Maximum $w$-cyclic holey group divisible packings with block size three and applications to optical orthogonal codes

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**Abstract**
In this paper we investigate combinatorial constructions for $w$-cyclic holey group divisible packings with block size three (3-HGDPs). For any positive integers $u, v, w$ with $u \equiv 0, 1 \pmod{3}$, the exact number of base blocks of a maximum $w$-cyclic 3-HGDP of type $(u, w^v)$ is determined. This result is used to determine the exact number of codewords in a maximum three-dimensional $(u \times v \times w, 3, 1)$ optical orthogonal code with at most one optical pulse per spatial plane and per wavelength plane.

**Keywords**
holey group divisible packing, maximum, optical orthogonal code, three-dimensional, $w$-cyclic

**1. Introduction and Preliminaries**

Let $K$ be a set of positive integers. A group divisible packing, denoted by $K$-GDP, is a triple $(X, \mathcal{G}, \mathcal{B})$, where $X$ is a finite set of points, $\mathcal{G}$ is a partition of $X$ into subsets (called groups) and $\mathcal{B}$ is a collection of subsets (called blocks) of $X$, each block has cardinality from $K$, such that any pair of $X$ from two different groups is contained in at most one block. The multiset $T = \{|G| : G \in \mathcal{G}\}$ is called the type of the $K$-GDP. If $\mathcal{G}$ contains $u_i$ groups of size $g_i$ for $1 \leq i \leq r$, we also denote the type by $g_1^{u_1}g_2^{u_2}...g_r^{u_r}$. If $K = \{k\}$, we write a $\{k\}$-GDP as $k$-GDP. If every pair from different groups appears in exactly one block, then the $K$-GDP is referred to as a group divisible design and denoted by $K$-GDD. A $K$-GDD of type $1^v$ is commonly called a pairwise balanced design, denoted by PBD($\nu$, $K$), where $\nu = |X|$.

A sub-GDD $(Y, \mathcal{H}, \mathcal{A})$ of a GDD $(X, \mathcal{G}, \mathcal{B})$ is a GDD satisfying that $Y \subseteq X$, $\mathcal{A} \subseteq \mathcal{B}$, and every group of $\mathcal{H}$ is contained in some group of $\mathcal{G}$. If a GDD has a missing sub-GDD, then we say that the GDD has a hole. (In fact, the sub-GDD as a hole may not exist.) If a GDD has several holes
which partition the point set of the GDD, then we call it a double GDD. Next we generalize the concept of double GDD to a formal definition of double group divisible packing.

Let \( u, v \) be positive integers and \( K \) be a set of positive integers. A double group divisible packing, denoted by \( K\text{-DGDP} \), is a quadruple \((X, \mathcal{G}, \mathcal{H}, B)\) satisfying that

1. \( X \) is a finite set of points;
2. \( \mathcal{G} = \{G_1, G_2, ..., G_u\} \) is a partition of \( X \) into \( u \) subsets (groups);
3. \( \mathcal{H} = \{H_1, H_2, ..., H_v\} \) is another partition of \( X \) into \( v \) subsets (called holes);
4. \( B \) is a collection of subsets (blocks) of \( X \) with cardinality from \( K \), such that each block meets each group and each hole in at most one point, and any two points from different groups and different holes appear together in at most one block.

Let \( G_i \cap H_j = R_{ij} \) for \( 1 \leq i \leq u \) and \( 1 \leq j \leq v \). The \( u \times v \) matrix \( T = ([R_{ij}]) \) is called the type of this design. If \( K = \{k\} \), we write a \( \{k\}\text{-DGDP} \) as \( k\text{-DGDP} \). If every two points from different groups and different holes appear in exactly one block, then the \( k\text{-DGDP} \) is a double group divisible design and denoted by \( K\text{-DGDD} \).

For \( 0 \leq e \leq v - 1 \), let \( c_e = (w, w, ..., w)^T \) be a column vector of length \( u \). A \( K\text{-DGDP} \) of type \((c_0, c_1, ..., c_{v-1})\) is usually called a holey group divisible packing and denoted by \( K\text{-HGDP} \) of type \((u, w^v)\). Note that a \( K\text{-HGDP} \) of type \((u, w^v)\) is equivalent to a \( K\text{-HGDP} \) of type \((v, w^u)\) because we can exchange the expression of groups and holes. For \( 0 \leq e \leq v - 1 \), let \( c_e = (w, w, ..., w, wt)^T \) be a column vector of length \( u + t + 1(0 \leq t < u) \). We call a \( K\text{-DGDP} \) of type \((c_0, c_1, ..., c_{v-1})\) an incomplete holey group divisible packing and denote by \( K\text{-IHGDP} \) of type \((u, t, w^v)\), by which we mean that the group of size \( w^v \) is regarded as the union of \( t \) disjoint parts of equal size \( w \) such that each pair of points from the \( t \) parts does not appear in any block. Especially, if \( t = 0, 1 \), a \( K\text{-IHGDP} \) of type \((u, t, w^v)\) is the same as a \( K\text{-HGDP} \) of type \((u, w^v)\). When \( w = 1 \), a \( K\text{-HGDP} \) of type \((u, 1^v)\) is often said to be a modified group divisible packing and denoted by \( K\text{-MGDP} \) of type \( v^u \).

Naturally we have the notions of a holey group divisible design (HGDD), an incomplete holey group divisible design (IHGDD), and a modified group divisible design (MGDD). Wei [24] first introduced the concept of HGDDs and gave a complete solution to the existence of 3-HGDDs.

**Theorem 1.1** (Wei [24]). There exists a 3-HGDD of type \((u, w^v)\) if and only if \( u, v \geq 3, (u - 1)(v - 1)w \equiv 0 \) (mod 2) and \( uv(u - 1)(v - 1)w^2 \equiv 0 \) (mod 6).

In what follows we always denote \( I_u = \{0, 1, ..., u - 1\} \) to be a set and denote by \( Z_v \) the additive group of integers in \( I_u \) (implying that the arithmetic is taken modulo \( u \)). Suppose that \((X, \mathcal{G}, \mathcal{H}, B)\) is a \( K\text{-DGDP} \) of type \((d_0, d_1, ..., d_{v-1})\), where \( d_e = (w_{g_1}, w_{g_2}, ..., w_{g_u})^T \) for \( 0 \leq e \leq v - 1 \). Let \( \pi \) be a permutation on \( X \). If \( \pi \) keeps \( \mathcal{G}, \mathcal{H}, \) and \( B \) invariant, then \( \pi \) is an automorphism of the \( K\text{-DGDP} \). The block set \( B \) is partitioned into equivalence classes called block orbits under the action of any automorphism \( \pi \). A set of base blocks is an arbitrary set of representatives for these block orbits of \( B \). In particular, if the \( k\text{-DGDP} \) admits an automorphism which is the product of every\( \sum_{i=1}^{u} g_i \) disjoint \( w \)-cycles, fixes every group and every hole, and leaves \( B \) invariant, then the design is said to be \( w \)-cyclic. Without loss of generality, we always construct a \( w \)-cyclic \( k\text{-DGDP} \) on the point set \( X = (\bigcup_{i=1}^{u} R_i) \times I_u \times Z_w \) with group set \( \mathcal{G} = \{G_i \times I_v \times Z_w, 1 \leq i \leq u\} \) and hole set \( \mathcal{H} = \{(I_v \times \bigcup_{i=1}^{u} R_i) \times \{j\} \times Z_w : 0 \leq j \leq v - 1\} \), where \( R_i, 1 \leq i \leq u \), are \( u \) pairwise disjoint sets with \( |R_i| = g_i \). In this case, the automorphism will be taken as \((a, b, c) \mapsto (a, b, c + 1) \mod (-, -, w) \) for \( a \in \bigcup_{i=1}^{u} R_i, b \in I_v \) and \( c \in Z_w \).
For positive integers $u, v, w, k$ with $u, v \geq k$, let $\Psi(u \times v \times w, k, 1)$ denote the largest possible number of base blocks of a $w$-cyclic $k$-HGDP of type $(u, w^v)$. A $w$-cyclic $k$-HGDP of type $(u, w^v)$ is called maximum if it contains $\Psi(u \times v \times w, k, 1)$ base blocks.

Based on the Johnson bound [13] for constant weight codes, Dai et al. [6] displayed an upper bound of $\Psi(u \times v \times w, k, 1)$, as stated in the following lemma.

\textbf{Lemma 1.2} (Dai et al. [6], Theorem 5.2). $\Psi(u \times v \times w, k, 1) \leq J(u \times v \times w, k, 1)$ holds for any positive integers $u, v \geq k$, where

$$J(u \times v \times w, k, 1) = \left\lfloor \frac{uv}{k} \left( \frac{(u-1)(v-1)w}{k-1} \right) \right\rfloor.$$ 

This paper is devoted to research on maximum $w$-cyclic 3-HGDPs. The main theorem is as follows.

\textbf{Theorem 1.3.} For all positive integers $u, v, w$ with $u \equiv 0, 1 \pmod{3}$ ($u, v \geq 3$), we have

$$\Psi(u \times v \times w, 3, 1) = \begin{cases} 6(w-1) & \text{if } (u, v) = (3, 3) \text{ and } w \equiv 0 \pmod{2}, \\ J(u \times v \times w, 3, 1) & \text{otherwise}. \end{cases}$$

In the rest of this section we prepare some preliminary results of close relevance to $k$-HGDPs.

If $\Psi(u \times v \times w, k, 1) = \Psi(u \times v \times w, k, 1)$, then we can check that a $w$-cyclic $k$-HGDP of type $(u, w^v)$ with $\Psi(u \times v \times w, k, 1)$ base blocks is equivalent to a $w$-cyclic $k$-HGDD of type $(u, w^v)$.

\textbf{Theorem 1.4} (Dai et al. [6], Theorem 4.11). There exists a $w$-cyclic 3-HGDD of type $(u, w^v)$ if and only if $u, v \geq 3, (u-1)(v-1)w \equiv 0 \pmod{2}$ and $uv(u-1)(v-1)w \equiv 0 \pmod{6}$ with the exceptions of $u = v = 3$ and $w \equiv 0 \pmod{2}$.

Combining Lemma 1.2 and Theorem 1.4 yields the following corollary.

\textbf{Corollary 1.5.} $\Psi(u \times v \times w, 3, 1) = J(u \times v \times w, 3, 1)$ if any of the following conditions is satisfied: (1) $u \equiv 1, 3 \pmod{6}, u, v \geq 3$ with the exceptions of $u = v = 3$ and $w \equiv 0 \pmod{2}$; (2) $u \equiv 0, 4 \pmod{6}, v \equiv 1 \pmod{2}$ and $w \equiv 1 \pmod{2}$; (3) $u \equiv 0, 4 \pmod{6}, v \geq 4$ and $w \equiv 0 \pmod{2}$.

Suppose that $(X, \mathcal{G}, \mathcal{H}, B)$ is a $K$-HGDD of type $(u, w^v)$. It is said to be semyclic, briefly by $K$-SCHGDD of type $(u, w^v)$, if it admits an automorphism which is the product of $u$ disjoint $wv$-cycles, fixes every group, leaves $\mathcal{G}$ and $\mathcal{H}$ invariant. Without loss of generality, for a $K$-SCHGDD of type $(u, w^v)$, we always identify $X$ with $I_u \times \mathbb{Z}_{wv}$, $\mathcal{G}$ with $\{[i] \times \mathbb{Z}_{wv} : i \in I_u\}$ and $\mathcal{H}$ with $\{I_u \times [i, v+i, \ldots, (w-1)v+i] : 0 \leq i \leq v-1\}$. In this case, the automorphism will be taken as $(i, x) \mapsto (i, x + 1) \mod(-, wv)$ for $i \in I_u$ and $x \in \mathbb{Z}_{wv}$. The existence of a $K$-SCHGDD of type $(u, w^v)$ implies that of a $w$-cyclic $k$-HGDD of type $(u, w^v)$, as indicated by the following result.

\textbf{Lemma 1.6} (Wang and Chang [21], Construction 4.1). If there exists a $K$-SCHGDD of type $(u, (gw)^v)$, then there exists a $w$-cyclic $k$-HGDD of type $(u, (gw)^v)$.
For $k = 3$, we record the existence results of 3-SCHGDDs which were studied by [7,8,20].

