MAXIMAL COMMUTATIVE SUBALGEBRAS, POISSON GEOMETRY AND HOCHSCHILD HOMOLOGY.

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ABSTRACT. A Poisson geometry arising from maximal commutative subalgebras is studied. A spectral sequence convergent to Hochschild homology with coefficients in a bimodule is presented. It depends on the choice of a maximal commutative subalgebra inducing appropriate filtrations. Its $E^2_{p,q}$-groups are computed in terms of canonical homology with values in a Poisson module defined by a given bimodule and a maximal commutative subalgebra.

1. Introduction. In the paper [11] Jean-Luc Brylinski introduces a spectral sequence convergent to Hochschild homology of an almost commutative algebra (filtered algebra whose associated graded algebra is commutative) and computes (in the case when the associated graded algebra is smooth) its $E^2_{p,q}$-groups in terms of canonical homology of the associated graded algebra. As the main applications serve there envelopping algebras of Lie algebras and algebras of differential operators of commutative algebras.

In this paper we show another general construction of almost commutative algebras. They arise always as an effect of choosing a maximal commutative subalgebra in an arbitrary associative algebra. We study the Poisson geometry related to this construction. It turns out that this geometry describes a nonlinear involutive distribution on the spectrum of the maximal commutative subalgebra. Every bimodule over the associative algebra defines a graded sheaf with a flat connection along this nonlinear involutive distribution. After a slight generalization of the result of Brylinski we construct a spectral sequence convergent to Hochschild homology of this almost commutative algebra with coefficients in an almost symmetric bimodule. We compute its $E^2_{p,q}$-groups in terms of canonical homology of the associated graded Poisson algebra with values in an associated graded Poisson module.

The canonical complex was investigated by Gelfand-Dorfman [2], Koszul [8], Brylinski [3], Huebschmann [6] and Fresse [2] with the relation to Poisson homology. The relations between canonical homology and Poisson geometry were discussed by Vaisman in [12]. The relation between Poisson algebras and Hochschild homology of enveloping algebras was investigated by Kassel [7].

2. The spectral sequence. All rings below are unital and all (bi)modules are unitary. Let $R$ be a noetherian commutative ring of characteristic 0. Unadorned tensor products mean tensor products over $R$. For an increasing filtration $F$, $F_{p-1} \subset F_p$, we denote its associated gradation $Gr$, $Gr = \bigoplus Gr_p$, $Gr_p = F_p/F_{p-1}$. For any graded abelian group $G$ we denote by $G_p$ its $p$-th homogeneous part.

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Definition 1. A \( \mathbb{Z} \)-filtered associative \( R \)-algebra \( A \) (resp. a \( \mathbb{Z} \)-filtered bimodule over \( A \), symmetric over \( R \)), \( F_{p-1}A \subseteq F_pA, \bigcap_{p \in \mathbb{Z}} F_pA = 0, \bigcup_{p \in \mathbb{Z}} F_pA = A \) (resp. \( F_{p-1}M \subseteq F_pM, \bigcap_{p \in \mathbb{Z}} F_pM = 0, \bigcup_{p \in \mathbb{Z}} F_pM = M \)), is called almost commutative (resp. almost symmetric) if its associated graded algebra \( \text{Gr}A \) (resp. associated graded bimodule \( \text{Gr}M \)) is commutative (resp. symmetric). This means that

\[
\begin{align*}
(1) & \quad F_{p_0}A \cdot F_{p_1}A \subseteq F_{p_0+p_1}A, \quad [F_{p_0}A, F_{p_1}A] \subseteq F_{p_0+p_1-1}A \\
(2) & \quad (\text{resp. } F_{p_0}A \cdot F_{p_1}M, F_{p_0}M \cdot F_{p_1}A \subseteq F_{p_0+p_1}A, \quad [F_{p_0}A, F_{p_1}M] \subseteq F_{p_0+p_1-1}M).
\end{align*}
\]

On the Hochschild complex \( C_\bullet(A, M) = \bigoplus_k C_k(A, M) \), \( C_k(A, M) = M \otimes A^\otimes_k \) we have an increasing filtration \( F_p \)

\[
F_pC_k(A, M) = \sum_{p_0 + \cdots + p_k \leq p} F_{p_0}M \otimes F_{p_1}A \otimes \cdots \otimes F_{p_k}A.
\]

This gives rise to a spectral sequence with \( E^1_{p,q} = H_{p+q}(\text{Gr}A, \text{Gr}M)_p \), the homogeneous part of degree \( p \) of the Hochschild homology \( H_{p+q}(\text{Gr}A, \text{Gr}M) \), converging to the Hochschild homology \( H_{p+q}(A, M) \). For \( M = A \) we obtain the spectral sequence of Brylinski as in \( \Box \).

