The Role of Non-quantized Fluxes in Coulombic and Casimir Scaling Regimes of the Thick Center Vortex Potentials

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ABSTRACT: A non-quantized flux is introduced claimed to be responsible for the Coulombic potential as well as confinement at intermediate distances. Our calculations show if uncorrelated vortices are not quantized with center elements, the asymptotic string tensions are lost while the ratio of the Coulombic strength and intermediate string tensions are in exact agreement with Casimir scaling. Using uncorrelated quantized and non-quantized fluxes with different contributions in the potential between static color sources result in the correct N-ality dependence at large distances and exact agreement with Casimir scaling at short and intermediate distances.
1 Introduction

Since Quantum Chromodynamics was accepted as a promising theory of hadrons, the interaction between heavy quarks has been investigated in different ways. The potential between static $q\bar{q}$ pair is an essential quantity to understand the phenomenology of the inter-quark interactions. Lattice calculations show a short-range Coulombic potential which represents the asymptotic freedom nature of the ultraviolet regime of QCD. In addition, a long-range linear term has been observed for the potential between a static quark and anti-quark which indicates an infrared confined theory [1, 2]. However, the behavior of the gluonic excitations -in this regime- is similar to the color singlet potential [3, 4]. Therefore, the potential between static color sources in representation $r$ might be written in the following form:

$$V_r(R) = -\frac{A_r}{R} + \sigma_r R + B_r,$$

(1.1)

where $A_r$ is the strength of the Coulombic potential, $\sigma_r$ is called the string tension and $B_r$ is a constant term. The Coulombic term dominates the short distances and is a result of one gluon exchange. Numerical calculations show that potentials in different representations are proportional to each other with their ratios given by the respective ratios of the eigenvalues of the corresponding quadratic Casimir operators. This hypothesis called Casimir scaling is exact in the two dimensional Yang-Mills theories [5] and might be written in the below form for the string tension ratios:

$$\sigma_r = \sigma_F \frac{C_r}{C_F},$$

(1.2)

where the index $F$ denotes for the fundamental representation and $C_r$ represents the eigenvalue of the quadratic Casimir operator in representation $r$.

The second term in eq. (1.1) shows confinement which means all asymptotic particles in nature are color singlets. Besides numerical calculations done on this unsolved problem of QCD, there are wide variety of phenomenological models which try to understand how the vacuum acts to confine quarks and also provide a scheme to prove the confinement
property. In these models, it is believed that the QCD vacuum is filled by some class of topological field configurations which cause the area law fall-off for large Wilson loops and a linear term in the inter-quark potentials. One of the candidates for these vacuum configurations is center vortex which is a tube (surface) of chromomagnetic flux. In the vortex picture suggested by 't Hooft in the late seventies [6], it was shown that there is a certain correspondence between the chromomagnetic flux of center vortices and \( z_n \) center elements i.e. the flux carried by vortices is quantized in terms of the center elements. Lattice simulations on thin vortices prove that the fluctuations in the number of center vortices are responsible for the asymptotic string tensions. In other words, if vortices are removed from the lattice configuration, the asymptotic string tensions vanish [7].

The generalization of the vortex picture to the thick center vortex model gives the correct \( N \)-ality dependence of the potentials at large distances and the qualitative agreement with Casimir scaling at intermediate distances called Casimir scaling regime [8]. Moreover, modifications done at short distances -in two different ways- have equipped the model with the Coulombic regime as well. Deldar \textit{et al.} showed by increasing the role of vortex fluxes piercing Wilson loops with contributions close to the trivial center element and also by fluctuating the vortex core size, the coulombic potential is produced [9]. Although the ratios of the coefficients are in agreement with Casimir scaling qualitatively, the ratio of string tensions differs the expected values especially for higher representations. For example, the ratios of the string tension reported for representations 10 and 15 of the \( SU(3) \) gauge group, are equal to 1.62(10) and 1.63(12), respectively while the expected Casimir ratios are 4.5 and 7. In a different work, the fluctuation of non-quantized, correlated closed magnetic flux lines has been considered by Neudecker \textit{et al.} to calculate the short range potential [10]. In this article, it is shown that the non-quantization condition is the only condition which must be obeyed by vacuum fluctuations to produce the Coulombic and intermediate linear potentials. Moreover, when the vacuum of the theory only consists of non-quantized uncorrelated objects, the asymptotic string tensions are lost. In this case, the behavior of the \( SU(3) \) theory at large distances is similar to the theory which does not contain non-trivial center elements. But, there is certainly a linear potential at intermediate distances and the representation dependence of the string tensions follows an exact Casimir scaling law.

