Excitations in the Higher Lattice Gauge Theory Model for Topological Phases I: Overview

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In this series of papers, we study a Hamiltonian model for 3+1d topological phases, based on a generalisation of lattice gauge theory known as “higher lattice gauge theory”. Higher lattice gauge theory has so called “2-gauge fields” describing the parallel transport of lines, just as ordinary gauge fields describe the parallel transport of points. In the Hamiltonian model this is represented by having labels on the plaquettes of the lattice, as well as the edges. In this paper we summarize our findings in an accessible manner, with more detailed results and proofs to be presented in the other papers in the series. The Hamiltonian model supports both point-like and loop-like excitations, with non-trivial braiding between these excitations. We explicitly construct operators to produce and move these excitations, and use these to find the loop-loop and point-loop braiding relations. These creation operators also reveal that some of the excitations are confined, costing energy to separate. This is discussed in the context of condensation/confinement transitions between different cases of this model. We also discuss the topological charges of the model and use explicit measurement operators to re-derive a relationship between the number of charges measured by a 2-torus and the ground-state degeneracy of the model on the 3-torus. From these measurement operators, we can see that the ground state degeneracy on the 3-torus is related to the number of types of linked loop-like excitations.

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I. INTRODUCTION

Outside of the phases of matter described by Landau symmetry breaking classification [1], there exist so-called topological phases of matter [2–4]. These topological phases, which include the celebrated fractional quantum Hall systems [5–9], are characterised by long-range entanglement between their local degrees of freedom [4, 10, 11]. While the fact that these phases cannot be described by symmetry breaking is itself interesting, topological phases can also possess rather unique properties as a result of this long-range entanglement. For example, these long-range entangled topological phases may have a ground state degeneracy even in the absence of additional symmetry [12, 13] (when also considering phases with enforced symmetry, the classification of topological phases becomes more rich and includes so-called symmetry protected and symmetry enriched topological phases [10]). This ground-state degeneracy depends on the topology of the manifold on which the topological phase is placed (e.g., such a phase may have no ground state degeneracy on the sphere, but have a degeneracy on the torus), with this degeneracy being resistant to local perturbations [12, 13]. This feature may allow such topological phases to serve as quantum memories [13–16], because encoding information in the topologically protected degenerate subspace...
makes the information resistant to local noise [13, 17].

Further intriguing properties of the long-ranged entangled topological phases are revealed when we consider excitations. In 2+1d, the entanglement structure allows these phases to support anyonic excitations, which are generalizations of the more familiar bosons and fermions, and can undergo non-trivial transformations when two anyons are passed around each other, even at large distances [18–20] (for the interested reader, we note that there are many works giving pedagogical introductions to anyon physics, such as Ref. [21], Ref. [22] and Appendix E in Ref. [23]). Because these transformations (called braiding relations) do not depend on local details, it is believed that these excitations can be used for fault-tolerant quantum computation [24, 25], should sufficiently stable phases and excitations be constructed.

In three spatial dimensions, while any point-like excitations must be fermionic or bosonic [22, 24, 26, 27], topological phases can admit loop-like excitations that undergo non-trivial transformations when a point-like or loop-like excitation is passed through a loop-like excitation [28, 29] (even the Aharonov-Bohm effect [30] can be thought of as braiding a point-like electron with a loop- or string-like magnetic flux tube).

In order to provide a setting where the unique properties of topological phases can be studied in detail, it is convenient to use exactly solvable toy models [13, 31–33]. While these models may not resemble those used to describe real materials [32], only describe the renormalization group fixed point of their phase [34], they provide representatives for a large class of phases of matter [32]. This means that such constructions can be used to probe (and attempt to classify [33]) which kinds of phases can exist. The toy models have Hamiltonians constructed out of commuting projector operators, so that the quasiparticle excitations can be found exactly. Of these constructions for 2+1d topological phases, two of the most successful are the Levin-Wen string-net model [33] and Kitaev’s Quantum Double model [13] (see also work on discrete gauge theory related to the Quantum Double model in Refs. [35] and [36]). The Kitaev Quantum Double class of models includes the toric code, which appears to have a practical application as a robust way to store qubits [13]. Indeed one approach to building quantum computers uses so-called surface codes, which take inspiration from the toric code [37] and which have recently been experimentally realized on a small scale [38, 39]. The string-net construction is more general than Kitaev’s Quantum Double model, and has been conjectured to cover all phases that can be represented by commuting projector models in 2+1d in the absence of an additional symmetry [40], when generalized appropriately from the original construction in Ref. [33] (see Refs. [31, 32, 41–43] for such generalizations). In both of these classes of models, it is well understood how to find the ground-state degeneracy [13, 44] and the properties of the excitations, such as braiding statistics [13, 32, 33] (as we will elaborate on shortly).

In the 2+1d commuting projector models, a useful way of obtaining information about the topological phase under study is to find the operators, known as ribbon operators, that create and move the quasiparticle excitations [13]. This approach was used in Ref. [13] to study the excitations in Kitaev’s Quantum Double model, and in Ref. [33] for the string-net model. As well as classifying the quasiparticles, these ribbon operators can be used to find the braiding relations of the quasiparticles, by taking appropriate commutation relations of the ribbon operators. Further, in Ref. [45] a method was developed for constructing operators to measure topological charge, which is a conserved charge that can exist without symmetry, by using closed ribbon operators. This method was used to identify confined charges and to demonstrate the condensation of certain charges during a condensation-confinement transition, in a modified version of the Quantum Double model introduced in that paper [45].

The models that we have discussed so far describe topological phases in two spatial dimensions. However, there are also many toy models for topological phases in three spatial dimensions. Existing commuting projector Hamiltonian models include the twisted gauge theory model [46–48], which is a generalized version of the Quantum Double model in 3+1d and is based on the Dijkgraaf-Witten topological field theory [49]; a class of models developed from Unitary G-crossed Braided Fusion Categories (UGxBFCs) [50]; the Walker Wang models [51–54], which are 3+1d generalizations of the Levin-Wen string-net models [51]; and the higher lattice gauge theory models [55], based on a generalization of lattice gauge theory (and related to the Yetter quantum field theory [56]).

For the twisted gauge theory model in particular, there has been significant study of the properties of the ground state [46] and the excitations, including their braiding properties [47, 48]. However the general approach to studying these 3+1d models has been different to the approach used for models in two spatial dimensions. While in the 2+1d case, the use of ribbon operators to obtain the properties of the excitations is common, in the 3+1d case an explicit construction of the ribbon and membrane operators (the higher dimensional counterparts to ribbon operators, which produce loop-like excitations) can be difficult. There are some examples of such explicit constructions for the twisted gauge theory models in three spatial dimensions, such as in Refs. [57] and [58], but less so for other models. Instead, indirect methods like dimensional reduction [28, 47] and tube algebras [48] are often used. These methods are certainly useful, but seem to offer a less complete picture of the excitations than a direct construction. Given the success of ribbon operator approaches in 2+1d, and these examples of membrane operators in the 3+1d twisted gauge theory model, we would like to be able to apply similar approaches to other 3+1d models. In this work we will do precisely that, with one of the models discussed above.
In this series of papers we will study the higher lattice gauge theory models introduced in Ref. [55], which can be defined in arbitrary dimension but which we will study in two and three spatial dimensions. These models are based on a generalization of lattice gauge theory, known as higher lattice gauge theory (we will describe both lattice gauge theory and higher lattice gauge theory in more detail in Sections 1C and 1D). In higher lattice gauge theory, the edges of a lattice (or more generally a graph) are labelled by group elements which describe the effect of parallel transport of points across the edges, while the plaquettes are labelled by elements in another group, which describe the parallel transport of paths across these plaquettes. This is a generalization of Kitaev’s Quantum Double model [13], which is based on lattice gauge theory and in which only the edges are labelled. In Ref. [55], where the Hamiltonian models based on higher lattice gauge theory were introduced, several of the properties of these constructions were explored. For example, the ground state degeneracy was given in terms of the partition function of a topological quantum field theory (TQFT), the Yetter TQFT, and also explicitly computed for some examples. Then in Ref. [59], the excitations were studied using a tube algebra approach, through which the loop-like excitations in the model were classified and the simple types were counted. Furthermore, it was shown in Ref. [59] that there is a relationship between the number of types of elementary excitation and the ground-state degeneracy of the model on a 3-torus. In addition, it was shown in Ref. [60] that higher gauge theory could lead to loop-like excitations with non-trivial loop braiding statistics, and the associated representations of the loop braid group were found (the loop braid group describes the motions of loops [61, 62]), although this was not done in the Hamiltonian model but instead from more geometric reasoning about the fluxes and gauge transforms involved. Until now, there was no explicit construction of these excitations in the Hamiltonian model using ribbon and membrane operators, and the braiding statistics of the excitations in the Hamiltonian model have not been found. We aim to address this and describe some of the other features of the excitations, in this work. That is, in this paper we will study these higher lattice gauge theory models [55] by explicitly constructing the membrane and ribbon operators for this model for the first time and applying them to find the other properties of the excitations. We note that these models are particularly interesting to study in this way because they share a similar structure to lattice gauge theory models, which helps with the difficult task of directly constructing ribbon and membrane operators, and yet still exhibit features not seen in ordinary (1-gauge) gauge theory models, as we elaborate on shortly.

Our main results in the 3+1d case are as follows. By using membrane and ribbon operators, we explicitly construct the excitations for this model, in a broad subset of the higher lattice gauge theory models. We find that the basic excitations are either loop-like or point-like and that some of the point-like excitations are confined, costing energy to separate a particle from its anti-particle. This is described in terms of a condensation-confinement transition between different higher lattice gauge theory models, during which some of the loop-like excitations condense out, becoming topologically trivial. This model therefore provides an example of condensation and confinement in a 3+1d setting, which we would not see in a lattice gauge theory model without some modification. Then, using our direct construction of the ribbon and membrane operators, we find the (loop)-braiding relations of our excitations, which describe what happens as we move point-like or loop-like excitations around and through each other. These braiding relations are described in terms of simple group-theoretic quantities. We find that the braiding is generally non-Abelian, so that our relations involve more than a simple accumulation of phase. Instead the excitations have an internal space, which undergoes a transformation under braiding. While the internal spaces are transformed, the excitations carry a conserved topological charge which is not changed by the braiding. This topological charge is of significant interest, and so we also consider the charges present in the higher lattice gauge theory model. Extending the methods of Ref. [45] to 3+1d, we construct operators that can measure the topological charge present in a region. These measurement operators are made from closed membrane and ribbon operators applied on the boundary of the region in question, and the topology of this boundary determines what types of charge we can resolve. For example, the charge associated to point-like objects is measured by putting a sphere around that charge. This is analogous to how the electric charge of an object may be measured by looking at the electric flux through a surface enclosing that object and applying Poisson’s law. On the other hand, loop-like excitations require a surface with handles in order to detect their loop-like character. As far as we know, this is the first time that this method has been applied in 3+1d using explicit membrane operators, although this method has some similarity to the tube algebra methods used in Ref. [59]. In the tube algebra method, the excitations are described in terms of the allowed degrees of freedom on the boundary of a given region, which is presumably equivalent to finding the allowed measurement operators for the surface forming the boundary of the region. Indeed, just like Ref. [59] we find that the number of different charges that can be measured by a torus is equal to the ground state degeneracy of the model when placed on a 3-torus.

A. Structure of This Series

Due to the large number of results obtained for the higher lattice gauge theory model, and the formidable length of an article expounding all of these ideas, we are publishing a series of papers of more manageable length. This work is the first of the series, so we feel that it would be valuable to provide a brief guide to the set of articles. In this work, we will provide a more informal and descriptive
overview of our main results for the 3+1d model. This includes the types of excitation found in the model and their properties. The braiding properties of the excitations are discussed and we describe which excitations are confined. We suggest that a general reader consider this work, before looking through the other papers in the series if they are interested in more detail, or are specifically interested in the 2+1d model.

In the second paper [63], we consider the 2+1d version of the higher lattice gauge theory model. Again we find the ribbon operators that produce the point-like excitations. Despite being in 2+1d, there are also loop-like excitations produced by membrane operators. We find these operators, and find that in certain cases the loop-like excitations can be described as domain walls between different symmetry sectors. This idea that the 2+1d models can describe symmetry enriched topological phases is further expanded on when we map a subset of these models to another construction for such phases, the symmetry enriched string-net model from Ref. [40]. We also consider the braiding relations of the point-like excitations, and construct measurement operators for topological charge.

In the final paper [64], we return to the 3+1d model to provide more detailed results. This includes an explicit presentation of the commutation relations of the ribbon and membrane operators that give us the braiding relations. We also directly construct the measurement operators for topological charge within a torus and a sphere, and find the point-like charge of the simple excitations of the model (including the loop-like ones).

B. Structure of This Paper

In the rest of the introduction, we describe the model introduced in Ref. [55] and introduce other important concepts from existing work. As we mentioned earlier, the model that we examine is based on an extension to lattice gauge theory, known as higher lattice gauge theory. To introduce the model, we shall first discuss lattice gauge theory in Section I C, then higher lattice gauge theory in Section I D. Then in Section I E we will use these ideas to motivate the Hamiltonian model from Ref. [55], and explicitly define the model. Throughout the paper, we will be placing different conditions on the model to examine special cases, which we describe in Section I F. In the last section of the introduction, Section I G, we describe what we mean by braiding statistics in 3+1d.

After discussing the background for our work, we then move on to a description of our results. In this paper, we will give an overview of our results without excessive detail or mathematical proofs, which will be presented in future papers [63, 64]. We start in Section II by using ideas from gauge theory to motivate the excitations and some of their properties. Then in the rest of the paper we will look at how these properties (and additional features) arise in our lattice model. In Section III we describe the excitations that we actually find in the lattice model, together with the operators that create and move them. Next we look at one of the significant features of this model: confinement. Confinement means that it costs energy to separate certain particles from their antiparticles. We explain how this could come about from a “condensation” transition between different higher lattice gauge theory models and how other excitations are affected by this process in Section IV. After this, in Section V we present the braiding relations of the excitations, describing the result of exchanging our excitations in various ways. Finally in Section VI we discuss topological charge, a type of conserved charge realised by topological phases, and point out a relation between the allowed values of this charge and the ground-state degeneracy of our model.

C. Lattice Gauge Theory

To describe a continuum gauge theory, the key ingredients are matter fields, gauge fields (which describe parallel transport of the matter fields) and gauge symmetry. As an example, we can consider conventional electrodynamics. In this case the matter fields describe charges, such as electrons, which couple to the usual gauge field. This gauge field describes parallel transport of the charged matter via the Aharonov-Bohm effect [30]. Finally we have a gauge symmetry, which gives us gauge transforms that appear to change the values of the gauge and matter fields. However states that are related by gauge transforms represent the same physical state and are simply different descriptions of the same physical system.

These ideas translate fairly simply to a lattice gauge theory model [65]. The first thing to consider is the physical space on which we consider the model, that is the lattice. Throughout this paper, we will use lattice in the more informal sense, referring to a collection of vertices and edges (and later plaquettes), without requiring a repeating structure (i.e. we consider a graph). Then we have to place the other ingredients of gauge theory into this discrete setting. The matter field is placed on the vertices of the lattice [65], while the gauge field is placed on the (directed) edges of the lattice and determines the result of transport of matter along the edges [65]. However for the purposes of this paper, we take the matter field to be trivial and do not discuss it further. This leaves us only with the gauge field, which is valued in some group $G$. The group structure means that two paths that lie end-to-end can be composed, with the field label of the resulting path given by group multiplication of the labels of the constituent paths [65], as shown in Figure 1. If we wish to combine two paths that point in opposite directions, we must first flip the direction of one of them, so that they align. This flipping procedure takes the group element to its inverse. Therefore, if in Figure 1 the path labelled $g_2$ pointed in the opposite direction, the combined path would have label $g_1 g_2^{-1}$.
1. Gauge Transforms

Having considered the fields, we now look at the gauge symmetry. The gauge symmetry is included through a set of local operators that each act on the degrees of freedom near a vertex. Each operator is labelled by the vertex it acts on and an element of $G$, so that the gauge transform for a vertex $v$ and element $x \in G$ is denoted by $A^x_v$ [13]. This transform affects the edges surrounding it by pre-multiplying the group element on each adjacent edge by $x$ if the edge is outgoing, and post-multiplying the element by $x^{-1}$ if the edge is ingoing [13, 65–67]. We give an example of the action of the vertex transform in Figure 2, from which we can see that this action is equivalent to adding an imaginary edge, labelled by $x$, to the vertex $v$ and parallel transporting the entire vertex along it.

![FIG. 2. The gauge transform on a vertex is equivalent to adding an imaginary edge at that vertex and then combining this edge into the diagram, or equivalently transporting the vertex along that edge.](image)

At this point, we would like to mention two particular properties of the vertex transform. Firstly, we see that the vertex transform only affects paths that start or end at that vertex. For example, in the top-left image in Figure 2 the path entering the vertex $v$ from the lower left and exiting $v$ through the lower right is labelled by the product $g_1g_2$. In the bottom-left image of Figure 2, which represents the state after the gauge transform, the same path is labelled by $g_1 x^{-1} x g_2 = g_1 g_2$. That is, the path label is unchanged by the gauge transform, because the path does not start or terminate at the vertex $v$. Secondly, note that applying two gauge transforms to the same vertex is the same as parallel transporting along two edges in sequence. This should be equivalent to parallel transport of the vertex across a single path composed of the two edges, and so is the same as applying a single vertex transform with a label obtained by combining the labels of the two edges (and so combining the labels of the two original transforms). If we first apply a vertex transform $A^x_v$ and then another transform $A^y_v$, the label of the combined path introduced by the transforms is $h g$ (it is $h g$ rather than $g h$, due to the fact that the vertex is parallel transported against the direction of the edge, as seen in Figure 2). Therefore we must have that $A^x_v A^y_v = A^{h g}_v$ [13].

2. Gauge-Invariants

Because states related by gauge transforms are equivalent, any physical quantity should be gauge-invariant. We can construct these gauge-invariant quantities from the closed loops of our lattice [66, 67]. Under a gauge transform, the group element assigned to a closed loop is at most conjugated by the vertex transforms [67]. Therefore the conjugacy class of that label is a gauge invariant quantity. For example, consider Figure 3 below, which shows the action of a vertex transform $A^x_v$ on a closed loop starting at $v$. Initially the group element associated to the closed loop is $g_1 g_2$. After applying the vertex transform it becomes $x g_1 g_2 x^{-1}$. This indicates that the group element is not generally a gauge-invariant quantity, but its conjugacy class is.

![FIG. 3. We consider the effect of a vertex transform $A^x_v$ on a closed loop starting at the vertex $v$. The path label in this case goes from $g_1 g_2$ to $x g_1 g_2 x^{-1}$. That is, the path label of the closed loop is conjugated by $x$ under the action of the transform.](image)

As an example of the importance of such closed loops, we can consider the case of electromagnetism. Here we have a $U(1)$ gauge symmetry, so that the edges in our lattice would be labelled by phases. There is a physical process where we take a charge $q$ around a closed loop in the presence of a magnetic field described by a vector potential $\vec{A}$. In the continuum theory this leads to the Aharanov-Bohm effect [30], where the wavefunction accumulates a phase of $\theta = q \oint \vec{A} \cdot d\vec{l}$. This phase is equivalent to the label we would give our closed loop in the lattice model.
Using Stoke’s theorem, the Aharonov-Bohm phase can be related to the magnetic flux through the surface enclosed by the loop. The phase is a gauge invariant quantity, as is required by the fact that this phase can be measured in interference experiments and thus is a physical quantity.

It is important to keep track of such gauge-invariant quantities, because they allow us to differentiate between physically distinct states. For instance, many states can be reduced to the trivial configuration, where every edge physically distinct states. For instance, many states can be reduced to the trivial configuration, where every edge is labelled by 1 (the identity in the group \( G \)), by applying gauge transforms. The state where the edges are all labelled by the identity describes trivial parallel transport, and so the states related to this trivial state by gauge transforms must also be trivial. This indicates that in these equivalent states the apparently non-trivial edge labels only describe a change of basis, rather than a physical change under parallel transport. On the other hand, if a state has any closed loops with non-trivial path label, then (because the conjugacy classes of closed path labels are gauge invariant) the state cannot correspond to this trivial state. Therefore in such a state the parallel transport across the edges must describe both a change of basis and some physical “flux”, analogous to the magnetic flux in electromagnetism, which differentiates it from the trivial case.

3. The Quantum Double Model

Having now considered lattice gauge theory, we can build a model for topological phases from it, known as Kitaev’s Quantum Double model [13]. The lattice represents the spatial dimensions of the models, while a Hamiltonian controls the time evolution. In order to construct the Hamiltonian, we first denote the gauge symmetry by an energetic constraint. We do this by adding an energy term to the Hamiltonian for each vertex, with the energy term enforcing the symmetry. We also energetically punish closed paths with non-trivial labels, by adding an energy term at each plaquette that punishes plaquettes with non-trivial boundary paths. The Hamiltonian is [13]

\[
H = - \sum_{\text{vertices, } v} A_v - \sum_{\text{plaquettes, } p} B_p.
\]

Here we have

\[
A_v = \frac{1}{|G|^2} \sum_{g \in G} A_v^g,
\]

where the \( A_v^g \) are the gauge transforms from earlier and \(|G|\) is the number of elements in the discrete group \( G \). \( A_v \) is therefore an average over all gauge transforms at vertex \( v \). The operator \( A_v \) is a projector [13], because

\[
A_v A_v = \frac{1}{|G|^2} \sum_{g \in G} \sum_{h \in G} A_v^g A_v^h = \frac{1}{|G|^2} \sum_{g \in G} \sum_{h \in G} A_v^{gh} = \frac{1}{|G|^2} \sum_{g \in G} A_v^g = A_v.
\]  

As a projector, \( A_v \) has eigenvalues of 0 and 1, with the eigenvalue of 1 corresponding to states which are gauge symmetric at that vertex (because the gauge transforms leave such states unchanged). \( A_v \) enters the Hamiltonian with a minus sign, so the gauge-invariant states are lower in energy.

The other term in the Hamiltonian, \( B_p \), acts on the edges around a plaquette \( p \). It leaves states where the boundary of the plaquette is labelled by the identity unchanged and returns zero for other states. As an example, consider Figure 4, which illustrates the action of the plaquette term \( B_p \) on a simple plaquette made from two edges (a bigon). In this case the boundary path label is given by \( g_1 g_2^{-1} \), and so the plaquette term returns the state if \( g_1 g_2 = 1_G \). \( B_p \) is clearly a projector just like the vertex term, with the eigenvalue of 1 corresponding to states with trivial flux around the plaquette (we say the plaquette satisfies flatness in these states). Again, \( B_p \) enters the Hamiltonian with a minus sign, so that these trivial flux states are lower in energy. The trivial flux label \( 1_G \) is in a conjugacy class on its own, meaning that it is unchanged by gauge transforms. This means that the operator \( B_p \) is built out of gauge-invariant quantities and therefore commutes with the gauge transforms. All of the terms in the Hamiltonian are projectors and they all commute, so this is an example of a commuting projector model. This structure to the Hamiltonian enables the model to be solved exactly. The excitations are charge-like (excitations of the vertex term), flux-like (primarily excitations of the plaquette term, though they may also excite a vertex term), or some combination of the two [13]. These excitations are called electric if they are charge-like, magnetic if they are flux-like and dyonic if they are a combination.

![Figure 4](image)

FIG. 4. The plaquette term \( B_p \) gives 1 if the closed path forming the boundary of the plaquette \( p \) is \( 1_G \) (i.e. if it is flat) and 0 otherwise. For the example plaquette shown in this figure, where the edges are in states labelled by \( g_1 \) and \( g_2 \), that means that acting with the plaquette term gives a non-zero result only if \( g_1 g_2 = 1_G \). We have seen how lattice gauge theory can be used to define a Hamiltonian model for topological phases, known as
Kitaev’s Quantum Double model [13]. This model is primarily defined in 2+1d, though it can also be extended to 3+1d (see e.g. Ref. [46], where Kitaev’s Quantum Double model is both extended to 3+1d and generalized). The higher lattice gauge theory model that we will consider in this paper is built in a similar way from a generalization of lattice gauge theory, called higher lattice gauge theory. Because of this, many of the features of Kitaev’s Quantum Double model will carry over to the higher lattice gauge theory model considered in this paper.

D. Higher Lattice Gauge Theory

We have now looked at the general principles of ordinary lattice gauge theory and its associated model for topological phases. Next we can consider higher lattice gauge theory [68], before moving on to its associated model for topological phases [55]. As stated before, higher lattice gauge theory is a generalization of lattice gauge theory [68, 69]. In lattice gauge theory we consider parallel transport along paths, and label paths by group elements to allow composition of paths [69]. That is, we have labelled geometric objects, the paths, with algebraic objects, the group elements. A natural generalization is to label more of the geometric objects. We still label the paths (now calling their associated gauge field the 1-gauge field), but now also consider labelling objects of a higher dimension: the surfaces. If paths describe the parallel transport of points, then surfaces describe the parallel transport of paths, that is of the 1-gauge fields [69]. We can view this pictorially as shown in Figure 5. The blue double arrow on the surface enclosed by the paths represents the transport of one path (the source) into another (the target) [55]. The two paths (source and target) start and end at the same points (vertices) and the start of these two paths is called the base-point of the surface.

