Non-Bayesian Social Learning on Random Digraphs With Aperiodically Varying Network Connectivity

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Abstract—In this article, we study non-Bayesian social learning on random directed graphs and show that under mild connectivity assumptions, all the agents almost surely learn the true state of the world asymptotically in time if the sequence of the associated weighted adjacency matrices belongs to Class $P^{\infty}$ (a broad class of stochastic chains that subsumes uniformly strongly connected chains). We show that uniform strong connectivity, while being unnecessary for asymptotic learning, ensures that all the agents’ beliefs converge to a consensus almost surely, even when the true state is not identifiable. We then provide a few corollaries of our main results, some of which apply to the variants of the original update rule, such as inertial non-Bayesian learning and learning via diffusion and adaptation. Others include the extensions of known results on social learning. We also show that if the network of influences is balanced in a certain sense, then asymptotic learning occurs almost surely even in the absence of uniform strong connectivity.

Index Terms—Distributed algorithms, multiagent systems, network theory (graphs), networked control systems, stochastic systems.

I. INTRODUCTION

The advent of social media and Internet-based sources of information has significantly influenced the way people learn about the world around them. For instance, while learning about political candidates, people gather information from news websites, as well as from their social circles.

To study the impact of social networks and external sources of information on the evolution of individual beliefs, several models of social dynamics have been proposed (see [1] and [2] for a detailed survey). The manner in which the modeled agents update their beliefs ranges from being naive as in [3], wherein an agent’s belief keeps shifting to the arithmetic mean of his/her neighbors’ beliefs, to being fully rational (Bayesian) as in [4] and [5]. For a survey on Bayesian learning, see [6].

However, as argued in [7], it is unlikely that real-world social networks consist of fully rational agents, because not only are Bayesian update rules computationally burdensome, but they also require all the agents to know the structure of the social network and the history of others’ private observations. Therefore, Jadabaie et al. [7] proposed a non-Bayesian model of social learning to model agents with limited rationality. This model assumes that the world (or the agents’ object of interest) is described by a set of possible states, of which only one state is true. With the objective of identifying the true state, each agent performs measurements on the state of the world and learns his/her neighbors’ most recent beliefs in every state. At every new time step, the agent updates his/her beliefs by incorporating his/her own latest observations in a Bayesian manner and others’ beliefs in a naive manner. With this update rule, all the agents almost surely (a.s.) learn the true state asymptotically in time, without having to learn the network structure or others’ private observations.

Notably, some of the non-Bayesian learning models inspired by Jadabaie et al. [7] have yielded efficient algorithms for distributed learning (for example, see [8]–[15], and see [16] for a tutorial). Furthermore, the model of [7] has motivated research on decentralized estimation [17]; cooperative device-to-device communications [18]; crowdsensing in mobile social networks [19]; manipulation in social networks [20]; impact of social media and fake news on social learning [21], [22]; and learning in the presence of malicious agents and model uncertainty [23].

However, most of the existing non-Bayesian learning models make two crucial assumptions. First, they assume a deterministic network topology. Second, they describe the network either by a time-invariant influence graph or by a sequence of influence graphs that are uniformly strongly connected. By contrast, real-world networks are not likely to satisfy either assumption. This motivates us to extend the model of Jadabaie et al. [7] to random directed graphs satisfying weaker connectivity criteria. To do so, we identify certain sets of agents called observationally self-sufficient sets. The collection of measurements obtained by any of these sets is at least as useful as that obtained by any other set of agents. We then introduce the concept of $\gamma$-epochs, which are periods of time over which the network is adequately well connected. We then derive our main result:
under the assumptions made in [7] on the agents’ prior beliefs and observation structures, if the sequence of the weighted adjacency matrices associated with the network belongs to a broad class of random stochastic chains called Class $\mathcal{P}^*$, and if these matrices are independently distributed, then our relaxed connectivity assumption ensures that all the agents a.s. learn the truth asymptotically in time.

The main contributions of this article are summarized as follows.

1) **Criteria for learning on random digraphs:** Our work extends the earlier studies on non-Bayesian learning to the scenario of learning on random digraphs, and as we will show, our assumption of recurring $\gamma$-epochs is weaker than the standard assumption of uniform strong connectivity. Therefore, our main result identifies a set of sufficient conditions for almost-sure asymptotic learning that are weaker than those derived in prior works. Moreover, our main result (see Theorem 1) does not assume almost-sure fulfillment of our connectivity criteria (see Assumption IV and Remark 1).

2) **Implications for distributed learning:** Since the learning rule (1) is an exponentially fast algorithm for distributed learning [16], [24], our main result significantly extends the practicality of the results in [7], [12], [25], and [26].

3) **A sufficient condition for consensus:** Theorem 2 shows how uniform strong connectivity ensures that all the agents’ beliefs converge to a consensus a.s. even when the true state is not identifiable.

4) **Results on related learning scenarios:** Section V provides sufficient conditions for almost-sure asymptotic learning in certain variants of the original model, such as learning via diffusion adaptation and inertial non-Bayesian learning.

5) **Methodological contribution:** The proofs of Theorems 1 and 2 illustrate the effectiveness of the less-known theoretical utility of Class $\mathcal{P}^*$ and absolute probability sequences.

Out of the available non-Bayesian learning models, we work on the proposed model in [7]. This choice is supported by experiments that have repeatedly shown that the variants of DeGroot’s model predict the evolution of beliefs in the real world better than models based solely on Bayesian rationality [27]–[29]. Moreover, DeGroot’s learning rule is the only rule that satisfies the psychological assumptions of imperfect recall, label neutrality, monotonicity, and separability [30].

**Related works:** We first describe the main differences between this article and our prior work [31].

1) The main result (Theorem 1) of [31] applies only to deterministic time-varying networks, while the main result (see Theorem 1) of this article applies to random time-varying networks. As we will show in Remark 1, the results of this article apply to certain random graph sequences that a.s. fall outside the class of deterministic graph sequences considered in [31].

2) In addition to the corollaries reported in [31], this article provides three corollaries of our main results that apply to random networks that are central to results in Sections V-A–V-C.

As for other related works, the authors in [32] and [33] make novel connectivity assumptions, but unlike our work, neither of them allows for arbitrarily long periods of poor network connectivity. The same can be said about [22] and [34], even though they consider random networks and impose connectivity criteria only in the expectation sense. Finally, we note that [35] and [36] come close to our work because the former proposes an algorithm that allows for aperiodically varying network connectivity, while the latter makes no connectivity assumptions. However, the sensor network algorithms proposed in [35] and [36] require each agent to have an actual belief and a local belief, besides using minimum-belief rules to update the actual beliefs. By contrast, the learning rule we analyze is more likely to mimic social networks because it is simpler and closer to the empirically supported DeGroot learning rule. Moreover, unlike our analysis, neither [35] nor [36] accounts for randomness in the network structure.

The rest of this article is organized as follows. We begin by defining the model in Section II. In Section III, we review Class $\mathcal{P}^*$, a special but broad class of matrix sequences that forms an important part of our assumptions. Next, Section IV establishes our main result. We, then, discuss the implications of this result in Section V. Finally, Section VI concludes this article.

**Notation:** We denote the set of real numbers by $\mathbb{R}$, the set of positive integers by $\mathbb{N}$, and define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any $n \in \mathbb{N}$, we define $[n] := \{1, 2, \ldots , n\}$.

We denote the vector space of $n$-dimensional real-valued column vectors by $\mathbb{R}^n$. We use the superscript notation $^T$ to denote the transpose of a vector or a matrix. All the matrix and vector inequalities are entrywise inequalities. Likewise, if $v \in \mathbb{R}^n$, then $|v| := |v_1| \cdot |v_2| \cdot \ldots \cdot |v_n|^2$, and if $v > 0$ additionally, then $\log(v) := \log(v_1) \cdot \log(v_2) \cdot \ldots \cdot \log(v_n)^2$. We use $I$ to denote the identity matrix (of the known dimension) and 1 to denote the column vector (of the known dimension) that has all the entries equal to 1. Similarly, 0 denotes the all-zero vector of the known dimension. We say that a vector $v \in \mathbb{R}^n$ is stochastic if $v \geq 0$ and $v^T \mathbf{1} = 1$, and a matrix $A$ is stochastic if $A$ is nonnegative and if each row of $A$ sums to 1, i.e., if $A \geq 0$ and $A^T \mathbf{1} = A \mathbf{1} = 1$. A stochastic matrix $A$ is doubly stochastic if each column of $A$ sums to 1, i.e., if $A \geq 0$ and $A^T \mathbf{1} = A \mathbf{1} = 1$. A sequence of stochastic matrices is called a stochastic chain. If $\{A(t)\}_{t=0}^\infty$ is a stochastic chain, then for any two times $t_1, t_2 \in \mathbb{N}_0$ such that $t_1 < t_2$, we define $A(t_2 : t_1) := A(t_2 - 1)A(t_2 - 2) \cdots A(t_1)$, and let $A(t_1 : t_1) := I$. If $\{A(t)\}_{t=0}^\infty$ is a random stochastic chain (a sequence of random stochastic matrices), then it is called an independent chain if the matrices $\{A(t)\}_{t=0}^\infty$ are $\mathcal{P}$-independent with respect to a given probability measure $P$.

II. PROBLEM FORMULATION

A. Non-Bayesian Learning Model

We now describe our non-Bayesian learning model, which extends the model proposed in [7] to random networks.

We let $\Theta$ denote the (finite) set of possible states of the world, and let $\theta^* \in \Theta$ denote the true state. We consider a social network of $n$ agents that seek to learn the identity of the true state with the help of their private measurements as well as their neighbors’ beliefs.
1) Beliefs and Observations: For each \(i \in [n]\) and \(t \in \mathbb{N}_0\), we let \(\mu_{i,t}\) be the probability measure on \((\Theta, 2^\Theta)\) such that \(\mu_{i,t}(\theta) := \mu_{i,t}(\{\theta\})\) denotes the degree of belief of agent \(i\) in the state \(\theta\) at time \(t\). In addition, for each \(\theta \in \Theta\), we let \(\mu_t(\theta) := \mu_{i,t}(\theta) \mu_{2,t}(\theta) \ldots \mu_{n,t}(\theta)\) for all \(t \in \mathbb{N}_0\).

