A non-diagrammatic calculation of the $\rho$ parameter from heavy fermions

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Abstract

A simple non-diagrammatic evaluation of the nondecoupling effect of heavy fermions on the Veltman’s $\rho$ parameter is presented in detail. This calculation is based on the path integral approach, the electroweak chiral Lagrangian formalism, and the Schwinger proper time method.

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Although the Standard Model (SM) of particle physics was in very good agreement with the data during the past thirty years or so, there are strong indications that it is just a low-energy effective theory. For example, we do not discover the Higgs particle; we do not know how to unify the SM with gravity, why the SM gauge group is $SU(3)_c \times SU(2)_L \times U(1)_Y$, and why there are 3 families of particles and so on; and the SM suffers the hierarchy problem, the unnaturalness problem, etc. At present there are many new physics extensions beyond the SM. Although we do not know whether nature really behaves like one of them or not, we can estimate their effects on the current electroweak precision observables. The Veltman’s $\rho$ parameter, defined by $m_W^2/(m_Z^2 c_W^2)$, is of particular interest among these observables [1]. In the SM the $\rho$ parameter is equal to 1 at tree level, which is protected by the custodial $SU(2)$ symmetry in the Higgs sector of the SM. This symmetry overlaps with the local $SU(2)_L$ gauge symmetry. Mass splittings in weak isospin doublets break this symmetry. Their effect leaks into radiative corrections which make $\rho$ differ slightly from 1. In model-independent studies the $\rho$ parameter is related to the $T$ parameter in the Peskin-Takeuchi’s formalism [2, 3], and is also related to the $\beta_1$ parameter in the electroweak chiral Lagrangian [4, 5, 6, 7, 8, 9, 10]. Whether the $\rho$ parameter is 1 or not is an indication whether the custodial symmetry is preserved or not in a generic theory. Current electroweak data fitting favors the $\rho$ parameter being close to 1, that is, $1.00989 \leq \rho^{\text{exp}} \leq 1.01026$ [11, 12], which puts stringent constraints on many new physics models. In literature the nondecoupling effects of a heavy fermion weak doublet or even an $n$-plet on the $\rho$ parameter were already obtained using the Feynman diagrammatic calculation [3, 4, 13]. In the present study we give a new calculational method of the contributions of heavy fermions to the $\rho$ parameter without referring to Feynman diagrams. The calculation itself is quite simple in the present method, although one needs some basic knowledge of the electroweak chiral Lagrangian and the Schwinger proper time method [14, 15, 16, 17].

We start by considering a weak doublet of heavy fermions $U$ and $D$, with masses $m_U$ and $m_D$ respectively, and representations of $SU(2)_L \times U(1)_Y$ as

$$\psi_L \equiv \begin{pmatrix} U \\ D \end{pmatrix}_L \sim (2,0), \quad U_R \sim (1,\frac{1}{2}), \quad D_R \sim (1,-\frac{1}{2}) \quad (1)$$

At some energy scale higher than the electroweak scale, the effective action of the heavy fermion sector can be written in a chiral invariant form as

$$S_{\text{eff}}[U,W_\mu^a,B_\mu,\bar{\psi},\psi] = \int d^4x \left[ \bar{\psi}_L (i\partial - g_2 \frac{\tau^a}{2} W_\mu^a) \psi_L + \bar{\psi}_R (i\partial - g_1 \frac{\tau^3}{2} B) \psi_R - \left( \bar{\psi}_L U M \psi_R + \bar{\psi}_R M U^\dagger \psi_L \right) \right] \quad (2)$$

where $\tau^a (a = 1,2,3)$ are Pauli matrices; $g_1$ and $g_2$ ($B_\mu$ and $W_\mu^a$) are the $U(1)_Y$ and $SU(2)_L$ gauge couplings (fields), respectively; the dimensionless unitary unimodular matrix $U(x)$ is the nonlinear realization of the Goldstone boson fields in the electroweak chiral Lagrangian; and the fermion mass matrix $M \equiv \text{diag}(m_U, m_D)$. The
where $S_U$ breaking, and its transformation under $SU(2)_L \times U(1)_Y$ is given by

