Kaluza-Klein Braneworld Cosmology
with Static Internal Dimensions

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We investigate the Kaluza-Klein braneworld cosmology from the point of view of observers on the brane. We first generalize the Shiromizu-Maeda-Sasaki (SMS) equations to higher dimensions. As an application, we study a \((4+n)\)-dimensional brane with \(n\) dimensions compactified on the brane, in a \((5+n)\)-dimensional bulk. By assuming that the size of the internal space is static, that the bulk energy-momentum tensor can be ignored, we determine the effect of the bulk geometry on the Kaluza-Klein braneworld. Then we derive the effective Friedmann equation on the brane. It turns out that the Friedmann equation explicitly depends on the equation of state, in contrast to the braneworld in a 5-dimensional bulk spacetime. In particular, in a radiation-dominated era, the effective Newton constant depends logarithmically on the scale factor. If we include a pressureless matter on the brane, this dependence disappears after the radiation-matter equality. This may be interpreted as stabilization of the Newton constant by the matter on the brane. Our findings imply that the Kaluza-Klein braneworld cosmology is quite different from the conventional Kaluza-Klein cosmology even at low energy.

§1. Introduction

Early universe models are usually motivated by theories developed to describe fundamental physics at high energies. String theory is a leading candidate for describing high energy physics and since it requires 10 dimensions to be consistent, it is natural to consider higher-dimensional universes. To reconcile a higher-dimensional universe with our empirically 4-dimensional universe, the conventional approach has been to resort to the Kaluza-Klein compactification. Another option, the braneworld scenario, was proposed about a quarter century ago,1,2 and it has been revived recently, due to the discovery of D-branes.3) In particular, Randall and Sundrum (RS) proposed an interesting framework,4,5 partly inspired by the Horava-Witten model.6) They formulated the universe as a domain wall in 5-dimensional anti-de Sitter (AdS) spacetime. Much work has been done on its cosmology (see Refs. 7–10) for reviews) and black hole physics.11–16) However, the RS framework with a codimension-one braneworld is insufficient to reach 10 dimensions. To go beyond five dimensions, while keeping the number of our spacetime dimensions at four, we need to consider either higher codimensions17,18 or Kaluza-Klein compactification on the brane. The former option, i.e. to realize a higher-codimension braneworld is difficult,
due to the strong self-gravity of the brane. In fact, a higher-codimension braneworld develops a severe singularity, except for codimension-two models. For this reason, no successful cosmological model is known. Even in the case of codimension-two models, it seems almost impossible to construct a consistent cosmological model, due to the subtlety involving the conical singularity.\textsuperscript{19)—22} The latter option, i.e. to consider a Kaluza-Klein cosmology on a brane,\textsuperscript{23)—26} is the topic of this paper.

One might think it is a trivial task to construct braneworld models with Kaluza-Klein compactification. Unfortunately, this is not true.\textsuperscript{27} In the case of the RS model, the bulk geometry is given and static. Hence, the cosmology on the brane is simply due to its motion in the bulk spacetime. In the case of Kaluza-Klein braneworlds, however, the bulk geometry is not known \textit{a priori}.\textsuperscript{28)—30} Moreover, as we require the internal space to be static, we might have to take into account the matter in the bulk, for instance in the form of fluxes. This makes it difficult to obtain the bulk geometry in most cases. In general, we have to solve for the bulk geometry and the brane motion at the same time, and explicit analytical examples are difficult to construct. (See, e.g. Ref. 31) for anisotropic 5D bulk-brane configurations, with problems similar in spirit to Kaluza-Klein braneworlds.) Although numerical methods seem to be necessary in general, an analytical approach would be useful, even if it is of limited capability.

In this paper, we take a first step in the analytical description of Kaluza-Klein braneworlds. Here, we do not attempt to derive the bulk geometry. Instead, we use the Shiromizu-Maeda-Sasaki (SMS) equation\textsuperscript{32} to analyze the Kaluza-Klein cosmology. Of course, this effective equation cannot be solved without knowing the projected Weyl tensor. For this reason, we employ the following strategy. We use the static nature of the internal space as a principle to constrain the unknown bulk geometry. We also assume that the bulk matter can be ignored, at least in the vicinity of the branes, in the regimes that we study. Then, we can determine the Friedman equation on the brane. Interestingly, the resultant Friedman equation is found to depend on the equation of state of the matter explicitly. In particular, the effective Newton constant varies logarithmically in the radiation-dominated stage. Thus the Kaluza-Klein braneworld cosmology appears to be quite different from the conventional Kaluza-Klein cosmology, even at low energy.

