Covariant $W$ Gravity & its Moduli Space from Gauge Theory

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Abstract

In this paper we study arbitrary $W$ algebras related to embeddings of $sl_2$ in a Lie algebra $g$. We will give a simple general formula for all $W$ transformations, which will enable us to construct the covariant action for general $W$ gravity. It turns out that this covariant action is nothing but a Fourier transform of the WZW action. The same general formula provides a ‘geometrical’ interpretation of $W$ transformations: they are just a homotopy contraction of ordinary gauge transformations. This is used to argue that the moduli space relevant to $W$ gravity is part of the moduli space of $G$-bundles over a Riemann surface.

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1. Introduction

One of the most dramatic discoveries concerning quantum gravity in two dimensions has been Polyakov’s observation that 2D gravity in the light-cone (or chiral) gauge has a $Sl(2, \mathbb{R})$ gauge symmetry [2]. The reason for the occurrence of this unexpected symmetry has become more clear from Polyakov’s ‘soldering procedure’ [3], which shows how two dimensional diffeomorphisms can be obtained from $Sl(2, \mathbb{R})$ gauge transformations. This procedure strongly hints at a connection between the gravitational action and the WZW action for a $Sl(2, \mathbb{R})$ gauge field, something which was indeed established by [3, 4], who showed that the gravitational action is related to the WZW action of a constrained $Sl(2, \mathbb{R})$ gauge field. The consequences of these results are quite far-reaching: i) they relate the correlators in chiral gravity to those in $Sl(2, \mathbb{R})$ gauge theory, ii) they pave the way to study renormalization effects in 2D quantum gravity [5], and iii) replacing $Sl(2, \mathbb{R})$ by some other non-compact real Lie group they open the possibility to study higher spin generalizations of ordinary 2D gravity, i.e. $W$ gravity theories.

In this paper we will concern ourselves with the last point. We will show that many of the results mentioned above for the case of ordinary gravity can be extended to the case of $W$ gravity. The $W$ algebras we will consider are quite general: they are related to $sl_2$ embeddings in a Lie algebra $g$ [17], which among others contain the standard $W_N$ algebras [18]. For these generalized $W$ algebras we will construct the chiral and covariant action, and show that both can be understood in terms of a WZW model based on the Lie group $G$ corresponding to $g$. For the chiral action this implies that correlation functions in chiral $W$ gravity can be computed from the operator product expansion of the Kac–Moody currents of the WZW theory based on $G$.

More importantly, having a clear understanding of the covariant action for $W$ gravity gives us information about the degrees of freedom of $W$ gravity. To appreciate this point, consider ordinary gravity. The metric in ordinary gravity has three degrees of freedom, the Liouville field $\phi$ and a complex Beltrami differential $\mu$. These parametrize a general metric via $ds^2 = e^{-2\phi}|dz + \mu d\bar{z}|^2$. Now imagine that we start working in the chiral gauge $\bar{\mu} = \phi = 0$ and construct the chiral gravitational action $\Gamma[\mu]$, and similar, working in the gauge $\mu = \phi = 0$, its partner of opposite chirality $\Gamma[\bar{\mu}]$. Suppose we want to couple these chiral actions in such a way that the resulting theory is invariant under diffeomorphisms. For ordinary gravity we know how to do this: we have to introduce the Liouville field $\phi$, and build out of this $\phi$ and $\mu, \bar{\mu}$ a metric as indicated above.
Given this metric we construct an action using only covariant quantities, which is such that it reduces to the chiral action $\Gamma[\mu]$ or $\Gamma[\bar{\mu}]$ if one takes the corresponding gauge. Since only covariant quantities are used, this action is then diffeomorphism invariant by construction. For $W$ gravity however, this procedure is not as clear cut as this. We can construct $\Gamma[\mu_i]$, the $W$ analogue of the chiral action $\Gamma[\mu]$, but it is not at all clear which fields we should introduce in order to construct out of these and the $\mu_i, \bar{\mu}_i$ the $W$ analogue of the metric, and possible higher spin fields, simply since we have no idea what these $W$ analogues are. But if we somehow can circumvent this problem of not knowing what the $W$ analogue of the metric is, and still construct a covariant action, the extra fields one has to introduce for $W$ gravity should arise automatically. In \[35\], such extra fields were found necessary to construct an action of matter covariantly coupled to the $W_3$ algebra.

This is in fact what we shall do in this paper. We will show that the covariant action for $W$ gravity can be obtained by Fourier transformation of the WZW action based on $G$. This Fourier transformation will turn out to be easy and exact to do, since the integrals involved are simple Gaussian integrals. The resulting action is closely related to the action of a gauged WZW theory \[27\], albeit that the subgroup one usually gauges is replaced by the $W$ algebra. This procedure will make it clear that the extra fields we have to introduce for $W$ gravity can be labeled by one extra group variable $G \in G$. Specializing to a particular $W$ algebra one can subsequently show that some of the degrees of freedom labeled by $G$ are non-propagating and can therefore be integrated out \[26\]. For example for ordinary gravity it will turn out that only the Cartan subgroup of $G = SL(2, \mathbb{R})$ labels a true degree of freedom, which is of course nothing but the Liouville field $\phi$. For other $W$ algebras one is generically left with the Cartan subgroup of $G$ plus more. Working out the action for these $W$ algebras one finds that the Cartan subgroup of $G$ gives rise to a Toda system coupled in some way to the parameters $\mu_i, \bar{\mu}_i$ of the $W$ algebra, something which might have been expected from the intimate relation between Toda theories and $W$ algebras \[21\]. But since the degrees of freedom for generic $W$ algebras are not labeled by the Cartan subgroup only, this Toda system will in general be supplemented by some extra piece. This structure of the covariant action was previously found in \[26\], where we studied $W$ algebras using Chern–Simons theory. The advantage of our present method is that: \(i\) it gives better insight in what the degrees of freedom for $W$ gravity are, \(ii\) it is also applicable to non-standard $W$ algebras, \(iii\) it gives a relation between the partition function for $W$ gravity and the partition function for WZW theory, and \(iv\) the constructed covariant action is manifestly invariant under left and right $W$ transformations throughout the
construction.

Another subject we will concern ourselves with in this paper regards the ‘moduli space for $W$ gravity’. Recall that a nontrivial ingredient of string theory in general and of quantum gravity in particular is that the space of metrics modulo Weyl transformations and diffeomorphisms is not trivial, but constitute the well known moduli space of Riemann surfaces. For ordinary gravity this space is well understood, but what the appropriate generalization of this space to $W$ gravity is, is still unclear. Topological field theory and the matrix model approach to 2D gravity seem to suggest that the moduli space for $W$ gravity is somehow related to the moduli space of flat $Sl(N, \mathbb{R})$ bundles [29]. In this paper we will show that with an appropriate definition of $W$ moduli space (or more precisely $W$ Teichmüller space), which essentially defines it to be the space of $W$ fields modulo $W$ transformations, this space can be computed for arbitrary $W$ gravity. It turns out that for the standard $W_N$ algebras, this space is a component of the moduli space of flat $Sl(N, \mathbb{R})$ bundles (as conjectured by E. Witten in [30]), and that for the other $W$ algebras it is a space whose interpretation remains unclear.

The computation involves several steps that are of independent interest. First of all, a formulation of $W$ algebras on arbitrary Riemann surfaces is given, thereby generalizing [31]. This is necessary, because the existence of moduli is intimately related to the fact that the Riemann surface has nontrivial topology. This formulation involves certain nontrivial $Sl(N, \mathbb{C})$ bundles over the Riemann surface, forcing us to extend the definition of the WZW action to the case of nontrivial bundles. This will enable us to write down chiral and covariant actions on arbitrary genus Riemann surfaces. Next, we show how one can obtain $W$ transformations from ordinary gauge transformations by a mathematical procedure called homotopy contraction, providing an alternative geometrical picture of $W$ transformations. To complete the proof of the relation between $W$ moduli space and flat $Sl(N, \mathbb{R})$ bundles, we show that the $W$ moduli space is in a natural way a subspace of the so-called moduli space of Higgs bundles [13, 15]. This generalizes the relation between metrics on a Riemann surface and Higgs bundles previously discussed by Hitchin.

This paper is organized as follows: in section 2 we show how the covariant action for ordinary gravity can be obtained from $Sl(2, \mathbb{R})$ WZW theory. In section 3 we review the construction of general $W$ algebras associated to $sl_2$ embeddings in certain non-compact Lie algebras $g$, and present a general formula for $W$ transformations using an operator $L$ that depends only on the $sl_2$ embedding. This general formulation is
used in section 4 to construct the chiral and covariant actions for general $W$ gravity. Finally, in section 5 we discuss the generalization of all this to higher genus Riemann surfaces and compute the moduli space for $W$ gravity.

In this paper we will only consider classical $W$ algebras, by which we mean Poisson algebras that describe the large $c$ behavior of quantum $W$ algebras, i.e. of all numerical coefficients of the quantum $W$ algebras we only keep the highest power of $1/c$. In the literature, one sometimes finds different definitions of a classical $W$ algebra, that are further reduced versions of the $W$ algebras we consider. All our computations will also be valid only to lowest order in $1/c$.

2. Induced Gauge Theory and Gravity in Two Dimensions

Consider some action $S(\text{matter}, g_{ab})$, describing a set of matter fields coupled to gravity, which is invariant under diffeomorphisms and Weyl transformations. Integrating out the matter from such a theory one obtains a gravitational induced action $\Gamma[g_{ab}]$. If the theory has no anomalies, $\Gamma[g_{ab}]$ reduces to an action on moduli space, since on the classical level the number of degrees of freedom of the metric equals the number of invariances. However, as is well known, the procedure of integrating out the matter cannot be done in both a Weyl and diffeomorphism invariant way, leading to a non-trivial $g$ dependence of $\Gamma[g_{ab}]$. Using a diffeomorphism invariant regulator to handle the matter integration one obtains the following expression for the induced gravitational action:

$$\Gamma[g_{ab}] = \frac{c_m}{96\pi} \int R \frac{1}{\Box} R,$$

a result first obtained by Polyakov [1]. Note that $c_m$, the central charge of the matter system, is the only remnant of the matter system we used to define the induced action.

In this paper we will concern ourselves with the generalization of the above result for ordinary gravity to the case of $W$ gravity. At first sight this seems rather difficult, since up till now it is by no means clear what the $W$ generalizations of the metric, diffeomorphisms, etc. are. We will circumvent these difficulties by first relating (2.1) to the action of a $Sl(2,\mathbb{R})$ WZW model. Then the generalization to $W$ gravity will consist of replacing $Sl(2,\mathbb{R})$ by some other non-compact real Lie group.
The fact that $\Gamma[g_{ab}]$ is somehow related to $Sl(2, \mathbb{R})$ WZW theory, might have been expected from the works of Polyakov [2] and Bershadsky and Ooguri [4], who showed that a similar relation exists for induced gravity in the chiral gauge [4]. Indeed it was Polyakov who first discovered that two dimensional gravity in the chiral gauge can be described in terms of a $Sl(2, \mathbb{R})$ current algebra [2]. The reason for the appearance of this $Sl(2, \mathbb{R})$ current algebra was further elucidated in [4], where it was shown that the partition function for gravity in the chiral gauge can be written in terms of a constrained WZW model:

$$\int D\mu \exp(-\Gamma[\mu]) = \int \frac{Dg}{(\text{gauge vol})} \delta(J^+ - 1) \exp(-kS^{-}_{\text{wzw}}(g)),$$

(2.2)

where $J_z = g^{-1} \partial g$, $k$ is related to the matter central charge $c_m$ via $c_m \sim -6k$, and we have to divide by the volume of the Borel subgroup $B^- = \exp(\epsilon T^-)$ under which the constrained WZW model is invariant. This result can be straightforwardly generalized to the case of chiral $W_N$ gravity (at least in the large $k$ limit) by considering a constrained $Sl(N, \mathbb{R})$ model instead of the $Sl(2, \mathbb{R})$ model.

We will take a different approach to the subject here, and study the covariant induced gravity theory. We will argue that also the partition function for this theory can be understood in terms of a $Sl(2, \mathbb{R})$ WZW model. The correspondence we will arrive at reads:

$$\int \frac{Dg_{ab}}{(\text{gauge vol})} \exp(-\Gamma[g_{ab}]) = \int \frac{Dg DG D\bar{g}}{(\text{gauge vol})} \delta(J^+_z - 1) \delta(\bar{J}^-_z - 1) \exp(-kS^{-}_{\text{wzw}}(gG\bar{g}^{-1})),$$

(2.3)

where on the l.h.s. we have to divide by the volume of the diffeomorphism group and on the r.h.s. we have to divide by the symmetry group of the left, right constrained WZW model. Also this formula can be easily generalized to the case of general $W$ gravity by replacing $Sl(2, \mathbb{R})$ by some other non-compact real Lie group, something which will be done in section 4.

Before we come to the above result (2.3), we will first explain how chiral induced gravity is related to chiral induced $Sl(2, \mathbb{R})$ gauge theory. We start by reviewing some generalities of chiral induced gauge theories. Next we show how the defining relation for $\Gamma[\mu]$ can be obtained from $Sl(2, \mathbb{R})$ gauge theory, which makes it possible to obtain an expression for $\Gamma[\mu]$. Then, to obtain the covariant action for induced gravity, we...
couple the left and right sector, i.e. $\Gamma[\mu]$ and $\Gamma[\bar{\mu}]$, in such a way that the resulting theory is invariant under diffeomorphisms, leading to (2.3).

2.1. Chiral Induced Gauge Theories

Given some action $S(\text{matter}, A_{\bar{z}})$, describing a chiral gauge field $A_{\bar{z}}$ coupled to some matter system, which is invariant under gauge transformations, we define the induced action for chiral gauge theory as:

$$\exp(-\Gamma[A_{\bar{z}}]) = \int D(\text{matter}) \exp(-S(\text{matter}, A_{\bar{z}})) = \langle \exp -\frac{1}{2\pi} \int d^2 z \text{Tr}(A_{\bar{z}}J_z) \rangle,$$

where $J_z^a$ are the Noether currents corresponding to the gauge symmetry. The operator product expansion

$$J_z^a(z)J_w^b(w) \sim f^{ab}_{\bar{c}}J_{\bar{c}}^c(z,w) + \frac{k}{(z-w)^2} \eta^{ab},$$

where $k$ is the level of the matter current algebra, implies the following differential equation for the induced action:

$$\left(\eta^{ab}\bar{\partial} + f^{ab}_{\bar{c}}A_{\bar{c}}(z,\bar{z})\right) \frac{\delta \Gamma[A_{\bar{z}}]}{\delta A_{\bar{z}}^a(z,\bar{z})} = \frac{k}{2\pi} \partial A_{\bar{z}}^a(z,\bar{z}).$$

The solution to this equation is well known [4]: it is given by $\Gamma[A_{\bar{z}}] = kS_{wzw}^+(g)$, where $A_{\bar{z}}$ is related to $g$ via $A_{\bar{z}} = g^{-1}\partial g$. So starting with any matter system coupled to a chiral gauge field $A_{\bar{z}}$, the induced gauge theory action will always be given by $kS_{wzw}^+(g)$, with $k$ the level of the matter current algebra. Note that if one defines

$$j_{\bar{z}}^{ind} = \frac{2\pi}{k} \frac{\delta \Gamma[A_{\bar{z}}]}{\delta A_{\bar{z}}},$$

(2.7) states that the pair $\{j_{\bar{z}}^{ind}, A_{\bar{z}}\}$ has vanishing curvature. In this way the WZW functional can be viewed upon as the solution of the zero-curvature condition.

\footnote{Such a parametrization for $A_{\bar{z}}$ is of course only generic if we are working on the plane, as we do here. The generalization of this to higher Riemann surfaces will be dealt with in section 5.}
In the definition of $\Gamma[A_{\bar{z}}]$ the gauge field $A_{\bar{z}}$ is treated as a classical background field. To ‘quantize’ the gauge field we consider the generating functional for $A_{\bar{z}}$ correlators:

$$\exp(-\Gamma[U_{\bar{z}}]) = \int DA_{\bar{z}} \exp \left( -\Gamma[A_{\bar{z}}] + \frac{k}{2\pi} \int d^2z \, \text{Tr}(A_{\bar{z}}U_{\bar{z}}) \right).$$

(2.8)

$\Gamma[U_{\bar{z}}]$, which can be seen as the Fourier transform of $\Gamma[A_{\bar{z}}]$, can be computed explicitly if one uses the Polyakov–Wiegman identity for the WZW action (see the Appendix):

$$S_{wzw}^+(gh) = S_{wzw}^+(g) + S_{wzw}^+(h) + \frac{1}{2\pi} \int d^2z \, \text{Tr}(g^{-1}\partial g \partial hh^{-1}).$$

(2.9)

For this, parametrize $A_{\bar{z}} = g^{-1}\partial g$ and $U_{\bar{z}} = h^{-1}\partial h$, and replace the measure $DA_{\bar{z}}$ in (2.8) by the Haar measure $Dg$. One has:

$$\exp(-\Gamma[U_{\bar{z}}]) = \exp(kS_{wzw}^-(h)) \int Dg \exp(-kS_{wzw}^+(gh^{-1})).$$

(2.10)

Since the Haar measure is invariant under the action of the gauge group, the result of the $g$ integration is independent of $h$, and we obtain $\Gamma[U_{\bar{z}}] = -kS_{wzw}^-(h)$. This result could also have been derived as follows: since in the large $k$ limit the semi-classical approximation to the path integral in (2.8) is exact, we solve for the stationary points of $\Gamma[A_{\bar{z}}] - \frac{k}{2\pi} \int d^2z \, \text{Tr}(A_{\bar{z}}U_{\bar{z}})$. Using the results of Appendix A one easily verifies that these are given by $g = h$. Plugging this back into (2.8) results in $\Gamma[U_{\bar{z}}] = -kS_{wzw}^-(h)$.

