ALGEBRAIZABLE WEAK LOGICS

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Abstract. We extend the framework of abstract algebraic logic to weak logics, namely logical systems which are not necessarily closed under uniform substitution. We interpret weak logics by algebras expanded with an additional predicate and we introduce a loose and strict version of algebraizability for weak logics. We study this framework by investigating the connection between the algebraizability of a weak logic and the algebraizability of its schematic fragment, and we then prove a version of Blok and Pigozzi’s Isomorphism Theorem in our setting. We apply this framework to logics in team semantics and show that the classical versions of inquisitive and dependence logic are strictly algebraizable, while their intuitionistic versions are only loosely so.

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Introduction

At least since Tarski’s 1936 article on logical notions, logic has often been understood as the subject studying those notions which are “invariant under all possible one-one transformations of the world onto itself” [29, p. 149]. Tarski’s view was inspired by Felix Klein’s 1872 Erlanger Programm, and expresses very neatly the philosophical idea of logic as the discipline with the most general character. While, for instance, metric geometry studies notions that are invariant under transformations that preserve distances, and topology studies notions invariant under continuous maps, logic deals with notions which are invariant under arbitrary bijections.

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Formally, this led to the technical definition of logics as consequence relations which are additionally closed under uniform substitutions.

As a matter of fact, Tarski’s approach can be seen as a key step in the transition from the “symbolic” algebraic logic of the 19th century (exemplified by the works of Boole, De Morgan, Jevons, Peirce, etc.) to the contemporary field of abstract algebraic logic. In fact, using the general notion of logic as a closure operator on the term algebra, Rasiowa made the first steps into the “mathematical” version of algebraic logic, and in particular she developed in An Algebraic Approach to Non-Classical Logics [27] a general theory of algebraization for implicative logics. Finally, the algebraic approach was put in its contemporary formulation by Blok and Pigozzi, who introduced in Algebraizable Logics [4] the notion of algebraizable logics and developed their general theory.

In recent years, however, there has been an increasing interest into systems which do possess a logical nature but fail nonetheless to be closed under uniform substitution in Tarski’s original strong sense. The field of modal logic is particularly rich of such examples: Buss’ pure provability logic [6], public announcement logic [19, 20] and other epistemic logics are all examples of this behaviour. Furthermore, propositional logics based on team semantics, such as inquisitive [12] and dependence logic [30], also do not satisfy Tarski’s requirement of closure under uniform substitution. We believe that this state of affairs is not a mistake requiring correction, but that it rather reflects the increasing plurality of logics and their aptness for applications. At the same time, however, the anomalous behaviour of these logical systems prevented so far a uniform abstract study, and did not allow for immediately applying facts and results from abstract algebraic logic, thus forcing scholars to reprove abstract results in these settings, or adapt standard techniques to their specific situation.

Motivated by these facts, we propose in this article a generalisation of the notion of algebraizable logics which breaks apart from Tarski’s original view of logics as being invariant under all substitutions. In other words, we want to study consequence operators that are invariant under some substitutions, but not necessarily all. We introduce the notion of weak logic, generalising previous definitions of Ciardelli and Roelofsen [12] and Punččochář [25], as a consequence relation which is invariant under all substitutions which map atomic variables to atomic variables. In other words, we relax Tarski’s constraint of invariance under arbitrary substitutions and we require that it holds only with respect to these so-called atomic substitutions (cf. Section 2 below).

We proceed in this article as follows. In Sections 2-5 we develop at length the key aspects of the theory of algebraizable weak logics. In Section 2 we define weak logics and introduce what we call expanded algebras as their corresponding algebraic notion. In Section 3 we introduce the notions of loose and strict algebraizability for weak logics and we show that the (loose or strict) equivalent algebraic semantics of a weak logic is unique, thus mirroring the classical result by Blok and Pigozzi for standard algebraizable logics. Then, in Section 4, we study the relation between a weak logic and its schematic fragment of consequences invariant under arbitrary substitutions. In particular, we provide a characterization of the algebraizability of weak logics in terms of the algebraizability of their schematic variant. Additionally, we introduce the notion of standard companion and generalise to this setting previous results from [3] and [1]. Finally, in Section 5 we develop on these results
and we prove a version of Blok and Pigozzi’s Isomorphism Theorem for strictly algebraizable weak logics. These sections thus establish the fundamentals of the theory of algebraizable weak logics and show that several key results from the field of abstract algebraic logic carry on to the setting without uniform substitution. On a partially separate line of inquiry, in Section 6 we take a small excursus explaining how to adapt the usual matrix semantics of propositional logics to the setting of weak logics. We show that (similarly to the standard setting) every weak logic admits a matrix semantics, and we describe the class of reduced models of (strictly) algebraizable weak logics. We put our abstract framework to the test in Section 7, where we apply it to the specific case of inquisitive and dependence logics. In particular, we build on previous results from [3, 11, 26] to show that the classical version of inquisitive and dependence logic is strictly algebraizable, while their intuitionistic versions are only loosely so. To our eyes, this indicates a significant difference between the classical version of inquisitive (dependence) logic – which can essentially be recasted as a theory over its schematic fragment – and its intuitionistic counterpart \( \text{InqI} \), which does not admit such reformulation.

Together with these results on inquisitive and dependence logic, we regard as the main contribution of the present work the fact that it provides a framework for reasoning about logical systems lacking uniform substitution. In particular, this work relates several algebraic studies on inquisitive logic [2, 3, 24, 25], dependence logic [26], and polyatomic logics [1], by showcasing all the algebraic semantics from these works as instances of what we call core semantics in the present article. Given the increasing popularity of logics without uniform substitution in the logic literature, we hope that our approach will be useful to researchers working in these areas also in the future.

1. Preliminaries

We recall in this section some basic facts concerning logics, algebras and model theory. We also fix some notation that we shall follow throughout the rest of the article. The following general context sets the framework of our work.

**Context 1.1.** Throughout this article we always let \( \text{Var} \) be a set of variables and we let \( \mathcal{L} \) be an algebraic (i.e., purely functional) signature (unless we specify otherwise). We denote by \( \text{Fm}_\mathcal{L} \) both the set of first-order terms over \( \text{Var} \) in the signature \( \mathcal{L} \) and the term algebra in the signature \( \mathcal{L} \) over \( \text{Var} \). We omit the index \( \mathcal{L} \) when it is clear from the context. Notice that, since we are often dealing with propositional logical systems, we often refer to elements of \( \text{Fm}_\mathcal{L} \) also as (propositional) formulas in the language \( \mathcal{L} \). These should not be confused with the first-order formulas in the signature \( \mathcal{L} \).

Given the language \( \mathcal{L} \), we recall the standard abstract Tarskian definition of (propositional) logic. We refer the reader to [16, §1] for more on consequence relations and logics, and for slight variations of these definitions.

**Definition 1.2.** A **(finitary) consequence relation** is a relation \( \vdash \subseteq \varphi(\text{Fm}) \times \text{Fm} \) such that, for all \( \Gamma \subseteq \text{Fm} \):

1. \( \Gamma \vdash \phi \) for all \( \phi \in \Gamma \);
2. if \( \Gamma \vdash \phi \) for all \( \phi \in \Delta \), and \( \Delta \vdash \psi \), then \( \Gamma \vdash \psi \);
3. if \( \Gamma \vdash \phi \) and \( \Gamma \subseteq \Delta \), then \( \Delta \vdash \phi \);
4. if \( \Gamma \vdash \phi \) then there is some \( \Delta \subseteq \Gamma \) such that \( |\Delta| < \aleph_0 \) and \( \Delta \vdash \phi \).
Remark 1.3. We notice that Condition 1.2(3) and Condition 1.2(4) are not always required in the definition of consequence relations. We assume Condition 1.2(3) for simplicity, and since we shall mostly deal with algebraizable logics, which are always monotone. The finitarity requirement from Condition 1.2(4) is also not necessary, and it is possible to study consequence relations and propositional logics where this condition fails. However, to simplify our treatment, we shall restrict attention to consequence relations that are finitary, in the sense of Condition 1.2(4). This reflects our elementary approach to the subject, as Condition 1.2(4) allows us to translate propositional systems into first-order logic and avoid the use of infinitary logical systems lacking compactness. We refer the interested reader to [16] for a treatment of non-finitary propositional logics.

Definition 1.4. A substitution is an endomorphism $\sigma : \text{Fm} \to \text{Fm}$ of the $\mathcal{L}$-term algebra. We denote by $\text{Subst}(\mathcal{L})$ the set of all substitutions in the language $\mathcal{L}$. If $p_1, \ldots, p_n \in \text{Var}$ and $\phi_1, \ldots, \phi_n$ are arbitrary $\mathcal{L}$-formulas, we denote by $\Gamma[\phi_1 \ldots \phi_n/p_1 \ldots p_n]$ the result of simultaneously substituting each $\phi_i$ for all occurrences of $p_i$ in the formulas in $\Gamma$.

Definition 1.5 (Logic). A consequence relation $\vdash$ is closed under uniform substitution if $\Gamma \vdash \phi$ entails $\sigma[\Gamma] \vdash \sigma(\phi)$ for all substitutions $\sigma \in \text{Subst}(\mathcal{L})$. A (standard) logic is a consequence relation $\vdash$ which is closed under uniform substitution.

Example 1.6. Obvious examples of standard logics are the classical propositional logic CPC and the intuitionistic propositional logic IPC. Non-examples of logics in this sense are first-order logic, as it is not a consequence relation of the propositional term algebra, or other higher-order systems.

In the context of abstract algebraic logic, one is often interested in the algebraic semantics of a propositional logic $\vdash$. This is provided by algebras, i.e., first-order structures in some purely functional language $\mathcal{L}$. We first fix some notation concerning first-order models.

Notation 1.7. Let $\mathcal{L}$ be a first-order language, not necessarily functional. We use Latin letters $A, B, \ldots$ both to denote first-order $\mathcal{L}$-structures and their underlying domain. When confusion may arise, we also write $\text{dom}(A)$ to refer to the underlying universe of $A$. For all function symbols $f \in \mathcal{L}$ and all relation symbols $R \in \mathcal{L}$, we write $f^A$ and $R^A$ for their interpretation in $A$. We use the same notations for symbols and their interpretation when it does not cause confusion. If $t(\bar{x})$ is a term in the language $\mathcal{L}$ (i.e., a propositional formula), we usually call its interpretation $t^A$ a polynomial. If $X \subseteq A$, we write $\langle X \rangle$ for the substructure of $A$ generated by $X$. The symbol $\models$ refers to the standard satisfaction symbol from first-order logic. We usually denote classes of structures by boldface font — both for arbitrary collections ($\mathbf{Q}, \mathbf{K}, \ldots$) and designated ones, e.g., the class of all Heyting algebras $\mathbf{HA}$ or the class of all Boolean algebras $\mathbf{BA}$.

We assume the reader is familiar with the usual constructions from model theory and universal algebra, and refer to [5,9] for background. We recall in particular the following notions of maps, as we shall need them in the rest of the article.

Definition 1.8. Let $h : A \to B$ be a function between two $\mathcal{L}$-structures $A$ and $B$, for some first-order language $\mathcal{L}$. We define the following notions:
(1) we say that \( h \) is a **homomorphism** if for every function symbol \( f \in \mathcal{L} \) we have
\[
h(f(a_1, \ldots, a_n)) = f(h(a_1), \ldots, h(a_n))
\]
and for every relation symbol \( R \in \mathcal{L} \),
\[
A \models R(a_1, \ldots, a_n) \iff B \models R(h(a_1), \ldots, h(a_n));
\]
we say that \( h \) is a **strong homomorphism** if it is a homomorphism and, additionally, we have that \( R^B = h[R^A] \) for every relation symbol \( R \in \mathcal{L} \);
(3) we say that \( h \) is a **strict homomorphism** if it is a homomorphism and, for every relation symbol \( R \in \mathcal{L} \),
\[
A \models R(a_1, \ldots, a_n) \iff B \models R(h(a_1), \ldots, h(a_n));
\]
(4) we say that \( h \) is an **embedding** if it is an injective strict homomorphism.

We write \( A \subseteq B \) if \( A \) is a substructure of \( B \), i.e., if the identity map \( \text{id} : A \to B \) is an embedding. We write \( A \cong B \) if \( A \) is isomorphic to \( B \). We say that \( B \) is a homomorphic image of \( A \) if there is a surjective homomorphism \( h : A \to B \); we say that \( B \) is a strong (resp. strict) homomorphic image of \( A \) if \( h \) is a strong (resp. strict) homomorphism.

**Remark 1.9.** We briefly explain the rationale behind the different notions of homomorphism. The notion of homomorphism from Definition 1.8 is standard from the literature in model theory and universal algebra (cf. [9, pp. 70-71], [5, p. 203]). The notion of strong homomorphism comes from [9, p. 321] and is motivated by the following observation. Let \( A \) be an \( \mathcal{L} \)-structure and let \( \mathcal{L}' \) consists of the functional symbols from \( \mathcal{L} \). If \( \theta \) is a congruence of the algebraic reduct of \( A \) we can consider the quotient \( \mathcal{L}'\)-structure \( A/\theta \) and expand it to a \( \mathcal{L} \)-structure by letting \( R^{A/\theta} = R^A/\theta \) for all relational symbols \( R \in \mathcal{L} \). Then, the projection map induced by this quotient is a strong homomorphism.

Finally, we take the notion of strict homomorphism from [7,14], and we stress that strict homomorphisms correspond to those quotients which are additionally compatible with the relational part of the vocabulary from \( \mathcal{L} \), in the sense that if \( (a_i, b_i) \in \theta \) for all \( 1 \leq i \leq n \) then for every relational symbol \( R \in \mathcal{L} \) we have that \( A \models R(a_1, \ldots, a_n) \) if and only if \( A \models R(b_1, \ldots, b_n) \). In this article we will mostly be dealing with strong homomorphisms, but we will consider strict homomorphisms in Section 6.

**Notation 1.10.** Let \( K \) be any class of first-order structures. We denote by \( \mathbb{I}(K) \) its closure under isomorphic copies, by \( \mathbb{S}(K) \) its closure under substructures, by \( \mathbb{P}(K) \) its closure under (direct) products and by \( \mathbb{P}_U(K) \) its closure under ultraproducts. Finally, we write \( \mathbb{H}(K) \) for its closure under strong homomorphic images, and \( \mathbb{H}_s(K) \) for its closure under strict homomorphic images.

**Definition 1.11.** A class of algebras \( K \) is a **quasivariety** if it is closed under the operators \( \mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_U \), i.e., if it is closed under isomorphic copies, subalgebras, products and ultraproducts. A class of algebras \( K \) is a **variety** if it is closed under \( \mathbb{H}, \mathbb{S}, \mathbb{P} \), i.e., if it is closed under homomorphic images, subalgebras and products. We denote by \( \mathbb{Q}(K) \) and \( \mathbb{V}(K) \) the quasivariety and the variety generated by \( K \), respectively.

Crucially, the closure of a class of structures under (some of) the operators \( \mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_U, \mathbb{H} \) is related to conditions pertaining its axiomatisability. Most famously, it is a fundamental result by Keisler and Shelah that a class of structures \( K \) is elementary
(i.e., first-order axiomatisable) if and only if \( K \) is closed under ultraproducts. In this work we are concerned with less general definability conditions. We introduce them by focusing on three special subclasses of first-order formulas.

**Notation 1.12.** We write \( \bar{x} \) as a shorthand for a sequence of variables \((x_0, \ldots, x_n)\). Also, as we often deal with equational classes of structures, we abide to the usual convention from universal algebra to distinguish the syntactical equality symbol, written \( \approx \), from the semantical equality symbol, which we denote by \( = \). To help the reader to distinguish when we talk of first-order formulas from when we deal with propositional ones, we use lowercase Greek symbols \( \phi, \psi, \ldots \) for the latter and uppercase Greek symbols \( \Phi, \Psi, \ldots \) for the former.

**Definition 1.13.** Let \( L \) be an arbitrary first-order signature, not necessarily functional. Then we define the following types of formulas:

1. an **equation** is a formula of the form \( \varepsilon \approx \delta \), where \( \varepsilon \) and \( \delta \) are two \( L \)-terms;
2. a **quasiequation** is a formula of the form \( \bigwedge_{1 \leq i \leq n} (\varepsilon_i \approx \delta_i) \rightarrow \varepsilon \approx \delta \), for some \( n < \omega \), where all \( \varepsilon_i, \delta_i \) and \( \varepsilon, \delta \) are \( L \)-terms;
3. a **basic Horn formula** is a formula of the form \( \Phi_i = \bigvee_{1 \leq i \leq n} \Psi_i \), where every \( \Psi_i \) is either an atomic formula or a negated atomic formula, and at most one \( \Psi_i \) is atomic;
4. a basic Horn formula is **strict** if exactly one of its disjuncts is atomic;
5. A **universal Horn formula** is a formula of the form \( \forall x_1 \ldots \forall x_m \left( \bigwedge_{1 \leq i \leq \ell} \Phi_i(\bar{x}) \right) \) for some \( m, \ell < \omega \), and where each \( \Phi_i \) for \( 1 \leq i \leq m \) is a basic Horn formula;
6. a universal Horn formulas is **strict** if all the basic Horn formulas \( \Phi_i \) occurring in it are strict.

The following classical results relate the closure under the operators \( I, S, P, P_U, H \) with different definability conditions. We refer the reader to [5; I: Thm. 2.23, Thm. 2.25; II: Thm. 11.9] for their proofs. Facts (1) and (2) are also known respectively as Birkhoff’s Theorem and Mal’tsev’s Theorem.

**Fact 1.14.** Let \( L \) be an algebraic signature and \( K \) a class of \( L \)-algebras, then:

1. \( K \) is a variety if and only if it is axiomatised by equations;
2. \( K \) is a quasivariety if and only if it is axiomatised by quasiequations.

Moreover, if \( L \) is any first-order signature and \( K \) is a class of \( L \)-structures, then:

1. \( K \) is closed under \( I, S, P \) and \( P_U \) if and only if it is axiomatised by universal Horn sentences.

In the light of the previous results, it is convenient to fix some shorthand notation to talk about (fragments of) first-order theories, and their classes of models.

**Notation 1.15.** Let \( L \) be an arbitrary first-order language. If \( T \) is a set of first-order sentences in \( L \), we write \( \text{Mod}(T) \) for the class of all \( L \)-structures \( A \) such that \( A \models T \). We write \( \text{Mod}(L) \) for the class of all \( L \)-structures. Notice that, if \( T \) is a set of formulas (e.g., a set of equations or quasiequations), then \( \text{Mod}(T) \) is the class of structures which models \( \forall x_1 \ldots \forall x_n(\Phi(\bar{x})) \), for every \( \Phi \in T \). On the other hand, let \( K \) be a class of \( L \)-structures. Then we write \( \text{Th}_{ul}(K) \) for the set of all \( L \)-sentences true in \( K \); \( \text{Th}_{ul}(K) \) — for the set of all universal Horn sentences true in
\(\Theta\) can always be the validity of quasiequations is preserved above. However if the relation systems and we provide several examples of them. Alongside, we define and algebras which are not necessarily closed under uniform substitution, we first identify the restricted class of atomic substitutions. If the consequence relation from first-order logic by the consequence relation of some suitable infinitary logic. Given our present interest in finitary logical systems replace the consequence relation from first-order logic by the consequence relation of (Weak Logic)

**Definition 2.2** (Weak Logic). A weak logic is a (finitary) consequence relation \(\vdash\) such that, for all atomic substitutions \(\sigma \in \text{At}(\mathcal{L})\), \(\Gamma \vdash \phi\) entails \(\sigma[\Gamma] \vdash \sigma(\phi)\).
Remark 2.3. A weak logic is thus a consequence relation which is closed under atomic substitution. Intuitively, this principle reifies the least prerequisite a consequence relation must satisfy in order to be characterizable as a logic: the validity of the consequences in a weak logic can depend on the logical complexity of its formulas, but not on the specific variables that occur in them. Philosophically, this can be interpreted as a weakening of the Bolzani-Tarskian notion of logicality.

Obviously, standard logics are weak logics. More poignantly, there are several examples of weak logics which are not standard logics and that have been extensively studied in the literature. Their existence and recognition constitutes the main motivation behind our interest for this class of consequence relations and for the abstract results of this article.

Example 2.4. Public Announcement Logic (PAL) [19] is an example of a modal logic that is not closed under uniform substitution [20]. However, it can be shown that PAL is closed under atomic substitution [20, §2.1] and it is therefore a weak logic. Introducing the proper syntax and semantics of PAL is out of scope of this paper, but we mention the following example from [20] to provide the reader with some intuition why uniform substitution fails. Given a set of agents $A$, the language of PAL extends the basic modal language with operators $K_i$, for all $i \in A$, and $\langle \phi \rangle$ for any formula $\phi$. The sentence $K_i \phi$ should be read as “agent $i$ knows that $\phi$” and $\langle \phi \rangle \psi$ as “after the truthful announcement of $\phi$ to all agents, $\psi$ holds”. Let the atoms of the language stand for facts – that is, sentences that can be truly uttered at any time. Consider then the principle:

\[(*) \quad p \rightarrow \langle p \rangle p \quad \text{(if $p$ is true, $p$ remains true after a truthful announcement)}\]

The schema $(*)$ is valid for facts, but in general does not hold if we substitute $p$ with a sentence talking about the epistemic state of an agent. Let $\phi$ be the sentence “Ljubljana became the capital of an independent Slovenia in 1991, and agent $j$ does not know this”, with translation $c \land \neg K_j c$. Now substituting $\phi$ for $p$ in $(*)$ gives us a Moorean sentence – after truthfully announcing $\phi$, agent $j$ learns that “Ljubljana became the capital of an independent Slovenia in 1991”, and thus the conclusion $\langle \phi \rangle \phi$ is no longer truthful.

Example 2.5. Logics based on team semantics, such as inquisitive and dependence logics [11,12,30], offer a rich supply of examples of weak logics. In Section 7 we will focus particularly on InqB, InqB*, InqI and InqI*, namely the classical and the intuitionistic versions of inquisitive and dependence logic. However, already now we can provide a conceptual motivation why InqB is not closed under uniform substitution. One of the main goals of InqB is to serve as a basis for a uniform treatment of both truth-conditional statements and questions in natural language. To that end, the intended semantics of InqB must establish when a piece of information supports a statement or settles a question rather than their truth conditions. We call the evidence an information state and represent it as a set of possible worlds.

