Connections between abelian sandpile models and the $K$-theory of weighted Leavitt path algebras

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Abstract
In our main result, we establish that any conical sandpile monoid $M = \text{SP}(E)$ of a directed sandpile graph $E$ can be realised as the $V$-monoid of a weighted Leavitt path algebra $L_k(F, w)$ (where $F$ is an explicitly constructed subgraph of $E$), and consequently, the sandpile group $\mathcal{G}(E)$ is realised as the Grothendieck group $K_0(L_k(F, w))$. Additionally, we describe the conical sandpile monoids which arise as the $V$-monoid of a standard (i.e., unweighted) Leavitt path algebra.

Keywords Abelian sandpile model · Sandpile monoid · Weighted Leavitt path algebra · Grothendieck group

Mathematics Subject Classification 16S88 · 05C57

1 Introduction

Your descendants will be like the dust of the earth, and you will spread out to the west and to the east, to the north and to the south.

Genesis 28:14

A part of this work was done when the second author was an Alexander von Humboldt Fellow at the University of Münster in the Winter of 2021. He would like to thank both institutions for an excellent hospitality.
The notion of sandpile models encapsulates how objects spread and evolve along a grid. The models were conceived in 1987 in the seminal paper [8] by Bak, Tang and Wiesenfeld as examples of self-organised criticality, or the tendency of physical systems to organise themselves without any input from outside the system, toward critical but barely stable states. The models have been used to describe phenomena such as forest fires, traffic jams, stock market fluctuations, etc. The book of Bak [7] describes how events in nature apparently follow this type of behaviour.

The mathematical formulation of the model is as follows. Consider a sandpile graph, namely a (finite, directed) graph $E$ with a distinguished sink vertex $s$ such that there is a (directed) path from any vertex of $E$ to $s$. Consider a collection of grains of sand placed on each vertex of the sandpile graph (a configuration). A vertex is unstable if it has the same or more grains of sand than the number of edges emitting from it. In this case the vertex topples by sending one grain along each edge emitting from the vertex to each neighbouring vertex. This toppling may cause neighbouring vertices to become unstable. The assumption is that grains arriving to the vertex $s$ vanish. A configuration is stable if no vertex is unstable. Two foundational results used in the theory are: the order of toppling does not matter (thus “abelian” sandpile models); and any configuration can be sequentially toppled to reach a unique stable configuration. This allows for the construction of a finite monoid associated to an abelian sandpile model. Let $\text{SP}(E)$ be the set of all stable configurations of a sandpile graph $E$. The “sum” of two configurations is interpreted as simply adding the number of grains of sand at each vertex corresponding to the two configurations. The operation of addition followed by stabilisation endows the set $\text{SP}(E)$ with the structure of a commutative monoid, called the sandpile monoid (see Definition 2.10 for a precise definition).

The smallest set of configurations of $\text{SP}(E)$ which is closed under adding any grains to them (i.e., the recurrent configurations) forms a group, called the sandpile group $G(E)$ associated to $E$. These algebraic structures constitute one of the main themes of the subject. In a major work [13], Dhar championed the use of $G(E)$ as an invariant which proved to capture many properties of the model. A more algebraic study of these monoids and their groups is carried out in [6, 11]. The books of Klivans [16] and Corry and Perkinson [12] give self-contained treatments of the subject of sandpile models.

In a different realm, the notion of Leavitt path algebras $L_k(E)$ associated to directed graphs $E$, with coefficients in a field $k$, were introduced in 2005 [1, 5]. These are a generalisation of algebras (denoted by $L_k(1,1+k)$) introduced by William Leavitt in 1962 [17] as a “universal” $k$-algebra $A$ of type $(1,1+k)$, so that $A \cong A^{1+k}$ as right $A$-modules, where $k \in \mathbb{N}^+$. The study of the commutative monoid $\mathcal{V}(B)$ of isomorphism classes of finitely generated projective right modules over a unital ring $B$ (with operation $\oplus$) goes back to the work of Grothendieck and Serre [18]. For a Leavitt path algebra $L_k(E)$, the monoid $\mathcal{V}(L_k(E))$ has received substantial attention due to two separate impetuses. On the one hand, these monoids are conical and refinement, and thus related to the realisation problem for von Neumann regular rings [14]. (The problem asks whether every countable conical refinement monoid can be realised as the monoid of a von Neumann regular ring [2, Section 7.2.3].) On the other hand, the group completion of $\mathcal{V}(L_k(E))$ is the Grothendieck group $K_0(L_k(E))$. This group has been used effectively as a complete invariant for a large class of graph $C^*$-algebras.
(these are the analytic versions of Leavitt path algebras). Thus there is a hope that this group could play a similar role in the algebraic setting as well. Whether $K_0$ classifies these algebras has remained one of the most important and yet elusive questions in the theory of Leavitt path algebras [2, Section 7.3.1]. We will touch on this question in Sect. 6.

In fact Leavitt established much more in [17] than mentioned in the previous paragraph; indeed, he showed that, for any $n, k \in \mathbb{N}$, there is a universal $k$-algebra $A$ of type $(n, n+k)$ (denoted $L_k(n, n+k)$) for which $A^n \cong A^{n+k}$ as right $A$-modules. When $n \geq 2$, this universal algebra is not realizable as a Leavitt path algebra. With this in mind, the notion of weighted Leavitt path algebras $L_k(E, w)$ associated to weighted graphs $(E, w)$ were introduced by the second author in 2011 ([15]; see [21] for a nice overview of this topic). The weighted Leavitt path algebras $L_k(E, w)$ provide a natural context in which all of Leavitt’s algebras (corresponding to any pair $n, k \in \mathbb{N}$) can be realised as a specific example. As well, the monoid $\mathcal{V}(L_k(E, w))$ and the corresponding Grothendieck group $K_0(L_k(E, w))$ have been completely described in the two works [15, 20].

In this article we tie the notions of sandpile models and weighted Leavitt path algebras together. We start in Sect. 2 by defining weighted graph monoids, and show that these include the sandpile monoids. We then show in Sect. 3 that any weighted graph monoid satisfies the Confluence Property. In particular, this property will allow us to identify the unit group of the weighted graph monoid (Proposition 3.3). With these preliminaries established, we move towards our main result by casting the theory of sandpile monoids in an abstract algebraic setting. To do so we utilise two foundational works of George Bergman: the Diamond Lemma [10], and the Universal Ring Construction [9]. The first work will allow us to re-establish that sandpile configurations have unique normal forms (Sect. 4), while the second will allow us to realise monoids of certain algebras as graph monoids (Sect. 5, see especially Theorem 5.7). In Sect. 6 we achieve our main result (Theorem 6.1). The key consequence of Theorem 6.1 is that, starting from a sandpile graph $E$ having a specified form (‘conical’), and considering $E$ as a weighted graph $(E, w)$ for an appropriate weight $w$, and considering a specific weighted subgraph $(F, w_r)$ of $E$, we obtain that $SP(E) \cong \mathcal{V}(L_k(F, w_r))$. Consequently, we conclude that $\mathcal{G}(E) \cong K_0(L_k(F, w_r))$ as well. Furthermore, we identify those sandpile monoids which arise as the $\mathcal{V}$-monoid of a standard (i.e., unweighted) Leavitt path algebra.

The connection established in Theorem 6.1 between sandpile monoids and weighted Leavitt path algebras allows us to naturally associate an algebra, a sandpile algebra, to the theory of sandpile models, thereby opening up an avenue by which to investigate sandpile models via the structure of the sandpile algebras, and vice versa.

## 2 Weighted graph monoids and sandpile monoids: basic properties

Throughout we write $\mathbb{N}$ for the set of non-negative integers, and $\mathbb{N}^+$ for the set of positive integers.

We start by setting some terminology and notation about graphs. A directed graph is a quadruple $E = (E^0, E^1, s, r)$, where $E^0$ and $E^1$ are sets and $s, r : E^1 \to E^0$ are
Definition 2.2  A finite directed graph $E$ is called a sandpile graph if $E$ has a unique sink (denote it by $s$), and for every $v \in E^0$ there is a path $p$ with $s(p) = v$ and $r(p) = s$.

A subset $H \subseteq E^0$ is said to be hereditary if for any $e \in E^1$, $s(e) \in H$ implies $r(e) \in H$. For a hereditary subset $H$ of $E$, we define the quotient graph $E/H$ as follows:

$$(E/H)^0 = E^0 \setminus H^0 \quad \text{and} \quad (E/H)^1 = \{e \in E^1 \mid r(e) \notin H\}.$$ 

The source and range maps of $E/H$ are the source and range maps restricted from the graph $E$. We have a natural (inclusion) morphism $\phi : E/H \to E$ which is injective on vertices and edges.

Definition 2.2  A weighted graph is a pair $(E, w)$, where $E$ is a graph and $w : E^1 \to \mathbb{N}^+$ is a map. If $e \in E^1$, then $w(e)$ is called the weight of $e$.

For each regular vertex $v$ in a weighted graph $(E, w)$ we set $w(v) := \max\{w(e) \mid e \in s^{-1}(v)\}$. This gives a map (called $w$ again) $w : E^0_{\text{reg}} \to \mathbb{N}^+$.

A weighted graph is called a vertex weighted graph if for each regular vertex $v$ in the graph, the weights of all edges emitting from $v$ are equal one to the other; i.e., if $w(e) = w(e')$ for all $e, e' \in s^{-1}(v)$. In this case $w(v)$ coincides with the weight of any edge emitting from $v$.

A vertex weighted graph is called a balanced weighted graph if $w(v) = |s^{-1}(v)|$ for all $v \in E^0_{\text{reg}}$. 

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Remark 2.3 Note that any (row-finite) directed graph can be given the structure of a balanced weighted graph by assigning \( w(v) = |s^{-1}(v)| \), for all \( v \in E_{\text{reg}} \). (By definition we are not required to assign a weight to any sinks in \( E \).)

In particular, suppose that \( H \) is a hereditary subset of the weighted graph \((E, w)\). We form the quotient graph \( E/H \). Then there are two natural ways to define a weight function on \( E/H \): restrict the weight function on \( E \) to \( E/H \), or impose the structure of a balanced weighted graph on \( E/H \). We note that if \( w \) is a balanced weight function on \( E \), then the restriction function of \( w \) to \( E/H \) is not in general the same as the balanced weight function on \( E/H \). In situations where additional clarity is warranted, we will write \((E/H, w_r)\) to indicate that the weight function being considered on \( E/H \) is the restriction function of the weight function \( w \) to \( E/H \).

We continue by setting some terminology and notation about monoids. Let \((M, +)\) be a commutative monoid. We define the algebraic pre-ordering on \( M \) by setting \( a \leq b \) if \( b = a + c \), for some \( c \in M \). A commutative monoid \( M \) is called conical if \( a + b = 0 \) implies \( a = b = 0 \), and is called cancellative if \( a + b = a + c \) implies \( b = c \) for all \( a, b, c \in M \). The monoid \( M \) is called refinement if, whenever \( a, b, c, d \in M \) have \( a + b = c + d \), then there are \( e_1, e_2, e_3, e_4 \in M \) such that \( a = e_1 + e_2, b = e_3 + e_4 \) and \( c = e_1 + e_3, d = e_2 + e_4 \). An element \( 0 \neq a \in M \) is called an atom if whenever \( a + b = c \) then \( b = 0 \) or \( c = 0 \). We say a commutative monoid is atom-cancellative if for any atom \( a \in M \), the equality \( a + m = a + m' \) implies \( m = m' \).

