On the Complementarity of F-theory, Orientifolds, and Heterotic Strings

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We study F-theory duals of six dimensional heterotic vacua in extreme regions of moduli space where the heterotic string is very strongly coupled. We demonstrate how to use orientifold limits of these F-theory duals to regain a perturbative string description. As an example, we reproduce the spectrum of a $T^4/\mathbb{Z}_4$ orientifold as an F-theory vacuum with a singular $K3$ fibration. We relate this vacuum to previously studied heterotic $E_8 \times E_8$ compactifications on $K3$.

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1. Introduction

The interest in duality focuses on the possibility of finding an alternative description of strong coupling phenomena by means of a map to a theory which is weakly coupled. Using this idea all known perturbative string theories can be related via duality transformations. One such theory, the type IIB string theory, is of particular interest due to a conjectured non-perturbative $SL(2, \mathbb{Z})$ self-duality \cite{1,2}, which takes the string coupling $g$ to $1/g$. From F-theory, this self-duality can be used to generate a rich class of non-perturbative vacua, exact up to corrections of order the string scale. The basic idea is to consider an artificial twelve dimensional space and compactify it on a Calabi–Yau manifold which is an elliptic fibration \cite{3}. Then, $\tau$ of the torus, with its natural $SL(2, \mathbb{Z})$ action, is identified with the complexified axion-dilaton scalar, $\tau = a + ie^{-\phi}$. In effect, the geometric machinery used in constructing Calabi–Yau manifolds generates the analyticity needed for exact quantum results.

There are some operational limits, however, in manipulating F-theory vacua. Many vacua of interest are represented via extremely degenerate geometries. For these geometries the non-geometric moduli coming from the F-theory 7–brane gauge bundles, e.g. Wilson lines, become hard to identify because the 7–branes themselves lie on very degenerate surfaces (for example complex hyperboloids which have degenerated to intersecting planes). Another issue with F-theory vacua is that they receive stringy $\alpha'$ corrections. It would be interesting to get a handle on these corrections, as this would allow for a more complete picture. Fortunately, F-theory vacua are enmeshed in a web of dualities which can resolve some of these issues. Of particular interest is a chain of dualities which relates some F-theory vacua to (perturbative) heterotic vacua. Sen considered this particular chain of dualities in eight dimensions, relating F-theory on $K3$ to heterotic string theory on $T^2$ \cite{4}. He demonstrated how one could move to a region of parameter space where the base of the $K3$ resembled a IIB orientifold. He used T-duality to relate this orientifold to Type I on $T^2$, then applied Heterotic–Type I duality \cite{5}. Each link in this chain, reliable in its own individual region of parameter space, yields non-perturbative information on the other links; and there is sufficient overlap for us to trust this information.

The chain of dualities connecting F-theory to heterotic vacua becomes much richer when we consider six dimensional compactifications \cite{6,7}. In six dimensions there exists a much broader range of perturbative and non-perturbative behavior over which to test the predictive power of these dualities. For example, for the $Spin(32)/\mathbb{Z}_2$ heterotic string on $K3$ the possibility of “small instantons” \cite{8} yielding extra non-perturbative gauge

\footnote{1 In his original paper, Vafa conjectured a duality between F-theory on an elliptically fibered $K3$ and the heterotic theory on $T^2$ \cite{3}.}
groups arises. These are easier understood in terms of D5-branes in the context of dual Type I compactifications such as $\mathbb{Z}_2$ orientifolds, the so called GP-models [9]. In a careful study [10], Sen showed how these orientifolds are in fact T-dual to limits of F-theory vacua involving an elliptic Calabi–Yau, $\mathcal{M}_1$, with base $\mathbb{P}^1 \times \mathbb{P}^1$. Type I/heterotic S-duality allows us to use this map to relate a $\text{Spin}(32)/\mathbb{Z}_2$ compactification on $K3$ to F-theory on $\mathcal{M}_1$. This duality is of particular interest because, as we will demonstrate below, F-theory treats both the perturbative and non-perturbative gauge enhancements of its dual heterotic compactification on an equal footing.

Let us examine the duality between F-theory on $\mathcal{M}_1$ and the heterotic string on $K3$ in terms of the eight dimensional duality mentioned previously. If we think of $\mathcal{M}_1$ as a $K3$ surface fibered over $\mathbb{P}^1$, then this eight dimensional duality maps F-theory on such a $K3$ fibration to a heterotic compactification involving a $T^2$ fibered over the same $\mathbb{P}^1$. Here the geometry of the $K3$ fibers encode the perturbative gauge group. There are two ways to pick the base $\mathbb{P}^1$ for $\mathcal{M}_1$, implying the existence of two different $K3$ fibrations. These two $K3$ fibrations lead to two different descriptions of the heterotic string on $K3$, with different perturbative gauge groups. From the point of view of F-theory, we can think of each of these strings in terms of a D3-brane wrapping one or the other of the $\mathbb{P}^1$ embedded in $\mathcal{M}_1$. Each will become weakly coupled when the $\mathbb{P}^1$ that it wraps becomes very small relative to the other $\mathbb{P}^1$. In this way we recover the S–duality of Duff, Minasian and Witten [11]. Of course, the F-theory limit is best understood when both $\mathbb{P}^1$s are large. This corresponds to treating both perturbative and non-perturbative gauge groups on an equal footing in either of the heterotic dual representations.

What happens when both $\mathbb{P}^1$s are small? The theory receives $\alpha'$ corrections large enough that F-theory is no longer valid. But the heterotic dual is not weakly coupled in any description! Fortunately, in certain regions of the complex structure moduli space where the $\tau$ parameter does not vary too much, we can think of the F-theory compactification in terms of orientifolds and get a good perturbative description. In addition, the orientifold description gives us a better handle on those hypermultiplet moduli which are not, strictly speaking, directly encoded in the Calabi–Yau 3-fold description of the F-theory vacuum (i.e., Wilson lines on the 7–branes) [3]. Here, we see the advertised complementarity between F-theory, orientifolds, and the heterotic string come in to play.

So far, we have only used this complementarity to examine possible perturbative and non-perturbative gauge groups. The heterotic theory exhibits another interesting class of non-perturbative effects when compactified on $K3$. If we look at $E_8 \times E_8$ heterotic vacua

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2 For a recent discussion of the hypermultiplet moduli space in the context of F-theory, see [12].
with instanton numbers \( (12 - n, 12 + n) \) there exists a strong coupling singularity \([11]\). Morrison and Vafa explained \([6]\) how these heterotic singularities correspond, in their F-theory dual vacua, to the collapse of an exceptional divisor in the F-theory base \([8]\). We shall only concern ourselves with such divisors of self-intersection number \(-1\) or \(-2\), but for a broader class of F-theory vacua.

Using the results of Witten \([13]\) (see also \([6,7]\)), we can interpret the shrunken divisors as follows. The collapse of a divisor with self-intersection \(-1\) leads to a phase transition involving a non-critical tensionless string carrying a rank eight current algebra. (This string is a D3-brane wrapped around the divisor in question). This leads to the possibility of a transition to a “Higgs” phase with one less six dimensional tensor multiplet and 29 extra hypermultiplets. The exceptional divisor becomes a generic point on the base, with no new singularity. If we take a generic F-theory 3-fold compactification, we can blow up any point on its base, as long as we preserve the Calabi–Yau condition. This transition was first understood by Ganor and Hanany \([14]\) (see also \([15]\)) in terms of the heterotic string theory as a small \(E_8 \times E_8\) heterotic instanton which shrinks and opens up the possibility, via a phase transition, of a ”Coulomb” branch with an extra \(\mathcal{N} = 1\) tensor multiplet associated with an M-theory 5-brane.

The collapse of a divisor with intersection number \(-2\) leads to very different physics \([6,15,13,7]\). First of all, the divisor is blown down to an \(A_1\) singularity in the base. There is no possibility of a phase transition; this is a true boundary in the moduli space of the relevant six dimensional theory. Second, because the local geometry of an \(A_1\) singularity is hyper-Kähler, the fibration will be trivial in a neighborhood of the collapsing divisor. This means that locally the theory behaves like IIB at an \(A_1\) singularity. Thus the D3-brane wrapped on this divisor yields a tensionless non-critical string with twice the supersymmetry. It will couple to an \(\mathcal{N} = 2\) six dimensional tensor multiplet with five scalars. In terms of the low energy six dimensional theory, this means that we need to tune both an \(\mathcal{N} = 1\) hypermultiplet and an \(\mathcal{N} = 1\) tensor multiplet to reach this boundary in moduli space. In the dual heterotic theory, there are two ways to reach this type of boundary in moduli space. As introduced above, we can understand this collapse as a strong coupling singularity. Alternatively, when the heterotic theory has M-theory 5-branes (the heterotic theory is no longer truly perturbative), the same type of boundary will be reached when 5-branes come together \([14,15]\). Neither of these scenarios is well understood in a perturbative string expansion, though, as there are large values of the string coupling involved. A new weakly coupled stringy description is necessary.

\[3\] For perturbative heterotic vacua, a total of 24 instantons is required to compensate for the curvature of \(K3\)
One way to get such a weakly coupled stringy description of the physics near a boundary in moduli space is to use the complementarity between F-theory and orientifolds. By this we mean that we will find an orientifold which describes a limit of F-theory on an elliptic Calabi–Yau threefold. In this limit, exceptional divisors of self-intersection number -2 in the base will shrink to zero size. This paper will study the limit above for a Calabi–Yau space, $\mathcal{M}_2$. We will denote the (singular) manifold, in which the exceptional divisors with self-intersection number -2 have collapsed, by $\overline{\mathcal{M}}_2$. This latter model will be shown to be the F-theory vacuum corresponding to a $T^4/\mathbb{Z}_4$ orientifold of Type IIB [16] [17]. In section 2, we will review the properties of this orientifold, as well as the $\mathbb{Z}_2$ orientifold. We will also focus on the T–duality which takes the general class of orientifolds of this type from a configuration with D9–branes and D5–branes (appropriate for Heterotic–Type I duality) to one with D7–branes (such as are found in F-theory). This will set the stage for section 3, where we will review the analysis of how a GP $\mathbb{Z}_2$ orientifold can be seen as a limit of $\mathcal{M}_1$ [10]. We will then show in section 4 how this leads to a natural construction of the space $\overline{\mathcal{M}}_2$, corresponding to the $T^4/\mathbb{Z}_4$ orientifold, and how the moduli spaces match.

Having found the proper match between F-theory compactification and orientifold, we will make use of the complementarity of these two descriptions. The boundaries in the $\mathcal{M}_2$ moduli space involving $\mathcal{N} = 2$ tensionless strings will be made evident in the weakly coupled orientifold string description. In section 5 we will return to the notion of using alternate $K3$ fibrations of the F-theory threefold, $\mathcal{M}_2$, to find weakly coupled stringy descriptions of different regions of its moduli space. In this manner, we will demonstrate how to recover a new description of $\overline{\mathcal{M}}_2$, such that its duality with the heterotic $E_8 \times E_8$ theory on $K3$ with instanton embedding (10,10) becomes evident.

