A REMARK ABOUT THE ANOMALIES OF CYCLIC HOLOMORPHIC PERMUTATION ORBIFOLDS

MARCEL BISCHOFF

Abstract. Using a result of Longo and Xu, we show that the anomaly arising from a cyclic permutation orbifold of order 3 of a holomorphic conformal net $\mathcal{A}$ with central charge $c = 8k$ depends on the “gravitational anomaly” $k \pmod{3}$. In particular, the conjecture that holomorphic permutation orbifolds are non-anomalous and therefore a stronger conjecture of Müger about braided crossed $S_n$-categories arising from permutation orbifolds of completely rational conformal nets are wrong. More general, we show that cyclic permutations of order $n$ are non-anomalous if and only if $3 \nmid n$ or $24 \mid c$. We also show that all cyclic permutation gaugings of $\text{Rep}(\mathcal{A})$ arise from conformal nets.

1. Orbifolds and anomalies

Conformal nets axiomatize chiral conformal field theory in the framework of algebraic quantum field theory using von Neumann algebras. There is a notion of a completely rational conformal net $[\text{Rep}(\mathcal{A})]$ whose representation category $\text{Rep}(\mathcal{A})$ is a modular tensor category. Let $\mathcal{A}$ be a holomorphic conformal net, i.e. a completely rational conformal net with trivial representation category $\text{Rep}(\mathcal{A}) \cong \text{Hilb}$. Here we denote by Hilb the trivial unitary modular tensor category of finite-dimensional Hilbert spaces. Let $G \leq \text{Aut}(\mathcal{A})$ be a finite group of automorphisms of the net $\mathcal{A}$, see [Xu00, Müg05]. Then it is well-known [KLM01,Müg05,Müg10,Bis16,Bis18] that there is a unique class $[\omega] \in H^3(G, \mathbb{T})$, such that the category of $G$-twisted representations of $\mathcal{A}$ denoted by $G^{\text{Rep}}(\mathcal{A})$ is tensor equivalent to the category $\text{Hilb}_G$ of $G$-graded finite-dimensional Hilbert spaces with associator given by $\omega$. More precisely, for every $g \in G$ there is an irreducible $g$-twisted representation $\beta_g$ localized in $\mathcal{I}$, which is unique up to conjugation by a unitary. Then $g \mapsto [\beta_g] \in \text{Out}(\mathcal{A}(\mathcal{I}))$ is a $G$-kernel and it follows that $\beta_g \beta_h = \text{Ad} u_{g,h} \beta_{gh}$ for unitaries $(u_{g,h})_{g,h \in G}$ and that $\omega : G \times G \times G \to \mathbb{T}$ defined by $\omega(g, h, k) \cdot 1 = u_{g,h} u_{gh,k} u_{g,hk}^{-1} \beta_g(u_{h,k})^{-1}$ is a cocycle. The

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class \([\omega]\) is called the \textbf{anomaly} of \(G\) and we say that \(G\) acts \textbf{non-anomalous} if \(\omega\) is a coboundary. Furthermore, the \textit{orbifold} or \textit{fixed point net} \(\mathcal{A}^G\) has a representation category \(\text{Rep}(\mathcal{A}^G)\) which is braided equivalent to the Drinfel’d center \(\mathcal{Z}(G, \omega) = \mathcal{Z}(\text{Hilb}^G_G)\).

We denote by \(S_n\) the symmetric group on \(\{1, \ldots, n\}\). Let \(\mathcal{A}\) be a holomorphic net and \(G \leq S_n\), then \(G\) acts by permutation on \(\mathcal{A}^G\). It seems to be widely believed that this action should be non-anomalous. But Johnson-Freyd argued that this conjecture is false \cite{JF17}, and we give a counter-example in the framework of conformal nets where the permutation action picks up what can be thought of a \textbf{gravitational anomaly}\footnote{cf. \cite[Section 1.4]{Wit07} for how this name might be justified, namely he asks that our \(k\) equals \(0\) (mod 3) in order for the chiral CFT to be dual to quantum gravity.} \(k \equiv c/8\) (mod 3) \(\in \mathbb{Z}_3\), where \(c\) is the central charge of \(\mathcal{A}\).

