) is the end point of the block \(s_i^*\).
Because the size of the edit distance between \( \bar{s} \) and \( s \) is bounded by \( d \), we have
\[
|r_i - \beta_i + 1| \leq d.
\]
This along with (21),
\[
|r'_i - \beta_i + 1| \leq |r'_i - r_i| + |r_i - \beta_i + 1| \leq \kappa + d \leq 2d.
\]
Therefore, \( r'_i \) is in \( [\beta_i + 1 - 2d, \beta_i + 1 + 2d] \) and \( (l'_i, r'_i) \) is a candidate interval for \( s^* \). Thus in this solution, every block \( s^*_i \) is mapped to one of its candidate intervals. Consider the last block, it is mapped to the interval \( [l'_{\sqrt{n}}, r'_{\sqrt{n}}] \). Let \( q = r'_{\sqrt{n}} \). By the definition of \( D \) functions, \( D_{\sqrt{n}}[q] \) is the cost of the best solution such that each block maps to one of its candidate interval, and the ending of the last interval is \( q \). Therefore,
\[
D_{\sqrt{n}}[q] \leq \sum_{i=1}^{\sqrt{n}} \text{ed} (s[l'_i, r'_i], s^*_i) \\
\leq (1 + 3\epsilon) \text{OPT}.
\]
By (23). (24)
After arrival of all blocks, in Algorithm 3 function \( D \) will be equal to \( D_{\sqrt{n}} \), and the algorithm returns
\[
\min_{r \in D} D[r] + (n - r + 1) = \min_{r \in D_{\sqrt{n}}} D_{\sqrt{n}}[r] + (n - r + 1) \\
\leq D_{\sqrt{n}}[q] + (n - q + 1) \\
\leq (1 + 3\epsilon) \text{OPT} + (n - q + 1) \quad \text{By (24).} \\
\leq (1 + 3\epsilon) \text{OPT} + (r_{\sqrt{n}} - q) \quad \text{Since } r_{\sqrt{n}} = n + 1. \\
\leq (1 + 3\epsilon) \text{OPT} + \kappa \quad \text{By (21).} \\
\leq (1 + 3\epsilon) \text{OPT} + \epsilon \cdot d \\
\leq (1 + 5\epsilon) \text{OPT}.
\]
Therefore the approximation ratio of the algorithm is bounded by \( (1 + 5\epsilon) \) and it proves the theorem.
\[\square\]

The above theorem immediately implies the following theorem.

**Theorem 4.2.** For any \( \epsilon > 0 \), there exists a streaming algorithm that finds \( (1 + \epsilon) \) approximation of the edit distance between \( s \) and \( \bar{s} \) using \( \tilde{O}(\sqrt{n}/\epsilon) \) memory.

**References**

[1] Abboud, A. and Backurs, A. (2017). Towards hardness of approximation for polynomial time problems. In Proceedings of the Eighth Innovations in Theoretical Computer Science Conference.

[2] Abboud, A., Backurs, A., and Williams, V. V. (2015). Tight hardness results for LCS and other sequence similarity measures. In FOCS.

[3] Abboud, A., Hansen, T. D., Williams, V. V., and Williams, R. (2016). Simulating branching programs with edit distance and friends or: a polylog shaved is a lower bound made. In STOC.
[4] Abboud, A. and Rubinstein, A. (2018). Fast and deterministic constant factor approximation algorithms for LCS imply new circuit lower bounds.

[5] Alves, C. E., Cáceres, E. N., and Song, S. W. (2006). A coarse-grained parallel algorithm for the all-substrings longest common subsequence problem. *Algorithmica*, 45(3):301–335.

[6] Andoni, A., Deza, M., Gupta, A., Indyk, P., and Raskhodnikova, S. Lower bounds for embedding edit distance into normed spaces. In *SODA 2003*.

[7] Andoni, A., Goldberger, A., McGregor, A., and Porat, E. Homomorphic fingerprints under misalignments: Sketching edit and shift distances. In *STOC 2013*.

[8] Andoni, A. and Krauthgamer, R. (2007). The computational hardness of estimating edit distance [extended abstract]. In *48th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2007), October 20-23, 2007, Providence, RI, USA, Proceedings*, pages 724–734.

[9] Andoni, A. and Krauthgamer, R. (2008). The smoothed complexity of edit distance. In *ICALP*, pages 357–369. Springer.

[10] Andoni, A., Krauthgamer, R., and Onak, K. (2010). Polylogarithmic approximation for edit distance and the asymmetric query complexity. In *FOCS*.

