CIRCUIT COVERS OF SIGNED EULERIAN GRAPHS

BO BAO, RONG CHEN, AND GENGHUA FAN

Abstract. A signed circuit cover of a signed graph is a natural analog of a circuit cover of a graph, and is equivalent to a covering of its corresponding signed-graphic matroid with circuits. It was conjectured that a signed graph whose signed-graphic matroid has no coloops has a 6-cover. In this paper, we prove that the conjecture holds for signed Eulerian graphs.

1. Introduction

Let $G$ be a graph. A signed graph is a pair $(G, \Sigma)$ with $\Sigma \subseteq E(G)$, each edge in $\Sigma$ is labelled by $-1$ and other edges are labelled by 1. The graph $G$ can be viewed as the signed graph $(G, \emptyset)$. A circuit is a connected 2-regular graph. A circuit $C$ of $G$ is balanced if $|C \cap \Sigma|$ is even, otherwise it is unbalanced. We say that a subgraph of $(G, \Sigma)$ is unbalanced if it contains an unbalanced circuit, otherwise it is balanced. Signed graphs is a special class of “biased graphs”, which was defined by Zaslavsky in [7, 8]. Just as biased graphs, there are two interesting classes of matroids, the class of signed-graphic matroids and the class of even-cycle matroids, associated with signed graphs, which in fact are special classes of “frame matroids” and “lifted-graphic matroids” associated with biased graphs, respectively.

A barbell is a union of two unbalanced circuits sharing exactly one vertex or a union of two vertex-disjoint unbalanced circuits together with a minimal path joining them. A signed circuit of $(G, \Sigma)$ is a balanced circuit or a barbell. We say the matroid with $E(G)$ as its ground set and with the set of all signed circuits as its circuit set is the signed-graphic matroid defined on $(G, \Sigma)$. We say that $(G, \Sigma)$ is flow-admissible if each element of $E(G)$ is in a circuit of its signed-graphic matroid, that is, each edge of $G$ is in a signed circuit of $(G, \Sigma)$.

Date: June 21, 2021.

Key words and phrases. signed graphs, Eulerian graphs, signed-circuit covers.

This research was partially supported by grants from the National Natural Sciences Foundation of China (No. 11971111 and No. 11971110) and NSFFP (2019J01645).
For a positive integer $k$, we say that a signed graph $(G, \Sigma)$ has a $k$-cover if there is a family $C$ of signed circuits of $(G, \Sigma)$ such that each edge of $G$ belongs to exactly $k$ members of $C$. For ordinary graphs $G$ (signed graph $(G, \Sigma)$ with $\Sigma = \emptyset$), a $k$-cover of $G$ is just a family of circuits which together covers each edge of $G$ exactly $k$ times. In [1], Bermond, Jackson and Jaeger proved that every bridgeless graph $G$ has a 4-cover. Fan [4] proved that every bridgeless graph $G$ has a 6-cover. Together it follows that every bridgeless graph $G$ has a $k$-cover, for every even integer $k$ greater than 2. The only left case that $k = 2$ is the famous Circuit Double Cover Conjecture: every bridgeless graph $G$ has a 2-cover, which is still open and believed to be very hard. It is somehow a surprise that it is even unknown whether there is an integer $k$ such that every signed graph $(G, \Sigma)$ has a $k$-cover.

Let $A$ and $B$ be two vertex-disjoint unbalanced circuits of length $2m + 1$. Let $G$ be the signed graph obtained from $A$ and $B$ by joining $A$ and $B$ with two internally disjoint paths of length $2m + 1$ such that the two paths form an unbalanced circuit. Then each signed circuit of $G$ is a barbell of $6m + 3$ edges. Any $k$-cover of $G$ contains $k|E(G)| = k(8m + 4) = 4k(2m + 1)$ edges, which must be divisible by $6m + 3 = 3(2m + 1)$. That is, $4k$ must be divisible by 3, which means that $k$ cannot be 2 or 4. Thus $G$ has neither 2-covers nor 4-covers. Consider the signed graph $H$ consisting of three unbalanced circuits of length $2m + 1$ with exactly one vertex in common. Then each signed circuit of $H$ is a barbell of $4m + 2$ edges. Any $k$-cover of $H$ contains $k|E(G)| = k(6m + 3) = 3k(2m + 1)$ edges, which must be divisible by $4m + 2 = 2(2m + 1)$. That is, $3k$ must be divisible by 2, which means that $k$ cannot be odd. Thus $H$ has no $k$-cover for any odd $k$. These counterexamples were first given by Fan [5], who also proposed the following conjecture.

**Conjecture 1.1.** Every flow-admissible signed graph has a 6-cover.

In this paper, we prove

**Theorem 1.2.** Conjecture [1.1] holds for signed Eulerian graphs.

In [3], Cheng, Lu, Luo, and Zhang proved that each signed Eulerian graph with an even number of negative edges has a 2-cover. We will prove Theorem [1.2] from a different aspect, and our proof does not rely on their result.

