On fundamental groups related to the Hirzebruch surface \( F_1 \)

Dedicated to Qi-keng LU on the occasion of his 80th birthday

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Abstract  Given a projective surface and a generic projection to the plane, the braid monodromy factorization (and thus, the braid monodromy type) of the complement of its branch curve is one of the most important topological invariants, stable on deformations. From this factorization, one can compute the fundamental group of the complement of the branch curve, either in \( \mathbb{C}^2 \) or in \( \mathbb{P}^2 \). In this article, we show that these groups, for the Hirzebruch surface \( F_{1,(a,b)} \), are almost-solvable. That is, they are an extension of a solvable group, which strengthen the conjecture on degeneratable surfaces.

Keywords: Hirzebruch surfaces, degeneration, generic projection, branch curve, braid monodromy, fundamental group, classification of surfaces

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1 Introduction

In the study of smooth algebraic surfaces of degree \( n \), which are embedded in \( \mathbb{P}^N \), one can consider the surface \( X \) as a branched cover of \( \mathbb{P}^2 \). In this case the branch locus, \( S_X \) in \( \mathbb{P}^2 \), plays a crucial role. It is, in general, singular and, if the projection \( X \rightarrow \mathbb{P}^2 \) is generic, the singularities are nodes and cusps. The significance of \( S_X \) (or of \( S \subset \mathbb{C}^2 \subset \mathbb{P}^2 \), a generic affine portion of \( S_X \)) arises when studying equivalence class of the braid monodromy factorization of the branch curve \( S_X \) (which is known to be the BMT invariant of the surface \( X \); see [2]). From this factorization one can induce the fundamental groups \( G = \pi_1(\mathbb{P}^2 - S_X) \) or \( G = \pi_1(\mathbb{C}^2 - S) \), which are stable on deformations. That is, if two surfaces have different fundamental groups, then they are not deformation equivalent. For surfaces \( X, Y \) denote \( X \stackrel{G}{\simeq} Y \) \( \Leftrightarrow \) \( G_X = G_Y \) and \( \overline{G}_X = \overline{G}_Y \); \( X \stackrel{\text{Diff}}{\simeq} Y \) \( \Leftrightarrow \) \( X \) is diffeomorphic to \( Y \); \( X \stackrel{\text{Def}}{\simeq} Y \) \( \Leftrightarrow \) \( X \) is deformation equivalent to \( Y \); and \( X \stackrel{\text{BMT}}{\simeq} Y \) \( \Leftrightarrow \) \( X \) and \( Y \) has the same BMT invariant.

It turns out that \( X \stackrel{\text{Def}}{\simeq} Y \Rightarrow X \stackrel{G}{\simeq} Y \), but the inverse direction is not correct (see [3]); and \( X \stackrel{\text{Def}}{\simeq} Y \Rightarrow X \stackrel{\text{BMT}}{\simeq} Y \Rightarrow X \stackrel{\text{Diff}}{\simeq} Y \) (and again, the inverse directions are not correct; see [3, 4]).

In this article, we take \( X \) to be the Hirzebruch surface \( F_1 \) which is the projectivization of the line bundle \( \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1} \). We then embed it into \( \mathbb{P}^n \) with respect to the linear system \( |aC + bE_0| \), where \( C, E_0 \) generate the Picard group of \( F_1 \), \( b > 1, a \geq 1 \). We show that \( G \) and \( \overline{G} \) can be computed when \( X = F_{1,(a,b)} \), which is the image of \( F_1 \) after the embedding w.r.t. the above linear system.

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It was conjectured\[2\] that \( G \) and \( \overline{G} \) are almost solvable in a large family of surfaces: that is, these groups are extensions of a solvable group by the symmetric group. So far, it was proven for \( V_p \) (the Veronese surface\[5\]) and \( X_{p,q} \) (the double-double covering of \( \mathbb{CP}^1 \times \mathbb{CP}^1\[6\]).

Our main result proves that \( X = F_{1,(a,b)} \) \((a \geq 1, b > 1)\) satisfies the conjecture. In particular, there exist a series \( 1 \triangleleft A_1 \triangleleft A_2 \triangleleft A_3 \triangleleft G \) s.t.

\[
G/A_3 \simeq S_{2ab+b^2}, \quad A_3/A_2 \simeq \mathbb{Z}, \quad A_2/A_1 \simeq (\mathbb{Z}_{b-2a})^{2ab+b^2-1}, \quad A_1 \simeq \begin{cases} \mathbb{Z}_2, & a \text{ odd } b \text{ even}, \\ 1, & \text{otherwise}, \end{cases}
\]

and a series \( 1 \triangleleft \overline{A}_1 \triangleleft \overline{A}_2 \triangleleft \overline{A}_3 \triangleleft \overline{G} \) where

\[
\frac{\overline{G}}{\overline{A}_3} = \frac{G}{A_3}, \quad \frac{\overline{A}_2}{\overline{A}_2} \simeq \mathbb{Z}, \quad m = 3ab - a - b + \frac{3b^2 - 3b}{2}, \quad \frac{\overline{A}_2}{A_2} = \frac{A_2}{A_1} = \overline{A}_1 = A_1.
\]

As noted, the significance of this article lies in the fact that \( G \) and \( \overline{G} \) are determined by the deformation type, since they are stable under deformation of the surface. Thus, computing \( G \) and \( \overline{G} \) explicitly (and the series of groups derived from them) can help us distinguish between non-deformation equivalent Hirzebruch surfaces.

Another important aspect of this article is the fact that it gives a general approach and another example of how to compute and deal with the fundamental groups \( G \) and \( \overline{G} \). So far, only a few examples of calculating these groups were presented (see \([7, 8]\)) and most of the calculations dealt with the Galois cover of such a degeneratable surface; especially with finding the fundamental group of this Galois cover (see \([9, 10]\)). Calculating \( G \) and \( \overline{G} \) is another step in understanding the whole structure of these groups with respect to surfaces which can be degenerated.

2 Hirzebruch surfaces and their degenerations

The Hirzebruch surfaces \( F_k \) \((k \geq 0)\) are given by the equation \( x_1y_1^k = x_2y_2^k \) in \( \mathbb{CP}^1 \times \mathbb{CP}^1\). However, the construction these days is as follows: the \( k \)-th Hirzebruch surface is the projectivization of the vector bundle \( \mathcal{O}_{\mathbb{CP}^1}(k) \oplus \mathcal{O}_{\mathbb{CP}^1} \).

Let \( \sigma \) be a holomorphic section of \( \mathcal{O}_{\mathbb{CP}^1}(k) \), and let \( E_0 \subset F_k \) denote the image of the section \((\sigma, 1)\) of \( \mathcal{O}_{\mathbb{CP}^1}(k) \oplus \mathcal{O}_{\mathbb{CP}^1} \). The curve \( E_0 \) is called a zero section of \( F_k \). All zero sections are homologous and hence define a divisor class which is independent of choice of \( \sigma \). Let \( C \) denote a fiber of \( F_k \). The Picard group of \( F_k \) is generated by \( E_0 \) and \( C \). It is elementary that \( E_0^2 = k \), \( C^2 = 0 \) and \( E_0 \cdot C = 1 \).

The surface \( F_0 \) is the quadric \( \mathbb{CP}^1 \times \mathbb{CP}^1 \), and \( F_1 \) is the blow-up of the plane \( \mathbb{CP}^2 \). For \( k > 0 \), the surface \( F_k \) contains a unique (irreducible) curve of negative self-intersection \(-k\). This curve is a section of the bundle; it is denoted \( E_\infty \) and is called the negative section or the section at infinity. We mention that it can be contracted to an isolated normal singularity, the resulting normal surface being the cone over the rational normal curve of degree \( k \). Zero sections are always disjoint to \( E_\infty \). Schematically, we describe \( F_k \) as in Figure 2.1.

Let \( F_0 \) be the \( k \)-th Hirzebruch surface. Let \( E_0, E_\infty, C \) be as in the introduction. For \( a, b \geq 1 \), or for \( a = 0 \) and \( k \geq 1 \), the divisor \( ac + bE_0 \) on \( F_k \) is very ample and thus defines an embedding \( f_{[aC+bE_0]} : F_k \rightarrow \mathbb{CP}^N \). Let \( F_{k(a,b)} = f_{[aC+bE_0]}(F_k) \subseteq \mathbb{CP}^N \). For \( k > 0 \), the map \( f_{[0:C+bE_0]} \) collapses the section at infinity to a point, so \( F_{k(0,b)} \) is the image of the cone over the rational normal curve of degree \( k \) with respect to a suitable embedding.
In [9], a degeneration to a union of $2ab + kb^2$ planes was constructed in the following configuration (in Figure 2.2, $k = 2, a = 2, b = 3$ was taken). Each triangle represents a plane and each inner edge represents an intersection line between planes.

This degeneration is obtained by using a technique developed by Moishezon-Robb-Teicher which they refer to as the D-construction. The D-construction is described (and proved to work) in [7]. Specific degeneration for the Hirzebruch surfaces using the D-construction is explained in [9, Section 2, Theorem 2.1.2]. The difference between the D-construction and other blow-up procedures for obtaining degenerations is that the D-construction can also be applied along a subvariety of codim 0 (see, for example, Step 2 below). The degeneration is obtained via the following steps.