3. Canonical homology.

Definition 2. A commutative graded algebra \( B = \bigoplus_{p \in \mathbb{Z}} B_p, B_p = 0 \) for \( p \ll 0 \) (resp. a symmetric graded bimodule \( N \) over a commutative graded algebra \( B \), \( N = \bigoplus_{p \in \mathbb{Z}} N_p, N_p = 0 \) for \( p \ll 0 \)), is called a Poisson (graded) algebra (resp. Poisson (graded) module) over a (graded) Poisson algebra \( B \) if there is given a Lie algebra structure on \( B \)

\[\{ -, - \} : B \otimes B \rightarrow B \]

(resp. a structure of a right module structure over the Lie algebra \( B; \{ -, - \} \))

\[\{ -, - \} : N \otimes B \rightarrow N,\]

(with

\[\{ B_{p_0}, B_{p_1} \} \subseteq B_{p_0+p_1-1} \quad (\text{resp. } \{ N_{p_0}, B_{p_1} \} \subseteq N_{p_0+p_1-1}),\]

if they are Poisson graded) such that for all \( b_0, b_1, b_2 \in B \)

\[\{ b_0, b_1 b_2 \} = \{ b_0, b_1 \} b_2 + b_1 \{ b_0, b_2 \}\]

(resp. for all \( n \in N, b_1, b_2 \in B \)

\[\{ nb_1, b_2 \} = \{ n, b_2 \} b_1 + n \{ b_1, b_2 \}, \quad \{ n, b_1 b_2 \} = \{ n, b_1 \} b_2 + b_1 \{ n, b_2 \}.\])

Definition 3. Let \( N \) be a Poisson module over a Poisson algebra \( B \). On the graded \( R \)-module \( C^\text{can}_\bullet(B, N) = \bigoplus_k C^\text{can}_k(B, N) \), \( C^\text{can}_k(B, N) = N \otimes_B \Omega^k_B \) one defines \( \Box \) the chain complex structure as follows:

\[\partial : C^\text{can}_k(B, N) \rightarrow C^\text{can}_{k-1}(B, N),\]

\[\partial(n \otimes_B db_1 \cdots db_k) =\]

\[\sum_{i=1}^k (-1)^{i-1} \{ n, b_i \} \otimes_B db_1 \cdots \hat{db}_i \cdots db_k\]
One verifies that the boundary operator $\partial$ is well defined and $\partial^2 = 0$. The homology $H^\text{can}_k(B, N)$ of this complex is called \textit{canonical homology} of the Poisson module $N$ over a Poisson algebra $B$. Note that if $B$ and $N$ are graded Poisson then $\partial$ is homogeneous of degree (-1). Therefore in the Poisson graded case $k$-th canonical chain and homology groups are graded in a canonical way.

4. The Hochschild-Kostant-Rosenberg isomorphism. We will use the simple observation \cite{2} that the Hochschild-Kostant-Rosenberg isomorphism (see \cite{1, 5, 9}) holds also in the case of coefficients in a symmetric bimodule $N$ over a smooth commutative algebra $B$. This means that the map

$$\beta : H_k(B, N) \to N \otimes_B \Omega^k_{B/R},$$

\hspace{1cm} (4)

is an isomorphism, with the inverse $\gamma$:

$$\gamma(n \otimes_B db_1 \cdots db_k) = \left[ \sum_{\sigma \in S_k} \text{sgn}(\sigma) n \otimes b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(k)} \right],$$

\hspace{1cm} (5)

where the square bracket denotes the Hochschild homology class of a cycle.

5. Hochschild and canonical homology. Given an almost commutative algebra $A$ (resp. an almost symmetric bimodule $M$ over $A$), $B = \text{Gr}A$ (resp. $N = \text{Gr}M$) has a canonical structure of a graded Poisson algebra (resp. a graded Poisson module over $B$) with the Poisson structure defined as follows: for $b_0 = a_0 + F_{p_0-1}A$, $a_0 \in F_{p_0}A$, $b_1 = a_1 + F_{p_1-1}A$, $a_1 \in F_{p_1}A$, $n = m + F_{p_0-1}M$, $m \in F_{p_0}M$

$$\{b_0, b_1\} = [a_0, a_1] + F_{p_0+p_1-2}A,$$

$$\{n, b_1\} = [m, a_1] + F_{p_0+p_1-2}M.$$

Moreover we have the following theorem generalizing Theorem 3.1.1 of \cite{1}. In the proof we follow the lines of the beautiful proof of \cite{1}, improving a little misprint in the original proof ($a_i$ instead of $a_0$ in the formula (II) in \cite{1}). At first sight this (spoiled) structure of the formula (II) makes our generalization impossible, but after this minor correction everything can be adapted verbatim.