We assume that the signal space (the space of privately observed signals) of each agent is finite. We let \(S_i\) denote the signal space of agent \(i\), define \(S := S_1 \times S_2 \times \cdots \times S_n\), and let \(\omega_t = (\omega_{1,t}, \ldots, \omega_{n,t}) \in S\) denote the vector of observed signals at time \(t\). Furthermore, we suppose that for each \(t \in \mathbb{N}\), the vector \(\omega_t\) is generated according to the conditional probability measure \(l(-|\theta)\) given that \(\theta^* = \theta\), i.e., \(\omega_t\) is distributed according to \(l(-|\theta)\) if \(\theta\) is the true state.

We now reproduce the assumptions made in [7].

1) \(\{\omega_t\}_{t \in \mathbb{N}}\) is an independent identically distributed (i.i.d.) sequence.

2) For every \(i \in [n]\) and \(\theta \in \Theta\), agent \(i\) knows \(l_i(-|\theta)\), the \(i\)th marginal of \(l(-|\theta)\) (i.e., \(l_i(s|\theta)\) is the conditional probability that \(\omega_{i,t} = s\) given that \(\theta\) is the true state).

3) \(l_i(s|\theta) > 0\) for all \(s \in S_i, i \in [n]\), and \(\theta \in \Theta\). We let \(l_0 := \min_{\theta \in \Theta} \min_{i \in [n]} \min_{s \in S_i} l_i(s|\theta) > 0\).

In addition, it is possible that some agents are unable to distinguish between certain states solely on the basis of their private measurements because these states induce the same conditional probability distributions on the agents’ signal spaces. This motivates the following definition.

Definition 1 (Observational equivalence [7]): Two states \(\theta_1, \theta_2 \in \Theta\) are said to be observationally equivalent from the point of view of agent \(i\) if \(l_i(-|\theta_1) = l_i(-|\theta_2)\).

For each \(i \in [n]\), let \(\Theta_i^* := \{\theta \in \Theta : l_i(-|\theta) = l_i(-|\theta^*)\}\) denote the set of states that are observationally equivalent to the true state from the viewpoint of agent \(i\). In addition, let \(\Theta^* := \bigcap_{i \in [n]} \Theta_i^*\) be the set of states that are observationally equivalent to \(\theta^*\) from every agent’s viewpoint. Since we wish to identify the subsets of agents that can collectively distinguish between the true state and the false states, we define two related terms.

Definition 2 (Observational self-sufficiency): If \(O \subset [n]\) is a set of agents such that \(\bigcap_{j \in [n] \setminus O} \Theta_j^* = \Theta^*\), then \(O\) is said to be an observationally self-sufficient set.

Definition 3 (Identifiability): If \(\Theta^* = \{\theta^*\}\), then the true state \(\theta^*\) is said to be identifiable.

2) Network Structure and the Update Rule: Let \(\{G(t)\}_{t \in \mathbb{N}_0}\) denote the random sequence of \(n\)-vertex directed graphs such that for each \(t \in \mathbb{N}_0\), there is an arc from node \(i \in [n]\) to node \(j \in [n]\) in \(G(t)\) if and only if agent \(i\) influences agent \(j\) at time \(t\). Let \(A(t) = (a_{ij}(t))\) be a stochastic weighted adjacency matrix of the random graph \(G(t)\), and for each \(i \in [n]\), let \(N_i(t) := \{j \in [n] \setminus \{i\} : a_{ij}(t) > 0\}\) denote the set of in-neighbors of agent \(i\) in \(G(t)\). We assume that at the beginning of the learning process (i.e., at \(t = 0\)), agent \(i\) has \(\mu_{i,0}(\theta) \in [0, 1]\) as his/her prior belief in state \(\theta \in \Theta\). At time \(t + 1\), she updates her belief in \(\theta\) as per the following:

\[
\mu_{i,t+1}(\theta) = a_{ii}(t) \text{BU}_{i,t+1}(\theta) + \sum_{j \in N_i(t)} a_{ij}(t) \mu_{j,t}(\theta)
\]

where “BU” stands for “Bayesian update” and

\[
\text{BU}_{i,t+1}(\theta) := l_i(\omega_{i,t+1}\theta) \mu_{i,t}(\theta) \sum_{\theta' \in \Theta} l_i(\omega_{i,t+1}\theta') \mu_{i,t}(\theta')
\]

Finally, we let \((\Omega, B, P^*)\) be the probability space such that \(\{\omega_t\}_{t \in \mathbb{N}_0}\) and \(\{A(t)\}_{t \in \mathbb{N}_0}\) are measurable w.r.t. \(B\), and \(P^*\) is a probability measure such that

\[
P^*(\omega_1 = s_1, \omega_2 = s_2, \ldots, \omega_r = s_r) = \prod_{t=1}^r l(s_t|\theta^*)
\]

for all \(s_1, \ldots, s_r \in S\) and all \(r \in \mathbb{N} \cup \{\infty\}\). In addition, we let \(E^*\) denote the expectation operator associated with \(P^*\).

B. Forecasts and Convergence to the Truth

At any time step \(t\), agent \(i\) can use her current set of beliefs to estimate the probability that she will observe the signals \(s_1, s_2, \ldots, s_k \in S_i\) over the next \(k\) time steps. This is referred to as the \(k\)-step-ahead forecast of agent \(i\) at time \(t\) and denoted by \(m_{i,t}^{(k)}(s_1, s_2, \ldots, s_k)\). We, thus, have

\[
m_{i,t}^{(k)}(s_1, s_2, \ldots, s_k) := \sum_{\theta \in \Theta} \prod_{r=1}^k l_i(s_r|\theta) \mu_{i,t}(\theta).
\]

We use the following notions of convergence to the truth.

Definition 4 (Eventual correctness [7]): The \(k\)-step-ahead forecasts of agent \(i\) are said to be eventually correct on a path \((A(0), \omega_1, A(1), \omega_2, \ldots)\) if along that path

\[
m_{i,t}^{(k)}(s_1, s_2, \ldots, s_k) \rightarrow \prod_{j=1}^k l_i(s_j|\theta^*) \quad \text{as} \quad t \rightarrow \infty.
\]

Definition 5 (Weak merging to the truth [7]): We say that the beliefs of agent \(i\) weakly merge to the truth on some path if, along that path, his/her \(k\)-step-ahead forecasts are eventually correct for all \(k \in \mathbb{N}\).

Definition 6 (Asymptotic learning [7]): Agent \(i \in [n]\) asymptotically learns the truth on a path \((A(0), \omega_1, A(1), \omega_2, \ldots)\) if, along that path, \(\mu_{i,t}(\theta^*) \rightarrow 1\) (and hence, \(\mu_{i,t}(\theta) \rightarrow 0\) for all \(\theta \in \Theta \setminus \{\theta^*\}\)) as \(t \rightarrow \infty\).

Note that if the beliefs of agent \(i\) weakly merge to the truth, it only means that agent \(i\) is able to estimate the probability distributions of his/her future signals/observations with arbitrary accuracy as time goes to infinity. On the other hand, if agent \(i\) asymptotically learns the truth, it means that in the limit as time goes to infinity, agent \(i\) rules out all the false states and correctly figures out that the true state is \(\theta^*\). In fact, it can be shown that asymptotic learning implies weak merging to the truth, even though the latter does not imply the former [7].

III. REVISITING CLASS P*: A SPECIAL CLASS OF STOCHASTIC CHAINS

Our next goal is to deviate from the standard strong connectivity assumptions for social learning [7], [12], [25], [26]. We first explain the challenges involved in this endeavor. To begin, we express (1) as [7, eq. (4)], i.e.,

\[
\mu_{t+1}(\theta) = A(t) \mu_t(\theta)
\]
\[ n \text{ with } m(s,t) := m_{i,i}(s,t) \text{ for all } s \in S_i \text{. Now, suppose } \theta = \theta^* \text{. Then, an extrapolation of the known results on non-Bayesian learning suggests that the right-hand side of (2) decays to 0 a.s. as } t \to \infty \text{. This means that for large values of } t \text{ (say } t \geq T_0), \text{ the dynamics (2) for } \theta = \theta^* \text{ can be approximated as } \mu_{t+1}(\theta^*) \approx A_t(\theta^*). \text{ Hence, we expect the limiting value of } \mu_t(\theta^*) \text{ to be closely related to } \lim_{t \to \infty} A_t : T_0^\infty, \text{ whenever the latter limit exists. However, without standard connectivity assumptions, it is challenging to gauge the existence of limits of backward matrix products.} \]

To overcome this difficulty, we use the notion of Class \( \mathcal{P}^* \) introduced in [37]. This notion is based on the Kolmogorov ingenious concept of absolute probability sequences, which we now define.

**Definition 7 (Absolute probability sequence [37]):** Let \( \{ A(t) \}_{t=0}^\infty \) be either a deterministic stochastic chain or a random process of independently distributed stochastic matrices. A deterministic sequence of stochastic vectors \( \{ \pi(t) \}_{t=0}^\infty \) is said to be an absolute probability sequence for \( \{ A(t) \}_{t=0}^\infty \) if \( \pi^T(I+1) \mathbb{E}[A(t)] = \pi^T(t) \) for all \( t \geq 0 \).

Note that every deterministic stochastic chain admits an absolute probability sequence [38]. Hence, every random sequence of independently distributed stochastic matrices also admits an absolute probability sequence.

Of interest to us is a special class of random stochastic chains that are associated with absolute probability sequences satisfying a certain condition. This class is defined as follows.

