Observables in a Noncommutative Approach to the Unification of Quanta and Gravity. A Finite Model

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Abstract

We further develop a noncommutative model unifying quantum mechanics and general relativity proposed in Gen. Rel. Grav. (2004) 36, 111-126. Generalized symmetries of the model are defined by a groupoid $\Gamma$ given by the action of a finite group on a space $E$. The geometry of the model is constructed in terms of suitable (noncommutative) algebras on $\Gamma$. We investigate observables of the model, especially its position and momentum observables. This is not a trivial thing since the model is based on a noncommutative geometry and has strong nonlocal properties. We show that, in the position representation of the model, the position observable is a coderivation of a corresponding coalgebra, “coparallely” to the well known fact that the momentum observable is a derivation of the algebra. We also study the momentum representation of the model. It turns out that, in the
case of the algebra of smooth, quickly decreasing functions on \( \Gamma \), the
model in its “quantum sector” is nonlocal, i.e., there are no nontrivial
coderivations of the corresponding coalgebra, whereas in its “gravity
sector” such coderivations do exist. They are investigated.

1 Introduction

Noncommutative geometry is not only a quickly developing branch of mathe-

matics \([7, 9, 17, 18]\), but also finds interesting applications in physics. Within
its conceptual framework several approaches have been elaborated to unify
relativity and quanta (see, for example, \([2, 3, 4, 6, 19, 20]\)), and noncommu-
tative methods more and more deeply penetrate into superstring or M-theories
(first indication \([8]\), first result, \([21]\), reviews \([14, 16]\)). In a series of works
\([10, 11, 12]\) we have proposed a scheme for unifying general relativity and
quantum mechanics based on a noncommutative algebra related to a trans-
formation groupoid which plays the role of generalized symmetries of the
model. In \([13]\) we have tested the scheme by constructing a simplified (but
still mathematically interesting) model and computing many of its details.
In the present work, we discuss observables of this model; in particular, posi-
tion and momentum observables. It is a common knowledge that the algebra
structure of observables is a crucial ingredient of the standard formulation of
quantum mechanics; our analysis has disclosed that its coalgebra structure
is also implicitly present in the standard approach.

The present paper is thus a continuation of \([13]\) (we will refer to it as to a
“previous work”), but, for the reader’s convenience, we shortly summarize its
main features and results. We construct our transformation groupoid in the
following way. Let \( \tilde{E} \) be a differential manifold and \( \tilde{G} \) a group acting smoothly
and freely on \( \tilde{E} \). In consequence, we have the bundle \(( \tilde{E}, \pi_M, M = \tilde{E}/\tilde{G})\).
We can think of it as of the frame bundle, with \( \tilde{G} \) the Lorentz group, over
space-time \( M \). Now, we choose a finite subgroup \( G \) of \( \tilde{G} \), a cross section
\( S : M \to \tilde{E} \) of the above bundle (it need not be continuous), and define
\( E = \cup_{x \in M} S(x)G \). The fact that \( G \) acts freely (to the right) on \( E \), allows
us to define the transformation groupoid structure on the Cartesian product
\( \Gamma = E \times G \) (for details see the previous work).

The choice of the cross section \( S : M \to \tilde{E} \) can be thought of as the choice
of a gauge for our model. To be more precise every \( \gamma \in \Gamma \) can be presented in
the form $\gamma = (S(x)g, \bar{g})$ where $g, \bar{g} \in G$. The set of all $\gamma$'s with the beginning at $p \in E$ is denoted by $\Gamma_p$ (a “fiber” of $\Gamma$ over $p$). We define the Hilbert space $L^2(\Gamma_p) = \{ u : \Gamma_p \to \mathbb{C} : \Sigma_{g \in G} |u(S(x)g_0, g)|^2 < \infty \}$ with the scalar product $\langle u, v \rangle = \Sigma_{g \in G} \bar{u}(S(x)g_0, g)v(S(x)g_0, g)$. If $L_h : L^2(\Gamma_p) \to L^2(\Gamma_p)$ is a left translation, it is straightforward to show that $L_h$ is a unitary operator with respect to the above scalar product. It transforms $S(x)$ into $S(x)h$ (for each fibre independently). In this sense, our choice of the gauge is unique up to unitary transformations.

The groupoid $\Gamma$ is a key structure of our model. It represents a space, the elements of which are symmetry operations of the model. The algebra $A = C^\infty(\Gamma, \mathbb{C})$ of smooth complex valued functions on $\Gamma$ (if necessary, we shall assume that they vanish at infinity) plays the role of an algebraic counterpart of this symmetry space. In the previous work, we have reconstructed geometry of the groupoid $\Gamma = E \times G$ (including generalized Einstein’s equations) in terms of this algebra. The Cartesian product structure of the groupoid $\Gamma$ has enabled us to consider its two natural components. By projecting the full geometry into the $E$-direction we recover the usual space-time geometry and, consequently, the standard general relativity. It is a remarkable fact that this can equivalently be achieved by suitably averaging elements of the algebra $A$ (see below Section 3). On the other hand, the regular representation $\pi_p : A \to \text{End}(\mathcal{H}_p)$ of the groupoid algebra $A$ in a Hilbert space $\mathcal{H}_p$, for $p \in E$, leads to the $G$-component of the model which can be considered as its quantum sector.

