THE VERSAL DEFORMATION OF CYCLIC QUOTIENT SINGULARITIES

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ABSTRACT. We describe the versal deformation of two-dimensional cyclic quotient singularities in terms of equations, following Arndt, Brohme and Hamm. For the reduced components the equations are determined by certain systems of dots in a triangle. The equations of the versal deformation itself are governed by a different combinatorial structure, involving rooted trees.

One of the goals of singularity theory is to understand the versal deformations of singularities. In general the base space itself is a highly singular and complicated object. Computations for a whole class of singularities are only possible in the presence of many symmetries. A natural class of surface singularities to consider consists of the affine toric singularities. These are just the cyclic quotient singularities. Their infinitesimal deformations were determined by Riemenschneider [8]. Explicit equations for the versal deformation are the result of a series of PhD-theses. Arndt [1] gave a recipe to find equations of the base space. This was further studied by Brohme [3], who proposed explicit formulas. Their correctness was finally proved by Hamm [6]. One of the objectives of this paper is to describe these equations.

Unfortunately it is difficult to find the structure of the base space from the equations. What one can do is to study the situation for low embedding dimension e. On the basis of such computations Arndt [1] conjectured that the number of irreducible components should not exceed the Catalan number $C_{e-3} = \frac{1}{e-3} \binom{2e-3}{e-3}$. This conjecture was proved in [11] using Kollár and Shepherd-Barron’s description [7] of smoothing components as deformation spaces of certain partial resolutions. It was observed by Jan Christophersen that the components are related to special ways of writing the equations of the singularity. In terms of his continued fractions, representing zero, these equations are given in [11 §2], and in terms of subdivisions of polygons in [11 Sect. 6]. A more direct way of operating with the equations was found by Riemenschneider [2]. We use it, and the combinatorics behind it, in this paper to describe the components.

From the toric picture one finds immediately some equations, by looking at the Newton boundary in the lattice of monomials:

$$z_{\varepsilon - 1} z_{\varepsilon + 1} = z_{\varepsilon}^{2e}, \quad 2 \leq \varepsilon \leq e - 1.$$ 

These form the bottom line of a pyramid of equations $z_{\delta-1} z_{\varepsilon+1} = p_{\delta, \varepsilon}$. In computing these higher equations choices have to be made. We derive $p_{\delta, \varepsilon}$ from $p_{\delta, \varepsilon-1}$ and $p_{\delta+1, \varepsilon}$. As $z_{\delta-1} z_{\varepsilon+1} = (z_{\delta-1} z_{\varepsilon}) (z_{\delta} z_{\varepsilon+1}) / (z_{\delta} z_{\varepsilon})$, we have two natural choices for $p_{\delta, \varepsilon}$:

$$\frac{p_{\delta, \varepsilon-1} p_{\delta+1, \varepsilon}}{p_{\delta+1, \varepsilon-1}} \quad \text{or} \quad \frac{p_{\delta, \varepsilon-1} p_{\delta+1, \varepsilon}}{z_{\delta} z_{\varepsilon}}.$$

We encode the choice by putting a white or black dot at place $(\delta, \varepsilon)$ in a triangle of dots. Only for certain systems of choices we can write down (in an easy way) enough deformations to fill a whole component. We call the corresponding triangles of dots sparse coloured triangles.
We prove that the number of sparse coloured triangles of given size is the Catalan number $C_{e-3}$.

For the computation of the versal deformation one also starts from the bottom line of the pyramid of equations. Due to the presence of deformation parameters, divisions which previously were possible, now leave a remainder. We describe Arndt’s formalism to deal with these remainders. One introduces new symbols, which in fact can be considered as new variables on the deformation space. Because they are independent of the $a_\varepsilon$, one obtains that the base spaces of different cyclic quotients with the same embedding dimension are isomorphic up to multiplication by a smooth factor, provided all $a_\varepsilon$ are large enough. Also here, in writing the equations, some choices have to be made. A particular system of choices was proposed by Brohme. To be able to handle the terms in the formulas, one needs a combinatorial description of them. It turns out that the number of terms grows rapidly, faster than the Catalan numbers, and a different combinatorial structure is needed. Hamm discovered how rooted trees can be used. We will describe the computation of the versal deformation for embedding dimension 7 and then introduce Hamm’s rooted trees, and give the equations in general in terms of these trees. We also describe the main steps in the proof that one really obtains the versal deformation.

Now that the equations are known, it is time to use them. We make a start here by showing that one recovers Arndt’s equations for the versal deformation of the cones over rational normal curves (the case that all $a_\varepsilon = 2$). Furthermore, we look at the reduced base space. We start by looking at an example. We then define an ideal, using sparse coloured triangles, which has the correct reduced components. We do not touch upon the embedded components, leaving this for further research.

As one will see, notation becomes rather heavy, with many levels of indices. Although \TeX{} allows almost anything, we have tried to restrict it to a minimum. One has to admire Arndt’s thesis \cite{Arndt}, written on a typewriter. At that time, \TeX{} was available, but Jürgen had already purchased an electronic typewriter for his Diplomarbeit. He decided to write the indices separately, diminish them with a photocopier and to glue them in the manuscript.

This paper is organised as follows. After a section introducing cyclic quotients and their infinitesimal deformations, we treat the case of embedding dimension 5 in detail. In Section 3 we define sparse coloured triangles and show how to describe the reduced components with them. In Section 4 we give the equations for the total space of the versal deformation: we describe Arndt’s results, do the case of embedding dimension 6, and formulate and sketch the proof of the general result in terms of Hamm’s rooted trees. In the last Section we discuss the reduced base space.

1. Cyclic quotient singularities

Let $G_{n,q}$ be the cyclic subgroup of $Gl(2, \mathbb{C})$, generated by $\left( \begin{array}{cc} \zeta_n & 0 \\ 0 & \zeta_q \end{array} \right)$, where $\zeta_n$ is a primitive $n$-th root of unity and $q$ is coprime to $n$. The group acts on $\mathbb{C}^2$ and on the polynomial ring $\mathbb{C}[u,v]$. The quotient $\mathbb{C}^2/G_{n,q}$ has a singularity at the origin, which is called the cyclic quotient singularity $X_{n,q}$. The quotient map is a map of affine toric varieties, given by the inclusion of the standard lattice $\mathbb{Z}^2$ in the lattice $N = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{n}(1,q)$, with as cone $\sigma$ the first quadrant. The dual lattice $M$ gives exactly the invariant monomials: $\mathbb{C}[M \cap \sigma^\vee] = \mathbb{C}[u,v]^{G_{n,q}}$. Generators of this ring are

$$z_\varepsilon = u^{i_\varepsilon} v^{j_\varepsilon}, \quad \varepsilon = 1, \ldots, e.$$
where the numbers $i_ε, j_ε$ are determined by the continued fraction expansion $n/(n - q) = [a_2, \ldots, a_ε-1]$ in the following way:

\[
i_ε = 0, \quad i_{ε-1} = 1, \quad i_{ε+1} + i_{ε-1} = a_ε i_ε,
\]

\[
j_1 = 0, \quad j_2 = 1, \quad j_{ε-1} + j_{ε+1} = a_ε j_ε.
\]

We also write $X[a]$ for $X_{n,q}$.

We exclude the case of the $A_k$ singularities and assume that the embedding dimension $ε$ is at least 4. The equations for $X[a]$ can be given in quasi-determinantal format [9]:

\[
\begin{pmatrix}
z_1 & z_2 & \cdots & z_ε-2 & z_ε-1 \\
z_2 & a_ε^{-2} & \cdots & a_ε^{-ε-2} & z_ε-1 \\
z_3 & \cdots & z_ε-1 & & z_ε \\
\end{pmatrix}
\]

We recall that the generalised minors of a quasi-determinant

\[
\begin{pmatrix}
f_1 & f_2 & \cdots & f_{k-1} & f_k \\
h_{1,2} & \cdots & h_{k-1,k} \\
g_1 & g_2 & \cdots & g_{k-1} & g_k \\
\end{pmatrix}
\]

are $f_i g_j - g_i \prod_{k=2}^{j-1} h_{ε,k+1} f_j$.

By perturbing the entries in the quasi-determinantal in the most general way one obtains the equations for the Artin component, the deformations which admit simultaneous resolution. We describe these deformations more concretely, following the notation of Brohme [3] (differing slightly from [11] in that the letters $s$ and $t$ are interchanged). We first remark that in general the first and the last extra term in a quasi-determinantal can be written in the matrix: just take $g_1 h_{1,2}$ as entry in the lower left corner and $f_k h_{k-1,k}$ as entry in the upper right corner. For a cyclic quotient this gives entries $z_2^{-a_ε^2-1}$ and $z_ε^{-a_ε^{-1}}$. We deform (1)

\[
\begin{pmatrix}
z_1 & z_2 & \cdots & z_ε-2 & z_ε-1 \\
z_2 & a_ε^{-2} & \cdots & a_ε^{-ε-2} & z_ε-1 \\
z_3 & \cdots & z_ε-1 & & z_ε \\
\end{pmatrix}
\]

where

\[
Z_ε^{(a_ε-2)} = z_ε^{-a_ε^{2}} + s_ε^{(1)}z_ε^{-3} + \cdots + s_ε^{(a_ε-2)},
\]

and

\[
Z_ε^{(a_ε^{-1})} = z_ε^{a_ε^{-1}} + s_ε^{(1)}z_ε^{-2} + \cdots + s_ε^{(a_ε^{-1})}.
\]

To obtain all infinitesimal deformations we add variables $s_ε^{(a_ε^{-1})}$ and write the perturbation

\[
Z_ε^{(a_ε-2)} = z_ε^{-a_ε^{2}} + s_ε^{(1)}z_ε^{-3} + \cdots + s_ε^{(a_ε-2)} + s_ε^{(a_ε^{-1})}z_ε^{-1},
\]

which gives the coordinates $s_ε^{(a)}, 1 ≤ a ≤ a_ε - 1, \varepsilon = 2, \ldots, \varepsilon - 1$ and $t_ε, \varepsilon = 3, \ldots, \varepsilon - 2$ on the vector space $T_{X[a]}^1$. We note in particular the polynomial equations

\[
z_ε^{-1}(z_ε+1 + t_ε+1) = (z_ε + t_ε)(z_ε^{a_ε^{-1}} + s_ε^{(1)}z_ε^{-2} + \cdots + s_ε^{(a_ε^{-1})}).
\]

To avoid special cases we make this formula valid for all $ε$ by introducing variables $t_2, t_{ε-1}$ and $t_ε$, which we set to zero.
2. Embedding dimension 5

The two components of the versal deformation of the cone over the rational normal curve of degree four are related to the two different ways of writing the equations. The largest, the Artin component, is obtained by deforming the \(2 \times 4\) matrix
\[
\begin{pmatrix}
z_1 & z_2 & z_3 & z_4 \\
z_2 & z_3 & z_4 & z_5
\end{pmatrix}.
\]
The equations can also be written as \(2 \times 2\) minors of the symmetric \(3 \times 3\) matrix
\[
\begin{pmatrix}
z_1 & z_2 & z_3 & z_4 \vphantom{\frac{1}{2}} \\
z_2 & z_3 & z_4 & z_5 \\
z_3 & z_4 & z_5 & z_6
\end{pmatrix},
\]
and perturbing this matrix gives as total space the cone over the Veronese embedding of \(\mathbb{P}^2\). Riemenschneider observed that this generalises to all cyclic quotients of embedding dimension 5 \cite{R}. One can even give the equations as quasi-determinants. For the Artin component we take as described above
\[
\begin{pmatrix}
z_1 & z_2 & z_3 & z_4^3 \vphantom{\frac{1}{2}} \\
z_2 & z_3^2 & z_4 & z_5^3 \\
z_3 & z_4 & z_5 & z_6
\end{pmatrix},
\]
and for the other component
\[
\begin{pmatrix}
z_1 & z_2 & z_3^2 & z_4^3 \vphantom{\frac{1}{2}} \\
z_2 & z_3^2 & z_4 & z_5^3 \\
z_3 & z_4 & z_5 & z_6
\end{pmatrix}.
\]
The meaning of the last symbol becomes clear if we write out the equations, which we have to do in order to generalise, as for higher embedding dimension only the Artin component has such a nice determinantal description.

