ON THE HOMOGENEOUS ERGODIC BILINEAR AVERAGES
WITH MÔBIUS AND LIOUVILLE WEIGHTS

E. H. EL ABDALAOUI

Abstract. It is shown that the homogeneous ergodic bilinear averages with Möbius or Liouville weight converge almost surely to zero, that is, if \( T \) is a map acting on a probability space \((X, \mathcal{A}, \nu)\), and \( a, b \in \mathbb{Z} \), then for any \( f, g \in L^2(X) \), for almost all \( x \in X \),

\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) f(T^{an}x) g(T^{bn}x) \xrightarrow{N \to +\infty} 0,
\]

where \( \nu \) is the Liouville function or the Möbius function. We further obtain that the convergence almost everywhere holds for the short interval with the help of Zhan’s estimation. Also our proof yields a simple proof of Bourgain’s double recurrence theorem. Moreover, we establish that if \( T \) is weakly mixing and its restriction to its Pinsker algebra has singular spectrum, then for any integer \( k \geq 1 \), for any \( f_j \in L^\infty(X) \), \( j = 1, \ldots, k \), for almost all \( x \in X \), we have

\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) \prod_{j=1}^{k} f_j(T_j^{n}x) \xrightarrow{N \to +\infty} 0,
\]

where \( T_j \) are some powers of \( T \), \( j = 1, \ldots, k \).

1. INTRODUCTION

The purpose of this short note is to establish that the homogeneous ergodic bilinear averages with Möbius or Liouville weight converge almost surely to zero. Our result, in some sense, extend Sarnak’s result which assert that the ergodic averages with Möbius or Liouville weight converge almost surely to zero [22]. Moreover, our proof allows us to obtain a simple proof of Bourgain’s double recurrence theorem [6].

The problem of the convergence almost everywhere (a.e.) of the ergodic multilinear averages was introduced by Furstenberg in [13]. Later, J. Bourgain proved

2010 Mathematics Subject Classification. Primary: 37A30; Secondary: 28D05, 5D10, 11B30, 11N37, 37A45.

Key words and phrases. multilinear ergodic averages, Furstenberg’s problem of a.e. convergence, Liouville function, Möbius function, Birkhoff ergodic theorem, Bourgain’s double recurrence theorem, Zhan’s estimation, Davenport-Hua’s estimation, Sarnak’s conjecture.

Friday 23rd June, 2017.
that the homogeneous ergodic bilinear averages converges almost surely \[6\]. Subsequently, I. Assani established that the convergence a.e. of the homogeneous ergodic multilinear averages holds if the restriction of the map to its Pinsker algebra has a singular spectrum. Assani’s proof is based essentially on Bourgain’s theorem \[6\] combined with Host’s joining theorem \[15\]. Very recently, E. H. el Abdalaoui proved that there is a subsequence for which the convergence a.e. of the ergodic multilinear averages holds \[1\]. For a recent survey on the Furstenberg’s problem on the ergodic multilinear averages, we refer to \[10\]. Let us mention also that C. Demeter in \[10\] obtained an alternative proof of Bourgain’s theorem \[6\].

For the convergence almost everywhere of the homogeneous ergodic bilinear averages with weight, I. Assani, D. Duncan, and R. Moore proved à la Weiner-Wintner that the exponential sequences \((e^{2\pi int})_{n\in\mathbb{Z}}\) are good weight for the homogeneous ergodic bilinear averages \[4\]. Subsequently, I. Assani and R. Moore showed that the polynomials exponential sequences \((e^{2\pi iP(n)})_{n\in\mathbb{Z}}\) are also uniformly good weights for the homogeneous ergodic bilinear averages \[5\]. One year later, I. Assani and P. Zorich proved independently that the nilsequences are uniformly good weights for the homogeneous ergodic bilinear averages. Their proof depend heavily on Bourgain’s theorem. Let us further notice that Zorich’s proof yields that if the ergodic multilinear averages converges a.e. then the nilsequences are a good weight for the ergodic multilinear averages.

Here, our goal is to prove that the Möbius and Liouville functions are a good weight for the homogeneous ergodic bilinear averages. Our proof follows closely Bourgain’s proof \[6\]. We thus apply Calderón transference principal in order to establish some kind of maximal inequality. Furthermore, we apply Assani’s result to prove that the Möbius and Liouville functions are a good weight for the homogeneous ergodic multilinear averages if the restriction of the map to its Pinsker algebra has singular spectrum.

Let us remind that Sarnak announced in his seminal paper \[22\] that the Möbius function is a good weight in \(L^2\) for the ergodic averages. In \[2\], the authors apply Davenport’s estimation combined with Etamedi’s trick \[12\] to obtain a simple proof of Sarnak’s result. Therein, they proved that the Möbius function is a good weight in \(L^1\) for the ergodic averages.

2. Notations and Tools

The Liouville function is defined for the positive integers \(n\) by

\[
\lambda = (-1)^{\Omega(n)},
\]

where \(\Omega(n)\) is the length of the word \(n\) is the alphabet of prime, that is, \(\Omega(n)\) is the number of prime factors of \(n\) counted with multiplicities. The Möbius function is given by

\[
\mu(n) = \begin{cases} 
1 & \text{if } n = 1; \\
\lambda(n) & \text{if } n \text{ is the product of } r \text{ distinct primes}; \\
0 & \text{if not}
\end{cases}
\]
These two functions are of great importance in number theory since the Prime Number Theorem is equivalent to

\[(2) \sum_{n \leq N} \lambda(n) = o(N) = \sum_{n \leq N} \mu(n).\]

Furthermore, there is a connection between these two functions and Riemann zeta function, namely

\[1 - \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \text{ for any } s \in \mathbb{C} \text{ with } \Re(s) > 1.\]

Moreover, Littlewood proved that the estimate

\[\left| \sum_{n=1}^{x} \mu(n) \right| = O\left( x^{1/2 + \varepsilon} \right) \quad \text{as } x \rightarrow +\infty, \quad \forall \varepsilon > 0\]

is equivalent to the Riemann Hypothesis (RH) ([24 pp.315]).

