A NOTE ON THE MOMENT OF COMPLEX WIENER-ITÔ INTEGRALS

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Abstract. For a sequence of complex Wiener-Itô multiple integrals, the equivalence between the convergence of the symmetrized contraction norms and that of the non-symmetrized contraction norms is shown directly by means of a new version of complex Mallivian calculus using the Wirtinger derivatives of complex-valued functions.

Keywords: Complex Wiener-Itô Integrals; Fourth Moment theorems; Ornstein-Uhlenbeck Operator.

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1. Introduction

Recently, fourth moment theorems are extended to the case of complex multiple stochastic integrals with different methods [2, 4, 6]. S.Campese [2] uses Stein’s method for a general context of Markov diffusion generators. [6] is essentially by reduction to the two-dimensional real-valued case. [4] is an adaption of the classical arguments by D. Nualart, G. Peccati and S. Ortiz-Latorre for the one-dimensional real-valued case in [9, 11]. That is to say, in [4] they show the five equivalent conditions by means of (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) ⇒ (i).

Since in the real case there is a direct and short proof [9, p100] for the equivalence between conditions (iii) and (iv), i.e., the convergence of the symmetrized contraction norms and that of the non-symmetrized contraction norms, the question naturally arises whether there is still a direct proof to that equivalence in the complex case. The key aim of this note is to give an affirmative answer to the above question.

To state the theorem we denote $H$ a complex separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$ and let $Z = \{ Z(h) : h \in H \}$ be a complex isonormal Gaussian process over $H$, i.e., the complexification of the classical real isonormal Gaussian process (see Example 1.9 of [8] or Definition 2.6 of [6]). The complex Wiener-Itô (multiple) integrals is an isometric mapping $I_{m,n}$ from $H^\otimes m \times H^\otimes n$ to $L^2(\Omega, \sigma(Z))$ (see Definition 2.10 of [6]). Now the theorem is stated as follows.

Theorem 1.1. Let $\{ F_k = I_{m,n}(f_k) \}$ with $f_k \in H^\otimes m \otimes H^\otimes n$ be a sequence of $(m, n)$-th complex Wiener-Itô multiple integrals, with $m$ and $n$ fixed and $m + n \geq 2$. Then the following statements are equivalent:

(iii) $\| f_k \otimes_i j f_k \|_{H^\otimes (2l-1-l)} \to 0$ and $\| f_k \otimes_i j h_k \|_{H^\otimes (2l-1-l)} \to 0$ for any $0 < i + j \leq l - 1$ where $l = m + n$ and $h_k$ is the kernel of $F_k$, i.e., $F_k = I_{n,m}(h_k)$.

(iv) $\| f_k \otimes i, j f_k \|_{H^\otimes (2l-1-l)} \to 0$ and $\| f_k \otimes i, j h_k \|_{H^\otimes (2l-1-l)} \to 0$ for any $0 < i + j \leq l - 1$. 

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The proof of the above theorem is a direct application of the following proposition which gives an expression of the fourth moment of a complex Wiener-Ito integral by means of the sum of the inner products of some symmetrized contractions.

**Proposition 1.2.** Suppose that $F = I_{m,n}(f)$ with $f \in \mathcal{S}^m \otimes \mathcal{S}^n$ and that $\hat{F} = I_{n,m}(h)$. Then

$$
\mathbb{E}[|F|^4] - 2(\mathbb{E}[|F|^2])^2 - \mathbb{E}[|F|^2]^2 
= 2 \sum_{r=1}^{l-1} [(l-r)!]^2 \langle \vartheta_r, \psi_r \rangle_{\mathcal{S}^{2l-2(r)}(\mathbb{C}^n)} + \sum_{r=1}^{l'-1} (2m-r)! (2n-r)! \langle \varsigma_r, \varphi_r \rangle_{\mathcal{S}^{2l-2(r)}(\mathbb{C}^n)},
\tag{1.1}
$$

where $l = m + n$, $l' = 2(m \wedge n)$ and

$$
\vartheta_r = \sum_{i+j=r} \frac{i}{m} \binom{m}{i} \binom{n}{j} i^{j} l^{j} f_{i,j} h, \tag{1.2}
$$

$$
\psi_r = \sum_{i+j=r} \frac{i}{m} \binom{m}{i} \binom{n}{j} i^{j} l^{j} f_{i,j} h, \tag{1.3}
$$

$$
\varsigma_r = \sum_{i+j=r} \frac{i}{m} \binom{m}{i} \binom{n}{j} i^{j} l^{j} f_{i,j} f, \tag{1.4}
$$

$$
\varphi_r = \sum_{i+j=r} \frac{i}{m} \binom{m}{i} \binom{n}{j} i^{j} l^{j} f_{i,j} f. \tag{1.5}
$$

