Godement–Jacquet $L$-function, some conjectures and some consequences

Amrinder Kaur and Ayyadurai Sankaranarayanan

Abstract. In this paper, we investigate the mean square estimate for the logarithmic derivative of the Godement–Jacquet $L$-function $L_f(s)$ assuming the Riemann hypothesis for $L_f(s)$ and Rudnick–Sarnak conjecture.

Keywords. Godement–Jacquet $L$-function, Rudnick–Sarnak conjecture, Hecke–Maass form, Riemann Hypothesis.

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1. Introduction

Let $n \geq 2$, and let $v = (v_1, v_2, \ldots, v_{n-1}) \in \mathbb{C}^{n-1}$. A Maass form [Gol06] for $SL(n, \mathbb{Z})$ of type $v$ is a smooth function $f \in L^2(SL(n, \mathbb{Z}) \backslash \mathcal{H}^n)$ which satisfies

1. $f(\gamma z) = f(z)$, for all $\gamma \in SL(n, \mathbb{Z})$, $z \in \mathcal{H}^n$,
2. $Df(z) = \lambda_D f(z)$, for all $D \in \mathfrak{D}^n$ where $\mathfrak{D}^n$ is the center of the universal enveloping algebra of $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{gl}(n, \mathbb{R})$ is the Lie algebra of $GL(n, \mathbb{R})$,
3. $\int_{(SL(n,\mathbb{Z}) \cap U) \backslash U} f(uz) \, du = 0$,

for all upper triangular groups $U$ of the form

$$U = \left\{ \begin{pmatrix} I_{r_1} & & * \\ & I_{r_2} & \\ & & \ddots \\ & & & I_{r_b} \end{pmatrix} \right\},$$

with $r_1 + r_2 + \cdots + r_b = n$. Here, $I_r$ denotes the $r \times r$ identity matrix, and $*$ denotes arbitrary real entries.

A Hecke–Maass form is a Maass form which is an eigenvector for the Hecke operators algebra. Let $f(z)$ be a Hecke–Maass form of type $v = (v_1, v_2, \ldots, v_{n-1}) \in \mathbb{C}^{n-1}$ for $SL(n, \mathbb{Z})$. Then it has the Fourier expansion

$$f(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z}) \backslash SL(n-1, \mathbb{Z})} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \frac{A(m_1, \ldots, m_{n-1})}{\prod_{j=1}^{n-1} |m_j|^{(n-2)/2}} \times W_f \left( M \cdot \begin{pmatrix} \gamma \\ z, v, \psi_1, \ldots, \psi_{n-1} \end{pmatrix}, \frac{m_{n-1}}{m_{n-1}} \right),$$

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where
\[
M = \begin{pmatrix}
m_1 \ldots m_{n-2} & \cdots & m_1 \\
\vdots & & \ddots \\
1 & & & m_1 m_2 \\
0 & \cdots & & 1
\end{pmatrix},
\]
\[
A(m_1, \ldots, m_{n-1}) \in \mathbb{C}, \quad A(1, \ldots, 1) = 1,
\]
\[
\psi_{1, \ldots, 1, \epsilon} = e^{2\pi i (u_1 + \cdots + u_{n-2} + \epsilon u_{n-1})},
\]
\[
U_{n-1}(\mathbb{Z}) \text{ denotes the group of } (n-1) \times (n-1) \text{ upper triangular matrices with 1s on the diagonal and an integer entry above the diagonal and } W_J \text{ is the Jacquet Whittaker function.}
\]

If \( f(z) \) is a Maass form of type \( (v_1, \ldots, v_{n-1}) \in \mathbb{C}^{n-1} \), then
\[
\tilde{f}(z) := f(w \cdot (z^{-1})^T \cdot w),
\]
is a Maass form of type \( (v_{n-1}, \ldots, v_1) \) for \( SL(n, \mathbb{Z}) \) called the dual Maass form. If \( A(m_1, \ldots, m_{n-1}) \) is the \( (m_1, \ldots, m_{n-1}) \)-Fourier coefficient of \( f \), then \( A(m_{n-1}, \ldots, m_1) \) is the corresponding Fourier coefficient of \( \tilde{f} \).