Here, we will need the following Davenport-Hua’s estimation [9, 17, Theorem 10.]: for each \( A > 0 \), for any \( k \geq 1 \), we have

\[(3) \max_{z \in T} \left| \sum_{n \leq N} z^n \lambda(n) \right| \leq C_A \frac{N}{\log A N} \text{ for some } C_A > 0.\]

This estimate has been generalized for the short interval by T. Zhan [26] as follows: for each \( A > 0 \), for any \( \varepsilon > 0 \), we have

\[(4) \max_{z \in T} \left| \sum_{N \leq n \leq N + M} z^n \lambda(n) \right| \leq C_{A,\varepsilon} \frac{M}{\log^2(M)} \text{ for some } C_{A,\varepsilon} > 0,\]

provided that \( M \geq N^{1/2 + \varepsilon} \).

Davenport-Hua’s estimation was extended by Green-Tao to the nilsequences setting. We refer to Theorem 1.1 in [15] for the exact estimation and for the definition of the nilsequences.

In our setting, we consider also the ergodic multilinear averages given by

\[\frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} f_i(T_i^n x),\]

where \( k \geq 2 \), \((X, B, \mu, T_i)_{i=1}^{k}\) are a finite family of dynamical systems where \( \mu \) is a probability measure, \( T_i \) are commuting invertible measure preserving transformations and \( f_1, f_2, \cdots, f_k \) a finite family of bounded functions. The bilinear case corresponds to \( k = 2 \).

The ergodic multilinear averages is said to be homogeneous if \( T_i, i = 1, \cdots k \), are the powers of some given map \( T \).

For the convergence a.e., J. Bourgain proved
**Theorem 2.1** (Bourgain’s double recurrence theorem [6]). Let \((X, \mathcal{A}, \mu, T)\) be an ergodic dynamical system, and \(T_1, T_2\) be powers of \(T\). Then, for any \(f, g \in L^\infty(X)\), for almost all \(x \in X\),
\[
\frac{1}{N} \sum_{n=1}^{N} f(T_1^nx)g(T_2^nx)
\]
converges.

Applying Host’s joining theorem [18] combined with Bourgain’s theorem (Theorem 2.1), I. Assani proved

**Theorem 2.2** (Assani’s singular multilinear recurrence theorem [3]). Let \((X, \mathcal{A}, \mu, T)\) be a weakly mixing dynamical system such that the restriction of \(T\) to its Pinsker algebra has singular spectrum, then, for all positive integers \(k\), for all \(f_i \in L^\infty(X)\), \(i = 1, \cdots, k\), for almost all \(x \in X\), we have
\[
\frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} f_i(T^{\alpha_i}x) \xrightarrow{N \to +\infty} \prod_{i=1}^{k} \int f_i(x) d\mu.
\]

3. **Some tools on the maximal ergodic inequalities and Calderón transference principle**

We say that the sequence of complex number \((a_n)\) is good weight in \(L^p(X, \mu)\), \(p \geq 1\) for linear case, if, for any \(f \in L^p(X, \mu)\), the ergodic averages
\[
\frac{1}{N} \sum_{j=1}^{N} a_jf(T^jx)
\]
converges a.e.. We further say that the maximal ergodic inequality holds in \(L^p(X, \mu)\) for the linear case with weight \((a_n)\) if, for any \(f \in L^p(X, \mu)\), the maximal function given by
\[
M(f)(x) = \sup_{N \geq 1} \left| \frac{1}{N} \sum_{j=1}^{N} a_jf(T^jx) \right|
\]
satisfy the weak-type inequality
\[
\lambda \mu \left\{ x : M(f)(x) > \lambda \right\} \leq C \| f \|_p,
\]
for any \(\lambda > 0\) with \(C\) is an absolutely constant.

It is well known that the classical maximal ergodic inequality is equivalent to the Birkhoff ergodic theorem [14].

The previous notions can be extended in the usual manner to the multilinear case. Let \(k \geq 2\), we thus say that \((a_n)\) is good weight in \(L^p_i(X, \mu)\), \(p_i \geq 1\), \(i = 1, \cdots, k\), with \(\sum_{i=1}^{k} \frac{1}{p_i} = 1\), if, for any \(f_i \in L^{p_i}(X, \mu)\), \(i = 1, \cdots, k\), the ergodic \(k\)-multilinear averages
\[
\frac{1}{N} \sum_{j=1}^{N} a_j \prod_{i=1}^{k} f_i(T_i^jx),
\]
converges a.e.. The maximal multilinear ergodic inequality is said to hold in $L^{p_i}(X,\mu)$, $p_i \geq 1$, $i = 1, \cdots, k$, with $\sum_{i=1}^{k} \frac{1}{p_i} = 1$, if, for any $f_i \in L^{p_i}(X,\mu)$, $i = 1, \cdots, k$, the maximal function given by

$$M(f_1, \cdots, f_k)(x) = \sup_{N \geq 1} \left| \frac{1}{N} \sum_{n=1}^{N} a_j \prod_{i=1}^{k} f_i(T_i^j x) \right|$$

satisfy the weak-type inequality

$$\lambda \mu \left\{ x : M(f)(x) > \lambda \right\} \leq C \prod_{i=1}^{k} \| f_i \|_{p_i},$$

for any $\lambda > 0$ with $C$ is an absolutely constant.

