Simulating waves on a horizontal liquid film entrained by a gas flow

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Abstract. Nonlinear waves on a horizontal liquid film surface are considered. The effect of adjacent gas flow is taken into account through the data on shear stress at the film-gas interface obtained by the Boussinesq model of turbulence. A model nonlinear equation for the film thickness deviation from the undisturbed level is used to simulate nonlinear wave modes. Weakly nonlinear steady-state travelling solutions of this equation with wave numbers located in the vicinity of neutral wave numbers are constructed analytically. The evolution of periodic perturbations with wave numbers lying in the depth of the linear instability region is also considered numerically. Several typical scenarios of their evolution have been identified.

1. Introduction and problem statement
The joint flow of a liquid layer and gas is a well-known problem in hydrodynamic theory and practice. Solving this problem often requires two stages of modeling: first, the gas stresses on the film surface are determined, and then the evolution of waves in the liquid is calculated. Dividing the problem into such stages is used by many authors (see, for example, [1–4]).

In the present work, the dynamics of nonlinear waves on a horizontal liquid film, driven by shear stresses of the gas flow is simulated. These stresses are obtained from the solution of the gas part of the problem, the algorithm of which is described in detail in [4]. The gas flow is assumed to be turbulent and occurs in a wide horizontal channel. The x-axis is directed horizontally, and the y-axis is oriented opposite to the direction of free fall acceleration g.

For this problem, an equation for film thickness deviation from the undisturbed level was obtained in [5] by using the approximation of a small but finite perturbation amplitude. In a reference frame moving relative to a solid wall with the velocity of liquid particles at the interface (for an undisturbed flow) after the corresponding coordinate transformation and film thickness deviation, this equation is represented as:

\[
\frac{\partial H}{\partial t} + 2H \frac{\partial H}{\partial x} - \frac{\partial^2 H}{\partial x^2} + \frac{\partial^4 H}{\partial x^4} + B \int_{-\infty}^{\infty} ik^2 \tau(k) \hat{H}(k, t) e^{ikx} dk = 0. \tag{1}
\]

Here \( B \) is the complex that contains the problem parameters (see [5]). The integral term contains Fourier components of the shape of the liquid film surface \( \hat{H}(k, t) \) and Fourier components of the shear stresses of the gas at the interface \( \tau(k) = \tau_r(k) + i \tau_i(k) \). In this paper, in contrast to quasi-turbulent models similar to the Benjamin model [4, 5], the coefficient of turbulent viscosity was taken...
into account not only when calculating the undisturbed flow, but also when finding perturbations of shear stresses of the gas at the interface.

When reducing the equation for film thickness deviation to the form (1), the characteristic scales for nondimensionalization were chosen so that the neutral wave number $k_n = 1$. At that, it was assumed (see [5]) that the value of $k_n$ corresponds to a well-defined value of $\tau_{im} = \tau_{im}(k_n) \equiv \tau_{im}(1)$. In this case, for the value of parameter $B$ the following ratio is valid (see [5]):

$$B = 2/\tau_{im}(1)$$

Recall that $\tau_{im}$ is the imaginary part of $\tau = \tau(1)$.

The aim of this work is to model the process of nonlinear waves’ generation on the surface of a horizontal layer entrained by a gas flow based on equation (1).

2. Results of nonlinear wave calculation with the model equation

The results presented below were obtained with the use of the data on stresses at the film-gas interface calculated by the Boussinesq turbulence model. Approaches and methodology similar to those used in [5, 6] for quasi-laminar models were applied. As it turns out, some of the results are qualitatively consistent with the results obtained on the Benjamin model [7]. For example, calculations show that for the corresponding parameter values, similar to [5, 6], the region of unstable wave numbers for linear periodic solutions to equation (1) for the Boussinesq turbulence model is in the interval $2/nk \approx 1$. In this model, linear steady-state traveling solutions also branch off from the trivial solution at points with neutral wave numbers $k_n = 1$. As it follows from (1), their phase velocity and frequency are:

$$c_0 = B\tau_{im}(1), \quad \omega_0 \equiv c_0$$

This solution has the form [6]:

$$H = A\exp[i(kx - \omega t)] + A^2A_{H2}\exp[2i(kx - \omega t)] + C.C.$$

(2)

Here $A$ is the modulus of the amplitude of the first harmonic, and $C.C.$ is the complex-conjugate expression. Solutions (2) have a phase velocity:

$$c \equiv \frac{\omega}{k} = c_0 + A_A A^2$$

where $k = 1 + A_A A^2$, $\omega = \omega_0 + A_A A^2$, $A_A = A_0 - c_0 A_2$. The coefficients $A_{H2}$, $A_2$, $A_0$ depend only on the first two Fourier harmonics of gas shear stress pulsations at the gas-film interface and on the parameter $B$.

