Birkhoff normal forms for Fourier integral operators II.

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Abstract. In this work we construct logarithms and Birkhoff normal forms for elliptic Fourier integral operators in the semi-classical limit, under more general assumptions than in a previous work by the first author. The methods are similar but slightly different.

Résumé. Dans ce travail nous construisons des logarithmes et des formes normales de Birkhoff pour des opérateurs intégraux de Fourier dans la limite semi-classique. Les hypothèses sont plus faibles que dans un travail antérieur du premier auteur et les méthodes sont semblables mais un peu différentes.

0. Introduction.

This work is a continuation of the work [Ia] of the first author. As in that paper we consider the problem of constructing a microlocal logarithm and a normal form of an elliptic semi-classical Fourier integral operator near a fixed point of the corresponding canonical transformation. In [Ia] the canonical transformation was assumed to be of real hyperbolic type and the purpose of the present work is to relax this assumption considerably, to what we think are the natural conditions.

As in [Ia] a motivation is to improve the analysis of quantized billiard ball maps near closed trajectories for boundary value problems. Then the associated canonical transformation is the corresponding classical Poincaré map. In the introduction of [Ia] one such problem in the case of scattering by obstacles was mentioned, where this canonical transformation is of hyperbolic type near its fixed point, but in many other cases the canonical transformation can be more arbitrary. (Such an improvement of spectral asymptotics for non-degenerate potential wells was obtained in [Sj] by means of a Birkhoff normal form.)

Another motivation for returning to this study comes from recent works on inverse spectral problems by Guillemin [Gui] and Zelditch [Ze1–3]. In these works, quantum normal forms of the whole (Laplace) operator were constructed in a neighborhood of the closed trajectory. We believe that it may often be sufficient and technically easier to work with normal forms of a corresponding monodromy operator, which coincides with the quantized billiard ball map in the case of boundary problems. For general semi-classical problems, this operator was recently introduced and studied in a more systematic fashion in [SjZw]. In particular one might arrive at very nice trace formulae by combining the basic trace formula of that paper with the normal form obtained in the present work.

It turned out to be somewhat difficult to extend the whole method of [Ia] to the more general situation here. In [Ia] a major idea was to use that the symplectic group is connected and work with deformations from the identity transformation to the given canonical transformation. In the hyperbolic case this can be done in a such a way that the

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intermediate transforms satisfy the assumptions for having a normal form. In the general case considered here, we encounter singular values for the deformation parameter where the conditions are not fulfilled. It seems possible to circumvent this difficulty by complexifying the deformation parameter and use corresponding slightly complex transforms. Eventually however we found a method allowing us to avoid deforming the differential of the canonical transformation but only the higher order part in its Taylor expansion. In this way we have families of objects which satisfy our assumptions everywhere.

Consider a semiclassical Fourier integral operator $A$ of order 0, with an associated canonical transformation $\kappa : \text{neigh}((0,0), \mathbb{R}^{2n}) \to \text{neigh}((0,0), \mathbb{R}^{2n})$ having $(0,0)$ as a fixed point. Assume $A$ is elliptic at $(0,0)$. The set of eigen-values of $d\kappa(0)$ is then closed under inversion $\lambda \to 1/\lambda$ and under complex conjugation. Assume that to the distinct $\lambda$ in the spectrum of $d\kappa(0,0)$, we can associate a logarithm $\mu = \log \lambda$ in such a way that inversion and complex conjugation correspond to the map $\mu \to -\mu$ and to complex conjugation respectively. (Notice that this assumption excludes the existence of negative eigen-values of $d\kappa(0,0)$.) Assume also that

$$\sum k_j \mu_j = 2\pi in, \quad k_j, n \in \mathbb{Z} \Rightarrow \sum k_j \mu_j = 0 \quad (0.1)$$

Then a real version the Lewis–Sternberg theorem (see [St], [Fr] and Theorem 1.3 below) tells us that we can write $\kappa(\rho) = \exp H_p(\rho) + \mathcal{O}(\rho^\infty)$ ($\rho = (x, \xi) \in \mathbb{R}^{2n}$) for some smooth and real-valued $p = \mathcal{O}(\rho^2)$. The first result of this work says that under the same assumptions, we can write $A \equiv e^{-iP/h}$ modulo an operator which vanishes to infinite order at $(x, \xi) = (0,0)$, $h = 0$, where $h > 0$ is the small semi-classical parameter (Theorem 3.2). Here $P$ is a semi-classical pseudodifferential operator of order 0 with $p$ as its leading symbol.

The second result is a straightforward extension of the normal form in [Sj] and says that under the non-resonance condition (4.2) below, the operator $P$ has a simple normal form in terms of certain action operators. (In (0.1) we do not have to be very precise concerning the enumeration of the logarithms of the eigen-values of the linearization as long as we have have one of each, modulo the sign, while in (4.2) we enumerate one rather specific half of these logs. Also notice that the combination of the two conditions takes the simple form (4.3).) We do not expect that the exclusion of negative eigen-values is a serious restriction, for if such eigen-values are present, we can find specially adapted and explicit metaplectic operators $M$ and apply our results to $MA$.

Both our results were obtained by the first author ([Ia]) under the assumption that $d\kappa(0,0)$ is of real-hyperbolic type.

For the reader’s convenience we have taken the pain review some well-known linear symplectic geometry in section 1, and in that section we also give a (probably well-known) proof of the real version of the Lewis-Sternberg theorem, which has a structure that we can follow in the proof of the corresponding quantum result for logarithms of Fourier integral operators.

It is beyond the scope of this work to consider convergence questions related to the perturbation series that appear in connection with normal forms. See for instance H. Ito [It] and G. Popov [Po1,2].

The plan of the paper is the following:
In section 1 we review some standard facts about linear symplectic geometry and add a few remarks for treating the real case. We also review a proof of a real version of the Lewis–Sternberg theorem, that we can later use as guideline for the proof of the quantum result.

In section 2, we introduce some notions of equivalence that are used in the formulation of the main results. These notions are essentially the same as in [Ia].

In section 3 we establish the main result about logarithms of Fourier integral operators.

In section 4 we give the ”Birkhoff” normal form for the logarithm i.e. for a certain $h$-pseudodifferential operator of order 0.

In section 5 we extend the results to the parameter dependent case. In many genuinely semi-classical problems we do not have any homogeneity, inferring that the Poincaré map is essentially energy independent, and then we cannot expect in general that our arithmetic conditions be fulfilled for all energies in some interval. Consequently it is of interest to know that the results are valid to infinite order at points where the conditions are fulfilled.

1. Review of some symplectic geometry.

In this section, we review some more or less well-known arguments that will later be extended to the quantized case. See [MeHa].

We start with the linear case and let $A : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be linear and symplectic in the sense that

$$\sigma(Ax, Ay) = \sigma(x, y), \quad x, y \in \mathbb{C}^{2n},$$

where $\sigma$ is the standard symplectic 2-form on $\mathbb{C}^{2n}$. We recall that this implies that $\det A = 1$, since $A$ will conserve the volume form $\sigma^n/n!$. When $n = 1$ the converse is also true.

Let $E_\lambda = \mathcal{N}((A - \lambda)^{2n}) = \text{Ker } ((A - \lambda)^{2n})$ be the generalized eigen-space associated to $\lambda \in \mathbb{C} \setminus \{0\}$.

**Proposition 1.1.** If $\lambda \mu \neq 1$, $\lambda, \mu \in \mathbb{C} \setminus \{0\}$, then $E_\lambda$ and $E_\mu$ are symplectically orthogonal: $E_\lambda \perp^{\sigma} E_\mu$.

**Proof.** Let $E^{(j)}_\lambda = E_\lambda \cap \mathcal{N}((A - \lambda)^j)$, so that

$$0 \neq E^{(1)}_\lambda \subset E^{(2)}_\lambda \subset \ldots \subset E^{(2n)}_\lambda = E_\lambda.$$

Define $E^{(k)}_{\mu}$ in the same way. Then for $x \in E^{(1)}_\lambda$, $y \in E^{(1)}_{\mu}$:

$$\sigma(x, y) = \sigma(\frac{1}{\lambda}Ax, \frac{1}{\mu}Ay) = \frac{1}{\lambda \mu} \sigma(Ax, Ay) = \frac{1}{\lambda \mu} \sigma(x, y),$$

and since $1/\lambda \mu \neq 1$, we get $\sigma(x, y) = 0$.

Assume that we have for some $m \geq 2$:

$$\sigma(x, y) = 0, \quad \text{for } x \in E^{(j)}_\lambda, \ y \in E^{(k)}_{\mu}, \ j + k \leq m. \quad (P_m)$$
We have just established \((P_2)\).

Let \(x \in E^{(j)}_\lambda\), \(y \in E^{(k)}_\mu\), \(j + k = m + 1\). Write \(A|_{E_\lambda} = \lambda + N\), \(A|_{E_\mu} = \mu + M\), where \(N, M\) are nilpotent with \(N : E^{(j)}_\lambda \to E^{(j-1)}_\lambda\), \(M : E^{(k)}_\mu \to E^{(k-1)}_\mu\). Then

\[
\sigma(x, y) = \sigma\left(\frac{1}{\lambda} Ax - \frac{1}{\lambda} Nx, \frac{1}{\mu} Ay - \frac{1}{\mu} My\right) \\
= \frac{1}{\lambda \mu} \sigma(x, y) + \sigma\left(-\frac{N}{\lambda} x, \frac{1}{\mu} Ay\right) + \sigma\left(\frac{1}{\lambda} Ax, -\frac{1}{\mu} My\right) + \sigma\left(\frac{1}{\lambda} Nx, \frac{1}{\mu} My\right).
\]

The last three terms vanish because of the induction assumption, and we get \(\sigma(x, y) = 0\), so we have proved \((P_{m+1})\). The proposition follows. 

We conclude that \(E_\lambda\) are isotropic if \(\lambda^2 \neq 1\) (i.e. \(\sigma\) vanishes on \(E_\lambda \times E_\lambda\)). We also see that \(E_1 \perp^\sigma E_\lambda\) for \(\lambda \neq 1\), \(E_{-1} \perp^\sigma E_\mu\) for \(\mu \neq -1\).

It follows that

\[
\mathbb{C}^{2n} = E_1 \oplus E_{-1} \oplus \bigoplus_{j=1}^k (E_{\lambda_j} \oplus E_{1/\lambda_j}),
\]

where \(\lambda_j, \lambda_j^{-1}\) and possibly \(1, -1\) denote the distinct eigen-values of \(A\) with \(\lambda_j \neq \pm 1\).

Moreover, all the \(\oplus\) and \(\bigoplus\) except the \(\oplus\)-s inside the parentheses indicate symplectically orthogonal decomposition.

For \(\lambda \not\in \{1, -1\}\), write \(A|_{E_\lambda} = \lambda + N_\lambda\) with \(N_\lambda\) nilpotent. \(\sigma|_{E_\lambda \times E_{1/\lambda}}\) is non-degenerate and if \(\sigma A\) denotes the symplectic transpose of \(A\), so that \(\sigma(Ax, y) = \sigma(x, \sigma Ay)\), then \(\sigma A = A\), and \(\sigma AA = 1\), and hence

\[
\sigma A = A^{-1}.
\]

Also notice that the \(E_\lambda\) are invariant subspaces for \(\sigma A\). On \(E_{1/\lambda}\) we have on the one hand \(A = 1/\lambda + N_{1/\lambda}\) and on the other hand,

\[
1 = \sigma AA = (\lambda + \sigma N_\lambda)\left(\frac{1}{\lambda} + N_{1/\lambda}\right) = (1 + \frac{1}{\lambda} \sigma N_\lambda)(1 + \lambda N_{1/\lambda})
\]

Hence,

\[
\lambda N_{1/\lambda} = -(\frac{1}{\lambda} \sigma N_\lambda) + (\frac{1}{\lambda} \sigma N_\lambda)^2 - (..)^3 + ... .
\]

On \(E_1\) we have \(A = 1 + N_1\), where \(N_1\) is nilpotent. The requirement that \(A\) be symplectic means that

\[
(1 + \sigma N_1)(1 + N_1) = 1.
\]

Similarly on \(E_{-1}\), we have

\[
(\sigma N_{-1} - 1)(N_{-1} - 1) = 1.
\]

Conversely consider a decomposition of \(\mathbb{C}^{2n}\) as in (1.2) into symplectically orthogonal spaces \(E_1, E_{-1}, E_{\lambda_j} \oplus E_{1/\lambda_j}\) with \(E_{\pm 1}\) symplectic, \(E_{\lambda_j}, E_{1/\lambda_j}\) isotropic for \(\lambda_j \neq 1, -1\) and
all the $1, -1, \lambda_j, 1/\lambda_j$ different. Let $A$ be an operator leaving all the $E_{(,.)}$ invariant, with $A|_{A_{\lambda}} = \lambda + N_{\lambda}$, $\lambda = \pm 1, \lambda_j, 1/\lambda_j$ and $N_{\lambda}$ nilpotent. Then $A$ will be symplectic if (1.3–5) hold.

