The characteristic function and entanglement of optical evolution

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Received 10 August 2006, in final form 14 September 2006
Published 27 October 2006
Online at stacks.iop.org/JPhysB/39/4605

Abstract
The master equation of the quantum optical density operator is transformed to the equation of the characteristic function. Parametric amplification and amplitude damping as well as phase damping are considered. The solution for the most general initial quantum state is obtained for parametric amplification and amplitude damping. The purity of the one-mode Gaussian system and the entanglement of the two-mode Gaussian system are studied.

1. Introduction
Quantum information with continuous variables (CV) [1, 2] is a flourishing field, as shown by the spectacular implementations of deterministic teleportation schemes [3–6], quantum key distribution protocols [7], entanglement swapping [5, 8], dense coding [9], quantum state storage [10] and quantum computation [11] processes in quantum optical settings. The crucial resource enabling a better-than-classical manipulation and processing of information is CV entanglement. In all such practical instances the information and entanglement contained in a given quantum state of the system, so precious for the realization of any specific task, is constantly threatened by the unavoidable interaction with the environment. Such an interaction entangles the system of interest with the environment, causing some amount of information to be scattered and lost in the environment. The overall process, corresponding to a non-unitary evolution of the system, is commonly referred to as decoherence [4, 5]. In this work we study the decoherences of the general states of continuous variable systems whose evolutions are ruled by optical master equations. We will consider the parametric amplification, amplitude damping and phase damping [12] of a general quantum CV state in the fashion of the quantum characteristic function. The main starting point for this research work was the result of Lindblad’s bounded generator of a completely positive quantum dynamical semigroup [13]. The quantum characteristic function has been used to treat the amplitude damping of one-mode squeezed states [14].
2. Time evolution of characteristic function

The density matrix obeys the following master equation [12, 13, 15]

\[ \frac{d\rho}{dt} = -\frac{i}{\hbar} [H, \rho] + (\mathcal{L}_1 + \mathcal{L}_2)\rho \]  

(1)

with the quadratic Hamiltonian

\[ H = \hbar \sum_{jk} \left( \eta_{jk} a_j^\dagger a_k^\dagger - \eta_{jk}^* a_j a_k \right) \]  

(2)

where \( \eta \) is a complex symmetric matrix. In the single-mode case, this Hamiltonian describes two-photon downconversion from an undepleted (classical) pump [15]. The full multi-mode model describes quasi-particle excitation in a BEC within the Bogoliubov approximation [16]. This item represents the parametric amplifier. While the amplitude damping is described by \( \mathcal{L}_1 \):

\[ \mathcal{L}_1\rho = \sum_j \Gamma_j (\bar{\pi}_j + 1)L[a_j]\rho + \bar{\pi}_j L[a_j^\dagger]\rho - w_j^* M[a_j]\rho - w_j M[a_j^\dagger]\rho, \]  

(3)

where the Lindblad super-operators are defined as \( L[\hat{o}]\rho \equiv 2\hat{o}\rho\hat{o}^\dagger - \hat{o}^\dagger\hat{o}\rho - \rho\hat{o}^\dagger\hat{o} \) and \( M[\hat{o}]\rho \equiv \hat{o}\rho - \rho\hat{o} \). The requirement of positivity of the density operator imposes the constraint \( |w_j|^2 \leq \bar{\pi}_j (\bar{\pi}_j + 1) \). At thermal equilibrium, i.e. \( w_j = 0 \), \( \bar{\pi}_j \) is equal to the average thermal photon number of the environment. If \( w_j \neq 0 \), then the bath \( j \) is said to be 'squeezed' [17]. \( \mathcal{L}_2 \) describes the phase damping

\[ \mathcal{L}_2\rho = \sum_j \gamma_j L[a_j^\dagger a_j]\rho. \]  

(4)

We now transform the density operator master equation to the diffusion equation of the characteristic function. We use the characteristic function because it is more convenient for our problem. One may use Glauber’s P-representation instead, but Glauber’s P-representation does not exist for some of the states (e.g. [14, 18]), although the generalized positive P-representation does always exist [19]. Any quantum state can be equivalently specified by its characteristic function. Every operator \( A \in \mathcal{B}(\mathcal{H}) \) is completely determined by its characteristic function \( \chi_A \) [20], where \( D(\mu) = \exp(\mu a^\dagger - \mu^* a) \) is the displacement operator, with \( \mu = [\mu_1, \mu_2, \ldots, \mu_s]^T \) and the total number of modes is \( s \). It follows that \( A \) may be written in terms of \( \chi_A \) as [21] \( A = \int \prod \frac{d\mu}{\pi} \chi_A(\mu) D(-\mu) \). The density matrix \( \rho \) can be expressed with its characteristic function \( \chi \). \( \chi = \text{tr}[\rho D(\mu)] \). Multiplying \( D(\mu) \) by the master equation then taking trace, the master equation of the density operator will be transformed to the diffusion equation of the characteristic function. It should be noted that the complex parameters \( \mu_j \) are not a function of time; thus \( \frac{d\chi}{dt} = \text{tr}[\frac{d \rho}{d t} D(\mu)] \), the parametric amplification part in the form of a characteristic function will be [15]

