A gradient flow for the prescribed Gaussian curvature problem on a closed Riemann surface with conical singularity

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Abstract

In this note, we prove that the abstract gradient flow introduced by Baird-Fardoun-Regbaoui is well-posed on a closed Riemann surface with conical singularity. Long time existence and convergence of the flow are proved under certain assumptions. As an application, the prescribed Gaussian curvature problem is solved when the singular Euler characteristic of the conical surface is non-positive.

Keywords: prescribed Gaussian curvature problem, conical singularity, gradient flow

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1. Introduction

Let \( \Sigma \) be a closed Riemann surface, \( g \) be a smooth metric and \( \kappa \) be its Gaussian curvature. If \( \tilde{g} = e^{2u}g \) for some smooth function \( u \), then the Gaussian curvature of \( \tilde{g} \) satisfies \( \tilde{\kappa} = e^{-2u} \left( \Delta_g u + \kappa \right) \), where \( \Delta_g \) is the Laplace-Beltrami operator. For a given function \( K : \Sigma \to \mathbb{R} \), can one find a metric \( \tilde{g} = e^{2u}g \) having \( K \) as its Gaussian curvature? This problem is equivalent to the solvability of the equation

\[
\Delta_g u + \kappa - Ke^{2u} = 0.
\]

Integration by parts and the Gauss-Bonnet formula imply that necessarily \( K \) must have the same sign as the topological Euler characteristic \( \chi(\Sigma) \) somewhere and in the case \( \chi(\Sigma) = 0 \), either \( K \) is identically zero or changes sign. It is natural to ask if this condition is also sufficient to guarantee a solution.

In the case \( \chi(\Sigma) < 0 \), via the method of upper and lower solutions, it was shown by Kazdan-Warner that if \( K \leq 0 \) and \( K \not\equiv 0 \), then (1) has a solution. Suppose that \( K \leq \sup_{\Sigma} K = 0 \), \( K \not\equiv 0 \), and \( \lambda \in \mathbb{R} \). Using a variational method, Ding-Liu proved the following: Replacing \( K \) by \( K + \lambda \) in (1), one finds some constant \( \lambda^* > 0 \) such that if \( 0 < \lambda < \lambda^* \), then (1) has at least two different solutions; if \( \lambda = \lambda^* \), then (1) has at least one solution; while if \( \lambda > \lambda^* \), then (1) has no solution. In the case \( \chi(\Sigma) = 0 \), the problem was completely solved. It was proved by Berger that if \( K \equiv 0 \) or \( K \) changes sign and \( \int \langle K e^{2u} dv, v \rangle < 0 \), where \( v \) is a solution of \( \Delta_g v = -\kappa \), then (1) has a solution. Later Kazdan-Warner pointed out that the above assumptions on \( K \) is also necessary. If \( \chi(\Sigma) > 0 \), \( \Sigma \) is either the projective space \( \mathbb{R}P^2 \) or the 2-sphere \( S^2 \). In the case of \( \mathbb{R}P^2 \),...
it was shown by Moser [31] that (1) has a solution provided that \( \sup_{\Sigma} K > 0 \) and \( K(p) = K(-p) \) for all \( p \in \mathbb{S}^2 \). While the problem on \( \mathbb{S}^2 \) is much more complicated and known as the Nirenberg problem. Moser’s result was extended by Chang-Yang [32] to reflected symmetric function \( K \) under further assumptions. For rotationally symmetric function \( K \), sufficient condition was given by Chen-Li [35] and Xu-Yang [37]. Concerning more general functions \( K \), we refer the reader to [10, 11, 14].

Also various flows have been employed to attack the problem. In [22], The Ricci flow was introduced by Hamilton to find a solution of (1), where \( K \) is a constant. His result was later completed by Chow [19]. The Calabi flow was investigated by Bartz-Struwe-Ye [34] and Struwe [34]. While in [35], Struwe used the Gaussian curvature flow to reprove Chang-Yang’s results [12]. For further developments of this flow, we refer the reader to Brendle [8, 9], Ho [23] and Zhang [45]. Assuming that the initial metric \( g \) has constant Gaussian curvature \( \kappa \). Baird-Fardoun-Regbaoui [2] proposed an abstract gradient flow, through which \( g(t) \) converges to a metric having the prescribed Gaussian curvature. This method solved (1) perfectly in the case \( \chi(\Sigma) \leq 0 \) and partially in the case \( \chi(\Sigma) > 0 \).

The same problem can be proposed on conical surfaces. We begin with basic definitions. Let \( \Sigma \) be a closed Riemann surface as before. A metric \( g \) is said to be a conformal metric having conical singularity of order \( \beta_i > -1 \) at \( p \in \Sigma \), if in a local holomorphic coordinate with \( z(p) = 0 \), there exists some function \( u \) which is continuous and \( C^2 \) away from zero such that

\[
g = e^{2u|z|^2}dz^2.
\]

If \( g \) has conical singularities of order \( \beta_i > -1 \) at \( p_i \in \Sigma, i = 1, \ldots, \ell \), we say that \( g \) represents a divisor \( \beta = \sum_{i=1}^\ell \beta_i p_i \). Then the pair \((\Sigma, \beta)\) is called a conical surface, and the corresponding singular Euler characteristic is written as

\[
\chi(\Sigma, \beta) = \chi(\Sigma) + \sum_{i=1}^\ell \beta_i,
\]

where \( \chi(\Sigma) \) is the topological Euler characteristic.

