ABSTRACT A three-neuron network model with mixed delays involving multiple discrete and distributed delays is considered in this paper. Taking the discrete and distribute delays as bifurcation parameters respectively, we investigate the stability of the system structure and the conditions for the generation of Hopf bifurcation from the perspective of the distribution of the root of the characteristic equation of the linearized system at the equilibrium state of the nonlinear system. The intervals of parameters that make the system stable and unstable are also given. In addition, when the conditions of Hopf bifurcation theorem are satisfied, the calculation formulas for determining the direction of Hopf bifurcation and the stability of bifurcating periodic solution are presented by means of the central manifold theorem and normal form method. Finally, numerical experiments are carried out to support the correctness of the theoretical results in different cases. It can be concluded that the oscillation and instability caused by delays obviously affect the stability of the network.

INDEX TERMS neural networks, stability, Hopf bifurcation, delay, periodic solution

I. INTRODUCTION

In recent years, neural network models have been widely used in engineering, especially in signal processing, image recognition, pattern recognition, remote sensing technology and so on. As we all know, the mathematical theory of neural network is the premise of its application. Owing to its nonlinear characteristics, neural network models often have rich dynamic properties. Most of the early work focused on the analysis and research of neural network dynamics of non-time delay, constant time delay and autonomous neural network models, and obtained many meaningful results, including equilibrium point, periodic solution, bifurcation and chaos [1]-[12].

In the circuit implementation of neural networks and the stickiness triggered between synapses of biological neural networks, there is a common phenomenon that the signal transmission has time delays, which is the primary origin of neural network oscillation and instability. Therefore, as one of the main subjects, the stability analysis of time-delay neural networks has been previously studied [4]-[17]. It was originally represented by the following two-neural networks with two discrete delays proposed by Olien and Bélair [4] in 1997.

$$x_i'(t) = -x_i(t) + \sum_{j=1}^{2} a_{ij} f(x_j(t - \tau_j)), i = 1, 2.$$ (1)

Afterward, Wei and Ruan [5] and Huang et al. [9] considered a simple two-neuron network model with two and four time delays, respectively. Song [8] designed a simplified bidirectional associative memory (BAM) with two delays using three neurons and investigated its stability at equilibrium and the properties of bifurcating periodic...
solutions. Yu and Cao [11] analyzed the dynamic characteristics of a class of four-neuron BAM network with four time delays. Xu et al. [17] discussed the dynamical properties about stability and Hopf bifurcation of a six-neuron BAM network model with six discrete delays. All of these indicate that the dynamical behavior of network system with multiple delays and neurons may be more complex and interesting for which it has a transcendent characteristic equation [18].

The early developed models with fixed delays can approach to the actual network well in a simple and small circuit. However, with the increasing circuit scale, the structure of neural network becomes complex. Meanwhile, due to the existence of a multitude of parallel channels and different numbers and lengths of neural axons in the network, the transmission between signals produces time-space effect, which makes it inappropriate to simulate the actual network system only involving discrete time-delay neural network. Therefore, the neural network model with distributed delay that can more fully describe the changes of network state comprehensively has emerged [18]-[28].

Particularly, in [18], a two-neuron network with two discrete and distributed delays was proposed as follows

\[
\begin{align*}
x_1'(t) &= -x_1(t) + a_{11}f_1(t-x_1(t)) + a_{12}f_2(t-x_2(t-	au_2)), \\
x_2'(t) &= -x_2(t) + a_{21}f_2(t-x_1(t-	au_1)) + a_{22}f_2(t-x_2(t-	au_2)) + a_{23}f_3(t-x_2(t-	au_3)), \\
x_3'(t) &= -x_3(t) + a_{31}f_3(t-x_1(t-	au_1)) + a_{32}f_2(t-x_2(t-	au_2)) + a_{33}f_3(t-x_2(t-	au_3)) + a_{34}f_4(t-x_2(t-	au_4)).
\end{align*}
\]

The stability and bifurcation analysis of system (2) were investigated by regarding the sum of the two discrete delays as a bifurcation parameter. Recently, Wang et al [23] gave the following three-neuron network model with multiple discrete and distributed delays

\[
\begin{align*}
x_1'(t) &= -x_1(t) + a_{11}f_1(t-x_1(t)) + a_{12}f_2(t-x_2(t-	au_2)), \\
x_2'(t) &= -x_2(t) + a_{21}f_2(t-x_1(t-	au_1)) + a_{23}f_2(t-x_2(t-	au_2)) + a_{24}f_3(t-x_2(t-	au_3)), \\
x_3'(t) &= -x_3(t) + a_{31}f_3(t-x_1(t-	au_1)) + a_{32}f_2(t-x_2(t-	au_2)) + a_{33}f_3(t-x_2(t-	au_3)) + a_{34}f_4(t-x_2(t-	au_4)).
\end{align*}
\]

They used \( \tau = \tau_1 + \tau_2 + \tau_3 + \tau_4 \) as bifurcation parameter to discuss the existence conditions of Hopf bifurcation, the direction of bifurcations and the corresponding stability of the bifurcating periodic solutions.

What the most network systems with mixed delays involving discrete and distributed delays appear to have in common is that the distributed delay of neuron only receives it from itself failing to consider its distributed transmission effect on other neurons, including systems (2) and (3). Inspired by the viewpoint, this work develops a mathematical model of a three-neuron network with mixed delays to describe the response of discrete delays between neurons and the influence of distributed transmission delays of one neuron by itself and to another. Our aim is to give a linear stability analysis for the three-neuron network with discrete time delays and distributed delays. Also, some criteria for analyzing the characteristics of dynamic behavior are performed to decide properties of the bifurcating periodic solutions.

The rest of this article is organized as follows. In Section II, we introduce a three-neuron network with discrete and distributed delays. In Section III, we discuss some conditions to ensure the stability of equilibrium and existence of Hopf bifurcation by considering the characteristic equation of the linearization. In Section IV, the properties of the bifurcating periodic solutions are decided by employing the normal form theory and center manifold theorem. In Section V, we present some examples and numerical simulations to support theoretical analysis. Finally, we come to a conclusion shown in Section VI.