2. Description of Orientifolds

We are interested in studying IIB orientifolds, such as those described in [16], [18], in terms of F-theory on elliptically fibered Calabi–Yau 3–folds. More specifically, we will extend the F-theory analysis [10] on the GP $\mathbb{Z}_2$ orientifolds [3] to the $\mathbb{Z}_4^1$ family of orientifolds of [17]. These orientifolds are the simplest generalization of the GP $\mathbb{Z}_2$ orientifolds which exhibit new behavior. In the 7–brane picture, which is most useful for connecting with the F-theory formalism, this new behavior is manifested in two ways. First, these $\mathbb{Z}_4$ orientifolds contain not only the $O\tau_2$ planes of the $\mathbb{Z}_2$ models ($O\tau_2$ planes are orientifold 7–planes, with a deficit angle $\pi$), but also orientifold points (actually orientifold 5–planes) and $O\tau_4$ planes ($O\tau_2$ planes with further identifications due to the presence of orientifold points on their world-volume). At strong coupling, the $O\tau_2$ planes can be resolved into
two 7–branes with a coupling dependent separation \[4]. We will be interested in how this analysis extends to the more general \(O7_4\) planes and orientifold points found in the \(\mathbb{Z}_4\) models. Second, these \(\mathbb{Z}_4\) models have closed string spectra which contain extra chiral tensor multiplets \[5\]. This complicates their relation to potential dual heterotic theories (which perturbatively contain no such extra tensors) and has been the basis for some interesting predictions \[17\]. One aim of this work is to put these predictions on a firmer footing.

### 2.1. The Transition to 7–branes

As mentioned earlier, in order to connect Type I orientifolds with D9-branes and D5–branes \[4,16,21,18\] to F-theory on Calabi–Yau 3–folds, we will look at dual models which contain only D7–branes, natural objects in F-theory. We do this by T–dualizing along the 67-plane \[6\], which leaves us with D7–branes along the 01234567 directions (call them 7–branes), and D7-branes along the 01234589 directions (we denote them 7’–branes). In addition, there will be orientifold 7–planes parallel to both the 7 and 7’–branes. It is useful to consider what happens when we make the transition from a 5-9 picture to a 7-7’ picture. Both pictures describe identical six dimensional \(\mathcal{N} = 1\) theories spanning the 012345 directions. Physically equivalent excitations, however, arise from quite different sources.

The map between 5-9 degrees of freedom and 7-7’ degrees of freedom is very simple for the open string spectrum. This map is just inherited from the T–duality transformations of the original \(T^4\). That is, the D5–branes become 7–branes and the D9–branes become 7’–branes. The coordinates of the D5–branes on the original \(T^4\), which form complete \(D = 6, \mathcal{N} = 1\) hypermultiplets, now split into two scalars each from the 89 coordinate of the 7–brane, and two scalars each from the Wilson lines of the 7–brane gauge theory around the 6 and 7 directions. There is further enhancement to matrix valued scalars when several 7–branes sit atop each other.

A minor subtlety arises when the orientifold projection of the underlying \(T^4\) includes elements which project out 1-cycles. Naively this could preclude Wilson lines on the 7–branes. These remain, however, because the same element which removes 1-cycles has a non-trivial action on the Chan-Paton gauge bundle living on the 7–branes. As one

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4 For a more general analysis of \(O7_2\) planes in F-theory see also \[19\].

5 In fact a subset of the \(\mathbb{Z}_4^A\) models \[16\] (as well as a subset of the GP models) were first discovered by Bianchi and Sagnotti \[20\] as early examples of models with extra chiral tensor multiplets.

6 By T-duality, we mean the T-duality inherited from the covering space \(T^4\) which in the smooth \(K3\) should correspond to T-duality as defined in ref. \[22\].
might expect, the combined geometric and gauge action of the orientifold group on the 7–brane Wilson lines yields exactly the same projection as we had for the 67 coordinates of the original D5–branes. Schematically, we can attribute the existence of this continuous, though disconnected, moduli space of flat connections on the 7–branes to the fact that they are pierced by orientifold planes. This allows for non-trivial monodromies of the 7–brane gauge bundle about the puncture points. Similarly, Wilson lines from the D9–branes become hypermultiplets for the 7’–branes, parameterizing their position in the 67 directions and their Wilson lines along the 89 directions.

The 67 T–duality operation is not nearly so simple for the closed string sector of our theory. For the untwisted sector there is again a legacy from the original $T^4$. T–duality mixes the components of the metric and various forms of the Type IIB theory aligned partly or fully along the compact directions. More subtle is understanding the fate of the twisted sectors after T–duality, as these are added only after the orientifold projections. There is strong evidence from anomaly analysis that twisted closed string modes localized at one fixed point undergo a discrete Fourier transform. They are mapped to a linear combination of modes localized at separate fixed points in the 67 plane. At this point, one might be tempted to forget about the 5-9 picture and just derive the complete 7-7’ spectrum by applying the appropriate orientifold projection to the Type IIB string on $T^4$. However, it is much simpler to understand the interplay of Higgsing patterns with blow-up modes of orbifold points in the 5-9 picture. So we will need to keep the relationship between the 5-9 picture and the 7-7’ picture in the back of our mind.

2.2. The Transformed GP Models

We will use the GP models to illustrate the relationship between the 5-9 picture and the 7-7’ picture. Here we start with IIB string theory on $T^4$. The 5-9 picture orientifold group is:

$$\{1, \Omega, \Omega R_{6789}, R_{6789}\}, \quad (2.1)$$

7 One might wonder why there are Wilson lines for the original D9–branes on a $T^4$ orientifold. Again these arise from the action of the orientifold on the Chan-Paton gauge bundle. The schematic picture for this moduli space for the gauge bundle is quite different from the 7-7’ case. In the 5-9 picture the presence of abelian instantons at the core of orientifold points indicates that we are discussing a moduli space of curved connections. There exist continuous moduli describing how abelian instantons are embedded relative to each other in the 9-brane gauge group (see ref. 23).

8 See for example Berkooz et. al. 23.
where $R$ is the reflection along the subscripted axes. As mentioned previously, this represents Type I strings on a $K3$ surface in the $T^4/\mathbb{Z}_2$ orbifold limit. The closed string sector can be easily computed. The graviton yields the $D = 6$ graviton and ten scalars describing the shape and size of the original $T^4$. It also yields three scalars from the singular two–cycles at each of the sixteen fixed points. The Ramond–Ramond two–form yields one $D = 6$ anti–self–dual and one self–dual tensor. The former joins with the graviton to form the bosonic part of the gravity multiplet while the later fills out a chiral $\mathcal{N} = 1$ tensor multiplet with the dilaton. The two-form also yields six scalars representing fluxes on the original $T^4$, and a scalar for each singular two–cycle. In total, the $D = 6$ theory thus has a gravity multiplet along with a tensor multiplet and 20 hypermultiplets (the 80 moduli for $K3$). Because of the curvature of $K3$, we also expect the 9–brane gauge bundle to have 24 instantons. Eight of these are realized as independent D5–brane units with $SU(2)$ gauge group, while the other 16 consist of abelian instantons located at the core of each $\mathbb{Z}_2$ fixed point [9,23] (see also ref. [25]).

Now we T–dualize along the 67 directions to get a 7–7’ orientifold. The new orientifold group is

$$\{1, \Omega(-1)^{F_L} R_{67}, \Omega(-1)^{F_L} R_{89}, R_{6789}\},$$

where $F_L$ is the left moving fermion number. Calculating the closed string spectrum is now slightly more complicated. The orientifold group in (2.2) can no longer be factorized into a subgroup acting exclusively on the worldsheet times a subgroup acting on the target space as in (2.1). The $R_{6789}$ element still gives us a $T^4/\mathbb{Z}_2$ orbifold limit of $K3$ as the underlying geometry. But now the $(-1)^{F_L} R_{67}$ element will freeze four of the $T^4$ metric moduli plus one modulus for each singular two–cycle, allowing us to factor $T^4$ as $T^2 \times T^2$. The NS–NS and R–R two–forms are odd under $\Omega(-1)^{F_L}$ but since four of the six two–cycles from $T^4$ and all sixteen of the singular two–cycles are odd under $R_{67}$, each of the two–forms will still yield $4 + 16$ scalars. Finally, the R–R zero–form and self–dual four–form each contribute one scalar.

Summarizing, in the untwisted closed string sector we get three hypermultiplets, each of which has two scalars from the $K3$ metric and two more scalars from the R–R and NS–NS two–forms. There is also one hypermultiplet with a scalar from the NS–NS two–form and one from each of the R–R zero, two, and four–form respectively. This last hypermultiplet is the “universal” hypermultiplet which contains the volume of the $K3$ (coming from the R–R four–form). Each of the sixteen twisted sectors will contribute one hypermultiplet with two geometric scalars from the metric, and one scalar each from the R–R and NS–NS two–forms, so–called “theta angles”. Note that this means that the fixed points of $T^4/\mathbb{Z}_2$ can not be completely resolved.

We now see how the hypermultiplet moduli space of this orientifold can be matched to the complex deformations of F-theory on a particular Calabi–Yau 3–fold, $\mathcal{M}_1$. Except
for the “universal” hypermultiplet, a special case in F-theory (and M-theory), every hypermultiplet can be split into a pair of complex scalars. One complex scalar comes from either geometric deformations of the orientifold or from a 7–brane position, and the other scalar from two–form fluxes or 7–brane Wilson lines. The first half of each hypermultiplet can be put in one-to-one correspondence with the 243 complex deformations of $M_1$. If we think of F-theory on $M_1$ as a limit of IIA as in $\text{[6]}$, then the second half of every hypermultiplet comes from RR-scalars. The origin of these scalars is less clear in F-theory. Our expectation is that they should all originate from Wilson lines in those portions of moduli space where all 7–branes lie on non-degenerate surfaces. Unfortunately, when matching with the $T^4/\mathbb{Z}_2$ orientifolds, we encounter just such degenerating surfaces. $O7^2$ planes crossing at right angles can be thought of as a degenerate complex hyperboloid; the two–form fluxes at their intersection are most likely associated with a Wilson line around this collapsed one–cycle of the hyperboloid.

Now that we understand the closed string sectors of the 5–9 and 7–7’ picture for the $\mathbb{Z}_2$ orientifolds, let us take a closer look at the open string sector. In the 5–9 picture when two 5–branes coincide, the gauge group is enhanced from $USp(2) \times USp(2)$ to $USp(4)$ $\text{[7]}$. How does this work for 7–branes? The analog of placing 5–branes together is again to have 7–branes coincide, but also to match their Wilson lines along the 67 directions. When the two $USp(2)$ Wilson lines are equal, the overall Wilson line is proportional to the invariant anti-symmetric matrix of $USp(4)$ and so does not break down the group. Thus as expected, the combination of equal 89 coordinates and 67 Wilson lines for 7–branes gives the same gauge group as for overlapping 5–branes in the 5-9 picture. This can be generalized for $USp(2)^n$ getting enhanced to $USp(2n)$ with $n$ overlapping 7–branes.