If \( \chi(\Sigma, \beta) \) is nonpositive, the problem can be solved in the variational framework as the case of smooth metrics. Precisely, it was shown by Troyanov [36] that if \( \chi(\Sigma, \beta) < 0 \), then any smooth negative function is the Gaussian curvature of a unique conformal metric \( \hat{g} \) representing \( \beta \). Recently this result has been improved by Zhu and the author [41] by using the variational method of Ding-Liu [20] and Borer-Galimberti-Struwe [7]. In particular, if we assume \( \chi(\Sigma, \beta) < 0 \), the background metric \( g \) has the Gaussian curvature \( \kappa = -1 \), and \( K \) is a smooth function satisfying \( \sup_{\Sigma} K = 0 \) and \( K \neq 0 \), then there exists a unique function

\[
u \in \mathcal{C} = C^2(\Sigma \setminus \{p_1, \ldots, p_\ell\}) \cap C^0(\Sigma) \cap W^{1,2}(\Sigma, g)
\]

such that the metric \( e^{2u}g \) has the Gaussian curvature \( K \); moreover, there exists some constant \( \lambda' > 0 \) such that when \( 0 < \lambda < \lambda' \), there exist at least two different functions \( u_1, u_2 \in \mathcal{C} \) such that \( e^{2u_1}g \) and \( e^{2u_2}g \) have the same Gaussian curvature \( K + \lambda \); when \( \lambda = \lambda' \), there exists at least one function \( u \in \mathcal{C} \) such that \( e^{2u}g \) has the Gaussian curvature \( K + \lambda' \); when \( \lambda > \lambda' \), there is no function \( u \in W^{1,2}(\Sigma, g) \) such that \( e^{2u}g \) has the Gaussian curvature \( K + \lambda \). The problem was completely solved by Troyanov [36] in the case \( \chi(\Sigma, \beta) = 0 \). Namely, there exists a flat metric \( g \) representing \( \beta \); moreover, a smooth function \( K \) is the Gaussian curvature of a metric \( \hat{g} \) conformal to \( g \) if and only if \( K \) changes sign and \( \int_\Sigma Kdv_{\hat{g}} < 0 \). If \( \chi(\Sigma, \beta) > 0 \), then the problem becomes
very subtle. There is much interesting work concerning this situation, see for examples Troyanov [36], McOwen [29], Chen-Li [16, 17, 18], Luo-Tian [27], Mondello-Panov [30], Bartolucci [3], Bartolucci-De Marchis-Malchiodi [4], Fang-Lai [21] and a very nice survey of Lai [25].

Again the Ricci flow is an elegant way to solve the problem on conical surfaces. Yin [42, 43, 44] established a basic theory in this regards, and proved long time existence and convergence of the flow when \( \chi(\Sigma, \beta) \leq 0 \). The convergence in the case \( \chi(\Sigma, \beta) > 0 \) was studied by Phong-Song-Sturm-Wang [32, 33]. Another approach for the Ricci flow was proposed by Mazzeo-Rubinstein-Sesum [28].

Our aim is to establish the gradient flow of Baird-Fardoun-Regbaoui [2] on conical surfaces. Assuming the background metric has a constant Gaussian curvature, we prove the long time existence of the flow. Moreover, when \( \chi(\Sigma, \beta) \leq 0 \), we obtain the convergence of the flow under additional assumptions. For the proof of our results, we follow the lines of Baird-Fardoun-Regbaoui [2]. Here the key point is the following observation: the functionals involved are still analytic if the background metric has conical singularity.

The remaining part of this note is organized as follows: In Section 2, we construct functional framework and give main results of this note; In Section 3, we prove the analyticity of functionals \( \mathcal{F} \) and \( L \), and calculate their gradients; In Section 4, we show the long time existence of the gradient flow; In Section 5, a sufficient condition for convergence of the flow will be discussed; In Section 6, we prove that when \( \chi(\Sigma, \beta) \leq 0 \), the flow converges to the desired solution of the problem.

2. Notations and main results

Let \( \Sigma \) be a closed Riemann surface, \( \beta = \sum_{i=1}^{\ell} \beta_i p_i \) be a divisor, \( \beta_i > -1 \) for all \( i \), and \( g \) be a conformal metric representing \( \beta \). Let \( \kappa : \Sigma \setminus \text{supp} \beta \to \mathbb{R} \) be the Gaussian curvature of \( g \), where \( \text{supp} \beta = \{p_1, \ldots, p_\ell\} \). From now on, we assume \( \kappa \) is a constant. Then the Gauss-Bonnet formula (see for example [36]) reads

\[
\kappa \text{Vol}_g(\Sigma) = \int_\Sigma \kappa dv_g = 2\pi \chi(\Sigma, \beta),
\]

where \( \chi(\Sigma, \beta) \) is defined as in [2], and \( dv_g \) denotes the volume element with respect to the conical metric \( g \). Clearly there exists a smooth metric \( g_0 \) such that

\[
g = \rho g_0,
\]

where \( \rho > 0 \) on \( \Sigma, \rho \in C^2(\Sigma \setminus \text{supp} \beta) \), and \( \rho \in L^r(\Sigma) \) for some \( r > 1 \). Let \( W^{1,2}(\Sigma, g) \) be the completion of \( C^\infty(\Sigma) \) under the norm

\[
\|u\|_{W^{1,2}(\Sigma, g)} = \left( \int_\Sigma (|\nabla_g u|^2 + u^2) dv_g \right)^{1/2},
\]

where \( \nabla_g \) denotes the gradient operator with respect to the metric \( g \). It was observed by Troyanov [36] that \( W^{1,2}(\Sigma, g) = W^{1,2}(\Sigma, g_0) \). In particular, \( W^{1,2}(\Sigma, g) \) is a Hilbert space, which is hereafter denoted by \( \mathcal{H} \), with an inner product

\[
\langle u, w \rangle_{\mathcal{H}} = \int_\Sigma (\nabla_g u \nabla_g w + uw) dv_g.
\]
Moreover, by the Sobolev embedding theorem for smooth Riemann surface $(\Sigma, g_0)$ and the Hölder inequality, one has

$$W^{1,2}(\Sigma, g) \hookrightarrow L^p(\Sigma, g), \quad \forall p > 1.$$ 

Let $\bar{g} = e^{2u}g$ be another conical metric representing $\beta$ and $K : \Sigma \setminus \text{supp} \beta \to \mathbb{R}$ be the Gaussian curvature of $\bar{g}$. Then $K$ satisfies point-wisely on $\Sigma \setminus \text{supp} \beta$,

$$K = e^{-2u}(\kappa + \Delta_g u),$$

where $\Delta_g$ denotes the Laplace-Beltrami operator with respect to the metric $g$. Obviously, if $u$ is a distributional solution of the equation

$$\Delta_g u + \kappa - K e^{2u} = 0, \quad (4)$$

then $u$ satisfies (3).

