Lower cone distribution functions and set-valued quantiles form Galois connections∗

Çağın Ararat† Andreas H. Hamel‡

December 18, 2018

Abstract

It is shown that the recently introduced lower cone distribution function together with set-valued multivariate quantiles generate a Galois connection between a complete lattice of closed convex sets and the intervall [0,1]. This generalizes the (not so well-known) corresponding univariate result. It is also shown that an extension of the lower cone distribution function and the set-valued quantile characterize the capacity functional of a random set extension of the original multivariate variable along with its distribution.

Keywords Galois connection, multivariate quantile, complete lattice, lower cone distribution function, random set

Mathematics Subject Classification 60A05, 62H05

1 Introduction

Several features of set-valued quantiles for multivariate random variables as introduced in [9] are investigated and extended. In particular, the lower cone distribution function from [9] is extended to a function on sets, and it is shown that this extension together with the set-valued quantile forms a Galois connection between ([0,1], ≤) and a complete lattice of closed convex sets ordered by ⊇. In the univariate case, a similar result is known (see [4, Remark 3.1]), but apparently not too popular under this label. For example, in the recent work [6], the property constituting the Galois connection is stated (formula (2), p. 5), but the Galois connection is neither mentioned, nor exploited.

∗In honour of Y. Kabanov on the occasion of his 70th birthday.
†Bilkent University, Department of Industrial Engineering, Ankara, Turkey, cararat@bilkent.edu.tr.
‡Free University of Bozen, Faculty of Economics and Management, Bozen-Bolzano, Italy, andreas.hamel@unibz.it.
Our approach turns two downsides of previous proposals for multivariate quantiles into upsides. First, by using the cone distribution function (instead of the joint distribution function even if the cone is $\mathbb{R}^d_+$), an arbitrary (vector) order can be dealt with, thus ‘the absence of a natural ordering of Euclidean spaces of dimension greater than one’ ([12, p. 214]) is turned into a huge potential for applications in statistical analysis where such an order relation very often is present by default. Second, by understanding quantile functions as functions mapping into complete lattices of sets, the fact that an inverse of a monotone function usually ‘defines only a correspondence, that is, a multi-valued or set-valued mapping’ ([6, p. 5]) is turned into a tool for a better understanding of multivariate quantiles.

It is shown that the set-valued quantiles characterize the distribution of a random set extension of the original random variable, thus the three objects “distribution of the random set,” “(extended) cone distribution function of the random vector,” and “lattice-valued quantile function” carry the same information. This is very much parallel to the univariate case (compare, for instance, [6, formula (4), p. 5]). The ordering cone enters the definition of the lattice of sets in which the random set extension of the original multivariate random variable takes its values.

2 Set-up

The framework and notation of [9] and, when it comes to concepts from set-valued convex analysis, [8] are used. In particular, we consider a preorder on $\mathbb{R}^d$ which is generated by a nonempty closed convex cone $\emptyset \neq C \subseteq \mathbb{R}^d$ by means of

$$y \leq_C z \iff z - y \in C$$

for $y, z \in \mathbb{R}^d$; $C$ is neither assumed to have a non-empty interior, nor be pointed, i.e., $C \cap (-C) = \{0\}$ is not assumed. Thus, the cases $C = \{0\}$ and $C = H^+(w) := \{z \in \mathbb{R}^d \mid w^T z \geq 0\}$ for $w \in \mathbb{R}^d \setminus \{0\}$ are not excluded. In the latter case, $C$ is a (homogeneous) halfspace and $\leq_C$ a total preorder. The (positive) dual of the cone $C$ is

$$C^+ = \{w \in C \mid \forall z \in C : w^T z \geq 0\}.$$ 

The bipolar theorem yields $C = C^{++} := (C^+)^+$ under the given assumptions. The set

$$\mathcal{G}(\mathbb{R}^d, C) = \{D \subseteq \mathbb{R}^d \mid D = \text{cl co} (D + C)\}$$

comprises the closed convex subsets of $\mathbb{R}^d$ which are stable under addition of $C$; the sum $A + B$ is understood in the Minkowski sense with $A + \emptyset = \emptyset + A$ for all $A \subseteq \mathbb{R}^d$. The pair $(\mathcal{G}(\mathbb{R}^d, C), \supseteq)$ is an order complete lattice with the following formulas for inf and sup (see, for example, [8])

$$\inf_{D \in \mathcal{D}} D = \text{cl co} \bigcup_{D \in \mathcal{D}} D, \quad \sup_{D \in \mathcal{D}} D = \bigcap_{D \in \mathcal{D}} D.$$
whenever $\mathcal{D} \subseteq G(\mathbb{R}^d, C)$.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $L^0_d$ the space of all equivalence classes of random variables with values in $\mathbb{R}^d$. For $X \in L^0_d$, $w \in \mathbb{R}^d \setminus \{0\}$, the function $F_{X,w}: \mathbb{R}^d \to [0,1]$ defined by $F_{X,w}(z) = \mathbb{P}\{X \in z - H^+(w)\} = \mathbb{P}\{w^T X \leq w^T z\}$ is called the $w$-distribution function of $X$. If $d = 1$, $C = \mathbb{R}_+$ and $w = 1$, then $F_{X,w}$ is just the usual cumulative distribution function (cdf) of the univariate $X$. The function $F_{X,C}: \mathbb{R}^d \to [0,1]$ defined by

$$F_{X,C}(z) = \inf_{w \in C^+} F_{X,w}(z) = \inf \{\mathbb{P}\{X \in z - H^+(w)\} \mid w \in C^+\}$$

is called the lower $C$-distribution function associated to $X$.