Let us use this analysis for a further understanding of the 7–brane gauge enhancement patterns. When $n$ 5–branes coincide with a $\mathbb{Z}_2$ orientifold point whose three metric blow-up modes are set to zero, there is a gauge group enhancement to $SU(2n)$. In the 7–7’ picture, this corresponds to placing 7–branes on an $O7_2$ plane. Note that each of these “planes” has 4 fixed points on its world–volume. For specific values of the Wilson lines, we can tune the metric blow-up mode doublets, along with one combination of the two–form “theta angles”, at each of the fixed points to get $SU(2n)$. There will be four values of the Wilson lines were we can do this, each of which needs a different linear combination of

\[9\) We thank C. Vafa for pointing this out to us.

\[10\) By 5–branes we mean a pair of D5–branes glued together by the orientifold group action. In general, we use the notation n-brane for an irreducible collection of Dn–branes, each of which is labeled by exactly one Chan-Paton factor.

\[11\) Our conventions for Sp groups are as follows. We will denote the rank n gauge algebra $Sp(n)$, but refer to the gauge group as $USp(2n)$ since we realize it in terms of $2n \times 2n$ unitary matrices.
the $SU(2)_R$ triplet of twisted sector fields at the four fixed points to be set to zero. For a comparison of this orientifold with F-theory, we would like to set the Wilson lines and two-form fluxes to zero, since they are not described directly as complex deformations in F-theory. This is not entirely possible, however, as the orientifold typically has non-zero two-form fluxes.

In the 5-9 picture, the GP $\mathbb{Z}_2$ models are Type I compactifications with no vector structure [23,26]. As such they have discrete NS–NS two–form “theta angles” [27,28] which come from the fact that the NS–NS two–form takes half-integer values in $H_2(K3, \mathbb{R})$. In fact, it is natural to expect these “theta angles” to have non-zero value. We can think of the GP $\mathbb{Z}_2$ orientifold in terms of an $\Omega$ projection on the $T^4/\mathbb{Z}_2$ orbifold compactification of IIB string theory. But we know what the IIB orbifold “theta angles” are [29]. They are precisely the values needed to obstruct “vector structure” in the Type I compactification. In the 7-7’ picture, these “theta angles” are no longer constrained to take discrete values, yet their background value still starts out non-zero. This makes it possible to find special gauge group enhancements in the moduli space of $\mathcal{M}_1$ which are not immediately obvious in the orientifold picture [10]. Further details of the duality of GP $\mathbb{Z}_2$ models with F-theory on the Calabi–Yau 3–fold $\mathcal{M}_1$ will be left to sections 3 and 5. We will now discuss the $\mathbb{Z}_4$ orientifold as it is the focus of our exploration of the relationship between F-theory and orientifolds.

2.3. $\mathbb{Z}_4$ Basics

In the previous section we used the $\mathbb{Z}_2$ orientifold to develop the tools necessary for understanding IIB orientifolds with 7-7’ branes. In addition, we understand how data about their gauge enhancement patterns can be extracted from previously known result for their 5-9 duals. d. For the $\mathbb{Z}_4$ orientifold, the orientifold group, after T–duality, is the product:

$$\left\{1, \Omega(-1)^{F_L} R_{67}, \Omega(-1)^{F_L} R_{89}, R_{6789}\right\} \times \{1, \alpha_4\}. \quad \text{(2.3)}$$

Here $\alpha_4$ has the following action:

$$\alpha_4 : \begin{cases} z_1 & = X^6 + iX^7 \to e^{\frac{\pi i}{4}} z_1, \\ z_2 & = X^8 + iX^9 \to e^{-\frac{\pi i}{4}} z_2. \end{cases} \quad \text{(2.4)}$$

This form illustrates how this orientifold is related to the $\mathbb{Z}_2$ orientifold. We take the $\mathbb{Z}_2$ orientifold, fix $\tau = i$ for the 67 and 89 tori, and then gauge a further $\mathbb{Z}_2$ [3].

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12 Here, $\alpha_4$ can be thought of as a $\mathbb{Z}_2$ action on this orientifold with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ structure. Of course, $\alpha_4$ has a $\mathbb{Z}_4$ action on the covering space. That is why we end up with an orientifold with a $\mathbb{Z}_2 \times \mathbb{Z}_4$ structure, where the $\mathbb{Z}_2$ comes from $\Omega R_{67}$.
If we look at one torus, say along the 67 directions, in the $\mathbb{Z}_2$ orientifold it will be wrapped by the 89 $O7_2$ planes and 7–branes, but will also have localized on it four $O7_2$ planes and eight 7′–branes. The $z_1$ coordinates of the $O7_2$ planes are:

$$z_1 = \frac{1}{2}, \quad \frac{i}{2}, \quad 0, \quad \frac{(i + 1)}{2}$$

(2.5)

Under the action of $\alpha_4$ the first two $O7_2$ planes are identified and the last two become $O7_4$ planes. The eight 7′–branes pair up to become four 7′–branes. Similarly, $\alpha_4$ leaves the other torus with one $O7_2$ plane, two $O7_4$ planes and four 7–branes.

Where an $O7_2$ plane intersects an $O7_2$ plane or an $O7_4$ plane we have an $A_1$ singularity. Furthermore, we also get an $A_3$ singularity where an $O7_4$ plane intersects an $O7_4$ plane. Note that because $\alpha_4$ acts on both $z_1$ and $z_2$ at the same time, an $O7_2$ plane will intersect another $O7_2$ plane at two distinct points. Thus we have six $A_1$ and four $A_3$ singularities as expected for the $T^4/\mathbb{Z}_4$ orbifold limit of $K3$.

From Gauss’ Law, it is clear that the $O7_4$ planes will have half the charge of the $O7_2$ planes. Thus in each torus we can cancel all the charge locally by placing one 7–brane (for each torus) at each of the $O7_4$ planes, and the remaining two at the $O7_2$ plane. In this situation there will be no dilaton gradient. Therefore we expect that the orientifold picture will be exact and any corrections that we get from an F-theory analysis should be trivial at this point in parameter space. Perturbations of the physics near the $O7_2$ plane has already been carried out in the context of the $\mathbb{Z}_2$ orientifold [10]. The key, then, is to understand what happens near an $O7_4$ plane. Before we start analyzing the situation using F-theory, we will first go over the spectra predicted from the orientifold analysis [16].

In the bulk a 7–brane carries on it an $SU(2)$ gauge group. It intersects with each of the four perpendicular 7′–branes in two distinct points where 7-7’ strings in a $(2, 2)$ representation live. Thus, as expected, the low energy excitations on the brane correspond to an $SU(2)$ gauge group with 16 fields in the fundamental representation. When $n$ 7–branes coalesce, we get an $USp(2n)$ gauge group with 16 fundamentals and one antisymmetric. At an $O7_2$ plane, this gauge group will get enhanced to $SU(2n)$ with 16 fundamental and two antisymmetric. Finally, at an $O7_4$ plane, $n$ 7–branes will give an $SU(2n) \times SU(2n)$ gauge group with eight fundamentals, one antisymmetric in each $SU(2n)$ as well as one $(2n, 2n)$ representation. To get a better handle on how this last pattern of gauge enhancement is affected by twisted closed string sectors, let us first look at the dual 5-9 picture.

If we take a 5–brane $USp(2n)$ unit and place it at a $\mathbb{Z}_4$ orientifold point, we will get an $SU(2n) \times SU(2n)$ gauge group [16]. By blowing up the orientifold point and solving the D-flatness conditions, we can higgs to a variety of gauge groups [30]. The $\mathbb{Z}_4$ orientifold
Figure 1. The gauge enhancement patterns for the $A_3$ orientifold point. The $y_i$ are represented by dots, and the "frozen" two-cycle $|y_2 - y_1|$ is represented by a line. The $\times$ represents the position of a D5–brane, the circles are drawn to indicate overlapping objects.

point has two blow-up modes. It consist of an $A_3$ singularity with one "frozen" cycle [31]. If we take the standard form [32,33] for the $A_3$ metric we have:

$$ds^2 = V^{-1}(dt - A \cdot dy)^2 + V dy \cdot dy,$$

where

$$V = \sum_{i=0}^{3} \frac{1}{|y - y_i|} \quad \text{and} \quad \nabla V = \nabla \times A. \quad (2.6)$$

We now set $y_1 = y_2$, leaving us with a pair of two–cycles whose sizes are determined by $|y_1 - y_0|$ and $|y_3 - y_2|$. Scenarios for the various intermediate gauge groups are shown in fig. 1.

To summarize, when $y_0$ and $y_3$ have generic values (case A) we are left with a special $\mathbb{Z}_2'$ orientifold point [16,31]. When the $USp(2n)$ unit is placed on this $\mathbb{Z}_2'$ point (case B)
we get a $USp(2n) \times USp(2n)$ gauge group. By setting one or the other of the blow-up modes to zero (cases C and C’), we can get $SU(2n) \times USp(2n)$ and $USp(2n) \times SU(2n)$ respectively. We can also tune the two blow-up modes ($y_3 = y_0$) so as to create a regular GP $\mathbb{Z}_2$ orientifold point separate from the $\mathbb{Z}_2'$ point. Placing 5–branes there will give us an $SU(2n)$ gauge group (case D). In the most singular limit, the original orientifold point, we recover the full $SU(2n) \times SU(2n)$ gauge group.

The twisted sectors of a $\mathbb{Z}_4$ orientifold point also include an $\mathcal{N} = 1$ tensor multiplet [16]. This type of multiplet contains a single scalar whose vev, we expect, will control the relative couplings of any product gauge group associated with placing 5-branes on the singularity. Combined with the tensor scalars from the other $\mathbb{Z}_4$ orientifold points, it will also control the relative couplings for the 9-brane gauge group.

Using the tools from section 2.3 we can now switch to the 7-7’ picture. The 5-9 analysis above implies that a cluster of $n$ 7–branes will enhance its world-volume gauge group from $USp(2n)$ to $SU(2n) \times SU(2n)$ when it is placed on an $O7_4$ plane, provided the correct linear combination of blow-up modes has been set to zero. Turning on the appropriate modes will then yield the subgroups listed above.

In the 7-7’ picture, each $\mathcal{N} = 1$ tensor multiplet gets its scalar component from metric deformations. To be more precise, this scalar component will control the area of a two-cycle. Naively, shrinking this two-cycle should lead to the appearance of a light non-critical $\mathcal{N} = 1$ string. However, this can not be the case as the tensor scalar vevs in the orientifold are naturally zero. The conformal field theory describing the orientifold does not exhibit any of the singular behaviour associated with non-critical strings. In fact, we can only get this type of behaviour if we tune “theta angles” to zero. This implies that the $\mathcal{N} = 1$ tensor multiplet, along with one of the hypermultiplets which modifies the “theta angles”, couples to an $\mathcal{N} = 2$ non-critical string. Because this happens for any value of the string coupling, we expect that in the F-theory description of the $\mathbb{Z}_4$ orientifold, the relevant two-cycles will have self-intersection number -2 (see ref. [13]).

3. F-theory Interpretation of the $T^4/\mathbb{Z}_2$ Orientifold

Let us start by briefly reviewing the F-theory realization of the GP $\mathbb{Z}_2$ orientifold. (For more details, we refer to [10].) Rather than starting from the orientifold and trying to derive the relevant F-theory compactification we will simply give the Calabi–Yau manifold, $\mathcal{M}_1$, on which F-theory is compactified and point out the correspondence with the $\mathbb{Z}_2$ model. This review will be useful as we go to the F-theory vacuum relevant for the $\mathbb{Z}_4$ orientifold as it is realized in terms of a $\mathbb{Z}_2$ orbifold of F-theory on $\mathcal{M}_1$. 

