ENDOFUNCTORS OF SINGULARITY CATEGORIES
CHARACTERIZING GORENSTEIN RINGS

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Abstract. In this paper, we prove that certain contravariant endofunctors of singularity categories characterize Gorenstein rings.

Let $\Lambda$ be a noetherian ring. Denote by $D_{sg}(\Lambda)$ the singularity category of $\Lambda$, that is, the Verdier quotient of the bounded derived category $D^b(\Lambda)$ of finitely generated (right) $\Lambda$-modules by the full subcategory consisting of bounded complexes of finitely generated projective $\Lambda$-modules. We are interested in the following question.

Question 1. What contravariant endofunctor of $D_{sg}(\Lambda)$ characterizes the Iwanaga–Gorenstein property of $\Lambda$?

In this paper we shall consider this question in the case where $\Lambda$ is commutative and Cohen–Macaulay.

Let $R$ be a commutative Cohen–Macaulay local ring of Krull dimension $d$. Denote by $\text{CM}(R)$ the category of (maximal) Cohen–Macaulay $R$-modules and by $\text{CM}_{st}(R)$ its stable category: the objects of $\text{CM}(R)$ are the Cohen–Macaulay $R$-modules, and the hom-set $\text{Hom}_{\text{CM}(R)}(M, N)$ is defined as $\text{Hom}_R(M, N)$, the quotient module of $\text{Hom}_R(M, N)$ by the submodule consisting of homomorphisms factoring through finitely generated projective (or equivalently, free) $R$-modules. The natural full embedding functor $\text{CM}(R) \to D^b(R)$ induces an additive covariant functor

$$\eta : \text{CM}(R) \to D_{sg}(R).$$

Furthermore, the assignment $M \mapsto \Omega^d \text{Tr} M$, where $\Omega$ and $\text{Tr}$ stand for the syzygy and transpose functors respectively (see [1, Chapter 2, §1] for details of the functors $\Omega$ and $\text{Tr}$), makes an additive contravariant functor

$$\lambda : \text{CM}(R) \to \text{CM}(R).$$

The following result gives a partial answer to Question 1.

Theorem 2. The following are equivalent:

1. The ring $R$ is Gorenstein.
(2) The functor $\eta$ is an equivalence (i.e. $\eta$ is full, faithful and dense).

(3) There exists a functor $\phi : \text{D}_{sg}(R) \to \text{D}_{sg}(R)$ such that the diagram

$$
\begin{array}{ccc}
\text{D}_{sg}(R) & \xrightarrow{\phi} & \text{D}_{sg}(R) \\
\eta & \uparrow & \eta \\
\text{CM}(R) & \xrightarrow{\lambda} & \text{CM}(R)
\end{array}
$$

of functors commutes up to isomorphism.

Proof. (1) $\Rightarrow$ (2): If $R$ is Gorenstein, then a celebrated theorem of Buchweitz [2, Theorem 4.4.1] implies that the functor $\eta$ is an equivalence.

(2) $\Rightarrow$ (3): When $\eta$ is an equivalence, we have a contravariant endofunctor $\phi = \eta\lambda\rho : \text{D}_{sg}(R) \to \text{D}_{sg}(R)$ of $\text{D}_{sg}(R)$, where $\rho$ stands for a quasi-inverse of $\eta$. Condition (3) holds for this functor $\phi$.

(3) $\Rightarrow$ (1): In the remainder of the proof, we will omit writing free summands. Let

$$
\pi : \text{mod} R \to \text{mod} R
$$

be the canonical functor from the category of finitely generated $R$-modules to its stable category, that is, the objects of $\text{mod} R$ are the finitely generated $R$-modules and the hom-set $\text{Hom}_{\text{mod} R}(M,N)$ is defined as $\text{Hom}_{R}(M,N)$.

Assume that there are a contravariant functor $\phi : \text{D}_{sg}(R) \to \text{D}_{sg}(R)$ and an isomorphism

$$
\Delta : \phi\eta \to \eta\lambda
$$

of functors from $\text{CM}(R)$ to $\text{D}_{sg}(R)$. Take a Cohen–Macaulay $R$-module $M$. It follows from [1, Proposition (2.21)] that there exists an exact sequence

$$
(2.1) \quad 0 \to F \to \text{Tr}\Omega\text{Tr}\Omega M \xrightarrow{f} M \to 0
$$

of finitely generated $R$-modules with $F$ free. The map $f$ induces a morphism

$$
\Theta : \text{Tr}\Omega\text{Tr}\Omega \to 1
$$

of functors from $\text{CM}(R)$ to $\text{CM}(R)$, where $I$ stands for the identity functor. Applying the $R$-dual functor $(-)^* = \text{Hom}_{R}(-,R)$ to (2.1) gives an exact sequence

$$
0 \to M^* \xrightarrow{f^*} (\text{Tr}\Omega\text{Tr}\Omega M)^* \xrightarrow{g^*} F^* \to \text{Tr}M \xrightarrow{h} \text{Tr}((\text{Tr}\Omega\text{Tr}\Omega M)) \to \text{Tr}F \to 0
$$

with $\pi(h) = \text{Tr}\pi(f)$; see [1, Lemma (3.9)]. Note that there is also an exact sequence

$$
0 \to M^* \xrightarrow{f^*} (\text{Tr}\Omega\text{Tr}\Omega M)^* \xrightarrow{g^*} F^* \to \text{Ext}^1_R(M,R) \to \text{Ext}^1_R((\text{Tr}\Omega\text{Tr}\Omega M),R).
$$

