Noncommutative $SO(n)$ and $Sp(n)$ Gauge Theories

L. Bonora$^{1,3}$, M. Schnabl$^{1,3}$, M.M. Sheikh–Jabbari$^2$, A. Tomasiello$^1$

bonora@sissa.it, schnabl@sissa.it,jabbari@ictp.trieste.it,tomasiel@sissa.it

1 Scuola Internazionale Superiore di Studi Avanzati, Trieste, Via Beirut 2, 34014 Trieste, Italy
2 The Abdus Salam International Centre for Theoretical Physics Strada Costiera 11, 34014 Trieste, Italy
3 INFN, Sezione di Trieste

Abstract

We study the generalization of noncommutative gauge theories to the case of orthogonal and symplectic groups. We find out that this is possible, since we are allowed to define orthogonal and symplectic subgroups of noncommutative unitary gauge transformations even though the gauge potentials and gauge transformations are not valued in the orthogonal and symplectic subalgebras of the Lie algebra of antihermitean matrices. Our construction relies on an antiautomorphism of the basic noncommutative algebra of functions which generalizes the charge conjugation operator of ordinary field theory. We show that the corresponding noncommutative picture from low energy string theory is obtained via orientifold projection in the presence of a non–trivial NSNS B–field.
1 Introduction

It is well-known that a constant NSNS two form B-field background can be gauged away in the perturbative type II string theories. The addition of D-branes drastically modifies this conclusion. The novel point is that the components of a constant B-field background which are parallel to a $Dp$-brane can not be gauged away anymore [1, 2, 3]. Studying the open strings ending on D-branes with B-field turned on, it has been shown that the worldvolume of these branes become noncommutative [2, 3, 4]. In addition, by computing open string states scattering amplitudes and extracting the massless poles contributions, one can show that the low energy effective theory describing the system is the noncommutative $U(1)$ ($NCU(1)$) theory [5, 6].

In the zero B-field background we know that the low energy effective theory of $n$ coincident $Dp$-branes is a $p+1$ dimensional supersymmetric (with 16 supercharges) $U(n)$ theory. The $U(1)$ part of this $U(n)$ basically represents the interactions of D-brane open strings with the bulk closed strings (supergravity fields). This $U(1)$ part, which is usually called the center of mass $U(1)$, decouples from the open strings dynamics and hence effectively we find an $SU(n)$ theory. However, for $n$ coincident D-branes in a constant B-field background, the above argument is modified by the fact that the brane worldvolume is a noncommutative Moyal plane. In this case we deal with a noncommutative version of $U(n)$ theory, namely, $NCU(n)$. This theory is obtained by replacing the usual products of fields by the star (Moyal) product [7]. However, in this case separating the center of mass (noncommutative) $U(1)$ is impossible. Intuitively, this corresponds to the fact that when we turn on a B-field, the open string left and right movers (holomorphic and anti-holomorphic parts) contribute unequally, but in the closed strings sector (bulk fields) left and right movers always appear on an equal footing. So, in view of the open–closed strings interactions, we cannot decouple $U(1)$ modes. They contribute a part, which, as expected, depends on the background B-field. Apart from this string theoretic reasoning, one can also understand this point through a gauge theory argument. Suppose we start with the usual $SU(n)$ gauge theory and make it noncommutative by replacing the products with star products. Then one can show that this theory will not be consistent: the noncommutative gauge transformations will not close to form a group. In other words, by performing a noncommutative $SU(n)$ gauge transformation we create extra terms which can consistently appear only in a noncommutative $U(n)$ theory [8].

As one can see from the above discussion, trying to define a noncommutative gauge theory corresponding to subgroup of $U(n)$ and a string/brane theory configuration that
corresponds to it, does not look very promising, at first sight. However we will show below that, to a certain extent, it is possible to find consistent noncommutative extension for gauge theories corresponding to certain subgroups of \( U(n) \). The main observation is that it is possible to define gauge transformations that close to form a subgroup of the group of \( NCU(n) \) gauge transformations even though the corresponding gauge potentials and gauge transformations are not valued in a classical Lie subalgebra of the unitary Lie algebra \( u(n) \). We handle this problem from two different points of view, one relying on purely gauge field theory considerations and the other on string theory. From the gauge theory point of view, we show that it is possible to impose constraints on the gauge potentials and the gauge transformation so that when the deformation parameter vanishes we recover the ordinary orthogonal and symplectic gauge theories. An essential role in defining the constraints is played by an operator which is a generalization of the charge conjugation operators for gauge theories.

From a string theory point of view, the analogous operator is that which in the ordinary cases (without background B-field) is the worldsheet parity (possibly plus spatial parity) which is responsible for orientifold projections. So, a natural framework from a string theory point of view seems to be an orientifold in the presence of D-branes with a B-field switched on.

The paper is organized as follows. In section 2, we briefly review the noncommutative gauge theories and show that, in general, the concept of noncommutative gauge group should be modified compared to commutative theories. We will argue that one should abandon the naive picture of connections taking their values in a Lie algebra of the gauge group. The generalization we need is based on an antiautomorphism, \( r \), in the corresponding \( C^\ast \)-algebra of functions. We show that this \( r \) map together with the matrix transposition is a realization of the charge conjugation operator we need to extract \( SO \) and \( Sp \) parts out of \( U(n) \).

