Exact solutions of hyperbolic systems of kinetic equations. Application to Verhulst model with random perturbation*

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Abstract

For hyperbolic first-order systems of linear partial differential equations (master equations), appearing in description of kinetic processes in physics, biology and chemistry we propose a new procedure to obtain their complete closed-form non-stationary solutions. The methods used include the classical Laplace cascade method as well as its recent generalizations for systems with more than 2 equations and more than 2 independent variables. As an example we present the complete non-stationary solution (probability distribution) for Verhulst model driven by Markovian coloured dichotomous noise.

Keywords: Master equations, hyperbolic systems, complete non-stationary solutions, kinetic processes, Verhulst model.

1 Introduction

This paper is devoted to a novel application of methods of explicit integration of hyperbolic linear systems of PDEs recently developed in \cite{18, 19, 20} to an important class of dynamical nonlinear systems driven by a coloured noise.

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Modelling dynamical systems, in which stochastic behaviour is involved, as a rule leads to nonlinear stochastic differential equations for the dynamical variable or sets of dynamical variables. The comprehensive statistical treatment of these variables may be formulated with the aid of Liouville stochastic equation for the probability distribution (see [8]).

**Example 1.** In this paper we consider as the simplest example the following one-dimensional dynamical system

\[ \dot{x} = p(x) + \alpha(t)q(x), \]

where \( x(t) \) is the dynamical variable, \( p(x) \), \( q(x) \) are given functions of \( x \), \( \alpha(t) \) is the random function with known statistical characteristics. The model (1) arises in different applications (see for example [8, 10] and bibliography therein). An important application of this model consists in study of noise-induced transitions in physics, chemistry and biology. The functions \( p(x), q(x) \) are often taken polynomial. For example, if we set \( p(x) = p_1x + p_2x^2 \), \( q(x) = q_2x^2 \), \( p_1 > 0 \), \( p_2 < 0 \), \( |p_2| > q_2 > 0 \), then the equation (1) describes the population dynamics when resources (nutrition) fluctuate (Verhulst model). In the following we will assume \( \alpha(t) \) to be binary (dichotomic) noise \( \alpha(t) = \pm 1 \) with switching frequency \( 2\nu > 0 \). As one can show (see [17, 11]), the averages \( W(x,t) = \langle \tilde{W}(x,t) \rangle \) and \( W_1(x,t) = \langle \alpha(t)\tilde{W}(x,t) \rangle \) for the probability density \( \tilde{W}(x,t) \) in the space of possible trajectories \( x(t) \) of the ODE satisfy a system of the form (11) (also called “master equations”):

\[
\begin{cases}
W_t + (p(x)W)_x + (q(x)W_1)_x = 0, \\
(W_1)_t + 2\nu W_1 + (p(x)W) + (q(x)W_1)_x = 0.
\end{cases}
\]

We suppose that the initial condition \( W(x,0) = W_0(x) \) for the probability distribution is nonrandom. This implies that the initial condition for \( W_1(x,t) \) at \( t = 0 \) is zero: \( W_1(x,0) = \langle \alpha(0)\tilde{W}(x,0) \rangle = \langle \alpha(0)\rangle W_0(x) = 0 \). The probability distribution \( W(x,t) \) should be nonnegative and normalized for all \( t \): \( W(x,t) \geq 0 \), \( \int_{-\infty}^{\infty} W(x,t) \, dx = 1 \).

**Example 2.** Let us consider the following dynamical system driven by two statistically independent Markovian dichotomous noises \( \alpha(t) \) and \( \beta(t) \):

\[ \dot{x} = p(x) + \alpha(t)q(x) + \beta(t)g(x). \]

The averaged probability density \( W(x,t) = \langle \tilde{W}(x,t) \rangle |_{\alpha,\beta} \) satisfies the following system of master equations:

\[
\begin{cases}
W_t + (p(x)W)_x + (q(x)W_1)_x + (g(x)P)_x = 0, \\
(W_1)_t + 2\nu W_1 + (p(x)W) + (q(x)W_1)_x + (g(x)Q)_x = 0, \\
(P)_t + 2\mu P + (p(x)P)_x + (q(x)Q)_x + \beta^2 (g(x)W)_x = 0, \\
(Q)_t + 2(\mu + \nu)Q + (p(x)Q)_x + \alpha^2 (q(x)P)_x + \beta^2 (g(x)W_1)_x = 0,
\end{cases}
\]
where the auxiliary functions $W_1(x,t)$, $P(x,t)$ and $Q(x,t)$ are some averages over realizations of the noises $\alpha(t)$, $\beta(t)$. They play the same auxiliary role as the function $W_1(x,t)$ in the system [2]. We suppose that $\alpha(t) = \pm \alpha$, $\beta(t) = \pm \beta$ for any $t$. The characteristic switching frequencies of these random noises are $2\nu$ and $2\mu$ respectively. We again suppose that the initial condition $W(x,0) = W_0(x)$ is non-random, so the initial conditions for $W_1(x,t)$, $P(x,t)$ and $Q(x,t)$ at $t = 0$ are zeros. The probability distribution $W(x,t)$ should be nonnegative and normalized for all $t$: $W(x,t) \geq 0$, $\int_{-\infty}^{\infty} W(x,t) \, dx \equiv 1$.

