BLACK HOLE TUNNELING ENTROPY
AND THE SPECTRUM OF GRAVITY

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ABSTRACT

The tunneling approach for entropy generation in quantum gravity is applied to black holes. The area entropy is recovered and shown to count only a tiny fraction of the black hole degeneracy. The latter stems from the extension of the wave function outside the barrier. In fact the semi-classical analysis leads to infinite degeneracy. Evaporating black holes leave then infinitely degenerate “planckons” remnants which can neither decay into, nor be formed from, ordinary matter in a finite time. Quantum gravity opens up at the Planck scale into an infinite Hilbert space which is expected to provide the ultraviolet cutoff required to render the theory finite in the sector of large scale physics.

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1. Introduction

Tunneling in quantum gravity can generate entropy\cite{1,2}. To understand how such an apparent violation of unitarity may arise, let us first consider a classical space-time background geometry with compact Cauchy hypersurfaces. If quantum fluctuations of the background are taken into account, quantum gravity leaves no “external” time parameter to describe the evolution of matter configurations in this background. Indeed, the solutions of the Wheeler-De Witt equation\cite{3}

\[ \mathcal{H}|\Psi\rangle = 0 \]  

(1)

where \( \mathcal{H} \) is the Hamiltonian density of the interacting gravity-matter system can contain no reference to such time when there are no contribution to the energy from surface terms at spatial infinity. This is due to the vanishing of the time displacement generator and even though the theory can be unambiguously formulated only at the semi-classical level, such a consequence of reparametrization invariance should have a more general range of validity.

To parametrize evolution, one then needs a “clock” which could correlate matter configurations to ordered sequences of spatial geometries. If quantum fluctuations of the metric field can be neglected, the field components \( g_{ij} \) at every point of space can always be parametrized by a classical time parameter, in accordance with the classical equations of motion. This classical time, which is in fact a function of the \( g_{ij} \), can be used to describe the evolution of matter and constitutes thus such a dynamical “clock” correlating matter to the gravitational field\cite{4}. This description is available in the semi-classical limit of (1) where the classical background evolving in time is represented by a coherent superposition of W.K.B. “forward” waves formed from eigenstates of (1). When quantum metric field fluctuations are taken into account, “backward” waves, which can be interpreted as flowing backwards in time, are unavoidably generated from (1) and the operational significance of the metric clock gets lost outside the domain of validity of the semi-classical approximation. Nevertheless, in domains of metric field configurations
where both forward and backward waves are present but where quantum fluctuations are sufficiently small, interferences with such “time reversed” semi-classical solutions will in general be negligible†. Projecting then out the backward waves restores the operational significance of the metric clock but the evolution marked by the correlation time is no more unitary: information has been lost in projecting these backward waves stemming from regions where quantum fluctuations of the clock are significant. This is only an apparent violation of unitarity which would be disposed of if the full content of the theory would be kept, perhaps eventually by reinterpreting backward waves in terms of the creation of “universe” quanta through a further quantization of the wavefunction (1).

This apparent violation of unitarity is particularly marked if the gravitational clock experiences the strong quantum fluctuations arising from a tunneling process. This can be illustrated from the simple analogy, represented in Fig.1, offered by a nonrelativistic closed system of total fixed energy $E$ where a particle in one space dimension plays the role of a clock for surrounding matter and tunnels through a large potential barrier. Outside the barrier, the clock is well approximated by semi-classical waves, but if on the left of the turning point one would take only forward waves, one would inevitably have on the right of the other turning point both forward and backward waves with large amplitudes compared with the original ones. The ratio between the squares of the forward amplitudes on the right and on the left of the barrier for a component of the clock wave with given clock energy $E_c$ is the inverse transmission coefficient $N_0(E_c)$ through the barrier and provides a measure of the apparent violation of unitarity.

In reference [2], it was shown that for a class of de Sitter type of space-times, which admit compact Cauchy hypersurfaces, tunneling could occur between a “wormhole” and an expanding universe. It was proven that, for sufficiently small cosmological constant, $\ln N_0$ played the role of a true thermodynamic entropy for the metric clock transferable to matter in reversible processes. Explicit evaluation

† For a recent discussion of related problems see reference [5].
gave for this tunneling entropy $\ln N_0 = A/4$ where $A$ is the area of the event horizon. Thus one recovered in this way the horizon thermodynamics of Gibbons and Hawking[^6].

The tunneling entropy $\ln N_0$ is in last analysis an effect of quantum fluctuations in quantum gravity. Therefore, despite the fact that no violation of unitarity would appear in a complete description including backward waves, this entropy should be expressible in terms of density of states of matter and gravity. Tunneling offers an interesting perspective in this direction because it enlarges the semi-classical wave function of space-time to include in its description the other side of the barrier. Unfortunately, for the space-times considered above, the other side is a wormhole, that is in the classical limit simply a point on the Euclidean section of the original manifold. This makes it illusory to describe in semi-classical terms the configurations of the wormhole side of the barrier*.

In the present paper, we shall show that for black hole geometries, the area entropy is also interpretable in terms of tunneling. Now, Cauchy hypersurfaces are not compact and evolution can be described by the Minkowskian time available from surface terms at spacelike infinity. However for spherically symmetric solutions, as long as no mass is brought into the system from infinity, the Minkowskian time appears as an irrelevant unmeasurable phase in the quantum state of the system and the above argument can be repeated. Tunneling amplitudes between spherically symmetric configurations with same total mass $M$ can still be searched for and the related tunneling entropy is still expected to be $\ln N_0$ and equal to $A/4$. This will indeed turn out to be the case but the crucial new feature which will emerge is that the black hole is not connected by tunneling to a wormhole (in fact, there are no such tunneling amplitudes), but to a large scale matter-gravity configuration describable in classical terms. These configurations are analogous to macroscopic collapsing states frozen just outside the Schwartzschild radius. We