This article is organized as follows: In section 2, thin and thick vortices are reviewed briefly and the potential induced by thick vortices is introduced. A non-quantized Gaussian flux is presented in section 3. The potential between static color sources is calculated by means of the uncorrelated non-quantized flux and discuss the features of the potentials at different regimes. In section 4, both quantized and non-quantized fluxes are contributed in the potential and the ratio of the coefficients in eq. (1.1) is reported.

2 Confinement from center vortices

In the original vortex idea, an operator \( B(\dot{C}) \) was introduced in \( SU(N) \) gauge theories which creates magnetic flux along loop \( \dot{C} \) and measures the electric flux through \( \dot{C} \). This operator is, in some sense, the dual of the Wilson loop operator creating electric flux along
and measuring magnetic flux through $C$. When loops $C$ and $\hat{C}$ are topologically linked to each other, the following commutation relation is satisfied:

$$W(C)B(\hat{C}) = zB(\hat{C})W(C)$$  \hspace{1cm} (2.1)

This relation results in an area-law falloff for the Wilson loop and a perimeter-law falloff for the 't Hooft loop in the confined phase of the gauge theory. Roughly speaking, similar to Aharonov-Bohm effect, a vortex linked topologically to the Wilson loop, multiplies the Wilson loop -in the representation $r$ of the $SU(N)$ gauge group- by a phase factor which is equal to the group center element:

$$W(C) \rightarrow (z_n)^k W(C),$$  \hspace{1cm} (2.2)

where $k$ is the N-ality of the representation and $z_n$ is given by the nth root of the unity:

$$z_n = \exp\left(\frac{2\pi in}{N}\right) \hspace{0.5cm} n = 0, 1, ..., N - 1.$$  \hspace{1cm} (2.3)

Therefore, Center vortices affect the Wilson loops with the same N-ality in the same way. It should be noted when vortices and the Wilson loop do not link, the Wilson loop is unaffected. So, if $f$ is the probability of piercing a plaquette on the Wilson loop by the $z$ vortex of the $SU(2)$ gauge group, the Wilson loop might be written as:

$$\langle W(C) \rangle = \prod_{x \in A} \left\{ (1 - f) + f(-I) \right\} \langle W_0(C) \rangle$$

$$= \exp \left\{ -\sigma(C)A \right\} \langle W_0(C) \rangle,$$  \hspace{1cm} (2.4)

where $\langle W_0(C) \rangle$ is the Wilson loop expectation value when no vortices pierce the minimal area of the Wilson loop. Numerical simulations prove that this term does not include an area-law falloff. Therefore, center elements do not contribute in $\langle W_0(C) \rangle$.

From eq. (2.4), the string tension is

$$\sigma = -\ln(1 - 2f).$$  \hspace{1cm} (2.5)

Then the thin vortices give the correct N-ality dependence of the potentials at large distances. But this equation is correct only when the vortex is completely outside or inside the minimal area of the loop. As lattice investigations on center vortices show a finite thickness of order 1 fm for physical center vortices, the vortex may overlap the perimeter of the Wilson loop. In this case, the center element in eq. (2.4) should be replaced by a group factor $G_j$ which parametrizes the influence of the vortex on the Wilson loop. This factor interpolates between 1 -when the vortex and the Wilson loop do not interact- and a center element -when the vortex is completely inside the Wilson loop:

$$G_j[\alpha] = \frac{1}{2j + 1} Tr \exp \left[ i\alpha L_3 \right],$$  \hspace{1cm} (2.6)

where $L_3$ is the $SU(2)$ diagonal generator in representation $j$ and $\alpha$ is the vortex flux distribution which depends on what fraction of the vortex core is enclosed by the loop. By considering the role of the group factor, eq. (2.4) is modified as the following:

$$\langle W(C) \rangle = \prod_{x \in A} \left\{ (1 - f) + fG_j[\alpha_C(x)] \right\} \langle W_0(C) \rangle.$$  \hspace{1cm} (2.7)
In an $SU(N)$ gauge group, there are $N - 1$ types of center vortices corresponding to the numbers of the group center elements. Vortices of type $n$ and $N - n$ are complex conjugates of one another and might be regarded as the same type but with chromomagnetic flux pointing in opposite directions. The potential between static quarks might be obtained from the Wilson loop by the following equation:

$$V(R) = - \lim_{T \to \infty} \frac{1}{T} \ln \langle W(C) \rangle.$$  \hspace{1cm} (2.8)