Example 3. We can also consider nonlinear dynamical systems of higher order:

\[
\begin{cases}
\dot{x} = f(x,y) + \alpha(t)q(x,y), \\
\dot{y} = d(x,y) + \beta(t)s(x,y),
\end{cases}
\]

(5)

where $f$, $g$, $d$, $s$ are given functions. We use the same conventions for the noises $\alpha(t)$, $\beta(t)$ and the averaged probability distributions as in Example 2. The master equations for the main average $P(x,y,t)$ and auxiliary averages $P_1(x,y,t)$, $Q(x,y,t)$ and $Q_1(x,y,t)$ become

\[
\begin{align*}
& P_t + (fP)_x + (gP)_x + (dP)_y + (sQ)_y = 0, \\
& (P_1)_t + 2\nu P_1 + (fP_1)_x + \alpha^2 (gP)_x + (dP_1)_y + (sQ_1)_y = 0, \\
& Q_t + 2\mu Q + (fQ)_x + (gQ_1)_x + (dQ)_y + \beta^2 (sP)_y = 0, \\
& (Q_1)_t + 2(\mu + \nu)Q_1 + (fQ_1)_x + \alpha^2 (gQ)_x + (dQ_1)_y + \beta^2 (sP)_y = 0,
\end{align*}
\]

(6)

The Cauchy initial value problem is formulated in the same way as above: $P(x,y,0) = P_0(x,y)$ is nonrandom; $P_1(x,y,t)$, $Q(x,y,t)$ and $Q_1(x,y,t)$ at $t = 0$ are zeros; $P(x,y,t)$ should be nonnegative and normalized for all $t$: $P(x,y,t) \geq 0$, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x,y,t) \, dx \, dy \equiv 1$.

Numerous publications (see [8] [10] for the bibliography) are devoted to solution of the system [2] asymptotically for $t \to \infty$, that is to stationary solutions. A number of important phenomena of noise-induced phase transitions with applications in physics, chemistry and biology were discovered in this stationary case.

There are only a few publications dealing with non-stationary solutions of [2]. We refer to [16] and the recent paper [3], where some (incomplete) exact solutions of the system [2] for particular forms of the functions $p(x)$ and $q(x)$ were obtained.

As we show in this paper, some interesting non-stationary kinetic equations (master equations) for probability distributions allow complete explicit closed-form solution of the general Cauchy initial value problem. These complete solutions are obtainable through a modification of the classical Laplace cascade method (see e.g. [4] [5] [6]). This method is applicable to hyperbolic systems with two first-order linear PDEs in the plane (as [2] above) or a single second-order linear PDE in the plane. A preliminary closed-form complete solution for [2] was obtained by this method in [19]. In that paper a much more general method of explicit integration, applicable to arbitrary hyperbolic higher-order linear systems (or a single higher-order linear
PDE) in the plane was developed. Later another generalization was proposed in [20], it gives closed-form complete solutions for some special class of second-order linear hyperbolic equations with more than two independent variables.

We give a brief account of the classical Laplace method as well as its new generalizations in Section 2. Section 3 is devoted to a detailed study of the system (2) for the simplest case of polynomial coefficients \( p(x) = p_1 x + p_2 x^2 \), \( q(x) = q_2 x^2 \), \( p_1 > 0 \), \( p_2 < 0 \), \( |p_2| > q_2 > 0 \) (Verhulst model). We show that for an infinite sequence of values of the switching frequency \( \nu \equiv p_1 \), \( \nu \equiv 2p_1 \), \( \nu \equiv 3p_1 \), …, the complete explicit solution of the Cauchy problem is obtainable by our methods.

In the final Section 4 we discuss future prospects and possible applications of our methods to more complicated systems of type (4), (6).

## 2 Explicit integration of hyperbolic systems

### 2.1 Laplace cascade method

We give here only a special form of this method suitable for our purpose, see [11] [12] [13] for more details.

