Parametric dependence of bound states in the continuum: a general theory

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Photonic structures with high-Q resonances are essential for many practical applications, and they can be relatively easily realized by modifying ideal structures with bound states in the continuum (BICs). When an ideal photonic structure with a BIC is perturbed, the BIC may be destroyed (becomes a resonant state) or may continue to exist with a slightly different frequency and a slightly different wavevector (if appropriate). Some BICs are robust against certain structural perturbations, but most BICs are nonrobust. Recent studies suggest that a nonnegative integer n can be defined for any generic nondegenerate BIC with respect to a properly defined set of structural perturbations. The integer n is the minimum number of tunable parameters needed to preserve the BIC for perturbations arbitrarily chosen from the set. Robust and nonrobust BICs have n = 0 and n ≥ 1, respectively. A larger n implies that the BIC is more difficult to find. If a structure is given by m real parameters, the integer n is the codimension of a geometric object formed by the parameter values at which the BIC exists in the m-dimensional parameter space. In this paper, we suggest a formula for n, give some justification for the general case, calculate n for different types of BICs in two-dimensional structures with a single periodic direction, and illustrate the results by numerical examples. Our study improves the theoretical understanding on BICs and provides useful guidance to their practical applications.

I. INTRODUCTION

In an open wave system, a bound state in the continuum (BIC) is an eigenmode with a localized wave field (localized in the open spatial directions) and a frequency inside the radiation continuum. An ideal BIC can be regarded as a resonant state with an infinite quality factor (Q-factor), and it gives rise to high-Q resonances when the structure and/or solution parameters (such as the wavevector) are perturbed. In recent years, many applications of BICs have been realized in photonics. Most of these applications rely on field enhancement or sharp features in scattering spectra caused by the BIC-induced high-Q resonances. From a theoretical point of view, it is important to understand how BICs are formed and what properties they possess. Existing studies have identified a number of physical mechanisms under which BICs can be found. Rigorous proofs for the existence of BICs are available for symmetry-protected BICs. In structures that are extended to infinity in one or two spatial directions, a BIC is characterized by its frequency and wavevector. It is known that some special BICs are surrounded by resonances whose Q-factors diverge with unusually high rates. Moreover, BICs in a periodic structure are polarization singularities in momentum space and they can be characterized by a topological charge defined using the polarization vector of the surrounding resonant states.

Another important theoretical question is concerned with the effect of structural perturbations on BICs. For a symmetry-protected BIC, it is clear that the BIC should still exist if the perturbation preserves the symmetry. This means that the BIC is robust with respect to symmetry-preserving perturbations. On the other hand, for a typical symmetry-breaking perturbation of magnitude δ, the BIC is destroyed and becomes a resonant state with an Q-factor proportional to 1/δ^2. However, for special symmetry-breaking perturbations, the BIC may become a resonant state with a larger Q-factor (proportional to 1/δ^4, 1/δ^6, ...). BICs unprotected by symmetry are also robust with respect to certain structural perturbations. As examples, we mention propagating BICs in periodic structures with both the up-down mirror symmetry and the in-plane inversion symmetry, and BICs in optical waveguides with lateral leakage channels. Even though these BICs are unprotected by symmetry, their robust existence still depends crucially on symmetry. Similar to the case of symmetry-protected
BICs, if the structural perturbation breaks the required symmetry, the BIC is usually but not always destroyed.

In recent works [31, 32], a new approach was developed to characterize nonrobust BICs. The idea is to determine the minimum number of tuning parameters needed to continuously follow the BIC. More precisely, assuming a BIC exists in a structure with dielectric function \( \varepsilon_s(\mathbf{r}) \) where \( \mathbf{r} = (x, y, z) \) is the position vector, we try to find the smallest integer \( n \), so that the BIC persists in the perturbed structure with dielectric function
\[
\varepsilon(\mathbf{r}) = \varepsilon_s(\mathbf{r}) + \delta F(\mathbf{r}) + \gamma_1 G_1(\mathbf{r}) + \ldots + \gamma_n G_n(\mathbf{r}),
\]
where \( F, G_1, \ldots, G_n \) are arbitrary perturbation profiles, \( \delta \) is an arbitrary small real number, \( \gamma_1, \ldots, \gamma_n \) are tuning parameters determined together with the BIC. If \( n = 0 \), i.e., no tuning parameters are necessary, then the BIC is robust. If \( n \geq 1 \), the BIC is nonrobust. A larger \( n \) implies that the BIC is more difficult to find. The number \( n \) also describes how the BIC depends on generic structural parameters [32]. If the structure depends on \( m \) parameters, we consider the geometric object \( P_{\text{BIC}} \) formed by the parameter values at which the BIC exists in the \( m \)-dimensional parameter space, then \( n = m - \text{dim}(P_{\text{BIC}}) \) is the codimension of \( P_{\text{BIC}} \). For example, if \( n = 1 \), the BIC forms a curve in the plane of two parameters and a surface in the 3D space of three parameters.