[^6]: One could deform the Euclidean section to open the point into a Planckian size universe but this would not change the conclusion.
shall call these configurations “achronons” because they are, for the outside observer deprived of any time dependent properties as a consequence of an infinite time dilation. Achronons of given mass will be shown to be quantum mechanically infinitely degenerate. Hence black holes in quantum gravity have also infinite degeneracy as their wave function is connected by tunneling to the achronon side of the barrier. This means of course that the number of states \(\exp A/4\) counted by the tunneling entropy is only a finite number of “surface” states out of an infinite set of internal states which cannot belong to the same finite Hilbert space as the matter surrounding the black hole in a finite volume. This mismatch will entirely modify the black hole evaporation process at its last stage. In fact the evaporation must stop when the black hole reaches the Planck scale, leaving a stable “planckon” remnant which can neither be created out of, nor decay into, ordinary matter in a finite external time. Such objects were introduced previously\(^7\) to avoid the violation of unitarity which would arise from a complete black hole evaporation\(^\dagger\). They follow here directly from the tunneling structure of the black hole-achronon wave function.

To strengthen these tentative conclusions, one should improve the present analysis in two respects. First, we have been restricted by the semi-classical treatment of quantum gravity and we can only surmise that the crucial element which came out of it, namely the infinite degeneracy of the black hole wave function, will survive the full quantum description. Second, although the semi-classical treatment defines unambiguously the achronon from the tunneling, we have not realised such a configuration in a genuine field theoretic way. We have only examplified its features by a phenomenological model, which although consistent, is too schematic to be directly physically relevant. Hopefully a more complete and realistic illustration of the achronon will secure the explicit construction of the black hole-achronon wave function.

Notwithstanding these limitations, the present approach gives strong support

\(^\dagger\) For a comprehensive review on recent attempts to solve the black hole unitarity puzzle, see reference \(8\).
to the planckon hypothesis. It would of course be of great interest to find at least some indirect evidence in favour of their existence. This is not totally impossible as planckons may have interesting cosmological and astrophysical consequences if they were present in the early universe as might be the case if primordial black holes played an important role in cosmogenesis\cite{9}. At a more fundamental level, they would provide, as a consequence of unitarity, a natural cut-off at the Planck size for the ultraviolet spectrum of Hilbert space of states describing large scale physics and are therefore expected to render quantum gravity expressible as a finite theory.

In the presentation of the paper, rather than deducing achronon configurations from the analysis of black hole tunneling amplitudes, we found it more convenient to motivate the latter by first introducing achronons as classical solutions of general relativity. This is exemplified in section 2 in a simple shell model. In section 3, we show how achronons surrounding a black hole can screen their temperature to zero or to a finite quantity if the black hole has, classically, a vanishingly small Schwartzschild radius. These properties are then used in section 4 to prove that eternal black hole are related by tunneling to achronon configurations. The inverse transmission coefficient is computed in the semi-classical limit and its relation to entropy is proven. Contact is made between the black hole tunneling entropy and the Gibbons-Hawking thermodynamics. In section 5, the infinite degeneracy of quantum black holes of given mass is established. The nature of the planckon remnants follows then from the evolution of the potential barrier during the black hole decay. Their properties are reviewed and their bearing on the spectrum and the scope of quantum gravity is discussed. Mathematical details are relegated to the Appendix.
2. The Achronon

Our basic action in four dimensional Minkowski space-time will be

\[ S = S_{\text{grav}} + S_{\text{matter}} \]  \hspace{1cm} (2)

where \( S_{\text{grav}} \) has the conventional form \((G = 1)\):

\[ S_{\text{grav}} = -\frac{1}{16\pi} \int \sqrt{-g} R d^4 x \]  \hspace{1cm} (3)

and \( S_{\text{matter}} \) contains sufficiently many free parameters to allow for the stress tensors considered below. In describing classical solutions, it should be kept in mind that they have to be interpreted as semi-classical solutions of (1). In particular, when using shells of infinite energy density, eventual smearing out by quantum spread should be understood.

Spherically symmetric solutions in general relativity are entirely determined in terms of the energy density function \( \sigma = T^0_0 \) and the radial pressure function \( p_1 = -T^1_1 \) in the coordinate system

\[ ds^2 = g_{00}(r) dt^2 - g_{11}(r) dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]  \hspace{1cm} (4)

where we have restricted ourselves to \( t \) independent configurations. The metric tensor is given by

\[ r_2[g^{11}(r_2) - 1] - r_1[g^{11}(r_1) - 1] = -2 \int_{r_1}^{r_2} dM(r) \]  \hspace{1cm} (5)

\[ dM(r) = 4\pi \sigma r^2 dr \]

and

\[ \frac{g_{00}(r_2)g_{11}(r_2)}{g_{00}(r_1)g_{11}(r_1)} = \exp \int_{r_1}^{r_2} 8\pi r(\sigma + p_1)g_{11} dr. \]  \hspace{1cm} (6)

For asymptotically flat solutions one chooses \( g_{00} = 1 \) at \( \infty \); in absence of black hole horizon, \( g_{11} > 0 \) and the solution considered is everywhere static. The other
components of the (diagonal) energy-momentum tensor $p_\theta = -T^\theta_\theta$ and $p_\phi = -T^\phi_\phi$ are determined by the Bianchi identities and can be expressed in terms of $p_1$ and $\sigma$:

$$p_\theta = p_\phi = \frac{1}{4} (\sigma + p_1) \left( \frac{8\pi r^2 p_1 + 2M(r)/r}{1 - 2M(r)/r} + \frac{1}{2} rp_1' + p_1 \right).$$  \hspace{1cm} (7)

Let us now consider a static spherically symmetric distribution of matter surrounded by an extended shell comprised between two radii $r_a$ and $r_b$. We define

$$\hat{\sigma} \equiv \int_{r_a}^{r_b} \sigma g_{11}^{1/2} \, dr, \quad \hat{p}_\theta \equiv \int_{r_a}^{r_b} p_\theta g_{11}^{1/2} \, dr, \quad \hat{p}_1 \equiv \int_{r_a}^{r_b} p_1 g_{11}^{1/2} \, dr. \hspace{1cm} (8)$$

