FRACTIONAL INTEGRALS ASSOCIATED WITH RADON TRANSFORMS

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Abstract. We obtain sharp $L^p$-$L^q$ estimates for fractional integrals generated by Radon transforms of the following three types: the classical Radon transform over the set of all hyperplanes in $\mathbb{R}^n$, the Strichartz transversal transform over only those hyperplanes, which meet the last coordinate axis, and the Radon transform associated with paraboloids. The method relies on a version of Stein’s interpolation theorem for analytic families of operators communicated by L. Grafakos.

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2010 Mathematics Subject Classification. Primary 42B20; Secondary 44A12.
Key words and phrases. Fractional integrals, Radon transforms, norm estimates.
1. Introduction

The present article grew up from our study of the Radon-type transforms

\[(Rf)(\theta, t) = \int_{\theta^\perp} f(y + t\theta) \, d\theta y, \quad (1.1)\]

\[(Tf)(x) = \int_{\mathbb{R}^{n-1}} f(y', x_n + x' \cdot y') \, dy', \quad (1.2)\]

\[(Pf)(x) = \int_{\mathbb{R}^{n-1}} f(x' - y', x_n - |y'|^2) \, dy', \quad (1.3)\]

arising in Analysis and applications. Here

\((\theta, t) \in S^{n-1} \times \mathbb{R}, \quad x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n) \in \mathbb{R}^n\) (similarly for \(y\)),

\(S^{n-1}\) is the unit sphere in \(\mathbb{R}^n\), \(\theta^\perp\) is the hyperplane orthogonal to \(\theta\) and passing through the origin, \(d\theta y\) denotes the Euclidean measure on \(\theta^\perp\), \(f\) is a sufficiently good function on \(\mathbb{R}^n\). We also write

\[(R_\theta f)(t) = (Rf)(\theta, t), \quad (Tx'f)(x_n) = (Tf)(x', x_n),\]

\[(P_{x'}f)(x_n) = (Pf)(x', x_n).\]

The first operator is the well-known hyperplane Radon transform; see, e.g., [12, 29], and references therein. The second operator integrates functions on \(\mathbb{R}^n\) over only those hyperplanes which meet the last coordinate axis. Following Strichartz [41], we call it the transversal Radon transform; see also [5, 26], [29, Section 4.13]. The third operator was considered by Christ [5] and the author [30]. It performs integration over shifted paraboloids. We call it the parabolic Radon transform. Different parametrizations of Radon transforms were discussed by Ehrenpreis [6]. Sharp \(L^p-L^q\) estimates for \(R\) were obtained by Oberlin and Stein [18]. They yield similar estimates for \(T\) and \(P\) because these three operators are intimately connected [5, 26, 29, 30].

The purpose of the present paper is two-fold. We plan to examine known \(L^p-L^q\) boundedness results for the operators \(R\), \(T\), and \(P\), which were previously obtained with the aid of Stein’s interpolation theorem for analytic families of operators [37]. Our second aim is to study the corresponding analytic families themselves, because they are of interest on their own right.

In fact, there exist many such families indexed by a complex parameter \(\alpha\) and associated with the afore-mentioned Radon transforms. They
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can be defined using the tools of single-variable Fractional Calculus and represented by convolutions of the form

\begin{align}
(1.6) & \quad R_\pm f = h_\pm^\alpha \ast R_0 f, \quad R_0^\alpha f = h_0^\alpha \ast R_\theta f, \quad R_s^\alpha f = h_s^\alpha \ast R_\theta f; \\
(1.7) & \quad T_\pm f = h_\pm^\alpha \ast T_0 f, \quad T_0^\alpha f = h_0^\alpha \ast T_\theta f, \quad T_s^\alpha f = h_s^\alpha \ast T_\theta f; \\
(1.8) & \quad P_\pm f = h_\pm^\alpha \ast P_0 f, \quad P_0^\alpha f = h_0^\alpha \ast P_\theta f, \quad P_s^\alpha f = h_s^\alpha \ast P_\theta f.
\end{align}

Here \( h_\pm^\alpha, h_0^\alpha, \) and \( h_s^\alpha \) are single-variable tempered distributions acting on \( R, T, \) and \( P \) in the last variable and defined by the formulas

\begin{align}
(1.9) & \quad h_\pm^\alpha(\tau) = \frac{\tau^{\alpha-1}}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)} \begin{cases} |\tau|^{\alpha-1} & \text{if } \pm \tau > 0, \\ 0 & \text{otherwise}; \end{cases} \\
(1.10) & \quad h_0^\alpha(\tau) = \frac{|\tau|^{\alpha-1}}{\gamma(\alpha)}, \quad \alpha \neq 1, 3, 5, \ldots; \\
(1.11) & \quad h_s^\alpha(\tau) = \frac{|\tau|^{\alpha-1}\text{sgn}(\tau)}{\gamma'(\alpha)}, \quad \alpha \neq 2, 4, 6, \ldots;
\end{align}

\( \gamma(\alpha) = 2\Gamma(\alpha) \cos(\alpha\pi/2), \quad \gamma'(\alpha) = 2i\Gamma(\alpha) \sin(\alpha\pi/2); \)

see, e.g., [8, Chapter I, Section 3], [7, 27, 33].

All these operators can be treated using the same ideas as \( R_+^\alpha, T_+^\alpha, \) and \( P_+^\alpha. \) For the sake of simplicity, we will be focusing only on these three operators and leave others to the interested reader. The corresponding one-dimensional convolution

\begin{align}
(1.12) & \quad (I_+^\alpha \omega)(t) = (h_+^\alpha \ast \omega)(t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \tau^{\alpha-1} \omega(t - \tau) d\tau,
\end{align}

is the well-known Riemann-Liouville fractional integral [33]. Fractional integrals \( R_0^\alpha f \) were introduced by V.I. Semyanistyi [34]. The operators \( R_0^\alpha, R_+^\alpha, \) and \( R_s^\alpha \) were also considered in [27] and [29, Section 4.9.3] in the framework of the Semyanistyi-Lizorkin space of Schwartz functions orthogonal to all polynomials (see Definition 2.1).

Main results. One of the main results can be stated as follows.

**Theorem 1.1.** Let \( 1 \le p, q \le \infty, \alpha_0 = \Re \alpha. \) The operators \( T_+^\alpha \) and \( P_+^\alpha, \) initially defined on Schwartz functions \( f \in S(\mathbb{R}^n) \) by analytic
continuation, extend as linear bounded operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if

$$\frac{1-n}{2} \leq \alpha_0 \leq 1, \quad p = \frac{n+1}{n+\alpha_0}, \quad q = \frac{n+1}{1-\alpha_0}. \tag{1.13}$$

This statement is a combination of Theorems 5.5 and 7.8. The case $\alpha = 0$ gives known results for the Radon transforms $T$ and $P$ [5, 30].

We also obtain a similar “if and only if” statement for the operator $R_+^\alpha$ in the cases $\alpha = 0$ and $0 < \Re \alpha < 1$; see Theorems 6.2 and 6.3. Theorem 6.2 represents an alternative version of the Oberlin-Stein boundedness result for the Radon transform $R$. Theorem 6.3, which includes the weak-type estimate for $p = 1$, is an analogue of the Hardy-Littlewood-Sobolev theorem for Riesz potentials; cf. [38, Chapter V, Section 1.2], [35, Theorem 0.3.2], [36, p. 189]).

For $(1-n)/2 \leq \Re \alpha \leq 0$, $\alpha \neq 0$, we show that the “if” part of Theorem 1.1 still holds for $R_+^\alpha$. The validity of the “only if” part for these values of $\alpha$ is an open problem. We conjecture that this problem has an affirmative answer, but to prove it, one needs a suitable modification of Stein’s interpolation theorem (see comments below).

Theorem 1.1 for $T_+^\alpha$ and $P_+^\alpha$, and the corresponding statement for $R_+^\alpha$ guarantee the existence of the $L^p$-$L^q$ bounded extensions of the operators provided by interpolation. It is natural to ask:

What are the explicit analytic formulas for these extensions?

Of course, the corresponding expressions can be described in terms of distributions. However, they belong to $L^q$, and we wonder how these $L^q$-functions look like pointwise. A similar question for solutions of the wave equation was studied in [24]. The case $\Re \alpha < 0$ is especially intriguing because our operators are not represented by absolutely convergent integrals and need a suitable $L^q$-regularization. We shall prove (see Theorem 8.1) that the latter can be performed by making use of the hypersingular integrals, generalizing the concept of Marchaud’s fractional derivative [16, 23, 33].

The plan of the paper is reflected in the Contents.

Comments and challenges.

1. One of the motivations for writing this article was the following observation. A remarkable $L^p$-$L^q$ boundedness result for the Radon transform (1.1) was obtained by Oberlin and Stein [18] by making use of the operator family $\{R_0^\alpha\}$ (up to a constant multiple) and Stein’s interpolation theorem [37, Theorem 1]. Although formal application of this theorem yields the correct result, justification of the applicability of the theorem does not seem to be trivial. The crux of the matter is that the assumptions of Stein’s theorem and its proof are given in
terms of simple functions (finite linear combinations of the characteristic functions of disjoint compact sets), and it is not so clear how to check the validity of these assumptions for the operator family \( \{R^\alpha_0\} \), which is defined in terms functions in \( S(\mathbb{R}^n) \), rather than simple ones.

Since the details related to implementation of simple functions were omitted in [18] and the results of [18] have already been used in other publications, it was natural to try to find an alternative approach and present it in full detail. To circumvent this obstacle, we invoke fractional integrals associated with the transversal transform \( T \). This way became possible thanks to the recent modification of Stein’s interpolation theorem due to Grafakos [10], who replaced simple functions by arbitrary smooth compactly supported functions. That was done in the case when operators under consideration take functions on \( \mathbb{R}^n \) to functions on \( \mathbb{R}^n \).

To treat the problem directly and thus cover the “only if” part for all \( (1 - n)/2 \leq \Re \alpha \leq 0 \), we need an interpolation theorem like that in [10], but for the case when the source space and the target space are not necessarily the same. In our case, the target space must be \( S^{n-1} \times \mathbb{R} \). To the best of my knowledge, such a more general interpolation theorem is not available in the literature and is highly desirable for the study of diverse analytic families arising in Analysis and its applications.

Regarding the general context of metric measure spaces, including the case when the source space and the target space may be different, one should mention a recent paper by Grafakos and Ouhabaz [11]. In this paper, simple functions are substituted by continuous compactly supported functions, not necessarily smooth. The interpolation theorem from [11] is not directly applicable to the operator families (1.6)-(1.8), because the smoothness of \( f \) is crucial for the their definition when \( \alpha \) is negative.

Note also that unlike [18], to minimize technicalities related to fractional powers of the Laplacian, we prefer to work with the analytic family \( \{R^\alpha_+\} \), rather than \( \{R^\alpha_0\} \).

2. Some \( L^p-L^q \) estimates for the localized modifications of \( P_{\pm}^\alpha f \) and \( Pf \) with a smooth cut-off function under the sign of integration were announced by Littman [15] and Tao [42], who also referred to Stein’s interpolation theorem in [37]. In contrast, our operators are not localized and we use another interpolation technique, which is based on [10].
Acknowledgements. The author is grateful to Professor Loukas Grafakos for useful discussions and sharing his knowledge of the subject. Special thanks go to Professor Daniel M. Oberlin for correspondence.

2. Preliminaries

2.1. Notation.
In the following, \( x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n) \in \mathbb{R}^n \); \( S^{n-1} \) is the unit sphere in \( \mathbb{R}^n \) with the surface area measure \( d\theta; \sigma_{n-1} \equiv \int_{S^{n-1}} d\theta = 2\pi^{n/2}/\Gamma(n/2) \) is the area of \( S^{n-1} \). The notation \( C(\mathbb{R}^n), C^\infty(\mathbb{R}^n) \), and \( L^p(\mathbb{R}^n) \) for function spaces is standard; \( \| \cdot \|_p = \| \cdot \|_{L^p(\mathbb{R}^n)} \); \( C_0(\mathbb{R}^n) = \{ f \in C(\mathbb{R}^n) : \lim_{|x| \to \infty} f(x) = 0 \} \); \( C_c^\infty(\mathbb{R}^n) \) is the space of compactly supported infinitely differentiable functions on \( \mathbb{R}^n \). A similar notation will be used for functions on the cylinder \( Z_n = S^{n-1} \times \mathbb{R} \). The corresponding \( L^p \)-norms will be denoted by \( \| \cdot \|_p \).

The notation \( \langle f, g \rangle \) for functions \( f \) and \( g \) is used for the integral of the product of these functions. We keep the same notation when \( f \) is a distribution and \( g \) is a test function.

If \( m = (m_1, \ldots, m_n) \) is a multi-index, then \( \partial^m = \partial_1^{m_1} \cdots \partial_n^{m_n} \), where \( \partial_i = \partial/\partial x_i \); \( \Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2 \) is the Laplace operator. The Fourier transform of a function \( f \in L^1(\mathbb{R}^n) \) is defined by

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{ix \cdot \xi} \, dx, \quad \xi \in \mathbb{R}^n,
\]

where \( x \cdot \xi = x_1 \xi_1 + \cdots + x_n \xi_n \). We denote by \( S(\mathbb{R}^n) \) the Schwartz space of \( C^\infty \)-functions which are rapidly decreasing together with their derivatives of all orders. The space \( S(\mathbb{R}^n) \) is equipped with the topology generated by the sequence of norms

\[
\| \varphi \|_k = \max_x (1 + |x|)^k \sum_{|m| \leq k} |(\partial^m \varphi)(x)|, \quad k = 0, 1, 2, \ldots.
\]

The Fourier transform is an isomorphism of \( S(\mathbb{R}^n) \). The space of tempered distributions, which is dual to \( S(\mathbb{R}^n) \), is denoted by \( S'(\mathbb{R}^n) \). The Fourier transform of a distribution \( f \in S'(\mathbb{R}^n) \) is a distribution \( \hat{f} \in S'(\mathbb{R}^n) \) defined by

\[
\langle \hat{f}, \psi \rangle = \langle f, \hat{\psi} \rangle, \quad \psi \in S(\mathbb{R}^n).
\]

The equality (2.3) is equivalent to

\[
\langle \hat{\varphi}, \varphi \rangle = (2\pi)^n \langle f, \varphi_1 \rangle, \quad \varphi \in S(\mathbb{R}^n), \quad \varphi_1(x) = \varphi(-x).
\]
The inverse Fourier transform of a function (or distribution) \( f \) is denoted by \( \hat{f} \).

Given a real-valued quantity \( X \) and a complex number \( \lambda \), we set \((X)^{\lambda}_{\mp} = |X|^\lambda \) if \( \pm X > 0 \) and \((X)^{\lambda}_{\pm} = 0\), otherwise. All integrals are understood in the Lebesgue sense. The letter \( c \), sometimes with subscripts, stands for a nonessential constant that may be different at each occurrence. We write \( \mathbb{N} \) for the set of all positive integers; \( \mathbb{Z}_+ \) denotes the set of all nonnegative integers; \( \mathbb{Z}_n^+ \) is the set of all multi-indices \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \), \( \gamma_i \in \mathbb{Z}_+ \) \((i = 1, 2, \ldots, n)\). The notation \( e_1, \ldots, e_n \) will be used for coordinate unit vectors in \( \mathbb{R}^n \); \( O(n) \) is the group of orthogonal transformation of \( \mathbb{R}^n \) equipped with the probability Haar measure.

2.2. The Semyanistyi-Lizorkin spaces. To circumvent some obstacles arising in the study of convolutions whose Fourier transforms have singularities, it may be convenient to reduce the space \( S(\mathbb{R}^n) \) of the test functions in a suitable way and expand the corresponding space of distributions. An idea of this approach originated in the work of Semyanistyi [34] who treated the Fourier multipliers having singularity at a single point \( x = 0 \). This idea was independently used by Helgason [13, p. 162]. It was extended by Lizorkin [14] and later by Samko [31] to more general sets of singularities; see also [29, 32, 33, 44] and references therein. In the present paper, the set of singularities is a either the hyperplane \( x_n = 0 \) or a single point \( x = 0 \).

**Definition 2.1.** For \( n \geq 2 \), we denote by \( \Psi(\mathbb{R}^n) \), the subspace of functions \( \psi \in S(\mathbb{R}^n) \) vanishing with all derivatives on the hyperplane \( x_n = 0 \). Let \( \Phi(\mathbb{R}^n) \) be the Fourier image of \( \Psi(\mathbb{R}^n) \). For \( n \geq 1 \), we denote by \( \Psi_0(\mathbb{R}^n) \) the subspace of functions in \( S(\mathbb{R}^n) \) vanishing with all derivatives at the origin \( x = 0 \). The Fourier image of \( \Psi_0(\mathbb{R}^n) \) will be denoted by \( \Phi_0(\mathbb{R}^n) \).