II. MODEL DESCRIPTION AND TRANSFORMATION

A continuous three-neuron network with mixed delays involving discrete and distributed delays is described as the following differential equations

\[
\begin{align*}
x_1'(t) &= -x_1(t) + a_{11}f_1(t-x_1(t)) + a_{12}f_2(t-x_2(t-	au_2)), \\
x_2'(t) &= -x_2(t) + a_{21}f_2(t-x_1(t-	au_1)) + a_{23}f_2(t-x_2(t-	au_2)) + a_{24}f_3(t-x_2(t-	au_3)), \\
x_3'(t) &= -x_3(t) + a_{31}f_3(t-x_1(t-	au_1)) + a_{32}f_2(t-x_2(t-	au_2)) + a_{33}f_3(t-x_2(t-	au_3)) + a_{34}f_4(t-x_2(t-	au_4)),
\end{align*}
\]

where \( a_i \) and \( a_{ij} (i=1,2,3; j=1,3) \) are connection weights of neurons; \( \tau \) is a discrete delay that interpreted as a time lag in the transmission of signals along the axons from neuron \( x_i(t) \) to \( x_j(t) \) and \( x_k(t) \) to others; \( f(i) \) is a continuously differentiable neuron excitation function satisfying \( f(0) = 0 \); \( F_i(t) (i=1,2) \) are distributed delay kernel functions defined on \([0, +\infty)\), which are used to measure the effect of distributed delays. We take expressions: weak kernel \( F_1(s) = a e^{-\alpha s} \) and kernel \( F_2(s) = \alpha s e^{-\alpha s} \), respectively, here \( \alpha > 0 \) is the mean delay of the kernel and one can see reference [23] for general expressions of kernel functions. It is worth noting that, in previous studies, most of the kernel functions are just expressed in the form of weak.

The simplified network architecture of system (4) is presented in Fig. 1. What is mainly different from system (3) is that the three neurons transmit signals with discrete time delay \( \tau \), and the neuron \( x_i \) receives a distributed delay input from itself and sends another to \( x_j \). In addition, in system (4), kernel functions and delays of the distribution terms are different.

For system (4), as a matter of convenience, we introduce...
the following three new variables
\[ x_1(t) = \int_{t}^{\infty} F_1(t - s)x_1(s - \tau_1)ds = \int_{t}^{\infty} e^{-\alpha(s - t)}x_1(s - \tau_1)ds, \]
\[ x_2(t) = \int_{t}^{\infty} F_2(t - s)x_2(s - \tau_2)ds = \int_{t}^{\infty} e^{-\alpha(s - t)}x_2(s - \tau_2)ds, \]
\[ x_3(t) = \int_{t}^{\infty} F_3(t - s)x_3(s - \tau_3)ds = \int_{t}^{\infty} e^{-\alpha(s - t)}x_3(s - \tau_3)ds. \]

Using the linear chain technique, system (4) can be transformed into the following equivalent one:
\[
\begin{align*}
x'_1(t) &= -x_1(t) + a_{11}f(x_1(t)) + a_{12}f(x_2(t - \tau)), \\
x'_2(t) &= -x_2(t) + a_{21}f(x_1(t)) + a_{22}f(x_2(t - \tau)), \\
x'_3(t) &= -x_3(t) + a_{31}f(x_1(t)) + a_{32}f(x_2(t - \tau)).
\end{align*}
\]
(5)

Sequentially, we take the transformation system (5) as the model object for analysis. For the sake of argument, we denote that \( \mathbb{N} \) is the set of all positive integers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), \( C([-1,0],\mathbb{R}^6) \) is the Banach space of continuous mapping from \([-1,0]\) into \( \mathbb{R}^6 \), and \( C^k([-1,0],\mathbb{R}^6) \) is the space of k order continuous derivative mapping from \([-1,0]\) into \( \mathbb{R}^6 \) throughout this paper.

III. STABILITY AND HOPF BIFURCATION ANALYSIS

Our work in this section is to analyze the stability and Hopf bifurcation of system (5). Actually, the equilibrium point \( (0,0,0) \) of system (5) is the constant solution of equations. Set \( x_i(t) = x_i^*(i = 1,2,\cdots,6) \) and substitute them into (5). According to \( f \in C^3 \), \( f(0) = 0 \), it is easy to obtain \( x_i^* = 0(i = 1,2,\cdots,6) \), that is, the origin is the equilibrium point of system (5). Accordingly, \( (0,0,0) \) is the equilibrium point of system (5).

At the right end of (5), using Taylor’s expansion to expand \( f(\cdot) \) at \( x^* \) and take its linear approximation part, we can obtain the following linear approximation system of (5) at the equilibrium point.

\[
\begin{align*}
x'_1(t) &= -x_1(t) + a_{11}bx_1(t) + a_{12}bx_2(t - \tau), \\
x'_2(t) &= -x_2(t) + a_{21}bx_1(t), \\
x'_3(t) &= -x_3(t) + a_{31}bx_1(t) + a_{32}bx_2(t - \tau). 
\end{align*}
\]
(6)

where \( b = f'(0) \). Its Jacobian matrix \( J \) and corresponding characteristic equation are as follows, respectively.

\[
J = \begin{pmatrix}
-1 & a_{12}be^{-\alpha\tau} & 0 & a_1b & 0 & 0 \\
0 & -1 & 0 & 0 & a_2b & 0 \\
\alpha e^{-\alpha\tau} & a_{12}be^{-\alpha\tau} & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & -\alpha 
\end{pmatrix},
\]
(7)

where
\[
a_5 = 3(1 + \alpha), \\
a_4 = 3 + 9\alpha + 3\alpha^2, \\
a_3 = 1 + 9\alpha + 9\alpha^2 + \alpha^3, \\
a_2 = 3\alpha + 9\alpha^2 + 3\alpha^3, \\
a_1 = 3\alpha^2 + 3\alpha^3, \\
a_0 = \alpha^3, \\
b_2 = -\alpha a_{21}a_{23}b^2, \\
b_1 = -\alpha^2 a_{10}a_{23}b^2 - \alpha a_{12}a_{23}b^2, \\
b_0 = -\alpha^2 a_{12}a_{23}b^2, \\
c_4 = -\alpha a_{11}b, \\
c_3 = -2\alpha a_{11}b - 2\alpha^2 a_{11}b, \\
c_2 = -\alpha a_{11}b - 4\alpha^2 a_{11}b - \alpha a_{13}b, \\
c_1 = -2\alpha^2 a_{11}b - 2\alpha^3 a_{11}b, \\
c_0 = -\alpha^2 a_{11}b.
\]

Now, the dynamic properties of the system (5) in some different cases will be analyzed.

