A novel observation in the BRST approach to a free spinning relativistic particle

S. Krishna\(^{(a)}\), D. Shukla\(^{(a)}\), R. P. Malik\(^{(a,b)}\)

\(^{(a)}\) Physics Department, Centre of Advanced Studies, Banaras Hindu University, Varanasi - 221 005, (U.P.), India
and

\(^{(b)}\) DST Centre for Interdisciplinary Mathematical Sciences, Faculty of Science, Banaras Hindu University, Varanasi - 221 005, India
e-mails: skrishna.bhu@gmail.com; dheerajkumarshukla@gmail.com; malik@bhu.ac.in

Abstract: For the newly proposed coupled (but equivalent) Lagrangians for the supersymmetric (SUSY) system of a one (0 + 1)-dimensional spinning relativistic particle, we derive the Noether conserved charges corresponding to its (super)gauge, Becchi-Rouet-Stora-Tyutin (BRST), anti-BRST and ghost-scale symmetry transformations. We deduce the underlying algebra amongst the continuous symmetry operators and corresponding conserved charges. We point out a novel observation that emerges, for this specific SUSY system, when we discuss it within the framework of BRST formalism. In particular, the requirement of the physicality criteria with the (anti-)BRST charges leads to a completely new observation because, as it turns out, one of the primary constraints does not annihilate the physical state of the theory. We have never come across this kind of result in the realm of BRST approach to usual gauge theories. We provide the physical and theoretical reasons behind the above observation.

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1 Introduction

The model of a spinning relativistic particle (SRP) presents a prototype example of a one (0 + 1)-dimensional (1D) supersymmetric (SUSY) system (embedded in a D-dimensional Minkowskian target spacetime supermanifold) which is characterized by the bosonic variables $x^\mu(\tau)(\mu = 0, 1, 2, \ldots, D - 1)$ and its fermionic (i.e. SUSY) partners $\psi^\mu(\tau)(\mu = 0, 1, 2, \ldots, D - 1)$ where $\tau$ is a monotonically increasing evolution parameter that describes the super world-line traced out by the motion of the SPR. This model is interesting as it respects the (super)gauge symmetries as well as the reparametrization symmetry in a very elegant manner. As a consequence, it provides a toy model for the SUSY gauge theories as well as supergravity theories. Its generalization leads to the emergence of reparametrization invariant models for the (super)strings, too (see, e.g. [1, 2] for details).

The model of SRP has been studied from many angles (see, e.g. [3-6]). The BRST analysis of this model has also been performed in [6] by exploiting its (super)gauge symmetries. However, only its BRST symmetries have been discussed and its proper anti-BRST transformations have been left untouched. In a very recent paper [7], we have exploited the theoretical arsenal of the augmented version of superfield formalism [8-11] to obtain a consistent set of (anti-)BRST transformations. The existence of anti-BRST symmetry is important as it has its mathematical origin in the concept of gerbes [12,13]. The full set of proper (i.e. off-shell nilpotent and absolutely anticommuting) (anti-)BRST symmetries are shown to be respected by a coupled (but equivalent) set of Lagrangians for our present theory (see, e.g. [7] for details).

The purpose of our present investigation is to discuss various continuous symmetries of the above coupled Lagrangians and derive corresponding conserved charges. We also discuss the underlying algebra amongst the symmetry operators and their corresponding charges in an explicit fashion. We demonstrate some novel features in the context of physicality condition for this model which have not been observed, hitherto, in the context of application of BRST formalism to any other gauge/reparametrization invariant theories. It turns out that the requirements of the annihilation of the physical states by the conserved (anti-)BRST charges do not produce the annihilation of these physical states by one of the primary constraints of the theory. This is a completely new observation in the realm of application of BRST formalism to a physical system with constraints.

One of the other novel observations for this model is the existence of Curci-Ferrari (CF)-type restriction which is the root cause behind the existence of coupled (but equivalent) Lagrangians as well as the absolute anticommutativity of the (anti-)BRST symmetries (and corresponding conserved charges). This restriction has been derived by exploiting the basic tenets of superfield approach [8-11] to BRST formalism. However, this restriction also emerges as an off-shoot of the equations of motion from the coupled (but equivalent) Lagrangians for our present system (cf. (8) below).

The central motivating factors behind our present investigation are as follows. First and foremost, we observe that the physicality criteria with conserved and nilpotent (anti-)BRST charges do not select out the physical state that is annihilated by one of the primary constrains of the theory. Second, within the framework of BRST formalism, for the first-time, we observe that the above specific primary constraint is expressed in terms of the (anti-)ghost variables of the theory which is totally different from our experience in
usual gauge theories. Third, our present attempt would, perhaps, provide some insights into application of BRST formalism to supergravity theories. Finally, our present investigation is our modest first-step towards our main goal of applying the BRST formalism to supersymmetric p-form ($p = 1, 2, 3, \ldots$) (non-)Abelian gauge theories.

Our present paper is organized as follows. In our Sec. 2, we recapitulate the bare essentials of (super) gauge symmetries and their generalizations to the (anti-)BRST symmetry transformations in the Lagrangian formulation where a coupled (but equivalent) set of the latter exist. Our Sec. 3 is devoted to a brief discussion of the constraints of our present SUSY system of SRP and generators constructed from them. In our Sec. 4, we dwell on the derivation of BRST charge and establish its superiority over the generator of our previous section. Our Sec. 5 deals with the derivation of anti-BRST charge. The ghost charge and the complete set of BRST algebra are deduced in our Sec. 6. Finally, we make some concluding remarks in our Sec. 7 where we compare and contrast the importance of (anti-)BRST charges and the generators (constructed from the first-class constraints of our present theory).