We next consider logarithms of a symplectic matrix $A$. Decompose $C^{2n}$ into generalized eigen-spaces as in (1.2). We construct $\log A$ with the same generalized eigen-spaces in the following way: On $E_1$ we have $A = 1 + N_1$ with $N_1$ nilpotent and we put

$$\log A = \log(1 + N_1) = N_1 - \frac{1}{2}N_1^2 + \frac{1}{3}N_1^3 + \ldots,$$

where the sum is finite, and in the following we always define the log of $1 + N$ in this way, when $N$ is nilpotent. If $\lambda \in \{\lambda_j, \lambda_j^{-1}, -1\}$, we choose $\mu = \mu(\lambda)$ with $\lambda = e^\mu$ in such a way that

$$\mu(\frac{1}{\lambda_j}) = -\mu(\lambda_j).$$

(1.6)

Write

$$A|_{E_\lambda} = \lambda + N_\lambda = \lambda(1 + \frac{1}{\lambda}N_\lambda),$$

and define on $E_{\lambda}$:

$$\log A = \mu + \log(1 + \frac{1}{\lambda}N_\lambda).$$

This gives a definition which only depends on a choice of logarithms of $\lambda_j, 1 \leq j \leq k$, and on $\log(-1)$ if $E_{-1}$ has positive dimension. It is easy to check that

$$\log A + \sigma \log A = (2k + 1)2\pi i\Pi_{-1},$$

(1.7)

for some integer $k$, where $\Pi_{-1}$ denotes the spectral projection onto $E_{-1}$. Of course we have that $\exp \log A = A$. Recall also that if $B + \sigma B = 0$, then $\exp B$ is a symplectic matrix.

Assume now in addition that $A$ is a real matrix: $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. Then $E_{\pm 1}$ become real in the sense that they are invariant under complex conjugation $\Gamma : (x, \xi) \mapsto (\bar{x}, \bar{\xi})$. The same holds for $E_\lambda$ if $\lambda$ is real. If $\lambda$ is not real, there are two possibilities:

1) $|\lambda| \neq 1$. Then $\overline{\lambda}$, $\frac{1}{\lambda}$, $\frac{1}{\lambda}$, $\overline{\lambda}$ are also eigen-values and

$$E_{\lambda} \oplus E_{\overline{\lambda}} \oplus E_{\frac{1}{\lambda}} \oplus E_{\frac{1}{\lambda}}$$

is the complexification of a real symplectic space.

2) $|\lambda| = 1$. Then $1/\lambda = \overline{\lambda}$ and $E_{\lambda} \oplus E_{\overline{\lambda}}$ is the complexification of a real symplectic space.

In all cases we have

$$A|_{E_{\overline{\lambda}}} = \Gamma(A|_{E_{\lambda}})\Gamma,$$

and it is easy to see that $\log A$ will enjoy the same property, provided that we choose $\mu(\lambda)$ in such a way that

$$\mu(\overline{\lambda}) = \overline{\mu(\lambda)}.$$

(1.8)

This is possible, if we assume that $A$ has no negative eigen-values. We get
Proposition 1.2. a) Let $A$ be a complex symplectic $2n$-matrix, and choose a value $\mu(\lambda)$ of the logarithm of each distinct eigen-value $\lambda$, different from 1 in such a way that (1.6) holds. Then we have a corresponding choice of $B = \log A$ with $e^B = A$, satisfying (1.7).

b) Assume in addition that $A$ is real and has no negative eigen-values. Then by choosing $\mu(\lambda)$ with the additional property (1.8), $B = \log A$ becomes a real matrix, and $^sB + B = 0$.

From now on, we work under the assumptions of b) above, so that $B = \log A$ is real and symplectically anti-symmetric. Consider the quadratic form

$$b(\rho) = \frac{1}{2} \sigma(\rho, B\rho).$$

(1.9)

Then

$$\langle db(\rho), t \rangle = \frac{1}{2} (\sigma(t, B\rho) + \sigma(\rho, Bt)) = \sigma(t, B\rho).$$

On the other hand

$$\langle db(\rho), t \rangle = \sigma(t, H_b(\rho)),$$

where $H_b$ denotes the Hamilton vector field associated to the function $b$, so

$$H_b(\rho) = B\rho,$$

(1.10)

and the fact that $B = \log A$ can be expressed by

$$A = \exp H_b.$$

(1.11)

We now consider the problem of finding the "logarithm" of a canonical transformation also in the non-linear case. We first proceed somewhat formally and let $p_s$ be a smooth real function depending smoothly on the real parameter $s$. Consider the corresponding canonical transformation

$$\kappa_{t,s} = \exp tH_{p_s}.$$  

(1.12)

We will later assume that $p_s$ vanishes to second order at some point $\rho_0$, and then the germ of $\kappa_{t,s}$ at $\rho_0$ will be well-defined for all real $t$. We differentiate the identity

$$\partial_t \kappa_{t,s}(\rho) = H_{p_s}(\kappa_{t,s}(\rho)),$$

with respect to $s$:

$$\partial_t(\partial_s \kappa_{t,s}(\rho)) - (\frac{\partial H_{p_s}}{\partial \rho}(\kappa_{t,s}(\rho))(\partial_s \kappa_{t,s}(\rho)) = (\partial_{s}H_{p_s})(\kappa_{t,s}(\rho)) = H_{\partial_s p_s}(\kappa_{t,s}(\rho)).$$

(1.13)

Notice that the differential $d\kappa_{t,s}(\rho)\nu = \frac{\partial \kappa_{t,s}}{\partial \rho}(\rho)\nu$ satisfies

$$\partial_t d\kappa_{t,s}(\rho) - \frac{\partial H_{p_s}}{\partial \rho}(\kappa_{t,s}(\rho)) \circ d\kappa_{t,s}(\rho) = 0, \quad d\kappa_{0,s}(\rho) = 1.$$  

(1.14)
Comparing the last two identities, we see that
\[ \partial_s \kappa_{t,s}(\rho) = \int_0^t d(\kappa_{t\to \tilde{t},s}(\rho))(H_{\partial_s p_s})(\kappa_{\tilde{t},s}(\rho))d\tilde{t}, \] (1.15)
which can also be written as
\[ \partial_s \kappa_{t,s} = \int_0^t (\kappa_{t\to \tilde{t},s})^* (H_{\partial_s p_s})d\tilde{t}, \] (1.16)
where we use standard notation: lower * for push forward and upper * for pull back.

Rewrite (1.16) as
\[ \partial_s \kappa_{t,s} = (\kappa_{t,s})^* \int_0^t (\kappa_{t\to \tilde{t},s})^* H_{\partial_s p_s}d\tilde{t}, \]
and notice that
\[ (\kappa_{t\to \tilde{t},s})^* H_{\partial_s p_s} = H(\kappa_{t\to \tilde{t},s})^* \partial_s p_s = H_{\partial_s p_s \circ \kappa_{t,s}}, \]
since \( \kappa_{t\to \tilde{t},s} \) is a canonical transformation. Then
\[ \partial_s \kappa_{t,s} = (\kappa_{t,s})^* \int_0^t H_{\partial_s p_s \circ \kappa_{t,s}}d\tilde{t} = (\kappa_{t,s})^* H \int_0^t \partial_s p_s \circ \kappa_{t,s}d\tilde{t}, \]
so
\[ \partial_s \kappa_{t,s} = (\kappa_{t,s})^* H q_{t,s}, \] (1.17)
where
\[ q_{t,s} = \int_0^t \partial_s p_s \circ \kappa_{t,s}d\tilde{t}. \] (1.18)

In the last formula we shall take \( t = 1 \) and consider a problem where \( \partial_s p_s \) will be the unknown. More precisely, let \( \kappa \) be a smooth canonical transformation: \( \text{neigh}(0, \mathbb{R}^{2n}) \to \text{neigh}(0, \mathbb{R}^{2n}) \) with \( \kappa(0) = 0 \). Let \( A := d\kappa(0) =: \kappa_0 \) have no negative eigen-values so that part b) of Proposition 1.2 applies. (More assumptions will be added later.) Let \( B \) be a real logarithm of \( A \) as in the proposition and define the quadratic form \( p_0 = b \) as in (1.9). Then
\[ \kappa_0 = \exp H p_0. \] (1.19)
We look for \( p(\rho) = p_0(\rho) + \mathcal{O}(\rho^3), \) so that
\[ \kappa(\rho) = \exp H p(\rho) + \mathcal{O}(\rho^\infty). \] (1.20)

Let \( \kappa_s, 0 \leq s \leq 1, \) be a smooth family of canonical transformations with
\[ \kappa_s(0) = 0, \ d\kappa_s(0) = d\kappa(0), \ \kappa_0 = d\kappa(0), \ \kappa_1 = \kappa. \] (1.21)
Then we look for a corresponding smooth family \( p_s(\rho) = p_0(\rho) + \mathcal{O}(\rho^3), \) with \( p_{s=0} = p_0 \) as above, such that
\[ \kappa_s(\rho) = \exp H p_s(\rho) + \mathcal{O}(\rho^\infty). \] (1.22)
Then \( p = p_1 \) will be a solution to our problem. Define \( q_s(\rho) = \mathcal{O}(\rho^3) \), by \( \partial_s \kappa_s = (\kappa_s)_s H_{q_s} \), or:

\[
(\kappa_s)^* \partial_s \kappa_s = H_{q_s}.
\]  

(1.23)

The discussion leading to (1.18) indicates that we should find \( p_s \) with the above properties, so that

\[
q_s(\rho) = \int_0^1 \partial_s p_s \circ \exp t H_{p_s}(\rho) dt + \mathcal{O}(\rho^\infty).
\]  

(1.24)

Let \( N \geq 2 \) and suppose that we have already found a smooth family \( p_s^{(N)}(\rho) = p_0(\rho) + \mathcal{O}(\rho^3) \) with \( p_0^{(N)} = p_0 \), so that

\[
q_s(\rho) = \int_0^1 \partial_s p_s^{(N)} \circ \exp t H_{p_s^{(N)}}(\rho) dt + \mathcal{O}(\rho^{N+1}).
\]  

\(_{(E_N)}\)

Notice that \( p_s^{(2)} = p_0 \) solves \((E_2)\) since \( q_s(\rho) = \mathcal{O}(\rho^3) \). Look for \( p_s^{(N+1)} = p_s^{(N)} + r_s^{(N+1)} \), with \( r_s^{(N+1)}(\rho) = \mathcal{O}(\rho^{N+1}) \). Then

\[
\exp t H_{p_s^{(N+1)}}(\rho) = \exp t H_{p_s^{(N)}}(\rho) + \mathcal{O}(\rho^N),
\]

\[
\exp t H_{p_s^{(N+1)}}(\rho) = \exp t H_{p_0}(\rho) + \mathcal{O}(\rho^2),
\]

and we get

\[
\int_0^1 \partial_s p_s^{(N+1)} \circ \exp t H_{p_s^{(N+1)}}(\rho) dt =
\]

\[
\int_0^1 \partial_s p_s^{(N)} \circ \exp t H_{p_s^{(N)}}(\rho) dt + \int_0^1 \partial_s r_s^{(N+1)} \circ \exp t H_{p_0}(\rho) dt + \mathcal{O}(\rho^{N+2}).
\]

If we write the remainder in \((E_N)\) as \(-v^{(N+1)}(\rho) + \mathcal{O}(\rho^{N+2})\), where \( v^{(N+1)} \) is a homogeneous polynomial of degree \( N + 1 \) depending smoothly on \( s \), we will get \((E_{N+1})\), if we can find \( \partial_s r_s^{(N+1)} \) as a homogeneous polynomial \( u^{(N+1)} \) of degree \( N + 1 \) depending smoothly on \( s \), such that

\[
\int_0^1 u^{(N+1)} \circ \exp t H_{p_0}(\rho) dt = v^{(N+1)}(\rho).
\]  

(1.25)

Consider a general linear map \( B : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n} \) with Jordan decomposition \( B = D + N \), where \( D \) is diagonalizable, \( = \text{diag}(d_j) \) with respect to a suitable basis, and \( N \) is nilpotent and commutes with \( D \). The action of the vector field \( Bx \cdot \partial_s \) on the space \((\mathbb{C}^{2n})^*\) of linear forms on \( \mathbb{C}^{2n} \) can then be identified with \( 'B \) in the natural way. Notice that \( 'B \) has the Jordan decomposition \( 'D + 'N \) and that \( 'D \) becomes \( \text{diag}(d_j) \) if we select the dual basis to the one where \( D \) is diagonal. Define

\[
' B^{(m)} = 'B \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes 'B \otimes 1 \otimes \ldots \otimes 1 + \ldots 1 \otimes \ldots \otimes 1 \otimes 'B,
\]

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as a linear endomorphism of the $m$-fold tensor product $((C^{2n})^*)^\otimes m$. We have the Jordan decomposition

$$tB^{(m)} = tD^{(m)} + tN^{(m)}$$

(1.26)

(where the first term to the right is diagonalizable, the second nilpotent and the two terms commute). The corresponding eigen-values are $d_{j_1} + .. + d_{j_m}$, for $j = (j_1, .., j_m) \in \{1,2,..,2n\}^{1,2,..,m}$. The three operators in (1.26) act naturally on the symmetric tensor product $((C^{2n})^*)^\otimes m$ and the decomposition (1.26) is still a Jordan one on that space. The eigen-values of $tB^{(m)}$ become

$$\beta_k = k_1 d_1 + .. + k_{2n} d_{2n}, \ k_1 + .. + k_{2n} = m.$$  

(1.27)

$((C^{2n})^*)^\otimes m$ is equal to the space $P^m_{\text{hom}}(C^{2n})$ of $m$-homogeneous polynomials on $C^{2n}$ and $tB^{(m)}$ is the action of $Bx \cdot \partial_x$ on that space.

