CLARKSON’S TYPE INEQUALITIES FOR POSITIVE $l_p$ SEQUENCES WITH $p \geq 2$

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ABSTRACT. For a fixed $1 \leq p < +\infty$ denote by $\| \cdot \|_p$ the usual norm in the space $l_p$ (or $L_p$). In this paper we prove that for all real numbers $p$ and $q$ such that $2 \leq p \leq q$ holds

$$2\|x\|_p^q + \|y\|_p^q \leq \|x + y\|_p^q + \|x - y\|_p^q$$

for all nonnegative sequences $x = \{x_n\}, y = \{y_n\} in l_p$ (or nonnegative functions $x, y$ in $L_p$). Note that the above inequality with $p = q \geq 2$ reduces to the well known Clarkson’s inequality. If in addition, holds $x_i \geq y_i$ for each $i = 1, 2, \ldots$ (or $x \geq y$ a.e. in $L_p$), then we establish an improvement of the above inequality.

1. INTRODUCTION

Let $(X, \Sigma, \mu)$ be a measure space with a positive Borel measure $\mu$. For any $0 < p < +\infty$ let $L^p = L^p(\mu)$ denotes the usual Lebesgue space consisting of all $\mu$-measurable complex-valued functions $f : X \to \mathbb{C}$ such that $\int_X |f|^p d\mu < +\infty$. Recall that for any $p \geq 2$ the usual norm $\| \cdot \|_p$ of $f \in L^p$ is defined as $\|f\|_p = \left(\int_X |f|^p d\mu\right)^{1/p}$. Further, the space $l_p$ is the space of all complex sequences $a = \{a_n\}$ such that $\sum_{n=1}^{+\infty} |a_n|^p < +\infty$; if $2 \leq p < +\infty$, then the usual $l_p$-norm is defined for $a \in l_p$ as $\|a\|_p := \left(\sum_{n=1}^{+\infty} |a_n|^p\right)^{1/p}$.

In [1] J. Clarkson introduced the concept of uniform convexity in Banach spaces and obtained that the spaces $L_p$ (or $l_p$) with $1 < p < +\infty$ are uniformly convex throughout the following inequalities.

**Theorem 1.1.** ([1] Theorem 2) Let $2 \leq p < +\infty$, and let $q = p/(p-1)$ be the conjugate of $p$. Then for all $x$ and $y$ in $L_p$ or $l_p$,

1. $2\|x\|_p^q + \|y\|_p^q \leq \|x + y\|_p^q + \|x - y\|_p^q$ \hspace{.5cm} (1.1)
2. $\|x + y\|_p^q + \|x - y\|_p^q \leq 2\|x\|_p^q + \|y\|_p^q$ \hspace{.5cm} (1.2)
3. $2\|x\|_p^q + \|y\|_p^q \leq \|x + y\|_p^q + \|x - y\|_p^q \leq 2^{p-1}(\|x\|_p^p + \|y\|_p^p)$ \hspace{.5cm} (1.3)

For $1 < p \leq 2$ these inequalities hold in the reverse sense.

Clarkson pointed out that for all values of $p$ the right hand side of (1.3) is equivalent to the left hand side, while (1.1) is equivalent to (1.2); to see this, set $x + y = u, x - y = v$ and reduce. Notice that the proof of (1.2) ([1] Proof of Theorem 2) is rather long and nontrivial. In this proof Clarkson deduced (1.3) from (1.1). A direct simple proof of the inequality (1.3), based on the classical Hőlder inequality was given in [2].

For each $1 \leq p < +\infty$ denote by $l^+_p$ the set of all nonnegative real sequences in $l_p$, and by $L^+_p$ the set of all nonnegative real-valued functions in $L_p$. In this note we prove an extension of the inequality (1.3) for the spaces $l^+_p$ (or $L^+_p$) with $p \geq 2$ as follows.

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Theorem 1.2. Let $2 \leq p \leq q < +\infty$. Then for all $x$ and $y$ in $l_p^+$ (or $L_p^+$) we have

$$2(\|x\|_p^q + \|y\|_p^q) \leq \|x + y\|_p^q + \|x - y\|_p^q.$$  

Remark 1.3. Note that for $p = q \geq 2$ the inequality (1.4) reduces to the Clarkson’s inequality on the left hand side of (1.3).

On the other hand, if $2 \leq p \leq q < +\infty$, then $1/p + 1/q = 1$ only for $p = q = 2$, and thus the inequality (1.4) cannot be derived from any Clarkson’s inequalities in Theorem 1.1.

The following result is basic for the proof of Theorem 1.2.