$$U(x) \to V_L(x)U(x)V_R^T(x)$$

(3)

where $V_L(x) = \exp\{i \frac{x^a}{2} \theta^a(x)\}$ and $V_R(x) = \exp\{i \frac{x^3}{2} \theta^0(x)\}$ with $\theta^a(x)$ and $\theta^0(x)$ being the $SU(2)_L$ and $U(1)_Y$ group parameters, respectively. To derive the contribution of the heavy fermions to the $\rho$ parameter, or equivalently the $\beta_1$ parameter in the electroweak chiral Lagrangian, we need to integrate out the heavy fermions above the electroweak scale which can be formulated as

$$\exp(iS_{EW}[U,W^a_\mu, B_\mu]) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp(iS_{\text{eff}}[U,W^a_\mu, B_\mu, \bar{\psi}, \psi])$$

(4)

where $S_{EW}$ denotes the nondecoupling contribution of heavy fermions to the effective action just at the electroweak scale. Following Ref.[7], we can make the chiral decomposition of the $U(x)$ field as follows:

$$U(x) = \xi_L^T(x)\xi_R(x)$$

(5)

where $\xi_L(x) = \exp\{i \frac{x^a}{2} \phi^a(x)\}$ and $\xi_R(x) = \exp\{i \frac{x^3}{2} \phi^0(x)\}$, and their transformation under $SU(2)_L \times U(1)_Y$ are

$$\xi_L(x) \to h(x)\xi_L(x)V_L^T(x), \quad \xi_R(x) \to h(x)\xi_R(x)V_R^T(x)$$

(6)

where $h(x) = \exp\{i \frac{x^3}{2} \theta_h(x)\}$ belongs to an induced hidden local $U(1)$ symmetry group. Now we make a special $SU(2)_L \times U(1)_Y$ chiral rotation as: $V_L(x) = \xi_L(x)$, $V_R(x) = \xi_R(x)$, under which $U(x)$ is rotated to be 1 and the explicit $U$-dependence of $S_{\text{eff}}$ disappears. Thus eq.(2) becomes

$$S_{\text{eff}}[V^a_\mu, V^0_\mu, \bar{\psi}_\xi, \psi_\xi] = \int d^4x \bar{\psi}_\xi(i\tau^a D^a + \bar{\psi} + \phi^0 - \Omega - m)\psi_\xi$$

(7)

where $m \equiv (m_U + m_D)/2$, the traceless matrix $\Omega \equiv \Delta m \cdot \tau^3$ with $\Delta m \equiv (m_U - m_D)/2$, and the rotated fields are defined as

$$\psi_\xi(x) = P_L\xi_L(x)\psi_L(x) + P_R\xi_R(x)\psi_R(x)$$

(8)

$$v_\mu(x) \equiv \frac{1}{2}[g_2 \frac{\tau^a}{2} V^a_\mu(x) + g_1 \frac{\tau^3}{2} V^0_\mu(x)]$$

(9)

$$a_\mu(x) \equiv \frac{1}{2}[g_2 \frac{\tau^a}{2} V^a_\mu(x) - g_1 \frac{\tau^3}{2} V^0_\mu(x)]$$

(10)

with

$$g_2 \frac{\tau^a}{2} V^a_\mu(x) \equiv \xi_L[g_2 \frac{\tau^a}{2} W^a_\mu(x) - i\partial_\mu]\xi_L^T$$

$$g_1 \frac{\tau^3}{2} V^0_\mu(x) \equiv \xi_R[g_1 \frac{\tau^3}{2} B_\mu(x) - i\partial_\mu]\xi_R^T$$

(11)
According to eq. (6), the transformations of the above rotated fields under $SU(2)_L \times U(1)_Y$ are given by
\[
\begin{align*}
\psi(x) &\to h(x)\psi(x) \\
g_2 \frac{\tau^a}{2} V_\mu^a &\to h(x)[g_2 \frac{\tau^a}{2} V_\mu^a - i\partial_\mu]h^\dagger(x) \\
g_1 \frac{\tau^3}{2} V_\mu^0 &\to h(x)[g_1 \frac{\tau^3}{2} V_\mu^0 - i\partial_\mu]h^\dagger(x)
\end{align*}
\]
(13) \hspace{1cm} (14) \hspace{1cm} (15)

