STATISTICAL INFERENCE FOR FUNCTIONAL LINEAR QUANTILE REGRESSION

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We propose inferential tools for functional linear quantile regression where the conditional quantile of a scalar response is assumed to be a linear functional of a functional covariate. In contrast to conventional approaches, we employ kernel convolution to smooth the original loss function. The coefficient function is estimated under a reproducing kernel Hilbert space framework. A gradient descent algorithm is designed to minimize the smoothed loss function with a roughness penalty. With the aid of the Banach fixed-point theorem, we show the existence and uniqueness of our proposed estimator as the minimizer of the regularized loss function in an appropriate Hilbert space. Furthermore, we establish the convergence rate as well as the weak convergence of our estimator. As far as we know, this is the first weak convergence result for a functional quantile regression model. Pointwise confidence intervals and a simultaneous confidence band for the true coefficient function are then developed based on these theoretical properties. Numerical studies including both simulations and a data application are conducted to investigate the performance of our estimator and inference tools in finite sample.

1. Introduction. Functional data analysis (FDA), extending the traditional data domain to curves, surfaces, and objects, has attracted a lot of attention in the past decades. Monographs, such as Ramsay and Silverman (2005), Ferraty and Vieu (2006), Horváth and Kokoszka (2012), Hsing and Eubank (2015) and Kokoszka and Reimherr (2017), exhibit a comprehensive review of the past and current topics in FDA. Among all the topics, functional linear models focusing on the mean response or the mean response function are arguably the most studied. Theoretical investigation on the asymptotic consistencies in coefficient function estimation and mean prediction has been mainly considered under two frameworks: the functional principal component analysis (FPCA) and the reproducing kernel Hilbert space (RKHS) frameworks. Examples for FPCA include Yao et al. (2005), Cai and Hall (2006), and Hall and Horowitz (2007). Examples for the RKHS approach include Yuan and Cai (2010), Cai and Yuan (2012), and Sun et al. (2018). In this paper, we shall study, under the RKHS framework, the functional linear quantile regression (FLQR) model where the modeling of the (conditional) quantile function of a scalar response against a functional covariate is of interest.

Comparing with various functional linear models, functional quantile regression models have been studied much less in the literature with some exceptions given below. Cardot et al. (2005) considered the convergence of a penalized B-spline coefficient function estimator under a functional quantile regression model. Ferraty et al. (2005) and

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Chen and Müller (2012) estimated the conditional quantile function through inverting an estimate of the conditional cumulative distribution of the response given the functional covariate. The difference was that the former used kernel estimation and the latter used a generalized functional linear model via FPCA. Kato (2012) considered a more thorough and direct approach. The conditional quantile function over a range of quantile indices was modeled through a function-on-function regression model against a functional covariate. The FPCA was used to estimate the coefficient function and the resulting estimator of the conditional quantile function was further monotonized to satisfy the monotonicity constraint over the quantile index. A minimax optimal rate of convergence was established for the estimators. Yu et al. (2016) considered a model similar to Cardot et al. (2005) but the basis functions used for estimating the coefficient function were generated from a so-called partial quantile regression technique that resembles the partial least squares. They also extended the approach to the composite quantile regression scenario (Zou and Yuan, 2008). Yao et al. (2017) considered quantile regression models with a functional covariate and a high dimensional predictor variable where the focus was on variable selection. Ma et al. (2019) considered a similar high dimensional model with multiple functional covariates involved and double penalization applied to select both functional and scalar variables. However, one common drawback in these papers is the lack of any rigorously derived weak convergence results and inference tools.

Such kind of inference theory has been studied for functional regression models with a mean response. To test on the nullity of the slope function in a functional linear model, Cardot et al. (2003) proposed two test statistics based on the norm of the empirical cross-covariance operator of the response and the functional predictor with a pre-selected number of FPCs of the predictor process. Müller and Stadtmüller (2005) derived the asymptotic distribution for the coefficient function estimate under a generalized functional linear regression model fitted by the FPCA, where the truncation point of the Karhunen-Loève expansion is assumed to increase at a certain rate of the sample size and randomness of FPCs is ignored. Cardot et al. (2007) developed a central limit theorem for the FPCA approach under a functional linear regression model where the predictor function can reside in a more general Hilbert space. Ferraty et al. (2007) and Ferraty et al. (2010) considered nonparametric functional regression models based on kernel estimation. They derived point-wise normality results for the regression function and developed bootstrap procedures for the construction of point-wise confidence intervals. A similar bootstrap procedure was proposed in González-Manteiga and Martínez-Calvo (2011) for functional linear models. Zhang and Chen (2007) considered statistical inference for functional linear models where the functional data are reconstructed by local polynomial kernel estimation and proposed a global Wald type of test statistic on the effect of functional covariates. To test the nullity of the slope function in a functional linear model, Hilgert et al. (2013) studied a Fisher-type nonadaptive test statistic corresponding to the projection of the response on the leading FPCs of the predictor process where the randomness of the FPCs are incorporated. Employing a sequential approximation by a series of functional principal components regression models, Lei (2014) proposed a global test procedure on the slope function in a functional linear model. Under the framework of RKHS, Shang and Cheng (2015a) developed a system of statistical inference tools for generalized functional linear models, which include confidence intervals/bands, prediction intervals, functional contrast tests, and global tests on slope functions. Extensions of their approach to a functional Cox model and a function-on-function regression model were considered respectively in Hao et al. (2021) and Dette and Tang (2021). Cuesta-Albertos et al. (2019) constructed goodness-of-fit
tests for the functional linear model with a scalar response based on marked empirical processes indexed by a randomly projected functional covariate.

Statistical inference for functional quantile regression models has been rarely studied, partly due to the fact that the quantile loss function is non-differentiable. To our best knowledge, only one informal treatment was attempted in Li et al. (2016) where an adjusted Wald test was proposed to test on whether a common coefficient function is shared across models for a given set of multiple quantile indices. However, a key assumption in their work is that the coefficient function can be represented by the expansion of a fixed number of functional principle components. This strong assumption essentially reduces the problem to the traditional quantile linear regression model. Then all the traditional theoretical tools can be directly applied to yield the root-n convergence rate of the coefficient estimate and an asymptotic chi-square distribution for the test statistic. The problem considered here is much harder and the treatment here is more rigorous than those in Li et al. (2016) from the following aspects. First, we don’t have a fixed dimension assumption for the coefficient function which we assume belongs to an infinite dimensional Sobolev space. It is well known that statistical inference on an infinite dimensional parameter space is a much harder problem. Second, we consider the model over a continuum range of quantile indices rather than the composite quantile regression scenario that only entertains a small number of quantile indices. Such a functional quantile regression model has only been studied by Kato (2012) with a minimax estimation consistency result.

Therefore, this paper aims to provide a systematic and rigorous study on the inference problem for a FLQR model with a scalar response and a functional covariate over a range of quantile indices. Recognizing the non-differentiability of the quantile loss function, we first employ a kernel density smoothing technique introduced in Fernandes et al. (2021) to consider a smoothed loss function. With the addition of a roughness penalty to the loss function, our coefficient function estimate is defined as the minimizer of this penalized and smoothed loss function. The representer theorem guarantees that this minimizer lies on a finite dimensional subspace (note that the finite dimension is not fixed but increases with n), although the Sobolev space of the original optimization problem is of infinite dimensions. Based on this we develop an efficient gradient descent algorithm to compute the coefficient function estimate. For the theoretical properties, we first show that the minimizer of the penalized and smoothed objective function provides a good approximation to the minimizer of the penalized quantile loss objective function with a negligible error. Then a functional Bahadur representation is derived for the smoothed FLQR model estimate. Based on the representation, the weak convergence of the coefficient function estimate to a Gaussian process is established. We then derive the pointwise confidence interval and the simultaneous confidence band for the coefficient function, as well as the confidence interval for the conditional quantile. The techniques we use in this paper are very different from those in the earlier work on penalized functional regression models, such as Yuan and Cai (2010) and Cai and Yuan (2012), where the alignment of the covariance function and the reproducing kernel plays an important role. Useful as they were in deriving the optimal estimation and prediction consistencies, the covariance alignment argument can be difficult to extend to the inference scenario. Here, we rely on the Banach fixed-point theorem to derive the functional Bahadur representation. The work here is also significantly different from that in Shang and Cheng (2015a) who studied the functional Bahadur representation for a generalized functional linear model. Comparing with the smooth and convex objective function in their model setting, the quantile loss function here is non-differentiable. Although the smoothed objective function trick makes the computation more tractable,
considerable effort is needed to fill in the gap between the smoothed approximation and the original objective settings.

In summary, our work makes the following contributions to FLQR models. First, we provide a smoothed approximation to the original FLQR model with a negligible approximation error. Second, we develop an efficient estimation procedure whose resulting coefficient function estimator is not only theoretically consistent but also delivers better accuracy than the existing methods in empirical experiments. Lastly and more importantly, we derive the functional Bahadur representation for the coefficient function estimator, based on which the inference tools, such as simultaneous confidence bands for the coefficient function and confidence intervals for the conditional quantile, are constructed. As far as we know, these inference tools are the first ones for a functional quantile regression model.

The rest of the paper is structured as follows. In Section 2 we introduce the smoothed version of FLQR and the gradient descent method to fit the smoothed FLQR with a roughness penalty. Theoretical properties such as consistency, convergence rate and weak convergence of our proposed estimator are discussed in Section 3. In Sections 4 and 5 we investigate finite sample performance of the proposed estimator as well as inference tools through empirical studies. Section 6 concludes the article. All technical proofs are delegated to the supplementary material.

2. Model and Estimation. Let $X$ denote a random function defined on $I$, a compact subset of $\mathbb{R}$ and $Y$ be a scalar random variable taking values in $Y$. Without loss of generality, we assume that $I = [0, 1]$. Let $Q_{Y|X}(\cdot|X)$ denote the conditional quantile of $Y$ given $X$. Throughout the paper the conditional quantile of $Y$ w.r.t. $X$ can be understood as a function of a collection of random variables $\{X(t) : 0 \leq t \leq 1\}$. Let $U$ be a given subset of $(0, 1)$ that is away from 0 and 1, that is, for some small $c_0 \in (0, 1/2)$, $U \subset [c_0, 1-c_0]$. We consider the following functional linear quantile regression (FLQR) model:

$$
Q_{Y|X}(\tau|X) = \alpha_0(\tau) + \int_0^1 \beta_0(t, \tau) X(t) dt, \quad \tau \in U.
$$

For each $\tau \in U$, let $\beta(\cdot, \tau) \in H^m(I)$, the $m$th-order Sobolev space defined by

$$
H^m(I) = \{h : I \rightarrow \mathbb{R} : h^{(j)} , j = 0, \ldots, m - 1 \text{ are absolutely continuous, and } h^{(m)} \in L^2(I)\}.
$$

The unknown parameter $\theta_0(\tau) \equiv \{\alpha_0(\tau), \beta_0(\cdot, \tau)\} \in \mathcal{H} \equiv \mathbb{R} \times H^m(I)$. We assume $m > 1/2$ to ensure that $H^m(I)$ is an RKHS.

Given an i.i.d sample, $(X_1, Y_1), \ldots, (X_n, Y_n)$, from the joint distribution of $(X, Y)$, we aim to carry out statistical inference on $\theta_0(\tau)$ in model (2.1). The regularized estimator of $(\alpha_0(\tau), \beta_0(\cdot, \tau))$ is defined by

$$
(\hat{\alpha}(\tau), \hat{\beta}(\cdot, \tau)) = \arg \min_{(\alpha, \beta) \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^n \rho_{\tau}(Y_i - \alpha - \int_0^1 \beta(t) X(t) dt) + \frac{\lambda}{2} J(\beta, \beta),
$$

where $\rho_{\tau}(u) = u|\tau - 1(u < 0))$ is the commonly used loss function in quantile regression (QR), and $J(\beta_1, \beta_2) = \int_0^1 \beta_1^{(m)}(t) \beta_2^{(m)}(t) dt$ is a roughness penalty for $\beta_1, \beta_2 \in H^m(I)$. We use $\lambda/2$ for simplifying future expressions when calculating Fréchet derivatives.

2.1. Smoothed FLQR. As pointed out by Fernandes et al. (2021), the first-order linear Gaussian approximation of the distribution of the standard QR estimator could fail in finite samples. They proposed a convolution-type smoother of the objective function
to yield a continuous QR estimator. In particular, the first term of the right-hand side of Equation (2.2) can be rewritten as

\[
\hat{R}(\theta; \tau) := \frac{1}{n} \sum_{i=1}^{n} \rho_{\tau}(\epsilon_i(\theta)) = \int \rho_{\tau}(u) d\hat{F}(u; \theta),
\]

where \(\epsilon_i(\theta) := Y_i - \alpha - \frac{1}{h} \beta(t) X_i(t) dt\) and \(\hat{F}(\cdot; \theta)\) denotes the empirical distribution function of \(\epsilon_i(\theta)\)'s. Since \(\hat{F}(\cdot; \theta)\) is not an absolutely continuous function, the mapping \(\theta \to \hat{R}(\theta; \tau)\) is not differentiable. Let \(k\) denote a smooth kernel function satisfying \(\int k(u) du = 1\) and \(h > 0\) be a bandwidth parameter shrinking towards 0 as the sample size increases. Then define \(k_h(v) = \frac{k(v/h)}{h}\). Instead of using the empirical distribution function, Fernandes et al. (2021) employed a kernel method to estimate the probability density function of \(\epsilon_i(\theta)\). In our context, the kernel density estimate is given by

\[
\hat{f}(v; \theta) = \frac{1}{n} \sum_{i=1}^{n} k_h(v - \epsilon_i(\theta))
\]

for any given \(\theta \in \mathcal{H}\). Hence the corresponding estimated distribution function is \(\hat{F}_h(u; \theta) = \int_{-\infty}^{u} \hat{f}(v; \theta) dv\). Replacing \(\hat{F}(u; \theta)\) with \(\hat{F}_h(u; \tau)\) in Equation (2.3), we obtain the smoothed empirical risk function:

\[
\hat{R}_h(\theta; \tau) := \int \rho_{\tau}(u) d\hat{F}_h(u; \theta) = \int \rho_{\tau}(u) \hat{f}_h(u; \theta) du.
\]

Then the corresponding regularized FLQR estimator is

\[
(\hat{\alpha}_{h\lambda}(\tau), \hat{\beta}_{h\lambda}(\cdot; \tau)) \in \arg \min_{(\alpha, \beta) \in \mathcal{H}} \hat{R}_h(\theta; \tau) + \frac{\lambda}{2} J(\beta, \beta).
\]

2.2. Representer theorem. \(H^m(\mathbb{I})\) is an RKHS when equipped with the (squared) norm

\[
\|f\|_{V_2}^2 = \sum_{l=0}^{m-1} \left\{ \int_0^1 f^{(l)}(t) dt \right\}^2 + \int_0^1 \left\{ f^{(m)}(t) \right\}^2 dt.
\]

for any \(f \in H^m(\mathbb{I})\). Let \(H_0(\mathbb{I}) = \{ \beta \in \mathcal{H} : J(\beta, \beta) = 0 \}\) be the null space of \(J\), i.e., the collection of all the \(\beta\)'s such that \(J(\beta, \beta) = 0\). Then it is a finite-dimensional linear subspace of \(H^m(\mathbb{I})\). Let \(\psi_1, \ldots, \psi_m\) be the basis functions of \(H_0(\mathbb{I})\). Denote by \(H_1(\mathbb{I})\) its orthogonal complement in \(H^m(\mathbb{I})\) such that \(H^m(\mathbb{I}) = H_0(\mathbb{I}) \oplus H_1(\mathbb{I})\). That is, for any \(\beta \in H^m(\mathbb{I})\), there exists a unique decomposition \(\beta = \beta_0 + \beta_1\) such that \(\beta_0 \in H_0(\mathbb{I})\) and \(\beta_1 \in H_1(\mathbb{I})\). Note that \(H_1(\mathbb{I})\) is also an RKHS with the inner product of \(H^m(\mathbb{I})\) restricted to \(H_1(\mathbb{I})\).

Let \(R\) and \(R_1: \mathbb{I} \times \mathbb{I} \to \mathbb{R}\) be the reproducing kernels of \(H^m(\mathbb{I})\) and \(H_1(\mathbb{I})\), respectively. Then we have \(J(\beta_1, \beta_1) = \|\beta_1\|_{\mathbb{I}}^2 = \|\beta_1\|_{V_2}^2\) for any \(\beta_1 \in H_1(\mathbb{I})\). According to Cucker and Smale (2002), \(RX_1(\cdot) := \int_0^1 R(\cdot, s) X_1(s) ds \in H^m(\mathbb{I})\) for \(i = 1, \ldots, n\). As shown in Shin and Lee (2016), the solution to (2.5) can be expressed as

\[
\hat{\beta}_{h\lambda}(t, \tau) = \sum_{i=1}^{n} d_i \psi_i(t) + \sum_{i=1}^{n} c_i \xi_i(t),
\]

where \(\xi_i(t) = \int_0^1 R_1(t, s) X_i(s) ds, i = 1, \ldots, n\). Let \(d = (d_1, \ldots, d_m)^\top, c = (c_1, \ldots, c_n)^\top\) and \(\Xi = (J(\xi_i, \xi_j))_{ij} \in \mathbb{R}^{n \times n}\). Therefore, solving (2.5) is reduced to minimizing

\[
Q_h(\alpha, d, c; \tau) := \frac{1}{n} \sum_{i=1}^{n} \ell_h \left( Y_i - \alpha - \sum_{l=1}^{m} d_l \int_0^1 X_i(t) \psi_l(t) dt - \sum_{j=1}^{n} c_j \xi_j(t) H^m(\mathbb{I}) \right) + \frac{\lambda}{2} c^\top \Xi c.
\]
with respect to \( \alpha, d \) and \( c \), where \( f_h(u; \tau) = \int \rho_r(v)k_h(v-u)dv \).