Proposition 1.4. Let $2 \leq p \leq q < +\infty$ and let $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ be nonnegative sequences in $l_p^+$ such that $u_i \geq v_i$ for each $i = 1, 2, \ldots, n$ (or $u$ and $v$ be functions in $L_p^+$ such that $u \geq v$ a.e. on $X$). Then

$$\|u + v\|_p^q + \|u - v\|_p^q \geq 2(\|u\|_p^q + 2^{q-2}\|v\|_p^q).$$

In particular, for $q = p \geq 2$, we have

$$\|u + v\|_p^p + \|u - v\|_p^p \geq 2(\|u\|_p^p + 2^{p-2}\|v\|_p^p).$$

As an immediate consequence, we obtain an improvement of the Clarkson’s inequality on the left hand side of (1.3) in the real case.

Corollary 1.5. For real numbers $x \geq y \geq 0$ and $q \geq 2$ holds

$$2(x^q + 2^{q-2}y^q) \leq (x + y)^q + (x - y)^q.$$  

The proofs of Proposition 1.4 and Theorem 1.2 are given in the next section and they are very simple and elementary.

2. PROOFS OF PROPOSITION 1.4 AND THEOREM 1.2

The following interesting lemma is basic in the proofs of Proposition 1.4.

Lemma 2.1. Let $2 \leq p \leq q < +\infty$ and let $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ be $n$-tuples of nonnegative real numbers such that $u_i \geq v_i$ for each $i = 1, 2, \ldots, n$. Define the function $\varphi$ from $[0, 1]$ into $\mathbb{R}$ by

$$\varphi(t) = \|u + tv\|_p^q + \|u - tv\|_p^q - 2^{q-1}(\|u\|_p^q + \|tv\|_p^q), \quad t \in [0, 1].$$

Then $\varphi$ is a nondecreasing function on $[0, 1]$.

Proof. Obviously, (2.1) may be written as

$$\varphi(t) = \left(\sum_{i=1}^{n} |u_i + v_i t|^p\right)^{q/p} + \left(\sum_{i=1}^{n} |u_i - v_i t|^p\right)^{q/p} - 2^{q-1}\left(\sum_{i=1}^{n} |u_i|^p\right)^{q/p} + \left(\sum_{i=1}^{n} |v_i|^p\right)^{q/p}.$$  

Note that the function $\varphi$ is continuous on $[0, 1]$ and differentiable on the set $D := (0, 1) \setminus \{\pm u_i / v_i : v_i \neq 0, i = 1, 2, \ldots, n\}$. Since for any fixed $1 \leq i \leq n$ hold $u_i + v_i t \geq 0$ and
Now inserting the inequality (2.4) into (2.3), we immediately obtain that for each \( t \in D \). Hence, the derivative of \( \varphi \) for \( t \in D \) is

\[
\varphi'(t) = q \left( \sum_{i=1}^{n} (u_i + v_i t)^p \right)^{(q/p) - 1} \cdot \left( \sum_{i=1}^{n} v_i (u_i + v_i t)^{p-1} \right)
- \left( \sum_{i=1}^{n} (u_i - v_i t)^p \right)^{(q/p) - 1} \cdot \left( \sum_{i=1}^{n} v_i (u_i - v_i t)^{p-1} \right)
- 2^{q-1} \left( \sum_{i=1}^{n} v_i^p \right)^{q/p} t^{q-1}.
\]

(2.3)

We will show that \( \varphi'(t) \geq 0 \) for each \( t \in D \). To show this, we use the well known inequality \((a+b)^r \geq a^r + b^r\) for real numbers \( a \geq 0, b \geq 0 \) and \( r \geq 1 \). Applying this inequality in the form \((u_i + v_i t)^r \geq (u_i - v_i t)^r + (2v_i t)^r\) for \( 1 \leq i \leq n \), with \( r \in \{p, p-1\} \), and using that \( q/p \geq 1 \), we estimate the first term on the right of (2.3) as

\[
\sum_{i=1}^{n} (u_i + v_i t)^p \geq \sum_{i=1}^{n} (u_i - v_i t)^p + \sum_{i=1}^{n} (2v_i t)^p
= \sum_{i=1}^{n} (u_i - v_i t)^p + 2^p \sum_{i=1}^{n} v_i^p
\]

(2.4)

Now inserting the inequality (2.4) into (2.3), we immediately obtain that for each \( t \in D \)

\[
\varphi'(t) \geq q \left( 2^{q-1} \left( \sum_{i=1}^{n} v_i^p \right)^{q/p} t^{q-1} - 2^{q-1} \left( \sum_{i=1}^{n} v_i^p \right)^{q/p} t^{q-1} \right) = 0.
\]

(2.5)

Therefore, \( \varphi'(t) \geq 0 \) for each \( t \in D \), and since \( \varphi \) is a continuous function on \([0, 1]\), we infer that \( \varphi \) is a nondecreasing function on \([0, 1]\), and the proof is completed.

\[\square\]

