ON STATIC HEDGING, REAL OPTIONS AND VALUATION OF CASH FLOWS WITH SKEWED DISTRIBUTIONS

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Abstract. Draft. We combine static hedging and real options valuation ideas to build a capital budgeting technique. Here one applies the market information of derivative prices on a traded ‘quasi twin security’ to benchmark a single-step stochastic cash stream. We provide a transparent, more or less closed-form solution for valuing these streams. The fundamental properties of this valuation rule are then studied.

The derivation of the pricing rule is developed in such a way as to generalize intuitive real option considerations to continuous state step-by-step. We also discuss some required mathematical finance machinery as well as results of Breeden-Litzenberger type.

1. Introduction

This is the first part of two papers in which we investigate a real options analysis type valuation technique involving non-parametric calibration and static hedging. However, we will not pay attention in identifying real options, per se.

In the literature there are several approaches to the valuation of stochastic cash flows closely related to this paper. See for instance [Glasserman 2003], [Jaeckeck 2002] on Monte Carlo method; [Rubinstein 1994], [Rubinstein 1998] for implied trees; [Ait-Sahalia&Lo 1998], [Bliss&Panigirtzoglou 2004], [Jackwerth&Rubinstein 1999] for implied state price density distributions; [Carr&Chou 1997], [Carr & al. 1999] for static hedging; [Black 1976], [Merton 1974], [Shreve 1998] for closed form solutions and PDEs; and [Gerber&Shiu 1994], [Liu & al. 2007] for transforms. See also [Ait-Sahalia&Duarte 2003], [Bakshi & al. 2003], [Brunner&Hafner 2003], [Derman&Kani 1994], [Dupire 1994], [Fengler 2005] for special shapes of the state price density surfaces, including volatility smile and [Buchen&Kelly 1996], [Stutzer 1996] for maximum entropy and estimation of risk-neutral densities.

The aim of this study is to analyze the reasonable values of a stochastic cash flow with a given priori continuous probability distribution. For cash flows corresponding to standard commonly traded securities the markets, of course, provide a natural benchmark. For example, the real options valuation (ROV) techniques provide insight in such a situation. Here we will present a method for evaluating in closed form the present value of cash flows with more or less arbitrary distribution. Such a value is required in various situations where no closely matching benchmark is available, e.g.
in capital budgeting problems, derivatives on less common securities, risk management and securitization of insurance contract bundles.

The risk-neutral pricing approach would be operational in the aforementioned task as well. One can value complicated single-step cash flows by applying e.g. the binomial approach, which follows the Cox-Ross-Rubinstein model, see [CRR 1979]. Taking this approach a step further, one may bypass some phases by directly computing the expected value of the discounted cash flow with respect to a risk-neutral measure, if convenient. However, the binomial approach is not necessarily easy to implement in the case of arbitrary (e.g. multimodal or fat tailed) distributions and does not typically provide a closed form solution (cf. [Georgiadis 2011]). For example, a bimodal distribution can be relevant in a project having two anticipated main scenarios. There are many approaches to ROV and no general method neatly covering all cases, see e.g. [Schulmerich 2010, pp. 59-62].

We will apply risk-neutral valuation here in the guise of 'continuous portfolios' containing Arrow-Debreu securities. We will assume the following pieces of information to be known: the risk-free bond prices, the distribution of a traded benchmark security \( S(T) \) at horizon, the values of calls on \( S(T) \) corresponding to all possible strike prices (statistically estimated), and the distribution of a single-step cash flow \( CF_T \). By using this information we will value the cash flow. This can be viewed as a non-parametric calibration technique. One practical upside in passing to the continuous state space, instead of considering trees (and the resulting numerical algorithms) is that one can apply rather simple calculus to analyze the model. On the other hand, the approach taken here does not work in atomistic probability spaces and some new phenomena arise compared to the lattice models applied in mainstream real options valuation.

The main points of these papers are threefold. First, we will develop a closed form value formula of a single step cash flow. This is accomplished by mimicking the distribution of the cash flow with respect to a very flexible option price system. We will combine static hedging arguments with real options philosophy (but without decision tree analysis) and build a kind of idealized continuous portfolio of Arrow-Debreu assets. The idea of the construction of the portfolio will be transparent, thus we hope that our method brings insight into the risk-neutral valuation of cash flows. The closed form solution developed here, unfortunately, involves some complicated functions. Our method applies, for instance, in pricing securities having skewed or otherwise distorted distributions in comparison with some standard models, or traded securities.

The estimation of the price of the the Arrow-Debreu securities, i.e. the state price density enables one to incorporate market information by non-parametric calibration. Thus, the theme of the paper can be summarized as non-parametrically calibrated real options. We are sure that the main ideas are more or less anticipated by specialists working with valuation. For example, consider the well-known paper [Breeden&Litzenberger 1978], where the second derivative of a European call option is connected to the state-price density. Although the paper is known for the above mentioned useful formula, a good portion (and lesser known part) of it is spent in
suggesting how to apply the notion in capital budgeting problems, and in fact this effectually precedes real options valuation. However, here we will study most importantly the properties of the valuation mechanism which include some subtle aspects.

Secondly, we will introduce the notion of treating 'continuous portfolios' employed in static hedging as signed Radon measures. This provides a convenient and precise way of defining the portfolios in question, and, on the other hand, some technical unpleasantries (e.g. distribution theory) related to static hedging are circumvented.

Thirdly, in the subsequent paper we will show that there is inherently some slack in the prices in the presented valuation method. We will also quantify this statement.

This paper is organized as follows. We begin by recalling the idea behind static hedging in Section 2. The motivation for doing this, instead of merely giving a reference, is that we wish to illustrate (and recall) how hedging can be accomplished in a rather general option price system, requiring very little structural assumptions, and that the idea is transparent. Namely, the notion of state price density due to Arrow is applied, furnished with simple mathematical arguments accessible to the general audience. Next, in Section 3.1 we show how to (re)construct a one-step cash flow approximately and in Section 4 we pass to the limit to obtain a continuous distribution by building a sharp continuous portfolio of Arrow-Debreu digital options yielding the same distribution. In real options valuation one takes the value of such portfolio as a proxy for the value of the cash flow (e.g. induced by a project).

The arguments in this paper do not involve deep methodologies of probability theory, although, some measure theory and analysis are required. For example Radon-Nikodym derivatives are applied. Since this paper involves general valuation problem rather than specific martingale pricing methods, we wish to give more attention to the financial motivations, rather than the mathematical technicalities (such as integrability issues, etc.).

1.1. Preliminaries. We refer to [Brealey & al. 2011], [Cochrane 2005], [Copeland & Antikarov 2001], [Föllmer & Schied 2011], [Hull 2003], [Shreve 1998] and [Carr & Chou 1997] for suitable background information.

Recall that an affine transform is a linear transform shifted by adding a vector.

Here we denote by $S(\omega, t), \omega \in \Omega, t \in [0, \infty)$, a stochastic process which is regarded as the value of a security $S$ at time $t$. This security is not necessarily a stock and we assume that the value of the security $S(\omega, t)$ is distributed according to a probability measure $\mathbb{P}_t$ which is usually absolutely continuous with respect to the Lebesgue measure $m$ on $\mathbb{R}$. Some stricter conditions are imposed on the value distribution case-by-case. Although there are potentially infinitely many times $t < T$, we will work essentially with a single-step model having only two times ($t < T$).

By abuse of notation we often write the pushforward measure $\mathbb{P}_* (A) = \mathbb{P}(S(T) \in A)$ simply as $\mathbb{P}(A)$ because usually we are not required to make a distinction between the spaces $(\Omega, \mathbb{F})$ and $(\mathbb{R}, \mathbb{P}_*)$. 
We denote by $|\rho|$ the variation (measure) of a signed measure $\rho$. This is given by

$$|\rho|(A) = \sup_{A_1, \ldots, A_k} \sum |\rho(A_i)|$$

where the supremum is taken over $\rho$-measurable partitions $A_1, \ldots, A_k$ of $A$. The total variation is $|\rho| = |\rho|(\Omega)$. If $f : \Omega \to \mathbb{R}$ is a $|\rho|$-measurable function and $A$ is a $|\rho|$-measurable set, we denote

$$\int_A f \, d\rho = \int_A f \, \frac{d\rho}{|\rho|} \, d|\rho|$$

whenever sensible. Recall that $\frac{d\mu}{d\nu}$ is the Radon-Nikodym derivative, determined by the condition

$$\mu(A) = \int_A \frac{d\mu}{d\nu} \, d\nu$$

for all $\nu$-measurable sets $A$. We will frequently apply the chain rule for Radon-Nikodym derivatives which is similar to the usual chain rule for differentiable functions. We refer to [Fremlin 2002] for in-depth treatment of measure theory.

