Lower bound for cyclic sums of Diananda type

Sergey Sadov∗

Abstract

Let $C = \inf (k/n) \sum_{i=1}^{n} x_i (x_{i+1} + \cdots + x_{i+k})^{-1}$, where the infimum is taken over all pairs of integers $n \geq k \geq 1$ and all positive $x_1, \ldots, x_{n+k}$ subject to cyclicity assumption $x_{n+i} = x_i, i = 1, \ldots, k$. We prove that $\ln 2 \leq C < 0.9305$. In the definition of the constant $C$ the operation $\inf_k \inf_n \inf_x$ can be replaced by $\lim_{k \to \infty} \lim_{n \to \infty} \inf_x$.

Keywords: Cyclic sums, Shapiro’s problem, Drinfeld’s constant

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1 Background and main theorem

Given integers $n \geq k \geq 1$ and a vector $\mathbf{x}$ with positive components $x_1, \ldots, x_n$ (we write $\mathbf{x} > 0$), let us define interval sums

$$t_{i,k} = \sum_{j=0}^{k-1} x_{i+j}$$

and the cyclic sum of Diananda type

$$S_{n,k}(\mathbf{x}) = \sum_{i=1}^{n} \frac{x_i}{t_{i+1,k}}.$$ 

Hereinafter we treat subscripts modulo $n$, that is, $x_{n+i} = x_i$ by definition.

For example,

$$S_{3,2}(\mathbf{x}) = \frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_1} + \frac{x_3}{x_1 + x_2}.$$ 

The sums $S_{n,2}$ are commonly referred to as Shapiro sums. Following [2] p. 217, we associate the name of P.H. Diananda with the more general sums $S_{n,k}$ because they have first occurred in his note [4].

∗E-mail: serge.sadov@gmail.com
The function $S_{n,k}(x)$ is homogeneous of degree zero in its vector argument. Let

$$A(n, k) = \inf_{x > 0} S_{n,k}(x). \quad (1)$$

The domain of the function $S_{n,k}(x)$ can be extended to allow zero values of some $x_i$; it suffices to require that $t_{i,k} > 0$ for all $i$. The value of inf is not affected.

For every $k = 1, 2, \ldots$ denote

$$B(k) = \inf_{n \geq k} \frac{k}{n} A(n,k). \quad (2)$$

Define

$$C = \inf_{k \geq 1} B(k). \quad (3)$$

**Theorem 1** Let $C^+ \approx 0.930498$ be the $y$-intercept of the common tangent to the graphs $y = e^{-x}$ and $y = x/(e^x - 1)$. Then

$$\ln 2 \leq C \leq C^+. \quad (4)$$

This is our main result. Let us put it in context.

If all $x_i$ are equal, then

$$S_{n,k}(x) = \frac{n}{k},$$

hence always

$$A(n,k) \leq \frac{n}{k}, \quad B(k) \leq 1.$$

For $k = 1$ we have

$$S_{n,1}(x) = \frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \geq n$$

by the inequality between arithmetic and geometric means (AM-GM). Thus $B(1) = 1$.

The inequality $S_{3,2}(x) \geq 3/2$ implying $A(3,2) = 3/2$ has been known since 1903 at latest (due to A.M. Nesbitt) — see Ref. [11, p. 440], which offers three proofs. H.S. Shapiro [15] proposed to prove that $A(n,2) \geq n/2$ for all $n$, i.e. that $B(2) = 1$. This conjecture was soon disproved [12]. The precise validity range for Shapiro’s conjectured inequality was determined through analytical and numerical labor over time span of more than 20 years: even $n \leq 12$ and odd $n \leq 23$. See review [3]. The actual value of $B(2)$ found
by V.G. Drinfeld [9] is slightly less than one: $B(2) = 0.989133 \ldots$. It equals the $y$-intercept of the common tangent to the graphs $y = g_1(x) = e^{-x}$ and $y = g_2(x) = 2/(e^{x/2} + e^{x})$.