In the next subsection we will consider the analogous case of induced chiral gravity. The operator product expansion of the matter stress energy tensor $T_{\text{mat}}$ gives a differential equation for the induced action $\exp(-\Gamma[\mu]) = \langle \exp(-\frac{1}{2\pi} \int \mu T_{\text{mat}}) \rangle$ quite similar to (2.6). We will show that this differential equation can be obtained from the zero-curvature condition for a $\text{Sl}(2, \mathbb{R})$ gauge field if we constrain the gauge field to a particular form. (The origin of these constraints can be traced back to the work of Drinfel’d and Sokolov [23], who showed how such a constrained gauge field in a natural way (via Hamiltonian reduction) gives rise to the second Gelfand-Dickii bracket [24], which exactly reproduces the classical form of W algebras.) Given this relation between the defining differential equation for $\Gamma[\mu]$ and the zero-curvature condition for a constrained gauge field, it will be straightforward to write down an expression for

\footnote{Here we neglect the Jacobian $|\frac{DA_{\bar{z}}}{Dg}|$ arising from the variable transformation, which can be computed exactly as $\det(\partial + A_{\bar{z}}) = \exp(2c_{V}S_{wzw}^+(g))$, where $c_{V}$ is the dual Coxeter number. This contribution can be ignored since we take the large $k$ limit.}
Before we come to this, we will spend a few words on how two dimensional diffeomorphisms follow from $Sl(2,\mathbb{R})$ gauge transformations if one considers constrained gauge fields, a construction due to Polyakov \[3\].

### 2.2. Diffeomorphisms from Gauge Transformations

Consider a constrained $Sl(2,\mathbb{R})$ gauge potential $A_z$ of the form

$$A_z = \Lambda + W = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix},$$

(2.11)

where $\Lambda$ is the constant matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $W$ is the matrix containing $T$, the only dynamical component of $A_z$. We will focus on gauge transformations that leave the form of $A_z$ invariant. As we will see in a moment, finding such gauge transformations is equivalent to finding a current $J_{\bar{z}}$, such that the pair $\{A_z, J_{\bar{z}}\}$ has vanishing field strength. One can easily solve for two of the three components of $J_{\bar{z}}$ from the zero-curvature equation

$$F(A_z, J_{\bar{z}}) = \partial J_{\bar{z}} - \bar{\partial} A_z + [A_z, J_{\bar{z}}] = 0.$$

(2.12)

The result is that $J_{\bar{z}}$ should be of the form

$$J_{\bar{z}} = \begin{pmatrix} \frac{1}{2} \partial \mu \\ \mu T - \frac{1}{2} \partial^2 \mu \\ -\frac{1}{2} \partial \mu \end{pmatrix}.$$

(2.13)

Substituting this in (2.12) leaves us with one equation, which is precisely the chiral Virasoro Ward-identity

$$\left(\bar{\partial} - \mu \partial - 2(\partial \mu)\right) T + \frac{1}{2} \partial^3 \mu = 0.$$

(2.14)

If we replace in this equation $\mu$ by $\epsilon$ and $\bar{\partial} T$ by $\delta \epsilon T$, we find precisely the transformation rule for an energy-momentum tensor with $c = 6$ under a general co-ordinate transformation. Going back to (2.12), we see that

$$\delta \epsilon A_z = D_{A_z}(J_{\bar{z}}(\epsilon)),$$

(2.15)
where $D_{Az} = \partial + [A_z, \cdot]$ is the covariant derivative, and $J_{\tilde{z}}(\epsilon)$ is (2.13) with $\mu$ replaced by $\epsilon$. Note that $J_{\tilde{z}}(\epsilon)$ is indeed the unique gauge transformation that leaves the form of $A_z$ invariant. This shows that co-ordinate transformations take the form of field dependent gauge transformations. This generalizes to arbitrary $W$ algebras as will be discussed in section 3.

2.3. Chiral Gravity from $Sl(2, \mathbb{R})$ Gauge Theory

As emphasized in [3] the above geometrical observation hints at a connection between the gravitational action and the action for induced $Sl(2, \mathbb{R})$ gauge theory. Consider the induced action $\exp(-\Gamma[\mu]) = \langle \exp(-\frac{1}{2\pi} \int \mu T_{\text{mat}}) \rangle$ for chiral gravity. Since $\Gamma[\mu]$ is the generating functional for the correlation functions of the matter stress energy tensor $T_{\text{mat}}$, the operator product expansion

$$T_{\text{mat}}(z)T_{\text{mat}}(w) \sim \frac{c_m/2}{(z-w)^4} + \frac{2T_{\text{mat}}(w)}{(z-w)^2} + \frac{\partial T_{\text{mat}}(w)}{z-w},$$

implies the following differential equation for $\Gamma[\mu]$:

$$\left( \bar{\partial} - \mu \partial - 2(\partial \mu) \right) \frac{\delta \Gamma[\mu]}{\delta \mu} = \frac{c_m}{24\pi} \partial^3 \mu. \quad (2.17)$$

Defining

$$T_{\text{ind}} = \frac{2\pi}{k} \frac{\delta \Gamma[\mu]}{\delta \mu}, \quad (2.18)$$

(where we made the identification $c_m \sim -6k$) we see that this defining relation for $\Gamma[\mu]$ is precisely the same as the chiral Virasoro Ward identity (2.14), obtained in the previous subsection by considering the zero-curvature condition for a constrained $Sl(2, \mathbb{R})$ gauge field.

As explained in section 2.1, the solution to this zero-curvature condition is given by the WZW functional. This implies that $\Gamma[T]$, which is the Fourier transform of $\Gamma[\mu]$, is simply given by $\Gamma[T] = -kS_{wzw}(g)$, where $g$ is such that $g^{-1} \partial g = \Lambda + W$ (see (2.11)). In turn, $\Gamma[\mu]$ can then be determined by Fourier transformation:

$$\exp(-\Gamma[\mu]) = \int DT \exp \left(-\Gamma[T] - \frac{k}{2\pi} \int d^2 \mu T \right) \quad (2.19)$$
\[ = \int DT D h \exp \left( -k S^+_{wzw}(h) + \frac{k}{2\pi} \int d^2 z \, \text{Tr}(g^{-1} \partial g h^{-1} \bar{\partial} h) - \frac{k}{2\pi} \int d^2 z \, \mu T \right) \]
\[ = \int D h \, \delta (J^+ - \mu) \exp \left( -k S^+_{wzw}(h) + \frac{k}{2\pi} \int d^2 z \, \text{Tr}(\Lambda h^{-1} \bar{\partial} h) \right) , \]

where \( J^z = h^{-1} \bar{\partial} h, \) and we used the Polyakov–Wiegman identity for the WZW action to go from the first to the second line. The ‘extra’ term in the exponent, \( \text{Tr}(\Lambda h^{-1} \bar{\partial} h), \) was previously found in \([3]\) and in a different form in \([39]\). Before we come to our final, more manageable result for \( \Gamma[\mu] \) note that the argument of the exponent in the last line can be written as \(-k S^+_{wzw}(h f), \) with \( f = \exp(-z \Lambda) \). Using the invariance of the Haar measure we thus find

\[ \exp(-\Gamma[\mu]) = \int D h \, \delta (\mu - (J^+ + 2z J^0 - z^2 J^-)) \exp(-k S^+_{wzw}(h)), \tag{2.20} \]

which explains the occurrence of the \( Sl(2, \mathbb{R}) \) current algebra in chiral induced gravity.

In order to obtain a more tangible expression for \( \Gamma[\mu] \) we take the large \( k \) limit of (2.19). In this limit \( \Gamma[\mu] \) is given by the semi-classical approximation to (2.19). As explained in Appendix A, the equation of motion for \( h \) has as its solution that \( h^{-1} \bar{\partial} h \) should be of the form (2.13). So \( \Gamma[\mu] = k S^+_{wzw}(h) - \frac{k}{2\pi} \int d^2 z \, \text{Tr}(\Lambda h^{-1} \bar{\partial} h), \) with \( J^z = h^{-1} \bar{\partial} h \) of the form (2.13). In (2.13) \( T \) is related to \( \mu \) via the chiral Virasoro Ward identity (2.14). This identity can in principle be used to determine \( T \) as a non-local function of \( \mu. \) In the next subsection we will see how a local expression for \( T \) can be obtained when one takes a convenient parametrization for \( \mu. \) [2].

**2.4. Local Expressions for the Induced Actions \( \Gamma[T] \) and \( \Gamma[\mu] \)**

We saw that the induced action \( \Gamma[\mu] \) can be written in terms of a WZW model, in which the group element \( h \) is constrained such that \( J^z = h^{-1} \bar{\partial} h \) is of the particular form (2.13). To find a more explicit form for the induced action we now want to solve the constraints imposed on \( h \). For this we take the following Gauss decomposition for \( h: \)

\[ h = \begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^\psi & 0 \\ 0 & e^{-\psi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ V & 1 \end{pmatrix} . \tag{2.21} \]

Demanding \( J^z \) to be of the form (2.13) gives the following identities for \( F, V, \psi: \) \( \psi = \frac{1}{2} \log \partial F \) and \( V = -\partial \psi, \) and \( F \) the only independent variable is related to \( \mu \) through
μ = ∂F/∂F. From (2.13) one obtains the following local expression for T:

\[ T = \frac{1}{2} \{ F, z \} = \frac{1}{2} \left( \frac{\partial^3 F}{\partial F^2} - \frac{3}{2} \left( \frac{\partial^2 F}{\partial F} \right)^2 \right), \quad (2.22) \]

i.e. the Schwarzian derivative of F. One easily verifies that this μ and T indeed solve the chiral Virasoro Ward identity (2.14). Inserting this solution for h back into the result Γ[μ] = kS^wzw + wzw(h) − k/2π \int d^2z Tr(Λh^{-1}∂h) we found previously, gives

\[ Γ[μ(F)] = \frac{k}{8\pi} \int d^2z \frac{\bar{∂}F}{(∂F)^3} \left( ∂^3 F∂F - (∂^2 F)^2 \right). \quad (2.23) \]

In a similar way we can find a local expression for Γ[T] = −kS^wzw(g), with g\(^{-1}\)∂g restricted to be of the form (2.11). Taking the above Gauss decomposition for g we find that the constraints on g are also solved by \( ψ = \frac{1}{2} \log ∂F, V = -∂ψ. \) Again T is the Schwarzian derivative of F, and Γ[T] becomes

\[ Γ[T(F)] = -\frac{k}{8\pi} \int d^2z \frac{\bar{∂}F}{(∂F)^3} \left( ∂^3 F∂F - 2(∂^2 F)^2 \right). \quad (2.24) \]

Note that Γ[T(F)] = −Γ[μ(F\(^{-1}\))], where F\(^{-1}\) is such that F(F\(^{-1}\)(z, \( \bar{z} \)), \( \bar{z} \)) = z. So in this picture Fourier transformation just amounts to taking the inverse function.

### 2.5. Covariant Gravity from Sl(2, IR)

Previously we demonstrated how diffeomorphisms arise from Sl(2, IR) gauge transformations if the gauge field is restricted to a particular form. This fact enabled us to compute the action for induced chiral gravity

\[ Γ[μ] = kS^wzw(h) - \frac{k}{2\pi} \int d^2z Tr(Λh^{-1}∂h), \quad (2.25) \]

where \( J_z = h^{-1}∂h \) is restricted as in (2.13). As explained diffeomorphisms are represented by gauge transformations: \( δ_h h = hX(ε) \), where X(ε) is (2.13) with μ replaced by ε. Note that this transformation rule for h reproduces the standard transformation rule for the Beltrami differential μ: \( δ_ε μ = ∂ε + ε∂μ - μ∂ε \). Given the behavior of the WZW
functional under gauge transformations, one easily verifies the following ‘anomalous’
transformation rule of the induced action:
\[ \delta \Gamma[\mu] = -\frac{k}{4\pi} \int d^2 z \epsilon \partial^3 \mu. \]

Instead of gauging the right handed sector which we did up till now, we could as
well gauge the left handed sector. Of course, all the results derived above for the former
gauge have their analogs for this new gauge. The induced action \( \Gamma[\bar{\mu}] \) is given by
\[ \Gamma[\bar{\mu}] = kS_{wzw}^{-} (\bar{h}) - \frac{k}{2\pi} \int d^2 z \text{Tr}(\bar{\Lambda} \bar{h}^{-1} \partial \bar{h}), \quad (2.26) \]
where \( \bar{h} \) is such that \( \bar{h}^{-1} \partial \bar{h} \) is given by the transpose of \( (2.13) \) with \( \mu, T \)
replaced by \( \bar{\mu}, \bar{T} \), and \( \bar{\Lambda} \) is the transpose of \( \Lambda \). Similarly, diffeomorphisms are represented by \( \delta \epsilon \bar{h} = \bar{h} \bar{X}(\epsilon) \),
where \( \bar{X}(\bar{\epsilon}) \) is the transpose of \( X(\epsilon) \) with \( \epsilon \) replaced by \( \bar{\epsilon} \). Under diffeomorphisms \( \Gamma[\bar{\mu}] \)
transforms as:
\[ \delta \Gamma[\bar{\mu}] = -\frac{k}{4\pi} \int d^2 z \bar{\epsilon} \partial^3 \bar{\mu}. \]

The main goal of this section is to construct out of the chiral actions \( \Gamma[\mu] \) and
\( \Gamma[\bar{\mu}] \) a covariant action \( S_{\text{cov}}(\mu, \bar{\mu}, \Phi) \) which is invariant under diffeomorphisms. Here \( \Phi \)
denotes some set of additional fields which may be needed to make the action invariant.
Stated differently, we are looking for a ‘local counterterm’ \( \Delta \Gamma[\mu, \bar{\mu}, \Phi] \) whose anomalous
behavior under diffeomorphisms exactly cancels that of the chiral actions. Then the
covariant action will be given by \( S_{\text{cov}} = \Delta \Gamma[\mu, \bar{\mu}, \Phi] + \Gamma[\mu] + \Gamma[\bar{\mu}] \). An interesting
consequence of this decomposition of \( S_{\text{cov}} \) into a left and right chiral action and a
mixed term \( \Delta \Gamma \) is that if we define the wave-function \( \Psi[\mu] = \exp(-\Gamma[\mu]) \), the partition
function for induced gravity can be written as the norm squared of \( \Psi[\mu] \):
\[ Z_{\text{grav}} \equiv \int \frac{Dg_{ab}}{\text{(gauge vol)}} \exp(-\Gamma_{\text{grav}}) = \langle \Psi, \Psi \rangle, \quad (2.27) \]
where \( \langle \rangle \) denotes the volume of the diffeomorphism group, and the inner-
product is defined as:
\[ \langle \Psi_1, \Psi_2 \rangle \equiv \int \frac{D\mu D\bar{\mu} D\Phi}{\text{(gauge vol)}} \exp(-\Delta \Gamma[\mu, \bar{\mu}, \Phi]) \Psi_1[\bar{\mu}] \Psi_2[\mu]. \quad (2.28) \]
From standard Fourier theory we then know that the partition function can also be writ-
ten as the norm squared of the Fourier transformed wave-function \( \Upsilon[T] = \exp(-\Gamma[T]) \),
for which the inner-product is given by:
\[ \langle \Upsilon_1, \Upsilon_2 \rangle \equiv \int \frac{DT D\bar{T} D\Phi}{\text{(gauge vol)}} \exp(-\Delta \Gamma[T, \bar{T}, \Phi]) \Upsilon_1[\bar{T}] \Upsilon_2[T], \quad (2.29) \]
where $\Delta \Gamma[T, \bar{T}, \Phi]$ is the Fourier transformed of $\Delta \Gamma[\mu, \bar{\mu}, \Phi]$ with respect to $\mu, \bar{\mu}$.

This second formulation is in fact more suitable, because, as we will see shortly, the covariant action $S_{\text{cov}}(T, \bar{T}, \Phi) = \Delta \Gamma[T, \bar{T}, \Phi] + \Gamma[T] + \Gamma[\bar{T}]$, is easier to derive. Recall that $\Gamma[T]$ is simply given by a restricted WZW functional and that diffeomorphisms are represented by the gauge transformations $\delta_{\epsilon}g = gX(\epsilon)$. Similarly, $\Gamma[\bar{T}] = -kS^+ (\bar{g})$, with $\bar{g}^{-1} \bar{\partial} \bar{g} = \bar{\Lambda} + \bar{W}$ (the transpose of (2.11) with $T$ replaced by $\bar{T}$), and diffeomorphisms are represented as $\delta_{\bar{\epsilon}}\bar{g} = \bar{g}X'(\bar{\epsilon})$. From all this it is obvious that if we introduce as our extra field $\Phi$ one more group variable $G$, which under diffeomorphisms transforms as:

$$\delta_{\epsilon, \bar{\epsilon}}G = -X(\epsilon)G + G\bar{X}(\bar{\epsilon}), \quad (2.30)$$

the covariant action we are looking for is given by $S_{\text{cov}}(T, \bar{T}, G) = -kS_{wzw}^{-w}(gG\bar{g}^{-1})$.

