A NOTE ON THE DYNAMICS OF LINEAR AUTOMORPHISMS
OF A MEASURE CONVOLUTION ALGEBRA

A. Baraviera, E. Oliveira and F. B. Rodrigues

Instituto de Matemática-UFRGS
Avenida Bento Gonçalves 9500 Porto Alegre-RS Brazil

Abstract. In this work we are going to study the dynamics of the linear
automorphisms of a measure convolution algebra over a finite group, \( T(\mu) = \nu \ast \mu \). In order to understand and classify the asymptotic behavior of this
dynamical system we provide an alternative to classical results, a very direct
way to understand convergence of the sequence \( \{\nu^n\}_{n\in\mathbb{N}} \), where \( G \) is a finite
group, \( \nu \in \mathcal{P}(G) \) and \( \nu^n = \nu \ast \ldots \ast \nu \), through the subgroup generated by his
support.

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tion, convergence, finite groups.

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1. Introduction

The space of probabilities on a metric space \( G \) (or more generally, Radon Mea-
urses) has two natural classes of linear automorphisms, the first one is the push
forward (induced by some fixed map \( f : G \to G \)), that has been extensively stud-
ied by Sigmund (in \cite{13}) and Komuro (in \cite{14}); more recently it also appears in
Kloeckner (in \cite{15}) for example, and just take in consideration the linear structure
of the space of measures. The second one, when \( G \) is a topological group, is based
on the convolution of two measures. In this case the space of Radon measures is
an infinite dimensional Banach algebra, with respect to the convolution operation,
that is, a Measure Convolution Algebra (see \cite{16} pg 73 and \cite{17}). Hence the other
natural linear automorphism is \( T(\mu) = \nu \ast \mu \), for a fixed measure \( \nu \). In this way we
propose to understand the topological dynamics of this map. The iteration of \( T \)
lead us to analyze the powers of convolutions of \( \nu \), since from basic properties of
the operation \( \ast \) we get that iterating \( n \) times the map \( T_{\nu} \) is the convolution \( \nu^n \ast \mu \).

The problem of study powers of convolution of probability measures has been
studied in several papers in the last few years and has several applications in
statistics and group theory (see \cite{3} and \cite{4}). In a general setting \( G \) is a compact
topological group, \( \mathcal{P}(G) \) is the set of all probability measures on \( G \) and \( \nu \in \mathcal{P}(G) \).

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The main goal of this paper is to establish direct conditions on the support of measure \( \nu \), what is quite natural from the ergodic point of view, to ensure convergence of the sequence \( \{\nu^n := \nu * ... * \nu\}_{n \in \mathbb{N}} \).

We study the asymptotic behavior of the sequence \( \{\nu^n\}_{n \in \mathbb{N}} \) on a finite group, with a complete description of the accumulation points of that sequence, that is, the limit sets of the dynamics \( T_\nu \). The main point in this note is that our presentation follows a dynamical point of view, and the main result is obtained with the use of Perron-Frobenius Theorem (see [5]).

We like to point out that our results on the convergence of power are not necessarily new, or a replacement of the classical literature, but just easier to compute and to apply. In the best of our knowledge, there is no direct way to extract this kind of characterization of the limit powers just from the necessary and sufficient, or just sufficient conditions for convergence, that we find on the previous works. More than that, our characterization make use of much more elementary results of analysis and algebra.

Many of the ideas developed here can be immediately applied to compact (or locally compact) topological groups, but the results will be more abstract, restricted and not computational.

1.1. Main result. In this text we present the following

**Theorem 1.** Let \( G = \{g_0, ..., g_{n-1}\} \) be a finite group. If \( \nu \in \mathcal{P}(G) \) is an acyclic probability and \( H \) is the subgroup generated by the support of \( \nu \), then

\[
\lim_{n \to \infty} \nu^n = \sum_{h \in H} \frac{1}{|H|} \delta_h.
\]

We also get an interesting result when the probability measure \( \nu \) is not acyclic, that is used in the last section in order to obtain a solution for the Choquet-Deny equation.

2. Proof of the main Theorem

We will always denote by \( (G = \{g_0, g_1, ..., g_{n-1}\}, \cdot) \) a finite group of order \( n \) where \( g_0 = e \) is the neutral element of the operation \( \cdot \).

We remind that the space of real continuous functions in \( G \), \( C(G, \mathbb{R}) \) is identified with \( \mathbb{R}^n \) and we denote a function \( f \in C(G, \mathbb{R}) \) by the row vector

\[
f(G) = (f(g_0), f(g_1), ..., f(g_{n-1})) \in \mathbb{R}^n.
\]

As usual, the dual of \( C(G, \mathbb{R}) \) is identified with \( (\mathbb{R}^n)^* \simeq \mathbb{R}^n \), is the space of signed measures over \( G \),

\[
C(G, \mathbb{R})' = \left\{ \mu = \sum_{i=0}^{n-1} p_i \delta_{g_i}, \ p = (p_0, p_1, ..., p_{n-1}) \in \mathbb{R}^n \right\}.
\]

in this work we denote

\[
\int_G f d\mu = \sum_{i=0}^{n-1} p_i f(g_i) = \langle f(G), p \rangle,
\]
where $\langle \cdot, \cdot \rangle$ is the usual scalar product.