We write a pyramid of equations. From the \(2 \times 4\) quasi-determinantal we get
\[
z_1z_5 = z_2^{a_2-1}z_3^{a_3-2}z_4^{a_4-1}
z_1z_4 = z_2^{(a_2-1)}z_3^2z_4^{a_4-1}
z_1z_3 = z_2^{a_2}z_3z_4z_5 = z_4^{a_4}
\]
and from the symmetric quasi-determinantal
\[
z_1z_5 = z_2^{a_2-2}(z_3^{a_3-1})^2z_4^{a_4-2}
z_1z_4 = z_2^{(a_2-1)}z_3^{a_3}z_4 = z_5^{a_5}
z_1z_3 = z_2^{a_2}z_3^{a_3}z_4z_5 = z_4^{a_4}
\]
The difference between these two systems of equations lies in the top line. Observe that
\[
z_1z_5 - z_2^{a_2-2}(z_3^{a_3-1})^2z_4^{a_4-2} = (z_1z_5 - z_2^{a_2-1}z_3^{a_3-2}z_4^{a_4-1}) + z_2^{a_2-2}z_3^{a_3-2}z_4^{a_4-2}(z_2z_4 - z_3^2).
\]
To describe the deformation we introduce the following polynomials:
\[
Z_{a_e-k_e}(z_e) = s_{a_e-k_e} + s_{a_e-k_e}^{-1} + \cdots + s_{a_e-k_e}^{(a_e-k_e)}.
\]
By deforming the first set of equations we obtain the Artin component:

\[ z_1 z_5 = \tilde{z}_2^{(a_2 - 1)} \tilde{Z}_3^{(a_3 - 2)} Z_4^{(a_4 - 1)} \]

\[ z_1 z_4 = \tilde{Z}_2^{(a_2 - 1)} Z_3^{(a_3 - 2)} z_3 \quad z_2 z_5 = (z_3 + t_3) \tilde{Z}_3^{(a_3 - 2)} Z_4^{(a_4 - 1)} \]

\[ z_1 (z_3 + t_3) = z_2 \tilde{Z}_2^{(a_2 - 1)} \quad z_2 z_4 = z_3 \tilde{Z}_3^{(a_3 - 2)} (z_3 + t_3) \quad z_3 z_5 = z_4 Z_4^{(a_4 - 1)} \]

The second set of equations leads to the other component:

\[ z_1 z_5 = \tilde{Z}_2^{(a_2 - 2)} (Z_3^{(a_3 - 1)})^2 Z_4^{(a_4 - 2)} \]

\[ z_1 z_4 = z_2 \tilde{Z}_2^{(a_2 - 2)} Z_3^{(a_3 - 1)} \quad z_2 z_5 = Z_3^{(a_3 - 1)} Z_4^{(a_4 - 2)} z_4 \]

\[ z_1 z_3 = z_2 \tilde{Z}_2^{(a_2 - 2)} z_2 \quad z_2 z_4 = z_3 \tilde{Z}_3^{(a_3 - 1)} \quad z_3 z_5 = z_4 Z_4^{(a_4 - 2)} z_4 \]

Together these two components constitute the versal deformation. They fit together to the deformation

\[ z_1 z_5 = \tilde{Z}_2^{(a_2 - 1)} Z_3^{(a_3 - 2)} Z_4^{(a_4 - 1)} + s_3^{(a_3 - 1)} \tilde{Z}_2^{(a_2 - 2)} Z_3^{(a_3 - 1)} Z_4^{(a_4 - 2)} \]

\[ z_1 z_4 = \tilde{Z}_2^{(a_2 - 1)} Z_3^{(a_3 - 1)} \quad z_2 z_5 = \tilde{Z}_3^{(a_3 - 1)} Z_4^{(a_4 - 1)} \]

\[ z_1 (z_3 + t_3) = z_2 \tilde{Z}_2^{(a_2 - 1)} \quad z_2 z_4 = Z_3^{(a_3 - 1)} (z_3 + t_3) \quad z_3 z_5 = z_4 Z_4^{(a_4 - 1)} \]

over the base space defined by the equations

\[ s_2^{(a_2 - 1)} s_3^{(a_3 - 1)} = t_3 s_3^{(a_3 - 1)} = s_3^{(a_3 - 1)} s_4^{(a_4 - 1)} = 0 \]

Here the factor \( \tilde{Z}_3^{(a_3 - 1)} \) is defined by the equation \( z_3 \tilde{Z}_3^{(a_3 - 1)} = (z_3 + t_3) Z_3^{(a_3 - 1)} \), which is possible because of the equation \( t_3 s_3^{(a_3 - 2)} = 0 \).

Remark. The equations for the versal deformation restrict (by setting the deformation variables to zero) to the equations of the singularity in the preferred form for the Artin component. A choice has to be made, and this one is sensible as the Artin component is the only component, which exists for all cyclic quotients. Observe also that the right hand side of the top equation in (3) is no longer a product. For the study of the non-Artin component, e.g., to determine adjacencies, the adapted equations are much better suited. We have therefore in general two tasks, to describe equations suited for each reduced component separately, and to give equations for the total versal deformation.

3. Equations for Components

The reduced components of the versal deformation are related to ways of writing the equations of the singularity, as shown in [4] and [11]. Here we give a description which first appeared in [2].

We have to write the equations \( z_{\delta-1} z_{\varepsilon+1} = p_{\delta, \varepsilon} \). Motivated by the case of embedding dimension 5 we want the right hand side of the equations to be of the form \( p_{\delta, \varepsilon} = \prod_\beta (z_\beta^{a_\beta - k_\beta})^{a_\beta} \). Here the \( k_\beta \) and \( a_\beta \) depend on \( \varepsilon - \delta \), but the formula should be in some sense universal, it should hold for all \( a_\beta \) large enough (for \( a_\beta - k_\beta \) has to be non-negative). The toric weight vectors \( w_\beta \in \mathbb{Z}^2 \) of the variables \( z_\beta \) should therefore satisfy the equations

\[ w_\delta + w_\varepsilon = \sum a_\beta (a_\beta - k_\beta) w_\beta , \]

the same equations as encountered by Jan Christophersen (see the Introduction of [4]).
We construct a pyramid of equations $z_{\delta-1}z_{\varepsilon+1} = p_{\delta,\varepsilon}$, where $2 \leq \delta \leq \varepsilon \leq e - 1$. We start from the base line containing the $z_{\varepsilon-1}z_{\varepsilon+1} = z_{\varepsilon}^{n_{\varepsilon}}$, and construct the next lines inductively. We have to make choices, which we encode in a subset $B(\triangle)$ of the set of pairs $(\delta, \varepsilon)$ with $1 < \delta < \varepsilon < e$. As $z_{\delta-1}z_{\varepsilon+1} = (z_{\delta-1}z_{\varepsilon})(z_{\delta}z_{\varepsilon+1})/(z_{\delta}z_{\varepsilon})$, we have two natural choices for $p_{\delta,\varepsilon}$: we take

$$P_{\delta\varepsilon} = \begin{cases} p_{\delta-1,\varepsilon+1}, & \text{if } (\delta, \varepsilon) \notin B(\triangle), \\ \frac{p_{\delta-1,\varepsilon+1}}{z_{\delta}z_{\varepsilon}} p_{\delta+1,\varepsilon-1}, & \text{if } (\delta, \varepsilon) \in B(\triangle). \end{cases}$$

We depict our set by a triangle $\triangle$ of the type

```
  ●
  ○
  ●
  ○
  ●
```

The dots correspond to equations $z_{\delta-1}z_{\varepsilon+1} = p_{\delta,\varepsilon}$ above the base line in the pyramid of equations. We colour a dot in $\triangle$ black if the corresponding point $(\delta, \varepsilon)$ is an element of $B(\triangle)$.

As the second line of equations always reads $p_{\varepsilon-1,\varepsilon} = z_{\varepsilon-1}^{n_{\varepsilon}-1}z_{\varepsilon}^{n_{\varepsilon}-1}$, $2 < \varepsilon \leq e - 1$, the lowest line of the triangle is coloured black. To characterise the coloured triangles, which give good equations, it suffices to consider only triangles $\triangle$, obtained by deleting this black line:

```
  ●
  ○
  ●
```

The original triangle will be referred to as extended triangle. We introduce some more terminology. A (broken) line $l_{\varepsilon}$, in both the triangle $\triangle$ and the extended triangle $\triangle$, is a line connecting all dots which have $\varepsilon$ as one of the coordinates:

```
  l_6
      ●
      ○
      ●
      ○
      ●
```

If necessary we specify a triangle by the coordinates of its vertex $(\delta, \varepsilon)$, as $\triangle_{\delta,\varepsilon}$. The height of a triangle $\triangle_{\delta,\varepsilon}$ is $\varepsilon - \delta - 1$. This is the number of horizontal lines and also the number of dots on the base line. A dot $(\alpha, \beta)$ in a triangle $\triangle_{\delta,\varepsilon}$ determines a sub-triangle $\triangle_{\alpha,\beta}$, standing on the same base line, of height $\beta - \alpha - 1$.

The crucial property for getting good equations is given in the following definition.

**Definition 3.1.** A coloured triangle $\triangle_{\delta,\varepsilon}$ is sparse, if for it and for every sub-triangle $\triangle_{\alpha,\beta}$ the number of black dots is at most the height of the triangle with equality if and only if its vertex is black.

Note that the example triangle above is sparse, whereas the following triangle is not sparse.

```
  ○
  ○
  ●
  ○
```
The relation \((\alpha, \gamma) \preceq (\beta, \delta)\), if \(\alpha \geq \beta\) and \(\gamma \leq \delta\) is a partial ordering. It means that \((\alpha, \gamma)\) lies (as black or white dot) in the triangle \(\triangle_{\beta, \delta}\).

**Lemma 3.2.** If two black dots lie in the region on or above a given line \(l_e\), then both of them lie on \(l_e\) or they are comparable in the partial ordering \(\preceq\).

**Proof.** Suppose on the contrary that the black dots \((\alpha, \gamma)\) and \((\beta, \delta)\) on or above \(l_e\) are not comparable in the partial ordering and that at least one of them lies strictly above \(l_e\). We may assume that \(\gamma \leq \delta\). This implies that \(\alpha < \beta\). The assumption that \((\alpha, \gamma)\) lies on or above \(l_e\) means that \(\alpha \leq \varepsilon \leq \gamma\) and likewise \(\beta \leq \varepsilon \leq \delta\). Furthermore, in one of these one has strict inequalities. Therefore \(\beta < \gamma\). The triangle \(\triangle_{\alpha, \gamma}\) contains exactly \(\gamma - \alpha - 1\) black dots, the triangle \(\triangle_{\beta, \delta}\) contains exactly \(\delta - \beta - 1\) black dots and their intersection is the triangle \(\triangle_{\beta, \gamma}\), which contains at most \(\gamma - \beta - 1\) dots. So the triangle \(\triangle_{\alpha, \delta}\), which has as vertex the supremum \((\alpha, \delta)\) of \((\alpha, \gamma)\) and \((\beta, \delta)\) in the partial ordering, contains at least \((\gamma - \alpha - 1) + (\delta - \beta - 1) - (\gamma - \beta - 1) = \delta - \alpha - 1\) black dots other than its vertex, contradicting sparsity. \(\square\)

**Theorem 3.3.** The number of sparse coloured triangles of height \(e - 4\) is the Catalan number \(C_{e-3} = \frac{1}{e-2} \binom{2(e-3)}{e-3}\).

**Proof.** Consider a sparse triangle \(\triangle_{2, e-1}\) and let \((2, \beta)\) be the highest black dot on the line \(l_2\). There are no black dots above the line \(l_2\), for according to Lemma 3.2 the dot \((2, \beta)\) should lie in the triangle of such a dot, implying that it lies on \(l_2\), but \((2, \beta)\) is the highest black dot on that line. The triangle \(\triangle_{\beta, e-1}\) can be an arbitrary sparse triangle. The sparse triangle \(\triangle_{3, \beta}\) determines the colour of the remaining dots on the line \(l_2\): proceeding inductively downwards, the dot \((2, \gamma)\) has to be black if and only if there are exactly \(\gamma - \beta\) black dots in the triangle \(\triangle_{2, \beta}\), not lying in \(\triangle_{2, \gamma}\); as \((2, \beta)\) is black, there are at least \(\gamma - \beta\) black dots in this complement.

This shows that the number \(C_n\), \(n = e - 3\), of sparse coloured triangles of height \(n - 1\) satisfies Segner’s recursion formula for the Catalan numbers

\[ C_{n+1} = C_0C_n + C_1C_{n-1} + \cdots + C_{n-1}C_1 + C_nC_0. \]

For more information on the Catalan numbers, see [10]. \(\square\)

**Remark.** The Catalan number \(C_{e-3}\) also counts the number of subdivisions of an \((e - 1)\)-gon in triangles. An explicit bijection is as follows. Mark, as in [11], a distinguished vertex and number the remaining ones from 2 to \(e - 1\). If the vertices \(\delta\) and \(\varepsilon\) are joined by a diagonal, then we colour the dot \((\delta, \varepsilon)\) black. Conversely, given a triangle \(\triangle_{2, e-1}\) we join the vertices \(\delta\) and \(\varepsilon\) by a diagonal, if the dot \((\delta, \varepsilon)\) is black. By Lemma 3.2 these diagonals do not intersect. We complete the subdivision with diagonals through the distinguished vertex. Sometimes it is easier to use subdivisions, but we will derive all facts we need directly from the combinatorics of sparse triangles.

To describe the equations we need the numbers \([k_2, \ldots, k_{e-1}]\) and \((\alpha_2, \ldots, \alpha_{e-1})\). These are indeed the continued fractions \([k_2, \ldots, k_{e-1}]\) representing zero [4] and the corresponding numbers satisfying \(\alpha_{e-1} + \alpha_{e+1} = k_e\). To define these numbers out of a triangle we inductively give weights to black dots.

**Definition 3.4.** Let \(\triangle_{2, e-1}\) be a sparse triangle. The weight \(w_{\delta, \varepsilon}\) of a black dot \((\delta, \varepsilon)\) is the sum of the weights of the dots lying in the sector above it, increased by one: \(w_{\delta, \varepsilon} = \)
$1 + \sum_{\alpha < \delta} w_{\alpha, \beta}$. For $2 \leq \varepsilon \leq e - 1$ we define $\alpha_\varepsilon$ as the sum of the weights of black dots above the line $l_\varepsilon$, increased by one:

$$\alpha_\varepsilon = 1 + \sum_{\alpha < \delta \leq \beta > \varepsilon} w_{\alpha, \beta}.$$ 

In particular, $\alpha_\varepsilon = 1$ if there are no black points above the line $l_\varepsilon$, so $\alpha_2 = \alpha_{e-1} = 1$. We set $k_\varepsilon$ to be the number of black dots on the line $l_\varepsilon$ in the extended triangle if $\alpha_\varepsilon = 1$ and this number minus 1 otherwise.

**Example.**

![Diagram](image)

**Remarks.** 1. We leave it as an exercise to prove that the so defined numbers $\alpha_\varepsilon$ and $k_\varepsilon$ satisfy $\alpha_{\varepsilon-1} + \alpha_{\varepsilon+1} = k_\varepsilon \alpha_\varepsilon$.