It is not known whether the classical maximal multilinear ergodic inequality ($a_n = 1$, for each $n$) holds for the general case $n \geq 3$. Nevertheless, we have the following Calderón transference principal in the homogeneous case,

**Proposition 3.1.** Let $(a_n)$ be a sequence of complex number and assume that for any $\phi, \psi \in L^2(\mathbb{Z})$, we have

$$\left\| \sup_{N \geq 1} \left| \frac{1}{N} \sum_{n=1}^{N} a_n (j + n) \psi(j-n) \right| \right\|_{L^1(\mathbb{Z})} < C \| \phi \|_{L^2(\mathbb{Z})} \| \psi \|_{L^2(\mathbb{Z})},$$

where $C$ is an absolutely constant. Then, for any dynamical system $(X, \mathcal{A}, T, \mu)$, for any $f, g \in L^2(X,\mu)$, we have

$$\left\| \sup_{N \geq 1} \left| \frac{1}{N} \sum_{n=1}^{N} a_n f(T^n x) g(T^{-n} x) \right| \right\|_{1} < C \| f \|_2 \| g \|_2.$$

We further have

**Proposition 3.2.** Let $(a_n)$ be a sequence of complex number and assume that for any $\phi, \psi \in L^2(\mathbb{Z})$, for any $\lambda > 0$, for any integer $J \geq 2$, we have

$$\left| \left\{ 1 \leq j \leq J : \sup_{N \geq 1} \left| \frac{1}{N} \sum_{n=1}^{N} a_n (j + n) \psi(j-n) \right| > \lambda \right\} \right| < C \frac{\| \phi \|_{L^2(\mathbb{Z})} \| \psi \|_{L^2(\mathbb{Z})}}{\lambda},$$

where $C$ is an absolutely constant. Then, for any dynamical system $(X, \mathcal{A}, T, \mu)$, for any $f, g \in L^2(X,\mu)$, we have

$$\mu \left\{ x \in X : \sup_{N \geq 1} \left| \frac{1}{N} \sum_{n=1}^{N} a_n f(T^n x) g(T^{-n} x) \right| > \lambda \right\} < C \frac{\| f \|_2 \| g \|_2}{\lambda}.$$

It is easy to check that Proposition 3.1 and 3.2 hold for the homogeneous $k$-multilinear ergodic averages, for any $k \geq 3$. Moreover, one may state and prove the finitary version where $\mathbb{Z}$ is replaced by $\mathbb{Z}/J\mathbb{Z}$ and the functions $\phi$ and $\psi$ with $J$-periodic functions. We refer to Proposition 14.1 in [11] for more details.

In finitary setting, for $M \geq 1, p \geq 1$ and $f$ a $M$-periodic function, we put

$$E_M(f) = \frac{1}{M} \sum_{j=0}^{M-1} f(j),$$
\[\|f\|_{L^p(\mathbb{Z}_M)} = \left( \frac{1}{M} \sum_{j=0}^{M} |f(j)|^p \right)^{\frac{1}{p}}.\]

For \( m \in \mathbb{Z}_M \), we denote by \([-m, m] \) the subset of \( \mathbb{Z}_M \) given by \( \{-m, -m + 1, \cdots, m - 1, m\} \).

4. MAIN RESULTS AND ITS PROOF

We start by stating our first main result.

**Theorem 4.1.** Let \((X, \mathcal{A}, \mu, T)\) be an ergodic dynamical system, and \(T_1, T_2\) be powers of \(T\). Then, for any \(f, g \in L^2(X)\), for almost all \(x \in X\),
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n)f(T_1^n x)g(T_2^n x) \xrightarrow{N \to +\infty} 0,
\]
where \(\nu\) is the Liouville function or the Möbius function.

Our second main result can be stated as follows:

**Theorem 4.2.** Let \((X, \mathcal{A}, \mu, T)\) be weakly mixing ergodic dynamical system, and \(T_1, T_2, \cdots, T_k\) be powers of \(T\), \(k \geq 2\). Assume that the spectrum of the restriction of \(T\) to its Pinsker algebra is singular. Then, for any \(f_j \in L^\infty(X), j = 1, \cdots, k\), for almost all \(x \in X\), we have
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) \prod_{j=1}^{k} f_j(T^n x) \xrightarrow{N \to +\infty} 0,
\]
where \(\nu\) is the Liouville function or the Möbius function.

For the proof of Theorem 4.2, we need the following criterion due to Katai-Bourgain-Sarnak-Ziegler [7], [20].

**Theorem 4.3** (Katai-Bourgain-Sarnak-Ziegler’s (KBSZ) criterion [7], [20]). Let \(f\) be an arithmetic bounded function and let \(\nu\) be a bounded multiplicative function. Assume that for all sufficiently large distinct primes \(p, q\) we have
\[
\frac{1}{N} \sum_{n=1}^{N} f(np)f(nq) \xrightarrow{N \to +\infty} 0.
\]
Then
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n)f(n) \xrightarrow{N \to +\infty} 0.
\]

**Proof of Theorem 4.2**. The proof goes by induction on \(k\). We further assume that for some \(i \in \{1, \cdots, k\}, \int f_i d\mu(x) = 0\). The case \(k = 1\) follows from Sarnak’s result. For \(k = 2\), put
\[F(n) = f_1(T_1^n x)f_2(T_2^n x),\]
where \(T_1, T_2\) are the powers of \(T\). Then, by Theorem 2.2 for almost all \(x \in X\), for all \(p \neq q\),
\[
\frac{1}{N} \sum_{n=1}^{N} F(np)F(nq) \xrightarrow{N \to +\infty} 0,
\]
Therefore, by KBSZ criterion (Theorem 4.3), we get, for almost all \(x \in X\),
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) F(n) \xrightarrow{N \to +\infty} 0,
\]
that is, for almost all \(x \in X\),
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) f_1(T_1^n x) f_2(T_2^n x) \xrightarrow{N \to +\infty} 0.
\]
But, for any \(f_1, f_2 \in L^\infty(X)\),
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) f_1(T_1^n x) f_2(T_2^n x)
= \frac{1}{N} \sum_{n=1}^{N} \nu(n) \left( f_1 - \int f_1 d\mu \right)(T_1^n x) f_2(T_2^n x) + \left( \int f_1 d\mu(x) \right) \frac{1}{N} \sum_{n=1}^{N} \nu(n) f_2(T_2^n x).
\]
Consequently, for any \(f_1, f_2 \in L^\infty(X)\), for almost all \(x \in X\),
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) f_1(T_1^n x) f_2(T_2^n x) \xrightarrow{N \to +\infty} 0.
\]
Now, assume that for almost all \(x \in X\), and for any \(\ell \leq k\), we have
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) \prod_{i=1}^{\ell} f_i(T_i^n x) \xrightarrow{N \to +\infty} 0,
\]
where \(T_i, i = 1, \ldots, \ell\) are the powers of \(T\). Then, by applying again Theorem 2.2, we see that for almost all \(x\),
\[
\frac{1}{N} \sum_{n=1}^{N} F(np) F(nq) \xrightarrow{N \to +\infty} 0,
\]
where
\[
F(n) = \prod_{i=1}^{k+1} f_i(T_i^n x).
\]
Hence, once again by KBSZ criterion (Theorem 4.3), it follows that for almost all \(x \in X\),
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) F(n) \xrightarrow{N \to +\infty} 0,
\]
whence, for almost all \(x \in X\),
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) \prod_{i=1}^{k+1} f_i(T_i^n x) \xrightarrow{N \to +\infty} 0.
\]
The proof of the theorem is complete. \(\square\)