Much less was known until now about lower bounds for the cyclic sums $S_{n,k}$ with $k \geq 3$. In [5] Diananda showed that

$$\frac{k}{n} A(n,k) \geq \frac{2(k+1)}{n}$$

for $n > 2(k+1)$. (In the region $2(k+1) < n < (2/\ln 2)(k+1)$ of the $(n,k)$-lattice this lower bound beats the estimate $(k/n) A(n,k) \geq \ln 2$ that follows from our Theorem[1]). Diananda also found a few cases where $A(n,k) = n/k$ with $k > 2$. They are listed with references in [8, p. 173] or [11, p. 445]. The cited results do not allow one to conclude that $B(k) > 0$ for $k \geq 3$. The only result to that effect was Diananda’s [4] simple lower bound $A(n,k) \geq n/k^2$, which implies that $B(k) \geq 1/k$. Compare: Theorem[1] says that in fact $B(k)$ are uniformly separated from zero.

A systematic study of cyclic sums in which numerators and denominators are overlapping interval sums is carried out in [1]. The closest in appearance to our inequality $A(n,k) \geq \text{const} \cdot (n/k)$, const $= \ln 2$, is Baston’s formula (in our notation)

$$\inf x \sum_{i=1}^{k} \frac{x_i}{t_{i,k}} = \frac{1}{1 + \frac{k-1}{n}} \quad (k \geq 2)$$

contained in his Theorem 1. Yet the presence of the summand 1 in the denominator on the right causes a striking contrast with our situation: the analog of $B(k)$,

$$\inf_{n, n \geq k} \inf_{x} \frac{k}{n} \sum_{i=1}^{k} \frac{x_i}{t_{i,k}},$$

equals zero unless $k = 1$!

Some further relevant citations can be found in Remarks to Theorems 2–4 below. For a detailed review of similar and other cyclic inequalities see [11, Ch. 16], particularly § 15 and further on.

In the following two sections we will establish $k$-dependent bounds for the individual constants $B(k)$ tighter than their common bounds in Theorem[1] the latter will easily follow. The lower and upper estimates are treated separately, since the methods of proof are different.

The final section of the paper answers in the affirmative a natural question whether the operations $\inf_{n, n \geq k}$ and $\inf_{k}$ in the definitions (2) of $B(k)$ and (3) of $C$ can be replaced, respectively, by $\lim_{n \to \infty}$ and $\lim_{k \to \infty}$. 

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2 Lower bounds for $B(k)$

**Theorem 2** The constants $B(k)$ are bounded below as follows:

$$B(k) \geq k(2^{1/k} - 1).$$

In other words, for any integers $n \geq k \geq 1$ and any $n$-dimensional vector $\mathbf{x} > 0$ the cyclic inequality

$$\frac{k}{n} \sum_{i=1}^{n} \frac{x_i}{t_{i+1,k}} \geq k(2^{1/k} - 1)$$

holds.

**Remark 2.1** The left inequality [1] of Theorem 1 follows since $k(2^{1/k} - 1) > \ln 2$ (indeed, $e^x - 1 > x$ ($x \neq 0$); take $x = k^{-1} \ln 2$).

**Remark 2.2** The numerical values of our lower bounds for $k \leq 7$ are listed below.

| $k$ | $k(2^{1/k} - 1)$ |
|-----|------------------|
| 2   | 0.82843          |
| 3   | 0.77976          |
| 4   | 0.75683          |
| 5   | 0.74349          |
| 6   | 0.73477          |
| 7   | 0.72863          |

In the case $k = 2$ our value is worse than the best lower estimate $B(2) \geq 0.922476 \ldots$ [7] known before Drinfeld’s exact result $B(2) = 0.989133 \ldots$, yet it is better than earlier attempts, e.g. $B(2) \geq 0.66046 \ldots$ due to R.A. Rankin [14]. This will be helpful to keep in mind when trying to improve our result.

The author is unaware of any published lower bound for $B(3)$ except for $B(3) \geq 1/3$, a particular case of Diananda’s inequality $B(k) \geq 1/k$.

**Proof.** Without loss of generality we may assume that $k | n$ ($k$ divides $n$). Indeed, given any $k$ and $n$, let $n' = kn$ and define the $n'$-dimensional vector $\mathbf{x}'$ as concatenation of $k$ copies of $\mathbf{x}$. Obviously, $k | n'$ and

$$\frac{1}{n'} \sum_{i=1}^{n'} \frac{x'_i}{t_{i+1,n'}} = \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{t_{i+1,k}}.$$

(The index arithmetic for $\mathbf{x}'$ is done modulo $n'$, unlike for $\mathbf{x}$.)