Case (1) \( \tau_1 = \tau_2 = 0 \)

Then the characteristic equation (7) becomes
\[
\lambda^6 + a_5\lambda^5 + (a_4 + c_1)\lambda^4 + (a_3 + c_2)\lambda^3 + (a_2 + c_3)\lambda^2 + (a_1 + c_4)\lambda + (a_0 + c_5) = 0.
\]
(8)

We denote \( p_i = a_i + b_i + c_i (i = 0,1,2) \) and
\[
\lambda^6 + p_5\lambda^5 + p_4\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 = 0.
\]
(9)
Apparently, (9) is a simplified form of (8) for \( \tau = 0 \). According to Routh-Hurwitz criterion, system (5) is asymptotically stable if and only if each order sequential principal minor of Hurwitz determinant is greater than zero, that is, the following conditions (H1) satisfy.

\[
\begin{align*}
\Delta_1 &= p_4 p_5 - p_3 > 0, \\
\Delta_2 &= p_5 (p_5 p_4 - p_3 p_2) + p_1 p_5 - p_6 ^2 > 0, \\
\Delta_3 &= -p_1 ^2 + 2 p_2 p_3 p_5 + p_1 p_4 - p_2 p_5 ^2 - p_2 ^2 p_3 ^2 - p_2 ^2 p_3 - p_2 p_3 ^2 > 0, \\
\Delta_4 &= -p_0 ^2 p_5 ^3 - 3 p_0 p_4 p_4 p_3 + 2 p_0 p_4 p_4 p_2 + p_0 p_3 p_3 ^2 + p_0 p_2 p_2 ^3 > 0, \\
\Delta_5 &= -p_0 ^2 p_4 p_4 p_3 - p_0 p_4 p_4 p_2 - 2 p_0 p_4 p_4 + p_0 p_3 p_3 ^2 + p_0 p_2 p_2 ^3 > 0, \\
\Delta_6 &= -p_0 ^2 p_4 p_4 p_3 - p_0 p_4 p_4 p_2 - 2 p_0 p_4 p_4 + p_0 p_3 p_3 ^2 + p_0 p_2 p_2 ^3 > 0.
\end{align*}
\]

Then the system (5) is asymptotically stable for \( \tau = \tau_1 = \tau_2 = 0 \).

We assume \( \tau = 0 \) below. Let \( \lambda = i \omega > 0 \) be a root of (8) and substitute it into (8). Separating real and imaginary parts, and then gives

\[
\begin{align*}
&-\omega^2 + (a_0 + c_0) \omega^2 - (a_2 + c_2) \omega^2 + (a_0 + c_0) \\
&+ (b_0 - b_2 \omega) \cos \omega \tau + b_0 \omega \sin \omega \tau = 0, \\
&a_0 \omega^2 - (a_1 + c_1) \omega^2 + (a_0 + c_0) \omega + b_0 \omega \cos \omega \tau \\
&- (b_0 - b_2 \omega) \sin \omega \tau = 0.
\end{align*}
\]

Solve the equations (10) and (11), we get

\[
\begin{align*}
\cos \omega \tau &= \frac{m_0 \omega^8 + m_0 \omega^6 + m_0 \omega^4 + m_0 \omega^2 + m_0}{m_0 \omega^8 + m_0 \omega^6 + m_0 \omega^4 + m_0 \omega^2 + m_0}, \\
\sin \omega \tau &= \frac{m_0 \omega^7 + m_0 \omega^5 + m_0 \omega^3 + m_0 \omega}{m_0 \omega^8 + m_0 \omega^6 + m_0 \omega^4 + m_0 \omega^2 + m_0},
\end{align*}
\]

where

\[
\begin{align*}
m_1 &= -b_2 ^2, \\
m_2 &= 2b_0 b_2 - b_2 ^2, \\
m_3 &= -b_0 ^2, \\
m_4 &= b_2, \\
m_5 &= -b_0 - (a_2 + c_2) b_2 + a_1 b_1, \\
m_6 &= -c_0 b_2 + b_1 b_2 - (a_2 + c_2) b_2, \\
m_7 &= -(a_2 + c_2) b_2 - (a_0 + c_0) b_2 + (a_0 + c_1) b_1, \\
m_8 &= (a_0 + c_0) b_2, \\
m_9 &= a_1 b_2 - b_1, \\
m_{10} &= -(a_1 + c_2) b_2 - a_1 b_2 + (a_0 + c_1) b_1, \\
m_{11} &= (a_1 + c_1) b_2 + (a_2 + c_2) b_2 - (a_2 + c_2) b_1, \\
m_{12} &= -(a_1 + c_1) b_2 + (a_1 + c_1) b_1.
\end{align*}
\]

Square both sides of (10) and (11) respectively, and then add them together to obtain

\[\omega^6 + k_1 \omega^2 + k_2 \omega^2 + k_3 \omega^2 + k_4 \omega^2 + k_5 \omega^2 + r = 0, \tag{14}\]

where

\[
\begin{align*}
k_1 &= -2(a_0 + c_0), \\
k_2 &= (a_0 + c_0)^2 + 2(a_0 + c_2) - 2(a_0 + c_1), \\
k_3 &= -2(a_0 + c_0) - 2(a_0 + c_2)(a_0 + c_4) + (a_0 + c_1)^2 \\
&+ 2(a_0 + c_1), \\
k_4 &= (a_2 + c_2)^2 + 2(a_0 + c_0)(a_0 + c_4) - 2(a_0 + c_1)(a_0 + c_3) - b_2^2, \\
k_5 &= -2(a_0 + c_0)(a_0 + c_4) + (a_0 + c_1)^2 + 2b_0 b_2 - b_2^2, \\
r &= (a_0 + c_1)^2 - b_2^2.
\end{align*}
\]

Let \( \omega = \omega^2 \), then (14) is equivalent to

\[
z^6 + k_1 z^4 + k_2 z^4 + k_3 z^2 + k_4 z^2 + k_5 z + r = 0. \tag{15}\]

**Lemma 1.** If \( r < 0 \), then (15) has at least one pure imaginary root.

**Proof.** Denote \( h_1(z) = z^6 + k_1 z^4 + k_2 z^4 + k_3 z^2 + k_4 z^2 + k_5 z + r \).

Obviously, \( h_1(0) = r < 0 \) and \( \lim h(z) = +\infty \). Therefore, there is \( \zeta_0 \in (0, +\infty) \) such that \( h(\zeta_0) = 0 \). □

Naturally, we suppose that \( \zeta(i = 1, 2, \cdots, 6) \) are all positive roots of (15), then \( \omega = \sqrt{\zeta} \in (i = 1, 2, \cdots, 6) \). By (12) and (13), we have a sequence \( \tau_i \in \mathbb{N}_0 \) described by

\[
\tau_i = \frac{1}{\omega_i} \left\{ \arctan \frac{m_1 \omega_i + m_{10} \omega_i + m_{11} \omega_i + m_{12} \omega_i}{m_0 \omega^2 + m_0 \omega^2 + m_0 \omega^2 + m_0} \right\} + j \pi
\]

with \( i = 1, 2, \cdots, 6 \). That is for \( j \in \mathbb{N}_0 \), \( (\tau_i, \omega_i) \) is a solution of equations (12) and (13), and \( \pm i \omega_i \) is a pair of pure imaginary roots of (7) with \( \tau = \tau_i \). In other words, (7) has no pure imaginary roots when \( \tau \neq \tau_i \). Further, we take \( \tau_0 = \tau_i \in \mathbb{N}_0 \) as \( \omega = \omega_i \).