Thus, the chiral symmetry $SU(2)_L \times U(1)_Y$ covariance of the unrotated fields has been transferred totally to the hidden symmetry $U(1)$ covariance of the rotated fields. We further find that $a_\mu(x)$ transforms covariantly: $a_\mu(x) \to h(x)a_\mu(x)h^\dagger(x)$, while $v_\mu(x)$ transforms as the “gauge field” of the hidden local $U(1)$ symmetry: $v_\mu(x) \to h(x)[v_\mu(x)+i\partial_\mu]h^\dagger(x)$. Accordingly, for any operator $O(x)$ which transforms covariantly under the hidden local symmetry: $O(x) \to h(x)O(x)h^\dagger(x)$, we can define its covariant derivative as: $d_\mu O(x) \equiv \partial_\mu O(x) - i[v_\mu(x), O(x)]$. It is important to note that the traces of some combinations of the rotated fields can be related to the terms of the electroweak chiral Lagrangian. First of all, from eqs.(11) and (12) it is straightforward to obtain:
\[
g_2 \frac{\tau^a}{2} V_\mu^a(x) - g_1 \frac{\tau^3}{2} V_\mu^0(x) = -i\xi_R X_\mu \xi_R^\dagger
\]
(16)

where $X_\mu \equiv U^\dagger(D_\mu U)$. Thus, the axial vector field $a_\mu = -\frac{i}{2}\xi_R X_\mu \xi_R^\dagger$. And we can also write $s$ in this covariant form: $s = \xi_R \Delta m \cdot \tau^3 \xi_R^\dagger$, since $\tau^3$ commutes with $\xi_R$ and $\xi_R^\dagger$. On the other hand, we have
\[
v_\mu = \frac{i}{2}\xi_R X_\mu \xi_R^\dagger - \xi_R g_1 \frac{\tau^3}{2} B_\mu \xi_R^\dagger + i\xi_R(\partial_\mu \xi_R^\dagger)
\]
(17)

which leads to the following two relations:
\[
\begin{align*}
 i\xi_R^\dagger v_\mu &= -\frac{1}{2}X_\mu \xi_R^\dagger - ig_1 \frac{\tau^3}{2} B_\mu \xi_R^\dagger - (\partial_\mu \xi_R^\dagger) \\
 -iv_\mu \xi_R &= \frac{1}{2}X_\mu + i\xi_R g_1 \frac{\tau^3}{2} B_\mu - (\partial_\mu \xi_R)
\end{align*}
\]
(18) \hspace{1cm} (19)
i.e.
\[
(\partial_\mu \xi_R^\dagger) + iv_\mu \xi_R = \xi_R (\frac{1}{2}X_\mu + ig_1 \frac{\tau^3}{2} B_\mu)
\]
(20)
\[
(\partial_\mu \xi_R) - iv_\mu \xi_R = \xi_R (\frac{1}{2}X_\mu + ig_1 \frac{\tau^3}{2} B_\mu)
\]
(21)

Thus, for any chiral rotated field $f \equiv \xi_R F \xi_R^\dagger$, we have
\[
d_\mu f = (D_\mu F) + \frac{1}{2}[X_\mu, F]
\]
\[
= \xi_R(\partial_\mu F) \xi_R^\dagger + \xi_R (\frac{1}{2}X_\mu + ig_1 \frac{\tau^3}{2} B_\mu) F \xi_R^\dagger + \xi_R F (-\frac{1}{2}X_\mu - ig_1 \frac{\tau^3}{2} B_\mu) \xi_R^\dagger
\]
\[
= \xi_R \left\{ (D_\mu F) + \frac{1}{2}[X_\mu, F] \right\} \xi_R^\dagger
\]
(22)
where \( D_{\mu} F \equiv \partial_{\mu} F + [g_{\mu} \frac{\tau^3}{2} B_{\mu}, F] \). In particular, if \( f = s = \xi_R \Delta m \cdot \tau^3 \xi_R^\dagger \), then \( F = \Delta m \cdot \tau^3 \) and we have

\[
d_{\mu}s = \xi_R \left\{ D_{\mu}(\Delta m \cdot \tau^3) + \frac{1}{2}[X_{\mu}, \Delta m \cdot \tau^3] \right\} \xi_R^\dagger = \frac{1}{2} \Delta m \xi_R[X_{\mu}, \tau^3] \xi_R^\dagger
\]  