1. D-construction along $C$ to get $F_{0(a,b)} \cup F_{k(a-1,b)}$.
2. D-construction along $F_{0(1,b)}$ to get $F_{0(1,b)} \cup F_{0(1,b)} \cup F_{k(a-2,b)}$.
3. Induction on the second step to get $\overbrace{F_{0(1,b)} \cup \cdots \cup F_{0(1,b)} \cup F_{k(0,b)}}^{\text{a times}}$ (see ?).
4. Degeneration of each $F_{0(1,b)}$ to a union of $2b$ planes in the following configuration (here $b = 3$).
5. D-construction on $F_{k(0,b)}$ to get $\overbrace{F_{1(0,b)} \cup \cdots \cup F_{0(1,b)} \cup F_{1(0,b)}}^{\text{k times}}$ ($F_{1(0,b)}$ is the Veronese surface $V_b$).
6. Degeneration of each $F_{1(0,b)}$ to a union of $b^2$ planes in the following configuration (here $b = 3$).

Note that in our case $k = 1$, so we are looking at the surface $F_{1(a,b)}$. We now describe in greater detail the degenerated object and its branch curve using the degeneration described earlier. $F_{1(a,b)}$ is degenerated to $\tilde{F}_{1(a,b)}$, a union of planes in the following configuration.
Each triangle represents a plane; each inner edge represents an intersection line between planes. The number of the planes is $2ab + b^2$; the number of intersection lines is $3ab - a + \frac{3b}{2}(b-1)$. We take a generic projection of $\tilde{F}_{1,(a,b)}$ onto $\mathbb{C}\mathbb{P}^2$ where each plane is projected onto $\mathbb{C}\mathbb{P}^2$. The ramification curve of this projection is the union of lines. The singular points of the ramification curve are represented by the vertices. The branch curve of $\tilde{F}_{1,(a,b)} \to \mathbb{C}\mathbb{P}^2$, denoted by $\tilde{S}_{(a,b)}$, is the image of the union of lines and its singular points are the images of the vertices and the intersection points in $\mathbb{C}\mathbb{P}^2$ of the images of any two of the intersection lines. Special notations of the vertices and the edges of the complex in Figure 2.5 (which represent $\tilde{S}_{(a,b)}$) will be given in Section 4.

Figure 2.5

3 \( \tilde{B}_n \) and \( \tilde{B}_n \)-groups
The aim of this section is to introduce a few facts about \( B_n \) and a certain quotient of it, which will serve us in the next section.

Definition 1. \( B_n, S_n \).

The braid group on \( n \) strings is

\[
B_n = \left\{ x_1, \ldots, x_{n-1} \mid [x_1, x_j] = 1, \quad |i - j| > 1 \right\}.
\]

Recall that the permutation group is

\[
S_n = \left\{ x_1, \ldots, x_{n-1} \mid [x_1, x_j] = 1, \quad |i - j| > 1, \quad \langle x_i, x_j \rangle = 1, \quad |i - j| - 1, \quad x^2 = 1 \right\}.
\]

So, \( \exists \) homomorphism \( \varphi : B_n \to S_n \). Denote by \( \delta \) the degree homomorphism \( \delta : B_n \to \mathbb{Z} \); denote \( P_n = \ker \varphi \), \( P_{n,0} = P_n \cap \ker \delta \).

We now recall another definition of \( B_n \).

Let \( D \) be a closed disk in \( \mathbb{R}^2 \), \( K \subset \text{Int}(D) \), \( K \) finite, \( n = |K| \). Recall that the braid group \( B_n[D,K] \) can be defined as the group of all equivalent diffeomorphisms \( \beta \) of \( D \) such that \( \beta(K) = K, \beta|_{\partial D} = \text{Id}|_{\partial D} \).

Definition 2. \( H(\sigma) \), half-twist defined by \( \sigma \).

Let \( a, b \in K \), and let \( \sigma \) be a smooth simple path in \( \text{Int}(D) \) connecting \( a \) with \( b \) s.t. \( \sigma \cap K = \{a, b\} \). Choose a small regular neighborhood \( U \) of \( \sigma \) contained in \( \text{Int}(D) \), s.t. \( U \cap K = \{a, b\} \). Denote by \( H(\sigma) \) the diffeomorphism of \( D \) which switches \( a \) and \( b \) by a counterclockwise 180
degree rotation and is the identity on \( D \setminus U \). Thus it defines an element of \( B_n[D,K] \), called the half-twist defined by \( \sigma \).

**Definition 3.** \( \tilde{B}_n \).

Let \( \tilde{B}_n \) be the quotient of \( B_n \) by the following commutator, \( \tilde{B}_n = B_n/\langle [x_2,(x_2)_{x_1}z_3] \rangle \), that is, by the commutator of two half-twists intersecting transversally.

**Lemma 1**\(^7\). Let \( x, y \in \tilde{B}_n \).

(i) If the endpoints of \( x \) and \( y \) are disjoint, then \( [x,y] = 1 \);

(ii) If the endpoints of \( x \) and \( y \) have one common endpoint, the \( \langle x,y \rangle = 1 \).

**Proof.** Let \( \tilde{\varphi} \) be the induced homomorphism from \( \varphi \), s.t. \( \tilde{\varphi} : \tilde{B}_n \rightarrow S_n \). Define \( \tilde{P}_n = \ker \tilde{\varphi} \), \( \tilde{P}_{n,0} = \ker \tilde{\varphi} \cap \ker \tilde{\delta} \) (where \( \tilde{\delta} : \tilde{B}_n \rightarrow \mathbb{Z} \)).

We cite now the main results of \([6, \text{Section 1}]\); see also \([11]\).

**Lemma 2.** Denote by \( x_i \) the image of the generator \( X_i \) in \( \tilde{B}_n \). Let \( s_1 = x_1^2, \mu = [x_1^2,x_2^2], \)

\( u_i = [x_i^{-1}]_{x_i+1} \) \( \forall 1 \leq i \leq n - 2, u_{n-1} = [x_{n-2}^2,x_{n-1}] \). So \( \tilde{P}_{n,0} \) is generated by \( u_1, \ldots, u_{n-1} \), and \( \tilde{P}_n \) is generated by \( s_1, u_1, \ldots, u_{n-1} \).

We also have the following:

\[
[u_i, u_j] = \begin{cases} 
1 & |i-j| > 1, \\
\mu & \text{otherwise},
\end{cases} \quad [s_1, u_i] = \begin{cases} 
1 & i \neq 2, \\
\mu & i = 2.
\end{cases}
\]

Moreover, \( \mu^2 = 1, \mu \in \text{Center}(\tilde{B}_n) \) and \( \langle \mu \rangle = [\tilde{P}_{n,0}, \tilde{P}_{n,0}] = [\tilde{P}_n, \tilde{P}_n] \). Therefore, \( \tilde{P}_{n,0} \) is solvable and \( Ab(\tilde{P}_n) \simeq \mathbb{Z}^n, Ab(\tilde{P}_{n,0}) \simeq \mathbb{Z}^{n-1} \).

We can also formulate the action of \( \tilde{B}_n \) on \( \tilde{P}_n \) by conjugation:

\[
(s_1)_{x_i} = \begin{cases} 
s_1 & i \neq 2, \\
s_1u_2^{-1} & i = 2,
\end{cases} \quad (u_j)_{x_i} = \begin{cases} 
u_j & |i-j| > 1, \\
u_iu_j & |i-j| = 1, \\
u_i^{-1}\mu & i = j.
\end{cases}
\]

Actually, this action of \( \tilde{B}_n \) on \( \tilde{P}_n \) is developed (see \([7]\)) to abstract groups with \( \tilde{B}_n \) actions similar to the action on \( \tilde{P}_n \) and \( P_{n,0} \). This is explained in the following properties.

**Definition 4.** **Adjacent half-twists.**

If \( x, y \in \tilde{B}_n \) are two half-twists whose endpoints have only one point in common (and they can intersect each other transversally), we say \( x \) and \( y \) are adjacent.

The following definitions, lemmas and propositions are taken from \([11]\).

**Definition 5.** **Polarized half-twists, polarization.**

We say that a half-twist \( X \in B_n \) (or \( \tilde{X} \) in \( \tilde{B}_n \)) is polarized if we choose an order on the endpoints of \( X \). The order is called the polarization of \( X \) or \( \tilde{X} \).

**Definition 6.** **Orderly adjacent.**

Let \( X,Y \) be two adjacent polarized half-twists in \( B_n \) (resp. in \( \tilde{B}_n \)). We say that \( X,Y \) are orderly adjacent if their common point is the "end" of one of them and the "origin" of another.

The following definition derives its motivation from the action of \( \tilde{B}_n \) on \( \tilde{P}_n \).

**Definition 7.** **\( \tilde{B}_n \)-group.**
A group $G$ is called a $\tilde{B}_n$-group if there exists a homomorphism $\tilde{B}_n \to \text{Aut}(G)$. We denote $(g)_\tau$ by $g_\tau$.

**Definition 8.**  Prime element, supporting half-twist (s.h.t.) corresponding central element.

Let $G$ be a $\tilde{B}_n$-group.

An element $g \in G$ is called a prime element of $G$ if there exists a half-twist $X \in B_n$ and $\tau \in \text{Center}(G)$ with $\tau^2 = 1$ and $\tau_b = \tau \forall b \in \tilde{B}_n$ such that

1. $g_{X^{-1}} = g^{-1}\tau$.
2. For every half-twist $Y$ adjacent to $X$ we have $g_{X^{-1}Y^{-1}X^{-1}} = g_{X}^{-1}g_{Y}^{-1}$, $g_{Y^{-1}X^{-1}} = g_{X}^{-1}g_{Y}$.
3. For every half-twist $Z$ disjoint from $X$, $g_Z = g$.

