ENERGY DISTRIBUTION IN MELVIN’S MAGNETIC UNIVERSE

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Abstract

We use the energy-momentum complexes of Landau and Lifshitz and Papapetrou to obtain the energy distribution in Melvin’s magnetic universe. For this space-time we find that these definitions of energy give the same and convincing results. The energy distribution obtained here is the same as we obtained earlier for the same space-time using the energy-momentum complex of Einstein. These results uphold the usefulness of the energy-momentum complexes.

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I. INTRODUCTION

The well-known Melvin’s magnetic universe [1] is a collection of parallel magnetic lines of force in equilibrium under their mutual gravitational attraction. This magnetic universe is described by an electrovac solution to the Einstein-Maxwell equations [1]. This space-time is invariant under rotation about, and translation along, an axis of symmetry. This is also invariant under reflection in planes comprising that axis or perpendicular to it. Wheeler [4] demonstrated that a magnetic universe could also be obtained in Newton’s theory of gravitation and showed that it is unstable according to elementary Newtonian analysis. Further Melvin [3] showed his universe to be stable against small radial perturbations and Thorne [5] proved the stability of the magnetic universe against arbitrary large perturbations. Thorne [5] further pointed out that the Melvin magnetic universe might be of great value in understanding the nature of extragalactic sources of radio waves and thus the Melvin solution to the Einstein-Maxwell equations is of immense astrophysical interest. Virbhadra and Prasanna [6] studied spin dynamics of charged massive test particles in this space-time. It is really tempting to investigate the energy distribution in the Melvin magnetic universe.

The energy-momentum localization or quasi-localization in general relativity remains an elusive problem. In a flat space-time the energy-momentum tensor $T^i_k$ satisfies the divergence relation $T^i_{k;k} = 0$. The presence of gravitation necessitates the replacement of an ordinary derivative by a covariant one, and therefore one has the covariant conservation laws $T^i_{k;k} = 0$. In a curved space-time the energy-momentum tensor of matter plus all non-gravitational fields does not satisfy $T^i_{k;k} = 0$; the contribution from the gravitational field is also required to construct an energy-momentum complex which satisfies a divergence relation like one has in a flat space-time. Attempts aimed at finding a quantity for describing distribution of energy-momentum due to matter, non-gravitational and gravitational fields resulted in various energy-momentum complexes, notably those proposed by Einstein, Landau and Lifshitz, Papapetrou and Weinberg. The physical meaning of these nontensorial (under general coordinate transformations) complexes have been questioned by some researchers (see references in [7]). There are suspicions that different energy-momentum complexes could give different energy distributions in a given space-time. The problems associated with energy-momentum complexes resulted in a number of researchers questioning the concept of energy-momentum localization. According to Misner et al. [8] the energy is localizable only for spherical systems. However, Cooperstock and Sarracino [9] refuted their viewpoint and stated that if the energy is localizable in spherical systems then it is also localizable for all systems. Bondi [10] wrote “In relativity a non-localizable form of energy is inadmissible, because any form of energy contributes to gravitation and so its location can in principle be found.” A large number of definitions of quasi-local mass have been proposed (see in [11], [12]). Bergqvist [13] furnished computations with many definitions of quasi-local masses for the Reissner-Nordström and Kerr space-times and came to the conclusion that no two of these definitions gave the same result.

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1This solution had been obtained earlier by Misra and Radhakrishna [2]. Melvin [3] mentioned that this solution is contained implicitly as a special case among the solutions obtained by Misra and Radhakrishna.
Virbhadra, his collaborators and some others [14] considered many asymptotically flat space-times and presented that several energy-momentum complexes give the same and acceptable results for a given space-time. Rosen and Virbhadra [15] investigated the Einstein-Rosen space-time (which is not asymptotically flat) and noted that many energy-momentum complexes give the same and persuading results for the energy and energy current densities. Fascinated by this interesting work some researchers studied few asymptotically non-flat space-times using different energy-momentum complexes and obtained appealing results (see [16]-[17]). Aguirregabiria et al. [18] proved that several energy-momentum complexes “coincide” for any Kerr-Schild class metric. Recently Virbhadra [20] showed that for a general non-static spherically symmetric metric of the Kerr-Schild class, the energy-momentum complexes of Einstein, Landau and Lifshitz, Weinberg and Papapetrou furnish the same result as Tod obtained using the Penrose quasi-local mass definition. These are definitely inspiring results and it is worth pursuing this research topic further. Recently there has been some other important papers on this topic [21].