Suppose we are given a 2 × 2 first-order linear system of PDEs

\[
\begin{pmatrix}
  v_1 \\
  v_2
\end{pmatrix}
= \begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
  v_1 \\
  v_2
\end{pmatrix}
+ \begin{pmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{pmatrix}
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix}
\]  

(7)

with \( a_{ij} = a_{ij}(x, y) \), \( b_{ij} = b_{ij}(x, y) \). We will suppose hereafter that (7) is strictly hyperbolic, i.e. the eigenvalues \( \lambda_1(x, y) \), \( \lambda_2(x, y) \) of the matrix \( (a_{ij}) \) are real and distinct. Let \( \hat{p}_1 = (p_{11}(x, y), p_{12}(x, y)) \), \( \hat{p}_2 = (p_{21}(x, y), p_{22}(x, y)) \) be the corresponding left eigenvectors: \( \sum_k p_{ik} a_{kj} = \lambda_i p_{ij} \). Form the following first-order differential operators \( \hat{X}_i = \hat{D}_x - \lambda_i \hat{D}_y \) (the characteristic vector fields) and change the initial unknown functions \( v_i \) to new characteristic functions \( u_i = \sum_k p_{ik} v_k \). Then the operator rescaling \( \hat{X}_i \rightarrow \gamma_i(x, y) X_i \) and gauge transformations \( u_i \rightarrow g_i(x, y) u_i \). It is easy to check that the gauge transformations to not change the Laplace invariants of the system \( h = \hat{X}_2(\alpha_{11}) - \hat{X}_1(\alpha_{22}) - \hat{X}_1 \hat{X}_2 \ln(\alpha_{12}) - \hat{X}_1(P) + \hat{P} \alpha_{11} + \alpha_{12} \alpha_{21} + (\alpha_{22} + \hat{X}_2(\ln \alpha_{12}) + P) Q \) and \( k = \alpha_{12} \alpha_{21} \). Here \( P(x, y) \) and \( Q(x, y) \) are the coefficients of the commutator

\[
[\hat{X}_1, \hat{X}_2] = \hat{X}_1 \hat{X}_2 - \hat{X}_2 \hat{X}_1 = P(x, y) \hat{X}_1 + Q(x, y) \hat{X}_2.
\]  

(9)
These invariants \( h(x, y) \) and \( k(x, y) \) are just the classical Laplace invariants (cf. \[1, 5, 6, 19\]) of the second-order scalar equation, obtained after elimination of \( u_2 \) from (8). Rescaling transformations of \( \hat{X}_i \) change the Laplace invariants multiplicatively: \( h \rightarrow \gamma_1 \gamma_2 h, \ k \rightarrow \gamma_1 \gamma_2 k \).

Starting from (8) one can obtain two different (inequivalent w.r.t. gauge transformations) second-order scalar equations, eliminating either \( u_1 \) or \( u_2 \) from (8). This observation gives rise to the Laplace cascade method of integration of strictly hyperbolic systems in characteristic form (8):

\((L)_1\) If \( k \) vanishes then either \( \alpha_{12} \) or \( \alpha_{21} \) vanishes so the system becomes triangular:

\[
\begin{align*}
\dot{X}_1 u_1 &= \alpha_{11} u_1, \\
\dot{X}_2 u_2 &= \alpha_{21} u_1 + \alpha_{22} u_2,
\end{align*}
\]

or

\[
\begin{align*}
\dot{X}_1 u_1 &= \alpha_{11} u_1 + \alpha_{12} u_2, \\
\dot{X}_2 u_2 &= \alpha_{22} u_2.
\end{align*}
\]

If we perform an appropriate change of coordinates \((x, y) \rightarrow (\overline{x}, \overline{y})\) (NOTE: for this we have to solve first-order nonlinear ODEs \( dy/dx = -\lambda_i(x, y) \), cf. Appendix in [7]) one can suppose \( \dot{X}_1 = \overline{D}_\overline{x}, \ \dot{X}_2 = \overline{D}_\overline{y} \) and obtain the complete solution of (10) in quadratures: if for example \( \alpha_{12} = 0 \), then

\[
\begin{align*}
u_1 &= Y(\overline{y}) \exp \left( - \int \alpha_{11} \, d\overline{x} \right), \\
u_2 &= \exp \left( - \int \alpha_{22} \, d\overline{y} \right) \left( X(\overline{x}) + \int Y(\overline{y}) \exp \left( \int \left( \alpha_{11} \, d\overline{x} - \alpha_{22} \, d\overline{y} \right) \right) \, d\overline{y} \right)
\end{align*}
\]

where \( X(\overline{x}) \) and \( Y(\overline{y}) \) are two arbitrary functions of the characteristic variables \( \overline{x}, \overline{y} \) respectively.