2 Preliminaries: (Super)gauge and (anti-)BRST symmetries in the Lagrangian formalism

We begin with the following first-order Lagrangian ($L_f$) for a one (0 + 1)-dimensional supersymmetric system of a massive spinning relativistic particle [6]

\[ L_f = p_\mu \dot{x}^\mu - \frac{e}{2} (p^2 - m^2) + \frac{i}{2} (\psi_\mu \dot{\psi}^\mu - \psi_5 \dot{\psi}_5) + i \chi (p_\mu \dot{\psi}^\mu - m \psi_5), \]

where $\dot{x}^\mu = (dx^\mu/d\tau)$ and $\dot{\psi}^\mu = (d\psi^\mu/d\tau)$ are the generalized “velocities” for the target space bosonic variable $x^\mu$ and its fermionic $(\psi_\mu \psi_\nu + \psi_\nu \psi_\mu = 0, \ \psi_\mu^2 = 0)$ counterpart $\dot{\psi}^\mu$ where $\tau$ is the parameter that describes the evolution of our present SUSY system. The canonical momenta $\Pi_\epsilon = (\partial L_f/\partial \dot{\epsilon}) \approx 0$ and $\Pi_\chi = (\partial L_f/\partial \dot{\chi}) \approx 0$ are the primary constraints on the theory and the secondary constraints $(p^2 - m^2) \approx 0$ and $(p_\mu \psi^\mu - m \psi_5) \approx 0$ emerge out from the equations of motion corresponding to the Lagrange multiplier bosonic and fermionic variables $\epsilon(\tau)$ and $\chi(\tau)$, respectively. These variables are the analogues of the vierbein and Rarita-Schwinger fields of the 4D supergravity theories. For our present SUSY theory, the above multiplier variables are the gauge- and super-gauge variables, too. All the fermionic variables (e.g. $\psi_\mu, \chi$) anticommute $(\psi_\mu \psi_\nu + \psi_\nu \psi_\mu = 0, \ \psi_\mu \chi + \chi \psi_\mu = 0, \ \psi_\mu^2 = 0, \ \chi^2 = 0)$ among themselves and they commute (i.e. $\psi_\mu x_\nu - x_\nu \psi_\mu = 0, \ \psi_\mu e - e \psi_\mu = 0, \ \chi x_\mu - x_\mu \chi = 0$, etc.) with all the bosonic variables of our present SUSY theory. In addition to the above variables, a fermionic $(\psi_5 \psi_\mu + \psi_\mu \psi_5 = 0, \ \psi_5 \chi + \chi \psi_5 = 0, \ etc.)$ variable $\psi_5(\tau)$ has been introduced to take care of the mass $m$ of the SUSY particle. The variable $\psi_5(\tau)$ satisfies: $\{\psi_5, \psi_5\} = 1$ as well as $\psi_5 x_\mu - x_\mu \psi_5 = 0, \ \psi_5 e - e \psi_5 = 0$, etc.

The above Lagrangian is endowed with the following local, continuous and infinitesimal (super)gauge symmetry transformations ($\delta_{sg}, \delta_g$) (see, e.g. [6]):

\[
\begin{align*}
\delta_{sg} x_\mu &= \kappa \psi_\mu, & \delta_{sg} p_\mu &= 0, & \delta_{sg} \psi_\mu &= i \kappa p_\mu, & \delta_{sg} \chi &= i \kappa, \\
\delta_{sg} \epsilon &= 2 \kappa \chi, & \delta_{sg} \psi_5 &= i \kappa m, & \delta_g x_\mu &= \xi p_\mu, & \delta_g p_\mu &= 0,
\end{align*}
\]
where \( \kappa \) and \( \xi \) are the (super)gauge infinitesimal parameters which are fermionic (i.e. \( \kappa^2 = 0, \kappa \psi_\mu + \psi_\mu \kappa = 0, \kappa \chi + \chi \kappa = 0, \) etc.) and bosonic (i.e. \( \xi^2 \neq 0, \xi \psi_\mu - \psi_\mu \xi = 0, \xi \chi - \chi \xi = 0, \) etc.) in nature, respectively. The generators (i.e. conserved charges) of the above classical continuous transformations can be calculated in a straightforward manner. These are *

\[
\begin{align*}
Q_{sg} &= k \left( p \cdot \psi - m \psi_5 \right), \\
Q_g &= \frac{\xi}{2} \left( p^2 - m^2 \right),
\end{align*}
\]

where the celebrated Noether’s theorem has been exploited in its full blaze of glory. The above charges are the generators of transformations (2) as can be checked by using the standard formula: \( \delta \chi \phi = \pm i \left[ \phi, Q_\lambda \right] \pm, (\lambda = sg, g) \) where the \((\pm)\) signs, as the subscripts on the square bracket, correspond to the (anti)commutator for the generic variable \( \phi \) being (fermionic) bosonic in nature (and belonging to the Lagrangian (1)).

At this juncture, there are a few remarks in order. First, we note that the conserved charges in (3) are unable to generate the symmetry transformations for the variables \( \epsilon(\tau) \) and \( \chi(\tau) \). Second, we point out that there is a reparametrization invariance in the theory because \( L_f \) remains invariant under: \( \delta_r x_\mu = \epsilon \dot{x}_\mu, \delta_r \psi_\mu = \epsilon \dot{\psi}_\mu, \delta_r \psi_\mu = \epsilon \dot{\psi}_\mu, \delta_r \chi = \frac{d}{d\tau}(\epsilon \chi), \delta_r e = \frac{d}{d\tau}(\epsilon e) \), where \( \epsilon \) is the infinitesimal transformation parameter in: \( \tau \rightarrow \tau' = \tau - \epsilon(\tau) \). However, it can be shown that this transformation is equivalent to the gauge transformation in specific limits [6]. Third, it can be shown that the commutator of two supergauge transformations is equivalent to a reparametrization transformation when the equations of motion, emerging from (1), are used (see, e.g. [3-5] for details).