Let us define two functionals $J : \mathcal{H} \to \mathbb{R}$, $L : \mathcal{H} \to \mathbb{R}$ by

$$J(u) = \frac{1}{2} \int_{\Sigma} |\nabla_g u|^2 d\nu_g + \kappa \int_{\Sigma} u d\nu_g,$$

$$L(u) = \frac{1}{2} \int_{\Sigma} K e^{2u} d\nu_g,$$

and a set of functions by

$$\mathcal{I} = \{ u \in \mathcal{H} : L(u) = \kappa \text{Vol}_g(\Sigma) = 2\pi \chi(\Sigma, \beta) \}.$$  

The gradients of $J$ and $L$, $\nabla J : \mathcal{H} \to \mathcal{H}$ and $\nabla L : \mathcal{H} \to \mathcal{H}$ are defined by

$$\langle \nabla J(u), w \rangle_{\mathcal{I}} = dJ(u)(w) = \frac{d}{dt} \bigg|_{t=0} J(u + tw),$$

$$\langle \nabla L(u), w \rangle_{\mathcal{I}} = dL(u)(w) = \frac{d}{dt} \bigg|_{t=0} L(u + tw)$$

respectively, where $u$ and $w$ are functions taken from $\mathcal{H}$. Hereafter we assume $K \not\equiv 0$. It follows that $\nabla L(u) \neq 0$ for all $u \in \mathcal{I}$. Thus $\mathcal{I}$ is a smooth hypersurface in $\mathcal{H}$. A unit normal on $\mathcal{I}$ is

$$N(u) = \frac{\nabla L(u)}{\|\nabla L(u)\|_{\mathcal{I}}}$$

for any $u \in \mathcal{I}$, where $\|\cdot\|_{\mathcal{I}} = \langle \cdot, \cdot \rangle_{\mathcal{I}}$. This allows us to consider the gradient of $J$ with respect to the hypersurface $\mathcal{I}$, which is defined by

$$\nabla J(u) = \nabla J(u) - \langle \nabla J(u), N(u) \rangle_{\mathcal{I}} N(u).$$

The gradient flow of $J$ with respect to the hypersurface $\mathcal{I}$ can be written as

$$\begin{cases}
\partial_t u = -\nabla \mathcal{I} J(u) \\
u(0) = u_0 \in \mathcal{I}.
\end{cases}$$

If the flow exists for all time and converges at infinity, then the limit function $u_{\infty}$ gives a distributional solution of (4). Our first result is an analog of (12, Theorem 1), namely
**Theorem 1.** Let $\Sigma$ be a closed Riemann surface, $\beta_i = \sum_{i=1}^{t} \beta_i p_i$ be a divisor with $\beta_i > -1$, $i = 1, \cdots, t$, and $g$ be a metric representing $\beta$. Let $\mathcal{F}$, $\mathcal{L}$ and $\mathcal{H}$ be defined by (5), (6) and (7) respectively. Suppose that the Gaussian curvature of $g$ is a constant $\kappa$, and that $K \in C^0(\Sigma)$ satisfies the condition

$$
\begin{align*}
\int_{\Sigma} K dv_g < 0 & \quad \text{when } \chi(\Sigma, \beta) < 0 \\
\int_{\Sigma} K dv_g < 0, \sup_{\Sigma} K > 0 & \quad \text{when } \chi(\Sigma, \beta) = 0 \\
\sup_{\Sigma} K > 0 & \quad \text{when } \chi(\Sigma, \beta) > 0.
\end{align*}
$$

Then for any $u_0 \in \mathcal{F}$, there exists a unique global solution $u \in C^\infty([0, \infty), \mathcal{H})$ of the gradient flow (11), satisfying $u(0) = u_0$. Moreover the energy identity

$$
\int_0^\infty \|\partial_s u(s)\|^2 ds + J(u(t)) = J(u_0).
$$

holds for all $t > 0$.

If $\chi(\Sigma, \beta) \leq 0$, then we have the convergence of the flow, an analog of [2], Theorem 2.

**Theorem 2.** Let $u_0 \in \mathcal{F}$ and $u : [0, \infty) \to \mathcal{H}$ be given as in Theorem [2]. In the case $\chi(\Sigma, \beta) = 0$, there exists a $u_\infty \in W^r(\Sigma, g) \cap C^\alpha(\Sigma)$ for some $r > 1$ and $0 < \alpha < 1$ such that $u(t)$ converges to $u_\infty$ in $\mathcal{H}$ as $t \to \infty$, moreover $u_\infty + \tau$ is a distributional solution of (13) for some constant $\tau$. In the case $\chi(\Sigma, \beta) < 0$, there exists a positive constant $\epsilon_0$ depending only on $K(x) = \max\{-K(x), 0\}$ and the conformal metric $g$ such that if $u_0$ satisfies

$$
e^{\gamma \|u_0\|_{\mathcal{H}}} \sup_{x \in \Sigma} K(x) \leq \epsilon_0,$$

where $\gamma > 1$ is a constant depending only on $g$, then $u(t)$ converges in $\mathcal{H}$ to a distributional solution $u_\infty$ of (13) as $t \to \infty$.