Let \( Y \) be a submonoid of \( M \) (so \( Y \) is closed under \(+\), and contains \( 0 \)). For \( a, b \in M \), write \( a \sim_Y b \) if there exist \( i, j \in Y \) such that \( a + i = b + j \) in \( M \). This is a congruence relation and thus one can form the quotient monoid \( M/\sim \) which we will denote by \( M/Y \). The congruence \( a \sim_Y b \) is equivalent to \( (a + Y) \cap (b + Y) \neq \emptyset \). Observe that \( a \sim_Y 0 \) in \( M \) for any \( a \in Y \).

A subset \( I \) of \( M \) is called an ideal of \( M \) if \( m + I \subseteq I \), for any \( m \in M \).

For a commutative monoid \( M \), the set

\[
Z(M) = \{ a \in M \mid a + b = 0, \text{ for some } b \in M \}
\]

is an abelian group, called the unit group of \( M \). Clearly \( Z(M) = 0 \) if and only if \( M \) is conical and \( Z(M) = M \) if and only if \( M \) is a group. We refer the reader to [22] for the general theory of commutative monoids and [24] for refinement commutative monoids.

The following simple lemma will be quite useful later, especially in the proof of Proposition 6.7. We note that this result is embedded in [4, Lemma 2.1, Proposition 2.2]; for completeness, we included a self-contained proof here.

Lemma 2.4 Any refinement monoid is atom-cancellative. Furthermore, any monoid which is finite, conical and refinement has no atoms.

Proof Let \( a, m, m' \in M \), where \( M \) is a refinement monoid and \( a \) is an atom such that \( a + m = a + m' \). Then there are \( e_1, e_2, e_3, e_4 \in M \) such that \( a = e_1 + e_2, m = e_3 + e_4 \) and \( a = e_1 + e_3, m' = e_2 + e_4 \). Since \( a \) is an atom, either \( e_1 = 0 \) or \( e_2 = 0 \). If \( e_1 = 0 \) then \( e_2 = e_3 = a \) and thus \( m = a + e_4 = m' \). If \( e_2 = 0 \) then \( e_1 = a \) and \( e_3 = 0 \) which gives that \( m = e_4 = m' \).
Now let $M$ be finite, conical and refinement. Suppose $a \in M$ is an atom. Consider the translation map $\phi_a : M \setminus \{0\} \to M \setminus \{0\}$, $m \mapsto a + m$. Since $a$ is an atom, the first part of the lemma implies $\phi_a$ is injective. Thus $\phi_a$ is surjective as $M \setminus \{0\}$ is finite. But $a$ cannot be in the image of $\phi_a$, a contradiction. \hfill \Box

Let $M$ be a commutative monoid. The group completion of $M$, denoted $\mathcal{G}(M)$, is the free abelian group generated by elements of the set $\{[a] | a \in M\}$ subject to the relations $[a + b] - [a] - [b]$, for any $a, b \in M$. The homomorphism $M \to \mathcal{G}(M)$; $m \mapsto [m]$ is initial among all such maps [18]. The Grothendieck group of a unital ring $A$, denoted $K_0(A)$, is by definition the group completion of the monoid $\mathcal{V}(A)$.

When the monoid $M$ is finite, the group completion $\mathcal{G}(M)$ is finite as well; indeed, $\mathcal{G}(M)$ is a subgroup of $M$. This fact has been noted elsewhere in the literature (and has been attributed to Cuntz), and has been verified in different ways (e.g., by analyzing the properly infinite elements of the monoid). As with the previous lemma, we include a self-contained proof.

**Lemma 2.5** Let $M$ be a finite commutative monoid. Then the group completion $\mathcal{G}(M)$ of $M$ is isomorphic to the smallest ideal of $M$. In particular, $\mathcal{G}(M)$ is finite.

**Proof** Let $I$ denote $\bigcap_{a \in M}(a + M)$. Since $M$ is finite, $t := \sum_{a \in M} a$ is a well-defined element of $M$, and clearly $t \in I$, so that $I \neq \emptyset$. It is easy to show that $I$ is an ideal of $M$. Let $J$ be any ideal of $M$. For each $a \in J$ and $m \in M$ we have $m + a \in J$, so $a + M \subseteq J$, so $I \subseteq J$. So $I$ is the smallest ideal of $M$.

We show that $I$ is a group. For each $x \in I$, the subset $I + x := \{t + x \mid t \in I\}$ is an ideal of $M$ contained in $I$, so $I + x = I$. In particular there exists $z \in I$ with $z + x = x$. As well, for each $y \in I$ there is $u \in I$ with $y = u + x$. Then $y + z = u + x + z = u + z + x = u + x = y$, so $z$ is an identity element for $I$. But $I + x = I$ also gives that there exists $x' \in I$ with $x' + x = z$. So $I$ is a group.

Define $\phi : M \to I$ by setting $\phi(m) = m + z$ for each $m \in M$. (Note that $m + z \in I$ as $z \in I$ and $I$ is an ideal of $M$.) Since $z + z = z$, $\phi$ is a monoid homomorphism. Let $\Gamma$ be any abelian group, and let $\psi : M \to \Gamma$ be a monoid homomorphism. Then easily $\psi = \psi |_I \circ \phi$. Thus $I \cong \mathcal{G}(M)$. \hfill \Box

**Example 2.6** We clarify some of the previous observations in the context of a specific example, one to which we will make reference often in the sequel. For $n, k \in \mathbb{N}^+$, let $M_{n,n+k}$ denote the finite commutative monoid

$$M_{n,n+k} := \{0, x, 2x, \ldots, nx, \ldots, (n + k - 1)x\}, \text{ with relation } (n + k)x = nx.$$ 

Then $M_{n,n+k}$ is clearly conical, and not cancellative. For $n \geq 2$ the unique atom of $M_{n,n+k}$ is $x$; on the other hand, $M_{1,1+k}$ contains no atoms. The monoid $M_{n,n+k}$ is refinement if and only if $n = 1$. (To show that $M_{n,n+k}$ is not refinement for $n \geq 2$, consider the equation $x + (n - 1)x = x + (n + k - 1)x$ in $M_{n,n+k}$.)

Finally, the smallest ideal of $M_{n,n+k}$ consists of the subset $\{nx, \ldots, (n + k - 1)x\}$. By Lemma 2.5 this is also then the group completion $\mathcal{G}(M_{n,n+k})$ of $M_{n,n+k}$. So we see that $\mathcal{G}(M_{n,n+k})$ is isomorphic to the cyclic group of order $k$: it is easy to show that the identity element of this group is $tx$, where $t$ is the unique multiple of $k$ appearing in the integer interval $[n, n + k - 1]$, and that $(t + 1)x$ is a generator for $\mathcal{G}(M_{n,n+k})$. 

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We complete this section by tying together the two previously presented themes, by associating certain monoids to weighted graphs. Once the specific description and basic properties of these graph monoids have been established, in the subsequent two sections we will look at the Confluence and the Reduction-Uniqueness properties, respectively.

**Definition 2.7** Let \((E, w)\) be a weighted graph. We assign \(w(v) = 1\) if \(v\) is a sink. The reduced graph monoid \(M(E, w)\) associated to \((E, w)\) is defined to be

\[
M(E, w) := \mathbb{F}_E / \left\langle w(v) v = \sum_{e \in s^{-1}(v)} r(e) \mid v \in E^0 \right\rangle.
\]  

(1)

Here \(\mathbb{F}_E\) is the free commutative monoid on the set \(E^0\) of vertices of \(E\). (If \(E^0\) is the empty set then we interpret \(\mathbb{F}_E\) as the zero monoid.)

Note that if \(s\) is a sink, then since the summation in (1) is over an empty set, the relation corresponding to \(s\) in \(M(E, w)\) reduces to: \(s = 0\). (This is why we use the terminology reduced graph monoid to describe \(M(E, w)\).)

**Example 2.8** Let \(n, k \in \mathbb{N}^+\). Let \(E_{n, n+k}\) denote the vertex weighted graph having one vertex \(v\), with \(n\) loops at \(v\), each having weight \(n + k\). Pictorially, \(E_{n, n+k}\) can be viewed as

\[
e_1: w(e_1) = n + k, \\
e_2: w(e_2) = n + k.
\]

So \(w(v) = n + k\), and

\[
M(E_{n, n+k}) := \mathbb{F}_v / \langle (n + k) v = nv \rangle \cong M_{n, n+k},
\]

where \(M_{n, n+k}\) is the monoid defined in Example 2.6.

**Remark 2.9** We note that even if \(E\) is finite, certainly \(M(E, w)\) might be infinite. As an easy example, if \(E\) is the graph with one vertex \(v\) and one loop \(e\) based at \(v\) having \(w(e) = 1\), then \(w(v) = 1\) and \(M(E, w) = \mathbb{F}_v / \langle 1v = v \rangle \cong \mathbb{N}\).

There is an explicit description of the congruence on \(\mathbb{F}_E\) given by the defining relations of \(M(E, w)\) in (1), as follows. For \(v \in E^0\), with weight \(w(v)\), define the “\(r\)-transform on \(\mathbb{F}_E\)” by setting

\[
r(w(v)v) := \sum_{e \in s^{-1}(v)} r(e).
\]
The nonzero elements of $\mathbb{F}_E$ can be written uniquely up to permutation as $\sum_{i=1}^{n} k_i v_i$, where $v_i$ are distinct vertices and $k_i \in \mathbb{N}^+$. Define a binary relation $\rightarrow_1$ on $\mathbb{F}_E$ by

$$\sum_{i=1}^{n} k_i v_i \rightarrow_1 \left( \sum_{i \neq j} k_i v_i \right) + (k_j - w(v_j)) v_j + r(w(v_j) v_j),$$

whenever $j \in \{1, \ldots, n\}$ and $k_j \geq w(v_j)$. (Clearly the transformation described by $\rightarrow_1$ models the toppling process at $v_j$ in the sandpile model.) Let $\rightarrow$ be the transitive and reflexive closure of $\rightarrow_1$ on $\mathbb{F}_E$. Namely

$$a \rightarrow b \text{ if } a = b, \text{ or } a = a_0 \rightarrow_1 a_1 \rightarrow_1 \cdots \rightarrow_1 a_k = b.$$  \hfill (3)

Finally, let $\sim$ be the congruence on $\mathbb{F}_E$ generated by the relation $\rightarrow$. That is, $a \sim b$ in case there is a string $a = a_0, a_1, \ldots, a_n = b$ in $\mathbb{F}_E$ such that $a_i \rightarrow_1 a_{i+1}$ or $a_{i+1} \rightarrow_1 a_i$ for each $0 \leq i \leq n - 1$. Then

$$M(E, w) = \mathbb{F}_E / \sim.$$  

To avoid cumbersome equivalence class notation, it is standard (but not technically correct) to denote the elements of $M(E, w)$ and the elements of $\mathbb{F}_E$ using the same symbols. For instance, we will sometimes write $a = b$ in $M(E, w)$ for elements $a, b \in \mathbb{F}_E$.

**Definition 2.10** For a sandpile graph $E$, the sandpile monoid $\text{SP}(E)$ has been described informally in the Introduction; we revisit that description, and provide here a formal definition. (See e.g. [6] or [11] for additional information). For $k_j \in \mathbb{N}$ we envision $k_j$ grains of sand sitting on each non-sink vertex $v_j$ of $E$. When the number of grains of sand sitting at $v_j$ is equal to or larger than the number of edges emitting from $v_j$, then $v_j$ fires one grain of sand along each edge emitted by $v_j$ to its adjacent vertices. Further, any grains which arrive at the sink $s$ are understood to vanish.