In a very recent work [7], we have generalized the “classical” Lagrangian (1) and corresponding “classical” symmetries to the “quantum” level within the framework of BRST formalism. The coupled (but equivalent) Lagrangians [that are consistent generalizations the Lagrangian (1)] are (anti-)BRST invariant. These Lagrangians, in explicit form, are as follows [7]

\[
\begin{align*}
L_b &= L_f - b \dot{\bar{\beta}} + b (\dot{\bar{\beta}} + 2 \bar{\beta} \beta) + i \dot{\bar{c}} (\dot{\bar{c}} + 2 \bar{\beta} \chi) + 2 i \beta \bar{c} \dot{\chi} \\
&\quad + 2 i \beta \bar{c} \chi - 2 e (\gamma \chi - \beta \dot{\beta}) + 2 \beta \gamma \bar{c} + \bar{\beta}^2 \beta^2 + 2 \bar{\beta} c \gamma, \\
L_b &= L_f + b \dot{\bar{\beta}} + b (\dot{\bar{\beta}} + 2 \bar{\beta} \beta) - i \dot{\bar{c}} (\dot{\bar{c}} + 2 \bar{\beta} \chi) - 2 i \beta \bar{c} \dot{\chi} \\
&\quad - 2 i \bar{\beta} c \chi - 2 e (\gamma \chi + \bar{\beta} \beta) + 2 \beta \gamma \bar{c} + \bar{\beta}^2 \beta^2 + 2 \bar{\beta} c \gamma,
\end{align*}
\]

where \( b \) and \( \bar{b} \) are the Nakanishi-Lautrup type auxiliary variables, \( (\bar{\beta})\beta \) are the bosonic (anti-)ghost variables in addition to the fermionic \( (c^2 = \bar{c}^2 = 0, \ c \bar{c} + \bar{c} c = 0) \) (anti-)ghost variables \( (\bar{c})c \). We note that the (anti-)ghost variables \( (\bar{c})c \) are the generalizations of the bosonic gauge parameter \( \xi \) and the (anti-)ghost variables \( (\bar{\beta})\beta \) are needed for the supergauge parameter \( \kappa \) [in transformations (2)] for the accurate derivation of the proper

*In the standard application of the Noether theorem, the (super)gauge parameters do not appear in the computation of the conserved charge. However, we have kept these parameters so that the connection between the generators (i.e. conserved charges) and the first-class constraints could be made clearly (in view of the seminal work done in Ref. [14]).
(anti-)BRST invariant Lagrangians (4) and (5). We also require a fermionic \((\gamma^2 = 0, \gamma \chi + \chi \gamma = 0, \gamma \psi^{\mu} + \psi_{\mu} \gamma = 0, \text{etc.})\) auxiliary variable \(\gamma\) in the theory for the complete analysis of the above Lagrangians within the framework of BRST formalism.

The above classical infinitesimal transformations (2) can be generalized to the proper (i.e. off-shell nilpotent and absolutely anticommuting) (anti-)BRST symmetry transformations at the quantum level. The fermionic “quantum” (anti-)BRST symmetry transformations, corresponding to the combined “classical” \((\delta_g + \delta_{sg})\) transformations (2), are as follows (see, e.g. [7] for details)

\[
\begin{align*}
\text{for } a \\ s_{ab} x_{\mu} &= \bar{c} p_{\mu} + \beta \psi_{\mu}, \\
\text{for } b \\ s_{ab} e &= \dot{c} + 2 \beta \dot{\chi}, \\
&= i \beta \bar{p}_{\mu}, \\
\text{for } c \\ s_{ab} \bar{c} &= -i \beta^2, \\
&= i \bar{b}, \\
\text{for } \gamma \\ s_{ab} \bar{c} &= i \beta \bar{b}, \\
&= i \beta \gamma, \\
\text{for } b \\ s_{ab} \gamma &= 0, \\
&= 0, \\
\text{for } \chi \\ s_{ab} \beta &= -i \gamma, \\
&= i \beta, \\
\text{for } \psi^{\mu} \\ s_{ab} p_{\mu} &= 0, \\
\end{align*}
\]

\[
\begin{align*}
\text{for } a \\ s_{ab} x_{\mu} &= c p_{\mu} + \beta \psi_{\mu}, \\
\text{for } b \\ s_{ab} e &= \dot{c} + 2 \beta \dot{\chi}, \\
&= i \beta p_{\mu}, \\
\text{for } c \\ s_{ab} \bar{c} &= -i \beta^2, \\
&= i b, \\
\text{for } \gamma \\ s_{ab} \bar{c} &= i \beta, \\
&= i \beta \gamma, \\
\text{for } b \\ s_{ab} \gamma &= 0, \\
&= 0, \\
\text{for } \chi \\ s_{ab} \beta &= i \beta, \\
&= -2 i \beta \gamma, \\
\text{for } \psi^{\mu} \\ s_{ab} p_{\mu} &= 0, \\
\end{align*}
\]

\[
\begin{align*}
\text{for } a \\ s_{ab} x_{\mu} &= \bar{c} p_{\mu} + \beta \psi^{\mu}, \\
\text{for } b \\ s_{ab} e &= \dot{c} + 2 \beta \dot{\chi}, \\
&= i \beta p_{\mu}, \\
\text{for } c \\ s_{ab} \bar{c} &= -i \beta^2, \\
&= i b, \\
\text{for } \gamma \\ s_{ab} \bar{c} &= i \beta, \\
&= i \beta \gamma, \\
\text{for } b \\ s_{ab} \gamma &= 0, \\
&= 0, \\
\text{for } \chi \\ s_{ab} \beta &= -i \gamma, \\
&= -2 i \beta \gamma, \\
\text{for } \psi^{5} \\ s_{ab} \psi^{5} &= i \beta m.
\end{align*}
\]

