SOME REMARKS ON SIGN CHANGING SOLUTIONS OF A QUASILINEAR ELLIPTIC EQUATION IN TWO VARIABLES

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Abstract. We consider planar solutions to certain quasilinear elliptic equations subject to the Dirichlet boundary conditions; the boundary data is assumed to have finite number of relative maximum and minimum values. We are interested in certain vanishing properties of sign changing solutions to such a Dirichlet problem. Our method is applicable in the plane.

1. INTRODUCTION AND PRELIMINARIES

In this paper we consider solutions of quasilinear second order elliptic partial differential equations of the form

\[ \nabla \cdot A(x, \nabla u) = B(x, \nabla u), \]

(1.1)

where \( A: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \) and \( B: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) are Carathéodory functions under certain structural conditions discussed in Section 1.2. A noteworthy example of such equations is the \( p \)-Laplace equation

\[ \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0, \]

(1.2)

where \( 1 < p < \infty \), which gives the Laplace equation when \( p = 2 \); we refer to [13].

The result of this note is the following. Let \( G \) be a bounded simply-connected Jordan domain in \( \mathbb{R}^2 \). Suppose \( u \) is a solution to (1.1) subject to the Dirichlet boundary condition

\[ u = g \text{ on } \partial G, \]

where \( g \in W^{1,p}(G) \cap C(\overline{G}) \). We assume minimal regularity conditions on \( \partial G \) so that every boundary point is regular, and hence \( u \in C(\overline{G}) \). If \( g|_{\partial G} \) has finite number of relative maximum and minimum values, and we assume further that for all \( x \in G \) there exists \( r_x > 0 \) such that for all \( r \leq r_x \) the set \( \{ z \in B_r(x) \subset G : u(z) = 0 \} \) is connected, then if \( u \) vanishes in some open subset of \( G \), then \( u \) vanishes identically in \( G \).

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For linear equations the study of certain vanishing properties, unique continuation in particular, is rather complete [10, 11]. The non-linear case, on the other hand, is more or less open, although there are some results; we refer to [1], [3], [4], [5], [6], [8], [9], [14], and [15]. In [15], in particular, some counterexamples are constructed in the case of $p = n$ in (1.3), $n \geq 3$, and $B \equiv 0$. More precisely, the author shows that there is a solution that vanishes in the lower half space $x_n < 0$ of $\mathbb{R}^n$ but does not vanish identically in $\mathbb{R}^n$.

Our approach in this note is based on the analysis of nodal domains, which are maximal connected components of the set

$$\{x \in G : u(x) \neq 0\},$$

and nodal lines

$$\{x \in G : u(x) = 0\},$$

which are the boundaries of nodal domains. Our main tool is to couple the strong maximum principle and the Harnack inequality with some topological arguments; this argument applies in the situation in which there are finite number of nodal domains.

The topological approach taken in the paper can be applied also to the nonlinear eigenvalue problem involving the $p$-Laplacian and to more general eigenvalue problems constituting the Fučik spectrum. We refer to a recent paper [12] for more details.

Finally, we want to refer to [2] since the framework and some ideas there are somewhat related to those taken in the present note.

1.1. Notation. Throughout, $G$ is a bounded simply-connected Jordan domain of $\mathbb{R}^2$. A domain is an open connected set in $\mathbb{R}^2$. We write $B_r(x) = B(x, r)$ for concentric open balls of radii $r > 0$ centered at some $x \in G$. We denote the closure, interior, exterior, and boundary of $E$ by $\overline{E}$, $\text{int}(E)$, $\text{ext}(E)$, and $\partial E$, respectively.

1.2. Structural assumptions. Let us specify the structure of $\mathcal{A}$ and $\mathcal{B}$ in (1.1): We shall assume that there are constants $0 < a_0 \leq a_1 < \infty$ and $0 < b_1 < \infty$ such that for all vectors $h$ in $\mathbb{R}^2$ and almost every $x \in G$ the following structural assumptions apply

$$\begin{align*}
\mathcal{A}(x, h) \cdot h & \geq a_0 |h|^p, \\
|\mathcal{A}(x, h)| & \leq a_1 |h|^{p-1}, \\
|\mathcal{B}(x, h)| & \leq b_1 |h|^{p-1},
\end{align*}$$

where $1 < p < \infty$. We do not assume the monotoneity or the homogeneity of the operator $\mathcal{A}$ since we do not consider existence or uniqueness problems.