If $p \in [0,1]$ and $w \in \mathbb{R}^d$, the set $Q_{X,w}(p) = \{z \in \mathbb{R}^d \mid F_{X,w}(z) \geq p\}$ is called the lower $w$-quantile, and the set

$$Q_{X,C}(p) = \{z \in \mathbb{R}^d \mid F_{X,C}(z) \geq p\} = \bigcap_{w \in C^+} Q_{X,w}(p)$$

(1)

is called the lower $C$-quantile of $X$. Clearly, $Q_{X,C}(0) = \mathbb{R}^d$ and by definition, the sets $Q_{X,C}(p)$ are nested, i.e., $Q_{X,C}(p_1) \supseteq Q_{X,C}(p_2)$ whenever $0 \leq p_1 \leq p_2 \leq 1$.

If $C = \{0\}$ and hence $C^+ = \mathbb{R}^d$, then $F_{X,C}$ is the Tukey depth function and $Q_{X,C}(p)$ is the corresponding depth region. In this case, it might happen that (for continuous distributions, for example) $F_{X,C}(z) \in [0,1/2]$ for all $z \in \mathbb{R}^d$ which also shows that Tukey’s depth function in case $d = 1$ is not a generalization of the univariate cdf which requires $C = C^+ = \mathbb{R}_+$ and has values in $[0,1]$, in general.

3 Lower cone distribution functions and quantiles

The first one is quasiconcavity; the following result is an extension of [11, Proposition 1] which works for the Tukey depth function (but even for an arbitrary positive measure).

**Proposition 1.** The lower $C$-distribution function $F_{X,C}$ is quasiconcave and monotone nondecreasing with respect to $\leq_C$.

**Proof.** Take $z^1, z^2 \in \mathbb{R}^d$, $s \in [0,1]$ and $w \in C^+ \setminus \{0\}$. Define $z(s) = sz^1 + (1-s)z^2$. Then either $z^1 \in z(s) - H^+(w)$ or $z^2 \in z(s) - H^+(w)$ or even both. This implies $z^1 - H^+(w) \subseteq z(s) - H^+(w)$ or $z^2 - H^+(w) \subseteq z(s) - H^+(w)$ or both and hence

$$\mathbb{P}\{X \in z(s) - H^+(w)\} \geq \min \{\mathbb{P}\{X \in z^1 - H^+(w)\}, \mathbb{P}\{X \in z^2 - H^+(w)\}\}.$$ 

Since this is true for each $w \in C^+ \setminus \{0\}$, $F_{X,C}$ is quasiconcave. If $y \leq_C z$, then $y - H^+(w) \subseteq z - H^+(w)$ for all $w \in C^+ \setminus \{0\}$, hence $\mathbb{P}\{X \in y - H^+(w)\} \leq \mathbb{P}\{X \in z - H^+(w)\}$ for all $w \in C^+ \setminus \{0\}$. This implies $F_{X,C}(y) \leq F_{X,C}(z)$. \qed
Corollary 1. The set $Q_{X,C}^{-}(p)$ is convex and satisfies $Q_{X,C}^{-}(p) + C \subseteq Q_{X,C}^{-}(p)$ for all $p \in [0,1]$.

Proof. Convexity follows since the upper level sets of quasiconcave functions are convex; see [13, Section 2.1], for instance. The second property is a consequence of the monotonicity of $F_{X,C}$. □

Proposition 2. The lower $C$-distribution function $F_{X,C}$ is upper semicontinuous.

Proof. Note that $F_{X,C}$ is upper semicontinuous if and only if its upper level set $Q_{X,C}^{-}(p)$ is closed for every $p \in [0,1]$. To show the latter, let $p \in [0,1], w \in C^{+}$. By (1), it enough to show that $Q_{X,w}^{-}(p)$ is a closed set. We have

$$Q_{X,w}^{-}(p) = \{ z \in \mathbb{R}^{d} \mid \mathbb{P}\{ w^{T}X \leq w^{T}z \} \geq p \}. $$

Note that the (usual) cumulative distribution function of $w^{T}X$ is right-continuous and increasing, hence it is also upper semicontinuous. Hence, its composition with the continuous mapping $z \mapsto w^{T}z$ gives the upper semicontinuous mapping $z \mapsto \mathbb{P}\{ w^{T}X \leq w^{T}z \}$. Since $Q_{X,w}^{-}(p)$ is an upper level set of this mapping, it follows that $Q_{X,w}^{-}(p)$ is a closed set, which finishes the proof of upper semicontinuity. □