We note that the Fourier coefficients \( A(m_1, \ldots, m_{n-1}) \) satisfy the multiplicative relations
\[
A(m_1 m'_1, \ldots, m_{n-1} m'_{n-1}) = A(m_1, \ldots, m_{n-1}) \cdot A(m'_1, \ldots, m'_{n-1}),
\]
if
\[
(m_1 \ldots m_{n-1}, m'_1 \ldots m'_{n-1}) = 1,
\]
\[
A(m, 1, \ldots, 1)A(m_1, \ldots, m_{n-1}) = \sum_{\prod_{i=1}^n c_i = m \atop c_1 m_1, c_2 m_2, \ldots, c_{n-1} m_{n-1}} A\left(\frac{m_1 c_n}{c_1}, \frac{m_2 c_1}{c_2}, \ldots, \frac{m_{n-1} c_{n-2}}{c_{n-1}}\right),
\]
and
\[
A(m_{n-1}, \ldots, m_1) = A(m_1, \ldots, m_{n-1}).
\]
1. Introduction

Definition 1.1. [Gol06] The Godement–Jacquet $L$-function $L_f(s)$ attached to $f$ is defined for $\Re(s) > 1$ by

$$L_f(s) = \sum_{m=1}^{\infty} \frac{A(m,1,\ldots,1)}{m^s} = \prod_{p} \prod_{i=1}^{\pi_n} (1 - \alpha_{p,i} p^{-s})^{-1},$$

where $\{\alpha_{p,i}\}, 1 \leq i \leq n$ are the complex roots of the monic polynomial

$$X^n + \sum_{j=1}^{n-1} (-1)^j A(1,\ldots,1,p,1,\ldots,1) X^{n-j} + (-1)^n \in \mathbb{C}[X],$$

and

$$A(1,\ldots,1,p,1,\ldots,1) = \sum_{1 \leq i_1 < \cdots < i_j \leq n} \alpha_{p,i_1} \cdots \alpha_{p,i_j}, \quad \text{for } 1 \leq j \leq n-1.$$

$L_f(s)$ satisfies the functional equation:

$$\Lambda_f(s) := \prod_{i=1}^{\pi_n} \pi^{-s+\frac{\lambda_i(vf)}{2}} \Gamma \left(\frac{s - \frac{\lambda_i(vf)}{2}}{2}\right) L_f(s) = \Lambda_f(1-s),$$

where $\tilde{f}$ is the Dual Maass form.

In the case of Godement–Jacquet $L$-function, Yujiao Jiang and Guangshi Lü [JiLu17] have studied cancellation on the exponential sum

$$\sum_{m \leq N} \mu(m) A(m,1) e^{2\pi i m \theta}$$

related to $SL(3,\mathbb{Z})$ where $\theta \in \mathbb{R}$.

Throughout the paper, we assume that $f$ is self dual i.e., $\tilde{f} = f$.

$\epsilon$, $\epsilon_1$ and $\eta$ always denote any small positive constants.

If $N_f(T)$ denotes the number of zeros of $L_f(s)$ in the rectangle mentioned below, then from the functional equation and the argument principle of complex function theory we have,

$$N_f(T) \sim c(n) T \log T,$$

where $c(n)$ is a non zero constant depending only on the degree $n$ of $L_f(s)$.

\(\begin{array}{c}
-1 + 2iT \\
2 + 2iT \\
-1 + iT \\
2 + iT
\end{array}\)

(i) The generalized Ramanujan conjecture:

It asserts that

$$|A(m,1,\ldots,1)| \leq d_n(m)$$
where $d_n(m)$ is the number of representations of $m$ as the product of $n$ natural numbers. The current best estimates are due to Kim and Sarnak [Kim03] for $2 \leq n \leq 4$ and Luo, Rudnick and Sarnak for $n \geq 5$

$$|A(m)| \leq m^{\frac{7}{64}}d(m),$$

$$|A(m, 1)| \leq m^{\frac{5}{2^3}}d_3(m),$$

$$|A(m, 1, 1)| \leq m^{\frac{9}{2^2}}d_4(m),$$

$$|A(m, 1, \ldots, 1)| \leq m^{\frac{1}{2} - \frac{1}{n^2+1}}d_n(m).$$

We note that the generalized Ramanujan conjecture is equivalent to

$$|\alpha_{p,i}| = 1 \quad \forall \text{ primes } p \text{ and } i = 1, 2, \ldots, n.$$ 

Other estimates are equivalent to

$$|\alpha_{p,i}| \leq p^{\theta_n} \quad \forall \text{ primes } p \text{ and } i = 1, 2, \ldots, n$$ 

where

$$\theta_2 := \frac{7}{64}, \quad \theta_3 := \frac{5}{14}, \quad \theta_4 := \frac{9}{22}, \quad \theta_n := \frac{1}{2} - \frac{1}{n^2+1} (n \geq 5).$$

(ii) Ramanujan’s generalized weak conjecture:

We formulate this conjecture as:

For $n \geq 2$, the inequality

$$|\alpha_{p,i}| \leq p^{\frac{1}{2} - \epsilon_1}$$

holds for some small $\epsilon_1 > 0$, for every prime $p$ and for $i = 1, 2, \ldots, n$. Of course, this weak conjecture holds good for $n = 2$. For $n \geq 3$, this conjecture is still open.