This note is an extension of an unpublished note (consisting essentially of Section 2) circulated in 2017. The results were announced April 15th, 2018 at the AMS Sectional Meeting at Vanderbilt University, Nashville, TN. Shortly after that, a more general result appeared in a preprint by Evans and Gannon \cite{EG18}.

\textbf{Acknowledgements.} The original note is based on communication with Theo Johnson-Freyd who told me that permutation orbifolds can be anomalous and gave me the counterexample arising from the \(E_8\) lattice \cite{JF17}. I am thankful for the communication and explaining me his work.

\section{Cyclic permutations of order 3}

\subsection{Twisted doubles of \(\mathbb{Z}_3\).} Recall that a unitary fusion category is called pointed if all simple objects are invertible. Pointed unitary fusion categories with \(\mathbb{Z}_3\)-fusion rules are classified by \(H^3(\mathbb{Z}_3, \mathbb{T}) = \{[\omega_i] : i \in \mathbb{Z}/3\mathbb{Z}\} \cong \mathbb{Z}_3\). Their Drinfel’d centers \(\mathcal{Z}(\mathbb{Z}_3, \omega_i) := \mathcal{Z}(\text{Hilb}^G_{\mathbb{Z}_3^i})\) are pointed. Indeed, it can be easily checked that they are braided equivalent to the pointed unitary modular tensor categories \(\mathcal{C}(G_i, q_i)\), respectively, where \((G_i, q_i)\) are the metric groups given in Table \ref{tab:metric_groups} and \(\mathcal{C}(G, q)\) is the braided fusion category associated to \((G, q)\), see Appendix \ref{app:metric_groups} in particular Proposition \ref{prop:braided_equivalent}.

\subsection{The anomaly.} By a conformal net we mean a diffeomorphism covariant net on the circle, see e.g. \cite{KL06}. Let \(\mathcal{A}\) be a holomorphic conformal net. Since \(\mathcal{A}\) is diffeomorphism covariant we can assign a central charge \(c > 0\). It is conjectured that if \(\mathcal{A}\) is holomorphic, then \(c \equiv 0\) (mod 8) and it is a theorem that \(c \equiv 0\) (mod 4) \cite{KLX05}. We will from now on assume that the central charge \(c\) of \(\mathcal{A}\) fulfills \(c \in 8\mathbb{N}\).
Table 1. Twisted doubles of $\mathbb{Z}_3$

| $i$ | $G_i$ | $q_i: G_i \to \mathbb{Q}/\mathbb{Z}$ |
|-----|-------|----------------------------------|
| 0   | $\mathbb{Z}_3 \times \mathbb{Z}_3$ | $q_0(x, y) \equiv xy/3 \pmod{1}$ |
| 1   | $\mathbb{Z}_9$ | $q_1(x) \equiv 4x^2/9 \pmod{1}$ |
| 2   | $\mathbb{Z}_9$ | $q_2(x) \equiv 8x^2/9 \pmod{1}$ |

$\mathcal{A}$ is holomorphic, then any tensor power $\mathcal{A}^\otimes n$ is holomorphic [KLM01]. Let $\sigma \in S_n$ be a permutation. Then there is an element $\sigma \in \text{Aut}(\mathcal{A}^\otimes n)$ given by

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n \mapsto x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)}$$

see e.g. [LX04].

**Proposition 2.1.** Let $\mathcal{A}$ be a diffeomorphism covariant holomorphic net with $c = 8k$, and let $\mathbb{Z}_3 \cong \langle \tau \rangle \leq \text{Aut}(\mathcal{A}^\otimes 3)$ be the group generated by the cyclic permutation $\tau = (123)$. Then the anomaly of $\langle \tau \rangle$ is $\omega^2_k$, i.e. $\langle \tau \rangle - \text{Rep}(\mathcal{A}^\otimes 3)$ is tensor equivalent to $\text{Hilb}_{\mathbb{Z}_3}^{\omega^2_k}$ and $\text{Rep}(\langle \mathcal{A}^\otimes 3 \rangle^{\langle \tau \rangle})$ is braided equivalent to $\mathbb{Z}(\mathbb{Z}_3, \omega^2_k)$.