Lemma 1. Let $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ be convergent sequences in $\mathbb{R}$ with limits $a, b \in \mathbb{R}$, respectively. If $a \neq b$, then,

$$\lim_{n \to \infty} 1_{(-\infty,a_n]}(b_n) = 1_{(-\infty,a]}(b).$$

Proof. Suppose that $a < b$ so that $1_{(-\infty,a]}(b) = 0$. Let $\varepsilon = \frac{b-a}{2}$. There exists $n_0 \in \mathbb{N}$ such that $|a_n - a| \leq \varepsilon$ and $|b_n - b| \leq \varepsilon$ for every $n \geq n_0$. In particular, $a_n \leq a + \varepsilon < b - \varepsilon \leq b_n$ so that $1_{(-\infty,a_n]}(b_n) = 0$ for every $n \geq n_0$. Hence, $\lim_{n \to \infty} 1_{(-\infty,a_n]}(b_n) = 0 = 1_{(-\infty,a]}(b)$. The case $a > b$ can be treated by similar arguments. □

Remark 1. The condition $a = b$ in Lemma 1 cannot be omitted. As a counterexample, let $a = b = 0$ and $a_n = -\frac{1}{n}, b_n = \frac{1}{n}$ for every $n \in \mathbb{N}$. Note that $1_{(-\infty,a]}(b) = 1$ and $1_{(-\infty,a_n]}(b_n) = 0$ for every $n \in \mathbb{N}$. Hence, $\lim_{n \to \infty} 1_{(-\infty,a_n]}(b_n) = 0 \neq 1 = 1_{(-\infty,a]}(b)$.

Proposition 3. If the distribution of $X$ under $\mathbb{P}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{d}$, then $F_{X,C}$ is continuous.

Proof. Let $B = C^{+} \cap S^{d-1}$, where $S^{d-1}$ is the unit sphere in $\mathbb{R}^{d}$. Note that $B$ is a base for $C^{+}$ in the sense that every $\tilde{w} \in C^{+} \setminus \{0\}$ can be written in the form $\tilde{w} = rw$ for some unique $r > 0$ and unique $w \in B$, and we have $F_{X,w}(z) = F_{X,w}(z)$ for every $z \in \mathbb{R}^{d}$. It follows that

$$F_{X,C}(z) = \inf_{w \in B} F_{X,w}(z) \quad (2)$$
for every \( z \in \mathbb{R}^d \). Moreover \( B \) is a compact set.

By Proposition 2, it suffices to show that \( F_{X,C} \) is lower semicontinuous. We fix \( p \in [0, 1) \) and show that the lower level set \( L(p) = \{ z \in \mathbb{R}^d \mid F_{X,C}(z) \leq p \} \) is closed. To that end, let \( (z_n)_{n \in \mathbb{N}} \) be a convergent sequence in \( L(p) \) with limit \( \bar{z} \in \mathbb{R}^d \). Let \( \epsilon > 0 \). By (2), for every \( n \in \mathbb{N} \), there exists \( w_n \in B \) such that \( F_{X,w_n}(z_n) < F_{X,C}(z_n) + \epsilon \leq p + \epsilon \). Since \( (w_n)_{n \in \mathbb{N}} \) is a sequence in the compact set \( B \), by Bolzano-Weierstrass theorem, there exists a convergent subsequence \( (w_{n_k})_{k \in \mathbb{N}} \) of it, say, with limit \( \bar{w} \in B \). Hence, \( \lim_{k \to \infty} w_{n_k}^T \bar{z}_k = \bar{w}^T \bar{z} \), and applying Lemma 1 gives

\[
\lim_{k \to \infty} 1_{(-\infty, w_{n_k}^T \bar{z}_k)}(w_{n_k}^T X(\omega)) = 1_{(-\infty, \bar{w}^T \bar{z})}(\bar{w}^T X(\omega))
\]

for every \( \omega \in \Omega \) such that \( \bar{w}^T X(\omega) \neq \bar{w}^T \bar{z} \). Note that \( \mathbb{P}\{\bar{w}^T X = \bar{w}^T \bar{z}\} = 0 \). Since \( A = \{ z \in \mathbb{R}^d \mid \bar{w}^T z = \bar{w}^T \bar{z} \} \) is a Borel subset of \( \mathbb{R}^d \) with zero Lebesgue measure and the distribution of \( X \) is absolutely continuous with respect to the Lebesgue measure, we have \( \mathbb{P}\{\bar{w}^T X = \bar{w}^T \bar{z}\} = 0 \). Hence, we may write

\[
\lim_{k \to \infty} 1_{(-\infty, w_{n_k}^T z_k)}(w_{n_k}^T X) = 1_{(-\infty, \bar{w}^T \bar{z})}(\bar{w}^T X) \text{ almost surely.}
\]