**Remark 4.4.** Of course, our proof yields that the convergence a.e. holds for any bounded multiplicative function \(\nu\). We deduce also that Theorem 4.2 is valid for the class of weakly mixing PID or the distal flows with the help of the recent result of Gutman-Huang-Shao-Ye [16] and Huang-Shao-Ye [19]. Obviously, if the answer
to Furstenberg’s question [13] is positive then Theorem 4.2 holds for the general case.

We move now to prove Theorem 4.1. For any \( \rho > 1 \), we will denote by \( I_\rho \) the set \( \{(\lfloor \rho^n \rfloor), n \in \mathbb{N}\} \). The maximal functions are defined by

\[
M_{N_0,\bar{N}}(f,g)(x) = \sup_{N_0 \leq N \leq \bar{N}} \left\{ \frac{1}{N} \sum_{n=1}^{N} \nu(n)f(T^nx)g(T^{-n}x) - \frac{1}{N_0} \sum_{n=1}^{N_0} \nu(n)f(T^nx)g(T^{-n}x) \right\},
\]

\[
M_{N_0}(f,g)(x) = \sup_{N \geq N_0} \left\{ \frac{1}{N} \sum_{n=1}^{N} \nu(n)f(T^nx)g(T^{-n}x) - \frac{1}{N_0} \sum_{n=1}^{N_0} \nu(n)f(T^nx)g(T^{-n}x) \right\}.
\]

Obviously,

\[
\lim_{\bar{N} \to +\infty} M_{N_0,\bar{N}}(f,g)(x) = M_{N_0}(f,g)(x).
\]

For the shift \( \mathbb{Z} \)-action, the maximal functions are denoted by \( m_{N_0,\bar{N}}(\phi,\psi) \) and \( m_{N_0}(\phi,\psi) \).

We start by proving the following:

**Theorem 4.5.** For any \( \rho > 1 \), for any \( f, g \in \ell^2(\mathbb{Z}) \), for any \( K \geq 1 \), we have

\[
\sum_{k=1}^{K} \left\| m_{N_k,N_{k+1}}(f,g) \right\|_{\ell^1(\mathbb{Z})} < C \sqrt{K} \left\| f \right\|_{\ell^2(\mathbb{Z})} \left\| g \right\|_{\ell^2(\mathbb{Z})},
\]

where \( \nu \) is the Liouville function or the Möbius function and \( C \) is an absolutely constant which depend only on \( \rho \).

The classical Calderón transference principal (see Proposition 3.1 and 3.2) allows us to obtain from Theorem 4.5 the following:

**Theorem 4.6.** Let \((X,A,T,\mu)\) be an ergodic dynamical system, and let \( f, g \in L^2(X,\mu) \). Then, for any \( \rho > 1 \), for any \( K \geq 1 \),

\[
\sum_{k=1}^{K} \left\| M_{N_k,N_{k+1}}(f,g) \right\|_{1} < 4C \sqrt{K} \left\| f \right\|_{2} \left\| g \right\|_{2},
\]

where \( \nu \) is the Liouville function or the Möbius function.

Let us give the proof of Theorem 4.6.

**Proof of Theorem 4.6.** The proof goes, as in the proof of Proposition 3.1 and 3.2. Let \( \bar{N} = N_{K+1} \) and \( J \gg \bar{N} \). Put

\[
\phi_x(n) = \begin{cases} 
   f(T^nx), & \text{if } n \in [-2\bar{N},2\bar{N}]; \\
   0, & \text{if not},
\end{cases}
\]

and

\[
\psi_x(n) = \begin{cases} 
   g(T^nx), & \text{if } n \in [-2\bar{N},2\bar{N}]; \\
   0, & \text{if not},
\end{cases}
\]

Then, by Theorem 4.5, we have

\[
\sum_{k=1}^{K} \left\| m_{N_k,N_{k+1}}(\phi_x,\psi_x) \right\|_{\ell^1(\mathbb{Z})} < C \left\| \phi_x \right\|_{\ell^2(\mathbb{Z})} \left\| \psi_x \right\|_{\ell^2(\mathbb{Z})}.
\]
We thus get
\[
\sum_{k=1}^{K} \left( \sum_{|j| \leq N} m_{N_k,N_{k+1}}(\phi_x, \psi_x)(j) \right) < C \sqrt{K} \|\phi_x\|_{\ell^2(Z)} \|\psi_x\|_{\ell^2(Z)},
\]
which can be rewritten as follows
\[
\sum_{k=1}^{K} \sum_{|j| \leq N} \left( M_{N_k,N_{k+1}}(f,g)(T^j x) \right) < C \sqrt{K} \left( \sum_{|n| \leq 2N} |f|^2(T^n x) \right)^{\frac{1}{2}} \left( \sum_{|n| \leq 2N} |g|^2(T^n x) \right)^{\frac{1}{2}}.
\]
Integrating and applying Hölder inequality we obtain
\[
\sum_{k=1}^{K} \left\| M_{N_k,N_{k+1}}(f,g) \right\|_1 < 4C \sqrt{K} \|f\| \|g\|_2,
\]
since $T$ is measure preserving, and this finish the proof of the theorem. \qed

We proceed now to the proof of Theorem 4.5. Our proof follows Bourgain’s arguments combined with Davenport-Hua estimation.