2. We note the following alternative way to compute the $\alpha_\varepsilon$ [3, Bemerkung 1.7]. As mentioned, $\alpha_\varepsilon = 1$ if there are no black dots above the line $l_\varepsilon$. For every other index $\varepsilon$ there exist unique $\beta < \varepsilon < \gamma$, such that the intersection of the lines $l_\beta$, $l_\varepsilon$ and $l_\gamma$ (in the extended triangle) consists only of black dots. Then $\alpha_\varepsilon = \alpha_\beta + \alpha_\gamma$. In fact, this is the way the numbers are determined from a subdivision of a polygon. We sketch a proof. Let $(\beta, \varepsilon)$ be the highest black dot on the left half-line of $l_\varepsilon$, and $(\varepsilon, \gamma)$ the highest black dot on the right half-line. Let $(\alpha, \delta)$ be a black dot above the line $l_\varepsilon$, such that $\triangle_{\alpha, \delta}$ contains no other black dots above $l_\varepsilon$. Then $(\beta, \gamma) \leq (\alpha, \delta)$, and if $(\beta, \gamma) \neq (\alpha, \delta)$, then $\triangle_{\alpha, \delta}$ does not contain enough dots. So if $\alpha_\varepsilon \neq 1$, then $(\beta, \gamma)$ is black. Black dots above the lines $l_\beta$, $l_\varepsilon$ and $l_\gamma$ can only lie in the sector with $(\beta, \gamma)$ as lowest point. One now computes $\alpha_\varepsilon = \alpha_\beta + \alpha_\gamma$.

We can now describe the equations belonging to a sparse triangle $\triangle_{2, e-1}$. To avoid cumbersome notation we only give the formula for the highest equation $z_1 z_\varepsilon = p_{2, e-1}$, but this implies by obvious changes the formula for each equation $z_\delta z_{\varepsilon+1} = p_{\delta, e}$, as such an equation is determined by its own sparse triangle $\triangle_{\delta, \varepsilon}$, giving its own $\alpha$ and $k$ values. We will specify in the text from which triangle a specific $\alpha_\varepsilon$ or $k_\varepsilon$ is computed, but we do not include this information in the notation.

**Proposition 3.5.** Let the triangle $\triangle_{2, e-1}$ determine the numbers $\alpha_\varepsilon$, $k_\varepsilon$, according to Definition [3.4]. Forming the equations by taking $p_{\delta, e} = \frac{p_{\delta, e-1} p_{e+1, e}}{z_{2, \varepsilon}}$ if the dot $(\delta, \varepsilon)$ is black and $p_{\delta, e} = \frac{p_{\delta, e-1} p_{e+1, e}}{p_{e+1, e-1}}$ otherwise, leads to the highest equation

$$z_1 z_\varepsilon = \prod_{\beta=2}^{e-1} \left(z_\beta^{a_\beta-k_\beta}\right)^{\alpha_\beta}.$$

**Proof.** We fix an index $\varepsilon$ and look at the $z_\varepsilon$-factor in $p_{2, e-1}$. The proof proceeds by induction on $\varepsilon$, i.e., on the height of the triangle. The base of the induction is formed by the equations $z_{\varepsilon-1} z_{\varepsilon+1} = z_{\varepsilon}^{a_\varepsilon}$, which correspond to empty extended triangles, with $\alpha_\varepsilon = 1$ and $k_\varepsilon = 0$.  


Suppose that the formula is proved for all \( p_{a,z} \) with \( \varepsilon - \delta < e - 3 \). There are two cases, depending on the colour of the dot (2, e - 1).

**Case 1:** the dot (2, e - 1) is white. Then we have to compare \( p_{2,e-1}, p_{2,e-2}, p_{3,e-1} \) and \( p_{3,e-2} \) and the values of \( \alpha_{e} \) and \( k_{e} \) computed from the corresponding triangles. There are several sub-cases. In the first two we assume that \( \varepsilon \neq 2, e - 1 \).

1.a: Suppose both (2, \( \varepsilon \)) and (\( \varepsilon, e - 1 \)) are black. A black dot above the line \( l_{e} \) should contain both these points in its triangle, so it can only be (2, e - 1), which however is assumed to be uncoloured. Therefore \( \alpha_{e} = 1 \) in all the relevant triangles. We have that there are \( k_{e} \) dots on the line \( l_{e} \) in the extended triangle \( \Delta_{a,e-1} \), \( k_{e} - 1 \) in \( \Delta_{2,e-2} \) and \( \Delta_{3,e-1} \), and \( k_{e} - 2 \) in \( \Delta_{3,e-2} \).

So indeed the \( z_{e} \) factor in \( p_{2,e-1} \) is equal to \( z_{e}^{\alpha_{e}-k_{e}} \cdot z_{e}^{\alpha_{e}-k_{e}+1} / z_{e}^{\alpha_{e}-k_{e}+2} = z_{e}^{\alpha_{e}-k_{e}} \).

1.b: Otherwise the segments (2, \( \varepsilon \)) - (2, e - 1) and (\( \varepsilon, e - 1 \)) - (2, e - 1) cannot both contain black dots. Suppose the first segment, on \( l_{2} \), is empty (in particular (2, \( \varepsilon \)) is white). Then the number of black dots on the line \( l_{e} \) is equal in both \( \Delta_{2,e-1} \) and \( \Delta_{3,e-1} \) being equal to \( k_{e} \) or \( k_{e} + 1 \) depending on the value of \( \alpha_{e} \), and one computes also the same value for \( \alpha_{e} \). Also \( \Delta_{3,e-2} \) and \( \Delta_{3,e-2} \) yield the same values \( \alpha_{e}' \) and \( k_{e}' \), so we get \( \left( z_{e}^{\alpha_{e}-k_{e}} \right)^{\alpha_{e}} \cdot \left( z_{e}^{\alpha_{e}-k_{e}+1} / z_{e}^{\alpha_{e}-k_{e}+2} \right)^{\alpha_{e}} = \left( z_{e}^{\alpha_{e}-k_{e}} / z_{e}^{\alpha_{e}} \right)^{\alpha_{e}} \).

1.c: Suppose that \( \varepsilon = 2 \) or \( \varepsilon = e - 1 \). Consider the first case. The monomial \( z_{2} \) does not occur in \( p_{3,e-1} \) and \( p_{3,e-2} \). Always \( \alpha_{2} = 1 \) and the number of dots on \( l_{2} \) is the same in both relevant triangles.

**Case 2:** the dot (2, e - 1) is black. We have to compare the values of \( \alpha_{e} \) and \( k_{e} \) in \( p_{2,e-1}, p_{2,e-2} \) and \( p_{3,e-1} \). Again we consider \( \varepsilon = 2, e - 1 \) separately.

2.a: Suppose both (2, \( \varepsilon \)) and (\( \varepsilon, e - 1 \)) are black. Then (2, e - 1) is the only black dot above the line \( l_{e} \), which makes \( \alpha_{e} = 2 \), whereas \( \alpha_{e} = 1 \) in both smaller triangles. The number of black dots on the line \( l_{e} \) is \( k_{e} + 1 \) in \( \Delta_{2,e-1} \), \( k_{e} \) in \( \Delta_{2,e-2} \), and \( k_{e} \) in \( \Delta_{3,e-1} \). So the \( z_{e} \) factor in \( p_{2,e-1} \) is equal to \( z_{e}^{\alpha_{e}-k_{e}} \cdot z_{e}^{\alpha_{e}-k_{e}+1} = \left( z_{e}^{\alpha_{e}-k_{e}} \right)^{2} \).

2.b: Suppose (2, \( \varepsilon \)) is black and (\( \varepsilon, e - 1 \)) not. Then all black points above \( l_{e} \) lie on \( l_{2} \), all having weight 1, and there is at least one of them between (2, \( \varepsilon \)) and (2, e - 1), as \( \varepsilon, e - 1 \) is not black. There are \( k_{e} + 1 \) points on \( l_{e} \) in \( \Delta_{2,e-1} \) and \( \Delta_{2,e-2} \) and \( k_{e} \) in \( \Delta_{3,e-1} \). The last triangle gives the value \( \alpha_{e} = 1 \) in \( p_{3,e-1} \), whereas \( \alpha_{e} > 1 \) in the first two, and the value from \( \Delta_{2,e-1} \) is one more than that from \( \Delta_{2,e-2} \). So the \( z_{e} \) factor in \( p_{2,e-1} \) is equal to \( \left( z_{e}^{\alpha_{e}-k_{e}} \right)^{2} \cdot z_{e}^{\alpha_{e}-k_{e}+1} = \left( z_{e}^{\alpha_{e}-k_{e}} \right)^{3} \).

2.c: Suppose both (2, \( \varepsilon \)) and (\( \varepsilon, e - 1 \)) are white. The number of dots on \( l_{e} \) is the same in all three relevant triangles. In the largest one \( \alpha_{e} > 1 \), so to get the same value for \( k_{e} \) we need that \( \alpha_{e} > 1 \) also in both other triangles. This means that the triangle \( \Delta_{3,e-2} \) has to have a dot above the line \( l_{e} \). Of the segments of \( l_{2} \) and \( l_{e-1} \) above \( l_{e} \) only one can contain black dots besides the vertex. Suppose (\( j, e - 1 \)) is the lowest dot of the segment on \( l_{e-1} \). The number of dots in \( \Delta_{j,e-1} \) on or under \( l_{e} \) is at most \( (e - 1 - \varepsilon - 2) + (\varepsilon - j - 1) = (e - 1 - j - 3) \). As the triangle contains exactly \( e - 1 - j - 2 \) dots other than the vertex, there has to be a dot above \( l_{e} \), which does not lie on \( l_{e-1} \) due to the choice of (\( j, e - 1 \)). To compute \( \alpha_{e} \) we have to look at the weights. With the convention that points above a triangle have weight 0, the inductive formula holds for points in a sub-triangle with summation over all points in the sector of the big triangle. We show by induction that for all points except the vertex (2, e - 1) the weight \( w_{i,j} \) computed from the big triangle, equals the sum of the weights \( w_{i,j}' \) from \( \Delta_{2,e-2} \) and \( w_{i,j}'' \) from \( \Delta_{3,e-1} \). Indeed, \( w_{i,j} = 1 + \sum w_{k,l} = 1 + \sum (k,l) \neq (2,e-1) w_{k,l} = 1 + \sum w_{k,l}' + 1 + \sum w_{k,l}'' = w_{i,j}' + w_{i,j}'' \).
The same computation shows that also the values of $\alpha_\varepsilon$ add. So indeed the $z_\varepsilon$ factor in $p_{2,e-1}$ is the product of those in $p_{2,e-2}$ and $p_{3,e-1}$.

2.d: If $\varepsilon = 2$, then $\alpha_2 = 1$ and the number of points on $l_2$ in $\triangle_{2,e-1}$ is $k_2$, whereas it is $k_2 - 1$ in $\triangle_{2,e-2}$. The $z_2$ factor in $p_{2,e-1}$ is $z_2^{a_2-k_2+1}/z_2 = z_2^{a_2-k_2}$. The case $\varepsilon = e - 1$ is similar. □

As in the case of embedding dimension 5 we can now deform. We perturb each term $z_\varepsilon^{a_\varepsilon-k_\varepsilon}$ to

$$Z_\varepsilon^{(a_\varepsilon-k_\varepsilon)} = z_\varepsilon^{a_\varepsilon-k_\varepsilon} + s_\varepsilon^{(1)} z_\varepsilon^{a_\varepsilon-k_\varepsilon-1} + \cdots + s_\varepsilon^{(a_\varepsilon-k_\varepsilon)}.$$ 

Here we write $s_\varepsilon^{(j)}$, as these variables are not quite the same as the coordinates $s_\varepsilon^{(j)}$ on $T_1$, specified by the equations [2]. The relation is the following: if $\alpha_\varepsilon > 1$, we set $t_\varepsilon = 0$, and $s_\varepsilon^{(j)} = s_\varepsilon^{(j)}$, but if $\alpha_\varepsilon = 1$ one has

$$(z_\varepsilon + t_\varepsilon)^{\varepsilon-1}(z_\varepsilon^{a_\varepsilon-k_\varepsilon} + s_\varepsilon^{(1)} z_\varepsilon^{a_\varepsilon-k_\varepsilon-1} + \cdots + s_\varepsilon^{(a_\varepsilon-k_\varepsilon)}) z_\varepsilon^{\alpha_\varepsilon+1} = (z_\varepsilon + t_\varepsilon)(z_\varepsilon^{a_\varepsilon-1} + s_\varepsilon^{(1)} z_\varepsilon^{a_\varepsilon-2} + \cdots + s_\varepsilon^{(a_\varepsilon-1)}).$$

This formula makes sense, as $\alpha_\varepsilon k_\varepsilon = \alpha_\varepsilon-1 + \alpha_{\varepsilon+1}$, so for $\alpha_\varepsilon = 1$ one has $k_\varepsilon = \alpha_\varepsilon-1 + \alpha_{\varepsilon+1}$.

**Proposition 3.6.** Let $\triangle_{2,e-1}$ be a sparse triangle. Put $t_\varepsilon = 0$, if $\alpha_\varepsilon > 1$. Now form the equations $z_\delta-1(z_\varepsilon+1 + t_\varepsilon+1) = P_{\delta,\varepsilon}$, starting from

$$P_{\varepsilon,\varepsilon} = (z_\varepsilon + t_\varepsilon)^{\varepsilon-1}(z_\varepsilon^{a_\varepsilon-k_\varepsilon} + s_\varepsilon^{(1)} z_\varepsilon^{a_\varepsilon-k_\varepsilon-1} + \cdots + s_\varepsilon^{(a_\varepsilon-k_\varepsilon)}) z_\varepsilon^{\alpha_\varepsilon+1},$$

if $\alpha_\varepsilon = 1$ and

$$P_{\varepsilon,\varepsilon} = (z_\varepsilon^{a_\varepsilon-k_\varepsilon} + s_\varepsilon^{(1)} z_\varepsilon^{a_\varepsilon-k_\varepsilon-1} + \cdots + s_\varepsilon^{(a_\varepsilon-k_\varepsilon)}) z_\varepsilon^{k_\varepsilon}$$

otherwise. Take $P_{\delta,\varepsilon} = P_{\delta-1,\varepsilon} P_{\delta+1,\varepsilon-1}$ if the dot $(\delta,\varepsilon)$ is black and $P_{\delta,\varepsilon} = P_{\delta-1,\varepsilon} P_{\delta+1,\varepsilon-1}$ otherwise. This gives the highest equation

$$z_1 z_\varepsilon = \prod_{\beta=2}^{e-1} (Z_\beta^{(a_\beta-k_\beta)})^{\alpha_\beta} \cdot$$

These equations define a flat deformation of the cyclic quotient singularity $X[a]$.