The integrals here are usually inessential, unless otherwise stated. Recall that the relative entropy of equivalent measures $\mathbb{P}_1$ and $\mathbb{P}_2$ is defined as follows:

$$H(\mathbb{P}_1 | \mathbb{P}_2) = \int \log \frac{d\mathbb{P}_1}{d\mathbb{P}_2} \, d\mathbb{P}_1.$$ 

This is a non-symmetric measure of distance between the probability measures is also known as Kullback-Leibler divergence.

We denote by $1_{S(T) \in A}(\omega)$ an indicator function, which has value 1 if $S(T) \in A$ and 0 otherwise. As usual, $(x)^+$ is short for max$(x, 0)$. The support of a probability distribution $\phi$ is denoted by supp$(\phi)$.

The short rate is denoted by $r \in (0, 1)$.

We will work in a simplistic framework with a security $S_t$, a risk-free bond $B_t$ with a simple (standard) behavior and a system of European call options on $S_t$ with typically a given time to maturity ($T$) but all possible strike prices $K$. The bond pays 1 unit of numeraire at the horizon $T$ and sometimes we will apply the model $B_t = e^{-r(T-t)}$.

The observed value of a (‘plain vanilla’) European call option $C_S$ on security $S$ is denoted by $\hat{V}_{C_S}(S(\omega, t), t, T, K)$. These values need to be at least partially interpolated or estimated, since there are obviously not uncountably many quotes publicly available. When we implicitly assume that the values come from a model we will drop the hats. Here $t \geq 0$ is time, $T$ is the time of maturity and $K$ is the strike price. The payoff of such an option at time $t = T$ is $(S(\omega, T) - K)^+$. Similar notations apply for a European put option $P_S$ with terminal payoff $(K - S(\omega, T))^+$.

The linear subspace $\mathcal{I}$ of $L^1(\mathbb{P}_1)$, the space of $\mathbb{P}_1$-measurable functions, consists of potential investment. Without specifying it accurately we will just assume that it is rich enough to support all the securities considered. Thus $\mathcal{I}$ can be seen as a subspace consisting of the payoff functions $f \in L^1(\mathbb{Q})$ with yield $f(S_T)$ at the maturity. Here $\mathbb{Q}$ is a risk-neutral measure arising from the postulated call options price system, which will be discussed shortly.
A word of warning: we often discuss matching a cash flow. We seldom hedge cash flows here by proper arbitrage. Instead, we typically build a matching cash flow distribution, a much weaker hedging notion.

Let us recall the Sharpe ratio which is here taken (slightly inaccurately) to be

$$\frac{\mu - r}{\sigma}$$

where $\mu$, $r$ and $\sigma$ are the variables of the BSM model. This can be viewed as the market price of risk.

1.2. The valuation method in a nutshell. We will price a single-step stochastic cash flow by building a portfolio with matching cash flow distribution. Thus the valuation involves statistical arbitrage. However, by no means is this kind of portfolio unique, nor is the price unique. Therefore, we are required to make some further specifications.

In an ideal situation one can find a traded twin security perfectly correlated with the cash flow under consideration, so that a simple portfolio consisting of the securities and risk-free bonds serves as a good benchmark for the cash flow.

In a less ideal situation we have non-perfectly but still highly correlated liquid security and European style call options on the security. We will use the option price information to patch some of the information lost due to imperfect correlation.

This is performed as follows. From the call option prices we may estimate the state price density of the security. This density can be interpreted as a system of Arrow-Debreu securities with infinitesimal prices. We will reweigh the A-D securities and reassemble them to obtain a portfolio with the same distribution as the cash flow. The estimate price of the portfolio will serve as a benchmark price for the cash flow.

This can be seen as pricing by state price density transformations. The particular way of transforming or the reassembly procedure is chosen in such a way that the corresponding prices meet some natural rationality conditions.

2. Some background and preparations

The main ideas discussed in this section can be considered classical, see [Bick 1982], [Breeden&Litzenberger 1978], [Brown&Ross 1991] and [Jarrow 1986]. This introduction is given for the sake of convenience, since the static hedging methodology is not easily found, at least in an explicit form, in the real options literature.

Our aim is to value stochastic cash flows by replicating their distribution. We will build a portfolio of Arrow-Debreu securities (AD securities) to accomplish this. Thus we are required to fix a price system for the AD securities. It is known that the prices of the AD securities are easily obtained from the strike price to option value function $K \mapsto C(K)$ and we shall recall shortly how this can be accomplished in our setting.

One problem in static hedging is that the markets do not even come close to being deep enough to allow realizing static hedges with infinite (even a continuum of) strike prices of call options.
Thus we choose a flexible approach, letting $\hat{V}_{C_S}$ represent an estimated value of call on a security $S$. This value does not necessarily follow any standard option pricing formula, nor does it necessarily represent the directly observed value of the options prices traded. It is rather a theoretical price estimated by a practitioner and/or a specialist.

The estimated value could be obtained for example by statistically estimating expected market values for the option with finite strikes and then interpolating the values corresponding to the rest of the (infinitely many) strikes by using the shape of the $K \mapsto C(K)$ curve known from the theory. The specific construction of the values $\hat{V}$ is not treated here in detail and we assume it is given. We refer to [Bertsimas & Popescu 2002], [Bondarenko 2003], [Fengler 2005], [Jackwerth & Rubinstein 1996], [Monteiro & al. 2011] and [Stutzer 1996] for examples of different approaches to reconstructing the function $K \mapsto C(K)$.

There are at least two ways how the market information can be incorporated in the valuation process. First, liquid options provide plenty of current data (volatility surfaces etc.), so that one would expect this information content to be passed on to the interpolated $K \mapsto C(K)$ function. Secondly, if the project to be valued has a traded twin security, ‘quasi twin security’, or otherwise a strongly related traded security (e.g. commodity), then the valuation could be based on the option price system on this underlying.

The details are included in the Appendix for the sake of convenience.

We begin by describing how a static hedge can be constructed for a European style option whose value is given by $f(S(T))$, where $T$ is the time of maturity. We follow an approach by e.g. Carr and Chou in [Carr & Chou 1997] and construct a portfolio of continuum of European call options with with same maturity $T$ and varying strike prices. Of course such a portfolio is not conceivable in the real world. This is the case with many other idealizations in mathematical finance, such as no transaction costs, continuous trading, no illiquidity issues, etc. However, several authors have taken this approach, because it arises as the limiting process of portfolios having options with only finitely many strikes. Static hedging enjoys some advantages compared to dynamic hedging. Namely, the transaction costs are smaller in the static hedging and the possible model specification error plays a smaller role.

In fact, it is important to observe that we do not require any information whatsoever about the stochastic behavior of the underlying security $S$, as long as the price system of the calls and the distribution of $S(T)$ are known. Therefore, if the project to be valued is strongly related to, say, a natural resource, then the observed prices of options on the resource could be applied without making much assumptions about the dynamics of the price process of the underlying.

2.1. On the Arrow-Debreu security. Let us recall some well-known ideas from the work from [Breeden & Litzenberger 1978]. Suppose that $\frac{\partial^2 V_{C_S}}{\partial K^2}$ exists and is continuous. Next we will consider a digital option on $S$, which pays 1 unit of numeraire if the underlying asset has exactly the price $K$ at the maturity, and pays nothing in the contrary case. Such an option is called
an Arrow-Debreu security. Since the probability of $S$ having the price $R$ at maturity is typically zero, the Arrow-Debreu security has typically price 0. However, they provide an effective tool for continuous decompositions of cash flows.

We will apply infinitesimal notation:

$$S_{AD}(S(t), t, T, K) = -\frac{\partial C_{digit}(S(t), t, T, K)}{\partial K} dK.$$  

This is a formal notation and the right hand side becomes sensible in the context of integration. The Arrow-Debreu security pays one unit of numeraire if $S_T = K$. The probability of this incident in our model is 0 and therefore one anticipates the value of the A-D security to be infinitesimal.

It is easy to convince oneself that the above formula holds by plotting the corresponding payoff function with $h$ in the definition of the derivative small. The plot results in a payoff with an asymptotically rectangular spike around $K$ and of height 1. This observation does not use hardly any assumptions about the model. See the appendix for more details.

2.2. Portfolios. We are required to work with portfolios including infinite amount of types of call options, or, typically Arrow-Debreu securities. For instance, we are interested in portfolios containing calls with infinitely (even uncountably) many different strike prices. In such a case the weight of one call in the portfolio is typically infinitesimal. Thus we are required to adopt a method for flexibly keeping track of the ‘continuous’ portfolios.