We remark that if $K(x) \leq 0$, then the hypothesis (14) is obviously satisfied by all $u_0 \in \mathcal{H}$. Finally, as an interesting application of Theorem [2], we have the following:

**Corollary 3.** Suppose $K \in C^0(\Sigma)$ and $\int_{\Sigma} K dv_g < 0$. If in addition $\sup_{x \in \Sigma} K(x) > 0$ in the case $\chi(\Sigma, \beta) = 0$, or $\sup_{x \in \Sigma} \max\{K(x), 0\}$ is sufficiently small in the case $\chi(\Sigma, \beta) < 0$, then there exists a conformal metric $\tilde{g}$ representing $\beta$ and having $K$ as its Gaussian curvature.

## 3. Preliminaries

In this section, we first show the analyticity of the functionals $J$ and $\mathcal{L}$, and then calculate their gradients.

**Lemma 4.** The functionals $J : \mathcal{H} \to \mathbb{R}$ and $\mathcal{L} : \mathcal{H} \to \mathbb{R}$ are analytic.

**Proof.** Let $u, h \in \mathcal{H}$ be fixed. Clearly $J$ has the following Taylor expansion (see for example Chang [13], Theorem 1.4 of Chapter 1)

$$
J(u + h) = \sum_{k=0}^{n} \frac{J^{(k)}(u)h^{(k)}}{k!} + R_n(u, h)h^{(n)},
$$

where $R_n(u, h)$ is the remainder term.
where $J^{(0)}(u) = J(u)$, $h^{(k)}$ stands for $(h, \cdots, h)$, $k = 0, 1, 2, \cdots$, and $R_n(u, h)$ satisfies

$$
R_n(u, h) = \int_0^1 \frac{(1-t)^{n-1}}{(n-1)!} \left[ J^{(n)}(u + th) - J^{(n)}(u) \right] dt.
$$

One easily computes when $n \geq 3$,

$$
J^{(n)}(u)h^{(n)} = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} J(u + t_1 h + \cdots + t_n h) \bigg|_{t_1=\cdots=t_n=0} = 0,
$$

$$
J^{(n)}(u + th)h^{(n)} = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} J(u + th + t_1 h + \cdots + t_n h) \bigg|_{t_1=\cdots=t_n=0} = 0.
$$

Hence we have

$$
\lim_{n \to \infty} R_n(u, h)h^{(n)} = 0.
$$

Combining (15) and (17), we conclude that $J : \mathcal{H} \to \mathbb{R}$ is analytic.

Similar to (15), we have

$$
\mathcal{L}(u + h) = \sum_{k=0}^n \frac{L^{(k)}(u)h^{(k)}}{k!} + R^L_n(u, h)h^{(n)},
$$

where $R^L_n(u, h)h^{(n)}$ is an analog of (16) with $J$ replaced by $\mathcal{L}$. In view of (6), we have for all $n \in \mathbb{N}$, $t \in [0, 1]$,

$$
\mathcal{L}^{(n)}(u)h^{(n)} = \int_\Sigma K e^{2n} h^n dv_g,
$$

$$
\mathcal{L}^{(n)}(u + th)h^{(n)} = \int_\Sigma K e^{2(n+th)} h^n dv_g.
$$

Clearly there holds for all $t \in [0, 1]$,

$$
\left| \left[ \mathcal{L}^{(n)}(u + th) - \mathcal{L}^{(n)}(u) \right] h^{(n)} \right| \leq \int_\Sigma |K| e^{2(|u|+|h|)} h^n dv_g
$$

$$
\leq \left( \int_\Sigma K^2 e^{4(|u|+|h|)} dv_g \right)^{1/2} \left( \int_\Sigma h^{2n} dv_g \right)^{1/2}.
$$

It follows that

$$
|R^L_n(u, h)h^{(n)}| \leq \left( \int_\Sigma K^2 e^{4(|u|+|h|)} dv_g \right)^{1/2} \left( \int_\Sigma h^{2n} dv_g \right)^{1/2} \frac{1}{n!
$$

$$
= \frac{1}{n!} \left( \int_\Sigma K^2 e^{4(|u|+|h|)} dv_g \right)^{1/2} \left( \int_\Sigma h^{2n} dv_g \right)^{1/2}
$$

$$
\leq \frac{1}{n!} \left( \int_\Sigma K^2 e^{4(|u|+|h|)} dv_g \right)^{1/2} \left( \int_\Sigma e^{\phi^2} dv_g \right)^{1/2}.
$$

Since $u$ and $h$ are fixed functions in $\mathcal{H}$, by a singular Trudinger-Moser inequality (36), Theorem 6), both $e^{2(|u|+|h|)}$ and $e^{\phi^2}$ belong to $L^p(\Sigma, g)$ for any $p > 1$. Note also $K \in C^0(\Sigma)$. Then it follows from (20) that

$$
\lim_{n \to \infty} R^L_n(u, h)h^{(n)} = 0.
$$
This together with (18) implies that $L : H \rightarrow \mathcal{H}$ is analytic. □

Let $I$ be an identity operator. We now define a map $(\Delta_g + I)^{-1} : L^2(\Sigma, g) \rightarrow \mathcal{H}$ in the following way. For any $f \in L^2(\Sigma, g)$, we say $u = (\Delta_g + I)^{-1}f \in \mathcal{H}$ provided that $(\Delta_g + I)u = f$. Though in our setting, the metric $g$ has conical singularity, the existence and uniqueness of $u$ follows from the Lax-Milgram theorem. Thus the map $(\Delta_g + I)^{-1}$ is well defined. Moreover $(\Delta_g + I)^{-1}$ is a linear map, which follows from the linearity of $\Delta_g + I$. Now we have