Assuming $p_1 = 0$, one may perform the thin shell limit $r_b \to r_a = R$ in these integrals by using $dM(r) = 4\pi \sigma R^2 \, dr$. From (5) and (7) one then gets

$$4\pi R \hat{\sigma} = (1 - 2m^-/R)^{1/2} - (1 - 2m/R)^{1/2} \hspace{1cm} (9)$$

$$8\pi R \hat{p}_\theta = \frac{1 - m/R}{(1 - 2m/R)^{1/2}} - \frac{1 - m^-/R}{(1 - 2m^-/R)^{1/2}} \hspace{1cm} (10)$$

$$\hat{p}_1 = 0 \hspace{1cm} (11)$$

where $m$ and $m^-$ are the values of $M(r)$ respectively at $r_b$ and $r_a$ and $m_s = m - m^-$ is thus the mass of the shell. Equations (9) and (10) are the standard result\textsuperscript{[10]}. As the radius $R$ approaches $2m$, these solutions become physically meaningless when $\hat{p}_\theta$ becomes greater than $\hat{\sigma}$; this violates indeed the “dominant energy condition”\textsuperscript{[11]}, implying the existence of observers for which the momentum flow of the classical matter becomes spacelike. In fact, the shell is mechanically unstable even before this condition is violated\textsuperscript{[12]}.

The divergence of $\hat{p}_\theta$ when $R \to 2m$ appears in (10) because of the vanishing denominator in (7). Equation (10) depends however crucially on the radial pressure being zero inside the shell. Relaxing this condition we see indeed that a finite value of $p_1$ multiplies in (7) the energy density $\sigma$ which becomes infinite in the thin shell
limit. It is in fact possible to avoid all singularities of the stress tensor as $R \rightarrow 2m$ by requiring $p_1$ inside the shell to satisfy, before performing the thin shell limit,

$$4\pi r^2 p_1 + \frac{M(r)}{r} = 0. \quad (12)$$

Inserting the solution of (12) back in (7), we get for the trace of the energy momentum tensor $T^\mu_\mu$ the equation of state

$$T^\mu_\mu = 2\sigma \quad (13)$$

which means that the source of the “Newtonian” force $T^0_0 - 1/2\delta^0_0 T^\mu_\mu$ due to the any inner part of the shell on the remainder vanishes. An alternate way to discover the solution (12) is precisely to impose the trace condition (13) in equation (7): the solution of this differential equation with $p_1(r_a)$ fixed (and equal to $-M(r_a)/4\pi(r_a)^3$) is equation (12).

This solution is unsatisfactory if the (extended) shell sits in an arbitrary background because of the finite discontinuity of the radial pressure across the shell boundaries which would lead to singularities in $p_\theta$. We may ensure continuity of the radial pressure by immersing the shell in suitable left and right backgrounds. To avoid reintroducing stress divergences when $r^b$ approaches $2M(r^b)$ these should satisfy $(\sigma + p_1) = 0$ at the shell boundaries. One can now perform the thin shell limit. The finite discontinuity of $p_1(r)$ at $r = R$ leads to

$$\hat{p}_\theta = -\frac{\hat{\sigma}}{2}, \quad \hat{p}_1 = 0 \quad (14)$$

instead of (10),(11) and $\hat{\sigma}$ is still given by (9). The dominant energy condition is satisfied everywhere, as is the “weak energy condition”[11] ensuring positivity of the energy density for any observer. Provided the background is smooth enough in the neighbourhood of the shell, no stress divergences will appear when it approaches the Schwartzschild radius.
Such static thin shells sitting outside the Schwartzschild radius but infinitesimally close to it will be referred to as “limiting shells”. The mass \( m_s \) of the limiting shell plus the mass \( m^- \) of the inner matter contribution is equal to the black hole mass whose horizon would be at the limiting radius \( r = 2m \). The striking feature of the region bounded by the limiting shell is that it gives rise to an infinitely large time dilation in the global Schwartzschild time. Indeed

\[
g_{00}(r) = (1 - \frac{2M(r)}{r}) \exp \left[ - \int_{r}^{\infty} 8\pi r'(\sigma + p_1)g_{11} \, dr' \right] \tag{15}
\]

and performing the explicit integration over the shell, we get in the region \( 0 \leq r < 2m \)

\[
g_{00}(r) = (1 - \frac{2M(r)}{r}) \left[ \frac{R - 2m}{R - 2m^-} \right] \exp \left[ -\mathcal{R} \int_{r}^{\infty} 8\pi r'(\sigma + p_1)g_{11} \, dr' \right] \tag{16}
\]

Here the radius \( R \) of the shell is taken at \( R = 2m + \epsilon \) where \( \epsilon \) is a positive infinitesimal and the symbol \( \mathcal{R} \) means that the integral is carried over the regular matter contribution only. Clearly, \( g_{00}(r) = O(\epsilon) \) for \( 0 \leq r < 2m \), \( t \) arbitrary.

This domain of space-time is characterized by a Killing vector which is light-like in the limit \( \epsilon \to 0 \). When the space-time geometry presents a 4-domain endowed with such a limiting light-like Killing vector, we shall call the domain an achronon. All spherically symmetric achronon configurations will exhibit an infinite dilation of the Schwartzschild time \( t \) with respect to the outside world, or equivalently, massless modes emitted by the achronon are infinitely redshifted. Classically, the achronon has the “frozen” appearance of a collapse at infinite Schwartzschild time. The difference is that it is also frozen in space-time. This is the reason why, in contradistinction to collapsing shells, stresses (14) were needed to build the purely static solution considered above. However, requiring exact staticity everywhere in space-time is mathematically convenient but perhaps a too stringent and physically unnecessary constraint. Thus our shell solution (even if extended to a finite width)
and the concomitant restriction on the background, should be viewed as a simple illustrative model. More elaborate achronon solutions will be discussed elsewhere.