Therefore, by dominated convergence theorem,

\[
\lim_{k \to \infty} F_{X,w_{n_k}}(z_{n_k}) = \lim_{k \to \infty} \mathbb{P}\{w_{n_k}^T X \leq w_{n_k}^T z_{n_k}\} = \lim_{k \to \infty} \mathbb{E} \left[ 1_{(-\infty, w_{n_k}^T z_{n_k})}(w_{n_k}^T X) \right] = \mathbb{E} \left[ 1_{(-\infty, \bar{w}^T \bar{z})}(\bar{w}^T X) \right] = \mathbb{P}\{\bar{w}^T X \leq \bar{w}^T \bar{z}\} = F_{X,\bar{w}}(\bar{z}).
\]

Hence, \( F_{X,\bar{w}}(\bar{z}) \leq p + \epsilon \). Since \( \epsilon > 0 \) is arbitrary, we conclude that \( F_{X,C}(\bar{z}) \leq F_{X,\bar{w}}(\bar{z}) \leq p \). So \( \bar{z} \in L(p) \). Hence, \( L(p) \) is a closed set. \( \square \)

There is dual way of writing \( Q_{X,C} \). The proof is prepared by the following lemma which should be known (and is implicitly part of the proof of [14, Theorem 2.11]). The result itself is inspired by [11, Propositions 2 & 6].

**Lemma 2.** For all \( w \in \mathbb{R}^d \setminus \{0\} \) and all \( z \in \mathbb{R}^d \) with \( \mathbb{P}\{X \in z - H^+(w)\} < p \) there is \( y \in z + \text{int } H^+(w) \) such that \( \mathbb{P}\{X \in y - \text{int } H^+(w)\} < p \).

**Proof.** Fix \( w \in \mathbb{R}^d \setminus \{0\} \) and \( z \in \mathbb{R}^d \) with \( \mathbb{P}\{X \in z - H^+(w)\} < p \). Take \( \bar{z} \in \mathbb{R}^d \) with \( w^T \bar{z} = 1 \) which exists since \( w \neq 0 \). Then \( s\bar{z} \in \text{int } H^+(w) \) for all \( s > 0 \). Define \( y_n = z + \frac{1}{n} \bar{z} \in z + \text{int } H^+(w) \) for \( n \in \mathbb{N} \). Then

\[
w^T y_n = w^T (z + \frac{1}{n} \bar{z}) = w^T z + \frac{1}{n},
\]
so $w^T y_{n+1} \leq w^T y_n$ and $\lim_{n \to \infty} w^T y_n = w^T z$. Since $s \mapsto F_{w^T X}(s)$ is right-continuous, it follows that
\[ P\{X \in z - H^+(w)\} = F_{w^T X}(w^T z) = \lim_{n \to \infty} F_{w^T X}(w^T y_n) < p, \]
so there is $\tilde{n} \in \mathbb{N}$ with
\[ F_{w^T X}(w^T y_{\tilde{n}}) = P\{X \in y_{\tilde{n}} - H^+(w)\} < p, \]

hence
\[ P\{X \in y_{\tilde{n}} - \text{int}\, H^+(w)\} \leq P\{X \in y_{\tilde{n}} - H^+(w)\} < p \]
which proves the claim with $y = y_{\tilde{n}}$.

\textbf{Theorem 1.} For all $p \in [0, 1]$,
\[
Q_{X,C}(p) = \bigcap_{w \in C^+} \bigcap_{y \in \mathbb{R}^d} \left\{ y + H^+(w) \mid P\{X \in y + H^+(w)\} > 1 - p \right\} 
= \bigcap_{w \in C^+} \bigcap_{y \in \mathbb{R}^d} \left\{ y + H^+(w) \mid P\{X \in y - \text{int}\, H^+(w)\} < p \right\}.
\]

\textbf{Proof.} The two expressions on the right hand side clearly coincide since $P\{X \in y + H^+(w)\} = 1 - P\{X \in y - \text{int}\, H^+(w)\}$.

First, assume $z \notin \bigcap_{w \in C^+} \bigcap_{y \in \mathbb{R}^d} \left\{ y + H^+(w) \mid P\{X \in y - \text{int}\, H^+(w)\} < p \right\}$. Then, there are $w \in C^+, y \in \mathbb{R}^d$ such that $P\{X \in y - \text{int}\, H^+(w)\} < p$ and $z \notin y + H^+(w)$. It follows $z \in y - \text{int}\, H^+(w)$ which implies $z - H^+(w) \subseteq y - \text{int}\, H^+(w)$, so
\[ P\{X \in z - H^+(w)\} \leq P\{X \in y - \text{int}\, H^+(w)\} < p, \]
hence $z \notin Q_{X,C}^-(p)$.

Therefore, $Q_{X,C}^-(p) \subseteq \bigcap_{w \in C^+} \bigcap_{y \in \mathbb{R}^d} \left\{ y + H^+(w) \mid P\{X \in y - \text{int}\, H^+(w)\} < p \right\}$.