**Proposition 2.2.** (cf. [29, Proposition 4.141]) A function \( \varphi \in S(\mathbb{R}^n) \) belongs to \( \Phi(\mathbb{R}^n) \) if and only if

\[
\int_{\mathbb{R}} \varphi(x', x_n) x_n^k \, dx_n = 0 \quad \text{for all} \quad k \in \mathbb{Z}_+ \quad \text{and all} \quad x' \in \mathbb{R}^{n-1}.
\]

The space \( \Phi_0(\mathbb{R}^n) \) consists of Schwartz functions orthogonal to all polynomials.

**Proposition 2.3.** (cf. [32, Theorem 2.33]) The spaces \( \Phi(\mathbb{R}^n) \) and \( \Phi_0(\mathbb{R}^n) \) are dense in \( L^p(\mathbb{R}^n) \), \( 1 < p < \infty \).
Proposition 2.4. (cf. [25, Lemma 3.11], [44]) Let $1 \leq p, q < \infty$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ coincide as $\Phi'$-distributions or as $\Phi'_0$-distributions, then they coincide almost everywhere on $\mathbb{R}^n$.

3. Two modifications of the Radon transform over hyperplanes

Let $\Pi_n$ be the set of all unoriented hyperplanes in $\mathbb{R}^n$. The Radon transform of a sufficiently good function $f$ on $\mathbb{R}^n$ is a function on $\Pi_n$ defined by the formula

\[(Rf)(h) = \int f(x) \, d_h x, \quad h \in \Pi_n,\]

where $d_h x$ stands for the induced Euclidean measure on $h$. To study this transform using the tools of Analysis, we need to endow $\Pi_n$ with parametrization which converts $Rf$ into a function on some measure space. Several different parametrizations of $\Pi_n$ are known; see, e.g., [29, Section 4.1]. The most common one (cf. [12]) is the parametrization of $h \in \Pi_n$ by the pair $(\theta, t) \in S^{n-1} \times \mathbb{R}$, so that

\[
h \equiv h(\theta, t) = \{x \in \mathbb{R}^n : x \cdot \theta = t\}.\]

One can think of $(\theta, t)$ as a point of the cylinder $Z_n = S^{n-1} \times \mathbb{R}$ equipped with the product measure $d\theta dt$. Since

\[
h(\theta, t) = h(-\theta, -t),\]

the equality (3.2) realizes a two-to-one correspondence between $Z_n$ and $\Pi_n$. Thus, every function $\varphi$ on $\Pi_n$ can be identified with an even function $\varphi(\theta, t)$ on $Z_n$, so that

\[
\varphi(\theta, t) = \varphi(-\theta, -t) \quad \forall (\theta, t) \in Z_n.\]

In the $(\theta, t)$-language, the Radon transform (3.1) has the form

\[
(Rf)(\theta, t) \equiv (R_{\theta} f)(t) = \int f(t\theta + y) \, d_{\theta^\perp} y,\]

where $\theta^\perp = \{x : x \cdot \theta = 0\}$ is the hyperplane orthogonal to $\theta$ and passing through the origin, and $d_{\theta^\perp} y$ denotes the Euclidean measure on $\theta^\perp$. Clearly,

\[
(Rf)(\theta, t) = (Rf)(-\theta, -t).\]
If \( f \) is a radial function, \( f(x) \equiv f_0(|x|) \), then

\[
(Rf)(\theta, t) = \sigma_{n-2} \int_0^\infty f_0(r)(r^2 - t^2)^{(n-3)/2} r dr
\]

for all \( \theta \in S^{n-1} \); see, e.g., [29, Lemma 4.17].

A certain inconvenience of the \((\theta, t)\)-parametrization is that the source space \( \mathbb{R}^n \) and the target space \( Z_n \) have different geometry and the harmonic analysis on \( Z_n \) is more involved than that on \( \mathbb{R}^n \).

An alternative parametrization can be used if we eliminate a subset of all hyperplanes which are parallel to the \( x_n \)-axis. This subset has measure zero and can be ignored in some considerations. The remaining set \( \tilde{\Pi}_n \) contains only those hyperplanes which meet the last coordinate axis. Every hyperplane \( h \in \tilde{\Pi}_n \) can be parametrized by a point \( x = (x', x_n) \in \mathbb{R}^n \), so that

\[
h = \{ y = (y', y_n) \in \mathbb{R}^n : y' \in \mathbb{R}^{n-1}, y_n = x' \cdot y' + x_n \}.
\]

The corresponding Radon-type transform has the form

\[
(Tf)(x) \equiv (T_{x'} f)(x_n) = \int_{\mathbb{R}^{n-1}} f(y', x_n + x' \cdot y') dy', \quad x \in \mathbb{R}^n.
\]

The dual transform \( T^* \), satisfying \( \langle Tf, g \rangle = \langle f, T^* g \rangle \), has a similar form

\[
(T^* g)(x) \equiv (T_{x'}^* g)(x_n) = \int_{\mathbb{R}^{n-1}} g(y', x_n - x' \cdot y') dy'.
\]

Clearly,

\[
(T^* g)(x', x_n) = (Tg)(-x', x_n) = (T\hat{g})(x', -x_n),
\]

where \( \hat{g}(y', y_n) = g(y', -y_n) \).

An advantage of \( T \) in comparison with \( R \) is that \( T \) maps functions on \( \mathbb{R}^n \) to functions on \( \mathbb{R}^n \). By the standard Calculus,

\[
(Rf)(\theta, t) = (Rf)(h) = \sqrt{1 + |x'|^2} (Tf)(x', x_n).
\]

Different parametrizations are related by

\[
\theta = \frac{x' - e_n}{\sqrt{1 + |x'|^2}}, \quad t = -\frac{x_n}{\sqrt{1 + |x'|^2}}
\]

\[
x' = -\frac{\theta'}{\theta_n}, \quad x_n = \frac{t}{\theta_n}.
\]
The corresponding transition mappings are defined on functions \( \varphi(x) \equiv \varphi(x', x_n) \) and \( \psi(\theta, t) \) by

\[
(\Lambda_0 \varphi)(\theta, t) = |\theta_n|^{-1} \varphi \left( \frac{-\theta' + t \theta_n}{\theta_n} \right),
\]

\[
(\Lambda_0^{-1} \psi)(x) = (1 + |x'|^2)^{-1/2} \psi \left( \frac{x' - e_n}{\sqrt{1 + |x'|^2}}, \frac{x_n}{\sqrt{1 + |x'|^2}} \right).
\]

**Lemma 3.1.** The following equalities hold:

\[
(R_f)(\theta, t) = (\Lambda_0 T_f)(\theta, t), \quad \theta_n \neq 0; \tag{3.17}
\]

\[
(T_f)(x) = (\Lambda_0^{-1} R_f)(x). \tag{3.18}
\]

Moreover, for every \( 1 \leq p < \infty \) and a real number \( \nu \),

\[
\int_{S_{n-1}} |t|^{\nu p} |(R_f)(\theta, t)|^p \, d\theta dt = 2 \int_{\mathbb{R}^n} \left| x_n \right|^{\nu p} |(T_f)(x)|^p \frac{1}{(1 + |x'|^2)^{(n-p+\nu p+1)/2}} \, dx. \tag{3.19}
\]

It is assumed that either side of the corresponding equality exists in the Lebesgue sense.

**Proof.** The equalities (3.17) and (3.18) follow from (3.12). To prove (3.19), we write the left-hand side as

\[
I \equiv \int_{\mathbb{R}} |t|^{\nu p} dt \int_{S_{n-1}} |\theta_n|^{-p} \left| (T_f) \left( \frac{-\theta' + t \theta_n}{\theta_n} \right) \right|^p \, d\theta.
\]

The integral over \( S^{n-1} \) can be converted into the integral over \( \mathbb{R}^{n-1} \) by the formula [29, Proposition 1.37 (i)]

\[
\int_{S_{n-1}} F(\theta) \, d\theta = \int_{\mathbb{R}^{n-1}} \left[ F \left( \frac{x' + e_n}{x' + e_n} \right) + F \left( \frac{x' - e_n}{|x' - e_n|} \right) \right] \frac{dx'}{(1 + |x'|^2)^{n/2}}.
\]

In our case,

\[
F(\theta) = |\theta_n|^{-p} \left| (T_f) \left( \frac{-\theta' + t \theta_n}{\theta_n} \right) \right|^p.
\]

Hence

\[
I = \int_{\mathbb{R}} |t|^{\nu p} dt \]

\[
\times \int_{\mathbb{R}^{n-1}} \left[ |(T_f)(-x' + t \sqrt{1 + |x'|^2} e_n)|^p + |(T_f)(x' - t \sqrt{1 + |x'|^2} e_n)|^p \right] \]

\[
\times \frac{dx'}{(1 + |x'|^2)^{(n-p)/2}}.
\]
Changing variables, we obtain
\[ I = 2 \int_{\mathbb{R}^n} \frac{|x_n|^{vp}}{(1 + |x'|^2)^{(n-p+vp+1)/2}} |(Tf)(x)|^p \, dx, \]
as desired. \[\square\]

Both transforms \( R \) and \( T \) are well defined on functions \( f \in L^1(\mathbb{R}^n) \) because by Fubini’s theorem,
\[ \int_{-\infty}^{\infty} (Rf)(\theta, t) \, dt = \int_{-\infty}^{\infty} (Tf)(x', x_n) \, dx_n = \int_{\mathbb{R}^n} f(x) \, dx. \] (3.21)

The Radon transforms \( R, T, \) and \( \tilde{T} \) are intimately related to the Fourier transform.

**Theorem 3.2.** Let \( f \in L^1(\mathbb{R}^n) \). Then
\[ (R_\theta f)^\wedge(\rho) = \hat{f}(\theta \rho), \quad \rho \in \mathbb{R}, \quad \theta \in S^{n-1}; \] (3.22)
\[ (T_{x'} f)^\wedge(\xi_n) = \hat{f}(-x' \xi_n, \xi_n), \quad (T_{x'}^* f)^\wedge(\xi_n) = \hat{f}(x' \xi_n, \xi_n). \] (3.23)

**Proof.** The statement (3.22) is well known in the literature as the Fourier slice theorem; see, e.g., [17, p. 11], [29, p. 129]. The equalities (3.23) have the same meaning for the transversal transform and its dual. The first equality in (3.23) can be found in [29, Lemma 4.143] and its proof is straightforward:
\[ (T_{x'} f)^\wedge(\xi_n) = \int_{-\infty}^{\infty} e^{ix_n \xi_n} \, dx_n \int_{\mathbb{R}^{n-1}} f(y', x_n + x' \cdot y') \, dy' \]
\[ = \int_{\mathbb{R}^n} f(y) e^{i\xi_n (y_n - x' \cdot y')} \, dy = \hat{f}(-x' \xi_n, \xi_n). \]

The second equality in (3.23) follows from the first one if we note that \((T^* f)(x', x_n) = (T f)(-x', x_n)\); cf. (3.10). \[\square\]

**Lemma 3.3.** [29, formulas (4.4.4), (4.13.6)] The following relations hold:
\[ \int_{\mathbb{R}^n} \frac{(Rf)(\theta, t)}{(1 + t^2)^{n/2}} \, d\theta dt = \sigma_{n-1} \int_{\mathbb{R}^n} \frac{f(x)}{(1 + |x|^2)^{1/2}} \, dx, \] (3.24)
\[ \int_{\mathbb{R}^n} \frac{(Tf)(x)}{(1 + |x|^2)^{n/2}} \, dx = \frac{\sigma_{n-1}}{2} \int_{\mathbb{R}^n} \frac{f(x)}{(1 + |x|^2)^{1/2}} \, dx. \] (3.25)
It is assumed that either side of the corresponding equality exists in the Lebesgue sense.

**Theorem 3.4.** If \( f \in L^p(\mathbb{R}^n), \, 1 \leq p < n/(n-1), \) then \((Rf)(\theta, t)\) and \((Tf)(x)\) are finite for almost all \((\theta, t) \in Z_n\) and \(x \in \mathbb{R}^n\), respectively. The left-hand sides of (3.24) and (3.25) do not exceed \(c\|f\|_p\), \(c = \text{const}\).

This statement follows from (3.24) and (3.25) by Hölder’s inequality.

**Remark 3.5.** The bound \( p < n/(n-1) \) in Theorem 3.4 is sharp. For example, if \( p \geq n/(n-1) \) and
\[
(3.26) \quad f_0(x) = (2 + |x|)^{-n/p}(\log(2 + |x|))^{-1} \quad (\in L^p(\mathbb{R}^n)),
\]
then \((Rf_0)(\theta, t) \equiv \infty\); see, e.g., [29, Theorem 4.28]. By (3.18), it follows that \(Tf_0 \equiv \infty\). The next statements provide additional information about the mapping properties of \(R\) and \(T\).

**Theorem 3.6.** [29, Theorem 4.32] Let
\[(3.27) \quad 1 \leq p < n/(n-1), \quad \nu = -(n-1)/p', \quad 1/p + 1/p' = 1.
\]
We introduce the weighted space
\[
(3.28) \quad L^p_\nu(Z_n) = \{ \varphi : ||\varphi||^\nu_{p,\nu} \equiv |||\varphi||^\nu_{p}(Z_n) < \infty \}.
\]
Then \( ||Rf||^\nu_{p,\nu} \leq A ||f||_p \), where
\[
(3.29) \quad A = ||R|| = \frac{2^{1/p} \pi^{(n-1)/2} \Gamma \left( \frac{1-n/p'}{2} \right)}{\Gamma \left( \frac{n}{2p} \right)}.
\]

The necessity of the condition \(\nu = -(n-1)/p'\) can be proved using the standard scaling argument \(x \to \lambda x, \lambda > 0\); see, e.g., the proof of Theorem 4.2 in the more general situation.

Theorem 3.6 yields the corresponding result for \(T\).

**Theorem 3.7.** Let \(1 \leq p < n/(n-1),\)
\[
u = -n(1-2/p), \quad \nu = -(n-1)/p',
\]
\[
(3.30) \quad L^p_\nu(\mathbb{R}^n) = \{ f : ||f||^\nu_{p,\nu} \equiv ||u(x)f||_p < \infty \}.
\]
Then
\[
(3.31) \quad ||Tf||^\nu_{p,\nu} \leq 2^{-1/p} A ||f||_p,
\]
where the constant \(A\) is defined by (3.29).
Proof. Set \( \nu = -(n - 1)/y' \) in (3.19).

Let us also consider the Radon transforms of functions \( f \in S(\mathbb{R}^n) \).

**Lemma 3.8.** If \( f \in S(\mathbb{R}^n) \), then \( T f \in C^\infty(\mathbb{R}^n) \). Moreover, the function
\[
x_n \to (T f)(x', x_n)
\]
belongs to \( S(\mathbb{R}) \) for each \( x' \in \mathbb{R}^{n-1} \), and the function
\[
x' \to (T f)(x', x_n)
\]
is tempered for each \( x_n \in \mathbb{R} \).

**Proof.** The first statement is obvious, because we can differentiate under the sign of integration in (3.9) infinitely many times. Furthermore, for any positive integers \( \ell \) and \( m \), there is a constant \( c_{\ell,m} \) such that
\[
|f(y', x_n + x' \cdot y')| \leq \frac{c_{\ell,m}}{(1 + |y'|^\ell)(1 + |x_n + x' \cdot y'|^m)}.
\]

Note that
\[
\frac{1}{1 + |x_n + x' \cdot y'|} = \frac{1 + |x_n + x' \cdot y' - x' \cdot y'|}{(1 + |x_n + x' \cdot y'|)(1 + |x_n|)}
\]
\[
\leq \frac{1}{1 + |x_n|} \left( 1 + \frac{|x' \cdot y'|}{1 + |x_n + x' \cdot y'|} \right) \leq \frac{1 + |x' \cdot y'|}{1 + |x_n|}
\]
\[
\leq \frac{1 + 2|x' \cdot y'|}{1 + |x_n|} \leq \frac{1 + |x'|^2 + |y'|^2}{1 + |x_n|}
\]
\[
\leq \frac{(1 + |x'|^2)(1 + |y'|^2)}{1 + |x_n|}.
\]

Hence
\[
(T f)(x', x_n) \leq c_{\ell,m} \frac{(1 + |x'|^2)^m}{(1 + |x_n|)^m} \int_{\mathbb{R}^{n-1}} \frac{(1 + |y'|^2)^m}{(1 + |y'|)^\ell} \, dy'
\]
\[
(3.32)
\]
\[
\leq \tilde{c}_{\ell,m} \frac{(1 + |x'|^2)^m}{(1 + |x_n|)^m}, \quad \tilde{c}_{\ell,m} = \text{const},
\]
if \( \ell \) is big enough in comparison with \( m \). A similar estimate holds for any derivative of \( T f \). This gives other statements of the lemma.

An analogue of Lemma 3.8 for the Radon transform \((R f)(\theta, t)\) is more involved and needs additional definitions.