3. Thermal Screening of a Black Hole

Up to now, we have considered achronons in a trivial space-time topology but they can also be introduced in the topology of an eternal black hole. An eternal black hole of mass $m_0$ contains two asymptotically flat Schwartzshild patches connected by a throat. It represents the maximal extension of the Schwartzschild solution which is singular at $r = 0$ and is dynamical outside the patches as seen from the well known Kruskal representation (Fig.2). One can add in the patches a static distribution of matter without changing the topology as long as outer horizons are avoided. We shall consider such distributions and we shall limit ourselves to matter configurations which are identical in both patches. Thus, achronons of mass $m - m_0$ surrounding a black hole of mass $m_0$ are defined in this topology by their matter distribution in a static patch.

Let us now consider such a achronon, possibly surrounded by static matter. Using the metric (4) in the static patches, $M(r)$ is defined in general from (5) for $r > 2m_0$ by

$$M(r) = m_0 + \int_{2m_0}^{r} 4\pi \sigma r'^2 dr'.$$  \hspace{1cm} (17)

The Kruskal metric is

$$ds^2 = dT^2 - dX^2 \frac{r'^2}{F^2(\xi)} - r^2(\xi) (d\theta^2 + \sin^2 \theta d\phi^2)$$ \hspace{1cm} (18)

which is related to the static metric (4) within a patch by

$$g_{11}^{1/2} dr = d\xi, \quad \xi = 0 \text{ at the horizon},$$ \hspace{1cm} (19)
\[ F(\xi) = \sqrt{X^2 - T^2}. \] (20)

A Cauchy hypersurface \( \Sigma_c \) represented in Kruskal coordinates by \( T = 0 \) (Fig.2) connects the space-time with Minkowskian signature to a solution of the Euclidean Einstein equations. The latter can be described by the metric (4) with \( t = -it_e \) and is periodic in the Euclidean time \( t_e \). The Euclidean period \( T^{-1} \) can then be computed from the metric (4) in the vicinity of the black hole horizon \( r_0 \):

\[
T = \frac{1}{4\pi} [g_{00}(r_0) g_{11}(r_0)]^{-1/2} \frac{dg_{00}(r)}{dr}\Big|_{r=r_0} \] (21)

and from (5) and (6) one gets

\[
T = \frac{1}{8\pi m_0} \exp \left[ - \int_{2m_0}^{\infty} 4\pi r'(\sigma + p_1) g_{11} dr' \right]. \] (22)

Comparing (22) with (15), one immediately sees that the inverse Euclidean period of a black hole surrounded by an achronon vanishes.

To illustrate this phenomenon consider an achronon of mass \( m - m_0 \) bounded by a limiting shell of mass \( m_s \leq m - m_0 \). The limiting shell sits at a radius \( R = 2m + \epsilon \) and we may rewrite (22) as

\[
T = \frac{1}{8\pi m_0} \lim_{\epsilon \to 0} \exp \left[ - \int_{2m_0}^{R+\epsilon} 4\pi r'(\sigma + p_1) g_{11} dr' \right] \exp \left[ - \int_{2m_0}^{R-\epsilon} 4\pi r'(\sigma + p_1) g_{11} dr' \right] \exp \left[ - \int_{R+\epsilon}^{\infty} 4\pi r'(\sigma + p_1) g_{11} dr' \right]. \] (23)

The first factor is easily evaluated in the limit \( \epsilon \to 0 \) and (23) yields

\[
T = \frac{1}{8\pi m_0} \left[ \frac{R - 2m}{R - 2m^-} \right]^{1/2} \exp \left[ - \mathcal{R} \int_{2m_0}^{\infty} 4\pi r'(\sigma + p_1) g_{11} dr' \right] \] (24)

where \( m^- = m - m_s \). A glance at (24) shows that the limiting inverse Euclidean
period when $\epsilon \to 0$ is indeed

$$T_{\epsilon \to 0} = 0$$  \hspace{1cm} (25)

We know, from the work of Gibbons and Hawking\[^7\] that $T$ in (21) is the temperature at infinity of quantum matter in the background of the classical gravity-matter system considered and is its equilibrium temperature in the energy conjugate to the static time $t$. In particular, when no matter surrounds the black hole, $T$ reduces to the usual black hole temperature $1/8\pi m_0$. We shall show in the following section that $T$ is also the equilibrium temperature, in the semi-classical limit of quantum gravity, of the interacting gravity-matter system itself. Thus, (25) implies that the thermal effects of a black hole of mass $m_0$ can be entirely screened by a achronon of mass $m - m_0$, as expected from the infinite redshift due to the achronon.

A different situation can however arise if the black hole has, classically, a vanishing small mass. Such an object, which we shall call a germ black hole, generates a non trivial topology. As long as the surrounding matter does not form a achronon, the value of $T$ tends to infinity when the mass $m_0$ of the germ tends to zero. But in the presence of an achronon the resulting Euclidean periodicity can take any value, depending on the limiting process. In particular, one may have

$$T_{\epsilon \to 0} = T_m$$  \hspace{1cm} (26)

where $T_s$ is the inverse Euclidean period of a black hole of mass $m$ surrounded by the same matter distribution as the corresponding achronon of mass $m - m_0$. This is exemplified in (24) by letting $R - 2m$ go to 0 as $Cm_0^2$ and tuning the constant $C$ to satisfy (26).