In section 3, we present our string theoretic arguments. First we discuss the issue of orientifold planes with a particular step function-like B-field, which passing through the O–plane changes the sign and is zero on the plane. Putting \( Dp \)-branes in this background parallel to \( Op \)-planes, we study the supersymmetries preserved by this system and we show that it is stable. Then, we proceed to find the low energy effective action for open strings attached to such \( Dp \)-branes. We show that actually it coincides with the results of our field theory arguments. Last section is devoted to remarks and further discussion.
2 \textbf{SO}(n) \textbf{ and } \textbf{Sp}(n): \textbf{noncommutative realization}

2.1 \textbf{Preliminary discussion}

Before we describe our proposal, it is useful to illustrate what are the problems connected with the realization of noncommutative orthogonal and symplectic groups. This subsection is pedagogical in nature and the reader may wish to skip it. The definition and treatment of noncommutative orthogonal and symplectic groups start in the next subsection.

First, let us discuss what we mean by gauge group in noncommutative geometry. Since we have in mind the case of $\mathbb{R}^d$, let us use the Moyal approach (even though some of the remarks which follow are independent of this assumption) and define our starting complex algebra $\mathcal{A}_\theta$ as the vector space $C^\infty(\mathbb{R}^d)$ endowed with product

$$f \ast g(x) \equiv f(x)e^{2\theta \mu \nu \partial_\mu \partial_\nu g(x)}.$$  \hspace{1cm} (2.1)

Let now $A$ be the connection (the rank is not necessarily 1; the indices are understood), which transforms under gauge transformation as $A_U = U^{-1} \ast (d + A \ast)U$, where $U^{-1} \ast U = 1$. The hermiticity condition $A^\dagger = -A$ is preserved if $U^\dagger = U^{-1}$; these $U$’s are called unitary automorphisms (of the module on which the gauge theory is constructed), but they are not functions on $\mathbb{R}^d$ valued in $U(n)$. It is true instead that the connection $A$ and the infinitesimal gauge transformations $\lambda$ are $u(n)$–valued functions on $\mathbb{R}^d$. Indeed, they are both antihermitean: being $\delta A = d\lambda + A \ast \lambda - \lambda \ast A$, the antihermiticity condition on $A$ is preserved if also $\lambda^\dagger = -\lambda$ holds, thanks to the relation

$$(f \ast g)^\dagger = g^\dagger \ast f^\dagger,$$  \hspace{1cm} (2.2)

which is a generalization of the $\ast$-algebra property to matrix–valued functions. For the same reason $F$, the field strength,

$$F_{\mu \nu} = \partial_{[\mu} A_{\nu]} + A_{\mu} \ast A_{\nu} - A_{\nu} \ast A_{\mu},$$  \hspace{1cm} (2.3)

is antihermitean too.

Now, when dealing with orthogonal and symplectic groups, something worse happens. The connection is no longer a function from $\mathbb{R}^d$ to Lie($G$); as we briefly discuss below, this is impossible to accomplish. But, what we said above should convince the reader that in noncommutative geometry this is not such a dramatic loss: finite gauge transformations are just unitary automorphisms of the module, and their infinitesimal counterparts are $U(n)$-valued for the simple reason that there is no product in the antihermiteicity condition. Our
theory is a noncommutative gauge theory which reduces in the commutative limit to the
desired one, and, what is most important, seems to have a physical origin in string theory.

Let us now discuss in some detail why one has to abandon the Lie\(G\)-valuedness of the
gauge potential. From the description above, one would hope to generalize \(SO(n)\) and \(Sp(n)\)
gauge groups in a simple way: just replace the complex (because of the \(i\) in (2.1)) algebra
we started with by a real (resp. quaternionic) one. Let us illustrate such an approach with
the example \(Sp(1) = SU(2)\). Could we find a deformation of the algebra \(\mathcal{A} \equiv C^\infty(\mathbb{R}^d, \mathbb{H})\) of
functions from \(\mathbb{R}^d\) to \(\mathbb{H}\) (quaternions), we would be done. Elements of \(\mathcal{A}\) can be expressed
as \(f = f_i \tau_i\). The idea of tensoring the usual Moyal product (2.1) with the matrix one (a
definition like \((f_i \tau_i) \ast (g_j \tau_j) \equiv (f_i \ast g_j) \, \tau_i \tau_j\)) is clearly too naive: this would not be a product
in \(\mathcal{A}\), since it does not close. If we start, indeed, from real \(f_i\) and \(g_j\), we get a complex \(f_i \ast g_j\)
(recall the \(i\) in (2.1), which cannot be dropped because it is needed for the property (2.2)
and hence for gauge symmetry).

Alternatively one can exhibit true products. The easiest try would be to consider the \(i\) in
(2.1) as one of the imaginary quaternions \(i, j, k\). In this way the property (2.2) remains valid,
but what is lost is associativity. Although this initial try is wrong, it illustrates the idea.
However, we have checked to first order that there is no associative deformation compatible
with (2.2). Even more, suppose one wants to take a pragmatic point of view and accept to
live with non associative products. Then one may look directly for deformed Moyal brackets
which satisfy Jacobi identity. It turns out that at first order such brackets do not exist.

In conclusion, if we want to find a realization of noncommutative orthogonal and symplec-
tic groups, we seem to be obliged to give up the familiar idea of Lie\(G\)-valued connections.

2.2 Noncommutative \(SO(n)\) and \(Sp(n)\)

We are now ready to illustrate our proposal for the generalization of orthogonal and symplec-
tic group.

