Divergence equations and uniqueness theorem of static black holes

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Abstract
Equations of divergence type in static spacetimes play a significant role in the proof of uniqueness theorems of black holes. We generalize the divergence equation originally discovered by Robinson in four dimensional vacuum spacetimes into several directions. We find that the deviation from spherical symmetry is encoded in a symmetric trace-free tensor $H_{ij}$ on a static timeslice. This tensor is the crux for the construction of the desired divergence equation, which allows us to conclude the uniqueness of the Schwarzschild black hole without using Smarr’s integration mass formula. In Einstein–Maxwell(-dilaton) theory, we apply the maximal principle for a number of divergence equations to prove the uniqueness theorem of static black holes. In higher ($n \geq 5$) dimensional vacuum spacetimes, a central obstruction for applicability of the current proof is the integration of the $(n-2)$-dimensional scalar curvature over the horizon cross-section, which has been evaluated to be a topological constant by the Gauss–Bonnet theorem for $n = 4$. Nevertheless, it turns out that the $(n-1)$-dimensional symmetric and traceless tensor $H_{ij}$ is still instrumental for the modification of the uniqueness proof based upon the positive mass theorem, as well as for the derivation of the Penrose-type inequality.

Keywords: black hole uniqueness, Einstein–Maxwell-dilaton theory, higher dimensions

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1. Introduction

In four spacetime dimensions, asymptotically flat and static black holes to vacuum Einstein’s equations are uniquely determined to be the Schwarzschild solution. A first proof was undertaken by Israel [1], assuming that the horizon is spherical, non-degenerate and connected. The authors in [2] were able to remove some technical conditions assumed in [1] such as spherical topology, although this turned out to be a consequence of the topology theorem of event horizon. Subsequently, Robinson [3] gave a considerably simplified proof that encompasses the previous works. All of these methods are based upon nonlinear ‘divergence equations’ built out of the quantities on the static timeslice. Integrating this divergence equation over the static timeslice, one gets inequalities involving mass, area and surface gravity of the horizon and it turns out that only the equalities are consistent. This leads to the spherical symmetry and therefore the metric is exhausted by the Schwarzschild solution. An alternative strategy proposed in [4] makes an elegant use of the positive mass theorem [5–7] and has been extended with suitable modifications into higher dimensions [8–13].

A notion closely linked to the black hole uniqueness is the Penrose inequality $M \geq \sqrt{A/(16\pi)}$ [14], where $A$ is the minimal area of the surface enclosing the apparent horizons and $M$ is the Arnowitt–Deser–Misner (ADM) mass of the asymptotically flat initial data set. Although the Penrose inequality is an important concept from the perspective of the cosmic censorship conjecture, its unequivocal proof is still lacking. Nevertheless, the Riemannian–Penrose inequality has been established for an asymptotically flat three-manifold of nonnegative scalar curvature foliated by evolving surfaces of inverse mean curvature flow [15–17]6. The monotonicity of the Geroch/Hawking quasi-local mass [20, 21] along the inverse mean curvature flow is the key property for the proof of the Riemannian–Penrose inequality. In this setting, the statement is also rigid, in the sense that equality is achieved if and only if the outside region of the apparent horizon is the Schwarzschild solution. This illustrates a fertile relationship between the uniqueness theorem of black holes and the Penrose inequality. Indeed, it has been argued that two inequalities in Israel’s proof correspond to the Penrose-type inequality and its ‘reversed’ version [22], and the concurrent rigidities of the two inequalities correspond to the spherically symmetric spacetime.

At the moment, the proof of the Riemannian–Penrose inequality based on the inverse mean curvature flow fails in higher dimensions. In order to understand in detail the relation to the Riemannian–Penrose inequality, it is instructive to validate the uniqueness theorem in higher dimensions following the arguments in [1–3]. The proof would offer a new insight into the corresponding flow in higher dimensions and would be much more intuitive than the one exploiting the positive mass theorem. However, it has been widely believed that the proofs in [1–3] do not have a simple generalization into higher dimensions, since the dimensionality of the spacetime enters the proof in the following manner. The proofs in [1–3] are based upon a divergence equation defined on a spacelike hypersurface which is asymptotically flat and terminates at the bifurcation surface of event horizon. This divergence equation gives rise to several inequalities involving integration of the scalar curvature for the induced metric on the horizon cross-section. In four dimensions, the Gauss–Bonnet theorem enables us to evaluate this quantity as a topological invariant. Obviously this is not possible in $n \geq 5$ dimensions. Moreover, the source term in the divergence equation involves the Cotton tensor for the spatial metric, which turns out to vanish in a static and spherically symmetric spacetime. Since the

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6 The conformal flow is another effective tool to prove the Riemannian–Penrose inequality [18]. The proof has also been extended to the case with a charge in [19].
Cotton tensor is an obstruction for the conformal flatness only for the three dimensional space, the existence of the desired divergence equation might be special to spatial dimension three. These unsettled issues motivate us to study more deeply the uniqueness proofs based on the divergence equations. In hindsight, it is surprising that there exists a useful formula of divergence type adapted to the proof of the uniqueness theorem. We therefore attempt to provide a systematic derivation of the divergence equation toward the spherical symmetry. The present reformulation has several advantages over the original treatment in [3]. Our formula in the vacuum case includes an additional free parameter, which allows us to conclude the uniqueness of the Schwarzschild solution without invoking the integrated mass formula that relates the mass, horizon area and surface gravity. The redundancy of the mass formula is a desirable presage when one tries to apply the proof for the non-asymptotically flat situation. For the Einstein–Maxwell theory, previous attempts for the uniqueness proof based upon divergence equation made a heavy use of the property of the symmetric coset space of the nonlinear sigma model. In contrast, our formulation does not rely on this property. In Einstein–Maxwell–dilaton theory, the most difficult problem was how to determine the value of the dilaton field at the horizon, which is not constant in general. We overcome this hindrance by discovering an entirely new divergence equation which is used to fix the dilaton field at the horizon in conjugation with the maximal principle. In addition, our formula in four dimensions does not make a direct appeal to the Cotton tensor to conclude the spherical symmetry. In place of the Cotton tensor, a central role is played by the symmetric and trace-free tensor constructed out of geometric quantities on a spatial hypersurface. Although our proof does not admit a straightforward higher-dimensional generalization, this tensorial field is of considerable help in modifying the proof based upon the positive mass theorem. This presents a more geometrically intuitive explanation for the spherical symmetry, rather than the Dirichlet boundary value problem defined on the underlying Euclidean space. The Penrose-type inequality in the static case is also derived from the diverse divergence equations in higher dimensions.

The present article is organized as follows. In the next section, we study the uniqueness proof of static vacuum black holes by extending the result of Robinson [3]. In section 3, we shall discuss the proof in the electrovacuum case. The Einstein–Maxwell-dilaton theory will be addressed in section 4. In section 5, we will discuss the uniqueness theorem in higher dimensions. Concluding comments are described in the final section 6.

2. Uniqueness theorem for vacuum black holes

Let us consider the solutions to the vacuum Einstein equations $R_{\mu\nu} = 0$. The most fundamental black hole solution is the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega_2^2,$$  

where $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the standard metric of a unit two-sphere. Here $M > 0$ is the ADM mass [23]. This metric is static, spherically symmetric and asymptotically flat. A regular event horizon locates at $r = 2M$.

Among the static black hole solutions in the asymptotically flat spacetimes, the Schwarzschild solution is the only vacuum solution which admits a regular horizon [1–4]. We discuss how the proof proceeds. When the spacetime admits a static Killing vector, the metric can be cast into the following form

$$ds^2 = -V^2(x)dt^2 + g_{ij}(x)dx^idx^j,$$
where $g_{ij}$ is the metric of the constant timeslice $\Sigma$. $V$ and $g_{ij}$ are independent of the time coordinate $t$. The event horizon locates at $V = 0$, where the Killing vector $\partial/\partial t$ becomes null. The vacuum Einstein equations $R_{\mu\nu} = 0$ decouple into

$$D^2 V = 0, \quad (2.3)$$

and

$$^{(3)}R_{ij} = \frac{1}{V} D_i D_j V, \quad (2.4)$$

where $D_i$ and $^{(3)}R_{ij}$ are the linear connection and the Ricci tensor associated with $g_{ij}$. Here and throughout this paper, we use the abbreviated notation $D^2 V = D_i D_j V$ and $(DV)^2 = |D_j V|^2 = D_i V D_j V$. From these equations, one sees that the scalar curvature for the space $(\Sigma, g_{ij})$ vanishes.

For later purpose, it turns out useful to locally foliate $\Sigma$ by the level set $S_V = \{V = \text{constant}\}$. Let us denote the unit normal to $S_V$ in $\Sigma$ by

$$n_i = \rho D_i V, \quad (2.5)$$

where $\rho \equiv (D^2 V)^{-1/2}$ stands for the lapse function. The induced metric and the extrinsic curvature of $S_V$ in $\Sigma$ are given respectively by

$$h_{ij} = g_{ij} - n_i n_j, \quad k_{ij} = h_{ik} D_k n_j. \quad (2.6)$$

We shall denote (twice) the mean curvature and the shear tensor of $S_V$ as

$$k \equiv h^{ik} k_{ik}, \quad \sigma_{ij} \equiv k_{ij} - \frac{1}{2} k h_{ij}. \quad (2.7)$$

The vacuum Einstein’s equations can be expressed in terms of these geometric objects. The equations which we need in our analysis are

$$n^i D_i \rho = \rho k, \quad (2.8)$$

Here $^{(2)}R$ is the scalar curvature for the first fundamental form $h_{ij}$. The former equation stems from (2.3), while the latter is derived from (2.4) and the Gauss equation.