![Figure 1](image1.png) ![Figure 2](image2.png)

**Figure 1.** Dependence of parameter $B$ on the “gas” parameter $k_g^{-1}$.

**Figure 2.** The dependence of the coefficient $A_k$ on the parameter $B$. 
The dependence of the parameter $B$ on the "gas" parameter $k_g^{-1}$ is illustrated in figure 1. Here $k_g = k v_g / u^*$ is the dimensionless wave number for the gas part of the problem, where $u^* = \sqrt{T_0 / \rho_g}$ is the dimensional dynamic velocity in the gas; $k = 2\pi / \lambda_{cm}$ is the dimensional wave number for the corresponding spatial period $\lambda_{cm}$ of the wavy wall, $v_g$ is the kinematic coefficient of gas viscosity (see details in [5]). When calculating the values of $B$, it was taken into account that $\tau_{im}(1) = \tau_{im}(k_g)$.

Figure 2 demonstrates the dependence of the coefficient $A_k$ on the parameter $B$. It follows that the correction to the wave number is negative ($A_k < 0$), i.e., the wave numbers of periodic steady-state travelling regimes of small but finite amplitude lie in the region of linear instability of the undisturbed flow. Thus, it is clear that the branching of the wave mode (2) occurs in a “soft” way.

Figure 3. The dependence of the coefficient $A_c$ on the parameter $B$.

It is clear from figure 3, that in the vicinity of the branching point, the deeper into the instability region, the smaller the phase velocity of weakly linear perturbations is in comparison with the velocity of neutral linear perturbations $c_0$ (the coefficient $A_c$ is negative). Traditionally, the family of steady-state travelling solutions that branches off from an unperturbed solution is called the first family.

To construct periodic solutions with wave numbers that lie quite far away from the neutral wave number $k_n = 1$, the problem is solved numerically. The solutions are represented as a spatial Fourier series. Their algorithm is described in [5, 6, 8]. The following results are taken at $k_g = 0.005$, and $B = 0.623$.

The evolution of perturbations in the region of wave numbers where only the first family of steady-state travelling waves exists is quite conservative for a wide range of models. The results for the Boussinesq turbulence model are not an exception. In this case, all initial perturbations evolve to the corresponding steady-state travelling solution from the first family. In this case, if the wave number of the first harmonic is near the upper boundary of the instability region ($0.9 \leq k \leq 1$), then the first harmonic significantly exceeds other harmonics, and the shape of the steady-state travelling wave is close to the sine wave.

Further into the instability region, the nature of evolution changes. Figures 4 and 5 show the development of a perturbation with the wave number $k = 0.6$. The solution approaches a steady-state trav-
elling solution (see figure 4), for which the second and third harmonics are comparable to the first one. In this case, only the first harmonic is in the region of instability. However, the second harmonic, being stable, is located quite close to the boundary of the instability region. When the solution enters the stationary mode, the competition of its own attenuation and nonlinear quadratic pumping from the first harmonic leads to a significant increase in the modulus of the second harmonic. As a result, higher harmonics also reach higher steady-state values and, therefore, the shape of the steady-state wave differs significantly from the sine wave (see figure 5). Figure 5 presents the wave profiles at various points in time. The x-axis is fit for two wavelengths. The velocity of a stationary travelling wave is $c = -1.34$. In this and in all the examples below the following initial data are used:

$$H_{1r} = 0.1, \quad H_{1i} = 0, \quad H_{2r} = 0.05, \quad H_{2i} = 0. \quad (3)$$

The initial values of the other harmonics are taken as zero. Quite far in the instability region ($k \leq 0.5$), the variety of scenarios for the evolution of solutions increases significantly and depends essentially on the initial data. Detailed demonstration of various types of perturbation evolution is impossible in this paper. Below are only two typical examples.