Taking the logarithmic derivative of $L_f(s)$, we have

$$-\frac{L'_f}{L_f}(s) := \sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m^s} = \sum_{m=1}^{\infty} \frac{\Lambda(m)a_f(m)}{m^s}$$

where $a_f(m)$ is multiplicative and

$$a_f(p^r) = \sum_{i=1}^{n} \alpha_{p,i}$$

for any integer $r \geq 1$.

In particular,

$$a_f(p) = \sum_{i=1}^{n} \alpha_{p,i} = A(p, 1, \ldots, 1).$$

(iii) Rudnick–Sarnak conjecture:

For any fixed integer $r \geq 2$,

$$\sum_{p} \frac{|a_f(p^r)|^2 (\log p)^2}{p^r} < \infty.$$
We know that this conjecture is true for \( n \leq 4 \). (See [Ki06, RuSa96].)

(iv) **Riemann hypothesis for** \( L_f(s) \):
It asserts that \( L_f(s) \neq 0 \) in \( \Re(s) > \frac{1}{2} \).

The aim of this paper is to establish:

**Theorem 1.1.** Ramanujan’s weak conjecture implies Rudnick–Sarnak conjecture.

**Remark 1.2.** Theorem 1.1 is indicated in [Ki06].

**Theorem 1.3.** Assume \( n \geq 5 \) be any arbitrary but fixed integer. Let \( \epsilon \) be any small positive constant and \( T \geq T_0 \) where \( T_0 \) is sufficiently large. Assume the Rudnick–Sarnak conjecture and Riemann hypothesis for \( L_f(s) \). Then the estimate:

\[
\int_T^{2T} \left| \frac{L'_f}{L_f}(\sigma_0 + it) \right|^2 dt \ll_{f,n,\epsilon,\eta} T(\log T)^{2\eta}
\]

holds for \( \frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon \) with \( \eta \) being some constant satisfying \( 0 < \eta < \frac{1}{2} \).

**Remark 1.4.** Since Rudnick–Sarnak conjecture is true for \( 2 \leq n \leq 4 \), Theorem 1.3 holds just with the assumption of Riemann hypothesis for \( L_f(s) \) whenever \( 2 \leq n \leq 4 \).

**Remark 1.5.** It is not difficult to see from our arguments that only assuming Riemann Hypothesis for \( L_f(s) \), Theorem 1.3 can be upheld for any \( \sigma_0 \) satisfying \( 1 - \frac{1}{n^2 + 1} + \epsilon \leq \sigma_0 \leq 1 - \epsilon \) by using the bound \( \theta_n = \frac{1}{2} - \frac{1}{n^2 + 1} \) of Luo, Rudnick and Sarnak.

It is also not difficult to see from our arguments that the generalized Ramanujan conjecture and the Riemann hypothesis for \( L_f(s) \) together imply the bound

\[
\int_T^{2T} \left| \frac{L'_f}{L_f}(\sigma_0 + it) \right|^2 dt \ll_{f,n,\epsilon} T
\]

(1.1)

to hold for any \( \sigma_0 \) satisfying \( \frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon \).

Though we expect the bound stated in Equation 1.1 to hold unconditionally for \( \sigma_0 \) in the said range, this seems to be very hard.

2. **Some Lemmas**

**Lemma 2.1.** If \( f(s) \) is regular and

\[
\left| \frac{f(s)}{f(s_0)} \right| < e^M \quad (M > 1)
\]

in \( |s - s_0| \leq r_1 \), then for any constant \( b \) with \( 0 < b < \frac{1}{2} \),

\[
\left| \frac{f'}{f}(s) - \sum \frac{1}{s - \rho} \right| \ll_b \frac{M}{r_1}
\]
in \(|s - s_0| \leq \left( \frac{1}{2} - b \right) r_1\), where \( \rho \) runs over all zeros of \( f(s) \) such that \(|\rho - s_0| \leq \frac{r_1}{2}\).