This paper is organised as follows. Definitions and results needed in the proof of Theorem [1.2] are given in Section 2. Theorem [1.2] will be proved in Section 4 by contradiction. All “small” signed Eulerian graphs occurring in Section 4 in the proof by contradiction are dealt with in Section 3.
2. Preliminaries

Let $G$ be a finite graph. Let $\text{loops}(G)$ denote the set of loops in $G$. Let $\Delta(G)$ and $\delta(G)$ be the maximal and minimal degree of $G$, respectively. For a positive integer $k$, let $V_k(G)$ be the subset of $V(G)$ consisting of degree-$k$ vertices of $G$. A subgraph $H$ of $G$ is spanning if $V(H) = V(G)$. In this paper, we will also use $H$ to denote its edge-set. For example, we will let $G \setminus H$ denote $G \setminus E(H)$. If exactly one component of $G$ has edges, then we say that $G$ is connected up to isolated vertices. Evidently, a connected graph is also connected up to isolated vertices, but the converse may not be true.

We say that $G$ is even if every vertex of $G$ is of even degree. If an even graph is connected, we say that it is Eulerian. A circuit $C$ of $G$ is non-separating if $G \setminus C$ is connected, otherwise, it is separating. A theta graph is a graph that consists of a pair of vertices joined by three internally vertex-disjoint paths. Let $C$ be a circuit-decomposition of an Eulerian graph $G$. Let $H$ be a graph with $C$ as its vertex set, where two vertices in $H$ are adjacent if and only if their corresponding circuits in $G$ have a common vertex. We say that $H$ is determined by $C$.

**Lemma 2.1.** Let $G$ be an Eulerian graph with $\Delta(G) \geq 4$. Let $C$ be a circuit of $G$. Then there is a circuit $C'$ of $G$ with $C \cap C' = \emptyset$ such that $G \setminus C'$ is connected up to isolated vertices.

**Proof.** Since $G$ is Eulerian, $G$ has a circuit-decomposition $C$ containing $C$. Let $H$ be the graph determined by $C$. Since $G$ is connected with $\Delta(G) \geq 4$, the graph $H$ is connected with at least two vertices. Let $T$ be a spanning tree of $H$. Since $T$ has at least two degree-1 vertices, $T$ has a degree-1 vertex, say $C''$, which is not $C$. Then $C''$ is the circuit as required by the lemma. 

**Lemma 2.2.** Let $G$ be a 2-connected graph with $|V(G)| \geq 3$. For any vertex $v$ of $G$, there is an edge $e$ of $G - v$ such that $G - V(e)$ is connected.

**Proof.** Let $C$ be a circuit of $G$ passing through $v$ with $|C|$ as large as possible. Evidently, $|C| \geq 3$ as $|V(G)| \geq 3$ and $G$ is 2-connected. Let $e$ be an edge of $C$ that is not incident with $v$. Then $G - V(e)$ is connected, otherwise we can find a longer circuit going through $v$. 

A set $\Sigma' \subseteq E(G)$ is a signature of $(G, \Sigma)$ if $(G, \Sigma)$ and $(G, \Sigma')$ have the same balanced circuits and the same unbalanced circuits. Evidently, for any edge-cut $C^*$ of $G$, the symmetric difference $\Sigma \Delta C^*$ is a signature of $(G, \Sigma)$. We say that $(G, \Sigma')$ is obtained from $(G, \Sigma)$ by switching. The following three lemmas are well-known results on signed graphs,
which will be frequently used in Section 3 without reference. Please refer to ([2], Lemma 3.5.), if the reader needs more detail about Lemma 2.3.

**Lemma 2.3.** All edges of a balanced signed subgraph of \((G, \Sigma)\) can be labelled by 1 by switching.

**Lemma 2.4.** Each signed theta-graph has a balanced circuit and cannot have exactly two balanced circuits.

**Lemma 2.5.** Every 2-edge-connected signed graph containing two edge-disjoint unbalanced circuits is flow-admissible.

In ([6], Theorem 4.2.), Máčajová and Škoviera proved that a flow-admissible signed Eulerian graph with an odd number of negative edges contains three edge-disjoint unbalanced circuits. On the other hand, since each unbalanced Eulerian signed graph with an even number of negative edges contains two edge-disjoint unbalanced circuits, we have

**Lemma 2.6.** A flow-admissible unbalanced signed Eulerian graph contains two edge-disjoint unbalanced circuits.

For simplicity, we will also use \(G\) to denote a signed graph defined on \(G\).

3. **Signed Eulerian graphs with special circuit decompositions**

Let \(k\) be a positive integer. Let \(kG\) be the graph obtained from \(G\) by replacing each edge in \(G\) with exactly \(k\) parallel edges. Consider a graph constructed as follows. For \(k \geq 3\), let \(G\) be a circuit of length \(k\) and \(N\) be a subdivision of \(2G\). Let \(C\) be a circuit of \(N\), we say that \(C\) is small if \(|V(C) \cap V_4(N)| = 2\), otherwise, \(C\) is long. When \(C\) is small, we also say that each vertex in \(V(C) \cap V_4(N)\) is an end of \(C\). Let \(e_1, e_2\) be edges in a small circuit of \(N\) such that \(\{e_1, e_2\}\) is not an edge-cut of \(N\). That is, \(\{e_1, e_2\}\) separates the two ends of the small circuit. We say that the signed graph obtained from \(N\) by labelling \(\{e_1, e_2\}\) by \(-1\) and all other edges by 1 is a necklace of length \(k\). Evidently, all small circuits in a necklace are balanced and all long circuits are unbalanced. Hence, the small circuits form a 1-cover in a necklace.