Similar to the real case [9, p97], the key idea of the proof of Proposition 1.2 is using the complex Mallivian calculus. We have to exploit a new version of complex Mallivian derivative $D$, its adjoint operator $\delta$ and a complex Ornstein-Uhlenbeck operator $L = \delta D$ which is distinct from the known versions of complex Mallivian calculus in [1] or [8].

2. Preliminaries: Concise Complex Malliavin Calculus

2.1. Malliavin derivative operators. The following definition of complex Malliavin derivatives which makes use of the Wirtinger derivatives of complex-valued functions is distinct from what the authors defined in [1] or [8] and is easier to use in our case.

**Definition 2.1.** Let $\mathcal{S}$ denote the set of all random variables of the form

$$
f(Z(\varphi_1), \cdots, Z(\varphi_m)), \tag{2.1}
$$

where $f \in C_0^\infty(\mathbb{C}^n)$ and $\varphi_i \in \mathcal{S}, i = 1, 2, \cdots, m$. If $F \in \mathcal{S}$, then the complex Malliavin derivatives of $F$ (with respect to $\zeta$) are the elements of $L^2(\Omega, \mathcal{F})$ defined by [4, 6]:

$$
DF = \sum_{i=1}^{m} \partial_i f(Z(\varphi_1), \cdots, Z(\varphi_m)) \varphi_i, \tag{2.2}
$$

$$
\bar{D}F = \sum_{i=1}^{m} \bar{\partial}_i f(Z(\varphi_1), \cdots, Z(\varphi_m)) \bar{\varphi}_i, \tag{2.3}
$$

where

$$
\partial_j f = \frac{\partial}{\partial z_j} f(z_1, \ldots, z_m), \quad \bar{\partial}_j f = \frac{\partial}{\partial \bar{z}_j} f(z_1, \ldots, z_m), \quad j = 1, \ldots, m.
$$
are the Wirtinger derivatives.

The above definition implies that $\overline{DF} = \bar{D}F$. The following proposition gives an integration by parts formula of complex Gaussian random variables, whose proof is straightforward. Please refer to Lemma 3.2 of [5] or Lemma 2.3 of [2].

**Proposition 2.2** (integration by parts formula). Suppose that $F \in \mathcal{S}$ and $h \in \mathcal{S}$, then we have the following integration by parts formula

$$
E[Z(h) \times F] = E[(h, DF)], \quad E[Z(h) \times \bar{F}] = E[(h, \bar{D}F)].
$$

It is routine to show that $D$ and $\bar{D}$ are closable from $L^p(\Omega)$ to $L^p(\Omega, \mathcal{S})$. Denote by $\mathbb{D}^{1,p}$ and $\bar{\mathbb{D}}^{1,p}$ the closure of $\mathcal{S}$ with respect to the Sobolev seminorm $\| \cdot \|_{1,p}$. The following proposition is an adaption of the real-valued case which gives a sufficient condition to check a random belonging to the domain $\mathbb{D}^{1,2}$ or $\bar{\mathbb{D}}^{1,2}$, please see for example [10].

**Proposition 2.3.** Let $\{F_n, n \geq 1\}$ be a sequence of random variable in $\mathbb{D}^{1,2}$ (resp. $\bar{\mathbb{D}}^{1,2}$) that converges to $F$ in $L^2(\Omega)$ and that

$$
\sup_n E[\|DF_n\|_p^2] < \infty, \quad (\text{resp. } \sup_n E[\|\bar{D}F_n\|_p^2] < \infty),
$$

then $F$ belongs to $\mathbb{D}^{1,2}$ (resp. $\bar{\mathbb{D}}^{1,2}$) and the sequence of derivatives $DF_n$ (resp. $\bar{D}F_n$) converges weakly to $DF$ (resp. $\bar{D}F$) in $L^2(\Omega, \mathcal{S})$.

By the chain rules of Wirtinger derivatives [2], we obtain the following chain rules of complex Malliavin derivatives.