\[ \frac{1}{2} \text{tr} \left\{ \sum_{jk} \left[ \eta_{jk} a_j^\dagger - \eta_{jk}^* a_j a_k \right] D(\mu) \right\} = -\sum_{jk} \left( \eta_{jk} \mu_j^* \frac{\partial \chi}{\partial \mu_k} + \eta_{jk}^* \mu_j \frac{\partial \chi}{\partial \mu_k^*} \right). \]  

(5)

The master equation can be transformed to the diffusion equation of the characteristic function, that is,
\[ \frac{\partial \chi}{\partial t} = -\sum_{jk} \left( \eta_{jk} \mu_j \frac{\partial \chi}{\partial \mu_k} + \eta_{jk}^* \mu_j^* \frac{\partial \chi}{\partial \mu_k^*} \right) - \frac{1}{2} \sum_j \Gamma_j \left\{ |\mu_j|^2 \frac{\partial \chi}{\partial |\mu_j|^2} + (2\pi \Gamma_j + 1) |\mu_j|^2 \right. \\
\left. - w_j^* \mu_j^2 - w_j \mu_j^2 \right\} + \frac{1}{2} \sum_j \gamma_j \frac{\partial^2 \chi}{\partial \theta_j^2}. \]  

Here we denote \( \mu_j \) as \( |\mu_j| e^{i\theta_j} \), and we should take care that the variables are \( \mu_j \) and \( \mu_j^* \) in the amplification while they are \( |\mu_j|, \theta_j \) in the damping.

3. Solutions of some special cases

Firstly, let us consider \( \Gamma_j = \gamma_j = 0 \) for all \( j \), the solution to the amplification is

\[ \chi(\mu, \mu^*, t) = \chi(\nu, \nu^*, 0), \]  

with

\[ (\nu, \nu^*) = (\mu, \mu^*) \begin{pmatrix} \cosh(|\eta|t) & -\eta^* \sinh(|\eta|t) \frac{|\eta|}{|\eta|} \\ -\sinh(|\eta|t) \frac{|\eta|}{|\eta|} & \cosh(|\eta|t) \end{pmatrix}, \]

where the matrix \( \cosh \) and \( \sinh \) functions are defined as \[22\]

\[ \cosh|\xi| = I + \frac{1}{2!} \xi \xi^* + \frac{1}{4!} (\xi \xi^*)^2 + \cdots, \]

\[ \sinh|\xi| \frac{|\xi|}{|\xi|} = \xi + \frac{1}{3!} \xi \xi^* \xi + \frac{1}{5!} (\xi \xi^*)^2 \xi + \cdots. \]  

Then suppose \( \eta = 0 \) and \( \gamma_j = 0 \) for all \( j \), the solution to the amplitude damping equation of \( \chi \) is

\[ \chi(\mu, t) = \chi(\mu e^{-\gamma_j t}, 0) \exp \left\{ -\sum_j (1 - e^{-\gamma_j t}) \left[ (\pi_j + \frac{1}{2}) |\mu_j|^2 - \frac{1}{2} w_j^* \mu_j^2 - \frac{1}{2} w_j \mu_j^2 \right] \right\}, \]

where \( \mu e^{-\gamma_j t} \) is the abbreviation of \( (\mu_1 e^{-\gamma_1 t}, \mu_2 e^{-\gamma_2 t}, \ldots, \mu_s e^{-\gamma_s t}) \). Next, suppose \( \eta = 0 \) and \( \Gamma_j = 0 \) for all \( j \), the solution to the phase equation of \( \chi \) then will be

\[ \chi(\mu, \mu^*, t) = \int dx \chi(\mu e^{ix}, \mu^* e^{-ix}, 0) \prod_j (2\pi \gamma_j t)^{-1/2} \exp \left( -\frac{x^2 j}{2 \gamma_j t} \right), \]

where \( \mu e^{ix} \) is the abbreviation of \( (\mu_1 e^{ix_1}, \mu_2 e^{ix_2}, \ldots, \mu_s e^{ix_s}) \). The simultaneous amplitude and phase damping (\( \eta = 0 \)) for any initial characteristic function is (for \( w_j = 0 \)) \[23\]

\[ \chi(\mu, \mu^*, t) = \int dx \chi(\mu e^{-\gamma_j t} - i \xi_j, \mu^* e^{-\gamma_j t} - i \xi_j, 0) \prod_j (2\pi \gamma_j t)^{-1/2} \]

\[ \times \exp \left[ -\frac{x^2 j}{2 \gamma_j t} - (1 - e^{-\gamma_j t}) \left( \pi_j + \frac{1}{2} \right) |\mu_j|^2 \right]. \]