However, the existing studies have covered only the cases \( n = 1 \) and \( n = 2 \) for some BICs in 2D structures with a single periodic direction [31, 32]. In this paper, we present a general theory for generic nondegenerate BICs. The rest of this paper is organized as follows. In Sec. II, we present a general theory including a formula for \( n \) and a brief justification. In Sec. III, we apply the general theory to BIC in 2D structures that are translationally invariant in one spatial direction and periodic in another direction. Some details are given for cases that have not been analyzed in previous works [31, 32]. Numerical examples for the new cases of Sec. III are presented in Sec. IV. The paper is concluded with a brief discussion in Sec. V.

II. GENERAL THEORY

To present the theory in a more concrete setting, we assume an electromagnetic BIC of frequency \( \omega_* \) exists in a lossless non-magnetic structure given by scalar dielectric function \( \varepsilon_s(\mathbf{r}) \). The structure may be translationally invariant and/or periodic in one or two directions, thus the BIC may possess a wavevector \( \alpha_* \), which contains at most two nonzero components. The electric field of the BIC can be written as
\[
\mathbf{E}_s(\mathbf{r}) = \Phi_*(\mathbf{r}) e^{i \alpha_* \cdot \mathbf{r}} \tag{2}
\]
where \( \Phi_* \) has the same invariant and/or periodic directions as \( \varepsilon_* \). For example, if the structure is a photonic crystal (PhC) slab with 2D periodicity in the \( xy \)-plane, then \( \alpha_* = (\alpha_*, \beta_*, 0) \), where \( \alpha_* \) and \( \beta_* \) are the Bloch wavenumbers.

The definition of BIC requires the existence of at least one radiation channel. This implies that the structure must have at least one open direction where outgoing and incoming waves with the same frequency \( \omega_* \) and wavevectors compatible with \( \alpha_* \) can propagate to and from infinity, respectively. For a PhC slab surrounded by air, the \( z \) variable (perpendicular to the slab) provides the open directions (as \( z \to \pm \infty \)), and the propagating diffraction orders (compatible with \( \alpha_* \)) serve as the radiation channels. For each radiation channel, we have at least one and at most two linearly independent scattering solutions. The case of two solutions in one radiation channel arises when the incident waves have different polarizations. Let us denote the electric fields of the independent scattering solutions by \( \mathbf{E}^{(s)}_k(\mathbf{r}) = \Psi_k(\mathbf{r})e^{i \alpha_* \cdot \mathbf{r}} \) for \( k = 1, 2, \ldots, N_{ss} \), where \( N_{ss} \) is the total number of such solutions.

In order to discuss the robustness of a BIC and determine the codimension \( n \) of nonrobust BICs, we must specify the conditions (most importantly, the symmetries) satisfied by the perturbation profiles \( F(\mathbf{r}) \) and \( G_j(\mathbf{r}) \), \( 1 \leq j \leq n \). The original structure, given by the dielectric function \( \varepsilon_s(\mathbf{r}) \), may have more symmetries than the perturbation profiles. For example, a PhC slab may have \( C_4 \) or \( C_6 \) symmetry, but robustness can be discussed for perturbations with \( C_4 \) symmetry only, where \( C_4 \) is the rotation by \( 2\pi/p \) about the \( z \)-axis. The symmetries in the original structure and the perturbations give rise to related symmetries in the BIC and the scattering solutions. Typically, this requires a proper scaling. For example, in a PhC slab with \( C_2 \) symmetry [same as the in-plane inversion symmetry, namely \( \varepsilon(\mathbf{r}) = \varepsilon(-x, -y, z) \)], the BIC and the scattering solutions can be scaled such that
\[
\begin{bmatrix}
\mathbf{E}_x(r) \\
\mathbf{E}_y(r) \\
\mathbf{E}_z(r)
\end{bmatrix} = \begin{bmatrix}
E_x(-x, -y, z) \\
E_y(-x, -y, z) \\
-E_z(-x, -y, z)
\end{bmatrix}, \tag{3}
\]
where \( E_x \) is the \( x \)-component of \( \mathbf{E}_* \) or \( \mathbf{E}^{(s)}_k \), \( \mathbf{E} \) is its complex conjugate, etc [29]. Notice that \( \Phi_* \) and \( \Psi_k \) also satisfy Eq. (3).

