Indirect-adaptive Model Predictive Control for Linear Systems with Polytopic Uncertainty

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Abstract

We develop an indirect-adaptive model predictive control algorithm for uncertain linear systems subject to constraints. The system is modeled as a polytopic linear parameter varying system where the convex combination vector is constant but unknown. Robust constraint satisfaction is obtained by constraints enforcing a robust control invariant. The terminal cost and set are constructed from a parameter-dependent Lyapunov function and the associated control law. The proposed design ensures robust constraint satisfaction and recursive feasibility, is input-to-state stable with respect to the parameter estimation error and it only requires the online solution of quadratic programs.

I. INTRODUCTION

In Model Predictive Control (MPC) the prediction model is exploited for evaluating feasibility and performance of the sequence of actions to be selected by the controller. In several cases, some of the model parameters are uncertain at design time, especially when a controller is deployed to multiple instances of the plant, such as in automotive, factory automation, and aerospace applications, where the control algorithm and its auxiliary functions need also to have low complexity and computational effort, due to stringent cost, timing, and validation requirements.

For the cases when model parameters are uncertain, robust MPC methods have been proposed, see, e.g., [4]–[8]. Some of the limitations of these methods are either in the computational cost, due to solving linear matrix inequalities (LMIs) at each control step [4]–[6], or in applying online to additive disturbances [7], or in imposing limitations on cost function and terminal set [8]. These limitations often arise due to considering the challenging case where uncertain parameters are constantly changing during system operation.

However, when the parameters are unknown but are constant or slowly varying over time, an alternative approach is to learn their values, resulting in adaptive control techniques that ensure safe operation during the learning phase, and improve performance, for instance in terms of stability or tracking,

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as the learning proceeds. Adaptive MPC algorithms have been recently proposed based on different methods, such as a comparison model [9], min-max approaches with open-loop relaxations [10], learning of constant offsets [11], and set membership identification [12]. Another class of adaptive MPC algorithms focuses on “dual objective control”, i.e., controlling the system while guaranteeing sufficient excitation for identifiability, see, e.g., [13]–[15], and references therein.

In [16], a MPC design allowing for the prediction model to be adjusted after deployment was proposed. In this paper we propose a MPC design that operates concurrently with a parameter estimation scheme, thus resulting in an indirect-adaptive MPC (IAMPC) approach, that retains constraints satisfaction guarantees and certain stability properties. Motivated by the case of the unknown but constant (or slowly varying) parameters and by the need to keep computational burden small for application in fast systems equipped with low cost microprocessors [2], [3], here we do not seek robust stability as in robust MPC, but rather robust constraint satisfaction and an input-to-state stable (ISS) closed-loop with respect to the estimation error. ISS will hold with only minimal assumptions on the estimates, and if the correct parameter value will be eventually estimated (possibly in finite time, see, e.g., [17]), by the definition of ISS the closed-loop will become asymptotically stable (AS). Constraint satisfaction holds even if the parameters keep changing.

For uncertain systems represented as polytopic linear difference inclusions (pLDI) we design a parameter-dependent quadratic terminal cost and a robust terminal constraint using a parameter-dependent Lyapunov function (pLF) [18] and its corresponding stabilizing control law. Robust constraint satisfaction in presence of parameter estimation error is obtained by enforcing robust control invariant set constraints [19]. A parameter prediction update law is also designed to ensure the desired properties. The proposed IAMPC allows uncertainty in the system dynamics, as opposed to additive disturbances/offsets as in [7], [11], and only solves quadratic programs (QPs), as opposed to robust MPC methods that require the online solution of LMIs [5], [6].

The paper is structured as follows. After the preliminaries in Section II in Section III we design the cost function that results in the unconstrained IAMPC to be ISS with respect to the estimation error. For constrained IAMPC, in Section IV we first design the terminal set that guarantees that the nominal closed-loop is AS. Then, we design robust constraints that ensure that the system constraints are satisfied and the IAMPC remains feasible even in presence of parameter estimation error. In Section V we combine the cost function and the constraints with a parameter estimate prediction update and we briefly discuss the required properties of the estimator, hence describing the complete IAMPC. In Section VI we show a numerical example. Conclusions and future developments are discussed in Section VII.

Notation: $\mathbb{R}$, $\mathbb{R}_{0+}$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{Z}_{0+}$, $\mathbb{Z}_+$ are the sets of real, nonnegative real, positive real, and integer,
nonnegative integer, positive integer numbers. We denote interval of numbers using notations like $\mathbb{Z}_{(a,b)} = \{ z \in \mathbb{Z} : a \leq z < b \}$. co$\{\mathcal{X}\}$ denotes the convex hull of the set $\mathcal{X}$. For vectors, inequalities are intended componentwise, while for matrices indicate (semi)definiteness, and $\lambda_{\text{min}}(Q)$ denotes the smallest eigenvalue of $Q$. By $[x]_i$ we denote the $i$-th component of vector $x$, and by $I$ and $0$ the identity and the “all-zero” matrices of appropriate dimension. $\| \cdot \|_p$ denotes the $p$-norm, and $\| \cdot \| = \| \cdot \|_2$. For a discrete-time signal $x \in \mathbb{R}^n$ with sampling period $T_s$, $x(t)$ is the state a sampling instant $t$, i.e., at time $t_k$, $x_{k|t}$ denotes the predicted value of $x$ at sample $t + k$, i.e., $x(t + k)$, based on data at sample $t$, and $x_{0|t} = x(t)$. A function $\alpha : \mathbb{R}_{0+} \to \mathbb{R}_{0+}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, $\alpha(0) = 0$; if in addition $\lim_{c \to \infty} \alpha(c) = \infty$, $\alpha$ is of class $\mathcal{K}_\infty$.

II. PRELIMINARIES AND PROBLEM DEFINITION

Details on the following standard definitions and results are, e.g., in [1, Appendix B].

Definition 1: Given $x(t + 1) = f(x(t), w(t))$, $x \in \mathbb{R}^n$, $w \in \mathcal{W} \subseteq \mathbb{R}^d$, a set $\mathcal{SS} \subset \mathbb{R}^n$ is robust positive invariant (RPI) for $f$ iff for all $x \in \mathcal{SS}$, $f(x, w) \in \mathcal{SS}$, for all $w \in \mathcal{W}$. If $w = \{0\}$, $\mathcal{SS}$ is called positive invariant (PI). □

Definition 2: Given $x(t + 1) = f(x(t), u(t), w(t))$, $x \in \mathbb{R}^n$, $u \in \mathcal{U} \subseteq \mathbb{R}^m$, $w \in \mathcal{W} \subseteq \mathbb{R}^d$, a set $\mathcal{SS} \subset \mathbb{R}^n$ is robust control invariant (RCI) for $f$ iff for all $x \in \mathcal{SS}$, there exists $u \in \mathcal{U}$ such that $f(x, u, w) \in \mathcal{SS}$, for all $w \in \mathcal{W}$. If $w = \{0\}$, $\mathcal{SS}$ is called control invariant (CI). □

Definition 3: Given $x(t + 1) = f(x(t))$, $x \in \mathbb{R}^n$, and a PI set $\mathcal{SS}$ for $f$, $0 \in \mathcal{SS}$, a function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}_{0+}$ such that there exists $\alpha_1, \alpha_2, \alpha_\Delta \in \mathcal{K}_\infty$ such that $\alpha_1(\|x\|) \leq \mathcal{V}(x) \leq \alpha_2(\|x\|)$, $\mathcal{V}(f(x)) - \mathcal{V}(x) \leq -\alpha_\Delta(\|x\|)$ for all $x \in \mathcal{SS}$ is a Lyapunov function for $f$ in $\mathcal{SS}$. □

Definition 4: Given $x(t + 1) = f(x(t), w(t))$, $x \in \mathbb{R}^n$, $w \in \mathcal{W} \subseteq \mathbb{R}^d$, and a RPI set $\mathcal{SS}$ for $f$, $0 \in \mathcal{SS}$, a function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}_+$ such that there exists $\alpha_1, \alpha_2, \alpha_\Delta \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}$ such that $\alpha_1(\|x\|) \leq \mathcal{V}(x) \leq \alpha_2(\|x\|)$, $\mathcal{V}(f(x)) - \mathcal{V}(x) \leq -\alpha_\Delta(\|x\|) + \gamma(\|w\|)$ for all $x \in \mathcal{SS}$, $w \in \mathcal{W}$ is an input-to-state stable (ISS) Lyapunov function for $f$ in $\mathcal{SS}$ with respect to $w$.