The corresponding estimator of the conditional \( \tau \)-th quantile of \( Y \) is given by \( \hat{Q}(\tau, X) = \hat{\alpha}_h(\tau) + \int_0^1 X(t)\beta_h(t, \tau)dt \). One possible issue of the above estimation method is that this estimator is not necessarily monotone nondecreasing with respect to \( \tau \), a property satisfied by the true conditional quantile of \( Y \). Here we adopt the strategy employed in Kato (2012) to monotonize the quantile estimator. More specifically, we postulate that \( U = [\underline{\tau}, \overline{\tau}] \) with \( 0 < \underline{\tau} < \overline{\tau} < 1 \). Given \( X = x \), we first define a distribution function \( \tilde{F}_U(y|x) = \frac{1}{\overline{\tau} - \underline{\tau}} \int_{-\infty}^{\overline{\tau}} \mathbb{1}\{\hat{Q}_{Y|X}(\tau|x) \leq y\}d\tau \) with the support \( \min_{\tau \in U} \hat{Q}_{Y|X}(\tau|x), \max_{\tau \in U} \hat{Q}_{Y|X}(\tau|x) \). Then a modified estimator of \( Q_{Y|X}(\tau|x) \) is defined by \( \hat{Q}_{Y|X}(\tau|x) = \tilde{F}_U^{-1}((\tau - \underline{\tau})/(\overline{\tau} - \underline{\tau})|X = x) \). Obviously \( \hat{Q}_{Y|X}(\tau|x) \) is a nondecreasing function of \( \tau \). Then we employ the “pool adjacent violators” algorithm to obtain the isotonized estimator denoted by \( \hat{Q}^I_{Y|X}(\tau|x) \). The final estimator of \( Q_{Y|X}(\tau|x) \) is given by a convex combination of \( \hat{Q}^I_{Y|X}(\tau|x) \) and \( \hat{Q}^I_{Y|X}(\tau|x) \). As shown in Kato (2012), this final estimator of the \( \tau \)-th quantile, denoted by \( \hat{Q}^I_{Y|X}(\tau|x) \) is always superior to the initial estimator in the following sense: for any \( q \geq 1 \),

\[
\left[ \int_U |\hat{Q}^I_{Y|X}(\tau|x) - Q_{Y|X}(\tau|x)|^q d\tau \right]^{1/q} \leq \left[ \int_U |\hat{Q}_{Y|X}(\tau|x) - Q_{Y|X}(\tau|x)|^q d\tau \right]^{1/q}.
\]

2.3. Gradient descent algorithm. To minimize (2.7), we consider a gradient descent (GD) algorithm. To facilitate calculations of the gradient of the objective function \( Q_h \), we rewrite \( \hat{R}_h(\theta; \tau) \) as

\[
\hat{R}_h(\theta; \tau) = (1 - \tau) \int_{-\infty}^0 \hat{F}_h(u|\theta)du + \tau \int_0^\infty [1 - \hat{F}_h(u|\theta)]du
\]

by the Tonelli’s theorem. Then the derivatives of \( Q_h(\alpha, d, c; \tau) \) are immediately available:

\[
\frac{\partial Q_h(\alpha, d, c; \tau)}{\partial \alpha} = \frac{1}{n} \sum_{i=1}^n \left[ K\left(-\frac{\hat{\epsilon}_i(\theta)}{h}\right) - \tau \right],
\]

\[
\frac{\partial Q_h(\alpha, d, c; \tau)}{\partial d_l} = \frac{1}{n} \sum_{i=1}^n \left[ K\left(-\frac{\hat{\epsilon}_i(\theta)}{h}\right) - \tau \right] \left( \int_0^1 X_i(t)\psi_l(t)dt \right),
\]

\[
\frac{\partial Q_h(\alpha, d, c; \tau)}{\partial c_i} = \frac{1}{n} \sum_{i=1}^n \left[ K\left(-\frac{\hat{\epsilon}_i(\theta)}{h}\right) - \tau \right] \Xi_i + \lambda \Xi c,
\]

where \( \hat{\epsilon}_i(\theta) \) is \( \epsilon_i(\theta) \) with \( \beta(t) \) replaced by \( \hat{\beta}_h(t, \tau) \) in (2.6), \( \Xi_i \) is the \( i \)th column of \( \Xi \) and \( K(u) = \int_{-\infty}^\infty k(v)dv \). Given these derivatives and an initial value, \((\hat{\alpha}^0, \hat{d}^0, \hat{c}^0)\), the GD update at the \((r+1)\)th iteration is given by

\[
\hat{\alpha}^{r+1} = \hat{\alpha}^r - \gamma_r \frac{\partial Q_h(\hat{\alpha}^r, \hat{d}^r, \hat{c}^r; \tau)}{\partial \alpha},
\]

\[
\hat{d}_l^{r+1} = \hat{d}_l^r - \gamma_r \frac{\partial Q_h(\hat{\alpha}^r, \hat{d}^r, \hat{c}^r; \tau)}{\partial d_l}, \quad l = 1, \ldots, m,
\]

\[
\hat{c}^r = \hat{c}^r - \gamma_r \frac{\partial Q_h(\hat{\alpha}^r, \hat{d}^r, \hat{c}^r; \tau)}{\partial c},
\]

where \( \gamma_r > 0 \) is the learning rate.
Selecting an appropriate learning rate is critical when implementing the GD algorithm. A line search method is commonly used to determine the learning rate. In particular, a fixed learning rate chosen from a tuning strategy or a decreasing sequence of learning rates is taken in practice. However, neither of these approaches is efficient in this context. Instead, we adopt the strategy proposed by Barzilai and Borwein (1988) to select the learning rate at each iteration. For \( r = 1, 2, \ldots \), the learning rate \( \gamma_r \) is chosen to minimize either \( ||a\delta^r - g^r||^2_2 \) or \( ||\delta^r - ag^r||^2_2 \) w.r.t. \( a \), where

\[
\delta^r = (\hat{\alpha}^r, (\hat{d}^r)^T, (\hat{c}^r)^T)^T - (\hat{\alpha}^{r-1}, (\hat{d}^{r-1})^T, (\hat{c}^{r-1})^T)^T \quad \text{and} \\
g^r = \nabla Q_h(\hat{\alpha}^r, \hat{d}^r, \hat{c}^r; \tau) - \nabla Q_h(\hat{\alpha}^{r-1}, \hat{d}^{r-1}, \hat{c}^{r-1}; \tau).
\]

As a result, ordinary least squares lead to two possible choices of the learning rate:

\[
\gamma_{1,r} = \frac{\langle \delta^r, \delta^r \rangle}{\langle \delta^r, g^r \rangle} \quad \text{and} \quad \gamma_{2,r} = \frac{\langle \delta^r, g^r \rangle}{\langle g^r, g^r \rangle}.
\]

Both of them are referred to as the BB rate in the article. Then the GD update proceeds as in (2.8) by replacing \( \gamma_r \) with either one in (2.9). Note that we need point estimates of all parameters at the first iteration to calculate \( \gamma_{1,1} \) and/or \( \gamma_{1,2} \), but we only have an initial value \((r = 0)\) at hand. Thus we suggest taking \( \gamma_0 \) to be 1 in (2.8) to obtain the point estimates at the first iteration.

As pointed out in He et al. (2021), the BB rate could be negative at some iterations and may vibrate drastically sometimes especially when \( \tau \) approaches to 0 or 1. To address these issues, we follow their suggestion to choose \( \gamma_r = \min\{\gamma_{1,r}, \gamma_{2,r}, 100\} \) if \( \gamma_{1,r} > 0 \) and 1 otherwise. Iterations of our proposed GD-BB algorithm will not be stopped until \( ||\nabla Q_h(\hat{\alpha}, \hat{d}, \hat{c}; \tau)||_2 \leq \text{tol} \), where \( \text{tol} > 0 \) is a predetermined threshold called the gradient tolerance in literature.

3. Theoretical Properties. Let \( R(\theta; \tau) \) and \( R_h(\theta; \tau) \) denote the expected loss function in (2.3) and (2.4), respectively. Note that \( \hat{\theta}_{h,\lambda}(\tau) \) actually estimates \( \theta_{0h}(\tau) = \arg\min_{(\alpha, \beta) \in \mathcal{H}} R_h(\theta; \tau) \). To investigate the consistency and weak convergence of \( \hat{\theta}_{h,\lambda}(\tau) \), we introduce the following objective function

\[
\ell_{h,\lambda} := R_h(\theta; \tau) + \frac{\lambda}{2} J(\beta, \beta)
\]

to bridge the gap between \( \hat{\theta}_{h,\lambda}(\tau) \) and \( \theta_0(\tau) \). Our main idea is to first establish the approximation error of the minimizer of (3.1), denoted by \( \theta_{0h,\lambda}(\tau) \), relative to \( \theta_0(\tau) \), and then quantify the estimation error of \( \hat{\theta}_{h,\lambda}(\tau) \) relative to \( \theta_{0h,\lambda}(\tau) \). We develop a key technical tool called the functional Bahadur representation of the estimator \( \hat{\theta}_{h,\lambda}(\tau) \) to derive its consistency and weak convergence.

3.1. Approximation error analysis. Let \( f(\cdot|x) \) be the conditional probability density function of \( Y \) given \( X = x \). For the existence of the regular conditional distribution of \( Y \) given \( X \), please refer to Kato (2012) for a more detailed discussion. To establish the desirable theoretical properties of the proposed estimator, we need the following regularity conditions.

Assumption A1. The conditional density and quantile of \( Y \) given \( X = x \) satisfy

(a) The conditional density \( f(y|x) \) is continuous and strictly positive over \( \mathbb{R} \times \text{supp}(X) \).
(b) There exists an integer $s \geq 1$ such that the $s$-th order derivative $f^{(s)}(\cdot|x)$ is uniformly continuous in the sense that

$$
\lim_{\epsilon \to 0} \sup_{(y,z) \in \mathbb{R} \times \text{supp}(X)} \sup_{t: |t| \leq \epsilon} |f^{(s)}(y + t|x) - f^{(s)}(y|x)| = 0,
$$

and that for $j = 0, 1, \ldots, s$, $\sup_{(y,z) \in \mathbb{R} \times \text{supp}(X)} |f^{(j)}(y|x)| < \infty$ and $\lim_{y \to \pm \infty} f^{(j)}(y|x) = 0$.

(c) $X$ is weighted-centered in the sense that $E \{ B_\tau(X) X(t) \} = 0$ for every $t \in \mathbb{I}$, where $B_\tau(X) = (\alpha_0(\tau) + \int_0^1 \beta_0(t, \tau) X(t) dt | X)$.

Furthermore, we assume that there exists a positive constant $C_1$ such that $C_1^{-1} \leq B_\tau(X) \leq C_1$ a.s.

**Remark 3.1.** Assumptions A1 (a) and (b) are fairly common in the literature of quantile regression; see Belloni et al. (2019), Chao et al. (2017) and Fernandes et al. (2021) for instance. Concerning the definition of $B_\tau(X)$, it is actually $F^\tau_{\epsilon}(X)$ in the context of a standard functional linear regression $Y = \alpha + \int \beta(t) X(t) dt + \epsilon$, where $\epsilon$ is independent of $X$.

**Assumption A2.** The kernel function $k$ and bandwidth $h$ satisfy

(a) The kernel function $k : \mathbb{R} \to \mathbb{R}$ is even, integrable, twice differentiable with bounded first and second derivatives, and satisfies $\int k(u) du = 1$ and $0 < \int_0^\infty K(u)(1 - K(u)) du < \infty$. Additionally, $\int |u^{s+1} k(u)| du < \infty$ for $s$ in (b) of Assumption A1, and $k$ is orthogonal to all non-constant monomials of degree up to $s$, i.e., $\int u^j k(u) du = 0$ for $j = 1, \ldots, s$, and $\int |u^{s+1} k(u)| du \neq 0$.

(b) $h = h_n = o(1)$ as $n \to \infty$.

Let $C(s, t; \tau) = E \{ B_\tau(X) X(t) X(s) \}$ be the weighted covariance function. We introduce the following inner product in $H^m(\mathbb{I})$:

$$
\langle \beta, \tilde{\beta} \rangle_1 = V(\beta, \tilde{\beta}) + \lambda J(\beta, \tilde{\beta}),
$$

where $V(\beta, \tilde{\beta}) = \int_0^1 \int_0^1 C(s, t; \tau) \beta(s) \tilde{\beta}(s) ds dt$. Here we suppress the dependence of $V$ on $\tau$ for ease of notation. Define an integral operator $C$ from $L^2(\mathbb{I})$ to $L^2(\mathbb{I})$: $\beta \mapsto (C \beta)(t) = \int_0^1 C(s, t; \tau) \beta(s) ds$. The following regularity condition ensures that $V$ is positive definite.

**Assumption A3.** $C(s, t; \tau)$ is continuous on $\mathbb{I} \times \mathbb{I}$. Furthermore, for any $\beta \in L^2(\mathbb{I})$, $C \beta = 0$ if and only if $\beta = 0$.

As illustrated in Shang and Cheng (2015a), $H^m(\mathbb{I})$ is an RKHS under $\langle \cdot, \cdot \rangle_1$. Denote the reproducing kernel function by $K(s, t)$. Then we define a proper inner product in $H$ such that it is a well-defined Hilbert space. Following the idea of Shang and Cheng (2015a), for any $\theta = (\alpha, \beta), \hat{\theta} = (\tilde{\alpha}, \tilde{\beta}) \in H$, define

$$
\langle \theta, \tilde{\theta} \rangle = E \left\{ B_\tau(X) \left[ \alpha + \int_0^1 X(t) \beta(t) dt \right] \left[ \tilde{\alpha} + \int_0^1 X(t) \tilde{\beta}(t) dt \right] \right\} + \lambda J(\beta, \tilde{\beta}).
$$

Obviously $\langle \theta, \tilde{\theta} \rangle = \langle \beta, \tilde{\beta} \rangle_1 + E \{ B_\tau(X) \} \alpha \tilde{\alpha}$. Given the fact that $V$ is positive definite under Assumption A3, $\langle \cdot, \cdot \rangle$ defined in (3.3) is a well-defined inner product. This indicates that $H$ is indeed a Hilbert space equipped with the norm $\langle \cdot, \cdot \rangle$ in (3.3).

Next we assume that there exists a sequence of basis functions in $H^m(\mathbb{I})$ that can simultaneously diagonalize both $V$ and $J$ in Equation (3.2).
ASSUMPTION A4. There exists a sequence of basis functions \( \{ \phi_\nu \}_{\nu \geq 1} \subset H^m(\mathbb{I}) \) such that \( \| \phi_\nu \|_{L^2} \leq C_\phi \nu^a \) holds uniformly over \( \nu \) for some constants \( a \geq 0, C_\phi > 0 \), and that

\[
V(\phi_\mu, \phi_\nu) = \delta_{\mu \nu}, \quad J(\phi_\mu, \phi_\nu) = \rho_\nu \delta_{\mu \nu} \quad \text{for any } \mu, \nu \geq 1.
\]

where \( \delta_{\mu \nu} = 1 \) if \( \mu = \nu \) and 0 otherwise, and \( \rho_\nu \) is a nondecreasing nonnegative sequence satisfying \( \rho_\nu \sim \nu^l \) for some constant \( l > a + 1/2 \). Furthermore, any \( \beta \in H^m(\mathbb{I}) \) admits the Fourier expansion \( \beta = \sum_{\nu=1}^{\infty} V(\beta, \phi_\nu) \phi_\nu \) with convergence in \( H^m(\mathbb{I}) \) under \( \langle \cdot, \cdot \rangle_1 \).

Let \( K_t(\cdot) := K(t, \cdot) \in H^m(\mathbb{I}) \) for any \( t \in \mathbb{I} \). Assumption A4 implies that there exists a sequence of real numbers \( a_\nu \) such that \( K_t = \sum_{\nu=1}^{\infty} a_\nu \phi_\nu \). Obviously, \( \phi_\nu(t) = \langle K_t, \phi_\nu \rangle = a_\nu(1 + \nu \rho_\nu) \) for any \( \nu \geq 1 \). It follows that \( K_t = \sum_{\nu=1}^{\infty} \frac{\phi_\nu(t)}{1 + \nu \rho_\nu} \phi_\nu \). By the Riesz representation theorem, there exists a linear operator \( W_\lambda \) from \( H^m(\mathbb{I}) \) to \( H^m(\mathbb{I}) \) such that \( \langle W_\lambda \beta, \beta \rangle = \lambda J(\beta, \beta) \) for all \( \beta, \bar{\beta} \in H^m(\mathbb{I}) \). By the definition of \( J \), \( W_\lambda \) is nonnegative and self-adjoint. For any \( \nu \geq 1 \), write \( W_\lambda \phi_\nu = \sum_{\mu=1}^{\infty} b_{\nu \mu} \phi_\mu \). From Assumption A4, we have \( \lambda \rho_\nu \delta_{\mu \nu} = \lambda J(\phi_\nu, \phi_\mu) = \langle W_\lambda \phi_\nu, \phi_\mu \rangle = b_{\mu}(1 + \nu \rho_\mu) \). Therefore, \( b_\mu = \lambda \rho_\nu / (1 + \nu \rho_\nu) \) if \( \mu = \nu \) and 0 otherwise. As a result, \( W_\lambda \phi_\nu = \frac{\lambda \rho_\nu}{1 + \nu \rho_\nu} \phi_\nu \).

**Proposition 3.1.** Suppose that Assumption A4 holds, then for any \( t \in \mathbb{I} \), we have

\[
K_t(\cdot) = \sum_{\nu=1}^{\infty} \frac{\phi_\nu(t)}{1 + \nu \rho_\nu} \phi_\nu(\cdot),
\]

and for any \( \nu \geq 1 \),

\[
(W_\lambda \phi_\nu)(\cdot) = \frac{\lambda \rho_\nu}{1 + \nu \rho_\nu} \phi_\nu(\cdot).
\]

For any \( x \in L^2(\mathbb{I}) \), let \( \eta(x) = \sum_{\nu=1}^{\infty} \frac{x_\nu}{1 + \nu \rho_\nu} \phi_\nu \), where \( x_\nu = \int_0^1 x(t) \phi_\nu(t) dt \). By Proposition 2.4 of Shang and Cheng (2015a), we can conclude that \( R_x = \left( \mathbb{E} \{ B_x(X) \} \right)^{-1}, \eta(x) \in \mathcal{H} \), satisfies \( \langle R_x, \theta \rangle = \alpha + \int_0^1 x(t) \beta(t) dt \) for any \( \theta = (\alpha, \beta) \in \mathcal{H} \). We need to emphasize that \( R_x \) depends on \( \lambda \) through the definition of \( \eta(x) \).

Let \( \ell_{n, h, \lambda}(\theta) = \hat{R}_h(\theta; \tau) + \frac{\lambda}{2} J(\beta, \beta) \). For notational convenience, denote \( \Delta \theta = (\Delta \alpha, \Delta \beta) \) and \( \Delta \theta_j = (\Delta \alpha_j, \Delta \beta_j) \) for \( j = 1, 2, 3 \). The first order Fréchet derivative of \( \ell_{n, h, \lambda}(\theta) \) w.r.t \( \theta \) is given by

\[
S_{n, h, \lambda}(\theta) \Delta \theta = \frac{1}{n} \sum_{i=1}^{n} \left[ \mathcal{R} \left( \frac{-\epsilon_i(\theta)}{h} \right) - \tau \right] \langle R_{X_i}, \Delta \theta \rangle + \lambda J(\beta, \beta).
\]

The second- and third-order Fréchet derivatives of \( \ell_{n, h, \lambda} \) can be shown to be, respectively,

\[
DS_{n, h, \lambda}(\theta) \Delta \theta_1 \Delta \theta_2 = \frac{1}{n} \sum_{i=1}^{n} k_h(-\epsilon_i(\theta)) \langle R_{X_i}, \Delta \theta_1 \rangle \langle R_{X_i}, \Delta \theta_2 \rangle + \lambda J(\Delta \beta_1, \Delta \beta_2)
\]

and

\[
D^2 S_{n, h, \lambda}(\theta) \Delta \theta_1 \Delta \theta_2 \Delta \theta_3 = \frac{1}{n} \sum_{i=1}^{n} h^{-2} k^{(1)}(-\epsilon_i(\theta)/h) \langle R_{X_i}, \Delta \theta_1 \rangle \langle R_{X_i}, \Delta \theta_2 \rangle \langle R_{X_i}, \Delta \theta_3 \rangle
\]

**ASSUMPTION A5.** The functional covariate \( X \) satisfies that there exists a constant \( s \in (0, 1) \) such that

\[
E \{ \exp(s \| X \|_{L^2}) \} < \infty,
\]
where \( \|X\|_2^2 = \int_0^1 X^2(t) dt \). We further assume that there exists a constant \( M_0 > 0 \) such that for any \( \beta \in H^m(\mathbb{I}) \),

\[
(3.7) \quad \mathbb{E} \left\{ \left\| \int_0^1 X(t) \beta(t) dt \right\|^4 \right\} \leq M_0 \left\{ \mathbb{E} \left\{ \left\| \int_0^1 X(t) \beta(t) dt \right\|^2 \right\} \right]^2
\]

Remark 3.2. It is easy to check that (3.6) holds if the support of \( X \) is a compact subset of \( L^2(\mathbb{I}) \) since \( \|X\|_2 \leq c_0 \) a.s. for some constant \( c_0 > 0 \). When \( X \) is a Gaussian process with a square-integrable mean function, Equation (3.6) holds for any \( s \in (0, 1/4) \) by Proposition 3.2 of Shang and Cheng (2015a), and Equation (3.7) is fulfilled for \( M_0 = 3 \); see Yuan and Cai (2010).

