Enumeration of plane unicuspidal curves of any genus via tropical geometry

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Abstract

We enumerate complex plane curves of any given degree and genus having one cusp and nodes as their singularities and matching appropriately many point constraints. The solution is obtained via the tropical enumerative geometry: We establish a correspondence between algebraic curves in count and certain plane tropical curves and compute how many algebraic curves tropicalize to a given tropical curve. To enumerate these tropical curve, we provide a version of Mikhalkin’s lattice path algorithm. The same approach enumerates unicuspidal curves of any divisor class and genus on many other toric surfaces. We also demonstrate how to enumerate real plane cuspidal curves.

Introduction

Counting of singular plane curves, or, equivalently, computation of degree of certain strata in the discriminant, is a classical task in enumerative geometry. Such problems as computation of degree of Severi varieties that parameterize curves of a given degree and genus on the plane or on another algebraic surface have attracted much attention due to a close relation to the Gromov-Witten theory. Geometrically, the degree of a Severi variety can be computed via enumeration of nodal curves of a given degree and genus passing through an appropriate collection of points in general position. This idea play a crucial role in the tropical geometry approach developed by Mikhalkin [19], which allows one to visualize algebraic curves in count via their tropical counterparts and then to solve the stated enumerative problem with simple combinatorial algorithms.

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Enumeration of curves with singularities more complicated than nodes appears to be rather more complicated, even if we allow just one ordinary cusp. We mention a few known results:

- It classically known that the variety $V_d(A_2)$ parameterizing plane curves of degree $d$ with an ordinary cusp as the only singularity is irreducible, has codimension 2 in the linear system $|O_{P^2}(d)| \simeq P^{d(d+3)/2}$ and degree $12(d-1)(d-2)$ (for a derivation of that formula see, for example, [2], Example in page 3174 or [17], Proposition 1.1 which rely on the classical algebraic geometry methods, or [16], §10 based on the theory of Thom polynomials).

- Closed formulas for degrees of varieties parameterizing curves having several singularities (including cusp) with total Milnor number $\leq 5$ can be found in [16], §10.

- Rational curves with one cusp in the plane (or on a del Pezzo surface) were enumerated in [27] and [4] via the study of the cuspidal stratum in the moduli spaces of stable maps of rational curves to the plane (or a del Pezzo surface).

Our main result is the enumeration of plane unicuspidal curves of any degree and genus. More precisely, let $V_{d,g}(A_2) \subset |O_{P^2}(d)|$ be the family of plane curves of degree $d \geq 3$ and genus $0 \leq g \leq \frac{(d-1)(d-2)}{2}$ with one ordinary cusp and $(d-1)(d-2) - d - 1$ nodes. Theorem 3.1 in Section 3 expresses $\deg V_{d,g}(A_2)$ via the sum of multiplicities of certain lattice subdivisions of the Newton polygon that all can be enumerated in an explicit lattice path algorithm. Moreover, this results holds for unicuspidal curves on a wide class of toric surfaces (including toric del Pezzo surfaces and even many singular toric surfaces). We use the tropical geometry approach in the spirit of [19] and [23]. Namely, we establish a correspondence between unicuspidal algebraic curves and so-called plane cuspidal curves and, in particular, we single out five combinatorial types of tropical cuspidal fragments that are tropical incarnations of a cuspidal singularity (see Theorem 2.3 in Section 2.2). Then we compute multiplicities of the cuspidal tropical curves, i.e., the number of unicuspidal algebraic curves tropicalizing to the given cuspidal tropical curve possessing a tropical cuspidal fragment of one of the three chosen types (see Section 2.3). In Section 3 we show that for the plane and some other toric surfaces the algebraic-tropical correspondence involves

\footnote{Particular cases when either cusp is the only singularity, or $g = 0$ were addressed in [23] and [9], though the treatment was incomplete in both the sources.}
tropical curves with cuspidal fragments of only aforementioned three types, and we provide a modification of Mikhalkin’s lattice path algorithm for enumeration of these cuspidal tropical curves. In Section 4 we demonstrate examples of enumeration and, in particular, show that there exists a two-dimensional linear subsystem in \(|O_{\mathbb{P}^2}(d)}\) containing at least \(4d^2 + O(d)\) real cuspidal curves. In addition, for the reader convenience, in Section 1 we provide all necessary details from the theory of tropical curves. In Appendix we discuss multiplicities of the tropical cuspidal fragments of the two remaining types.

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## 1 Preliminaries

Here we remind some basic stuff in tropical geometry adapted to our setting, introduce notations, and provide related auxiliary statement. Almost all details can be found in [15, 19, 23].

### 1.1 Definitions and notations

(1) We use the complex field \(\mathbb{C}\) and the field \(\mathbb{K}\) of locally convergent complex Puiseux series. For an element \(a = \sum_{r \geq r_0} a_r t^r \in \mathbb{K}\), \(a_r \in \mathbb{C}^*\), denote

\[
\text{val}(a) = -r_0, \quad \text{ini}(a) = a_{r_0}, \quad \text{Init}(a) = a_{r_0} t^{r_0} .
\]

Let \(F = \sum_{\omega \in P \cap \mathbb{Z}^n} a_{\omega} z^\omega \in \mathbb{K}[z], \ z = (w_1, ..., w_n)\), have Newton polytope \(P\). It yields a tropical polynomial

\[
N(x) = N_F(x) = \max_{\omega \in P \cap \mathbb{Z}^n} ((\omega, x) + \text{val}(a_\omega)), \quad N : \mathbb{R}^n \to \mathbb{R},
\]

and its Legendre dual, valuation function \(\nu = \nu_N : P \to \mathbb{R}\ting the graph defines a subdivision \(\Sigma_\nu\) of \(P\) into linearity domains which all are convex lattice polytopes. One can write

\[
F(z) = \sum_{\omega \in P \cap \mathbb{Z}^n} (a_\omega^0 + O(t^{\geq 0})) t^{\nu(\omega)} z^\omega ,
\]
where \( a_0^\omega \in \mathbb{C} \), \( a_0^\omega \neq 0 \) for all \( \omega \) vertices of the subdivision \( \Sigma_\nu \). Given a face \( \delta \) of the subdivision \( \Sigma_F \), we write

\[
F_\delta(z) = \sum_{\omega \in \delta \cap \mathbb{Z}^n} a_0^\omega z^\omega, \quad \text{ini}(F_\delta)(z) = \sum_{\omega \in \delta \cap \mathbb{Z}^n} a_0^\omega \mu(\omega) z^\omega \in \mathbb{C}[z],
\]

\[
\text{Ini}(F_\delta)(z) = \sum_{\omega \in \delta \cap \mathbb{Z}^n} a_0^\omega t^{\nu(\omega)} z^\omega \in \mathbb{K}[z].
\]

\( \text{(2)} \) All lattice polyhedra lie in Euclidean spaces \( \mathbb{R}^N \) with a fixed integral lattice \( \mathbb{Z}^N \subset \mathbb{R}^N \). We consider these spaces as \( \mathbb{Z}^N \otimes \mathbb{R} \) and denote by \( \mathbb{R}^N \mathbb{Z} \).

For a convex lattice polygon \( P \subset \mathbb{R}^2_\mathbb{Z} \), by Tor\((P)\) we denote the complex toric surface associated with \( P \), by \( \mathcal{L}_P \) the tautological line bundle over Tor\((P)\), by \( |\mathcal{L}_P| \) the linear system generated by the non-zero global sections (equivalently, by the monomials \( x^i y^j \), \((i,j) \in P \cap \mathbb{Z}^2 \)). We also use the notation \( \mathcal{L}_P(-Z) \) for \( \mathcal{L}_P \otimes \mathcal{J}_{Z/\text{Tor}(P)} \), where \( \mathcal{J}_{Z/\text{Tor}(P)} \) is the ideal sheaf of a zero-dimensional subscheme \( Z \subset \text{Tor}(P) \).

We call a reduced irreducible curve \( C \) in a toric surface \( X \) peripherally unibranch if, for any toric divisor \( D \subset X \), the divisor \( n^*(D) \subset \hat{C} \) is concentrated at one point, where \( n : \hat{C} \to C \) is the normalization. Respectively we call a curve \( C \) peripherally smooth, if it is smooth along its intersection with each toric divisor.

\( \text{(3)} \) For a vector \( v \in \mathbb{Z}^n \), resp. a lattice segment \( \sigma \subset \mathbb{R}^n_\mathbb{Z} \), by \( \|v\|_Z \), resp. \( \|\sigma\|_Z \) we denote the lattice length. More generally, for an \( m \)-dimensional lattice polytope \( P \subset \mathbb{R}^n_\mathbb{Z} \), \( n \geq m \), we denote by \( \|P\|_Z \) its \( m \)-dimensional lattice volume (i.e., the ratio of the Euclidean \( m \)-dimensional volume of \( P \) and the minimal Euclidean volume of a lattice simplex inside the affine \( m \)-dimensional subspace of \( \mathbb{R}^n_\mathbb{Z} \) spanned by \( P \)).

\( \text{(4)} \) We always take the standard basis in \( \mathbb{R}^2_\mathbb{Z} \) and identify \( \Lambda^2(\mathbb{R}^2_\mathbb{Z}) \simeq \mathbb{R}_\mathbb{Z} \) by letting

\[
\sigma \wedge \tau = \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}, \quad \tau = (a_1, a_2), \quad \tau = (b_1, b_2) \in \mathbb{R}^2_\mathbb{Z}.
\]

\subsection*{1.2 Plane tropical curves}

\( \text{(1)} \) A \emph{plane tropical curve} is a pair \((\Gamma, h)\), where

- \( \Gamma \) is either isometric to \( \mathbb{R} \), or is a finite connected metric graph without bivalent vertices, whose set \( \Gamma^0 \) of vertices in nonempty, the set of edges \( \Gamma^1 \) contains a subset \( \Gamma^1_\infty \neq \emptyset \) consisting of edges isometric to \([0, \infty)\) (called ends), while \( \Gamma^1 \setminus \Gamma^1_\infty \) consists of edges isometric to compact segments in \( \mathbb{R} \) (called finite edges),
• $h : \Gamma \to \mathbb{R}^2$ is a proper continuous map such that $h$ is nonconstant, affine-integral on each edge of $\Gamma$, and, at each vertex $V$ of $\Gamma$, one has the balancing condition

$$\sum_{E \in \Gamma^1, V \in E} \overline{a}_V(E) = 0,$$

where $\overline{a}_V(E)$ is the image under the differential $D(h\big|_E)$ of the unit tangent vector to $E$ emanating from its endpoint $V$.

We call $\overline{a}_V(E)$ the directing vector of $E$ (centered at $V$).

Given $V \in \Gamma^0$, the directing vectors $\overline{a}_V(E)$ over all edges $E$ incident to $V$, being positively rotated by $\frac{\pi}{2}$, form a convex lattice polygon $P(V)$ dual to $V$. The multi-set $\Delta(\Gamma, h) \subset \mathbb{R}_{\geq 0}^2$ of vectors $\{\overline{a}_V(E) : E \in \Gamma^1_{\infty}, V \in \Gamma^0, V \in E\}$ is called the degree of $(\Gamma, h)$. The vectors of $\Delta(\Gamma, h)$ sum up to zero (i.e., $\Delta = \Delta(\Gamma, h)$ is balanced). The degree $\Delta$ is called primitive if it consists of primitive vectors (i.e., vectors of lattice length 1). Positively rotated by $\frac{\pi}{2}$, they can be combined into a convex lattice polygon $P(\Delta)$, called the Newton polygon of $(\Gamma, h)$. The degree $\Delta$ is called nondegenerate if $\dim P(\Delta) = 2$.

To each edge $E \in \Gamma^1$ we assign the weight

$$w(E) = \left\|\overline{a}_V(E)\right\|_Z$$

and a geometric genus $g(E) \geq 0$. To each vertex $V$ of $(\Gamma, h)$ we assign the arithmetic genus $p_a(V)$, the number of interior integral points in the dual polygon $P(V)$, and a geometric genus

$$0 \leq g(V) \leq p_a(V)$$

The genus of $(\Gamma, h)$ is

$$g(\Gamma, h) = b_1(\Gamma) + \sum_{E \in \Gamma^1} g(E) + \sum_{V \in \Gamma^0} g(V).$$

A vertex $V \in \Gamma^0$ is called flat if

$$\dim \text{Span} \{\overline{a}_V(E) : E \in \Gamma^1, V \in E\} = 1.$$

(2) In the preceding notations, the image $h(\Gamma) \subset \mathbb{R}^2$ is a closed rational finite one-dimensional polyhedral complex without uni- and bivalent vertices. Each edge $e \subset h(\Gamma)$ is assigned a weight $w(e)$ which is the sum of the weights of the edges of $\Gamma$ intersecting $h^{-1}(x)$, where $x \in e$ is a generic point.
With these weights, \( h(\Gamma) \) becomes balanced, and hence a plane tropical curve which we call an embedded plane tropical curve induced by \((\Gamma, h)\) and denote by \( h_*(\Gamma) \). The embedded plane tropical curve \( h_*(\Gamma) \) is a corner locus of a tropical polynomial \( N : \mathbb{R}^2 \to \mathbb{R} \) with the Newton polygon \( P = P(\Delta(\Gamma, h)) \).

The dual valuation function \( \nu_N : P \to \mathbb{R} \) defines a subdivision \( \Sigma \) of \( P \) into convex lattice polygons, and this subdivision is completely determined by \( h_*(\Gamma) \). The polygons of \( \Sigma \) are in bijective duality with the vertices of \( h_*(\Gamma) \) so that the number of sides of a polygon in \( \Sigma \) equals to the valency of the dual vertex of \( h_*(\Gamma) \), while the edges of \( \Sigma \) are in bijective duality with the edges of \( h_*(\Gamma) \) so that the lattice length of an edge of \( \Sigma \) equals the weight of the dual edge of \( h_*(\Gamma) \). Furthermore, the duality inverts the incidence relation.

(3) A marked plane tropical curve is a triple \((\Gamma, p, h)\), where \( p \) is an ordered subset of \( n \geq 1 \) distinct points of \( \Gamma \). Suppose that \( h \) is injective on \( p \). In this case, we introduce a reduced marked plane tropical curve \((\Gamma_{\text{red}}, p_{\text{red}}, h_{\text{red}})\).

If there are no two edges \( E', E'' \in \Gamma^1 \) incident to the same vertex and such that \( h(E') \cap h(E'') \) is infinite (i.e., a compact segment or a ray), we define \((\Gamma_{\text{red}}, p_{\text{red}}, h_{\text{red}}) = (\Gamma, p, h)\). Otherwise, we perform the following procedure until we end up with a curve possessing the above property, which then will be denoted \((\Gamma_{\text{red}}, p_{\text{red}}, h_{\text{red}})\). The procedure consists in the following elementary steps: given two edges \( E', E'' \in \Gamma^1 \) incident to a vertex \( V \in \Gamma^0 \) such that \( h(E') \cap h(E'') \) is infinite,

- we respectively identify \( h^{-1}(h(E') \cap h(E'')) \cap E' \) with \( h^{-1}(h(E') \cap h(E'')) \cap E'' \) into one edge \( \hat{E} \),
- set \( \overline{a}_V(\hat{E}) = \overline{a}_V(E') + \overline{a}_V(E'') \) and \( w(\hat{E}) = w(E') + w(E'') \),
- if \( h(E') \subsetneq h(E'') \), \( V' \neq V \) the second vertex of \( E' \), \( \hat{V} \) the second vertex of \( \hat{E} \), and \( \hat{E}' \) is the closure of \( E'' \setminus h^{-1}(h(E') \cap h(E'')) \), we set \( g(\hat{E}) = g(E'), g(\hat{V}) = g(V') \), and \( g(\hat{E}') = g(E'') \),
- if \( h(E') = h(E'') \) is a ray, we set \( g(\hat{E}) = g(E') + g(E'') \),
- if \( h(E') = h(E'') \) is a compact segment, \( V' \neq V'' \) the second vertices of \( E', E'' \), respectively, we set

\[
g(\hat{E}) = g(E') + g(E''), \quad g(\hat{V}) = g(V') + g(V'')
\]

- if \( h(E') = h(E'') \) is a compact segment, \( V' \) a common second vertex of \( E', E'' \), we set \( g(\hat{E}) = g(E') + g(E'') + 1, g(\hat{V}) = g(V') \).
It follows from the construction that \( g(\Gamma) = g(\Gamma_{\text{red}}) \). Denote by \( p_{\text{red}} \) the image of \( p \) in \( \Gamma_{\text{red}} \).

Suppose that \( \Gamma_{\text{red}} \) either is trivalent, or all but one of its vertices are trivalent, while the remaining one is four-valent. Let \( |p_{\text{red}}| = |\Delta_{\text{red}}| + b_1(\Gamma_{\text{red}}) - 1 \). A curve \((\Gamma, p, h)\) is called regular, if

- \( \Gamma^0_{\text{red}} \cap p_{\text{red}} = \emptyset \) and \( h_{\text{red}} \) is injective on \( p_{\text{red}} \),
- each connected component of \( \Gamma_{\text{red}} \setminus p_{\text{red}} \) either is trivalent, simply connected, and contains one end of \( \Gamma_{\text{red}} \), or contains one four-valent vertex and two ends of \( \Gamma_{\text{red}} \), and is simply connected, or contains one four-valent vertex, one end of \( \Gamma_{\text{red}} \), has \( b_1 = 1 \), and its complement to the four-valent vertex is simply connected.

It is easy to see that, under the regularity condition, the closure \( \overline{K} \) of each connected component \( K \) of \( \Gamma_{\text{red}} \setminus p_{\text{red}} \) possesses a unique regular orientation of all its edges such that:

- each end of \( \overline{K} \) is oriented toward infinity,
- each edge of \( \overline{K} \) incident to a point \( p \in p_{\text{red}} \) is emanating from \( p \),
- each vertex \( V \in K \cap \Gamma^0_{\text{red}} \) is incident to exactly two incoming edges.

The regular orientation of \((\Gamma_{\text{red}}, p_{\text{red}}, h_{\text{red}})\) induces an orientation on the edges of the closures of components of \( \Gamma \setminus p \) so that among edges incident to the same vertex \( V \in \Gamma^0 \) that are mapped to the same line, at most one is incoming to \( V \).

(4) The following statement is due to Mikhalkin [19, Proposition 2.23, Corollary 2.24 and Lemma 4.20].

**Lemma 1.1.** (1) Let \( \text{Def}(\Gamma, p, h) \) be the germ of the deformation space of \((\Gamma, p, h)\), in which \((\Gamma, p)\) keeps its combinatorial type, and \( h \) keeps the differentials along all edges. Then

\[
\dim \text{Def}(\Gamma_{\text{red}}, p_{\text{red}}, h_{\text{red}}) \leq |\Delta| + b_1(\Gamma_{\text{red}}) - 1 + n.
\]

(2) Suppose that

\[
n = |p| = |\Delta| + g - 1
\]

and \( h(p) \subset \mathbb{R}^2 \) is a configuration of \( n \) points in general position (cf. [19, Section 4.2]). Then \((\Gamma, p, h)\) is trivalent, reduced, regular, \( g(\Gamma) = b_1(\Gamma) \), and \( p \cap \Gamma^0 = \emptyset \).
We shall use also the following characterization of combinatorial types of codimension one.

**Lemma 1.2.** In the notations of Lemma 1.1(1), suppose that

- \( n = |\Delta| + g - 2 \),
- \((\Gamma, h)\) either is not trivalent, or not reduced, or \( g(\Gamma) > b_1(\Gamma) \),
- \( h(p) \subset \mathbb{R}^2 \) is a configuration of \( n \) points in general position.