The half-twist $X$ (or $X$) is called the supporting half-twist of $g$ ($X$ is the s.h.t. of $g$).

The element $\tau$ is called the corresponding central element.

**Lemma 4.** Let $G$ be a $\tilde{B}_n$-group.

Let $g$ be a prime element in $G$ with supporting half-twist $X$ and corresponding central element $\tau$. Then

1. $g_X = g_{X^{-1}} = g^{-1}\tau$, $g_X^2 = g$.
2. $g_Y = g\tau$, $\forall Y$ consecutive half-twist to $X$.
3. $[g, g_Y^{-1}] = \tau$, $\forall Y$ consecutive half-twist to $X$.

**Definition 9.** Polarized pair.

Let $G$ be a $\tilde{B}_n$-group, $h$ a prime element of $G$, $X$ its supporting half-twist. If $X$ is polarized, we say that $(h, X)$ (or $(h, X)$) is a polarized pair with central element $\tau$, $\tau = hh_{X}^{-1}$.

**Definition 10.** Coherent pairs, anti-coherent pairs.

We say that two polarized pairs $(h_1, X_1)$ and $(h_2, X_2)$ are coherent (anti-coherent) if $\exists b \in \tilde{B}_n$ such that $(h_1)^b = h_2$, $(X_1)^b = X_2$, and $b$ preserves (reverses) the polarization.

**Proposition 1.** Let $(h, X)$ be a polarized pair, $h \in G$, $X \in \tilde{B}_n$. Let $\tilde{T}$ be a polarized half-twist in $\tilde{B}_n$. Then there exists a unique prime element $g \in G$ such that $(g, \tilde{T})$ and $(h, X)$ are coherent.

**Definition 11.** $L_{h, X}(\tilde{T})$.

Let $(h, X)$ be a polarized pair $\tilde{T} \in \tilde{B}_n$. $L_{h, X}(\tilde{T})$ is the unique prime element s.t. $(L_{(h, X)}(\tilde{T}), \tilde{T})$ is coherent with $(h, X)$.

In fact, one can prove that $\tilde{P}_n$ (as a $\tilde{B}_n$-group) has a prime element, and $\tilde{P}_{n, 0}$ is generated by the orbit of this prime element.

**Lemma 5.** Let $X_1, X_2$ be 2 consecutive half-twists in $B_n$. Let $u = (X_2^2)_{X_1^{-1}}X_2^{-2}$. Then $u \in \tilde{P}_{n, 0}$, $u$ is a prime element in $\tilde{P}_n$ (considered as a $\tilde{B}_n$-group), and $X_1$ is the supporting half-twist of $u$.

**Lemma 6.** $\tilde{P}_{n, 0}$ is a primitive $\tilde{B}_n$-group generated by the $\tilde{B}_n$-orbit of a prime element $u = X^2Y^{-2}$, where $X, Y$ are adjacent half-twists in $\tilde{B}_n$, $\tilde{T} = X^2Y^{-2}$ is a supporting half-twist for $u$.

We shall also cite from [11] the criterion for prime elements in $\tilde{B}_n$-groups; we will not use it directly, but rather implicitly, when quoting, in Section 4, the results for the $\tilde{B}_n$-groups (see Lemma 18).
Proposition 2. Assume $n \geq 5$. Let $G$ be a $\tilde{B}_n$-group, and let $(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_{n-1})$ be a standard base of $\tilde{B}_n$. Let $S$ be an element of $G$ with the following properties:

(0) $G$ is generated by $\{S_b, b \in \tilde{B}_n\}$;

(1) $S_{\tilde{X}_1^2-\tilde{X}_2^{-1}}^{-1} = S_{\tilde{X}_1}^{-1} S_{\tilde{X}_2^{-1}}^{-1} = S_{\tilde{X}_1}^{-1} S_{\tilde{X}_2}^{-1}$;

(2) For $\tau = SS_{\tilde{X}_1^{-1}}$, $T = S_{\tilde{X}_2^{-1}}$, we have $\tau_{\tilde{X}_1} = \tau$, $\tau_T = \tau^{-1}_{\tilde{X}_1}$;

(3) $S_{\tilde{X}_j} = S$, $\forall j \geq 3$; $S_c = S$, where $c = [\tilde{X}_1^2, \tilde{X}_2^2]$.

Then $S$ is a prime element of $G$, $\tilde{X}_1$ is a supporting half-twist of $S$ and $\tau$ is the corresponding central element. In particular, $\tau^2 = 1$, $\tau \in \text{Center}(G)$, $\tau_b = \tau$, $\forall b \in \tilde{B}_n$.

4 Calculation of the fundamental group

In this section we will calculate the fundamental group of the complement of the branch curve of $F_{1,(a,b)}$. This computation requires explicit knowledge of the braid monodromy factorization (BMF) technique. This knowledge can be found at [1, 12, 13]. However, we recall the main definitions regarding the braid monodromy factorization related to a curve $S$.

Definition 12. The braid monodromy w.r.t. $S, \pi, u$.

Let $S$ be a curve, $S \subseteq \mathbb{C}^2$. Let $\pi : S \to \mathbb{C}^1$ be defined by $\pi(x, y) = x$. We denote $\deg \pi$ by $m$. Let $N = \{x \in \mathbb{C}^1 \mid \pi^{-1}(x) < m\}$. Take $u \notin N$, $u$ real, s.t. $\Re(x) < u$, $\forall x \in N$. Let $\mathbb{C}_u^1 = \{(u, y)\}$. There is a natural defined homomorphism

$$\pi_1(\mathbb{C}^1 - N, u) \xrightarrow{\varphi} B_m[\mathbb{C}_u^1, \mathbb{C}_u^1 \cap S],$$

which is called the braid monodromy w.r.t. $S, \pi, u$, where $B_m$ is the braid group. We sometimes denote $\varphi$ by $\varphi_u$. Note that in this definition we regard $B_m$ as the group of diffeomorphisms, as described in the previous section.

Denote the generator of the center of $B_n$ as $\Delta^2$. We recall Artin’s theorem on the presentation of $\Delta^2$ as a product of braid monodromy elements of a geometric-base (a base of $\pi_1 = \pi_1(\mathbb{C}^1 - N, u)$ with certain properties; see [14] for definitions).

Theorem 1. Let $S$ be a curve transversal to the line in infinity, and $\varphi$ is a braid monodromy of $S, \varphi : \pi_1 \to B_m$. Let $\delta_i$ be a geometric (free) base (g-base) of $\pi_1$. Then $\Delta^2 = \prod \varphi(\delta_i)$. This product is also defined as the braid monodromy factorization (BMF) related to a curve $S$.

Since $\tilde{S}_{F_{1,(a,b)}}$, which is the branch curve of the degenerated surface $\tilde{F}_{1,(a,b)}$, is a line arrangement, we can compute the braid monodromy factorization as in [14]. In order to compute the braid monodromy factorization of $\tilde{S}_{F_{1,(a,b)}}$, we use the regeneration rules. The regeneration methods are actually, locally, the reverse process of the degeneration method. When regenerating a singular configuration consisting of lines and conics, the final stage in the regeneration process involves doubling each line, so that each point of $K$ (which is the set of points in the disk, that is $\mathbb{C}_u^1 \cap \tilde{S}_{F_{1,(a,b)}}$) corresponding to a line labelled $i$ is replaced by a pair of points, labelled $i$ and $i'$. The purpose of the regeneration rules is to explain how the braid monodromy behaves when lines are doubled in this manner.

Let $F_{1,(a,b)}$, $a, b > 1$ be the Hirzebruch surface embedded w.r.t. the linear system $aC + bE_0$. As shown, $F_{1,(a,b)}$ could be degenerated into a union of $2ab + b^2$ planes in the following arrangement:
We shall give a special presentation of $B_n$, from which we will induce an injection of $\tilde{B}_n$ to $G = \pi_1(\mathbb{C}^2 - S_{F_1(a,b)})$.

**Remark 1.** From now on, we denote by $\tilde{S}_{F_1(a,b)}$ the branch curve of $F_1(a,b)$ (in $\mathbb{C}P^2$), and by $S_{F_1(a,b)}$ a generic affine portion of it (in $\mathbb{C}^2$).

Let $a, b$ be integers $b > 1, a \geq 1$; $n = 2ab + b^2$. Let $s_{ij} = (i, j), t_{ij} = (i + \frac{1}{2}, j)$ in $\mathbb{R}^2$. Let $K_{a,b}$ be the set in $\mathbb{R}^2$ consisting of the points $s_{ij}, t_{ij}, 1 \leq j \leq b, 1 \leq i \leq a + b$; so $\sharp K_{a,b} = 2ab + b^2$.

Let $D$ be a large disk in $\mathbb{R}^2$ containing $K_{a,b}$. Consider the oriented line segments $\vec{x}_{ij} = [s_{ij}, t_{ij}], 1 \leq j \leq b, \ 1 \leq i \leq a + b; \vec{y}_{ij} = [t_{ij}, s_{i+1,j}], 1 \leq j \leq b, 1 \leq i \leq a + j - 1; \vec{z}_{ij} = [s_{ij}, t_{i+1,j}], 1 \leq j \leq b - 1, 1 \leq i \leq a +j$. Consider $B_n = B_n[D, K_{a,b}]$. Let $X_{ij}, Y_{ij}, Z_{ij}$ be polarized half-twists in $B_n$ by the oriented segments $\vec{x}_{ij}, \vec{y}_{ij}, \vec{z}_{ij}$ respectively. Let $Z_{ij} = Z_{ij}$, when $i = a + j, 1 \leq j \leq b - 1$. We define $Z_{ij}$ for $1 \leq i \leq a + j - 1$, inductively:

$$Z_{ij} = (Z_{i+1,j})_{X_{i+1,j+1}}Y_{i,j+1}Y_{i,j}^{-1}X_{i,j}^{-1}.$$

**Proposition 3.** $B_n$ can be finitely presented as follows:

**Generators:**

- $X_{ij}, 1 \leq j \leq b, 1 \leq i \leq a + j$.
- $Y_{ij}, 1 \leq j \leq b, 1 \leq i \leq a + j - 1$.
- $Z_{ij}, i = a + j, 1 \leq j \leq b - 1$.