From now on we assume that $n = k\nu$ with $\nu \geq 1$ an integer. We fix the vector argument $\mathbf{x}$ for the rest of the proof. Set

$$r_j = \frac{t_{k(j+1),k}}{t_{kj+1,k}}, \quad j = 0, \ldots, \nu - 1.$$
(When \( j = \nu - 1 \) the denominator equals \( t_{1,k} \))

We will derive lower estimates for the partial sums

\[
s_j = \sum_{i=1}^{k} \frac{x_{jk+i}}{t_{jk+i+1,k}}, \quad j = 0, \ldots, \nu - 1.
\]

Consider \( j = 0 \) for simplicity of notation. We have

\[
x_i = t_{i,2k-i+1} - t_{i+1,2k-i}.
\]

Also for \( i = 1, \ldots, k \)

\[
t_{i+1,k} \leq t_{i+1,2k-i}.
\]

Hence

\[
s_0 \geq \sum_{i=1}^{k} \frac{t_{i,2k-i+1} - t_{i+1,2k-i}}{t_{i+1,2k-i}} = -k + \sum_{i=1}^{k} \frac{t_{i,2k-i+1}}{t_{i+1,2k-i}}.
\]

Since

\[
\prod_{i=1}^{k} \frac{t_{i,2k-i+1}}{t_{i+1,2k-i}} = \frac{t_{1,2k}}{t_{k+1,k}} = \frac{t_{1,k} + t_{k+1,k}}{t_{k+1,k}} = 1 + r_0,
\]

the AM-GM inequality yields

\[
s_0 \geq k \left( (1 + r_0)^{1/k} - 1 \right).
\]

Similar inequalities hold for \( s_j \) with \( j = 1, \ldots, \nu - 1 \) in place of \( j = 0 \).

Introduce the function

\[
f_k(t) = k \left( (1 + e^t)^{1/k} - 1 \right).
\]

Let \( r_j = e^{t_j} \). From the above we obtain

\[
S_{n,k}(x) = \sum_{j=0}^{\nu-1} s_j \geq \sum_{j=0}^{\nu-1} f_k(t_j).
\]

The function \( f_k(t) \) is convex. A simple way to see it is to factor the derivative as follows

\[
f_k'(t) = \frac{(1 + e^t)^{1/k}}{1 + e^{-t}},
\]

where the numerator increases while the denominator decreases, so \( f_k''(t) > 0 \).

Since \( \prod_{j=0}^{\nu-1} r_j = 1 \), the Jensen inequality yields

\[
\frac{S_{n,k}(x)}{\nu} \geq f_k \left( \frac{1}{\nu} \sum_{0}^{\nu-1} t_j \right) = f_k(0) = k(2^{1/k} - 1).
\]

This is precisely the claimed inequality. \( \square \)
3 Upper bounds for $B(k)$

In this section we use Drinfeld’s [9] construction to prove upper estimates for $B(k)$, $k = 2, 3, \ldots$. For $k = 2$ we just get Drinfeld’s constant. It is conceivable that for every $k \geq 3$ this construction provides a minimizing sequence $x^{(n)}$, too, but we are unable to prove it. Drinfeld’s proof (for $k = 2$) does not extend to $k > 2$.

As a preparation to Theorem 3 below let us introduce the family of functions

$$g_k(x) = \frac{k(1 - e^{-x/k})}{e^x - 1}, \quad k \geq 1, \quad (5)$$

and study some of their properties. We set $g_k(0) = 1$ to make $g_k(x)$ continuous (in fact, real analytic). Note that $g_1(x) = e^{-x}$. Denote also

$$g_{\infty}(x) = \lim_{k \to \infty} g_k(x) = \frac{x}{e^x - 1}.$$

**Lemma 1** The functions $g_{\infty}(x)$, $p(x) = (1 - e^{-x})/x$, and $g_k(x)$ for any $k > 0$ are positive, decreasing and convex.

**Proof.** Note that $g_k(x) = g_{\infty}(x/k)p(x)$. By the Leibniz Rule the class of positive, decreasing and convex functions is closed under multiplication. It is also closed under rescaling of the independent variable. Hence it suffices to give a proof for $g_{\infty}(x)$ and $p(x)$. The only nontrivial task is to check convexity. For $g_{\infty}$ we have

$$g''_{\infty}(x) = \frac{e^{2x}(x - 2) + e^x(x + 2)}{(e^x - 1)^3} = \frac{\cosh \frac{x}{2} \left( \frac{x}{2} - \tanh \frac{x}{2} \right)}{2 \sinh^3 \frac{x}{2}} > 0.$$