The flatness is proved explicitly in [3] 2.1.2 and [3] 2.2. It is of course a consequence of the inductive definition of the polynomials $P_{\delta,\varepsilon}$.

In fact, one gets in this way exactly all reduced components of the versal deformation. This was proved in [1] using Kollár and Shepherd-Barron’s description [7] of smoothing components as deformation spaces of certain partial resolutions. A more elementary (but not easier) approach would be to use the equations for the base space of the versal deformation, which we describe in the next section.

4. Versal deformation

In this section we derive the equations for the versal deformation. We have to write the pyramid of equations, as in the example of embedding dimension five. The base line consist of the equations [2]. These equations are lacking in symmetry: when introducing the deformation variables $t_\varepsilon$, say in the quasi-determinantal, there is a choice of writing them in the upper or the lower row. Arndt [1] formally symmetrises by setting $t_\varepsilon = x_\varepsilon + t_\varepsilon$. We go one step further and replace $t_\varepsilon$ by two deformation variables. This makes that our deformation is
versal, but no longer miniversal. Furthermore, there is no \( t_2 \) and \( t_{e-1} \), but in order to avoid special cases, we allow the index \( \varepsilon \) in \( t_\varepsilon \) to take the values 2 and \( e-1 \).

We start from the equations \( z_\varepsilon \cdot z_{\varepsilon+1} = a_\varepsilon \), which we deform into
\[
(4) \quad (z_\varepsilon - l_\varepsilon - 1)(z_{\varepsilon+1} - r_\varepsilon + 1) = z_\varepsilon^{a_\varepsilon} + \sigma_\varepsilon^{(1)} z_\varepsilon^{a_\varepsilon-1} + \cdots + \sigma_\varepsilon^{(a_\varepsilon)}.
\]
We abbreviate \( z_\varepsilon - r_\varepsilon = R_\varepsilon \) and \( z_\varepsilon - l_\varepsilon = L_\varepsilon \). The minus sign is introduced to simplify the conditions for divisibility by \( R_\varepsilon \) or \( L_\varepsilon \), which will be the main ingredient in our description of the base space. We write the equation (4) shortly as
\[
L_{\varepsilon-1} R_{\varepsilon+1} = Z_{\varepsilon}^{(00)}.
\]

As written, we do not even get an infinitesimal deformation: one needs \( \sigma_\varepsilon^{(a_\varepsilon)} \equiv 0 \) modulo the square of the maximal ideal of the parameter space. Comparison with equation (2) shows that \( Z_{\varepsilon}^{(00)}(z_\varepsilon) \) (we use this notation to emphasise that we consider \( Z_{\varepsilon}^{(00)} \) as polynomial in \( z_\varepsilon \)) has to be divisible by \( R_\varepsilon \), i.e, \( Z_{\varepsilon}^{(00)}(r_\varepsilon) = 0 \). This gives an equation with non-vanishing linear part. We could as well require divisibility by \( L_\varepsilon \). This gives an equation \( Z_{\varepsilon}^{(00)}(l_\varepsilon) = 0 \) with the same linear part.

In fact, we shall assume both conditions, \( Z_{\varepsilon}^{(00)}(r_\varepsilon) = 0 \) and \( Z_{\varepsilon}^{(00)}(l_\varepsilon) = 0 \). This yields then one equation with non-vanishing linear part, and one equation, not involving \( \sigma_\varepsilon^{(a_\varepsilon)} \) at all, which factorises:
\[
Z_{\varepsilon}^{(00)}(l_\varepsilon) - Z_{\varepsilon}^{(00)}(r_\varepsilon) = (l_\varepsilon - r_\varepsilon) \sigma_\varepsilon^{(11)} = 0.
\]
This formula defines \( \sigma_\varepsilon^{(11)} \), which should not be confused with one of the variables in equation (4). Those variables do not play a prominent role in the computations to come. They are important for the monodromy covering of the versal deformation, as noted by Riemenschneider and studied by Brohme [3]. There is a large covering, which induces the monodromy covering of each reduced component, obtained by considering the \( \sigma_\varepsilon^{(i)} \) as elementary symmetric functions in new variables. For details we refer to [3].

We have to give the other equations. They will have the form
\[
L_{\delta-1} R_{\varepsilon+1} = P_{\delta,\varepsilon}.
\]

The polynomials \( P_{\delta,\varepsilon} \) will be well defined modulo the ideal \( J \), generated by the equations of the base space. To describe them we perform division with remainder.

\textbf{Definition 4.1.} We inductively define polynomials \( Z_{\varepsilon}^{(ij)} \) in the variable \( z_\varepsilon \), starting from
\[
Z_{\varepsilon}^{(00)} = z_\varepsilon^{a_\varepsilon} + \sigma_\varepsilon^{(1)} z_\varepsilon^{a_\varepsilon-1} + \cdots + \sigma_\varepsilon^{(a_\varepsilon)},
\]
by division by \( L_\varepsilon \)
\[
Z_{\varepsilon}^{(ij)} = L_\varepsilon Z_{\varepsilon}^{(i+1,j)} + \sigma_\varepsilon^{(i+1,j)},
\]
and by \( R_\varepsilon \)
\[
Z_{\varepsilon}^{(ij)} = Z_{\varepsilon}^{(i,j+1)} R_\varepsilon + \sigma_\varepsilon^{(i,j+1)}.
\]

Note that \( \sigma_\varepsilon^{(i+1,j)} = Z_{\varepsilon}^{(ij)}(l_\varepsilon) \), and \( \sigma_\varepsilon^{(i,j+1)} = Z_{\varepsilon}^{(ij)}(r_\varepsilon) \). From the equations (5) or (6) we obtain by substituting that
\[
\sigma_\varepsilon^{(i+1,j)} - \sigma_\varepsilon^{(i,j+1)} = (l_\varepsilon - r_\varepsilon) \sigma_\varepsilon^{(i+1,j+1)}.
\]

The condition \( Z_{\varepsilon}^{(00)}(l_\varepsilon) = Z_{\varepsilon}^{(00)}(r_\varepsilon) = 0 \) translates into \( \sigma_\varepsilon^{(10)} = \sigma_\varepsilon^{(01)} = 0 \) and we can write
\[
Z_{\varepsilon}^{(00)} = L_\varepsilon Z_{\varepsilon}^{(10)} = Z_{\varepsilon}^{(01)} R_\varepsilon.
\]
The next line in the pyramid of equations can now be computed:

\[ L_{\varepsilon-2}R_{\varepsilon+1} = \frac{(L_{\varepsilon-2}R_{\varepsilon})(L_{\varepsilon-1}R_{\varepsilon+1})}{L_{\varepsilon-1}R_{\varepsilon}} = \frac{Z_{\varepsilon-1}^{(00)}Z_{\varepsilon}^{(00)}}{L_{\varepsilon-1}R_{\varepsilon}} = Z_{\varepsilon-1}^{(10)}Z_{\varepsilon}^{(01)}. \]

For the higher lines we do not quite proceed as before, when describing the components. Computing with \( L_{\delta-1}R_{\varepsilon+1} = (L_{\delta-1}R_{\varepsilon})(L_{\delta}R_{\varepsilon+1})/(L_{\delta}R_{\varepsilon}) \) would be too complicated. Instead we take the asymmetric approach

\[ Z_{\varepsilon} \]

and we now have to use the division with remainder \( Z_{\varepsilon-1}^{(01)} = L_{\varepsilon-1}Z_{\varepsilon}^{(11)} + \sigma_{\varepsilon-1}^{(11)} \) of equation \( (5) \) to get

\[ L_{\varepsilon-3}R_{\varepsilon+1} = \frac{Z_{\varepsilon-2}^{(10)}Z_{\varepsilon-1}^{(11)}Z_{\varepsilon}^{(01)}}{L_{\varepsilon-1}} + \frac{Z_{\varepsilon-2}^{2\sigma_{\varepsilon-1}^{(11)}Z_{\varepsilon}^{(01)}}}{L_{\varepsilon-1}}. \]

This is not the final formula, as we can pull out a factor \( L_{\varepsilon-2} \) from \( Z_{\varepsilon-2}^{(10)} \) and \( R_{\varepsilon} \) from \( Z_{\varepsilon}^{(01)} \) by division with remainder. Doing this successively and then using \( L_{\varepsilon-2}R_{\varepsilon} = L_{\varepsilon-1}Z_{\varepsilon-1}^{(10)} \) gives us

\[ L_{\varepsilon-3}R_{\varepsilon+1} = \frac{Z_{\varepsilon-2}^{(10)}Z_{\varepsilon-1}^{(11)}Z_{\varepsilon}^{(01)}}{L_{\varepsilon-1}} + \frac{L_{\varepsilon-2}Z_{\varepsilon}^{(20)}\sigma_{\varepsilon-1}^{(11)}Z_{\varepsilon}^{(01)}}{L_{\varepsilon-1}} + \frac{L_{\varepsilon-2}Z_{\varepsilon}^{(20)}\sigma_{\varepsilon-1}^{(11)}Z_{\varepsilon}^{(01)}}{L_{\varepsilon-1}}.
\]

Further steps are not possible. For the formula to be polynomial we need that the last two summands vanish. We obtain the equations

\[ \lambda_{\varepsilon-2,\varepsilon-1} := \sigma_{\varepsilon-2}^{(20)}\sigma_{\varepsilon-1}^{(11)} = 0, \quad \rho_{\varepsilon-1,\varepsilon} := \sigma_{\varepsilon-1}^{(11)}\sigma_{\varepsilon}^{(02)} = 0 \]

in the deformation variables.

**Example** (embedding dimension 5). The computations up to now suffice. We get, modulo the ideal of the base space, the same equations as equations \( (3) \) in Section 2. To translate in the notation used there, note that there are no variables \( t_2 \) and \( t_4 \), so we set \( l_2 = r_2 = l_4 = r_4 = 0 \), and we take \( l_3 = 0, r_3 = -t_3 \). One gets \( Z_2^{(20)} = Z_2^{(a_2-k)} \) and \( Z_4^{(20)} = Z_4^{(a_4-k)} \), \( \sigma_2^{(02)} = s_2^{(a_2-1)} \) and \( \sigma_4^{(02)} = s_4^{(a_4-1)} \). For \( \varepsilon = 3 \) we find

\[ Z_3^{(00)} = Z_3^{(01)}R_3 = Z_3^{(a_3-1)}(z_3 + t_3) = L_3Z_3^{(10)} = z_3Z_3^{(a_3-1)} \]

and

\[ Z_3^{(01)} = L_3Z_3^{(11)} + \sigma_3^{(11)} = z_3Z_3^{(a_3-2)} + s_3^{(a_3-1)}. \]
The formula (8) gives $\tilde{Z}_3^{(a_3-1)}$ as factor in the second summand of the right-hand side of the equation $z_1z_3 = P_{2,4}$, but the difference with $Z_3^{(a_3-1)}$, as given in the equations (3), lies in the ideal of the base space. Note that in general

$$Z_\varepsilon^{(i+1,j)} - Z_\varepsilon^{(i,j+1)}$$

$$= (Z_\varepsilon^{(i+1,j+1)}R_\varepsilon + \sigma_\varepsilon^{(i+1,j+1)}) - (L_\varepsilon Z_\varepsilon^{(i+1,j+1)} + \sigma_\varepsilon^{(i+1,j+1)}) = (l_\varepsilon - r_\varepsilon)Z_\varepsilon^{(i+1,j+1)}.$$  

The factor $\sigma_3^{(11)}$ in the second summand gives that we can use the equation $(l_3 - r_3)\sigma_3^{(11)} = t_3\delta^{(a_3-1)} = 0$.

We obtained the equations (9) as necessary condition to find a polynomial $P_{\varepsilon-1,\varepsilon+1}$. We observe that they could be computed before computing $P_{\varepsilon-1,\varepsilon+1}$, as they are the result of suitable substitutions in the right hand side of the equations of the previous line: $\lambda_{\varepsilon-1,\varepsilon} = \sigma_{\varepsilon-1,\varepsilon}^{(20)}\sigma_\varepsilon^{(11)}$ is gotten by setting $z_{\varepsilon-1} = l_{\varepsilon-1}$ and $z_{\varepsilon} = l_\varepsilon$ in $P_{\varepsilon-1,\varepsilon} = Z_\varepsilon^{(10)}R_\varepsilon^{(01)}$, while $P_{\varepsilon,\varepsilon+1}$ gives $\rho_{\varepsilon,\varepsilon+1}$ by $z_\varepsilon = r_\varepsilon$ and $z_{\varepsilon+1} = r_{\varepsilon+1}$.

To find the versal deformation in general one has to proceed in the same way for the higher lines of the pyramid. Arndt has shown that this works. As the proof is only written in his thesis [1], we sketch it here.

**Theorem 4.2.** Let $z_{\delta-1}z_{\varepsilon+1} = p_{\delta,\varepsilon}$, $2 \leq \delta \leq \varepsilon \leq e-1$, be the quasi-determinantal equations for a cyclic quotient singularity $X$ of embedding dimension $e$. There exists a deformation $L_{\delta-1}R_{\varepsilon+1} = P_{\delta,\varepsilon}$ of these equations over a base space, whose ideal $J$ has $\dim T_X^2 = (e-2)(e-4)$ generators, being $(l_\varepsilon - r_\varepsilon)\sigma_\varepsilon^{(11)}$ for $3 \leq \varepsilon \leq e-2$, $\lambda_{\delta,\varepsilon}$ for $2 \leq \delta < \varepsilon \leq e-2$, and $\rho_{\delta,\varepsilon}$ for $3 \leq \delta < \varepsilon \leq e-1$. The polynomials $P_{\delta,\varepsilon}$ can be determined inductively, followed by $\lambda_{\delta,\varepsilon} = P_{\delta,\varepsilon}|_{z_{\beta}=l}\beta$ and $\rho_{\delta,\varepsilon} = P_{\delta,\varepsilon}|_{z_{\beta}=r}\beta$, where $\delta \leq \beta \leq \varepsilon$. This deformation is versal.