Let us now specify our boundary conditions in terms of these geometric quantities. The curvature invariant $K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ is easily computed as

$$K = \frac{8}{V^2 \rho^2} \left[ k_{ij} k^{ij} + k^2 + \frac{2}{\rho^2} (D \rho)^2 \right], \quad (2.9)$$

where $D_i$ is the linear connection for $h_{ij}$. The finiteness of $K$ at the horizon $V = 0$ imposes the boundary conditions

$$k_{ij}|_{V=0} = 0, \quad D_i \rho|_{V=0} = 0. \quad (2.10)$$

The second condition represents the 0th law, i.e. equilibrium condition, of black hole thermodynamics, since $\rho_0 \equiv \rho|_{V=0}$ corresponds to the inverse of surface gravity of the event horizon. In what follows, we assume that the event horizon is nonextremal ($0 < \rho_0 < \infty$) and connected.

The boundary conditions at infinity are fixed by the asymptotic flatness

$$V \sim 1 - \frac{M}{r}, \quad g_{ij} \sim \left( 1 + \frac{2M}{r} \right) \delta_{ij}, \quad (2.11)$$
where $M(>0)$ is the ADM mass. In terms of geometric quantities of $S_V$, the asymptotic boundary conditions (2.11) translate into

$$k \sim \frac{2}{r}, \quad \rho \sim \frac{r^2}{M}, \quad \sigma_{ij} = O(1/r^6).$$

(2.12)

By the maximum/minimum principle, (2.3) fixes the range of $V$ as

$$0 \leq V < 1.$$  

(2.13)

In the original work of Israel [1], the global foliation of $\Sigma = \mathbb{R} \times S_V$ has been postulated. Henceforth, the topology of the cross section at infinity $S_{V=1}$ and at the horizon $S_{V=0}$ must be topologically homeomorphic, i.e. they are both $S^2$. In the present formulation, by contrast, we shall use equations (2.8), (2.10) and (2.12) only at the evaluation of surface integrals either at infinity or at the horizon. Thus, the local existence of the foliation at each neighborhood is sufficient for our purpose. This is a main advantage of the current prescription.

### 2.1. Robinson’s proof

Let us first recapitulate the argument in Robinson’s short letter [3], where it was pointed out without derivation that there exists a powerful identity

$$D_i \left[-2f^R_1(V) \frac{D^i \rho}{\rho^3} + \frac{f^R_2(V)}{\rho^5} \frac{D^i V}{V} \right] = \frac{1}{4} \rho^2 f^R_1(V) \rho^4 C^{ijk} C_{ijk} + 3 f^R_1(V) \left| \frac{D^i \rho}{\rho} - \frac{4V}{1-V^2} D^i V \right|^2.$$  

(2.14)

where $f^R_1, f^R_2$ are given by

$$f^R_1(V) = \frac{c_1 V^2 + c_2}{V(1-V^2)^3}, \quad f^R_2(V) = -\frac{2c_1}{(1-V^2)^3} + \frac{6(c_1 V^2 + c_2)}{(1-V^2)^4}.$$  

(2.15)

Here $C_{ijk} = 2D_j R^{(3)}_{ijk} - (3) R / 4 g_{ijk}$ is the Cotton tensor, whose vanishing is equivalent to the conformal flatness of the three surface $\Sigma$. $c_1$ and $c_2$ are arbitrary constants. To ensure $f^R_1(V) \geq 0$, the constants $c_1$ and $c_2$ are chosen to take values in

$$c_1 + c_2 \geq 0, \quad c_2 \geq 0.$$  

(2.16)

For this range of parameters, the right hand side of (2.14) is nonnegative. Using the Stokes theorem, one can transform the volume integral of (2.14) over the spatial slice $\Sigma$ into surface integral. The surface integral can be estimated by using (2.8), (2.10) and (2.11), giving rise to

$$0 \leq -\frac{\pi}{2M} \left[ (c_1 + c_2) - \left[ -4\pi \chi c_2 \rho_0^{-1} + (6c_2 - 2c_1) \rho_0^{-3} A_H \right] \right],$$  

(2.17)

where $A_H, \chi$ are respectively the area and the Euler number for the cross-section $B$ of the horizon

$$A_H = \int_B dS, \quad \chi = \frac{1}{4\pi} \int_B (2) R dS.$$  

(2.18)

One sees that the choice $c_1 = 1$ and $c_2 = 0$ satisfies (2.16), for which one obtains the relation

$$A_H \geq \frac{\pi}{4M \rho_0^3}.$$  

(2.19)

For the choice $c_1 = -1, c_2 = 1$, one gets

$$A_H \leq \frac{1}{2} \pi \rho_0^2 \chi.$$  

(2.20)
This is obviously compatible only for $\chi > 0$. Since the compact, orientable and connected two-surface of positive Euler number is inevitably a sphere, one obtains $\chi = 2$. This is consistent with the general results on the topology of stationary black holes [24, 25]. Thus (2.20) is reduced to

$$A_H \leq \rho_0^2 \pi.$$  (2.21)

We have obtained two inequalities (2.19) and (2.21), but we need one more relation between physical/geometrical parameters $(M, \rho_0, A_H)$. This can be obtained by integrating $D^2 V = 0$ over $\Sigma$, yielding Smarr’s integrated mass formula [26]

$$\rho_0 M = \frac{1}{4\pi} A_H.$$  (2.22)

This relation arises from the integrability of the Killing equation $(\nabla^\nu \nabla_\nu \xi^\mu = -R^\mu_{\nu\rho} \xi^\nu)$ for the static Killing vector $\xi = \partial/\partial t$.

Eliminating $\rho_0$ by use of (2.22) and (2.19) yields the ordinary Penrose inequality

$$A_H \leq 16\pi M^2,$$  (2.23)

while (2.21) gives the reversed inequality

$$A_H \geq 16\pi M^2.$$  (2.24)

Compatibility demands that the equality must hold. One thus deduces from (2.14) to conclude that

$$C_{ijk} = 0, \quad \frac{D\rho}{\rho} = \frac{4V}{1 - V^2} D\rho V = 0.$$  (2.25)

By the Lindblom identity [27]

$$C_{ijk} C^{ijk} = \frac{8}{V^4 \rho^2} \left( \sigma_{ij}\sigma^{ij} + \frac{(D\rho)^2}{2\rho^2} \right),$$  (2.26)

it turns out that the foliation $S_V = \{ V = \text{constant} \}$ is shear-free in $\Sigma$ and the second equation in (2.25) is solved to give $\rho(V) = 4M/(1 - V^2)^2$, where the integration constant has been settled by the asymptotic flatness. Equation (2.8) then implies that the scalar curvature for the two-dimensional metric $\hat{h}_{ij} = (M\rho)^{-1} h_{ij}$ is a positive constant, implying that $\hat{h}_{ij}$ is the standard metric of a unit sphere. It turns out that the spacetime is spherically symmetric. A change of variable $r = 2M/(1 - V^2)$ casts the metric into the Schwarzschild solution (2.1).

This completes the proof.

In contrast to Israel’s original method [1], Robinson’s proof does not assume the foliation $\Sigma = \mathbb{R} \times S_V$ throughout the domain of outer communications. Accordingly, one does not a priori put any restrictions to the topology of black holes. If the horizon is not spherical, the foliation $S_V = \{ V = \text{constant} \}$ obviously fails to cover the whole of domain of outer communications. This would be a nice property when one tries to generalize the proof in higher dimensions, since the possibility for the topology of higher dimensional black holes are much richer.

2.2. Generalization of divergence equation

In Robinson’s proof, the divergence equation (2.14) plays a central role. Its effectiveness to deduce the black hole uniqueness without expending considerable effort is an appealing
feature. At the same time, it remains enigmatic why this kind of desirable equations exists at all. Also, it seems unlikely that one can generalize the proof in the presence of electromagnetic field in this original formulation, without specifying the functional relationship between the norm of the Killing vector and the electrostatic potential. The presence of the Cotton tensor within the formula is not convenient as well, if one tries to generalize the present scheme to higher dimensions. Finally, we would like to relinquish the Smarr relation since analogous integrated mass formulas do not exist in the asymptotically (A)dS case.

Motivated by these issues, let us try to generalize the equation (2.14). Inspecting (2.14), we wish to find a current \( J^i \) satisfying the divergence type equation

\[
D_i J^i = \text{(terms of a definite sign)}.
\]  

(2.27)
The right hand side of this equation consists of a sum of the tensorial norms which vanish for the Schwarzschild solution. We find that the candidates of this kind are the symmetric tensor of the form

\[
H_{ij} \equiv D_i D_j V - \frac{2V}{\rho^2 (1 - V^2)} g_{ij} + \frac{6V}{1 - V^2} D_i V D_j V,
\]

(2.28)

and the vector field

\[
H_i \equiv \frac{D_i \rho}{\rho} - \frac{4V}{1 - V^2} D_i V.
\]

(2.29)
These spatial tensors satisfy

\[
H_{ij} D^j V = -\rho^{-2} H_j, \quad H^i_i = 0.
\]

(2.30)
The vanishing of the tensor \( H_{ij} \) for the Schwarzschild solution can be easily deduced by decomposing (2.28) into geometric quantities of \( S_V \) as

\[
H_{ij} = \rho^{-1} \sigma_{ij} - 2\rho^{-2} n_i (D_j \rho) + \frac{1}{2\rho} \left( k - \frac{4V}{\rho(1 - V^2)} \right) (h_{ij} - 2n_in_j).
\]

(2.31)
It follows that the tensor \( H_{ij} \) encodes the deviation from spherical symmetry. (See the discussion after (2.26), and also (5.10) for \( n \geq 5 \) dimensional case. We also refer to (2.50) and (5.30) for further geometric meaning of \( H_{ij} \).) This can be also seen by computing the Cotton tensor, which is now expressed as

\[
C_{ijk} = \frac{2}{V^2} (2H_{k[i} D_{j]} V + \rho^{-2} H_{[ij]k}).
\]

(2.32)
Therefore our current aim is to show \( H_{ij} = 0 \) under the present boundary conditions.