Consider the map

$$P^m_{\text{hom}}(C^{2n}) \ni u \mapsto \int_0^1 u \circ \exp(tB) dt \in P^m_{\text{hom}}(C^{2n}),$$

(1.28)

which is equal to

$$\int_0^1 \exp(t'B^{(m)}) dt.$$

(1.29)

The Jordan decomposition (1.26) gives a similar decomposition of (1.29). The eigen-values of (1.29) are therefore given by

$$\int_0^1 e^{t\beta_k} dt = \begin{cases} 1 \text{ for } \beta_k = 0 \\ \frac{e^{\beta_k} - 1}{\beta_k} \text{ for } \beta_k \neq 0, \ k \in N^{2n}, \ |k| = m. \end{cases}$$

(1.30)

We conclude that the map (1.28) is invertible for a given $m$ precisely when for all $k \in N^{2n}$ with $|k| = m$:

$$k_1 d_1 + .. + k_{2n} d_{2n} \in 2\pi i Z \Rightarrow k_1 d_1 + .. + k_{2n} d_{2n} = 0.$$  

(1.31)

Now return to the equation (1.25), where $p_0(\rho) = b(\rho) = \frac{1}{2}\sigma(\rho, B\rho)$ and $B$ is the logarithm of the real symplectic matrix $A = d\kappa(0)$, obtained under the assumptions of Proposition 1.2, part b). The eigen-values of $B$ are then of the form $0$ with some possibly vanishing multiplicity and $\mu_j, -\mu_j \neq 0$ with equal multiplicity $> 0$. Here we arrange so that all the eigen-values are distinct for instance by taking either $\Re \mu_j > 0$ or with $\Re \mu_j = 0$ and $\Im \mu_j > 0$. We also recall that our set of eigen-values is closed under complex conjugation. The assumption that (1.31) holds for all $m$, then amounts to the assumption that

$$\sum k_j \mu_j \in 2\pi i Z \Rightarrow \sum k_j \mu_j = 0,$$

(1.32)

for all $k_1, .., k_r \in Z$. Here $r \leq n$ is the number of distinct $\mu_j$. (We could also have chosen to repeat the eigen-values according to their multiplicity without changing (1.32).)
We have practically finished the proof of the following version of a theorem of Lewis–Sternberg:

**Theorem 1.3.** Let \( \kappa : \text{neigh}(0, \mathbb{R}^{2n}) \to \text{neigh}(0, \mathbb{R}^{2n}) \) be a smooth canonical transformation. Assume that \( d\kappa(0) \) has no negative eigen-values. Let the distinct eigen-values of \( d\kappa(0) \) be 1 (possibly with multiplicity 0) and \( \lambda_j, \lambda_j^{-1}, 1 \leq j \leq r \) with \( |\lambda_j| > 1 \) or with \( |\lambda_j| = 1 \) and \( 0 < \arg \lambda_j < \pi \). Choose \( \mu_j \) with \( \lambda_j = e^{\mu_j} \), in such a way that \( \overline{\lambda}_j \) corresponds to \( \pi_j \) and let \( B = \log A \) be given as in part b of Proposition 1.2. Assume that (1.32) holds, and let \( p_0(\rho) = b(\rho) \) be given in (1.9).

Then there exists \( p(\rho) \in C^\infty(\text{neigh}(0, \mathbb{R}^{2n}); \mathbb{R}) \) such that \( p(\rho) = p_0(\rho) + O(\rho^3) \) and

\[
\kappa(\rho) = \exp H_p(\rho) + O(\rho^\infty), \tag{1.33}
\]

\( p \) is uniquely determined by these properties (for a given choice of \( p_0 \)).

This result (at least the existence part) is extremely close to a corresponding one for complex canonical transformations, due to Lewis–Sternberg ([St], Theorem 1 and Corollary 1.1) and clearly stated in [Fr], Théorème V.1.

**End of the proof.** We establish the existence of \( p \). Let \( \kappa_s, 0 \leq s \leq 1 \) be a smooth family of canonical transformations with \( \kappa_1 = \kappa, \kappa_s(0) = 0, d\kappa_s(0) = d\kappa(0) \) and with \( \kappa_0 \) linear (= \( d\kappa(0) \)). Define \( q_s \) by (1.23). The preceding discussion gives us a smooth family \( p_s(\rho) \in C^\infty(\text{neigh}(0, \mathbb{R}^{2n}); \mathbb{R}) \) with \( p_{s=0} = p_0 \), such that if \( \tilde{\kappa}_s = \exp H_{p_s} \), then

\[
\tilde{\kappa}_s^* \partial_s \tilde{\kappa}_s = H_{q_s} + O(\rho^\infty). \tag{1.34}
\]

A simple computation shows that (1.23,34) can be written as

\[
\partial_s \kappa_s^{-1}(\rho) = -H_{q_s}(\kappa_s^{-1}(\rho)), \quad \partial_s \tilde{\kappa}_s^{-1}(\rho) = -H_{q_s}(\tilde{\kappa}_s^{-1}(\rho)) + O(\rho^\infty), \tag{1.35}
\]

and we also know that \( \kappa_0^{-1} = \tilde{\kappa}_0^{-1} \). It follows that

\[
\kappa_s^{-1}(\rho) = \tilde{\kappa}_s^{-1} + O(\rho^\infty),
\]

and hence that

\[
\kappa_s(\rho) = \tilde{\kappa}_s(\rho) + O(\rho^\infty).
\]

Taking \( s = 1 \) gives (1.33) with \( p = p_1 \).

We next prove the uniqueness of the Taylor expansion of \( p \). Let \( \tilde{p} \) have the same properties as \( p \), so that

\[
\exp H_{\tilde{p}}(\rho) = \exp H_p(\rho) + O(\rho^\infty), \quad \tilde{p}(\rho) = p(\rho) + O(\rho^3).
\]

Assume that \( \tilde{p} - p \) does not vanish to infinite order, so that \( \tilde{p} = p + r \), where \( r(\rho) = O(\rho^m) \), \( r(\rho) \neq O(\rho^{m+1}) \), for some \( 3 \leq m \in \mathbb{N} \).

Put \( p_s = (1-s)p + s\tilde{p} = p + sr, 0 \leq s \leq 1, \) so that \( p_0 = p, p_1 = \tilde{p} \), and define \( \kappa_s = \exp H_{p_s} \), so that

\[
\kappa_1(\rho) = \kappa_0(\rho) + O(\rho^\infty). \tag{1.36}
\]
For this family, define $q_s$ by (1.23). Then (1.24) holds and since $\partial_s p_s = r$, we have

$$q_s = \int_0^1 r \circ \exp(t H_{p_0}) dt + \mathcal{O}(\rho^{m+1}).$$

The previous discussion shows that the integral has a non-zero Taylor polynomial of degree $m$:

$$q_s(\rho) = \tilde{q}(\rho) + \mathcal{O}(\rho^{m+1}), \quad 0 \neq \tilde{q} \in \mathcal{P}_m^{\text{hom}}.$$  \hfill (1.37)

From (1.35), we conclude that

$$\kappa^{-1}_1(\rho) - \kappa^{-1}_0(\rho) = -H_{\tilde{q}}(\kappa^{-1}_0(\rho)) + \mathcal{O}(\rho^m),$$

which contradicts (1.36). The proof is complete.

2. Notions of equivalence.

As in [Ia], our results will be valid "to infinite order at (0,0)" and in this section we review the corresponding notions of equivalence. Using these notions we also develop a very rudimentary functional calculus for functions of several pseudodifferential operators.

If $V_j \subset \mathbb{R}^n$ are open neighborhoods of 0 and $v_j \in C^\infty(V_j)$, we say that $v_1$ and $v_2$ are equivalent; $v_1 \equiv v_2$, if $v_1 - v_2$ vanishes to infinite order at 0: $v_1(x) - v_2(x) = \mathcal{O}(x^\infty)$. This is clearly an equivalence relation and the equivalence classes can be identified with the corresponding formal Taylor expansions.

With $V = V_j \subset \mathbb{R}^n$ as above, let $S^0_{\text{cl}}(V)$ denote the space of functions $a(x; h)$ in $C^\infty(V)$ depending on the semi-classical parameter $h \in [0, h_0]$ for some $h_0 > 0$, such that

$$a(x, h) \sim \sum_{j=0}^{\infty} h^j a_j(x), \quad h \to 0,$$  \hfill (2.1)

for some sequence of $a_j \in C^\infty(V)$. We say that $a^{(k)} \in S^0_{\text{cl}}(V_k)$, $k = 1, 2$ are equivalent and write $a^{(1)} \equiv a^{(2)}$, if $a_j^{(1)} \equiv a_j^{(2)}$ for the corresponding coefficients in (2.1). Equivalently we can say that $a^{(k)}$ are equivalent if $a^{(1)}(x; h) - a^{(2)}(x; h) = \mathcal{O}((x, h)^\infty)$.

If $m > 0$ is a smooth weight function on $V$, we define $S^0(V, m)$ to be the space of smooth functions $a$ on $V$ such that for all multi-indices $\alpha$, we have $|\partial^{\alpha}_x a(x)| \leq C_{\alpha, a} m(x)$. Let $S^0_{\text{cl}}(V, m)$ be the subspace of $S^0_{\text{cl}}$ for which (2.1) holds in $S^0(V, m)$.

We next pass to the case of pseudodifferential operators. Recall that if $p(x, \xi)$ belongs to an appropriate symbol class of functions on $\mathbb{R}^{2n}$, then we define the corresponding $h$-Weyl quantization $P = P^w(x, hD_x)$ by:

$$P u(x) = \frac{1}{(2\pi)^n} \iint e^{i(x-y) \cdot \theta/h} p(\frac{x+y}{2}, \theta) u(y) dy d\theta.$$

Recall that $p$ is called the Weyl-symbol of $P$. (See for instance [DiSj].) Let $S^0(\mathbb{R}^{2n})$ denote the space of smooth functions that are bounded together with all their derivatives. If
Let \( \kappa : \text{neigh}(0, \mathbb{R}^2n) \to \text{neigh}(0, \mathbb{R}^{2n}) \) be a canonical transformation which maps 0 to 0. Then there exist \( N \in \mathbb{N} \) and a non-degenerate phase function \( \phi(x, y, \theta) \) in \( \text{neigh}(0, \mathbb{R}^{n+n+N}) \) such that the graph of \( \kappa \) in a neighborhood of \( (0, 0) \) coincides with the image of the local diffeomorphism:

\[
C_{\phi} := \{(x, y, \theta); \phi_\theta(x, y, \theta) = 0\} \ni (x, y, \theta) \mapsto (x, \phi'_x; y, -\phi'_y).
\] (2.3)

Here we recall that a smooth real-valued function is called a non-degenerate phase function (in the sense of Hörmander) if \( d\phi_\theta, \ldots, d\phi_{\theta_N} \) are linearly independent on the set \( C_{\phi} \) above, which then becomes a \( 2n \)-dimensional smooth sub-manifold. When discussing the relation between phases and symbols with canonical transformations, it is tacitly understood that the point \( x = 0, y = 0, \theta = 0 \) corresponds to \( \kappa(0) = 0 \) under the map (2.3).

Let \( \kappa \) be as above and let \( \phi \) be a corresponding generating phase. Let \( \bar{\kappa} : \text{neigh}(0, \mathbb{R}^{2n}) \to \text{neigh}(0, \mathbb{R}^{2n}) \) be a second canonical transformation (with the tacit convention that it also maps 0 to 0). It is easy to see that \( \bar{\kappa} \equiv \kappa \) if and only if \( \bar{\kappa} \) has a generating phase \( \bar{\phi} \) which is equivalent to \( \phi \).