The density matrix then can be obtained by making use of the operator integral.
4. The parametric amplifier and the amplitude damping

The diffusion equation (6) now has $\gamma_j = 0$ for all $j$. Suppose the solution to the diffusion equation is

$$\chi(\mu, \mu^*, t) = \chi(v, v^*, 0) \exp \left[ \frac{1}{2} (v, -v^*) \begin{pmatrix} 1 & \beta \\ \beta^* & \alpha^* \end{pmatrix} (v, -v)^T \right]$$

$$= \frac{1}{2} (\mu, -\mu^*) \begin{pmatrix} 1 & \beta^* \\ \beta & \alpha \end{pmatrix} (\mu^*, -\mu)^T,$$

where $v = \mu M + \mu^* N$, with $M$ and $N$ being time varying matrices. $\alpha$ and $\beta$ are constant matrices and $\alpha^j = \alpha, \beta = \beta^T$. $M$ and $N$ are the solutions of the following matrix equations:

$$\frac{dM}{dt} = -\eta^* N - \frac{\Gamma}{2} M,$$  \hspace{1cm} (14)

$$\frac{dN}{dt} = -\eta M - \frac{\Gamma}{2} N,$$  \hspace{1cm} (15)

where $\Gamma = \text{diag}(\Gamma_1, \Gamma_2, \ldots, \Gamma_s)$. $\alpha$ and $\beta$ are the solution of the following matrix equations

$$2\eta^* + 2\alpha^* \eta - \Gamma^* \beta - \beta^* \Gamma + \Gamma w + w \Gamma = 0,$$  \hspace{1cm} (16)

$$\Gamma^* \alpha + \alpha \Gamma - 2\eta^* \beta - 2\beta^* \eta - \Gamma (\eta + \frac{1}{2}) - (\eta + \frac{1}{2})^* \Gamma = 0,$$  \hspace{1cm} (17)

where $w = \text{diag}(w_1, w_2, \ldots, w_s), \pi = \text{diag}(\pi_1, \pi_2, \ldots, \pi_s)$. The constant matrices $\alpha$ and $\beta$ can be worked out as the solution of linear algebraic equations (16) and (17). What is left is to solve matrix equations (14) and (15). There are two situations in which the equations are solvable. The first case is that all the modes undergo the same amplitude damping, that is $\Gamma_1 = \Gamma_2 = \cdots = \Gamma_s$, thus $\Gamma = \Gamma_1 I_s$. $\Gamma$ commutes with any matrix. Equations (14) and (15) have the solution

$$M = e^{-i\Gamma_1 t} \cosh^{\ast}(|\eta| t),$$

$$N = -e^{-i\Gamma_1 t} \sinh(|\eta| t)|\eta|.$$

The solution to the one-mode situation is simple. In the two-mode situation, $M$ and $N$ can be further simplified. As $\eta$ is a symmetric matrix, we can express it as $\eta = \eta_0 \sigma_0 + \eta_1 \sigma_1 + \eta_2 \sigma_2$, where $\sigma_0 = I_2, \sigma_1, \sigma_2 (i = 1, 2, 3)$ are Pauli matrices. The $\sigma_2$ term is cancelled by the symmetry of $\eta$. By using the algebra of Pauli matrices, we arrive at the following results:

$$\cosh(|\eta| t) = \sigma_0 \cosh(\sqrt{(C + A)/2} t) \cosh(\sqrt{(C - A)/2} t)$$

$$+ \sinh(\sqrt{(C + A)/2} t) \sinh(\sqrt{(C - A)/2} t) \vec{\sigma} \cdot \vec{b},$$

$$\sinh(|\eta| t) = \frac{1}{2} \sigma_0 [\sinh(\sqrt{(C + B) t})/\sqrt{(C + B)} + \sinh(\sqrt{(C - B) t})/\sqrt{(C - B)}] \eta$$

$$+ \frac{1}{2} [\sinh(\sqrt{(C + B) t})/\sqrt{(C + B)} - \sinh(\sqrt{(C - B) t})/\sqrt{(C - B)}] \vec{\sigma} \cdot \vec{b} \eta.$$  \hspace{1cm} (20)