Result 1: Given $x(t + 1) = f(x(t))$, $x \in \mathbb{R}^n$, and a PI $\mathcal{SS}$ for $f$, $0 \in \mathcal{SS}$, if there exists a Lyapunov function for $f$ in $\mathcal{SS}$, the origin is asymptotically stable (AS) for $f$ with domain of attraction $\mathcal{SS}$. Given $x(t + 1) = f(x(t), w(t))$, $x \in \mathbb{R}^n$, $w \in \mathcal{W} \subseteq \mathbb{R}^d$, and a RPI $\mathcal{SS}$ for $f$, $0 \in \mathcal{SS}$, if there exists a ISS Lyapunov function for $f$ in $\mathcal{SS}$, the origin is ISS for $f$ with respect to $w$ with domain of attraction $\mathcal{SS}$.

We consider the uncertain constrained discrete-time systems with sampling period $T_s$,

\[
x(t + 1) = \sum_{i=1}^{\ell} [\xi_i, A_i x(t) + Bu(t)], \quad (1a)
\]

\[
x(t + 1) \in \mathcal{X}, \quad u \in \mathcal{U} \quad (1b)
\]
where $A_i \in \mathbb{R}^{n \times n}$, $i \in \mathbb{Z}_{[1, \ell]}$ and $B$ are known matrices of appropriate size, and $\mathcal{X} \subseteq \mathbb{R}^n$, $\mathcal{U} \subseteq \mathbb{R}^m$ are constraints on system states and inputs. In (1), the uncertainty is associated to $\xi \in \Xi \subseteq \mathbb{R}^\ell$, which is unknown and constant or changing slowly with respect to the system dynamics, and $\Xi = \{\xi \in \mathbb{R}^\ell : 0 \leq [\xi]_i \leq 1, \sum_{i=1}^\ell [\xi]_i = 1\}$. We call $\xi$ convex combination vector, since it describes a convex combination of the “vertex systems” $f_i(x, u) = A_i x + B u$, $i \in \mathbb{Z}_{[1, \ell]}$.

Assumption 1: An estimator is computing the estimate $\hat{\xi}(t)$ of $\xi$ such that $\hat{\xi}(t) \in \Xi$ for all $t \in \mathbb{Z}_{0+}$.

We denote by $\tilde{\xi}(t) = \hat{\xi}(t) - \xi$, the estimation error at time $t$ for which it holds that $\tilde{\xi}(t) + \xi \in \Xi$. Given $\xi \in \Xi$, for shortness we write $\check{\xi} \in \tilde{\Xi}(\xi)$, where $\tilde{\Xi}(\xi) = \{\check{\xi} \in \mathbb{R}^\ell : \exists \tilde{\xi} \in \Xi, \check{\xi} = \tilde{\xi} - \xi\}$ is the set of possible estimation error vectors. Assumption 1 is what is required from the estimator for the development in this paper to hold. Some comments on how to design estimators that satisfy Assumption 1 are given later, in Section V-A.

Remark 1: The trajectories produced by (1a) are a subset of those of the pLDI

$$x(t + 1) \in \text{co}\{A_i x(t) + B u(t)\}_{i=1}^\ell. \quad (2)$$

The pLDI (2) is equivalent to (1a) if a varying parameter vector, i.e., $\check{\xi}(t)$, is considered.

Consider the finite time optimal control problem

$$\mathcal{V}_{\xi_t}^{\text{MPC}}(x(t)) = \min_{U_t} x'_{N|t} \mathcal{P}(\xi_{N|t}) x_{N|t} + \sum_{k=0}^{N-1} x'_{k|t} Q x_{k|t} + u'_{k|t} R u_{k|t} \quad (3a)$$

s.t. $x_{k+1|t} = \sum_{i=1}^\ell [\xi_{k|t}]_i A_i x_{k|t} + B u_{k|t}$ \quad (3c)

$u_{k|t} \in \mathcal{U}$, $x_{k|t} \in \mathcal{X}$ \quad (3d)

$(x_{k|t}, u_{k|t}) \in \mathcal{C}_{xu}$ \quad (3e)

$x_{N|t} \in \mathcal{X}_N$ \quad (3f)

$x_{0|t} = x(t)$ \quad (3g)

where $N \in \mathbb{R}_+$ is the prediction horizon, $Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times m}$, $Q, R > 0$, $\mathcal{P}(\xi) \in \mathbb{R}^{n \times n}$, $\mathcal{P}(\xi) > 0$, for all $\xi \in \Xi$, $\mathcal{C}_{xu} \subseteq \mathcal{X} \times \mathcal{U}$, $U_t = [u_{0|t} \ldots u_{N-1|t}]$ is the sequence of control inputs along the prediction horizon, and $\xi^N_t = [\xi_{0|t} \ldots \xi_{N|t}] \in \Xi^{N+1}$ is a sequence of predicted parameters, not necessarily constant. Let $U^*_t = [u^*_{0|t} \ldots u^*_{N-1|t}]$ be the solution of (3) at $t \in \mathbb{Z}_{0+}$.

Problem 1: Given (1) and an estimator producing the sequence of estimates $\{\xi(t)\}_t$ such that $\xi(t) \in \Xi$ for all $t \in \mathbb{Z}_{0+}$ according to Assumption 1, design the sequence of predicted convex combination vectors $\xi^N_t$, the terminal cost $\mathcal{P}(\xi)$, the robust terminal set $\mathcal{X}_N$, and the robust constraint set $\mathcal{C}_{xu}$ in (3) so that the
IAMPC controller that at any $t \in \mathbb{Z}_{0+}$ solves (3) and applies $u(t) = u_{0|t}^*$ achieves: (i) ISS of the closed-loop with respect to $\tilde{\xi}_{0|t} = \tilde{\xi} - \xi_{0|t}$, (ii) robust satisfaction of the constraints including when $\tilde{\xi}_{0|t} \neq 0$, (iii) guaranteed convergence of the runtime numerical algorithms and computational load comparable to a (non-adaptive) MPC.

In Problem [1] (i) is concerned with conditions on the behavior during the estimation transient and when the estimate has converged, (ii) is concerned with the system safety in terms of constraints satisfaction, and (iii) is concerned with computational requirements, especially due to recent applications to fast systems [2], [3].