(23)

and hence,

\[
d_{\mu}(d^{\mu}s) = \xi_R \left\{ \frac{1}{2} \Delta m [(D_{\mu}X_{\mu}), \tau^3] + \frac{1}{4} \Delta m [X_{\mu}, [X_{\mu}, \tau^3]] \right\} \xi_R^\dagger
\]  

(24)

Using the above relations, we can obtain the following identities:

\[
\text{tr}(s_{\mu}s_{a\mu}) = -\frac{1}{4} \Delta m^2 \text{tr}(\tau^3 X_{\mu})\text{tr}(\tau^3 X_{\mu}) + \frac{1}{4} \Delta m^2 \text{tr}(X_{\mu}X_{\mu})
\]  

(25)

\[
\text{tr}[(d_{\mu}s)(d^{\mu}s)] = -\text{tr}[s_{\mu}(d^{\mu}s)]
\]  

(26)

which are relevant to the \( \beta_1 \) term in the electroweak chiral Lagrangian. Now, we proceed to integrate out the heavy fermion fields to get their contributions to the low-energy electroweak effective action. The integral measure of the heavy fermions remains unchanged under the special chiral rotation, since these fermions have an anomaly-free assignment of gauge charges. Thus, eq.(4) can be written as

\[
\exp(iS_{\text{EW}}[U,W_{\mu}^a,B_{\mu}]) = \int D\bar{\psi} \xi D\psi \xi \exp \left[ i \int d^4x \bar{\psi}_{\xi} (i\partial_{\xi} + \frac{s}{v} + s) \psi_{\xi} \right]_M
\]  

\[
= \int D\bar{\psi} \xi D\psi \xi \exp \left[ - \int d^4x \bar{\psi}_{\xi} (\partial_{\xi} - i\gamma_{5} - s + m) \psi_{\xi} \right]_E
\]  

(27)

where the subscripts \( M \) and \( E \) indicate that the expressions are written respectively in the Minkowski spacetime and in the Euclidean spacetime (See Appendix A for the conversion relations of quantities in these two spaces). Here and henceforth, we mostly work in the Euclidean spacetime and, for convenience, we drop the subscript \( E \) in the expressions until further notice. Eq.(27) gives

\[
iS_{\text{EW}}[U,W_{\mu}^a,B_{\mu}] = \ln \text{Det}(D + m) = \text{Tr} \ln(D + m)
\]  

(28)

where \( D \equiv \partial_{\xi} - i\gamma_{5} - s \). Since the imaginary part of the fermion determinant corresponds to the Wess-Zumino-Witten anomaly term [18, 19, 20], for any anomaly-free underlying model we only need to consider the calculation of the real part of the fermion determinant. Using the Schwinger proper time formula [14], we have

\[
\text{Re Tr} \ln(D + m) = \frac{1}{2} \text{Tr} \ln((D^\dagger + m)(D + m))
\]  

\[
= -\frac{1}{2} \lim_{\Lambda \to \infty} \int d^4x \int \frac{d\tau}{\tau} \text{tr}_{c,f} \langle x | e^{-\tau(D^\dagger + m)(D + m)} | x \rangle
\]  

