PARTICLE ACCELERATION BY SLOW MODES IN STRONG COMPRESSIBLE MAGNETOHYDRODYNAMIC TURBULENCE, WITH APPLICATION TO SOLAR FLARES

BENJAMIN D. G. CHANDRAN
Center for Magnetic Reconnection Studies and Department of Physics and Astronomy, University of Iowa, Iowa City, IA 52242; benjamin-chandran@uiowa.edu

ABSTRACT

Energetic particles that undergo strong pitch-angle scattering and diffuse through a plasma containing strong compressible MHD turbulence undergo diffusion in momentum space with diffusion coefficient \( D_p \). If the rms turbulent velocity is of the order of the Alfvén speed \( v_A \), the contribution to \( D_p \) from slow-mode eddies is \( \frac{2\pi^2 v_A^2}{9f \ln\left(\frac{v_A}{D_p}\right)} \), where \( f \) is the outer scale of the turbulence, \( \gamma \approx 0.577 \) is Euler’s constant, and \( D_p \) is the spatial diffusion coefficient of energetic particles, which is assumed to satisfy \( D_p \ll 4\pi \). The energy spectrum of accelerated particles is derived for this value of \( D_p \), taking into account Coulomb losses and particle escape from the acceleration region with an energy-independent escape time. Slow modes in the \( D_p \ll 4\pi \) limit are an unlikely explanation for electron acceleration in solar flares to energies of 10–100 keV, because for solar flare conditions, the predicted acceleration times are too long, and the predicted energy spectra are too hard. The acceleration mechanism discussed in this paper could in principle explain the relatively hard spectra of gyrosynchrotron-emitting electrons in the 100–5000 keV range, but only if \( D_p \ll 4\pi \) for such particles.

Subject headings: acceleration of particles — MHD — Sun: flares

1. INTRODUCTION

The first part of this paper treats stochastic particle acceleration by slow modes in compressible magnetohydrodynamic (MHD) turbulence assuming efficient pitch-angle scattering. The second part presents the energy spectra of accelerated particles, taking into account Coulomb losses and assuming that particles escape the acceleration region on an energy-independent timescale. The third part of the paper shows that slow modes in the efficient pitch-angle scattering limit are not a viable explanation for electron acceleration in solar flares to energies of 10–100 keV, because for solar flare conditions, the predicted acceleration times are too long, and the predicted energy spectra are too hard. The acceleration mechanism discussed in this paper could in principle explain the relatively hard spectra of gyro synchrotron-emitting electrons in the 100–5000 keV range, but only if \( D_p \ll 4\pi \) for such particles.

If the small-scale waves are Alfvén waves, the transport equation describing the energetic-particle distribution function \( f \) is

\[
\frac{\partial f}{\partial t} + \frac{\partial}{\partial p} \left( p^3 w \right) \cdot \nabla f = \nabla \cdot \left( D \hat{b} \cdot \nabla f \right)
+ (\nabla \cdot w) \frac{\partial f}{\partial p} \\
+ \frac{\partial}{\partial p^3} \left( p^5 D \frac{\partial f}{\partial p} \right)
\]

(Skilling 1975), where \( p \) is the momentum, \( \hat{b} \) is the magnetic-field unit vector,

\[
w = u + \left( \frac{3}{2} (1 - \mu^2) \nu_+ - \nu_- \right) v_A \hat{b}
\]

is an effective “wave-frame” velocity at which the cosmic rays are advected, \( u \) is the large-scale plasma velocity, \( v_A \) is the Alfvén speed, \( \nu_+ \) and \( \nu_- \) are, respectively, the pitch-angle scattering rates associated with waves traveling parallel to and antiparallel to the magnetic field, \( \mu \) is the pitch-angle cosine, \( \langle \ldots \rangle \) indicates an average over \( \mu \),

\[
D_p = v^2 \frac{1 - \mu^2}{2(\nu_+ + \nu_-)}
\]

is the diffusion coefficient for particle motion along the
magnetic field,

\[ D_{pp}^{(s)} = 4\gamma^2 m^2 v_A^2 \left( \frac{1 - \mu^2}{2} \right) \frac{\nu_+ \nu_-}{\nu_+ + \nu_-} \]  

(4)

is the momentum diffusion coefficient associated with the small-scale waves, and \( \gamma \) is the relativistic Lorentz factor of an energetic particle. Equations (2) and (4) show that either \( \mathbf{w} \neq \mathbf{u} \) or \( D_{pp}^{(s)} \neq 0 \) (Schlickeiser 2002). Nevertheless, for simplicity the momentum diffusion associated with small-scale waves is ignored \( (D_{pp}^{(s)} \to 0) \), and at the same time, inconsistently, it is assumed that \( \mathbf{w} = \mathbf{u} \). This approximation has been standard in studies of particle acceleration by velocities on scales \( \gg \lambda_{mfp} \) (Bykov & Toptygin 1982, 1990, 1993; Ptuskin 1988; Dolginov & Silant’ev 1990; Katz & Stehlik 1991); it reduces equation (1) to

\[ \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f = \mathbf{v} \cdot \left( D_{pp} \mathbf{b} \cdot \nabla \mathbf{v} \right) + (\mathbf{v} \cdot \mathbf{u}) \frac{\partial f}{\partial \rho}. \]  