Let $p$ be an arbitrary statement without inquisitive content, e.g., “It is raining in Glasgow”. Assume that $p$ holds in the possible worlds $a$ and $b$, i.e., the information state $\{a, b\}$ supports $p$ (see Fig. 1). We form the polar question $?p$ – “Is it raining in Glasgow?”, and model it as the set of alternatives $\{a, b\}$ and $\{c, d\}$. Let’s check the validity of Double Negation Elimination (DNE) – $\neg \neg q \rightarrow q$; we interpret negation as the complement of the union of alternatives. Thus any information state supporting
\[ \neg\neg p \text{ will support the statement } p \text{ as well (Fig. } 1(\text{c})\text{). However, this is not the case for questions – e.g., the state } \{ b, d \} \text{ supports } \neg\neg ?p, \text{ but does not settle } ?p \text{ as the possible worlds } b \text{ and } d \text{ do not agree on a same answer. Hence we can conclude that the schema } \text{DNE} \text{ is valid only for statements without inquisitive content, i.e., for propositional atoms.}

Actually, \text{InqB} \text{ is a concrete example of a wider class of weak logics – a double negation atoms logic or DNA-logic. A DNA-logic (also negative variant of an intermediate logic [10, 22]) is a set of formulas } L^\neg = \{ \phi_0, \ldots, \neg p_n/p_0, \ldots, p_n : \phi \in L \}, \text{ where } L \text{ is an intermediate logic, namely a logic comprised between IPC and CPC. It can be proved (see e.g. [10, Prop. 3.2.15]) that DNA-logics are closed under atomic substitutions. However, for any DNA-logic } L \neq CPC \text{ it is the case that } \neg\neg p \rightarrow p \in L^\neg, \text{ but } (\neg\neg (p \vee \neg p) \rightarrow p \vee \neg p) \notin L^\neg, \text{ showing that DNA-logics are not standard logics. We also notice that DNA-logics can be further generalised to } \chi \text{-logics, defined in [26], which provide yet another non-trivial example of weak logics.}

We briefly mention the following natural notions, although we will not use them in the rest of the paper. If } \Vdash \text{ is a weak logic we know that it is at least closed under all atomic substitutions } \sigma \in \text{At}(L), \text{ but in general there could be more substitutions for which the logic } \Vdash \text{ is closed. We call such substitutions } \text{admissible} \text{ for } \Vdash .

**Definition 2.6 (Admissible Substitutions).** Let } \Vdash \text{ be a weak logic. The set of } \text{admissible substitutions } AS(\Vdash) \text{ is the set of all substitutions } \sigma \text{ such that, for all sets of formulas } \Gamma \cup \{ \phi \} \subseteq \text{Fm}, \text{ } \Gamma \Vdash \phi \text{ entails } \sigma[\Gamma] \Vdash \sigma(\phi).

**Remark 2.7.** As noticed above, we immediately have that } \text{At}(L) \subseteq AS(\Vdash). \text{ However, in stark contrast with the set of atomic substitutions, determining the set of admissible substitutions of a weak logic is in principle much harder. An example of such a characterization can be given for the case of inquisitive logic } \text{InqB}; \text{ one can in fact verify that } \sigma \in AS(\text{InqB}) \text{ if and only if } \sigma \text{ is a classical substitution, namely if for all } p \in \text{Var}, \sigma(p) \equiv_{\text{InqB}} \psi \text{ where } \psi \text{ is a disjunction-free formula.}
Even if in weak logics we cannot freely substitute formulas in place of variables, we often want to consider the subset of formulas for which this is possible. We refer to this subset as the core of a logic.

**Definition 2.8** (Core of a Logic). The core of a weak logic $\vdash$ is the set $\text{core}(\vdash)$ of all formulas $\psi \in Fm$ such that for all sets of formulas $\Gamma \cup \{ \phi \}$ we have that:

$$\Gamma \vdash \phi \implies \Gamma[\psi/p] \vdash \phi[\psi/p],$$

where $p \in \text{Var}$ is any atomic variable.

**Remark 2.9.** Equivalently, we can say that $\psi$ is a core formula of $\vdash$ if and only if for all $p \in \text{Var}$ the substitution $\sigma$ such that $\sigma|\text{Var} \setminus \{p\} = \text{id}_{\text{Var}}$ and $\sigma(p) = \psi$ is admissible. Clearly, we always have that $\text{Var} \subseteq \text{core}(\vdash)$.

### 2.2. Expanded Algebras

In order to make sense of weak logics from an algebraic perspective, we need to refine the usual algebraic semantics from abstract algebraic logic in order to handle the failure of uniform substitution. To this end, we introduce expanded algebras as the expansion of standard algebras by an extra predicate symbol.

**Definition 2.10** (Expanded Algebra). Let $A$ be an $\mathcal{L}$-algebra and $P$ a unary predicate, an expanded algebra is a structure in the vocabulary $\mathcal{L} \cup \{P\}$. We denote the interpretation $P^A$ also by $\text{core}(A)$, and we refer to it as the core of the expanded algebra $A$.

**Remark 2.11.** Essentially, expanded algebras are first-order structures with exactly one predicate symbol of arity 1, and arbitrary many functional symbols. Since the relational part of the language consists of only one predicate, we can always assume without loss of generality that it consists of the same symbol, so that we always regard any two expanded $\mathcal{L}$-algebras as structures in the same vocabulary. We often talk simply of algebras and expanded algebras when the vocabulary $\mathcal{L}$ is clear from the context.

Since expanded algebras are first-order structures, we can apply the definition of maps from Definition 1.8 in their setting. In particular, we recall that a strong homomorphism $h : A \to B$ of two expanded algebras is a $\mathcal{L}$-algebra homomorphism such that $h[\text{core}(A)] = \text{core}(B)$. Then, we recall from Notation 1.10 that $\mathbb{H}(K)$ indicates the closure of $K$ under strong homomorphisms. We can then extend the notions of quasivarieties and varieties to the setting of expanded algebras.

**Definition 2.12.** A class of expanded algebras $K$ is a quasivariety if it is closed under $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{P}_U$. A class of expanded algebras $K$ is a variety if it is closed under $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$. We denote by $\mathbb{Q}(K)$ the quasivariety generated by $K$ and by $\mathbb{V}(K)$ the variety generated by $K$.

**Notation 2.13.** Let $K$ be a class of expanded $\mathcal{L}$-algebras, then we write $K|\mathcal{L}$ for the class of its $\mathcal{L}$-reducts. If it is clear from the context, we sometimes simply write $K$ also for $K|\mathcal{L}$.

**Remark 2.14.** We notice the following: if $K$ is a quasivariety of expanded $\mathcal{L}$-algebras, then its $\mathcal{L}$-reducts $K|\mathcal{L}$ form a quasivariety of $\mathcal{L}$-algebras. However, if $K$ is a quasivariety of $\mathcal{L}$-algebras, then it is not the case that an arbitrary expansion of the algebras in $K$ gives rise to a quasivariety of expanded algebras. To obtain a
quasivariety of expanded algebras we need to consider the generated quasivariety $Q(K/\mathcal{L})$. We shall consider in 3.3 later some cases when this additional step is not necessary, namely the case when a quasivariety of algebras determines uniquely a quasivariety of expanded algebras.

Crucially, expanded algebras allow us to define a more fine-grained consequence relation than the one we introduced in Definition 1.17. The key idea is to use the core of the algebra to restrict the possible valuation of the atomic variables of the language. To our knowledge, the idea of restricting the possible valuations of the atomic variables to a specific subset of an algebra first appeared in [2].

**Definition 2.15.** The *expanded term algebra* in the signature $\mathcal{L}$ is the structure $\text{Fm}_\mathcal{L}$ augmented with a core $\text{core}(\text{Fm}_\mathcal{L}) = \text{Var}$. We often write $(\text{Fm}, \text{Var})$ and omit both the index $\mathcal{L}$ and its signature operations when the language is clear from the context. A homomorphism from $\text{Fm}_\mathcal{L}$ to an expanded $\mathcal{L}$-algebra $A$ is a core assignment, i.e., it is a map $h : \text{Fm}_\mathcal{L} \rightarrow A$ such that $h(p) \in \text{core}(A)$ for all $p \in \text{Var}$. We write $\text{Hom}^c(\text{Fm}_\mathcal{L}, A)$ for the set of core assignments from $\text{Fm}_\mathcal{L}$ into $A$.

**Remark 2.16.** We notice that, alternatively, one could also consider the expansion of the term algebra $\text{Fm}$ with the core defined by letting $\text{core}(\text{Fm}_\mathcal{L}) = \text{core}(\models)$. This means that the endomorphisms of the term algebra are all the admissible substitutions of $\models$, and not only the atomic substitutions. As this makes only for a minor generalisation of our results, we stick to the former definition and always consider the atomic formulas as the underlying core of the term algebra.

**Definition 2.17** (Core Semantics). Let $K$ be a class of expanded algebras and $\Theta \cup \{ \varepsilon \approx \delta \}$ a set of equations, then the *equational core-consequence relative to $K$* is defined as follows:

$$\Theta \models_K \varepsilon \approx \delta \iff \text{for all } A \in K, h \in \text{Hom}^c(\text{Fm}, A),
\text{if } h(\varepsilon_i) = h(\delta_i) \text{ for all } \varepsilon_i \approx \delta_i \in \Theta, \text{ then } h(\varepsilon) = h(\delta).$$

We then write $\models_K \bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \rightarrow \varepsilon \approx \delta$ if $\bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \models_K \varepsilon \approx \delta$. We usually write $A \models^c \bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \rightarrow \varepsilon \approx \delta$ in place of $\models^c_{\text{Fm}} \bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \rightarrow \varepsilon \approx \delta$. The related notions for equations are defined analogously.

**Remark 2.18.** Crucially, if $\Theta$ is a finite set of equations $\{ \varepsilon_i \approx \delta_i : i \leq n \}$, then we have that $\bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \models_K \varepsilon \approx \delta$ holds if and only if

$$K \models \forall x_0, \ldots, \forall x_m \left( \bigwedge_{i \leq n} \text{core}(x_i) \land \bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \rightarrow \varepsilon \approx \delta \right).$$

In other words, when $\Theta$ is finite the core semantics over a class of expanded algebra can be encoded in terms of the standard first-order consequence relation $\models$, exactly as in the case of the relation $\models^c$. As we stressed in Remark 1.18, if the relation $\models^c$ is not finitary then one could provide a similar translation into a suitable infinitary logic and replace quasivarieties by generalised quasivarieties.

By Definition 2.17, in core semantics atomic variables are always assigned to core elements and, as a result of this feature, arbitrary formulas are always interpreted inside the subalgebra generated by the core. This motivates the interest in core-generated structures and in quasivarieties which are generated by these structures.
Definition 2.19. An expanded algebra \( A \) is core-generated if \( A = \langle \text{core}(A) \rangle \). If \( Q \) is a class of algebras, we write \( Q_{CG} \) for its subclass of core-generated structures. A quasivariety \( Q \) of expanded algebras is core-generated if it is generated by its subclass of core-generated expanded algebras, i.e., \( Q = Q(Q_{CG}) \). Similarly, a variety \( V \) of expanded algebras is core-generated if \( V = V(V_{CG}) \).

The following proposition shows that unrestricted substitutions interacts nicely with core-generated quasivarieties in the context of core semantics. In a sense, the next result shows that core semantics in core-generated classes of structures approximates as much as possible the standard consequence relations defined over quasivarieties.

Lemma 2.20. Let \( A \) be a core-generated expanded algebra and let \( h \in \text{Hom}(\text{Fm}, \ A) \), then there are a core assignment \( g \in \text{Hom}^c(\text{Fm}, A) \) and a substitution \( \sigma \) such that 
\[
\begin{align*}
\forall \phi \in \text{Fm} . \quad h(\phi) &= g(\sigma(\phi)) \quad \text{for all } \phi \in \text{Fm}.
\end{align*}
\]

Proof. Let \((p_i)_{i<\omega}\) be an enumeration of \( \text{Var} \). Since \( A \) is core-generated, we have that for all \( p_i \in \text{Var} \) there is a polynomial \( t_i \) such that \( h(p_i) = t_i(x_0, \ldots, x_{n_i}) \) with \( x_j \in \text{core}(A) \) for all \( j \leq n_i \). Let now \( \{q^i_j : i < \omega, j \leq n_i\} \) be another enumeration of \( \text{Var} \) and define \( g \in \text{Hom}^c(\text{Fm}, A) \) such that \( g(q^i_j) = x^i_j \) for all \( i < \omega, j \leq n_i \). In particular, this means that for all formulas \( \psi(p_0, \ldots, p_m) \in \text{Fm} \) we have 
\[
\begin{align*}
&h(\psi(p_0, \ldots, p_m)) = g(\psi(t_0(q^0_0, \ldots, q^0_{m_0}), \ldots, t_m(q^m_0, \ldots, q^m_{m_m}))).
\end{align*}
\]

By construction we have that \( g \) is a core assignment. Let \( \sigma \) be the substitution defined by letting \( \sigma(p_i) = t_i(q^i_0, \ldots, q^i_{n_i}) \) for all \( p_i \in \text{Var} \), then from the display above we derive that \( h(\phi) = g(\sigma(\phi)) \) for all \( \phi \in \text{Fm} \).

Proposition 2.21. Let \( Q \) be a core-generated quasivariety of expanded algebras, then \( \sigma(\Theta) \models_Q \sigma(\varepsilon \approx \delta) \) for all \( \sigma \in \text{Subst}(\mathcal{L}) \) holds if and only if \( \Theta \models_Q \varepsilon \approx \delta \).

Proof. We consider the left-to-right direction. Suppose that \( \Theta \not\models_Q \varepsilon \approx \delta \). Since \( Q \) is core-generated, there is a core-generated expanded algebra \( A \in Q \) and some \( h \in \text{Hom}(\text{Fm}, A) \) such that \( A \models_h \Theta \) and \( A \not\models_h \varepsilon \approx \delta \) (recall Definition 1.17).

From Lemma 2.20 it follows that there is a core assignment \( g \in \text{Hom}^c(\text{Fm}, A) \) and a substitution \( \sigma \) such that \( h(\phi) = g(\sigma(\phi)) \) for all \( \phi \in \text{Fm} \). Then it follows that 
\[
\begin{align*}
A \models g(\sigma(\Theta)) \quad \text{but} \quad A \not\models g(\sigma(\varepsilon \approx \delta)), \quad \text{whence} \quad \sigma(\Theta) \not\models_Q \sigma(\varepsilon \approx \delta).
\end{align*}
\]

We consider the right-to-left direction. Suppose that \( \sigma(\Theta) \not\models_Q \sigma(\varepsilon \approx \delta) \), thus we can find an algebra \( A \in Q \) and an assignment \( h \in \text{Hom}^c(\text{Fm}, A) \) such that 
\[
\begin{align*}
A \models_h \sigma(\Theta) \quad \text{and} \quad A \not\models_h \sigma(\varepsilon \approx \delta).
\end{align*}
\]

Then we have that \( h \circ \sigma \in \text{Hom}(\text{Fm}, A) \), \( A \models_{h \circ \sigma} \Theta \) and \( A \not\models_{h \circ \sigma} \varepsilon \approx \delta \), whence \( \Theta \not\models_Q \varepsilon \approx \delta \).

Since core semantics only looks at the substructure \( \langle \text{core}(A) \rangle \) of an expanded algebra \( A \), it make sense to consider extensions of \( A \) that preserve the core.

Definition 2.22. We say that \( B \) is a core superalgebra of \( A \) if \( B \in \text{Mod}(\mathcal{L}) \), \( A \subseteq B \) and \( \text{core}(A) = \text{core}(B) \). If \( K \) is class of algebras, we write \( \mathcal{C}(K) \) for the class of all core superalgebras of elements of \( K \).

The next lemma and the following proposition establish some fundamental facts about the relation \( \models^c \). Importantly, they show that the validity of quasiequations is preserved both under the operators \( \mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_U \), and also under the core superstructure operator \( \mathcal{C} \) from Definition 2.22. Additionally, we also show that
equations are preserved under strong homomorphisms. The following lemma is essentially a rephrasing of Remark 2.18.

**Lemma 2.23.** Let $K$ be a class of expanded algebras, let $\bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \rightarrow \alpha \approx \beta$ be a quasiequation and let $V$ be all the variables occurring in it, then

$$\bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \models_K \alpha \approx \beta \iff \bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \land \bigwedge_{x \in V} \text{core}(x) \models_K \alpha \approx \beta.$$  

**Proof.** This follows immediately from the fact that an assignment $h \in \text{Hom}(Fm, A)$ is a core assignment if and only if $h(x) \in \text{core}(A)$ for all $x \in \text{Var}$.

**Proposition 2.24.** Let $K$ be a class of expanded algebras, then the following hold:

1. Let $\bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \rightarrow \varepsilon \approx \delta$ be a quasi-equation, then for all $O \in \{\mathbb{I}, S, P, P_U, C\}$ we have that

$$\bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \models_C \varepsilon \approx \delta \implies \bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \models_{\mathbb{O}(K)} \bigwedge_{i \leq n} \varepsilon \approx \delta;$$

2. if $\models_K \varepsilon \approx \delta$ then $\models_{\mathbb{H}(K)} \varepsilon \approx \delta$.

**Proof.** For Claim (1) recall that the validity of universal Horn formulas is preserved by the operations $\mathbb{O} \in \{\mathbb{I}, S, P, P_U\}$. Then Lemma 2.23 implies that the validity of formulas in core semantics is preserved by $\mathbb{O} \in \{\mathbb{I}, S, P, P_U\}$. Preservation of validity by $C$ follows immediately by the definitions of core semantics and core superstructure. Claim (2) is immediate by the definition of core semantics and the fact that, if $B \in \mathbb{H}(K)$, then there is $A \in K$ and a surjective homomorphism $h : A \rightarrow B$ such that $\text{core}(B) = h[\text{core}(A)].$

We conclude this section by showing a version of Maltsev’s Theorem for the setting of core-generated quasivarieties, i.e., we prove that every core-generated quasivariety is axiomatised by its validities under core semantics.

**Definition 2.25.** We define the following notions.

1. For any set $T$ of quasiequations (or equations), we let $\text{Mod}^c(T)$ be the class of expanded algebras $A$ such that $A \models^c T$ and $\text{Mod}_{\text{CG}}^c(T)$ for its subclass of core-generated models.
2. For any class $K$ of expanded algebras, we denote by $\text{Th}_{qe}^c(K)$ the set of all quasiequations true in $K$ under core semantics, and we denote by $\text{Th}_{qe}^c(K)$ the set of all equations true in $K$ under core semantics.

The following proposition is an immediate corollary of Maltsev’s Theorem (cf. Fact 1.14) and Proposition 2.21.

**Proposition 2.26.** Let $Q$ be a quasivariety of expanded algebras and let $A$ be a core-generated expanded algebra, then $A \in Q_{\text{CG}}$ if and only if $A \models^c \text{Th}_{qe}^c(Q)$.

**Proof.** The direction from left-to-right follows immediately by the definition of $\text{Th}_{qe}^c(Q)$. We consider the direction from right-to-left. Let $A$ be core-generated and suppose $A \notin Q_{\text{CG}}$. It follows that $A \notin Q$ and so by Maltsev’s Theorem $A \not\models \text{Th}_{qe}(Q)$. Let $\bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \rightarrow \alpha \approx \beta \in \text{Th}_{qe}(Q)$ be such that $A \not\models \bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \rightarrow \alpha \approx \beta$. By Proposition 2.21 it follows that there is a substitution $\sigma$ such that $A \not\models^c \sigma(\bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \rightarrow \alpha \approx \beta)$. Now, since the relation $\models_Q$ is closed under uniform substitution, it follows in particular that $\sigma(\bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \rightarrow \alpha \approx \beta) \in \text{Th}_{qe}(Q)$. 

Moreover, since obviously $\text{Th}_{\mathsf{qe}}(Q) \subseteq \text{Th}_{\mathsf{qe}}^{\varepsilon}(Q)$, we obtain that $\sigma(\bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \rightarrow \alpha \approx \beta) \in \text{Th}_{\mathsf{qe}}^{\varepsilon}(Q)$. We conclude that $A \not \vdash_{\varepsilon} \text{Th}_{\mathsf{qe}}^{\varepsilon}(Q)$.

3. Algebraizability of Weak Logics

In this section we use quasivarieties of expanded algebras to provide three different notions of algebraizability for the setting of weak logics, which we respectively call loose algebraizability, strict algebraizability and fixed-point algebraizability. We proceed as follows: firstly, we review the notion of algebraizability for the setting of standard logics in Section 3.1, then in Section 3.2 we introduce the loose notion of algebraizability for weak logics, in Section 3.3 we consider the strict version of algebraizability for weak logics, and finally in Section 3.4 we focus on the very specific notion of fixed-point algebraizability.

3.1. Algebraizability of Standard Logics

We first review in this section the notion of algebraizability for the setting of standard logics. Recall that we denote by $\mathsf{Fm}$ and $\mathsf{Eq}$ respectively the set of formulas and equations in the signature $\mathcal{L}$. We shall refer to two maps, called transformers $\tau : \mathsf{Fm} \to \varphi(\mathsf{Eq})$ and $\Delta : \mathsf{Eq} \to \varphi(\mathsf{Fm})$. We say that $\tau$ and $\Delta$ are structural if for all substitutions $\sigma \in \text{Subst}(\mathcal{L})$, $\tau(\sigma(\phi)) = \sigma(\tau(\phi))$ and $\sigma(\Delta(\varepsilon, \delta)) = \Delta(\sigma(\varepsilon), \sigma(\delta))$, where we let $\sigma(\varepsilon, \delta) = (\sigma(\varepsilon), \sigma(\delta))$. We say that the transformers $\tau, \Delta$ are finitary if for all $\phi \in \mathsf{Fm}$ and $\varepsilon \approx \delta \in \mathsf{Eq}$, $|\tau(\phi)| < \aleph_0$ and $|\Delta(\varepsilon, \delta)| < \aleph_0$. For any set of formulas $\Gamma \subseteq \mathsf{Fm}$, we let $\tau(\Gamma) = \bigcup_{\phi \in \Gamma} \tau(\phi)$ and for all set of equations $\Theta \subseteq \mathsf{Eq}$ we let $\Delta(\Theta) = \bigcup_{\varepsilon \approx \delta \in \Theta} \Delta(\varepsilon, \delta)$.