The information about the distribution of different types of calls can be conveniently decoded as a Radon-Nikodym derivative of a measure. Thus we will think of portfolios as signed Radon measures, denoted by $\rho$. Typically $\rho$ is a measure on the real line and absolutely continuous with respect to the Lebesgue measure. If the Radon-Nikodym derivative $\frac{d\rho}{dm}$ (defined $|\rho|$-a.e.) is positive at a point $x$ this means that the portfolio has a long position on the call corresponding to parameter $x$, say, the strike price, and the relative weight of the long position is given by $\frac{d\rho}{dm}(x)$. Similarly, if $\frac{d\rho}{dm}(x)$ is negative, then the portfolio has a short position on the call corresponding to parameter $x$ and moreover the relative weight of the position is the absolute value of $\frac{d\rho}{dm}(x)$. It is useful to think of relative weights as being analogous to probability densities, only the sign may vary according to the short/long position. There is no particular reason why $\rho$ could not also be a mixture of the following form

$$\rho(A) = \int_A w(t) \, dm(t) + \sum_{t \in A \cap C} w(t)$$

where $C \subset \mathbb{R}$ is a countable set.

In fact, one advantage of using such approach to static hedging is that the possibility of non-trivial point masses yields flexibility in constructing the non-differentiable pay-off function by means of integration. Another way to achieve the same effect is to use distribution theory.

Of course it is not possible to hold a portfolio with infinitely many (kinds of) financial instruments. However, one should think of the calculations involving $\rho$ as the unique theoretical limit resulting from approximations with
portfolios containing finitely many instruments. Although, the instruments are then thought of being infinitely divisible. The total variation \(|\rho|\) of the portfolio measure \(\rho\) is the total mass of all (short/long) positions. In order to keep the framework financially realistic, we would prefer to assume that \(|\rho| < \infty\). To this end, it seems sufficient to assume that the state price density is integrable with respect to \(\rho\), that is, the portfolio has a unique finite value (positive or negative). For example, the value of a European call has the representation \(\int S_{AD}(K) \frac{d\rho}{dK}\) where \(\rho(A) = \int_A (x - K)^+ \, dx\).

2.3. Arbitrage. Here we assume for convenience that \(P_1(S(T) > 0) = 1\) and that \(P_1\) is atomless.

Observe that in order for the price system on \(I\) to be arbitrage free, it must satisfy \(P_1(S(T) \in L) = 1\) where \(L \subset \mathbb{R}\) is a subset such that the right hand side of (A.2) is strictly positive for \(K \in L\). Indeed, otherwise there is a set \(M \subset \mathbb{R}\) with \(P_1(S(T) \in M) > 0\) such that the portfolio \(\rho\) of A-D securities with strikes in \(M\) (i.e. \(\rho(A) = \int_{A \cap M} 1 \, dm\)) has value at most 0 at time \(t = 0\). Still,

\[ P_1(V_\rho(T) = 1) = P_1(S(T) \in M) > 0 \text{ and } P_1(V_\rho(T) < 0) = 0 \]

by the assumptions.

It follows that in an arbitrage free market the function \(\hat{V}_{CS}\) exists, is continuous and strictly increasing.

Secondly,

\[ B_t = \int_{\mathbb{R}} S_{AD}(S(t), t, T, K) \]

Considering the integral as a portfolio (an infinite bundle of AD securities) we see that it pays 1 unit of numeraire at time \(T\), no matter what. Therefore the present value identity is justified.

Thirdly,

\[ S(t) = \int_{\mathbb{R}} K S_{AD}(S(t), t, T, K). \]

Observe here that the integral is bundling AD securities of strike \(K\), weighted by \(K\), so that the value of the portfolio at time \(T\) will be \(S(T)\).

These conditions state a clearly necessary requirement of arbitrage free markets, namely that if the risk-free bond and security are both reconstructed as portfolios of A-D securities, then the prices remain the same.

These kind of postulates actually characterize the absence of arbitrage opportunities. In our opinion this provides a good intuition to understanding arbitrage-free pricing. In fact, we will base our definition of arbitrage in the present setting (i.e. static hedging with one time step and continuous portfolios) on the above state price densities. We say that no arbitrage opportunities exist, if, given a portfolio \(P\) which pays at time \(T\) the amount \(a + bS(T) + f(S(T))\),

the value of this portfolio at time \(0 \leq t \leq T\) is

\[ V_P(t) = ae^{-(T-t)r_0} + bS(t) + \int \frac{\partial^2 V_{CS}(S(t), t, T, K)}{\partial^2 K} \, d\rho. \]
Here $\rho$ is a portfolio such that
\[
\int_{\{\omega: S(T) \in A\}} f(S(\omega, T)) \, dP_1(\omega) = \int_A 1 \, d\rho \quad \text{for all measurable } A \subset \mathbb{R}.
\]

Now, suppose that $P$ is such a portfolio with $V_P(t) \leq 0$. By redefining $f_1(S(T)) = a + bS(T) + f(S(T))$ we may consider $P$ purely in terms of $\rho$, so that $a = b = 0$, and organizing $P$ this way does not affect its price according to the above assumptions. Since $V_P(0) \leq 0$, we obtain by (2.5) and the first assumptions that either $|\rho|(L) = 0$ for some $L \subset \mathbb{R}$ with $P_1(S(T) \in L) = 1$, or there is $A \subset \mathbb{R}$ such that $\int_{\{\omega: S(T) \in A\}} f_1(S(\omega, T)) \, dP_1(\omega) < 0$. In the first case $P_1(\hat{V}_P(T) > 0) = 0$ and in the second case $P_1(\hat{V}_P(T) < 0) > 0$, so that $\rho$ is not a risk-free portfolio.

As already stated, throughout we will assume that no arbitrage opportunities exist.

2.4. State-price densities and risk-neutral measure. Next we will look more closely into how the system of option prices and the risk-neutral measure are related. See [Bick 1982], [Brown & Ross 1991] and [Jarrow 1986] for related work.

The above identities provide us with a linear pricing rule. Namely, that the price of a European style derivative with a payoff function $f$, so that the yield at maturity $T$ will be $f(S(T))$, is at time $t$ given by
\[
V(t) = \int f(K)S_{AD}(S(t), t, T, K) \, dK.
\]

When thinking about the contents of portfolios this is an intuitive tool. However, the integral on the right hand side is admittedly awkward. The ‘physical’ AD securities having ‘infinitesimal’ value can be replaced by the more abstract but notationally more attractive state price density concept defined as follows:

(2.6)\[ q(K) = \frac{\partial^2V_{CS}(S(t), t, T, K)}{\partial^2K}. \]

So, the incidents $S(T) = K$ are thought of as states of the world and $q$ at time $t$ is the relative price density of these states, i.e. $q(K) \, dK = S_{AD}(S(t), t, T, K)$. Thus the integrals involving the cumbersome AD security unit $S_{AD}(S(t), t, T, K)$ can be rewritten in terms of $q$.

Recall that there are various approaches to the estimation of the state price density given market data on call options. An easy semi-parametric method is also outlined in the second part of this paper.

Returning to formula (2.5), we observe that a measure $Q$ defined by
\[
Q(A) = \frac{1}{B_t} \int_A q(K) \, dK
\]
is a probability measure. Moreover, it is risk-neutral because the above linear pricing rule can be reformulated as
\[
V(t) = \int f(K)S_{AD}(S(t), t, T, K) = \int f(K)q(K) \, dK = B_tE_Q(f(S(T))).
\]
Therefore calculating with portfolios of AD securities, which is frequently done here, is risk-neutral pricing in disguise. However, we would like to maintain a connection to portfolios of assets, instead of discussing merely measures.

2.5. **Real option valuation and static hedging.** Roughly speaking, the classical real options valuation amalgamates decision tree analysis and an application of financial derivatives pricing theory. In this paper we concentrate on the latter aspect and not on the real options found in investment analysis (cf. [Copeland & Antikarov 2001]).

Instead, we will apply the valuation paradigm of the real options analysis. This comprises of finding benchmarks or twin securities among traded securities (often financial options), and, traditionally, applying the risk-neutral pricing machinery to stochastic cash flows (often arising as a result of decision tree analysis or identification of real options). Actually, we will focus mostly on the latter mentioned part, bearing the first one in mind.

One way to model a stochastic cash flow is to consider it as a complicated derivative on an underlying security. The distribution characteristics of the cash flow then depend on the payoff function $f$ and the distribution of the underlying security $S$. The choice of the physical security $S$ to be used as a benchmark deserves some attention. This may be in particular fruitful in the case where there is a strong statistical relationship with the cash flow and the benchmark candidate. The approach taken here allows for example using a modification of a ‘quasi-twin-security’ by reshaping its distribution through the choice of $f$.