For each scattering solution, we define an integer index. First, we consider the linear operator \( \mathcal{L} \), such that the BIC and the scattering solutions (with \( e^{i \alpha_* \cdot \mathbf{r}} \) removed) satisfy
\[
\mathcal{L} \Phi_* = 0, \quad \mathcal{L} \Psi_k = 0, \tag{4}
\]
for \( k = 1, \ldots, N_{ss} \). The inhomogeneous equation
\[
\mathcal{L} \mathbf{u} = \mathbf{f} \tag{5}
\]
usually has outgoing solutions, since \( \mathbf{f} \) in the right hand side serves as a source. If we insist that Eq. (5) has a solution that decays to zero rapidly in the open direction(s), then
\[
\int_\Omega \nabla_k \cdot \mathbf{f} \, d\mathbf{r} = \int_\Omega \nabla_k \cdot \mathbf{u} \, d\mathbf{r} = \int_\Omega \nabla_k \cdot u \, d\mathbf{r} = 0, \tag{6}
\]
where $\Omega$ covers one period in the periodic direction and a unit length in the invariant direction. The index $I_k$ is defined as the number of real constraints on $f$ for the left hand side above being zero. Since $\Psi_k$ is complex, we usually have $I_k = 2$. However, if $f$ has some useful symmetry related to the symmetry of the BIC and the perturbation profiles, $I_k$ may be reduced to 1 or 0. Moreover, if there are two scattering solutions in the same radiation channel, they should be chosen so that the constraints are independent.

If a BIC exists in the perturbed structure with the dielectric function given in Eq. (1), it should have a frequency near $\omega_s$ and a wavevector near $\alpha_s$. The degrees of freedom $N_{wv}$ of the BIC wavevector, is the number of components in $\alpha_s$ that will vary independently. Usually $N_{wv} = 2$, but if the BIC is generic and $\alpha_s = 0$, then, as we will show in Sec. III, the perturbed BIC also has a zero wavevector, thus $N_{wv} = 0$. If $\alpha_s$ has just one nonzero component, the wavevector of the perturbed BIC may have one or two nonzero components depending on the symmetry, thus $N_{wv}$ can be 1 or 2, respectively.

Based on the above definitions, we believe that the integer $n$ of a generic nondegenerate BIC is

$$n = \sum_{k=1}^{N_{ss}} I_k - N_{wv}. \quad (7)$$

To find $n$, we actually need to show that a BIC exists in the perturbed structure with the minimum of $n$ tunable parameters. To do so, we expand the desired BIC [with frequency $\omega$, wavevector $\alpha$, electric field $E(r) = \Phi(r)e^{i\alpha \cdot r}$] and the tunable parameters in power series of $\delta$:

$$\Phi = \Phi_s + \delta \Phi_1 + \delta^2 \Phi_2 + \cdots, \quad (8)$$

$$\omega = \omega_s + \delta \omega_1 + \delta^2 \omega_2 + \cdots, \quad (9)$$

$$\alpha = \alpha_s + \delta \alpha_1 + \delta^2 \alpha_2 + \cdots, \quad (10)$$

$$\gamma_j = \delta \gamma_{j,1} + \delta^2 \gamma_{j,2} + \cdots, \quad j = 1, \ldots, n. \quad (11)$$

The leading order gives $L\Phi_s = 0$, i.e., the original equation satisfied by the BIC. For $l \geq 1$, the $O(\delta^l)$ equation can be written as

$$L\Phi_l = f_l \quad (12)$$

where $f_l$ is related to $\omega_l$, $\alpha_l$, $\gamma_{j,l}$, $\Phi_s$, ..., $\Phi_{l-1}$, $\varepsilon_s$ and the perturbation profiles.