In the rest of this section, we will always let \(G\) denote a 2-connected flow-admissible signed Eulerian graph with \(\delta(G) \geq 4\), and \(C\) a circuit-decomposition of \(G\), and let \(H\) be the graph determined by \(C\). We say that \(C\) is optimal if it satisfies the following properties:
(CD1) $C$ is chosen with the number of unbalanced circuits as large as possible.

(CD2) subject to (CD1), $C$ is chosen with $|C|$ as large as possible.

In the rest of this section, we will always assume that $C$ is optimal. For any $C \in C$, we say that $C$ is a balanced vertex of $H$ if $C$ is a balanced circuit of $G$, otherwise it is unbalanced.

The following lemma follows immediately from (CD1), (CD2), and Lemma 2.6.

**Lemma 3.1.** For every pair of adjacent vertices $C_i$ and $C_j$ in $H$, if $C_i$ is balanced, we have

1. $1 \leq |V_G(C_i) \cap V_G(C_j)| \leq 2$,
2. $C_i \cup C_j$ is balanced when $C_j$ is balanced, and
3. $C_i \cup C_j$ is not flow-admissible when $C_j$ is unbalanced.

**Lemma 3.2.** For every pair of adjacent unbalanced vertices $C_i$ and $C_j$ in $H$, if $|V_G(C_i) \cap V_G(C_j)| \geq 3$, then $C_i \cup C_j$ is a necklace.

**Proof.** Since $C_i$ and $C_j$ are unbalanced, for any circuit decomposition $C'$ of $C_i \cup C_j$, either all circuits in $C'$ are balanced or at least two of them are unbalanced. If $C_i \cup C_j$ has an unbalanced circuit avoiding some vertex in $V_4(C_i \cup C_j)$, then $C_i \cup C_j$ can be decomposed into at least three circuits and two of which are unbalanced, which is not possible as $C$ is optimal. So each circuit in $C_i \cup C_j$ avoiding a vertex in $V_4(C_i \cup C_j)$ is balanced. Hence, $C_i \cup C_j$ is a necklace. \qed

We say that $G$ is cover-decomposable if $G$ can be decomposed into two proper edge-disjoint flow-admissible signed Eulerian subgraphs.

**Lemma 3.3.** If $H$ is isomorphic to a graph pictured as Figure 1 and $G$ has no balanced loops, then $G$ is cover-decomposable or has a 6-cover.

![Figure 1](image)

**Figure 1.** All degree-3 vertices are balanced, and others are unbalanced. All $f_i$’s are loops of $G$. 

Proof. Assume otherwise. Assume that $H$ is isomorphic to the graph pictured as Figure 1(d). For any $1 \leq i < j \leq 3$, when $\left|V(C_i) \cap V(C_j)\right| \leq 2$, it is obvious that $C_i \cup C_j$ has a 1-cover; when $\left|V(C_i) \cap V(C_j)\right| \geq 3$, it follows from Lemma 3.2 that $C_i \cup C_j$ is a necklace, so $C_i \cup C_j$ has a 1-cover too. Then $G$ has a 2-cover. So $H$ is isomorphic to a graph pictured as Figure 1(a)-(c). Note that, $1 \leq \left|V_G(C_i) \cap V_G(C_j)\right| \leq 2$ when $C_i$ is balanced by Lemma 3.1. When some $C_i$ is a loop, implying that $G$ is isomorphic to the graph pictured as Figure 1(b) or (c), since each pair of adjacent circuits intersect in at most 2 vertices, there are a few cases to check that $G$ has a 6-cover. So no $C_i$ is a loop. When $\left|V_G(C_i) \cap V_G(C_j)\right| = 1$ for all $1 \leq i < j \leq 3$, since $C_1 \cup C_2 \cup C_3$ is isomorphic to a $2K_3$-subdivision, combined the fact that all $f_i$ are unbalanced loops, it is easy to see that $G$ has a 6-cover. Hence, $\left|V_G(C_i) \cap V_G(C_j)\right| \geq 2$ for some $1 \leq i < j \leq 3$.

Assume that $H$ is isomorphic to a graph pictured as Figure 1(a). By Lemma 3.1(1) and symmetry we may assume that $V_G(C_2) \cap V_G(C_3) = \{u, v\}$. Let $C$ be the circuit of $C_2 \cup C_3$ that is incident to neither $f_2$ nor $f_3$. Since $C$ is balanced by Lemma 3.1(2), $G \setminus C$ is not connected otherwise $G$ is cover-decomposable, so $V_G(C_1) \cap V_G(C_2 \cup C_3) \subseteq V_G(C) - \{u, v\}$. When $\left|V_G(C_1) \cap V_G(C_2)\right| = \left|V_G(C_1) \cap V_G(C_3)\right| = 1$, the graph $G$ has a 2-cover. When $\left|V_G(C_i) \cap V_G(C_j)\right| = 2$ for some $2 \leq i \leq 3$, since $G \setminus \text{loops}(G)$ is balanced by Lemma 3.1(2), there is a non-separating balanced circuit $C'$ contained in $C \cup C_i$, implying that $G$ is cover-decomposable. Hence, $H$ is isomorphic to a graph pictured as Figure 1(b) or (c).