**Lemma 5.** The gradients of $J$ and $L$ at $u \in \mathcal{H}$ are calculated by

$$\nabla J(u) = u - (\Delta_g + I)^{-1}(u - \kappa), \quad (21)$$

$$\nabla L(u) = (\Delta_g + I)^{-1}(Ke^{2\kappa}). \quad (22)$$

**Proof.** On one hand, integration by parts gives

$$\langle \nabla J(u), w \rangle_{\mathcal{H}} = \int_\Sigma (\nabla_g \nabla J(u) \nabla_g w + \nabla J(u) w) dv_g = \int_\Sigma (\Delta_g + I) \nabla J(u) w dv_g. \quad (23)$$

On the other hand,

$$dJ(u)(w) = \frac{d}{dt} \bigg|_{t=0} J(u + tw) = \int_\Sigma (\nabla_g u \nabla_g w + \kappa w) dv_g = \int_\Sigma (\Delta_g u + \kappa) w dv_g. \quad (24)$$

Combining (18), (23) and (24), we have

$$(\Delta_g + I) \nabla J(u) = \Delta_g u + \kappa = (\Delta_g + I)u - (u - \kappa),$$

which leads to

$$(\Delta_g + I)(\nabla J(u) - u) = -(u - \kappa).$$

Then (21) follows immediately.

To calculate $\nabla L(u)$, we firstly have an analog of (23),

$$\langle \nabla L(u), w \rangle_{\mathcal{H}} = \int_\Sigma (\Delta_g + I) \nabla L(u) w dv_g.$$

Secondly we have

$$dL(u)(w) = \frac{d}{dt} \bigg|_{t=0} L(u + tw) = \int_\Sigma K e^{2\kappa} w dv_g.$$

Finally, in view of (9), we obtain (22). \qed

4. **Long time existence and energy identity**

In this section, we prove Theorem 1 by following the lines of Baird-Fardoun-Regbaoui [2].

**Proof of Theorem 1** By (22), we have $\nabla L(u) \neq 0$ for all $u \in \mathcal{H}$ since $K \neq 0$. We set

$$\mathcal{F}(u) = -\nabla J(u) + \langle \nabla J(u), \nabla L(u) \rangle_{\mathcal{H}} \frac{\nabla L(u)}{||\nabla L(u)||_{\mathcal{H}}^2}, \quad (25)$$

7
By Lemma 4 and the fact $\nabla L(u) \neq 0$ for all $u \in \mathcal{H}$, we conclude that $F \in C^\infty(\mathcal{H}, \mathcal{H})$. Thus from the classical Cauchy-Lipschitz theorem (13), Theorem 1.9 of Chapter 1), there exists some $T > 0$ such that $u \in C^\infty([0, T); \mathcal{H})$ is a solution of

$$
\begin{aligned}
\partial_t u &= F(u) \\
u(0) &= u_0 \in \mathcal{J},
\end{aligned}
$$

(26)
or equivalently (11). In view of (25), we have

$$
\|F(u)\|_{\mathcal{H}} \leq 2\|\nabla J(u)\|_{\mathcal{H}}.
$$

This together with (21) leads to

$$
\|F(u)\|_{\mathcal{H}} \leq C\|u\|_{\mathcal{H}} + C.
$$

Here and in the sequel, we often denote various constants by the same $C$. This together with the equation (26) implies that

$$
\partial_t\|u\|_{\mathcal{H}}^2 = \langle \partial_t u, u \rangle_{\mathcal{H}} \leq C\|u\|_{\mathcal{H}}^2 + C,
$$

which leads to

$$
\partial_t \left( e^{-Ct}\|u(t)\|_{\mathcal{H}}^2 \right) \leq Ce^{-Ct}.
$$

Integrating this inequality from 0 to $t < T$, one has

$$
\|u(t)\|_{\mathcal{H}} \leq (1 + \|u_0\|_{\mathcal{H}})e^{CT/2}.
$$

(27)

It follows from (27) that $u$ can be extended for all $t \in [0, \infty)$.

By (25) and (26), we calculate

$$
\partial_t L(u(t)) = 2\langle \nabla L(u(t)), \partial_t u \rangle_{\mathcal{H}} = \langle \nabla L(u(t)), F(u) \rangle_{\mathcal{H}} = 0.
$$

Then we have for all $t \in [0, \infty)$,

$$
L(u(t)) = L(u_0) = 2\pi\chi(\Sigma, \beta)
$$

and thus $u(t) \in \mathcal{J}$. We now prove the energy identity (13). By (10),

$$
\|\partial_t u\|_{\mathcal{H}}^2 = -\langle \nabla J(u), \partial_t u \rangle_{\mathcal{H}} = \langle \nabla J(u), N(u) \rangle_{\mathcal{H}} \langle N(u), \partial_t u \rangle_{\mathcal{H}}.
$$

Noting that

$$
\langle N(u), \partial_t u \rangle_{\mathcal{H}} = \|\nabla L(u)\|^{-1}_{\mathcal{H}} \partial_t L(u) = 0,
$$

we have

$$
\|\partial_t u\|_{\mathcal{H}}^2 = -\langle \nabla J(u), \partial_t u \rangle_{\mathcal{H}} = -\partial_t J(u).
$$

(28)

Integrating (28) from 0 to $t$, we obtain

$$
\int_0^t \|\partial_t u(s)\|_{\mathcal{H}}^2 ds = J(u_0) - J(u(t)).
$$

This ends the proof of the Theorem. □
5. A sufficient condition for convergence

In this section, we shall prove that if the solution $u(t)$ of (11) is uniformly bounded in $H$, then the flow must converge in $H$. Precisely we have the following:

**Proposition 6.** Let $u : [0, \infty) \to H$ be the solution of (11). Suppose that for all $t \in [0, \infty)$, there exists a constant $C_0$ satisfying

$$\|u(t)\|_H \leq C_0.$$  \hspace{1cm} (29)

Then there exists some function $u_\infty \in W^{2,2}(\Sigma, g) \cap C^0(\Sigma)$ for some $r > 1$ and $0 < \alpha < 1$, such that $u(t)$ converges to $u_\infty$ in $H$ as $t \to \infty$. Moreover, if $\chi(\Sigma, \beta) \neq 0$, then $u_\infty$ is a solution of (4); if $\chi(\Sigma, \beta) = 0$, then $u_\infty + c$ is a solution of (4) for some constant $c$.

