On the idempotents of Hecke algebras

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Abstract

We give a new construction of primitive idempotents of the Hecke algebras associated with the symmetric groups. The idempotents are found as evaluated products of certain rational functions thus providing a new version of the fusion procedure for the Hecke algebras. We show that the normalization factors which occur in the procedure are related to the Ocneanu–Markov trace of the idempotents.

1 Introduction

It was observed by Jucys \cite{Jucys} that the primitive idempotents of the symmetric group $S_n$ can be obtained by taking certain limit values of the rational function

$$\Phi(u_1, \ldots, u_n) = \prod_{1 \leq i < j \leq n} \left( 1 - \frac{(i,j)}{u_i - u_j} \right),$$

where $u_1, \ldots, u_n$ are complex variables and the product is calculated in the group algebra $\mathbb{C}[S_n]$ in the lexicographical order on the pairs $(i,j)$. A similar construction, now commonly referred to as the fusion procedure, was developed by Cherednik \cite{Cherednik}, while complete proofs were given by Nazarov \cite{Nazarov}. A simple version of the fusion procedure establishing its equivalence with the Jucys–Murphy construction was recently found by one of us in \cite{IsaevMolev}; see also \cite{Cherednik} Ch. 6 for applications to the Yangian representation theory and more references. In more detail, let $T$ be a standard tableau associated with a partition $\lambda$ of $n$ and let $c_k = j - i$, if the element $k$ occupies the cell of the tableau in row $i$ and column $j$. Then the consecutive evaluations

$$\Phi(u_1, \ldots, u_n)|_{u_1 = c_1, u_2 = c_2, \ldots, u_n = c_n}$$

are well-defined and this value yields the corresponding primitive idempotent $E^\lambda_T$ multiplied by the product of the hooks of the diagram of $\lambda$. The left ideal $\mathbb{C}[S_n] E^\lambda_T$ is the
irreducible representation of $\mathfrak{S}_n$ associated with $\lambda$, and the $\mathfrak{S}_n$-module $\mathbb{C}[\mathfrak{S}_n]$ is the direct sum of the left ideals over all partitions $\lambda$ and all $\lambda$-tableaux $T$.

Our aim in this paper is to derive an analogous version of the fusion procedure for the Hecke algebra $\mathcal{H}_n = \mathcal{H}_n(q)$ associated with $\mathfrak{S}_n$. The procedure goes back to Cherednik [2], while detailed proofs relying on $q$-versions of the Young symmetrizers were given by Nazarov [14]; see also Grime [4] for its hook version. We use a different approach based on the formulas for the primitive idempotents of $\mathcal{H}_n$ in terms of the Jucys–Murphy elements. These formulas derived by Dipper and James [3] generalize the results of Jucys [9] and Murphy [12] for $\mathfrak{S}_n$.

The main result of this paper is an explicit formula for the orthogonal primitive idempotents of $\mathcal{H}_n$. These formulas derived by Dipper and James [3] generalize the results of Jucys [9] and Murphy [12] for $\mathfrak{S}_n$. The normalization factors in the expressions for the Hecke algebra idempotents turn out to be related to the Oceana–Markov trace of the idempotents.

2 Idempotents of $\mathcal{H}_n$ and Jucys–Murphy elements

Let $q$ be a formal variable. The Hecke algebra $\mathcal{H}_n$ over the field $\mathbb{C}(q)$ is generated by the elements $T_1, \ldots, T_{n-1}$ subject to the defining relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$

$$T_i T_j = T_j T_i \quad \text{for } |i - j| > 1,$$

$$T_i^2 = 1 + (q - q^{-1}) T_i.$$

Given a reduced decomposition $w = s_{i_1} \ldots s_{i_r}$ of an element $w \in \mathfrak{S}_n$ in terms of the generators $s_i = (i, i + 1)$, set $T_w = T_{i_1} \ldots T_{i_r}$. Then $T_w$ does not depend on the reduced decomposition, and the set $\{T_w \mid w \in \mathfrak{S}_n\}$ is a basis of $\mathcal{H}_n$ over $\mathbb{C}(q)$.