The structural conditions (1.3) result in Hölder continuity of a weak solution to (1.1), and moreover in the Harnack inequality and the strong maximum principle, we refer to Serrin [20].
We could also allow for the following structural conditions

\[
\begin{aligned}
A(x, h) \cdot h &\geq a_0 |h|^p, \\
|A(x, h)| &\leq a_1 |h|^{p-1}, \\
|B(x, h)| &\leq b_0 |h|^p + b_1 |h|^{p-1}
\end{aligned}
\]

(1.4)

where \(0 < b_0 < \infty\) and \(1 < p < \infty\). In this case local Hölder continuity and the Harnack inequality for a locally bounded weak solution of (1.1) follow from Trudinger [21].

We do not consider the case in which \(A\) and \(B\) may depend on \(u\), or, for that matter, aim at the most general structure in (1.3) or (1.4). We will only need that solutions of (1.1) are continuous, and satisfy the Harnack inequality and the strong maximum principle.

1.3. Some elements of the plane topology. We recall a few facts about the topology of planar sets; a good reference is [18]. Let \(\Omega\) be any domain in \(\mathbb{R}^2\). A Jordan arc is a point set which is homeomorphic with \([0, 1]\), whereas a Jordan curve is a point set which is homeomorphic with a circle. By Jordan’s curve theorem a Jordan curve in \(\mathbb{R}^2\) has two complementary domains, and the curve is the boundary of each component. One of these two domains is bounded and this domain is called the interior of the Jordan curve. A domain whose boundary is a Jordan curve is called a Jordan domain.

As a related note, it is well known that the boundary of a bounded simply-connected domain in the plane is connected. In the plane a simply-connected domain \(\Omega\) can be defined by the property that all points in the interior of any Jordan curve, which consists of points of \(\Omega\), are also points of \(\Omega\). A Jordan arc with one end-point on \(\partial \Omega\) and all its other points in \(\Omega\), is called an end-cut. If both end-points are in \(\partial \Omega\), and the rest in \(\Omega\), a Jordan arc is said to be a cross-cut in \(\Omega\). A point \(x \in \partial \Omega\) is said to be accessible from \(\Omega\) if it is an end-point of an end-cut in \(\Omega\). Accessible boundary points of a planar domain are aplenty: The accessible points of \(\partial \Omega\) are dense in \(\partial \Omega\) [18, p. 162].

We recall a few facts about connected sets and \(\varepsilon\)-chains. If \(x\) and \(y\) are distinct points, then an \(\varepsilon\)-chain of points joining \(x\) and \(y\) is a finite sequence of points

\[
x = a_1, a_2, \ldots, a_k = y
\]

such that \(|a_i - a_{i+1}| \leq \varepsilon\), for \(i = 1, \ldots, k - 1\). A set of points is \(\varepsilon\)-connected if every pair of points in it can be joined by an \(\varepsilon\)-chain of points in the set. A compact set \(F\) in \(\mathbb{R}^2\) is connected if and only if it is \(\varepsilon\)-connected for every \(\varepsilon > 0\) [18 Theorem 5.1, p. 81]. If a connected set of points in \(\mathbb{R}^2\) intersects both \(\Omega\) and \(\mathbb{R}^2 \setminus \Omega\) it intersects \(\partial \Omega\) [18 Theorem 1.3, p. 73].