(1) The following holds

(i) either \((\Gamma, p, h)\) is trivalent, reduced, and regular, and \( g(\Gamma) = b_1(\Gamma) + 1 \),

(ii) or \((\Gamma, p, h)\) is reduced, regular, has one four-valent vertex, while the other vertices are trivalent, and \( g(\Gamma) = b_1(\Gamma) \),

(iii) or \((\Gamma, p, h)\) is not reduced, but \((\Gamma_{\text{red}}, p_{\text{red}}, h_{\text{red}})\) is trivalent, regular and satisfies \( g = g(\Gamma_{\text{red}}) = b_1(\Gamma_{\text{red}}) + 1 \) and \( |\Delta_{\text{red}}| = |\Delta| \),

(iv) or \((\Gamma, p, h)\) is not reduced, but \((\Gamma_{\text{red}}, p_{\text{red}}, h_{\text{red}})\) is trivalent, regular, satisfies \( g = g(\Gamma_{\text{red}}) = b_1(\Gamma_{\text{red}}) \), and \( |\Delta_{\text{red}}| = |\Delta| - 1 \),

(v) or \((\Gamma, p, h)\) is not reduced, has at most one four-valent vertex and the others trivalent, and satisfies \( g(\Gamma) = b_1(\Gamma) \), the curve \((\Gamma_{\text{red}}, p_{\text{red}}, h_{\text{red}})\) is regular, all but one of its vertices are trivalent, one vertex is four-valent with exactly one pair of incident edges lying on one line, and it holds \( g(\Gamma_{\text{red}}) = b_1(\Gamma_{\text{red}}) \).

(2) Furthermore, in all cases (i)-(iv), \( p \cap \Gamma^0 = \emptyset \), the intersection of \( \Gamma_{\text{red}}^0 \) with the set of points \( x \in \mathbb{R}^2 \) such that \( |h^{-1}_{\text{reg}}(x)| > 1 \) is empty, and in the cases (i)-(iii), (v) the \( h\)-images of distinct ends of \( \Gamma \) do not lie on the same ray.

**Proof.** It immediately follows from Lemma 1.1 that \( p \cap \Gamma^0 = \emptyset \) and \( g \leq b_1(\Gamma) + 1 \).

Suppose that \( g = b_1(\Gamma) + 1 \). Then \( n = |\Delta| + b_1(\Gamma) - 1 \). Again applying Lemma 1.1 we get that \((\Gamma, p, h)\) is trivalent, reduced, and regular; hence, this case fits the conditions of item (i).

Suppose that \( g = b_1(\Gamma) \), and \( \Gamma \) is reduced, but not trivalent. Let us show that \( \Gamma \) cannot have a vertex of valency \( v \geq 5 \), or a pair of vertices of valency \( v > 3 \). Assume that For, we use the idea of the proof of [23, Lemma 2.2], but in a different setting. We embed the deformation space \( \text{Def}(\Gamma, h) \) into...
\( \mathbb{R}^{e(\Gamma)} \), where \( e(\Gamma) = |\Gamma^1| \), namely, for each edge \( E \in \Gamma^1 \) we take the value in \( \mathbb{R} \cong \mathbb{R}^2 / \text{Span} \{ \overline{a}_V(E) \} \) determined by the line containing \( h(E) \). Assuming that \( \Gamma \) contains vertices as above, we shall show that

\[
\dim \text{Def}(\Gamma, h) \leq |\Delta| + g - 3 . \tag{2}
\]

Given a generic vector \( \overline{a} \in \mathbb{R}^2 \), orient the edges of \( \Gamma \) so that their \( h \)-images form acute angles with \( \overline{a} \). This, in particular defines a partial order on \( \Gamma^0 \), which we extend to a total order. We can suppose that, for some vertex \( V^* \in \Gamma^0 \) of valency \( v(V^*) \geq 4 \), there are at least two edges merging to \( V^* \) and at least two edges emanating from \( V^* \). Then we estimate \( \dim \text{Def}(\Gamma, h) \) from above as follows:

- count all ends oriented from the infinity,
- go through the set \( \Gamma^0 \) ordered as above and, at each vertex \( V \in \Gamma^0 \), we do not add new parameters if there are at least two edges merging to \( V \), and we add one new parameter corresponding to some of the emanating edges if there is only one edge merging to \( V \).

After that we perform the same estimation with respect to the orienting vector \( -\overline{a} \), sum up these two bounds, and using the above assumption on the vertices of \( \Gamma \) and the Euler characteristic relation

\[
2|\Delta| + 2b_1(\Gamma) - 2 = |\Delta| + \sum_{V \in \Gamma^0} (v(V) - 2) ,
\]

obtain

\[
2 \dim \text{Def}(\Gamma, h) \leq |\Delta| + (|\Gamma^0| - 1)
= |\Delta| + \sum_{V \in \Gamma^0} (v(V) - 2) - \sum_{V \in \Gamma^0} (v(V) - 3) - 1 . \tag{3}
\]

Hence,

\[
\leq (2|\Delta| + 2b_1(\Gamma) - 2) - 3 = 2|\Delta| + 2b_1(\Gamma) - 5
\]

and \( \text{(2)} \), which excludes the above assumption on the valency of vertices of \( \Gamma \). That is, \( \Gamma \) has one four-valent vertex while the others are trivalent. The regularity follows immediately, since otherwise one would encounter either a bounded component of \( \Gamma \setminus p \), hence a restriction to the position of the points \( h(p) \), or a simply connected component of \( \Gamma \setminus p \) with one four-valent vertex and one end, hence again a restriction to the position of the points \( h(p) \) in contradiction to the generality assumption.

If \( (\Gamma, h) \) is not reduced, but \( \Gamma_{\text{red}} \) is trivalent. Then either \( (\Gamma_{\text{red}}, p_{\text{red}}, h_{\text{red}}) \) is obtained by collapsing at least one cycle of \( \Gamma \), thus, \( b_1(\Gamma_{\text{red}}) \leq b_1(\Gamma) - 1 \),
or by merging at least one pair of ends of \( \Gamma \), thus, \( |\Delta_{\text{red}}| \leq \Delta - 1 \). Then the above argument leads either to the case (iii), or to (iv).

Suppose that \((\Gamma, h)\) is not reduced, and \(\Gamma_{\text{red}}\) is not trivalent. Again the above considerations leave the only possibility of the assertion (v).

Now we prove statement (2). The condition \( p \cap \Gamma^0 = \emptyset \) is immediate. The remaining conditions we demonstrate in the most difficult case (ii), while the other cases can be settled in the same manner. So, assume that there are \( V \in \Gamma^0, x \in \Gamma \setminus \Gamma^0 \) such that \( h(V) = h(x) \). We identify \( V \) and \( x \) into a vertex \( V' \) and obtain a curve \( \Gamma' \) with \( b_1(\Gamma') = b_1(\Gamma) + 1 \), which either has a 5-valent vertex and a 4-valent one, or has a 6-valent vertex, while the other vertices are trivalent. In both the cases we shall derive (2).

In the former case, we apply the estimation procedure with an orienting vector \( \overline{a} \) such that the 4-valent vertex has two incoming and two outgoing incident edges. Note that the 5-valent vertex has at least two incoming and at least two outgoing edges. Then

\[
2 \dim \text{Def}(\Gamma, h) \leq 2 \dim \text{Def}(\Gamma', h')
\]

\[\text{cf. (3)}\]

\[= |\Delta| + \sum_{V \in (\Gamma')^0} (v(V) - 2) - \sum_{V \in (\Gamma')^0} (v(V) - 3) - 2 \]

\[= 2|\Delta| + 2g - 5 \implies \dim \text{Def}(\Gamma, h) \leq |\Delta| + g - 3.\]

In the latter case, we pick a generic point \( O \in \mathbb{R}^2 \) close to \( h(V') \), order the vertices of \( \Gamma' \) by the growing distance of their \( h \)-images from \( O \), and respectively orient the edges of \( \Gamma' \) (in particular, all ends are oriented to infinity). Denote by \( (\Gamma')^0_1, (\Gamma')^0_2 \) the sets of trivalent vertices with one or two emanating incident edges, respectively. Similarly to the previous estimations we have

\[
\dim \text{Def}(\Gamma', h') \leq 2 + |(\Gamma')^0_2|.
\]

Since

\[
|(\Gamma')^0_1| + 2|(\Gamma')^0_2| = |(\Gamma')^1| - 6,
\]

\[
|(\Gamma')^0_1| + |(\Gamma')^0_2| + 1 + |\Delta| - |(\Gamma')^1| = 1 - g' = -g,
\]

we get

\[
|(\Gamma')^0_2| = |\Delta| + g - 5,
\]

and hence

\[
\dim \text{Def}(\Gamma', h') \leq 3 + (|\Delta'| + g - 5) = |\Delta'| + g - 3
\]

as required. \( \square \)
Definition 1.3. Let \((\Gamma, h)\) be a plane tropical curve, \(V \in \Gamma^0\) its trivalent vertex. Define the Mikhalkin multiplicity of \(V\) by

\[ \mu(V) = \left| \mathbf{a}_V(E_1) \land \mathbf{a}_V(E_2) \right|, \]

where \(E_1, E_2 \in \Gamma^1\) are some two incident to \(V\) edges.

1.3 Tropicalization of algebraic curves over a non-Archimedean field

1.3.1 Embedded tropical limit

Let \(\Delta \subset \mathbb{Z}^2 \setminus \{0\}\) be a nondegenerate, primitive, balanced multiset, \(C \in |\mathcal{L}_{P(\Delta)}|_{\mathbb{R}}\) a reduced, irreducible curve of genus \(g\), which does not hit intersection points of toric divisors. In particular, it can be given by an equation

\[ F(x, y) = \sum_{(i,j) \in P(\Delta) \cap \mathbb{Z}^2} t^{\nu'(i,j)} (a_{ij}^0 + O(t^{>0})) x^i y^j, \]

where \(a_{ij}^0 \in \mathbb{C}, (i,j) \in P(\Delta) \cap \mathbb{Z}^2\), and \(a_{ij}^0 \neq 0\) as the coefficient of \(x^i y^j\) in \(F\) does not vanish (for instance, when \((i,j)\) is a vertex of \(P(\Delta)\)). We then define a convex, piecewise-linear function \(\nu : P(\Delta) \to \mathbb{R}\), whose graph is the lower part of the \(\text{conv}\{(i,j,\nu'(i,j)), (i,j) \in P(\Delta) \cap \mathbb{Z}^2\} \subset \mathbb{R}_3^\mathbb{Z}\). Via a parameter change \(t \mapsto t^M\), we can make \(\nu(P(\Delta) \cap \mathbb{Z}^2) \subset \mathbb{Z}\). Denote by \(\Sigma\) the subdivision of \(P(\Delta)\) into linearity domains of \(\nu\), which are convex lattice polygons \(P_1, \ldots, P_m\). We then have

\[ F(x, y) = \sum_{(i,j) \in P(\Delta) \cap \mathbb{Z}^2} t^{\nu(i,j)} (c_{ij}^0 + O(t^{>0})) x^i y^j, \]

where \(c_{ij}^0 \neq 0\) for \((i,j)\) a vertex of some of the \(P_1, \ldots, P_m\). This data defines a flat family \(\mathcal{X} \to \mathbb{C}\) with the toric threefold \(\mathcal{X} = \text{Tor}(\mathcal{O}(\nu))\), where

\[ \mathcal{O}(\nu) = \{(i,j,c) \in \mathbb{R}_3^\mathbb{Z} : (i,j) \in P(\Delta), c \geq \nu(i,j)\} \]

is the overgraph of \(\nu\), the central fiber \(\mathcal{X}_0\) splits into the union of toric surfaces \(\text{Tor}(P_k), 1 \leq k \leq m\), and the other fibers are isomorphic to \(\text{Tor}(P(\Delta))\). The evaluation of the parameter \(t\) turns the given curve \(C\) into an inscribed family of curves

\[ C^{(t)} \subset \mathcal{X}_t, \quad C^{(t)} \in |\mathcal{L}_{P(\Delta)}|, \quad t \in (\mathbb{C}, 0) \setminus \{0\}, \]
which closes up to a flat family over \(( \mathbb{C}, 0)\) with the central element

\[
C^{(0)} = \bigcup_{k=1}^{m} C^{(0)}_k,
\]

where

\[
C^{(0)}_k = \left\{ F^{(0)}_k(x, y) := \sum_{(i,j) \in \triangle_k \cap \mathbb{Z}^2} c^{(0)}_{ij} x^i y^j = 0 \right\} \in |\mathcal{L}_{P_k}|, \quad 1 \leq k \leq m.
\]

The function \(\nu : P(\Delta) \to \mathbb{R}\) defines an embedded plane tropical curve \(\tau(C)\), whose support is the closure of the valuation image of \(C\).

We define the embedded tropical limit of \(C\) to be the collection \((\tau(C), \{C^{(0)}_k\}_{k=1,\ldots,m})\) (where \(C^{(0)}_1, \ldots, C^{(0)}_m\) are called limit curves), cf. [23, Section 2].

1.3.2 Parameterized tropical limit

Let \(n : \hat{C} \to C\) be the normalization, or, equivalently, the family

\[
n_t : \hat{C}^{(t)} \to C^{(t)} \hookrightarrow X_t, \quad t \in (\mathbb{C}, 0) \setminus \{0\},
\]

where each \(\hat{C}^{(t)}\) is a smooth curve of genus \(g\) (cf. [24] or [6]). We also assume that \(\hat{C}\) contains a configuration \(\mathbf{w}\) of \(n\) marked points which form \(n\) disjoint families \(w_i(t) \in \hat{C}^{(t)}, t \in (\mathbb{C}, 0) \setminus \{0\}\), projecting to disjoint families in \(C^{(t)}\) that avoid singularities (also denoted \(w_i(t)\), no confusion will arise), and, furthermore, the valuation image of \(\mathbf{w}\) consists of \(n\) distinct points of \(T(C)\). The family [15] admits (after a suitable untwist \(t \mapsto t^M\)) a flat extension to \((\mathbb{C}, 0)\) with the central element \(n_0 : \hat{C}^{(0)} \to \mathbf{X}_0, \hat{C}^{(0)}\) a nodal curve of arithmetic genus \(0\) (see, for instance [11, Theorem 1.4.1]) and such that \((n_0)_* \hat{C}^{(0)} = C^{(0)}\). We contract the part \(\hat{C}^{(0)}_{fin}\) of \(\hat{C}^{(0)}\) mapped to a finite set, obtaining the map \(n'_0 : (\hat{C}^{(0)})' \to C^{(0)}\); we assume also that the sections \(p_i(t), i = 1, \ldots, n\), close up at \(n\) distinct points of \((\hat{C}^{(0)})'\).

We will use in the sequel the parameterized tropical limit of \((n : \hat{C} \to C, \mathbf{w})\), which consists of some plane marked tropical curve \((\Gamma, \mathbf{p}, h)\) and the pair \((n'_0 : (\hat{C}^{(0)})' \to C^{(0)}, \mathbf{w}(0))\) (cf. [26, Section 2]). The combinatorics of \(\Gamma\) includes the description of vertices equipped with nonnegative genera and of their incident edges equipped with nonnegative weights:

- the set of vertices \(\Gamma^0\) splits into two disjoint subsets, \(\Gamma^0_{car}\) corresponding to the components of \((\hat{C}^{(0)})'\) and \(\Gamma^0_{pt}\) corresponding to the intersection
points of distinct components of \((\tilde{C}(0))'\); to each vertex \(C' \in \Gamma_{\text{cur}}^0\) we assign the genus \(g(C')\), to each vertex \(z \in \Gamma_{\text{pt}}^0\) we assign the genus 0 if it is not in the image of \(\tilde{C}_{\text{fin}}(0)\), and assign the arithmetic genus of the component of \(\tilde{C}_{\text{fin}}(0)\) that maps to \(z\) otherwise.

- we then form a bipartite graph on the sets \(\Gamma_{\text{cur}}^0\) and \(\Gamma_{\text{pt}}^0\); if \(C'\) is a component of \((\tilde{C}(0))'\), \(C \subset \tilde{C}(0)\) its normalization, and \(z \in C'\) an intersection point of \(C'\) with some other components of \((\tilde{C}(0))'\), then \(C' \in \Gamma_{\text{cur}}^0\) and \(z \in \Gamma_{\text{pt}}^0\) are joined in \(\Gamma\) by a set of edges bijectively corresponding to the components of the divisor \((n|_{\Gamma})^*(\mathbf{n}'(z)) \subset C\); 

- ends (noncompact edges) of \(\Gamma\) emanate from each element \(C' \in \Gamma_{\text{cur}}^0\) mapped to \(\text{Tor}(\Delta_k)\) such that \(\Delta_k \cap \partial P(\Delta)\) contains \(r \geq 1\) sides \(\sigma_1, \ldots, \sigma_r\) of \(\Delta_k\), namely, we attach to \(C'\) a set of ends bijectively corresponding to the components of the divisor \((n|_{\Gamma})^*(\text{Tor}(\sigma_1) \cup \ldots \cup \text{Tor}(\sigma_r))\);

- the weight of an edge joining \(C' \in \Gamma_{\text{cur}}^0\) and \(z \in \Gamma_{\text{pt}}^0\) with \(z \in C' \cap (\mathbb{C}^*)^2 \subset \text{Tor}(\Delta_k)\) is zero; the weight of an edge joining \(C' \in \Gamma_{\text{cur}}^0\) and \(z \in \Gamma_{\text{pt}}^0\) with \(z \in C' \cap \text{Tor}(\sigma), \sigma\) a side of \(\Delta_k\), is the degree of the corresponding component of the divisor \((n|_{\Gamma})^*(\text{Tor}(\sigma))\); the weight of an end attached to \(C' \in \Gamma_{\text{cur}}^0\) is the degree of the corresponding components of the divisor \((n|_{\Gamma})^*(\text{Tor}(\sigma_1) \cup \ldots \cup \text{Tor}(\sigma_r))\).

The map \(h\) takes \(\Gamma\) onto \(T(C)\) so that it is affine-integral on each edge of \(\Gamma\). More precisely, we let \(h(C') = V_k\) for each element \(C' \in \Gamma_{\text{cur}}^0\) mapped by \(n'\) to \(\text{Tor}(\Delta_k)\) with \(\Delta_k\) dual to the vertex \(V_k\) of \(\tau(C)\). An element \(z \in \Gamma_{\text{pt}}^0\) joined in \(\Gamma\) by edges with \(C_1', C_2' \in \Gamma_{\text{cur}}^0\) such that \(h(C_1') = h(C_2') = V_k\) is sent by \(h\) to \(V_k\) as well as all the edges in \(\Gamma\) joining \(z\) with \(C_1', C_2'\). An element \(z \in \Gamma_{\text{pt}}^0\) joined in \(\Gamma\) by edges with \(C_1', C_2' \in \Gamma_{\text{cur}}^0\) such that \(h(C_1') = V_{k_1}, h(C_2') = V_{k_2}, k_1 \neq k_2\), is mapped to an interior point of the edge of \(\tau(C)\) with endpoints \(V_{k_1}, V_{k_2}\), the position of \(h(z)\) is determined by the original curve \(C \subset \text{Tor}_{\mathbb{Z}}(P(\Delta))\) via modifications (see Section 2.3.3), but for the moment we do not precise it and choose \(h(z)\) generic in the edge \([V_{k_1}, V_{k_2}]\). The end of \(\Gamma\) attached to \(C' \in \Gamma_{\text{cur}}^0\) and associated with a point \(z \in C' \cap \text{Tor}(\sigma_i)\) (see the notation of the preceding item) is mapped to the end of \(\tau(C)\) dual to \(\sigma_i\). At last, both the differentials of \(h\) and the lengths of non-contracted edges of \(\Gamma\) are determined by their weights and the lengths of their images.

To adjust the constructed object \((\Gamma, \mathbf{p}, h)\) to the definition in Section 1.2 we remove bivalent vertices respectively gluing the two incident edges into
one withe genus inherited from the removed vertex, and contract the edges of \( \Gamma \) mapped to points, again assigning the genus of each contracted fragment to its image.