Therefore, the potential energy between static color sources induced by vortices is

$$V(R) = - \sum_{m=\pm \infty} \ln \left\{ 1 - \sum_{n=1}^{N-1} f_n \left( 1 - \text{Re} G_r [\alpha^0_c(x_m)] \right) \right\},$$  \hspace{1cm} (2.9)

and eq. (2.6) is generalized as follows:

$$G_r [\alpha^{(n)}] = \frac{1}{d_r} T r \exp \left[ i \alpha^{(n)} . \bar{H} \right],$$  \hspace{1cm} (2.10)

where $d_r$ is the dimension of representation $r$ and $H_i$’s, $\{ i = 1, \ldots, N - 1 \}$ are the generators spanning the Cartan sub-algebra.

3 Non-quantized flux and the thick center vortex potentials

In the original thick center vortex model, the QCD vacuum consists of uncorrelated fluxes which are quantized with non-trivial center elements. So, every vortex profile ansatz must satisfy the following conditions [8]:

1. Vortices which pierce the plane far outside the loop do not affect the loop. In other words, for fixed $R$, as $x \to \infty$, $\alpha \to 0$.

2. If the vortex core is completely contained within the loop, the vortex flux would get the maximum value. In this case, $G_r (\alpha) = z_n$ which represents the dependence of the vortex flux to center elements. For the $SU(3)$ gauge group, the maximum value of the vortex flux ($\alpha_{\text{max}}$) equals to $\frac{4\pi}{\sqrt{3}}$.

3. As $R \to 0$ then $\alpha \to 0$.

In an attempt to produce Coulombic part in the model, Neudecker et. al showed that correlated non-quantized fluxes are able to produce the Coulombic behavior of the potentials. The profile function of the Coulombic fluctuations is given by a step function which varies by a Gaussian distribution in respect to their width [10]. In this section, it is shown that even uncorrelated non-quantized fluxes are able to create Coulombic potential and these vacuum fluctuations are responsible for the intermediate linear potential as well.

A non-quantized flux might be considered as the following:

$$\alpha(x) = \frac{\alpha_{\text{max}}}{2} \left( \tanh \left[ \frac{\alpha}{R} (x + \frac{R}{2}) \right] - \tanh \left[ \frac{\alpha}{R} (x - \frac{R}{2}) \right] \right),$$  \hspace{1cm} (3.1)
Table 1. The potential ratios obtained from the non-quantized flux of eq. (3.1). Casimir ratios of a few representations of the SU(3) gauge group are represented in the second row. The third row shows the ratio of Coulombic strengths. The ratio of string tensions and the constant terms are given in the forth and fifth rows, respectively. The fit error are given in the brackets. It is observed that the potential ratios are in exact agreement with Casimir scaling.

| Representations | Fund. | 8     | 6     | 10    | 27    | 15s   |
|----------------|-------|-------|-------|-------|-------|-------|
| Casimir Scaling | 1     | 2.25  | 2.50  | 4.50  | 6.00  | 7.00  |
| \( \frac{A_r}{A_F} \) | 1     | 2.25(1)| 2.50(1)| 4.50(2)| 6.00(3)| 7.00(4) |
| \( \frac{\sigma_r}{\sigma_F} \) | 1     | 2.2483(4)| 2.4977(5)| 4.49(3)| 5.982(1)| 6.974(1) |
| \( \frac{B_r}{B_F} \) | 1     | 2.250(8)| 2.500(9)| 4.50(2)| 6.00(2)| 7.00(2) |

where \( R \) is the quark anti-quark separation and \( \dot{a} \) is considered as a free parameter. Figure 1 represents the flux given in eq. (3.1) versus \( x \) -the location of the vacuum fluctuations- for \( \dot{a} = 0.05 \) and the three values of \( R \). It is observed that for every quark separation, the flux reaches its maximum value at \( x = 0 \) which is equal to \( \frac{4\pi}{\sqrt{3}} \tanh \frac{\dot{a}}{2} \approx 0.18 \). It should be mentioned that \( \frac{\sigma_{\text{max}}}{\sigma} \) in eq. (3.1) which is equal to \( \frac{2\pi}{\sqrt{3}} \) in the SU(3) gauge group is just a coefficient and does not contain any quantization meaning.