Problem [1] requires robust constraint satisfaction (as in [4], [5], [7]) and ISS, i.e., a proportional effect of the estimation error on the closed-loop Lyapunov function. The rationale for seeking ISS rather than robust stability as in [4], [5] is that, when the unknown parameters do not change or change slowly, a “well designed” estimator should eventually converge to correct value, and hence, by ISS definition, the closed-loop becomes AS. However, ISS holds regardless of the estimator convergence as long as Assumption [1] is satisfied, as well as robust constraint satisfaction, which has to be guaranteed also in presence of estimation error and thus is guaranteed even if the parameters change. While this paper focuses on a control design independent from the estimator design, the dependency of the closed-loop performance on estimation error is captured by the expansion term in the ISS Lyapunov function.

Consider the linear parameter-varying (LPV) system

$$x(t + 1) = \ell \sum_{i=1}^{\ell} [\xi(t)]_i A_i x(t) + B u(t), \quad (4)$$

where for all $t \in \mathbb{Z}_+$, $\xi(t) \in \Xi$, the parameter-dependent (linear) control law

$$u = \kappa(\xi)x = \left( \sum_{i=1}^{\ell} [\xi]_i K_i \right) x, \quad (5)$$

and the parameter-dependent (quadratic) function

$$V_{\xi}(x) = x' P(\xi) x = x' \left( \sum_{i=1}^{\ell} [\xi]_i P_i \right) x, \quad (6)$$

where $P_i > 0$, $i \in \mathbb{Z}_{[1,\ell]}$.

**Definition 5 ([18]):** A function (6) such that $V_{\xi(t+1)}(x(t+1)) - V_{\xi(t)}(x(t)) \leq 0$, for all $\xi(t), \xi(t+1) \in \Xi$, where equality holds only if $x = 0$, is a parameter-dependent Lyapunov function for (4) in closed-loop with (5).
By [5], [16], [18], given $Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times m}$, $Q, R > 0$, any solution $G_i, S_i \in \mathbb{R}^{n \times n}$, $S_i > 0$, $E_i \in \mathbb{R}^{m \times n}$, $i \in \mathbb{Z}_{[1, \ell]}$, of

$$
\begin{bmatrix}
G_i + G'_i - S_i & (A_i G_i + B E_i)' & 0 & 0 \\
(A_i G_i + B E_i) & S_i & 0 & 0 \\
0 & 0 & R^{-1} & 0 \\
E_i & 0 & 0 & Q^{-1}
\end{bmatrix} > 0, \forall i, j \in \mathbb{Z}_{[1, \ell]}.
$$

is such that (5), (6) where $P_i = S_i^{-1}$, $K_i = E_i G_i^{-1}$, $i \in \mathbb{Z}_{[1, \ell]}$, satisfy

$$
V(x(t+1), \xi(t+1)) - V(x(t), \xi(t)) \leq -x(t)'(Q + \kappa(\xi(t))'R\kappa(\xi(t)))x(t)', \forall \xi(t), \xi(t+1) \in \Xi
$$

for the closed-loop (4), (5).

Assumption 2: For the given $A_i, i \in \mathbb{Z}_{[1, \ell]}$, $B$, $Q$, $R$, (7) admits a feasible solution

The LMI (7) is a relaxation of those in [4], [5] since it allows for a parameter-dependent Lyapunov function and a parameter-dependent linear control law. Thus, Assumption 2 is more relaxed of and implied by the existence of an (unconstrained) stabilizing linear control law for the uncertain system (1a), see, e.g., [4], [5]. Indeed, if the vertex systems are such that the uncertainty is too large, (7) may be infeasible, similarly to the case where a stabilizing controller for an uncertain system does not exist. However, since (7) is used here for design, such situation will be recognized before controller execution and corrective measures, such as improving the engineering of the plant or resorting to other control techniques can be actuated.

By using (7) only for design, as opposed to [4], [5], the proposed method solves online only QPs, which makes it feasible also for applications with fast dynamics and low-cost microcontrollers [2], [3].

Remark 2: Here we consider the case where $B$ in (4) is independent of the uncertain parameters due to the limited space, as this allows to shorten several derivations. The expanded derivations related to the case of uncertain $B$ will be included in future/extended versions of this work.

III. UNCONSTRAINED IAMPC: ISS PROPERTY

We start from the unconstrained case, $\mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^m$.

A. Stability with parameter prediction along the horizon

We begin by considering a simpler case where $\xi_{k|t} = \tilde{\xi}(t+k), k \in \mathbb{Z}_{[0,N]}$, where it is possible that $\xi_{k_1|t} \neq \xi_{k_2|t}$, for $k_1, k_2 \in \mathbb{Z}_{[0,N]}$. This corresponds to controlling an LPV system with preview on the parameters for $N$ steps in the future, but no information afterwards.

Lemma 1: Let Assumption 2 hold and consider (4) and the MPC that at $t \in \mathbb{Z}_{0+}$ solves (3) where $\mathcal{X}_N = \mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^m$, $C_{xu} = \mathbb{R}^{n+m}$, $U = \mathbb{R}^m$, $\xi_{k|t} = \tilde{\xi}(t+k)$, and $P(\xi), \kappa(\xi)$ are from (7). Then,
the origin is AS for the closed loop with domain of attraction $\mathbb{R}^n$ for every sequence $\{\bar{\xi}(t)\}_t$, such that $\bar{\xi}(t) \in \Xi$, for all $t \in \mathbb{R}_{0+}$.

Proof: We follow the proofs for unconstrained MPC extended to time-varying systems, see, [1] Sec.2.4. For (4) in closed-loop with (5) designed by (7), (8) holds, where $V$ in (6) is designed also by (7). Thus, in (3), $V_{\xi_t^N}^{\text{MPC}}(x)$, is lower and upper bounded by class $\mathcal{K}_\infty$ functions, $\alpha(||x(t)||) = \lambda_{\min}(Q)||x(t)||^2$, $\bar{\alpha}(||x(t)||) = \psi v ||x(t)||^2$, where $\psi \in \mathbb{R}_+$ and $v = \max_{i \in \mathbb{Z}_{[0,\ell_t]}} \lambda_{\max}(P_i)$, for any $\xi_t^N \in \Xi^{N+1}$, see [1] Sec. 2.4.5. From $x(t)$, let $U^*(t) = [u_{0|t}^* \ldots u_{N-1|t}^*]$ be the optimal solution of (5). At $t+1$ from $x(t+1) = x_{t+1}$, $[u_{0|t+1} \ldots \bar{u}_{N-1|t+1}]$ where $\bar{u}_{k|t+1} = u_{k+1|t}^*$ for $k \in \mathbb{Z}_{[0,N-2]}$, $\bar{u}_{N-1|t+1} = \sum_{i=1}^\ell [\xi_{N-1|t+1}]_i K_i$, has cost $\bar{J} = V_{\xi_t^N}^{\text{MPC}}(x(t)) - x(t)'Qx(t)$, due to (8) and $\xi_{k|t+1} = \xi_{k+1|t}$, for all $k \in \mathbb{Z}_{[0,N-1]}$. Since $V_{\xi_{t+1}^N}^{\text{MPC}}(x(t+1)) \leq \bar{J}$,

\[
V_{\xi_{t+1}^N}^{\text{MPC}}(x(t+1)) - V_{\xi_t^N}^{\text{MPC}}(x(t)) \leq -x(t)'Qx(t) \\
\leq -\lambda_{\min}(Q)||x(t)||^2 = \alpha_\Delta(||x(t)||),
\]

and $\alpha_\Delta \in \mathcal{K}_\infty$. Thus, $V_{\xi_t^N}^{\text{MPC}}(x(t))$ is a Lyapunov function for the closed-loop system for any $\xi_t^N$ such that $\xi_{t|k} = \bar{\xi}(t+k)$, and hence the origin is AS in $\mathbb{R}^n$.