In this setting, if $\Delta_n = \{ p \in \mathbb{R}^n \mid p_i \in [0,1] \text{ and } \sum_{i=0}^{n-1} p_i = 1 \}$ then
\[
\mathcal{P}(G) = \left\{ \sum_{i=0}^{n-1} p_i \delta_{g_i} \in C(G, \mathbb{R})' \mid p \in \Delta_n \right\}.
\]

If $\nu = \sum_{i=0}^{n-1} p_i \delta_{g_i}$ and $\mu = \sum_{i=0}^{n-1} q_i \delta_{g_i}$ we define the convolution between them as
\[
(\nu * \mu)(f) = \int_G f d(\nu * \mu) = \int_G \int_G f(gh) d\nu(g) d\mu(h).
\]

Defining $f(G^2)$ as
\[
f(G^2) = \begin{bmatrix} f(g_0 g_0) & \cdots & f(g_0 g_{n-1}) \\ \vdots & \ddots & \vdots \\ f(g_{n-1} g_0) & \cdots & f(g_{n-1} g_{n-1}) \end{bmatrix}
\]
we get an characterization of the convolution in coordinates.

**Lemma 2.** If $\nu = \sum_{i=0}^{n-1} p_i \delta_{g_i} \simeq p$ and $\mu = \sum_{i=0}^{n-1} q_i \delta_{g_i} \simeq q$ then
\[
(\nu * \mu)(f) = \langle q, f(G^2) \cdot p \rangle.
\]

**Proof.** Indeed,
\[
(\nu * \mu)(f) = \int_G \int_G f(gh) d\nu(g) d\mu(h) = \int_G \sum_{i=0}^{n-1} p_i f(g_i h) d\mu(h)
\]
\[
= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} q_i p_j f(g_i q_j) = \langle q, f(G^2) \cdot p \rangle.
\]

Since, $\mathcal{P}(G)$ is an affine space of codimension 1 in $C(G, \mathbb{R})'$ we know that $\nu(G \times G)$ is given by an bi-stochastic matrix. In order to get the next result we define a new matrix obtained by a measure $\nu \simeq (p_0, \ldots, p_{n-1}) \in \mathcal{P}(G)$. If we denote,
\[
G^{-1} \times G = \begin{bmatrix} g_0^{-1} g_0 & \cdots & g_0^{-1} g_{n-1} \\ \vdots & \ddots & \vdots \\ g_{n-1}^{-1} g_0 & \cdots & g_{n-1}^{-1} g_{n-1} \end{bmatrix},
\]
then
\[
\nu(G^{-1} \times G) = \begin{bmatrix} \nu(g_0^{-1} g_0) & \cdots & \nu(g_0^{-1} g_{n-1}) \\ \vdots & \ddots & \vdots \\ \nu(g_{n-1}^{-1} g_0) & \cdots & \nu(g_{n-1}^{-1} g_{n-1}) \end{bmatrix},
\]
where $\nu(g_i^{-1} g_j) = p_m$ if $g_i^{-1} g_j = g_m$. 
Lemma 3. Given $\nu, \mu \in \mathcal{P}(G)$, then

$$\nu \ast \mu = \mu \cdot \nu(G^{-1} \times G).$$

Proof. If $\nu = \sum_{i=0}^{n-1} p_i \delta_{g_i}$ and $\mu = \sum_{i=0}^{n-1} q_i \delta_{g_i}$, we set $\nu \ast \mu = \sum_{k=0}^{n-1} a_k \delta_{g_k}$.

From the Lemma 2 we know that,

$$a_k = \sum_{g_i g_j = g_k} p_i q_j \{q_i p_j \mid g_j = g_i^{-1} g_k\}.$$

Since the equation $g_i g_j = g_k$ has an unique solution for a fixed $k$ and for each $i$ we have $j(i,k)$ well determined. It allows us to write,

$$a_k = q_0 \cdot p_j(0,k) + \ldots + q_{n-1} \cdot p_j(n-1,k).$$

Using matrices we have

$$\begin{bmatrix} a_0 & \cdots & a_{n-1} \end{bmatrix} = \begin{bmatrix} q_0 & \cdots & q_{n-1} \end{bmatrix} \begin{bmatrix} p_j(0,0) & \cdots & p_j(n-1,0) \\ \vdots & \ddots & \vdots \\ p_j(0,n-1) & \cdots & p_j(n-1,n-1) \end{bmatrix} + \ldots + \begin{bmatrix} q_0 & \cdots & q_{n-1} \end{bmatrix} \begin{bmatrix} p_j(0,n) & \cdots & p_j(n,0) \\ \vdots & \ddots & \vdots \\ p_j(n,n-1) & \cdots & p_j(n,n-1) \end{bmatrix}.$$

and we get the first formula

$$\nu \ast \mu = \mu \cdot \nu(G^{-1} \times G).$$

Thus, if we desire to compute the powers of the convolution $\nu \ast \nu$ we have

$$\nu^n := \nu \ast \ldots \ast \nu = \nu \cdot \nu(G^{-1} \times G)^m,$$

so we can estimate the long time behavior of $\nu^n$ from the powers of the matrix $\nu(G^{-1} \times G)$.