Conversely, assume
\[ \bar{z} \notin Q_{X,C}^-(p) = \bigcap_{w \in C^+} Q_{X,w}^- (p) = \bigcap_{w \in C^+} \left\{ z \in \mathbb{R}^d \mid F_{X,w}(z) \geq p \right\} \]
Then, there is $w \in C^+$ such that $F_{X,w}(\bar{z}) = P\{X \in \bar{z} - H^+(w)\} < p$. Lemma 2 yields $\bar{y} \in \bar{z} + \text{int}\, H^+(w)$ satisfying $P\{X \in \bar{y} - \text{int}\, H^+(w)\} < p$. If
\[ \bar{z} \in \bigcap_{w \in C^+} \bigcap_{y \in \mathbb{R}^d} \left\{ y + H^+(w) \mid P\{X \in y - \text{int}\, H^+(w)\} < p \right\} \]
would be true, then also $\bar{z} \in \bar{y} + H^+(w)$ and
\[ \bar{z} \in (\bar{y} - \text{int}\, H^+(w)) \cap (\bar{y} + H^+(w)), \]
which is a contradiction. So, \( \bar{z} \not\in \bigcap_{w \in C^+} \bigcap_{y \in \mathbb{R}^d} \{ y + H^+(w) \mid \mathbb{P} \{ X \in y - \text{int} \ H^+(w) \} < p \} \).

This shows \( Q^{-}_{X,C}(p) \supseteq \bigcap_{w \in C^+} \bigcap_{y \in \mathbb{R}^d} \{ y + H^+(w) \mid \mathbb{P} \{ X \in y - \text{int} \ H^+(w) \} < p \} \). \( \square \)

Theorem 1 immediately confirms that the sets \( Q^{-}_{X,C}(p) \) are closed and convex since they are intersections of closed halfspaces. Together with Corollary 1, this yields \( Q^{-}_{X,C}(p) \in \mathcal{G}(\mathbb{R}^d, C) \) which means that \( Q^{-}_{X,C} \) can be seen as a function mapping \([0, 1]\) into the complete lattice \((\mathcal{G}(\mathbb{R}^d, C), \supseteq)\) (“\( \supseteq \)” has to be read as “\( \leq \)”).

The nestedness of the sets \( Q^{-}_{X,C}(p) \) now means that \( p \mapsto Q^{-}_{X,C}(p) \) is non-decreasing considered as a function into the complete lattice. It also implies

\[
Q^{-}_{X,C}(p) = \sup_{0 \leq q < p} Q^{-}_{X,C}(q) = \bigcap_{0 \leq q < p} Q^{-}_{X,C}(q)
\]

where the supremum is understood in \((\mathcal{G}(\mathbb{R}^d, C), \supseteq)\). In this sense, \( Q^{-}_{X,C} \) is left continuous. To summarize, the quantile function \( p \mapsto Q^{-}_{X,C}(p) \) is the non-decreasing, left-continuous \( \mathcal{G}(\mathbb{R}^d, C) \)-valued inverse of the lower \( C \)-distribution function \( F_{X,C} \) (in the sense of, e.g., [5, Definition 1]). This provides a complete analog to the univariate case.

**Remark 2.** The left-continuity of \( Q^{-}_{X,C} \) yields \( Q^{-}_{X,C}(1) = \bigcap_{0 \leq q < 1} Q^{-}_{X,C}(q) \), and this set can be non-empty. Since \( Q^{-}_{X,C}(0) = \mathbb{R}^d \) is the obvious choice, \( Q^{-}_{X,C} \) is defined well-defined on \([0, 1]\) by (1). Even for the univariate case, it has been observed that ‘leaving out the probabilities 0 and 1 is artificial’ ([4, Remark 3.1]).

## 4 Galois connections

Let \( \mathcal{P}(\mathbb{R}^d) \) denote the power set of \( \mathbb{R}^d \) (including \( \emptyset \)). The function \( F_{X,C}^\diamond : \mathcal{P}(\mathbb{R}^d) \to [0, 1] \) defined by

\[
F_{X,C}^\diamond(D) = \inf_{z \in D} F_{X,C}(z)
\]

is called the inf-extension of \( F_{X,C} \) where \( F_{X,C}^\diamond(\emptyset) = +\infty \) is understood. We shall verify a few of its properties which are basically inherited from \( F_{X,C} \).

**Proposition 4.** (a) \( F_{X,C}^\diamond \) is monotone in \( X^1 \leq_{C} X^2 \) \( \mathbb{P} \)-a.s. implies \( F_{X^2,C}^\diamond(D) \leq F_{X^1,C}(D) \) for all \( D \in \mathcal{P}(\mathbb{R}^d) \).

(b) \( F_{X,C}^\diamond \) is monotone in \( D \): \( A \supseteq B \) implies \( F_{X,C}^\diamond(A) \leq F_{X,C}^\diamond(B) \) for \( A, B \in \mathcal{P}(\mathbb{R}^d) \).

(c) \( F_{X,C}^\diamond(\{z\} + C) = F_{X,C}^\diamond(\{z\}) = F_{X,C}(z) \) for all \( z \in \mathbb{R}^d \).