**Sketch of proof.** To find $P_{\delta,\varepsilon}$ we have to express the product $L_{\delta-1}R_{\varepsilon+1}$ in the local ring in terms of variables with indices between $\delta$ and $\varepsilon$. We assume that we already have the equations $L_{\beta-1}R_{\gamma+1} = P_{\beta,\gamma}$ for $\gamma - \beta < \varepsilon - \delta$, and also the base equations formed from them. Let $I_{\delta,\varepsilon}$ be the ideal of all these equations. Obviously $P_{\delta,\varepsilon}$ has to satisfy

$$L_{\beta}R_{\gamma}P_{\delta,\varepsilon} \equiv P_{\beta+1,\varepsilon}P_{\delta,\gamma-1} \mod I_{\delta,\varepsilon}$$

for all $\beta, \gamma$, and it can be determined from any of these equations. The other ones then follow. For the actual computation (following [3]) we use $\beta = \varepsilon - 1$ and $\gamma = \varepsilon$, but now we take $\beta = \delta, \gamma = \varepsilon$, so the right hand side of equation (10) becomes $P_{\delta,\varepsilon-1}P_{\delta+1,\varepsilon}$. We perform successively division with remainder by $L_{\beta}$ and find

$$P_{\delta,\varepsilon-1} = \sum_{\beta=\delta}^{\varepsilon-1} P_{\delta,\varepsilon-1}^{(\beta)}L_{\beta} ,$$

without remainder because of the equation $\lambda_{\delta,\varepsilon-1}$. Now we use the congruences

$$L_{\beta}P_{\delta+1,\varepsilon} \equiv L_{\delta}P_{\beta+1,\varepsilon} ,$$

whose validity one sees upon multiplying with $R_{\varepsilon+1}$. We conclude that

$$P_{\delta,\varepsilon-1}P_{\delta+1,\varepsilon} \equiv L_{\delta}(\sum P_{\delta,\varepsilon-1}^{(\beta)}P_{\beta+1,\varepsilon}) .$$

Likewise, from

$$P_{\delta+1,\varepsilon} = \sum Q_{\delta+1,\varepsilon}^{(\beta)}R_{\beta} ,$$

...
we get, using \( \rho_{\delta+1, \varepsilon} \), that

\[
P_{\delta, \varepsilon} \cdot P_{\delta+1, \varepsilon} \equiv R_{\varepsilon} \left( \sum Q_{\delta+1, \varepsilon}^{[\beta]} P_{\delta, \beta-1} \right).
\]

Arndt proves that, if a polynomial is divisible by \( L_{\delta} \) and by \( R_{\varepsilon} \), then it is divisible by the product \( L_{\delta} R_{\varepsilon} \). To check the statement it suffices to do it for the special fibre (according to [II 1.2.2]). One notes that the ideal \( I_{\delta, \varepsilon} \) defines a flat deformation of the product of a certain cyclic quotient singularity in the variables \( z_{\delta}, \ldots, z_{\varepsilon} \) with a smooth factor of the remaining coordinates, so \( z_{\delta} = u^{n'}, z_{\varepsilon} = v^{n'} \) for a certain \( n' \). Here indeed it holds, that if a polynomial is divisible by \( u^{n'} \) and by \( v^{n'} \), then it is divisible by the product \((uv)^{n'}\). Therefore there exists a polynomial \( P_{\delta, \varepsilon} \) with \( L_{\delta} R_{\varepsilon} P_{\delta, \varepsilon} = P_{\delta, \varepsilon} P_{\delta+1, \varepsilon} \).

We do not know very much about \( P_{\delta, \varepsilon} \). We know that over the Artin component \( Z_{\varepsilon-1}^{0(1)} \) is divisible by \( L_{\varepsilon-1} \), so we can define inductively \( P_{\delta, \varepsilon}^{AC} = P_{\delta, \varepsilon-1}^{AC} Z_{\varepsilon}^{0(1)} / L_{\varepsilon-1} \). Restricted to the Artin component, the difference between the so defined \( P_{\delta, \varepsilon}^{AC} \) and \( P_{\delta, \varepsilon} \) from above lies in the restriction of the ideal \( I_{\delta, \varepsilon} \). By induction the elements of this ideal extend in the correct way, so we can use them to change \( P_{\delta, \varepsilon} \), so that its restriction is equal to \( P_{\delta, \varepsilon}^{AC} \).

To show flatness we lift the relations. On the Artin component we have the quasi-determinantal relations, which come in two types, depending on the use of the bottom or top line of the quasi-matrix. We give the lift for one type, the other being similar. On the Artin component one has the relation

\[
L_{\gamma-1}(L_{\delta-1} R_{\varepsilon+1} - P_{\delta, \varepsilon}) = L_{\delta-1}(L_{\gamma-1} R_{\varepsilon+1} - P_{\gamma, \varepsilon}) + \frac{P_{\gamma, \varepsilon}}{R_{\gamma}}(L_{\delta-1} R_{\gamma} - P_{\delta, \gamma-1})
\]

so using an expansion like (11) for \( P_{\gamma, \varepsilon} \) we find modulo the ideal \( I_{\delta, \varepsilon} \) the relation

\[
L_{\gamma-1}(L_{\delta-1} R_{\varepsilon+1} - P_{\delta, \varepsilon}) \equiv L_{\delta-1}(L_{\gamma-1} R_{\varepsilon+1} - P_{\gamma, \varepsilon}) + \sum Q_{\gamma, \varepsilon}^{[j]}(L_{\delta-1} R_{\beta} - P_{\delta, \beta-1}).
\]

For versality one needs firstly the surjectivity of the map of the Zariski tangent space of our deformation to \( T_X^1 \), and secondly the injectivity of the obstruction map \( \text{Ob}: (J/mJ)^* \to T_X^2 \), where \( J \) is the ideal of the base space. That we cover all possible infinitesimal deformations, is something we have already said and used; for a proof (which requires an explicit description of \( T_X^1 \)), see [II], [I] or [III]. We neither give here an explicit description of \( T_X^2 \) (see [I]). For the map \( \text{Ob} \) one starts with a map \( l: J/mJ \to O_X \) and exhibits the following function on relations: consider a relation \( r \), i.e., \( \sum f_i r_j = 0 \), which lifts to \( \sum F_i R_i = \sum g_j q_j \), where the \( g_j \) are the generators of the ideal \( J \). Then \( \text{Ob}(l)(r) = \sum l(g_j) q_j \in O_X \). From our description of the relations we see that the equation \( \rho_{\delta+1, \varepsilon} \) occurs for the first time when lifting the relation

\[
L_{\delta}(L_{\delta-1} R_{\varepsilon+1} - P_{\delta, \varepsilon}) = L_{\delta-1}(L_{\delta} R_{\varepsilon+1} - P_{\delta+1, \varepsilon}) + \frac{P_{\delta+1, \varepsilon}}{R_{\delta+1}}(L_{\delta-1} R_{\delta+1} - P_{\delta, \delta}).
\]

This more or less shows that one really needs all equations for the base space. \( \square \)

Note that this result indeed determines the ideal of the base space, but does not give explicit formulas for specific generators. Looking at the equations, say for the total space, one might recognise the numbers \([1, 2, 1]\) and \([2, 1, 2]\), suggesting that an explicit formula can be somehow given in terms of Catalan combinatorics. To show that the situation is more complicated, we will derive the equations of the next line.

We compute the polynomial \( P_{\varepsilon-3, \varepsilon} \). It will be more complicated than formula (8), so better notation is desirable to increase readability. Following Brohme [3] we use a position system. In stead of the complicated symbols \( Z_{\varepsilon-1}^{(ij)} \) and \( \sigma_{\varepsilon-1}^{(ij)} \) we write only the upper index \( ij \); the
lower index is not needed, if we write factors with the same \( \varepsilon - \gamma \) below each other. To distinguish between \( Z_{\varepsilon-\gamma}^{ij} \) and \( \sigma_{\varepsilon-\gamma}^{ij} \) we write the \( ij \) representing \( Z_{\varepsilon-\gamma}^{ij} \) in bold face.

**Example.** The symbol

\[
\begin{array}{c}
30 & 20 & 10 & 03 \\
20 & 11 \\
11 & 12 \\
\end{array}
\]

represents the monomial \( Z_{\varepsilon-3}^{(30)} \sigma_{\varepsilon-2}^{(11)} Z_{\varepsilon-2}^{(20)} \sigma_{\varepsilon-1}^{(12)} Z_{\varepsilon-1}^{(10)} Z_{\varepsilon}^{(03)} \).

A factor \( L_{\varepsilon-\gamma} \) in the denominator will be represented by \( \overline{L} \), whereas an extra factor \( L_{\varepsilon-\gamma} \) in the numerator will be written in bold face. We start from \( P_{\varepsilon-3,\varepsilon-1}Z_{\varepsilon}^{(01)}/L_{\varepsilon-1} \), being the sum of two terms. These will be transformed using the division with remainder (5) and (6) and previous equations. One gets some terms, which occur in the final answer, and some terms, which will be transformed again. Terms that will be transformed, are written in italics, and should be considered erased, when transformed. So \( P_{\varepsilon-3,\varepsilon-1}Z_{\varepsilon}^{(01)}/L_{\varepsilon-1} \) is, modulo previous equations, equal to the sum of the not italicised terms up to a line in italics, plus the terms in that line (disregarding all text in between). The final result consists of a polynomial of 8 terms, (P.1) – (P.8), and two types of terms with \( L_{\varepsilon-1} \) in the denominator, called (L.1) – (L.4) and (R.1) – (R.4). Now we start:

\[
\begin{array}{c}
(\text{I.1})
10 & 11 & 01 & 01 & \overline{L} & + & (\text{I.2})
20 & 10 & 02 & 01 \\
(\text{P.1})
10 & 11 & 11 & 01 & + & (\text{P.2})
20 & 10 & 12 & 01 \\
10 & 11 & 01 & \overline{L}
(\text{I.3})
(\text{I.4})
11 & 12 & \overline{L}
\end{array}
\]

Now take out factors \( L_{\varepsilon-2} \) and \( R_{\varepsilon} \), giving \( L_{\varepsilon-1}Z_{\varepsilon-1}^{(10)} \) and two remainders:

\[
\begin{array}{c}
(\text{P.3})
10 & 21 & 10 & 02 & 11 & + & (\text{P.4})
20 & 20 & 10 & 02 & 11 & 12 \\
(\text{I.5})
10 & 01 & \overline{L}
(\text{I.6})
20 & 12 & \overline{L} \\
(\text{R.1})
10 & 21 & 11 & 02 & \overline{L}
(\text{R.2})
20 & 20 & 11 & 12 & 02 \\
\end{array}
\]
Taking out $L_{\varepsilon-3}$ and $R_{\varepsilon}$ from (I.5) and (I.6) gives $Z_{\varepsilon-2}^{(10)}Z_{\varepsilon-1}^{(01)}$ and two remainders:

\[
\begin{array}{cccccc}
20 & 10 & 01 & 02 \\
\hline
(I.7) & 21 & 11 & T & + & (I.8) & 20 & 12 & T \\
+ & (L.1) & 20 & 21 & 11 & + & (L.2) & 30 & 20 & 12 & T \\
\hline
L & 20 & 11 & 02 & + & (R.3) & 20 & 12 & 02 \\
\hline
(R.3) & 21 & 11 & 02 & + & (R.4) & 20 & 12 & 02 \\
\end{array}
\]

Now we follow the steps of the computation of $P_{\varepsilon-2,\varepsilon}$.

\[
\begin{array}{cccccc}
20 & 10 & 11 & 02 \\
\hline
(P.5) & 21 & 11 & + & (P.6) & 30 & 10 & 11 & 02 \\
\hline
20 & 10 & 02 \\
\hline
(I.9) & 21 & 11 & + & (I.10) & 20 & 11 & T \\
\hline
20 & 20 & 10 & 03 \\
\hline
(P.7) & 21 & 11 & + & (P.8) & 30 & 20 & 10 & 03 \\
\hline
20 & 02 \\
\hline
(L.3) & 20 & 11 & + & (L.4) & 20 & 12 & T \\
\hline
L & 20 & 20 & 02 \\
\hline
(R.5) & 21 & 11 & 03 & + & (R.6) & 30 & 20 & 10 & 03 \\
\end{array}
\]

**Remarks.** 1. It is easy to see that the terms (L.i) vanish modulo the ideal $J$: the terms (L.1) and (L.2) vanish because of the equation $\lambda_{\varepsilon-3,\varepsilon-1}$, and the terms (L.3) and (L.4) both vanish by equation $\lambda_{\varepsilon-2,\varepsilon-1}$.

The terms (R.i) are more difficult. Taken together, (R.2) and (R.5) vanish by equation $\rho_{\varepsilon-2,\varepsilon}$. The term (R.1) vanishes by $\rho_{\varepsilon-1,\varepsilon}$, as does less evidently (R.3). For (R.6) one uses $\lambda_{\varepsilon-2,\varepsilon-1}$. The term (R.4) is the most complicated. We multiply $\rho_{\varepsilon-2,\varepsilon}$ with $\sigma_{\varepsilon-2}^{20}$ to get $20\ 12\ 02 + 20\ 11\ 03$, in which the second summand vanishes by equation $\lambda_{\varepsilon-2,\varepsilon-1}$.