Example 4. We consider $G = (\mathbb{Z}_3, +)$ and $\nu = (1/3, 1/4, 5/12)$. So

$$G^{-1} \times G = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad \nu(G^{-1} \times G) = \begin{bmatrix} 1/3 & 1/4 & 5/12 \\ 5/12 & 1/3 & 1/4 \\ 1/4 & 5/12 & 1/3 \end{bmatrix}.$$

Definition 5. A stochastic matrix $A = (a_{ij})$ is called primitive if there is $N \in \mathbb{N}$ such that all the entries of the matrix $A^N$ are positive.

Definition 6. A matrix $A$ with non-negative entries is called doubly-stochastic if its rows and columns sum 1.

The following will be very useful in what follows

Theorem 7. (Perron-Frobenius) If $A$ is $n \times n$ primitive and doubly stochastic matrix, then

$$\lim_{m \to \infty} A^m = \frac{1}{n} J,$$

where $J = (a_{ij})$, $a_{ij} = 1$ for all $i, j$. 
Definition 8. Let $G$ be a finite abelian group of order $n$. We say that $G$ is finitely generated if there exist $g_1, \ldots, g_k \in G$ such that for all $g \in G$ we have that $g = g_1^{r_1} \cdots g_k^{r_k}$, with $r_j \in \{0, 1, \ldots, n\}$.

We remember the definition of the support of a given measure. Let $G$ a finite group and $\nu = (p_0, \ldots, p_{n-1}) \in \mathcal{P}(G)$. The support of $\nu$ is the set 
$$\text{supp}(\nu) = \{g_i \in G : \nu(g_i) = p_i > 0\}.$$ 
We will denote by $H$ the subgroup of $G$ generated by $\text{supp}(\nu)$, i.e., 
$$H = \langle \text{supp}(\nu) \rangle.$$

In order to get the next result we need a new definition and a Lemma. We start with the definition:

Definition 9. (Acyclic) Given $\nu \in \mathcal{P}(G)$, we define the set $Z_+^{(\nu)}$ by 
$$Z_+^{(\nu)} = \{g_{i_1} \cdots g_{i_m} : g_{i_k} \in \text{supp}(\nu)\}.$$ 
Let $H$ the subgroup of $G$ generated by $\text{supp}(\nu)$. We say that $\nu$ is a acyclic probability measure if there exist $N \in \mathbb{N}$ such that $Z_+^{(\nu)} = H$. In particular, $Z_+^{(\nu)} = \text{supp}(\nu)$.

We would like to observe that, the acyclic property is similar to [4] for probabilities in matrices, but in that case the convergence is given by a rank theorem.

Example 10. (a) Let $g \in G$ an element of order 2 and $\nu = \delta_g$. In this case $H = \{e, g\}$ and 
$$Z_+^{(\nu)} = \begin{cases} 
e, & \text{if } m \text{ is even} \\ g, & \text{if } m \text{ is odd.} \end{cases}$$

From it follows that $\nu$ is not acyclic probability measure.

(b) Let $H$ a cyclic group generated by $g$ and $\nu = \alpha \delta_e + (1 - \alpha) \delta_g$, $0 < \alpha < 1$. Then $\nu$ is a acyclic. In fact, if $H = \{e, g, \ldots, g^{n-1}\}$, then 
$$Z_+^{(\nu)} = \{e^n, e^{n-1}g, e^{n-2}g^2, \ldots, e^1g^{n-1}\} = H.$$

Example 11. Let $G$ be a finite abelian group of order $n$ and $\nu \in \mathcal{P}(G)$. We can make the identification $\nu = \sum p_i \delta_{g_i} \simeq p = (p_0, \ldots, p_{n-1})$. If $Z_+^{(p)} = \{g, h\}$ and $H = \langle g^{-1}h \rangle$, then $\nu$ is a acyclic. In fact, to see it we only need to notice that 
$$g^{n-k}h^k = (g^{-1}h)^k.$$

Example 12. Let $H = \langle g_1, \ldots, g_k \rangle$ be a finitely generated abelian subgroup of $G$ and $\nu \in \mathcal{P}(G)$. If $\nu \in \mathcal{P}(G)$ is such 
$$Z_+^{(\nu)} = \{e, g_1, \ldots, g_k\}$$
$\nu$ is a acyclic.

Remark 13. If we have that $|G| = n$ and the support of $\nu$ has more than $\frac{n^2}{2} + 1$ elements, then $\nu$ is acyclic, in particular $\nu(G^{-1} \times G)$ primitive.
Proposition 14. Let $G = \{g_0, \ldots, g_{n-1}\}$ a finite group, $\nu \in \mathcal{P}(G)$ acyclic and $H = \langle Z_+(\nu) \rangle$. The matrix $\nu(G^{-1} \times G) = (\nu(g_i^{-1} g_j))_{i,j}$ satisfies $\lim_{n \to \infty} \nu(G^{-1} \times G)^n = B$, where $B$ is the matrix given by
\[
\begin{pmatrix}
\frac{1}{|H|} J & 0 & \cdots & 0 \\
0 & \frac{1}{|H|} J & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{|H|} J
\end{pmatrix},
\]
where $0$ is the null matrix of order $|H|$ and $J$ is the matrix of order $|H|$, with all the coefficients equal to 1.