As the first step, let us make an ansatz for the current \( J^i \) to be the following form

\[
J^i = f_1 (V) g_1 (\rho) D^i \rho + f_2 (V) g_2 (\rho) D^i V,
\]

(2.33)
where \( f_{1,2}(V) \) and \( g_{1,2}(\rho) \) are functions of each argument which will be fixed below. This separable form of the current is the same as (2.14). The divergence of this current is computed as

\[
D_i J^i = (f'_1 g_1 + f_2 g'_2) D^i \rho D_i \rho + f_1 g'_1 (D\rho)^2 + f_1 g_1 (D\rho)^2 \rho + f'_2 g_2 (DV)^2,
\]

(2.34)
where the prime denotes the differentiation with respect to the single variable of the corresponding function. Using
\[ D^2 \rho = -\rho^3 |D_i D_i V|^2 + \frac{1}{V} D^i V D_i \rho + \frac{3}{\rho} (D^i \rho)^2, \tag{2.35} \]
equation (2.34) is rewritten into
\[ D_i J' = f_i(V) \rho^3 g_i(\rho) \left[ -|H_{ij}|^2 + \frac{|H_{ij}|^2}{\rho^2} \left( 3 + \frac{\rho g_1'(\rho)}{g_1(\rho)} \right) \right] + H_i D^i V S_1 + S_2, \tag{2.36} \]
where
\[ S_1 = \frac{\rho g_1(\rho) V f_i(V)}{1 - V^2} \left[ 1 - \frac{V^2}{V} \left( \frac{1}{f_i(V)} \right) + 12 + \frac{8 \rho g_1'(\rho)}{g_1(\rho)} + \frac{1 - V^2 f_2(V) g_2'(\rho)}{V f_1(V) g_1(\rho)} \right], \tag{2.37} \]
\[ S_2 = \frac{4V}{(1 - V^2)^2} \rho^2 \left[ \left( \frac{1 - V^2 f_2(V) g_2(\rho)}{V^2 f_1(V)} \right) - \frac{8 \rho g_1'(\rho)}{g_2(\rho)} \left( 3 + \frac{2 \rho g_2'(\rho)}{g_1(\rho)} \right) \right]. \tag{2.38} \]
Now we would like to render the right-hand side of (2.36) to have a definite sign and to vanish for the Schwarzschild solution. Since we cannot control the signs of the last two terms in (2.36), we require \( S_1 = S_2 = 0 \). Inferring from the last term of (2.37), one needs either (i) \( g_2'(\rho) \propto g_1(\rho) \) or (ii) \( f_2(V) \propto \rho^{-c} f_1(V) \) to render the equations decoupled. In this paper we will focus on the former case\(^7\), for which
\[ g_1(\rho) = -c \rho^{-(c+1)}, \quad g_2(\rho) = \rho^{-c}, \tag{2.39} \]
where \( c \) is an integration constant. Substituting these back into (2.37) and (2.38) yields two first-order linear differential equations
\[ f_1(V) + f_2(V) + \frac{1 + (3 - 8c) V^2}{V(1 - V^2)} f_1(V) = 0, \quad f_2(V) + \frac{8c(1 - 2c) V^2}{(1 - V^2)^2} f_1(V) = 0. \tag{2.40} \]
These equations are easily solved to give
\[ f_1(V) = \frac{(1 - V^2)^{1-c}}{V} \left[ a + b(1 - V^2) \right], \quad f_2(V) = \frac{2}{(1 - V^2)^c} \left[ a(2c - 1) + 2bc(1 - V^2) \right], \tag{2.41} \]
where \( a \) and \( b \) are integration constants. Using
\[ \left| D_{ij} VH_{ij} \right|^2 = \frac{1}{2 \rho^2} \left| H_{ij} \right|^2 - \frac{3}{2 \rho^2} \left| H_{ij} \right|^2, \tag{2.42} \]
we finally arrive at an improved divergence equation
\[ D_i J' = \frac{c f_i(V)}{2 \rho^2} \left[ 2 \rho^2 D_{ij} VH_{ij} - H_{ij} g_{ij} \right]^2 + (2c - 1) \left| H_{ij} \right|^2. \tag{2.43} \]
That this formula contains three tunable parameters is the key to the present proof for the uniqueness. By writing the first term in terms of the Cotton tensor (2.32), this recovers Robinson’s equation (2.14) for \( c = \frac{3}{2} \), and equations (2.12) and (2.13) in [2] for \( c = 1/2 \). The right-hand side of the above equation becomes positive semi-definite, provided\(^8\)
\[ f_1(V) \geq 0, \quad c \geq \frac{1}{2}, \tag{2.44} \]
\( ^7 \) Even if we adopt the option (ii), the final conclusion is identical. We shall not attempt to follow this route.
\( ^8 \) We also need \( 0 < \rho < \infty \) in the interior of \( \Sigma \) for the right-hand side of (2.43) to be well-defined. This can be shown by applying the maximal principle to (2.35), as demonstrated in [8].
The condition $f(V) \geq 0$ for $0 \leq V < 1$ is assured by
\[ a > 0, \quad a + b \geq 0. \tag{2.45} \]

By Stokes’ theorem, integration of (2.43) over $\Sigma$ is transformed into the inequality for surface integrals
\[ \int_\Sigma D_i J^i d\Sigma = \int_{S^\infty} J_i n^i dS - \int_B J_i n^i dS \geq 0, \tag{2.46} \]
where $S^\infty$ is the two-surface at infinity and $B$ denotes the bifurcation two-surface of the horizon. Upon using (2.8), (2.10) and (2.11), we end up with
\[ 0 \leq a[A_H - \pi(4M)^{1-c}] + (a + b)c \rho^0_{\chi^{\rho^2_0 - 2A_H]}]. \tag{2.47} \]
This inequality holds for any values of $a$, $b$ and $c$ satisfying (2.44) and (2.45), if and only if the pair of inequalities
\[ \pi \frac{(4M)}{\rho^0_{\chi^{\rho^2_0 - 2A_H}}} \leq \frac{1-c}{2} \chi \tag{2.48} \]
is satisfied.

The case for $c = 1$ gives $\chi \geq 2$, implying that only the spherical topology ($\chi = 2$) is allowed. By setting $\chi = 2$, one sees that the only case for $A_H = \pi \rho^0_{\chi^{\rho^2_0 - 2A_H}}$ consistent with (2.48) for $c = 1$. Hence, the inequality (2.46) is converted to an equality, and then (2.43) implies $H_i = 0$. Therefore, the spacetime admits spherical symmetry, as desired.

We have shown the uniqueness of the Schwarzschild solution without using Smarr’s formula (2.22). The underlying reason behind this is that one can choose $c = 1$ for which the ADM mass disappears from (2.48). This would not have been possible in the original form of Robinson’s equation (2.14), since it corresponds to $c = 2$. Even though this freedom does not offer any advantages in the proof of Schwarzschild solution, the unnecessity of the Smarr formula is a desirable precursor to the proof of uniqueness theorem via divergence equation in the asymptotically (A)dS case.

It is worth commenting that the inequality $\chi \geq 2$ obtained here is the stronger than the one derived by the variational formula [24]. In particular this rules out explicitly the topology of $\mathbb{RP}^2$ surface which has $\chi = 1$ but is unorientable.

We also dispensed with the Cotton tensor in the source term of our divergence equation (2.43), since the obstruction for spherical symmetry is completely encoded in $H_i$. This property continues to be valid in higher dimensions, as we will see in section 5.

From the separable form (2.33) of the current $J^i$, the geometric meaning is less obvious and it is hard to read off the spherical symmetry. This is rectified by recasting the current $J^i$ into a more suggestive form
\[ J^i = -[(1 - V^2)^2 \rho]^{-c} \left[ \frac{c}{V}(1 - V^2)[a + b(1 - V^2)]H^i + 2aD^iV \right]. \tag{2.49} \]
It is important to observe that $H_i$ defined in (2.29) is given by the derivative of $(1 - V^2)^2 \rho$. For the Schwarzschild solution, this is indeed constant $(1 - V^2)^2 \rho = 4M$. Because of this fact along with the equations of motion $D^iV = 0$, the conservation of the current for the spherical symmetry becomes compelling.

We also remark that $H_{ij} = 0$ is equivalent to the condition that $\zeta \equiv (1 - V^2)^{-3}D_iV$ is a conformal Killing vector on $\Sigma$:
\[ D_i\zeta_j + D_j\zeta_i - \frac{2}{3}D_k\zeta^k g_{ij} = \frac{2}{(1 - V^2)^3}H_{ij}. \]  
\hline
(2.50)
\hline

For \( H_{ij} = 0 \), this conformal Killing vector field corresponds to the dilatation vector field, which is always present in the spherically symmetric space. Namely, the conformally flat space \( ds^2 = dr^2/f(r) + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \) admits a dilatational conformal Killing vector of the form \( \zeta_i = (r/\sqrt{f(r)}) D_i r \). This provides another geometric meaning to \( H_{ij} \) as an obstacle for the existence of the conformal Killing vector corresponding to the dilatation, in a space of nonnegative scalar curvature.

3. Einstein–Maxwell theory

This section discusses the Einstein–Maxwell system described by the Lagrangian
\[ L = R - F_{\mu\nu}F^{\mu\nu}. \]  
\hline
(3.1)
\hline

Here \( F_{\mu\nu} = F_{[\mu\nu]} \) is the Faraday tensor. Consider the static spacetime (2.2) and assume that the Maxwell field is invariant under the action generated by the static Killing vector \( L\partial/\partial t \).

Performing the electromagnetic duality rotation if necessary, the Bianchi identity \( dF = 0 \) brings the Maxwell field to be electric
\[ F = dt \wedge d\psi, \]  
\hline
(3.2)
\hline

where \( \psi \) is an electrostatic potential and we can work in a gauge in which \( \psi \) is \( t \)-independent.