Using the Polyakov–Wiegman identity for the WZW functional, one now easily computes $\Delta \Gamma[T, \bar{T}, G]$ to be:

$$\Delta \Gamma[T, \bar{T}, G] = \frac{k}{2\pi} \int d^{2}z \left( (\bar{\Lambda} + \bar{W})G^{-1}\partial G \right) - \frac{k}{2\pi} \int d^{2}z \left( (\Lambda + W)\bar{\partial}GG^{-1} \right)$$

$$+ \frac{k}{2\pi} \int d^{2}z \left( (\Lambda + W)G(\bar{\Lambda} + \bar{W})G^{-1} \right) - kS_{wzw}^{-w}(G). \quad (2.31)$$

$\Delta \Gamma[\mu, \bar{\mu}, G]$ can now be obtained from this by Fourier transformation. In the chiral case this was a non-trivial, although not impossible procedure, but here things are much more simple since the integral over $T, \bar{T}$ is a Gaussian integral, and can thus easily be carried out. Indeed, if we parametrize $G$ by the Gauss decomposition:

$$G = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \begin{pmatrix} e^{\phi} & 0 \\ 0 & e^{-\phi} \end{pmatrix} \begin{pmatrix} 1 & -\bar{\omega} \\ 0 & 1 \end{pmatrix}, \quad (2.32)$$

the saddle-point equations for $T, \bar{T}$ become

$$T = \omega^{2} - \partial\omega + \mu e^{-2\phi},$$

$$\bar{T} = \bar{\omega}^{2} - \bar{\partial}\bar{\omega} + \mu e^{-2\phi}. \quad (2.33)$$

Substituting these expressions back leads to the following result for $\Delta \Gamma[\mu, \bar{\mu}, G]$ \cite{25,26}

$$\Delta \Gamma[\mu, \bar{\mu}, G] = \frac{k}{2\pi} \int d^{2}z \left[ \partial\phi \bar{\partial}\bar{\phi} + \omega(2\bar{\partial}\phi + \partial\mu) + \bar{\omega}(2\partial\phi + \bar{\partial}\bar{\mu}) + \mu \omega^{2} + \bar{\mu}\bar{\omega}^{2} + 2\omega\bar{\omega} - (1 - \mu\bar{\mu})e^{-2\phi} \right]. \quad (2.34)$$

\footnote{Note that $S_{\text{cov}}$ cannot just be any function of $gG\bar{g}^{-1}$ since for the chiral gauges $G = 1, T = 0$ or $G = 1, \bar{T} = 0$ the covariant action should reduce to the corresponding chiral action.}
From (2.34) we recognize that $\omega, \bar{\omega}$ are auxiliary fields. Replacing also these fields by their equations of motion gives:

$$\Delta \Gamma[\mu, \bar{\mu}, \phi] = S_L[\phi, \mu, \bar{\mu}] + K[\mu, \bar{\mu}].$$

(2.35)

Here

$$S_L = \frac{k}{4\pi} \int d^2 z \sqrt{-\hat{g}} \left( \hat{g}^{ab} \partial_a \phi \partial_b \phi + 4 e^{-2\phi} + \phi \hat{R} \right),$$

(2.36)

is the well known Liouville action, the metric $\hat{g}$ is defined by $ds^2 = |dz + \mu d\bar{z}|^2$, and $K[\mu, \bar{\mu}]$ reads

$$K[\mu, \bar{\mu}] = \frac{k}{4\pi} \int d^2 z (1 - \mu \bar{\mu})^{-1} \left( \partial \mu \bar{\mu} - \frac{1}{2} \mu (\bar{\partial} \mu)^2 - \frac{1}{2} \bar{\mu} (\partial \mu)^2 \right).$$

(2.37)

$\Delta \Gamma[\mu, \bar{\mu}, \phi]$ as given in (2.35) is known as the Quillen–Belavin–Knizhnik anomaly [28]. It ‘covariantizes’ the product of the wave-functions $\Psi[\mu] \Psi[\bar{\mu}]$. In conclusion we find that Polyakov’s result for the covariant induced gravity action (2.1) admits the following decomposition:

$$\frac{c_m}{24\pi} \int R \frac{1}{\Box} R = \Delta \Gamma[\mu, \bar{\mu}, \phi] + \Gamma[\mu] + \Gamma[\bar{\mu}].$$

(2.38)

In section 4 the above results for ordinary gravity will be generalized to the case of $W$ gravity, simply by replacing $Sl(2, \mathbb{R})$ by some other real, non-compact Lie group. We have kept our notation quite general so that (2.31) is in fact already the correct formula for general $W$ gravity; one only has to choose some $G$ and some $\Lambda$ and then the Fourier transform of (2.31) gives the local counterterm which covariantizes the product of the chiral wave-function for general $W$ gravity times its partner of opposite chirality. Before we end this section we will make a few more comments on the partition function of covariant induced gravity.

2.6. THE PARTITION FUNCTION OF COVARIANT INDUCED GRAVITY

In the previous subsection we already argued that the partition for covariant induced gravity could be seen as the norm squared of the chiral wave-function $\Psi[\mu] = \exp(-\Gamma[\mu])$, which is a solution of the chiral Virasoro Ward identity, when one takes a suitable definition for the inner-product (2.28). Equivalently, the partition function could be written as the modulus squared of the Fourier transformed wave-function...
\[ \Upsilon[T] = \exp(-\Gamma[T]), \] again for some suitable definition of the inner-product (2.28). In both cases this choice of the inner-product is such that

\[ Z_{\text{grav}} = \int \frac{D(\text{Fields})}{(\text{gauge vol})} \exp(-S_{\text{cov}}(\text{Fields})), \tag{2.39} \]

where we have to divide by the volume of the diffeomorphism group. Now looking back at the case where our fields were \( T, \bar{T}, G \), we find that the partition function can be written as:

\[ Z_{\text{grav}} = \int \frac{DgDGD\bar{g}}{(\text{gauge vol})} \delta(J^+_z - 1) \delta(J^-_z - 1) \exp(-kS_{wzw}^- (gG\bar{g}^{-1})), \tag{2.40} \]

where \( J_z = g^{-1} \partial g \), \( \bar{J}_z = \bar{g}^{-1} \partial \bar{g} \) and now \((\text{gauge vol})\) denotes the volume of the symmetry group of the left, right constrained WZW model. Note that due to the delta functions in (2.40) this symmetry group is not simply \( G \times G \). What we do have is invariance under \( g \to gB^-, \ G \to (B^-)^{-1}GB^+, \ \bar{g} \to \bar{g}B^+, \) where \( B^-(B^+) \) is the Borel subgroup of lower (upper) triangular matrices, since these transformations do not alter the arguments of the delta functions. Using this \( B^- \) invariance in the left sector and the \( B^+ \) invariance in the right sector we can bring \( g \) in the form such that \( g^{-1} \partial g = \Lambda + W \), and similarly, bring \( \bar{g} \) in the form such that \( \bar{g}^{-1} \partial \bar{g} = \bar{\Lambda} + \bar{W} \). Once this is done (2.40) reduces to

\[ \int \frac{DTDGD\bar{T}}{(\text{gauge vol})} \exp(-S_{\text{cov}}(T, \bar{T}, G)). \tag{2.41} \]

The residual gauge invariances of (2.41) are those which leave the form of \( g^{-1} \partial g \) and \( \bar{g}^{-1} \partial \bar{g} \) invariant, and as we know from section 2.2 these are just the diffeomorphisms, represented as \( g \to gX(\epsilon) \) and \( \bar{g} \to \bar{g}\bar{X}(\bar{\epsilon}) \). So altogether we have shown that (2.40) indeed gives the partition function for covariant gravity.

On the other hand, (2.40) can also be trivially computed to be

\[ Z_{\text{grav}} = \int DG \exp(-kS_{wzw}^- (G)) \equiv Z_{wzw}. \tag{2.42} \]

So we come to the remarkable conclusion that, at least in the large \( k \) limit, the partition function for covariant induced gravity equals the partition function of the \( SL(2, \mathbb{R}) \) WZW model.

\[
\text{Again we neglect a Jacobian in going from } g \text{ to } T, \text{ which now equals } \det(\partial^3 + 2T\partial + \partial T) = \exp(-2\Gamma[T]). \text{ As before, this contribution can be discarded in the large } k \text{ limit.}
\]
This equivalence could have been anticipated from our previous work \cite{26}, see also \cite{25}, in which the covariant action was constructed from Chern–Simons theory. In this theory one works with wave-functionals $\Psi$ satisfying the Gauss law constraint $F(A_z, A_{\bar{z}})\Psi = 0$. Here $A_z, A_{\bar{z}}$ obey the Poisson brackets: $\{ A^a_z(z), A^b_{\bar{z}}(w) \} = \frac{2\pi i}{k} \eta^{ab} \delta(z - w)$. The precise form of the wave-functional $\Psi$ depends on which ‘polarization’ we choose. By this we mean that we have to divide the set $(A^a_z, A^a_{\bar{z}})$ into two subsets. One subset will contain fields $X_i$ and the other will consist of derivatives $\frac{\delta}{\delta X_i}$. For instance if one works in the so called ‘standard’ polarization, in which the $A^a_z$ label the fields and correspondingly the $A^a_{\bar{z}}$ are represented as derivatives w.r.t. these fields, the solution to the Gauss law constraint is given by $\Psi[A_z] = \exp(-kS_{wzw}(g))$, with $A_z = g^{-1} \partial g$, a result already mentioned in section 2.1.

Furthermore, in Chern–Simons theory there is a natural inner-product for the wave-functional $\Psi$ given by:

$$\langle \Psi | \Psi \rangle = \int \frac{DA_z DA_{\bar{z}}}{(\text{gauge vol})} \exp\left( \frac{k}{2\pi} \int d^2z \text{Tr}(A_z A_{\bar{z}}) \right) \Psi[A_z] \Psi[A_{\bar{z}}],$$

(2.43)

where we divide by the volume of the gauge group. Using $\Psi[A_z] = \exp(-kS^-_{wzw}(\bar{g}))$, with $A_{\bar{z}} = \bar{g}^{-1} \partial \bar{g}$, this inner-product can be worked out as (see also \cite{37}):

$$\langle \Psi | \Psi \rangle = \int DG \exp(-kS^-_{wzw}(G)) \equiv Z_{wzw},$$

(2.44)

where we once more invoked the Polyakov–Wiegman formula, to change variables from $g, \bar{g}$ to $G = g \bar{g}^{-1}$. The point is now that instead of the standard polarization one can choose a different ‘mixed’ polarization in which the fields are taken from both the $A^a_z$ and the $A^a_{\bar{z}}$. In \cite{26} we showed that such a non-standard polarization can lead to the situation in which the Gauss law constraint for the wave-functional reduces to the Virasoro Ward identity (2.14), and in which the above inner-product (2.43) reduces to the inner-product defined in the previous subsection (2.28). This then gives the equivalence of the partition functions for covariant gravity and $SL(2, \mathbb{R})$ WZW theory.
3. Review of $W$ algebras

In section 2.2, we saw how imposing constraints on an $Sl(2, \mathbb{R})$ connection $A_z$ naturally led to the Virasoro algebra (2.14). This algebra basically arose, because the constraints enabled us to solve some of the zero-curvature equations (2.12) explicitly. In this section we will describe a generalization of this procedure, leading to $W$ algebras as a generalization of the Virasoro algebra. The relation between zero-curvature equations and $W$ algebras has been worked out for the $W_N$ algebras in [38]. A different approach is to use Hamiltonian reduction to construct $W$ algebras [4, 17, 23, 22].

The main new result of this section is the introduction of an operator $L$ that enables us to write down many explicit expressions, e.g. (3.9) for $A_{\bar{z}}$, and that will play a prominent role in section 5 where we discuss global aspects of $W$ algebras. The $W$ algebras discussed here all result from embeddings of $sl_2$ in a Lie-algebra $g$. We believe that the reason for the occurrence of these $sl_2$ algebras is that they describe the identification of $sl_2$ rotations in the tangent space to the two-dimensional world-sheet with certain gauge transformations, a picture first advocated by Polyakov who called this procedure ‘soldering’ [3]. Generic $W$ algebras from $sl_2$ embeddings were first studied in [17]. For a more detailed account of $W$ algebras, see [20].

Starting with a non-compact semisimple real Lie-algebra $g$, we would like to impose some set of constraints on $A_z$ in order to find an interesting constrained algebra. However, such constraints will not in general enable us to solve part of the zero-curvature equations, or enable us to solve them only at the cost of introducing infinite power series or negative powers of components of $A_z$, and this is certainly not what one wants. To avoid this, one has to choose these constraints in a special way. To avoid the presence of denominators containing components of $A_z$ or $A_{\bar{z}}$, it would be nice if the zero-curvature equations that one wants to solve at each instant were of the form

$$\text{linear combination of certain components of } A_z = \text{function of } A_z \text{ and the remaining components of } A_{\bar{z}}.$$  \hspace{1cm} (3.1)

Clearly, in this case one only would have to solve some linear set of equations for the components of $A_z$ each time, thereby avoiding unpleasant denominators. The only way linear combinations of components of $A_z$ can enter the zero-curvature equations, is when part of $A_z$ is constant. Let us therefore try to impose the constraint $A_z = \Lambda + W$, where $\Lambda$ denotes a constant element of $g_C$ and $W$ contains the fields that will generate
the constrained algebra in the end. In this case we can write (2.12) in the form (3.1),

\[- \text{ad}_\Lambda (A_\bar{z}) = \partial A_\bar{z} - \bar{\partial} W + [W, A_\bar{z}]\].

(3.2)

The right hand side of this equation will, in general, contain the same components of $A_\bar{z}$ as the left hand side, even when we restrict our attention to only a few components of this zero-curvature equation. The occurrence of such equations can be prevented if the Lie algebra comes with a gradation. Recall that a gradation is a decomposition of the Lie algebra as $\mathfrak{g} = \bigoplus_{\alpha \in I} \mathfrak{g}_\alpha$, where $I$ is a finite set of real numbers ($\mathfrak{g}_\alpha = \{0\}$ for $\alpha \notin I$) and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha + \beta}$. An arbitrary element $X \in \mathfrak{g}$ can be decomposed as $X = \sum_i X_i$ with $X_i \in \mathfrak{g}_{\alpha_i}$, $X_i \neq 0$. Let $\maxdeg(X)$ denote $\max_i \alpha_i$ and $\mindeg(X) = \min_i \alpha_i$. If $\mindeg(\Lambda) > 0$ and $\mindeg(\Lambda) > \maxdeg(W)$, then the left hand side of (3.2) will only contain components of $A_\bar{z}$ of lower degree than the components of $A_\bar{z}$ occurring on the right hand side of (3.2). In this case we can express components of $A_\bar{z}$ in terms of components of higher degree. Therefore, many of the components of $A_\bar{z}$ can be expressed in terms of others if we first solve the zero-curvature equations of highest degree, and then take equations of lower and lower degree. Clearly, the left hand side of (3.2) does not change if we add an element of $\ker(\text{ad}_\Lambda)$ to $A_\bar{z}$. Therefore, once we fix the part of $A_\bar{z}$ that is annihilated by $\text{ad}_\Lambda$, the remaining part of $A_\bar{z}$ can be expressed in terms of these independent variables. In the case of the Virasoro algebra this independent component is $\mu$.

The bilinear product on $\mathfrak{g}$ is given by the Killing form $(X, Y) = \text{Tr}(\text{ad}_X \text{ad}_Y) = \frac{1}{2h} \text{tr}(\text{ad}_X \text{ad}_Y)$, normalized such that the longest roots of $\mathfrak{g}$ have length squared two: $h$ is the dual Coxeter number and Tr represents the ‘ordinary’ trace. The Lie algebra $\mathfrak{g}$ has a natural involution, the Cartan involution, which we will denote by $^\dagger$, and satisfies $(X, X^\dagger) \geq 0$. In the case of $\mathfrak{sl}_N$ this corresponds to taking the transpose of a matrix in the fundamental representation. In general, if we decompose the Lie algebra as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the maximal compact subalgebra of $\mathfrak{g}$, the Cartan involution is $-1$ on $\mathfrak{k}$ and $+1$ on $\mathfrak{p}$.

If we restrict the zero-curvature equation (3.2) to the complement of $\text{im}(\text{ad}_\Lambda)$ in $\mathfrak{g}$, then the left hand side of (3.2) vanishes, and the equations that remain are analogous to (2.14). These equations are the Ward identities of the constrained algebra, and at the same time provide the Poisson brackets. To get a good constrained algebra, the independent components in $A_\bar{z}$ must be in one-to-one correspondence with the components of $W$. The correspondence is that a gauge transformation with parameter $A_\bar{z}$ should generate the same transformations as the ‘charge’ $Q = \int dz (W, A_\bar{z})$ generates.
with respect to the Poisson bracket. Therefore, an independent component \(m(z, \bar{z})X\) of \(A_\bar{z}\) corresponds to the component \((X, W)\) of \(W\). The independent components of \(A_\bar{z}\) span \(\ker(\text{ad}_\Lambda)\), and this implies that for a proper one-to-one correspondence with components of \(W\) we need

\[
(\ker(\text{ad}_\Lambda)(\mathfrak{g}))^\perp \bigoplus \mathfrak{g}_W = \mathfrak{g}, \quad \dim(\mathfrak{g}_W) = \dim(\ker(\text{ad}_\Lambda)(\mathfrak{g})),
\]

(3.3)

where \(\mathfrak{g}_W\) is the subspace of \(\mathfrak{g}\) spanned by \(W\) as we let its independent components vary. The zero-curvature equation (3.2) consists of precisely two pieces: if we write \(F = F_1 + F_2\), with \(F_1 \in \text{im}(\text{ad}_\Lambda)(\mathfrak{g})\) and \(F_2 \in \mathfrak{g}_W\), then \(F_1 = 0\) are the equations that can be used to express \(A_\bar{z}\) in terms of a set of independent components, that are in one-to-one correspondence with \(\ker(\text{ad}_\Lambda)\), and \(F_2 = 0\) are the equations that provide the Poisson brackets of the constrained algebra. The decomposition of the zero-curvature equations in two pieces requires \(\text{im}(\text{ad}_\Lambda) \oplus \mathfrak{g}_W = \mathfrak{g}\), and consistency of this equation with (3.3) gives an extra constraint on \(\Lambda\), namely

\[
\ker(\text{ad}_\Lambda)(\mathfrak{g})^\perp \bigoplus \text{im}(\text{ad}_\Lambda)(\mathfrak{g}) = \mathfrak{g}.
\]

(3.4)

We expect that if this condition is satisfied, one will get a good constrained algebra. This algebra does not necessarily contain the Virasoro algebra. This will only be the case if the constraints are compatible with conformal invariance, which will impose extra conditions on the choice of \(\Lambda\). The connection \(A_\bar{z}\) transforms as a spin 1 field under co-ordinate transformations, whereas a constant like \(\Lambda\) transforms as a spin 0 field. Therefore one cannot just put \(A_\bar{z}\) equal to \(\Lambda + W\). The only way to get things right is to change the conformal weight of \(A_\bar{z}\) so that \(\Lambda\) will have spin 0. Changing the conformal weight of \(A_\bar{z}\) is in one-to-one correspondence with adding improvement terms to the Sugawara energy-momentum tensor of the type \(-\text{Tr}(H_0 \partial J)\), where \(H_0\) is an arbitrary element of the Cartan subalgebra of \(\mathfrak{g}\). These define, in turn, just a gradation of \(\mathfrak{g}\), where the decomposition \(\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha\) is precisely the decomposition of \(\mathfrak{g}\) in eigenspaces of \(\text{ad}_{H_0}\). Such an improvement term will therefore change the conformal weight of \(X_\alpha \in \mathfrak{g}_\alpha\) from 1 to \(1 - \alpha\). This shows that we need a gradation of \(\mathfrak{g}\) such that \(\Lambda\) will be homogeneous of degree 1. If we write

\[
W = \sum_{i=1}^n W^i(z, \bar{z})X_i,
\]

(3.5)

where \(X_i \in \mathfrak{g}_{\alpha_i}\), then \(W^i(z, \bar{z})\) will have weight \(s_i = 1 - \alpha_i\). Let us summarize the conditions we have obtained so far, so as to obtain a proper constrained algebra:
• Λ is constant and homogeneous of degree one with respect to some gradation of \( g \).