[11] Andoni, A. and Onak, K. (2009). Approximating edit distance in near-linear time. In *STOC*.

[12] Backurs, A. and Indyk, P. (2015). Edit distance cannot be computed in strongly subquadratic time (unless SETH is false). In *STOC*.

[13] Bansal, N., Lewenstein, M., Ma, B., and Zhang, K. (2010). On the longest common rigid subsequence problem. *Algorithmica*, 56(2):270–280.

[14] Bar-Yossef, Z., Jayram, T., Krauthgamer, R., and Kumar, R. (2004). Approximating edit distance efficiently. In *FOCS*.

[15] Batu, T., Ergun, F., and Sahinalp, C. (2006). Oblivious string embeddings and edit distance approximations. In *SODA*.

[16] Belazzougui, D. and Zhang, Q. (2016). Edit distance: Sketching, streaming, and document exchange.

[17] Bellman, R. (1957). Dynamic programming (dp).

[18] Boroujeni, M., Ehsani, S., Ghodsi, M., HajiAghayi, M., and Seddighin, S. (2018). Approximating edit distance in truly subquadratic time: Quantum and MapReduce. In *SODA*.

[19] Boroujeni, M. and Seddighin, S. (2019). Improved MPC algorithms for edit distance and Ulam distance. In *SPAA*.

[20] Bringman, K. and Künemann, M. (2018). Multivariate fine-grained complexity of longest common subsequence. In *SODA*.

[21] Bringmann, K., Grandoni, F., Saha, B., and Williams, V. V. (2016). Truly sub-cubic algorithms for language edit distance and RNA-folding via fast bounded-difference min-plus product. In *FOCS*, pages 375–384. IEEE.

[22] Bringmann, K. and Kunnam, M. (2015). Quadratic conditional lower bounds for string problems and dynamic time warping. In *FOCS*. 

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[23] Chakraborty, D., Das, D., Goldenberg, E., Koucky, M., and Saks, M. (2018). Approximating edit distance within constant factor in truly sub-quadratic time. In *FOCS*.

[24] Chakraborty, D., Goldenberg, E., and Koucký, M. (2016). Streaming algorithms for embedding and computing edit distance in the low distance regime. In *Proceedings of the Forty-eighth Annual ACM Symposium on Theory of Computing, STOC ’16*, pages 712–725, New York, NY, USA. ACM.

[25] Charikar, M., Geri, O., Kim, M. P., and Kuszmaul, W. (2018). On estimating edit distance: Alignment, dimension reduction, and embeddings. In *ICALP*, pages 34:1–34:14. Dagstuhl.

[26] Chen, L., Goldwasser, S., Lyu, K., Rothblum, G. N., and Rubinstein, A. (2019). Fine-grained complexity meets IP=PSPACE. In *SODA*, pages 1–20. SIAM.

[27] Crochemore, M., Iliopoulos, C. S., Pinzon, Y. J., and Reid, J. F. (2001). A fast and practical bit-vector algorithm for the longest common subsequence problem. *Information Processing Letters*, 80(6):279–285.

[28] Crochemore, M., Landau, G. M., and Ziv-Ukelson, M. (2003). A subquadratic sequence alignment algorithm for unrestricted scoring matrices. *SIAM Journal on Computing*, 32(6):1654–1673.

[29] de Monvel, J. B. (1999). Extensive simulations for longest common subsequences. *The European Physical Journal B-Condensed Matter and Complex Systems*, 7(2):293–308.

[30] Ergün, F. and Jowhari, H. (2008). On distance to monotonicity and longest increasing subsequence of a data stream.

[31] Gál, A. and Gopalan, P. (2007). Lower bounds on streaming algorithms for approximating the length of the longest increasing subsequence.

[32] Garofalakis, M. and Kumar, A. (2003). Correlating XML data streams using tree-edit distance embeddings. In *PODS*, pages 143–154. ACM.

[33] Gold, O. and Sharir, M. (2017). Dynamic time warping and geometric edit distance: Breaking the quadratic barrier. In *ICALP*. Dagstuhl.

[34] Goldenberg, E., Krauthgamer, R., and Saha, B. (2019). Sublinear algorithms for gap edit distance. In *FOCS*. IEEE. in press.

[35] Gopalan, P., Jayram, T. S., Krauthgamer, R., and Kumar, R. (2007). Estimating the sortedness of a data stream.

[36] Gusfield, D. (1997). *Algorithms on strings, trees and sequences: computer science and computational biology*. Cambridge University Press.