Since we are focusing on the asymptotically flat spacetime satisfying the null convergence condition, the topological censorship holds and therefore the domain of outer communication is simply connected \([25, 31]\), for which the global existence of \( \psi \) is assured. The electrovacuum Einstein’s equations then read
\[ D^2 V = \frac{1}{V} (D\psi)^2, \]  
\hline
(3.3)
\hline

and
\[ ^{(3)}R_{ij} - \frac{1}{V} D_i D_j V = \frac{2}{V^2} \left( -D_i\psi D_j\psi + \frac{1}{2} (D\psi)^2 g_{ij} \right). \]  
\hline
(3.4)
\hline

The Maxwell equation gives
\[ D_i (V^{-1} D^i \psi) = 0. \]  
\hline
(3.5)
\hline

Expressed in terms of geometric quantities for the foliation \( \Sigma \simeq \mathbb{R} \times \mathcal{S}_V \), the relevant Einstein’s equations are
\[ n^i D_i \rho = \rho k - \frac{\rho^2}{V} \left( (n^i D_i \psi)^2 + (D\psi)^2 \right), \]  
\hline
(3.6)
\hline

At infinity, the metric and the gauge field behave as
\[ V \sim 1 - \frac{M}{r}, \quad g_{ij} \sim \left( 1 + \frac{2M}{r} \right) \delta_{ij}, \quad \psi \sim \frac{Q}{r}. \]  
\hline
(3.7)
\hline
9 In general, if the four-dimensional spacetime admits a non-null Killing vector \( \xi \), the Maxwell field satisfies
\[ \mathcal{L}_\xi F = \Psi \ast F, \]  
where \( \ast F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \) and \( \Psi \) being constant \([28, 29]\). We refer the readers to \([30]\) for an attempt to get rid of the assumption of symmetry inheritance \( \mathcal{L}_\xi F = 0 \) in the uniqueness proof.
where $M$ is the ADM mass and $Q$ is the electric charge which is taken to be positive without loss of generality. In terms of $k$ and $\rho$, the asymptotic conditions can be translated as (2.12). We assume that these conserved charges strictly obey the Bogomol'ny inequality [32, 33]

$$M > Q. \quad (3.8)$$

The boundary conditions at the event horizon $V = 0$ can be determined by requiring the curvature invariants $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ and $F_{\mu\nu}F^{\mu\nu}$ to remain finite, leading to

$$k_{ij}|_{V=0} = 0, \quad D_i \rho|_{V=0} = 0, \quad n^i D_i \psi|_{V=0} = 0, \quad D_i \psi|_{V=0} = 0. \quad (3.9)$$

The constancy of $\rho$ and $\psi$ over the horizon means the zeroth law of black hole thermodynamics. The nonextremality condition of the horizon amounts to $0 < \rho_0 = \rho|_{V=0} < \infty$. An example of the black hole solutions is the Reissner–Nordström solution

$$ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 d\Omega_2^2, \quad \psi = \frac{Q}{r}, \quad (3.10)$$

where $f(r) = 1 - 2M/r + Q^2/r^2$. The uniqueness of (3.10) was first addressed by Israel [34] under the assumption that the horizon is connected, nondegenerate and spherical. Later works [35–38] removed some of these assumptions. We are now going to show the uniqueness of the (3.10), based on the divergence equation. A primary utility of this scheme is that divergence equations are suitable for adaptation of the maximum/minimum principle.

From the field equations, one can derive following two divergence equations

$$D_i(D^i V - V^{-1} \psi D^i \psi) = 0, \quad (3.11)$$

$$D_i \left( \frac{V^2 + \psi^2}{V} D^i \psi - 2 \psi D^i V \right) = 0. \quad (3.12)$$

Upon integrating (3.5), (3.11) and (3.12) over the static timeslice $\Sigma$, one easily obtains

$$4\pi Q + \int_B n^i D_i \psi V dS = 0, \quad (3.13)$$

$$4\pi M - \rho_0^{-1} A_H + \psi_0 \int_B n^i D_i \psi V dS = 0, \quad (3.14)$$

$$4\pi Q - 2\rho_0^{-1} \psi_0 A_H + \psi_0^2 \int_B n^i D_i \psi V dS = 0. \quad (3.15)$$

The above equations are combined to give Smarr’s mass formula

$$A_H = 4\pi \rho_0 (M - Q \psi_0), \quad (3.16)$$

as well as

$$\psi_0 = \frac{M - \sqrt{M^2 - Q^2}}{Q}, \quad (3.17)$$

where the sign in front of the square root in $\psi_0$ has been chosen to ensure $A_H = 4\pi \rho_0 \sqrt{M^2 - Q^2} > 0$.

Of crucial importance to the present proof is the following equation

$$D^2 G_1 - D_i \left[ \log(V \psi^{-2}) \right] D^i G_1 = 0, \quad (3.18)$$
where $G_1(V, \psi) \equiv \psi^{-1}(1 - V^2) + \psi$. Here note that (3.5) and the maximum/minimum principle [39] tell us $0 < \psi < \psi_0$ and then $|D_i \log(V \psi^{-2})| < \infty$. Hence, the maximum/minimum principle can be applied to (3.18), implying that $G_1(V, \psi)$ never acquires the global maximum/minimum in the interior of $\Sigma$. From the present boundary conditions, the values of $G_1(V, \psi)$ at the horizon and at infinity both coincide to give $2M/Q$, and hence $G_1(V, \psi)$ is constant ($=2M/Q$) throughout $\Sigma$. Then $\psi$ depends only on $V$ and is given by

$$\psi = \frac{1 - \beta(V)}{q},$$  \hspace{1cm} (3.19)

where for notational simplicity we have introduced

$$\beta(V) \equiv \sqrt{1 - q^2(1 - V^2)}, \hspace{1cm} q \equiv Q/M < 1.$$  \hspace{1cm} (3.20)

One can also derive the relation (3.19) by several fashions. Israel exploited a divergence equation which depends explicitly on physical parameters $M, Q$ to conclude (3.19) [34]. We also find another useful equation

$$D^2 G_2 - \frac{1}{V} DV D_2 G_2 = 0, \hspace{1cm} G_2(V, \psi) \equiv V^2 - (\psi - \psi_1)^2,$$  \hspace{1cm} (3.21)

where $\psi_1$ is an arbitrary constant. If one chooses $\psi_1 = (1 + \psi_0^2)/(2\psi_0)$, the values of $G_2$ at infinity and at horizon are equal. By the maximum/minimum principle, $G_2(V, \psi)$ is constant, yielding (3.19).

Since the functional dependence of $\psi$ on $V$ has been specified, the field equations are now simplified to

$$D^2 V = \frac{q^2 V}{\beta(V)^2} (DV)^2,$$  \hspace{1cm} (3.22)

and

$$(^3 R_{ij} = \frac{1}{V} D_i D_j V + \frac{q^2}{\beta(V)^2} [(DV)^2 g_{ij} - 2DV D_j V].$$  \hspace{1cm} (3.23)

As in the vacuum case, let us define a three dimensional symmetric tensor

$$H_{ij} \equiv D_i D_j V + \frac{V}{1 - V^2} \frac{(\beta(V) + 1)(4\beta(V) - 1)}{\beta(V)^2} D_i V D_j V - \frac{V(1 + \beta(V))}{(1 - V^2)\beta(V)} (DV)^2 g_{ij},$$  \hspace{1cm} (3.24)

and a vector field

$$H_i \equiv \frac{D_i \beta}{\rho} - \frac{V(\beta(V) + 1)(3\beta(V) - 1)}{(1 - V^2)\beta(V)^2} D_i V.$$  \hspace{1cm} (3.25)

These tensors satisfy the relations identical to the vacuum case

$$H_0 D^i V = -\rho^{-2} H_i, \hspace{1cm} H'_i = 0.$$  \hspace{1cm} (3.26)

When $Q = 0$, these tensors reduce to (2.28) and (2.29). In terms of local quantities of $S_V$, we have

$$H_{ij} = \rho^{-1} \sigma_{ij} - 2\rho^{-2} n_i(D_j) + \frac{1}{2\rho} \left[ k - \frac{2V(1 + \beta(V))}{\rho\beta(V)(1 - V^2)} \right] (h_{ij} - 2n_i n_j).$$  \hspace{1cm} (3.27)

Hence, $H_{ij} = 0$ is a condition for the spherical symmetry. This will be also apparent by looking at the Cotton tensor $C_{ijk} = 2D_i(^3 R_{jk} - (^3 R/4)g_{ijk})$, which is given in terms of $H_{ij}$ as
It then suffices to show $H_{ij} = 0$ for the uniqueness of the Reissner–Nordström solution (3.10).

We wish to find the current conservation equation of the separable type (2.27). Using

$$D^2 \rho = -\rho^3 |D_i D_j V|^2 + \frac{3}{\rho} (D \rho)^2 + D_j \rho D_i V \left( \frac{1}{V} + \frac{2q^2 V}{\beta(V)^2} \right) + \frac{2q^4 V^2}{\rho \beta(V)^2},$$

the divergence of $J'$ culminates in a separable form (2.33) with respect to $\rho$ and $V$, if we choose

$$g_1(\rho) = -c \rho^{-(c+1)}, \quad g_2(\rho) = \rho^{-c},$$

where $c$ is a constant. With this choice, we are left with two linear first-order equations which are easily solved to give

$$f_1(V) = \frac{1}{V(1 - V^2)^{2c-1}} \left[ \frac{1 + \beta(V)^2}{\beta(V)^c} \left[ a(1 + \beta(V)) + 2b(1 - V^2) \right] \right],$$

$$f_2(V) = \frac{1 + \beta(V)^2}{(1 - V^2)^2 \beta(V)^{2c+1}} \left[ -a \beta(V)(1 + \beta(V)) + c(3\beta(V) - 1) \left[ a(1 + \beta(V)) + 2b(1 - V^2) \right] \right],$$

where $a$ and $b$ are integration constants. We thus conclude that the following divergence equation holds

$$D_i J^i = \frac{c f_1(V)}{2\rho^2} \left[ 2\rho^2 D_i V H_{ij} - H_{[i} \delta_{j]} \right] + (2c - 1) |H_i|^2.$$

This is of the same form as in the vacuum case (2.43). The right hand side of this equation becomes nonnegative, provided

$$c \geq \frac{1}{2}, \quad a \geq 0, \quad a + \frac{2}{1 + \sqrt{1 - q^2}} b \geq 0.$$  (3.34)

The last two conditions ensure $f_1(V) \geq 0$.