• \( \ker(\text{ad}_\Lambda)^\dagger \oplus \text{im}(\text{ad}_\Lambda) = g \).

• \( g_W \oplus \text{im}(\text{ad}_\Lambda) = g \), and \( \dim(g_W \cap g_{\alpha}) = \dim(\ker(\text{ad}_\Lambda) \cap g_{-\alpha}) \).

• \( \min\{s_i\} > 0 \), so that \( 1 = \deg(\Lambda) > \max\deg(W) \).

The third condition guarantees that \( W \) admits a decomposition of the type (3.5).

At this stage we can already derive a general relation between the \( s_i \) of the constrained algebra, and the dimension of \( g \). Let \( n_\alpha = \dim(\ker(\text{ad}_\Lambda) \cap g_{\alpha}) \), \( m_\alpha = \dim(\text{im}(\text{ad}_\Lambda) \cap g_{\alpha}) \) and \( d_\alpha = \dim(g_{\alpha}) \). The fact that \( \Lambda \) is homogeneous of degree 1 gives the relation \( d_\alpha = n_\alpha + m_{\alpha+1} \), and (3.4) implies \( d_\alpha = n_\alpha + m_{-\alpha} \). Subtracting these two identities gives \( m_{\alpha+1} = m_{-\alpha} \). An immediate consequence is that \( \sum_\alpha (2\alpha + 1)m_{-\alpha} = 0 \), as \( (2\alpha + 1) \) is odd under \( -\alpha \to \alpha + 1 \). Furthermore \( \deg(X) = -\deg(X^\dagger) \) implies \( d_\alpha = d_{-\alpha} \), so that \( \sum_\alpha \alpha d_\alpha = 0 \). Combining all this we have

\[
\sum_i (2s_i - 1) = \sum_\alpha n_{-\alpha}(1 - 2\alpha) = \sum_\alpha n_\alpha(1 + 2\alpha) = \sum_\alpha (d_\alpha + m_{-\alpha})(1 + 2\alpha) = \sum_\alpha d_\alpha = \dim g.
\]

(3.6)

This hints at the existence of an underlying \( sl_2 \) structure, as the dimension of a spin \((s - 1)\) representation is precisely \((2s - 1)\). Equation (3.6) would follow trivially if we would have decomposed \( g \) with respect to some \( sl_2 \) subalgebra. Later we will see how this \( sl_2 \) structure emerges, once we require that the constrained algebra contains the Virasoro algebra.

The choice of \( W \) is severely restricted by the third of the four conditions mentioned above. Although the subspace \( g_W \subset g \) has to fulfill the above criteria, one can still choose different \( g_W \). Different choices of \( g_W \) will not affect the weight spectrum \( \{s_i\} \), but will correspond to basis transformations of the constrained algebra. Different choices of \( g_W \) are in the literature known as different Drinfeld-Sokolov gauges. One particular simple choice is the ‘highest weight gauge’, corresponding to choosing \( g_W = \ker(\text{ad}_{\Lambda^-}) \), for a particular \( \Lambda^- \) that we will define later. In this gauge all the fields \( W^i(z, \bar{z}) \) will automatically be primary with respect to the Virasoro subalgebra. However, in some cases it is advantageous to choose a different gauge, for instance one in which the non-linear algebra formed by the \( W^i \) is manifestly quadratic \[23, 26\]. In \[26\] this quadratic form was necessary to be able to obtain a covariant action for \( W_N \) gravity from Chern–Simons theory.
Given such a choice of subspace $g_W$, it will turn out to be very convenient to introduce an operator $L$ which plays the role of the inverse of $\text{ad}_\Lambda$. More precisely, let $g_W^\perp$ be the orthocomplement of $g_W$ in $g$, then $\text{ad}_\Lambda$ defines an invertible linear operator from $g_W^\perp \to (g_W^\perp)^\dagger$. Let $L$ be the inverse of this operator, extended by 0 to a linear operator $g \to g$. Let us also introduce a set of projection operators in the Lie algebra $g$, called $\Pi_i, \Pi_k, \Pi_i^\dagger$ and $\Pi_k^\dagger$, being the orthogonal projections in $g$ onto the subspaces $(g_W^\perp)^\dagger, g_W^\dagger, g_W^\perp$ and $g_W$ respectively. The subscripts $i$ and $k$ refer to image and kernel, as e.g. $\Pi_i$ is almost the projection onto the image of $\text{ad}_\Lambda$. However, as we allow for other than the highest weight gauges, one should keep in mind that $\Pi_i$ need not be exactly equivalent to the projection onto the image of $\text{ad}_\Lambda$.

Using the operator $L$, it is possible to derive an expression for $A_\bar{z}$ where we express it in terms of its independent components. If we apply $\Pi_i$ to (3.2) we find

$$-\Pi_i \circ \text{ad}_\Lambda A_\bar{z} = \epsilon \partial \Pi_i A_\bar{z} + \epsilon \Pi_i [W, A_\bar{z}],$$

(3.7)

where we introduced a ‘small’ parameter $\epsilon$. We will solve this equation perturbatively in $\epsilon$ and later put $\epsilon = 1$ (and see if that makes sense). To lowest order in $\epsilon$, the right hand side vanishes and $A_\bar{z} = F$, where $F$ is an arbitrary element of ker $\text{ad}_\Lambda$. The first order term of $A_\bar{z}$ satisfies the equation

$$-\Pi_i \circ \text{ad}_\Lambda A_\bar{z}^{(1)} = \partial \Pi_i F + \Pi_i [W, F].$$

(3.8)

By definition, $L \circ \Pi_i = L$, and therefore this equation is solved by $A_\bar{z}^{(1)} = -L(\partial F + [W, F])$. Proceeding with higher orders we find in precisely the same way that $A_\bar{z}^{(k+1)} = -L(\partial + \text{ad}_W)(A_\bar{z}^{(k)})$. This shows that

$$A_\bar{z} = \frac{1}{1 + \epsilon L(\partial + \text{ad}_W)} F.$$  

(3.9)

The operator $L(\partial + \text{ad}_W)$ always lowers the degree, as $L$ has degree $-1$, $\partial$ has degree 0, and $W$ consists of elements of degree less than one only. This implies that this operator is nilpotent, and that it makes perfect sense to put $\epsilon = 1$ in (3.9). The independent components of $A_\bar{z}$ are given by an arbitrary element $F$ of ker $\text{ad}_\Lambda$.

This enables us to extract the Poisson brackets for the constrained algebra. In the same way as we did in the beginning of this section for the Virasoro algebra, we replace
\( \partial W \) in the zero-curvature equation by \( \delta_F W \), and we get

\[
\delta_F W = (\partial + \text{ad}_{\Lambda} + \text{ad}_W) \frac{1}{1 + L(\partial + \text{ad}_W)} F. \tag{3.10}
\]

Restricting this to a component \( W^i(z, \bar{z}) \) of \( W \), we find an expression for \( \delta_F W^i \), and the Poisson brackets can then be read off from the identity

\[
\delta_F W^i = \left\{ \int dz Q(z), W^i(w) \right\}_{PB}, \tag{3.11}
\]

where \( Q \) satisfies \( \delta Q / \delta W = F \).

We still have not answered the question whether or not this algebra contains a Virasoro algebra. This is most easily studied in the highest weight gauge, so we take \( g_W = \ker \text{ad}_{\Lambda^-} \), where \( \Lambda^- \) is yet to be specified. A natural candidate for the spin 2 energy-momentum tensor is given by the Sugawara-like expression

\[
T = \frac{1}{2}(\Lambda + W, \Lambda + W) - (\rho, \partial W), \tag{3.12}
\]

where \( \rho \in \Pi^1_k H \), with \( H \) the Cartan subalgebra of \( g \), represents an arbitrary ‘improvement’ term. To compute the Poisson bracket of \( T \) with an arbitrary field \( W^i \), we have to compute (3.10) with \( F \) chosen in such a way that it generates small \( T \)-transformations. In other words, \( F \) must satisfy

\[
F = \frac{\delta Q}{\delta W} = \frac{\delta (T(z)\epsilon(z))}{\delta W} = \Lambda \epsilon + (\Pi_k W)\epsilon + \rho \epsilon', \tag{3.13}
\]

where we used that \((W, W) = (\Pi_k W, \Pi_k W)\). From (3.13) we read off that we must choose \( F = \Lambda \epsilon + (\Pi_k W)\epsilon + \rho \epsilon' \). Our next task is to compute (3.10) for this \( F \). It turns out that the calculations simplify a great deal if we demand that \( \{ \Lambda, -\Lambda, \Lambda^- \} \) form a \( sl_2 \) algebra, i.e.

\[
\begin{align*}
[\Lambda, \Lambda^-] &= -\Lambda, \\
[\Lambda, \Lambda] &= \Lambda, \\
[\Lambda^-, \Lambda] &= -\Lambda^-.
\end{align*} \tag{3.14}
\]

We do not know whether it is possible to identify a Virasoro algebra without making these assumptions. If (3.14) are satisfied, it is clear that we have essentially decomposed
in representations of this $sl_2$ algebra, and kept one field $W^i$ for every representation. In the highest weight gauge, one precisely keeps the ‘highest weight’ of each representation, namely the one annihilated by $\Lambda^-$.

Assuming the validity of (3.14), one can step by step compute $A_{z}(F)$, with $F = \Lambda \epsilon + (\Pi_k W) \epsilon + \rho \epsilon'$: $L(\partial + \text{ad}_W)(F) = (L\Lambda) \epsilon' - (\Pi_k W) \epsilon$, $(L(\partial + \text{ad}_W))^2(F) = (L^2 \Lambda) \epsilon'' = -\Lambda^- \epsilon''$, and all higher powers of $L(\partial + \text{ad}_W)$ acting on $F$ vanish. This shows that

$$A_{z}(F) = \Lambda \epsilon + (\Pi_k W) \epsilon + \rho \epsilon' + (\Pi_k^i W) \epsilon - (L\Lambda) \epsilon' + (L^2 \Lambda) \epsilon''$$

Substituting this in the expression for $\delta F W$ gives

$$\delta_{F} W = -\Lambda^- \epsilon'' + \rho \epsilon'' + (W - [\rho - L\Lambda, W]) \epsilon' + W' \epsilon.$$ (3.16)

This implies that $T$ indeed generates a Virasoro algebra with nontrivial central extension, and that all other $W^i$ are primary with respect to this energy-momentum tensor, with conformal weight given by $1 - \text{eigenvalue}(\rho - L\Lambda)$. An exception form the $W^i$ that live in the Cartan subalgebra of $g$, as they in general will have a background charge with respect to $T$, determined by the choice of $\rho$. Only if $\rho = 0$ will this background charge vanishes, and will the $W^i$ that live in the Cartan subalgebra really have conformal weight one. We also see that the gradation of the Lie algebra that gives the right conformal weight assignments, is given by the element $H_0 = \rho - L\Lambda$ of the Cartan subalgebra. Because $[H_0, \Lambda] = \Lambda$, $\Lambda$ is indeed homogeneous of degree one with respect to this gradation.

As for the central charges of these algebras, they are fixed to a particular value. Arbitrary central charges can easily be obtained by imposing precisely the same constraints on $kA_z$ instead of $A_z$. This is equivalent to imposing the constraints on the currents of a level-$k$ Kac-Moody algebra. The new Poisson brackets can be obtained from the ones we have constructed so far by rescaling $W \rightarrow W/k$, $T \rightarrow T/k$, $\Lambda \rightarrow \Lambda/k$, $\Lambda^- \rightarrow k\Lambda^-$ and $L \rightarrow kL$. After rescaling (3.12) and (3.16) one can easily extract the central charge for the $W$ algebra, it is given by

$$c = -12(\Lambda, \Lambda^-)k.$$ (3.17)

Note that to obtain the usual $W_N$ algebra, one starts with $g = SL_N$ and usually
takes

\[ \Lambda = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}, \tag{3.18} \]

which can be compactly notated as \( \Lambda = \sum_{i=1}^{N-1} e_{i,i+1} \), where \( e_{i,j} \) is the matrix which has a one on its \( i^{th} \) row and its \( j^{th} \) column and is zero everywhere else. An easy computation shows that \( (3.14) \) is satisfied, if \( \Lambda = \frac{1}{2} \sum_{i=1}^{N-1} i(N - i) e_{i+1,i} \). The corresponding \( \mathfrak{sl}_2 \)-embedding is a special case of a ‘principal’ \( \mathfrak{sl}_2 \) embedding \cite{18}. Plugging this \( \Lambda \) and \( \Lambda^{-1} \) back into \( (3.17) \) gives \( c = -k(N^3 - N) \).

So far we have imposed several constraints in order that the constrained algebra will have certain desired properties. If we omit some of these constraints, one may still end up with a good algebra, although it will not necessarily have all the properties described above. For instance, if we do not require the existence of an \( \mathfrak{sl}_2 \) subalgebra, the constrained algebra is still ok, but it need not have a Virasoro subalgebra. In other cases one can impose new constraints on the \( W \) algebras and reduce them even further, as \( \text{e.g.} \) in \cite{8}. We will in this paper not concern us with such types of algebras, and focus our attention only on those that can be obtained from a \( \mathfrak{sl}_2 \) embedding, although some of the results may also be applicable to more exotic cases. On the other hand, if a \( W \) algebra contains the Virasoro algebra it automatically has some kind of \( \mathfrak{sl}_2 \) structure \cite{9}, so that this restriction does not seem to be very severe.

4. The Action of Chiral and Covariant \( W \) Gravity

In the previous section we explained how general (classical) \( W \) algebras can be obtained by considering constrained gauge fields \( A_z = \Lambda + W \), where \( \Lambda \) and \( W \) obey certain criteria. Given a proper choice for \( g_W = \Pi_k^l g \), we showed that the solution \( A_{\bar{z}} \) of the zero-curvature condition \( F(A_z, A_{\bar{z}}) = 0 \) is given by

\[ A_{\bar{z}} = \frac{1}{1 + L(\partial + \text{ad}_W)} F(\mu_i), \tag{4.1} \]
where $F(\mu_i)$ is an element of the kernel of $\text{ad}_\Lambda$. Here the $\mu_i$ are the parameters of the $W$ algebra dual to the fields $W_i$. They are the generalization of the Beltrami differential $\mu$ which appeared for the case of ordinary gravity. To be more precise, the zero-curvature condition for the constrained gauge field $A_z$ splits into two parts: if we write $F = F_1 + F_2$, with $F_1 \in \Pi_g$ and $F_2 \in \Pi_g^\dagger$, then $F_1 = 0$ are the equations leading to (4.1) and $F_2 = 0$ are the equations which encode the generalized $W$ transformations (3.10):

$$\delta \epsilon W = D_{A_z}(X(\epsilon)),$$

(4.2)

where $X(\epsilon)$ is (4.1) with the $\mu_i$ in $F$ replaced by $\epsilon_i$. In this section we will consider the chiral and covariant action for these generalized $W$ algebras.

4.1. The Chiral Action for General $W$ Gravity

In the spirit of section 2 the induced action for general $W$ gravity can be defined as:

$$\exp(-\Gamma[\mu_i]) = \left\langle \exp\left(-\frac{i}{2\pi} \int d^2 z \sum_i \mu_i W_i^\text{mat}\right) \right\rangle,$$

(4.3)

where the $W_i^\text{mat}$ satisfy the operator product expansions of the generalized $W$ algebra. These operator product expansions give differential equations for the chiral induced action $\Gamma[\mu_i]$, as we explained in detail in section 2. Since these differential equations can be obtained from the zero-curvature condition for a constrained gauge field, and since the solution to this zero-curvature condition is given by the WZW functional, we have as before that $\Gamma[W_i]$, the Fourier transformed of $\Gamma[\mu_i]$, is given by $\Gamma[W_i] = -kS_{wzw}^-(g)$, where $g$ is restricted such that $g^{-1}\partial g = \Lambda + W$, and $k$ is related to the matter central charge via $c_m \sim -12(\Lambda, \Lambda^-)k$ as shown in section 3. Repeating calculation (2.19) of section 2.3, we compute $\Gamma[\mu_i]$ from this by Fourier transformation:

$$\exp(-\Gamma[\mu_i]) = \int D\epsilon \exp(-\Gamma[W_i] - \frac{k}{2\pi} \int d^2 z \text{Tr}(WF(\mu_i)))$$

$$= \int Dh \delta(\Pi_k(F(\mu_i) - fJ_zJ^{-1})) \exp(-kS_{wzw}^+(h)),$$

(4.4)

where $J_z = h^{-1}\partial h$ and $f = \exp(-z\Lambda)$. We thus find that the correlation functions of chiral $W$ gravity can be obtained from WZW theory, if one identifies the $\mu_i$ with Kac-Moody currents in the way dictated by the delta-function in (4.4).