**Proof.** See Lemma \( \alpha \) in Section 3.9 of [TiHe86] or see [RaSa91].

**Lemma 2.2.** Let \( N_f^*(T) \) denote the number of zeros of \( L_f(s) \) in the region \( 0 \leq \sigma \leq 1, 0 \leq t \leq T \). Then,
\[
N_f^*(T + 1) - N_f^*(T) \ll n \log T.
\]

**Proof.** Let \( n(r_1) \) denote the number of zeros of \( L_f(s) \) in the circle with centre \( 2 + iT \) and radius \( r_1 \). By Jensen’s theorem,
\[
\int_0^3 n(r_1) \frac{dr_1}{r_1} = \frac{1}{2\pi} \int_0^{2\pi} \log |L_f(2 + iT + 3e^{i\theta})| d\theta - \log |L_f(2 + iT)|.
\]

From the functional equation, we observe that
\[
|L_f(s)| \ll f t^A \quad \text{for} \quad -1 \leq \sigma \leq 5 \quad \text{where} \quad A \quad \text{is some fixed positive constant},
\]
and hence we have,
\[
\log |L(2 + iT + 3e^{i\theta})| \ll A \log T.
\]

Note that
\[
\left| 1 - \frac{\alpha_{p,i}}{p^{\sigma + i\theta}} \right| \geq 1 - \frac{|\alpha_{p,i}|}{p^2} \geq 1 - \frac{p_2^2}{p^2} = 1 - \frac{1}{p^2}.
\]

Thus we have,
\[
|L_f(2 + it)| = \prod_p \prod_{i=1}^n \left( 1 - \frac{\alpha_{p,i}}{p^{\sigma + it}} \right)^{-1} \leq \prod_p \prod_{i=1}^n \left( 1 - \frac{1}{p^2} \right)^{-1} \leq \left( \frac{\zeta(3)}{2} \right)^n \ll_n 1.
\]
Therefore,
\[ \int_0^3 \frac{n(r_1)}{r_1} dr_1 \ll A \log T + A \ll \log T, \]
\[ \int_0^3 \frac{n(r_1)}{r_1} dr_1 \geq \int_{\sqrt{5}}^{3} \frac{n(r_1)}{r_1} dr_1 \geq n(\sqrt{5}) \int_{\sqrt{5}}^{3} \frac{dr_1}{r_1} \geq c.n(\sqrt{5}). \]

Hence,
\[ N^*_f(T + 1) - N^*_f(T) \ll_n \log T. \]

**Lemma 2.3.** Let \( a_m (m=1,2,\ldots,N) \) be any set of complex numbers. Then
\[
\int_T^{2T} \left| \sum_{m=1}^{N} a_m e^{-it} \right|^2 dt = \sum_{m=1}^{N} |a_m|^2 \left( T + O(m) \right).
\]

**Lemma 2.4.** Let \( b_m \) be any set of complex numbers such that \( \sum m \left| b_m \right|^2 \) is convergent. Then
\[
\int_T^{2T} \left| \sum_{m=1}^{\infty} b_m e^{-it} \right|^2 dt = \sum_{m=1}^{\infty} |b_m|^2 \left( T + O(m) \right).
\]

**Proof.** See [MoVa74] or [Ram79] for Montgomery and Vaughan theorem.

Hereafter, \( Y \geq 10 \) is an arbitrary parameter depending on \( T \) which will be chosen suitably later. Also, \( \sigma_0 \) satisfies the inequality \( \frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon \) for any small positive constant \( \epsilon \).

**Lemma 2.5.** For \( \frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon \), we have
\[
\sum_{m > \frac{Y}{2} (\log Y)^2} m|\Lambda_f(m)|^2 e^{-\frac{Y^2}{m^2}} \ll 1.
\]

**Proof.** We have,
\[
\sum_{m > \frac{Y}{2} (\log Y)^2} m|\Lambda_f(m)|^2 e^{-\frac{Y^2}{m^2}} \ll \sum_{m > \frac{Y}{2} (\log Y)^2} \frac{m|\Lambda_f(m)|^2 e^{-\frac{Y^2}{m^2}}}{m^{2\sigma_0}} \ll Y^2 \sum_{m > \frac{Y}{2} (\log Y)^2} \frac{|\Lambda_f(m)|^2 e^{-\frac{Y^2}{m^2}}}{m^{1 + 2\sigma_0}}.
\]
Since $\frac{m}{Y} \geq \frac{1}{2} (\log Y)^2$ for $m \geq \frac{Y}{2} (\log Y)^2$, we have $e^{\frac{m}{Y}} \gg Y^B$ for any large positive constant $B$. Therefore,

$$\sum_{m > \frac{Y}{2} (\log Y)^2} \frac{m|\Lambda_f(m)|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}} \ll \frac{Y^2}{Y^B} \sum_{m > \frac{Y}{2} (\log Y)^2} \frac{|\Lambda_f(m)|^2}{m^{1+2\sigma_0}} \ll 1.$$ 