(5)

Equation (5) is used to calculate the momentum diffusion coefficient arising from the large-scale turbulence. It seems plausible that a nonzero \( D_{pp} \) would approximately add to the momentum diffusion coefficient associated with the large-scale turbulence calculated from equation (5), but future work is needed to verify this. The spatial diffusion caused by the large-scale turbulence is neglected. It is assumed that the rms large-scale turbulent velocity is comparable to \( v_A \) and that the ratio \( \beta \) of thermal-to-magnetic pressure is \( \ll 1 \). The power spectrum of the slow modes is assumed to follow the theory of Lithwick & Goldreich (2001), which has received support from direct numerical simulations (Cho & Lazarian 2002). It is also assumed that \( v_A \lambda_{min} < D_{p} < v_A / l \), where \( \lambda_{min} \) is the length along the magnetic field of the smallest slow-mode eddies under consideration (\( \lambda_{min} \) is either the length of the eddies at the dissipation scale or several times \( \lambda_{mfp} \), whichever is larger) and \( l \) is the length of the largest (outer-scale) eddies. Since the time for an eddy of length \( \lambda_{ij} \) (measured along the magnetic field) to be randomized is \( \lambda_{ij}/v_A \) (Lithwick & Goldreich 2001), the inequality \( v_A \lambda_{min} < D_{p} < v_A / l \) implies that the largest eddies are randomized before a particle diffuses through them and that a particle diffuse through eddies of length \( \lambda_{min} \) before such eddies are randomized. The assumption \( D_{p} > v_A \lambda_{min} \) requires that the particle speed \( v \) satisfy \( v > v_A \), since \( D_{p} \sim v_A \lambda_{mfp} \) and \( \lambda_{min} > \lambda_{mfp} \).

The procedure used to calculate the momentum diffusion coefficient \( D_{p} \) arising from the large-scale velocities is to average over the slow-mode fluctuations, neglect spatial variations in the averaged distribution function \( f_0 \), and approximate the eddies on scales comparable to \( l \) as a uniform background field \( B_0 \). The eddies on scales \( \ll l \) have velocities \( \ll v_A \) and magnetic perturbations \( \ll B_0 \) and can be treated using quasi-linear theory. It is found that

\[ \frac{\partial f_0}{\partial t} = \frac{1}{p^2} \frac{\partial}{\partial p} \left( p^2 D_{p} \frac{\partial f_0}{\partial p} \right), \]  

(6)

with

\[ D_{p} \approx 2p^2 v_A^2 \frac{\gamma}{\gamma - 1} \left[ \ln \left( \frac{v_A}{D_{p}} \right) + 2\gamma - 3 \right], \]  

\[ v_A \lambda_{min} < D_{p} < v_A / l, \]  

(7)

where \( \gamma \approx 0.577 \) is Euler’s constant. The corresponding acceleration timescale \( p^2 / D_{p} \) is of the order of the large-eddy turnover time \( l / v_A \) and depends only weakly through the \( \ln \left( v_A / D_{p} \right) \) term on \( p \) and the particle species. An approximate version of equation (7) is obtained using phenomenological arguments in § 2.3.

Section 3 presents the time-dependent spectrum of accelerated particles assuming an ad hoc model of particle escape from the acceleration region with an energy-independent escape time. Section 4 gives the steady state spectrum for electron acceleration taking into account Coulomb losses, again assuming an energy-independent escape time. Section 5 compares the predicted spectra and acceleration times with observations of electrons in solar flares. The results of the paper are summarized in § 6. Much of the notation used in the paper is defined in Table 1.

### Table 1

| Notation | Meaning |
|----------|---------|
| \[ \lambda_{mfp} \] | Scattering mean free path of energetic particles |
| \[ \lambda_{thermal} \] | Coulomb mean free path of thermal protons |
| \[ \lambda_{ij} \] | Length of a turbulent eddy measured along the magnetic field |
| \[ \lambda_{ij}^t \] | Width of a turbulent eddy measured across the magnetic field |
| \[ r_g \] | Energetic-particle gyroradius |
| \[ N(E) \] | Energy-dependent accelerated particles per unit energy |
| \[ B \] | Magnetic field |
| \[ d_l \] | Length along magnetic field of eddies at the dissipation scale |
| \[ \Delta \] | Stirring scale (outer scale) of turbulence |
| \[ v_A \] | Alfvén speed |
| \[ c_s \] | Sound speed |
| \[ \beta \] | Ratio of thermal to magnetic pressure |
| \[ u \] | Turbulent velocity |
| \[ \epsilon \] | Energetic-particle velocity |
| \[ D_{pp} \] | Diffusion coefficient for motion along the magnetic field |
| \[ D_{pp}^{(s)} \] | Momentum diffusion coefficient |
| \[ \tau_{acc} \] | Escape timescale |
| \[ \tau_{acc}^{(s)} \] | Acceleration timescale |
| \[ \tau \] | Turnover time |
| \[ \gamma \] | Euler’s constant, \( \sim 0.577 \) |
field, like slow modes in compressible MHD turbulence (see § 2.2). The wave frequency, polarization, and velocity divergence when \( k_z \ll k \) are given by