**Proposition 2.4.** (Chain rule) If $\varphi : \mathbb{C}^m \to \mathbb{C}$ is a continuously differentiable function with bounded partial derivatives and if $F = (F_1, \ldots, F_m)$ is a random vector whose components are elements of $\mathbb{D}^{1,2} \cap \bar{\mathbb{D}}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2} \cap \bar{\mathbb{D}}^{1,2}$ and

$$
D\varphi(F) = \sum_{j=1}^{m} \partial_j \varphi(F) DF^j + \bar{\partial}_j \varphi(F) \bar{D}F^j, \quad (2.4)
$$

$$
\bar{D}\varphi(F) = \sum_{j=1}^{m} \partial_j \varphi(F) \bar{D}F^j + \bar{\partial}_j \varphi(F) \bar{D}F^j. \quad (2.5)
$$

**Remark 2.5.** To compare with Theorem 15.34 of [8, p238], we find that our definitions of complex Malliavin derivative are different with Janson’s Definition 15.26 [8, p236].

We define the divergence operators $\delta$ and $\bar{\delta}$ as the adjoint of $D$ and $\bar{D}$ respectively, with the domains $\text{Dom}(\delta)$ and $\text{Dom}(\bar{\delta})$ the subsets of $L^2(\Omega, \mathcal{S})$ composed of those elements $u$ such that there exists a constant $c > 0$ verifying for all $F \in \mathcal{S}$,

$$
|E[(DF, u)]| \leq c \|F\|, \quad (\text{resp. } |E[(\bar{D}F, u)]| \leq c \|F\|).
$$

If $u \in \text{Dom}(\delta)$ or $u \in \text{Dom}(\bar{\delta})$, then $\delta u$ and $\bar{\delta} u$ are the unique element of $L^2(\Omega)$ given respectively by the following duality formula: for all $F \in \mathcal{S},$

$$
E[(\delta u) \times F] = E[(u, DF)], \quad (\text{resp. } E[(\bar{\delta} u) \times \bar{F}] = E[(u, \bar{D}F)]). \quad (2.6)
$$
2.2. Complex Ornstein-Uhlenbeck operators. We define complex Ornstein-Uhlenbeck operators which are different with that in [8].

Definition 2.6. Complex Ornstein-Uhlenbeck operators are defined as

\[ L = \delta D, \quad \bar{L} = \bar{\delta} \bar{D}. \] (2.7)

Proposition 2.7. Suppose that \( I_{m,n}(f) \) is the complex Wiener-Itô integral of \( f \) with respect to \( Z \) for any \( f \in \mathcal{S}_m \otimes \mathcal{S}_n \). Then we have that

\[ D(I_{m,n}(f)) = mI_{m-1,n}(f), \quad \bar{D}(I_{m,n}(f)) = nI_{m,n-1}(f), \] (2.8)

\[ L(I_{m,n}(f)) = mI_{m,n}(f), \quad \bar{L}(I_{m,n}(f)) = nI_{m,n}(f). \] (2.9)

Proof. First, we claim that a complex Hermite polynomials \( J_{m,n}(z, \rho) \) satisfies that

1) partial derivatives:

\[ \frac{\partial}{\partial z} J_{m,n}(z, \rho) = mJ_{m-1,n}(z, \rho), \quad \frac{\partial}{\partial \rho} J_{m,n}(z, \rho) = nJ_{m,n-1}(z, \rho), \] (2.10)

\[ \frac{\partial}{\partial \rho} J_{m,n}(z, \rho) = -mnJ_{m-1,n-1}(z, \rho). \] (2.11)

2) recursion formula:

\[ J_{m+1,n}(z, \rho) = zJ_{m,n}(z, \rho) - n\rho J_{m,n-1}(z, \rho), \] (2.12)

\[ J_{m,n+1}(z, \rho) = zJ_{m,n}(z, \rho) - m\rho J_{m-1,n}(z, \rho). \] (2.13)

In fact, about Eq.(2.10), please refer to Theorem 12 (D) of [7] or Proposition A.6 of [5]. Eq.(2.11) is obtained by taking partial derivative \( \frac{\partial}{\partial \rho} \) in both sides of the generating function of complex Hermite polynomials. Eq.(2.12)-(2.13) are shown in Theorem 12 (C) of [7] and [5, p15].