**Lemma 2.6.** Assuming Rudnick–Sarnak conjecture and taking $Y$ sufficiently large, we have

$$\sum_{m \leq \frac{Y}{2} (\log Y)^2} \frac{|\Lambda_f(m)|^2}{m^{2\sigma_0}} e^{-\frac{2m}{Y}} \ll (\log Y)^2.$$ 

**Proof.** Note that

$$\sum_{m \leq \frac{Y}{2} (\log Y)^2} \frac{|\Lambda_f(m)|^2}{m^{2\sigma_0}} e^{-\frac{2m}{Y}} \leq \sum_{p \leq \frac{Y}{2} (\log Y)^2} \frac{(|\log p|^2)}{p^{2\sigma_0}} + \sum_{r=2}^{[\log Y (\log 2)^{r-1}]} \sum_{p} \frac{(|\log p|^2)}{(p^r)^{2\sigma_0}},$$ 

and

$$|a_f(p)| = \left| \sum_{i=1}^{n} a_{p,i} \right| = |A(p, 1, \ldots, 1)|.$$

We have,

$$\sum_{m \leq Y} \frac{c_m}{m^l} = \int_{1}^{Y} d \left( \sum_{m \leq u} c_m \right) \frac{u^l}{u^l} = \sum_{m \leq u} c_m \frac{Y}{u^l} \left| Y \int_{1}^{Y} (-1) \sum_{m \leq u} c_m \frac{u^l}{u^{l+1}} du. \right.$$

From Remark 12.1.8 of [Gol06], we have

$$\sum_{m_{1}^{n-1}m_{2}^{n-2}\ldots m_{n-1} \leq Y} |A(m_1, m_2, \ldots, m_{n-1})|^2 \ll f Y.$$ 

Therefore,

$$\sum_{m \leq Y} |A(m, 1, \ldots, 1)|^2 \leq \sum_{m_{1}^{n-1}m_{2}^{n-2}\ldots m_{n-1} \leq Y} |A(m_1, m_2, \ldots, m_{n-1})|^2 \ll f Y.$$
Taking $l = 2\sigma_0$ and $c_m = |A(m, 1, \ldots, 1)|^2$,

$$\sum_{m \leq \frac{Y}{2}(\log Y)^2} \frac{|A(m, 1, \ldots, 1)|^2}{m^{2\sigma_0}} \ll 1.$$ 

Hence,

$$\sum_{p \leq \frac{Y}{2}(\log Y)^2} \frac{(\log p)^2 |a_f(p)|^2}{p^{2\sigma_0}} \ll (\log Y)^2 \sum_{m \leq \frac{Y}{2}(\log Y)^2} \frac{|A(m, 1, \ldots, 1)|^2}{m^{2\sigma_0}} \ll (\log Y)^2.$$

By Rudnick–Sarnak conjecture and the bound $|\alpha_{p,i}| \leq p^{\theta_n}$ with $\theta_n = \frac{1}{2} - \frac{1}{n^2 + 1}$,

$$\sum_{r \geq 2} \sum_p \frac{(\log p)^2 |a_f(p^r)|^2}{p^r}$$

converges (as in proof of Theorem 1.1) and in particular,

$$\sum_{r = 2}^{\left\lceil \frac{\log Y}{\log 2} \right\rceil + 1} \sum_p \frac{(\log p)^2 |a_f(p^r)|^2}{p^r} \ll 1.$$

Therefore,

$$\sum_{m \leq \frac{Y}{2}(\log Y)^2} \frac{|\Lambda_f(m)|^2}{m^{2\sigma_0}} \ll (\log Y)^2.$$ 

3. Proof of Theorem 1.1

Assuming $|\alpha_{p,i}| \leq p^{\theta_n}$ with $\theta_n = \frac{1}{4} - \epsilon_1$, we need to prove that for every integer $n \geq 5$ and for every integer $r \geq 2$,

$$\sum_p \frac{(\log p)^2 |a_f(p^r)|^2}{p^r} < \infty.$$ 

It is enough to show that

$$\sum_{r = 2}^{\infty} \sum_p \frac{(\log p)^2 |a_f(p^r)|^2}{p^r} < \infty.$$ 

Using

$$a_f(p^r) := \sum_{i=1}^n \alpha_{p,i}^r \quad \text{and} \quad |\alpha_{p,i}| \leq p^{\theta_n}$$
we get,