Here $C = |\eta_0|^2 + |\eta_1|^2 + |\eta_3|^2, A = |\eta_3^2 - \eta_1^2 - \eta_2^2|, B = \sqrt{C^2 - A^2}$. $\vec{b}$ is a unit vector and it is equal to $(\eta_0 \eta_1^2 + \eta_1 \eta_0^2, i\eta_3 \eta_1^3 - i\eta_3 \eta_1, \eta_0 \eta_1^3 + \eta_3 \eta_0^3)/B$. Thus the characteristic function is
simplified. If the initial state is Gaussian, the correlation matrix of the time-dependent state can be obtained explicitly.

The second case is that $\eta$ is a real matrix while the amplitude damping can be different for each mode. The solutions of equations (14) and (15) will be simplified. If the initial state is Gaussian, the correlation matrix of the time-dependent state

$$M = \frac{1}{2} \left[ \exp \left( -\eta t - \frac{\Gamma t}{2} \right) + \exp \left( \eta t - \frac{\Gamma t}{2} \right) \right],$$

$$N = \frac{1}{2} \left[ \exp \left( -\eta t - \frac{\Gamma t}{2} \right) - \exp \left( \eta t - \frac{\Gamma t}{2} \right) \right].$$

For a two-mode situation, these $M$ and $N$ can be simplified to

$$M = \frac{1}{2} e^{-C t} \left[ \cosh(B_1 t) \sigma_0 - \sinh(B_1 t) \vec{\sigma} \cdot \vec{b}_1 \right] + \frac{1}{2} e^{-C t} \left[ \cosh(B_2 t) \sigma_0 - \sinh(B_2 t) \vec{\sigma} \cdot \vec{b}_2 \right],$$

$$N = \frac{1}{2} e^{-C t} \left[ \cosh(B_1 t) \sigma_0 - \sinh(B_1 t) \vec{\sigma} \cdot \vec{b}_1 \right] - \frac{1}{2} e^{-C t} \left[ \cosh(B_2 t) \sigma_0 - \sinh(B_2 t) \vec{\sigma} \cdot \vec{b}_2 \right],$$

where $C_{1,2} = \pm \eta_0 + \frac{1}{2} (\Gamma_1 + \Gamma_2)$, $B_{1,2} = \sqrt{\eta_1^2 + \left( \frac{1}{4} (\Gamma_1 - \Gamma_2) \pm \eta_3 \right)^2}$, $\vec{b}_{1,2} = (\pm \eta_1, 0, \pm \eta_3 + \frac{1}{4} (\Gamma_1 - \Gamma_2))/B_{1,2}$.

### 5. One-mode Gaussian system

In this section and the next, we will apply the solutions of the previous section to Gaussian states. In the case of the parametric amplifier and amplitude damping, we consider the case of $\Gamma_1 = \Gamma_2 = \cdots = \Gamma_i = \Gamma$ (hereafter $\Gamma$ is a number, not the matrix used in the previous sections). If the initial state is Gaussian, its characteristic function has the form of $\chi(\mu, \mu^*, 0) = \exp \left[ \mu m^T(0) - \mu^* m^T(0) - \frac{1}{2} (\mu, -\mu^*) \gamma(0)(\mu^*, -\mu)^T \right]$, the state will remain Gaussian in later evolution. The complex correlation matrix (CM) $\gamma(0)$ should be chosen in such a fashion that the initial state is physical. The time evolution of the complex CM is

$$\gamma(t) = e^{-\frac{\Gamma t}{2}} \left[ \begin{array}{c} \cosh(|\eta|t) \frac{\eta^* \sinh(|\eta|t)}{|\eta|} \\ \frac{\sinh(|\eta|t)}{|\eta|} \cosh(|\eta|t) \end{array} \right] \left[ \begin{array}{c} \gamma(0) \\ \alpha^* \beta \end{array} \right]$$

$$\times \left[ \begin{array}{c} \cosh(|\eta|t) \frac{\eta^* \sinh(|\eta|t)}{|\eta|} \\ \frac{\sinh(|\eta|t)}{|\eta|} \cosh(|\eta|t) \end{array} \right] + \left[ \begin{array}{c} \alpha^* \beta \end{array} \right].$$

The time evolution of the complex first moment $m$ is

$$m(t) = e^{-\frac{\Gamma t}{2}} \left[ m(0) \cosh(|\eta|t) + m^*(0) \frac{\sinh(|\eta|t)}{|\eta|} \right].$$