Lastly, we recall the following topological property [18 p. 159]. A subset \(E\) of \(\mathbb{R}^2\) is said to be locally connected at any \(x \in \mathbb{R}^2\) if for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that any two points of \(B_\delta(x) \cap E\)
are joined by a connected set in \( B_\varepsilon(x) \cap E \). A set is uniformly locally connected, if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that all pairs of points, \( x \) and \( y \), for which \( |x - y| < \delta \) can be joined by a connected subset of diameter less than \( \varepsilon \). All convex domains and, more generally, Jordan domains are uniformly locally connected \([15, \text{Theorem 14.1, p. 161}]\). However, simply-connected domains are not necessarily locally connected.

2. Vanishing properties and nodal domains

We may interpret equation \((1.1)\) in the weak sense. We recall that it follows from the structural assumptions \((1.3)\) (or \((1.4)\)) that a weak solution to \((1.1)\) is H"older continuous and satisfies the following Harnack inequality. We refer to Serrin \([20]\). The proof is based on the Moser iteration method \([16]\).

**Theorem 2.1** (Harnack’s inequality). Suppose \( u \) is a non-negative solution to \((1.1)\) in \( B_{3r} \subset G \). Then

\[
\sup_{B_r} u \leq C \inf_{B_r} u,
\]

where \( C = C(p, a_0, a_1, b_1) \).

Having the structure \((1.4)\) in \((1.1)\), Theorem 2.1 can be found in Trudinger \([21]\). Moreover, in this case we shall assume that a weak solution \( u \) is locally bounded.

We also point out the following important property, the strong maximum principle, which can be deduced from the Harnack inequality. We refer to a monograph by Pucci and Serrin \([19]\) on maximum principles.

**Theorem 2.2** (Strong maximum principle). Suppose \( u \) is a non-constant solution to \((1.1)\) in \( G \). Then \( u \) cannot attain its maximum at an interior point of \( G \).

We shall make use of the fact that if \( u \) is a solution to \((1.1)\), then \( -u + c, c \in \mathbb{R} \), is also a solution to an equation similar to \((1.1)\). Hence Harnack’s inequality and the strong maximum principle apply to both \( u \) and \( -u + c \).

Our main result is the following theorem. We assume minimal regularity conditions on \( \partial G \) so that every boundary point is regular, and hence \( u \in C(\overline{G}) \).

**Theorem 2.3.** Suppose \( u \) is a solution to the equation \((1.1)\) under structural conditions \((1.3)\) (or \((1.4)\), and \( u \) locally bounded) in a bounded simply-connected Jordan domain \( G \) of \( \mathbb{R}^2 \) subject to the Dirichlet condition

\[
u = g \quad \text{on} \quad \partial G,
\]

where \( g \in W^{1,p}(G) \cap C(\overline{G}) \). We assume further that for all \( x \in G \) there exists \( r_x > 0 \) such that for all \( r \leq r_x \) the set \( \{ z \in B_r(x) \subset G : u(z) = g \} \)
is connected. If \( g|_{\partial G} \) has finite number of relative maximum and minimum values, and if \( u \) vanishes in some open subset of \( G \), then \( u \) vanishes identically in \( G \).

We could also state the result as follows: If \( u \) is a constant in some open subset of \( G \), then \( u \) is identically constant in \( G \). In what follows, however, we stick to the classical formulation by dealing with a vanishing solution.

The crux of the proof is to study so-called nodal domains. A maximal connected component, i.e. one that is not a strict subset of any other connected set, of the set \( \{ x \in G : u(x) \neq 0 \} \) is called, in what follows, a nodal domain. We denote these components by

\[
N^+_i = \{ x \in G : u(x) > 0 \}, \quad \text{and} \quad N^-_j = \{ x \in G : u(x) < 0 \},
\]

where \( i, j = 1, 2, \ldots \). We remark that if \( u \) is, for instance, a solution to the \( p \)-Laplace equation (1.2) it is not known whether the number of nodal domains is finite.

In the proof of the following key lemma, we make use of the fact that \( G \) is a Jordan domain, and more precisely, \( G \) is uniformly locally connected at every \( x \in \partial G \), we refer to Section 1.3.

**Lemma 2.4.** Let the hypothesis of Theorem 2.3 be satisfied. Then the number of nodal domains, \( N^+_i \) and \( N^-_j \), is finite.