![Figure 5](image1.png)

**FIG. 5.** Just as a path is associated to parallel transport of points, so is a surface associated to parallel transport of a path. The initial position of the path is called the source and the final position is called the target.

In the same way that we can compose paths that lie end-to-end, so may we combine adjacent surfaces. In fact, surfaces can be composed in two ways. Firstly, they may be combined vertically [68, 69], as shown in Figure 6. Vertical composition corresponds to the case where we perform two parallel transportations of a path (the top path in Figure 6) in sequence (first moving it to the middle position in the figure and then to the bottom). We can combine these two steps to describe the two parallel transportations as parallel transport along a single, combined, surface.

![Figure 6](image2.png)

**FIG. 6.** Consider two surfaces over which we can sequentially transport a path, such as the ones in the left side of the figure. In this case we first transport the top path over the upper surface (across the arrow) to the middle location (the straight path) and then over the lower surface to the bottom position. We can express the same process as transport over a single surface, made from a combination of the two individual surfaces, as shown in the right figure. This is called vertical composition of the surfaces.

We may also combine the surfaces horizontally [68, 69], as shown in Figure 7. This horizontal combination corresponds to the case where we have two paths lying end to end, which we can parallel transport separately. However we can also combine the two paths into one, before transporting them across a single surface.

![Figure 7](image3.png)

**FIG. 7.** In addition to vertical composition of surfaces, as shown in Figure 6, we can consider horizontal composition of two surfaces that lie side by side. In this case, the individual surfaces describe parallel transport of two paths (A-B and B-C) which can be composed, while the combined surface describes the parallel transport of the paths after they have been composed (into A-C). While we show the resulting surface as a simple bigon (2-gon) for convenience, the composition does not change the shape of the constituent surfaces on the lattice, and so the point B remains on the boundary of the combined surface. The two points labelled B in the right-hand side are the same and so should be glued together.

As a special case of horizontal combination, we have the case where the first path is not parallel transported across any surface. This lets us combine a surface with a path. As an example, such a situation is shown in Figure 8. In Figure 8, we combine the edge that runs from A to B with a surface, by treating the edge and its inverse (the inverse is the same edge, but with reversed direction) as bounding an infinitesimally thin surface and then using horizontal composition. This process of combining a surface with a
path is known as whiskering [69], and can also be thought of as moving the base-point of a surface (in the case shown in Figure 8, the base-point of the surface is initially at B, but is moved to A, where in each case the base-point is represented by the blue dot).

![Diagram](image)

**FIG. 8.** Here we consider “whiskering” a surface (with specified source and target) along a path. Just as for horizontal composition, the shape of the final surface should match the combined shape of the path and surface, though we have represented the final surface as a simple bigon. The point B remains on the boundary of the final surface, in both the source and target. Note that the base-point (which is represented by the blue dot) of the surface changes from B in the first image to A in the second.

Given this geometric structure for the paths and surfaces, we must now determine the appropriate algebraic structure with which to label it. As before, we label the edges with elements of a group $G$ (the 1-gauge field), to enable composition. The label assigned to a directed path is called the 1-holonomy of that path [55]. We also label the plaquettes with elements of another group, $E$ [55, 68, 69]. Labelling every plaquette in this way gives us a field which is valued in $E$ at each plaquette, with composition rules allowing us to find the label for an arbitrary surface. We refer to this field as the 2-gauge field and call the label associated to a surface with given source and target a 2-holonomy [55]. As we will see shortly, parallel transport will involve various mappings between the groups $E$ and $G$.

Armed with the algebraic objects which describe our fields, we go back to our diagram defining parallel transport across a surface (Figure 5) and label it, giving the situation shown in Figure 9. Figure 9 then describes how parallel transport of a 1-gauge field across a surface depends on the 2-gauge field at that surface.

![Diagram](image)

**FIG. 9.** We can label the geometric objects in Figure 5 with algebraic quantities from the groups $G$ and $E$. The paths are labelled by elements of $G$ and the surfaces by elements of $E$. The parallel transport of a path over a surface labelled by $e \in E$ results in the path element $g$ gaining a factor of $\partial(e)$, where $\partial$ is a group homomorphism from $E$ to $G$.

In Figure 9, $\partial$ is a group homomorphism from $E$ to $G$ (i.e. a map that preserves the group multiplication), so that for $e \in E$, $\partial(e) \in G$ [69]. As we will now show, this is required to be a group homomorphism in order to make the vertical composition diagram consistent. We map vertical composition of two surfaces onto the group multiplication, with the multiplication of the labels in $E$ done from right to left, following the convention in [55]. An example of this is shown in Figure 10. Requiring the label of the bottom path to be the same on both sides of Figure 10 gives the consistency condition $\partial(e_2 \cdot e_1) = \partial(e_2) \partial(e_1)$, which is the condition for a group homomorphism. This ensures that the effect of transporting the edge along one surface, labelled by $e_1$, and then another surface, labelled by $e_2$, is the same as transporting the edge along the combined surface (labelled by $e_2 e_1$). It is worth noting that in Ref. [69] the horizontal composition, rather than the vertical composition, is described by the group multiplication.

![Diagram](image)

**FIG. 10.** Vertical composition of two 2-holonomies

We need a second group-theoretic quantity to describe whiskering [55, 69] (see Figure 8). Whiskering can be thought of as parallel transport of the base-point of a surface along an edge. Because parallel transport of objects along paths is described by the group $G$, we need some way for $G$ to act on $E$. We therefore introduce the map $\triangleright$. $\triangleright$ is a map from $G$ to the endomorphisms on $E$ [55, 69] (endomorphisms are homomorphisms from a group to itself). That is, given an element $g$ of $G$, the object $g \triangleright$ is then a map from $E$ to itself. We write $g \triangleright e$ as acting on an element $e$ of $E$ as $g \triangleright e$. $g \triangleright e$ is then also an element of $E$. In Figure 11, we see how this map $\triangleright$ is involved in whiskering. If we move the base-point of a surface (initially B in the figure) across an edge labelled by $g$, against the direction of the edge, then the surface label changes from $e$ to $g \triangleright e$. On the other hand, moving the end-point (C in the figure), rather than the base-point, has no effect on the surface label.

For the diagrams that we have considered to give a consistent theory, the different ways of combining the elements of the diagram must be consistent. One consequence of
FIG. 11. Whiskering is described by the map \( \cdot \). Requiring the lower path from \( A \) to \( C \) to have the same label after combining the edge with the surface via whiskering gives the condition that \( g \partial(e) h = \partial(g \cdot e) g h \), so that \( \partial(g \cdot e) = g \partial(e) g^{-1} \).

This is that we can find the result of horizontal composition of two surfaces, by combining the rules for vertical composition and whiskering. Consider Figure 12, which shows a diagram involving the horizontal composition of two surfaces (on the top line), where the left and right surfaces are labelled by group elements \( e_L \) and \( e_R \) respectively. We can reproduce this horizontal composition with a series of other manipulations, which take us the other way around the diagram. These other processes involve changing the start-point and end-point of surfaces, as well as vertical composition, all of which we already know how to perform. Applying these manipulations (as explained in Figure 12), we find that the label resulting from horizontal composition must be \( e_L [g \partial \cdot e_R] \).

In addition to allowing us to define horizontal composition without introducing additional algebraic objects, requiring the consistency of various diagrams enforces certain restrictions on the algebraic objects we already have. For example, if we have a diagram with three surfaces to combine, the order in which we combine the surfaces should not matter. This restricts our multiplication of surface labels to be associative. Because our vertical composition is described by group multiplication in the group \( E \), this associativity is immediately guaranteed by the group properties without any additional conditions on the group. However there are additional constraints that must be satisfied by the maps \( \partial \) and \( \cdot \). Requiring the consistency of whiskering with vertical composition of surfaces and composition of paths (see Figures 13 and 14 or Ref. [69] for more detail) gives us the following conditions for all \( g, h \in G \) and \( e, f \in E \) [55]:

\[
\begin{align*}
g \cdot (ef) &= (g \cdot e) (g \cdot f) \quad (2) \\
g \cdot (h \cdot e) &= (gh) \cdot e. \quad (3)
\end{align*}
\]

These are the conditions for a group action of \( G \) on \( E \). That is, these conditions mean that \( \cdot \) is a homomorphism from \( G \) to the endomorphisms on \( E \), where endomorphisms are group homomorphisms from \( E \) to itself. Furthermore, because these endomorphisms are invertible (from Equation 3, \( g^{-1} \cdot \) is the inverse of \( g \cdot \)), they are automorphisms.

As illustrated in Ref. [69] (though note the different conventions used in this reference), consistency of whiskering with other diagrams (see Figures 11 and 15) also demands that [68, 70]

\[
\begin{align*}
\partial(g \cdot e) &= g \partial(e) g^{-1} \quad (4) \\
\partial(e) &= f = e f e^{-1}. \quad (5)
\end{align*}
\]

These two conditions are known as the Peiffer conditions [55]. The algebraic structure \((G, E, \partial, \cdot)\) satisfying all of these conditions (Equations 2, 3, 4 and 5 in addition to \( \partial \) being a group homomorphism) is known as a crossed module. In order to familiarize the reader with these crossed modules, we describe a handful of examples here.

**Example 1:**

One example of a crossed module is \((G,G,\text{id},\text{ad})\), where \( G \) is any finite group, \( \text{id} \) is the identity map and ad maps \( g \in G \) to conjugation by \( g \) [55]. That is, we have \( \partial(e) = e \) and \( g \cdot e = geg^{-1} \). This clearly satisfies the first Peiffer condition because

\[
\partial(g \cdot e) = g \cdot e = geg^{-1} = g \partial(e) g^{-1}.
\]

It also satisfies the second condition as

\[
\partial(e) \cdot f = e \cdot f = e f e^{-1}.
\]

**Example 2:**

Another example is \((G,\{e\},\partial \rightarrow 1_G, \cdot \rightarrow \text{id})\) [55]. That is, we take the group \( E \) to be trivial. Then \( \partial \) maps the element of \( E \) to the identity of \( G \) and \( \cdot \) is the identity map on \( E \) (clearly these are the only allowed \( \partial \) and \( \cdot \) when \( E \) is \( \{e\} \)). We have that

\[
\partial(g \cdot e) = 1_G \cdot g = gg^{-1} = g1_G g^{-1} = g \partial(e) g^{-1}
\]

so the first Peiffer condition is satisfied. Furthermore

\[
\partial(e) \cdot f = 1_E = e f e^{-1},
\]

because the only element of \( E \) is the identity, so the second Peiffer condition is also satisfied. This special case recovers lattice gauge theory, because the surfaces all have trivial label and so we can just neglect to label them.

**Example 3:**

A third, more interesting example is \((Z_2, Z_3, \partial \rightarrow 1_{G}, \cdot \rightarrow \text{id})\) [55]. We take the elements of \( G = Z_2 \) to be \( 1_G \) and \(-1_G \) and the elements of \( E = Z_3 \) to be \( 1_E, \omega_E \) and \( \omega_E^2 \). Then we define \( \cdot \) by \( 1_G \cdot e = e \) and \(-1_G \cdot e = e^{-1} \) (where \( \omega_E^{-1} = \omega_E^2 \)). This satisfies the requirement of having a group structure on the elements of \( G \) (as described in Equation 2) because applying two \(-1_G \cdot \) maps in sequence gives

\[
\begin{align*}
-1_G \cdot (-1_G \cdot e) &= -1_G \cdot e^{-1} = e \\
&= 1_G \cdot e \\
&= (-1_G \cdot 1_G) \cdot e,
\end{align*}
\]
while the other conditions for the group structure involve $1_G \triangleright$ and are satisfied because $1_G \triangleright$ is the identity map. The individual maps $g \triangleright$ are also endomorphisms as required. For $-1_G \triangleright$ we have

$$-1_G \triangleright (e_1 e_2) = e_2^{-1} e_1^{-1}$$

$$= e_1^{-1} e_2^{-1}$$

$$= (-1_G \triangleright e_1)(-1_G \triangleright e_2),$$

where we used the fact that $E$ is Abelian to swap the order of multiplication in the second line. This indicates that $-1_G \triangleright$ is a group homomorphism on $E$. $1_G \triangleright$ is also a homomorphism because it is the identity map. Therefore $\triangleright$ is indeed a group action of $G$ on $E$. Next, we will check that the Peiffer conditions are satisfied. We have $\partial(g \triangleright e) = 1_G = \partial(e) gg^{-1} = g \partial(e) g^{-1}$ (using $\partial(e) = 1_G$ and the fact that the group $G$ is Abelian). Finally $\partial(e) \triangleright f = 1_G \triangleright f = f = fe e^{-1} = e fe^{-1}$, where we used that $E$ is Abelian. Because all of the consistency conditions are satisfied, this is indeed a valid crossed module. This $(Z_2, Z_3, \partial \rightarrow 1_G, \triangleright)$ crossed module can be generalized slightly by replacing $Z_3$ with $Z_n$, where $n$ is an odd integer, with $-1_G \triangleright$ still acting as inversion.

FIG. 12. The requirement that the different ways of combining a diagram must be consistent means that we can express the horizontal composition of two surfaces in terms of whiskering and vertical composition. Consider the top-left image, consisting of a left and right surface with label $e_L$ and $e_R$ respectively. We wish to combine these two surfaces to obtain the surface in the top-right image. There are two ways around the diagram that lead from the top-left to the top-right image, and these are required to give the same result for the label of the final surface. The first way is horizontal composition of the two surfaces, which gives us an unknown label for the final surface that we wish to find. The second way around the diagram involves the other processes that we do have algebraic expressions for. We can therefore use this to find the label resulting from horizontal composition. The first step (represented by the downwards arrow) is to whisker the right-hand surface so that it has the same base-point (A) as the left surface. This gives the right surface a label $g_L \triangleright e_R$, where $g_L$ is the label of the path from the original base-point B to A, as shown in the bottom-left image. The next step is to move the end-point of the right-surface from C to B, to match the end-point of the left surface, as shown in the bottom-right image. This has no effect on the label of the surface. Then the target of the right surface is the same as the source of the left surface (the path from A to B with label $g_L$) and so the two surfaces can be combined via vertical composition, giving a surface with label $e_L[g_L \triangleright e_R]$ as shown in the centre-right image. Finally, moving the end-point from B to C (the original end-point of the right surface) gives us the same surface that we would have from horizontal composition, with label $e_L[g_L \triangleright e_R]$. This is therefore the label resulting from horizontal composition.
FIG. 13. Requiring consistency of whiskering with the vertical composition of surfaces demands that the map $g \triangleright$, for arbitrary $g \in G$, is a group homomorphism on $E$. That is $(g \triangleright f)(g \triangleright e) = g \triangleright (fe)$ for $g \in G$ and $e, f \in E$. This can be seen from the figure, because consistency demands that the diagram commute, i.e. the two routes from the top-left image to the bottom-right image should give the same result.

FIG. 14. We require that whiskering is consistent with the composition of edges. That is, combining two edges together and then whiskering a surface along the combined edge should give the same result as whiskering that surface by one edge and then the other in sequence. This gives us the mathematical condition $g_1 \triangleright (g_2 \triangleright e) = (g_1 g_2) \triangleright e$, which is the condition that the map $\triangleright : G \to \text{End}(E)$ be a group homomorphism on $G$. 
FIG. 15. The second Peiffer condition can be derived from demanding that this figure be consistent. Starting at the top-right diagram, we may combine the left and right parts of the diagram to obtain the top-left diagram, using the rules for horizontal composition given in Figure 12. We can then use vertical composition to obtain the bottom-left diagram, which should have a label of $efe^{-1}$. However, we could also have performed vertical composition on the top-right diagram to obtain the bottom-right diagram, before whiskering to obtain the bottom-left diagram. In that case the surface label is $\partial(e) \triangleright f$. Consistency therefore demands that $efe^{-1} = \partial(e) \triangleright f$.

1. Gauge Transforms

Now that we have considered the fields and the parallel transport rules, it is time to look at the gauge transforms. We have gauge transforms associated to the more familiar 1-gauge field in addition to transforms associated to the 2-gauge field [55]. The 1-gauge transforms are similar to the gauge transforms in ordinary lattice gauge theory. As before, we label the 1-gauge transforms associated to a vertex $v$ by $A^g_v$, where $g$ is an element of $G$ and we have one such transform for each element of $G$. $A^g_v$ acts on the degrees of freedom near the vertex $v$ in a way equivalent to parallel transport of the vertex $v$ along an edge of label $g^{-1}$ (or $g$ if we transport the vertex against the direction of the edge, as in Figure 16). The effect on the edges around the vertex is therefore the same as in the lattice gauge theory case and, just as in the lattice gauge theory case, only paths that start or terminate on the vertex are affected by the gauge transform (see Section I C 1 and Figure 2 in particular). The only difference is that now we must also consider parallel transport of surfaces along the edge, so that the vertex transform also affects the surface labels. This parallel transport can be performed by adding a new edge and vertex, which we proceed to combine with the rest of the lattice (so that the lattice is the same at the end as it was before the transform, apart from changes to the group labels), as illustrated in Figure 16. In the last step we relabel the vertex $v'$ to $v$ in order to match the original vertex. We can recognise the middle diagram in Figure 16 as the whiskering diagram (see Figure 11), so combining the edge with the plaquette gives us a $g \triangleright$ action on the plaquette label. This tells us that any surface with base-point at the vertex on which we apply the transform must be acted on by $g \triangleright$. On the other hand, surfaces not based at that vertex are left unaffected [55]. In summary, the 1-gauge transform acts on an edge $i$ or plaquette $p$ according to [55]

$$A^g_v : g_i \rightarrow \begin{cases} \ g g_i \ & \text{if } v \text{ is the start of } i \\ g_i g^{-1} \ & \text{if } v \text{ is the end of } i \\ g_i \ & \text{otherwise} \end{cases}$$

$$A^g_v : e_p \rightarrow \begin{cases} g \triangleright e_p \ & \text{if } v \text{ is the base-point of } p \\ e_p \ & \text{otherwise} \end{cases}$$

(6)

In addition to these 1-gauge transforms, we also have 2-gauge transforms, which act on an edge and the surfaces that adjoin it [55]. The 2-gauge transform on an edge $i$ and labelled by an element $e \in E$ (denoted by $A^e_i$) acts like parallel transport of the edge along a surface labelled
by $e$ (this follows from the action defined in [55]). Recall that for the ordinary gauge transforms we added an edge and combined it with the diagram, as shown in Figure 16. For the 2-gauge transforms, we instead add a surface and combine the surface with the rest of the diagram, as shown in Figure 17. Note that in Figure 17, the base-point of each surface is also the start of edge $i$, which results in the simple expression for the edge transform given in that figure. Due to subtleties regarding how the edge transform acts on surfaces whose base-points do not match the start of $i$, we postpone an explicit formula for the general action of the edge transform until Section 1E.

2. Gauge-Invariants

Recall that in ordinary lattice gauge theory we could build gauge-invariant quantities out of closed loops. What are the appropriate quantities for higher lattice gauge theory? We can build gauge-invariants from the closed loops as before, but also from closed surfaces. The closed loops work much as before, except that we need to modify the group element that labels them to account for parallel transport of paths over surfaces. Given a closed loop made of two paths, as shown in Figure 18, to work out the group element for the loop, we need to transport the paths so that they are in the same location. This is necessary because the two paths may be defined with different gauge choices, with the conversion between the gauge choices performed by parallel transport. To obtain a gauge invariant, we will need to ensure that the two paths are described in the same gauge. The relevant transport is shown in Figure 18.

FIG. 16. The effect of a vertex transform $A_v^0$ on a plaquette attached to the vertex $v$ depends on the base-point of that plaquette. If the base-point is at $v$, as in this case, the plaquette label obtains a $g\triangleright$ action, because the vertex transform is equivalent to parallel transport of the vertex $v$, and so of the base-point (i.e. the vertex transform is like whiskering the plaquette, as we see in the middle diagram). If the base-point of the plaquette is not at $v$, then the plaquette is not whiskered and its label is left unchanged.

FIG. 17. A 2-gauge transform $A^e_i$ on an edge $i$, with initial label $g_i$, acts like parallel transport of that edge across an additional surface of label $e$. In the middle picture, the new surface points upwards, out of the plane of the other two surfaces. In the third picture, we combine this new surface with the others and then relabel the edge $i'$ (the target of the additional surface) to $i$.

FIG. 18. The gauge invariant associated to plaquettes is modified to account for parallel transport of the paths over the plaquette.
ciated to the closed loop in Figure 18, from $g_1g_2^{-1}$ to $\partial(e_2)g_1g_2^{-1}$. This quantity is the 1-flux or 1-holonomy for the closed loop. For a general surface, we replace $g_1g_2^{-1}$ with the label of the boundary of the surface. For a plaquette $p$, with boundary label $g_p$, the 1-flux is given by $\partial(e_p)g_p$, and we refer to this quantity as $H_1(p)$ [55]. This label can be changed only within a conjugacy class by either the vertex transforms (as in Figure 16) or the edge transforms (as in Figure 17), so those conjugacy classes are gauge invariant quantities [55].

As an example, we can consider acting on the diagram in Figure 18 with a vertex transform. This gives us the situation shown in Figure 16. From that figure, we see that the plaquette holonomy, which is initially given by $\partial(e)g_1g_2^{-1}$, transforms as

$$\partial(e)g_1g_2^{-1} \rightarrow \partial(g \cdot e)gg_1g_2^{-1}g^{-1} = g\partial(e)gg_1g_2^{-1}g^{-1} \quad \text{(using the Peiffer conditions)} = g\partial(e)g_1g_2^{-1}g^{-1},$$

which is only conjugation. This indicates that the conjugacy class of this group element is a gauge-invariant quantity, in a similar way to the closed cycle in lattice gauge theory.

In addition to closed paths, we have closed surfaces, with their own gauge-invariants. In this case, the gauge invariant assigned to a closed surface is simply the group label (2-gauge label) assigned to that closed surface, which may be obtained by using the rules for composing surfaces if that closed surface is comprised of multiple plaquettes. The 2-gauge label for a closed surface, which we call the 2-flux of that surface, is only changed within certain equivalence classes by the gauge transforms [55]. Again, the identity element is in a class on its own, so that trivial 2-flux is preserved by the transforms [55]. An important thing to note is that this is only strictly true when the system satisfies certain additional constraints [55], which we will discuss in Section I F.

In the same way that the 1-flux on a closed loop determines the result of a process where we move a charge around the loop, the 2-flux of a closed surface corresponds to a transport process. For a sphere at least, we can measure this 2-flux by nucleating a small loop at the base-point of that surface, before passing it over the surface and then contracting it again, as indicated in Figure 19. This reflects the fact that a spherical closed surface (which can be built from a series of open surfaces) can have empty source and target, and so can represent a transport process where we nucleate the loop at the start and collapse it at the end. For a surface such as a torus, with non-contractible cycles, the corresponding transport process may not involve nucleation and collapse.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure19.png}
\caption{The 2-holonomy of a surface (in this case a sphere) can be measured by a transport process. A small loop is created at the base-point (the small red sphere), then dragged over the surface (the larger blue sphere), as indicated by the arrow.}
\end{figure}

E. Hamiltonian Model

Having considered higher lattice gauge theory, we are now in a position to consider the Hamiltonian model based on it (as described in [55]). However we must first look in more detail at how we interpret the geometric meaning of our 2-holonomy in the case of a fixed lattice. When we label a surface element, that element corresponds to parallel transport of a particular path along that surface. It does not immediately describe the parallel transport of a different path across the same surface. For example consider a square, with different paths denoted as the source or target, shown in Figure 20. In the left diagram, the surface corresponds to the case where we deform the top edge (which is the source for the surface) into the bottom three (which form the target). However, as indicated in the right diagram, we could also transport the left edge into the right three over the same surface. Despite corresponding to the same square in space, the label in $E$ associated with these two parallel transports is different in general. This means that when setting up the lattice, we need to define which path the surface describes parallel transport of. That is, for each plaquette we need to define the source and target [55]. This is analogous to how we define a direction for each edge. We also need a way of manipulating the label to describe parallel transport of other paths. This is just like how we can find the group element associated to travelling an edge in the opposite direction to its arrow, in that case by simply inverting the group element. This kind of manipulation of the surfaces is necessary, for example, to consider the gauge transforms, which we saw acted by parallel transport. It is also necessary for combining surfaces, because (as described in Section I D) vertical composition of two surfaces requires the target of the first surface to match the source of the second.

Having seen that a given plaquette can correspond to different transport processes depending on the source and
target, we need a way of defining a “canonical” position for the source and target paths of every plaquette when we set up the lattice (we refer to this, along with the orientation of the edges, as the “dressing” of the lattice, and we consider the effect of changing this dressing in Appendix A). This is done via a so-called branching structure. Every vertex in our lattice is assigned a unique integer index [55]. Edges point from the lower indexed vertex to the higher indexed one. Then for a given plaquette, the lowest indexed vertex on that surface is the base-point for that surface (recall that the base-point is the start of the source and target paths for the surface) [55]. This base-point is adjacent to two other vertices on that surface. The source travels from the base-point to the lower indexed of these two vertices and then all the way around the surface until it reaches the other vertex. The target is just the path from the base-point to that other vertex (in the opposite direction from the source). This is illustrated in Figure 21, where the path 0-2-1-3 is the source, while the path 0-3 is the target.