More formally, $\text{SP}(E)$ is the monoid defined by generators and relations as

$$\text{SP}(E) := \mathbb{F}_E / \left\langle s = 0; |s^{-1}(v)| v = \sum_{e \in s^{-1}(v)} r(e), \text{ for } v \in E^0 \setminus \{s\} \right\rangle.$$  

Furthermore, the sandpile group $\mathcal{G}(E)$ is defined by setting

$$\mathcal{G}(E) := \text{ the smallest ideal of } \text{SP}(E).$$

**Example 2.11** Let $G$ be the graph pictured here.

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So $G$ has one sink $s$, and one non-sink vertex $x$, with $n$ loops based at $x$, and $k$ edges from $x$ to $s$. Thus $G$ is a sandpile graph. Moreover,

$$SP(G) = \mathbb{F}_{x,s}/\langle s = 0; (n + k)x = nx + ks \rangle \cong M_{n,n+k},$$

where $M_{n,n+k}$ is the monoid defined in Example 2.6.

**Remark 2.12** *(A key connection)* Let $E$ be a sandpile graph. Considering $E$ as a balanced weighted graph (so $w(v) = \lfloor s^{-1}(v) \rfloor$ for each non-sink vertex $v$ of $E$), and defining $w(s) = 1$ for the sink $s$, Definitions 2.7 and 2.10 immediately give

$$SP(E) = M(E, w).$$

So every sandpile monoid is the monoid of a (balanced) weighted graph. The converse is certainly not true: as one justification, we will show below that the monoid $SP(E)$ is finite for any finite graph $E$, while $M(E, w)$ need not be finite in general (see Remark 2.9).

Of course it is not coincidental that the monoid $M(E, w)$ of Example 2.8 is the same as the monoid $SP(G)$ of Example 2.11. We will make the connection precise in Theorem 6.1.

**Remark 2.13** Let $E$ be a sandpile graph. Suppose $v$ is an irrelevant vertex in $E$, and denote by $w$ the vertex $r(e)$, where $e$ is the unique edge having $s(e) = v$. By “collapsing” $E$ at $v$ we mean eliminating $v$ and $e$ from $E$, and, for any edge $f$ in $E$ having $r(f) = v$, we assign $r(f) = w$. Let $G$ be the graph gotten from $E$ by continuing to collapse at each vertex until no irrelevant vertices remain. Then clearly $G$ is a reduced sandpile graph, and $SP(E) \cong SP(G)$. In particular, when studying the structure of $SP(E)$ we may assume without loss that $E$ is reduced.

### 3 The Confluence Property

In this section we show that the reduced graph monoids $M(E, w)$ have a certain “confluence” property. Informally, this means that whenever two elements $a, b$ are equal when viewed in the quotient monoid $M(E, w) = \mathbb{F}_E/\sim$, then there is a common element in $\mathbb{F}_E$ to which both $a$ and $b$ flow. The proof of the Confluence Lemma we present here is similar to the proof of the corresponding result in the case of the graph monoid $ME$ for an unweighted graph $E$ [5, Section 6.3], but with the added complexity of having weights and sinks in the relations.

**Lemma 3.1** *(Confluence Lemma)* Let $(E, w)$ be a weighted graph and $M(E, w)$ its associated monoid. For $a, b \in \mathbb{F}_E$, we have $a = b$ in $M(E, w)$ (i.e., $a \sim b$ in $\mathbb{F}_E$) if and only if there exists $c \in \mathbb{F}_E$ such that $a \rightarrow c$ and $b \rightarrow c$.

In particular, any sandpile monoid satisfies this “Confluence Property”.

**Proof** The sufficiency follows directly from the definition of $\sim$, as it is the symmetric and transitive closure of the relation $\rightarrow_1$. 

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To establish the necessity, we start by observing the following fact. Let \( a \) be an element of \( \mathbb{F}_E \setminus \{0\} \) such that \( kv \) appears in its presentation, where \( v \) is a vertex having weight \( k \). Writing \( a = kv + a' \) then by definition (2) a transformation \( \rightarrow_1 \) can take either the form \( a \rightarrow_1 r(kv) + a' \) or \( a \rightarrow_1 kv + a'' \), where \( a' \rightarrow_1 a'' \).

Now suppose \( a = b \) in \( M(E,w) \). Then there is a string \( a = a_0, a_1, \ldots, a_n = b \) in \( \mathbb{F}_E \) such that \( a_i \rightarrow_1 a_{i+1} \) or \( a_i \rightarrow_1 a_i \), where \( 0 \leq i \leq n - 1 \). We argue by induction on \( n \). If \( n = 0 \), then \( a = b \) and there is nothing to prove. Suppose the statement holds for strings of length \( n - 1 \) and let \( a = a_0, a_1, \ldots, a_n = b \) be a string of length \( n \). By induction, there is a \( c \) such that \( a_0 \rightarrow_1 c \) and \( a_{n-1} \rightarrow_1 c \). We consider two cases:

If \( b = a_n \rightarrow_1 a_{n-1} \), then clearly \( a_0 \rightarrow_1 c \) and \( b \rightarrow_1 c \) and we are done.

If \( a_{n-1} \rightarrow_1 b \), then by the definition (2), there is a vertex \( v \) with weight \( k \) such that \( kv \) appears in the presentation of \( a_{n-1} \), i.e., \( a_{n-1} = kv + a'_{n-1} \) and \( b = r(kv) + a'_{n-1} \). Since \( a_{n-1} \rightarrow_1 c \) there is a string of transformations

\[
a_{n-1} = kv + a'_{n-1} = c_0 \rightarrow_1 c_1 \rightarrow_1 \cdots \rightarrow_1 c_l = c. \tag{4}
\]

We consider two cases.

**Case 1.** Going along the displayed string (4) if there is no transformation of this \( kv \) in each of the \( c_i \), \( 0 \leq i \leq l \), then \( kv \) appears in the presentation of each of the \( c_i \) and therefore we have a chain

\[
a'_{n-1} = c_0' \rightarrow_1 c_1' \rightarrow_1 \cdots \rightarrow_1 c_l', \tag{5}
\]

where \( c_i = kv + c_i' \), \( 0 \leq i \leq l \).

Now we apply one more transformation on \( c_l = kv + c_l' \rightarrow_1 r(kv) + c_l' \). Since \( b = r(kv) + a'_{n-1} \), applying the same transformations (5) we have

\[
b = r(kv) + a'_{n-1} \rightarrow_1 r(kv) + c_1' \rightarrow_1 \cdots \rightarrow_1 r(kv) + c_{l-1}' \rightarrow_1 r(kv) + c_l'.
\]

Therefore \( a \rightarrow_1 c_l = kv + c_l' \rightarrow_1 r(kv) + c_l' \) and \( b = r(kv) + c_l' \) and we are done.

**Case 2.** Suppose \( 0 \leq s \leq l - 1 \) is the first instance in the string (4) that \( \rightarrow_1 \) transforms this \( kv \) to \( r(kv) \). Thus we have a chain

\[
a'_{n-1} = c_0' \rightarrow_1 c_1' \rightarrow_1 \cdots \rightarrow_1 c_s', \tag{6}
\]

where \( c_i = kv + c_i' \) and

\[
a_{n-1} = kv + c_0' \rightarrow_1 kv + c_1' \rightarrow_1 \cdots \rightarrow_1 kv + c_s' \rightarrow_1 r(kv) + c_s' = c_{s+1} \rightarrow_1 \cdots \rightarrow_1 c_l = c.
\]

Since \( b = r(kv) + a'_{n-1} \), applying the same transformations (6) we have

\[
b = r(kv) + a'_{n-1} = r(kv) + c_0' \rightarrow_1 \cdots \rightarrow_1 r(kv) + c_s' = c_{s+1} \rightarrow_1 \cdots \rightarrow_1 c_l = c.
\]
Therefore $a \to c$ and $b \to c$ and we are done.

The final statement follows from Remark 2.12.

Let $(E, w)$ be a weighted graph and $H$ a hereditary subset of $E$. Consider $(H, w)$, the weighted subgraph of $(E, w)$ consisting of all vertices of $H$ and all edges emitting from these vertices, with the same weights as in $E$. Observe that there is a well-defined monoid homomorphism $M(H, w) \to M(E, w); a \mapsto a$. For $a, b \in M(H, w)$ if $a = b$ in $M(E, w)$, an application of the Confluence Lemma 3.1 shows there is a $c \in F_E$ such that $a \to c$ and $b \to c$. But since $H$ is hereditary, all the transformations occur already in $H$, and thus $a = b$ in $M(H, w)$. Thus we can consider $M(H, w)$ as a submonoid of $M(E, w)$. Consequently:

**Remark 3.2** Let $(E, w)$ be a weighted graph. Let $S$ denote the subset of $E^0$ consisting of those vertices which do not connect to any cycle in $E$. Then clearly $S$ is hereditary. So by the above observation we can consider the monoid $M(S, w)$ as a submonoid of $M(E, w)$. Furthermore, we note for later use that the graph $E/S$ is easily seen to contain no sinks.

As another application of the Confluence Lemma 3.1, we determine the unit group of reduced graph monoids.

**Proposition 3.3** Let $(E, w)$ be a finite weighted graph, $M(E, w)$ its associated reduced graph monoid and $S$ the set of all vertices in $E$ which do not connect to any cycle in $E$. Then

$$Z(M(E, w)) = M(S, w).$$

Furthermore, if $(E, w)$ is a finite vertex weighted graph, then

$$M(E, w)/Z(M(E, w)) = M(E, w)/M(S, w) \cong M(E/S, w_r),$$

where $w_r$ is the restriction of the weight function $w$ to $E/S$.

**Proof** By Remark 3.2 we have $M(S, w) \subseteq M(E, w)$. If $S$ is not empty, then any path emitting from an element $v$ in $S$ can be extended to a path that ends in a sink, as $E$ is finite and $v$ is not connected to a cycle. An easy induction (on the maximum length of the paths connecting to sinks) shows that $kv = 0$, for some $k \in \mathbb{N}^+$. Thus $M(S, w) \subseteq Z(M(E, w))$.

Conversely, let $a \in Z(M(E, w))$. Then $a + b = 0$ in $M(E, w)$ for some $b \in M(E, w)$. The Confluence Lemma 3.1 implies that $a + b \to 0$ in $F_E$. Seeking a contradiction, suppose that there is a vertex $w \in E^0 \setminus S$ appearing in the presentation of $a$. Since $w$ is connected to a cycle, any possible transformations of $w$ or its multiple would after finitely many applications of $\to_1$ give a vertex on a cycle in the presentation, and subsequently any further transformations always contain a vertex on a cycle. Thus $a + b$ cannot be transformed to 0, and so $a \in M(S, w)$.

The monoid isomorphism given in (7) (which is valid for any hereditary subset of $E$, of which $S$ is a particular case) is easy to check, and is left to the reader. □
We can now establish the following corollary, which will be used to connect sandpile monoids to weighted Leavitt path algebras in Theorem 6.1.

**Corollary 3.4** Let \( M(E, w) \) be the reduced graph monoid associated to a finite weighted graph \((E, w)\) and \( S \) the set of vertices in \( E \) which do not connect to any cycle.

(i) \( M(E, w) \) is a group if and only if \( E \) is acyclic.

(ii) \( M(E, w) \) is conical if and only if every non-sink vertex in \( S \) has weight one.

**Proof** (i) As noted above for arbitrary monoids, \( M(E, w) \) is a group if and only if \( M(E, w) = Z(M(E, w)) \). The claim follows now from Proposition 3.3.