**Theorem 1.7** (Wang et al. [20], Theorem 1.4). There exists a 3-SCHGDD of type $(u, w^o)$ if and only if $u, v ≥ 3, (u − 1)(v − 1)w ≡ 0 \pmod{2}$ and $u(u − 1)(v − 1)w ≡ 0 \pmod{6}$ except when (1) $u ≡ 3, 7 \pmod{12}, w ≡ 1 \pmod{2}$ and $v ≡ 2 \pmod{4}$; (2) $u = 3, w ≡ 1 \pmod{2}$ and $v ≡ 0 \pmod{2}$; (3) $u = v = 3$ and $w ≡ 0 \pmod{2}$; (4) $(u, w, v) ∈ \{(5, 1, 4), (6, 1, 3)\}$; (5) $u ≡ 11 \pmod{12}$, either $w ≡ 3 \pmod{6}$ and $v ≡ 2 \pmod{4}$, or $w ≡ 1, 5 \pmod{6}$ and $v ≡ 10 \pmod{12}$.

In this paper, we concentrate on combinatorial constructions for maximum $w$-cyclic 3-HGDPs of type $(u, w^o)$. Our main purpose is to determine the value of $Ψ(u × v × w, 3, 1)$ for all positive integers $u, v, w$ where $u ≡ 0, 1 \pmod{3}$ ($u, v ≥ 3$). The rest of the paper is organized as follows. In Section 2, we present a few recursive constructions for $w$-cyclic $k$-DGDPs and show some special applications to produce maximum $w$-cyclic 3-HGDPs. In Section 3, we shall determine the number of base blocks of maximum $w$-cyclic 3-HGDPs of type $(u, w^o)$ when $u ≡ 0 \pmod{3}$. In Section 4, we construct two types of auxiliary designs, namely incomplete HGDPs and semicyclic $K$-GDDs. In Section 5 we pay attention to the case $u ≡ 1 \pmod{3}$ and finally prove the main theorem. The existence of maximum $w$-cyclic 3-HGDPs of type $(u, w^o)$ not only is interesting in its own right, but also has nice applications in three-dimensional optical orthogonal codes (OOC) with certain restrictions. We consider this application in Section 6. We give brief concluding remarks in Section 7. Note that a full version of the paper is available at https://arxiv.org/abs/2006.06921, where we provide an appendix on many direct constructions for small designs which will be utilized in some of the subsequent proofs.

# 2 RECURSIVE CONSTRUCTIONS

In this section, some recursive constructions for $w$-cyclic $k$-DGDPs will be given. Then several special constructions yielding maximum $w$-cyclic 3-HGDPs will be displayed.

**Construction 2.1.** Suppose that the following designs exist:

1. a $w$-cyclic $k$-DGDP of type $(d_0, d_1, ..., d_{v−1})$ with $b$ base blocks where $d_i = (wm_1, wm_2, ..., wm_s)^T$ for $0 ≤ e ≤ v − 1$;
2. a $w$-cyclic $k$-IHGDP of type $(m_i + t, t, w^o)$ with $c_i$ base blocks for each $1 ≤ i ≤ s − 1$.

Then there exists a $w$-cyclic $k$-IHGDP of type $(\sum_{i=1}^s m_i + t, t, w^o)$ with $b + \sum_{i=1}^{s−1} c_i$ base blocks. Furthermore, if there exists a $w$-cyclic $k$-IHGDP of type $(m_s + t, t, w^o)$ with $c_s$ base blocks, then there exists a $w$-cyclic $k$-IHGDP of type $(\sum_{i=1}^s m_i + t, t, w^o)$ with $b + \sum_{i=1}^s c_i$ base blocks.

**Proof.** Let $R_i, 1 ≤ i ≤ s$, be $s$ pairwise disjoint sets with $|R_i| = m_i$. By assumption we have a $w$-cyclic $k$-DGDP $(X, G, H, F)$ of type $(d_0, d_1, ..., d_{v−1})$ with $b$ base blocks, where $X = \left(\bigcup_{i=1}^s R_i\right) × I_v × Z_w, G = \{R_i × I_v × Z_w : 1 ≤ i ≤ s\}$ and $H = \{(\bigcup_{i=1}^s R_i) × \{j\} × Z_w : j ∈ I_v\}$. Let $F$ be the collection of all base blocks of this DGDP.

Let $E$ be a set of size $t$ such that $E × I_v × Z_w$ is disjoint from $X$. For each group $R_i × I_v × Z_w, 1 ≤ i ≤ s − 1$, construct a $w$-cyclic $k$-IHGDP of type $(m_i + t, t, w^o)$ with $c_i$
base blocks on \((R_i \cup E) \times I_v \times Z_w\) with group set \([\{x\} \times I_v \times Z_w : x \in R_i]\) \cup \([E \times I_v \times Z_w]\) and hole set \([\{R_i \cup E\} \times \{j\} \times Z_w : j \in I_v]\). Denote by \(A_i\) the collection of all base blocks of this design. Let \(A = (\bigcup_{i=1}^{s-1} A_i) \cup \mathcal{F}\). It is readily checked that \(A\) forms \(b + \sum_{i=1}^{s-1} c_i\) base blocks of a \(w\)-cyclic \(k\)-IHGDP of type \((\sum_{i=1}^{s} m_i + t, m_s + t, w^v)\) on \(((\bigcup_{i=1}^{s-1} R_i) \cup E) \times I_v \times Z_w\) with group set \([\{x\} \times I_v \times Z_w : x \in \bigcup_{i=1}^{s-1} R_i] \cup ((R_s \cup E) \times I_v \times Z_w)\) and hole set \([((\bigcup_{i=1}^{s} R_i) \cup E) \times \{j\} \times Z_w : j \in I_v]\).

Furthermore, construct a \(w\)-cyclic \(k\)-IHGDP of type \((m_s + t, t, w^v)\) with \(c_i\) base blocks on \((R_s \cup E) \times I_v \times Z_w\) with group set \([\{x\} \times I_v \times Z_w : x \in R_s]\) \cup \([E \times I_v \times Z_w]\) and hole set \([\{R_s \cup E\} \times \{j\} \times Z_w : j \in I_v]\). Denote \(A_s\) the collection of all base blocks of this design. Let \(C = (\bigcup_{i=1}^{s} A_i) \cup \mathcal{F}\). It is readily checked that \(C\) forms \(b + \sum_{i=1}^{s} c_i\) base blocks of a \(w\)-cyclic \(k\)-IHGDP of type \((\sum_{i=1}^{s} m_i + t, t, w^v)\).

Sometimes we do not consider if a design admits an automorphism group. So we list a corollary to Construction 2.1 as follows.

**Corollary 2.2.** Suppose that the following designs exist:

1. a \(k\)-DGDP of type \((d_0, d_1, \ldots, d_{v-1})\) with \(b\) blocks where \(d_e = (w^{m_1}, w^{m_2}, \ldots, w^{m_e})^T\) for \(0 \leq e \leq v - 1\);
2. a \(k\)-IHGDP of type \((m_i + t, t, w^v)\) with \(c_i\) blocks for each \(1 \leq i \leq s - 1\).

Then there exists a \(k\)-IHGDP of type \((\sum_{i=1}^{s} m_i + t, m_s + t, w^v)\) with \(b + \sum_{i=1}^{s-1} c_i\) blocks.

Furthermore, if there exists a \(k\)-IHGDP of type \((m_s + t, t, w^v)\) with \(c_s\) blocks, then there exists a \(k\)-IHGDP of type \((\sum_{i=1}^{s} m_i + t, t, w^v)\) with \(b + \sum_{i=1}^{s} c_i\) blocks.

Suppose that \((X, \mathcal{G}, \mathcal{B})\) is a \(K\)-GDP of type \([w^{m_1}, w^{m_2}, \ldots, w^{m_r}]\). If there is a permutation \(\pi\) on \(X\) that is the product of \(\sum_{i=1}^{r'} m_i\) disjoint \(w\)-cycles, fixes every group and leaves \(\mathcal{B}\) invariant, then the \(K\)-GDP is said to be \(w\)-cyclic. A \(w\)-cyclic \(K\)-GDP of type \(w^u\) is called semicircular and denoted by \(K\)-SCGDP of type \(w^u\). Without loss of generality, we may construct a \(K\)-SCGDP on the point set \(X = (\bigcup_{i=1}^{r'} R_i) \times Z_w\) with group set \(\mathcal{G} = \{R_i \times Z_w : 1 \leq i \leq r\}\), where \(|R_i| = m_i\). And the automorphism will be taken as \((a, b) \mapsto (a, b + 1) \mod(w, w)\) for \(a \in \bigcup_{i=1}^{r'} R_i\) and \(b \in Z_w\).

In some cases a semicircular \(K\)-GDD can be obtained from a semicircular \(L\)-HGDD; we show this by an obvious lemma.

**Lemma 2.3.** Let \(K\) be a set of positive integers and \(u \notin K\). A \(K\)-SCHGDD of type \((u, 1^v)\) can be viewed as a \(K \cup \{u\}\)-SCGDD of type \(v^u\) with one base block of size \(u\).

**Construction 2.4.** Suppose that there exists a \(w\)-cyclic \(K\)-GDD of type \([w^{m_1}, w^{m_2}, \ldots, w^{m_r}]\) with \(b_k\) base blocks for each \(k \in K\). If there exists an \(h\)-cyclic \(l\)-HGDP of type \((k, h^u)\) with \(c_k\) base blocks for each \(k \in K\), then there exists an \(hw\)-cyclic \(l\)-DGDP of type \((d_0, d_1, \ldots, d_{v-1})\) with \(\sum_{k \in K} b_k c_k\) base blocks, where \(d_e = (h^{m_1}, h^{m_2}, \ldots, h^{m_r})^T\) for \(0 \leq e \leq v - 1\).

**Proof.** Start from the given \(w\)-cyclic \(K\)-GDD of type \([w^{m_1}, w^{m_2}, \ldots, w^{m_r}]\) defined on \(X = (\bigcup_{i=1}^{r'} R_i) \times Z_w\) with group set \([R_i \times Z_w : 1 \leq i \leq r]\), where \(|R_i| = m_i\). Let \(\mathcal{F}\) be the collection of the base blocks.
For each block \( B \in \mathcal{F} \) and \( |B| = k \), construct an \( h \)-cyclic \( l \)-HGDP of type \((k, h^v)\) with \( c_k \) base blocks on \( Y = B \times I_v \times Z_w \) with group set \( \{ [x] \times I_v \times Z_w : x \in B \} \) and hole set \( \{ B \times [j] \times Z_h : j \in I_l \} \). Denote the family of base blocks by \( \mathcal{A}_B \). Let \( \mathcal{A} = \bigcup_{B \in \mathcal{F}} \mathcal{A}_B \).

For each \((a, b, c, d) \in Y\), define a mapping \( \tau : (a, b, c, d) \mapsto (a, c, b + wd) \). Then define \( \tau(A) = \{ \tau(x) : x \in A \} \) and \( \mathcal{C} = \bigcup_{A \in \mathcal{A}} \tau(A) \). It is readily checked that \( \mathcal{C} \) forms all base blocks of an \( hw \)-cyclic \( l \)-DGDP of type \((d_0, d_1, \ldots, d_{v-1})\) on \( (\bigcup_{i=1}^r R_i) \times I_v \times Z_{hw} \) with group set \( \{ R_i \times I_v \times Z_{hw} : 1 \leq i \leq r \} \) and hole set \( \{ (\bigcup_{i=1}^r R_i) \times [j] \times Z_{hw} : j \in I_l \} \). It is immediate that the number of base blocks equals \( \sum_{k \in K} b_k c_k \).

**Corollary 2.5.** Suppose that there exists a \( K \)-SCGDD of type \((w^v)\) with \( b_k \) base blocks for each \( k \in K \). If there exists a \( h \)-cyclic \( l \)-HGDP of type \((u, h^k)\) with \( c_k \) base blocks for each \( k \in K \), then there exists an \( hw \)-cyclic \( l \)-HGDP of type \((u, (hw)^v)\) with \( \sum_{k \in K} b_k c_k \) base blocks.

**Construction 2.6.** Suppose that the following designs exist:

1. a \( K \)-HGDD of type \((u, w^v)\) which has \( b_k \) blocks of size \( k \) for each \( k \in K \);
2. an \( l \)-SCGDP of type \( w^k \) with \( e_k \) base blocks of size \( k \) for each \( k \in K \).