Now for $p(x)$: if $x < 0$, then we write $p(x) = \tilde{p}(-x)$, where $\tilde{p}(y) = (e^y - 1)/y$ has the Maclaurin series with positive coefficients, hence convex when $y > 0$. And if $x > 0$, then we calculate

$$p''(x) = \frac{2e^{-x}}{x^3} \left( e^x - \left( 1 + x + \frac{x^2}{2} \right) \right),$$

and conclude that $p''(x) > 0$ by the Maclaurin expansion, again. $\square$

**Lemma 2** For a fixed real $x \neq 0$ the function $k \mapsto k(1 - e^{-x/k})$ is increasing.

**Proof.** The claim is immediately clear, for both signs of $x$, by writing

$$k(1 - e^{-x/k}) = \int_0^x e^{-y/k} \, dy.$$

$\square$
Lemma 3 The function \( k \mapsto g_k(x) \) is increasing for any fixed \( x > 0 \) and decreasing for any fixed \( x < 0 \). In particular, for \( k > 1 \) the inequalities

\[
g_k(x) > g_1(x) = e^{-x} \quad \text{if} \quad x > 0
\]

and

\[
g_k(x) < e^{-x} \quad \text{if} \quad x < 0
\]

hold.

Proof. Apply Lemma 1, taking into account sign of \( e^x - 1 \). \(\square\)

Let \( h_k(x) \) be the **convex minorant** of the function \( \min(g_1(x), g_k(x)) \). From Lemmas 1 and 3 it follows that \( h_k(x) \) is of the form

\[
h_k(x) = \begin{cases} 
g_k(x), & x \leq a_k, \\
g_k(x) + \lambda_k x, & a_k < x < b_k, \\
e^{-x}, & x \geq b_k,
\end{cases}
\]

where \( a_k < 0 < b_k \) are the abscissas of the tangency points of the common tangent to the graphs \( y = g_k(x) \) and \( y = e^{-x} \). The parameters \( a_k, b_k, \gamma_k \) and \( \lambda_k \) are uniquely determined by the condition that \( h_k(x) \) be continuous and differentiable. A simple way to find them numerically is as follows (we omit the subscript \( k \) to lighten notation). The tangent to the graph \( y = e^{-x} \) at \((b, e^{-b})\) is \( y = e^{-b}(1 + b - x) \). It is also tangent to \( y = g(x) \) at \((a, g(a))\), hence \(-e^{-b} = g'(a)\) and \( g(a) = -g'(a)(1 + b - a)\). Eliminating \( b \) from the last two equations leads to the equation for the single unknown \( a \):

\[
\frac{g(a)}{g'(a)} - a + 1 = \ln(-g'(a)). \quad (6)
\]

Now \( \lambda = g'(-a) \), \( b = -\ln \lambda \), and \( \gamma = -\lambda(1 + b) \).

Clearly, \( \gamma_k < 1 \) and \( \lambda_k < 0 \). The pointwise monotonicity of the family \( g_k(x) \) stated in Lemma 3 implies that as \( k \) increases, \( \gamma_k \) and \( |\lambda_k| \) decrease. As \( k \to \infty \), the parameters \( a_k, b_k, \gamma_k \) and \( \lambda_k \) tend to their limits corresponding to the convex minorant of \( \min(g_1(x), g_\infty(x)) \). Therefore the constant \( C^+ \) in Theorem 1 is the monotone limit

\[
C^+ = \gamma_\infty = \lim_{k \to \infty} \gamma_k.
\]

Consequently, the upper bound in Theorem 1 (left inequality in (4)) follows from more precise estimates in the next theorem.
Theorem 3. Let $k \geq 2$, the function $g_k(x)$ be defined by (5), and $\gamma_k$ be (as defined above) the $y$-intercept of the common tangent to the graphs $y = e^{-x}$ and $y = g_k(x)$. The constant $B(k)$ defined in (2) satisfies the inequality

$$B(k) \leq \gamma_k.$$ 

Remark 3.1. Some numerical values of our upper bounds (found by solving Eq. (6)) are listed below. The limit value $\gamma_{\infty} = C^+$ is included for convenience of comparison.

| $k$ | 2   | 3     | 4     | 10    | 100   | 1000  | $\infty$ |
|-----|-----|-------|-------|-------|-------|-------|---------|
| $\gamma_k$ | 0.98913 | 0.97793 | 0.96994 | 0.94983 | 0.93272 | 0.93072 | 0.930498 |