It might be interesting to investigate the behavior of the group factor using the above non-quantized flux. Figure 2 shows \( \Re G_r[\alpha] \) versus the position of the vortex core \( x \) for the fundamental representation of the SU(3) gauge group with \( \dot{a} = 0.05 \) and two values of \( R \). For both \( R = 2 \) and \( R = 15 \), only some fluctuations are observed near the trivial center element. As only the trivial center element exists in the vacuum, one might expect to have screening for every representation.

Using non-quantized flux distribution of eq. (3.1), the inter-quark potential might be calculated for various representations of the SU(3) gauge group. The upper diagram of figure 3 shows the potential of representations 3, 6, 8, 10, 15s and 27 versus \( R \) by choosing \( \dot{a} = 0.05 \) and \( \dot{f} = 0.2 \).1 The Coulombic and the linear parts are clearly observed at short and intermediate distances, respectively. At large distances -as expected, the 3-ality dependence of the potentials has been lost which means the potential of all representations has been screened due to the vortex removal. Therefore, removing quantized fluxes which is equivalent to the center elements removal, does not affect the behavior of the potentials at short and intermediate distances. So, one might claim that non-quantized fluxes originating from vacuum fluctuations are responsible for Coulombic potential as well as the linear intermediate potential.

Fitting the data obtained from the non-quantized fluxes in the thick center vortex model with eq. (1.1) has been pictured in the lower diagram of figure 3 for the quark separation between \( R = 1 \) and \( R = 20 \). The result of the fit has been represented in table 1. The second row of this table contains Casimir ratios of the representations given in the first row. The ratio of the Coulombic strength, string tension and the constant term of each representation to that of the fundamental one are given in the third, forth and

\(^1\)The definition of \( \dot{f} \) will be given in the next section where the notation of this probability will be changed to \( f_c \).
the fifth rows, respectively. The numbers in brackets represent the errors of the fit. The comparison between the potential ratios and Casimir ratios shows an exact agreement with Casimir scaling. In the next section, both quantized and non-quantized fluxes contribute in a modified formula for the inter-quark potential to restore the 3-ality dependence of the potentials.

4 Contribution of quantized and non-quantized fluxes in the potentials

To have a model to be able to predict the inter-quark potential in every quark separation, it was suggested to add an additional term in the potential of eq. (2.9) [10]. This term which represents vacuum fluctuations, modifies the $SU(3)$ potential as the following:

$$V(R) = - \sum_{m=\pm \infty}^{m=+\infty} \ln \{(1 - 2f - fc) + 2fReG_r[\alpha(x)] + fcReG_r[\alpha_c(x)]\},$$ (4.1)

where $fc$ is the probability that a vacuum fluctuation with the distribution given by $\alpha_c$-originating at $x$- occurs on a plaquette on the Wilson loop. As defined in section 2, $f$ is the probability that an interaction happens between the vortex carrying the flux $\alpha$ and any plaquette on the Wilson loop.

The potential of eq. (4.1) has been checked by the non-quantized flux introduced in the previous section and a quantized flux as the following:

$$\alpha(x) = \frac{2\pi}{\sqrt{3}} \{\tanh[d(x + \frac{R}{2})] - \tanh[d(x - \frac{R}{2})]\} ,$$ (4.2)

where $d$ is defined as follows:

$$d = \begin{cases} R^{-1} & \text{for } R < a^{-1} \\ a & \text{for } R \geq a^{-1} \end{cases} ,$$ (4.3)

and $a$ is proportional to the inverse of the vortex thickness. So, when the vortex thickness is greater than the Wilson loop spatial extent, $d$ equals to the inverse of the quark separation. Otherwise, when the vortex thickness is smaller than or equal to the quark separation, $d$ is equal to $a$. Therefore, one might say that the parameter $d$ gives the cross section of the interaction between the vortex and the Wilson loop. This parameter floats between the inverse of the vortex thickness and the spatial extent of the Wilson loop.