\textbf{Theorem 1.} Assume that the above filtrations are bounded and exhaustive, and $B = \text{Gr}A$ is smooth over $R$. Then for any $q \geq 0$ the Hochschild-Kostant-Rosenberg isomorphism induces an isomorphism of complexes

$$\beta : (E^1_{p,q}(A, M), d^1_{p,q}) \to (C^\text{can}_{p+q}(B, N)_p, \partial),$$

\hspace{1cm} (6)

where $d^1_{p,q} : E^1_{p,q}(A, M) \to E^1_{p-1,q}(A, M)$ is the differential in the spectral sequence.

In particular

$$E^2_{p,q}(A, M) \cong H^\text{can}_{p+q}(B, N)_p.$$  

\hspace{1cm} (7)

\textit{Proof.} It is enough to prove that $\beta \circ d^1 \circ \gamma = \partial$. Now, the $R$-module $C^\text{can}_k(B, N)_p = (N \otimes_B \Omega^k_{B/R})_p$ is generated by elements of the form $n \otimes_B db_1 \cdots db_k$, with $n \in N$, $d^1_{p,q}$ defined by $d^1_{p,q} = d_{p,q}$. The excision property of the spectral sequence implies that $\beta$ is an isomorphism of all terms of the spectral sequence.
where \( n = m + F_{p_0-1}M, m \in F_{p_0}M, b_i = a_i + F_{p_i-1}A, a_i \in F_{p_i}A, p_0 + \cdots + p_k = p. \)

First, we have

\[
\gamma(n \otimes_B dB_1 \cdots dB_k) = \left[ \sum_{\sigma \in \mathfrak{S}_k} \text{sgn}(\sigma) \ n \otimes b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(k)} \right].
\]

The cycle on the right hand side lives in \( C_k(B, N)_p \) and lifts to the chain

\[
\sum_{\sigma \in \mathfrak{S}_k} \text{sgn}(\sigma) m \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)} \in F_p C_k(A, M).
\]

Its Hochschild boundary is the sum of three terms (I), (II), (III), with

\[
\text{(I)} = \sum_{\sigma \in \mathfrak{S}_k} \text{sgn}(\sigma) \ m a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \cdots \otimes a_{\sigma(k)};
\]

\[
\text{(II)} = \sum_{\sigma \in \mathfrak{S}_k} \sum_{1 \leq i < k} \text{sgn}(\sigma)(-1)^i m \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i)} a_{\sigma(i+1)} \otimes \cdots \otimes a_{\sigma(k)};
\]

\[
\text{(III)} = \sum_{\sigma \in \mathfrak{S}_k} \text{sgn}(\sigma)(-1)^k a_{\sigma(1)} m \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k-1)}.
\]

Since the chain \( \Box \) is a lift of a Hochschild cycle in \( C_k(B, N)_p \) its Hochschild boundary lives in \( F_{p-1} C_{k-1}(A, M) \). Now we are to compute the image of this Hochschild boundary in \( \text{Gr}_{p-1} C_{k-1}(A, M) = C_{k-1}(B, N)_{p-1} \).

First, transforming \( \sigma \in \mathfrak{S}_k \) to \( \sigma \tau \), where \( \tau \) is a cyclic permutation, we can rewrite (I) as follows

\[
\text{(I)} = \sum_{\sigma \in \mathfrak{S}_k} (-1)^{k+1} \text{sgn}(\sigma) m a_{\sigma(k)} \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k-1)}.
\]

Since

\[
\text{(I)} + \text{(III)} = \sum_{\sigma \in \mathfrak{S}_k} (-1)^{k+1} \text{sgn}(\sigma) \ [m, a_{\sigma(k)}] \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k-1)}
\]

we have (I)+(III) \( \in F_{p-1} C_{k-1}(A, M) \); its image in \( \text{Gr}_{p-1} C_{k-1}(A, M) = C_{k-1}(B, N)_{p-1} \) is equal to

\[
\sum_{\sigma \in \mathfrak{S}_k} (-1)^{k+1} \text{sgn}(\sigma) \ \{n, b_{\sigma(k)}\} \otimes b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(k-1)}.
\]