In the present work, we develop the model by considering its observables, especially its position and momentum observables. This is not a trivial thing. The model is based on a noncommutative algebra $A$ and to determine these two observables means to disentangle its local features (essentially rooted in the center of the algebra $A$) from its nonlocal properties.

The organization of our material runs as follows. In Section 2, we present the preliminaries of the model, and study some properties of the subalgebra $A_{proj} \subset A$ which is isomorphic to the center of $A$, and serves to reconstruct the standard geometry of space-time $M$. In Section 3, we formulate the eigenvalue equation for observables of the model, and show that by averaging functions belonging to $A_{proj}$ we obtain functions on $M$. The position observable is discussed in Section 4. We show that, in the position representation of the model, the position observable is a coderivation of a corresponding coalgebra, whereas the momentum observable is a derivation of the algebra.
(as it is well known). This remains valid in the usual quantum mechanics (see below, Example 1).

In Section 5, we study derivations and coderivations of the group algebra and its dual in the case of a finite group, and, in Section 6, we extend this analysis to the algebra and the dual coalgebra on the groupoid $\Gamma = E \times G$. To do so we limit ourselves to the case $M = \mathbb{R}^n$, and consider the space $S$ of smooth, quickly decreasing functions on $\Gamma$ (the Schwarz space). It can be equipped with the algebra structure in two ways which gives us two algebras, $S_1$ and $S_2$, on $\Gamma$. We then extend the space $S$ on $\Gamma$ to a distribution space $S'$ and obtain, correspondingly, two coalgebras $S'_1$ and $S'_2$ on $\Gamma$. We show that $S'_2$ defines the position representation of our model, and $S'_1$ its momentum representation. We also demonstrate that a coderivation of the coalgebra $S'_2$ is the position observable in the position representation, whereas a coderivation of the coalgebra $S'_1$ is the position observable in the momentum representation. It turns out that in the $G$-component of the model there is no localization (there exist no non-trivial corresponding coderivations). This remains in agreement with the “noncommutative paradigm” of this aspect of the model. In the $E$-component of the model, the corresponding coderivations do exist and the local properties are preserved.

Finally, in Section 7, we present the position and momentum observables in the elegant language of a sheaf of algebras on the groupoid $\Gamma$.

2 Preliminaries

Let, as usually, $\Gamma = E \times G$, with $G$ a finite group, be a transformation groupoid. We shall consider the algebra $A = C_0^\infty(\Gamma, \mathbb{C})$ of smooth, complex valued functions on $\Gamma$ vanishing at infinity, i.e., functions vanishing at infinity on every connected component of $\Gamma$ diffeomorphic with $M$. Hermitian elements of this algebra are candidates for being observables of the model. However, the true observable should leave some traces in space-time $M$ where it could be registered by a measuring device. This is guaranteed by the following construction.

Let $C_0^\infty(M)$ denote the algebra of smooth functions on $M$ vanishing at infinity. We define

$$A_{\text{proj}} = pr^*(C_0^\infty(M))$$

where $pr = \pi_M \circ \pi_E$. $A_{\text{proj}}$ is an algebra without unit. Let $(C_0^\infty(M))_M$ denote
the set of complex functions \( f : M \rightarrow \mathbb{C} \) such that, for any such function and for any \( x \in M \), there exists an open neighborhood \( U \) of \( x \) and a function \( \phi \in C^\infty_0(M) \) satisfying the condition \( f|U = \phi|U \). We will say that functions of \((C^\infty_0(M))_M\) are localized to \( M \).

**Lemma 2.1** \((C^\infty_0(M))_M = C^\infty(M)\) where \( C^\infty(M) \) denotes the algebra of all smooth functions on \( M \).

**Proof** The inclusion \( C^\infty(M) \subset (C^\infty_0(M))_M \) is obvious. It is enough to show that every continuous function \( f : M \rightarrow \mathbb{C} \) belongs to \((C^\infty_0(M))_M\). But this is indeed the case. For any \( x \in M \) there exists an open neighborhood \( U \) of \( x \) and a function \( \phi \in C^\infty_0(M) \) such that \( \phi|U = 1 \); of course \( \phi f \in C^\infty_0(M) \), and \( f|U = \phi f|U \). \( \square \)

Although the algebra \( A_{proj} \) does not contain constant functions, we can always – on the strength of the above lemma – recover them locally. Let us also notice that \((A_{proj})_\Gamma\) is a subalgebra of the algebra \((C^\infty_0(\Gamma, \mathbb{C}))_\Gamma\). Therefore, we can safely assume that the Hermitian elements of \( A_{proj} \) represent observables of the model.

Let us now consider the regular representation \( \pi_p : A \rightarrow \mathcal{B}(\mathcal{H}_p) \) of the algebra \( A \) in the Hilbert space \( \mathcal{H}_p = L^2(\Gamma_p) \) given by

\[
(\pi_p(a)\psi)(\gamma) = (a \ast \psi)(\gamma) = \sum_{\gamma_1 \in \Gamma_p} a(\gamma \circ \gamma_1^{-1})\psi(\gamma_1)
\]

for \( \gamma \in \Gamma_p \). Let further \( I_p : L^2(G) \rightarrow L^2(\Gamma_p) \) be the obvious isomorphism of Hilbert spaces. For every \( a \in A \) we define

\[
\bar{\pi}_p(a) = I_p^{-1} \circ \pi_p(a) \circ I_p.
\]

Clearly, \( \bar{\pi}_p(a) \in \mathcal{B}(L^2(G)) \).