If Eq. (12) has a solution that decays rapidly in the open direction(s) and has the same symmetry as the original BIC, we obtain $\sum_{k=1}^{N_{ss}} I_k$ constraints on $f_l$ from the $N_{ss}$ scattering solutions. In addition, similar to Eq. (6), we have

$$\int_{\Omega} \Phi_s \cdot f_l \, dr = 0. \quad (13)$$

The above always gives one real constraint. Therefore, we have the total of $1 + \sum_{j=1}^{N_{ss}} I_j$ real equations for solving $\omega_l$, $\alpha_l$ and $\gamma_{j,l}$ for $1 \leq j \leq n$. The total number of real unknowns is $1 + N_{wv} + n$. We choose $n$ to satisfy Eq. (7), so that the total number of equations is exactly the total number of unknowns. Since the unknowns appear in $f_l$ linearly with coefficients only related to $\varepsilon_s$, $\Phi_s$, $F$ and $G_j$, we have a real square matrix $A$ and a real vector $b_l$ such that

$$A \begin{bmatrix} \omega_l \\ \alpha_l \\ \vdots \\ \gamma_{n,l} \end{bmatrix} = b_l, \quad (14)$$

where $\alpha_l$ includes only $N_{wv}$ components. The matrix $A$ depends on the BIC, the scattering solutions and the perturbation profiles, but it does not depend on the previous iterations $\Phi_l$ for $1 \leq s < l$. Importantly, Eq. (6), with $f$ replaced by $f_l$, and Eq. (13) are sufficient conditions for Eq. (8) to have a solution that decays in the open direction(s) and preserves the symmetry of the original BIC [23, 52]. Therefore, if $A$ is invertible, we can solve $\omega_l$, $\alpha_l$ and $\gamma_{j,l}$ from Eq. (14), then solve $\Phi_l$ from Eq. (12). This implies that we can iteratively find the BIC and the tuning parameters through the power series.

The theory is applicable to generic BICs defined as those for which the matrix $A$ is invertible. The perturbation profiles should be arbitrary except for some specified conditions (such as the symmetry). Even for a generic BIC, if the perturbations are improperly chosen, the matrix $A$ can be non-invertible, then a BIC may not exist in the perturbed structure. On the other hand, there are also non-generic BICs for which the matrix $A$ is always non-invertible for any choice of the perturbation profiles.

### III. BICS IN 2D PERIODIC STRUCTURES

In this section, we consider BICs in 2D lossless dielectric structures that are invariant in $x$, periodic in $y$ with period $L$, bounded in $z$ by $|z| < d$, and surrounded by air. The dielectric function $\varepsilon(y, z)$ of such a structure is real and satisfies

$$\varepsilon(y + L, z) = \varepsilon(y, z), \quad \forall (y, z) \in \mathbb{R}^2, \quad (15)$$

$$\varepsilon(y, z) = 1, \quad |z| > d. \quad (16)$$

BICs in 2D structure with 1D periodicity have been investigated by many authors [33, 53]. Very often, one assumes that the structure has the following additional symmetry:

$$\varepsilon(y, z) = \varepsilon(-y, z), \quad \forall (y, z) \in \mathbb{R}^2, \quad (17)$$

$$\varepsilon(y, z) = \varepsilon(y, -z), \quad \forall (y, z) \in \mathbb{R}^2. \quad (18)$$

In recent works [33, 52], the codimension $n$ for some BICs in 2D periodic structures with the up-down mirror symmetry, i.e., Eq. (15), has been determined. In the following, we apply the general theory of Sec. III to all cases with or without the symmetry conditions (17) and (18).
Let \( \varepsilon_s(y, z) \) be the dielectric function of a specific periodic structure [satisfying Eqs. (15) and (16)], in which there is a nondegenerate BIC with frequency \( \omega_s \) and wavevector \( \alpha_s = (\alpha_s, \beta_s, 0) \), where \( \alpha_s \) and \( \beta_s \) are real wavenumbers in the \( x \) and \( y \) directions, respectively. Due to the periodicity in \( y \), we can assume \( \beta_s \in (-\pi/L, \pi/L] \). For simplicity, we consider BICs with \( \omega_s \) satisfying
\[
\sqrt{\alpha_s^2 + \beta_s^2} < \frac{\omega_s}{c} < \sqrt{\alpha_s^2 + \left(\frac{2\pi}{L} - |\beta_s|\right)^2},
\]
(19)
where \( c \) is the speed of light in vacuum. The above condition implies that only the zeroth diffraction order is propagating and all other diffraction orders (corresponding to the \( y \)-wavenumber \( \beta_s + 2m\pi/L \) for \( m \neq 0 \)) are evanescent.