**Figure 4.** Dependences of modules of the first $H_1$ (1), second $H_2$ (2), third $H_3$ (3) and fourth $H_4$ (4) Fourier harmonics on time at $k = 0.6$.

**Figure 5.** Surface profiles at $k = 0.6$ in different time moments: $t = 50$ (1), $t = 100$ (2).

**Figure 6.** Dependences of modules of the first $H_1$, second $H_2$ (2), third $H_3$, fourth $H_4$ (4) and fifth $H_5$ Fourier harmonics on time at $k = 0.3$.

**Figure 7.** Surface profiles at $k = 0.3$ in different time moments: $t = 50$ (1), $t = 100$ (2), $t = 150$ (3).
Figures 6 and 7 show an example of evolution with higher harmonics eventually becoming pre-dominant in comparison with the first one. In this case, the initial perturbation (3) has a wave number \( k = 0.3 \). After the transition process \( (0 \leq t \leq 75) \), all odd harmonics tend to zero. Modules of even harmonics reach their stationary values (see figure 6). Curves of harmonics modules \( H_1, H_3, H_5 \) are not numbered there. The limit profiles are shown in figure 7. Here, the spatial interval is also of two wavelengths \( 2\lambda \) \( (\lambda = 2\pi / k) \). It is seen that four wavelengths of the limit profile fit into this interval. As a result, the solution converges to a steady-state travelling wave mode with a wave number \( k = 0.6 \). And given that its phase velocity is \( c = -1.34 \), the initial perturbation is obviously attracted to the same solution as the limit mode shown in figures 4, 5.

Information about the evolution of the perturbation with a wave number \( k = 0.2 \) is presented in figures 8 and 9. Figure 8 shows the dependence of the modules of the first five harmonics on time. In this case, the transition process (for the used initial values (3) it takes a period of time \( 0 \leq t \leq 140 \)) ends with their attainment of constant values. As you can see from the figure, the limit values of the first three harmonics are close to each other, the third being the largest of them. The phase velocity of the limit wave mode in this case is \( c = 7.61 \). The wave profiles for three time points \( (t = 100, 200, \) and \( 300) \) are given in figure 9. Since the first harmonic is not the predominant one, two local maxima and minima are observed on the wavelength, in contrast to the previous cases.

**Conclusion**

Nonlinear waves on the surface of a horizontal liquid film entrained by a turbulent gas flow have been considered. In the case of small Reynolds numbers, the problem is reduced to the consideration of a nonlinear integro–differential equation for the film thickness deviation from the undisturbed level. It takes into account the effect of gas through the data on shear stresses at the film–gas interface, obtained by the Boussinesq turbulence model. For this model, the character of wave modes branching from the undisturbed flow has been analyzed. Weakly nonlinear steady-state traveling solutions with wave numbers located in the vicinity of neutral wave numbers have been analytically found. Similar to quasi-laminar models, the branching pattern for the Boussinesq turbulence model is found to be soft.

The evolution of periodic perturbations has been considered numerically. It is demonstrated how the behavior of such perturbations changes deeper into the linear instability region at a fixed parameter \( B \) and identical initial data. Several typical scenarios of their evolution have been identified. It is
shown how some modes with wave numbers lying quite deep in the instability region of the undisturbed flow are attracted to steady-state traveling solutions.

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References
[1] Dietze G F and Ruyer-Quil C 2013 J Fluid Mech 722 348–93
[2] Tseluiko D and Kalliadasis S 2011 J Fluid Mech 673 19–59
[3] Vellingiri R, Tseluiko D, Savva N and Kalliadasis S 2013 Int. J. Multiph. Flow 56 93–104
[4] Vozhakov I S, Arkhipov D G and Tsvelodub O Yu 2015 Therm. and Aeromech. 22 191–202
[5] Tsvelodub O Yu and Arkhipov D G 2017 J Appl Mech and Techn Phys 2017 58 619–28
[6] Tsvelodub O Yu 2019 Thermophysics and Aeromechanics 26 861–7
[7] Benjamin T B 1959 J Fluid Mech 6 161
[8] Tsvelodub O Yu and Bocharov A A 2017 Chaos, Solitons and Fractals 104 580