Assume that $V_G(C_2) \cap V_G(C_3) = \{u, v\}$. Since exactly one of $\{C_2, C_3\}$ is unbalanced, there is a $(u, v)$-path $P$ of $C_2 \cup C_3$ such that a circuit in $C_2 \cup C_3$ is unbalanced if and only if it contains $P$. Since all degree-3 vertices in Figure 1 are balanced, $P$ is not incident to $f_2$ or $f_3$. Let $C$ be the unique balanced circuit of $C_2 \cup C_3$ that is not incident to $f_2$ or $f_3$. Since $(C_2 \cup C_3) - C$ and $f_3$ are unbalanced, $G \setminus C$ is not connected, so $V_G(C_1) \cap V_G(C_2 \cup C_3) \subseteq V_G(C) - \{u, v\}$. When $C_1 \cup C_2$ is a necklace, implying that $H$ is isomorphic to a graph pictured as Figure 1(c) by Lemma 3.1 there is a non-separating small circuit $C'$ of the necklace $C_1 \cup C_2$ with $C' \subseteq C_1 \cup (C_2 - P)$. Since $(C_2 \cup C_3) - C$ and $f_3$ are unbalanced, $G \setminus C'$ is flow-admissible, so $G$ is cover-decomposable as $C'$ is balanced. Hence, by Lemma 3.1(1) or Lemma 3.2 we have $\left|V_G(C_1) \cap V_G(C_i)\right| \leq 2$ for each $2 \leq i \leq 3$. Moreover, since $V_G(C_1) \cap V_G(C_2 \cup C_3) \subseteq V_G(C) - \{u, v\}$, repeatedly using a similar strategy, we can find a 6-cover of $G$ or a non-separating balanced circuit $C$ such that $G \setminus C$ is flow-admissible, a contradiction.
By symmetry we may therefore assume that $|V_G(C_1') \cap V_G(C_3')| = 1$ for each $1 \leq i \leq 2$. Set $m = |V_G(C_1') \cap V_G(C_2')| \geq 2$. When $m = 2$, by simple computation, the lemma holds. So $m \geq 3$. By Lemmas 3.1 and 3.2, $H$ is isomorphic to the graph pictured as Figure 1(c) and $C_1 \cup C_2$ is a necklace of length $m$. Assume that $G$ is a counterexample to the lemma with $|V(G)|$ as small as possible. When $C_3$ does not share a vertex with a small circuit $C$ of $C_1 \cup C_2$, delete $C$ and identify its two ends as a new vertex. Let $G'$ be the new graph. Then $G'$ is cover-decomposable or has a 6-cover by the choice of $G$, so is $G$ since $G$ is balanced. Hence, $C_3$ intersects all small circuits of $C_1 \cup C_2$. Moreover, since $m \geq 3$ and $|V_G(C_i') \cap V_G(C_3')| = 1$ for each $1 \leq i \leq 2$, there are edge-disjoint long circuits $C_1', C_2'$ of $C_1 \cup C_2$ with $|V_G(C_1') \cap V_G(C_3')| = 2$ and $|V_G(C_2') \cap V_G(C_3')| \geq 1$. Since $C_1', C_2'$ are unbalanced and $C_1' \cap C_2' = C_1 \cup C_2$, the graph determined by $\{C_1', C_2', C_3, \{f_3\}\}$ isomorphic to the graph pictured as Figure 1(c). Since $|V_G(C_1') \cap V_G(C_3')| = 2$, the lemma holds by similar analysis in the second paragraph of the proof.

Let $C$ be a separating circuit of a graph $G$ with $u, v \in V(C)$. Let $P$ be an $(u, v)$-path on $C$. For a component $G'$ of $G \setminus C$, if $V(G') \cap V(P) \neq \emptyset$ we say that $G'$ intersects $P$; if $V(G') \cap V(C) \subseteq V(P) - \{u, v\}$ we say that $G'$ properly intersects with $P$.

**Lemma 3.4.** Let $C$ be a separating circuit of $G$ such that all components of $G \setminus C$ are unbalanced. Let $C'$ be a circuit-component of $G \setminus C$ with $\{u, v\} = V(C) \cap V(C')$. Let $P_1$ and $P_2$ be the $(u, v)$-paths of $C$. When $C$ is balanced or $G \setminus C$ has at least three components, one of the following holds.

1. $G$ is cover-decomposable, or
2. $G \setminus C$ has exactly three components, none of which is flow-admissible and one of which properly intersects with $P_i$ for each $1 \leq i \leq 2$.