**Lemma 2.2** \( L^{-1}_{g_0} \bar{\pi}_p(a) L_{g_0} = \bar{\pi}_{pg_0}(a) \) for every \( g_0 \in G \).

**Proof** Let \( \psi_p \in L^2(\Gamma_p) \) and \( \psi \in L^2(G) \). We have \( \psi_p = I_p(\psi) \), and we compute

\[
(\pi_{pg_0}(a)\psi_{pg_0})(\tilde{\gamma}) = (a \ast \psi_{pg_0})(\tilde{\gamma}) = \sum_{\gamma_1 \in \Gamma_{g_0}} a(\tilde{\gamma} \circ \gamma_1^{-1})\psi_{pg_0}(\gamma_1) = \sum_{g_1 \in G} a(pg_0g_1, g_1^{-1}g)\psi(g_1) = \sum_{g_1' \in G} a(pg_1', g_1'^{-1}g_0g)\psi(g_0^{-1}g_1').
\]

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In the last line we have made the substitution $g_0 g_1 \mapsto g'_1$. Now, let $\tilde{\psi} = L_{g_0} \psi$, and we compute

$$
\pi_p(a) \tilde{\psi}_p(\gamma) = \sum_{\gamma_1 \in \Gamma_p} a(\gamma \circ \gamma_1^{-1}) \tilde{\psi}(\gamma_1)
$$

$$
= \sum_{g_1 \in G} a(pg_1, g_1^{-1} g) \tilde{\psi}(p, g_1)
$$

$$
= \sum_{g_1 \in G} a(pg_1, g_1^{-1} g_0 g_1) \tilde{\psi}(g_1)
$$

In the last line we have substituted $g \mapsto g_0 g$. We thus have

$L^{-1}_{g_0}(\text{left hand side}) = (\text{right hand side})$. □

We now define the norm in the algebra $A$ in the following way

$$
||a|| = \sup_{p \in E} ||\pi_p(a)||.
$$

The above lemma can be written in the form

$$
\pi_{s(x)g} = L_{g}^{-1} \circ \pi_{s(x)} \circ L_{g},
$$

hence

$$
||\pi_{s(x)g}(a)|| = ||\pi_{s(x)}(a)||.
$$

We see that supremum is taken over $M$, i.e.,

$$
||a|| = \sup_{x \in M} ||\pi_{s(x)}(a)||.
$$

For a fixed $x \in M$ we have a finally dimensional vector space (of matrices). In such a space all norms are equivalent. In the matrix representation we write

$$
\pi_{s(x)}(a) = (M_a(x))_{i,j},
$$

and

$$
||a|| = \sup \sum_{i,j} |M_a(x)|_{i,j}.
$$

Since functions $a$ vanish at infinity, this supremum is finite. Now, we complete the algebra $\mathcal{A}$ in the above norm to the $C^*$-algebra. From now on we shall always consider the algebra $\mathcal{A}$ as a $C^*$-algebra.

A function $\psi \in L^2(\Gamma)$ is said to be $G$-invariant if

$$
\psi(p_1, g_1) = \psi(p_2, g_2),
$$

when there exists $g \in G$ such that $p_2 = p_1 g$. The set of all $G$-invariant functions will be denoted by $L^2_G(\Gamma)$. $L^2_G(\Gamma)$ is evidently isomorphic with $L^2(M)$. The fact that $\mathcal{A}$ is a $C^*$-algebra allows us to employ in our model the algebraic quantization method.
3 Eigenvalue Equation

Let $a$ be a Hermitian element of $A_{\text{proj}}$, and $\psi \in L^2_G(\Gamma)$. The eigenvalue equation for the observable $a$ assumes the form

$$\pi_p(a)\psi = \lambda_p \psi$$

where $\lambda_p$ is the eigenvalue of $\pi_p(a)$. Here we have $p \in \pi^{-1}_M(x), x \in M$. For simplicity we consider a nondegenerate case. We compute

$$(\pi_p(a)\psi)(\gamma) = (\psi * a)(\gamma) = \sum_{g_1 \in G} \psi(p, g_1)a(pg_1, g_1^{-1}g) = \sum \tilde{\psi}(x)\tilde{a}(x) = |G|\tilde{\psi}(x)\tilde{a}(x)$$

where $\gamma \in \Gamma_p$, and we have introduced the abbreviations: $\tilde{\psi}(x) = \psi(p, g)$ for every $g \in G$, and $\tilde{a}(x) = a(p, g)$ for every $p \in \pi^{-1}_M(x)$ and $g \in G$. If we further denote

$$(\pi_p(a)\psi)(x) = (\pi_p(a)\psi)(\gamma),$$

where $\gamma \in \Gamma_p$ and $p \in \pi^{-1}_M(x)$, then we finally have

$$(\pi_p(a)\psi)(x) = |G|\tilde{\psi}(x) \cdot \tilde{a}(x).$$

Hence, the eigenvalue of this observable is

$$\lambda_p = |G| \cdot \tilde{a}(x)$$

for every $p \in \pi^{-1}_M(x)$.