With \( \phi, \kappa \) as above we consider a Fourier integral operator of order 0:

\[
Uu(x) = I(a, \phi)u(x) = h^{-\frac{n+N}{2}} \int \int e^{i(\phi(x, y, \theta) + W(x, y, \theta; h)})a(x, y, \theta; h)u(y)dyd\theta,
\] (2.4)

where \( a \in S_{\text{cl}}^0 \) has its support in a sufficiently small neighborhood of \( (0, 0, 0) \). In this paper we only consider Fourier integral operators that are elliptic at \( (0, 0, 0) \) in the sense that \( a_0(0, 0, 0) \neq 0 \). In order to normalize things, we will always assume that \( \phi(0, 0, 0) = 0 \). If \( \kappa \) is the canonical transformation generated by \( \phi \), we say that \( \kappa \) is the canonical transformation associated to \( U \). Thanks to the ellipticity assumption, \( \kappa \) is uniquely determined by \( U \) in some neighborhood of 0. We also recall the fundamental theorem about Fourier integral operators, namely that if \( \psi(x, y, w) \) is a second phase which generates \( \kappa \) and if \( a \) has support in a sufficiently small neighborhood of \( (0, 0, 0) \), then there exists a classical symbol \( b(x, y, w; h) \) of order 0 with support in a small neighborhood of \( (0, 0, 0) \), such that \( I(b, \psi) \) (formed as in (2.4) with \( N \) replaced by the dimension of \( w \)-space) is equal to \( I(a, \phi) \).

Let \( \tilde{\kappa} \) be a second canonical transformation with \( \tilde{\kappa} \equiv \kappa \). Let \( \tilde{\phi} \equiv \phi \) be a corresponding generating phase. We say that \( \tilde{U} = I(\tilde{a}, \tilde{\phi}) \) is equivalent to \( U \) and write \( \tilde{U} \equiv U \), if \( \tilde{a} \equiv a \). It is a standard exercise in Fourier integral operator theory to verify that this definition of equivalence does not depend on the choice \( \phi \). It is also easy to show that the definition is stable under composition in the natural way.
Below we will also need some functional calculus. First we consider exponentials of pseudodifferential operators. Let \( p \sim \sum_{j=0}^{\infty} p_j(x, \xi) h^j \) in \( S^0_0(\mathbb{R}^{2n}, 1) \), and assume that \( p_0 \) is real-valued with \( p_0(0, 0) = 0, p'_0(0, 0) = 0 \). Then \( e^{-itp} = \sum_{j=0}^{\infty} p_j(x, hD(h))/h \) is well-defined for all complex \( t \) (even without the reality assumption on \( p_0 \)) and for real \( t \) we get a Fourier integral operator. If \( \chi \in C_0^\infty(\mathbb{R}^{2n}) \) is equal to 1 near 0, then up to an operator whose distribution kernel is rapidly decreasing together with all its derivatives, we have that \( \chi^u e^{-itP/h} \chi^u \) is a Fourier integral operator as above, with the associated canonical transformation \( \kappa_t = \exp tH_p \), whose equivalence class does not depend on the choice of \( \chi \).

It is also easy to see that if \( \chi \) be the sequence of polynomials in \( \partial_s \), we shall denote by \( \kappa_t \) the sequence of polynomials in \( \partial_s \), and that this sequence defines naturally an equivalence class of pseudodifferential operators. Let \( \kappa_t \) be a sequence of polynomials in \( \partial_s \), and more generally \( \kappa_t \) be the sequence of polynomials in \( \partial_s \), then for real \( t \), we have \( e^{-itP/h} \equiv e^{-it\bar{P}/h} \) (in the sense that we have equivalence for the corresponding truncated operators).

Finally we discuss a very primitive pseudodifferential functional calculus. Let \( P_\kappa = p_\kappa(x, hD; h), k = 1, \ldots, N_0 \) be a commuting family of pseudodifferential operators with \( p_\kappa \in S^0_0(\mathbb{R}^{2n}, ((x, \xi))^m) \) (with the standard notation \( ((x, \xi)) = (1 + |(x, \xi)|^2)^{1/2} \)) and assume that the leading symbols \( p_{\kappa,0} \) vanish at \( (0, 0) \). Let \( F(\iota_1, \ldots, \iota_{N_0}; h) \in S^0_0(\text{neigh}(0, \mathbb{R}^{N_0})) \). Let \( F_N \) be the sequence of polynomials in \( \iota_1, \ldots, \iota_{N_0}, h \) obtained by taking the Taylor polynomials of order \( N \) of the first \( N \) terms in the asymptotic expansion of \( F \), so that

\[
F - F_N = \mathcal{O}((\iota, h)^N), \ (\iota, h) \to 0.
\]

Then it is easy to see that \( F_N(P_1, \ldots, P_N; h) = q_N(x, hD_x; h) \), where

\[
q_N(x, \xi; h) - q_M(x, \xi; h) = \mathcal{O}((x, \xi, h)^k(N,M)), \text{ where } k(N, M) \to \infty, N, M \to \infty,
\]

and that this sequence defines naturally an equivalence class of pseudodifferential operators that we shall denote by \( F(P_1, \ldots, P_{N_0}; h) \).

3. Logarithms of Fourier integral operators.

Let \( U_s, 0 \leq s \leq 1 \) be a smooth family of elliptic Fourier integral operators of order 0, associated to a fixed canonical transformation \( \kappa : \text{neigh}(0, \mathbb{R}^{2n}) \to \text{neigh}(0, \mathbb{R}^{2n}) \) with \( \kappa(0) = 0 \). We represent \( U_s \) by

\[
U_s u(x) = h^{-\frac{n+N}{2}} \int e^{it\phi(x,y,\theta)} u_s(x, y, \theta; h) u(y) dyd\theta, \quad (3.1)
\]

where \( u_s \in S^0_0 \) and more generally \( \partial^k u_s \in S^0_0 \) for all \( k \in \mathbb{N} \) is a smooth family of classical symbols of order 0, defined in \( \text{neigh}((0, 0, 0); \mathbb{R}^{2n+N}) \) and \( \phi \) is a real phase function which is non-degenerate in the sense of Hörmander [Hö] and generates \( \kappa \), so that \( C_\phi \ni (x, y, \theta) \mapsto (x, \phi'_x; y, -\phi'_y) \) is a local diffeomorphism, where \( C_\phi \subset \mathbb{R}^{2n+N} \) is the sub-manifold given by \( \phi'_\theta(x, y, \theta) = 0 \). To normalize things, we assume that \( \phi'(0, 0, 0) = 0 \) and that

\[
\phi(0, 0, 0) = 0. \quad (3.2)
\]

Notice that this last assumption does not depend on the choice of phase in the representation (3.1). In the following, we shall use the equivalence relations “\( \equiv \)”, defined in section2.
We define the "logarithmic derivative" of our family, to be the smooth family of pseudodifferential operators \( Q_s \) given by

\[
Q_s \equiv U_s^{-1}hD_sU_s. \quad (3.3)
\]

\( Q_s \) and more generally \( \partial^k_s Q_s \) is a smooth family of classical pseudodifferential operators defined in \( \text{neigh}((0,0);\mathbb{R}^{2n}) \). (We made an arbitrary choice of the order of the factors in (3.3), if \( \tilde{Q}_s \equiv (hD_sU_s)U_s^{-1} \), then we get a new pseudodifferential operator which is related to \( Q_s \) by the intertwining relation \( U_sQ_s \equiv \tilde{Q}_sU_s \).)

The family \( U_s \) is determined uniquely by \( U_0 \) and its logarithmic derivative:

**Lemma 3.1.** Let \( V_s \) be a second family of Fourier integral operators with the same properties as \( U_s \) and associated to the same canonical transformation \( \kappa \). Assume that \( U_s^{-1}hD_sU_s \equiv V_s^{-1}hD_sV_s \) and that \( U_0 \equiv V_0 \). Then \( U_s \equiv V_s \).

**Proof.** Let \( U \) be a fixed elliptic Fourier integral operator associated to \( \kappa \), so that \( U_s \equiv UA_s, V_s \equiv UB_s \), where \( A_s, B_s \) are smooth families of pseudodifferential operators. Then

\[
U_s^{-1}hD_sU_s \equiv A_s^{-1}hD_sA_s,
\]

and similarly for \( V_s \), so we get

\[
A_s^{-1}hD_sA_s \equiv B_s^{-1}hD_sB_s, \quad A_0 \equiv B_0. \quad (3.4)
\]

From this we conclude first that \( A_s \) and \( B_s \) have equivalent principal symbols, then equivalent sub-principal symbols and so on, so \( A_s \equiv B_s \) and hence \( U_s \equiv V_s \). #

**Remark.** We have \( hD_sU_s \equiv U_sQ_s \), hence \( hD_sU_s^* \equiv -Q_s^*U_s^* \) for the adjoint operators, so

\[
hD_s(U_s^*U_s) + (Q_s^*(U_s^*U_s) - (U_s^*U_s)Q_s) \equiv 0.
\]

\( U_s^*U_s \) is a smooth family of elliptic pseudodifferential operators and we conclude

a) If \( U_s \) are unitary (up to equivalence), then \( Q_s \) are self-adjoint (up to equivalence).

b) If \( U_s \) is unitary for one value of \( s \) and \( Q_s \) are self-adjoint for all \( s \), then \( U_s \) is unitary for all \( s \) (again up to equivalence).

We now assume for a while that

\[
U_s = U_{1,s}, \quad \text{where} \quad U_{t,s} \equiv e^{-itP_s/h}, \quad (3.5)
\]

and \( P_s \) is a smooth family of pseudodifferential operators with the leading symbol \( p(x,\xi) \) independent of \( s \), so that \( \kappa \equiv \exp Hp \) and \( p(0,0) = 0 \) (thanks to (3.2) and \( p'(0,0) = 0 \) (since \( \kappa(0,0) = (0,0) \)). We shall derive a simple formula for the logarithmic derivative: Start with

\[
hD_tU_{t,s} - P_sU_{t,s} \equiv 0, \quad U_{0,s} \equiv 1, \quad (3.6)
\]
and recall that $P_s$ and $U_{t,s}$ commute. Apply $hD_s$ to this relation:

$$hD_t(hD_s U_{t,s}) + P_s(hD_s U_{t,s}) \equiv -(hD_s P_s) U_{t,s},$$

which implies

$$hD_t(\tilde{U}_{t-s} hD_s U_{t,s}) \equiv \tilde{U}_{t-s} (P_s + hD_s) hD_s U_{t,s} \equiv -\tilde{U}_{t-s} (hD_s P_s) U_{t,s}.$$ 

Integrate this from $\tilde{t} = 0$ to $\tilde{t} = t$:

$$hD_s U_{t,s} \equiv -\frac{i}{\hbar} \int^t_0 \tilde{U}_{t-\tilde{t},s}(hD_s P_s) U_{\tilde{t},s} d\tilde{t} \equiv -\int^t_0 \tilde{U}_{t-\tilde{t},s}(\partial_s P_s) U_{\tilde{t},s} d\tilde{t}.$$ 

Taking $t = 1$, we get the promised formula:

$$U^{-1}(hD_s U_s) \equiv -\int^1_0 U_{-t,s}(\partial_s P_s) U_{t,s} dt,$$ 

under the assumption (3.5).

Let $\kappa : \text{neigh}(0, \mathbb{R}^n) \to \text{neigh}(0, \mathbb{R}^{2n})$ be a canonical transformation as in Theorem 1.3 (so that (1.32) holds), and choose $p = p_0 + \mathcal{O}(\rho^3)$ satisfying (1.33):

$$\kappa(\rho) = \exp H_p(\rho) + \mathcal{O}(\rho^\infty).$$ 

Recall that $p$ is uniquely determined modulo $\mathcal{O}(\rho^\infty)$ by $\kappa$ and the choice of the quadratic form $p_0$ with $\exp H_{p_0} = d\kappa(0)$.

Let $U$ be an elliptic Fourier integral operator of order 0 associated to the canonical transformation $\kappa$. We look for a pseudodifferential operator $P$ with leading symbol $p$ such that

$$U \equiv e^{-iP/h}$$

Let $P_0$ be a pseudodifferential operator with leading symbol $p$ and put

$$U_0 \equiv e^{-iP_0/h}. $$

Let $[0, 1] \ni s \mapsto U_s$ be a smooth family of Fourier integral operators as above, all associated to $\kappa$ (modulo equivalence) and with $U_{s=0} = U_0$, $U_1 = U$. We look for a corresponding smooth family of pseudodifferential operators $P_s$, with leading symbol $p$, such that $P_{s=0} = P_0$, and

$$U_s \equiv e^{-iP_s/h}.$$ 

Then $P = P_1$ will solve (3.9).