Then there exists a \( w \)-cyclic \( l \)-HGDP of type \((u, w^v)\) with \( \sum_{k \in K} b_k e_k \) base blocks.

**Proof.** Start from the given \( K \)-HGDD of type \((u, w^v)\) defined on \( X = I_u \times I_v \) with group set \( \mathcal{G} = \{ [i] \times I_v : i \in I_u \} \) and hole set \( \mathcal{H} = \{ I_u \times [j] : j \in I_l \} \). Denote the family of blocks by \( \mathcal{F} \). For each \( B \in \mathcal{F} \) and \( |B| = k \), construct on \( B \times Z_w \) an \( l \)-SCGDP of type \( w^k \) with \( e_k \) base blocks and with group set \( \{ [x] \times Z_w : x \in B \} \). Let \( \mathcal{A}_B \) be the family of base blocks of this design. Define \( \mathcal{A} = \bigcup_{B \in \mathcal{F}} \mathcal{A}_B \). It is readily checked that \( \mathcal{A} \) forms \( \sum_{k \in K} b_k e_k \) base blocks of a \( w \)-cyclic \( l \)-HGDP of type \((u, w^v)\) on \( I_u \times I_v \times Z_w \) with group set \( \mathcal{G} = \{ [i] \times I_v \times Z_w : i \in I_u \} \) and hole set \( \mathcal{H} = \{ I_u \times [j] \times Z_w : j \in I_l \} \).

**Construction 2.7.** Let \( c_e = (g_1, g_2, \ldots, g_n)^T \) for \( 0 \leq e \leq vw - 1 \) and \( d_f = (w_g, w_g, \ldots, w_g)^T \) for \( 0 \leq f \leq v - 1 \). If there exist a \( k \)-DGDD of type \((d_0, d_1, \ldots, d_{v-1})\) with \( b \) blocks and a \( k \)-DGDP of type \((c_0, c_1, \ldots, c_{w-1})\) with \( c \) blocks, then there exists a \( k \)-DGDP of type \((c_0, c_1, \ldots, c_{w-1})\) with \( b + cv \) blocks.

**Proof.** Let \( R_i \) for \( 1 \leq i \leq n \), be \( n \) pairwise disjoint sets with \( |R_i| = g_i \). By assumption, we may construct a \( k \)-DGDD \((X, \mathcal{G}, \mathcal{H}, B)\) of type \((d_0, d_1, \ldots, d_{v-1})\) with \( b \) blocks, where \( X = (\bigcup_{i=1}^n R_i) \times I_v \times I_w \), \( \mathcal{G} = \{ R_i \times I_v \times I_w : 1 \leq i \leq n \} \), and \( \mathcal{H} = \{ (\bigcup_{i=1}^n R_i) \times [j] \times I_w : j \in I_l \} \).

For each hole \( H = (\bigcup_{i=1}^n R_i) \times [j] \times I_w \), \( 0 \leq j \leq v - 1 \), we construct a \( k \)-DGDP of type \((c_0, c_1, \ldots, c_{w-1})\) with \( c \) blocks on \( H \) with group set \( \{ R_i \times [j] \times I_w : 1 \leq i \leq n \} \) and hole set \( \{ (\bigcup_{i=1}^n R_i) \times [j] \times I_w : l \in I_w \} \). Denote the set of all blocks by \( \mathcal{A}_H \). Let \( \mathcal{A} = \bigcup_{H \in \mathcal{H}} \mathcal{A}_H \). It is easily checked that \( \mathcal{A} \) forms \( b + cv \) blocks of the desired design on \( (\bigcup_{i=1}^n R_i) \times I_w \) with group set \( \{ R_i \times I_{vw} : 1 \leq i \leq n \} \) and hole set \( \{ (\bigcup_{i=1}^n R_i) \times [j] : j \in I_{vw} \} \).

If we apply the previous recursive constructions and choose appropriate ingredient designs, then we produce maximum \( w \)-cyclic 3-HGDPS. To end this section we show three special cases of constructions to produce 3-HGDPS of type \((u, w^v)\) with \( J(u \times v \times w, 3, 1) \) base blocks.
Lemma 2.8. Let $u, v, g$ be even, $w$ be odd, $g \equiv 0 \pmod 3$ and $v \equiv s \pmod g$. Suppose that a $w$-cyclic 3-DGDD of type $(d_0, d_1, \ldots, d_{u-1})$ exists, where $d_e = (gw, gw, \ldots, gw, sw)^T$ is a column vector of length $\frac{v-s+g}{6}$ for $0 \leq e \leq u - 1$. If there exists a $w$-cyclic 3-HGDP of type $(k, w^u)$ with $J(k \times u \times w, 3, 1)$ base blocks for each $k \in [g, s]$, then there exists a $w$-cyclic 3-HGDP of type $(v, w^u)$ with $J(v \times u \times w, 3, 1)$ base blocks.

Proof. By assumption, there exists a $w$-cyclic 3-DGDD of type $(d_0, d_1, \ldots, d_{u-1})$, which has $\frac{uv(u-1)(v-s)(v+s-g)}{6}$ base blocks. Apply Construction 2.1 with a $w$-cyclic 3-HGDP of type $(k, w^u)$ with $J(k \times u \times w, 3, 1)$ base blocks for $k \in [g, s]$ to produce a $w$-cyclic 3-HGDP of type $(v, w^u)$. It has

$$\frac{uv(u-1)(v-s)(v+s-g)}{6} + \frac{v-s}{g} \cdot \frac{gu(g-1)(u-1)w-gu}{6} + \frac{su(s-1)(u-1)w-su}{6} \cdot \frac{vu((v-1)(u-1)w-1)}{6} = J(v \times u \times w, 3, 1)$$

base blocks.

Lemma 2.9. Let $u, v$ be even, $mg$ be odd and $u(u-1)m \equiv 0 \pmod 3$. If there exist

1. a $\{3, 5, v\}$-SCGDD of type $g^v$ containing one base block of size $v$; and
2. an $m$-cyclic 3-HGDP of type $(u, m^v)$ with $J(u \times v \times m, 3, 1)$ base blocks.

Then there exists an $mg$-cyclic 3-HGDP of type $(u, (mg)^v)$ with $J(u \times v \times mg, 3, 1)$ base blocks.

Proof. By assumption, there exists a $\{3, 5, v\}$-SCGDD of type $g^v$ with one base block of size $v$. Let the SCGDD have $a$ base blocks of size 3 and $b$ base blocks of size 5. It is immediate that $3a + 10b = \frac{v(v-1)(g-1)}{2}$. Apply Corollary 2.5 with an $m$-cyclic 3-HGDP of type $(u, m^v)$ with $J(u \times v \times m, 3, 1)$ base blocks and an $m$-cyclic 3-HGDD of type $(u, 5^v)$ for each $k \in \{3, 5\}$ which exists by Theorem 1.4. Then we obtain an $mg$-cyclic 3-HGDP of type $(u, (mg)^v)$. It has

$$\left[ \frac{uvm(u-1)(v-1) - uv}{6} \right] + a \times mu(u-1) + b \times \frac{10mu(u-1)}{3} = J(u \times v \times mg, 3, 1)$$

base blocks.

Lemma 2.10. Let $u, v, g$ be even, $w$ be odd, $ug \equiv 0 \pmod 3$ and $v \equiv t \pmod g$. Suppose that there exists a 3-GDD of type $g^{\frac{v-t}{g}}$. If there exists a $w$-cyclic 3-HGDP of type $(u, w^v)$ with $J(u \times k \times w, 3, 1)$ base blocks for each $k \in [g, t]$, then there exists a $w$-cyclic 3-HGDP of type $(u, w^v)$ with $J(u \times v \times w, 3, 1)$ base blocks.

Proof. By assumption, there exists a 3-GDD of type $g^{\frac{v-t}{g}}$ with $\frac{(v-t)(v+t-g)}{6}$ blocks, which is also a $\{3, g, t\}$-SCGDD of type $1^v$ containing $\frac{(v-t)(v+t-g)}{6}$ blocks of size $3, \frac{v-t}{g}$

...
blocks of size $g$ and one block of size $t$. Apply Corollary 2.5 with a $w$-cyclic 3-HGDP of type $(u, w^k)$ with $J(u \times k \times w, 3, 1)$ base blocks for $k \in \{g, t\}$ and a $w$-cyclic 3-HGDD of type $(u, w^3)$ which exists from Theorem 1.4. Then we obtain a $w$-cyclic 3-HGDP of type $(u, w^h)$. The number of base blocks equals
\[
\frac{(v-t)(v+t-g)}{6}u(u-1)w + \frac{v-t}{g} \times \frac{ugw(u-1)(g-1) - ug}{6}
\]
\[+ \left[ \frac{ut((u-1)(t-1)w-1)}{6} \right] \]
\[= \left[ \frac{uw((u-1)(v-1)w-1)}{6} \right] = J(u \times v \times w, 3, 1). \]

\section{CASE $u \equiv 0 \pmod{3}$}

In this section, we shall determine the number of base blocks of a maximum $w$-cyclic 3-HGDP of type $(u, w^h)$ when $u \equiv 0 \pmod{3}$.

\textbf{Lemma 3.1} (Gallant et al. [9], Theorem 2). A 3-SCGDD of type $m^n$ exists if and only if $n \geq 3$ and (1) $n \equiv 1, 3 \pmod{6}$ when $m \equiv 1, 5 \pmod{6}$; (2) $n \equiv 1 \pmod{2}$ when $m \equiv 3 \pmod{6}$; (3) $n \equiv 0, 1, 4, 9 \pmod{12}$ when $m \equiv 2, 10 \pmod{12}$; (4) $n \equiv 0, 1 \pmod{3}$ when $m \equiv 3 \pmod{12}$; (5) $n \equiv 3$ when $m \equiv 0 \pmod{12}$; (6) $n \equiv 0, 1 \pmod{4}$ when $m \equiv 6 \pmod{12}$.

\textbf{Lemma 3.2.} $\Psi(3 \times 3 \times w, 3, 1) \leq 6(w-1)$ for $w \equiv 0 \pmod{2}$.

\textbf{Proof.} Let $X = I_3 \times I_3 \times Z_w$, $G = \{i \times I_3 \times Z_w : i \in I_3\}$ and $H = \{I_3 \times j \times Z_w : j \in I_3\}$. Suppose that $(X, G, H, B)$ is a $w$-cyclic 3-HGDP of type $(3, w^3)$ with at least $6w - 5$ base blocks. Obviously, its base blocks can be partitioned into the following six possible types:

1. $\{(0, 0, *), (1, 1, *), (2, 2, *)\}$,
2. $\{(0, 0, *), (1, 2, *), (2, 1, *)\}$,
3. $\{(0, 1, *), (1, 0, *), (2, 2, *)\}$,
4. $\{(0, 1, *), (1, 2, *), (2, 0, *)\}$,
5. $\{(0, 2, *), (1, 0, *), (2, 1, *)\}$,
6. $\{(0, 2, *), (1, 1, *), (2, 0, *)\}$.

Using the Pigeonhole principle guarantees that there exists a type containing at least $w$ base blocks. However, it is easy to check that every type contains at most $w$ base blocks since the base blocks of this type omitting the first coordinate produce a 3-SCGDP of type $w^3$. As a result, we have a type containing exactly $w$ base blocks which form a 3-SCGDD of type $w^3$. But Lemma 3.1 shows that there is no 3-SCGDD of type $w^3$ for any $w \equiv 0 \pmod{2}$. A contradiction occurs. The conclusion then follows.

\textbf{Lemma 3.3.} $\Psi(3 \times 3 \times w, 3, 1) = 6(w-1)$ for $w \equiv 0 \pmod{2}$.

\textbf{Proof.} By Theorem 1.1, there exists a 3-HGDD of type $(3, 1^3)$ with six blocks. Apply Construction 2.6 with a 3-SCGDP of type $w^3$ with $w-1$ base blocks which exists from
[21, 22, Theorem 5.18] to obtain a \(w\)-cyclic 3-HGDP of type \((3, w^3)\) with \(6(w - 1)\) base blocks. So the conclusion follows by Lemma 3.2.