2. The term (P.8) lies in the ideal, so one could leave it out. However, to have a general formula it is better to keep it.
3. Arndt [1] gives a slightly different, more symmetric result (without showing his computation). He has also 8 terms, (P.8) is missing and (P.4) is replaced by two terms, which together are equivalent to it, modulo the base ideal:

\[
\begin{array}{cccc}
20 & 20 & 01 & 02 \\
11 & 12 \\
\end{array}
+ \begin{array}{cccc}
20 & 20 & 02 & 02 \\
11 & 11 \\
\end{array}
\]

In our computation we work systematically from the right to the left. Once we take out a factor \(L^\gamma R^\varepsilon\) and replace it by \(P^{\gamma+1,\varepsilon-1}\), we basically repeat an earlier computation. Brohme [3] has given an inductive formula for the resulting terms (P.i) and the remainder terms (R.i). The problem lies in showing that the remainder terms lie in the ideal of the base space. This problem was solved by Martin Hamm [6] on the basis of a more direct, combinatorial description of the occurring terms. Each term is represented by a rooted tree, which we draw horizontally (Hamm puts as usually the root at the top). Accordingly we call for length of a tree, what is usually called its height.

We consider as example the term (P.7). We first draw the tree such that each vertex directly correspond to a position in the symbol for (P.7) above, but then we transform it so that the bottom line is straight. This will be the way we draw all trees in the sequel.

We explain how to compute the numbers \(ij\) in the symbol from the tree, with the highest node at distance \(\gamma\) from the root giving \(Z^{(ij)}_{\varepsilon-\gamma}\), and the other nodes \(\sigma^{(ij)}_{\varepsilon-\gamma}\). Given a tree \(T\), the resulting polynomial in these variables will be denoted by \(P(T)\). We write \(\lambda(T)\) for the corresponding term in \(\lambda_{\delta,\varepsilon}\), obtained by putting \(z_\beta = l_{\beta}\), and \(\rho(T)\) for the term obtained with \(z_\beta = r_{\beta}\).

**Definition 4.3.** Let \(T\) be a rooted tree. To each node \(a \in T\) we associate two numbers, \(i\) and \(j\), as follows. The second number \(j\) is the number of child nodes of \(a\). Let \(p(a)\) be the parent of \(a\); if there exists a node \(b\) lying directly above \(a\), let \(p(b)\) be its parent. Then the number \(i\) is the number of nodes between \(p(a)\) and \(p(b)\) (with \(p(a)\) and \(p(b)\) included), or the number of nodes above \(p(a)\) (with \(p(a)\) included), if there is no node lying above \(a\).

We also represent the remainder terms (R.i) by a tree. Such a term has the form \(L^\gamma R_{\delta,\varepsilon}^{(i)}/L^{\varepsilon-1}\). The tree will give \(R_{\delta,\varepsilon}^{(i)}\). If a tree \(T\) is given, we write \(R(T)\) for this polynomial. As example we consider (R.5). We observe that its symbol contains the same pairs of numbers as (P.7), except that some on the top are missing, and that the root represents \(\sigma^{(ij)}_{\varepsilon}\) instead of \(Z^{(ij)}_{\varepsilon}\). We represent it by the following tree.

To the open nodes (except the root) we do not attach numbers \(ij\), but these nodes do contribute to the numbers \(ij\) for the other nodes.
Example. For $P_{\varepsilon-3,\varepsilon}$ we find the following trees

and we have the following trees for the remainder.

We now characterise the tree representing terms $P(T)$ in the polynomial $P_{\delta,\varepsilon}$, which is obtained as above from $P_{\delta,\varepsilon-1}Z_{\varepsilon}^{(01)}/L_{\varepsilon-1}$, by working systematically from the right to the left. Let $a \in T$ be a node in a rooted tree, then $T(a)$ is the (maximal) subtree with $a$ as root.

**Definition 4.4.** An $\alpha$-tree is a rooted tree satisfying the following property: if two nodes $a$ and $b$ have the same parent, and if $b$ lies above $a$, then the subtree $T(b)$ is shorter than $T(a)$. By $A(k)$ we denote the set of all $\alpha$-trees of length $k$.

**Theorem 4.5.** The polynomial $P_{\delta,\varepsilon}$ is given by

$$P_{\delta,\varepsilon} = \sum_{T \in A(\varepsilon-\delta+1)} P(T) .$$

This Theorem claims two things: that the $\alpha$-trees give exactly the polynomial terms in the computation, and secondly that this is really the polynomial $P_{\delta,\varepsilon}$ we are after. To show the second part we have to prove that the remainder terms lie in the ideal generated by the equations of the base space. As we have seen with (R.4) above, the use of the equations leads to terms, which do not occur in the computation itself. We have to characterise the corresponding trees. This leads to the concept of $\gamma$-tree (Hamm has also $\beta$-trees [6]). We postpone the definition, and first consider a sub-class of the $\alpha$-trees.

**Definition 4.6.** An $\alpha\gamma$-tree is an $\alpha$-tree, whose root has at least two child nodes and the subtree of the highest child of the root is unbranched (this is the chain of open dots in our pictures). Let $AC(k, l)$ be the set of all $\alpha\gamma$-trees of length $k$, such that the unbranched subtree (with the root included) has length $l$.

**Lemma 4.7.** Modulo the equations $\lambda_{\gamma,\varepsilon-1}$ one has

$$P_{\delta,\varepsilon-1}Z_{\varepsilon}^{(01)}/L_{\varepsilon-1} = \sum_{T \in A(\varepsilon-\delta)} P(T) + \sum_{T \in AC(\varepsilon-\delta,\varepsilon-\gamma)} L_{\gamma}R(T)/L_{\varepsilon-1} .$$

*Proof.* We compute as for $P_{\varepsilon-3,\varepsilon}$. We first consider the rest terms $R(T)$. Such a term comes about from writing $Z_{\varepsilon}^{(0,k)} = Z_{\varepsilon}^{(0,k+1)}R_{\varepsilon} + \sigma_{\varepsilon}^{(0,k+1)}$. We replace $L_{\gamma}R_{\varepsilon}$ by $P_{\gamma+1,\varepsilon-1}$. The first term of $P_{\gamma+1,\varepsilon-1}$ is given by an unbranched tree, it is 10 11 · · · 11 01, and writing
\[ Z_{\varepsilon - 1}^{(01)} = L_{\varepsilon - 1} Z_{\varepsilon - 1}^{(11)} + \sigma_{\varepsilon - 1}^{(11)} \] leads to a term \( P(T) \) with the same tree as \( R(T) \) (with the only difference that all nodes are denoted by black dots).

We are left to show that the polynomial terms are exactly those represented by \( \alpha \)-trees. This is done by induction. We can construct an \( \alpha \)-tree of length \( \varepsilon - \delta - 1 \) by taking a root, an \( \alpha \)-tree of length \( \varepsilon - \delta - 2 \) as lowest subtree, and as its complement an arbitrary \( \alpha \)-tree of length at most \( \varepsilon - \delta - 1 \) (conversely, given an \( \alpha \)-tree of length \( \varepsilon - \delta \), the lowest subtree starting from the root, but not including it, is \( \alpha \)-tree of length \( \varepsilon - \delta - 1 \), while its complement has length at most \( \varepsilon - \delta - 1 \)). Doing this in all possible ways gives all \( \alpha \)-trees of length \( \varepsilon - \delta \). All these trees occur by our construction: in all monomials of \( P_{\delta, \varepsilon - 1} Z_{\varepsilon - 1}^{(01)} / L_{\varepsilon - 1} \) we simultaneously take out factors \( L_{\gamma} \), until we finally are left with \( \lambda_{\delta, \varepsilon - 1} Z_{\varepsilon - 1}^{(01)} / L_{\varepsilon - 1} \). Each \( L_{\gamma} R_{\varepsilon} \) is replaced by \( P_{\gamma + 1, \varepsilon - 1} \), and here we repeat the same computation as for \( P_{\gamma + 1, \varepsilon - 1} \), except that the upper index of \( Z_{\varepsilon - 1}^{(0k)} \) is different. This means that we place all possible trees of length at most \( \varepsilon - \gamma - 1 \) above the given tree.

\[ \square \]

**Remark.** The above proof gives an inductive formula for the number of \( \alpha \)-trees of length \( k \):

\[
\#A(k) = \#A(k - 1) \cdot \sum_{i=0}^{k-1} \#A(i) .
\]

One has \( \#A(0) = 1 \), \( \#A(1) = 1 \), \( \#A(2) = 2 \), \( \#A(3) = 8 \) and \( \#A(4) = 96 \). As we have seen in the example, some of the terms lie in the ideal \( J \), generated by the equations of the base space. For \( k = 4 \) already 55 of the 96 terms lie in \( J \), leaving “only” 41 terms. Still this number is considerably larger than the relevant Catalan number (14 in this case).

As already mentioned, the use of \( \rho \)-equations brings us outside the realm of \( \alpha \)-trees. We retain some properties, which are automatically satisfied for \( \alpha \)-trees. The definition becomes rather involved.

**Definition 4.8.** A \( \gamma \)-tree of length \( k \) is a rooted tree satisfying the following properties:

(i) there is only one node at distance \( k \) from the root, and it lies on the bottom line,
(ii) the number of child nodes of a node at distance \( d \) from the root is at most \( k - d \),
(iii) a node \( a \) has a child node, if there exists a node \( b \) lying above \( a \) with the same parent,
(iv) the root has at least two child nodes and the subtree of the highest child of the root is unbranched.

By \( G(k, l) \) we denote the set of all \( \gamma \)-trees of length \( k \), such that the unbranched subtree (with the root included) has length \( l \).

**Example.** We consider the term (R.4) above. The sum of the following two terms

\[
\begin{array}{c}
\quad \\
\quad \\
\quad \\
\quad \\
\end{array} + \begin{array}{c}
\quad \\
\quad \\
\quad \\
\quad \\
\end{array}
\]

is a multiple of the equation \( \rho_{\varepsilon - 2, \varepsilon} \), and the second graph is not an \( \alpha \)-tree.

We have to show that the sum of all remainder terms (i.e., the sum of the \( R(T) \) over all \( \alpha \gamma \)-trees) lies in the ideal \( J \) generated by the base equations. We do this by showing that the sum of \( R(T) \) over all \( \gamma \)-trees lies in the ideal, as does the sum over all \( \gamma \)-trees which are not \( \alpha \)-trees.
Lemma 4.9. The sum $\sum_{T \in G(\varepsilon-\delta, \varepsilon-\gamma)} R(T)$ lies in the ideal generated by the $\lambda$-equations.

Proof. Let $T$ be a $\gamma$-tree, which is not an $\alpha$-tree. Then there exists a node $a$, such that the subtree $T(a)$ is an $\alpha$-tree, but directly above $a$ lies a node $b$ with the same parent, such that the bottom line of the subtree $T(b)$ is at least as long as $T(a)$. Denote by $R(T|T(a))$ the monomial obtained by only multiplying the factors of $R(T)$ corresponding to the nodes lying in $T(a)$. We claim that $R(T|T(a)) = \lambda(T(a))$. We have to compute the numbers $ij$. The second number, of child nodes, is determined by $T(a)$ only. The number $i$ also coincides in $R(T)$ and $P(T(a))$, except when $c$ is a node without nodes above it in $T(a)$. Then its value in $R(T)$ is one more than in $P(T(a))$, so the same as in $\lambda(P(a))$, proving the claim. Replacing $T(a)$ in $T$ by another $\alpha$-tree of the same length gives a another $\gamma$-tree, which is not an $\alpha$-tree. So the sum of $R(T)$ over all $\gamma$-trees, differing only in the $\alpha$-tree with root $a$, is a multiple of a $\lambda$-equation. If $T$ has several such subtrees, we consider all possible replacements, and get the product of $\lambda$-equations. \qed

The next task is to find terms of $\rho$-equations in a given tree. For this we introduce the operation of taking away one child node at each highest node. This can be done for any $\gamma$-tree.

Definition 4.10. Let $T$ be a $\gamma$-tree. We determine inductively a subtree $G(T)$ with the same root as $T$ by the following condition: if $a_1, \ldots, a_p$ are the nodes in $G(T)$ at distance $d$ from the root, then they have the same child nodes in $G(T)$ as in $T$, except for the highest node $a_p$, where we take away the highest child node.

Example.

$\begin{array}{cccc}
\bullet & \rightarrow & \bullet & \rightarrow \\
\circ & \rightarrow & \circ & \rightarrow \\
\end{array}$

Lemma 4.11. Suppose $G(T)$ is an $\alpha$-tree. Then $R(T|G(T)) = \rho(G(T))$ if and only if the number of child nodes in $T$ is at least 1 for every highest lying node in $G(T)$.

The sum $\sum R(T)$ over all trees satisfying the conditions of the lemma lies in the ideal generated by the $\rho$-equations. If the number of child nodes in $T$ is at least 1 for every highest lying node in $G(T)$, but $G(T)$ is not an $\alpha$-tree, then one can find as before a term of a $\lambda$-equation.

The most difficult case is when the condition on the number of child nodes is not satisfied. An example is the remainder term (R.6), which is represented by the last two pictures in the previous example. The term contains a factor, which is a term in a $\lambda$-equation, but the corresponding dots are not connected by an edge. There is a way to connect the edges differently, bringing a $\lambda$-term into evidence. For this we refer to [6 pp. 30–40]. We conclude:
Lemma 4.12. The sum $\sum_{T \in \mathcal{G}(\varepsilon - \delta, \varepsilon - \gamma)} R(T)$ lies in the ideal generated by the $\lambda$ and $p$-equations.

Together with Lemma 4.9 this shows that the remainder $\sum_{T \in \mathcal{A}\mathcal{G}(\varepsilon - \delta, \varepsilon - \gamma)} R(T)$ lies in the ideal $J$, thereby concluding the proof of Theorem 4.5.