**Definition 3.9.** A function \( g \) on \( S^n-1 \) is called differentiable if the corresponding homogeneous function
\[
\tilde{g}(x) = g(x/|x|)
\]
is differentiable in the usual sense on \( \mathbb{R}^n \setminus \{0\} \). The derivatives of a function \( g \) on \( S^{n-1} \) will be defined as restrictions to \( S^{n-1} \) of the
derivatives of $\tilde{g}(x)$, namely,
\begin{equation}
(3.33) \quad (\partial^\gamma \tilde{g})(\theta) = (\partial^\gamma \tilde{g})(x)|_{x=\theta}, \quad \gamma \in \mathbb{Z}_+^n, \quad \theta \in S^{n-1}.
\end{equation}

**Definition 3.10.** We denote by $S(Z_n)$ the space of functions $\varphi(\theta, t)$ on $Z_n = S^{n-1} \times \mathbb{R}$, which are infinitely differentiable in $\theta$ and $t$ and rapidly decreasing as $t \to \pm \infty$ together with all derivatives. The topology in $S(Z_n)$ is defined by the sequence of norms
\begin{equation}
(3.34) \quad ||\varphi||_m = \sup_{|\gamma|+j \leq m} (1 + |t|)^m |(\partial^\gamma \partial^j_t \varphi)(\theta, t)|, \quad m \in \mathbb{Z}_+.
\end{equation}

The subspace of even functions in $S(Z_n)$ will be denoted by $S_e(Z_n)$.

**Theorem 3.11.** [29, Theorem 4.47], [12, Chapter I, Theorem 2.4] The Radon transform $f \to Rf$ is a linear continuous map from $S(\mathbb{R}^n)$ into $S_e(Z_n)$.

## 4. Radon-type fractional integrals $R_+^\alpha f$ and $T_+^\alpha f$

In this section we consider fractional integrals of the form
\begin{align}
(4.1) \quad (R_+^\alpha f)(\theta, t)) &= \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^n} (t-x \cdot \theta)_+^{\alpha-1} f(x) \, dx, \\
(4.2) \quad (T_+^\alpha f)(x) &= \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^n} (x_n - y_n)^{\alpha-1} f(y', y_n + x' \cdot y') \, dy, \\
(4.3) \quad (T_+^\alpha g)(x) &= \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^n} (x_n - y_n)_{-1}^{\alpha-1} g(y', y_n - x' \cdot y') \, dy.
\end{align}

If $\Re \alpha > 0$ and $f$ is good enough, these integrals are absolutely convergent. The limiting case $\alpha = 0$ in (4.1) and (4.2) yields $Rf$ and $Tf$, respectively. The limiting case $\alpha = 0$ in (4.3) gives $T^* f$, as in (3.10). One can readily see that
\begin{align}
\langle T_+^\alpha f, g \rangle &= \langle f, T_+^\alpha g \rangle \\
\langle T_+^\alpha g, x', x_n \rangle &= (T_+^\alpha \tilde{g})(x', -x_n), \quad \tilde{g}(y', y_n) = g(y', -y_n).
\end{align}

Given two functions $\varphi(x) \equiv \varphi(x', x_n)$ and $\psi(\theta, t)$, denote
\begin{align}
(4.5) \quad (\Lambda_\alpha \varphi)(\theta, t) &= |\theta_n|^{\alpha-1} \varphi \left( -\frac{\theta'}{\theta_n}, t \right), \\
(4.6) \quad (\Lambda_\alpha^{-1} \psi)(x) &= (1 + |x'|^2)^{\alpha-1/2} \psi \left( \frac{x' - e_n}{\sqrt{1 + |x'|^2}}, -\frac{x_n}{\sqrt{1 + |x'|^2}} \right);
\end{align}
cf. (3.15), (3.16) for $\alpha = 0$.

**Lemma 4.1.** The following relations hold:

\begin{align}
(R^\alpha_+ f)(\theta, t) &= (\Lambda_\alpha T^\alpha_+ f)(\theta, t), \quad \theta_n \neq 0; \\
(T^\alpha_+ f)(x) &= (\Lambda_\alpha^{-1} R^\alpha_+ f)(x).
\end{align}

Moreover, for every $1 \leq p < \infty$ and a real number $\nu$,

\begin{equation}
\int_{Z_n} |t|^\nu (R^\alpha_+ f)(\theta, t)|^p d\theta dt = 2 \int_{\mathbb{R}^n} |x_n|^\nu (T^\alpha_+ f)(x)|^p dx.
\end{equation}

It is assumed that either side of the corresponding equality exists in the Lebesgue sense.

**Proof.** By (3.13),

\begin{align}
(R^\alpha_+ f)(\theta, t) &= \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^n} (t - y \cdot \theta)^{\alpha-1} f(y) dy \\
&= \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^n} \left( \frac{x_n + x' \cdot y' - y_n}{\sqrt{1 + |x'|^2}} \right)^{\alpha-1} f(y) dy \\
&= \frac{(1 + |x'|^2)^{(1-\alpha)/2}}{\Gamma(\alpha)} \int_{\mathbb{R}^n} (x_n - y_n)^{\alpha-1} f(y', y_n + x' \cdot y') dy \\
&= (1 + |x'|^2)^{(1-\alpha)/2} (T^\alpha_+ f)(x).
\end{align}

This gives (4.8). The equality (4.7) is a consequence of (4.8). The proof of (4.9) is an almost verbatim copy of the proof of (3.19). □

### 4.1. Fractional integrals of $L^p$-functions.

Consider the weighted space

$$L^p_\nu(Z_n) = \left\{ \varphi : ||\varphi||_{p,\nu} = \left( \int_{Z_n} |t|^\nu (|\varphi(\theta, t)|)^p d\theta dt \right)^{1/p} < \infty \right\},$$

where $Z_n = S^{n-1} \times \mathbb{R}$, $1 \leq p < \infty$, and $\nu$ is a real number. Equivalently,

\begin{equation}
||\varphi||_{p,\nu} = \left( \sigma_{n-1} \int_{-\infty}^{\infty} |t|^\nu dt \int_{O(n)} \varphi(\gamma e_1, t)|^p d\gamma \right)^{1/p}.
\end{equation}

**Theorem 4.2.** Let $0 < \alpha < 1$,

\begin{equation}
1 \leq p < \frac{n}{n - 1 + \alpha}, \quad \nu = -\alpha - \frac{n - 1}{p'}.
\end{equation}
$1/p + 1/p' = 1$. Then
\begin{equation}
||R^α_+ f||_{p,ν} \leq c_α ||f||_p, \quad c = c(α, n, p) = \text{const}.
\end{equation}

The conditions (4.11) are sharp. There is a function $\tilde{f} \in L^p(\mathbb{R}^n)$ with $p \geq n/(n - 1 + α)$, such that $(R^α_+ \tilde{f})(x) \equiv \infty$.

**Proof.** Denote $ϕ(θ, t) = (R^α_+ f)(θ, t)$ and consider the cases $t > 0$ and $t < 0$ separately. For $t > 0$, changing variables $θ = γe_1$, $γ \in O(n)$, and $x = tγy$, we get
\begin{equation}
ϕ(γe_1, t) = \frac{t^{α+n-1}}{Γ(α)} \int_{\mathbb{R}^n} f(tγy) (1 - y_1)_{+}^{α-1} dy.
\end{equation}

For $t < 0$, we similarly have
\begin{equation}
ϕ(γe_1, t) = \frac{|t|^{α+n-1}}{Γ(α)} \int_{\mathbb{R}^n} f(|t|γy) (-1 - y_1)_{+}^{α-1} dy.
\end{equation}

Splitting the integral $\int_{-∞}^{∞}$ in (4.10) in two pieces, we write
\begin{equation}
||ϕ||_{p,ν} \sim_ρ \frac{σ_{n-1}^{1/p}}{Γ(α)} (I_1 + I_2)^{1/p},
\end{equation}

where
\begin{align*}
I_1^{1/p} &\leq \int_{\mathbb{R}^n} (1 - y_1)_{+}^{α-1} \left( \int_0^∞ \int_{O(n)} t^{(α+n-1+ν)p} |f(tγy)|^p dγ dt \right)^{1/p} dy \\
&= c_1 ||f||_p, \quad c_1 = σ_{n-1}^{-1/p} \int_{\mathbb{R}^n} (1 - y_1)_{+}^{α-1} |y|^{-n/p} dy
\end{align*}

(here we have used the equality $ν = -α - (n - 1)/p'$). Similarly,
\begin{align*}
I_2^{1/p} &\leq \int_{\mathbb{R}^n} (-1 - y_1)_{+}^{α-1} \left( \int_{-∞}^0 \int_{O(n)} |t|^{(α+n-1+ν)p} |f(|t|γy)|^p dγ dt \right)^{1/p} dy \\
&= c_2 ||f||_p, \quad c_2 = σ_{n-1}^{-1/p} \int_{\mathbb{R}^n} (-1 - y_1)_{+}^{α-1} |y|^{-n/p} dy.
\end{align*}

One can easily check that $c_1$ and $c_2$ are finite. Hence (4.12) follows.

The necessity of the equality $ν = -α - (n - 1)/p'$ can be proved using the scaling argument. Specifically, let $f_λ(x) = f(λx)$, $λ > 0$. Then
\begin{equation}
||f_λ||_p = λ^{-n/p} ||f||_p, \quad ||R^α_+ f_λ||_{p,ν} = λ^{-ν-n-α+1/p'} ||R^α_+ f||_{p,ν}.
\end{equation}
Hence, if \( \| R_+^\alpha f \|_{\tilde{L}^p} \leq c \| f \|_p \) with \( c \) independent of \( f \), then, necessarily, \( \nu = -\alpha - (n - 1)/p' \).

Let us show the necessity of the bound \( p < n/(n - 1 + \alpha) \). Consider, for example,

\[
\tilde{f}(x) = (2 + |x|)^{-n/p}(\log(2 + |x|))^{-1} \quad (\in L^p(\mathbb{R}^n)).
\]

This function is radial. We set \( \tilde{f}(x) \equiv f_0(|x|) \). Hence, by (3.7),

\[
(R_+^\alpha \tilde{f})(\theta, t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1}(R\tilde{f})(\theta,s)ds
\]

\[
= \frac{\sigma_{n-2}}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1}ds \int f_0(r)(r^2-s^2)^{(n-3)/2}rdr.
\]

Suppose \( t < 0 \) and change variables \( t = -\tau, s = -\zeta \). We obtain

\[
(R_+^\alpha \tilde{f})(\theta,-\tau) = \frac{\sigma_{n-2}}{\Gamma(\alpha)} \int_{\zeta}^{\infty} (\zeta-\tau)^{\alpha-1}d\zeta \int f_0(r)(r^2-\zeta^2)^{(n-3)/2}rdr
\]

\[
= \frac{\sigma_{n-2}}{\Gamma(\alpha)} \int f_0(r)k(r,\tau)dr, \quad k(r,\tau) = r \int (\zeta-\tau)^{\alpha-1}(r^2-\zeta^2)^{(n-3)/2}d\zeta.
\]

If \( n \geq 3 \), then \( (r^2-\zeta^2)^{(n-3)/2} \geq (r-\zeta)^{(n-3)/2}r^{(n-3)/2} \),

\[
k(r,\tau) \geq c (r-\tau)^{(n-3)/2+\alpha}r^{(n-1)/2},
\]

and

\[
(R_+^\alpha \tilde{f})(\theta,-\tau) \geq c \int_{\tau}^{\infty} (r-\tau)^{(n-3)/2+\alpha}r^{(n-1)/2}/(2 + r)^{n/p} \log(2 + r) \quad dr \equiv \infty
\]

if \( p \geq n/(n - 1 + \alpha) \). Recall that the constant \( c \) may be different at each occurrence. If \( n = 2 \), then

\[
(r^2-\zeta^2)^{-1/2} \geq (r-\zeta)^{-1/2}(2r)^{-1/2}, \quad k(r,\tau) \geq c (r-\tau)^{\alpha-1/2}r^{1/2},
\]

and the conclusion is the same. \( \square \)

**Theorem 4.3.** Let \( 0 < \alpha < 1, \)

\[
1 \leq p < \frac{n}{n-1+\alpha}, \quad \mu = -n \left(1 - \frac{2}{p}\right), \quad \nu = -\alpha - \frac{n-1}{p'},
\]

\( 1/p + 1/p' = 1 \). We define

\[
u\]

\[
\frac{|x_n|^\nu}{(1+|x'|^2)^{\mu/2}}.
\]
(4.16) \[ L_p^\nu(\mathbb{R}^n) = \{ f : \|f\|_{p,u} \equiv \|u(x)f\|_p < \infty \}. \]

Then
\[ ||T_+^\alpha f||_{p,u} \leq c \|f\|_p, \quad c = c(\alpha, n, p) = \text{const}. \]

The assumptions (4.15) are sharp. There exists a function \( \tilde{f} \in L^p(\mathbb{R}^n) \) with \( p \geq n/(n - 1 + \alpha) \), such that \( (T_+^\alpha \tilde{f})(x) \equiv \infty \).

**Proof.** Set \( \nu = -\alpha - (n - 1)/p' \) in (4.9) and make use of Theorem 4.2, combined with Lemma 4.1. \( \square \)

### 4.2. Fractional integrals of smooth functions.

It is convenient to write \( T_+^\alpha f \) and \( T_+^\ast g \) in the form
\[
(T_+^\alpha f)(x', x_n) = (h_+^\alpha * T_{x'} f)(x_n), \quad (T_+^\ast g)(x', x_n) = (h_-^\ast * T_{x'}^\ast g)(x_n),
\]
or
\[
(T_+^\alpha f)(x', x_n) = (I_+^0 T_{x'} f)(x_n), \quad (T_+^\ast g)(x', x_n) = (I_+^0 T_{x'}^\ast g)(x_n);
\]

cf. (1.7), (1.12). Similarly,
\[
(R_+^\alpha f)(\theta, t) = (I_+^0 R_\theta f)(t).
\]

The following equalities can be easily checked for functions \( \omega \in S(\mathbb{R}) \):

(4.19) \[ I_+^\alpha I_+^\beta \omega = I_+^{\alpha+\beta} \omega, \quad \text{Re} \alpha > 0, \quad \text{Re} \beta > 0; \]

(4.20) \[ (\pm d/dt)^{\ell} (I_+^\alpha \omega)(t) = (I_+^\alpha \omega)(t), \quad \text{Re} \alpha > 0, \quad \ell \in \mathbb{N}; \]

(4.21) \[ (I_+^\alpha \omega)(t) = (\pm 1)^{\ell} (I_+^{\alpha+\ell} \omega^{(\ell)})(t), \quad \text{Re} \alpha > 0, \quad \ell \in \mathbb{N}, \]

where \( \omega^{(\ell)}(t) = (d/dt)^{\ell} \omega(t) \). The equality (4.21) can be used for analytic continuation of \( I_+^\alpha \omega \) to the domain \( \text{Re} \alpha > -\ell \). It follows that, \( I_+^\alpha \omega \) extend to \( \alpha \in \mathbb{C} \) as entire functions of \( \alpha \) with preservation of the properties (4.19)-(4.21).

These facts extend to the operators \( T_+^\alpha, T_+^\ast, \) and \( R_+^\alpha \).

**Lemma 4.4.** Let \( f \in S(\mathbb{R}^n) \).

(i) For each \( x \in \mathbb{R}^n \), \( (T_+^\alpha f)(x) \) extends as an entire function of \( \alpha \), so that
\[
(4.22) \lim_{\alpha \to 0} (T_+^\alpha f)(x) = (T f)(x),
\]

where \( (T f)(x) \) is the transversal Radon transform (1.2).

(ii) For \( \text{Re} \alpha > 0 \) and \( \ell \in \mathbb{N} \) the following relations hold:
\[
(4.23) (d/dx_n)^{\ell} (T_+^{\alpha+\ell} f)(x) = (T_+^\alpha f)(x),
\]
\[
(4.24) (T_+^\alpha f)(x) = (T_+^{\alpha+\ell} \partial_n^\ell f)(x).
\]
Proof. By Lemma 3.8, \((Tf)(x',x_n)\) belongs to \(S(\mathbb{R})\) in the \(x_n\)-variable for each \(x' \in \mathbb{R}^{n-1}\). Hence, owing to the properties of the Riemann-Liouville fractional integrals, \((T_+^{α}f)(x',x_n) = (I_+^{α}T_{x'}f)(x_n)\) extends as an entire function of \(α\) satisfying (4.22). Other statements of the lemma hold for the same reason.

An analogue of Lemma 4.4 holds for \(T_+^{α}g\). For the fractional integral \(R_+^{α}f\) we have the following statement.