The following notion of algebraizability was introduced by Blok and Pigozzi in their seminal article [4].

**Definition 3.1.** A pair of transformers in the language $\mathcal{L}$ is a pair of functions $\tau : \mathsf{Fm} \to \varphi(\mathsf{Eq})$ and $\Delta : \mathsf{Eq} \to \varphi(\mathsf{Fm})$. We say that $\tau$ and $\Delta$ are structural if for all substitutions $\sigma \in \text{Subst}(\mathcal{L})$, $\tau(\sigma(\phi)) = \sigma(\tau(\phi))$ and $\sigma(\Delta(\varepsilon, \delta)) = \Delta(\sigma(\varepsilon), \sigma(\delta))$, where we let $\sigma(\varepsilon, \delta) = (\sigma(\varepsilon), \sigma(\delta))$. We say that the transformers $\tau, \Delta$ are finitary if for all $\phi \in \mathsf{Fm}$ and $\varepsilon \approx \delta \in \mathsf{Eq}$, $|\tau(\phi)| < \aleph_0$ and $|\Delta(\varepsilon, \delta)| < \aleph_0$. For any set of formulas $\Gamma \subseteq \mathsf{Fm}$, we let $\tau(\Gamma) = \bigcup_{\phi \in \Gamma} \tau(\phi)$ and for all set of equations $\Theta \subseteq \mathsf{Eq}$ we let $\Delta(\Theta) = \bigcup_{\varepsilon \approx \delta \in \Theta} \Delta(\varepsilon, \delta)$.

The following notion of algebraizability was introduced by Blok and Pigozzi in their seminal article [4].

**Definition 3.2 (Algebraizability).** A (standard) logic $\vdash$ is algebraizable if there are a quasivariety of (standard) algebras $Q$ and structural transformers $\tau : \mathsf{Fm} \to \varphi(\mathsf{Eq})$ and $\Delta : \mathsf{Eq} \to \varphi(\mathsf{Fm})$ such that:

- $(A1)$ $\Gamma \vdash \phi$ $\iff$ $\tau(\Gamma) \models Q \tau(\phi)$
- $(A2)$ $\Delta[\Theta] \vdash \Delta(\varepsilon, \delta)$ $\iff$ $\Theta \models Q \varepsilon \approx \delta$
- $(A3)$ $\phi \vdash_{Q} \Delta[\tau(\phi)]$
- $(A4)$ $\varepsilon \approx \delta \equiv_{Q} \Delta[\tau(\varepsilon, \delta)]$.

If these conditions are met, then we say that $Q$ is an equivalent algebraic semantics of $\vdash$, and that $(Q, \tau, \Delta)$ witnesses the loose algebraizability of $\vdash$.

**Remark 3.3.** The restriction to quasivarieties of algebras stems from the fact that we are exclusively considering finitary consequence relations $\vdash$. See [16, §3] for an extension of this definition to generalised quasivarieties and non-finitary logics.

We recall the following two important facts about algebraizability. We refer the reader to [16, Prop. 3.12, Thm. 3.37] for a proof of these results.

**Fact 3.4.** Let $\vdash$ be a standard logic, then:

1. If 3.2(A1) and 3.2(A4), or 3.2(A2) and 3.2(A3) hold, then $\vdash$ is algebraized by $(Q, \tau, \Delta)$.
(2) if $\vdash$ is algebraizable, then there are two finitary transformers $\tau, \Delta$ witnessing this fact.

It is a key property of algebraizability that the equivalent algebraic semantics of a standard logic is unique. See [16] for a proof of this result.

**Fact 3.5.** If the tuples $(Q_0, \tau_0, \Delta_0)$ and $(Q_1, \tau_1, \Delta_1)$ both witness the algebraizability of a standard logic $\vdash$, then:

1. $Q_0 = Q_1$;
2. $\Delta_0(\varepsilon, \delta) \models \Delta_1(\varepsilon, \delta)$, for all $\varepsilon \approx \delta \in Eq$;
3. $\tau_0(\phi) \equiv_{Q_0} \tau_1(\phi)$ with $i \in \{0, 1\}$, for all $\phi \in Fm$.

### 3.2. Loose Algebraizability of Weak Logics

By using the core consequence relation $\models^c$ in place of the standard one $\models$, we can provide a first version of the notion of algebraizability in the setting of weak logics. Obviously, we write $\Gamma_0 = \Gamma_1$ as a shorthand for $\Gamma_0 \models_k \Gamma_1$ and $\Gamma_1 \models_k \Gamma_0$.

**Definition 3.6.** A weak logic $\vdash$ is loosely algebraizable if there is a core-generated quasivariety of expanded algebras $Q$ and two structural transformers $\tau : Fm \rightarrow \omega(\text{Eq})$ and $\Delta : \text{Eq} \rightarrow \nu(\text{Fm})$ such that:

(W1) $\Gamma \models \phi \iff \tau(\Gamma) \models_{Q} \tau(\phi)$
(W2) $\Delta[\Theta] \models \Delta(\varepsilon, \delta) \iff \Theta \models_{Q} \varepsilon \approx \delta$
(W3) $\phi \models \Delta[\tau(\phi)]$
(W4) $\varepsilon \approx \delta \models_{Q} \tau[\Delta(\varepsilon, \delta)]$.

If these conditions are met, we then say that $Q$ is a loose algebraic semantics of $\vdash$, and that the pair $(Q, \tau, \Delta)$ witnesses the loose algebraizability of $\vdash$.

To show this definition is indeed robust, we adapt the proof of the uniqueness of the equivalent algebraic semantic of standard logics [16, Thm. 3.17] to the case of loosely algebraizable weak logics.

**Theorem 3.7.** If both the tuples $(Q_0, \tau_0, \Delta_0)$ and $(Q_1, \tau_1, \Delta_1)$ witness the algebraizability of $\vdash$, then:

1. $Q_0 = Q_1$;
2. $\Delta_0(\varepsilon, \delta) \models \Delta_1(\varepsilon, \delta)$, for all $\varepsilon \approx \delta \in Eq$;
3. $\tau_0(\phi) \equiv_{Q_0} \tau_1(\phi)$ with $i \in \{0, 1\}$, for all $\phi \in Fm$.

**Proof.** Notice that the two witnesses of algebraizability give rise to two different consequence relations, which we shall denote by $\models_{Q_0}(\Delta_0)$ and $\models_{Q_1}$.

Clause (2): $\Delta_0(\varepsilon, \delta) \models \Delta_1(\varepsilon, \delta)$, for all $\varepsilon \approx \delta \in Eq$.

We prove $\Delta_0(\varepsilon, \delta) \models \Delta_1(\varepsilon, \delta)$. Let $\phi \in \Delta_1(\varepsilon, \delta)$, then we clearly have that:

$\tau_0(\phi(\varepsilon, \delta)), \phi(\varepsilon, \delta) \equiv \phi(\varepsilon, \delta) \models_{Q_0} \tau_0(\phi(\varepsilon, \delta))$.

By 3.6(W2) it follows:

$\Delta_0(\tau_0(\phi(\varepsilon, \delta))), \Delta_0(\phi(\varepsilon, \delta)) \models \Delta_0(\tau_0(\phi(\varepsilon, \delta)))$.

hence, by 3.6(W3):

(a) $\phi(\varepsilon, \delta), \Delta_0(\phi(\varepsilon, \delta)) \models \phi(\varepsilon, \delta)$. 


Now, we also have that $\varnothing \models_{Q_1} \varepsilon \approx \varepsilon$, hence by $\phi \in \Delta_1(\varepsilon, \varepsilon)$ and 3.6(W2), we obtain:

(b) $\varnothing \not\vdash \phi(\varepsilon, \varepsilon)$.

Moreover, it also follows that $\varepsilon \approx \delta \models_{Q_0} \phi(\varepsilon, \delta)$, hence by 3.6(W2):

(c) $\Delta_0(\varepsilon, \delta) \models_{Q_0} \Delta_0(\phi(\varepsilon, \varepsilon), \phi(\varepsilon, \delta))$.

Finally, by (a), (b) and (c), it follows that $\Delta_0(\varepsilon, \delta) \not\vdash \phi(\varepsilon, \delta)$, hence $\Delta_0(\varepsilon, \delta) \not\vdash \Delta_1(\varepsilon, \delta)$. The converse direction is proven analogously.

Clause (1): $Q_0 = Q_1$.

We first prove that $Q_0$ and $Q_1$ satisfy the same quasi-equations under core semantics. We show only that $\text{Th}_{\text{eq}}^c(Q_0) \subseteq \text{Th}_{\text{eq}}^c(Q_1)$, as the other direction follows analogously. Let $\bigwedge_{i \leq n} \varepsilon_i = \Delta_i \rightarrow \varepsilon \approx \delta \in \text{Th}_{\text{eq}}^c(Q_0)$, then it follows that

$\bigwedge_{i \leq n} \varepsilon_i = \Delta_i \rightarrow \varepsilon \approx \delta \models_{Q_0} \varepsilon \approx \delta$ and this yields that $\bigcup_{i \leq n} \Delta_0(\varepsilon_i, \delta_i) \models_{Q_0} \Delta_0(\varepsilon, \delta)$ by 3.6(W2).

By point (3) above, it follows that $\bigcup_{i \leq n} \Delta_1(\varepsilon_i, \delta_i) \models_{Q_0} \Delta_1(\varepsilon, \delta)$, hence by 3.6(W2) we get $\bigwedge_{i \leq n} \varepsilon_i = \Delta_i \models_{Q_0} \varepsilon \approx \delta$. The latter finally entails $\bigwedge_{i \leq n} \varepsilon_i = \Delta_i \rightarrow \varepsilon \approx \delta \in \text{Th}_{\text{eq}}^c(Q_1)$ and thus $\text{Th}_{\text{eq}}^c(Q_0) \subseteq \text{Th}_{\text{eq}}^c(Q_1)$. By reasoning analogously we obtain that $\text{Th}_{\text{eq}}^c(Q_1) \subseteq \text{Th}_{\text{eq}}^c(Q_0)$, hence $\text{Th}_{\text{eq}}^c(Q_0) = \text{Th}_{\text{eq}}^c(Q_1)$. It then follows by Proposition 2.26 above that $(Q_0)_{\text{CG}} = (Q_1)_{\text{CG}}$ and since both $Q_0$ and $Q_1$ are core-generated $Q_0 = Q_1$.

Clause (3): $\tau_0(\phi) \equiv_{Q_0}^c \tau_1(\phi)$ with $i \in \{0, 1\}$, for all $\phi \in \text{Fm}$.

By Clause (1), it suffices to prove that $\tau_0(\phi) \equiv_{Q_0}^c \tau_1(\phi)$. By 3.6(W3), we have $\Delta_0(\tau_0(\phi)) \not\models_{Q_0} \Delta_1(\tau_1(\phi))$ and by Clause (3) this is equivalent to $\Delta_0(\tau_0(\phi)) \not\models_{Q_0} \Delta_0(\tau_1(\phi))$. It then follows by 3.6(W2) that $\tau_1(\phi) \equiv_{Q_0}^c \tau_2(\phi)$.

We have thus established that every algebraizable weak logic has a unique equivalent algebraic semantics, up to equivalence under the core consequence relation. We conclude this section by proving an analogue of Fact 3.4 for weak logics.

**Proposition 3.8.** Let $\models$ be a weak logic, then:

(1) if 3.6(W1) and 3.6(W4), or 3.6(W2) and 3.6(W3) hold, then $\models$ is algebraized by $(Q, \tau, \Delta)$;

(2) if $\models$ is algebraizable, then there are two finitary transformers $\tau, \Delta$ witnessing this fact.

**Proof.** We prove (1). Suppose $\models$ is a weak logic and let $(Q, \tau, \Delta)$ satisfy 3.6(W1) and 3.6(W4). We verify that 3.6(W2) and 3.6(W3) hold as well. By 3.6(W4) $\Theta \models_{Q}^c \varepsilon \approx \delta$ is equivalent to $\tau[\Delta(\Theta)] \models_{Q}^c \tau(\Delta(\varepsilon, \delta))$, which by 3.6(W1) is equivalent to $\Delta(\Theta) \models_{Q}^c \Delta(\varepsilon, \delta)$, proving 3.6(W2). Also, for any formula $\phi$, we have that $\tau(\phi) \equiv_{Q}^c \tau(\phi)$, hence by 3.6(W4) $\tau(\Delta(\tau(\phi))) \models_{Q}^c \tau(\phi)$ and by 3.6(W1) $\Delta(\tau(\phi)) \not\models_{Q}^c \phi$, proving 3.6(W3). If $(Q, \tau, \Delta)$ satisfies 3.6(W2) and 3.6(W3), then we proceed analogously.

We prove (2). For any $\phi \in \text{Fm}$, we have by 3.6(W3) that $\phi \not\models_{Q}^c \Delta[\tau(\phi)]$, thus by $\models$ being finitary there is some $\tau_0(\phi) \subseteq \tau(\phi)$ such that $|\tau_0(\phi)| < \aleph_0$ and $\phi \not\models_{Q}^c \Delta[\tau_0(\phi)]$. Moreover, for any equation $\varepsilon \approx \delta \in \text{Eq}$, we have by 3.6(W4) that $\varepsilon \approx \delta \equiv_{Q}^c \tau(\Delta(\varepsilon, \delta))$ we obtain by finitariness a finite subset $\Delta_0(\varepsilon, \delta) \subseteq \Delta(\varepsilon, \delta)$ such that $\varepsilon \approx \delta \equiv_{Q}^c \tau(\Delta_0(\varepsilon, \delta))$. Finally, it follows by the choice of $\tau_0, \Delta_0$ that $\Delta[\tau(\phi)] \not\models_{Q}^c \Delta(\tau_0(\phi))$, hence $\tau(\phi) \equiv_{Q}^c \tau_0(\phi)$. Similarly, from $\tau(\Delta(\varepsilon, \delta)) \models_{Q}^c \tau(\Delta_0(\varepsilon, \delta))$ we obtain $\Delta_0(\varepsilon, \delta) \not\models_{Q}^c \Delta(\varepsilon, \delta)$. Thus $\tau_0, \Delta_0$ together with $Q$ and $\Sigma$ witness the algebraizability of $\models$.
Remark 3.9. As witnessed by the previous results, loose algebraizability satisfies the same uniqueness property of standard algebraizability. However, we believe this is too weak of a notion, as it does not really meet the fundamental intuition behind algebraizability. In fact, in contrast to the matrix semantics of logics (which we shall explore later in Section 6), the fundamental aspect of algebraizability is that it allows us to translate logical systems into algebras, and not simply into first-order structures. We achieve this by considering strict algebraizability in the next section.

3.3. Strict Algebraizability of Weak Logics

We introduce strict algebraizability as a refined notion of algebraizability for weak logical systems. In fact, as we mentioned in Remark 3.9, the problem with loose algebraizability is that it relates weak logics to (universal Horn) classes of first-order structures, and not really to algebras. To avoid this issue and deal exclusively with algebras, we need to look for ways in which one can, so to speak, eliminate the core predicate \( \text{core}(A) \). The key idea is to restrict attention to classes of expanded algebras in which the core is already definable in the functional part of the signature. Furthermore, since we are dealing with quasivarieties and we want the validity of formulas to be preserved by the operators \( I, S, P \) and \( P_U \), we exclusively consider definability by means of equations. We make this explicit in the following definitions.

Notation 3.10. If \( A \) is an \( L \)-algebra and \( \Sigma(x) \) a set of equations in the variable \( x \), we let \( \Sigma(A) = \{ a \in A : A \models \varepsilon(a) \approx \delta(a) \text{ for all } \varepsilon \approx \delta \in \Sigma \} \).

Definition 3.11. An expanded algebra \( A \) is said to have an equationally definable core if there is a finite set of equations \( \Sigma \) in the variable \( x \) such that \( \text{core}(A) = \Sigma(A) \). A class of expanded algebras \( K \) is said to have a (uniformly) equationally definable core if there is a finite set of equations \( \Sigma \) such that \( \text{core}(A) = \Sigma(A) \) for all \( A \in K \).

The key reason that explains our interest in quasivarieties of expanded algebras with an equationally definable core is that, given an algebra \( A \) and a finite set of equations \( \Sigma \), there is a unique way to expand \( A \) into an expanded algebra with core defined by \( \Sigma \). As the following proposition shows, this provides us with a canonical expansion of \( Q \) into a quasivariety of expanded algebras with core defined by \( \Sigma \).

Proposition 3.12. Let \( Q \) be a quasivariety of \( L \)-algebras and \( \Sigma(x) \) a finite set of equations, then the class of structures \( (A, \text{core}(A)) \) with \( A \in Q \) and \( \text{core}(A) = \Sigma(A) \) is a quasivariety of expanded algebras.

Proof. Let \( Q \) be a quasivariety of \( L \)-algebras and let \( K \) be the class of structures \( (A, \text{core}(A)) \) with \( A \in Q \) and \( \text{core}(A) = \Sigma(A) \). Consider the first-order formula

\[
\Phi := \forall x (\text{core}(x) \rightarrow \alpha(x) \approx \beta(x)) \land \forall x (\alpha(x) \approx \beta(x) \rightarrow \text{core}(x)).
\]

Clearly, \( \Phi \) is a conjunction of universal Horn sentences, thus by Fact 1.14 it is preserved under the quasivariety operators \( I, S, P \) and \( P_U \). It follows that \( \text{core}(A) = \Sigma(A) \) for all \( A \in Q(K) \) and so \( K \) is already a quasivariety.

Remark 3.13. Crucially, the previous proposition cannot be extended to varieties of expanded algebras. In particular, if \( K \) is a class of expanded algebras with \( \text{core}(A) = \Sigma(A) \) for all \( A \in K \), it is not necessarily the case that \( \text{core}(A) = \Sigma(A) \) for all \( A \in H(K) \). For example, consider the equation \( x \approx \neg x \), let \( A \) be any Boolean
algebra with size $|A| \geq 2$ and let $B$ be the trivial one-element Boolean algebra. Clearly there is a surjective homomorphism $h : A \to B$, but at the same time we also have that $\Sigma(B) = B$ and $\Sigma(A) = \emptyset$, showing that $\text{core}(B) \neq h(\text{core}(A))$. This shows that we cannot replace quasivarieties by varieties in the statement of Proposition 3.12. We shall see later that this is possible in the restricted setting of fixed-point algebraizability.

It is easy to find concrete examples of quasivarieties of expanded algebras whose core is equationally definable. We list here some examples which are determined by some very basic equations.

Example 3.14. Let $M$ be a monoid in $\mathcal{L} = (\cdot, e)$ and define $\text{core}(M) = \{ x \in M : M \models x^n \approx e \}$, then $M$ is an expanded algebra with core defined by $\Sigma = \{ x^n \approx e \}$.

Example 3.15. Let $\text{HA}$ be the variety of Heyting algebras and for all $A \in \text{HA}$ let $\text{core}(A) = A_\approx = \{ x \in A : A \models x \approx \neg \neg x \}$, i.e., the core of $A$ is its subset of regular elements. Let $\text{ML}$ be the variety of all Medvedev algebras, then $\text{ML}$ is generated by its subclass of core-generated Heyting algebras $A$ with core $A_\approx$ (cf. [2, 3] and Fact 7.16 below). As we shall see in Section 7, this class plays an important role in the semantics of classical inquisitive propositional logic.

We define the strict version of algebraizability for weak logics by requiring that the core-generated algebras corresponding to a weak logic $\models$ have the core defined by a finite set of equations $\Sigma$. This definition is significantly stronger than the one we introduced in the previous section, and it essentially reduces the weak logic $\models$ to the relative consequence relation $\models_Q$ of the corresponding quasivariety of algebras.

Definition 3.16. A weak logic $\models$ is strictly algebraizable if it is loosely algebraized (in the sense of Definition 3.6) by $(Q, \tau, \Delta)$ and, additionally, there is a finite set of equations $\Sigma$ defining the core of the expanded algebras in $Q$. We then say that $\models$ is strictly algebraized by $(Q, \Sigma, \tau, \Delta)$.

We can then strengthen Theorem 3.7 to the present context of core-generated quasivarieties with a definable core.

Theorem 3.17. If both the tuples $(Q_0, \Sigma_0, \tau_0, \Delta_0)$ and $(Q_1, \Sigma_1, \tau_1, \Delta_1)$ witness the algebraizability of $\models$, then:

1. $(Q_0, \Sigma_0, \tau_0, \Delta_0)$ and $(Q_1, \Sigma_1, \tau_1, \Delta_1)$ witness the algebraizability of $\models$.

Proof. The clauses (1), (3) and (4) follow immediately from Theorem 3.7, so we prove (2). First, notice that by (1) we have $Q_0 = Q_1$, so let $Q = Q_1$, $i \in \{0, 1\}$. Let $\alpha_0 \approx \beta_0 \in \Sigma_0$, then $\models_{Q_0} \alpha_0 \approx \beta_0$ hence by 3.6(W2) we obtain that $\emptyset \models_{\Sigma_0} \Delta_0(\alpha_0 \approx \beta_0)$ and thus $\emptyset \models \Delta_1(\alpha_0 \approx \beta_0)$. It follows that $\models_{Q_1} \alpha_0 \approx \beta_0$, meaning that $(A, \text{core}_1(A)) \models_{Q_1} \alpha_0 \approx \beta_0$ for all $\alpha_0 \approx \beta_0 \in \Sigma_0$. This shows that $\text{core}_1(A) \subseteq \{ x \in A : \alpha_0(x) \approx \beta_0(x) \} = \text{core}_0(A)$. The other direction is proven analogously.

The following corollary follows immediately from Proposition 3.8.

Corollary 3.18. Let $\models$ be a standard logic, then:

1. if 3.6(W1) and 3.6(W4), or 3.6(W2) and 3.6(W3) hold, then $\models$ is algebraized by $(Q, \Sigma, \tau, \Delta)$;
(2) if \( \vdash \) is algebraizable, then there are two finitary transformers \( \tau, \Delta \) witnessing this fact.

### 3.4. Fixed-Point Algebraizability of Weak Logics

We introduce a third alternative definition of algebraizability for weak logics, which can be seen as a refinement of strict algebraizability. We first define the following notion of selector term, essentially from [1].

**Notation 3.19.** Let \( t(x) \) be unary \( \mathcal{L} \)-term, then we define recursively \( t^1(x) = t(x) \) and \( t^{n+1}(x) = t(t^n(x)) \) for all \( n < \omega \).