The canonical sources and targets, which are fixed by the branching structure, form a reference that we can use to unambiguously define a state (as we need to specify the sources and targets whenever we define a 2-holonomy). The branching structure defines the precise parallel transport process associated to the plaquette by default, and the group label given to the plaquette corresponds to this process. However sometimes we need to know how to perform parallel transport over the surface, but with a different source and target to the ones given by our branching structure. In particular, we will often need to compose two surfaces. The vertical composition described in Section 1D relies on the target of one surface being the same as the source of the other, and so we may need to manipulate the sources and targets of different surfaces to get them to line up. There are certain rules that let us imagine changing the source and target [55]. These rules will tell us what label the surface would have if it had that source and target. The first thing we can do is to swap the source and target. The resulting plaquette label is just inverted [55], as shown in Figure 22.

Next, we can move the base-point around. We can either move it along the plaquette (as shown in Figure 23), or away from the plaquette (as shown in Figure 24) [55]. In either case, the surface label changes from its original label $e_p$ to $g(t)^{-1} \triangleright e_p$, where $t$ is the path along which we move the base-point and $g(t)$ is the group element assigned to that path [55].
Another useful manipulation of the source and target is to move the base-point of our surface along the boundary of that surface, which adds or removes edges from the start of the source (and removes or adds those edges to the target). This results in a $\triangledown$ action on the surface label.

We can also move the end-point, either along the plaquette, or away from it [55]. An example of the latter case is shown in Figure 25. Either way, the plaquette label is unchanged [55]. This latter move (as shown in Figure 25) allows us to add additional edges to the boundary of the surface, though these additional edges enclose no area. Though in Figure 25 the edges are added near the end-point, we can move the end-point around the surface and so we can add these additional edges anywhere on the surface's boundary. If these edges are not added at the end-point, the added edges appear twice in the source or target and are travelled in opposite directions for their two appearances, meaning that they do not contribute to the path element of the source and target. If they are added at the end-point, they contribute equally to the end of the source and target. Either way, their contribution to the group element associated to the surface boundary (the 1-gauge value assigned to the path around the surface) cancels. For example, in Figure 25 adding the edge of label $x$ to the end-point takes the path label of the boundary from

$$g(\text{boundary}) = g(\text{source})g(\text{target})^{-1}$$

to

$$g(\text{source})xg(\text{target})x^{-1} = g(\text{source})xx^{-1}g(\text{target})^{-1}$$
$$= g(\text{source})g(\text{target})^{-1}$$
$$= g(\text{boundary}),$$

whereas adding such an edge in the middle of the source or target would lead to similar cancellation within $g(\text{source})$ or $g(\text{target})$.

The fact that the label is unchanged when we move the end-point of the source and target means that we do not need to keep track of all of the information specifying a surface in order to be able to assign that surface a group label. This motivates us to consider a change of notation. Rather than specify the source and the target as two paths, with an arrow between them to highlight the parallel transport, we simply combine the source with the target by moving the end-point all the way along the target (so that the new source is now the original source composed with the inverse of the original target, and the new target is an empty path). This means that we now just have one path all the way around the surface. To specify this, we only need the start of that path (the base-point) and its orientation. Rather than draw an arrow, we indicate this as a circulation, as shown in Figure 26. We note that this circulation labelling scheme is not used in the original paper on the higher lattice gauge theory model [55], but we will use it throughout this paper for its convenience.

Having set up the lattice with its branching structure, we can discuss the Hilbert space and the Hamiltonian. The lattice represents the spatial dimensions of the model (in this series of papers we will consider the model in two or three spatial dimensions), while the temporal dimension is continuous and time evolution is controlled by the Hamiltonian. As already alluded to, we label each edge
of the lattice with an element of group \( G \) and each plaquette with an element of group \( E \) [55]. Labelling every edge and plaquette gives a configuration (or colouration). These configurations then form a basis for the Hilbert space, so that a general state is a linear combination of the different labellings of the lattice.

To motivate the Hamiltonian considered by Bullivant et al. [55], we can take the same approach as when we considered a model based on lattice gauge theory. We first denote the gauge symmetries to energetic constraints, by averaging over the transforms. Then we can implement these constraints in the Hamiltonian as

\[
H = - \sum_{\text{vertices}, v} A_v - \sum_{\text{edges}, i} A_i + ...
\]

Here \( A_v \) is the average over gauge transforms at the vertex \( v \):

\[
A_v = \frac{1}{|G|} \sum_{g \in G} A_v^g.
\]  

(7)

We saw how the gauge transform \( A_v^g \) acts on the degrees of freedom in the lattice in Figure 16, and again in Equation 6. Restating Equation 6 here, the action of the vertex transform on an edge label \( g_i \) or plaquette label \( e_p \) is

\[
A_v^g : g_i \rightarrow \begin{cases} 
gg_i & \text{if } v \text{ is the start of } i 
g_i g^{-1} & \text{if } v \text{ is the end of } i 
g_i & \text{otherwise} \end{cases}
\]

\[
A_v^g : e_p \rightarrow \begin{cases} 
g \triangleright e_p & \text{if } v \text{ is the base-point of } p 
eq e_p & \text{otherwise} \end{cases}
\]

As with Kitaev’s Quantum Double model, the vertex transforms satisfy \( A_v^g A_v^h = A_v^{gh} \) for any \( g, h \in G \). We can see that the action of the transforms on the edges satisfies this relationship from the fact that this action is the same as in the lattice gauge theory case. The relationship also holds for the action on the plaquettes, because if \( v \) is the start-point of a plaquette \( p \) then we have

\[
A_v^g A_v^h : e_p = g \triangleright (h \triangleright e_p) = (gh) \triangleright e_p,
\]

where in the last step we used Equation 3 from Section 1D, while if \( v \) is not the start-point then the plaquette is unaffected. Either way \( A_v^g A_v^h \triangleright e_p = A_v^{gh} \triangleright e_p \). The relation \( A_v^g A_v^h = A_v^{gh} \) results in \( A_v \) being a projector [55], just as for the equivalent term in Kitaev’s Quantum Double model (see Equation 1), where the eigenvalue of one corresponds to configurations which are gauge-symmetric at that vertex, as we will see shortly. We can absorb vertex transforms into the corresponding vertex energy term, by which we mean that \( A_v^x A_v = A_v \) for any \( x \in G \), as we can demonstrate by expanding the vertex term and then using the algebra of the vertex transforms:

\[
A_v^x A_v = A_v \frac{1}{|G|} \sum_{g \in G} A_v^g \\
= \frac{1}{|G|} \sum_{g \in G} A_v^{xg} \\
= \frac{1}{|G|} \sum_{x' = xg \in G} A_v^{x'} \\
= A_v.
\]

This means that the states (such as the ground states) which satisfy \( A_v |\psi\rangle = |\psi\rangle \) are invariant under the individual vertex transforms, rather than just the energy term:

\[
A_v^g |\psi\rangle = A_v^g A_v |\psi\rangle = A_v |\psi\rangle = |\psi\rangle.
\]

Therefore such states are gauge-invariant, as we claimed earlier.

In a similar way to the vertex terms, the edge term \( A_i \) is the average over 2-gauge transforms at the edge \( i \) [55]:

\[
A_i = \frac{1}{|E|} \sum_{e \in E} A_i^e.
\]  

(8)

Because we earlier described the action of \( A_i^e \) in terms of parallel transport of the edge \( i \), we now need to address how this works on our lattice, where the various possible base-points and orientations of the surfaces around \( i \) may make this parallel transport more complicated. The situation shown in Figure 17, which we used to introduce the edge transform, corresponds to the special case where the base-point of both plaquettes is at the start of the same edge \( i \). If the base-point of either surface is not at the start of the edge, we have to move the base-point so that it is, using the rules introduced earlier in this section. We may also need to flip the direction of the surface. Then we perform the gauge transform \( A_i^e \) and finally move the base-point back (and undo any flips, so that we return to the canonical sources and targets). This results in an additional \( g \triangleright \) action on the factor gained by the plaquette label \( e_p \), where \( g \) depends on the edge values around the plaquette in a way that we explain shortly, so that generally \( e_p \) becomes \( e_p (g \triangleright e^{-1}) \) or \( (g \triangleright e)e_p \) [55]. While this action can look fairly complicated, the essence is that the edge transform fluctuates the plaquette labels surrounding an edge, as well as changing the edge label itself.

This is similar to how the 1-gauge transform at a vertex fluctuates the edges around the vertex (along with any plaquettes based at that vertex).

Now we want to describe the action of the edge operator resulting from this procedure algebraically. The edge operator acts on an edge \( i \) and neighbouring plaquettes.
Consider the path around one of the plaquettes, starting at the base-point \( v_0 \) of the plaquette and travelling along its boundary, aligned with its orientation. This path reaches the edge at a vertex that we call \( v_i \), as shown in the left picture of Figure 27. The path up to this point is denoted by \( g(v_0 - v_i) \) \[55\]. Now consider the path starting at the base-point of the plaquette, but travelling the opposite way around the plaquette. At some point this path will reach the other vertex on the edge, which we denote \( v_{i+1} \). This path is denoted by \( g(v_0 - v_{i+1}) \), where the overline is used to indicate that this path travels the “wrong way” around the plaquette \[55\].

The meaning of the overline notation is further illustrated in Figure 28, by looking at different paths to the same vertex. Then the action of the edge transform on each edge \( i' \) and plaquette \( p \) is \[55\]:

\[
\mathcal{A}_i^e : g_{i'} \rightarrow \begin{cases} \partial(e)g_{i'} & \text{if } i = i' \\ g_{i'} & \text{otherwise} \end{cases}
\]

\[ (9) \]

\[
\mathcal{A}_i^e : e_p \rightarrow \begin{cases} e_p(g(v_0 - v_i) \triangleright e^{-1}) & \text{if } i \text{ is on } p \text{ and aligned with } p \\ (g(v_0 - v_{i+1}) \triangleright e)e_p & \text{if } i \text{ is on } p \text{ and aligned against } p \\ e_p & \text{otherwise.} \end{cases}
\]

The paths involved in the cases where the edge is aligned or anti-aligned with the plaquette are indicated in Figure 27. Note that in either case the path terminates at the source of edge \( i \), where the source of an edge is the vertex attached to that edge and which the edge points away from (while the target of the edge is the vertex attached to the edge and which the edge points towards). This means that we can replace \( v_i \) (in the aligned case) or \( v_{i+1} \) (in the anti-aligned case) in the expression for the paths in Equation 9 with this source \( s(i) \).

The edge terms can be combined in the same way as the vertex terms \[55\]:

\[
\mathcal{A}_i^e \mathcal{A}_i^f = \mathcal{A}_i^{ef}.
\]

As with the vertex terms, this means that

\[
\mathcal{A}_i^e \mathcal{A}_i = \mathcal{A}_i.
\]

This leads to the energy term \( \mathcal{A}_i \) being a projector \[55\], with the eigenvalue of one corresponding to states that are 2-gauge-symmetric at that edge, as we will see shortly. The minus sign with which this term enters the Hamiltonian ensures that the energy term favours these gauge-symmetric states. To see that these lower energy states (such as the ground states) are invariant not just under the edge energy terms, but also under the individual 2-gauge transforms \( \mathcal{A}_i^e \), note that the lower energy states satisfy \( \mathcal{A}_i |\psi\rangle = |\psi\rangle \) by definition and so

So far we have considered energy terms that enforce the 1-gauge symmetry and 2-gauge symmetry. Now we add terms that depend on quantities that are invariant under the two types of gauge transform. By building these terms from gauge-invariant quantities, we guarantee that the new terms commute with the gauge transforms. Recall from Section 1D2 that there are gauge-invariant quantities associated to the closed cycles of the lattice. In particular, whether a cycle has a trivial group element or not is invariant under gauge transforms. We can therefore energetically punish cycles that have non-trivial 1-flux. We do this with an energy term at each plaquette, which gives one if the plaquette has trivial flux and zero if the flux is non-trivial. As explained in Section 1D2, the 1-flux for a plaquette with label \( e_p \) and path label...
$g_p$ for its boundary is given by $\partial(e_p)g_p$. The plaquette term therefore acts as $\delta(\partial(e_p)g_p, 1_G)$. An example of the plaquette energy term is shown in Figure 29.

$$B_p = \delta(\partial(e_p)g_p g_t^{-1}, 1_G)$$

FIG. 29. The plaquette term checks that the flux through a plaquette is trivial. In this case, the boundary of the plaquette is made of two edges, labelled by $g_s$ and $g_t$, and the boundary label is $g_s g_t^{-1}$. The flux through the plaquette is therefore $\partial(e_p)g_s g_t^{-1}$, where $e_p$ is the plaquette label, and the plaquette term checks whether this expression for the flux is trivial.

The plaquette terms enter the Hamiltonian with a minus sign, so that

$$H = - \sum_{vertices, v} A_v - \sum_{edges, i} A_i - \sum_{plaquettes, p} B_p + ...$$

Including these terms with a minus sign ensures that the lowest energy states have trivial flux on the plaquettes. This trivial flux condition is called fake-flatness [55], due to its analogy with the flatness condition of ordinary lattice gauge theory (which enforces that all cycles are trivial). We call plaquettes that satisfy the fake-flatness condition, such as the plaquette shown in Figure 30, fake-flat.

Finally, we consider the gauge-invariant quantity associated to a blob $b$, $\hat{e}(b)$, the identity element or $1_G$, as shown in Figure 31. We denote the blob term associated to a blob $b$ by $B_b$. Note that in this series of papers we will consider the cases where the lattice is in two or three spatial dimensions (so the overall model is 2+1d or 3+1d), and in the former case the lattice has no blobs and so we will have no blob energy terms.

The blob term also enters the Hamiltonian with a minus sign, so that the full Hamiltonian is given by [55]

$$H = - \sum_{vertices, v} A_v - \sum_{edges, i} A_i - \sum_{plaquettes, p} B_p - \sum_{blobs, b} B_b$$

FIG. 31. The blob energy term $B_b$ checks whether the total surface label of the blob $b$, $\hat{e}(b)$, is the identity element or not. This surface label must be determined by using the rules for combining surface elements from Sections 1D and 1E to combine the plaquettes on the boundary of the blob. For example, when $\triangleright$ is trivial the surface label is a product of the plaquette labels (with inverses if the orientation of the plaquette needs to be reversed to match the overall surface).

We note that when we take $E$ to be the trivial group (so that we consider lattice gauge theory), the terms $A_i$ and $B_b$ become trivial. In addition, the vertex transforms and plaquette terms become equivalent to the corresponding terms from Kitaev’s Quantum Double model. Therefore we recover that model in this case, indicating that the higher lattice gauge theory model is a generalization of the Quantum Double model [55].

F. Some Special Cases and Consistency

We now wish to consider the conceptual issues that we mentioned in the previous section. For the most general crossed modules, the Hamiltonian model has certain
inconsistencies. As an example of what we mean, consider the surface holonomy of a plaquette, \( e_p \). We can move the base-point of the plaquette around the plaquette, so consider moving it all the way around the plaquette. This induces a change to the surface label of the plaquette, given by \( e_p \xrightarrow{g_p^{-1} \triangleright} e_p \) [55], where \( g_p \) is the path label of the boundary of the plaquette, as shown in Figure 32. The base-point is back to the same position, and the surface appears to be the same, yet the label may have changed. The label does stay constant if the configuration is fake-flat, meaning that it satisfies the plaquette energy term defined in Section 1E). In that case, the boundary label satisfies \( g_p^{-1} = \partial(e_p) \) and so \( g_p^{-1} \triangleright e_p = \partial(e_p) \triangleright e_p = e_p e_p^{-1} = e_p \) [55], where we used the Peiffer condition Equation 5 in the second step. However if fake-flatness is not satisfied, then we cannot guarantee that the plaquette label is unchanged.

This is not the only issue arising from violating fake-flatness: as we describe in Section A1 in the Appendix, the edge energy term also appears to become inconsistent with changes to the branching structure of the lattice. One approach for dealing with this problem is to enforce fake-flatness on the level of the Hilbert space, as a hard constraint rather than an energy term. This is the case most closely considered by the authors of the original paper on this model [55]. However, another possibility is to take \( \triangleright \) trivial, so that the base-point of the plaquette loses any meaning, but allow fake-flatness violations. In this case the model loses some of its complexity, due to the 1-gauge field having no way to act on the 2-gauge field. If we use this condition, then all of the energy terms commute naturally, with no need to restrict the Hilbert space. Some additional consequences of taking \( \triangleright \) trivial are that \( E \) must be Abelian and that \( \partial \) maps to the centre of \( G \). The first condition, that \( E \) is Abelian, comes from the second Peiffer condition (Equation 5 in Section 1D), because \( \partial(e) \triangleright f = e f e^{-1} \iff f = e f e^{-1} \) so that any elements of \( E \) commute with each-other. The second condition, that \( \partial \) maps to the centre of \( G \), comes from the first Peiffer condition (Equation 4 in Section 1D), as \( \partial(g \triangleright e) = g \partial(e) g^{-1} \iff \partial(e) = g \partial(e) g^{-1} \) so that \( \partial(e) \) commutes with all elements of \( G \). In this paper, both of these special cases (\( \triangleright \) trivial and restricting to fake-flat configurations) are considered. We also consider the case where \( E \) is Abelian and \( \partial \) maps to the centre of \( G \), but we do not enforce fake-flatness on the level of the Hilbert space or require \( \triangleright \) to be trivial. This is the most interesting case that we consider, but also the most difficult to deal with due to the non-commutativity of energy terms. This case is still more tractable than the fully general case (i.e., the case where we do not make any restrictions on either the crossed module or the Hilbert space), not just because the groups are simpler algebraically, but also because the inconsistency under changes to the base-point that we considered above is not as generic. If a plaquette \( p \) with label \( e_p \) violates fake-flatness because the boundary label \( g_p \) of the plaquette differs from \( \partial(e_p)^{-1} \) only by an element \( \partial(e) \in \partial(E) \), then moving the base-point of the plaquette around \( p \) results in the plaquette label \( e_p \)

\[
(\partial^{-1})^{-1} e_p = (\partial^{-1}) e_p = (e^{-1} e_p) e_p (e^{-1} e_p)^{-1}
\]

which is just \( e_p \) because \( E \) is Abelian. This case, where the boundary label differs from \( \partial(e_p)^{-1} \) by an element in \( \partial(E) \), is significant because it occurs when the fake-flatness violation is caused by a change to the plaquette label \( e_p \), rather than changes to the edge labels. Such flatness-violating changes to the plaquette labels occur for a whole class of ribbon operators (the confined blob ribbon operators we describe in Section IV). This means that the inconsistencies considered previously are less prevalent in the case where \( E \) is Abelian. We will see other simplifications that occur due to \( E \) being Abelian and \( \partial \) mapping to the centre of \( G \) throughout the paper. Because we will refer to these restrictions, along with the other cases that we have considered in this section, many times in the following text, we summarize all of them in Table I.

![FIG. 32. Moving the base-point of a plaquette all the way around the plaquette transforms the surface label from \( e_p \) to \( g_p = g_s g_t^{-1} \).](image)

| Case | \( E \) | \( \partial(E) \) | Full Hilbert Space |
|------|--------|----------------|------------------|
| 1    | Abelian| Trivial        | centre(\( G \))  |
| 2    | Abelian| General        | centre(\( G \))  |
| 3    | General| General        | No               |

**TABLE I.** A summary of the special cases of the model

### G. Braiding Relations in 3+1d

One of the important features that we are interested in is the braiding of the various excitations that we find in the higher lattice gauge theory model. While we anticipate that most readers will have at least some familiarity with the concept of braiding in 2+1d, it may be useful to give an overview of braiding in 3+1d, particularly where loop-like excitations are involved. In 2+1d, topological phases support excitations with exchange statistics that generalize the familiar Fermi and Bose statistics [18–20], and which are described by the (coloured) braid group. On the other hand, in 3+1d the point-like particles can only have Fermi or Bose statistics [22, 26, 27]. However,
the presence of loop-like excitations means that we can still have interesting braiding statistics. The motion of loops can be described by the (coloured) loop braid group \([60–62]\) (considered under different names and contexts in papers such as Refs. \([71–73]\)). The braid and loop braid groups are both examples of motion groups \([74, 75]\), which describe the motions of arbitrary objects, up to homotopy \([61]\) (formally the objects should return to the same positions, perhaps swapping positions). The loop braid group is generated by a few simple motions. Firstly consider braiding that involves only two loop-like excitations, and for simplicity suppose that those two excitations are stacked vertically, as shown in Figure 33. Then if we want to move the lower loop up past the upper loop, there are several ways to do this. The first way, shown in Figure 34, is to simply move the lower loop around and past the upper one, so that neither loop passes through the other \([60, 62]\). We refer to this motion as a permutation move, because such moves generate the permutation group (symmetric group) \([61]\) that would describe the motions of point particles in 3+1d. The second way, shown in the left side of Figure 35, is to move the lower loop through the upper loop \([60, 62]\). We call this a braiding move and say that the lower loop has braided through the upper loop. The third way, shown in the right side of Figure 35, is to pull the lower loop over the upper loop, which we can also think of as the upper loop passing through the lower loop (and so is the inverse of the previous motion). Another difference from point particles is that loops have an orientation, and so we must also allow a move that flips this orientation, as shown in Figure 36. Then any motion of the two loops can be performed by a series of such moves (we say that these generate the loop braid group for the two excitations) \([60, 62]\). More generally, we can have any number of loops, and the different motions of this set of loops can be performed using these pairwise moves. Generally we are interested in comparing two motions that result in the same final position of all of the excitations. For example, we could compare the result of the permutation move in Figure 34 to the braiding move in Figure 35, or we could compare a motion that returns all particles to their initial positions to the trivial motion. Making a comparison between states where the excitations have the same final position is useful because it separates the topological content of the model from the geometric details. As we explain in Section V, we will be considering processes that involve the production of the loop-like excitations from the ground state, rather than just the movement of existing excitations, but the same principles discussed in this section hold regardless.

In addition to loop-like particles, we will find point-like particles in the lattice model. While braiding between two point-like particles in 3+1d is bosonic or fermionic \([26, 27]\) as described earlier (and is exclusively bosonic in this model), the point-like excitations may braid non-trivially with the loop-like excitations of the model. In order to move a point-like particle past a loop excitation, we can either move the point-like excitation around the loop-like excitation (analogous to Figure 34 for two loops, if we replace the initially lower loop \(A\) with a point-like excitation), or we can move the point-like excitation through the loop-like one (analogous to the left-side of Figure 35, if we replace the lower loop \(A\) with a point-like excitation).

II. PROPERTIES FROM GAUGE THEORY PICTURE

Before we discuss the excitations that we find in the model in great mathematical detail, it will be instructive to give a more qualitative description of the excitations that we expect, using ideas from higher gauge theory. As a starting point, we shall give a brief description of the picture in ordinary gauge theory. In ordinary gauge theory, we find objects which we call “electric charges” and “magnetic fluxes”, which we will describe shortly. A clear exposition on these objects and their properties, in the 2+1d case, is given by Preskill’s lecture notes on topological quantum computation \([76, \text{Chapter 9}]\) and an early description of non-Abelian magnetic fluxes is given in Ref. \([77]\) (see also Refs. \([35]\) and \([36]\)). Here we will instead examine the 3+1d case, as described in (for example) Ref. \([78]\).

Electric charges are point particles labelled by irreducible representations of the group \(G\). To see how these charges arise from our model, we look at our vertex gauge transforms, as defined in Figure 2. The gauge transforms \(A^\alpha_\nu\) at a particular vertex form a group, with the product \(A^\alpha_\nu A^\beta_\mu = A^{\gamma\delta}_\nu\), which is isomorphic to \(G\). Our Hilbert
A citation which transforms under the irrep transforms under the gauge transforms. We denote an ex-

irrep of we expect to find excitations that carry some non-trivial vertex, then that vertex will be excited. This means that transforms as some other, non-trivial, irrep at a particular unexcited vertex terms. On the other hand, if a state states described by the trivial irrep at each vertex have described by the trivial irrep. From this we see that the irreps are therefore exactly those in the same position but flipped over [62].

space then splits into subspaces that transform as irreducible representations (irreps) of \( G \) under the action of the gauge transforms at every vertex. These irreps are related to the energy of the states. The states that are unexcited (lower energy) at each vertex are those states for which \( A_v = 1 \) at each vertex \( v \) (a vertex is excited in a state if \( A_v = 0 \) for that state). Such states are gauge invariant, meaning that they are invariant under the individual vertex transforms, as discussed in Section 1E. These gauge-invariant states are therefore exactly those described by the trivial irrep. From this we see that the states described by the trivial irrep at each vertex have unexcited vertex terms. On the other hand, if a state transforms as some other, non-trivial, irrep at a particular vertex, then that vertex will be excited. This means that we expect to find excitations that carry some non-trivial irrep of \( G \), with this irrep describing how the excitations transform under the gauge transforms. We denote an excitation which transforms under the irrep \( R \) by \((R, a, b)\), where \( a, b \) are the matrix indices of the representation and describe an internal space. We define such an excitation by the action of the vertex operator on it:

\[
A_v^g \cdot (R, a, b) = \sum_c [D_R(g^{-1})]_{ac}(R, c, b),
\]

where \( D_R(g) \) is the matrix representation of element \( g \). The label \( g^{-1} \), rather than \( g \), is used to ensure that the action of the \( A_v^g \) satisfies the composition rule \( A_v^g A_v^h = A_v^{gh} \). We could equally have defined the action of \( A_v^g \) to be right multiplication by \( D_R(g) \) instead, which would also satisfy the composition law. This transformation under the vertex transforms can also be used to tell us something about the transport properties of the excitation. In Section 1D1 we explained that the vertex transforms are equivalent to parallel transport. Therefore Equation 13 tells us how we expect these excitations to behave under parallel transport over an edge labelled by \( g^{-1} \). Looking at Equation 13, we see that there is mixing between states defined by different matrix indices, while the irrep is unchanged. This suggests that the electric charges carry some conserved charge labelled by the representation, while the matrix indices describe some non-conserved details.