**Remark 2.5.** The extra assumption, for all \( x \in G \) there exists \( r_x > 0 \) such that for all \( r \leq r_x \) the set \( \{ z \in B_r(x) \subset G : u(z) = 0 \} \) is connected, in Theorem 2.3 can be omitted in Lemma 2.4.

**Proof of Lemma 2.4.** We note first that \( u \) vanishes on all nodal lines in \( G \), i.e. on \( \partial N^+_i \cap \partial G \) and \( \partial N^-_j \cap \partial G \). Hence by the strong maximum principle each nodal line meets the boundary of \( G \).

By the strong maximum principle the set \( \partial N^+_i \cap \partial G \) contains a global maximum of \( u \) in \( N^+_i \). We then show that such maximum point \( x_0 \in \partial N^+_i \) is also a relative maximum of \( g \) on \( \partial G \): Let \( x_0 \in \partial G \) be a maximum point of \( u \) on some fixed nodal domain \( N^+_i \). We shall then apply local connectedness of \( G \) at every boundary point \( x \in \partial G \) (Section 1.3). We claim next that there exists \( \delta_{x_0} > 0 \) such that \( B_{\delta_{x_0}}(x_0) \cap G \), for each \( \delta < \delta_{x_0} \), contains only points of \( N^+_i \). But assume, for now, that this is not the case. Hence for each \( \delta < \delta_{x_0} \) there exists \( \tilde{x} \in B_{\delta}(x_0) \cap G \) such that \( \tilde{x} \) belongs to some other nodal domain than \( N^+_i \), say, \( N^-_j \) or \( u(\tilde{x}) = 0 \). Obviously, we need to consider only the former case.

We write \( u(x_0) = \max_{x \in N^+_i} u(x) = \sigma > 0 \). There exists a positive \( \tilde{\delta} < \delta_{x_0} \) such that

\[
u(x) > \frac{\sigma}{2}
\]

for every \( x \in \partial N^+_i \cap \partial G \cap B_{\tilde{\delta}}(x_0) \) (it can be verified that such points exist since \( G \) is a Jordan domain). Moreover, since there exists a point
\[ \tilde{x} \in B_\delta(x_0) \cap G \text{ such that } \tilde{x} \in N_j^- \text{ and } G \text{ is locally connected at every } x \in \partial G, \text{ there must exist also a point } \tilde{x} \in B_\delta(x_0) \cap G \text{ so that } u(\tilde{x}) = 0. \text{ For small enough } \delta \text{ this is not possible since } u \text{ is continuous and } u(x_0) > 0. \]

We have therefore obtained that there exists a positive \( \delta_{x_0} \) such that \( B_\delta(x_0) \cap G, \delta < \delta_{x_0} \), contains only points of \( N_i^+ \).

It follows that the inequality

\[ u(x) \leq u(x_0) \]

is valid both for every \( x \in B_\delta(x_0) \cap \partial N_i^+ \) and for every \( x \in B_\delta(x_0) \cap \partial G \) (in fact \( B_\delta(x_0) \cap \partial N_i^+ = B_\delta(x_0) \cap \partial G \) as sets). Hence each maximum point \( x_0 \in \partial N_i^+ \) constitutes a relative maximum of \( g \) on \( \partial G \). An analogous reasoning applies for minima and relative minima on \( N_j^- \) and \( \partial G \), respectively.

Since \( g \) is assumed to possess only finite number of relative maxima and minima on \( \partial G \), the number of nodal domains must be finite. \( \square \)

Our idea in the proof of the preceding lemma has certain similarity to that of Lemma 1.1 in Alessandrini [2] where the number of interior critical points was considered to solution of linear equations.

The following proof resembles the argument presented in a recent paper by the authors, we refer to [12] for more details.

**Proof of Theorem 2.3.** We assume for contradiction that

(A) \( u \) vanishes in a maximal open set \( D \subset G \) but is not identically zero in \( G \).