The organization of this paper is as follows. In §2, we consider a braneworld model with bulk matter in an arbitrary number of dimensions and derive the effective SMS equations on the brane. In §3, we apply the SMS equations to a \((5 + n)\)-dimensional Kaluza-Klein braneworld model and derive the effective Friedman equation on the brane by imposing the stability of the internal space. We conclude in §4.

§2. SMS effective equation in \((d+1)\)-dimensions

In this section, we derive the effective gravitational equations on the brane for an arbitrary number of dimensions. To be as general as possible, we also include a bulk energy-momentum tensor.
The action we consider is
\[ S = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-\tilde{g}} \left[ R - 2\Lambda - \sigma \int d^d x \sqrt{-g} + S_m \right], \quad \Lambda = -\frac{d(d-1)}{2\ell^2} \] (2.1)
where \( \kappa^2, \ell \) and \( \sigma \) are the gravitational coupling constant, the scale of the bulk curvature radius and the tension of the brane, respectively. We assume a negative cosmological constant in the bulk. Here, \( S_m \) represents the action for the matter both in the bulk and on the brane. The \((d+1)\)-dimensional and \(d\)-dimensional metrics are represented by \( \tilde{g} \) and \( g \), respectively.

We consider a \(d\)-dimensional brane with \((d-4)\) compactified dimensions. To describe the bulk spacetime, we can use Gaussian normal coordinates, so that the metric takes the form
\[ ds^2 = dy^2 + g_{\mu\nu}(y,x^\mu)dx^\mu dx^\nu, \] (2.2)
and the brane position is \( y = 0 \) in this coordinate system. One can deduce the effective equation on the brane following SMS. The extrinsic curvature is defined as
\[ K_{\mu\nu} = -\frac{1}{2} \frac{\partial}{\partial y} g_{\mu\nu} \equiv -\frac{1}{2} g_{\mu\nu,y}. \] (2.3)
Using the extrinsic curvature, we can write down the Einstein equations in \((d+1)\)-dimensions as
\[ (d+1)G^y = -\frac{1}{2} R + \frac{1}{2} K^2 - \frac{1}{2} K^{\alpha\beta} K_{\alpha\beta} = \frac{d(d-1)}{2\ell^2} + \kappa^2 T^y, \] (2.4)
\[ (d+1)G^y = -\nabla_\lambda K_{\mu}^\lambda + \nabla_\mu K = \kappa^2 T^y, \] (2.5)
\[ (d+1)G^\mu\nu = G^\mu\nu + (K^\mu\nu - \delta^\mu\nu K)_{,y} - KK^\mu\nu + \frac{1}{2} \delta^\mu\nu \left( K^2 + K^{\alpha\beta} K_{\alpha\beta} \right) = \frac{d(d-1)}{2\ell^2} \delta^\mu\nu + \kappa^2 S^\mu\nu \delta(y) + \kappa^2 T^\mu\nu, \] (2.6)
where \( G_{\mu\nu} \) is the \(d\)-dimensional Einstein tensor, and \( T_{\mu\nu}, T_{y\mu}, \) and \( T_{yy} \) are the components of the bulk energy-momentum tensor. Here \( \nabla_\mu \) denotes the covariant derivative with respect to the metric \( g_{\mu\nu} \), and \( S_{\mu\nu} = -\sigma g_{\mu\nu} + t_{\mu\nu} \) is the energy-momentum tensor on the brane, where \( t_{\mu\nu} \) is the energy-momentum tensor of the brane matter excluding the tension. Then, the junction conditions are given by
\[ [K^\mu\nu - \delta^\mu\nu K] \big|_{y=0} = \frac{\kappa^2}{2} (\sigma \delta^\mu\nu + t^\mu\nu), \] (2.7)
where we have assumed \( Z_2 \)-symmetry. Combining Eqs. (2.4) and (2.6), we have
\[ -\frac{1}{d-1} \left( R_{\mu\nu} - \frac{1}{d} g_{\mu\nu} R \right) = \frac{1}{d-1} \left[ K_{\mu\nu,y} - g_{\mu\nu} K_{,y} - KK_{\mu\nu} + 2K^{\lambda}_{\mu} K_{\lambda\nu} \right] + \frac{1}{d(d-1)} g_{\mu\nu} K^2 + \frac{1}{d} g_{\mu\nu} K^{\alpha\beta} K_{\alpha\beta} - \frac{1}{\ell^2} g_{\mu\nu} \]
\[ -\frac{\kappa^2}{d-1} \left( T_{\mu\nu} - \frac{d-2}{d} g_{\mu\nu} T^y \right). \] (2.8)
The trace of this equation gives
\[ K_y = K^\alpha\beta K_{\alpha\beta} - \frac{d}{\ell^2} + \kappa^2 \frac{d - 2}{d - 1} T^y y - \frac{\kappa^2}{d - 1} T^\mu \mu \] \tag{2.9}