Since the $U_s$ are associated to the same canonical transformation, the logarithmic derivative

$$Q_s \equiv U_s^{-1} hD_s U_s,$$ 

(3.12)
will be of order \(-1\) (i.e. \(O(h^{1+})\)) with Weyl symbol:

\[
Q_{s}(p; h) \sim hq_{s,1}(p) + h^2q_{s,2}(p) + \ldots.
\]  

(3.13)

Motivated by (3.7) we shall first look for a smooth family \(P_{s}\) with leading symbol \(p\) and \(P_{s=0} = P_{0}\), such that

\[
Q_{s} \equiv -\int_{0}^{1} e^{itP_{s}/h}(\partial_{s}P_{s})e^{-itP_{s}/h}dt.
\]  

(3.14)

Denoting the Weyl symbol of \(P_{s}\) by the same letter,

\[
P_{s}(p; h) = p(p) + hp_{s,1}(p) + h^2p_{s,2}(p) + \ldots,
\]  

(3.15)

we first see that \(p_{s,1}\) should solve

\[
q_{s,1}(p) = -\int_{0}^{1} (\partial_{s}p_{s,1}) \circ \exp(tH_{p})dt + O(p^{\infty}).
\]  

(3.16)

As in the proof of Theorem 1.3, we see that (3.16) has a unique solution \(\partial_{s}p_{s,1} \pmod{O(p^{\infty})}\) and since \(p_{0,1}\) is given by the choice of \(P_{0}\), we get a unique choice of \(p_{s,1}\).

Proceeding inductively, we assume that we have found \(P_{s}^{(m)}\) with symbol

\[
P_{s}^{(m)}(p; h) = \sum_{j=0}^{m} h^{j}p_{s,j}(p) + \sum_{m+1}^{\infty} h^{j}p_{0,j}(p),
\]  

(3.17)

where \(p_{s,0} = p\) and \(h^{j}p_{0,j}\) are the terms in the asymptotic expansion of \(P_{0}(p; h)\), such that

\[
-\int_{0}^{1} e^{itP_{s}^{(m)}/h}(\partial_{s}P_{s}^{(m)})e^{-itP_{s}^{(m)}/h}dt \equiv Q_{s} + h^{m+1}R_{m+1,s},
\]  

(3.18)

where \(R_{m+1,s}\) is of order 0 with leading symbol \(r_{m+1,s}\). We just saw how to obtain this for \(m = 1\).

If \(A\) is a pseudodifferential operator of order 0, we see that \(e^{itP_{s}^{(m)}/h}Ae^{-itP_{s}^{(m)}/h}\) will change by an operator of order \(\leq -(m + 1)\) if we modify \(P_{s}^{(m)}\) by an operator of order \(\leq -(m + 1)\), for instance by passing to \(P_{s}^{(m+1)}\). It follows that

\[
e^{itP_{s}^{(m+1)}/h}\partial_{s}P_{s}^{(m+1)}e^{-itP_{s}^{(m+1)}/h} = e^{itP_{s}^{(m)}/h}\partial_{s}P_{s}^{(m+1)}e^{-itP_{s}^{(m)}/h} + O(h^{m+2}).
\]

To get \(P_{s}^{(m+1)}\) satisfying (3.18) with \(m\) replaced by \(m + 1\), it suffices to have

\[
\int_{0}^{1} e^{itP_{s}^{(m)}/h}\partial_{s}(P_{s}^{(m+1)} - P_{s}^{(m)})e^{-itP_{s}^{(m)}/h}dt \equiv h^{m+1}R_{m+1,s} + O(h^{m+2})
\]  

(3.19)

(with the same \(R_{m+1,s}\) as in (3.18)), which gives for the leading symbols

\[
\int_{0}^{1} (\partial_{s}P_{s,m+1}) \circ \exp(tH_{p})dt \equiv r_{m+1,s},
\]  

(3.20)
Again this has a unique solution $\partial_s p_{s,m+1}$, and our induction procedure can be continued and gives a solution $P_s$ to (3.14).

Let $\tilde{U}_s = e^{-iP_s/h}$. Then by construction $\tilde{U}_0 = U_0$, $\tilde{U}_s^{-1} hD_s \tilde{U}_s \equiv U_s^{-1} hD_s U_s$, and Lemma 3.1 implies that $\tilde{U}_s \equiv U_s$ and in particular that

$$U \equiv e^{-iP/h}, \quad P = P_1. \quad (3.21)$$

This gives the existence part of the following

**Theorem 3.2.** Let $\kappa : \text{neigh}(0, \mathbb{R}^{2n}) \to \text{neigh}(0, \mathbb{R}^{2n})$ be a smooth canonical transformation as in Theorem 1.3 and choose $\mu_j, p_0$ as there, so that (1.32) holds. Let $p \in C^\infty(\text{neigh}(0, \mathbb{R}^{2n}); \mathbb{R})$ be the unique function mod $O(\rho^\infty)$ of the form $p = p_0 + O(\rho^3)$ with $\kappa(\rho) = \exp H_p(\rho) + O(\rho^\infty)$.

Let $U$ be an elliptic Fourier integral operator of order 0 associated to $\kappa$. Then there exists a pseudodifferential operator $P^w(x, hD_x; h)$ with symbol

$$P(\rho; h) \sim p(\rho) + hp_1(\rho) + ..., \quad (3.22)$$

such that

$$U \equiv e^{-iP/h}. \quad (3.23)$$

$P$ is uniquely determined modulo ”$\equiv$” and up to an integer multiple of $2\pi h$ by (3.23) and the choice of $p_0$.

It remains to prove the uniqueness modulo ”$\equiv$”. Let $\tilde{P}^w(x, hD_x; h)$ be another operator with the same properties;

$$\tilde{P}(\rho; h) \sim \tilde{p}(\rho) + h\tilde{p}_1(\rho) + ..., \quad \tilde{p} = p_0 + O(\rho^3). \quad (3.24)$$

Then we must have $\kappa(\rho) = \exp \tilde{H}_p(\rho) + O(\rho^\infty)$ and from the uniqueness part of Theorem 1.3, we conclude that $\tilde{p} = p + O(\rho^\infty)$.

Put $P_s = (1 - s)P + s\tilde{P}, 0 \leq s \leq 1$, so that $P_0 = P, P_1 = \tilde{P}$ and define $U_s = e^{-iP_s/h}$. For this family, define $Q_s$ by (3.12). If $\tilde{P} \neq P$, let $1 \leq m \leq \infty$ be the smallest integer with

$$\tilde{p}_m - p_m \neq O(\rho^\infty). \quad (3.25)$$

If $m = 1$, we may also assume that $\tilde{p}_1 - p_1$ is not $\equiv$ to an integer multiple of $2\pi$. From (3.14), we see that

$$Q_s \sim \sum_{j=1}^\infty h^j q_{s,j},$$

with $q_{s,j}(\rho) = O(\rho^\infty)$ for $1 \leq j \leq m - 1$, and with

$$q_{m,s} = -\int_0^1 (\tilde{p}_m - p_m) \circ \exp(tH_p) dt + O(\rho^\infty). \quad (3.26)$$
When \( m = 1 \), \( q_{m,s} \) is not \( \equiv \) to an integer multiple of \( 2\pi \). From the invertibility of the map (1.28), we conclude that

\[
q_m := q_{m,0} = q_{m,s} + O(\rho^\infty), \quad q_m \neq O(\rho^\infty).
\]  

(3.26)

Let \( W_s, 0 \leq s \leq 1 \) be a smooth family of Fourier integral operators which solves

\[
Q_s \equiv W_s^{-1}hD_sW_s, \quad W_0 = 1.
\]  

(3.27)

If \( q_{0,s} \) had been 0 rather than just \( O(\rho^\infty) \), the \( W_s \) would have been pseudodifferential operators, so in general they are equivalent to such operators:

\[
W_s \equiv R^w_s(x, hD_x; h), \quad \text{where} \quad R_s(\rho; h) \sim \sum_{j=0}^{\infty} h^j r_{j,s}(\rho),
\]  

(3.28)

satisfying

\[
r_{0,s}(\rho)^{-1} \partial_s r_{0,s}(\rho) = iq_1
\]

when \( m = 1 \) and

\[
r_{0,s}(\rho) = 1, \ r_{j,s} = 0 \ \text{for} \ 1 \leq j \leq m - 2, \ \partial_s r_{m-1,s} = iq_m,
\]

when \( m \geq 2 \). In other words,

\[
r_{0,s}(\rho) = e^{isq_1(\rho)}, \ \text{when} \ m = 1,
\]

\[
R_s(\rho; h) = 1 + isq_m h^{m-1} + O(h^m), \ \text{when} \ m \geq 2.
\]

In both cases, we have \( R_1 \neq 1 \), so

\[
W_1 \neq 1.
\]  

(3.29)

If we put \( \tilde{U}_s = U_0W_s \), we see that \( \tilde{U}_0 = U_0 \) and that

\[
Q_s \equiv \tilde{U}_s^{-1}hD_s\tilde{U}_s \equiv U_s^{-1}hD_sU_s.
\]

By Lemma 3.1 we conclude that \( \tilde{U}_s \equiv U_s \) and in particular,

\[
U_0W_1 = \tilde{U}_1 \equiv U_1.
\]

Since \( W_1 \neq 1 \), this contradicts the assumption that \( U_1 \equiv U_0 \), and the proof of Theorem 3.2 is complete.

\[\#\]

Remark. Up to equivalence we have that \( U \) is unitary iff \( P \) is self-adjoint:

\[
U^*U \equiv 1 \Leftrightarrow P^* \equiv P.
\]

Indeed (3.23) gives

\[
(U^*)^{-1} \equiv e^{-iP^*/h},
\]

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and it suffices to apply the uniqueness statement in Theorem 3.2.

4. Birkhoff normal forms.

To get a normal form for the Fourier integral operator in Theorem 3.2, it suffices to get the quantized "Birkhoff" normal form of the operator $P$. For simplicity we shall make a non-resonance assumption, and simply recall how this was done in [Sj] in a slightly less general setting (in the spirit of works of Bellissard–Vittot, Graffi–Paul and others cited there). The extension to the present case is however completely immediate.

Let $P \sim p(\rho) + hp_1(\rho) + ..$ be as in (3.22), with $p$ real, $p(0) = 0$, $p'(0) = 0$. Put $p_0(\rho) = \frac{1}{2} \langle p''(0) \rho, \rho \rangle$ and let $B$ be the corresponding fundamental matrix, so that

\[ p_0(\rho) = \frac{1}{2} \sigma(\rho, B \rho), \quad \sigma B = -B. \quad (4.1) \]

Let $\mu_j, -\mu_j$ and possibly 0 be the distinct eigen-values of $B$. We recall that Theorem 3.2 was obtained under the assumption (1.32). We add a non-resonance assumption, and for that purpose we temporarily change the notation slightly and denote by $\mu_j, -\mu_j, 1 \leq j \leq n$ all the eigen-values of $B$, possibly repeated according to their multiplicity. Assume

\[ \sum_{1}^{n} k_j \mu_j = 0, \quad k_j \in \mathbb{Z} \Rightarrow k_1 = .. = k_n = 0. \quad (4.2) \]

This implies that the $\mu_j$ are distinct and $\neq 0$, so $B$ has the $2n$ distinct eigen-values $\mu_j, -\mu_j$, $1 \leq j \leq n$, which is in agreement with the earlier notation with $r = n$.

Notice that (1.32) and (4.2) combine into the single condition

\[ \sum_{1}^{n} k_j \mu_j \in 2\pi i \mathbb{Z}, \quad k_j \in \mathbb{Z} \Rightarrow k_1 = .. = k_n = 0, \quad (4.3) \]

which does not change if we modify the choice of the $\mu_j$ by some multiples of $2\pi i$.

Let $e_1, .., e_n, f_1, .., f_n \in \mathbb{C}^{2n}$ be a basis of eigen-vectors of $B$, associated to $\mu_1, .., \mu_n$, $-\mu_1, .., -\mu_n$. Then $\sigma(e_j, e_k) = \sigma(f_j, f_k) = 0$ and $\sigma(f_j, e_k) = 0$ for $j \neq k$. We can arrange so that

\[ \sigma(f_j, e_k) = \delta_{j,k}, \]

and then we have a symplectic basis in $\mathbb{C}^{2n}$. The corresponding coordinates $x_j, \xi_j$ given by $\mathbb{C}^{2n} \ni \rho = \sum_{1}^{n}(x_j e_j + \xi_j f_j)$ will be symplectic, and in these coordinates, we get

\[ p_0(\rho) = \sum_{1}^{n} \mu_j x_j \xi_j, \quad (4.4) \]

with the Hamilton field

\[ H_{p_0} = \sum_{1}^{n} \mu_j (x_j \partial_{x_j} - \xi_j \partial_{\xi_j}). \quad (4.5) \]
If we consider $H_{p_0} : \mathcal{P}_{\text{hom}}^m \to \mathcal{P}_{\text{hom}}^m$, we see that the monomials $x^\alpha \xi^\beta$, $|\alpha| + |\beta| = m$ form a basis of eigen-vectors and

$$H_{p_0}(x^\alpha \xi^\beta) = \mu \cdot (\alpha - \beta)x^\alpha \xi^\beta,$$

(4.6)

where $\mu = (\mu_1, \ldots, \mu_n)$. The assumption (4.2) implies that $\mu \cdot (\alpha - \beta) = 0$ precisely when $\alpha = \beta$, so if we let the set of resonant polynomials $\mathcal{R}_{\text{hom}}^m \subset \mathcal{P}_{\text{hom}}^m$ be the space of linear combinations of all the $x^\alpha \xi^\beta = (x_1 \xi_1)^{\alpha_1} \cdots (x_n \xi_n)^{\alpha_n}$ with $2|\alpha| = m$, we see that $H_{p_0}$ induces a bijection from $\mathcal{P}_{\text{hom}}^m / \mathcal{R}_{\text{hom}}^m$ into itself.