The maximal open set \( D \) is formed as follows: for every \( x \in G \) for which there exists an open neighborhood such that \( u \equiv 0 \) on this neighborhood we denote by \( B(x,r_x) \), \( r_x = \sup \{ t > 0 : u|_{\partial B(x,t)} \equiv 0 \} \), the maximal open neighborhood of \( x \) where \( u \) vanishes identically. Then the maximal open set \( D \) is simply the union of all such neighborhoods. We pick a connected component of \( D \), still denoted by \( D \).

It is worth noting that (A) implies that the boundary data function \( g \) changes sign at least once on \( \partial G \).

By Lemma 2.4 there exist positive \( M^+ \), \( M^- < \infty \) such that we may index the nodal domains \( i = 1, \ldots, M^+ \) and \( j = 1, \ldots, M^- \).

Each nodal domain is simply-connected which can be seen as follows. Suppose that \( N_i^+ \) is not simply-connected for some \( i \in \{1, \ldots, M^+\} \). Then there exists a Jordan curve \( \gamma \subset N_i^+ \) with its interior \( S_\gamma \) and \( S_\gamma \) contains points which do not belong to the fixed nodal domain \( N_i^+ \). Moreover, \( S_\gamma \subset G \) since \( G \) is assumed to be simply-connected. It follows that the set \( E = \{ x \in S_\gamma \setminus N_i^+ : u(x) \leq 0 \text{ or } u(x) > 0 \} \) is non-empty. If \( \tilde{E} = \{ x \in S_\gamma : u(x) < 0 \text{ or } u(x) > 0 \} \) was empty, then \( u(x) = 0 \) for all \( x \in E \), and \( u(x) > 0 \) for all \( x \in S_\gamma \setminus E \). This is
impossible by Harnack’s inequality, Theorem 2.1. Hence $N_i^+$ is simply-connected.

We consider next the case in which $\tilde{E} = \{ x \in S_\gamma : u(x) < 0 \text{ or } u(x) > 0 \} \subset E$ is non-empty. It suffices to consider only the points at which $u < 0$ (the points at which $u > 0$ are handled in the same way); this set is still denoted by $\tilde{E}$. The set $\tilde{E}$ is open and each component of $\tilde{E}$ is a subset of some nodal domain $N_j^-$. This contradicts the fact that each nodal line meets $\partial G$. Hence $N_i^+$ is simply-connected.

An analogous, symmetric, reasoning applies to $N_j^-$. Hence $\partial N_i^+$ and $\partial N_j^-$ are connected as the boundaries of simply-connected domains, and thus continua, i.e. compact connected sets with at least two points, for each $i$ and $j$.

Suppose next there exists a point $x \in \partial D \cap G$ and its neighborhood $B_\delta(x)$, $\delta > 0$, such that $\overline{B}_\delta \subset G$ and $B_\delta(x) \cap \text{ext}(D)$ contains only points of either $N_i^+$ or $N_j^-$ for some $i$ and $j$, i.e. points at which either $u > 0$ or $u < 0$. Assume, without loss of generality, that $B_\delta(x) \cap \text{ext}(D)$ contains points of $N_i^+$ only. Then $u \geq 0$ on $B_\delta(x)$ and by Harnack’s inequality, Theorem 2.1, $u \equiv 0$ on $B_{\delta/2}(x)$, which contradicts the maximality of the set $D$, and hence also the antithesis (A). In this case our claim follows.

By the preceding reasoning it is sufficient to consider the following situation. For any $x \in \partial D \cap G$ and for any $\delta < \delta_0$, $\delta_0 > 0$, the neighborhood $B_\delta(x) \subset G$ contains points of the nodal domains $N_i^+$ and $N_j^-$ for some indices $i$ and for some indices $j$.

