Some operator inequalities via convexity

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ABSTRACT
In this article, we employ a standard convexity argument to obtain new and refined inequalities related to the matrix mean of two accretive matrices, the numerical radius and the Tsallis relative operator entropy.

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1. Introduction

Let \( \mathcal{B}(\mathcal{H}) \) denote the algebra of all bounded linear operators acting on a Hilbert space \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \). When \( \mathcal{H} \) is finite dimensional, we identify \( \mathcal{B}(\mathcal{H}) \) with the algebra \( \mathcal{M}_n \) of all complex \( n \times n \) matrices. An operator \( A \in \mathcal{B}(\mathcal{H}) \) is said to be positive (denoted by \( A \geq 0 \)) if \( \langle Ax, x \rangle \geq 0 \) for all \( x \in \mathcal{H} \), and is said to be strictly positive (denoted by \( A > 0 \)) if \( A \) is positive and invertible. The Gelfand map \( f(t) \mapsto f(A) \) is an isometrically \( * \)-isomorphism between the \( C^* \)-algebra \( C(\text{sp}(A)) \) of continuous functions on the spectrum \( \text{sp}(A) \) of a self-adjoint operator \( A \) and the \( C^* \)-algebra generated by the identity operator \( I \) and \( A \). If \( f, g \in C(\text{sp}(A)) \), then \( f(t) \geq g(t) \) \( (t \in \text{sp}(A)) \) implies that \( f(A) \geq g(A) \).

Extending the definition of positive matrices, accretive matrices are defined as those matrices \( A \in \mathcal{B}(\mathcal{H}) \) such that \( \Re A > 0 \), where \( \Re A \) is the real part of \( A \in \mathcal{B}(\mathcal{H}) \), defined by \( \Re A = \frac{A + A^*}{2} \).

Given a matrix monotone function \( f : (0, \infty) \to (0, \infty) \) with \( f(1) = 1 \), and two accretive matrices \( A, B \), there is a matrix mean associated with \( f \), denoted by \( \sigma_f \) or \( \sigma \), defined as [1]

\[
A \sigma B = A^{\frac{1}{2}} f \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}. \tag{1}
\]
In this formula, if $X$ is any matrix with no eigenvalue in $(-\infty, 0]$, and if $f$ is analytic in a domain containing the eigenvalues of $X$, the quantity $f(X)$ is defined via the Dunford integral

$$f(X) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(zI - X)^{-1} \, dz,$$

where $\Gamma$ is a closed simple curve in $\mathbb{C}$ surrounding the eigenvalues of $X$, and lying in the domain of analyticity of $f$. The above Dunford integral has been recently simplified for matrices with no eigenvalues in $(-\infty, 0]$ using the weighted Harmonic mean $!, defined for two such matrices $A, B$ as $A!tB = ((1 - t)A^{-1} + tB^{-1})^{-1}, 0 \leq t \leq 1$. Indeed, when $X$ does not have any eigenvalue in $(-\infty, 0]$, it has been recently shown in [1] that

$$f(X) = \int_{0}^{1} I_{t}X \, d\nu_{f}(t),$$

(2)

where $f : (0, \infty) \to (0, \infty)$ is a matrix monotone function satisfying $f(1) = 1$, for some probability measure $\nu_{f}$ on $[0, 1]$. It is well known that when $A, B$ are accretive, then $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ does not have any eigenvalue in $(0, \infty)$, [2]. Hence, (2) applies when $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$.

It should be noted that the identity in (1) is an extension of the same identity known for strictly positive matrices.

Using (1) and (2), it can be easily seen that [1]

$$A\sigma_{f}B = \int_{0}^{1} A!_{t}B \, d\nu_{f}(t),$$

(3)

where $\nu_{f}$ is the probability measure characterizing the matrix monotone function $f$ associated with $\sigma$.

When $A, B$ are accretive, it has been shown that [1]

$$\Re(A\sigma_{f}B) \geq (\Re A) \, \sigma_{f} (\Re B)$$

(4)

for any matrix mean $\sigma$ (defined via (1)). This last inequality is the extension of the corresponding inequalities known for the geometric and harmonic means of accretive matrices, [2, 3].