**Definition 3.20.** Let \( Q \) be a quasivariety of \( \mathcal{L} \)-algebras and \( \delta(x) \) be a unary term in \( \mathcal{L} \), then we say that \( \delta(x) \) is a selector term for \( Q \) if \( Q \models \delta(x) \approx \delta(x) \).

**Remark 3.21.** Selector terms essentially identifies the fixed points of polynomials in \( Q \). If there is some \( n < \omega \) such that for all \( m < \omega \) we have \( \delta^{n+m}(x) = \delta^n(x) \), then in particular we obtain

\[
\delta^n(\delta^n(a)) = \delta^{n+n}(a) = \delta^n(a)
\]

for all \( A \in Q \). Thus \( \delta^n \) is a selector term for \( Q \).

**Definition 3.22.** A weak logic \( \vdash \) is fixed-point algebraizable if it is strictly algebraized by a tuple \( (Q, \Sigma, \tau, \Delta) \) (as in 3.16) and additionally \( \Sigma = \{ x \approx \delta(x) \} \) for some selector term \( \delta(x) \).

The key idea in the previous definition is that the core of the expanded algebras corresponding to the logic \( \vdash \) is not simply defined by a finite sets of equations, but by one single equation characterising it as the set of fixed points of a certain polynomial. The main motivation lies in the fact that fixed-point algebraizability behaves very well in the context of varieties, as the following result shows (this is essentially [18, Thm. 19] and [1, Thm. 3.14]).

**Proposition 3.23.** Let \( V \) be a variety of \( \mathcal{L} \)-algebras and \( \Sigma(x) = \{ \delta(x) \approx x \} \) for some selector term \( \delta \), then the class of structures \( (A, \text{core}(A)) \) with \( A \in V \) and \( \text{core}(A) = \Sigma(A) \) is a variety of expanded algebras.

**Proof.** Let \( V \) be a variety of \( \mathcal{L} \)-algebras and let \( K \) be the class of structures \( (A, \text{core}(A)) \) with \( A \in V \) and \( \text{core}(A) = \Sigma(A) \). By Proposition 3.12 it follows that \( K \) is a quasivariety. We show that \( K \) is also closed under homomorphic images. Let \( A \in V \) and consider a surjective homomorphism \( h : A \to B \), it suffices to show that \( \text{core}(B) = h[\text{core}(A)] \). Since \( \text{core}(A) = \Sigma(A) \) and \( \text{core}(B) = \Sigma(B) \), it follows immediately by the fact that homomorphic images preserve the validity of equations that \( h[\text{core}(A)] \subseteq \text{core}(B) \). Consider now some \( b \in \text{core}(B) \) and notice that since \( h \) is surjective there is some \( a \in A \) with \( h(a) = b \). Then we have that \( h(\delta(a)) = \delta(h(a)) = h(a) \), given that \( h(a) = b \in \text{core}(B) = \Sigma(B) \). Then, since \( \delta \) is a selector term we have \( A \models \delta(a) \approx \delta^2(a) \) and so \( a \in \text{core}(A) \). \( \blacksquare \)

**Remark 3.24.** We notice that most of our examples of algebraizable weak logics are actually fixed-point algebraizable. However, in the present work we shall not focus on the special features of fixed-point algebraizability, as our interest is rather in the more general properties of loose and strict algebraizability. We refer the reader to [1] for an in-depth study of logics arising from selector terms.
4. Schematic Fragment and Standard Companions

We start in this section to investigate the relation between the loose and the strict version of algebraizability. In particular, we introduce the notion of schematic fragment of a weak logic and relate the algebraizability of a weak logic to the standard algebraizability of its schematic fragment.

4.1. The Schematic Fragment of a Weak Logic

Given a weak logic $\vdash$, it is natural to ask if we can associate to it some specific standard logical system. The study of negative variants of intermediate logics led to the notion of schematic fragment, which was originally introduced in [22, p. 545] (under the name of standardization) and further investigated in [3, 12]. Here we generalise it to arbitrary weak logics.

**Definition 4.1 (Schematic Fragment).** Let $\vdash$ be a weak logic, we define its **schematic fragment** $\text{Schm}(\vdash)$ as follows:

$$\text{Schm}(\vdash) := \{(\Gamma, \phi) : \forall \sigma \in \text{Subst}(\mathcal{L}), \sigma[\Gamma] \vdash \sigma(\phi)\}$$

and we then also write $\Gamma \vdash_s \phi$ if $(\Gamma, \phi) \in \text{Schm}(\vdash)$.

It is clear from the definition that $\text{Schm}(\vdash)$ is the largest standard logic contained in $\vdash$. We use the schematic fragment of a weak logic to relate loose and strict algebraizability with standard algebraizability. To this end, we first introduce the following notion of (finite) representability of a weak logic.

**Notation 4.2.** Let $\Gamma \subseteq \text{Fm}$ be a set of formulas, we let $\text{At}[\Gamma]$ be the closure of $\Gamma$ under all atomic substitutions $\sigma \in \text{At}(\mathcal{L})$. If $\Theta$ is a set of equations, we similarly denote by $\text{At}[\Theta]$ the closure of $\Theta$ under atomic substitutions.

**Definition 4.3.** We say that a weak logic $\vdash$ is **representable** if there is a set of formulas $\Lambda$ in one variable such that for all $\Gamma \cup \{\phi\} \subseteq \text{Fm}$:

$$\Gamma \vdash \phi \iff \Gamma \cup \text{At}[\Lambda] \vdash_s \phi.$$ 

We say that $\vdash$ is **finitely representable** if the condition above holds for some finite set $\Lambda \subseteq \text{Fm}$.

**Lemma 4.4.** If a weak logic $\vdash$ is loosely algebraizable, then its schematic fragment $\text{Schm}(\vdash)$ is algebraizable.

**Proof.** Let $(Q, \tau, \Delta)$ witness the loose algebraizability of $\vdash$. In particular $Q$ is a quasivariety of expanded algebra, and thus the collection of its algebraic reducts $Q|\mathcal{L}$ is a quasivariety of algebras. We claim that $(Q|\mathcal{L}, \tau, \Delta)$ witnesses the algebraizability of $\text{Schm}(\vdash)$. Notice that by Fact 3.4 it suffices to verify that $(Q|\mathcal{L}, \tau, \Delta)$ satisfies 3.2(A1) and 3.2(A4).

Firstly, we consider 3.2(A1):

$$\Gamma \vdash_s \phi \iff \forall \sigma \in \text{Subst}(\mathcal{L}), \sigma[\Gamma] \vdash \sigma(\phi)$$

(by definition)

$$\iff \forall \sigma \in \text{Subst}(\mathcal{L}), \tau(\sigma[\Gamma]) \models_{Q|\mathcal{L}} \tau(\sigma(\phi))$$

(by 3.6(W2))

$$\iff \forall \sigma \in \text{Subst}(\mathcal{L}), \sigma(\tau[\Gamma]) \models_{Q|\mathcal{L}} \sigma(\tau(\phi))$$

(by structurality)

$$\iff \tau[\Gamma] \models_{Q|\mathcal{L}} \tau(\phi)$$

(by Proposition 2.21).

Next, consider 3.2(A4). To this end, suppose that $\tau(\Delta(x, y)) \not\equiv_{Q|\mathcal{L}} x \approx y$, then by Lemma 2.21 and the fact that $Q$ is core-generated, we get that there exists a
We first show that (1) entails (2). Obviously if

\[ \vdash \tau \]

Proof.\hspace{1cm}\[ \Box \]

\[ \vdash \tau \]

Let (\( \Delta(x,y) \)) such that \( \tau(\Delta(x,y)) \neq \tau(x) \approx y \). Then, we obtain by structurality that \( \tau(\Delta(\sigma(x),\sigma(y))) \neq Q \sigma(x) \approx \sigma(y) \), contradicting the algebraizability of \( \vdash \). This completes the proof. \[ \Box \]

The following theorem allows us to relates loose and strict algebraizability of a weak logic to the standard algebraizability of their schematic fragment.

**Theorem 4.5.** For a weak logic \( \vdash \), the following are equivalent:

1. \( \vdash \) is strictly algebraizable;
2. \( \vdash \) is loosely algebraizable and \( \vdash \) is finitely representable;
3. \( \text{Schm}(\vdash) \) is algebraizable and \( \vdash \) is finitely representable.

**Proof.** We first show that (1) entails (2). Obviously if \( \vdash \) is strictly algebraizable then it is also loosely algebraizable. We show that it is also finitely representable. Let \((Q, \Sigma, \tau, \Delta)\) be the strict algebraic semantics of \( \vdash \). In particular we have that \( \text{core}(A) = \Sigma(A) \) for all \( A \in Q \). We then have the following equivalences:

\[ \begin{align*}
\Gamma \vdash \phi & \iff \tau[\Gamma] \models Q \tau(\phi) \\
& \iff \tau[\Gamma] \cup \text{At}[\Sigma] \models Q \tau(\phi) \tag{by Lemma 2.23}
\end{align*} \]

and thus \( \Lambda = \Delta(\Sigma) \) witnesses the fact that \( \vdash \) is finitely representable.

The direction from (2) to (3) follows immediately by Lemma 4.4.

Finally, we show that (3) entails (1). Suppose that \( \text{Schm}(\vdash) \) is algebraized by \((Q, \tau, \Delta)\) and that \( \vdash \) is finitely represented via a set of formulas \( \Lambda \). Let \( \text{core}(A) = \tau[\Lambda](A) \) for all \( A \in Q \) and consider \( Q' := Q(\text{core}(A), A \in Q) \). We then derive the following equivalences:

\[ \begin{align*}
\Gamma \vdash \phi & \iff \Gamma \cup \text{At}[\Lambda] \vdash_s \phi \\
& \iff \tau[\Gamma] \cup \tau[\text{At}[\Lambda]] \models Q \tau(\phi) \tag{by assumption}
\end{align*} \]

which establish 3.6(W1). We next verify that 3.6(W4) holds as well. Suppose that \( \tau(\Delta(x,y)) \neq Q x \approx y \), then since \( \text{core}(A) = \tau[\Lambda](A) \) we obtain that

\[ \tau(\Delta(x,y)) \cup \tau[\Lambda] \neq Q' \{ x \approx y \} \cup \tau[\Lambda] \]

which contradicts the fact that \( \text{Schm}(\vdash) \) is algebraized by \((Q, \tau, \Delta)\).

\[ \Box \]

The following corollary is an immediate consequence of the proof of the previous theorem and lemma.

**Corollary 4.6.** A weak logic \( \vdash \) is strictly algebraized by \((Q, \Sigma, \tau, \Delta)\) if and only if it is represented by \( \Delta[\Sigma] \) and \( \text{Schm}(\vdash) \) is algebraized by \((Q, \tau, \Delta)\).
4.2. The Lattice of Standard Companions

The previous results established that, if $\vdash$ is strictly algebraizable, then there is a finite set $\Lambda$ witnessing its finite representability. Essentially, this means that the consequences in $\vdash$ can be encoded as logical consequences in its schematic fragment $\vdash_s$, modulo the set of formulas $\Lambda$. However, the schematic fragment of $\vdash$ is not necessarily the only standard logic which bears this property. We study what are the standard logics that, up to $\Lambda$, share the same consequences.

**Definition 4.7.** Let $\vdash$ be a strictly algebraizable weak logic which is finitely representable by $\Lambda$ and with witness $(Q, \Sigma, \tau, \Delta)$. A standard logic $\vdash$ is a standard companion of $\vdash$ if the following conditions hold:

1. $\Gamma \vdash \phi \iff \Gamma \cup \text{At}[\Lambda] \vdash \phi$;
2. $\vdash$ is strictly axiomatised by some quasivariety $K$ together with $\Sigma$ and the two transformers $\tau$ and $\Delta$.

We then denote by $\text{St}(\vdash)$ the family of all standard companions of $\vdash$.

**Remark 4.8.** We provide some explanations of the previous definition. Requirement 4.7(1) is essentially the same condition of representability from Definition 4.3, but with respect to an arbitrary standard logic. Thus condition (1) identifies those standard logics that, up to $\Lambda$, deliver the same weak logic $\vdash$. However, this condition by itself is quite weak, and thus we focus on logics that satisfy also condition 4.7(2), i.e., that can be algebraized via the same transformers $\tau$ and $\Delta$ as $\vdash$. Notice that, in the specific cases of negative variants and polyatomic logics, these families had already been identified and studied in [3] and [1], respectively.

Our underlying intuition is that the family of standard companions defined above must give rise to a corresponding notion on the side of quasivarieties of algebras. We define it as follows.

**Definition 4.9.** Let $Q$ be a core-generated quasivariety of expanded algebras with core defined by a finite set of equations $\Sigma(x)$. Let $K$ be an arbitrary quasivariety of algebras (in the same signature as $Q$), then we say that $K$ is a standard companion of $Q$ if, for all $A \in K$, we have $\langle \Sigma(A) \rangle \in Q$.

The next results provides an important bridge between standard companion of weak logics, and standard companions of expanded quasivarieties.

**Lemma 4.10.** Let $\vdash$ be a weak logic strictly algebraized by $(Q, \Sigma, \tau, \Delta)$, and let $\vdash$ be a standard logic. Then $\vdash \in \text{St}(\vdash)$ if and only if $\vdash$ is algebraized by $(K, \tau, \Delta)$ for some $K$ standard companion of $Q$.

**Proof.** We first prove the left-to-right direction. By definition $\vdash$ is algebraized by some $(K, \tau, \Delta)$, where $\tau$ and $\Delta$ also witness the strict algebraizability of $\vdash$. In particular, notice that by definition of strict representability and Corollary 4.6 we have that $\Lambda = \Delta[\Sigma]$. Now, we consider $K$ as a class of expanded algebras with core $\Sigma(A)$ for all $A \in K$ and we let $C = Q(\{\Sigma(A) : A \in K\})$. Clearly, we have by Proposition 2.26 that $\Theta \models^e_C \varepsilon \approx \delta$ holds if and only if $\Theta \models^e_K \varepsilon \approx \delta$ does. Also, we have that $\Theta \models^e_K \varepsilon \approx \delta$ is equivalent to $\Theta \models^e_Q \varepsilon \approx \delta$ since $\vdash \in \text{St}(\vdash)$ and $\Lambda = \Delta[\Sigma]$. Thus we obtain that $\text{Th}^e_C(Q) = \text{Th}^e_Q(Q)$ and so $C = Q$.

We consider the right-to-left direction. Suppose $\vdash$ is algebraized by $(K, \tau, \Delta)$ and let $K$ be a standard companion of $Q$, we show that $\vdash$ is a standard companion of
\[\vdash.\] We have the following equivalences:

\[
\begin{align*}
\Gamma \vdash \phi & \iff \tau(\Gamma) \vdash Q \tau(\phi) & \text{(by 3.6(W1))} \\
& \iff \tau(\Gamma) \models K \tau(\phi) & \text{(by Definition 4.9 and Proposition 2.26)} \\
& \iff \tau(\Gamma) \cup \text{At}[\Sigma] \models K \tau(\phi) & \text{(by Lemma 2.23)} \\
& \iff \Gamma \cup \text{At}(\text{At}[\Sigma]) \vdash \phi & \text{(by 3.2(A2) and 3.2(A3))} \\
& \iff \Gamma \cup \text{At}(\Delta[\Sigma]) \vdash \phi & \text{(by structurality)}
\end{align*}
\]

and thus, since \( \Lambda = \Delta[\Sigma] \) (as we argued in the previous direction), it follows that \( \vdash \) is a standard companion of \( \models \). This completes our proof.

The correspondence provided by the previous lemma motivates the following definition. Intuitively, while the schematic fragment identifies the maximal standard logic contained in weak logic, the following notion characterises the maximal quasivariety with core defined by \( \Sigma \) that has the same core-generated substructures of a quasivariety \( Q \). The following definition essentially refines the notion of core superalgebra (Definition 2.22) in the setting with definable core.

**Definition 4.11.** We say that \( B \) is a \( \Sigma \)-superalgebra of \( A \) if \( A \subseteq B \) and \( \Sigma(A) = \Sigma(B) \). If \( Q \) is a class of algebras, then we write \( C_{\Sigma}(Q) \) for the class of all \( \Sigma \)-superalgebras of elements of \( Q \).

**Lemma 4.12.** Suppose \( K \) is a quasivariety of expanded algebras with core defined by \( \Sigma \), then \( C_{\Sigma}(K) \) is also a quasivariety of expanded algebras.

**Proof.** We notice that \( A \in C_{\Sigma}(K) \) if and only if \( A \models \forall x(\text{core}(x) \rightarrow \Sigma(x)) \), \( A \models \forall x(\Sigma(x) \rightarrow \text{core}(x)) \) and, for all quasiequations \( \bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \rightarrow \varepsilon \approx \delta \in \text{Th}_{\text{eq}}(K) \) we have

\[
A \models \forall x_1, \ldots, \forall x_n \left( \bigwedge_{i \leq n} \text{core}(x_i) \land \bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \rightarrow \varepsilon \approx \delta \right).
\]

Since the above sets of formulas form a Horn class, it follows immediately by [5, 2.23] that \( C_{\Sigma}(K) \) is closed under \( \mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_U \).

We conclude this section with the following characterisation of the family of standard companion of a strictly algebraizable weak logic \( \models \). This generalises the previous results from [1, 3] to our abstract setting.

**Notation 4.13.** Let \( \tau : \text{Fm} \rightarrow \text{Eq} \) and let \( Q \) be a quasivariety, then we write \( \text{Log}_\tau(Q) \) for the set of all pairs \( (\Gamma, \phi) \) such that \( \tau(\Gamma) \models Q \tau(\phi) \).

**Theorem 4.14.** Let \( \models \) be a weak logic algebraized by \( (Q, \Sigma, \tau, \Delta) \), then \( \text{St}(\models) \) with the subset ordering \( \subseteq \) forms a bounded lattice with maximal element \( \text{Schm}(\models) \) and minimal element \( \text{Log}_{\tau}(C_{\Sigma}(Q)) \).

**Proof.** Let \( \vdash_0, \vdash_1 \in \text{St}(\models) \), then \( \vdash_0 \cap \vdash_1 \) is a standard logic and it is straightforward to verify that \( \vdash_0 \cap \vdash_1 \in \text{St}(\models) \). Similarly, if \( \vdash_0, \vdash_1 \in \text{St}(\models) \) and they are algebraized respectively by \( (K_0, \tau, \Delta) \) and \( (K_1, \tau, \Delta) \), then \( K_0 \cap K_1 \) is a quasivariety, and the logic \( \vdash_2 \) defined by letting

\[
\Gamma \vdash_2 \phi \iff \tau(\Gamma) \models_{K_0 \cap K_1} \tau(\phi)
\]

is the supremum of \( \vdash_0 \) and \( \vdash_1 \) in \( \text{St}(\models) \). This shows that \( (\text{St}(\models), \subseteq) \) is a lattice.

We show that \( \text{Schm}(\models) \) is maximal in \( \text{St}(\models) \). Recall that, by Corollary 4.6, we have that \( \text{Schm}(\models) = \text{Log}_{\tau}(Q) \). Now, if \( \vdash \in \text{St}(\models) \) then \( \Gamma \vdash \phi \) entails \( \Gamma \cup \Lambda \vdash \phi \), and
thus we have $\Gamma \vDash \phi$. This shows that $\vdash \subseteq \Vdash_s$. Since $\vdash$ and $\text{Schm}(\Vdash)$ are both closed under uniform substitution, it then follows that $\vdash \subseteq \text{Schm}(\Vdash)$.

We show that $\text{Log}_\tau(\Sigma(Q))$ is minimal in $\text{St}(\Vdash)$. If $\vdash \in \text{St}(\Vdash)$, then by Lemma 4.10 $\vdash$ is algebraized by $(K, \tau, \Delta)$, for some standard companion $K$ of $Q$. By Definition 4.11, it follows that $K \subseteq \Sigma(Q)$ and so $\text{Log}_\tau(\Sigma(Q)) \subseteq \text{Log}_\tau(K) = \vdash$.

Remark 4.15. We notice that, if $\Vdash$ is fixed-point algebraized by $(Q, \Sigma, \tau, \Delta)$ as in 3.22, then $\Theta \models \alpha \approx \beta$ if and only if $\sigma_\delta(\Theta) \models \sigma_\delta(\alpha) \approx \sigma_\delta(\beta)$, where $\sigma_\delta$ is the substitution obtained by letting $\sigma_\delta(x) = \delta(x)$ for all $x \in \text{Var}$. Then, since $\Vdash_s$ is the logic $\text{Log}_\tau(Q)$, we obtain that $\Gamma \vdash \phi$ if and only if $\sigma_\delta(\Gamma) \Vdash_s \sigma_\delta(\phi)$. This correspondence showcases a special case where the equations identifying the core of the expanded algebra induce a translation between a logic and its schematic fragment. We refer the reader to [1] for an in-depth study of such translations and to [1, 23] for the relation between such translations and corresponding adjunctions between quasivarieties of algebras.

4.3. Example of Bridge Theorem: Deduction Theorem

As stressed by Font in [16, p. 160], “Bridge theorems and transfer theorems are the ultimate justification of abstract algebraic logic”. We show that this motivation remains applicable in our modified setting of weak logics. The key observation is that in virtue of Theorem 4.5 we do not have to come up with novel algebraic descriptions of logical properties, but we can rely on the established characterisation of such properties for standard logics.

To showcase a concrete example, we look at one bridge theorem for strictly algebraizable weak logics, i.e., we consider the case of strictly algebraizable logics with a deduction-detachment theorem. We start by recalling the definition of this property and formulas we can rely on the established characterisation of such properties for standard logics.

**Definition 4.16.** A weak logic $\vdash$ has the deduction-detachment theorem (DDT) if there exists a finite set of formulas $I(x, y)$ such that for every set of formulas $\Gamma$ we have

$$\Gamma \cup \{\phi\} \vdash \psi \iff \Gamma \vdash I(\phi, \psi).$$

**Definition 4.17.** A quasivariety $K$ has equationally definable principal relative congruences (EDPRC) if there is a finite set of equations $\Theta(x, y, z, v)$ such that, for every $A \in K$ and $a, b, c, d \in A$:

$$(a, b) \in C_{K}(c, d) \iff A \models \Theta(c, d, a, b),$$

where $C_{K}(c, d)$ is the $K$-congruence of $A$ induced by the equation $c \approx d$.