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Consider the following elliptically fibered Calabi–Yau hypersurface, $\mathcal{M}_1$, with base $\mathbb{P}^1 \times \mathbb{P}^1$,

$$Y^2 = X^3 + X f(z, w) + g(z, w)$$

(3.1)

where $(z, w)$ are the coordinates on the base, and $f(z, w)$ and $g(z, w)$ are of bi-degree 8 and 12 respectively [6]; the above equation defines a torus for fixed $(z, w)$. The fiber has singularities where the discriminant, $\Delta = 4f^3 + 27g^2$, vanishes. Since we want to compare this F-theory vacuum with the $\mathbb{Z}_2$ orientifold, which is a type IIB compactification, we are interested in finding configurations for which the string coupling is constant. In F-theory this coupling, complexified as $a + ie^{-\phi}$, is identified with the $\tau$ parameter of the torus in eq. (3.1). By studying the modular invariant $j(\tau)$-function we can deduce properties of $\tau$ and hence of the coupling constant. In terms of $f$ and $g$ we have that $j(\tau)$ can be expressed as [10],

$$j(\tau) = \frac{4 \cdot (24f)^3}{4f^3 + 27g^2}.$$  

(3.2)

Thus, constant $j(\tau)$ implies that $f^3/g^2 = \text{const}$. We can satisfy this condition by choosing [4],

$$f = \left(\prod_{i=1}^{4}(z - z_i)(w - w_i)\right)^2$$

$$g = \left(\prod_{i=1}^{4}(z - z_i)(w - w_i)\right)^3$$

(3.3)

Let us compare this with the orientifold picture. This choice of $f$ and $g$ corresponds to a configuration clustering the 24 D7–branes in groups of six around the four points $z_i$ (and similarly for the D7–branes in the $w$-plane). If we count two D7–branes in each of the clusters as making up one orientifold plane [4], this leaves 16 D7–branes on each $\mathbb{P}^1$, just what we expect from the $\mathbb{Z}_2$ model. Unfortunately, we do not have an exact correspondence. In F-theory, the configuration above describes colliding $D_4$ singularities, a situation which heralds the presence of “tensionless strings” [34]. On the orientifold side, the $\mathbb{Z}_2$ model has non-zero “theta-angles” which drive the theory away from this critical point.

Let us consider, therefore, a more generic situation from the orientifold point of view. For this we move sets of 7–branes off the orientifold planes. Theoretically, we could move off sixteen independent D7–branes in each $\mathbb{P}^1$ [4]. The orientifold group action, however, pairs these up into eight groups of two, each contributing an $SU(2)$ gauge group (see the

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13 We can think of $T^2/\mathbb{Z}_2$ as $\mathbb{P}^1$, the 16 D7–branes come from 32 D7–branes on the covering space.
discussion in section 2.2). We therefore would like to find a choice of \( f, g \) corresponding to this \((SU(2) \times SU(2)')^8\) and containing objects which become orientifold planes at weak coupling.

The F-theory gauge groups are given in terms of the singularities of the elliptic fiber. This singularity structure is encoded in the behavior of \( \Delta, f, g \). In order to obtain the \((SU(2) \times SU(2)')^8\), we need eight \( A_1 \) singularities in each of the \( z, w \) planes. This implies

\[
\Delta \sim \prod_{i=1}^{8} (z - z_i)^2 (w - w_i)^2,
\]

with \( f, g \) non-vanishing as \( z \to z_i, w \to w_i \). This is obtained by making the following choices \cite{10},

\[
\begin{align*}
    f &= \eta - 3h^2 \\
    g &= h(\eta - 2h^2) \\
    \eta &= C \prod_{i=1}^{8} (z - z_i)(w - w_i) \\
    h &= \prod_{i=1}^{4} (z - \tilde{z}_i)(w - \tilde{w}_i)
\end{align*}
\]

for which

\[
\Delta = C^2 \prod_{i=1}^{8} (z - z_i)^2 (w - w_i)^2 (4\eta - 9h^2).
\]

This choice of \( h \) and \( \eta \) is motivated by the need to recover an orientifold at weak coupling. Taking \( C \to 0 \) sends \( j(\lambda) \to \infty \) almost everywhere, see eq. (3.2). This implies that up to an \( SL(2, \mathbb{Z}) \) transformation we have weak coupling almost everywhere. Note that the last factor in eq. (3.6) yields pairs of 7–branes centered about the zeros of \( h \) and separated by a distance of order \( C \). These are the orientifold planes \cite{10}. Deformations of \( h \) are mapped to the blow-up modes of the type IIB orientifold. By moving the \( z_i, w_i \) around (i.e. the locations of the 7–branes in the orientifold picture), we can enhance the symmetry further. In particular, if \( n \) of the \( z_i \) coincide one gets an \( Sp(n) \) algebra, and if it happens at the orientifold plane there is the possibility of further enhancement \cite{10}.

4. F-theory Interpretation of the \( T^4/\mathbb{Z}_4 \) Orientifold

Let us next turn to the type IIB orientifold \( T^4/\mathbb{Z}_4 \) and in particular the realization (and extension) of the orientifold in terms of F-theory. As above, we will mainly discuss
the F-theory compactification and where appropriate, compare with the \( \mathbb{Z}_4 \) orientifold. In section 2 we showed that one can construct a T-dual version of the original Gimon-Johnson \( \mathbb{Z}_4 \) orientifold as a \( \mathbb{Z}_2 \) orbifold of the GP-model. Thus we are naturally lead to build the corresponding F-theory vacuum as a \( \mathbb{Z}_2 \) orbifold of \( \mathcal{M}_1 \).

4.1. Construction of the F-theory orbifold

Starting with our elliptically fibered Calabi–Yau with base \( \mathbb{P}^1 \times \mathbb{P}^1 \) we construct an orbifold, \( \overline{\mathcal{M}}_2 = \mathcal{M}_1/\mathbb{Z}_2 \), using the \( \mathbb{Z}_2 \) quotient \((z, w) \rightarrow (-z, -w)\). There are four \( \mathbb{Z}_2 \) fixed points in the base, or \( A_1 \) singularities, and hence four fixed tori in the Calabi–Yau manifold. Each fixed torus will contribute one Kähler deformation, from a 2-cycle living on a \( \mathbb{P}^1 \) of the blown-up torus, and one complex structure deformation, from a 3-cycle built out of a family of \( \mathbb{P}^1 \)s over a 1-cycle of the torus. As we will show in section 4.3, there are 123 complex deformations invariant under the \( \mathbb{Z}_2 \) quotient. If we extend our notion \( \overline{\mathcal{M}}_2 \), beyond its strict definition as an orbifold with unresolved fixed tori, to the surface were the singularities are slightly resolved, then the total number of complex structure deformations is \( h_{2,1}(\overline{\mathcal{M}}_2) = 123 + 4 = 127 \). Similarly, the four new Kähler deformations join the three inherited from \( \mathcal{M}_1 \) to give \( h_{1,1}(\overline{\mathcal{M}}_2) = 3 + 4 = 7 \).

To understand where the fixed tori come from we study the definition of the manifold as a hypersurface in a toric variety. (For a more detailed discussion of toric geometry in relation to F-theory, see for example \[6,7,38,39\] ) A hypersurface in (weighted) projective space is defined using a scaling relation (also known as a \( \mathbb{C}^* \) action),

\[
x_i \rightarrow \lambda^{k_i}, \quad p(x_i) \rightarrow \lambda^d p(x_i),
\]

on a defining polynomial, \( p(x_i) = 0 \), of degree \( d = \sum_i k_i \). In a toric variety, there are more coordinates and hence a larger number of scaling relations. In particular, the elliptic fibration over a base \( \mathbb{P}^1 \times \mathbb{P}^1 \), \( \mathcal{M}_1 \), can be described as a hypersurface in a toric variety with seven homogeneous coordinates and three \( \mathbb{C}^* \) actions \([3]\);

\[
(s, t, u, v, X, Y, Z) \rightarrow (\lambda_1 s, \lambda_1 t, \lambda_2 u, \lambda_2 v, (\lambda_1 \lambda_2)^4 \lambda_3^2 X, (\lambda_1 \lambda_2)^6 \lambda_3^3 Y, \lambda_3 Z).
\]

By a rescaling of \( \lambda_3 \) we set \( Z = 1 \). We define our inhomogeneous coordinates as \( z = s/t, w = u/v \). By setting \( |\lambda_i| = 1 \), this implies in particular the existence of three discrete

\[\text{\textsuperscript{14}}\] A general treatment of F-theory on orbifolds has yet to be done. See \[7,36,37\] for some more examples of such constructions.
identifications, each one associated with the $\mathbb{C}^*$ actions above. This can be written in a more compact notation as

$$
g_1 : (\mathbb{Z}_{12} : 0, 0, 1, 1, 4, 6, 0), \quad g_2 : (\mathbb{Z}_{12} : 1, 1, 0, 0, 4, 6, 0), \quad g_3 : (\mathbb{Z}_6 : 0, 0, 0, 2, 3, 1),
$$

where $(\mathbb{Z}_d : a_1, ..., a_7)$ implies

$$(s, ..., z) \rightarrow (\alpha^{a_1} s, ..., \alpha^{a_7} z), \quad \sum_i a_i = 0 \ (\text{mod} \ d), \quad \alpha^d = 1.$$

In this notation we can express our $\mathbb{Z}_2$ action as follows, $\tilde{g} : (\mathbb{Z}_2 : 1, 0, 1, 0, 0, 0, 0)$. Combining the various actions we then find the following fixed points

$$
\tilde{g} : (\mathbb{Z}_2 : 1, 0, 1, 0, 0, 0, 0), \quad s = u = 0, \quad g_1^6 \tilde{g} : (\mathbb{Z}_2 : 0, 1, 1, 0, 0, 0, 0), \quad t = u = 0,
$$
$$
g_2^6 \tilde{g} : (\mathbb{Z}_2 : 1, 0, 0, 1, 0, 0, 0), \quad s = v = 0, \quad (g_1 g_2)^6 \tilde{g} : (\mathbb{Z}_2 : 0, 1, 0, 1, 0, 0, 0), \quad t = v = 0.
$$

From the definition of the inhomogeneous coordinates, $(z, w)$ we see that the four $\mathbb{Z}_2$ fixed points in the base are given by $(z, w) = \{(0, 0), (0, \infty), (\infty, 0), (\infty, \infty)\}$.

4.2. Comparing F-theory with the type IIB orientifold

How does the spectrum of this F-theory vacuum, $\overline{M}_2$, compare with the type IIB $T^4/\mathbb{Z}_4$ orientifold? As shown in [17], it is possible to Higgs the gauge symmetry completely in the orientifold theory. This gives us a theory with $n_H = 128$ hypermultiplets, $n_T = 1 + 4 = 5$ tensor multiplets, and $n_V = 0$ vector multiplets. Except for the $\mathcal{N} = 1$ tensor multiplet inherited from six-dimensional supergravity, the tensor multiplets act like $\mathcal{N} = 2$ multiplets formed by one $\mathcal{N} = 1$ tensor multiplet and one $\mathcal{N} = 1$ hypermultiplet. Thus, four of the 128 hypermultiplets are on a different footing.