In the real options valuation the formal reduction of a cash flow distribution to a portfolio of derivatives is often circumvented by taking an expectation of the discounted cash flow with respect to a risk neutral probability measure, and further, this is often implemented in practice by using a binomial tree and numerical approximation.

Our pricing method falls clearly in the category of risk-neutral valuation but the emphasis here is on the flexible use of benchmarks. This enables taking into account basic market price information regarding the traded security, as well as other factors, such as liquidity.

Recall that the Sharpe ratio is the numerical value expressing the risk premium. Our method also takes into account the ‘shape of the risk premium’ and can be thought of as a non-parametric pricing method.

There are of course problematic issues involved in benchmarking real, highly illiquid cash flows with traded assets. Also, the risk-free rate of bonds differs considerably from the WACC relevant in valuing a project. These matters will not be solved here but we note that typically they are not taken into account in risk-neutral valuation, either.

2.6. **Incompleteness of markets and MAD.** Above we pointed out some problematic issues with using benchmarks. In general capital budgeting the incompleteness of the markets is even a more grave issue. There is typically a component of project specific risk which cannot be hedged, contracted, or straight-forwardly priced by means of risk neutral pricing.
In our model the probability space supporting financial options is different from the one supporting project cash flows. Thus we will witness here a rather extreme form of incompleteness; the project cash flow and financial options are formally in different worlds. There is a binding map between the probability spaces, playing the part of an analogy, encoding all the correlation information.

Technically, we could amalgamate these distinct probability spaces but this would complicate matters, and, as it is, there is a clean distinction between the financial option events and project events.

Since we cannot price the cash flows purely on arbitrage arguments we need to impose some extraneous principle. The required principle is the Marketed Asset Disclaimer (MAD) frequently employed in ROV. Under this paradigm one may benchmark project cash flows by theoretical constructs, such as the BSM model price of options, even if the option component of the cash flow is not traded and is not actually based on any identifiable underlying security. Although this appears a violent way to price assets, the principle is less drastic if applied in such a way that the priced cash flows are highly correlated to the constructed benchmark portfolios.

3. THE FRAMEWORK

Suppose that \( \phi_1, \phi_2 : \mathbb{R} \to [0, \infty) \) are continuous density distributions of a given cash flow \( CF \) and security \( S \), respectively, both at maturity \( T \). We denote by \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \) the corresponding probability measures. Assume that \( \phi_1^{-1}((0, \infty)) \) and \( \phi_2^{-1}((0, \infty)) \) are open intervals and \( \min \text{supp}(\phi_1) = 0 \). Thus there is a continuous increasing function \( K : \mathbb{R} \to \mathbb{R} \) such that \( \mathbb{P}_2(K(M)) = \mathbb{P}_1(M) \) for any Lebesgue measurable subset \( M \subset \mathbb{R} \). We will assume that the function \( k \mapsto q(k) \) is uniformly continuous and by our previous considerations it will be \( L^1(\mathbb{P}_2) \)-integrable.

The measures \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \) are not connected, a priori, in any way. This has to do with the fact that in ROV one does not really hedge cash flows of projects by traded instruments, instead, one benchmarks by trying to match the present value distributions. The purpose of \( K \) is to pair up the distributions in a suitable way.

In the sequel we denote by \( \psi : \Sigma \to \Sigma \) the set-valued mapping given by \( \psi(M) = K(M) \), the image of \( M \in \Sigma \) under the mapping \( K \). The function \( K \) can be constructed by means of (and for the purpose of) numerical integration.

To replicate the distribution of \( CF \) we will buy a suitably weighted portfolio \( \rho \) of Arrow-Debreu securities with different strike prices. Our aim is to build a portfolio \( \rho \) whose value \( \hat{V}_\rho \) at time \( t = T \) in the event \( S(T) = K(x) \) satisfies

\[
V_\rho|_{S(T)=K(x)} = \frac{d\rho}{dm}(K(x)) = CF(x, T) \quad \mathbb{P}_1 - \text{a.s. } x \in \mathbb{R}.
\]

3.1. Approximating the cash flow with finite portfolios. Let us assume that \( \phi_1 \) is continuous and \( \phi_1^{-1}((0, \infty)) \) and \( \phi_2^{-1}((0, \infty)) \) are a bounded open intervals. In such a case we may approximate in distribution the cash flow \( CF \) with a finite portfolio \( \rho \) of cash-or-nothing calls. Let \( \epsilon > 0 \). Then there is \( n \in \mathbb{N} \) with partition \( x_1 < x_2 < \ldots < x_n \) of the support of \( \phi_1 \) such
that \( \int_{x_k}^{x_{k+1}} \phi_1 \, dx = 1/n \), and \( x_{k+1} - x_k < \epsilon \) for \( k = 1, \ldots, n - 1 \). Similarly we let \( y_1 < y_2 < \ldots < y_n \) be the partition of the support of \( \phi_2 \) (possibly \( y_i = -\infty \) and/or \( y_n = \infty \)) such that \( \int_{y_k}^{y_{k+1}} \phi_2 \, dx = 1/n \) for \( k = 1, \ldots, n - 1 \).

Clearly

\[
\left\| C_T - \sum_{k=1}^{n-1} \frac{x_k + x_{k+1}}{2} 1_{[x_k, x_{k+1}]} \right\|_{L^\infty(\mathbb{P}_1)} < \epsilon.
\]

Motivated by this observation, we will build a portfolio on the security side to match the above linear combination of indicator functions. We form a portfolio \( \rho_d \) of cash-or-nothing calls by including in \( \rho_d \) for each \( k = 1, \ldots, n - 1 \) a position

\[
\frac{x_{k+1} + x_k}{2} (C_{\text{digit}}(S(t), t, y_k) - C_{\text{digit}}(S(t), t, y_{k+1})).
\]

That is, \( \rho_d \) is a discrete measure defined by \( \rho_d(1) = \frac{x_1 + x_2}{2} \), and

\[
\rho_d(k + 1) = \frac{x_{k+1} + x_k}{2} + \frac{x_{k+2} + x_{k+1}}{2} = \frac{x_{k+2} - x_k}{2}
\]

for \( k = 1, \ldots, n - 1 \). Then \( \rho_d(k) \) describes the volume of cash-or-nothing calls with strike price \( y_k \) in the portfolio \( \rho_d \).

Eventually we will pass on to the limit \( n \to \infty \) in order to match the distribution of \( C_T \) sharply (and in a more general setting). Then the portfolio measure will not be a discrete one. As a midway step towards continuous portfolios, we will note how the above \( \rho_d \) can be realized in financial terms by means of Arrow-Debreu securities. We denote by \( \rho \) this version of the portfolio containing \( \frac{dp}{dm}(y) \)-many A-D securities of strike \( y \). Here one defines \( \rho \) by

\[
\rho(A) = \sum_{k=1}^{n-1} \frac{x_k + x_k}{2} \int_{[y_k, y_{k+1}]} 1_A \, dm_2.
\]

Indeed, note that the arbitrage-free value of the instrument

\[
C_{\text{digit}}(S(t), t, y_k) - C_{\text{digit}}(S(t), t, y_{k+1})
\]

is \( \int_{y_k}^{y_{k+1}} q(k) \, dk \). The price of the described portfolio at time \( t \) is

\[
(3.1) \quad V_\rho = \sum_{k=1}^{n-1} \frac{x_{k+1} + x_k}{2} \int_{y_k}^{y_{k+1}} q(k) \, dk.
\]

4. Constructing a cash-flow-distribution-equivalent portfolio

Suppose that \( C_T = x \in \mathbb{R} \). We recall that the density of such incident is \( \phi_1(x) \). In practice it must be estimated from data; call the estimate \( \hat{\phi}_1(x) \). The corresponding incident on the security side is \( K(x) \) and its (estimated) density is \( \hat{\phi}_2(K(x)) \). In the case of such incident the corresponding A-D security returns 1 (and not \( x \)). Therefore we will compensate by using the weight \( x \) on the A-D security corresponding to the incident \( K(x) \). Secondly,

\[
K(x + \Delta x) - K(x) + o(\Delta x) = \Delta x \frac{\hat{\phi}_1(x)}{\hat{\phi}_2(K(x))} = \Delta x \frac{dp_1}{dm}(x)/\frac{dp_2}{dm}(K(x)),
\]
since $K$ is measure preserving. Here the fractions on the right hand side are the Radon-Nikodym derivatives of the measures. Thus, in (4.1) we will regard $K$ as a non-trivial weak solution to the differential equation

\[ K'(x) = \frac{\hat{\phi}_1(x)}{\hat{\phi}_2(K(x))}, \quad K(\min \text{supp}(\hat{\phi}_1)) = \min \text{supp}(\hat{\phi}_2) \]

where $x \in \text{supp}(\phi_1)$. If $F_1$ and $F_2$ are the corresponding cumulative distributions, then $K(x) = F_2^{-1}(F_1(x))$.