**Lemma 4.5.** Let \(f \in S(\mathbb{R}^n)\).

(i) For each \((θ,t) \in Z_n\), \((R_+^{α}f)(θ,t)\) extends as an entire function of \(α\), so that

\[
\lim_{α \to 0} (R_+^{α}f)(θ,t) = (Rf)(θ,t).
\]

(ii) For \(Re\ α > 0\) and \(ℓ \in \mathbb{N}\) the following relations hold:

\[
(d/dt)^ℓ(R_+^{α+ℓ}f)(θ,t) = (R_+^{α}f)(θ,t),
\]

\[
(R_+^{α}f)(θ,t) = (I_+^{α+ℓ}R_θ^{ℓ}f)(t),
\]

where

\[
(R_θ^{ℓ}f)(t) = (d/dt)^ℓ(Rf)(θ,t)
\]

\[
= \begin{cases} 
R[Δ^m f](θ,t), & \text{if } ℓ = 2m, \\
\sum_{k=1}^n θ_k[R[Δ^m f]](θ,t), & \text{if } ℓ = 2m + 1,
\end{cases}
\]

Proof. By Theorem 3.11, \((Rf)(θ,t)\) belongs to \(S(\mathbb{R})\) in the \(t\)-variable for each \(θ \in \mathbb{R}^{n-1}\). Hence \((R_+^{α}f)(θ,t) = (I_+^{α}R_θf)(t)\) extends as an entire function of \(α\) satisfying (4.25). Other statements hold for the same reason. The differentiation formula (4.28) can be found in [12, p. 3], [29, p. 135]. □

The following definition will be used in the next sections.

**Definition 4.6.** For \(f \in S(\mathbb{R}^n)\) and \(α \in \mathbb{C}\), we define \((T_+^{α}f)(x)\) and \((R_+^{α}f)(θ,t)\) by the formulas

\[
(T_+^{α}f)(x) = a.c. (I_+^{α}T_{x'}f)(x_n),
\]

\[
(R_+^{α}f)(θ,t) = a.c. (I_+^{α}R_θf)(t),
\]

where “a.c.” denotes analytic continuation. For each \(ℓ \in \mathbb{N}\), this analytic continuation can be realized in the domain \(Re\ α > -ℓ\) by (4.24) and (4.27), respectively. Equivalently,

\[
(T_+^{α}f)(x) = (h_+^{α} * T_{x'}f)(x_n) = (h_+^{α}(y_n), (T_{x'}f)(x_n - y_n)),
\]

\[
(R_+^{α}f)(θ,t) = (h_+^{α} * R_θf)(t) = (h_+^{α}(s), (R_θf)(t - s)),
\]
where \( h_\alpha^+ \) is regarded as the \( S'(\mathbb{R}) \)-distribution.

We recall that \((T_{x'}f)(\cdot)\) and \((R_\theta f)(\cdot)\) belong to \( S(\mathbb{R}) \) and the right-hand sides of (4.31) and (4.32) are infinitely differentiable tempered function of \( x_n \) and \( t \), respectively; cf. e.g., [7, p. 26, Lemma 2.5], [43, p. 84].

4.2.1. Fractional integrals in the Semyanisty-Lizorkin space. Let \( \Phi(\mathbb{R}^n) \) be the Semyanisty-Lizorkin space, and let \( \Phi_0(\mathbb{R}) \) be a similar space on the real line; see Section 2.2. It is known (see, e.g., [33, Section 8.2]) that the Riemann-Liouville operators \( I_\alpha^\pm, \alpha \in \mathbb{C} \), are automorphisms of \( \Phi_0(\mathbb{R}) \) and the following relations hold for the Fourier transforms:

\[
(I_\alpha^\pm \omega)^\wedge (\tau) = (\mp i \tau)^{\pm \alpha} \hat{\omega}(\tau).
\]

Here the branch of the multi-valued power function is chosen so that

\[
(\mp i \tau)^{-\alpha} = \exp \left( -\alpha \log |\tau| \pm \frac{\alpha \pi i}{2} \operatorname{sgn} \tau \right).
\]

Note also that the transversal Radon transform \( T \) and its dual \( T^* \) are automorphisms of \( \Phi(\mathbb{R}^n) \); cf. [29, Theorem 4.142]. By (4.17), it follows that the operators \( T_\alpha^\pm \) and \( T_\alpha^* \) are automorphisms of \( \Phi(\mathbb{R}^n) \), too.

Given a function \( f(x) \equiv f(x', x_n) \) on \( \mathbb{R}^n \), we denote by \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) the Fourier transforms of \( f \) in the \( x' \)-variable and the \( x_n \)-variable, respectively. Let also \( I_2^\lambda \omega, \lambda \in \mathbb{C} \), be the Riesz potential operator, which can be defined on functions \( \omega \in \Phi_0(\mathbb{R}) \) by the formula

\[
(I_2^\lambda \omega)^\wedge (\tau) = |\tau|^{-\lambda} \hat{\omega}(\tau).
\]

**Theorem 4.7.** (cf. [29, Theorem 4.145]) If \( f \in \Phi(\mathbb{R}^n) \), then for any complex \( \alpha \) and \( \beta \),

\[
T_+^\alpha T_+^\beta f = (2\pi)^{n-1} I_2^\alpha I_2^\beta f,
\]

\[
T_+^\beta T_+^\alpha f = (2\pi)^{n-1} I_2^\beta I_2^\alpha f.
\]

**Proof.** Let \( g = T_+^\beta f \). Then \( g \in \Phi(\mathbb{R}^n) \) and (3.23) yields

\[
(\mathcal{F}_2 T_+^\alpha g)^\wedge (\xi_n) = (I_2^\alpha T_{x'} g)^\wedge (\xi_n) = (-i \xi_n)^{-\alpha} \hat{g}(-x' \xi_n, \xi_n).
\]
Here

\[
\hat{g}(-x'\xi_n, \xi_n) = (\mathcal{F}_1\{\mathcal{F}_2[g(y', \cdot)](\xi_n)\})(-x'\xi_n) = (\mathcal{F}_1\{\mathcal{F}_2[(T^* f)(y', \cdot)](\xi_n)\})(-x'\xi_n) = (i\xi_n)^{-\beta} (\mathcal{F}_1[\hat{f}(y'\xi_n, \xi_n)])(-x'\xi_n) = (i\xi_n)^{-\beta} \int_{\mathbb{R}^{n-1}} \hat{f}(y'\xi_n, \xi_n) e^{-ix'\xi_n} dy' = (2\pi)^{n-1} \frac{(i\xi_n)^{-\beta} |\xi_n|^{1-n}}{\xi_n} \int_{\mathbb{R}^{n-1}} \hat{f}(z', \xi_n) e^{-ix'z'} dz'.
\]

Hence, by (4.38),

\[
(\mathcal{F}_2 T^{\alpha}_+ T^\beta \ast f)(\xi_n) = (2\pi)^{n-1} (-i\xi_n)^{-\alpha} (i\xi_n)^{-\beta} |\xi_n|^{1-n} (\mathcal{F}_2[f(y', \cdot)])(\xi_n).
\]

This gives (4.36). The proof of (4.37) is similar. □

Setting \(\alpha = \beta = (1-n)/2\) and \(\alpha = \beta = 0\), we obtain the following corollary.

**Corollary 4.8.** Let \(f \in \Phi(\mathbb{R}^n)\). Then

\[
(4.39) \quad T^{\frac{1-n}{2}}_+ T^{\frac{1-n}{2}}_+ f = (2\pi)^{n-1} f;
\]

\[
(4.40) \quad TT^* f = T^* T f = (2\pi)^{n-1} I_2^{n-1} f.
\]

**Remark 4.9.** The equality (4.39) can be easily understood if we notice that

\[
(-i\xi_n)^{-\alpha} (i\xi_n)^{-\alpha} = |\xi_n|^{-2\alpha}.
\]

Furthermore, it will be proved (see Lemma 5.2) that (4.39) extends to all \(f \in L^2(\mathbb{R}^n)\). An analogue of the second equality in (4.40) for the Radon transform \(R\) is well known and has the form

\[
(4.41) \quad R^* R f = c_n I^{n-1} f, \quad c_n = 2^{n-1} \pi^{n/2-1} \Gamma(n/2),
\]

where \(I^{n-1} f\) is the Riesz potential of order \(n - 1\) in \(\mathbb{R}^n\) and \(R^*\) is the dual Radon transform, which averages \(R f\) over all hyperplanes passing through \(x\); see, e.g., [29, Proposition 4.37].
5. \textit{L}^p-\textit{L}^q \textit{estimates of} \( T_\alpha^+ f \) \textit{and interpolation}

5.1. \textbf{Some preparations.} Let \( f \in L^p(\mathbb{R}^n) \). If \( \text{Re} \alpha > 0 \), then, by Theorem 4.3, \( T_\alpha^+ f \) exists a.e. as an absolutely convergent integral provided \( 1 \leq p < n/(n - 1 + \text{Re} \alpha) \), and this bound is sharp. If \( \text{Re} \alpha \leq 0 \), the integral \( T_\alpha^+ f \) diverges. However, if \( f \in S(\mathbb{R}^n) \), then \( T_\alpha^+ f \) can be defined by analytic continuation; see Definition 4.6.

We wonder, for which \( \alpha \), \( p \), and \( q \) the operator \( T_\alpha^+ : S(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n) \) extends as a linear bounded operator (we denote the extended operator by \( T_\alpha^+ \)) mapping \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \).

To answer this question, one may be tempted to make use of Stein’s interpolation theorem for analytic families of operators [37, 3, 9, 39]. However, the assumptions of this theorem and its proof are given in terms of simple functions, which are not smooth. It is unclear how to define our operators on such functions in the case \( \text{Re} \alpha < 0 \), when the integrals are divergent, and check the validity of the assumptions of the theorem; cf. Definition (4.29), which is given in terms of function belonging to \( S(\mathbb{R}^n) \).

The following theorem, which is applicable to our situation, can be found in Grafakos [10].

Given \( 0 < p_0, p_1 \leq \infty \) and \( 0 < q_0, q_1 \leq \infty \), let

\[
\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}; \quad 0 < \theta < 1.
\]

Denote

\( S = \{ z \in \mathbb{C} : 0 < \text{Re} z < 1 \}, \quad \bar{S} = \{ z \in \mathbb{C} : 0 \leq \text{Re} z \leq 1 \} \).

\textbf{Theorem 5.1.} [10, Theorem 5.5.3] Let \( T_z, z \in \bar{S} \), be a family of linear operators satisfying the following conditions.

(A) For each \( z \in \bar{S} \), the operator \( T_z \) maps \( C^\infty_c(\mathbb{R}^n) \) to \( L^1_{\text{loc}}(\mathbb{R}^n) \).

(B) For all \( \varphi, \psi \in C^\infty_c(\mathbb{R}^n) \), the function

\[
A(z) = \int_{\mathbb{R}^n} (T_z \varphi)(x) \psi(x) \, dx
\]

is analytic in \( S \) and continuous on \( \bar{S} \).

(C) There exist constants \( \gamma \in [0, \pi) \) and \( s \in (1, \infty] \), such that for any \( \varphi \in C^\infty_c(\mathbb{R}^n) \) and any compact subset \( K \subset \mathbb{R}^n \),

\[
\log ||T_z \varphi||_{L^s(K)} \leq C e^{\gamma |\text{Im} z|}
\]

for all \( z \in \bar{S} \) and some constant \( C = C(\varphi, K) \).
There exist constants $B_0, B_1,$ and continuous functions $M_0(\gamma), M_1(\gamma)$ satisfying
\begin{equation}
M_0(\gamma) + M_1(\gamma) \leq \exp(c e^{\tau|\gamma|})
\end{equation}
with some constants $c \geq 0$ and $0 \leq \tau < \pi$, such that
\begin{equation}
||T_{i\gamma}f||_{q_0} \leq B_0 M_0(\gamma) ||f||_{p_0}, \quad ||T_{i+i\gamma}f||_{q_1} \leq B_1 M_1(\gamma) ||f||_{p_1}
\end{equation}
for all $\gamma \in \mathbb{R}$ and all $f \in C^\infty_c(\mathbb{R}^n)$.

Then for all $f \in C^\infty_c(\mathbb{R}^n)$,
\begin{equation}
||T_{\theta}f||_q \leq B_\theta M_\theta ||f||_p,
\end{equation}
where $B_\theta = B_0^{1-\theta} B_1^\theta$ and
\begin{equation}
M_\theta = \exp \left\{ \frac{\sin(\pi \theta)}{2} \int_{-\infty}^{\infty} \left[ \frac{\log M_0(\gamma)}{\cosh(\pi \gamma) - \cos(\pi \theta)} + \frac{\log M_1(\gamma)}{\cosh(\pi \gamma) + \cos(\pi \theta)} \right] d\gamma \right\}.
\end{equation}

Our aim is to apply Theorem 5.1 to the analytic family $\{T_\alpha\}_{\alpha \in \mathbb{C}}$. First, we examine the cases $\Re \alpha = (1-n)/2$ and $\Re \alpha = 1$.

**Lemma 5.2.** For the operator $T_+^{(1-n)/2+i\gamma}$, $\gamma \in \mathbb{R}$, initially defined by (4.29), there exists a unique linear bound extension
\begin{equation}
T_+^{(1-n)/2+i\gamma} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),
\end{equation}
such that for every $f \in L^2(\mathbb{R}^n)$,
\begin{equation}
||T_+^{(1-n)/2+i\gamma}f||_2 \leq c_n e^{\gamma |\gamma|/2} ||f||_2, \quad c_n = 2^{n/2} \pi^{(n-1)/2}.
\end{equation}
Moreover, in the case $\gamma = 0$, the operator $c_n^{-1} T_+^{(1-n)/2}$ is unitary.

**Proof.** We recall that the Fourier transform of the $S'$-distribution $h_+^\alpha$, $\alpha \in \mathbb{C}$, is computed as
\begin{equation}
\hat{h}_+^\alpha(\xi_n) = (-i \xi_n)^{-\alpha} \equiv \exp \left( -\alpha \log |\xi_n| + \frac{\alpha \pi i}{2} \text{sgn} \xi_n \right);
\end{equation}
cf. (4.34). This formula can be found in different forms in many sources; see, e.g., [33, pp. 98, 137], [8, Chapter II, Section 2.3], [7, Chapter I, Section 2]. Suppose $\Re \alpha \leq 0$. This assumption guarantees absolute convergence of all integrals in our calculations below. By the Plancherel formula, (4.31) and (3.23) yield
\( (T_+^\alpha f)(x) = \frac{1}{2\pi} \langle (h_+^\alpha(y_n)), ((T_x f)(x_n + y_n))\rangle \)
\[ = \frac{1}{2\pi} \langle (-i\xi_n)^{-\alpha}, e^{-ix_n\xi_n} (T_x f)^\wedge (\xi_n) \rangle \]
\[ = \frac{1}{2\pi} \langle (-i\xi_n)^{-\alpha}, e^{-ix_n\xi_n} \hat{f}(-x'\xi_n, \xi_n) \rangle. \]

Hence, by the Parseval equality,
\[ F^{-1}[(\hat{f}(-x')\xi_n, \xi_n)](x_n), \]
where \( F^{-1} \) stands for the inverse Fourier transform in the last variable. Note that if \( \alpha_0 = Re \alpha \) and \( \gamma = Im \alpha \), then, by (5.9),
\[ \|(-i\xi_n)^{-\alpha}\| = \|\xi_n\|^{-\alpha_0} \exp\left(-\frac{\gamma\pi}{2} \text{sgn} \xi_n\right). \]

Hence, by the Parseval equality,
\[ \int_{-\infty}^{\infty} \|\hat{T_+^\alpha f}(x', x_n)\|^2 dx_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|(-i\xi_n)^{-\alpha} \hat{f}(-x'\xi_n, \xi_n)\|^2 d\xi_n \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\xi_n\|^{-2\alpha_0} \exp(-\gamma\pi \text{sgn} \xi_n) \|\hat{f}(-x'\xi_n, \xi_n)\|^2 d\xi_n \]
\[ = \frac{e^{-\gamma\pi}}{2\pi} \int_0^{\infty} r^{-2\alpha_0} \|\hat{f}(-x'r, r)\|^2 dr + \frac{e^{\gamma\pi}}{2\pi} \int_0^{\infty} r^{-2\alpha_0} \|\hat{f}(x'r, -r)\|^2 dr. \]

Now integration in \( x' \) gives
\[ \|T_+^\alpha f\|_2^2 = \frac{\cosh\gamma\pi}{\pi} \int_{\mathbb{R}^n} |y|^{-2\alpha_0 + 1-n} |\hat{f}(y)|^2 dy. \]

If \( \alpha_0 = (1-n)/2 \), then
\[ \|T_+^{(1-n)/2 + i\gamma} f\|_2^2 = \frac{\cosh\gamma\pi}{\pi} \|\hat{f}\|_2^2 = (2\pi)^n \cosh\gamma\pi \|f\|_2^2. \]

Because \( |\cosh\gamma\pi| \leq e^{\gamma|\pi|} \), the latter gives (5.8) for \( f \in S(\mathbb{R}^n) \). Now the statements of the lemma follow from the density of \( S(\mathbb{R}^n) \) in \( L^2(\mathbb{R}^n) \). If \( \gamma = 0 \) then (5.10) gives
\[ \|T_+^{(1-n)/2 + i\gamma} f\|_2 = c_n \|f\|_2, \quad c_n = 2^{n/2} \pi^{(n-1)/2}. \]

Hence the operator \( c_n^{-1}T_+^{(1-n)/2} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) is an isometry. To show that it is unitary, it remains to note that it is “onto”. Indeed, by (4.39), if \( f \in \Phi(\mathbb{R}^n) \), then
\[ f = (2\pi)^{1-n}T_+^{(1-n)/2} \hat{T}_+^{(1-n)/2} f. \]
where, by (4.4),
\[ \| T_+^{-(1-n)/2} f \|_2 = \| T_+^{-(1-n)/2} \dot{f} \|_2 = c_n \| \dot{f} \|_2 = c_n \| f \|_2. \]

Because \( \Phi(R^n) \) is dense in \( L^2(R^n) \) (see Proposition 2.3), the result follows. \( \Box \)

**Remark 5.3.** The last statement of Lemma 5.2 about unitarity of the operator \( c_n^{-1} T_+^{-(1-n)/2} \) is not needed for interpolation. However, it is of interest on its own right. In the close situation, operators of this kind occur in inverse problems for the wave equation; cf. [4, Proposition 1].