**Lemma 3.4.** There exists a 3-HGDP of type \((4, 1^6)\) with \(J(4 \times 6 \times 1, 3, 1)\) blocks.

**Proof.** We construct the desired design on \(X = Z_4 \times I_6\) with group set \(G = \{[i] \times I_6: i \in Z_4\}\) and hole set \(H = \{Z_4 \times [j]: j \in I_6\}\). All 56 blocks can be obtained by developing the following 14 base blocks by \((+1, -) \mod (4, -)\) successively.

\[
\begin{align*}
&\{(0, 0), (1, 2), (2, 1)\}, \{(0, 0), (1, 3), (2, 2)\}, \{(0, 0), (1, 4), (3, 1)\}, \{(0, 0), (1, 5), (3, 2)\}, \\
&\{(0, 0), (2, 3), (3, 4)\}, \{(0, 0), (2, 4), (3, 5)\}, \{(0, 0), (2, 5), (3, 3)\}, \{(0, 1), (1, 2), (3, 4)\}, \\
&\{(0, 1), (1, 3), (2, 5)\}, \{(0, 1), (1, 4), (2, 3)\}, \{(0, 1), (1, 5), (3, 3)\}, \{(0, 1), (2, 2), (3, 5)\}, \\
&\{(0, 2), (1, 3), (3, 4)\}, \{(0, 2), (1, 4), (3, 5)\}.
\end{align*}
\]

\(\square\)

**Lemma 3.5.** Let \(w \equiv 1 \mod 2\). Then there exists a \(w\)-cyclic 3-HGDP of type \((u_1^w, v)\) with \(J(\times \times v w_1^w, 3, 1)\) base blocks for \(u_1 \in \{6, 12\}\) and \(v \in \{4, 6, 10\}\), \((2)(u, v) \in \{(8, 12), (14, 12)\}\).

**Proof.** When \((u, v, w) = (6, 6, 3)\), the desired design can be obtained from Example B.8 in the appendix directly.

For other parameters, by Theorem 1.7 there exists a 3-SCHGDD of type \((v, 1^w)\) which can be viewed as a \([3, v]\)-SCGDD of type \(w^v\) with \(\frac{v(v - 1)(w - 1)}{6}\) base blocks of size 3 and one base block of size \(v\) by Lemma 2.3. From Lemma 3.4, there exists a 3-HGDP of type \((4, 1^6)\), also of type \((6, 1^4)\), with \((J(6 \times 6 \times 1, 3, 1))\) blocks. For the rest parameters, Lemmas B.1-B.5 in the appendix provide 3-HGDPs of type \((u_1^w, v)\) with \(J(\times \times v w_1^w, 3, 1)\) base blocks. Apply Lemma 2.9 by taking parameters \((g, v, u, m) = (w, v, u, 1)\) to yield a \(w\)-cyclic 3-HGDP of type \((u_1^w, v)\) with \(J(\times \times v w, 3, 1)\) base blocks.

\(\square\)

**Lemma 3.6.** Let \(w \equiv 1 \mod 2\). There exists a \(w\)-cyclic 3-HGDP of type \((6, v^w)\) with \(J(6 \times v \times w, 3, 1)\) base blocks for \(v \in \{8, 14\}\).

**Proof.** For \(v \in \{8, 14\}\), by Lemma A.1 in the appendix, there exists a \([3, 6]\)-HGDD of type \((6, 1^v)\) with \(5v(v - 2)\) blocks of size 3 and \(v\) blocks of size 6, where the blocks of size 6 form a parallel class. Apply Construction 2.6 with a 3-SCGDD of type \(w^v\) with \(w\) base blocks which exists from Lemma 3.1 and a 3-SCGDP of type \(w^6\) with \(5w - 1\) base blocks to obtain a \(w\)-cyclic 3-HGDP of type \((6, v^w)\) with \(w \times 5v(v - 2) + v \times (5w - 1) = J(\times \times v w, 3, 1)\) base blocks, where the required 3-SCGDP of type \(w^6\) can be obtained from [21, Lemma 5.1].

\(\square\)

**Theorem 3.7** (Colbourn et al. [5], Main Theorem). Let \(g, t,\) and \(w\) be positive integers. Then there exists a 3-GDD of type \(g^tw^1\) if and only if all the following conditions are satisfied:

1. \(t \geq 3\), or \(t = 2\) with \(w = g;\)
2. \(w \leq g(t - 1);\)
3. \(g^2(t - 1) + w \equiv 0 \mod (2);\)
4. \(g(t - 1) + w \equiv 0 \mod (2);\)
5. \(g^2(t - 1) + 2gtw \equiv 0 \mod (6).\)

**Lemma 3.8.** Let \(v \in \{6, 12\}, v \equiv 0 \mod (2)(v \geq 4)\) and \(w \equiv 1 \mod 2\). Then there exists a \(w\)-cyclic 3-HGDP of type \((u, w^v)\) with \(J(\times \times v w, 3, 1)\) base blocks.
Proof. For \( v \in \{4, 6, 8, 10, 14\} \), the desired designs exist from Lemmas 3.5 and 3.6. Note that a 3-HGDP of type \((u, v)\) is also that of type \((w, i^u)\).

Let \( g = 4 \) if \( v \equiv 0 \pmod{4} \) \((v \geq 12)\) and let \( g = 6 \) if \( v \equiv 6 \pmod{12} \) \((v \geq 18)\). From Theorem 1.7 and Lemma 1.6 there exists a \( w \)-cyclic 3-HGDD of type \((u, (gw)^v)\), also of type \( \left(\frac{v}{g}, (gw)^v\right) \), for \( u \in \{6, 12\} \) and odd \( w \). By Lemma 3.5 there exists a \( w \)-cyclic 3-HGDP of type \((u, w^8)\), also of type \( w^v\) \((u)\)

Let \( g = 4 \) if \( vv \equiv 0 \pmod{4} \) \((12) \equiv \geq \) and let \( g = 6 \) if \( vv \equiv 6 \pmod{12} \) \((18) \equiv \geq \). From Theorem 1.7 and Lemma 1.6 there exists a \( w \)-cyclic 3-HGDD of type \((u, (gw)^v)\), also of type \( \left(\frac{v}{g}, (gw)^v\right) \), for \( u \in \{6, 12\} \) and odd \( w \). By Lemma 3.5 there exists a \( w \)-cyclic 3-HGDP of type \((u, w^8)\), also of type \( w^v\) \((u)\)

Let \( g = 4 \) if \( vv \equiv 0 \pmod{6} \) \((v \equiv 2 \pmod{12} \) \((v \geq 26)\) and let \( g = 6 \) if \( vv \equiv 10 \pmod{12} \) \((v \geq 22)\). By Theorem 3.7 there exists a 3-GDD of type \( g^\pm t^1\). For \( u \in \{6, 12\} \) and odd \( w \), apply Lemma 2.10 to obtain a \( w \)-cyclic 3-HGDP of type \((u, w^i)\) with \( J(u \times v \times w, 3, 1) \) base blocks, where the required maximum \( w \)-cyclic 3-HGDPs of types \((u, w^8)\) and \((u, w^i)\) exist by the previous arguments.

Lemma 3.9.

\[
\Psi(u \times v \times w, 3, 1) = \begin{cases} 
6(w - 1) & \text{if } (u, v) = (3, 3) \text{ and } w \equiv 0 \pmod{2}, \\
J(u \times v \times w, 3, 1) & \text{if } (u, v) = (3, 3) \text{ and } w \equiv 1 \pmod{2}, \\
J(u \times v \times w, 3, 1) & \text{if } u \equiv 0 \pmod{3}, u, v \geq 3, (u, v) \neq (3, 3) \text{ and } w \geq 1.
\end{cases}
\]

Proof. The assertion holds for \((u, v) = (3, 3)\) and \(w \equiv 0 \pmod{2}\) by Lemma 3.3. Then, by Corollary 1.5, we only need to handle the case that \(u \equiv 0 \pmod{6}\), \(v \equiv 0 \pmod{2}\), and \(w \equiv 1 \pmod{2}\). Lemma 1.2 shows \(\Psi(u \times v \times w, 3, 1) \leq J(u \times v \times w, 3, 1)\).

For \(u \in \{6, 12\}\), by Lemma 3.8 we have \(\Psi(u \times v \times w, 3, 1) = J(u \times v \times w, 3, 1)\).

For \(u \equiv 0 \pmod{6}\) and \(u \geq 18\), by Theorem 1.7, there exists a 3-SCHGDD of type \(\left(\frac{u}{6}, (6w)^v\right)\), which is also a \( w \)-cyclic 3-HGDD of type \(\left(\frac{u}{6}, (6w)^v\right)\) by Lemma 1.6. There exists a \( w \)-cyclic 3-HGDD of type \((6, w^i)\) by Lemma 3.8. Apply Lemma 2.8 by taking parameters \((g, s, v, u, w) = (6, 6, u, v, w)\) to obtain a \( w \)-cyclic 3-HGDD of type \((u, w^i)\) with \( J(u \times v \times w, 3, 1) \) base blocks. So we have \(\Psi(u \times v \times w, 3, 1) = J(u \times v \times w, 3, 1)\).

4 | AUXILIARY DESIGNS

In this section, we study two types of auxiliary designs, namely incomplete HGDPs and semicyclic \(K\)-GDDs, which will play important roles in constructing \( w \)-cyclic 3-HGDPs of type \((u, w^i)\) when \(u \equiv 1 \pmod{3}\).

4.1 | Incomplete holey group divisible packings

In this subsection we focus on incomplete 3-HGDPs of type \((u, t, w^i)\) which contain the largest possible number of blocks. Let \(u, v, t\) be even. It is not difficult to check that the number of blocks of a 3-IHGDP of type \((u, t, w^i)\) does not exceed
\[ \Theta(u, t, v, w) = \begin{cases} \frac{vw^2(u - t)(u + t - 1)(v - 1)}{6} & \text{if } w \text{ is even}, \\ \frac{vw(u - t)((u + t - 1)(v - 1)w - 1)}{6} & \text{if } w \text{ is odd}. \end{cases} \] (1)

Furthermore, if \( w \) is even, then a 3-IHGDP of type \( (u, t, w^v) \) with \( \Theta(u, t, v, w) \) blocks is actually a 3-IHGDD. In Section 5 we shall employ 3-IHGDPs of type \( (u, t, 1^v) \) with \( \Theta(u, t, v, 1) \) blocks to produce maximum 3-HGDPs of type \( (u, 1^v) \), which is \( w \)-cyclic with \( w = 1 \).

**Lemma 4.1.** Let \( u, v, t \) be even. If there exist a 3-IHGDP of type \( (u, t, 1^v) \) with \( \Theta(u, t, v, 1) \) blocks and a 3-HGD of type \( (t, 1^v) \) with \( J(t \times v \times 1, 3, 1) \) blocks, then there exists a 3-IHGDP of type \( (u, 1^v) \) with \( J(u \times v \times 1, 3, 1) \) blocks.

**Proof.** By assumption, there exists a 3-IHGDP of type \( (u, t, 1^v) \) with \( \Theta(u, t, v, 1) \) blocks. Apply Corollary 2.2 with a 3-HGD of type \( (t, 1^v) \) with \( J(t \times v \times 1, 3, 1) \) blocks to obtain a 3-IHGDP of type \( (u, 1^v) \). It has \( (u - t)\left(\frac{uv^2(v - 1)}{6} + \frac{tv((t - 1)(v - 1) - 1)}{6}\right) = J(u \times v \times 1, 3, 1) \) blocks. \( \square \)

**Lemma 4.2.** Let \( m, t, v \) be even. Suppose that there exists a 3-IHGDD of type \( (u, (mw)^v) \).

1. Let \( w \) be odd. If there exists a 3-IHGDP of type \( (m + t, t, w^v) \) with \( \Theta(m + t, t, v, w) \) blocks, then there exist a 3-IHGDP of type \( (um + t, m + t, w^v) \) with \( \Theta(um + t, m + t, v, w) \) blocks and a 3-IHGDP of type \( (um + t, m + t, w^v) \) with \( \Theta(um + t, t, v, w) \) blocks.

2. Let \( w \) be even. If there exists a 3-IHGDD of type \( (m + t, t, w^v) \), then there exist a 3-IHGDD of type \( (um + t, m + t, w^v) \) and a 3-IHGDD of type \( (um + t, t, w^v) \).