**Proof.** (a) follows from the monotonicity of \( X \mapsto F_{X,C}(z) \). (b) follows by construction of \( F_{X,C}^\diamond \). (c) follows from the monotonicity of \( z \mapsto F_{X,C}(z) \). \( \square \)

The following proposition prepares a new feature.
Proposition 5. For every $D \in \mathcal{P}(\mathbb{R}^d)$,

$$F_{X,C}^\wedge(\text{cl} \text{ co } (D + C)) = F_{X,C}^\wedge(D).$$

Proof. By the definition of $F_{X,C}^\wedge$, it is certainly true. Since $F_{X,C}$ is monotone with respect to $\leq_C$ (see Proposition 1)

$$\forall z \in C, \forall x \in D: F_{X,C}(x) \leq F_{X,C}(x + z),$$

hence $F_{X,C}^\wedge(\{x\}) \leq F_{X,C}^\wedge(\{x\} + C)$ and therefore, $F_{X,C}^\wedge(D) \leq F_{X,C}^\wedge(D + C)$. This gives $F_{X,C}^\wedge(D) = F_{X,C}^\wedge(D + C)$.

Next, take $z^1, \ldots, z^m \in D$ and $s_1, \ldots, s_m \in [0, 1]$ such that $\sum_{i=1}^m s_i = 1$. Set $z = \sum_{i=1}^m s_i z^i$. The quasiconcavity of $F_{X,C}$ yields

$$F_{X,C}(z) \geq \min \{F_{X,C}(z^1), \ldots, F_{X,C}(z^m)\} \geq F_{X,C}^\wedge(D).$$

This proves $F_{X,C}^\wedge(\text{co } D) = F_{X,C}^\wedge(D)$.

Finally, take a sequence $(z^n)_{n \in \mathbb{N}}$ in $D$ which converges to some $z \in \mathbb{R}^d$. Then, the upper semicontinuity of $F_{X,C}$ produces

$$F_{X,C}(z) \geq \limsup_{n \to \infty} F_{X,C}(z^n) \geq \limsup_{n \to \infty} F_{X,C}^\wedge(D) = F_{X,C}^\wedge(D),$$

hence $F_{X,C}(\text{cl } D) \geq F_{X,C}^\wedge(D)$ which gives “=.” □

As a result of Proposition 5, it is enough to consider the inf-extension $F_{X,C}^\wedge$ as a function from $\mathcal{G}(\mathbb{R}^d, C)$ rather than the whole power set $\mathcal{P}(\mathbb{R}^d)$, which we do in the sequel.

Corollary 2. For every $\mathcal{D} \subseteq \mathcal{G}(\mathbb{R}^d, C)$,

$$F_{X,C}^\wedge(\inf_{D \in \mathcal{D}} D) = \inf_{D \in \mathcal{D}} F_{X,C}^\wedge(D).$$

(3)

Proof. Since $D \subseteq \inf_{D \in \mathcal{D}} D = \text{cl} \text{ co } \cup_{D \in \mathcal{D}} D$ for all $D \in \mathcal{D}$, “≤” certainly is true. Take $z \in \cup_{D \in \mathcal{D}} D$. Then, there is $D' \in \mathcal{D}$ with $z \in D'$, hence $F_{X,C}(z) \geq \inf_{D \in \mathcal{D}} F_{X,C}(D)$ which in turn implies $F_{X,C}(\cup_{D \in \mathcal{D}} D) \geq \inf_{D \in \mathcal{D}} F_{X,C}^\wedge(D)$. The previous proposition now produces $F_{X,C}^\wedge(\text{cl } \text{ co } \cup_{D \in \mathcal{D}} D) = \inf_{D \in \mathcal{D}} F_{X,C}^\wedge(D)$.

Equation (3) means that $F_{X,C}^\wedge$ preserves infima (meets) as a function from $\mathcal{G}(\mathbb{R}^d, C)$ to $[0, 1]$. This property has been called “inf-stability” in [2].

Proposition 6. For every $D \in \mathcal{G}(\mathbb{R}^d, C)$ and $p \in [0, 1]$,

$$Q_{X,C}^-(p) \supseteq D \iff p \leq F_{X,C}^\wedge(D).$$
Proof. Straightforward, by using the definitions of $F_{X,C}^{\hat{\phi}}$ and $Q_{X,C}^{-}$.

Proposition 6 establishes the fact that $F_{X,C}^{\hat{\phi}}$ and $Q_{X,C}^{-}$ form a Galois connection between the two complete lattices $(\mathcal{G}([0, 1], \leq))$ and $(\mathcal{G}([0, 1], \leq))$ where $F_{X,C}^{\hat{\phi}}$ is the upper adjoint and $Q_{X,C}^{-}$ the lower adjoint; results from the theory of Galois connections (see, for example, [3, Chapter 7]) can be used. For example, it follows that

$$\forall p \in [0, 1]: \quad Q_{X,C}^{-}(p) = \inf \left\{ D \in \mathcal{G}(\mathbb{R}^d, C) \mid F_{X,C}^{\hat{\phi}}(D) \geq p \right\}$$

(4)

$$\forall D \in \mathcal{G}(\mathbb{R}^d, C): \quad F_{X,C}^{\hat{\phi}}(D) = \sup \left\{ p \in [0, 1] \mid D \subseteq Q_{X,C}^{-}(p) \right\}.$$  

(5)

This means that $F_{X,C}^{\hat{\phi}}$ and $Q_{X,C}^{-}$ determine each other; they carry the same information.