Reversing (4), we have [1]

$$\Re(A\sigma_{f}B) \leq \sec^{2}(\alpha)(\Re A) \, \sigma_{f} (\Re B),$$

(5)

where $A, B$ are sectorial matrices, with sectorial index $\alpha$. Recall that a matrix $A$ is said to be sectorial with sectorial index $\alpha$, if $W(A) \subset S_{\alpha}$, where $W(A)$ is the numerical range of $A$ and $S_{\alpha}$ is the sector in the complex plane defined by

$$S_{\alpha} = \{ z \in \mathbb{C} : \Re z > 0, |\Im z| \leq \tan(\alpha)\Re z \}; 0 \leq \alpha < \frac{\pi}{2},$$

where $\Im z$ is the imaginary part of the complex number $z$.

One main goal of this article is to find refinements of (4) and (5), via convex functions. Further inequalities for accretive matrices involving Hermite-Hadamard inequalities will be presented too.
However, it turns out that this approach can be applied also to obtain some new versions of the numerical radius and Tsallis relative entropy inequalities. Recall that when $A \in \mathcal{B}(\mathcal{H})$, the numerical radius of $A$ is defined by

$$\omega(A) = \sup\left\{ |\langle Ax, x \rangle| : x \in \mathcal{H}, \|x\| = 1 \right\}. \tag{6}$$

This quantity has its applications in operator theory, where considerable research has been conducted to find optimal bounds for the numerical radius, due to the difficulty in computing the exact value of $\omega(A)$. Among the most accurate upper bounds for $\omega(A)$ is the celebrated result of Kittaneh [4, Equation (8)] stating that

$$\omega(A) \leq \frac{1}{2} \left\| A^* \right\| + |A|, \tag{6}$$

where $\| \cdot \|$ is the usual operator norm, and $|A| = (A^*A)^{1/2}$. We will present a new generalized form of this inequality.

The Tsallis relative operator entropy is defined for two strictly positive operators $A$, $B$ by

$$T_t(A|B) := A^{1/2} \ln_t \left( A^{-1/2}BA^{-1/2} \right) A^{1/2} = \frac{A^t_t B - A}{t}$$

where $\ln_t x = \frac{x^{t-1}}{t}$ and $0 < t \leq 1$; [5]. This converges to the relative operator entropy:

$$\lim_{t \to 0} T_t(A|B) = A^{1/2} \log \left( A^{-1/2}BA^{-1/2} \right) A^{1/2} =: S(A|B).$$

The key tool in our proofs is the following well known inequalities for convex functions. The reader is referred to [6] for further related results to Lemma 1.1.

**Lemma 1.1 ([7]):** If $f : J \to (0, \infty)$ is a convex function, then

$$f((1-t)a + tb) \leq f(a) + tf(b),$$

and

$$(1-t)f(a) + tf(b) \leq f((1-t)a + tb) + 2R \left( f\left( \frac{a+b}{2} \right) - f\left( \frac{a}{2} \right) \right) \tag{7}$$

where $a, b \in J, 0 \leq t \leq 1, r = \min\{t, 1-t\}, and R = \max\{t, 1-t\}.$

To state the following proposition, we need to remind the reader of interpolational means. For a symmetric operator mean $\sigma$ (in the sense that $A\sigma B = B\sigma A$), a parametrized operator mean $\sigma_t$ ($t \in [0,1]$) is called an interpolational path for $\sigma$ (or Uhlmann’s interpolation for $\sigma$) if it satisfies the following properties for the strictly positive operators $A, B$:

(c1) $A\sigma_0 B = A$ (here we recall the convention $T^0 = I$ for any positive operator $T$),
$A\sigma_t B = B$, and $A\sigma_\frac{1}{2} B = A\sigma B$;

c2) $(A\sigma_\alpha B)\sigma (A\sigma_\beta B) = A\sigma_{\frac{\alpha + \beta}{2}} B$ for all $\alpha, \beta \in [0,1].$
(c3) the map $\alpha \in [0,1] \mapsto A\sigma_\alpha B$ is norm continuous for each $A$ and $B$.