**Definition 8 (Class \( \mathcal{P}^* \) [37]):** We let \( \text{(Class-)} \mathcal{P}^* \) be the set of random stochastic chains that admit an absolute probability sequence \( \{ \pi(t) \}_{t=0}^\infty \) such that \( \pi(t) \geq p^* 1 \) for some scalar \( p^* > 0 \) and all \( t \in \mathbb{N}_0 \).

Remarkably, in scenarios involving a linear aggregation of beliefs, if \( \{ \pi(t) \}_{t=0}^\infty \) is an absolute probability sequence for \( \{ A(t) \}_{t=0}^\infty \), then \( \pi(t) \) denotes the Kolmogorov centrality or social power of agent \( i \) at time \( t \), which quantifies how influential the \( i \)-th agent is relative to other agents at time \( t \) [30], [37]. In view of Definition 8, this means that if a stochastic chain belongs to Class \( \mathcal{P}^* \), then the expected chain describes a sequence of influence graphs, in which the social power of every agent exceeds a fixed threshold \( p^* > 0 \) at all times.

**Example 1:** Suppose \( A(t) = A_e \) for all even \( t \in \mathbb{N}_0 \) and \( A(t) = A_0 \) for all odd \( t \in \mathbb{N}_0 \) with

\[
A_e := \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad A_0 := \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}.
\]

Then, one may verify that the alternating sequence

\[
\begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix}^T, \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}^T, \begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix}^T, \ldots
\]

is an absolute probability sequence for the chain \( \{ A(t) \}_{t=0}^\infty \).

Hence, \( \{ A(t) \}_{t=0}^\infty \in \mathcal{P}^* \).

Let us now add a zero-mean independent noise sequence \( \{ W(t) \}_{t=0}^\infty \) to the original chain, where for all even \( t \in \mathbb{N}_0 \), the matrix \( W(t) \) is the all-zero matrix (and, hence, a degenerate random matrix), and for all odd \( t \in \mathbb{N}_0 \), the matrix \( W(t) \) is uniformly distributed on \( \{ W_0, -W_0 \} \).

Then, by [37, Th. 5.1], the random stochastic chain \( \{ A(t) + W(t) \}_{t=0}^\infty \) belongs to Class \( \mathcal{P}^* \) because the expected chain \( \{ E[A(t) + W(t)] \}_{t=0}^\infty \) \( \{ A(t) \}_{t=0}^\infty \) belongs to Class \( \mathcal{P}^* \).

**Remark 1:** Interestingly, Example 1 illustrates that a random stochastic chain may belong to Class \( \mathcal{P}^* \) even though almost every realization of the chain lies outside Class \( \mathcal{P}^* \). To elaborate, consider the setup of Example 1, and let \( \tilde{A}(t) := A(t) + W(t) \).

Observe that \( A_0 - W_0 = I \), which means that for any \( B \in \mathbb{N} \) and \( t_1 \in \mathbb{N}_0 \), the probability that \( A(t) + W(t) = I \) for all odd \( t \in \{ t_1, \ldots, t_1 + 2B - 1 \} \) is \((1/2)^B > 0 \). Since \( \{ W(t) \}_{t=0}^\infty \) are independent, it follows that for \( \mathbb{P} \)-almost every realization \( \{ \tilde{A}(t) \}_{t=0}^\infty \) of \( \{ A(t) \}_{t=0}^\infty \), there exists a time \( \tau \in \mathbb{N}_0 \) such that \( (\tau + 2B : \tau) \) and \( \{ \tilde{A}(t) \}_{t=0}^\infty \), we can use induction along with Definition 7 to show that \( \pi^T_{\tilde{A}(\tau + 2B)} = \pi^T_{\tilde{A}(\tau)} \). Thus, \( \pi^T_{\tilde{A}(\tau)} = \pi^T_{\tilde{A}(\tau + 2B)} = \pi^T_{\tilde{A}(\tau)} \leq 1/2 \leq 1/2 \). Since the second entry of \( \pi^T_{\tilde{A}(\tau)} \leq 1/2 \), and since \( B \) is arbitrary, it follows that there is no lower bound \( p^* > 0 \) on the second entry of \( \pi^T_{\tilde{A}(\tau)} \).

Hence, \( \{ \tilde{A}(t) \}_{t=0}^\infty \notin \mathcal{P}^* \), implying that \( \mathbb{P} \)-almost no realization of \( \{ \tilde{A}(t) \} \) belongs to Class \( \mathcal{P}^* \).

We now turn to a noteworthy subclass of Class \( \mathcal{P}^* \): the class of uniformly strongly connected chains (see [37, Lemma 5.8]). Following is the definition of this subclass (reproduced from [37]).

**Definition 9 (Uniform strong connectivity):** A deterministic stochastic chain \( \{ A(t) \}_{t=0}^\infty \) is said to be uniformly strongly connected if:

1. there exists \( \delta > 0 \) such that for all \( i, j \in [n] \) and all \( t \in \mathbb{N}_0 \), either \( a_{ij}(t) \geq \delta \) or \( a_{ij}(t) = 0 \);
2. \( a_{ii}(t) > 0 \) for all \( i \in [n] \) and all \( t \in \mathbb{N}_0 \);
3. there exists a constant \( B \in \mathbb{N} \) such that for the sequence of graphs \( \{ G(t) = ([n], E(t)) \}_{t=0}^\infty \), where \( E(t) := \{(i,j) \in [n]^2 : a_{ij}(t) > 0 \} \), has the property that the graph

\[
G(k) := \left[ \begin{array}{c} [n] \cup \bigcup_{q=kB} E(q) \end{array} \right]
\]

is strongly connected for every \( k \in \mathbb{N}_0 \).

Owing to the last requirement above, uniformly strongly connected chains are also called \( B \)-strongly connected chains or simply \( B \)-connected chains. Essentially, a \( B \)-connected chain describes a time-varying network that may or may not be connected at every time instant but is guaranteed to be connected over bounded time intervals that occur periodically.

Besides uniformly strongly connected chains, we are interested in another subclass of Class \( \mathcal{P}^* \): the set of independent balanced chains with feedback property (see [37, Th. 4.7]).
Definition 10 (Balanced chains): A stochastic chain \( \{A(t)\}_{t=0}^{\infty} \) is balanced if there exists \( \alpha \in (0, \infty) \) with
\[
\sum_{i \in C} \sum_{j \in [n] \setminus C} a_{ij}(t) \geq \alpha \sum_{i \in [n] \setminus C} \sum_{j \in C} a_{ij}(t)
\]
for all sets \( C \subseteq [n] \) and all \( t \in \mathbb{N}_0 \).

Definition 11 (Feedback property): Let \( \{A(t)\}_{t=0}^{\infty} \) be a random stochastic chain, and let \( F_t := \sigma(A(0), \ldots, A(t-1)) \) for all \( t \in \mathbb{N} \). Then, \( \{A(t)\}_{t=0}^{\infty} \) is said to have feedback property if there exists \( \delta > 0 \) such that
\[
E[a_{ii}(t)a_{jj}(t) | F_t] \geq \delta E[a_{ij}(t) | F_t] \quad \text{a.s.}
\]
for all \( t \in \mathbb{N}_0 \) and all distinct \( i, j \in [n] \).

Intuitively, a balanced chain is a stochastic chain, in which the total influence of any subset of agents \( C \subseteq [n] \) on the complement set \( \bar{C} := [n] \setminus C \) is neither negligible nor tremendous when compared with the total influence of \( C \) on \( A \). As for the feedback property, we relate its definition to the strong feedback property, which has a clear interpretation.

Definition 12 (Strong feedback property): We say that a random stochastic chain \( \{A(t)\}_{t=0}^{\infty} \) has the strong feedback property with feedback coefficient \( \delta \) if there exists \( \delta > 0 \) such that \( a_{ii}(t) \geq \delta \) a.s. for all \( i \in [n] \) and all \( t \in \mathbb{N}_0 \).

Intuitively, a chain with the strong feedback property describes a network, in which all the agents’ self-confidence is always above a certain threshold.

To see how the strong feedback property is related to the (regular) feedback property, note that by [37, Lemma 4.2], if \( \{A(t)\}_{t=0}^{\infty} \) has feedback property, then the expected chain \( \{E[A(t)]\}_{t=0}^{\infty} \) has the strong feedback property. Thus, a balanced independent chain with feedback property describes a network, in which the complementary sets of agents influence each other to comparable extents, and every agent’s mean self-confidence is always above a certain threshold.

Remark 2: It may appear that every stochastic chain belonging to Class \( P^* \) is either uniformly strongly connected or balanced with feedback property, but this is not true. Indeed, one such chain is described in Example 1, wherein we have \( A(t) + W(t) = A_t \) for even \( t \in \mathbb{N}_0 \), which implies that (3) is violated at even times. Hence, \( \{A(t) + W(t)\}_{t=0}^{\infty} \) is not a balanced chain. As for uniform strong connectivity, recall from Remark 1 that \( P^*-\text{almost every realization of } \{A(t) + W(t)\}_{t=0}^{\infty} \) lies outside Class \( P^* \). Since Class \( P^* \) is a superset of the class of uniformly strongly connected chains (see [37, Lemma 5.8]), it follows that \( \{A(t) + W(t)\}_{t=0}^{\infty} \) is a.s. not uniformly strongly connected.

Remark 3: Touri and Nedić [39] provide the examples of subclasses of Class \( P^* \) chains that are not uniformly strongly connected, such as the class of doubly stochastic chains. For instance, let \( D \subseteq \mathbb{R}^{n \times n} \) be any finite collection of doubly stochastic matrices such that \( D \cap D \) and let \( \{A(t)\}_{t=0}^{\infty} \) be a sequence of i.i.d. random matrices, each of which is uniformly distributed on \( D \). Then, \( \{A(t)\}_{t=0}^{\infty} \), being a doubly stochastic chain, belongs to Class \( P^* \) (see [39]). Now, for any \( B \in \mathbb{N} \) and \( t_1 \in \mathbb{N}_0 \), the probability that \( A(t) = I \) for all \( t \in \{t_1, \ldots, t_1 + B - 1\} \)

IV. MAIN RESULT AND ITS DERIVATION

We first introduce a network connectivity concept called \( \gamma \)-epoch, which plays a key role in our main result.