We lay emphasis on the fact that, only for the combined “classical” \((\delta_g + \delta_{sg})\) transformations, the off-shell nilpotent “quantum” BRST symmetries exist [6]. We note that the above transformations are off-shell nilpotent (i.e. \(s^2 = 0\)) and their absolute anticommutativity \((s_s s_a + s_a s_s = 0)\) is guaranteed only when the following Curci-Ferrari (CF) type restriction, emerging from the superfield formalism [7], namely;

\[
b + \bar{b} + 2 \beta \bar{\beta} = 0,
\]

is satisfied. For instance, it can be checked explicitly that \(\{s_s, s_a\} x_{\mu} = 0\) and \(\{s_s, s_a\} e = 0\) are true if and only if CF-type restriction (8) is obeyed. Furthermore, this restriction is found to be (anti-)BRST invariant (i.e. \(s_{(a)b} b + \bar{b} + 2 \beta \bar{\beta} = 0\)) and it can (besides superfield formalism [7]) be also obtained from the equations of motion: \(b = -\dot{e}/2 - \beta \bar{\beta}, \bar{b} = \dot{e}/2 - \beta \beta\) that emerge from the coupled (but equivalent) Lagrangians (4) and (5).

3 Constraint analysis: A brief sketch

As pointed out earlier, the starting Lagrangian (1) is endowed with the first-class constraints: \(\Pi_e \approx 0, \Pi_{\chi} \approx 0, (p^2 - m^2) \approx 0, (p \cdot \psi - m \psi^5) \approx 0\) where we have used the notation \(p \cdot \psi = p_{\mu} \psi^\mu\) and the standard notation of “\(\approx\)” as weakly zero in the language of Dirac’s prescription for the constraint analysis. These constraints are responsible for the (super)gauge transformations (2). However, as pointed out earlier, the conserved Noether charges (3) are unable to generate the (super)gauge transformations for \(\chi\) and \(e\) (which is a drawback in the Noether theorem for continuous symmetries for our present SUSY system).
One can write the generators of transformations (2), in the language of the first-class constraints†, as: (see, e.g. [14] for the detail discussions)

\[ G^{(sg)} = \Pi_e (2 \kappa \chi) - i \Pi_e \kappa + \kappa (p \cdot \psi - m \psi_5), \quad G^{(g)} = \Pi_e \dot{\xi} + \frac{\xi}{2} (p^2 - m^2). \]

(9)

It can be readily checked that, for the generic variables \( \phi \), we have the following:

\[ \delta \chi \phi = \pm i [\phi, G^{(\lambda)}]_{(\pm)}, \quad \lambda = g, sg, \]

(10)

where (±) signs, as the subscripts on the square bracket, correspond to (anti)commutator for \( \phi \) being (fermionic) bosonic in nature. The (±) signs, in front of the square bracket, are chosen judiciously (see, e.g. [15]). In the explicit computations of (10), the following basic canonical (anti)commutators have to be exploited:

\[ [e, \Pi_e] = i, \quad \{\psi_5, \psi_5\} = 1, \quad \{\chi, \Pi_e\} = i, \quad \{\psi_\mu, \psi_\nu\} = -\eta_{\mu\nu}, \]

(11)

which emerge from the starting Lagrangian (1) (for \( \hbar = 1 \)). The rest of the (anti)commutators are zero for all the variables of our present SUSY system.

With the help of the beautiful relations in (9) and (10), we are able to be consistent with the Dirac’s prescription for the quantization of system with constraints. For instance, at this stage, we are able to prove the time-evolution invariance of the first-class constraints of the theory. In other words, the conditions \( G^{(\lambda)} |_{phys} = 0 \), (where \( |phys> \) are the physical states of the theory), is able to lead to the following [cf. (9)].

\[ \dot{\Pi}_e |_{phys} = 0 \Rightarrow -\frac{1}{2} (p^2 - m^2) |_{phys} = 0, \]
\[ \dot{\Pi}_\chi |_{phys} = 0 \Rightarrow i(p \cdot \psi - m\psi_5) |_{phys} = 0, \]
\[ \Pi_e |_{phys} = 0, \quad \Pi_\chi |_{phys} = 0, \]

(12)

in a clear and consistent manner at the same time. This can be shown by using the equations of motion, derived from Lagrangian (1), which imply the following

\[ \frac{d}{d\tau} \left( \frac{\partial L_f}{\partial \dot{\chi}} \right) = \frac{\partial L_f}{\partial \chi} \Rightarrow \dot{\Pi}_\chi = \frac{\partial L_f}{\partial \dot{\chi}} = i(p \cdot \psi - m\psi_5), \]
\[ \frac{d}{d\tau} \left( \frac{\partial L_f}{\partial \dot{e}} \right) = \frac{\partial L_f}{\partial e} \Rightarrow \dot{\Pi}_e = \frac{\partial L_f}{\partial \dot{e}} = -\frac{1}{2} (p^2 - m^2). \]

(13)

Ultimately, we note that the generators \( G^{(\lambda)}(\lambda = g, sg) \) (which are written in an ad-hoc fashion) are able to provide a consistent set of conditions on the physical states which are in total agreement with Dirac’s prescription for the quantization of system with constraints. However, a question still remains to be answered. We have to provide a theoretical basis for the derivation of generators (9) together by exploiting the basic principles of theoretical physics. This is what precisely we try to do in our next section by exploiting the BRST formalism.

