CRITICAL GROUPS OF ITERATED CONES

GOPAL GOEL AND DAVID PERKINSON

Abstract. Let $G$ be a finite graph, and let $G_n$ be the $n$-th iterated cone over $G$. We study the structure of the critical group of $G_n$ arising in divisor and sandpile theory.

1. Introduction

The critical group $K(G)$ of a connected graph $G$ is the torsion part of the cokernel of its discrete Laplacian (details appear below). It is known as the degree-zero part of the Picard group or as the Jacobian of $G$ in the divisor theory of graphs ([1]). It is isomorphic to the sandpile group of $G$ from statistical physics ([4]) and to the group of parking functions of $G$ from combinatorics ([5]). The $n$-th iterated cone over $G$, denoted $G_n$, is the join of $G$ and the complete graph on $n$-vertices, $K_n$, formed by connecting each vertex of $G$ with each vertex of $K_n$ by an undirected edge. Our main result is Theorem 1, which provides a description of the structure of $K(G_n)$ as an abelian group.

The question of the structure of $K(G_n)$ was addressed previously in [2]. In that paper, Theorem A provides a short exact sequence for $K(G_n)$ (cf. our Corollary 3) and Corollary B computes the order of $K(G)$ in terms of the characteristic polynomial of the Laplacian of $G$ (cf. our Theorem 1 (3)). We give new short and direct proofs of both of these results. We also give a partial answer to question 1.2 of [2], which asks when the short exact sequence splits. (See the discussion after Corollary 5 below.)

Acknowledgements. We are grateful to David Zureick-Brown for presenting the problem of determining the structure of $K(G_n)$ to us. We thank Collin Perkinson for comments on the exposition.

2. Main Results

Let $G$ be an Eulerian digraph. As a special case, $G$ could be an undirected graph. Loops and multiple edges are allowed. We assume that $G$ is connected with finite vertex set $V$ and finite edge multiset $E$. We write $(v, w)$ for a directed edge starting at $v$ and ending at $w$. The main Eulerian property we need is that the indegree and outdegree are equal at each vertex. Letting $ZV$ denote the free abelian group on the vertices, the (discrete) Laplacian of $G$ is the homomorphism $L: ZV \rightarrow ZV$ determined by $L(v) = \text{outdeg}(v) v - \sum_{(v, w) \in E} w$ for each $v \in V$. We assume the vertices are ordered so that we can identify $L$ with a $k \times k$ matrix where $k := |V|$. Then $L = D - A^t$ where $D$ is the diagonal matrix of the outdegrees of the vertices and $A^t$ is the transpose of the directed adjacency matrix of $G$. The $i, j$-th entry of $A$ is the number of edges from the $i$-th vertex to the $j$-th vertex. The image of $L$ lies in the kernel of the “degree” homomorphism $\delta: ZV \rightarrow ZV$ determined by $\delta(v) = 1$ for each $v \in V$. The critical group of $G$ is

$$K(G) := \ker \delta / \text{im} L.$$
Fixing any vertex \( u \in V \), there is an isomorphism
\[
\text{cok}(L) \to \mathcal{K}(G) \oplus \mathbb{Z}
\]
\[
f \mapsto (f - \delta(f)\chi_u, \delta(f)),
\]
where \( \chi_u \in \mathbb{Z}^V \) is the indicator function for \( u \). It is well-known (e.g., via the matrix-tree theorem) that since \( G \) is connected, the rank of \( L \) is \( k - 1 \), and hence \( \mathcal{K}(G) \) is finite. Deleting the row and column corresponding to \( u \) from the matrix \( L \) gives the reduced Laplacian \( \tilde{L} \) of \( G \), and since \( G \) is Eulerian ([3, Theorem 12.1]), there is an isomorphism
\[
\mathcal{K}(G) \approx \text{cok}(\tilde{L})
\]
over \( \mathbb{Z} \).

**Theorem 1.** Let \( G \) be an Eulerian digraph with \( k \) vertices and Laplacian \( L \). Let \( G_n \) be the \( n \)-th cone over \( G \) where \( n \geq 2 \).

1. Let \( 1 \) be the \( k \times k \) matrix whose entries are all 1, and let \( I_k \) be the \( k \times k \) identity matrix. Then
\[
\mathcal{K}(G_n) \approx (\mathbb{Z}/(n + k)\mathbb{Z})^{n-2} \oplus \text{cok}(nI_k + L + 1).
\]
2. The group \( \text{cok}(nI_k + L + 1) \) has a subgroup isomorphic to \( \mathbb{Z}/(n + k)\mathbb{Z} \).
3. ([2, Corollary B]) The order of the critical group of \( G_n \) is
\[
|\mathcal{K}(G_n)| = \left| p_L(-n) \right| n(n + k)^n - 1
\]
where \( p_L \) is the characteristic polynomial of \( L \).