Second, suppose \( f = h^{\otimes m} \otimes \bar{h}^{\otimes n} \) with \( h \in \mathfrak{h} \). Denote \( \rho = ||h||^2 \) and \( \bar{s} = (s_1, \ldots, s_k) \). Then we obtain that

\[ D(I_{m,n}(f)) = D(J_{m,n}(Z(h), \rho)) = mJ_{m-1,n}(Z(h), \rho)h(\cdot) = mI_{m-1,n}(h^{\otimes m-1} \otimes \bar{h}^{\otimes n})h(\cdot) = mI_{m-1,n}(Z(h), \rho). \]

Denote \( G = I_{m-1,n}(h^{\otimes m-1} \otimes \bar{h}^{\otimes n}) \), then we have that \( DG = nI_{n-1,m-1}(h^{\otimes n-1} \otimes \bar{h}^{\otimes m-1})h \) and that

\[ L(I_{m,n}(f)) = m\delta(Gh) = m[\delta(GZ(h)) - \langle h, DG \rangle] = m[\delta(Z(h))J_{m-1,n}(Z(h), \rho) - n\rho J_{m,n-1}(Z(h), \rho)] = mJ_{m,n}(Z(h), \rho) = mI_{m,n}(f). \]

Similarly, we have that \( \bar{D}(I_{m,n}(f)) = nI_{m,n-1}(\bar{h}^{\otimes n-1} \otimes \bar{h}^{\otimes m-1})h \) and that \( \bar{L}(I_{m,n}(f)) = nI_{m,n}(f). \)
Finally, by means of density arguments (or the polarization technique), it is easily to show that (2.8)-(2.9) hold.

\[\square\]

### 3. Proof of the main theorems

To compare Lemma 2.3 of [4] with our findings, we list it as follows.

**Lemma 3.1.** Suppose that \( F = I_{m,n}(f) \) with \( f \in \mathcal{H}^* \) and that \( \tilde{F} = I_{n,m}(h) \). Then
\[
E[|F|^4] - 2(E[|F|^2])^2 - |E[F^2]|^2
= \sum_{0 < j < l} \left( \binom{m}{i} \binom{n}{j} \right) (m!n!)^2 \|f \otimes_{i,j} f\|^2_{\mathcal{H}^*} + \sum_{r=1}^{l-1} ((l-r)!)^2 \|\psi_r\|^2_{\mathcal{H}^*} + 2(2m)!(2n)!(2m-n)!(m+n)!\|\varphi_r\|^2_{\mathcal{H}^*},
\]
where \( l = m + n + l' = 2(m \land n) \) and \( \psi_r, \varphi_r \) are as in Proposition 1.2.

**Proof of Proposition 1.2.** We divide the proof into several steps.

Step 1: We claim that
\[
\frac{1}{m} E[|F|^2 \|DF\|^2_{\mathcal{H}}] = \left( E[|F|^2]\right)^2 + \sum_{r=1}^{m+n-1} \left( (m+n-r)! \langle \vartheta_r, \psi_r \rangle_{\mathcal{H}^*} \right),
\]
(3.1)

In fact, it follows from the product formula of complex Wiener-Itô integrals [3] and the Fubini theorem that
\[
\frac{1}{m} \|DF\|^2_{\mathcal{H}} = m \|I_{m-1,n}(f)\|^2_{\mathcal{H}}
= m \sum_{i=0}^{m-1} \sum_{j=0}^{n} \left( \binom{m-1}{i} \binom{n}{j} \right) (m-1)! I_{m-1-i-j,n-1-m+n-i-j} (f \otimes_{i,j} h)
= m \sum_{i=1}^{m} \sum_{j=0}^{n} \left( \binom{m-1}{i} \binom{n}{j} \right) (i-1)! I_{m-n-i-j,n-1-m+n-i-j} (f \otimes_{i,j} h)
= E[|F|^2] + \sum_{r=1}^{m+n-1} I_{m+n-r,m+n-r}(\vartheta_r).
\]
(3.2)

On the other hand, we can obtain that
\[
|F|^2 = \sum_{i=0}^{m} \sum_{j=0}^{n} \left( \binom{m}{i} \binom{n}{j} \right) i!j! I_{m+n-i-j,m+n-i-j} (f \otimes_{i,j} h)
= E[|F|^2] + \sum_{r=0}^{m+n-1} I_{m+n-r,m+n-r}(\psi_r).
\]
(3.3)