**Proof.** It is enough to show that $\mathcal{C} := \text{Rep}(\langle \mathcal{A}^\otimes 3 \rangle^{\langle \tau \rangle})$ is braided equivalent to $\mathcal{C}(G_{2k}, q_{2k})$. But this follows from [LX04, Theorem 6.3e] which gives that the spins in $\text{Rep}(\langle \mathcal{A}^\otimes 3 \rangle^{\langle \tau \rangle})$ coming from twisted sectors $\alpha_i$ are $h_i = i/3 + 8k/9$ for $i = 0, 1, 2$ and then $q(\alpha_i) \equiv h_i \pmod{1}$ by the spin-statistic theorem [GL96]. This readily identifies $\mathcal{C}$ to be braided equivalent with $\mathcal{C}(G_{2k}, q_{2k})$. \qed

**Example 2.2.** Let $\mathcal{A}_{E_8}$ be the conformal net associated with the even lattice $E_8$ [DX06]. Then $\langle \tau \rangle - \text{Rep}(\mathcal{A}_{E_8}^\otimes 3)$ is tensor equivalent to $\text{Hilb}_{\mathbb{Z}_3}^{\omega^2_k}$ with $[\omega^2]$ a generator of $H^3(\mathbb{Z}_3, \mathbb{T})$. Thus $\text{Rep}(\langle \mathcal{A}_{E_8}^\otimes 3 \rangle^{\langle \tau \rangle})$ is braided equivalent to $\mathbb{Z}(\mathbb{Z}_3, \omega^2)$.

**Example 2.3.** Let $\mathcal{A}$ be a holomorphic net with central charge $c = 8k$. Let $S_3 \leq \text{Aut}(\mathcal{A}^\otimes 3)$ be the group of all permutations. Since $H^3(S_3, \mathbb{T})) \cong H^3(\mathbb{Z}_3, \mathbb{T}) \oplus H^3(\mathbb{Z}_2, \mathbb{T})$, where the isomorphism comes from restriction, it follows that $S_3$ is anomalous unless $k = 0 \pmod{3}$. In particular, $\text{Rep}(\langle \mathcal{A}_{E_8}^\otimes 3 \rangle^{S_3})$ is braided equivalent to $\mathbb{Z}(S_3, \tilde{\omega})$ for some $[\tilde{\omega}] \in H^3(S_3, \mathbb{T})$ of order 3.

In particular, the conjecture by Müger [Tur10, Appendix 5, Conjecture 6.3] that states that for every completely rational conformal net $\mathcal{A}$ the category of $S_n$-twisted representations $S_n - \text{Rep}(\mathcal{A}^\otimes n)$ up to tensor equivalence depends only on the modular tensor category $\text{Rep}(\mathcal{A})$ is wrong.
3. Cyclic holomorphic orbifolds

The argument can be generalized to arbitrary cyclic extensions and we get the following result.

**Proposition 3.1.** Let $A$ be a holomorphic net with central charge $c = 8k$ for some $k \in \mathbb{N}$. Let $\alpha$ be a cyclic permutation of order $n$ on $A^\otimes m$. Then the action of $\langle \alpha \rangle \cong \mathbb{Z}_n$ on $A^\otimes m$ is non-anomalous if and only if $3 \nmid n$ or $24 \mid c$.

3.1. Cyclic homomorphic twisted orbifolds. We have the following application of Proposition 3.1. If $A$ is holomorphic and $G \leq \text{Aut}(A)$ non-anomalous we can form the so-called twisted orbifold $A//G$ as described in [Bis18] by lifting the $G$-kernel given by $G\text{-Rep}(A)$ to a homorphism $G \hookrightarrow \text{Aut}(A(I))$ or in other words by choosing a trivialization.

