Josephson current in a superconductor–ferromagnet–superconductor junction with in-plane ferromagnetic domains

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We study a diffusive superconductor–ferromagnet–superconductor (SFS) junction with in-plane ferromagnetic domains. Close to the superconducting transition temperature, we describe the proximity effect in the junction with the linearized Usadel equations. We find that properties of such a junction depend on the size of the domains relative to the magnetic coherence length. In the case of large domains, the junction exhibits transitions to the $\pi$ state, similarly to a single-domain SFS junction. In the case of small domains, the magnetization effectively averages out, and the junction is always in the zero state, similarly to a superconductor–normal metal–superconductor (SNS) junction. In both those regimes, the influence of domain walls may be approximately described as an effective spin-flip scattering. We also study the inhomogeneous distribution of the local current density in the junction. Close to the $0\!-\!\pi$ transitions, the directions of the critical current may be opposite in the vicinity of the domain wall and in the middle of the domains.

I. INTRODUCTION

Substantial progress has been made recently in understanding the physical properties of nanoscopic layered structures composed of superconducting (S) and ferromagnetic (F) materials (for a review, see Refs. 1,2). In such systems, the coexistence of superconducting and magnetic correlations may lead to a variety of interesting physical effects. The exchange splitting of the Fermi level$^3$ in the ferromagnet breaks the spin degeneracy, and Andreev-reflected Cooper pairs acquire a finite momentum, which produces oscillations of the Cooper-pair wave function. By tuning the geometric and electronic parameters, one can realize SFS junctions in which the superconducting order parameter is of opposite sign in the two S electrodes.$^4$ Such a $\pi$ state manifests itself in a reversal of the sign of the critical current as the thickness of the ferromagnets is varied.$^1$ The characteristic thickness at which the first $0\!-\!\pi$ transition occurs is of the order of the magnetic coherence length $\xi_h$. In the diffusive limit, which is realized in most experimentally fabricated SF heterostructures, $\xi_h$ is given by $\sqrt{D/\hbar}$ where $D$ denotes the diffusion constant and $\hbar$ is the ferromagnetic exchange energy. Therefore, the experimental observation of such $0\!-\!\pi$ transitions in nanoscale devices requires a low exchange energy $\hbar$. This stringent condition was achieved using weak ferromagnetic CuNi or PdNi alloys.$^5$–$^8$

The physics of single-domain SFS junctions (including the effect of spin-flip$^9$–$^{11}$ and spin-orbit scattering$^3,12,13$) is now well understood. However, in some experiments the $0\!-\!\pi$ transition points may deviate from standard predictions$^8,14$ or even be absent.$^{15,16}$ There is no consensus on the interpretation of such deviations. It may be attributed either to the presence of a magnetically dead layer at the interface between the superconductor and the ferromagnet,$^8,17$ or to a domain structure or inhomogeneities in the ferromagnetic layer. The domain structure crucially depends on the nature of the ferromagnet: strong ferromagnets consist of well-defined magnetic domains whose spatial extension may be reduced by the proximity effect.$^{18}$–$^{21}$ In weakly ferromagnetic alloys, on the other hand, the magnetization may fluctuate on short length scales without forming domains.$^{22}$

Theoretically, SFS junctions with inhomogeneous magnetization have been studied recently in different setups.$^{23}$–$^{28}$ However, in most works (except for Refs. 27,28) only domains along the junction were studied (quasi-one-dimensional geometry), while in the experimental realizations of SFS junctions with thin F layers the domain structure is more likely to form in the plane of the F film.

Motivated by the experimental progress on $\pi$ junctions, we study a model of a diffusive SFS junction with in-plane domains (so that the domain walls are orthogonal to the S and F layers, see Fig. 1). This geometry has been studied previously by Volkov and Anishchanka within the macroscopic approach of London equations.$^{28}$ Our model is different from the one studied in Ref. 27: in that work, the Neel domain walls are considered, and the junction is brought to the regime with only the long-range triplet component contributing to the Josephson current. In our model, the domain walls are taken to be sharp, and no long-range triplet component appears for domains with antiparallel magnetization.

