Partial Selection for Successive Cancellation List Decoding of Polar Codes
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Abstract—Polar codes have attracted a lot of attention during past few years and have been adopted as a coding scheme for 5G standard. Successive-cancellation list (SCL) decoder provides high level error-correction performance for polar codes, but the implementation complexity grows rapidly with the increase of the list size. Since the computation cost of sorting, many works focus on reducing the sorting complexity for SCL decoder. In this paper, we propose a partial selection method for SCL which directly reduce the number of input elements of sorting network without performance loss. Compared with SCL decoder, the proposed method has up to 95% less sorter size on average, and the performance loss is negligible.

Index Terms—Polar codes, successive cancellations list decoding, sorter.

I. INTRODUCTION

POLAR codes [1], as the first class of capacity achievable codes, are applied as the coding scheme of the control channels in enhanced mobile broadband (eMBB) in the 5G wireless communication system [11]. Successive cancellation (SC) algorithm is proposed in [1] with low complexity; however, it has a big performance gap compared with ML bound. In [2], SC list (SCL) decoder is proposed which significantly narrows down the performance gap. However, a large list number is required for SCL algorithm in order to achieve a relative better performance, which increases the decoding latency and resource overhead for realization.

In order to decrease the decoding latency, an SC and ML hybrid decoder is proposed [3], [4]. But the complexity is extremely high for implementation. Another method is to increase the degree of decoding parallelism by introducing specific node types, in which multiple bits are decoded simultaneously. To be specific, some simplified path metric (PM) calculations of special nodes (Rate-1, Rate-0, SPC and Rep) are adopted in [8]. For multi-bit parallel decoding, the number of splitting pathes grows exponentially with the number of information bits in the node, resulting in high computation complexity of path metric sorting.

PM sorting is studied in many literatures. Bitonic sorting network, with easy implementation and short delay, can be adopted in SCL decoder to select paths. By taking advantage of the relation of the input value, [5], [6] proposed a simplified bitonic sorter which in which redundant comparators are omitted. But the simplified bitonic sorter is difficult to be adopted for multi-bit parallel decoding. A radix sorting based state machine is proposed in [7], which has lower complexity for high channel quality. [9], [10] significantly reduce the sorting network size by limiting the number of splitting pathes in Rate-1 and SPC node. But this method introduces slight performance loss and cannot process the node other than Rate-1 and SPC. Therefore, further investigation is needed to reduce the complexity of sorting network.

This paper focus on reducing sorting network size of SCL decoder. Since the input LLRs for SCL nodes are partially ordered, some paths are more likely to be valid than others. Here, we propose a partial selection method for polar SCL (PS-SCL) decoder, which uses the partial ordered property to find the paths more likely to be correct. Besides, a pattern based selection method is also proposed to further decrease the computation complexity.

The rest of the paper is organized as follows. Section II provides a background on polar codes and its decoding algorithms. In Section III, we propose an partial selection based SCL polar decoder. Numerical results will be shown in section IV. Finally, conclusions are drawn in Section V.

II. PRELIMINARIES

A. Polar Codes

Polar code is a linear block code recursively concatenated by Kronecker power. An \( (N,K) \) polar code can be represented as

\[
x = uG^{\otimes n}
\]

(1)

where \( u = \{u_0,u_1,\ldots,u_{N-1}\} \) is the input vector, \( x = \{x_0,x_1,\ldots,x_{N-1}\} \) is the encoded vector, \( G = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \) and \( \otimes n \) denotes nth Kronecker power.

The input vector \( u \) is comprised of \( K \) information bits and \( N-K \) frozen bits. The frozen bits are usually set to be a predefined value (typically 0) known by the decoder. Vector \( x \) is transmitted through the channel, and the decoder receivers the Logarithmic Likelihood Ratio (LLR) vector \( y = \{y_0,y_1,\ldots,y_{N-1}\} \) for decoding.

B. Successive Cancellation (SC) and Successive Cancellation List (SCL) decoding

SC decoding can be represented as a binary tree search, and each node represents a sub codeword [8]. The LLRs which is represented as \( \alpha \) go through the nodes from parents to children, and the hard estimates \( \beta \) pass from child to parent. The left and right message \( \alpha' \) and \( \alpha'' \) are calculated as [8]

\[
\alpha'_i = \ln \left( \frac{1+e^{-\alpha_i^1+\delta_i}}{e^{-\alpha_i^1}+e^{-\delta_i}} \right) \\
\alpha''_i = \alpha_{i+2^{r-1}} + (1-2\beta_i)\alpha_i
\]

(2)
While \( \beta \) is computed as
\[
\beta_i = \begin{cases} 
\beta_i^l \oplus \beta_{i+1}^r, & \text{if } i < 2^{n-1} \\
\beta_{i-2^{n-1}}^r, & \text{otherwise}
\end{cases}
\]
(3)

Where \( s \) denotes the node stage in binary tree, \( \oplus \) denotes the bitwise XOR, and \( i \) represent the index for current decoded bit in vector \( u \). Due to the data dependency, each node receives \( \alpha \) first, than calculates \( \alpha' \), hard decides \( \beta' \), calculates \( \alpha'' \), hard decides \( \beta'' \), and finally sends \( \beta \).