By making use of Stokes’ theorem and inserting the boundary conditions (3.7), (3.9) with (3.6), one gets

$$0 \leq a \left[ -4\pi M^{1-c} + A_H (1 - q^2)^{-(c+1)/2} \left( 1 + \sqrt{1 - q^2}^2 \rho_0^{2(1+c)} \right) \right]$$

$$+ \left[ a(1 + \sqrt{1 - q^2}) + 2b \right] c(1 - q^2)^{-c/2} \left( 1 + \sqrt{1 - q^2}^2 \rho_0^{2(1+c)} \right)^{-c}$$

$$\times \left[ 2\pi \chi - A_H (1 - q^2)^{-1} \left( 1 + \sqrt{1 - q^2}^2 \rho_0^{-2} \right) \right].$$

Here, $\chi$ is the Euler characteristic (2.18) of the horizon cross-section. The necessary and sufficient condition for which this inequality holds for any values of $a$, $b$, and $c$ satisfying (3.34) is

$$4\pi \left( \frac{M(1 + \sqrt{1 - q^2})^2}{\rho_0 \sqrt{1 - q^2}} \right)^{1-c} \leq A_H \left( \frac{(1 + \sqrt{1 - q^2})^2}{\rho_0 \sqrt{1 - q^2}} \right)^2 \leq 2\pi \chi.$$  (3.36)

The case for $c = 1$ immediately gives $\chi \geq 2$, and hence we set $\chi = 2$ in the hereafter. Then this inequality for $c = 1$ is consistent only if the equality holds, that is
\[ A_H = \frac{4\pi \rho_0^2 (1 - q^2)}{(1 + \sqrt{1 - q^2})^2}. \]  

(3.37)

On account of \( H_{ij} = 0 \), the spacetime is spherically symmetric and therefore the Reissner–Nordström solution (3.10) is singled out.

Note that we did use Smarr’s formula (3.16) to derive (3.19), in contrast to the vacuum case. Nonetheless, we have nowhere used the property of the symmetric coset space, in contrast to the argument in [36]. In this sense, the present proof seems to have a potentially broader applicability.

4. Einstein–Maxwell-dilaton theory

We next consider the four-dimensional Einstein–Maxwell-dilaton gravity described by Lagrangian

\[ L = R - 2g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - e^{-2\alpha \phi} F_{\mu\nu} F^{\mu\nu}, \]  

(4.1)

where \( \alpha (> 0) \) is a coupling constant. Let us focus on the static spacetimes, where the metric (2.2), the Maxwell field and the dilaton field are all invariant under the orbit of static Killing field. Here we assume that the Maxwell field is electric \( F = -d\psi \wedge dt \), where \( \psi \) is the \( t \)-independent electrostatic potential. Note that the absence of the magnetic potential is a genuine restriction, since the equations of motion arising from the action (4.1) do not admit a duality rotation. Under these conditions, Einstein’s equations are simplified to

\[ D^2 V = \frac{1}{V} e^{-2\alpha \phi} (D\psi)^2, \]  

(4.2)

\[ (3) R_{ij} = \frac{1}{V} D_i D_j V - \frac{2}{V^2} e^{-2\alpha \phi} \left( D_i \psi D_j \psi - \frac{1}{2} g_{ij} (D\psi)^2 \right) + 2 D_i \phi D_j \phi. \]  

(4.3)

The scalar curvature of the three-dimensional space reads

\[ (3) R = 2 V^{-2} e^{-2\alpha \phi} (D\psi)^2 + 2 (D\phi)^2. \]

The Maxwell and dilaton equations of motion are given by

\[ D_i (e^{-2\alpha \phi} V^{-1} D\psi) = 0, \]  

(4.4)

\[ D_i (V D^i \phi) - \frac{\alpha}{V} e^{-2\alpha \phi} (D\psi)^2 = 0. \]  

(4.5)

The finiteness of curvature invariants at the horizon requires

\[ k_{ij} |_{r=0} = 0, \quad \rho |_{r=0} = \rho_0, \quad \psi |_{r=0} = \psi_0, \quad \phi, \partial_{\nu} \phi, D_i \phi \text{ are finite}, \]  

(4.6)

where \( \rho_0 (0 < \rho_0 < \infty) \) and \( \psi_0 \) are constants, representing the inverse of the surface gravity and the electrostatic potential at the horizon. Let us emphasize that the value of the dilaton field is not necessarily constant over the horizon, i.e. \( D_i \phi |_{r=0} \neq 0 \) in general.

The boundary conditions at infinity are

\[ g_{ij} \sim \left( 1 + \frac{2M}{r} \right) \delta_{ij}, \quad V \sim 1 - \frac{M}{r}, \quad \psi \sim \frac{Q}{r}, \quad \phi \sim \frac{\phi_\infty}{r}. \]  

(4.7)

Here \( M \) is the ADM mass, \( Q \) is the electric charge and \( \phi_\infty \) is a constant. From the positive mass theorem [40, 41], the mass is bounded from below by the charge as
\[
M \geq \frac{Q}{\sqrt{1 + \alpha^2}}. \quad (4.8)
\]

From now on, we assume that this inequality is strictly satisfied.

A static and spherically symmetric black-hole solution to this theory was discovered by Gibbons and Maeda [42], whose metric boils down to

\[
\mathrm{d}s^2 = -(1 - \frac{r_+}{r})(1 - \frac{r_-}{r})^{1 - \alpha^2} \mathrm{d}t^2 + \frac{(1 - \frac{r_-}{r})^{-1}}{(1 - \frac{r_+}{r})^{1 + \alpha^2}} \mathrm{d}r^2 + r^2 \left(1 - \frac{r_+}{r} \right)^{1 + \alpha^2} \mathrm{d}\Omega^2, \quad (4.9)
\]

with

\[
\phi = \frac{\alpha}{1 + \alpha^2} \log \left(1 - \frac{r_-}{r} \right), \quad \psi = \frac{\sqrt{r_+ r_-}}{\sqrt{1 + \alpha^2 r}}. \quad (4.10)
\]

Here \(r_+ (> r_- > 0)\) is a locus of the event horizon and \(r_-\) corresponds to the curvature singularity. These two parameters are related to the ADM mass \(M\) and the electric charge \(Q\) by

\[
r_\pm = \frac{1 + \alpha^2}{1 \pm \alpha^2} \left( M \pm \sqrt{M^2 - (1 - \alpha^2)Q^2} \right). \quad (4.11)
\]

The uniqueness property of this solution was first addressed in [43] for a coupling constant \(\alpha = 1\), by finding a suitable Witten spinor on a spatial timeslice. Subsequently, [44] extended the proof to encompass the case of arbitrary \(\alpha\) using the conformal positive mass theorem. These techniques were also worked out in higher dimensions [10, 11]. In the context of divergence equations, the most difficult issue is how to assess the value of the dilaton field at the event horizon. In this section, we overcome this obstacle by finding yet another divergence equation. We develop a new proof based only upon divergence equations.

One can rewrite some of the equations of motion into the divergence type

\[
D_i \left( D^j \psi \right) = 0, \quad (4.12)
\]

\[
D_i \left[ (1 + \alpha^2) \right] e^{-2\alpha \phi} \psi^2 + V^2 \right) D^j \psi - 2 \psi D^j V - 2 \alpha V \psi D^j \phi = 0, \quad (4.13)
\]

\[
D_i \left( V D^j \phi - \alpha D^j V \right) = 0. \quad (4.14)
\]

Integration of (4.4) and (4.12)–(4.14) gives

\[
4\pi M - A H \phi_0^{-1} \rho_0 + \frac{\psi_0}{\rho_0} \int_B e^{-2\alpha \phi} \frac{\partial \psi}{V} \mathrm{dS} = 0, \quad (4.15)
\]

\[
4\pi Q + \frac{1}{\rho_0} \int_B e^{-2\alpha \phi} \frac{\partial \psi}{V} \mathrm{dS} = 0, \quad (4.16)
\]

\[
-4\pi Q + 2\psi_0 \rho_0^{-1} A H - \frac{1 + \alpha^2}{\rho_0} \psi_0^2 \int_B e^{-2\alpha \phi} \frac{\partial \psi}{V} \mathrm{dS} = 0, \quad (4.17)
\]

\[
-4\pi \phi_\infty - 4\pi M + \alpha A H \rho_0^{-1} = 0. \quad (4.18)
\]
These equations are combined to give

$$A_W = 4\pi \rho_0(M - Q\nu_0), \quad \psi_0 = \frac{Q}{M + \sqrt{M^2 - Q^2(1 - \alpha^2)}}, \quad \phi_\infty = -\alpha Q\psi_0.$$ (4.19)

It follows that $\phi_\infty$ is a secondary charge, since it is specified by $M$ and $Q$.

To proceed, let us define [44]

$$\Phi_{\pm 1} = \frac{1}{2} \left[ e^{\alpha \phi} V \pm \frac{e^{-\alpha \phi}}{V} - (1 + \alpha^2) e^{-\alpha \phi} \right],$$

$$\Phi_0 = \frac{1}{\sqrt{1 + \alpha^2}} e^{-\alpha \phi} \psi,$$ (4.20)

$$\Psi_{\pm 1} = \frac{1}{2} \left( e^{-\phi/\alpha} V \pm e^{\phi/\alpha} V^{-1} \right),$$ (4.22)

satisfying the constraints

$$\Phi_{+1}^2 - \Phi_{-1}^2 + \Phi_0^2 = 1, \quad \Psi_{+1}^2 - \Psi_{-1}^2 = 1.$$ (4.23)

These scalars span the homogeneous coordinates of $SO(1, 1) \times SL(2, \mathbb{R})/SO(1, 1)$, which corresponds to the nonlinear sigma model of static Einstein–Maxwell–dilaton theory. The equations of motions for these scalars read

$$D_0(VD_0\Phi_{-1/0/+1}) = VG(\Phi)\Phi_{-1/0/+1}, \quad D_0(VD_0\Psi_{\pm 1}) = VG(\Psi)\Psi_{\pm 1},$$ (4.24)

where

$$G(\Phi) = |D_0\Phi_{-1}|^2 - |D_0\Phi_0|^2 - |D_0\Phi_{+1}|^2, \quad G(\Psi) = |D_0\Psi_{-1}|^2 - |D_0\Psi_{+1}|^2.$$ (4.25)

From the field equations, the scalars $G_\pm = 1 - V e^{\alpha \phi} \pm \sqrt{1 + \alpha^2} \psi$ obey

$$D^2 G_\pm + \left( \mp \sqrt{1 + \alpha^2} \psi D_0\phi - \alpha D_0\phi \right) D^0 G_\pm = 0.$$ (4.26a)

These are elliptic differential equations for which the maximum/minimum principle can be applied [39]. Since $G_\pm \to 0$ at infinity and $G_\pm \to (1 \pm \sqrt{1 + \alpha^2} \psi_0) > 0$ at the horizon, we see that $G_\pm$ never attains zero inside $\Sigma$. We therefore have $\Phi_{+1} - 1 = \frac{1}{2} V^{-1} e^{-\phi/\alpha} G_+ G_- > 0$ inside $\Sigma$.