Energy distribution in Melvin’s universe was earlier computed [17] using Einstein’s complex. In this paper we obtain energy distribution in Melvin’s magnetic universe using the definitions of Landau and Lifshitz, and Papapetrou. We wish to check whether or not we get the same result as we obtained earlier using the energy-momentum complex of Einstein. In this paper we use geometrized units where gravitational constant $G = 1$ and the speed of light in vacuum $c = 1$. The convention used in this paper is that Latin indices take values from 0 to 3 and Greek indices values from 1 to 3, comma indicates ordinary derivative and semi-colon covariant derivative.

II. MELVIN’S MAGNETIC UNIVERSE

The Einstein-Maxwell equations are very well-known in the literature. These are given by

$$R_{i}^{k} - \frac{1}{2} g_{i}^{k} R = 8\pi T_{i}^{k}, \quad (1)$$

$$\frac{1}{\sqrt{-g}} \left( \sqrt{-g} F^{ik} \right)_{,k} = 4\pi J^{i}, \quad (2)$$

$$F_{ij,k} + F_{j,k,i} + F_{k,i,j} = 0, \quad (3)$$

where the energy-momentum tensor of the electromagnetic field is

$$T_{i}^{k} = \frac{1}{4\pi} \left[ -F_{im}F^{km} + \frac{1}{4} g_{i}^{k} F_{mn}F^{mn} \right]. \quad (4)$$

$J^{i}$ is the electric current density vector and $R_{i}^{k}$ is the Ricci tensor. Melvin (see in [24]) obtained electrovac solution $(J^{i} = 0)$ to these equations, which is expressed by the line element

$$ds^{2} = \Lambda^{2} \left[ dt^{2} - dr^{2} - r^2 d\theta^2 \right] - \Lambda^{-2} r^2 \sin^2 \theta d\phi^2 \quad (5)$$
and the Cartan components of the magnetic field

\[ H_r = \Lambda^{-2} B_o \cos \theta, \]
\[ H_\theta = -\Lambda^{-2} B_o \sin \theta, \]  

(6)

where

\[ \Lambda = 1 + \frac{1}{4} B_o^2 r^2 \sin^2 \theta. \]  

(7)

\( B_o \equiv B_o \sqrt{G/c^2} \) is the magnetic field parameter and this is a constant in the solution given above. The non-zero components of the energy-momentum tensor are [17]

\[ T^0_1 = -T^2_2 = \frac{B_o^2 \left( 1 - 2 \sin^2 \theta \right)}{8 \pi \Lambda^4}, \]
\[ T^0_0 = -T^3_3 = \frac{B_o^2}{8 \pi \Lambda^4}, \]
\[ T^1_2 = -T^2_1 = \frac{2 B_o^2 \sin \theta \cos \theta}{8 \pi \Lambda^4}. \]  

(8)

To get meaningful results using these energy-momentum complexes one is compelled to use “Cartesian” coordinates (see [19] and [20]). Then the line element (5) is transformed to “Cartesian” coordinates \( t, x, y, z \) using the standard transformation

\[ r = \sqrt{x^2 + y^2 + z^2}, \]
\[ \theta = \cos^{-1} \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right), \]
\[ \phi = \tan^{-1}(y/x). \]  

(9)