Here $\gamma_2$ is nothing but Drinfeld’s constant. Besides it, the only other previously reported estimate of this sort seems to be that due to J.C. Boarder and D.E. Daykin [2, Table 2, row ‘a/bcd’] (reproduced in [11] as Eq. (27.41), p. 453): \( \inf_n A(n, 3)/n \leq 0.32598 \), implying \( B(3) \leq 0.97794 \). Our estimate, with more digits than in the table above, is \( B(3) \leq 0.977927986 \ldots \). Since \( \gamma_3/3 > 0.32598 - 0.5 \times 10^{-5} \), within the accuracy of 5 significant digits we can not claim an improvement over [2]. However the method of [2] is entirely numerical and based on calculation of bounds for \( A(n, 3)/n \) for finitely many $n$, while in the proof below we let $n \to \infty$.

Proof. Fix $k \geq 2$. Given an $\epsilon > 0$ we will find an integer $n > k$ and an $n$-dimensional vector $x$ such that $S_{n,k}(x) < \gamma_k + \epsilon$.

The point $(0, \gamma_k)$ is a convex combination of the points $(a_k, g_k(a_k))$ and $(b_k, e^{-b_k})$ with some coefficients $\mu_k$ and $\mu'_k = 1 - \mu_k$,

$$\mu_k a_k + \mu'_k b_k = 0, \quad \mu_k g_k(a_k) + \mu'_k e^{-b_k} = \gamma_k.$$ 

Let us choose rational $\mu_* = p/q \in (0, 1)$ sufficiently close to $\mu_k$ and real $a_*, b_*$ sufficiently close to $a_k, b_k$ respectively so that

$$\mu_* a_* + \mu'_* b_* = 0, \quad \mu_* g_k(a_*) + \mu'_* e^{-b_*} < \gamma_k + \frac{\epsilon}{2}.$$ 

From now on $\mu_*, a_*, b_*$ are assumed fixed. We write $\mu_*$ as a fraction generally not in the lowest terms,

$$\mu_* = \frac{m}{n}.$$ 

Later it will be important to allow $n$ be as large as we please. We may and will assume that the numerator $m$ and denominator $n$ are divisible by $k$. 

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Let us now describe construction of \( x \) assuming \( n \) and \( m = \mu n \) given; a specific choice of \( n \) will be made afterwards.

Denote \( m' = n - m = \mu' n \). Define the \( n \)-dimensional vector \( x \) as follows:

\[
\begin{align*}
x_{jk} &= e^{jb_*}, & \text{if } 1 \leq j < m'/k; \\
x_i &= 0, & \text{if } 1 \leq i < m', k \nmid i; \\
x_i &= e^{-a_*(n-i)/k}, & \text{if } m' \leq i \leq n.
\end{align*}
\]

The sequence \( x_i \) of length \( n \) consists of two parts. It is sparse when \( i < m' \); only one in every \( k \) consecutive terms is nonzero, and those nonzero terms are increasing. For \( m \leq i \leq n \), all terms are nonzero; they form a decreasing geometric sequence.

Note that the formula \( x_{jk} = e^{jb_*} \), which is the definition when \( j < m'/k \), continues to hold for \( j = m'/k \). Indeed, it follows from the equality \( (m'/k)b_* = -a_*(n-m')/k \), which is true since \( \mu' b_* = -\mu a_* \).

Let us compute nonzero terms in the sum \( S_{n,k}(x) \). For \( i = jk < m' \) we have

\[
\frac{x_i}{t_{i+1,k}} = \frac{x_i}{x_{i+k}} = e^{-b_*}.
\]

For \( m' \leq i \leq n - k - 1 \) (there are \( m-k \) such \( i \)'s) we get

\[
\frac{x_i}{t_{i+1,k}} = \left( \sum_{j=1}^{k} e^{a_*/k} \right)^{-1} = \frac{1 - e^{-a_*/k}}{e^{a_*/k} - 1} = \frac{g_k(a_*)}{k}.
\]

For the remaining \( k \) terms (with \( i = n-k, \ldots, n-1 \)) a convenient closed form expression is not available. The rough estimate \( t_{i+1,k} > x_{i+1} = e^{a_*/k}x_i \) will suffice. Thus

\[
\sum_{i=n-k}^{n-1} \frac{x_i}{t_{i+1,k}} < ke^{-a_*/k}.
\]

In total,

\[
S_{n,k}(x) < \frac{m'}{k} e^{-b_*} + \frac{m-k}{k} g_k(a_*) + ke^{-a_*/k}.
\]