Also, the following components of the Weyl tensor are relevant:
\[ C_{y\mu y\nu} = -\frac{1}{d - 1} \left( R_{\mu\nu} - \frac{1}{d} g_{\mu\nu} R \right) + \frac{d - 2}{d - 1} K_{\mu\nu,y} - \frac{d - 2}{d(d - 1)} g_{\mu\nu} K_y + \frac{d - 3}{d - 1} K^\lambda K_{\lambda\nu} \\
+ \frac{1}{d - 1} K K_{\mu\nu} + \frac{1}{d} g_{\mu\nu} K^\alpha\beta K_{\alpha\beta} - \frac{1}{d(d - 1)} g_{\mu\nu} K^2. \] \tag{2.10}

The above components of the Weyl tensor can be rewritten by using Eqs. (2.8) and (2.9) as
\[ C_{y\mu y\nu} = K_{\mu\nu,y} - g_{\mu\nu} K_y + K^\lambda K_{\lambda\nu} + g_{\mu\nu} K^\alpha\beta K_{\alpha\beta} - \frac{d - 1}{\ell^2} g_{\mu\nu} \\
+ \kappa^2 \frac{d - 2}{d} g_{\mu\nu} T^y y - \frac{\kappa^2}{d - 1} \left( T_{\mu\nu} + \frac{d - 2}{d} g_{\mu\nu} T^\alpha_\alpha \right). \] \tag{2.11}

Thus off the brane, using these components of the Weyl tensor, Eq. (2.6) is expressed as
\[ G_{\mu\nu} = -C_{y\mu y\nu} - K^\lambda K_{\lambda\nu} + K K_{\mu\nu} + \frac{1}{2} g_{\mu\nu} K^\alpha\beta K_{\alpha\beta} - \frac{1}{2} g_{\mu\nu} K^2 + \frac{(d - 1)(d - 2)}{2\ell^2} g_{\mu\nu} \\
+ \frac{d - 2}{d - 1} \kappa^2 \left( T_{\mu\nu} - \frac{1}{d} g_{\mu\nu} T^\alpha_\alpha \right) + \frac{d - 2}{d} \kappa^2 T^y y g_{\mu\nu}, \] \tag{2.12}

where we stress that the term $T^\alpha_\alpha$ is the trace defined with respect to the $d$-dimensional metric $g$, not the full trace defined with respect to $\tilde{g}$. Eliminating the extrinsic curvature by using the junction conditions (2.7), and assuming the RS-type relation
\[ \kappa^2 \sigma = \frac{2(d - 1)}{\ell}, \] \tag{2.13}

we finally obtain the $d$-dimensional generalization of the SMS equations,
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -E_{\mu\nu} + 8\pi G t_{\mu\nu} + \kappa^4 \pi_{\mu\nu} \\
+ \frac{d - 2}{d - 1} \kappa^2 \left( T_{\mu\nu} - \frac{1}{d} g_{\mu\nu} T^\alpha_\alpha \right) + \frac{d - 2}{d} \kappa^2 T^y y g_{\mu\nu}, \] \tag{2.14}

where we have defined the Newton constant in $d$ dimensions by
\[ 8\pi G = \frac{(d - 2)\kappa^2}{2\ell}, \] \tag{2.15}

the tensor
\[ \pi_{\mu\nu} = \frac{1}{4(d - 1)} t_{\mu\nu} - \frac{1}{4} t_\mu t_\lambda - \frac{1}{8} g_{\mu\nu} t^\alpha\beta t_{\alpha\beta} - \frac{1}{8(d - 1)} g_{\mu\nu} t^2, \] \tag{2.16}
and the projected Weyl tensor
\[ E_{\mu\nu} = C_{\mu\nu\sigma}\big|_{y=0}. \] (2.17)

These results are also obtained in Refs. 33) and 34).