\((L)_2\) If \( k \neq 0 \), transform the system into a second-order scalar equation eliminating \( u_2 \) from (8): from the first equation

\[
u_2 = (\dot{X}_1 u_1 - \alpha_{11} u_1)/\alpha_{12},
\]

substitute this expression into the second equation obtaining

\[
\dot{L} u_1 = \dot{X}_2 \frac{1}{\alpha_{12}} (\dot{X}_1 u_1 - \alpha_{11} u_1) - \alpha_{21} u_1 - \frac{u_2}{\alpha_{12}} (\dot{X}_1 u_1 - \alpha_{11} u_1) = 0.
\]

Now, using the commutator relation (9), we can represent \( \dot{L} u_1 \) as

\[
\dot{L} u_1 = (\dot{X}_1 \dot{X}_2 + \beta_1 \dot{X}_1 + \beta_2 \dot{X}_2 + \beta_3) u_1 = (\dot{X}_1 + \beta_2)(\dot{X}_2 + \beta_1) u_1 - hu_1 = 0.
\]

From this form we see that this equation is equivalent to another \( 2 \times 2 \) system

\[
\mathcal{H}_{(1)} : \begin{cases}
\dot{X}_2 u_1 = -\beta_1 u_1 + u_2, \\
\dot{X}_1 u_2 = hu_1 - \beta_2 u_2.
\end{cases}
\]

This new system (we will call it \( X_1 \)-transformed system) has the same characteristic form (8) with different coefficients in the right-hand side. It also has new Laplace invariants \( h_{(1)}, k_{(1)} \), and it turns out that \( k_{(1)} \) equals to the invariant \( h \) of the original system. So if we have \( k_{(1)} = h = 0 \), we solve this new system in quadratures and using the same differential substitution (12) we obtain the complete solution of the original equation \( \dot{L} u = 0 \).

\((L)_3\) If again \( k_{(1)} \neq 0 \), apply this \( X_1 \)-transformation several times, obtaining a sequence of \( 2 \times 2 \) characteristic systems \( \mathcal{H}_{(2)}, \mathcal{H}_{(3)}, \ldots \). If on any step we get \( k_{(m)} = 0 \),
0, we solve the corresponding system in quadratures and, using the differential substitutions (12), obtain the complete solution of the original system. Alternatively one may perform $\hat{X}_2$-transformations, eliminating $u_1$ instead of $u_2$ on step ($L_2$). In fact this $\hat{X}_2$-transformation is a reverse of the $\hat{X}_1$-transformation up to a gauge transformation (see [1]). So we have (infinite in general) chain of systems

$$\ldots \hat{X}_2 \mathcal{H}_{(-2)} \hat{X}_2 \mathcal{H}_{(-1)} \hat{X}_2 \mathcal{H} \hat{X}_1 \mathcal{H}_{(1)} \hat{X}_1 \mathcal{H}_{(2)} \hat{X}_1 \ldots$$

and the corresponding chain of their Laplace invariants

$$\ldots, k_{(-3)}, k_{(-2)}, k_{(-1)}, k, k_{(1)} = h, k_{(2)}, k_{(3)}, \ldots$$

We do not need to keep the invariants $h_{(i)}$ in (15) since $k_{(i)} = h_{(i-1)}$. If on any step we have $k_{(N)} = 0$ then the chains (14) and (15) can not be continued: the differential substitution (12) is not defined; precisely on this step the corresponding system (8) is triangular and we can find its complete solution as well as the complete solution for any of the systems of the chain (14).

As one may prove (see e.g. [4]) if the chain (14) is finite in both directions (i.e. we have $k_{(N)} = 0$, $k_{(-K)} = 0$ for some $N \geq 0$, $K \geq 0$) one may even obtain a quadrature-free expression for the general solution of the original system:

$$u_1 = c_0 F + c_1 F' + \ldots + c_N F^{(N)} + d_0 \tilde{G} + d_1 \tilde{G}' + \ldots + d_{K+1} \tilde{G}^{(K+1)};$$

$$u_2 = e_0 F + e_1 F' + \ldots + e_N F^{(N)} + f_0 \tilde{G} + f_1 \tilde{G}' + \ldots + f_{K+1} \tilde{G}^{(K+1)},$$

with definite $c_i(\mathfrak{p}, \mathfrak{q})$, $d_i(\mathfrak{p}, \mathfrak{q})$, $e_i(\mathfrak{p}, \mathfrak{q})$, $f_i(\mathfrak{p}, \mathfrak{q})$ and two arbitrary functions $F(\mathfrak{p})$, $G(\mathfrak{q})$ of the characteristic variables. Vice versa: existence of (a priori not complete) solution of the form (16) with arbitrary functions $F$, $G$ of characteristic variables implies $k_{(s)} = 0$, $k_{(-r)} = 0$ for some $s \leq N$, $r \leq K$. So minimal differential complexity of the answer (16) (the number of terms in it) is equal to the number of steps necessary to obtain vanishing Laplace invariants in the chains (14), (15) and consequently triangular systems. Complete proofs of these statement may be found in [4, t. 2], [5, 6] for the case $\hat{X}_1 = \hat{D}_x$, $\hat{X}_2 = \hat{D}_y$, for the general case cf. [6, p. 30] and [1].

We give a detailed example of application of this method in Section 3.