Taking the large $k$ limit of (4.4) we find (using the results of Appendix A) that $\Gamma[\mu_i] = kS_{wzw}^+(hf)$, where $h$ is such that $h^{-1}\partial h$ is given by the r.h.s. of (4.1). So
4.2. The Covariant Action for General $W$ Gravity

Recall from section 2.5 that in order to find the covariant action for general $W$ gravity, we have to construct a local counterterm whose anomalous behavior cancels that of the chiral actions. As in the case of ordinary gravity this local counterterm is most easily found if we work with the Fourier transformed actions $\Gamma[W]$ instead of the $\Gamma[\mu]$. In fact, (2.31), derived in section 2.5 for the case of ordinary gravity, is already the correct formula for the local counterterm for general $W$ gravity, once we take the appropriate choice for $\Lambda$, $W$ and $G$. So the local counterterm is given by:

$$\Delta \Gamma[W', W^i, G] = \frac{k}{2\pi} \int d^2z \, Tr((\Lambda + W(G^{-1})\partial G) - \frac{k}{2\pi} \int d^2z \, Tr((\Lambda + W)\bar{\partial}GG^{-1})$$

$$+ \frac{k}{2\pi} \int d^2z \, Tr((\Lambda + W)G(\bar{\Lambda} + \bar{W}G^{-1}) - kS_{wzw}(G). \quad (4.5)$$

Notice the close resemblance of this expression with the action for a gauged WZW theory [27]. In fact, apart from a term $\frac{k}{2\pi} \int d^2z \, Tr((\Lambda + W)(\bar{\Lambda} + \bar{W}))$, it is precisely the action for a gauged WZW theory, where the gauge fields are of a particular restricted form, such that one does not gauge a subgroup, but one actually gauges the $W$ algebra. An important difference, however, is that the covariant action is invariant under both left and right $W$ transformations, whereas the gauged WZW model only is invariant under left and right gauge transformations simultaneously.

The local counterterm $\Delta \Gamma[\mu, \bar{\mu}, G]$ can now be obtained from (4.3) by Fourier transformation with respect to $\mu, \bar{\mu}$. As said before this Fourier transformation is simple and exact to do, since the $W$ fields appear algebraically and at most quadratic in (4.3). One easily verifies that the saddle-point equations for the $W$ fields are:

$$\Pi_k \left( \bar{\partial}GG^{-1} - G(\bar{\Lambda} + \bar{W})G^{-1} + F(\mu_i) \right) = 0,$$

$$\Pi^i_k \left( G^{-1}\partial G + G^{-1}(\Lambda + W)G - \bar{F}(\bar{\mu}_i) \right) = 0. \quad (4.6)$$

Unfortunately, there is no general formula for the solutions of these saddle-point equations which is valid for all generalized $W$ algebras. We can give a general formula for the case of the standard $W_N$ algebras, valid for all $N$, but for the non-standard $W$ algebras the solutions to (4.6) should be determined for each case separately.
For standard $W_N$ we take $Sl(N, \mathbb{R})$ as our gauge group, and we constrain the gauge field as follows [23]:

$$A_z = \Lambda + W = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ W_N & W_{N-1} & W_{N-2} & \cdots & W_2 & 0 \end{pmatrix}.$$  \tag{4.7}

Note that $\Pi_k g$ is spanned by $\{e_{i,N}\}$, for $i = 1, \ldots, N - 1$, where $e_{i,j}$ is the matrix which has a 1 on its $i^{th}$ row and its $j^{th}$ column and is zero everywhere else. $F(\mu_i)$ can be conveniently parametrized by $F(\mu_i) = \sum_{i=2}^{N} \mu_i \Lambda_i^{-1}$. In this case $G$ admits a decomposition in terms of three subgroups: $G = \Pi_k \Gamma_y G \Pi_k G$, which in terms of matrices looks like

$$G = \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & * \end{pmatrix} \begin{pmatrix} 1 & * \\ \vdots & \ddots \vdots \\ \vdots & \vdots \ddots \vdots \\ 1 & * \end{pmatrix}, \tag{4.8}$$

and which is crucial to be able to solve the saddle-point equations (4.6). One finds

$$W = (\Pi_y G) \Pi_k \left( \bar{F}(\bar{\mu}_i) - \bar{\Lambda}_G \right) (\Pi_y G)^{-1},$$

$$\bar{W} = (\Pi_y G)^{-1} \Pi_k \left( F(\mu_i) - \bar{\Lambda}_G \right) (\Pi_y G), \tag{4.9}$$

where $\Lambda_G = G^{-1} \Lambda G + G^{-1} \partial G$ and a similar definition for $\bar{\Lambda}_G$. Substituting this solution for the $W$ fields back into the local counterterm gives:

$$\Delta \Gamma[\mu_i, \bar{\mu}_i, G] = \frac{k}{2\pi} \int d^2 z \ Tr(\Lambda G \bar{\Lambda}_G^{-1}) + \frac{k}{2\pi} \int d^2 z \ Tr(\bar{\Lambda} G^{-1} \partial G) - \frac{k}{2\pi} \int d^2 z \ Tr(\Lambda \bar{\partial} G G^{-1}) - \frac{k}{2\pi} \int d^2 z \ Tr(\Pi_k (F(\mu_i) - \bar{\Lambda}_G) \Pi_y G \Pi_k^\dagger (\bar{F}(\bar{\mu}_i) - \Lambda_G) \Pi_y G^{-1}) - k S_{wzw}(G). \tag{4.10}$$

The total covariant action is given by the sum of this local counterterm and the chiral action constructed in the previous subsection together with its partner of opposite chirality. This covariant action is invariant under the transformations [26]

$$\delta G = -X(\epsilon) G,$$

$$\delta \Pi_k F(\mu_i) = \Pi_k \left( \partial X(\epsilon) + [A_z, X(\epsilon)] \right), \tag{4.11}$$

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where $X(\epsilon)$ is still given by (4.1), but with the $W$ fields in (4.1) replaced by the solutions to the saddle-point equations (4.9), and $A_\bar{z}$ is (4.1) with the $W$ fields given by $W_i = \frac{2\pi}{k} \frac{\delta \Gamma[\mu]}{\delta \mu_i}$. These transformations form a $W_N$ algebra by construction.

5. Global Aspects of $W$ Algebras

So far we have restricted our attention to $W$ algebras and actions that were defined on the complex plane. If we want to answer questions like 'what is the moduli-space of $W$ algebras', we have to discuss $W$ algebras on higher genus Riemann surfaces, as on the complex plane one can choose globally well-defined co-ordinates, and it is sufficient to express everything in terms of these co-ordinates. One need not bother about the transformation properties of the different objects one encounters under change of co-ordinates, and generally there are no moduli. In this section we will make a start of the study of $W$ algebras on general Riemann surfaces. In previous sections we have seen how one can use the formulation of $W$ algebras in terms of zero-curvature equations and special gauge-transformations to obtain the chiral and covariant actions for $W$ gravity, and we will now also use the same formulation to go to arbitrary Riemann surfaces.

Suppose that we have some $G$-bundle $P$ over a Riemann surface $\Sigma$, equipped with a connection that in certain local complex co-ordinates can be written as $A = A_z dz + A_{\bar{z}} d\bar{z}$. We want, as before, to impose constraints on $A_z$, and in particular to set certain components of $A_z$ equal to some constant value. However, as $A_z$ transforms as a one-form under co-ordinate transformations, and not as a function, one cannot impose these constraints globally. The same problem already exists on the algebraic level, where one wants to put a spin one current equal to a constant, which is not compatible with conformal invariance. In the latter case, the problem is resolved by improving the energy momentum tensor, which changes the spins of the currents. Luckily this ‘soldering’ procedure has a geometrical counterpart, which amounts to twisting a trivial $G$-bundle into a non-trivial one. For the sake of simplicity, we will restrict our attention to the case $G = Sl(N,\mathbb{C})$ in this section, and assume that we are working in a highest weight gauge and that there is no background charge $\rho$ in (3.12). Let us first describe how the twisting works.
5.1. THE TWISTED BUNDLE

Improving the energy-momentum tensor with a term $-\text{Tr}(H_0\partial J)$ amounts to changing the gradation of $g$ from the trivial one into the one determined by $H_0$. Suppose that $H_0 = \text{diag}(d_1, \ldots, d_N)$ in the fundamental representation of $sl_N$, so that the degree of $E_{\alpha_i}$ is $d_{i+1} - d_i$ for each simple root $\alpha_i$. Let $K$ be the holomorphic cotangent bundle $T^*_\Sigma$ of the Riemann surface, and let $E_{ij}$ denote the line bundle $K^{-d_i}$, $i, j = 1 \ldots N$. Given (locally) a section $s$ of the $N^2$-dimensional vector bundle $E = \bigoplus_{i,j=1}^N E_{ij}$, one can consider the function $\text{det}(s_{ij}) : \Sigma \to \mathbb{C}$, which is well defined because $\sum d_i = 0$. The sections $s$ with $\text{det}(s) = 1$ are sections of a principal $Sl(N,\mathbb{C})$ bundle $P_c$, with reduced structure group $GL(1,\mathbb{C})$. This is precisely the twisted bundle we need. To see this, consider the adjoint bundle $\text{ad}(P_c)$. This bundle has an alternative description as $sl(V)$, the bundle of traceless endomorphisms of $V$, sometimes also denoted by $\text{End}_0(V)$, where $V$ is the vector bundle

$$V = K^{-d_1} \bigoplus \ldots \bigoplus K^{-d_N}. \quad (5.1)$$

A connection on $P_c$ is locally a one-form with values in $\text{ad}(P_c)$. The $dz$ part of such a connection looks like $a_{ij}dz$, where $a_{ij}$ is an $N \times N$ matrix; $a_{ij}$ transforms as a section of $K \otimes \text{hom}(K^{-d_j}, K^{-d_i}) \simeq K^{1+d_j-d_i}$. The improvement term $-\text{Tr}(H_0\partial J)$ changes the spin of $a_{ij}$ from 1 to $1 + d_j - d_i$, and we see that by twisting the principal $Sl(N,\mathbb{C})$ bundle $P_c$ we have achieved the same thing. However, a connection is only locally an $\text{ad}(P_c)$-valued one-form, and globally $a_{ij}$ does not transform as a section of $K^{1+d_j-d_i}$. Only the difference between two connections transforms globally in the proper way, because such a difference is a global $\text{ad}(P_c)$-valued one-form. This shows that we should impose constraints on the difference of two connections, rather than on the connection itself. Before discussing the form of such constraints, we will now first consider WZW actions for connections on non-trivial bundles, which are necessary to write down generalizations of the actions we have considered so far.

5.2. GENERALIZED WZW ACTION

In section 2 we saw that the WZW action arises as the induced action for a gauge

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*for noninteger $d_i$ one has to choose some appropriate root of $K$, the precise choice is not important here.

†The space of connections is an affine space modeled on the vector space $\Omega^1(\Sigma; \text{ad}(P_c))$. 

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field coupled to some matter system. In this section we will take a specific matter system, namely we will study a gauge field coupled to a chiral fermion \( \psi \). This \( \psi \) transform as a section of \( V \otimes \langle \bar{K} \rangle \frac{1}{2} \), and \( A_z \) is part of a connection of \( P_z \). The action

\[
S(\psi, A_z) = \frac{1}{2} \int d^2 z \text{Tr} \left( \psi^\dagger (\partial + A_z) \psi \right)
\]  

(5.2)

is gauge invariant under \( \psi^\dagger \rightarrow \psi^\dagger h, \psi \rightarrow h^{-1} \psi, \) and \( A \rightarrow h^{-1} Ah + h^{-1} \partial h \). The two-point function \( G(A_z; z, w) = \langle \psi(z) \psi^\dagger(w) \rangle \) satisfies \( (\partial + A_z(z))G(A_z; z, w) = \pi \delta(z - w) \). From this one finds the following rule for the change of \( G \) under a gauge transformation \( h: G(A_z^h; z, w) = h(z)^{-1} G(A_z; z, w) h(w) \). For \( z \rightarrow w \) \( G \) behaves as

\[
G \sim \frac{1}{z - \bar{w}} + \chi_z(z) - A_z(z) \frac{z - w}{z - \bar{w}} + \text{terms that vanish as } z \rightarrow w.
\]  

(5.3)

Under the gauge transformation \( A_z \rightarrow A_z^h = h^{-1} \partial h + h^{-1} A_z h \) we find, by expanding \( h(w) = h(z) + (z - \bar{w}) \partial h(z) + (z - w) \partial h(z) + \ldots \), that \( \chi_z \rightarrow h^{-1} \chi_z h + h^{-1} \partial h, i.e. \chi_z \) transforms as a connection. Furthermore, if locally \( A_z = 0 \) we know that \( G \) is exactly given by \( 1/(z - \bar{w}) \). Combining these facts we deduce that the curvature of the connection one-form \( A_z dz + \chi_z d\bar{z} \) must vanish. To express this fact in terms of the current \( J_z \), we must first define what we mean by \( J_z \). The naive definition \( J_z = \lim_{z \rightarrow w} G(A_z; z, w) \) does not work, due to the singularity of \( G \) as \( z \rightarrow w \). This means we have to regularize \( J_z \), and a standard way of doing this is by using point splitting regularization: one defines \( J_z = \lim_{z \rightarrow w} (G(A_z; z, w) - G_0(A_z; z, w)) \) where \( G_0 \) is some function that has the same singular behavior as \( G \). This means that

\[
G_0 \sim \frac{1}{z - \bar{w}} + B_z^0(z) - A_z(z) \frac{z - w}{z - \bar{w}} + \text{terms that vanish as } z \rightarrow w.
\]  

(5.4)

Here, \( B_z^0(z) \) is some fixed field that transforms as the \( dz \)-part of a connection. Unfortunately, one cannot just take \( B_z^0 \) equal to zero, because zero is not a globally well-defined connection. The current \( J_z \) is

\[
J_z = \chi_z - B_z^0, \tag{5.5}
\]

and the Ward-identity (expressing the fact that the current \( J_z \) is not conserved) reads

\[
D_{A_z} J_z = \bar{\partial} A_z + D_{A_z} B_z^0. \tag{5.6}
\]
This identity should come from some kind of generalized WZW action. The WZW action \( S_{\text{wzw}}(g) \) is really a functional on the space of gauge transformations, and on the complex plane this space is isomorphic to maps from the plane to \( G \). The terms \( g^{-1} \partial g \) in the WZW action are just \( 0^g \), where \( 0 \) is the trivial connection on the complex plane, and the superscript \( g \) refers as usual to a gauge transformation. Even on the complex plane one sees that the WZW action is not invariant if one chooses a different trivialization of the (trivial) bundle \( P_c \). If one chooses a different trivialization, related to the first one via a gauge transformation \( h \), then \( 0^g \to (0^h)^{h^{-1}gh} \), because \( g \) transforms as a section of the adjoint bundle \( \text{Ad}(P_c) \). However, \( (0^h)^{h^{-1}gh} = (gh)^{-1} \partial (gh) \), and as is well known, \( S_{\text{wzw}}(gh) \neq S_{\text{wzw}}(g) \). As the WZW action should not depend on the choice of trivialization of \( P_c \), there is something wrong with the identification of \( g^{-1} \partial g \) with \( 0^g \). Actually, there is another possibility, which turns out to be the right one, namely to identify \( g^{-1} \partial g \) with \( 0^g - 0^h \). Under a change of trivialization

\[
0^g - 0 \to (0^h)^{h^{-1}gh} - (0^h) = h^{-1}(0^g - 0)h, \tag{5.7}
\]

and the WZW action is invariant, because the \( h \) and \( h^{-1} \) cancel each other inside the traces in the WZW action.

If \( P_c \) is a non-trivial bundle, one cannot take \( 0 \) as a well-defined connection, and it must be replaced by some other, fixed connection \( B = B_0^0 dz + \bar{B}_0^0 d\bar{z} \). It turns out that we must require \( B \) to be flat. As \( A_z \) is identified with \( g^{-1} \partial g \), we must replace \( g^{-1} \partial g \) by a function \( A_z(A_z) \) determined by requiring the curvature of the connection one-form \( A(A_z) = A_z dz + A_\bar{z} d\bar{z} \) to vanish. This leads to the following definition of the WZW action

\[
kS_{\text{wzw}}^-(A; B) = \frac{k}{8\pi} \int_\Sigma \text{Tr}((A - B^0) \wedge \ast(A - B^0)) - \frac{k}{12\pi} \int_M \text{Tr}(\tilde{A} - \tilde{B}^0)^3, \tag{5.8}
\]

where \( \partial M = \Sigma \), and \( \tilde{A}, \tilde{B}^0 \) denote flat extensions of \( A, B^0 \) on a bundle \( \tilde{P}_c \) on \( B \) that restricts to \( P_c \) on \( \Sigma \). The \( \ast \) is the Hodge star on the Riemann surface \( \Sigma \), where we assume that some metric compatible with the complex structure on \( \Sigma \) is given. That an extension \( \tilde{P}_c \) of \( P_c \) exists can be seen as follows:\footnote{\( \text{Ad}(P_c) \) is defined as \((P_c \times G)/G\), where one the \( G \)-action on \( P_c \times G \) is given by the standard left action on \( P_c \) and by the adjoint action on \( G \).} it is sufficient to construct a complex vector bundle \( \tilde{V} \) over \( B \), that extends \( V \) \((5.1)\). The vector bundle \( V \) is the pull-back of a universal vector bundle over a certain grassmannian \( G_r \). The map we
use to pull back this universal vector bundle maps $\Sigma$ into an element of the second homology of the grassmannian, and it is precisely this element of $H_2(Gr)$ that gives the obstruction to construct an extension $\tilde{V}$. Because this element is essentially the first Chern class of $V$, and the first Chern class of $V$ vanishes, we know $\Sigma$ maps to zero in $H_2(Gr)$ and an extension $\tilde{V}$ indeed exists. It may seem surprising that one needs an additional connection $B$ to write down the WZW action, but this is necessary if one wants to write down an action for chiral fermions only. It also appears when considering determinant bundles associated to operators such as $D_{A_z}$.