**Proof of Theorem 4.5** We proceed by using the finitary method as in [11]. Let $J, N$ be positive integers as in the proof of Theorem 4.6, $f, g \in \ell^2(\mathbb{Z})$ and $\delta > 0$. Put $L = 2(J + N) + 1$, and denote by $\mathcal{F}$ the discrete Fourier transform on $\mathbb{Z}_L$.

We recall that $\mathcal{F}(f)(\chi) = \sum_{n \in \mathbb{Z}_L} f(n) \chi(-n)$, for $\chi \in \hat{\mathbb{Z}}_L$, we still denote by $f$ the function of finite support on $\mathbb{Z}$ and the $L$-periodic function associated to $f$. For any $j \in \mathbb{Z}$, put
\[
\nu_j(n) = \nu(j - n), \quad n \in \mathbb{Z}.
\]
Of course $\nu$ is extended to the negative integers $\mathbb{Z}_-$ in the usual fashion.

Obviously, we have
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) f(j + n) g(j - n) = \left( f * \left( \frac{1}{N} \nu_j \cdot g \cdot 1_{[-N,j]} \right) \right)(2j),
\]
where $*$ is the operation of convolution given by
\[
(a * b)(j) = \sum_{n \in \mathbb{Z}} a(j - n) b(n), \quad \forall a, b \in \ell^2(\mathbb{Z}) \quad \text{and} \quad \forall j \in \mathbb{Z}.
\]
It is easy to state the previous formula in the finitary setting. Furthermore, by a standard arguments, we can rewrite (7) as follows
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) f(j + n) g(j - n) = \mathcal{F}^{-1}(\mathcal{F}(f) \cdot \mathcal{F}(G_j))(2j),
\]
where $G = \frac{1}{N} \nu_j \cdot g \cdot 1_{[-N,j]}$, $j \in [-J, J] \subset \mathbb{Z}_L$ and $n \in [-N, N] \subset \mathbb{Z}_L$.

Therefore
\[
\left| \frac{1}{N} \sum_{n=1}^{N} \nu(n) f(j + n) g(j - n) \right| = \left| \frac{1}{L} \int_{\mathbb{Z}_L} \mathcal{F}(f)(\chi) \mathcal{F}(G_j)(\chi)(2j) d\chi \right|
\]
\[
\leq \frac{1}{L} \int_{\mathbb{Z}_L} |\mathcal{F}(f)(\chi)| |\mathcal{F}(G_j)(\chi)| d\chi.
\]
Now, applying Cauchy-Schwarz inequality, we get

\[
\int_{\mathbb{Z}_L} |\mathcal{F}(f)(\chi)||\mathcal{F}(G_j)(\chi)|d\chi
\]

(10) \[= \int_{\mathbb{Z}_L} \left| \sum_{n \in \mathbb{Z}_L} f(n)\chi(-n) \right| \frac{1}{N} \sum_{n=1}^{N} \nu(n)g(j-n)\chi(-n) d\chi.\]

But, a straightforward computations gives

\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n)f(j+n)g(j-n)\]

\[\leq \left( \frac{1}{L} \int_{\mathbb{Z}_L} \left| \sum_{n \in \mathbb{Z}_L} f(n)\chi(-n) \right|^2 d\chi \right) \left( \frac{1}{L} \int_{\mathbb{Z}_L} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)g(j-n)\chi(n) \right|^2 d\chi \right).\]

Integrating, we see that

\[
\mathbb{E}_{\mathbb{Z}_L} \left( \frac{1}{N} \sum_{n=1}^{N} \nu(n)f(j+n)g(j-n) \right)^2 \mathbb{1}_{[-J,J]}(j)
\]

\[\leq \frac{1}{L} \|f\|_2^2 \mathbb{E}_{\mathbb{Z}_L} \left( \int \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)g(j-n)\chi(n) \right|^2 d\chi \cdot \mathbb{1}_{[-J,J]}(j) \right)\]

(11) \[\leq \|f\|_2^2 \int \mathbb{E}_{\mathbb{Z}_L} \left( \frac{1}{N} \sum_{n=1}^{N} \nu(n)g(j-n)\chi(n) \right)^2 d\chi\]

We thus need to estimate the RHS of the inequality (11). For that, write

\[
\mathbb{E}_{\mathbb{Z}_L} \left( \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)g(j-n)\chi(n) \right|^2 \right) = \mathbb{E}_{\mathbb{Z}_L} \left( \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)(U^{-n}g)(j)\chi(n) \right|^2 \right),
\]

where \(U\) is the Koopman operator of the shift map \(S\). Consequently, by the spectral theorem, we have

\[
\left\| \frac{1}{N} \sum_{n=1}^{N} \nu(n)g(S^{-n}j)\chi(n) \right\|_{L^2(\mathbb{Z}_L)} = \left\| \frac{1}{N} \sum_{1 \leq n \leq N} \nu(n)\lambda^n\chi(n) \right\|_{L^2(\sigma_\mu)},
\]

where \(\sigma_\mu\) is the spectral measure of \(g\). Hence, by Davenport-Hua’s estimation \(^3\), for each \(A > 0\), we get

\[
\left\| \frac{1}{N} \sum_{n=1}^{N} \nu(n)g(S^{-n}j)\chi(n) \right\|_{L^2(\mathbb{Z}_L)} \leq \frac{C_A}{(\log(N))^A} \|g\|_{L^2(\mathbb{Z}_L)},
\]

(12) \[
\left\| \frac{1}{N} \sum_{n=1}^{N} \nu(n)g(S^{-n}j)\chi(n) \right\|_{L^2(\mathbb{Z}_L)} \leq \frac{C_A}{(\log(N))^A} \|g\|_{L^2(\mathbb{Z}_L)},
\]

where \(C_A\) is a constant that depends only on \(A\).