By Lemma [1] the MPC based on (3) with perfect preview along the horizon is stabilizing. Next we account for the effect of the parameter estimation error.

B. ISS with respect to parameter estimation error

Consider now the case relevant to Problem [1] where $\tilde{\xi}(t)$ is constant, i.e., $\bar{\xi}(t) = \bar{\xi}$, for all $t \in \mathbb{Z}_{0+}$, unknown, and being estimated. Thus, $\xi_{0|t} = \bar{\xi} - \xi_{0|t}$ is the error in the parameter estimate, which may be time-varying, and $\bar{\xi}_{0|t} \in \Xi(\xi_{0|t})$. The parameter estimation error induces a state prediction error

\[
\varepsilon_x = \sum_{i=1}^\ell [\bar{\xi}_{0|t}]_i A_i x - \sum_{i=1}^\ell [\xi_{0|t}]_i A_i x = \sum_{i=1}^\ell [\bar{\xi}_{0|t}]_i A_i x.
\]

Indeed,

\[
||\varepsilon_x|| = \left\| \sum_{i=1}^\ell [\bar{\xi}_{0|t}]_i A_i x \right\| \leq \left\| \sum_{i=1}^\ell [\bar{\xi}_{0|t}]_i A_i \right\| \cdot ||x|| \\
\leq \left\| \sum_{i=1}^\ell ||[\bar{\xi}_{0|t}]_i A_i|| \cdot ||x|| \right\| \leq \gamma_A ||\bar{\xi}_{0|t}||_1 ||x||
\]

where $\gamma_A = \max_{i=1, \ldots, \ell} ||A_i||$.

Consider the value function $V_{\xi_t^N}^{\text{MPC}}$ of (3), the following result is straightforward from [1].

Result 2: For every compact $\mathcal{X}_L \subseteq \mathbb{R}^n$, the value function of (3), where $\mathcal{P}(\xi)$ is designed according to (7), is Lipschitz-continuous in $x \in \mathcal{X}_L$, that is, there exists $L \in \mathbb{R}_+$ such that for every $x_1, x_2 \in \mathcal{X}_L$,
\[ \| \mathcal{V}_{\xi_N}^{\text{MPC}}(x_1) - \mathcal{V}_{\xi_N}^{\text{MPC}}(x_2) \| \leq L \| x_1 - x_2 \|, \text{ for every } \xi_N \in \Xi^{N+1}. \]

Result 2 follows directly from the fact that for every \( \xi_N \in \Xi^{N+1} \), \( \mathcal{V}_{\xi_N}^{\text{MPC}} \) is piecewise quadratic \(^1\) and hence it is Lipschitz continuous in any compact set \( \mathcal{X}_L \). Thus, for any \( \mathcal{X}_L \subseteq \mathbb{R}^n \) and \( \xi_N \in \Xi^{N+1} \), there exists a Lipschitz parameter \( L_{\xi_N} \in \mathbb{R}^+ \). Since \( \Xi^{N+1} \) is compact, i.e., closed and bounded, there exists a maximum of \( L_{\xi_N} \in \mathbb{R}^+ \) for \( \xi_N \in \Xi^{N+1} \). Such maximum is the Lipschitz constant \( L \).

**Lemma 2:** Let \( \xi_{k-1|t+1} = \xi_{k|t} \), for all \( k \in \mathbb{Z}_{[1,N]} \), \( t \in \mathbb{Z}_{0+} \). Then, there exists \( \gamma_L > 0 \), such that for every \( x \in \mathcal{X}_L \),

\[ \mathcal{V}_{\xi_{t+1}}^{\text{MPC}}(x(t+1)) \leq \mathcal{V}_{\xi_{t+1}}^{\text{MPC}}(x(t)) - \lambda_{\min}(Q) \| x(t) \|^2 + \gamma_L \| \hat{\xi}_0[t] \| \| x(t) \|. \quad \text{(11)} \]

**Proof:** By Lipschitz continuity of \( \mathcal{V}_{\xi_t}^{\text{MPC}}(x) \),

\[ \mathcal{V}_{\xi_{t+1}}^{\text{MPC}}(x(t+1)) - \mathcal{V}_{\xi_{t+1}}^{\text{MPC}}(x_1[t]) \leq L \| \varepsilon_x(t) \| \leq \gamma_A L \| \hat{\xi}_0[t] \| \| x(t) \|. \]

Also, due to the result of Lemma \(^1\)

\[ \mathcal{V}_{\xi_{t+1}}^{\text{MPC}}(x_{1|t}) - \mathcal{V}_{\xi_{t+1}}^{\text{MPC}}(x(t)) \leq -\lambda_{\min}(Q) \| x \|^2 \]

Thus,

\[ \mathcal{V}_{\xi_{t+1}}^{\text{MPC}}(x(t+1)) \leq \mathcal{V}_{\xi_{t+1}}^{\text{MPC}}(x(t)) - \lambda_{\min}(Q) \| x \|^2 + \gamma_L \| \hat{\xi}_0[t] \| \| x(t) \| \]

where \( \gamma_L = \gamma_A L. \)

**Theorem 1:** Let Assumptions \(^1\) \(^2\) hold, and \( \xi_{k|t+1} = \xi_{k+1|t} \), for all \( k \in \mathbb{Z}_{[0,N-1]} \), and all \( t \in \mathbb{Z}_{0+} \). For the MPC that at any step solves \(^3\) where \( \mathcal{P}(\xi) \) is designed according to \(^7\), \( \mathcal{X}_N = \mathcal{X} = \mathbb{R}^n \), \( \mathcal{U} = \mathbb{R}^m \), \( \mathcal{C}_{xu} = \mathbb{R}^{n+m} \), \( \mathcal{U} = \mathbb{R}^m \), \( \mathcal{V}_{\xi_N}^{\text{MPC}}(x) \) is an ISS-Lyapunov function with respect to the estimation error \( \hat{\xi}_0[t] = \tilde{\xi} - \xi_0[t] \in \hat{\Xi}(\xi_0[t]) \) for \(^1\) in closed loop with the MPC based on \(^3\) in any \( \mathcal{X}_\eta \subseteq \mathcal{X}_L \), where \( \mathcal{X}_\eta \) is RPI with respect to \( \hat{\xi}_0[t] \) for the closed loop.