\[
\sum_{r=2}^{\infty} \sum_{p} \frac{(\log p)^2 |a_f(p^r)|^2}{p^r} \leq \sum_{r=2}^{\infty} \sum_{p} \frac{(\log p)^2 \left( \sum_{i=1}^{n} p^{\theta_{p^r}} \right)^2}{p^r} \\
= \sum_{r=2}^{\infty} \sum_{p} \frac{(\log p)^2 n^2 p^{2r \theta_{p^r}}}{p^r} \leq n^2 \sum_{p} (\log p)^2 \sum_{r=2}^{\infty} \frac{p^{2r (\frac{1}{2} - \epsilon_1)}}{p^r} \\
= n^2 \sum_{p} (\log p)^2 \sum_{r=2}^{\infty} \frac{1}{p^{2r + 2r \epsilon_1}} = n^2 \sum_{p} (\log p)^2 \frac{p^{-(1+4\epsilon_1)}}{1 - p^{-(\frac{1}{2} + 2\epsilon_1)}} = n^2 \sum_{p} (\log p)^2 \frac{1}{p^{\frac{1}{2} + 2\epsilon_1} \left( p^{\frac{1}{2} + 2\epsilon_1} - 1 \right)} \ll_{n, \epsilon_1} 1.
\]

This proves Theorem 1.1.

4. Proof of Theorem 1.3

First, we wish to approximate \( \frac{L'(s)}{L_f(s)} \) uniformly for \( \frac{1}{2} < \sigma_0 \leq \sigma \leq \sigma_1 < 1 \) when \( T \leq t \leq 2T \). We assume throughout below the Riemann hypothesis for \( L_f(s) \).

From the work of Godement–Jacquet [GoJa06], it is known that the function \( L_f(s) \) is of finite order in any bounded vertical strip. Hence, we can very well assume that

\[
L_f(s) \ll T^A = e^{A \log T}
\]

for \(-1 \leq \sigma \leq 2, T \leq t \leq 2T \) and \( A \) some fixed positive constant.

Taking \( s_0 = 2 + it \) with \( t \in \mathbb{R} \), we have

\[
L_f(2 + it) = \prod_{p} \prod_{i=1}^{n} \left( 1 - \frac{\alpha_{p,i}}{p^{2 + it}} \right)^{-1}.
\]
Observe that
\[
\left|1 - \frac{\alpha_{p,i}}{p^{s+it}}\right| \leq 1 + \frac{|\alpha_{p,i}|}{p^2} \\
\leq 1 + \frac{p^{\theta_n}}{p^2} \\
= 1 + \frac{1}{p^{2-\theta_n}} \\
\leq 1 + \frac{1}{p^2}
\]
because $\theta_n \leq \frac{1}{2}$ for $n \geq 2$.

Therefore,
\[
|L_f(2+it)| \geq \prod_{p} \prod_{i=1}^{n} \left(1 + \frac{1}{p^2}\right)^{-1} \\
= \prod_{p} \left(1 + \frac{1}{p^2}\right)^{-n} \\
= \prod_{p} \left(\frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^2}}\right)^n \\
= \left(\frac{\zeta(3)}{\zeta\left(\frac{3}{2}\right)}\right)^n
\]
which is a constant depending only on $n$. Therefore, $L_f(2+it) \neq 0 \forall \ t \in \mathbb{R}$.

Hence from Lemma 2.1, with $r = 12$, $s_0 = 2 + iT$, $f(s) = L_f(s)$, $M = A\log T$, we obtain
\[
-\frac{L'_f}{L_f}(s) = \sum_{|s-s_0| \leq 6} \frac{1}{s-\rho} + O(\log T).
\]

For $|s-s_0| \leq 3$ and so in particular for $-1 \leq \sigma \leq 2$, $t = T$, replacing $T$ by $t$ in the particular case, we obtain
\[
-\frac{L'_f}{L_f}(s) = \sum_{|\rho-s_0| \leq 6} \frac{1}{s-\rho} + O(\log t).
\]

Any term occurring in $\sum_{|\gamma| \leq 1} \frac{1}{s-\rho}$ but not in $\sum_{|s-s_0| \leq 6} \frac{1}{s-\rho}$ is bounded and the number of such terms does not exceed
\[
N_f^+(t+6) - N_f^+(t-6) \ll \log t,
\]
where $N_f^+(t)$ is the number of zeros of $L_f(s)$ in the region $0 \leq \sigma \leq 1$ and $0 \leq t \leq T$. Thus, we get
\[
-\frac{L'_f}{L_f}(s) = \sum_{|\gamma| \leq 1} \frac{1}{s-\rho} + O(\log t).
\]
Assume $\frac{1}{2} < \sigma < 1$ and $T \leq t \leq 2T$, then

$$\sum_{m=1}^{\infty} \frac{A_f(m)}{m^s} e^{-\frac{m}{T}} = -\frac{1}{2\pi i} \int_{2-\infty}^{2+i\infty} \frac{L'_f(s+w)}{L_f(s)} \Gamma(w) Y^w dw.$$ 