**Proof.** Assume that (1) is not true. Without loss of generality we may assume that $C' = \{e, f\}$. Since $C'$ is unbalanced, we may assume that $P_1 \cup \{e\}$ and $P_2 \cup \{x\}$ are balanced for some $x \in \{e, f\}$. Since $G \setminus C$ has at least two components, besides $C'$, some component of $G \setminus C$ intersects with some $P_i$, say $P_2$. Since $C$ is balanced or $G \setminus C$ has at least three components, $G \setminus (P_1 \cup \{e\})$ has two edge-disjoint unbalanced circuits. Since (1) does not hold, $G \setminus (P_1 \cup \{e\})$ is disconnected, so there exists some non-flow-admissible component of $G \setminus C$ properly intersecting with $P_1$. Repeating the analysis, there is also a non-flow-admissible component of $G \setminus C$ properly intersecting with $P_2$. So $G \setminus C$ has at least three components.
Let $G_i$ be the union of the components of $G \setminus C$ that properly intersects with $P_i$ for each $1 \leq i \leq 2$. Then $G_1$ and $G_2$ are not flow-admissible. Assume that $G_1$ is disconnected. Since $G_1$ contains two edge-disjoint unbalanced circuits, $G_1 \cup P_1 \cup \{x\}$ and $G_2 \setminus (G_1 \cup P_1 \cup \{x\})$ are flow-admissible, implying that (1) holds. Hence, $G_1$ is connected, so is $G_2$ by symmetry. Besides $C'$, $G_1$ and $G_2$, assume that $G \setminus C$ has another component $G_3$. Since $G_3$ is unbalanced and intersects $V(P_1)$ and $V(P_2)$ by the definition of $G_1$ and $G_2$ and the fact that $G$ is 2-connected, both $G_1 \cup P_1 \cup \{f\}$ and $G_2 \setminus (G_1 \cup P_1 \cup \{f\})$ are flow-admissible, a contradiction. So $G \setminus C$ has exactly three components $C'$, $G_1$ and $G_2$, that is, (2) holds. 

For an $(u, v)$-path $P$ of $G$, we say that $P$ is pendant if $u \in V_1(G)$, $v \notin V_1(G) \cup V_2(G)$ and all internal vertices of $P$ are in $V_2(G)$.

**Lemma 3.5.** Let $H$ be a tree with a unique vertex $C$ of degree at least three, all leaf vertices are unbalanced, and all pendant paths have at most two edges. When $C$ is balanced, $V_2(H) = \emptyset$. When $C$ is unbalanced, all degree-2 vertices of $H$ are balanced triangles and leaf vertices that are adjacent to degree-2 vertices are loops. Then $G$ is cover-decomposable or has a 6-cover.

**Proof.** Assume that the lemma is not true. Since $G$ has a 6-cover when each component of $G \setminus C$ is a loop, there is a vertex $C'$ in $H$ adjacent to $C$ with $|C'| \geq 2$. Set $m = |V_2(C) \cap V_2(C')|$. Since $G$ is 2-connected and $\delta(G) \geq 4$, we have $m \geq 2$.

We claim that $C'$ is balanced or $|C'| \neq 2$. Assume otherwise. Then $C'$ is a component of $G \setminus C$ as all degree-2 vertices of $H$ are balanced. Since $|C'| = 2$, we have $m = 2$. Let $\{u, v\} = V_2(C') \cap V_2(C)$, $P_1$ and $P_2$ be the $(u, v)$-paths of $C$. By Lemma 3.4, $G \setminus C$ has exactly three components $C'$, $G_1$ and $G_2$, where $G_1$ and $G_2$ properly intersect $P_1$ and $P_2$, respectively. When $C \cup G_1$ is a necklace, there is a small circuit $D$ of $C \cup G_1$ such that $G \setminus D$ is connected. Since $C'$ and $G_2$ are unbalanced, $G \setminus D$ is flow-admissible, so $G$ is cover-decomposable. Hence, $G_1$ is an unbalanced circuit of size at most 2 or $G_1$ consists of a balanced triangle and a loop, so is $G_2$ by symmetry. By simple computation, $G$ is cover-decomposable or has a 6-cover.

Assume that $C'$ is balanced. Then $C' \in V_2(H)$ is a triangle. So $C$ is unbalanced and $|V_2(C) \cap V_2(C)| = 2$ by Lemma 3.1. Let $u, v, P_1, P_2$ be defined as above. Let $e$ be the loop incident with $C'$ and $f$ the edge in $C'$ whose ends are $u, v$. Since $C$ is unbalanced, $P_1 \cup \{f\}$ is balanced and $P_2 \cup \{f\}$ is unbalanced. Evidently, (a) a component of $G \setminus C$ properly intersects with $P_1$, otherwise $P_1 \cup \{f\}$ and its complement
are flow-admissible; and (b) no component of $G \setminus C$ intersects $P_2 - \{u, v\}$, otherwise the union $G'$ of $P_2 \cup \{f\}$ and all components of $G \setminus C$ intersecting $P_2 - \{u, v\}$ and $G' \setminus G'$ are flow-admissible. Then $P_2 \cup (C' - \{f\}) \cup \{e\}$ and its complement are flow-admissible, a contradiction.

We may therefore assume that $C'$ is unbalanced with $|C'| \geq 3$, implying that $C$ is unbalanced by Lemma 3.1. By the choice of $C'$, for each component $G'$ of $G \setminus C$, either $G'$ is a loop or $|G'| \geq 3$. When $|G'| \geq 3$, $C \cup G'$ is a necklace by Lemma 3.2. Let $D$ be a small circuit of $C \cup C'$. Since $G \setminus D$ has two edge-disjoint unbalanced circuits, $G \setminus D$ is disconnected, so a component $G_D$ of $G \setminus C$ properly intersects in $C \cap D$. Since $C \cup C'$ has three small circuits, $G_D$ is the unique component of $G \setminus C$ properly intersecting in $C \cap D$ and $C \cup C'$ has exactly three small circuits, implying $|C'| = 3$, otherwise $G$ is cover-decomposable. When $G_D$ is not a loop, there is a small circuit $D'$ of $C \cup G_D$ such that $G' \setminus D'$ is connected, so $G$ is cover-decomposable. Hence, $G_D$ is a loop. By the choice of $C'$, each component $G'$ of $G \setminus C$ that is not a loop is an unbalanced triangle. When $C'$ is the unique component of $G \setminus C$ that is not a loop, $G$ has a 3-cover. When there is another component $G_1$ of $G \setminus C$ that is not a loop, let $D$ be a small circuit of $C \cup C'$ intersecting $G_1$. Let $G'$ be the union of $D \cup G_1$ and the loop incident with $D$. Then $G'$ and $G \setminus G'$ are flow-admissible, so $G$ is cover-decomposable.