Now we consider the case \( \alpha = 1 + i\gamma, \gamma \in \mathbb{R} \), when
\[
(T_+^{1+i\gamma} f)(x) = \frac{1}{\Gamma(1+i\gamma)} \int_{\mathbb{R}^n} (x_n - y_n)^{i\gamma} f(y', y_n + x' \cdot y') \, dy.
\]

Note that
\[
|\Gamma(1+i\gamma)|^2 = \frac{\pi \gamma}{\sinh(\pi \gamma)};
\]
see, e.g., [1]. This gives the following statement.

**Lemma 5.4.** For any \( \gamma \in \mathbb{R} \) and \( f \in L^1(\mathbb{R}^n) \),
\[
(5.11) \quad \| T_+^{1+i\gamma} f \|_{\infty} \leq e^{\pi |\gamma|/2} \| f \|_1.
\]

5.2. **The main theorem.** Below we apply Theorem 5.1 to the operator family \( \{ T_+^\alpha \} \).

**Theorem 5.5.** Suppose \( 1 \leq p, q \leq \infty \), \( \alpha_0 = Re \alpha \). The operator \( T_+^\alpha \), initially defined by (4.29) on functions \( f \in S(\mathbb{R}^n) \), extends as a linear bounded operator \( T_+^\alpha \) from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) if and only if
\[
(5.12) \quad \frac{1-n}{2} \leq \alpha_0 \leq 1, \quad p = \frac{n+1}{n+\alpha_0}, \quad q = \frac{n+1}{1-\alpha_0}.
\]

**Proof.** The proof consists of two parts related to the “if” statement and the “only if” statement, respectively.

**Part I.** We define
\[
(5.13) \quad T_z = T_+^{\alpha(z)}, \quad \alpha(z) = \frac{1+n}{2} z + \frac{1-n}{2}.
\]

Then \( 0 \leq Re z \leq 1 \) corresponds to \( (1-n)/2 \leq Re \alpha \leq 1 \). In our case,
\[
p_0 = q_0 = 2, \quad p_1 = 1, \quad q_1 = \infty.
\]
If for real \( \alpha \in [(1-n)/2, 1] \) we set \( \alpha = ((1+n)/2)\theta + (1-n)/2 \), then \( \theta = (2\alpha + n - 1)/(n+1) \), and (5.1) yields
\[
1/p = (\alpha + n)/(n+1), \quad 1/q = (1-\alpha)/(n+1).
\]
The latter agrees with (5.12).

To apply Theorem 5.1, we must show, step by step, that the operator family \( T_z = T_+^{\alpha(z)} \) meets the conditions (A)-(D). This important technical part of the proof relies on the material of the previous sections.

(A) Let \( \varphi \in C^\infty_c(\mathbb{R}^n) \). Fix any integer \( \ell > (n-1)/2 \). Then for all \( (1-n)/2 \leq \text{Re} \alpha \leq 1 \),

\[
(T_+^\alpha \varphi)(x) = (T_+^{\alpha + \ell} \partial_\nu^\ell \varphi)(x) = \frac{1}{\Gamma(\alpha + \ell)} \int_0^\infty s^{\alpha + \ell - 1} A_x(s) \, ds,
\]

where, by (4.2),

\[
A_x(s) = \int_{\mathbb{R}^{n-1}} (\partial_n^\ell \varphi)(y', x_n - s + x' \cdot y') \, dy' = (T \partial_n^\ell \varphi)(x', x_n - s).
\]

Hence, by (3.32), for sufficiently large \( m \in \mathbb{N} \) there is a constant \( c_m \) such that

\[
|A_x(s)| \leq c_m \frac{(1 + |x'|^2)^m}{(1 + |x_n - s|)^m} \leq c_m \frac{(1 + |x'|^2)^m (1 + |x_n|)^m}{(1 + s)^m}.
\]

It follows that

\[
|(T_+^\alpha \varphi)(x)| \leq \frac{c_m (1 + |x'|^2)^m (1 + |x_n|)^m}{|\Gamma(\alpha + \ell)|} \int_0^\infty s^{\alpha + \ell - 1} ds = \tilde{c}_m = \text{const},
\]

if \( m \) is large enough. Hence \( T_+^\alpha \varphi \in L^1_{\text{loc}}(\mathbb{R}^n) \) for all \( (1-n)/2 \leq \text{Re} \alpha \leq 1 \).

(B) For any \( \varphi, \psi \in C^\infty_c(\mathbb{R}^n) \), we have

\[
A(z) = \int_{\mathbb{R}^n} (T_z \varphi)(x) \psi(x) \, dx = \int_{\text{supp}(\psi)} F(x, z) \, dx,
\]

where

\[
F(x, z) = (T_z \varphi)(x) \psi(x) = (T_+^{\alpha(z)} \varphi)(x) \psi(x)
\]

is an entire function of \( z \) (see Lemma 4.4(i)), which is locally integrable in the \( x \)-variable for each \( z \) in any compact subset of \( \bar{S} \); see (A). Using known facts about analyticity of functions represented by integrals (see, e.g., [29, Lemma 1.3]), we obtain (B).

(C) We make use of the (5.14), which gives

\[
||T_+^\alpha \varphi||_{L^\infty(\mathbb{K})} \leq \frac{C}{|\Gamma(\alpha + \ell)|}, \quad C = C(K, \varphi).
\]
Now, let us revert to the notation of the interpolation theorem. If \( \alpha = \alpha(z) \) and \( T_z \varphi = T_+^{\alpha(z)} \varphi \), then for all \( 0 \leq \text{Re} z \leq 1 \), (5.15) gives
\[
||T_z \varphi||_{L^\infty(K)} \leq \frac{C}{|\Gamma(\zeta)|}, \quad \zeta = \frac{1+n}{2} z + \frac{1-n}{2} + \ell.
\]
If \( z = x + iy \), then \( \zeta = a + ib \), where
\[
a = \frac{1+n}{2} x + \frac{1-n}{2} + \ell, \quad b = \frac{1+n}{2} y; \quad 0 \leq x \leq 1.
\]
It is known that if \( a_1 \leq a \leq a_2 \) and \( |b| \to \infty \), then
\[
(5.16) \quad |\Gamma(\zeta)| = \sqrt{2\pi} |b|^{a-1/2} e^{-\pi |b|/2} [1 + O(1/|b|)],
\]
where the constant implied by \( O \) depends only on \( a_1 \) and \( a_2 \); see, e.g., [2, Corollary 1.4.4]. Taking into account that \( 1/\Gamma(\zeta) \) is an entire function and using (5.16), after simple calculations we obtain an estimate of the form
\[
\log ||T_z \varphi||_{L^\infty(K)} \leq \begin{cases} c_1 & \text{if } |y| \leq 10, \\ c_2 + c_3 \log |y| + c_4 |y| & \text{if } |y| \geq 10, \end{cases}
\]
for some positive constants \( c_1, c_2, c_3, c_4 \). This estimate yields (5.3) for any \( \gamma > 0 \). Thus verification of the assumption (C) in Theorem 5.1 is complete.

(D) Let us check (5.4). By (5.13),
\[
T_{\gamma} = T_+^{(1-n)/2+i\gamma(1+n)/2}, \quad T_{1+i\gamma} = T_+^{1+i\gamma(1+n)/2}.
\]
Hence the results of Lemmas 5.2 and 5.4 can be stated as
\[
||T_{\gamma} \varphi||_2 \leq B_0 M_0(\gamma)||\varphi||_2, \quad ||T_{1+i\gamma} \varphi||_{\infty} \leq B_1 M_1(\gamma)||\varphi||_1,
\]
with some constants \( B_0, B_1 \). These estimates give (5.4).

Thus, by Theorem 5.1, the “if” part of Theorem 5.5 is proved for \( T_+^\alpha \) when \( \alpha \) is real. If \( \alpha \) has a nonzero imaginary part, say, \( t \), the above reasoning can be repeated almost verbatim if we re-define \( T_z \) in (5.13) by setting \( T_z = T_+^{\alpha(z+it)} \).

Part II. Let us prove the “only if” part. First we show that the left bound \( \text{Re} \alpha = (1 - n)/2 \) in (5.12) is sharp. Suppose the contrary, assuming for simplicity that \( \alpha \) is real. Then there is a triple \( (p_0, q_0, \alpha_0) \) with \( 1 \leq p_0 < q_0 \leq \infty \) and \( \alpha_0 < (1 - n)/2 \), such that \( T_+^{\alpha_0} \) is bounded from \( L^{p_0}(\mathbb{R}^n) \) to \( L^{q_0}(\mathbb{R}^n) \). Interpolating the triples \( (p_0, q_0, \alpha_0) \) and
Let \( \alpha \) extends to all \( \text{too}. \) Let Theorem 5.6. The above reasoning shows that the right bound \( \|\| \) that is, \( \|\| \) of \( \alpha \) \((1-28 B. RUBIN\) and let \( \lambda \) \(T \) operator \( \lambda \) argument. Specifically, suppose, for example, that \( \alpha \) \((5.17) \) assuming that \( (Tf) \) \(\lambda \to \infty \) and \(LAB \to \infty \). Specifically, suppose, for example, that \( \alpha \) \((5.17) \) is true for all \( \lambda \) \((n-2) \) and \( \alpha_0 = (1-n)/2, 0 < \varepsilon < (1+n)/2, \) we obtain

\[
 p_{(1-n)/2} = \frac{2(1+n)}{1+n-2\varepsilon}, \quad q_{(1-n)/2} = \frac{2(1+n)}{1+n+2\varepsilon}.
\]

The latter agrees with the known \( L^2-L^2 \) boundedness of \( T_+^{(1-n)/2} \) only if \( \varepsilon = 0 \), that is, \( \alpha_0 = (1-n)/2. \)

The necessity of \( p \) and \( q \) in (5.12) can be proved using the scaling argument. Specifically, suppose, for example, that \( \alpha \) is real. Abusing notation, let \( \lambda = (\lambda_1, \lambda_2) \), \( \lambda_1 > 0, \lambda_2 > 0 \), and denote

\[
 (A_\lambda f)(x) = f(\lambda_1 x', \lambda_2 x_n), \quad (B_\lambda F)(x) = \frac{\lambda_1^{1-n}}{\lambda_2^2} F \left( \frac{\lambda_2}{\lambda_1} x', \lambda_2 x_n \right).
\]

Then \( T_+^\alpha A_\lambda f = B_\lambda T_+^\alpha f \). This equality is straightforward if \( \alpha > 0 \) and extends to all \( \alpha \in \mathbb{C} \) by analyticity. We have

\[
 ||A_\lambda f||_p = \lambda_1^{(1-n)/p}\lambda_2^{-1/p}||f||_p, \quad ||B_\lambda F||_q = \lambda_1^{1-n/(n-1)/q}\lambda_2^{-\alpha-n/q}||F||_q.
\]

If \( ||T_+^\alpha f||_q \leq c||f||_p \) is true for all \( f \in L^p \), then it is true for \( A_\lambda f \), that is, \( ||T_+^\alpha A_\lambda f||_q \leq c||A_\lambda f||_p \) or \( ||B_\lambda T_+^\alpha f||_q \leq c||A_\lambda f||_p \). The latter is equivalent to

\[
 \lambda_1^{1-n+(n-1)/q}\lambda_2^{-\alpha-n/q}||T_+^\alpha f||_q \leq c\lambda_1^{(1-n)/p}\lambda_2^{-1/p}||f||_p.
\]

Assuming that \( \lambda_1 \) and \( \lambda_2 \) tend to zero and to infinity, we conclude that the last inequality is possible only if

\[
 p = \frac{n+1}{n+\alpha}, \quad q = \frac{n+1}{1-\alpha}.
\]

The above reasoning shows that the right bound \( \Re \alpha = 1 \) is sharp, too.

Now, the proof of Theorem 5.5 is complete. \( \square \)

**Theorem 5.6.** Let \( p \) and \( q \) be defined by

\[
 p = \frac{n+1}{n+\Re \alpha}, \quad q = \frac{n+1}{1-\Re \alpha}.
\]

and let \( f \in L^p(\mathbb{R}^n) \). In the cases \( 0 < \Re \alpha < 1 \) and \( \alpha = 0 \), the \( L^q \) functions \((T_+^\alpha f)(x)\) and \((T f)(x) = (T_+^\alpha f)(x)|_{\alpha=0}\), the existence of which is guaranteed by Theorem 5.5, coincide a.e. with \((T_+^\alpha f)(x)\) and \((T f)(x)\), respectively.
Proof. In the case $0 < \text{Re}\alpha < 1$, it suffices to assume that $\alpha$ is real. Because

$$\theta = (n + 1)/(n + \alpha) < n/(n - 1 + \alpha),$$

the operator $T^\alpha_+$ obeys the conditions of Theorem 4.3. Thus, by Theorems 5.5 and 4.3, we have two linear bounded operators

$$T^\alpha_+ : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n), \quad T^\alpha_- : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n),$$

where $u(x) = |x_n|^{\nu}(1 + |x'|^2)^{-n/2}$. Hence both operators are bounded from $L^p(\mathbb{R}^n)$ to $L^1(K)$ for any compact set $K \subset \mathbb{R}^n$ away from the hyperplane $x_n = 0$. Let $\{f_k\} \subset S(\mathbb{R}^n)$ be a sequence approximating $f$ in the $L^p$-norm. Because $T^\alpha_+$ was defined as an extension of $T^\alpha_-$ from $S(\mathbb{R}^n)$, $(T^\alpha_-f_k)(x) = (T^\alpha_-f_k)(x)$. Then, by linearity,

$$T^\alpha_+ f - T^\alpha_- f = T^\alpha_+(f - f_k) - T^\alpha_-(f - f_k),$$

and therefore

$$||T^\alpha_+ f - T^\alpha_- f||_{L^1(K)} \leq ||T^\alpha_+(f - f_k)||_{L^1(K)} + ||T^\alpha_-(f - f_k)||_{L^1(K)} \leq c_1||f - f_k||_p + c_2||f - f_k||_p$$

for some constants $c_1$ and $c_2$ depending on $K$. Assuming $k \to \infty$, we obtain $||T^\alpha_+ f - T^\alpha_- f||_{L^1(K)} = 0$. This gives $(T^\alpha_+ f)(x) = (T^\alpha_- f)(x)$ for almost all $x \in K$, and therefore for almost all $x \in \mathbb{R}^n$.

If $\alpha = 0$, a similar reasoning relies on Theorems 5.5 and 3.7. \qed

6. $L^p$-$L^q$ estimates of $R^\alpha_+ f$

Theorem 5.5 yields the following result for the operator $R^\alpha_+$.