Proof. The prove of this result follows form the lemmas below. \[\square\]

Remark 15. Let us consider an acyclic probability $\nu \in \mathcal{P}(G)$, and the subgroup $H$ generated by $Z_+(\nu)$. We suppose that $|H| = n$. We can take the equivalence classes determined by $H$ in $G$, i.e,
\[gH = \{gh : h \in H\}.\]
We know that $G$ can be written as a disjoint union of the equivalence classes determined by $H$, and as $G$ is finite $H$ is also a finite group. Then we can write $G$ as follows
\[G = \{e, h_1, \ldots, h_k, g_1h_1, \ldots, g_1, g_1h_k, g_2h_1, \ldots, g_2, g_2h_k, \ldots, g_nh_1, \ldots, g_nh_k\} = \{H, g_1H, \ldots, g_nH\},\]
where $g_iH \cap g_jH = \emptyset$ for $i \neq j$. Then we have that the matrix $A = \nu(G^{-1} \times G)$ is given by
\[A = \begin{pmatrix}
\nu(H^{-1} \times H) & \nu(H^{-1}g_1 \times H) & \cdots & \nu(H^{-1}g_n \times H) \\
\nu(H^{-1}g_1^{-1} \times H) & \nu(H^{-1}g_1^{-1}g_1 \times H) & \cdots & \nu(H^{-1}g_1^{-1}g_1 \times H) \\
\vdots & \vdots & \ddots & \vdots \\
\nu(H^{-1}g_1^{-1} \times H) & \cdots & \cdots & \nu(H^{-1}g_1^{-1} \times H)
\end{pmatrix},\]
where the blocks in the diagonal are always the matrix $\nu(H^{-1} \times H)$.

Lemma 16. The blocks $\nu(H^{-1}g_i^{-1} \times g_jH)$, $\nu(H^{-1}g_i^{-1} \times H)$ and $\nu(H^{-1} \times g_jH)$ are always the null matrix for $i \neq j$.

Proof. Take the block $\nu(H^{-1}g_i^{-1} \times g_jH)$ and notice that
\[\nu((h_i^{-1}g_1^{-1})(g_2h_j)) > 0 \Leftrightarrow (h_i^{-1}g_1^{-1})(g_2h_j) \in Z_+(\nu) \subset H \Rightarrow g_1^{-1}g_2 \in H \Rightarrow g_1H = g_2H,\]
but it is a contradiction, since $g_1H \cap g_2H = \emptyset$. By analogous computations we have the result for the others cases. \[\square\]

By Lemma 10 we have that the powers of the matrix $\nu(G^{-1} \times G)$ are given by
\[\nu(G^{-1} \times G)^n = \begin{pmatrix}
\nu(H^{-1} \times H)^n & 0 & \cdots & 0 \\
0 & \nu(H^{-1} \times H)^n & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \nu(H^{-1} \times H)^n
\end{pmatrix}.
\]
Lemma 17. The matrix $\nu(H^{-1} \times H)$ is primitive.

Proof. Let us consider the matrix $A = (a_{ij})_{i,j} := (\nu(h_i^{-1}h_j))_{i,j}$. Then we notice that

$$a_{ij} > 0 \iff h_i^{-1}h_j \in Z_+(\nu)$$

$$\iff \exists \bar{h} \in Z_+(\nu) \text{ such that } h_i^{-1}h_j = \bar{h}$$

$$\iff h_j = h_i\bar{h}, \quad \bar{h} \in Z_+(\nu).$$

It implies that $a_{ij} > 0$ if and only if $h_j \in L_{h_i}(Z_+(\nu))$. As $L_{h_i}$ is a bijection, in each line we have $|Z_+(\nu)|$ positive coefficients. Consider now $A^2$, which we will denote by $A^2 = (a^2_{ij})_{i,j}$. Then we have that

$$a^2_{ij} > 0 \iff \sum_{k=0}^{n-1} \nu(h_i^{-1}h_k)\nu(h_k^{-1}h_j)$$

$$\iff \exists k \in \{0, ..., n-1\} \text{ such that } \nu(h_i^{-1}h_k)\nu(h_k^{-1}h_j) > 0$$

$$\iff \nu(h_i^{-1}h_k) > 0 \text{ and } \nu(h_k^{-1}h_j) > 0$$

$$\iff \exists h', h'' \in Z_+(\nu) \text{ such that } h_k = h_ih', \quad h_j = h_kh''$$

$$\iff h_j = h_ih'h''$$

$$\iff h_j \in L_{h_i}(Z_+(\nu)^2).$$

Again, we can see that $A^2$ has $|Z_+(\nu)^2|$ positive coefficients. Following by induction, if $A^n = (a^n_{ij})_{i,j}$, then

$$a^n_{ij} > 0 \iff h_j \in L_{h_i}(Z_+(\nu)^n).$$

As $\nu$ is acyclic we have, from Definition 9 that there exists $N \in \mathbb{N}$, such that for $n > N$

$$a^n_{ij} > 0 \iff h_j \in L_{h_i}(Z_+(\nu)^n) = h_iH = H.$$

It implies that for $n > N$ the matrix $A^n = (a^n_{ij})_{i,j}$ has $|H|$ coefficients positive in each line. As the matrix $A$ has order $|H|$ we see that $A$ is primitive.