It immediately follows from the construction that \((\Gamma, p, h)\) is a plane tropical curve, \(g(\Gamma) = g\), and \(h_*(\Gamma, h) = \tau(C)\).

## 2 Correspondence between unicuspidal algebraic curves and unicuspidal tropical curves

Throughout this section, \( \Delta \) always means a nondegenerate, primitive, balanced multi-set in \( \mathbb{R}_2^2 \).

### 2.1 Statement of the enumerative problem

Define the arithmetic genus by \( p_a(P(\Delta)) = |\text{Int}(P(\Delta)) \cap \mathbb{Z}^2| \). Assuming that \( p_a(P(\Delta)) \geq 1 \), for any \( 0 \leq g < p_a(P(\Delta)) \), denote by \( V_{\Delta,g}(A_2) \) the family of irreducible curves \( C \in |L_{P(\Delta)}| \) on the toric surface \( \text{Tor}(P(\Delta)) \) that have genus \( g \) and precisely one singular local branch, and this branch is of multiplicity 2.

**Lemma 2.1.** Suppose that \( |\Delta| \geq 5 \) and there exists a subset \( Q \subset (\partial P(\Delta) \cap \mathbb{Z}^2) \) such that \( Q = \text{conv}(Q) \) is a nondegenerate quadrangle without parallel edges. Then \( V_{\Delta,g}(A_2) \neq \emptyset \) is of pure dimension \(|\Delta| + g - 2\), and a generic element of any component of \( V_{\Delta,g}(A_2) \) is a curve with one ordinary cusp and \( p_a(P(\Delta)) - g - 1 \) nodes.

Under the hypotheses of Lemma 2.1, we can correctly pose the problem. For any \( 0 \leq g < p_a(P(\Delta)) \), compute \( \text{deg} V_{\Delta,g}(A_2) \).

For the case of \( \text{Tor}(P(\Delta)) \) a smooth projective surface and \( p_a(P(\Delta)) - g \) obeys a certain upper bound (for example, \( p_a(P(\Delta)) - g < 2(m - 1) \) for degree \( m \) plane curves), the formulas for \( \text{deg} V_{\Delta,g}(A_2) \) can be obtained by methods of [16] (see [16, Section 10.2] with examples covering the domain \( p_a(P(\Delta)) - d \leq 4 \)). At the other extreme, the plane rational curves of any degree \( d \) with one cusp and \( \frac{d(d-3)}{2} \) nodes have been enumerated in [27].

We solve the problem for a wide class of toric surfaces, including the plane, quadric, toric del Pezzo, Hirzebruch surfaces, for any divisor class, satisfying conditions of Lemma 2.1 and for all genera.

The solution is given via tropical geometry in the style of [19] and [23]: namely, we consider the problem over the field \( K \) of locally convergent complex Puiseux series, choose \(|\Delta| + g - 2\) points in a special position in \( 14 \)
$(\mathbb{K}^*)^2 \subset \text{Tor}_K(P(\Delta))$, prove the correspondence theorem, which describes the plane tropical curves (called further on unicuspidal) obtained via the tropicalization of the counted algebraic curves, and computes the multiplicities of these tropical curves (i.e., the number of algebraic curves tropicalizing to the given tropical curve). Moreover, we provide a suitable analogue of the Mikhalkin’s lattice path algorithm allowing one, in a purely combinatorial way, to enumerate the unicuspidal tropical curves and compute their multiplicities.

**Proof of Lemma 2.1.** First of all notice that any curve in $V_{\Delta,g}(A_2)$ admits a deformation into a curve (also belonging to $V_{\Delta,g}(A_2)$) that has an ordinary cusp and $p_a(P(\Delta)) - g - 1$ nodes as its singularities. This follows from [21, Theorem 1.1] (restated with the same proof for curves on an arbitrary normal surface). namely, for a reduced curve $C \in |L_{P(\Delta)}|$ define the number of virtual cusps (cf., [18]) to be

$$c_{vir}(C) = \sum_{z \in \text{Sing}(C)} (\kappa(C, z) - 2\delta(C, z))$$

(the definition of $\kappa$- and $\delta$-invariant can be found, for instance, in [11, Section I.3.4]). The sufficient condition for the required deformation reads as $c_{vir} \leq -KC - 4$ ($K$ being the canonical class of the surface). In our situation, $c_{vir} = 1$ and $-KC = |\Delta| \geq 5$; hence, the condition holds.

We can triangulate $P(\Delta) \setminus Q$ by diagonals. The resulting subdivision of $P(\Delta)$ is convex, i.e., lifts to a graph of a convex piecewise-linear function. We, furthermore, orient the adjacency tree so that $Q$ is a source, then extend the induced partial order on the polygons of the subdivision up to some total order. Notice that, for each triangle, at most two edges of the adjacency graph are incoming.

By Lemma 2.4 below, there exists a rational peripherally unibranch curve $C_Q \in |L_{Q}|$, which is peripherally smooth, has a unique singular branch in $(\mathbb{C}^*)^2 \subset \text{Tor}(Q)$, and this branch is of multiplicity 2. Following the chosen above total order, we extend $C$ to a collection of rational peripherally unibranch curves for each triangle of the subdivision, each time matching the points on the common toric divisors determined by the preceding curves. By the patchworking theorem [22, Theorem 2.4] (see also Section 2.4 below) there exists a rational curve $\mathbb{P}^1 \to C_{P(\Delta)} \in |L_{P(\Delta)}|$, which is immersed everywhere but at one point of $\mathbb{P}^1$ that is mapped to a singular branch of multiplicity 2 in the torus $(\mathbb{C}^*)^2 \subset \text{Tor}(P(\Delta))$. Hence, $V_{\Delta,g}(A_2) \neq \emptyset$.

It follows from [10, Theorem in Section 6.1] that

- for any $1 \leq g < p_a(P(\Delta))$, a generic curve $C \in V_{\Delta,0}(A_2)$ admits a de-
formation into a curve $C' \in V_{\Delta,0}(A_2)$ by smoothing out any prescribed number of nodes, while keeping the rest of singularities,

- for any $0 \leq g < p_a(P(\Delta))$, the family of curves of genus $g$ in $|L_{P(\Delta)}|$ having one cusp and $p_a(P(\Delta)) - 1 - g$ nodes as its singularities, is smooth of expected dimension $|\Delta| + g - 2$.

Indeed, the sufficient condition for that asserted in [10, Theorem in Section 6.1] requires the number of cusps to be less than $-KC = |\Delta| \geq 5$.

So, it remains to prove that $V_{\Delta,0}(A_2) \neq \emptyset$. By Lemma 2.4 below, there exists a rational peripherally unibranch curve $C_Q \in |L_Q|$, which is peripherally smooth, has a unique singular branch in $(\mathbb{C}^*)^2 \subset \text{Tor}(Q)$, and this branch is of multiplicity 2. Following the chosen above total order, we extend $C$ to a collection of rational peripherally unibranch curves for each triangle of the subdivision, each time matching the points on the common toric divisors determined by the preceding curves. By the patchworking theorem [22, Theorem 2.4] (see also Section 2.4 below), there exists a rational curve $C_{P(\Delta)} \in |L_{P(\Delta)}|$, which has a unique singular local branch, and this branch is of multiplicity 2. Hence, $V_{\Delta,0}(A_2) \neq \emptyset$.

\[\square\]

### 2.2 Tropicalization

**Definition 2.2.** A plane tropical curve $(\Gamma, h)$ of degree $\Delta$ and genus $g \geq 0$ is called cuspidal if it contains one of the following cuspidal fragments (assumed to be a small neighborhood of the subset of $\Gamma$ described below)

- (A) a four-valent vertex of genus 0, whose incident edges are mapped by $h$ to four distinct lines,

- (B) a flat trivalent vertex of genus 0,

- (C) a bounded edge of genus 1,

- (D) a flat trivalent vertex and a four-valent vertex connected by two edges,

- (E) a trivalent vertex of genus 1,

while the remaining part is reduced and trivalent with the zero genera of all edges and vertices. In case (B), we call the cuspidal tropical curve exceptional if two of the edges incident to the flat vertex are ends of $\Gamma$. Then the third incident edge is bounded (since its weight is $> 1$ and $\Delta$ is primitive), and we include to the cuspidal fragment also that multiple edge and its second endpoint.
Theorem 2.3. Let $\Delta \subset \mathbb{Z}^2 \setminus \{0\}$ be a nondegenerate, primitive, balanced multiset satisfying the hypotheses of Lemma 2.1, $0 \leq g < p_a((P(\Delta)) - 1$, $n = |\Delta| + g - 2$, and $(n : \hat{C} \to C \mapsto \text{Tor}_K(P(\Delta)), w)$ an $n$-marked parameterized curve such that $C \in V_{\Delta, g}(A_2)$ and $x = \text{val}(w)$ is a configuration of $n$ distinct point in $\mathbb{R}^2$ in general position. Then the parameterized tropical limit of $C$ (in the sense of Section 1.3.2) consists of a regular cuspidal tropical curve $(\Gamma, p, h)$ of degree $\Delta$ and genus $g$ such that $h(p) = x$, and of the pair $(n'_0 : (\hat{C}^{(0)})' \to C^{(0)}, w(0))$ such that, for each trivalent vertex $V$ of $\Gamma$ outside the cuspidal tropical fragment, the corresponding limit curve $C^{(0)}_i$ in the toric $\text{Tor}(T)$ ($T$ being the triangle of the subdivision of $P(\Delta)$ dual to the trivalent vertex $h(V_i)$ of $\tau(C)$) is rational, nodal, peripherally unibranch and smooth. Furthermore, let $z_c \in C$ be the cuspidal singular point, $z_c(t) \in X_t$, $t \in (\mathbb{C}, 0)$, the corresponding section of the family $X \to C$ (see Section 1.3.7). Then $z_c(0) \in X_0$ belongs to the union of the subvarieties $\text{Tor}(S) \subset X$, where $S$ runs over the cells of the subdivision of $P(\Delta)$ dual to the cells of the reduced cuspidal fragment.

Proof. The statement follows from Lemma 1.2. First, we construct a parameterized tropical limit of $C$. By construction, the corresponding plane marked tropical curve $(\Gamma, p, h)$ has genus $g' \leq g$ and degree $\Delta'$ either equal to $\Delta$, or obtained from $\Delta$ by replacing certain disjoint submultisets of $\Delta$ by sums of their vectors. Observe that the case when $g' = g$, $\Delta' = \Delta$, and $(\Gamma, p, h)$ has only bivalent and non-flat trivalent vertices, all of genus zero, is not possible: by [23, Lemmas 3.5 and 3.9] a deformation of such a tropical limit yields a nodal curve. Thus, in view of the general position of the configuration $x$, $(\Gamma, p, h)$ must be of one of the types listed in Lemma 1.2(1):

- In case (i), we obtain either an elliptic non-flat trivalent vertex (cuspidal fragment E), or an elliptic edge (cuspidal fragment C).
- In case (ii) we obtain a rational four-valent vertex (cuspidal fragment A).
- In case (iii) we obtain a flat cycle (cuspidal fragment D).
- In case (iv) we obtain a flat trivalent vertex incident to two ends of $\Gamma$ (cuspidal fragment B).
- In case (v) we obtain a flat rational trivalent vertex incident to at least two bounded edges (cuspidal fragment B).
Finally, we notice that, from the above cited \cite[Lemmas 3.5 and 3.9]{23} follows that the tropicalization of the cuspidal singularity of $C$ lies in the cuspidal tropical fragment, except may be for the case of Lemma \cite[1.2]{1.2}liv). Below, in Lemma \cite[2.6]{2.6} we show that this case does not occur. \hfill $\square$

2.3 Multiplicities of tropical cuspidal fragments

Theorem \cite[2.10]{2.10} (Section \cite[2.4]{2.4} below) expresses the multiplicity of each unicuspidal tropical curve $(\Gamma, p, h)$ appearing in Theorem \cite[2.3]{2.3} (i.e., the number of unicuspidal algebraic curves tropicalizing to the given tropical curve) as the product of factors associated with the non-flat rational trivalent vertices and of a factor $\mu_c$ associated with its tropical cuspidal fragment $(\Gamma_c, p_c, h_c)$. In the following Sections \cite[2.3.1-2.3.4]{2.3.1-2.3.4} we compute the factors associated with cuspidal fragments of type A, B, and C. In Section \cite[3]{3} we show that, in most interesting cases, tropical curves with cuspidal fragments of type D and E do not appear at all. The treatment of types D and E can be found in Appendix.

2.3.1 Multiplicity of a four-valent vertex (fragment A)

Consider the cuspidal fragment $(\Gamma_c, h_c)$ consisting of an unmarked four-valent vertex $V$. Denote by $E_1, E_2$ the edges incident to $V$ and regularly oriented towards $V$ (while the two other edges $E_3, E_4$ are oriented outwards).

Denote by $\Delta(\Gamma_c, h_c)$ the multiset of vectors $\overline{a}_V(E_i), i = 1, 2, 3, 4$ and let $Q = P(\Delta(\Gamma_c, h_c))$. Denote also by $D_i$ the toric divisor in Tor$(Q)$ corresponding to $E_i$, $i = 1, 2, 3, 4$.

**Lemma 2.4.** Let $M \subset |L_Q|$ be the family of peripherally unibranch rational curves $C \in |L_Q|$ having a singular local branch centered in $(\mathbb{C}^*)^2 \subset \text{Tor}(Q)$. Then

(1) $M = \emptyset$ if the $h$-images of some two edges of $\Gamma$ lie on the same line, and in this case we set $\mu_c(\Gamma_c, h_c) = 0$. Otherwise $M$ is smooth of dimension 2 and consists of peripherally smooth curves with a unique singular local branch centered in the big torus $(\mathbb{C}^*)^2 \subset \text{Tor}(Q)$; furthermore, this singular branch is of type $A_2$.

(2) Let $M \neq \emptyset$, and let $z_1 \in D_1, z_2 \in D_2$ be generic points. Then the intersection of $M$ with the linear system $|L_Q(-z_1-z_2)| \subset |L_Q|$ is transversal.

(3) The number of intersection points $M \cap |L_Q(-z_1-z_2)|$ in the item (2) equals $\mu_c(\Gamma_c, h_c) \cdot (w(E_1)w(E_2))^{-1}$. where

$$
\mu_c(\Gamma_c, h_c) = |\overline{a}_V(E_1) \wedge \overline{a}_V(E_2)|.
$$
Proof. Up to an automorphism of \( \mathbb{Z}^2 \), we can assume that \( Q \) has vertices \((p, 0), (q, 0), (0, m), (r, s)\), where \( m, p, q, r, s > 0, q > p \geq m, q > r + s \). Let \( n : \mathbb{P}^1 \to C \in \mathcal{M} \). Denoting an affine coordinate on \( C \subset \mathbb{P}^1 \) by \( t \), we can suppose that

\[
\begin{align*}
n^{*} \operatorname{Tor}([ (p, 0), (0, m) ]) &= 0, \quad n^{*} \operatorname{Tor}([ (p, 0), (q, 0) ]) &= 1, \\
n^{*} \operatorname{Tor}([ (0, m), (r, s) ]) &= \infty, \quad n^{*} \operatorname{Tor}([ (r, s), (q, 0) ]) &= \tau \in \mathbb{C} \setminus \{0, 1\}.
\end{align*}
\]

Then, in the corresponding affine coordinates \( x, y \) in the torus \((\mathbb{C}^*)^2 \subset \operatorname{Tor}(Q)\), the curve \( C \) is parameterized by

\[
x = \frac{\alpha t^m}{(t - \tau)^s}, \quad y = \frac{\beta t^p (t - 1)^{q - p}}{(t - \tau)^{q - r}}, \quad \alpha, \beta \in \mathbb{C}^* .
\]

These formulas define a three-dimensional family, whose generic member is immersed in \((\mathbb{C}^*)^2\): indeed, the singular branch condition in \((\mathbb{C}^*)^2\) for some \( t = t^* \in \mathbb{C}^* \setminus \{0, 1, \tau\} \) reads

\[
\frac{dx}{dt}(t^*) = 0, \quad \frac{dy}{dt}(t^*) = 0 \quad \iff \quad \begin{cases} (m - s)t^* - m \tau = 0 \\ (qs - m(q - r))t^* - (ps - m(q - r)) = 0 \end{cases}
\]

which is a nontrivial condition on \( \tau \). Furthermore, if \( Q \) has a pair of parallel edges, i.e., either \( m = s \), or \( ps - m(q - r) = 0 \), one obtains from the above relations \( \tau = 0 \) contrary to the assumption \( \tau \in \mathbb{C}^* \setminus \{0, 1\} \), and hence, \( M = \emptyset \). Similarly, if \( Q \) is degenerate, i.e., the points \((0, m), (r, s), (q, 0)\) are collinear, then \( qs - m(q - r) = 0 \), that is \( \tau = \infty \), again a contradiction. Otherwise, we get uniquely defined values in \( \mathbb{C}^* \setminus \{0, 1\} \)

\[
t^* = \frac{ps - m(q - r)}{qs - m(q - r)}, \quad \tau = \frac{(m - s)(ps - m(q - r))}{m(qs - m(q - r))} .
\]

Furthermore, in the latter case, one can easily check that the intersection points of \( C \) with the toric divisors are smooth. For the divisor \( \operatorname{Tor}([ (p, 0), (q, 0) ]) \) this reduces to the relation

\[
\left. \frac{dx}{dt} \right|_{t=1} = \frac{(m - s) - m \tau}{(1 - \tau)^{s+1}} \neq 0 ,
\]

which immediately follows from [5]. The same argument works for the other toric divisors via an \( \mathbb{Z}^2 \)-automorphism placing another edge of \( Q \) on the horizontal axis.
Thus, $M \simeq (\mathbb{C}^*)^2$. If $q > 1$ is the order of the singular branch $B$ of $C \in M$, then \cite{14} Inequality (5) reads (in the notation of \cite{14}, $Z = \emptyset$ and $W$ is the set of intersection points of $C$ with the toric divisors):

$$-CK_{\text{Tor}}(Q) \geq 2 + (-CK_{\text{Tor}}(Q) - 4) + (q - 1) + 1 \implies q \leq 2,$$

and hence $q = 2$ as asserted in Lemma. Let us show that this branch $B$ is of type $A_2$, arguing on the contrary. Suppose that $C = C_{\alpha_0,\beta_0} \in M$ is given by \cite{7} with coefficients $\alpha_0, \beta_0$, and its singular branch $B$ is centered at some point $z \in (\mathbb{C}^*)^2$ and is of type $A_{2k}$, $k \geq 2$. The action of $(\mathbb{C}^*)^2$ on $M$ defines a section of the bundle $\mathbb{P}T(\mathbb{C}^*)^2$ that to each point $z' \in (\mathbb{C}^*)^2$ assigns the tangent line to the singular branch $B'$ centered at $z'$ of the corresponding curve $C' \in M$. Let $(L, z) \subset ((\mathbb{C}^*)^2, z)$ be the germ of an integral curve to this section, and let

$$\alpha = \varphi(\tau), \quad \beta = \psi(\tau), \quad \tau \in (\mathbb{C}, 0), \quad \varphi(0) = \alpha_0, \quad \psi(0) = \beta_0,$$

be the corresponding curve germ in the space of parameters $\alpha, \beta$. Consider the one-dimensional family $C_{\alpha, \beta} \subset M$, $\alpha = \varphi(\tau)$, $\beta = \psi(\tau)$, $\tau \in (\mathbb{C}, 0)$, and apply to it \cite{14} Inequality (5) taking into account the order of the branch $B$ is 2 and the intersection multiplicity of $B$ with $(L, z)$ is at least 4:

$$-CK_{\text{Tor}}(Q) \geq 2 + (-CK_{\text{Tor}}(Q) - 4 + 2) + 1 = -CK_{\text{Tor}}(Q) + 1,$$

a contradiction.