Definition 13 (\( \gamma \)-epoch): For a given \( \gamma > 0 \) and \( t_s, t_f \in \mathbb{N} \) satisfying \( t_s < t_f \), the time interval \( [t_s, t_f] \) is a \( \gamma \)-epoch if, for each \( i \in [n] \), there exists an observationally self-sufficient set of agents \( O_i \subseteq [n] \) such that, for any \( j \in O_i \), there exists \( t \in \{t_s + 1, \ldots, t_f\} \) satisfying \( a_{jj}(t) \geq \gamma \) and \( (A(t) : t_s)_{ji} \geq \gamma \). In addition, if \( [t_s, t_f] \) is a \( \gamma \)-epoch, then \( t_f - t_s \) is the epoch duration.

As an example, if \( n \geq 9 \) and if the sets \( \{2, 5, 9\} \) and \( \{7, 9\} \) are observationally self-sufficient, then Fig. 1 illustrates the influences of agents 1 and 1 in the \( \gamma \)-epoch [0, 5].

Intuitively, \( \gamma \)-epochs are time intervals over which every agent strongly influences an observationally self-sufficient set of agents, whose self-confidence is guaranteed to be above a certain threshold at the concerned time instants.

We now list the assumptions underlying our main result.

1) Recurring \( \gamma \)-epochs: There exist constants \( \gamma > 0 \) and \( B \in \mathbb{N} \), and an increasing sequence \( \{t_k\}_{k=1}^{\infty} \subseteq \mathbb{N} \) such that \( t_{2k} - t_{2k-1} \leq B \) for all \( k \in \mathbb{N} \), and
\[
\sum_{k=1}^{\infty} |P^* ([t_{2k-1}, t_{2k}])| \quad \text{is a } \gamma \text{-epoch} \quad = \infty.
\]

This means that the probability of occurrence of a \( \gamma \)-epoch of bounded duration does not vanish too fast with time. Note, however, that \( t_{2k+1} - t_{2k} \) (the time elapsed between two consecutive candidate \( \gamma \)-epochs) may be unbounded.
2) Existence of a positive prior: There exists an agent \(j_0 \in [n]\) such that \(\mu_{j_0} (\theta^*) > 0\), i.e., the true state is not ruled out entirely by every agent. We assume without loss of generality that \(j_0 = 1\).

3) Initial connectivity with the agent with the positive prior: There a.s. exists a random time \(T < \infty\) such that \(E^\pi [\log (A(T : 0)]_{11} > -\infty\) for all \(i \in [n]\).

4) Class \(\mathcal{P}^*\): \(\{ A(t) \}_{t=0}^\infty \in \mathcal{P}^*\), i.e., the sequence of weighted adjacency matrices of the networks belongs to Class \(\mathcal{P}^*\) w.r.t. the probability measure \(\mathbb{P}^*\).

5) Independent chain: \(\{ A(t) \}_{t=0}^\infty\) is a \(\mathbb{P}^*\)-independent chain.

6) Independence of observations and network structure: The sequences \(\{ \omega_t \}_{t=1}^\infty\) and \(\{ A(t) \}_{t=0}^\infty\) are \(\mathbb{P}^*\)-independent of each other.

The main results are as follows:

**Theorem 1:** Suppose that the sequence \(\{ A(t) \}_{t=0}^\infty\) and the agents’ initial beliefs satisfy Assumptions II–VI.

i) If \(\{ A(t) \}_{t=0}^\infty\) either has the strong feedback property or satisfies Assumption I, then every agent’s beliefs weakly merge to the truth \(\mathbb{P}^*\)-a.s.

ii) If Assumption I holds and \(\theta^*\) is identifiable, then all the agents asymptotically learn the truth \(\mathbb{P}^*\)-a.s.

Theorem 1 applies to stochastic chains belonging to Class \(\mathcal{P}^*\) and, hence, to scenarios in which the social power (Kolmogorov centrality) of each agent exceeds a fixed positive threshold at all times in the expectation sense (see Section III). While Part (i) identifies the recurrence of \(\gamma\)-epochs as a sufficient connectivity condition for the agents’ forecasts to be eventually correct, Part (ii) asserts that if \(\gamma\)-epochs are recurrent and if the agents’ observation methods enable them to collectively distinguish the true state from all other states, then they will learn the true state asymptotically a.s.

Note that the sufficient conditions provided in Theorem 1 do not include uniform connectivity. However, it turns out that uniform strong connectivity as a connectivity criterion is sufficient not only for almost-sure weak merging to the truth but also for ensuring that all the agents asymptotically agree with each other a.s., even when the true state is not identifiable. We state this result formally as follows.

**Theorem 2:** Suppose Assumption II holds, and suppose \(\{ A(t) \}_{t=0}^\infty\) is deterministic and uniformly strongly connected; then, all the agents’ beliefs converge to a consensus \(\mathbb{P}^*\)-a.s., i.e., for each \(\theta \in \Theta\), there exists a random variable \(C_\theta \in [0, 1]\) such that \(\lim_{t \to \infty} \mu_t (\theta) = C_\theta 1\) a.s.

Before proving Theorems 1 and 2, we look at the effectiveness of the concepts of Section III in analyzing the social learning dynamics studied in this article. We begin by noting the following implication of Assumption IV: there exists a deterministic sequence of stochastic vectors \(\{ \pi(t) \}_{t=0}^\infty\) and a constant \(\rho > 0\) such that \(\{ \pi(t) \}_{t=0}^\infty\) is an absolute probability sequence for \(\{ A(t) \}_{t=0}^\infty\), and \(\pi(t) \geq \rho 1\) for all \(t \in \mathbb{N}_0\).

### A. Using Absolute Probability Sequences and the Notion of Class \(\mathbb{P}^*\) to Analyze Social Learning

1) **Linear Approximation of the Update Rule:** Consider the update rule (2) with \(\theta = \theta^*\). Note that the only nonlinear term in this equation is

\[ u(t) := \text{diag} (\ldots, a_{ii} (t) \left\{ \frac{1}{m_{ii} (\omega_{i, t+1})} \right\} \ldots, \ldots) \mu_t (\theta^*). \]

Therefore, in case \(\lim_{t \to \infty} u(t) = 0\), the resulting dynamics for large \(t\) would be \(\mu_t (\theta^*) \approx A(t) \mu_t (\theta^*)\), which is approximately linear and, hence, easier to analyze. This motivates us to use the following trick: we could take the dot product of each side of (2) with a nonvanishing positive vector \(q(t)\) and, then, try to show that \(q^T (t) u(t) \to 0\) as \(t \to \infty\). In addition, since \(\{ A(t) \}_{t=0}^\infty \in \mathcal{P}^*\), we could simply choose \(q(t) = \pi(t)\) as our sequence of nonvanishing positive vectors.

Before using this trick, we need to take suitable conditional expectations on both the sides of (2) so as to remove all the randomness from \(A(t)\) and \(a_{ij}(t)\). To this end, we define \(B_t := \sigma (\omega_1, \ldots, \omega_t, A(0), \ldots, A(t))\) for each \(t \in \mathbb{N}\) and obtain the following from (2):

\[ E^\pi [\mu_{t+1} (\theta^*) | B_t] - A(t) \mu_t (\theta^*) = E^\pi [u(t) | B_t], \]

where we used that \(\mu_t (\theta^*)\) is measurable w.r.t. \(B_t\). We now use the said trick as follows: we left-multiply both the sides of the above equation by the nonrandom vector \(\pi^T (t+1)\) and obtain

\[ \pi^T (t+1) E^\pi [u(t) | B_t] = \pi^T (t+1) E^\pi [\pi_{t+1} (\theta^*) | B_t] - \pi^T (t+1) A(t) \mu_t (\theta^*). \]

Here, we use the definition of absolute probability sequences (see Definition 7); we replace \(\pi^T (t+1)\) with \(\pi^T (t+2) E^\pi [A(t+1)]\) in the first term on the right-hand side. Consequently

\[ \pi^T (t+1) E^\pi [u(t) | B_t] \]

\[ = \pi^T (t+2) E^\pi [A(t+1)] E^\pi [\pi_{t+1} (\theta^*) | B_t] - \pi^T (t+1) A(t) \mu_t (\theta^*) \]

\[ \overset{(a)}{=} E^\pi [\pi^T (t+2) A(t+1) \mu_{t+1} (\theta^*) | B_t] - \pi^T (t+1) A(t) \mu_t (\theta^*) \]

(4)

where (a) follows from Assumptions V and VI (for details, see [40, Lemma 8], the extended version of this article). Now, to prove that \(\lim_{t \to \infty} u(t) = 0\), we could begin by showing that the left-hand side of (4) (i.e., \(\pi^T (t+1) E^\pi [u(t) | B_t]\)) goes to 0 as \(t \to \infty\). As it turns out, this latter condition is already met: according to Lemma 2 (on the next page), the right-hand side of (4) vanishes as \(t \to \infty\) as a result

\[ \lim_{t \to \infty} \pi^T (t+1) E^\pi [u(t) | B_t] = 0 \] a.s.

Equivalently

\[ \sum_{i=1}^n \pi_i (t+1) a_{ii} (t) E^\pi \left\{ \frac{1}{m_{ii} (\omega_{i, t+1})} - 1 \right\} \mu_{i,i} (\theta^*) \to 0 \]
as $t$ goes to infinity, where we have used that $a_{ti}(t)$ and $\mu_i(\theta^*)$ are measurable w.r.t. $B_t$. To remove the summation from the above limit, we use the lower bound in Lemma 5 to argue that every summand in the above expression is nonnegative. Thus, for each $i \in [n]$, we have

$$\lim_{t \to \infty} \pi_i(t+1)a_{ti}(t)E^*\left[\frac{l_i(\omega_i,t+1|\theta^*)}{m_i(\omega_i,t+1)} - 1 \mid B_t\right] \mu_i(t+1) = 0$$

a.s. More compactly, $\lim_{t \to \infty} \pi_i(t+1)E^*[a_{ti}(t) \mid B_t] = 0$ a.s. Here, Class $\mathcal{P}^*$ plays an important role: since $\pi(t+1) \geq p^*1$, the multiplicand $\pi_i(t+1)$ can be omitted, i.e.,

$$\lim_{t \to \infty} E^*[u_i(t) \mid B_t] = 0 \text{ a.s.} \quad (5)$$

We have, thus, shown that $\lim_{t \to \infty} E^*[u(t) \mid B_t] = 0$ a.s. With the help of some algebraic manipulation, we can now show that $\lim_{t \to \infty} u(t) = 0$ a.s. (see [40, Lemma 6] for details).