There were some attempts to generalize Laplace transformations for higher-order systems or the number of independent variables larger than 2, both in the classical time [9, 14, 15] and in the last decade [2, 19]. As one can show, all of them essentially try to triangulize the given system in some sense. A general definition of “generalized factorization” (triangulation) comprising all known practical methods was given in [18]. Unfortunately the theoretical considerations of [18] did not provide any algorithmic way of establishing generalized factorizability of a given higher-order operator or a given higher-order system. Below we present other approach for search of “generalized factorizations” resulting in explicit complete solution of some classes of hyperbolic systems.

6
2.2 Generalized Laplace cascade method for \( n \times n \) hyperbolic systems in the plane

Any \( n \times n \) first-order linear system

\[
(v_i)_x = \sum_{k=1}^{n} a_{ik}(x, y)(v_k)_y + \sum_{k=1}^{n} b_{ik}(x, y)v_k
\]

(17)

with strictly hyperbolic matrix \((a_{ik})\) (i.e. with real and distinct eigenvalues of this matrix) is equivalent to a system in characteristic form

\[
\hat{X}_i u_i = \sum_k \alpha_{ik}(x, y)u_k,
\]

(18)

as a straightforward calculation similar to that in the beginning of Section 2.1 immediately shows.

Our generalization of the Laplace transformations consists in the following.

\( \mathcal{L}_1 \) For a given \( n \times n \) characteristic system (18) choose one of its equations with a non-vanishing off-diagonal coefficient \( \alpha_{ik} \neq 0 \), find \( u_k = (\hat{X}_i u_i - \sum_{s \neq k} \alpha_{is} u_s)/\alpha_{ik} \) and substitute this expression into all other equations of the system. We obtain one second-order equation

\[
\hat{X}_k(\frac{1}{\alpha_{ik}}(\hat{X}_i u_i - \sum_{s \neq k} \alpha_{is} u_s)) - \sum_{p \neq k} \alpha_{kp} u_p - \frac{\alpha_{kk}}{\alpha_{ik}}(\hat{X}_i u_i - \sum_{s \neq k} \alpha_{is} u_s) = 0
\]

(19)

and \( n - 2 \) first-order equations

\[
\hat{X}_j u_j - \sum_{s \neq k} \alpha_{js} u_s - \frac{\alpha_{jk}}{\alpha_{ik}}(\hat{X}_i u_i - \sum_{s \neq k} \alpha_{is} u_s) = 0
\]

(20)

for \( j \neq i, k \).

\( \mathcal{L}_2 \) The second step consists in rewriting the system (19), (20) in the following form with slightly modified unknown functions \( \overline{u}_j = u_j + \rho_j(x, y)u_i, \ j \neq i, k, \overline{u}_i \equiv u_i \), new coefficients \( \beta_{pq}(x, y) \) but the same characteristic operators \( \hat{X}_p \):

\[
\hat{X}_i(\frac{1}{\alpha_{ik}}(\hat{X}_k \overline{u}_i - \sum_{s \neq k} \beta_{is} \overline{u}_s)) - \sum_{p \neq k} \beta_{kp} \overline{u}_p - \frac{\beta_{kk}}{\alpha_{ik}}(\hat{X}_k \overline{u}_i - \sum_{s \neq k} \beta_{is} \overline{u}_s) = 0,
\]

(21)

\[
\hat{X}_j \overline{u}_j - \sum_{s \neq k} \beta_{js} \overline{u}_s - \frac{\beta_{jk}}{\alpha_{ik}}(\hat{X}_k \overline{u}_i - \sum_{s \neq k} \beta_{is} \overline{u}_s) = 0, \ j \neq i, k.
\]

(22)

As one can prove this is always possible in a unique way.

\( \mathcal{L}_3 \) Introducing \( \overline{u}_k = \frac{1}{\alpha_{ik}}(\hat{X}_k \overline{u}_i - \sum_{s \neq k} \beta_{is} \overline{u}_s) \) rewrite (21), (22) as the transformed characteristic system

\[
\begin{aligned}
\hat{X}_i \overline{u}_k &= \sum_p \beta_{kp} \overline{u}_p, \\
\hat{X}_k \overline{u}_i &= \sum_{s \neq k} \beta_{is} \overline{u}_s + \alpha_{ik} \overline{u}_k, \\
\hat{X}_j \overline{u}_j &= \sum_s \beta_{js} \overline{u}_s, \ j \neq i, k.
\end{aligned}
\]

(23)
The reason of doing such generalized Laplace transformation consists in the fact that after it (or after a chain of such transformations) one may obtain a triangular system \((\mathcal{L}_3), (\mathcal{L}_2), (\mathcal{L}_1)\) obtain the complete solution of the original \(n \times n\) hyperbolic system.

Cf. \cite{19} for the details, an example and the proof of correctness of the step \((\mathcal{L}_2)\).

2.3 Explicit solution of equations with more than two independent variables

The method described in this Section is based on an idea given by Ulisse Dini in 1902 (cf. \cite{20} for references and details). We will limit here to a simple example showing the idea of the method.