Let us fix the intersection points of $C \in M$ with the toric divisors $\text{Tor}(e_1)$ and $\text{Tor}(e_2)$, where $e_1 = [(p, 0), (q, 0)]$, $e_2 = [(p, 0), (0, m)]$. This means that the polynomial defining $C$ has truncations

$$(x^{p_1} + ay^{m_1})^d \quad \text{and} \quad x^p \left( \frac{x^b}{b} + 1 \right)^{q-p}$$

on the edges $e_1, e_2$ of $Q$, respectively, with given constants $a, b \in \mathbb{C}^*$ and $d = \gcd(m, p)$, $m_1 = \frac{m}{d}$, $p_1 = \frac{p}{d}$. Plugging the expressions \cite{7} and \cite{8} into these truncations, we obtain

$$m_1(q - p) = \frac{|\vec{e}_1 \wedge \vec{e}_2|}{\|e_1\|_Z \cdot \|e_2\|_Z}$$

solutions for the pair $(\alpha, \beta) \in (\mathbb{C}^*)^2$ in accordance with the assertion of Lemma. Similarly we treat the other choices of pairs of edges of $Q$. \hfill \Box

\footnote{This, in fact, is a reformulation of \cite{12} Theorem 2.}
2.3.2 Modification of the tropical limit and exceptional cuspidal tropical curves

In what follows we shall use modifications of the embedded and parameterized tropical limits in the sense of [23, Sections 3.5 and 3.6] and [15, Sections 2.5.8 and 2.5.9].

Given an embedded tropical limit \((\tau(C), \{C^{(0)}_k\}_{k=1,...,m})\) as in Section 1.3.1, suppose that two polygons \(P_i, P_j\) share a common side \(\sigma\), and let \(z \in \text{Tor}(\sigma)\) be a common point of the limit curves \(C^{(0)}_i \subset \text{Tor}(P_i), C^{(0)}_j \subset \text{Tor}(P_j)\). Applying to \(F(x, y) \in \mathbb{K}[x, y]\) a suitable automorphism of the torus \((\mathbb{K}^*)^2\) and multiplication by a monomial, we can place the edge \(\sigma\) on the vertical axis in \(\mathbb{RZ}^2\) and set \(\nu|_\sigma \equiv 0\) and \(\nu|_{\partial P(\Delta)\setminus \sigma} > 0\). Let \(b \in \mathbb{C}^*\) be the \(y\)-coordinate of \(z\) in \(\mathbb{C}^* \subset \text{Tor}(\sigma)\). The modification of the embedded tropical limit at the point \(z\) is the following operation:

- first, introduce the polynomial \(\tilde{F}(x, y) = F(x, y + b) \in \mathbb{K}[x, y]\); it induces a convex piecewise linear function on its Newton polygon \(\tilde{\nu} : P(\tilde{F}) \to \mathbb{R}\);

- suppose in addition that the polynomials \(F^{(0)}_i(x, y), F^{(0)}_j(x, y)\) defining the limit curves \(C^{(0)}_i, C^{(0)}_j(x, y)\) are not divisible by \(y - b\); then the subdivision of \(P(\tilde{F})\) into the linearity domains of \(\tilde{\nu}\) contains a fragment \(\tilde{P}\), containing the origin and bounded by the horizontal axis and the Newton diagrams at the origin for the polynomials \(F^{(0)}_i(x, y + b)\) and \(F^{(0)}_j(x, y + b)\) (see Figure 1(a) and [23, Section 3.5] and [15, Section 2.5.8] for more details);

- the subdivision of \(\tilde{P}\) and the corresponding limit curves determined by \(\tilde{F}\) are called a local modification of the embedded tropical limit at the point \(z\) along the toric divisor \(\text{Tor}(\sigma)\).

Geometrically, this means that perform an extra toric degeneration of the fibers of the family \(X \to \mathbb{C}\) in a neighborhood of the point \(z \in \text{Tor}(\sigma) \subset X_0\) (cf. [24, Section 2]).

Remark 2.5. The limit curves in the modification must satisfy the preassigned conditions imposed by the limit curves \(\{C^{(0)}_k\}_{k=1,...,m}\), which are the
fixed intersection points and intersection multiplicities with the toric divisors associated with those segments on $\partial \tilde{P}$ that do not lie on the horizontal axis. We shall see in Sections 2.3.3 and A1 that the fragment of a plane tropical curve dual to the subdivision of $\tilde{P}$ is regularly oriented (in the sense of Section 1.2(3)) when assuming that the edges dual to the above segments in $\partial \tilde{P}$ are oriented inwards.

Here we demonstrate the use of local modifications as defined above in the following lemma, while in Section 2.3.3 we use a slightly more general version of local modifications.

**Lemma 2.6.** Under the hypotheses of Theorem 2.3, an exceptional cuspidal tropical curve $(\Gamma, p, h)$ of degree $\Delta$ and genus $g$ cannot be a tropicalization of any curve $C \in V_{\Delta,g}(A_2)$ passing through $w$.

**Proof.** We argue on the contrary, assuming that such a curve $C \in V_{\Delta,g}(A_2)$ exists.

Note that the curve $(\Gamma_{\text{red}}, p_{\text{red}}, h_{\text{red}})$ is trivalent of genus $g$, without flat vertices, and having degree $\Delta'$ obtained from $\Delta$ by replacing two equal primitive vectors $\pi, \pi$ by one vector $2\pi$. Since $n = |\Delta'| - 1 + g$, and $h_*(\Gamma) = \tau(C)$ passes through $n$ points in general position, by [19 Proposition 4.3]
(or Lemma 2.2) the dual subdivision of $P(\Delta)$ consists of triangles and parallelograms. The boundary of $P(\Delta)$ is subdivided into segments of lattice length 1 and one segment $\sigma$ of lattice length 2, and there is a unique triangle $T$ (dual to the trivalent vertex of $\Gamma_{\text{red}}$ incident to the (unique) double end of $(\Gamma_{\text{red}}, h_{\text{red}})$) either containing $\sigma$ as a side, or having a parallel, congruent side joined with $\sigma$ via a sequence of parallelograms (see Figure 1(b)). It follows from Theorem 2.3 that the limit curves for all triangles different from $T$ are rational, peripherally unibranch and smooth, while the limit curve $C_T$ for the triangle $T$ is rational and unibranch along the two toric divisors associated with the incline sides of $T$ (see Figure 1(b)). Placing (via a suitable automorphism of $\mathbb{Z}^2$) $T$ on the plane $\mathbb{R}^2$ as shown in Figure 1(c), we obtain a parametrization of $C_T$ in the form

$$x = \alpha t^r, \quad y = \beta t^s R(t), \quad \alpha, \beta \in \mathbb{C}^*, \quad \text{deg } R = 2,$$

and hence $C_T \subset \text{Tor}(T)$ is immersed outside its two intersection points with the toric divisors $D_1 = \text{Tor}([(s, 0), (0, r)])$, $D_2 = \text{Tor}([(s + 2, 0), (0, r)])$. Hence, the tropical limit $z_c(0)$ of the cuspidal singularity belongs to one of the above toric divisors.

If $C_T$ is smooth at the intersection points with the toric divisors $D_1, D_2$, then we obtain the desired contradiction. Indeed, a local modification at $z_c(0) \in D_1$ along $D_1$ would consist of the triangle $\text{conv}\{(-1, 0), (1, 0), (0, r)\}$ and a rational, peripherally smooth, nodal curve in the corresponding toric surface (see details in Section 3.5), and none of the points of that curve can be a limit of a cuspidal singularity.

Suppose that $C_T$ is singular at the intersection point $z_c(0)$ with $D_1$. By Bézout bound, this is a singularity of order 2, i.e., of type $A_{2k}$. The local modification at the point $z_c(0) \in D_1$ along $D_1$ results (after a horizontal shift) in the triangle $T_1 = \text{conv}\{(0, 0), (3, 0), (1, 2k+2, 0)\}$ (see Figure 1(d)) and a rational limit curve $C_{T_1}$ unibranch along the toric divisors associated with the incline sides of $T_1$. As above one can show that $C_{T_1}$ is immersed outside the intersection point $z_1 \in \text{Tor}([(1, r), (3, 0)])$. If $r$ is odd then the lattice length of $[(1, r), (3, 0)]$ is 1; hence $C_{T_1}$ is smooth at $z_1$, which again yields the desired contradiction. Suppose that $r$ is even. Then $C_{T_1}$ can have at $z_1$ a singularity $A_{2k}$ ($k \geq 1$) transversal to the toric divisor. A local modification at $z_1$ along $\text{Tor}([(1, r), (3, 0)])$ leads to the triangle $T_2 = \text{conv}\{((0, 0), (1, 2), (2k + 2, 0)\}$ (see Figure 1(e)) and a rational curve $C_{T_2} \subset \text{Tor}(T_2)$. We claim that $C_{T_2}$ is immersed, and hence the desired contradiction. Indeed, its affine part $C_{T_2} \cap \mathbb{C}^2$ is an image of $\mathbb{C}^*$, which is naturally projected onto the torus $\mathbb{C}^*$ in the horizontal axis, and then no ramifications are possible as required. □
2.3.3 Multiplicity of a flat trivalent vertex (fragment B)

Suppose that \((\Gamma, p, h)\) contains a tropical cuspidal fragment \((\Gamma_c, h_c)\) which is a neighborhood of a flat trivalent vertex \(V\). Denote the weights of the incident edges by \(m, m_1, m_2\), where \(m = m_1 + m_2\) (see Figure 2(a)). It follows from Lemma 2.6 that the edge of weight \(m\) and at least one of the other edges are bounded (see Figure 2(a)). We will consider the lower fragment shown in Figure 2(a), since, in the upper one, \(m_1 = 1\), and one can append a trivalent vertex of Mikhalkin multiplicity 1 without affecting the multiplicity of the whole fragment. The \(h\)-image of this fragment is dual in the subdivision of \(P(\Delta)\) to the union of three polygons: triangles \(T_1, T_2\) dual to the trivalent vertices \(h(V_1), h(V_2)\) and a trapeze \(T\) dual to \(h(V_3)\) (see Figure 2(b)). Let \(e_1 = T_1 \cap T\) and \(e_2 = T_2 \cap T\) be the parallel bases of \(T\). The corresponding limit curves \(C_i \subset \text{Tor}(T_i)\), \(i = 1, 2\), are rational, nodal, peripherally unibranch and smooth, while the limit curve \(C \subset \text{Tor}(T)\) splits into two irreducible components, \(m_1 C'\), where \(C' \simeq \mathbb{P}^1\) intersects only the toric divisors \(\text{Tor}(e_1), \text{Tor}(e_2)\), and \(C''\) rational, nodal, peripherally unibranch and smooth, disjoint from \(\text{Tor}(e_1)\) and intersecting \(\text{Tor}(e_2)\) at the same point \(z\) as \(C'\). Note that, in the parameterized tropical limit, the component \(m_1 C'\) is an image of an \(m_1\)-multiple cover \(n : \mathbb{P}^1 \to C'\) ramified at the intersection points with \(\text{Tor}(e_1), \text{Tor}(e_2)\).

The local modification at the point \(z\) along the edge \(e_2\) yields a polygons \(\tilde{P}\) of one of the shapes shown in Figure 2(c,d,e) according as \(m_1 > m_2\), \(m_1 = m_2\), or \(m_1 < m_2\). The slopes of the piecewise linear function \(\tilde{\nu}\) (see Section 2.3.2) restricted on the edges

\[
[(0,0), (1, m_1)], [(1, m_1), (2, m)], [(2, m), (3, 0)]
\]

of \(\tilde{P}\) are induced by the slopes of the function \(\nu\) on the polygons \(T, T_1, T_2\), respectively, and hence the graph of \(\tilde{\nu}\) restricted to the union of the above three edges does not lie in one plane. This implies, in particular, that \(\tilde{P}\) is necessarily subdivided, and the point \((1, m_1)\) is a vertex of the subdivision.

The local tropical limit associated with \(\tilde{P}\) meets the following requirements: the union \(C^{(0)}_{\tilde{P}}\) of the limit curves has arithmetic genus zero and it develops a cusp (and few nodes) in the deformation along the family \(\{C^{(t)}\}_{t \in (\mathbb{C}, 0)}\).

To obtain a more precise information on the considered local modification, we perform an extra transformation. Observe that the coefficients of \(\text{Ini}(\tilde{F}^{e_2})\) of \(x^2 y^j\), \(0 \leq j < m\), contain \(t\) to a positive power. From the convexity of the function \(\nu_{\tilde{F}}\) we then derive that the exponents of \(t\) in the coefficients of \(x y^j\), \(0 \leq j < m_1\), in \(\text{Ini}(\tilde{F}^{e_1})\) are greater that that for \(x y^{m_1}\). Hence, for
Figure 2: Cuspidal flat vertex and its modification, I
some $\zeta(t)$ vanishing at $t = 0$, the corrected polynomial $\widetilde{F}(x, y)$ obtained by the coordinate change

$$(x, y) \mapsto (x, y + \zeta(t)),$$  

(9)

does not contain the monomial $xy^{m_1-1}$, while it determines the same shape of the polygon $\widetilde{P}$. The meaning of this extra transformation is that it excludes the segment $[(1, 0), (1, m_1)]$ as an element of a possible subdivision of $\widetilde{P}$, since, otherwise, one would encounter at least two intersection points of the limit curves with the toric divisor $\text{Tor}([(1, 0), (1, m_1)])$ and hence the jump of the arithmetic genus of the union of the limit curves in the modification contrary to the condition that it must be zero.

Next, we note that a subdivision of $\widetilde{P}$ into triangles, or into triangles and parallelograms is not possible, since in such a case, no cuspidal singularity can develop in the deformation $\{C(t)\}_{t \in (\mathbb{C}, 0)}$, and this can be established using the argument applied in a similar situation in the proof of Lemma 2.6. Thus, there must be a quadrangle $Q$ without parallel sides and a corresponding limit curve having a cuspidal singularity in the big torus $(\mathbb{C}^*)^2 \subset \text{Tor}(Q)$, and we claim that the only possible subdivision in case $m_1 > m_2$ is either as shown in Figure 2(f) or as shown on one of the Figures 2(g,j,k,l), in case $m_1 = m_2$ is as shown in Figure 2(h), in case $m_1 < m_2$ is as shown in Figure 2(i).

Furthermore, the case depicted in Figure 2(g,j,k,l) can be neglected, since they do not satisfy the convexity condition for the function $\tilde{\nu}$ (we leave this elementary geometry exercise to the reader).

We are now in a position to determine all possible multiplicities of the considered fragment. Namely, the multiplicity we are looking for counts in how many ways one can find a tuple of limit curves corresponding to the trivalent vertices $V_1, V_2, V_3$ and to a pair of a trivalent and a four-valent vertex of the local modification. The answer depends on the induced regular orientation on the fragment shown in Figure 2(a) and on the fragment dual to a suitable subdivision shown in Figures 3(f,g,i,l).

Suppose that no horizontal edge contains a marked point. Then possible orientations are shown in Figure 3(a) (here orientations of non-horizontal edges may be chosen other way, provided that precisely two edges are oriented towards a non-flat trivalent vertex). Denote by $m'_1, m'_2, m'_3, m'_4$ the weights of the non-horizontal edges oriented inside the fragment. The induced orientation on the fragment dual to the subdivision of $\widetilde{P}$ in all the cases is as shown in Figure 3(b). Applying [23, Lemmas 3.5 and 3.9] and
Figure 3: Cuspidal flat vertex and its modification, II
Lemma 2.4 we obtain the multiplicities
\[ \mu(V_1) \mu(V_2) \mu(V_3) / m'_1 m'_2 m'_3 m'_4 \cdot \mu_c(\Gamma_c, h_c), \]
where \( \mu_c(\Gamma_c, h_c) \) equals
\[ (m + m_2) m_1 / mm_2, \quad m + m_2 / m, \quad m + m_2 / m_2 \]
for the upper, middle, and lower oriented fragments shown in Figure 3(a), respectively.

Suppose now that the fragment contains a marked point on a horizontal edge of the cuspidal fragment \((\Gamma_c, h_c)\) (shown by asterisk in Figures 3(d,e,f)). Here three of the non-horizontal edges are oriented from outside, and we denote their weights by \(m'_1, m'_2, m'_3\). As compared to the preceding case, the computation of the multiplicity requires also the modification of the marked point condition (cf. \cite{15} Section 2.5.9). Analytically, this modification means an additional equation on \(\text{Ini}(\zeta)\) for the parameter \(\zeta\) that appeared in the coordinate change (9). The considered marked point corresponds to a point \(w = (\xi(t), b + \eta(t))\) in the given configuration \(w\), where without loss of generality we can assume \(\eta(t)\) to be of an arbitrarily high order in \(t\). After the coordinate changes \((x, y) \mapsto (x, y + b), (x, y) \mapsto (x, y + \eta)\) performed above, we obtain the point \(\tilde{w} = (\xi(t), \eta(t) - \zeta)\), and its tropical image appears on one of the edges of the four-valent vertex of the fragment shown in Figure 3(b): in accordance with the original position shown in Figures 3(d,e,f), the new position is shown by asterisk in Figure 3(c). The condition to pass through \(\tilde{w}\) yields
\[ \hat{F}(\xi(t), \eta(t) - \zeta) = 0. \]
(11)
The minimal exponent of \(t\) in the left-hand side of (11) must occur twice, and it happens at the two vertices of one of the incline sides of the quadrangle (see Figure 2(d)), namely, for the side is dual to the edge containing the marked point (see Figure 3(c)). Let
\[ a_{1,m_1} t^\alpha + \text{h.o.t.,} \quad a_{2,0} t^\beta + \text{h.o.t.,} \quad a_{2,m} + \text{h.o.t.,} \quad a_{3,0} t^\gamma + \text{h.o.t.} \]
be the coefficients of \(\hat{F}\) at the vertices of \(Q\), where \(a_{1,m_1}, a_{2,0}, a_{2,m}, a_{3,0} \in \mathbb{C}^*\) are determined by the limit curves associated with the subdivision shown in Figure 2(d). Relation (11) yields one of the equations
\[ \left\{ \begin{array}{l}
  a_{1,m_1} t^\alpha (\eta(t) - \zeta)^{m_1} + a_{2,0} t^\beta \xi(t) + \text{h.o.t.} = 0, \\
  a_{1,m_1} t^\alpha + a_{2,m} \xi(t)(\eta(t) - \zeta)^{m_2} + \text{h.o.t.} = 0, \\
  a_{2,m} (\eta(t) - \zeta)^m + a_{3,0} t^\gamma \xi(t) + \text{h.o.t.} = 0
\end{array} \right. \]
(12)
in accordance with the case shown in Figures 3(d,e,f), respectively. Correspondingly, we obtain \( m_1, m_2, \) or \( m \) solutions for \( \text{Ini}(\zeta) \). Combining this result with the count of the limit curves related to subdivision shown in Figure 2(d), we finally obtain the multiplicity

\[
\mu(V_1)\mu(V_2)\mu(V_3) \cdot \mu_c(\Gamma_c, h_c),
\]

where \( \mu_c(\Gamma_c, h_c) \) equals

\[
\frac{(m + m_2)\cdot m_{1}}{m_{2}} \cdot \frac{m + m_2}{m} \cdot \frac{m + m_2}{m_2} \quad (13)
\]

in accordance with the cases shown in Figures 3(d,e,f), respectively.