**Proof.** By assumption, there exists a 3-HGD of type \( (u, (mw)^v) \) with \( \frac{uv^2(v - 1)}{6}m^2w^2 \) blocks. Apply Corollary 2.2 with a 3-IHGDP of type \( (m + t, t, w^v) \) which has \( \Theta(m + t, t, v, w) \) blocks to obtain a 3-IHGDP of type \( (um + t, m + t, w^v) \). It is easy to check from Equation (1) that the number of blocks equals \( \frac{uv^2(v - 1)m^2w^2}{6} + (u - 1) \times \Theta(m + t, t, v, w) = \Theta(um + t, m + t, v, w) \).

Also, we further obtain a 3-IHGDP of type \( (um + t, t, w^v) \) with \( \Theta(um + t, t, v, w) \) blocks. In the case that \( w \) is even, we can readily check that the produced 3-IHGDPs become 3-IHGDDs. This completes the proof. \( \square \)

**Lemma 4.3.** Let \( u, t, v \) be even. If there exist a 3-IHGDD of type \( (u, t, w^v) \) and a 3-IHGDP of type \( (u, t, 1^w) \) with \( \Theta(u, t, v, 1) \) blocks, then there exists a 3-IHGDP of type \( (u, t, 1^w) \) with \( \Theta(u, t, v, 1) \) blocks.
Proof. Since there exists a 3-IHGDD of type \((u, t, w^v)\) with \(\frac{vw^2(u-t)(u+t-1)(v-1)}{6} + v \times \frac{w(u-t)((u+t-1)(w-1)-1)}{6}\) blocks, apply Construction 2.7 with a 3-IHGDP of type \((u, t, 1^w)\) with \(\Theta(u, t, w, 1)\) blocks to obtain a 3-IHGDP of type \((u, t, 1^w)\). It has

\[
\frac{vw^2(u-t)(u+t-1)(v-1)}{6} + v \times \frac{w(u-t)((u+t-1)(w-1)-1)}{6} = \Theta(u, t, vw, 1)
\]

blocks. □

Now we construct two small examples of 3-IHGDPs of type \((u, t, w^v)\) with \(\Theta(u, t, v, w)\) blocks.

**Example 4.4.** We construct a 3-IHGDD of type \((8, 2, 4^v)\) with \(\Theta(8, 2, 4, 4)\) blocks on \(Z_6 \times Z_4 \times Z_4\) with group set \(\{(i) \times Z_4 \times Z_4 : i \in Z_6\} \cup \{(a, b) \times Z_4 \times Z_4\}\) and hole set \(\{(Z_6 \cup \{a, b\}) \times \{j\} \times Z_4 : j \in Z_4\}\). All 1728 blocks are obtained by developing the following 36 base blocks by \((-1, +1, +1) \mod (6, 4, 4)\) successively, where \(a + 2 = a\) and \(b + 2 = b\).

\[
\begin{align*}
\{(0, 0, 0), (2, 1, 0), (a, 2, 0)\}, & \quad \{(0, 0, 0), (a, 1, 1), (1, 2, 1)\}, & \quad \{(0, 0, 0), (1, 3, 0), (a, 2, 1)\}, \\
\{(0, 0, 0), (1, 2, 0), (a, 1, 2)\}, & \quad \{(0, 0, 0), (1, 1, 0), (a, 2, 3)\}, & \quad \{(0, 0, 0), (3, 2, 0), (a, 1, 3)\}, \\
\{(0, 0, 0), (3, 1, 0), (a, 2, 2)\}, & \quad \{(0, 0, 0), (5, 1, 0), (a, 3, 1)\}, & \quad \{(0, 0, 0), (1, 1, 2), (a, 3, 2)\}, \\
\{(0, 0, 0), (a, 3, 0), (1, 1, 1)\}, & \quad \{(0, 0, 0), (1, 2, 3), (a, 3, 3)\}, & \quad \{(1, 0, 0), (3, 1, 1), (a, 2, 2)\}, \\
\{(0, 0, 0), (4, 1, 0), (5, 3, 2)\}, & \quad \{(0, 0, 0), (b, 1, 0), (5, 3, 3)\}, & \quad \{(0, 0, 0), (2, 2, 0), (4, 3, 1)\}, \\
\{(0, 0, 0), (5, 2, 0), (2, 1, 2)\}, & \quad \{(0, 0, 0), (b, 2, 0), (5, 3, 0)\}, & \quad \{(0, 0, 0), (3, 1, 1), (5, 3, 1)\}, \\
\{(0, 0, 0), (2, 2, 1), (3, 1, 3)\}, & \quad \{(0, 0, 0), (5, 2, 1), (1, 3, 1)\}, & \quad \{(0, 0, 0), (3, 1, 2), (1, 1, 3)\}, \\
\{(1, 0, 0), (b, 1, 0), (5, 3, 1)\}, & \quad \{(0, 0, 0), (4, 1, 1), (b, 3, 2)\}, & \quad \{(0, 0, 0), (3, 3, 0), (4, 1, 2)\}, \\
\{(0, 0, 0), (3, 3, 1), (b, 2, 3)\}, & \quad \{(0, 0, 0), (2, 3, 1), (5, 2, 3)\}, & \quad \{(0, 0, 0), (b, 2, 2), (5, 1, 3)\}, \\
\{(0, 0, 0), (4, 2, 1), (b, 1, 2)\}, & \quad \{(0, 0, 0), (3, 2, 2), (b, 1, 3)\}, & \quad \{(0, 0, 0), (b, 3, 0), (3, 2, 3)\}, \\
\{(0, 0, 0), (5, 1, 2), (1, 3, 3)\}, & \quad \{(1, 0, 0), (b, 2, 0), (5, 1, 2)\}, & \quad \{(1, 0, 0), (5, 1, 1), (b, 3, 3)\}, \\
\{(0, 0, 0), (5, 1, 1), (3, 3, 3)\}, & \quad \{(0, 0, 0), (b, 1, 1), (4, 2, 2)\}, & \quad \{(1, 0, 0), (5, 1, 0), (3, 2, 3)\}.
\end{align*}
\]

**Example 4.5.** We construct a 3-IHGDP of type \((8, 2, 1^v)\) with \(\Theta(8, 2, 4, 1)\) blocks on \(X = I_8 \times Z_4\) with group set \(G = \{i\} \times Z_4 \times Z_4 \cup \{6, 7 \times Z_4\}\) and hole set \(H = \{i\} \times \{j\} : j \in Z_4\). All 104 blocks are obtained by developing the following 26 base blocks by \((-1, +1, +1) \mod (6, 4, 4)\) successively.

\[
\begin{align*}
\{(0, 0), (6, 1), (1, 2)\}, & \quad \{(0, 0), (6, 2), (2, 3)\}, & \quad \{(0, 0), (4, 2), (6, 3)\}, & \quad \{(1, 0), (6, 1), (3, 3)\}, \\
\{(2, 0), (6, 1), (5, 3)\}, & \quad \{(1, 0), (6, 2), (5, 3)\}, & \quad \{(2, 0), (6, 2), (4, 3)\}, & \quad \{(3, 0), (6, 1), (4, 3)\}, \\
\{(3, 0), (5, 2), (6, 3)\}, & \quad \{(1, 0), (3, 1), (2, 3)\}, & \quad \{(0, 0), (2, 1), (1, 3)\}, & \quad \{(1, 0), (2, 1), (3, 2)\}, \\
\{(1, 0), (7, 2), (4, 3)\}, & \quad \{(2, 0), (7, 1), (3, 3)\}, & \quad \{(2, 0), (4, 1), (7, 2)\}, & \quad \{(2, 0), (5, 1), (4, 2)\}, \\
\{(0, 0), (7, 1), (2, 2)\}, & \quad \{(0, 0), (1, 1), (4, 3)\}, & \quad \{(1, 0), (5, 1), (7, 3)\}, & \quad \{(1, 0), (7, 1), (5, 2)\}, \\
\{(3, 0), (4, 2), (5, 3)\}, & \quad \{(0, 0), (5, 1), (7, 2)\}, & \quad \{(0, 0), (3, 2), (7, 3)\}, & \quad \{(0, 0), (4, 1), (5, 3)\}, \\
\{(0, 0), (3, 1), (5, 2)\}, & \quad \{(3, 0), (4, 1), (7, 3)\}.
\end{align*}
\]

**Lemma 4.6.** There exists a 3-IHGDP of type \((20, 8, 1^v)\) with \(\Theta(20, 8, 4, 1)\) blocks.

Proof. There exist a 3-HGDD of type \((3, 6^v)\) and a 3-IHGDP of type \((8, 2, 1^v)\) from Theorem 1.1 and Example 4.5, respectively. Apply Lemma 4.2 by taking parameters
Lemma 4.7. Let \((x, y, z) \in \{(10, 4, 16), (16, 4, 16), (20, 8, 16), (10, 4, 20)\}\). Then there exists a 3-IHGDP of type \((x, y, 1^w)\) with \(\Theta(x, y, z, 1)\) blocks.

Proof. By Theorem 1.1, there exist 3-HGDDs of types \((4, 4^4)\) and \((4, 4^5)\) which are also 3-IHGDDs of types \((4, 1, 4^4)\) and \((4, 1, 4^5)\), respectively. There exists a 3-IHGDD of type \((8, 2, 4^4)\) from Example 4.4. Apply Lemma 4.2 by taking parameters \((u, v, w, m, t)\) in the second column of Table 1. Then we obtain a 3-IHGDD of type \((x, y, w^v)\), where \(x = um + t, y = m + t\) and \(z = vw\). The needed 3-IHGDDs of type \((x, y, w)\) exist from Theorem 1.1. Furthermore, 3-IHGDPs of type \((x, y, 1^w)\) with \(\Theta(x, y, w, 1)\) blocks exist, for which we list the sources in the third column of Table 1 (see Lemma C.1 in the appendix). Then by Lemma 4.3, we obtain the desired 3-IHGDPs of type \((x, y, 1^w)\).

4.2 | K-SCGDDs

In this subsection we establish the existence of a class of \{3, 5, \}-SCGDDs of type \(w^v\), which will be applied in Lemma 2.9 to produce maximum \(w\)-cyclic 3-HGDPs. Note that Lemma 2.3 shows the relationship between \(K\)-SCHGDDs and \(L\)-SCGDDs, so we form \{3, 5, \}-SCGDDs of type \(w^v\) by constructing \{3, 5\}-SCHGDDs of type \((v, 1^w)\).

Cyclotomic cosets play an important role in direct constructions for SCHGDDs. Let \(p \equiv 1 \pmod{n}\) be a prime and \(w\) be a primitive element of \(\mathbb{Z}_p\). Let \(C_0^n\) denote the multiplicative subgroup \(\{w^i : 0 \leq i < (p - 1)/n\}\) of the \(n^{th}\) powers in \(\mathbb{Z}_p\) and \(C_j^n\) denote the coset of \(C_0^n\) in \(\mathbb{Z}_p\backslash\{0\}\), that is, \(C_j^n = w^j \cdot C_0^n, 0 \leq j \leq n - 1\).

Lemma 4.8 (Feng et al. [7], Lemma 6.8). Let \(p \geq 5\) be a prime. There exists an element \(x \in \mathbb{Z}_p\) such that \(x \in C_1^2, x + 1 \in C_1^2,\) and \(x - 1 \in C_0^2\).

Lemma 4.9. There exists a \{3, 5\}-SCHGDD of type \((8, 1^w)\) for any prime \(p \geq 5\).

Proof. Let \(\omega\) be a primitive element of \(\mathbb{Z}_p\) where \(p \geq 5\). The desired design will be constructed on \(X = \mathbb{Z}_8 \times \mathbb{Z}_p\) with group set \(G = \{(i) \times Z_p : i \in I_8\}\) and hole set \(H = \{I_8 \times \{j\} : j \in Z_p\}\). By Lemma 4.8, we may take \(x \in Z_p\) such that \(x \in C_1^2, x + 1 \in C_1^2,\) and \(x - 1 \in C_0^2\).

When \(p \equiv 1 \pmod{4}\), we construct twelve initial base blocks of size 3 and two initial base blocks of size 5. All base blocks are generated by multiplying the second coordinate by 3.

| \((x, y, z)\) | \((u, v, w, m, t)\) | \((x, y, 1^w)\) |
|---|---|---|
| \((10, 4, 16)\) | \((3, 4, 4, 3, 1)\) | Lemma C.1 |
| \((16, 4, 16)\) | \((5, 4, 4, 3, 1)\) | Lemma C.1 |
| \((20, 8, 16)\) | \((3, 4, 4, 6, 2)\) | Lemma 4.6 |
| \((10, 4, 20)\) | \((3, 5, 4, 3, 1)\) | Lemma C.1 |
of these initial base blocks by $\omega^{2r}$, $0 \leq r \leq (p - 3)/2$. The initial base blocks are listed in four cases.