In addition to the electric charges, we anticipate magnetic fluxes. Recall from Section 1C2 that fluxes are associated with closed paths that have non-trivial labels. In that section, we drew an analogy to how a magnetic field leads to a non-trivial Aharonov-Bohm effect for taking a charge around a closed loop. In a 2+1d model, flux can be created by a point particle, which we can think of as being similar to a magnetic field penetrating our surface at a point. If this point particle generates a flux, then this flux should be measured by a closed loop that encloses the particle. Therefore we can describe this particle with the closed loop that measures the flux. However, a point particle cannot be sensibly described by a closed loop in a 3+1d topological theory. This is because any closed loop around a point particle can be smoothly deformed away from that particle and contracted to nothing, i.e. to a path with a trivial label. Therefore the label of the path cannot be a topological quantity if the excitation generating the flux is a point particle. Instead the magnetic flux particles should be closed loops (flux tubes). The flux generated by such a tube can be measured by a closed path that links with the flux tube, as shown in Figure 37. Then there is no way to smoothly contract the measurement path without it intersecting with the flux tube, and so the label of the path can be a topological quantity.

We can label a flux tube by the group element of the closed path that measures the flux. This label will describe how a charge would evolve as it travels along that closed path. However, the value we assign to a path will depend on the start-point of that path. To see this, we consider taking a particular closed path that links with that excitation and then changing its start-point, as shown in Figure 38. In order to traverse the new path, we must first travel the path joining the start-points, then the original closed path, and then back along the path joining the start-points. This means that the label of the new path is given by conjugating the group element of
FIG. 37. The magnetic excitation (thick red torus) is measured by a non-trivial closed loop linking with it, such as the thin green torus. This closed loop begins at some start-point (shown as a yellow sphere) and the value of the flux that we measure depends on which start-point we choose, but the conjugacy class of the flux does not.

the old path by the element associated to the path between the start-points. We therefore see that this new path has a label that is different from the label of the original path, but which lies in the same conjugacy class. This means that the conjugacy class describes the excitation in a robust way, but to obtain a full description of the flux tube we must also specify its element within that conjugacy class and the start-point from which we measure that value.

FIG. 38. With the start-point (s.p) at the grey (higher) sphere, the flux label is \( h \), however with the s.p at the yellow (lower) sphere the flux is \( g(t)hg(t)^{-1} \).

In addition to giving us the excitations that we would expect for a lattice gauge theory model, this picture will give us the braiding relations. We already established how an electric excitation should transform as it moves through space. We can now consider taking an electric excitation, labelled by an irrep \( R \), on a closed path that links with a flux tube labelled by \( h \), as shown in Figure 39. If this closed path is the one used to define the flux, then the path label is given by the flux label of the flux tube, \( h \). Therefore the excitation \((R, a, b)\) should become \( \sum_c [D^R(h)]_{ac}(R, c, b) \) after the motion. On the other hand, if the path has a different start-point to the defining path of the flux, we should replace \( h \) with some other element in the conjugacy class of \( h \), in order to describe the transport of the charge and the start-point as well as around the defining loop of the flux. For instance, if we were to first take the charge on some path \( t \) to get from a new start-point to that of the flux, then take the charge around the defining path of the flux before finally returning it along \( t^{-1} \) to its initial position, then we would have

\[
(R, a, b) \rightarrow \sum_c \sum_d \sum_e [D^R(g(t))]_{ac}[D^R(h)]_{cd} \times [D^R(g(t)^{-1})]_{de}(R, e, b)
\]

\[
= \sum_e [D^R(g(t)hg(t)^{-1})]_{ae}(R, e, b)
\]

from which we see that the charge experiences the same transform as if it travelled around a flux of \( g(t)hg(t)^{-1} \), as expected from our rules for changing the start-points of fluxes.

FIG. 39. Schematic view of braiding a charge through a loop. The red line tracks the motion of the charge.

We can also obtain the braiding relations of the magnetic fluxes using this picture. Consider the case where we have two flux tubes, which we define with the same start-point. We want to keep track of the measurement paths and the flux labels, as we move the fluxes around. We consider exchanging the two flux loops by pushing one through the other, as shown in Figure 40. When we move a flux loop, the measurement path (which is associated with the flux label) moves with it so that the measurement path and flux tube remain linked (we can imagine the flux tube dragging the measurement path with it). For example, in the top-right part of Figure 40, which shows the situation after we perform the braiding move, we see that the measurement path for the blue flux tube (which is originally lower than the red flux tube) is pulled through
the (now lower) red flux tube. This new deformed path carries the original flux label \((h\) in Figure 40) and so this flux label is now associated to a process where we pull a charge through the lower loop, then around the upper one and then back through the lower loop, rather than a process where we simply braid the charge around the upper loop. We want to define our fluxes with respect to our original measurement paths, in order to find the labels associated to the original measurement processes and so to find the change to the system under braiding. That is, we want to find the labels associated to the original measurement paths \((\alpha\) and \(\beta\) in Figure 40). To do this, we need to write the original paths in terms of the new deformed ones, for which we know the path labels. This will allow us to obtain the labels of the original paths and so tell us the result of braiding our fluxes.

Looking at Figure 40, we see that \(\beta\) is the path originally associated to the upper (red) flux, with label \(g\). When we deform space to push the lower (blue) flux tube through the upper (red) flux tube, this path \(\beta\) is deformed to \(\beta'\). This means that the label of this new path, \(\beta'\), is equal to the original label of path \(\beta\), i.e. \(g\). However this path is equivalent to (i.e. can be smoothly deformed into) the original path \(\alpha\) around the old lower flux. So we have \(\beta' = \alpha\) and so the new label of the path \(\alpha\) is \(g\) (which is now associated with the red flux tube). On the other hand consider the path \(\alpha\) originally associated with the blue (initially lower) flux. When we move the fluxes, this path is deformed into \(\alpha'\) and it keeps its label of \(h\), as indicated in the upper right diagram. We want to write \(\alpha'\) in terms of our old paths \(\alpha\) and \(\beta\). To do this we note that \(\alpha'\) can be smoothly deformed into another path \(\alpha''\),
which is equal to the path $\alpha \beta \alpha^{-1}$ obtained by traversing $\alpha$ then $\beta$ and then $\alpha$ in reverse, as shown in the bottom-right figure. Therefore we have that $\alpha' = \alpha \beta \alpha^{-1}$. This means that $\beta = \alpha^{-1} \alpha' \alpha$. Using the fact that $\alpha$ now has the label $g$ and $\alpha'$ has the label $h$, we see that $\beta$ has the label $g^{-1} hg$. We can write this braiding relation in the following way. We start with $(h, g)$, where the first symbol in brackets is the label given to path $\alpha$ and the second is the one given to path $\beta$. Then under braiding we have $(h, g) \rightarrow (g, g^{-1} hg)$, where on both sides the first symbol refers to the value of path $\alpha$ and the second to the value of path $\beta$, rather than giving the label of a particular one of the excitations (blue or red). If we instead keep track of the labels of each tube, we see that $h \to g^{-1} hg$ for the blue tube and $g \to g$ for the red tube. Therefore we see that the label of one of our flux tubes is conjugated by the label of the other one under our braiding.

We have so far described the results we expect from ordinary lattice gauge theory, but not higher lattice gauge theory. However we can use very similar ideas to explore higher lattice gauge theory. Our vertex terms still have the same algebra as in ordinary gauge theory. Namely, we have a group isomorphic to $G$ at each vertex. Therefore we expect electric charges labelled by irreps of $G$, just as with lattice gauge theory. Again, under parallel transport the charges will transform according to this irrep. This suggests that our electric excitations will be largely unchanged when compared to those from ordinary lattice gauge theory. Similarly, we expect magnetic flux tubes, just as in lattice gauge theory. However there is some subtlety in considering the braiding between these two types of excitation. This is because the lattice does not satisfy flatness in the ground state, but instead fake-flatness. This means that deforming a path over an unexcited region causes the path label to pick up a factor of the form $\partial(e)$, rather than remain unchanged. As we discussed earlier, moving an electric excitation through space, such as when we braid it around a magnetic flux tube, causes it to transform according to the label of the path traversed. This suggests that the result of braiding an electric charge around such a tube depends on the precise path chosen for the braiding, not just its homotopy class, implying that the braiding relation is not topological. As we shall discuss further in Section IV A, the resolution to this is that any electric excitation that is sensitive to such factors of $\partial(e)$ must be confined (i.e. cost energy to separate from its antiparticle), and so is not topological. In addition, the fact that we have fake-flatness rather than flatness indicates that a closed path may have a non-trivial label $\partial(e)$ even in the ground state (and we would have to measure the surface enclosed by that path to check that fake-flatness is satisfied). This implies that fluxes with label in $\partial(E)$ cannot be distinguished from trivial fluxes just by measuring the closed path. Furthermore, deforming the measurement path for a magnetic flux tube will change the label measured by an element of $\partial(E)$. Therefore, when we talk about the flux of a magnetic excitation, we should only define it up to elements in $\partial(E)$. This leads to magnetic excitations with label in $\partial(E)$ becoming topologically trivial, in a sense that we explain in Section IV B.

In addition to the vertex gauge transforms we also have the 2-gauge (or edge) transforms, which again have a group structure. These operators form a group isomorphic to $E$ for each edge: $A_i^e \cdot A_j^f = A_k^e$. Therefore we expect to find edge excitations that are labelled by irreps of the group $E$. Recall that the 2-gauge transform is equivalent to parallel transport of an extended object (a line object) over a surface. Therefore an object which transforms as a particular irrep under the 2-gauge transform should also transform as that irrep under parallel transport over a surface. Because of the fact that this transport is over a surface rather than a path, we expect our “2-gauge charges” to be extended objects. In fact, we will find that these 2-gauge charges are loop-like objects. Then when we transport a loop, labelled by an irrep $\mu$ and matrix indices $a$ and $b$, over a surface labelled by $c$, we should obtain the transformation $(\mu, a, b) \rightarrow \sum_i \sum_j [D^a_i]_ac$ $(\mu, c, b)$. However there is some subtlety to this. Recall that we established in the introduction that every surface element has a base-point, which we can change by whiskering. If we change the base-point we change the label of that surface from $c$ to some $g \mapsto e$ for some $g \in G$. Then how do we know if we should have $[D^a_i]$ or $[D^a_i(g \mapsto e)]$ in our transformation when we transport the loop? That is, where should we take the base-point of our surfaces? The answer is that, just as our flux excitations are defined with respect to some start-point, so must our 2-gauge charges. That is, when we define our 2-gauge charges, as well as giving the irrep and its matrix indices, we give a special point on the lattice, which we call the start-point. When we move the loop over a surface, we always take the label of that surface with respect to the start-point of our loop excitation. This start-point is particularly important for these loop excitations, because the action of the group $G$ on the start-point, which changes the surface label by some $g \mapsto e$ map, enables $G$ to affect the loop excitations. This action can even change the irrep labelling a 2-gauge charge loop, suggesting that the irrep is not a conserved quantity. Instead there is some mixing within certain classes of irreps, which we term $\mapsto$-Rep classes of irreps of $E$, with the irreps in a particular class being related by the action of $\mapsto$. When $E$ is Abelian we define the classes with the equivalence relation

$$\mu_1 \sim \mu_2 \iff \mu_1(e) = \mu_2(g \mapsto e) \text{ for some } g \in G$$

and when $E$ is non-Abelian we must generalize this to account for irreps related by conjugation. We therefore see that $\mapsto$ plays a significant role in determining how the excitations behave.

Just as we have magnetic fluxes that are associated with non-trivial loops, we established in the introduction that we have “2-fluxes” associated to non-trivial closed surfaces. We expect excitations corresponding to the 2-flux
of a sphere (as the blob terms in our Hamiltonian correspond to spheres on the surface of each blob) to be point excitations, because we can shrink a sphere to enclose just a single point. Just as the fluxes are labelled by elements of \( G \), the 2-fluxes (or blob excitations) should be labelled by elements of \( E \). In the same way that we must define a closed path to specify a flux, we must specify a closed surface to specify our 2-flux. Recalling that every surface must be defined with a base-point, we see that we must choose a base-point for our 2-flux excitation, which we call the start-point of the excitation. From here on we use start-point to refer to privileged vertices related to the excitations, while base-point is used to refer to the base-point of a surface. It happens that for the 2-flux excitations the base-point of our measurement surface is the start-point of the excitation. Moving the base-point of a surface along a path labelled by \( g \) changes the surface label from \( e \) to \( g^{-1} \triangleright e \), so similarly changing the start-point of our 2-flux changes its label by this \( g^{-1} \triangleright \) action. We also expect the 2-flux excitation to transform in this way as it moves along a path. This tells us that the 2-flux label is not conserved, but rather each group element belongs in a class of elements related by the \( \triangleright \) action. The equivalence relation defining such a class is that two elements \( e, f \in E \) satisfy

\[
e \sim f \iff \exists g \in G \text{ such that } e = g \triangleright f.
\]  

These “\( \triangleright \)-classes” are then the quantities that are conserved under motion.

As this picture gives us the transport properties of these excitations, we can obtain the braiding relations as well. We know how our \( E \)-valued loops, the 2-gauge charges, transform under transport over a surface. Therefore we know how they will transform when we pull them over one of our 2-fluxes, as a 2-flux corresponds to a particular closed surface with a particular label. For a loop excitation \( \mu, a, b \) and a 2-flux \( e \), defined with the same start-point, the loop excitation becomes \( \sum_{\alpha}[D^\mu(e^{-1})]_{\alpha}(\mu, c, b) \) when it is pulled over the 2-flux. We can also work out how the 2-fluxes braid with ordinary fluxes. We know how the 2-fluxes transform when they move along edges. The 2-flux label changes from \( e \) to \( g(t) \triangleright e \) from moving along a path \( t \). Therefore when moving a 2-flux labelled by \( e \) around an ordinary magnetic flux labelled by \( h \), the 2-flux becomes \( h \triangleright e \) (or \( h^{-1} \triangleright e \), depending on the orientation of the magnetic flux).

This picture therefore tidily describes several types of simple excitation that we expect to find. However, as explained in Ref. [60], there are more complicated excitations as well. We may expect loop particles that generate both a 2-flux and a 1-flux. The non-trivial magnetic 1-flux is associated to a closed path that links with the excitation, while the 2-flux corresponds to a spherical surface enclosing that excitation. In Ref. [60], the braiding relation between two such loops is established, using geometric arguments. The authors look at the situation where they braid two such excitations labelled by \( (g,e) \) and \( (h,f) \), where the first label of each pair gives the magnetic flux and the second the 2-flux. When the excitation labelled by \( (g,e) \) is pushed through the one labelled by \( (h,f) \), the excitations should transform under braiding to become \( (h^{-1}gh,h^{-1} \triangleright e) \) and \( (h,ef[h^{-1} \triangleright e^{-1}]) \). As will be explained in Section V, we do indeed find such excitations with these braiding statistics.

Having discussed the excitations we will find in terms of electric charges, magnetic fluxes and their 2-gauge equivalents, we can now look in more detail at how these arise in our Hamiltonian model. We will describe the operators that produce and move these excitations and will also see features that are not apparent from this brief discussion, including which energy terms the fluxes and charges excite and which of the excitations are confined (cost energy to separate from their antiparticle).

### III. Excitations

The aim of this study is to find the excitations of the higher lattice gauge theory model and their properties. In the previous section, we gave arguments about these properties from gauge theory. However we need a fuller and more rigorous description of the excitations. A significant feature of this model is that we can explicitly find the operators to produce the excitations in various broad cases. Here we will explicitly construct these operators for the higher lattice gauge theory model for the first time, and use the operators to study the properties of the excitations directly. We find four types of excitation in our model, with a rough correspondence to the four energy terms of our Hamiltonian. In 3+1d, two of these types of excitation are point-like and two types are loop-like.

#### A. Electric Excitations

The first type of excitation we construct is called the electric excitation and is primarily associated to the vertex terms of our Hamiltonian. These electric excitations therefore correspond to the “electric charges” that we described at the start of Section II. In order to create these electric excitations, we measure the group element associated to some path on our lattice and apply weights depending on the result. In order to measure a path element, we take the product of edge elements along the path, with inverses if the orientation of the edge is against the orientation of the path. For example, considering the path shown in Figure 41, the edges labelled by \( g_2 \) and \( g_3 \) are aligned against the path, and so those group elements appear with an inverse in the path element. In order to measure each possible value of the path label, we apply an operator of the form

\[
S^\alpha(t) = \sum_{g \in G} \alpha_g \delta(\hat{g}(t), g)
\]  

(16)
where \( \dot{g}(t) \) is the path element for path \( t \) and \( \alpha_g \) is a coefficient (or weight) for the element \( g \). This operator can excite the two vertex terms at the ends of the path \( t \). The operator in Equation 16, which measures the path element and gives a weight to each possibility, is our first example of a ribbon operator [13]. These ribbon operators produce point-like excitations in pairs and move them around (to the ends of the ribbon), and so are the central objects for manipulating the point-like excitations of commuting projector models. We call the start of path \( t \) the start-point of the operator, and it can be thought of as the position where the pair of excitations is created before the excitations are moved.

\[
g(t) = g_1 g_2^{-1} g_3^{-1} g_4
\]

FIG. 41. An electric ribbon operator measures the value of a path and assigns a weight to each possibility, creating excitations at the two ends of the path. In order to find the group element associated to the path, we must first find the contribution of each edge to the path. In this example, the edges along the path are shown in black. Some of the edges are anti-aligned with the path and so we must invert the elements associated to these edges to find their contribution to the path. This is represented by the grey dotted lines, which are labelled with the contribution of each edge to the path.

At this point we would like to note that these electric excitations are equivalent to the electric excitations found in Kitaev’s Quantum Double model [13], even up to the precise form of the ribbon operator that creates them. As we go on we will find that several features of the Quantum Double model (which is based on lattice gauge theory) carry over to the higher lattice gauge theory model. However we will also see important distinctions between the two models. For instance, as we describe in Section IV A (and prove in Ref. 63), in the section called “Energy of Ribbon and Membrane Operators in 2+1d”, although the proof holds in both 2+1d and 3+1d, some of our electric excitations are confined, meaning that the two excitations at the end of the electric ribbon operator cost energy to separate.

Any set of weights \( \alpha_g \) that we choose in Equation 16 will give a valid ribbon operator. Varying the weights therefore takes us through a space of these electric ribbon operators. A particularly useful basis for this space has the weights described using representations of \( G \), as we anticipated in Section II. Each basis element is labelled by an irreducible representation (irrep) of the group \( G \), along with two matrix indices. Then for the irrep \( R \) and the indices \( a \) and \( b \), the corresponding basis ribbon operator is given by

\[
\hat{S}^{R,a,b}(t) = \sum_{g \in G} [D^R(g)]_{ab} \delta(\dot{g}(t), g),
\]

where \( D^R(g) \) is the matrix representation of the element \( g \) in the irrep \( R \). The operator labelled by the identity irrep is then the identity operator

\[
1 = \sum_{g \in G} \delta(g, \dot{g}(t))
\]

and so does not produce any excitations. Any other basis operator (i.e. an operator labelled by a non-trivial irrep) does produce excitations at the two ends of the ribbon.

In addition to determining which operators excite the vertices, this basis is a good choice for examining the properties of the electric excitations. The excitations of the model carry a set of conserved charges, called topological charge. The excitations are then split into sectors that have different values of topological charge. In 3+1d, there are multiple types of topological charge, corresponding to different measurement surfaces, as we explain in Section VI (and in more detail in Ref. 64). In this work we will consider the charge measured by a sphere, which captures the point-like character of an excitation or set of excitations, and the charge measured by a torus, which captures the loop-like character. The electric excitations are point-like and so we are interested in the point-like charge, or the charge measured by a sphere. As we show in Ref. 64 (in the section entitled “The Point-Like Charge of Simple Excitations”), the point-like topological charge for (non-confined) pure electric excitations is labelled by the irreps of the group \( G/\partial(E) \) (the quotient removes the confined excitations) and the basis operators given above transport definite values of topological charge. That is, in the operator above \( R \) labels a conserved charge, while the matrix indices \( a \) and \( b \) describe some internal space to the sector (as we expect from our discussion of gauge theory in Section II). In particular, when \( R \) is the trivial irrep the ribbon operator transports the vacuum charge, as we require from the fact that the ribbon operator is just the identity.

B. Blob Excitations

The next point-like excitations that we find are called blob excitations, because they primarily correspond to violations of the blob terms of the Hamiltonian. These excitations are therefore associated to the non-trivial closed surfaces we discussed in Section I D 2 of the introduction. That is, the blob excitations correspond to non-trivial “2-fluxes” on a sphere. The Hamiltonian includes a blob term for each blob (3-cell), which enforces that a particular product of the surface elements around the blob gives the identity element, \( 1_E \). To violate this condition
We can therefore consider multiplying one of the plaquette labels on a blob by an element \( e \) of \( E \). However, each plaquette is shared by two blobs, one on either side of the plaquette. Changing a plaquette’s label will therefore excite both of the adjacent blobs. We can try to correct this by changing another plaquette on this second blob. However that plaquette will in turn be connected to a third blob, which will become excited, as illustrated in Figure 42. This means that we simply move the second excitation from the second to the third blob. We therefore see that we produce these blob excitations in pairs, just as with the electric excitations. The series of plaquettes that we had to change to produce the excitations will form a string that passes through the centres of the blobs (3-cells). The operator that changes the plaquette labels is therefore another of our ribbon operators.

![Image of plaquette excitations](image)

**FIG. 42.** We consider a series of blobs on the ground state (left-most image). In the ground state, all of the blob terms are satisfied, which we represent here by colouring the blobs blue. Changing the label of the plaquette between blobs 1 and 2 excites both adjacent blobs, as can be seen in the middle image (we represent excited blobs by colouring them orange). Multiplying another plaquette label on blob 2 to try to correct it just moves the right-hand excitation from blob 2 to blob 3 (right-most image). In each step, the plaquettes whose labels we changed are shown in red and their orientations are indicated by an arrow.

We have so far been somewhat vague about the precise action of our blob ribbon operator. This is because the action depends on which special case (as described in Section II F) that we take. In the simplest case, where \( \triangleright \) is trivial (Case 1 in Table I), the action is fairly simple. We choose an element of \( E \) to label the operator. We also choose a path on the dual lattice, which passes between the centres of blobs just as a path on the direct lattice passes between vertices. This path cuts through the plaquettes that separate the blobs (analogous to direct paths passing along edges), such as the red plaquettes in the example in Figure 42. The choice of element \( e \in E \) and path \( r \) gives us a blob ribbon operator \( B^r(r) \). The action of this operator is to just multiply all of the plaquettes pierced by this dual path by \( e \) or \( e^{-1} \), depending on the orientation of these plaquettes relative to the ribbon (where the orientation of the plaquette is obtained from its circulation using the right-hand rule). This results in the two blobs at the end of the path being excited, as we discussed for the example in Figure 42. As we explain in Section IV A, some of the blob ribbon operators (those labelled by an element \( e \in E \) for which \( \partial(e) \) is not \( 1_G \)) also excite the plaquettes pierced by the ribbon operator, resulting in the corresponding blob excitations being confined.