**Theorem 6.1.** Suppose $1 \leq p, q \leq \infty$, $\alpha_0 = \text{Re}\alpha$. The operator $R^\alpha_+$, initially defined by (4.30) on functions $f \in S(\mathbb{R}^n)$, extends as a linear bounded operator $R^\alpha_+$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if

$$\frac{1 - n}{2} \leq \alpha_0 \leq 1, \quad p = \frac{n + 1}{n + \alpha_0}, \quad q = \frac{n + 1}{1 - \alpha_0}.$$

**Proof.** We recall that for $\text{Re}\alpha > 0$, (4.7) yields

$$\theta(a) = |\theta_n|^{\alpha-1}(T^\alpha_+ f)\left(\frac{-\theta'}{\theta_n}, \frac{t}{\theta_n}\right), \quad \theta_n \neq 0.$$  

For $f \in S(\mathbb{R}^n)$, let

$$u(\theta, t) = a.c. (R^\alpha_+ f)(\theta, t) \quad \text{and} \quad v(x) = a.c. (T^\alpha_+ f)(x)$$

be analytic continuations of the integrals $R^\alpha_+ f$ and $T^\alpha_+ f$ from the domain $\text{Re}\alpha > 0$ to the entire complex plane. By Lemmas 4.4 and
4.5, these analytic continuations are well defined and represent smooth functions. Hence (6.2) extends analytically to all $\alpha \in \mathbb{C}$ and we get

$$u(\theta, t) = |\theta_n|^{\alpha - 1}v\left(-\frac{\theta'}{\theta_n}, \frac{t}{\theta_n}\right), \quad \theta_n \neq 0, \quad \alpha \in \mathbb{C}.$$ 

As in the proof of (3.19), for any $q \geq 1$ we have

$$(||u|_q^\sim|^q) = \int_{\mathbb{R}} dt \int_{S^{n-1}} |\theta_n|^{\alpha - 1}v\left(-\frac{\theta'}{\theta_n}, \frac{t}{\theta_n}\right)|^q d\theta = 2 \int_{\mathbb{R}^n} |v(x)|^q \frac{(1+|x'|^2)^{(n+(\alpha_0-1)q+1)/2}}{dx}.$$ 

In particular, if $q = (n+1)/(1-\alpha_0)$, as in (6.1), then $||u|_q^\sim = 2^{1/q}||v|_q^\sim$. In other words, for any $f \in S(\mathbb{R}^n)$ we have

$$(6.4) \quad ||a.c. R_+^\alpha f||_q^\sim = 2^{1/q}||a.c. T_+^\alpha f||_q = 2^{1/q}||T_+^\alpha f||_q.$$ 

By Definition 4.6 and Theorem 5.5, it follows that extension of (6.4) to all $f \in L^p(\mathbb{R}^n)$ gives

$$||R_+^\alpha f||_q^\sim \leq 2^{1/q}c ||f||_p, \quad p = (n+1)/(n+\alpha_0),$$

as desired. \hfill \Box

We observe that unlike Theorem 5.5, Theorem 6.1 does not include the “only if” part. As we shall see below, the values for $p$ and $q$ may differ from those in (6.1).

6.1. The Oberlin-Stein theorem. The following statement, due to Oberlin and Stein, is contained in Theorem 1 of the paper [18] (set $q = r$ in this theorem). Below we obtain it as a consequence of our result for $Tf$.

**Theorem 6.2.** Let $1 \leq p, q \leq \infty$. For $n \geq 2$, the inequality

$$(6.5) \quad ||Rf||_q^\sim \leq c ||f||_p, \quad c = \text{const},$$

holds if and only if $1 \leq p \leq (n+1)/n$ and $1/q = n/p - n + 1$.

**Proof.** First, note that $(n+1)/n < n/(n-1)$, and therefore, by Theorem 3.4, both $Rf$ and $Tf$ exist a.e. Further, if we make use of (3.19) with $\nu = 0$ and $p$ replaced by $q = n + 1$, we obtain

$$\int_{\mathbb{Z}_n} ||(Rf)(\theta, t)||^{n+1} dtd\theta = 2 \int_{\mathbb{R}^n} ||(Tf)(x)||^{n+1} dx.$$
By Theorem 6.1, this gives the endpoint estimate in (6.5). The result for all $1 \leq p \leq (n + 1)/n$ then follows by interpolation, taking into account that $\|Rf\|_q = \sigma_{n-1} \|f\|_1$ by (3.21).

The necessity of the relations $1/q = n/p - n + 1$ and $p \leq (n + 1)/n$ was justified in [18]. For the sake of completeness, we perform this justification in detail in a slightly different way, using the scaling argument. Denote $f_{\lambda}(x) = f(\lambda x)$, $\lambda > 0$. Then $(Rf_{\lambda})(\theta, t) = \lambda^{1-n}(Rf)(\theta, \lambda t)$,

$$
\|f_{\lambda}\|_p = \lambda^{-n/p} \|f\|_p, \quad \|Rf_{\lambda}\|_q = \lambda^{1-n-1/q} \|Rf\|_q.
$$

If $\|Rf\|_q \leq c \|f\|_p$ for all $f \in L^p(\mathbb{R}^n)$, then, replacing $f$ by $f_{\lambda}$, we obtain

$$
\lambda^{1-n-1/q} \|Rf\|_q \leq \lambda^{-n/p} \|f\|_p,
$$

which gives

$$
(6.6) \quad 1 - n - 1/q = -n/p.
$$

To prove the necessity of the bound $p \leq (n + 1)/n$, we set $\tilde{f}_{\lambda}(x) = f(\lambda x', x_n)$. Then, as above, $\|\tilde{f}_{\lambda}\|_p = \lambda^{(1-n)/p} \|f\|_p$, and (3.19) (with $p$ replaced by $q$) yields

$$
(||R\tilde{f}_{\lambda}||_q^q)^q = 2 \int_{\mathbb{R}^n} \frac{|(T\tilde{f}_{\lambda})(x)|^q}{(1 + |x'|^2)^{(n-q)/2} dx}.
$$

Note that $(T\tilde{f}_{\lambda})(x) = \lambda^{1-n}(Tf)(x'/\lambda, x_n)$. Hence, changing variables, we obtain

$$
(||R\tilde{f}_{\lambda}||_q^q)^q = 2\lambda^{(n-1)(1-q)} \int_{\mathbb{R}^n} \frac{|(Tf)(x', x_n)|^q}{(1 + |x'|^2)^{(n-q)/2} dx}.
$$

If $\|Rf\|_q \leq c \|f\|_p$ for all $f \in L^p(\mathbb{R}^n)$, then, setting $f = \tilde{f}_{\lambda}$, we obtain

$$
\lambda^{(n-1)(1-q)/q} \left(2 \int_{\mathbb{R}^n} \frac{|(Tf)(x', x_n)|^q}{(1 + |x'|^2)^{(n-q)/2} dx} \right)^{1/q} \leq \lambda^{(1-n)/p} \|f\|_p,
$$

or

$$
(6.7) \quad \lambda^s A^{1/q}(\lambda) \leq \|f\|_p, \quad s = (n - 1)(1 - q)/q + (n - 1)/p,
$$

where $A(\lambda)$ stands for the expression in brackets. Let us pass to the limit in (6.7) as $\lambda \to 0$ assuming $f$ to be good enough. If $s < 0$, then the left-hand side of (6.7) tends to infinity. Hence, necessarily, $s \geq 0$, which is equivalent to $q \leq p'$. Combining the last inequality with (6.6), we obtain $p \leq (n + 1)/n$. \qed
6.2. **The Hardy-Littlewood-Sobolev theorem for \( R^\alpha_+ \).** The next theorem resembles the celebrated Hardy-Littlewood-Sobolev theorem for Riesz potentials; cf. [38, Chapter V, Section 1.2], [35, Theorem 0.3.2], [36, p. 189]. We use the notation \( m\{\cdot\} \) for the Lebesgue measure of the corresponding set and assume, for simplicity, that \( \alpha \) is real-valued.

**Theorem 6.3.** Let \( n \geq 2, 0 < \alpha < 1, 1 \leq p \leq \infty; \ 1/p + 1/p' = 1 \).

(i) If \( p = 1 \), then \( R^\alpha_+ \) is an operator of weak \((1, q)\)-type with \( 1/q = 1 - \alpha \), that is,

\[
m\{(\theta, t) : |(R^\alpha_+ f)(\theta, t)| > \lambda \} \leq c \left( \frac{\|f\|_1}{\lambda} \right)^{1/(1-\alpha)} \text{ for all } \lambda > 0.
\]

(ii) The inequality

\[
\|R^\alpha_+ f\|_q^* \leq c \|f\|_p, \quad c = \text{const},
\]

holds if and only if

\[
1 < p \leq p_\alpha, \quad p_\alpha = \frac{n+1}{n+\alpha}, \quad \frac{1}{q} = 1 - \alpha - \frac{n}{p'}.
\]

**Proof.** The proof consists of three steps. First we prove (i), then the “if” part of (ii), and then the “only if” part of (ii).

**STEP I.** Let us prove (i). We proceed as in [38, Chapter V, Section 1.3] with minor changes related to the specifics of our object. It suffices to prove (6.8) for \( \|f\|_1 = 1 \). Indeed, if for such an \( f \),

\[
m\{(\theta, t) : |(R^\alpha_+ f)(\theta, t)| > \lambda \} \leq c \left( \frac{1}{\lambda} \right)^q \text{ for all } \lambda > 0,
\]

then, for arbitrary \( f \in L^1(\mathbb{R}^n) \) with \( \|f\|_1 \neq 0 \) and \( \lambda \) replaced by \( \lambda/\|f\|_1 \) we have

\[
m\{(\theta, t) : |(R^\alpha_+[f/\|f\|_1])(\theta, t)| > \lambda/\|f\|_1 \} \leq c \left( \frac{\|f\|_1}{\lambda} \right)^q.
\]

The latter coincides with (6.8) by dilation argument.

Let \( \mu \) be a fixed positive constant to be specified later. We set

\[
(R^\alpha_+ f)(\theta, t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \eta^{\alpha-1} (R_\theta f)(t-\eta) d\eta = k_1 * R_\theta f + k_\infty * R_\theta f,
\]

where

\[
k_1(\eta) = \frac{1}{\Gamma(\alpha)} \begin{cases} \eta^{\alpha-1} & \text{if } \eta \leq \mu, \\ 0 & \text{if } \eta > \mu; \end{cases} \quad k_\infty(\eta) = \frac{1}{\Gamma(\alpha)} \begin{cases} 0 & \text{if } \eta \leq \mu, \\ \eta^{\alpha-1} & \text{if } \eta > \mu. \end{cases}
\]
As in [38, Chapter V, Section 1.3], we replace $\lambda$ by $2\lambda$ in the left-hand side of (6.8), which gives

$$m\{((\theta,t) : |(R_{1}^{\alpha} f)(\theta,t)| > 2\lambda\} \leq m\{((\theta,t) : |(k_{1} \ast R_{0} f)(t)| > \lambda\} + m\{((\theta,t) : |(k_{\infty} \ast R_{0} f)(t)| > \lambda\} = m_{1}\{\cdot\} + m_{\infty}\{\cdot\}.$$ 

Thus it suffices to estimate $m_{1}\{\cdot\}$ and $m_{\infty}\{\cdot\}$. Taking into account (3.21) and the assumption $\|f\|_{1} = 1$, we have

$$\|k_{1} \ast R_{0} f\|_{1} \leq 1 = \Gamma(\alpha) \int_{\mathbb{R}^{n}} \left| \int_{0}^{\mu} \eta^{\alpha-1}(R_{0} f)(t-\eta)d\eta \right| d\theta dt \leq \frac{\mu^{\alpha}}{\Gamma(\alpha + 1)} \|R_{0} f\|_{1} = \frac{\mu^{\alpha} \sigma^{n-1}}{\Gamma(\alpha + 1)} \|f\|_{1} = c_{1}\mu^{\alpha}.$$ 

However, $\|k_{1} \ast R_{0} f\|_{1} \geq \lambda m_{1}\{\cdot\}$. Hence

$$\text{(6.11)} \quad m_{1}\{\cdot\} \leq c_{1}\frac{\mu^{\alpha}}{\lambda}.$$ 

Furthermore, by (3.21),

$$\|k_{\infty} \ast R_{0} f\|_{\infty} \leq \frac{1}{\Gamma(\alpha)} \left\| \int_{\mathbb{R}^{n}} \eta^{\alpha-1}(R_{0} f)(t-\eta)d\eta \right\|_{\infty} \leq \frac{\mu^{\alpha-1}}{\Gamma(\alpha)} \left\| \int (R_{0} f)(\eta)d\eta \right\|_{L^{\infty}(S^{n-1})} = \frac{\mu^{\alpha-1}}{\Gamma(\alpha)} \|f\|_{1} = c_{2}\mu^{\alpha-1}.$$ 

Now we choose $\mu$ so that $c_{2}\mu^{\alpha-1} = \lambda$, i.e., $\mu = (c_{2}/\lambda)^{1/(1-\alpha)}$. Then $m_{\infty}\{\cdot\} = 0$ and, by (6.11),

$$m\{((\theta,t) : |(R_{1}^{\alpha} f)(\theta,t)| > 2\lambda\} = m_{1}\{\cdot\} \leq c_{3} \lambda^{-1/(1-\alpha)} = c_{3}\left(\frac{\|f\|_{1}}{\lambda}\right)^{q}.$$ 

This gives (6.8).

STEP II. Let us prove the “if” part of (ii). It relies on the connection between $R_{1}^{\alpha} f$ and $T_{1}^{\alpha} f$. Recall that by Theorems 4.2 and 4.3, both integrals exist a.e. provided $p < n/(n - 1 + \alpha)$. We meet this condition because

$$p_{\alpha} = (n + 1)/(1 + \alpha) < n/(n - 1 + \alpha).$$
By (4.7), $R^\alpha_+ f = \Lambda_\alpha T^\alpha_+ f$, whence (use (4.9) with $\nu = 0$ and $p$ replaced by $q_\alpha = (n + 1)/(1 - \alpha)$)

$$\int_{\mathbb{Z}_n} |(R^\alpha_+ f)(\theta, t)|^{q_\alpha} d\theta d\theta = 2 \int_{\mathbb{R}^n} |(T^\alpha_+ f)(x)|^{q_\alpha} dx.$$ 

This gives the endpoint estimate in (6.9) for $p = p_\alpha$ and $q = q_\alpha$ because $T^\alpha_+ = \mathcal{T}^\alpha_+$ and $\mathcal{T}^\alpha_+$ is bounded from $L^{q_\alpha}(\mathbb{R}^n)$ to $L^{p_\alpha}(\mathbb{R}^n)$; see Theorems 5.6 and 5.5. Combining the endpoint estimate with (i) and making use of the Marcinkiewicz interpolation theorem, we obtain (6.9) for all $1 < p \leq p_\alpha$, $1/q = 1 - \alpha - n/p'$.

**STEP III.** Let us prove the “only if” part of (ii). The necessity of the bounds $p = p_\alpha$ and $q = q_\alpha$ can be proved as in Theorem 6.2. Specifically, let $f_\lambda(x) = f(\lambda x)$, $\lambda > 0$. Then $(R^\alpha_+ f_\lambda)(\theta, t) = \lambda^{1-n-\alpha}(R^\alpha_+ f)(\theta, \lambda t)$,

$$||f_\lambda||_p = \lambda^{-n/p}||f||_p, \quad ||R^\alpha_+ f_\lambda||_q = \lambda^{1-n-\alpha-1/q}||R^\alpha_+ f||_q.$$ 

If $||R^\alpha_+ f||_q \leq c ||f||_p$ for all $f \in L^p(\mathbb{R}^n)$, then, replacing $f$ by $f_\lambda$, we obtain

$$\lambda^{1-n-\alpha-1/q}||R^\alpha_+ f||_q \leq \lambda^{-n/p}||f||_p,$$

which gives

$$(6.12) \quad 1 - n - \alpha - 1/q = -n/p \quad \text{or} \quad 1/q = 1 - \alpha - n/p'.$$

To obtain another relation between $p$ and $q$, we set $\tilde{f}_\lambda(x) = f(\lambda x', x_n)$. Then $||\tilde{f}_\lambda||_p = \lambda^{(1-n)/p}||f||_p$, and (4.9) (with $p$ replaced by $q$) yields

$$||(R^\alpha_+ \tilde{f}_\lambda)|^{q_\alpha}_q = 2 \int_{\mathbb{R}^n} |(T^\alpha_+ \tilde{f}_\lambda)(x)|^{q_\alpha} (1 + |x'|^2)^{(n+(\alpha-1)q+1)/2} dx.$$ 

Note that $(T^\alpha_+ \tilde{f}_\lambda)(x) = \lambda^{1-n}(T^\alpha_+ f)(x'/\lambda, x_n)$. Hence, changing variables, we obtain

$$||(R^\alpha_+ \tilde{f}_\lambda)|^{q_\alpha}_q = \lambda^{(n-1)(1-q)} A(\lambda), \quad A(\lambda) = 2 \int_{\mathbb{R}^n} |(T^\alpha_+ f)(x', x_n)|^{q_\alpha} (1 + |x'|^2)^{(n+(\alpha-1)q+1)/2} dx.$$ 

If $||R^\alpha_+ f||_q \leq c ||f||_p$ for all $f \in L^p(\mathbb{R}^n)$, then $||R^\alpha_+ \tilde{f}_\lambda||_q \leq c ||\tilde{f}_\lambda||_p$ for all $\lambda > 0$, and we have $\lambda^{(n-1)(1-q)/q} A^{1/q}(\lambda) \leq \lambda^{(1-n)/p}||f||_p$. This gives

$$(6.13) \quad \lambda^s A^{1/q}(\lambda) \leq ||f||_p, \quad s = (n - 1)(1 - q)/q + (n - 1)/p.$$ 

Let us pass to the limit in (6.13) as $\lambda \to 0$ assuming $f$ to be good enough. If $s < 0$, then the left-hand side of (6.13) tends to infinity, which gives contradiction. Hence $s \geq 0$, that is, $q \leq p'$. Combining the last inequality with (6.12), we obtain $p \leq p_\alpha$, as desired.
To complete the proof, it remains to show that (6.9) fails if $p = 1$. This can be done using the approximation argument; cf. [38, p. 119], where similar reasoning was applied to the Riesz potentials in $\mathbb{R}^n$.