(ii) We have that \( M(E, w) \) is conical if and only if \( Z(M(E, w)) = 0 \), so that by Proposition 3.3 we get \( M(S, w) = 0 \). Suppose the weight of each vertex in \( S \) is 1. Thus the relation given in Display (1) for a non-sink vertex \( v \in S \) reduces to \( v = \sum_{e \in s^{-1}(v)} r(e) \), whereas for a sink \( s \), we have \( s = 0 \). Thus for \( v \in S \), an easy induction (on the maximum length of the paths connecting to sinks) shows that \( v = 0 \) in \( M(S, w) \). On the other hand if \( M(E, w) \) is conical, then \( M(S, w) = 0 \). Suppose \( v \in S \). Since \( v = 0 \) in \( M(S, w) \), the Confluence Lemma 3.1, \( v \to 0 \). However, no transformation can be applied to \( v \) if the weight of \( v \) is larger than 1. This completes the proof. \( \square \)

**Remark 3.5** We note that if we consider a finite graph \( E \) as a balanced weighted graph, then Corollary 3.4(ii) says that \( M(E, w) \) is conical precisely when every non-sink vertex in \( S \) is irrelevant.

**Remark 3.6** For those readers who are familiar with the Confluence Lemma of [5, Section 6.3], we note that although the Confluence Lemma is a key step used in [5] to establish that \( M_E \) is a refinement monoid (for unweighted graphs), and although we have established a corresponding Confluence Lemma for weighted graphs (Lemma 3.1), in general \( M(E, w) \) need not be refinement.

### 4 Unique normal form: the Diamond Lemma

In addition to the Confluence Property, one of the main features of sandpile monoids is that their elements stabilise (i.e., are reduction-finite, see below) to a unique normal form. In particular, sandpile monoids are necessarily finite. Although any monoid of the form \( M(E, w) \) has been shown to satisfy the Confluence Property in Lemma 3.1, not every monoid of the form \( M(E, w) \) has the property that each of its elements stabilises (nor do they need to be finite, as observed previously).

**Example 4.1** Let \( E \) be the graph

![Graph](image)

on which we define the vertex weighting \( w(u) = w(v) = 2 \). Then the relations of reduced graph monoids give \( 2u = u + 2v \) and \( 2v = u + v \) in \( M(E, w) \). One
observes that the element $2u$ never stabilises; specifically, we can continue to apply
the two relations to the element $2u$ and produce any element of the monoid of the form
$2u + nv$ for $n \in \mathbb{N}^+$.

We note further that the monoid $M(E, w)$ is finite (equal to $\{0, u, v, 2u, 2v\}$), and
conical, and has atoms $u$ and $v$. But $M(E, w)$ is not a refinement monoid, as an analysis
of the equation $u + u = u + 2v$ bears out.

**Example 4.2** Let $E$ be the graph pictured here.

![Diagram of a graph](image)

We point out that the size of $M(E, w)$ is highly dependent on the weight $w$ assigned
to the edges of $E$. If we assign weight $w_1(e) = 1$ to each edge (i.e., the unweighted case), then $w_1(u) = w_1(v) = w_1(z) = 1$, and in this situation $M(E, w_1)$ is infinite.
(Note: the monoid $M_E$ of this particular unweighted graph has been analysed as
Example $E_1^3$ in [3, Section 4].) Similarly, if we assign weight $w_2(e) = 2$ to each
edge, then $w_2(u) = w_2(v) = w_2(z) = 2$, and in this situation $M(E, w_2)$ is infinite as well. However, if we assign weight $w_3(e) = 3$ to each edge, then $w_3(u) = w_3(v) = w_3(z) = 3$, and in this situation $M(E, w_3)$ is finite, indeed $|M(E, w_3)| = 27$. This conclusion will follow from Theorem 6.1, as we will see that $M(E, w_3)$ is isomorphic to the sandpile monoid $SP(G)$ of an appropriate graph $G$.

We more fully justify each of these statements in Remark 5.8 below.

We will employ Bergman’s Diamond Lemma to re-establish that the elements of
sandpile monoids have unique normal forms. Although a proof of the uniqueness of
normal forms has appeared in the literature (see e.g. [6, Section 9]), we include a proof
here for two reasons. First, while previous proofs have made oblique reference to a
“Jordan–Hölder type” “Diamond Lemma” result stemming from Newman [19], we
will show that Bergman’s Diamond Lemma can be applied quite easily and directly
here. Second, with Example 4.2 as context, we will play up exactly where in this
analysis the specific weight function is utilised.

We briefly remind the reader of the setting (see [10]). Let $\mathbb{F}_X$ be the free commutative
monoid generated by a nonempty set $X$. Let $R$ be a set of pairs of the form $\sigma = (W_\sigma, f_\sigma)$, where $W_\sigma, f_\sigma \in \mathbb{F}_X \setminus \{0\}$. The set $R$ is called a reduction system for $\mathbb{F}_X$. For any $\sigma \in R$ and $A \in \mathbb{F}_X$, let $r_{A+\sigma} : \mathbb{F}_X \to \mathbb{F}_X$ denote a map that sends $A + W_\sigma$ to $A + f_\sigma$
and fixes all other elements of $\mathbb{F}_X$. The maps $r_{A+\sigma} : \mathbb{F}_X \to \mathbb{F}_X$ are called reductions.
For $a, b \in \mathbb{F}_X$, we write $a \rightarrow b$ if there is a sequence $r_1, r_2, \ldots, r_i$ of reductions,
such that $r_i \circ \cdots \circ r_1(a) = b$. An element $a \in \mathbb{F}_X$ is called reduction-finite if for every
infinite sequence $r_1, r_2, \ldots, r_i$ of reductions, $r_i$ acts trivially on $r_{i-1} \circ \cdots \circ r_1(a)$, for all
sufficiently large $i$. We call an element $a \in \mathbb{F}_X$ reduction-unique if it is reduction-finite,
and if its images under all final sequences of reductions are the same. This
unique image is called the normal form of the element $a$. 

\[ \text{Springer} \]
For $\sigma \neq \tau$ in $R$, we call a configuration in which $W_\sigma = A + B$ and $W_\tau = B + C$ (for some $A, B, C \in \mathbb{F}_X$) an overlap ambiguity of $R$. Such an overlap ambiguity is called resolvable if there exist compositions of reductions, $r$ and $r'$, such that $r(f_\sigma + C) = r'(A + f_\tau)$. Similarly, for $\sigma \neq \tau$ in $R$, we call a configuration in which $W_\sigma = B$ and $W_\tau = A + B + C$ (for some $A, B, C \in \mathbb{F}_X$) an inclusion ambiguity. Such an inclusion ambiguity is called resolvable if there exist compositions of reductions, $r$ and $r'$, such that $r(A + f_\sigma + C) = r'(f_\tau)$.

In order to use Bergman’s Diamond Lemma result, it is necessary that $\mathbb{F}_X$ be equipped with a semigroup partial ordering $\leq$, i.e., if $x < x'$ then $x + y < x' + y$, for all $x, x', y \in \mathbb{F}_X$. Furthermore, we need the partial ordering $\leq$ to be compatible with $R$, namely, that $f_\sigma < W_\sigma$ for all $\sigma \in R$.

Bergman’s Diamond Lemma in the setting of commutative monoids [10, Theorem 1.2, Sections 9.1, 10.3] says the following. Suppose $\leq$ is a semigroup partial ordering on $\mathbb{F}_X$ compatible with the reduction system $R$, and that $\leq$ has the descending chain condition. Then all ambiguities of $R$ are resolvable if and only if all elements of $\mathbb{F}_X$ are reduction-unique under $R$.

The main obstacle in trying to utilise Bergman’s machinery in practice lies in establishing that the ambiguities are resolvable. We are in position to place sandpile monoids in the setting of Bergman’s Diamond Lemma. We will see that neither type of ambiguity arises in the context of sandpile relations, and thus once the partial order on the monoid is established, the fact that these monoids have reduction-unique forms will follow quite easily.

**Proposition 4.3** Suppose $E$ is a sandpile graph. Then every element of the sandpile monoid $\text{SP}(E)$ is reduction-unique.

**Proof** Let $E$ be a sandpile graph with the unique sink $s$ and consider $E$ as a balanced weighted graph by assigning $w(v) = |s^{-1}(v)|$ for all $v \in E^0 \setminus \{s\}$.

Let $\mathbb{F}_E$ denote the free commutative monoid on the set $E^0$. Let $R$ be the set of pairs of the form $\sigma = (W_v, f_v)$, where

$$W_v = w(v)v, \quad \text{and} \quad f_v = \sum_{e \in s^{-1}(v)} r(e), \quad (8)$$

for every non-sink vertex $v$.

We employ here an idea quite similar to one presented in [6]. Let $D = \max\{w(v) | v \in E^0\}$. (Set $D = 2$ in case this maximum value is 1; i.e., in case each non-sink vertex in $E$ is irrelevant.) Let $n$ denote $|E^0|$, and let $\ell_v$ denote the length of the shortest path which connects $v$ to the sink $s$. Note that $\ell_v < n$ for all $v \in E^0$, as no shortest length path can contain a cycle. The assignment $v \mapsto D^{n-\ell_v}$, for any $v \in E^0$, induces a homomorphism of monoids $p: \mathbb{F}_E \to \mathbb{N}$, where for $x = \sum_{v \in E^0} k_v v \in \mathbb{F}_E$, we define

$$p(x) := \sum_{v \in E^0} k_v D^{n-\ell_v}.$$
We check that $p(W_{\sigma}) < p(f_{\sigma})$ for the relations defined in (8). By definition we have $p(W_{\sigma}) = w(v)D^{n-\ell_v}$ and $p(f_{\sigma}) = \sum_{e \in s^{-1}(v)} D^{n-\ell_{r(e)}}$.

It is here that we use the assumption that $w(v) = |s^{-1}(v)|$. Since $w(v) \leq D$ and $\ell_{r(f)} = \ell(v) - 1$ for some $f \in s^{-1}(v)$, we obtain

$$p(W_{\sigma}) = w(v)D^{n-\ell_v} = |s^{-1}(v)|D^{n-\ell_v} < D \cdot D^{n-\ell_v} + \sum_{e \in s^{-1}(v) \setminus \{f\}} D^{n-\ell_{r(e)}} = \sum_{e \in s^{-1}(v)} D^{n-\ell_{r(e)}} = p(f_{\sigma}).$$

We now define an order $\geq$ on $F_E$ by setting

$$x \geq y \text{ in case } x \rightarrow y,$$

where the binary relation $\rightarrow$ is defined as in Displays (2) and (3). The displayed inequality above yields that $x > y$ implies $p(x) < p(y)$, and so $\geq$ is a partial order on $F_E$. By the definition of the ordering, it is immediate that $\geq$ is a semigroup partial order which is compatible with relations (8). The fortunate, obvious observation in this setting is that for $v \neq v' \in E^0$, the expressions $W_v = w(v)v$ and $W_{v'} = w(v')v'$ in $F_E$ involve neither overlap ambiguities nor inclusion ambiguities.

Finally, we have $p(x) \leq (\sum_{v \in E^0} k_v)D^n$, for all $x = \sum_{v \in E^0} k_vv \in F_E$. This guarantees that the partial ordering on $F_E$ defined above has the descending chain condition.

Thus Bergman’s Diamond Lemma guarantees that the monoid generated by $F_E$ subject to relations (8) is reduction-unique. Since $SP(E)$ is this monoid by identifying the sink to be zero, $SP(E)$ is reduction-unique, as desired. \hfill \Box

**Remark 4.4** We note that the sandpile monoid $SP(E)$ of a sandpile graph $E$ must be finite. Indeed $|SP(E)|$ is the product of the positive integers $|s^{-1}(v_j)|$ taken over all non-sink vertices $v_j$ of $E$. This is clear because each element of $SP(E)$ corresponds to the (unique) reduced form of each of the elements in the quotient description of $SP(E)$ as guaranteed by Proposition 4.3.