We must modify the action of the blob ribbon operators slightly if \( \triangleright \) is non-trivial, as in Cases 2 and 3 of Table I. When \( \triangleright \) is non-trivial, we must keep track of the base-points of the plaquettes that we want to change. We first move all of their base-points to a common location at the start of our operator (for example, the base-point of the first plaquette that we want to change), which we call the start-point. Then we multiply each plaquette by \( e \) or \( e^{-1} \) before moving their base-point back to its original location. Recall from the introduction that moving the base-point of a plaquette along path \( t \) changes its label from \( e_p \) to \( g(t)^{-1} \triangleright e_p \) (see Figure 11 for a reminder). The total change to \( e_p \) is therefore

\[
e_p \rightarrow g(t)^{-1} \triangleright e_p \quad \text{move base-point}
\]

\[
\rightarrow [g(t)^{-1} \triangleright e_p] e^{-1}( \text{postmultiply by } e^{-1})
\]

\[
\rightarrow g(t) \triangleright ([g(t)^{-1} \triangleright e_p] e^{-1}) \quad \text{move base-point back}
\]

\[
e_p g(t) \triangleright e^{-1}
\]

where \( t \) is the path from the base-point \( v_0(p) \) of plaquette \( p \) to the start-point of our operator. We write this as

\[
e_p \rightarrow e_p [g(s.p - v_0(p))^{-1} \triangleright e^{-1}]
\]

where \( g(s.p - v_0(p)) \) is the path from the start-point of our operator to the base-point of \( p \). Note that, depending on the orientation of our operator, instead of postmultiplying by \( g(s.p - v_0(p))^{-1} \triangleright e^{-1} \) we may need to pre-multiply by \( g(s.p - v_0(p))^{-1} \triangleright e \). Because of the key part that the start-point plays in this more complicated action, the start-point vertex may be excited by the ribbon operator. This results from the fact that the vertex transform at the start-point does not commute with the ribbon operator when \( \triangleright \) is non-trivial, because the vertex transform affects the path element \( g(s.p - v_0(p)) \). While we leave a more detailed discussion and proof of this for Ref. [64], we can understand this somewhat intuitively using the geometric interpretation discussed in Section II. The blob excitations correspond to non-trivial 2-fluxes, and these 2-fluxes must be defined with respect to a base-point, which is the start-point of the ribbon operator. Moving this base-point induces a \( g \triangleright \) action on the label of the 2-flux. Then, because applying a vertex transform at a vertex is analogous to parallel transport of that vertex (see Section ID), the vertex transform also has a similar \( g \triangleright \) action on the label of the 2-flux produced by the ribbon operator. The ribbon operators which are invariant under this action therefore commute with the vertex transforms and so with the vertex energy term at the start-point. However generic ribbon operators are not invariant under this action, which leads to some of them exciting the start-point. Specifically, given a linear combination \( \sum_e \alpha_e B^r(e) \) of group-labelled blob ribbon operators, the start-point is not excited if the coefficients \( \alpha_e \) are a function of \( \triangleright \)-class, so that \( \alpha_g = \alpha_{g \triangleright f} \) for all pairs \( g \in G \) and \( f \in E \). On the other hand, the start-point is definitely excited if the sum of the coefficients in each \( \triangleright \)-class is zero (i.e., \( \sum_{g \in G} \alpha_{g \triangleright e} = 0 \) for all \( e \in E \)). Ribbon operators that do not satisfy either of these conditions do not produce eigenstates of the vertex term when acting
on the ground state, but can be written as a linear combination of operators that excite the vertex and operators that do not.

C. \( E \)-Valued Loops

In this model, we find two main types of loop-like excitation, in addition to the point-like excitations that we already discussed. Whereas point-like excitations are produced by ribbon operators, which act on a path or ribbon, loop-like excitations are produced by so-called membrane operators, which act across a general two-dimensional surface. The first loop-like excitations that we find, which we call \( E \)-valued loop excitations, are produced by membrane operators which act primarily on the surface labels. These membrane operators measure the label of a surface in our lattice, using the rules for combining surface elements given in Sections ID and IE. Then they apply a weight, depending on the result measured. This procedure is very similar to that used for the electric excitations (as described by Figure 41 and Equation 17), except that we measure a surface rather than a path. Indeed, if the electric excitations are charges for the 1-gauge field, then the \( E \)-valued loops are “2-flux charges” corresponding to the higher gauge (2-gauge) field. This means that they can interact with our 2-fluxes, in a way somewhat analogous to the interaction of ordinary charges and fluxes, as will be discussed more closely when we examine braiding. In the same way as with the electric excitations, the weights describe a space of operators and an appropriate basis is given using irreps, this time of the group \( E \). For an irrep \( \mu \) and matrix indices \( a \) and \( b \), the operator acting on a membrane \( m \) is given by

\[
I^{\mu,a,b}(m) = \sum_{e \in E} [D^\mu(e)]_{ab} \delta(\hat{e}(m), e),
\]

where \( \hat{e}(m) \) is the total surface element of membrane \( m \). This surface element can be written in terms of the labels of the plaquettes making up the membrane as

\[
\hat{e}(m) = \prod_{\text{plaquettes } p \in m} g(s.p(m) - v_0(p)) \triangleright e_p^\sigma_p
\]

where \( v_0(p) \) is the base-point of the plaquette \( p \), \( s.p(m) \) is the base-point with respect to which we measure the surface label (and which we call the start-point of the membrane) and \( \sigma_p \) is +1 if the plaquette’s orientation matches that of \( m \) and −1 if it is anti-aligned with \( m \). Note that when \( E \) is non-Abelian, the order of the product in 19 is non-trivial under the \( \triangleright \) action, as indicated in Figure 43. If \( \mu \) is the trivial irrep, then the membrane operator is instead the identity operator, so the operator does not produce any excitations.

FIG. 43. We consider applying an \( E \)-valued membrane on the shaded membrane \( m \) in a fragment of the three-dimensional lattice. The membrane operator measures the surface label of the membrane, with a weight for each possible label. When measuring a surface, if \( \triangleright \) is non-trivial we must specify the base-point of that surface. The base-point of the surface measured by the membrane operator (the yellow dot) is called the start-point of the membrane. A non-trivial \( E \)-valued membrane operator excites the edges (solid blue lines) on the boundary of the membrane, and may also excite the start-point of the membrane.

As with the blob excitations, there are some features that depend on which special case from Section IF that we look at. As we explained in Section IF, when \( \triangleright \) is trivial, the Peiffer conditions (Equations 4 and 5 in Section ID) imply that \( E \) is Abelian. This is because we require \( \partial(e) \triangleright f = e e^{-1} f \) for all \( e, f \in E \) from the second Peiffer condition (Equation 5). When \( \triangleright \) is trivial, this condition becomes \( f = e e^{-1} \), which implies that \( E \) is Abelian. In this case, the irreps are all one dimensional, so that we can drop the matrix indices \( a \) and \( b \) in Equation 18.

On the other hand, when \( \triangleright \) is non-trivial, \( E \) may be non-Abelian and so generally we must include the matrix indices. In addition, the membrane operator has an additional feature. When we measure the surface label, we must choose a base-point for that surface. We call this base-point the start-point of our membrane operator. Much as with the blob excitations, this start-point can be excited by the operator, which reflects the non-trivial transformation undergone by the operator when we move the start-point (due to the connection between vertex transforms and parallel transport). Similarly to the blob ribbon operator, the start-point is not excited if the membrane operator is made of a linear combination \( \sum_\alpha \alpha_\epsilon \hat{e}(\epsilon(m), e) \) whose set of coefficients \( \alpha_\epsilon \) is a function of \( \triangleright \)-class (i.e., \( \alpha_\epsilon = \alpha_{\epsilon \triangleright \epsilon} \)). On the other hand, the start-point may be excited if the coefficients transforms non-trivially under the \( \triangleright \) action, as we describe in more detail in Refs. [63] and [64]. The start-point excitation is significant because it can carry a (point-like) topological charge, which must be balanced by a point-like charge on the loop itself, as we prove explicitly in Ref. [64], in the Section called “The Point-Like Charge of Simple Excita-
tions”. Furthermore, as we describe in Ref. [64] in the section called “Ribbon and Membrane Operators in the Fake-Flat Case”, it seems that this point-like charge can be confined for certain membrane operators (meaning it costs energy to separate the start-point charge from the point-like charge carried by the loop), in which case the charge drags a line of excited edges between the start-point and the loop excitation. We note that this confinement can only occur when $E$ is non-Abelian, so it does not occur for Cases 1 and 2 in Table I.

D. Magnetic Excitations

The final type of elementary excitation is the magnetic excitation, named so due to its correspondence to the magnetic excitations in Kitaev’s Quantum Double model [13] and its analogy to magnetic flux. In Section II we discussed how the presence of magnetic excitations results in closed loops linking with the magnetic flux tubes having non-trivial path label. We therefore expect these magnetic excitations to be created by changing the edge labels of the lattice in order to introduce this non-trivial flux. In this section, we will see in more detail how this is done.

The magnetic loop excitation is primarily associated with excitation of the plaquette energy terms, due to its nature as a flux tube. Recall from Section I E (see Figure 29) that the plaquette term checks that the 1-flux of the plaquette is trivial, where the 1-flux is given by the product of edge elements around the boundary multiplied by the image under $\partial$ of the plaquette’s surface element. In order to create an elementary flux excitation, we want to excite the fewest number of plaquettes by changing edge labels (changing the plaquette label could also excite the plaquette, but this results in blob excitations as we saw in Section III B). We therefore consider trying to excite a plaquette by changing one of the edges on that plaquette. However in three spatial dimensions the edge will generally be shared by multiple plaquettes. Therefore changing the edge label will excite all of the plaquettes surrounding this edge, not just the one we wished to excite. We can try to fix one of these additional excited plaquettes by changing the label of another edge on that plaquette, but this will in turn excite the other plaquettes surrounding that edge. We can repeat this process, but we can never get rid of all of the additional excitations. When we change a series of edge labels in this way, the result is a set of excited plaquettes that lie on a closed loop which intersects the centres of these plaquettes. For example, in the first step when we change just one edge, we excite all of the plaquettes on a closed loop around that edge. Upon changing additional edge labels, we deform this loop, as shown in Figure 44, but always get a closed string of excited plaquettes (unless we collapse the loop to nothing). This means that the magnetic excitation is indeed loop-like, as we predicted earlier. The edges that we have to change to produce the magnetic excitation are bisected by a membrane bounded by the excited loop, as shown in Figure 45. We call the membrane cutting these edges the dual membrane, because the membrane bisects the edges of the lattice rather than lying on the lattice itself. The fact that we must change degrees of freedom across a membrane in order to produce a general magnetic excitation means that the creation operator is a membrane operator, which we call the magnetic membrane operator.

Now we should consider more concretely the action of the membrane operator. The features of the magnetic excitation depend strongly on the special case that we take. We first consider the trivial case (Case 1 in Table I). Then the operator to produce the excitation is analogous to the 2+1d magnetic ribbon operator from Kitaev’s Quantum Double model [13]. Just like the ribbon operator from the Quantum Double model, the magnetic membrane operator is labelled by a group element $h$ of $G$. When the group $G$ is Abelian, the action is simple. We just multiply each of the affected edges by the element $h$ or its inverse, depending on the orientation of the edge. On the other hand, when the group is non-Abelian the action is slightly more complicated. Instead of multiplying each edge by $h$, we must multiply by some element in the same conjugacy class as $h$. To determine which element this is, we must first endow our operator with a privileged point, called the start-point. Furthermore, we must specify a path from this start-point to each edge we want to change. Denoting the path to edge $i$ by $t_i$, the edge label $g_i$ of $i$ transforms as $g_i \rightarrow g(t_i)^{-1}hg(t_i)g_i$ or
$g_i \rightarrow g_i g(t_i)^{-1} h^{-1} g(t_i)$ depending on the orientation of that edge. This action is shown in Figure 45. The paths involved in this action lie on a second membrane, which we call the direct membrane, so the support of the membrane operator actually lies on both the direct and dual membranes, which we sometimes refer to together as the thickened membrane. This is analogous to how the ribbon operators in Kitaev’s Quantum Double model act on a ribbon [13], which is a thickened string.

As with the blob and $E$-valued loop excitations, the start-point of the magnetic membrane operator may be excited. As mentioned previously, the magnetic excitations in this model are analogous to the magnetic excitations from Kitaev’s Quantum Double model [13], and the potential start-point excitation is also present for the magnetic ribbon operators in that model (see for example Ref. [79]). We can interpret the start-point of the magnetic membrane operators in this model, as well as the start-points of the magnetic ribbon operators in Kitaev’s Quantum Double model, in terms of gauge theory. Recall from Section II that whenever we measure a flux, we must do so with respect to a certain start-point. The start-point of the membrane operator (or ribbon operator for the Quantum Double model) is the start-point with respect to which the flux label of the corresponding excitation is well-defined. Suppose that we have a flux $h$ when measured with respect a certain start-point. If we were to measure the flux from a different start-point, the flux that we measure would have the form $\hat{g}(t) h \hat{g}(t)^{-1}$, where $\hat{g}(t)$ is a path element operator for which we are not generally in an eigenstate (even if there are no excitations present other than the magnetic flux). Note that the element $\hat{g}(t) h \hat{g}(t)^{-1}$ is still in the conjugacy class of $h$, indicating that this conjugacy class is independent of the start-point even if the flux element is not. This interpretation of the start-point is then connected to whether the start-point of a magnetic membrane operator is excited. The vertex transform at a vertex acts like parallel transport of that vertex, and so we can think of the vertex transform at the start-point as moving that start-point. As we described in Section II, moving the start-point results in conjugation of the flux label. In order to diagonalize the vertex term at the start-point, we must therefore take a linear combination of magnetic membrane operators with different labels in the conjugacy class of $h$ (so that we are considering a state in a superposition of different flux labels), as we will prove in Ref. [64]. If the start-point is unexcited, it means that the membrane operator is not sensitive to changes to the start-point from which we measure the flux, which occurs when the membrane operator produces an equal combination of fluxes in the conjugacy class (i.e. the coefficients of the linear combination are the same for each element in the conjugacy class). This equal combination of elements in the conjugacy class produces a flux tube with a trivial point-like charge (or in the 2+1d case such as the Quantum Double model, a pair of excitations that can be annihilated, as described in Ref. [76]).

So far this consideration of the magnetic excitations has all been in the case where $D$ is trivial. As mentioned earlier, the form of the membrane operator depends quite strongly on which case we look at. In the case where $D$ is non-trivial but we restrict to fake-flat configurations (Case 3 in Table I), we cannot include the magnetic excitations at all, because the magnetic excitations violate the plaquette terms and hence break fake-flatness. The most interesting case is the remaining special case we mentioned in Section IF (Case 2 in Table I), where we loosen the restrictions on the crossed module without throwing out the non-fake-flat configurations. Specifically, we require that $D$ maps to the centre of $G$ and that $E$ is Abelian. In this case, we are allowed to keep the magnetic excitations, though their operators must be modified. We briefly describe this modification here, but for a full and detailed description, please see Ref. [64]. The new membrane operators act on edge elements in the same way as described above in the $D$ trivial case, but they also act on the plaquette elements around the membrane. The dual membrane that cuts edges also cuts through plaquettes. These plaquettes have base-points that can be either on the direct membrane or away from it (on the other side of the dual membrane). If a cut plaquette has its base-point on the membrane, then it is acted on with a $D$ action. This action on a plaquette $p$, with label $e_p$, is
given by \( e_p \rightarrow (g(t)^{-1}h g(t)) \triangleright e_p \), where \( g(t) \) is the path from the start-point of the operator to the base-point of the plaquette being changed. Any plaquette whose base-point is away from the direct membrane is left unaffected. An example of this action is shown in Figure 46. This action on the plaquettes is analogous to how the vertex transform affects plaquettes based at that vertex, but not plaquettes that are not based at the vertex. The difference between the action of the membrane operator and the vertex transforms is that the vertex transform acts on all plaquettes whose base-points are that vertex, but the magnetic membrane operator acts only on plaquettes whose base-points lie on the direct membrane and which are also cut by the dual membrane. In this sense, the vertex transform is like a closed magnetic membrane operator, whose dual membrane encloses that vertex and whose direct membrane is just the vertex itself.

In addition to this \( \triangleright \) action, we have to include certain blob ribbon operators into the membrane operator, with these ribbons running from a special blob defined in the membrane operator, which we call blob 0, to the plaquettes of the direct membrane. In the \( \triangleright \) trivial case (Case 1 of Table I), the plaquettes around the membrane were excited and the start-point could also be excited. In this more general case (Case 2 of Table I) the special blob, blob 0, may also be excited, as may the edges and blobs surrounding the membrane. The word “may” is important for the edges and blobs at the boundary of the membrane, because near an excited plaquette, the edge and blob terms cease to commute and also become inconsistent with changes to the branching structure of the lattice. This effect would only become worse if we lifted the condition on \( \partial \). In that case, the blob ribbon operators that we add would generally be confined, which could lead to plaquette excitations away from the boundary (although we expect this to be true only for certain labels of the membrane operator) and this is the reason that we do not consider the fully general case. In Case 2 the problematic plaquette excitations are restricted to the boundary of the membrane, and any topological quantities can be measured far from this boundary (if the membrane is sufficiently large), so the inconsistencies from the plaquette excitations are not important.

Due to the extra features of the magnetic membrane operator in this case, the magnetic excitation may carry both an ordinary 1-flux and a 2-flux, as we discussed in Section II. Recall that a non-trivial 2-flux indicates a closed surface with a non-trivial label. We can see that the loop excitation must be associated to a non-trivial surface when blob 0 of the membrane operator is excited. This is because blob 0 being excited indicates that this blob carries a non-trivial 2-flux, which must be balanced by a 2-flux belonging to the loop-like excitation itself.

**IV. CONDENSATION AND CONFINEMENT**

A. Confinement

Many of the excitations that we have discussed, particularly in the \( \triangleright \) trivial case, are familiar from Kitaev’s Quantum Double model [13]. However two important features of the higher lattice gauge theory model that are not present in the Quantum Double model are confinement and the related phenomenon of condensation. These concepts are seen in other commuting projector models, such as the ones considered in Ref. [45], but here the ideas are found in a 3+1d setting. Confinement refers to an energetic cost to separating particles (or growing and moving loop-like excitations). Non-confined excitations can move without an energy cost, as represented by the membrane operators (which produce loop-like excitations) only having an energy cost associated to the length of the loop they create and ribbon operators (which produce point particles) having a fixed cost independent of length. However for some excitations this is not the case and there is an energy cost associated to the length of a ribbon operator or the area of a membrane operator. In this case, we say that the excitations created by that operator are confined.

In this model, we find that some of the point particles are confined. This means that when we create a pair of point excitations it costs energy to separate them. Firstly, some of the blob excitations are confined. Recall from Section III B that the blob ribbon operator multiplies plaquettes pierced by a ribbon by an element of \( E \). This element takes the form \( g \triangleright e \), where \( g \) is some element which varies from plaquette to plaquette and whose precise value is not important for this discussion (as we shall see shortly). In addition to exciting the blobs at the ends of the ribbon, this action may excite the plaquettes that are pierced by the ribbon. Recall from Section I E that the plaquette term (see Figure 29) does not just depend on edge elements, but also on the surface label of that plaquette. In particular it depends on the surface label through the map \( \partial \), as illustrated by Figure 29. Recall from Section I D that \( \partial \) is a group homomorphism from \( E \) to \( G \), so it associates each element of \( E \) to an element of \( G \) (and in particular an element of the image of \( \partial \), \( \partial(E) \)). For a plaquette \( p \), the plaquette term projects onto states for which the plaquette label \( e_p \) and the path label \( g_p \) of the boundary of the plaquette are related by \( \partial(e_p)g_p = 1_G \). Therefore if our ribbon operator changes \( \partial(e_p) \) for the plaquettes that it pierces, it will excite those plaquettes. Because the blob ribbon operator labelled by an element \( e \in E \) multiplies the plaquette label \( e_p \) of any plaquette \( p \) pierced by the ribbon by an element of the form \( g \triangleright e \) (or the inverse), the ribbon operator changes \( \partial(e_p) \) when \( \partial(g \triangleright e) \) is non-trivial. Because of the Peiffer condition Equation 4, \( \partial(g \triangleright e) = g \partial(e)g^{-1} \), and so \( \partial(g \triangleright e) \) is the identity element when \( \partial(e) = 1_G \) regardless of the value of \( g \). That means that the blob ribbon operator will change \( \partial(e_p) \) for each plaquette \( p \) pierced by the ribbon.
precisely when $\epsilon$ is outside the kernel of $\partial$ (and so will excite these plaquettes). This means that the blob ribbon operators whose labels are outside the kernel will excite a series of plaquettes along the ribbon, with the energy cost associated to this growing with the length of the ribbon. That is, the blob excitations produced by ribbon operators with label outside the kernel of $\partial$ are confined, while those produced by ribbon operators with label in the kernel are not. Because the mechanism of this confinement is via the plaquette terms (which enforce fake-flatness), if we restrict to fake-flat configurations (Case 3 in Table 1), we must not allow the confined blob excitations, which violate fake-flatness. Note that it may at first seem surprising that certain violations of the projector terms (the blob terms here) are associated with violations of other projector terms, given that these projectors commute. However commutation and independence are not the same. A simple counterexample is the pair of projectors $P_1$ and $P_2 = 1 - P_1$. These commute and so share eigenvectors, but the eigenvalues of these eigenvectors are always different for the two projectors, indicating that the projectors are not independent.

We also find that some of the electric excitations are confined. In the case of the electric excitations, the mechanism of this confinement is the excitation of the edges measured by the electric ribbon operator. These edges may be excited by the ribbon operator because the edge energy terms change the path element that the electric ribbon measures, by an element of $\partial(E)$ (the image of $\partial$). Therefore, if the electric ribbon operator is sensitive to the path element changing by a factor in $\partial(E)$, then the associated electric excitation is confined (or contains a confined and unconfined component). We can be more precise by using some ideas from group theory. Recall that a useful basis for the electric ribbon operators has basis elements labelled by irreps of $G$. These basis elements are guaranteed to be either purely unconfined or purely confined (i.e. they diagonalize the edge energy terms along the ribbon), rather than containing components of both types. To see this, note that $\partial(E)$ is a normal subgroup of $G$ from the first Peiffer condition, Equation 4. Then given an irrep $R$ of $G$ labelling one of the basis operators, we can determine whether the corresponding excitations are confined by finding the restriction of $R$ to the subgroup $\partial(E)$ (i.e. we only evaluate the irrep $R$ on elements of $\partial(E)$, and treat this as a representation of the subgroup). When we restrict the irrep $R$ to this subgroup, we obtain a generally reducible representation of this subgroup. When we decompose this
representation into irreps of $\partial(E)$, by Clifford’s theorem \cite{Clifford1911} the constituent irreps will all be related by conjugation. This means that if one of the irreps of $\partial(E)$ found by restricting $R$ is the trivial irrep, then they all are (this result can also be obtained by Schur’s Lemma in the case where $\partial(E)$ is in the centre of $G$). If $R$ branches to the trivial irrep in this way, the electric excitation is not confined. On the other hand if $R$ does not branch to the trivial irrep then the excitation is confined, with all of the edges along the ribbon being excited.

We also found that some of the $E$-valued loop-like excitations can be confined in a certain sense when $E$ is non-Abelian, as described in Section III C. This confinement does not give an energy cost to growing the loop-like excitation, but instead it costs energy to move the excitation away from the start-point of the membrane. As we discussed in Section III C, it seems like the point-like charge carried by the loop excitation is confined, rather than the loop-like charge.

As well as the energy cost, confined and non-confined excitations have another important distinction. Normally our membrane and ribbon operators are topological. By this we mean that deforming the membrane or ribbon, while keeping all excitations fixed, does not affect the result of acting with the membrane or ribbon operator. This is true provided that the deformation does not cause the membrane or ribbon to cross an excitation, as we prove in Refs. \cite{Semenoff1984} and \cite{Semenoff1985} (in sections entitled “The Topological Nature of Ribbon Operators” and “The Topological Nature of Ribbon and Membrane Operators”). In particular, acting with a closed and contractible ribbon or membrane operator is trivial. However with confined excitations, the corresponding ribbon or membrane operators cannot be topological. The confined ribbon operators have an energy cost associated to the length of the ribbon of a ribbon operator, so that deforming them in a way that changes the length of the ribbon changes the energy cost. This clearly means that the state resulting from the action of a deformed ribbon operator must be different from the state resulting from the action of the original operator, because these states can have different energies. Therefore the ribbon operators for confined excitations are not topological. In the case of the membrane operators producing the particular confined loop-like excitations in this model, the membrane operator is still partially topological: we can deform the membrane without affecting the action of the operator, but we must keep the location of the excited edges fixed (and so the location of the confining string is fixed). This is again because it is the point-like charge that is confined (and so the motion of the point-like charge is not topological).

**B. Condensation**

The phenomenon of confinement is closely related to a process known as condensation. Consider a topological model with some set of topological charges. By deforming the Hamiltonian, we may find that some of the non-trivial topological charges from the old Hamiltonian are present in the ground state of the deformed Hamiltonian. Because the ground state corresponds to the topological vacuum, this means that the excitations which carried the previously non-trivial charges now carry the trivial charge, although they still remain energetic. We say that those excitations condense \cite{Semenoff1984, Semenoff1985}. When this occurs, any excitations that braided non-trivially with those excitations in the original become confined in the deformed model \cite{Semenoff1984, Semenoff1985} (this is a bit of a simplification, but is sufficient to describe this model). This process is known as a condensation-confinement transition. While this is fairly well understood in 2+1d, there has been comparatively little study of such transitions in the 3+1d case (examples of work in this area include Refs. \cite{Semenoff1984} and \cite{Semenoff1985}), and so it is interesting to see how condensation and confinement arise in this 3+1d model.