**Proof:** By Lemma \(^2\) we have that

\[ \mathcal{V}_{\xi_{t+1}}^{\text{MPC}}(x(t+1)) \leq \mathcal{V}_{\xi_{t+1}}^{\text{MPC}}(x(t)) - \lambda_{\min}(Q) \| x \|^2 + \gamma_L \| \hat{\xi}_0[t] \| \| x(t) \|. \]

Due to the norm equivalence in finite dimensional spaces, for a \( p \)-norm, there exists \( \gamma_p \) such that \( \| \hat{\xi}_0[t] \| \leq \gamma_p \| \xi_0[t] \| \) for every \( \hat{\xi}_0[t] \in \mathbb{R}^n \). Hence,

\[ \mathcal{V}_{\xi_{t+1}}^{\text{MPC}}(x(t+1)) \leq \mathcal{V}_{\xi_{t+1}}^{\text{MPC}}(x(t)) - \lambda_{\min}(Q) \| x \|^2 + \gamma_L \gamma_p \| \hat{\xi}_0[t] \| \| x(t) \|. \]

Since \( \mathcal{X}_\eta \subseteq \mathcal{X}_L \) and \( \mathcal{X}_L \) is compact, there exists a finite \( \gamma_\eta > 0 \) such that \( \| x \| \leq \gamma_\eta \) for all \( x \in \mathcal{X}_\eta \).
Then, for all $x \in \mathcal{X}_\eta$, where $\mathcal{X}_\eta$ is RPI with respect to $\tilde{\xi}_{0|t} \in \tilde{\Xi}(\xi_{0|t})$,

$$Y_{\xi_{N+1}}^{\text{MPC}}(x(t+1)) \leq Y_{\xi_{t}}^{\text{MPC}}(x(t)) - \lambda_{\text{min}}(Q)\|x(t)\|^2 + \gamma_{\text{ISS}}\|\tilde{\xi}_{0|t}\|$$  \hspace{1cm} (12)

and the closed-loop is ISS with respect to $\tilde{\xi}_{0|t}$, with ISS Lyapunov function $Y_{\xi_{t}}^{\text{MPC}}$ and ISS gain $\gamma_{\text{ISS}} = \gamma_{L}\gamma_{p}\gamma_{\eta}$.

Theorem 1 requires the existence of a RPI set $\mathcal{X}_\eta$ contained in $\mathcal{X}_L$ because that is where the value function is Lipschitz-continuous. For $\eta > 0$, $\mathcal{X}_\eta \subseteq \mathcal{X}_L$ such that for all $\xi^{N} \in \Xi^{N+1}$, $Y_{\xi_{N}}^{\text{MPC}} \leq \eta$ and for all $x \in \mathcal{X}_\eta$, $Y_{\xi_{N}}^{\text{MPC}} - \lambda_{\text{min}}(Q)\|x\|^2 \leq \eta - 2\gamma_{L}\|x\|$ is RPI, because of Lemma 2 and $\|\tilde{\xi}\|_1 \leq 2$, for all $\tilde{\xi} \in \tilde{\Xi}(\xi)$.

For the case of constrained IAMPC where $\mathcal{X}$ is compact, for ensuring constraint satisfaction we need to construct a compact RPI set $\mathcal{X}_\eta \subseteq \mathcal{X}_L$, so that we can define $\mathcal{X}_L = \mathcal{X}_\eta$, and Result 2 holds in $\mathcal{X}_\eta$. Next, we focus on the case of $\mathcal{X}, \mathcal{U}$ compact and how to build $\mathcal{X}_\eta$ for constrained IAMPC.

IV. Constrained IAMPC: Robust Constraints

By designing the terminal cost from a pLF as in Section III the closed loop of (1) with the unconstrained IAMPC that solves (3) is ISS with respect to $\tilde{\xi}_{0|t}$. Next, we consider constrained IAMPC, i.e., $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^m$.

Assumption 3: $\mathcal{X}, \mathcal{U}$ are compact sets, $0 \in \text{int}(\mathcal{X})$, $0 \in \text{int}(\mathcal{U})$.

Under Assumption 3, we first show that for the LPV system (4) with perfect preview along the prediction horizon, i.e., $\xi_{k|t} = \tilde{\xi}(t+k)$, for all $k \in \mathbb{Z}_{[0,N]}$, the closed-loop recursively satisfies the constraints and is AS. Then, we enforce constraint satisfaction when $\tilde{\xi}_{0|t} \neq 0$. These objectives are achieved by designing $\mathcal{X}_N$ and $C_{xu}$ in (3), respectively.

A. Terminal set design for nominal terminal constraint

Consider (4) where $\xi(t)$ is known at $t \in \mathbb{Z}_{0+}$ and the control law (5) resulting in the closed-loop LPV system

$$x(t+1) = \sum_{i=1}^{\ell} [\xi(t)]_i (A_i + BK_i)x(t).$$  \hspace{1cm} (13)

The trajectories of (13) are contained in those of the pLDI

$$x(t+1) \in \text{co}\{(A_i + BK_i)x(t)\}_{i=1}^{\ell}. \hspace{1cm} (14)$$

For (14) in closed loop with (5) designed by (7) subject to (1b), in (16) it was shown that the maximum constraint admissible set $\mathcal{X}^\infty \subseteq \tilde{\mathcal{X}}$, where $\tilde{\mathcal{X}} = \{x \in \mathcal{X} : \kappa(\xi)x \in \mathcal{U}, \forall \xi \in \Xi\}$ is polyhedral, finitely determined and has non-empty interior with $0 \in \text{int}(\mathcal{X}^\infty)$. $\mathcal{X}^\infty$ is RPI for (13) for all $\xi \in \Xi$, and is
the limit of a sequence of backward reachable sets. Let \( \mathcal{X}_{xu} \) be a given set of feasible states and inputs \( \mathcal{X}_{xu} \subseteq \mathcal{X} \times \mathcal{U} \), \( 0 \in \text{int}(\mathcal{X}_{xu}) \) and let

\[
\mathcal{X}^{(0)} = \{ x : (x, K_i x) \in \mathcal{X}_{xu}, \forall i \in \mathbb{Z}_{[1, \ell]} \}
\]

\[
\mathcal{X}^{(h+1)} = \{ x : (A_i + BK_i)x \in \mathcal{X}^{(h)}, \forall i \in \mathbb{Z}_{[1, \ell]} \} \cap \mathcal{X}^{(h)}
\]

\[
\mathcal{X}^{\infty} = \lim_{h \to \infty} \mathcal{X}^{(h)}.
\] (15)

Due to the finite determination of \( \mathcal{X}^{\infty} \) there exists a finite \( \tilde{h} \in \mathbb{Z}_{0+} \) such that \( \mathcal{X}^{(\tilde{h}+1)} = \mathcal{X}^{(\tilde{h})} = \mathcal{X}^{\infty} \), i.e., the limit in (15) is reached in a finite number of iterations.

**Lemma 3:** Consider (4) and the MPC that at \( t \in \mathbb{Z}_{0+} \) solves (3) where \( \mathcal{X} \subset \mathbb{R}^n \), \( \mathcal{U} \subset \mathbb{R}^m \), \( C_{xu} = \mathbb{R}^{n+m} \), \( \xi_{k|t} = \bar{\xi}(t + k) \), \( \mathcal{P}(\xi) \), \( \kappa(\xi) \) are designed according to (7) and \( \mathcal{X}_N = \mathcal{X}^{\infty} \), where \( \mathcal{X}^{\infty} \) is from (15). At a given \( t \in \mathbb{Z}_{0+} \), let \( x(t) \in \mathcal{X}, \xi_{t}^N \in \mathcal{X}^{N+1} \) be such that (3) is feasible. Then, (3) is feasible for any \( \tau \geq t \), i.e., \( \mathcal{X}_f(\xi_{t}^N) = \{ x \in \mathcal{X} : (3) \) feasible for \( x_{0|t} = x, \xi_{k|t} = \xi_k \in \mathcal{X}, k \in \mathbb{Z}_{[0,N]} \} \) is a PI set, and the origin is AS in \( \mathcal{X}_f(\xi_{t}^N) \).