### 2.3.4 Multiplicity of an elliptic edge (fragment C)

Let \((\Gamma, p, h)\) have a bounded open edge of genus 1 and weight \( m \) as its cuspidal tropical fragment \((\Gamma_c, p_c, h_c)\). We extend \((\Gamma_c, p_c, h_c)\) to a fragment \((\Gamma', p', h')\) by adding the endpoints \(V_1, V_2 \in \Gamma^0\) of this edge and other edges of \(\Gamma\) incident to \(V_1, V_2\) (see Figure 4(a)). By Theorem 2.3, the vertices \(V_1, V_2\) are trivalent, and we can suppose that the triangles \(T_1, T_2\) of the subdivision of the Newton polygon \(P\) of \((\Gamma, p, h)\) that are dual to \(h(V_1), h(V_2)\), respectively, share a common side \(S\) dual to the edge \(e = h(\Gamma_c)\), see Figure 4(b). By Theorem 2.3, the limit curves \(C_1 \subset \text{Tor}(T_1), C_2 \subset \text{Tor}(T_2)\) are nodal rational, smooth and unibranch along the toric divisor \(\text{Tor}(e)\). Thus, performing a modification as in Section 2.3.3, we obtain a new fragment dual to the lattice triangle \(T = \text{conv}\{(0,0), (0,2), (m,1)\}\) (cf. [23, Section 3.5] and [15, Section 2.5.8]), and the limit curve \(C \subset \text{Tor}(T)\) must have a singularity more complicated than a node (a tropical limit of the cusp) in the torus \((\mathbb{C}^*)^2 \subset \text{Tor}(T)\) and genus \(\leq 1\). By [23, Lemma 3.9], a rational curve with Newton triangle \(T\) must be nodal, and hence, \(C\) is elliptic. Furthermore, \(C\) cannot be immersed, since an immersed singularity deforms into a cusp (and may be other singularities) with a jump of the genus.

**Lemma 2.7.** Let \(m \geq 3\), and \(T\) be the triangle with vertices \((0,0), (0,2),\) and \((m,1)\). Then an elliptic curve \(C \subset \text{Tor}(T)\), defined by a polynomial \(F(x,y)\) with the Newton polygon \(T\), has at most one singular local branch. Furthermore,

(i) the family \(M \subset |\mathcal{L}_T|\), parameterizing elliptic curves having one singular branch, is smooth of dimension 3;
Figure 4: Cuspidal elliptic edge and its modification

(ii) $M$ intersects transversally with the linear system $|L_T(-z_1-z_2)| \subset |L_T|$ of curves passing through the points $z_1, z_2$ chosen generically in the toric divisors

$$\text{Tor}([(0,2),(m,1)]), \text{Tor}([(0,0),(m,1)]) \subset \text{Tor}(T),$$

respectively.

Proof. Clearly, the singularities of $C \in M$ are of types $A_k, k \geq 1$. Suppose that they are $A_{2i}, A_{2j}$ (singular branches) and some $A_{k_1}, ..., A_{k_s}$ ($s \geq 0$). Following the proof of [21, Theorem 1.1], we can show that the dimension of the germ at $C$ of the equisingular family does not exceed

$$\dim |L_T| - \delta(C) - 2 = m + 2 - (m - 2) - 2 = 2,$$  \hspace{1cm} (14)$$

where $\delta(C)$ is the total $\delta$-invariant of $C$. To prove this, we choose projections of the germ of $|L_T|$ at $C$ to versal deformation bases of the singular points of $C$ and consider the germ at $C$ of the family $M_1 \subset |L_T|$, which is the intersection of pull-backs of the equiclassical strata in the versal deformations of $A_{2i}$ and $A_{2j}$, and the equigeneric strata in the versal deformations of the rest of singularities (see details in [8]). The germ at $C$ of the equisingular
stratum is contained in $M_1$. According to [8, Section 4.3] and [21, Section 2] the tangent cone to $M_1$ can be identified with $H^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D - D_0 - D_{ec}))$, where $\tilde{C} \rightarrow C$ is the normalization, $D$ is the pull-back of $c_1(\mathcal{O}_C \otimes \mathcal{L}_T)$, $D_0$ is the double-point divisor, $D_{ec}$ is the pull-back of the singular points of type $A_{2i}$ and $A_{2j}$. Here $g(\tilde{C}) = 1,$

$$\deg(D - D_0 - D_{ec}) = C^2_{\text{Tor}(T)} - 2\delta(C) - 2 = 2m - 2(m - 2) - 2 = 2$$

$$> 2g(\tilde{C}) - 2 = 0,$$

and hence, (14) follows.

However the dimension of the equisingular stratum must be at least 3 in view of the action of the torus $(\mathbb{C}^*)^2$ and the shifts $x \mapsto x + a$, $a \in \mathbb{C}$.

A similar computation for $C$ with one singular branch proves statements (i) and (ii) as well. \hfill \Box

**Definition 2.8.** For a pair of integers $1 \leq p \leq q$, introduce the convex lattice polygons

$$\sigma_{p,q}(n) = \text{conv} \left\{ (i, j) \in \mathbb{Z}^2 : i, j \geq 0, pi + qj \leq n \right\}, \quad n \geq 1.$$ 

For a pair of convex polygons $P_1, P_2 \subset \mathbb{R}_2^2$, denote by $\langle P_1, P_2 \rangle$ their mixed area. Put $\theta(1) = \theta(2) = 0$, $\theta(3) = 2$ and, for $k \geq 2$,

$$\theta(2k + 1) = 2(2k + 1)(\langle \sigma_{2,3}(k), \sigma_{2,3}(k + 1) \rangle + \langle \sigma_{1,2}(k - 1), \sigma_{1,2}(k) \rangle),$$

$$\theta(2k) = 4k(\langle \sigma_{2,3}(k - 2), \sigma_{2,3}(k - 1) \rangle + \langle \sigma_{1,2}(k - 1), \sigma_{1,2}(k) \rangle).$$

**Lemma 2.9.** Let $m \geq 1$, and let $T$ be the triangle with vertices $(0,0)$, $(0,2)$, and $(m,1)$. There are exactly $\theta(m)$ polynomials $F(x,y) = y^2 - 2yf(x) + 1$ with Newton polygon $T$ and such that

- $f(x)$ is a monic polynomial of degree $m$ with the vanishing coefficient of $x^{m-1}$,
- the curve defined by $F$ in the toric surface $\text{Tor}(T)$ is elliptic and has a singular local branch.

Moreover, each of the above curves has one ordinary cusp and $(m-3)$ nodes.
Proof. The equation \( F(x, y) = 0 \) yields \( y = f(x) \pm \sqrt{f(x)^2 - 1} \), and hence a singular branch of \( C = \{ F = 0 \} \) corresponds to a root of \( f(x)^2 - 1 \) of multiplicity \( 2s + 1 \), \( 1 \leq s < m/2 \), while other singularities correspond to roots of even multiplicity. Hence, \( f(x)^2 - 1 = (x - \xi)^2 Q(x)^2 R_3(x) \), where \( \xi \in \mathbb{C}, \deg Q = m - s - 2, \deg R_3 = 3 \), where \( Q(\xi)R_3(\xi) \neq 0 \). Note that \( R_3(x) \) has no multiple roots, since otherwise the curve \( C \) would be rational, but a rational curve with the Newton polygon \( T \) must be nodal [23 Lemma 3.5]. In particular, this yields that \( \gcd(Q, f) = 1 \). We will canonically normalize \( F(x, y) \) by substitution of \( x + \xi \) for \( x \).

There are no solutions for \( m \leq 2 \). If \( m = 3 \), we obtain two solutions \( f(x) = x^3 \pm 1 \).

Suppose that \( m > 3 \). Since \( \gcd(f - 1, f + 1) = 1 \), we have for \( m = 2k + 1, k \geq 2 \),

\[
\begin{align*}
\text{either} & \quad \begin{cases} 
  f(x) - 1 = x^{2s+1}S_{k-s}(x)^2 \\
  f(x) + 1 = T_{k-1}(x)^2R_3(x) 
\end{cases} \\
\text{or} & \quad \begin{cases} 
  f(x) + 1 = x^{2s+1}S_{k-s}(x)^2 \\
  f(x) - 1 = T_{k-1}(x)^2R_3(x) 
\end{cases} \\
\text{or} & \quad \begin{cases} 
  f(x) - 1 = x^{2s+1}S_{k-s-1}(x)^2R_2(x) \\
  f(x) + 1 = T_k(x)^2R_1(x) 
\end{cases} \\
\text{or} & \quad \begin{cases} 
  f(x) + 1 = x^{2s+1}S_{k-s-1}(x)^2R_2(x) \\
  f(x) - 1 = T_k(x)^2R_1(x) 
\end{cases} 
\end{align*}
\]

where \( R, S, T \) stand for monic polynomials with the subindex designating the degree, pairwise coprime in each system \([15] - [18] \), and similarly for \( m = 2k, k \geq 2 \),

\[
\begin{align*}
\text{either} & \quad \begin{cases} 
  f(x) - 1 = x^{2s+1}S_{k-s-1}(x)^2R_1(x) \\
  f(x) + 1 = T_{k-1}(x)^2R_2(x) 
\end{cases} \\
\text{or} & \quad \begin{cases} 
  f(x) + 1 = x^{2s+1}S_{k-s-1}(x)^2R_1(x) \\
  f(x) - 1 = T_{k-1}(x)^2R_2(x) 
\end{cases} \\
\text{or} & \quad \begin{cases} 
  f(x) - 1 = x^{2s+1}S_{k-s-2}(x)^2R_3(x) \\
  f(x) + 1 = T_k(x)^2 
\end{cases} \\
\text{or} & \quad \begin{cases} 
  f(x) + 1 = x^{2s+1}S_{k-s-2}(x)^2R_3(x) \\
  f(x) - 1 = T_k(x)^2 
\end{cases} 
\end{align*}
\]
where $R, S, T$ stand for monic polynomials with the subindex designating the degree, pairwise coprime in each of the systems (19)-(22).

Let us analyze system (15). Differentiating both the equations, we obtain

\[ x^{2s}S_{k-s}((2s+1)S_{k-s} + 2xS'_{k-s}) = T_{k-1}(2T'_{k-1}R_3 + T_{k-1}R'_3) . \]

It follows, that $s = 1$ and that both $S_{k-1}(x)$ and $T_{k-1}$ have no multiple roots, that is, all solutions represent curves with an ordinary cusp and $(m - 3)$ nodes. Furthermore,

\[
\begin{align*}
3S_{k-1} + 2xS'_{k-1} &= (2k + 1)T_{k-1} \\
2T'_{k-1}R_3 + T_{k-1}R'_3 &= (2k + 1)x^2S_{k-1}
\end{align*}
\]

Plugging $S_{k-1} = x^{k-1} + \alpha_1 x^{k-2} + \ldots + \alpha_{k-1}, T_{k-1} = x^{k-1} + \beta_1 x^{k-2} + \ldots + \beta_{k-1}$, we obtain from the former equation in (23)

\[ \beta_i = \frac{2k + 1 - 2i}{2k + 1} \alpha_i, \quad i = 1, \ldots, k - 1 , \]

which we together with $R_3 = x^3 + \gamma_1 x^2 + \gamma_2 x + \gamma_3$ plug to the second equation and, subsequently equating the coefficients of $x^{k-1}, \ldots, x^2, x, 1$, obtain the following system of quasihomogeneous equations, where each variable $\alpha_i, \gamma_i$ has weight $i, i \geq 1$:

\[
\sum_{r \geq 1, \ s \geq 0 \atop r + s = i} a_{rs} \gamma_r \alpha_s, \quad i = 1, \ldots, k - 1 ,
\]

\[
0 = \sum_{r \geq 1, \ s \geq 0 \atop r + s = i} a_{rs} \gamma_r \alpha_s, \quad i = k, k + 1 ,
\]

where $a_{rs} > 0$ for all $r, s$, and we set $\alpha_0 = 1, \alpha_i = 0$ as $i \geq k$. Finally, we add one more equation coming from the comparison of the constant terms in (15)

\[ 2 = \frac{9}{(2k + 1)^2} \alpha^2_{k-1} \gamma_3 . \]

Substituting expressions (24) to (25) and (26) several times, we end up with the quasihomogeneous equations

\[
\sum_{r,s,t \geq 0 \atop r + 2s + 3t = i} a_{rst} \gamma_r^2 \gamma_s^2 \gamma_t^2 = 0, \quad i = k, k + 1 ,
\]

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\[
\sum_{r,s,t \geq 0 \atop r+2s+3t=2k+1} a_{rst} \gamma_1^r \gamma_2^s \gamma_3^t = 2, \tag{28}
\]
with all coefficients \(a_{rst}\) positive. Consider solutions to the system (27) as intersections of two curves on a suitable toric surface. Notice that there are no intersections on toric divisors, since otherwise one would get a one-parameter family of solutions to the system (27), (28) contrary to Lemma 2.7. Hence all intersection are in the big torus, and they all are simple again by Lemma 2.7. Passing to the variables \(\gamma_2 \gamma_1^{-2}, \gamma_3 \gamma_1^{-3}\), we obtain in (27) two polynomial equations with Newton polygons \(\sigma_{2,3}(k)\) and \(\sigma_{2,3}(k+1)\), respectively, which yield, by Bernstein-Koushnirenko theorem \(\text{[3]}\), \(\langle \sigma_{2,3}(k), \sigma_{2,3}(k+1) \rangle\) solutions. Each of these solutions yields \((2k+1)\) values for \(\gamma_1\) from (28).

In the same manner, we treat systems (16), (21), and (22), obtaining respectively \((2k+1)\langle \sigma_{2,3}(k), \sigma_{2,3}(k+1) \rangle\) solutions in the first case and \(2k\langle \sigma_{2,3}(k-2), \sigma_{2,3}(k-1) \rangle\) solutions in each of the two remaining cases.

The same approach we explore for system (17). Equating the derivatives of the right-hand sides, we get
\[
x^{2s}Q_{k-s-1}(((2s+1)R_2 + xR_2')S_{k-s-1} + 2xR_2S_{k-s-1}') = T_k(2T_k'R_1 + T_kR_1')
\]
and, as an immediate consequence, \(s = 1\) and the facts that \(S_{k-2}\) and \(T_k\) have only simple roots. Furthermore,
\[
\begin{cases}
(2s+1)R_2 + xR_2'S_{k-s-1} + 2xR_2S_{k-s-1}' = (2k+1)T_k \\
2T_k'R_1 + T_kR_1' = (2k+1)x^2Q_{k-2}
\end{cases}
\]
Substituting
\[
S_{k-2} = x^{k-2} + \alpha_1 x^{k-3} + \ldots + \alpha_{k-2}, \quad T_k = x^k + \beta_1 x^{k-1} + \ldots + \beta_k,
\]
\[
R_2 = x^2 + \gamma_1 x + \gamma_2, \quad R_1 = x + \delta_1
\]
to the above system, we, first, obtain
\[
\beta_i = \frac{2k+1-2i}{2k+1} \alpha_i + \sum_{r,s \geq 1 \geq 0 \atop r+s=i} a_{rs}\alpha_r \gamma_s, \quad i = 1, \ldots, k,
\]
where \(a_{rs} > 0\) for all \(r,s\), we set \(\alpha_r = 0\) as \(r \geq k-1\) and \(\gamma_s = 0\) as \(s \geq 3\), and then
\[
\frac{4i(2k+1-i)}{2k+1} \alpha_i = \sum_{r,s,t \geq 0, t < i \atop r+s+t=i} a_{rst} \gamma_r \delta_s \alpha_t, \quad i = 1, \ldots, k-2, \tag{29}
\]
\[ 0 = \sum_{r,s,t \geq 0, \, t < i \atop r + s + t = i} a_{rst} \gamma_r \delta_s \alpha_t, \quad i = k - 1, k, \quad (30) \]

where \( a_{rst} > 0 \) for all \( r, s, t \), and we set \( \delta_0 = \gamma_0 = \alpha_0 = 1, \, \delta_r = 0 \) as \( r \geq 2, \gamma_s = 0 \) as \( s \geq 3, \alpha_t = 0 \) as \( t \geq k - 1 \). Iteratively plugging the right-hand sides of \((29)\) to \((30)\), we exclude there \( \alpha_i \)'s and come to the system

\[ 0 = \sum_{r,s,t \geq 0, \, t < i \atop r + 2s + t = i} b_{rst} \gamma_1^{-r} \gamma_2^{-s} \delta_1^t, \quad i = k - 1, k, \quad b_{rst} > 0 \text{ for all } r, s, t, \]

which together with the constant term relation in \((17)\)

\[ 2 = \beta_2^k \delta_1 \]

gives in total \( 2k[\sigma_{1,2}(k-1), \sigma_{1,2}(k)] \) solutions. Similarly we solve the systems \((18), (21), \text{ and } (22)\), completing the proof of Lemma. \(\Box\)

We are now in a position to determine the multiplicity \( \mu_c(\Gamma_c, p_c, h_c) \); more precisely, we count how many compatible triples \((C_1, C_2, C)\), where \( C_1, C_2 \) are rational, peripherally smooth and unibranch, \( C \) as in Lemma \(2.9\) satisfy appropriate initial data. Compatibility means that the intersection points of \( C \) with the toric divisors \( \text{Tor}([0,0], (1,m)), \text{Tor}([1,m], (2,0)] \) are determined by \( C_1, C_2 \), respectively. The initial data depend in the regular orientation of the fragment induced from \((\Gamma, p, h)\).

If \( \Gamma_c \) does not contain a marked point, i.e., \( p_c = \emptyset \), then the regular orientation of the fragment \((\Gamma', h')\) is as shown in Figure \((4d)\) (up exchange of \( V_1, V_2 \)). Denote by \( m_1, m_2, m_3 \) the weights of the outer edges oriented towards \( V_1 \) or \( V_2 \). It then follows from \([23, \text{Lemma 3.9}]\) and Lemma \(2.7\) that the number of required triples \((C_1, C_2, C)\) equals

\[ \frac{\mu(V_1) \mu(V_2)}{m_1 m_2 m_3} \cdot \mu_c(\Gamma_c, h_c), \quad \text{where} \quad \mu_c(\Gamma_c, h_c) = \frac{\theta(m)}{m}. \quad (31) \]

If \( \Gamma_c \) contains a marked point, then the regular orientation looks like in Figure \((4c)\). Denote by \( m_1, m_2 \) the weights of the outer edges oriented towards \( V_1 \) or \( V_2 \). In addition to \([23, \text{Lemma 3.9}]\) and Lemma \(2.9\) we apply the result of \([13, \text{Section 2.3.9}]\) that takes into account the condition to pass through the given marked point. It follows that the number of required triples \((C_1, C_2, C)\) equals

\[ \frac{\mu(V_1) \mu(V_2)}{m_1 m_2} \cdot \mu_c(\Gamma_c, p_c, h_c), \quad \text{where} \quad \mu_c(\Gamma_c, p_c, h_c) = \frac{\theta(m)}{m}. \quad (32) \]
2.4 Patchworking

The following statement is complementary to Theorem 2.3.

**Theorem 2.10.** Let \( \Delta \subset \mathbb{Z}^2 \setminus \{0\} \) be a nondegenerate, primitive, balanced multiset satisfying the hypotheses of Lemma 2.1, \( 0 \leq g < p_a((P(\Delta))^2) - 1 \), \( n = |\Delta| + g - 2 \), and \( w \) a configuration of \( n \) distinct points in \((\mathbb{R}^*)^2\) such that \( x = \text{val}(n(w)) \) is a set of \( n \) distinct point in \( \mathbb{R}^2 \) in general position. Let \((\Gamma, \mathbf{p}, h)\) be a plane cuspidal \( n \)-marked tropical curve of degree \( \Delta \) and genus \( g \) such that \( h(p) = x \) and its cuspidal tropical fragment \((\Gamma_c, \mathbf{p}_c, h_c)\) is of type \( A \), \( B \), or \( C \). Assume in addition that, in case \( B \), the curve \((\Gamma, \mathbf{p}, h)\) is not exceptional. Then the number of curves \( C \in V_{\Delta, g}(A_2) \) passing through \( w \) and tropicalizing to \((\Gamma, \mathbf{p}, h)\) equals

\[
\mu_c(\Gamma_c, \mathbf{p}_c, h_c) \cdot \prod_{V \in \Gamma^0 \setminus \Gamma_c} \mu(V) ,
\]

where the value \( \mu_c(\Gamma_c, \mathbf{p}_c, h_c) \) should be appropriately chosen from formulas (6), (10), (13), (31), or (32).