The following result, due to Blok and Pigozzi, is one classical bridge theorem from abstract algebraic logic. We refer the reader to [16, 3.85] for a proof of this result.

**Theorem 4.18** (Blok-Pigozzi). Let $\vdash$ be a standard logic with equivalent algebraic semantics $Q$, then $\vdash$ has a deduction-detachment theorem if and only if $Q$ has EDPRC.

In the case of a strictly algebraizable weak logic $\Vdash$, local representability allows us to transfer the property of having the DDT from $\Vdash$ to $\Vdash_s$. More precisely, we can prove the following proposition.
Proposition 4.19. Let $\vdash$ be a strictly algebraizable weak logic, then $\vdash$ has DDT if and only if $\models_s$ has DDT.

Proof. We first prove the right-to-left direction. Suppose $\models_s$ has DDT, and notice that since $\vdash$ is strictly algebraizable then it is finitely represented by a set of formulas $\Lambda$ (by 4.5). We have the following equivalences:

\[
\Gamma \cup \{\phi\} \vdash_s \psi \iff \Gamma \cup \{\phi\} \cup \text{At}[\Lambda] \models_s \psi \quad \text{(by representability)}
\]
\[
\iff \Gamma \cup \text{At}[\Lambda] \models_s I(\phi, \psi) \quad \text{(by assumption)}
\]
\[
\iff \Gamma \vdash I(\phi, \psi) \quad \text{(by representability)}
\]

which establish that $\vdash$ has DDT.

Consider now the left-to-right direction. We assume that $\vdash$ has DDT and that this is witnessed by the finite set of formulas $I(x, y)$, then:

\[
\Gamma \cup \{\phi\} \models_s \psi \iff \forall \sigma \in \text{Subst}(\mathcal{L}), \sigma(\Gamma \cup \{\phi\}) \vdash \sigma(\psi) \quad \text{(by Definition 4.1)}
\]
\[
\iff \forall \sigma \in \text{Subst}(\mathcal{L}), \sigma(\Gamma) \vdash I(\sigma(\phi), \sigma(\psi)) \quad \text{(by assumption)}
\]
\[
\iff \Gamma \models_s I(\phi, \psi) \quad \text{(by Definition 4.1)}
\]

which proves our claim.

Corollary 4.20. Let $\vdash$ be a strictly algebraizable weak logic with equivalent algebraic semantics $Q$, then $\vdash$ has a DDT if and only if $Q$ has EDPRC.

Proof. Immediate from Proposition 4.19 and Theorem 4.18.

Remark 4.21. The previous results showcase one example where one can obtain results about strictly algebraizable weak logics by investigating their schematic fragments. In particular, we did not need to come up with a novel algebraic characterisation for the presence of a deduction-detachment theorem in a strictly algebraizable weak logic $\vdash$, and we could simply rely on Theorem 4.5 and the classical bridge theorem by Blok and Pigozzi. This shows that strict algebraizability provides an especially strong and well-behaved notion in the setting of weak logics. Furthermore, we point out that we deem the underlying reason for this phenomena to ultimately be the presence of a strong version of the Isomorphism Theorem in the case of strict algebraizability (in contrast to loose algebraizability). We shall investigate this in Section 5.3.

5. The Isomorphism Theorem

Blok and Pigozzi’s Isomorphism Theorem [16, §3.5] is possibly the most important single result on algebraizability, as it relates the algebraizability of a logic to the existence of an isomorphism between the lattice of deductive filters of the logic and the lattice of congruences of the corresponding class of algebras. We consider in this section analogues of this result for the setting of algebraizable weak logics. Firstly, in Section 5.1 we recall the isomorphism theorem for standard algebraizable logic. In Section 5.2 we provide a partial analogue of this result for loosely algebraizable weak logics, and then in Section 5.3 we prove a stronger result for the case of strictly algebraizable weak logics.
5.1. Standard Algebraizable Logics

We start by providing some preliminary definitions needed to state Blok and Pigozzi’s result. We first recall the notion of deductive filter and fix some notation for algebraic congruences.

Definition 5.1. For any algebra $A$ and standard logic $\vdash$, we say that $F \subseteq A$ is a *deductive filter* of $\vdash$ on $A$ if:
\[
\Gamma \vdash \phi \implies \forall h \in \text{Hom}(\text{Fm}, A), h[\Gamma] \subseteq F \text{ entails } h(\phi) \in F;
\]
and we let $\text{Fi}_{\vdash}(A)$ be the set of all deductive filters of $\vdash$ on $A$.

Remark 5.2. We notice that $\text{Fi}_{\vdash}(A)$ together with the subset ordering forms a lattice. Moreover, if $F$ is a deductive filter and $\sigma$ an endomorphism of $A$, then $\sigma^{-1}(F)$ is also a deductive filter. We denote by $\text{Fi}^+_\vdash(A)$ the lattice expansion ($\text{Fi}_{\vdash}, \subseteq, \{\sigma^{-1} : \sigma \in \text{End}(\mathcal{L})\}$). We refer the reader to [16, §2.3] for proofs of these facts.

Notation 5.3. Given any algebra $A$ we let $\text{Con}(A)$ be the set of all congruences $\theta$ over $A$, and $\text{Con}_Q(A)$ be the set of all $Q$-congruences of $A$, i.e., those congruences $\theta$ over $A$ such that $A/\theta \in Q$.

Remark 5.4. Similarly to the case of deductive filters, it is possible to verify that $\text{Con}_Q(A)$ forms a lattice under the subset ordering and that it is closed under inverse endomorphisms of $A$. We then write $\text{Con}^+_Q(A)$ for this lattice expansion, i.e., $\text{Con}^+_Q(A) = (\text{Con}_Q(A), \subseteq, \{\sigma^{-1} : \sigma \in \text{End}(A)\})$.

Finally, we introduce syntactical and semantical theories as follows.

Definition 5.5. If $\vdash$ is a standard logic, then we denote by $\text{Th}(\vdash)$ the set of all (syntactic) theories over $\vdash$, i.e., all sets $\Gamma \subseteq \text{Fm}$ such that $\Gamma \vdash \phi$ entails $\phi \in \Gamma$. If $Q$ is a quasivariety, then $\text{Th}(\models_Q)$ denotes the set of (semantical) theories over $Q$, i.e., the sets of equations $\Theta \subseteq \text{Eq}$ such that $\Theta \models_Q \alpha \approx \beta$ entails $\alpha \approx \beta \in \Theta$.

Remark 5.6. We notice that $\text{Th}(\vdash)$ forms a lattice under the subset relation, and that it is additionally closed under inverse substitutions. We refer by $\text{Th}^+(\vdash)$ to this lattice expansion, namely $\text{Th}^+(\vdash) = (\text{Th}(\vdash), \subseteq, \{\sigma^{-1} : \sigma \in \text{Subst}(\mathcal{L})\})$. Similarly, also $\text{Th}^+(\models_Q)$ forms a lattice under the subset relation and is also closed under inverse substitutions. We let $\text{Th}^+(\models_Q) = (\text{Th}(\models_Q), \subseteq, \{\sigma^{-1} : \sigma \in \text{Subst}(\mathcal{L})\})$.

These properties follows from the easily verifiable fact that the syntactic theories over $\vdash$ are exactly the deductive filter of $\vdash$ over $\text{Fm}$, while the semantic theories over $\models_Q$ are the $Q$-congruences of $\text{Fm}$.

Blok and Pigozzi’s isomorphism theorem for standard logics provides a criterion to determine if a logic is algebraized by a quasivariety based on their associated lattices of filters and congruences. For a proof of the following theorem we refer the reader to [16, §3.5].

Theorem 5.7 (Isomorphism Theorem). Let $\vdash$ be a standard logic and $Q$ a quasivariety, then the following are equivalent:
1. $\vdash$ is algebraizable with equivalent algebraic semantics $Q$;
2. $\text{Fi}^+_\vdash(A) \cong \text{Con}^+_Q(A)$, for any algebra $A$;
3. $\text{Th}^+(\vdash) \cong \text{Th}^+(\models_Q)$. 

Remark 5.8. We clarify what are the underlying witnesses in the previous theorem. On the one hand, let $\vdash$ be an algebraizable logic with equivalent algebraic semantics $(Q, \tau, \Delta)$. Then the associated isomorphism $F^+_\vdash(A) \cong \text{Con}_Q(A)$ is given by the following map from filters to congruences:

$$\theta(\_): F^+_\vdash(A) \rightarrow \text{Con}_Q(A)$$

$$G \mapsto \theta_G := \{(a, b) \in A^2 : \Delta^A(a, b) \subseteq G\}$$

and the following map from congruences to filters

$$F(\_): \text{Con}_Q(A) \rightarrow F^+_\vdash(A)$$

$$\eta \mapsto F_\eta := \{a \in A : \tau^A(a) \subseteq \eta\}.$$ 

which can then be shown to be inverse of each other. On the other hand, suppose $\Omega : \text{Th}^+(\vdash) \cong \text{Th}^+(|=Q)$ is an isomorphism. Then the two transformers $\tau$ and $\Delta$ are defined as follows:

$$\tau(x) = \sigma_x(\Omega(\text{Cn}_\vdash(x)))$$

$$\Delta(x, y) = \sigma_{x, y}(\Omega^{-1}(\text{Cn}_Q(x \approx y))).$$

where $\text{Cn}_\vdash$ and $\text{Cn}_Q$ denote respectively the closure consequence operators on the logic $\vdash$ and the quasivariety $Q$. $\sigma_x$ is the substitution sending every variable to $x$, and $\sigma_{x, y}$ is the substitution sending every variable but $y$ to $x$.

5.2. Loosely Algebraizable Weak Logics

We prove in this section a (partial) version of the isomorphism theorem for loosely algebraizable weak logics. To this end, we start by introducing a version of deductive filters and congruences relative to core semantics.

Definition 5.9. For any expanded algebra $A$ and weak logic $\vdash$, we say that $F \subseteq A$ is a core filter of $\vdash$ over $A$ if:

$$\Gamma \vdash \phi \implies \forall h \in \text{Hom}^c(Fm, A), h[\Gamma] \subseteq F \text{ entails } h(\phi) \subseteq F;$$

and we denote the set of core-filter of $A$ with respect to $\vdash$ by $F^+_\vdash(A)$.

Remark 5.10. Notice that, if $A$ is an expanded algebra and $\theta$ a congruence of the algebraic reduct of $A$, then the structure $A/\theta$ is simply the quotient of the algebraic reduct of $A$ by $\theta$ with core$(A/\theta) = \text{core}(A)/\theta$. This corresponds to viewing $A/\theta$ as a strong homomorphic image of $A$ (cf. Definition 1.8 and Remark 1.9).

Definition 5.11. Given an expanded algebra $A$ and a quasivariety $Q$ of expanded algebras, a congruence $\theta \in \text{Con}(A)$ is said to be a core $Q$-congruence if $A/\theta \in Q$.

We write $\text{Con}_Q(A)$ for the set of all core $Q$-congruences over $A$.

Lemma 5.12. Let $A$ be a core-generated expanded algebra, then:

1. if $G \subseteq A$ is a core filter of $\vdash$, then it is a deductive filter of its schematic fragment $\vdash_s$;
2. if $\theta$ is core $Q$-congruence of $A$, then it is also a $Q|\mathcal{L}$-congruence.

Proof. Clause (2) follows immediately from the definition of core $Q$-congruence. We thus consider clause (1). Let $G \subseteq A$ be a core filter of $\vdash$, and let $\Gamma \vdash_s \phi, h[\Gamma] \subseteq G$ for some $h \in \text{Hom}(Fm, A)$. Now, since $A$ is core-generated, by Lemma 2.20 we can find a substitution $\sigma$ and a core assignment $g$, such that $g(\sigma(x)) = h(x)$ for all $x \in \text{Var}$. In particular, it follows that $g[\sigma[\Gamma]] \subseteq G$. Now, since $\Gamma \vdash_s \phi$, it follows by
uniform substitution that $\sigma[\Gamma] \models_{\mathcal{L}} \sigma(\varphi)$ and therefore $\sigma[\Gamma] \models \sigma(\varphi)$. Now, since $g$ is a core assignment and $G$ a core morphism of $A$, then the following hold:

1. if $F$ is a core filter, then $\sigma^{-1}(F)$ is also a core filter;
2. if $\theta \in \text{Con}_Q(A)$ then $\sigma^{-1}(\theta) \in \text{Con}_Q(A)$.

Therefore, $\text{Fi}_\mathcal{L}(A)$ and $\text{Con}_Q(A)$ are two lattices commuting with all inverse strong endomorphisms of $A$.

**Proof.** The fact that $\text{Fi}_\mathcal{L}(A)$ and $\text{Con}_Q(A)$ are lattices under the subset ordering is easily verified, whence we simply prove (1) and (2).

We prove (1). Suppose $\Gamma \models \varphi$ and let $h \in \text{Hom}_c(\mathcal{Fm}, A)$ be such that $h[\Gamma] \subseteq \sigma^{-1}(F)$. Then $\sigma(h[\Gamma]) \subseteq F$ and, since $\sigma \circ h \in \text{Hom}_c(\mathcal{Fm}, A)$, it follows that $\sigma(h[\varphi]) \subseteq F$ and thus $h(\varphi) \in \sigma^{-1}(F)$.

We prove (2). The fact that $\sigma^{-1}(\theta) \in \text{Con}(A)$ follows from Remark 5.4. We next show that $A/\sigma^{-1}(\theta) \in Q$. Consider a quasiequation $\bigwedge_{i \leq n} \epsilon_i \approx \delta_i \rightarrow \alpha \approx \beta \in \text{Th}_c^e(Q)$. Since $A/\theta \in Q$, we have that $A/\theta \models^c \bigwedge_{i \leq n} \epsilon_i \approx \delta_i \rightarrow \alpha \approx \beta$. Now, suppose that $\epsilon_i/\sigma^{-1}(\theta) = \delta_i/\sigma^{-1}(\theta)$ for all $i \leq n$, then it follows that $\sigma(\epsilon_i)/\theta = \sigma(\delta_i)/\theta$ for all $i \leq n$. Since $\sigma$ is a strong endomorphism of $A$ we obtain that $\sigma(\alpha)/\theta = \sigma(\beta)/\theta$. It then follows that $\alpha/\sigma^{-1}(\theta) = \beta/\sigma^{-1}(\theta)$. This shows that $A/\sigma^{-1}(\theta) \models^c \bigwedge_{i \leq n} \epsilon_i \approx \delta_i \rightarrow \alpha \approx \beta$ and thus, since $A$ is core generated, $A \in Q$. We conclude that $\sigma^{-1}(\theta) \in \text{Con}_Q(A)$.

As expanded algebras augment standard algebras by an additional core predicate, so we expand $\text{Fi}_\mathcal{L}(A)$ and $\text{Con}_Q(A)$ to capture core filters and congruences.

**Definition 5.14.** For any core-generated expanded algebra $A$, we write $\widehat{\text{Fi}}_{\mathcal{L}}(A)$ for the structure

$$(\text{Fi}_{\mathcal{L}}(A), \text{Fi}_\mathcal{L}(A), \subseteq, \{\sigma^{-1} : \sigma \in \text{End}(A)\})$$

and we write $\widehat{\text{Con}}_Q(A)$ for the structure

$$(\text{Con}_Q(A), \text{Con}_Q(A), \subseteq, \{\sigma^{-1} : \sigma \in \text{End}(A)\}).$$

**Remark 5.15.** Thus, the key intuition in the context of weak logic is to consider *together* both arbitrary filters and core filters, and arbitrary $Q$-congruences and core $Q$-congruences. This is possible in the setting of core-generated algebras because of Lemma 5.12, which makes sure that core filters are deductive filters, and core congruences are congruences. Then, by Lemma 5.13 we additionally have that strict endomorphisms also preserve core filters and core congruences.

**Theorem 5.16.** Let $\models$ be a weak logic and suppose it is loosely algebraized by $(Q, \tau, \Delta)$, then for every core-generated expanded algebra $A$ there is an isomorphism $\Omega : \widehat{\text{Fi}}_{\mathcal{L}}(A) \cong \widehat{\text{Con}}_Q(A)$.

**Proof.** Firstly, we define the following map from filters to congruences:

$$\theta(\cdot) : \text{Fi}_{\mathcal{L}}(A) \longrightarrow \text{Con}_Q(A)$$

where $G \mapsto \theta_G := \{(a, b) \in A^2 : \Delta^A(a, b) \subseteq G\}$
and the following map from congruences to filters

\[ F(\cdot) : \text{Con}_Q(A) \rightarrow \text{F}_\tau(A) \]

\[ \eta \mapsto F_\eta := \{ a \in A : \tau^A(a) \subseteq \eta \}. \]

Now, notice that since \( \models \) is loosely algebraized by \((Q, \tau, \Delta)\), then it follows from Lemma 4.4 that also \( \models_\tau \) is algebraized by \((Q, \tau, \Delta)\). Since the two maps above are exactly the same that occur in the proof of the Isomorphism theorem for standard logics (cf. Remark 5.8) it follows immediately that they describe an isomorphism \( \text{F}_\tau(A) \cong \text{Con}_Q(A) \). It thus suffices to show that \( \theta(\_\_) \) sends core filters to core congruences, and that \( F(\_\_) \) sends core congruences to core filters.

First, let \( G \) be a core filter over \( A \), we show that \( \theta_G \) is a core congruence of \( A \). Consider \( A/\theta_G \) and notice that this is well-defined because \( \theta_G \) is a congruence. Now, consider the expansion of \( A/\theta_G \) defined by letting \( \text{core}(A/\theta_G) = \text{core}(A)/\theta_G \), we claim that \( (A/\theta_G, \text{core}(A)/\theta_G) \in Q \). Let \( \bigcup_{i \leq n} \Delta(\epsilon_i) = \delta_i \models q_0 = q_\beta \) and suppose \( h \in \text{Hom}^+(\text{Fm}, A) \) is such that \( h(\epsilon_i) = h(\delta_i) \) for all \( i \leq n \). By algebraizability we have that \( \bigcup_{i \leq n} \Delta(\epsilon_i) = \delta_i \models \Delta(\alpha, \beta) \) and so \( h(\Delta(\alpha, \beta)) \subseteq G \). It follows that \( (\alpha, \beta) \in \theta_G \) and so \( A/\theta_G \models^c \alpha \approx \beta \). This shows that \( (A/\theta_G, \text{core}(A)/\theta_G) \models^c \bigcup_{i \leq n} \Delta(\epsilon_i) = \delta_i \rightarrow \alpha \approx \beta \). Since \( Q \) is a core-generated quasivariety of expanded algebras, it follows from Proposition 2.26 that \( (A/\theta_G, \text{core}(A)/\theta_G) \in Q \). We conclude that the map \( \theta(\_\_) \) sends core filters to core congruences.

Next, we show that for any core congruence \( \theta \) over \( A \) the filter \( F_\theta \) is a core filter. Suppose \( \Gamma \models^c \phi \) and \( h \in \text{Hom}^+(\text{Fm}, A) \) is such that \( h[\Gamma] \subseteq F_\theta \). Then, we have by definition that \( \tau(c) \subseteq \theta \) for every \( c \in h[\Gamma] \). By algebraizability notice that we have \( \tau[\Gamma] \models^c \tau(\phi) \), thus we obtain \( h(\tau(\phi)) = \tau(h(\phi)) \subseteq \theta \) and so \( h(\phi) \in F_\theta \). This shows that \( F_\theta \) is a core deductive filter and completes the proof.

Exactly as in the standard setting, it is possible to specialise the previous isomorphism to the specific case of filters and congruences of the (expanded) algebra of formulas \text{Fm}. Firstly, we define syntactic and semantic theories in the setting of weak logics.

**Definition 5.17.** If \( \models \) is a weak logic then we denote by \( \text{Th}(\models) \) the set of all (syntactic) theories over \( \models \), i.e., all sets \( \Gamma \subseteq \text{Fm} \) such that if \( \models \phi \) then \( \phi \in \Gamma \). If \( Q \) is a core-generated quasivariety of expanded algebras, then \( \text{Th}(\models_Q) \) denotes the set of (semantic) core theories over \( Q \), i.e., those sets of equations \( \Theta \subseteq \text{Eq} \) such that \( \Theta \models^c Q \alpha \approx \beta \) entails \( \alpha \approx \beta \in \Theta \).

**Remark 5.18.** It is straightforward to verify that, for \( \models \) a weak logic, the syntactic theories from 5.17 are exactly the core filters of \( \models \) over \( (\text{Fm}, \text{Var}) \). Similarly, if \( Q \) is a quasivariety of expanded algebras, then the semantic core theories over \( Q \) are exactly the core congruences of \( (\text{Fm}, \text{Var}) \). Since \( (\text{Fm}, \text{Var}) \) is a core-generated algebra, we obtain as in Definition 5.14 an expanded lattice \( \overline{\text{Th}}(\models) \) defined by

\[ (\text{Th}(\models_\tau), \text{Th}(\models), \subseteq, \{ \sigma^{-1} : \sigma \in \text{Subst}(\mathcal{L}) \}) \]

and an expanded lattice \( \overline{\text{Th}}(\models_Q) \) defined by

\[ (\text{Th}(\models_Q), \text{Th}(\models_Q), \subseteq, \{ \sigma^{-1} : \sigma \in \text{Subst}(\mathcal{L}) \}). \]
This clearly corresponds to the fact that theories over $\vdash$ are also theories in $\models_s$, and theories over $\models_{cQ}$ are also theories over $\models_Q$. Additionally, since the strong endomorphisms of $(\text{FM}, \text{Var})$ are exactly the atomic substitutions, it follows from Lemma 5.13 that inverse atomic substitutions preserve elements in $\text{Th}(\vdash)$ and $\text{Th}(\models_{cQ})$.

**Corollary 5.19.** Let $\vdash$ be a weak logic and suppose it is loosely algebraized by $(Q, \tau, \Delta)$, then there is an isomorphism $\Omega : \hat{\text{Th}}(\vdash) \cong \hat{\text{Th}}(\models_{cQ})$.