The possibility of constructing, in presence of a germ, an achronon with the same behaviour in Euclidian time as a genuine black hole will be the key to the tunneling between achronons and black holes.
4. The Black Hole Tunneling Entropy

We first review\footnote{For a more detailed discussion see reference [2].} and generalize to the present case the description of tunneling in quantum gravity, obtained in reference [2] for geometries with de Sitter topology. Consider in general two spacelike hypersurfaces $\Sigma_1$ and $\Sigma_2$ which are turning points in superspace (or turning hypersurfaces) along which solutions of the Minkowskian classical equations of motion for gravity and matter meet a classical solution of their Euclidean extension. $\Sigma_1$ and $\Sigma_2$ are thus the boundaries of a region $E$ of Euclidean space-time defined by the Euclidean solution. If $E$ can be continuously shrunk to zero one can span $E$ by a continuous set of hypersurfaces $\tau = \text{constant}$ such that $\tau \equiv \tau_1$ on $\Sigma_1$ and $\tau \equiv \tau_2$ on $\Sigma_2$. These $\tau = \text{constant}$ surfaces define a coordinate system which we shall call synchronous; the Euclidean metric in $E$ can be written in the form
\begin{equation}
    ds^2 = N^2(\tau, x_k) d\tau^2 + g_{ij}(\tau, x_k) dx^i dx^j
\end{equation}
where $N(\tau, x_k)$ is a lapse function. The Euclidean action $S_e$ over $E$, from $\Sigma_1$ to $\Sigma_2$, is obtained by analytic continuation from the Minkowskian action (2) and can be written as
\begin{align}
    S_e(\Sigma_2, \Sigma_1) &= \int_E \Pi^{ij} \partial_\tau g_{ij} d^4x + \int_E \Pi^a \partial_\tau \phi_a d^4x - \int_E \frac{d}{d\tau}(g_{ij} \Pi^{ij}) d^4x \\
    &- \frac{1}{8\pi} \int_E \partial_k [(\partial_j N) g^{kj} \sqrt{g^{(3)}}] d^4x.
\end{align}
Here $\Pi^{ij}$ and $\Pi^a$ are the Euclidean momenta conjugate to the gravitational fields $g_{ij}$ and to the matter fields $\phi_a$; $g^{(3)}$ is the three dimensional determinant.

On the turning hypersurfaces $\Sigma_1$ and $\Sigma_2$, all field momenta ($\Pi^{ij}, \Pi^a$) are zero in the synchronous system and the third term in (28) vanishes. The last term in (28) also vanishes if the hypersurfaces $\Sigma_1$ and $\Sigma_2$ are compact (which was the case...
considered in reference [2]) but may receive contributions from infinity otherwise. We shall have to consider here the case where the two non compact turning hypersurfaces merge at infinity so that the Euclidean action $S_e(\Sigma_1, \Sigma_2)$ does not get contributions in $\mathcal{E}$ from the last term in (28). The classical Minkowskian solution in the space-time $\mathcal{M}_1$ containing $\Sigma_1$ can be represented quantum mechanically by a “forward wave” solution $\Psi(g_{ij}, \phi_a)$ of the Wheeler-de Witt equation (1) in the semi-classical limit. At $\Sigma_1$, this wave function enters, in the WKB limit, the Euclidean region $\mathcal{E}$ and leaves it at $\Sigma_2$ to penetrate a new Minkowskian space-time $\mathcal{M}_2$. The tunneling of $\Psi(g_{ij}, \phi_a)$ through $\mathcal{E}$ engenders in addition to the “forward wave” solution a time reversed “backward wave”. The inverse transmission coefficient $N_0$ through the barrier measures the ratio of the norms of the forward waves at $\Sigma_2$ and $\Sigma_1$. For large $N_0$ one may write in the synchronous system

$$N_0 = \exp \left[ -2 \int_{\mathcal{E}} \Pi^{ij} \partial_r g_{ij} \, d^4x + \int_{\mathcal{E}} \Pi^a \partial_r \phi_a \, d^4x \right]. \quad (29)$$

As all surface terms in (28) vanish in this system, (29) can be rewritten in the coordinate invariant form

$$N_0 = \exp \left[ 2 S_e(\Sigma_1, \Sigma_2) \right]. \quad (30)$$

Consider now an eternal black hole of mass $m$ surrounded by a spherically symmetric distributions of matter, the same in both static patches. Compare this classical solution of general relativity to another one consisting of an achronon of mass $m - m_0(\epsilon)$ surrounded by the same matter distribution and screening a germ black hole of mass $m_0(\epsilon) \to 0$ to the same Euclidean period. Both solutions are thus characterized by the same total mass $M$, the same matter distribution of mass $M - m$ outside the radius $2m + \eta$, $\eta$ infinitesimal† and the same Euclidean period $\mathcal{T}^{-1}$. We shall identify $\mathcal{M}_1$ with the achronon solution and $\mathcal{M}_2$ with the black

† For the shell model of section 2, one may take $\eta = \epsilon$ as $g_{00}$ and $R - 2m$ are of the same order of magnitude. For sake of generality we do not impose this relation here.
hole one. We label by $\Sigma_{c}^{B.H.}$ and $\Sigma_{c}^{A}$ respectively the turning hypersurfaces in the black hole and in the achronon geometries.

$\Sigma_{c}^{B.H.}$ and $\Sigma_{c}^{A}$ can be represented in Kruskal coordinates by hypersurfaces $T = 0$ and are depicted in Fig.3. They belong to Euclidean sections of these solutions $\mathcal{E}^{B.H.}$ and $\mathcal{E}^{A}$ which can be described by Euclidean Kruskal metrics (17) with Euclidean time $T_{e} = iT$ or by static coordinates (4) with a periodic Euclidean time $t_{e} = it$; for both solutions the period has the same value $T^{-1}$. It is clear, from the static coordinate description, that the two Euclidean space-time geometries $\mathcal{E}^{B.H.}$ and $\mathcal{E}^{A}$ coincide for $r > 2m + \eta$ but, while the Euclidean black hole terminates at $r = 2m$, the achronon solution has an extra “needle” in the region $0 < r < 2m$ whose 4-volume is of order $\epsilon$.