As we can see, the observable $a \in \mathcal{A}$ indeed leaves a trace in space-time $M$. In the previous work we have shown that the transition from the noncommutative geometry on the groupoid $\Gamma$ to the classical geometry on the manifold $M$ can be done with the help of a suitable averaging procedure of elements of the groupoid algebra. The following lemma establishes the equivalence of this averaging method with the one using the subalgebra $A_{\text{proj}}$.

**Lemma 3.1** The averaging of a function belonging to $A_{\text{proj}}$ gives a function of $C_0^\infty(M)$.
Proof Let $A_a$ be the matrix representation of $a \in \mathcal{A}$. Then its averaging is $\langle A_a \rangle = \frac{1}{|G|} \text{Tr}(A_a)$. If $a \in \mathcal{A}_{\text{proj}}$ then $a = pr^*f$ for some $f \in C_0^\infty(M)$. By averaging we obtain

$$\langle A_{pr^*f} \rangle = \frac{1}{|G|} \cdot f \cdot \text{Tr}[1] = f.$$ 

Here $[1]$ denotes the matrix with all entries equal to one. Of course, $\text{Tr}[1] = |G|$. □

4 Position Operator as a Coderivation

In this section we consider the position observable of our model. The projection $pr : \Gamma \to M$, $pr = \pi_M \circ \pi_E$, which is clearly connected with localization in space-time $M$, is not a numerical function (it has no values in $\mathbb{R}$ or $\mathbb{C}$), and consequently it does not belong to the algebra $\mathcal{A}$). However, in the four-dimensional case, if we choose a local coordinate map $x = (x^\mu)$, $\mu = 0, 1, 2, 3$, in $M$ then the projection $pr$ determines four observables in the domain $D_x$ of $x$

$$pr_\mu = x^\mu \circ pr$$

We thus have the system of four position observables $pr = (pr_0, pr_1, pr_2, pr_3)$. For a fixed $\mu$ one has $pr_\mu \in \mathcal{A}_{\text{proj}}|_{D_x}$, and it is Hermitian.

Let us notice that the projection $pr : \Gamma \to M$ contains, in a sense, the information about all possible local observables $pr_\mu$. This can be regarded as a "noncommutative formulation" of the fact that there is no absolute position but only the position with respect to a local coordinate system.

From the previous work (Section 7) it follows that in the matrix representation of the algebra $\mathcal{A}$ one has

$$(\pi_q(pr_1))(\xi) = \xi^T \cdot M_{pr_1} = x^1 \cdot \xi,$$

$x \in M$, in the local map, where $\xi^T$ is $\xi \in \mathbb{C}^n$ transposed, and $M_{pr_1}$ denotes the matrix corresponding to the projection $pr_1$. We see that the position observable in the "quantum sector" of our model has the same form as in the ordinary quantum mechanics. This indicates that we are working in the position representation of the model.
We shall now demonstrate the connection between the position operator and the coderivation of a coalgebra. Let \((A, +, m, \cdot)\) be an associative (not necessarily commutative) algebra over the field of complex numbers \(\mathbb{C}\). Here \(m : A \otimes A \to A\) is a product map. We define the dual space \(A^* = \text{Hom}(A, \mathbb{C})\) which has the structure of a coalgebra with the coproduct \(\Delta : A^* \to A^* \otimes A^*\) given by

\[
\Delta(\varphi)(f, g) = \varphi(m(f \otimes g)),
\]

\(g, f \in A, \varphi \in \mathbb{C}^*\). We assume that \((A \otimes A)^* \subset A^* \otimes A^*\). This is always true for finite dimensional algebras. Rather than considering a completion of the tensor product \(A^* \otimes A^*\) (see [1, Chapters 5-6]), we shall slightly modify the coproduct definition (in Section 6).

We recall that a derivation \(X\) of the algebra \(A\) is, by definition, a linear map \(X : A \to A\) satisfying the Leibniz rule

\[
X \circ m = m \circ (X \otimes \text{id}_A) + m \circ (\text{id}_A \otimes X).
\]

The set \(\text{Der}_A\) of all derivations of the algebra \(A\) is a \(A\)-module.

Let \(X^* : A^* \to A^*\) be a linear mapping satisfying the following condition

\[
\Delta \circ X^* = (X^* \otimes \text{id}_{A^*}) \circ \Delta + (\text{id}_{A^*} \otimes X^*) \circ \Delta,
\]

which we shall call the co-Leibniz rule. For \(\varphi \in A^*\) it can be written in the form

\[
\Delta(X^*(\varphi)) = X^*(\varphi(1)) \otimes \varphi(2) + \varphi(1) \otimes X^*(\varphi(2)).
\]

**Definition 4.1** Let \(A\) be an associative not necessarily commutative algebra, and \(A^*\) its dual coalgebra. A coderivation of the coalgebra \(A^*\) is a linear map \(X^* : A^* \to A^*\) satisfying the co-Leibniz rule.