The potentials of different representations of the $SU(3)$ gauge group has been pictured in the upper diagram of figure 4 by choosing $a = \hat{a} = 0.05$, $f = 0.001$ and $fc = 0.02$. The Coulombic potential is clearly observed at short distances and the potentials of the representations with the same 3-ality behave the same at large distances: zero 3-ality representations $(8, 10, 27)$ are screened and the potential of representations $6$ and $15_s$ has become parallel to that of the fundamental one. The fit of the data obtained from the modified thick center vortex potential to eq. (1.1) for $R \in [1, 20]$ is pictured in the lower diagram of figure 4 and the result of this fit could be viewed in table 2. It is observed that the ratio of the potential of each representation to that of the fundamental one obeys the Casimir scaling law with good precision.
5 Conclusion

Thick center vortex model was suggested as a simple phenomenological model for investigating the potential between static color sources. The original model was able to predict the N-ality dependence of the potentials correctly. Also, a qualitative agreement with Casimir scaling was observed at different works done by different vortex flux distributions.

In this paper, it has been shown that uncorrelated vacuum fluctuations considered as Gaussian fluxes have an independent role from center vortices in the QCD vacuum. These fluctuations could cause the intermediate linear potentials which are in exact agreement with Casimir scaling. In addition, the same fluctuations are responsible for the Coulombic potentials at short distances. Our calculations demonstrate that the ratio of the Coulombic strength of each representation to that of the fundamental one obeys Casimir scaling law exactly. In fact, our attempt in producing the Coulombic behavior of the potentials via uncorrelated non-quantized fluxes implies that the works done by Deldar and Neudecker in two different methods are the same physically; Because non-quantized vortices only consist of some fluctuations around the trivial center element. Therefore, trivial center element has a key role in producing confinement at intermediate distances and the short distance Coulombic behavior. Moreover, a quantized flux has been introduced with a floating free parameter i.e. it is defined as the cross section of the Wilson loop and the vortex interaction. Introducing such a parameter prevents the potentials from being concave. It has been shown when both uncorrelated quantized and non-quantized vortices exist in the QCD vacuum, a well-defined potential is achieved at every quark separation which fits the formula of the inter-quark potential very well. Although there might be some correlations between the vortices in the vacuum, uncorrelated quantized and non-quantized vortices are able to give interesting and accurate results in the simple thick center vortex model. Therefore, one might claim that this model is a suitable scheme for calculating potentials quantitatively. Now, as all the topological field configurations in the vacuum -including quantized and non-quantized vortices- seem to be uncorrelated in this model, the search for a single object to have both quantized and non-quantized properties inside, appears to be easier than before.

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Figure 1. Distribution of vacuum fluctuations of eq. (3.1) for $\hat{a} = 0.05$ and the three values of $R$. It is observed that the fluctuations reach the maximum value at $x = 0$ for every value of the quark separation.

Figure 2. The $SU(3)$ group factor for the two values of $R$ by choosing $\hat{a} = 0.05$. For every quark separation, only some fluctuations are observed around the trivial center element.
Figure 3. Upper diagram: The potential of a few representations of the $SU(3)$ gauge group by choosing $\hat{a} = 0.05$ and $\hat{f} = 0.2$ for $R \in [1, 100]$. The Coulombic potential is clearly visible at short distance as well as a large intermediate linear part. Due to the absence of the center element in the vacuum, all the potentials have screened at large distance. The lower diagram: The fit of the data calculated in the thick center vortex model to eq. (1.1) in the range $R \in [1, 20]$. The ratio of the parameters obtained from the fit has been presented in table 1.
Figure 4. Upper diagram: The same as figure 3 but using the modified potential and both quantized and non-quantized fluxes in the range $R \in [1, 300]$ with parameters $a = \hat{a} = 0.05$, $f = 0.001$ and $f_c = 0.02$. The behavior of the potentials at large distances depends on the 3-ality of the representations: Representations 8, 10 and 27 with 3-ality=0 have been screened and the potentials of representations 6 and 15s have become parallel to that of the fundamental one. Coulombic potential at short distances and a linear part at intermediate distances are clearly observed. Lower diagram: The fit of the above data to eq. (1.1) in the range $R \in [1, 20]$. The ratio of the parameters obtained from the fit has been presented in table 2.