In our concrete case, this can be easier described. Namely, $\text{Rep}((A^\otimes m)\langle \alpha \rangle)$ is braided equivalent to $C(\mathbb{Z}_n \times \hat{\mathbb{Z}}_n, q_{st})$ with the quadratic form $q_{st}(g, \chi) = \chi(g)$, such that the Lagrangian subgroup $\mathbb{Z}_n \times \{\chi_0\}$ gives $A^\otimes m$. We have a second Lagrangian subgroup $\{0\} \times \hat{\mathbb{Z}}_n$ which gives a new holomorphic net $A^\otimes m//\langle \alpha \rangle$ which is the twisted orbifold net $A^\otimes m//\langle \alpha \rangle$ of $A^\otimes m$ with respect to $\langle \alpha \rangle$. Thus we have:

**Proposition 3.2.** Let $A$ be a holomorphic net with central charge $c = 8k$ for some $k \in \mathbb{N}$. Let $\alpha$ be a cyclic permutation of order $n$ on $A^\otimes m$. If $3 \nmid n$ or $24 \mid c$, we have a holomorphic net given by the twisted orbifold $A^\otimes m//\langle \alpha \rangle$.

**Example 3.3.** $(A_{E_8} \otimes A_{E_8})//\langle \tau_2 \rangle$ is isomorphic to $A_{D_{16}^+}$.

3.2. Determining the anomalies. We now proceed to prove Proposition 3.1. Let $A$ be a holomorphic net and let $\tau_n$ be the cyclic permutation

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n \mapsto x_2 \otimes x_3 \otimes \cdots \otimes x_1.$$ 

Then $\tau_n$ yields an inner symmetry $\tau_n \in \text{Aut}(A^\otimes n)$, see e.g. [LX04].

For $G \leq \text{Aut}(A)$ and $g \in G$ we denote by $\text{Rep}(A)^G_g$ the category of representations coming from restrictions of $g\text{-Rep}(A)$.

**Lemma 3.4.** Let $A$ be a holomorphic net with central charge $c = 8k$ for some $k \in \mathbb{N}$.

(1) For $3 \nmid n$ the spectrum of $h_{\alpha}$ with $\alpha \in \text{Rep}((A^\otimes n)\langle \tau_n \rangle)_{\tau_n}$ is $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\} \pmod{1}$.

(2) For $n = 3m$ the spectrum of $h_{\alpha}$ with $\alpha \in \text{Rep}((A^\otimes n)\langle \tau_n \rangle)_{\tau_n}$ is $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\} \pmod{1}$ if and only if $c \equiv 0 \pmod{24}$.
(3) For \( k = \pm 1 \equiv \frac{c}{8} \pmod{3} \) there is an \( \alpha \in \text{Rep}((A \otimes n)^{(\tau_n)}) \) with \( h_\alpha \equiv -\frac{k}{3n} \pmod{1} \).

Proof. Let \( n = 3\ell \pm 1 \). Then using [LX04, Theorem 6.3e] we have
\[
h_\beta = i + \frac{n^2 - 1}{24n} c = \frac{i + k(3\ell \pm 2)}{n},
\]
thus we have (1). Now let \( n = 3\ell \), then
\[
h_\beta = i + \frac{n^2 - 1}{24n} c = \frac{i + 9\ell^2 - 1}{9\ell} k
= \begin{cases} 
\frac{i - m}{n} \pmod{1} & k = 3m \\
\frac{-3(m - i) + 1}{3n} \pmod{1} & k = 3m \pm 1.
\end{cases}
\]
Thus we have (2) and since \( 3 \nmid 3(m - i) \pm 1 \) we get (3). \( \square \)

Proposition 3.5. Let \( A \) be a holomorphic net. If \( 3 \nmid n \) then \( \langle \tau_n \rangle \sim \text{Rep}(A) \) is tensor equivalent to \( \text{Hilb}_{\mathbb{Z}_n} \).