†It will be noted that the derivation of the expression for generator in Ref. [14] is not for a SUSY system. For instance, the origin of the factor \( (2\kappa \chi) \), associated with the primary constraints \( \Pi_e \) in \( G^{(sg)} \), is not discussed with any theoretical backing.
4  Conserved BRST charge: As a generator

We can address the above question within the framework of BRST formalism. To elaborate it, we note that the Lagrangians (4) and (5) transform as follows:

\[
\begin{align*}
s_b L_b &= \frac{d}{d\tau} \left[ \frac{1}{2} c (p^2 + m^2) + \frac{1}{2} \beta (p \cdot \psi + m \psi_5) + b (\dot{c} + 2 \beta \chi) \\
&\quad + 2 \gamma c \chi - 2 \beta \beta^2 \chi - 2 \beta \dot{\beta} \dot{c},
\right]
\end{align*}
\]

\[
\begin{align*}
s_b L_\bar{b} &= \frac{d}{d\tau} \left[ \frac{1}{2} c (p^2 + m^2) + \frac{1}{2} \beta (p \cdot \psi + m \psi_5) + 2 i \eta \beta \gamma \\
&\quad - 2 b \beta \chi - \bar{b} (\dot{c} + 2 \beta \chi) + 2 \beta \dot{\bar{\beta}} \bar{c}
\right]
\end{align*}
\]

(14)

under the BRST symmetry transformations (7). Thus, it is crystal clear that \( L_b \) respects a perfect BRST symmetry but, under the very same BRST transformations, \( L_\bar{b} \) changes to a total derivative plus terms that are zero on the hyper super world-line where the CF-type restriction (8) is valid. In other words, second and third terms of (14) vanish due to the validity of (8) in the infinitesimal variation (i.e. \( s_b L_\bar{b} \)) of \( L_\bar{b} \).

Exploiting the standard techniques of Noether’s theorem in the context of action principle, it can be seen that the following BRST charge:

\[
Q_b = \frac{1}{2} c (p^2 - m^2) + \beta (p \cdot \psi - m \psi_5) + b (\dot{c} + 2 \beta \chi) + \beta^2 (\ddot{c} + 2 \beta \chi),
\]

(15)

is conserved (i.e. \( \dot{Q}_b = 0 \)) when we use the following equations of motion

\[
\begin{align*}
\dot{\psi}_\mu &= \chi p_\mu, \\
\dot{p}_\mu &= 0, \\
\dot{x}_\mu &= e p_\mu - i \chi \psi_\mu, \\
\dot{\psi}_5 &= \chi m,
\end{align*}
\]

\[
\begin{align*}
\ddot{b} + 2 (\gamma \chi + \beta \dot{\beta}) + \frac{1}{2} (p^2 - m^2) = 0, \\
b \beta + i \dot{c} \chi - e \dot{\beta} + \beta \beta^2 + c \gamma = 0,
\end{align*}
\]

\[
\begin{align*}
\ddot{c} + 2 \beta \chi + 2 \beta \dot{\chi} + 2 i \beta \gamma = 0, \\
b \beta + \dot{\beta} + e \beta + \beta \beta^2 + \gamma \ddot{c} + i \chi \dot{c} = 0,
\end{align*}
\]

\[
\begin{align*}
\ddot{\bar{c}} + 2 \dot{\beta} \chi + 2 \dot{\beta} \dot{\chi} + 2 i \dot{\beta} \gamma = 0, \\
\dot{\bar{c}} \beta + e \chi + \beta \dot{c} = 0,
\end{align*}
\]

\[
\begin{align*}
(p \cdot \psi - m \psi_5) + 2 \beta \dot{\bar{c}} - 2 \beta \dot{\bar{c}} - 2 i e \gamma = 0, \\
\dot{\bar{c}} \frac{\dot{\bar{c}}}{2} = - (b + \beta \bar{\beta}),
\end{align*}
\]

(16)

that emerge out from the Lagrangian \( L_b \). We note that the BRST charge \( Q_b \) is superior to the generator \( G^{(\lambda)} \) [cf. (9)] because (i) it generates all the transformations of equation (7), (ii) it produces all the constraints (and their time-evolution invariance) through the physicality criteria where we demand that the true physical states are those that are annihilated by the BRST charge \( Q_b \), and (iii) there is a fundamental principle (i.e. Noether’s theorem) involved in its precise derivation.

To elaborate on the above statements, it can be explicitly checked that the analogue of equation (10), with the BRST charge (15), produces all the transformations (7) for the dynamical variables. The transformations for the auxiliary variables are obtained by the requirements of nilpotency and anticommutativity with anti-BRST symmetries (which we
discuss in the next section). Finally, the physicality requirement with the conserved and nilpotent BRST charge $Q_b$, namely;

$$Q_b |_{phys} = 0,$$

produces all the appropriate constraints (and their time-evolution invariance) on the physical states. Thus, we note that $Q_b$ is superior to the generators $G^{(\lambda)}$ [written in equation (9)], as we derive $Q_b$ by using the Noether’s theorem which is a fundamental principle. We discuss the constraint analysis, in more detail, in the next section. We close this section with the remark that (17) does not lead to $\Pi_\chi |_{phys} = 0$. In this sense, the generators $G^{(\lambda)}$ are also, in some sense, superior to $Q_b$.