## 5 Weighted Leavitt path algebras

In this section we briefly recall the notion of weighted Leavitt path algebras. These are algebras associated to weighted graphs (see Sect. 2). We refer the reader to [15, 21] for a detailed analysis of these algebras.

**Definition 5.1** Let $(E, w)$ be a weighted graph and $k$ a field. The free $k$-algebra generated by \{ $v, e_i, e_i^* \mid v \in E^0, e \in E^1, 1 \leq i \leq w(e)$ \} subject to relations

(i) $uv = \delta_{uv} u$, where $u, v \in E^0$,
(ii) $s(e)e_i = e_i = e_i r(e), r(e)e_i^* = e_i^* = e_i^* s(e)$, where $e \in E^1, 1 \leq i \leq w(e)$,
(iii) $\sum_{e \in s^{-1}(v)} e_i e_j^* = \delta_{ij} v$, where $v \in E^0_{\text{reg}}$ and $1 \leq i, j \leq w(v)$,
(iv) $\sum_{1 \leq i \leq w(v)} e_i^* f_i = \delta_{ef} r(e)$, where $v \in E^0_{\text{reg}}$ and $e, f \in s^{-1}(v)$,
is called the weighted Leavitt path algebra of \((E, w)\), and denoted \(L_k(E, w)\). In relations (iii) and (iv) we set \(e_i\) and \(e_i^*\) to be zero whenever \(i > w(e)\).

Note that if the weight of each edge is 1, then \(L_k(E, w)\) reduces to the usual Leavitt path algebra \(L_k(E)\) of the graph \(E\). Also, if \(E\) is the empty graph then \(L_k(E, w)\) is defined to be the zero ring.

**Example 5.2** Weighted Leavitt path algebras were originally conceived in [15] in order to provide a context in which to generalise the algebras \(L_k(n, n + k)\) constructed by Leavitt mentioned in the Introduction. Specifically, let \(E_{n,n+k}\) denote the weighted graph consisting of one vertex \(v\) and \(n\) loops of weight \(n + k\) at \(v\). Then it is shown in [15, Example 5.5] that

\[
L_k(E_{n,n+k}) \cong L_k(n, n + k).
\]

In particular, let \((E, w)\) be the weighted graph consisting of one vertex and one loop of weight \(n\). Then the weighted Leavitt path algebra of \((E, w)\) is isomorphic to the Leavitt path algebra of a graph with one vertex and \(n\) loops, which in turn is the Leavitt algebra \(L_k(1, n)\).

**Remark 5.3**

(1) Let \(n, k \in \mathbb{N}^+\), and let \((E, w)\) denote the weighted graph \(E_{n,n+k}\) described in Example 5.2. So \((E, w)\) has one vertex \(v\), and \(n\) loops at \(v\) each having weight \(n + k\). Let \((E', w')\) denote the weighted graph \(E_{n+k,n}\). So \((E', w')\) has one vertex \(v'\), and \(n + k\) loops at \(v'\) each having weight \(n\). Because of the symmetry involved in the definition of weighted Leavitt path algebras, it is easy to show that \(L_k(E, w) \cong L_k(E', w')\) as \(k\)-algebras. Our choice here to focus on the \(n\) loops of weight \(n + k\) point of view is to make the connection we will establish in Theorem 6.1 below between sandpile monoids and weighted Leavitt path algebras more transparent. We note, however, that the algebras \(L_k(E, w)\) and \(L_k(E', w')\) are different if these are viewed as \(\mathbb{Z}\)-graded \(k\)-algebras in the standard Leavitt path algebra \(\mathbb{Z}\)-grading.

(2) The previous paragraph notwithstanding, we caution the reader that in general one cannot cavalierly modify the edges and weights and expect that the corresponding Leavitt path algebras will be isomorphic. For instance, let \((F, w)\) be the graph with one vertex \(v\), and two loops at \(v\) each having weight 2; and let \((F', w')\) be the graph with one vertex \(v'\) and four loops at \(v'\) each having weight 1. Then \(L_k(F, w) \ncong L_k(F', w')\). (Indeed, \(M(F, w) \ncong M(F', w')\) as monoids, since the latter is finite, while the former is infinite. Since these two monoids are not isomorphic, neither can the two weighted Leavitt path algebras be isomorphic; see Theorem 5.7 below.)

For a ring \(A\) with identity, the commutative monoid \(\mathcal{V}(A)\) is defined as the set of isomorphism classes of finitely generated projective right \(A\)-modules equipped with direct sum as the binary operation. The group completion of this monoid is the Grothendieck group \(K_0(A)\) of \(A\). We refer the reader to the book of Magurn for a comprehensive treatment of these ideas [18].

The Leavitt algebras \(L_k(n, n + k)\) can be produced from Bergman’s machinery of the Universal Construction of Rings [9], which subsequently thereby also describes
the structure of the $\mathcal{V}$-monoids of these algebras. Let $k$ be a field, $A$ a $k$-algebra and let $P$ and $Q$ be nonzero finitely generated projective right $A$-modules. Then there is a $k$-algebra $B$, with an algebra homomorphism $A \to B$ such that there is a universal isomorphism $i : P \to Q$, where $\overline{M} = M \otimes_A B$ for any right $A$-module $M$ ([9, p. 38 and Theorem 3.3]). Bergman’s Theorem [9, Theorem 5.2] states that $\mathcal{V}(B)$ is the quotient of $\mathcal{V}(A)$ modulo the relation $[P] = [Q]$.

Using this, starting with a field $A = k$, and the finitely generated projective $A$-modules $P = k^n$ and $Q = k^{n+k}$, Bergman’s machinery applied to this data gives that $B = L_k(n, n+k)$, and subsequently that

$$\mathcal{V}(L_k(n, n+k)) \cong M_{n,n+k} = \{0, x, 2x, \ldots, (n+k-1)x | (n+k)x = nx\}.$$ 

We note that the process of explicitly identifying the $\mathcal{V}$-monoid $\mathcal{V}(B)$ is extremely delicate, requiring the full use of the powerful tools of Bergman developed in [9]. Indeed, while Leavitt had established in [17] that the monoid of free modules over $L_k(n, n+k)$ is isomorphic to $M_{n,n+k}$, it was not established until Bergman’s fundamental work fifteen years later in [9] that $M_{n,n+k}$ actually represents the full $\mathcal{V}$-monoid of $L_k(n, n+k)$.

\textbf{Remark 5.4} It is germane to point out that the original observation which led us to connect sandpile monoids to the $\mathcal{V}$-monoids of weighted Leavitt path algebras was made in the context of the monoid $M_{n,n+k}$. Specifically, $M_{n,n+k}$ arises on the one hand (as noted in Example 5.2 together with the previous paragraph) as the $\mathcal{V}$-monoid of the most elementary type of vertex weighted graph, namely the vertex weighted graph having exactly one vertex. On the other hand (as noted in Example 2.11), $M_{n,n+k}$ also arises as the sandpile monoid of the most elementary type of sandpile graph, namely the sandpile graph having exactly one non-sink vertex. Once this connection was realised, the roadmap to connect general sandpile monoids and weighted Leavitt path algebras became evident.

Although the computation of the $\mathcal{V}$-monoid can be carried out for the Leavitt path algebra of arbitrary (finite) weighted graphs (see e.g. [20]), for the current work we need only describe the $\mathcal{V}$-monoid of weighted Leavitt path algebras associated to vertex weighted graphs, which we now do.

\textbf{Definition 5.5} Let $(E, w)$ be a row-finite vertex weighted graph. We define the graph monoid of $(E, w)$ to be

$$M_{(E, w)} := \mathbb{F}_E / \left\langle w(v) v = \sum_{e \in s \cdot (v)} r(e) \mid v \in E_{\text{reg}}^0 \right\rangle.$$ 

\textbf{Remark 5.6} While clearly similar one to the other, the definitions of the monoids $M(E, w)$ (Definition 2.7) and $M_{(E, w)}$ are not identical. Specifically, there is no relation in $M_{(E, w)}$ whose left-hand side involves any sinks which may exist in $E$; while in $M(E, w)$, there is a relation of the form $1 \cdot s = 0$ for each sink in $E$. Formally,

$$M(E, w) \cong M_{(E, w)}/\langle s = 0 \mid s \in E_{\text{sink}}^0 \rangle.$$
So if $E$ contains no sinks, then $M(E, w) = M_{(E, w)}$.

In particular, for a directed (unweighted) row-finite graph $E$ we have the graph monoid

$$M_E \cong \mathbb{F}_E / \left\langle v = \sum_{e \in \Delta^{-1}(v)} r(e) \mid v \in E^0_{\text{reg}} \right\rangle$$

originally defined in [5]. See also [2, Definition 1.4.2].

Recall that for a commutative monoid $M$, an element $1 \in M$ is called a distinguished element if for any element $m \in M$, there exist $k \in \mathbb{N}$ and $m' \in M$ such that $m + m' = k1$.

In particular, let $(E, w)$ be a finite vertex weighted graph, and let $M_{(E, w)}$ be the monoid described in Definition 5.5. Then for each choice of integers $0 < k_v < w(v)$ (for each non-sink vertex $v$ of $E$), the element $\sum_{v \in E^0} k_v v$ of $M_{(E, w)}$ is easily seen to be a distinguished element of $M_{(E, w)}$.

In [5], Ara, Moreno, and Pardo built Leavitt path algebras $L_k(E)$ via the machinery presented by Bergman in [9]. They showed that $L_k(E)$ can be obtained by Bergman’s universal construction of rings, which thereby allowed them to conclude in particular that $\mathcal{V}(L_k(E)) \cong M_E$ [5, Theorem 3.5]. Similar arguments were used in [15, 20, 21] to construct weighted Leavitt path algebras, and thereby realise their corresponding monoids, via this same machinery. Here we describe in detail how to produce weighted Leavitt path algebras via the Bergman machinery, in the situation where one starts with a monoid of the form $M_{(E, w)}$. Additionally, we will show how to modify the resulting algebra in such a way that the regular module of the algebra corresponds to whatever distinguished element of $M_{(E, w)}$ one might choose.

**Theorem 5.7** Let $k$ be a field, $(E, w)$ a finite vertex weighted graph, and $M_{(E, w)}$ the monoid described in Definition 5.5. For each choice of integers $0 < k_v < w(v)$ (for each non-sink vertex $v$ of $E$), let $1$ denote the distinguished element $\sum_{v \in E^0} k_v v$ of $M_{(E, w)}$. Consider the vertex weighted graph $(F, w)$ obtained from $E$ by adjoining a path of length $k_v - 1$ to each vertex $v$ in $E$, and assign weight $1$ to each newly adjoined vertex. Then there is a natural monoid isomorphism

$$\mathcal{V}(L_k(F, w)) \cong M_{(E, w)},$$

via which $[L_k(F, w)]$ corresponds to $1$. In particular, for any finite vertex weighted graph $(E, w)$, there is a natural isomorphism

$$\mathcal{V}(L_k(E, w)) \cong M_{(E, w)},$$

via which $[L_k(E, w)]$ corresponds to $\sum_{v \in E^0} v$.

**Proof** Since $F$ is an enlarged version of the graph $E$, we can name the corresponding vertices in $F$ by $v^0$ in case $v \in E^0$, and name the vertices on the path of length $k_v - 1$ connected to the vertex $v^0$ by $v^i$, $1 \leq i \leq k_v - 1$. (For any vertex $v$ having $k_v = 1$ we do not adjoin any new edges to $v$, but we do relabel $v$ as $v^0$.)