Example (The base space for $e = 6$, see [11, 3]). There are 8 equations, which read as

\[ (l_3 - r_3)\sigma_{3}^{(11)}, \quad (l_4 - r_4)\sigma_{4}^{(11)}, \]
\[ \sigma_{2}^{(20)}\sigma_{3}^{(11)}, \quad \sigma_{3}^{(20)}\sigma_{3}^{(11)}, \quad \sigma_{3}^{(11)}\sigma_{4}^{(02)}, \quad \sigma_{4}^{(11)}\sigma_{5}^{(02)}, \]
\[ \sigma_{2}^{(20)}\sigma_{3}^{(11)}\sigma_{4}^{(11)} + \sigma_{2}^{(30)}\sigma_{3}^{(11)}\sigma_{5}^{(12)}, \quad \sigma_{3}^{(11)}\sigma_{4}^{(12)}\sigma_{5}^{(02)} + \sigma_{3}^{(21)}\left(\sigma_{4}^{(11)}\right)^{2}\sigma_{5}^{(03)}. \]

We note the relations

\[ \sigma_{3}^{(20)} - \sigma_{3}^{(11)} = (l_3 - r_3)\sigma_{3}^{(21)}, \quad \sigma_{4}^{(02)} - \sigma_{4}^{(11)} = (r_4 - l_4)\sigma_{4}^{(12)}. \]

We can take $\sigma_{2}^{(20)}, \sigma_{3}^{(11)}, l_3 - r_3, \sigma_{3}^{(21)}, l_4 - r_4, \sigma_{4}^{(11)}, \sigma_{4}^{(12)}, \sigma_{5}^{(02)}$ and $\sigma_{5}^{(03)}$ as independent coordinates. The relation with the coordinates in Section 1, see formula (2), is the following: $l_3 - r_3 = t_3, \sigma_{3}^{(11)} = s_{3}^{(a_3 - 1)}, \sigma_{3}^{(12)} = s_{3}^{(a_3 - 2)}$. Also for $\varepsilon = 2$ and $\varepsilon = 5$ it is simple: $\sigma_{2}^{(i)} = s_{2}^{(a_2 - i - j)}$, but for $\varepsilon = 4$ it is more complicated: $r_4 - l_4 = t_4, \sigma_{4}^{(11)} = s_{4}^{(a_4 - 1)}$, while $\sigma_{4}^{(12)} = s_{4}^{(a_4 - 2)}$, where $z_{\varepsilon}^{(\nu)}$ is defined as following [11, 5.1.1], see also [3, p. 38], by the equality

\[ Z_{\varepsilon}^{(00)} = (z_{\varepsilon} + t_{\varepsilon}) \sum_{\nu=0}^{a_{\varepsilon} - 1} s_{\varepsilon}^{(\nu)} z_{\varepsilon}^{a_{\varepsilon} - 1 - \nu} = z_{\varepsilon} \sum_{\nu=0}^{a_{\varepsilon} - 1} s_{\varepsilon}^{(\nu)} (z_{\varepsilon} + t_{\varepsilon})^{a_{\varepsilon} - 1 - \nu}, \]

where we put $s_{\varepsilon}^{(0)} = 1$. This implies that

\[ z_{\varepsilon}^{(\nu)} = \sum_{\mu=0}^{\nu} \binom{a_{\varepsilon} - 2 - \mu}{a_{\varepsilon} - 2 - \nu} (-t_{\varepsilon})^{\mu - \nu} s_{\varepsilon}^{(a_{\varepsilon} - 1 - \mu)}. \]

The primary decomposition gives five reduced components and one embedded component. The five components are parametrised by the five sparse coloured triangles of height 2.

- $\sigma_{3}^{(11)} = \sigma_{4}^{(11)} = 0$
- $\sigma_{2}^{(20)} = l_3 - r_3 = \sigma_{3}^{(11)} = \sigma_{4}^{(12)} = 0$
- $\sigma_{3}^{(11)} = \sigma_{3}^{(21)} = r_4 - l_4 = \sigma_{5}^{(02)} = 0$
- $\sigma_{2}^{(20)} = \sigma_{2}^{(30)} = l_3 - r_3 = r_4 - l_4 = \sigma_{4}^{(11)} = \sigma_{5}^{(02)} = 0$
- $\sigma_{2}^{(20)} = l_3 - r_3 = \sigma_{3}^{(11)} = r_4 - l_4 = \sigma_{5}^{(02)} = \sigma_{5}^{(03)} = 0$

The embedded component is supported at $\sigma_{2}^{(20)} = l_3 - r_3 = \sigma_{3}^{(11)} = r_4 - l_4 = \sigma_{4}^{(11)} = \sigma_{5}^{(02)} = 0$, which is the locus of singularities of embedding dimension 6.
The primary decomposition, given in the example above, holds if all \( a_\varepsilon \) are large enough, meaning that \( a_\varepsilon \geq \max(k_\varepsilon) \), where the \( k_\varepsilon \) depend on the possible triangles. By openness of versality one deduces the structure for all cyclic quotient singularities of the given embedding dimension. The formulas for the base space and the total space of the deformation hold in all cases, with suitable interpretations. We note that \( Z_{\varepsilon}^{(ij)} \) is a monic polynomial in \( z_{\varepsilon} \) of degree \( a_\varepsilon - i - j \), obtained by division with remainder. Therefore \( Z_{\varepsilon}^{(ij)} = 1 \) if \( i + j = a_\varepsilon \) and \( Z_{\varepsilon}^{(ij)} = 0 \) if \( i + j > a_\varepsilon \). For the remainder terms \( \sigma_{\varepsilon}^{(ij)} \) we find therefore that \( \sigma_{\varepsilon}^{(ij)} = 1 \) if \( i + j = a_\varepsilon + 1 \) and \( \sigma_{\varepsilon}^{(ij)} = 0 \) if \( i + j > a_\varepsilon + 1 \). By using these values the formulas hold. Also the description of the reduced components holds in general, if one takes an equation \( 1 = 0 \) to mean that the component is absent.

Example (The cone over the rational normal curve [11][8]).

**Proposition 4.13.** For the cone over the rational normal curve of degree \( e - 1 \) the versal deformation is given by the equations \( L_{\delta-1}R_{\varepsilon+1} = P_{\delta,\varepsilon} \) with \( P_{\varepsilon,\varepsilon} = Z_{\varepsilon}^{(00)} \) and

\[
P_{\delta,\varepsilon} = Z_{\delta}^{(10)}Z_{\varepsilon}^{(01)} + \sum_{\gamma=1}^{\varepsilon-\delta-1} \sigma_{\varepsilon-\gamma}^{(11)}Z_{\delta+\gamma}^{(10)}
\]

for \( \varepsilon - \delta > 0 \).

**Proof.** We derive the formula from Hamm’s description of the equations (i.e., Theorem 4.5). All terms in the equations containing \( Z_{\varepsilon}^{(ij)} \) with \( i + j > 2 \) and \( \sigma_{\varepsilon}^{(ij)} \) with \( i + j > 3 \) are absent. We characterise the remaining \( \alpha \)-trees. We have of course a simple chain, giving rise to the first term in the formula. Suppose we have a factor \( \sigma_{\varepsilon-\gamma}^{(11)} \) coming from a node \( a \) on the bottom line. Then there is a node lying directly above it, having the same parent. The unique child node of \( a \) has \( i = 2 \). If it has itself a child node, then necessarily \( i + j = 21 \), and there is a node lying above it. This process continues until we come to an end node with \( ij = 20 \). If the parent of \( a \) is not the root, then necessarily \( ij = 12 \) for it, so there is a node lying above it. For the node above \( a \) we find then that \( i = 2 \), and there lies a whole chain above the chain starting with this node. In this way we proceed to the root, which has \( ij = 02 \). We find that the tree has the following shape: from each node on the right of the node \( a \) originates a chain of maximal length. An example is

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```

Finally we observe that the lowest lying child node of the root or of a node with \( ij = 12 \) cannot have \( i = 2 \), and that the next to last node on the left cannot have \( ij = 12 \), so there has to be a node \( a \) with \( ij = 11 \).

Consider now \( P(T) \), if \( T \) is not a chain. The only node with \( ij = 10 \) lies as end-node on the highest chain, so indeed \( P(T) = \sigma_{\varepsilon-\gamma}^{(11)}Z_{\delta+\gamma}^{(10)} \). □

It follows that

\[
\lambda_{\delta,\varepsilon} = \sum_{\gamma=0}^{\varepsilon-\delta-1} \sigma_{\varepsilon-\gamma}^{(11)}\sigma_{\delta+\gamma}^{(20)}
\]
and

\[ \rho_{\delta, \varepsilon} = \sigma_{\delta}(\Omega(1)) \sigma_{\varepsilon}(\Omega(2)) + \sum_{\gamma=1}^{\varepsilon-\delta-1} \sigma_{\varepsilon-\gamma}(\Omega(1)) \sigma_{\delta+\gamma}. \]

With \( l_\varepsilon - r_\varepsilon = t_\varepsilon, \sigma_{\varepsilon}(\Omega(1)) = s_\varepsilon, \sigma_{\varepsilon}(\Omega(2)) = s_\varepsilon + t_\varepsilon \) and \( \sigma_{\varepsilon}(\Omega(2)) = s_\varepsilon - t_\varepsilon \) we get the same formulas as Arndt gives [1, 5.1.4].

Note that \( (\sigma_{\varepsilon}(\Omega(1)))^3 = \sigma_{\varepsilon}(\Omega(1)) \rho_{\varepsilon-1, \varepsilon+1} - \sigma_{\varepsilon}(\Omega(1)) \rho_{\varepsilon-1, \varepsilon} \), so \( \sigma_{\varepsilon}(\Omega(1)) \) lies in the radical of the ideal for \( 3 < \varepsilon < e - 1 \); for \( 2 < \varepsilon < e - 2 \) one has a formula with \( \lambda \)-equations. So indeed the Artin component is the only component, if \( e > 5 \).

Other applications of the explicit equations include

- the discriminant of the components and adjacencies, studied by Christophersen [4] and Brohme [3].
- embedded components. For low embedding dimension Brohme found all components. He made a general conjecture [3, 4.4].

5. Reduced base space

The ideal \( J \) of the base space is described explicitly by Theorems \( 4.2 \) and \( 4.5 \). We have to determine the radical \( \sqrt{J} \) of this ideal. We are able to do this explicitly for low embedding dimension, and formulate a conjecture in general. We prove that the proposed ideal describes the reduced components. The combinatorics involved resembles that described by Jan Christophersen in his thesis [5]. To prove the conjectural part one has to show that the monomials we give below, really lie in \( \sqrt{J} \), something we do not do here.

**Example** \( (e = 6 \) continued\). We multiply \( \rho_{3,5} \), the last one of the 8 equations for the base space, by \( \sigma_{4}(\Omega(1)) \). Then the first summand contains the factors \( \sigma_{4}(\Omega(1)) \sigma_{5}(\Omega(2)) \) so lies in the ideal \( J \). Therefore also the second term \( \sigma_{3}(\Omega(1)) \left( \sigma_{4}(\Omega(1)) \right)^3 \sigma_{5}(\Omega(3)) \) lies in \( J \), and \( \sigma_{3}(\Omega(1)) \sigma_{4}(\Omega(2)) \sigma_{5}(\Omega(3)) \) lies in the radical \( \sqrt{J} \). Then also the first summand of \( \rho_{3,5} \) lies in \( \sqrt{J} \). If we multiply \( \lambda_{2,4} \) with \( \sigma_{4}(\Omega(1)) \), then the second summand lies in \( J \). We find that the first summand of \( \lambda_{2,4} \) lies in the radical, so also the second summand. One has \( \sigma_{3}(\Omega(1)) \left( \sigma_{3}(\Omega(2)) - \sigma_{3}(\Omega(1)) \right) = \sigma_{3}(\Omega(1)) (l_3 - r_3) \sigma_{3}(\Omega(1)) \), which lies in the ideal, so not only the second summand \( \sigma_{3}(\Omega(1)) \sigma_{3}(\Omega(2)) \sigma_{4}(\Omega(1)) \), but also \( \sigma_{4}(\Omega(2)) \left( \sigma_{3}(\Omega(1)) \right)^2 \sigma_{4}(\Omega(1)) \) and therefore \( \sigma_{4}(\Omega(2)) \left( \sigma_{3}(\Omega(1)) \right)^2 \sigma_{4}(\Omega(1)) \) lie in \( \sqrt{J} \). We find the following equations

\[
\begin{align*}
(l_3 - r_3) \sigma_{3}(\Omega(1)), & \quad (l_4 - r_4) \sigma_{4}(\Omega(1)), \\
\sigma_{2}(\Omega(1)) \sigma_{3}(\Omega(1)) & \quad \sigma_{3}(\Omega(1)) \sigma_{4}(\Omega(1)), \\
\sigma_{2}(\Omega(1)) \sigma_{3}(\Omega(1)) & \quad \sigma_{3}(\Omega(1)) \sigma_{4}(\Omega(1)), \\
\sigma_{2}(\Omega(1)) \sigma_{3}(\Omega(1)) & \quad \sigma_{3}(\Omega(1)) \sigma_{4}(\Omega(1)).
\end{align*}
\]

This ideal is not reduced, as it contains \( \left( \sigma_{3}(\Omega(1)) \right)^2 \sigma_{4}(\Omega(1)) = \sigma_{3}(\Omega(1)) \sigma_{3}(\Omega(2)) \sigma_{4}(\Omega(2)) \sigma_{4}(\Omega(1)) - \sigma_{3}(\Omega(1)) (l_3 - r_3) \sigma_{3}(\Omega(1)) \sigma_{4}(\Omega(1)) \), but not \( \sigma_{3}(\Omega(1)) \sigma_{4}(\Omega(1)) \). But it is easy to find the reduced components from the given equations.