5 Anti-BRST charge: Physicality criteria

We focus now on the Lagrangians $\bar{L}_b$ and $L_b$ and discuss their anti-BRST symmetry invariance under the transformations (6). In fact, it can be readily checked [7] that we have the following transformations for the Lagrangian $L_b$ and $\bar{L}_b$, namely;

$$s_{ab} L_b = \frac{d}{d\tau} \left[ \frac{1}{2} \bar{c} (p^2 + m^2) + \frac{1}{2} \bar{\beta} (p \cdot \psi + m \psi_5) - \bar{b} (\dot{\bar{c}} + 2 \bar{\beta} \chi) + 2 \gamma \bar{c} \chi + 2 \beta \bar{\beta} \gamma + 2 \bar{b} \bar{\beta} \chi \right],$$

$$s_{ab} \bar{L}_b = \frac{d}{d\tau} \left[ \frac{1}{2} \bar{c} (p^2 + m^2) + \frac{1}{2} \bar{\beta} (p \cdot \psi + m \psi_5) + 2 i e \bar{\beta} \gamma + 2 \bar{b} \bar{\beta} \chi + b (\dot{\bar{c}} + 2 \bar{\beta} \chi) - 2 \bar{b} \bar{\beta} \bar{c} \right] + (2 i \bar{\beta} \gamma) (b + \bar{b} + 2 \beta \bar{\beta}).$$

Thus, we note that $\bar{L}_b$ has a perfect symmetry under the nilpotent transformations $s_{ab}$ because it transforms to a total time derivative. On the contrary, under the very same symmetry transformations $s_{ab}$, the Lagrangian $L_b$ transforms to a total time derivative plus terms that are zero on the constrained hyper super world-line where the CF-type restriction $b + \bar{b} + 2 \beta \bar{\beta} = 0$ is satisfied. In other words, second and third terms, in the variation $s_{ab} L_b$, would vanish due to the validity of (anti-)BRST invariant CF-type restriction (8).

According to Noether’s theorem, the invariance of action $S = \int d\tau L_b$, under continuous and nilpotent symmetry transformations $s_{ab}$, leads to the derivation of nilpotent (i.e. $Q_{ab}^2 = 0$) and conserved (i.e. $\dot{Q}_{ab} = 0$) charge $Q_{ab}$, in an accurate fashion, as follows

$$Q_{ab} = \frac{1}{2} \bar{c} (p^2 - m^2) + \bar{\beta} (p \cdot \psi - m \psi_5) - \bar{b} (\dot{\bar{c}} + 2 \bar{\beta} \chi) - \bar{\beta}^2 (\dot{\bar{c}} + 2 \beta \chi).$$

The conservation law (i.e. $\dot{Q}_{ab} = 0$) of this charge $Q_{ab}$ can be proven by exploiting the following Euler-Lagrange equations of motion

$$\dot{x}_\mu = e p_\mu - i \chi \psi_\mu, \quad \dot{\psi}_\mu = \chi p_\mu, \quad \dot{b} = 2 (\bar{\gamma} \chi - \bar{\beta} \bar{\beta}) + \frac{1}{2} (p^2 - m^2),$$

$$\bar{b} \bar{\beta} - i \bar{\epsilon} \chi + e \bar{\beta} + \beta \bar{\beta}^2 + \gamma \bar{c} = 0,$$

$$e \chi - \beta \bar{c} + \bar{\beta} \bar{c} = 0,$$

$$\dot{p}_\mu = 0,$$
that emerge out from $L_b$.

Exploiting the generator equation (10) appropriately, it is straightforward to check that the conserved charge $Q_{ab}$ is the generator of the transformations $s_{ab}$ [cf. (6)]. It is worthwhile to point out that $Q_{ab}$ generates transformations $s_{ab}$ only for the dynamical variables of the theory. Such transformations for the auxiliary variables (e.g. $b, \bar{b}, \gamma$, etc.) are obtained by the requirements of nilpotency, absolute anticommutativity and (anti-)BRST invariance of the CF-type restriction (8). Furthermore, the physicality criteria (17) with the anti-BRST charge $Q_{ab}$ leads to the annihilation of all the physical state $|\text{phys}\rangle$ by the operator forms of the first-class constraints $\Pi_c \approx 0$, $(p^2 - m^2) \approx 0$, $(p \cdot \bar{\psi} - m\bar{\psi}) \approx 0$. This observation is consistent with the requirements of Dirac’s prescription for quantization of physical systems with any arbitrary kinds of constraints.

To corroborate the above statements, it is essential to use the explicit expressions for $Q_{ab}$ and equations of motion (20) that have been derived from $L_b$. In fact, the physicality criteria $Q_{ab}|\text{phys}\rangle = 0$ implies the following

$$ (p^2 - m^2)|\text{phys}\rangle = 0 \Rightarrow \Pi_c |\text{phys}\rangle = 0,$$

$$ (p \cdot \bar{\psi} - m\bar{\psi})|\text{phys}\rangle = 0 \Rightarrow \Pi_\bar{c} |\text{phys}\rangle = 0,$$

$$ \bar{b}|\text{phys}\rangle = 0 \Rightarrow \Pi_e |\text{phys}\rangle = 0,$$

(21)

We note that $\Pi_\bar{c}|\text{phys}\rangle = 0$ is not produced by the physicality criteria $Q_{(a)b}|\text{phys}\rangle = 0$ with the conserved and nilpotent (anti-)BRST charges $Q_{(a)b}$. We elaborate on this new feature as well as compare and contrast the importance of $G^{(\lambda)}$ and $Q_{(a)b}$ in our “conclusions” section (see, Sec. 7 below). Thus, we clearly note that the (anti-)BRST charges $Q_{(a)b}$ are theoretically more appealing than the straightforward calculations of generators $G^{(\lambda)}$ ($\lambda = g, sg$) [cf. (9)] as there is a consistent theoretical basis for their derivations.