**Proof.** We closely follow the ideas of [23]. Note that the number (33) counts in how many ways the given curve \((\Gamma, \mathbf{p}, h)\) can be enhanced to an admissible tropical limit, that is, the tropical limit which matches the given configuration \( w \) and includes the modification of the extended cuspidal tropical fragment as defined in Section 2.3.1, 2.3.3, and 2.3.4, the modifications along multiple edges outside the cuspidal tropical fragment, and the choice in the conditions to pass through a fixed point in \( w \). Indeed (cf. [15, Section 2.5.7, 2.5.8, and 2.5.9], [19, Sections 4 and 5], and [23, Section 3.7]), following the order determined by the regular orientation of \( \Gamma \setminus \mathbf{p} \), we obtain

\[
\mu_c(\Gamma_c, \mathbf{p}_c, h_c) \cdot \prod_{V \in \Gamma^0 \setminus \Gamma_c} \mu(V) \cdot \left( \prod_{E \in \Gamma^1 \setminus (\Gamma_\infty \cup \Gamma_c)} m(E) \cdot \prod_{E \in \Gamma^1 \setminus (\Gamma_\infty \cup \Gamma_c)} m(E) \right)^{-1}
\]

choices for the admissible collections of limit curves including the modifications of the cuspidal fragment. The modifications along the bounded edges outside the bounded fragment multiply the latter value by

\[
\prod_{E \in \Gamma^1 \setminus (\Gamma_\infty \cup \Gamma_c)} m(E) ,
\]

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and the choice in the conditions to pass through a given point of \( w \) provide an additional factor

\[ \prod_{E \in \Gamma \setminus \Gamma_1 \setminus \Gamma_\infty \cap p \neq \emptyset} m(E). \]

Then we apply the patchworking statement in [22, Theorem 2.4] to conclude that an enhanced tropical limit as above lifts to a unique family of curves \( C(t) \subset X_t \simeq \text{Tor}(P(\Delta)), t \in (\mathbb{C}, 0) \setminus \{0\} \), (cf. Sections 1.3.1 and 1.3.2) such that \( C(t) \in V_{\Delta, \varphi}(A_2) \) and \( c(t) \supset w(t) \) as \( t \in (\mathbb{C}, 0) \setminus \{0\} \).

We only make few remarks. The limit curves in [22, Theorem 2.4] assumed to have semiquasihomogeneous singularities on the toric divisors, but the proof literally extends to the case of Newton nondegenerate singularities (as those which occur in Section 2.3.3). In Lemma 2.4 and consequently, in modifications in Section 2.3.3, the deformation in a neighborhood of the singular branch of the limit curve along the family \( C(t), t \in (\mathbb{C}, 0) \), in not equisingular, but equiclassical\(^4\), that is, preserving a cuspidal singular branch and the total \( \delta \)- and \( \kappa \)-invariant in a neighborhood of each singular point in the torus \((\mathbb{C}^*)^2 \subset \text{Tor}(Q), Q \) being the quadrangle from either Lemma 2.4 or in the subdivisions shown in Figure 2(f,g). This means, for instance, that the limit curve may have a cuspidal singular branch and also several smooth branches centered at the same point. We notice that the equiclassical deformation base is smooth in this case by [7, Theorem 27], and the curves \( C(t), t \neq 0 \) have nodes and one cusp as their singularities by Lemma 2.1 and due to the general position of the configurations \( w(t) \subset \text{Tor}(P(\Delta)), t \neq 0 \). At last, the required transversality conditions for [22, Theorem 2.4] follow from the two facts:

- the existence of a regular orientation on \( \Gamma \setminus bp \) and on the tropical curves that appear in the modifications (cf. Remark 2.5), which yields a partial order on the set of all limit curves including modifications, and which can be extended to a linear order;

- moving along the set of all limit curves in the aforementioned linear order, each time the next limit curve, matching the conditions imposed by all the preceding limit curves, is determined up to a finite choice, and any choice corresponds to a transverse intersection of a certain equisingular (or equiclassical) family having an expected dimension with the linear system coming from the imposed conditions (see [23, Lemmas 3.5 and 3.9], Lemmas 2.4(2) and 2.7(ii)). □

\(^4\)For precise definition of the equiclassical deformations, see [8, Page 433, item (3), and Section 6].
3 Lattice path algorithm

Theorems 2.3 and 2.10 reduce the enumeration of unicuspidal algebraic curves in a toric surface to enumeration of unicuspidal plane tropical curves passing through an appropriate configuration of points in the plane, provided, that the counted algebraic curves tropicalize to unicuspidal tropical curves having a cuspidal fragment of type A, B, or C. Here we present a version of Mikhalkin’s lattice path algorithm [19, Section 7.2] adapted to our setting and solving the stated enumerative problem of computing \( \deg V_{\Delta,g}(A_2) \) in the following sense: we exhibit a configuration of points in \( \mathbb{R}^2 \) such that the unicuspidal plane tropical curves of a given degree and genus passing through the chosen configuration have a cuspidal fragment of type A, B, or C, and we reduce the enumeration of these tropical curves to a finite combinatorial problem of enumeration of lattice paths and related subdivisions of the Newton polygon.

Let \( \Delta \subset \mathbb{Z}^2 \setminus \{0\} \) be a nondegenerate, primitive, balanced multiset satisfying the hypotheses of Lemma 2.1 and such that the Newton polygon \( P(\Delta) \) is \( h \)-transversal, i.e., its intersection with any vertical line \( x - i, i \in \mathbb{Z} \), is either empty, or an integral point, or a lattice segment. Let \( 0 \leq g < p_a((P(\Delta)) - 1 \) and \( n = |\Delta| + g - 2 \).

Fix a linear functional

\[ \lambda : \mathbb{R}^2_+ \rightarrow \mathbb{R}, \quad \lambda(x, y) = x - y \varepsilon, \quad 0 < \varepsilon \ll 1. \] (34)

It defines a linear order on the set of integral points \( P(\Delta) \cap \mathbb{Z}^2 \):

\[ (i, j) < (i', j') \iff i < i' \text{ or } \begin{cases} i = i', \\ j > j' \end{cases} \]

A \( \lambda \)-path of length \( m \geq 1 \) in \( P(\Delta) \) is a broken line with vertices \( (\omega_0, ..., \omega_m) \subset P(\Delta) \cap \mathbb{Z}^2 \) such that \( \omega_i < \omega_{i+1} \) for all \( i = 0, ..., m - 1 \), and

\[ \omega_0 = \omega_{\min} := \min(P(\Delta) \cap \mathbb{Z}^2), \quad \omega_m = \omega_{\max} := \max(P(\Delta) \cap \mathbb{Z}^2). \]

Introduce also the linear functional \( \lambda(x, y) = \varepsilon x + y. \) For a given \( \lambda \)-path \( \pi \), any line \( L_a = \{ \lambda = a \} \) with \( \lambda(\omega_{\min}) \leq a \leq \lambda(\omega_{\max}) \) splits into two rays

\[ L_a^+(\pi) = \{ q \in L_a : \lambda^+(q) \geq \lambda^+(L_a \cap \pi) \}, \quad L_a^-(\pi) = \{ q \in L_a : \lambda^+(q) \leq \lambda^+(L_a \cap \pi) \}. \]

Set

\[ \Pi^+(\pi) = \bigcup_{\lambda(\omega_{\min}) \leq a \lambda(\omega_{\max})} L_a^+(\pi), \quad \Pi^- (\pi) = \bigcup_{\lambda(\omega_{\min}) \leq a \lambda(\omega_{\max})} L_a^-(\pi). \]
The following algorithm constructs a subdivision of $P(\Delta)$ into convex lattice polygons:

(i) Choose a $\lambda$-path $\pi_0$ of length $n$.

(ii) Define two lattice paths $\pi_0^+ = \pi_0^- = \pi_0$, then construct two finite sequences of $\lambda$-paths $\{\pi_i^+\}_{i \geq 0}$ and $\{\pi_i^-\}_{i \geq 0}$ following the recipe described in the next item.

(iii) Given a $\lambda$-path $\pi_i^+, i \geq 0$, with vertices $(v_0, v_1, ..., v_m)$, take

$$k = \min\{1 \leq s \leq m : v_{k-1}, v_k, v_{k+1} \text{ are not collinear},$$

$$\text{and } [v_{k-1}, v_{k+1}] \subset \Pi^\pm(\pi_i^+)\} ,$$

then perform one of the following operations

(a) either define $\pi_{i+1}^+$ by the sequence of vertices $(v_0, ..., v_{k-1}, v_k, v_{k+1}, ..., v_m)$;

(b) or define $\pi_{i+1}^+$ by the sequence of vertices $(v_0, ..., v_{k-1}, v_k, v_{k+1}, ..., v_m)$, where $\text{conv}\{v_{k-1}, v_k, v_{k+1}, v_k'\}$ is a parallelogram and $v_k' \in P(\Delta)$;

(c) or define $\pi_{i+1}^+$ by the sequence of vertices $(v_0, ..., v_{k-1}, v_k', v_{k+1}, ..., v_m)$, where $\text{conv}\{v_{k-1}, v_k, v_{k+1}, v_k'\}$ is a nondegenerate quadrangle without parallel sides and $v_k' \in P(\Delta)$;

(d) or define $\pi_{i+1}^+$ by the sequence of vertices $(v_0, ..., v_{k-1}, v_k', v_{k+1}, ..., v_m)$, where $\text{conv}\{v_{k-1}, v_k, v_{k+1}, v_k'\}$ is a trapeze and $v_k' \in P(\Delta)$.

Add either the triangle $P_i^+ = \text{conv}\{v_{k-1}, v_k, v_{k+1}\}$ in case (a), or the quadrangle $P_i^+ = \text{conv}\{v_{k-1}, v_k, v_{k+1}, v_k'\}$ in cases (b)-(d) to the list of elements of the subdivision. If such $k$ does not exist, then $\pi_i^+$ is called terminal, and the sequence $\{\pi_s^+\}_{0 \leq s \leq i}$ is completed.

(iv) The same operations we perform with $\pi_i^-$ replacing everywhere plus by minus.

The outcome of the above procedure is called an admissible subdivision of $P(\Delta)$ if

- the union of all obtained polygons $P_i^\pm, i \geq 0$, equals $P(\Delta)$, each all points of $\partial P(\Delta) \cap \mathbb{Z}^2$ are vertices of that subdivision;

- at most one polygon $P_i^\pm, i \geq 0$, is not a triangle or a parallelogram;

- if all the polygons $P_i^\pm, i \geq 0$, are triangles or parallelograms, then there exists an edge of the subdivision of lattice length $\geq 3$;
• the subdivision is dual to an irreducible plane tropical curve in the following sense: first, we take the embedded plane tropical curve dual to the subdivision (we show later that it does exist), then resolve all self-intersection points dual to the parallelograms; the resulting graph must be connected.

An admissible subdivision we convert into a marked admissible subdivision: if it contains a polygon $P_i^k$, $i \geq 0$, different from a triangle or a parallelogram, we mark this polygon, otherwise we mark one of the edges of the subdivision of lattice length $\geq 3$.

To a pair $(\pi_0, \Sigma)$, where $\pi_0$ is a $\lambda$-path of length $n$, $\Sigma$ a marked admissible subdivision of $P(\Delta)$ obtained in the previous procedure starting with $\pi_0$, we assign the following multiplicity $\mu(\pi_0, \Sigma)$:

- if $\Sigma$ consists only of triangles and parallelograms, we set $\mu(\pi_0, \Sigma)$ to be the product of lattice areas of all the triangles times $\frac{\theta(m)}{m}$, where $m$ is the lattice length of the marked edge, $\theta$ introduced in Definition 2.8;

- if $\Sigma$ contains a nondegenerate quadrangle $Q = \text{conv}\{v_{k-1}, v_k, v_{k+1}, v'_k\}$ without parallel sides (in the notation of item (c) above), then $\mu(\pi_0, \Sigma)$ equals the product of lattice areas of all triangles times the lattice area of the triangle $\text{conv}\{v_{k-1}, v_k, v_{k+1}\}$;

- if $\Sigma$ contains a trapeze $Q = \text{conv}\{v_{k-1}, v_k, v_{k+1}, v'_k\}$ with bases of lengths $m > m_1$, then $\mu(\pi_0, \Sigma)$ equals the product of lattice areas of all triangles times $\frac{m+2m_1}{m_2}$ if $v_k$ is incident to the long parallel side, or $\frac{(m+m_2)m_1}{mm_2}$ otherwise (here $m_2 = m - m_1$).

**Theorem 3.1.** Let $\Delta \subset \mathbb{Z}^2 \setminus \{0\}$ be a nondegenerate, primitive, balanced multiset satisfying the hypotheses of Lemma 2.7 and such that the Newton polygon $P(\Delta)$ is $h$-transversal. Let $0 \leq g < p_a((P(\Delta)) - 1$, $n = |\Delta| + g - 2$, and $\lambda$ be given by (34). Then

$$\deg V_{\Delta, g}(A_2) = \sum_{(\pi_0, \Sigma)} \mu(\pi_0, \Sigma),$$

where $\pi_0$ runs over all $\lambda$-paths in $P(\Delta)$ of length $n$, and $\Sigma$ runs over all marked admissible subdivisions of $P(\Delta)$ arising from $\pi_0$.

**Proof.** (1) The condition of the primitivity and $h$-transversality yields then the following observation:
Each vector $\mathbf{\pi} \in \Delta$ either equals $(\pm 1, 0)$, or satisfies $\text{pr}_v \mathbf{\pi} = \pm 1$, where $\text{pr}_v$ is the projection on the vertical axis in $\mathbb{R}^2_Z$.

Fix a configuration $\mathbf{x}$ of $n$ points on the line $L_{1,-\varepsilon} = \{y = -\varepsilon x\}$:

$$x_i = (M_i - M_\varepsilon), \quad i = 1, \ldots, n, \quad 0 < M_1 \ll M_2 \ll \ldots \ll M_n.$$  \hspace{1cm} (35)

These points are in a tropically general position (see [19, Theorem 2]). If $(\Gamma, p, h)$ is a cuspidal tropical curve of degree $\Delta$ and genus $g$ such that $h(p) = \mathbf{x}$, then by Lemma 1.2 the points $x_i, \ i = 1, \ldots, n$, lie on edges of the embedded curve $h_*(\Gamma)$, and $(\Gamma, p, h)$ is regular. Furthermore, by Lemma 2.6 we can suppose that $(\Gamma, p, h)$ is not exceptional, and hence all ends of $h_*(\Gamma)$ are of weight 1, and are in one-to-one correspondence with the vectors of $\Delta$.

(2) Assume that the cuspidal tropical fragment of $(\Gamma, p, h)$ is of type C, D, or E. Then $h_*(\Gamma)$ can be regarded as an image of a parameterized tropical curve of degree $\Delta$ and genus $g - 1$, and it is dual to a subdivision of $P(\Delta)$ into triangles and parallelograms. This is the setting of the original Mikhalkin’s lattice path algorithm [19, Section 7.2] that enumerates all the considered embedded plane tropical curves via their dual subdivisions as we described above with the only options (a) and (b) in item (iii).

Now we underline further important observations (see [15, Section 2.5.6], [19, Section 7.2] for the former one and [13, Proposition 2.6 and its proof] for the latter one):

(O2) The edges of $h_*(\Gamma) \setminus \mathbf{x}$ are regularly oriented so that their directing vectors project positively or negatively on $\nabla \lambda^+ \perp$ according as they lie above or below the line $L_{1,-\varepsilon}$ (see Figure 5(a)).

(O3) For any vertex of $h_*(\Gamma)$,

$$\max |\text{pr}_v \mathbf{\pi}| \leq \max |\text{pr}_v \mathbf{\bar{b}}|,$$

where $\mathbf{\bar{a}}$ runs over the directing vectors of incoming edges and $\mathbf{\bar{b}}$ runs over the directing vectors of outgoing edges, which is straightforward from the balancing condition (see Figure 5(a)).

An immediate consequence of (O1) and (O3) is

(O4) Each triangle in the dual to $h_*(\Gamma)$ subdivision $\Sigma$ of $P(\Delta)$ has a vertical side. Each edge of $\Sigma$ of lattice length $\geq 3$ is vertical.

For the cuspidal fragment of type D, the lattice triangle dual to the $h$-image of the four-valent vertex of $\Gamma$ has height $\geq 2$ to the base dual to the edge containing the image of the flat cycle (see Figure 8(c)), where the
(a) \hspace{0.5cm} (b) \hspace{0.5cm} (c)

Figure 5: Construction of a tropical curve via the lattice path algorithm

triangle is marked as \( T_1 \), and Lemma A1(1) in Appendix). In turn, for the
cuspidal fragment of type E, the lattice triangle dual to the \( h \)-image of the
elliptic vertex must contain at least two interior integral points (cf. Lemma
A4(1) in Appendix). Combining this with (O1), (O3), and (O4), we derive
that

\[(O5) \text{ The curve } (\Gamma, p, h) \text{ has no cuspidal fragment of type D or E.}\]

(3) Assume that the cuspidal fragment of \((\Gamma, p, h)\) is of type A or B.
Then \( h^*(\Gamma) \) contains a four-valent vertex \( V \) dual a nondegenerate quadrangle
different from a parallelogram. Since the preceding steps of the algorithm
added only triangles and parallelograms to the set of tiles of the subdivision,
the incoming edges incident to \( V \) are oriented as described in (O2). We
claim that the edges emanating from \( V \) also obey (O2). Indeed, otherwise,
we would encounter one of the situations depicted in Figure 5(b), but then
the outgoing edge satisfying (O2) would have the directing vector \( \tilde{\tau} \) with
\[|\text{pr}_x \tilde{\tau}| \geq 2.\]
This, however is not possible, since the rest of \( P(\Delta) \) should
be filled with triangles and parallelograms, which due to (O3) would lead
to a violation of the condition (O1) on the degree. Thus, the only possible
appearance of a four-valent vertex satisfies (O2) (see Figure 5(c)), which
matches the options (c) and (d) in step (iii) of the algorithm.

Notice also that (O2) yields that \( h^*(\Gamma) \cap L_{1,-\epsilon} = x \). In particular, this
means that the connected components of \( L_{1,-\epsilon} \setminus x \) are dual to a \( \lambda \)-ordered
sequence of integral points in \( P(\Delta) \) (the left and the right rays are dual to
\( \omega_{\min} \) and \( \omega_{\max} \), respectively), while the edges of \( h^*(\Gamma) \) passing through the
points of \( x \) are dual segments joining the corresponding points in the above
sequence. Thus, one obtains a \( \lambda \)-path as defined above.

(4) Finally, the formulas for the multiplicities of the marked admissible
subdivisions are straightforward consequences of formula (33). We only
notice that the middle orientation in Figure 3(a) is forbidden by (O2), and, when the long base of the trapeze is a part of \( \pi_0 \), then the multiplicity of the subdivision corresponds either to the multiplicity of the tropical curve with a fragment shown in Figure 3(f), or to the sum of multiplicities of the curves containing the fragments shown in Figures 3(d,e) (of course, with the same result).

\[ \square \]

4 Examples

(1) Figure 6 demonstrates a lattice path enumeration of the number of plane cuspidal cubics passing through 7 points in general position. The result is

\[ 6 + 3 + 6 + 3 + 3 + 3 = 24 \]

in agreement with the known classical formula \( 12(d-1)(9D-2) \) for the degree of the family of plane curves of degree \( d \) with a cusp as the only singularity (see Introduction).