For F-theory on $\overline{M}_2$, a generic choice of complex structure gives no gauge enhancements, and hence $n_V = 0$. Furthermore, the number of complex structure deformations is related to the number of hypermultiplets by $n_H = h_{2,1} + 1$, where the extra contribution comes from the volume of the Calabi–Yau manifold [5]. Since $h_{2,1} = 127$, this agrees with the orientifold analysis. The total number of tensor multiplets is given by $n_T = h_{1,1}(\text{Base}) - 1$ [4]. Since $h_{1,1}(\text{Base}) = 6$, we find that the spectrum of F-theory compactified on $\overline{M}_2$ is in agreement with that of the type IIB $T^4/\mathbb{Z}_4$ orientifold.

This agreement becomes even more natural, if we study the correspondence between the tensor multiplets and the fixed points in the base of $\overline{M}_2$ in more detail. In resolving an $A_1$ singularity, we replace the $\mathbb{Z}_2$ fixed point by a two-cycle, whose self-intersection number is $-2$. This is a different phenomenon than that of blowing up a regular point in the base,
in which we obtain an exceptional divisor with self-intersection number -1. In particular, there is a non-toric complex structure deformation associated to the $A_1$ singularity, by which the singularity can be deformed. Thus, we have an effective $\mathcal{N} = 2$ tensor multiplet containing an $\mathcal{N} = 1$ hypermultiplet in addition to the usual $\mathcal{N} = 1$ tensor. In all, resolving the four $A_1$s give us four $\mathcal{N} = 2$ tensor multiplets just as in the type IIB $T^4/\mathbb{Z}_4$ orientifold. (For a similar discussion of $A_1$ singularities in F-theory, see [6,13].)

4.3. Blow-ups, deformations and gauge enhancement

Having shown that the spectra agree, we now turn to a more detailed comparison between the models. In particular, we want to study how the fixed point deformations and the gauge enhancement in the type IIB orientifold arise on the F-theory side. In section 2.3, we described the generic configuration of the $T^4/\mathbb{Z}_4$ orientifold. It had four 7–branes and four 7'–branes, each with an $SU(2)$ gauge group on its world–volume. In this section we show how the complex structure deformations of $\mathcal{M}_2$ can be tuned to get appropriate $O7$ planes along with the branes carrying the $SU(2)^4 \times SU(2)'^4$ gauge group. We will then sketch how further tuning can place these branes on $O7_4$ planes and how the gauge enhancement patterns in fig. 1 can take place. We leave most of the technical details to Appendix A.

To tune the complex structure of $\mathcal{M}_2$, we first need to determine how it descends from that of $\mathcal{M}_1$. We keep only deformations left invariant by the $\mathbb{Z}_2$ orbifold action. In terms of the defining equation (3.1), we restrict $f, g$ to terms which are $\mathbb{Z}_2$ invariant. This reduces the number of binomials from 81 and 169 to 41 and 85 for $f(z, w)$ and $g(z, w)$, respectively. As before, we can rescale the defining equation by an overall factor, which removes one degree of freedom. Although the $SL(2, \mathbb{C})$ reparameterization of each of the $\mathbb{P}^1$ has been broken by the $\mathbb{Z}_2$ action there is one “rescaling” that can be done. A one-parameter subgroup of the original $SL(2, \mathbb{C})$ leaves $\tau = i$ invariant. We are thus left with 123 parameters, were we have momentarily neglected the additional contribution to the number of complex structure deformations from the four $\mathbb{Z}_2$ fixed points in the base.

In order to have an $(SU(2) \times SU(2)')^4$ and respect the quotient symmetry, we take

\[
\eta = C \prod_{i=1}^{4} (z^2 - z_i^2)(w^2 - w_i^2) \\
h = Q(z, w)(z^2 - ˜z^2)(w^2 - ˜w^2)
\]  

(4.6)

where $Q(z, w)$ is of bi-degree two, and invariant under the $\mathbb{Z}_2$ action. This form for $\eta$ gives, after identification, the four 7–branes and 7'–branes we require. As was the case in

\[\text{We will use } O7 \text{ plane to refer to both the } O7_2 \text{ and } O7_4 \text{ orientifold planes.} \]
section 3, $h$ controls the $O7$ planes. There are branes making up $O7_2$ planes about $z = \tilde{z}$ and $w = \tilde{w}$. $Q(z, w)$ is related to the $O7_4$ planes and their intersections. Note that we have chosen the most generic form for $h$ consistent with the $\mathbb{Z}_2$ quotient.

In [16], and as discussed in section 2, it was shown that out of the 128 hypermultiplets sixteen are associated to closed string sector of the $T^4/\mathbb{Z}_4$. Two of them come from the untwisted sector and the other fourteen from blow-up modes for the fixed points. Of these, ten come from the $\mathbb{Z}_2$ twisted sector of the $\mathbb{Z}_2$ and $\mathbb{Z}_4$ fixed points. An additional four come from the other twisted sector of the $\mathbb{Z}_4$ fixed points, appearing together with the four tensor multiplets and forming four effective $\mathcal{N} = 2$ tensor multiplets (one tensor + five scalars).

Let us account for these blow-up modes in our F-theory compactification. The blow-up modes for the $\mathbb{Z}_4$ orientifold are encoded in deformations of the form of $h(z, w)$ given in (4.6), in analogy with the situation for the $\mathbb{Z}_2$ orientifold. The six blow-up modes of the $\mathbb{Z}_2$ points come from mixing the three factors of $h$ in (4.6) consistent with the F-theory $\mathbb{Z}_2$ quotient. $Q(z, w)$ controls the one deformation for each of the $\mathbb{Z}_4$ orientifold points. The last four deformations, the ones which pair up with the tensor multiplets, come from the non-toric deformations.

To realize one of these non-toric deformations we make a change of coordinates such that the $A_1$ singularity at $z = w = 0$ is given by

$$a \cdot b = c^2, \quad \text{where} \quad a = z^2, \quad b = w^2, \quad c = zw.$$  \hspace{1cm} (4.7)

Then we can deform the $\mathbb{Z}_2$ quotient singularity at $a = b = c = 0$ in the base of the elliptic Calabi–Yau,

$$a \cdot b = c^2 - \lambda_{11}^2.$$  \hspace{1cm} (4.8)

The expression for the deformations of the other $A_1$ singularities can be found in appendix A.

Now that we understand how the various $O7$ planes and fixed point deformations of the $T^4/\mathbb{Z}_4$ orientifold appear in F-theory, we examine enhancement patterns for the 7–branes (a similar analysis holds for the 7’–branes). When $n$ coinciding 7–branes are located away from an $O7$ plane, we have an $USp(2n)$ gauge symmetry just as in the $\mathbb{Z}_2$ orientifold. In F-theory this is obtained by identifying $n$ of the $z_i$’s, e.g. $z_1 = \ldots = z_n$ in the defining equation for $\mathcal{M}_2$ (4.6). The discriminant then takes the form

$$\Delta \sim (z^2 - z_1^2)^{2n},$$  \hspace{1cm} (4.9)

which at $z = z_1$ gives an $A_{2n-1}$ singular elliptic fiber. For a generic choice of $h$, this singularity is non-split and the gauge group has an $Sp(n)$ algebra [35]. This is the F-theory description of scenario A in the $\mathbb{Z}_4$ orientifold (see fig. 1).

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The matter content can be deduced in analogy with Sen’s analysis for F-theory on $\mathcal{M}_1$ corresponding to the $\mathbb{Z}_2$ orientifold \cite{10}. We know that only $n - 1$ moduli are involved in enhancing the symmetry from $USp(2^n)$ to $USp(2n)$. They correspond to separating the $n$ 7-branes. This is done by higgsing $USp(2n)$ using matter transforming as $(n(2n - 1) - 1)$ for which $Sp(n) \to Sp(1)^n$ and we are left with $(n - 1)$. Thus, there is one $n(2n - 1) - 1$ of $Sp(n)$.

In order to study the situation of $n$ 7-branes approaching an $O7^+$ plane we let $z_1 \to 0$. The details of this analysis can be found in appendix A. We find complete correspondence between the various gauge enhancement patterns in the $\mathbb{Z}_4$ orientifold as given in fig. 1, and F-theory on $\overline{\mathcal{M}}_2$. The crux of the matter can be understood as follows. From eq. (4.7), one can clearly see that the divisors corresponding to $z = constant$ can also be defined by $a = constant$, except when $z = 0$. For this case, the corresponding divisor can be thought of, using eq. (4.8), as the sum of two divisors with defining equations $a = 0, c = \pm \lambda_{11}$. When we take $z_1 \to 0$, $\Delta$ now has two divisors with $A_{2n-1}$ singularities. This yields the expected product gauge groups.

5. F-theory on elliptic Calabi–Yau 3-folds and the dual heterotic theory on $K3$

In the previous section we established the relation between the type IIB orientifold on $T^4/\mathbb{Z}_4$ and F-theory compactified on the orbifold $\overline{\mathcal{M}}_2 = \mathcal{M}_1/\mathbb{Z}_2$. We would now like to understand the dual heterotic descriptions of this model. In particular, we are interested in the role of the heterotic $E_8 \times E_8$ string compactified on $K3$ with instanton embedding $(10,10)$ and four extra $\mathcal{N} = 1$ tensor multiplets, the conjectured dual to the $T^4/\mathbb{Z}_4$ orientifold (see ref. \cite{7}). We will demonstrate that F-theory on $\overline{\mathcal{M}}_2$ is dual to a strong coupling limit of this heterotic $E_8 \times E_8$ theory on $K3$ with instanton embedding $(10,10)$. The heterotic theory can be described as M-theory compactified on $K3 \times (S^1/\mathbb{Z}_2)$ with four M-theory 5-branes located in pairs at two points in the $K3$. Surprisingly, one can reformulate this strongly coupled theory as a new weakly coupled heterotic theory.

5.1. The $(12,12)$ instanton embedding and its F-theory dual

The first step in our demonstration will be a review of the relationship between the heterotic $E_8 \times E_8$ string compactified on $K3$ with instanton embedding $(12,12)$ and its dual F-theory compactification on $\mathcal{M}_1$.