\[
\omega = \frac{c_s v_A |k_z|}{\sqrt{c_s^2 + v_A^2}} \left[ 1 + O\left(\frac{k_z}{k}\right)^2 \right] \tag{8}
\]

\[
u_{\|} \sim k \frac{c_s}{c_s^2 + v_A^2} k_z, \tag{9}
\]

\[
u_{\perp} = 0 \tag{10}
\]

\[
k \cdot u = \frac{v_A^2}{c_s^2 + v_A^2} \left[ 1 + O\left(\frac{k_z}{k}\right)^2 \right] k_z u_z \tag{11}
\]

(Lithwick & Goldreich 2001), where \( u \) is the fluctuating velocity, \( k \) is taken to lie in the \((x, z)\)-plane, and \( c_s \) is the sound speed. Equations (9) and (10) imply that the slow-wave velocity is approximately aligned with the background magnetic field. The total pressure perturbation (magnetic plus thermal) vanishes to second order in \( k_z/k \) (Lithwick & Goldreich 2001). An important property of slow waves with \( k_z \ll k \) when \( \beta \ll 1 \) (and thus \( c_s \ll v_A \)) follows from equations (9) and (11): \( k \cdot u \sim k_z |u| \).

### 2.2. Slow-Mode Eddies in Strong Anisotropic MHD Turbulence

The anisotropy of MHD turbulence has been studied by many authors.\(^1\) In this paper, it is assumed that the ambient plasma is stirred at a scale \( l \) and that the rms turbulent velocity is comparable to \( v_A \). In this case, the rms magnetic fluctuation at scale \( l \), denoted \( B_l \), is comparable to any mean magnetic field in the system. The stirring excites an inertial range of turbulent fluctuations extending from the large scale \( l \) to a much smaller dissipation scale. Within any box of dimension \( \lesssim l \), the fluctuations can be decomposed into the three MHD wave polarizations with respect to the average magnetic field direction within the box, \( B_{\text{local}} \). For example, when \( \beta \ll 1 \), the velocity fluctuations aligned with \( B_{\text{local}} \) are associated with slow waves. The slow-wave fluctuations within the box can be thought of as a collection of nested eddies, where an eddy is simply a volume of some specified width \( \lambda_1 \) measured across the magnetic field and length \( \lambda_1 \) measured along the magnetic field, for which the velocity variation across the width of the eddy is comparable to the velocity variation along the length of the eddy. For values of \( \lambda_1 \), in the inertial range, slow-mode eddies are elongated along \( B_{\text{local}} \), with

\[
\lambda_1 \sim \lambda_2^{2/3} l^{1/3} \tag{12}
\]

and the rms velocity variation across a slow-mode eddy is

\[
u_{\lambda_1} \sim v_A \left(\frac{\lambda_1}{l}\right)^{1/3} \tag{13}
\]

(Lithwick & Goldreich 2001; Cho & Lazarian 2002). Equation (12) means that the velocity varies more rapidly across \( B_{\text{local}} \) than along \( B_{\text{local}} \). Suppose the vector separation between two points is \( r \), with \( r \) in the inertial range of the turbulence, and let the rms velocity difference between the two points be \( \delta u \). If \( r \perp B_{\text{local}} \), equation (13) implies that \( \delta u \sim v_A (r/l)^{1/3} \); if \( r \parallel B_{\text{local}} \), equations (12) and (13) imply that \( \delta u \sim v_A (r/l)^{1/2} \). Slow-mode eddies are randomized in a time \( \sim \lambda_1/v_A \) because of the mixing of slow modes by Alfvén-mode eddies (Lithwick & Goldreich 2001).

The dissipation scale in the directions perpendicular to \( B_{\text{local}} \), denoted \( d_\perp \), and the corresponding parallel scale, \( d_\parallel = d_\perp l^{1/3} \), are given by

\[
d_\perp \sim r_{\text{q, thermal}}, \quad \beta \ll 1, \tag{14}
\]

\[
d_\parallel \sim \lambda_{\text{thermal}}, \quad \beta \gg 1 \tag{15}
\]

(Lithwick & Goldreich 2001), where \( \lambda_{\text{thermal}} \) is the collisional mean free path of thermal ions, and ion-neutral friction and radiative cooling are ignored. Equation (14) is surprising in that linear slow magnetosonic waves are strongly damped in low-\( \beta \) plasmas on scales smaller than the collisional mean free path: i.e., when \( \beta \ll 1 \), the damping time of slow modes with \( \lambda_1 < \lambda_{\text{thermal}} \) is \( \sim \lambda_1/c_s \), comparable to the linear wave period. The reason that turbulent slow-mode eddies can persist on parallel scales less than \( \lambda_{\text{thermal}} \) when \( \beta \gg 1 \) is that the cascade of slow-mode eddies is controlled by the Alfvén modes, and the cascade time \( \lambda_1/v_A \) is much less than the linear slow-wave damping time (Lithwick & Goldreich 2001).