We say that $u \in C^\infty(\text{neigh}(0, \mathbb{R}^{2n}))$ is resonant if its Taylor expansion at $0$ only contains resonant polynomials. Since $p_0$ is real it is easy to see that the space of resonant smooth functions is closed under complex conjugation. We also see that $u$ is resonant iff $\exists \tilde{f} \in C^\infty(\text{neigh}(0, \mathbb{C}^n))$ with $\tilde{f}(\rho) = \mathcal{O}(\rho^\infty)$ such that

$$u(x) = f(x_1 \xi_1, \ldots, x_n \xi_n) + \mathcal{O}(\rho^\infty).$$

Considering Taylor expansions it is easy to get (cf [Sj]):

**Lemma 4.1** For every $v \in C^\infty(\text{neigh}(0, \mathbb{R}^{2n}))$, $\exists u \in C^\infty(\text{neigh}(0, \mathbb{R}^{2n}))$ unique up to a resonant function, such that

$$H_{p_0} u = v + r,$$

where $r$ is resonant. If $v = \mathcal{O}(\rho^m)$, we can find $u, r$ with the same property.

As for $H_p$ we only give the corresponding existence statement:

**Lemma 4.2.** For every $v \in C^\infty(\text{neigh}(0, \mathbb{R}^{2n}))$, $\exists u \in C^\infty(\text{neigh}(0, \mathbb{R}^{2n}))$, such that

$$H_p u = v + r,$$

where $r$ is resonant. If $v = \mathcal{O}(\rho^m)$, we can choose $u, r$ with the same property.

Notice that since $p$ is real, if $v$ is real, we can take $u, r$ real. The classical Birkhoff normal form is then given in

**Proposition 4.3.** $\exists$ a smooth canonical transformation $\kappa : \text{neigh}(0, \mathbb{R}^{2n}) \to \text{neigh}(0, \mathbb{R}^{2n})$, such that $\kappa(\rho) = \rho + \mathcal{O}(\rho^2)$, and

$$p \circ \kappa = p_0 + r,$$

where $r$ is resonant and $\mathcal{O}(\rho^3)$.

**Proof.** If $q \in C^\infty(\text{neigh}(0, \mathbb{R}^{2n}); \mathbb{R})$, $q(\rho) = \mathcal{O}(\rho^{m+1})$, with $m \geq 2$, then we see that $\exp H_q(\rho) = \rho + H_q(\rho) + \mathcal{O}(\rho^{2m-1})$. Let first $q_3 = \mathcal{O}(\rho^3)$ solve $H_p(q_3) = (p - p_0) - r_3$, where $r_3(\rho) = \mathcal{O}(\rho^3)$ is resonant. Let $\kappa_2(\rho) = \exp H_{q_3}(\rho)$. Then

$$p(\kappa_2(\rho)) = p(\rho + H_{q_3}(\rho) + \mathcal{O}(\rho^3)) = p_0(\rho) + r_3(\rho) + \mathcal{O}(\rho^4) =: \tilde{p}(\rho) + r_3(\rho).$$

(We used Lemma 1.1 in [Sj],.) Now repeat the argument with $p$ replaced by $\tilde{p}$ and find $\kappa_3 = \exp H_{q_4}$ et c. Finally, we choose $\kappa$ with $\kappa(\rho) \sim \kappa_2 \circ \kappa_3 \circ \kappa(\ldots) \circ \ldots$. See for instance [Sj] for more details.
We next review the quantum normal form of a pseudodifferential operator. Let $P_0 = p_0^w(x, hD_x)$. If $A$ is (equivalent to) a pseudodifferential operator with symbol $a \sim a_0(\rho) + \hbar a_1(\rho) + \ldots$, we say that $A$ is resonant if every $a_j$ is resonant. Since $a_j$ is resonant precisely when $H_{p_0}a_j = \mathcal{O}(\rho^\infty)$ and $[P_0, A]$ has the symbol $\frac{1}{2i}H_{p_0}a$, we see that $A$ is resonant if $[P_0, A] \equiv 0$. (Later we shall also recall that $A$ is resonant precisely when it is equivalent to a function of the elementary action operators.)

With $p$ as above, let $P = P^w$ be a pseudodifferential operator with leading symbol $p$, so that $P(\rho; h) \sim p(\rho) + \hbar p_1(\rho) + \ldots$. Let $\kappa$ be as in Proposition 4.3 and let $U$ be a corresponding elliptic Fourier integral operator that we choose to be microlocally unitary near 0. Then $U^{-1}PU \equiv \tilde{P}$, where $\tilde{P}$ has the leading symbol $\tilde{p} = p_0 + r$ with $r = \mathcal{O}(\rho^3)$ resonant. We drop the tilde and continue the reduction of $"P = \tilde{P}"$ by means of conjugation with pseudodifferential operators. We look for a pseudodifferential operator $Q = Q^w$ of order 0, such that

$$e^{iQ}Pe^{-iQ} = P_0 + R,$$

where $R$ is resonant. Here the left hand side can also be written

$$e^{iQ}Pe^{-iQ} = e^{i\text{ad}_Q} = P + i\text{ad}_Q P + \frac{(i\text{ad}_Q)^2}{2}P + \ldots,$$

where the sum is asymptotic in $h$, since $\text{ad}_Q P$ is of order $\leq -k$. We look for $Q$ with symbol $q_0 + \hbar q_1 + \ldots$. The leading symbol of $i\text{ad}_Q P$ is $hH_{q_0}p = -hH_pq_0$, so we first choose $q_0$ so that

$$H_pq_0 = p_1 + r_1,$$

with $r_1$ resonant. Then the first two terms in the asymptotic expression of the operator (4.8) become resonant. The choice of $q_1$ will influence the $h^2$ term in the symbol of $e^{iQ}Pe^{-iQ}$ only via the term $i\text{ad}_Q P$, and to make the $h^2$ term resonant, leads to a new equation of the same type as (4.9). It is clear that this construction can be iterated and we find $Q$ so that (4.7) holds with $R$ resonant.

If the original symbol $P$ is self-adjoint, then the new "$P = \tilde{P} = U^{-1}PU"$ will also be self-adjoint and hence have a real-valued symbol. We can then find $Q$ in (4.7) self-adjoint, because of the observation that if $A, B$ are self-adjoint, then $i\text{ad}_A B$ is self-adjoint, so if $Q, P$ are self-adjoint, then all terms of the last expression in (4.8) have the same property. Consequently, in each step of the computation, we will encounter an equation of the form $H_pq_k = \hat{p}_k + r_k$, with $\hat{p}_k$ real-valued, and we then choose the solution $q_k$ and the resonant remainder $r_k$ to be real. This means that $e^{-iQ}$ will be unitary. If $V = Ue^{-iQ}$, we finally obtain for the original $P$, that

$$V^{-1}PV \equiv P_0 + R,$$

where $R$ is resonant of order 0 with leading symbol $r = \mathcal{O}(\rho^3)$. Summing up we have

**Theorem 4.4.** Let $p(\rho) = p_0(\rho) + \mathcal{O}(\rho^3)$ be real-valued with $p_0(\rho) = \frac{1}{2}\sigma(\rho, B\rho)$, where $B$ is symplectically anti-symmetric satisfying the non-resonance condition (4.2). Let $P$ be a pseudodifferential operator with leading symbol $p$. Then there exists an elliptic Fourier integral operator $V$ associated to the canonical transformation $\kappa$ in Proposition 4.3, such that

$$V^{-1}PV \equiv P_0 + R$$

(4.10)
where $R$ is a resonant pseudodifferential operator of order $\leq 0$ with leading symbol $= \mathcal{O}(\rho^3)$. If $P$ is self-adjoint, we can choose $V$ to be unitary.

When applying this to $U$ and $P$ in Theorem 3.2, we notice that

$$V^{-1}UV \equiv e^{-iV^{-1}PV/h}, \quad (4.11)$$

which can be viewed as a quantum normal form for our Fourier integral operator $U$.

In the appendix to this section, we review that under the non-resonance assumption (4.2), there are real symplectic coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ such that

$$p_0(\rho) = \sum_{1}^{n_{hc}} \left( \alpha_j (x_{2j-1} \xi_{2j-1} + x_{2j} \xi_{2j}) - \beta_j (x_{2j-1} \xi_{2j}-x_{2j} \xi_{2j-1}) \right)$$

$$+ \sum_{2n_{hc}+1}^{2n_{hc}+n_{hr}} \mu_j x_j \xi_j + \sum_{2n_{hc}+n_{hr}+1}^{n} \nu_j \frac{1}{2}(x_j^2 + \xi_j^2), \quad (4.12)$$

where $\nu_j \in \mathbb{R}$ are non-vanishing with distinct values of $|\nu_j|$, $\mu_j > 0$ are distinct, and $\alpha_j, \beta_j > 0$ with $\alpha_j + i\beta_j$ distinct. We have the corresponding resonant "actions":

$$\left\{ \begin{array}{l}
\iota_j = x_{2j-1} \xi_{2j-1} + x_{2j} \xi_{2j}, \\
\iota_{j+n_{hc}} = x_{2j-1} \xi_{2j} - x_{2j} \xi_{2j-1}, \quad 1 \leq j \leq n_{hc}, \\
\iota_j = x_j \xi_j, \quad 2n_{hc} + 1 \leq j \leq 2n_{hc} + n_{hr}, \\
\iota_j = \frac{1}{2}(x_j^2 + \xi_j^2), \quad 2n_{hc} + n_{hr} + 1 \leq j \leq 2n_{hc} + n_{hr} + n = n.
\end{array} \right. \quad (4.13)$$

A resonant function is one which can be written $f(\iota_1, \ldots, \iota_n) + \mathcal{O}(\rho^\infty)$ for some smooth function $f$, and using the simple functional calculus of section 2, we see that a pseudodifferential operator $R$ of order 0 is resonant iff $R \equiv F(I_1, \ldots, I_n; h)$, where $F(\iota; h)$ is a classical symbol of order 0 and $I_j = \iota_j^0(x, hD_x; h)$ is the corresponding commuting family of quantized actions. (We refer to [DiSj] and references there to the original work of B. Helffer and D. Robert, for more elaborate functional calculi.) Combining this with Theorem 4.4 and (4.11), we get

$$V^{-1}UV \equiv e^{-iF(I_1, \ldots, I_n; h)/h}, \quad (4.14)$$

where

$$F(\iota; h) \sim \sum_{0}^{\infty} F_j(\iota)h^j, \quad (4.15)$$

and

$$F_0(\iota) = \sum_{1}^{n_{hc}} (\alpha_j \iota_j - \beta_j \iota_{n_{hc}+j}) + \sum_{2n_{hc}+1}^{2n_{hc}+n_{hr}} \mu_j \iota_j + \sum_{2n_{hc}+n_{hr}+1}^{n} \nu_j \iota_j + \mathcal{O}(\iota^2), \quad (4.16)$$
Appendix. Review of the real normal form for the quadratic part.

Here we review some standard facts. See also [Ze2], [It]. Let $B : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the symplectically anti-symmetric matrix of Proposition 1.2, case b. We make the non-resonance assumption (4.2), so that all the eigen-values of $B$ are simple and $\neq 0$. Recall from section 1 that they can be grouped into families of 2 or 4 according to the following 3 cases:

**Case 1.** $\mu > 0$ is an eigen-value. Then $-\mu$ is also an eigen-value. Let $e, f$ be corresponding real eigen-vectors with $\sigma(f, e) = 1$, spanning a real symplectic space of dimension 2. A point in this subspace can be written $\rho = xe + \xi f$, so that $(x, \xi)$ become symplectic coordinates, and we get $p_0(\rho) = b(\rho) = \frac{1}{2}\sigma(\rho, B\rho) = \frac{1}{2}\sigma(xe + \xi f, \mu xe - \mu \xi f) = \mu x\xi$. The corresponding resonant action is $x\xi$.