The following lemma proposed by Ruan and Wei [29] is of great importance in determining the distributions of the roots of (7).

**Lemma 2[29]:** Consider the exponential polynomial

\[
P(\lambda, e^{i \omega_1}, \cdots, e^{i \omega_m}) = \lambda^2 + p_1(\lambda) e^{i \omega_1} + \cdots + p_n(\lambda) e^{i \omega_m}
\]

where \( \tau = \sqrt{\omega_1, \cdots, \omega_m} \) and \( \omega_j(i = 1, 2, \cdots, m) \) are constants. As \( \tau, \omega_1, \cdots, \omega_m \) vary, the sum of the order of the zeros of \( P(\lambda, e^{i \omega_1}, \cdots, e^{i \omega_m}) \) on the right half plane can change only if a zero appears on or crosses the imaginary axis.

Denote \( \lambda(\tau) = \chi(\tau_i) + i \omega(\tau) \) as the root of (8) satisfying \( \chi(\tau_i) = 0 \). Let us substitute \( \lambda(\tau) \) into (8) and differentiate with respect to \( \tau \), then it yields
Thus, the transversality condition is arrived.
By (21) and (22), we have a sequence \( \{ r_{i_0}^j \}_{j \in \mathbb{N}_0} \) described by
\[
\tau_i^j = \frac{1}{\alpha_i} \arctan \left( \frac{m_{i_0}^j \alpha_{i_0}^j + m_{i_0}^j \alpha_{i_0}^j + m_{i_0}^j \alpha_{i_0}^j + m_{i_0}^j \alpha_{i_0}^j}{m_{i_0}^j \alpha_{i_0}^j + m_{i_0}^j \alpha_{i_0}^j + m_{i_0}^j \alpha_{i_0}^j + m_{i_0}^j \alpha_{i_0}^j + m_{i_0}^j} \right) + \frac{j \pi}{\alpha_i}, \quad i = 1, 2, \ldots, 6; \quad j \in \mathbb{N}_0.
\]
That is for \( j \in \mathbb{N}_0, (\tau_i^j, \alpha_i) \) is a solution of (21) and (22) and \( \pm i \alpha_j \) is a pair of pure imaginary roots of (18) with \( \tau_i = \tau_i^j \). In other words, (18) has no pure imaginary roots when \( \tau_i \neq \tau_i^j \). We take \( \tau_0 = \min \{ \tau_0^j \}, \alpha_0 = \alpha_0^j \).

Denote \( \lambda(\tau_i) = \chi(\tau_i) + i \alpha_0(\tau_i) \) as the root of (18) satisfying \( \chi(\tau_i^j) = 0, \alpha_0(\tau_i^j) = \alpha_0, j \in \mathbb{N}_0 \). Let us substitute \( \lambda(\tau_i) \) into (18) and differentiate with respect to \( \tau_i \), then it yields
\[
\left( \frac{d \lambda(\tau_i)}{d \tau_i} \right)^{-1} = \frac{1}{c_1 \lambda^3 + c_2 \lambda^2 + c_3 \lambda^2 + c_4 \lambda^2 + c_5 \lambda^2} \left[ (6 \lambda^2 + 5 a_i \lambda^4 + 2 a_i \lambda^2 + 2 (a_i + b_i) \lambda + (a_i + b_i)) e^{\tau_i} + (4 c_i \lambda^3 + 3 c_i \lambda^3 + 2 c_i \lambda^3 + 2 c_i \lambda^3) \right] - \frac{\tau_i}{\lambda},
\]
thus
\[
\text{Re} \left( \frac{d \lambda(\tau_i)}{d \tau_i} \right)^{-1} \right|_{\tau_i = 0} = \frac{1}{N} \left( \beta_1^* \cos \alpha_0 \tau_i^j + \beta_2^* \sin \alpha_0 \tau_i^j + \beta^* \right),
\]
where
\[
N' = \left( c_1 \alpha_0^3 - c_2 \alpha_0^3 \right)^2 + \left( c_4 \alpha_0^3 - c_5 \alpha_0^3 \right)^2,
\]
\[
\beta_1' = \left[ 5 a_i \alpha_0^3 - 3 a_i \alpha_0^3 + (a_i + b_i) \right] \left( c_4 \alpha_0^3 - c_5 \alpha_0^3 \right) + 6 a_i \alpha_0^3 - 4 a_i \alpha_0^3 + 2 (a_i + b_i) \overline{\alpha_0} \left( c_4 \alpha_0^3 - c_5 \alpha_0^3 \right) + (a_i + b_i) \overline{\alpha_0} \left( c_4 \alpha_0^3 - c_5 \alpha_0^3 \right),
\]
\[
\beta_2' = \left[ 5 a_i \alpha_0^3 - 3 a_i \alpha_0^3 + (a_i + b_i) \right] \left( c_4 \alpha_0^3 - c_5 \alpha_0^3 \right) + 6 a_i \alpha_0^3 - 4 a_i \alpha_0^3 + 2 (a_i + b_i) \overline{\alpha_0} \left( c_4 \alpha_0^3 - c_5 \alpha_0^3 \right) + (a_i + b_i) \overline{\alpha_0} \left( c_4 \alpha_0^3 - c_5 \alpha_0^3 \right),
\]
\[
\beta' = \left( -3 c_i \alpha_0^3 + c_i \right) \left( c_4 \alpha_0^3 - c_5 \alpha_0^3 \right) + \left( -4 c_i \alpha_0^3 + 2 c_i \alpha_0^3 \right) + \left( -3 c_i \alpha_0^3 + c_i \right) \left( c_4 \alpha_0^3 - c_5 \alpha_0^3 \right).
\]

We give the assumption below (H3):
\[
\beta_1'^* \cos \alpha_0 \tau_i^j + \beta_2'^* \sin \alpha_0 \tau_i^j + \beta' \neq 0 (i = 1, 2, \ldots, 6; \ j \in \mathbb{N}_0),
\]
then
\[
\text{Re} \left( \frac{d \lambda(\tau_i)}{d \tau_i} \right)^{-1} \right|_{\tau_i = 0} \neq 0.
\]
Thus, the transversality condition is also arrived. Hence, we have the following main results for this case.