In our class of models, we can consider the process of condensation by constructing two models related by such a transition. That is, we consider cases where we can turn the confinement on and off by changing a parameter in the Hamiltonian. We cannot change the groups $G$ and $E$, which also fix the Hilbert space, but we can change the map $\partial$. When $E$ is Abelian we can construct a model with no confinement, described by the crossed module $(G, E, \partial \rightarrow 1_G, \triangleright)$, and called the uncondensed model. Here $\partial \rightarrow 1_G$ indicates that in the uncondensed model $\partial$ maps to the identity of $G$. The condition that $E$ be Abelian is required from the second Peiffer condition, Equation 5, which enforces that $\partial(e) \triangleright f = e f e^{-1}$ for every pair of elements $e$ and $f$ in $E$. Because $\partial(e) = 1_G$ for the uncondensed model and $1_G \triangleright$ is always the trivial map, the second Peiffer condition ensures that conjugation is also trivial: $e f^{-1} = 1_G \triangleright f = f$. Starting from this uncondensed model we can “turn on” $\partial$ (while keeping the groups fixed), by moving into a model described by a crossed module where $\partial$ maps to some other subgroup of $G$. This can be done by interpolating between the two Hamiltonians, because the two models have the same Hilbert space. During this process, some of the excitations condense. When this occurs, the condensing excitations may still cost energy in the new model, but they carry trivial charge in this condensed phase. We therefore refer to such excitations in the condensed phase as condensed excitations. These condensed excitations can be produced by an operator local to the excitation. For loop-like excitations this means that a condensed loop-like excitation can be produced with an operator that only acts near the loop, rather than on an entire membrane. Indeed, finding such an operator to produce the loop excitation is one way to show that the excitation is condensed.

In this model, some of the loop-like excitations are indeed condensed. Recall from Section III D that the magnetic excitations are labelled by elements of the group $G$. If that element is in the image of $\partial$, then the exci-
tation is condensed. This can be seen from considering the magnetic excitations in light of the plaquette term. We discussed in Section II that the magnetic excitations are associated with closed loops that have non-trivial label. However, the plaquette energy term enforces that the label of closed loops in our lattice match the image under ∂ of the surface element bounded by the loop, rather than enforcing that the path label is the identity element. Therefore the ground state contains closed loops with all values in the image of ∂. A magnetic membrane operator with label in the image of ∂ modifies the labels of closed loops that link with the excitation only by multiplication by another element of ∂(E), and so results in closed loop values already found in the ground state. Therefore the topological charges of the corresponding magnetic excitations (which we measure with closed paths) belong in the ground state, and so these charges have been condensed (in the uncondensed model, only 1G is in the image of ∂). However note that this does not mean that these magnetic excitations are not excitations at all. Changing the path label by an element of ∂(E) (as done by these magnetic excitations) still leads to an excitation, because the path labels do not match the surface enclosed by the loop (and so do not satisfy fake-flatness, implying the presence of a plaquette excitation). In Ref. [64] (in a section entitled “Condensation of Magnetic Excitations from the Membrane Operator Picture”) we explicitly show that the action of a condensed membrane operator on the ground state is equivalent to the action of a (confined) blob ribbon operator around the boundary of the membrane operator, which is local to the excitation, confirming that the corresponding loop-like excitation is condensed.

In a similar way, some of the E-valued loops are condensed. This is because some of the membrane operators that produce them are equivalent (when acting on the ground state) to an electric ribbon operator acting around the boundary of that membrane. The membrane operators for the E-valued loops measure the surface element of the membrane that they are placed on. However, in the ground state the group element e_m assigned to a surface m is related to the path around the boundary of the surface (labelled by g(boundary)) by ∂(e_m)g(boundary) = 1_G, due to the plaquette terms. This suggests that we can measure the surface element just by examining the boundary, but because this expression involves only ∂(e_m), this correspondence between the surface and boundary does not fully fix the value e_m. The surface element can be split into a part that describes the image under ∂ of that element and a part in the kernel of ∂, with the former part fixed by the boundary label. This means that the image under ∂ of the surface element can be measured by looking at the boundary of the surface, but not the part in the kernel. Therefore, if a membrane operator is only sensitive to the former part of the surface element, and not to the part in the kernel, then it is equivalent to an operator that simply measures the boundary path element (i.e. an electric ribbon operator applied around the boundary). In this case, the corresponding loop excitation can be produced by an operator that only acts near the loop itself, indicating that the loop cannot carry non-trivial loop topological charge and is condensed. We can be more precise about this notion of sensitivity to the kernel by using the irrep basis for the membrane operators. Recall from Section III C that the E-valued loops are labelled by irreps of E, as seen in Equation 18. We can construct an irrep of the kernel of ∂ (which is a subgroup of E) by restricting the irrep of E to the kernel. This results in a (generally reducible) representation of ker(∂), which will branch to several copies of one of the irreps of the kernel. The reason that the representation only branches to copies of one of the irreps of the kernel, rather than branching to multiple distinct irreps, is that the kernel is always central in E due to the second Peiffer condition, Equation 5, so Schur’s Lemma applies (meaning that the matrix representation of any group element in the kernel must be a scalar multiple of the identity). If this resulting irrep is trivial then the excitation is condensed, while if the irrep is non-trivial then the excitation is not condensed. More concisely, for an excitation produced by a membrane operator labelled by an irrep µ of E, if the matrix representation satisfies \( D^\mu(e_K) = 1 \) for all e_K in the kernel then the excitation is condensed.

V. BRAIDING

In Section I G we discussed the idea of braiding statistics in 3+1d. Now we look at the particular realisation of braiding that is found in this model. First we will give an idea of how braiding relations may be obtained from our membrane operators, then we will describe our results.

To understand how we can obtain the braiding relations from the ribbon and membrane operators, we should further discuss the interpretation of these operators. Ribbon operators can be thought of as creating a pair of excitations and then separating them along the ribbon. The ribbon operator therefore encodes the result of moving an excitation. Similarly a membrane operator can be thought of as nucleating a small loop and moving it into the final position of the excitation, as well as changing its size and shape. Therefore we should be able to find the result of braiding by applying successive membrane operators. In particular the braiding is related to the commutation relations of the membrane operators. For instance, consider the commutation relation shown in Figure 47, which we will see relates to loop-loop braiding. The images represent membrane operators, which are displaced horizontally to indicate an order of operators (although the membranes intersect in space). In the first line, we first act with an operator that produces a loop excitation (the green loop) and then act with an operator that creates another loop (shown in red) and moves it through that first loop. This performs a braiding move, specifically the one from the left side of Figure 35. In the second line of Figure 47, we first act with the operator that creates and moves the red loop through empty space, before creating the other loop excitation. In
this case no braiding move occurs. Comparing these two lines (that is, working out the commutation relation for the two membrane operators) therefore lets us compare the situation with braiding to the one without, but where the excitations have the same final positions in each case. This latter point is important because ensuring that the excitations have the same final positions in the two cases isolates the effect of braiding from any other effects.

There are some subtleties when determining the braiding relations, which mean that we have to take care when interpreting the commutation relations between membrane operators. The formulae given in this section will refer to “same-site” braiding. This refers to the case where the start-points of the operators (as defined when discussing the excitations in Section III) are in the same location. In this case the excitations involved have a definite fusion product. Fusion is where we look at two topological charges close together and consider their combined charge, in an analogous way to looking at two electrons and finding their total spin. Generally those electrons will be in a superposition of states with total spin zero and one, and in a similar way two anyons can fuse to multiple products. In a non-Abelian anyon theory, where two anyons may fuse to multiple different types of anyon, the braiding of two particles depends not only on the charges of the two particles, but also which charge they fuse to [21, Preliminaries]. To find the braiding relations we therefore want to consider the case where they have definite total charge, otherwise the braiding relations are not well-defined (which in practice is reflected in this model by the presence of operator labels in the braiding relations). Using this same-site case for the braiding means that the excitations involved have a well-defined combined topological charge, and so is a useful case to consider. In simple cases, where the fusion is Abelian (i.e. where two charges only have one fusion channel), this requirement for “same-site” braiding is lifted (and this is usually reflected by the start-point of the corresponding membrane or ribbon operators being irrelevant to the commutation relation). While we only discuss the same-site case here, we will give more general results in Ref. [64].

Having discussed the method for finding the braiding, we now discuss our results. Firstly, we note that any braiding of two point-like excitations (which we term permutation) must be trivial in the 3+1d model. This is because in this model the permutation is implemented by ribbon operators that connect the initial and final positions of the excitations. In 3+1d, the ribbon operators that permute the two excitations do not need to intersect and so do not interact in any way (i.e. they commute), which leads to trivial (bosonic) exchange statistics. That is, the result of moving a point-like excitation around another one is the same as moving that excitation through empty space (and then producing the other point-like particle). In a similar way, exchange involving loop-like excitations moving past each-other (which we refer to as permutation), rather than through each-other (which we will refer to as loop-braiding or just braiding) is trivial for the same reason. That is, under permutation each excitation transforms just as if it were moving through empty space.

Next we consider the braiding where we move a point-like or loop-like excitation through a loop-like excitation, as shown in Figure 39 for the point-like case. First consider the case where ▽ is trivial. In this case, the excitations split into two separate sets, with non-trivial braiding only within each set. Firstly we have the excitations that are labelled by objects related to G, namely the electric and magnetic excitations. We have non-trivial point-loop braiding between the electric and magnetic excitations, where the magnetic excitation labelled by a group element h acts on the electric excitation, labelled by an irrep R and indices a and b, by multiplication by the matrix \( D_R^G(h) \). That is, given an electric ribbon operator \( S^{R,a,b}(t) = \sum_g [D_R^G(g)]_{ab} \hat{g}(t,g) \) and a magnetic membrane operator \( C^b(m) \), commuting the operators gives the following braiding relation

\[
S^{R,a,b}(t) \to \sum_c [D^R(h)]_{ac} S^{R,c,b}(t).
\]

We see that this results in mixing of the operators labelled by different indices, but not of operators labelled by different irreps. This is because R labels a conserved charge, whereas the matrix indices only represent some internal space to the topological sector, as we mentioned in Section III A when defining the irrep basis for the electric ribbon operators. Note that when the irrep is one dimensional, such as when the group G is Abelian, the braiding relation results in the accumulation of a simple phase of \( R(h) \), because the matrices belonging to a 1d unitary representation are just phases. We should also mention that the braiding relation depends on the orientation of the loop-like excitation and the direction with which the point-like particle is moved through the loop.
If either orientation was reversed, then we would replace $h$ with its inverse in the braiding relation. This braiding relation is natural from the gauge theory perspective and indeed is equivalent to the one that we predicted in Section II. The magnetic excitations result in closed paths with non-trivial label (the measurement paths from that section) and the electric excitations are charges which transform when moving around these closed paths.

We also find non-trivial loop-loop braiding between the magnetic excitations when the group $G$ is non-Abelian. When one loop, labelled by $g$, is passed through another, labelled by $h$, the first loop has its label conjugated, so that it becomes $h^{-1}gh$ (or $hgh^{-1}$ depending on the relative orientations of the loops). The other label is left unchanged. This braiding is just as we would expect for the braiding of the magnetic excitations in ordinary lattice gauge theory, as explained in Section II. This indicates that, while confinement is controlled by $\partial$, it is $\triangleright$ that describes how the braiding differs from the ordinary lattice gauge theory case.

Next we consider the other set of excitations, those labelled by objects corresponding to the group $E$. In this set the only non-trivial braiding, whether point-loop or loop-loop, is the braiding between the point-like blob excitations and the $E$-valued loop excitations. When a blob excitation labelled by an element $e \in E$ passes through a loop excitation labelled by the 1D irrep $\alpha$ of $E$, a phase of $\alpha(e)$ (or the inverse phase, depending on the orientation of the loop and direction of braiding) is accumulated. When $\triangleright$ is trivial, the group $E$ is Abelian, so all of the irreps are 1D and so the braiding transformation is only a phase.

The braiding relations are a little different when $\triangleright$ is non-trivial. If we restrict to fake-flat configurations (Case 3 from Table I), we have to throw out the magnetic excitations (and the blob excitations labelled by elements outside the kernel of $\partial$) and the only non-trivial braiding is between the blob excitations and the $E$-valued loops. When we pass a blob excitation labelled by $e$ through a loop labelled by the irrep $\mu$ and indices $a$ and $b$, the loop transforms by multiplication by the matrix $D^\mu(e)$ or the inverse. On the other hand, if we take our other special case (Case 2 in Table I), where $\triangleright$ maps from an Abelian $E$ to the centre of $G$, then the braiding is richer than in the $\triangleright$ trivial case. While braiding not involving the magnetic excitations is the same as in the $\triangleright$-trivial case, the magnetic excitations now braid non-trivially with all of the types of excitation. To obtain the braiding relations we have to combine the magnetic excitation with the $E$-valued loop excitation, giving us an excitation we call a higher-flux loop excitation. This is because the magnetic excitation now carries a 2-flux, but this flux is not well-defined unless we also apply an $E$-valued membrane operator on the same region of space (which fixes the 2-flux). Then the combined membrane operator produces an excitation labelled by a pair of group elements $(g, \tilde{e})$, where $g$ is in $G$ and gives the 1-flux, while $\tilde{e}$ is in $E$ and gives the 2-flux.

Here $g$ is simply the label of the magnetic membrane operator, but $\tilde{e}$ is related to the label $e$ of the additional $E$-valued membrane operator by $\tilde{e} = e[g^{-1} \triangleright e^{-1}]$, as we explain in more detail in Ref. [64]. When a pair of these higher-flux excitations braid, we have

$$(g, \tilde{e}_2), (h, \tilde{e}_1) \rightarrow (hgh^{-1}, h \triangleright \tilde{e}_2), (h, \tilde{e}_1 \tilde{e}_2[h \triangleright \tilde{e}_2^{-1}]) .$$

This braiding then matches the prediction of Ref. [60], where the result is argued from geometric grounds.

Finally, when a blob excitation labelled by $e$ is passed through a higher-flux excitation labelled by $(h, f)$, the labels of the excitations change in two ways. Firstly the label $f$ of the pinned $E$-valued loop is changed by multiplication by $e$ or by $h \triangleright e^{-1}$ (depending on orientation), which induces a change in the 2-flux $\tilde{f}$ of the excitation by multiplication by $e[h^{-1} \triangleright e^{-1}]$ or $e[h \triangleright e^{-1}]$. Secondly, the blob ribbon operator labelled by $e$ is acted on by the magnetic operator, so that $e \rightarrow h^{-1} \triangleright e$ or $h \triangleright e$ (again depending on orientation). From this we can see that the product $\tilde{f}e$ of the 2-fluxes of the two excitations is preserved by the braiding.

## VI. TOPOLOGICAL CHARGE

So far, we have alluded to the idea of the excitations carrying a conserved “topological” charge, without spending much time actually discussing this charge and the values it can take. In this section we will explain in more detail what we mean by a topological charge and how we can measure it. Topological charge is a conserved quantity associated with the excitations of the model (the anyons), while the ground state is said to have the trivial charge, or vacuum charge. The charge held within a region can only be changed by moving that charge from inside the region to outside or moving charge from the outside in. This means that the charge in a region can only be changed by operators that connect the inside of the region to the outside. We note that there is no need for a symmetry to enforce this conservation. Topological charge is conserved on the level of the Hilbert space and can be defined without reference to any Hamiltonian (although in this case we must define the vacuum charge in another way). We can measure the topological charge associated to a region using operators on the boundary of that region, which is reminiscent of the way that we can determine the electric charge in a region by measuring the divergence of the electric field on the boundary of that region. For example, we can measure the charge associated to a loop excitation using a torus that encloses that loop, as shown in Figure 48. Any operator that would move topological charge from inside a region to outside it must cross the boundary of that region, and so can be detected by the measurement operator on the surface.

In order to identify the operators that measure the topological charge, we first consider the characteristics that
we require such operators to have. While the topological charges are properties of the Hilbert space, the Hamiltonian picks out a certain set of charges, such that the ground state has the trivial charge. Then, because a measurement operator should not change the charge in any region, we require that the measurement operators do not create any excitations. Therefore we enforce that the measurement operator commutes with the Hamiltonian. Another property we require is that smoothly deforming the measurement operator without crossing any excitations should preserve the measured charge, because the ground state has trivial charge. Following the method of Bombin and Martin-Delgado [45], we construct such measurement operators using closed ribbon and closed membrane operators (the latter because we consider the 3+1d case, whereas Ref. [45] considered a 2+1d model). This may sound restrictive, but all operators in this model can be expressed in terms of the ribbon and membrane operators, and the ribbon operators that commute with the energy terms must be closed. However we may need to take additional steps to guarantee that the closed operators commute with all of the energy terms (because there may be some obstruction to closing the operator without producing some excitations). Our approach is as follows. We take the surface on which we want to measure the charge. Then we apply as many independent closed membrane and ribbon operators that act on that surface as possible, and construct all possible sums of these operators that commute with the Hamiltonian. The space of such operators corresponds to the space of possible topological charges. We note that the step of choosing the surface is important, not just because it determines where we want to measure the charge, but also because the set of charges to be measured depends on the topology of the measurement surface. A spherical measurement surface measures a different set of topological charges from a toroidal surface for example. A spherical measurement surface cannot distinguish a loop-like excitation from a point-like one, and for this reason we say that a spherical surface measures the point-like charge of an excitation (or set of excitations). In order to determine the loop-like charge of an excitation, we must use a toroidal surface, such as the one shown in Figure 48 (or a surface of higher genus, although we will not consider these in this work). We are therefore most interested in the torus charge, but we will consider the sphere charge in Ref. [64], where in particular we explicitly find the sphere charge of the simple excitations.

To understand how we apply the independent closed membrane and ribbon operators on the measurement surface, consider the case of a torus, shown in Figure 49. We apply ribbon operators around the two non-contractible cycles of the torus, and membrane operators on the torus itself. Any other closed ribbon operators that we could apply would not be independent (i.e. could be reduced to operators of the types already considered by deformation or other means), or would leave excitations on the surface. This means that the only operators we have are an electric and a blob ribbon operator running around each independent non-contractible cycle and a magnetic and $E$-valued membrane operator over the torus itself. This gives us six labels, one for each operator (if we label each operator with a group element, rather than using the irrep description for the $E$-valued membrane and electric ribbon operators). We then construct a linear combination over all possible labels, with coefficients chosen so that the sum commutes with the Hamiltonian. The set of allowed coefficients gives us the space of measurement operators. The size of this space tells us how many different charges we have that can be measured by a toroidal surface. Unfortunately, not being able to construct the magnetic excitations for a general crossed module (Case 3 in Table I) prevents us from constructing all of our charge measurement operators in every case. However when $E$ is Abelian and $\partial$ maps from an Abelian group to the centre of $G$ (Case 2 in Table I, of which Case 1 is a subset), we are able to explicitly construct the measurement operators. In this case we find that the number of different charges measured by a toroidal surface matches the ground-state degeneracy of the 3-torus. This is perhaps to be expected for a topological phase; in topological quantum field theories (TQFTs) there is a correspondence between the partition function associated to a closed (four-dimensional) manifold $M \times S^1$ and the dimension of the Hilbert space associated to the open manifold $M \times I$ (where $I$ is the interval) [88]. Then the ground state degeneracy of the higher lattice gauge theory model on a (three-dimensional) manifold $M$ is equal to the partition function of the Yetter TQFT [55, 56] on a manifold $M \times S^1$. Taking $M$ to be the 3-torus $T^3 = S^1 \times T^2$, we may therefore perhaps expect a relationship between the ground-state degeneracy of the 3-torus (which matches the partition function of the TQFT on $T^3 \times S^1 = S^1 \times T^2 \times S^1 = S^3 \times T^2$) and dimension of the space for the degrees of freedom on thickened 2-torus $I \times T^2$ (which should match the size of the space associated to $I \times T^2 \times S^1 = I \times T^4$ in the TQFT). This in turn should match the number of independent measurement operators that we can apply on the toroidal measurement surface (which is really a thickened torus). This correspondence between ground-state degeneracy and the charges for this model was also discovered in Ref. [59], using a different method (employing tube algebra). Indeed, in Ref. [64], we explicitly construct the projection opera-
tors to the different values of topological charge and find that these projection operators are labelled by the same objects labelling the simple modules of the tube algebra found in Ref [59].

FIG. 49. We apply closed ribbon operators on the two cycles of the torus (the thin red and yellow loops) and a membrane operator over the surface itself (the thicker green torus)

One important thing to note is that the torus surface can measure loops that link with either of the cycles of the torus, as shown in Figure 50. This means that the general object measured by the surface is a pair of linked loops (or a set of objects that can be fused into such a pair), as shown in Figure 50. In particular, we note that the number of charges measured by the torus (and so the ground state degeneracy) is not equal to the number of distinct loop-like excitations, but instead the number of link-like excitations (counting those obtained by fusion with point-like excitations as well). This suggests that it is important to consider the linking of loop-like excitations when studying 3+1d topological phases. This reinforces work by other authors [28, 58] that shows that so-called three-loop (or necklace) braiding, which is braiding of two loops while both are linked to a third loop, is useful for characterising 3+1d topological phases.

VII. CONCLUSION

In this paper we constructed the membrane and ribbon operators that produce the excitations of the higher lattice gauge theory model in three broad cases and used these operators to obtain various properties of the excitations. The model has two classes of excitations: those present in ordinary lattice gauge theory model in three broad cases and used these operators to obtain various properties of the excitations. The model has two classes of excitations: those present in ordinary lattice gauge theory and those that involve the surface holonomy, which only appear when we consider higher lattice gauge theory. The character of the excitations depends on which of the special cases we consider. When the map $\partial$ from our crossed module is trivial, the first class of excitation, those familiar from ordinary lattice gauge theory, are largely unchanged by the move to higher lattice gauge theory. The excitations related to the surface holonomy are also simple, with Abelian fusion rules. Furthermore the braiding between the two sectors is trivial, with non-trivial braiding only within the sectors (i.e. between the $G$-valued excitations and between the $E$-valued excitations). However, unlike in ordinary lattice gauge theory, some of the point-like excitations are confined, costing additional energy to separate pairs of excitations after producing them, while some of the loop-like excitations are condensed and carry trivial topological charge. We discussed how this arises from a condensation-confinement transition between higher lattice gauge theory models described by crossed modules with the same groups $G$ and $E$ but different maps $\partial$ between the groups.

We saw that the excitations are more interesting when $\partial$ is non-trivial. In particular, we looked at the case where the group $E$ labelling the 2-flux is Abelian and where the map $\partial$ maps to the centre of the group $G$. In this case, even though the group $E$ is Abelian, the excitations related to 2-flux have an internal space controlled by $\partial$. In addition, there is non-trivial braiding between excitations from the two different sectors. In particular, it is sensible to consider loop excitations, which we called higher-flux loops, built from a combination of excitations from the two sectors. These excitations carry both an ordinary magnetic flux along paths that link with the loop and a 2-flux associated to a surface enclosing the loop. The higher-flux loops have potentially non-trivial braiding with excitations from both sectors. We found the braiding relations of these higher-flux loops and discovered that the braiding between two higher-flux loops matches a braiding scheme for loop-like excitations described in Ref. [60].

In addition to using the membrane operators to find the braiding properties of the various excitations, we have discussed the explicit construction of the membrane operators to construct operators to measure the topological charge in a region. The charge measurement operators are operators with support on or near the sur-
face bounding a region, with different surfaces having different potential topological charges. A related idea is the fact that in 3+1d there are both point-like and loop-like excitations, with associated point-like and loop-like charge. While loop-like excitations may possess a point-like charge, which may be measured by enclosing the excitation in a sphere, they also possess loop-like charge, which can only be measured by surfaces with non-contractible handles such as the torus. This approach of explicitly constructing the charge measurement operators (which we present more fully in Ref. [64]) could allow us to measure the charge of collections of excitations and so provide a tool to consider fusion of topological charge.

There are many interesting avenues of research that would build on the results we’ve obtained in this work, either directly or indirectly. Firstly, and most obviously, there are some results for the higher lattice gauge theory model that we have so far not been able to obtain. We have considered certain special cases (described by Table I) and while there are some conceptual issues associated with taking the most general case (as described in Section I F), there may still be a way to work around these issues. It is possible that there are some features exhibited by the general case that we have not been able to study in our special cases.

It would also be interesting to further study the different types of topological charge possible in 3+1d topological phases, perhaps in a more general and conceptual setting than this specific model. In this work we have mostly considered the simple excitations, and only considered measurement operators for the topological charge within a sphere and torus. We believe that it would be useful to do more work with the charges themselves, either in this model or more generally. There are several questions that could be explored in this direction. For example, which surfaces do we need to consider to obtain all unique charges? How do we consider the fusion of charges that are measured by surfaces other than simple spheres? For instance, considering two tori, we could fuse the enclosed charges by bringing the tori together and stacking them on top of each other, so that a single torus can enclose both. After some preliminary calculations, we found that, in the higher lattice gauge theory model at least, this leads to consistency conditions for the “threading flux” passing up through the two tori, which must be satisfied in order to be able to fuse them, but a complete calculation is left for future work. Another way to combine two torus charges would be to bring the two tori side-by-side, so that a 2-handled torus would be needed to enclose both. For more general surfaces, there could be even more ways of fusing charges.