**Relations:**

\[
\forall \text{ two generators } a, b \text{ of the above which are adjacent, } \langle a, b \rangle = 1.
\]

\[
\forall \text{ two generators } c, d \text{ which are disjoint } [c, d] = 1.
\]

\[
\forall j \in (1, \ldots, b - 1), i = a + j : [X_{i,j}, Z_{ij}Y_{i-1,j}Z_{ij}^{-1}] = 1.
\]

**Proof.** This is a standard consequence of the usual presentation of $B_n[D, K_{a,b}]$ (see [14]). The formulas define inductively a polarization for each $Z_{ij}$. One can check that it coincides with the given polarization of $Z_{ij}$, i.e., corresponds to the ordered pair $(s_{ij}, t_{i,j+1})$.

Denote by $x_{ij}, y_{ij}, z_{ij}$ the images of $X_{ij}, Y_{ij}, Z_{ij}$ in $\tilde{B}_n$. Thus we get a representation of $\tilde{B}_n$. We consider $\{x_{ij}, y_{ij}, z_{ij}\}$ with polarization introduced above.

**Definition 13.** Let $G$ be a primitive $\tilde{B}_n$-group generated by the orbit of a prime element $B_{1,1}$ supported by the half-twist $Y_{1,1}$. According to Proposition 1, $\forall$ polarized half-twist $t \in \tilde{B}_n$, $\exists$ unique prime element $L_{B_{1,1},Y_{1,1}}(t) \in G$, s.t. the pair $\{L_{B_{1,1},Y_{1,1}}(t), t\}$ is coherent with $\{B_{1,1}, Y_{1,1}\}$.

Define $$A_{ij} = L_{B_{1,1}, Y_{1,1}}(x_{ij}), \quad B_{ij} = L_{B_{1,1}, Y_{1,1}}(y_{ij}), \quad C_{ij} = L_{B_{1,1}, Y_{1,1}}(z_{ij}).$$

**Remark 2.** Looking at [6, Remark 6], one gets the formulas for the $\tilde{B}_n$-action on $G$ in terms of $\{x_{ij}, y_{ij}, z_{ij}; A_{ij}, B_{ij}, C_{ij}; i, j = \ldots\}$. In particular, we see that $G$ is generated by $\{A_{ij}, B_{ij}, C_{ij}\}$ (because $G$ is generated by the $\tilde{B}_n$-orbit of $B_{1,1}$).

Denote by $\tilde{S}_{a,b} := \tilde{S}_{F_1(a,b)}$ the degenerated branch curve of $F_1(a,b)$. We define now a planar 2-complex, to represent the polygon in Figure 4.1.

**Definition 14.** We use a planar 2-complex $K(a,b)$ defined as follows: $K(a,b) \subset \mathbb{R}^2$. Define $P$, the polygon whose vertices are $(0,0), (a,0), (a+b, b), (0,b)$. So the vertices of $K(a,b)$
are the points \( \{ \omega_{r,k} = (r,k) \}_{r,k \in \mathbb{Z}} \). The edges of \( K(a,b) \) are the straight line segments of the following three types:

(a) “diagonal”: \( [\omega_{r,k}, \omega_{r+1,k+1}] \), \( 0 \leq k \leq b-1 \), \( 0 \leq r \leq a+k-1 \);

(b) “vertical”: \( [\omega_{r,k}, \omega_{r,k+1}] \), \( 0 \leq k \leq b-1 \), \( 0 \leq r \leq a+k-1 \);

(c) “horizontal”: \( [\omega_{r,k}, \omega_{r+1,k}] \), \( 0 \leq k \leq b \), \( 0 \leq r \leq a+k-1 \).

The 2-simplices of \( K(a,b) \) are the triangles \( \Delta \{ \omega_{r,k}, \omega_{r+1,k}, \omega_{r+1,k+1} \} \) and \( \Delta \{ \omega_{r,k}, \omega_{r,k+1}, \omega_{r+1,k+1} \} \).

![Figure 4.1](image)

**Definition 15.** The vertices \( \omega_{r,k} \) that are not on the boundary of \( P \) will be called 6-point; the vertices \( \omega_{0,0}, \omega_{a,0} \) will be called 2-point; and all the other vertices \( \omega_{r,k} \) on the boundary of \( P \) s.t. \( (r,k) \neq (0,b), (a+b,b) \) will be called 3-point.

**Definition 16.** (1) Consider \( B_m = B_m[D,K] \), where \( D \) is a large disk in \( \mathbb{C}^1 \), centered at \( (0) \) and \( K = \{ q_{r,k}^{(c)} \mid \varepsilon = 1, 2, 3, \delta = 0, 1 \text{ s.t.: for } \varepsilon = 1, 1 \leq k \leq b, 1 \leq r \leq a+k-1; \text{ for } \varepsilon = 2, 1 \leq k \leq b, 1 \leq r \leq a+k-1; \text{ for } \varepsilon = 3, 1 \leq k \leq b-1, 1 \leq r \leq a+k; \} \) are real points such that \( q_{r,k}^{(c)} \), \( q_{r,k}^{(c)'} \) are very close to each other, and \( q_{r,k}^{(c)} < q_{r,k}^{(c)'} \) if either \( k < k' \) or \( k = k' \) and \( r < r' \) or \( k = k' \), \( r = r' \) and \( \varepsilon < \varepsilon' \), or \( k = k' \), \( r = r' \), \( \varepsilon = \varepsilon' \) and \( \delta < \delta' \).

The points of \( K \) that we associate with the non-boundary edges of \( K(a,b) \) are as follows: \( q_{r+k,0}^{(1)}, q_{r+1,1}^{(1)} \) correspond to the diagonal edge \( \omega_{r-1,k-1}, \omega_{r,k} \); \( q_{r+k,0}^{(2)}, q_{r+1,1}^{(2)} \) correspond to the vertical edge \( \omega_{r,k-1}, \omega_{r,k} \); \( q_{r+k,0}^{(3)}, q_{r+1,1}^{(3)} \) correspond to the horizontal edge \( \omega_{r-1,k}, \omega_{r,k} \).

As was indicated earlier, during the regeneration process, each line doubles itself, and thus each point of \( \mathbb{C}^1 \cap \tilde{S}_{a,b} \) is replaced by a pair of points, which are \( q_{r,k}^{(c)} \) and \( q_{r,k}^{(c)'} \).

(2) Let

\[
m_{r,k} = \begin{cases} 12 & \text{if } \omega_{r,k} \text{ is a 6-point,} \\ 4 & \text{if } \omega_{r,k} \text{ is a 3-point,} \\ 2 & \text{if } \omega_{r,k} \text{ is a 2-point.} \end{cases}
\]

Denote by \( K_{r,k} \) the subset of \( K \) consisting of the points associated with the non-boundary edges of \( K(a,b) \) which meet at \( \omega_{r,k} \). Clearly, \( 2K_{r,k} = m_{r,k} \).

(3) Denote \( f_{r,k} : B_{m_{r,k}} \to B_m[D,K] \) an embedding of \( B_{m_{r,k}} \) into \( B_m[D,K] \) corresponding to a connection below the real axis of the points of \( K_{r,k} \) by consecutive simple paths (see [14]). Clearly, each \( B_{m_{r,k}} \) is either \( B_{12}, B_4 \) or \( B_2 \).

From each 6/3/2-point, relations between the generators of the fundamental group \( \pi_1(\mathbb{C}^2 - S_{F_{k,(a,b)}}) \) can be induced. These relations are written with the same notations as in [6]. We refer the readers to this article. However, we state a few of the main results.
Consider $K \subset D$, $K = \{ q^{(c)}_{r,k}\delta \}$. Take a point $u$ on $\partial D$ below the real axis. Using small (positively oriented) circles around the points $q^{(c)}_{r,k}\delta$ and connecting these circles by (straight) simple lines with $u$, we obtain a geometric base $\{ \gamma^{(c)}_{r,k}\delta \}$ for $\pi_1(D - K, u)$.