Let us take the following equation:

\[
Lu = (\hat{\mathcal{D}}_x \hat{\mathcal{D}}_y + x \hat{\mathcal{D}}_x \hat{\mathcal{D}}_z - \hat{\mathcal{D}}_z)u = 0.
\]  

(24)

It has three independent derivatives \(\hat{\mathcal{D}}_x, \hat{\mathcal{D}}_y, \hat{\mathcal{D}}_z\), so the Laplace method is not applicable. On the other hand its principal symbol splits into product of two first-order factors:

\[
\xi_1 \xi_2 + x \xi_1 \xi_3 = \xi_1 (\xi_2 + x \xi_3).
\]

This is no longer a typical case for hyperbolic operators in dimension 3; we will use this special feature in introducing two characteristic operators \(\hat{X}_1 = \hat{\mathcal{D}}_x, \hat{X}_2 = \hat{\mathcal{D}}_y + x \hat{\mathcal{D}}_z\). We have again a nontrivial commutator \([\hat{X}_1, \hat{X}_2] = \hat{\mathcal{D}}_z = \hat{X}_3\). The three operators \(\hat{X}_i\) span the complete tangent space in every point \((x, y, z)\). Using them one can represent the original second-order operator in one of two partially factorized forms:

\[
L = \hat{X}_2 \hat{X}_1 - \hat{X}_3 = \hat{X}_1 \hat{X}_2 - 2 \hat{X}_3.
\]

Let us use the first one and transform the equation into a system of two first-order equations:

\[
Lu = 0 \iff \begin{cases}
\hat{X}_1 u = v, \\
\hat{X}_3 u = \hat{X}_2 v.
\end{cases}
\]

(25)

Here comes the difference with the classical case \(\text{dim} = 2\): we can not express \(u\) as we did in \((12)\). But we have another obvious possibility instead: cross-differentiating the left hand sides of \((25)\) and using the obvious identity \([\hat{X}_1, \hat{X}_3] = [\hat{\mathcal{D}}_x, \hat{\mathcal{D}}_z] = 0\) we get \(\hat{X}_1 \hat{X}_2 v = \hat{\mathcal{D}}_x (\hat{\mathcal{D}}_y + x \hat{\mathcal{D}}_z)v = \hat{X}_3 v = \hat{\mathcal{D}}_z v\) or \(0 = \hat{\mathcal{D}}_x (\hat{\mathcal{D}}_y + x \hat{\mathcal{D}}_z)v - \hat{\mathcal{D}}_z v = (\hat{\mathcal{D}}_x \hat{\mathcal{D}}_y + x \hat{\mathcal{D}}_x \hat{\mathcal{D}}_z)v = (\hat{\mathcal{D}}_y + x \hat{\mathcal{D}}_z) \hat{\mathcal{D}}_x v = \hat{X}_2 \hat{X}_1 v\).

Since we have now another second-order equation which is “naively” factorizable we easily find its complete solution:

\[
v = \int \phi(x, xy - z) \, dx + \psi(y, z)
\]

where \(\phi\) and \(\psi\) are two arbitrary functions of two variables each; they give the general solutions of the equations \(\hat{X}_2 \phi = 0, \hat{X}_1 \psi = 0\).
Now we can find $u$:

$$u = \int \left( v \, dx + (\hat{D}_y + x \hat{D}_z) v \, dz \right) + \theta(y),$$

where an extra free function $\theta$ of one variable appears as a result of integration in (25).

So we have seen that such Dini transformations (24) in some cases may produce a complete solution in explicit form for a non-trivial three-dimensional equation (23). This explicit solution can be used to solve initial value problems for (24).

## 3 Verhulst model

Here we describe in detail the procedure of solution for the system (2).

The characteristic operators and left eigenvectors of this $2 \times 2$-system are simple:

$$\hat{X}_i = \hat{D}_t - \lambda_i \hat{D}_x, \quad \lambda_{1,2} = -p(x) \pm q(x),\quad p_{11} = p_{21} = p_{22} = 1,\quad p_{12} = -1.$$  

The characteristic system (8) for the new characteristic functions $u_1 = W - W_1$, $u_2 = W + W_1$ is

$$\begin{align*}
\hat{X}_1 u_1 &= -(p_x - q_x + \nu) u_1 + \nu u_2, \\
\hat{X}_2 u_2 &= \nu u_1 - (p_x + q_x + \nu) u_2.
\end{align*}$$

(26)

The Laplace invariants are $h = \nu^2 - [p_{xx} q^2 (p + q) + p_x^2 q^2 - p_x q_x q(3p + q) - q_{xx} p q(p + q) - q_x^2 p(2p + q)]/q^2$, $k = \nu^2$, so if $\nu$, $p(x)$ and $q(x)$ satisfy a second-order differential relation $h = 0$, one can solve (2) in quadratures. Especially simple formulas may be obtained for polynomial $p(x) = p_1 x + p_2 x^2$, $q(x) = q_2 x^2$: in this case $k = \nu^2$, $h = h_{(-2)} = \nu^2 - p_1^2$ so if $\nu = p_1$, one may solve (2) explicitly. It is convenient at this point to use the dimensionless variable $\tau = \nu t$; so we have to change $t \mapsto \tau$, $\nu \mapsto 1$, $p_1 \mapsto 1$ and change $p_2$, $q_2$ respectively. For simplicity we will still use the same notations $p_2$, $q_2$.