The solution of the characteristic function for a one-mode system is characterized by equation (18) and equation (19), where $\eta$ is simply a complex number now, and (for $\Gamma \neq 2|\eta|$)

$$\alpha = \alpha^* = \frac{\Gamma}{\Gamma^2 - 4|\eta|^2} \left[ \Gamma \left( \eta + \frac{1}{2} \right) + \eta^* w + \eta w^* \right],$$

$$\beta = \frac{1}{\Gamma^2 - 4|\eta|^2} \left[ 2\Gamma \eta \left( \eta + \frac{1}{2} \right) + (\Gamma^2 - 2|\eta|^2) w + 2\eta^2 w^* \right].$$
The result reduces to that of [22] when \( w = 0, \pi = 0 \) and \( \eta \) is real. The degree of mixedness of a quantum state \( \rho \) is characterized by means of the so-called purity \( \mu_p = \text{Tr}\rho^2 \). With the characteristic function, we have \( \mu_p = \int \frac{2m}{\pi} \left| \chi(\mu, \mu^*) \right|^2. \) For a one-mode Gaussian state, it reads

\[
\mu_p(t) = \frac{1}{2\sqrt{(\text{Re} \gamma(t))^2 - |\gamma_2(t)|^2}} \quad \text{with} \quad \gamma(t) = \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix}.
\] (30)

We now consider the evolution of the coherent state, squeezed state and thermal state. The coherent state is given by \( \exp[m(0)a^\dagger - m^*(0)a] |0\rangle \), the complex CM is \( \gamma(0) = \text{diag}\left\{ \frac{1}{2}, \frac{1}{2} \right\} \). The squeezed state is given by \( \exp[(\zeta a^2 - \zeta^* a^\dagger)/2] |0\rangle \) where \( \zeta = r \exp[i\varphi] \), the complex CM is \( \gamma(0) = \frac{1}{2} [\cosh(2r) \sigma_0 + \sinh(2r) \cos \varphi \sigma_1 + \sinh(2r) \sin \varphi \sigma_2] \). The thermal state is given by \( \sum_{n=0}^{\infty} N/(N + 1)^{n} |n\rangle \) with \( N \) being the average photon number; the complex CM is \( \gamma(0) = \text{diag}\left\{ N + \frac{1}{2}, N + \frac{1}{2} \right\} \).

For \( \Gamma > 2|\eta|^2 \), all these states (and other Gaussian initial states) will tend to a Gaussian state with the complex CM \( \gamma(\infty) = \left[ \begin{array}{c} \beta \\ \beta^* \end{array} \right] \) after a sufficiently large evolution time. The ultimate purity is

\[
\mu_p(\infty) = \frac{1}{2} \left\{ \frac{\Gamma^2}{\Gamma^2 - 4|\eta|^2} \left( \pi + \frac{1}{2} \right)^2 - |w|^2(\Gamma^2 - 2|\eta|^2) - \eta^2 w^2 - \eta^2 w^* \right\}^{-\frac{1}{2}}. \quad (31)
\]

When the phase angle of the amplification \( \eta \) is equal to the phase angle of the ‘squeezed’ environment \( w \), the ultimate purity is maximized, which is

\[
\mu_{p \text{ max}}(\infty) = \frac{1}{2} \left\{ \frac{\Gamma^2}{\Gamma^2 - 4|\eta|^2} \left( \pi + \frac{1}{2} \right)^2 - |w|^2 \right\}^{-\frac{1}{2}}. \quad (32)
\]

The ultimate purity is a monotonically decreasing function of the amplification \( |\eta|^2 \).

For \( \Gamma < 2|\eta|^2 \), we consider the case of \( w = 0 \) and \( \eta \) is real and positive for simplicity. When the initial state is a thermal state with \( \gamma(0) = \text{diag}\left\{ N + \frac{1}{2}, N + \frac{1}{2} \right\} \) (coherent state is the special case of \( N = 0 \) when the CM is concerned). Denoting \( \cosh r_0 = 2\eta/\sqrt{4\eta^2 - \Gamma^2} \); then

\[
\gamma(t) = e^{-\Gamma t} \left( N + \frac{1}{2} \right) \left[ \cosh(2\eta t) \sigma_0 + \sinh(2\eta t) \sigma_1 + (\pi + \frac{1}{2}) \sinh r_0 [e^{-\Gamma t} (\cosh(2\eta t) r_0 + \cosh(2\eta t) r_0^*) + \cosh(2\eta t) r_0^* (\cosh(2\eta t) r_0 + \cosh(2\eta t) r_0^*) - (\sinh r_0 \sigma_0 + \cosh r_0 \sigma_1) \right].
\] (33)