Now, the line element in \( t, x, y, z \) coordinates becomes

\[ ds^2 = \Lambda^2 dt^2 - \Lambda^2 (dx^2 + dy^2 + dz^2) + \left( \Lambda^2 + \frac{1}{\Lambda^2} \right) \frac{(xdy - ydx)^2}{x^2 + y^2}. \]

(10)

The determinant of the metric tensor is given by

\[ g = -\Lambda^4 \]  

(11)

and the non-zero contravariant components of the metric tensor are

\[ g^{00} = \Lambda^{-2}, \]
\[ g^{11} = -\frac{\Lambda^{-2} x^2 + \Lambda^2 y^2}{x^2 + y^2}, \]
\[ g^{12} = -\left( \Lambda^2 - \frac{1}{\Lambda^2} \right) \frac{xy}{x^2 + y^2}, \]
\[ g^{22} = -\frac{\Lambda^{-2} y^2 + \Lambda^2 x^2}{x^2 + y^2}, \]
\[ g^{33} = -\Lambda^{-2}. \]  

(12)
III. THE LANDAU AND LIFSHITZ ENERGY-MOMENTUM COMPLEX

The energy-momentum complex of Landau and Lifshitz \[22\] is

\[ L_{ij} = \frac{1}{16\pi} S^{ikjl,kl} \]  

where

\[ S^{ikjl} = -g(g^{ij}g^{kl} - g^{il}g^{kj}) \]  

\( L_{ij} \) is symmetric in its indices. \( L^{00} \) is the energy density and \( L^{0\alpha} \) are the momentum (energy current) density components. \( S^{mjnk} \) has symmetries of the Riemann curvature tensor. The expression

\[ P^i = \int \int \int L^{0i}dx^1dx^2dx^3 \]  

(15)

gives the energy \( P^0 \) and the momentum \( P^\alpha \) components. Thus the energy \( E \), after applying the Gauss theorem, is given by the expression

\[ E_{LL} = \frac{1}{16\pi} \int \int S^{00\beta\alpha} \mu_\beta dS \]  

(16)

where \( \mu_\beta \) is the outward unit normal vector over an infinitesimal surface element \( dS \). In order to calculate the energy component for Melvin’s universe expressed by the line element (10) we need the following non-zero components of \( S^{ikjl} \)

\[ S^{0101} = \frac{x^2 + y^2\Lambda^4}{x^2 + y^2}, \]

\[ S^{0102} = \frac{xy(\Lambda^4 - 1)}{x^2 + y^2}, \]

\[ S^{0202} = \frac{y^2 + x^2\Lambda^4}{x^2 + y^2}, \]

\[ S^{0303} = -1. \]  

(17)

Equation (13) with equations (14) and (17) gives

\[ L^{00} = \frac{1}{8\pi} B^2\Lambda^3. \]  

(18)

For a surface given by parametric equations \( x = r\sin\theta\cos\phi, \ y = r\sin\theta\sin\phi, \ z = r\cos\theta \) (where \( r \) is constant) one has \( \mu_\beta = \{x/r, y/r, z/r\} \) and \( dS = r^2\sin\theta d\theta d\phi \). Using equations (17) in (16) over a surface \( r = \text{const.} \), we obtain

\[ E_{LL} = \frac{1}{6} B_o^2 r^3 + \frac{1}{20} B_o^4 r^5 + \frac{1}{140} B_o^6 r^7 + \frac{1}{2520} B_o^8 r^9. \]  

(19)
IV. THE ENERGY-MOMENTUM COMPLEX OF PAPAPETROU

The symmetric energy-momentum complex of Papapetrou [23] is

\[ \Omega^{ij} = \frac{1}{16\pi} N^{ijkl} \eta_{kl} \]  

where

\[ N^{ijkl} = \sqrt{-g} \left( g^{ij} \eta^{kl} - g^{ik} \eta^{jl} + g^{kl} \eta^{ij} - g^{jl} \eta^{ik} \right) \]

and the Minkowski metric

\[ \eta^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

\( \Omega^{00} \) and \( \Omega^{a0} \) are the energy and momentum density components. The energy and momentum components are given by

\[ P^i = \int \int \int \Omega^i \ dx^1 dx^2 dx^3. \]  