**Proof.** It follows immediately from Theorem 5.16 and Remark 5.18.

**Remark 5.20.** A key aspect of the Blok and Pigozzi’s Isomorphism Theorem (Theorem 5.7) is that one can recover the two transformers $\tau$ and $\Delta$ from the isomorphism between $\text{Fi}^+_{\vdash}(A)$ and $\text{Con}^+_{Q}(A)$, thus showing algebraizability. We indicate however that this same fact is not as clear in the setting of loose algebraizability. In fact, in the proof of the Isomorphism Theorem (cf. [16, 3.5]) the two transformers are defined as follows:

$$
\tau(x) = \sigma_x(\Omega(\text{Cn}_{\vdash}(x)))
$$

$$
\Delta(x, y) = \sigma_{x,y}(\Omega^{-1}(\text{Cn}_Q(x \approx y))); 
$$

where $\text{Cn}_{\vdash}$ and $\text{Cn}_Q$ denote respectively the closure consequence operators on the logic $\vdash$ and the quasivariety $Q$, while $\sigma_x$ is a substitution sending every variable to $x$, and $\sigma_{x,y}$ is one sending every variable but $y$ to $x$. If one simply mimics this proof in the setting of weak logics and lets, for example,

$$
\tau(x) = \sigma_x(\Omega(\text{Cn}_{\vdash}(x)))
$$

then crucially it could be that structurality fails. In fact, we could that $x \vdash \delta(x)$ but also $\phi \not\vdash \delta(\phi)$, witnessing the failure of uniform substitution. Let $\sigma$ be the substitution sending every variable to $\phi$, then we have

$$
\delta(\phi) \in \text{Cn}_{\vdash}(\sigma(\text{Cn}_{\vdash}(x))) \text{ and } \delta(\phi) \notin \text{Cn}_{\vdash}(\sigma(x)),
$$

and therefore $\Omega(\text{Cn}_{\vdash}(\sigma(\text{Cn}_{\vdash}(x)))) \neq \Omega(\text{Cn}_{\vdash}(\sigma(x)))$. Clearly, $\Omega(\text{Cn}_{\vdash}(\sigma(x))) = \Omega(\text{Cn}_{\vdash}(\phi))$, simply by definition of $\sigma$. Assume additionally that $\Omega$ commutes with closure operators and substitution, i.e., that

$$
\Omega(\text{Cn}_{\vdash}(\sigma(\Gamma))) = \text{Cn}_Q(\sigma(\Omega(\text{Cn}_{\vdash}(\Gamma))))
$$

for all sets of formulas $\Gamma$ and substitutions $\sigma$. Then we obtain that

$$
\Omega(\text{Cn}_{\vdash}(\sigma(\text{Cn}_{\vdash}(x)))) = \text{Cn}_Q(\sigma(\Omega(\text{Cn}_{\vdash}(\text{Cn}_{\vdash}(x)))))
$$

$$
= \text{Cn}_Q(\sigma(\Omega(\text{Cn}_{\vdash}(x)))) 
$$

$$
= \text{Cn}_Q(\sigma_{x}(\Omega(\text{Cn}_{\vdash}(x)))) 
$$

$$
= \text{Cn}_Q(\tau(\phi))
$$

and therefore $\text{Cn}_Q(\tau(\phi)) \neq \Omega(\text{Cn}_{\vdash}(\phi))$. This indicates that the usual proof of the isomorphism theorem does not work in the setting of loosely algebraizable weak logics. We leave it as a pointer for future works if there is a version of the isomorphism theorem in the setting of loosely algebraizable weak logics.
5.3. Strictly Algebraizable Weak Logics

In contrast to Remark 5.20, in the case of strictly algebraizable weak logics, we can use the fact that they are finitely representable to derive a full version of the isomorphism theorem. We first introduce some preliminary definitions.

Definition 5.21 (Closure Operators). Let $A$ be an expanded algebra, $\vdash$ a weak logic, and $Q$ a core-generated quasivariety of expanded algebras. We denote by $C_{n_{\Lambda}}$ the closure operator on $A$ defined by letting $C_{n_{\Lambda}}(X)$ be the smallest core filter of $A$ with respect to $\vdash$ that contains $X \subseteq A$. Similarly, we denote by $C_{n_{Q}}$ the closure operator on $A^2$ defined by letting $C_{n_{Q}}(R)$ be the smallest congruence in $\text{Con}_{Q}(A)$ containing $R \subseteq A^2$.

Remark 5.22. Recall that, in the expanded term algebra $(\text{Fm}, \text{Var})$, the core filters coincide exactly with the syntactical theories and the core congruences coincide with the semantical theories. In this case we recover the more intuitive definition of the closure operators $C_{n_{\Lambda}}$ and $C_{n_{Q}}$, namely, for $\Theta \subseteq \text{Eq}$ and $\Gamma \subseteq \text{Fm}$:

$$C_{n_{\Lambda}}(\Gamma) = \{ \phi \in \text{Fm} : \Gamma \vdash \phi \}$$

$$C_{n_{Q}}(\Theta) = \{ \alpha \approx \beta \in \text{Eq} : \Theta \models_{Q} \alpha \approx \beta \}.$$

In the context of strictly algebraizable weak logics, it is convenient to work with the following notions of $\Lambda$-filters and $\Sigma$-congruences.

Definition 5.23. For any algebra $A$ and standard logic $\vdash$, we say that $F \subseteq A$ is a $\Lambda$-filter over $A$ with respect to $\vdash$ if

$$\Gamma \cup \text{At}[\Lambda] \vdash \phi \implies \forall h \in \text{Hom}(\text{Fm}, A), h[\Gamma] \subseteq F \text{ entails } h(\phi) \in F.$$

We denote the set of $\Lambda$-filter of $A$ with respect to $\vdash$ by $\Lambda\text{Fi}_{\vdash}(A)$. We let $\Lambda\text{Th}(\vdash)$ be the set of all syntactical theories $\Gamma$ over $\vdash$ such that $\text{At}[\Lambda] \subseteq \Gamma$.

Definition 5.24. Given an expanded algebra $A$, a quasivariety of expanded algebras $Q$ and a finite set of equations $\Sigma$, a congruence $\theta \in \text{Con}_{Q\mid \Sigma}(A)$ is said to be a $\Sigma$-congruence if $\Sigma(a) \subseteq \theta$ for all $a \in \text{core}(A)$. We write $\Sigma\text{Con}_{Q\mid \Sigma}(A)$ for the set of all $Q$-congruences of $A$ which are also $\Sigma$-congruences. Similarly, we let $\Sigma\text{Th}(\vdash_Q)$ denote the the set of all semantical theories $\Theta$ such that $\text{At}[\Sigma] \subseteq \Theta$.

Remark 5.25. It is clear from the definition of $\Lambda$-filters and the monotonicity of $\vdash$ that the $\Lambda$-filters of $\vdash$ on $A$ are a special kind of deductive filters of $\vdash$. Similarly, $\Sigma$-congruences are clearly a special kind of congruences. One can also verify that $\Lambda$-filters and $\Sigma$-congruences form lattices under the subset relation but, crucially, they are not necessarily closed under arbitrary inverse endomorphism.

Lemma 5.26. Let $\vdash \vdash$ be a weak logic, then $\vdash$ is finitely represented by $\Lambda$ if and only if $\text{Fi}_{\vdash}(A) = \Lambda\text{Fi}_{\vdash}(A)$ for any core-generated expanded algebra $A$.

Proof. We first prove the left-to-right direction. Suppose $\vdash$ is finitely represented by $\Lambda$. Let $F \in \text{Fi}_{\vdash}(A)$, $\Gamma \cup \text{At}[\Lambda] \vdash_f \phi$ and $h \in \text{Hom}(\text{Fm}, A)$ be such that $h[\Gamma] \subseteq F$. By the finite representability of $\vdash$ we have that $\Gamma \vdash \phi$. Since $A$ is core-generated, by Lemma 2.20 there are a core-assignment $g \in \text{Hom}^c(\text{Fm}, A)$ and a substitution $\sigma \in \text{Subst}$ such that $h[\Gamma \cup \{ \phi \}] = g[\sigma(\Gamma \cup \{ \phi \})]$. Then it follows that $g(\sigma(\Gamma)) \subseteq F$ and so since $\Gamma \vdash \phi$ we obtain from the definition of core filters that $h(\phi) = g(\sigma(\phi)) \in F$. This shows that $\text{Fi}_{\vdash}(A) \subseteq \Lambda\text{Fi}_{\vdash}(A)$.
Conversely, let $F \in \Delta F_{\mathsf{imp}}(A)$. Suppose $\Gamma \vdash \phi$ and let $h \in \text{Hom}^{c}(\mathsf{Fm}, A)$ be such that $h[\Gamma] \subseteq F$. By finite representability we obtain that $\Gamma \cup \text{At}[A] \vdash_{s} \phi$, thus it follows from $F \in \Delta F_{\mathsf{imp}}(A)$ that $h(\phi) \in F$. This shows that $\Delta F_{\mathsf{imp}}(A) \subseteq F_{\mathsf{imp}}^{c}(A)$ and thus proves that $F_{\mathsf{imp}}^{c}(A) = \Delta F_{\mathsf{imp}}(A)$.

We now prove the right to left direction. Since the claim holds for all core-generated expanded algebras, in particular it holds for the expanded term algebra $(\text{Th}, \text{Fi}, \Lambda)$, thus proves that $\text{Th}(\vdash) = \Lambda \text{Th}(\vdash_{s})$. We thus derive the following equivalences:

$$
\Gamma \vdash \phi \iff C_{\mathsf{imp}}^{c}(\phi) \subseteq C_{\mathsf{imp}}(\Gamma)
$$

$$
\iff C_{\mathsf{imp}}(\phi \cup \text{At}[A]) \subseteq C_{\mathsf{imp}}(\Gamma \cup \text{At}[A])
$$

$$
\iff \Gamma \cup \text{At}[A] \vdash_{s} \phi,
$$

which give us finite representability via $\Lambda$.

\begin{lemma}
Let $Q$ be a core-generated quasivariety, then $\text{core}(A) = \Sigma(A)$ entails $\text{Con}_{Q}(A) = \Sigma \text{Con}_{Q|L}(A)$ for any core-generated expanded algebra $A$.
\end{lemma}

\begin{proof}
Suppose that $\text{core}(A) = \Sigma(A)$ for all $A \in Q$. First, let $\theta \in \text{Con}_{Q}(A)$, then by assumption we have that $\text{core}(A/\theta) = \Sigma[A/\theta]$ and so $\Sigma(a) \subseteq \theta$ for all $a \in \text{core}(A)$, showing $\theta \in \Sigma \text{Con}_{Q|L}(A)$. Conversely, suppose $\theta \in \Sigma \text{Con}_{Q|L}(A)$, then $\text{core}(A/\theta) = \Sigma[A/\theta]$ and $A/\theta \in Q|L$. Since $\text{core}(A) = \Sigma(A)$ we have that $(A/\theta, \Sigma(A/\theta)) \in Q$, and so it follows that $\theta \in \text{Con}_{Q}(A)$.
\end{proof}

We obtain the following Isomorphism Theorem for strictly algebraizable weak logics. This further motivates the centrality of strictly algebraizable weak logics.

\begin{theorem}
Let $\vdash$ be a weak logic and $Q$ a core-generated quasivariety of expanded $L$-algebras with core defined by $\Sigma$. The following are equivalent:

1. $\vdash$ is strictly algebraized by $(Q, \Sigma, \tau, \Delta)$;
2. for every core-generated expanded algebra $A$ there is an isomorphism $\Omega : \hat{\text{Fm}}(A) \cong \hat{\text{Con}}_{Q}(A)$, where additionally $\hat{\text{Fm}}(A) = \Delta \text{Fi}_{\text{imp}}(A)$ and $\text{Con}_{Q}(A) = \Sigma \text{Con}_{Q|L}(A)$;
3. there is an isomorphism $\Omega : \text{Th}(\vdash) \cong \text{Th}(\vdash_{s})$, where additionally $\text{Th}(\vdash_{s}) = \Lambda \text{Th}(\vdash)$ and $\text{Th}(\vdash_{s}) = \Sigma \text{Th}(\vdash_{s})$.
\end{theorem}

\begin{proof}
Direction from (1) to (2) follows immediately from Theorem 5.16 and Lemmas 5.26, 5.27. Direction from (2) to (3) follows from Remark 5.18. It remains to show that (3) entails (1).

Firstly, notice that the isomorphism $\Omega : \text{Th}(\vdash) \cong \text{Th}(\vdash_{s})$ induces an isomorphism $\Omega : \text{Th}^{+}(\vdash_{s}) \cong \text{Th}(\vdash_{s})$, thus by the standard isomorphism theorem it immediately follows that $\vdash_{s}$ is algebraized by $(Q, \tau, \Delta)$, where

$$
\tau(x) = \sigma_{x}(\Omega(C_{\mathsf{imp}}^{c}(x)))
$$

$$
\Delta(x, y) = \sigma_{x,y}(\Omega^{-1}(C_{\mathsf{imp}}^{c}(x \approx y))),
$$

as in Remark 5.8. Now, since $\Omega$ is an isomorphism, the empty syntactic theory in $\vdash$ must be mapped to the empty theory in core semantics, i.e., $\Omega(C_{\mathsf{imp}}(\emptyset)) = C_{\mathsf{imp}}(\emptyset)$. By assumption, we have that $C_{\mathsf{imp}}(\emptyset) = C_{\mathsf{imp}}(A)$ and $C_{\mathsf{imp}}(\emptyset) = C_{\mathsf{imp}}(\emptyset)$. Thus, we obtain that $\Omega(C_{\mathsf{imp}}(\Lambda)) = C_{\mathsf{imp}}(\emptyset)$. Let $\sigma_{x}$ be the substitution sending every...
variable to \( x \) and \( \sigma_\delta \) the substitution sending \( x \) to \( \delta \) and \( \sigma_\alpha,\beta \) the substitution sending \( x \) to \( \alpha \) and \( y \) to \( \beta \). Then we obtain that:

\[
\sigma_x(\Omega(\text{Cn}_{\text{w}}(\bigcup_{\delta \in \Lambda} \sigma_\delta(x)))) = \sigma_x(\text{Cn}_Q(\bigcup_{\alpha \approx \beta \in \Sigma(x)} \sigma_\alpha,\beta(x \approx y))) \\
\subseteq \text{Cn}_Q(\bigcup_{\alpha \approx \beta \in \Sigma(x)} \sigma_\alpha,\beta(x \approx y))
\]

as \( \sigma_x \) is an atomic substitution. In turn, this shows that \( \tau(\Lambda) \equiv Q \Sigma \) and therefore also \( \Delta(\Sigma) \vdash \Lambda \). Also, we obtain by the assumptions and Lemma 5.26 that \( \Lambda \) witnesses the finite representability of \( \vdash \), thus it follows from Corollary 4.6 that \( \vdash \) is strictly algebraized by \( (Q, \Sigma, \tau, \Delta) \).

6. Aside: Matrix Semantics

In the previous sections we have focused on loosely and strictly algebraizable weak logics, thus extending the notion of algebraizability to the setting of logics without uniform substitution. However, in the setting of abstract algebraic logic, algebraizability makes only for one of several properties, and it could be seen as the concept carving out the most well-behaved family of logical systems. On the converse direction, in this section we work towards an increased level of generality and we study the matrix semantics of arbitrary weak logics. In fact, while not every logic is algebraizable, it can be shown that every logic admits a matrix semantics (see e.g., [16, Th. 4.16]). Furthermore, Dellunde and Jansana [14] provided a characterisation of the class of matrices of a (possibly infinitary) logic in terms of some model-theoretic results for first-order logic without equality. We show in this section that similar results can be proved in the context of weak logics. We prove in Section 6.1 that every weak logic is complete with respect to a suitable class of so-called bimatrices, and we show in Section 6.2 that Dellunde and Jansana’s results are still applicable in our setting.

6.1. Completeness of Matrix Semantics

We briefly recall the matrix semantics for standard logics. Intuitively, the idea is to work with first-order structures with a predicate \( \text{truth}(A) \) which encodes the “truth set” of the algebra \( A\).

**Definition 6.1.** A (logical) matrix of type \( \mathcal{L} \) is a pair \( (A, \text{truth}(A)) \) where \( A \) is an \( \mathcal{L} \)-algebra and \( \text{truth}(A) \subseteq \text{dom}(A) \).

Matrices induce a consequence relation over propositional formulas analogously as classes of algebras do. However, notice that here we work directly with propositional formulas (i.e., terms in the language \( \mathcal{L} \)) and not with equations. This corresponds to the fact that these logics do not necessarily correspond to quasiequational theories over classes of algebras.

**Notation 6.2.** For notational convenience, in this section we denote logics and weak logics by \( L, L_0, L_1, \ldots \). We then write \( \Gamma \vdash_L \phi \) if \( (\Gamma, \phi) \in L \). If \( L \) is a weak logic, then we write \( \Gamma \vdash_L \phi \) if \( (\Gamma, \phi) \in L \).
Definition 6.3. Let $K$ be a class of $\mathcal{L}$-matrices and let $\Gamma \cup \{\phi\}$ be a set of propositional formulas, then we let
\[
\Gamma \models_{K} \phi \iff \text{for all } A \in K, \ h \in \text{Hom}(Fm, A), \ 
\text{if } h[\Gamma] \subseteq \text{truth}(A), \text{ then } h(\phi) \in \text{truth}(A).
\]

Given a logic $L$, we say that $(A, \text{truth}(A))$ is a model of $L$ and write $(A, \text{truth}(A)) \models L$ if, for every $\Gamma \cup \{\phi\} \subseteq \mathcal{L}$, $\Gamma \vdash_{L} \phi$ entails $\Gamma \models_{K} \phi$. For set of formulas $\Gamma$ and a matrix $A$, we write $A \models \Gamma$ if $\models_{(A)} \Gamma$.

We refer the reader to [16, §4] for a detailed study of matrix semantics in the context of standard propositional logics. In particular, [16, Th. 4.16]) states that every logic is complete with respect to a class of matrices. We show that one can obtain the same result in the setting of weak logics. First, we extend the matrix semantics to the setting of weak logics by introducing a further predicate, i.e., by viewing them as structures in an algebraic language $\mathcal{L}$ augmented by two unary predicates, one for the core set of $A$ and one for the truth set of $A$.

Definition 6.4. The tuple $(A, \text{truth}(A), \text{core}(A))$ is a (logical) bimatrix of type $\mathcal{L}$ if $A$ is a $\mathcal{L}$-algebra, $\text{truth}(A) \subseteq \text{dom}(A)$ and $\text{core}(A) \subseteq \text{dom}(A)$.

Notation 6.5. As in the case of expanded algebras, we write $\text{Hom}^{e}(Fm, A)$ for the set of all assignments $h : Fm \rightarrow A$ such that $h[\text{Var}] \subseteq \text{core}(A)$.

Bimatrices induce a consequence relation analogous to that of expanded algebras by restricting attention to assignments over core elements.

Definition 6.6. Let $K$ be a class of $\mathcal{L}$-bimatrices and let $\Gamma \cup \{\phi\}$ be a set of propositional formulas, then we let
\[
\Gamma \models^{e}_{K} \phi \iff \text{for all } A \in K, \ h \in \text{Hom}^{e}(Fm, A), \ 
\text{if } h[\Gamma] \subseteq \text{truth}(A), \text{ then } h(\phi) \in \text{truth}(A).
\]

Given a weak logic $L$, we say that $(A, \text{truth}(A), \text{core}(A))$ is a model of $L$, and write $(A, \text{truth}(A), \text{core}(A)) \models L$ if, for every $\Gamma \cup \{\phi\} \subseteq \mathcal{L}$, $\Gamma \vdash_{L} \phi$ entails $\Gamma \models^{e}_{K} \phi$. For set of formulas $\Gamma$ and a bimatrix $A$, we write $A \models^{e} \Gamma$ if $\models^{e}_{(A)} \Gamma$.

Thus, the main intuition behind bimatrices is the same of expanded algebras: we add a new predicate specifying the core of the matrix in order to consider only the assignments sending atomic formulas to elements of the core. As shown by the following proposition, bimatrices give rise to several weak logics. As we stressed already before (cf. Remark 1.3), the finitary requirement in the following definition is not necessary per se, but we need it as we are focusing on finitary logical systems.

Definition 6.7. Let $K$ be a class of bimatrices in language $\mathcal{L}$, then $\text{Log}(K)$ is the set of all pairs $(\Gamma, \phi)$ with $\Gamma \cup \{\phi\} \subseteq Fm$ such that, for some finite $\Gamma_{0} \subseteq \Gamma$, we have that $\Gamma_{0} \models^{e}_{K} \phi$. We say that a weak logic $L$ is complete with respect to $K$ if $L = \text{Log}(K)$.

Proposition 6.8. Let $K$ be a class of bimatrices, then $\text{Log}(K)$ is a weak logic.

Proof. Let $L = \text{Log}(K)$, then the Conditions (1), (2) and (3) from the definition of consequence relation 1.2 are immediately valid by Definition 6.6. Additionally, if $\Gamma \vdash_{L} \phi$, then by definition there is some finite $\Gamma_{0} \subseteq \Gamma$ such that $\Gamma_{0} \models^{e}_{K} \phi$, and therefore $\Gamma_{0} \vdash_{L} \phi$, which shows that $L$ is also finitary.
Finally, we show that $L$ is closed under atomic substitutions. Suppose $\Gamma \models \phi$ but $\sigma[\Gamma] \not\models_{K} \phi$ for some atomic substitution $\sigma$. Then there are a bimatrix $M \in K$ and a core assignment $h : \text{Fm} \to M$ such that $h[\sigma[\Gamma]] \subseteq \text{truth}(M)$ but $h(\sigma(\phi)) \not\in \text{truth}(M)$. Now, since $h$ is a core assignment and $\sigma$ an atomic substitution, it follows that $h \circ \sigma \in \text{Hom}(\text{Fm}, A)$, contradicting $\Gamma \models_{K} \phi$. 

By the previous proposition we have that every class of bimatrices determines a weak logic. Additionally, we can also show that every weak logic is complete with respect to a class of bimatrices.

**Definition 6.9.** For every weak logic $\models$ and set of propositional formulas $\Gamma \subseteq \text{Fm}$, we let $M_{\models}^L$ be the bimatrix with domain $\text{dom}(M_{\models}^L) = \text{Fm}$ and predicates $\text{truth}(M_{\models}^L) = \text{Cn}_{\models}(\Gamma)$ and $\text{core}(M_{\models}^L) = \text{Var}$. We let $\text{M}_{\models}$ be the class of all bimatrices $M_{\models}^L$ for $\Gamma \subseteq \text{Fm}$.