**Proposition 4.1** Let \(X : A \to A\) be an endomorphism of an algebra \(A\) as a linear space, and \(X^* : A^* \to A^*\) its adjoint endomorphism, i.e.,

\[
(X^*(\varphi))(a) = \varphi(X(a)),
\]

\(\varphi \in A^*, a \in A\). \(X\) satisfies the Leibniz rule if and only if \(X^*\) satisfies the co-Leibniz rule.
Proof Let us suppose that $X$ satisfies the Leibniz rule, then
\[
(\Delta(X^*\varphi))(a_1, a_2) = (X^*\varphi)(a_1 \cdot a_2) = \varphi(X(a_1 \cdot a_2)) = \varphi(X(a_1)a_2) + \varphi(a_1X(a_2)) = \Delta\varphi(X(a_1), a_2) + \Delta\varphi(a_1, X(a_2)) = (((X^* \otimes \text{id}_{A^*}) \circ \Delta + (\text{id}_{A^*} \otimes X^*) \circ \Delta))(\varphi))(a_1, a_2),
\]
and similarly in the other direction. \(\square\)

Example 4.1 Let us consider the algebra $\mathcal{A} = (L^1(\mathbb{R}), \ast)$, where $\ast$ denotes the convolution. The Fourier transformation of $f \in L^1(\mathbb{R})$ gives
\[
\hat{f}(x) = \int_{\mathbb{R}} e^{-ipx} f(p)dp.
\]
We have
\[
\int_{\mathbb{R}} e^{-ipx} \frac{d}{dp} f(p)dp = ix \int_{\mathbb{R}} e^{-ipx} f(x)dx
\]
or, in the abbreviated form,
\[
\hat{f}'(x) = ix \cdot \hat{f}(x),
\]
$x \in \mathbb{R}$.

The dual of $\mathcal{A} = L^1(\mathbb{R})$ is $\mathcal{A}^* = L^\infty(\mathbb{R})$ (with the pointwise multiplication). Let us denote
\[
(X^*\varphi)(x) = x\varphi(x)
\]
for $\varphi \in L^\infty(\text{bf R})$. $X^*$ is an operator adjoint to the operator $X : \mathcal{A} \to \mathcal{A}$ given by $X = -i\frac{d}{dp}$.

Since $\mathbb{R}$ is an Abelian group, as a coproduct in $L^\infty(\mathbb{R})$, we have
\[
\Delta \varphi(x_1, x_2) = \varphi(x_1 + x_2)
\]
for $\varphi \in L^\infty(\mathbb{R})$, $x_1, x_2 \in \mathbb{R}$. It can be readily checked that $X_1^*$ is a coderivation of the coalgebra $L^\infty(\mathbb{R})$. This coderivation is called the position operator in the position representation.

The above example suggests that in noncommutative generalizations of quantum mechanics we can treat coderivations of suitable coalgebras as counterparts of the position operator (in the position representation). We will explore this possibility in the following sections.
5 Derivations and Coderivations on a Finite Group

In this section, we apply considerations of the previous section to the quantum mechanics on a finite group $G$. In this case, we have the group algebra $H = \mathbb{C}G$ of formal linear combinations of elements of $G$ (which corresponds to the momentum representation) and the algebra $H^* = \mathbb{C}(G)$ of complex functions on $G$ with pointwise multiplication (corresponding to the position representation of the model). Let us notice that both $H$ and $H^*$ are bialgebras with coproducts being conjugations of products of the corresponding dual algebras. We assume that $G$ is a non-Abelian group.

Let us consider the Fourier transformation of a finite group $G$

$$\mathcal{F} : G \rightarrow \prod_{d_\lambda} M_{d_\lambda}(\mathbb{C}) \equiv \mathcal{M},$$

here $d_\lambda$ is the dimension of the representation from the class $\lambda$, $\lambda \in \hat{G}$ where $\hat{G}$ is the dual object of $G$, given by

$$\mathcal{F}(g) = (T_\lambda(g))_{g \in G}$$

with $T_\lambda(g) \in M_{d_\lambda}(\mathbb{C})$. This is extended by linearity to the whole of the group algebra $\mathcal{F} : \mathbb{C}G \rightarrow \mathcal{M}$ [15, Chapter 12]. As it is well known, $\mathcal{F}$ is the isomorphism of algebras. We thus can change the algebra $\mathbb{C}G$ into the corresponding matrix algebra. The latter algebra has only inner derivations with $\text{dim}(\text{Inn}(M_n(\mathbb{C}))) = n^2 - 1$.

**Lemma 5.1** Let $\mathcal{A}$ be a unital algebra, $J$ an ideal defined by a central idempotent, i.e., $J = e\mathcal{A} = \mathcal{A}e$, $e^2 = e$, $e \in Z(\mathcal{A})$, and $D$ a derivation of the algebra $\mathcal{A}$. Then

$$DJ \subset J, \quad De = 0.$$  

**Proof** The fact that $e$ is idempotent implies

$$De = D(e^2) = (2De)e \in J.$$

From $1 = e + e'$, where $e' = 1 - e$ is also an idempotent, it follows that $ee' = e'e = 0$, and we have

$$D1 = De + De' = 0.$$
Therefore,
\[ De = -De', \]
and
\[ D(ae) = (Da)e \in J \]
for any \( a \in A \). Thus \( DJ \subset J \). \( \square \)

**Lemma 5.2** Any isomorphism of algebras determines the isomorphism of the corresponding derivation spaces in such a way that the inner derivations are transformed into the inner derivations. \( \square \)

**Theorem 5.1** In the algebra \( C_G \) there are only inner derivations, and in the algebra \( C(G) \) there are no nonzero derivations.