As we later will be interested in non-perturbative effects in the heterotic $E_8 \times E_8$ theory on $K3$, let us consider the corresponding situation in terms of M-theory on $K3 \times (S^1/\mathbb{Z}_2)$ \cite{11}, (see fig. 2 a,b)). The fundamental string is represented in terms of a
membrane, stretching between the “end-of-the-world” 9-branes. At each of the ends, the boundary of the membrane is a string, which carries a level one $E_8$ current algebra $[40]$. The tension of the string is proportional to the distance between the 9-branes, the interval $S^1/\mathbb{Z}_2$ which in M-theory units we denote by $R$; hence as $R$ decreases we obtain the weakly coupled $E_8 \times E_8$ heterotic string. Let us denote this string by $het_1$. In addition, there exists a second heterotic string given by an M-theory 5-brane wrapping the $K3$ $[11]$. We will denote that string by $het_2$. The two heterotic strings are related by a duality due to the electric-magnetic duality in M-theory between membranes and 5-branes $[41]$. This duality is manifested in six dimensions by on one hand wrapping a membrane on $S^1/\mathbb{Z}_2$ and then reducing on $K3$, and on the other hand by wrapping a 5-brane on $K3$ and reducing it on $S^1/\mathbb{Z}_2$. In this way one can see that the coupling for $het_1$, $\lambda_1$, is related to the coupling for $het_2$, $\lambda_2$ by

$$\lambda_2^2 = (\lambda_2^2)^{-1} \propto R/V; \quad (5.1)$$

where $V$ is the volume of the $K3$, and both $R$ and $V$ are expressed in M-theory units (for more details see ref. $[11]$). We see that in, analogy to the weakly coupled $het_1$ with small $R$ relative to $V$, there is a weakly coupled dual heterotic string when the volume of the $K3$ becomes small relative to $R$. When $R$ and $V$ are of comparable size, neither string description is valid and we turn to F-theory for a better description.

In the dual F-theory on the elliptic $\mathcal{M}_1$ the picture above is given in terms of the divisors of the base $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$ (see fig 2 b,d). Following $[6]$, we assign an area $a_d$ to the $\mathbb{P}^1$ fiber of $F_0$ and similarly an area $a_h$ to the $\mathbb{P}^1$ base. In terms of the divisor classes $[D_s] = [D_t]$ and $[D_u] = [D_v]$ (the projective coordinates $(s, t)$ label one $\mathbb{P}^1$ and $(u, v)$ the other) these areas can be expressed as:

$$a_d = \text{area}(D_s) = \text{area}(D_t), \quad a_h = \text{area}(D_u) = \text{area}(D_v). \quad (5.2)$$

The area of a divisor, $D_{x_j}$, is computed by considering the intersection of $D_{x_j}$ with the general Kähler class, $K = a_d D_v + a_h D_s$, given that $D_{x_i} \cdot D_{x_j} = 0$ unless $D_{x_i}$ and $D_{x_j}$ are neighboring divisors in which case $D_{x_i} \cdot D_{x_j} = 1$ (see fig. 2c,d)).

In order to identify this F-theory vacuum with that of the heterotic string, we first observe that we have two types of D-strings. They are obtained by wrapping D3-branes on elements of either of the divisor classes $[D_u], [D_s]$. For these D-strings, the tension is given in terms of the area of the wrapped divisor. These D-strings are the dual heterotic strings in the six-dimensional heterotic theory. This allows us to identify the six-dimensional heterotic coupling constant, $\lambda_1$, in terms of F-theory variables as $[6]$

$$\lambda_1^2 = a_h/a_d. \quad (5.3)$$
Figure 2.

a) Heterotic $E_8 \times E_8$ string $het_1$ on $K3 \times S^1/\mathbb{Z}_2$ with instanton embedding $(12, 12)$.

b) Magnetic dual $E_8 \times E_8$ heterotic string $het_2$ on dual $\hat{K}3 \times \hat{S}^1/\mathbb{Z}_2$ background.

c) Divisors for the base $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$ of the (3,243) Calabi–Yau dual to fig. 2a), where $\lambda_1 \propto \text{area}(D_v) = \text{area}(D_u)$.

d) The same base gives the F-theory dual to fig. 2b) with $\lambda_2 \propto \text{area}(D_s) = \text{area}(D_t)$.

Since the overall volume of the Calabi–Yau in the context of F-theory is a hypermultiplet, and the size of the elliptic fiber is frozen, we can fix the remaining “effective” Kähler parameter by choosing $a_ha_d = 1$. Thus,

$$\lambda_1 = a_h, \quad \lambda_2 = a_d.$$  \hfill (5.4)

We will next consider situations in which we blow up the base at generic points. Much
of the above analysis carries through, with the obvious modification of the Kähler class such that it now depends on the exceptional divisors from the blown-up \( \mathbb{P}^1 \)s. Also, \( a_{h,d} \) will now be associated with new divisor classes, defined so as to contain only elements of self-intersection number 0. There is a natural correspondence between divisors of self-intersection number 0, or rather the D3-branes which wrap these divisors, and the (dual) heterotic strings \[6\]. As discussed in the introduction, the remaining exceptional divisors with self-intersection number -1 and -2 correspond to \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) tensionless strings respectively, in the limit that the area of the given divisor goes to zero \[16\].

Typically, some of the original divisors of \( F_0 \) will have modified self-intersection numbers after we blow up the base (see for example Fig. 3 c,d)). This change is interpreted in the dual heterotic strings, defined above, as follows. If we consider the D-string whose coupling depends on \( a_h \), it will have as a target space an \( E_8 \times E_8 \) K3 compactification with instanton numbers \((12 + n_s, 12 + n_t)\) where \( n_{s,t} \) are the self-intersection numbers of the divisors \( D_{s,t} \). A similar story follows for \( a_d \). Note that for the \((12, 12)\) compactification that we have been studying this means that all the relevant divisors have self-intersection number zero, and are thus appropriate for defining the areas \( a_{h,d} \).

### 5.2. The \((10, 10)\) instanton embedding and its F-theory dual

The next step in our demonstration will be to construct a new manifold, \( \mathcal{M}_2 \), which we will use as an F-theory compactification to produce a dual six dimensional model for the heterotic \( E_8 \times E_8 \) theory compactified on K3 with instanton embedding \((10, 10)\). Using \[6\], (see also \[39\]) we obtain \( \mathcal{M}_2 \) by blowing up the base \( F_0 \) of \( \mathcal{M}_1 \) at four points. We can use \( \mathcal{M}_2 \) to represent either of the electric-magnetic dual \((10, 10)\) models, shown in fig 3a),b). These figures represent these models in terms of M-theory on \( K3 \times S^1/\mathbb{Z}_2 \), as they are inherently strongly coupled. The crux of our demonstration will be to exhibit \( \overline{\mathcal{M}_2} \) as a limit of \( \mathcal{M}_2 \). Both of these elliptic fibrations correspond to the same Calabi–Yau three-fold with Hodge numbers \((7,127)\), but the moduli space of \( \overline{\mathcal{M}_2} \) is the subset of \( \mathcal{M}_2 \) with four unresolved \( A_1 \) singularities in base. We will demonstrate how these singularities appear as we take the limit.

The heterotic \( E_8 \times E_8 \) theory compactified on K3 with instanton embedding \((10, 10)\) can be obtained from the \((12,12)\) instanton embedding by shrinking two \( E_8 \) instantons on each of the “end-of-the-world” 9-branes. Taking the zero-size limit of an \( E_8 \)-instanton yields a new phase of the theory in which an M-theory 5-brane is detached from the 9-brane \[14,15\]. This 5-brane carries an \( \mathcal{N} = 1 \) tensor multiplet whose scalar component parameterizes the position of the 5-brane relative to the 9-brane from which it emanated.

\[16\] For the case of divisors with self-intersection number \(-n, n > 2\), see ref. \[13\].
In addition, this scalar component will enter in the gauge kinetic terms of the low-energy six dimensional supergravity action in a fashion determined by anomalies (for more details, see refs. [42,43]). We label the scalars corresponding to the four M-theory 5-branes in the (10,10) model $\phi_i$, $i = 1, \ldots, 4$. Note that these 5-branes also have coordinates in $K3$ inherited from their parent $E_8$ small instantons.
From the generic situation for the \((10,10)\) model, we now want to tune parameters so as to reach a point in the moduli space of this model which is connected to the \(T^4/\mathbb{Z}_4\) orientifold. One immediate problem arises in matching to this orientifold. As we will later demonstrate, the \((10,10)\) model is only weakly coupled in the limit where the scalars \(\phi_i\) are small. This means that we expect to have four light non-critical \(N=1\) strings in the theory. We know from section 2, however, that the \(T^4/\mathbb{Z}_4\) orientifold is in a region of moduli space associated with four light non-critical \(N=2\) strings. This means that if the \((10,10)\) model is the dual of this orientifold, it must be strongly coupled. In order to describe this let us turn to F-theory.

To study the heterotic \((10,10)\) model we start with the F-theory description of the heterotic \((12,12)\) model with coupling \(\lambda_1 = a_h\), and blow up two points on the divisors \(D_s\) and \(D_u\) to get \(\mathcal{M}_2\) as shown in Fig. 3c). This gives us two of the divisors with self-intersection -2 necessary for \(N = 2\) non-critical strings. We produce the other two such divisors by locating these blow-ups pairwise on \(D_v\) and \(D_u\). As can be seen from fig. 3a), this last operation corresponds to placing two pairs of M-theory 5-branes at identical \(K3\) positions. Figs. 3b) and 3d) illustrate how the construction looks almost identical starting from the dual \((12,12)\) model with coupling \(\lambda_2 = a_d\).

It is interesting to contrast the origins of the exceptional divisors in the two electric-magnetic dual \((10,10)\) models of figs. 3a) and 3b). For fig. 3a) shrinking the exceptional divisors \(D_s\) and \(D_t\) corresponds to strong coupling singularities inside each end-of-the-world nine-brane, and shrinking the exceptional divisors \(D_v\) and \(D_u\) corresponds to \(N = 2\) non-critical strings appearing from overlapping 5-branes as in ref. [44]. Fig. 3b) gives us the complimentary picture where these two separate phenomena are exchanged!

To continue our quest to link \(\mathcal{M}_2\), the F-theory dual of the electric and magnetic dual \((10,10)\) models, with \(\overline{\mathcal{M}}_2\), the F-theory description of the \(T^4/\mathbb{Z}_4\) orientifold, we need only stare at Figs. 3c) and 4c) (or alternatively 3d) and 4d)). Clearly, blowing down the exceptional divisors \(D_{u,v,s,t}\) to \(A_1\) singularities does the job. Let us examine how the relevant couplings behave. As we mentioned earlier, blowing up the base \(F_0\) of \(\mathcal{M}_1\), will change the Kähler class of the base. The new Kähler class \((D_i\) are the exceptional divisors from our blow-ups) is

\[
K = a_h (D_s + D_1 + D_4) + a_d (D_u + D_1 + D_3) - \sum_{i=1}^{4} \phi_i (D_i),
\]

where \(D_i\) are the exceptional divisors from our blow-ups. This gives the areas of the divisors of interest as

\[
\begin{align*}
\text{area}(D_s) &= a_d - \phi_1 - \phi_4, & \text{area}(D_t) &= a_d - \phi_2 - \phi_3, \\
\text{area}(D_u) &= a_h - \phi_1 - \phi_3, & \text{area}(D_v) &= a_d - \phi_2 - \phi_4, & \text{area}(D_i) &= \phi_i.
\end{align*}
\]

(5.6)
and a volume for the base proportional to

\[ \sum_{i=1}^{4} \phi_i^2. \]  

(5.7)

Requiring that all the areas be positive implies that \( a_h \) and \( a_d \) are bounded from below by the vevs \( \phi_i \). Thus, as we asserted before, for either of the dual (10, 10) models to be weakly coupled requires the \( \phi_i \)'s to be small (the couplings \( \lambda_{1,2} \) are still proportional to \( a_h, a_d \)), which will certainly not be the case if we take the limit of \( M_2 \) which matches \( \overline{M}_2 \).