We now identify, at finite $\eta$, $\Sigma_{1}$ with $\Sigma_{c}^{A}$ and consider instead of a second turning hypersurface $\Sigma_{2}$ a hypersurface $\Sigma_{c}^{B.H.}$ which lies in $\mathcal{E}^{B.H.}$ and is such that $r > 2m + \eta$ everywhere on it. $\Sigma_{c}^{B.H.}$ is then contained in the intersection of $\mathcal{E}^{B.H.}$ and of $\mathcal{E}^{A}$. When $\eta \to 0$, $\Sigma_{c}^{B.H.}$ can be taken arbitrarily close to $\Sigma_{c}^{B.H.}$ and we shall prove in the Appendix that all gravitational momenta on $\Sigma_{c}^{B.H.}$ in a synchronous system vanish in this limit. We may then identify $\Sigma_{c}^{B.H.}$ with $\Sigma_{2}$. The region $\mathcal{E}$ is thus contained in the needle $0 < r < 2m + \eta$ of $\mathcal{E}^{A}$. Because of the Kruskal twofold symmetry $\Sigma_{c}^{A}$ is mapped onto itself by a Euclidean time rotation of half a period and thus $\mathcal{E}$ spans only half the needle 4-volume. From (28), we learn that the inverse transmission coefficient $N_{0}$ is simply the exponential of the total Euclidean action of the needle. Although the limiting 4-volume of the needle vanishes, the action is computable as the difference between the Euclidean action of the black hole $S_{e}^{B.H.}$ over $\mathcal{E}^{B.H.}$ and of the achronon $S_{e}^{A}$ over $\mathcal{E}^{A}$. This difference is finite and well defined by cutting off the two spaces at an arbitrary radius $r_{c}$ greater than $2m$ as the two geometries and the two actions coincide for all $r > r_{c}$. We thus write

$$N_{0} = \exp[S_{e}^{B.H.} - S_{e}^{A}].$$

To evaluate these actions we take advantage of the covariance to express them in terms of the static coordinate system with gravitational and matter momenta
everywhere vanishing. Thus only the last surface integral in (28) contributes now to the action and can be expressed as

\[ S_e = \int \frac{d}{dr} \left[ \frac{-r^2}{4T} [g_{00}g_{11}]^{-1/2} \frac{dg_{00}}{dr} \right] dr. \] (32)

Using (21) and the fact that the integrand is the same at \( r_c \) for \( S_e^{B.H.} \) and for \( S_e^A \), we get

\[ N_0 = \exp \left[ 4\pi m^2 - 4\pi m_0^2(\epsilon \to 0) \right] \] (33)

or, as \( m_0 \) vanishes in the limit,

\[ N_0 = \exp A/4 \] (34)

where \( A = 16\pi m^2 \) is the area of the event horizon of the black hole.

We have thus learned that black holes are related by quantum tunneling to another classical solution for gravity and matter, namely to an achronon*. Each achronon is connected through a “potential” barrier to a corresponding black hole of mass \( m \). Let us tentatively take boundary conditions in field space by assigning pure forward waves to achronons; the relative probability of finding a black hole with respect to an achronon is then \( N_0 \), since in the classical limit interferences between black holes propagating forward or backward in time must be negligible.

Consider then two distinct achronons surrounded by matter distributions such that the total mass \( M \) is the same for both classical solutions but \( m \) and the surrounding mass \( M - m \) need not be the same. Each achronon is related by tunneling to its corresponding black hole and the ratio of inverse transmission coefficients \( N_0 \) between the two achronon-black hole configurations is, from (34),

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* One might have thought that a single point on \( E^{B.H.} \) would represent, in the classical limit, a wormhole-like turning hypersurface. This is not the case as there is no continuous mapping of \( \Sigma^{B.H.} \) onto a point. More generally, one can show that no lower manifold contained in \( E^{B.H.} \) constitutes a turning hypersurface.
equal to $\exp \Delta A/4$, where $\Delta A$ is the change in black hole area. From the differential Killing identity of reference [13], or equivalently from the variation of the integrated constraint equation over a static patch[2], we have

$$\delta M - T \frac{\delta A}{4} = \delta \lambda H_{\text{matter}}$$

(35)

where $\lambda$ labels the explicit dependence of the matter Hamiltonian $H_{\text{matter}}$ deduced from the action (2) on all (non gravitational) “external” parameters. As we are considering only contributions with fixed total mass $M$, it follows from (35) that matter configurations with neighbouring energies in a static patch of the black hole would be Boltzmann distributed at the global temperature $T$ provided achronons with different mass are taken to be equally probable. This is indeed a consistent assumption as all configurations describing achronons and surrounding matter with fixed total mass $M$ have, from (35) with $A = 0$, the same total energy and may be described by a microcanonical ensemble. Thus the temperature of the static patch is indeed $T$. Therefore (35) also implies that $\delta A/4$ is the differential entropy of the black hole and that the latter is in thermal equilibrium with the surrounding matter at the temperature $T$. As the entropy must be an intrinsic property of the black hole, not only is equilibrium a consequence of the chosen boundary conditions in field space but the converse is also true: the temperature obtained directly from (35) with entropy identified as $A/4$ must agree at equilibrium with the thermal distribution generated from the field boundary conditions. This justifies a posteriori the above choice of boundary conditions†.

The tunneling approach to the horizon entropy and temperature[1],[2] applied here to the black hole differs from the analysis based on the Euclidean periodicity of Green’s functions[6] in two respects. On the one hand, the present approach yields the thermal spectrum, and then the entropy, from the backreaction of the thermal matter on the gravitational field, in contradistinction to the Green’s function approach. On the other hand however, the thermal matter considered here is

† up to changes which would not alter the probability ratios in the large $N_0$ limit
taken in the classical limit while the Green's function describes genuine quantum radiation. Both methods fall short of a fully consistent quantum treatment of the backreaction problem but the interpretation of the horizon entropy from tunneling will permit us to uncover the quantum states building the black hole entropy; in fact, we shall see that the number of states \( \exp A/4 \) count only a minute fraction of the full black hole degeneracy.