2.2.1 The \(r\) antiautomorphism

To start with, we will work in a setting in which \(\theta\) has to be thought of as a parameter.
Accordingly, we will consider \(\mathcal{A}_\theta\) as an algebra of (possibly formal) power series in \(\theta\). This
algebra has an anti–automorphism \(r\) defined by

\[ (,)^r : f(x, \theta) \mapsto f^r(x, \theta) \equiv f(x, -\theta). \]  
\[ (2.4) \]
This map reduces to the identity on the generators $x^\mu$ and reverses the order in the product:

$$(x_1^\mu \ast \ldots \ast x_n^\mu)_r = (x_n^\mu)^* \ast \ldots \ast (x_1^\mu)^*.$$  

### 2.2.2 Orthogonal and symplectic constraints

First of all, we consider our groups as subgroups of $U(n)$. In other words we keep the usual antihermicity condition on the $u(n)$–valued connections $A$ and gauge transformations $\lambda$. To fix our conventions we will use Greek letters for space-time indices and $i$ and $j$ for matrix (group) indices. Here, for later use, we write down explicitly the hermiticity condition:

\begin{align*}
A_{ij}^*(x,\theta) &= -A_{ji}(x,\theta) \\
\lambda_{ij}^*(x,\theta) &= -\lambda_{ji}(x,\theta). \quad (2.5)
\end{align*}

Our defining condition for the $NCSO(n)$ connections and gauge transformations is to take the gauge connections and transformations satisfying the following constraints:

\begin{align*}
A_{ij}^r(x,\theta) &= -A_{ji}(x,\theta) \\
\lambda_{ij}^r(x,\theta) &= -\lambda_{ji}(x,\theta) \quad (2.6)
\end{align*}

Let us comment on these constraints. First of all, it is easy to see that they are preserved by gauge transformations. One can see it componentwise. Alternatively, rewrite (2.6) in the concise form $A = -(A^t)^r$ and $\lambda = -(\lambda^t)^r$, i.e. $t$ is the matrix transposition. Define $(.)^r \equiv (.)^{rt}$; one can show that the $rt$ map enjoys the (2.2) property, with $rt$ replacing $\dagger$. The proof is now formally similar to the usual one for $U(n)$: $(\lambda \ast A - A \ast \lambda)^rt = A^{rt} \ast \lambda^{rt} - \lambda^{rt} \ast A^{rt} = -(\lambda \ast A - A \ast \lambda)$.

The second comment we wish to make is that the constraints we introduced are natural if one recalls that in noncommutative gauge theories the map $-(\cdot)^{rt}$ is nothing but complex conjugation; our theory is the charge-conjugation invariant version of the usual one. More explicitly, as discussed in [9], indeed the charge conjugation operator is

$$A^c = -A^{rt}. \quad (2.7)$$

One can write an explicit solution of (2.6) as:

$$A_\mu = \frac{1}{2}(A_\mu - A^{\mu r}_\mu) = \frac{1}{2}(A_\mu + A^c_\mu). \quad (2.8)$$

This notation may be ambiguous and we hasten to specify that when (2.8) is used we
understand that $A_\mu$ transforms with gauge parameter $\Lambda = \frac{1}{2}(\lambda - \lambda^\dagger)$. More precisely, our $A_\mu$ enjoys the noncommutative gauge transformations generated by $\Lambda$:

$$A_\mu \rightarrow A'_\mu = U_*^{-1}(\Lambda) * A_\mu * U_*(\Lambda) - U_*^{-1}(\Lambda) * \partial_\mu U_*(\Lambda),$$

(2.9)

where

$$U_*(\Lambda) \equiv 1 + i\Lambda - \frac{1}{2}\Lambda * \Lambda + ....$$

(2.10)

$$U_*^{-1}(\Lambda) = U_*(-\Lambda), \ U_*^{-1} * U_* = 1.$$

As we see it is immediate that our $NCSO(n)$ gauge fields are charge conjugation invariant.

Thirdly, we anticipated above that under the (2.6), connections and gauge parameters do not turn out to be so(n)–valued. Nevertheless (2.6) introduces restrictions on the matrix functions $A_{ij}$. To see what they are, let us write (2.6) more explicitly

$$A_{ij}(x,\theta) = -A_{ji}(x,-\theta)$$

$$\lambda_{ij}(x,\theta) = -\lambda_{ji}(x,-\theta)$$

(2.11)

Inserting a power expansion in $\theta$ for $A$

$$A^\mu(x,\theta) = A_0^\mu(x) + i\theta^\nu A_1^{\mu\rho}(x) + \ldots,$$

(2.12)

we see that (2.6) implies that $A_0, A_2, \ldots$ are antisymmetric and $A_1, A_3 \ldots$ symmetric. The hermiticity condition (2.5) imposes that all the coefficients $A_0, A_1, \ldots$ be real. The same conclusions hold for the power expansion of $\lambda$.

Up to now, $A_0, A_1, \ldots$ are unrestricted, except for the just mentioned constraint. However, if we want to make connection with string theory, $A_1, A_2, \ldots$ are expected not to introduce new degrees of freedom, but to be functionally dependent on $A_0$. The simplest proposal is to regard them as given by the Seiberg–Witten map [7]:

$$A^\mu(A_0) = A_0^\mu - \frac{i}{4}\theta^{\nu\rho}\{A_0^\nu, \partial_{\rho}A_0^\mu + F_{0\mu}\} + O(\theta^2);$$

(2.13)

(the presence of $i$ is due the fact that Seiberg and Witten use hermitean connections rather than anti-hermitean ones, as we do). This is indeed consistent: the term linear in $\theta$ is

\footnote{In the ordinary commutative case, this is the way to ‘reduce’ a unitary connection to an orthogonal one, [10], Prop.6.4.}
symmetric if the constant part is antisymmetric. In fact, one can also see that the next term is antisymmetric, and so on; so we have complete accord with (2.12).

This is related to the further subtle issue of fixing $\theta$ to a particular value. In this case, of course the approach we have taken so far – considering $\theta$ as a formal parameter – loses its validity, and the very definition of $r$ is in jeopardy. However, thanks to the fact that $A_1, A_2, \ldots$ depend on $A_0$, even when one puts $\theta$ to a particular value, $A$ is not the most general $U(n)$ field; our constraint becomes more involved but is still there. If we invert the map to obtain $A_0(A)$, the constraint can be formulated simply as

$$A_0(A) = -A_0(A). \quad (2.14)$$

So we could say that our theory is the image of the Seiberg–Witten map restricted to the $SO(n)$ case.