(1) $2, x + 2 \in C_0^2$.

$$\{(0, 0), (1, x), (2, 1)\}, \quad \{(0, 0), (2, x), (5, x + 1)\}, \quad \{(1, 0), (3, x - 1), (7, x)\}$$

$$\{(4, 0), (1, 1), (7, x)\}, \quad \{(2, 0), (6, x), (7, x + 1)\}, \quad \{(3, 0), (5, x), (7, x + 1)\}$$

$$\{(0, 0), (3, 1), (5, x)\}, \quad \{(4, 0), (5, x), (6, x + 1)\}, \quad \{(1, 0), (5, 1), (6, x + 1)\}$$

$$\{(0, 0), (6, -1), (7, x)\}, \quad \{(3, 0), (2, x), (6, x + 1)\}, \quad \{(4, 0), (2, x), (3, x + 1)\}$$

$$\{(0, -x), (4, 0), (5, 1 - x), (2, 1), (7, 2)\}, \quad \{(0, 0), (1, 1), (3, x + 1), (4, x + 2), (6, x)\}.$$

(2) $2, x + 2 \in C_i^2$.

$$\{(3, 1), (5, x), (7, 0)\}, \quad \{(1, 0), (5, 1), (6, x + 1)\}, \quad \{(3, 0), (2, x), (6, x + 1)\}$$

$$\{(4, 0), (5, x), (6, 1)\}, \quad \{(4, 1 - x), (2, 0), (3, 1)\}, \quad \{(1, 0), (2, x), (5, x + 1)\}$$

$$\{(2, 0), (6, x), (7, 1)\}, \quad \{(0, 0), (1, x), (2, x + 1)\}, \quad \{(0, 0), (3, 1 - x), (5, 1)\}$$

$$\{(0, 0), (6, -1), (7, x - 1)\}, \quad \{(1, 0), (3, x - 1), (7, -1)\}, \quad \{(4, 0), (1, 1), (7, 1 - x)\}$$

$$\{(0, 0), (1, 1), (3, x + 1), (4, x + 2), (6, x)\}, \quad \{(0, 1), (4, 0), (5, 1 - x), (2, 2), (7, -x)\}.$$

(3) $2 \in C_i^2, x + 2 \in C_0^2$.

$$\{(1, 0), (5, 1), (6, x + 1)\}, \quad \{(3, 1), (5, x), (7, 0)\}, \quad \{(3, 0), (2, x), (6, x + 1)\}$$

$$\{(4, 0), (1, 1), (7, 1 - x)\}, \quad \{(4, 0), (5, x), (6, 1)\}, \quad \{(4, 1 + x), (2, 0), (3, 1)\}$$

$$\{(1, 0), (2, x), (5, x + 1)\}, \quad \{(2, 0), (6, x), (7, 1)\}, \quad \{(1, 0), (3, x - 1), (7, -1)\}$$

$$\{(0, 0), (1, x), (2, x + 1)\}, \quad \{(0, 0), (3, 1 - x), (5, 1)\}, \quad \{(0, 0), (6, -1), (7, -x - 1)\}$$

$$\{(0, 0), (1, 1), (3, x + 1), (4, x + 2), (6, x)\}, \quad \{(0, 1), (4, 0), (5, 1 - x), (2, 2), (7, -x)\}.$$

(4) $2 \in C_0^2, x + 2 \in C_i^2$.

$$\{(0, 0), (1, x), (2, x + 1)\}, \quad \{(0, 0), (3, 1), (5, 2)\}, \quad \{(0, 0), (6, -1), (7, x - 1)\}$$

$$\{(1, 0), (2, x), (5, x + 1)\}, \quad \{(2, 0), (6, x), (7, 1)\}, \quad \{(1, 0), (3, x - 1), (7, -1)\}$$

$$\{(1, 0), (5, 1), (6, x + 1)\}, \quad \{(3, 1), (5, x), (7, 0)\}, \quad \{(3, 0), (2, x), (6, x + 1)\}$$

$$\{(4, 0), (1, 1), (7, 1 - x)\}, \quad \{(4, 0), (2, x), (3, x + 1)\}, \quad \{(4, 0), (5, x), (6, x + 1)\}$$

$$\{(0, 0), (1, 1), (3, x + 1), (4, x + 2), (6, x)\}, \quad \{(0, 1), (4, 0), (5, 1 - x), (2, 2), (7, -x)\}.$$

When $p \equiv 3 \pmod{4}$, we construct six initial base blocks of size 3 and one initial base block of size 5. All base blocks can be obtained by successively developing these initial base blocks by $(4, -) \pmod{8, -}$ and multiplying the second coordinate by $\omega^{2r}$, $0 \leq r \leq (p - 3)/2$. We list the initial base blocks by considering two cases.

Case 1. $x + 2 \neq p$. Four subcases are distinguished.

(1) $-2 \in C_1^2, x + 2 \in C_0^2$.

$$\{(0, 0), (1, x), (2, x - 1)\}, \quad \{(0, 0), (3, x), (5, -1)\}, \quad \{(1, 0), (2, 1), (5, x)\}$$

$$\{(1, 0), (3, -x - 1), (7, -x)\}, \quad \{(2, 0), (3, x), (6, x + 1)\}, \quad \{(0, 0), (6, 1), (7, 1 - x)\}$$

$$\{(0, 0), (1, 1), (3, x + 2), (6, x), (4, x + 1)\}.$$

(2) $-2, x + 2 \in C_1^2$.

$$\{(0, 0), (1, x - 1), (2, x + 1)\}, \quad \{(0, 0), (3, x), (5, x + 1)\}, \quad \{(0, 0), (6, -x - 1), (7, 1)\}$$

$$\{(1, 0), (2, x + 1), (5, x - 1)\}, \quad \{(1, 0), (3, -1), (7, x - 1)\}, \quad \{(2, 0), (3, x - 1), (6, -1)\}$$

$$\{(0, 0), (1, x + 2), (3, 1), (6, x + 1), (4, x)\}.$$
\[(3) \quad -2 \in C_0^2, x + 2 \in C_1^2.\]
\[
\{(1, 0), (2, 1), (5, x)\}, \quad \{(0, 0), (3, x - 1), (5, -1)\}, \quad \{(0, 0), (6, 1), (7, 1 - x)\},
\{(0, 0), (1, x), (2, x - 1)\}, \quad \{(1, 0), (3, -x - 1), (7, -x)\}, \quad \{(2, 0), (3, x), (6, x - 1)\};
\{(0, 0), (1, 1), (3, x + 2), (6, x), (4, x + 1)\}.
\]

\[(4) \quad -2, x + 2 \in C_0^2.\]
\[
\{(0, 0), (1, x), (2, 1)\}, \quad \{(0, 0), (3, -1), (6, x)\}, \quad \{(1, 0), (2, x + 2), (5, x + 1)\},
\{(1, 0), (3, 1), (7, -x)\}, \quad \{(0, 0), (3, x + 2), (5, 1)\}, \quad \{(0, 0), (6, x), (7, x + 1)\};
\{(0, 0), (1, 1), (3, x + 1), (6, x + 2), (4, x)\}.
\]

Case 2. \(x + 2 = p\). It yields that \(-2 \equiv x \pmod{p}\) and \(-2 \in C_1^2\). We consider two subcases.

(1) \(x - 2 \in C_1^2.\)
\[
\{(0, 0), (1, 1 - x), (2, -1)\}, \quad \{(0, 0), (3, x), (5, 1)\}, \quad \{(0, 0), (6, x - 1), (7, 1)\},
\{(1, 0), (2, x - 1), (5, x + 1)\}, \quad \{(1, 0), (3, -1), (7, x)\}, \quad \{(2, 0), (3, x), (6, x + 1)\};
\{(0, 0), (1, x - 1), (3, 1), (6, x + 1), (4, x)\}.
\]

(2) \(x - 2 \in C_0^2.\)
\[
\{(1, 0), (2, 1), (5, x)\}, \quad \{(0, 0), (3, x), (5, -1)\}, \quad \{(0, 0), (6, 1), (7, x - 1)\},
\{(1, 0), (3, x), (7, 1)\}, \quad \{(0, 0), (1, x), (2, x - 1)\}, \quad \{(2, 0), (3, x), (6, x - 1)\};
\{(0, 0), (1, 1), (3, x - 1), (6, x), (4, x + 1)\}.
\]

Semicyclic HGDDs are closely related to cyclic holey difference matrices (CHDMs). A \((k, w; t)\)-CHDM is a \(k \times w(t - 1)\) matrix \(D = (d_{ij})\) with entries from \(Z_{wt}\) such that for any two distinct rows \(x\) and \(y\), the difference list \(L_{xy} = \{d_{i,j} - d_{i',j} : j \in I_{w(t-1)}\}\) contains each integer of \(Z_{wt} \setminus S\) exactly once, where \(S = \{0, t, ..., (w - 1)t\}\) is a subgroup of order \(w\) in \(Z_{wt}\). A \((k, t)\)-CDM (cyclic difference matrix) is a \(k \times t\) matrix \(D = (d_{ij})\) with entries from \(Z_t\) such that for any two distinct rows \(x\) and \(y\), the difference list \(\{d_{i,j} - d_{i',j} : j \in I_t\}\) contains each integer of \(Z_t\) exactly once. Note that a \((k, t; 1)\)-CHDM is equivalent to a \((k, t)\)-CDM.

**Lemma 4.10** (Wang and Yin [23] and Yin [25]). A \(k\)-SCGDD of type \(m^k\) is equivalent to a \((k, m)\)-CDM. And a \(k\)-SCHGDD of type \((k, w')\) is equivalent to a \((k, wt; w')\)-CHDM.

**Lemma 4.11** (Abel and Ge [1], Theorem 4.2). A \((5, g)\)-CDM exists for any integer \(g\) which satisfies any of the following conditions: (1) \(\gcd(g, 6) = 1\), (2) \(g \in \{15, 27, 39, 51\}\).

**Lemma 4.12.** Let \(v \equiv 1, 5 \pmod{6}\) and \(v = 3^i\) for \(i \geq 3\). Then there exists a \((5, v; 1)\)-CHDM.

**Proof.** When \(v \equiv 1, 5 \pmod{6}\), there exists a \((5, v)\)-CDM by [4, Theorem 2.1]. When \(v = 3^i\) and \(i \geq 3\), by [10, Lemma 3.10] there exists a \((5, 3^i)\)-CDM. Thus the conclusion follows. \(\square\)
The following constructions are slight generalizations of several constructions in [7]. Here we omit their proofs and refer the reader to parallel proofs in [7]. A K-GDP of type \(w^u\) is said to be cyclic, if it admits an automorphism forming a cycle of length \(w^u\). If the length of each orbit in the cyclic K-GDP of type \(w^u\) is \(w^u\), then the K-GDP is called strictly cyclic.

**Construction 4.13** (Feng et al. [7], Construction 3.1). If there exist a K-SCHGDD of type \((n, (gw)^{t})\) and a K-SCHGDD of type \((n, g^{w^u})\), then there exists a K-SCHGDD of type \((n, g^{w^u})\).

**Construction 4.14** (Feng et al. [7], Construction 3.4). If there exist a K-SCHGDD of type \((n, w^t)\) and a \((k, v)\)-CDM for each \(k \in K\), then there exists a K-SCHGDD of type \((n, (wv)^{t})\).

**Construction 4.15** (Feng et al. [7], Construction 3.2). Suppose that there exist a K-SCGDD of type \(g^u\) and an L-SCHGDD of type \((k, w^t)\) for each \(k \in K\). Then there exists an L-SCHGDD of type \((n, (wv)^{t})\).

**Construction 4.16** (Feng et al. [7], Construction 3.3). If there exist a strictly cyclic k-GDD of type \(w^u\) and a K-MGDD of type \(k^u\), then there exists a K-SCHGDD of type \((u, w^u)\).

**Theorem 4.17** (Abel et al. [2], Hanani [12]). For any \(v \equiv 2 \pmod{6}\) and \(v \geq 14\), there exists a PBD \((v, \{3, 4, 5\})\).