Second, transforming \( \sigma \in \mathfrak{S}_k \) to \( \sigma s_h \), where \( s_h \) is a transposition which exchanges \( h \) and \( (h + 1) \), we can rewrite (II) as follows

\[
\text{(II)} = \frac{1}{2} \sum_{\sigma \in \mathfrak{S}_k} \sum_{1 \leq i < k} \text{sgn}(\sigma)(-1)^h m \otimes a_{\sigma(1)} \otimes \cdots \otimes [a_{\sigma(h)}, a_{\sigma(h+1)}] \otimes \cdots \otimes a_{\sigma(k)}
\]

which also lives in \( F_{p-1} C_{k-1}(A, M) \); its image in \( \text{Gr}_{p-1} C_{k-1}(A, M) = C_{k-1}(B, N)_{p-1} \) is equal to

\[
\frac{1}{2} \sum_{\sigma \in \mathfrak{S}_k} \sum_{1 \leq i < k} \text{sgn}(\sigma)(-1)^h n \otimes b_{\sigma(1)} \otimes \cdots \otimes \{b_{\sigma(h)}, b_{\sigma(h+1)}\} \otimes \cdots \otimes b_{\sigma(k)}.
\]

Adding (III) and (II) we obtain \((d^1 \circ \gamma)(n \otimes_B dB_1 \cdots dB_k)\). It remains to apply \( \beta \) to this.
Now, for the image of the sum (11) under $\beta$ notice that all $\sigma$’s with $\sigma(k) = i$ fixed, give the same value for $\beta(\{n, b_{\sigma(k)}\} \otimes b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(k-1)})$ equal to

$$\frac{1}{(k-1)!}(-1)^{k-i}\{n, b_i\} \otimes_B \widehat{db}_1 \cdots \widehat{db}_i \cdots \widehat{db}_{\sigma(k)}.$$ 

Since there are $(k-1)!$ such permutations, substituting this value, common for each $\sigma$ with $\sigma(k) = i$, to the image of (10) we obtain

$$\sum_{1 \leq i \leq k} (-1)^{i-1}\{n, b_i\} \otimes_B \widehat{db}_i \cdots \widehat{db}_{\sigma(k)}.$$ 

Next, for the image of (10) under $\beta$ notice that all pairs $(\sigma, h)$ with the set $\{\sigma(h), \sigma(h+1)\}$ equal to a fixed set $\{i, j\}$ (say $i < j$), give the same value for $\beta(\text{sgn}(\sigma)(-1)^{i+j} n \otimes_{\sigma(1)} \cdots \otimes \{b_{\sigma(h)}, b_{\sigma(h+1)}\} \otimes \cdots \otimes b_{\sigma(k)})$ equal to

$$\frac{1}{(k-1)!}(-1)^{i+j}n \otimes_B d\{b_i, b_j\} \widehat{db}_i \cdots \widehat{db}_j \cdots \widehat{db}_k.$$ 

Since there are $2(k-1)!$ such pairs, substituting this value, common for each $(\sigma, h)$ with the set $\sigma(h) = i$ equal to a fixed set, to the image of (10) we obtain

$$\sum_{1 \leq i, j \leq k} (-1)^{i+j}n \otimes_B d\{b_i, b_j\} \widehat{db}_i \cdots \widehat{db}_j \cdots \widehat{db}_k.$$ 

Taking the sum of (11) and (12) we obtain

$$(\beta \circ d^1 \circ \gamma)(n \otimes_B \widehat{db}_1 \cdots \widehat{db}_k) = \partial(n \otimes_B \widehat{db}_1 \cdots \widehat{db}_k).$$

6. Maximal commutative subalgebras and almost commutative algebras.

**Definition 4.** Let $C$ be a maximal commutative subalgebra of $A$ and $M$ be a bimodule over $A$. For $c \in C$, $m \in M$ we define an operation $\text{ad}_c(m) := [c, m]$ and an increasing $\mathbb{N}$-filtration on $M$

$$(13) \quad \text{F}^C_p M := \{m \in M \mid \forall c \in C \quad \text{ad}^{p+1}_c(m) = 0\}.$$ 

Since for all $c, c' \in C$ $[\text{ad}_c, \text{ad}_{c'}] = \text{ad}_{[c,c']}$, the multilinear map

$$(14) \quad (c_0, \ldots, c_p) \mapsto \text{ad}_{c_0} \cdots \text{ad}_{c_p}(m)$$

is symmetric in $(c_0, \ldots, c_p)$. Therefore (14) is a symmetric $(p+1)$-linear form in $(c_0, \ldots, c_p)$ corresponding, via the polarization formula, to the homogeneous polynomial of degree $(p+1)$ in $c$

$$(15) \quad c \mapsto \text{ad}^{p+1}_c(m).$$

Therefore the filtration $\text{F}^C$ can be rewritten equivalently as follows

$$(16) \quad \text{F}^C_p M := \{m \in M \mid \forall c_{0}, \ldots, c_{p} \in C \quad \text{ad}_{c_0} \cdots \text{ad}_{c_p}(m) = 0\}.$$ 