(29)
where $\text{tr}_{c,f,l}$ denotes taking trace with respect to color (or technicolor), flavor, and Lorentz indices, and the operator $(D^\dagger + m)(D + m)$ in the exponential can be further simplified as follows:

$$(D^\dagger + m)(D + m) = E - \nabla^2 + m^2 \quad (30)$$

with

$$E \equiv -2ms - 2im\phi \gamma_5 + \frac{i}{4} [\gamma^\mu, \gamma^\nu] R_{\mu\nu} + \gamma^\mu d_\mu s + i\gamma^\mu \{ s, a_\mu \} \gamma_5 + s^2 \quad (31)$$

$$\nabla_\mu \equiv \partial_\mu - iv_\mu - ia_\mu \gamma_5 \quad (32)$$

$$R_{\mu\nu} \equiv i[\nabla_\mu, \nabla_\nu] = V_{\mu\nu} + (d_\mu a_\nu - d_\nu a_\mu) \gamma_5 - i[a_\mu, a_\nu] \quad (33)$$

$$V_{\mu\nu} \equiv i[\partial_\mu - iv_\mu, \partial_\nu - iv_\nu] = \partial_\mu v_\nu - \partial_\nu v_\mu - i[v_\mu, v_\nu] \quad (34)$$

$$d_\mu s \equiv \partial_\mu s - i[v_\mu, s] \quad (35)$$

Thus eq.(29) can be written as:

$$\text{Re} \; \text{Tr} \ln(D + m) = -\frac{1}{2} \lim_{\Lambda \to \infty} \int d^4x \int_0^{\infty} \frac{d\tau}{\tau} e^{-\tau m^2} \text{tr}_{c,f,l} \langle x | e^{-\tau(E - \nabla^2)} | x \rangle \quad (36)$$

where $\langle x | e^{-\tau(E - \nabla^2)} | x \rangle$ is the so-called heat kernel, and it can be expanded in powers of $\tau$ (and in powers of momenta as well) using the Seely-DeWitt expansion formula [15] as

$$\langle x | e^{-\tau(E - \nabla^2)} | x \rangle = \frac{1}{16\pi^2} \left\{ \frac{1}{\tau^2} - \frac{E}{\tau} + \left( \frac{1}{2} E^2 - \frac{1}{6} [\nabla_\mu, [\nabla_\mu, E]] - \frac{1}{12} R_{\mu\nu} R^{\mu\nu} \right) \right. \quad (37)$$

$$+ \tau \left( -\frac{1}{6} E^3 + \frac{1}{12} ([\nabla_\mu, [\nabla_\mu, E]] E + E[\nabla_\mu, [\nabla_\mu, E]] \right)$$

$$+ [\nabla_\mu, E][\nabla_\mu, E]) + \frac{\tau^2}{24} E^4 \right\} + \mathcal{O}(\tau^3)$$

From the structure of $E$ and $\nabla_\mu$, we find that only the traces of the following terms in eq.(37) have contributions to the coefficient of $\text{tr}(s a_\mu s a^\mu)$:

$$\text{tr}(\frac{1}{2} E^2) \ni 4s a_\mu s a^\mu \quad (38)$$

$$\text{tr}(\frac{\tau}{6} E^3) \ni -16\tau m^2 s a_\mu s a^\mu \quad (39)$$

$$\text{tr}(\frac{\tau}{12} ([\nabla_\mu, [\nabla_\mu, E]] E + E[\nabla_\mu, [\nabla_\mu, E]] + [\nabla_\mu, E][\nabla_\mu, E]) \ni \frac{8}{3} \tau m^2 s a_\mu s a^\mu \quad (40)$$

$$\text{tr}(\frac{\tau^2}{24} E^4) \ni \frac{16}{3} \tau^2 m^4 s a_\mu s a^\mu \quad (41)$$
where \( \text{tr}_l \) stands for taking trace with respect to Lorentz spinor indices. Combing eqs (38-41), we obtain

\[
\text{tr}_l \langle x | e^{-\tau(E - \nabla^2)} | x \rangle \ni \frac{1}{16\pi^2} (4 - \frac{40}{3} \tau m^2 + \frac{16}{3} \tau^2 m^4) s_a \mu s^\mu \tag{42}
\]

which, by subsequently taking trace with respect to color (or technicolor) and flavor indices, leads to

\[
\text{tr}_{c,f,l} \langle x | e^{-\tau(E - \nabla^2)} | x \rangle \ni \frac{N_c}{16\pi^2} (4 - \frac{40}{3} \tau m^2 + \frac{16}{3} \tau^2 m^4) \text{tr}_f(s_a \mu s^\mu) \tag{43}
\]