To apply the theory of Sec. III, we first consider the degrees of freedom \( N_{\text{wv}} \) for the BIC wavevector. If both \( \alpha_s \) and \( \beta_s \) are nonzero, a BIC in the perturbed structure should have wavenumbers \( \alpha \) near \( \alpha_s \) and \( \beta \) near \( \beta_s \), and thus \( N_{\text{wv}} = 2 \). If \( \alpha_s \neq 0 \) and \( \beta_s = 0 \), we claim that the perturbed BIC must have \( \beta = 0 \) and thus \( N_{\text{wv}} = 1 \). This is true only when the BIC in the unperturbed structure is generic, so that there is only one BIC (near the original one) can be found in the perturbed structure. Recall that a BIC is a special point in a band of resonant states (below and above the periodic layer, respectively) corresponding to the zeroth diffraction order.

In Table I, we list the codimension \( N \) for different kinds of BICs according to the symmetry of the structure and the zero pattern of the wavevector. In the first column of Table I, the reflection symmetries in \( y \) and \( z \) are denoted as \( y \leftrightarrow -y \) and \( z \leftrightarrow -z \), respectively. The first row shows the zero pattern of the wavevector \( (\alpha_s, \beta_s) \), where \( \Box \) denotes a nonzero entry. The last two rows summarize the results already obtained in previous works [31, 32]. In the following, we give some justification for the new results listed in the table.

First, we consider the case of no symmetry. The perturbation profiles \( F \) and \( G_j \) for \( j \leq j \leq n \), do not need to satisfy Eq. (17) or (18), but they must be real functions of \( y \) and \( z \), periodic in \( y \) with period \( L \), and vanish for \( |z| > d \). Since the BIC in the unperturbed structure satisfies condition (19), there are two radiation channels (below and above the periodic layer, respectively) corresponding to the zeroth diffraction order.

For a generic BIC with \( \alpha_s \neq 0 \) and \( \beta_s \neq 0 \), we need to consider both polarizations in each radiation channel. Thus, the total number of independent scattering solutions is \( N_{ss} = 4 \). The constraint for each scattering solution is a complex condition, thus the index for each scattering solution is \( I_k = 2 \). Therefore, the codimension of the BIC is \( n = 4 \times 2 - 2 = 6 \).

If \( \alpha_s = 0 \) and \( \beta_s = 0 \), the BIC propagates along the \( x \) axis, has a vectorial field \( E = \Phi_s e^{i\alpha_s x} \) with \( \Phi_s \), depending on \( y \) and \( z \) only. Moreover, the BIC can be scaled such that the \( x \) component of \( \Phi_s \) is pure imaginary and the \( y \) and \( z \) components of \( \Phi_s \) are real [32]. Similarly, we can scale the scattering solutions \( E_k^{(y)} = \Psi_k e^{i\alpha_s x} \), so that the \( y \) and \( z \) components of \( \Psi_k \) are real and the \( x \) component of \( \Psi_k \) is pure imaginary. As a result, the integral condition \( \int_\Omega \Psi_k \cdot f \, dx = 0 \) gives one real equation, and \( I_k = 1 \). Since \( \Omega \) is a 3D domain with a unit length in \( x \), the above integral on \( \Omega \) is identical to the integral on the 2D cross section of \( \Omega \) given by \( 0 < y < L \) and \( -\infty < z < \infty \). Since \( N_{ss} = 4 \) and \( N_{\text{wv}} = 1 \), we have \( n = 4 \times 1 - 1 = 3 \).

The BIC with \( \alpha_s = 0 \) and \( \beta_s \neq 0 \) is a scalar mode in the \( E \) or \( H \) polarization. The electric or magnetic field has only one nonzero component (the \( x \) component). Since the fields for opposite polarizations are orthogonal, we can consider only the scattering solutions in the same polarization as the BIC. Therefore, \( N_{ss} = 2 \). Since \( I_k = 2 \) and \( N_{\text{wv}} = 1 \), we have \( n = 2 \times 2 - 1 = 3 \).

If the BIC is a standing wave with \( \alpha_s = \beta_s = 0 \), we have \( I_k = 1 \) and \( N_{ss} = 2 \) for the same reasons given above. Therefore, \( n = 2 \times 1 - 0 = 2 \).