Therefore, for each $x \in \mathbb{R}$ the $\mathbb{P}_1$-random variable $x 1_{[x,x+\Delta x]}$ can be replicated via $K: \mathbb{R} \to \mathbb{R}$, up to precision $o(\Delta x)$, by the $\mathbb{P}_2$-random variable

\[ x(\hat{C}_\text{digit}(S(T), T, T, K(x)) - \hat{C}_\text{digit}(S(T), T, T, K(x + \Delta x))). \]

We write the hats for estimates, since the prices of derivatives are not quoted continuously for strikes and therefore have to be estimated by some sort of interpolation. In our portfolio we will buy (infinitesimal) A-D securities continuously for strikes and therefore have to be estimated by some sort of interpolation. In our portfolio we will buy (infinitesimal) A-D securities continuously for strikes and therefore have to be estimated by some sort of interpolation.

We have not framed the problem precisely enough in order to state that estimate of $V$ converges to the real value when the other estimates converge to their respective model value. Although, this appears reasonable. We are not claiming that any of the estimates are unbiased and the formulation of this statement would require, again, more precise framing.

The above valuation formula will be further discussed in the subsequent article in a more general framework.

### 4.1. Some basic properties of the valuation mechanism.

**Proposition 4.1.** The values of the approximating finite portfolios in (3.1) converge to the value (4.2).

**Proof.** Let $\{\{x_k^{(n)}: 1 \leq k \leq n\}\}_{n}$ be increasing partitions of the support of $\phi_1$ such that

\[ \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} \phi_1(t) \, dt = 1/n \text{ for } n \in \mathbb{N} \text{ and } 1 \leq k < n. \]

It follows from the properties of $\phi_1$ that $\sup_k x_{k+1}^{(n)} - x_k^{(n)} \to 0$ as $n \to \infty$. By compactness considerations we observe that

\[ \min \{\phi_2(y): K(x_2^{(n)}) = y_2^{(n)} \leq y \leq y_{n-1}^{(n)} = K(x_{n-1}^{(n)})\} \]
exists and is non-zero. Therefore the function
\[x \mapsto \frac{\phi_1(x)}{\phi_2(K(x))},\]
defined on \([x_2^{(n)}, x_{n-1}^{(n)}]\) is uniformly continuous. This means that
\[\lim_{n \to \infty} \sup_{\max(x_k^{(n)}, x_i^{(n)}) \leq x \leq \min(x_{k+1}^{(n)}, x_{i-1}^{(n)})} \left| \frac{K'(x) - x_{k+1}^{(n)} - x_k^{(n)}}{y_{k+1}^{(n)} - y_k^{(n)}} \right| = 0\]
for \(i \in \mathbb{N}\).

Note that by the integrability of \(q\) we have that
\[\frac{1}{y_{k+1}^{(n)} - y_k^{(n)}} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} (x_{k+1}^{(n)} - x_k^{(n)}) q(K(x)) \, dx \to 0\]
as \(n \to \infty\) for every \(k\) (including cases \(k = 1, n - 1\)).

Since \(q\) is integrable and \(\text{supp}(\phi_1)\) bounded, we have by (4.3) that
\[\lim_{n \to \infty} \sum_{k=2}^{n-2} \frac{x_{k+1}^{(n)} + x_k^{(n)}}{2} \left| \int_{y_k^{(n)}}^{y_{k+1}^{(n)}} q(K) \, dK - \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} \frac{x_{k+1}^{(n)} - x_k^{(n)}}{y_{k+1}^{(n)} - y_k^{(n)}} q(K(x)) \, dx \right| = 0\]
and
\[\lim_{n \to \infty} \sum_{k=2}^{n-2} \frac{x_{k+1}^{(n)} + x_k^{(n)}}{2} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} \frac{x_{k+1}^{(n)} - x_k^{(n)}}{y_{k+1}^{(n)} - y_k^{(n)}} q(K(x)) \, dx = \int x \frac{\phi_1(x)}{\phi_2(K(x))} q(K(x)) \, dx.\]
from which the claim follows. \(\square\)

**Proposition 4.2** (Idempotency). A cash flow with the same distribution as the benchmark security has the same value as the security. This is to say, if \(\phi_1 = \phi_2\), then \(\hat{V}_\rho(t) = S(t)\) for \(0 \leq t \leq T\).

**Proof.** Observe that \(K\) is necessarily an identical mapping and \(\frac{\phi_1(x)}{\phi_2(K(x))} = 1\) for \(\mathbb{P}_1\text{-a.e.} \, x\). By using the definition of (the absence of) arbitrage we obtain that
\[\hat{V}_\rho = \int \mathbb{R} x \, q(K(x)) \, dx = S(t).\] \(\square\)

**Proposition 4.3** (Monotonicity w.r.t. stochastic dominance.). Suppose that \(\mathbb{P}_{1a}\) and \(\mathbb{P}_{1b}\) are probability measures describing the distribution of a cash flow \(CF\) such that \(\mathbb{P}_{1a} \leq \mathbb{P}_{1b}\), that is \(F_a(t) \geq F_b(t)\) for \(t \in \mathbb{R}\) where
\[F_a(t) = \mathbb{P}_{1a}(CF_T \in (-\infty, t]) \quad \text{and} \quad F_b(t) = \mathbb{P}_{1b}(CF_T \in (-\infty, t]).\]

Let \(\phi_a, \phi_b, K_a\) and \(K_b\) be similar as above, corresponding to pairs \((\mathbb{P}_{1a}, \mathbb{P}_2)\) and \((\mathbb{P}_{1b}, \mathbb{P}_2)\), respectively. Then \(\hat{V}_{\rho_a} \leq \hat{V}_{\rho_b}\).
Proof. We will follow the finite approximation of the portfolios as follows. Let \((x_k)_{k \in \mathbb{Z}}, (y_k)_{k \in \mathbb{Z}}\) and \((z_k)_{k \in \mathbb{Z}}\) be increasing sequences of \(\mathbb{R}\) such that \(F_a(x_k) = F_b(y_k) = F_2(S(T) \in (-\infty, z_k))\) for \(k \in \mathbb{Z}\). Then \(x_k \leq y_k\) by the assumption. Therefore it is clear that

\[
\sum_{k \in \mathbb{Z}} \frac{x_{k+1} + x_k}{2} \int_{z_k}^{z_{k+1}} q(K) \, dK \leq \sum_{k \in \mathbb{Z}} \frac{y_{k+1} + y_k}{2} \int_{z_k}^{z_{k+1}} q(K) \, dK.
\]

\[\square\]

Along the same lines one can show that if \(\mathbb{P}_{1a} \prec \mathbb{P}_{1b}\), then \(\hat{V}_{\rho_a} < \hat{V}_{\rho_b}\). We note that the fact that \(K\) is increasing is essential here.

**Proposition 4.4 (Continuity).** Assume that \(q\) is bounded and that \(CF_n\) converges to \(CF\) in \(L^1(\mathbb{P}_1)\) norm. Then \(\hat{V}_{\rho_n} \rightarrow \hat{V}_\rho\) as \(n \rightarrow \infty\).

**Proof.** Indeed, let \(\sup_K q(K) = C\). By inspecting the finite approximation we first see that if \(CF_a \geq CF_b\ \mathbb{P}_1\)-almost surely, then \(\hat{V}_{\rho_a} \leq \hat{V}_{\rho_b} + C \int x(\phi_a(x) - \phi_b(x)) \, dx\). Here

\[
\int x(\phi_a(x) - \phi_b(x)) \, dx = \int x(dF_a - dF_b) = \mathbb{E}_{\mathbb{P}_1}(CF_a) - \mathbb{E}_{\mathbb{P}_1}(CF_b).
\]

On the other hand the valuation operator depends only on the distribution of the cash flow, thus

\[
\hat{V}_{\rho_a} \leq \hat{V}_{\rho_b} + C\mathbb{E}_{\mathbb{P}_1}(|CF_a - CF_b|)
\]

for any cash flows \(CF_a\) and \(CF_b\). \[\square\]

4.1.1. **Modigliani-Miller separation.** The pricing formula is not linear, that is, if \(CF_1\) and \(CF_2\) are cash flows with respective AD portfolios \(\rho_1\) and \(\rho_2\), then the portfolio \(\rho\) resulting from the combined cash flow \(CF_1 + CF_2\) typically satisfies \(\rho \neq \rho_1 + \rho_2\) and typically \(\hat{V}_\rho \neq \hat{V}_{\rho_1} + \hat{V}_{\rho_2}\). This is due to the fact that the valuation mechanism, on the cash flow side to be priced, depends only on the distribution of the cash flow. Indeed, consider for instance equally distributed cash flows \(CF_a, CF_b\) and \(CF_c\) such that \(CF_a + CF_b\) has zero variance and \(CF_b + CF_c\) has non-zero variance.