**Theorem 6.10.** Every weak logic $\models$ is complete with respect to the class $\text{M}_{\models}$.

**Proof.** We need to show that $\models = \text{Log}(K)$. Firstly, suppose towards contradiction that $\Gamma \models \phi$ but $\Gamma \not\models_{\text{M}_{\models}} \phi$. Then there is a bimatrix $M_{\models}^L$ and a core assignment $h : \text{Fm} \to M_{\models}^L$ such that $h[\Gamma] \subseteq \text{truth}(M_{\models}^L)$ and $h(\phi) \not\in \text{truth}(M_{\models}^L)$. Consider now the substitution $\sigma$ defined letting $\sigma(x) = h(x)$ for all $x \in \text{Var}$. Notice in particular that this is well-defined because the domain of $M_{\models}^L$ is $\text{Fm}$. Since $h$ is a core assignment and $\text{core}(M_{\models}^L) = \text{Var}$, it follows that $\sigma$ is an atomic substitution, thus we obtain that $\sigma[\Gamma] \models \sigma(\phi)$. Now, since $h[\Gamma] \subseteq \text{truth}(M_{\models}^L)$ it follows in particular that $\Delta \models \sigma[\Gamma]$ and so by transitivity $\Delta \models \sigma(\phi)$. Since $\sigma(\phi) = h(\phi) \not\in \text{truth}(M_{\models}^L)$, this contradicts the definition of $\text{truth}(M_{\models}^L)$. It follows that $\Gamma \models_{\text{M}_{\models}} \phi$.

Conversely, suppose $\Gamma \models_{\text{M}_{\models}} \phi$ and let $\text{id}_{\text{Fm}} : \text{Fm} \to M_{\models}^L$ be the identity map. Then clearly $\text{id}_{\text{Fm}}[\text{Var}] \subseteq \text{Var}$ and by definition $\text{id}_{\text{Fm}}[\Gamma] \subseteq \text{truth}(M_{\models}^L)$. Since $\Gamma \models_{\text{M}_{\models}} \phi$ we then obtain $\phi = \text{id}_{\text{Fm}}(\phi) \in \text{truth}(M_{\models}^L)$, showing $\phi \in \text{Cn}_{\models}(\Gamma)$ and thus $\Gamma \models \phi$. 

6.2. Connections to Model Theory without Equality

In the previous section we have established that every class of bimatrices defines a weak logic and, conversely, that every weak logic is complete with respect to a class of bimatrices. Here we next consider what is exactly the class of all bimatrices defined by a weak logic $L$, i.e., the class of all bimatrices $M$ such that $M \models_{\models} \Gamma$ entails $M \models \phi$ whenever $\Gamma \models \phi$. In the standard context this issue was first considered by Czelakowski [13], who characterized the class of matrices complete with respect to a logic. Here we follow however the later work of Dellunde and Jansana in [14], which provided a novel proof of Czelakowski’s result by employing the fact that (finitary) propositional logics can be translated into Horn theories without equality. We start by reviewing this translation, which is essentially a generalisation of what we already considered in Remark 2.18.

**Proposition 6.11.** Let $\mathcal{L}$ be an algebraic language, $\Gamma \cup \phi \subseteq \text{Fm}$ and $|\Gamma| < \aleph_0$, then we can translate the consequence relations from Definition 6.3 and Definition 6.6 as follows:

(a) let $K$ be a class of matrices then:

$$\Gamma \models_{K} \phi \iff K \models \forall x_0, \ldots, x_n \left( \bigwedge_{\gamma \in \Gamma} \text{truth}(\gamma(x)) \rightarrow \text{truth}(\phi(\bar{x})) \right);$$
(b) let \( K \) be a class of bimatrices, then:

\[
\Gamma \models_\mathcal{K} \phi \iff K \models \forall x_0, \ldots, \forall x_n \left( \bigwedge_{\gamma \in \Gamma} \text{truth}(\gamma(\bar{x})) \land \bigwedge_{i \leq n} \text{core}(x_i) \to \text{truth}(\phi(\bar{x})) \right).
\]

**Proof.** This follows immediately from the definition of \( \models_\mathcal{K} \) and \( \models_\mathcal{C} \).

It follows from the previous proposition that standard logics in the language \( \mathcal{L} \) can be encoded by Horn theories in \( \mathcal{L} \cup \{\text{truth}\} \), while weak logics can be encoded in \( \mathcal{L} \cup \{\text{truth}, \text{core}\} \). This motivates the following definitions.

**Notation 6.12.** Let \( \Gamma \cup \{\phi\} \subseteq \text{Fm} \), then we write \( \Phi(\Gamma, \phi) \) for the first-order formula

\[
\forall x_0, \ldots, \forall x_n \left( \bigwedge_{\gamma \in \Gamma} \text{truth}(\gamma(\bar{x})) \land \bigwedge_{i \leq n} \text{core}(x_i) \to \text{truth}(\phi(\bar{x})) \right).
\]

**Definition 6.13.** Let \( \models \) be a weak logic, then we let \( \text{Horn}(\models) \) be the Horn theory obtained by letting \( \Phi(\Gamma, \phi) \in \text{Horn}(\models) \) whenever \( \Gamma \models \phi \). We write \( \text{Mod}(\models) \) for the class of structures \( \text{Mod}(\text{Horn}(\models)) \).

While Czelakowski’s original approach in [13] was specifically tailored to logical matrices, Dellunde and Jansana considered arbitrary model classes axiomatized by Horn theories without equality, thus making it possible to apply their results to the setting of bimatrices and expanded algebras. We recall from Notation 1.10 that the operator \( \mathbb{H}_s \) refers to the closure under strict homomorphic images, and thus \( \mathbb{H}_{s}^{-1}(K) \) is the class of all structures \( A \) that are preimage under some strict homomorphism of some structure in \( K \). We stress that Dellunde and Jansana’s result (from [14]) hold in the setting without the equality symbol. In particular, when in the rest of this section we consider Horn formulas, we always restrict attention to Horn formulas not containing the equality symbol. For clarity, we introduce the following notation.

**Notation 6.14.** We write \( \mathcal{L}^- \) to refer to a fixed first-order language without the equality symbol \( \approx \). Also, we write \( \mathcal{L}^- \) for the collections of all formulas in this language (which clearly do not contain the equality symbol).

**Theorem 6.15** (Dellunde, Jansana). Let \( K \) be a class of \( \mathcal{L}^- \)-structures, then the following are equivalent:

1. \( K \) is axiomatised by strict universal Horn formulas in \( \mathcal{L}^- \);
2. \( K \) is closed under \( \mathbb{H}^{-1}_s, \mathbb{H}_s, \text{S}, \text{P}, \text{P}_U \) and contains a trivial structure;
3. \( K = \mathbb{H}_{s}^{-1}\mathbb{H}_s\text{SPP}_U(K_0) \) for some class \( K_0 \) of \( \mathcal{L} \)-structures containing a trivial structure.

As we mentioned in Fact 1.14, the validity of Horn formulas is always preserved under the operators \( \mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_U \). From the previous theorem it follows that strict Horn formulas in a language without equality are also preserved under the operators \( \mathbb{H}^{-1}_s \) and \( \mathbb{H}_s \). From this fact and the previous theorem we immediately obtain as a corollary the following characterization of the class of models \( \text{Mod}(\models) \), where \( \models = \text{Log}(\mathcal{K}) \) for some class of bimatrices \( \mathcal{K} \).

**Corollary 6.16.** Let \( \mathcal{K} \) be a class of bimatrices and let \( T = \text{Horn}(\text{Log}(\mathcal{K})) \), then:

\[
\text{Mod}(T) = \mathbb{H}_{s}^{-1}\mathbb{H}_s\text{SPP}_U(\mathcal{K}').
\]

where \( \mathcal{K}' \) is \( \mathcal{K} \) together with some trivial bimatrices.
Notice that, by Theorem 6.10, every weak logic $\vdash$ is complete with respect to a class of bimatrices $\mathcal{M}_k$ and so the previous theorem applies to all weak logics $\vdash$ and provides us with a characterization of the class of all bimatrices defined by $\vdash$.

Importantly, however, one can see that the class of all matrices $\text{Mod}(T)$ above does not meet the intuition about the “right” semantics of a (weak) logic, and will contain several pathological examples. For example, in the case of the standard logic CPC, the class Mod(CPC) is not the class of Boolean algebras, but rather the class $\mathbb{H}_n^<(\mathcal{B}\mathcal{A})$ (for example, the so-called “benzene ring” is a matrix model of CPC, but is not a Boolean algebra, cf. [16, Ex. 4.79]). In order to identify the non-pathological models of a propositional logic, Dellunde and Jansana focused in [14] to the so-called reduced structures. Since they work at the general level of model theory without equality, it is again straightforward to adapt their results to our current setting of bimatrices.

**Definition 6.17.** Let $\mathcal{L}$ be any first-order structure, $M$ a $\mathcal{L}$-structure and $X \subseteq M$, then we write $\mathcal{L}(X)$ for the set of all $\mathcal{L}$ formulas with parameters in $X$.

**Definition 6.18.** Let $M$ be a first-order structure, $D \subseteq M$ and $\bar{a} \in M^{<\omega}$, we let the type without equality of $\bar{a}$ over $D$ in $M$ be the following set of equality-free formulas:

$$\text{tp}_M(\bar{a}/D) = \{\phi(x) \in \mathcal{L}^-(D) : M \models \phi(\bar{a})\}$$

Then, the Leibniz congruence is the relation $\sim_*$ on $A$ defined by letting, for $a, b \in M$:

$$a \sim_* b \iff \text{tp}_M(a/M) = \text{tp}_M(b/M).$$

We say that a model $M$ is reduced if the Leibniz congruence $\sim_*$ over $M$ is the identity.

**Remark 6.19.** As shown in [14], $\sim_*$ is the largest non-trivial congruence relation on $M$, meaning that it is the largest congruence $\theta$ over its algebraic reduct such that, for any relation symbol $R \in \mathcal{L}$, if $(a_1, b_1) \in \theta$ for all $1 \leq i \leq n$ then $A \models R(a_1, \ldots, a_n)$ if and only if $A \models R(b_1, \ldots, b_n)$. As we stressed already in Remark 1.9, the projections induced by congruences respecting this conditions are always strict homomorphisms. Additionally, since $\sim_*$ is the greatest such congruence, any strict homomorphism from $M/\sim_*$ to some further structure $N$ must be the identity.

**Notation 6.20.** Let $M$ be an $\mathcal{L}^-$-structure and let $\sim_*$ be the Leibniz congruence over $M$, then we write $M^*$ for the quotient structure $M/\sim_*$. For any class operator $\mathcal{O}$, we let $\mathcal{O}^*(C) := \{A^* : A \in \mathcal{O}(C)\}$, and for any class $\mathcal{K}$ we let $\mathcal{K}^* = \{A^* : A \in \mathcal{K}\}$. Finally, we let $\text{Mod}^*(T) = \{A^* : A \in \text{Mod}(T)\}$.

**Remark 6.21.** By Theorem 6.15 the validity of (strict) universal Horn formulas in a language without equality is closed by $\mathbb{H}_n$, and thus $\text{Mod}^*(T) \subseteq \text{Mod}(T)$.

From Theorem 6.15 then one immediately obtain the following theorem [14, Thm. 18] and corollary. Notice that our formulation differs from the original one as Dellunde and Jansana assume that closure operators are already closed under isomorphic copies.

**Theorem 6.22** (Dellunde, Jansana). Let $\mathcal{K}$ be a class of reduced $\mathcal{L}^-$-structures, then the following are equivalent:

(1) $\mathcal{K}$ is the class of reduced models of a $\mathcal{L}^-$-universal Horn theory;
(2) $K$ is closed under the operators $\Pi^*, S^*, P^*, P_U^*$;
(3) $K = \Pi^* S^* \Pi P^*_U(K_0)$ for some class $K_0$ of $L^-$-structures.

Corollary 6.23. Let $K$ be a class of reduced bimatrices, $T = \text{Horn}(\text{Log}(K))$, then:
\[
\text{Mod}^*(T) = \Pi^* \Pi P^*_U(K).
\]

Thus, since every weak logic is complete with respect to a class of bimatrices, the previous corollary provides us with a characterization of the class of reduced bimatrices defined by any weak logic $\models$. We conclude our abstract study of matrix semantics by showing that, in the case of a loosely algebraizable weak logic $\models$, its reduced core-generated bimatrices coincide with the core-generated expanded algebras of its equivalent algebraic semantics. We first define these notions and notice that Proposition 2.21 easily extends to this setting.

Definition 6.24. Let $M$ be a bimatrix, we say that $M$ is core-generated if $M = \langle \text{core}(M) \rangle$. We then write $\text{Mod}_{CG}(\models)$ for the subclass of $\text{Mod}(\models)$ consisting only of core-generated structures. We then let $\text{Mod}_{CG}^*(\models) = (\text{Mod}(\models))^*$.

Proposition 6.25. Let $A$ be a core-generated bimatrix, then $\sigma(\Theta) \models_{\langle A \rangle} \sigma(\varepsilon \equiv \delta)$ for all $\sigma \in \text{Subst}(L)$ holds if and only if $\Theta \models_{\langle A \rangle} \varepsilon \equiv \delta$.

Proof. This follows by reasoning as in Lemma 2.20 and Proposition 2.21.

We can then characterise the reduced, core-generated bimatrices of a loosely algebraizable weak logics $\models$ by directly applying the standard version of this result. We recall the following classical result from [16, Thm. 4.60].

Fact 6.26. Let $\models$ be an algebraizable standard logic with equivalent algebraic semantics $(Q, \tau, \Delta)$. Then $(A, \text{truth}(A)) \in \text{Mod}_{CG}^*(\models)$ if and only if $A \in Q$ and $\text{truth}(A) = \{a \in A : A \models^{\varepsilon} \tau(a)\}$.

Corollary 6.27. Let $\models$ be a loosely algebraizable weak logic with equivalent algebraic semantics $(Q, \tau, \Delta)$. Then $(A, \text{truth}(A), \text{core}(A)) \in \text{Mod}_{CG}^*(\models)$ if and only if $(A, \text{core}(A)) \in Q_{CG}$ and $\text{truth}(A) = \{a \in A : A \models^{\varepsilon} \tau(a)\}$.

Proof. If $\models$ is loosely algebraizable, then by Fact 6.26 it follows that $A \models \text{core}(A)) \in Q_{CG}$ if and only if $(A, \text{core}(A)) \in Q$. Moreover, it follows from Proposition 6.25 that $(A, \text{truth}(A)) \in \text{Mod}_{CG}^*(\models)$ if and only $(A, \text{truth}(A), \text{core}(A)) \in \text{Mod}_{CG}^*(\models)$, and from Fact 6.26 that $A \in Q \models \text{truth}(A) = \{a \in A : A \models^{\varepsilon} \tau(a)\}$ if and only $(A, \text{truth}(A)) \in \text{Mod}_{CG}^*(\models)$ if and only $(A, \text{truth}(A)) \in \text{Mod}_{CG}^*(\models)$.

7. APPLICATION: INQUISTIVE AND DEPENDENCE LOGICS

We turn in this section to one application of the abstract machinery that we studied, i.e., the case of (propositional) inquisitive and dependence logics. In fact, these logical systems make for two interesting examples of logics where uniform substitution fails, but which have been studied from the algebraic point of view. In particular, an algebraic semantics for the classical version of inquisitive logic $\text{InqB}$ was introduced in [2, 3], although some preliminary inquiry into the subject was provided already in [28]. Such semantics was later generalised in [26] to the intuitionistic logics $\text{InqI}$ and $\text{InqI}^\circ$, and to both the classical and intuitionistic version of dependence logic $\text{InqB}^{\circ}$ and $\text{InqI}^{\circ}$. Since these logical systems do not satisfy the
rule of uniform substitution, it has so far been an open question whether such semantics are in any sense unique. The notion of algebraizability of weak logics that we have introduced in this article provides us with a framework to make sense of this question. In this section, we build on these previous works and relate them to the notion of algebraizability from the present article. More precisely, we prove that the classical versions of inquisitive and dependence logic \( \text{InqB} \) and \( \text{InqB} \otimes \) are strictly algebraizable, while their intuitionistic versions \( \text{InqI} \) and \( \text{InqI} \otimes \) are only loosely so.

### 7.1. Inquisitive and Dependence Logic

We introduce in this section the intuitionistic and classical propositional variants of inquisitive and dependence logic. We refer the reader to [22] and [12] for the original presentation of classical propositional inquisitive logic, to [30] for classical propositional dependence logic, and to [11] for their intuitionistic versions. We make explicit in the following remark the language in which we formulate these logics. Notice that while our presentation follows essentially [11], our notation is as in [26].

**Context 7.1.** We let \( L_{\text{IPC}} \) be the propositional language \( L_{\text{IPC}} = \{ \land, \lor, \to, \bot, \top \} \), i.e., \( L_{\text{IPC}} \) is simply the usual language of propositional intuitionistic logic. With some abuse of notation, we denote by \( L_{\text{IPC}} \) also the set of all propositional formulas recursively built from \( \text{Var} \) in this syntax. Also, we let \( L_{\otimes \text{IPC}} \) be the propositional language \( L_{\otimes \text{IPC}} = \{ \land, \lor, \otimes, \to, \bot, \top \} \), namely the expansion of \( L_{\text{IPC}} \) by a novel tensor disjunction operation \( \otimes \). With some abuse of notation, we denote by \( L_{\otimes \text{IPC}} \) also the set of all propositional formulas recursively built from \( \text{Var} \) in the syntax \( L_{\otimes \text{IPC}} \).

**Notation 7.2.** We define the inquisitive operation \( \phi^? := \phi \lor \neg \phi \). We treat negation as a defined operation and we define it by \( \neg \phi := \phi \to \bot \). Also, we write \( \phi \leftrightarrow \psi \) as an abbreviation for \( \phi \to \psi \land \psi \to \phi \).

**Definition 7.3.** A formula of \( L_{\otimes \text{IPC}} \) is **standard** if it does not contain the symbol \( \lor \). We write \( L_{\text{CL}} \) for the signature \( L_{\text{CL}} = \{ \land, \to, \bot, \top \} \) and for the sets of formulas determined by it. Similarly, we write \( L_{\otimes \text{CL}} \) for the signature \( L_{\otimes \text{CL}} = \{ \land, \to, \otimes, \bot, \top \} \) and for the sets of formulas determined by it.

We recall briefly the standard semantics of inquisitive and dependence logic, i.e., their **team** (or **state**) semantics. Intuitively, while in the standard semantics of classical propositional logic a formula is evaluated by a truth-table, i.e., by an assignment of atomic variables into \( \{0, 1\} \), in the classical version of team semantics it is evaluated by a set of such assignments. Similarly, while in the standard semantics of intuitionistic propositional logic a formula is evaluated at a node of a poset, in the intuitionistic version of team semantics it is evaluated by a set of such nodes. We make this idea precise by the following definitions from [11].

**Definition 7.4.** A Kripke frame is a partial order \( \mathfrak{A} = (W, R) \). A Kripke model is a pair \( \mathcal{M} = (\mathfrak{A}, V) \), where \( \mathfrak{A} \) is a Kripke frame and \( V : W \to \wp(\text{Var}) \) a valuation of atomic formulas such that, if \( p \in V(w) \) and \( wRv \), then \( p \in V(v) \). We say that a Kripke frame (resp. model) \( \mathfrak{A} = (W, R) \) is **classical** if \( R \) is the identity relation.

**Definition 7.5.** Let \( \mathcal{M} = (W, R, V) \) be an Kripke model. A **team** is a subset \( t \subseteq W \) of the set of possible world. A team \( s \) is an **extension** of a team \( t \) if \( s \subseteq R[t] \).
Remark 7.6. Crucially, an element in a Kripke model $(\mathcal{F}, V)$ is essentially a classical assignment, and we write $w(p) = 1$ if and only if $p \in V(w)$. This reflects the main underlying intuition teams are essentially sets of assignments.

We can next define team semantics of the formulas in $L^\otimes_{IPC}$, and thus of the classical and inquisitive variant of inquisitive and dependence logic.

Definition 7.7. Let $\mathcal{M} = (W, R, V)$ be a Kripke model. The notion of a formula $\phi \in L^\otimes_{IPC}$ being true in a team $t \subseteq W$ is defined as follows:

- (a) the system $\Gamma \vdash_{\text{InqI}}$ of intuitionistic inquisitive logic is the consequence relation in the language $L^\otimes_{IPC}$ over the class of all Kripke frames $\mathcal{F}$, namely,
  \[
  \Gamma \vdash_{\text{InqI}} \phi \iff \mathcal{M}, t \models \forall \mathcal{T} \text{ entails } \mathcal{M}, t \models \phi \text{ for all Kripke frames } \mathcal{M} \text{ and teams } t \subseteq W;
  \]

- (b) the system $\Gamma \vdash_{\text{InqB}}$ of classical inquisitive logic is the consequence relation in the language $L^\otimes_{IPC}$ over the class of all classical Kripke frames $\mathcal{F}$, namely,
  \[
  \Gamma \vdash_{\text{InqB}} \phi \iff \mathcal{M}, t \models \Gamma \text{ entails } \mathcal{M}, t \models \phi \text{ for all classical Kripke frames } \mathcal{M} \text{ and teams } t \subseteq \mathcal{M};
  \]

- (c) the system $\Gamma \vdash_{\text{InqB}^\otimes}$ of intuitionistic dependence logic is the consequence relation in the language $L^\otimes_{IPC}$ over the class of all Kripke frames $\mathcal{F}$, namely,
  \[
  \Gamma \vdash_{\text{InqB}^\otimes} \phi \iff \mathcal{M}, t \models \Gamma \text{ entails } \mathcal{M}, t \models \phi \text{ for all Kripke frames } \mathcal{M} \text{ and teams } t \subseteq \mathcal{M};
  \]

- (d) the system $\Gamma \vdash_{\text{InqB}^\otimes}$ of classical dependence logic is the consequence relation in the language $L^\otimes_{IPC}$ over the class of all classical Kripke frames $\mathcal{F}$, namely,
  \[
  \Gamma \vdash_{\text{InqB}^\otimes} \phi \iff \mathcal{M}, t \models \Gamma \text{ entails } \mathcal{M}, t \models \phi \text{ for all classical Kripke frames } \mathcal{M} \text{ and teams } t \subseteq \mathcal{M};
  \]

Remark 7.9. Notice that in [11] the previous systems were introduced simply as the sets of validities of the consequence relations from Definition 7.8. However this does not really make a difference, since these logics are all finitary and satisfy the deduction theorem. In particular, the fact that each of the previous consequence relations is finitary follows from [11, Cor. 4.22], while the deduction theorem is essentially [11, Prop. 4.3]. It then follows that, for any $L \in \{\text{InqB, InqB}^\otimes, \text{InqI, InqI}^\otimes\}$,

\[
\Gamma \vdash_L \phi \iff \models_L \bigwedge_{\psi \in \Gamma_0} \psi \rightarrow \phi \text{ for some finite } \Gamma_0 \subseteq \Gamma,
\]

showing that each consequence relation $\vdash_L$ is determined by its set of validities.
The next proposition makes it explicit that these logics are proper examples of weak logic. The failure of uniform substitution in these logics has been pointed out since their introduction in [11,12,30]. We provide details of the following proposition (which essentially develops Example 2.5) for completeness of exposition, and since the lack of uniform substitution provides the key motivation of our abstract work.