The Jucys-Murphy elements $y_1, \ldots, y_n$ of $\mathcal{H}_n$ are defined inductively by

$$y_1 = 1, \quad y_{k+1} = T_k y_k T_k \quad \text{for } k = 1, \ldots, n - 1. \quad (2)$$

These elements satisfy

$$y_k T_m = T_m y_k, \quad m \neq k, k - 1.$$

In particular, $y_1, \ldots, y_n$ generate a commutative subalgebra of $\mathcal{H}_n$. The elements $y_k$ can be written in terms of the elements $T_{(i,j)} \in \mathcal{H}_n$, associated with the transpositions $(i \ j) \in \mathfrak{S}_n$ as follows:

$$y_k = 1 + (q - q^{-1}) (T_{(1,k)} + T_{(2,k)} + \cdots + T_{(k-1,k)}).$$

Hence, the normalized elements $(y_k - 1)/(q - q^{-1})$ specialize to the Jucys–Murphy elements for $\mathfrak{S}_n$ as $q \to 1$; see [9], [12], [3].

For any $k = 1, \ldots, n$ we let $w_k$ denote the unique longest element of the symmetric group $\mathfrak{S}_k$ which is regarded as the natural subgroup of $\mathfrak{S}_n$. The corresponding elements $T_{w_k} \in \mathcal{H}_n$ are then given by $T_{w_1} = 1$ and

$$T_{w_k} = T_1 (T_2 T_1) \cdots (T_{k-2} \ldots T_1)(T_{k-1} T_{k-2} \cdots T_1)$$

$$= (T_1 \ldots T_{k-2} T_{k-1})(T_1 \ldots T_{k-2}) \cdots (T_1 T_2) T_1, \quad k = 2, \ldots, n. \quad (3)$$
We point out the following properties of the elements $T_{w_k}$ which are easily verified by induction with the use of (3) and (4):

$$T_{w_k} T_j = T_{k-j} T_{w_k}, \quad 1 \leq j < k \leq n,$$

$$T_{w_k}^2 = y_1 y_2 \cdots y_k, \quad k = 1, \ldots, n. \quad (5)$$

Following [14], for each $i = 1, \ldots, n - 1$ set

$$T_i(x, y) = \frac{T_i y - T_i^{-1} x}{y - x} = T_i + \frac{q - q^{-1}}{x - 1}, \quad (6)$$

where $x$ and $y$ are complex variables. We will regard the $T_i(x, y)$ as rational functions in $x$ and $y$ with values in $H_n$. It is well-known that they satisfy the relations

$$T_i(x, y) T_{i+1}(x, z) T_i(y, z) = T_{i+1}(y, z) T_i(x, z) T_{i+1}(x, y), \quad (7)$$

(the Yang–Baxter equation), and

$$T_i(x, y) T_i(y, x) = \frac{(x - q^2 y)(x - q^{-2} y)}{(x - y)^2}. \quad (8)$$

Lemma 2.1. We have the identities

$$T_{w_k} T_j(x, y) = T_{k-j}(x, y) T_{w_k}, \quad 1 \leq j < k \leq n, \quad (9)$$

and

$$T_{w_{k+1}} T_2(u, \sigma_{k-1}) \cdots T_k(u, \sigma_1) T_{w_k}^{-1} = T_{w_k} T_1(u, \sigma_{k-1}) \cdots T_{k-1}(u, \sigma_1) T_{w_{k-1}}^{-1} T_k, \quad (10)$$

where $1 \leq k < n$ and $u, \sigma_1, \ldots, \sigma_{k-1}$ are complex parameters.