6 Ghost symmetry: Conserved ghost charge and BRST algebra

It can be readily checked that if the (anti-)ghost variables ($\bar{c})c$ and ($\bar{\beta})\beta$ undergo the following scale transformations [with $\Omega$ as a global (i.e. spacetime independent) scale parameter] :

$$ c \rightarrow e^{+\Omega} c, \quad \bar{c} \rightarrow e^{-\Omega} \bar{c}, \quad \beta \rightarrow e^{+\Omega} \beta, \quad \bar{\beta} \rightarrow e^{-\Omega} \bar{\beta},$$

$$ (x_\mu, \psi_\mu, e, \chi, b, \bar{b}, \gamma) \rightarrow e^0 (x_\mu, \psi_\mu, e, \chi, b, \bar{b}, \gamma),$$

(22)

the (anti-)BRST invariant Lagrangians (4) and (5) remain invariant. The infinitesimal version of the ghost transformations (22) are as follows

$$ s_{gh}c = c, \quad s_{gh}\bar{c} = -\bar{c}, \quad s_{gh}\beta = \beta,$$

$$ s_{gh}\bar{\beta} = -\bar{\beta}, \quad s_{gh}(x_\mu, \psi_\mu, e, \chi, b, \bar{b}, \gamma) = 0,$$

(23)
which are derived after setting $\Omega = 1$ for the sake of brevity. The numbers in the exponents denote the ghost numbers of the variables. We further note that the ghost numbers for $(x_\mu, \psi_\mu, \epsilon, \chi, b, \bar{b}, \gamma)$ are zero. As a result, these variables do not transform under $s_{gh}$.

At this stage, a few comments are in order as far as the symmetry operators $\delta_g, \delta_{sg}, s_b, s_{ab}$ and $s_{gh}$ are concerned. First, it can be explicitly checked that $\delta_g$ and $\delta_{sg}$ are independent because $[\delta_g, \delta_{sg}] = 0$ when they operate on any arbitrary variable of the Lagrangian (1). Second, the symmetry operators $s_b, s_{ab}$ and $s_{gh}$ obey the following algebra, namely:

$$
\begin{align*}
 s_b^2 &= 0, & s_{ab}^2 &= 0, & \{s_b, s_{ab}\} &= 0,
 [s_{gh}, s_b] &= +s_b, & [s_{gh}, s_{ab}] &= -s_{ab}.
\end{align*}
$$

(24)

We would like to point out that, for the proof of absolute anticommutativity property of (anti-)BRST symmetry transformations (i.e. $s_b s_{ab} + s_{ab} s_b = \{s_b, s_{ab}\} = 0$), one has to exploit the (anti-)BRST invariant CF-type restriction (8) in an explicit fashion.

According to Noether’s theorem, the invariance of the Lagrangians (4) and (5) under the infinitesimal version of ghost-scale transformations (22) lead to the derivation of the following conserved ghost charge ($Q_{gh}$):

$$
Q_{gh} = 2 i \beta \bar{c} \chi + 2 i \bar{\beta} c \chi - 2 e \beta \bar{\beta}.
$$

(25)

The conservation law (i.e. $\dot{Q}_{gh} = 0$) of the above charge can be proven by exploiting the appropriate equations of motion derived from the Lagrangians (4) and (5). The conserved charges $Q_g$ and $Q_{sg}$ [cf. (3)] commute with each other (i.e. $[Q_g, Q_{sg}] = 0$). The rest of the conserved charges $Q_b, Q_{ab}$ and $Q_{gh}$ obey the following standard BRST algebra

$$
\begin{align*}
 Q_b^2 &= 0, & Q_{ab}^2 &= 0, & \{Q_b, Q_{ab}\} &= 0, \\
 i [Q_{gh}, Q_b] &= +Q_b, & i [Q_{gh}, Q_{ab}] &= -Q_{ab},
\end{align*}
$$

(26)

which establishes the nilpotency of (anti-)BRST charges $Q_{(a)b}$ and the fact that the ghost numbers for the nilpotent (anti-)BRST charges $Q_{(a)b}$ are $(\mp 1)$, respectively.

The above proper BRST algebra (26) can be obtained by using the canonical (anti)commutators, derived from the Lagrangians (4) and (5). There is a simpler way to derive these relations where the generator equation (10) plays an important role. For instance, the following is true [if we use (16) and (20)]:

$$
\begin{align*}
 s_b Q_b &= -i \{Q_b, Q_b\} = 0, & s_{ab} Q_{ab} &= -i \{Q_{ab}, Q_{ab}\} = 0, \\
 s_b Q_{ab} &= -i \{Q_{ab}, Q_b\} = 0, & s_{ab} Q_b &= -i \{Q_b, Q_{ab}\} = 0, \\
 s_{gh} Q_b &= +i [Q_{gh}, Q_b] = Q_b, & s_{gh} Q_{ab} &= +i [Q_{gh}, Q_{ab}] = -Q_{ab},
\end{align*}
$$

(27)

where the l.h.s. uses the transformations (6), (7) and (23) and expressions (15) and (19) in their explicit forms. The above relations capture the BRST algebra (26). We would like to lay emphasis on the fact that, in the proof of anticommutativity $s_b Q_{ab} \equiv s_{ab} Q_b = \{Q_b, Q_{ab}\} = 0$, we have to use (8).
7 Conclusions

In our previous paper [7] and in our present investigation, we have observed many novel features associated with the model of a spinning relativistic particle when it is considered within the framework of BRST formalism. For instance, the existence of the coupled (but equivalent) Lagrangians, derivation of the proper (i.e. off-shell nilpotent and absolutely anticommuting) (anti-)BRST symmetries, emergence of the (anti-)BRST invariant CF-type restriction, etc., are some of the completely new results for this model which are elucidated, for the first-time, in our present and earlier [7] works.