However, a Modigliani-Miller type separation of value holds. Namely, if a cash flow \(CF_0\) (with density \(\phi_0\)) is combined with a risk free bond worth \(ae^{-r(t-t)}\) at time \(t\), modeled on the space \((\Omega, \mathbb{P}_1)\), then the value of the combined cash flow \(CF\) obeys the pricing rule presented.

We note that since a risk-free bond does not have a continuous density and thus no associated continuous transform \(K\), the valuation formula, per se, is not sensible. Therefore, it is required to assume some liberties in defining the formula in this special case: We define \(\phi(a + x) = \phi_0(x), K(a + x) = K_0(x)\) for \(x > 0\) and \(\frac{\phi(x)}{\phi_0(K(x))} = 0\) otherwise.
\begin{align*}
\hat{V} &= \int_0^\infty x \frac{\phi(x)}{\phi_2(K(x))} q(K(x)) \, dx \\
&= \int_a^\infty x \frac{\phi(x)}{\phi_2(K(x))} q(K(x)) \, dx \\
&= \int_0^\infty (a + s) \frac{\phi(a + s)}{\phi_2(K(a + s))} q(K(a + s)) \, ds \\
&= \int_0^\infty a \frac{\phi(a + s)}{\phi_2(K(a + s))} q(K(a + s)) \, ds + \int_0^\infty x \frac{\phi_0(x)}{\phi_2(K_0(x))} q(K_0(x)) \, dx \\
&= a \int_0^\infty K_0'(x) q(K_0(x)) \, dx + \hat{V}_0.
\end{align*}

The left term on the last line is \( \int_0^\infty q(K) \, dK = aB_t \) by the elementary properties of \( q \).

Observe that if instead we approximate the bond with a cash flow with density \( \phi_b(t) = \frac{1}{\Delta t} 1_{[a, a+\Delta t]} \) (and respective \( K \)), then we have

\begin{align*}
\hat{V}_{\rho_b} &= \int_0^\infty x \frac{\phi_b(x)}{\phi_2(K(x))} q(K(x)) \, dx \\
&= \frac{1}{\Delta t} \int_a^{a+\Delta t} xK'(x) q(K(x)) \, dx \to a \int_0^\infty q(K) \, dK.
\end{align*}

as \( \Delta t \to 0^+ \).

Similar considerations yield that if \( CF_2 = cCF_1, \ c \geq 0 \) a constant, then the value of \( CF_2 \) is \( c \) times the value of \( CF_1 \). This means that the valuation mechanism is stable in transforming a single cash flow by affine transforms, i.e. multiplying a cash flow with a positive constant and adding a (multiple of) risk-free asset.

These role of the connections \( K \) will be further discussed in the subsequent article.

4.1.2. A thought experiment and Sharpe ratios. The above observation suggests studying Sharpe ratio in connection with the valuation method as it does not change if a security is transformed linearly (excluding multiplication by 0).

Assume that we have two distinct risk-neutral pricing worlds (e.g. BSM worlds). Suppose that these models, call them \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \), contain securities \( S_1 \) and \( S_2 \), respectively. These models have probability measures \( \mathbb{P}_1, \mathbb{P}_2 \) and some parameters, e.g. the risk-free rates \( r_1, r_2 \), the drift parameters \( \mu_1, \mu_2 \) of the corresponding securities, the implied volatility parameters \( \sigma_1, \sigma_2 \) and possibly others. We will also assume that both the models admit some state price measure (not necessarily a density).

In the previous subsection we considered affine transformation of cash flows. Now, let us assume that the distributions of \( S_1(T) \) and \( S_2(T) \) coincide, up to a linear transformation. We will apply our pricing rule to price \( S_1 \) at time \( t = 0 \) by using the distribution and the state price system related to \( S_2 \).
Since the distributions of the securities coincide up to the linear transform, the binding map \( K: \Omega_1 \rightarrow \Omega_2 \) satisfies \( S_2(T)(K(\omega)) = c(S_1(T)(\omega)) \mathbb{P}_1\text{-a.s.} \) \((\omega \in \Omega_1)\). By inspection, the pricing rule attributes to \( S_1 \) at time \( t = 0 \) the value \( \mathbb{E}_{\mathbb{Q}_2}(S_1(T)(K^{-1}(\omega_1))) / c \).

Note that if we were dealing with BSM worlds, then the parameters \( c\mu_1 \) and \( \mu_2 \) coincide, as do \( c\sigma_1 \) and \( \sigma_2 \) due to the postulated coincidence of the distributions. If we additionally require the pricing rule \( \hat{V} \) to be canonical in the sense that it values \( S_1 \) by its risk-neutral price in \( \mathcal{M}_1 \), then, in the case of BSM models, the parameters \( c\mu_1 = \mu_2 \) and \( c\sigma_1 = \sigma_2 \) already fixed, we obtain that \( cS_1(0) = S_2(0) \) from distribution shapes and by invoking Girsanov type representation of the risk-neutral measures, \( r_1 = \sigma_2 \). This is to say, the Sharpe ratios \( \frac{\mu_1 - r_1}{\sigma_1} \) and \( \frac{\mu_2 - r_2}{\sigma_2} \) coincide. Observe that we applied very minimally the properties of the BSM model, mainly the fact that there are only three determining parameters for the physical and risk-neutral distributions.

Motivated by these considerations, there is a natural, almost linear transform that one can perform on a cash flow. Namely, a kind of \textit{Sharpean operation} defined as follows:

- Given an unimodally distributed cash flow \( CF \) we subtract from it a risk-free asset \( R_{CF} \) (possibly with negative sign) so that the support of the density distribution of \( CF - R_{CF} \) has infimum 0.
- Next, we normalize the cash flow by dividing it with the standard deviation \( \sigma_{CF} \) of \( CF \).
- The resulting expected value \( \mathbb{E}\left(\frac{CF - R_{CF}}{\sigma_{CF}}\right) \) can be used in comparisons of cash flows.

It is quite obvious how this operation resembles the Sharpe ratio, thus the terminology. The expected value above is intended to be, roughly speaking, a 'first-order' proxy attribute for the signature of the cash flow. Note that the Sharpean operation is invariant under scaling with a non-zero constant and bundling up with a risk-free asset. If the cash flow is the return of an optimal portfolio as in CAPM, then the resulting expected value can be thought of as the market price of risk.

Observe that \( R_{CF} \) does not depend linearly on \( CF \) and thus the operation is not linear. Also note that one runs into troubles with discontinuities in the formation of \( R_{CF} \) if we dispense with the unimodality assumption.

In practice, one does not typically find a perfectly correlated traded twin security for a project. One plausible way around this obstruction is benchmarking a project with an affinely adjusted 'pseudo twin security' \( S \) having the following properties:

(i) The NPV of the project and the security have the same beta.
(ii) The NPV of the project and the security yield the same value in the Sharpean operation.

Some more subtle profile matching issues are discussed in the sequel.

5. Discussion

Potential applications. We have framed the valuation problem here as a capital budgeting problem with a single-step stochastic cash flow. A natural
project valuation problem where the project cash flow is highly correlated with a commodity would be for example the valuation of mining, oil drilling or a natural resources extraction project where the resource in question is liquid and has a rich system of derivatives traded on it. The fact that we have dealt with a single-step cash flows is not really that restrictive since we may sometimes apply the methods to different steps separately. Our method could possibly be combined with the usual ROV techniques to handle more complicated situations with dependencies between the steps.

Moreover, instead of a ROV technique, one may also consider our method as a tool for the valuation of financial options and securities. An obvious application would be pricing securities which are highly correlated with an index, say SP500, but the estimated returns distribution exhibits some extra skew and/or fat tails compared to the benchmark index. Thus this involves estimating in a given time step the return distribution of the security and of the index ($\phi_1$ and $\phi_2$, respectively) and the risk-neutral density of the index ($q$). As a short cut to avoiding non-parametric estimation of $\phi_2$ and $q$ one could merely estimate the BSM parameters and then apply the densities given by the model, so that only $\phi_1$ must be non-parametrically estimated. Of course, any of the three can be non-parametrically or parametrically estimated, depending on the setting.

Benchmarking financial securities against other such securities does not necessarily differ considerably from benchmarking projects against securities. In both cases the return distributions may look different and the correlations may be imperfect and our valuation method can be performed in a similar way.