**Proof.** Order the vertices of \( G_n \) so that the cone vertices appear at the end. The reduced Laplacian for \( G_n \) is then, in block form,
\[
\tilde{L}_n = \begin{bmatrix}
I_k + L - 1 \\
1 \\
(n + k)I_{n-1} - 1
\end{bmatrix},
\]
where each \( 1 \) denotes a matrix of 1s (with dimensions inferred from context). Since \( G \) is Eulerian, all row and column sums of \( L \) are 0. Perform the following operations in order on \( \tilde{L}_n \):

1. Subtract the last column from all other columns.
2. Add all but the last row to the last row.
3. Add the last row to all rows but the last.

The result is the block matrix
\[
M := \begin{bmatrix}
I_k + L + 1 & 0 & 0 \\
0 & (n + k)I_{n-2} & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]
Then \( \text{cok}(M) \approx \text{cok}(\tilde{L}_n) \approx \mathcal{K}(G_n) \), and Part [1] follows.

For Part [2] first note that since \( G_n \) is connected, \( \tilde{L}_n \) has full rank, and hence so does \( A := nI_k + L + 1 \). Consider the homomorphism \( \phi : \mathbb{Z} \to \text{cok}(A) \) sending 1 to the all-ones vector \( \vec{1} \). If \( \ell \in \ker \phi \), then there is a vector \( \vec{v} \in \mathbb{Z}^k \) such that \( A\vec{v} = \ell \cdot \vec{1} \). However, \( A\vec{1} = (n + k) \cdot \vec{1} \). Since \( A \)
has full rank and $\vec{v}$ is an integer vector, it follows that $n + k$ divides $\ell$ and $\vec{v}$ is a constant vector. Hence, $\ker \phi$ is generated by $n + k$.

Finally, for Part 3, note that $|K(G_n)| = \det(M) = (n + k)^{n-2}|\det(nI_k + L + 1)|$. Let $r_1, \ldots, r_k$ be the rows of $nI_k + L$. For each $i = 1, \ldots, k$, we use the identity $r_1 + \cdots + r_k = n\vec{1}$ to substitute for $r_i$ and use the fact that the determinant is an alternating multilinear function of the rows of a matrix to get

$$p_L(-n) = \det(nI_k + L) = \det(r_1, \ldots, r_k) = n \det(r_1, \ldots, \vec{1}, \ldots, r_k),$$

where $\vec{1}$ appears in the $i$-th component. Then

$$\det(nI_k + L + 1) = \det(r_1 + \vec{1}, \ldots, r_k + \vec{1})$$

$$= \det(r_1, \ldots, r_k) + \sum_{i=1}^{k} \det(r_1, \ldots, \vec{1}, \ldots, r_k)$$

$$= (n + k)\frac{p_L(-n)}{n}.$$

The result follows.

Remark 2. Part 3 of the theorem also holds in the case $n = 1$, i.e., for the (first) cone over $G$. The reduced Laplacian of $G_1$ is $I_k + L$. Therefore, $K(G_1) \approx \text{cok}(I_k + L)$, and $|K(G_1)| = |\det(I_k + L)| = |p_L(-1)|$.

As an immediate corollary of Theorem 1 we have the following:

Corollary 3. ([2, Theorem A]) There is an exact sequence,

$$0 \rightarrow (\mathbb{Z}/(n + k)\mathbb{Z})^{n-1} \rightarrow K(G_n) \rightarrow H_n \rightarrow 0$$

where $H_n$ is a group of order $|p_L(-n)|/n$.

In [2], the orders of $H_n$ and $K(G_n)$ are stated in terms of the characteristic polynomial of the endomorphism of $\ker \delta$ obtained from our Laplacian by restriction. Calling that characteristic polynomial $P_G$, we have $P_G(x) = p_L(x)/x$.

Question 1.2 of [2] asks when the exact sequence in Corollary 3 splits. By Theorem 1, $(\mathbb{Z}/(n + k)\mathbb{Z})^{n-2}$ always splits off of $K(G_n)$, and the exact sequence of Theorem A splits exactly when $\mathbb{Z}/(n + k)\mathbb{Z}$ is a direct summand of $\text{cok}(nI_k + L + 1)$. The latter will depend, for instance, on comparing the prime factorization of $n + k$ to the primary decomposition of the abelian group $\text{cok}(nI_k + L + 1)$. It would be interesting if much more could be said in answer to the question for arbitrary $G$.

References
1. M. Baker and S. Norine, Riemann-Roch and Abel-Jacobi Theory on a Finite Graph, Adv. Math. 215 (2007), 766–788.
2. M. V. Brown, J. S. Morrow, and D. Zureick-Brown, Chip-firing Groups of Iterated Cones, Linear Algebra Appl. 556 (2018), 46–54.
3. S. Corry and D. Perkinson, Divisors and Sandpiles, American Mathematical Society, 2018.
4. D. Dhar, The Abelian Sandpile and Related Models, Physica A 263 (1999), no. 4, 4–25.
5. A. Postnikov and B. Shapiro, Trees, Parking Functions, Syzygies, and Deformations of Monomial Ideals, Trans. Amer. Math. Soc. 356 (2004), no. 8, 3109–3142 (electronic).
Portland, OR
E-mail address: gopal.krishna.goel@gmail.com

Reed College, Portland, OR
E-mail address: davidp@reed.edu