The domain structure introduces an additional length scale: the domain size $a$. As one can expect, we find that the effect of inhomogeneous magnetization depends strongly on the relative magnitude of $a$ and $\xi_h$. In the limit of small domains, $a \ll \xi_h$, the exchange field effectively averages out, and the critical current of a single nonmagnetic SNS junction is retrieved. In the opposite limit of large domains $a \gg \xi_h$, the influence of domain walls is localized to their vicinity and produces only a small correction to the current of a single-domain SFS junction. Between those limits, the supercurrent shows either a damped oscillatory behavior as a function of the junction thickness (for large domains $a > a_c \approx 0.83 \xi_h$), or a monotonic exponential decay (for smaller domains $a < a_c$). In the former case, the multidomain
The paper is organized as follows. In Section II we compute the superconducting Green functions and the Josephson current density for the multidomain SFS junction. Section III is devoted to the analysis of the total Josephson current. In Section IV we discuss the spatial distribution of the current density. Finally, in Section V we summarize our conclusions.

II. A MODEL FOR THE MULTIDOMAIN SFS JUNCTION

We assume that the ferromagnetic layer is strongly disordered, and the motion of electrons is diffusive. In this regime, the Green functions are given by the solutions to the Usadel equations. To simplify the calculations, we further assume that the junction is close to the superconducting critical temperature . In this case, the superconducting correlations are weak so that the Usadel equations can be linearized, and the current-phase relation is sinusoidal

\[ J = J_c \sin \varphi, \]

where \( \varphi = 2\chi \) is the superconducting phase difference across the junction and \( J_c \) is the critical current. The sign of \( J_c \) determines if the junction is in the “zero” phase or in the “\( \pi \)” phase.

In this paper, we consider a SFS junction with in-plane ferromagnetic domains of opposite magnetization. We introduce a coordinate system with the F layer in the \( yz \) plane (Fig. 1). The \( x \) axis is directed along the junction, and the SF interfaces correspond to the coordinates \( x = 0, d \). The domain walls are taken to be normal to the \( y \) axis. The origin of the \( y \) axis is chosen at the interface between two domains. The system is invariant under translation along the \( z \) axis. Our further calculations will be equally applicable to either the system with two domains of width \( a \) (see Fig. 1) or the \( 2a \)-periodic multidomain case (the same setup periodically repeated in the \( y \) direction).

The (nonlinear) Usadel equation in the ferromagnetic layer takes the form (we follow the conventions used in Ref. 32)

\[ D \nabla (\hat{g} \nabla \hat{g}) - \omega [\hat{\tau}_3 \hat{\sigma}_0, \hat{g}] - i [\hat{\tau}_3 (h \cdot \hat{\sigma}), \hat{g}] = 0. \]

where \( D \) denotes the diffusion constant and the system of units with \( h = k_B = \mu_B = 1 \) is chosen. The Green function \( \hat{g} \) is a matrix in the Nambu \( \otimes \) spin space, \( \hat{\tau}_\alpha \) and \( \hat{\sigma}_\alpha \) denote the Pauli matrices respectively in Nambu (particle-hole) and spin space, \( \omega = (2n + 1) \pi T \) are the Matsubara frequencies and \( h \) is the exchange field in the ferromagnet. The Usadel equation is supplemented with the normalization condition for the quasiclassical Green function

\[ \hat{g}^2 = \hat{1} = \hat{\tau}_0 \hat{\sigma}_0. \]
For simplicity, we assume that the superconductors are much less disordered than the ferromagnet, and then we can impose the rigid boundary conditions at the SF interfaces,

$$\hat{g} = \frac{1}{\sqrt{\omega^2 + \Delta^2}} \left( \begin{array}{cc} \omega & \Delta \varepsilon e^{\pm i\chi} \\ -\Delta \varepsilon e^{\mp i\chi} & -\omega \end{array} \right)_{Nambu} \otimes \hat{\sigma}_0,$$

where $\Delta$ denotes the superconducting order parameter, and the different signs refer respectively to the boundary conditions at $x = 0$ and $x = d$. At the (transparent) interface between the two ferromagnetic domains, we impose the continuity of the Green functions and their derivatives.