After the necessary transformation, described in Section 2.2, we obtain the following quadrature-free expression for the complete solution of the system (2):

$$W = \frac{q_2}{x^2} \left[ F'(\tau) - F(\tau) + G'(\gamma) - G(\gamma) \right],$$

$$W_1 = \frac{1}{x^3} \left[ -q_2 x G'(\gamma) + (1 + p_2 x) G(\gamma) + q_2 F'(\tau) \right] + (1 + p_2 x) F(\tau),$$

(27)

where $\tau = -t + \ln \frac{x}{1 + (p_2 + q_2)x}$, $\gamma = -t + \ln \frac{x}{1 + (p_2 - q_2)x}$ are the characteristic variables ($\hat{X}_2 \tau = 0$, $\hat{X}_1 \gamma = 0$) and $F$, $G$ are two arbitrary functions of the corresponding characteristic variables.

For the case $\nu^2 \neq p_1^2$ we can compute other Laplace invariants of the chain (15): $h_{(1)} = h_{(-3)} = \nu^2 - 4p_1^2$, $h_{(2)} = h_{(-4)} = \nu^2 - 9p_1^2$, $h_{(3)} = h_{(-5)} = \nu^2 - 16p_1^2$, etc., so for the fixed $p(x) = p_1 x + p_2 x^2$, $q(x) = q_2 x^2$ and $\nu = \pm p_1$, $\nu = \pm 2p_1$, $\nu = \pm 3p_1$, ... one can obtain closed-form quadrature-free complete solution of the system (2),

with increasing complexity of the answer (16).
Now we demonstrate how the formulas (27) may be used to solve the Cauchy initial value problem. For this set $\tau = 0$ inside the variables $\overline{x}$, $\overline{y}$ and equate $W(x,0) = W_0(x)$, $W_1(x,0) = 0$. Since now $\overline{x} = \ln \frac{x}{1+(p_2+q_2)x}$, $\overline{y} = \ln \frac{x}{1+(p_2-q_2)x}$, one can express the derivatives $F' = dF/d\overline{x}$, $G' = dG/d\overline{y}$ as $F' = \frac{dF}{dx} \frac{dx}{d\overline{x}}$, $G' = \frac{dG}{dx} \frac{dx}{d\overline{y}}$ and obtain from (27) a system of two linear ODEs for $F(x) = F(\overline{x}(x))$, $G(x) = G(\overline{y}(x))$. It may be solved explicitly (see an explanation of this fact in Section 4) for any $W_0(x)$:

$$F(x) = \frac{1}{2q_2(1+(p_2+q_2)x)} \left[ -x \int_{c_0}^{x} \frac{W_0(\theta)}{\theta} d\theta + (1+q_2x) \int_{c_1}^{x} W_0(\theta) d\theta \right],$$

$$G(x) = \frac{1}{2q_2(1+(p_2-q_2)x)} \left[ x \int_{c_0}^{x} \frac{W_0(\theta)}{\theta} d\theta + (q_2x-1) \int_{c_1}^{x} W_0(\theta) d\theta \right].$$

(28)

Perform now the inverse substitution $F(\overline{x}) = F(x(\overline{x}))$, $G(\overline{y}) = G(x(\overline{y}))$ (for $\tau = 0$) to find the “true” functions $F(\overline{x})$, $G(\overline{y})$ suitable for substitution into (27) for any $\tau$. This final form of the explicit solution of the Cauchy problem is:

$$W(x, \tau) = \frac{1}{2q_2x^2} \left[ I_1(\dot{y}) - I_1(\dot{x}) \right] + \frac{W_0(\dot{y})}{2(e^\tau(1+(p_2-q_2)x) - x(p_2-q_2))^2}$$

$$+ \frac{W_0(\dot{x})}{2(e^\tau(1+(p_2+q_2)x) - x(p_2+q_2))^2},$$

$$W_1(x, \tau) = \frac{I_1(\dot{y}) - I_1(\dot{x})}{2q_2x^2} \left[ (e^{-\tau} - 1)p_2x - 1 \right] + \frac{e^{-\tau}}{2q_2x^2} \left[ I_2(\dot{y}) - I_2(\dot{x}) \right]$$

$$- \frac{W_0(\dot{y})}{2(e^\tau(1+(p_2-q_2)x) - x(p_2-q_2))^2} + \frac{W_0(\dot{x})}{2(e^\tau(1+(p_2+q_2)x) - x(p_2+q_2))^2},$$

where $\dot{x} = \left( \frac{x}{e^\tau(1+(p_2+q_2)x) - x(p_2+q_2)} \right)$, $\dot{y} = \left( \frac{x}{e^\tau(1+(p_2-q_2)x) - x(p_2-q_2)} \right)$, $I_1(z) = \int_{c_0}^{z} W_0(\theta) d\theta$, $I_2(z) = \int_{c_1}^{z} W_0(\theta) d\theta$, $c_0$ and $c_1$ may be chosen arbitrary.