Examples of interpolational means are numerous, but the arithmetic, geometric and harmonic means are the most common. These are defined respectively as follows

$$A\nabla_t B = (1 - t)A + tB, A\sharp_t B = A^{\frac{1}{2}} \left( A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \right)^t A^{\frac{1}{2}}, A!_t B = ((1 - t)A^{-1} + tB^{-1})^{-1}.$$  

It is well known that $!_t \leq \sharp_t \leq \nabla_t, 0 \leq t \leq 1$; where for two symmetric means $\sigma, \tau$, the inequality $\sigma \leq \tau$ means that $A\sigma B \leq A\tau B$ for all positive operators $A, B$.

The following proposition then applies.

**Proposition 1.1:** Let $A, B \in \mathcal{B}(\mathcal{H})$ be two strictly positive operators. Then

$$f(t) = \langle A\sigma_t B, x \rangle$$

is convex on $[0,1]$. Further, if $\sigma \leq \sharp$, then $f$ is log-convex.

**Proof:** It is well-known that the arithmetic mean is the biggest one among symmetric means, see [8, Equation (2.2)]. This, together with [9, Lemma 4], implies that

$$f\left(\frac{t + s}{2}\right) = \langle A\sigma_{\frac{t + s}{2}} B, x \rangle$$

$$= \langle (A\sigma_t B) \sigma (A\sigma_s B) x, x \rangle$$

$$\leq \langle A\sigma_t B, x \rangle \sigma \langle A\sigma_s B, x \rangle$$

$$\leq \frac{1}{2} \left( \langle A\sigma_t B, x \rangle + \langle A\sigma_s B, x \rangle \right)$$

$$= \frac{1}{2} (f(t) + f(s)).$$

On the other hand, if $\sigma \leq \sharp$, then the above computations show that

$$f\left(\frac{t + s}{2}\right) \leq f(t)\sigma f(s) \leq f(t)\sharp f(s),$$

which is equivalent to log-convexity of $f$. $\blacksquare$

**Remark 1.1:** Letting $A = I$ in Proposition 1.1 implies that the function $f(t) = \langle B^t x, x \rangle$ is log-convex on $[0,1]$, for any strictly positive operator $B$. Applying Lemma 1.1 to this function implies the following multiplicative refinements of the celebrated McCarthy inequalities [10]

$$\langle B^t x, x \rangle \leq \left( \frac{\langle B^\frac{1}{2} x, x \rangle}{\langle Bx, x \rangle^{\frac{1}{2}}} \right)^{2r} \langle Bx, x \rangle^t$$

and

$$\langle Bx, x \rangle^t \leq \left( \frac{\langle Bx, x \rangle^{\frac{1}{2}}}{\langle B^\frac{1}{2} x, x \rangle} \right)^{2R} \langle B^t x, x \rangle,$$

where $0 \leq t \leq 1, r = \min\{t, 1 - t\}$ and $R = \max\{t, 1 - t\}$. 
2. Main results

As mentioned earlier in the introduction, we will be interested in presenting new and refined inequalities for accretive matrices, the numerical radius and the Tsallis relative operator entropy. These are the three parts of our main results.

2.1. Accretive versions

In this part of the paper, we present new inequalities for accretive matrices. We emphasize that this approach has never been tackled in the literature to treat accretive matrices, yet it was used for positive ones. In what follows, the notation $\sigma_t$ is implicitly understood to be the interpolational paths of the symmetric mean $\sigma$.

**Theorem 2.1:** Let $A, B \in \mathcal{M}_n$ be two accretive matrices. If $0 \leq t \leq 1$, then

$$\Re (A \nabla_t B) \leq \Re (A\sigma_t B) + 2R (\Re (A \nabla B) - (\Re A)\sigma (\Re B)),$$

where $R = \max\{t, 1 - t\}$.