It is now easy to introduce similar definitions for noncommutative $Sp(n)$. One imposes in this case the condition $JA^r = -A^r J$, where $J = \epsilon \otimes \text{Id}_n$, where $\epsilon = i\sigma_2$. This constraint is preserved by gauge transformations that satisfy the same condition.

One could think that the group $SU(n)$ could be tackled in a similar way: by defining a constraint like $\text{Tr}(A + A^{r^t}) = \text{Tr}(A + A^r) = 0$. However, this would not be gauge invariant. So, even by using the $r$ map, it is not possible to define a $NCSU(n)$ gauge theory.

### 2.2.3 Field theory

To define a Yang–Mills $NCSO(n)$ theory, let $A = A(x, \theta)$ satisfy the constraint (2.6). The action is the usual one

$$S = -\frac{1}{4} \int d^d x F_{ij}^{\mu\nu} F_{ji}^{\mu\nu}, \quad (2.15)$$

where $F$ is defined as

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} + A_{\mu} \ast A_{\nu} - A_{\nu} \ast A_{\mu}. \quad (2.16)$$

The action (2.15) is naturally gauge invariant under $NCSO(n)$ and positive. It reduces to the usual one for $SO(n)$ in the $\theta = 0$ case.

It is rather straightforward to introduce matter fields in this context in a coherent way. For example, suppose we want to introduce fermions in the adjoint representation. Let us consider a generalization of the Seiberg-Witten map to such fields.

Let $\psi_0$ be an ordinary spinor in the adjoint representation, which therefore under an ordinary gauge transformations transforms as follows

$$\delta_{\lambda_0} \psi_0 = [\psi_0, \lambda_0]. \quad (2.17)$$
Then it is reasonable to postulate the following noncommutative gauge transformation for the corresponding noncommutative field:

$$\delta \lambda \psi = \psi * \lambda - \lambda * \psi ,$$  \hspace{1cm} (2.18)

where $\lambda = \lambda_0 + \lambda'(\lambda_0, A_0)$, $A = A_0 + A'(A_0)$ and $\psi = \psi_0 + \psi'(\psi_0, A_0)$; the primed fields are first order in $\theta$. We want to find a function $\psi(\psi_0, A_0)$ which transform as (2.18) when the corresponding $\psi_0$ transform as (2.17). This amounts to satisfying the equation

$$\psi(\psi_0, A_0) + \delta \lambda \psi(\psi_0, A_0) = \psi(\psi_0 + \delta \lambda_0 \psi_0, A_0 + \delta \lambda_0 A_0) .$$ \hspace{1cm} (2.19)

The solution to first order in $\theta$ is

$$\psi(\psi_0, A_0) = \psi_0 - \frac{i}{2} \theta^{\mu \nu} \left( \{A_0 \mu, \partial_\nu \psi_0\} + \frac{1}{2} \{[\psi_0, A_0 \mu], A_0 \nu\} \right) + O(\theta^2) .$$ \hspace{1cm} (2.20)

It is easy to see that our noncommutative orthogonal constraint

$$\psi^{rt} = -\psi$$ \hspace{1cm} (2.21)

is consistent with this map. Therefore such spinors form a representation of $NCSO(n)$. In a similar way one can introduce also the fundamental representation. The action terms containing these matter fields are the usual one with ordinary product replaced by the noncommutative one, and will not be written down here.

### 3 Orientifolds and $B$ field.

We want now to derive the gauge theory we described above from a brane configuration in string theory in the limit $\alpha' \to 0$.

In the commutative case, gauge theories with orthogonal or symplectic groups are realized as low energy effective actions of branes on orientifold planes in type I theories. Since noncommutativity is achieved by a non zero $B$ field, and this vanishes on the orientifold plane, one may deem our search hopeless. As an example let us consider type IB theory, in which case the gauge groups of the branes are the ones we are looking for. However a standard argument tells us that the $B$ field is absent, except for a quantized background $[11]$; for a comment on the quantized $B$ and noncommutativity see $[12]$.

Our proposal however is more subtle.
For the sake of definiteness, we will consider type IA theory. This can be obtained in two ways: as T-dual of type IB, or as an orientifold of type IIA theory. In the second way, it is from the very beginning a 10d theory if the initial IIA is; in the first, it is of course compactified in at least one direction, and one can make contact with the other approach by sending this radius to infinity. Either way, we obtain a 10D theory (and no quantization on $B$).

First we start with a brief review of some issues in IA theory. The symmetry of IIA theory by which one orientifolds is $P_9 \cdot \Omega$, where $P_9 : x^9 \to -x^9$ is a spacetime reflection and $\Omega : \sigma \to \pi - \sigma$ is worldsheet parity \[13\]. So, the orientifold plane is an eight-plane located at $x^9 = 0$. Physics in the $x^9 > 0$ region is locally the same as the IIA one. However, strings always have an image on the other side, and as a consequence spacetime fields are reflected as

$$\Phi(x^1, \ldots, x^8, x^9) = \pm \Phi(x^1, \ldots, x^8, -x^9),$$

(3.1)

the sign is determined by the $\Omega$ parity. So, in particular, the RR charges of image branes have a relative $\pm$ according to their dimensionality. To obtain the gauge groups we are looking for, namely $SO$ and $Sp$ groups, we have to put branes and their mirrors on the orientifold plane (otherwise gauge symmetry would be broken to $[U(n/2)]^2$); so, as far as we are concerned only branes whose mirrors have the same RR charge survive – the others meet their antibranes and annihilate. The surviving ones are 0, 4, and 8-branes; the gauge group on them is $SO$, $Sp$ and $SO$ respectively. From the low energy effective theory point of view, as stated in the literature, \[18\], the orientifold projection corresponds to charge conjugation operator.