**Theorem 3:** Consider the system (5) for \( \tau = \tau_2 = 0 \), Assume that the conditions (H1), (H3) and \( r' < 0 \) hold, then the following statements are true

1. If \( \tau_i \in [0, \tau_{i_0}] \), then the solution \( x' \) of system (5) is asymptotically stable;
2. If \( \tau_i > \tau_{i_0} \), then the solution \( x' \) of system (5) is unstable;
3. The system (5) undergoes a Hopf bifurcation at the equilibrium \( x' \) when \( \tau_i = \tau_i^j, (i = 1, 2, \ldots, 6; \ j \in \mathbb{N}_0) \).

**Case (3)** \( \tau = \tau_i = 0 \)

Then the characteristic equation (7) becomes
\[
\lambda^6 + a_i \lambda^4 + (a_i + c_i) \lambda^3 + (a_i + c_i) \lambda^2 + (a_i + c_i) \lambda + (a_i + c_i) = 0.
\]

Obviously, (26) and (8) are essentially the same characteristic equation, so the discussion will not be repeated.

**IV. PROPERTY OF HOPF BIFURCATION**

In Section III, the conditions for different cases to guarantee that system (5) undergoes Hopf bifurcation and Hopf bifurcation values are obtained. In this section, we will discuss the direction and the period of bifurcating periodic solutions by employing the normal form theory and center manifold theorem introduced by Hassard et al. [30]. Next, we only take the first case \( \tau = \tau_2 = 0 \) in Section III as an example, and the discussion of other cases is similar.

Let \( u_i(t) = x_i(t) \) and \( \tau = \tau_2 + \mu, \mu \in R \). Then system (5) takes a functional differential equation in \( C([-1, 0], \mathbb{R}^6) \)
\[
\dot{u}_i(t) = L_\mu(u_i(t)) + F(\mu, u_i(t)),
\]
where \( u_i(\theta) = (u_i(t) \mid t \in [-1, 0]) \). The linear operator \( L_\mu : C([-1, 0], \mathbb{R}^6) \rightarrow \mathbb{R}^6 \) and the nonlinear operator \( F(\mu, \cdot) : C([-1, 0], \mathbb{R}^6) \rightarrow \mathbb{R}^6 \) are given, respectively, by
\[
L_\mu(\phi) = (\tau_2 + \mu) A \Phi(0) + (\tau_2 + \mu) B \Phi(-1),
\]
and
\[
F(\mu, \phi) = (\tau_2 + \mu) \cdot \begin{cases} \xi_{31} \phi_1^*(0) + \xi_{32} \phi_2^*(0) + \eta_{31} \xi_1^*(0) + \xi_{32} \phi_2^*(0) + H.O.T. \\ \xi_{31} \phi_1^*(0) + \xi_{32} \phi_2^*(0) + H.O.T. \\ \xi_{31} \phi_1^*(0) + \xi_{32} \phi_2^*(0) + H.O.T. \\ 0 \\ 0 \\ 0 \end{cases},
\]
where
\[
\xi_j = \frac{1}{6} a_i f^{(n)}(0), \eta_j = \frac{1}{6} a_i f^{(n)}(0), (i, j = 1, 2, 3),
\]
and
For another, assume that $q'(s)$ is the eigenvector of $A'(0)$ corresponding to the eigenvalue $-io\tau_i$. Similarly, we have $A'(0)q'(s) = -io\tau_i q'(s)$.

Let $D \in C$ and $q'(s) = D(1, q_1', q_2', q_3', q_4', q_5') e^{io\tau_i}$, where

$$q_1' = \frac{a_1 b e^{io\tau_i}}{1 - io}, \quad q_2' = 0, \quad q_3' = \frac{a_1 b}{\alpha + io}, \quad q_4' = \frac{a_1 a_2 b^2 e^{io\tau_i}}{(\alpha + io)^2},$$

such that $\langle q'(s), q(\theta) \rangle = 1$. Then, we have

$$\langle q'(s), q(\theta) \rangle = \bar{q}'(0)q(0) - \int_{-\infty}^{\infty} \bar{q}'(\xi)q(\xi)d\xi$$

$$= \bar{D}(1, q_1', q_2', q_3', q_4', q_5')(1, q_1, q_2, q_3, q_4, q_5)^T$$

$$- \int_{-\infty}^{\infty} \bar{D}(1, q_1', q_2', q_3', q_4', q_5') e^{-i(\xi - \theta)\tau_i} d\eta(\theta)(1, q_1, q_2, q_3, q_4, q_5)^T$$

$$= \bar{D}(1 + \bar{q}_1 q_1 + \bar{q}_2 q_2 + \bar{q}_3 q_3 + \bar{q}_4 q_4 + \bar{q}_5 q_5)$$

$$+ \int_{-\infty}^{\infty} \bar{D}(1, q_1', q_2', q_3', q_4', q_5') e^{i(\xi - \theta)\tau_i} d\eta(\theta)(1, q_1, q_2, q_3, q_4, q_5)^T$$

Thus, we can take $D$ as

$$D = \begin{pmatrix}
1 + \bar{q}_1 q_1 + \bar{q}_2 q_2 + \bar{q}_3 q_3 + \bar{q}_4 q_4 + \bar{q}_5 q_5 \\
+ \tau e^{-i\theta} \bar{b}(a_1 \bar{q}_2 + a_2 q_1 + a_3 q_5, \bar{q}_2)
\end{pmatrix}^{-1}$$

In addition, by $\langle y(s), A\phi \rangle = \langle A'y(s), \phi \rangle$ and $Aq(\theta) = -io\tau_iq(\theta)$, we can get

$$-io\tau_i q'(s) = \langle q', Aq \rangle = \langle A'q', \phi \rangle = \langle -io\tau_i q', \phi \rangle = io\tau_i \langle q', \phi \rangle$$

Hence, $\langle q'(s), \bar{q}(\theta) \rangle = 0$.