### 2.3. Phenomenological Estimate of \( D_p \)

Since only that part of the turbulence with \( \lambda_1 \gg \lambda_{\text{mfp}} \) is considered, and since the momentum diffusion associated with small-scale waves is ignored, the time derivative of a particle’s momentum induced by the large-scale velocity \( u \) is given by

\[
\frac{dp}{dt} = -\frac{\mathbf{v} \cdot \mathbf{u}}{3} p \tag{16}
\]

(Plasunsk 1988). For \( \beta \ll 1 \), the rms velocity divergence of slow-mode eddies of width \( \lambda_1 \) satisfies

\[
\left\langle |\mathbf{v} \cdot \mathbf{u}|^2 \rightangle^{1/2} \sim \frac{u_{\lambda_1}}{\lambda_1} \tag{17}
\]

(Lithwick & Goldreich 2001). The rms contribution to \( dp/dt \) from eddies of length \( \lambda_1 \) is thus

\[
\left(\frac{dp}{dt}\right)_{\lambda_1} \sim \frac{p v_A}{3 \sqrt{\lambda_1} l}. \tag{18}
\]

If a particle interacts coherently with an eddy of length \( \lambda_1 \) for a time \( \Delta t \), it incurs a random momentum increment of rms magnitude \( \Delta p \sim (dp/dt) \Delta t \). The contribution to \( D_p \) from eddies of size \( \lambda_1 \) is \( \sim (\Delta p)^2 / \Delta t \). When \( d_{\text{min}} v_A \ll D_1 \ll l v_A \), particles are confined within eddies with \( \lambda_1 > D_1/v_A \) throughout the time \( \lambda_1/v_A \) required for the eddies to be randomized in the turbulent flow. For such eddies, \( \Delta t \sim \lambda_1/v_A \). Each eddy size between \( D_1/v_A \) and \( l \) makes a contribution to \( D_p \) of \( \sim p^2 v_A/(9 l) \). The contribution

\[^{1}\text{See, e.g., Montgomery & Turner 1981; Shebalin, Matthaeus, & Montgomery 1983; Higdon 1984, 1986; Oughton, Priest, & Matthaeus 1994; Sridhar & Goldreich 1994; Goldreich & Sridhar 1995, 1997; Montgomery & Matthaeus 1995; Ghosh & Goldstein 1997; Matthaeus, Oughton, & Ghosh 1998; Spangler 1999; Bhattacharjee & Ng 2001; Cho & Vishniac 2000; Maron & Goldreich 2001; Milano et al. 2001; Lithwick & Goldreich 2001; Cho & Lazarian 2002.}\]
to $D_p$ from all such eddies is

$$D_p \approx \frac{p^2 v_A}{9l} \ln \left( \frac{v_A}{D_p} \right), \quad d_{\text{min}} \ll \frac{D_p}{v_A} \ll l,$$

in approximate agreement with equation (30) below. The contribution to $D_p$ from eddies with $\lambda_1 < D_p/v_A$ can be neglected.\(^2\)

2.4. Derivation of $D_p$ in the Quadratic Approximation

Particle acceleration by slow-mode eddies on scales $l \ll A$ is now treated analytically for $\beta \lesssim 1$. A scale $l$ is introduced with $\lambda_{\text{min}} \ll l \ll A$, and an approximation is made in which eddies with $l < \lambda_1 < l$ are replaced with a uniform magnetic field $B_{\text{local}}$, so that $B_{\text{local}}$ is everywhere approximately along $\hat{z}$. In terms of the Fourier transform of the velocity $\mathbf{u}(k, t)$, and in accord with equations (12) and (13), the spectrum of the (homogeneous) turbulence is taken to be (Goldreich & Sridhar 1995; Lithwick & Goldreich 2001; Cho & Lazarian 2002)

$$\langle \mathbf{u}(k_1, t_1) \cdot \mathbf{u}(k_2, t_2) \rangle = P(k_1)e^{-\gamma|t_1-t_2|}\delta(k_1 + k_2),$$

with

$$P(k) = \frac{v_A^2}{6\pi} k_1^{-1/3} k_1^{-10/3} \left( \frac{k_1^{-1/3}}{k_1^{-1/3}} \right)$$

for ($p$)$^{-1} < k_1 < d_1^{-1}$ [if $P(k) = 0$ otherwise],

$$g(x) = e^{-|x|},$$

and

$$\gamma_k = k_\perp^2 l^{-1/3} v_A \sim \frac{v_A}{\lambda_1},$$

where $k_\perp = (k_1^2 + k_2^2)^{1/2}$. Given equation (17), it is assumed that

$$\langle [\mathbf{k}_1 \cdot \mathbf{u}(k_1, t_1)] [\mathbf{k}_2 \cdot \mathbf{u}(k_2, t_2)] \rangle = -k_\perp^2 P(k_1)e^{-\gamma|t_1-t_2|}\delta(k_1 + k_2).$$