Since $\text{Ext}^1_R((\text{Tr}\Omega\text{Tr}\Omega M),R) = 0$ by [1, Theorem (2.17)] and since $\text{Tr}F$ is free, we obtain an exact sequence

$$
0 \to \text{Ext}^1_R(M,R) \to \text{Tr}M \xrightarrow{h'} \text{Tr}((\text{Tr}\Omega\text{Tr}\Omega M)) \to 0
$$

such that $\pi(h') = \pi(h)$. Taking the $d$-th syzygies of $\text{Ext}^1_R(M,R)$ and $\text{Tr}((\text{Tr}\Omega\text{Tr}\Omega M))$ and using the horseshoe lemma, we get an exact sequence of Cohen–Macaulay $R$-modules

$$
(2.2) \quad 0 \to \Omega^d\text{Ext}^1_R(M,R) \to \lambda M \xrightarrow{f} \lambda(\text{Tr}\Omega\text{Tr}\Omega M) \to 0
$$
with \( \pi(\ell) = \lambda \pi(f) \). Note that for each short exact sequence \( \sigma : 0 \to X \alpha \to Y \beta \to Z \to 0 \) of Cohen–Macaulay \( R \)-modules, the image of \( \sigma \) by the canonical functor \( \pi \) is sent by \( \eta \) to an exact triangle \( X \alpha \to Y \beta \to Z \to 0 \). Hence, \( \eta \) sends (2.2) to an exact triangle

\[
\eta \Omega^d \text{Ext}^1_R(M, R) \to \eta \lambda M \xrightarrow{\eta \lambda(\Theta M)} \eta(\text{Tr}\Omega \text{Tr} M) \to \eta M
\]

in \( D_{sg}(R) \). We have a commutative diagram

\[
\begin{array}{ccc}
\eta \lambda M & \xrightarrow{\eta \lambda(\Theta M)} & \eta(\text{Tr}\Omega \text{Tr} M) \\
\Delta M \uparrow \cong & \cong & \Delta(\text{Tr}\Omega \text{Tr} M) \\
\phi \eta M & \xrightarrow{\phi \eta(\Theta M)} & \phi \eta(\text{Tr}\Omega \text{Tr} M)
\end{array}
\]

of morphisms in \( D_{sg}(R) \), and the exact sequence (2.1) induces an isomorphism

\[
\eta(\Theta M) : \eta(\text{Tr}\Omega \text{Tr} M) \to \eta M
\]

in \( D_{sg}(R) \). Therefore \( \eta \Omega^d \text{Ext}^1_R(M, R) \) is isomorphic to \( \eta M \) in \( D_{sg}(R) \), which means that the \( R \)-module \( \text{Ext}^1_R(M, R) \) has finite projective dimension. Thus, letting \( M := \Omega^d k \), where \( k \) denotes the residue field of \( R \), shows that \( \text{Ext}^{d+1}_R(k, R) \) has finite projective dimension. If \( \text{Ext}^{d+1}_R(k, R) = 0 \), then \( R \) is Gorenstein. If \( \text{Ext}^{d+1}_R(k, R) \neq 0 \), then the \( R \)-module \( k \) has finite projective dimension, which implies that \( R \) is regular, so that \( \text{Ext}^{d+1}_R(k, R) = 0 \), a contradiction. Consequently, in either case \( R \) is a Gorenstein ring.

Now the proof of the theorem is completed. \( \square \)

**Remark 3.** In the proof of the theorem, the assumption that the ring \( R \) is commutative is used to deduce the Gorensteinness of \( R \) from the fact that the \( R \)-module \( \text{Ext}^{d+1}_R(k, R) \) has finite projective dimension. For a noncommutative ring \( \Lambda \) with Jacobson radical \( J \) and an integer \( n \), the \( n \)-th Ext group \( \text{Ext}_n^\Lambda(\Lambda/J, \Lambda) \) of the right \( \Lambda \)-modules \( \Lambda/J \) and \( \Lambda \) is not necessarily semisimple as a left \( \Lambda \)-module.

We end this paper by stating a direct consequence of the theorem.

**Corollary 4.** Suppose that \( R \) is artinian. Then \( R \) is Gorenstein if and only if the transpose functor \( \text{Tr} : \text{mod} R \to \text{mod} R \) extends to the singularity category \( D_{sg}(R) \).

**REFERENCES**

[1] Maurice Auslander and Mark Bridger, *Stable module theory*, Memoirs of the American Mathematical Society, No. 94, American Mathematical Society, Providence, R.I., 1969. MR0269685 (42 #4580)

[2] R.-O. Buchweitz, *Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings*, Preprint (1986), http://hdl.handle.net/1807/16682.