Information Thermodynamics on Causal Networks

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Nonequilibrium relations for small thermodynamic systems such as molecular motors have been intensively investigated in these decades. The second law of thermodynamics (\textit{i.e.}, $\langle \sigma \rangle \geq 0$) can be derived from the Jarzynski equality (or the integrated fluctuation theorem):

$$\langle \exp[-\sigma] \rangle = 1,$$

where $\sigma$ is the stochastic entropy production and $\langle \cdots \rangle$ describes the ensemble average.

On the other hand, in the presence of feedback control by Maxwell demon, the second law seems to be violated, \textit{i.e.}, $\langle \sigma \rangle$ can be negative. For such cases, we have a generalized second law $\langle \sigma \rangle \geq \langle \Delta I \rangle$ and a generalized Jarzynski equality

$$\langle \exp[-\sigma + \Delta I] \rangle = 1,$$

where $\langle \Delta I \rangle$ is mutual information that is exchanged between the system and the feedback controller \cite{1,2}. Although the relations are applicable to nonequilibrium dynamics with a single information exchange, the general theory has been elusive for more complex cases, in which multiple systems exchange information many times.

Here, we study a system that is involved in a complex information flow induced by multiple other systems. Characterizing the interaction of multiple systems by a causal network (\textit{i.e.}, Bayesian network such as Fig. 1), we obtain a new generalization of the Jarzynski equality

$$\langle \exp[-\sigma + \Theta] \rangle = 1,$$

which leads to a new generalized second law $\langle \sigma \rangle \geq \langle \Theta \rangle$. Here, $\langle \Theta \rangle$ is a quantity that consists of the transfer entropy \cite{3} and the exchanged mutual information between multiple systems. For special cases, Eq. (3) reduces to Eqs. (1) and (2) where the dynamics are characterized by Figs. 2 and 3, respectively.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{Figure1.png}
\caption{An example of the Bayesian network.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{Figure2.png}
\caption{A Bayesian network that describes the Markov process.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{Figure3.png}
\caption{A Bayesian network with a feedback controller described by $y_1$.}
\end{figure}

\begin{thebibliography}{9}
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