Note also that from the above reasoning

$$\frac{L'_f}{L_f}(s) \ll \log t \quad \text{on any line } \sigma \neq \frac{1}{2}.$$ 

Also,

$$\frac{L'_f}{L_f}(s) \ll \frac{\log t}{\min(|t-\gamma|)} + \log t \quad \text{uniformly for } -1 \leq \sigma \leq 2.$$ 

From Lemma 2.2, we observe that each interval $(j, j+1)$ contains values of $t$ whose distance from the ordinate of any zero exceeds $\frac{A}{\log j}$, there is a $t_j$ in any such interval for which

$$\frac{L'_f}{L_f}(s) \ll (\log t)^2 \quad \text{where } -1 \leq \sigma \leq 2 \text{ and } t = t_j.$$ 

Applying Cauchy’s residue theorem to the rectangle, we get

$$\frac{1}{2\pi i} \left( \int_{2-\infty}^{2+i\infty} \frac{L'_f(s+w)}{L_f(s)} \Gamma(w) Y^w dw + \sum_{-t_j < \gamma < t_j} \Gamma(\rho - s) Y^{\rho-s} \right).$$ 

In the sum appearing on the right hand side above, zeros $\rho$ are counted with its multiplicity if there are any multiple zeros. The integrals along the horizontal lines tend to zero as $j \to \infty$ since gamma function decays exponentially and $Y$ is going to be at most a power of $T$ only, so that

$$\sum_{m=1}^{\infty} \frac{A_f(m)}{m^s} e^{-\frac{m}{T}} = \frac{1}{2\pi i} \int_{\frac{1}{2}-\infty}^{\frac{1}{2}+i\infty} \frac{L'_f(s+w)}{L_f(s)} \Gamma(w) Y^w dw \frac{L'_f}{L_f}(s) - \sum_{\rho} \Gamma(\rho - s) Y^{\rho-s}.$$
4. Proof of Theorem 1.3

Note that $\Gamma(w) \ll e^{-A|v|}$ so that the integral on $\Re(w) = \frac{1}{4} - \sigma$ is

\[
\ll \int_{-\infty}^{\infty} e^{-A|v|} \log(|t + v| + 2) Y^{\frac{1}{4} - \sigma} dv
\]

\[
\ll \int_{0}^{2t} e^{-A|v|} \log(10|t| + 2) Y^{\frac{1}{4} - \sigma} dv + \left( \int_{-\infty}^{0} + \int_{2t}^{\infty} \right) e^{-A|v|} \log(|v| + 10) Y^{\frac{1}{4} - \sigma} dv
\]

\[
\ll Y^{\frac{1}{4} - \sigma} \log T + Y^{\frac{1}{4} - \sigma}
\]

\[
\ll Y^{\frac{1}{4} - \sigma} \log T.
\]

Note that for $\frac{1}{2} < \sigma_0 \leq \sigma \leq \sigma_1 < 1$,

\[
|\Gamma(\rho - s)| < A_1 e^{-A_2|\gamma - t|}
\]

uniformly for $\sigma$ in the said range. Therefore,

\[
\sum_{\rho} |\Gamma(\rho - s)| < A_1 \sum_{\rho} e^{-A_2|\gamma - t|} = A_1 \sum_{m=1}^{\infty} \sum_{m-1 \leq \gamma \leq m} e^{-A_2|t - \gamma|}.
\]

The number of terms in the inner sum is

\[
\ll \log(|t| + m) \ll \log |t| + \log(m + 1)
\]

and hence

\[
\sum_{\rho} |\Gamma(\rho - s)| \ll \sum_{m=1}^{\infty} e^{-A_2m} (\log |t| + \log(m + 1)) \ll \log T;
\]

\[
\frac{1}{2} \ll \sum_{\rho} \Gamma(\rho - s) Y^{\sigma - s} \ll Y^{\frac{1}{4} - \sigma} \log T.
\]

Thus for $\frac{1}{2} < \sigma_0 \leq \sigma \leq \sigma_1 < 1$, we have

\[
\frac{L'_f(s)}{L_f(s)} = \sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m^s} e^{-\frac{m}{T}} + O_f(Y^{\frac{1}{4} - \sigma} \log T).
\]

Thus for $\frac{1}{2} + \epsilon \leq \sigma_0 \leq \epsilon$ and $T \leq t \leq 2T$, we obtain

\[
\left| \frac{L'_f(s)}{L_f(s)}(\sigma_0 + it) \right|^2 \ll \left( \sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m^{\sigma_0 + it}} \right)^2 + \left( Y^{\frac{1}{2} - \sigma_0} \log T \right)^2.
\]