Suppose the contrary, that is, there is a constant $c$, such that

$$ \| R^\alpha_{+} f \|_{1/(1-\alpha)} \leq c \| f \|_1 $$

for all $f \in L^1(\mathbb{R}^n)$.

We choose $f$ to be a mollifier

$$ \omega_\varepsilon(x) = \begin{cases} \frac{C}{\varepsilon^n} \exp \left( -\frac{\varepsilon^2}{\varepsilon^2 - |x|^2} \right), & |x| \leq \varepsilon, \\ 0, & |x| > \varepsilon, \end{cases} $$

where $C$ is chosen so that $\int_{\mathbb{R}^n} \omega_\varepsilon(x) \, dx = 1$. Then

$$ (R^\alpha_{+} \omega_\varepsilon)(\theta, t) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} (t - \eta)^{\alpha-1} (R^\omega_\varepsilon)(\theta, \eta) \, d\eta. $$

Clearly, $\omega_\varepsilon(x) = \varepsilon^{-n} \omega_1(x/\varepsilon)$ and $(R^\omega_\varepsilon)(\theta, \eta) = \varepsilon^{-1}(R^\omega_1)(\theta, \eta/\varepsilon)$. We set $\omega_1(x) = \omega(|x|)$. Then for all $\theta \in S^{n-1}$,

$$ (R^\omega_1)(\theta, \eta) = \psi(\eta), \quad \psi(\eta) = \sigma_{n-2} \int_0^\infty \omega(r)(r^2 - \eta^2)^{(n-3)/2} r \, dr; $$

see (3.7). Thus (6.16) becomes

$$ (R^\alpha_{+} \omega_\varepsilon)(\theta, t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} * \psi_\varepsilon, \quad \psi_\varepsilon(\eta) = \varepsilon^{-1} \psi(\eta/\varepsilon). $$

The family $\{ \psi_\varepsilon \}_{\varepsilon > 0}$ is an approximate identity. To see that, we observe that by (3.21),

$$ ||\psi||_{L^1(\mathbb{R})} = \int_{\mathbb{R}} (R^\omega_1)(\theta, \eta) \, d\eta = ||\omega_1||_{L^1(\mathbb{R}^n)} = 1. $$

Further, $\psi$ is a monotone decreasing function of $|\eta|$. The latter is obvious for $n = 3$ and can be easily checked for $n > 3$, when differentiation yields $\psi'(\eta) < 0$ for $\eta > 0$. In the case $n = 2$, $\eta > 0$, we first integrate by parts to get $\psi(\eta) = -2 \int_\eta^\infty \omega'(r)(r^2 - \eta^2)^{1/2} \, dr$ and then differentiate, which gives $\psi'(\eta) = 2\eta \int_\eta^\infty \omega'(r)(r^2 - \eta^2)^{-1/2} \, dr < 0$.

Now applying the approximation to the identity machinery [38, Theorem 2(b), p. 63] to the function

$$ t^\alpha_{+} = t^\alpha_{+} \chi(0, 1) + t^\alpha_{+} \chi(1, \infty) \in L^1(\mathbb{R}) + L^q(\mathbb{R}), \quad q > 1/(1 - \alpha), $$
we conclude that \((R^+ \omega_\varepsilon)(\theta, t)\) converges to \(t_+^{1-\alpha}/\Gamma(\alpha)\) a.e. on \(\mathbb{R}\), and hence for almost all \((\theta, t) \in Z_n\). By Fatou’s lemma and the assumption (6.14) it follows that
\[
\int_{\mathbb{R}} |t_+^{\alpha-1}/\Gamma(\alpha)|^{1/(1-\alpha)} dt = \frac{1}{\Gamma(\alpha)} \lim_{\varepsilon \to 0} (R^+ \omega_\varepsilon)(\theta, t) ||_{1/(1-\alpha)} \leq \frac{1}{\Gamma(\alpha)} \lim_{\varepsilon \to 0} ||R^+ \omega_\varepsilon||_{1/(1-\alpha)} \leq c ||\omega_\varepsilon||_1 = c,
\]
which is impossible, because the integral on the left-hand side diverges.

\[\square\]

Remark 6.4. As we pointed out in Introduction, it is an interesting open problem to obtain necessary and sufficient conditions of the \(L^p-L^q\) boundedness of the extended operator \(R_+^\alpha\) for all complex \(\alpha\). The above theorems contain such conditions only for \(\alpha = 0\) and \(0 < Re \alpha < 1\). As we could see, the theory of the Radon-type fractional integrals \(R_+^\alpha f\), associated with the classical Radon transform \(R\) is more complicated than that for \(T_+^\alpha f\), associated with the transversal Radon transform. One of the explanations of this phenomenon might be that \(R_+^\alpha\) can be scaled only in the radial direction, whereas \(T_+^\alpha\) enjoys bi-parametric dilations (in the \(x'\)-variable and in the \(x_n\)-variable).

7. Parabolic Radon-type fractional integrals

In this section we consider the parabolic Radon transform
\[
(Pf)(x) = \int f(x' - y', x_n - |y'|^2) dy', \quad x = (x', x_n) \in \mathbb{R}^n,
\]
and the corresponding fractional integral
\[
(P_+^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int (y_n - |y'|^2)^{\alpha-1} f(x - y) dy.
\]
The integral (7.1) resembles integration over the shifted paraboloid
\[\pi_x = \pi_0 + x, \quad \pi_0 = \{y = (y', y_n) : y_n = -|y'|^2\}.\]
However, \((Pf)(x)\) differs from the usual surface integral
\[
\int_{\pi_x} f(y) d\sigma(y) = \int f(x' - y', x_n - |y'|^2) (1 + 4|y'|^2)^{1/2} dy'
\]
by the Jacobian factor, which is suppressed in our consideration.

Our aim is to obtain sharp \(L^p-L^q\) estimates for \(Pf\) and \(P_+^\alpha f\).
7.1. The parabolic Radon transform. If \( f \in L^1(\mathbb{R}^n) \), then, by Fubini’s theorem,

\[
(7.4) \quad \int_{-\infty}^{\infty} (Pf)(x', x_n) dx_n = \int_{\mathbb{R}^n} f(x) dx,
\]

whence \((Pf)(x', x_n)\) is finite for all \( x' \in \mathbb{R}^{n-1} \) and almost all \( x_n \in \mathbb{R} \).

There is a remarkable connection between \( P \) and the transversal Radon transform \( T \) in (1.2). This connection was pointed out in [5, Lemma 2.3] and used in [30]. For the sake of completeness, below we describe it in detail. Let

\[
(7.5) \quad (B_1 f)(x) = f(x', x_n - |x'|^2), \quad (B_2 F)(x) = F(2x', x_n - |x'|^2).
\]

The corresponding inverse maps have the form

\[
(7.6) \quad (B_1^{-1} u)(x) = u(x', x_n + |x'|^2), \quad (B_2^{-1} v)(x) = v\left(\frac{x'}{2}, x_n + \frac{|x'|^2}{4}\right).
\]

One can readily see that

\[
(7.7) \quad \|B_1 f\|_p = \|f\|_p, \quad \|B_2 F\|_q = 2^{(1-n)/q} \|F\|_q.
\]

**Lemma 7.1.** The equality

\[
(7.8) \quad Pf = B_2TB_1f,
\]

holds provided that either side of it exists in the Lebesgue sense.

**Proof.** We write the left-hand side as

\[
(Pf)(x) = \int_{\mathbb{R}^{n-1}} f(y', x_n - |x' - y'|^2) dy'
\]

\[
= \int_{\mathbb{R}^{n-1}} f(y', x_n - |x'|^2 - |y'|^2 + 2x' \cdot y') dy'.
\]

Hence

\[
(B_2^{-1} Pf)(x) = (Pf)\left(\frac{x'}{2}, x_n + \frac{|x'|^2}{4}\right) = \int_{\mathbb{R}^{n-1}} f(y', x_n - |y'|^2 + x' \cdot y') dy'.
\]

On the other hand,

\[
(TB_1 f)(x) = \int_{\mathbb{R}^{n-1}} (B_1 f)(y', x' \cdot y' + x_n) dy' = \int_{\mathbb{R}^{n-1}} f(y', x_n - |y'|^2 + x' \cdot y') dy',
\]

as above. This gives the result. \( \Box \)
Lemma 7.2. Let $v(x) = (1 + |x'|^2 + (x_n + |x'|^2)^{-1/2}$. Then
\begin{equation}
\int_{\mathbb{R}^n} (Pf)(x)u(x)dx = \frac{\sigma_{n-1}}{2^n} \int_{\mathbb{R}^n} f(x)v(x)dx,
\end{equation}
provided that either side of this equality exists in the Lebesgue sense.

Proof. The formula (7.9) follows from (3.25) and (7.8). \qed

Lemma 7.3. If $1 \leq p < \frac{n}{n/(n-1)}$, then $(Pf)(x)$ is finite for almost all $x \in \mathbb{R}^n$ and the absolute value of the left-hand side of (7.9) does not exceed $c||f||_p$, $c = \text{const}$.

This statement follows from (7.9) by Hölder's inequality.

Remark 7.4. The bound $p < n/(n-1)$ in Lemma 7.3 is sharp. Indeed, suppose $p \geq n/(n-1)$ and let $f_0(\in L^p(\mathbb{R}^n))$ be a function for which $Tf_0 \equiv \infty$; see Remark 3.5. The function $f_\ast = B_1^{-1}f_0$ belongs to $L^p(\mathbb{R}^n)$ by (7.7). By (7.8), it follows that $Pf_\ast = B_2TB_1f_\ast \equiv \infty$.

7.2. Fractional integrals $P^\alpha f$. Elementary properties. The following lemma is a generalization of (7.8).

Lemma 7.5. Let $B_1$ and $B_2$ be the mappings (7.5). If $\Re \alpha > 0$, then
\begin{equation}
P^\alpha f = B_2T^\alpha_+ B_1 f,
\end{equation}
provided that either side of this equality exists in the Lebesgue sense.

Proof. Owing to (7.6),
\begin{align*}
(B_2^{-1}P^\alpha_+ f)(x) &= \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^n} (y_n - |y'|^2)^{\alpha-1} \left(\frac{x'}{2} - y', x_n + \frac{|x'|^2}{4} - y_n\right)dy, \\
&= \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^{n-1}} dy' \int_0^\infty s^{\alpha-1}f \left(\frac{x'}{2} - y', x_n + \frac{|x'|^2}{4} - s - |y'|^2\right)ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1}ds \int_{\mathbb{R}^{n-1}} f(z', x_n - s + x' \cdot z' - |z'|^2)dz'. \tag{7.11}
\end{align*}

On the other hand,
\begin{align*}
(T^\alpha_+ B_1 f)(x) &= \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^n} (x_n - y_n)^{\alpha-1}(B_1f)(y', y_n + x' \cdot y')dy \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1}ds \int_{\mathbb{R}^{n-1}} f(y', x_n - s + x' \cdot y' - |y'|^2)dy',
\end{align*}
which coincides with (7.11). \qed
Theorem 4.3 gives the following boundedness result for $P_{+}^{\alpha}f$.

**Lemma 7.6.** Let $0 < \alpha < 1$,

$$1 \leq p < \frac{n}{n-1+\alpha}, \quad \nu = -\alpha - \frac{n-1}{p'}, \quad \mu = -n \left(1 - \frac{2}{p}\right),$$

$1/p + 1/p' = 1$. We define

$$L_{p}^{v}(\mathbb{R}^{n}) = \{ f : \| f \|_{p,v} \equiv \| v(x)f \|_{L_{p}^{p}(\mathbb{R}^{n})} < \infty \}, \quad v(x) = \frac{|x_n - |x'|^{2}|^{\nu}}{(1+4|x'|^{2})^{\mu/2}}.$$

Then

$$\| P_{+}^{\alpha}f \|_{p,v} \leq c \| f \|_{p}, \quad c = \text{const.}$$

**Proof.** Let $u(x) = |x_n|^{\nu}(1+|x'|^{2})^{-\mu/2}$, as in Theorem 4.3. By (7.10) and (7.6),

$$\| T_{+}^{\alpha}f \|_{p,u} = \| u(x)B_{-1}^{1}P_{+}^{\alpha}B_{1}^{-1}f \|_{p} = \int_{\mathbb{R}^{n}} \frac{|x_n|^{\nu}|x_n - |x'|^{2}|^{\nu}}{(1+|x'|^{2})^{\mu p/2}} \left| (P_{+}^{\alpha}B_{1}^{-1}f) \left( \frac{x'}{2}, x_n + \frac{|x'|^{2}}{4} \right) \right|^{p} dx.$$

Changing variables, we write the last expression as

$$\int_{\mathbb{R}^{n}} \left| (P_{+}^{\alpha}B_{1}^{-1}f)(y) \frac{|x_n - |x'|^{2}|^{\nu}}{(1+4|x'|^{2})^{\mu/2}} \right|^{p} dy = \| P_{+}^{\alpha}B_{1}^{-1}f \|_{p,v}.$$

Hence, setting $\varphi = B_{1}^{-1}f$, we obtain

$$\| P_{+}^{\alpha}\varphi \|_{p,v} = \| T_{+}^{\alpha}B_{1}\varphi \|_{p,u} \leq 2^{-1/p}c_{\alpha} \| B_{1}\varphi \|_{p} = 2^{-1/p}c_{\alpha} \| \varphi \|_{p};$$

cf. (7.7). This gives (7.13), up to notation. □

**Lemma 7.7.** Let $f \in S(\mathbb{R}^{n})$. The following statements hold.

(i) For each $x \in \mathbb{R}^{n}$, $(P_{+}^{\alpha}f)(x)$ extends as an entire function of $\alpha$. Moreover,

$$\lim_{\alpha \to 0}(P_{+}^{\alpha}f)(x) = (Pf)(x),$$

where $(Pf)(x)$ is the parabolic Radon transform (7.1).

(ii) If $\text{Re} \alpha > 0$, then for any multi-index $m$,

$$\partial^{m}P_{+}^{\alpha}f = P_{+}^{\alpha}\partial^{m}f.$$

(iii) For any positive integer $k$,

$$P_{+}^{\alpha}f = P_{+}^{\alpha+k}\partial_{n}^{k}f = \partial_{n}^{k}P_{+}^{\alpha+k}f.$$
To prove (i), we have

\[
(P_+^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^{n-1}} dy' \int_{-\infty}^{\infty} (y_n - |y'|^2)^{\alpha-1} f(x' - y', x_n - y_n) dy_n
\]

(7.17) \[= \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} A_x(s) ds; \]

\[A_x(s) = \int_{\mathbb{R}^{n-1}} f(x' - y', x_n - |y'|^2 - s) dy'. \]

Because the function \(A_x(s)\) is smooth, rapidly decreasing, and satisfies \(A_x(0) = (Pf)(x)\), the result follows; cf. [8, Chapter I, Section 3.2], [29, Section 2.5]. In (ii) we simply differentiate under the sign of integration. The first equality in (7.16) can be obtained using integration by parts. The second equality is the result of differentiation:

\[\partial_n^k P_+^{\alpha+k} f = \partial_n^k T_+^\alpha P_+^\alpha f = P_+^\alpha f. \]

\(\Box\)

7.3. The main theorem for \(P_+^\alpha f\).

**Theorem 7.8.** Suppose \(1 \leq p, q \leq \infty\), \(\alpha_0 = \text{Re} \alpha\).

(i) The operator \(P_+^\alpha\) initially defined on functions \(\varphi \in S(\mathbb{R}^n)\) by analytic continuation, extends as a linear bounded operator \(P_+^\alpha : L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)\) if and only if

\[\frac{1-n}{2} \leq \alpha_0 \leq 1, \quad p = \frac{n+1}{n+\alpha_0}, \quad q = \frac{n+1}{1-\alpha_0}. \]

In particular, the parabolic Radon transform \(P\) extends as a linear bounded operator \(P : L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)\) if and only if \(p = (n+1)/n\) and \(q = n+1\).

(ii) In the cases \(0 < \text{Re} \alpha < 1\) and \(\alpha = 0\), the \(L^q\)-functions \((P_+^\alpha f)(x)\) and \((Pf)(x)\) coincide a.e. with absolutely convergent integrals \((P_+^\alpha f)(x)\) and \((Pf)(x)\), respectively.