One can check that $\dot{x} < \dot{y}$ for all $t \geq 0$, $x \geq 0$.

We get an especially simple form of this solution for the initial distribution $W_0(x) = \delta(x-x_*)$ with some fixed initial state $x(0) = x_* > 0$:

$$W(x, \tau) = \frac{\delta(\dot{x} - x_*)}{2(e^\tau(1+(p_2+q_2)x) - x(p_2+q_2))^2} + \frac{\delta(\dot{y} - x_*)}{2(e^\tau(1+(p_2-q_2)x) - x(p_2-q_2))^2}$$

$$+ \frac{H(\dot{y} - x_*) - H(\dot{x} - x_*)}{2q_2x^2}.$$

Here $H(z) = \int_{-\infty}^{z} \delta(\theta) d\theta$ is the Heaviside function.

According to the standard formula $\delta'(\phi(x)) = \delta(\phi^{-1}(0))/\phi'(\phi^{-1}(0))$ one gets

$$\frac{\delta(\dot{x} - x_*)}{2(e^\tau(1+(p_2+q_2)x) - x(p_2+q_2))^2} = \frac{\delta(x - \frac{e^{\tau}x}{1-(p_2+q_2)(e^\tau-1)x})}{2e^\tau},$$

$$\frac{\delta(\dot{y} - x_*)}{2(e^\tau(1+(p_2-q_2)x) - x(p_2-q_2))^2} = \frac{\delta(x - \frac{e^{\tau}x}{1-(p_2-q_2)(e^\tau-1)x})}{2e^\tau},$$

$$\frac{H(\dot{y} - x_*) - H(\dot{x} - x_*)}{2q_2x^2} = \frac{\delta(x - \frac{e^{\tau}x}{1-(p_2+q_2)(e^\tau-1)x})}{2e^\tau}.$$
\[ \frac{\delta(\hat{y} - x_*)}{2(e^\tau(1 + (p_2 - q_2)x) - x(p_2 - q_2))^2} = \frac{\delta(x - \frac{e^\tau x_*}{1-(p_2-q_2)(e^\tau-1)x_*})}{2e^\tau}, \]

so we see that this simple solution (and consequently the complete solution) obviously obeys the necessary physical requirements of positivity and normalization: \( W(x, t) \geq 0, \int_{-\infty}^{\infty} W(x, t) \, dx \equiv 1. \) Asymptotically, for \( \tau \to \infty, \) this solution exponentially fast converges to the stationary probability distribution \( W_\infty(x) = 0 \) outside the interval \( \frac{1}{|p_2-q_2|} < x < \frac{1}{|p_2+q_2|} \) and \( W_\infty(x) = \frac{1}{2q_2 x^2} \) inside this interval.

### 4 Concluding remarks and future prospects

There is an algorithmic possibility to obtain closed-form solutions of the Cauchy problem for the more complicated cases \( \nu = mp_1, m = 2,3,4,\ldots \) in the Verhulst model. The respective classical form (16) is algorithmically obtainable with the methods of Section 2.1. Since the orders of derivations of \( F(\tau), G(y) \) in the right-hand sides of (16) are proportional to the integer coefficient \( m \) in the relation \( \nu = mp_1 \), directly assigning \( W(x, 0) = W_0(x), W_1(x, 0) = 0 \) in this formula for \( \tau = 0 \) will result in a linear system of ODEs for \( F(\tau), G(y) \) of high order with nonconstant coefficients. Much more efficient is to use the transformations (12) directly: simply recalculate the Cauchy data for the new functions \( u_2 \) on step (\( L_2 \)), using (13), until we get (after \( m \) steps) a triangular system, solve this system for the recalculated Cauchy data and then use the inverse \( \hat{X}_2 \)-transformations to get the solution of the original system. This also explains why we could find the solution (28) in the case \( \nu = p_1 \) in Section 3.

Methods, described in Sections 2.2, 2.3 suggest that one can also investigate systems (14), (10) and classify completely integrable cases for special forms of their coefficients \( p(x), g(x), f(x,y), q(x,y), d(x,y), s(x,y) \) and switching frequencies \( \mu, \nu \). Systematic investigation of such integrable cases will be reported in subsequent publications.

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