The compositions

$$[F_{X,C}^{\hat{\phi}} \circ Q_{X,C}^{-}] (p) = F_{X,C}^{\hat{\phi}} (Q_{X,C}^{-}(p)) = \inf_{z \in Q_{X,C}^{-}(p)} F_{X,C}(z)$$

$$[Q_{X,C}^{-} \circ F_{X,C}^{\hat{\phi}}] (D) = Q_{X,C}^{-} (F_{X,C}^{\hat{\phi}}(D)) = \{ z \in \mathbb{R}^d \mid F_{X,C}(z) \geq F_{X,C}^{\hat{\phi}}(D) \}$$

can be defined which are usually called closure and kernel operator of the Galois connection; here, it might be better to call the first the kernel and the second the closure operator; their corresponding properties follow from the theory of Galois connections.

5 The simulation result

The main question in this section is how the quantile function characterizes the distribution. In the univariate case, one can show that the quantile, taken at a random variable uniformly distributed over $[0, 1]$, produces a random variable which has the cumulative distribution function that defines the quantile (compare, for example, [7, Lemma A.19], the “simulation lemma”). In our setting, quantiles are sets, so plugging in a random variable with values in $[0, 1]$ produces a random set.

Let $U : \Omega \to [0, 1]$ be standard uniform random variable and define a function $\mathcal{X} : \Omega \to \mathcal{G}(\mathbb{R}^d, C)$ by

$$\mathcal{X}(\omega) = Q_{X,C}^{-}(U(\omega))$$

for every $\omega \in \Omega$. To be able to talk about the distribution of $\mathcal{X}$ under $\mathbb{P}$, we first view $\mathcal{X}$ as a measurable function by equipping $\mathcal{G}(\mathbb{R}^d, C)$ with the $\sigma$-algebra constructed below.

For each $K \subseteq \mathbb{R}^d$, let

$$\mathcal{D}^K = \{ D \in \mathcal{G}(\mathbb{R}^d, C) \mid D \cap K = \emptyset \}$$

and

$$\mathcal{D}_K = \{ D \in \mathcal{G}(\mathbb{R}^d, C) \mid D \cap K \neq \emptyset \},$$
which is the complement of $D^K$ in $G(\mathbb{R}^d, C)$. Let us denote by $K$ the set of all compact subsets of $\mathbb{R}^d$. Note that the collection $\{D^K \mid K \in \mathcal{K}\}$ is a $\pi$-system on $G(\mathbb{R}^d, C)$ since $D^{K_1} \cap D^{K_2} = D^{K_1 \cup K_2}$ and $K_1 \cup K_2 \in \mathcal{K}$ for every $K_1, K_2 \in \mathcal{K}$. Let $\mathcal{B}(G(\mathbb{R}^d, C))$ be the $\sigma$-algebra generated by $\{D^K \mid K \in \mathcal{K}\}$, called the Borel $\sigma$-algebra on $G(\mathbb{R}^d, C)$; the reader is referred to [10, Section 1.1] for a detailed discussion. Clearly, $\mathcal{B}(G(\mathbb{R}^d, C))$ is also generated by $\{D^K \mid K \in \mathcal{K}\}$.

We shall establish the measurability of $X$ with respect to $\mathcal{B}(G(\mathbb{R}^d, C))$.

**Lemma 3.** The function $X : \Omega \to G(\mathbb{R}^d, C)$ is measurable with respect to $\mathcal{A}$ and $\mathcal{B}(G(\mathbb{R}^d, C))$.

**Proof.** Let $K \in \mathcal{K}$. Note that

$$\{X \in D_K\} = \{\omega \in \Omega \mid Q_{X,C}(U(\omega)) \cap K \neq \emptyset\}$$

$$= \{\omega \in \Omega \mid \{z \in \mathbb{R}^d \mid F_{X,C}(z) \geq U(\omega)\} \cap K \neq \emptyset\}$$

$$= \{\omega \in \Omega \mid \exists z \in K : F_{X,C}(z) \geq U(\omega)\}$$

$$= \left\{\omega \in \Omega \mid \inf_{z \in K} F_{X,C}(z) \geq U(\omega)\right\}$$

$$= \left\{U \leq \inf_{z \in K} F_{X,C}(z)\right\} \in \mathcal{A}$$

since $U$ is measurable with respect to the Borel $\sigma$-algebra on $[0,1]$ and $\mathcal{A}$. Hence, by [1, Proposition I.2.3], it follows that $\{X \in D\}$ for every $D \in \mathcal{B}(G(\mathbb{R}^d, C))$, that is, $X$ is measurable. \qed