[2] proposed SCL decoding algorithm to improve the error correction performance. The key idea is keep not only most reliable decoded sequence but also some suboptimal decoded sequences. To this purpose, a path metric (PM) is associated to each path and update at every new estimation as a cost function. For each decoding step, SCL stores a reliability metric \( PM_l^t \) for each path \( l \) that is updated for every estimated bit \( i \) according to:
\[
PM_l^t = \begin{cases} 
PM_l^{t-1} + |\alpha_i|, & \text{if } \hat{u}_{i} \neq \frac{1}{2}(1 - \text{sgn}(\alpha_i)), \\
PM_l^{t-1}, & \text{otherwise,}
\end{cases}
\]
(4)

Where \( l \) is the path index, \( \alpha_i \) is the calculated LLR value of bit \( i \) at path \( l \), and \( \hat{u}_i \) is the estimate of bit \( i \) at path \( l \).

C. Simplified SCL

In [9], some special nodes with constituent bits are identified, and the candidate codewords corresponding to these nodes can be directly generated without traversing the binary tree. To achieve this goal, a chase decoding providing a list of candidate paths and satisfying node parity check formulas is used. \( \beta_j \) denotes a candidate output codeword of target node. When starting from a source path \( l \) with reliability \( PM_l^{t-1} \), the reliability of the path corresponding to the output codeword \( \beta_j \) is
\[
PM_{l,j}^t = PM_{l,j}^{t-1} + \sum_{i=0}^{N_v-1} |\beta_j^i[i] - h(\alpha_v^i[i])| |\alpha_v^i[i]|
\]
(5)

Where \( h() \) denotes hard-decision, \( t \) denotes decoding step index, \( N_v \) denotes the node size and \( \alpha_v^i[i] \) is the input LLR values for the node.

III. PARTIAL SELECTION (PS) BASED SCL POLAR DECODER

A. Path metric probability table

For SSCL decoder, the path metrics (PMs) in \( t - 1 \) step are sorted in ascending order. If the next node for decoding contains \( s \) information bits, there will be \( 2^s \) branches split from its parent path. We put PMs of these split branches into a PM table row by row. Then, sort the elements in each row in ascending order and obtain matrix \( PM' \). \( PM_{l,j}^t \) denotes the elements in \( l \)-th row and \( j \)-th column for \( t \)-th node. \( PM_l^{t-1} \) denotes the \( l \)-th PM obtained from \( (t-1) \)-th step. Derived from (5) that the elements in \( PM' \) has the following properties:
\[
PM_{l,j}^t = PM_{l,j}^{t-1} + \sum_{i=0}^{N_v-1} |\beta_j^i[i] - h(\alpha_v^i[i])| |\alpha_v^i[i]|
PM_l^{t-1} < PM_{l+1}^{t-1}
\]
(6)

\[
PM_{l,j}^t < PM_{l,j+1}^t
\]
(7)

\( l = 0, 1, 2, ..., L - 1, \ j = 0, 1, 2, ... \)

It should be noted that the \( PM' \) is only used for illustrate the proposed method, and in next subsection we will show that the sorting operation to get \( PM' \) is unnecessary for real implementation.

Although we dont know the exact relationship of size between elements in \( PM' \). By (6) and (7), it can be found that the elements in the top left corner have a larger probability lower than the elements in the lower right corner. To verify that, numerical simulations are done to check the probability that each path is selected by a normal SCL decoder. These probabilities are written into a table called PM probability (PMP) table.

Fig. 1 shows a PMP table for \( L = 32 \) as an example. The node size \( N_v \) is fixed to 4 in Fig. 1(a) and 8 in Fig. 1(b). The probabilities of different type of nodes are drawn together. The unclassed nodes (not belong to Rate-1, Rate-0 and SPC) will be decoded by an ML decoder. Different colors represent different probabilities. The results show that SCL decoder is more likely to select the PMs in the top left corner on the ordered PM table, so a sorter for partial PMs has potential to achieve comparable performance with a sorter for all PMs.