**Proof:** We know that the function

$$f(t) = \langle (\Re A)\sigma_t (\Re B)x, x \rangle$$

is a convex function on $[0, 1]$, since $\Re A, \Re B > 0$. On the other hand, it has been shown in [1, Proposition 5.1] that

$$\Re (A\sigma_t B) \geq (\Re A)\sigma_t (\Re B).$$

These tools, together with Lemma 1.1 imply that

$$\langle ((1 - t)\Re A + t\Re B) x, x \rangle \leq \left\langle \left( \left( (\Re A)\sigma_t (\Re B) + 2R \left( \frac{(\Re A) + (\Re B)}{2} - (\Re A)\sigma (\Re B) \right) \right) x, x \right\rangle \right.$$
**Proof:** We know that for two sectorial matrices $A$, $B$ with sectorial index $\alpha$,

$$\Re (A \sigma_t B) \leq \sec^2(\alpha) (\Re A \sigma_t \Re B)$$

holds [1, Proposition 5.2]. Now, applying the same argument as in the proof of Theorem 2.1, we infer that

$$\frac{1}{\sec^2(\alpha)} \langle \Re (A \sigma_t B) x, x \rangle$$

$$\leq \langle (\Re A) \sigma_t (\Re B) x, x \rangle$$

$$\leq \left( (1 - t) (\Re A) + t (\Re B) - 2r \left( \frac{\Re A + (\Re B)}{2} - (\Re A) \sigma (\Re B) \right) \right) x, x \rangle.$$

\[\square\]

**Remark 2.1:** If in Theorems 2.1 and 2.2, $A$ and $B$ are positive matrices, we get

$$2r \left( A \nabla B - A \sigma B \right) \leq A \nabla_t B - A \sigma_t B \leq 2R \left( A \nabla B - A \sigma B \right),$$

which is a generalization of [11, Corollary 3.1].

For the harmonic mean, the following interesting refinement holds, refining [3, Lemma 2.3].

**Theorem 2.3:** Let $A, B \in \mathcal{M}_n$ be two accretive matrices. If $0 \leq t \leq 1$, then

$$\Re (A \! t B) \geq \left( \left( \Re A \right) \! t (\Re B) \right)^{-1} - 2r \left( \left( \Re A \right) ! (\Re B)^{-1} - (\Re (A ! B))^{-1} \right)^{-1}$$

$$\geq (\Re A) \! t (\Re B).$$

where $r = \min\{t, 1 - t\}$.

**Proof:** If $f$ is operator convex, then

$$g(t) = \langle f \left( (1 - t) A + t B \right) x, x \rangle$$

is convex on $[0, 1]$. This, together with Lemma 1.1 imply that [12]

$$f \left( (1 - t) A + t B \right) \leq (1 - t) f(A) + tf(B) - 2r \left( \frac{f(A) + f(B)}{2} - f \left( \frac{A + B}{2} \right) \right).$$

On the other hand,

$$f \left( \frac{A + B}{2} \right) \leq \frac{f(A) + f(B)}{2},$$

which implies

$$f \left( (1 - t) A + t B \right) \leq (1 - t) f(A) + tf(B) - 2r \left( \frac{f(A) + f(B)}{2} - f \left( \frac{A + B}{2} \right) \right)$$

$$\leq (1 - t) f(A) + tf(B).$$
Since \( f(T) = (\mathfrak{N} T^{-1})^{-1} \) is operator convex [13, Theorem 2.2], we get
\[
(\mathfrak{N} \left( \left( (1-t)A + tB \right)^{-1} \right))^{-1} \leq (1-t)(\mathfrak{N} A)^{-1} + t(\mathfrak{N} B)^{-1} - 2r \left( \frac{(\mathfrak{N} A)^{-1} + (\mathfrak{N} B)^{-1}}{2} - \left( \mathfrak{N} \left( \left( \frac{A + B}{2} \right)^{-1} \right) \right)^{-1} \right)
\]
\[
\leq (1-t)(\mathfrak{N} A)^{-1} + t(\mathfrak{N} B)^{-1}.
\]
By replacing \( A \) and \( B \) by \( A^{-1} \) and \( B^{-1} \), respectively, and then taking inverse, we infer that
\[
\mathfrak{N} \left( \left( (1-t)A^{-1} + tB^{-1} \right)^{-1} \right) \geq \left( (1-t)(\mathfrak{N} A)^{-1} + t(\mathfrak{N} B)^{-1} - 2r \left( \frac{(\mathfrak{N} A)^{-1} + (\mathfrak{N} B)^{-1}}{2} - \left( \mathfrak{N} \left( \left( \frac{A^{-1} + B^{-1}}{2} \right)^{-1} \right) \right)^{-1} \right) \right)^{-1}
\]
which is equivalent to
\[
\mathfrak{N} (A!_t B) \geq \left( ((\mathfrak{N} A)!_t(\mathfrak{N} B))^{-1} - 2r \left( ((\mathfrak{N} A)!_t(\mathfrak{N} B))^{-1} - (\mathfrak{N} (A!t B))^{-1} \right) \right)^{-1}
\]
\[
\geq (\mathfrak{N} A)!_t(\mathfrak{N} B).
\]
This completes the proof.