5. From Achronons to Planckons

The entropy \( A/4 \) which can be exchanged reversibly from a black hole to ordinary matter was rederived in the preceding section from the existence of a “potential barrier” between a black hole of mass \( m \) and an achronon of the same mass. This was done in the context of an eternal black hole admitting a Kruskal twofold symmetry with two achronons separated by an infinitesimal throat, each imbedded in the surrounding geometry of a static space emerging from the eternal black hole throat. Within each space black hole-achronon states are in thermal equilibrium with their surroundings. We are therefore led to picture a black hole-achronon state, in the semi-classical limit, as a quantum superposition of two coherent (normalized) states, \( |B.H.\rangle \) and \( |A.\rangle \) representing respectively a classical black hole and a classical achronon. The relative weight of the two states in thermal equilibrium is approximately, up to a phase, \( \exp(-A/8) \). It follows from detailed balance at equilibrium between radiated matter and the black hole that the same superposition should hold for a the black hole who would only emit (and not receive) thermal radiation at the equilibrium temperature. As a black hole formed from collapse indeed emits such a thermal flux, we infer that its state \( |C\rangle \) should contain an achronon component with the same weight as in thermal equilibrium. We thus write

\[
|C\rangle = |B.H.\rangle + \exp(-A/8)|A.\rangle.
\]  

(36)

To a single black hole configuration one may associate many distinct classical achronon configurations. In the shell model, for instance, there are infinitely many
distinct classical matter configurations of the same total mass $m$. The argument is however much more general and infinite quantum degeneracy of the achronon is a direct consequence of the infinite time dilation. Indeed, the Hamiltonian $H$ is of the form

$$H = \int \sqrt{g_{00}} K(\phi_a, g_{ij}, \Pi_a, \Pi_{ij}) \, d^3x$$

and all its eigenvalues are squashed towards zero by the Schwarzschild time dilation factor $\sqrt{g_{00}}$, thus generating an infinite number of orthogonal zero energy modes on top of the original achronon. In the classical limit, the phase space of zero-energy solutions becomes infinite and the Killing identity (35) with $A = 0$ confirms that the modes give no contribution to the achronon mass $m$. The same conclusions can be arrived at by considering the wave equation instead of the canonical Hamiltonian. For example, the scalar wave equation is

$$\frac{1}{\sqrt{g}} \partial_{\mu} \sqrt{g} g^{\mu \nu} \partial_{\nu} \Phi = 0.$$  

Thus the frequency of any mode is proportional to $\sqrt{g_{00}}$ and vanishes in the limit $g_{00} \to 0$. By imposing boundary conditions at the surface, $\phi$ may be expanded in creation and annihilation operators for the above modes, thus realizing the degenerate spectrum.

The infinity of zero energy modes around any background implies an infinite degeneracy of achronons of given mass and thus an infinity of distinct quantum black hole states of the same mass differing by the achronon component of their wave function. This infinite degeneracy of the quantum black hole provides the reservoir from which are taken the finite number of “surface” quantum states $\exp A/4$ counted by the area entropy $A/4$ transferable reversibly to outside matter.

Except for providing a rational for the large but finite testable entropy of the black hole, achronons do not modify the behaviour of large macroscopic black holes. However when their mass is reduced by evaporation and approaches the Planck mass the barrier disappears and quantum superposition completely mixes
the two components. Of course, this means that both the description in terms of semiclassical configurations and of tunneling disappears. What remains however as a consequence of unitarity, is the infinity of distinct orthogonal quantum states available which have no counterpart in the finite number of decayed states. The quantum black hole has become a planckon\textsuperscript{[7]}, that is a planckian mass object with infinite degeneracy. Causality and unitarity prevent the decay (and the production) in a finite time of such object\textsuperscript{[7]}, and the argument applies to the “parent achronon” as well. Indeed, if a state $|A_i\rangle$ of finite size and mass $m$ decays, or is produced, within a finite time $\tau$ in an approximately flat space-time, the total number of possible final states is limited by the number $\mathcal{N}$ of orthogonal states with total mass $m$ in a volume $\tau^3$. From unitarity, the degeneracy $\nu(m)$ of the states $|A_i\rangle$ is at most $\mathcal{N}$. Thus if $\nu(m) \to \infty$, the time $\tau$ tends to infinity. Thus, achronons and in particular planckons can neither decay nor be formed in a finite time.

As recalled in the introduction, planckons are a solution to the unitarity puzzle\textsuperscript{*} arising from black hole evaporation and may have played a crucial role in seeding our universe and its large scale structure. At a more fundamental level they have far reaching implications on the spectrum of quantum gravity. The opening at the Planck size of an infinite number of states, an unavoidable consequence of the existence of planckons, may appear as a horrendous complication which could make quantum gravity definitely unmanageable but hopefully the converse may be true. Indeed planckons should make quantum gravity ultraviolet finite. The Hilbert space of physical states available to macroscopic observer must be orthogonal to the infinite set of states describing planckian bound states. Their wave function at planckian scales where planckon configurations are concentrated are therefore expected to be vanishingly small. In this way, planckons would provide the required short distance cut-off for a consistent field theoretic description of quantum gravity within our universe while leaving the largest part of its information content hidden at the Planck scale.

\textsuperscript{*} see also reference[14].
An operational formulation of quantum gravity applicable within our universe and based on conventional four dimensional gravity, may thus well be within reach. But it is nevertheless tempting to dwell upon the further significance of the picture that emerges. The sudden widening of the spectrum of physical states at the Planck scale strongly suggests that the relative scarcity of states which describe large distance physics (as compared to the Planck size) is due to the fact that the existence of observables whose correlations survive at macroscopic range is contingent on the notions of scale and metric. The appearance of these concepts in the organization of long distance physics, at the cost of relegating most of the information to the Planck scale, would imply that scale should be absent from a fundamental description of the physical world. Consistency may then ultimately require a unified theory, of which string theory is perhaps a precursor, which by eliminating the gravitational scale from the basic formulation would render obsolete the use of a standard of length or of time.