We observe, for the first-time, that the physicality criteria with the conserved and nilpotent (anti-)BRST charges do not produce the annihilation of the physical states by one of the primary constraints of the theory. In other words, we find that \( Q(a)_{b|\text{phys}} = 0 \) does not imply \( \Pi_\chi|\text{phys} >= 0 \), in our present model, due to the fact that \( \Pi_\chi = 2i\beta c \) (and/or \( \Pi_\chi = -2i\beta c \)) is accurately expressed only in terms of the (anti-)ghost variables (cf. \( L_b \)). As a consequence, it is but natural to find that the primary constraint \( (\Pi_\chi|\text{phys} >= 0 \) does not appear\(^1\) from the physicality criteria \( Q(a)_{b|\text{phys}} = 0 \). The other way of saying this fact is that \( \Pi_\chi \) is nothing but the (anti-)ghost variables which are not the physical objects of our present theory. As a result, \( \Pi_\chi|\text{phys} >= 0 \) does not ensue from the above requirements of the physicality criteria (i.e. \( Q(a)_{b|\text{phys}} = 0 \)).

The deeper theoretical reason behind the above riddle is the fact that the kinetic term for \( \chi \) variable does not exist (i.e. \( \chi^2 = 0 \)) unlike the case of the bosonic gauge variable where it does [i.e. \( (-e^2/2) = b \dot{e} + b^2/2 \)]. In fact, this is the reason that, ultimately, in the expression for \( Q_b \), we have a term \( b(\dot{c} + 2\beta \chi) \) [cf. equation (15)] and \( Q_b|\text{phys} >= 0 \) implies that \( b|\text{phys} >= 0 \) which is equivalent to the statement \( \Pi_e|\text{phys} >= 0 \) (as \( b = \Pi_e \)). No such thing happens for \( \Pi_\chi \) as there is no analogue of ‘\( b \)’ in the expression for \( Q_b \) as the momentum for \( \chi \). Whether this features is a decisive property of a SUSY gauge theory, within the framework of BRST formalism, we do not know at present. It is an open problem for us for the future investigations.

We observe that the conditions \( G^{(\lambda)}|\text{phys} >= 0 \) leads to the annihilation of the physical states by the primary as well as secondary constraints together [cf. (12)]. However, the expression for the \( G^{(\lambda)} \) is somewhat ad-hoc. In fact, the expression for \( G^{(\lambda)} \) \( (\lambda = g, sg) \) has been written because we already know the symmetry transformations (2) and the first-class constraints of the theory. It has not been derived from any basic principles. On the contrary, the physicality criteria with (anti-)BRST charges \( (Q(a)_{b|\text{phys}} = 0) \) do not yield \( \Pi_\chi|\text{phys} >= 0 \) but \( Q(a)_{b} \) are derived from the basic principles. Logically one can explain the non-existence of \( \Pi_\chi|\text{phys} >= 0 \), within the framework of BRST formalism, because \( \Pi_\chi = 2i\beta c \) (or \( \Pi_\chi = -2i\beta c \)) is expressed only in terms of the (anti-)ghost variables which are non-physical. Thus, we note that \( G^{(\lambda)} \) and \( Q(a)_{b} \) have their own virtues and vices.

It would be very interesting endeavor for us to apply the BRST approach to find out the problems of quantization associated with the pseudoclassical description of the massive spinning particle in odd and any arbitrary dimension of spacetime [16-18]. Furthermore, we plan to apply our formalism to the issues of quantization related with the field theo-

\(^1\)The other way of saying this fact is that \( \Pi_\chi \) is nothing but the (anti-)ghost variables which are not the physical objects of our present theory. As a result, the constraint condition \( \Pi_\chi|\text{phys} >= 0 \) does not ensue from the above requirements of the physicality criteria \( (Q(a)_{b|\text{phys}} = 0) \).
retic models of Chern-Simons theories with P-T invariance (see, e.g. [19,20] and references therein). It would be very important for us to apply the superfield formalism [8-11] to the description of above models so that we could find out the proper (anti-)BRST symmetries for these theories. This exercise will enable us to obtain the (anti-)BRST invariant Lagrangians (and Lagrangian densities) of the above theories which, in turn, would lead to the discussions about the quantization issues within the framework of BRST formalism.

It would be a very challenging problem to apply the key ideas of superfield and BRST formalisms to other phenomenologically realistic SUSY models of gauge theories so that some novel features could be explored in physical four dimensions of spacetime. Further, we plan to address the problem of anomalies associated with the spinning particles [21] within the framework of BRST formalism at the quantum level. It is gratifying to state, at this juncture, that we have already shown the time-evolution invariance of the CF-type restriction within the framework of Hamiltonian formalism in our recent publication [22]. We are currently deeply involved with all the above cited problems and our results will be reported in our future publications [23].

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