It is interesting that Theorem 2.3 implies the following refinement of (4).

**Corollary 2.1:** Let \( A, B \) be accretive matrices, and let \( \sigma \) be a matrix mean, characterized by the matrix monotone function \( f \). Then
\[
\mathfrak{N}(A \sigma B) \geq \int_0^1 \left( ((\mathfrak{N} A)!_t(\mathfrak{N} B))^{-1} - 2r \left( ((\mathfrak{N} A)!_t(\mathfrak{N} B))^{-1} - (\mathfrak{N} (A!t B))^{-1} \right) \right) \, d\nu(t)
\]
\[
\geq \mathfrak{N} A \sigma \mathfrak{N} B.
\]

**Proof:** This follows from Theorem 2.3, upon integration, then using (3).

In the next result, we present the accretive version of the first inequality in Lemma 1.1.

**Theorem 2.4:** Let \( f : (0, \infty) \to (0, \infty) \) be matrix concave with \( f(1) = 1 \), and let \( A, B \) be two accretive matrices with \( W(A), W(B) \subset S_\alpha \) for some \( 0 \leq \alpha < \pi/2 \). Then for any \( 0 \leq t \leq 1 \)
\[
\mathfrak{N} \left( f(A) \nabla_t f(B) \right) + 2r \sec^2(\alpha) \left( f(\mathfrak{N} (A \nabla B)) - f(\mathfrak{N} A) \nabla f(\mathfrak{N} B) \right) \leq \sec^2(\alpha) \mathfrak{N} f(A) \nabla_t(B),
\]
where \( r = \min\{t, 1-t\} \).
**Proof:** If \( f : (0, \infty) \to (0, \infty) \) is matrix concave with \( f(1) = 1 \) and \( A \) is accretive, then \([1, \text{Proposition 7.1}]\)

\[
f(\Re A) \leq \Re f(A).
\]

Furthermore, if \( W(A) \subset S_\alpha \) for some \( 0 \leq \alpha < \pi / 2 \), then \([1, \text{Proposition 7.2}]\)

\[
\Re f(A) \leq \sec^2(\alpha)f(\Re A).
\]

Now, assume that \( 0 \leq t \leq 1/2 \). Then using \([1, \text{Propositions 7.1, 7.2}]\), we have

\[
\Re f(A \nabla_t B)
= \Re f((1 - t)A + tB)
\geq f((1 - t)\Re A + t\Re B)
= f\left((1 - 2t)\Re A + 2t\frac{\Re A + \Re B}{2}\right)
\geq (1 - 2t)f(\Re A) + 2tf\left(\frac{\Re A + \Re B}{2}\right)
= (1 - t)f(\Re A) + tf(\Re B) + 2t\left(f\left(\frac{\Re A + \Re B}{2}\right) - \frac{f(\Re A) + f(\Re B)}{2}\right)
\geq \frac{1}{\sec^2(\alpha)}\left((1 - t)f(\Re A) + tf(\Re B)\right) + 2t\left(f\left(\frac{\Re A + \Re B}{2}\right) - \frac{f(\Re A) + f(\Re B)}{2}\right)
= \frac{1}{\sec^2(\alpha)}\Re\left(f(A)\nabla_if(B)\right) + 2t\left(f\left(\Re (A \nabla_t B)\right) - f(\Re A)\nabla_i f(\Re B)\right).
\]