2) Analysis of One-Step-Ahead Forecasts: Interestingly, the result $\lim_{t \to \infty} u(t) = 0$ a.s. can be strengthened further to comment on one-step-ahead forecasts, as we now show. Recall that $\pi(t) \geq p^*1$ for all $t \in \mathbb{N}_0$. Since $\log(\mu_i(\theta^*)) \leq 0$, this means that the following inequalities hold a.s.:

$$p^* \lim_{t \to \infty} \inf_{i=1}^{n} \log(\mu_i(t+1|\theta^*)) = \lim_{t \to \infty} p^*1^T \log(\mu_i(\theta^*)) \geq \lim_{t \to \infty} p^*1^T \log(\mu_i(\theta^*)) > -\infty$$

where the last step follows from Lemma 2. This is possible only if $\lim_{t \to \infty} \log(\mu_i(t+1|\theta^*)) > -\infty$ a.s. for each $i \in [n]$, which implies that $\lim_{t \to \infty} \mu_i(t+1|\theta^*) > 0$ a.s., that is, there is a.s. exist random variables $\delta > 0$ and $T' \in \mathbb{N}_0$ such that $\mu_i(t+1|\theta^*) > \delta$ a.s. for all $t \geq T'$. Since $\lim_{t \to \infty} u_i(t) = 0$ a.s., it follows that $\lim_{t \to \infty} u_i(t) = 0$ a.s. i.e.,

$$\lim_{t \to \infty} a_{ii}(t) \left(\frac{l_i(\omega_i,t+1|\theta^*)}{m_i(\omega_i,t+1|\theta^*)} - 1\right) = 0 \text{ a.s.}$$

Multiplying the above limit by $-m_i(t+1|\omega_i,t+1)$ yields $\lim_{t \to \infty} a_{ii}(t)(m_i(t+1|\omega_i,t+1) - l_i(\omega_i,t+1|\theta^*)) = 0$ a.s. We now perform some simplification (see [40, Lemma 9]) to show that

$$\lim_{t \to \infty} a_{ii}(t)(m_i(t,s) - l_i(s|\theta^*)) = 0 \text{ a.s. for all } s \in S_i \quad (6)$$

Therefore, if there exists a sequence of times $\{t_k\}_{k=1}^{\infty}$ with $t_k \uparrow \infty$ such that the $i$th agent’s self-confidence $a_{ii}(t)$ exceeds a fixed threshold $\gamma > 0$ at times $\{t_k\}_{k=1}^{\infty}$, then (6) implies that its one-step-ahead forecasts sampled at $\{t_k\}_{k=1}^{\infty}$ converge to the true forecasts, i.e., $\lim_{k \to \infty} m_i(t_k,s) = l_i(s|\theta^*)$ a.s.

The following lemma generalizes (6) to $h$-step-ahead forecasts. Its proof is based on the induction and elementary properties of conditional expectations (see [40] for the proof).

**Lemma 1**: For all $h \in \mathbb{N}_0$, $s_1, s_2, \ldots, s_h \in S_i$, and $i \in [n]$, we have

$$\lim_{t \to \infty} a_{ii}(t)\left(m_{i,j}^{(h)}(s_1, s_2, \ldots, s_h) - \prod_{r=1}^{h} l_i(s_r|\theta^*)\right) = 0 \text{ a.s.}$$

3) Asymptotic Behavior of the Agents’ Beliefs: As it turns out, Lemma 2, which we used above to analyze the asymptotic behavior of $u(t)$, is a useful result based on the idea of absolute probability sequences. We prove this lemma as follows.

**Lemma 2**: Let $\theta \in \Theta^*$. Then, the following limits exist and are finite: $P^*$-a.s.: $\lim_{t \to \infty} \pi^T(t)\mu_i(\theta)$, $\lim_{t \to \infty} \pi^T(t + 1)A(t)\mu_i(\theta)$, and $\lim_{t \to \infty} \pi^T(t)\log(\mu_i(\theta^*))$. As a result, $E^*\left[\pi^T(t + 2)A(t + 1)\mu_{i+1}(\theta^*) \mid B_t\right] - \pi^T(t + 1)A(t)\mu_i(\theta^*)$ approaches $0$ a.s. as $t \to \infty$.

**Proof**: Let $B^*_t := \sigma(A(0), \ldots, A(t - 1), \omega_1, \ldots, \omega_t)$ for all $t \in \mathbb{N}$. Observe that $E[\cdot | B^*_t] = E[E[\cdot | B^*_t] | B_t]$ because $B^*_t \subset B_t$ for all $t \in \mathbb{N}$. Therefore, if we take the conditional expectation $E[\cdot | B^*_t]$ on both the sides of (2), then the lower bound in Lemma 5 can be shown to imply that

$$E^*[\pi^T(t + 1)\mu_{i+1}(\theta) | B^*_t] \geq E^*[A(t)\mu_i(\theta)] \quad (7)$$

a.s. for all $t \in \mathbb{N}$ (see [40] for more details). Multiplying both the sides of (7) by $\pi^T(t + 1)$ on the left results in

$$\pi^T(t + 1)E^*[\pi^T(t + 1)\mu_{i+1}(\theta) | B^*_t] \geq \pi^T(t + 1)E^*[A(t)\mu_i(\theta)] \quad (8)$$

where the last step follows from the definition of absolute probability sequences (see Definition 7). Since $\{\pi(t)\}_{t=0}^{\infty}$ is a deterministic sequence, it follows from (8) that

$$E^*[\pi^T(t + 1)\mu_{i+1}(\theta) | B^*_t] \geq \pi^T(t)\mu_i(\theta) \quad \text{a.s.} \quad (9)$$

We have, thus, shown that $\{\pi^T(t)\mu_i(\theta)\}_{t=1}^{\infty}$ is a submartingale w.r.t. the filtration $\{B_t\}_{t=1}^{\infty}$. Since it is also a bounded nonnegative sequence (because $0 \leq \pi(t) \mu_i(\theta) \leq 1$), it follows that $\{\pi^T(t)\mu_i(\theta)\}_{t=1}^{\infty}$ is a bounded nonnegative submartingale. Hence, $\lim_{t \to \infty} \pi^T(t)\mu_i(\theta)$ exists and is finite $P^*$-a.s.

The almost-sure existence of $\lim_{t \to \infty} \pi^T(t + 1)A(t)\mu_i(\theta)$ and $\lim_{t \to \infty} \pi^T(t)\log(\mu_i(\theta^*))$ can be proved using similar submartingale arguments (see [40]). Having shown that $\lim_{t \to \infty} \pi^T(t + 1)A(t)\mu_i(\theta)$ exists a.s., we use the dominated convergence theorem for conditional expectations (see [41, Th. 5.5.9]) to prove the last assertion of the lemma (see [40]).

We now use the above observations to prove Theorems 1 and 2. We omit some minor details, which are available in [40].

B. Proof of Theorem 1

We prove each assertion of the theorem one by one.

**Proof of (i)**: If $\{A(t)\}_{t=0}^{\infty}$ has the strong feedback property, then by Lemma 1, for all $s \in S_i$, $h \in \mathbb{N}$, and $i \in [n]$, we have

$$m_{i,j}^{(h)}(s_1, s_2, \ldots, s_h) - \prod_{r=1}^{h} l_i(s_r|\theta^*) \prod_{r=1}^{\infty} 0 \text{ a.s.}$$

which proves (i).

Therefore, let us now ignore the strong feedback property and suppose that Assumption 1 holds. Let $D_k$ denote the event that $[t_{2k-1}, t_{2k}]$ is a $\gamma$-epoch. Since $\{A(t)\}_{t=0}^{\infty}$ are independent, and since $\sum_{k=1}^{\infty} \Pr(D_k) = \infty$, we know from the second Borel-Cantelli lemma that $\Pr(D_k \text{ infinitely often}) = 1$ a.s. In other words, infinitely many $\gamma$-epochs occur a.s. Therefore, for each $k \in \mathbb{N}$, suppose that the $k$th $\gamma$-epoch is the random time interval $[T_{2k-1}, T_{2k}]$. Then, Definition 13 can be used to show that for each $i \in [n]$, there a.s. exist an observationally self-sufficient
set \( \{ \sigma_1(\cdot), \ldots, \sigma_i(\cdot) \} \subset [n] \) and times \( T_{i,k} \subseteq \{ T_{2k-1}, \ldots, T_{2k} \} \) such that \((A(\tau_{i,k}(\cdot) : T_{2k-1}^i)) \sigma_i(\cdot) \geq \gamma \) and \( a(\sigma_i(\cdot), \tau_{i,k}(\cdot)) \geq \gamma \) for each \( q \in [r_i] \). Thus, for any \( q \in [r_i] \), the self-confidence of agent \( \sigma_i(\cdot) \) exceeds the threshold \( \gamma > 0 \) at certain times. Hence, by Lemma 1, we have

\[
m_{i,q}(\sigma_i(\cdot), \tau_{i,k}(\cdot))(s_1, \ldots, s_h) \to \prod_{p=1}^h I_{\sigma_i(\cdot)}(s_p|\theta^*)
\]
a.s. for all \( q \in [r_i] \) as \( k \to \infty \), which means that the forecasts of each agent in \( \{ \sigma_i(\cdot) : q \in [r_i] \} \) are asymptotically accurate along a sequence of times. Now, making accurate forecasts is possible only if agent \( \sigma_i(\cdot) \) rules out every state that induces on \( S_{\Theta}(\cdot) \) (the agent’s signal space) a conditional probability distribution other than \( I_{\sigma_i(\cdot)}(\cdot|\theta^*) \). Such states are contained in \( \Theta \cap \Theta_{\sigma_i(\cdot)}^* \). Thus, for every state \( \theta \notin \Theta_{\sigma_i(\cdot)}^* \), we have \( \mu_{\sigma_i(\cdot), \tau_{i,k}(\cdot)}(\theta) \to 0 \) a.s. as \( k \to \infty \) (alternatively, we may repeat the arguments used in [7, proof of Proposition 3] to prove that

\[
m_{i,q}(\sigma_i(\cdot), \tau_{i,k}(\cdot))(\theta) \to 0 \text{ a.s. as } k \to \infty.
\]