Thanks to Lemma 3, $X$ is a random variable taking values in $G(\mathbb{R}^d, C)$. Hence, its distribution under $P$ is the probability measure $P \circ X^{-1}$ on $(G(\mathbb{R}^d, C), \mathcal{B}(G(\mathbb{R}^d, C)))$ defined by

$$P \circ X^{-1}(D) = P\{X \in D\}$$

for every $D \in \mathcal{B}(G(\mathbb{R}^d, C))$. Since $\{D^K \mid K \in \mathcal{K}\}$ is a $\pi$-system which generates the $\sigma$-algebra $\mathcal{B}(G(\mathbb{R}^d, C))$, the distribution of $X$ is determined by its values on this $\pi$-system; see [1, Proposition I.3.7], for instance. Since $P\{X \in D^K\} = 1 - P\{X \in D_K\}$ for every $K \in \mathcal{K}$, the distribution of $X$ is also determined by the so-called capacity functional $T_X : \mathcal{K} \to [0,1]$ defined by

$$T_X(K) = P\{X \in D_K\} = P\{X \cap K \neq \emptyset\}$$

for each $K \in \mathcal{K}$.

**Theorem 2.** The inf-extension $F_{X,C} : G(\mathbb{R}^d, C) \to [0,1]$ and the distribution of the set-valued random variable $X$ determine each other.
Proof. Let $K \in \mathcal{K}$. Following the calculation in the proof of Lemma 3, we have

$$T_X(K) = \mathbb{P}\left\{U \leq \inf_{z \in K} F_{X,C}(z)\right\} = \inf_{z \in K} F_{X,C}(z)$$

since $U$ has the standard uniform distribution. Next, by Proposition 5,

$$\inf_{z \in K} F_{X,C}(z) = F_{\Delta}^\alpha(K) = F_{\Delta}^\alpha(\text{cl co } (K+C)).$$

Since $T_X(K) = F_{\Delta}^\alpha(\text{cl co } (K+C))$ for every $K \in \mathcal{K}$ and $T_X$ determines the distribution of $\mathcal{X}$, it follows that $F_{\Delta}^\alpha$ determines the distribution of $\mathcal{X}$.

Conversely, let $A \in \mathcal{G}(\mathbb{R}^d, C)$. The calculations in Lemma 3 as well as the above calculations can be repeated when the compact set $K$ is replaced with $A$, which yields

$$\mathbb{P}\{\mathcal{X} \in D_A\} = \inf_{z \in A} F_{X,C}(z) = F_{\Delta}^\alpha(A).$$

Hence, the distribution of $\mathcal{X}$ determines $F_{\Delta}^\alpha$ as well. \qed

Theorem 2 together with (4), (5) implies that the lower $C$-quantile $Q_{X,C}^-$, the inf-extension $F_{\Delta}^\alpha$, the capacity functional $T_X$, and the distribution $\mathbb{P} \circ \mathcal{X}^{-1}$ determine each other.

References

[1] Çınlar E, Probability and Stochastics. Springer Science and Business Media, 2011

[2] Crespi G, Hamel AH, Rocca M, Schrage C, Set relations and approximate solutions in set optimization, arXiv:1812.03300, 2018

[3] Davey BA, Priestley HA, Introduction to Lattices and Order. Cambridge University Press, 2nd edition, 2002

[4] Doering A, Dewitt B, Self-adjoint operators as functions II: quantum probability, arXiv:1210.574v2, 2012 (2nd version Dec. 2013)

[5] Drapeau S, Hamel AH, Kupper M, Complete duality for quasiconvex and convex set-valued functions, Set-Valued and Variational Analysis, 24(2), 253-275, 2016

[6] Faugeras OP, Rüschendorf L, Markov morphisms: a combined copula and mass transportation approach to multivariate quantiles, Mathematica Applicanda, 45(1), 21-63, 2017

[7] Föllmer H, Schied A, Stochastic Finance: An Introduction in Discrete Time. Walter de Gruyter, 3rd edition, 2011

11
[8] Hamel AH, Heyde F, Löhne A, Rudloff B, Schrage C, *Set optimization—a rather short introduction*. In: Hamel, A.H., Heyde, F., Löhne, A., Rudloff, B., Schrage, C. (eds.), Set optimization and applications – the state of the art. From set relations to set-valued risk measures, Springer-Verlag Berlin 2015, pp. 65-141

[9] Hamel AH, Kostner D, *Cone distribution functions and quantiles for multivariate random variables*, Journal of Multivariate Analysis, 167, 97-113, 2018

[10] Molchanov I, Theory of Random Sets. Springer, 2nd edition, 2017

[11] Rousseeuw PJ, Ruts I, *The depth function of a population distribution*, Metrika 49(3), 213-244, 1999

[12] Serfling R, *Quantile functions for multivariate analysis: approaches and applications*, Statistica Neerlandica 56, 214-232, 2002

[13] Zălinescu C, Convex Analysis in General Vector Spaces. World Scientific, 2002

[14] Zuo Y, Serfling R, *General notions of statistical depth function*, Annals of Statistics 28(2), 461-482, 2000