Close to the critical temperature $T_c$, we linearize the Usadel equations (2), (3) around the solution for the normal metal state $\hat{g} = \hat{\tau}_3 \hat{\sigma}_0 \text{sgn}(\omega)$. The linearized Green function then takes the form

$$\hat{g} = \left( \begin{array}{cc} \sigma_0 \text{sgn}(\omega) & f_0 \sigma^\alpha \\ -f_0^\dagger \sigma^\alpha & -\sigma_0 \text{sgn}(\omega) \end{array} \right),$$

where the scalar $f_0$ (respectively $f_0^\dagger$) and vector $\mathbf{f}$ (respectively $\mathbf{f}^\dagger$) components of the anomalous Green functions obey the linear equations

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f^{(1)}_{\pm} - \lambda_{\pm}^2 f^{(1)}_{\pm} = 0,$$

with

$$\lambda_{\pm} = \left[ 2 \frac{|\omega| \mp ih \text{sgn}(\omega)}{D} \right]^{1/2}.$$

The projections of the anomalous Green function along the direction of the exchange field (“parallel” components) are defined as $f^{(1)}_{\pm}(x,y) = f^{(1)}_{\pm}(x) \cdot \mathbf{e}_z$ (we assume that the ferromagnetic exchange field $\mathbf{h}$ is aligned in the direction $\mathbf{e}_z$, see Fig. 1). Note that there is no perpendicular (“long-range triplet”$^{33}$) component of the vector part of the Green function, since the magnetizations of the domains are collinear. We have used the invariance under translation along the $z$ direction. The superscript 1 refers to domains with field along $\mathbf{e}_z$ and in the following we will use the superscript 2 for domains with the field along $-\mathbf{e}_z$. Similar equations hold for $f^{2(1)}_{\pm}$ with $\lambda_{\pm} \leftrightarrow \lambda_{\mp}$.

It is convenient to write the solutions to those equations in the form

$$f^{1,2}_{\pm}(x,y) = f^{(1,2)}_{\pm\text{Bulk}}(x) + \delta^{1,2}_{\pm}(x,y),$$

where $f^{1,2}_{\pm\text{Bulk}}$ are the solutions of Eq. (6) for a single-domain SFS junction with the magnetization along $\mathbf{e}_z$ (respectively $-\mathbf{e}_z$). Since the equations (6) are linear, the correction $\delta(x,y)$ is also a solution to the same equations with the boundary conditions

$$\partial_y \delta^{1,2}_{\pm}(x,y = \mp a) = 0,$$

$$\delta^{1,2}_{\pm}(x = \{0,d\},y) = 0,$$

$$\delta^{1}_{\pm}(x,y = 0) - \delta^{2}_{\pm}(x,y = 0) = \Delta f^{\pm\text{Bulk}}(x),$$

$$[\partial_y] \delta^{1}_{\pm}(x,y = 0) - [\partial_y] \delta^{2}_{\pm}(x,y = 0) = 0.$$

Here $\Delta f^{\pm\text{Bulk}} = f^{2}_{\pm\text{Bulk}}(x) - f^{1}_{\pm\text{Bulk}}(x)$ is the difference of the bulk Green functions in the two domains. For the two-domain junction, the first condition imposes zero current at the interface with vacuum, the second condition ensures the continuity of the Green functions at the SF interfaces. Finally, the last two conditions reflect the continuity of the Green function and its derivatives at the interface between the two domains. It can be easily shown from symmetry considerations that this set of boundary conditions can also be applied to a periodic multidomain SFS junction with domains of width $2a$.

The condition (10) allows us to express $\delta^{1,2}_{\pm}$ in the form of the Fourier series

$$\delta^{1,2}_{\pm} = \sum_{n=1}^{\infty} \sin \left( \frac{\pi n}{d} x \right) A^{1,2}_{n\pm}(y).$$

For each $n$ we solve

$$\partial_y^2 A^{1}_{n\pm} = \gamma^2_{n\pm} A^{1}_{n\pm}.$$
with
\[ \gamma_n = \sqrt{\left(\frac{\pi n}{d}\right)^2 + \lambda_n^2}. \]