The state is a squeezed thermal state, with purity

\[
\mu_p(t) = \frac{1}{2} \left\{ \left| N' e^{(2\rho - \Gamma)^t} + \pi' e^{\rho t} (e^{(2\rho - \Gamma)^t} - 1) \right| \left| N' e^{-(2\rho + \Gamma)^t} + \pi' e^{-\rho t} (1 - e^{-(2\rho + \Gamma)^t}) \right| \right\}^{-\frac{1}{2}}, \quad (34)
\]

where \( N' = N + \frac{1}{2}, \pi' = (\pi + \frac{1}{2}) \sinh r_0 \). For large \( t \), \( \mu_p(t) \approx \frac{1}{2} \exp[-(\eta - \Gamma/2)t]/\sqrt{\pi^2 + 2N'e^{-\rho t}} \). The purity is a decreasing function of the amplification \( \eta \) at large \( t \). When the initial state is a squeezed state with the squeezed parameter \( \zeta = r \exp[i\varphi] \), we consider the simple case of \( \varphi = 0 \), then

\[
\mu_p(t) = \frac{1}{2} \left\{ \left[ \frac{1}{2} e^{(2\rho - \Gamma)^t} + \pi' e^{\rho t} (e^{(2\rho - \Gamma)^t} - 1) \right] \left[ \frac{1}{2} e^{-(2\rho + \Gamma)^t} + \pi' e^{-\rho t} (1 - e^{-(2\rho + \Gamma)^t}) \right] \right\}^{-\frac{1}{2}}. \quad (35)
\]

For large \( t \), \( \mu_p(t) \approx \frac{1}{2} \exp[-(\eta - \Gamma/2)t]/\sqrt{\pi^2 + 2N'e^{-\rho t}/2} \).
6. Two-mode Gaussian system

The algebra for equations of $\alpha$ and $\beta$ in a two-mode system is complicated in a general situation. To investigate the entanglement property of the amplifier, we will set $\eta_0 = \eta_1 = 0$ which corresponds to no single-mode amplification. Thus $\eta_1 = \eta_1 \sigma_1$, which corresponds to two-mode amplification. The solution is simplified to (with $\Gamma_1 = \Gamma_2 = \Gamma$, and the two modes undergo the same noise ($\vec{\eta} = \vec{\eta}_0 \mathbf{L}_2$) for simplicity)

$$\alpha = \frac{\Gamma}{\Gamma^2 - 4|\eta_1|^2} \begin{bmatrix} \eta_1 w_b + \eta_1^* w_a^* \\ \eta_1 w_a + \eta_1^* w_b^* \end{bmatrix}$$

(36)

$$\beta = \frac{1}{\Gamma(\Gamma^2 - 4|\eta_1|^2)} \begin{bmatrix} (\Gamma^2 - 2|\eta_1|^2) w_a + \eta_1^* w_b^* \\ 2\eta_1 \Gamma (\vec{\eta}_0 + \frac{i}{2}) \end{bmatrix} \begin{bmatrix} \eta_1 w_b + \eta_1^* w_a^* \end{bmatrix}$$

(37)

where we have denoted $w = \text{diag}(w_a, w_b)$ for the two modes $a$ and $b$. If the initial state is Gaussian, the state will remain Gaussian in later evolution.

For $\Gamma > 2|\eta_1|$, the state will tend to a Gaussian state which is characterized by the residue complex CM $\gamma'(\infty)$ after a sufficiently large evolution time. The Peres–Horodecki criterion [24, 25] for separability of the state is an inequality on the real parameter CM $\gamma_{re}(\infty)$, which is

$$\gamma_{re}(\infty) = \begin{bmatrix} \gamma_a & \gamma_b \\ \gamma_b^* & \gamma_a \end{bmatrix} = L \begin{bmatrix} \alpha & \beta^* \\ \beta & \alpha^* \end{bmatrix} L^\dagger,$$

(38)

with

$$L = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & i \\ 1 & 0 & -1 \\ 0 & i & 0 \end{bmatrix}$$

(39)

Thus we have $\gamma_a = \alpha_a \sigma_0 - \text{Im} \beta_a \sigma_1 + \text{Re} \beta_a \sigma_3$, $\gamma_b = \alpha_b \sigma_0 - \text{Im} \beta_b \sigma_1 + \text{Re} \beta_b \sigma_3$, $\gamma_c = \text{Re} \alpha_c \sigma_0 - \text{Im} \alpha_c \sigma_1 + \text{Re} \beta_c \sigma_3$, where we have denoted

$$\alpha = \begin{bmatrix} \alpha_a & \alpha_c \\ \alpha_c^* & \alpha_b \end{bmatrix}, \beta = \begin{bmatrix} \beta_a & \beta_c \\ \beta_c^* & \beta_b \end{bmatrix}$$