Having specified that we have branes stuck on the orientifold plane, let us now analyze more in detail the consequence of a (special) background $B$ field. Since we are really interested in its components parallel to the orientifold, we set $B_{\mu 9} = 0, \mu = 1 \ldots 8$. As for the remaining components, bearing in mind that the $B$ field is odd under worldsheet parity \[14\] from (3.1) we learn that $B_{\mu \nu}(x^1, \ldots, x^8, x^9) = -B_{\mu \nu}(x^1, \ldots, x^8, -x^9)$. So, we will consider a configuration $B_{\mu \nu} = b_{\mu \nu} f(x^9)$, where $f$ is odd in $x^9$. It is certainly true that the $B$ field is zero on the orientifold, so this would seem hopeless; but strings which end on the branes can stretch also outside, and the usual statement that their interaction with $B$ is a boundary term is, in general, true only when $B$ is constant. The interaction term equals ($\Sigma$ is the worldsheet of the open string)

$$\int_{\Sigma} B_{\mu \nu} dx^\mu \wedge dx^\nu = \int_{\Sigma} d(B_{\mu \nu} x^\mu dx^\nu) - \int_{\Sigma} dB_{\mu \nu} \wedge x^\mu dx^\nu$$

$$= \int_{\partial \Sigma} B_{\mu \nu} x^\mu dx^\nu - \int_{\Sigma} \partial_\rho B_{\mu \nu} dx^\rho \wedge x^\mu dx^\nu.$$

(3.2)
In the usual $B = \text{constant}$ case, the first term of the final expression is the boundary term which is responsible for noncommutativity, while the second vanishes. In our case, the situation is different: the first term is now zero, due to the vanishing of the $B$ field on the orientifold plane (branes are on the orientifold, so $\partial \Sigma \subset O8$), but the second is

$$\int_{\Sigma} \partial_{9} f(x^9)b_{\mu\nu}x^\mu dx^\nu \wedge dx^9. \quad (3.3)$$

We choose $f$ to be the step function $\epsilon(x^9)$, which is in fact the easiest field configuration one can think of in this case. The factor $\partial_{9} f(x^9)$ becomes a $\delta(x^9)$, and this makes the integral to reduce of dimensionality and concentrate on the orientifold \{x^9 = 0\}:

$$\int_{\Sigma \cap O8} b_{\mu\nu}x^\mu dx^\nu. \quad (3.4)$$

Now, as $\Sigma \cap O8 \supset \partial \Sigma$, this provides a boundary term which has exactly the form of the one which usually accounts for noncommutativity.

This situation would seem at first to be strangely discontinuous with respect to what happens as soon as one separates branes; in that case, one would be tempted intuitively to consider boundary conditions which vary. For instance, for a string going from one brane to its image, one would write boundary conditions $(g_{\mu\nu}\partial_n + \epsilon(y)b_{\mu\nu}\partial_y)x^\nu = 0$, where the worldsheet is to be thought of as the upper half plane, $y$ is a coordinate on the real axis, and $\partial_n$ denotes normal derivative. This is not right: one has to remember that both $P_9$ and $\Omega$ have been applied. In detail: the $\Omega P_9$ symmetry is expected to leave the closed string background $B_{\mu\nu} = \epsilon(x^9)b_{\mu\nu}$ invariant; to do so, since $x^9$ is reversed, the parameter $b$ should be reversed as well. The operation $z \rightarrow -\bar{z}, b \rightarrow -b$ is compatible with the usual boundary conditions $(g_{\mu\nu}\partial_n + b_{\mu\nu}\partial_y)x^\nu = 0$, and not with the naive ones.

One can see that all our arguments can also be extended to the case with lower dimensional orientifold planes. On the contrary they cannot be extended to the type IB case. However we do not exclude that noncommutative gauge theories may arise in this case too.

### 3.1 Supersymmetry argument.

In this subsection we give another argument, based on supersymmetry, for the stability of a system of $Dp$-branes on top of an orientifold plane ($Op$-plane with $p \leq 8$) in the presence of a step function-like $B$ field. To illustrate our reasoning we first consider the zero $B$ field case.
Suppose that $Q_L$ and $Q_R$ represent the 32 supercharges of type II theories, i.e. $Q_L$ and $Q_R$ have the same (or opposite) ten dimensional chiralities for IIB (or HIA) theory. Introducing a $Dp$-brane, half of these supersymmetries are preserved. The corresponding conserved generators are given by the following linear combination of $Q_L$ and $Q_R$:

$$Q = \epsilon_L Q_L + \epsilon_R Q_R,$$

(3.5)

where

$$\Gamma^{01\ldots p} \epsilon_L = \epsilon_R.$$

(3.6)

Now, let us consider an $Op$-plane parallel to the $Dp$-brane. To study the stability of this system one way is to check whether the supercharges (3.3), or a portion of them, are preserved under the orientifold projection. The $Op$-plane we are interested in is characterized by invariance under $\Omega P_T$ operator, where $P_T$ is the parity in all the directions transverse to the $Op$-plane. Under $\Omega$, $Q_L$ and $Q_R$, which correspond to supersymmetries of closed string left and right movers are interchanged. On the other hand under $P_T$, equation (3.6) is reversed, namely $\Gamma^{01\ldots p} \epsilon_R = \epsilon_L$. So, altogether

$$(\Omega P_T) Q = Q.$$

In other words the presence of the parallel $Op$-plane does not break supersymmetry any further and exactly the same supersymmetry as for a $Dp$-brane is preserved. Hence the whole system is stable. One can extend the above argument to an $Op$-plane parallel to a $D(p-4)$-brane. Again this system is stable (it preserves some supersymmetry) however in this case 8 supercharges survive.