We call $D^C A := \bigcup_p \text{F}^C_p A$ (resp. $D^C M := \bigcup_p \text{F}^C_p M$) the **differential hull** of $C$ in $A$ (resp. of the centralizer of $C$ in $M$). Note that $D^C M$ inherits the filtration from $M$ and has the same associated gradation.

The following example justifies the name of the differential hull.
Example 1. Let us take two left $A$-modules $P$ and $Q$, and form the $A$-bimodule $M := \text{Hom}_R(P, Q)$. Then by (16) we have
\[ F_p^C M = \text{Diff}^{C/R}_p(P, Q), \quad \text{Gr}^C_p M = \text{Smbl}^{C/R}_p(P, Q), \]
where $\text{Diff}^{C/R}_p(P, Q)$ (resp. $\text{Smbl}^{C/R}_p(P, Q)$) denotes the $C$-bimodule of differential operators of order $p$ from $P$ to $Q$ (resp. the symmetric $C$-bimodule of their principal symbols).

Theorem 2. Given a maximal commutative subalgebra $C$ of $A$, the above filtration makes $D^CA$ almost commutative and a bimodule $D^CM$ over $D^CA$ almost symmetric.

Proof. Let $m \in F^C_{p_0} M$, $a \in F^C_{p_1} A$.
On the right hand side of the identities
\[ \frac{1}{(p_0 + p_1 + 1)!} \text{ad}^{p_0+p_1+1}_{c}(ma) = \sum_{i+j=p_0+p_1+1} \frac{1}{i!} \text{ad}^{i}_{a}(m) \frac{1}{j!} \text{ad}^{j}_{c}(a) \]
\[ \frac{1}{(p_0 + p_1 + 1)!} \text{ad}^{p_0+p_1+1}_{c}(am) = \sum_{i+j=p_0+p_1+1} \frac{1}{i!} \text{ad}^{i}_{c}(a) \frac{1}{j!} \text{ad}^{j}_{a}(m) \]
at least one of the two factors in every summand is zero, since either $i \geq p_0 + 1$ or $j \geq p_1 + 1$. This proves that
\[ F^C_{p_0} M \cdot F^C_{p_1} A, \quad F^C_{p_1} A \cdot F^C_{p_0} M \subset F^C_{p_0+p_1} M. \]

Observe now that $F^C_0 A = C$, since $C$ is a maximal commutative subalgebra in $A$, and $[F^C_0 M, C] = 0$. Next, if $c \in C$, $m \in F^C_p M$ then
\[ \text{ad}^{p+1-i}_{c}(\text{ad}^{i}_{c}(m)) = \text{ad}^{p+1}_{c}(m) = 0 \]
which means that
\[ \text{ad}^{i}_{c}(F^C_{p} M) \subset F^C_{p-i} M. \]

On the right hand side of the identity
\[ \frac{1}{(p_0 + p_1)!} \text{ad}^{p_0+p_1+1}_{c}([m, a]) = \sum_{i+j=p_0+p_1} \left[ \frac{1}{i!} \text{ad}^{i}_{a}(m), \frac{1}{j!} \text{ad}^{j}_{c}(a) \right] \]
all summands are zero, since the following implications hold:
\[ i < p_0 \Rightarrow j > p_1 \Rightarrow \text{ad}^{j}_{c}(a) = 0, \]
\[ i = p_0 \Rightarrow j = p_1 \Rightarrow \text{ad}^{j}_{c}(m) \in F^C_0 M, \quad \text{ad}^{j}_{c}(a) \in F^C_0 A = C, \]
\[ i > p_0 \Rightarrow \text{ad}^{j}_{c}(m) = 0. \]

This proves that
\[ [F^C_{p_0} M, F^C_{p_1} A] \subset F^C_{p_0+p_1-1} M. \]

Taking $M = A$ in (17) and (18) we see that the filtration $F^C$ makes $D^CA$ almost commutative and, also by (17) and (19), $D^CM$ almost symmetric over $D^CA$. □
7. Poisson geometry of the differential hull. To discuss the Poisson geometry arising from the differential hull we need to generalize the notion of an involutive distribution and a sheaf with a flat connection along an involutive distribution.