From the Bianchi identity satisfied by the Einstein tensor, we can deduce the following constraint equation on the tensors that appear on the right-hand side of (2.14):
\[ \nabla^\mu E_{\mu\nu} = \frac{\kappa^4}{4} \nabla^\mu \pi_{\mu\nu} + \frac{d-2}{d-1} \kappa^2 \nabla^\mu T_{\mu\nu} - \frac{d-2}{d} \kappa^2 \nabla^\nu T_{\nu\alpha} + \frac{d-2}{d} \kappa^2 \nabla^\nu T_{\nu\gamma}. \] (2.18)

Here, we have assumed the conservation of the energy-momentum tensor for the matter on the brane,
\[ \nabla^\mu t_{\mu\nu} = 0; \] (2.19)
i.e., we do not allow energy exchange between the brane and the bulk (as studied, e.g., in Ref. 35) in the context of 5D brane cosmology). We also have the conservation law for the bulk matter, which can be decomposed as
\[ 0 = \partial_y T^y_y - K T^y_y + K^\mu_y T^\nu_\mu + \nabla_\mu T^\mu_y, \] (2.20)
\[ 0 = \partial_y T^y_\mu - K T^y_\mu + \nabla_\nu T^\nu_\mu. \] (2.21)

Of course, the above equations do not form a closed system, because we do not know \( E_{\mu\nu} \). In other words, without knowing the bulk geometry, we cannot solve the effective Einstein equations (2.14). However, we may regard the SMS equation as an initial value equation. More precisely, once \( E_{\mu\nu} \) is obtained from the SMS equation, we can solve the \((d+1)\)-dimensional Einstein equations along the \( y \)-direction to obtain the bulk geometry. In this picture, unless we impose some conditions on the properties of the spacetime, there will be too many allowed bulk solutions, and most of the solutions will be physically meaningless.

In the next section, we attempt to solve the effective Einstein equations to study the cosmology of Kaluza-Klein braneworlds from the SMS equation by assuming that the internal dimensions are static and that the bulk energy-momentum tensor can be ignored on the brane. Given these conditions, we show that, similarly to the case of the cosmology of a codimension-one brane in an empty 5D bulk,\(^{36,37}\) the effective Friedmann equations can be solved, up to a constant of integration. However, in contrast to the 5D bulk, we do not derive a bulk geometry associated with this cosmology.

\section*{§3. Kaluza-Klein braneworld cosmology}

To realize a braneworld in higher dimensions, it seems natural to consider a codimension-one brane with internal dimensions that are compactified \( \textit{à la} \) Kaluza-
Klein, in short a Kaluza-Klein braneworld. Here, for simplicity, we consider $n$ internal dimensions compactified on a torus. The brane thus represents a $(4+n)$-dimensional spacetime embedded in a $(5+n)$-dimensional spacetime. Since we wish to study the cosmology, we consider metrics of the form

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j + b^2(t)\delta_{\alpha\beta}dz^\alpha dz^\beta,$$

(3.1)

where $x^i$ are the three ordinary spatial coordinates and $z^\alpha$ are the internal coordinates. For simplicity, we assume that there is a single scale factor, $b$, characterizing the size of the internal dimensions. The scale factor $a$ is the usual scale factor for the external space.

We can imagine two kinds of matter, that in the bulk and that on the brane. The bulk matter is important to realize a well-behaved geometry in the bulk.\textsuperscript{39),40) However, for simplicity, we suppose here that we can ignore it for the cosmology on the brane. Hereafter, we consider only the matter on the brane. (We note a recent work in which a similar analysis is considered for a 6D Kaluza-Klein brane embedded in a 7D bulk spacetime.\textsuperscript{38) That work takes into account a bulk energy-momentum tensor and the possibility of brane-bulk energy exchange.) Because of the symmetries, the energy-momentum tensor is restricted to be of the form

$$t_{\mu\nu} = (\rho, Pa^2\delta_{ij}, Qb^2\delta_{\alpha\beta}) ,$$

(3.2)

where $\rho$ is the energy density, $P$ the external pressure and $Q$ the internal pressure. Similarly, the projected Weyl tensor is of the form

$$E_{\mu\nu} = (e, \chi a^2\delta_{ij}, \xi b^2\delta_{\alpha\beta}) .$$

(3.3)