For the cases with non-zero $B$ field, we follow a similar discussion. For definiteness let us consider a rank one $B$ field, which is non-zero along $p - 1$-th and $p$-th directions; generalization to other cases is straightforward. The portion (half) of supersymmetry, which is preserved by the $Dp$–brane, is given by

$$\Gamma^{01\ldots p} \left( \frac{1}{\sqrt{1 + b^2}} - \frac{b}{\sqrt{1 + b^2}} \Gamma^{p-1,p} \right) \epsilon_L = \epsilon_R.$$

(3.7)

Now, we introduce the $Op$-plane, which again acts by $\Omega P_T$ projection. Noting that

i. $\Gamma^{01\ldots p}$ and $(1 - \Gamma^{p-1,p}b)$ are commuting,

ii. $(1 - \Gamma^{p-1,p}b)(1 + \Gamma^{p-1,p}b) = (1 + b^2)1$ and

iii. under $\Omega P_T$, $b$ is reflected to $-b$,
Ω $P_T$, acting on (3.7), will lead to the same equation with $\epsilon_L$ and $\epsilon_R$ interchanged. Therefore, there are 16 supercharges invariant under the Ω $P_T$ transformation. Hence, our system formed by parallel $Dp$-branes in the background $B$ field introduced earlier, in the presence of a parallel $Op$-plane, is stable.

The above argument can also be understood in a more intuitive way. A $Dp$-brane in a $b$ field (say $bp_{−p}$) background can be treated as a bound state of a $Dp$- and $D(p−2)$-branes, with $(p−2)$-branes having their worldvolume along the 01...$p−2$ directions, while $bp_{−p}$ represents their distribution density in the $p−1$ and $p$ directions [16]. If we put this system in front of an $Op$-plane, as above, $(p−2)$-branes are reflected to anti-$(p−2)$-branes; however since at the same time we also change the $b$ to $−b$, the image of a $(p,p−2)$ bound state remains the same object, which is known to preserve 16 supercharges.

Another argument (from which in fact (3.7) could follow) that supersymmetry is preserved can be obtained via a T duality transformation in a direction parallel to the branes. Suppose for example, in the $O8$ case, that our $p$–branes extend in the 3 and 4 directions, and that there is a non-vanishing $b^{34} ≡ b$ field. Let us perform a T-duality in the direction 3; by considering the boundary conditions before and after the transformation, it is easy to see that the branes get transformed to lower dimensional tilted branes, which extend in the direction $x^4 − bx^3$. It is less trivial to understand what happens to the orientifold plane: to see what happens, we have to consider the transformation $ΩP_9$ and interpret it in terms of the dual coordinates $x_D$:

$$
\begin{align*}
\begin{array}{ccc}
x^\mu & \xrightarrow{T_3} & x^\mu_D \\
ΩP_9 & \downarrow & \downarrow ΩP_9 \\
ΩP_9 x^\mu & \xrightarrow{T_3} & (ΩP_9 x^\mu)_D.
\end{array}
\end{align*}
$$

The map $ΩP_9$ is what we have to find, and is what defines the orientifold on the dual side. Let us consider the action on $x^3$; in this case $x^3_D = x^3(z) − x^3(\bar{z})$. The map $ΩP_9$ acts on the original expansion by sending $z → −\bar{z}, b → −b$ (and of course $x^9 → −x^9$), and the $ΩP_9$ map, on $x^3_D$, does the same but with an extra overall minus sign. This is not, however, the map $ΩP_9 P_3$. On this side of the duality, $ΩP_9 P_3$ (contrary to $ΩP_9$) would indeed not touch $b$, since it is a number not a field as seen from $ΩP_9 P_3$ : it is in fact the angular coefficient by which the brane is tilted. To understand what the map really is, we have to act on the mixed coordinates $x^3 + bx^4, x^4 − bx^3$. These have an expansion which contains only pure combinations $(z^{−n} − \bar{z}^{−n}), (z^{−n} + \bar{z}^{−n})$ respectively, and so in terms of the latter it is easily seen that $ΩP_9$ is $ΩP_9 P_{x^3+bx^4}$. This means that the orientifold plane is tilted along the $x^4 − bx^3$ coordinate, and so is parallel to the brane. The mirror brane is now necessarily also parallel.
to them, and the whole system preserves supersymmetry.

3.2 Correlation functions.

To show that actually the field theories we described above arise as low energy effective actions on branes, we will now follow similar steps to the usual $U(n)$ case.

Low energy effective actions are found by computing scattering of strings corresponding to the various effective fields. In our case, incoming states have to be accompanied by orientifold projectors $(1 + \Omega P_T)/2$. This means that any vertex $V_{(s)}$ has to be changed in the combination $1/2(V_{(s)} + V_{\Omega P_T(s)})$, where the second term is the vertex that creates the $\Omega P_T|s\rangle$ state. Let us specialize to the gauge bosons, in which case the vertex, in the $-1$ picture, is $V(z) = \xi^i\cdot(\psi + \bar{\psi})e^{ikx}$, where we define $\xi^{ij\mu} = \xi^{\mu ij}$. In the $B = 0$ case, since $\Omega P_T(\xi \cdot b_{-1/2}|0,k\rangle) = -\xi^t \cdot b_{-1/2}|0,k\rangle$, correlation functions become

$$\langle \frac{1}{2} (V - V^t)(y_1) \ldots \frac{1}{2} (V - V^t)(y_n) \rangle,$$

and give the usual result: amplitudes are obtained by substituting in the usual ones $\xi$ with $\xi - \xi^t$, i.e. keeping only the antisymmetric part of $\xi$.