Since \( 0 \leq t \leq 1/2 \), we have \( \min\{t, 1 - t\} = t \). Consequently, the above inequality implies

\[
\Re\left(f(A)\nabla_if(B)\right) + 2t\sec^2(\alpha)\left(f\left(\Re (A \nabla_t B)\right) - f(\Re A)\nabla_i f(\Re B)\right) \leq \sec^2(\alpha)\Re f(A \nabla_t B),
\]

where \( r = \min\{t, 1 - t\} \).

To show the validity of this inequality when \( 1/2 \leq t \leq 1 \), replace \( A \) by \( B \) and \( B \) by \( A \) in the above inequality to obtain

\[
\Re\left(f(B)\nabla_r f(A)\right) + 2t\sec^2(\alpha)\left(f\left(\Re (A \nabla B)\right) - f(\Re A)\nabla_r f(\Re B)\right) \leq \sec^2(\alpha)\Re f(B \nabla_r A),
\]

where \( r = \min\{\gamma, 1 - \gamma\} \) and \( 0 \leq \gamma \leq 1/2 \). Noting that \( X\nabla_\gamma Y = Y\nabla_{1-\gamma} X \) for any \( X, Y \) implies

\[
\Re\left(f(A)\nabla_{1-\gamma} f(B)\right) + 2\gamma\sec^2(\alpha)\left(f\left(\Re (A \nabla B)\right) - f(\Re A)\nabla f(\Re B)\right)
\leq \sec^2(\alpha)\Re f(A \nabla_{1-\gamma} B),
\]

when \( 0 \leq \gamma \leq 1/2 \). Now if \( 1/2 \leq t \leq 1 \), we may apply the above inequality using \( \gamma = 1 - t \) to obtain the desired inequality. This completes the proof. \( \blacksquare \)
Theorem 2.5 (Hermite-Hadamard inequality for accretive matrices): Let $f : (0, \infty) \to (0, \infty)$ be matrix concave with $f(1) = 1$, and let $A$, $B$ be two accretive matrices with $W(A), W(B) \subset S_\alpha$ for some $0 \leq \alpha < \pi / 2$. Then

$$\Re \left( \frac{f(A) + f(B)}{2} \right) \leq \sec^2(\alpha) \int_0^1 \Re f ((1 - t)A + tB) \, dt \leq \sec^4(\alpha) \Re f \left( \frac{A + B}{2} \right).$$

**Proof:** Observe that

$$\Re f \left( \frac{A + B}{2} \right) = \Re \left( \frac{(1 - t)A + tB + (1 - t)B + tA}{2} \right) \geq \frac{1}{\sec^2(\alpha)} \Re f ((1 - t)A + tB) + \Re f ((1 - t)B + tA) \geq \frac{1}{\sec^4(\alpha)} \Re f(A) + \Re f(B),$$

where we have used [1, Theorem 7.2] twice to obtain the first and second inequalities. Equivalently,

$$\Re \left( \frac{f(A) + f(B)}{2} \right) \leq \sec^2(\alpha) \Re \left( \frac{f ((1 - t)A + tB) + f ((1 - t)B + tA)}{2} \right) \leq \sec^4(\alpha) \Re f \left( \frac{A + B}{2} \right).$$

By taking the integral over $0 \leq t \leq 1$, we reach the desired result. ■

### 2.2. Numerical radius inequalities

It has been shown in [14, Theorem 2] that if $A \in \mathcal{B}(\mathcal{H})$, then

$$\omega^{2p}(A) \leq \left\| (1 - t)\|A^*\|^{2p} + t\|A\|^{2p} \right\|,$$  \hspace{1cm} (9)

for any $0 \leq t \leq 1$ and $p \geq 1$. The next result improves the inequality (9).