**Case 2.** $\mu$ is an eigen-value with $\text{Re} \mu, \text{Im} \mu > 0$. Then $-\mu, \mu, -\mu, \mu$ are also eigen-values, and we let $e, f, \overline{e}, \overline{f}$ be corresponding eigen-vectors. We have

$$\sigma(e, \overline{e}) = 0, \sigma(e, \overline{f}) = 0, \sigma(f, \overline{f}) = 0,$$

and if $e$ is fixed, we can choose $f$ so that

$$\sigma(f, e) = 1.$$  \hspace{1cm} (A.1)

$e, f$ and $\overline{e}, \overline{f}$ span 2-dimensional complex symplectic subspaces that are complex conjugate to each other, while $e, \overline{e}, f, \overline{f}$ span a 4-dimensional symplectic space which is the complexification of a real symplectic space (of real dimension 4).

Writing $\rho = ze + \zeta f + w\overline{e} + \omega \overline{f}$, we get

$$b(\rho) = \frac{1}{2}\sigma(\rho, B\rho) = \mu z\zeta + \overline{\mu} w\omega.$$ \hspace{1cm} (A.2)

The resonant action terms are $z\zeta$ and $w\omega$.

To get the real canonical form, we write

$$e = \frac{1}{\sqrt{2}}(e_1 + ie_2), \quad f = \frac{1}{\sqrt{2}}(f_1 - if_2),$$

with $e_j, f_j$ real. Using this in (A.1,2), we get

$$\sigma(e_j, e_k) = \sigma(f_j, f_k) = 0, \quad \sigma(f_j, e_k) = \delta_{j,k},$$ \hspace{1cm} (A.3)

so $e_1, e_2, f_1, f_2$ is a symplectic basis in the real symplectic space mentioned above.

We also have the inverse relations

$$e_1 = \frac{1}{\sqrt{2}}(e + \overline{e}), \quad e_2 = \frac{1}{i\sqrt{2}}(e - \overline{e}),$$

$$f_1 = \frac{1}{\sqrt{2}}(f + \overline{f}), \quad f_2 = \frac{i}{\sqrt{2}}(f - \overline{f}).$$

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Write
\[ \rho = ze + \zeta f + w\varpi + \omega \bar{f} = \sum_{j=1}^{2} x_j e_j + \sum_{j=1}^{2} \xi_j f_j, \]
so that \((x, \xi)\) are real symplectic coordinates on our symplectic 4-space. Then
\[ z = \frac{1}{\sqrt{2}}(x_1 - ix_2), \quad w = \frac{1}{\sqrt{2}}(x_1 + ix_2), \]
\[ \zeta = \frac{1}{\sqrt{2}}(\xi_1 + i\xi_2), \quad \omega = \frac{1}{\sqrt{2}}(\xi_1 - i\xi_2), \]
and using this in (A.3), we get
\[ b(\rho) = \alpha(x_1 \xi_1 + x_2 \xi_2) - \beta(x_1 \xi_2 - x_2 \xi_1), \quad \text{(A.5)} \]
with \(\mu = \alpha + i\beta\).

The resonant actions can also be written
\[ z\zeta = \frac{1}{2}((x_1 \xi_1 + x_2 \xi_2) + i(x_1 \xi_2 - x_2 \xi_1)) \]
\[ w\omega = \frac{1}{2}((x_1 \xi_1 + x_2 \xi_2) - i(x_1 \xi_2 - x_2 \xi_1)). \]

The (resonant) real-valued functions (on the real symplectic 4-space above) which only depend on \(z\zeta, w\omega\) are precisely the functions of \(x_1 \xi_1 + x_2 \xi_2, x_1 \xi_2 - x_2 \xi_1\) modulo \(O(\rho^\infty)\). Notice that these two functions Poisson commute.

**Case 3.** \(\mu \neq 0\) is an eigen-value with \(\text{Re} \mu = 0\). Then \(\overline{\mu} = -\mu\) is also an eigen-value. If \(e\) is an eigen-vector corresponding to \(\mu\), then \(\overline{e}\) will be an eigen-vector corresponding to \(\overline{\mu}\) and \(\sigma(e, \overline{e}) \in i\mathbb{R} \setminus \{0\}\). Possibly after permuting \(\mu\) and \(-\mu\) and after normalization, we can assume that
\[ \frac{1}{i} \sigma(e, \overline{e}) = 1. \]

\(e, \overline{e}\) span a 2-dimensional symplectic subspace which is the complexification of a corresponding real 2-dimensional space. Let \(f = i\overline{e}\), so that \(\sigma(f, e) = 1\). Writing \(\rho = ze + \zeta f\), we see that \(z, \zeta\) are complex symplectic coordinates, and \(b(\rho) = \mu z\zeta\).

Write \(e = \frac{1}{\sqrt{2}}(e_1 + ie_2)\) with \(e_1, e_2\) real, so that \(f = \frac{1}{\sqrt{2}}(e_2 + ie_1)\). Then we see that \(e_1, e_2\) is a real symplectic basis. Also notice that
\[ e_1 = \frac{1}{\sqrt{2}}(e - if), \quad e_2 = \frac{1}{\sqrt{2}}(f - ie), \]
so if
\[ \rho = ze + \zeta f = xe_1 + \xi e_2, \]
we see that \(x, \xi\) are real symplectic coordinates on our symplectic 2-space and
\[ b(\rho) = \frac{\mu}{2i}(x^2 + \xi^2). \quad \text{(A.6)} \]
The resonant action \( z \zeta \) becomes \( \frac{1}{2}(x^2 + \xi^2) \) times a constant factor.

5. Parameter dependent case.

In some applications (for instance when dealing with an energy dependent monodromy operator ([SjZw])) our Fourier integral operator will depend smoothly on some real parameter \( s \), and then it may happen that the non-resonance condition is fulfilled for one value of \( s \), say for \( s = 0 \) but not everywhere in any neighborhood of that point. In this section we show that the previous results still apply if we require them to hold only to infinite order with respect to \( s \) at \( s = 0 \). We do this by checking the earlier constructions step by step.

Let \( A = A_s \) be a real symplectic \( 2n \)-matrix depending smoothly on \( s \in \text{neigh}(0, \mathbb{R}) \), such that \( A_s \) satisfies the assumptions of Proposition 1.2, case b. Then \( \log A_0 \) can be extended to a smooth family of real matrices \( B_s = \log A_s \) with \( e^{B_s} = A_s \), \( \sigma B_s + B_s = 0 \). (The construction of \( B = \log A \) can be reformulated by writing \( B = f(A) \), where \( f(z) \) is a suitable holomorphic branch of the logarithm, defined near the spectrum of \( A \). We take \( B_s = f(A_s) \) for the same \( f \).)

Let \( \kappa^s(\rho), s \in \text{neigh}(0, \mathbb{R}) \) be a smooth family of canonical transformations with \( \kappa^s(0) = 0 \) and assume that \( \kappa = \kappa^0 \) fulfills the assumptions of Theorem 1.3, so that

\[
\kappa^0(0) = \exp H_{p^0}(\rho) + \mathcal{O}(\rho^\infty),
\]

where \( p^0 \) is unique modulo \( \mathcal{O}(\rho^\infty) \) once its quadratic part \( p^0_0 \) has been fixed in accordance with Proposition 1.2.b. We want to extend \( p^0 \) to a smooth real-valued family \( p^s \), with

\[
\kappa^s(\rho) = \exp H_{p^s}(\rho) + \mathcal{O}((s, \rho)\infty).
\]

Define \( q^s = \mathcal{O}(\rho^2) \) as in (1.23), so that

\[
(\kappa^s)^* \partial_s \kappa^s = H_{q^s},
\]

and consider the problem analogous to (1.24):

\[
q^s(\rho) = \int_0^1 \partial_s p^s \circ \exp t H_{p^s}(\rho) dt + \mathcal{O}((s, \rho)\infty).
\]

Putting \( s = 0 \), we get a unique solution \( (\partial_s p^s)_{s=0} = \mathcal{O}(\rho^2) \) modulo \( \mathcal{O}(\rho^\infty) \). If we differentiate \( k \) times we get

\[
\int_0^1 (\partial_s^{k+1} p^s) \circ \exp t H_{p^s}(\rho) dt = \partial_s^k q^s(\rho) + F_k(p^s, \ldots, \partial_s^k p^s, \rho) + \mathcal{O}((s, \rho)\infty),
\]

and if \( p^0, \ldots, (\partial_s^k p^s)_{s=0} = \mathcal{O}(\rho^2) \) already have been determined, we get \( (\partial_s^{k+1} p^s)_{s=0} = \mathcal{O}(\rho^2) \) from this equation. It is then clear that (5.4) has a solution which is unique modulo \( \mathcal{O}((s, \rho)\infty) \). Let \( \tilde{\kappa}^s = \exp H_{p^s} \). Then

\[
(\kappa^s)^* \partial_s \kappa^s = (\tilde{\kappa}^s)^* (\partial_s \tilde{\kappa}^s) + \mathcal{O}((s, \rho)\infty), \quad \tilde{\kappa}^0 = \kappa^0.
\]
and as in the proof of Theorem 1.3, we see that (5.2) holds.

We next look at corresponding families of Fourier integral operators and we start by extending the equivalence notions of section 2 to the parameter dependent case. If $V_j \subset \mathbb{R}^n$ are open neighborhoods of 0 and $I_j \subset \mathbb{R}$ are open intervals containing 0, we say that $v_j \in C^\infty(I_j \times V_j)$, $j = 1, 2$ are equivalent if they are equivalent in the sense of section 2 with $V_j$ there replaced by $I_j \times V_j$. Similarly, we define equivalence for symbols $a^{(j)} \in S^0_{cl}(I_j \times V_j)$ and the corresponding notion for pseudodifferential operators.

Two canonical transformations $\kappa_{j,s} : \text{neig}(0, \mathbb{R}^{2n}) \to \text{neig}(0, \mathbb{R}^{2n})$ depending smoothly on $s \in \text{neig}(0, \mathbb{R})$ with $\kappa_{j,s}(0) = 0$, are said to be equivalent, if $\kappa_{1,s}(\rho) = \kappa_{2,s}(\rho) + O((s, \rho)^\infty)$. (We write $\equiv$ for the parameter version of equivalence also.) Again $\kappa_{1,s} \equiv \kappa_{2,s}$ iff $\kappa_{1,s}^{-1} \equiv \kappa_{2,s}^{-1}$.

If we parametrize $\kappa_s$ by a non-degenerate phase $\phi_s(x, y, \theta)$ depending smoothly on $s$ (and with $(x, y, \theta) = (0, 0, 0)$ corresponding to $\kappa_s(0) = 0$) then $\kappa_s$ is equivalent to smooth family $\kappa_s$ iff we can parametrize $\kappa_s$ by a phase $\tilde{\phi}_s(x, y, \theta)$ which is equivalent to $\phi_s(x, y, \theta)$.

Consider a family $U_s = I(a_s, \phi_s)$ of elliptic Fourier integral operators as in (2.4), associated to a smooth family of canonical transformations $\kappa_s$ as above. Assume $\phi_s(0, 0, 0) = 0$. We say that $U_s$ is equivalent to a second family $\tilde{U}_s$, if $\tilde{U}_s$ has an associated family of canonical transformations $\tilde{\kappa}_s$ with $\tilde{\kappa}_s(0) = 0$ and we can represent $\tilde{U}_s = I(\tilde{a}_s, \tilde{\phi}_s)$, with $\tilde{\phi}_s(0, 0, 0) = 0$ and with $\tilde{a}_s \equiv a_s$, $\tilde{\phi}_s \equiv \phi_s$ in the sense of families. (We then have $\kappa_s \equiv \kappa_s$.)

Let $U_s, V_s$ be two families of elliptic Fourier integral operators as above, with $U_0 = V_0$ and $U_s^{-1}hD_sU_s \equiv V_s^{-1}hD_sV_s$ (in the sense of families). Then $U_s \equiv V_s$. In fact, let $\kappa_s, \tilde{\kappa}_s$ be the associated canonical transformations and write $U_s^{-1}hD_sU_s = -P_s$, so that $P_s$ is a smooth family of pseudodifferential operators of order 0 with real leading symbol $p_s(\rho) = O(\rho^2)$. Then $\kappa_s$ satisfies $\partial_s \kappa_s(\rho) = (\kappa_s)_*(H_{p_s}(\rho))$. Similarly $\partial_s \tilde{\kappa}_s(\rho) = (\tilde{\kappa}_s)_*(H_{\tilde{p}_s}(\rho))$, where $\tilde{p}_s \equiv p_s$ and $\tilde{\kappa}_0 = \kappa_0$, so it follows that $\kappa_s \equiv \kappa_s$.