**Proof.** By (13) and (29), there exists a constant $C$ depending only on $C_0$ and $\kappa$ such that

$$\int_0^\infty \|\partial_t u(s)\|_{2,\gamma}^2 ds \leq \mathcal{F}(u_0) + C.$$  \hspace{1cm} (30)

As a consequence, there is a sequence $t_j \to \infty$ satisfying

$$\|\partial_t u(t_j)\|_H = \|\nabla H \mathcal{F}(u(t_j))\|_{2,\gamma} \to 0$$

as $j \to \infty$. Since $\|u(t_j)\|_H \leq C_0$ for all $j$, there would be some $u_\infty \in H$ such that up to a subsequence,

$$u(t_j) \to u_\infty \text{ weakly in } H$$

$$u(t_j) \to u_\infty \text{ strongly in } L^q(\Sigma, g), \ \forall q > 1.$$  \hspace{1cm} (31)

Moreover, the singular Trudinger-Moser inequality (36, Theorem 6) implies that for any $\gamma > 0$, there exists some constant $C$ depending only on $\gamma$ and the conical metric $g$ such that

$$\int_\Sigma e^{\gamma u(t_j)} dv_g \leq C.$$  \hspace{1cm} (32)

**Claim 1.** There holds $u_\infty \in \mathcal{J}$.

To see this, we have by the mean value theorem

$$\int_\Sigma K(e^{2u(t_j)} - e^{2u_\infty}) dv_g = \int_\Sigma K e^\xi (2u(t_j) - 2u_\infty) dv_g,$$

where $\xi$ lies between $2u(t_j)$ and $2u_\infty$. Clearly $e^\xi \leq e^{2u(t_j)} + e^{2u_\infty}$. Thus in view of (32), we estimate

$$\left| \int_\Sigma K(e^{2u(t_j)} - e^{2u_\infty}) dv_g \right| \leq 2 \sup_\Sigma |K| \left( \int_\Sigma (e^{4u(t_j)} + e^{4u_\infty}) dv_g \right)^{1/2} \left( \int_\Sigma (u(t_j) - u_\infty)^2 dv_g \right)^{1/2} \leq C \left( \int_\Sigma (u(t_j) - u_\infty)^2 dv_g \right)^{1/2}.$$

This together with (31) and the fact that $u_j \in \mathcal{J}$ leads to

$$\int_\Sigma K e^{2u} dv_g = \lim_{j \to \infty} \int_\Sigma K e^{2u(t_j)} dv_g = 2\pi \chi(\Sigma, \beta).$$
Hence $u_\infty \in \mathcal{J}$ and thus Claim 1 follows.

**Claim 2.** There holds $\nabla^{\mathcal{J}} \mathcal{F}(u_\infty) = 0$ and $u(t_j) \to u_\infty$ in $\mathcal{H}$ as $j \to \infty$.

In view of (10), one has
\[
\nabla^{\mathcal{J}} \mathcal{F}(u(t)) = \nabla \mathcal{F}(u(t)) - \langle \nabla \mathcal{F}(u(t)), \nabla \mathcal{L}(u(t)) \rangle_{\mathcal{H}} \frac{\nabla \mathcal{L}(u(t))}{\|
abla \mathcal{L}(u(t))\|_{\mathcal{H}}}. \tag{33}
\]

We first prove that $\nabla^{\mathcal{J}} \mathcal{F}(u(t_j))$ converges to $\nabla^{\mathcal{J}} \mathcal{F}(u_\infty)$ weakly in $\mathcal{H}$ as $j \to \infty$. To see this, it suffices to prove that as $j \to \infty$,
\[
\nabla \mathcal{F}(u(t)) \to \nabla \mathcal{F}(u_\infty) \quad \text{weakly in } \mathcal{H}, \tag{34}
\]
\[
\nabla \mathcal{L}(u(t)) \to \nabla \mathcal{L}(u_\infty) \quad \text{weakly in } \mathcal{H}, \tag{35}
\]
\[
\langle \nabla \mathcal{F}(u(t)), \nabla \mathcal{L}(u(t)) \rangle_{\mathcal{H}} \to \langle \nabla \mathcal{F}(u_\infty), \nabla \mathcal{L}(u_\infty) \rangle_{\mathcal{H}}, \tag{36}
\]
\[
\|
abla \mathcal{L}(u(t))\|_{\mathcal{H}} \to \|
abla \mathcal{L}(u_\infty)\|_{\mathcal{H}}. \tag{37}
\]

In view of (21), we have
\[
\nabla \mathcal{F}(u(t)) = u(t) - (\Delta_g + I)^{-1}(u(t) - \kappa). \tag{38}
\]

For any $\phi \in \mathcal{H}$, one calculates
\[
\langle (\Delta_g + I)^{-1}(u(t_j) + \kappa), \phi \rangle_{\mathcal{H}} = \int_{\Sigma} \nabla g \left( (\Delta_g + I)^{-1}(u(t_j) + \kappa) \right) \nabla g \phi \, dv_g
\]
\[+ \int_{\Sigma} (\Delta_g + I)^{-1}(u(t_j) + \kappa) \phi \, dv_g
\]
\[= \int_{\Sigma} (\Delta_g + I) \left( (\Delta_g + I)^{-1}(u(t_j) + \kappa) \right) \phi \, dv_g
\]
\[= \int_{\Sigma} (u(t_j) + \kappa) \phi \, dv_g.
\]

This together with (30), (31) and (38) leads to (34).