In the present case, the analogy keeps working: now correlation functions after orientifolding are obtained from the ones before by the rule $\xi \to \xi - \xi^t$. Thus, for instance, for the gauge three point function the result is proportional to

$$\text{Tr} \left\{ (\xi_1 - \xi^t_1) \cdot p_2 (\xi_2 - \xi^t_2) \cdot (\xi_3 - \xi^t_3) + (\xi_2 - \xi^t_2) \cdot p_3 (\xi_3 - \xi^t_3) \cdot (\xi_1 - \xi^t_1) + (\xi_3 - \xi^t_3) \cdot p_1 (\xi_1 - \xi^t_1) \cdot (\xi_2 - \xi^t_2) \right\} e^{-\frac{i}{2}b^a_{\mu}\theta_{\mu
u}\rho^\nu e} + (1 \leftrightarrow 2);$$

inner products are understood with respect to the open string metric. This is the same amplitude one finds starting from a noncommutative gauge theory, but with the additional constraint $\xi = -\xi^t$; thus it coincides with the field theory we have suggested.

As in theories arising from a non-orientifold case, there are two descriptions of the system that are equivalent at least in a perturbative sense, one is noncommutative and the other commutative, $\Omega P_T$; the commutative one in the present case is an ordinary $SO(n)$ gauge theory. So it is reasonable that, as we said, our theory is the image of a commutative $SO(n)$ gauge theory under the Seiberg-Witten map.
4 Discussions and Remarks

In this paper we have tackled the problem of noncommutative gauge theories, for groups other than $U(n)$. We have argued that to obtain the noncommutative extension of a gauge theory, in general, the usual interpretation of gauge symmetries as local internal symmetries should be modified. More precisely one must focus on an “NC Lie-algebra of gauge transformations”, rather than on the corresponding algebra of space-time independent transformations. Elaborating on the $NCU(n)$ group of gauge transformations, we have showed that actually one can extract some $NC$-subgroups of it. In particular we have discussed $NCSO(n)$ and $NCSp(n)$ gauge theories. As it is clear from our construction, in the commutative limit, $\theta \to 0$, we recover the usual $SO$ and $Sp$ theories. Physically our method is based on the charge conjugation operation in gauge theories. Noting the fact that the $SO(n)$ subgroup of the commutative $U(n)$ theory can be constructed by simply choosing the gauge field to be in the charge conjugation invariant subgroup, it follows that one must restrict the gauge transformations to the same subgroup. The same idea also works for the noncommutative case. But, of course, in the $NCU(n)$ case one has to consider the proper charge conjugation operator $[9]$. Therefore, as we see, for the particular case of $NCSO(2)$, this theory is not equivalent to $NCU(1)$, although in the commutative case they are the same theory. The main difference between these two may be that the first, $NCSO(2)$, is invariant under the charge conjugation, but the other is not. So, in this way it seems more reasonable to consider the $(NCSO(2) + \text{fermions})$ as the proper noncommutative version of QED. From this example we also learn that given a commutative gauge theory, its noncommutative extension is not unique. Another special case is $NCSp(1)$. This theory can be treated as the noncommutative version for an $SU(2)$ gauge theory, and since there is no consistent way of finding noncommutative deformed $SU$ theories in general, $NCSp(1)$ seems to be very interesting.

From the string theory side of our construction, since the orientifold projection corresponds to the charge conjugation operator in the low energy effective theory, we have guessed that a string theory environment that might give rise to effective $NCSO(n)$ and $NCSp(n)$ theories should contain an orientifold plane in a B-field background. Then, calculating the corresponding scattering amplitudes, we have checked that this is indeed the case. This provides a further support to our definition of $NCSO(n)$ and $NCSp(n)$ gauge theories.

Naturally there are lots of interesting issues related to these theories, e.g. studying renormalizability, which we do not deal with here but postpone to future research. However, we would like to end this paper with some remarks. The first concerns chiral anomalies.
In [19, 20, 21] the problem of anomaly cancellation in noncommutative $U(n)$ Yang–Mills theories was analyzed. It was shown that anomaly cancellation can occur only by matching anomaly coefficients from opposite chiralities. The main reason (although not the only one) is that the $U(1)$ factor does not decouple as in ordinary theories, therefore we cannot define a noncommutative $SU(n)$ theory (as discussed in the introduction). This fact motivated in part the research reported in this paper. One question we would like to answer is: do there exist noncommutative (non $U(n)$) gauge theories in which a more subtle cancellation mechanism for anomalies exists, with, in particular, anomaly–free representations for chiral fermions? In this paper we have introduced new noncommutative gauge theories with more general gauge groups. However, as far as anomaly cancellation is concerned, the situation is no better, for example, in orthogonal theories than in $U(n)$ theories. The reason is that connections and gauge transformations have a nontrivial symmetric part. Therefore we are led back to the same conclusion as for noncommutative $U(n)$ theories. It would seem that noncommutative Yang–Mills theories are definitely more anomalous than ordinary theories.

A second remark concerns the implications of $SO(n)$ being the Lorentz group in commutative $n$–dimensional spacetimes. Recently, its noncommutative extension has been considered by gauging the $NCU(1, D − 1)$ instead of the corresponding $SO$ symmetry [22]. As a result, a complexified gravity theory was found. Using our definition of noncommutative $SO(n)$ gauge theory one can take, in regards to this problem a different attitude. One can address the formulation of gravity theories on the Moyal plane, by using the $NCSO(1, D − 1)$ gauge group of transformations. This may lead to a more reasonable gravity theory, since it must correspond to the usual gravity theory when $\theta \to 0$. However, we expect that again in this case we will deal with some complexified gravity.