Next, we consider the case with reflection symmetry in \( y \). Notice that we assume \( \varepsilon_s \), \( F \), and \( G_j \) all satisfy symmetry condition (17). In addition, \( F \) and \( G_j \) must be real and periodic as before. Since the structure is invariant in \( x \), Eq. (17) is identical to the in-plane inversion symmetry \( \varepsilon(x, y, z) = \varepsilon(-x, -y, z) \) or \( C_2 \) symmetry (rotation by 180° about the \( z \) axis). As we have mentioned in Sec. III this symmetry allows us to scale the BIC and the scattering solutions, so that their \( x \) and \( y \) components are \( PT \)-symmetric and their \( z \) components are anti-\( PT \)-symmetric, as in Eq. (3) [32]. For \( x \)-invariant structures, \( \Phi_s \) and \( \Psi_k \) are functions of \( y \) and \( z \) only, then the \( PT \)-symmetric solutions are

| \( (\alpha_s, \beta_s) \) | (\( \square, \square \)) | (\( \square, 0 \)) | (\( 0, \square \)) | (\( 0, 0 \)) |
|---|---|---|---|---|
| No symmetry | 6 | 3 | 3 | 2 |
| \( y \leftrightarrow -y \) | 2 | 1 | 1 | 0 |
| \( z \leftrightarrow -z \) | 2 | 1 | 1 | 1 |
| \( y \leftrightarrow -y, z \leftrightarrow -z \) | 0 | 0 | 0 | 0 |
and anti-$\mathcal{PT}$ symmetry conditions are
\[
\begin{bmatrix} \mathcal{P}_x(-y, z) \\ \mathcal{P}_y(-y, z) \\ -\mathcal{P}_z(-y, z) \end{bmatrix} = \begin{bmatrix} P_x(y, z) \\ P_y(y, z) \\ -P_z(y, z) \end{bmatrix} \tag{22}
\]
where $P_x$, $P_y$ and $P_z$ are the components of $\Phi_*$ or $\Psi_*$ and $\mathcal{P}_x$ is the complex conjugate of $P_x$.

For a generic BIC with $\alpha_* \neq 0$ and $\beta_* \neq 0$, we have $N_{ss} = 4$ and $N_{sv} = 2$ as before. The $\mathcal{PT}$ and anti-$\mathcal{PT}$ symmetry reduces the index $I_k$ from 2 to 1. Therefore, the codimension of the BIC is $n = 4 \times 1 - 2 = 2$.

For a BIC with $\alpha_* \neq 0$ and $\beta_* = 0$, we have $N_{ss} = 4$ and $N_{sv} = 1$. Since the BIC and the scattering solutions can be scaled such that their $x$ components are pure imaginary and their $y$ and $z$ components are real, we have
\[
\begin{bmatrix} P_x(y, z) \\ P_y(y, z) \\ P_z(y, z) \end{bmatrix} = \begin{bmatrix} iQ_x(y, z) \\ P_y(y, z) \\ P_z(y, z) \end{bmatrix}, \tag{23}
\]
where $Q_x$, $P_y$ and $P_z$ are real. The $\mathcal{PT}$ and anti-$\mathcal{PT}$ symmetry, i.e., Eq. (22), is obtained under a different scaling which can be compensated by multiplying a constant $C$ of unit magnitude to the left hand side of Eq. (22). Therefore,
\[
\begin{bmatrix} iQ_x(y, z) \\ P_y(y, z) \\ P_z(y, z) \end{bmatrix} = C \begin{bmatrix} -iQ_x(-y, z) \\ P_y(-y, z) \\ -P_z(-y, z) \end{bmatrix}. \tag{24}
\]
The above implies that $C$ is real and can only be 1 or $-1$. Among the four scattering solutions, two have $C = 1$ and the other two have $C = -1$. If the scattering solution has a different value of $C$ with the BIC, then the index is $I_k = 0$. Therefore, $n = 1 + 1 + 0 + 0 - 1 = 1$.

The BIC with $\alpha_* = 0$ and $\beta_* \neq 0$ is a scalar mode, thus $N_{ss} = 2$. Using the scaling for $\mathcal{PT}$ and anti-$\mathcal{PT}$ symmetry, we have $I_k = 1$. Therefore $n = 2 \times 1 - 1 = 1$.