In view of (22),
\[
\nabla \mathcal{L}(u(t)) = (\Delta_g + I)^{-1}(K e^{2u(t)}). \tag{39}
\]

For any $\phi \in \mathcal{H}$, one has as $j \to \infty$,
\[
\langle (\Delta_g + I)^{-1}(K e^{2u(t)}), \phi \rangle_{\mathcal{H}} = \int_{\Sigma} K e^{2u(t)} \phi \, dv_g \to \int_{\Sigma} K e^{2u_\infty} \phi \, dv_g = \langle (\Delta_g + I)^{-1}(K e^{2u_\infty}), \phi \rangle_{\mathcal{H}}.
\]

This together with (39) leads to (35).

Let $f_j = (\Delta_g + I)^{-1}(K e^{2u(t)})$, or equivalently $(\Delta_g + I)f_j = K e^{2u(t)}$. Then standard elliptic estimates lead to that $f_j$ is bounded in $W^{2,r}((\Sigma, g))$ for some $r > 1$ and thus pre-compact in $\mathcal{H}$. Up to a subsequence one may assume $(\Delta_g + I)^{-1}(K e^{2u(t)})$ converges to $(\Delta_g + I)^{-1}(K e^{2u_\infty})$ in $\mathcal{H}$.
Similarly as before, one calculates

\[
\langle \nabla J(u(t_j)), \nabla L(u(t_j)) \rangle = \int_{\Sigma} \nabla g(\Delta_g + I)^{-1}u(t_j)\nabla g(\Delta_g + I)^{-1}(Ke^{2u(t_j)})dv_g + \int_{\Sigma} (\Delta_g + I)^{-1}u(t_j)(\Delta_g + I)^{-1}(Ke^{2u(t_j)})dv_g
\]

\[
= \int_{\Sigma} u(t_j)(\Delta_g + I)^{-1}(Ke^{2u(t_j)})dv_g
\]

\[
\to \int u_0(\Delta_g + I)^{-1}(Ke^{2u_0})dv_g
\]

\[
= \langle \nabla J(u_0), \nabla L(u_0) \rangle_{H^1}.
\]

This is exactly (36). As for (37), one has a strong estimate

\[
\|\nabla L(u(t_j))\|_{H^1}^2 = \int_{\Sigma} Ke^{2u(t_j)}(\Delta_g + I)^{-1}(Ke^{2u(t_j)})dv_g
\]

\[
\to \int Ke^{2u_0}(\Delta_g + I)^{-1}(Ke^{2u_0})dv_g
\]

\[
= \|\nabla L(u_0)\|_{H^1}^2.
\]

Therefore we have proved (34)-(37), and thus \(\nabla^\ast J(u(t_j))\) converges to \(\nabla^\ast J(u_0)\) weakly in \(H^1\).

As a consequence

\[
\|\nabla^\ast J(u(t_j))\|_{H^1}^2 \leq \lim_{j \to \infty} \langle \nabla J(u(t_j)), \nabla^\ast J(u(t_j)) \rangle_{H^1} \leq \lim_{j \to \infty} \|\nabla J(u(t_j))\|_{H^1} \|\nabla^\ast J(u(t_j))\|_{H^1} = 0.
\]

This immediately leads to \(\nabla^\ast J(u(t_j))\) converges in \(H^1\) to \(\nabla^\ast J(u_0) = 0\). It follows from (40) that \(\nabla L(u(t_j))\) converges in \(H^1\) to \(\nabla L(u_0)\). Therefore, in view of (34) and (35), we obtain \(u_j\) converges in \(H^1\) to \(u_0\). This concludes Claim 2.

By (33), (38) and (39), the equation \(\nabla^\ast J(u_0) = 0\) is equivalent to

\[
\Delta_g u_0 + \kappa = c_0 Ke^{2u_0}
\]

(41)

for some constant \(c_0\). By elliptic estimates, we conclude that \(u_0 \in W^{2,r}(\Sigma, g) \cap C^\alpha(\Sigma)\) for some \(r > 1\) and \(0 < \alpha < 1\). If \(\chi(\Sigma, \beta) \neq 0\), then we have by integrating (41), the Gauss-Bonnet formula and Claim 1

\[
2\pi\chi(\Sigma, \beta) = \int_{\Sigma} Kdv_g = c_0 \int_{\Sigma} Ke^{2u_0}dv_g = 2\pi\chi(\Sigma, \beta)c_0.
\]

It follows that \(c_0 = 1\) and \(u_0\) is a distributional solution of (4). If \(\chi(\Sigma, \beta) = 0\), then \(\kappa = 0\). Multiplying both sides of (41) by \(e^{-u_0}\), we have

\[
-\int_{\Sigma} e^{-u_0}|\nabla u_0|^2dv_g = c_0 \int_{\Sigma} Kdv_g,
\]

which together with (12) implies that \(c_0 > 0\). Then \(u_0 + \log c_0\) is a distributional solution of (4).

Repeating the same argument of (2), Pages 25-27), one can derive a Lojasiewicz-Simon inequality and then use it to obtain

\[
\lim_{j \to \infty} \|u(t_j) - u_0\|_{H^1} = 0.
\]

This completes the proof of the proposition. □
6. Convergence of the flow

In this section, we prove Theorem \(2\) by using Proposition \(6\). The key point is to prove that \(\|u(t)\|_{H^k} \leq C\) for all \(t \in [0, \infty)\) under appropriate conditions.