For $B_{\text{local}} \propto \hat{z}$, equation (5) becomes

$$\frac{df}{dt} + u \cdot \nabla f = D_1 \frac{\partial^2 f}{\partial z^2} + (\nabla \cdot u) \frac{p}{3} \frac{\partial f}{\partial p}.$$  

The distribution function is written as the sum of two parts:

$$f = f_0 + f_1,$$  

where $f_0 = f$, $f_1 = f - \langle f \rangle$, and angled brackets denote an average over an ensemble of realizations of the turbulence.

For simplicity, it is assumed that $\langle u \rangle = 0$ and $f_0 = 0$. The ensemble average of equation (25) is then

$$\frac{df_0}{dt} = -\langle u \cdot \nabla f \rangle + \left< \left( \nabla \cdot u \right) \frac{p}{3} \frac{\partial f_0}{\partial p} \right>.$$  

Subtracting equation (27) from equation (25) yields

$$\frac{df_1}{dt} - D_1 \frac{\partial^2 f_1}{\partial z^2} = \left( \nabla \cdot u \right) \frac{p}{3} \frac{\partial f_0}{\partial p},$$

in which products of fluctuating quantities have been neglected. Such products are small compared to $|\partial f_1 / \partial t|$, since the slow-mode velocity is almost aligned with $z$, $u \ll v_A$ for eddies on scales $\ll A$, and $|\partial f_1 / \partial t| \sim v_A |\partial f_1 / \partial z|$. On solving for $f_1$ and substituting into equation (27), one recovers equation (6) with

$$D_p = \frac{p^2}{9} \int d^3 k \int_0^\infty dt \kappa^2 P(k) e^{-\gamma_k |\kappa^2 \kappa^2 T| \tau}.$$  

When $d_{\text{min}} \ll (D_p/v_A) \ll l_\parallel$, one obtains $f = f_1$ is obtained by taking $l_\parallel \rightarrow 0$ in equation (30).

3. TIME-DEPENDENT ENERGY SPECTRUM OF ACCELERATED PARTICLES ASSUMING AN ENERGY-INDEPENDENT ESCAPE TIME AND NO COULOMB LOSSES

In this section the energy spectrum of accelerated particles is presented for $p > p_0$, where $p_0$ corresponds to a particle velocity of a few times $v_A$. Coulomb collisions are neglected, and particle escape from the acceleration region is modeled in an ad hoc fashion by adding a loss term to the right-hand side of equation (6):

$$\frac{df}{dt} = \frac{1}{p^2} \frac{\partial}{\partial p} \left( p^2 D_p \frac{\partial f}{\partial p} \right) = -\frac{f}{\tau_{\text{esc}}},$$

where the escape time $\tau_{\text{esc}}$ is assumed to be momentum-independent. In equation (31) and throughout this section, the subscript on $f_0$ is dropped, with the understanding that $f$ now refers to the distribution function obtained after averaging over the turbulent fluctuations. Equation (31) is solved for $p \geq p_0$ and $t \geq 0$ subject to the boundary conditions

$$f(p = p_0, \ t \geq 0) = A,$$  

with $A$ constant,  

$$f(p > p_0, \ t = 0) = 0,$$  

$$f(p \rightarrow \infty, \ t \geq 0) = 0.$$  

Equation (32) would apply, at least approximately, if Coulomb collisions kept the distribution at low energies fixed and only a small minority of the particles were accelerated. It is assumed that

$$D_p = \frac{p^2}{\tau_{\text{esc}}},$$

if slow-mode eddies were oscillatory, as opposed to repeatedly randomized, then eddies larger than $D_p/v_A$ would contribute less to $D_p$ than eddies of length $D_p/v_A$, and the logarithmic term would disappear from eq. (19). However, although $\nabla \cdot u$ is oscillatory within any infinitesimal fluid element (since the density does not change secularly in time), the average value of $\nabla \cdot u$ over the volume of a slow-mode eddy is randomized in the time $\lambda_1 / v_A$, since the eddy is completely shredded and mixed with its neighbors in a time $\lambda_1 / v_A$ by Alfven modes. In addition, an energetic particle moves from one fluid element within an eddy of length $\lambda_1$ to a new fluid element that is chosen randomly from the surrounding eddy volume of width $\lambda_1$ and length $\lambda_1$ in a time $\lambda_1 / v_A$ because of cross-field diffusion and the turbulent mixing of fluid elements by Alfven modes.
with $\tau_{\text{acc}}$ independent of $p$. From equation (30) it can be seen that $\tau_{\text{acc}}$ for acceleration by slow-mode eddies in compressible MHD turbulence is of the order of $l/v_A$ and independent of $p$ and particle species, aside from a possible weak momentum and species dependence arising from the $\ln(l/v_A/D_0)$ term. The solution to equations (31)–(34) is (Kardashev 1962; Schlickeiser 2002)\(^3\)