[37] Haeupler, B., Rubinstein, A., and Shahrasbi, A. (2019). Near-linear time insertion-deletion codes and (1+ε)-approximating edit distance via indexing. In *STOC*, pages 697–708.

[38] Hajiaghayi, M., Seddighin, M., Seddighin, S., and Sun, X. (2019a). Approximating lcs in linear time: Beating the $\sqrt{n}$ barrier. In *SODA 2019*.

[39] Hajiaghayi, M., Seddighin, S., and Sun, X. (2019b). Massively parallel approximation algorithms for edit distance and longest common subsequence. In *SODA*.
Hajiaghayi, M., Seddighin, S., and Sun, X. (2019c). Massively parallel approximation algorithms for edit distance and longest common subsequence. In *SODA 2019*.

Hunt, J. W. and Szymanski, T. G. (1977). A fast algorithm for computing longest common subsequences. *Communications of the ACM*, 20(5):350–353.

Indyk, P. (2001). Algorithmic applications of low-distortion geometric embeddings. In *FOCS*.

Jayaram, R. and Saha, B. (2017). Approximating language edit distance beyond fast matrix multiplication: Ultralinear grammars are where parsing becomes hard! In *ICALP*, pages 19:1–19:15. Dagstuhl.

Koucky, M. and Saks, M. (2019). Constant factor approximations to edit distance on far input pairs in nearly linear time. *arXiv preprint arXiv:1904.05459*.

Kuszmaul, W. (2019). Dynamic time warping in strongly subquadratic time: Algorithms for the low-distance regime and approximate evaluation. *arXiv preprint arXiv:1904.09690*.

Landau, G. M., Myers, E. W., and Schmidt, J. P. (1998). Incremental string comparison. *SIAM Journal on Computing*, 27(2):557–582.

Leiserson, C. E., Rivest, R. L., Cormen, T. H., and Stein, C. (2001). *Introduction to algorithms*, volume 6. MIT press Cambridge, MA.

Liben-Nowell, D., Vee, E., and Zhu, A. (2005). Finding longest increasing and common subsequences in streaming data.

Masek, W. J. and Paterson, M. S. (1980). A faster algorithm computing string edit distances. *Journal of Computer and System Sciences*, 20(1):18–31.

Ostrovsky, R. and Rabani, Y. (2005). Low distortion embeddings for edit distance. In *STOC*, pages 218–224. ACM.

Rubinstein, A. and Brakensiek, J. (2019). Constant-factor approximation of near-linear edit distance in near-linear time. *arXiv preprint arXiv:1904.05390*.

Rubinstein, A. and Song, Z. (2019). Reducing approximate longest common subsequence to approximate edit distance. *arXiv preprint arXiv:1904.05451*.

Runbinstein, A., Seddighin, S., Song, Z., and Sun, X. (2019). Approximation algorithms for LCS and LIS with truly improved running times. In *FOCS*. IEEE. in press.

Saha, B. (2015). Language edit distance and maximum likelihood parsing of stochastic grammars: Faster algorithms and connection to fundamental graph problems. In *FOCS*, pages 118–135. IEEE.

Saha, B. (2017). Fast & space-efficient approximations of language edit distance and RNA folding: An amnesic dynamic programming approach. In *FOCS*, pages 295–306. IEEE.

Saks, M. E. and Seshadhri, C. (2013). Space efficient streaming algorithms for the distance to monotonicity and asymmetric edit distance. In *SODA*, pages 1698–1709.

Savitch, W. J. (1970). Relationships between nondeterministic and deterministic tape complexities. *Journal of computer and system sciences*, 4(2):177–192.
[58] Sun, X. and Woodruff, D. P. (2007). The communication and streaming complexity of computing the longest common and increasing subsequences. In Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2007, New Orleans, Louisiana, USA, January 7-9, 2007, pages 336–345.
A Omitted proofs

Theorem 2.1. For given two strings $s$ and $\bar{s}$ of length $n$, suppose we have a random access to the characters of $s$ and $\bar{s}$ using a oracle. Then $\text{lcs}(s, \bar{s})$ and $\text{ed}(s, \bar{s})$ can be computed using $O(\log^2 n)$ memory.

Proof. It is known that the LCS and the edit distance problems are in Non-deterministic Logarithmic-space (NL) complexity class. This means that we can solve these problem using a non-deterministic Turing machine with a memory of $O(\log n)$. Savitch’s theorem [57] says that every problem in NL can be solved using a deterministic Turing matching with a memory of $O(\log^2 n)$, which implies the theorem. \qed