**Proposition 7.10.** Let \( L \in \{ \text{InqB}, \text{InqB}^\oplus, \text{InqI}, \text{InqI}^\oplus \} \), then \( \models_L \) is a weak logic, and in particular it is not closed under uniform substitution.

**Proof.** Let \( L \in \{ \text{InqB}, \text{InqB}^\oplus, \text{InqI}, \text{InqI}^\oplus \} \), then the fact that \( \models_L \) satisfies Conditions (1)–(3) from Definition 1.2 follows from the definition of the semantic consequence relation \( \models \) from Definition 7.7. Condition (4) follows from Remark 7.9 (thus in particular from [11, Cor. 4.22]). Finally, the fact that \( \models_L \) is closed under atomic substitution is immediate to verify using the team semantics from above.

It remains to show that \( \models_L \) is not closed under uniform substitution. We prove this rigorously only for \( \text{InqB} \) and \( \text{InqB}^\oplus \) and mention an example that applies also to \( \text{InqI} \) and \( \text{InqI}^\oplus \) below in Remark 7.11. Suppose thus \( L \in \{ \text{InqB}, \text{InqB}^\oplus \} \), we formalize the model that we sketched in Example 2.5. Let \( \mathfrak{S} = (\{ a, b, c, d \}, =) \) be a classical Kripke frame, and let \( V : \text{Var} \to \wp(\mathfrak{S}) \) be such that \( V(p) = \{ a, b \} \).

Firstly, we use the team semantics from Definition 7.7 to show that \( \mathfrak{S}, \{ b, d \} \models \neg \neg ?p \). Suppose on the contrary that there is a nonempty team \( t \subseteq \{ b, d \} \) such that \( \mathfrak{S}, t \models \neg ?p \). Then since \( t \neq \emptyset \) we have in particular that either \( \mathfrak{S}, \{ b \} \models ?p \) or \( \mathfrak{S}, \{ d \} \models ?p \), which is clearly false. We thus conclude that \( \mathfrak{S}, \{ b, d \} \models \neg \neg ?p \). Additionally, we notice that \( \mathfrak{S}, \{ b, d \} \not\models ?p \). In fact, \( \mathfrak{S}, \{ b, d \} \not\models p \) since \( d(p) = 1 \), and \( \mathfrak{S}, \{ b, d \} \not\models \neg p \) since \( b(p) = 1 \). We showed that \( \neg \neg ?p \not\models ?p \) in any of the logics \( L \in \{ \text{InqB}, \text{InqB}^\oplus, \text{InqI}, \text{InqI}^\oplus \} \).

However, suppose that \( \mathfrak{S}, t \models \neg p \) for some Kripke frame \( \mathfrak{S} = (W, R) \) and \( t \subseteq W \). This in particular means that for every \( w \in t \) we have that \( \mathfrak{S}, \{ w \} \not\models \neg p \), which entails that \( w(p) = 1 \). It follows that \( w(p) = 1 \) for all \( w \in t \), showing \( \mathfrak{S}, t \models \neg p \). In other words, we proved that \( \neg p \models_L p \) in any of the logics \( L \in \{ \text{InqB}, \text{InqB}^\oplus, \text{InqI}, \text{InqI}^\oplus \} \). We conclude that all these logics are not closed under the non-atomic substitution \( p \mapsto ?p \).

\[ \square \]

**Remark 7.11.** We mention a counterexample to uniform substitution that applies to all \( L \in \{ \text{InqB}, \text{InqB}^\oplus, \text{InqI}, \text{InqI}^\oplus \} \). One can verify that:

\[ \models_L (p \to (q \lor r)) \to ((p \to q) \lor (p \to r)), \]

but the result of the substitution \( p \to q \lor r \) is not a validity of these logics:

\[ \not\models_L ((q \lor r) \to (q \lor r)) \to (((q \lor r) \to q) \lor ((q \lor r) \to r)). \]

We refer the reader to [11, 4.5] for an explanation of this fact and a lengthier discussion of the failure of uniform substitution in propositional inquisitive and dependence logic.

### 7.2. Strict Algebraizability of InqB and InqB^⊕

We prove in this section the strict algebraizability of the classical versions of inquisitive and dependence logic. We recall some important facts and definitions.

**Definition 7.12.** A Heyting algebra \( H \) is a bounded distributive lattice with an operation \( \to \) such that for all \( a, b, c \in H \):

\[ a \land b \leq c \iff a \leq b \to c. \]
The negation of $x \in H$ is $\neg x := x \rightarrow \bot$. An element $x \in H$ is regular if $x = \neg \neg x$.

We write $H_\prec$ for the subset of regular elements of $H$, and we say that a Heyting algebra $H$ is regular if $H = \langle H_\prec \rangle$. We say that a variety $V$ of Heyting algebras is regularly generated if it is core-generated with $\text{core}(H) = H_\prec$ for all $H \in V$.

**Remark 7.13.** We recall that an intermediate logic is a standard logic $L$ such that $\text{IPC} \subseteq L \subseteq \text{CPC}$, where $\text{IPC}$ denotes the intuitionistic propositional calculus, and $\text{CPC}$ the classical propositional calculus. Every intermediate logic $L$ is algebraized by the variety of Heyting algebras $\text{Var}(L)$ defined by

$$\text{Var}(L) = \{ H \in \text{HA} : \models \phi \approx 1 \text{ for all } \phi \in L \}$$

and by the transformers $\tau(\phi) = (\phi \approx 1)$ and $\Delta(\alpha, \beta) = \alpha \leftrightarrow \beta$ (cf. [16, Ex. 3.34]).

The following two definitions are less standard. First we recall the DNA-logics from Example 2.5. Negative variants were originally considered in [22].

**Definition 7.14.** Let $L$ be an intermediate logic, then its negative variant $L^-$ is the set of formulas $L^-=\{\phi|\neg p_0, \ldots, \neg p_n/p_0, \ldots, p_n : \phi \in L\}$. A DNA-logic is the negative variant of some intermediate logic.

Additionally, we recall Medvedev’s logic of finite problems, which was originally introduced by Medvedev in [21]. This is an intermediate logic defined in terms of validity in a specific class of Kripke frames. We direct the reader to [8, §2.2] for reference to the Kripke semantics of intermediate logics.

**Definition 7.15.** We recall that the intermediate logic $\text{ML}$, known as Medvedev’s logic of finite problems, is the logic of all Kripke frames of the form $(\varphi^+(s), \supseteq)$, where $|s| < \aleph_0$ and $\varphi^+(s) = \varphi(s) \setminus \{\emptyset\}$.

The next fact collects together some results by Ciardelli [10] on the schematic variant of $\text{InqB}$ and the characterization of regularly generated varieties from [3]. From these it is then immediate to prove the strict algebraizability of $\text{InqB}$.

**Fact 7.16.**

1. $ML^- = \text{InqB}$ and $\text{Schm(}\text{InqB}) = \text{ML}$;
2. $\text{Var}(\text{ML})$ is regularly generated;
3. let $\tau(\phi) = (\phi \approx 1)$ for all $\phi \in \text{Fm}$, then for all $\Gamma \cup \{\phi\} \subseteq \text{L}_{\text{IPC}}$, $\Gamma \models_{\text{InqB}} \phi$ is equivalent to $\tau(\Gamma) \models_{\text{Var}(\text{ML})} \tau(\phi)$, where $\text{core}(H) - H_{\prec}$ for all $H \in \text{Var}(\text{ML})$.

**Proof.** Considering Clause (1), both $\text{ML}^- = \text{InqB}$ and $\text{Schm(}\text{InqB}) = \text{ML}$ were proved by Ciardelli in [10]. Clause (2) follows immediately from (1) together with Proposition 4.17 from [3]. Clause (3) is essentially, the main algebraic completeness result for inquisitive logic shown in [2] and [3]. However, in these articles completeness is formulated with respect to the intermediate logics $\text{KP}$ and $\text{ND}$, thus we explain how to obtain completeness with respect to $\text{Var}(\text{ML})$ as stated in (3). Firstly, by Theorem 3.32 in [3] we have that $\Gamma \models_{\text{InqB}} \phi$ is equivalent to $\Gamma \models_{\text{Var}(\text{ND})} \phi$, where $\text{ND}$ is a specific intermediate logic. Additionally, it follows from [3, Thm. 5.9] that $\text{ND}$ and $\text{ML}$ contain the same regularly generated algebras. It then follows from Proposition 2.26 that $\Gamma \models_{\text{InqB}} \phi$ is equivalent to $\Gamma \models_{\text{Var}(\text{ML})} \phi$.

**Theorem 7.17.** $\text{InqB}$ is strictly algebraizable.
Proof. Let $\tau(\phi) = \phi \approx 1$, $\Delta(x,y) = x \leftrightarrow y$ and $\Sigma = \{x \equiv \neg \neg x\}$. We prove that $(\text{Var}(\text{ML}), \Sigma, \tau, \Delta)$ algebraizes $\text{InqB}$. Firstly, by Fact 7.16 above we have that $\text{Var}(\text{ML})$ is core-generated with core defined by $\Sigma$. Then, by Proposition 3.8 it suffices to show that $(\text{Var}(\text{ML}), x \equiv \neg \neg x, \phi \approx 1, x \leftrightarrow y)$ satisfies 3.6(W1) and 3.6(W4). By Fact 7.16(3), Condition 3.6(W1) follows immediately. Moreover, for all $H \in \text{Var}(\text{ML})$ and $x,y \in H$, we have that $x = y$ if and only if $x \leq y$ and $y \leq x$. This is equivalent to $H \models x \rightarrow y \equiv 1$ and $H \models y \rightarrow x \equiv 1$. It follows that $x \approx y \equiv_{\text{Var}(\text{ML})} \{x \rightarrow y \equiv 1, y \rightarrow x \equiv 1\}$, showing 3.6(W4) holds. It follows that $\text{InqB}$ is strictly algebraizable. 

To extend this result to dependence logic, we firstly need to introduce a suitable notion of dependence algebras. We refer the reader to [5, p. 57] for the definition of subdirectly irreducible algebras.

**Definition 7.18.** A $\text{InqB}^\circ$-algebra $A$ is a structure in the signature $\mathcal{L}_{\text{IPC}}^\circ$ satisfying the following conditions:

(a) $A \models \{\lor, \land, \rightarrow, \bot\} \in \text{Var}(\text{ML})$,
(b) $A \models \{\otimes, \land, \rightarrow, \bot\} \in \text{BA}$,
(c) $A \models \forall x \forall y \forall z (x \otimes (y \lor z) \approx (x \otimes y) \lor (x \otimes z))$,
(d) $A \models \forall x \forall y \forall z \forall k ((x \rightarrow z) \rightarrow (y \rightarrow k) \approx (x \otimes y) \rightarrow (z \otimes k))$.

We then let $\text{InqBAlg}^\circ$ be the variety of all $\text{InqB}^\circ$-algebras and $\text{InqBAlg}^\circ_{\text{FRSI}}$ be its subclass of finite, regular and subdirectly irreducible elements.

**Remark 7.19.** Our definition is essentially from [26, 2.2], with the difference that here we assume that the equations hold in the full algebra and not only in the subalgebra generated by the core. Since our results deal with core semantics and core-generated structure, this does not affect the validity of the results from [26].

The previous class of algebras was shown to provide a sound and complete semantics of $\text{InqB}^\circ$. We recall the following fact from [26, 2.15, 3.20] and use it to show that $\text{InqB}^\circ$ is strictly algebraizable. Notice that, since $A \models \{\lor, \land, \rightarrow, \bot\} \in \text{Var}(\text{ML})$, for $A \in \text{InqBAlg}^\circ$, it follows that the notion of regular elements and the subset $A_\approx$, are welldefined in this context.

**Fact 7.20.** Let $\tau(\phi) = (\phi \approx 1)$ for all $\phi \in \text{Fm}$, then for all $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\text{IPC}}^\circ$, $\Gamma \models^\circ_{\text{InqB}^\circ} \phi$ is equivalent to $\tau(\Gamma) \models^\circ_{\text{InqBAlg}^\circ_{\text{FRSI}}} \tau(\phi)$, where $\text{core}(A) = A_\approx$ for all $A \in \text{InqBAlg}^\circ_{\text{FRSI}}$.

**Theorem 7.21.** $\text{InqB}^\circ$ is strictly algebraizable.

Proof. This follows from Fact 7.20 analogously to Theorem 7.17, by $\forall(\text{InqBAlg}^\circ_{\text{FRSI}})$, $\tau(\phi) := (\phi \approx 1)$, $\Delta(x,y) = x \rightarrow y \land y \rightarrow x$ and $\Sigma = \{x \equiv \neg \neg x\}$. 

7.3. Loose Algebraizability of $\text{InqI}$ and $\text{InqI}^\circ$

We show in this section that $\text{InqI}$ and $\text{InqI}^\circ$ are both loosely algebraizable, but not strictly so. The loose algebraizability of these logics is essentially a corollary of the algebraic completeness result from [26]. We start by introducing the algebraic semantics for $\text{InqI}$ and $\text{InqI}^\circ$ described in [26]. We review the following definitions. As in Definition 7.18 we slightly modify the definitions from [26] so that the equations defining $\text{InqI}$-algebras and $\text{InqI}^\circ$-algebra are valid in the entire structure and not only in the subalgebra generated by the core. As stressed in Remark 7.19, this does not affect the validity of the results from [26].
Definition 7.22. A Browerian semilattice \( B \) is a bounded join-semilattice with an additional operation \( \rightarrow \) such that, for all \( a, b, c \in B \):

\[
a \land b \leq c \iff a \leq b \rightarrow c,
\]

and we write \( \text{BS} \) for the class of all Browerian semilattices.

Definition 7.23. An \( \text{InqI} \)-algebra \( A \) is is an expanded algebra in the signature \( \mathcal{L}_{\text{IPC}} \) satisfying the following conditions:

(a) \( A[\{\lor, \land, \to, \bot\}] \in \mathcal{HA} \),
(b) \( \text{core}(A)[\{\land, \to, \bot\}] \in \text{BS} \),
(c) \( A \models \forall x \forall y \forall a \left( \text{core}(a) \to (a \to (x \lor y) = (a \to x) \lor (a \to y)) \right) \);

and we then write \( \text{InqIAlg} \) for the class of all \( \text{InqI} \)-algebras.

Definition 7.24. An \( \text{InqI}^\circ \)-algebra \( A \) is is an expanded algebra in the signature \( \mathcal{L}_{\text{IPC}}^\circ \) satisfying the following conditions:

(a) \( A[\{\lor, \land, \to, \bot\}] \in \mathcal{HA} \),
(b) \( \text{core}(A)[\{\land, \to, \bot\}] \in \mathcal{HA} \),
(c) \( A \models \forall x \forall y \forall a \left( \text{core}(a) \to (a \to (x \lor y) = (a \to x) \lor (a \to y)) \right) \),
(d) \( A \models \forall x \forall y \forall z \left( (x \land (y \lor z)) = ((x \land y) \lor (x \land z)) \right) \),
(e) \( A \models \forall x \forall y \forall z \forall k \left( (x \to k) = (z \land y) \lor (z \land k) \right) \);

and we then write \( \text{InqIAlg}^\circ \) for the class of all \( \text{InqI}^\circ \)-algebras.

We next recall the algebraic completeness result for intuitionistic inquisitive and dependence logic. This follows immediately from [26, 2.15, 3.20]. From this, it is immediately to show that both \( \text{InqI} \) and \( \text{InqI}^\circ \) are loosely algebraizable.

Fact 7.25. Let \( \tau : \text{Fm} \to \text{Eq} \) be defined by \( \tau(x) = (x \approx 1) \), then the following completeness results hold:

(1) For all \( \Gamma \cup \{ \phi \} \subseteq \mathcal{L}_{\text{IPC}} \), \( \Gamma \models_{\text{InqI}} \phi \) is equivalent to \( \tau(\Gamma) \models_{\text{InqIAlg}} \tau(\phi) \).

(2) For all \( \Gamma \cup \{ \phi \} \subseteq \mathcal{L}_{\text{IPC}}^\circ \), \( \Gamma \models_{\text{InqI}^\circ} \phi \) is equivalent to \( \tau(\Gamma) \models_{\text{InqIAlg}^\circ} \tau(\phi) \).

Theorem 7.26. \( \text{InqI} \) and \( \text{InqI}^\circ \) are loosely algebraizable.

Proof. Consider the logic \( \text{InqI} \). Let \( Q = Q(\text{InqIAlg}_{\mathcal{CG}}) \), \( \tau(\phi) = (\phi \approx 1) \) and \( \Delta(\alpha, \beta) = ((\alpha \to \beta) \land (\beta \to \alpha)) \). Notice that, since every \( H \in Q \) is clearly a Heyting algebra, it follows that \( \tau(\Delta(\alpha, \beta)) = (\alpha \leftrightarrow \beta \approx 1) \) is equivalent to \( \alpha \approx \beta \). Thus this establishes Def. 3.6(W4). Moreover, we have by the algebraic completeness 7.25(1) and Proposition 2.24 that

\[
\Gamma \models_{\text{InqIAlg}} \phi \iff \tau(\Gamma) \models_{Q} \tau(\phi),
\]

which establishes Def. 3.6(W1). It thus follows from Proposition 3.8 that \( (Q, \tau, \Delta) \) loosely algebraizability of \( \text{InqI} \). The loose algebraizability of \( \text{InqI}^\circ \) follows by a parallel argument.

While the loose algebraizability of \( \text{InqI} \) and \( \text{InqI}^\circ \) follows straightforwardly from [26], the fact that they are not strictly algebraizable is substantially more subtle. In principle, to show that a logic is not strictly algebraizable one should test infinitely many equations to see whether any of them defines the core of the algebras in the corresponding quasivariety of expanded algebras, which is clearly a no-go. However, the failure of strict algebraizability in \( \text{InqI} \) and \( \text{InqI}^\circ \) is witnessed by a simpler pattern, namely, we can find two inquisitive (dependence) algebras \( H, K \) which share their algebraic reduct, but have different cores. Clearly, if a logic is
strictly algebraizable this cannot happen, as the core is uniquely determined by a finite set of equations \( \Sigma \). We make this intuition precise in the following proof.

**Theorem 7.27.** InqI and InqI\(^\oplus\) are not strictly algebraizable.

**Proof.** We prove this for InqI, as the proof easily adapts to InqI\(^\oplus\). Firstly, notice that if \((Q, \Sigma, \tau, \Delta)\) strictly algebraizes InqI, then clearly \((Q, \tau, \Delta)\) is also a witness of its loose algebraizability. It then follows from Theorem 7.26 and Theorem 3.7 that \( Q = Q(\text{InqI} \text{Alg}_{CG}) \). Now, by directly checking the definition of InqI-algebras from Definition 7.23 (or, alternatively, by applying the categorical duality between finite Kripke frames and finite, well-connected, core-generated, InqI-algebras from [26]), one can verify that the expanded algebras \( H_0 \) and \( H_2 \) from Fig. 2 (with circles indicating which elements are in the core) belong to \( \text{InqI} \text{Alg}_{CG} \). Furthermore, since \( H_1 \leq H_2 \), it also follows that \( H_1 \in Q(\text{InqI} \text{Alg}_{CG}) \). Crucially, \( H_0 \) and \( H_1 \) have the same algebraic reduct but different subsets of core elements. In particular, assuming \((Q, \Sigma, \tau, \Delta)\) strictly algebraizes InqI, we obtain that \( \text{core}(H_0) = \Sigma(H_0) = \Sigma(H_1) = \text{core}(H_1) \), contradicting \( \text{core}(H_0) \neq \text{core}(H_1) \). It follows that InqI is not strictly algebraizable. The same argument also works for InqI\(^\oplus\), as it suffices to notice that the core subsets of \( H_0, H_1, H_2 \) can all be expanded to form Heyting algebras with an additional disjunction \( \otimes \) and satisfying the axioms from Definition 7.24.

**Corollary 7.28.** InqI and InqI\(^\oplus\) are not finitely representable.

**Proof.** Immediate from Theorem 7.26, Theorem 7.27, and Theorem 4.5.

**Question 7.29.** We notice that in the proof of Theorem 7.27 we were in a sense lucky, i.e., we proved the impossibility of strict algebraizability by showcasing two expanded algebras with the same algebraic reduct and different cores. Must such situation always happen whenever a weak logic is loosely algebraizable but not strictly so? More precisely, consider the following properties of a weak logic \( \vdash \):

(i) \( \vdash \) is loosely algebraized by a tuple \((Q, \tau, \Delta)\);
(ii) \( \vdash \) is not strictly algebraizable;
(iii) if \( A \models L = B \models L \) then \( \text{core}(A) = \text{core}(B) \).

Is it possible to find a weak logic \( \vdash \) which satisfies the properties (i)-(iii) from above?

**Question 7.30.** We notice that the argument for the loose algebraizability of InqI and InqI\(^\oplus\) easily generalises to the entire class of intermediate inquisitive
(and dependence) logics from [26]. Now, as mentioned in Fact 7.16, the schematic fragment of \texttt{InqB} is \texttt{ML}. Moreover, as pointed out in [15], it is a corollary of the main result from [17] that the schematic fragment of \texttt{InqI} is \texttt{ML} as well. It thus follows by our Theorem 4.5 that the strictly algebraizable intermediate inquisitive logic are exactly those obtained by closure under modus ponens of sets of the form \texttt{ML∪At(φ)} for some univariate formula \(φ \in \mathcal{L}_{IPC}\). Is it possible to refine this characterisation? For example, can one provide a semantical characterisation of the intermediate inquisitive and dependence logics which are strictly algebraizable? We leave this and the previous question as pointers for future investigations.

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