Let $C$ be a smooth commutative $R$-algebra and let $\Theta^{C/R} = \text{Hom}_C(\Omega^1_{C/R}, C)$ denote the relative tangent module of $C$ over $R$.

Every $f \in \text{Hom}_C(\Sym^p C, \Omega^1_{C/R}, C) = \Sym^p_C \Theta^{C/R}$ can be regarded as a symmetric $R$-linear $C$-valued $p$-form on $C$, which is a derivation with respect to every linear argument. Using this fact we can define the canonical graded Poisson algebra structure of the $R$-algebra $\Sym_C \Theta^{C/R} = \bigoplus_{p \geq 0} \Sym^p_C \Theta^{C/R}$ of polynomial functions on the relative cotangent bundle of the scheme $\text{Spec}C$ over $R$ as follows: for $f_i \in \text{Hom}_C(\Sym^p C, \Omega^1_{C/R}, C)$, $i = 0, 1$,

$$\begin{align*}
(fof_1)(c_1, \ldots, c_{p_0+p_1}) := \\
&= \frac{1}{(p_0 + p_1)!} \sum_{\sigma \in \Sigma_{p_0+p_1}} f_0(c_{\sigma(1)}, \ldots, c_{\sigma(p_0)}) f_1(c_{\sigma(p_0+1)}, \ldots, c_{\sigma(p_0+p_1)}),
\end{align*}$$

$$\begin{align*}
\{f_0, f_1\}(c_1, \ldots, c_{p_0+p_1-1}) := \\
&= \frac{1}{(p_0 + p_1 - 1)!} \sum_{\sigma \in \Sigma_{p_0+p_1-1}} (p_0 f_0(c_{\sigma(1)}, \ldots, c_{\sigma(p_0-1)}), f_1(c_{\sigma(p_0)}, \ldots, c_{\sigma(p_0+p_1-1)}))
\end{align*}$$

Definition 5. A nonlinear involutive distribution on a scheme $\text{Spec}C$ over $R$ is a graded Poisson subalgebra $B$ of the algebra $\Sym_C \Theta^{C/R}$ of polynomial functions on the relative cotangent bundle of the scheme $\text{Spec}C$ over $R$ such that $B_0 = C$. A graded sheaf with a flat connection along the nonlinear involutive distribution $B$ is a graded Poisson module $N$ over $B$ together with a structure of a graded Poisson module over $B$ on $\bigotimes_N \Sym_C \Theta^{C/R}$ and an embedding of $N$ as a graded Poisson submodule of $\bigotimes_N \Sym_C \Theta^{C/R}$.

Example 2. Let $L \subset \Theta^{C/R} = \text{Der}_R(C, C)$ be a finitely generated projective $C$-submodule which is also an $R$-Lie-subalgebra. It describes an involutive distribution on $\text{Spec}C$ over $R$. The $C$-linear embedding $L \hookrightarrow \Theta^{C/R}$ is equivalent to the grading preserving embedding of graded $C$-algebras

$$\Sym_C L \hookrightarrow \Sym_C \Theta^{C/R}.$$  

The Lie subalgebra structure defined on the image of $L$ a structure of a graded Poisson subalgebra uniquely determined by the following brackets for $\theta \in \Theta^{C/R}$, $c \in C$, $l \in L$

$$\{\theta, c\} = \theta(c), \quad \{\theta, l\} = [\theta, l].$$

This shows that an involutive distribution is an instance of a nonlinear involutive distribution. On the other hand, every nonlinear involutive distribution of the form $\Sym_C L$ for some finitely generated projective $C$-module $L$ defines on $L$ a structure of an involutive distribution. Assume now that $N_0$ is a $C$-module equipped with a flat connection along an involutive distribution $L \subset \Theta^{C/R}$

$$\nabla : N_0 \otimes_R L \to N, \quad n_0 \otimes l \mapsto \nabla l n_0.$$
\[\nabla_c n_0 = c \nabla_l n_0, \quad \nabla_l (n_0 c) = (\nabla_l n_0) c + n_0 l(c), \quad [\nabla_l, \nabla_l] = \nabla_{[l,r]}.
\]

Then the brackets
\[(26) \quad \{n_0 \otimes 1, 1 \otimes c\} = 0, \quad \{n_0 \otimes 1, 1 \otimes l\} = -\nabla_l n_0 \otimes 1\]
determine uniquely structures of graded Poisson modules over \(\text{Sym}_C L\) on \(N := N_0 \otimes_C \text{Sym}_C L\) and \(N_0 \otimes_C \text{Sym}_C \Theta^{C/R}\), and an embedding (of graded Poisson modules over \(\text{Sym}_C L\)) \(N \hookrightarrow N_0 \otimes_C \text{Sym}_C \Theta^{C/R}\). In this way \(N\) becomes a graded sheaf with a flat connection along the nonlinear involutive distribution \(\text{Sym}_C L\).