On the other hand, since the influence of agent \( i \) on agent \( \sigma_j(\cdot) \) over the time interval \( [T_{2k-1}, T_{2k}] \) exceeds \( \gamma \) (i.e., \( (A(\tau_{i,k}(\cdot) : T_{2k-1})) \sigma_j(\cdot) \geq \gamma \) \), it follows from Lemma 4 that \( \mu_{\sigma_j(\cdot), \tau_{i,k}(\cdot)}(\theta) \) is lower bounded by a multiple of \( \mu_{T_{2k-1}}(\theta) \) [40]. Considering the limit obtained above, this is possible only if \( \lim_{k \to \infty} \mu_{T_{2k-1}}(\theta) = 0 \) a.s. for all \( \theta \in \Theta \cap \Theta_{\sigma_i(\cdot)}^* \), and \( q \in [r_i] \), i.e., \( \lim_{k \to \infty} \mu_{T_{2k-1}}(\theta) = 0 \) a.s. for all \( \theta \in \cup_{q \in [r_i]}(\Theta \cap \Theta_{\sigma_i(\cdot)}^*) \). Since \( \{ \sigma_i(\cdot) : q \in [r_i] \} \) is an observationally self-sufficient set, it follows that \( \cup_{q \in [r_i]}(\Theta \cap \Theta_{\sigma_i(\cdot)}^*) = \Theta \cap \Theta_{\sigma_i(\cdot)}^* \) and, hence, that \( \lim_{k \to \infty} \mu_{T_{2k-1}}(\theta) = 0 \) a.s. for all \( \theta \in \Theta \cap \Theta_{\sigma_i(\cdot)}^* \). Since \( i \in [n] \) is arbitrary, this further implies that \( \lim_{k \to \infty} \mu_{T_{2k-1}}(\theta) = 0 \) a.s. for all \( \theta \notin \Theta_{\sigma_i(\cdot)}^* \). Hence, \( \lim_{k \to \infty} \sum_{\theta \in \Theta_{\sigma_i(\cdot)}^*} \mu_{T_{2k-1}}(\theta) = 1 \) a.s.

To convert the above subsequence limit to a limit of the sequence \( \{ \omega_{i,t} \}_{t=1}^\infty \) according to the true probability distribution \( I_{\omega_i(\cdot)}(\cdot|\theta^*) \). That is, agent \( i \) asymptotically rules out all those states that generate signals according to distributions that differ from the one associated with the true state. Since agent \( i \) knows that each of the remaining states generates \( \{ \omega_{i,t} \}_{t=1}^\infty \) according to \( I_{\omega_i(\cdot)}(\cdot|\theta^*) \), this implies that agent \( i \) estimates the true distributions of his/her forthcoming signals with arbitrary accuracy as \( t \to \infty \), i.e., his/her beliefs weakly merge to the truth. This is proved formally using straightforward algebraic computations (see [40]).

**Proof of (iii):** Next, we note that if Assumption I holds and \( \theta^* \) is identifiable, then

\[
\lim_{t \to \infty} \mu_T(\theta^*) = \lim_{t \to \infty} \sum_{\theta \in \Theta^*} \mu_T(\theta) = \lim_{t \to \infty} \sum_{\theta \in \Theta^*} \mu_T(\theta) = 1
\]
a.s., where the last step follows from (9). This proves (ii).

### C. Proof of Theorem 2

To begin, suppose \( \{ A(t) \}_{t=0}^\infty \) is a deterministic uniformly strongly connected chain, and let \( B \) denote the constant satisfying Condition 3 in Definition 9. Then, one can easily verify that Assumptions I and III hold (see [40, proof of Lemma 3] for a detailed verification). Moreover, \( \{ A(t) \}_{t=0}^\infty \in P^* \) [37, Lemma 5.8]. Thus, Assumptions I–VI hold (the last two of them hold trivially), implying that (9) holds, which proves that \( c_0 = 0 \) for all \( \theta \in \Theta \setminus \Theta_{\sigma_i(\cdot)}^* \). Therefore, we restrict our subsequent analysis to the states belonging to \( \Theta_{\sigma_i(\cdot)}^* \), and we let \( \theta \) denote a generic state in \( \Theta_{\sigma_i(\cdot)}^* \).

Since we aim to show that all the agents’ beliefs converge to a consensus, we first show that their beliefs attain synchronization as time goes to \( \infty \) (i.e., \( \lim_{t \to \infty} (\mu_{i,t}(\theta) - \mu_{j,t}(\theta)) = 0 \) a.s. for all \( i, j \in [n] \)) and then show that the agents’ beliefs converge to a steady state a.s. as time goes to \( \infty \).

1) **Synchronization:** To achieve synchronization asymptotically in time, the quantity \( \max_{i,j \in [n]} (\mu_{i,t}(\theta) - \mu_{j,t}(\theta)) \) is arbitrary, this further implies that (9) holds, which proves that \( c_0 = 0 \) for all \( \theta \in \Theta \setminus \Theta_{\sigma_i(\cdot)}^* \). Therefore, we restrict our subsequent analysis to the states belonging to \( \Theta_{\sigma_i(\cdot)}^* \), and we let \( \theta \) denote a generic state in \( \Theta_{\sigma_i(\cdot)}^* \).

Remarkably, the function \( V_\sigma(\cdot, \cdot) \) is comparable in magnitude with the difference function \( d(x) := \max_{i \in [n]} (x_i - \min_{i \in [n]} x_i) \). To be specific, [40, Lemma 10] shows that for each \( k \in \mathbb{N} \)

\[
\frac{(p^* / 2)^2}{d(x)} dx \leq \sqrt{V_\sigma(x, k)} \leq d(x).
\]

As a result, just like \( V_\sigma \), the difference function \( d(\cdot) \) behaves like a Lyapunov function for linear dynamics on a time-varying network described by \( \{ A(t) \}_{t=0}^\infty \). To elaborate, \( V_\sigma \) behaving like a Lyapunov function means that, for the linear dynamics \( x(k + 1) = A(k)x(k) \) with \( x(0) \in \mathbb{R}^n \) as the initial condition, there exists a constant \( \kappa \in (0, 1) \) such that

\[
V_\sigma(x((q + 1)B), (q + 1)B) \leq (1 - \kappa)^q V_\sigma(x(0), 0)
\]

for all \( q \in \mathbb{N} \) (see [37, eq. (5.18)]). As shown in [40, this inequality can be combined with (10) to obtain a similar inequality for the function \( d(\cdot) \). This inequality is

\[
d(x((q_0 + 1)B)) \leq \alpha d(x(0))
\]

where \( q_0 \in \mathbb{N} \) is sufficiently large and \( \alpha < 1 \) is a constant that depends on \( p^* \), \( \kappa \), and \( q_0 \). We write this compactly as

\[
d(x(T_0)) \leq \alpha d(x(0)),
\]

where \( T_0 := (q_0 + 1)B \). Equivalently,

\[
d(A(T_0) : x(0)) \leq \alpha d(x(0)) \text{ for all the initial conditions } x(0) \in \mathbb{R}^n.
\]

Now, given any \( r \in \mathbb{N} \), by the definition of uniform strong connectivity the truncated chain \( \{ A(t) \}_{t=r}^\infty \) is also \( B \)-strongly connected. Therefore, the above inequality can be generalized
to
\[ d(A(T_0 + rB : rB)x_0) \leq \alpha d(x_0). \] (11)

With the help of some algebra involving the row stochasticity of \( \{A(t)\}_{t \geq 0} \) [40, Lemma 11] transforms (11) into the following inequality, where \( t_1, t_2 \in \mathbb{N} \) and \( t_1 < t_2 \):
\[ d(A(t_2 : t_1)x_0) \leq \alpha^{\frac{t_2 - t_1}{\gamma_0}} - 2d(x_0). \] (12)

For the linear dynamics \( x(k + 1) = A(k)x(k) \), (12) implies that \( d(x(k)) \rightarrow 0 \) as \( k \rightarrow \infty \). Since we need a similar result for the nonlinear dynamics (2), we first recast (2) into an equation involving backward matrix products (such as \( A(t_2 : t_1) \)) and then use (12) to obtain the desired limit. The first step yields the following equation, which is straightforward to prove by induction [26]
\[ \mu_{t+1}(\theta) = A(t + 1 : 0)\mu_0(\theta) + \sum_{k=0}^{t} A(t + 1 : k + 1)\rho_k(\theta) \] (13)
where \( \rho_k(\theta) \) is the vector with entries
\[ \rho_{i,k}(\theta) := a_{ii}(k) \left( \frac{l_i(\omega_{i,k+1})[\theta]}{m_{i,k}(\omega_{i,k+1})} - 1 \right) \mu_{i,k}(\theta). \]

We now apply (\cdot) to both sides of (13) so that we can make effective use of (12). We do this as follows:
\[ d(\mu_{t+1}(\theta)) \]
\[ \leq d(A(t + 1 : 0)\mu_0(\theta)) + \sum_{k=0}^{t} d(A(t + 1 : k + 1)\rho_k(\theta)) \]
\[ \leq \alpha^{\frac{t+1}{\gamma_0}} - 2d(\mu_0(\theta)) + \sum_{k=0}^{t} \alpha^{\frac{k+1}{\gamma_0}} - 2d(\rho_k(\theta)). \] (14)

In (14), the inequality (b) follows from (12), and (a) follows from the fact that \( d(x + y) \leq d(x) + d(y) \) for all \( x, y \in \mathbb{R}^n \).