Proof of Proposition 1.4. We first consider the case \( 2 \leq p \leq q < +\infty \) related to the space \( l_p \). By the continuity of \( l_p \)-norm, it is enough to prove the inequality (1.5) for two arbitrary \( n \)-tuples \( u = (u_1, u_2, \ldots, u_n) \) and \( v = (v_1, v_2, \ldots, v_n) \) of nonnegative real numbers with \( u_i \geq v_i \) for each \( i = 1, 2, \ldots, n \). Since the function \( \varphi \) defined by (2.1) from Lemma 2.1
increases on $[0, 1]$, we have
\[
\|u + v\|_p^q + \|u - v\|_p^q \geq \|u\|_p^q + \|v\|_p^q = \varphi(1) \geq \varphi(0) = 2\|u\|_p^q - 2^{q-1}\|u\|_p^q,
\]
or equivalently,
\[
\|u + v\|_p^q + \|u - v\|_p^q \geq 2(\|u\|_p^q + 2^{q-2}\|v\|_p^q),
\]
as desired.

In order to prove that the inequality (2.8) can be extended for any two functions $u, v \in L^+_p$ such that $u \geq v$ a.e. in $L^+_p$, it is enough to apply a standard argument that the set of all measurable simple functions $s : X \to [0, +\infty)$ forms a dense subset in $L_p$, and so the result follows by the continuity of the norm (see [1, Proof of Theorem 2]; also cf. [3, Proof of Theorem 2.3]). □

For the proof of Theorem 1.2 we will need still two following lemmas.

**Lemma 2.2.** Let $A, a, B, b$ and $r \geq 1$ be nonnegative real numbers such that $A \geq B$ and $a > b$. Then
\[
(A + a)^r + (B + b)^r > (A + b)^r + (B + a)^r.
\]

**Proof.** Put $c = a + b$, and define the function $f$ from $[c/2, c]$ into $\mathbb{R}$ by
\[
f(t) = (A + t)^r + (B + c - t)^r, \quad t \in \left[\frac{c}{2}, c\right].
\]
Then
\[
f'(t) = r((A + t)^{r-1} - (B + c - t)^{r-1}), \quad t \in \left[\frac{c}{2}, c\right].
\]
Since $A \geq B$, we see that $A + t > B + c - t \geq 0$ for $t \in (c/2, c]$; so, for such a $t$, we have $(A + t)^{r-1} - (B + c - t)^{r-1} > 0$. This shows that $f$ is an increasing function on $(c/2, c] = ((a + b)/2, a + b]$. This together with the fact that $(a + b)/2 < b < a \leq a + b$ yields $f(a) > f(b)$, which is actually the inequality (2.8). □

**Lemma 2.3.** Let $r \geq 1$ and let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be $n$-tuples of nonnegative real numbers. Define sequences $x' = (u_1, u_2, \ldots, u_n)$ and $y' = (v_1, v_2, \ldots, v_n)$ as
\[
u_i = x_i \text{ and } v_i = y_i \text{ if } x_i \geq y_i
\]
\[
u_i = y_i \text{ and } v_i = x_i \text{ if } x_i < y_i
\]
for each $i = 1, 2, \ldots, n$. Then
\[
\left(\sum_{i=1}^n u_i\right)^r + \left(\sum_{i=1}^n y_i\right)^r \geq \left(\sum_{i=1}^n x_i\right)^r + \left(\sum_{i=1}^n y_i\right)^r.
\]

**Proof.** Without loss of generality, we can suppose that
\[
S := \left(\sum_{i=1}^n x_i\right)^r \geq \left(\sum_{i=1}^n y_i\right)^r := T.
\]
Clearly, in order to prove the inequality (2.12), it is sufficient to show that if for some index $k$ holds $x_k < y_k$, then after the interchange of $x_k$ and $y_k$ the sum on the right hand side of
increases. Assume that \( x_k < y_k \) for some \( k \in \{1, 2, \ldots, n\} \), and denote new sums by

\[
S' := \left( \sum_{i \neq k} x_i + y_k \right)^r = \left( \sum_{i=1}^n x_i - x_k + y_k \right)^r
\]

(2.14)

\[
T' := \left( \sum_{i \neq k} y_i + x_k \right)^r = \left( \sum_{i=1}^n y_i - y_k + x_k \right)^r.
\]

(2.15)

If we denote \( A = \sum_{i=1}^n x_i - x_k \) and \( B = \sum_{i=1}^n y_i - y_k \), then from (2.13) and \( x_k < y_k \) we see that \( A > B \geq 0 \). Now applying Lemma 2.2 with \( a = y_k \) and \( b = x_k \), we find that

\[
S' + T' = (A + y_k)^r + (B + x_k)^r > (A + x_k)^r + (B + y_k)^r
\]

(2.16)

\[
= \left( \sum_{i=1}^n x_i - x_k + x_k \right)^r + \left( \sum_{i=1}^n y_i - y_k + y_k \right)^r = S + T.
\]

Hence, repeating the above procedure successfully for all indices \( k \in \{1, 2, \ldots, n\} \) such that \( x_k < y_k \), we obtain our inequality (2.12).