\(\square\) Springer
It is easy to show that
\[
\phi: M_{(F, w)} \to M_{(E, w)} \quad \text{via} \quad v^t \mapsto v
\]
induces an isomorphism of monoids with \(\phi(\sum_{v \in F^0} v) = 1\).

Next we use Bergman’s universal construction machinery of [9] to show that \(M_{(F, w)}\) can be realised as the monoid \(\mathcal{V}(L_k(F, w))\) with \(\sum_{v \in F^0} v \in M_{(F, w)}\) corresponding to \([L_k(F, w)] \in \mathcal{V}(L_k(F, w))\). Together with the just-noted monoid isomorphism \(\phi\), this will give the desired result.

Let \(A_0 = \prod_{v \in F^0} k\). Thus the monoid \(\mathcal{V}(A_0) \cong \bigoplus_{v \in F^0} \mathbb{N}\) is the free commutative monoid generated by vertices of the graph \(F\), with \([A_0] = \sum_{v \in F^0} v \mapsto (1, 1, \ldots, 1)\). We will use Bergman’s [9, Theorems 3.1 and 3.2] to construct a universal ring \(A\) from \(A_0\) which identifies certain projective modules of \(A_0\). Then [9, Theorem 5.2] guarantees \(\mathcal{V}(A)\) is isomorphic to \(\mathcal{V}(A_0)\) modulo the identifications of these projective modules. The projective modules over \(A_0\) will be chosen to give us precisely the relations of Definition 5.5, so that \(\mathcal{V}(A) \cong M_{(F, w)}\). On the other hand we will observe that the ring \(A\) is precisely \(L_k(F, w)\).

Let \(\{v_1, \ldots, v_m\}\) be the set of all the vertices in \(F\) which emit edges. Consider the following finitely generated projective \(A_0\)-modules: \(P = \bigoplus_{v_1} (A_0 v_1)\), and \(Q = \bigoplus_{v \in F^1} A_0 r(v)\). (We are identifying \(v_i\) with the \(i\)-th unit vector in \(A_0 = \prod_{v \in F^0} k\).) Using Bergman’s [9, Theorem 3.1], one obtains a ring \(A_0\) and universal homomorphisms \(i: A_0 \otimes A_0 P \to A_0 \otimes A_0 Q\) and \(\tilde{i}: A_0 \otimes A_0 Q \to A_0 \otimes A_0 P\). Next, Bergman’s [9, Theorem 3.2] applies to obtain the ring \(A_1\), where extensions of \(i\) and \(\tilde{i}\) (call them again by \(i\) and \(\tilde{i}\)) over \(A_1\) gives \(1 - i\tilde{i} = 0\) and \(1 - \tilde{i}i = 0\). Thus we have \(A_1 = A_0 (i, i^{-1}: \overline{P} \cong \overline{Q})\) with a universal isomorphism \(i: \overline{P} := A_1 \otimes A_0 P \to \overline{Q} := A_1 \otimes A_0 Q\).

Continuing to follow the Bergman construction (specifically, the proofs of [9, Theorem 3.1 and 3.2]) shows that \(A_1\) is \(L_k(X_1, w)\), where \(X_1\) is a graph with the same vertices as \(F\) and where \(v_1\) emits the same edges as in \(F\) and other vertices do not emit any edges. Namely, if \(\{e_1, \ldots, e_s\}\) is all the edges which are emitted from \(v_1\) with \(n = w(v_1)\) then the right multiplication by the matrix \(Y = (e_{ij})_{1 \leq j \leq s, 1 \leq i \leq n}\), where \(e_{ij} = (e_i)_{j}\), gives the map
\[
\bar{i}: \overline{P} = w(v_1)(A_1 v_1) \quad \to \quad \overline{Q} = \bigoplus_{v \in F^1, s(e) = v_1} A_1 r(e),
\]
while \(X = (Y^t)^t\), where \(^t\) is the transpose operation, gives \(i^{-1}\). Now [9, Theorem 5.2] guarantees that \(\mathcal{V}(A_1)\) is obtained from \(\mathcal{V}(A_0)\) by adding the relation \([P] = [Q]\). Translating this to our setting, we get that \(\mathcal{V}(A_1)\) is the monoid generated by the set \([v] | v \in F^0\) subject to the relation \(w(v_1)[v_1] = \sum_{v \in F^1, s(e) = v_1} r(e)\).

We repeat this process to cover the whole graph. To be precise, let \(A_k = L_k(X_k, w)\), \(k \geq 1\), where \(X_k\) is the graph with the same vertices as \(F\), but only the first \(k\) vertices \(\{v_1, \ldots, v_k\}\) emit structured edges. By induction, \(\mathcal{V}(A_k)\) is a commutative monoid generated by \([v] | v \in F^0\) subject to the relation \(w(v_i)[v_j] = \sum_{v \in F^1, s(e) = v_j} r(e)\).
where $1 \leq i \leq k$. Then $A_{k+1} = A_k \langle i, i^{-1} : \overline{P} \cong \overline{Q} \rangle$ with $P = \bigoplus_{w(v_k+1)} A_k v_{k+1}$ and $Q = \bigoplus_{e \in F_1 | s(e) = v_{k+1}} A_k r(e)$. So one more application of [9, Theorem 5.2] gives that $\mathcal{V}(A_{k+1})$ is the monoid generated by all the vertices of $F$ subject to relations corresponding to $\{v_1, \ldots, v_{k+1}\}$. Thus after repeating this process $m$ times we arrive at the monoid $\mathcal{V}(A_0)$ subject to exact same relations of Definition 5.5. Putting these together we have $\mathcal{V}(L_k(F, w)) \cong M(F, w) \cong M(E, w)$, as desired.

The final statement is simply the case where $k_v$ is chosen to be 1 for all non-sink vertices $v$ of $E^0$. We note that a proof of the final statement appears as [15, Theorem 5.21] (without mentioning the distinguished element).

For any finite graph $E$ we let $A_E$ denote the adjacency matrix of $E$, i.e., $(a_{ij}) \in M_{|E^0| \times |E^0|}$ where $a_{ij}$ is the number of edges from $v_i$ to $v_j$.

**Remark 5.8** For a finite graph $E$, let $I'_w$ be the weighted identity matrix of $(E, w)$, defined to be the $|E^0| \times |E^0|$ matrix $(b_{ij})$, where $b_{ij} = 0$ for $i \neq j$ and $b_{ii} = w(v_i)$.

Denote by $N^t$ and $I_w$, respectively, the matrices obtained from $A_E$ and $I'_w$ by first taking the transpose, and then removing the columns corresponding to sinks, respectively.

Since the Grothendieck group of $L_k(E, w)$ is the group completion of the commutative monoid $M(E, w)$, we obtain that

$$K_0(L_k(E, w)) \cong \text{coker} \left( N^t - I_w : \mathbb{Z}^{E^0_{\text{reg}}} \to \mathbb{Z}^{E^0} \right).$$

With this information in hand, we now clarify the statements made in Example 4.2. The graph $E$ is

![Graph E](image)

For the balanced vertex weighting $w_1$ of $E$, and the balanced vertex weighting $w_2$ of $E$ (in which each vertex has weight 1, resp. weight 2), clearly we have

$$N^t - I_{w_1} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N^t - I_{w_2} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

It is straightforward to show that the cokernel of each of these matrices (when viewed as a linear transformation from $\mathbb{Z}^3$ to $\mathbb{Z}^3$) is isomorphic to $\mathbb{Z}$. Since this cokernel is the group completion of the associated monoid, we conclude by Lemma 2.5 that each of $M(E, w_1)$ and $M(E, w_2)$ is infinite.

On the other hand, for the balanced vertex weighting $w_3$ of $E$, we will show in Example 6.3 that $M(E, w_3)$ has 27 elements, and thus (again using Lemma 2.5) the
corresponding group completion must be finite. Here

\[ N^t - I_{w_3} = \begin{pmatrix} -3 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}, \]

and the cokernel can be easily shown (e.g., using Smith normal form) to be the cyclic group \( \mathbb{Z}_8 \).

Let \((E, w)\) be any vertex weighted graph. If \(E\) has no sinks, then \(M(E, w) = M(E, w)\). On the other hand, if \(E\) does contain sinks, then one may construct the weighted graph \(E'\) by attaching a loop to each sink, and defining the weight \(w'\) on \(E'\) to be \(w\) for any edge \(e \in E^1\), and \(w'(e) = 1\) for any added loop. Then clearly \(M(E, w) = M(E', w')\). Consequently, by the Confluence Lemma 3.1 together with Theorem 5.7, we get

**Corollary 5.9** Let \((E, w)\) be a weighted graph. Then \(\mathcal{V}(L_k(E, w))\) has the Confluence Property.

### 6 Sandpile monoids and weighted Leavitt path algebras: the connection

Let \(E\) be a row-finite graph, and let \(S\) denote the (hereditary) subset of vertices of \(E\) which do not connect to any cycle. Consider \(E\) as a balanced weighted graph \((E, w)\), so that \(w(v) = |s^{-1}(v)|\) for all \(v \in E^0\), and consider \((E/S, w_r)\) as a weighted subgraph of \((E, w)\) as described in Remark 2.3.

We are now in position to realise the conical sandpile monoids as the \(\mathcal{V}\)-monoids of weighted Leavitt path algebras.

**Theorem 6.1** Let \(E\) be a sandpile graph, and let \(S\) denote the set of vertices of \(E\) which do not connect to any cycle. Let \(w\) denote a balanced weighting on \(E\). Then

\[ \text{SP}(E)/\text{Z}(\text{SP}(E)) \cong \mathcal{V}(L_k(E/S, w_r)). \]  

In particular \(\text{SP}(E)\) is conical if and only if any vertex not connected to a cycle is irrelevant. In this case

\[ \text{SP}(E) \cong \mathcal{V}(L_k(E/S, w_r)) \text{ and } \mathcal{G}(E) \cong K_0(L_k(E/S, w_r)). \]

**Proof** By Remark 2.12, \(\text{SP}(E) \cong M(E, w)\). By Proposition 3.3, \(\text{SP}(E)/\text{Z}(\text{SP}(E)) \cong M(E/S, w_r)\). By the definition of \(S\) we see that \(E/S\) has no sinks. So \(M(E/S, w_r) = M(E/S, w_r)\). But by Theorem 5.7 applied to the graph \(E/S\), we get \(M(E/S, w_r) \cong \mathcal{V}(L_k(E/S, w_r))\). This gives the desired isomorphism (9).

By definition, the Grothendieck group \(K_0(A)\) of any associative unital ring \(A\) is the group completion of \(\mathcal{V}(A)\). So the second part of the statement follows from Remark 2.12 and Lemma 2.5 (as \(\mathcal{G}(E)\) is by definition the smallest ideal of \(\text{SP}(E)\)). □
The statement that the conicality of a sandpile monoid is equivalent to the property that vertices not connected to cycles are irrelevant has been established previously in a number of articles, see e.g. [23, Proposition 4.2.5] and [6, Proposition 5.7].

Here are a few examples which clarify how Theorem 6.1 plays out in some specific important situations.

**Example 6.2** Let $G$ denote the graph of Example 2.11. So $S = \{s\}$. Then $(G/S, w_r)$ is the graph $E_{n,n+k}$ (defined in Example 2.8) having one vertex $v$, and $n$ loops at $v$ each of weight $n + k$. So by Example 5.2 we have that $L_k(G/S, w_r) \cong L_k(n, n + k)$. Then Theorem 6.1 (re)establishes that

$$V(L_k(n, n + k)) \cong V(L_k(G/\{s\}, w_r)) \cong SP(G) \cong M_{n, n+k}$$

as expected.