**Proof:** First we show that \( \mathcal{X}_f(\xi_{t}^N) \subseteq \mathcal{X} \) is PI for the closed-loop system, i.e., if \( x(t) \in \mathcal{X}_f(\xi_{t}^N) \), then \( x(t+1) \in \mathcal{X}_f(\xi_{t+1}^N) \), for all \( \xi(t + 1 + N) \in \mathcal{X} \). Since \( x(t) \in \mathcal{X}_f(\xi_{t}^N) \), there exists \( U^*(t) = [u_{0|t}^* \ldots u_{N-1|t}^*] \) optimal (and feasible) for (3). From \( x(t+1) = x_{1|t} \), given \( \xi_{t+1}^N \), consider \( \tilde{U} = [\bar{u}_{0|t+1} \ldots \bar{u}_{N-1|t+1}] \), where \( \bar{u}_{i|t+1} = u_{i+1|t}^* \) for \( i \in \mathbb{Z}_{[0,N-2]} \), \( \bar{u}_{N-1|t+1} = \kappa(\xi_{t+1}^N)x_{N|t} \). By \( \xi_{k|t} = \bar{\xi}(t + k) \), it holds \( \xi_{k|t+1} = \xi_{k+1|t} \), and the trajectory generated by \( \tilde{U} \) is such that \( \bar{x}_{k|t+1} = x_{k+1|t}, \bar{x}_{k|t+1} \in \mathcal{X}, \bar{u}_{k|t+1} \in \mathcal{U} \) for all \( k \in \mathbb{Z}_{[0,N-1]} \). Since \( \mathcal{X}_N = \mathcal{X}^{\infty}, x_{N-1|t+1} = x_{N|t} \in \mathcal{X}^{\infty} \subseteq \mathcal{X} \) and \( \kappa(\xi_{N-1|t+1})x_{N|t} \in \mathcal{U} \) for all \( \xi_{N-1|t+1} \in \mathcal{X} \), hence (3d) is satisfied. Also, \( x_{N|t+1} \in \mathcal{X}^N \), because \( \mathcal{X}^N = \mathcal{X}^{\infty} \) is PI for (4) in closed loop with (5) for all \( \xi \in \mathcal{X} \). Thus, also (3f) is satisfied and \( \tilde{U} \) is feasible from \( x(t+1) \) for any \( \xi_{t+1}^N \) that is admissible according to the assumptions, and \( x(t+1) \in \mathcal{X}_f(\xi_{t+1}^N) \). Hence, \( \mathcal{X}_f(\xi_{t}^N) \) is PI. AS follows by the same arguments of Lemma 1 with \( V_{\xi}^{\text{MPC}} \) as Lyapunov function, and with \( \mathcal{X}_f(\xi_{t}^N) \) as domain of attraction.

Next, we ensure robust satisfaction of (3d), (3f), in presence of estimation error \( \bar{\xi}_{0|t} \neq 0 \).

**B. Robust constraints design**

In order to ensure robust constraint satisfaction in the presence of parameter estimation error we design the constraint (3e) from a RCI set for the pLDI (2), whose trajectories include those of (13). Based on Definition 2 let \( C \subseteq \mathcal{X} \) be a convex set such that for any \( x \in C \) there exists \( u \in \mathcal{U} \) such that \( A_i x + Bu \in C \) for all \( i \in \mathbb{Z}_{[1, \ell]} \). Given \( C \), we design \( C_{xu} \) in (3e) as

\[
C_{xu} = \{ (x, u) \in C \times \mathcal{U}, A_i x + Bu \in C, \forall i \in \mathbb{Z}_{[1, \ell]} \},
\] (16)

that is, the state-input pairs that result in states within the RCI set for any vertex system of the pLDI (2).
Lemma 4: Consider (3) where $\mathcal{X}_N = \mathbb{R}^n$, and $\mathcal{C}_{xu}$ in (3e) is defined by (16). If $x(t) \in \mathcal{C}$, (3) is feasible for all $\tau \geq t$, for any $\xi^N_\tau \in \Xi^{N+1}$ and any $\bar{\xi}_0|_\tau \in \tilde{\Xi}(\xi_0|_\tau)$.

Proof: Due to the definition of $\mathcal{C}_{xu}$, for all $x \in \mathcal{C}$ there exists $u \in \mathcal{U}$ such that $(x, u) \in \mathcal{C}_{xu}$. Thus, if $(x(t), u(t)) \in \mathcal{C}_{xu}$, by convexity $x(t + 1) = \sum_{i=1}^\ell \bar{\xi}_i A_i x(t) + B u(t) \in \mathcal{C}$ for all $\xi^N_0|_\tau + \bar{\xi}_0|_\tau = \bar{\xi} \in \Xi$, i.e., (3e) and hence (3d) are satisfied for any actual $\bar{\xi} \in \Xi$. Since $x(t + 1) \in \mathcal{C}$, the reasoning can be repeated proving robust feasibility for any $\tau \geq t$.

$\mathcal{C}$ can be computed as the maximal RCI set for (2) from the sequence [19].

$$\mathcal{C}^{(0)} = \mathcal{X},$$
$$\mathcal{C}^{(h+1)} = \{x : \exists u \in \mathcal{U}, \forall i \in \mathbb{Z}_{[0,\ell]} \} \cap \mathcal{C}^{(h)}.$$  

The maximal RCI set in $\mathcal{X}$ is the fixpoint of (17), i.e., $\mathcal{C}^\infty \equiv \mathcal{C}^{(h)}$ such that $\mathcal{C}^{(h+1)} = \mathcal{C}^{(h)}$, and is the largest set within $\mathcal{X}$ that can be made invariant for (2) with inputs in $\mathcal{U}$.

Based on Lemma 4 by $\mathcal{C}_{xu}$ in (16) we obtain constraint satisfaction even when $\bar{\xi}_0|_\tau \neq 0$. However, the maximal RCI set does not guarantee that the terminal constraint can be satisfied, that is, it may not be possible to reach $\mathcal{X}_N$ in $N \in \mathbb{Z}_+$ steps for all $x \in \mathcal{C}$ by trajectories such that $(x_k|_t, u_k|_t) \in \mathcal{C}_{xu}$. Furthermore, for Lemma 3 to hold, the control inputs generated by (5) that make $\mathcal{X}_N$ PI for (13) must be feasible for (3), that is, $(x, \kappa(\xi)x) \in \mathcal{C}_{xu}$ for every $x \in \mathcal{X}_N$, $\xi \in \Xi$.

To guarantee satisfaction of the terminal constraint, the horizon $N$ needs to be selected such that for every $x \in \mathcal{C}$ and $\xi^N \in \Xi^{N+1}$, there exists $[u(0) \ldots u(N-1)]$ such that for (4) with $x(0) = x$, $\xi(k) = \xi_k$ for all $k \in \mathbb{Z}_{[0,N]}$, $(x(k), u(k)) \in \mathcal{C}_{xu}$ for all $k \in \mathbb{Z}_{[0,N-1]}$, and $x(N) \in \mathcal{X}_N$. Let

$$SS^{(0)} = \mathcal{X}_N,$$
$$SS^i_{(h+1)} = \{x \in \mathcal{X} : \exists u \in \mathcal{U}, A_i x + B u \in SS^i \},$$
$$SS^{(h+1)} = \bigcap_{i=1}^\ell SS^i_{(h+1)}.$$ 

The set $SS^{(h)}$ is such that for any $x(0) \in SS^{(h)}$, given any $\xi^{h-1} \in \Xi^h$, there exists a sequence $[u(0) \ldots u(h-1)]$ such that for (4) with $x(0) = x$ and $\xi(k) = \xi_k$ for all $k \in \mathbb{Z}_{[0,N]}$, $(x(k), u(k)) \in \mathcal{C}_{xu}$ and $x(h) \in \mathcal{X}_N$.