Similarly, we obtain

$$D^2 \left( e^{\phi/\alpha} - 1 \right) + D^0 \left( 2 \log V - \frac{\phi}{\alpha} \right) D_0 \left( e^{\phi/\alpha} - 1 \right) = 0.$$ (4.27)

By virtue of $e^{\phi/\alpha} - V \to 0$ at infinity and $e^{\phi/\alpha} - V \to e^{\phi/\alpha}$ at the event horizon, the minimum of $V^{-1} e^{\phi/\alpha} - 1$ must be attained at infinity, and then in the interior of $\Sigma$ we have $e^{\phi/\alpha} - V > 0$. This means that $\Psi_{+1} - 1 = \frac{1}{2} V^{-1} e^{-\phi/\alpha} (e^{\phi/\alpha} - V)^2$ is strictly positive in the interior of $\Sigma$.

Since the conditions $\Phi_{+1}, \Psi_{+1} > 1$ have been demonstrated, we now move on to show that two of the inhomogeneous coordinates $(V, \phi, \psi)$ of $SO(1, 1) \times SL(2, \mathbb{R})/SO(1, 1)$ are linearly dependent. In particular, our current aim is to demonstrate that $\Phi_{+1}/\Psi_{+1}$ and $\Phi_0^2/(\Psi_{+1}^2 - 1)$ are constants. Guided by the argument to derive (2.43), we consider the divergence equation with the following current
\[ J' = V[f_1(\Phi_+)g_1(\Psi_+)D'\Phi_+ + f_2(\Phi_+)g_2(\Psi_+)D'\Psi_+]. \] (4.28)

By choosing
\[ f_1(\Phi_+) = \frac{(\Phi_+ + \sqrt{\Phi_+^2 - 1})^{-c}}{\sqrt{\Phi_+^2 - 1}}, \quad f_2(\Phi_+) = -\frac{(\Phi_+ + \sqrt{\Phi_+^2 - 1})^{-c}}{\sqrt{\Phi_+^2 - 1}}, \]
\[ g_1(\Psi_+) = (\Psi_+ + \sqrt{\Psi_+^2 - 1})^c, \quad g_2(\Psi_+) = \frac{(\Psi_+ + \sqrt{\Psi_+^2 - 1})^c}{\sqrt{\Psi_+^2 - 1}}, \] (4.29)
where \( c \) is a constant, some amount of algebra shows that the following equation holds
\[ D_jJ' = -V\left(\frac{\Phi_+ + \sqrt{\Phi_+^2 - 1}}{\Phi_+ + \sqrt{\Phi_+^2 - 1}}\right)^c \left[\frac{\Phi_+\sqrt{\Phi_+^2 - 1}}{\Phi_+^2 + \Phi_0^2 - 1} D_j\Phi_0 - \frac{\Phi_0\Phi_+}{\Phi_+^2 - 1} D_j\Phi_+\right]^2 \]
\[ + \frac{c}{\sqrt{\Phi_+^2 - 1}} D_j\Psi_+ - \frac{2\sqrt{\Psi_+^2 - 1}}{\Phi_+^2 - 1} D_j\Phi_+ \right]^2. \] (4.30)

For \( c > 0 \), the right-hand side of this equation becomes negative semi-definite. We can thus derive the inequality
\[ \int \int D_jJ'd\Sigma = \int_{S^\infty} J'n_dS - \int_B J'n_dS \leq 0. \] (4.31)

Our present boundary conditions tell us that both of the surface integrals at infinity and at horizon vanish, regardless of the value of \( c(>0) \). This means that the equality is saturated and the each piece on the right-hand side of (4.30) vanishes independently, leading to
\[ \frac{\Phi_+}{\Psi_+} = 1, \quad \frac{\Phi_0}{\Phi_+^2} = \frac{4(1 + \alpha^2)\psi_0}{(1 + \alpha^2)\psi_0^2}, \] (4.32)
where the integration constants have been fixed by asymptotic value. These equations give
\[ V^2 = \frac{e^{2\alpha/\alpha} - [1 - (1 + \alpha^2)\psi_0^2]e^{4\alpha^2\phi}}{(1 + \alpha^2)\psi_0^2}, \quad \phi = \frac{1 - e^{4\alpha^2\phi}}{(1 + \alpha^2)\psi_0}. \] (4.33)

It is worth commenting that the value of the dilaton field at the horizon is constant
\[ \phi|_{\psi=0} = \frac{\alpha}{1 + \alpha^2} \log[1 - (1 + \alpha^2)\psi_0^2]. \] (4.34)

Since the functional dependence of \( (V, \phi, \psi) \) has been established, our remaining task is to show the spherical symmetry following the prescription given in the foregoing sections. At this final step of the proof, it is of advantage to consider the conformal transformation
\[ g_\theta = e^{2\alpha\phi}g_\theta, \quad \hat{V} = Ve^{\alpha\phi}, \quad \hat{\psi} = \sqrt{1 + \alpha^2}\psi. \] (4.35)
With the relation (4.33), one can check that the following relations are satisfied

\[ \dot{R}_j = \frac{1}{V} \dot{D}_i \dot{D}_j \dot{V} - \frac{2}{V^2} \left( \dot{D}_i \dot{V} \dot{D}_j - \frac{1}{2} \dot{g}_{ij} (\dot{D} \dot{V})^2 \right), \]

\[ \dot{D}_i (V^{-1} \dot{D} \dot{V}) = 0, \quad \frac{1 - \dot{V}^2}{\psi} + \dot{\psi} = \psi_0^{-1} + \psi_0. \]  

(4.36)

In terms of these new variables, the boundary conditions at infinity reduce to

\[ \dot{V} \sim 1 - \frac{r_+ + r_-}{2r}, \quad \dot{g}_{ij} \sim \left( 1 + \frac{r_+ + r_-}{r} \right) \delta_{ij}, \quad \dot{\psi} \sim \frac{\sqrt{r_+ r_-}}{r}, \]  

(4.37)

where \( r_\pm \) is given by (4.11). The event horizon is located at \( \dot{V} = 0 \) where \( \dot{\psi} \) and \((\dot{D} \dot{V})^2 \) are constants. Since these conditions are exactly the same as in the electrovacuum case in section 3, the solution must be spherical, i.e.

\[ \dot{V} = \left( 1 - \frac{r_+}{r} \right) \left( 1 - \frac{r_-}{r} \right), \quad \dot{g}_{ij} dx^i dx^j = \frac{dr^2}{V^2} + r^2 d\Omega_2^2, \quad \dot{\psi} = \frac{\sqrt{r_+ r_-}}{r}. \]  

(4.38)

and hence \( e^{(1 + \alpha^2) \phi/\alpha} = 1 - \dot{\psi} \dot{\psi}_{0} = 1 - r_- / r \). Going back to the original frame by \( g_{ij} = e^{2 \alpha \phi} \dot{g}_{ij} \), we recover the Gibbons–Maeda solution (4.9).

5. Higher dimensions

It is natural to inquire whether our algorithm is applicable in higher dimensions. For simplicity, we focus on solutions to the \( n \)-dimensional vacuum Einstein equations \( \mathcal{R}_{\mu \nu} = 0 \), which reduce in the static spacetime \( ds^2 = -V^2(x) dt^2 + g_{ij}(x) dx^i dx^j \) to

\[ D^2 V = 0, \quad (n-1) R_j = \frac{1}{V} D_i D_j V, \]  

(5.1)

where \((n-1) R_j \) is the Ricci tensor for the \((n-1)\)-dimensional spatial metric \( g_{ij} \). Assuming the local foliation \( \mathcal{S}_V = \{ V = \text{constant} \} \) of constant timeslice \( \Sigma \), the following relations are satisfied

\[ n^i D_i \rho = \rho k, \quad (n-2) R = \frac{2k}{V \rho} + k^2 - k_{ij} k^{ji}. \]  

(5.2)

Here \( n_i = \rho D_i V \) is the outward pointing unit normal to \( \mathcal{S}_V \) in \( \Sigma \) with \( \rho \) being the lapse function and we shall denote the induced metric of \( \mathcal{S}_V \) as \( h_{ij} = g_{ij} - n_i n_j \) as before. \( k_{ij} \) is the extrinsic curvature of \( \mathcal{S}_V \) in \( \Sigma \) and it splits up into the trace-free part and the trace part

\[ k_{ij} = \sigma_{ij} + \frac{k}{n-2} h_{ij}, \quad k = h^{ij} k_{ij}. \]  

(5.3)

The boundary conditions at the horizon are

\[ k_{ij} \big|_{V=0} = 0, \quad \rho \big|_{V=0} = \rho_0, \]  

(5.4)

where \( \rho_0 \) is a positive constant. The asymptotic flatness is

\[ V \sim 1 - \frac{m}{r^{n-3}}, \quad g_{ij} \sim \left( 1 + \frac{2m}{(n-3)r^{n-3}} \right) \delta_{ij}, \]  

(5.5)

where \( m \) corresponds to the ADM mass up to a constant. In terms of lapse and mean curvature, (5.5) is translated into
\[ \rho \sim \frac{\rho^{n-2}}{(n-3)m}, \quad k \sim \frac{n-2}{r}. \]  
(5.6)