**Lemma 4.18.** Let \(w \equiv 1, 5 \pmod{6}\) and \(w \geq 5\). Then there exists a \(\{3, 5\}\)-SCHGDD of type \((v, 1^{w^u})\) for \(v \in \{8, 14, 20\}\).

**Proof.** First we let \(v = 8\). Let \(w = p_1^{a_1}p_2^{a_2}...p_i^{a_i}\) be its prime factorization, where \(a_i \geq 1\) and \(p_i \geq 5\) is a prime, \(1 \leq i \leq t\). Start from a \(\{3, 5\}\)-SCHGDD of type \((8, 1^{p_i})\) which exists from Lemma 4.9. Apply Construction 4.14 with \((k, q)\)-CDM for \(k = 3, 5\) and \(q \in \{p_2, ..., p_t\}\) which exist from Lemmas 3.1, 4.10 and 4.11. Then we obtain a \(\{3, 5\}\)-SCHGDD of type \((8, q^{p_i})\). Apply Construction 4.13 with a \(\{3, 5\}\)-SCHGDD of type \((8, 1^{q^{p_i}})\) which exists by Lemma 4.9 to obtain a \(\{3, 5\}\)-SCHGDD of type \((8, 1^{q^{p_i}})\). Repeating this process will produce a \(\{3, 5\}\)-SCHGDD of type \((8, 1^{w^u})\). Then let \(v \in \{14, 20\}\). By Theorem 4.17 there exists a PBD\((v, \{3, 4, 5\})\) which is also a \(\{3, 4, 5\}\)-GDD of type \(1^v\). When \(w \equiv 1, 5 \pmod{6}\) and \(w \geq 5\), by Lemma 4.12 there exists a \((5, w; 1)\)-CHDM, which is a 5-SCHGDD of type \((5, 1^w)\) from Lemma 4.10. A 3-SCHGDD of type \((k, 1^w)\) for \(k \in \{3, 4\}\) exists by Theorem 1.7. Then apply Construction 4.15 to obtain a \(\{3, 5\}\)-SCHGDD of type \((v, 1^{w^u})\).

**Example 4.19.** We construct a \(\{3, 5\}\)-SCHGDD of type \((5, 1^w)\) on \(X = I_5 \times Z_9\) with group set \(G = \{\{i\} \times Z_9 : i \in I_5\}\) and hole set \(H = \{I_5 \times \{j\} : j \in Z_9\}\). We have two base blocks of size 5 : \{\{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)\}, \{(0, 4), (1, 8), (2, 7), (3, 6), (4, 5)\}\}. Then we list 20 base blocks of size 3 as follows.

\[
\begin{align*}
(0, 0, (2, 1), (4, 2)) & , \quad (0, 0, (3, 1), (2, 8)) & , \quad (0, 0, (1, 2), (2, 5)) & , \quad (0, 0, (1, 3), (3, 7)) & , \\
(0, 0, (4, 3), (1, 7)) & , \quad (0, 0, (2, 4), (1, 8)) & , \quad (0, 0, (3, 4), (4, 8)) & , \quad (0, 0, (1, 5), (3, 6)) & , \\
(0, 0, (3, 5), (2, 7)) & , \quad (0, 0, (4, 5), (2, 6)) & , \quad (0, 0, (1, 6), (4, 7)) & , \quad (0, 0, (4, 6), (3, 8)) & , \\
(1, 0, (2, 2), (3, 6)) & , \quad (1, 0, (4, 2), (2, 6)) & , \quad (1, 0, (3, 5), (2, 7)) & , \quad (1, 0, (2, 4), (4, 8)) & , \\
(1, 0, (4, 4), (3, 8)) & , \quad (1, 0, (3, 5), (4, 7)) & , \quad (2, 0, (3, 3), (4, 6)) & , \quad (2, 0, (4, 3), (3, 6)) & .
\end{align*}
\]
Lemma 4.20. Let \( w = 3^r \) with \( r \geq 2 \). Then there exists a \([3,5]-\text{SCHGDD}\) of type \((v, 1^w)\) for \( v \in \{8, 14, 20\} \).

Proof.  

1. Let \( v = 8 \). When \( r \in \{2, 3\} \), the required \([3,5]-\text{SCHGDDs}\) of type \((8, 1^w)\) are constructed directly in Lemma A.2 in the appendix.

   When \( r = 4 \), by [15, Lemma 2.5] there exists a strictly cyclic 5-GDD of type \( 1^8 \). By Lemma 4.9 there exists a \([3,5]-\text{SCHGDD}\) of type \((8, 1^5)\) which is also a \([3,5]-\text{MGDD}\) of type \( 5^8 \). Then apply Construction 4.16 to obtain a \([3,5]-\text{SCHGDD}\) of type \((8, 1^8)\).

   When \( r \geq 5 \), we use induction on \( r \). Assume that there exists a \([3,5]-\text{SCHGDD}\) of type \((8, 1^3)\). By Lemma 4.10 there exists a \([3,5]-\text{MGDD}\) of type \( 3^8 \). Then apply Construction 4.16 to obtain a \([3,5]-\text{SCHGDD}\) of type \((8, 1^8)\).

2. Let \( v \in \{14, 20\} \). By Theorem 4.17 there exists a PBD\((v, [3, 4, 5])\) which is a \([3,4,5]-\text{GDD}\) of type \( 1^v \). For \( k \in \{3, 4\} \), there exists a \([3,5]-\text{SCHGDD}\) of type \((k, 1^v)\) from Theorem 1.7. Apply Construction 4.14 with the given \([3,5]-\text{SCHGDD}\) of type \((8, 1^v)\) to obtain a \([3,5]-\text{SCHGDD}\) of type \((8, 1^v)\). Then apply Construction 4.13 with a \([3,5]-\text{SCHGDD}\) of type \((5, 1^v)\) to get a \([3,5]-\text{SCHGDD}\) of type \((8, 1^v)\).

\[ \square \]

Lemma 4.21. Let \( v \in \{8, 14, 20\} \). For \( w = 3^r \) with \( r \geq 2 \) and \( w \equiv 1, 5 \pmod{6} \) with \( w \geq 5 \), there exists a \([3,5,v]-\text{SCGDD}\) of type \( w^v \) containing one base block of size \( v \).

Proof. The conclusion follows from Lemmas 2.3, 4.18, and 4.20.

\[ \square \]

Lemma 4.22. There exists a \([3,4]-\text{SCGDD}\) of type \( 3^8 \) in which the blocks of size 4 form a parallel class.

Proof. The design is constructed on \( X = I_8 \times Z_3 \) with group set \( \{[i] \times Z_3 : i \in I_8\} \). We have two base blocks of size 4 : \( \{(0, 0), (1, 0), (2, 0), (3, 0)\}, \{(4, 0), (5, 0), (6, 0), (7, 0)\} \). Next we list 24 base blocks of size 3 as follows.

\[
\begin{align*}
&\{(0, 0), (4, 0), (1, 1)\}, \{(0, 0), (6, 0), (5, 1)\}, \{(0, 0), (5, 0), (7, 2)\}, \{(0, 0), (7, 0), (4, 1)\}, \\
&\{(0, 0), (2, 1), (5, 2)\}, \{(0, 0), (6, 1), (3, 2)\}, \{(0, 0), (1, 2), (4, 2)\}, \{(0, 0), (3, 1), (6, 2)\}, \\
&\{(0, 0), (7, 1), (2, 2)\}, \{(3, 0), (7, 0), (4, 2)\}, \{(2, 0), (4, 1), (6, 2)\}, \{(2, 0), (4, 0), (5, 2)\}, \\
&\{(2, 0), (6, 0), (7, 1)\}, \{(1, 0), (5, 0), (6, 1)\}, \{(1, 0), (7, 1), (3, 2)\}, \{(1, 0), (2, 1), (6, 2)\}, \\
&\{(1, 0), (6, 0), (7, 2)\}, \{(3, 0), (6, 0), (4, 1)\}, \{(2, 0), (5, 0), (3, 1)\}, \{(1, 0), (7, 0), (5, 2)\}, \\
&\{(3, 0), (4, 0), (5, 1)\}, \{(1, 0), (4, 1), (2, 2)\}, \{(2, 0), (7, 0), (3, 2)\}, \{(1, 0), (3, 1), (5, 1)\}. \\
\end{align*}
\]

\[ \square \]

5 | CASE \( u \equiv 1 \pmod{3} \) AND PROOF OF THE MAIN RESULT

In this section we consider the maximum \( w \)-cyclic 3-HGDPs of type \((u, w^v)\) when \( u \equiv 1 \pmod{3} \). Then we prove our main result in Theorem 1.3.
Example 5.1. We construct a 3-HGDP of type $(4, 1^4)$ with $J(4 \times 4 \times 1, 3, 1)$ blocks on $I_4 \times I_4$ with group set $\{[i] \times I_4 : i \in I_4\}$ and hole set $\{I_4 \times [j] : j \in I_4\}$. All 21 blocks are listed as follows.

\[
\{(0, 0), (1, 1), (3, 2)\}, \{(0, 0), (2, 1), (1, 2)\}, \{(0, 0), (3, 1), (2, 3)\}, \{(0, 0), (2, 2), (1, 3)\}, \\
\{(1, 1), (3, 1), (0, 3)\}, \{(1, 0), (0, 2), (2, 3)\}, \{(1, 0), (2, 2), (3, 3)\}, \{(2, 0), (0, 1), (1, 3)\}, \\
\{(2, 0), (1, 1), (3, 3)\}, \{(2, 0), (3, 1), (1, 2)\}, \{(2, 0), (3, 2), (0, 3)\}, \{(3, 0), (0, 1), (2, 2)\}, \\
\{(3, 0), (1, 1), (0, 2)\}, \{(3, 0), (2, 1), (0, 3)\}, \{(3, 0), (1, 2), (3, 3)\}, \{(0, 1), (1, 2), (0, 3)\}, \\
\{(0, 1), (2, 2), (0, 3)\}, \{(2, 1), (0, 2), (3, 3)\}, \{(2, 1), (3, 2), (1, 3)\}, \\
\{(3, 1), (0, 2), (1, 3)\}.
\]

Lemma 5.2. Let $u \in \{4, 10, 16\} \text{ and } v \in \{4, 8, 10, 14, 16, 20\}$. There exists a 3-HGDP of type $(u, 1^v)$ with $J(u \times v \times 1, 3, 1)$ blocks.

Proof. For $u = 4$ and $v = 4, 8, 14$, the desired designs exist from Example 5.1 and Lemmas B.6, B.7 in the appendix.

For other parameters, apply Lemma 4.1 by taking parameters $(u, t, v)$ in Table 2 to yield a 3-HGDP of type $(u, 1^v)$ with $J(u \times v \times 1, 3, 1)$ blocks, where the needed 3-IHGDPs of type $(u, t, 1^v)$ and 3-HGDPs of type $(t, 1^v)$ exist from the sources listed in the fourth and fifth column, respectively. Again note that a 3-HGDP of type $(u, 1^v)$ is equivalent to a 3-HGDP of type $(v, 1^u)$.

Lemma 5.3. Let $u \in \{4, 10, 16\} \text{ and } v \in \{8, 14, 20\}$. There exists a 3-cyclic 3-HGDP of type $(u, 3^v)$ with $J(u \times v \times 3, 3, 1)$ base blocks.