**Proof.** Let \(f \in S(\mathbb{R}^n)\). If \(\text{Re} \alpha > 0\), then by (7.10), \(P_+^\alpha f = B_2 T_+^\alpha B_1 f\). By Lemmas 4.4 and 7.7, this equality extends analytically to all \(\alpha \in \mathbb{C}\), and the analytic continuation represents a smooth function on \(\mathbb{R}^n\). Here we take into account that \(B_1\) and \(B_2\) are diffeomorphisms of \(S(\mathbb{R}^n)\). We
keep the notation \( P_+^\alpha \) and \( T_+^\alpha \) for the corresponding analytic continuations. Then, by (7.7) and Theorem 5.5,

\[
||P_+^\alpha f||_q = ||B_2 T_+^\alpha B_1 f||_q = 2^{(1-n)/q} ||T_+^\alpha B_1 f||_q \leq c ||B_1 f||_p = c ||f||_p,
\]

as desired. The proof of (ii) mimics the reasoning from Theorem 5.6. □

8. Explicit formulas for operators defined by interpolation

We restrict our consideration to the operators \( T_+^\alpha \). Similar reasoning is applicable to the operators \( R_+^\alpha \), \( P_+^\alpha \), and their modifications. The cases \( \text{Re}\alpha > 0 \) and \( \alpha = 0 \), when the extended operators \( R_+^\alpha \) and \( P_+^\alpha \) coincide a.e. with the corresponding absolutely convergent integrals were mentioned in Theorems 5.6 and 7.8. The most intriguing are the remaining cases, when these integrals diverge.

We invoke the regularization technique, which amounts to the concept of Marchaud’s fractional derivative [16]. The latter is well known in Fractional Calculus and has proved to be useful for regularization of divergent integrals with power singularity; see, e.g., [23, 24, 25, 29, 32, 33]. For the sake of simplicity, we restrict to the case of real \( \alpha \in [(1-n)/2, 0) \). The same idea can be applied to the case \( \text{Im}\alpha \neq 0 \), but the formulas look a bit more complicated; cf. [16, 25].

Let us recall some known facts. The Marchaud fractional derivative \( D_+^\alpha \varphi \) of order \( \alpha > 0 \) of a function \( \varphi : \mathbb{R} \to \mathbb{C} \) is defined by

\[
(8.1) \quad (D_+^\alpha \varphi)(x) = \frac{1}{\kappa_\ell(\alpha)} \int_0^\infty (\Delta_\ell^t \varphi)(x) \frac{dt}{t^{1+\alpha}}, \quad x \in \mathbb{R}.
\]

Here \( (\Delta_\ell^t \varphi)(x) \) is the finite difference

\[
(8.2) \quad (\Delta_\ell^t \varphi)(x) = \sum_{j=0}^\ell \binom{\ell}{j} (-1)^j \varphi(x - jt), \quad \ell > \alpha,
\]
and the normalizing constant $\kappa_\ell(\alpha)$ has the form

\[
(8.3) \quad \kappa_\ell(\alpha) \equiv \int_0^\infty \frac{(1-e^{-v})^\ell}{v^{\alpha+1}} \, dv
\]

\[
= \begin{cases} 
\Gamma(-\alpha) \sum_{j=1}^\ell \binom{\ell}{j} (-1)^j j^\alpha, & \alpha \neq 1, 2, \ldots, \ell - 1, \\
\frac{(-1)^{1+\alpha}}{\alpha!} \sum_{j=1}^\ell \binom{\ell}{j} (-1)^j j^\alpha \log j, & \alpha = 1, 2, \ldots, \ell - 1. 
\end{cases}
\]

If $\alpha = m$ is a positive integer and $\varphi$ is good enough, then $(D_+^\alpha \varphi)(x) = \varphi^{(m)}(x)$ is the usual $m$th derivative of $\varphi$. One can show that

\[
(8.4) \quad D_+^\alpha I_+^\alpha f = \lim_{\varepsilon \to 0} D_+^{\alpha, \varepsilon} I_+^{\alpha} f = f,
\]

where

\[
(8.5) \quad (D_+^{\alpha, \varepsilon} \varphi)(x) = \frac{1}{\kappa_{\ell}(\alpha)} \int_\varepsilon^\infty \frac{\Delta_+^\ell \varphi(x)}{t^{1+\alpha}} \, dt, \quad \varepsilon > 0,
\]

is the truncated Marchaud fractional derivative. The proof of (8.4) relies on the representation of $D_+^{\alpha, \varepsilon} I_+^{\alpha} f$ as an approximate identity

\[
(8.6) \quad (D_+^{\alpha, \varepsilon} I_+^{\alpha} f)(x) = \int_0^\infty \lambda_{\ell, \alpha}(\eta) f(x - \varepsilon \eta) \, d\eta,
\]

where the averaging kernel $\lambda_{\ell, \alpha}(\eta)$ is defined by

\[
(8.7) \quad \lambda_{\ell, \alpha}(\eta) = \frac{1}{\eta \Gamma(1+\alpha) \kappa_{\ell}(\alpha)} \sum_{j=0}^\ell \binom{\ell}{j} (-1)^j (\eta - j)^\alpha,
\]

and has the following properties

\[
(8.8) \quad \int_0^\infty \lambda_{\ell, \alpha}(\eta) \, d\eta = 1, \quad \lambda_{\ell, \alpha}(\eta) = \begin{cases} 
O(\eta^{\alpha-1}) & \text{if } \eta < 1, \\
O(\eta^{\alpha-\ell-1}) & \text{if } \eta > 1;
\end{cases}
\]

see, e.g., [22, p. 182] [29, pp. 50, 51]. If $f \in L^p(\mathbb{R})$, $p \in [1, \infty)$, then the expression in (8.6) tends to $f$ as $\varepsilon \to 0$ in the $L^p$-norm and in the almost everywhere sense. If, moreover, $f \in C_0(\mathbb{R})$, the limit exists in the sup-norm. The result is independent of the choice of the integer $\ell > \alpha$.

Let us proceed to regularization of $T_+^{\alpha} f$. Below we use the above results for Marchaud’s fractional derivative with $\alpha$ replaced by $-\alpha$ and
obtain explicit formulas for $\mathcal{T}_+^\alpha f$ in the case $(1-n)/2 \leq \alpha < 0$. Suppose $f \in L^p(\mathbb{R}^n)$, $p = (n + 1)/(n + \alpha)$, and set

$$\varphi = \mathcal{T} f \quad \psi = \mathcal{T}_+^\alpha f.$$ 

Recall that by Theorem 5.5,

$$\varphi \in L^r(\mathbb{R}^n), \quad r = (n+1)/n, \quad \text{and} \quad \psi \in L^q(\mathbb{R}^n), \quad q = (n+1)/(1-\alpha).$$

Moreover, by Theorem 5.6, $\varphi = \mathcal{T} f$ can be written in the integral form

$$\varphi(x) = (T f)(x) = \int_{\mathbb{R}^{n-1}} f(y', x_n + x' \cdot y') \, dy'.$$

**Theorem 8.1.** Let $f \in L^p(\mathbb{R}^n)$, $p = (n+1)/(n+\alpha)$, $q = (n+1)/(1-\alpha)$, $(1-n)/2 \leq \alpha < 0$. Then the function $\mathcal{T}_+^\alpha f \in L^q(\mathbb{R}^n)$, determined by Theorem 5.5, can be represented by the difference hypersingular integral

$$(\mathcal{T}_+^\alpha f)(x', x_n) = \frac{1}{\kappa(-\alpha)} \int_{0}^{\infty} \left[ \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j (Tf)(x', x_n - jt) \right] \frac{dt}{t^{1-\alpha}},$$

in which $\kappa(-\alpha)$ is defined by (8.3), $\ell > -\alpha$, and $\int_{0}^{\infty} (\ldots) = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} (\ldots)$.

The limit exists in the $L^q$-norm with respect to the $x_n$-variable for almost all $x' \in \mathbb{R}^{n-1}$. It also exists for almost all $x \in \mathbb{R}^n$.

**Example 8.2.** Let $n = 2$, when

$$p = 3/(2 + \alpha), \quad q = 3/(1 - \alpha), \quad -1/2 \leq \alpha < 0.$$

Choosing $\ell = 1$, we obtain

$$(\mathcal{T}_+^\alpha f)(x', x_n) = \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} \frac{(Tf)(x', x_n - t) - (Tf)(x', x_n)}{t^{1-\alpha}} \, dt.$$

**Proof of Theorem 8.1.** Recall that $\varphi = \mathcal{T} f = T f \in L^r(\mathbb{R}^n)$, $\psi = \mathcal{T}_+^\alpha f \in L^q(\mathbb{R}^n)$. We consider $\varphi$ and $\psi$ as single-variable functions $\varphi_{x'}(x_n) \equiv \varphi(x', x_n)$ and $\psi_{x'}(x_n) \equiv \psi(x', x_n)$. Clearly, $\varphi_{x'}(\cdot) \in L^r(\mathbb{R})$ and $\psi_{x'}(\cdot) \in L^q(\mathbb{R})$ for almost all $x' \in \mathbb{R}^{n-1}$. Denote

$$((\Lambda_\varepsilon \psi_{x'})(x_n)) = \int_{0}^{\infty} \lambda_{\varepsilon,-\alpha}(\eta) \psi_{x'}(x_n - \varepsilon \eta) \, d\eta, \quad \varepsilon > 0.$$

Our nearest aim is to show that

$$(D_{+,\varepsilon}^-\alpha \varphi_{x'})(x_n) = (\Lambda_\varepsilon \psi_{x'})(x_n),$$

(8.10)
where $\mathbb{D}^{-\alpha}_{+,\varepsilon}$ stands for the truncated Marchaud fractional derivative (8.5) of order $-\alpha$ and $\lambda_{\varepsilon,-\alpha}(\eta)$ is the averaging kernel (8.7) with $\alpha$ replaced by $-\alpha$. The next step will be passage to the limit in (8.10) as $\varepsilon \to 0$.

Let $\Phi_0(\mathbb{R})$ be the Semyanistyi-Lizorkin space of test functions on the real line; see Definition 2.1. Because $\mathbb{D}^{-\alpha}_{+,\varepsilon}\varphi_{x'} \in L^r(\mathbb{R})$ and $\Lambda_{\varepsilon}\psi_{x'} \in L^q(\mathbb{R})$ for almost all $x' \in \mathbb{R}^{n-1}$, then, by Proposition 2.4, it is enough to prove (8.10) in a weak sense, i.e.,

$$\langle \mathbb{D}^{-\alpha}_{+,\varepsilon}\varphi_{x'}, \omega \rangle = \langle \Lambda_{\varepsilon}\psi_{x'}, \omega \rangle, \quad \omega \in \Phi_0(\mathbb{R}).$$

(8.11)

We have

$$\langle \mathbb{D}^{-\alpha}_{+,\varepsilon}\varphi_{x'}, \omega \rangle = \langle \mathbb{D}^{-\alpha}_{+,\varepsilon}[Tf(x', \cdot)], \omega \rangle = \langle (Tf)(x', \cdot), \tilde{\omega} \rangle,$$

(8.12)

where $\tilde{\omega} = \mathbb{D}^{-\alpha}_{-,\varepsilon}\omega \in \Phi_0(\mathbb{R})$.

By Lemma 2.3, there is a sequence $\{f_k\} \subset \Phi(\mathbb{R}^n)$, converging to $f$ in the $L^p$-norm. Then, by Theorem 5.5 and the equality $Tf = Tf$, the sequence $\{Tf_k\}$ converges to $Tf$ in $L^r(\mathbb{R}^n)$, that is,

$$\|Tf - Tf_k\|_r = \int_{\mathbb{R}^{n-1}} dx' \int_{\mathbb{R}} |(Tf)(x', x_n) - (Tf_k)(x', x_n)|^r \, dx_n \to 0, \quad k \to \infty.$$

It follows that there is a subsequence $\{k_j\}$, such that

$$\int_{\mathbb{R}} |(Tf)(x', x_n) - (Tf_{k_j})(x', x_n)|^r \, dx_n \to 0, \quad j \to \infty,$$

for almost all $x' \in \mathbb{R}^{n-1}$. Hence, for almost all $x' \in \mathbb{R}^{n-1}$, by Hölder’s inequality we obtain

$$\left| \langle (Tf)(x', \cdot), \tilde{\omega} \rangle - \langle (Tf_{k_j})(x', \cdot), \tilde{\omega} \rangle \right|$$

$$\leq \left( \int_{\mathbb{R}} |(Tf)(x', x_n) - (Tf_{k_j})(x', x_n)|^r \, dx_n \right)^{1/r} \|\tilde{\omega}\|_{L^r} \to 0, \quad j \to \infty,$$

and (8.12) can be continued:

$$\langle \mathbb{D}^{-\alpha}_{+,\varepsilon}\varphi_{x'}, \omega \rangle = \lim_{j \to \infty} \langle (Tf_{k_j})(x', \cdot), \tilde{\omega} \rangle$$

$$= \lim_{j \to \infty} \langle I^\alpha_+ I^-_+[(Tf_{k_j})(x', \cdot)], \tilde{\omega} \rangle$$

$$= \lim_{j \to \infty} \langle I^\alpha_+ [(Tf_{k_j})(x', \cdot)], I^-_+ \mathbb{D}^{-\alpha}_{-,\varepsilon}\omega \rangle$$

$$= \lim_{j \to \infty} \langle (T^\alpha_+ f_{k_j})(x', \cdot), \mathbb{D}^{-\alpha}_{-,\varepsilon} I^-_+\omega \rangle,$$
where
\[ (D_{-\varepsilon}^{-\alpha} \varphi)(x) = \frac{1}{\varepsilon^{\alpha}} \int_0^\infty \left[ \sum_{j=0}^\ell \binom{\ell}{j} (-1)^j \varphi(x + jt) \right] \frac{dt}{t^{1-\alpha}}. \]

Note that $D_{-\varepsilon}^{-\alpha}$ preserves the space $\Phi_0(\mathbb{R})$ and the operators $I_{-\varepsilon}^{-\alpha}$ and $D_{-\varepsilon}^{-\alpha}$ commute in this space. Note also that $I_{-\varepsilon}^{\alpha} \rightarrow f \ast k_j$ because $f_{k_j} \in \Phi(\mathbb{R}^n)$.

Furthermore, since $\{f_k\}$, as a subsequence of $\{f_k\}$, converges to $f$ in the $L^p$-norm, by Theorem 5.5 it follows that $T_{+}^{\alpha} f_{k_j} \rightarrow T_{+}^{\alpha} f$ in the $L^q$-norm. Hence the last limit can be written as $\langle (T_{+}^{\alpha} f)(x'), D_{-\varepsilon}^{-\alpha} I_{-\varepsilon}^{-\alpha} \omega \rangle$ or $\langle \psi \cdot, D_{-\varepsilon}^{-\alpha} I_{-\varepsilon}^{-\alpha} \omega \rangle$. The composition $D_{-\varepsilon}^{-\alpha} I_{-\varepsilon}^{-\alpha} \omega$ can be transformed by the formula
\[ (D_{-\varepsilon}^{-\alpha} I_{-\varepsilon}^{-\alpha} f)(x) = \int_0^\infty \lambda_{\ell,-\alpha}(\eta) f(x + \varepsilon \eta) d\eta, \]
which is a modification of (8.6). Hence, we continue:

\[
\begin{align*}
\langle D_{+,-\varepsilon}^{-\alpha} \varphi_{x'}, \omega \rangle &= \int_{\mathbb{R}} \psi_{x'}(x_n) \, dx_n \int_0^\infty \lambda_{\ell,-\alpha}(\eta) \omega_{x'}(x_n + \varepsilon \eta) d\eta \\
&= \int_0^\infty \lambda_{\ell,-\alpha}(\eta) d\eta \int_{\mathbb{R}} \psi(x', x_n) \omega_{x'}(x_n + \varepsilon \eta) \, dx_n \\
&= \int_0^\infty \lambda_{\ell,-\alpha}(\eta) d\eta \int_{\mathbb{R}} \psi(x', x_n - \varepsilon \eta) \omega_{x'}(x_n) \, dx_n \\
&= \int_{\mathbb{R}} (\Lambda_{\varepsilon} \psi_{x'})(x_n) \omega_{x'}(x_n) \, dx_n.
\end{align*}
\]
This gives (8.11), and therefore (8.10).

To complete the proof, it remains to apply the standard machinery of approximation to the identity to the right-hand side of (8.10) in the $x_n$-variable, assuming $x'$ fixed. Taking into account that $\psi_{x'}(\cdot) \in L^q(\mathbb{R})$ and using the properties (8.8) of the averaging kernel $\lambda_{\ell,-\alpha}(\eta)$, we conclude that for almost all $x' \in \mathbb{R}^{n-1}$, $(\Lambda_{\varepsilon} \psi_{x'})(x_n) \rightarrow \psi_{x'}(x_n)$ in the norm of the space $L^q(\mathbb{R})$ and for almost all $x_n \in \mathbb{R}$. Because the Marchaud fractional derivative $(D_{+}^{-\alpha} \varphi_{x'})(x_n) = \lim_{\varepsilon \to 0} (D_{+,-\varepsilon}^{-\alpha} \varphi_{x'})(x_n)$ coincides with the right-hand side of (8.9), we are done. $\square$
9. Conclusion

Many other Radon-type fractional integrals are known in harmonic analysis and integral geometry. Such integrals are associated with Radon transforms on constant curvature spaces, matrix spaces, and diverse homogeneous spaces of Lie groups; see, e.g., [19, 20, 21, 27, 28, 40]. The corresponding $L^p$-$L^q$ estimates are of great interest. Perhaps some readers will be inspired to develop interpolation tools that would be applicable to analytic families of this kind.

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