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 which is restated here in a slightly different way.

**Theorem 4.1.** Every flow-admissible signed Eulerian graph has a 6-cover.

**Proof.** Assume that the result is not true. Let $G$ be a counterexample with $|V(G)|$ as small as possible. Evidently,

4.1.1. • $G$ is unbalanced with $\delta(G) \geq 4$;

• $G$ has no balanced loops; and

• $G$ is not cover-decomposable, in particular, if $C$ is a non-separating balanced circuit of $G$, then $G \setminus C$ is not flow-admissible.

4.1.2. $G$ is 2-connected.

**Subproof.** Assume otherwise. There are edge-disjoint Eulerian subgraphs $G_1, G_2$ of $G$ with $|E(G_1)|, |E(G_2)| \geq 2$, with $\{v\} = V(G_1) \cap V(G_2)$, and with $E(G) = E(G_1) \cup E(G_2)$. Since $G$ is a minimal counterexample and not cover-decomposable, $G_1$ and $G_2$ are unbalanced.
Let $G_i^+$ be a signed graph obtained from $G_i$ by adding an unbalanced loop $e_i$ incident with $v$ for each integer $1 \leq i \leq 2$. Since $G_1^+$ and $G_2^+$ are flow-admissible, both of them have 6-covers by the choice of $G$. Since $|V(G_1) \cap V(G_2)| = 1$, we can obtain a 6-cover of $G$ by combining 6-covers of $G_1^+$ and $G_2^+$, a contradiction. 

Let $\mathcal{C}$ be an optimal circuit decomposition of $G$ and $H$ the graph determined by $\mathcal{C}$. Since $G$ is connected, so is $H$. By Lemma 2.6, at least two members of $\mathcal{C}$ are unbalanced. Hence, by Lemma 3.2, $|V(H)| \geq 3$ and the following holds. If a block of $H$ contains exactly one cut-vertex of $H$, we say the block is a leaf block.

**4.1.3.** Each balanced vertex of $H$ is a cut-vertex, in particular, each vertex in a leaf block of $H$ that is not a cut-vertex is unbalanced.

By 4.1.3 or the third part of 4.1.1 for any vertex $C$ of $H$, all components of $G \setminus C$ are unbalanced. For a subgraph $H'$ of $H$, each vertex $v \in V(H')$ is labeled by a circuit $C_v$ in $\mathcal{C}$. We say that the subgraph $G' = \bigcup_{v \in V(H')} E(C_v)$ corresponds to $H'$.

**4.1.4.** Let $e$ be a cut-edge of $H$ whose ends are $C_i$ and $C_j$. If $e$ is not a leaf edge and $H - \{C_i, C_j\}$ has exactly two components, then $C_i$ or $C_j$ is unbalanced.

**Subproof.** Assume to the contrary that $C_i$ and $C_j$ are balanced. Let $G_1$ and $G_2$ be the subgraphs of $G$ corresponding to the two components of $H - \{C_i, C_j\}$ with $V(G_1) \cap V(G_i) \neq \emptyset$. It follows from 4.1.3 that $G_1, G_2$ are unbalanced. Moreover, since $G$ is 2-connected, by Lemma 3.1 we have $|V_G(C_i) \cap V_G(C_j)| = 2$. Let $u \in V_G(G_1) \cap V_G(C_i)$ and $v \in V_G(G_2) \cap V_G(C_j)$. Since $|V_G(C_i) \cap V_G(C_j)| = 2$, the graph $C_i \cup C_j$ has a circuit $C$ avoiding $u$ and $v$ such that $(C_i \cup C_j) \setminus C$ is connected up to isolated vertices. Since $H - \{C_i, C_j\}$ has exactly two components, $G \setminus C$ is connected, so $G \setminus C$ is flow-admissible. Moreover, since $C_i \cup C_j$ is balanced by Lemma 3.1, $C$ is balanced, so $G$ is cover-decomposable, a contradiction.

**4.1.5.** For any separating circuit $C \in \mathcal{C}$, if $G'$ is a component of $G \setminus C$ that is not flow-admissible, then one of the following holds.

1. $G'$ is an unbalanced circuit such that $|G'| \leq 2$ or $C \cup G'$ is a necklace. In particular, when $C$ is balanced, $|G'| \leq 2$.

2. $G'$ consists of a loop and a balanced triangle.