Proof. For $v \in \{8, 14, 20\}$, by Lemmas 4.22 and D.1 in the appendix, there exists a $[3, 4, 6]$-SCGDD of type $3^v$ with $\frac{3v^2-8v+16}{6}$ base blocks of size 3, two base blocks of size 4 and $\frac{v-8}{6}$ base blocks of size 6. Apply Corollary 2.5 by taking $h = 1$ and $w = 3$ to yield a desired

| Row | $(u, v)$ | $(u, t, v)$ | $(u, t, 1^v)$ | $(t, 1^v)$ |
|-----|----------|-------------|---------------|-------------|
| 1   | (10, 4)  | (10, 4, 4)  | Lemma C.1     | Example 5.1 |
| 2   | (16, 4)  | (16, 4, 4)  | Lemma C.1     | Example 5.1 |
| 3   | (20, 4)  | (20, 8, 4)  | Lemma 4.6     | Lemma B.6   |
| 4   | (10, 8)  | (10, 4, 8)  | Lemma C.3     | Lemma B.6   |
| 5   | (16, 8)  | (16, 4, 8)  | Lemma C.4     | Lemma B.6   |
| 6   | (10, 10) | (10, 4, 10) | Lemma C.2     | Row 1       |
| 7   | (10, 14) | (10, 4, 14) | Lemma C.3     | Lemma B.7   |
| 8   | (16, 14) | (16, 4, 14) | Lemma C.4     | Lemma B.7   |
| 9   | (10, 16) | (10, 4, 16) | Lemma 4.7     | Row 2       |
| 10  | (16, 16) | (16, 4, 16) | Lemma 4.7     | Row 2       |
| 11  | (20, 16) | (20, 8, 16) | Lemma 4.7     | Row 5       |
| 12  | (10, 20) | (10, 4, 20) | Lemma 4.7     | Row 3       |
3-cyclic 3-HGDP, where the required 3-HGDPs of types \((u, 1^3), (u, 1^4), \) and \((u, 1^6)\) exist by Theorem 1.1, Lemmas 5.2 and 3.9, respectively. The number of base blocks equals
\[
\frac{3v^2 - 8v + 16}{6} \times u(u - 1) + 2 \times \frac{6u^2 - 8u - 1}{3} + \frac{v - 8}{6} \times u(5u - 6) = \frac{3uv(u - 1)(v - 1) - uv - 4}{6} = J(u \times v \times 3, 3, 1).
\]

**Lemma 5.4.** Let \(u \in \{4, 10, 16\}, v \in \{4, 8, 10, 14, 16, 20\}\) and \(w \equiv 1 \pmod{2}\). Then there exists a \(w\)-cyclic 3-HGDP of type \((u, w^v)\) with \(J(u \times v \times w, 3, 1)\) base blocks.

**Proof:**

(a) For \(w = 1\) and the assumed \(u, v\), by Lemma 5.2, there exists a 3-HGDP of type \((u, 1^3)\) with \(J(u \times v \times 1, 3, 1)\) blocks.

(b) For \(w = 3\) and the assumed \(u, v\), when \(v \in \{8, 14, 20\}\), a 3-cyclic 3-HGDP of type \((u, 3^v)\) with \(J(u \times v \times 3, 3, 1)\) base blocks exists from Lemma 5.3.

(c) For other assumed parameters, apply Lemma 2.9 by taking parameters \((g, v, u, m)\) in the second column of Table 3. The required \(\{3, 5, v\}\)-SCGDDs of type \(g^v\) containing one base block of size \(v\) exist, for which the sources are listed in the fourth column. Maximum \(m\)-cyclic 3-HGDPs of type \((u, m^v)\) exist from the sources in the last column. Finally we produce \(w\)-cyclic 3-HGDPs of type \((u, w^v)\) with \(J(u \times v \times w, 3, 1)\) base blocks.

**Lemma 5.5.** Let \(m \geq 3, w \equiv 1 \pmod{2}, v \equiv 0 \pmod{2}\) and \(s \in \{4, 8\}\). There exists a \(w\)-cyclic 3-DGDD of type \((d_0, d_1, ..., d_{v-1})\) where \(d_e = (6w, 6w, ..., 6w, sw)^T\) is a column vector of length \(m + 1\) for \(0 \leq e \leq v - 1\).

**Proof:** For the assumed parameters, apply Construction 2.4 with a 3-GDD of type \(6^m s^1\) whose existence is guaranteed by Theorem 3.7 and a \(w\)-cyclic 3-HGDD of type \((3, w^v)\) from Theorem 1.4. It is not difficult to learn that we obtain a \(w\)-cyclic 3-DGDD of type \((d_0, d_1, ..., d_{v-1})\).

**Lemma 5.6.** Let \(u \in \{4, 10, 16\}, v \equiv 0 \pmod{2} \pmod{4}\) and \(w \equiv 1 \pmod{2}\). Then there exists a \(w\)-cyclic 3-HGDP of type \((u, w^v)\) with \(J(u \times v \times w, 3, 1)\) base blocks.

### Table 3 Parameters for Lemma 5.4

| Row | \((u, w^v)\) | \((g, v, u, m)\) | SCGDD | \((u, m^v)\) |
|-----|-------------|----------------|-------|-------------|
| 1   | \(v \in \{4, 10, 16\}, w \equiv 1 \pmod{2}, w \geq 3\) | (w, v, u, 1) | Theorem 1.7 | Lemma 5.2 |
| 2   | \(v \in \{8, 14, 20\}, w \equiv 1, 5 \pmod{6}, w \geq 5\) | (w, v, u, 1) | Lemma 4.21 | Lemma 5.2 |
| 3   | \(v \in \{8, 14, 20\}, w = 3^r, r \geq 2\) | (w, v, u, 1) | Lemma 4.21 | Lemma 5.2 |
| 4   | \(v \in \{8, 14, 20\}, w = 3^w', w' \equiv 1, 5 \pmod{6}\) | \((w', v, u, 3^r)\) | Lemma 4.21 | (b), Row 3 |

\((r \geq 1, w' \geq 5)\)
Proof. For \( v \in \{4, 8, 10, 14, 16, 20\} \), the required designs exist by Lemma 5.4.

Next note that \( u \in \{4, 10, 16\} \) and \( w \equiv 1 \pmod{2} \). For \( v \equiv 0 \pmod{6} \) and \( v \geq 6 \), by Lemma 3.9 there exists a \( w \)-cyclic 3-HGDP of type \((v, w^u)\), also of type \((u, w^v)\), with \( J(u \times v \times w, 3, 1) \) base blocks.

For \( v \geq 22 \), let \( s = 8 \) if \( v \equiv 2 \pmod{6} \) and \( s = 4 \) if \( v \equiv 4 \pmod{6} \). By Lemma 5.5 there exists a \( w \)-cyclic 3-DGDD of type \((d_0, d_1, \ldots, d_{u-1})\), where \( d_e = (6w, 6w, \ldots, 6w, sw)^T \) is a column vector of length \( \frac{v-s+6}{6} \) for \( 0 \leq e \leq u-1 \). For \( k \in \{6, s\} \), there exists a \( w \)-cyclic 3-HGDP of type \((k, w^u)\) with \( J(k \times u \times w, 3, 1) \) base blocks from Lemmas 3.8 and 5.4. Apply Lemma 2.8 by taking parameters \((g, s, v, u, w)\) in Table 4 to obtain a desired \( w \)-cyclic 3-HGDP of type \((v, w^u)\), also of type \((u, w^v)\).

Lemma 5.7. \( \Psi(u \times v \times w, 3, 1) = J(u \times v \times w, 3, 1) \) where \( u \equiv 1 \pmod{3} \) and \( u, v \geq 3 \).

Proof. By Corollary 1.5, we only need to treat the case that \( u \equiv 4 \pmod{6} \), \( w \equiv 1 \pmod{2} \), and \( v \equiv 0 \pmod{2} \), \( v \geq 4 \).

For \( u \in \{4, 10, 16\} \), the required designs exist by Lemma 5.6.

For \( u \geq 22 \), by Lemma 5.5 there exists a \( w \)-cyclic 3-DGDD of type \((d_0, d_1, \ldots, d_{u-1})\), where \( d_e = (6w, 6w, \ldots, 6w, 4w)^T \) is a column vector of length \( \frac{u+2}{6} \) for \( 0 \leq e \leq u-1 \). And there exist maximum \( w \)-cyclic 3-HGDPs of type \((4, w^u)\) and \((6, w^v)\) from Lemmas 5.6 and 3.9, respectively. Apply Lemma 2.8 by taking parameters \((g, s, v, u, w)\) in Table 4 to obtain a desired \( w \)-cyclic 3-HGDP of type \((u, w^v)\) with \( J(u \times v \times w, 3, 1) \) base blocks.

Now we prove our main result.

Proof of Theorem 1.3. Combining Lemmas 3.9 and 5.7 completes the proof.

6 | APPLICATION

An OOC is a family of \((0, 1)\) sequences with good correlation properties. It was introduced as signature sequences to facilitate multiple access in optical fibre networks (see, e.g. \([3,17,18]\)). OOCs have been found applications in wide ranges such as mobile radio, frequency-hopping spread-spectrum communications, radar, sonar signal designs \([11]\), collision channels without feedback \([16]\), and neuromorphic networks \([19]\).

Let \( u, v, w, k, \) and \( \lambda \) be positive integers, where \( u, v, \) and \( w \) are the number of spatial channels, wavelengths, and time slots, respectively. A \textit{three-dimensional} \((u \times v \times w, k, \lambda)\) OOC (briefly 3-D \((u \times v \times w, k, \lambda)\)-OOC), \( C \), is a family of \( u \times v \times w(0, 1) \) arrays (called \textit{codewords}) of Hamming weight \( k \) satisfying that for any two arrays \( A = [a(i,j,l)] \), \( B = [b(i,j,l)] \in C \) and any integer \( \tau \),
\[
\sum_{i=0}^{u-1} \sum_{j=0}^{v-1} \sum_{l=0}^{w-1} a(i, j, l)b(i, j, l + \tau) \leq \lambda,
\]

where either \( A \neq B \) or \( \tau \neq 0 \pmod{w} \), and the arithmetic \( l + \tau \) is reduced modulo \( w \).

A \( v/w \) plane is usually called a spatial plane, and a \( u/w \) plane is called a wavelength plane. There are several classes of 3-D OOCs of particular interest (see [14]), for which the following three additional restrictions on the placement of pulses are often placed within the arrays of a 3-D OOC to simplify practical implementation.

- **One-pulse per plane (OPP) restriction**: Each codeword contains exactly one optical pulse per spatial plane.
- **At most one-pulse per spatial plane (AM-OPPSP) restriction**: For any array in \( \mathcal{C} \), the element 1 appears at most once in each spatial plane.
- **At most one-pulse per wavelength plane (AM-OPPWP) restriction**: For any array in \( \mathcal{C} \), the element 1 appears at most once in each wavelength plane.

A 3-D OOC with both AM-OPPSP and AM-OPPWP properties is simply denoted by AM-OPPSP/WP 3-D OOC. The number of codewords of a 3-D OOC is called its size. An AM-OPPSP/WP 3-D \( (u \times v \times w, k, 1) \)-OOC of the largest possible size is said to be maximum. Dai et al. established the equivalence between a maximum AM-OPPSP/WP 3-D \( (u \times v \times w, k, 1) \)-OOC and a maximum \( w \)-cyclic \( k \)-HGDP of type \( (u, w^v) \).

**Theorem 6.1** (Dai et al. [6], Theorem 5.1). A maximum AM-OPPSP/WP 3-D \( (u \times v \times w, k, 1) \)-OOC is equivalent to a maximum \( w \)-cyclic \( k \)-HGDP of type \( (u, w^v) \).

By Theorems 1.3 and 6.1, the size of a maximum AM-OPPSP/WP 3-D \( (u \times v \times w, 3, 1) \)-OOC is determined for any integers \( u, v, w \) with \( u \equiv 0, 1 \pmod{3} \).

**Theorem 6.2.** A maximum AM-OPPSP/WP 3-D \( (u \times v \times w, 3, 1) \)-OOC contains \( \Psi(u \times v \times w, 3, 1) \) codewords for positive integers \( u, v, w \) with \( u \equiv 0, 1 \pmod{3} \) and \( u, v \geq 3 \), where

\[
\Psi(u \times v \times w, 3, 1) = \begin{cases} 
6(w - 1) & \text{if } (u, v) = (3, 3) \text{ and } w \equiv 0 \pmod{2}, \\
J(u \times v \times w, 3, 1) & \text{otherwise}.
\end{cases}
\]

**7 | CONCLUDING REMARKS**

In this paper the exact number of base blocks of a maximum \( w \)-cyclic 3-HGDP of type \( (u, w^v) \) is determined for any positive integers \( u, v, w \) with \( u \equiv 0, 1 \pmod{3} \). In the process, we generalize and adapt many standard recursive constructions in design theory. Then the solution to the problem reduces to direct constructions for a handful of small ingredient designs. Most of the direct constructions are produced by combining appropriate automorphism groups with computer programs search. Applying the construction methods in this paper, we may hopefully resolve the remaining case \( u \equiv 2 \pmod{3} \) on the condition that necessary auxiliary small designs are all constructed. However, many of the required designs are not easy to be produced in this case. More powerful recursions and efficient direct constructions are expected to solve completely the existence problem of maximum \( w \)-cyclic 3-HGDPS of type \( (u, w^v) \).
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