A full set of relations between $\{ \gamma^{(c)}_{r,k}\delta \}$ can be described, corresponding to the braid monodromy factorization (see [14] for a formula computing the BMF of a generic line arrangement—which is actually the factorization on which we perform the regeneration process to get the following):

$$\Delta^2 = \varepsilon(a, b) = \prod_{r,k} C(r,k) \mathcal{H}(r,k),$$

where $\mathcal{H}(r,k)$ are the factorizations induced from the 6/3/2-points-$\omega_{r,k}$ (see Appendix). $C(r,k)$ are the factorizations that we get from the parasitic intersection of the branch curves (see [6, Chapter 2] or [14]). We get a presentation of $\pi_1(\mathbb{C}^2 - S_{F_1,(a,b)})$ by using the Van-Kampen Theorem [15] which says that from each factor from $\varepsilon(a,b)$, a relation between $\{ \gamma^{(c)}_{r,k}\delta \}$ can be induced. Taking a braid which is a half-twist that corresponds to a path $\sigma$ from $q^{\varepsilon_1}_{r_1,k_1}\delta_1$ to $q^{\varepsilon_2}_{r_2,k_2}\delta_2$ via $u$, we let $\delta_1$ (resp. $\delta_2$) be the path from $u$ to $q^{\varepsilon_1}_{r_1,k_1}\delta_1$ (resp. $q^{\varepsilon_2}_{r_2,k_2}\delta_2$) along $\sigma$, going around $q^{\varepsilon_1}_{r_1,k_1}\delta_1$ (resp. $q^{\varepsilon_2}_{r_2,k_2}\delta_2$) and coming back to $u$ along the same path, respectively. Let $A$ and $B$ be the homotopy classes of a loop around $q^{\varepsilon_1}_{r_1,k_1}\delta_1$ (resp. $q^{\varepsilon_2}_{r_2,k_2}\delta_2$) along $\delta_1$ (resp. $\delta_2$). $A$ (resp. $B$) is a conjugation of $\gamma^{\varepsilon_1}_{r_1,k_1}\delta_1$ (resp. $\gamma^{\varepsilon_2}_{r_2,k_2}\delta_2$).

By the Van Kampen Theorem, we have one of the following relations in $\pi_1(\mathbb{C}^2 - S_{F_1,(a,b)})$ (fixed according to the type of singularity, from which we have the path $\sigma$):

1) $A = B$, if the singularity is a branch point,
2) $|A,B| = ABA^{-1}B^{-1} = e$ if it is a node,
3) $\langle A,B \rangle = ABAB^{-1}A^{-1}B^{-1} = e$ if it is a cusp.

**Definition 17.** Let

$$\ell_{r,k}^{(1)} = \begin{cases} 1 - k, & \text{for } r \geq k, \\ 1 - r, & \text{for } r < k, \end{cases} \quad \ell_{r,k}^{(2)} = k - 1, \quad \ell_{r,k}^{(3)} = 0.$$

(Evidently, $\ell_{r+1,k}^{(3)} = \ell_{r,k}^{(3)}$, $\ell_{r,k+1}^{(2)} = \ell_{r,k}^{(2)} + 1$; $\ell_{r+1,k}^{(1)} = \ell_{r,k}^{(1)} - 1$). Let $e^{(c)}_{r,k\delta} = (\gamma^{(c)}_{r,k\delta})(\rho^{(c)}_{r,k\delta})e^{(c)}_{r,k}$, where $\rho^{(c)}_{r,k}$ is the half-twist in $B_m[D,K]$ defined by the segment $[q^{(c)}_{r,k\delta}, q^{(c)}_{r,k\delta}]$.

**Definition 18.** Denote by $G$ the group defined by $\varepsilon(a,b)$; more precisely, the quotient of the free group generated by $\{ e^{(c)}_{r,k}\delta \}$, modulo relations (we call them $R\varepsilon$) induced from 6/3/2-points, and the relation induces from the parasitic intersections, for all $r,k$ (see [6, Chapter 3] for those relations or in the Appendix).

By the definition of $\varepsilon(a,b)$ (braid monodromy factorization for $S_{F_1,(a,b)}$), we have $G \simeq \pi_1(\mathbb{C}^2 - S_{F_1,(a,b)}, u)$. Let $E^{(c)}_{r,k\delta}$ be the images of $e^{(c)}_{r,k\delta}$ in $G$.

**Proposition 4.** $\exists$ homomorphism $\tilde{\alpha} : \hat{B}_n \rightarrow G$ which is defined by:

$$\tilde{\alpha}(x_{ij}) = E^{(1)}_{ij0}, \quad \tilde{\alpha}(y_{ij}) = E^{(2)}_{ij0}, \quad \forall i,j,$$

$$\tilde{\alpha}(z_{ij}) = E^{(3)}_{ij0}, \quad \text{where } i = a + j;$$

moreover, $\tilde{\alpha}(z_{ij}) = E^{(3)}_{ij0}, \quad \forall (i,j) \in \text{Vertices}(K(a,b)), \; i \neq a + j.$

**Proof.** See [6, Proposition 8]. See the induced relations for each 2/3/6-point and explanation why $\hat{B}_n$ can be embedded in $G$ in the Appendix.
Let $E_{rk} = E_{rk0}^\perp B$ be the subgroup of $G$ generated by $\{ E_{rk} \}$. It follows from Proposition 4 that $B = \alpha(B_n)$. Let $P = \alpha(P_n)$, $P_0 = \alpha(P_{n,0})$ (where $P_{n,0} = \ker(P_n \to Ab(B_n))$, $P_{n,0}$ is the image of $P_{n,0}$ in $B_n$). From [6, Theorem 1] or from Lemma 6, it follows that $P_{n,0}$ is a primitive $\tilde{B}_n$-group with prime element $u = (y_{11}^2, x_{12}^{-1}, x_{21}^{-2}) (x_{21}, y_{11})$ are two adjacent half-twists in $\tilde{B}_n$, and s.h.t. equal to $y_{11}$. Denote $c = [y_{11}^2, x_{21}^{-2}]$. We get from [6, Theorem 1] that $c^2 = 1$, $c \in \text{Center}(\tilde{B}_n)$ and $c$ generates $\tilde{P}_n$ and $\tilde{P}_{n,0}$. Denoting $\eta_{1,1} = \tilde{e}(u) = (E_{2,1}^{(2)})^2, \eta_{1,1} = \eta_{2,1}$, $\mu = \tilde{e}(c) = (E_{1,1}^{(2)})^2, (E_{2,1}^{(1)})^2$, we get that $\eta_{1,1}$ is a prime element of $\eta_{2,1}$ with s.h.t. $y_{11}, \mu^2 = 1$, $\mu \in \text{Center}(\tilde{B}_n)$ and $\mu$ generates $\tilde{P}_n$ and $\tilde{P}_{n,0}$. Using the polarization of $X_{ij}, Y_{ij}, Z_{ij}$ and Proposition 3, we can find $\forall t \in \{x_{ij}, y_{ij}, z_{ij}\}$ (the generators of $\tilde{B}_n$) and the pair $\{ (x_{ij}, y_{ij}, z_{ij}) \} \in \text{Vertices}(K(a, b))$, $i, j > 1, i \neq a + j$ unique $L_{\{y_{11}, y_{11}\}}(t) \in \tilde{P}_0$ s.t. the pair $\{ L_{\{y_{11}, y_{11}\}}(t), \} \in \text{coherent with } \{ \eta_{1,1}, y_{11} \}$. 

**Definition 19.** Recall that $u = (y_{11}^2, x_{21}^{-1}, x_{21}^{-2})$, $\eta_{1,1} = \tilde{e}(u)$. Define

$$
\xi_{i,j} = L_{\{x_{11}, y_{11}\}}(x_{ij}), \quad \eta_{i,j} = L_{\{y_{11}, y_{11}\}}(y_{ij}), \quad \zeta_{i,j} = L_{\{y_{11}, y_{11}\}}(z_{ij}).
$$

**Lemma 7.** $\mu \in \text{Center}(G)$. 

**Proof.** See [6, Lemma 16].

**Definition 20.** Let $d_{rk} = E_{rk1}^*(E_{rk0})^{-1}, v_{rk} = E_{rk1}^*(E_{rk0})^{-1}, h_{rk} = E_{rk1}^*(E_{rk0})^{-1}$ with $d, v, h$ corresponding to “diagonal”, “vertical”, “horizontal”. Clearly, $G$ is generated by $\{d_{rk}, v_{rk}, h_{rk}; r, k = \ldots \}$ and $B$. Denote by $H$ the subgroup of $G$ generated by the $B$-(or $B_n$-) orbit of $\eta_{1,1}$. 

**Lemma 8.** 1) $H$ is a primitive $\tilde{B}_n$-group with prime element $\eta_{1,1}$, s.h.t. $y_{11}$, $\eta_{1,1}$ is actually a prime element of $G$ with s.h.t. $y_{11}$ (i.e., $\eta_{1,1} \cdot (\eta_{1,1})^{-1} \in \text{Center}(G)$). 

**Proof.** As in [6, Lemma 7].

**Definition 21.** Using the polarization of $X_{ij}, Y_{ij}, Z_{ij}$, we find $\forall t \in \{x_{ij}, y_{ij}, z_{ij}\}$ $\exists L_{\{y_{11}, y_{11}\}}(t) \in H$ s.t. the pair $\{ L_{\{y_{11}, y_{11}\}}(t), \} \in \text{coherent with } \{ \eta_{1,1}, y_{11} \}$. Define

$$
a_{ij} = L_{\{y_{11}, y_{11}\}}(x_{ij}), \quad b_{ij} = L_{\{y_{11}, y_{11}\}}(y_{ij}), \quad c_{ij} = L_{\{y_{11}, y_{11}\}}(z_{ij}).
$$

**Remark 3.** $\xi_{ij}, \eta_{ij}, \zeta_{ij}$ $(a_{ij}, b_{ij}, c_{ij})$ coincide with $A_{ij}, B_{ij}, C_{ij}$ introduced in Definition 13 for an arbitrary primitive $B_n$-group $G$, when this $G$ is replaced by $\tilde{P}_0$ (resp. $H$), and $\{B_{11}, Y_{11}\}$ is replaced by $\{y_{11}, y_{11}\}$ (resp. $\{v_{11}, y_{11}\}$). Therefore, replacing $A_{ij}, B_{ij}, C_{ij}$ by $\xi_{ij}, \eta_{ij}, \zeta_{ij}$ (resp. $a_{ij}, b_{ij}, c_{ij}$), we obtain formulas expressing the $B_n$-action on $\tilde{P}_0$ (resp. on $H$). In particular, $\tilde{P}_0$ (resp. $H$) is generated by $\{\xi_{ij}, \eta_{ij}, \zeta_{ij}\}$ (resp. $\{a_{ij}, b_{ij}, c_{ij}\}$).