Lemma 18. Let $\mu, \nu \in \mathcal{P}(G)$ and $\sigma$ a permutation on $G$. Then we have that

$$\mu \cdot \nu((\sigma(G))^{-1} \times \sigma(G)) = \mu \cdot \nu(G^{-1} \times G).$$

Proof. We notice that the convolution does not depend on the order of the group, then

$$\mu \cdot \nu((\sigma(G))^{-1} \times \sigma(G)) = \mu \ast \nu = \mu \ast \nu(G^{-1} \times G).$$

We would like to observe that, $B$ is also doubly stochastic and always has 1 as an eigenvalue.

Remark 19. The main fact used in the Lemma 18 was the fact that the integral does not change under permutation of the group $G$.

Using Lemma 18 and Proposition 14, and making some permutation on the elements one can easily conclude that,
Proposition 20. Let \( G = \{g_0, \ldots, g_{n-1}\} \) be a finite group, \( \nu \in \mathcal{P}(G) \) an acyclic probability and \( H = \langle Z_+(\nu) \rangle \). The matrix \( \nu(G^{-1} \times G) = (\nu(g_i^{-1}g_j))_{i,j} \) satisfies
\[
\lim_{n \to \infty} \nu(G^{-1} \times G)^n = B, \text{ where } B \text{ is the matrix given by }
\]
\[
b_{ij} = \begin{cases} 0, & \text{if } g_i^{-1}g_j \notin H \\ \frac{1}{|H|}, & \text{if } g_i^{-1}g_j \in H \end{cases}
\]
Applying Proposition 20 we get the main result:

Theorem 21. Let \( G = \{g_0, \ldots, g_{n-1}\} \) be a finite group. If \( \nu \in \mathcal{P}(G) \) is an acyclic probability \( H = \langle Z_+(\nu) \rangle \) is the subgroup generated by the support of \( \nu \), then
\[
\lim_{n \to \infty} \nu^n = \sum_{h \in H} \frac{1}{|H|} \delta_h.
\]

Example 22. Take \( \tilde{G} \) a finite abelian group and \( a, b \in \tilde{G} \) such that \( a^2 = e, b^3 = e \) and \( \nu = p = \alpha \delta_e + (1-\alpha) \delta_0, 0 < \alpha < 1 \) and \( G = \{e, a, b, ab, ab^2\} \). So we have
\[
\nu(G^{-1} \times G) = \begin{pmatrix} \alpha & 0 & 0 & 0 & (1-\alpha) & 0 \\ 0 & \alpha & 0 & (1-\alpha) & 0 & 0 \\ (1-\alpha) & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & (1-\alpha) \\ 0 & 0 & (1-\alpha) & 0 & \alpha & 0 \\ 0 & (1-\alpha) & 0 & 0 & \alpha & 0 \end{pmatrix}.
\]
In that case \( Z_+(\nu) = \{e, b\} \) and \( \langle Z_+(\nu) \rangle = \{e, b, b^2\} \), and by Theorem 20 we have that
\[
\lim_{n \to \infty} \nu(G^{-1} \times G)^n = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}.
\]
Then we have that \( \lim_{n \to \infty} \nu^n = \frac{1}{3}(\delta_e + \delta_b + \delta_{b^2}) \).

Remark 23. If the probability \( \nu \in \mathcal{P}(G) \) is not acyclic, then there exists a finite number of subsets of \( \langle Z_+(\nu) \rangle \), let us say \( K_1, \ldots, K_l \) such that for each \( n \in \mathbb{N}, Z_+(\nu)^n = K_i \) for some \( i \in \{1, \ldots, l\} \). Following the same computations made to get Theorem 20 is possible to show that the sequence \( \{\nu^n\}_{n \in \mathbb{N}} \) has \( l \) accumulation points and each of these accumulation points is a uniform probability measure supported on a set \( K_j \).

Remark 24. Let \( G_1, G_2 \) finite groups and \( \phi : G_1 \to G_2 \) a homomorphism of groups. It is easy to see that the push forward map \( \phi_* : \mathcal{P}(G_1) \to \mathcal{P}(G_2) \) given by \( \phi_* (\mu)(A) = \mu(\phi^{-1}(A)) \) for all \( A \subset G_2 \), satisfies the following
\[
\phi_* (\nu * \mu) = \phi_* (\nu) * \phi_* (\mu).
\]
It implies that \( \lim_{n \to \infty} \phi_* (\nu^n) = \lim_{n \to \infty} (\phi_* (\nu))^n \).
The next proposition guarantee the density of the set of acyclic probability. It shows how big is the set of acyclic probabilities in the sense of the topology of $\mathcal{P}(G)$.

**Proposition 25.** Let $\nu_0 \in \mathcal{P}(G)$, where $G$ is a finite group. Given $\varepsilon > 0$, there exists $\tilde{\nu} \in \mathcal{P}(G)$ such that $\tilde{\nu}$ is a acyclic and $d(\tilde{\nu}, \nu_0) < \varepsilon$, i.e., the set of acyclic probabilities is dense in $\mathcal{P}(G)$.

**Proof.** Let $\varepsilon > 0$ and $\nu_0 = p = \sum_{i=0}^{k-1} p_i \delta_{h_i}$, with $Z_+(p) = \{ g \in G : \nu(g) > 0 \}$ and $H = \langle Z_+(p) \rangle = \{ h_0, ..., h_{k-1} \}$.