where \( N_c \) is the color (or technicolor) degree of freedom of the heavy fermions under consideration. From eq. (26) we see that \( \text{tr}[(d_\mu s)(d^\mu s)] \) and \( \text{tr}[s\mu d_\mu (d^\mu s)] \) differ only by opposite sign, and thus we may recognize them as just one independent term. Likewise, by analyzing the structure of \( E \) and \( \nabla_\mu \), we find that only the traces of the following terms in eq.(37) give contributions to the coefficient of \( \text{tr}[(d_\mu s)(d^\mu s)] \):

\[
\text{tr}_l \left( \frac{1}{2} E^2 \right) \ni 2(d_\mu s)(d^\mu s) \tag{44}
\]

\[
\text{tr}_l \left( \frac{\tau}{12} ([\nabla_\mu, [\nabla^\mu, E]] E + E[[\nabla_\mu, [\nabla^\mu, E]] + [\nabla_\mu, E][[\nabla^\mu, E]] \right) \ni -\frac{4}{3} \tau m^2 (d_\mu s)(d^\mu s) \tag{45}
\]

Combing the above equations gives

\[
\text{tr}_l \langle x | e^{-\tau(E - \nabla^2)} | x \rangle \ni \frac{1}{16\pi^2} (2 - \frac{4}{3} \tau m^2)(d_\mu s)(d^\mu s) \tag{46}
\]

which further implies

\[
\text{tr}_{c,f,l} \langle x | e^{-\tau(E - \nabla^2)} | x \rangle \ni \frac{N_c}{16\pi^2} (2 - \frac{4}{3} \tau m^2) \text{tr}_f[(d_\mu s)(d^\mu s)] \tag{47}
\]

Now, taking eqs. (13) and (17) into account, we finally arrive at:

\[
\text{tr}_{c,f,l} \langle x | e^{-\tau(E - \nabla^2)} | x \rangle \ni \frac{N_c}{16\pi^2} \left\{ (4 - \frac{40}{3} \tau m^2 + \frac{16}{3} \tau^2 m^4) \text{tr}_f(s_a \mu s^\mu) + (2 - \frac{4}{3} \tau m^2) \text{tr}_f[(d_\mu s)(d^\mu s)] \right\} \tag{48}
\]

In the following we will re-use the subscripts \( E \) and \( M \) to stand for, respectively, the quantities in the Euclidean space and those in the Minkowski space, and we will use \( \text{tr} \) as a short for \( \text{tr}_f \), i.e. taking trace over flavor space. Substituting eq.(48) into
eq. (36), we obtain

\[ \text{Re Tr ln}(D + m) \]

\[ \geq -\frac{1}{2} \int d^4 x_E \lim_{\Lambda \to \infty} \int_1^\infty \frac{d\tau}{\tau} e^{-\tau m^2} \frac{N_c}{16\pi^2} \left\{ (4 - \frac{40}{3} \tau m^2 + \frac{16}{3} \tau^2 m^4) \text{tr}(s\mu_s s\mu) \\
+ (2 - \frac{4}{3} \tau m^2) [\text{tr}[(d\mu_s)(d\mu_s)]] \right\}_E \]

\[ = -\frac{1}{2} \frac{N_c}{16\pi^2} \int d^4 x_E \lim_{\Lambda \to \infty} \left\{ [4(-\gamma - \ln \frac{m^2}{\Lambda^2}) - 8] \text{tr}(s\mu_s s\mu) \\
+ [2(-\gamma - \ln \frac{m^2}{\Lambda^2}) - \frac{4}{3}] \text{tr}[(d\mu_s)(d\mu_s)] \right\}_E \]

\[ = \frac{i}{2} \frac{N_c}{16\pi^2} \int d^4 x_M \lim_{\Lambda \to \infty} \left\{ [4(-\gamma - \ln \frac{m^2}{\Lambda^2}) - 8] \text{tr}(s\mu_s s\mu) \\
+ [2(-\gamma - \ln \frac{m^2}{\Lambda^2}) - \frac{4}{3}] \text{tr}[(d\mu_s)(d\mu_s)] \right\}_M \]