To obtain the equation for \( \mathcal{A}_n \), one needs to substitute \( \gamma_n \rightarrow \gamma_{n\pm} \). We can solve those equations for each Fourier component \( n \) with the boundary conditions provided by (9), (11) and (12). The solution is given by
\[ \delta_\pm = \frac{\Delta}{|\omega|} \sum_{n=1}^{\infty} \sin \left(\frac{\pi n x}{d}\right) \frac{2\pi n \cosh \gamma_{n\pm}(y + a) \gamma_{n\mp} a}{\cosh \gamma_{n\pm} \gamma_{n\mp}} \gamma_{n\mp} \tanh \gamma_{n\mp} a + \gamma_{n\pm} \tanh \gamma_{n\pm} a \left( \frac{1}{\gamma_{n\mp}^2} - \frac{1}{\gamma_{n\pm}^2} \right) (e^{i\chi} - (-1)^n e^{-i\chi}) \quad (16) \]

In the second domain, the correction \( \delta_\pm \) is given by the same formula with the replacement of \( y, \gamma_{\pm} \) by \(-y, \gamma_{\mp}\). The bulk Green functions are given by \( 26 \)
\[ f_{\pm}^{\dagger}{\text{Bulk}} = \frac{\Delta}{|\omega|} \left[ \frac{\sinh \lambda_{\pm} x}{\sinh \lambda_{\mp} d} e^{-i\chi} + \frac{\sinh \lambda_{\pm} (d - x)}{\sinh \lambda_{\mp} d} e^{i\chi} \right], \quad (17) \]

and \( f_{\pm}^{\dagger}{\text{Bulk}} = f_{\mp}^{\dagger}{\text{F Bulk}} \). Finally, note that \( f_{\pm}^{\dagger}{\text{Bulk}} \) and \( \delta_\pm \) are given by the same expressions (16), (17) with the replacement of \( \chi \) by \(-\chi\).

The last step will be to compute the Josephson current density using the formula\(^2\)
\[ J = ieN(0)D \pi T \sum_{\omega = -\infty}^{\infty} \frac{1}{2} \text{Tr} \left( \hat{\gamma}_3 \hat{\sigma}_0 \hat{g} \nabla \hat{g} \right), \quad (18) \]

where \( N(0) \) is the density of states in the normal metal phase (per one spin projection) and the trace has to be taken over the Nambu and spin indices. The current density can be explicitly rewritten for the linearized \( \hat{g} \)
\[ J = -ieN(0)D \pi T \sum_{\omega = -\infty}^{\infty} \left[ \sum_{\sigma = \pm} \frac{1}{2} (f_{\sigma} \nabla f_{\sigma}^{\dagger} - f_{\sigma}^{\dagger} \nabla f_{\sigma}) \right]. \quad (19) \]

The symmetry of translation along the \( z \) direction implies that the current remains in the \( xy \) plane. Using the expression for the Green functions (16) and (17), we can obtain a general expression for the current density (which is too cumbersome to be reproduced here). This expression involves two contributions. The first one is produced exclusively by the bulk Green functions (17) and corresponds to a homogeneous ferromagnetic interlayer. The second contribution is due to the correction (16) and reflects the influence of the domain structure. The current resulting from this contribution is not uniform in space. The characteristic decay scale of this correction as a function of the distance from the domain interface is given by \( \Re \left( \frac{1}{\gamma_{n\pm}} \right) \sim \min(\xi_T, \xi_h, d) \), where \( \xi_T = \sqrt{D/2\pi T_c} \) and \( \xi_h = \sqrt{D/h} \) are the thermal and magnetic coherence lengths, respectively. Far from the interface between the domains \((y \gg \min(\xi_T, \xi_h, d))\), the correction (16) vanishes and we recover locally the single-domain SFS current. Thus we expect the properties of the junction to be very different in the two limits of small \([a \ll \min(\xi_T, \xi_h, d)]\) and large \([a \gg \min(\xi_T, \xi_h, d)]\) domains.