For \( j = 1, 2, 3 \), let \( R^{(j)}(\theta; \tau) \) and \( R_h^{(j)}(\theta; \tau) \) be the \( j \)th order Fréchet derivative operators of \( R(\theta; \tau) \) and \( R_h(\theta; \tau) \), respectively. Denote by \( S \) the set \( \mathcal{H} \times [\tau, \bar{\tau}] \) to which \((\theta, \tau)\) belongs. An auxiliary norm is introduced for technical purpose: \( \|\theta\|_2 = |\alpha| + \|\beta\|_{L^2} \) for any \( \theta = (\alpha, \beta) \in \mathcal{H} \). Before we present the theorem concerning the approximation error of \( \hat{\theta}_{0h, \lambda}(\tau) \), we first establish the following properties of the expected surrogate loss function \( R_h(\theta; \tau) \).

Lemma 3.1. Suppose Assumptions A1, A2 and A5 hold. Then

(i) \( \sup_{(\theta, \tau) \in S} \left| \frac{R_h(\theta; \tau) - R(\theta; \tau)}{h + 1} \right| = O(1) \);

(ii) \( \sup_{(\theta, \tau) \in S} \left| \frac{R^{(1)}(\theta; \tau) - R^{(1)}(\theta; \tau)}{h + 1} \right| = O(1) \|\theta\| \) for any \( \theta \in \mathcal{H} \).

(iii) \( \sup_{(\theta, \tau) \in S} \left| \frac{R^{(2)}(\theta; \tau) \theta_2 - R^{(2)}(\theta; \tau) \theta_2}{h + 1} \right| = o(1) \|\theta\| \|\theta\| \) for any \( \theta, \theta_2 \in \mathcal{H} \).

(iv) \( \sup_{(\theta, \tau) \in \mathcal{H} \times [\tau, \bar{\tau}]} \left| \frac{R^{(2)}(\theta + \theta, \tau) \theta \theta_2 - R^{(2)}(\theta; \tau) \theta \theta_2}{\|\theta\|} \right| = O(1) \|\theta\| \|\theta\| \) for any \( \theta, \theta_2 \in \mathcal{H} \).

The following theorem establishes the approximation error of the minimizer of \( \ell_{h\lambda} \) relative to \( \theta_0(\tau) \).

Theorem 3.1. Suppose that Assumptions A1 to A5 hold, and there exists some constant \( C_2 > 0 \) such that \( \sup_{\tau \in [\tau, \bar{\tau}]} J(\beta_0(\cdot, \tau), \beta_0(\cdot, \tau)) \leq C_2 \) Then there exists a unique minimizer of \( \ell_{\lambda} \), denoted by \( \hat{\theta}_{0h, \lambda}(\tau) \), for every \( \tau \in [\tau, \bar{\tau}] \). Furthermore, \( \|\theta_{0h, \lambda}(\tau) - \theta_0(\tau)\| = O(\lambda^{1/2}) \) uniformly over \( \tau \in [\tau, \bar{\tau}] \), provided that \( h = O(\lambda^{1/2}) \).

3.2. Bahadur representation for smoothed FLQR. Next we study the estimation error of \( \hat{\theta}_{h\lambda}(\tau) \). Denote \( \omega = \lambda^{1/(2l)} \), where \( l \) is specified in Assumption A4. Before laying out the main theorem, we first prove that \( \hat{\theta}_{h\lambda}(\tau) \) is a consistent estimator of \( \theta_0(\tau) \) and the convergence rate is established.

Theorem 3.2 (Convergence rate). Suppose that Assumptions A1 to A5 hold, and that the following rate conditions on \( \omega \) (or equivalently, \( \lambda \)) and \( h \) are satisfied

\[
(3.8) \quad h = O\left(\omega^{1/(s+1)}\right), \quad \omega = o(1), \quad n^{-1/2} \omega^{-1} h^{-2} = o(1), \quad n^{-1/2} \omega^{-(a+1)-(2l-2a-1)/(4m)} (h^{-1} \log n)(\log \log n)^{1/2} = o(1).
\]

Then \( \hat{\theta}_{h\lambda}(\tau) = (\hat{\alpha}_h(\tau), \hat{\beta}_h(\cdot, \tau)) \) is the unique solution to (2.5) and for every \( \tau \in [\tau, \bar{\tau}] \),

\[
\|\hat{\theta}_{h\lambda}(\tau) - \theta_0(\tau)\| = O_P(r_n), \text{ where } r_n = (n\omega)^{-1/2} + \omega^l.
\]
Now we are ready to present our main theorem: the Bahadur representation for the FLQR model under the RKHS framework. Similar results have been established in classic semi-/non-parametric regression; see Shang (2010), Shang and Cheng (2013), and Cheng and Shang (2015). Our result is the first one in functional quantile regression.

**Theorem 3.3 (Functional Bahadur representation).** Suppose that the assumptions of Theorem 3.2 are met. Then as \( n \to \infty \), \( \|\hat{\theta}_{h,\lambda}(\tau) - \theta_0(\tau) + S_{n,h,\lambda}(\theta_0(\tau))\| = O_P(a_n) \) for every \( \tau \in [\underline{\tau}, \overline{\tau}] \), where
\[
a_n = n^{-1/2} \omega^{-\frac{4m_0+6m-1}{4m}} r_n (n^{-1} \log n) (\log \log n)^{1/2} + 2h^{-1}r_n^2
\]

The Bahadur representation implies the following pointwise limiting distribution of the estimated slope function defined by Equation (2.5). This result will be used to construct pointwise confidence intervals for \( \beta_0(\cdot, \tau) \).

**Corollary 3.1.** Suppose that the assumptions of Theorem 3.3 are satisfied, and \( \sup_{\nu \geq 1} \|\phi_{\nu}\| \leq C_{\phi} \nu^\beta \) where the constants \( a \) and \( C_{\phi} \) are defined in Assumption A4. Furthermore, we assume \( \sum \nu \phi_{\nu}^2 (t) (1 + \lambda \nu) \tau \) \( \omega^{-(2a+1)} \), \( n^{1/2}a_n = o(1) \), \( n^{1/2}h^{s+1} = o(1) \) and \( n\omega^{2a+1}(\log(\omega^{-1}))^{-4} \to \infty \), as \( n \to \infty \). Let \( s_n = \text{Var} \{ \sum_{\nu = 1}^{n} [K(-\epsilon_i(\theta_0(\tau))/h) - \tau] \eta(X_i(t)) \} \). Then we have for any fixed \( t \in \mathbb{I} \) and any fixed \( \tau \in \mathcal{U} \),
\[
\frac{n}{s_n} [\hat{\beta}_{h,\lambda}(t, \tau) - \beta_0(t, \tau) + (W_\lambda \beta_0(\cdot, \tau))(t)] \xrightarrow{d} N(0,1).
\]

Additionally, if \( n s_n^{-1} [W_\lambda \beta_0(\cdot, \tau)](t) = o(1) \), then
\[
\frac{n}{s_n} [\hat{\beta}_{h,\lambda}(t, \tau) - \beta_0(t, \tau)] \xrightarrow{d} N(0,1).
\]

**Remark 3.3.** As explained in Remark 3.1, if the underlying true model is given by \( Y_i = \alpha + \int \beta(t)X_i(t)dt + \epsilon \), where \( \epsilon \) is independent of \( X \), \( B_\tau(X) = f_i(F^{-1}_\epsilon(t)) \). In this scenario, based on the proof Corollary 3.1, \( \hat{\beta}_{h,\lambda}(t, \tau) \) is immediately available:
\[
P \left( \beta_0(t_0, \tau) \in \left[ \hat{\beta}_{h,\lambda}(t_0, \tau) + \pm \frac{z_{1/2}}{\sqrt{n s_n^{-1} \text{Var}(f_i(F^{-1}_\epsilon(t))) - \sum \nu \phi_{\nu}^2 (t_0) \lambda \nu} \right] \right) \to 1 - \xi
\]
as \( n \to \infty \), where \( z_{1/2} \) is the \( (1 - \xi/2) \)-quantile of \( N(0,1) \), if \( n s_n^{-1} [W_\lambda \beta_0(\cdot, \tau)](t_0) = o(1) \).

Besides its limiting distribution at a fixed point \( t \in \mathbb{I} \), we can even establish weak convergence of \( \hat{\beta}_{h,\lambda}(\cdot, \tau) \) in the Hilbert space \( H^m(\mathbb{I}) \) equipped with the inner product \( V(\cdot, \cdot) \) for any given \( \tau \in \mathcal{U} \). Define \( \kappa_n^2 := (\tau - \tau^2)\mathbb{E} [\int_0^1 X_i(t) \phi_{\nu}(t)dt]^2 \).

**Theorem 3.4.** Suppose that the assumptions of Theorem 3.3 are satisfied, and \( \sup_{\nu \geq 1} \|\phi_{\nu}\| \leq C_{\phi} \nu^\beta \) where the constants \( a \) and \( C_{\phi} \) are defined in Assumption A4. Furthermore, \( \sum \nu \phi_{\nu}^2 (t) (1 + \lambda \nu) \tau \) \( \omega^{-(2a+1)} \) holds uniformly for \( t \in \mathbb{I} \), \( n^{1/2}a_n = o(1) \), \( n^{1/2}h^{s+1} = o(1) \), \( n\omega^{2a+1}(\log(\omega^{-1}))^{-4} \to \infty \), as \( n \to \infty \). In addition, the true slope function \( \beta_0(\cdot, \tau) \) can be written as \( \beta_0 = \sum \nu b_{\nu} \phi_{\nu} \) satisfying \( \sum \nu b_{\nu}^2 \phi_{\nu}^2 < \infty \), and \( \Gamma(s, t) = \lim_{n \to \infty} \omega^{2a+1} \sum \nu \frac{s^2 \phi_{\nu}(s) \phi_{\nu}(t)}{(1 + \lambda \nu)^2} \) finitely exists for any \( s, t \in [0, 1] \). Then \( n^{1/2} \omega^{2a+1/2} \hat{\beta}_{h,\lambda}(t, \tau) - \beta_0(t, \tau) \] converges weakly to a mean zero Gaussian process \( \mathcal{G}(t) \) in the Hilbert space \( H^m(\mathbb{I}) \) equipped with the inner product \( V(\cdot, \cdot) \), whose covariance function is \( \Gamma(s, t) \).
Remark 3.4. We leverage Theorem 3.4 to construct an \((1 - \alpha)\) simultaneous confidence band (SCB) for \(\beta(\cdot, \tau)\). In particular, we need to find a critical value \(q_\alpha\) that approximately satisfies \(P(\sup_{t \in [1]} \mathcal{G}(t) \leq q_\alpha) = 1 - \alpha\). We employ the Chebfun package in Matlab to find the eigenvalues \(\rho_\nu\)'s and the corresponding eigenfunctions \(\phi_\nu\)'s by solving the integro-differential equations (2.7) of Shang and Cheng (2015a). One can refer to Driscoll et al. (2008) for more details on the implementation of this open source package. After estimating the covariance function \(\Gamma(s, t)\) of \(\mathcal{G}(t)\), we generate independent sample paths of \(\mathcal{G}(t)\) to estimate the critical value \(q_\alpha\).

3.3. Confidence interval for conditional quantile. In this section, we discuss confidence intervals for conditional quantile. In particular, given any nonrandom \(x_0 \in L^2(I)\), we aim to construct a confidence interval for the true \(\tau\)-th conditional quantile of \(Y\) given \(X = x_0\). Obviously \(Q_{Y|X}(\tau|x_0) = \alpha_0(\tau) + \int_0^1 x_0(t) \beta_0(t, \tau) dt\). A direct plug-in point estimate of \(Q_{Y|X}(\tau|x_0)\) is \(\hat{Q}_{Y|X}(\tau|x_0) = \hat{\alpha}_{h, \lambda}(\tau) + \int_0^1 x_0(t) \hat{\beta}_{h, \lambda}(t, \tau) dt\). Let \(\sigma_0^2(x_0) = [E \{B_\nu(X)\}]^{-1} + \sum \nu (x_0^2)_{(1 + \nu, \tau)}\), where \(x_0^\nu = \int_0^1 x_0(t) \phi_\nu(t) dt, \nu \geq 1\). The following theorem provides a valid \((1 - \xi)\) confidence interval for the true conditional quantile.

Theorem 3.5. Suppose that the assumptions of Theorem 3.3 are satisfied. Furthermore, \(n^{1/2} \sigma_{x_0} = o(1), \|R_{x_0}\| \ll \sigma_n(x_0), n^{1/2} \sigma_{x_0} = o(1)\) and \(n \omega^{-2n+1} \log(\omega^{-1})^{-4} \to \infty\), as \(n \to \infty\). Let \(s_{2n}^2 = \text{Var} \left( \sum_{i=1}^n \| \mathcal{K} \{ -\epsilon_i(\theta_0(\tau))/h \} - \tau \| (R_{x_0}, R_{X_i}) \right)\). Then

\[
\frac{n}{s_{2n}^2} \left[ \hat{Q}_{Y|X}(\tau|x_0) - Q_{Y|X}(\tau|x_0) + \int_0^1 x_0(t) \{ W_\lambda(\beta_0(\cdot, \tau)) \}(t) dt \right] \dot{\rightarrow} N(0, 1).
\]

Additionally, if the true slope function \(\beta_0(\cdot, \tau)\) can be written as \(\beta_0(\cdot, \tau) = \sum \nu b_\nu \phi_\nu\) satisfying \(\sum \nu b_\nu^2 \nu^2 < \infty\) and \(n \omega^{-d} = o(1)\), then \(\frac{n}{s_{2n}^2} \int_0^1 x_0(t) \{ W_\lambda(\beta_0) \}(t) dt = o(1)\). It follows that

\[
\frac{n}{s_{2n}^2} \left[ \hat{Q}_{Y|X}(\tau|x_0) - Q_{Y|X}(\tau|x_0) \right] \dot{\rightarrow} N(0, 1).
\]

Hence the \((1 - \xi)\) confidence interval for \(Q_{Y|X}(\tau|x_0)\) is

\[
[\hat{Q}_{Y|X}(\tau|x_0) \pm n^{-1} s_{2n} \xi / 2]
\]

4. Simulation study. In this section, we carry out simulation studies to evaluate the finite-sample performance of the proposed method in estimating the coefficient function in FLQR. We further investigate the estimated confidence intervals and the SCB proposed in Section 3.

4.1. Estimation of \(\beta(\cdot, \tau)\). We first compare the performance of the estimated coefficient function through the FPCA approach proposed by Kato (2012) and our proposed approach under the RKHS framework. In particular, the functional covariate is generated as \(X_i(t) = \sum_{k=1}^{50} \xi_k \psi_k \) for \(i = 1, \ldots, n\), where \(\xi_k = 4 \cdot (-1)^{k+1} \cdot k^{-2}\), and \(\psi_1 = 1\) and \(\psi_k = \sqrt{2} \cos((k - 1)\pi t)\) for \(k \geq 2\) are the eigenfunctions of \(X\). The random coefficients \(\xi_k\)'s are i.i.d generated from uniform distribution on \([-\sqrt{3}, \sqrt{3}]\). Figure 1 displays the trajectories of 10 randomly selected functional covariates. The response is generated as \(y_i = \alpha + \int_0^1 X_i(t) \beta(t) dt + \sigma_i\), where the intercept \(\alpha = 0.1\) and the coefficient function \(\beta(t) = e^{-t}\). We consider two different designs, normal distribution of mean 0 and student t distribution with 3 degrees of freedom, for the random noise \(\epsilon\) to accommodate both light and heavy tails. Different values of \(\sigma\)'s are chosen to deliver
two signal-to-noise ratios (SNR): 10 and 5. 200 independent Monte Carlo simulations are run for each design, where \( n = 200 \) curves are generated.

**Figure 1.** Trajectories of the functional covariate from 10 randomly selected subjects in one simulation run.

**Figure 2.** True slope function \( \beta(t) \) and estimates from FPAC and RKHS based methods. In each panel, the solid black line represents the estimate from the FPCA-based method, while the dashed blue line and the dotted red line represent the true coefficient function and the estimate from the proposed method, respectively.

To implement our proposed method, we take \( m = 2 \) for the choice of \( H^m(\mathbb{I}) \) and adopt the Silverman (1986)’s rule-of-thumb (ROT) bandwidth to choose \( h \) in kernel smoothing.
for saving computational cost. More specifically, we first fit a standard quantile regression of $Y$ with respect to $z_{ij} = \int X_i(t)g_j(t)dt$, where $g_1 = 1$, $g_2(t) = t - 0.5$ and $g_j(t) = \xi_{j-2}(t)$ for $j = 3, \ldots, 2 + n$. Then the ROT bandwidth is given by $h_{\text{ROT}} = 1.06\hat{s}n^{-1/5}$, where $\hat{s}$ is the minimum of the estimated standard error and the interquartile range (divided by 1.39) of the residuals from fitting the standard quantile regression. To choose the other tuning parameter $\lambda$, we employ the five-fold cross validation, where the smoothed empirical loss function $\hat{R}_h$ defined in Equation (2.5) is used to compare the performance of the estimators generated from different $\lambda$'s. The optimal $\lambda$ that minimizes this empirical loss turns out to be around $1 \times 10^{-6}$.

We consider three different quantile levels ($\tau = 0.25, 0.5, 0.75$) to compare these two approaches. In particular, for each $\tau$, we define the mean squared integration error of the estimated slope function $\hat{\beta}(\cdot, \tau)$ as $\text{MISE}(\hat{\beta}(\cdot, \tau)) = \int_0^1 (\hat{\beta}(t, \tau) - \beta(t))^2 dt$. Table 1 summarizes MISEs for the estimated slope functions based on the FPCA method and the RKHS method in different scenarios across 200 simulation runs.

| Distribution | SNR 10 | SNR 5 |
|--------------|-------|-------|
| Normal       |       |       |
| FPCA         | 1.00 (1.06) | 0.55 (0.57) |
| RKHS         | 1.47 (1.42) | 0.29 (0.30) |
| FPCA         | 1.53 (1.60) | 0.33 (0.32) |
| RKHS         | 1.00 (1.06) | 0.55 (0.57) |
| t3           |       |       |
| FPCA         | 2.73 (1.73) | 0.95 (0.98) |
| RKHS         | 2.57 (1.44) | 0.53 (0.56) |
| FPCA         | 2.85 (2.08) | 0.64 (0.73) |
| RKHS         | 1.72 (1.53) | 0.58 (0.56) |
| t3           |       |       |
| FPCA         | 0.88 (1.05) | 0.37 (0.34) |
| RKHS         | 0.61 (0.72) | 0.19 (0.20) |
| FPCA         | 0.81 (0.93) | 0.15 (0.16) |
| RKHS         | 1.39 (1.40) | 0.31 (0.35) |
| t3           |       |       |
| FPCA         | 1.72 (1.53) | 0.58 (0.56) |
| RKHS         | 1.39 (1.40) | 0.31 (0.35) |
| FPCA         | 1.82 (1.52) | 0.28 (0.32) |
| RKHS         | 1.00 (1.06) | 0.55 (0.57) |

Table 1. Summary of averages and standard errors of MISE ($\times 10^{-2}$) for the estimated slope functions based on the FPCA method and the RKHS method in different scenarios across 200 simulation runs.

| $\tau$ | SNR = 10 | SNR = 5 |
|--------|----------|---------|
|        | 0.1      | 0.5     | 0.9     |
|        | 0.1      | 0.5     | 0.9     |

Table 2. Coverage probabilities of $\beta_0(t, \tau)$ at $t = 0.1, 0.5$ and 0.9 and $\tau \in \{0.25, 0.5, 0.75\}$.