Let us demonstrate that (5.8) indeed satisfies the Ward-identity (5.6), if we identify the two $B_0^0$’s with each other. Consider a small variation $A \to A + \delta A$. From the zero-curvature equation $dA + A \wedge A = 0$ we find $d\delta A + A \wedge \delta A + \delta A \wedge A = 0$, with similar equations for $\tilde{A}$. Using this we compute

$$\delta \text{Tr}(\tilde{A} - \tilde{B})^3 = 3\text{Tr}(\delta A \wedge (\tilde{A} - \tilde{B})^0) = 3\text{Tr}(A \wedge \delta A \wedge (\tilde{A} - \tilde{B})^0 + \delta \tilde{A} \wedge A \wedge (\tilde{A} - \tilde{B})^0) -$$

$$\delta \tilde{A} \wedge \tilde{A} \wedge \tilde{A} + \delta \tilde{B}^0 \wedge \tilde{B}^0 \wedge \tilde{B}^0) = 3\text{Tr}(-d(\delta A) \wedge (\tilde{A} - \tilde{B})^0 + \delta \tilde{A} \wedge (d\tilde{A} - d\tilde{B}^0))$$

$$= -3\text{Tr}(d(\delta A) \wedge (\tilde{A} - \tilde{B}^0))). \quad (5.9)$$

This gives for the total variation of the WZW action

$$k\delta S_{wzw}^{-}(A; B^0) = \frac{k}{4\pi} \int_{\Sigma} \text{Tr}\left(\delta A \wedge (1 + \ast)(A - B^0)\right) = \frac{k}{2\pi} \int_{\Sigma} d^2z \text{ Tr}(\delta A_z (A - B^0)_z), \quad (5.10)$$

because $(1 \pm \ast)/2$ are precisely the operators that define the complex structure. This shows that this action indeed solves the Ward-identity (5.6) with $J_\bar{z} = 2\pi \frac{\delta S_{wzw}(A_z)}{\delta A_z}$, if we identify the $\tilde{B}^0$'s with each other.

Having defined a generalized WZW action, it is interesting to see whether this action shares some of the properties of the ordinary WZW action. Using a calculation similar as (5.9), one can verify the following version of the Polyakov-Wiegmann formula

$$kS_{wzw}^{-}(A; B) = kS_{wzw}^{-}(A; C) + kS_{wzw}^{-}(C; B) + \frac{k}{2\pi} \int_{\Sigma} d^2z \text{ Tr}((A - C)_z (C - B)_z), \quad (5.11)$$

from which the usual Polyakov-Wiegmann formula follows by putting $A = (gh)^{-1}d(gh)$, $C = h^{-1}dh$ and $B = 0$ for a trivial bundle $P_c$. Another issue is whether $S_{wzw}(A; B)$
depends on the choice of extension \( \tilde{A} \) and \( \tilde{B} \). Choosing a different extension will change the action by a term \( \frac{k}{12\pi} \int_M \text{Tr}(\tilde{A} - \tilde{B})^3 \), where now \( \Sigma \subset B \) and \( \partial B = \emptyset \). Let \( \mathcal{U} \) denote the space of flat connections \( \tilde{A} \) on such a \( B \) such that \( \tilde{A} |_{\Sigma} = A \), and consider the function \( r_{\tilde{B}} : \mathcal{U} \to \mathbb{C} \) given by \( r_{\tilde{B}}(\tilde{A}) = \frac{k}{12\pi} \int \text{Tr}(\tilde{A} - \tilde{B})^3 \). From the identity

\[
\text{Tr}((\tilde{A} - \tilde{B})^3 + (\tilde{B} - \tilde{C})^3 + (\tilde{C} - \tilde{A})^3) = 3\text{Tr}(d((\tilde{A} - \tilde{B}) \wedge (\tilde{B} - \tilde{C})))
\]

(5.12)

it follows that \( r_{\tilde{B}}(\tilde{A}) + r_{\tilde{C}}(\tilde{B}) = r_{\tilde{C}}(\tilde{A}) \). This implies that \( r_{\tilde{B}}(\tilde{A} + \delta \tilde{A}) - r_{\tilde{B}}(\tilde{A}) \) is of third order in \( \delta \tilde{A} \), and therefore that \( r_{\tilde{B}} \) is locally constant on \( \mathcal{U} \), i.e. \( r_{\tilde{B}} \) descends to a map \( \pi_0(\mathcal{U}) \to \mathbb{C} \). We see that the WZW action is invariant under a continuous change of the choice of extension. To find out whether or not \( k \) is quantized is not very easy as it requires knowledge of \( \pi_0(\mathcal{U}) \). However, in the case that \( \mathcal{U}/G \) is connected, where \( G \) is the space of gauge transformations acting on \( \mathcal{U} \), one can say a little bit more, using the fact that in this case all connected components of \( \mathcal{U} \) can be reached from a fixed one using gauge transformations. To do this, one has to take a slightly different look at the function \( r_{\tilde{B}} \). For any group \( G \) one can write down an element of \( H^3(G) \) by extending the three-form \( \omega(X,Y,Z) = k \frac{1}{12\pi} \text{Tr}(X[Y,Z]) \) on the Lie algebra of \( G \) all over the group \( G \). One can choose \( k \) such that \( \omega \) defines actually an (possibly trivial) element of \( H^3(G,\mathbb{Z}) \). This three-form is invariant under the adjoint action of \( G \), and therefore defines an element of \( \tilde{\omega} \in H^3(\text{Ad}(\tilde{P}_c),\mathbb{Z}) \) which restricts to \( \omega \) on each fiber. A simple computation now shows that \( r_{\tilde{B}}(\tilde{B}^g) = \int_M g^*\tilde{\omega} \), which is an integer, because \( g^*\tilde{\omega} \) is an element of integral cohomology and evaluating such an element on a three manifold without boundary always gives an integer. We conclude that \( k \) must sometimes be restricted to those values for which \( \tilde{\omega} \) is an element of integral cohomology (so that upon quantizing the model everything is independent of the choice of extension), but if for instance \( \tilde{\omega} = 0 \) in \( H^3(\text{Ad}(\tilde{P}_c),\mathbb{Z}) \) for a \( k \neq 0 \), \( k \) can be taken arbitrarily.

5.3. \( W \) Algebras on Arbitrary \( \Sigma \)

In section 3 we saw how \( W \) algebras can be constructed by imposing the constraint \( A_z = \Lambda + W \) on the connection one-form \( A_z \). On a general surface these constraints can only be imposed locally; when relating different co-ordinate patches, the connection one-form transforms as \( A_z \to \frac{dz}{dz'}(h^{-1}A_zh + h^{-1}\partial_zh) \), which does not preserve the constraint \( A_z = \Lambda + W \). However, due to the special structure of the bundles for which \( A_z \) is a connection, one can always choose a trivialization such that \( h \) is a diagonal matrix. For these gauge transformations, the special form of \( A_z \) is preserved up to the
term \( \frac{d^2}{dh^2} h^{-1} \partial_z h \). We can get rid of this term by imposing the constraints on \( A_z - B_z \) instead of \( A_z \), where \( B_z \) is some fixed connection, in the same spirit as we did in the previous paragraphs. Now the constraints are preserved: \( \Lambda \) was homogeneous of degree one with respect to the gradation of \( \mathbf{g} \) and does not transform when going from one co-ordinate patch to another, while the components of \( W \) transform as fields of certain spins that are also determined by the gradation. A natural choice for \( B \) is the connection \( \nabla = \nabla_z + \nabla_{\bar{z}} \) that comes directly from the Levi-Civita connection associated to a fixed metric on the Riemann surface. Let us consider what happens in this case, and impose the constraint

\[
D_{A_z} = \nabla_z + \Lambda + W. \tag{5.13}
\]

We can now repeat the same steps as those that lead to (3.9). An important difference is that \( \nabla_z \) and \( \nabla_{\bar{z}} \) need not commute with each other, and the answer contains the curvature \( R_{\bar{z}z} = [\nabla_z, \nabla_{\bar{z}}] \). Besides this, nothing new happens and one finds that

\[
\bar{D}_{A_z} = \bar{\partial} + \text{ad} \left( \frac{1}{2} \partial \log \rho \right), \tag{5.14}
\]

Since \( \nabla_z \) has degree zero, \( L(\nabla_z + \text{ad}_W) \) is nilpotent and this expression is well defined. It is instructive to work out the zero-curvature equation for the case where \( G = \text{Sl}(2, \mathbb{R}) \), and to impose the same constraints as in section 2 (2.11). In this case the vector bundle \( V \) in (1.1) is \( V = K^{-\frac{1}{2}} \oplus K^{+\frac{1}{2}} \). Working in isothermal co-ordinates where \( ds^2 = \rho dz d\bar{z} \), one finds that

\[
D_{A_z} = \partial + \text{ad} \left( \frac{1}{2} \partial \log \rho \right), \tag{5.15}
\]

and from (5.14) that

\[
\bar{D}_{A_z} = \bar{\partial} + \text{ad} \left( \frac{1}{2} \partial \mu + \frac{1}{2} \mu \partial \log \rho \right) \left( \tfrac{1}{2} \partial \log \rho \right), \tag{5.16}
\]

The remaining zero-curvature equation, which is the generalization of (2.14), reads

\[
(\bar{\partial} - \mu \partial - 2(\partial \mu) T = -\frac{1}{2}(\partial - \partial \log \rho) \partial(\partial + \partial \log \rho) \mu + \frac{1}{2} \partial \partial \log \rho \mu - \frac{1}{2} \partial \partial \partial \log \rho \mu \right), \tag{5.17}
\]

which can be rewritten as

\[
(\bar{\partial} - \mu \partial - 2(\partial \mu) T = -\frac{1}{2}(\partial^3 + 2\mathcal{R} \partial + (\partial \mathcal{R}) \mu + \frac{1}{2} \partial \mathcal{R} \mu). \tag{5.18}
\]
where we introduced the projective connection $R = \partial^2 \log \rho - \frac{1}{2} (\partial \log \rho)^2$. This form of the Virasoro Ward identity is almost identical to the form of the Virasoro Ward identity on arbitrary Riemann surfaces as derived in [31], although there one works with an holomorphic projective connection and the last term of (5.18) is therefore absent. The precise form of [31] is recovered if one takes an appropriate regularization of the induced action for gravity, see appendix B. The form of the Virasoro Ward-identity (2.14) on the plane can be recovered by replacing $T$ by $T + R/2$, a fact already observed in [19].

Given this construction of $W$ algebras on an arbitrary Riemann surface $\Sigma$, it is straightforward to write down $W$ transformations on a Riemann surface, and to construct the chiral and covariant actions for $W$ gravity; one simply follows the construction in sections 1 and 2, and replaces the WZW actions by their generalization (5.8). The choice of 'base point' $B^0$ in (5.8) is not really important, though one should realize that one cannot in general take it to be equal to $\nabla$, because $\nabla$ is not in general flat. However, when one constructs the full covariant action, one will see that this expression is independent of the choice of base-point $B^0$. As an example of this procedure, we will in Appendix B compute the covariant action for gravity once more, but now on an arbitrary Riemann surface.

5.4. The Moduli Space for $W$ Gravity

The moduli space for $W$ gravity is in principle given by the quotient of the space of $W$ fields by the space of $W$ transformations. In our case the space of $W$ fields is given by the set of operators $M = \{\nabla_z + \Lambda + W\}$. Each operator $D' \in M$ defines an anti-holomorphic structure on the bundle $V$ (5.1). Such an anti-holomorphic structure is determined by defining what the local anti-holomorphic sections of the vector bundle are. In the anti-holomorphic structure corresponding to $D'$ these are just the local sections $s$ that satisfy $D's = 0$. The space of anti-holomorphic structures $M$ must be divided by the set of $W$ transformations. To do this, introduce the following equivalence relation on $M$: two operators $D'_1, D'_2 \in M$ are equivalent, $D'_1 \sim D'_2$, if there is a gauge transformation $g \in G_c$ relating the two, $D'_1 = (D'_2)^g$. The moduli space we are looking for is the space $\mathcal{M}_W = M/\sim$. The transformations that relate two different $D'$ are what one might call global $W$ transformations. The infinitesimal transformations of this type are precisely the $W$ transformations considered previously. Note that the equivalence relation $D'_1 \sim D'_2$ is not generated by the action of a group on $M$, as the
precise form of the gauge transformation relating two different $D'$. Thus, we cannot view $\mathcal{M}_W$ as the quotient of some space by a group action, and this makes the study of $\mathcal{M}_W$ somewhat more difficult. One of the things we would in particular like to compute is the dimension of the $\mathcal{M}_W$, or equivalently, of its tangent space. If one were to consider the full set of anti-holomorphic structures on $V$ modulo gauge transformations, \textit{i.e.} the space $\mathcal{M} = \{\nabla + A_z\}/\mathcal{G}_c$, the tangent space $T_{D'}\mathcal{M}$ at $D' \in \mathcal{M}$ is given by the $(1,0)$-cohomology of the short complex

$$0 \xrightarrow{D'} \Omega^0(\Sigma; \text{ad}(P_c)) \xrightarrow{D'} \Omega^{1,0}(\Sigma; \text{ad}(P_c)) \xrightarrow{D'} 0. \quad (5.19)$$

Here, $\Omega^{p,q}(\Sigma; \text{ad}(P_c))$ denotes the space of $(p,q)$-forms with values in $\text{ad}(P_c)$. To compute the tangent space $T_{D'}\mathcal{M}_W$ for $\mathcal{M}_W$, we should replace this complex by some kind of $W$ complex containing the $W$ transformations. There is an interesting connection between the two, which we will now explain. This connection relies heavily on the existence of the operator $L$ that was defined as the inverse of $\text{ad}_A$ in section 2. Because $L$ is an operator of degree $-1$, it provides us with an ‘integration’ operator

$$\Omega^{1,0}(\Sigma; \text{ad}(P_c)) \xrightarrow{L} \Omega^0(\Sigma; \text{ad}(P_c)). \quad (5.20)$$

As an analogy one might think of the operation of integrating over the $n$th co-ordinate in $\mathbb{R}^n$, which maps $p$-forms on $\mathbb{R}^n$ to $(p-1)$-forms on $\mathbb{R}^{n-1}$. This latter operator can be used to show that the cohomology of $\mathbb{R}^n$ is the same as the cohomology of $\mathbb{R}^{n-1}$, by constructing a so-called homotopy-equivalence between the de Rham complexes for $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$. Here we can perform a similar construction using the ‘integration’ operator $L$. Defining the two operators

$$f_0 = 1 - L \circ D', \quad f_1 = 1 - D' \circ L, \quad (5.21)$$

we can construct the following commutative diagram

$$\begin{array}{cccc}
0 & \xrightarrow{D'} & \Omega^0(\Sigma; \text{ad}(P_c)) & \xrightarrow{D'} \Omega^{1,0}(\Sigma; \text{ad}(P_c)) & \xrightarrow{D'} 0 \\
\downarrow f_0 & & \downarrow f_1 & & \\
0 & \xrightarrow{D'} & f_0(\Omega^0(\Sigma; \text{ad}(P_c))) & \xrightarrow{D'} f_1(\Omega^{1,0}(\Sigma; \text{ad}(P_c))) & \xrightarrow{D'} 0
\end{array} \quad (5.22)$$

which gives actually a homotopy equivalence of complexes\footnote{Indeed, denoting by $f$ both maps $f_0$ and $f_1$, we have $1 - f = L \circ D' + D' \circ L$, so that $L$ is precisely an homotopy operator as defined in \cite{33}.}, implying that the cohomology of both complexes in (5.22) is the same. The next step is to iterate this
construction a number of times, until the complex does not change anymore. Let us 
denote the corresponding limit complex, if it exists, by 

$$
0 \xrightarrow{D'} f_0^\infty(\Omega^0(\Sigma; \text{ad}(P_c))) \xrightarrow{D'} f_1^\infty(\Omega^{1,0}(\Sigma; \text{ad}(P_c))) \xrightarrow{D'} 0.
$$

(5.23)

Using the properties of $L$ one can show that a sufficient condition for the limit complex 
to exist is that the operator $L \circ (D' - \text{ad}_{\Lambda})$ is nilpotent, and that in that case 

$$
f_0^\infty = (1 - L \circ D')^\infty = \frac{1}{1 + L(D' - \text{ad}_{\Lambda})} \circ \Pi_k,
$$

$$
f_1^\infty = (1 - D' \circ L)^\infty = \frac{1}{1 + (D' - \text{ad}_{\Lambda})L}.
$$

(5.24)

Specializing to the case of $W$ algebras, we take $D' = \nabla_z + \Lambda + W$ and find, upon 
comparing the limit complex with (5.14), that the limit complex precisely contains the 
$W$ transformations, and is the $W$ complex we were looking for. To illustrate how this 
works in practice, we take again $G = SL(2, \mathbb{R})$ as we did in section 5.3. The limit 
complex is reached by applying $f_0$ and $f_1$ two times. The operator $L$ is given by 

$$
L : \begin{pmatrix}
  p^0 & p^+ \\
  p^- & -p^0
\end{pmatrix} = \begin{pmatrix}
  -p^+/2 & 0 \\
  p^0/2 & p^+
\end{pmatrix},
$$