(2) Furthermore, we now derive the formula \( \deg V_d(A_2) = 12(d-1)(d-2) \) for any \( d \geq 4 \) using Theorem 3.1. Like in the first example, each lattice path
\( \pi_0 \) has vertices in all but two integral points in the Newton triangle \( T_d = \text{conv}\{(0,0), (d,0), (0,q)\} \) (we call these two integral point missing points). If the missing points \((i', j'), (i'', j'')\) satisfy either \(|i' - i''| \geq 2\), or \(|i' - i''| = 1\) and \(0 < j' < d - i'\), \(0 < j'' < d - i''\), or \(i' = i''\) and \(|j' - j''| \geq 2\), or \(i' = i'' = 0\), or \(j' = i'\) and \(j'' = d - i''\), or \(j' = 0\) and \(j'' = d - i''\), then the lattice path does not induce an admissible subdivision of \( T_d \). The remaining cases and admissible subdivisions are shown in Figure 7 (where missing points are designated by bullets, and the remaining part of \( T_d \) is covered by lattice triangles of lattice area 1):

(a) If the missing points are \((i, j), (i, j + 1)\), where \(1 \leq i \leq d - 3\) and \(1 \leq j \leq d - i - 2\) (see Figure 7(a)), then we have a unique subdivision of \( T_d \) with an edge \([([i, j - 1], (i, j + 2)]\) dual to a cuspidal fragment of type C. The multiplicity of the subdivision equals 6 (see formulas (31), (32) and Theorem 2.10), and the total contribution of all such subdivisions equals \(6 \cdot \frac{(d-3)(d-2)}{2} = 3(d-3)(d-2)\).

(b) If the missing points are \((i, d - i), (i, d - i - 1)\), \(1 \leq i \leq d - 2\) (see Figure 7(b)), then an admissible subdivision contains a trapeze \(\text{conv}\{([i, d - i], (i, d - i - 1), (i - 1, j), (i - 1, j + 1)]\}, \) where \(0 \leq j \leq d - i\). Any such subdivision has multiplicity 3 (see the middle formula in (13) and Theorem 2.10), and their total contribution equals \(3 \cdot \frac{(d-2)(d+3)}{2} = \frac{3}{2}d^2 + \frac{3}{2}d - 9\).

(c) If the missing points are \((i, 0), (i, 1)\), where \(1 \leq i \leq d - 4\), we have several types of admissible subdivisions:

- either containing a trapeze spanned by the points \((i, 0), (i, 2), (i + 1, j), (i + 1, j + 1)\) as \(3 \leq j \leq d - i\) (see Figure 7(b)); the multiplicity of such a subdivision is 3 as in item (b);

- or containing the quadrangle \(\text{conv}\{((i - 1, 0), (i, 0), (i, 2), (i + 1, 3)]\} \) (see Figure 7(d)); the multiplicity of this subdivision is 1 (see Lemma 2.4(3) and Theorem 2.10);

- or containing the quadrangle \(\text{conv}\{((i - 1, 0), (i, 0), (i + 1, 2), (i + 1, 3)]\} \) (see Figure 7(e)); the multiplicity of this subdivision is 2 (cf. the preceding case);

- or containing the trapeze \(\text{conv}\{((i - 1, 0), (i, 0), (i + 1, 1), (i + 1, 2)]\}; the multiplicity of this subdivision is 6 (see the last formula in (10) and Theorem 2.10).
The total contribution of these four types of admissible subdivisions amounts to

\[ 3 \cdot \frac{(d-5)(d-4)}{2} + (1 + 2 + 6)(d-4) = \frac{3}{2}d^2 - \frac{9}{2}d - 6. \]

(d) If the missing points are \((i, 0), (i, 1)\), where \(i = d-3\) or \(d-2\), we have subdivisions containing either the trapeze \(\text{conv}\{(d-4, 0), (d-3, 0), (d-2, 1, d-2, 2)\}\) (see Figure 7(g)), or the quadrangle \(\text{conv}\{(d-3, 0), (d-2, 0), (d-2, 2), (d-1, 1)\}\) (see Figure 7(h)), and having multiplicity 6 or 3, respectively.

(e) The remaining options for the pair of missing points are \((i, j), (i + 1, d-i - 1)\), where \(1 \leq i \leq d-2, 1 \leq j \leq d-i - 1\) (see Figure 7(i)), and \((i, j), (i-1, 0)\), where \(1 \leq i, \leq d-2, 1 \leq j \leq d-i - 1\) (see Figure 7(j)); in both cases the multiplicity of the subdivision equals 6, which results in the total contribution \(6(d-1)(d-2)\).

The sum of the above contributions equals \(12(d-1)(d-2)\).

(3) The next example exhibits a real version of the preceding enumeration.

**Theorem 4.1.** For any \(d \geq 3\), there exists a two-dimensional linear subsystem in \(|\mathcal{O}_{\mathbb{F}_2}(d)|\) that contains at least \(c(d)\) real cuspidal curves, where \(c(d)\) is a positive function satisfying \(c(d) \sim 4d^2 + O(d)\) as \(d \to \infty\).

**Proof.** We intend to show that the two-dimensional linear system of curves of degree \(d\) passing through the configuration of \(n = \frac{d(d+3)}{2} - 2 \) points

\[ w = \{(t^{-M_i}, s_i t^{M_i})\}_{i=1,...,n}, \tag{36} \]

where \(s_i = \pm 1\) will be specialized later and which always tropicalizes to configuration \(\hat{\mathcal{M}}\), meets the requirements of Theorem. For, we go through the computations of the preceding example and check how many real solutions occur among all complex ones. Since we are interested only in the leading term of the asymptotics, we focus only on the subdivisions as shown in Figures 7(a,b,c,i,j). The question reduces to counting real solutions in finding limit curves associated with a given subdivision and limit curves associated with modifications, and in equations for conditions to pass through fixed points in \(w\).

First, it is easy to see that all limit curves associated with the considered subdivisions are real. Indeed, up to a constant factor, all the coefficients \(a_\omega\),
Figure 7: Enumeration of unicuspidal curves
\( \omega \in \mathcal{T}_d \cap \mathbb{Z}^2 \) can be found from the condition to pass through the configuration \( \omega \) (see details in [23, Section 3.7, formula (3.7.27)] or [15, Section 2.5.7]) and from the known structure of the limit curves associated with non-unit triangles and quadrangles, and all these conditions reduce to linear equations.

Thus, we analyze modifications of the tropical limit and modified fixed point conditions.

- In case shown in Figure 7(a), both limit curves of the modification are real (see the second paragraph in the proof of Lemma 2.9). The condition to pass through a point of \( \omega \) amounts to taking the cubic root (cf. equations (12) and [15, Section 2.3.9]). Hence, the number of real solutions is the third part of the number of complex ones, i.e., \( d^2 + O(d) \).

- In the case shown in Figures 7(b,c), a non-linear algebraic equation pops up only in computing the limit curve of the modification associated with a quadrangle as shown in Figure 2(h) with \( m = 2, m_1 = 1 \); here we again take the cubic root, and hence the third part of all complex solutions appears to be real, that is, \( d^2 + O(d) \).

- Similarly to the previous item, in the cases shown in Figure 7(i,j), the third part of the limit curves in the modification is real. So, we analyze the fixed point conditions. The cuspidal fragments dual to the considered subdivisions are of types depicted in Figures 3(d,e,f). Due to the condition (35) to the parameters \( M_i, i = 1, \ldots, n \), the marked point is located much closer to the vertex \( V_2 \) than to \( V_3 \) (in the notation of Figure 2(a)) in case of Figure 7(i), and vice versa in case of Figure 7(j). Thus, in the latter case, we are in a position shown in Figure 3(d,e), where both the fixed point conditions are linear, and hence here we encounter \( d^2 + O(d) \) real solutions.

- In case of Figure 7(i), the fixed point condition requires taking square root: see the last equation in (12), which in the considered case reads

\[
a(\eta(t) - \zeta)^2 + bt^{\gamma + M_i} + \text{h.o.t.} = 0, \quad a = (\gamma - s_{i_1})^{d-1}a_{i-1,i}, \quad b = a_{i-1,0},
\]

(37)

where \( i_1 \) is the number of the segment \([(i, j-1), (i, j+1)]\) in the lattice path. Now we define the signs \( s_i \) in (36). The number of the incline segment \([(i-1,0), (i, d-i)]\) in the lattice path is \( m(i) = \frac{1}{2}(2d - i + 3) \), and we set \( s_{m(i)} = 1 \) for all \( i = 1, \ldots, d - 2 \), while the remaining \( s_k = -1 \). In particular, \( a = a_{i, j+1} \) in (37). Applying [23, Formula
we get that the signs of the coefficients \( a_{ik} \) are the same for all \( 0 \leq k \leq d - i \), and \( a_{i-1,0} \) has an opposite sign. Hence (37) has two real solutions, which finally yields \( d^2 + O(d) \) additional real cuspidal curves.

The bound asserted in Theorem follows. \( \square \)

Appendix

A1. Multiplicity of a flat cycle (fragment D)

(1) Let \((\Gamma, p, h)\) have a flat cycle as its cuspidal tropical fragment \((\Gamma_c, p_c, h_c)\) consisting of the vertex \( V \), a flat trivalent vertex and two edges joining them (see Figure 8(a), where \( m_1, m_2, m \) denote the weights of the edges incident to the flat trivalent vertex). Since \( m \geq 2 \) and \( \Delta \) is primitive, all the edges incident to the flat trivalent vertex are bounded, and we extend \((\Gamma_c, p_c, h_c)\) to a fragment \((\Gamma', p', h')\) by adding one more trivalent vertex \( V_2 \) as shown in Figure 8(b). The fragment of the subdivision of \( P(\Delta) \) dual to \( h' \) consists of two triangles \( T_1, T_2 \) sharing a common side \( e \) (see Figure 8(c)). Denote by \( V_1, \text{red} \) the vertex of \( h' \) dual to the triangle \( T_1 \). The limit curves \( C_1 \subset \text{Tor}(T_1), C_2 \subset \text{Tor}(T_2) \) are rational, \( C_2 \) is peripherally smooth and unibranch, while \( C_1 \) has two local branches centered at the same point of \( \text{Tor}(e) \) and it is unibranch along the two other toric divisors. The following lemma describes the geometry of \( C_1 \).

Lemma A1. Let \( T \) be a nondegenerate lattice triangle with sides \( \sigma_1, \sigma_2, \sigma_3 \). Suppose that \( m = \| \sigma_1 \|_Z = m_1 + m_2, m_1 \geq m_2 \geq 1 \). Let \( M \subset \mathcal{M}_{0,4}(\text{Tor}(T), \mathcal{L}_T) \) be the family of isomorphism classes of maps \( \mathbf{n} : \mathbb{P}^1 \rightarrow \text{Tor}(T) \) of \( \mathbb{P}^1 \) with four distinct marked points \( p_1, p'_1, p_2, p_3 \in \mathbb{P}^1 \) such that

\[
\mathbf{n}(p_1) = \mathbf{n}(p'_1) \in \mathbb{C}^* \subset \text{Tor}(\sigma_1), \quad \mathbf{n}(p_i) \in \mathbb{C}^* \subset \text{Tor}(\sigma_i), \; i = 2, 3,
\]

and

\[
\mathbf{n}^* \text{Tor}(\sigma_1) = m_1 p_1 + m_2 p'_1, \quad \mathbf{n}^* \text{Tor}(\sigma_i) = \| \sigma_i \|_Z \cdot p_i, \; i = 2, 3.
\]

The following holds:

(1) If \( \| T \|_Z = m \) then \( M = \emptyset \); if \( \| T \|_Z = rm, \; r \geq 2 \), then \( M \) is isomorphic to the union of disjoint copies of \((\mathbb{C}^*)^2\) whose number equals

\[
\begin{cases} 
    r - 1 & \text{if } m_1 > m_2, \\
    \left\lceil \frac{r}{2} \right\rceil & \text{if } m_1 = m_2 = \frac{m}{2}. 
\end{cases}
\]  \hspace{1cm} (38)
Except for the case

\[ \|\sigma_i\| \equiv 0 \mod 2, \; i = 1, 2, 3, \quad \text{and} \quad m_1 = m_2 = \frac{m}{2}, \quad (39) \]

each irreducible component of \( M \) parameterizes birational maps onto immersed curves, whose two local branches centered on \( \text{Tor}(\sigma_1) \) intersect each other with multiplicity \( m_2 \). In the case (39), all but one component of \( M \) parameterizes maps as above, and one component parameterizes double ramified coverings \( n : \mathbb{P}^1 \to \mathbb{C}' \hookrightarrow \text{Tor}(\mathcal{T}) \) with \( 2\mathbb{C}' \in |\mathcal{L}_\mathcal{T}| \) and \( \mathbb{C}' \) immersed, unibranch along the toric divisors, and the ramification is at \( \mathbb{C}' \cap \text{Tor}(\sigma_i), \; i = 2, 3. \)

(2) Under the condition \( \|\mathcal{T}\| \geq 2m \), given two points \( z_i \in \mathbb{C}^* \subset \text{Tor}(\sigma_i), \; z_j \in \mathbb{C}^* \subset \text{Tor}(\sigma_j), \; 1 \leq i < j \leq 3 \), each component of \( M \) transversally intersects in \( \mathcal{M}_{0,4}(\text{Tor}(\mathcal{T}), \mathcal{L}_\mathcal{T}) \) with the family \( \{n(p_i) = z_i, \; n(p_j) = z_j\} \) in \( \mathcal{M}_{0,4}(\text{Tor}(\mathcal{T}), \mathcal{L}_\mathcal{T}) \), we can identify \( \mathcal{T} \) with the triangle \( \text{conv}\{(p, 0), (p + m, 0), (0, r)\} \), where \( p \geq 0, \; r > 0, \; p + m > r \), and \( \sigma_1 = [(p, 0), (p + m, 0)] \). If \( \|\mathcal{T}\| = m \), i.e., \( r = 1 \), then \( n^* \text{Tor}(\sigma_1) \) is one point, and hence \( M = \emptyset \).

Suppose that \( r \geq 2 \). Then a map \( n : \mathbb{P}^1 \to \text{Tor}(\mathcal{T}) \) as asserted in Lemma 41 can be given by

\[ x = at^r, \; y = bt^{P(t-1)m_1(t-\lambda)m_2}, \quad t \in \mathbb{C}, \quad (41) \]

with some \( \lambda \in \mathbb{C} \setminus \{0, 1\} \) and arbitrary \( a, b \in \mathbb{C}^* \). Since \( x(1) = x(\lambda) \), we get \( \lambda = \exp \frac{2\pi k\sqrt{-1}}{r} \) with \( 1 \leq k \leq r - 1 \), and for each value of \( k \), the family of such parameterizations is isomorphic to \( (\mathbb{C}^*)^2 \). If \( m_1 > m_2 \), then the parameterizations are in one-to-one correspondence with the elements of \( M \). If \( m_1 = m_2 \) and \( \lambda \neq -1 \), then the parameterizations associated with the data \( (a, b, \lambda) \) and \( (a, b\lambda^{P+m}, \lambda) \) define the same element of \( M \). If \( m_1 = m_2 \) and \( \lambda = -1 \), but \( p \) is odd, then the parameterizations associated with the data \( (a, b, -1) \) and \( (a, -b, -1) \) define the same element of \( M \), and here \( (\mathbb{C}^*)^2/(\{a, b\} \sim (a, -b) \} \simeq (\mathbb{C}^*)^2 \). Hence, the formula (38).
The map germs \( n : (\mathbb{P}^1, 1) \to \text{Tor}(\mathcal{T}) \) and \( n : (\mathbb{P}^1, \lambda) \to \text{Tor}(\mathcal{T}) \) are given by
\[
\begin{align*}
x &= a + ar(t - 1) + O((t - 1)^2), \\
y &= b(1 - \lambda)m_2(t - 1)^{m_1} + O((t - 1)^{m_1+1})
\end{align*}
\tag{42}
\]
and
\[
\begin{align*}
x &= a + ar\lambda^{r-1}(t - \lambda) + O((t - \lambda)^2), \\
y &= b\lambda^p(\lambda - 1)^{m_1}(t - \lambda)^{m_2} + O((t - \lambda)^{m_2+1})
\end{align*}
\tag{43}
\]
If either (39) does not hold, or (39) holds but \( \lambda \neq -1 \), then formulas (42) and (43) yield the birationality of \( n \) and the asserted intersection number of these two branches. If (39) holds (in particular, \( r, p, m \) are even) and \( \lambda = -1 \), then formula (41) turns into
\[
\begin{align*}
x &= a(t^2)^{r/2}, \\
y &= b(t^2)^{p/2}(t^2 - 1)^{m/2}, \\
t &\in \mathbb{C},
\end{align*}
\]
which yields the double covering as asserted in Lemma.

Let
\[
d_1 = \gcd(p, r), \ p' = \frac{p}{d_1}, \ r' = \frac{r}{d_1}, \ d_2 = \gcd(p + m, r), \ p'' = \frac{p}{d_2}, \ r'' = \frac{r}{d_2}.
\]
Then the conditions \( n(p_i) = z_i, n(p_j) = z_j \) amount in the following systems of equations in unknowns \( a, b \)
\[
\begin{align*}
a &= \alpha, \\
a' &= \beta b', \\
a'' &= \beta b''.
\end{align*}
\]
for \( (i, j) = (1, 2), (1, 3), \) or \( (2, 3), \) respectively, with some \( \alpha, \beta \in \mathbb{C}^* \). In each case, the number of solutions \( (a, b) \) is given by the lower value in (40), while in the case \( m_1 = m_2, \lambda = -1, \) and \( p \equiv 1 \mod 2 \), one has to identify solutions \( (a, b) \) and \( (a, -b) \). The transversality of the intersection follows from the fact that each of the above systems has only simple solutions in \((\mathbb{C}^*)^2\). \( \square \)

(2) Next, we perform a local modification at the point \( z = C_1 \cap C_2 \) along the edge \( e = T_1 \cap T_2 \) following the recipe of Section 2.3.2 and we obtain a fragment of the modified subdivision inscribed either into the non-convex quadrangle \( Q \) with vertices \( (0, 0), (3, 0), (1, m_1), (2, m) \) if \( m_1 < m_2 \), or into the triangle \( T = \text{conv}\{(0,), (3, 0), (2, m)\} \) if \( m_1 = m_2 = \frac{m}{2} \) (see Figure 8(d,e)).
Suppose that \( m_1 < m_2 \). It follows from Theorem 2.3 that the union of the limit curves corresponding to the subdivision of \( Q \) induced by the modified polynomial must be of arithmetic genus zero and it must contain a local singular branch. Getting rid of the monomial \( xy^{m_1-1} \) in the modified polynomial and applying the argument used in Section 2.3.3 we derive that the only possible subdivision of \( Q \) is as shown in Figure 8(f), while the limit curve \( C' \subset \text{Tor}(Q') \) is nodal, rational, and \( C'' \subset \text{Tor}(Q'') \) is rational with a unique singular branch in \((\mathbb{C}^*)^2 \subset \text{Tor}(Q'')\). Observe that the curves \( C_1, C_2 \) determine the intersection points of \( C', C'' \) with the toric divisors corresponding to the incline sides of \( Q \). Taking into account possible orientations on \( \Gamma' \setminus p' \) induced by the regular orientation of \( \Gamma' \) in case \( p_c = \emptyset \) (two upper graphs in Figure 8(g)) and applying [23, Lemma 3.9] and Lemma A1, we obtain that the number of the tuples \( (C_1, C_2, C', C'') \) matching the initial data equals

\[
\mu(V_2)\tilde{m}^{-1}\mu_c(\Gamma_c, p_c, h_c),
\]  

where \( \tilde{m} \) is the product of the weights of the outer edges of \((\Gamma', p', h')\) oriented inward, and

\[
\mu_c(\Gamma_c, p_c, h_c) = \frac{\mu(V_{1,\text{red}})(\mu(V_{1,\text{red}}) - m)(m + m_2)m_1}{m^2}.
\]

If \( p_c \neq \emptyset \), i.e., there is a marked point on one of the edges incident to the flat trivalent vertex (see two lower graphs in Figure 8(g)), then we take into account the marked point condition as in [13, Section 2.3.9] and also sum up the multiplicities arising when the marked point lies on the upper and lower edge of the flat cycle. Then the final result coincides with (44), (45).