Without loss of generality, we may assume that $\tilde{\kappa}_s = \kappa_s$. Let $W_s$ be some fixed elliptic family of Fourier integral operators associated to $\kappa_s$, then

$$U_s \equiv A_sW_s, \ V_s \equiv B_sW_s, \ (5.5)$$

where $A_s, B_s$ are smooth families of pseudodifferential operators. We get

$$U_s^{-1}hD_sU_s \equiv W_s^{-1}(A_s^{-1}(hD_sA_s))W_s + W_s^{-1}hD_sW_s, \ V_s^{-1}hD_sV_s \equiv W_s^{-1}(B_s^{-1}(hD_sB_s))W_s + W_s^{-1}hD_sW_s.$$  

It follows that

$$A_s^{-1}hD_sA_s \equiv B_s^{-1}hD_sB_s, \ A_0 \equiv B_0,$$

and then as in the proof of Lemma 3.1, that $A_s \equiv B_s$ and hence that

$$U_s \equiv V_s. \ (5.6)$$

We can now prove

**Theorem 5.1.** Let $\kappa_s : \text{neig}(0, \mathbb{R}^{2n}) \to \text{neig}(0, \mathbb{R}^{2n})$, $s \in \text{neig}(0, \mathbb{R})$ be a smooth family of canonical transformations with $\kappa_s(0) = 0$ and assume that $\kappa_0$ satisfies the assumptions of Theorem 1.3. Let $U_s = I(a_s, \phi_s)$ be a corresponding smooth family of elliptic
Fourier integral operators of order 0 with \( \phi_s(0,0,0) = 0 \). Choose \( \mu_j \) and \( p_0 = p_0^0 \) as there, so that (1.32) holds. Then by that theorem and Theorem 3.2, there exists a real-valued smooth function \( p^0 = p_0^0 + \mathcal{O}(\rho^3) \) (uniquely determined mod (\( \mathcal{O}(\rho^\infty) \)), such that \( \kappa^0(\rho) = \exp H_{\rho_0}(\rho) + \mathcal{O}(\rho^\infty) \) and a corresponding pseudodifferential operator \( P^0 \) with leading symbol \( p^0 \), so that \( U^0 \equiv e^{-iP^0/h} \). (\( P^0 \) is uniquely determined up to equivalence and an integer multiple of \( 2\pi h \).)

\( P^0 \) can be extended to a smooth family of pseudodifferential operators \( P^s \) so that

\[
U^s \equiv e^{-iP^s/h} \quad (5.7)
\]

in the sense of families. Moreover, the family \( P^s \) is unique up to equivalence for families and an integer multiple of \( 2\pi h \). The leading symbol \( p^s \) satisfies (5.2).

**Proof.** Let \( Q^s \equiv (U^s)^{-1}hD_s U^s \) be the logarithmic derivative. We first look for \( P^s \) solving

\[
Q^s \equiv -\int_0^1 e^{itP^s/h}(\partial_s P^s)e^{-itP^s/h} dt. \quad (5.8)
\]

As in section 3, we first determine \( (\partial_s P^s)_{s=0} \). Then we can write

\[
hD_s(e^{-itP^s/h}) = R_{t,s}e^{-itP^s/h}, \quad (5.9)
\]

where \( R_{t,s} \) is a well-defined smooth family of 0th order pseudodifferential operators for \( 0 \leq t \leq 1, s = 0 \). We can then differentiate (5.8) once with respect to \( s \) and get for \( s = 0 \):

\[
\partial_s Q^s \equiv \int_0^1 e^{itP^s/h}\left[\frac{i}{h}R_{t,s}, \partial_s P^s\right]e^{-itP^s/h} dt - \int_0^1 e^{itP^s/h}(\partial^2_s P^s)e^{-itP^s/h} dt. \quad (5.10)
\]

From this we determine \( (\partial^2_s P^s)_{s=0} \). Then \( (\partial_s R_{t,s})_{s=0} \) is well-defined in (5.9) and we can differentiate (5.10) once more et c and determine \( (\partial^k_s P^s)_{s=0} \) for all \( k \). This means that we get a solution of (5.8) and we also see that this solution is unique modulo equivalence for families. From this we also get the uniqueness of \( P^s \) in the theorem, for if \( P^s \) is as in the theorem, then it has to satisfy (5.8).

It remains to show that \( P^s \) in (5.8) solves (5.7). For that, we put

\[
V^s = e^{-itP^s/h},
\]

so that by (5.8) and the earlier arguments of section 3:

\[
Q^s \equiv (V^s)^{-1}(hD_s V^s).
\]

Since \( Q^s \) is also the log-derivative of the family \( U^s \) and \( U^0 = V^0 \), we conclude as in (5.6), that \( U^s \equiv V^s \), and the proof is complete.

We end by indicating how to extend the Birkhoff normal form to the parameter dependent case. Let

\[
P^{(s)} \sim p^{(s)}(\rho) + h p_1^{(s)}(\rho) + \ldots, \quad (s, \rho) \in \text{neigh}(0, \mathbf{R} \times \mathbf{R}^{2n}), \quad (5.11)
\]
be smooth in $(s, \rho)$ with $p^{(s)}(\rho)$ real-valued. Assume that

$$p_{0}^{(0)}(\rho) = \frac{1}{2} \langle (p^{(0)})''(0) \rho, \rho \rangle$$

satisfies the non-resonance condition (4.2). Then in suitable complex linear symplectic coordinates, we have

$$p_{0}^{(0)}(\rho) = \sum_{1}^{n} \mu_{j} x_{j} \xi_{j}, \quad (5.12)$$

and if we allow the coordinates to depend smoothly on $s$, we get

$$p_{0}^{(s)}(\rho) = \sum_{1}^{n} \mu_{j}^{(s)} x_{j} \xi_{j}, \quad (5.13)$$

where $\mu_{j}^{(s)}$ depend smoothly on $s$ and $\mu_{j}^{(0)} = \mu_{j}$. From the appendix of section 4, it follows that we can find a real linear canonical transformation $\kappa_{0}^{(s)}$, depending smoothly on $s$ such that

$$p_{0}^{(s)} \circ \kappa_{0}^{(s)}(\rho) = \sum_{1}^{n} \mu_{j}^{(s)} x_{j} \xi_{j}, \quad (5.14)$$

where the coordinates $(x, \xi)$ are now independent of $s$. After composing $P^{(s)}$ with $\kappa_{0}^{(s)}$ we can assume that we have (5.13) with coordinates $x, \xi$ that are independent of $s$. Notice however that the non-resonance condition (4.2) may be violated for $s \neq 0$ arbitrarily close to 0.

We say that a function $r = r_{s}(x, \xi) \in C^{\infty}(\text{neigh}(0, \mathbb{R} \times \mathbb{R}^{2n}))$ is resonant if $H_{p_{0}^{(0)}} \equiv 0$ in the sense of families. Notice that this definition does not change, if we replace $H_{p_{0}^{(0)}}$ by $H_{p_{0}^{(s)}}$ (provided we have (5.13) in $s$-independent coordinates). Also notice $r$ is resonant if we have $r_{s} \equiv f_{s}(x_{1} \xi_{1}, .., x_{n} \xi_{n})$ for some smooth family $f_{s}$. The extension of this definition to the case of pseudodifferential operators is immediate. We next extend Lemma 4.2:

**Lemma 5.2.** For every $v = u^{(s)}(x, \xi) \in C^{\infty}(\text{neigh}(0, \mathbb{R} \times \mathbb{R}^{2n}))$ there exist $u = u^{(s)}$ and $r = r^{(s)}$ in $C^{\infty}(\text{neigh}(0, \mathbb{R} \times \mathbb{R}^{2n}))$ with $r^{(s)}$ resonant, such that

$$H_{p^{(s)}} u^{(s)} = v^{(s)} + r^{(s)}. \quad (5.15)$$

If $v = O(s^{k} \rho^{m})$, then we can choose $u, r$ with the same property.

**Proof.** Lemma 4.2 gives a solution $u^{(0)}, r^{(0)}$ for $s = 0$. Differentiate (5.15) with respect to $s$:

$$H_{p^{(s)}} \partial_{s} u^{(s)} = \partial_{s} v^{(s)} - \{ \partial_{s} p^{(s)}, u^{(s)} \} + \partial_{s} r^{(s)}, \quad (5.16)$$

and put $s = 0$. Let $(\partial_{s} u^{(s)})_{s=0}, (\partial_{s} r^{(s)})_{s=0}$ be the corresponding solutions to this equation, given by Lemma 4.2, then differentiate (5.16) etc. In this way, we get the Taylor series expansion of $u^{(s)}, r^{(s)}$ with respect to $s$, and the lemma follows.

#
Proposition 5.3. There exists a smooth family of canonical transformations \( \kappa_s : \text{neigh}(0, \mathbb{R}^{2n}) \to \text{neigh}(0, \mathbb{R}^{2n}) \) with \( \kappa_s(\rho) = \rho + \mathcal{O}(\rho^2) \) and

\[
p^{(s)} \circ \kappa_s = p^{(s)}_0 + r^{(s)},
\]

where \( r^{(s)} \) is resonant and \( \mathcal{O}(\rho^3) \).

The proof is essentially identical to that of Proposition 4.3. The treatment of the operators goes through without any changes, and we get

Theorem 5.4. Let \( P^{(s)} \) denote also the \( h \)-Weyl quantization of the symbol in (5.11). Let \( \tilde{p}^{(s)}_0 \) be given by (5.13) in the coordinates for which (5.12) holds and let \( \tilde{P}^{(s)}_0 \) be the corresponding quantization. Then there exists a smooth family of elliptic Fourier integral operators \( V = V^{(s)} \) associated to \( \kappa_0 \circ \kappa_s \) (cf (5.14) and Proposition 5.3) such that

\[
(V^{(s)})^{-1} P^{(s)} V^{(s)} \equiv P^{(s)}_0 + R^{(s)}
\]

in the sense of families, where \( R^{(s)} \) is a resonant pseudodifferential operator of order \( \leq 0 \) and with leading symbol \( = \mathcal{O}(\rho^3) \). If \( P^{(s)} \) is self-adjoint, then we can choose \( V^{(s)} \) unitary (microlocally near 0).

References.

[Bi] G.D. Birkhoff, Dynamical Systems, volume IX. A.M.S. Colloquium Publications, New York, 1927.

[DiSj] M. Dimassi, J. Sjőstrand, Spectral asymptotics in the semi-classical limit, London Math Soc. Lecture Note Ser. 268, Cambridge Univ. Press, 1999.

[Fr] J.-P. Françoise, Propriétés de généricité des transformations canoniques, pp 216–260 in Geometric dynamics. Proceedings, Rio de Janeiro, 1981, J. Palis Jr, editor, Springer LNM 1007.

[Gui] V. Guillemin, Wave trace invariants, Duke Math. J., 83(2)(1996), 287–352.

[Gus] F.G. Gustavsson, On constructing formal integrals of a Hamiltonian system near an equilibrium point, Astrophys. J., 71(1966), 670–686.

[Hö] L. Hörmander, The analysis of linear partial differential operators, I–IV, Grundlehren, Springer, 256(1983), 257(1983), 274(1985), 275(1985).

[Ita] A. Iantchenko, La forme normale de Birkhoff pour un opérateur intégral de Fourier, Asymptotic Analysis, 17(1)(1998), 71–92.

[It] H. Ito, Integrable symplectic maps and their Birkhoff normal forms, Tôhoku Math.J., 49(1997), 73–114.

[MeHa] K.R. Meyer, G.R. Hall, Introduction to Hamiltonian dynamical systems and the N-body problem, Applied Math. Sci. 90, Springer Verlag, 1992.

[Po1] G. Popov, Invariant torii, effective stability, and quasimodes with exponentially small error terms I. Birkhoff normal forms, Ann. Henri Poincaré 1(2)(2000), 223–248.
[Po2] G. Popov, Invariant torii, effective stability, and quasimodes with exponentially small error terms II. Quantum Birkhoff normal forms, Ann. Henri Poincaré 1(2)(2000), 249–279.

[Sj] J. Sjöstrand, Semi-excited states in non-degenerate potential wells, Asymptotic Analysis, 6(1992), 29–43.

[SjZw] J. Sjöstrand, M. Zworski, Quantum monodromy and semi-classical trace formulae, J. Math. Pures et Appl., to appear.

[St] S. Sternberg, Infinite Lie groups and formal aspects of dynamics, J. of Math. and Mechanics, 10(3)(1961), 451–476.

[Ze1] S. Zelditch, Wave invariants at elliptic closed geodesics, Geom. Funct. Anal., 7(1997), 145–213.

[Ze2] S. Zelditch, Wave invariants for non-degenerate closed geodesics, Geom. Funct. Anal., 8(1998), 179–217.

[Ze3] S. Zelditch, Spectral determination of analytic bi-axisymmetric plane domains, Geom. Funct. Anal., 10(3)(2000), 628–677.