Proof. Relation (9) is immediate from (5), while (10) is deduced from

$$(T_k \cdots T_2 T_1) T_j(x, y) = T_{j-1}(x, y) (T_k \cdots T_2 T_1), \quad 2 \leq j \leq k,$$

by taking into account the identity

$$T_{w_k}^{-1} T_{w_{k+1}}^{-1} T_k T_{w_k} = T_k \cdots T_2 T_1$$

implied by (3) and (4).

Now we recall the construction of the orthogonal primitive idempotents for the Hecke algebra from [3]. We will identify a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of $n$ with its diagram which is a left-justified array of rows of cells such that the top row contains $\lambda_1$ cells, the next row contains $\lambda_2$ cells, etc. A cell outside $\lambda$ is called addable to $\lambda$ if the union of $\lambda$ and the cell is a diagram. A tableau $T$ of shape $\lambda$ (or a $\lambda$-tableau $T$) is obtained by filling in the cells of the diagram bijectively with the numbers $1, \ldots, n$. A tableau $T$ is called standard if its entries increase along the rows and down the columns. If a cell occurs in row $i$ and column $j$, its $q$-content will be defined as $q^{2(j-i)}$. 

3
In accordance to [3], a set of orthogonal primitive idempotents \( \{ E^\lambda_T \} \) of \( \mathcal{H}_n \), parameterized by partitions \( \lambda \) of \( n \) and standard \( \lambda \)-tableaux \( T \) can be constructed inductively by the following rule. Set \( E^\lambda_T = 1 \) if \( n = 1 \), whereas for \( n \geq 2 \),

\[
E^\lambda_T = E^\mu_U (y_n - \rho_1) \cdots (y_n - \rho_k) \frac{(\sigma - \rho_1) \cdots (\sigma - \rho_k)}{\lambda_{\alpha}},
\]

where \( U \) is the tableau obtained from \( T \) by removing the cell \( \alpha \) occupied by \( n \), \( \mu \) is the shape of \( U \), and \( \rho_1, \ldots, \rho_k \) are the \( q \)-contents of all addable cells of \( \mu \) except for \( \alpha \), while \( \sigma \) is the \( q \)-content of the latter. In particular, if \( \lambda \) and \( \lambda' \) are partitions of \( n \), then

\[
E^\lambda_T E^{\lambda'}_{T'} = \delta_{\lambda\lambda'} \delta_{TT'}, E^\lambda_T
\]

for arbitrary standard tableaux \( T \) and \( T' \) of shapes \( \lambda \) and \( \lambda' \), respectively. Moreover,

\[
\sum_\lambda \sum_T E^\lambda_T = 1,
\]

summed over all partitions \( \lambda \) of \( n \) and all standard \( \lambda \)-tableaux \( T \).

In what follows we will omit the superscript \( \lambda \) and write simply \( E_T \) instead of \( E^\lambda_T \).

Given a standard \( \lambda \)-tableau \( T \) and \( k \in \{1, \ldots, n\} \), we set \( \sigma_k = q^{2(j-i)} \) if the element \( k \) of \( T \) occupies the cell in row \( i \) and column \( j \). Then

\[
y_k E_T = E_T y_k = \sigma_k E_T. \tag{12}
\]

Furthermore, given a standard tableau \( U \) with \( n - 1 \) cells, the corresponding idempotent \( E_U \) can be written as

\[
E_U = \sum_T E_T, \tag{13}
\]

summed over all standard tableaux \( T \) obtained from \( U \) by adding one cell with entry \( n \).

Exactly as in the case of the symmetric group \( \mathfrak{S}_n \) (see [10]), this relation can be used to derive the following alternative form of (11). Consider the rational function

\[
E_T(u) = E_U \frac{u - \sigma_n}{u - y_n}, \tag{14}
\]

in a complex variable \( u \) with values in \( \mathcal{H}_n \). Then this function is regular at \( u = \sigma_n \) and the corresponding value coincides with \( E_T \):

\[
E_T = E_U \frac{u - \sigma_n}{u - y_n} \bigg|_{u=\sigma_n}. \tag{15}
\]