(3) Suppose now that \( m_1 = m_2 = \frac{m}{2} \), which corresponds to the modification associated with the triangle \( T \) in Figure 8(e). Denote the incline edges of \( T \) by

\[ e_1 = [(0,0), (2,m)], \quad e_2 = [(2,m), (3,0)]. \]

If \( C_1 \) is reduced (i.e., is not doubly covered as in Lemma A1 under condition (39)), then the data for the modified tropical limit induced by \( C_1, C_2 \) are:

- one point on the toric divisor \( \text{Tor}(e_2) \),
- two distinct points on the toric divisor \( \text{Tor}(e_1) \), and at each one the sought limit curves has one branch intersecting \( \text{Tor}(e_1) \) with multiplicity \( \frac{m}{2} \).
Figure 8: Cuspidal flat cycle and its modification, I
Getting rid of the monomial \(x^2y^{m-1}\) by an extra coordinate change (cf. Section 2.3.3), we exclude the segment \([(2,0),(2,m)]\) as an element of a possible subdivision of \(T\). Thus, no any subdivision of \(T\) is possible, and the available limit curves \(C_T \in \text{Tor}(T)\) are described in the following lemma.

Lemma A2. Let \(M \subset |L_T|\) be the family of rational curves \(C_T\) having a singular branch outside \(\text{Tor}(e_1) \cup \text{Tor}(e_2)\), meeting \(\text{Tor}(e_1)\) in two given distinct points \(z'_1, z''_1 \in \mathbb{C}^* \subset \text{Tor}(e_1)\), where \(C_T\) is unibranch and intersect \(\text{Tor}(e_1)\) with multiplicity \(\frac{m}{2}\), and meeting \(\text{Tor}(e_2)\) in a given point \(z_2 \in \mathbb{C}^* \subset \text{Tor}(e_1)\). Then \(M\) is isomorphic to \(\frac{3}{4}m^2\) disjoint copies of \(\mathbb{C}\); each element of \(M\) is immersed except for one point in \(\text{Tor}(T) \setminus (\text{Tor}(e_1) \cup \text{Tor}(e_2))\), where it has a singular branch of of type \(A_2\).

Proof. Suppose that \(C_T \in M\). By a shift \(y \mapsto y + b\), we can place a singular branch of \(C_T\) on the \(x\)-axis. Then \(C_T\) admits a parametrization

\[
x = \alpha t^{m/2}(t-1)^{m/2}, \quad y = \beta \frac{S(t)}{t(t-1)}, \quad \alpha, \beta \in \mathbb{C}^*,
\]

where \(S(t)\) is a monic cubic polynomial. Since \(x'(t) = 0\) yields \(t = \frac{1}{2}\), the singular branch condition can be expressed as \(y(\frac{1}{2}) = y'(\frac{1}{2}) = 0\), which implies \(S(t) = (t-\gamma)(t-\frac{1}{2})^2\). Setting \(t = t' + \frac{1}{2}\), we obtain a parametrization of the singular branch in the form

\[
x - \frac{1}{2m} = \left((t')^2 - \frac{1}{4}\right)^{m/2} - \frac{1}{2m}, \quad y = \left(t' - \gamma + \frac{1}{2}\right) \left((t')^2 - \frac{1}{4}\right)^{-1},
\]

which yields its type \(A_2\). The point conditions on \(\text{Tor}(e_1) \cup \text{Tor}(e_2)\) amount to a system of equations

\[
\alpha \beta^{m/2} = \xi_1, \quad \alpha \beta^{m/2}(1-\gamma)^{m/2} = \xi_2, \quad \alpha = \xi_3 \beta^m,
\]

where \(\xi_3 \neq 0\) and either \(\xi_1 \neq \pm \xi_2\), or \(\xi_1 = -\xi_2\) when \(m\) is even, or \(\xi_1 = \xi_2\) when \(m\) is odd, which directly follows from formulas (42), (43). It is easy to see that this system has \(\frac{3}{4}m^2\) solutions, all matching the requirements of Lemma. The whole family \(M\) is obtained from these solutions by shifts \(y \mapsto y + b, \ b \in \mathbb{C}\).

Suppose that \(C_1\) is not reduced. Using the argument from the proof of Lemma 2.16, one can show that the curve \(C_T\) cannot be unibranch at the point \(C_1 \cap \text{Tor}(e_1)\). Thus, possible curves \(C_T\) are as described in the following lemma.

---

[3]
Lemma A3. Let \( M \subset |L_T| \) be the family of rational curves \( C_T \) having a singular branch outside \( \text{Tor}(e_1) \cup \text{Tor}(e_2) \), meeting \( \text{Tor}(e_1) \) in a given point, where it has two local branches each one intersecting \( \text{Tor}(e_1) \) with multiplicity \( \frac{m}{2} \), and meeting \( \text{Tor}(e_2) \) in a given point. Then \( M \) is isomorphic to

\[
N = \begin{cases} 
\frac{3}{2}m^2, & \text{if } m \equiv 0 \mod 4, \\
\frac{3}{2}m(m - 2), & \text{if } m \equiv 2 \mod 4 
\end{cases} 
\tag{47}
\]

disjoint copies of \( \mathbb{C} \); each element of \( M \) is immersed except for one point in \( \text{Tor}(T) \setminus (\text{Tor}(e_1) \cup \text{Tor}(e_2)) \), where it has a singular branch of order 2.

Proof. As in the proof of Lemma A1, we can suppose that \( C_T \in M \) has a singular branch on the \( x \)-axis, and it admits a parametrization (46). Similarly, the singular branch occurs only for \( t = \frac{1}{2} \), and we necessarily obtain \( S(t) = (t - \gamma)(t - \frac{1}{2})^2 \). The given data along the toric divisors \( \text{Tor}(e_1), \text{Tor}(e_2) \) amounts to a system of equations

\[
\alpha \beta^{m/2} \gamma^{m/2} = \xi_1, \quad \alpha \beta^{m/2}(1 - \gamma)^{m/2} = -\xi_1, \quad \alpha = \xi_2 \beta^m
\]

with some \( \xi_1, \xi_2 \in \mathbb{C}^\ast \). The number of solutions appears to be \( 2N \), where \( N \) is given by (47). At last, we notice that the involution \((\alpha, \beta, \gamma) \leftrightarrow (\alpha, -\beta, 1-\gamma)\) interchanges parameterizations that define the same curve \( C_T \). \( \square \)

Summarizing the above computations we derive that in case \( m_1 = m_2 = m/2 \), the multiplicity of the cuspidal fragment \( (\Gamma_c, p_c, h_c) \) appears to be

\[
\begin{cases} 
\frac{3}{2} \mu(V_{1, \text{red}})(\mu(V_{1, \text{red}} - m - 2) & \text{if (39) holds and } m \equiv 2 \mod 4, \\
\frac{3}{2} \mu(V_{1, \text{red}})(\mu(V_{1, \text{red}}) - m) & \text{otherwise.} 
\end{cases} 
\tag{48}
\]

A2. Multiplicity of an elliptic trivalent vertex (fragment E)

Suppose that the cuspidal fragment \( (\Gamma_c, h_c) \) of \( (\Gamma, p, h) \) is a non-flat trivalent vertex of genus 1.

Lemma A4. Let \( T \) be a nondegenerate lattice triangle, \( M \subset |L_T| \) the family of elliptic curves that are unibranch along the toric divisors and have at least one singular local branch in the big torus \((\mathbb{C}^\ast)^2 \subset \text{Tor}(T)\).

(1) If \( |\text{Int}(T) \cap \mathbb{Z}^2| \leq 1 \) then \( M = \emptyset \).

(2) If \( |\text{Int}(T) \cap \mathbb{Z}^2| \geq 2 \), then \( M \) is either empty, or is isomorphic to several disjoint copies of \((\mathbb{C}^\ast)^2 \) parameterizing curves, which are smooth along the toric divisors and which have exactly one singular local branch in \((\mathbb{C}^\ast)^2 \); furthermore, this singular branch has order 2.
(3) Under the preceding assumption, chose two sides $\sigma_1, \sigma_2$ of $T$ and fix points $z_i \in \mathbb{C}^* \subset \text{Tor}(\sigma_i)$, $i = 1, 2$. Then $M$ transversally intersects with the linear system $|L_T(-z_1 - z_2)| \subset |L_T|$.

**Proof.** Note that if $|\text{Int}(T) \cap \mathbb{Z}^2| \leq 1$ then either there are no elliptic curves in $|L_T|$, or all elliptic curves are smooth. Thus, $M = \emptyset$.

Suppose that $|\text{Int}(T) \cap \mathbb{Z}^2| \geq 2$ and $M \neq \emptyset$. In view of the $(\mathbb{C}^*)^2$-action, each component of $M$ has dimension $\geq 2$. Let us show that a generic curve $C$ of a component $M_0$ of $M$ has in $\text{Tor}(T)$ either a unique singular branch of order $\leq 3$, or two singular branches both of order 2. Indeed, let $m_1, ..., m_s \geq 2$ ($s \geq 1$) be orders of all singular branches of $C$. Then we apply [14, Inequality (5)], which in our situation reads as follows (here we denote by $\sigma_1, \sigma_2, \sigma_3$ the sides of $T$):

\[
\sum_{i=1}^{3} ||\sigma_i||_Z \geq \sum_{i=1}^{3} (||\sigma_i||_Z - 1) + \sum_{i=1}^{s} (m_i - 1) + 1 \implies \sum_{i=1}^{s} (m_i - 1) \leq 2,
\]

and hence the claim follows.

Now suppose that $C$ has a singular branch of order 3 (which then must be in $(\mathbb{C}^*)^2$), or two singular branches of order 2 (and at least one of them in $(\mathbb{C}^*)^2$). Since $C \in M_0$ is generic, the germ $(M_0, C)$ is equisingular, thus, is contained in the germ $(M_0^{ec}, C)$ of the equiclassical stratum (see details in [8, 21]). Then, in the same manner as in the proof of Lemma 2.7 and additionally using [12, Inequality (21)] for the case of a singular branch on a toric divisor, we conclude that the tangent cone to the considered equiclassical stratum can be identified with $H^0(\widetilde{C}, \mathcal{O}\_{\widetilde{C}}(D - D_0 - D_{ec}))$, where $n : \widetilde{C} \rightarrow C$ is the normalization, $D = n^*(c_1(\mathcal{O}_C \otimes \mathcal{L}_\Delta))$, $D_0$ is the double-point divisor, $D_{ec}$ is the pull-back of the centers of singular local branches of $C$ and the intersection points of $C$ with the toric divisors, counted with the total multiplicity

\[
\deg D_{ec} = \sum_i (m_i - 1) + \sum_{i=1}^{3} (||\sigma_i||_Z - 1) .
\]

Since

\[
\deg D_0 = 2 \sum_{z \in \text{Sing}(C)} \delta(C, z) = C^2 + CK_{\text{Tor}(T)} = \deg c_1(\mathcal{O}_C \otimes \mathcal{L}_\Delta) + K_{\text{Tor}(T)} C ,
\]

and

\[
\sum_{i=1}^{3} ||\sigma_i||_Z = -K_{\text{Tor}(T)} C ,
\]

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we have
\[
\deg(D - D_0 - D_{ec}) = -K_{\text{Tot}(T)}C - \sum_i (m_i - 1) - \sum_{i=1}^3 (\|\sigma_i\|z - 1)
\]
\[
= 3 = \sum_{i=1}^s (m_i - 1) > 0 = 2g(\widehat{C}) - 2 ,
\]
and hence
\[
H^1(\widehat{C}, \mathcal{O}_{\widehat{C}}(D - D_0 - D_{ec})) = 0 \quad (49)
\]
and
\[
h^0(\widehat{C}, \mathcal{O}_{\widehat{C}}(D - D_0 - D_{ec})) = \deg(D - D_0 - D_{ec}) - g(\widehat{C}) + 1 = 3 - \sum_{i=1} (m_i - 1) .
\]
Due to the fact that the germ of \(M\) at \(C\) is at least two-dimensional, we derive that
\[
s = 1, \quad m_1 = 2, \quad \text{and} \quad \dim M = 2 ,
\]
that is \(M\) is the union of disjoint smooth orbits of the \((\mathbb{C}^*)^2\)-action.

The claim (3) is straightforward from the preceding result. \(\square\)

Recall that a deformation of an isolated plane curve singularity is called equiclassical if it preserves both the total \(\delta\)- and total \(\kappa\)-invariant (see [8, Page 433, item (3)]). The following statement is contained in [8, Section 5].

**Lemma A5.** Let \(k \geq 2\), and let \(B_{2k}(S, z) \subset \mathcal{O}_{\mathbb{C}^2, z}, B_{2k} \simeq (\mathbb{C}^{2k}, 0)\) be a versal deformation base of a plane curve singularity \((S, z)\) of type \(A_{2k}\). Denote by \(B_{2k}^{ee}(S, z) \subset B_{2k}(S, z)\) the equiclassical stratum. Then \(B_{2k}^{ee}(S, z)\) is an irreducible germ of a complex variety of codimension \(k - 1\), its tangent cone \(TB_{2k}^{ee}(S, z)\) can be identified with the linear space

\[
J_{S, z}^{ee} := \{ \varphi \in B_{2k}(S, z) : \text{ord}\varphi \bigg|_{S, z} \geq 2k + 1 \}
\]
of codimension \(k - 1\), a generic element of \(B_{2k}^{ee}(S, z)\) has \(k - 1\) nodes and one ordinary cusp.

We define the multiplicity \(\text{mult}(B_{2k}^{ee}(S, z))\) of the equiclassical stratum as follows. Let \(\Lambda_t, t \in (\mathbb{C}, 0)\), be a family of affine subspaces of \(B_{2k}(S, z)\) of dimension \(k - 1\) such that \(\Lambda_0 \) intersects \(TB_{2k}^{ee}(S, z)\) transversally at the origin, and \(\Lambda_t, t \neq 0\), intersects \(B_{2k}^{ee}(S, z)\) in its generic elements. Set

\[
\text{mult}(B_{2k}^{ee}(S, z)) = |\Lambda_t \cap B_{2k}^{ee}(S, z)| \quad \text{for} \ t \neq 0 .
\]
Define the multiplicity of the considered cuspidal tropical fragment \( \mu_c(\Gamma_c, h_c) \) to be zero if \(|\text{Int}(T \cap \mathbb{Z}^2)| \leq 1\), and, in case \(|\text{Int}(T \cap \mathbb{Z}^2)| \geq 2\),
\[
\mu_c(\Gamma_c, h_c) = \sum_{C \in M^*} \text{mult}(B^c_{2k}(C, z)),
\]
(50)
where \(M^*\) is the (finite) set of elements of \(M\) given by polynomials having equal coefficients at the vertices of \(T\), \((C, z)\) is a singular local branch of \(C \in M^*\), and the type of \((C, z)\) is \(A_{2k}\).

**Remark A6.** From Lemma 2.9, one can extract an explicit formula for \(\mu_c(\Gamma_c, h_c)\) in the particular case of \((\Gamma_c, h_c)\) dual to the triangle \(T = \text{conv}\{(0, 0), (0, 2), (m, 1)\}\), but we do not present here a generalization to arbitrary triangles \(T\).

### A3. Patchworking: full version
We present here an extension of Theorem 2.10 to arbitrary cuspidal plane tropical curves.

**Theorem A7.** Let \(\Delta \subset \mathbb{Z}^2 \setminus \{0\}\) be a nondegenerate, primitive, balanced multiset satisfying the hypotheses of Lemma 2.1, \(0 \leq g < p_a((P(\Delta)) - 1\), \(n = |\Delta| + g - 2\), and \(w\) a configuration of \(n\) distinct points in \((\mathbb{K}^*)^2\) such that \(x = \text{val}(n(w))\) is a set of \(n\) distinct points in \(\mathbb{R}^2\) in general position. Let \((\Gamma, p, h)\) be a plane cuspidal \(n\)-marked tropical curve of degree \(\Delta\) and genus \(g\) such that \(h(p) = x\). Assume in addition that, in case B, the curve \((\Gamma, p, h)\) is not exceptional. Then the number of curves \(C \in \mathcal{V}_{\Delta, g}(A_2)\) passing through \(w\) and tropicalizing to \((\Gamma, p, h)\) equals the value (33), where \(\mu_c(\Gamma_c, p_c, h_c)\) should be appropriately chosen from formulas (6), (10), (13), (31), (32), (48), or (50).

**Proof.** As in the proof of Theorem 2.10, i.e., the patchworking statement [22, Theorem 2.4], is the main ingredient, and the extra remarks in the proof of Theorem 2.10 complete the argument. We only make two additional comments related to the newly allowed cuspidal tropical fragments of type D and E.

If the cuspidal tropical fragment is of type D, then by Lemma A1(1), one can encounter a limit curve that is a double covering
\[
n : \mathbb{P}^1 \to C' \hookrightarrow \text{Tor}(T)
\]
(51)
ramified at two points on toric divisors of \(\text{Tor}(T)\). In this case, it is more convenient to work with the parameterized tropical limit. Furthermore,
since the other limit curve are birational images of $\mathbb{P}^1$, they can be treated as in the proof of Theorem 2.10. For the parameterized limit curve (51) the only difference may be the trasversality condition, but it is provided by Lemma A1(2).

Let the cuspidal fragment be of type $E$, and let $M$ be the family from Lemma A4. Choose a curve $C \in M$. For each singular point $z \in \text{Sing}(C)$ and each intersection point $z = C \cap D_i$ with the toric divisor $D_i = \text{Tor}(\sigma_i)$, $i = 1, 2, 3$, take the space $\mathcal{B}_z = \mathcal{O}_{\text{Tor}(T),z}/m_z^d$, where $m_z \subset \mathcal{O}_{\text{Tor}(T),z}$ is the maximal ideal and $d \gg 0$. Furthermore, for each point $z \in \text{Sing}(C)$ away from the singular branch of $C$ take the subspace $\mathcal{B}^{eg}_z \subset \mathcal{B}_z$ parameterizing local equigeneric deformations (i.e., preserving the $\delta$-invariant, see [8, Page 433, item (2)]), for the point $z_c \in \text{Sing}(C)$, the center of the singular branch of $C$, take the subspace $\mathcal{B}^{ec}_z \subset \mathcal{B}_z$ parameterizing local equiclassical deformations, for each point $z_i = C \cap D_i$, $i = 1, 2, 3$, take the subspace $\mathcal{B}^{tan}_z \subset \mathcal{B}_z$ parameterizing deformations preserving the intersection multiplicity at the intersection point with $D_i$ (while the intersection point may move along $D_i$). We have a natural embedding of the germ of $|L_{\text{Tor}(T)}|$ at $C$ into $\prod_z \mathcal{B}_z$, where $z$ runs over $\text{Sing}(C) \cup \{z_1, z_2, z_3\}$. We claim that the image of that germ intersects transversally in $\prod_z \mathcal{B}_z$ with the product

$$\prod_{z \in \text{Sing}(C) \setminus \{z_c\}} \mathcal{B}^{eg}_z \times \mathcal{B}^{ec}_z \times \prod_{i=1}^3 \mathcal{B}^{tan}_{z_i}.$$ 

Indeed, this claim amounts to the $h^1$-vanishing (49) established in the proof of Lemma A4. Note that the germ of the family $M$ at $C$ is the preimage of the considered intersection in $|L_T|$. Combining this with the statement of Lemma A4(3), we obtain that the choice of am admissible enhanced tropical limit (as in the proof of Theorem 2.10), containing a curve $C \in M$, together with the choice in the conditions to pass through the configuration $w$ yields $\text{mult}(\mathcal{B}^{eg}_{2k}(C, z))$ families $\{C^{(t)}\}_{t \in (\mathbb{C}, 0) \setminus \{0\}}$ of curves of genus $G$ with nodes and one cusp as required.

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