Then we define $a = \min \{ p_i : p_i > 0 \}$ and $\bar{\varepsilon} = \frac{1}{2} \min \{ \varepsilon, a \}$. So we consider the measure $\tilde{\nu} = \bar{p} = \sum_{i=0}^{k-1} \bar{p}_i \delta_{h_i}$, where

$$\bar{p}_i = \begin{cases} \frac{\varepsilon}{k - |Z_+(p)|}, & \text{if } p_i = 0 \\ \frac{p_i - \varepsilon}{|Z_+(p)|}, & \text{if } p_i > 0. \end{cases}$$

Obviously $\tilde{\nu} \in \mathcal{P}(G)$ and as

$$d(\nu_0, \tilde{\nu}) = \sum_i |p_i - \bar{p}_i| = \sum_{p_i > 0} \frac{\varepsilon}{k - |Z_+(p)|} + \sum_{p_i > 0} \frac{\varepsilon}{|Z_+(p)|} = 2\bar{\varepsilon} < \varepsilon,$$

so we get the result. \(\square\)

3. **Application: Dynamics of $T_\nu$**

We start this section with the basic properties of $T_\nu$.

**Proposition 26.** The map $T_\nu(\mu) = \mu * \nu$ is continuous in the weak topology, linear and its fixed points satisfy the Choquet-Deny equation, $\mu * \nu = \mu$.

This claims are long time knowned from the literature (see [16] pg 73, [1] and [2]), thus what remains is to understand the asymptotic behavior of $T_\nu$.

**Theorem 27.** Let $G = \{ g_0, ..., g_{n-1} \}$ be a finite group. If $\nu \in \mathcal{P}(G)$ is an acyclic probability, $H = \langle Z_+(\nu) \rangle$ is the subgroup generated by the support of $\nu$, the $w$–limit set, here denoted by $L_\omega(\mu)$, that is the set of accumulation point of his orbit, is

$$L_\omega(\mu) = \sum_{h \in H} \frac{1}{|H|} \delta_h * \mu,$$

linear on $\mu$. Moreover, $\mu$ is a recurrent point of the dynamics, that is, $\mu \in L_\omega(\mu)$, only if $\mu$ is solution of the Choquet-Deny equation

$$\tilde{\nu} * \mu = \mu,$$

where $\tilde{\nu} = \lim_{n \to \infty} \nu^n$. 
Example 28. We consider $G = (\mathbb{Z}_3, +)$ and $\nu = (1/3, 1/4, 5/12)$. So

$$G^{-1} \times G = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad \nu(G^{-1} \times G) = \begin{bmatrix} 1/3 & 1/4 & 5/12 \\ 5/12 & 1/3 & 1/4 \\ 1/4 & 5/12 & 1/3 \end{bmatrix}.$$ 

To find the fixed points for $T_\nu$, we need to solve the following equation,

$$[q_0 \quad q_1 \quad q_2] = [q_0 \quad q_1 \quad q_2] \cdot \begin{bmatrix} 1/3 & 1/4 & 5/12 \\ 5/12 & 1/3 & 1/4 \\ 1/4 & 5/12 & 1/3 \end{bmatrix}.$$ 

By linear algebra we have that there is only one solution for the above equation and it is given by $\mu_0 = \frac{1}{3}(\delta_0 + \delta_1 + \delta_2)$. So the unique fixed point is $\mu_0$.

We also have that $\mu$ is recurrent only if $\lim_{n \to \infty} \nu^n * \mu = \mu$. But $\lim_{n \to \infty} \nu^n = \mu_0$, and it implies that $\mu$ is recurrent only if

$$[q_0 \quad q_1 \quad q_2] = [q_0 \quad q_1 \quad q_2] \cdot \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix},$$

and solving this equation the unique possibility is $\mu = \mu_0$.

Using Theorem 14 we will try to find conditions for two measures have the same $\omega$-limit, where $\nu = \sum_i p_i \delta_{g_i}$ is a acyclic. First we observe that if $\mu = \sum_i q_i \delta_{g_i} \in \mathcal{P}(G)$, then $\omega(\mu) = \{\mu \cdot B\}$. If we identify $\mu$ with the vector $q = \sum_i q_i e_i$ in $\mathbb{R}^n$, where $\{e_i\}_{0 \leq i \leq n-1}$ is the canonical basis of $\mathbb{R}^n$, we have that

$$q \cdot B = \left(\sum_i q_i e_i\right) \cdot B = \sum_i q_i \left(e_i \cdot B\right).$$