6.1. The null case

Proof of Theorem \(2\) in the null case. Suppose \(\chi(\Sigma, g) = 0\). Since \(\kappa\) is a constant, it follows from the Gauss-Bonnet formula that \(\kappa = 0\). In view of (21), one calculates
\[
\Delta_g u(t) = (\Delta_g + I)\nabla J(u(t)).
\]
Integration by parts gives
\[
\int_{\Sigma} \nabla J(u(t)) dv_g = 0,
\]
which leads to
\[
\langle \nabla J(u(t)), 1 \rangle_{H^k} = 0. \tag{42}
\]
In view of (22), we have
\[
K e^{2\theta(t)} = (\Delta_g + I)\nabla L(u(t)).
\]
Since \(u(t) \in \mathcal{F}\), we have by integrating by parts
\[
\int_{\Sigma} \nabla L(u(t)) dv_g = \int_{\Sigma} K e^{2\theta(t)} dv_g = 0.
\]
Hence
\[
\langle \nabla L(u(t)), 1 \rangle_{H^k} = 0. \tag{43}
\]
It follows from (42) and (43) that
\[
\partial_t \int_{\Sigma} u(t) dv_g = \int_{\Sigma} \partial_t u dv_g = \langle \partial_t u, 1 \rangle_{H^k} = 0.
\]
Then there exists a constant \(C\) such that
\[
\int_{\Sigma} u(t) dv_g \equiv C.
\]
Using the Poincare inequality, we obtain
\[
\int_{\Sigma} u^2 dv_g \leq C \int_{\Sigma} |\nabla_g u|^2 dv_g + C. \tag{44}
\]
By (13), there holds \(J(u(t)) \leq J(u_0)\), or equivalently
\[
\int_{\Sigma} |\nabla_g u|^2 dv_g \leq \int_{\Sigma} |\nabla_g u_0|^2 dv_g. \tag{45}
\]
Combining (44) and (45), we obtain
\[
\|u(t)\|_{H^k} \leq C
\]
for some constant \(C\). This together with Proposition \(6\) completes the proof of the theorem in the case \(\chi(\Sigma, g) = 0\).
6.2. The negative case

We first have a Poincaré inequality on conical surfaces.

Lemma 7. For all $u \in \mathcal{H}$, there holds

$$\int_{\Sigma} u^2 \, dv_g \leq \frac{1}{\lambda_g(\Sigma)} \int_{\Sigma} |\nabla_g u|^2 \, dv_g + \frac{1}{\Vol_g(\Sigma)} \left( \int_{\Sigma} u \, dv_g \right)^2,$$

where

$$\lambda_g(\Sigma) = \inf_{u \in \mathcal{H}, \int_{\Sigma} u \, dv_g \neq 0} \frac{\int_{\Sigma} |\nabla_g u|^2 \, dv_g}{\int_{\Sigma} u^2 \, dv_g}.$$ (46)

Proof. Applying a direct method of variation to (46), one finds a function $u_0 \in \mathcal{H}$ satisfying

$$\int_{\Sigma} u_0^2 \, dv_g = 1 \quad \text{and} \quad \lambda_g(\Sigma) = \int_{\Sigma} |\nabla_g u_0|^2 \, dv_g > 0.$$

Denote

$$\overline{u} = \frac{1}{\Vol_g(\Sigma)} \int_{\Sigma} u \, dv_g.$$

By the definition of $\lambda_g(\Sigma)$, we have for all $u \in \mathcal{H}$,

$$\int_{\Sigma} |u - \overline{u}|^2 \, dv_g \leq \frac{1}{\lambda_g(\Sigma)} \int_{\Sigma} |\nabla_g u|^2 \, dv_g.$$

Noting that

$$\int_{\Sigma} 2\overline{u}(u - \overline{u}) \, dv_g = 2\overline{u} \int_{\Sigma} (u - \overline{u}) \, dv_g = 0,$$

we obtain

$$\int_{\Sigma} u^2 \, dv_g = \int_{\Sigma} (u - \overline{u})^2 + \overline{u}^2 + 2\overline{u}(u - \overline{u}) \, dv_g$$

$$= \int_{\Sigma} (u - \overline{u})^2 \, dv_g + \overline{u}^2 \Vol_g(\Sigma)$$

$$\leq \frac{1}{\lambda_g(\Sigma)} \int_{\Sigma} |\nabla_g u|^2 \, dv_g + \frac{1}{\Vol_g(\Sigma)} \left( \int_{\Sigma} u \, dv_g \right)^2.$$

This gives the desired result. \qed

Next we have the following singular Trudinger-Moser inequality.

Lemma 8. There exist two constants $C$ and $\beta$ depending only on $(\Sigma, g)$ such that for all $u \in \mathcal{H}$,

$$\int_{\Sigma} e^{2u} \, dv_g \leq C \exp \left( \beta \int_{\Sigma} |\nabla_g u|^2 \, dv_g + \frac{2}{\Vol_g(\Sigma)} \int_{\Sigma} u \, dv_g \right).$$ (47)

Proof. Note that $g$ is a conical metric. The inequality (47) follows from that of Troyanov [36], Theorem 6) (see also Zhu [46] for a critical version). \qed
We remark that (47) is a weak version of Trudinger-Moser inequality. For related strong versions, we refer the reader to recent works [1, 26, 38, 39, 40] and the references therein.

Proof of Theorem 2 in the negative case. Having Lemmas 7 and 8 in hand, we can prove an analog of ([2], Lemma 2) by using the same method, and then repeating the argument of the proof of ([2], Part (ii) of Theorem 2), we conclude the theorem in the case \( \chi(\Sigma, \beta) < 0 \). □

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