**Definition 22.** $\forall x_{ij}, y_{ij}, z_{ij}$, let $\tilde{a}_{ij} = \tilde{e}(x_{ij}), \tilde{b}_{ij} = \tilde{e}(y_{ij}), \tilde{c}_{ij} = \tilde{e}(z_{ij})$. 

**Remark 4.** We have, by [6, Remark 30], the following:

$$
d_{r+1,k+1} = (d_{rk})_{x_{r+1}y_{r+1}z_{r+1} \beta_{r+1,k+1}}, \quad h_{r+1,k} = (h_{rk})_{x_{r+1}y_{r+1}z_{r+1} \beta_{r+1,k+1}}, \quad v_{r+1,k} = (v_{rk})_{x_{r+1}y_{r+1}z_{r+1} \beta_{r+1,k+1}}.
$$

**Remark 5.** By [6, Remark 31], we have

$$
d_{r+1,k} = \tilde{y}_{r+1}^2(v_{rk})_{x_{r+1}y_{r+1}z_{r+1} \beta_{r+1,k+1}}, \quad d_{1,k+1} = \tilde{y}_{1,k}(h_{1,k})_{x_{1,k} \beta_{1,k+1}}, \quad d_{1,k+1} = \tilde{y}_{1,k}(h_{1,k})_{x_{1,k} \beta_{1,k+1}}.
$$
Proposition 5. Let $\lambda(k) = \frac{k(k-1)}{2}$. We have
\[ h_{r,k} = c_{r,k}^{-k-1}\mu\lambda(k), \quad \forall \ r, k, \quad d_{r,k} = a_{r,k}^{-k-1}k\mu\lambda(k-r), \quad \forall \ r, k, \]
\[ v_{r,k} = b_{r,k}^{-r-1}\mu\lambda(r), \quad \forall \ r < a. \]

Proof. See [6, Proposition 10].

Proposition 6. $v_{a,k} = 1$, $\forall 0 \leq k \leq b$.

Proof. By the definition, $v_{a,0} = E^{(2)}_{a+1,2,0}(E^{(2)}_{a+1,2,0})^{-1}$, but $\omega_{a,0}$ is a 2-point, and the induced relation from it is $\gamma^{(2)}_{a,0} = \gamma^{(2)}_{a,0}$ or $1 = E^{(2)}_{a+1,2}(E^{(2)}_{a+1,2})^{-1}$. By the relation $v_{r,k+1} = (v_{r,k})^{-1}_{\omega_{a,1,k}z_{r,k}^{-1}z_{r+1,k}^{-1}h_{r+1,k+1}}$, we can see that $v_{a,k} = 1$ $\forall 0 \leq k \leq b$.

Proposition 7. For $(r, k) \in \{(a + 1, 2), (a + 2, 3), \ldots, (a + b - 1, b - 1)\}$, $I$,
\[ v_{r,k} = (E^{(3)}_{r,k-1})^{-2}h_{r,k-1}^{-1}(h_{r,k-1})^{-1}(E^{(2)}_{r,k+1})^{-1}(E^{(2)}_{r,k+1})^{-1}. \]

Proof. Assume $(r, k) = (a + 1, 2)$. The proof for the other points is the same.

We have by the relations induced from the 3-point $\omega_{a+1,2}$:
\[ E^{(3)}_{a+1,2,0} = (E^{(2)}_{a+1,2,0})(E^{(3)}_{a+1,2,0})^{-1}(E^{(3)}_{a+1,2,0})^{-1}, \]
or
\[ v_{a+2,1} = (E^{(3)}_{a+1,2,0})^{-2}h_{a+1}^{-1}(E^{(3)}_{a+1,2,0})^{-1}(E^{(3)}_{a+1,2,0})^{-1}. \]

By abuse of notation, we last the last index from the $E_{\ldots}$.

We know that $\eta_{r,k}$ (for $(r, k) \in I$) is a prime element with s.h.t. $y_{r,k}$ and a central element $\mu$.

So it can be proven (see [5, Claim 5.5]) that $\eta_{r,k} = (E^{(3)}_{r,k-1})^{-2}(E^{(3)}_{r,k-1})^{-1}$ or
\[ \mu\eta_{r,k}^{-1} = (E^{(3)}_{r,k-1})^{-2}(E^{(3)}_{r,k-1})^{-1}. \]

(1)

So we have
\[ v_{r,k} = (E^{(3)}_{r,k-1})^{-2}h_{r,k-1}^{-1}(h_{r,k-1})^{-1}(E^{(3)}_{r,k-1})^{-1}(E^{(3)}_{r,k-1})^{-1}. \]
\[ = h_{r,k-1}^{-1}(E^{(3)}_{r,k-1})^{-2}(E^{(3)}_{r,k-1})^{-1}(h_{r,k-1})^{-1}(E^{(3)}_{r,k-1})^{-1}. \]
\[ = h_{r,k-1}^{-1}\mu^{-1}(h_{r,k-1})^{-1}. \]

(2)

We compute now $(h_{r,k-1})^{-1}(E^{(3)}_{r,k-1})^{-1}$. We know that $(c_{r,k-1})^{-1}(E^{(3)}_{r,k-1})^{-1} = c_{r,k-1}b_{r,k}$ ([7, IV.6.3]) and $(\zeta_{r,k-1})^{-1}(E^{(3)}_{r,k-1})^{-1} = \zeta_{r,k-1}^{-1}(E^{(3)}_{r,k-1})^{-1}$. So
\[ (h_{r,k-1})^{-1}(E^{(3)}_{r,k-1})^{-1} = (\mu\eta_{r,k})^{-1}(c_{r,k-1}^{-1}(E^{(3)}_{r,k-1})^{-1} \zeta_{r,k-1}^{-1})(E^{(3)}_{r,k-1})^{-1}. \]

(see Proposition 5)

(3)
We substitute the expressions we found in Proposition 5, (1), (3) in (2), and we get (for \((r, k) \in I\)):

\[
v_{r,k} = (\mu \nu)^{\lambda(k-1)k-2} c_{r,k-1}^{1-k} \cdot \mu v_{r,k}^{-1} (\mu \nu)^{\lambda(k-1)} c_{r,k-1}^{k-1} b_{r,k}^{-1},
\]

\[
\begin{align*}
\mu^2 = 1 & \implies \mu^{\lambda(2)+1} \nu^{\lambda(k-1)} c_{r,k-1}^{1-k} \eta_{r,k} c_{r,k-1}^{k-1} b_{r,k}^{-1} \nu^{k-2} c_{r,k-1}^{k-2}, \\
\nu^2 = 1 & \implies \nu^{\lambda(2)+1} \mu^{\lambda(k-1)} c_{r,k-1}^{1-k} \eta_{r,k} c_{r,k-1}^{k-1} b_{r,k}^{-1} \mu^{k-2} c_{r,k-1}^{k-2}.
\end{align*}
\]

\[
\begin{align*}
\left[ c_{r,k-1}, \eta_{r,k} \right] & = \nu, \\
\left[ \zeta_{r,k-1}, b_{r,k} \right] & = \nu, \\
\mu^2 = \nu^2 & = 1
\end{align*}
\]

\[
\begin{align*}
\forall k, (k-2)(k-1) & \equiv 0 \pmod{2},
\end{align*}
\]

Now assume that \(a < r < b + a\), \(r - a + 1 \leq k \leq b\). Denote \(k' = r - a + 1\). So by using

\[
v_{r,k+1} = (v_{r,k})_{x_{r,k}^{-1} x_{r+1,k}^{-1} x_{r+1,k+1}}^{-1}
\]

we see that

\[
v_{r,k} = \mu^{\lambda(k'-2)+1} \nu^{\lambda(k')} b_{r,k}^{-1} \eta_{r,k}^{-1}. \tag{4}
\]

**Remark 5.** (1) As in [6], we can consider a 3-point \(\omega_{r,b}\), \(r < a\) and see that \(b_{1,1}^{-2} = (\mu \nu)^{\lambda(b+1)} \nu\) (see [6, Proposition 11, (1)]).

(2) If \(b\) is odd, then \(\mu = \nu [6, Proposition 11, (3)]\).

Consider now the 3-point \(\omega_{a,b}\). We know that \(v_{a,b} = 1\), but \(d_{a,b} = a_{a,b}^{-b} b_{a,b}^{b-a} (\mu \nu)^{\lambda(b-a)}\).

**Proposition 8.** (1) If \(a \neq b\), then \(\mu(\mu \nu)^{\lambda(b-a+1)} = (b_{1,1}^{-1} a_{1,1}^{-1} b_{1,1}^{-1} a_{1,1}^{-1})\).

(2) If \(a = b\), then \(\mu = \eta_{1,1} = 1\).