It implies that $L_\omega(\mu) = \sum_i q_i \omega(\delta_{g_i})$. So, to determine the $\omega$-limit of a measure it is enough to determine the $\omega$-limit of the measures $\delta_{g_i}$, for all $g_i \in G$. Then we notice that if $H = \langle Z_+ \rangle$, $|H| = k$, $|G| = |H|l$, $\bar{\mu} = (q_0, \ldots, q_{n-1})$, $\mu = \delta_{g_0}$, and if we write

$$\alpha_0 = \sum_{i=0}^{k-1} q_i, \quad \alpha_1 = \sum_{i=k}^{2k-1} q_i, \ldots, \quad \alpha_l = \sum_{i=n-k-1}^{n-1} q_i,$$

$$\mu \cdot B = \bar{\mu} \cdot B \quad \Leftrightarrow \quad \left(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k}, 0, \ldots, 0\right) = \left(\frac{1}{k} \alpha_0, \frac{1}{k} \alpha_0, \ldots, \frac{1}{k} \alpha_0, \frac{1}{k} \alpha_1, \ldots, \frac{1}{k} \alpha_1, \ldots, \frac{1}{k} \alpha_l, \ldots, \frac{1}{k} \alpha_l\right)$$

$$\Leftrightarrow \sum_{i=0}^{k-1} q_i = 1, \quad \sum_{i=k}^{2k-1} q_i = 0, \ldots, \quad \sum_{i=n-k-1}^{n-1} q_i = 0.$$
It implies that $L_\omega(\delta_{g_0}) = L_\omega(\mu)$, if and only if $\sum_{i=0}^{k-1} q_i = 1$, where $\mu = (1, 0, ..., 0)$.

By the same argument used above we can see that

$\begin{align*}
L_\omega(\delta_{g_i}) &= L_\omega(\delta_{g_0}) \text{ for } 0 \leq i \leq k - 1, \\
L_\omega(\delta_{g_k}) &= L_\omega(\delta_{g_{k+1}}) \text{ for } k \leq i \leq 2k - 1, \\
L_\omega(\delta_{g_{k+n}}) &= L_\omega(\delta_{g_{k(n-1)+1}}) \text{ for } n - k - 1 \leq i \leq n - 1,
\end{align*}$

and from it follows that $L_\omega(\mu) = \sum_{i=0}^{m-1} q_i L_\omega(\delta_{g_i}) = \sum_{j=0}^{l} \alpha_j L_\omega(\delta_{g_{jk}})$, and if $\tilde{\mu} = (q_0, ..., q_{n-1})$ and we take $\mu = \delta_{g_i}$, with $mk \leq i \leq mk - 1$,

$\begin{align*}
L_\omega(\tilde{\mu}) &= L_\omega(\delta_{g_i}) \Leftrightarrow \alpha_m = 1, \text{ and } \alpha_j = 0 \text{ for } j \neq m.
\end{align*}$

Finally, given $\mu = (q_0, ..., q_{n-1})$ and $\mu' = (q'_0, ..., q'_{n-1})$,

$\begin{align*}
L_\omega(\mu) = L_\omega(\mu') &\Leftrightarrow \sum_{i=0}^{k-1} q_i = \sum_{i=0}^{k-1} q'_i, \\
2k-1 &\sum_{i=k}^{2k-1} q_i = \sum_{i=k}^{2k-1} q'_i, ..., \\
2k-1 &\sum_{i=n-k-1}^{n-1} q_i = \sum_{i=n-k-1}^{n-1} q'_i.
\end{align*}$

**Definition 29.** Let $\nu \in \mathcal{P}(G)$ a acyclic probability measure and $\eta \in \mathcal{P}(G)$. We call the basin of $\eta$ the set

$\{ \mu \in \mathcal{P}(G) : \lim_{n \to \infty} T^n_\nu(\mu) = \eta \}.$

**Example 30.** Let’s go back to the Example 22 where $G = \{e, a, b^2, ab, b, ab^2\}$, in that particular situation, $\nu = p = \alpha \delta_0 + (1 - \alpha) \delta_5$, $0 < \alpha < 1$ and we can rewrite $G$ as $\{e, b, b^2, a, ab, ab^2\}$ as in Lemma 18.

Then, given $\mu = (q_0, ..., q_5)$ and $\mu' = (q'_0, ..., q'_5)$, we have

$\begin{align*}
L_\omega(\mu) = L_\omega(\mu') &\Leftrightarrow 2 q_0 = 2 q'_0, \text{ and } 5 q_i = 5 q'_i \\
\text{For instance, if } \mu' = (\frac{1}{12}, \frac{1}{12}, 0, \frac{1}{12}, 0, \frac{1}{12}) \text{ we have}
\end{align*}$

$\begin{align*}
\lim_{n \to \infty} T^n_\nu(\mu') &= \frac{1}{3} \left(q_0' + q_1' + q_2', ..., q_0 + q_1 + q_2, q_3 + q_4 + q_5, ..., q_0 + q_1 + q_2\right) \\
&= \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right) \\
&= \omega(\mu').
\end{align*}$

So, the basin of attraction of $\eta = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})$, that is,

$\{ \mu = (q_0, ..., q_5) \mid \lim_{n \to \infty} T^n_\nu(\mu) = \eta \}$

is given by

$\begin{align*}
\begin{cases}
q_0 + q_1 + q_2 &= \frac{3}{4} \\
q_3 + q_4 + q_5 &= \frac{1}{4} \\
q_0, ..., q_5 &\in [0, 1]
\end{cases}
\end{align*}$
that is a convex region of hyperplane in $\mathbb{R}^6$ of dimension 4, more precisely

\[
\begin{align*}
q_0 &= \frac{3}{4} - a - b \\
q_1 &= a, q_2 = b \\
q_3 &= \frac{1}{4} - c - d \\
q_4 &= c, q_5 = d \\
a + b &\leq \frac{3}{4}, c + d \leq \frac{1}{4} \\
a, b, c, d &\in [0, 1]
\end{align*}
\]

is the basin of attraction of $\eta = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)$.