Substituting (3.2) and (3.3) into the left side of (3.1) and using the orthogonality properties of multiple integrals, we have that (3.1) holds.
In fact, the product formula and the Fubini theorem implies that
\[ \text{any} \quad \text{Cauchy-Schwarz inequality imply that as} \quad \text{multiple integrals, we have that} \]
\[ \left( \sum_{r=1}^{2(m,n)-1} (2m - r)!(2n - r)! \langle \xi_r, \varphi_r \rangle \right)^2 \leq 2 \sum_{r=1}^{2(m,n)-1} (2m - r)!(2n - r)! \langle \xi_r, \varphi_r \rangle \text{.} \]  
(3.4)

In fact, for the function \( g \), we take \( \chi_{[0,1]}(z) \) a cut-off function. For any \( p \geq 1 \), \( g \in C^\infty_c(\mathbb{R}) \) and \( g, \partial g, \partial^2 g \) converge to \( g, \partial g, \partial^2 g \) respectively in the sense of \( L^p(\mu) \) with \( F \sim \mu \). The chain rule, i.e., Proposition 2.4, implies that
\[ D(g_n(F)) = \partial g_n(F)DF + \partial^2 g_n(F)D^2F \text{.} \]  
(3.7)

Substituting (3.5) and (3.6) into the left side of (3.4) and using the orthogonality properties of multiple integrals, we have that (3.4) holds.

Step 3: By approximation, we claim that for any Wiener-Ito integral \( F = I_{m,n}(f) \),
\[ D(\hat{F} F^2) = 2 |F|^2 DF + F^2 D\hat{F} \text{.} \]  
(3.7)

where \( \chi_A(\cdot) \) the index function of a set \( A \) and \( k(x) = e^{-\frac{1}{x^2}}\chi_{(0,1)}(x) \) a cut-off function. For any \( p \geq 1 \), \( g \in C^\infty_c(\mathbb{R}) \) and \( g, \partial g, \partial^2 g \) converge to \( g, \partial g, \partial^2 g \) respectively in the sense of \( L^p(\mu) \) with \( F \sim \mu \). The chain rule, i.e., Proposition 2.4, implies that
\[ D(g_n(F)) = \partial g_n(F)DF + \partial^2 g_n(F)D^2F \text{.} \]  
(3.7)

Then we obtain (3.7) by Proposition 2.3.

Step 4: It follows from Proposition 2.7, the dual relation and the chain rule that
\[ \mathbb{E}[|F|^4] = \mathbb{E}[F^{2}|F^{2}|] \]
\[
\begin{align*}
\text{COMPLEX MALLIAVIN CALCULUS} & = \frac{1}{m} \mathbb{E} [\delta D F \times \bar{F} F^2] \\
& = \frac{1}{m} \mathbb{E} [(D F, D(\bar{F} F^2))_B] \\
& = \frac{1}{m} [2 \| D F \|_B^2 \times |F|^2 + (D F, D F)_B \times \bar{F}^2].
\end{align*}
\]

By substituting (3.1) and (3.4) into the above equality displayed, we obtain (1.1). \qed

**Proof of Theorem 1.1.** (iii) implies (iv) is elementary. Now suppose that (iv) holds. Then the Cauchy-Schwarz inequality implies that as \( k \to \infty \),

\[
|\langle \varphi_r, \psi_r \rangle_B^{2(1-r)}| \leq \sum_{i+j=r}^{2} \sum_{i'+j'=r}^{2} \frac{i}{m} \binom{m}{i} \binom{n}{j} i! j! \left( \binom{m}{i'} \binom{n}{j'} i'! j'! \right) \left( \| f_k \bar{\otimes}_{i,j} h_k \|_{B^{2(1-r)}} \right)^2 \left( \| f_k \bar{\otimes}_{i',j'} h_k \|_{B^{2(1-r)}} \right)^2 \to 0.
\]

In the same way, we can obtain that as \( k \to \infty \),

\[
|\langle \varsigma_r, \varphi_r \rangle_N^{2(1-r)}| \to 0.
\]

Proposition 1.2 combining with the above two equalities displayed implies that as \( k \to \infty \),

\[
\mathbb{E}[|F|^4] - 2(\mathbb{E}[|F|^2])^2 - |\mathbb{E}[F^2]|^2 \to 0,
\]

which implies that (iii) holds from Lemma 3.1. \qed

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