**Proof.** (1) By Remark 5,

\[
1 = \tilde{x}_{a,b}^{-2}(a_{a,b}^{-1} b_{a,b} b_{a,b}^{b-a} (\mu \nu)^{\lambda(b-a)}) \eta_{a,b}^{-1} \tilde{x}_{a,b}^{-2}(\mu \nu)^{\lambda(b-a)} \eta_{a,b}^{-1}.
\]

\[
\begin{align*}
\tilde{x}_{a,b}^{-2}(\tilde{x}_{a,b}^{-2} g_{a,b}^{-1}) & = \mu \eta_{a,b}^{-1}, \\
\tilde{x}_{a,b}^{-2} g_{a,b}^{-1} & = \mu \eta_{a,b}^{-1}, \\
\eta_{a,b} & = (\xi_{a,b} g_{a,b}^{-1})^{-1}.
\end{align*}
\]

So \(\exists \gamma \in \tilde{B}_{a,b}\) s.t. \((b_{a,b})_{\gamma} = b_{1,1}, (\eta_{a,b})_{\gamma} = \eta_{1,1}\). Applying it, we obtain what we wanted.

(2) By Remark 5, we have

\[
1 = \tilde{x}_{a,b}^{-2}(\tilde{x}_{a,b}^{-2} g_{a,b}^{-1}) = \mu \eta_{a,b}^{-1}
\]

or \(\mu = \eta_{a,b}\). By the same argument as in (1), \(\mu = \eta_{1,1}\). By (6), we see that \(\tilde{x}_{a,b}^{-2}(\tilde{x}_{a,b}^{-2} g_{a,b}^{-1}) = 1\), or \(\eta_{a,b} = 1\); that is, \(\mu = \eta_{1,1} = 1\).
Proposition 9. If \( a \neq b \), then \((b_1,1^{-1})^{a-b}\eta_{1,1}^{-1} = (\nu)\lambda(b-a)\).

Proof. By (5), \( \eta_{a,b} = (a_{a,b}^{-1}b_{a,b}^{-1}a_{a,b}b_{a,b}^{-1}(\nu))_{y_{a,b}} = p_{a,b}^{-1}h_{a,b} (\nu)\lambda(b-a)\). Applying \( \gamma \) from above, we are done.

Note that if \( a = b \), we get \( b_{1,1}^2 = \nu\lambda(b+1) \).

Proposition 10. If \( b \) is even, \( a \) is odd, then \( \nu = 1 \); otherwise \( \mu = \nu = 1 \).

Proof. We will first prove a lemma.

Lemma 9. \( \forall \ r, \ a < r < a + b \), we have \( b_{1,1}^{-1}b_{1,1}^{-1} = \mu \lambda(\nu)\lambda(b-a) \).

Proof. By Remarks 4 and 5, we have from the 3-point \( \omega_r \) \((k' = r - a + 1)\):

\[
\mu^{(k'-2)} + \lambda(k')_{b_{r,b}}^{\nu(k')_{b_{r,b}}^{\nu-\nu-1} = \nu_{r,b}}(\omega_r)^{-1}(\mu)\lambda(b-r)\nu_{r,b}^{-1}
\]

\[
\Rightarrow \mu^{(k'-2)} + \lambda(k')_{b_{r,b}}^{\nu(k')_{b_{r,b}}^{\nu-\nu-1} = \nu_{r,b}}(\omega_r)^{-1}(\mu)\lambda(b-r)\nu_{r,b}^{-1}
\]

\[
\Rightarrow \nu_{r,b}^{-1(k'-1) + k' - 1} = \mu^{(k'-2)} + r - b_{r,b} \lambda(k') + b - (\mu)\lambda(b-r)
\]

\[
\Rightarrow k' = r - a + 1 \Rightarrow \nu_{r,b}^{-1} = \mu^{(r-a-1)}(\nu)\lambda(b-r)\lambda(b-r+1)
\]

\( \forall \ r, \exists \gamma_r \in \tilde{B}_n \), s.t. \( \gamma_{r,b} = \gamma_{1,1} \), \( \gamma_{b_r,b} = b_{1,1} \). Apply it, and we are done.

Assume \( b \) is odd. So we know that \( \mu = \nu \) (By Remark 6). If \( a = b \), then \( \nu = 1 \) (by Proposition 9). Else, \( a \neq b \). So from Lemma 9, set \( r = a + 1 \), and we get \( \eta(b_1,1^{-1})^{a-b}\eta_{1,1}^{-1} = \mu \).

From Proposition 9, if \( \mu = \nu \), \( b_{1,1}^{-1}b_{1,1}^{-1} = 1 \), so \( \mu = 1 \) \( \Rightarrow \nu = 1 \).

Assume now that \( b \) is even. From Lemma 9, when setting \( r = a + 1 \), we get

\[
(b_1,1^{-1})^{a-b} = \nu(\mu)\lambda(b-a).
\]  

(7)

If \( a = b \), then \( \eta_{1,1}^{-1} = \nu \); but \( \eta_{1,1} = 1 \), so \( \mu = \nu = 1 \). Else \( a \neq b \), we have by Proposition 9, \( b_{1,1}^{-1}b_{1,1}^{-1} = (\nu)\lambda(b-a) \). So we have \( \nu = 1 \) when \( b \) is even. Assume now that \( a \) is also even (and \( a \neq b \)). By Proposition 8, we get

\[
(b_1,1^{-1})^{a-b} = \mu(\mu)\lambda(b-a+1)
\]  

(8)

or (substituting \( \nu = 1 \)), we have the set of equations:

\[
\begin{cases}
(b_1,1^{-1})^{a-b} = \mu \cdot \mu \lambda(b-a+1) \\
(b_1,1^{-1})^{a-b} = \mu \lambda(b-a)
\end{cases}
\]

Thus,

\[
\mu \cdot \mu \lambda(b-a+1) = \mu \lambda(b-a) \Rightarrow \mu \cdot \mu \lambda(b-a) = 1 \Rightarrow b-a \text{ is even} \Rightarrow \mu = 1.
\]

As in [6], we define a \( \tilde{B}_n \)-group \( G_0(n) \) as the subgroup of \( G(n) \) generated by \( u_1, \ldots, u_{n-1} \); \( G_0(n) \) is \( \tilde{B}_n \)-isomorphic to \( \tilde{F}_{n,0} \) (recall that \( n = 2ab + b^2 \)).

**Definition 23.** \( G_0(n) \) is a group with

Generators:

\[
M_0 = \{ A_{ij}, 1 \leq j \leq b, 1 \leq i \leq a + j; \ B_{ij}, 1 \leq j \leq b, 1 \leq i \leq a + j - 1; \ C_{ij}, a < i < a + b, j = i - a \}.
\]
Relations:

(1) \( \forall a, b \in M_0 \) which are adjacent, \([a, b] = \tau\), where \( \tau \) is independent of (such) \( a, b \), \( \tau^2 = 1 \), \( \tau_d = \tau \), \( \forall d \in M_0 \).

(2) If \( a, b \in M_0 \) are not adjacent, then \([a, b] = 1\).

For each \( d \in M_0 \) we introduce the notion of supporting half-twist from \( \tilde{B}_n \) (resp. \( B_n \)) as follows: for \( d = A_{ij} \), it will be \( x_{ij} \) (resp. \( X_{ij} \)); for \( d = B_{ij} \), it will be \( y_{ij} \) (resp. \( Y_{ij} \)); for \( d = C_{ij} \), \( i = j + a \), it will be \( z_{ij} \) (resp. \( Z_{ij} \)).

We say that \( a, b \in M_0 \) are adjacent if their supporting half-twists are adjacent.

The \( \tilde{B}_n \)-action on \( G_0(n) \) in terms of \( \tilde{M} = \{x_{ij}, y_{ij}\} \cup \{z_{ij} \mid a < i < a + b\} \) and \( M_0 \) is defined in [6, Remark 6]. We consider the elements of

\[
\tilde{M} = \tilde{M} \cup \{z_{ij} \mid (i, j) \in \text{Vertices}(K(a, b))\}, \quad i, j \geq 1 \text{ and if } a < i < a + b, \text{ then } j \neq i - a
\]
as polarized half-twists, and define a larger subset of \( G_0(n) : \tilde{M}_0 \); when \( M_0 \subset \tilde{M}_0 \) s.t.:

\[
\tilde{M}_0 = M_0 \cup \{C_{ij} \mid (i, j) \in \text{Vertices}(K(a, b))\}, \quad i, j \geq 1 \text{ and if } a < i < a + b, \text{ then } j \neq i - a.
\]

We start with the pair \( \{B_{1,1}, y_{1,1}\} \). Then \( \forall t \in \tilde{M}_1 \), define \( L_0(t) \in \tilde{M}_0 \) as the unique element \( L_{\{B_{1,1}, y_{1,1}\}}(t) \) s.t. \( \{B_{1,1}, y_{1,1}\} \) and \( \{L_{\{B_{1,1}, y_{1,1}\}}(t), t\} \) are coherent. The definition of a \( \tilde{B}_n \)-action on \( G_0(n) \) is such that \( L_0(x_{ij}) = A_{ij}, L_0(y_{ij}) = B_{ij}, L_0(z_{ij}) = C_{ij} \) where \( a < i < a + b, j = i - a \).

So for \( t \in \tilde{M}_1 \), we have \( L(t) \in M_0 \).

Define \( C_{ij} = L_0(z_{ij}) \).