Moreover, the traceless property of $E_{\mu\nu}$ implies the relation $-e + 3\chi + n\xi = 0$. The components of the quadratic tensor in the energy-momentum tensor (2.16) are given by

$$\pi_{00} = \frac{1}{8(3+n)} \left[ (n+2)\rho^2 - 3n(P - Q)^2 \right] ,$$

(3.4)

$$\pi_{ij} = \frac{1}{8(3+n)} \left[ (n+2)\rho^2 + 2\rho(2P + nQ) + n(P - Q)(P - 3Q) \right] a^2 \delta_{ij} ,$$

(3.5)

$$\pi_{\alpha\beta} = \frac{1}{8(3+n)} \left[ (n+2)\rho^2 + 6\rho P + 2(n-1)\rho Q \right. \right.$$

$$\left. \left. + 3\left(n(P - Q)^2 - 2Q^2 + 2PQ\right) \right] b^2 \delta_{\alpha\beta} . \right.$$

(3.6)

Substituting the metric (3.1) and the tensors (3.2), (3.3) and (3.4)–(3.6) into the effective Einstein equations (2.14), we find

$$3H_a^2 + 3nH_a H_b + \frac{n(n-1)}{2} H_b^2 = 8\pi G\rho + \frac{\kappa^4}{8(3+n)} \left[ (n+2)\rho^2 - 3n(P - Q)^2 \right] - e ,$$

(3.7)
where we have defined the Hubble parameters $H(2.19)$, which is given here by

$$\dot{H}_a - 3H_a^2 - 2nH_a H_b - n\dot{H}_b - \frac{n(n+1)}{2}H_b^2$$

$$= 8\pi G P + \frac{\kappa^4}{8(3+n)} [(n+2)\rho^2 + 2\rho (2P + nQ) + n(P - Q)(P - 3Q)] - \chi , \quad (3.8)$$

$$-3\dot{H}_a - 6H_a^2 - 3(n - 1)H_a H_b - (n - 1)\dot{H}_b - \frac{n(n-1)}{2}H_b^2$$

$$= 8\pi G Q + \frac{\kappa^4}{8(3+n)} [(n+2)\rho^2 + 6\rho P + 2(n - 1)\rho Q$$

$$+ 3(n(P - Q)^2 - 2Q^2 + 2PQ)] - \xi . \quad (3.9)$$

In addition to the above equations, we need the conservation law for the matter (2.19), which is given here by

$$\dot{\rho} + 3H_a(\rho + P) + nH_b(\rho + Q) = 0 . \quad (3.10)$$

The constraint equation for the projected Weyl tensor (2.18) in the absence of the bulk matter can be written as

$$\dot{e} + 3H_a(e + \chi) + nH_b(e + \xi)$$

$$+ \frac{3n}{4(3+n)}\kappa^4(P - Q) \left[ \dot{P} - \dot{Q} - H_a(\rho + P) - H_b(\rho + Q) \right] = 0 , \quad (3.11)$$

where we have defined the Hubble parameters $H_a = \dot{a}/a$ and $H_b = \dot{b}/b$. In order to explicitly integrate these equations, we assume simple equations of state for the anisotropic fluid, namely $P = w\rho$ and $Q = v\rho$, where $w$ and $v$ are constants. Then Eqs. (3.7)–(3.9) become

$$3H_a^2 + 3nH_a H_b + \frac{n(n-1)}{2}H_b^2$$

$$= 8\pi G \rho + \frac{\kappa^4}{8(3+n)} \{ 2 + n(1 - 3(v - w)^2) \} \rho^2 - e , \quad (3.12)$$

$$-2\dot{H}_a - 3H_a^2 - 2nH_a H_b - n\dot{H}_b - \frac{n(n+1)}{2}H_b^2$$

$$= 8\pi G \rho + \frac{\kappa^4}{8(3+n)} \{ 2 + 4w +$$

$$+ n(1 + 2v + 3w^2 - 4vw + w^2) \} \rho^2 - \chi , \quad (3.13)$$

$$-3\dot{H}_a - 6H_a^2 - 3(n - 1)H_a H_b - (n - 1)\dot{H}_b - \frac{n(n-1)}{2}H_b^2$$

$$= 8\pi G \rho + \frac{\kappa^4}{8(3+n)} \{ 2 - 2v - 6v^2 + 6w + 6vw +$$

$$+ n(1 + 2v + 3v^2 - 6vw + 3w^2) \} \rho^2 - \frac{1}{n}(e - 3\chi) , \quad (3.14)$$

and the conservation law reduces to

$$\dot{\rho} + 3(1 + w)H_a \rho + n(1 + v)H_b \rho = 0 . \quad (3.15)$$
The constraint (3.11) can be written
\[
\dot{e} + 3H_a(e + \chi) + H_b((n + 1)e - 3\chi) + \frac{3n}{4(3 + n)}\kappa^4 [(w - v)^2 \rho \dot{\rho} + (H_a(1 + w) - H_b(1 + v))(w - v)\rho^2] = 0.
\tag{3.16}
\]