### 3 Fusion formulas for primitive idempotents

For \( k = 1, \ldots, n - 1 \) introduce the elements of \( \mathcal{H}_n \) by

\[
Y_k(\sigma_1, \sigma_2, \ldots, \sigma_k; u) = T_{w_k} T_k(\sigma_1, u) T_{k-1}(\sigma_2, u) \cdots T_1(\sigma_k, u) T_{w_{k+1}}^{-1}, \tag{16}
\]

where \( \sigma_1, \sigma_2, \ldots, \sigma_k \) and \( u \) are complex parameters.
Lemma 3.1. Let $\mathcal{U}$ be a standard tableau with $k$ cells and the $q$-contents $\sigma_1, \sigma_2, \ldots, \sigma_k$. Then

$$E_{\mathcal{U}} Y_k(\sigma_1, \ldots, \sigma_k; u) = (u - \sigma_1) \left( \prod_{j=1}^{k} \frac{(u - q^2 \sigma_j) (u - q^{-2} \sigma_j)}{(u - \sigma_j)^2} \right) E_{\mathcal{U}} (u - y_{k+1})^{-1}. \quad (17)$$

Proof. We start with representing (17) in the form

$$(u - \sigma_1)^{-1} E_{\mathcal{U}} (u - y_{k+1}) = E_{\mathcal{U}} T_{w_{k+1}} T_1(u, \sigma_k) \ldots T_k(u, \sigma_1) T_{w_k}^{-1}, \quad (18)$$

where we have used [3] and taken into account the fact that $E_{\mathcal{U}}$ commutes with $y_{k+1}$. Now we prove (18) by induction. For $k = 1$ we have

$$(u - \sigma_1)^{-1}(u - T_1^2) = T_1 \cdot T_1(u, \sigma_1),$$

which is true, as $\sigma_1 = 1$. Due to (9) and (10), the right hand side of (18) can be written in the form

$$E_{\mathcal{U}} T_k(u, \sigma_k) T_{w_{k+1}} T_2(u, \sigma_{k-1}) \ldots T_k(u, \sigma_1) T_{w_k}^{-1} = E_{\mathcal{U}} T_k(u, \sigma_k) T_{w_{k-1}} T_1(u, \sigma_{k-1}) \ldots T_{k-1}(u, \sigma_1) T_{w_{k-1}}^{-1} T_k.$$

Using (13), we can write $E_{\mathcal{U}} = E_{\mathcal{U}} E_{\mathcal{V}}$, where $\mathcal{V}$ is the tableau obtained from $\mathcal{U}$ by removing the cell occupied by $k$. Hence, the right hand side of (18) becomes

$$E_{\mathcal{U}} E_{\mathcal{V}} T_k(u, \sigma_k) T_{w_k} T_1(u, \sigma_{k-1}) \ldots T_{k-1}(u, \sigma_1) T_{w_{k-1}}^{-1} T_k = E_{\mathcal{U}} T_k(u, \sigma_k) \left( E_{\mathcal{V}} T_{w_k} T_1(u, \sigma_{k-1}) \ldots T_{k-1}(u, \sigma_1) T_{w_{k-1}}^{-1} \right) T_k = (u - \sigma_1)^{-1} E_{\mathcal{U}} T_k(u, \sigma_k)(u - y_{k}) T_k.$$

The last equality holds by the induction hypothesis. Now we represent $T_k(u, \sigma_k)$ in the form

$$T_k(u, \sigma_k) = \frac{T_k \sigma_k - T_k^{-1} u}{\sigma_k - u} = T_k + \frac{(q - q^{-1}) u}{\sigma_k - u}.$$

This gives

$$E_{\mathcal{U}} T_k(u, \sigma_k)(u - y_k) T_k = E_{\mathcal{U}} \left( T_k + \frac{(q - q^{-1}) u}{\sigma_k - u} \right) (u - y_k) T_k = E_{\mathcal{U}} \left( -u (q - q^{-1}) T_k + u T_k^2 - y_{k+1} \right) = E_{\mathcal{U}} (u - y_{k+1}),$$

thus completing the proof. \[\blacksquare\]

