TOTAL VERTEX IRREGULARITY STRENGTH OF FORESTS

MARCEL ANHOLCER, MICHAL KAROŃSKI, AND FLORIAN PFENDER

ABSTRACT. We investigate a graph parameter called the total vertex irregularity strength ($tvs(G)$), i.e. the minimal $s$ such that there is a labeling $w : E(G) \cup V(G) \rightarrow \{1, 2, \ldots, s\}$ of the edges and vertices of $G$ giving distinct weighted degrees $wt_G(v) := w(v) + \sum_{e \ni v} w(e)$ for every pair of vertices of $G$. We prove that $tvs(F) = \lceil (n_1 + 1)/2 \rceil$ for every forest $F$ with no vertices of degree 2 and no isolated vertices, where $n_1$ is the number of pendant vertices in $F$. Stronger results for trees were recently proved by Nurdin et al.

1. Introduction

Let us consider the simple undirected graph $G = (V(G), E(G))$ without loops, without isolated edges and with at most one isolated vertex. We assign a label (a natural positive number) to every edge (denoted by $w(e)$ for all $e \in E(G)$) and to every vertex (denoted by $w(v)$ for all $v \in V(G)$). We will refer to such a labeling as a total weighting of $G$. For every vertex $v \in V(G)$, we define its weighted degree as

$$wt_G(v) = \sum_{e \ni v} w(e) + w(v).$$

We call the labeling $w$ irregular if for each pair of vertices, their weighted degrees are distinct. In [3], a new graph parameter called total vertex irregularity strength ($tvs(G)$) was defined as the smallest integer $s$ such that there exists a total weighting of $G$ with integers $\{1, 2, \ldots, s\}$ that is irregular. This parameter is similar to the irregularity strength of $G$ ($s(G)$), introduced in [4] (see also [2], [5], [6] and [7]), where only weights on the edges are allowed.

In [3], several bounds and exact values of $tvs(G)$ were established for different types of graphs. In particular, the authors proved that for every graph $G$ with $n$ vertices and $m$ edges, the following bounds hold.

$$\left\lceil \frac{n + \delta(G)}{\Delta(G) + 1} \right\rceil \leq tvs(G) \leq n + \Delta(G) - 2\delta(G) + 1,$$

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where $\Delta(G)$ and $\delta(G)$ are the maximum and the minimum degree of $G$, respectively.

Recently, a much stronger upper bound on $\text{tvs}(G)$ has been established in [1]. Namely, for every graph $G$ with $\delta(G) > 0$,

$$\text{tvs}(G) \leq \left\lceil \frac{3n}{\delta} \right\rceil + 1.$$  

One should also mention that in [3], exact values of $\text{tvs}(G)$ for stars, cliques and prisms are given. Furthermore it is shown that for every tree $T$ without vertices of degree two, the following bounds hold.

$$\left\lceil \frac{n_1+1}{2} \right\rceil \leq \text{tvs}(T) \leq n_1,$$  

where $n_1$ is the number of pendant vertices of $T$.

Our main result stated below shows that the trivial lower bound in (2) is the true value of $\text{tvs}(T)$. Note the easy observation that for trees with less than 4 vertices, the equality $\text{tvs}(T) = \left\lceil \frac{n_1+1}{2} \right\rceil$ holds.

**Theorem 1.** For every forest $F$ with $n_1$ vertices of degree one and with no vertices of degree two and no isolated vertices,

$$\text{tvs}(F) = \left\lceil \frac{n_1+1}{2} \right\rceil.$$  

The proof of Theorem 1 is given in the next section.

**Remark.** Recently, stronger results for trees were proved by Nurdin et al. ([8]). However we decided to publish our paper for two reasons. Firstly, we consider more general case of forests, not only trees. Secondly, we use different proof technique.

2. PROOF OF THEOREM 1

Let us consider a forest $F$. Denote by $V_i$ the set of vertices of degree $i$, and let $n_i = |V_i|$. So $V_1$ is the set of pendant vertices and we call edges incident to pendant vertices pendant edges. Denote by $C_{ij}$ the set of its vertices of degree $i$ with exactly $j$ pendant neighbors, where $i, j \geq 0$, and let $n_{ij} = |C_{ij}|$.

Assume that in a total weighting of $F$ we use labels (weights) from the set $\{1, 2, \ldots, s\}$. Then, the lowest and the highest weighted degree of a pendant vertex $v \in V_1$ can be 2 and $2s$, respectively. Since there are $n_1$ such vertices, and each vertex has to have different total weight, the lower bound of (2) trivially follows and extends to all forests, i.e.,

$$\text{tvs}(F) \geq \left\lceil \frac{n_1+1}{2} \right\rceil.$$  

To prove Theorem 1 it is sufficient to construct an irregular labeling of $F$ using elements from the set $\{1, 2, \ldots, s\}$ only, where $s := \left\lceil \frac{n_1+1}{2} \right\rceil$.  