(5.25)

and if we represent an arbitrary element of $\Omega^0(\Sigma; \text{ad}(P_c))$ by $\begin{pmatrix}
  \epsilon^0 \\
  \epsilon^-
\end{pmatrix}$, and an 
element of $\Omega^{1,0}(\Sigma; \text{ad}(P_c))$ by $\begin{pmatrix}
  a^0 & a^+ \\
  a^- & -a^0
\end{pmatrix}$, we find the following diagram, where $\gamma = \partial \log \rho$:

$$
\begin{array}{ccc}
\begin{pmatrix}
  \epsilon^0 \\
  \epsilon^-
\end{pmatrix} & \xrightarrow{D'} & \begin{pmatrix}
  a^0 & a^+ \\
  a^- & -a^0
\end{pmatrix} \\
\begin{pmatrix}
  \frac{1}{2} \partial \epsilon^+ + \frac{1}{2} \epsilon^+ \gamma \\
  \epsilon^+ T - \partial \epsilon^0 - \frac{1}{2} \partial \epsilon^+ - \frac{1}{2} \epsilon^+ \gamma
\end{pmatrix} & \xrightarrow{D'} & \begin{pmatrix}
  \frac{1}{2} \partial a^+ \\
  a^- - (\partial - \gamma)a^0 + Ta^+ - \frac{1}{2} \partial a^+
\end{pmatrix} \\
\begin{pmatrix}
  \frac{1}{2} \partial \epsilon^+ + \frac{1}{2} \epsilon^+ \gamma \\
  \epsilon^+ T - \frac{1}{2} \partial^2 \epsilon^+ - \frac{1}{2} \partial (\epsilon^+ \gamma) - \frac{1}{2} \partial \epsilon^+ - \frac{1}{2} \epsilon^+ \gamma
\end{pmatrix} & \xrightarrow{D'} & \begin{pmatrix}
  0 \\
  a^- - (\partial - \gamma)(a^0 + (\frac{1}{2} \partial - T)a^+) - 0
\end{pmatrix}
\end{array}
$$

Working out the action of $D'$ in the last line we find 

$$
D' \begin{pmatrix}
  \frac{1}{2} \partial \epsilon^+ + \frac{1}{2} \epsilon^+ \gamma \\
  \epsilon^+ T - \frac{1}{2} \partial^2 \epsilon^+ - \frac{1}{2} \partial (\epsilon^+ \gamma) - \frac{1}{2} \partial \epsilon^+ - \frac{1}{2} \epsilon^+ \gamma
\end{pmatrix} = \begin{pmatrix}
  0 & 0 \\
  \delta_\epsilon T & 0
\end{pmatrix},
$$

(5.26)
where $\delta_{\epsilon} T = -\frac{1}{2}(\partial - \gamma)\partial(\partial + \gamma)\epsilon^\perp + 2\partial\epsilon^\perp T + \epsilon^\perp \partial T$ which indeed describes the transformation of $T$ under a co-ordinate transformation.

Altogether we reach the remarkable conclusion that $W$ transformations are nothing but a homotopic contraction of ordinary gauge transformations. Under a homotopy equivalence the cohomology does not change, and therefore the Riemann-Roch theorem can be applied to the $W$ complex (5.23) to give

$$\dim H^{1,0} - \dim H^{0,0} = (g - 1) \dim G = (g - 1)(N^2 - 1). \quad (5.27)$$

This is a useful formula which we need to prove the following: for genus $g > 1$, $\mathcal{M}_W = M/\sim = M^{hol}/\sim$, where $M^{hol} = \{\nabla_z + \Lambda + W \mid \nabla_{\bar{z}} W = 0\}$. In other words, the fields $W$ can always be made holomorphic using a global $W$ transformation. To prove this, it is sufficient to show that if we write down an even further reduced complex containing $D' \in M^{hol}$ and only those $W$ transformations that preserve the condition $D' \in M^{hol}$, this complex still has the same cohomology as (5.23). It might happen that in this way one misses certain connected components of $\mathcal{M}_W$, but that is not a problem here: $\mathcal{M} = \mathcal{M}_W$ is connected, because we are working with bundles of a fixed topological type.

The infinitesimal gauge transformations that preserve the condition $\nabla_{\bar{z}} W = 0$, are given by the $\epsilon$ satisfying

$$\nabla_{\bar{z}}(\nabla_z + \text{ad}_\Lambda + \text{ad}_W)\epsilon = 0. \quad (5.28)$$

If we choose a metric of constant curvature $R_{z\bar{z}} = [\nabla_z, \nabla_{\bar{z}}]$, then $L(R_{z\bar{z}})$ is proportional to the Lie-algebra element $\Lambda^-$ that defines a highest weight gauge (3.14). This shows that $[L(R_{z\bar{z}}), W] = 0$ and

$$[\nabla_z + \Lambda + W, \nabla_{\bar{z}} - L(R_{z\bar{z}})] = 0 \quad (5.29)$$

for the $W$ that satisfy $\nabla_{\bar{z}} W = 0$. Note that (5.29) gives a solution to the zero-curvature equations for these $W$. The $\epsilon$ that satisfy (5.28) must also be of the form $\epsilon = (1 + L(\nabla_z + \text{ad}_W))^{-1}F$ with $F \in \Pi_{k\mathfrak{g}}$ in order to preserve the form of $W$. If we substitute this in (5.28) and use the fact that $[L(R_{z\bar{z}}), \delta W] = 0$, (5.28) can be rewritten as

$$(\nabla_{\bar{z}} - L(R_{z\bar{z}}))(\nabla_z + \text{ad}_\Lambda + \text{ad}_W)\frac{1}{1 + L(\nabla_z + \text{ad}_W)} F = 0 \Leftrightarrow$$

$$(\nabla_z + \text{ad}_\Lambda + \text{ad}_W)(\nabla_{\bar{z}} - L(R_{z\bar{z}}))\frac{1}{1 + L(\nabla_z + \text{ad}_W)} F = 0 \Leftrightarrow$$
\[
(\nabla_z + \text{ad}_\Lambda + \text{ad}_W) \frac{1}{1 + L(\nabla_z + \text{ad}_W)} \nabla_{\bar{z}} F = 0. \tag{5.30}
\]

Locally, \(\nabla_{\bar{z}} F\) can be written as \(\sum_\alpha f_\alpha(\bar{z}) G_\alpha(z)\), where the \(f_\alpha\) are linearly independent antiholomorphic functions, and the \(G_\alpha\) are holomorphic sections with respect to \(\nabla_{\bar{z}}\) of \((\Pi_k \text{ad}(P_c)) \otimes \bar{K}\). Substituting this in (5.30) yields

\[
\sum_\alpha f_\alpha(\nabla_z + \text{ad}_\Lambda + \text{ad}_W) \frac{1}{1 + L(\nabla_z + \text{ad}_W)} G_\alpha = 0. \tag{5.31}
\]

Because the \(f_\alpha\) are linearly independent, each \(G_\alpha\) must satisfy \((\nabla_z + \text{ad}_\Lambda + \text{ad}_W)(1 + L(\nabla_z + \text{ad}_W))^{-1} G_\alpha = 0\). Locally, there are a finite number of solutions \(G_\alpha\) to this equation. Globally, such \(G_\alpha\) do not exist, as \((\Pi_k \text{ad}(P_c)) \otimes \bar{K}\) is a direct sum of line bundles \(K^r\) with \(r < 0\) (upon identifying \(\bar{K}\) with \(K^{r-1}\)), and these do not have any global holomorphic sections. Therefore (5.31) implies that \(\nabla_{\bar{z}} F = 0\). \(F\) is a section of a direct sum of line bundles \(K^r\) with \(r \leq 0\). These do, for genus \(g > 1\), not have global holomorphic sections unless \(r = 0\), in which case the only holomorphic sections are the constant ones. The piece of \(F\) which transforms as a section of \(K^0\) is precisely the piece that has degree zero with respect to the gradation of the Lie algebra. Let us denote the subalgebra in which this piece of \(F\) lives by \(\mathfrak{g}_0 = \Pi_k \Pi_k^\dagger_\mathfrak{g}\). For a constant \(F \in \mathfrak{g}_0\) the parameter \(\epsilon\) of the gauge transformation is given by \(\epsilon = (1 + L(\nabla_z + \text{ad}_W))^{-1} F = F\), and the gauge transformation reads \(\delta W = [W, F]\). The reduced complex we were looking for is

\[
0 \xrightarrow{D'} \mathfrak{g}_0 \xrightarrow{D'} T_{D'} M^{\text{hol}} \xrightarrow{D'} 0. \tag{5.32}
\]

To show that the cohomology of this complex agrees with that of (5.23) we need only compute the difference \(\dim H^{1,0} - \dim H^{0,0}\) of (5.32). In (5.32) only finite dimensional spaces occur, and therefore the index \(\dim H^{1,0} - \dim H^{0,0}\) equals \(\dim T_{D'} M^{\text{hol}} - \dim \mathfrak{g}_0\). The dimension of \(M^{\text{hol}}\) equals \(\sum_i H^0_{\nabla_{\bar{z}}} (\Sigma; K^{s_i})\), where \(s_i\) are the spins of the different components of \(W\). The dimension of \(H^0_{\nabla_{\bar{z}}} (\Sigma; K^r)\) equals \((2r - 1)(g - 1)\) for \(r > 1\), and \(g\) for \(r = 1\). Thus we find

\[
\dim H^{1,0} - \dim H^{0,0} = \sum_{i, s_i > 1} (g - 1)(2s_i - 1) + \sum_{i, s_i = 1} g - \dim \mathfrak{g}_0
\]

\[
= \sum_i (g - 1)(2s_i - 1) = (g - 1) \dim G, \tag{5.33}
\]

\(^*\text{That } \mathfrak{g}_0 \text{ is actually a subalgebra is related to the fact that } \mathfrak{g}_0 \text{ contains the Kac-Moody symmetries that survive the reduction to the } W \text{ algebra. One can in principle impose further constraints on the } W \text{ algebra so as to get rid of these residual Kac-Moody symmetries [8], but we will not do that here.}\)
which indeed agrees with (5.27). In the last line we used (3.4). Altogether this proves that $\mathcal{M}_W = M^\text{hol}/\sim$, and so we have a simple finite dimensional model of $W$ moduli space at our disposal.

Although the dimensions computed so far strongly hint that $W$ moduli space has something to do with the moduli space of $SL(N, \mathbb{R})$-bundles, it is at this stage not clear what the precise relation, if it exists, should be. The zero-curvature equations associate a flat connection to all operators $D' = \nabla_z + \text{ad}_\Lambda + \text{ad}_W$, but a priori these flat connections are flat $SL(N, \mathbb{C})$ connections, and it is not easy to see whether they can be written as flat $SL(N, \mathbb{R})$ connections using an appropriate gauge transformation. The main difficulty is that $SL(N, \mathbb{R})$ is a non-compact group, and therefore one cannot simply use the Narasimhan-Seshadri theorem (see also [12]), which essentially states that for compact groups the space of anti-holomorphic structures on an associated vector bundle modulo complexified gauge transformations is the same as the space of flat connections modulo ordinary gauge transformations. In this theorem, the anti-holomorphic structure is required to satisfy a certain condition called stability, and this condition is not valid for the special bundles under consideration.

There exists an extension of the work of Narasimhan-Seshadri where the compact group is replaced by the general linear group $GL(N, \mathbb{C})$. This is the theory of Higgs bundles [13, 15], and this seems to be the natural setting for $W$ moduli space. A Higgs bundle is a pair consisting of a holomorphic vector bundle $V$ and a holomorphic section $\theta \in H^0(\Sigma; \text{End}(V) \otimes K)$. In our case we are interested in the situation where $V$ is given by (5.1), the holomorphic structure is given by the operator $\nabla_z$, the group $GL(N, \mathbb{C})$ is reduced to $SL(N, \mathbb{C})$ and $\theta = \Lambda + W$, where $W$ is holomorphic. The group $GL(N, \mathbb{C})$ acts in a natural way on Higgs bundles, and one can define a moduli space for Higgs bundles by identifying two that are equivalent under a $GL(N, \mathbb{C})$-transformation. To obtain a good moduli space one has to impose a condition on the Higgs bundle that is also called stability. A Higgs-bundle is called stable if for every holomorphic subbundle $V' \subset V$ that satisfies $\theta(V') \subset V' \otimes K$, the slope $\mu(V')$ of $V'$ is smaller than the slope $\mu(V)$ of $V$. The slope is defined as the first Chern class divided by the rank of the bundle.

Let us see whether the Higgs bundle with $\theta = \Lambda + W$ is stable. The slope of $V$ vanishes, and therefore every subbundle $V'$ with $\theta(V') \subset V' \otimes K$ must have a negative slope for stability. The $sl_2$-algebra (3.14) acts via left multiplication on the vector bundle $V$. Under this action the $N$-dimensional representation furnished by $V$ decomposes in a direct sum of irreducible $sl_2$-representations, $V = \bigoplus_{i=1}^n V_i$, of
spin $j_i$, and $V_i \simeq K^{-j_i} \bigoplus K^{1-j_i} \bigoplus \ldots \bigoplus K^{j_i}$. The slope of each of the $V_i$ is zero, as they have vanishing Chern class. $\Lambda$ preserves $V_i$ and all subbundles of $V_i$ of the type $K^{-j_i} \bigoplus K^{1-j_i} \bigoplus \ldots \bigoplus K^{j_i-\tau}$ for some $\tau > 0$. These all have strictly negative slope, and therefore the only problematic subbundles of $V$ are direct sums of the $V_i$, as these are the only holomorphic subbundles preserved by $\Lambda$ that have a nonnegative slope. The same bundles are also the bundles that might threaten the stability of $(V, \theta)$ with $\theta = \Lambda + W$. Now there is a component of $W$ for every irreducible $sl_2$ representation in $(\bigoplus V_i) \otimes (\bigoplus V_i)$, except for one overall trivial representation. A component $W$ corresponding to an irreducible subrepresentation of $V_i \otimes V_i'$ mixes between the bundles $V_i$ and $V_i'$. Therefore if sufficiently many of these components are nonzero, no direct sum of $V_i$'s will be invariant under $\Lambda + W$ anymore, and all proper holomorphic subbundles, if they exist, will have negative slope.

Another, equivalent way to express this condition is to demand that $\ker \text{ad}_W(g_0) = 0$, so that $g_0$ acts faithfully on $M^{\text{hol}}$. If we therefore define $M^{\text{hol}}_{\text{red}} = \{ \nabla_z + \Lambda + W \mid \nabla_z W = 0 \wedge \ker \text{ad}_W(g_0) = 0 \}$, then the quotient space $M^{\text{hol}}_{\text{red}}/g_0$ has no singularities, and it is naturally a subspace of the moduli space of stable Higgs bundles, of complex dimension $(g - 1) \dim(G)$.

The dimension of the moduli space of Higgs bundles is $2(g - 1) \dim(G)$, which is twice as large as the dimension of the $W$ moduli space. These correspond to flat irreducible $SL(N, \mathbb{C})$ bundles over the Riemann surface $\Sigma$ [13, 16]. One might wonder which property characterizes the flat $SL(N, \mathbb{C})$ connections that correspond to points in the $W$ moduli space. For general $W$ algebras we do not know the answer to this question, but for the ‘standard’ $W$ algebras the answer is, that only those flat $SL(N, \mathbb{C})$ connections which are reducible to a flat $SL(N, \mathbb{R})$ connections can correspond to points in the $W$ moduli space. To prove this, we use lemma 3.20 in [13]. This lemma states that a Higgs bundle $(V, \theta)$ corresponds to a flat real connection if and only if there exists a bilinear symmetric form $S(u, v)$ on $V \otimes V_C$, where $V_C$ is the Higgs bundle $(V, -\theta)$, such that

$$\bar{\partial}S(u, v) = S((\nabla_z + \theta)u, v) + S(u, (\nabla_z - \theta)v). \quad (5.34)$$

For the standard $W$ algebras such a symmetric form $S$ exists. The vector bundle $V$ is in this case

$$V = \bigoplus_{i=1}^{N} V_i, \quad V_i \simeq K^{-N-i} \quad (5.35)$$
and the symmetric form $S$ is given by

$$S(u, v) = \sum_{l=1}^{N} u_l \cdot v_{N+1-l}. \quad (5.36)$$

It is an easy exercise to show that (5.34) indeed holds for (5.36). Putting everything together we conclude that, for standard $W$ algebras, $W$ moduli space is a component of the moduli space of flat irreducible $Sl(N, \mathbb{R})$ connections. This parametrization of a component of the moduli space of flat $Sl(N, \mathbb{R})$ connections in terms of certain Higgs bundles is similar to the one mentioned by Hitchin [14]. The relevant component is specified by the topological type of the real vector bundle on which the flat $Sl(N, \mathbb{R})$ connection lives. To really construct this flat connection explicitly, one needs to know the so-called Hermitian-Yang-Mills metric on the Higgs bundle, see [14]. This metric, and the associated flat connection are very easy to describe if $W = 0$. In that case one picks a constant curvature metric on the Riemann surface, and uses the metric this induces on $\tilde{K}$ to construct a metric on $V$. This is already the Hermitian-Yang-Mills metric and the corresponding flat connection is

$$D = \nabla_z + \nabla_{\bar{z}} + \Lambda - L(R_{z\bar{z}}). \quad (5.37)$$

The real vector bundle for which this defines an $Sl(N, \mathbb{R})$ connection is given by the bundle left invariant by an involution of $V$ that commutes with $D$. The involution $\sigma : V \to V$ is given by sending $u \in K^r \to \bar{u} \in \bar{K}^r \simeq K^{-r}$, where the metric is used to identify $\bar{K}$ with $K^{-1}$.