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition of $n$. We will use the conjugate partition $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ so that $\lambda'_j$ is the number of cells in the $j$-th column of $\lambda$. If $\alpha = (i, j)$ is a cell of $\lambda$, then the corresponding hook is defined as $h_\alpha = \lambda_i + \lambda'_j - i - j + 1$ and the content is $c_\alpha = j - i$. Set

$$f(\lambda) = \prod_{\alpha \in \lambda} \frac{q^{c_\alpha}}{[h_\alpha]_q}, \quad (19)$$
where we have used the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$ 

Suppose that $\mathcal{T}$ is a standard $\lambda$-tableau. As before, for each $k \in \{1, \ldots, n\}$ we let $\sigma_k$ denote the $q$-content $q^{2(j-i)}$ of the cell $(i,j)$ occupied by $k$ in $\mathcal{T}$. Consider the rational function

$$F_n(u) = \frac{u - \sigma_n}{u - \sigma_1} \prod_{k=1}^{n-1} \frac{(u - \sigma_k)^2}{(u - q^2\sigma_k)(u - q^{-2}\sigma_k)}.$$ 

Lemma 3.2. The rational function $F_n(u)$ is regular at $u = \sigma_n$ and

$$F_n(\sigma_n) = f(\mu)^{-1} f(\lambda),$$

where $\mu$ denotes the shape of the standard tableau obtained from $\mathcal{T}$ by removing the cell occupied by $n$.

Proof. It is clear that $F_n(u)$ depends only on the shape $\mu$ and does not depend on the standard tableau $\mathcal{U}$ obtained from $\mathcal{T}$ by removing the cell occupied by $n$. Therefore, we may assume that $\mathcal{U}$ is the row tableau obtained by writing the elements $1, \ldots, n-1$ into the cells of $\mu$ consecutively by rows starting with the top row. Suppose that the rows of $\mu$ are

$$\mu_1 = \cdots = \mu_{p_1} > \mu_{p_1+1} = \cdots = \mu_{p_2} > \cdots > \mu_{p_{s-1}+1} = \cdots = \mu_{p_s}$$

for some integers $p_1, \ldots, p_s$ such that $1 \leq p_1 < p_2 < \cdots < p_s$ and some $s \geq 1$. With this notation, $F_n(u)$ can be written in the form

$$F_n(u) = (u - \sigma_n) \prod_{i=1}^{s} \frac{(u - q^{2\mu_{p_i}-2p_i})}{(u - q^{2\mu_{p_i+1}-2p_i})},$$

where we set $p_0 = 0$ and $\mu_{p_s+1} = 0$. Possible values of the $q$-content $\sigma_n$ are $\sigma_n = q^{2\mu_{p_j}+1-2p_j}$ for $j = 0, 1, \ldots, s$. Hence, for a fixed value of $j$ the factor $u - \sigma_n$ cancels, and so $F_n(\sigma_n)$ is well-defined and can be expressed in the form

$$F_n(\sigma_n) = (q^{2\mu_{p_j+1} - q^{2\mu_{p_j}+2}}) \prod_{\alpha \in \mu} (1 - q^{2h_\alpha}) \prod_{\alpha \in \lambda} (1 - 2^{-h_\alpha})^{-1}, \quad (20)$$

which is verified by a simple calculation. On the other hand, $f(\lambda)$ can be represented as

$$f(\lambda) = q^{b(\lambda)} (1 - q^2)^n \prod_{\alpha \in \lambda} (1 - q^{2h_\alpha})^{-1}, \quad b(\lambda) = \sum_{i \geq 1} \lambda_i(\lambda_i - 1).$$

Therefore, the expression in (20) equals $f(\mu)^{-1} f(\lambda)$, as required. 