Three boxes in different colors are used to select partial elements in the PMP table, and the sums of the probabilities are listed in Table I. The number \( x, y, z \) after the blue and red boxes represent that the boxes have three levels, each of which keeps \( x, y \) and \( z \) PMs. The ratio between three levels is set as 1:1:2. From the table, it is found that stair-stepping box is more efficient than a rectangular box for selecting the correct PMs.

\[\begin{array}{c}
\text{fig1} 1 \text{PMP table for all types of nodes together, N = 64, K_{payload} = 32, K_{CRC} = 11 and E_b/N_0 = 3dB. (a) Node size N_v is fixed to 4. (b) Node size N_v is fixed to 8.}
\end{array}\]
TABLE I: Probability for Different Path Selection Methods

| Node size | Probability for paths selected by SCL decoder |
|-----------|---------------------------------------------|
| Blue box(4,2,1) | 98.26332% |
| Green box(4) | 99.33933% |
| Red box(8,4,2) | 99.97014% |

B. A fast calculation of the PM table

In Fig.1, it is found that only the first few columns are needed for PS-SCL. In this part, we will illustrate how to get the smallest 1, 2, 4 or 8 path metrics without sorting all the elements.

As (5) shows, the updated PM value for the t step can be calculated as a summation of the PM value in t − 1 step and the BM (Branch Metric) value in . The BM is calculated as

\[ BM_{i,j}^t = \sum_{i=0}^{N_v-1} |\beta_{i,j}^t[i] - h(\alpha_i^t[i])| \, |\alpha_i^t[i]| \]  
(8)

In which, t is the operation step, l is the path index, \(N_v\) is the size of the node and i is the bit index. \(\alpha_i^t[i]\) is the input LLR of the i-th bit in subpath l, \(\beta_{i,j}^t[i]\) denotes the i-th bit in sequence \(\beta_{i,j}\), and \(\beta_{i,j}\) denotes the j-th candidate output codeword for l-th path. \(h(.)\) denotes hard-decision operation.

In the following text, we take \(N_v = L_s = 4\) as an example to describe the BM selection procedure. The received LLR vector of a node is \(\alpha^t[i] (i = 0, 1, 2, 3)\), and its absolute value vector is

\[ a_i = |\alpha^t[\pi(i)]| (i = 0, 1, 2, 3) \]  
(9)

where \(\pi(i) (i = 0, 1, 2, 3)\) represents a permutation for \(|\alpha^t[i]|\) in ascending order, which satisfies:

\[ 0 \leq a_0 \leq a_1 \leq a_2 \leq a_3 \]  
(10)

For all the combination of the information bits, the BMs are calculated as:

\[ BM_j = \sum_{i=0}^{3} |\beta_j[\pi(i)] - h(\alpha^t[\pi(i)])| a_i \]  
(11)

If the information bit number \(N_i = 4\) (Rate-1 node), all candidate results for BMs are listed as follows:

\[
\begin{align*}
0, & \quad a_2, & \quad a_3, & \quad a_0 + a_1 + a_3, \\
0, & \quad a_0 + a_2, & \quad a_0 + a_3, & \quad a_0 + a_2 + a_3, \\
0, & \quad a_1 + a_2, & \quad a_1 + a_3, & \quad a_1 + a_2 + a_3, \\
0, & \quad a_0 + a_1 + a_2, & \quad a_2 + a_3, & \quad a_0 + a_1 + a_2 + a_3,
\end{align*}
\]

Here, a directed graph in Fig.2 is used to indicate the relationships between the PMs, in which an arrow pointing from A to B indicates that A is larger than B. Starting from any element in the graph, all other elements that can be reached are

smaller than the element itself. So, we can find the smallest 8, 4, 2 or 1 elements by traversing the directed graph. For \(N_i = 3\) (SPC node), similar graph can be used to find the specific smallest elements. The results are shown in table II. For \(N_i < 3\), there are only 4 or 2 candidate BMs, so 4-sorter or 2-sorter can be used directly to find the smallest 1 or 2 BMs.