$$
\begin{align*}
\mathcal{N}(E) &= 4\pi m p f(p) \bigg|_{p=\sqrt{2}mE},
\end{align*}
$$

where $m$ is the particle mass. In the nonrelativistic limit, $\mathcal{N}(E)$ approaches the power law $E^{-q}$ for $E < p_{\text{esc}}^{\text{max}}/2m$, with $q = 1 + (9 + 4\chi)^{1/2}/4$; equivalently, a specified power law $N(E) \propto E^{-q}$ corresponds to $\chi = 4q^2 - 2q - 2$. In the ultrarelativistic limit, $\mathcal{N}(E)$ approaches the power law $E^{-q}$ for $E < p_{\text{esc}}^{\text{max}}$, with $q = [1 + (9 + 4\chi)^{1/2}/4]$. For a specified $q$, $\chi = 4q^2 + 2q - 2$. If escape is neglected, then $\chi = 0$; this corresponds to $q = 1$ in the nonrelativistic limit and $q = \frac{1}{2}$ in the ultrarelativistic limit.

4. STEADY STATE ENERGY SPECTRUM OF ACCELERATED ELECTRONS WITH COULOMB LOSSES AND AN ENERGY-INDEPENDENT ESCAPE TIME

In the presence of Coulomb losses, steady electron injection at energy $E_0$, and an energy-independent escape time $\tau_{\text{esc}}$, the energy spectrum of superthermal electrons $\mathcal{N}(E)$ satisfies the Fokker-Planck equation

$$
\frac{\partial \mathcal{N}}{\partial t} = \frac{\partial^2 \mathcal{N}}{\partial E^2} (D_E \mathcal{N}) - \frac{\partial}{\partial E} (A_E \mathcal{N}) - \frac{N}{\tau_{\text{esc}}} + S\delta(E - E_0)\Theta(t)
$$

(Park & Petrov 1995), where in the nonrelativistic limit,

$$
D_E = \frac{p^2 D_p}{m_e},
$$

$$
A_E = \frac{m_e}{p^2} \frac{d}{dp} (pD_E) - \frac{\alpha_e m_e^2 e^3}{p},
$$

$$
\alpha_e = 6 \times 10^{-3} \left( \frac{n}{10^{10} \text{ cm}^{-3}} \right) \text{ s}^{-1},
$$

$n$ is the electron density, $S$ is a constant, and $\Theta(t)$ is the Heaviside function. Assuming

$$
D_p = \frac{p^2}{\tau_{\text{esc}}},
$$

with $\tau_{\text{acc}}$ independent of $p$, the steady state ($t \to \infty$) solution to equation (41) for $E > E_0$ is

$$
\mathcal{N}(E) = c_1 \chi^{3/2} M \left( \hat{a}, \hat{b}, |\beta| \chi^{-3/2} \right)
$$

Fig. 1.—Distribution function $f$ at $t = t_{\text{acc}}/10$ (solid line), $t = t_{\text{acc}}/2$ (dotted line), and $t = 2t_{\text{acc}}$ (dashed line) for $t_{\text{acc}} = 4t_{\text{esc}}$, when Coulomb losses are neglected.
(Park & Petrosian 1995, eq. [71]), where \( c_1 \) is a constant,

\[
x = \frac{E}{m_e c^2},
\]

\[
a = \frac{1}{2} \left( \frac{\sqrt{9 + 4\chi}}{6} \right),
\]

\[
b = \frac{1}{3} \left( \frac{\sqrt{9 + 4\chi}}{6} \right),
\]

\[
\gamma = \frac{1}{6} \left( \frac{\sqrt{9 + 4\chi}}{6} \right),
\]

\[
\beta = \frac{\alpha \gamma \tau_{\text{acc}}}{6\sqrt{2}}.
\]

Equation (46) is illustrated in Figure 2 for the parameters \( E_0 = 0.002m_e c^2 \), \( n = 2 \times 10^9 \text{ cm}^{-3} \), \( \chi = 10 \), and \( \tau_{\text{acc}} = 5 \text{ s} \).

For \( x \gg \beta^{2/3} \), Coulomb losses become unimportant, and the spectrum obtains the same steady state power law as in § 3. For \( x \ll \beta^{2/3} \), Coulomb losses cause the spectrum to steepen relative to its high-energy power law (Hamilton & Petrosian 1992).

5. APPLICATION TO ELECTRON ACCELERATION IN SOLAR FLARES

In this section it is shown that the acceleration mechanism discussed in this paper is probably not responsible for electron acceleration in solar flares in the 10–100 keV range. On the other hand, the predicted energy spectrum is roughly consistent with the energy spectra of electrons in the 100–5000 keV range, although it is not clear that \( D_B \ll \hbar c_A \) for such particles.