**Example 6.3** Let $(E, w_3)$ be the weighted graph described in Example 4.2. Now consider the sandpile graph $T$ pictured here.

Then clearly as graphs $E = T/S$. But $w_3$ on $E = T/S$ is precisely $w_r$ inherited from $T$, where $w$ is the balanced weight function. So $M(E, w_3) = SP(T)$. In particular, by Theorem 6.1, $SP(T)$ arises as the $\mathcal{V}$-monoid of the Leavitt path algebra of the weighted graph $(E, w_3)$.

Remark 4.4 gives that the size of this monoid is $3 \times 3 \times 3 = 27$.

We further emphasize the dependency of the sandpile monoid and sandpile group on the weighting of the graph by presenting one more example of a weight function on $E$. We define the weight function $\overline{w}$ on $E$ by setting $\overline{w}(v) = \overline{w}(z) = 2$ and $\overline{w}(u) = 3$. Now consider the sandpile graph $\overline{T}$ pictured here.

Then $E = T/S$; and $\overline{w}$ is exactly $w_r$ inherited from $\overline{T}$. So in this situation we get that $|M(E, \overline{w})| = |SP(\overline{T})| = 12$; and, perhaps intriguingly, we in this case that $\mathcal{G}(\overline{T})$ is the trivial group $\{0\}$. 
Example 6.4 Let $E$ be a reduced sandpile graph with $|\text{SP}(E)| = p$, a prime number. We show that either $\text{SP}(E) \cong \mathbb{Z}_p$, or $\text{SP}(E) \cong M_{p-l,p} \cong \mathcal{V}(L_k(E_{p-l,p}))$, for some $1 \leq l < p$.

Since the number of elements of the sandpile monoid is prime, by Remark 4.4 the graph $E$ must have only two vertices, namely a sink $s$ and a non-sink vertex $v$ which connects to the sink.

If all the edges from $v$ are connected to the sink, then by the relations that define a sandpile monoid, $\text{SP}(E) \cong M(E, w) \cong \langle v \mid p v = 0 \rangle$, where $(E, w)$ is the balanced weighted graph. Clearly then $\text{SP}(E) \cong \mathbb{Z}_p$. We note that in this case $\text{SP}(E)$ is not conical (as, e.g., $1 + (p - 1) = 0$), so cannot be realised as the $\mathcal{V}$-monoid of any associative ring.

On the other hand, if $E$ consists of $p-l$ loops at $v$ and $l$ edges from $v$ to $s$, the relations of the monoid of the balanced weighted graph gives $\text{SP}(E) \cong M(E \setminus \{s\}, w_r) \cong \langle v \mid p v = (p-l)v \rangle = M_{p-l,p}$. But this is exactly $\mathcal{V}(L_k(E_{p-l,p}))$ by Example 6.2.

Consequently, if $|\text{SP}(E)|$ is prime, then either $\text{SP}(E)$ is not conical, or, in case $\text{SP}(E)$ is conical, then by the observation made in Example 2.6, $\mathfrak{G}(E)$ is a cyclic group of order $p-l$ for some $1 \leq l < p$.

Now let $F$ be the graph

\begin{center}
\begin{tikzpicture}
  \node (v1) at (0,0) {$v_1$};
  \node (v2) at (1,1) {$v_2$};
  \node (v3) at (-1,1) {$v_3$};
  \draw[-latex] (v1) to (v3);
  \draw[-latex] (v1) to (v2);
  \draw[-latex] (v2) to (v3);
  \draw[-latex] (v3) to (v2);
\end{tikzpicture}
\end{center}

The graph $F$ appears in [3, Example 3.8], where it is shown that $|\mathcal{V}(L_k(F))| = 5$, and that $K_0(L_k(F))$ is isomorphic to the non-cyclic group $\mathbb{Z}_2 \times \mathbb{Z}_2$. In particular, $F$ is a graph for which $\mathcal{V}(L_k(F))$ is finite, but for which $\mathcal{V}(L_k(F))$ is not isomorphic to $\text{SP}(E)$ for any sandpile graph $E$.

Example 6.5 Let $E$ be a weighted cycle. That is, $E$ is a vertex weighted graph $(E, w)$ for which $E^0 = \{v_1, v_2, \ldots, v_m\}$, $E^1 = \{e_1, e_2, \ldots, e_m\}$ with $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$ for $1 \leq i \leq m-1$, $s(e_m) = v_m$, $r(e_m) = v_1$, and $w(e_i) = w(s(e_i))$ is some positive integer for each $1 \leq i \leq m$. We assume (for reasons described below) that $w(v_j) \geq 2$ for at least one $v_j$.

Consider the unweighted directed graph $F := \widehat{E}^{\text{op}}$. Here $\widehat{E}$ denotes the (unweighted) directed graph obtained from $E$ by replacing any edge in $E$ of weight $w(e) > 1$ with $w(e)$ edges; and $\text{op}$ stands for the opposite of the graph, i.e. the graph with all edge orientations reversed. Comparing the generators and relations of the algebra $L_k(E, w)$ with those of $L_k(F)$ shows that $L_k(E, w) \cong L_k(F)$ as $k$-algebras.

Therefore by Theorem 5.7 we get

\[ M(E, w) \cong \mathcal{V}(L_k(E, w)) \cong \mathcal{V}(L_k(F)) \cong M_F. \]

Now consider the graph $G$ defined as follows. The vertex set of $G$ is $G^0 := \{s, v_1, v_2, \ldots, v_m\}$. The vertex $s$ is a sink in $G$. The edge set $G^1$ consists of the
following. As in $E$, for each $v_i \in G^0$, $s^{-1}(v_i)$ includes the edge $e_i$. In addition, in $G$, each $s^{-1}(v_i)$ also includes $w(v_i) - 1$ edges in $G^1$ from $v_i$ to $s$. The assumption that $w(v_j) \geq 2$ for some $j$ ensures that there is at least one path from each $v_i$ to $s$; so $G$ is indeed a sandpile graph.

Then by considering the generators and relations of the indicated monoids, we get

$$SP(G) \cong \mathcal{V}(L_k(E, w)) \cong \mathcal{V}(L_k(F)). \quad (10)$$

In particular, $SP(G)$ arises as the $\mathcal{V}$-monoid of an unweighted Leavitt path algebra.

For any positive integer $n$ let $C_n$ denote the monoid

$$C_n = \{0, x, 2x, \ldots, (n - 1)x\} \text{ with relation } nx = x.$$ 

So $C_n = M_{1,n}$ when $n \geq 2$. Then each of the three displayed monoids in (10) is isomorphic to the monoid $C_N$, where $N = w(v_1) \cdot w(v_2) \cdots w(v_m)$. We note that if $n = 1$ then the monoid $C_1$ is infinite; thus the condition $w(v_i) \geq 2$ for some $i$ is necessary to ensure that this monoid is finite.

As a specific instance of the ideas presented in Example 6.5, suppose $(E, w)$ is the weighted cycle indicated below, where $E^0 = \{a, b, c\}$, $w(a) = w(b) = 2$ and $w(c) = 1$. The associated (unweighted) graphs $F = \hat{E}^{op}$ and $G$ are pictured as well. Since $N = 2 \cdot 2 \cdot 1 = 4$ we get that each of the monoids in (10) is isomorphic to $C_4$. Of course this is not hard to see directly: note that in $M(E, w)$ we have $2a = b$, $2b = c$, and $c = a$, and thus $M(E, w) = \{0, a, 2a, 3a\}$, where $4a = a$.

In Theorem 6.1 we established the ubiquity of weighted Leavitt path algebras in the context of sandpile monoids, namely, that every conical sandpile monoid $SP(G)$ arises as the $\mathcal{V}$-monoid of a suitably defined weighted Leavitt path algebra. Perhaps surprisingly, the unweighted Leavitt path algebras (i.e., the classical Leavitt path algebras) turn out to be quite sparse in this context. We will establish the precise statement in Proposition 6.9.
We now assume that $G$ denotes a conical sandpile graph; i.e., $G$ contains a unique sink $s$, and every vertex in $G$ connects to $s$, and (by Theorem 6.1) every relevant vertex connects to a cycle.

For context, we note that it is easy to find examples of reduced conical sandpile graphs $G$ for which there exists $v \in G^0 \setminus \{s\}$ and $x \in \mathbb{P}_G$ having $x \neq v$ in $\mathbb{P}_G$, but $x = v$ in $\text{SP}(G)$. For instance, let $v, w \in G^0$ for which $s^{-1}(w) = \{e, f_1, \ldots, f_k\}$ where $k \geq 1$ and $r(e) = v$ and $r(f_i) = s$ for all $1 \leq i \leq k$. Then $x := (k + 1)w \neq v$ in $\mathbb{P}_G$ (even if $v = w$), because $v$ is an atom in $\mathbb{P}_G$. But $x = v$ in $\text{SP}(G)$.

We will now show that the example described in the previous paragraph essentially describes the only possible such configurations.

**Lemma 6.6** Let $G$ be a reduced conical sandpile graph. Let $\mathbb{P}_G$ denote the free abelian monoid on the set $G^0$. Suppose $v \in G^0$ and suppose there exists a nonzero element $x \in \mathbb{P}_G$ for which $x \neq v$ in $\mathbb{P}_G$, but $x = v$ in $\text{SP}(G)$. Then there exists a vertex $u_v \in G^0 \setminus \{s\}$ for which

$$s^{-1}(u_v) = \{e_v, e_1, \ldots, e_{n_{u_v}}\},$$

where $n_{u_v}$ denotes $|s^{-1}(u_v)|$, $r(e_v) = v$ and $r(e_i) = s$ for all $1 \leq i \leq n_{u_v} - 1$. (Note that we allow $u_v = v$, and $e_v$ to be a loop at $v$.)

**Proof** If $x = v$ in $\text{SP}(G)$ but $x \neq v$ in $\mathbb{P}_G$, then by the definition of $\sim$ (see Display (3)) there exists a sequence of positive length $x = a_0 \sim a_1 \sim \cdots \sim a_{i-1} \sim a_i \sim \cdots \sim a_n = v$ where $a_{i-1} \sim a_i$ denotes either $\rightarrow$ or $\leftarrow$. We focus on the last move in the sequence, $a_{n-1} \sim a_n = v$. Because $G$ is reduced, each non-sink vertex of $G$ emits at least two edges, so the $\sim$ connecting $a_{n-1}$ to $v$ cannot be $\leftarrow$.

So we must have $a_{n-1} \rightarrow v$. That is, for some $u_v \in G^0 \setminus \{s\}$ we have that $n_{u_v}u_v$ is in $\text{supp}(a_{n-1})$, and

$$(a_{n-1} - n_{u_v}u_v) + \sum_{\{e \in E_1 \mid s(e) = u_v, r(e) \neq s\}} r(e) = v.$$

Since $v$ is an atom in $\mathbb{P}_G$, there are two possibilities:

**Case 1**: $a_{n-1} - n_{u_v}u_v = v$, and $\sum_{\{e \in E_1 \mid s(e) = u_v, r(e) \neq s\}} r(e) = 0$; or

**Case 2**: $a_{n-1} - n_{u_v}u_v = 0$, and $\sum_{\{e \in E_1 \mid s(e) = u_v, r(e) \neq s\}} r(e) = v$.

The equation $\sum_{\{e \in E_1 \mid s(e) = u_v, r(e) \neq s\}} r(e) = 0$ of Case 1 would give that all edges $e$ for which $s(e) = u_v$ have $r(e) = s$; since $n_{u_v} \geq 2$ this would give $n_{u_v}u_v = 0$ in $\text{SP}(G)$, which would violate that $G$ is conical. So Case 1 cannot occur.