Using the Gauss theorem, the energy \( E_P \) for a stationary metric is thus given by the expression

\[ E_P = \frac{1}{16\pi} \int \int \int N^{0[a}\eta^{b]} \mu_\alpha dS. \]

To find the energy component of the line element [10], we require the following non-zero components of \( N^{ijkl} \)

\[ N^{0011} = -(1 + \frac{x^2 + y^2 \Lambda^4}{x^2 + y^2}), \]
\[ N^{0012} = \frac{xy(\Lambda^4 - 1)}{x^2 + y^2}, \]
\[ N^{0022} = -(1 + \frac{y^2 + x^2 \Lambda^4}{x^2 + y^2}), \]
\[ N^{0033} = -2. \]  

Equations (24) in equation (20) give the energy density component

\[ \Omega^{00} = \frac{1}{8\pi} B^2 \Lambda^3. \]  

Thus we find the same energy density as we obtained in the last Section, we use Eq. (24) in (23) over a 2-surface (as in the last Section) and obtain

\[ E_P = \frac{1}{6} B_0^2 r^3 + \frac{1}{20} B_0^4 r^5 + \frac{1}{140} B_0^6 r^7 + \frac{1}{2520} B_0^8 r^9. \]
This result is expressed in geometrized units ($G = 1$ and $c = 1$). In the following we restore $G$ and $c$ and get

$$E_P = \frac{1}{6}B_0^2r^3 + \frac{1}{20}Gc^4B_0^4r^5 + \frac{1}{140}G^2c^8B_0^6r^7 + \frac{1}{2520}G^3c^{12}B_0^8r^9. \quad (27)$$

The first term $\frac{1}{6}B_0^2r^3$ is the known classical value of energy and the rest of the terms are general relativistic corrections. The general relativistic terms increase the value of energy.

V. DISCUSSION AND SUMMARY

The subject of energy-momentum localization in the general theory of relativity has been very exciting and interesting; however, it has been associated with some debate. Misner et al. [8] argued that to look for a local energy-momentum is looking for the right answer to the wrong question. However, they further mentioned that energy is localizable but only for spherical systems. Cooperstock and Sarracino [9] fully disagreed with them and argued that if the energy is localizable in spherical systems then it is also localizable for all systems. Bondi [10] wrote that a nonlocalizable form of energy is inadmissible in general theory of relativity. There prevails scepticism that different energy-momentum complexes could give unacceptable different energy distributions for a given space-time. However, buttressed by the remarkable results obtained by Virbhadra, his collaborators (Rosen, Parikh, Chamorro and Aguirregabiria) and some others (who demonstrated with several examples that for a particular space-time many energy-momentum complexes give the same and acceptable energy distribution), this research topic is rejuvenated. Recently Virbhadra [20] stressed that although the energy-momentum complexes are non-tensorial (under general coordinate transformations), these do not violate the principle of general covariance as the equations describing the conservation laws with these objects (for example, $L^{ik}_{,k} = 0$, $\Omega^{ik}_{,k} = 0$) are true in any coordinates systems.

In this paper we obtained the energy distribution in Melvin’s magnetic universe. We used the energy-momentum complexes of Landau and Lifshitz, and Papapetrou. Both definitions give the same results ($L^{00} = \Omega^{00}$, $E_{LL} = E_P$). We also note that the results obtained here is the same as we obtained earlier using the energy-momentum complex of Einstein. The first term in the energy expression (see equations (19) and (26)) is the well-known classical value for the energy of the uniform magnetic field and the other terms are general relativistic corrections. The general relativistic corrections increase the value of the energy. These results uphold the importance of the energy-momentum complexes and oppose the prevailing “folklore” against them.

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