This leaves us with a puzzle. In section 2, we described a formulation of the \( T^4/\mathbb{Z}_4 \) orientifold with 5-branes and 9-branes. In this formulation the bulk fields describe a Type I theory for which it is possible to have the string coupling everywhere strong. This implies a weakly coupled dual \( Spin(32)/\mathbb{Z}_2 \) heterotic theory (for an actual construction of this dual, see ref. \[45\]). We just showed, however, that the two heterotic dual D-strings one can construct are both strongly coupled in the orientifold limit. The answer to this puzzle lies in the final step of our demonstration.

5.3. A new heterotic string theory

For the last step in our demonstration, we will now show how to relate the coordinates used in section 4 to describe F-theory on \( \overline{M}_2 \), with the coordinates for \( M_2 \) which connect naturally with the (10, 10) model. From fig. 4c),d) we see that in terms of the divisors \( D_i, i = 1, \ldots, 4, \overline{M}_2 \) looks very much like an F-theory which could have a weakly coupled dual heterotic theory. (Recall that although the four \( A_1 \) singularities are there the physics from the perspective of the full F-theory is non-singular.) We propose a different heterotic string theory in which the (dual) strings are obtained in much the same way as that in which \( het_{1,2} \) were obtained to describe M-theory on \( K3 \times S^1/\mathbb{Z}_2 \).

Let us observe that for F-theory on \( \overline{M}_2 \), as shown in fig. 4c),d), the divisors \( D_i \) all have self-intersection number 0 (this number was raised from -1 when the \( A_1 \)'s were blown down). Also, the presence of the \( A_1 \) singularities signals that in this limit, \( D_1 \cdot D_3 = \frac{1}{2} \).

It is natural to suspect that the divisors \( 2D_i \) (the extra factor of two guarantees integer intersection numbers) can be associated with two new divisor classes with areas \( \tilde{a}_h \) and \( \tilde{a}_d \) such that in the limit where the exceptional divisors \( D_{u,v,s,t} \) shrink to zero size we have\[17\]

\[ \tilde{a}_d = \text{area}(2D_3) = \text{area}(2D_4), \quad \tilde{a}_h = \text{area}(2D_1) = \text{area}(2D_2). \]  

(5.8)

\[17\] The factors of two are really introduced because we are interested in the generic divisor class. Recall that in the orbifold \( \mathbb{P}^1(z) \times \mathbb{P}^1(w)/((z,w) \rightarrow (-z,-w)) \), the divisors \( z = \hat{z} \neq 0, \infty \) and \( w = \hat{w} \neq 0, \infty \) intersect twice, while \( [z = 0] \cdot [w = 0] = 1/2 \).
Figure 4.

a) Same as fig. 3a), except that we have placed the M-theory 5-branes on top of each other and gone to a strong coupling point on both 9-branes.

b) Same as fig. 3b), except that we have placed the M-theory 5-branes on top of each other and gone to a strong coupling point on both 9-branes (not the same 5-branes as in fig. 4a).

c) Same base as in fig. 3c), were have blown down all the \(-2\) divisors to produce \(A_1\) singularities. Note that the blown-down \(D_u\) adds \(\frac{1}{2}\) to the self-intersection numbers of \(D_{1,2}\) and similarly for the other \(-2\) divisors.

d) Same base as fig. 4c). This base describes F-theory duals to both figs. 4a) and 4b).

These new divisor classes would then provide us, upon wrapping D3-branes on them, with two new D-strings \(\tilde{\eta}_{1,2}\) with couplings as in eq. (5.4)

\[ \tilde{\lambda}_1 \propto \tilde{a}_h, \quad \tilde{\lambda}_2 \propto \tilde{a}_d. \]  \hspace{1cm} (5.9)

To make this more precise, we can write down these two new divisor classes by per-
forming a change of basis on the Kähler moduli space of \( \mathcal{M}_2 \). The new divisor classes can be represented as

\[
\begin{align*}
[\tilde{D}_h] &= [2D_1 + D_s + D_u] = [2D_2 + D_t + D_v], \\
[\tilde{D}_d] &= [2D_3 + D_t + D_u] = [2D_4 + D_s + D_v],
\end{align*}
\]

and the Kähler class can be rewritten as

\[
K = \frac{1}{2} \left( \tilde{a}_h(\tilde{D}_d) + \tilde{a}_d(\tilde{D}_h) - a_s(D_s) - a_t(D_t) - a_u(D_u) - a_v(D_v) \right). \tag{5.11}
\]

Taking the limit \( \mathcal{M}_2 \to \overline{\mathcal{M}}_2 \), a quick computation shows that the volume of the base for \( \overline{\mathcal{M}}_2 \) is proportional to

\[
\tilde{a}_h\tilde{a}_d. \tag{5.12}
\]

Thus, in F-theory on \( \overline{\mathcal{M}}_2 \) we have a vacuum with a base very similar to that of the F-theory dual for the (12, 12) model.

The conjectured dual heterotic theory on \( \tilde{K}3 \) would be that of a theory which, in terms of M-theory on \( \tilde{K}3 \times \tilde{S}^1/\mathbb{Z}_2 \), has a heterotic string \( \tilde{het}_1 \) obtained by wrapping the membrane around \( \tilde{S}^1/\mathbb{Z}_2 \). A second heterotic string, \( \tilde{het}_2 \), is obtained by wrapping the 5-brane on \( \tilde{K}3 \). Finally, because of the electric-magnetic duality in M-theory between membranes and 5-branes, the two heterotic strings are dual. The two couplings are given in terms of the volume and radius, in M-theory units, of the \( \tilde{K}3 \) and \( \tilde{S}^1/\mathbb{Z}_2 \) respectively,

\[
\tilde{\lambda}_1^2 = (\tilde{\lambda}_2^2)^{-1} = \tilde{R}/\tilde{V}. \tag{5.13}
\]

There is of course a crucial difference between \( \overline{\mathcal{M}}_2 \) and \( \mathcal{M}_1 \), the orbifolding! This is best understood by looking at the change of coordinates implied in eq.\( \text{(5.11)} \). The exceptional divisor \( D_1 + \frac{1}{2}D_s + \frac{1}{2}D_u \) is the divisor which looks like \( D_1 \) when we shrink \( D_{s,t,u,v} \), but it is half a member of the divisor class \([\tilde{D}_h]\). This means that our new coordinates are double valued. Thus the single valued domain when the divisors \( D_{s,t,u,v} \) are small is really \( (\mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_2 \) with slightly blown up \( A_1 \) singularities, exactly our description for \( \overline{\mathcal{M}}_2 \). Also, notice that the volume we compute for \( \overline{\mathcal{M}}_2 \) in eq.\( \text{(5.12)} \) is exactly half the volume one would get using the Kähler class, in section 5.1, for the (12, 12) model.

5.4. The new heterotic string in the (11, 11) model

It is interesting to note that we can understand the existence of the new heterotic strings, \( \tilde{het}_{1,2} \), in terms of the strongly coupled heterotic \( E_8 \times E_8 \) theory compactified on
The dual F-theory vacuum is that of an elliptic Calabi–Yau with base $F_0$ blown up at two points. It can be reached from $\mathcal{M}_2$ in two ways by either blowing down $D_1$ and $D_2$, or by blowing down $D_3$ and $D_4$.

It is sufficient to study the scenario where $D_1$ and $D_2$ are blown down. At this point, all the remaining exceptional divisors, $D_{s,t,u,v,3,4}$ have self-intersection number -1. There are now three different ways to shrink two divisors and obtain a model with a base $F_0$; i) shrink $D_3$ and $D_4$, ii) shrink $D_u$ and $D_v$ and finally, iii) shrink $D_s$ and $D_t$. In terms of the (11,11) heterotic model, these represent three separate methods to recover the (12,12) model with its two dual strings. Each method preserves two out of three dual strings. These strings are associated with the divisor classes:

$$[D_s + D_4], \quad [D_u + D_3], \quad \text{and} \quad [D_s + D_u] \quad (5.14)$$

The first two strings are familiar to us as descendants of $het_{1,2}$ in the (10,10) model. They represent the heterotic string and wrapped 5-brane on the (11,11) background. The third string is an entirely new object, whose heterotic origins should correspond to a bound state of the last two. After blowing up the points to get $D_1$ and $D_2$ it will correspond to $\tilde{het}_1$. Similarly, we can see $\tilde{het}_2$ in the (11,11) model, reached by blowing down the exceptional divisors $D_{3,4}$.

6. Conclusions

We have shown that the F-theory description of a IIB $T^4/\mathbb{Z}_4$ orientifold is given in terms of an F-theory orbifold, $\overline{\mathcal{M}}_2 = \mathcal{M}_1/\mathbb{Z}_2$ where $\mathcal{M}_1$ is the Calabi–Yau vacuum used to describe the F-theory corresponding to the GP $\mathbb{Z}_2$ orientifold. The appearance of an F-theory orbifold in the process has interesting implications, beyond the immediate scope of the specific models involved.

If we look at the volume formula, eq.(5.7), for the base of the Calabi-Yau $\mathcal{M}_2$, we see that for fixed volume, the tensor scalars describing the Kähler moduli space are constrained to sit on a hyperboloid. This is entirely consistent with the $SO(1,n_T)$ (here $n_T = 5$) structure of the six-dimensional supergravity tensor scalar moduli space described in refs. [47,42,43]. We recover a heterotic description of the six-dimensional theory when move very far out along one of the branches of the hyperboloid. This corresponds to shrinking the divisors $D_i$, with self-intersection -1, to recover the (12,12) heterotic model.

18 This model was studied in detail in [38].
What we have discovered is another type of limiting process, different from the one we just described, which will also recover a heterotic description.

We found that when we shrink two-cycles with self-intersection -2, a process akin to shrinking two-cycles of self-intersection -1 can happen. We can rewrite our F-theory Calabi-Yau vacuum as the orbifold (with action of order 2) of another Calabi-Yau model whose base contains shrinking two-cycles with self-intersection -1. So not only can we recover a perturbative heterotic picture when we move far out along the branches of the hyperboloid constraining the scalars of the \( \mathcal{N} = 1 \) tensor multiplets. But we can also recover a perturbative heterotic picture when we are near some of the points in the interior of the hyperboloid, where two-cycles of self-intersection -2 shrink down. The implication is that models were two-cycles with self-intersection \(-n \) \((n > 2) \) shrink down, should also have a well-defined perturbative heterotic description. The key would be to rewrite the corresponding F-theory Calabi-Yau as the orbifold, with elements of order \( n \), of another Calabi-Yau with shrinking two-cycles of self-intersection -1 and then to connect this later Calabi-Yau, via orientifolds, to a perturbative heterotic description. This is a fine illustration of the complementarity of F-theory, orientifold, and heterotic vacua in string theory.

**Note Added:** After this work was completed there appeared a paper which studies the type IIB orientifold \( T^4/\mathbb{Z}_4 \) [13].

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Appendix A.

In this appendix we present a more detailed analysis of the deformations of F-theory on $\mathcal{M}_2$ corresponding to the blow-ups of the fixed points of the $\mathbb{Z}_4$ orientifold. We also study gauge enhancement for F-theory on $\mathcal{M}_2$.