We point out that there exist a fixed index pair $(s, t) \in \{1, \ldots, M^+ \} \times \{1, \ldots, M^- \}$ and $\delta_0 > 0$ such that each $B_\delta(x)$ contains points of $N_s^+$ and $N_t^-$, but there might be also points of other nodal domains in $B_\delta(x)$, for every $\delta < \delta_0$; this is a consequence of the fact that the number of nodal domains is finite in our case. We reason as follows: We consider a point $x \in \partial D \cap G$ and $B_\delta(x)$, $\delta < \delta_0$. We then select a decreasing sequence $\{\delta_i\}$ such that $\delta_i < \delta_0$ and $\lim_{i \to \infty} \delta_i = 0$. For each $\delta_i$, we may pick a pair of nodal domains, which we write

$$a_i := (N_{s(\delta_i)}^+, N_{t(\delta_i)}^-),$$

such that $B_\delta(x)$ contains points of both nodal domains. Since the number of all possible nodal domain pairs is finite, there exists a pair which appears infinitely many times in the sequence $\{a_i\}$. We may hence choose this fixed pair $(N_s^+, N_t^-)$, where $s(\delta_i) = s$ and $t(\delta_i) = t$, for some subsequence $\{\delta_{i_j}\}$ such that $\lim_{j \to \infty} \delta_{i_j} = 0$. It can be seen from this reasoning that the same pair occurs in any neighborhood $B_\delta(x)$, $\delta < \delta_0$.

We shall next base our reasoning on some topological arguments. We write $\partial D_A = \{ x \in \partial D : x \text{ is accessible from } D \}$,

$$\partial N_{i,A}^+ = \{ x \in \partial N_i^+ : x \text{ is accessible from } N_i^+ \},$$
and correspondingly \( \partial N_{j,A}^- \). By [15] accessible boundary points \( \partial D_A \), \( \partial N_{i,A}^+ \), and \( \partial N_{j,A}^- \) are dense in \( \partial D, \partial N_{i}^+ \), and \( \partial N_{j}^- \), respectively.

We will describe a selection process which gives a pair of points \( x_1 \) and \( x_2 \) such that \( x_1, x_2 \in \partial D_A \cap G \) and that the associated spherical neighborhoods \( B_\delta(x_1) \) and \( B_\delta(x_2) \), \( \delta < \delta_0 \), contain points of the same nodal line \( \partial N_s^+ \), \( s \in \{1, \ldots, M^+\} \) fixed. Moreover, it is assumed that \( \overline{B}_\delta(x_1) \cap \overline{B}_\delta(x_2) = \emptyset \), and that \( \overline{B}(x_1), \overline{B}(x_2) \subset G \). This procedure is as follows: We select a finite sequence of points \( \{x_i\} \), each \( x_i \in \partial D_A \cap G \). As pointed out earlier, for each \( x_i \) there exists \( \delta_0 > 0 \) such that the spherical neighborhood \( B_\delta(x) \), \( \delta < \delta_0 \), contains points of \( N_s^+ \) and \( N_t^- \) for some \( s \in \{1, \ldots, M^+\} \) and \( t \in \{1, \ldots, M^-\} \). Since the number of all possible nodal domain pairs as described above is finite, after finite number of steps the sequence \( \{x_i\} \) will contain a pair of points, denoted \( x_1 \) and \( x_2 \), which have the aforementioned properties.

We then select \( x_3 \in B_\delta(x_1) \cap \partial N_{s,A}^+ \) and \( x_4 \in B_\delta(x_2) \cap \partial N_{s,A}^+ \).

We connect \( x_1 \) to \( x_2 \) by a cross-cut \( \gamma_D \) in \( D \), and \( x_3 \) to \( x_4 \) by a cross-cut \( \gamma_{N_s^+} \) in \( N_s^+ \). We remark that \( x_3 \), and analogously \( x_4 \), is accessible in \( N_s^+ \) with a line segment (consult, e.g., Remark 3.3 in [12]). Also \( x_1 \), and analogously \( x_2 \), is accessible in \( D \) with a line segment. We fix such line segments to access the points \( x_1, x_2, x_3, \) and \( x_4 \). In this way the line segments constitute part of the cross-cut \( \gamma_D \) and \( \gamma_{N_s^+} \), respectively.