Since we do not know \(e, \chi\) or \(\xi\), we cannot solve the above equations. We need to know the bulk geometry in order to obtain \(e, \chi\) and \(\xi\) in general. Here, instead of deriving the bulk geometry, we impose the condition of the stability of the internal space; that is, we set \(b = 1\). In this case, there may be a singularity in the bulk. However, if we allow the existence of matter in the bulk, it is reasonable to expect that the bulk geometry can be made regular through a suitable choice of the bulk matter. Determining the kind of matter necessary for this purpose, though, is an unsolved problem. Here we assume that the internal space is static and seek an effective Friedman equation on the brane. Under this assumption, Eqs. (3.12) – (3.16) reduce to
\[
3H_a^2 = 8\pi G \rho + \frac{\kappa^4}{8(3 + n)} \left\{ 2 + n(1 - 3(v - w)^2) \right\} \rho^2 - e,
\tag{3.17}
\]
\[
-2\dot{H}_a - 3H_a^2 = 8\pi G w \rho + \frac{\kappa^4}{8(3 + n)} \left\{ 2 + 4w + n(1 + 2v + 3v^2 - 4vw + w^2) \right\} \rho^2 - \chi,
\tag{3.18}
\]
\[
-3\dot{H}_a - 6H_a^2 = 8\pi G v \rho + \frac{\kappa^4}{8(3 + n)} \left\{ 2 - 6v^2 + 6w + 6vw + n(1 + 2v + 3(v - w)^2) \right\} \rho^2 - \frac{1}{n}(e - 3\chi),
\tag{3.19}
\]
and
\[
\dot{\rho} + 3(1 + w)H_a \rho = 0,
\tag{3.20}
\]
\[
\dot{e} + 3H_a(e + \chi) + \frac{3n}{4(3 + n)}\kappa^4 [(w - v)^2 \rho \dot{\rho} + H_a(1 + w)(w - v)\rho^2] = 0.
\tag{3.21}
\]

We seek the effective Friedman equation in the Kaluza-Klein braneworld. For this purpose, it is seen from Eq. (3.17) that we need to know \(e\), which is a component of the projected Weyl tensor, which encodes some information concerning the bulk geometry. For general \(w\), the solution of Eq. (3.20) is
\[
\rho = a^{-3(1+w)},
\tag{3.22}
\]
where we have absorbed a constant factor into the scale factor by rescaling it. This is a standard result. Eliminating \(\dot{H}\) and \(H^2\) from Eqs. (3.17)–(3.19), we obtain
\[
-n\xi = 3\chi - e = 8\pi G \frac{n}{2 + n}(3w - 2v - 1)\rho + \frac{\kappa^4 n}{4(3 + n)}v(1 - 3w + 3v)\rho^2.
\tag{3.23}
\]
Thus, the components of the projected Weyl tensor are related to the matter on the brane. Substituting (3.23) into the constraint equation for the dark radiation (3.21)
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gives the equation
\[ \dot{e} + 4H_a e = -8\pi G \frac{n}{2 + n} (3w - 2v - 1)H_a \rho \\
- \kappa^4 \frac{n}{4(3 + n)} (1 - 3w + 3v) \{v + 3(1 + w)(w - v)\} H_a \rho^2. \tag{3.24} \]

This equation can be integrated easily. Note that the cases \( w = 1/3 \) and \( w = -1/3 \) need to be considered separately.