The \((n-1)\)-dimensional tensor quantities which we wish to show them to vanish are

\[ H_{ij} = D_iD_jV = -\frac{2}{n-3} \frac{V(DV)^2}{1-V^2} g_{ij} + \frac{2(n-1)}{n-3} V \frac{1}{1-V^2} D_iV D_jV, \]  
(5.7)

and

\[ H_i = \frac{D_i\rho}{\rho} - \frac{2(n-2)}{n-3} \frac{V}{1-V^2} D_iV. \]  
(5.8)

These quantities satisfy

\[ H_{ij}D^jV = -\rho^{-2}H_i, \quad H^i_i = 0. \]  
(5.9)

In the geometric notation using the data on \(S_V\), we have

\[ H_{ij} = \rho^{-1}\sigma_{ij} - \frac{2}{\rho^2} n_i(D_j)\rho + \frac{1}{(n-2)\rho} [h_{ij} - (n-2)n_in_j] \left( k - \frac{2(n-2)}{n-3} \frac{V}{\rho(1-V^2)} \right). \]  
(5.10)

Let us now demonstrate that \( H_{ij} = 0 \) indeed implies that the space is spherically symmetric, i.e. it admits an isometry of \(SO(n-1)\). Suppose that \( H_{ij} = 0 \) holds. Then \( H_i = 0 \) is readily solved to give

\[ [(n-3)\rho]^{n-2} \left( \frac{1}{2} - \frac{V^2}{2} \right)^{n-2} = m, \]  
(5.11)

where the integration constant has been determined by the asymptotic condition. Next, \( \sigma_{ij} = 0 \) implies that \( k = \frac{1}{2\rho} \partial_\rho h_{ij} = \frac{1}{n-2} k h_{ij} \). Integrating this equation by use of (5.11), we are led to

\[ h_{ij} = [(n-3)m\rho]^{2/(n-2)} \tilde{h}_{ij}, \]  
where \( \tilde{h}_{ij} \) is a metric which is independent of \( V \). From the vacuum Einstein equations, \( \tilde{h}_{ij} \) is the Einstein metric of positive curvature. The asymptotic flatness requires that this must be a standard metric of a unit sphere. It follows that \( H_{ij} \) represents a deviation from the spherical symmetry also in higher dimensions.

We shall not attempt to derive in detail the divergence equation, but only show the final results, since the procedure is completely parallel with the four dimensional case. Starting with the separable ansatz (2.33), we have a higher dimensional version of the divergence equation

\[ D_iJ^i = \frac{(n-3)c}{2\rho^2} f_1(V) \left[ 2\rho^2D_iV H_{ik} - \frac{2}{n-2} H_i g_{ik} \right]^2 + 2 \left( c - \frac{n-3}{n-2} \right) |H|^2, \]  
(5.12)

where \( g_1(\rho) = -(n-3)c\rho^{-(1+c)}, g_2(\rho) = \rho^{-c} \) and

\[ f_1(V) = \frac{1}{V(1-V^2)^{\frac{n-3}{2+c}}}, \quad f_2(V) = \frac{2}{(1-V^2)^{\frac{n-3}{2+c}}} \left[ \alpha\{c(n-2) - (n-3)\} + bc(n-2)(1-V^2) \right]. \]  
(5.13)

Alternatively, the current \( J^i \) can be again put into a more useful form

\[ J^i = -(n-3)[(1-V^2)^{\frac{n-3}{2+c}}]^{-c} \left[ \frac{c}{V} (1-V^2) [a + b(1-V^2)] H^i + 2aD^iV \right]. \]  
(5.14)
The right-hand side of (5.12) becomes positive semi-definite if
\[ a \geq 0, \quad a + b \geq 0, \quad c \geq \frac{n - 3}{n - 2}. \]  
(5.15)

It is worth commenting that the right-hand side of (5.12) is expressed by means of the tensor \( H_{ij} \) only. This term is not expressible by the higher dimensional Cotton tensor \( C_{ijkl} = 2D_{ij}((n-1)R_{jk} - \frac{1}{2(n-2)}(n-1)Rg_{jk}) \) (note that this is not conformally invariant unless \( n = 4 \)), since the Weyl tensor \((n-1)C_{ijkl} \) of \((\Sigma, g_{ij})\) becomes relevant as
\[ C_{ijk} = (n-1)C_{ijkl} \frac{D^lV}{V} - \frac{2}{(n-3)V^2}[(n-2)DH_{ijk} - \rho H_{ij}g_{jk}]. \]  
(5.16)

### 5.1. Surface integral

As discussed above, the uniqueness of the Schwarzschild black hole follows, provided that one can show \( H_{ij} = 0 \) under our boundary conditions. The integration of (5.12) over the spatial slice \( \Sigma \) boils down to
\[ 0 \leq 2a(n-3)\left[ -\frac{(n-3)}{2} (2m)^{\frac{1}{c+1}} \Omega_{n-2} + A_H \rho_0^{-(1+c)} \right] + c(a + b) \rho_0^{1-c} \left[ \frac{(n-3)}{2} \int_B (n-2)RdS - 2(n-2)A_H \rho_0^{-2} \right]. \]  
(5.17)

This inequality holds for any values of \( a, b \) and \( c \) satisfying (5.15), if and only if the pair of inequalities
\[ \left( \frac{(n-3)}{2} \rho_0 \right)^{c+1} (2m)^{\frac{1}{c+1}} \Omega_{n-2} \leq A_H \leq \frac{(n-3)}{4(n-2)} \int_B (n-2)RdS \]  
(5.18)

is satisfied.

Combining the former inequality with the Smarr relation
\[ (n-3)m\Omega_{n-2}\rho_0 = A_H, \]  
(5.19)

for any \( c \), one obtains the Penrose-type inequality
\[ A_H \leq \Omega_{n-2} (2m)^{\frac{n-2}{n-4}}. \]  
(5.20)

We wish to show that, in (5.18), the at-most-right-hand side coincides with the at-most-left-hand side, which then results in the equalities. Unfortunately, the value of \( \int_B (n-2)RdS \) cannot be evaluated in higher dimensions in general, since it is not a topological invariant. Only the lower bound of \( \int_B (n-2)RdS \) is obtainable. For instance, the case of \( c = n - 3 \) gives rise to a lower bound
\[ (n-2)(n-3)^{n-3}\Omega_{n-2} \leq \left( \frac{2}{\rho_0} \right)^{n-4} \int_B (n-2)RdS. \]  
(5.21)

To obtain further insight, let us rewrite the second inequality in (5.18) into a more recognizable form. For this purpose, let us define the analogue of the Yamabe constant by
\[ y_H \equiv Y_H^{-1/2}, \quad Y_H \equiv \frac{\int_B (n-2)RdS}{A_H^{n-2}}, \quad Y_H^n \equiv (n-2)(n-3)^{n-4} \Omega_{n-2}^{2/(n-2)}. \]  
(5.22)
In terms of $y_H$, the Smarr relation (5.19) recasts the latter inequality of (5.18) into (note that the exponent of $y_H$ is different from [22])

$$\Omega_{n-2}(2m)^{\frac{n-1}{n-3}} \leq A_H y_H^{\frac{n-1}{n-3}}.$$  \hspace{1cm} (5.23)

If one can show

$$y_H \leq 1,$$  \hspace{1cm} (5.24)

inequalities (5.20) and (5.23) are consistent only if $y_H = 1$, yielding spherical symmetry. Recall that the currently only proof of the uniqueness of the higher dimensional Schwarzschild solution [9] is based on the positive mass theorem. We speculate that the new argument using the divergence equations might be useful in order to obtain (5.24).

5.2. Penrose inequality

In the previous subsection, we have derived the Penrose-type inequality (5.20) by evaluating the surface integral arising from the divergence equation (5.12). The appearance of the Penrose-type inequality rather than the reversed inequality is interesting and this might be helpful for the construction of the suitable flow in higher dimensions. As far as the Penrose inequality is concerned, we can derive it in several different fashions.

Setting $g_2(\rho) = 0$ in the separable ansatz (2.33) and repeating the identical procedure, one can derive the following inequality

$$D^2 G - \frac{1}{V} D^V D_V G = \frac{n-3}{2(n-2)} \left( \frac{\rho_0}{\rho} \right)^{\frac{n-3}{n-1}} \left[ 2 \rho^2 D_H V H_{ijkl} - \frac{2}{n-2} H_{ijkl} \right]^2 \geq 0,$$  \hspace{1cm} (5.25)

where we have defined $G(V, \rho) \equiv V^2 + (\rho_0/\rho)^{\frac{n-3}{n-1}} - 1$. This is the equation for which the maximum principle can be applied [39], so that $G$ does not admit a maximum in the interior of $\Sigma$. Since $G = 0$ both at infinity and at horizon, we have $G \leq 0$ throughout $\Sigma$. Since $G$ is expanded at infinity as

$$G = \frac{1}{\rho^{n-3}} (-2m + [(n - 3)m\rho_0]^{\frac{n-1}{n-3}} + O(1/\rho^{n-2}),$$  \hspace{1cm} (5.26)

we conclude $(n - 3)m\rho_0 \leq (2m)^{\frac{n-1}{n-3}}$. Multiplying $\Omega_{n-2}$ and using Smarr’s formula (5.19), we readily obtain the Penrose inequality (5.20).

5.3. Modification of the proof based on the positive mass theorem

As discussed in previous subsections, the quantity $\int_B (n-2)R dS$ is a primary obstruction of the present scheme in higher dimensions. Nevertheless, our formulation developed here is of use also for the uniqueness proof based upon the positive mass theorem [4, 9].