Since the boundary \( \partial N_s^+ \) is connected it is also \( \varepsilon \)-connected for every \( \varepsilon > 0 \). Hence for each \( \varepsilon > 0 \) the points \( x_1 \) and \( x_3 \) can be joined by an \( \varepsilon \)-chain \( \{a_1, \ldots, a_k\} \subset \partial N_s^+ \cap G \) such that

\[
x_1 = a_1, a_2, \ldots, a_{k-1}, a_k = x_3.
\]

We consider a collection of open balls \( \{B_{\delta_s}(a_i)\}_{i=1}^k \), \( a_i \in \partial N_s^+ \cap G \), such that \( B_{\delta_s}(a_i) \subset G \), and a domain \( U_s^1 \) which is defined to be

\[
U_s^1 = \bigcup_{i=1}^k B_{\varepsilon_s}(a_i).
\]

Since \( U_s^1 \) is a domain there exists a Jordan arc, \( \gamma_{x_1x_3}^\varepsilon \), connecting \( x_1 \) to \( x_3 \) in \( U_s^1 \). Correspondingly, the points \( x_2 \) and \( x_4 \) can be joined by an \( \varepsilon \)-chain in \( \partial N_s^+ \) and we obtain a domain \( U_s^2 \) and a Jordan arc \( \gamma_{x_2x_4}^\varepsilon \), connecting \( x_2 \) to \( x_4 \) in \( U_s^2 \).

It is worth noting that we have selected \( \gamma_{x_1x_3}^\varepsilon \) and \( \gamma_{x_2x_4}^\varepsilon \) such that either of them does not intersect \( \gamma_D \) or \( \gamma_{N_s^+} \), save the points \( x_1 \) and \( x_2 \), and \( x_3 \) and \( x_4 \), respectively. This is possible because of the line segment construction described above.

From the preceding Jordan arcs we obtain a Jordan curve \( \Gamma^\varepsilon \), and by slight abuse of notation we write it as a product

\[
\Gamma^\varepsilon = \gamma_{x_1x_3}^\varepsilon \cdot \gamma_{N_s^+} \cdot \gamma_{x_2x_4}^\varepsilon \cdot \gamma_D.
\]
The Jordan curve $\Gamma^\varepsilon$ divides the plane into two disjoint domains, and $\Gamma^\varepsilon$ constitutes the boundary of both domains. We consider the bounded domain, denoted by $T^\varepsilon$, enclosed by $\Gamma^\varepsilon$. See Figure [1].

We next deal with the Jordan domain $T^\varepsilon$. There exists at least one point $y \in T^\varepsilon$ such that $u(y) < 0$, i.e., $y \in N^-_{j_0}$ for some fixed $j_0 \in \{1, \ldots, M^-\}$. Assume that this is not the case: then $u(x) \geq 0$ for every $x \in T^\varepsilon$. As $\gamma_D$ is part of the boundary $T^\varepsilon$ contains also points of $D$, and hence $u$ vanishes at such points. By Harnack’s inequality, Theorem 2.1, $u \equiv 0$ in $T^\varepsilon$. This is, however, impossible since $\gamma_{N^+}^j$ is part of the boundary of $T^\varepsilon$, thus $u > 0$ on a sufficiently small neighborhood of a point in $\gamma_{N^+}^j$.

In an analogous way, it is possible to show that there exists a point $z \in N^-_{j_0} \cap (G \setminus T^\varepsilon)$. We stress that it is crucial that the selected points $z$ and $y$ belong to the same nodal domain $N^-_{j_0}$. It is worth noting here that by the strong maximum principle $\Gamma^\varepsilon$ cannot enclose the nodal domain $N^-_{j_0}$ containing the point $y$, and therefore $G \setminus T^\varepsilon$ must also contain points of $N^-_{j_0}$. We then connect $z$ and $y$ in $N^-_{j_0}$ by a Jordan arc $\gamma_{zy}$. Observe that $u(x) < 0$ for every $x \in \gamma_{zy}$.