A simple and fairly standard flare model is considered, as depicted in Figure 3, with features drawn from the works of, e.g., Sturrock (1966), LaRosa, Moore, & Shore (1994), Tsuneta (1996), Miller, LaRosa, & Moore (1996), Tsuneta et al. (1997), Miller et al. (1997), and Aschwanden (2002). The flare arises from reconnection between open magnetic field lines above a pair of footpoints. The reconnection layer is assumed to be Petschek-like, so that the reconnection rate is sufficiently fast. Most of the magnetic energy is released outside (downstream) of the reconnection layer, where the magnetic tension associated with newly reconnected field lines accelerates plasma away from the reconnection site in a “slingshot” action. It is assumed that turbulence is generated in this magnetized outflow with rms velocity \( \sim v_A \) and that the turbulence is similar to strong homogeneous MHD turbulence (§ 2.2). Electrons accelerated by the turbulence stream toward the solar surface, evaporating chromospheric plasma and emitting hard bremsstrahlung X-rays at the pair of footpoints that anchor the magnetic flux tube occupied by the energetic electrons. The evaporated chromospheric plasma rises, filling a magnetic-flux–tube loop with hot, relatively dense plasma that emits soft X-rays. As additional field lines reconnect, the hard–X-ray footpoints move progressively outward, and the top of the soft–X-ray loop rises. Time-of-flight measurements typically place the electron acceleration region well above the tops of the soft–X-ray loops and also above the hard–X-ray emission that has been observed above the soft–X-ray loops in a number of flares (Aschwanden 2002).

The electron energy spectrum varies between flares and between the footpoints and above-the-looptop regions. If \( N(E) \) is taken to be a power law \( E^{-q} \) in the 10–100 keV range, then a typical value of \( q \) is 4, although values between 1.5 and \( \sim 10 \) have been observed (Alexander & Metcalf...
It has been hypothesized that microturbulence in the acceleration region, e.g., whistler turbulence, causes $\lambda_{\text{amp}}$ for energetic electrons to be much shorter than the Coulomb mean free path (Miller et al. 1997). If $D_p \ll l_{\text{WA}}$, as assumed in this paper, then the time to diffuse out of the region of strongest turbulence, $l^2/p$, is much longer than the acceleration time, $\tau_{\text{acc}} \sim l/v_A$. On the other hand, the plasma undergoes a bulk flow at speed $v_{\text{AW}}$ away from the reconnecton site that advects electrons out of the region of strongest turbulence on a timescale $l/v_A$. For particles with $D_p \ll l_{\text{WA}}$, transport out of the turbulent region is dominated by this advection, and the escape time $\tau_{\text{esc}}$ is then $\sim l/v_A \sim 0.5$ s, independent of electron energy (E. Blackman 2003, private communication). If $d_{\min}v_A \ll D_p \ll l_{\text{WA}}$, then particle acceleration by slow modes is approximately described by equation (30). Taking $l_{\text{AW}}^i \rightarrow l$ in equation (30) and $D_p = l_{\text{WA}}/10$, one finds that $\tau_{\text{esc}} \sim p^2/D_p \sim 10l/v_A \sim 5$ s, and $\chi_{\text{esc}} = 10$.

Coulomb losses are the dominant energy losses for electrons in solar flares with $E < 100$ keV (Hamilton & Petrosian 1992). If escape is modeled with an ad hoc loss term as in equation (41), then the energy spectrum of electrons accelerated by slow modes approaches the form depicted in Figure 2. For energies above a few keV, Coulomb losses can be neglected and $\mathcal{N}(E) \propto E^{-q}$, with $q = 2$. Such a spectrum is significantly harder than typical solar flare spectra in the 10–100 keV range, as discussed above. For energies above a few keV, the time-dependent spectrum behaves approximately as in §3, with $\mathcal{N}(E) \propto E^{-2}$ for $E < E_{\text{max}}$, with $E_{\text{max}} = p_{\text{max}}^2/2m \times e^{(2q/\tau_{\text{acc}})}/\sqrt{q+4\chi} = e^{14l/\tau_{\text{acc}}}$.

The time required for $E_{\text{max}}$ to increase from 5 to 500 keV is then $\sim 0.3\tau_{\text{acc}} \sim 1.5$ s. This time is more than a factor of 10 larger than the energization time implied by time-of-flight measurements for the flare of 1991 December 15, 19:32 UT (Aschwanden 2002). The inability of the acceleration mechanism described in this paper to explain the observed steep electron spectra and short acceleration times makes it an unlikely explanation for electron acceleration in solar flares in the 10–100 keV range.

The predicted spectra are roughly consistent with the energy spectra of electrons in the 100–5000 keV range inferred from microwave gyrosynchrotron emission (Kundu et al. 2001). The electrons responsible for this emission may have considerably longer acceleration times than the electrons involved in X-ray bursts. The acceleration mechanism discussed in this paper could explain these high-energy electrons, but only if $D_p \ll l_{\text{WA}}$ for electrons in this energy range.