(40)

The separable criterion takes the form of $\text{det} \gamma_a \text{det} \gamma_b + \left(\frac{1}{4} - |\text{det} \gamma_c|\right)^2 - \text{tr}(\gamma_a J \gamma_b J \gamma_c^\dagger J) \geq \frac{1}{4} (\text{det} \gamma_a + \text{det} \gamma_b)$ [24], with $J = i \sigma_2$. Hence in the form of $\alpha$ and $\beta$, it will be

$$\text{det} \gamma_a \text{det} \gamma_b + \left(\frac{1}{4} - |\text{det} \gamma_c|\right)^2 - \text{tr}(\gamma_a \sigma_3 \gamma_b \sigma_3 \gamma_c^\dagger \sigma_3) \geq \frac{1}{4} (\text{det} \gamma_a + \text{det} \gamma_b),$$

(41)

where

$$\gamma_c' = \begin{bmatrix} \alpha_i & \beta_i^* \\ \beta_i & \alpha_i^* \end{bmatrix}, \quad i = a, b, c.$$  

(42)

We now consider the situation of $w = 0$ for simplicity. Then $\alpha_a = \alpha_b = \frac{\Gamma^2 (\eta_0 + \frac{1}{2})}{\Gamma^2 - 4|\eta_1|^2}, \alpha_c = 0$; $\beta_c = \frac{2\Gamma (\eta_0 + \frac{1}{2})}{\Gamma^2 - 4|\eta_1|^2}, \beta_a = \beta_b = 0$, the CM $\gamma_{re}(\infty)$ can be transformed to the standard form [25] by local rotation. The standard form CM is $\gamma_c^\dagger(\infty) = \frac{\Gamma (\eta_0 + \frac{1}{2})}{\Gamma^2 - 4|\eta_1|^2} (\Gamma \sigma_0 \otimes \sigma_0 + 2|\eta_1| \sigma_1 \otimes \sigma_3)$. The inseparability criterion reads $\alpha_a = |\beta_i| < \frac{1}{2}$, which is $\Gamma \eta_0 < |\eta_1| < \Gamma / 2$.  

(43)
The possible entanglement appears only when $\overline{n}_0 < \frac{1}{2}$. The entanglement of formation (EoF) of the inseparable state will be [26, 27]

$$E_f = g\left(\Delta \left(2\alpha_a - 2|\eta_1|\right)\right) = g\left(\Delta \left(\frac{\Gamma(2\overline{n}_0 + 1)}{\Gamma + 2|\eta_1|}\right)\right).$$

where $\Delta(z) = (z + z^{-1} - 2)/4$ and $g(x) = (x + 1) \log_2(x + 1) - x \log_2 x$ is the bosonic entropy function. $g(x)$ is a monotonically increasing function of $x$; thus the entanglement is a monotonically increasing function of the amplification $|\eta_1|$ (note that $\alpha_a - |\beta_c| < \frac{1}{2}$).

The entanglement tends to its supremum when $\overline{n}_0 = 0$ and $|\eta_1| \to \Gamma/2$. The supremum is $E^\text{sup}_f = g\left(\frac{1}{8}\right) = 0.5662$.

For $\Gamma < 2|\eta_1|$, we consider the case of $w = 0$ and $\eta_1$ is real and positive for simplicity. From equations (20) and (21), the time evolution solution of the complex CM will be

$$\gamma(t) = e^{-\Gamma t} \left[ \cosh(\eta_1 t) \alpha_0 \otimes \sigma_0 + \sinh(\eta_1 t) \sigma_1 \otimes \sigma_1 \right] \gamma(0) - (\alpha_a \alpha_0 \otimes \sigma_0 + \beta_c \sigma_1 \otimes \sigma_1) \cosh(\eta_1 t) + \alpha_a \sigma_0 \otimes \sigma_0 + \beta_c \sigma_1 \otimes \sigma_1,$$

where $\beta_c$ is real and positive. If the initial state is a two-mode squeezed thermal state with the real CM $\gamma_\text{re}(0) = (N + \frac{1}{2}) \cosh 2r \sigma_0 \otimes \sigma_0 + \sinh 2r \sigma_1 \otimes \sigma_1$, then $\gamma_\text{re}(t) = \gamma_\text{re1}(t) \sigma_0 \otimes \sigma_0 + \gamma_\text{re2}(t) \sigma_1 \otimes \sigma_1$ with

$$\gamma_\text{re1}(t) = e^{-\Gamma t} \left[ \left(N + \frac{1}{2}\right) \cosh(2\eta_1 t + r) - \alpha_a \cosh(2\eta_1 t) - \beta_c \sinh(2\eta_1 t) \right] + \alpha_a$$

$$\gamma_\text{re2}(t) = e^{-\Gamma t} \left[ \left(N + \frac{1}{2}\right) \sinh(2\eta_1 t + r) - \alpha_a \sinh(2\eta_1 t) - \beta_c \cosh(2\eta_1 t) \right] + \beta_c.$$