\(^1\)Recall that \(\sigma_\mu\) is a finite measure on the circle determined by its Fourier transform given by \(\hat{\sigma_\mu}(n) = \langle U^n g, g \rangle\). In our case the spectral measure \(\sigma_\mu\) is absolutely continous with respect to the uniform spectral measure \(\frac{1}{L} \sum_{j=0}^{L-1} \delta_{\frac{2\pi j}{L}}\).
Combining (11) and (12), we can rewrite (11) as follows

\[
E_{E_L} \left( \left\| \frac{1}{N} \sum_{n=1}^{N} \nu(n)f(j+n)g(j-n) \right\|_{\ell^2(Z_L)}^2 \| \mathbb{I}_{[-J,J]}(j) \right) \leq \frac{C^2_A}{(\log(N))^2} \| f \|_{\ell^2(Z_L)}^2 \| g \|_{\ell^2(Z_L)}^2.
\]

Form this, it follows that for any \( k = 1, \ldots, K \), we have

\[
\sum_{N_k \leq N \leq N_{k+1}} E_{E_L} \left( \left\| \frac{1}{N} \sum_{n=1}^{N} \nu(n)f(j+n)g(j-n) \right\|_{\ell^2(Z_L)}^2 \| \mathbb{I}_{[-J,J]}(j) \right) \leq \sum_{N_k \leq N \leq N_{k+1}} \frac{C^2_A}{(\log(N))^2} \| f \|_{\ell^2(Z_L)}^2 \| g \|_{\ell^2(Z_L)}^2.
\]

Whence

\[
\left\| \sup_{N_k \leq N \leq N_{k+1}} \left\| \frac{1}{N} \sum_{n=1}^{N} \nu(n)f(j+n)g(j-n) \right\|_{\ell^2(Z_L)} \right\|_{\ell^2(Z_L)} \leq \sqrt{\sum_{N_k \leq N \leq N_{k+1}} \frac{C^2_A}{(\log(N))^2} \| f \|_{\ell^2(Z_L)}^2 \| g \|_{\ell^2(Z_L)}^2},
\]

Thus by the elementary inequality \( \sqrt{a+b} \leq \sqrt{a} + \sqrt{b}, a, b \geq 0 \), it follows that

\[
\sum_{k=1}^{K} \left\| \sup_{N_k \leq N \leq N_{k+1}} \left\| \frac{1}{N} \sum_{n=1}^{N} \nu(n)f(j+n)g(j-n) \right\|_{\ell^2(Z_L)} \right\|_{\ell^2(Z_L)} \leq \sum_{k=1}^{K} \sum_{N_k \leq N \leq N_{k+1}} \frac{C_A}{(\log(N))^2} \| f \|_{\ell^2(Z_L)} \| g \|_{\ell^2(Z_L)}.
\]

Applying again Cauchy-Schwarz inequality, we conclude that

\[
\sum_{k=1}^{K} \left\| \sup_{N_k \leq N \leq N_{k+1}} \left\| \frac{1}{N} \sum_{n=1}^{N} \nu(n)f(j+n)g(j-n) \right\|_{\ell^2(Z_L)} \right\|_{\ell^2(Z_L)} \leq C \sqrt{K} \| f \|_{\ell^2(Z_L)} \| g \|_{\ell^2(Z_L)}.
\]

In the same manner we can see from (13) that we have

\[
\sum_{k=1}^{K} \left\| \frac{1}{N_k} \sum_{n=1}^{N_k} \nu(n)f(j+n)g(j-n) \right\|_{\ell^2(Z_L)} \leq \sum_{k=1}^{K} \frac{C_A}{(\log(N_k))^2} \| f \|_{\ell^2(Z_L)} \| g \|_{\ell^2(Z_L)} \leq C \| f \|_{\ell^2(Z_L)} \| g \|_{\ell^2(Z_L)}.
\]

Therefore, by the triangle inequality, we get

\[(13) \sum_{k=1}^{K} \left\| m_{N_k,N_{k+1}}(f,g)(j) \|_{\ell^2(Z_L)} \right\|_{\ell^2(Z_L)} \leq C \sqrt{K} \| f \|_{\ell^2(Z_L)} \| g \|_{\ell^2(Z_L)}.
\]
According to our notations, we can rewrite (13) as follows:

\[
\sum_{k=1}^{K} \left\| m_{N_k, N_{k+1}}(f, g) \cdot \mathbb{1}_{[-\bar{J}, \bar{J}]}(j) \right\|_{\ell^1(\mathbb{Z}_{(2(J+N)+1)})} \leq C. \sqrt{K} \left\| f \right\|_{\ell^2(\mathbb{Z}_{(2(J+N)+1)})} \left\| g \right\|_{\ell^2(\mathbb{Z}_{(2(J+N)+1)})},
\]

Letting \( \bar{J} \to +\infty \), we thus conclude that we have

\[
\sum_{k=1}^{K} \left\| m_{N_k, N_{k+1}}(f, g) \right\|_{\ell^1(\mathbb{Z})} < C. \sqrt{K} \left\| f \right\|_{\ell^2(\mathbb{Z})} \left\| g \right\|_{\ell^2(\mathbb{Z})},
\]

and the proof of the theorem is complete. \( \square \)

**Remark 4.7.** An alternative proof to the previous proof, as in the Bourgain’s proof, can be obtained by using the Fourier transform instead of the discrete Fourier transform to obtain the same inequalities. We remind that the Fourier transform is defined on abelian group \( G \) by

\[
\mathcal{F}(f)(\chi) = \int_{G} f(g) \overline{\chi(g)} dg,
\]

where \( \chi \) is a character of \( G \). For a nice account on the discrete Fourier transform and related topics we refer to [23]. For the classical Fourier analysis on groups, we refer to [21].

Now, we are able to give the proof of our main result Theorem 4.1.