### III. CRITICAL CURRENT

Experimentally, in SFS hybrid junctions, the measurable quantity is the total current flowing through the junction, that is along the \( x \)-axis. Since \( \nabla \cdot J = 0 \), the total current is conserved along the \( x \) direction. We can therefore compute it at \( x = 0 \), and we find
\[ J_c = i e N(0) D \pi T \sum_{\omega > 0} \frac{\Delta^2}{\omega^2 \sinh \lambda_{\pm} d} + \frac{16 \pi^2}{a d^2 \xi_h^2} \sum_{n=1}^{\infty} \left[ \frac{(-1)^{n-1} n^2}{(\gamma_n + \gamma_n) \gamma_{n+} \gamma_{n-} \coth \gamma_{n+} + \gamma_{n-} \coth \gamma_{n-}} \right] \quad (20) \]

with
\[ I_0 = \frac{4 e N(0) DS \pi T}{d}, \quad (21) \]

and \( S \) the area of the junction. The first term is the critical current for a single-domain SFS junction with a damped oscillatory dependence on the F-layer thickness (for a review, see Refs. 1, 34 and 35). It can be either positive
FIG. 2: Critical current $J_c/J_{SNS}$ vs. junction length $d/\xi_h$ for $a = 0.6 \cdot \xi_h$ (dotted line), $1.6 \cdot \xi_h$ (dashed line) and $\infty$ (solid line). We take $T = 100$.

(zero state of the junction) or negative ($\pi$ state). The second term reflects the influence of the domain structure. The critical current (20) depends on the three dimensionless parameters: $a/\xi_h$, $d/\xi_h$, and $\xi_T/\xi_h$. For some values of the parameters, the critical current (20) computed numerically is plotted in Fig. 2. Depending on the values of the parameters, it shows either an exponential decay or an exponential decay with oscillations, as a function of $d$.

Note that in most experimental situations $\xi_T \gg \xi_h$, because the ferromagnetic exchange energy exceeds by far the superconducting critical temperature. In the following we will refer to this situation as the high-field limit. In this limit, the summation over $\omega$ in Eq. (20) can be performed analytically $\sum_{\omega > 0} \Delta^2/\omega^2 = \Delta^2/(8T^2)$, and the deviation $\delta J_c$ from the critical current of a single-domain SFS junction [the second term in Eq. (20)] is expressed in terms of the reduced variables $n^* = d\sqrt{2}/\pi\xi_h$ and $a^* = \pi a/\xi_h$:

$$
\frac{\delta J_c}{J_0} = - \frac{\Delta^2}{2T^2} \frac{n^4}{a^*} \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{(n^2 + n^*^2)^{3/2}} \text{Re} \left[ \sqrt{n^2 + i n^*} \coth (a^* \sqrt{n^2 - i n^*}) \right].
$$

In the limit of large $d$, the asymptotic behavior of this expression may be estimated as an integral (in the variable $z = n/n^*$)

$$
\frac{\delta J_c}{J_0} = - \frac{\Delta^2}{2T^2 a^*} \int_{-\infty}^{\infty} dz \frac{e^{iz n^* z^2}}{(z^4 + 1)^{3/2} \left[ \sqrt{z^2 + i \coth (a^* n^* \sqrt{z^2 - i})} + \sqrt{z^2 - i \coth (a^* n^* \sqrt{z^2 + i})} \right]},
$$

which is, in turn, determined to the exponential precision by the singularities of the integrand in the complex plane. Remarkably, the contribution from the poles at $(\pm i)^{1/2}$ cancels exactly the first term (single-domain contribution) of Eq. (20). For sufficiently large $d$, to the exponential precision, the critical current is then given by

$$
J_c \propto e^{-\lambda d}, \quad \lambda = - \frac{i \sqrt{2z_0}}{\xi_h},
$$

where $z_0$ is the singularity of the integrand with the smallest positive imaginary part. Note that $z_0$ is now a function of one dimensionless parameter $\alpha = a^* n^* = \sqrt{2a}/\xi_h$.