We will often use the well-known fact that in every tree $T$ with maximum degree $\Delta \geq 1$ the numbers $n_j(T)$ of vertices of degree $j$ satisfy the equation

$$n_1(T) = 2 + \sum_{j=3}^{\Delta} (j - 2)n_j(T).$$

(4)

Further, for forests $F$ with $n \geq 3$ and $n_2 \leq 1$, we have

$$2n_{30} + n_{43} + 2n_{44} \leq n_1 - 2,$$

with equality only for $K_{1,4}$ and $P_3$. This can be seen as follows. For the sake of contradiction, assume that $F$ is a minimal counter example to the inequality. Then $n_{43} = 0$ as otherwise we could delete a pendant vertex adjacent to a vertex of class $C_{43}$ and receive a smaller counter example. Further, $n_{44} = 0$ as otherwise we can delete a vertex of $C_{44}$ and its neighbors and either receive a smaller counterexample, or $F$ itself is a $K_{1,4}$, which is not a counterexample. Now delete all pendant vertices from $F$ to construct a forest $F'$ with $n'_{30} \geq n_{30}$ and $n' = n - n_1$ vertices. Then

$$2n_{30} \leq 2n'_{30} \leq n' - 2 = n - n_1 - 2 \leq n_1 + n_2 - 4 \leq n_1 - 3.$$

As Theorem 1 is easily verified for $K_{1,4}$ and $P_3$, we may later work with the inequality

$$2n_{30} + n_{43} + 2n_{44} \leq n_1 - 3.$$  

(5)

Label all non-pendant edges with $s$. Next, we will label the pendant edges. First, label half the isolated edges (rounded down) with 1 and the remaining isolated edges with $s$. Let now $v \in C_{kj}$ for some $k \geq 3$ and $j \geq 1$, and we will label the incident pendant edges. Order the values $\{1, \ldots, s - 1\}$ as a list $S = (1, s - 1, 2, s - 2, 3, s - 3, \ldots)$.

(i) If $j$ is even, label $j/2$ pendant edges with $s$.

(ii) If $j$ is odd, label $(j - 1)/2$ (variant 1) or $(j + 1)/2$ (variant 2) pendant edges with $s$.

(iii) The remaining pendant $2i + \delta$ ($0 \leq \delta \leq 1$) edges are labeled from $S$, where we use the first $2i$ and the last $\delta$ values in $S$, which have not previously been used on non-isolated edges.

During the process, choose variant 1 and variant 2, so that the number of pendant edges labeled $s$ is maximized but at most $s$, so there are $s - 2$ or $s - 1$ pendant edges labeled with a number less than $s$.

For notation, we write $C_{kj}^i$ for the vertices of variant $i$ in $C_{kj}$, and $n_{kj}^i$ for their number.
Note that in this labeling, regardless of the labels in \( \{1, \ldots, s\} \) we give to the vertices themselves, vertices in \( C_1 \) have total weights between 2 and 2s, and all other vertices have total weights of at least 2s + 2. Label every pendant vertex incident to a non-isolated edge labeled with a number less than s with 1. This guarantees that the total weights of all these vertices are different. The remaining vertices in \( C_1 \) can now be weighted greedily one-by-one.

For the weight range \( 2s + 2 \leq wt(v) \leq 3s \), only vertices in \( C_{31}^1 \cup C_{32} \cup C_{33} \) play a role. Vertices \( v \in C_{32}^2 \) have total weight \( 2s + w(v) \), all others have weight \( 2s + w(e) + w(v) \), where \( e \) is a pendant edge with label other than s incident to v. We can set \( w(v) = 1 \) for at most \( s - 1 - n_{33}^2 \) vertices \( v \in (C_{31}^1 \cup C_{32} \cup C_{33}^1) \), giving them all different total weights in this weight range, and greedily choose weights for the vertices in \( C_{33}^2 \) to fill out the remaining weights. Note that \( n_{33}^2 \leq \frac{n_3 n_4 + 1}{2} \leq \frac{n_1 + 3}{6} \leq \frac{s + 1}{3} \), so this is possible.