The BIC with $\alpha_* = \beta_* = 0$ is a scalar standing wave. If it is generic, it has the opposite even/odd parity in $y$ with the scattering solutions of the same polarization. Therefore, the BIC is symmetry-protected and $n = 0$.

In Refs. [31, 32], detailed derivations are given for the results listed in the last two rows of Table I. For the two cases considered in this section (no symmetry and reflection symmetry in $y$), similar derivations can be worked out following the brief discussions above. In Table I, the results are shown only for BICs satisfying condition [19]. Using the general formula [17], it is not difficult to calculate the codimension $n$ for BICs that fail to satisfy [19].

IV. NUMERICAL RESULTS

To validate the theory developed in the previous sections, we use a highly accurate boundary integral equation method [40] to analyze two periodic structures with broken up-down mirror symmetry. The first example is a slab with a periodic array of air holes shown in Fig. II(a). The thickness and dielectric constant of the slab are $2d = L$ and $\varepsilon_1 = 11.56$, respectively, where $L$ is period in the $y$ direction. The medium above and below the slab (i.e. for $|z| > d$) is air. The boundary of the air hole is perturbed from a circle of radius $a$, such that the reflection symmetry in $y$ is preserved and the reflection symmetry in $z$ is broken. The perturbed boundary consists of two independent real parameters $\delta$ and $\gamma$ which correspond to the height (relative to $a$) of the bump and the dents shown in Fig. II(a). The precise formula of the boundary is
\[
y = a\rho(s) \cos(s), \quad z = a\rho(s) \sin(s), \quad 0 \leq s < 2\pi, \tag{25}\]
where $\rho(s) = 1 + \delta g(s) + \gamma [g(s - 0.2) + g(s + 0.2)]$, and $g(s) = \exp(-100|s - \pi/2|^2)$. For $a = 0.3L$ and $\delta = \gamma = 0$, the air holes are circular and the structure has an $E$-polarized propagating BIC with $\alpha_* = 0$, $\beta_* = 0.21728(2\pi/L)$ and $\omega_* = 0.44746(2\pi c/L)$. The electric field distribution (real part of the $x$ component) of

FIG. 1. (a) One period of the slab with an array of distorted air holes (for $\delta = 0.1$ and $\gamma = -0.1$) breaking the reflection symmetry in $y$. (b) Electric field pattern of a propagating BIC in a slab with circular air holes of radius $a = 0.3L$. (c) Parameter curve of the BIC. (d) and (e) Wavenumber $\beta$ and frequency $\omega$ of the BIC depending on parameters $\delta$ and $\gamma$. (f) Electric field pattern of the BIC at $\delta = 0.1$ and $\gamma \approx -0.07$. 
the BIC is shown in Fig. 1(b).

According to the theory of Sec. MI for perturbations that preserve the reflection symmetry in $y$, the scalar propagating BIC should have codimension $n = 1$. Therefore, for the fixed $a = 0.3L$, the BIC should form a curve in the $\delta\gamma$ plane. This is confirmed by the numerical result shown in Fig. 1(c). The Bloch wavenumber $\beta$ and frequency $\omega$ of this BIC are shown as functions of $\delta$ and $\gamma$ in Figs. 1(d) and 1(e), respectively. For $\delta = 0.1$, the BIC is obtained with $\gamma \approx -0.07$ for $\beta = 0.21789(2\pi/L)$ and $\omega = 0.44754(2\pi c/L)$. Its electric field distribution (the real part of $E_z$) is shown in Fig. 1(f). Moreover, since its codimension is 1, this BIC should form a surface in the space of three parameters. To illustrate this, we simply allow $a$ to be the third parameter. In Fig. 2 we show a surface of this BIC in the $\delta\gamma\alpha$ space for $0.28 \leq \alpha/L \leq 0.36$. Notice that this surface includes the vertical axis at $\delta = \gamma = 0$. It simply means that if the air holes are circular, the BIC is robust with respect to changes in the radius.