4.2. Confidence intervals/bands for $\beta(\cdot, \tau)$. In this section we study the performance of the pointwise confidence intervals introduced in Remark 3.3 and the SCB in Remark 3.4. Panel (a) in Figure 3 displays the coverage probabilities of the pointwise confidence intervals for normally distributed errors with SNR = 10. For all of these three quantile levels, the true coverage probabilities are always close to the nominal level 95%. Panels (b) - (d) depict the estimated coefficient function based on our proposed method and the corresponding pointwise confidence intervals and the SCB from one random instance for the three quantile levels. The estimated coefficient function provides a reasonable estimate of the true coefficient function for all three quantile levels and the SCB fully covers the true slope function. Table 2 details the coverage probability of $\hat{\beta}(t, \tau)$ for
β_0(t, τ) at \( t = \{0.1, 0.5, 0.9\} \) and \( τ = \{0.25, 0.5, 0.75\} \). Even with a moderate number of training curves (\( n = 200 \)), the true coverage probabilities are close to the nominal level 95% for both of these two signal to noise levels.

**Figure 3.** (a): The true coverage probability of the pointwise confidence intervals across 200 simulations for normally distributed errors with SNR = 10. (b)-(d): In each panel, the solid black line represents the true slope function, while the dashed red line represents the estimated slope function from one random simulation run. The solid blue lines and the dashed green lines represent 95% pointwise confidence intervals and a 95% SCB, respectively.

### 4.3. Inference on the conditional quantile.

We investigate the confidence intervals for conditional quantile proposed in Section 3.3. In addition to the training sample that consists of \( n = 200 \) or 400 curves and scalar responses, we generate a new observation \( x_0(t) \) and its corresponding response variable \( y_0 \); this new observation pair is fixed across \( B = 200 \) simulation runs. To evaluate the coverage probability of the constructed confidence interval for the conditional quantile given \( x_0 \), in each simulation run we implement the proposed estimation procedure on the independently generated training sample to generate a 95% confidence interval. The true coverage probability is approximated by the frequency of covering the true conditional quantile of \( y_0 \) given \( x_0 \) across 200 simulations. Table 3 summarizes the coverage probabilities of the constructed confidence interval in different designs. Regardless of sample size or SNR levels, the true coverage probability is always close to the nominal level 95%. When the SNR is fixed, increasing the size of the training sample can yield confidence intervals for the conditional quantile with a true coverage probability closer to the nominal level.

| \( τ \) | \( n = 200 \) | \( n = 400 \) |
|-------|----------------|----------------|
|       | SNR = 10       | SNR = 5        | SNR = 10       | SNR = 5   |
| 0.25  | 0.940          | 0.940          | 0.950          | 0.950     |
| 0.5   | 0.945          | 0.945          | 0.950          | 0.955     |
| 0.75  | 0.930          | 0.920          | 0.940          | 0.940     |

**Table 3.** Coverage probabilities of conditional quantile.
5. **Real example.** In this section, we apply the proposed method to one real-world example to demonstrate its performance.

Alzheimer’s disease (AD) is the most common cause of dementia, a general term for the impaired cognitive abilities such as memorizing, thinking, or making decisions that interferes with daily life. According to Ferreira and Busatto (2011), clinicians usually leverage a history of progressive and characteristic cognitive decline and the presence of objective cognitive deficits in the diagnosis of AD. Examinations leveraging neuroimaging plays a critical role in the diagnostic investigation of dementia. In particular, these examinations can not only facilitate identification of non-AD pathological processes that are related to cognitive decline but also are useful to pinpoint informative biomarkers that can help to enhance the diagnosis of AD. Substantial studies have shown correlations between quantitative measures collected by magnetic resonance brain imaging and AD progression (Ledig et al., 2018).

![Figure 4](image.png)

*Figure 4. (a): Histogram of the MMSE scores from $n = 214$ subject. (b): The trajectories of FA values measured at 83 different locations along CC from 10 randomly selected subjects.*

In this study, we obtained the data from the ongoing Alzheimer’s Disease Neuroimaging Initiative (ADNI), that unites researchers working to define the progression of AD. Particularly, the ADNI aims to determine the relationships between clinical, cognitive, imaging, genetic, and biochemical biomarkers over the whole range of AD progression. In this dataset, the functional covariate $X$ is fractional anisotropy (FA) collected by diffusion tensor imaging. The FA values were measured at 83 locations along the corpus callosum (CC) fiber tract for each subject. Panel (b) in Figure 4 depicts the FA trajectories of 10 randomly selected subjects. The scalar response of interest, $Y$, is the mini-mental state examination (MMSE) scores. We chose MMSE as the response since it is one of the most widely used test of cognitive functions for assessing the level of dementia a patient may have. Various functional regression models have been proposed to model the relations between the MMSE and FA trajectories; see Zhu et al. (2012) and Tang et al. (2021) for instance. Panel (a) in Figure 4 displays the histogram of the MMSE. Obviously the distribution of $Y$ is highly left skewed, hence a simple summary statistic like sample mean cannot adequately describe the (conditional) distribution of $Y$ given $X$. We are concerned about using functional linear quantile regression to...
provide a comprehensive characterization of the conditional distribution of the MMSE in this study.

After removing 3 subjects with missing values in FA trajectories, \( n = 214 \) subjects were retained in our analysis. To compare the prediction performance of the FPCA-based method by Kato (2012) and our RKHS-based method, a training set with 80% of all subjects were randomly selected and the remaining subjects constituted the test set. To better assess prediction accuracy, we randomly splitted the whole dataset 500 times with this ratio. Table 4 summarizes the mean prediction error, which is the average loss function \( \rho_\tau(\cdot) \) calculated on the test set, and the standard errors across the 500 splits for these two methods. Our proposed method outperforms its counterpart in terms of prediction accuracy in all of these three quantile levels.

| Method | \( \tau = 0.25 \) | \( \tau = 0.5 \) | \( \tau = 0.75 \) |
|--------|-----------------|-----------------|-----------------|
| FPCA   | 2.63 (1.46)     | 2.41 (0.95)     | 0.80 (0.25)     |
| RKHS   | 0.90 (0.11)     | 1.01 (0.11)     | 0.69 (0.08)     |

Table 4. Summary of averages and standard errors of prediction errors based on the FPCA method and the RKHS method across the 500 random splits.

Figure 5 depicts the estimated slope functions from minimizing Equation (2.7) for the above three quantile levels. In addition, the corresponding 95% pointwise confidence intervals for each estimated slope function are also displayed in Figure 5. Note \( t \) denotes the rescaled CC in Figure 4. We follow the idea of Yao et al. (2005) to employ the one-curve-leave-out analysis to obtain pointwise confidence intervals, which are depicted in Figure 5. These intervals suggest that the FA trajectory has a time-vary effect on the conditional quantile of the MMSE scores, and this effect becomes weaker at the middle range of the CC fiber tract.

![Figure 5](image-url)  
**Figure 5.** Estimated coefficient functions and the corresponding 95% pointwise confidence intervals for \( \tau \in \{0.25, 0.5, 0.75\} \).

6. Conclusions. In this paper we employ the kernel convolution technique to smooth the ordinary loss function in FLQR. Under the framework of RKHS, we establish a functional Bahadur representation for the minimizer of the regularized and
smoothed empirical risk function. Then we develop inference tools such as pointwise confidence intervals and a SCB for the coefficient function. With the aid of the representor theorem, we design a gradient descent algorithm to facilitate estimation of the intercept and the coefficient function in FLQR. This algorithm is essentially different from that based on FPCA proposed by Kato (2012). Furthermore, empirical studies demonstrate that our proposed estimator outperforms the latter in finite sample.

Traditional theoretical work on FLQR was focused on estimation consistency and/or convergence rates. For instance, Kato (2012) studied the minimax rate in the estimation of the coefficient function and the conditional quantile function. Yao et al. (2017) established the convergence rate of the estimated coefficient function in a partially FLQR model where a functional covariate as well as a finite-dimensional covariate is considered. In contrast, little work has been done to develop statistical inferences for this model. Though Li et al. (2016) is an exception, this work imposes strong assumptions on the coefficient function. In contrast, mild conditions are required for it under the theoretical framework developed in this paper.

APPENDIX A: PROOF OF LEMMAS AND THEOREMS

The section contains the proofs of the lemma and theorems in the main manuscript.

A.1. Proof of Lemma 3.1. Under Assumption A1, we obtain a Taylor expansion

\[ f(u + hz|x) = \sum_{j=0}^{s} f^{(j)}(u|x) \frac{(hz)^j}{j!} + \frac{(hz)^s}{(s-1)!} \int_0^1 (1-w)^{s-1} \left[ f^{(s)}(u + whz|x) - f^{(s)}(u|x) \right] dw. \]

Recall that for any \( x \in \text{supp}(X) \),

\[ E[k_h(u - Y)|x] - f(u|x) = \int k_h(u - y)f(y|x)dy - f(u|x) = \int k(z) [f(u + hz|x) - f(u|x)] dz \]

\( (A1) \)

\[ = \int_0^1 (1-w)^{s-1} \int \frac{(hz)^s}{(s-1)!}k(z) \left[ f^{(s)}(u + whz|x) - f^{(s)}(u|x) \right] dzdw \]

(i) As the check function can be written as

\[ \int \rho_\tau(u)dG(u) = (1 - \tau) \int_{-\infty}^0 G(v)dv + \tau \int_0^\infty [1 - G(u)]du \]

for any cdf \( G \). Therefore,

\[ R(\theta; \tau) = \int \left\{ (1 - \tau) \int_{-\infty}^0 \int_{-\infty}^{v+(R_s, \theta)} f(u|x)du dv + \tau \int_0^\infty \int_{v+(R_s, \theta)} f(u|x)du dv \right\} dP_X(x). \]

Similarly, we have

\[ R_h(\theta; \tau) = \int \left\{ (1 - \tau) \int_{-\infty}^0 \int_{-\infty}^{v+(R_s, \theta)} E[k_h(u - Y)|x]du dv + \tau \int_0^\infty \int_{v+(R_s, \theta)} E[k_h(u - Y)|x]du dv \right\} dP_X(x). \]

Under Assumption A2, integrating (A1) yields

\[ L_1 := \left| \int_{-\infty}^0 \int_{-\infty}^{v+(R_s, \theta)} \{ E[k_h(u - Y)|x] - f(u|x) \} du dv \right| \]
and 

Hence, for some constant $C$, this establishes (i).

(ii) By the Lebesgue dominated convergence theorem, we have

$$R^{(1)}(\theta; \tau)\theta_1 = E \left[ (F(\langle R_X, \theta \rangle | X) - \tau) \langle R_X, \theta_1 \rangle \right] = \int \left[ \int_{-\infty}^{\langle R_X, \theta \rangle} f(y|x) dy - \tau \right] \langle R_X, \theta_1 \rangle dP_X(x),$$

and

$$R_h^{(1)}(\theta; \tau)\theta_1 = E \left\{ \left[ K \left( \frac{\langle R_X, \theta \rangle - Y}{h} \right) - \tau \right] \langle R_X, \theta_1 \rangle \right\} = \int \left\{ \int_{-\infty}^{\langle R_X, \theta \rangle} E[k_h(u - Y|x)] du - \tau \right\} \langle R_X, \theta_1 \rangle dP_X(x).$$

As $f(u)du = 0$ and $\int u^s k(u)du < \infty$, by integrating (A1) we have

$$L_2 := \left| \int_{-\infty}^{\langle R_X, \theta \rangle} E[k_h(u - Y|x)] - f(u|x) du \right|$$

$$= \left| \int_0^1 (1 - w)^{s-1} \int \frac{(hw)^s}{(s-1)!} k(z) \int_{-\infty}^{\langle R_X, \theta \rangle} [f(s)(u + whz|x) - f(s)(u|x)] du dv dz dw \right|$$

$$= \left| \int_0^1 (1 - w)^{s-1} \int \frac{(hw)^s}{(s-1)!} k(z) \left[ f(s-1)(\langle R_X, \theta \rangle + whz|x) - f(s-1)(\langle R_X, \theta \rangle|x) \right] dz \right|$$

$$\leq Ch^{s+1},$$

uniformly given the fact that $f(s)$ is bounded.

It follows that

$$R_h^{(1)}(\theta; \tau)\theta_1 - R^{(1)}(\theta; \tau)\theta_1 \leq Ch^{s+1} E[|\langle R_X, \theta_1 \rangle|]$$

$$\leq Ch^{s+1} E \left[ \langle R_X, \theta_1 \rangle^4 \right]^{1/4} \leq Ch^{s+1} \|\theta_1\|$$

by the Cauchy-Schwarz inequality and Lemma S.3 of Shang and Cheng (2015b). This establishes (ii).
(iii) The second order Fréchet derivative operator of $R(\theta; \tau)$ satisfies

$$R^{(2)}(\theta; \tau) \theta_1 \theta_2 = E \left[ f(\langle R_X, \theta \rangle | X) \langle R_X, \theta_1 \rangle \langle R_X, \theta_2 \rangle \right] = \int f(\langle R_s, \theta \rangle | x) \langle R_s, \theta_1 \rangle \langle R_s, \theta_2 \rangle dP_X(x),$$

and, analogously,

$$R^{(2)}_h(\theta; \tau) \theta_1 \theta_2 = E \left[ k_h(\langle R_X, \theta \rangle - Y) \langle R_X, \theta_1 \rangle \langle R_X, \theta_2 \rangle \right] = \int E [k_h(\langle R_s, \theta \rangle - Y | x)] \langle R_s, \theta_1 \rangle \langle R_s, \theta_2 \rangle dP_X(x).$$

Setting $u = \langle R_s, \theta \rangle$. In light of (A1), we have

$$|E [k_h(u - Y)|x] - f(u|x)| \leq Ch^s \int |z^s k(z)| \sup_{(y,x) \in \mathbb{R} \times \text{supp}(X)} \sup_{t:|t| \leq h^z} \left| f(s)(y + t|x) - f(s)(y|x) \right| dz$$

$$= o(h^s)$$

under Assumption A1 and Assumption A2.

Then following the proof of part (ii), it is easy to verify that

$$|R^{(1)}_h(\theta; \tau) \theta_1 \theta_2 - R^{(1)}(\theta; \tau) \theta_1 \theta_2| = o(h^s) \|\theta_1\| \|\theta_2\|.$$ This establishes (iii).

(iv) Note that

$$R^{(2)}_h(\theta; \tau) \theta_1 \theta_2 = E \left[ k_h(\langle R_X, \theta \rangle - Y) \langle R_X, \theta_1 \rangle \langle R_X, \theta_2 \rangle \right] = \int k(z) \int f(\langle R_s, \theta \rangle + h z|x) \langle R_s, \theta_1 \rangle \langle R_s, \theta_2 \rangle dP_X(x) dz.$$

Since $f(\cdot|x)$ is Lipschitz under Assumption A1 (d),

$$\left| R^{(2)}_h(\theta + \theta^*; \tau) \theta_1 \theta_2 - R^{(2)}_h(\theta; \tau) \theta_1 \theta_2 \right| \leq C \int |k(z)| \int \langle R_X, \theta^* \rangle \|\langle R_X, \theta_1 \rangle \| \|\langle R_X, \theta_2 \rangle \| dP_X(x) dz$$

$$\leq CE \|\langle R_X, \theta^* \rangle\| \|\langle R_X, \theta_1 \rangle\| \|\langle R_X, \theta_2 \rangle\|$$

$$\leq C \left[ E \left( \langle R_X, \theta^* \rangle^4 \right) \right]^{1/4} \left[ E \left( \langle R_X, \theta_1 \rangle^4 \right) \right]^{1/4} \left[ E \left( \langle R_X, \theta_2 \rangle^4 \right) \right]^{1/4}$$

$$\leq C \|\theta^*\| \|\theta_1\| \|\theta_2\|$$

uniformly in $(\theta, h, \theta^*, \tau)$.

**A.2. Proof of Theorem 3.1.** Recall that $R(\theta; \tau) = \int \rho_r(u) E \left[ d\hat{F}(u; \theta) \right]$, where the expectation is taken with respect to the joint distribution of $(X,Y)$. Since $\hat{F}(u; \theta) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}(\epsilon_i(\theta) \leq u)$, $E \hat{F}(u; \theta) = P(\epsilon(\theta) \leq u)$, where $\epsilon(\theta) = Y - \langle R_X, \theta \rangle$. Hence

$$\frac{dE \hat{F}(u; \theta)}{du} \equiv f_\epsilon(u; \theta),$$

where $f_\epsilon(\cdot, \theta)$ denotes the marginal density of $\epsilon(\theta) := Y - \langle R_X, \theta \rangle$. It follows that $R(\theta; \tau) = \int \rho_r(u) f_\epsilon(u; \theta)$. Similarly, $\hat{R}_h(\theta; \tau) = \frac{1}{n} \sum_{i=1}^n \int \rho_r(u) k_h(u - \epsilon_i(\theta)) du$ and $R_h(\theta; \tau) = E [\hat{R}_h(\theta; \tau)] = \int \rho_r(u) E \left[ k_h(u - \epsilon(\theta)) \right] du$. Let $\tilde{K}_h(u; \theta) = E [k_h(u - \epsilon(\theta))]$, where the expectation is taken with respect to the distribution of $\epsilon(\theta)$. In other words, $\tilde{K}_h(u; \theta) = \int f_\epsilon(u; \theta) k_h(u - v) dv$ and $R_h(\theta; \tau) = \int \rho_r(u) \tilde{K}_h(u; \theta) du$. So

$$\ell_{h\lambda}(\theta) = \int \rho_r(u) \tilde{K}_h(u; \theta) du + \frac{1}{2} J(\beta, \beta)$$

based on the definition of $\ell_{h\lambda}$ in (3.1).
The first-order Fréchet derivative of \( \ell_{h\lambda} \) w.r.t \( \theta \) satisfies that for any \( \Delta \theta = (\Delta \alpha, \Delta \beta) \in \mathcal{H} \),

\[
S_{h\lambda}(\theta) \Delta \theta \equiv \mathcal{D}_\theta \ell_{h\lambda}(\theta) \Delta \theta = \int \rho_\tau(u)D_{\theta}K_h(u; \theta) \Delta \theta du + \langle P_\lambda \Delta \theta, \Delta \theta \rangle,
\]

where \( P_\lambda(\theta) = (0, W_{\lambda \beta}) \in \mathcal{H} \) and satisfies that \( \langle P_\lambda(\theta), \bar{\theta} \rangle = (W_{\lambda \beta}, \bar{\beta}) \) for any \( \bar{\theta} = (\tilde{\alpha}, \tilde{\beta}) \in \mathcal{H} \) from Proposition 2.5 of Shang and Cheng (2015a). The second-order Fréchet derivative of \( \ell_{h\lambda} \) is

\[
D^2_{\theta} \ell_{h\lambda}(\theta) \Delta \theta_1 \Delta \theta_2 = \int \rho_\tau(u)D^2_{\theta}K_h(u; \theta) \Delta \theta_1 \Delta \theta_2 du + \langle P_\lambda \Delta \theta_1, \Delta \theta_2 \rangle.
\]

For notational brevity, we shall drop \( \tau \) from \( \theta_{0h}(\tau) \) if the quantile level is clear from the context. Since \( \theta_{0h} \) minimizes \( R_h(\theta; \tau) \), \( S_{h\lambda}(\theta_{0h}) = P_{\lambda_0} \).