IV. NUMERICAL RESULTS

Numerical simulation was conducted to test the error correction performance of the PS-SCL decoder. All simulation results are for QPSK modulated random codewords transmitted over an AWGN channel. The polar codes are constructed
distribution becomes more dispersed when node size grows, loss turns to be negligible. Because in Fig. 1, the probability increase the partial selection level to \((8, 4, 2)\), the performance loss of the PS-SCL decoder with level \((4, 2, 1)\) compared with normal SCL decoder is clearly shown. The total numbers of CAS units for some different levels, and the number of kept PMs for the 3 levels is written as level \((x, y, z)\). The ratio of the three levels is found by (13) from [12]. The total numbers of CAS units for some different node size \(N_v\) is fixed to 4. The PS-SCL method uses a stair-stepping selections with 3 different levels, and the number of kept PMs for the 3 levels is written as level \((x, y, z)\). The ratio of the three levels is set as 1:1:2. For example, if PS-SCL32 is used with \((4, 2, 1)\), the kept PMs in ordered PM table should be the first 4 PMs from row 0 to row 7, the first 2 PMs from row 8 to row 15 and the first 1 PM from row 16 to row 31. The results show that at least for BLER up to 1e-4, the loss of PS-SCL decoder with level \((4, 2, 1)\) compared with normal SCL decoder is negligible.

In Fig.4, the node size \(N_v\) is fixed to 8. For the PS-SCL decoder with level \((4, 2, 1)\), as the list size changes from 4 to 32, its performance loss grows from 0.02dB to 0.3dB. If we increase the partial selection level to \((8, 4, 2)\), the performance loss turns to be negligible. Because in Fig. 1, the probability distribution becomes more dispersed when node size grows, more PMs are needed to be selected to avoid performance loss.

If a bitonic sorter is chosen, for \(2L_{in}\) input PMs, the total number of compare-and-select (CAS) units can be calculated by (13) from [12]. The total numbers of CAS units for some SCL decoders are shown in Table III.

\[
c_{BT} = \frac{L_{in}}{2} \log_2 L_{in} + 1 + \log_2 L_{in} + 2
\]

Compared with normal SCL, the proposed PS-SCL has 95% less CAS units for node size 4 and 99% less CAS units for node size 8. Compared with Fast-SSCL, PS-SCL has 65% less CAS units on average.

V. CONCLUSION

In this work, a partial selection method is proposed to reduce the complexity of sorting network for polar SCL decoder. The method takes advantage of the partial ordered property of path metrics (PMs) to make an ordered PM table, and uses a stair-stepping box selection on PM table to find the PMs that are more likely to be selected by a normal SCL decoder. Furthermore, a fast calculation method is proposed to find these PMs easily. Compared with normal SCL decoder, the PS-SCL can reduce the compare-and-select (CAS) units of a bitonic sorter up to 95%. Numerical results show that the performance loss of the PS-SCL is negligible with a suitable selection box size. When list size or node size grows larger, a larger selection box size is needed to avoid error correction performance loss. Next, we will focus on designing different selecting boxes for different nodes in the future.

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TABLE II: Smallest Elements of Different Node Type

| \(N_v\) | Syndrome | Smallest 8 | Smallest 4 | Smallest 2 | Smallest 1 |
|---|---|---|---|---|---|
| 4 | 0 - | 0, a0, a1, a0+a1, a2, a0+a2, \(\min(a_3, a_0 + a_2, a_1 + a_2, a_0 + a_1 + a_2)\), \(\text{submin}(a_3, a_0 + a_3, a_1 + a_2, a_0 + a_1 + a_2)\) | 0, a0, a1, \(\min(a_0 + a_1, a_2)\) | 0, a0 | 0 |
| 3 | \(\sum_{i=0}^{3} h(a[\pi(i)]) = 0\) | 0, a0+a1, a0+a2, \(\min(a_0 + a_1, a_2)\) | 0, a0+a1, 0, a0+a1 | 0, a0+a1 | 0 |
| 3 | \(\sum_{i=0}^{3} h(a[\pi(i)]) = 1\) | 0, a0, a1, a2, \(\min(a_0 + a_1, a_2)\) | 0, a0, a1, 0, a0 | a0, a1 | a0 |

TABLE III: Number of CAS Units for Different Decoding Algorithm

| List | PS-SCL, \(N_v = 4, (4, 2, 1)\) | PS-SCL, \(N_v = 8, (8, 4, 2)\) | SCL, \(N_v = 4\) | SCL, \(N_v = 8\) | Fast-SSCL in [10], \(N_v = 4\) | Fast-SSCL in [10], \(N_v = 8\) |
|---|---|---|---|---|---|---|
| L = 4 | 24 | 80 | 672 | 28160 | 80 | 80 |
| L = 8 | 80 | 240 | 1792 | 67584 | 240 | 672 |
| L = 16 | 240 | 672 | 4608 | 159744 | 672 | 1792 |
| L = 32 | 672 | 1792 | 11520 | 372736 | 1792 | 4608 |