First, we consider the generic case, \( w \neq 1/3, -1/3 \). The solution in this case is given by
\[ e = -\frac{3C}{a^4} + 8\pi G \frac{n}{2 + n} \frac{1 + 2v - 3w}{1 - 3w} a^{-3(1 + w)} \]
\[ + \kappa^4 \frac{n}{8(3 + n)} \frac{(1 - 3w + 3v)}{(1 + 3w)} \{v + 3(1 + w)(w - v)\} a^{-6(1 + w)}, \tag{3.25} \]

where \( C \) is a constant of integration which can be interpreted as dark radiation.\(^{37,41} \)

Substituting Eq. (3.25) into Eq. (3.17), we obtain the effective Friedman equation,
\[ H_a^2 = \frac{8\pi G_{\text{eff}}}{3} \rho + A \rho^2 + \frac{C}{a^4}, \tag{3.26} \]

where the coefficients are given by
\[ G_{\text{eff}} = \frac{2(1 - 3w - nv)}{(2 + n)(1 - 3w)} G, \tag{3.27} \]
\[ A = \frac{2 + 6w + n(1 + 2v + 3(v - w)^2)}{24(3 + n)(1 + 3w)} \kappa^4. \tag{3.28} \]

The above equations include the well-known 5D case, corresponding to \( n = 0 \) and for which \( G_{\text{eff}} = G \) and \( A = \kappa^4/36 \). By contrast, in higher dimensions, the effective Newton constant that we have defined depends on the equation of state. This means that, provided that our assumptions are valid, the Kaluza-Klein braneworld cosmology is different from the conventional Kaluza-Klein cosmology, even at low energies.

For \( w = 0 \), we have to assume \( nv < 1 \) in order to have a positive effective Newton constant. It should be noted that the effective Newton constant becomes negative in the regime satisfying \( 0 < nv < 1 \) and \( (1 - nv)/3 < w < 1/3 \). This indicates some transient instability around the matter-radiation equality.

Before proceeding, however, it is important to note that in addition to the assumption that the internal space is static, we have also assumed that the bulk matter can be neglected and that there is no explicit coupling between the matter on the brane and the matter in the bulk. If we relax one or more of these assumptions, the conclusion will be altered.

Now, we consider the radiation-dominated case, \( w = 1/3 \). In this case, the solution reads
\[ e = -\frac{3C}{a^4} + 8\pi G \frac{2n}{2 + n} \frac{\log a}{a^3} - \kappa^4 \frac{n}{16(3 + n)} v(9v - 4)a^{-8}. \tag{3.29} \]
Note that there appears a logarithmic correction in the above. Substituting Eq. (3.29) into Eq. (3.17), we obtain the effective Friedman equation (3.26), with the coefficients given by

\[ G_{\text{eff}} = G \left( 1 - \frac{2n}{2 + n} v \log \frac{a}{a_*} \right), \tag{3.30} \]

\[ A = \kappa^4 \frac{12 + n(4 + 9v^2)}{144(3 + n)}, \tag{3.31} \]

where \( a_* \) is a constant of integration corresponding to the dark radiation component \( C \). Remarkably, the effective Newton constant depends logarithmically on the scale factor. This interesting behavior of the cosmological evolution exists only during a radiation-dominated stage. Furthermore, depending on the value of \( a_* \), the effective Newton constant may become negative. This implies that the dark radiation component should be chosen appropriately in order to realize a sensible cosmology on the brane. We point out that this behavior contrasts with that in the case of the Brans-Dicke cosmology, in which the effective Newton constant begins to depend on time after pressureless matter starts to dominate. In a sense, dark matter stabilizes the effective gravitational coupling constant. The relevant constraint comes from nucleosynthesis:\(^{42},^{43}\)

\[ \frac{\Delta G_{\text{eff}}}{G_{\text{eff}}} = 0.01^{+0.20}_{-0.16}, \tag{3.32} \]

at one-sigma confidence level. It is easy to see that this constraint is satisfied for a sufficiently large \( a_* \).

Finally, we consider the case \( w = -1/3 \). The solution is given by

\[ e = -\frac{3C}{a^4} + 8\pi G \frac{n}{2 + n} (1 + v)a^{-2} + \kappa^4 \frac{n}{12(3 + n)} (2 + 3v)^2 \log \frac{a}{a_*} , \tag{3.33} \]

where again a logarithmic term appears. Substituting Eq. (3.33) into Eq. (3.17), we obtain the coefficients in the effective Friedman equation as

\[ G_{\text{eff}} = \frac{2 - nv}{2 + n} G, \tag{3.34} \]

\[ A = \kappa^4 \frac{6 + n(2 - 6v - 9v^2)}{72(3 + n)} - \kappa^4 \frac{n}{36(3 + n)} (2 + 3v)^2 \log \frac{a}{a_*}, \tag{3.35} \]

where \( a_* \) is the constant of integration mentioned above. In this case, we have a logarithmic scale factor dependence in the coefficient of \( \rho^2 \). This can have some influence at high energy. Its effect on the inflationary scenario may be interesting.