The next we compute the coordinates describing the center manifold $C_0$ at $\mu = 0$ by the same notation as Hassard et al. [30]. For the solution $u_t$ of the system (27), let
\[ z(t) = \begin{bmatrix} q^* \\ u_t \end{bmatrix}, \quad (40) \]

and

\[ W(t, \theta) = u_t(\theta) - 2 \text{Re} \{ z(t)q(t) \}. \quad (41) \]

Then on the center manifold \( C_0 \), we have

\[ W(t, \theta) = W(z(t), \overline{z}(t), \theta) = W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta) z \overline{z} + W_{02}(\theta) \frac{z^2}{2} + W_{30}(\theta) \overline{z}^3 + \cdots, \quad (42) \]

where \( z \) and \( \overline{z} \) are local coordinates for the center manifold \( C_0 \) in the direction of \( q \) and \( q^* \). Since \( \mu = 0 \), then from (34) and (36), we can get

\[ z(t) = \begin{bmatrix} q^* \\ u_t \end{bmatrix} = \begin{bmatrix} q^* \\ A(0)u_t + R(0)u_t \end{bmatrix} = \begin{bmatrix} A'q^* \\ u_t \end{bmatrix} + \begin{bmatrix} q^* \\ R(0)u_t \end{bmatrix} = \begin{bmatrix} A'q^* \\ u_t \end{bmatrix} + q^* \left( F(0, u_t) \right), \quad (43) \]

where

\[ \begin{aligned}
 g(z, \overline{z}) &= g_{20}(\theta) \frac{z^2}{2} + g_{11}(\theta) z \overline{z} + g_{02}(\theta) \frac{z^2}{2} + g_{21}(\theta) \frac{z^2}{2} \overline{z} + \cdots. \quad (45) \\
 \end{aligned} \]

From (45) and (46), we can get the result of \( g(z, \overline{z}) \) as follows

\[ g(z, \overline{z}) = \tau \mathcal{D} \left[ \frac{1}{2} \left( q_{11} z_{11}^2 + q_{12} z_{12} + q_{12} \overline{z}_{12} + q_{12} \overline{z}_{12} \right) + \frac{1}{2} \left( q_{12} \overline{z}_{12} + q_{12} \overline{z}_{12} + q_{12} \overline{z}_{12} \right) + \right] \]

Comparing the coefficients with equation (45), we can obtain

\[ g_{20} = 2 \tau_1 \mathcal{D} \left( 2 \tau_1 q_{11}^2 + 2 \tau_1 q_{12} + 2 \tau_1 q_{12} \right) + 2 \tau_2 \mathcal{D} \left( 2 \tau_2 q_{11} q_{12} + q_{11} \right); \]

\[ g_{11} = 2 \tau_1 \mathcal{D} \left( 2 \tau_1 q_{11} q_{12} + 2 \tau_1 q_{12} q_{12} \right) + 2 \tau_2 \mathcal{D} \left( 2 \tau_2 q_{11} q_{12} + q_{11} \right); \]

\[ g_{02} = 2 \tau_1 \mathcal{D} \left( 2 \tau_1 q_{11} q_{12} + 2 \tau_1 q_{12} q_{12} \right) + 2 \tau_2 \mathcal{D} \left( 2 \tau_2 q_{11} q_{12} + q_{11} \right); \]

\[ g_{21} = 2 \tau_1 \mathcal{D} \left( 2 \tau_1 q_{11} q_{12} + 2 \tau_1 q_{12} q_{12} \right) + 2 \tau_2 \mathcal{D} \left( 2 \tau_2 q_{11} q_{12} + q_{11} \right); \]

Since \( W_{20}(\theta) \) and \( W_{11}(\theta) \) in \( g_{21} \) are still unknown, we need to compute them. From (34), (40) and (41), we have...
\[ W(t, \theta) = \dot{z} - 2 \text{Re}\{z(t)q(\theta)\} \]
\[ = \left\{ \begin{array}{cl}
AW - 2 \text{Re}\{\overline{q}(0)F_0 q(\theta)\}, & \theta \in [-1, 0), \\
AW - 2 \text{Re}\{\overline{q}(0)F_0 q(\theta)\} + F_1, & \theta = 0,
\end{array} \right.
\]
\[ = AW + H(z(t), \tau(t), \theta), \]
where
\[ H(z(t), \tau(t), \theta) = \frac{\overline{q}(0)}{2} + H_{11}(\theta)z + H_{10}(\theta) + \frac{z^2}{2} + H_{20}(\theta) + \cdots. \]

On the other hand, note that on the center manifold \( C_0 \) near to the origin
\[ \dot{W}(z, \overline{z}) = W_I \dot{z} + W_E \overline{z}. \]