Introduce the rational function $\Psi(u_1, \ldots, u_n)$ in complex variables $u_1, \ldots, u_n$ with values in $\mathcal{H}_n$ by the formula

$$\Psi(u_1, \ldots, u_n) = \prod_{k=1, \ldots, n-1} (T_k(u_1, u_{k+1}) T_{k-1}(u_2, u_{k+1}) \cdots T_1(u_k, u_{k+1})) \cdot T_{w_n}^{-1}.$$ 

As before, we let $\lambda$ be a partition of $n$ and let $\mathcal{T}$ be a standard $\lambda$-tableau.
Theorem 3.3. The idempotent $E_T$ can be obtained by the consecutive evaluations

$$E_T = f(\lambda) \cdot \Psi(u_1, \ldots, u_n) \bigg|_{u_1=\sigma_1, u_2=\sigma_2, \ldots, u_n=\sigma_n},$$

where the rational functions are regular at the evaluation points at each step.

Proof. We argue by induction on $n$. For $n \geq 2$ we let $\mathcal{U}$ denote the standard tableau obtained from $T$ by removing the cell occupied by $n$ and let $\mu$ be the shape of $\mathcal{U}$. Applying Lemma 3.2 and the induction hypothesis, we can write the right hand side of (21) in the form

$$F_n(\sigma_n) E_{\mathcal{U}} Y_{n-1}(\sigma_1, \ldots, \sigma_{n-1}; u_n) \bigg|_{u_n=\sigma_n},$$

where the elements $Y_{n-1}(\sigma_1, \ldots, \sigma_{n-1}; u_n)$ are defined in (16). The proof is completed by the application of Lemma 3.1 and relation (15).

Example 3.4. Using (21), for $n = 3$ and $\lambda = (2, 1)$ we get

$$E_T = \frac{1}{[3]_q} T_1(\sigma_1, \sigma_2) T_2(\sigma_1, \sigma_3) T_1(\sigma_2, \sigma_3) (T_1 T_2 T_1)^{-1}.$$ (22)

In particular,

$$\sigma_1 = 1, \quad \sigma_2 = q^2, \quad \sigma_3 = q^{-2} \quad \text{for} \quad T = \begin{array}{ccc} 1 & 2 \\ 3 & \end{array}$$

and

$$\sigma_1 = 1, \quad \sigma_2 = q^{-2}, \quad \sigma_3 = q^2 \quad \text{for} \quad T = \begin{array}{ccc} 1 & \frac{3}{2} \\ \end{array}.$$ (23)

Note that (22) can be reduced to the fusion formulas contained in [3, p. 106].

Example 3.5. For $n = 4$ and $\lambda = (2^2)$ the idempotent $E_T$ is obtained by evaluating the rational function

$$\frac{1}{[3]_q[2]_q^2} T_1(u_1, u_2) T_2(u_1, u_3) T_1(u_2, u_3) T_3(u_1, u_4) T_2(u_2, u_4) T_1(u_3, u_4) T_{w_4}^{-1}$$

consecutively at $u_1 = \sigma_1$, $u_2 = \sigma_2$, $u_3 = \sigma_3$, and $u_4 = \sigma_4$. We have

$$\sigma_1 = 1, \quad \sigma_2 = q^2, \quad \sigma_3 = q^{-2}, \quad \sigma_4 = 1 \quad \text{for} \quad T = \begin{array}{ccc} 1 & 2 \\ 3 & 4 \end{array}$$

and

$$\sigma_1 = 1, \quad \sigma_2 = q^{-2}, \quad \sigma_3 = q^2, \quad \sigma_4 = 1 \quad \text{for} \quad T = \begin{array}{ccc} 1 & 3 \\ 2 & 4 \end{array}.$$ (23)
By [14, Lemma 2.1], the product $T_2(u_1, u_2)T_3(u_1, u_4)T_2(u_2, u_4)$ is equal to

\[
((T_2u_2 - T_2^{-1}u_1) T_3(T_2u_4 - T_2^{-1}u_2) + (q - q^{-1}) u_1((q - q^{-1}) u_2 T_2 + u_2 - u_1))
\]

\[
\frac{(u_2 - u_1)(u_4 - u_2)}{(u_2 - u_1)(u_4 - u_1)(u_4 - u_2)}
\]

and it is regular for $u_1 = q^{\pm 2}u_2$ at $u_1 = u_4$. It was shown in [14] that such considerations can be extended to the general expression (21) to prove that it is regular in the limits $u_i \to \sigma_i$.