Proof. Since there is a \( \tau_n \)-twisted representation \( \beta \) with \( h_\beta = 0 \pmod{1} \) from Lemma [A,3] it follows that \( \text{Rep}((A \otimes n)^{(\tau_n)}) \) is braided equivalent to \( C(\mathbb{Z}_n \times \hat{\mathbb{Z}}, q) \) and because the Lagrangian subgroup lives in the zero graded part we have \( \langle \tau_n \rangle \sim \text{Rep}(A) \) is tensor equivalent to \( \text{Hilb}_{\mathbb{Z}_n} \) again by Lemma [A,3]. \( \square \)

Proposition 3.6. Let \( A \) be a holomorphic net of central charge \( c = 8k \) and \( n = 3m \) for some \( m, k \in \mathbb{N} \).

(1) \( \text{Rep}((A \otimes n)^{(\tau_n)}) \) is braided equivalent to \( C(\mathbb{Z}_{9m} \otimes \mathbb{Z}_m, q_\pm) \) with
\[
q_\pm(x, y) = \frac{\pm x^2}{9m} + \frac{y^2}{m}.
\]
for \( k \equiv \pm 1 \pmod{3} \).

(2) \( \text{Rep}((A \otimes n)^{(\tau_n)}) \) is braided equivalent to \( C(\mathbb{Z}_{3m} \otimes \hat{\mathbb{Z}}_{3m}, q) \) for \( k \equiv 0 \pmod{3} \).

(3) \( \langle \tau_n \rangle \sim \text{Rep}(A) \) is tensor equivalent to \( \text{Hilb}_{\mathbb{Z}_n}^{\omega_k} \) with \( [\omega_k] = -km[\omega_0] \)
for a generator \( [\omega_0] \) of \( H^3(\mathbb{Z}_n, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}_n \).

In Figure [1] we demonstrate the twisted fusion rules depending on \( k \) in an example.

Proof. (2) is proved as before. We note that the cocycle has order three, since \( \tau_{3\ell} \) equals \( \tau_\ell \) on \( (A \otimes 3)^{\otimes \ell} \) since \( A \otimes 3 \) has central central charge \( c = 24k \). So there are only two choices for the cocycle which are distinguished by the values of \( h \), see Appendix [A] which proves (1) and (3). \( \square \)
Corollary 3.7. Let $\mathcal{A}$ be a holomorphic net. The action of $\langle \tau_n \rangle \cong \mathbb{Z}_n$ on $\mathcal{A}^{\otimes n}$ is non-anomalous if and only if $3 \nmid n$ or $24 \mid c$.

In particular, we have proven Proposition 3.1, since any cyclic permutation in $\mathcal{A}^{\otimes m}$ is conjugate to $\tau_n \otimes \text{id}$.

4. ALL GAUGINGS FOR CYCLIC PERMUTATION ORBIFOLDS

Given a unitary modular tensor category $\mathcal{C}$ we can consider $\mathcal{C}^{\mathbb{Z}_n}$ which has a categorical action of any subgroup $G \leq S_n$. Recently, T. Gannon and C. Jones have showed [GJ18] that certain obstructions vanish and that therefore such a symmetry can always be gauged, i.e. there is a with the categorical action compatible $G$-crossed braided extension $\mathcal{C} \rtimes G \supset \mathcal{C}^{\mathbb{Z}_n}$. The equivariantization $(\mathcal{C} \rtimes G)^G$ is a new unitary modular tensor category, which correspond to gauging. If $\mathcal{C} = \text{Rep}(\mathcal{A})$ for a rational conformal net, then $G \rtimes \text{Rep}(A^{\otimes n})$ (where $G$ acts by permutations) is a $G$-crossed braided extension and $\text{Rep}((A^{\otimes n})^G)$ is a special gauging.

Using cyclic orbifolds of rational (not necessarily holomorphic) nets, we show that if a unitary modular tensor category $\mathcal{C}$ is realized by conformal nets, then all $\mathbb{Z}_n$-permutation gaugings of $\mathcal{C}$ are realized.