The Jordan arc $\gamma_{zy}$ as a connected set intersects $\Gamma^\varepsilon$ at least at one point. We then distinguish the following four possible cases for the point of intersection: The point of intersection is contained in

1. $\gamma_D$,
2. $\gamma_{N^+}^j$,
3. $\gamma_{x_1x_3}^\varepsilon$,
4. $\gamma_{x_2x_4}^\varepsilon$.

In the case (1) and (2) we have reached a contradiction as $u(x) = 0$ for every $x \in \gamma_D$ and $u(x) > 0$ for every $x \in \gamma_{N^+}^j$, respectively.

Consider the case (3) and case (4). We denote the point of intersection by $x_\varepsilon$ for every $\varepsilon > 0$. We can select an appropriate subsequence $\{x_{\varepsilon_j}\}_{j=1}^\infty$, $\lim_{j \to \infty} \varepsilon_j = 0$, such that for each $j$ either $x_{\varepsilon_j} \in U_{\varepsilon_j}^1$ or $x_{\varepsilon_j} \in U_{\varepsilon_j}^2$. We assume, without loss of generality, that $x_{\varepsilon_j} \in U_{\varepsilon_j}^1$. The sequence $\{x_{\varepsilon_j}\}$ is clearly bounded, and hence there exists a subsequence, still denoted $\{x_{\varepsilon_j}\}$, such that $\lim_{j \to \infty} x_{\varepsilon_j} = x_0$. Observe that each $x_{\varepsilon_j} \in B_{2\varepsilon_j}(a_m)$ for some $a_m \in \partial N_{s_j}^+ \cap G$ in the $\varepsilon_j$-chain. We note that $u(a_m) = 0$. Moreover, if there existed $\delta_0$ and a subsequence, still denoted $\{x_{\varepsilon_j}\}$, such that $|u(x_{\varepsilon_j})| \geq \delta_0 > 0$ for every $x_{\varepsilon_j}$, this would contradict with uniform continuity of $u$ (note that $u$ is uniformly continuous on compact subsets of $G$). We hence have that $u(x_0) = \lim_{j \to \infty} u(x_{\varepsilon_j}) = 0$. 9
In conclusion, we have reached a contradiction since $u(x_0) = 0$ but, on the other hand, $x_0 \in \gamma_{zy}$ and hence $u(x_0) < 0$.

All four cases (1)–(4) lead to a contradiction. Hence antithesis (A) is false, thus the claim follows. \hfill \□

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Jordan domain $T_\varepsilon$ and Jordan curve $\gamma_{zy}$ (dotted line) connecting $z$ to $y$ in $N_{j_0}$.}
\end{figure}

Let us discuss our extra assumption in Theorem 2.3.

Remark 2.6. We mention that the extra assumption, for all $x \in G$ there exists $r_x > 0$ such that for all $r \leq r_x$ the set \{ $z \in \overline{B}_r(x) : u(z) = 0$ \} is connected, could be replaced with the assumption that the set has finitely many components.

Remark 2.7. The extra assumption is closely related to the concept of topological monotonicity or quasi-monotonicity introduced by Whyburn in [22]; we also refer to Astala et al. [7, 20.1.1, pp. 530 ff].

Let us try to clarify the role of this assumption in the proof of the preceding theorem. We fix there the point $x_1 \in \partial D_A \cap G$, its neighborhood $B_\delta(x_1)$, and the point $x_3 \in B_\delta(x_1) \cap \partial N^+_{\varepsilon,A}$ (similarly $x_2 \in \partial D_A \cap G$, $B_\delta(x_2)$, and $x_4 \in B_\delta(x_2) \cap \partial N^+_{\varepsilon,A}$). At $x_1$ and $x_3$ the function $u$ is known to vanish. Using the extra assumption in Theorem 2.3 we may conclude that there indeed exists a continuum $C_\delta$ that connects $x_1$ to $x_3$ in $B_\delta(x_1)$ so that $u(x) = 0$ for every $x \in C_\delta$, or in other words, that the set $B_\delta(x_1) \cap \partial N^+_{\varepsilon,A}$ is connected.

In [12, Remark 4.4] possible spiral-like behavior that illustrates the role of our extra assumption is discussed in more detail.

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