**Theorem 2.6:** Let $A \in \mathcal{B}(\mathcal{H})$ and let $0 \leq t \leq 1$. Then for any $p \geq 1$,

$$\omega^{2p}(A) \leq \left\| (1 - t)\|A^*\|^{2p} + t\|A\|^{2p} - 2r \left( \frac{\|A\|^{2p} + \|A^*\|^{2p}}{2} - \left( \frac{\|A\|^{p} + \|A^*\|^{p}}{2} \right)^2 \right) \right\|,$$

where $r = \min\{t, 1 - t\}$.

**Proof:** Since the function

$$f(t) = \langle |A|^{2pt}x, x \rangle \langle |A^*|^{2p(1-t)}x, x \rangle$$
is log-convex on \((0, \infty)\), it is convex. Hence, by Lemma 1.1, we obtain

\[
\langle |A|^{2pt}x, x \rangle \langle |A^*|^{2p(1-t)} x, x \rangle + 2r \left( \frac{\langle |A|^p x, x \rangle + \langle |A^*|^p x, x \rangle}{2} - \langle |A|^p x, x \rangle \langle |A^*|^p x, x \rangle \right)
\]

\[
\leq (1 - t) \langle |A^*|^p x, x \rangle + t \langle |A|^p x, x \rangle,
\]

where \( r = \min\{t, 1 - t\} \) and \( 0 \leq t \leq 1 \). By applying the arithmetic-geometric mean inequality, we infer that

\[
\langle |A|^p x, x \rangle \langle |A^*|^{2p(1-t)} x, x \rangle \leq \left( \langle 1 - t |A^*|^{2p} + t |A|^2p \rangle x, x \right)
\]

\[
+ 2r \left( \langle |A|^p x, x \rangle \langle |A^*|^{p} x, x \rangle - \left( \frac{|A|^2p + |A^*|^{2p}}{2} x, x \right) \right)
\]

\[
\leq \left( \langle 1 - t |A^*|^{2p} + t |A|^2p \rangle x, x \right)
\]

\[
+ 2r \left( \left( \frac{|A|^p + |A^*|^p}{2} x, x \right)^2 - \left( \frac{|A|^2p + |A^*|^{2p}}{2} x, x \right) \right)
\]

\[
\leq \left( \langle 1 - t |A^*|^{2p} + t |A|^2p \rangle x, x \right)
\]

\[
+ 2r \left( \left( \frac{|A|^p + |A^*|^p}{2} x, x \right)^2 - \left( \frac{|A|^2p + |A^*|^{2p}}{2} x, x \right) \right),
\]

where in the last inequality we used the Hölder-McCarthy inequality stating that \( \langle |A| x, x \rangle^r \leq \langle |A|^r x, x \rangle \) for \( r \geq 1 \). Using the inequality [15]

\[
|\langle Ax, x \rangle|^2 \leq \langle |A|^{2t} x, x \rangle \langle |A^*|^{2(1-t)} x, x \rangle
\]

then the Hölder-McCarthy inequality again, we obtain

\[
|\langle Ax, x \rangle|^{2p} \leq \left( \langle |A|^{2t} x, x \rangle \langle |A^*|^{2(1-t)} x, x \rangle \right)^p
\]

\[
\leq \langle |A|^{2pt} x, x \rangle \langle |A^*|^{2p(1-t)} x, x \rangle.
\]

Consequently

\[
|\langle Ax, x \rangle|^{2p} \leq \left( (1 - t) |A^*|^{2p} + t |A|^2p \right) x, x
\]

\[
+ 2r \left( \left( \frac{|A|^p + |A^*|^p}{2} x, x \right)^2 - \left( \frac{|A|^2p + |A^*|^{2p}}{2} x, x \right) \right).
\]
By taking the supremum over all unit vector \( x \in \mathcal{H} \) we get the desired result. □

**Remark 2.2:** Theorem 2.6 extends the celebrated inequality [4, Equation (8)]

\[
\omega(A) \leq \frac{1}{2} \| | A^* | + | A | \| ,
\]

by setting \( t = 1/2 \).

### 2.3. Tsallis relative operator entropy

We consider \( t \)-logarithmic function \( \ln_t x := \frac{x^t - 1}{t} \) defined for \( x > 0 \) and \( t > 0 \).