Apparently, the Kaluza-Klein cosmology we have obtained is different from the conventional Kaluza-Klein cosmology. In particular, the effective Friedman equation depends on the equation of state of the matter explicitly. This result should hold in more general Kaluza-Klein spacetimes. This can be understood as follows. The constraint equation (2.18) in the absence of the bulk matter reads

\[ \nabla^\mu E_{\mu\nu} = \kappa^4 \nabla^\mu \pi_{\mu\nu}, \tag{3.36} \]
Hence, in general, the projected Weyl tensor is affected by the energy-momentum tensor on the brane. If the brane were isotropic and homogeneous, the matter part would have the additional property $\nabla^\mu \pi_{\mu\nu} = 0$. In our example, this can be seen by setting $P = Q$ in Eq. (3.11). The effect of matter would then not appear in Eq. (3.36). This is related to Birkoff’s theorem. Because of the spherical-like symmetry, one does not see the details of the matter content but, rather, the dark radiation which is independently conserved. By contrast, in the case of Kaluza-Klein braneworlds, this no longer happens, because of the anisotropy of the brane. First of all, $\pi_{\mu\nu}$ is no longer conserved, i.e., $\nabla^\mu \pi_{\mu\nu} \neq 0$. Furthermore, the trace of $E_{\mu\nu}$ in the 4-dimensional spacetime is no longer zero; it is equal to the minus of the trace of $E_{\mu\nu}$ in the internal space. With the assumption that the internal space is static, this modifies the effective Friedmann equation, even at low energies.

Because the SMS equation is not a closed system, we cannot formulate cosmological perturbation theory without further information. As noted above, we may regard the SMS equations as giving “initial data” to solve the bulk equations. Of course, in general, there would be a singularity in the bulk. However, if one introduces some bulk matter, it may be possible to obtain a non-singular geometry in the bulk. With some explicit bulk matter, one can also explicitly formulate the cosmological perturbation theory for the corresponding Kaluza-Klein braneworld. Since there is the possibility of a deviation from Newton’s law at low energy, it would be interesting to carry out this program explicitly.

§4. Conclusion

We have considered a $(d+1)$-dimensional gravitational system with bulk matter and a $d$-dimensional brane, and derived the effective $d$-dimensional Einstein equations on the brane. As an application, we have studied the cosmology of the Kaluza-Klein braneworld with $n$ internal toroidal dimensions. By ignoring the bulk matter and stipulating that the internal space be static, we have obtained a closed set of equations, from which we have been able to derive the effective Friedmann equation. We have found that the resultant Friedmann equation explicitly depends on the equation of state. In particular, if the universe is dominated by radiation, a resonant contribution to the projected Weyl tensor imparts a time variation to the effective gravitational coupling constant. This time dependence disappears after the radiation-matter equality, which can be interpreted as a stabilization of the Newton constant by the matter on the brane. It should be emphasized that the Kaluza-Klein braneworld cosmology is quite different from the conventional Kaluza-Klein cosmology, even at low energy. This contrasts with the fact that the conventional Friedmann equation can be recovered at low energy in the RS braneworld cosmology. Hence, it is important to investigate whether Newtonian gravity can be recovered on the Kaluza-Klein braneworld. This question has already been studied in some specific 6D models based on flux compactifications.

We have also discussed the method to obtain the bulk geometry from the brane data. It seems that it would always be possible to adjust $E_{\mu\nu}$ such that the braneworld has a static internal space during the cosmic expansion. Of course, in general, the
geometry of the bulk will be unphysical, and it will perhaps contain singularities. However, assuming the presence of matter in the bulk, there is a chance to have a non-singular bulk. Admittedly, it is a non-trivial problem to find appropriate matter that can stabilize the braneworld without any pathology in the bulk. An alternative approach is to start from known bulk solutions in which a brane is embedded. This method has been used very recently to study the cosmology of Kaluza-Klein branes, with one internal dimension, in 6D bulk solutions of Einstein-Maxwell equations.46,47

There are many other issues to be explored. It would be interesting to formulate the quantum creation of the Kaluza-Klein braneworld as has been done in RS models.48,49 It is also important to understand the low-energy description of the Kaluza-Klein braneworld50–56 and the Kaluza-Klein corrections.57,58 It would be intriguing to consider the born-again scenario in higher dimensions.59,60 We leave these issues for future studies.

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