To this aim, let us first recall the uniqueness argument by [4, 9], where the positive mass theorem has been ingeniously exploited to prove the conformal flatness of the constant times-lapse. We first illustrate that the conformal transformation is imperative here. In terms of the isotropic coordinates, the higher-dimensional Schwarzschild metric can be written as

$$ds^2 = -\left[ 1 - (\bar{r}_0/\bar{r})^{n-3} \right]^2 d\bar{t}^2 + \left[ 1 + (\bar{r}_0/\bar{r})^{n-3} \right]^{4/(n-3)} (d\bar{r}^2 + \bar{r}^2 d\Omega_{n-2}^2),$$  \hspace{1cm} (5.27)
where $\tilde{r}_0 = (m/2)^{1/(n-3)}$. By setting $V = \pm [1 - (\tilde{r}_0/r)^{n-3}]/[1 + (\tilde{r}_0/r)^{n-3}]$, the metric (5.27) is written as
\[
d s^2 = -V^2 d t^2 + \left(\frac{2}{1 \pm V}\right)^{4/(n-3)} (d \tilde{r}^2 + \tilde{r}^2 d \Omega^2_{n-2}).
\] (5.28)

This form of the metric manifests the conformal flatness of the constant timeslice. Bearing this form of the metric in mind, the authors in [4, 9] considered two sort of the conformal transformations to $(\Sigma, g_{ij})$ as
\[
\hat{g}_{ij}^\pm = \Omega_{\pm}^2 g_{ij}, \quad \Omega_{\pm} = \left(\frac{1 \pm V}{2}\right)^{2/(n-3)}.
\] (5.29)

One can easily check that each manifold $(\Sigma_{\pm}, \hat{g}_{ij}^\pm)$ is asymptotically Euclidean with the vanishing ADM mass and the vanishing scalar curvature. Glue these manifolds at $V = 0$ and consider the complete Riemannian manifold $\tilde{\Sigma} = \Sigma_+ \cup \Sigma_- \cup \{\infty\}$. By positive mass theorem [5–7], $\tilde{\Sigma}$ is flat.

The next step in [9] is to embed the event horizon into the Euclid space $\mathbb{E}^{n-1}$. Considering the local foliation of $\Sigma_+$ by $\mathcal{S}_v^+ = \{v \equiv 2/(1 + V) = \text{constant}\}$ slice, the event horizon is located at $v = 2$. It is easy to see that this surface $\mathcal{S}_{v=2}$ is totally umbilic, viz, its second fundamental form is proportional to the first fundamental form with a constant coefficient. By the Gauss curvature decomposition, this kind of surface is maximally symmetric and of positive curvature, i.e. the induced metric on $\mathcal{S}_{v=2}$ is a round sphere. Thus, the event horizon appears spherical when embedded in $\mathbb{E}^{n-1}$. Finally, one can conclude the spherical symmetry outside of the horizon by noting that $v = 2/(1 + V)$ obeys a Laplace equation on $\mathbb{E}^{n-1}$, whose Dirichlet boundary value problem is unique.

Let us point out that the whole procedure in the previous paragraph can be by-passed. Once again, our tensorial quantity $H_{ij}$ provides more direct information on the underlying geometry. An important fact here is that the Ricci tensor $\hat{R}_{ij}^\pm$ for the conformally transformed metric (5.29) must also vanish, when $(\Sigma_{\pm}, \hat{g}_{ij}^\pm)$ is shown to be flat by the positive mass theorem. A simple calculation reveals
\[
\hat{R}_{ij}^\pm = \frac{1 \mp V}{V(1 \mp V)} H_{ij}.
\] (5.30)

It therefore follows that $H_{ij}$ defined in (5.7) is nothing but the Ricci tensor for the conformally transformed metric, up to a scalar function. From (5.10), the condition $H_{ij} = 0$ implies $\sigma_{ij} = \mathcal{D}_i \rho = 0$. The spherical symmetry on and outside the horizon immediately follows from the Gauss curvature decomposition formula in the space of the vanishing Weyl tensor. This standpoint is more geometric than previous analysis based on the Dirichlet boundary value problem.

6. Summary and final remarks

We made an extensive study on the uniqueness theorems of static black holes in the context of divergence equations. Following the strategy laid out in [1–3], we have generalized these arguments into various directions. In the four dimensional vacuum case, our formula (2.43) contains three tunable parameters, which allow us to conclude the spherical symmetry without resorting to the integrated mass formula. Using our divergence formula, one can also show $\chi \geqslant 2$. This is the inequality that is stronger than ever explored and excludes explicitly the real
projective space. Furthermore, our tensorial field $H_{ij}$ defined in (2.28) enjoys a geometrically clear meaning, i.e. it describes the obstruction for the spherical symmetry (2.31), the obstruction for the existence of the dilatation conformal Killing vector (2.50), as well as the obstruction for the conformal Ricci flatness (5.30). We expect that the discussion laid out in this paper will be applied for the stationary case, e.g. for the divergence equation in [46].

Our formulation is also robust in the four dimensional Einstein–Maxwell theory. For instance, one can apply the maximum/minimum principle to divergence-type equations to conclude that the electrostatic potential is a function of the norm of the static Killing vector. We believe that our formulation is insensitive to matter fields, as long as the material equations of motion are of divergence type. This is indeed the case for a theory with a conformally coupled scalar field [45].

As we have verified, this advantage is optimized in Einstein–Maxwell-dilaton theory. We found another divergence equation (4.30), according to which we obtain the functional relationships for the norm of the Killing vector, electrostatic potential and dilaton field. This represents the effectiveness of the present scheme, since it has been a long standing problem how to fix the value of the scalar field at the horizon. However, we do not know to what extent the coset representation comes into play for the existence of this type of divergence equation (4.30). It remains an intriguing issue to explore the case in which the scalar space is not symmetric nor homogeneous.

In higher dimensions, our divergence formula always encounters an intractable term $y_H$ given in (5.22). Although this limits the validity of the present strategy, it is still useful to obtain the Penrose-type inequality and for the modification of the uniqueness proof based upon the positive mass theorem. Interestingly, the bound (5.24) is the condition for the Penrose inequality for the time-symmetric Einstein–Maxwell initial data sets in higher dimensions, modulo some additional assumptions [47]. This line of study is also worth exploring.

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References

[1] Israel W 1967 Phys. Rev. 164 1776
[2] Müller Zum Hagen H, Robinson D C and Seifert H J 1973 Gen. Relativ. Gravit. 4 53
[3] Robinson D C 1977 Gen. Relativ. Gravit. 8 696
[4] Bunting G L and Masood-ul-Alam A K M 1987 Gen. Relativ. Gravit. 19 147
[5] Schoen R and Yau S T 1979 Commun. Math. Phys. 65 45
[6] Schoen R and Yau S T 2017 arXiv:1704.05490
[7] Witten E 1981 Commun. Math. Phys. 80 381
[8] Hwang S 1998 Geometriae Dedicata. 71 5
[9] Gibbons G W, Ida D and Shiromizu T 2003 Prog. Theor. Phys. Suppl. 148 284
[10] Gibbons G W, Ida D and Shiromizu T 2002 Phys. Rev. Lett. 89 041101
[11] Gibbons G W, Ida D and Shiromizu T 2002 Phys. Rev. D 66 044010
[12] Rogatko M 2003 Phys. Rev. D 67 084025
[13] Kunduri H K and Lucietti J 2018 Class. Quantum Grav. 35 054003
[14] Penrose R 1973 Ann. New York Acad. Sci. 224 125
[15] Jang P S and Wald R M 1977 J. Math. Phys. 18 41
[16] Huisken G and Ilmanen T 2001 J. Differ. Geom. 59 353
[17] Jang P S 1979 Phys. Rev. D 20 834
[18] Bray H 2001 J. Differ. Geom. 59 177
[19] Khuri M, Weinstein G and Yamada S 2017 J. Differ. Geom. 106 451
[20] Hawking S 1968 J. Math. Phys. 9 598
[21] Geroch R 1973 Ann. New York Acad. Sci. 224 108
[22] Mizuno R, Ohashi S and Shiromizu T 2010 Phys. Rev. D 81 044030
[23] Arnowitt R L, Deser S and Misner C W 2008 Gen. Relativ. Gravit. 40 1997
[24] Hawking S W 1972 Commun. Math. Phys. 25 152
[25] Chrusciel P T and Wald R M 1994 Class. Quantum Grav. 11 L147
[26] Smarr L 1973 Phys. Rev. Lett. 30 71
[27] Smarr L 1973 Phys. Rev. Lett. 30 521 (erratum)
[28] Lindblom L 1988 J. Math. Phys. 29 436
[29] Michalski H and Wainwright J 1975 Gen. Relativ. Gravit. 6 289
[30] Ray J R and Thompson E L 1975 J. Math. Phys. 16 345
[31] Tod P 2007 Gen. Relativ. Gravit. 39 111
[32] Galloway G J 1995 Class. Quantum Grav. 12 L99
[33] Gibbons G W and Hull C M 1982 Phys. Lett. 109B 190
[34] Nozawa M and Shiromizu T 2014 Nucl. Phys. B 887 380
[35] Israel W 1968 Commun. Math. Phys. 8 245
[36] Müller Zum Hagen H and Robinson D C 1974 Gen. Relativ. Gravit. 5 61
[37] Simon W 1984 Gen. Relativ. Gravit. 17 761
[38] Ruback P 1988 Class. Quantum Grav. 5 L155
[39] Masood-ul-Alam K M 1992 Class. Quantum Grav. 9 L53
[40] Gilbarg D and Trudinger N S 1977 Elliptic Partial Differential Equations of Second Order (Berlin: Springer)
[41] Gibbons G W, Kastor D, London L A J, Townsend P K and Traschen J H 1994 Nucl. Phys. B 416 850
[42] Nozawa M 2011 Class. Quantum Grav. 28 175013
[43] Gibbons G W and Maeda K I 1988 Nucl. Phys. B 298 741
[44] Masood-ul-Alam A K M 1993 Class. Quantum Grav. 10 2649
[45] Mars M and Simon W 2003 Adv. Theor. Math. Phys. 6 279
[46] Tomikawa Y, Shiromizu T and Izumi K 2017 Prog. Theor. Exp. Phys. 2017 033E03
[47] Simon W 1983 Gen. Relativ. Gravit. 16 465
[48] Lopes de Lima L, Girao F, Lozorio W and Silva J 2016 Class. Quantum Grav. 33 035008