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APPENDIX

In computing the tunneling amplitude from the achronon turning hypersurface \( \Sigma_c^A \) to the black hole hypersurface \( \Sigma_{c.B.H.} \), we have, in section 4, replaced \( \Sigma_{c.B.H.} \) by a hypersurface \( \Sigma_{c'}^{B.H.} \) which lay in the intersection of the Euclidean sections of the achronon and of the black hole solutions \( \mathcal{E}^A \) and \( \mathcal{E}^{B.H.} \) but could be taken arbitrarily close to \( \Sigma_{c.B.H.} \) in the limit \( \eta \to 0 \). In this way the existence of a classical Euclidean motion from \( \Sigma_c^A \) to \( \Sigma_{c'}^{B.H.} \) was self evident and the computation of the tunneling Eq (31) was straightforward by identifying in the limit \( \Sigma_{c'}^{B.H.} \) to \( \Sigma_{c.B.H.} \). It must be shown however that the limit is smooth enough so that the momenta that flows between \( \Sigma_c^A \) and \( \Sigma_{c'}^{B.H.} \) in a synchronous system vanishes indeed on the latter hypersurface when \( \eta \to 0 \). This is proven below.

Let us take, in the vicinity of the black hole throat a Euclidean Kruskal coordinate system

\[
 ds_c^2 = \frac{dT^2 + dX^2}{F'^2(\xi)} + r^2(\xi) (d\theta^2 + \sin^2 \theta d\phi^2) \quad (A.1)
\]

where

\[
 F(\xi) = \sqrt{X^2 + T_c^2}. \quad (A.2)
\]

Here \( \xi \) is defined by (19) so that the Euclidean static coordinate system can be written as

\[
 ds_c^2 = g_{00}(r) dt_c^2 + d\xi^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (A.3)
\]

where

\[
 \frac{2\pi TF(\xi)}{F'(\xi)} = \frac{1}{2} g_{00}(\xi) \quad (A.4)
\]

Consider a hypersurface \( \Sigma(T_c^0) \) which coincide with the \( T_c = T_c^0 \) hypersurface in an interval \(|X| < X_{max}\) where \( X_{max} \) is determined by the solution of

\[
 F'^{-1}(\xi) = g_{00}^{1/2}(\xi). \quad (A.5)
\]

Thus at \(|X_{max}|\) the lapse functions of static and Kruskal systems are equal. We
then complete the hypersurface $\Sigma(T^0_e)$ by matching at $|X_{max}|$ hypersurfaces of constant $t_e$.

For every $T^0_e > 0$, we have $r > 2m$ and hence one can always find a positive $\eta$ such that the hypersurface $\Sigma(T^0_e)$ lays in the intersection of $E^A$ and $E^{B.H.}$. We may thus choose $\Sigma^{B.H.}$ to coincide with $\Sigma(T^0_e)$. We must then show that the momentum flow through $\Sigma(T^0_e)$ vanishes in the limit $T^0_e \to 0$. More precisely, defining

$$ I \equiv \int_{\Sigma(T^0_e)} \Pi^{ij} g_{ij} d^3x, \quad (A.6) $$

we must prove that

$$ \lim_{T^0_e \to 0} I = 0 \quad (A.7) $$

as $I$ is indeed the surface integral in (28) which has to vanish in order to validate the tunneling result (30).

The integral $I$ receives only contributions from the region $-X_{max} < X < X_{max}$ and using

$$ \Pi^{ij} = \frac{\sqrt{g^{(3)}}}{32\pi N} [g^{im} g^{jn} - g^{ij} g^{mn}] \partial_{\tau} g_{mn} \quad (A.8) $$

where $N$ is the lapse function, one has

$$ I = - \int_{0}^{X_{max}} F'(\xi) \left[ \frac{\partial}{\partial T_e} \left( \frac{r^2(\xi)}{F'(\xi)} \right) \right] dX \quad (A.9) $$

Using (A.4) and (A.5), one gets

$$ |I| = \frac{1}{T^0_e} \int_{T^0_e}^{\frac{1}{T^0_e}} \frac{1}{\sqrt{F^2_e - (T^0_e)^2}} \left[ \frac{\partial}{\partial \xi} \left( \frac{r^2(\xi)}{F'(\xi)} \right) \right] dF < \frac{1}{T^0_e} \int_{T^0_e}^{\frac{1}{T^0_e}} \frac{1}{\sqrt{F^2 - (T^0_e)^2}} dF \quad (A.10) $$
where \( A \) is a positive number. Therefore, as \( T_e^0 \to 0 \),

\[
|I| < -A T_e^0 \ln T_e^0
\]  

(A.11)

and Eq (A.7) follows.
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FIGURE CAPTIONS.

Figure 1. Tunneling of a nonrelativistic “clock”.
The energy of the clock $E_c$ is represented by the dashed line. On the left of the turning point $a$ the clock is well represented by a forward wave only depicted here by a single arrow. On the right of the turning point $b$ the amplification of the forward wave and the large concomitant backward wave are indicated.

Figure 2. The Kruskal representation of a black hole, eventually surrounded by static matter.
The dashed straight line is the Euclidean axis $T_e$. The dashed circle is the analytic continuation in Euclidean time of the solid hyperbolae representing trajectories $r =$constant in the static patches I and III. These are separated from the dynamical regions II and IV by the horizons $r = 2m_0$ where lay the past and future singularities $r = 0$ depicted by the dashed hyperbolae. The Schwartzschild time $t$ run on opposite directions on the two hyperbolae $r =$constant and the Euclidean time $t_e$ spans the period $T^{-\infty}$ on the analytically continued circle.

Figure 3. Black hole-achronon tunneling.
The figure represents the Euclidean sections of an achronon and of its corresponding black hole. Each point is a 2-sphere and the circles span the Euclidean time $t_e$. The achronon geometry is depicted by thick lines and the black hole geometry by thin lines in the region where it differs from the first. The picture is not on scale as there are no 3-dimensional Euclidean imbedding of these surfaces.

The curve $a$ is the turning hypersurface $\Sigma_c^A$. The curve $b$ is the turning hypersurface $\Sigma_{c.B.H.}$. The curve $c$ is the hypersurface $\Sigma_{c.B.H.}$ which lay in the intersection of $E^{B.H.}$ and $E^A$ and tends to $\Sigma_{c.B.H.}$ in the limit $\eta \to 0$. 