Thus,

\[
\int_{T}^{2T} \left| \frac{L'_f(s)}{L_f(s)}(\sigma_0 + it) \right|^2 dt \ll f_t \int_{T}^{2T} \left( \sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m^{\sigma_0 + it}} \right)^2 dt + Y^{1 - 2\sigma_0} T (\log T)^2.
\]
We note that
\[
\left| \sum_{m=1}^{\infty} \frac{\Lambda_f(m)e^{-\frac{m}{T}\sigma_0}}{m^{\sigma_0+it}} \right|^2 \ll \sum_{m \leq \frac{T}{2}(\log Y)^2} \frac{\Lambda_f(m)e^{-\frac{m}{T}\sigma_0}}{m^{\sigma_0+it}} + \sum_{m > \frac{T}{2}(\log Y)^2} \frac{\Lambda_f(m)e^{-\frac{m}{T}\sigma_0}}{m^{\sigma_0+it}},
\]
and hence
\[
\int_T^{2T} \left| \frac{L'(f)(\sigma_0 + it)}{L_f(\sigma_0 + it)} \right|^2 \, dt \ll \int_T^{2T} \left| \sum_{m \leq \frac{T}{2}(\log Y)^2} \frac{\Lambda_f(m)e^{-\frac{m}{T}\sigma_0}}{m^{\sigma_0+it}} \right|^2 \, dt + \int_T^{2T} \left| \sum_{m > \frac{T}{2}(\log Y)^2} \frac{\Lambda_f(m)e^{-\frac{m}{T}\sigma_0}}{m^{\sigma_0+it}} \right|^2 \, dt + Y^{1-2\sigma_0}T(\log T)^2.
\]

By Montgomery–Vaughan theorem (Lemmas 2.3 and 2.4) and Lemma 2.5, we get
\[
\int_T^{2T} \left| \frac{L'(f)(\sigma_0 + it)}{L_f(\sigma_0 + it)} \right|^2 \, dt \ll_f \sum_{m \leq \frac{T}{2}(\log Y)^2} \frac{|\Lambda_f(m)|^2 e^{-\frac{2m}{T}}}{m^{2\sigma_0}} (T + O(m))
\]
\[
+ \sum_{m > \frac{T}{2}(\log Y)^2} \frac{|\Lambda_f(m)|^2 e^{-\frac{2m}{T}}}{m^{2\sigma_0}} (T + O(m)) + Y^{1-2\sigma_0}T(\log T)^2
\]
\[
\ll_f T \sum_{m \leq \frac{T}{2}(\log Y)^2} \frac{|\Lambda_f(m)|^2 e^{-\frac{2m}{T}}}{m^{2\sigma_0}} + \sum_{m \leq \frac{T}{2}(\log Y)^2} \frac{|\Lambda_f(m)|^2 e^{-\frac{2m}{T}}}{m^{2\sigma_0}}
\]
\[
+ T \sum_{m > \frac{T}{2}(\log Y)^2} \frac{|\Lambda_f(m)|^2 e^{-\frac{2m}{T}}}{m^{2\sigma_0}} + \sum_{m > \frac{T}{2}(\log Y)^2} \frac{|\Lambda_f(m)|^2 e^{-\frac{2m}{T}}}{m^{2\sigma_0}}
\]
\[
+ Y^{1-2\sigma_0}T(\log T)^2.
\]

By Lemmas 2.5 and 2.6, we obtain
\[
\int_T^{2T} \left| \frac{L'(f)(\frac{1}{2} + \epsilon + it)}{L_f(\frac{1}{2} + \epsilon + it)} \right|^2 \, dt \ll_{f,n,\epsilon} T(\log Y)^2 + Y(\log Y)^4 + Y^{1-2\sigma_0}T(\log T)^2.
\]

We choose \( Y = \exp\{(\log T)^{\eta}\} \) with any \( \eta \) satisfying \( 0 < \eta < \frac{1}{2} \) so that we obtain
\[
\int_T^{2T} \left| \frac{L'(f)(\sigma_0 + it)}{L_f(\sigma_0 + it)} \right|^2 \, dt \ll_{f,n,\epsilon,\eta} T(\log T)^{2\eta}.
\]

This proves Theorem 1.3.

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Amrinder Kaur
School of Mathematics and Statistics
University of Hyderabad
Hyderabad - 500046, Telangana, India.
email: amrinder1kaur@gmail.com

Ayyadurai Sankaranarayanan
School of Mathematics and Statistics
University of Hyderabad
Hyderabad - 500046, Telangana, India.
e-mail: sank@uohyd.ac.in