**Proof of Theorem 4.1.** By a standard argument, we may assume that the map \( T \) is ergodic. Let us assume also that \( f, g \) are in \( L^\infty(X, \mu) \). Therefore, by Theorem 4.6, it is easily seen that

\[
\frac{1}{K} \sum_{k=1}^{K} \left\| M_{N_k, N_{k+1}}(f, g) \right\|_{\ell^1} \xrightarrow{K \to +\infty} 0.
\]

Hence, by the same arguments as in [25] and [10], we see that for almost every point \( x \in X \), we have

\[
\frac{1}{|\rho^m|} \sum_{n=1}^{[\rho^m]} \nu(n) f(T^n x) g(T^{-n} x) \xrightarrow{m \to +\infty} 0,
\]
since the $L^2$-limit is zero by Green-Tao theorem \cite{15} Theorem 1.1 combined with Chu’s result \cite{8}. It follows that if $[\rho^m] \leq N < [\rho^m+1] + 1$, then

$$\left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)f(T^nx)g(T^{-n}x) \right|$$

$$= \left| \frac{1}{N} \sum_{n=[\rho^m]}^{[\rho^m]} \nu(n)f(T^nx)g(T^{-n}x) + \frac{1}{N} \sum_{n=[\rho^m]+1}^{N} \nu(n)f(T^nx)g(T^{-n}x) \right|$$

$$\leq \left| \frac{1}{[\rho^m]} \sum_{n=1}^{[\rho^m]} \nu(n)f(T^nx)g(T^{-n}x) \right| + \frac{\|f\|_{\infty}\|g\|_{\infty}}{[\rho^m]} (N - [\rho^m] - 1)$$

$$\leq \left| \frac{1}{[\rho^m]} \sum_{n=1}^{[\rho^m]} \nu(n)f(T^nx)g(T^{-n}x) \right| + \frac{\|f\|_{\infty}\|g\|_{\infty}}{[\rho^m]} ([\rho^m+1] - [\rho^m]).$$

Letting $m$ goes to infinity, we get

$$\left| \frac{1}{\rho^m} \sum_{n=1}^{[\rho^m]} \nu(n)f(T^nx)g(T^{-n}x) \right| \to 0,$$

and

$$\frac{\|f\|_{\infty}}{[\rho^m]} ([\rho^m+1] - [\rho^m]) \to \|f\|_{\infty}\|g\|_{\infty}(\rho - 1).$$

For any $\rho > 1$. Letting $\rho \to 1$ we conclude that

$$\frac{1}{N} \sum_{n=1}^{N} \nu(n)f(T^nx)g(T^{-n}x) \to 0, \quad \text{a.e.,}$$

To finish the proof, notice that for any $f, g \in L^2(X, \mu)$, and any $\varepsilon > 0$, there exist $f_1, g_1 \in L^\infty(X, \mu)$ such that $\|f - f_1\|_2 < \sqrt{\varepsilon}$, and $\|g - g_1\|_2 < \sqrt{\varepsilon}$. Moreover, by Cauchy-Schwarz inequality, we have

$$\left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)(f - f_1)(T^nx)(g - g_1)(T^{-n}x) \right|$$

$$\leq \frac{1}{N} \sum_{n=1}^{N} |(f - f_1)(T^nx)|(g - g_1)(T^{-n}x)|$$

$$\leq \left( \frac{1}{N} \sum_{n=1}^{N} |(f - f_1)(T^nx)|^2 \right)^{\frac{1}{2}} \left( \frac{1}{N} \sum_{n=1}^{N} |(g - g_1)(T^n x)|^2 \right)^{\frac{1}{2}}$$

Applying the ergodic theorem, it follows that for almost all $x \in X$, we have

$$\limsup_{N \to +\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)(f - f_1)(T^nx)(g - g_1)(T^{-n}x) \right| \leq \varepsilon.$$
Whence, we can write

\[
\limsup_{N \to +\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n) f(T^n x) g(T^{-n} x) \right|
\]

\[
\leq \limsup_{N \to +\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n) f_1(T^n x) g(T^{-n} x) \right| + \limsup_{N \to +\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n) f(T^n x) g_1(T^{-n} x) \right|
\]

\[
\leq \epsilon + \limsup_{N \to +\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n) f_1(T^n x) g(T^{-n} x) \right| + \limsup_{N \to +\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n) f(T^n x) g_1(T^{-n} x) \right|
\]

We thus need to estimate

\[
\limsup_{N \to +\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n) f_1(T^n x) g(T^{-n} x) \right|
\]

and

\[
\limsup_{N \to +\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n) f(T^n x) g_1(T^{-n} x) \right|
\]

In the same manner we can see that

\[
\limsup_{N \to +\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n) f_1(T^n x) (g - g_1)(T^{-n} x) \right|
\]

\[
\leq \limsup_{N \to +\infty} \left( \frac{1}{N} \sum_{n=1}^{N} |f_1(T^n x)|^2 \right)^{\frac{1}{2}} \limsup_{N \to +\infty} \left( \frac{1}{N} \sum_{n=1}^{N} |(g - g_1)(T^n x)|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \|f_1\|_2 \|g - g_1\|_2
\]

\[
\leq \left( \|f\|_2 + \sqrt{\varepsilon} \right) \sqrt{\varepsilon}
\]

This gives

\[
\limsup_{N \to +\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n) f_1(T^n x) g(T^{-n} x) \right|
\]

\[
\leq \left( \|f\|_2 + \sqrt{\varepsilon} \right) \sqrt{\varepsilon} + \limsup_{N \to +\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n) f_1(T^n x) g_1(T^{-n} x) \right|
\]

\[
\leq \left( \|f\|_2 + \sqrt{\varepsilon} \right) \sqrt{\varepsilon} + 0
\]

Summarizing, we obtain the following estimates

\[
\limsup_{N \to +\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n) f(T^n x) g(T^{-n} x) \right|
\]

\[
\leq \epsilon + \left( \|f\|_2 + \sqrt{\varepsilon} \right) \sqrt{\varepsilon} + \left( \|g\|_2 + \sqrt{\varepsilon} \right) \sqrt{\varepsilon}
\]
Since $\varepsilon > 0$ is arbitrary, we conclude that for almost every $x \in X$,
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) f(T^n x) g(T^{-n} x) \xrightarrow{N \to +\infty} 0.
\]
This complete the proof of the theorem. \hfill $\square$

As a consequence, we have proved Theorem 2.1.

**Remark 4.8.** Notice that our proof yields that the convergence almost sure holds for the short interval. Thanks to Zhan’s estimation (equation (11)).

We end this section by stating the following conjecture.