By analogy with a single-domain SFS junction with spin-flip scattering, the real and imaginary parts of $\lambda^2$ may be interpreted as an effective magnetic field and an effective spin-flip rate $38$,

$$
\lambda^2 = - \frac{2i}{\xi_h} + \frac{4\Gamma_{sf}(\xi)}{D} \xi_h, \quad \xi_h = \sqrt{\frac{D}{\hbar(\xi_h)}},
$$

Therefore the effective field and spin-flip rate can be found as

$$
h^{(\xi)} = h(\Im(z_0^2)), \quad \Gamma^{(\xi)}_{sf} = \frac{\hbar}{2} \Re(z_0^2).
$$

In the following, we discuss the limits of large and small domain sizes.
A. Limit of large domains: $a \gg \xi_h$

We consider the limit of large domains, $a \gg \xi_h$, with the assumption of the strong exchange field, $\xi_T \gg \xi_h$. In this regime, the damped oscillations of the critical current at large $d$ are determined by the solutions to the equation

$$\sqrt{z^2 + i \coth(\alpha \sqrt{z^2 - i})} + \sqrt{z^2 - i \coth(\alpha \sqrt{z^2 + i})} = 0$$

with the smallest positive imaginary part. At $\alpha = \sqrt{2a/\xi_{h}} \gg 1$, one of the arguments of $\coth(\alpha \sqrt{z^2 \pm 1})$ must be close to $\pm i\pi/2$. Expanding around this point, we obtain

$$z_0^2 = i - \frac{\pi^2}{4a^2} + \frac{(1-i)\pi^2}{4\alpha^2} + \ldots$$

This translates into the reduced effective field

$$h^{(\text{eff})} \approx h \left[ 1 - \frac{\pi^2}{8\sqrt{2}} \left( \frac{\xi_h}{a} \right)^3 \right]$$

and the effective spin-flip rate

$$\Gamma_{sf}^{(\text{eff})} \approx \frac{\pi^2}{16} \left( \frac{\xi_h}{a} \right)^2 \frac{h}{16a^2}. \quad (29)$$

Thus, to the leading order in $(\xi_h/a)$, the effect of domain walls reduces to an effective spin-flip rate, which increases the period of $0-\pi$ transitions as a function of $d$ and simultaneously decreases the overall decay length of the critical current (see Fig. 2, dashed line, for an illustration).

B. Limit of small domains: $a \ll \xi_h, d, \xi_T$

In the limit of small domains $a \ll \xi_h, d, \xi_T$, we can calculate a perturbative correction to the critical current by expanding (20) in $a$. To the lowest order in $a$, we obtain (without assuming the high-field limit),

$$\frac{J_c}{I_0} = \frac{J_{SNS}}{I_0} - \frac{2a^2d^2}{3\xi_h^2} \sum_{\omega > 0} \frac{\Delta^2}{\omega^2} \left[ \frac{\lambda_0 d \cosh \lambda_0 d - \sinh \lambda_0 d}{\lambda_0 d \sinh^2 \lambda_0 d} \right],$$

where $\lambda_0^2 = \frac{\lambda_2^2 + \lambda_0^2}{2} = \frac{2\omega d^2}{B}$ does not contain the exchange energy $h$, and $J_{SNS} = J_c(h = 0)$. This expression reveals that in the limit $a \rightarrow 0$ the multidomain SFS junction behaves like a SNS junction: the exchange field is averaged out when the domain width is small. Note also that the correction arising from a finite domain width is always negative: the amplitude of the current is decreased compared to the SNS case.
A more accurate approximation may be obtained in the high-field limit \( \xi_T \gg \xi_h \) by the asymptotic estimate of the oscillating sum described earlier in this Section. To the second order in \( a \), the solution to the equation (27) is given by \( z_0 = -\frac{a^2}{3} \), which translates into

\[
h^{(\text{eff})} = 0, \quad \Gamma^{(\text{eff})}_{sf} \approx \frac{1}{3} \left( \frac{a}{\xi_h} \right)^2 \frac{h}{\hbar} = \frac{h^2 a^2}{3D} \]

This expression for \( \Gamma^{(\text{eff})}_{sf} \) agrees with the general estimate for the effective spin-flip rate obtained by Ivanov and Fominov\(^\text{32} \) for SF structures with inhomogeneous magnetization.

Note that for sufficiently small \( a \), the equation (27) has a solution with real \( z_0^2 \) corresponding to a pure decay (without oscillations) of the critical current. The dependence of the critical current on \( d \) is then purely decaying, without 0–\( \pi \) oscillations (Fig. 2, dotted line).