We will now show that \( \lim_{t \rightarrow \infty} d(\mu_{t+1}(\theta)) = 0 \) a.s. Observe that the first term on the right-hand side of (14) vanishes as \( t \rightarrow \infty \), since \( \alpha < 1 \). To show that the second term also vanishes, we use some arguments of [26] in the following.

Note that by Theorem 1(i), for all \( i \in [n] \) and \( \theta \in \Theta^* \), we have
\[ l_i(\omega_{i,t+1})[\theta] - m_{i,t}(\omega_{i,t}) = l_i(\omega_{i,t+1})[\theta^*] - m_{i,t}(\omega_{i,t}) \rightarrow 0 \]
\( \text{a.s. as } t \rightarrow \infty. \) It now follows from the definition of \( \rho_k(\theta) \) that \( \lim_{k \rightarrow \infty} \rho_k(\theta) = 0 \) a.s. for all \( \theta \in \Theta^* \). Thus, \( \lim_{k \rightarrow \infty} d(\rho_k(\theta)) = 0 \) a.s. for all \( \theta \in \Theta^* \).

Next, note that \( \sum_{k=0}^{t} \alpha^{\frac{k+1}{\gamma_0}} - 2d(\rho_k(\theta)) = \alpha^{\frac{t+1}{\gamma_0}} - 2d(\rho_k(\theta)) \to 0 \) as \( t \to \infty \). Since \( \lim_{k \rightarrow \infty} d(\rho_k(\theta)) = 0 \) a.s. we have \( \lim_{k \rightarrow \infty} \sum_{k=0}^{t} \alpha^{\frac{k+1}{\gamma_0}} - 2d(\rho_k(\theta)) = 0 \) a.s. by the Toeplitz lemma. Thus, (14) now implies that \( \lim_{t \rightarrow \infty} d(\mu_{t+1}(\theta)) = 0 \) a.s. for all \( \theta \in \Theta^* \), i.e., synchronization is attained as \( t \to \infty \).

2) Convergence to a Steady State: We now show that \( \lim_{t \rightarrow \infty} \mu_{t+1}(\theta) \) exists a.s. for each \( i \in [n] \) because \( \lim_{t \rightarrow \infty} \pi^T(t)\mu_i(\theta) \) exists a.s. by Lemma 2. Formally, we a.s.

\[ \lim_{t \rightarrow \infty} \mu_{i,t}(\theta) = \lim_{t \rightarrow \infty} \left( \mu_{i,t}(\theta) \sum_{j=1}^{n} \pi_j(t) \right) \]
\[ = \lim_{t \rightarrow \infty} \sum_{j=1}^{n} \pi_j(t) (\mu_{j,t}(\theta) + (\mu_{i,t}(\theta) - \mu_{j,t}(\theta))) \]
\[ = \lim_{t \rightarrow \infty} \sum_{j=1}^{n} \pi_j(t)\mu_{j,t}(\theta) = \lim_{t \rightarrow \infty} \pi^T(t)\mu_i(\theta) \]
which exists a.s. Here, (a) holds because asymptotic synchronization implies that \( \lim_{t \rightarrow \infty} (\mu_{i,t}(\theta) - \mu_{j,t}(\theta)) = 0 \) a.s.

We have, thus, shown that \( \lim_{t \rightarrow \infty} \mu_i(\theta) \) exists a.s. for all \( \theta \in \Theta^* \) and that \( \lim_{t \rightarrow \infty} |\mu_i(\theta) - \mu_j(\theta)| = 0 \) a.s. for all \( i, j \in [n] \) and \( \theta \in \Theta^* \). It follows that for each \( \theta \in \Theta^* \), \( \lim_{t \rightarrow \infty} \mu_i(\theta) = C_{\theta} \) a.s. for some scalar random variable \( C_{\theta} \). This concludes the proof of the theorem.

V. APPLICATIONS

We now establish a few useful implications of Theorem 1, some of which are either known results or their extensions. The proofs of all these implications can be found in [40].

A. Learning in the Presence of Link Failures

In the context of learning on random graphs, the following question arises naturally: Is it possible for a network of agents to learn the true state of the world when the underlying influence graph is affected by random communication link failures? For simplicity, let us assume that there exists a constant stochastic matrix \( A \) such that \( a_{ij}(t) \), which denotes the degree of influence of agent \( j \) on agent \( i \) at time \( t \), equals 0 if the link \( (j, i) \) has failed and \( A_{ij} \) otherwise. Then, if the link failures are independent across time, the following result answers the question raised.

**Corollary 1:** Let \( ([n], E) \) be a strongly connected directed graph, whose weighted adjacency matrix \( A = (A_{ij}) \) satisfies \( A_{ii} > 0 \) for all \( i \in [n] \). Consider a system of \( n \) agents satisfying the following criteria:

1) Assumption II holds.

2) The influence graph at any time \( t \in \mathbb{N} \) is given by \( G(t) = ([n], E - F(t)) \), where \( F(t) \subset E \) denotes the set of failed links at time \( t \) and \( \{F(t)\}_{t=0}^{\infty} \) are independently distributed random sets.

3) The sequences \( \{\omega_{i,t}\}_{t=0}^{\infty} \) and \( \{F(t)\}_{t=0}^{\infty} \) are independent.

4) At any time step, any link \( e \in E \) fails with a constant probability \( \rho \in (0, 1) \). However, the failure of \( e \) may or may not be independent of the failure of other links.

5) The probability that \( G(t) \) is connected at time \( t \) is at least \( \sigma > 0 \) for all \( t \in \mathbb{N}_0 \).

Then, under the update rule (1), all the agents learn the truth asymptotically a.s.

B. Inertial Non-Bayesian Learning

In real-world social networks, it is possible that some individuals affected by psychological inertia cling to their prior beliefs.
in such a way that they do not incorporate their own observations in a fully Bayesian manner. This idea is closely related to the notion of prejudiced agents that motivated the popular Friedkin–Johnsen model in [42]. To describe the belief updates of such inertial individuals, we modify the update rule (1) by replacing the Bayesian update term $BU_{i,t+1}(\theta)$ with a convex combination of $BU_{i,t+1}(\theta)$ and the $i$th agent’s previous belief $\mu_{i,t}(\theta)$, i.e.,

$$
\mu_{i,t+1}(\theta) = a_i(t)(\lambda_i(t)\mu_{i,t}(\theta) + (1 - \lambda_i(t))BU_{i,t+1}(\theta))
+ \sum_{j \in N_i(t)} a_{ij}(t)\mu_{j,t}(\theta)
$$

(15)

where $\lambda_i(t) \in [0,1]$ denotes the degree of inertia of agent $i$ at time $t$. As it turns out, Theorem 1 implies that even if all the agents are inertial, they will still learn the truth asymptotically a.s. provided the inertias are all bounded away from 1.

**Corollary 2:** Consider a network of $n$ inertial agents, whose beliefs evolve according to (15). Suppose that for each $i \in [n]$, the sequence $\{\lambda_i(t)\}_{t=0}^{\infty}$ is deterministic. Furthermore, suppose that $\lambda_{\text{max}} := \sup_{i \in [n]} \max_{t \geq 0} \lambda_i(t) < 1$ and that Assumptions II–VI hold. Then, assertions (i) and (ii) of Theorem 1 are true.

**Remark 4:** Interestingly, Corollaries 1 and 2 imply that non-Bayesian learning (both inertial and non-inertial) occurs a.s. on a sequence of independent Erdős–Rényi random graphs, provided that the edge probabilities of these graphs are uniformly bounded away from 0 and 1 (i.e., if $\rho(t)$ is the edge probability of $G(t)$, then there should exist constants $0 < \delta < \eta < 1$ such that $\delta \leq \rho(t) \leq \eta$ for all $t \in \mathbb{N}_0$). This is worth noting because a sequence of Erdős–Rényi networks is a.s. not uniformly strongly connected, which can be proved by using arguments similar to those used in Remarks 2 and 3.

**C. Learning via Diffusion and Adaptation**

Let us extend our discussion to another variant of the original update rule (1). As per this variant, known as learning via diffusion and adaptation [12], every agent combines the Bayesian updates of his/her own beliefs with the most recent Bayesian updates of his/her neighbor’s beliefs (rather than combining the Bayesian updates of his/her own beliefs with his/her neighbors’ previous beliefs). As one might guess, this modification results in faster convergence to the truth in the case of static networks, as shown empirically in [12].

For a network of $n$ agents, the time-varying analog of the update rule proposed in [12] can be stated as

$$
\mu_{i,t+1}(\theta) = \sum_{j=1}^{n} a_{ij}(t)BU_{j,t+1}(\theta)
$$

(16)

for all $i \in [n]$, $t \in \mathbb{N}_0$ and $\theta \in \Theta$. On the basis of (16), we now generalize the theoretical results of [12] and establish that diffusion adaptation a.s. leads to asymptotic learning even when the network is time varying or random, provided that it satisfies the assumptions stated earlier.

**Corollary 3:** Consider a network $\mathcal{H}$ described by the rule (16), and suppose that the sequence $\{A(t)\}_{t=1}^{\infty}$ and the agents’ initial beliefs satisfy Assumptions II–VI. Then, assertions (i) and (ii) of Theorem 1 hold.