(49)

where we have used eqs. (62-64), (56), (57) and \( \int d^4 x_E = i \int d^4 x_M \). Together with eqs. (25) and (26), eq. (49) gives

\[ \text{Re Tr ln}(D + m) \]

\[ \geq \frac{i}{2} \frac{N_c}{16\pi^2} \int d^4 x_M \lim_{\Lambda \to \infty} \left\{ [4(-\gamma - \ln \frac{m^2}{\Lambda^2}) - 8] \frac{1}{4} \Delta m^2 \text{tr}(\tau^3 X_\mu) \text{tr}(\tau^3 X^\mu) \\
+ [2(-\gamma - \ln \frac{m^2}{\Lambda^2}) - \frac{4}{3}] \frac{1}{2} \Delta m^2 \text{tr}(\tau^3 X_\mu) \text{tr}(\tau^3 X^\mu) \right\} \]

\[ = \frac{i}{2} \frac{N_c}{24\pi^2} \Delta m^2 \int d^4 x_M \text{tr}(\tau^3 X_\mu) \text{tr}(\tau^3 X^\mu) \]

(50)

where, as expected, the divergence term \((-\gamma - \ln \frac{m^2}{\Lambda^2})\) has been exactly canceled by the two parts. Thus, eq. (50) gives the nondecoupling contribution of the heavy fermions to the electroweak effective action. Comparing this term with the \( \beta_1 \) term of the standard electroweak chiral Lagrangian leads to,

\[ \mathcal{L}_0' = \frac{1}{4} \beta_1 f^2 [\text{tr}(\tau^3 X_\mu)]^2 = \frac{N_c}{24\pi^2} \Delta m^2 [\text{tr}(\tau^3 X_\mu)]^2 \]

(51)

which, together with the well-known relations, \( 4/f^2 = e^2/(s^2 c^2 M_Z^2) \) and \( \Delta m = (m_U - m_D)/2 \), implies,

\[ \beta_1 = \frac{N_c}{96\pi^2} \frac{e^2}{s^2 c^2} \frac{(m_U - m_D)^2}{M_Z^2} \]

(52)

where \( c \equiv \cos \theta_W \), \( s \equiv \sin \theta_W \). Since \( \rho - 1 = \alpha T = 2\beta_1 \), we finally obtain

\[ \rho = 1 + \frac{N_c}{48\pi^2} \frac{e^2}{s^2 c^2} \frac{(m_U - m_D)^2}{M_Z^2} \]

(53)
Eqs. (52) and (53) exactly coincide with the Feynman-diagram calculated results in previous works [3, 4, 13].

In summary, we have presented a new nondiagrammatic calculation method of the nondecoupling effect of heavy fermions on the $\rho$ parameter. As we have seen, a crucial step of the present method is figuring out the coefficients of $\text{tr}(s\sigma_\mu s\sigma^\mu)$ and $\text{tr}[(d_\mu s)(d^\mu s)]$ in $\text{Re Tr} \ln(D + m)$. In the ordinary power counting of the QCD chiral Lagrangian, the scalar source term $s$ is regarded as of order $p^2$, and these two trace terms are of order $p^6$ and their coefficients were rarely considered in literature. However, in the power counting of the present case, $s$ is just a constant, i.e. the mass splitting of the heavy fermions, and thus these two trace terms are of order $p^2$ and must give contributions to the $\beta_1$ term in the electroweak chiral Lagrangian. Our method is also applicable to the case of several heavy fermion doublets with mass mixing, and the generalization of this calculation would be straightforward.