□

**Proof of Theorem 1.2.** We first consider the case \( 2 \leq p \leq q < +\infty \) related to the sequence space \( l_p \). By the continuity of \( l_p \)-norm, it is enough to prove the inequality (1.4) for any two nonnegative \( n \)-tuples \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \). Because of symmetry, without loss of generality we can suppose that \( \|x\|_p \geq \|y\|_p \). Accordingly, we can assume that \( x_1 \geq y_1 \). This fact, together with Lemma 2.3 with \( x_i^p \) and \( y_i^p \) instead of \( x_i \) and \( y_i \), respectively, and with \( u_i, v_i \), defined by (2.11) \( (i = 1, 2, \ldots, n) \), yields that

\[
\left( \sum_{i=1}^n |u_i|^p \right)^{q/p} + \left( \sum_{i=1}^n |v_i|^p \right)^{q/p} \geq \left( \sum_{i=1}^n |x_i|^p \right)^{q/p} + \left( \sum_{i=1}^n |y_i|^p \right)^{q/p}
\]

\[
= \|x\|_p^q + \|y\|_p^q.
\]

(2.17)

Obviously, the expression \( \|x + y\|_p^q + \|x - y\|_p^q \) is invariant under the interchange \( x_i \) with \( y_i \) for any \( i = 1, 2, \ldots, n \), and therefore,

\[
\|x + y\|_p^q + \|x - y\|_p^q = \left( \sum_{i=1}^n |u_i + v_i|^p \right)^{q/p} + \left( \sum_{i=1}^n |u_i - v_i|^p \right)^{q/p}.
\]

(2.18)

Further, since \( q \geq 2 \), the inequality (1.5) in Theorem 1.2 implies that

\[
\left( \sum_{i=1}^n |u_i + v_i|^p \right)^{q/p} + \left( \sum_{i=1}^n |u_i - v_i|^p \right)^{q/p} \geq 2 \left( \sum_{i=1}^n |u_i|^p \right)^{q/p} + \left( \sum_{i=1}^n |v_i|^p \right)^{q/p}.
\]

(2.19)

Clearly, the relations (2.17), (2.18) and (2.19) immediately imply the inequality (1.4), that is,

\[
\|x + y\|_p^q + \|x - y\|_p^q \geq 2(\|x\|_p^q + \|y\|_p^q).
\]

(2.20)

Finally, recall that the inequality (1.4) may be extended to the space \( L_p^x \), by applying the same argument as noticed at the end of proof of Theorem 1.2.

□

**Remark 2.4.** Note that the method in our proof of Theorem 1.2 cannot be applied for the proof of Clarkson’s inequalities (1.1) and (1.2). For example, Clarkson’s inequality (1.1) in the real case may be written as

\[
|x + y|^q + |x - y|^q \geq 2(|x|^p + |y|^p)^{q-1}, \quad x, y \in \mathbb{R}, p = q/(q - 1) \geq 2.
\]

(2.21)
Assuming that $|x| \leq |y|$ and setting $c = |x|/|y| \leq 1$, the inequality (2.21) is equivalent to
\begin{equation}
(1 + c)^q + (1 - c)^q \geq 2(1 + c^p)^{q-1}, \quad 0 \leq c \leq 1.
\end{equation}

Next for a fixed $c \in [0, 1]$, consider the function $\psi$ from $[0, 1]$ into $\mathbb{R}$ defined by
\begin{equation}
\psi(t) = (1 + ct)^q + (1 - ct)^q - 2(1 + c^p t^p)^{q-1}, \quad t \in [0, 1]
\end{equation}
A direct calculation gives
\begin{equation}
\psi'(t) = qc(1+ct)^{q-1}q - qc(1-ct)^{q-1}q - 2p(q-1)(1+c^p t^p)^{q-2}c^p t^{p-1}, \quad t \in [0, 1],
\end{equation}
which by using that $p(q-1) = q$ and setting $s = ct$, can be written as
\begin{equation}
\psi'(t) = \chi(s) = qc((1+s)^{q-1} - (1-s)^{q-1} - 2(1+s^p)^{q-2}s^{p-1}), \quad s \in [0, c].
\end{equation}

However, direct calculations show that for several values of $p$ and $q$ with $2 \leq p \leq q$ and a fixed $0 < c \leq 1$, the function $\chi(s)$ takes positive and negative values on the segment $[0, c]$. Thus, generally, the function $\psi(t)$ is not monotone on $[0, c]$.

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