**Subproof.** When $G'$ is a circuit, since $\delta(G) \geq 4$, by Lemmas 3.1 and 3.2 (1) holds. Assume that $G'$ is not a circuit. When $G'$ consists of exactly two edge-disjoint circuits that share exactly one vertex, since
C only shares vertices with the balanced circuit of $G'$ and $\delta(G) \geq 4$, the unbalanced circuit $C'$ in $G'$ has at most two edges. When $|C'| = 2$, there is a non-separating balanced circuit of $G$ contained in $C \cup C'$, a contradiction. So $C'$ is a loop. By Lemma 3.1 and 4.1.2 the balanced circuit in $G'$ is a triangle, so (2) holds. Hence, we may assume that $\Delta(G') \geq 6$ or $|V_4(G')| \geq 2$.

Since $G'$ is not flow-admissible, by switching we may assume that there is a unique edge $e$ of $G'$ labelled by $-1$ and all other edges in $G'$ are labelled by 1. When $e$ is a loop, let $v$ be the end of $e$, and $B$ a block of $G' \setminus \{e\}$ containing $v$, and let $C'$ be a circuit of $B$ containing $v$; otherwise, let $\{v\} = \emptyset$, and $B$ the block containing $e$, and let $C'$ be a circuit of $B$ with $e \in C'$. If possible, we may further assume that $C'$ is chosen with $V_G(C') \cap V_2(G') \neq \emptyset$. By Lemma 2.1, there is a circuit $C_1$ of $G' \setminus \text{loops}(G')$ with $C' \cap C_1 = \emptyset$ such that $G' \setminus C_1$ is connected up to isolated vertices. Since $C_1$ is balanced and $G \setminus C_1$ has two edge-disjoint unbalanced circuits, $G \setminus C_1$ is not connected. Hence, $V_G(C) \cap V(G') \subseteq V_G(C_1)$ and $\emptyset \neq V_2(G') \subseteq V_G(C_1)$ as $e$ is the only edge in $G'$ which has a chance to be a loop. By the choice of $C'$, the set $V_2(G')$ is contained in another block $B'$ of $G'$ with $B \neq B'$ as $C'$ contains no vertex in $V_2(G')$. Since $V_G(C) \cap V(G') \subseteq V(B')$ and $G$ is 2-connected, $|B| = 1$, a contradiction to the choice of $B$.  

4.1.6. For any $C \in \mathcal{C}$, the graph $G \setminus C$ has at most two components.

Subproof. Assume that $G \setminus C$ has three components. Since each component $G'$ of $G \setminus C$ is unbalanced and $G \setminus G'$ is flow-admissible, $G'$ is not flow-admissible. By 4.1.5 $H$ is a tree with $C$ as a unique vertex of degree at least three, and all its pendant paths have at most two edges. When $C$ is balanced, 4.1.4 implies that $V_2(H) = \emptyset$. Hence, by 4.1.5 and Lemma 3.5 $G$ is cover-decomposable or has a 6-cover, a contradiction.

4.1.7. For any balanced vertex $C$ of $H$, each degree-1 vertex of $H$ adjacent with $C$ is a loop of $G$.

Subproof. Let $C'$ be a degree-1 vertex of $H$ adjacent with $C$. Assume that $C'$ is not a loop of $G$. Then $|C'| = |V_G(C) \cap V_G(C')| = 2$ by 4.1.5. It follows from Lemma 3.4 and 4.1.6 that $G$ is cover-decomposable, a contradiction.

4.1.8. $H$ is not a tree.

Subproof. Assume otherwise. By 4.1.6 $H$ is a path. Evidently, at most one vertex in $V_2(H)$ is unbalanced, otherwise, $G$ is cover-decomposable. By 4.1.4 no balanced vertices of $H$ are adjacent, so $|V(H)| \leq 5$.  

Moreover, if $|V(H)| \geq 4$, then exactly one vertex in $V_2(H)$ is unbalanced. Assume that $H$ has two adjacent vertices $C_1, C_2$ with $|V_G(C_1) \cap V_G(C_2)| \geq 3$. Then $C_1 \in V_1(H)$, $|V(H)| \leq 4$ and $C_1 \cup C_2$ is a necklace by Lemma 3.2. Let $C_3$ be the other vertex adjacent to $C_2$ in $H$. When $V_G(C_2) \cap V_G(C_3)$ is in a small circuit of $C_1 \cup C_2$, the graph $G$ has a 6-cover. When $V_G(C_2) \cap V_G(C_3)$ is not in a small circuit of $C_1 \cup C_2$, implying $|V_G(C_2) \cap V_G(C_3)| = 2$, since $V_G(C_1) \cap V_G(C_3) = \emptyset$, the graph $C_1 \cup C_2$ can be decomposed to two long circuits $C'_1, C'_2$, where both share exactly one vertex with $C_3$. Note that the circuit decomposition $(C - \{C_1, C_2\}) \cup \{C'_1, C'_2\}$ is still optimal. Hence, the graph determined by $(C - \{C_1, C_2\}) \cup \{C'_1, C'_2\}$ is isomorphic to a graph pictured as Figure 1 (c) or (d). Lemma 3.3 implies that $G$ is cover-decomposable or has a 6-cover. Therefore, combined with Lemma 3.1 we can assume that every pair of adjacent vertices in $H$ share at most two vertices in $G$. Note that each degree-1 vertex of $H$ adjacent to a balanced vertex is a loop by 4.1.7. By simple computation, $G$ has a 6-cover, a contradiction.

\[\square\]

4.1.9. $H$ is not 2-connected and whose leaf blocks are isomorphic to $K_2$.