**Remark 5:** The proof of Corollary 3 (provided in [40]) enables us to infer the following: it is possible for a network of agents following the original update rule (1) to learn the truth asymptotically a.s. despite certain agents not taking any new measurements at some of the time steps (which effectively means that their self-confidence is set to zero at those time steps). This could happen, for instance, when some of the agents intermittently lose contact with their external sources of information and, therefore, depend solely on their neighbors for updating their beliefs at the corresponding time instants. As a simple example, consider a chain $\{A(t)\}_{t=0}^{\infty} \in P^* \cap \mathbb{R}^{n \times n}$, an increasing sequence $\{\tau_k\}_{k=0}^{\infty} \in \mathbb{N}_0$ with $\tau_0 := 0$, and a chain of permutation matrices, $\{P(k)\}_{k=1}^{\infty} \subset \mathbb{R}^{n \times n}$ such that $P(k) \neq I_n$ for any $k \in \mathbb{N}$. Then, the chain $A(0), \ldots, A(\tau_1 - 1), P^T(1)A(\tau_1), P(1), A(\tau_1 + 1), \ldots, A(\tau_2 - 1), P^T(2)A(\tau_2), P(2), A(\tau_2 + 1), \ldots$ can be shown to belong to Class $P^*$ even though $P_i(t) = 0$ for some $i \in [n]$ and infinitely many $k \in \mathbb{N}$. In particular, if, in addition, $\{A(t)\}_{t=0}^{\infty}$ satisfies Assumption I and $\{\tau_k\}_{k=0}^{\infty}$ have been chosen such that $\tau_{k-1} < \tau_{2k-1} < \tau_{2k} \leq \tau_k$ for each $k \in \mathbb{N}$, then it can be shown that even the modified chain satisfies Assumption I. In this case, the assertions of Theorem 1 apply to the modified chain. Moreover, the modified chain violates condition 2 of Definition 9, and hence, it is not a uniformly strongly connected chain. The upshot is that the intermittent negligence of external information combined with the violation of standard connectivity criteria does not preclude almost-sure asymptotic learning.

**D. Learning on Deterministic Time-Varying Networks**

We now provide some corollaries of Theorem 1 that apply to deterministic time-varying networks. These corollaries are based on the following lemma:

**Lemma 3:** Let $\{A(t)\}_{t=0}^{\infty}$ be deterministic and uniformly strongly connected. Then, Assumptions I, III, and IV hold.

An immediate consequence of Lemma 3 and Theorem 1 is the following result.

**Corollary 4:** Suppose that Assumption II holds and that $\{A(t)\}_{t=0}^{\infty}$ is a deterministic $B$-connected chain. Then, all the agents’ beliefs weakly merge to the truth a.s. In addition, all the agents’ beliefs converge to a consensus a.s. If, in addition, $\theta^*$ is identifiable, then the agents asymptotically learn $\theta^*$ a.s.

Note that Corollary 4 is a generalization of the main result (Theorem 2) of [25], which imposes on $\{A(t)\}_{t=0}^{\infty}$ the additional restriction of double stochasticity.

Besides uniformly strongly connected chains, Theorem 1 also applies to balanced chains with strong feedback property, since these chains too satisfy Assumption IV.

**Corollary 5:** Suppose that Assumptions II and III hold, and that $\{A(t)\}_{t=0}^{\infty}$ is a balanced chain with strong feedback property. Then, the assertions of Theorems 1 and 2 apply.

Essentially, Corollary 5 states that if every agent’s self-confidence is always above a minimum threshold and if the total influence of any subset $S$ of agents on the complement set
$S = [n] \setminus S$ is always comparable to the total reverse influence (i.e., the total influence of $S$ on $S$), then asymptotic learning takes place a.s. under mild additional assumptions.

It is worth noting that the following established result (see [26, Th. 3.2]) is a consequence of Corollaries 4 and 5. Its proof is straightforward and can be found in [40].

**Corollary 6 (Main result of [26]):** Suppose that $\{A(t)\}_{t=0}^{\infty}$ is a deterministic stochastic chain such that $A(t) = \eta(t)A + (1 - \eta(t))I$, where $\eta(t) \in (0, 1]$ is a time-varying parameter and $A = (A_{ij})$ is a fixed stochastic matrix. Furthermore, suppose that the network is strongly connected at all times, that there exists $\gamma > 0$ such that $A_{ii} \geq \gamma$ for all $i \in [n]$ (resulting in $a_{ii}(t) > 0$ for all $i \in [n]$ and $t \in \mathbb{N}_0$), and that $\mu_{k=0}(\theta^*) > 0$ for some $j_0 \in [n]$. Then, the one-step-ahead forecasts of all the agents are eventually correct a.s. In addition, suppose $\sigma := \inf_{t \in \mathbb{N}_0} \eta(t) > 0$. Then, all the agents converge to a consensus a.s. If, in addition, $\theta^*$ is identifiable, then all the agents asymptotically learn the truth a.s.

Finally, we note through the following example that uniform strong connectivity is not necessary for almost-sure asymptotic learning on time-varying networks.

**Example 2:** Let $n = 6$, let $\{2, 3\}$ and $\{5, 6\}$ be observationally self-sufficient sets, and suppose $\mu_{1,0}(\theta^*) > 0$. Let $\{A(t)\}_{t=0}^{\infty}$ be defined by $A(0) = \frac{1}{2} \mathbf{1} \mathbf{1}^T$ and

$$A(t) = \begin{cases} A_e, & \text{if } t = 2^k2^{3k+1} \text{ for some } k \in \mathbb{N}_0 \\ A_o, & \text{if } t = 2^k + 1 \text{ for some } k \in \mathbb{N}_0 \\ I, & \text{otherwise} \end{cases}$$

where

$$A_e := \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/8 & 1/2 & 3/8 & 0 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/8 & 3/8 & 1/2 \\ 0 & 0 & 0 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$

and

$$A_o := \begin{pmatrix} 1/3 & 0 & 0 & 0 & 1/3 & 1/3 \\ 0 & 3/8 & 3/8 & 1/4 & 0 & 0 \\ 0 & 1/6 & 1/2 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/4 & 1/4 \\ 1/2 & 0 & 0 & 0 & 3/8 & 1/8 \end{pmatrix}.$$  

Then, it can be verified that $\{A(t)\}_{t=0}^{\infty}$ is a balanced chain with strong feedback property. In addition, our choice of $A(0)$ ensures that Assumption III holds with $T = 1$. Moreover, we can verify that Assumption I holds with $t_{2k-1} = 2^k$ and $t_{2k} = 2^k + 1$ for all $k \in \mathbb{N}$. Therefore, by Corollary 5, all the agents asymptotically learn the truth a.s. This happens even though $\{A(t)\}_{t=0}^{\infty}$ is not $B$-connected for any finite $B$ (which can be verified by noting that $\lim_{k \to \infty} (2^{2k+1} - 2^{2k}) = \infty$).

**Remark 6:** Note that by Definition 10, balanced chains embody a certain symmetry in the influence relationships between the agents. Hence, the above example shows that asymptotic learning can be achieved even when some network connectivity is traded for influence symmetry.

**VI. CONCLUSION**

In this article, we extended the well-known model of non-Bayesian social learning [7] to study social learning over random directed graphs satisfying connectivity criteria that are weaker than uniform strong connectivity. We showed that if the sequence of weighted adjacency matrices associated with the network belongs to Class $P^*$, implying that no agent’s social power ever falls below a fixed threshold in the average case, then the occurrence of infinitely many $\gamma$-epochs (periods of sufficient connectivity) ensures almost-sure asymptotic learning. We then showed that our main result, besides generalizing a few known results, has interesting implications for related learning scenarios such as inertial learning or learning in the presence of link failures. We also showed that our main result subsumes time-varying networks described by balanced chains, thereby suggesting that influence symmetry aids in social learning. In addition, we showed how uniform strong connectivity guarantees that all the agents’ beliefs a.s. converge to a consensus even when the true state is not identifiable. This means that although periodicity in network connectivity is not necessary for social learning, it yields long-term social agreement, which may be desirable in certain situations.

In addition to the above results, our techniques can be useful to tackle the following problems:

1) **Log-linear learning:** In the context of distributed learning in sensor networks, it is well known that under standard connectivity criteria, log-linear learning rules (in which the agents linearly aggregate the logarithms of their beliefs instead of the beliefs themselves) also achieve almost-sure asymptotic learning but exhibit greater convergence rates than the learning rule that we have analyzed [8], [11]. We, therefore, believe that one can obtain a result similar to Theorem 1 by applying our Class $P^*$ techniques to analyze log-linear learning rules.

2) **Learning on dependent random digraphs:** As there exists a definition of Class $P^*$ for dependent random chains [37], one may be able to extend the results of this article to comment on learning on dependent random graphs. Regardless of the potential challenges involved in this endeavor, our intuition suggests that recurring $\gamma$-epochs (which ensure a satisfactory level of communication and belief circulation in the network) in combination with the Class $P^*$ requirement (which ensures that every agent is influential enough to make a nonvanishing difference to others’ beliefs over time) should suffice to achieve almost-sure asymptotic learning.

In future, we would like to derive a set of connectivity criteria that are both necessary and sufficient for asymptotic non-Bayesian learning on random graphs. Yet another open problem is to study asymptotic and nonasymptotic rates of learning in terms of the number of $\gamma$-epochs occurred.
APPENDIX

TECHNICAL LEMMAS

Lemma 4: Given \( t, B \in \mathbb{N} \) and \( \Delta \in [B] \), for all \( i, j \in [n] \) and \( \theta \in \Theta \), we have

\[
\mu_{j,t+\Delta}(\theta) \geq \left( A(t + \Delta : t) \right)_{ji} \left( \frac{1}{n} \right)^B n \mu_{i,t}(\theta).
\]

Lemma 5: There exists a constant \( K_0 < \infty \) such that

\[
0 \leq \mathbb{E}^* \left[ \frac{1}{n} \mu_{i,t}(\theta) - 1 \right] B_1 \leq K_0
\]

\( \mathbb{P}^*-a.s. \) for all \( \theta \in \Theta \), \( i \in [n] \) and \( t \in \mathbb{N}_0 \). Moreover, the second inequality above holds for all \( \theta \in \Theta \).

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