Substituting (36) and (44) into (49), and comparing the coefficients of the above equation with those of (49), we yield
\[ \begin{align*}
(A(0) - 2i\omega_1)W_0(\theta) &= -H_{20}(\theta), \\
A(0)W_{11}(\theta) &= -H_{11}(\theta),
\end{align*} \]
(50)
From (47), we know that for \( \theta \in [-1, 0) \)
\[ H(z(t), \tau(t), \theta) = -q(\theta)g_{20}(\overline{q}(0) - q^*(0))F_0q(\theta) \]
\[ = -g(z, \overline{z})q(\theta) - \overline{g}(z, \overline{z})q(\theta). \]
(51)
Comparing the coefficients in (51) with those in (48), we have
\[ H_{20}(\theta) = -q(\theta)g_{20}(\overline{q}(0) + q(0)), \]
\[ H_{11}(\theta) = -q(\theta)g_{11}(\overline{q}(0) + q_{11}). \]
(52)
From (50), (52) and the definition of \( A(0) \), we obtain
\[ W_{20}(\theta) = 2i\omega_1 + \overline{g}_{20}(q(0) + q_{20}). \]
(53)
Then, since \( q(\theta) = q(0)e^{i\omega_1 \theta} \), we have
\[ W_0(\theta) = \frac{i\overline{g}_{20} q(0)}{\omega_1} e^{i\omega_1 \theta} + \frac{i\overline{g}_{11} q(0)}{3 \omega_1} e^{i\omega_1 \theta} + E_2 e^{i\omega_1 \theta}. \]
(54)
where \( E_2 \) is a constant vector. Similarly,
\[ W_{11}(\theta) = -\frac{i\overline{g}_{11} q(0)}{\omega_1} e^{i\omega_1 \theta} + \frac{i\overline{g}_{11} q(0)}{3 \omega_1} e^{i\omega_1 \theta} + E_2, \]
(55)
where \( E_2 \) is also a constant vector. Then, let us compute the values of \( E_1 \) and \( E_2 \). If we take \( \theta = 0 \) at (50), we obtain
\[ \int_{-1}^{0} d\eta(0, \theta)W_{20}(\theta) = 2i\omega_1 W_{20}(0) - H_{20}(0), \]
(56)
and
\[ \int_{-1}^{0} d\eta(0, \theta)W_{11}(\theta) = -H_{11}(0). \]
(57)
From (52), we can get
\[ \begin{align*}
H_{20}(0) &= -g_{20}q(0) - \overline{g}_{20}q(0) + 2\tau, \\
H_{11}(0) &= -g_{20}q(0) - \overline{g}_{20}q(0) + 2\tau \left( \begin{array}{c}
\xi_{11} q_3^2 + \xi_{12} q_4 e^{i\omega_1 \tau} \\
\xi_{21} q_3^2 + \xi_{22} q_4 e^{i\omega_1 \tau}
\end{array} \right). \]
(58)
and
\[ \begin{align*}
H_{11}(0) &= -g_{20}q(0) - \overline{g}_{20}q(0) + 2\tau \left( \begin{array}{c}
\xi_{11} q_3^2 + \xi_{12} q_4 e^{i\omega_1 \tau} \\
\xi_{21} q_3^2 + \xi_{22} q_4 e^{i\omega_1 \tau}
\end{array} \right). \]
(59)
Substituting (54) and (58) into (56) and noticing that
\[ \left( i \omega_1 - \int_{-1}^{0} e^{i\omega_1 \theta} d\eta(0, \theta) \right)q(0) = 0, \]
and
\[ \left( -i \omega_1 - \int_{-1}^{0} e^{-i\omega_1 \theta} d\eta(0, \theta) \right)q(0) = 0. \]
We can obtain
\[ \Gamma_1' E_1 = 2b_1', \]
where
\[ \begin{align*}
\Gamma_1' &= \begin{pmatrix}
2i\omega + 1 & -a_1 b & 0 & -a_2 b \\
0 & 2i\omega + 1 & 0 & 0 & -a_1 b \\
-a_1 b & -a_2 b & 2i\omega + 1 & 0 & 0 \\
-\alpha & 0 & 0 & 2i\omega + \alpha & 0 \\
0 & 0 & 0 & 2i\omega + \alpha & 0
\end{pmatrix}, \\
\end{align*} \]
\[ b_1' = \begin{pmatrix}
\xi_{11} q_3^2 + \xi_{12} q_4 e^{i\omega_1 \tau} \\
\xi_{21} q_3^2 + \xi_{22} q_4 e^{i\omega_1 \tau}
\end{pmatrix}. \]
(62)
It follows that
\[ E_1 = \frac{2}{R_1} \left( E_{11}^{(1)}, E_{12}^{(2)}, E_{13}^{(3)}, E_{14}^{(4)}, E_{15}^{(5)}, E_{16}^{(6)} \right)^T, \]
where \( E_{1j}^{(j)} (j = 1, 2, \ldots, 6) \) is the six-order determinants obtained by substituting the \( j \)-th column of \( \Gamma_1' \) by \( b_1' \) and \( R_1 = \det(\Gamma_1') \).
Similarly, substituting (55) and (59) into (57), we can easily get
\[ \Gamma_2' E_2 = 2b_2', \]
where
\[ \begin{align*}
\Gamma_2' &= \begin{pmatrix}
-1 & a_1 b & 0 & a_2 b & 0 & 0 \\
0 & -1 & 0 & 0 & a_2 b & 0 \\
-a_1 b & a_2 b & -1 & 0 & 0 & 0 \\
\alpha & 0 & 0 & -\alpha & 0 & 0 \\
0 & 0 & 0 & -\alpha & 0 & 1 \\
\alpha & 0 & 0 & 0 & 0 & -\alpha
\end{pmatrix}, \\
b_2' &= \begin{pmatrix}
\xi_{11} q_3^2 + \xi_{12} q_4 e^{i\omega_1 \tau} \\
\xi_{21} q_3^2 + \xi_{22} q_4 e^{i\omega_1 \tau}
\end{pmatrix}. \]
It follows that

\[ E_j = \frac{2}{R_j} \left( E_j^{(1)}, E_j^{(2)}, E_j^{(3)}, E_j^{(4)}, E_j^{(5)}, E_j^{(6)} \right)^T, \]

where \( E_j^{(i)} (j = 1, 2, \ldots, 6) \) is the six-order determinants obtained by substituting the \( j \)-th column of \( F_j' \) by \( b_j' \) and \( R_j = \det \left( F_j' \right) \).

Thus, based on above derivation, we have

\[
\begin{align*}
\mu_2 &= -\frac{\Re \left[ c_1(0) \right]}{\Re \left[ \lambda'(\tau_0) \right]}, \\
\kappa_2 &= 2 \Re \left[ c_1(0) \right], \\
T_2 &= -\frac{1}{\omega_0} \left( \Im \left[ c_1(0) \right] + \mu_2 \Im \left[ \lambda'(\tau_0) \right] \right). \\
\end{align*}
\] (64)

**Theorem 4:** For any critical value \( \tau_i \), the following conclusions are true.

(i) \( \mu_i \) determines the direction of the Hopf bifurcation: if \( \mu_i > 0 \) (resp. <0), then the Hopf bifurcation is forward (resp. backward);

(ii) \( \kappa_2 \) determines the stability of the bifurcating periodic solutions on the center manifold: if \( \kappa_2 < 0 \) (resp. >0), then the bifurcating periodic solutions are orbitally asymptotically stable (resp. unstable);

(iii) \( T_2 \) determines the period of bifurcating periodic solution: if \( T_2 > 0 \) (resp. \( T_2 < 0 \)), then the period increases (resp. decreases).

**V. NUMERICAL SIMULATIONS**

In this section, we give two quantized systems and perform some simulations to support the above conclusions.

**Example 1:** A particular system is considered as follows

\[
\begin{align*}
x_1'(t) &= -x_1(t) - 0.04 \tanh(x_1(t)) + 1.8 \tanh(x_2(t - \tau)), \\
x_2'(t) &= -x_2(t) - 0.2 \tanh(x_1(t)), \\
x_3'(t) &= -x_3(t) + \tanh(x_1(t - \tau)) - 1.8 \tanh(x_2(t - \tau)), \\
x_4'(t) &= -0.1x_4(t) + 0.1x_1(t - \tau), \\
x_5'(t) &= -0.1x_5(t) + x_5(t), \\
x_6'(t) &= -0.1x_6(t) + 0.1x_2(t - \tau). \\
\end{align*}
\] (65)

Obviously, the solution of system (65) is origin stable.

\[ x' = (0, 0, 0, 0, 0, 0). \]

We consider the case for \( \tau_1 = \tau_2 = 0 \).

When \( \tau = 0 \), the system (65) is an ordinary differential equation model. By calculation, we have \( \Delta_1 = 3.3000 \), \( \Delta_2 = 10.9824 \), \( \Delta_3 = 17.4719 \), \( \Delta_4 = 4.6874 \), \( \Delta_5 = 0.1939 \), \( \Delta_6 = 8.9966 \times 10^4 \), that is all \( \Delta_i (i = 1, 2, \ldots, 6) > 0 \) and condition (H1) holds. From Theorem 1, the solution of system (65) is proved to be asymptotically stable. Its convergence graphs are shown in Fig. 2.