6. SUMMARY

The momentum diffusion coefficient $D_p$ arising from slow modes in strong compressible anisotropic MHD turbulence is calculated for the case in which the rms turbulent velocity is $v_A$, the pitch-angle scattering of energetic particles is very efficient ($d_{\min}v_A \ll D_p \ll l_{\text{WA}}$), the energetic particle speed $v > v_A$, and $\beta \ll 1$. It is found that

$$D_p \approx \frac{2p^3v_A}{9l} \left[ \ln \left( \frac{l_{\text{WA}}}{D_p} \right) + 2\gamma - 3 \right],$$

where $\gamma \approx 0.577$ is Euler’s constant. Aside from a possible weak momentum dependence associated with the $\ln(l_{\text{WA}}/D_p)$ term, $D_p \propto p^2$, implying that the acceleration time $\tau_{\text{acc}} = p^2/D_p$ is independent of $p$. In addition, $\tau_{\text{acc}}$ is of the order of $l/v_A$ and approximately independent of particle species.

Slow modes in the $D_p \ll l_{\text{WA}}$ limit are an unlikely explanation for electron acceleration in solar flares to energies of 10–100 keV, because for solar flare conditions, the predicted acceleration times are too long, and the predicted energy spectra are too hard. The acceleration mechanism discussed in this paper could in principle explain the relatively hard spectra of gyrosynchrotron-emitting electrons in the 100–5000 keV range, but only if $D_p \ll l_{\text{WA}}$ for such particles.

I thank Eliot Quataert, Steve Spangler, Torsten Ensslin, Eric Blackman, Ted LaRosa, and Vahe Petrosian for valuable input. I also thank the referee, R. Schlickeiser, for his very helpful comments. This work was supported by NSF grant AST 00-98086 and DOE grants DE-FG02-01ER54658 and DE-FC02-01ER54651.
Hamilton, R., & Petrosian, V. 1992, ApJ, 398, 350
Higdon, J. C. 1984, ApJ, 285, 109
———. 1986, ApJ, 309, 342
Jokipii, J. R. 1966, ApJ, 146, 480
Kardashev, N. 1962, Soviet Astron., 6, 317
Katz, M., & Stehlik, M. 1991, Ap&SS, 183, 259
Kulsrud, R., & Ferrari, A. 1971, Ap&SS, 12, 302
Kulsrud, R., & Pearce, W. 1969, ApJ, 156, 445
Kundu, M., Nindos, A., White, S., & Grechnev, V. 2001, ApJ, 557, 880
LaRosa, T., Moore, R., & Shore, S. 1994, ApJ, 425, 856
Lithwick, Y., & Goldreich, P. 2001, ApJ, 562, 279
Maron, J., & Goldreich, P. 2001, ApJ, 554, 1175
Matthaeus, W., Oughton, S., & Ghosh, S. 1998, Phys. Rev. Lett., 81, 2056
Milano, L., Matthaeus, W., Dmitruk, P., & Montgomery, D. C. 2001, Phys. Plasmas, 8, 2673
Miller, J., LaRosa, T., & Moore, R. 1996, ApJ, 461, 445
Montgomery, D., & Matthaeus, W. 1995, ApJ, 447, 706
Montgomery, D., & Turner, L. 1981, Phys. Fluids, 24, 825
Oughton, S., Priest, E., & Matthaeus, W. 1994, J. Fluid Mech., 280, 95
Park, B. 1995, Ph.D. thesis, Stanford Univ.
Park, B., & Petrosian, V. 1995, ApJ, 446, 699
Parker, E. 1957, Phys. Rev., 107, 830
Petrosian, V. 2002, preprint (astro-ph/0207482)
Ptuskin, V. 1988, Soviet Astron. Lett., 14, 255
Ramaty, R. 1979, in AIP Conf. Proc. 56, Particle Acceleration Mechanisms in Astrophysics, ed. J. Arons, C. McKee, & C. Max (New York: AIP), 135
Schlickeiser, R. 1984, A&A, 136, 227
———. 2002, Cosmic Ray Astrophysics (Berlin: Springer)
Schlickeiser, R., & Miller, J. 1998, ApJ, 492, 352
Shebalin, J. V., Matthaeus, W., & Montgomery, D. 1983, J. Plasma Phys., 29, 525
Skilling, J. 1975, MNRAS, 172, 557
Spangler, S. R. 1999, ApJ, 522, 879
Sridhar, S., & Goldreich, P. 1994, ApJ, 432, 612
Sturrock, P. 1966, Nature, 211, 695
Tsuneta, S. 1996, ApJ, 456, 840
Tsuneta, S., Masuda, S., Kosugi, T., & Sato, J. 1997, ApJ, 478, 787
Yan, H., & Lazarian, A. 2002, Phys. Rev. Lett., 89, 281102