**Conjecture.** Let $\nu$ be a aperiodic bounded multiplicative function and $l \geq 2$ a positive integer. If $T_1, \cdots, T_l$ are commuting measure preserving transformations acting on the same probability space $(X, A, \mu)$, then for all $f_1, \cdots, f_l \in L^\infty(X, \mu)$, for almost all $x \in X$, we have
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) \prod_{j=1}^{k} f(T_j^n x) \xrightarrow{N \to +\infty} 0.
\]

We remind that $\nu$ is a aperiodic multiplicative function if
\[
\nu(mn) = \nu(m) \nu(n), \quad \text{for all } m, n \in \mathbb{N} \text{ such that } m \wedge n = 1; \text{ and}
\]
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(an + b) \xrightarrow{N \to +\infty} 0, \quad \text{for all } a, b \in \mathbb{N}.
\]

**Acknowledgment.** The author wishes to express his thanks to XiangDong Ye and Benjamin Weiss for a stimulating conversations on the subject. He is also thankful to Nalini Anantharaman and to university of Strasbourg, IRMA, where a part of this paper was written. The author wishes also to thank Wilfrid Gangbo and the university of UCLA where the paper was revised, for the invitation and hospitality.

**References**

[1] E. H. el Abdalaoui, On the pointwise convergence of multiple ergodic averages, [arXiv:1406.2608v2 [math.DS]].

[2] E. H el Abdalaoui, J. Kulaga-Przymus, M. Lemanczyk & T. de la Rue, The Chowla and the Sarnak conjectures from ergodic theory point of view, Discrete Contin. Dyn. Syst., 37 (2017), no. 6, 2899-2944.

[3] I. Assani, Multiple recurrence and almost sure convergence for weakly mixing dynamical systems, Israel J. Math. 103 (1998), pp. 111-124.

[4] I. Assani, D. Duncan, and R. Moore, Pointwise characteristic factors for Wiener-Wintner double recurrence theorem, Erg. Th. and Dyn. Sys. volume 36, issue 04, pp. 1037-1066, [arXiv:1402.7094].

[5] I. Assani, R. Moore, Extension of Wiener-Wintner double recurrence theorem to polynomials, to appear on Journal d’Analyse Mathematique, [arXiv:1409.0463] [arXiv:1408.3064].

[6] J. Bourgain, Double recurrence and almost sure convergence, J. Reine Angew. Math. 404 (1990), pp. 140-161.

[7] J. Bourgain, P. Sarnak, T. Ziegler, Disjointness of Moebius from horocycle flows, From Fourier analysis and number theory to Radon transforms and geometry, 67-83, Dev. Math., 28, Springer, New York, 2013.

[8] Q. Chu, Convergence of weighted polynomial multiple ergodic averages, Proc. Amer. Math. Soc. 137 (2009), no. 4, 1363-1369.
[9] H. Davenport, *On some infinite series involving arithmetical functions. II*, Quart. J. Math. Oxf. 8 (1937), 313–320.
[10] C. Demeter, Pointwise convergence of the ergodic bilinear Hilbert transform, Illinois J. Math. 51.4 (2007), pp. 1123-1158.
[11] C. Demeter, T. Tao, C. Thiele, Maximal multilinear operators. Trans. Amer. Math. Soc. 360 (2008), no. 9, 4989-5042.
[12] N. Etemadi, An elementary proof of the strong law of large numbers. Z. Wahrsch. Verw. Gebiete 55 (1981), no. 1, 119-122.
[13] H. Furstenberg, Recurrence in ergodic theory and combinatorial number theory, M. B. Porter Lectures, Princeton University Press, Princeton, N.J., 1981.
[14] A. Garcia, *Topics in almost everywhere convergence*, Markham Publishing Company, Chicago, 1970.
[15] B. Green, T. Tao, The M"obius function is strongly orthogonal to nilsequence, *Ann. of Math.* (2) 175 (2012), no. 2, 541-566.
[16] Y. Gutman, W. Huang, S. Shao, and X. Ye, Almost sure convergence of the multiple ergodic average for certain weakly mixing systems, [arXiv:1612.02873v1 [math.DS]].
[17] L. K. Hua, Additive theory of prime numbers, Translations of Mathematical Monographs, Vol. 13 American Mathematical Society, Providence, R.I. 1965.
[18] B. Host, Mixing of all orders and pairwise independent joinings of systems with singular spectrum, Israel J. Math. 76 (1991), no. 3, 289-298.
[19] W. Huang, S. Shao and X. Ye, Pointwise convergence of multiple ergodic averages and strictly ergodic models, [arXiv:1406.5593]
[20] I. Kátai, Some remarks on a theorem of H. Daboussi, Math. Pannon. 19 (2008), no. 1, 71-80.
[21] Rudin, Walter *Fourier analysis on groups*. Reprint of the 1962 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1990.
[22] P. Sarnak, M"obius randomness and dynamics. Not. S. Afr. Math. Soc. 43 (2012), no. 2, 89-97.
[23] E. M. Stein and R. Shakarchi, *Fourier analysis, An introduction*, Princeton Lectures in Analysis, 1, Princeton University Press, Princeton, NJ, 2003.
[24] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Second edition. Edited and with a preface by D. R. Heath-Brown. The Clarendon Press, Oxford University Press, New York, 1986.
[25] J-P. Thouvenot, La convergence presque s\^ure des moyennes ergodiques suivant certaines sous-suites d’entiers (d’apr\`es Jean Bourgain). (French) [Almost sure convergence of ergodic means along some subsequences of integers (after Jean Bourgain)] Séminaire Bourbaki, Vol. 1989/90. Astérisque No. 189-190 (1990), Exp. No. 719, 133/153.
[26] T. Zhan, Davenport’s theorem in short intervals. Chin. Ann. of Math., 12B(4) 1991, 421-431.
[27] P. Zorin-Kranich, A nilsequence wiener wintner theorem for bilinear ergodic averages, [arXiv:1504.04647]

Université de Rouen-Mathématiques, Labo. de Math Raphael SALEM UMR 6085 CNRS, Avenue de l’Université, BP.12, 76801 Saint Etienne du Rouvray - France.
E-mail address: elhoucein.elabdalaoui@univ-rouen.fr