When \( \tau \neq 0 \), we can have \( r = -1.1878 \times 10^{-5} < 0 \) and (15) has only one positive root \( z_0 = 0.0248 \). That is Lemma 1 is true. Also \( a_0 = 0.1576 \) and \( \tau_0 = 5.3614 \) are easily obtained. Meanwhile \( \beta_1 \cos a_0 \tau_0 + \beta_2 \sin a_0 \tau_0 + \beta = 6.7360 \times 10^{-5} \neq 0 \) satisfies the condition (H2). According to Theorem 2, the solution \( x' \) of system (65) is asymptotically stable for \( \tau \in [0, \tau_0) \) and is unstable for \( \tau > \tau_0 \). These simulations are illustrated in Fig. 3 - Fig. 6.

From the quantitative analysis and computations in Section IV, one can obtain that \( \mu_2 = 5.5084 \), \( \kappa_2 = -0.1878 \) and \( T_2 = -0.0793 \). Therefore, by Theorem 4, the periodic solution is forward and orbitally asymptotically stable, and the period decreases.

**Example 2:** Let us take \( a_{11} = -2 \), \( \alpha = 1 \) and keep the other values of example 1 unchanged. Another system is considered as follows

\[
\begin{align*}
x_1'(t) &= -x_1(t) - 2 \tanh(x_1(t)) + 1.8 \tanh(x_2(t - \tau)), \\
x_2'(t) &= -x_2(t) - 0.2 \tanh(x_1(t)), \\
x_3'(t) &= -x_3(t) + \tanh(x_1(t - \tau)) - 1.8 \tanh(x_2(t - \tau)), \\
x_4'(t) &= -0.1x_4(t) + 0.1x_1(t - \tau), \\
x_5'(t) &= -x_5(t) + x_5(t), \\
x_6'(t) &= -x_6(t) + x_1(t - \tau). \\
\end{align*}
\] (66)

Also, the equilibrium \( x' \) of system (66) is the origin \( (0, 0, 0, 0, 0, 0) \). We consider the case for \( \tau_1 = \tau_2 = 0 \).

When \( \tau_1 = 0 \), (H1) holds and the solution of system (66) is asymptotically stable, its convergence trend is similar to Fig. 2.

When \( \tau_1 \neq 0 \), by calculations, we can obtain that \( r' = -2.1504 < 0 \), \( a_{10} = 1.0790 \) and \( \tau_{10} = 1.3737 \). Meanwhile, (24) has only one positive root \( z_0 = 1.1643 \) and (H3) \( \beta_1 \cos a_{10} \tau_0 + \beta_2 \sin a_{10} \tau_0 + \beta' \neq 0 \) holds. By Theorem 3, the equilibrium \( x' \) of system (66) is asymptotically stable for \( \tau_1 \in [0, \tau_{10}) \) and is unstable for \( \tau_1 > \tau_{10} \). These simulations are shown in Fig. 7 - Fig. 10.
FIGURE 2. The system (65) is asymptotically stable for $\tau = \tau_1 = \tau_2 = 0$.

FIGURE 3. The solution of system (65) for $\tau = 5.1 < \tau_0$, $x_i(i = 1, 2, \cdots, 6)$ converges to zero.

FIGURE 4. The system (65) is asymptotically stable at the equilibrium for $\tau = 5.1 \leq \tau_0$.

FIGURE 5. The solution of system (65) for $\tau = 5.5 > \tau_0$, $x_i(i = 1, 2, \cdots, 6)$ occurs period.
FIGURE 6. The system (65) is unstable at the equilibrium for $r = 5.5 > r_m$ and bifurcating periodic solution occurs.

FIGURE 7. The solution of system (66) for $r_1 = 1.3 < r_m$. $x_i (i = 1, 2, \ldots, 6)$ converges to zero.

FIGURE 8. The system (66) is stable at the equilibrium for $r_1 = 1.3 < r_m$.

FIGURE 9. The solution of system (66) for $r_1 = 1.4 > r_m$. $x_i (i = 1, 2, \ldots, 6)$ occurs period.
bifurcating parameter, we obtain that the zero solution of the system is asymptotically stable for \( \tau \in [0, \tau_0) \) and unstable for \( \tau > \tau_0 \). The system undergoes a Hopf bifurcation at origin at \( \tau = \tau_1' \). Secondly, setting \( \tau = \tau_2 = 0 \) and taking \( \tau_1 \) as a bifurcating parameter, we discuss the stability of the system at origin and existence of Hopf bifurcation. It is obtained that the zero solution of the system is asymptotically stable for \( \tau_1 \in [0, \tau_{10}) \) and unstable for \( \tau_1 > \tau_{10} \). The system undergoes a Hopf bifurcation at origin at \( \tau_1 = \tau_1' \). We find that when \( \tau = \tau_1 = 0 \), the discussion is the same as the first case, so there is no repetition. In addition, when the conditions of Hopf bifurcation theorem are satisfied for \( \tau_1 = \tau_2 = 0 \), we give the calculation formulas to determine the direction of Hopf bifurcation and the stability of bifurcating periodic solutions with the help of the central manifold theorem and normal for methods, so as to accurately characterize the existence and stability of periodic solutions. The experimental results show that the oscillation and instability caused by time delays obviously affect the stability of the network.

It can be seen that the advantage of our model is to use continuous distributed delay instead of point delay or discrete delay to more accurately describe the changes of network state and highlight the interaction mechanism between different neurons. Apparently, the treatment of distributed delays will increase the dimension of a network. However, the stability and bifurcation of high-dimensional neural network systems with delays are still limited, because the analysis of the distribution of higher-order exponential polynomial roots with multiple time delays is quite difficult. On the other hand, with the further increase of bifurcating parameters, how it affects the bifurcating solutions, that is, the global bifurcating, need to be further studied. The authors would like to thank the editors and the referees for their helpful suggestions incorporated into this paper.

VI. CONCLUSION

Discrete and distributed delays are usually inevitable in neural networks. Considering the response of discrete delays between neurons and the influence of the distributed transmission delays of neurons themselves and other neurons, this paper constructs a three-neuron network model with mixed delays involving multiple discrete and distributed delays, which is different from the existing models, to describe the transmission state among three neurons. We use two different kernel functions to deform the model equivalently, instead of considering only one kernel function as in a large number of previous literatures. According to different time delays, we discuss the stability of the system and the conditions for the generation of Hopf bifurcation from the perspective of the distribution of the roots of the characteristic equations of the linearized system at the equilibrium state of the nonlinear system. So as to the ranges of parameters for the system to remain stable and unstable are determined. Firstly, setting two distributed delays \( \tau_1 = \tau_2 = 0 \) and regarding the discrete delay \( \tau \) as a

![Figure 10](image-url)

**FIGURE 10.** The system (66) is unstable at the equilibrium for \( \tau_1 = 1.4 > \tau_{10} \) and bifurcating periodic solution occurs.

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