As an example, consider ordinary gravity. In that case $G = Sl(2, \mathbb{R})$, and $V = K^{-\frac{1}{2}} \oplus K^{\frac{1}{2}}$. The involution $\sigma$ is in local co-ordinates with metric $ds^2 = \rho dzd\bar{z}$ given by

$$\sigma : \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right) \to \left( \begin{array}{c} \bar{u}_2 / \sqrt{\rho} \\ \bar{u}_1 / \sqrt{\rho} \end{array} \right). \quad (5.38)$$

The corresponding real bundle is one of Euler class $2(g - 1)$, and the $W$ moduli space as defined here is the Teichmüller space of $\Sigma$, which is indeed closely related to the moduli space of Riemann surfaces, and has occurred before in studies of 2D quantum gravity [25]. For details, see [13].

It is a very interesting problem to characterize the flat $Sl(N, \mathbb{C})$-bundles that are related to the moduli space of the nonstandard $W$ algebras. We have checked for
a few cases that it is impossible to construct a symmetric bilinear form satisfying (5.34) for those \( W \) algebras, and therefore they do not correspond to flat \( SL(N, \mathbb{R}) \) connections. If we replace \( V \) by \( SL(V) \), and consider the Higgs bundle \( (SL(V), \theta) \) with \( \theta : SL(V) \to SL(V) \otimes K \) given by \( \theta(X) = [\Lambda + W, X] \), then the existence of symmetric bilinear form satisfying (5.34) for this Higgs bundle would show that the flat \( SL(N, \mathbb{C}) \) connections for these \( W \) algebras are always reducible to a flat \( g \) connection, where \( g \) is some Lie algebra whose complexification is \( sl(N, \mathbb{C}) \). However, for the cases we checked it was impossible to construct a symmetric form \( S \) for \( (SL(V), \theta) \) either, and therefore it is still unclear what precisely characterizes the nonstandard \( W \) moduli spaces.

Instead of looking at \( M^{hol}_{red}/g_0 \), one could also look at different ‘strata’ of \( M^{hol} \), by defining \( M^{hol}_k = \{ \nabla_z + \Lambda + W \mid \nabla_z W = 0 \wedge \dim \ker \text{ad}_W(g_0) = k \} \). The space \( \mathcal{M}_{W,k} = M^{hol}_k/g_0 \) is presumably related to ‘singular’ configurations of \( W \) fields, and deserves some further study as well.

The simplest nonstandard \( W \) algebra is \( W^{(2)}_3 \). For this \( W \) algebra, \( \Lambda \) and \( W \) are given by

\[
\Lambda = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} J & 0 & 0 \\ G^+ & -2J & 0 \\ T & G^- & J \end{pmatrix},
\]

where \( J \) has spin 1, \( G^+, G^- \) have spin 3/2 and \( T \) has spin 2. The space \( M^{hol} \) has dimension \( 8g - 7 \), and \( g_0 \) consists of the constant matrices

\[
X = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & -2\epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \quad \delta_e W = [W, X] = \begin{pmatrix} 0 & 0 & 0 \\ 3\epsilon G^+ & 0 & 0 \\ 0 & -3\epsilon G^- & 0 \end{pmatrix}.
\]

From this we see that \( M^{hol}_{red} = \{ J, G^+, G^-, T \mid G^+ \neq 0 \text{ or } G^- \neq 0 \} \), and that \( \mathcal{M}_{W,0} \) is topologically the product of \( \mathbb{C}^{4g-3} \) and a weighted projective space of dimension \( 4g - 5 \). Clearly, \( \mathcal{M}_{W,1} \) is topologically a vector space of dimension \( 4g - 3 \). We see that for this \( W \) algebra the moduli space is non trivial.

The discussion of \( W \) moduli space in this section has so far been limited to genus \( g > 1 \). Most of the analysis can also be carried through for genus \( g = 0, 1 \). The main difference with \( g > 1 \) is that for the latter case, \( W \) moduli space is in a natural way a subspace of the moduli space of stable Higgs bundles. For \( g = 0, 1 \) this is no longer the case, because the Higgs bundles one obtains for \( g = 0, 1 \) are not stable any more, the reason for this being the fact that the first Chern class of \( K \) is given by
$c_1(K) = 2(g - 1)$, and changes sign at $g = 1$. Let us briefly indicate what the moduli spaces for $g = 0, 1$ look like.

For $g = 0$, the line bundle $K^r$ with $r > 0$ has no global holomorphic sections. Therefore $M^{hol}$ is contains only $D' = \nabla + \Lambda$ and has dimension 0. The gauge transformations that act trivially on $\nabla + \Lambda$ are given by $\delta \Lambda = 0 = [\Lambda, F]$, where $F$ is an arbitrary holomorphic section of $\Pi_k \text{ad}(P_c) \simeq \oplus_i K^{1-s_i}$. For genus $g = 0$ the line bundle $K^{1-s_i}$ has $(2s_i - 1)$ holomorphic sections, and the dimension of the space of gauge transformations that act on $M^{hol}$ equals $\sum_i (2s_i - 1) = \dim G$. This shows that $\dim H^{1,0} - \dim H^{0,0} = -\dim G$, in agreement with the Riemann-Roch theorem (5.23), which is valid for arbitrary genus.

For $g = 1$, the line bundles $K^r$ are all trivial and have precisely one holomorphic section. If $d_W$ denotes the number of generators of the $W$ algebras, then $\dim M^{hol} = d_W$, and the space of gauge transformations that acts on $M^{hol}$ also has dimension $d_W$. These gauge transformations act on $M^{hol}$ via $\delta W = [\Lambda + W, (1 + \text{Ad}_W)^{-1}F]$, where $F$ is an arbitrary holomorphic section of $\Pi_k \text{ad}(P_c)$. Again, (5.23) is satisfied. A natural candidate for the moduli space is in this case

$$\mathcal{M}_W = \{\nabla + \Lambda + W | \dim C_g(\Lambda + W) = \text{rank } g\} \sim,$$

where $C_g(X)$ is the centralizer of $X$, i.e. the set of elements of $g$ that commute with $X$, and $g$ should be identified with the holomorphic sections of $\text{ad}(P_c) \otimes K$. The dimension of the genus 1 moduli space equals the rank of $g$.

Actually, what we really have been computing up till now is $W$ Teichmüller space rather than $W$ moduli space. For ordinary gravity, the latter is given by the quotient of the former by the action of the modular group. This suggests that a good candidate for $W$ moduli space is also to consider the quotient of the $W$ Teichmüller space by the action of the modular group (cf. [30]). It is an interesting problem to investigate these spaces in some more detail, and to try to generalize the framework for ordinary gravity coupled to matter, where one expresses correlation functions in terms of modular invariant combinations of conformal blocks, to the case of matter theories coupled to $W$ gravity.
Acknowledgement

We would like to thank Jaap Kalkman for stimulating discussions and useful comments. This work was financially supported by the Stichting voor Fundamenteel Onderzoek der Materie (FOM).

A. Appendix

In this Appendix we give the definitions and some of the properties of the WZW actions $S_{\pm}^{wzw}$:

$$S_{\pm}^{wzw}(g) = \frac{1}{4\pi} \int_{\Sigma} d^2z \, \text{Tr}(g^{-1} \partial gg^{-1} \bar{\partial}g) \pm \frac{1}{12\pi} \int_{B} \text{Tr}(g^{-1} dg)^3,$$

(A.1)
satisfying the following Polyakov–Wiegen identities [8]:

$$S_{\pm}^{wzw}(gh) = S_{\pm}^{wzw}(g) + S_{\pm}^{wzw}(h) + k \frac{1}{2\pi} \int d^2z \, \text{Tr}(g^{-1} \bar{\partial}g \bar{\partial}hh^{-1}),$$

$$S_{\mp}^{wzw}(gh) = S_{\mp}^{wzw}(g) + S_{\mp}^{wzw}(h) + k \frac{1}{2\pi} \int d^2z \, \text{Tr}(g^{-1} \partial g \bar{\partial}hh^{-1}).$$

(A.2)

The equations of motion resulting from these WZW actions were already studied in section 5.2 (see (5.10)). If we view $S_{\mp}^{wzw}(g)$ as a function of $A \bar{z} = g^{-1} \partial g$, the equation of motion is:

$$2\pi \frac{\delta S_{\mp}^{wzw}(A)}{\delta A_{\bar{z}}} = J_{\bar{z}} = 0,$$

(A.3)

where $J_{\bar{z}}$ is such that the pair $(A_{\bar{z}}, J_{\bar{z}})$ have vanishing curvature, so $J_{\bar{z}} = g^{-1} \bar{\partial}g$. Equivalently, if we take $S_{\pm}^{wzw}(h)$ to be a function of $A_{\bar{z}} = h^{-1} \bar{\partial}h$, we have:

$$2\pi \frac{\delta S_{\pm}^{wzw}(A)}{\delta A_{\bar{z}}} = J_{\bar{z}} = 0,$$

(A.4)

where now $F(J_{\bar{z}}, A_{\bar{z}}) = 0$. For the action $kS_{\pm}^{wzw}(h) - \frac{k}{2\pi} \int d^2z \, \text{Tr}(\Lambda h^{-1} \bar{\partial}h)$, where $(h^{-1} \bar{\partial}h)^+$ is held fixed and should be identified with $\mu$, this implies that the equations of motion can only be satisfied if $h^{-1} \partial h$ is of the form:

$$h^{-1} \partial h = \begin{pmatrix} 0 & 1 \\ * & 0 \end{pmatrix}.$$

(A.5)
From section 3 we know that \( \ast \) should be identified with a spin two field, and that the solution for \( A_\bar{z} = h^{-1} \bar{\partial} h \) is now that \( A_\bar{z} \) is as in (2.13), since this is the unique answer such that (A.3) and \( A_\bar{z} \) have vanishing curvature, as we argued in section 2.2.

B. Appendix

In this appendix we will sketch how some of the results of section 2 generalize to arbitrary Riemann surfaces, using the generalized WZW functional (5.8). The chiral action \( \Gamma[T] \) was in section 2 shown to be given by \( -k S_{wzw}(g) \), where \( g^{-1} \partial g = \Lambda + W \). This has a natural generalization, namely \( \Gamma[T] = -k S_{wzw}(A; B) \), where \( A \) is the flat connection consisting of (5.13) and (5.16), and \( B \) is a fixed flat reference connection, representing a particular regularization, as explained in section 5.2. If \( B \) locally has the form

\[
B = \left( \begin{array}{cc}
B^0_z & B^+_z \\
B^-_z & -B^0_\bar{z}
\end{array} \right) dz + \left( \begin{array}{cc}
B^0_\bar{z} & B^+_{\bar{z}} \\
B^-_{\bar{z}} & -B^0_z
\end{array} \right) d\bar{z},
\]

(B.1)

then the Ward-identity satisfied by \( \Gamma[T] \) is given by (5.18), with \( \mu \) replaced by \( B^+_{\bar{z}} - 2 \pi k \delta \Gamma[T] / \delta T \). By Fourier transformation, we define the induced action \( \Gamma[\mu] \)

\[
\exp(-\Gamma[\mu]) = \int DT \exp \left( -\Gamma[T] - \frac{k}{2\pi} \int d^2 z \left( \mu - B^+_{\bar{z}} \right) (T - B^-_z) \right),
\]

(B.2)

which differs from (2.19) by the explicit appearance of \( B \) in the definition of Fourier transformation. If we define \( T(\mu) \) by requiring the connection (5.15) + (5.16) to be flat, i.e. we view \( A \) as a flat connection depending on \( \mu \) rather than \( T \), then in a saddle point approximation the action \( \Gamma[\mu] \) is simply given by

\[
\Gamma[\mu] = \Gamma[T(\mu)] + \frac{k}{2\pi} \int d^2 z (\mu - B^+_{\bar{z}}) (T(\mu) - B^-_z).
\]

(B.3)

The 'induced' action \( \Gamma[\mu] \) satisfies the Ward identity

\[
(\bar{\partial} - \mu \partial - 2(\partial \mu)) \frac{\delta \Gamma[\mu]}{\delta \mu} = \frac{c}{24\pi} (\partial^3 + 2 \mathcal{R}' \partial + (\partial \mathcal{R}')) \mu - \frac{c}{24\pi} \bar{\partial} \mathcal{R}',
\]

(B.4)

where \( \mathcal{R}' = \partial^2 \log \rho - \frac{1}{2} (\partial \log \rho)^2 - 2B^-_z \). If we choose \( B^-_z \) in such a way so that \( \mathcal{R}' \) is a holomorphic projective connection, then we find precisely the Ward identity on
a general Riemann surface as found in [31]. The induced action $\Gamma[\mu]$ on arbitrary Riemann surfaces has been studied from different points of view in [32].

To construct the covariant action, we also need the left-moving sector of the theory, i.e. the actions $\Gamma[\bar{T}]$ and $\Gamma[\bar{\mu}]$. To construct these, we need to impose constraints on the $\bar{z}$-component of a connection $A_2$. Using the isomorphism between $\bar{K}$ and $K^{-1}$, one finds that one has to impose the following constraint on $A_2,\bar{z}$

$$\bar{D}_{A_2,\bar{z}} = \bar{\partial} + \left( \begin{array}{cc} 0 & \bar{T}/\rho \\ \rho & 0 \end{array} \right),$$

(B.5)

and

$$D_{A_2,\bar{z}} = \partial + \left( \begin{array}{cc} * & * \\ \bar{\mu}\rho & * \end{array} \right)$$

(B.6)

is such that $A_2$ has vanishing curvature. Now $\Gamma[\bar{T}] = -kS_w^-(B; A_2)$, and $\Gamma[\bar{\mu}]$ is the Fourier transform of $\Gamma[T]$. In order to able to construct a covariant action, we must use the same background connection $B$ in both the left and right moving sector. Repeating arguments similar to those in section 2, one finds that the complete covariant action for gravity is simply given by

$$S_{\text{cov}}(T, \bar{T}, G) = -kS_w^-(A_G^1; A_2),$$

(B.7)

which does not depend on the choice of $B$ anymore, and is manifestly invariant under both left and right diffeomorphisms. To extract $\Delta \Gamma[T, \bar{T}, G]$ from this covariant action, we make repeatedly use of the generalized Polyakov-Wiegmann identity (5.11) to obtain

$$S_{\text{cov}}(T, \bar{T}, G) = -kS_w^-(A_1; B) - kS_w^-(B^G; B) - kS_w^-(B; A_2)$$

$$- \frac{k}{2\pi} \int d^2z \text{Tr}((A_1 - B)_{\bar{z}}(B - A_2^{-1})_{\bar{z}})$$

$$- \frac{k}{2\pi} \int d^2z \text{Tr}((A^G_1 - B)_{\bar{z}}(B - A_2)_{\bar{z}})$$

$$- \frac{k}{2\pi} \int d^2z \text{Tr}((A_1 - B)_{\bar{z}}G(A_2 - B)_{\bar{z}}G^{-1}).$$

(B.8)

In $-kS_w^-(A_1; B)$ and $-kS_w^-(B; A_2)$ we recognize $\Gamma[T]$ and $\Gamma[\bar{T}]$, and the remainder of $S_{\text{cov}}$ in (B.8) is $\Delta \Gamma[T, \bar{T}, G]$. The ‘local counterterm’ $\Delta \Gamma[\mu, \bar{\mu}, G]$ can be obtained from $\Delta \Gamma[T, \bar{T}, G]$ by Fourier transformation, using the same Fourier transformation as in (B.2). If we parametrize $G$ locally by the Gauss decomposition

$$G = \left( \begin{array}{cc} 1 & 0 \\ \omega & 1 \end{array} \right) \left( \begin{array}{cc} e^{\phi} & 0 \\ 0 & e^{-\phi} \end{array} \right) \left( \begin{array}{cc} 1 & -\bar{\omega}/\rho \\ 0 & 1 \end{array} \right),$$

(B.9)
we find the following expression for $\Delta \Gamma[\mu, \bar{\mu}, G]$ on an arbitrary Riemann surface

$$\Delta \Gamma[\mu, \bar{\mu}, G] = \frac{k}{2\pi} \int d^2 z \left[ \partial \phi \bar{\partial} \phi + \omega (2 \partial \phi + (\partial + \partial \log \rho) \mu) + \bar{\omega} (2 \partial \phi + (\bar{\partial} + \partial \log \rho) \bar{\mu}) 
+ \mu \omega^2 + \bar{\mu} \bar{\omega}^2 + 2 \omega \bar{\omega} - (1 - \mu \bar{\mu}) e^{-2\phi} - \phi \bar{\partial} \partial \log \rho \right]$$

$$+ \text{extra piece}, \quad \text{(B.10)}$$

where the extra piece contains the terms which depend on the reference connection $B$:

$$\text{extra piece} = \frac{k}{2\pi} \int d^2 z \ Tr(B_z \left( \begin{array}{cc} 0 & -\mu \\ \rho & 0 \end{array} \right) + \left( \begin{array}{cc} \partial \log \rho/2 & 1 \\ -\bar{\mu} \rho & -\partial \log \rho/2 \end{array} \right) B_z)$$

$$- \frac{k}{2\pi} \int d^2 z \ Tr(B_z B_z) + \frac{k}{\pi} \int d^2 z \ B_z^{-} B_z^{+}. \quad \text{(B.11)}$$

This expression can be further reduced by integrating out $\omega$ and $\bar{\omega}$, and the result one obtains agrees precisely with the local counterterms that have been constructed previously in [34], provided one chooses $B_z^{-}$ and $B_z^{+}$ in such a way that in the Ward-identity (B.4) for $\Gamma[\mu]$ and in its analogue for $\Gamma[\bar{\mu}]$ only holomorphic and anti-holomorphic projective connections occur. This shows that generalized WZW actions provide a general and powerful framework to construct covariant actions on arbitrary Riemann surfaces.
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