But the equations given in Case 2 describe exactly the desired configuration of vertices and edges in $G$.

**Proposition 6.7** Let $G$ be a reduced conical sandpile graph. Suppose in addition that $\text{SP}(G)$ is a refinement monoid. Then there is a partition $\mathcal{P}$ of $G^0 \setminus \{s\}$ for which the graph $G$ has the following form.
Let $P = \{v_1, v_2, \ldots, v_t\}$ be an equivalence class in $\mathcal{P}$. Then each $v_i$ emits all but one of its edges to the sink $s$, and this one not-to-the-sink edge $e_i$ has $s(e_i) = v_i$ for each $1 \leq i \leq t$, $r(e_i) = v_{i-1}$ for each $2 \leq i \leq t$, and $r(e_1) = v_t$.

Denote by $G_P$ the subgraph of $G$ having $G^0_P = \{s, v_1, \ldots, v_t\}$ and $G^1_P = \bigcup_{v_i \in P} s^{-1}(v_i)$. For equivalence classes $P \neq Q \in \mathcal{P}$, $G^0_P \cap G^0_Q = \{s\}$ and $G^1_P \cap G^1_Q = \emptyset$. Then $G$ is the union of the subgraphs $\{G_P \mid P \in \mathcal{P}\}$.

**Proof** By Lemma 2.4, the monoid $SP(G)$ contains no atoms. In particular each vertex $v \in G^0 \setminus \{s\}$ can be written as the sum of two nonzero elements in $SP(G)$. Since $v$ is an atom in $\mathbb{F}_G$ we have that the hypotheses of Lemma 6.6 are satisfied. So for each $v \in G^0 \setminus \{s\}$ there exists a vertex $u_v$ with the indicated configuration. By considering this configuration, if a vertex $u \in G^0 \setminus \{s\}$ arises as $u_v$ for some vertex $v$, then $v$ is unique. (In other words, if $u \in G^0 \setminus \{s\}$ has $u = u_v = u_{v'}$ for vertices $v, v'$ in $G^0 \setminus \{s\}$, then $v = v'$.) Since such a configuration must occur for every $v \in G^0 \setminus \{s\}$, by the Pigeon Hole Principle applied to $G^0 \setminus \{s\}$ (recall that by definition any sandpile graph is finite) we have that each $u \in G^0 \setminus \{s\}$ appears exactly once as $u_v$ for some (unique) $v \in G^0 \setminus \{s\}$. Now partition $G^0 \setminus \{s\}$ using the transitive closure of this relationship. □

**Corollary 6.8** Suppose $G$ is a conical sandpile graph for which $SP(G)$ is a refinement monoid. Then

$$SP(G) \cong \bigoplus_{i=1}^t C_{n_i},$$

for some integers $n_1, \ldots, n_t \geq 2$. Conversely, any monoid of the form $\bigoplus_{i=1}^t C_{n_i}$ with $n_1, \ldots, n_t \geq 2$ is a conical sandpile monoid which is also a refinement monoid.

**Proof** This follows directly from the observations made in Example 6.5, together with Remark 2.13 and Proposition 6.7. □

**Proposition 6.9** Let $E$ be a finite graph. Let $L_k(E)$ denote the (unweighted) Leavitt path algebra. Then the monoid $\mathcal{V}(L_k(E))$ is isomorphic to the sandpile monoid $SP(G)$ of a sandpile graph $G$ if and only if $\mathcal{V}(L_k(E)) \cong \bigoplus_{i=1}^t C_{n_i}$ for some positive integers $n_1, \ldots, n_t$.

**Proof** If $\mathcal{V}(L_k(E)) \cong \bigoplus_{i=1}^t C_{n_i}$, then $\mathcal{V}(L_k(E)) \cong SP(G)$ as in Corollary 6.8, where the graph $G$ is explicitly described in Proposition 6.7.

Conversely, we note that the $\mathcal{V}$-monoid of any ring is conical, while the $\mathcal{V}$-monoid of the (unweighted) Leavitt path algebra $L_k(E)$ of any graph $E$ is refinement (see e.g. [2, Theorem 3.6.8]). So if $\mathcal{V}(L_k(E)) \cong SP(G)$ for some sandpile graph $G$ then Corollary 6.8 gives the result. □

**Remark 6.10** By Proposition 6.9 we know the structure of the $\mathcal{V}$-monoid of any unweighted Leavitt path algebra whose $\mathcal{V}$-monoid is a finite sandpile monoid. We discuss the specific case where $\mathcal{V}(L_k(E)) \cong C_n$ (i.e., that there is only one term in the direct sum given in Proposition 6.9). By [2, Theorem 3.1.10 or Lemma 6.3.14],

\[ \text{ Springer} \]
\( \nabla(L_k(E)) \cong C_n \) yields that \( L_k(E) \) is a purely infinite simple algebra. Moreover, \( K_0(L_k(E)) \cong \mathbb{Z}_{n-1} \). Fix any isomorphism \( \varphi : K_0(L_k(E)) \rightarrow \mathbb{Z}_{n-1} \), and let \( t \) denote \( \varphi([L_k(E)]) \).

Let \( R_n \) denote the graph having one vertex and \( n \) loops at that vertex. It is well known that \( K_0(M_\ell(L_k(R_n))) \cong \mathbb{Z}_{n-1} \), for each \( \ell \in \mathbb{N}^+ \), and that under this isomorphism \([M_\ell(L_k(R_n))] \mapsto \ell \in \mathbb{Z}_{n-1} \).

Therefore \( K_0(L_k(E)) \cong K_0(M_t(L_k(R_n))) \), via a map that takes \([L_k(E)]\) to \([M_t(L_k(R_n))]\). Let \( R_{n,t} \) denote the standard graph having \( L_k(R_{n,t}) \cong M_t(L_k(R_n)) \). Indeed, \( R_{n,t} \) is the graph \( R_n \) with a path of length \( t - 1 \) attached to the vertex; this is exactly the construction described in the proof of Theorem 5.7. It is well-known that \( \det(I - A_{R_{n,t}}) \) is negative.

If \( E \) has the property that \( \det(I - A_E) < 0 \), then by the Classification Theorem of purely infinite simple Leavitt path algebras [2, Theorem 6.3.32], we obtain that \( L_k(E) \) is isomorphic to \( M_t(L_k(R_n)) \).

On the other hand, if \( E \) has the property that \( \det(I - A_E) > 0 \), then as of the writing of this article we do not know whether \( L_k(E) \) is necessarily isomorphic to \( M_t(L_k(R_n)) \). We note that this barrier lies at the heart of the longstanding Algebraic Kirchberg Phillips Question for Leavitt Path Algebras of Finite Graphs, see e.g. [2, Question 6.3.3]. (Note: the situation \( \det(I - A_E) = 0 \) cannot occur for such \( E \).)

We conclude this article by presenting the following natural definition. An investigation into various properties of these structures will be taken up in forthcoming work.

**Definition 6.11** Let \( k \) be a field. We call a \( k \)-algebra \( A \) a **sandpile \( k \)-algebra** in case there exists a conical sandpile graph \( E \) for which

\[
A \cong L_k(E/S, w_r),
\]

where \( S \) denotes the set of vertices of \( E \) which do not connect to a cycle, and \( w_r \) is the restriction of the balanced weighting on \( E \) to \( E/S \).

In particular if \( E \) is a sandpile graph such that each non-sink vertex connects to a cycle, then considering \( E \) as a balanced weighted graph, \( L_k(E/[s], w_r) \) is a sandpile \( k \)-algebra (where \( s \) denotes the unique sink of \( E \)). Among other things, the collection of sandpile \( k \)-algebras provides an umbrella (significantly smaller than the umbrella provided by the collection of weighted Leavitt path \( k \)-algebras) under which all of Leavitt’s \( k \)-algebras \( L_k(n, n+k) \) (for any pair \( n, k \in \mathbb{N}^+ \)) may be realised.

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**References**

1. Abrams, G., Aranda Pino, G.: The Leavitt path algebra of a graph. J. Algebra **293**, 319–334 (2005)
2. Abrams, G., Ara, P., Siles Molina, M.: Leavitt Path Algebras. Lecture Notes in Mathematics, vol. 2191. Springer, London (2017)
3. Abrams, G., Anh, P.N., Louly, A., Pardo, E.: The classification question for Leavitt path algebras. J. Algebra **320**, 1983–2026 (2008)
4. Ara, P., Goodearl, K.: Tame and wild refinement monoids. Semigroup Forum 91, 1–27 (2015)
5. Ara, P., Moreno, M.A., Pardo, E.: Nonstable K-theory for graph algebras. Algebr. Represent. Theory 10, 157–178 (2007)
6. Babai, L., Toumpakari, E.: A structure theory of the sandpile monoid for directed graphs (2010). http://people.cs.uchicago.edu/~laci/REU10/evelin.pdf
7. Bak, P.: How Nature Works. Oxford University Press, Oxford (1997)
8. Bak, P., Tang, C., Weisenfeld, K.: Self-organized criticality: an explanation of 1/f noise. Phys. Rev. Lett. 59, 381–384 (1987)
9. Bergman, G.M.: Coproducts and some universal ring constructions. Trans. Amer. Math. Soc. 200, 33–88 (1974)
10. Bergman, G.M.: The Diamond lemma for ring theory. Adv. Math. 29, 178–218 (1978)
11. Chapman, S., Garcia, R., García-Puente, L., Malandro, M., Smith, K.: Algebraic and combinatorial aspects of sandpile monoids on directed graphs. J. Combin. Theory Ser. A 120, 245–265 (2013)
12. Corry, S., Perkinson, D.: Divisors and Sandpiles: An Introduction to Chip-Firing. American Mathematical Society, Providence (2018)
13. Dhar, D.: Self-organized critical state of sandpile automaton models. Phys. Rev. Lett. 64(14), 1613–1616 (1990)
14. Goodearl, K.R.: von Neumann Regular Rings, 2nd edn. Krieger Publishing, Malabar (1991)
15. Hazrat, R.: The graded structure of Leavitt path algebras. Israel J. Math. 195, 833–895 (2013)
16. Klivans, C.J.: The Mathematics of Chip-Firing. CRC Press, Boca Raton (2019)
17. Leavitt, W.G.: The module type of a ring. Trans. Amer. Math. Soc. 103, 113–130 (1962)
18. Magurn, B.: An Algebraic Introduction to K-Theory, Encyclopedia of Mathematics and its Applications, vol. 87. Cambridge University Press, Cambridge (2002)
19. Newman, M.H.A.: On theories with a combinatorial definition of “equivalence”. Ann. Math. 43(2), 223–243 (1942)
20. Preusser, R.: The V-monoid of a weighted Leavitt path algebra. Israel J. Math. 234, 125–147 (2019)
21. Preusser, R.: Weighted Leavitt path algebras, an overview, arXiv:2109.00434
22. Rosales, J.C., García-Sánchez, P.A.: Finitely Generated Commutative Monoids. Nova Science, Commack (1999)
23. Toumpakari, E.: On the abelian sandpile model. Thesis (Ph.D.) The University of Chicago (2005)
24. Wehrung, F.: Refinement Monoids, Equidecomposability Types, and Boolean Inverse Semigroups. Lecture Notes in Mathematics, vol. 2188. Springer, Cham (2017)

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