For the weight range \( 3s + 1 \leq wt(v) \leq 4s \), only vertices in

\[
C_3 \cup C_{41}^1 \cup C_{42} \cup C_{43} \cup C_{44} \cup C_{55}^1
\]

play a role. Label all vertices \( v \in C_{41}^1 \cup C_{42} \cup C_{43}^2 \cup C_{55}^1 \) with \( w(v) = s \), giving them total weight \( wt(v) = 4s + w(e) \), and thus pairwise different weights outside the weight range we currently consider. We give the remaining (at most) \( \max\{1, n_{31}^1 + n_{32} + n_{33} - s + 1\} \) vertices from \( C_{31}^1 \cup C_{32} \cup C_{33}^1 \) weight \( w(v) = s \), giving them all different total weights \( 3s + w(e) \) in this range. Vertices \( v \in C_{30} \cup C_{43}^1 \cup C_{44} \) have total weight \( 3s + w(v) \), so one can greedily fit them into the remaining weights of the range, provided there is enough room. If \( n_{31} + n_{32} + n_{33} \geq s \), then

\[
n_{30} + n_{43}^1 + n_{44} + \max\{1, n_{31}^1 + n_{32} + n_{33} - s + 1\} \\
\leq n_3 + n_4 - s + 1 \leq n_1 - s - 1 \leq s - 1.
\]

If, on the other hand, \( n_{31} + n_{32} + n_{33} \leq s - 1 \), then

\[
n_{30} + n_{43}^1 + n_{44} + \max\{1, n_{31}^1 + n_{32} + n_{33} - s + 1\} \\
= n_{30} + n_{43}^1 + n_{44} + 1 \leq \left\lfloor \frac{n_1}{2} \right\rfloor = s - 1.
\]

Thus, there is enough room to fit all the mentioned vertices.

For the weight range \( wt(v) \geq 4s + 1 \), only vertices in \( C_j^3, j \geq 4 \) play a role. By (4), there are at most s such vertices. We have dealt with vertices in \( C_4 \setminus C_{40} \) already, so all remaining vertices have \( wt(v) \geq 4s + w(e) \). Thus, we have at each vertex \( s \) choices for the total weight, which is enough to allow us to greedily pick the values \( w(v) \) to complete the irregular total weighting. \( \square \)
3. Final remarks

Note that in the proof of Theorem 1, the total weight $2s + 1$ was not used. With this observation we can prove a slight generalization.

**Theorem 1.** For every forest $F$ on $n$ vertices, with $n_0 = 0$ and $n_2 \leq 1$, we have

$$tv_s(F) = \lceil \frac{n_1 + 1}{2} \rceil.$$

**Proof.** The proof is the same as above, with one extra observations. Just note in the end, that since a vertex $v$ of degree 2 is incident to an edge of label $s$ (either a non-pendant edge or a pendant edge with this label by the construction), we can choose $w(v)$ such that $wt(v) = 2s + 1$. \hfill $\square$

As a consequence we have the following corollary.

**Corollary 3.1.** Let $T$ be a binary tree. Then $tv_s(T) = \lceil \frac{n_1 + 1}{2} \rceil$.

**References**

[1] Anholcer M., Kalkowski M., Przybyło J.: "A new upper bound for the total vertex irregularity strength of graphs". Discrete Mathematics, 309 (2009) 6316-6317.

[2] Baril J.-L., Kheddouci H., Togni O., The Irregularity Strength of Circulant Graphs. Discrete Mathematics, 304 (2005), 1-10.

[3] Baca M., Jendrol S., Miller M., Ryan J, On Irregular Total Labelings. Discrete Mathematics 307 (2007), 1378 - 1388.

[4] Chartrand G., Jacobson M.S., Lehel J., Oellermann O.R., Ruiz S., Saba F.,Irregular Networks. Congressus Numerantium 64 (1988), 187 - 192.

[5] Faudree R.J., Lehel J.,Bound on the Irregularity Strength of Regular Graphs. Colloquia Mathematica Societatis János Bolyai 52, Combinatorics, Eger (Hungary). North - Holland, Amsterdam 1987, 239 - 246.

[6] Frieze A., Gould R.J., Karoński M., Pfender F., On graph irregularity strength. Journal of Graph Theory 41, 120-137.

[7] Nierhoff T., A Tight Bound on the Irregularity Strength of Graphs. SIAM Journal on Discrete Mathematics Vol. 13 (2000), No 3, 313 - 323.

[8] Nurdin, Baskoro E.T., Salman A.N.M., Gaos N.N., On the Total Vertex Irregularity Strength of Trees, Discrete Mathematics 310 (2010), 3043-3048.
Current address, M. Anholcer: Poznań University of Economics, Faculty of Informatics and Electronic Economy, Poznań, Poland
E-mail address: m.anholcer@ue.poznan.pl

Current address, M. Karoński: Adam Mickiewicz University, Faculty of Mathematics and Computer Science, Poznań, Poland, and Emory University, Department of Mathematics and Computer Science, Atlanta, GA, USA
E-mail address: karonski@amu.edu.pl

Current address, F. Pfender: Universität Rostock, Institut für Mathematik, Rostock, Germany
E-mail address: Florian.Pfender@uni-rostock.de