Our first, rough conjecture is that each summand of the equations \( \lambda_{\delta, \varepsilon}, \rho_{\delta, \varepsilon} \) lies in the radical \( \sqrt{J} \). Let us look at \( \rho_{\varepsilon-3, \varepsilon} \). We note that \( (P.4) \) and \( (P.5) \) yield the same term, being \( 21 \ 03 \ 12 \) and \( 21 \ 11 \ 03 \) respectively. As we have the equation \( 21 \ 11 \ 03 \) in the radical, these terms do not contribute new equations. As \( (P.8) \) itself already lies in the ideal, we are left
with 5 terms (a Catalan number!). One computes that indeed each summand lies in the radical. We look at the term in \( \rho_{\varepsilon-3,\varepsilon} \), coming from (P.6):

\[
\begin{array}{cccc}
31 & 11 & 12 & 03 \\
20 & 12 & . \\
11
\end{array}
\]

We claim that it is associated to the extended triangle

\[
\bullet \\
\bullet \quad \circ \\
\bullet \quad \bullet \\
\bullet 
\]

The easiest way to see this is via the numbers \( k_\varepsilon \) and \( \alpha_\varepsilon \), being \([k] = \{3, 1, 2, 2\}\) and \((\alpha) = (1, 3, 2, 1)\) in this case. One sees that there are \( \alpha_\varepsilon \) factors \( \sigma_\varepsilon^{(ij)} \), and they all have \( i + j = k_\varepsilon + 1 \). The other terms can be parametrised in the same way by the other extended triangles. The same picture parametrises the term in \( \lambda_{\varepsilon-3,\varepsilon} \), coming from (P.6):

\[
\begin{array}{cccc}
40 & 20 & 21 & 12 \\
20 & 12 & . \\
11
\end{array}
\]

In the radical we find \( 31 \quad 11 \quad 12 \quad 03 \quad 40 \quad 11 \quad 12 \quad 12 \). For the last term we compute as follows:

\[
\sigma_{\varepsilon-1}^{(12)}(\sigma_{\varepsilon-1}^{(21)} - \sigma_{\varepsilon-1}^{(12)}) = \sigma_{\varepsilon-1}^{(12)}(l_\varepsilon - r_{\varepsilon-1})\sigma_{\varepsilon-1}^{(22)} = (\sigma_{\varepsilon-1}^{(11)} - \sigma_{\varepsilon-1}^{(02)})\sigma_{\varepsilon-1}^{(22)},
\]

and we observe that the term contains the factors \( \sigma_{\varepsilon-2}^{(20)} \) and \( \sigma_{\varepsilon-2}^{(11)} \).

We can now make our conjecture more precise. As remarked before, we do not quite get the radical \( \sqrt{J} \) of the ideal of the base space, but an intermediate ideal, obtained from the summands in the generators of \( J \). As variables we use \( l_\varepsilon - r_\varepsilon \), and the \( \sigma_\varepsilon^{(ij)} \), which are connected by the relations

\[
\sigma_\varepsilon^{(i+1,j)} - \sigma_\varepsilon^{(i,j+1)} = (l_\varepsilon - r_\varepsilon)\sigma_\varepsilon^{(i+1,j+1)}.
\]

**Conjecture.** For the ideal \( J \) of the base space of the versal deformation of a cyclic quotient singularity of embedding dimension \( e \) and its radical \( \sqrt{J} \), and its radical \( \sqrt{J} = \sqrt{J'} \supset J' \supset J \), where \( J' \) is the ideal generated by \( (l_\varepsilon - r_\varepsilon)\sigma_\varepsilon^{(11)} \), for \( 2 < \varepsilon < e - 1 \) and monomials \( \lambda(\Delta_{\delta,\varepsilon}) \), \( 2 \leq \delta < \varepsilon < e - 1 \), and \( \rho(\Delta_{\delta,\varepsilon}) \), \( 2 < \delta < \varepsilon \leq e - 1 \), parametrised by sparse coloured triangles \( \Delta_{\delta,\varepsilon} \), of the form \( \prod_{\beta=\delta}^{\varepsilon} \sigma_\varepsilon^{(i_{\beta},j_{\beta})} \). The numbers \( i_\beta, j_\beta \) are determined as follows: if \( \alpha_\beta > 1 \), then in both \( \lambda(\Delta_{\delta,\varepsilon}) \) and \( \rho(\Delta_{\delta,\varepsilon}) \)

\[
i_\beta = \# \{ \text{black dots on right half-line } l_\varepsilon \}
\]

\[
 j_\beta = \# \{ \text{black dots on left half-line } l_\varepsilon \}
\]

but if \( \alpha_\beta = 1 \), then in \( \lambda(\Delta_{\delta,\varepsilon}) \)

\[
i_\beta = \# \{ \text{black dots on right half-line } l_\varepsilon \} + 1
\]

\[
 j_\beta = \# \{ \text{black dots on left half-line } l_\varepsilon \}
\]

and in \( \rho(\Delta_{\delta,\varepsilon}) \)

\[
i_\beta = \# \{ \text{black dots on right half-line } l_\varepsilon \}
\]

\[
 j_\beta = \# \{ \text{black dots on left half-line } l_\varepsilon \} + 1
\]
Example.

One has $\lambda(\triangle_{\delta,\varepsilon}) = 40 \ 11 \ 22 \ 11 \ 13$ and $\rho(\triangle_{\delta,\varepsilon}) = 31 \ 11 \ 22 \ 11 \ 04$.

Remark. The generators of $J'$ correspond to certain terms in generators of $J$, so there is a special subclass of $\alpha$-trees, counted by the Catalan numbers. It would be interesting to characterise them. The five trees of length 3 can be seen from the previous pictures. We now list all 14 trees of length 4.

Inductive proofs about the reduced components often use the procedure of blowing up and blowing down [11, 1.1]. The term comes from the analogy with chains of rational curves on a smooth surface, which can be described by continued fractions. For sparse coloured triangles it means the following [3, Lemma 1.8].

Blowing up is a way to obtain an extended triangle of height $e - 2$ from an extended triangle of height $e - 3$. Choose an index $2 \leq \varepsilon \leq e$. We define a shift function $s: \{2, \ldots, e - 1\} \rightarrow \{2, \ldots, \varepsilon - 1\} \cup \{\varepsilon + 1, \ldots, e\}$ by $s(\beta) = \beta$ if $\beta \in \{2, \ldots, \varepsilon - 1\}$ and $s(\beta) = \beta + 1$ if $\beta \in \{\varepsilon, \ldots, e - 1\}$. The blow-up $\text{Bl}_\varepsilon(\triangle)$ of $\triangle$ at the index $\varepsilon$ is the triangle with $(s(\beta), s(\gamma)) \in B(\text{Bl}_\varepsilon(\triangle))$ if and only if $(\beta, \gamma) \in B(\triangle)$, and from the points on the line $l_\varepsilon$ only $(\varepsilon - 1, \varepsilon)$ and
\((\varepsilon, \varepsilon + 1)\) are black. If \(\varepsilon = 2\), then only \((2, 3)\) is black, while only \((\varepsilon - 1, \varepsilon)\) is black if \(\varepsilon = e\).

By deleting the base line we get the blow-up \(\text{Bl}_\varepsilon(\Delta)\). In terms of pictures this means that one moves the sector, bounded by \(l^\varepsilon\) and \(l^{\varepsilon-1}\) with lowest point \((\varepsilon - 1, \varepsilon)\), one position up, and moves the arising two triangles sideways, to make room for a new line \(l^\varepsilon\), which has no black dots in \(\text{Bl}_\varepsilon(\Delta)\). If \(\varepsilon = 2\) or \(\varepsilon = e\) one just adds an extra line without black dots to the triangle.

**Example.**

![Diagram](image)

The inverse process is called blowing down at \(\varepsilon\). This is possible at \(\varepsilon\), for \(2 < \varepsilon < e\), if the dot \((\varepsilon - 1, \varepsilon + 1)\) is black; by lemma 3.2 the line \(l^\varepsilon\) does not contain black dots. Actually, if \(l^\varepsilon\) is empty, but \(\alpha^\varepsilon > 1\), i.e., there are black dots above it, then it follows that \((\varepsilon - 1, \varepsilon + 1)\) is black; otherwise there cannot be enough black dots in a triangle with black vertex on the lowest level.

**Proposition 5.1.** The ideal \(J'\) has \(C_{e-3} = \frac{1}{2} (2^{2e-3}) \) reduced components.

**Proof.** If \(\sigma^\varepsilon_{(1)} = 0\) for all \(2 < \varepsilon < e - 1\), the equations are satisfied: if a triangle \(\Delta\) contains black dots, there has to be at least one on the base line, at \((\varepsilon - 1, \varepsilon + 1)\), so \(\sigma^\varepsilon_{(1)} = 0\) for that \(\varepsilon\); an equation \(\lambda(\Delta)\) for an empty triangle ends with a \(\sigma^\varepsilon_{(1)}\) for some \(\varepsilon < e - 1\), and \(\rho(\Delta)\) starts with a \(\sigma^\varepsilon_{(1)}\) for some \(\varepsilon > 2\). So the Artin component is a component.

Suppose now that there exists an \(\varepsilon\) with \(\sigma^\varepsilon_{(1)} \neq 0\). Let \(J'_\varepsilon\) be the saturation of \(J'\) by \(\sigma^\varepsilon_{(1)}\), i.e., \(J'_\varepsilon = \cup(J' : (\sigma^\varepsilon_{(1)})^i)\). It yields the equation \(l^\varepsilon - r^\varepsilon = 0\), so \(\sigma^\varepsilon_{(20)} = \sigma^\varepsilon_{(11)} = \sigma^\varepsilon_{(02)}\). We conclude that \(\sigma^\varepsilon_{(20)} = \sigma^\varepsilon_{(11)} = 0\) and \(\sigma^\varepsilon_{(11)} = \sigma^\varepsilon_{(20)} = 0\) (if \(\varepsilon = 2\) or \(\varepsilon = e - 1\) the statements have to be modified somewhat). As \(\sigma^\varepsilon_{(20)} - \sigma^\varepsilon_{(11)} = \sigma^\varepsilon_{(21)} (l^\varepsilon - r^\varepsilon - 1)\), one has \(\sigma^\varepsilon_{(21)} (l^\varepsilon - r^\varepsilon - 1) \in J'_\varepsilon\), and likewise \(\sigma^\varepsilon_{(12)} (l^\varepsilon - r^\varepsilon + 1) \in J'_\varepsilon\).

Consider a monomial \(\lambda(\Delta_{\beta,\gamma})\) or \(\rho(\Delta_{\beta,\gamma})\), containing \(\sigma^\varepsilon_{(1)}\) (or \(\sigma^\varepsilon_{(20)}\)) if \(\beta = \varepsilon\), or \(\sigma^\varepsilon_{(02)}\) if \(\varepsilon = \gamma\). Then \(\Delta_{\beta,\gamma} = \text{Bl}_\varepsilon(\Delta_{\beta,\gamma-1})\), and the monomial in question is obtained from \(\lambda(\Delta_{\beta,\gamma-1})\) or \(\rho(\Delta_{\beta,\gamma-1})\) by leaving the \(\sigma^\varepsilon_{(i)}\) unchanged for \(\delta < e - 1\), replacing \(\sigma^\varepsilon_{(i)}\) by \(\sigma^\varepsilon_{(i+1)}\), inserting \(\sigma^\varepsilon_{(1)}\), replacing \(\sigma^\varepsilon_{(i)}\) by \(\sigma^\varepsilon_{(i+1)}\) and \(\sigma^\varepsilon_{(j)}\) by \(\sigma^\varepsilon_{(j+1)}\) for \(\delta > \varepsilon\). We claim that the polynomials considered so far, together with the monomials, not involving \(\sigma^\varepsilon_{(ij)}\), \(\sigma^\varepsilon_{(ij)}\) and \(\sigma^\varepsilon_{(ij)}\) at all, generate the ideal \(J'_\varepsilon\). It follows then that this ideal, up to renaming the coordinates as above, and up to some linear equations, is an ideal of the same type as \(J'\), but one embedding dimension lower. By induction we conclude that \(J'_\varepsilon\) describes components, parametrised by sparse coloured triangles, blown up at \(\varepsilon\). By varying \(\varepsilon\), with \(\sigma^\varepsilon_{(11)} \neq 0\), we obtain all components (except the Artin component, which we already have).

It remains to prove the claim. The not yet considered generators of \(J'\) come in two types, those containing \(\sigma^\varepsilon_{(ij)}\) with \(i + j > 2\), and those not containing a \(\sigma^\varepsilon_{(ij)}\) at all, but ending with \(\sigma^\varepsilon_{(1)}\) or starting with \(\sigma^\varepsilon_{(1)}\). Regarding the first type, we prove that such a monomial is a
multiple of one of the claimed generators, by induction on the length of the monomial. For this we note that the claim holds for the monomial if and only if it holds for the monomial, obtained by blowing down the triangle at $\delta$ with $\delta \neq \varepsilon - 1, \varepsilon + 1$. The base of the induction is the case of monomials containing $\sigma_{(20)}(\varepsilon-1), \sigma_{(11)}(\varepsilon-1), \sigma_{(11)}(\varepsilon+1)$ or $\sigma_{(21)}(\varepsilon+1)$. As to the second type, we consider those starting with $\sigma_{(ij)}$. If the term is of the form $\lambda(\Delta \varepsilon + 1, \gamma)$, then it starts with $\sigma_{(j0)}(\varepsilon+1)$. One has $\sigma_{(j0)}(\varepsilon+1) = \sigma_{(j0)}(\varepsilon-1,1) + \sigma_{(j1)}(l_{\varepsilon+1} + r_{\varepsilon+1})$. The term obtained by replacing $\sigma_{(j0)}(\varepsilon+1)$ by $\sigma_{(j1)}(\varepsilon+1)$, is one of our generators. We are left with monomials, starting with $\sigma_{(j)}(\varepsilon-1,1)$; such monomials also come from $\rho(\Delta \varepsilon + 1, \gamma)$. For those the claim is again shown by induction, using blowing down. □

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