**Proof** (1) Let \( J_{\lambda_0} \subset M \) be an ideal of \( C_G \) such that
\[ J_{\lambda_0} = \{(0,0,\ldots,A_{\lambda_0},0,\ldots,0) : A_{\lambda_0} \in M_{\lambda_0}\}. \]
If \( \bar{D} \) is a derivation of the algebra \( M \) then, on the strength of Lemma 5.1, \( \bar{D}J_{\lambda_0} \subset J_{\lambda_0} \). Therefore \( J_{\lambda_0} \) is isomorphic with \( M_{\lambda_0}(C) \) which has only inner derivations. Hence
\[ \bar{D}|_{J_{\lambda_0}} = \text{ad}B_{\lambda_0} \]
where \( B_{\lambda_0} \in J_{\lambda_0} \).

Let now \( B = (B_\lambda)_{\lambda \in \hat{G}} \) be the sequence as above. Therefore, \( \bar{D} = \text{ad}B \) and, on the strength of Lemma 5.2, in the algebra \( C_G \) there exist only inner derivations.

(2) The element \( \delta_{g_0} \) is central idempotent in \( C(G) \), i.e., \( \delta_{g_0} \cdot \delta_{g_0} = \delta_{g_0} \), and such elements form a basis \( \{\delta_g\} \) in \( C(G) \). Therefore, \( D(a) = 0 \) for every \( a \in C(G) \) and any derivation \( D \) of \( C(G) \). \( \square \)

To sum up, in the bialgebra \( C_G \) there exist only (nonzero) derivations, and in the bialgebra \( C(G) \) only (nonzero) coderivations.

Our goal is now to discuss the position observable on a finite group \( G \). It is given by a coderivation of the coalgebra \( C(G) \), and could be found as an adjoint of the derivation of the algebra \( C_G \). But the latter algebra has only nonzero inner derivations; they are of the form
\[ X_{g_0}(g) = (\text{ad}g_0)(g) = g_0g - gg_0 \]
with \( g_0 \in G \). Therefore, if \( f \in \mathbb{C}(G) \), we dually have

\[
(X_{g_0}^*)(f)(g) = f(X_{g_0}g) = f(g_0g) - f(gg_0).
\]

The eigenvalue equation assumes the form

\[
(X_{g_0}^*)f = \lambda \cdot f
\]

where \( \lambda \in \mathbb{C} \), or

\[
f(g_0g) - f(gg_0) = \lambda \cdot f(g).
\]

This equation has non-trivial solutions. For instance, it can be easily seen that the eigenspace corresponding to the eigenvalue \( \lambda = 0 \) is the space of central functions.\(^1\) Therefore, we can have a well determined localization on a finite group. This remains in agreement with the fact that \( \mathbb{C}(G) \) is a commutative algebra.

### 6 Localization on a Transformation Groupoid

In this section, we extend the above analysis to the case of the transformation groupoid \( \Gamma = E \times G \) where \( G \) is a finite (non-Abelian) group. To do this we limit our considerations to the case when the base space \( M = \mathbb{R}^n \). We extend the space of functions on \( \Gamma \) to the distribution space on \( \Gamma \). Let then \( \mathcal{S} = \mathcal{S}(\Gamma, \mathbb{C}) \) be the space of smooth, quickly decreasing functions on \( \Gamma \), called also the Schwarz space (we recall that the Schwarz space on \( \mathbb{R}^n \) is the vector space \( \mathcal{S} \) of smooth functions on \( \mathbb{R}^n \) such that for every \( \phi \in \mathcal{S} \), \( \phi \) and its derivatives decrease more rapidly than any power of \( 1/|x| \), \( x \in \mathbb{R}^n \), when \( |x| \) goes to infinity, [5, p. 474]). In the following we consequently use the matrix representation of \( \mathcal{S} \). We thus have

\[
\mathcal{S} = \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{M}_n(\mathbb{C}).
\]

A (nonunital) algebra structure in \( \mathcal{S} \) can be introduced in two ways:

\(^1\)We remind that a function \( f \) is central if \( f(g) = f(g_0g_0^{-1}) \) for any \( g, g_0 \in G \).
1. 

\[ m_1[(f \otimes A) \otimes (g \otimes B)] = fg \otimes A \cdot B, \]

where \( f, g \in \mathcal{S}(\mathbb{R}^n) \), \( A, B \in M_n(\mathbb{C}) \), and \( f \) and \( g \) are multiplied pointwise, and \( A \) and \( B \) are multiplied in the usual matrix way; we will write \( \mathcal{S}_1 = (\mathcal{S}, m_1) \).

2. 

\[ m_2[(f \otimes A) \otimes (g \otimes B)] = f \ast g \otimes A \cdot B \]

where \( \ast \) denotes the usual convolution of functions; in this case we write \( \mathcal{S}_2 = (\mathcal{S}, m_2) \).

We should distinguish two kinds of derivations of the above algebras:

1. **Vertical derivations**, \( X_A, A \in M_n(\mathbb{C}) \), are of the form

   \[ X_A = \text{id} \otimes \text{ad}A, \]

   or, for \( f \in \mathcal{S}(\mathbb{R}^n) \),

   \[ X_A(f \otimes B) = [1 \otimes A, f \otimes B] \]

   on the basis elements and extended by linearity.