Define \( T_1(\theta) = \theta - S_{h\lambda}(\theta_0 + \theta) \) for \( \theta \in \mathcal{H} \), and let \( S_{0\lambda}(\theta_0) = P_{\lambda_0} \). Write \( T_1(\theta) \) as \( T_1(\theta) + S_{0\lambda}(\theta_0) - S_{0\lambda}(\theta_0) \). So \( \|T_1(\theta)\| \leq \|T_1(\theta) + S_{0\lambda}(\theta_0)\| + \|S_{0\lambda}(\theta_0)\| \). For the first term, we have

\[
T_1(\theta) + S_{0\lambda}(\theta_0) = \theta - S_{h\lambda}(\theta_0 + \theta) + S_{0\lambda}(\theta_0)
\]

\[
= \theta - \int \rho_\tau(u)D_{\theta}K_h(u; \theta + \theta_0) du - P_{\lambda}(\theta + \theta_0)
\]

\[
+ \int \rho_\tau(u)D_{\theta}f_c(u; \theta_0) du + P_{\lambda}(\theta_0)
\]

\[
= \theta - \int \rho_\tau(u)[D_{\theta}K_h(u; \theta + \theta_0) - D_{\theta}K_h(u; \theta_0)] du
\]

\[
- \int \rho_\tau(u)D_{\theta}K_h(u; \theta_0) du - P_{\lambda}(\theta_0) + \int \rho_\tau(u)D_{\theta}f_c(u; \theta_0) du
\]

\[
= \theta - \int \rho_\tau(u) \left[ \int_0^1 D^2_{\theta}K_h(u; s\theta + \theta_0) \theta ds \right] du
\]

\[
- \int \rho_\tau(u)[D_{\theta}K_h(u; \theta_0) - D_{\theta}f_c(u; \theta_0)] du - P_{\lambda}(\theta_0).
\]

By the proof of Proposition 3.5 of Shang and Cheng (2015a), we know that \( [\int \rho_\tau(u)D^2_{\theta}f_c(u; \theta_0) du + P_{\lambda}] \theta = \theta \). So

\[
T_1(\theta) + S_{0\lambda}(\theta_0) = \int \rho_\tau(u)D^2_{\theta}f_c(u; \theta_0) \theta du + P_{\lambda} \theta - \int \rho_\tau(u) \left[ \int_0^1 D^2_{\theta}K_h(u; s\theta + \theta_0) \theta ds \right] du
\]

\[
- \int \rho_\tau(u)[D_{\theta}K_h(u; \theta_0) - D_{\theta}f_c(u; \theta_0)] du - P_{\lambda} \theta
\]

\[
= \int \rho_\tau(u) \left( \int_0^1 [D^2_{\theta}f_c(u; \theta_0) \theta - D^2_{\theta}K_h(u; s\theta + \theta_0) \theta ds \right) du
\]

\[
- \int \rho_\tau(u)[D_{\theta}K_h(u; \theta_0) - D_{\theta}f_c(u; \theta_0)] du =: I + II.
\]

For I, we have

\[
\left\| \int \rho_\tau(u)[D^2_{\theta}K_h(u; s\theta + \theta_0) - D^2_{\theta}f_c(u; \theta_0)] du \right\|_{op}
\]

\[
= \left\| \int \rho_\tau(u)[D^2_{\theta}K_h(u; s\theta + \theta_0) - D^2_{\theta}f_c(u; s\theta + \theta_0) + D^2_{\theta}f_c(u; s\theta + \theta_0) - D^2_{\theta}f_c(u; \theta_0)] du \right\|_{op}
\]
\[
\leq \left\| \int \rho_r(u) [D^2_\theta \tilde{K}_h(u; \theta) - D^2_\theta f_x(u; \theta)] du \right\|_{\text{op}}
\]
\[
+ \left\| \int \rho_r(u) [D^2_\theta f_x(u; \theta) - D^2_\theta f_x(u; \theta)] du \right\|_{\text{op}}
\]

Note that
\[
\left\| \int \rho_r(u) [D^2_\theta \tilde{K}_h(u; \theta) - D^2_\theta f_x(u; \theta)] du \right\|_{\text{op}}
= \sup_{\|\Delta \theta_1\| = \|\Delta \theta_2\| = 1} \left| \int \rho_r(u) D^2_\theta \tilde{K}_h(u; \theta) \Delta \theta_1 \Delta \theta_2 du - \int \rho_r(u) D^2_\theta f_x(u; \theta) \Delta \theta_1 \Delta \theta_2 du \right|
\]

For any \( \Delta \theta_1, \Delta \theta_2 \) with \( \|\Delta \theta_1\| = \|\Delta \theta_2\| = 1 \), based on the proof of (iii) of Lemma 3.1, we have
\[
\left| \int \rho_r(u) D^2_\theta \tilde{K}_h(u; \theta) \Delta \theta_1 \Delta \theta_2 du - \int \rho_r(u) D^2_\theta f_x(u; \theta) \Delta \theta_1 \Delta \theta_2 du \right|
= \left| \int E [k_h((R_x, s \theta + \theta_0) - Y|x)] (R_x, \Delta \theta_1) (R_x, \Delta \theta_2) dP_X(x)
- \int f((R_x, s \theta + \theta_0)|x) (R_x, \Delta \theta_1) (R_x, \Delta \theta_2) dP_X(x) \right|
= o(h^s)E (\|\langle R_x, \Delta \theta_1 \rangle (R_x, \Delta \theta_2)\|) \leq o(h^s) \left[ E_X(\|\langle R_x, \Delta \theta_1 \rangle\|^2) \right]^{1/2} \left[ E_X(\|\langle R_x, \Delta \theta_2 \rangle\|^2) \right]^{1/2}
\]

By Lemma S.3 of Shang and Cheng (2015b), the last term is bounded from above by a constant if (3.7) in Assumption A5 is met. Similarly, for any \( \Delta \theta_1, \Delta \theta_2 \) with \( \|\Delta \theta_1\| = \|\Delta \theta_2\| = 1 \)

\[
\left| \int \rho_r(u) D^2_\theta f_x(u; \theta) \Delta \theta_1 \Delta \theta_2 du - \int \rho_r(u) D^2_\theta f_x(u; \theta) \Delta \theta_1 \Delta \theta_2 du \right|
= \left| \int f((R_x, s \theta + \theta_0)|x) (R_x, \Delta \theta_1) (R_x, \Delta \theta_2) dP_X(x)
- \int f((R_x, \theta_0)|x) (R_x, \Delta \theta_1) (R_x, \Delta \theta_2) dP_X(x) \right|
\leq C_3 E (\|\langle R_x, \theta \rangle (R_x, \Delta \theta_1) (R_x, \Delta \theta_2)\|) \leq C_4 \|\theta\|
\]

with some positive constant \( C \) independent of \( n \). Hence \( \|I\| \leq C[o(h^s) + \|\theta\|] \cdot \|\theta\| \) uniformly over \( \tau \in [\underline{\tau}, \overline{\tau}] \).

We treat II next. For any \( \Delta \theta \) with \( \|\Delta \theta\| = 1 \), by the proof of (ii) of Lemma 3.1, we have
\[
\left| \int \rho_r(u) D^2_\theta \tilde{K}_h(u; \theta) \Delta \theta du - \int \rho_r(u) D^2_\theta f_x(u; \theta) \Delta \theta du \right|
= \left| \int \left\{ \int_{-\infty}^{(R_x, \theta)} E [k_h(u - Y|x)] du - \tau \right\} (R_x, \Delta \theta) dP_X(x)
- \int \left[ \int_{-\infty}^{(R_x, \theta)} f(y|x) dy - \tau \right] (R_x, \theta_1) dP_X(x) \right|
\leq C h^{s+1} E (\|\langle R_x, \Delta \theta \rangle\|) = O(h^{s+1}).
\]
It follows that there exists some $C > 0$ such that $\|T_1(\theta) + S_{\Omega}(\theta_0)\| \leq C[O(h^s) + \|\theta\|] \cdot \|\theta\| + C h^{s+1}$.

Lastly, we handle $S_{\Omega}(\theta_0) = P_{\lambda}(\theta_0)$. Write $\beta_0 = \sum_\nu b_\nu^0 \phi_\nu$. According to Proposition 3.1, $W_{\lambda} \beta_0 = \sum_\nu b_\nu^0 \frac{\lambda \rho_\nu}{1 + \lambda \rho_\nu} \phi_\nu$. Hence $\|S_{\Omega}(\theta_0)\|^2 = (W_{\lambda} \beta_0, W_{\lambda} \beta_0)_1 = \sum_\nu |b_\nu^0|^2 \frac{\lambda \rho_\nu^2}{(1 + \lambda \rho_\nu)^2} (1 + \lambda \rho_\nu) = \lambda \sum_\nu |b_\nu^0|^2 \rho_\nu (1 + \lambda \rho_\nu) = o(\lambda)$. The last equation holds since $J(\beta_0, \beta_0) = \sum_\nu |b_\nu^0|^2 \rho_\nu < \infty$, and $\sum_\nu |b_\nu^0|^2 \rho_\nu (1 + \lambda \rho_\nu) = o(1)$ as $\lambda \to 0$ by the dominated convergence theorem. Therefore, $\|S_{\Omega}(\theta_0)\| = o(\lambda^{1/2})$. Let $B(\rho) = \{ \theta \in \mathcal{H} : \|\theta\| \leq \rho \}$ be the ball in $\mathcal{H}$ of radius $\rho$. If we take $r_{1n} = C \max(\lambda^{1/2}, h_1^{s+1})$, which is of the order $\lambda^{1/2}$ if $h = O(\lambda^{(2s+2)})$. Previous results indicate that $T_1(B(r_{1n})) \subset B(r_{1n})$.

Next we show that $T_1$ is a contraction mapping. For any $\theta_j = (\alpha_j, \beta_j) \in B(r_{1n})$ for $j = 1, 2$, Taylor’s expansion yields that

$$T_1(\theta_1) - T_1(\theta_2)$$

$$= \theta_1 - \theta_2 - \int \rho_r(u) D_0 K_h(u; \theta_1 + \theta_0) du + \int \rho_r(u) D_0 K_h(u; \theta_2 + \theta_0) du$$

$$- P_{\lambda}(\theta_1 + \theta_0) + P_{\lambda}(\theta_2 + \theta_0)$$

$$= \theta_1 - \theta_2 - P_{\lambda}(\theta_1 - \theta_2) - \int \rho_r(u) \left[ \int_0^1 D_0^2 K_h(u; \theta_2 + \theta_0 + s(\theta_1 - \theta_2))(\theta_1 - \theta_2) ds \right] du$$

$$= \int \rho_r(u) D_0^2 f_c(u; \theta_0)(\theta_1 - \theta_2) du + P_{\lambda}(\theta_1 - \theta_2) - P_{\lambda}(\theta_1 - \theta_2)$$

$$- \int \rho_r(u) \left[ \int_0^1 D_0^2 K_h(u; \theta_2 + \theta_0 + s(\theta_1 - \theta_2))(\theta_1 - \theta_2) ds \right] du$$

$$= \int \rho_r(u) \left\{ \int_0^1 \left[ D_0^2 f_c(u; \theta_0) - D_0^2 K_h(u; \theta_2 + \theta_0 + s(\theta_1 - \theta_2)) \right] (\theta_1 - \theta_2) ds \right\} du.$$

By similar arguments as in the above analysis of $T_1(\theta)$, we have

$$\|T_1(\theta_1) - T_1(\theta_2)\|$$

$$\leq \left\| \int \rho_r(u) \left[ \int_0^1 D_0^2 K_h(u; \theta_2 + \theta_0 + s(\theta_1 - \theta_2)) - D_0^2 f_c(u; \theta_2 + \theta_0 + s(\theta_1 - \theta_2)) ds \right] du \right\| \|\theta_1 - \theta_2\|$$

$$+ \left\| \int \rho_r(u) \left[ \int_0^1 D_0^2 f_c(u; \theta_2 + \theta_0 + s(\theta_1 - \theta_2)) - D_0^2 f_c(u; \theta_0) ds \right] du \right\| \|\theta_1 - \theta_2\|$$

$$= o(h^s)\|\theta_1 - \theta_2\| + \left\| \int \rho_r(u) \left[ \int_0^1 D_0^2 f_c(u; \theta_2 + \theta_0 + s(\theta_1 - \theta_2)) - D_0^2 f_c(u; \theta_0) ds \right] du \right\| \|\theta_1 - \theta_2\|.$$

Note that for any $\Delta \theta_1, \Delta \theta_2$ with $\|\Delta \theta_1\| = \|\Delta \theta_2\| = 1$, 

$$\int \rho_r(u) \left[ \int_0^1 D_0^2 f_c(u; \theta_2 + \theta_0 + s(\theta_1 - \theta_2))\Delta \theta_1 \Delta \theta_2 - D_0^2 f_c(u; \theta_0)\Delta \theta_1 \Delta \theta_2 ds \right] du$$

$$= \left| \int \left[ \int_0^1 f((R_x, \theta_2 + \theta_0 + s(\theta_1 - \theta_2))|x) - f((R_x, \theta_0)|x) ds \right] (R_x, \Delta \theta_1, \Delta \theta_2) dP_X(x) \right.$$ 

$$\leq C_0 \int_0^1 E \left\{ ||\langle R_X, \theta_2 + s(\theta_1 - \theta_2) \rangle|| \langle R_X, \Delta \theta_1 \rangle \langle R_X, \Delta \theta_2 \rangle \right\} ds$$

$$\leq C_0' \int_0^1 E \left\{ ||\langle R_X, \theta_2 + s(\theta_1 - \theta_2) \rangle|^4 \right\}^{1/4} ds \times [E \left\{ ||\langle R_X, \Delta \theta_1 \rangle|^4 \right\}^{1/4} E \left\{ ||\langle R_X, \Delta \theta_2 \rangle|^4 \right\}^{1/4}]$$

$$\leq C_0' \|\theta_2\| + \|\theta_1 - \theta_2\| < 1/2.$$
because as $n \to \infty$, $\|\theta_2\| + \|\theta_1 - \theta_2\| = O(r_{1n}) = o(1)$. For the same reason, $o(h^s)\|\theta_1 - \theta_2\| < 1/2\|\theta_1 - \theta_2\|$.

Therefore, $T_1$ is contraction mapping on $B(r_{1n})$. By contraction mapping theorem, there exists uniquely an element $\theta'_{\lambda h} \in B(r_{1n})$ such that $T_1(\theta'_{\lambda h}) = \theta'_{\lambda h}$. Define $\theta_{0h,\lambda} = \theta'_{\lambda h} + \theta_0$. Then we have $S_{h, \lambda}(\theta_{0h,\lambda}) = 0$ and $\|\theta_{0h,\lambda} - \theta_0\| \leq r_{1n}$. This completes the proof.

**A.3. Proof of Theorem 3.2.** Define the operator $T_2(\theta) = \theta - [DS_{h, \lambda}(\theta_{0h,\lambda})]^{-1}S_{n, h, \lambda}(\theta_{0h,\lambda} + \theta)$ for $\theta \in \mathcal{H}$. We first that the operator $DS_{h, \lambda}(\theta_{0h,\lambda})$ is invertible. For any $\Delta \theta_1, \Delta \theta_2 \in \mathcal{H}$,

$$
\|DS_{h, \lambda}(\theta_{0h,\lambda}) - DS_{h, \lambda}(\theta_0)\| \Delta \theta_1 \Delta \theta_2
\leq |R^{(2)}_h(\theta_{0h,\lambda}; \tau)\Delta \theta_1 \Delta \theta_2 - R^{(2)}_h(\theta_0; \tau)\Delta \theta_1 \Delta \theta_2|
+ |R^{(2)}_h(\theta_0; \tau)\Delta \theta_1 \Delta \theta_2 - R^{(2)}_h(\theta_0; \tau)\Delta \theta_1 \Delta \theta_2|
\leq C\|\theta_{0h,\lambda} - \theta_0\| \|\Delta \theta_1\| \|\Delta \theta_2\| + o(h^s)\|\Delta \theta_1\| \|\Delta \theta_2\|.
$$

The last equation holds due to (iv) and (iii) of Lemma 3.1, respectively. By Theorem 3.1, we have that the operator norm of the difference of $DS_{h, \lambda}(\theta_{0h,\lambda})$ and $DS_{h, \lambda}(\theta_0)$ is strictly less than 1/2. As shown in the proof of Theorem 3.1, $DS_{h, \lambda}(\theta_0) = id$. Therefore $DS_{h, \lambda}(\theta_{0h,\lambda})$ is invertible, and the norm of $[DS_{h, \lambda}(\theta_{0h,\lambda})]^{-1}$ is between 2/3 and 2.