Acknowledgements

We are grateful to M. Bianchi, C.S.Chu, E.Gava and T. Krajewski for useful discussions. M.M. Sh-J. would like to thank N. Nekrasov for helpful remarks and K. Narain and A. Kashani-Poor for discussions. The research of L.B., M.S. and A.T. was partially supported by EC TMR Programme, grant FMRX-CT96-0012, and by the Italian MURST for the program “Fisica Teorica delle Interazioni Fondamentali”. The work of M.M. Sh-J. was partly supported by the EC contract no. ERBFMRX-CT 96-0090.
References

[1] E. Witten, ”Bound States of Strings and $p$-branes”, *Nucl. Phys. B460* (1996) 335, *hep-th/9510135*.

[2] Y.-K. E. Cheung, M. Krogh, ”Noncommutative Geometry From 0-Branes in a Background $B$ Field”, *Nucl. Phys. B528* (1998) 185, *hep-th/9803031*.

F. Ardalan, H. Arfaei, M. M. Sheikh-Jabbari, ”Mixed Branes and M(atrix) Theory on Noncommutative Torus”, *hep-th/9803067*.

F. Ardalan, H. Arfaei, M. M. Sheikh-Jabbari, ”Noncommutative Geometry form Strings and Branes”, *JHEP 02* (1999) 016, *hep-th/9810072*.

[3] C-S. Chu and P-M. Ho, ”Noncommutative Open Strings and D-branes”, *Nucl. Phys. B550* (1999) 151, *hep-th/9812219*.

[4] F. Ardalan, H. Arfaei, M. M. Sheikh-Jabbari, ”Dirac Quantization of Open String and Noncommutativity in Branes”, *Nucl. Phys. B576* (2000) 578, *hep-th/9906161*.

C.-S. Chu, P.-M. Ho, ”Constrained Quantization of Open Strings in Background $B$ and Noncommutative D-Branes”, *Nucl. Phys. B568* (2000) 447, *hep-th/9906192*.

M. M. Sheikh-Jabbari and A. Shirzad, ”Boundary Conditions as Dirac Constraints ’, *hep-th/9907053*.

[5] M. R. Douglas, C. Hull, ”D-branes and Noncommutative Torus”, *JHEP 9802* (1998) 008, *hep-th/9711165*.

[6] M. M. Sheikh-Jabbari, ”Super Yang-Mills Theory on Noncommutative Torus From Open Strings Interactions”, *Phys. Lett. B450* (1999) 119, *hep-th/9810173*.

[7] N. Seiberg, E. Witten, ”String Theory and Noncommutative Geometry”, *JHEP 09* (1999) 032, *hep-th/9908142*.

[8] A. Armoni, ”Comments on Perturbative Dynamics of Noncommutative Yang-Mills Theory”, *hep-th/0005208*.

[9] M. M. Sheikh-Jabbari, ”Discrete Symmetries (C,P,T) in Noncommutative Field Theories”, *hep-th/0001167*, To appear in PRL.

[10] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, vol.I, New York 1963.
[11] M. Bianchi, A. Sagnotti, *Phys. Lett.* **B247** (1990) 517; *Nucl. Phys.* **B361** (1991) 519.
M. Bianchi, E. Gava, J. F. Morales, K. S. Narain, ”D-strings in unconventional type I vacuum configurations”, *Nucl. Phys.* **B547** (1999) 96, [hep-th/9811013].
Z. Kakushadze, ”Geometry of Orientifolds with NSNS B-flux”, [hep-th/0001212].

[12] C-S. Chu, ”Noncommutative Open String: Neutral and Charged”, [hep-th/0001144].

[13] E. G. Gimon, J. Polchinski, ”Consistency Conditions for Orientifolds and D-Manifolds”, *Phys. Rev.* **D54** (1996) 1667, [hep-th/9601038].

[14] J. Polchinski, ”TASI Lectures on D-Branes”, [hep-th/9611050].

[15] M. M. Sheikh-Jabbari, ”Noncommutative Super Yang-Mills Theories with 8 Supercharges and Brane Configurations”, [hep-th/0001089].

[16] M. M. Sheikh-Jabbari, ”More on Mixed Boundary Conditions and D-branes Bound States”, *Phys. Lett.* **B425** (1998) 48, [hep-th/9712199].
H. Arfaei, M. M. Sheikh-Jabbari, ”Mixed Boundary Conditions and Brane-String Bound States”, *Nucl. Phys.* **B526** (1998) 278, [hep-th/9709054].

[17] M. M. Sheikh-Jabbari, ”Open Strings in a $B$ field Background as Electric Dipoles”, *Phys. Lett.* **B455** (1999) 129, [hep-th/9901080].

[18] M. B. Green, J. H. Schwarz and E. Witten, ”Superstring theory”, *Cambridge, Uk: Univ. Pr. (1987) 469 P. (Cambridge Monographs On Mathematical Physics)*.

[19] F. Ardalan, N. Sadooghi, ”Axial Anomaly in Noncommutative QED on $\mathbb{R}^4$”, [hep-th/0002143].

[20] J. M. Gracia-Bondia and C. P. Martin, “Chiral gauge anomalies on noncommutative $\mathbb{R}^4$,” *Phys. Lett.* **B479** (2000) 321 [hep-th/0002171].

[21] L. Bonora, M. Schnabl and A. Tomasiello, “A note on consistent anomalies in noncommutative YM theories,” [hep-th/0002210].

[22] Ali H. Chamseddine, ”Complexified Gravity in Noncommutative Spaces”, [hep-th/0005222].