Theorem 2: Consider (3), let $\bar{h} \in \mathbb{Z}_{0+}$ be such that $\mathcal{C}^{(h+1)} = \mathcal{C}^{(h)} = \mathcal{C}$ in (17), and let $\mathcal{C}_{xu}$ be defined by (16). Let $\mathcal{X}_N = \mathcal{X}_I^\infty$ from (15), where $\mathcal{X}_{xu} = \mathcal{C}_{xu}$, and $N \in \mathbb{Z}_{0+}$ be such that $SS^{(N)} \supset \mathcal{C}$. If $x(t) \in \mathcal{C}$ at $t \in \mathbb{Z}_{0+}$, and $\xi^N_t \in \Xi^{N+1}$, $\bar{\xi}_0|_\tau \in \tilde{\Xi}(\xi_0|_\tau)$ for all $\tau \geq t$, (3) is feasible for all $\tau \geq t$. If there exists $t \in \mathbb{Z}_{0+}$ such that $\xi^N_k = \bar{\xi}(\tau + k)$ for all $\tau \geq t$, $k \in \mathbb{Z}_{[0,N]}$, (1) in closed-loop with the MPC that solves (3) is also
AS in $C$.

Proof: Since $C \subseteq SS^{(N)}$ for every $x(t) \in C$ and $\xi_t^N \in \Xi^{N+1}$, there exists an input sequence of length $N$ such that $(x_k|t, u_k|t) \in C_{xu}$ for all $k \in \mathbb{Z}_{[0,N-1]}$, and $x_N|t \in X_N$. Due to Lemma 4 for $u_{0|t}$ such that $(x_0|t, u_0|t) \in C_{xu}$, $\sum_{i=1}^{\tau} [\xi_0|t + \xi_0|i] A_i x_{0|i} + Bu_0|t \in C$, for every $\xi_0|t \in \Xi(\xi_0|t)$. Furthermore, if $\xi^N_\tau = \tilde{\xi}^N_\tau \in \Xi^{N+1}$ for all $\tau \geq t$, AS in $C$ of (1) in closed-loop with the MPC that solves (3) follows from Lemma 3 noting that if $x_N|t \in X_N$, $x_{N-1}|t+1 = x_{N}|t$, and $u_{N-1}|t+1 = \kappa(\xi_{N}|t) x_{N}|t$, then $(x_{N-1}|t+1, u_{N-1}|t+1) \in C_{xu}$, i.e., (5) is admissible in $X_N$ with respect (3b), which follows from computing $X_N$ by (15) with $X_{xu} = C_{xu}$.

In the definition of $SS^{(h)}$, i.e., (18), the parameter sequence $\xi^h$ is known. This is due to enforcing the terminal set only with respect to the nominal dynamics, while the robust invariance of $C$ and choosing $N$ so that $SS^{(N)} \supseteq C$ guarantee that at a successive step, even in presence of a parameter estimation error which causes a prediction error, the terminal set will still be reachable in $N$ steps.

There are alternative ways to compute $C$, other than as the maximal RCI. For instance, an RPI set can be constructed from (14) by including additional constraints in (7). Then, for given $N \in \mathbb{Z}_+$, $C$ can be obtained as $N$-step backward reachable set of such RPI. In this case the MPC horizon $N$ becomes a free design parameter. Such a procedure is not fully described here due to the limited space.

Theorem 2 ensures robust feasibility of (5), robust satisfaction of (15), and nominal asymptotic stability, in the sense that the closed loop is AS if there exists $t \in \mathbb{Z}_{0+}$ such that $\xi_{k|t} = 0$, for all $\tau \geq t$, $k \in \mathbb{Z}_{[0,N]}$. Next, we combine Theorem 1 and 2.

V. INDIRECT-ADAPTIVE MPC: COMPLETE ALGORITHM

The last design element in (3) is the construction of the parameter prediction vector $\xi^N_t$.

Since $\bar{\xi}$ in (1) is assumed to be constant or slowly varying, an obvious choice would be $\xi_{k|t} = \xi(t)$, for all $k \in \mathbb{Z}_{[0,N]}$, for all $t \in \mathbb{Z}_{0+}$. However, this choice violates the assumption of Theorem 1 (and implicitly those of Lemmas 1 and 3) that requires $\xi_{k|t+1} = \xi_{k+1|t}$, for all $k \in \mathbb{Z}_{[0,N-1]}$, $t \in \mathbb{Z}_{0+}$. Such an assumption is required because if the entire parameter prediction vector $\xi^N_t$ suddenly changes, the value function $V^\text{MPC}_N$ may not be decreasing. This is due to using the pLF only as terminal cost, as opposed to enforcing it along the entire horizon (5), (6), which then requires the solution of LMIs in real-time.

Thus, we introduce a $N$-step delay in the parameter prediction,

$$\xi_{k|t} = \xi(t - N + k), \forall k \in \mathbb{Z}_{[0,N]}.$$  \hspace{1cm} (19)

Due to (19), at each time $t$, the new estimate is added as last element of $\xi^N_t$, i.e., $\xi_{N|t} = \xi(t)$ and $\xi_{k|t} = \xi_{k+1|t-1}$, for all $k \in \mathbb{Z}_{[0,N-1]}$, $t \in \mathbb{Z}_{0+}$. We can now state the complete IAMPC strategy and its main result.
Theorem 3: Consider (1), where $\bar{\xi} \in \Xi$, in closed loop with the IAMPC controller that at every $t \in \mathbb{Z}_{0+}$ solves (3), where $P(\xi)$ defined by (6) and $\kappa(\xi)$ defined by (5) are from (7), $C$ and $X_N$ are designed according to Theorem 2 and $\xi^N_t \in \Xi^{N+1}$ is obtained from (19), where $\xi(t) \in \Xi$ for all $t \in \mathbb{Z}_{0+}$. If for some $t \in \mathbb{Z}_{0+}$, $x(t) \in C$, the closed-loop satisfies (1b), and (3) is recursively feasible for any $\tau \geq t$. Furthermore, the closed loop is ISS in the RPI set $C$ with respect to $\tilde{\xi}_0|\tau = \bar{\xi} - \xi_0|\tau$, i.e., the $N$-steps delayed estimation error $\tilde{\xi}_0|\tau = \bar{\xi} - \xi(t - N)$.

Proof: The proof follows by combining Theorem 1 with Theorem 2. By Theorem 2, $C$ is RCI, and if $x(t) \in C$, (3) is feasible for all $\tau \geq t$, for any $\xi^N_t \in \Xi^{N+1}$ that satisfies (19), since (19) implies that $\xi_{k|\tau} = \xi_{k+1|\tau-1}$, for all $k \in \mathbb{Z}_{[0, N-1]}$. Thus, by (16) enforced in (3), $C \subseteq X$ is a compact RPI for the closed-loop system, and hence (1b) is satisfied for all $\tau \geq t$. Since $\mathcal{V}^\text{MPC}_{\xi}$ is piecewise quadratic for every $\xi^N \in \Xi^{N+1}$, by taking $\mathcal{X}_0 = \mathcal{X}_L = C$, which is RPI for the closed-loop system and compact since $C \subseteq X$, the existence of a Lipschitz constant $L$ is guaranteed. Hence, Theorem 1 holds within $C$, proving ISS with respect to $\tilde{\xi}_0|\tau = \bar{\xi} - \xi_0|\tau = \bar{\xi} - \xi(t - N)$, i.e., the delayed estimation error.