Rewrite $T_2$ as

$$
T_2(\theta) = -[DS_{h, \lambda}(\theta_{0h,\lambda})]^{-1}DS_{n, h, \lambda}(\theta_{0h,\lambda})\theta - DS_{h, \lambda}(\theta_{0h,\lambda})\theta
- [DS_{h, \lambda}(\theta_{0h,\lambda})]^{-1}S_{n, h, \lambda}(\theta_{0h,\lambda} + \theta) - S_{n, h, \lambda}(\theta_{0h,\lambda}) - DS_{h, \lambda}(\theta_{0h,\lambda})\theta
- [DS_{h, \lambda}(\theta_{0h,\lambda})]^{-1}S_{n, h, \lambda}(\theta_{0h,\lambda})
:= I_1 + I_2 + I_3.
$$

Next we treat these three terms to verify that $T_2$ is a contraction mapping when its domain is restricted to a specific ball. Let $K_h(\cdot) = K(\cdot/h)$. To deal with $I_3$, we define $O_i = [K_h(-\epsilon_i(\theta_{0h,\lambda}))/\tau]RX$, and $O_i = O_i - E(O_i)$ for $i = 1, \ldots, n$. Since $S_{h, \lambda}(\theta_{0h,\lambda}) = 0$,

$$
E\|S_{n, h, \lambda}(\theta_{0h,\lambda})\|^2 = E\|S_{h, \lambda}(\theta_{0h,\lambda}) - S_{h, \lambda}(\theta_{0h,\lambda})\|^2 = n^{-1}E\|O_i\|^2
\leq n^{-1}E\|K_h(-\epsilon(\theta_{0h,\lambda}))/\tau\|^2 \|RX\|^2.
$$

To find the term on the right-hand side, we employ the law of iterated expectations. In particular, given $X = x$,

$$
E\{K_h(-\epsilon(\theta_{0h,\lambda}))/\tau\|^2|X = x\} = E[K_h^2((RX, \theta_{0h,\lambda}) - Y)|X = x]
= 2\tau E[K_h((RX, \theta_{0h,\lambda}) - Y)|X = x] + \tau^2.
$$

Integration by parts leads to

$$
E[K_h((RX, \theta_{0h,\lambda}) - Y)|X = x]
= \int K_h((RX, \theta_{0h,\lambda}) - y)f(y|x)dy
= \int k_h((RX, \theta_{0h,\lambda}) - y)F(y|x)dy
= F((RX, \theta_{0h,\lambda})|x) + \int [F((RX, \theta_{0h,\lambda}) - hz|x) - F((RX, \theta_{0h,\lambda})|x)]k(z)dz
\tau + O(h^{s+1}).
$$
The last equation holds due to Theorem 3.1 and Assumptions A1 and A2. Let $K^*(z) = 2k(z)\overline{K}(z) = \frac{d}{dz}\overline{K}^2(z)$, thus $\int K^*(z)dz = \lim_{z \to \infty} \overline{K}^2(z) = 1$. For the same reason, we have

$$E[\overline{K}^2_h(\langle R_X, \theta_{0h,\lambda} \rangle - Y)|X = x]$$

$$= h^{-1} \int K^* \left( \frac{\langle R_X, \theta_{0h,\lambda} \rangle - y}{h} \right) F(y|x)dy$$

$$= \tau + O(h^{\alpha+1}) + \int [F(\langle R_X, \theta_{0h,\lambda} \rangle - hz|x) - F(\langle R_X, \theta_{0h,\lambda} \rangle|x)]K^*(z)dz$$

$$= \tau + O(h^{\alpha+1}) - h[f(\langle R_X, \theta_{0h,\lambda} \rangle|x) + O(h)] \int zK^*(z)dz$$

$$= \tau + O(h^{\alpha+1}) - h[f(\langle R_x, \theta_0 \rangle|x) + O(h) + O(h^{\alpha+1})] \int zK^*(z)dz$$

$$= \tau - hB(x) \int zK^*(z)dz + O(h^2).$$

By the proof of Theorem 3 of Fernandes et al. (2021), $\int zK^*(z)dz = 2\int_0^\infty \overline{K}(z)[1 - \overline{K}(z)]dz < \infty$. In summary, $E[\overline{K}^2_h(\langle R_X, \theta_{0h,\lambda} \rangle - \tau)^2|X = x] = \tau - \tau^2 - 2hB(x) \int_0^\infty \overline{K}(z)[1 - \overline{K}(z)]dz + O(h^2)$. Since $E[\|R_X\|^2] = O(\omega^{-1})$ by Lemma S.4 of Shang and Cheng (2015b), $E[\|S_{n,h\lambda}(\theta_{0h,\lambda})\|^2] \leq Cn^{-1}(1 - \tau)E[\|R_X\|^2] = O((n\omega)^{-1})$. This indicates that with probability approaching one, $\|S_{n,h\lambda}(\theta_{0h,\lambda})\| \leq C_3(n\omega)^{-1/2}$ for some sufficiently large constant $C_3$. Let $r_{2n} = 4C_3(n\omega)^{-1/2}$ and $\mathbb{B}(r_{2n}) = \{\theta \in \mathcal{H} : \|\theta\| \leq r_{2n}\}$.

Next we handle $I_1$. Let $T = (Y, X)$ denote the data variable. Define $A_n = \cap_{i=1}^n A_{ni}$, where

$$A_{ni} = \{\|X_i\|_{L^2} \leq C \log n\},$$

and $C$ is a positive constant satisfying $C \log n > 1$ and $C \geq \max(\sup_{u \in \mathbb{R}} |k(u)|, \sup_{u \in \mathbb{R}} |k^{(1)}(u)|)$. Because of Assumption A5, we can choose a sufficiently large $C$ such that $P(A_{ni})$ converges to 1 as $n \to \infty$. Furthermore, we can even ensure $P(A_{ni}) = O(n^{-1})$ holds uniformly over $i = 1, \ldots, n$ for a enough large $C$.

Let $T = (Y, X)$ denote the data variable. To deal with the first term $I_1$, we define a function

$$\psi(T_i; \theta) = k_h(-\epsilon_i(\theta_{0h,\lambda}))\langle R_X, \theta \rangle \mathbb{1}(A_{ni}).$$

For any $\theta_j = (\alpha_j, \beta_j) \in \mathcal{H}, j = 1, 2$, we have

$$|\psi(T_i; \theta_1) - \psi(T_i; \theta_2)| = |k_h(-\epsilon_i(\theta_{0h,\lambda}))|\langle R_X, \theta_1 - \theta_2 \rangle \mathbb{1}(A_{ni})$$

$$\leq (Ch^{-1}) \|\alpha_1 - \alpha_2\| + \|X_i\|_{L^2} \|\beta_1 - \beta_2\|_{L^2} \mathbb{1}(A_{ni})$$

$$\leq (C^2h^{-1} \log n) \|\theta_1 - \theta_2\|_2.$$  

Let $\psi_n = (T_i; \theta) = C^{-2}(h^{-1} \log n)^{-1} \psi(T_i; \theta)$. Then $\psi_n$ satisfies that for any $\theta, \tilde{\theta} \in \mathcal{H}$,

$$|\psi_n(T_i; \theta) - \psi_n(T_i; \tilde{\theta})| \leq \|\theta - \tilde{\theta}\|_2.$$  

By Lemma 3.1 of Shang and Cheng (2015a), for any $\theta \in \mathcal{H}$, $\|\theta\|_2 \leq \kappa \omega^{-(2a+1)/2} \|\theta\|$ for some constant $\kappa > 0$. Let $d_n = \kappa \omega^{-(2a+1)/2}$ and $p_n = d_n^{-2}\lambda^{-1}$. For any $\theta \in \mathcal{H} \setminus \{0\}$, let $	ilde{\theta} = (\tilde{\alpha}, \tilde{\beta}) := \theta/(d_n \|\theta\|)$. Obviously $\|\tilde{\theta}\|_2 \leq 1$. In other words, $|\tilde{\alpha}| + \|\tilde{\beta}\|_{L^2} \leq 1$. We can further show that $\lambda J(\beta, \beta) \leq \|\beta\|^2 = d_n^{-2}$, which implies that $J(\beta, \beta) \leq d_n^{-2}\lambda^{-1} = p_n$. If we define $\mathcal{F}_{q_n} = \{\theta = (\alpha, \beta) \in \mathcal{H} : |\alpha| \leq 1, \|\beta\|_{L^2} \leq 1, J(\beta, \beta) \leq q_n\}$, where $q_n \geq 1$, then
\( \tilde{\theta} \in \mathcal{F}_{p_n} \). By Lemma 3.4 of Shang and Cheng (2015a), we have for any \( \theta \in \mathcal{B}(r_{2n}) \), with probability approaching one,

\[
\left\| \sum_{i=1}^{n} [\psi_n(T_i; \tilde{\theta}) R_{X_i} - E \{ \psi_n(T_i; \tilde{\theta}) R_{X_i} \}] \right\| \leq C_5 (n^{1/2} p_n^{1/(4m)} + 1)(\omega^{-1} \log n)^{1/2},
\]

for some positive constant \( C_5 \). As a result,

\[
\left\| \sum_{i=1}^{n} [\psi_n(T_i; \tilde{\theta}) R_{X_i} - E \{ \psi_n(T_i; \tilde{\theta}) R_{X_i} \}] \right\| \leq \]

\[
C_5 \omega^{-(a+1)} (n^{1/2} p_n^{1/(4m)} + 1)(C^2 h^{-1} \log n)(\log \log n)^{1/2} \| \theta \|
\]

with probability approaching 1. On the other hand, by Cauchy-Schwarz inequality, we have

\[
\| E [k_h(-\epsilon_i(\theta_{0h,\lambda}))(R_{X_i}, \theta) R_{X_i} \mathbb{1}(A_{ni}^c)] \|
\]

\[
= \sup_{\| \theta \| = 1} E [k_h(-\epsilon_i(\theta_{0h,\lambda}))(R_{X_i}, \theta) (R_{X_i}, \theta_1) \mathbb{1}(A_{ni}^c)]
\]

\[
\leq Ch^{-1} \sup_{\| \theta \| = 1} \{ E \| (R_{X_i}, \theta) \| \cdot (R_{X_i}, \theta_1) \mathbb{1}(A_{ni}) \}
\]

\[
\leq Ch^{-1} E \| (R_{X_i}, \theta) \|^4 \sup_{\| \theta \| = 1} E \| (R_{X_i}, \theta_1) \|^4 P(A_{ni}^c)^{1/2}
\]

\[
\leq Ch^{-1} \| \theta \| P(A_{ni}^c)^{1/2} = o(1) \| \theta \|
\]

where the last equation holds since \( P(A_{ni}^c) = O(1/n) \) uniformly over \( i = 1, \ldots, n \) and \( n^{-1/2} h^{-1} = o(1) \).

Now we are able to determine the order of \( I_1 \). Actually, with probability approaching 1, we have

\[
\| DS_{n,h,\lambda}(\theta_{0h,\lambda}) \theta - DS_{h,\lambda}(\theta_{0h,\lambda}) \theta \|
\]

\[
\leq n^{-1} \left\| \sum_{i=1}^{n} [k_h(-\epsilon_i(\theta_{0h,\lambda}))(R_{X_i}, \theta) R_{X_i} \mathbb{1}(A_{ni}) - E \{ k_h(-\epsilon_i(\theta_{0h,\lambda}))(R_{X_i}, \theta) R_{X_i} \mathbb{1}(A_{ni}) \}] \right\|
\]

\[
+ n^{-1} \sum_{i=1}^{n} E [k_h(-\epsilon_i(\theta_{0h,\lambda}))(R_{X_i}, \theta) R_{X_i} \mathbb{1}(A_{ni}^c)]
\]

\[
= O_P \left( n^{-1/2} \omega^{-(a+1)} \frac{2l - 2a - 1}{4m} (h^{-1} \log n)(\log \log n)^{1/2} \| \theta \| + o(1) \| \theta \| \right)
\]

\[
= o_P(1) \| \theta \|
\]

where the last equation follows by the assumption (3.8). Hence, with probability approaching one, we have \( \| DS_{n,h,\lambda}(\theta_{0h,\lambda}) \theta - DS_{h,\lambda}(\theta_{0h,\lambda}) \theta \| \leq r_{2n}/18 \) for any \( \theta \in \mathcal{B}(r_{2n}) \).

Lastly, we treat \( I_2 \). Taylor’s expansion leads to that

\[
S_{n,h,\lambda}(\theta_{0h,\lambda} + \theta) - S_{n,h,\lambda}(\theta_{0h,\lambda}) - DS_{n,h,\lambda}(\theta_{0h,\lambda}) \theta
\]

\[
= \int_0^1 \int_0^1 D^2 S_{n,h,\lambda}(\theta_{0h,\lambda} + s \theta) (s \theta) ds' ds.
\]

We assume that \( \cap_{i=1}^{n} A_{ni} \) holds for the rest of the proof. For an arbitrary \( \theta \in \mathcal{H} \setminus \{ 0 \} \), let \( \bar{\theta} = \theta/(\| \theta \|) \), so \( \bar{\theta} \in \mathcal{F}_{p_n} \). Let

\[
\psi_n(T_i; \theta) = \frac{\| R_{X_i} \| \cdot \langle R_{X_i}, \bar{\theta} \rangle}{\sqrt{2C_R(C \log n)^{2\omega - (2a+1)/2} \mathbb{1}(A_{ni})}}, i = 1, \ldots, n,
\]
where $C_R$ is a constant such that $\|RX\|^2 \leq C_R[1 + ((C \log n)^2 \omega^{-2a+1})$ based on Lemma S.4 of Shang and Cheng (2015b). Hence $\psi_n$ satisfies Lipschitz continuity given in (3.3) of Shang and Cheng (2015a). Then by Lemma 3.4 of Shang and Cheng (2015a), we have, with probability approaching 1, for any $\theta \in \mathcal{H} \setminus \{0\}$,

$$n^{-1/2} \left\| \sum_{i=1}^{n} [\psi_n(T_i; \tilde{\theta})RX_i - E_T\{\psi_n(T_i; \tilde{\theta})RX_i\}] \right\| \leq C_6p_n^{1/(4m)}(\omega^{-1} \log \log n)^{1/2},$$

for some positive constant $C_6$. This implies that with probability approaching one,

$$\left\| \sum_{i=1}^{n} [\|RX_i\| \cdot \langle RX_i, \theta \rangle \mathbb{1}(A_{ni})RX_i - E_T\{\|RX_i\| \cdot \langle RX_i, \theta \rangle \mathbb{1}(A_{ni})RX_i\}] \right\| \leq C_7n^{1/2} \omega^{-(2a+1) - \frac{2-2a-1}{4m}} (\log n)^2(\log \log n)^{1/2} \|\theta\|$$

for some large constant $C_7$. Note that this inequality also holds for $\theta = 0$. On the other hand, by Lemma S.3 and S.4 of Shang and Cheng (2015b) and Cauchy inequality,

$$E_T\{\|RX\| \cdot |\langle RX, \theta \rangle|^2 \} \leq (E_T\{|\langle RX, \theta \rangle|^2 \})^{1/2} \cdot (E_T\{|\langle RX, \theta \rangle|^4 \})^{1/4} \leq C_8\omega^{-1/2} \|\theta\|^2$$

for some large constant $C_8$. Therefore, on $\cap_{i=1}^{n} A_{ni}$, for any $\theta \in \mathbb{B}(r_{2n})$ and some $s, s' \in [0, 1]$, we have

$$\|D^2S_{n, \theta, \lambda}(\theta_{0h, \lambda} + ss'\theta)\theta d\theta ds'\|$$

$$= n^{-1} \left\| \sum_{i=1}^{n} h^{-2}k^{(1)}(-\epsilon_i(\tilde{\theta})/h)\langle RX_i, \theta \rangle^2 RX_i \right\|$$

$$\leq n^{-1} \sup_{a \in \mathbb{R}} \left\| \sum_{i=1}^{n} \|RX_i\| \cdot |\langle RX_i, \theta \rangle|^2 \right\|$$

$$\leq Cn^{-1}h^{-2}\sum_{i=1}^{n} \left\| \|RX_i\| \cdot \langle RX_i, \theta \rangle \mathbb{1}(A_{ni})RX_i - E_T\{\|RX_i\| \cdot \langle RX_i, \theta \rangle \mathbb{1}(A_{ni})RX_i\}, \theta \} \right\|$$

$$+ Ch^{-2}E_T\{\|RX_i\| \cdot |\langle RX_i, \theta \rangle|^2 \mathbb{1}(A_{ni})\}$$

$$\leq C' \left\{ n^{-1} \omega^{-2(a+1) - \frac{2-2a-1}{4m}} (\log n)^2h^{-2}(\log \log n)^{1/2} + n^{-1/2}h^{-2}\omega^{-1} \right\} \times \|\theta\|$$

$$\leq \|\theta\|/18,$$

where the last inequality is obtained from assumption (3.8). Combining the above results, we have that, with probability approaching one, for any $\theta \in \mathbb{B}(r_{2n})$,

$$T_2(\theta) \leq \|I_1\| + \|I_2\| + \|I_3\| \leq 2(r_{2n}/4 + r_{2n}/18 + r_{2n}/18) = 13r_{2n}/18.$$
Using the same arguments as in the analysis of the terms $I_2$ and $I_1$, we can show that with probability approaching one,
\[ \|I_4\| \leq O(n^{-\omega} \cdot 2^{(a+1)} - \frac{2^{(a+1)} \cdot (\log n)^2 \cdot h^{-2} \cdot (\log n)}{4m} + n^{-\omega} \cdot (\log n)^{1/2} + n^{-\omega} \cdot (\log n)^{1/2}) \|\theta_1 - \theta_2\| \]
\[ \leq \|\theta_1 - \theta_2\|/3, \]
and
\[ \|I_5\| = O(n^{-\omega} \cdot 2^{(a+1)} - \frac{2^{(a+1)} \cdot (\log n)^2 \cdot h^{-2} \cdot (\log n)}{4m} + n^{-\omega} \cdot (\log n)^{1/2}) \|\theta_1 - \theta_2\| + o(1) \|\theta_1 - \theta_2\| \]
\[ \leq \|\theta_1 - \theta_2\|/3. \]
It follows that $\|T_2(\theta_1) - T_2(\theta_2)\| \leq 2\|\theta_1 - \theta_2\|/3$ with probability approaching 1. Therefore, $T_2$ is a contraction mapping from $\mathbb{B}(r_{2n})$ to itself. By the contraction mapping theorem, there must exist a unique element $\tilde{\theta} \in \mathbb{B}(r_{2n})$ such that $T_2(\tilde{\theta}) = \tilde{\theta}$, which implies that $S_{n,h,\lambda}(\theta_{0h,\lambda} + \tilde{\theta}) = 0$. That is to say, $\tilde{\theta}_{h,\lambda}$ is the minimizer of the loss function $\ell_{n,h,\lambda}(\theta)$. Furthermore, by Theorem 3.1, with probability approaching one,
\[ \|\tilde{\theta}_{h,\lambda} - \theta_0\| \leq \|\tilde{\theta}_{h,\lambda} - \theta_{0h,\lambda}\| + \|\theta_{0h,\lambda} - \theta_0\| \leq r_{2n} + r_{1n} = O((n\omega)^{-1/2} + \omega^4) = O(r_n). \]
This completes the proof.

**A.4. Proof of Theorem 3.3.** By Theorem 3.2, there exists some sufficiently large $M > 0$ such that, with probability approaching one, $\|\tilde{\theta}_{h,\lambda} - \theta_0\| \leq Mr_n$. Let $\theta = \tilde{\theta}_{h,\lambda} - \theta_0$ for notational convenience. In the rest of the proof we assume that $\|\theta\| \leq Mr_n$ because the probability of its complement is negligible. Let $d_n = \kappa \cdot M \omega^{-2(a+1)/2} r_n$ and $\tilde{d} = d_n^{-1} \cdot \theta$. Let $p_n = \kappa^{-2} \omega^{-1-2m}$, where $\kappa$ is defined in the proof of Theorem 3.2. Note that $p_n \geq 1$ for sufficiently large $n$ as $\omega$ decays to 0 and $1 - 2m = 1 - 2(k - a) < 0$. We can show that $\|\theta\| \leq Mr_n$ indicates that $\tilde{\theta} \in \mathcal{F}_{p_n}$. Actually, by Lemma 3.1 of Shang and Cheng (2015a), $\|\tilde{\theta}\|_2 = \tilde{d}^{-1}\|\theta\|_2 \leq \tilde{d}^{-1}\kappa \omega^{-2(a+1)/2}\|\theta\| \leq \tilde{d}^{-1}\|\theta\|_2 \leq \tilde{d}^{-1}\kappa \omega^{-2(a+1)/2} Mr_n = 1$. Thus $|\alpha| \leq 1$ and $\|\beta\|_2 \leq 1$. Additionally,
\[ J(\tilde{\beta}, \tilde{\beta}) = d_n^{-2} \lambda^{-1} (\lambda J(\beta, \beta)) \leq d_n^{-2} \lambda^{-1} \|\theta\|_2^2 \leq d_n^{-2} \lambda^{-1} (Mr_n)^2 = \kappa^{-2} \omega^{-1-2m} = p_n. \]
Therefore, $\tilde{\theta} \in \mathcal{F}_{p_n}$.