Proposition 4.1. Consider the unitary modular tensor category $\mathcal{C} = \text{Rep}(\mathcal{A})$ for a rational conformal net $\mathcal{A}$.

Then any unitary $\mathbb{Z}_n$-crossed braided extension $\mathcal{C} \rtimes \mathbb{Z}_n$ of $\mathcal{C}^{\mathbb{Z}_n}$ where $\mathbb{Z}_n$ acts by cyclic permutations on $\mathcal{C}^{\mathbb{Z}_n}$ is realized as $\mathbb{Z}_n\text{-Rep}(\mathcal{B})$ for some conformal net $\mathcal{B}$ and $\mathbb{Z}_n \hookrightarrow \text{Aut}(\mathcal{B})$.

In particular, any gauging of the cyclic permutation on $\mathcal{C}^{\mathbb{Z}_n}$ are realized by a conformal net $\mathcal{B}^{\mathbb{Z}_n}$.

Proof. There are $n$ distinguished extensions $\mathcal{C} \rtimes \mathbb{Z}_n$ [EMJP18, Lemma 2.3]. One is realized by the cyclic permutation orbifold [LX04,KLX05] $\mathbb{Z}_n \hookrightarrow \langle \tau_n \rangle \leq \text{Aut}(\mathcal{A}^{\otimes})$. Let $[\omega] \in H^3(\mathbb{Z}_n, \mathbb{T}) \cong \mathbb{Z}_n$. Since $Z(\text{Hilb}_{\mathbb{Z}_n}^\omega)$ is pointed, by [Bis18, Theorem 3.6] there is a conformal net associated with a lattice $\mathcal{A}_L$ realizing $Z(\mathbb{Z}_n, \omega)$. Then there is a $\mathbb{Z}_n$-simple current extension $\mathcal{B}_\omega \supset \mathcal{A}_L$ and $\mathbb{Z}_n \hookrightarrow \text{Aut}(\mathcal{B}_\omega)$, such that $\mathbb{Z}_n\text{-Rep}(\mathcal{B}_\omega) \cong \text{Hilb}_{\mathbb{Z}_n}^\omega$.

Finally, $\Delta(\mathbb{Z}_n)\text{-Rep}(\mathcal{A}^{\mathbb{Z}_n} \otimes \mathcal{B}_\omega)$ with $\Delta(\mathbb{Z}_n) \subset \mathbb{Z}_n \times \mathbb{Z}_n$ the diagonal subgroup gives all $\mathbb{Z}_n$-crossed braided extensions by varying the class $[\omega]$ using [Bis18, Proposition 3.4].

We note that the reconstruction program asks if for any unitary modular tensor category $\mathcal{C}$ there is a conformal net realizing it. In this perspective, the $H^3(\mathbb{Z}_n, \mathbb{T})$ freedom in gauging of cyclic permutations does not give any obstructions.
Figure 1. $\mathbb{Z}_9 \oplus \mathbb{Z}_9$ and $\mathbb{Z}_{27} \oplus \mathbb{Z}_9$ fusion rules for $\text{Rep}((\mathcal{A}^{\otimes 9})^{(\tau_9)})$ for central charge $c \equiv 0 \pmod{24}$ and $c \equiv 8 \pmod{24}$, respectively. The gravitational anomaly $\exp\left(\frac{2\pi i k}{3}\right)$ with $k = 0, 1$, respectively, twists the torus.
Appendix A. Lagrangian extensions

A premetric group $(A, q)$ consists of a finite abelian group $A$ which we see as an additive group and a quadratic form $q: A \rightarrow \mathbb{Q}/\mathbb{Z}$, i.e. $q(na) = n^2 q(a)$ for all $a \in A$ and $n \in \mathbb{Z}$ and $\partial q(a, b) = q(a + b) - q(a) - q(b)$ is a bicharacter. A metric group is a premetric group $(A, q)$ with $\partial q$ non-degenerate. A morphism $\tau: (A_1, q_1) \rightarrow (A_2, q_2)$ is a homomorphism $\tau: A_1 \rightarrow A_2$ with $q_1 = q_2 \circ \tau$.