Based on Theorem 3, from any initial state $x(t) \in C$, the closed-loop system robustly satisfies the constraints for any admissible estimation error, and the expansion term in the ISS Lyapunov function is proportional to the norm of the delayed parameter estimation error. Thus, if the parameter estimate converges at time $t^*$ and such value is maintained for all $t \geq t^*$, for all $t \geq t^* + N$, $\tilde{\xi}_t^N = 0$ and hence the closed-loop is AS. However, note that ISS holds regardless of such convergence. Finally, note that at runtime, the IAMPC solves only a QP as a standard (non-adaptive) linear MPC. Thus, the following corollary derives immediately from Theorem 3.

Corollary 1: The IAMPC designed according to Theorem 3 solves Problem 1.

The requirement of solving only QP during execution in a significant reduction of computational load and code complexity with respect to robust MPC based on LMIs [4]–[6]. The drawback is on ensuring ISS versus the robust stability in [4]–[6]. However, here we still guarantee robust constraint satisfaction.

A. Comments on parameter estimator design

The ISS property established in Theorem 3 implies that when the estimator converges to the true parameter value the closed-loop becomes AS, but it is more general than that. In fact, ISS ensures that, even if the estimate never converges, the expansion term in the Lyapunov function, and hence the ultimate bound on the state, is proportional the estimation error. Thus, ISS allows to state properties that are parametric in the estimation error, and hence hold regardless of the convergence of the estimator. Because of this and because the IAMPC design does not require a specific choice for the estimator design, we have called the IAMPC as independent of the estimator. On the other hand, it is required for the estimator to
provide \( \xi(t) \in \Xi \), for all \( t \in \mathbb{Z}_{0+} \), as per Assumption 1. To enforce Assumption 1 one can always design an estimator that produces the unconstrained estimate \( \varrho \in \mathbb{R}^\ell \), while the IAMPC uses its projection onto \( \Xi \), i.e., \( \xi = \text{proj}_\Xi(\varrho) \). By using the \( \varrho \in \mathbb{R}^\ell \) in the estimator update equation and providing \( \xi = \text{proj}_\Xi(\eta) \) to the controller, this amounts to a standard estimator with an output nonlinearity. Thus, the convergence conditions will be the same as those for standard estimators, in particular identifiability, and persistent excitation. Guaranteeing the persistent excitation in closed-loop systems is currently an active area of research also in MPC, see, e.g., [13]–[15]. While it is certainly an interesting future research direction to merge the IAMPC developed here with some of the above techniques, it is worth remarking again that for the ISS property in Theorem 3 to hold, this is not necessary as convergence is not required.

As regards to identifiability, a subject that is worth a brief discussion is the case where the true value of the parameter \( \bar{\xi} \) is not uniquely define, which may be due to the polytopic representation (1) of the uncertain system. Given the actual system matrix \( \bar{A} \) the set \( \bar{\xi}(\bar{A}) = \{ \bar{\xi} \in \Xi : \sum_{i=1}^\ell [\bar{\xi}]_i A_i = \bar{A} \} \) may have cardinality greater than 1. In this case we can provide a slightly modified ISS Lyapunov function.

**Corollary 2:** Let the assumptions of Theorem 1 hold and \( \varepsilon(\xi, \bar{A}) = \min_{\bar{\xi} \in \bar{\xi}(\bar{A})} \| \bar{\xi} - \bar{\xi} \| \). Then for (1) in closed loop with the MPC based on (3), \( V_{\xi(t+1)}^{\text{MPC}}(x(t) + 1) \leq V_{\xi(t)}^{\text{MPC}}(x(t)) - \lambda_{\text{min}}(Q\|x(t)\|^2 + \gamma_{\text{ISS}} \cdot \varepsilon(\xi, \bar{A})) \), i.e., \( V_{\xi(t)}^{\text{MPC}} \) is an ISS Lyapunov function with respect to \( \varepsilon(\xi, \bar{A}) \).

The proof follows directly by the fact that Theorem 1 and all subsequent results only use the difference between the predicted and actual system state, which is the same for all \( \bar{\xi} \in \bar{\xi}(\bar{A}) \). Thus (12) holds for all values \( \bar{\xi} \in \bar{\xi}(\bar{A}) \), which means that it has to hold for the smallest expansion term, which is \( \varepsilon(\xi, \bar{A}) \).

Loosely, this means that while formulated on the convex combination vector for computational purposes, the expansion term is a function of the difference between the estimated and actual system matrix.
VI. NUMERICAL SIMULATIONS

First, we show some simulations on a numerical example. We consider (1), where $\ell = 5$, and $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_2 = 1.1 \cdot A_1$, $A_3 = 0.6 \cdot A_1$, $A_4 = \begin{bmatrix} 0.9 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$, $A_5 = \begin{bmatrix} 0.95 & 0 \\ 0 & 1.02 \end{bmatrix}$, and $B = [-0.035 -0.905]'$. While being only second order system, this example is challenging because some of the dynamics are stable and some unstable, and the system matrices are in some cases significantly different. In fact, for a similar system, [16] showed that without proper cost function adaptation, the closed-loop may not be AS even when the perfect model is estimated. The constraints are defined by (1b), where $X = \{x \in \mathbb{R}^2 : |[x]_i| \leq 15, i = 1,2\}$, $U = \{u \in \mathbb{R} : |[u]| \leq 10\}$. We have implemented a simple estimator that computes the least squares solution $\bar{\kappa}(t)$ based on past data window of $N_m$ steps and applies a first order filter on the projection of $\bar{\kappa}(t)$ onto $\Xi$, i.e., $\xi(t+1) = (1 - \varsigma)\xi(t) + \varsigma \cdot \text{proj}_\Xi(\bar{\kappa}(t))$, where $\varsigma \in \mathbb{R}_{(0,1)}$, and $[\xi(0)]_i = 1/\ell$, $i \in \mathbb{Z}_{[1,\ell]}$. Such simple estimator satisfies Assumption 1 because projection, summation and the guarantee that the result is a convex combination vector. Also, the least square problem can be regularized by a term based on $\kappa(t-1)$, and the projection can be computed by solving a simple QP. In the simulations we use the QP solver in [20], for both projection and MPC control computation. We design the controller according to Theorem 2, where $C = \mathcal{C}^\infty$, and we select $N_m = 3$ and $N = 8$, which is the smallest value such that $SS^{(N)} \supseteq \mathcal{C}^\infty$ by [13]. Figure 1 shows the simulations where the initial conditions are the vertices of $C$ and for each initial condition, 4 different simulations with different (random) values of $\xi \in \Xi$ are executed. Figure 2 compares the cases where $\varsigma = 1/2$ and $\varsigma = 1/20$, i.e., fast versus slow estimation, thus showing the impact of the estimation error on the closed-loop behavior.
VII. CONCLUSIONS AND FUTURE WORK

We have proposed an indirect-adaptive MPC that guarantees robust constraint satisfaction, recursive feasibility, and ISS with respect to the parameter estimation error, yet has computational requirements similar to standard MPC.

The IAMPC can be easily modified to handle uncertainty also in the input-to-state matrix $B$, to exploit non-maximal yet faster to compute RCI sets, and to account for additive disturbances. Future works will detail these, as well considering tracking and designs resulting in a different ISS expansion term providing AS even in the presence of a small-but-non-zero error in the parameter estimate.

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