2. **Horizontal derivations** are linearly generated by

   \[ X_k = D_k \otimes \text{id}_{M_n(\mathbb{C})} \]

   where \( k = 0, 1, \ldots, n \) and \( D_k = \frac{\partial}{i \partial x_k} \), i.e.,

   \[ X_k(g \otimes B) = \frac{1}{i} \frac{\partial g}{\partial x_k} \otimes B, \]

   and are extended by linearity.

As a distribution space we assume

\[ \mathcal{S}' = \mathcal{S}'(\mathbb{R}^n) \otimes M_n(\mathbb{C}) \]

where \( \mathcal{S}'(\mathbb{R}^n) \) is the dual of \( \mathcal{S}(\mathbb{R}^n) \) (but \( \mathcal{S}' \) is not the dual of \( \mathcal{S} \)). The space \( \mathcal{S}' \) has no algebra structure. However, if we slightly modify the usual condition
for coproduct, we can introduce in it (in two ways) the coalgebra structure. The usual coproduct would be the mapping $\Delta : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n) \otimes S'(\mathbb{R}^n)$, whereas we assume $\Delta_i : S'(\mathbb{R}^n) \to (S(\mathbb{R}^n) \otimes S(\mathbb{R}^n))'$, $i = 1, 2$, (which is also valid in the usual approach for finally dimensional cases), and for $T \in S'(\mathbb{R}^n)$ we define
\[(\Delta_i T)(f \otimes g) = T(m_i(f \otimes g))\]
where $m_i$ are restricted to the first factors in the corresponding tensor products, or $\Delta_i = m_i^*$, i.e., our coproduct is a dual homomorphism of linear spaces. This shows that our definition is very natural. However, we must accordingly adapt the associativity condition. We introduce the following notation
\[\Delta_i \bar{\otimes} \text{id} : (S(\mathbb{R}^n) \otimes S(\mathbb{R}^n))' \to (S(\mathbb{R}^n) \otimes S(\mathbb{R}^n) \otimes S(\mathbb{R}^n))'\]
where $\Delta_i \bar{\otimes} \text{id} = (m_i \otimes \text{id})^*$ and, analogously, $\text{id} \bar{\otimes} \Delta_i = (\text{id} \otimes m_i)^*$. With this notation our associativity condition reads
\[(\Delta_i \bar{\otimes} \text{id}) \circ \Delta_i = (\text{id} \bar{\otimes} \Delta_i) \circ \Delta_i.\]
It can be easily checked that the above defined coproducts satisfy this condition.

We define the coproduct on $M_n(\mathbb{C})$ in the usual way; in the basis $\{E_{ij}\}$ we have
\[\Delta_{M_n(\mathbb{C})} = E_{ij} \otimes E_{ij},\]
and we extend this by linearity.

Finally, we set
\[\bar{\Delta}_i = \Delta_i \otimes \Delta_{M_n(\mathbb{C})}.\]
As in Section 4, coderivations of $S'$ are defined to be linear mappings of $S'$ adjoint to the derivations of $S$. With the coproduct defined as above, this requires a slight modification of the co-Leibniz rule. If $X$ is a derivation of $S(\mathbb{R}^n)$ then $X^*$ is a coderivation of $S'(\mathbb{R}^n)$, i.e., it satisfies the co-Leibniz rule
\[\Delta \circ X^* = (X^* \bar{\otimes} \text{id}) + (\text{id} \bar{\otimes} X^*)\]
where we define
\[X^* \bar{\otimes} \text{id} = (X \otimes \text{id})^*\]
and analogously for \( \text{id} \otimes X^* \). Remembering that the distributional derivative of \( T \) is defined to be
\[
(D_k T)(f) = -T(D_k f),
\]
we have \( X = D_k : \mathcal{S}_1(\mathbb{R}^n) \to \mathcal{S}_1(\mathbb{R}^n) \) with \( D_k = \frac{1}{i} \frac{\partial}{\partial x_k} \), and \( X^* : \mathcal{S}'_1(\mathbb{R}^n) \to \mathcal{S}'_1(\mathbb{R}^n) \) with \( X^* = -D_k \). It can be readily checked that \(-D_k \) satisfies the above modified co-Leibniz rule.

Now, we consider the distributional Fourier transform \( \mathcal{F}^* : \mathcal{S}'_1(\mathbb{R}^n) \to \mathcal{S}'_2(\mathbb{R}^n) \) which is given by
\[
(\mathcal{F}^* T)(f) = T(\mathcal{F} f)
\]
where \( \mathcal{F} f \) is to be understood as
\[
(\mathcal{F} f)(x) = \int_{\mathbb{R}^n} f(t)e^{-itx}d\mu_k(x)
\]
with \( d\mu_k(x) = dx/(2\pi)^{n/2} \), normalized Lebesgue measure. \( \mathcal{F} \) is a continuous algebra isomorphism of \( \mathcal{S}_1(\mathbb{R}^n) \) into \( \mathcal{S}_2(\mathbb{R}^n) \), and \( \mathcal{F}^* \) is a continuous linear isomorphism of \( \mathcal{S}'_1(\mathbb{R}^n) \) into \( \mathcal{S}'_2(\mathbb{R}^n) \) which is additionally a coalgebra map, i.e., it satisfies the condition
\[
\Delta_2 \circ \mathcal{F}^* = (\mathcal{F}^* \otimes \mathcal{F}^*) \circ \Delta_1.
\]

One can check that, for \( X^* = -D_k \), one has
\[
\mathcal{F}^* (X^* T) = x_k(\mathcal{F}^* T),
\]
or, for \( \hat{T} \in \mathcal{S}'_2(\mathbb{R}^n) \),
\[
\tilde{X}^* \hat{T} = x_k \cdot \hat{T}.
\]
It can be also verified that \( \tilde{X}^* \hat{T} \) is a coderivation of \( \mathcal{S}'_2(\mathbb{R}^n) \).