The following is well-known, see eg. [JS93] and [EGNO15].

**Proposition A.1.** Given a metric group $(A, q)$ there is an up to braided equivalence unique unitary modular tensor category denoted by $\mathcal{C}(A, q)$ such that the braiding $c_{x_a, x_a} = c_q(g) \cdot 1_{x_a \otimes x_a}$ and thus the twist $\theta_{x_a} = \exp(2\pi i q(a))$ for all $a \in A$.

Conversely, given a pointed unitary modular tensor category $\mathcal{C}$, the finite set $A = \text{Irr}(\mathcal{C})$ is an abelian group under the tensor product and the braiding $c$ defines a quadratic form $q(g) \cdot 1_{x_g \otimes x_g} = c_{x_g, x_g}$ for every $g = [x_g] \in G$. Then $\mathcal{C}$ is braided equivalent to $\mathcal{C}(A, q)$.

We define $H^3(A, \mathbb{T})_{ab} = \ker(\psi^*)$, where $\psi^*: H^3(G, \mathbb{T}) \rightarrow \text{Hom}(\Lambda^3 G, \mathbb{T})$ is given by

$$[\psi^*([\omega])](x, y, z) = \prod_{\omega \in \mathbb{S}_3} \omega(\sigma(x), \sigma(y), \sigma(z))^{\text{sign}(\sigma)}.$$ 

The Drinfel’d center $Z(A, \omega) = Z(\text{Hilb}_{\omega}^\mathbb{A})$ is pointed if and only if $[\omega] \in H^3(A, \mathbb{T})_{ab}$ [MN01 Corollary 3.6], see also [Ng03 Proposition 4.1].

Let $\hat{B}$ be an abelian group. A Lagrangian extension of $\hat{B}$ is a triple $(A, q, \iota)$ consisting of a metric group $(A, q)$ with $|A| = |\hat{B}|^2$ and a monomorphism $\iota: (\hat{B}, 0) \hookrightarrow (A, q)$ of premetric groups. The isomorphism classes of Lagrangian extensions of $\hat{B}$ form an abelian group $\text{Lex}(\hat{B})$ via the multiplication $(A_1, q_1, \iota_1) \boxplus (A_2, q_2, \iota_2)$, see [DS18] for details. Given a Lagrangian extension $(A, q, \iota)$ of $\hat{B}$ we obtain a Lagrangian algebra $L = \iota(\hat{B})$ in $\mathcal{C}(A, q)$ and $\mathcal{C}(A, q)_L = \mathcal{C}(A, q)_B$ is naturally isomorphic to $\text{Hilb}_{\omega}^\mathbb{A}$ for some $[\omega] \in H^3(B, \mathbb{T})_{ab}$ and the map $(A, q, \iota) \rightarrow [\omega]$ gives an isomorphism $\text{Lex}(\hat{B}) \rightarrow H^3(B, \mathbb{T})_{ab}$ of abelian groups.

**Example A.2.** Let $A$ be an abelian group and $\hat{A} = \text{Hom}(A, \mathbb{Q}/\mathbb{Z})$ the dual group. Then $(A \times \hat{A}, q_{st}, \iota)$ is an Lagrangian extension of $A$, where $q_{st}(a, \chi) = \chi(a)$ and $\iota: A \rightarrow A \otimes \hat{A}$ is the canonical inclusion. Note that the isomorphism class of $(A \times \hat{A}, q_{st}, \iota)$ is the unit under $\boxplus$ and thus correspond to the trivial cohomology class in $H^3(A, \mathbb{T})$. 