We conclude this section by showing that taking an appropriate limit in Theorem 3.3 as $q \to 1$ we can recover the respective formulas of [10] for the primitive idempotents of the symmetric group $S_n$.

Take the parameters $x$ and $y$ in (6) in the form $x = q^2u$ and $y = q^2v$. Since $T_i \to s_i$, for the limit value of $T_i(x, y)$ we have

\[
T_i(x, y) = T_i + \frac{q^{u-v}}{v-u} \to s_i \varphi_{i,i+1}(u, v),
\]

where

\[
\varphi_{i,j}(u, v) = 1 - \frac{(i \, j)}{u-v}.
\]

Using (24) we can calculate the corresponding limit for the element (16) to get

\[
Y_k(\sigma_1, \sigma_2, \ldots, \sigma_k; u) \to \varphi_{1,k+1}(c_1, u)\varphi_{2,k+1}(c_2, u) \ldots \varphi_{k,k+1}(c_k, u),
\]

where $\sigma_m = q^{2c_m}$. Clearly, the normalization factor $f(\lambda)$ specializes to the inverse of the product of the hooks of $\lambda$, and so the substitution of (25) into (21) leads to the main result of [10].

4 The Ocneanu–Markov trace of the idempotents

The purpose of this section is to calculate the Ocneanu–Markov trace of the idempotents $E_T$ which turns out to be related to the normalization factor $f(\lambda)$ defined in (19).

Definition 4.1. For any given standard tableau $T$ with $n$ cells, its quantum dimension is defined as

\[
\text{qdim } T = T_{r_n}(E_T),
\]

where $T_{r_n} : \mathcal{H}_n \to C$ is the Ocneanu–Markov trace; see e.g. [7].

The Ocneanu–Markov trace $T_{r_n}$ can be defined as the composition of the maps

\[
T_{r_n} = T_1T_2 \ldots T_n.
\]
The linear maps $\text{Tr}_{m+1} : \mathcal{H}_{m+1} \to \mathcal{H}_m$ from the Hecke algebra $\mathcal{H}_{m+1}$ to its natural subalgebra $\mathcal{H}_m$ are determined by the following properties, where $Q \in \mathbb{C}$ is a fixed parameter, while $X, Y \in \mathcal{H}_m$ and $Z \in \mathcal{H}_{m+1}$:

\begin{align*}
\text{Tr}_{m+1}(XZY) &= X\text{Tr}_{m+1}(Z)Y, \quad \text{Tr}_{m+1}(X) = QX, \\
\text{Tr}_{m+1}(T^{\pm 1}_m XT^{\pm 1}_m) &= \text{Tr}_{m}(X), \quad \text{Tr}_{m+1}(T_m) = 1, \\
\text{Tr}_m \text{Tr}_{m+1}(T_m Z) &= \text{Tr}_m \text{Tr}_{m+1}(ZT_m). \\
\end{align*}

Our calculation of (26) is based on the approach of [6]. The following statement can be found in that paper.

**Proposition 4.2.** Consider the rational function in $u$ with values in the Hecke algebra $\mathcal{H}_m$ which is defined by

$$Z_{m+1}(u) = \text{Tr}_{m+1}(u - y_{m+1})^{-1}, \quad y_{m+1} \in \mathcal{H}_{m+1},$$

where $\mathcal{H}_m$ is regarded as a subalgebra of $\mathcal{H}_{m+1}$. Then,

$$Z_{m+1}(u) = \frac{lQ + u - 1}{tu(u - 1)} \left( \prod_{k=1}^{m} \frac{(u - y_k)^2}{(u - q^2y_k)(u - q^{-2}y_k)} - \frac{(1 - lQ)(u - 1)}{lQ + u - 1} \right),$$

where $l = q - q^{-1}$.