For $i = 1, \ldots, n$, define $A_{n,i} = \{\|X_i\|_2^2 \leq C \log n\}$, where $C$ is a positive constant satisfying $C \log n > 1$ and $C \sup_{n \in \mathbb{N}} |k(u)|$. Denote $\cap_{i=1}^n A_{n,i}$ by $A_n$. Because of Assumption A5, we can choose a sufficiently large $C$ such that $P(A_n)$ converges to 1 as $n \to \infty$. Furthermore, we can even ensure that
\[ n^{1/2} \omega^{-1/2} (1 (A_{n,i}^c))^{1/4} = o(p_n^{1/(4m)} \omega^{-1} \log n)^{1/2}) \]
for a large enough $C$.

Let $D_n = (C^4 \log n \cdot h^{-1})^{-1} d_n^{-1}$. We define $\psi(T_i; \theta) = K(-\epsilon_i(\theta + \theta_0)/h) - K(-\epsilon_i(\theta_0)/h)$ and $\psi_n(T_i; \tilde{\theta}) = D_n \psi(T_i; d_n \tilde{\theta}) 1(A_{n,i})$. Then, for any $\tilde{\theta}_1, \tilde{\theta}_2 \in \mathcal{F}_{p_n}$, we have
\[ |\psi_n(T_i; \tilde{\theta}_1) - \psi_n(T_i; \tilde{\theta}_2)| = D_n |\psi(T_i; d_n \tilde{\theta}_1) - \psi(T_i; d_n \tilde{\theta}_2)| 1(A_{n,i}) \]
\[ \leq D_n \sup_{u \in \mathbb{R}} |k(u)| \cdot h^{-1} \cdot d_n \cdot |R_{X_i} - R_{X_i} - \tilde{\theta}_2| 1(A_{n,i}) \]
\[ \leq D_n (\epsilon_i)^{-1} d_n (\log n) \||\tilde{\theta}_1 - \tilde{\theta}_2||_2 = \|\tilde{\theta}_1 - \tilde{\theta}_2\|_2. \]
It follows by Lemma 3.4 of Shang and Cheng (2015a), that there exists a constant $C_3$ such that for sufficiently large $n$, with probability approaching one, 

$$\|H_n(\bar{\theta})\| = \frac{1}{\sqrt{n}} \left\| \sum_{i=1}^{n} [\psi_n(T_i; \bar{\theta})R_{X_i} - E_T \{\psi_n(T; \bar{\theta})R_X\}] \right\| \leq C_3(p_n^{1/(4m)}\|\bar{\theta}\|_2^2 + n^{-1/2})(\omega^{-1} \log n)^{1/2}$$

$$\leq C_3(p_n^{1/(4m)} + n^{-1/2})(\omega^{-1} \log n)^{1/2},$$

where $\gamma = 1 - 1/(2m)$. On the other hand, by Cauchy’s inequality and the mean value theorem, we have 

$$\|E_T \{\psi(T_i; d_n \bar{\theta})R_{X_i}1(A_{ni}^c)\}\| \leq E_T \{ |\psi(T_i; d_n \bar{\theta})| \|R_{X_i}\| \cdot 1(A_{ni}^c) \}$$

$$\leq C_4 d_n E \{(1 + \|X_i\|_{L^2})\|R_{X_i}\| \cdot 1(A_{ni}^c) \}$$

$$\leq C_4 d_n E_T \{(1 + \|X_i\|_{L^2})^{4} n^{1/4} P(A_{ni}^c)^{1/4} E \{\|R_{X_i}\|^{2}\} \}^{1/2} \leq C_4 d_n E_T \{(1 + \|X_i\|_{L^2})^{4} n^{1/4} P(A_{ni}^c)^{1/4} (\log n)^{-1/2} \}$$

As a result, based on the choice of $C$, there exists some constant $C' > 0$, 

$$D_n n^{1/2} E_T \{\psi(T_i; d_n \bar{\theta})R_{X_i}1(A_{ni}^c)\} \leq C' n^{1/2} \omega^{-1/2} n P(A_{ni}^c)^{1/4} (\log n)^{-1/2}$$

Note that $1(A_{ni}) = 1$ on $A_n$. Therefore, with probability approaching one, 

$$n^{-1/2} D_n \left\| \sum_{i=1}^{n} [\psi(T_i; \theta)R_{X_i} - E_T \{\psi(T; \theta)R_X\}] \right\|$$

$$= \left\| n^{-1/2} \sum_{i=1}^{n} [\psi_n(T_i; \bar{\theta})R_{X_i} - E_T \{\psi_n(T; \bar{\theta})R_X\}] - D_n n^{1/2} E_T \{\psi(T_i; d_n \bar{\theta})R_{X_i}1(A_{ni}^c)\} \right\| \leq \|H_n(\bar{\theta})\| + D_n n^{1/2} \|E_T \{\psi(T_i; d_n \bar{\theta})R_{X_i}1(A_{ni}^c)\}\|$$

$$\leq C_5 p_n^{1/(4m)} (\omega^{-1} \log n)^{1/2}.$$ 

Then with probability approaching one, we have 

$$\|S_{n,h\lambda}(\theta + \theta_0) - S_{n,h\lambda}(\theta_0) - E_T \{S_{n,h\lambda}(\theta + \theta_0) - S_{n,h\lambda}(\theta_0)\}\|$$

$$= n^{-1/2} \left\| \sum_{i=1}^{n} [\psi(T_i; \theta)R_{X_i} - E_T \{\psi(T; \theta)R_X\}] \right\| \leq C_5 n^{-1/2} p_n^{1/(4m)} D_n^{-1} (\omega^{-1} \log n)^{1/2}$$

$$= C_5 \kappa C^2 \lambda^2 M n^{-1/2} \omega^{\frac{4m + 4d - 1}{4m}} r_n(n^{-1} \log n)(\log n)^{1/2}.$$ 

Note that $\theta$ satisfies $S_{n,h\lambda}(\theta + \theta_0) = 0$. Additionally, from the proof of Theorem 3.1 we know that $\int \rho_{\tau}(u) D_{\lambda} f_\lambda(u; \theta_0) \theta du + \lambda \theta = \theta$. Recall that $K_\lambda(u; \theta) = E[k_h(u - \epsilon(\theta))]$, where the expectation is taken with respect to the distribution of $\epsilon(\theta)$. It follows that 

$$E_T \{S_{n,h\lambda}(\theta + \theta_0) - S_{n,h\lambda}(\theta_0)\}$$
\[ S_{h\lambda}(\theta + \theta_0) - S_{h\lambda}(\theta_0) = P_{\lambda}(\theta) + \int \rho_\tau(u)D_{\theta}K_{h}(u; \theta + \theta_0)du - \int \rho_\tau(u)D_{\theta}K_{h}(u; \theta_0)du \]
\[ = P_{\lambda}(\theta) + \int \rho_\tau(u)D_{\theta}^2f_\tau(u; \theta_0)\theta du - \int \rho_\tau(u)D_{\theta}^2f_\tau(u; \theta_0)\theta du \]
\[ + \int \rho_\tau(u)D_{\theta}K_{h}(u; \theta + \theta_0)du - \int \rho_\tau(u)D_{\theta}K_{h}(u; \theta_0)du \]
\[ = \theta + \int \rho_\tau(u)D_{\theta}K_{h}(u; \theta + \theta_0)du - \int \rho_\tau(u)D_{\theta}K_{h}(u; \theta_0)du - \int \rho_\tau(u)D_{\theta}^2f_\tau(u; \theta_0)\theta du \]
\[ = \theta + \int \rho_\tau(u)\left[ \int_0^1 D_{\theta}^2K_{h}(u; \theta_0 + s\theta)\theta ds \right] du - \int \rho_\tau(u)D_{\theta}^2f_\tau(u; \theta_0)\theta du \]
\[ = \theta - \int \rho_\tau(u)\left( \int_0^1 [D_{\theta}^2f_\tau(u; \theta_0) - D_{\theta}^2K_{h}(u; s\theta + \theta_0)]\theta ds \right) du \]

From the proof of Theorem 3.1 and \( h = O(\omega^{k/(s+1)}) \), we know that
\[ \left\| \int \rho_\tau(u) \left( \int_0^1 [D_{\theta}^2f_\tau(u; \theta_0) - D_{\theta}^2K_{h}(u; s\theta + \theta_0)]\theta ds \right) du \right\| \leq C|\omega^*| + \|\theta\| \cdot \|\theta\| \leq 2h^{-1}r_n^2 \]
uniformly over \( \tau \in [\bar{\tau}, \tilde{\tau}] \) for sufficiently large \( n \). Therefore, as \( n \to \infty \), with probability approaching one,
\[ \|\theta + S_{n,h\lambda}(\theta_0)\| \leq C_{5}\omega^*C^2Mn^{-1/2}\omega^{-\frac{4m+6m-1}{4}}r_n(h^{-1} \log n)(\log \log n)^{1/2} + 2h^{-1}r_n^2. \]

This completes the proof.

**A.5. Proof of Corollary 3.1.** Let \( \zeta_i = K_i(-\epsilon_i(\theta_0)/h) - \tau, \tau, i = 1, \ldots, n \). By the definition of \( K_i \), we have
\[ \|K_i\|^2 = \left( \sum_{\nu} \frac{\phi_\nu(t)}{1 + \lambda \rho_\nu} \sum_{\nu} \frac{\phi_\nu(t)}{1 + \lambda \rho_\nu} \phi_\nu \right)_1 \leq C_K \sum_{\nu} \frac{\nu^{2\alpha}}{1 + \lambda \rho_\nu} \leq \sum_{\nu} \frac{\nu^{2\alpha}}{1 + c_p \lambda \nu^2} \leq C_K \omega^{-(2\alpha+1)}, \]
where \( C_K \) is a constant that only depends on \( C_\phi \) and \( c_p \). Then it follows that
\[ |\hat{\beta}_{h,\lambda}(t) - \beta_0(t, \tau) + \frac{1}{n} \sum_{i=1}^n \zeta_i \eta(X_i)(t) + (W_{\lambda}\beta_0(\cdot, \tau))(t)| \]
\[ = \langle K_t, \hat{\beta}_{h,\lambda} - \beta_0(\cdot, \tau) + \frac{1}{n} \sum_{i=1}^n \zeta_i \eta(X_i) + (W_{\lambda}\beta_0(\cdot, \tau)) \rangle_1 \]
\[ \leq \|K_t\|_1 \left\| \hat{\beta}_{h,\lambda} - \beta_0(\cdot, \tau) + \frac{1}{n} \sum_{i=1}^n \zeta_i \eta(X_i) + (W_{\lambda}\beta_0(\cdot, \tau)) \right\|_1 \]
\[ = O_P(a_n \omega^{-(2\alpha + 1)/2}), \]
where the last equation is obtained from Theorem 3.3.
Let $K_h(\cdot) = \overline{K}(\cdot/h)$. We first employ the law of iterated expectations to find $E\zeta_i^2[\eta(X_i)(t)]^2$. In particular, given $X = x$, 

$$E \{ K_h((-\epsilon(\theta_0)) - \tau^2)X = x \} = E [K_h^2((R_x, \theta_0) - Y)|X = x] 
- 2\tau E [K_h((R_x, \theta_0) - Y)|X = x] + \tau^2.$$ 

Integration by parts leads to

$$E [K_h((R_x, \theta_0) - Y)|X = x] 
= \int K_h((R_x, \theta_0)|y) f(y|x)dy 
= \int k_h((R_x, \theta_0)|y) F(y|x)dy 
= F((R_x, \theta_0)|x) + \int [F((R_x, \theta_0) - hz|x) - F((R_x, \theta_0)|x)] k(z)dz 
= \tau + O(h^{s+1}).$$

The last equation holds due to the definition of $\theta(\tau)$ and Assumptions A1 and A2. Let $K^*(z) = 2k(z)\overline{K}(z) = \frac{d}{dz} K^2(z)$, thus $\int K^*(z)dz = \lim_{z \to \infty} K^2(z) = 1$. For the same reason, we have

$$E [K_h^2((R_x, \theta_0) - Y)|X = x] 
= h^{-1} \int k^*(\frac{(R_x, \theta_0)|y}{h}) F(y|x)dy 
= \tau + O(h^{s+1}) + \int [F((R_x, \theta_0)|x) - hz] k^*(z)dz 
= \tau + O(h^{s+1}) - h[f((R_x, \theta_0)|x) + O(h)] \int zK^*(z)dz 
= \tau + O(h^{s+1}) - h[f((R_x, \theta_0)|x) + O(h) + O(h^{s+1})] \int zK^*(z)dz 
= \tau - hB(x) \int zK^*(z)dz + O(h^2).$$

By the proof of Theorem 3 of Fernandes et al. (2021), $\int zK^*(z)dz = 2\int_0^\infty K(z)|1 - \overline{K}(z)|dz < \infty$. In summary, $E \{ K_h((-\epsilon(\theta_0)) - \tau^2)|X = x \} = \tau - \tau^2 - 2hB(x) \int_0^\infty K(z)|1 - \overline{K}(z)|dz + O(h^2)$ uniformly for $x$. Therefore, by Assumption A1 (c),

$$E\zeta_i^2[\eta(X_i)(t)]^2 \leq C_1 E\zeta_i^2B(X_i)[\eta(X_i)(t)]^2 
\leq C_1 (\tau - \tau^2) \sum_{\nu} \frac{E(B(X_i)X_{\nu}^2)}{1 + \lambda_{\nu}^2} \phi_{\nu}^2(t) 
\leq \sum_{\nu} \frac{\phi_{\nu}^2(t)}{1 + \lambda_{\nu}^2} \geq \sigma^2 \omega^{-2a+1}$$

for some positive constant $\sigma^2$. Next we find the $E\zeta_i[\eta(X_i)(t)]$. It is actually $R_h^{(1)}(\theta_0; \tau)\theta_1$ if we take $\theta_1 = (0, K_1)$. By the proof of (ii) of Lemma 3.1, we have

$$|R_h^{(1)}(\theta_0; \tau)\theta_1 - R^{(1)}(\theta_0; \tau)\theta_1| \leq Ch^{s+1}E[|(R_x, \theta_1)|] 
\leq Ch^{s+1} \left[ E \left( (R_x, \theta_1)^4 \right) \right]^{1/4} 
\leq Ch^{s+1} \|\theta_1\| = Ch^{s+1} \|K_1\|_1$$
by the Cauchy-Schwarz inequality and Lemma S.3 of Shang and Cheng (2015b). On the other hand, as we’ve shown above, \(|K_i| \leq C K \omega^{-(2a+1)/2}\). Since \(R^{(1)}(\theta_0; \tau) \theta_1 = 0\), \(\{E_{[\eta(X_i)(t)]}^2 = O(h^{2a+2}) \omega^{-(2a+1)}\). On the other hand, since \(h = o(1)\), \(\{E_{[\eta(X_i)(t)]}^2 = o(E_{[\eta(X_i)(t)]}^2)\). Hence we conclude that \(s_n^2 \approx n \omega^{-(2a+1)}\).

We check the Lindeberg’s condition to establish the central limit theorem for the triangular array \(\sum_{i=1}^n \{\eta(X_i)(t) - R^{(1)}_h(\theta_0; \tau) \theta_1\}\), where \(\theta_1 = (0_K) \in \mathcal{H}\). Rearranging \(\eta(X_i)(t) - R^{(1)}_h(\theta_0; \tau) \theta_1\) leads to
\[
\left| \sum_{i=1}^n \{\eta(X_i)(t) - R^{(1)}_h(\theta_0; \tau) \theta_1\} \right| \leq \sum_{i=1}^n \left| \eta(X_i)(t) - R^{(1)}_h(\theta_0; \tau) \theta_1\right| \\
\leq \sum_{i=1}^n \left| \eta(X_i)(t) - \left[ E\left[\eta(X_i)(t)\right] \right] \right|.
\]

Obviously, there exists a positive constant \(C\) that is independent of \(n\) such that \(|\zeta_i| \leq C\) a.s. As shown above, \(|\langle R_X, \theta_1 \rangle| \leq \|\eta(X)\|_1\|\mathcal{R}_i\|_1 \leq C K \omega^{-(2a+1)/2}\|\eta(X)\|_1\). Furthermore, by Lemma S.4 of Shang and Cheng (2015b), \(\|\eta(X)\|_1 \leq C_R \|X\|_{L^2}^2 \omega^{-(2a+1)}\) for some constant \(C_R > 0\).