Lemma A.3. Let \((G, q, \iota)\) be a Lagrangian extension of \(\mathbb{Z}_n\) and consider the map \(p: G \to G/\mathbb{Z}_n \cong \mathbb{Z}_n\). If there is a \(x \in G\) with \(p(x)\) a generator and \(q(x) = 0\), then \((G, q) \cong (\mathbb{Z}_n \times \hat{\mathbb{Z}}_n, q_\text{st})\) and \(\mathcal{C}(G, q)_{\iota(\mathbb{Z}_n)}\) is tensor equivalent to \(\text{Hilb}_{\mathbb{Z}_n}\).

Proof. We claim that the order \(\text{ord}(x)\) of \(x\) is \(n\). One the one hand, it is a multiple of \(n\). On the other hand, \(q(mx) \equiv 0 \pmod{1}\) and thus \(L' = \langle x \rangle\) is an isotropic subspace of \((G, q)\) and thus \(\text{ord}(x) \leq n\). Then \(\chi(n) = q(x + \iota(n)) = q(x + \iota(n)) - q(x) - q(\iota(n)) = \partial(\iota(n), x)\) defines a character \(\chi: \mathbb{Z}_n \to \mathbb{Q}/\mathbb{Z}\) and \(\langle \chi \rangle = \hat{\mathbb{Z}}_n\) because \(q\) is non-degenerate. Finally, \(\iota(m) + nx \mapsto (m, n\chi)\) gives an isomorphism of metric groups \((G, q) \to (\mathbb{Z}_n \times \hat{\mathbb{Z}}_n, q_\text{st})\). \(\square\)

Example A.4. Let \(n = 3m\) and consider the following Lagrangian extension of \(\hat{\mathbb{Z}}_{3m} = \{\chi_j\}\) with \(\chi_j: \mathbb{Z}_{3m} \to \mathbb{Q}/\mathbb{Z}\) with \(\chi_j(x) = \frac{x^2}{3m}\). We define Lagrangian extensions \((A_\pm, q_\pm, \iota_\pm)\) and \((A_0, q_0, \iota_0)\)

\[
A_\pm = \mathbb{Z}_{3m} \oplus \mathbb{Z}_m \quad q_\pm(x, y) = \pm \frac{x^2}{9m} \mp \frac{y^2}{m}
\]

\[
A_0 = \hat{\mathbb{Z}}_{3m} \oplus \mathbb{Z}_{3m} \quad q_0(\chi, x) = \chi(x)
\]

with \(\iota_0\) the canonical embedding \(\iota_0: \hat{\mathbb{Z}}_3 \to \hat{\mathbb{Z}}_3 \oplus \mathbb{Z}_3 = A_0\). Then with \(j = (9m - 3, 1)\) is a simple current of order \(3m\) and we have the short exact sequence

\[
\{0\} \longrightarrow \hat{\mathbb{Z}}_{3m} \xrightarrow{\iota_+: \chi \mapsto j} A_\pm = \mathbb{Z}_{3m} \oplus \mathbb{Z}_m \longrightarrow (\mathbb{Z}_{3m} \oplus \mathbb{Z}_m)/\langle j \rangle \longrightarrow \{0\}.
\]

We have the relations \((A_\pm, q_\pm, \iota_\pm) \boxtimes (A_\pm, q_\pm, \iota_\pm) = (A_\mp, q_\mp, \iota_\mp)\) and \((A_+, q_+, \iota_+) \boxtimes (A_-, q_-, \iota_-) = (A_0, q_0, \iota_0)\) which gives a subgroup of \(\text{Lex}(\hat{\mathbb{Z}}_{3m})\) isomorphic to \(\mathbb{Z}_3\). Let \([\omega]\) be the cohomology class associated with \((A_\pm, q_\pm)\), then \([\omega]\) is \([\omega] = \pm m[\omega] \in [\omega]) = H^3(\mathbb{Z}_{3m}, \mathbb{Z})\). This are the cocycle arising in Proposition 3.6.

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Department of Mathematics, Morton Hall 321, 1 Ohio University, Athens, OH 45701, USA

E-mail address: bischoff@ohio.edu

E-mail address: marcel@localconformal.net