A Relations Between Minkowski Space and Euclidean Space

To fix the notation, in this appendix we briefly review the relations between quantities in the Minkowski space and those in the Euclidean space. We use a mostly minus metric for the Minkowski space, $(g_{\mu\nu})_M = \text{diag}(1, -1, -1, -1)$, and we use a positive metric for the Euclidean space, $(g_{\mu\nu})_E = \text{diag}(1, 1, 1, 1)$. The coordinates in these two space are related by:

$$x^0_M = -ix^0_E, \quad x^i_M = x^i_E \quad (i = 1, 2, 3)$$ (54)

which gives $\int d^4x_M = -i \int d^4x_E$. From the correspondence $p_\mu \sim \partial/(\partial x^\mu)$, we accordingly obtain the conversion relation of the momenta: $(p_0)_M = i(p_0)_E, (p_i)_M = (p_i)_E \quad (i = 1, 2, 3)$, which by raising the indices gives,

$$(p^0)_M = i(p^0)_E, \quad (p^i)_M = -(p^i)_E \quad (i = 1, 2, 3)$$ (55)

The vector source field $v^\mu(x)$ and the axial vector source field $a^\mu(x)$ are ordered to be converted exactly the same way as $p^\mu$. Furthermore, a minus sign appears in the relation of the inner products of two vectors in these two spaces, $(g_{\mu\nu}p^\mu p^\nu)_M = (p^0p^0 - p^i p^i)_M = -(p^0 p^0 - p^i p^i)_E = -(g_{\mu\nu}p^\mu p^\nu)_E$, which suggests us to let the scalar source field $s(x)$ be converted by: $s_M = -s_E$. From these relations, we have

$$\text{tr}(s\sigma_\mu s\sigma^\mu) \bigg|_M = -\text{tr}(s\sigma_\mu s\sigma^\mu) \bigg|_E$$ (56)

$$\text{tr}[(d_\mu s)(d^\mu s)] \bigg|_M = -\text{tr}[(d_\mu s)(d^\mu s)] \bigg|_E$$ (57)

On the other hand, since the Dirac matrices satisfy the Clifford algebra, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, with $g^{\mu\nu}$ different in these two spaces, their conversion relations are given by

$$(\gamma^0)_M = (\gamma^0)_E, \quad (\gamma^k)_M = i(\gamma^k)_E \quad (k = 1, 2, 3)$$ (58)
In the Minkowski space, \((\gamma^0)_M\) is set to be hermitian, while the other 3 Dirac matrices are anti-hermitian. And thus \((\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3)_M\) is hermitian. As a result, in the Euclidean space all the Dirac matrices \((\gamma^\mu)_E\) are hermitian, so is \((\gamma_5 \equiv \gamma^0\gamma^1\gamma^2\gamma^3)_E\). Moreover, it is easy to check the relation:

\[(\gamma_5)_M = (\gamma_5)_E \quad (59)\]

From eqs. (55) and (58), we have

\[(\phi)_M = (\gamma^0 p^0 - \gamma^k p^k)_M = i(\gamma^0 p^0 + \gamma^k p^k)_E = i(\phi)_E \quad (60)\]

Likewise, \((\bar{\phi})_M = i(\bar{\phi})_E\), \((\psi)_M = i(\psi)_E\), and \((\bar{\psi})_M = i(\bar{\psi})_E\). And we let the other quantities \(\psi\), \(\bar{\psi}\), and \(m\) keep unchanged in these two spaces. Thus, finally we obtain

\[
\exp \left[ i \int d^4x \bar{\psi}(i\partial + \gamma^0 p^0 + \gamma_5 s - m)\psi \right]_M \\
= \exp \left[ - \int d^4x \bar{\psi}(\partial - i\gamma^0 p^0 + \gamma_5 s + m)\psi \right]_E \quad (61)
\]

## B Necessary Integral Formulas

In this appendix we list the integral formulas needed in the text as follows:

\[
\lim_{\Lambda \to \infty} \int_{1/\Lambda^2}^\infty \frac{d\tau}{\Lambda^2} e^{-\tau m^2} = \lim_{\Lambda \to \infty} \left( -\gamma - \ln \frac{m^2}{\Lambda^2} \right) \quad (62)
\]

\[
\lim_{\Lambda \to \infty} \int_{1/\Lambda^2}^\infty d\tau e^{-\tau m^2} = \frac{1}{m^2} \quad (63)
\]

\[
\lim_{\Lambda \to \infty} \int_{1/\Lambda^2}^\infty d\tau \tau e^{-\tau m^2} = \frac{1}{m^4} \quad (64)
\]

where \(\gamma\) is the Euler-Mascheroni constant, \(\gamma \approx 0.5772\).

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