The second example is also a slab with a periodic array of air holes, but the boundaries of the air holes are perturbed to break both reflection symmetries in $y$ and $z$. The slab has the same thickness $2d = L$ and the same dielectric constant $\varepsilon_1 = 11.56$. The boundary of the air hole is also given by Eq. (23), but $\rho(s)$ is now given by

$$
\rho(s) = 1 + \delta g(s) + \sum_{j=1}^{3} \gamma_j \left[ g \left( s - \frac{j\pi}{10} \right) + g \left( s + \frac{j\pi}{10} \right) \right],
$$

and $g(s)$ is given by $g(s) = \exp(-100|s - 2\pi/3|^2)$. Since the codimension of a scalar propagating BIC in a periodic structure without symmetry is $n = 3$, we have introduced three tunable parameters $\gamma_j$ for $1 \leq j \leq 3$. However, if the BIC is a standing wave (in such a structure without symmetry), the codimension is $n = 2$. In Fig. 2(a), we show a unit cell of the periodic structure for $a = 0.3L$, $\delta = 0.1$, $\gamma_1 = 0.06456$, $\gamma_2 = -0.00524$, and $\gamma_3 = -0.006$.

In the structure with circular air holes of radius $a = 0.3L$, there is a BIC (a symmetry-protected standing wave) with frequency $\omega = 0.52044(2\pi c/L)$. Its electric field is shown in Fig. 3(b). For $\delta > 0$, we keep $\gamma_3 = 0$ and tune $\gamma_1$ and $\gamma_2$ to follow the BIC as $\delta$ is increased from 0, and show the numerical results in Fig. 3. The three parameters as shown as functions of $\delta$ in Figs. 3(a), 3(b) and 3(c), respectively. The frequency and Bloch wavenumber of the BIC are shown in Fig. 3(d).

To preserve a propagating BIC in such a periodic structure without symmetry, three tuning parameters are needed. Starting from the same propagating BIC studied in the first example (for circular air hole with radius $a = 0.3L$), we calculate $\gamma_1$, $\gamma_2$ and $\gamma_3$ to follow the BIC as $\delta$ is increased from 0, and show the numerical results in Fig. 4. The three parameters as shown as functions of $\delta$ in Figs. 4(a), 4(b) and 4(c), respectively. The frequency and Bloch wavenumber of the BIC are shown in Fig. 4(d).

For $\delta = 0.1$, the BIC has $\beta = 0.21673(2\pi/L)$ and $\omega = 0.44735(2\pi c/L)$, and is obtained at $\gamma_1 = -0.04357$, $\gamma_2 = -0.00952$, and $\gamma_3 = 0.01385$. In Fig. 4(e), we show the quality factor, $Q = -0.5\text{Re}(\omega)/\text{Im}(\omega)$, of nearby resonant modes. The electric field of the BIC (the real part of $E_z$) is shown in Fig. 4(f).
FIG. 4. (a), (b) and (c) Tuned parameters $\gamma_1$, $\gamma_2$ and $\gamma_3$ as functions of $\delta$ for a propagating BIC in a slab with distorted air holes breaking both reflection symmetries in $y$ and $z$. (d) Wavenumber $\beta$ and frequency $\omega$ of the BIC as functions of $\delta$. (e) $Q$ factor of the resonant modes near the BIC for $\delta = 0.1$. (f) Electric field pattern of the BIC for $\delta = 0.1$.

V. CONCLUSION

For theoretical interest and practical applications, it is important to find out what will happen to a BIC when the structure is perturbed. Although it typically becomes a resonant state of finite $Q$-factor, the BIC may persist either because it is robust against a class of perturbations, or the perturbation contains a sufficient number of tunable parameters. We have proposed a general formula for the minimum number $n$ of tunable parameters needed to preserve a generic nondegenerate BIC, and calculated $n$ for BICs in 2D structures that are invariant in one spatial direction and periodic in another. The integer $n$ is only defined when conditions (typically some symmetry) on structural perturbations are properly specified. For $n = 0$ and $n \geq 1$, the BIC is robust and nonrobust, respectively. A larger $n$ means that the BIC is difficult to find. A different point of view is to consider structures depending on a number of parameters. The set of parameter values at which a BIC exists form a geometric object in the parameter space, and the codimension of that geometric object is exactly $n$.

Although we have only verified our results for BICs in 2D structures with a single periodic direction, the general formula (7) is proposed for all generic nondegenerate electromagnetic BICs. In fact, we believe the general formula for $n$ is valid for any classical or quantum wave system. However, the current theory is only applicable to generic BICs that guarantee $n$ can be defined. Under a structural perturbation, a generic BIC either becomes a resonant state or continues its existence with slightly different frequency and wavevector. In contrast, a non-generic BIC has the additional possibility of splitting into two or more generic BICs. It is worthwhile to extend our theory to non-generic BICs, since they have additional interesting properties and valuable applications.

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