**Definition 24.** Using the \( \tilde{B}_n \)-action on \( G_0(n) \), we define canonically the semi-direct product \( G_0(n) \rtimes \tilde{B}_n \). Let \( u = y_{1,1}^2 x_{2,1}^2 \in \tilde{F}_n, 0 \subset \tilde{B}_n \). Let \( N(a, b) \) be the normal subgroup of \( G_0(n) \rtimes \tilde{B}_n \), normally generated by the elements:

\[
n_1 = B_{1,1}^b u^{2-a} c(t)^{\lambda(a - b)};
n_2 = c(t)^b;
n_3 = \left( B_{1,1} u^{-1} \right)^a u^{-1} c(t)^{\lambda(a - b + 1)};
n_4 = \left( B_{1,1} u^{-1} \right)^a u^{-1} \cdot \tau(c(t))^{\lambda(a - b)}
\]

(when \( c = \{x^2, y^2\} \), \( x, y \) are any two adjacent half-twists in \( \tilde{B}_n \); \( \lambda(k) = \frac{k(k - 1)}{2} \)).

Note that the elements in \( N(a, b) \) are defined according to the relations found in Proposition 10((7), (8)) and Remark 6.

Define \( G(a, b) = (G_0(n) \rtimes \tilde{B}_n)/N(a, b) \). So as in [6, Proposition 32], one can prove that \( \pi_1(C^2 - S_{F_1,(a,b)}) \simeq G(a, b) \).

Define \( \psi_{a,b} : G(a, b) \to S_n \), by \( \psi_{a,b}(a, \beta) = \psi(\beta) \), where \( \psi : \tilde{B}_n \to S_n \) is the homomorphism to the symmetric group, induced from the standard homomorphism \( B_n \to S_n \). Let \( H_{a,b} = \ker \psi_{a,b} \), \( (H_{a,b})_0 = \ker(H_{a,b} \to Ab(G(a, b))) \), or, in other words, if \( Ab_{a,b} = \text{abelization map of } G(a, b) \), then \( (H_{a,b})_0 = \ker \psi_{a,b} \cap \ker Ab_{a,b} \). Note that \( G(a, b)/H_{a,b} \simeq S_n \). Also define \( \overset{\sim}{\psi}_{a,b} : \pi_1(C^2 - S_{F_1,(a,b)}) \to S_n \), and let \( \overset{\sim}{\theta}_{a,b} = \ker \overset{\sim}{\psi}_{a,b} \). In the same way as above, we define \( \overset{\sim}{\theta}_{a,b} \) and \( (\overset{\sim}{\theta}_{a,b})_0 \).

So we have the following
This Appendix describes the braid monodromy factorizations induced from the regeneration of 5 points from Proposition 10. 3) follows from the definition of $\mathcal{N}(a,b)$ and the following facts: $n_1 = B_{1,1}^h u^{-b_1} c(\tau^r)^{\lambda b_{10}} = (B_{1,1} u^{-1})^b u^2 c(\tau^r)^{\lambda b_{10}}$ and

\[
\mathbb{Z}^2/\langle a, b, (a-b, 1) \rangle = \mathbb{Z}^2/\langle (b, 2), (a-b, 1) \rangle = \mathbb{Z}^2/\langle (b-2a, 0), (0, 1) \rangle = \mathbb{Z}_{b-2a}.
\]

As in [6, p. 74], one can consider the projective case $\pi_1(\mathbb{C} \mathbb{P}^2 - \mathcal{S}_{\mathcal{F}_{1, (a,b)}}) \simeq G(a,b)/(y_{10}^2 \cdot U)$, where $2n_1 = \deg \mathcal{S}_{\mathcal{F}_{1, (a,b)}} = 6ab - 2a - 2b + 3b^2$, $U \in (H_{a,b})_0$. From the definition of $\mathcal{H}_{a,b}$, $(\mathcal{H}_{a,b})_0$ it follows that they coincide with the images of $H_{a,b}$ and $(H_{a,b})_0$ in $G(a,b)/(y_{10}^2 \cdot U) = \mathcal{G}(a,b)$. So by the same arguments as in [6], we have $\mathcal{H}_{a,b}/(\mathcal{H}_{a,b})_0 \simeq \mathbb{Z}_{m_1}$, $(\mathcal{H}_{a,b})_0 \simeq (H_{a,b})_0$, so $\text{Ab}(\mathcal{H}_{a,b})_0 \simeq (\mathbb{Z}_{b-2a})^{n-1}$, and

\[
\mathcal{H}_{a,b} \simeq (\mathcal{H}_{a,b})_0 \simeq (H_{a,b})_0 \simeq \left\{ \begin{array}{ll} \mathbb{Z}_2, & \text{b even, a odd}, \\ 1, & \text{else}. \end{array} \right.
\]

Thus, there exists a series $1 \prec (H_{a,b})_0 \prec (H_{a,b})_0 \prec (H_{a,b})_0 \prec G(a,b)$, s.t.

\[
G(a,b)/H_{a,b} \simeq S_n, \quad H_{a,b}/(H_{a,b})_0 \simeq \mathbb{Z}, \quad (H_{a,b})_0/(H_{a,b})_0' \simeq (\mathbb{Z}_{b-2a})^{n-1},
\]

and

\[
(H_{a,b})_0' \simeq \left\{ \begin{array}{ll} \mathbb{Z}_2, & \text{b even, a odd}, \\ 1, & \text{else}, \end{array} \right.
\]

and a series $1 \prec (\mathcal{H}_{a,b})_0' \prec (\mathcal{H}_{a,b})_0 \prec \mathcal{H}_{a,b} \prec \mathcal{G}(a,b)$ s.t.

\[
\mathcal{G}(a,b)/\mathcal{H}_{a,b} \simeq G(a,b)/H_{a,b}, \quad \mathcal{H}_{a,b}/(\mathcal{H}_{a,b})_0 \simeq \mathbb{Z}_{m_1}, \quad (\mathcal{H}_{a,b})_0/(\mathcal{H}_{a,b})_0' \simeq (H_{a,b})_0/(H_{a,b})_0',
\]

and $(\mathcal{H}_{a,b})_0' \simeq (H_{a,b})_0'$.

5 Appendix

This Appendix describes the braid monodromy factorizations induced from the regeneration of each point and the induced relations from it.

For computing explicitly the braid monodromy factorizations $\mathcal{H}(r,k)$ induced from the 6/3/2-points $\omega_{r,k}$, we use the results of [6].

For $(r,k) = (0,0), (a,0)$, the vertex $\omega_{r,k}$ is a 2-point on the edge $L_j$ (a point which is on the intersection of two planes). Therefore, the braid monodromy factorization of the regenerated neighborhood of the vertex $\omega_{r,k}$ is $\mathcal{H}(r,k) = \mathbb{Z}_{j,j'}$.

For $(r,k)$ s.t. $\omega_{r,k}$ are on the boundary of $P$ and $(r,k) \neq (0,b), (a+b,b), (0,0), (a,0)$, $\omega_{r,k}$ is a 3-point (a point that lies on the intersection of three planes), such that locally it looks like one of the following configurations:
Consider the first and the third cases (where the line $L_j$ is regenerated first). Then the braid monodromy factorization of the regenerated neighborhood of the vertex $\omega_{r,k}$ is

$$\mathcal{H}(r, k) = Z_{i, j}^{(3)} \tilde{Z}_{j' (i)}$$

when $Z_{i, j}^{(3)} = Z_{3, j}^{3} Z_{i, j}^{3} (Z_{i, j}^{3}) z_{i, j'}$.

Consider the second and the fourth cases (where the line $L_i$ is regenerated first). Then the braid monodromy factorization of the regenerated neighborhood of the vertex $\omega_{r,k}$ is

$$\mathcal{H}(r, k) = Z_{j, i'}^{(3)} \tilde{Z}_{i' (j)}.$$

In both cases, $\tilde{Z}_{j' (i)}$ is represented by the following path:

For $(r, k)$ such that $\omega_{r,k}$ are not on the boundary of $P$, $\omega_{r,k}$ is a 6-point. Assume that locally it looks like the following configuration (when the lines are numerated locally):

Then the braid monodromy factorization of the regenerated neighborhood of the vertex $\omega_{r,k}$ is:

$$\mathcal{H}(r, k) = Z_{i', j'}^{(3)} \tilde{Z}_{6, 6'} Z_{3', 6'}^{(2)} (Z_{2', 6'}^{(2)})^* Z_{4', 6'}^{(2)} (Z_{2', 6'}^{(2)})^* (\hat{F}(\hat{F})_{\rho-1})^* Z_{5, 6'}^{(3)}$$

$$\left( \prod_{i=6', 6''} (Z_{3, i}^{2})^* \right) Z_{3', 3'}^{(3)} \prod_{i=6', 6''} (Z_{3, i}^{2}) \tilde{Z}_{1, 1'},$$

where $Z_{i', j'}^{(2)} = Z_{i', j'}^{2} Z_{i, j'}^{2}$, $()^*$ is the conjugation by the braid induced from the motion:
18

FRIEDMAN Michael & TEICHER Mina

and \( \bar{Z}_{11'}, \bar{Z}_{66'} \) are

\[
p = Z_{22'}, Z_{55'}, \quad \bar{F} = Z_{22'}^{(3)} Z_{44', 55'}^{(3)} Z_{33'}^{(3)} Z_{22', 55'}^{(3)} Z_{33'}^{(3)} Z_{22', 55'}^{(3)} , \text{ where } Z_{33'}, Z_{33'} \text{ are:}
\]

By the Van-Kampen Theorem\[^{15}\], we can see that we get a triple relation \((\langle A, B \rangle = e)\) for each pair of generators whose corresponding lines (from which they are created) induce a common triangle in the complex \(K(a, b); \) and we get a double (commutation) relation \((\lbrack A, B \rbrack = e)\) for each pair of generators whose corresponding lines does not induce a common triangle in the complex. This is the basis for the embedding of \( \bar{B}_n \) in \( G \). For more details, see [6].

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