Another sensible direction for future study would be to extend our approach of utilizing explicit construction of membrane operators, to more general models for topological phases. In particular, it is known that twisted gauge theory models can have non-trivial three-loop braiding statistics [28, 47]. It has been claimed that these models cover all phases that can be produced from bosonic degrees of freedom and that result in bosonic point excitations (i.e. phases with trivial braiding between point-like particles) [89]. Because these models have a similar structure to the model considered in this paper, the application of our methods to the twisted gauge theory model seems feasible. By looking at other models, we may be able to see if the relation between the ground state degeneracy of the 3-torus and the number of charges measured by a torus surface holds more generally than just in the higher lattice gauge theory model, as we may expect.

Finally, it would be interesting to study the condensation-confinement transitions in 3+1d topological phases in more detail. We have seen some examples in the higher lattice gauge theory model, but in the cases where $E$ is Abelian (which we studied in more detail) the pattern of condensation is rather simple. The current understanding of condensation in 3+1d is perhaps incomplete, particularly when it comes to loop-like excitations condensing, and the examples considered in this work may be useful in studying this process.

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**Appendix A: Consistency of the Higher-Lattice Gauge Theory Model Under Changes to the Branching Structure**

As we discussed in Section I E, when we define a lattice for the higher-lattice gauge theory model, we must give the lattice a branching structure. This determines the orientations of the edges of the lattice as well as the orientation and base-point of each plaquette, which we call the “dressing” of the lattice. In order for the model to be topological, it must somehow be resilient to changes to these details. In this section we will show that there is a certain sense in which the energy terms are invariant under changes to the branching structure (to be more precise, a branching structure forbids local oriented loops, and has an ordering of vertices, but we will not make these restrictions for the altered dressing). To understand what we mean, let us first consider an example of how we can change the dressing...
of the lattice. In Section I C, we described the notion of parallel transport across an edge of the lattice. The label of the edge describes the result of parallel transport and the orientation of the edge describes the direction of the parallel transport that gives this result. If we were to perform parallel transport in the opposite direction, we would expect the inverse transformation. This suggests that there is a natural operation in which we can take an edge with a given orientation and a label $g$ and reverse the orientation of the edge, while simultaneously changing the label to $g^{-1}$, without changing the physical meaning of the label. In a similar way, in Section I E we described how reversing the orientation of a surface should change its label from $e$ to $e^{-1}$. We can also move the base-point of a plaquette from $v_1$ to $v_2$, which should change its label from $e$ to $g(v_1 - v_2)^{-1} \triangleright e$, where $g(v_1 - v_2)$ is the group element assigned to the path along which we move the base-point. We can treat these transformations as maps between two copies of our Hilbert space, where each Hilbert space corresponds to different dressings of the lattice. We extend the action of these transformations to states where the edges or plaquettes are labelled by linear combinations of the group elements (and to states where the different degrees of freedom are entangled) in the sensible way, to make these transformations linear. Then the transformations, which involve changes to the dressing of a lattice (orientations and base-points) as well as to the labels attached to the different cells of the lattice, are maps that connect states from different Hilbert spaces that should have the same physical content, but have different descriptions. Each of the Hilbert spaces is equipped with a Hamiltonian of the form given by Equation 12 from Section I E. If the states related by the transformations do indeed contain the same physical content, then the energy of a state should be preserved under these transformations. That is, if we first apply the Hamiltonian on a state in one of the Hilbert spaces and then apply a transformation to change the dressing of the lattice, this should give us the same result as applying the transformation then the Hamiltonian on the new Hilbert space. Equivalently, given a transformation $\hat{T}$ that changes the dressing, then the Hamiltonian $\hat{H}_1$ in the original space should be related to the Hamiltonian $\hat{H}_2$ in the second space by $\hat{T}^{-1}\hat{H}_2\hat{T} = \hat{H}_1$. If this is true then we have a way to move between the different dressed lattices, while preserving the structure endowed on the space by the Hamiltonian. This demonstrates the desired resilience of the higher lattice gauge theory model to changes to the dressing of the lattice. In this section we will show that the Hamiltonian does indeed have this property, subject to certain caveats related to fake-flatness and the map $\triangleright$.

1. Reversing the Orientation of an Edge

Let us start by considering the procedure where we reverse the orientation of an edge and simultaneously invert the label of that edge. We denote this edge flipping transformation on an edge $i$ by $\hat{P}_i$. We wish to show that this map preserves each energy term individually (which is a slightly stronger condition than just preserving the Hamiltonian). Suppose that we start in one copy of our Hilbert space $\mathcal{H}_1$, and $\hat{P}_1$ maps to another copy $\mathcal{H}_2$ of the Hilbert space. Then for any vertex, edge, plaquette or blob in our lattice we have an energy term $\hat{O}_1$ acting on the first Hilbert space, and a corresponding energy term $\hat{O}_2$ acting on the second Hilbert space, where these energy terms are defined in Section I E. We wish to show that the energy is preserved under the transformation $\hat{P}_1$, by which we mean that $\hat{P}_1\hat{O}_1 = \hat{O}_2\hat{P}_1$. Thinking in terms of the eigenstates, if this is true then for a state $|\psi\rangle$ which is an eigenstate of the energy term $\hat{O}_1$, with eigenvalue $\lambda$, the state $\hat{P}_1|\psi\rangle$ reached by flipping the edge $i$ is an eigenstate of the equivalent energy term $\hat{O}_2$ with the same eigenvalue, $\lambda$. Because the map $\hat{P}_1$ is invertible (we can undo $\hat{P}_1$ by flipping the edge back and inverting the edge element again, so $\hat{P}_1$ is in fact its own inverse), we can also write the condition as $\hat{O}_1 = \hat{P}_1^{-1}\hat{O}_2\hat{P}_1$. Therefore, when we say that the energy is preserved by the transformation, we mean that changing the orientation of the edge, applying an energy term and then changing the orientation back has the same net effect as simply applying the corresponding energy term without flipping the edge.

We first wish to show that the edge flipping procedure is consistent with the vertex terms $A_v$. Each vertex term is an equal sum of the vertex transforms: $A_v = \frac{1}{|G|} \sum_{g \in G} A^g_v$. Therefore, if we can show that the vertex transforms $A^g_v$ are invariant under $\hat{P}_1$, this will also be true for the vertex terms $A_v$. The vertex transform on a vertex $v$ only has support on the edges that are adjacent to that vertex (and neighbouring plaquettes, although the action on plaquettes is independent of edge orientation). Therefore the vertex transform is only sensitive to the edge-flipping procedure on adjacent edges. Recall that the action of the vertex transform on an adjacent edge $i$, labelled by $g_i$, is

$$A^g_v : g_i = \begin{cases} xg_i & \text{if } i \text{ points away from } v \\ g_ix^{-1} & \text{if } i \text{ points towards } v. \end{cases}$$

The action of the vertex transform on edge $i$ is only sensitive to the orientation of edge $i$ itself and not to the orientation of any other edges. Therefore we only need to consider the effect of flipping edge $i$ itself. We have
\[ \hat{P}_i^{-1} A_i^\varepsilon \hat{P}_i : g_i = \hat{P}_i^{-1} A_i^\varepsilon : g_i^{-1} \]
\[ = \hat{P}_i^{-1} : \begin{cases} g_i x^{-1} & \text{if } i \text{ originally pointed away from (now towards) } v \\ x g_i^{-1} & \text{if } i \text{ originally pointed towards } v \end{cases} \]
\[ = \begin{cases} x g_i & \text{if } i \text{ points away from } v \\ g_i x^{-1} & \text{if } i \text{ points towards } v, \end{cases} \]

from which we see that the action of the vertex transform is preserved under the edge-flipping procedure \( P_i \).

Next we consider the plaquette and blob energy terms. These terms only involve the edges through path elements (the path around the boundary of the plaquette for the plaquette term, and paths between the base-points of surfaces for the blob term). When we construct a path element from the edge elements, each edge contributes the group element it would have if it pointed along the path. That is, if the edge is aligned with the path the edge contributes its group element, but if the edge is anti-aligned with the path then it instead contributes the inverse of its group element. This means that path elements are invariant under the flipping procedure. If an edge with label \( g_i \) already points along the path then it contributes \( g_i \) to the path. If we flip the edge, then we change its label to \( g_i^{-1} \). However then the flipped edge points against the path, so it contributes \((g_i^{-1})^{-1} = g_i\) to the path due to the way that we calculate path elements. Similarly, if the edge originally points against the path then it contributes \( g_i^{-1} \). If we flip it, then we change its label to \( g_i^{-1} \). However the edge then points along the path, so it still contributes \( g_i^{-1} \). The path elements being invariant under the edge flipping procedure in this way means that both the plaquette and blob energy terms are similarly invariant under this procedure.

This leaves us to consider the edge energy terms. The energy term for an edge \( i \) is \( A_i \), which is made up of a sum of edge transforms: \( A_i = \frac{1}{|E|} \sum_{e \in E} A_i^e \). Let us consider how an edge transform is affected by the edge flipping procedures. Recall that the edge transform \( A_i^e \) acts on the edge \( i \) and on neighbouring plaquettes. The support of \( A_i^e \) also includes some of the other edges along the boundaries of the adjacent plaquettes, because the action of \( A_i^e \) on an adjacent plaquette \( p \) is

\[ A_i^e : e_p = \begin{cases} e_p [g(v_0(p) - v_i) \triangleright e^{-1}] & \text{if } i \text{ points along the boundary of } p \\ (g(v_0(p) - v_{i+1}) \triangleright e) e_p & \text{if } i \text{ points against the boundary,} \end{cases} \]

which depends on the path element \( g(v_0(p) - v_i) \) or \( g(v_0(p) - v_{i+1}) \). Flipping the orientation of the edges along these paths will not affect the action on the edge transform, because the path elements are invariant under the flipping operation, as we just saw. However the action of \( A_i^e \) directly depends on the orientation of \( i \) itself. Therefore we need to consider whether \( A_i^e \) is invariant under the edge-flipping operation on edge \( i \). First consider the action of \( A_i^e \) on the edge \( i \) itself. If the edge initially has label \( g_i \), then the edge transform acts as \( A_i^e : g_i = \partial(e) g_i \). Therefore

\[ P_i^{-1} A_i^e P_i : g_i = P_i^{-1} A_i^e : g_i^{-1} \]
\[ = P_i^{-1} : \partial(e) g_i^{-1} \]
\[ = g_i \partial(e)^{-1} \]
\[ = g_i \partial(e)^{-1} g_i^{-1} g_i \]
\[ = \partial(g_i \triangleright e^{-1}) g_i \]
\[ = A_i^{g_i \triangleright e^{-1}} : g_i, \]

where we used the Peiffer condition Equation 4 to write \( \partial(g_i \triangleright e^{-1}) = g_i \partial(e) g_i^{-1} \). We therefore see that, unlike the vertex transforms, the individual edge transforms are not generally invariant under the edge-flipping procedure. However, this does not mean that the edge energy term itself is not invariant. We have, at least for the action on the edge...
On the other hand if we act with $P_i$ the plaquette, and so change which path element defines this action on the adjacent plaquettes. When we flip the edge, we reverse the relative orientation of $i$.

In order to determine whether the action on the plaquettes also satisfies this relationship, consider Equation A1 which suggests that the energy term is invariant under the procedure, even if the individual transforms are not. However we have only shown this for the action on the edges, and it must also hold for the plaquettes in order to obtain the operator relation $P_i^{-1}A_i^eP_i = A_i^{g,i\triangleright e^{-1}}$ (i.e. we must also satisfy the relationship $P_i^{-1}A_i^eP_i : e_p = A_i^{g_i\triangleright e^{-1}} : e_p$ for all plaquettes $p$).

In order to determine whether the action on the plaquettes also satisfies this relationship, consider Equation A1 which defines this action on the adjacent plaquettes. When we flip the edge, we reverse the relative orientation of $i$ and the plaquette, and so change which path element $g(v_0(p) - v_i)$ or $g(v_0(p) - v_i)$ appears in the expression for the action on the plaquette (i.e. we change whether the path travels along the circulation of the plaquette or against it). Furthermore, note that the vertex $v_i$ or $v_{i+1}$ which is the end-point of the path is always the source of the edge $i$, the vertex which the edge points away from. When we flip the edge, we exchange the source and target of the edge (where the target is the vertex the edge points towards), which therefore changes the end-point of the path that appears in the edge transform. These changes are shown in Figure 51. Denoting the original source of the edge $i$ by $s(i)$ and the original target of $i$ by $t(i)$, the action of the edge transform on an adjacent plaquette (without flipping the edge) is

$$A_i^e : e_p = \begin{cases} e_p[g(v_0(p) - s(i)) \triangleright e^{-1}] & i \text{ points along the boundary of } p \\ [g(v_0(p) - t(i)) \triangleright e] e_p & i \text{ points against the boundary.} \end{cases}$$

On the other hand if we act with $P_i^{-1}A_i^eP_i$, we have

$$P_i^{-1}A_i^eP_i : e_p = \begin{cases} [g(v_0(p) - t(i)) \triangleright e] e_p & i \text{ originally points along the boundary of } p \\ e_p[g(v_0(p) - t(i)) \triangleright e^{-1}] & i \text{ originally points against the boundary.} \end{cases}$$

Note that if $\triangleright$ is trivial, we can forget the path elements and so we get $P_i^{-1}A_i^eP_i = A_i^{e^{-1}}$ and the overall edge term will commute with the flipping procedure. On the other hand, if $\triangleright$ is non-trivial we need to consider the path elements more closely. Looking at Figure 51, we see that when $i$ points along the boundary of $p$, we have

$$g(v_0(p) - t(i))g^{-1}(v_0(p) - s(i))^{-1} = g(\text{boundary}(p))^{-1}.$$

Provided that the plaquette satisfies fake-flatness, so that $g(\text{boundary}(p))^{-1} = \partial(e_p)$, this means that

$$g(v_0(p) - t(i)) = g(\text{boundary}(p))^{-1}g(v_0(p) - s(i))g_i$$

and so

$$P_i^{-1}A_i^eP_i : e_p = [g(v_0(p) - t(i)) \triangleright e] e_p$$

$$= [\partial(e_p)g(v_0(p) - s(i))g_i \triangleright e] e_p.$$
FIG. 51. The action of an edge transform applied on edge \( i \) on an adjacent plaquette \( p \) depends on the orientation of that edge. If we flip the edge, then the paths which appear in the action of the edge transform change. In the top line, we consider the case where the edge \( i \) initially points along the plaquette’s orientation. The path which appears in the action of the edge transform is then \( v_0(p) - s(i) \). If we flip the edge and then apply the edge transform, then instead the path \( v_0(p) - s(i) \) from the right side appears in the edge transform. Because we have flipped the edge, the source of the edge after it is flipped is the target of the edge before the flip. Therefore the path \( v_0(p) - s(i) \) that appears in the edge transform after the flip is the same path as \( v_0(p) - t(i) \) (green path on the left side) before the flip. In the lower line, we show the analogous situation when the edge initially points against the plaquette’s orientation.

In a similar way, if \( i \) points against the boundary of the plaquette, we see from Figure 51 that

\[
g(v_0(p) - t(i))g_i^{-1}g(v_0(p) - s_i)^{-1} = g(\text{boundary}(p)).
\]

Again, if the plaquette satisfies fake-flatness, so that \( g(\text{boundary}(p)) = \partial(e_p)^{-1} \), then (following the same steps as before)

\[
P_i^{-1}A_i^e P_i : e_p = e_p g(v_0(p) - t(i)) \triangleright e^{-1}
\]

\[
= e_p [\partial(e_p)^{-1}g(v_0(p) - s_i)g_i] \triangleright e^{-1}
\]

\[
= e_p e_p^{-1} ([g(v_0(p) - s_i)g_i] \triangleright e^{-1}) e_p
\]

\[
= (g(v_0(p) - s_i) \triangleright (g_i \triangleright e)^{-1}) e_p
\]

\[
= A_i^{g \triangleright e^{-1}} ; e_p.
\]

We can therefore see that, for the action on every degree of freedom (and so for the operators themselves), \( P_i^{-1}A_i^e P_i = A_i^{g \triangleright e^{-1}} \) and so \( P_i^{-1}A_i P_i = A_i \). However, note that (unless \( \triangleright \) is trivial) this relies on the plaquettes around the edge satisfying fake-flatness. If fake-flatness is not satisfied near the edge and \( \triangleright \) is not trivial then we cannot say that the edge energy term is invariant under the edge flipping procedure. This indicates that the energy of the edge energy term
is not independent of the supposedly arbitrary dressing of the lattice and so we do not consider the energy of the edge to be well-defined (although note that if we fix a branching structure and never consider changing it, the energy can still be defined). However in the ground state, or more generally in regions where fake-flatness is satisfied, the energy of the edge is well-defined. We are typically interested in the ground state, and states with a few excitations. For such states, the edge energy is only ill-defined near these excitations. This does not affect the topological quantities of the theory, such as the braiding relations or topological charges, because these can be measured far away from the affected regions, where the theory is still fully consistent.

2. Reversing the Orientation of a Plaquette

Having considered the procedure of flipping an edge and inverting the corresponding edge label, and having shown that this is consistent with the various energy terms, we now consider the analogous procedure where we reverse the orientation of a plaquette and invert its label. We denote this operation by \( Q_p \) for a plaquette \( p \). First consider the vertex transform, \( A_v^p \). Apart from the edges, which are unaffected by the orientation of plaquettes, this transform acts only on plaquettes with base-point at \( v \). We have

\[
A_v^p : e_p = \begin{cases} 
  g \triangleright e_p & \text{if } v_0(p) = v \\
  e_p & \text{otherwise,}
\end{cases}
\]

where \( v_0(p) \) is the base-point of plaquette \( p \). The action of the vertex transform on plaquette \( p \) can only be affected by flipping the plaquette \( p \) itself (it is not affected by flipping other plaquettes). We have

\[
Q_p^{-1} A_v^p Q_p : e_p = Q_p^{-1} A_v^p : e_p^{-1}
\]

\[
= Q_p^{-1} : \begin{cases} 
  g \triangleright e_p^{-1} & \text{if } v_0(p) = v \\
  e_p^{-1} & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
  g \triangleright e_p & \text{if } v_0(p) = v \\
  e_p & \text{otherwise}
\end{cases}
\]

\[
= A_v^p : e_p,
\]

from which we see that the vertex transforms (and so the vertex energy term) are invariant under the plaquette flipping procedure.

Next consider the plaquette term. Recall that the plaquette term for a plaquette \( p \) is

\[
B_p = \delta(\partial(e_p)g(\text{boundary}(p)), 1_G).
\]

If we flip the orientation of the plaquette, then we also reverse the orientation of its boundary, so that \( g(\text{boundary}(p)) \) is inverted. Therefore

\[
Q_p^{-1} B_p Q_p = \delta(\partial(e_p^{-1})g(\text{boundary}(p))^{-1}, 1_G)
\]

\[
= \delta(g(\text{boundary}(p))^{-1}, \partial(e_p))
\]

\[
= \delta(1_G, \partial(e_p)g(\text{boundary}(p)))
\]

\[
= B_p,
\]

so that the plaquette energy term is preserved by the plaquette flipping procedure.

The blob energy term is similarly preserved by \( Q_p \). Recall that the blob energy term checks that the blob 2-holonomy is equal to the identity of group \( E \), where the blob 2-holonomy is a product of the plaquette elements around the boundary of the blob. For a blob \( B \), the 2-holonomy is

\[
H_2(B) = \prod_{p \in \text{bd}(B)} g(v_0(B) - v_0(p)) \triangleright e_p^\sigma_p.
\]
where \( \sigma_p \) is 1 or \(-1\) depending on the orientation of the plaquette \( p \), \( v_0(p) \) is the base-point of plaquette \( p \) and \( v_0(B) \) is the base-point of the blob \( B \). Flipping the orientation of plaquette \( p \) swaps \( \sigma_p \) between the two, which cancels with the inverse from the flipping procedure itself (in a similar way to how the contribution of edge elements to a path was invariant under the edge flipping procedure), so that the blob 2-holonomy, and thus the blob energy term, is preserved by the plaquette flipping orientation.

This just leaves the edge energy terms to consider. Recall that the action of the edge transform \( A^e_i \) on an adjacent plaquette \( p \) depends on the relative orientation of the edge \( i \) and the plaquette \( p \). We have

\[
A^e_i : e_p = \begin{cases} 
  e_p \left[ g(v_0(p) - s(i)) \triangleright e^{-1} \right] & i \text{ points along the boundary of } p \\
  [g(v_0(p) - s(i)) \triangleright e] e_p & i \text{ points against the boundary.}
\end{cases}
\]

Swapping the orientation of the plaquette does not change the source of \( i \), unlike the edge flipping procedure. However it does change the relative orientation of the plaquette and the edge. The orientation of the plaquette determines which path between the base-point of the plaquette and source of the edge we use. When the edge aligns with the orientation of the plaquette we use the path \( (v_0(p) - s(i)) \) that also aligns with the orientation of the plaquette, and when the edge is anti-aligned with the plaquette we use the path \( (v_0(p) - s(i)) \) that travels against the orientation of the plaquette. Therefore flipping the plaquette changes which of these paths we use. However when we flip the plaquette, the path that previously was aligned with the plaquette is now anti-aligned with the plaquette. This means that \( (v_0(p) - s(i)) \) before the flip is equal to \( (v_0(p) - s(i)) \) afterwards (and vice-versa), as we show in Figure 52.

These two effects cancel, and therefore flipping the orientation of the plaquette has no net effect on the path that appears in the edge transform (we swap whether we should use the aligned or anti-aligned path, but also swap which one is which). In addition, reversing the orientation of the plaquette changes whether we use pre-multiplication (by \( g(v_0(p) - s(i)) \triangleright e \)) or post-multiplication (by \( g(v_0(p) - s(i)) \triangleright e^{-1} \)). Using this, we see that

\[
Q_p^{-1} A^e_i Q_p : e_p = Q_p^{-1} A^e_i : e_p^{-1}
= Q_p^{-1} : \begin{cases} 
  [g(v_0(p) - s(i)) \triangleright e] e_p^{-1} & i \text{ originally points along the boundary of } p \\
  e_p^{-1} [g(v_0(p) - s(i)) \triangleright e^{-1}] & i \text{ originally points against the boundary of } p
\end{cases}
= \begin{cases} 
  e_p [g(v_0(p) - s(i)) \triangleright e^{-1}] & i \text{ points along the boundary of } p \\
  [g(v_0(p) - s(i)) \triangleright e] e_p & i \text{ points against the boundary}
\end{cases}
= A^e_i : e_p,
\]

so the edge transforms are invariant under the plaquette flipping procedure. This means that every energy term is invariant under this procedure.