The question remains to be answered of whether do exist nonzero vertical derivations, i.e., the derivations of the form
\[
\tilde{Y}^* = \text{id} \otimes Y^*
\]
where \( Y^* \) is a coderivation of the coalgebra \( M_n(\mathbb{C})^* \)? Clearly, the answer is “no” since there do not exist nonzero derivations of \( M_n(\mathbb{C})^* \) (with the pointwise multiplication).
Now, we should collect the results and conclusions of this lengthy analysis. First, we make this clear that by the position representation of our model we understand the representation in which the position operator has the form of the multiplication by a coordinate, and by the momentum representation the one in which the momentum operator has the form of the multiplication by a coordinate. In this sense, the coalgebra $S'_2$ defines the position representation of our model, and the coalgebra $S'_1$ its momentum representation, and we have

**Corollary 6.1**

1. The operator $\tilde{X}_k^* \otimes \text{id}_{M_n(C)}$ is a coderivation of the coalgebra $S'_2$, and it is the position observable in the position representation.

2. The operator $X^* \otimes \text{id}_{M_n(C)}$ is a coderivation of the coalgebra $S'_1$, and it is the position observable in the momentum representation. $\blacksquare$

We can see that the localization on the groupoid $\Gamma$ comes only from the “horizontal component” of our model which reflects essentially the space-time geometry; whereas its “vertical component”, representing the quantum sector of the model, is entirely nonlocal.

### 7 A Sheaf Structure on the Groupoid

In this section, we show how could one elegantly describe, by exploring a sheaf structure on the transformation groupoid $\Gamma$, the position and momentum observables of our model. On the Cartesian product $\Gamma = E \times G$ there exists the natural product topology; however, we shall consider a weaker topology in which the open sets are of the form $\pi_E^{-1}(U)$ where $U$ is open in the manifold topology $\tau_E$ on $E$. Every such open set is also open in the topology $\tau_E \times \tau_G$. Indeed, every such set is given by $\pi_E^{-1}(U) = U \times G$.

Let $\mathcal{A}$ be a functor which associates with an open set $U \times G$ the involutive noncommutative algebra $\mathcal{A}(U \times G)$ of smooth compactly supported complex valued functions with the ordinary addition and the convolution multiplication. As it can be easily seen, $\mathcal{A}$ is a sheaf of noncommutative algebras on the topological space $(\Gamma, \pi_E^{-1}(\tau_E))$.

The projection $pr: \Gamma \to M$ can be locally interpreted as a set of (local) cross sections of the sheaf $\mathcal{A}$ (i.e. as a set of position observables). Indeed, for the domain $D_x$ of any coordinate map $x = (x^0, x^1, x^2, x^3)$, the composition
\( x \circ pr = (x^0 \circ pr, x^1 \circ pr, x^2 \circ pr, x^3 \circ pr) \) is a set of such local cross sections of \( A \) on the open set \( \pi^{-1}_E(D_x \times G) \). The global mapping \( pr : \Gamma \to M \) is not a cross section of \( A \).

Let us notice that to a measurement result which is not a number but a set of numbers there does not correspond a single observable but rather a set of observables, i.e., a set of (local) cross sections of the sheaf \( A \).

Now, we define the \textit{derivation morphism} of the sheaf \( A \) over an open set \( U \in \pi^{-1}_E(\tau_E) \) as a family of mappings \( X = (X_W)_{W \subset U} \) such that \( X_W : \mathcal{A}(W) \to \mathcal{A}(W) \) is a derivation of the algebra \( \mathcal{A}(W) \), and for any \( W_1, W_2 \) open and \( W_1 \subset W_2 \subset U \), the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{A}(W_2) & \xrightarrow{X(W_2)} & \mathcal{A}(W_2) \\
\downarrow & & \downarrow \\
\mathcal{A}(W_1) & \xrightarrow{X(W_1)} & \mathcal{A}(W_1)
\end{array}
\]

where \( \rho^{W_2}_{W_1} \) is the known restriction homomorphism. The family of all derivation morphisms indexed by open sets is a sheaf of \( \mathcal{Z}(\mathcal{A}) \)-modules where \( \mathcal{Z}(\mathcal{A}) \) denotes the sheaf of centers of the algebras \( \mathcal{A}(U), U \in \pi^{-1}_E(\tau_E) \).

Components of the momentum observable \( \partial_\mu \) are cross sections of the sheaf of \( \mathcal{Z}(\mathcal{A}) \)-modules of derivations of the sheaf \( \mathcal{A} \) over domains of coordinate maps, and the representation \( \pi_U : \mathcal{A}(U) \to \pi_U(\mathcal{A}(U)) \), where \( U \in \pi^{-1}_E(\tau_E) \), transfers the sheaf structure from the groupoid \( \Gamma \) to the family of operator algebras over the topological space \((\Gamma, \pi^{-1}_E(\tau_E))\).


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