**Proof.** From the definition of the Jucys-Murphy elements (2) we deduce the identity

$$\frac{1}{u - y_{m+1}} = T_m \frac{1}{u - y_m} T^{-1}_m + \frac{1}{u - y_m} \left( T^{-1}_m + \frac{lu}{u - y_{m+1}} \right) \frac{y_m}{(u - y_m)}.$$

Applying the map $\text{Tr}_{m+1}$ to both sides of (29) and using (27) we get

$$\frac{(u - q^2y_m)(u - q^{-2}y_m)}{(u - y_m)^2} Z_{m+1}(u) = Z_m(u) + \frac{l(1 - Ql)y_m}{(u - y_m)^2}.$$

For all $k = 1, \ldots, m + 1$ introduce the function $\bar{Z}_k(u)$ by

$$Z_k(u) = \bar{Z}_k(u) + (Q - l^{-1})u^{-1}.$$

This gives the relation

$$\bar{Z}_{m+1}(u) = \frac{(u - y_m)^2}{(u - q^2y_m)(u - q^{-2}y_m)} \bar{Z}_m(u).$$

Solving this recurrence relation with the initial condition

$$\bar{Z}_1(u) = \text{Tr}_1(u - y_1)^{-1} - (Q - l^{-1})u^{-1} = \frac{tQ + u - 1}{tu(u - 1)},$$

we come to (28).

The normalization factor $f(\lambda)$ defined in (19) and the quantum dimension (26) turn out to be related as shown in the following proposition. As before, we let $\lambda$ be a partition of $n$, and $\mathcal{T}$ a standard $\lambda$-tableau.
Proposition 4.3. We have the relation

\[ f(\lambda) = \text{qdim } T \prod_{k=1}^{n} \sigma_k \left( Q + \frac{\sigma_k - 1}{q - q^{-1}} \right)^{-1}. \]

Proof. Using (14) and (15) we get

\[ \text{Tr}_n(E_T) = \text{Tr}_n E_T(u)|_{u=\sigma_n} = E_{U}(u-\sigma_n)\text{Tr}_n(u-y_n)^{-1}|_{u=\sigma_n}. \]

Using equations (28) and taking into account (12) we obtain

\[
\begin{align*}
\text{Tr}_n(E_T) &= \frac{1}{\sigma_n} \left( Q + \frac{\sigma_n - 1}{l} \right) E_{U} \\
& \times \frac{u-\sigma_n}{u-1} \left( \prod_{k=1}^{n-1} \frac{(u-\sigma_k)^2}{(u-q^2\sigma_k)(u-q^{-2}\sigma_k)} - (u-1) \frac{1-lQ}{lQ+u-1} \right)|_{u=\sigma_n} = \\
&= \frac{1}{\sigma_n} \left( Q + \frac{\sigma_n - 1}{l} \right) E_{U} F_n(\sigma_n).
\end{align*}
\]

Applying the maps \( \text{Tr}_k \) consequently, we finally obtain

\[ \text{qdim } T = \text{Tr}^n(E_T) = \text{Tr}_1 \text{Tr}_2 \ldots \text{Tr}_n(E_T) = \prod_{m=1}^{n} \frac{1}{\sigma_m} \left( Q + \frac{\sigma_m - 1}{l} \right) F_m(\sigma_m). \]

The statement now follows from Lemma 3.2.

The following corollary is immediate from Proposition 4.3.

Corollary 4.4. The Ocneanu–Markov trace \( \text{Tr}^n(E_T) \) of the idempotent \( E_T \) depends only on the shape \( \lambda \) of \( T \) and does not depend on \( T \).

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