The inseparable criterion for the state is $\gamma_\text{re1}(t) - \gamma_\text{re2}(t) < \frac{1}{2}$, that is

$$\left(N + \frac{1}{2}\right) \exp[-(2\eta_1 + \Gamma)t - 2r] + (\alpha_a - \beta_c)[1 - \exp[-(2\eta_1 + \Gamma)t]] < \frac{1}{2}.$$  

The EoF of the inseparable state will be

$$E_f(t) = g\left(\Delta(2\gamma_\text{re1}(t) - 2\gamma_\text{re2}(t))\right).$$
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1.5

2

2.5

3

0

0.5

1

1.5

2

0

0.5

1

1.5

2

0

0.5

1

1.5

2

0

0.5

1

1.5

2

Figure 2. Entanglement of formation of the two-mode amplification and damping, with the environment noise $n_0 = 0.6$. The initial state is a vacuum state.

![Graph](image)

Figure 3. The ultimate entanglement of formation with respect to the amplification damping ratio $\eta_1/\Gamma$ and environment noise $n_0$.

When $t \to \infty$, EoF saturates at

$$E_f(\infty) = g \left( \frac{\Delta}{\Gamma + 2 \eta_1} \left[ 2n_0 + 1 \right] \right).$$

Inequality (48) reduces to $\alpha_0 - \beta_\epsilon < \frac{1}{2}$. It can be further simplified to

$$\eta_1 > \max \left\{ \frac{n_0 \Gamma}{2}, \frac{\Gamma}{2} \right\}. \quad (50)$$

EoFs for $n_0 = 0.4, 0.6$ are shown in figure 1 and 2. We can see that EoF is a monotonically increasing function of the amplification $\eta_1$, EoF saturates at $E_f(\infty) = g \left( \frac{\Gamma}{\Gamma + 2 n_0} \left[ 2n_0 + 1 \right] \right)$. $E_f(\infty)$ has the same expression for $\eta_1 > \frac{\Gamma}{2}$ and $\eta_1 < \frac{\Gamma}{2}$ (see (44)). $E_f(\infty)$ is shown in figure 3 (for any Gaussian initial state, not only for a vacuum initial state). The inseparable criterion
inequalities (43) and (50) can be combined as
\[ \eta_1 / \Gamma > \pi_0. \]  

(51)

7. Conclusion

The master equation of the quantum continuous variable system is converted to the equation of the quantum characteristic function. It turns out to be a linear partial differential equation about the characteristic function. The time evolution solution can be obtained exactly for any initial quantum optical state in several cases. The solvable cases include (1) parametric amplification [22], (2) amplitude damping with thermal or squeezed noise [17], (3) simultaneous amplitude and phase damping together with thermal noise [23], (4) simultaneous multi-mode parametric amplification and amplitude damping with thermal or squeezed noise when each mode undergoes the same strength of damping and (5) simultaneous multi-mode real parametric amplification and amplitude damping with thermal or squeezed noise.

The applications to one-mode and two-mode Gaussian initial conditions are investigated. In a one-mode Gaussian system, the purity of the evolution state is monotonically decreasing with the amplification. In a situation of less amplification (|\eta| < \Gamma/2), the ultimate purity is maximized when the phase of the amplification \eta matches with the phase of the ‘squeezed’ environment \psi. In a two-mode Gaussian system, the entanglement of formation monotonically increases with the two-mode amplification. In the situation of less amplification (|\eta_1| < \Gamma/2), the supremum of the entanglement of formation is given. In the situation of over amplification (|\eta_1| > \Gamma/2), after a sufficiently large time, the entanglement of formation saturates. The ultimate entanglement of formation is given as a function of the amplification damping ratio and the noise. It is independent of the initial Gaussian state. The ultimate state is separable when the amplification damping ratio is greater than the noise.

In a real experiment, parametric amplification and damping may occur successively. Thus a combination of the above solutions will represent most of the optical evolution system.

Acknowledgements

Funding by the National Natural Science Foundation of China (under grant nos 10575092, 10347119), Zhejiang Province Natural Science Foundation (under grant no RC104265) and AQSIQ of China (under grant no 2004QK38) is gratefully acknowledged.

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