So

\[
\frac{k}{n} S_{n,k}(x) < \mu' e^{-b_*} + \mu g_k(a_*) + \delta/n,
\]

where

\[
\delta = k^2 e^{-a_*/k} - kg_k(a_*)
\]
does not depend on \( n \). We choose \( n \) so as to make

\[
\frac{\delta}{n} < \frac{\epsilon}{2}.
\]

Recalling (7), we obtain

\[
\frac{k}{n} S_{n,k}(x) < \gamma_k + \epsilon.
\]

Since \( \epsilon \) is arbitrary, we conclude that \( B(k) \leq \gamma_k \).

\[\Box\]

4 Greatest lower bounds \( B(k) \) and \( C \) as limits

**Theorem 4** The constants \( B(k) \) and \( C \) defined in (2) and (3) respectively can be expressed as limits. Specifically,

(a) for every integer \( k \geq 1 \)

\[
B(k) = \lim_{n \to \infty} \frac{k}{n} A(n,k);
\]

(b) the sequence \( B(k) \) is nonincreasing and (therefore)

\[
C = \lim_{k \to \infty} \downarrow B(k).
\]

**Remark 4.1** It has been known since the early period of investigation of the original Shapiro’s conjecture, that \( A(n,2)/n \) is not monotone in \( n \).

**Remark 4.2** Part (a) of this Theorem was established for Shapiro sums \( (k = 2) \) by R.A. Rankin [13] and for very general cyclic sums — by K. Goldberg [10]. To make our exposition self-contained, we still give a proof, which is a generalization (albeit obvious) of Rankin’s and more explicit than Goldberg’s.

**Proof.** (a) We assume \( k \geq 1 \) fixed once for all.

Given an \( \epsilon > 0 \), we will find \( N \) such that

\[
\frac{A(n,k)}{n} < \frac{B(k)}{k} + \epsilon.
\]

whenever \( n \geq N \).

There exist an \( m > k \) and an \( m \)-dimensional vector \( x \) for which

\[
\frac{S_{m,k}(x)}{m} < \frac{B(k)}{k} + \frac{\epsilon}{2}.
\]
Fix an arbitrary \((m - 1)\)-tuple \((\xi_1, \ldots, \xi_{m-1})\) of positive numbers. Define the vectors ("\(r\)-extensions of \(x\) by \(\xi\)\): 
\[
y^{(r)} = (x_1, \ldots, x_m, \xi_1, \ldots, \xi_r).
\]
Let
\[
M = \max_{0 \leq r \leq m-1} S_{m+r,k}(y^{(r)})
\]
and choose \(N\) big enough to make \(M/N < \epsilon/2\).

Now, given \(n = m\nu + r\), \(\nu \geq 1\), \(0 \leq r \leq m - 1\), we construct an \(n\)-dimensional vector \(x'\) as concatenation of \((\nu - 1)\) copies of \(x\) followed by \(y^{(r)}\). It is readily seen that
\[
S_{n,k}(x') = (\nu - 1)S_{m,k}(x) + S_{m+r,k}(y^{(r)}).
\]
Therefore, if \(n \geq N\), then
\[
\frac{A(n, k)}{n} \leq \frac{S_{n,k}(x')}{n} < \frac{(\nu - 1)m}{n} \left( \frac{B(k)}{k} + \frac{\epsilon}{2} \right) + \frac{M}{n} < \frac{B(k)}{k} + \frac{\epsilon}{2} + \frac{\epsilon}{2},
\]
as required.

(b) Given a \(k\nu\) dimensional vector \(x\), define a \((k+1)\nu\) dimensional vector \(x'\) as follows: for \(j = 0, 1, \ldots, \nu - 1\)
\[
x'_{(k+1)j+r} = \begin{cases} 
x_{kj+r} & \text{if } 1 \leq r \leq k, \\
0 & \text{if } r = k + 1.
\end{cases}
\]
Then
\[
S_{(k+1)\nu,k+1}(x') = S_{k\nu,k}(x).
\]
Taking \(\inf_x\) we conclude:
\[
A((k + 1)\nu, k + 1) \leq A(k\nu, k).
\]
By part (a),
\[
\lim_{\nu \to \infty} \frac{A(k\nu, k)}{\nu} = B(k),
\]
and the inequality \(B(k + 1) \leq B(k)\) follows. \(\square\)
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