A.1. Blowing up the orientifold points

The six $\mathbb{Z}_2$ fixed points in the orientifold picture correspond to points in the base at

$$((\tilde{z}, \tilde{\bar{w}}), (\tilde{z}, -\tilde{\bar{w}}), (0, \tilde{\bar{w}}), (\infty, \tilde{\bar{w}}), (\tilde{z}, 0), (\tilde{z}, \infty)), \quad (A.1)$$

while, to match with the $A_1$ singularities, the four $\mathbb{Z}_4$ fixed points are at,

$$(0, 0), (\infty, 0), (0, \infty), (\infty, \infty). \quad (A.2)$$

We know, from ref. [10], that these fixed points must be associated with the crossings of the zeroes of $h(z, w)$. We can match with the fixed points above by choosing $Q(z, w) = zw$ in eq. (4.6). This gives us the initial form for $h$ where the orientifold points are blown down:

$$h_0 = zw(z^2 - \tilde{z}^2)(w^2 - \tilde{w}^2) \quad (A.3)$$

Most of the blow-up modes in the $\mathbb{Z}_4$ orientifold can now be encoded via deformations of $h_0(z, w)$, in analogy with the situation for the $\mathbb{Z}_2$ orientifold. The analysis for the $\mathbb{Z}_2$ points is exactly the same as in the $\mathbb{Z}_2$ orientifold. For the points $(\tilde{z}, \tilde{\bar{w}})$ and $(\tilde{z}, -\tilde{\bar{w}})$ we have the following deformations

$$\delta h = zw \left( \frac{\alpha_{11}}{2} ((z - \tilde{z})(w - \tilde{\bar{w}}) + (z + \tilde{z})(w + \tilde{\bar{w}})) \right) \quad (A.4)$$

$$\delta h = zw \left( \frac{\alpha_{12}}{2} ((z - \tilde{z})(w + \tilde{\bar{w}}) + (z + \tilde{z})(w - \tilde{\bar{w}})) \right). \quad (A.5)$$

The deformations of $h_0$ associated to the points $(0, \tilde{\bar{w}})$, $(\infty, \tilde{\bar{w}})$ and $(\tilde{z}, 0)$, $(\tilde{z}, \infty)$, respectively, are given by

$$\delta h = w(z^2 - \tilde{z}^2) \left( \frac{\beta_{11}}{2} ((w - \tilde{\bar{w}}) + (w + \tilde{\bar{w}})) + \frac{\gamma_{11}}{2} (z^2(w - \tilde{\bar{w}}) + \tilde{z}^2(w + \tilde{\bar{w}})) \right) \quad (A.6)$$

$$\delta h = w(z^2 - \tilde{z}^2) \left( \frac{\beta_{12}}{2} ((z - \tilde{\bar{w}}) + (z + \tilde{\bar{w}})) + \frac{\gamma_{12}}{2} (w^2(z - \tilde{\bar{w}}) + \tilde{w}^2(z + \tilde{\bar{w}})) \right). \quad (A.6)$$

The $\mathbb{Z}_2$ twisted sector blow-up modes of the orientifold $\mathbb{Z}_4$ points (those not linked with any tensors) are inherited directly from the structure of $\mathcal{M}_1$. These deformations for the crossing zeroes of $h_0$ at $(0, 0)$, $(0, \infty)$, $(\infty, 0)$, and $(\infty, \infty)$ are, respectively,

$$h = (z^2 - \tilde{z}^2)(w^2 - \tilde{w}^2)(zw + \delta_{11}1 + \delta_{12}w^2 + \delta_{21}z^2 + \delta_{22}w^2z^2) \quad (A.6)$$
As discussed, these four points also have non-toric deformations which come in pairs with the $\mathcal{N} = 1$ tensor multiplets. To realize these deformations we make a change of coordinates given by

$$a \cdot b = c^2, \quad \text{where} \quad a = z^2, \quad b = w^2, \quad c = zw.$$  \hspace{1cm} (A.7)

The second set of blow-up modes for the $\mathbb{Z}_4$ orientifold points is then given by the deformation of the four $A_1$ singularities as follows

$$(a - \lambda_{12}) \cdot (b - \lambda_{21}) + \lambda_{12} \lambda_{21} - \lambda_{22}^2 (a \cdot b) = (c - \lambda_{11})(c + \lambda_{11}).$$  \hspace{1cm} (A.8)

\section*{A.2. Gauge Enhancements}

Given the above analysis, we now want to describe the gauge enhancement occurring when a collection of 7-branes are aligned with an $O7_4$ plane using F-theory. We will illustrate how to get the various enhancement patterns related to those in fig. 1 (which only describes enhancements in terms of the 5-9 picture). Our starting point is situation A, where the collection of 7-branes is in the bulk and carries a $USp(2n)$ gauge group. As in section 4.3, we can collect several 7-branes on top of each other by setting $z_1 = \ldots = z_n$ in the defining equation for $\mathcal{M}_2$ (4.6). If we use our new coordinates $(a, b, c)$, and define $a_i = z_i^2, \tilde{a} = z^2, b_i = w_i^2, \tilde{b} = \tilde{w}^2$, this collection of 7-branes yields a discriminant of the form

$$\Delta \sim (a - a_1)^{2n}$$

which generically is non-split [35], giving us the requisite $USp(2n)$ gauge group.

Next, we would like the collection of 7-branes to approach an $O7_4$ plane. If we rewrite $h_0$ as

$$h_0 = c(a - \tilde{a})(b - \tilde{b})$$

then the $O7_4$ planes are located near $a = 0, \infty$ and $b = 0, \infty$. For the purposes of this discussion, we will consider the $O7_4$ plane near the $a = 0$. To be able to study this case in the most detail, we have to deform the base of $\mathcal{M}_2$. The relevant deformations of $h_0$ for $a = 0$ can be read off from (A.3) and (A.4). We define the deformed $h$ as

$$h_d = (a - \tilde{a})\{\delta_{12}b^2 + b(c + \beta_{11} + \delta_{11} - \delta_{12}\tilde{b}) - \tilde{b}(c + \delta_{11})\},$$

\[19\] Recall that the complex structure deformation of the $A_1$ singularity accounts for two of the four scalars in the corresponding hypermultiplet. The remaining two are R-R and NS-NS two-form fluxes. We argued that in the 7–7' picture of the $\mathbb{Z}_4$ orientifold, the NS-NS scalar is non-zero due to a non-vanishing two-form flux, while the remaining three scalars are zero. Thus, we do not have a direct correspondence between this non-zero scalar and that of a complex structure deformation of the $A_1$. However, we will assume that as far as the gauge symmetry is concerned the effect is the same, as long as we avoid complicated singularities which would give rise to tensionless strings.
For $a = 0$ there are two $A_1$ singularities in the base located at $b = 0$ and $b = \infty$, respectively. To deform only with respect to these singularities we restrict eq. (A.8) such that

$$\lambda_{11} = \lambda, \quad \lambda_{12} = \lambda', \quad \lambda_{21} = \lambda_{22} = 0. \quad (A.12)$$

We can now consider what happens when the collection of 7-branes approaches the deformed $O7_4$ described above. We do this by letting $\alpha_1 \to \lambda'$. Inserting this in our eqn. (3.6) for the discriminant, $\Delta$, and using eqns. (A.8), (A.12) and (A.11) we have

$$\Delta = b^{-2n}(c - \lambda)^2(n + \lambda)^2 \tilde{\eta}^2(a, b)\{-9h_d^2(a, b, c) + \mathcal{O}(c - \lambda)(c + \lambda)\}, \quad (A.13)$$

where $\tilde{\eta}(a, b) = C \prod_{i=n+1}^4(a - a_i) \prod_{j=1}^4(b - b_j)$. Thus, by tuning one parameter ($a_1 \to \lambda'$), the discriminant takes the form

$$\Delta \sim \left(\frac{(c - \lambda)(c + \lambda)}{b}\right)^{2n} \quad (A.14)$$

The divisor $a = a_1$ has split into two separate divisors, and we get an enhancement to an $A_{2n-1} \times A_{2n-1}$ singularity locus. For generic $h_d$ the gauge group is $USp(2n) \times USp(2n)$. This corresponds to configuration B in fig. 1. We get matter transforming as $(2n, 2n)$; after higgsing $Sp(n) \times Sp(n)$ to $Sp(n)$ using $(2n, 2n)$, the remaining matter is in the $(n(2n-1) - 1) + 1$ representation of $Sp(n)$.

We can choose our deformations such that $h$ becomes a perfect square, for either of $c = \pm \lambda$ or for both. This corresponds to either (or both) of the $A_{2n-1}$ singularities to be split $[35]$. Let us rewrite $h_d$ as

$$h_d = (a - \tilde{a})(c - \mu)(b + \nu)^2 - (c + \tilde{\mu})(b + \tilde{\nu})^2 \quad (A.15)$$

where $\nu, \tilde{\nu}$ and $\mu, \tilde{\mu}$ are functions of the $\beta$’s and $\delta$’s in eq. (A.11). By setting either $\mu = \lambda$ or $\tilde{\mu} = -\lambda$, we get a perfect square for $c = -\lambda$ or for $c = \lambda$. In either of those cases, $h$ is a perfect square and we get an enhancement to $SU(2n) \times Sp(n)$. The only matter consistent with this enhancement is $(2n, 2n) + (n(2n-1), 1)$. This corresponds to configurations C and C’ in fig 1. If we let $\mu = \lambda$ and $\tilde{\mu} = -l$, then $h$ is a perfect square for both $c = -\lambda$ and for $c = \lambda$. We get a further enhancement to $SU(2n) \times SU(2n)$. The matter is in the $(2n, 2n) + (n(2n-1), 1) + (1, n(2n-1))$ representations. This is in agreement with the situation in configuration E in fig 1.

Finally, if we relax the condition $a_1 = \lambda'$ there is still a possibility for enhancement. When $a = a_1 \neq \lambda'$ we can use eq.(A.8) to write $b$ as a quadratic expression in $c$. Then we can rewrite $h_d$ as a quartic in $c$. Tuning two parameters will make $h_d(c)$ a perfect square,
and the $A_{2n-1}$ singularity in eq. (A.3) will now be split. Thus, exactly as in configuration D in fig 1, tuning two parameters gives us an $SU(2n)$ gauge group.

We have found a complete correspondence between the various gauge enhancement patterns in the $\mathbb{Z}_4$ orientifold on one hand and in F-theory on $\overline{\mathcal{M}}_2$ on the other. Note that if we keep $h = h_0$ and set $a_1 = 0$, then our F-theory model will be even more singular. It will have both $\mathcal{N} = 1$ and $\mathcal{N} = 2$ tensionless strings! This does not match with the $\mathbb{Z}_4$ orientifold. The reason is that in F-theory we have set all relevant scalars to zero, but in the orientifold some scalars (B-field fluxes) are non-zero. This is not apparent with the set of variables we are using to describe F-theory, which only describe two out of the four scalars in any hypermultiplet and hence miss these scalars. Fortunately, since we keep the first two scalars non-zero for the relevant hypermultiplets, our analysis is not compromised (a similar issue arises already in ref. [10]).
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