5.1. On the justification of the presented valuation technique. One problematic issue with the presented valuation technique is that we are dealing with completely distinct probability spaces, one supporting ‘financial options’ and the other supporting the stochastic cash flow arising e.g. from the capital budgeting problem. Here we assume, somewhat optimistically, that there exists a benchmark in traded securities highly correlated with the cash flow to be priced, or at least a traded ‘pseudo twin security’, and that there is abundant option price information on the traded asset.

In the case of perfect correlation the financial security provides an unrealistically good benchmark. Also, the valuation mechanism behaves particularly well in such a case, as was discussed in section 4.1.1. However, in the absence of perfect correlation the cash flow distributions are not necessarily, at least a priori, of similar shape, and, apart from that, one could argue that the price of the financial security has very little to do with the price of the cash flow. If the correlation is very loose, then this is truly a valid concern. Similar problem appears in the the real option valuation literature. Namely, a typical method in pricing a cash flow is constructing a binomial lattice approximation of the cash flow and computing its value by taking the expected value with respect to risk neutral probabilities. This can be somewhat justified by the fact that a similar pricing scheme has been very successfully applied in pricing financial options. However, one should note that the risk neutral pricing is based on an hedging argument and in the case of a real option or a more general project cash
flow there is no underlying asset traded. In fact, the ‘driving process’ associated to the binomial lattice can be more or less impossible to witness as a real-world index. This problem of private risk is often sweed under the rug by imposing Marketed Asset Disclaimer. Also, in the literature one does not frequently encounter warnings about the fact that the construction of a suitable binomial lattice is by no means unique, nor is the resulting value. This is so because neither is a risk-neutral measure unique (e.g. on single-step trinomial model), nor the discretization procedure.

Still, using the financial securities as a benchmark in unison with the related pricing machinery appears a most natural and attractive idea, especially in the absence of panacea to the wide range of investment analysis problems. So, can this intuitive correspondence be somehow justified?

Equilibrium valuation models often have a close connection to statistical hedging in the sense that the attributes describing assets may more or less efficiently predict the prices and thus there is some variance reduction. Benchmarking an asset or cash flow with a highly correlated asset appears to have the same effect. Therefore our method, statistical hedging and equilibrium valuation models are not necessarily that dissimilar. On the other hand, our model does not seem compatible with the CAPM, since it cannot handle the distinction between market risk and idiosyncratic risk well. This problem can be alleviated by selecting the benchmark asset to be in a sense close to the market portfolio and compatible with the CAPM, if possible.

Let us consider some behavioral considerations about the factual pricing process. It is plausible to presuppose that a pricing decision is typically based on existing financial theory and methods, or an ad hoc decision by the decision maker (DM). Also, we are probably not far from the truth if we assume that the theory-based decision typically exploits rather standard considerations appearing in well known books (not the latest state of the art papers) and that DM has very limited resources dwelling on the inner makings of the cash flow. Instead, the cash flow is likely to be priced by considering the mutual relations between simple superficial characteristics such as expected value, variance, beta, WACC, Sharpe ratio, etc. The investors are often also compelled applying the industrial standards.

In a sense, the situation resembles the use of APT model for financial securities (or projects). We note that a continuous density distribution of a cash flow is an infinite dimensional object, whereas the APT model contains low-dimensional information about securities.

We argue that the cash flow appears to the DM, phenomenologically speaking, as a finite dimensional vector consisting of some commonly applied characteristics. The DM performs the valuation, ceteris paribus, based on a model having these characteristics as its parameters.

By invoking a very weak version of the equilibrium argument applied in justifying the APT, one may argue that if the the cash flow opportunity in question was traded, then its value should adjust to a level determined somehow by its simple characteristics. What we have done here in this paper is, roughly, bundling up cash flows with high correlation and the same distribution. These constitute simple superficial characteristics. Also, there is a delicate detail about the equilibrium consideration. Namely, APT
by its construction is linear whereas the pricing rule \( \hat{V}_\rho \) is not (although it is monotone and continuous). We leave it to the reader to decide whether or not the proposed pricing rule would be natural enough to result in the convergence of the prices in the case where the cash flow opportunities were commonly traded.

As already mentioned, we have presented a closed form real options valuation technique. The novel ingredient is that the pricing rule arranges traded security states to replicate the cash flow distribution, incorporating market risk premium information of non-parametric and non-linear nature. The calibration used here in pricing the cash flows conveys, intuitively speaking, infinite-dimensional information whereas parametric calibration methods convey information whose dimensionality is bounded by the number of free parameters.

**A. Appendix**

A.1. **More on the Arrow-Debreu security.** These securities can be constructed via the analysis of digital options. A digital call option pays 1 unit of numeraire if the underlying asset exceeds the strike at maturity and otherwise it pays 0. By statically hedging we obtain an approximation for the the price of a digital call at time \( t \):

\[
\frac{1}{h}(V_{CS}(S(t), t, T, K) - V_{CS}(S(t), t, T, K + h)).
\]

Indeed, one forms a portfolio by buying a call with strike \( K \) and shorts a call with a bit higher strike, and then scales the portfolio weight so that the portfolios value at maturity increases from 0 to 1 as the price of the underlying shifts from \( \leq K \) to \( \geq K + h \).

Similarly we obtain cash-or-nothing put value approximation:

\[
\frac{1}{h}(V_{PS}(S(t), t, T, K + h) - V_{PS}(S(t), t, T, K)).
\]

Letting \( h \) tend to 0 we obtain the exact prices of cash-or-nothing options:

\[
C_{\text{digi}}(S(t), t, T, K) = -\frac{\partial V_{CS}(S(t), t, T, K)}{\partial K}
\]

and

\[
P_{\text{digi}}(S(t), t, T, K) = \frac{\partial V_{PS}(S(t), t, T, K)}{\partial K},
\]

respectively. Here a straightforward arbitrage argument is applied involving approximations of the derivatives by piecewise linear payoff functions and the continuity of the derivatives. By using digital options we obtain in a similar manner approximations for the Arrow-Debreu security:

\[
S_{AD}(S(t), t, T, K) \approx C_{\text{digi}}(S(t), t, T, K) - C_{\text{digi}}(S(t), t, T, K + h)
\]

and

\[
S_{AD}(S(t), t, T, K) \approx P_{\text{digi}}(S(t), t, T, K) - P_{\text{digi}}(S(t), t, T, K - h).
\]

These values tend to 0 as \( h \) tends to 0, which is sensible, since \( \mathbb{P}_1(S(T) = K) = 0 \) in our model, but not fruitful in view of performing further analysis.
of cash flows. Therefore we take a derivative and pass on to an infinitesimal notation:

\[(A.1) \quad S_{AD}(S(t), t, T, K) = -\frac{\partial C_{digit}(S(t), t, T, K)}{\partial K} dK.\]

This is a formal notation and the right hand side becomes sensible in the context of integration. Similarly we obtain that

\[S_{AD}(S(t), t, T, K) = \frac{\partial P_{digit}(S(t), t, T, K)}{\partial K} dK.\]

Next we recall the formulae of \(C_{digit}\) and \(P_{digit}\), thus

\[(A.2) \quad S_{AD}(S(t), t, T, K) = \frac{\partial^2 \widehat{V}_{CS}(S(t), t, T, K)}{\partial^2 K} dK = \frac{\partial^2 \widehat{V}_{Ps}(S(t), t, T, K)}{\partial^2 K} dK.\]

The right hand side equation follows also immediately from the put call parity and a similar observation appears e.g. in [Carr&Chou 1997]. In the Black-Scholes setting these identities are well known but the interesting fact is that they are not model specific.

A.2. More on the state price density. Note that at \(0 \leq t \leq T\) the value of the security \(1_{S(T) \in [a,b]}\) coincides with \(\int_a^b \frac{\partial V_{CS}(S(t), t, T, K)}{\partial K} dK\). Indeed, this is verified by considering approximations of suitable cash-or-nothing calls, similarly as above.

The adoption of the measure \(Q\) (instead of continuous density) allows more flexibility of the option price system. For example, suppose that the weak derivative \(\frac{\partial V_{CS}(S(t), t, T, K)}{\partial K}\) exists for all \(K\) and it vanishes as \(K \to \infty\). Assuming that the weak derivative is non-decreasing and recalling the definition of \(q\), one may define (or recover) \(Q\) in terms of the weak derivative by

\[(A.3) \quad \int_a^b \frac{\partial V_{CS}(S(t), t, T, K)}{\partial K} = B_t E_Q(1_{S(T) \in [a,b]}), \quad \text{for} \ a < b.\]

Note that \(Q\) considered on \(\mathbb{R}\) is necessarily a Borel measure and by Dynkin’s theorem it is the unique measure satisfying the above identity for all pairs \(a < b\).