Subproof. Assume otherwise. When $H$ is not 2-connected, let $B$ be a leaf block of $H$ that is not isomorphic to $K_2$, and $v$ be the unique cut-vertex of $H$ in $V(B)$. When $H$ is 2-connected, let $B = H$ and $v$ any vertex of $B$. By Lemma 2.2, there is an edge $e$ in $B - v$ such that $B - V_H(e)$ is connected, so $H - V_H(e)$ is also connected. Without loss of generality assume that $C_1$ and $C_2$ are the ends of $e$. Then $C_1 \cup C_2$ and $G \backslash (C_1 \cup C_2)$ are connected. Since $C_1 \cup C_2$ is flow-admissible by 4.1.3 the graph $G \backslash (C_1 \cup C_2)$ is not flow-admissible. Since $H$ is not isomorphic to the graph pictured as Figure 1 (d) by Lemma 3.3, $H$ has exactly three unbalanced vertices and exactly two leaf blocks, one of which is $B$ that is isomorphic to $K_3$ and the other is isomorphic to $K_2$. Let $C_1C_2C_3 \ldots C_n$ be a longest path in $H$. It follows from 4.1.4 that $n = 4$. By 4.1.7, the circuit $C_4$ is a loop of $G$. That is, $H$ is isomorphic to the graph pictured as Figure 1 (c). Hence, $G$ is cover-decomposable or has a 6-cover by Lemma 3.3 a contradiction.

\[\square\]

Let $B$ be a block of $H$ with $|V(B)| \geq 3$. By 4.1.8 and 4.1.9 such $B$ exists and $B$ is not a leaf block. When $H$ has two blocks that are not isomorphic to $K_2$, it follows from 4.1.3 and 4.1.9 that $G$ is cover-decomposable. Hence, $B$ is the unique block of $H$ that is not isomorphic to $K_2$. By 4.1.3 each vertex in $B$ that is not a cut-vertex of $H$ is unbalanced.
Let $u \in V(B)$ be a cut vertex of $H$. When $u$ is unbalanced or $H$ has two pendant paths using $u$, let $H_1$ be the union of all pendant paths containing $u$, and $G_1$ the subgraph of $G$ corresponding to $H_1$. Since $|V(B)| \geq 3$, by Proposition 4.1.3 and Proposition 4.1.9, both $G_1$ and $G \backslash G_1$ are flow-admissible, a contradiction. Hence, $u$ is balanced and $H$ has exactly one pendant path using $u$. By the arbitrary choice of $u$, all cut-vertices of $H$ in $B$ are balanced. Using a similar strategy, all vertices in $V_2(H) \setminus V(B)$ are balanced. Combined with Proposition 4.1.4, we have $V_2(H) \setminus V(B) = \emptyset$. That is, each pendant path of $H$ has exactly one edge. By Proposition 4.1.7, each vertex in $V_1(H)$ is a loop of $G$.

When there is a vertex in $V(B)$ that is not a cut-vertex of $H$, let $v$ denote such a vertex. Otherwise, let $v$ be any vertex of $B$. By Lemma 2.2, there is an edge $e \in B - v$ such that $B - V(e)$ is connected. Let $H_1$ be the union of $e$ and all pendant paths of $H$ using an end of $e$, and $G_1$ be the subgraph of $G$ corresponding to $H_1$. Since each vertex in $B$ that is not a cut-vertex of $H$ is unbalanced, $H_1$ contains two unbalanced vertices, so $G_1$ is flow-admissible. Since $H - V(H_1)$ is connected and has an unbalanced vertex, $H$ is isomorphic to a graph pictured as Figure 1 (a) or (b). Lemma 3.3 implies that $G$ is cover-decomposable or has a 6-cover, a contradiction. \[\square\]

Acknowledgements

We thank the referees for their careful reading of this manuscript and their detailed comments.

References

[1] J. C. Bermond, B. Jackson, F. Jaeger, Shortest covering of graphs with cycles, J. Combin. Theory Ser. B, 35 (1983) 297-308.
[2] R. Chen, M. DeVos, D. Funk, I. Pivotto, Graphic representation of graphic frame matroids, Graphs and Combin., 31 (2015), 2075-2086.
[3] J. Cheng, Y. Lu, R. Luo, C. Q. Zhang, Shortest circuit cover of signed graphs, J. Combin. Theory Ser. B, 134 (2019), 164-178.
[4] G. Fan, Integer flows and cycle covers, J. Combin. Theory Ser. B, 54 (1992) 113-122.
[5] G. Fan, Flows and circuit covers in signed Graphs, Lectures in NSFC Tianyuan Summer School, Jinhua, 2018.
[6] E. Mácajová, M. Skoviera, Nowhere-zero flows on signed Eulerian graphs, SIAM J. Discrete Math., 31 (2017), 1937-1952.
[7] T. Zaslavsky, Biased graphs. I. Bias, balanced, and gains, J. Combin. Theory Ser. B, 47 (1989), 32-52.
[8] T. Zaslavsky, Biased graphs. II. The three matroids. J. Combin. Theory Ser. B, 51 (1991), 46-72.
Center for Discrete Mathematics, Fuzhou University, Fuzhou, P. R. China.

Email address: tomcat0830@163.com (B. Bao)

Email address: rongchen@fzu.edu.cn (R. Chen)

Email address: fan@fzu.edu.cn (G. Fan)