**Theorem 3.** Let \(C\) be a smooth commutative \(R\)-subalgebra of an associative \(R\)-algebra \(A\) and let \(M\) be an arbitrary \(R\)-symmetric \(A\)-bimodule. Then \(\text{Gr}^C A\) is a nonlinear involutive distribution on a scheme \(\text{Spec} C\) over \(R\) and \(\text{Gr}^C M\) is a graded sheaf with a flat connection along \(\text{Gr}^C A\).

**Proof.** Since an \(A\)-bimodule \(M\) is symmetric as an \(R\)-bimodule and \(C\) is a commutative \(R\)-subalgebra in \(A\) \(M\) is a left module over \(C \otimes_R C\), where \((c_0 \otimes c_1)m := c_0 m c_1\). Let us consider the kernel \(I\) of the multiplication map \(C \otimes_R C \to C\). This is an ideal in \(C \otimes_R C\) generated by elements of the form \(c \otimes 1 - 1 \otimes c\). Since \((c \otimes 1 - 1 \otimes c)m = \text{ad}_c(m)\) the filtration \((10)\) gives rise, for any \(k \leq p\), to the embedding
\[(27) \quad \text{Gr}_p^C M \hookrightarrow \text{Hom}_C(I^k/I^{k+1}, \text{Gr}_{p-k}^C M),
\]
\[m + F_{p-1}^C M \mapsto ((c_1 \otimes 1 - 1 \otimes c_1) \cdots (c_k \otimes 1 - 1 \otimes c_k) + I^{k+1} \mapsto (-1)^k \frac{k!}{k!} \text{ad}_{c_1} \cdots \text{ad}_{c_k}(m) + F_{p-k-1}^C M).
\]

Since for \(C\) smooth over \(R\) one has the isomorphism of symmetric \(C\)-bimodules
\[(28) \quad \text{Sym}_C^k \Omega^1_{C/R} \xrightarrow{\cong} I^k/I^{k+1}
\]
and \(\text{Gr}_0^C M\) is a symmetric \(C\)-bimodule the latter embedding can be rewritten as
\[(29) \quad \text{Gr}_p^C M \hookrightarrow \text{Hom}_C(\text{Sym}_C^k \Omega^1_{C/R}, \text{Gr}_{p-k}^C M) = \text{Gr}_{p-k}^C M \otimes_C \text{Sym}_C^k \Theta^{C/R}.
\]

In particular, since \(\text{Gr}_0^C A = C\), we obtain the embedding
\[(30) \quad \text{Gr}_p^A \hookrightarrow \text{Hom}_C(\text{Sym}_C^p \Omega^1_{C/R}, C) = \text{Sym}_C^p \Theta^{C/R}.
\]

Then the embeddings \((30)\) define a grading preserving embedding of graded Poisson algebras
\[(31) \quad \text{Gr}^C A \hookrightarrow \text{Sym}_C \Theta^{C/R}.
\]

The canonical structure of a graded module over the graded algebra \(\text{Sym}_C \Theta^{C/R}\) on \(\text{Gr}_0^C M \otimes_C \text{Sym}_C \Theta^{C/R}\) is consistent through \((31)\) with the following structure of a graded Poisson module over \(\text{Gr}^C A\) on \(\text{Gr}_0^C M \otimes_C \text{Sym}_C \Theta^{C/R}\) identified with \(\bigoplus_{p \geq 0} \text{Hom}_C(\text{Sym}_C^p \Omega^1_{C/R}, \text{Gr}_0^C M)\): for all \(s_0 \in \text{Hom}_C(\text{Sym}_C^p \Omega^1_{C/R}, \text{Gr}_0^C M)\) and \(f_1 \in \text{Hom}_C(\text{Sym}_C^p \Omega^1_{C/R}, C)\)
\[(32) \quad (s_0 f_1)(c_1, \ldots, c_{p_0+p_1}) := \frac{1}{(p_0 + p_1)!} \sum_{\sigma \in \mathfrak{S}_{p_0+p_1}} s_0(c_{\sigma(1)}, \ldots, c_{\sigma(p_0)}) f_1(c_{\sigma(p_0+1)}, \ldots, c_{\sigma(p_0+p_1)}),
\]
\[(33) \quad \{s_0, b_1\}(c_1, \ldots, c_{p_0+p_1-1}) :=
\]
\[
\frac{1}{(p_0 + p_1 - 1)!} \sum_{\sigma \in S_{p_0+p_1-1}} (p_0 s_0(c_{\sigma(1)}, \ldots, c_{\sigma(p_0)}, b_1(c_{\sigma(p_0)}), \ldots, c_{\sigma(p_0+p_1-1)})) \\
+ p_1 \{ s_0(c_{\sigma(p_1)}, \ldots, c_{\sigma(p_0+p_1-1)}), b_1(c_{\sigma(1)}, \ldots, c_{\sigma(p_1)-1}, -) \},
\]