**FIG. 52.** The paths used for the action of an edge transform on a plaquette depend on the orientation of the plaquette with respect to the edge. We may therefore think that flipping the orientation of the plaquette will change the path used for the action of the edge transform. However we can see that while we do swap whether the path \( v_0(p) - s(i) \) or \( v_0(p) - s(i) \) is used, we also swap the definition of these paths. If the plaquette is initially aligned with the edge and we swap the orientation of the plaquette, then the path \( v_0(p) - s(i) \) before the flip is the same as the path \( v_0(p) - s(i) \) after the flip (and vice-versa).
3. Moving the Base-Point of a Plaquette

The final procedure to consider is changing the base-point of a plaquette. We denote the procedure that moves the base-point of plaquette $p$ from a vertex $v_1$ to a vertex $v_2$ by $E_p(v_1 \rightarrow v_2)$. In cases where the precise path by which we move the base-point (rather than just the end-points) is important, we will state that this is the case. Under the operation $E_p(v_1 \rightarrow v_2)$, as well as changing the base-point of the plaquette, we must change the plaquette’s label from $e_p$ to $g(v_1 - v_2)^{-1} e_p$, where $g(v_1 - v_2)$ is the group element assigned to the path along which we move the base-point. In the case where $>$ is trivial, the base-point of a plaquette is irrelevant, but in the more general cases the base-point of a plaquette affects the action of all of the energy terms. We first consider a vertex transform. The vertex transform at a vertex $v$ affects any plaquette whose base-point is at the vertex $v$. Furthermore, it affects path elements which start or end at the vertex $v$. This is relevant because the transformation of the plaquette label under $E_p(v_1 \rightarrow v_2)$ depends on the path element $g(v_1 - v_2)$, which is affected by vertex transforms at $v_1$ and $v_2$. This means that the vertex transforms at $v_1$ and $v_2$ might not commute with the procedure for moving the base-point. First consider the vertex transform $A^x_{v_1}$. We have $A^x_{v_1} : e_p = x \triangleright e_p$. On the other hand, we have

\[
E_p^{-1}(v_1 \rightarrow v_2) A^x_{v_1} E_p(v_1 \rightarrow v_2) : e_p = E_p^{-1}(v_1 \rightarrow v_2) A^x_{v_1} : g(v_1 - v_2)^{-1} \triangleright e_p
\]

where in the last line the vertex transform leaves the plaquette element unchanged because the base-point of $p$ is no longer at $v_1$. However the path element for $v_1 - v_2$ has been changed by the action of the vertex transform from its original value of $g(v_1 - v_2)$ to $x g(v_1 - v_2)$. This means that when we move the base-point back along the path, we pick up a factor of $x g(v_1 - v_2)$ acting on the plaquette, rather than just a factor of $g(v_1 - v_2)$. This gives us

\[
E_p^{-1}(v_1 \rightarrow v_2) : g(v_1 - v_2)^{-1} \triangleright e_p = (x g(v_1 - v_2)) \triangleright (g(v_1 - v_2)^{-1} \triangleright e_p) = x \triangleright e_p,
\]

so that

\[
E_p^{-1}(v_1 \rightarrow v_2) A^x_{v_1} E_p(v_1 \rightarrow v_2) : e_p = A^x_{v_1} : e_p.
\]

We therefore see that the vertex transforms (and so the vertex energy term) at $v_1$ are invariant under changing the base-point. Now consider the vertex transforms at $v_2$, the position of the base-point after we move it. If we do not move the base-point, we have $A^x_{v_2} : e_p = e_p$. On the other hand, if we do move the base-point we have:

\[
E_p^{-1}(v_1 \rightarrow v_2) A^x_{v_2} E_p(v_1 \rightarrow v_2) : e_p = E_p^{-1}(v_1 \rightarrow v_2) A^x_{v_2} : g(v_1 \rightarrow v_2)^{-1} \triangleright e_p
\]

where the vertex transform affects the plaquette label because the base-point is at $v_2$ after the action of $E_p(v_1 \rightarrow v_2)$. However the vertex transform also changes the path element $g(v_1 \rightarrow v_2)$ to $g(v_1 \rightarrow v_2) x^{-1}$. Therefore

\[
E_p^{-1}(v_1 \rightarrow v_2) : x \triangleright [g(v_1 \rightarrow v_2)^{-1} \triangleright e_p] = (g(v_1 \rightarrow v_2) x^{-1}) \triangleright (x \triangleright [g(v_1 \rightarrow v_2)^{-1} \triangleright e_p]) = e_p,
\]

from which we see that

\[
E_p^{-1}(v_1 \rightarrow v_2) A^x_{v_2} E_p(v_1 \rightarrow v_2) : e_p = A^x_{v_2} : e_p.
\]

This means that the vertex energy term at $v_2$ is also unaffected by our procedure for changing the base-points of plaquettes, and so all of the vertex transforms are preserved by this procedure.

Next consider the plaquette energy terms. Moving the base-point of a plaquette also affects the boundary of that plaquette, as shown in Figure 53. We see that under $E_p(v_1 \rightarrow v_2)$, the boundary of plaquette $p$ transforms as

\[g(\text{boundary}(p)) \rightarrow g(v_1 \rightarrow v_2)^{-1} g(\text{boundary}(p)) g(v_1 \rightarrow v_2).\]
Therefore the plaquette holonomy \(H_1(p) = \partial(e_p)g(\text{boundary}(p))\) becomes

\[
\partial(g(v_1 \rightarrow v_2)^{-1} \triangleright e_p)g(v_1 \rightarrow v_2)^{-1}g(\text{boundary}(p))g(v_1 \rightarrow v_2) = g(v_1 \rightarrow v_2)^{-1} \partial(e_p)g(v_1 \rightarrow v_2)g(v_1 \rightarrow v_2)^{-1}g(\text{boundary}(p))g(v_1 \rightarrow v_2) = g(v_1 \rightarrow v_2)^{-1} \partial(e_p)g(\text{boundary}(p))g(v_1 \rightarrow v_2),
\]

where we used the Peiffer condition Equation 4 to write \(\partial(g(v_1 \rightarrow v_2)^{-1} \triangleright e_p) = g(v_1 \rightarrow v_2)^{-1} \partial(e_p)g(v_1 \rightarrow v_2)\). We see that the plaquette holonomy is merely conjugated by a path element, which preserves the identity element. Therefore the energy term (which checks if the plaquette holonomy is equal to the identity) is unaffected by the base-point changing procedure.

**FIG. 53.** When we change the base-point of a plaquette, we also change the boundary of that plaquette. In the left image we show a case where the base-point is moved from a vertex \(v_1\) to \(v_2\), along a path \(v_1 - v_2\) (green) that leaves the plaquette. That is, we whisker the plaquette. We can see that the new boundary (grey dotted line) is \((v_1 - v_2)^{-1} \cdot (\text{boundary}) \cdot (v_1 - v_2)\), where (boundary) is the original boundary of the plaquette (black). In the right image, we instead show a case where the base-point of the plaquette is moved along the boundary of the plaquette. In this case, it is still true that the boundary after moving the base-point is \((v_1 - v_2)^{-1} \cdot (\text{boundary}) \cdot (v_1 - v_2)\), although this time there is some cancellation between sections of the original boundary and \((v_1 - v_2)\).

The next energy term to consider is the blob term. This checks that the blob 2-holonomy is the identity, where the blob 2-holonomy is

\[
H_2(B) = \prod_{p \in Bd(B)} g(v_0(B) - v_0(p)) \triangleright e_p^{\sigma_p},
\]

where \(\sigma_p\) depends on the orientation of the plaquette. The contribution of a particular plaquette \(p\) to the blob 2-holonomy is \(g(v_0(B) - v_0(p)) \triangleright e_p^{\sigma_p}\). If we change the base-point of this plaquette from \(v_1\) to \(v_2\) then we must change the label \(e_p\) to \(g(v_1 \rightarrow v_2)^{-1} \triangleright e_p\) due to the effect of \(E_p\), but we must also change the base-point \(v_0(p)\) that appears in the expression \(g(v_0(B) - v_0(p)) \triangleright e_p^{\sigma_p}\) along the same path (the same path so that the resulting surface is the same as the original). This means that when we change the base-point of plaquette \(p\), its contribution to the blob 2-holonomy transforms as

\[
g(v_0(B) - v_0(p)) \triangleright e_p^{\sigma_p} \rightarrow g(v_0(B) - v_2) \triangleright (g(v_1 \rightarrow v_2)^{-1} \triangleright e_p^{\sigma_p})
\]
\[
= (g(v_0(B) - v_1)g(v_1 \rightarrow v_2)) \triangleright (g(v_1 \rightarrow v_2)^{-1} \triangleright e_p^{\sigma_p})
\]
\[
= g(v_0(B) - v_1) \triangleright e_p^{\sigma_p},
\]
so that the contribution of the plaquette to the blob 2-holonomy is unchanged by moving the base-point of the plaquette.

The final energy terms to consider are the edge energy terms. There are several cases to consider. We need to consider edge terms that directly affect the plaquette whose base-point is being moved (i.e. the edges on the boundary of that plaquette), but also any edge transforms along the path on which we move the plaquette’s base-point. First consider the latter kind of edge, which is not on the original boundary of the plaquette, but is on the path along which we move the base-point. If we move the base-point of the plaquette before acting with the edge transform, then the path becomes part of the boundary of the plaquette. That is, the plaquette is whiskered along the path through which we move the base-point. The action of the edge transform, as defined in Equation 9, does not account for the plaquette), but also any edge transforms along the path on which we move the plaquette’s base-point. First consider the case where we move the base-point along the boundary of the plaquette and \( i \) is one of the edges on the path along which we move the base-point. If we move the base-point of the plaquette before acting with the edge transform, then the latter kind of edge, which is not on the original boundary of the plaquette, but is on the path along which we move the base-point, as shown in Figure 55. In this case, the edge transform affects the path element before we move the base-point, as shown in Figure 55. We will therefore not prove this consistency here.

Next we consider the case where we move the base-point along the boundary of the plaquette and \( i \) is one of the edges on the plaquette. Recall that the action of the edge transform \( \mathcal{A}_i^p \) on an adjacent plaquette is

\[
\mathcal{A}_i^p : e_p = \begin{cases} 
    e_p [g(v_0(p) - s(i)) \triangleright e^{-1}] & \text{\( i \) points along the boundary of } p \\
    g(v_0(p) - s(i)) \triangleright e \triangleright e_p & \text{\( i \) points against the boundary.}
\end{cases}
\]

Now consider \( E_p(v_1 \rightarrow v_2)^{-1} \mathcal{A}_i^p E_p(v_1 \rightarrow v_2) \), where \( v_1 \) is the initial base-point of the plaquette \( p \), \( v_0(p) \). Then

\[
E_p(v_1 \rightarrow v_2)^{-1} \mathcal{A}_i^p E_p(v_1 \rightarrow v_2) : e_p = E_p(v_1 \rightarrow v_2)^{-1} \mathcal{A}_i^p : g(v_1 - v_2)^{-1} \triangleright e_p
\]

\[
= E_p(v_1 \rightarrow v_2)^{-1} \mathcal{A}_i^p : \left\{ \begin{array}{ll}
    (g(v_1 - v_2)^{-1} \triangleright e_p) [g(v_2 - s(i)) \triangleright e^{-1}] & \text{\( i \) points along the boundary of } p \\
    [g(v_2 - s(i)) \triangleright e] (g(v_1 - v_2)^{-1} \triangleright e_p) & \text{\( i \) points against the boundary.}
\end{array} \right.
\]

Then, if the edge \( i \) is not on the path \( v_1 \rightarrow v_2 \), so that the label of the path is unaffected by the edge transform, then moving the base-point of the plaquette back gives

\[
E_p(v_1 \rightarrow v_2)^{-1} \mathcal{A}_i^p E_p(v_1 \rightarrow v_2) : e_p
\]

\[
= \begin{cases} 
    g(v_1 - v_2) \triangleright ((g(v_1 - v_2)^{-1} \triangleright e_p) [g(v_2 - s(i)) \triangleright e^{-1}]) & \text{\( i \) points along the boundary of } p \\
    g(v_1 - v_2) \triangleright ([g(v_2 - s(i)) \triangleright e] (g(v_1 - v_2)^{-1} \triangleright e_p)) & \text{\( i \) points against the boundary.}
\end{cases}
\]

\[
= \begin{cases} 
    e_p \left( (g(v_1 - v_2)g(v_2 - s(i)) \triangleright e^{-1}) & \text{\( i \) points along the boundary of } p \\
    ((g(v_1 - v_2)g(v_2 - s(i))) \triangleright e \triangleright e_p & \text{\( i \) points against the boundary.}
\end{cases}
\]

We then need to consider the path elements involved in the expression above. There are various cases, as shown in Figure 54. We see that \( g(v_1 - v_2)g(v_2 - s(i)) = g(v_1 - s(i)) \) if the edge is aligned with the boundary of \( p \) and \( g(v_1 - v_2)g(v_2 - s(i)) = g(v_1 - s(i)) \) if it is anti-aligned. Therefore

\[
E_p(v_1 \rightarrow v_2)^{-1} \mathcal{A}_i^p E_p(v_1 \rightarrow v_2) : e_p = \begin{cases} 
    e_p \left( (g(v_1 - v_2)g(v_2 - s(i)) \triangleright e^{-1}) & \text{\( i \) points along the boundary of } p \\
    ([g(v_1 - v_2)g(v_2 - s(i))] \triangleright e \triangleright e_p & \text{\( i \) points against the boundary.}
\end{cases}
\]

\[
= \mathcal{A}_i^p : e_p,
\]

and so the action of the edge transform is preserved.

Next we consider the case where \( i \) is not only on the plaquette, but is also one of the edges on the path along which we move the base-point, as shown in Figure 55. In this case, the edge transform affects the path element before we
return it to its original position. If the edge $i$ points along $v_1 - v_2$, then the edge transform acts on the path element $g(v_1 - v_2) = g(v_1 - s(i))g, g(t(i) - v_2)$ as

$$\mathcal{A}_i^c: g(v_1 - v_2) = \mathcal{A}_i^c: \begin{cases} g(v_1 - s(i))g, g(t(i) - v_2) & i \text{ points along the boundary of } p \\ g(v_1 - s(i))g, g(t(i) - v_2) & i \text{ points against the boundary of } p \end{cases}$$

$$= \begin{cases} g(v_1 - s(i))\partial(e)g, g(t(i) - v_2) & i \text{ points along the boundary of } p \\ g(v_1 - s(i))\partial(e)g, g(t(i) - v_2) & i \text{ points against the boundary of } p, \end{cases}$$

where we split the path $g(v_1 - v_2)$ into parts and then used the action of the edge transform on the edge $i$. Then we want to write this again in terms of $g(v_1 - v_2)$, so we introduce a factor of the identity in the form of $g(v_1 - s(i))^{-1}g(v_1 - s(i))$ (or the equivalent with $g(v_1 - s(i)))$ to obtain

$$\mathcal{A}_i^c: g(v_1 - v_2) = \mathcal{A}_i^c: \begin{cases} g(v_1 - s(i))\partial(e)g(v_1 - s(i))^{-1}g(v_1 - s(i))g, g(t(i) - v_2) & i \text{ points along the boundary of } p \\ g(v_1 - s(i))\partial(e)g(v_1 - s(i))^{-1}g(v_1 - s(i))g, g(t(i) - v_2) & i \text{ points against the boundary of } p \end{cases}$$

$$= \begin{cases} \partial(g(v_1 - s(i)) \triangleright e)g, g(v_1 - v_2) & i \text{ points along the boundary of } p \\ \partial(g(v_1 - s(i)) \triangleright e)g, g(v_1 - v_2) & i \text{ points against the boundary of } p. \end{cases}$$

(A2)

In addition, when $i$ is on the path $v_1 - v_2$, then the path $v_1 - s(i)$ (or $v_1 - s(i)$) along the boundary is not $(v_1 - v_2) \cdot (v_2 - s(i))$ (or $(v_1 - v_2) \cdot (v_2 - s(i)))$, as we can see from Figure 55. Instead $(v_1 - v_2) \cdot (v_2 - s(i)) = (\text{boundary}(p)) \cdot (v_1 - s(i))$ if $i$ points along the boundary and $(v_1 - v_2) \cdot (v_2 - s(i)) = (\text{boundary}(p))^{-1} \cdot (v_1 - v_2)$ if the edge $i$ points against the boundary. If the plaquette satisfies fake-flatness, then $g(\text{boundary}(p))^{-1} = \partial(e_p)$. We can then use this to find the action of $E_p(v_1 \rightarrow v_2)^{-1} \mathcal{A}_i^c E_p(v_1 \rightarrow v_2)$ on the plaquette label. We have

FIG. 54. The action of the edge transform on a plaquette $p$ depends on various paths, which change when we change the base-point of the plaquette. In this figure, we show two examples. In the left case, the orientation of the edge on which we apply the transform (i) matches the orientation of the plaquette (shown in blue). Because these orientations match, the path which appears in the edge transform is $v_0(p) - s(i)$, which has the same orientation as the plaquette. We see that if the original base-point is $v_1$ and we move it to $v_2$ along the path $v_1 - v_2$, where the edge $i$ is not on this path, then $v_1 - s(i) = (v_1 - v_2) \cdot (v_2 - s(i))$. Similarly, in the right of the figure we show a case where the edge points against the orientation of the plaquette, so that the path $v_0(p) - s(i)$, which has the opposite orientation to the plaquette, appears in the edge transform. In this case $v_1 - s(i) = (v_1 - v_2) \cdot (v_0(p) - s(i))$. Note that there are two other cases, where we flip the orientation of the plaquette in each of these images, where the paths obey similar relations.
FIG. 55. In this figure, we consider the case where the path on which we move the base-point includes the edge \( i \) on which we apply the edge transform. In the left case, the edge \( i \) has the same orientation as the plaquette. In this case, the path \((v_1 - v_2) \cdot (v_2 - s(i))\) includes the entire boundary of the plaquette, so \( (v_1 - v_2) \cdot (v_2 - s(i)) = (\text{boundary}) \cdot (v_1 - s(i)) \). In the right image, we consider the case where the edge and plaquette have opposite orientations. In this case \((v_1 - v_2) \cdot (v_2 - s(i)) = (\text{boundary})^{-1} \cdot (v_1 - s(i))\).

\[
E_p(v_1 \to v_2)^{-1} A_p^i E_p(v_1 \to v_2) : e_p = E_p(v_1 \to v_2)^{-1} A_p^i : g(v_1 - v_2)^{-1} \triangleright e_p
\]

\[
= E_p(v_1 \to v_2)^{-1} : \begin{cases} [g(v_1 - v_2)^{-1} \triangleright e_p] [g(v_2 - s(i)) \triangleright e^{-1}] & \text{i points along the boundary of } p \\ [g(v_2 - s(i)) \triangleright e] [g(v_1 - v_2)^{-1} \triangleright e_p] & \text{i points against the boundary of } p \end{cases}
\]

Then because the edge \( i \) is on the path \( v_1 - v_2 \), so that the label of this path is altered by the edge transform according to Equation A2, moving the base-point of the plaquette back gives

\[
E_p(v_1 \to v_2)^{-1} A_p^i E_p(v_1 \to v_2) : e_p = \begin{cases} (\partial(g(v_1 - s(i)) \triangleright e) g(v_1 - v_2)) \triangleright ((g(v_1 - v_2)^{-1} \triangleright e_p) [g(v_2 - s(i)) \triangleright e^{-1}]) & \text{i points along the boundary of } p \\ (\partial(g(v_1 - s(i)) \triangleright e) g(v_1 - v_2)) \triangleright ((g(v_2 - s(i)) \triangleright e) (g(v_1 - v_2)^{-1} \triangleright e_p)) & \text{i points against the boundary of } p \end{cases}
\]

We can then use the Peiffer condition Equation 5, to remove the factor of \( \partial(g(v_1 - s(i)) \triangleright e) \) in \( [\partial(g(v_1 - s(i)) \triangleright e) g(v_1 - v_2)] \) in favour of conjugation of the entire expression by \( g(v_1 - s(i)) \triangleright e \) (and similar for the other orientation):

\[
E_p(v_1 \to v_2)^{-1} A_p^i E_p(v_1 \to v_2) : e_p
\]

\[
= \begin{cases} [g(v_1 - s(i)) \triangleright e] [g(v_1 - v_2) \triangleright ((g(v_1 - v_2)^{-1} \triangleright e_p) [g(v_2 - s(i)) \triangleright e^{-1}])] [g(v_1 - s(i)) \triangleright e^{-1}] & \text{i points along the boundary of } p \\ [g(v_1 - s(i)) \triangleright e] [g(v_1 - v_2) \triangleright ((g(v_1 - v_2)^{-1} \triangleright e_p) [g(v_2 - s(i)) \triangleright e^{-1}])] [g(v_1 - s(i)) \triangleright e^{-1}] & \text{i points against the boundary of } p \end{cases}
\]

where in the last line we used the group homomorphism property \( g \triangleright (e_1 e_2) = (g \triangleright e_1)(g \triangleright e_2) \) to distribute the action \( g(v_1 - v_2) \triangleright e \) across the terms in the curved brackets. Next we use the relationships

\[
g(v_1 - s(i)) = g(\text{boundary}(p))^{-1} g(v_1 - v_2) g(v_2 - s(i)) = \partial(e_p) g(v_1 - v_2) g(v_2 - s(i))
\]

and
\[ g(v_1 - s(i)) = g(\text{boundary}(p))g(v_1 - v_2)g(v_2 - s(i)) = \partial(e_p^{-1})g(v_1 - v_2)g(v_2 - s(i)) \]

(where in each case the latter equality is obtained from fake-flatness of the plaquette) to write

\[
E_p(v_1 \to v_2)^{-1}\mathcal{A}_i^iE_p(v_1 \to v_2) : e_p
\]

\[
= \begin{cases} 
[g(v_1 - s(i)) \triangleright e] [e_p[[\partial(e_p^{-1})g(v_1 - s(i)) \triangleright e^{-1}]] [g(v_1 - s(i)) \triangleright e^{-1}] & \text{i points along the boundary of } p \\
[g(v_1 - s(i)) \triangleright e] [[\partial(e_p)g(v_1 - s(i)) \triangleright e]e_p] [g(v_1 - s(i)) \triangleright e^{-1}] & \text{i points against the boundary}
\end{cases}
\]

\[
= \begin{cases} 
[g(v_1 - s(i)) \triangleright e] [e_pe_p^{-1}[g(v_1 - s(i)) \triangleright e^{-1}] e_p] [g(v_1 - s(i)) \triangleright e^{-1}] & \text{i points along the boundary of } p \\
[g(v_1 - s(i)) \triangleright e] [e_p[g(v_1 - s(i)) \triangleright e]e_p^{-1}e_p] [g(v_1 - s(i)) \triangleright e^{-1}] & \text{i points against the boundary}
\end{cases}
\]

where we again used the Peiffer condition Equation 5, this time on the factor \(\partial(e_p)\). Then we can simplify the expression to give

\[
E_p(v_1 \to v_2)^{-1}\mathcal{A}_i^iE_p(v_1 \to v_2) : e_p
\]

\[
= \begin{cases} 
[[v_1 - s(i)) \triangleright e][g(v_1 - s(i)) \triangleright e^{-1}]e_p[g(v_1 - s(i)) \triangleright e^{-1}] & \text{i points along the boundary of } p \\
[e_pe_p^{-1}g(v_1 - s(i)) \triangleright e^{-1}]e_p & \text{i points against the boundary}
\end{cases}
\]

\[
= \begin{cases} 
e_p [g(v_1 - s(i)) \triangleright e^{-1}] & \text{i points along the boundary of } p \\
[e_p[g(v_1 - s(i)) \triangleright e]e_p^{-1}e_p] & \text{i points against the boundary}
\end{cases}
= \mathcal{A}_i^i : e_p.
\]

This means that again the action of the edge transform, and therefore the edge energy term, is invariant under moving the base-point. We also need to consider the case where \(i\) points against the path \(v_1 - v_2\). The calculation for this case is very similar. Rather than go through it, we will justify the fact that this case works by noting that the edge energy term is invariant under flipping the edge (from previous calculations in this section). Similarly, the base-point moving procedure is invariant under such a flip. Therefore, we can always flip the edge so that it points along \(v_1 - v_2\) before we apply either the edge transform or move the base-point of the plaquette. This means that if \(i\) points in the wrong way, we can apply \(P_i\) first. Then, using the above argument for the case where \(i\) points along the path, we have

\[
\mathcal{A}_i^i P_i = E_p(v_1 \to v_2)^{-1}\mathcal{A}_i^iE_p(v_1 \to v_2)P_i
\]

which implies that \(\mathcal{A}_i^i = E_p(v_1 \to v_2)^{-1}\mathcal{A}_i^iE_p(v_1 \to v_2)\).

We have therefore shown that each of the procedures for changing the structure of the lattice (flipping the orientations of edges and plaquettes or moving the base-point of a plaquette) is consistent with each of the energy terms. It is important to note that to show this for some energy terms we had to require that fake-flatness was satisfied in the region where the energy operator has support, at least when \(\triangleright\) is non-trivial. This further shows the importance of fake-flatness in ensuring the consistency of the higher-lattice gauge theory model.

4. Use of the Re-Branching Procedures for Other Proofs

In addition to demonstrating the consistency of the higher lattice gauge theory model, these procedures for changing the dressing of the lattice will be useful when considering the ribbon or membrane operators and their commutation relations with the energy terms, as we do in Refs. [63] and [64]. When we consider such commutation relations, the dressing of the lattice determines the action of both the energy terms and the ribbon operators, which means we need to consider several different cases for each commutation relation, depending on the branching structure in the region of the membrane operator and energy term. However, we can avoid this by instead showing that the membrane and ribbon operators are invariant under changing the dressing of the lattice in the same way as the energy terms are. In this case, we can demonstrate the commutation relations for one choice of the branching structure. Then, because the operators are invariant under changes to the dressing, the commutation relation will hold for all choices of branching structure. Let \(X\) be a series of these re-branching operations, and suppose a commutation relation between two operators \(O_1\) and \(O_2\) holds when acting on states that are defined when the lattice has a particular structure.
Then the relation also holds for the branching structure produced by acting with $\hat{X}$ on these states. For example, suppose the operators $\hat{O}_1$ and $\hat{O}_2$ commute for one choice of branching structure. Then for any state $|\psi\rangle$ defined on the original lattice, $\hat{O}_1 \hat{O}_2 |\psi\rangle = \hat{O}_2 \hat{O}_1 |\psi\rangle$. If the operators are consistent with the re-branching then $X^{-1} \hat{O}_1 X = \hat{O}_1$. Therefore

$$X^{-1} \hat{O}_1 X X^{-1} \hat{O}_2 X |\psi\rangle = X^{-1} \hat{O}_2 X X^{-1} \hat{O}_1 X |\psi\rangle \implies X^{-1} \hat{O}_1 \hat{O}_2 X |\psi\rangle = X^{-1} \hat{O}_2 \hat{O}_1 X |\psi\rangle \implies \hat{O}_1 \hat{O}_2 X |\psi\rangle = \hat{O}_2 \hat{O}_1 X |\psi\rangle,$$

so that the commutation relation also holds for the state $X |\psi\rangle$, that is for states defined on the lattice with the altered branching structure.

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