**Lemma 2.1:** The \( t \)-logarithmic function \( \ln_t x \) is convex in \( t \) if \( x \geq 1 \), and concave in \( t \) if \( 0 < x \leq 1 \).

**Proof:** We set the function \( f(y) := y(\log y)^2 - 2y \log y + 2y - 2 \) for \( y > 0 \). Since \( f(1) = 0 \) and \( f''(y) = (\log y)^2 \geq 0 \), we have \( f(y) \leq 0 \) for \( 0 < y \leq 1 \) and \( f(y) \geq 0 \) for \( y \geq 1 \).

By simple calculations, we have

\[\frac{d^2}{dt^2} (\ln_t x) = \frac{1}{t^3} \left\{ x^t (\log x^t)^2 - 2x^t \log x^t + 2x^t - 2 \right\}.\]

By putting \( y := x^t \) in the above, we thus get \( \frac{d^2}{dt^2} (\ln_t x) \geq 0 \) if \( x \geq 1 \) and \( \frac{d^2}{dt^2} (\ln_t x) \leq 0 \) if \( 0 < x \leq 1 \). □

**Proposition 2.1:** If \( 0 \leq A \leq B \), then we have

\[T_{(1-t)a+tb}(A|B) \leq (1 - t)T_a(A|B) + tT_b(A|B)\]

for all \( 0 < a, b \leq 1 \) and \( 0 \leq t \leq 1 \). If \( 0 \leq B \leq A \), then the reverse inequality above holds.

**Proof:** Since \( \ln_t x \) is convex in \( t \) for \( x \geq 1 \), we have

\[\ln_{(1-t)a+tb} x \leq (1 - t) \ln_a x + t \ln_b x\]

for all \( 0 < a, b \leq 1 \), \( 0 \leq t \leq 1 \) and \( x \geq 1 \). Putting \( x := A^{-1/2} BA^{-1/2} \) and then multiplying \( A^{1/2} \) to the both sides in the inequality, we get the desired result. For the case \( 0 < x \leq 1 \), we also obtain the result similarly. □

**Theorem 2.7:** For \( 0 < t \leq 1 \) and \( A, B \geq 0 \), we have

\[2r \left( \frac{B - A + S(A|B)}{2} - 2(A \# B - A) \right) \leq (1 - t)S(A|B) + t(B - A) - T_t(A|B) \leq 2R \left( \frac{B - A + S(A|B)}{2} - 2(A \# B - A) \right),\]

where \( r := \min\{t, 1 - t\} \) and \( R := \max\{t, 1 - t\} \).
Proof: We consider \( f(t) := \ln_t x \) for \( 0 < t \leq 1 \) and \( x > 0 \). Since \( f(0) = \lim_{t \to 0} \frac{x^t - 1}{t} = \log x \), \( f(1/2) = 2(\sqrt{x} - 1) \) and \( f(1) = x - 1 \), by Lemma 1.1, we have

\[
2r \left( \frac{x - 1 + \log x}{2} - 2 (\sqrt{x} - 1) \right) \leq (1 - t) \log x + t(x - 1) - \ln_t x \leq 2R \left( \frac{x - 1 + \log x}{2} - 2 (\sqrt{x} - 1) \right).
\]

From Kubo-Ando theory, we obtain the desired results.

Note that we consequently obtain a natural result \( T_{1/2}(A|B) = 2(A^\#_t B - A) \) if we take \( t = 1/2 \) in the above theorem.

Theorem 2.8: Let \( A, B \in B(H) \) be two strictly positive operators. If \( t \leq s \), then

\[ T_t(A|B) \leq T_s(A|B). \]

Proof: We know that if \( g \) is a convex function, then the function

\[ h(t) = \frac{g(t) - g(a)}{t - a} \]

is increasing. This means that the function

\[ f(t) = \left( \frac{A^\#_t B - A}{t} \right)_x, x = \langle T_t(A|B)x, x \rangle \]

is increasing, thanks to Proposition 1.1.

We remark that some similar results for the generalized relative operator entropy (but not Tsallis relative operator entropy) are given in [16, Theorem 1].

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