where \{-, -\} on the right hand side is the bracket \(Gr^C M \otimes_R Gr^C A \to Gr^C M\).

Then the embeddings (29) for \(k = p\) define an embedding of graded Poisson modules over \(Gr^C A\)

\(Gr^C M \hookrightarrow Gr^C M \otimes_C Sym_C Gr^C/R.\)

\(\square\)

There are also interesting examples of non-smooth maximal commutative subalgebras.

**Example 3.** Let \(A = M_n(k)\) be the \((n \times n)\) matrix algebra over an algebraically closed field \(k\) of characteristic 0. Among its many non-isomorphic maximal commutative subalgebras one has the following two extremes:

1) Diagonal subalgebra \(C \cong k^n = k \times \cdots \times k\). It is smooth (semisimple) and the filtration stabilizes at \(C\), hence \(D^C A = C\), \(Gr^C A = C\).

2) Subalgebra \(C \cong k[x]/(x^{n+1})\) generated by the nilpotent \((n \times n)\) Jordan block. Then the filtration is exhaustive, \(D^C A = Diff^r C\), the algebra of differential operators of the commutative \(k\)-algebra \(C\), and \(Gr^C A\) is isomorphic as a graded Poisson algebra to the polynomial algebra of the \(n\)th infinitesimal neighborhood of the nilpotent cone in \(sl_n^2\) with its canonical Kirillov-Kostant-Souriau Poisson structure and the grading such that \(\deg(e) = 0, \deg(h) = 1, \deg(f) = 2\).

**8. Deformation of the differential hull to a nonlinear involutive distribution.** According to the construction of Gerstenhaber we apply a slight generalization of the Rees algebra to our almost commutative algebra \(D^C A\) and an almost symmetric bimodule \(D^C M\).

**Definition 6.** For any bimodule \(M\) over an associative \(R\)-algebra \(A\), and a maximal commutative subalgebra \(C\) of \(A\), we define

\[
\text{Def} M := \sum_p F_p^C M \cdot t^p \subset M[t] = M \otimes_R R[t].
\]

By Theorem 3, according to [4], we have the following corollary.

**Corollary 1.** In the situation of Definition 6 the following holds.

1) \(\text{Def} A\) is a subalgebra in \(A[t]\) and \(\text{Def} M\) is a sub-bimodule of a \(\text{Def} A\)-bimodule \(M[t]\).

2) At \(t = 1\) \(\text{Def} A\) (resp. \(\text{Def} M\)) specializes to the almost commutative algebra \(D^C A\) (resp. to the almost symmetric \(\text{Def} A\)-bimodule \(\text{Def} M\)).

3) At \(t = 0\) \(\text{Def} A\) (resp. \(\text{Def} M\)) specializes to the nonlinear involutive distribution \(Gr^C A\) (resp. to a graded sheaf with a flat connection along \(Gr^C A\)) on the scheme \(\text{Spec} C\) over \(R\).

**9. Hochschild homology of the differential hull.** As a corollary of Theorem 1 and Theorem 2 we obtain the following

**Theorem 4.** Let \(C\) be a maximal commutative subalgebra of an associative \(R\)-algebra \(A\) and \(M\) be a bimodule over \(A\). Then
1) $C$ defines filtrations $F^C$ making $D^C A$ almost commutative and $D^C M$ almost symmetric over $D^C A$, such that $\text{Gr}^C A$ is a graded Poisson algebra and $\text{Gr}^C M$ a graded Poisson module over $\text{Gr}^C A$.

2) These filtrations give rise to a spectral sequence converging to the Hochschild homology

$$E^r_{p,q} \Rightarrow H_{p+q}(D^C A, D^C M).$$

3) If $\text{Gr}^C A$ is smooth over $R$ then

$$E^2_{p,q} \cong H^{con}_{p+q}(\text{Gr}^C A, \text{Gr}^C M)_p.$$ 

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