### C. 0–\( \pi \) phase diagram

Between the two regimes of small and large domains, there is a phase transition as a function of \( a/\xi_h \) corresponding to a bifurcation of the real solution \( z_0^2 \) to Eq. (27) at smaller \( a \) to complex solutions at larger \( a \). For \( a/\xi_h \) smaller than the critical value, the critical current decays as a function of \( d \) without oscillations (always in the 0 phase). For \( a/\xi_h \) larger than the critical value, the dependence on \( d \) is damped oscillatory, qualitatively similar to a single-domain SFS junction.

Numerically, we find the critical value \( a_c/\xi_h \approx 0.83 \). The full 0–\( \pi \) phase diagram in the high-field limit is plotted in Fig. 3. Periodic 0–\( \pi \) transitions (as a function of \( d \)) above \( a_c/\xi_h \) and zero phase below \( a_c/\xi_h \) illustrate our discussion. The absence of the 0–\( \pi \) transitions in the case of small domains may explain why in some experimental SFS junctions the \( \pi \) state is absent.\(^\text{15,16} \)

For completeness, in Fig. 4 we also plot the locus of solutions \( z_0^2 \) to Eq. (27) in the complex plane for all values of \( \alpha \) (in the units \( \Gamma^{(\text{eff})}_{sf}/h \) and \( h^{(\text{eff})}/h \)). The corresponding real and imaginary parts of \( \lambda \) determining the \( d \) dependence of the critical current (24) are plotted in the inset.

### IV. LOCAL CURRENT DENSITY

Since the system does not have a translational symmetry along the \( y \) direction, the Josephson current forms a nontrivial pattern in the \( x-y \) plane. In Fig. 5 we present plots of the current density (proportional to \( \sin \varphi \)) at two different points of the phase diagram: in the zero phase and in the \( \pi \) phase.

Those inhomogeneous patterns may be qualitatively understood on the basis of interpreting the domain walls as producing an effective spin-flip scattering. Different regions of the ferromagnet may be attributed different effective
spin-flip rates, depending on their distance from the domain wall. The effective spin-flip processes renormalize the decay coefficient $\lambda$ in (24) and, therefore, different parts of the junction experience $0$–$\pi$ transitions at different values of $d$. This can be clearly seen in Fig. 6 depicting the current density near a $0$–$\pi$ transition. While the neighborhood of the domain wall is in the $0$ phase, the region near the free boundaries (at $y = \pm a$) are in the $\pi$ phase. This situation resembles a model studied by Buzdin et al.\textsuperscript{29}: a system of alternating zero and $\pi$ junctions. In that work, an intermediate equilibrium phase difference was predicted, depending on the ratio between the junction widths and the magnetic coherence length. Even though our model cannot lead to such a $\varphi$-junction (we consider linearized Usadel equations and therefore obtain a purely sinusoidal current-phase relation with only two possible equilibrium phases $0$ or $\pi$), at low temperatures such a SFS system with domains could possibly produce a $\varphi$-state.

V. SUMMARY

In this work we consider a Josephson SFS junction consisting of domains with opposite magnetization connected “in parallel”. As a function of the junction thickness, the critical current may exhibit either a decaying oscillating or a purely decaying behavior, depending on the domain width. The effect of domain walls in this geometry may be approximated as an effective spin-flip scattering, together with a renormalization of the effective magnetic field. This behavior is different from that in SFF’S junctions with the domains connected “in series” studied in Ref. 26. In that SFF’S setup, the domain structure lead to a gradual reduction of the $\pi$ phase (at a non-parallel configuration
of the two domains), so that the relative fraction of the zero phase increases as a function of the mismatch in the magnetization directions. In the present work, however, we do not consider the case of an arbitrary angle between the two magnetizations, because of the complexity of the problem.

We expect that in a realistic geometry of domains both effects of the spin-flip scattering and of the reduction of the $\pi$ phase take place simultaneously, and our findings from this work and from Ref. 26 may help to qualitatively describe the $0-\pi$ phase diagram of real SFS junctions with inhomogeneous ferromagnets.

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31. We define the spin-flip scattering rate by $\Gamma_{sf} = 1/(2\tau_{sf})$, as in Refs. 32,36,37.