For any \(\epsilon > 0\), we choose \(\tilde{C}\) sufficiently large such that \(\tilde{C} \sigma_t / K \log(1/\omega) > C\). Considering the fact that \([R^{(1)}_h(\theta_0; \tau) \theta_1] = o(s_n^2/n)\), we can obtain that
\[
ns_n^{-2}E\left\{\left[\zeta \langle R_X, \theta_1 \rangle - R^{(1)}_h(\theta_0; \tau) \theta_1\right]^2 \right\} \leq ns_n^{-2}E\left\{\zeta \left[\langle R_X, \theta_1 \rangle - R^{(1)}_h(\theta_0; \tau) \theta_1\right]^2 \right\} \geq \epsilon^2 s_n^2.
\]

where the last equation holds because \(n \omega^{2a+1}(\log(\omega^{-1}))^{-4} \to \infty\) as \(n \to \infty\). Then by the Lindeberg’s CLT, we have \(s_n^{-1} \sum_{i=1}^n \{\zeta_i \eta(X_i)(t) - E[\zeta_i \eta(X_i)(t)]\} \xrightarrow{d} N(0, 1)\). Moreover, \(s_n^{-1} n \mathbb{E}[\zeta_i \eta(X_i)(t)] = O(\sqrt{n} \omega^{(2a+1)/2}) = o(1)\). Therefore, following the condition that \(n a_n \omega^{-(2a+1)/2} / s_n \approx n^{1/2} a_n = o(1)\), we get that as \(n \to \infty\),
\[
\frac{n}{s_n} [\hat{\beta}_{h,\lambda} - \beta_0(t, \tau)] + (W_{\lambda}(\beta_0(\cdot, \tau))(t))
\]

\[
\approx s_n^{-1} \sum_{i=1}^n \{(-\zeta_i \eta(X_i)(t) + E[\zeta_i \eta(X_i)(t)]) - ns_n^{-1} E[\zeta_i \eta(X_i)(t)] + O_P(n a_n \omega^{-(2a+1)/2} / s_n)\}
\]

\[
\approx s_n^{-1} \sum_{i=1}^n \{\zeta_i \eta(X_i)(t) - E[\zeta_i \eta(X_i)(t)]\} + o(1) + o_P(1) \xrightarrow{d} N(0, 1).
\]
A.6. Proof of Theorem 3.4. By the proof of Corollary 3.1, we have that for an arbitrary \( \tau \in \mathbb{I} \),
\[
n^{1/2} \omega^{a+1/2} \sup_{t \in \mathbb{I}} |\hat{\beta}_{h,\lambda}(t, \tau) - \beta_0(t, \tau) + \tilde{S}_n(\theta_0)(t) + (W_{\lambda}(\cdot, \tau))(t)| = o_P(1),
\]
where
\[
\tilde{S}_n(\theta_0)(t) = \frac{1}{n} \sum_{i=1}^{n} \left[ \int_{-\infty}^{(R_{X,i} \cdot \theta_0)} k_h(u-Y_i)du - \tau \right] (R_{X,i} \cdot \theta_1) - R^{(1)}_{h}(\theta_0; \tau) \theta_1
\]
with \( \theta_1 = (0, K_i) \). Since \( \beta_0 = \sum_\nu b_\nu \phi_\nu \), by Cauchy-Schwarz inequality, we have
\[
(W_{\lambda}(\cdot, \tau))(t) = \sum_\nu b_\nu \frac{\lambda \rho_\nu}{1 + \lambda \rho_\nu} \phi_\nu(t)
\]
\[
\leq \lambda \left( \sum_\nu b_\nu^2 \rho_\nu^2 \right)^{1/2} \left( \sum_\nu \frac{\phi_\nu^2(t)}{(1 + \lambda \rho_\nu)^2} \right)^{1/2}
\]
\[
\asymp \lambda \omega^{-(a+1/2)}
\]
uniformly over \( t \in \mathbb{I} \). It follows by the condition \( n^{1/2} \omega = o(1) \) that
\[
(A2) \quad n^{1/2} \omega^{a+1/2} \sup_{t \in \mathbb{I}} |\hat{\beta}_{h,\lambda}(t, \tau) - \beta_0(t, \tau) + \tilde{S}_n(\theta_0)(t)| = o_P(1),
\]
To show weak convergence of \( n^{1/2} \omega^{a+1/2} [\hat{\beta}_{h,\lambda}(. , \tau) - \beta_0(\cdot , \tau)] \) to some Gaussian process in \( H^m(\mathbb{I}) \), we only need to verify that \( J_n(t) := n^{1/2} \omega^{a+1/2} \tilde{S}_n(\theta_0)(t) \) converges weakly to the Gaussian process \( G \) in \( H^m(\mathbb{I}) \) equipped with the inner product \( V(\cdot, \cdot) \). Note that \( \{\phi_\nu\} \) is a complete orthonormal basis of \( H^m(\mathbb{I}) \) by Assumption A4. Then by Theorem 1.8.4 of Van der Vaart and Wellner (1996), it is equivalent to prove that (i) \( J_n(\cdot) \) is asymptotically finite-dimensional and (ii) \( V(J_n, \phi_\nu) \) converges in distribution to \( V(G, \phi_\nu) \) for each \( \nu \geq 1 \). Direct calculations lead to
\[
\sum_\nu V(J_n, \phi_\nu)^2 = \sum_\nu \frac{\omega^{2a+1}}{(1 + \lambda \rho_\nu)^2} \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \zeta_i X_{ih} - E(\zeta_i X_{ih}) \right] \right\}^2,
\]
where \( \zeta_i = R(-\epsilon_i(\theta_0)/h) - \tau \) and \( X_{ih} = \int_0^1 X_i(t) \phi_\nu(t) dt \) for \( i = 1, \ldots, n \). As \( |\zeta_i| \leq C \) holds almost surely for some positive constant \( C > 0 \),
\[
E \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \zeta_i X_{ih} - E(\zeta_i X_{ih}) \right] \right\}^2
\]
\[
= \text{Var} \left( \zeta_i X_{ih} \right) \leq E(\zeta_i^2 X_{ih}^2) \leq C^2 E(X_{ih}^2)
\]
\[
\leq E \|X_i\|_{L^2}^2 \cdot \int_0^1 \phi_\nu^2 dt
\]
\[
\leq v^{2a}.
\]
Furthermore, we have \( \sum_\nu \frac{\omega^{2a+1} v^{2a}}{(1 + \lambda \rho_\nu)^2} < \infty \). Then, for any \( \epsilon > 0 \) and \( \delta > 0 \), there exists some \( \nu_0 \) such that \( \sum_{\nu \geq \nu_0} \frac{\omega^{2a+1} v^{2a}}{(1 + \lambda \rho_\nu)^2} < \epsilon \delta \). For any \( \epsilon > 0 \) and \( \delta > 0 \), we have by the Markov inequality,
\[
\limsup_n \frac{1}{n} \sup_{\nu \geq \nu_0} V(J_n, \phi_\nu)^2 \geq \epsilon
\]
\[
\limsup_n \frac{1}{n} \sup_{\nu \geq \nu_0} E \left[ \frac{\sum_{\nu \geq \nu_0} V(J_n, \phi_\nu)^2}{\epsilon} \right] < \delta.
\]
This shows (i) that $J_n$ is asymptotically finite-dimensional.

Since $\eta(X_i) = \sum_{\nu} \frac{X_{i\nu}}{1+\lambda \rho_{\nu}} \phi_{\nu}$,

$$V(J_n, \phi_{\nu}) = \frac{\omega^{a+1/2}}{1 + \lambda \rho_{\nu}} \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [\xi_i X_{i\nu} - \mathbb{E} \langle \zeta_i X_{i\nu} \rangle] \right\}.$$ 

Using the idea of proving Corollary 3.1, we can easily show that the Lindberg's condition is satisfied for the triangular array $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} [\xi_i X_{i\nu} - \mathbb{E} \langle \zeta_i X_{i\nu} \rangle].$ Let $\sigma_n^2 = \mathbb{V} \xi_n$. Note that $\sigma_n^2 \approx 1$. Its proof is similar to the argument of $\mathbb{V} \langle \zeta_i (R_{X_i}, \theta_1) \rangle$ in the proof of Corollary 3.1. Actually, as $n \to \infty$, $\sigma_n^2$ will converge to $\psi_n^2 := (\tau - \tau^2)\mathbb{E} \left[ \int_0^1 X_i(t) \phi_{\nu}(t) dt \right]^2$.

Then $V(J_n, \phi_{\nu}) \xrightarrow{d} N \left( 0, \frac{\psi_n^{2a+2}}{(1+\lambda \rho_{\nu})^2} \right)$. Namely, $V(J_n, \phi_{\nu})$ converges in distribution to $V(G, \phi_{\nu})$. Then (ii) finite-dimensional convergence is verified.

This completes the proof.

**A.7. Proof of Theorem 3.5.** We first treat the main term $\hat{Q}_{Y|X}(\tau|x_0) - Q_{Y|X}(\tau|x_0)$. Note that for a fixed $\tau \in \mathcal{U}$,

$$\hat{Q}_{Y|X}(\tau|x_0) - Q_{Y|X}(\tau|x_0) = \hat{\alpha}_{h,\lambda} + \int_0^1 x_0(t) \hat{\beta}_{h,\lambda}(t, \tau) dt - \alpha_0(\tau) - \int_0^1 x_0(t) \beta(t, \tau) dt$$

$$= \langle R_{x_0}, \hat{\alpha}_{h,\lambda} - \theta_0 \rangle$$

$$= \langle R_{x_0}, \hat{\alpha}_{h,\lambda} - \theta_0 + S_{n,h,\lambda}(\theta_0) \rangle - \langle R_{x_0}, S_{n,h,\lambda}(\theta_0) \rangle.$$ 

The first term is bounded by $\|R_{x_0}\| \cdot \|\hat{\alpha}_{h,\lambda} - \theta_0 + S_{n,h,\lambda}(\theta_0)\|$, which is of order $O_P(a_n \sigma_n(x_0))$ under the condition $\sigma_n(x_0) \ll \|R_{x_0}\|$. We next derive the asymptotic distribution of the second term.

Recall that $S_{n,h,\lambda}(\theta_0) \Delta \theta = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \mathcal{R} \left( \frac{\xi_i \theta_0}{h} \right) - \tau \right] \langle R_{X_i}, \Delta \theta \rangle + \lambda \mathbb{J}(\beta_0, \Delta \beta)$ for any $\Delta \theta \in \mathcal{H}$. Let $\zeta = \mathcal{R}(\xi_i \theta_0/h) - \tau, i = 1, \ldots, n$. Then

$$\langle R_{x_0}, S_{n,h,\lambda}(\theta_0) \rangle$$

$$= \frac{1}{n} \sum_{i=1}^{n} \zeta_i \mathbb{E} \{ B(X) \}^{-1} + \langle \eta(x_0), \eta(X_i) \rangle_1 + \langle P_{\lambda}, \theta_0, R_{x_0} \rangle$$

$$= \frac{1}{n} \sum_{i=1}^{n} \zeta_i Z_i + \int_0^1 x_0(t) (W_{\lambda} \beta_0)(t) dt,$$

where $Z_i = \mathbb{E} \{ B(X) \}^{-1} + \langle \eta(x_0), \eta(X_i) \rangle_1$ for $i = 1, \ldots, n$. To establish the CLT for the second term, we need to calculate $s_n^2 = \mathbb{V} \left\{ \sum_{i=1}^{n} \mathbb{E} \{ B(X) \}^{-1} + \langle \eta(x_0), \eta(X_i) \rangle_1 \right\}$ first. We employ the law of iterated expectations to find $\mathbb{E} \{ B(X) \}^{-1}$. As shown in the proof of Corollary 3.1, $\mathbb{E} \{ B(X) \}^{-1} \approx 2h B(x) \int_0^1 \mathcal{K}(z) [1 - \mathcal{K}(z)] dz + O(h^2)$ uniformly for $x$. By the Fourier expansions of $\eta(x_0)$ and $\eta(X_i)$, we have $\langle \eta(x_0), \eta(X_i) \rangle_1 = \sum_{\nu} \frac{X_{i\nu}}{1+\lambda \rho_{\nu}}$, where $x_0^\nu = \int_0^1 x_0(t) \phi_{\nu}(t) dt$ and $X_{i\nu} = \int_0^1 X_i(t) \phi_{\nu}(t) dt$.
for \( \nu \geq 1 \). Therefore, by Assumption A1 (c),
\[
E \zeta_i Z_i^2 \leq C_1 E \zeta_i^2 B(X_i) Z_i^2 \\
\asymp C_1 (\tau - \tau^2) E \{ B(X) \} [E \{ B(X) \}]^{-2} \\
+ 2[E \{ B(X) \}]^{-1}(\eta(x_0), \eta(X_i))_1 + (\eta(x_0), \eta(X_i))^2)
\]
\[
= C_1 (\tau - \tau^2) \left( [E \{ B(X) \}]^{-1} + \sum_\nu \frac{|x_\nu|^2}{(1 + \lambda^\nu)^2} \right) \\
\asymp \sigma_n^2(x_0).
\]

Next we find the \( E[\eta(x_0), \eta(X_i)] \). It is actually \( R^{(1)}_h(\theta_0; \tau) \theta_1 \) if we take \( \theta_1 = ([E \{ B(X) \}]^{-1}, \eta(x_0)) \). By the proof of (ii) of Lemma 3.1, we have
\[
|R^{(1)}_h(\theta_0; \tau) \theta_1 - R^{(1)}(\theta_0; \tau) \theta_1| \leq Ch^{s+1}E(|\{R_X, \theta_1\}|) \\
\leq Ch^{s+1} \left[ E \left( \langle R_X, \theta_1 \rangle^2 \right) \right]^{1/4} \\
\leq Ch^{s+1}||\theta_1|| = Ch^{s+1}||R_x||
\]
by the Cauchy-Schwarz inequality and Lemma S.3 of Shang and Cheng (2015b).

On the other hand, \( ||R_{x_0}|| \asymp \sigma_n(x_0) \) by the condition. Since \( R^{(1)}(\theta_0; \tau) \theta_1 = 0, \{E[\eta(x_0), \eta(X_i)]\}^2 = O(h^{2s+2}) \sigma_n^2(x_0). \) On the other hand, since \( h = o(1), \{E[\zeta_i Z_i]\}^2 = o(E[\zeta_i^2 Z_i^2]). \) Hence we conclude that \( \sigma_n^2 > 0 \) that does not depend on \( n \), such that \( \sigma_n^2 \geq \sigma_n^2 \) for all \( n \geq 1 \).

We check the Lindeberg’s condition to establish the central limit theorem for the triangular array \( \sum_{i=1}^n \{\zeta_i Z_i - R^{(1)}_h(\theta_0; \tau) \theta_1\} \), where \( \theta_1 = R_{x_0} = ([E \{ B(X) \}]^{-1}, \eta(x_0)) \in \mathcal{H}. \) Rearranging \( \zeta_i Z_i - R^{(1)}_h(\theta_0; \tau) \theta_1 \) leads to
\[
\zeta_i Z_i - R^{(1)}_h(\theta_0; \tau) \theta_1 = \left[ \int_{-\infty}^{\langle R_X, \theta_1 \rangle} k_h(u - Y_1)du - \tau \right] \langle R_X, \theta_1 \rangle - \\
\int \left\{ \int_{-\infty}^{\langle R_x, \theta_1 \rangle} \left( E[k_h(u - Y|x)|]du - \tau \right) \langle R_x, \theta_1 \rangle dP_x(x) \right\}.
\]

Obviously, there exists a positive constant \( C \) that is independent of \( n \) such that \( |\zeta| \leq C \) a.s. As shown in the proof of Corollary 3.1, \( |Z_i| = |\langle R_X, \theta_1 \rangle| \leq [E \{ B(X) \}]^{-1} + C^{1/2}_R \|\eta(x_0)\|_1 \|X\|_L^2 \omega^{-(2a+1)/2} \) for some constant \( C_R > 0 \). Therefore, \( |Z_i|^2 \leq (1 + C_R \omega^{-(2a+1)} \|X\|_L^2)^2 \|R_{x_0}\|^2. \)

Since \( \log(\omega^{-1}) = O(\log n) \), we can choose \( \tilde{C} \) sufficiently large such that \( \tilde{C} \log n / s > C \) and \( \omega^{-(2a+1)} n^{-\tilde{C}} = o(1). \) Using the same idea of the proof of Corollary 3.1, we have for any \( \epsilon > 0, \) as \( n \) approaching \( \infty \),
\[
n \epsilon^2 s_n^2 E \{ |\zeta_i Z_i - R^{(1)}_h(\theta_0; \tau) \theta_1|^2 \} \leq P(|\zeta_i Z_i - R^{(1)}_h(\theta_0; \tau) \theta_1|^2 \geq \epsilon^2 s_n^2) \}
\]
\[
\leq \left| R_{x_0} \right|^2 \left( E \{ (1 + C_R \omega^{-(2a+1)} \|X\|_L^2)^2 \} \right)^{1/2}. \left( P(|\zeta_i|^2 Z_i^2 \geq \epsilon^2 s_n^2) \right)^{1/2} \\
\leq O(\omega^{-(2a+1)}) \left[ P(s^2 \|X\|_L^2 \geq \frac{s^2 \epsilon^2 s_n^2}{C_R} \left( \frac{s^2 \epsilon^2 s_n^2}{C_R \log n} - 1 \right) \right]^{1/2} \\
\leq O(\omega^{-(2a+1)}) \left[ P(s \|X\|_L^2 \geq \frac{s^2 \epsilon^2 s_n^2}{C_R \log n} \left( \frac{s^2 \epsilon^2 s_n^2}{C_R \log n} - 1 \right) \right]^{1/2}
\]
where the last equation follows by \( s_{2n}^2 \sim n \sigma_n^2(x_0) \sim n \| R_{x_0} \|^2 \), the choice of \( \tilde{C} \) and \( n \omega^{2a+1}(\log n)^{-4} \rightarrow \infty \). Thus the Lindberg’s condition is fulfilled. As a result, \( s_{2n}^{-1} \sum_{i=1}^{n} \{ \zeta Z_i - E[\zeta Z_i] \} \xrightarrow{d} N(0, 1) \). Moreover, \( s_{2n}^{-1} n E[\zeta Z_i] = O(\sqrt{n} \sigma_n^{-1}(x_0) \cdot h^{s+1} \| R_{x_0} \|) = O(\sqrt{n} h^{s+1}) = o(1) \). Therefore, following the condition that \( n a_n \sigma_n(x_0) / s_{2n} \sim n^{1/2} a_n = o(1) \), we get that as \( n \rightarrow \infty \),

\[
\frac{n}{s_{2n}} \left[ \hat{Q}_{Y|X}(\tau|x_0) - Q_{Y|X}(\tau|x_0) + \int_0^1 x_0(t) (W_\lambda(\beta_0))(t) \, dt \right]
\]

\[
= s_{2n}^{-1} \sum_{i=1}^{n} \{ \zeta Z_i - E[\zeta Z_i] \} + n s_{2n}^{-1} E[\zeta Z_i] + O_P(n a_n \sigma_n(x_0) / s_{2n})
\]

\[
= s_{2n}^{-1} \sum_{i=1}^{n} \{ \zeta Z_i - E[\zeta Z_i] \} + o(1) + o_P(1) \xrightarrow{d} N(0, 1).
\]

Recall that if \( \beta_0(\cdot, \tau) = \sum_\nu b_\nu \phi_\nu \), \( W_\lambda \beta_0 = \sum_\nu b_\nu \frac{\lambda_\rho_\nu}{1 + \lambda_\rho_\nu} \phi_\nu \). By Cauchy inequality the bias term satisfies that

\[
\left| \int_0^1 x_0(t) (W_\lambda(\beta_0))(t) \, dt \right|
\]

\[
= \left| \sum_\nu b_\nu \frac{\lambda_\rho_\nu}{1 + \lambda_\rho_\nu} \int_0^1 x_0(t) \phi_\nu(t) \, dt \right|
\]

\[
= \left| \sum_\nu b_\nu \frac{\lambda_\rho_\nu}{1 + \lambda_\rho_\nu} x_0^0 \right|
\]

\[
\leq \lambda \left( \sum_\nu b_\nu^2 \rho_\nu^2 \right)^{1/2} \left( \sum_\nu \frac{|x_0^0|^2}{(1 + \lambda_\rho_\nu)^2} \right)^{1/2}.
\]

On the other hand, since \( s_{2n}^2 \geq \sigma_0^2 n \sigma_n^2(x_0) \geq \sigma_0^2 n \sum_\nu \frac{|x_0^0|^2}{(1 + \lambda_\rho_\nu)^2} \) and \( \sum_\nu b_\nu^2 \rho_\nu^2 < \infty \) and \( n \omega^d = o(1) \), we have

\[
(n / s_{2n}) \left| \int_0^1 x_0(t) (W_\lambda(\beta_0))(t) \, dt \right|
\]

\[
\leq \frac{n}{\sqrt{\sigma_0^2 n \sum_\nu \frac{|x_0^0|^2}{(1 + \lambda_\rho_\nu)^2}}} \cdot \lambda \left( \sum_\nu b_\nu^2 \rho_\nu^2 \right)^{1/2} \left( \sum_\nu \frac{|x_0^0|^2}{(1 + \lambda_\rho_\nu)^2} \right)^{1/2}
\]

\[
= O(n^{1/2} \lambda) = o(1).
\]

Hence,

\[
(n / s_{2n}) \left[ \hat{\alpha}_{h, \lambda} + \int_0^1 x_0(t) \hat{\beta}_{h, \lambda}(t, \tau) \, dt - \alpha_0(\tau) - \int_0^1 x_0(t) \beta(t, \tau) \, dt \right] \xrightarrow{d} N(0, 1)
\]

as \( n \rightarrow \infty \).
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