Actually the next theorem shows this always happens.

**Proposition 31.** Let $\nu = p \in \mathcal{P}(G)$ a acyclic probability measure and $H = \langle Z_+(p) \rangle$, with $|H| = k$ and $|G| = |H|^l$. Given $\eta \in \mathcal{P}(G)$ with

\[
\eta = \left(\frac{q_0}{k}, \frac{q_1}{k}, \ldots, \frac{q_{n-1}}{k}\right)
\]

The basin of $\eta$ is a convex subset of a hyperplane of dimension $\frac{n(k-1)}{k}$ in $\mathbb{R}^n$.

**Proof.** To prove the convexity of the basin of a given $\eta$ we only need to notice that if $\mu_1, \mu_2 \in \mathcal{P}(G)$ and $0 \leq \alpha \leq 1$, then

\[
T_\nu(\alpha \mu_1 + (1 - \alpha)\mu_2) = (\alpha \mu_1 + (1 - \alpha)\mu_2) \cdot \nu(G^{-1} \times G)
= \alpha \mu \cdot \nu(G^{-1} \times G) + (1 - \alpha)\mu_2 \cdot \nu(G^{-1} \times G).
\]

Hence, if $\mu_1, \mu_2$ are in the basin of $\eta$, the

\[
\lim_{n \to \infty} T_\nu^n(\alpha \mu_1 + (1 - \alpha)\mu_2) = \alpha \lim_{n \to \infty} T_\nu^n(\mu_1) + (1 - \alpha) \lim_{n \to \infty} T_\nu^n(\mu_2)
= \alpha \eta + (1 - \alpha) \eta = \eta.
\]

To prove the second part of the theorem we notice that if $\mu = (q'_0, q'_1, \ldots, q'_{n-1})$ is in the basin of $\eta$, then $\lim_{n \to \infty} T_\nu^n(\mu) = \eta$ if and only if

\[
\begin{align*}
\frac{1}{k} \sum_{i=0}^{k-1} q'_i &= q_0 \\
\vdots \\
\frac{1}{k} \sum_{i=n-k}^{n-1} q'_i &= q_{n-1} \\
q_0, \ldots, q_{n-1} &\in [0, 1].
\end{align*}
\]

If we forget the restriction $\sum_{i=0}^{n-1} q'_i = 1$ and $q_0, \ldots, q_{n-1} \in [0, 1]$, we have a linear system which has $n$ variables and $l$ linearly independents equations. Then its space of solution is given by an hyperplane of dimension

\[
n - l = n - \frac{n}{k} = \frac{n(k - 1)}{k}.
\]
Then the solution of system above is a convex set given by the intersection of a
hyperplane of dimension \(\frac{n(n-1)}{2}\) with the simplex \(\Delta_n = \{(x_0, ..., x_{n-1}) : \sum_i x_i = 1, \ x_i \in [0, 1]\}\).

Thus we have a complete characterization of the limit set of this dynamics, but
the general behaviour of this dynamical systems is given by Example 28. Indeed
we can prove the following:

**Theorem 32.** There is an open and dense set \(\mathcal{O} \subset \mathcal{P}(G)\) such that, for all \(\nu \in \mathcal{O}\),

\[ L_\omega(\mu) = \{\nu_0 * \mu\}, \forall \mu \in \mathcal{P}(G), \]

where \(\nu_0 = \frac{1}{|G|} \sum_{i=0}^{|G|-1} \delta_{g_i}\), which is the unique fixed point of \(T_\nu\).

**Proof.** Consider the set,

\[ \mathcal{O} = \{\nu = \sum_{i=0}^{|G|-1} p_i \delta_{g_i} \in \mathcal{P}(G) \mid p_i > 0\}. \]

Thus, for \(T_\nu\) with \(\nu \in \mathcal{O}\), \(\nu\) is an acyclic probability \(H = G\), and Theorem 27
claims that the \(w\)-limit set by \(T_\nu\), here denoted by \(L_\omega(\mu)\), is

\[ L_\omega(\mu) = \frac{1}{|G|} \sum_{i=0}^{|G|-1} \delta_{g_i} * \mu. \]

Moreover, \(\mu\) is a recurrent point of the dynamics, that is, \(\mu \in L_\omega(\mu)\), only if \(\mu\) is
solution of the Choquet-Deny equation

\[ \nu_0 * \mu = \mu, \]

where \(\nu_0 = \lim_{n \to \infty} \nu^n = \frac{1}{|G|} \sum_{i=0}^{G-1} \delta_{g_i}\). From the theory of doubly stochastic matrices,
we know that the only solution is \(\mu = \nu_0\), since every fixed point is recurrent, there
is just one of them. Thus, we just need to prove that \(\mathcal{O}\) is an open and dense set,
but it is trivial because its complementary set is

\[ \mathcal{O}^c = \{\nu = \sum_{i=0}^{G-1} p_i \delta_{g_i} \in \mathcal{P}(G) \mid \exists p_i = 0\}, \]

is an finite union algebraic sets in \(\mathbb{R}^{|G|}\), so it is closed with empty interior, what
conclude the proof.  

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E-mail address: baravi@mat.ufrgs.br
E-mail address: oliveira.elismar@gmail.com
E-mail address: fagnerbernardini@gmail.com