The above uniqueness observation can also be reversed; suppose that \(\Psi: L^1(Q) \to \mathbb{R}\) is a linear continuous pricing rule satisfying

\[\Psi(1_{S(T) \in [a,b]}) = B_t Q(S(T) \in [a,b]), \quad \text{for} \ a < b.\]

Then by the above reasoning and the fact that the simple functions are dense in \(L^1(Q)\), we obtain that

\[\Psi(f) = B_t E_Q(f) \quad \text{for} \ f \in L^1(Q).\]

In fact, equations \((2.6)\), \((A.3)\) and Fubini’s theorem suggest characterizing \(Q\) by

\[\int_{0^+}^{K_0} d\widehat{V}_{CS}(K) - K_0 \lim_{h \to 0^+} \int_0^h \frac{d\widehat{V}_{CS}(K)}{h} = B_t \int_{0^+}^{K_0} (K_0 - K) dQ(K)\]

for \(K_0 > 0\).
A.3. Linear pricing rule. Let us assume more specifically that the price system of $\mathcal{I}$ admits a pricing rule $\Psi: \mathcal{I} \to \mathbb{R}$ satisfying the following conditions:

(i) $\Psi$ is linear.
(ii) If $f \leq g$ $\mathbb{P}$-a.s. then $\Psi(f) \leq \Psi(g)$.
(iii) $1_{S(T) \in A} \in \mathcal{I}$ and $\Psi(1_{S(T) \in A}) > 0$ if $\mathbb{P}(S(T) \in A) > 0$.
(iv) If in $\mathcal{I}$ $\lim_{n \to \infty} \|f_n\|_{L^2(\mathbb{P})} = 0$, then $\lim_{n \to \infty} \Psi(f_n) = 0$.
(v) $\Psi(1) = B_t$

Here the philosophy is that $\Psi$ is thought to reproduce all the prices at time $t$, $\Psi(S(T)) = S(t)$, $\Psi(C_S(K)) = \hat{V}_{C_S}(t, K)$, etc.

Observe that one may again define a probability measure by putting $\mathbb{Q}(A) = \Psi(1_A)/B_t$, which will be equivalent to $\mathbb{P}$ according to (ii) and (iii). The $\sigma$-additivity follows by using conditions (i) and (iv). It is easy to check by using simple functions that $\Psi(f) = \mathbb{E}_Q(f)/B_t$ for $f \in \mathcal{I} \cap L^2(\mathbb{P})$.

If $\mathcal{I} \subset L^2(\mathbb{P})$, then it follows that the existence of the above pricing rule $\Psi$ guarantees NA in the sense of section 2.3.

Proposition A.1. If the above pricing rule is imposed and all vanilla calls are included in $\mathcal{I}$, closed in $L^2(\mathbb{P})$, and the pricing rule respects the true prices, then the derivative $\partial V_{C_S}(S(t), t, T, K)$ exists and is continuous and the second derivative exists in the weak sense.

Moreover, if $\mathcal{I}$ additionally is considered as a subspace of $\text{M}(\mathbb{R})$, via identification $\mu(A) = \int_A f \, d\mathbb{P}$, and $\Psi$ is in (iv) weakly sequentially continuous and $\frac{d\mu}{dm}$ is continuous, then the state price density exists and is continuous.

Proof. Since $\mathbb{P} << m$, by using condition (iv) and the construction of AD securities from Section 2.3 we see that

$$\frac{\partial V_{C_S}(S(t), t, T, K)}{\partial K} = \lim_{h \to 0^+} \frac{\Psi((x - (K + h))^+) - \Psi((x - K)^+)}{h}$$

and this coincides with $-\Psi(1_{[K, \infty)})$. It is clear that $K \mapsto \Psi(1_{[K, \infty)})$ is decreasing and has the weak derivative $-\frac{d\mu}{dm}$.

Let us check the latter part. Let $x > 0$ and $g: \mathbb{R} \to \mathbb{R}$ be a continuous function. Select a sequence $x_n \to x$, and, by using the previous part, a decreasing sequence $h_k \to 0^+$ such that $q(x_k) - C_{\text{a}y}(x_k - h_k) - C_{\text{a}y}(x_k + h_k) \to 0$. Then the sequence of measures $\mu_k(A) = \frac{1}{h_k} \int_A 1_{[x - h_k, x + h_k]} - 1_{[x - h_k, x + h_k]} \, d\mathbb{P}$ converges weakly to 0. The assumptions yield that $q(x_k) \to q(x)$. □

A.4. Some Greeks and implied parameters. Let us assume that $\mathbb{P}_2(S(T) > 0) = 1$ and that the call price $\hat{V}_{C_S}$ has all higher order cross derivatives. Here $B_t = e^{-r(T - t)}$. Then we may verify the following facts:

By using

$$\frac{d}{dt} e^{r(T-t)} \int_0^\infty q(k) \, dk = 0$$

we obtain with simple calculations that

$$\hat{r} = e^{r(T-t)} \int_0^\infty \frac{\partial^2 \hat{V}_{C_S}}{\partial t \partial K} |_{K=0}$$
from which we infer

\[(A.4) \quad \lim_{K \to 0^+} \frac{\partial \hat{V}_{CS}}{dK} = -e^{-r(T-t)}.\]

Therefore it is possible, at least in theory, to observe the risk free interest rate in the option price system by studying the derivative with respect to the strike price.

On the other hand, the failure of \[(A.4)\] suggests a specification error of the model. For example, if

\[\lim_{K \to 0^+} \frac{\partial \hat{V}_{CS}}{dK} = -\lim_{K \to 0^+} \int K \frac{\partial^2 \hat{V}_{CS}}{\partial^2 K} dK > -e^{-r(T-t)},\]

then a possible explanation to this is that \(S_{AD}(S(t), t, T, 0) > 0\).

Default risk detection: The above condition reads

\[(A.5) \quad \hat{Q}(S(T) = 0) = e^{-\hat{r}(T-t)} + \lim_{K \to 0^+} \frac{\partial \hat{V}_{CS}}{dK} > 0.\]

If \(S\) is an equity in such a case, then, by using the equivalence of the measures \(Q\) and \(P_1\), the price system suggests that the probability of the total failure of \(S\) is strictly positive. Unfortunately, \[(A.5)\] does not provide us with exact information on the physical probability of the default, unless \(dP_1^Q\) is explicitly known, in which case \(P(S(T) = 0) = Q(S(T) = 0) / dP_1^{dP_2}|_{S(T)=0}.\)

This interpretation of the formula \[(A.5)\] should be compared to the observed volatility smile.

Similarly, by using the observation

\[\hat{S}(t) = \int_0^\infty K \frac{\partial^2 \hat{V}_{CS}}{\partial^2 K} dK = \left. \int_0^\infty K \frac{\partial^2 \hat{V}_{CS}}{\partial^2 K} dK \right|_{K=0^+} \hat{V}_{CS},\]

we obtain under some mild assumptions about the derivative in the right-most term that \(S(t) = \lim_{K \to 0^+} \hat{V}_{CS}(S(t), t, T, K)\). This again requires the condition \(\hat{Q}(S(T) = 0) = 0\) and it serves as a ‘reality check’, since the conclusion is most intuitive.

In the Black-Scholes model implied volatility could be estimated by finding the value of \(\sigma\) such that, among other relevant parameters, it predicts the correct price of the current value of the derivatives. However, here the model is not assumed known, since we are essentially working with observed prices of derivatives. In particular, the dynamics of the underlying security could differ considerably from the Black-Scholes model.

Therefore we suggest a non-parametric approach to measuring the implied volatility which somewhat resembles the relative entropy of measures.

Suppose that \(P_1\) and \(P_2\) are equivalent measures. Then we denote by

\[d(P_1, P_2) = \int \left| \log \frac{dP_1}{dP_2} \right| (dm/dP_1 + dm/dP_2)^{-1} dm\]

a symmetric distance of the measures \(P_2\) and \(P_1\). We suggest that this is a plausible distance notion for \(P_1\) and \(P_2\), which are considered as the measure of \(S(T)\) and the state-price density measure, respectively. This distance measure is clearly symmetric unlike the relative entropy of measure. It is evident that \(|\log dP_1/dP_2| = |\log dP_2/dP_1|\) but some explanation regarding
the weight \((dm/d\mathbb{P}_1 + dm/d\mathbb{P}_2)^{-1}\) is in order. Namely, in the usual symmetrized KL divergence one takes the arithmetic average of the densities. However, at this point we would like to take a frequentist approach to the probability densities. Thus the densities correspond to the number of incidents at a point in a given period of time (i.e. speed of accumulation) and it is known that the harmonic average is the right method for computing the average of speeds. Another, more standard alternative is, of course, to use the relative entropy.

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