Weak convergence of the sequences of homogeneous Young measures associated with a class of oscillating functions

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Abstract

We consider the sequences of oscillating functions which are sums of the functions having continuously differentiable inverses. We show that if the total slopes of the elements of these sequences form monotonic sequence then the sequence of the respective Young measures understood as elements of the Banach space of scalar valued measures is weakly convergent. It is also shown that the limit is a homogeneous Young measure with the density (with respect to the Lebesgue measure) being the weak $L^1$ sequential convergence of the densities of the underlying Young measures.

Keywords: Young measures, weak convergence of measures, density of a Young measure, total slope of oscillating function

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1 Introduction

Minimization of functionals with nonconvex integrands is an important problem both from theoretical and practical, including engineering applications, points of view. The fact that in this case the considered functional, although bounded from below, usually does not attain its infimum, is the source of the main difficulty when dealing with this kind of task. It is described in the following example attributed to Oscar Bolza and Laurence Chisholm Young.

Example 1.1. (see [8]) Find the minimum of the functional

$$J(u) = \int_0^1 [u^2 + (\frac{du}{dx})^2 - 1]^2 dx,$$

with boundary conditions $u(0) = 0 = u(1)$.

It can be shown that $\inf J = 0$.

Consider the function

$$u(x) = \begin{cases} x, & \text{for } x \in [0, \frac{1}{2}) \\ \frac{1}{2} - x, & \text{for } x \in [\frac{1}{2}, \frac{3}{4}) \\ x - 1, & \text{for } x \in [\frac{3}{4}, 1]. \end{cases}$$
Then the sequence \( u_n(x) := \frac{1}{n}u(nx) \) is the minimizing sequence for \( J \), that is it satisfies the condition \( J(u_n) \to \inf J \). However, we have \( \inf J \neq J(\lim u_n) \) because if the limit \( u_0 \) of \( (u_n) \) were the function satisfying the equality \( \inf J = J(u_0) \), it would have to satisfy simultaneously two mutually contradictory conditions: \( u_0 \equiv 0 \) and \( \frac{du_0}{dx} = \pm 1 \) a.e. (with respect to the Lebesgue measure on \([0,1]\)). This means that \( J \) does not attain its infimum. The elements of the minimizing sequence are wildly oscillating functions.

Further motivating examples can be found, among others, in [9] and [13].

Generally, the problem can be dealt with in two ways. The first one is convexification the original functional (more precisely: quasiconvexification). This procedure saves the infimum of the functional, but it has two main drawbacks. Namely, computing explicit form of the (quasi)convex envelope is usually very difficult in practice. Further, it erases some important information concerning the behaviour of the minimizing sequences: calculating a weak* accumulation point of the minimizing sequence is calculation a limit of the sequence with integral elements. The integrands are compositions of the Carathéodory function with highly oscillating elements of the minimizing sequence. Thus, in some sense, only the mean values are taken into account, see for example [12] and references cited there.

Another way is connected with the discovery of Laurence Chisholm Young, first published in [14]. He observed that the weak* limit of a sequence with elements being composition of a continuous function with oscillating functions is in fact a set function. This set function can be looked at as a mean summarizing the spatial oscillatory properties of minimizing sequences. Thus not all the information concerning behavioral characteristics of the phase involved is lost while passing to the limit.

More precisely, let there be given:

- \( \mathbb{R}^d \supset \Omega \) – nonempty, bounded open set of the Lebesgue measure \( M > 0 \);
- \( K \subset \mathbb{R}^l \) – compact;
- \( (u_n) \) – a sequence of functions from \( \Omega \) to \( K \), convergent to some function \( u_0 \): weakly* in \( L^\infty \) or weakly in \( L^p \), \( p \geq 1 \);
- \( \varphi \) – an arbitrary continuous real valued function on \( \mathbb{R}^l \).

Then the continuity of \( \varphi \) yields the norm boundedness of the sequence \( (\varphi(u_n)) \) of compositions. By the Banach-Alaoglu theorem we infer the existence of the converging to some function \( g \) the subsequence of \( (\varphi(u_n)) \). However, in general not only \( g \neq \varphi((u_0)) \), but \( g \) is not even a function with domain in \( \mathbb{R}^l \). It is, as Laurence Chisholm Young first proved for the special case in 1937, a family \( (\nu_x)_{x \in \Omega} \) Borel probability measures defined on Borel \( \sigma \)-algebra of subsets of \( \mathbb{R}^l \), with supports contained in \( K \), satisfying for each continuous \( \varphi \) and any
integrable function \( w \) the condition

\[
\lim_{n \to \infty} \int_{\Omega} \varphi(u_n(x)) w(x) dx = \int_{\Omega} \int_{K} \varphi(s) \nu_x(ds) w(x) dx := \varphi(x) w(x) dx,
\]

where

\[
\varphi(x) := \int_{\mathbb{R}} \varphi(\lambda) \nu_x(d\lambda).
\]

Thus the family \( (\nu_x)_{x \in \Omega} \) can be regarded as the 'generalized weak*-limit' of the sequence \( (\varphi(u_n)) \). Young himself called them 'generalized curves'; today we call them Young measures (or relaxed controls in control theory) associated with the sequence \( (u_n) \).

In the very important special case, where for any \( x \in \Omega \) there holds \( \nu_x = \nu \) (that is the family \( (\nu_x)_{x \in \Omega} \) is a one-element only), we use the term homogeneous Young measure. Majority of the examples of the Young measures that can be found in the literature are homogeneous ones, see for example \([8, 9, 13]\).

We can also look at Young measures from another point of view. Namely, they can be regarded as linear functionals acting on \( L^1(\Omega, C(K)) \) – the space of vector-valued, Bochner integrable functions defined on \( \Omega \) and taking values in the Banach space of continuous real-valued functions defined on a compact set \( K \). This enables us to associate Young measure with any Borel function \( f : \Omega \to K \), see Theorem 3.1.6 in \([13]\). Using this approach one may study Young measures in general, with Young measures associated with measurable functions being specific examples. We refer the reader to \([13]\) for detailed exposition of this point of view.

Although calculating an explicit form of the Young measure is usually very difficult, it turns out that quite often it can be done relatively easily, when functions under consideration are bounded oscillating ones. This is possible if the latter of the approaches mentioned above is adopted. As it has been observed in \([10]\) for quite large class of oscillating functions Young measures associated with them are homogeneous, absolutely continuous with respect to the Lebesgue measure with densities being merely the sums of the absolute values of the Jacobians of the inverses of their invertible parts. These sums (called later the total slopes, see Definition \([3.1]\)) are very important: different functions with equal total slopes have the same Young measure. Thus in many cases (for example for the sequence \( f_n(t) = \sin(2n\pi t), t \in (0, 1), n \in \mathbb{N} \)) the sequence of the associated Young measures is constant and it suffices to calculate only the Young measure associated with \( f_1 \).

In \([11]\) these observations has been broadened. Namely, it has been proved that the weak convergence of the sequence of homogeneous Young measures
(understood as elements of the Banach space rca(\(K\)) with densities is equivalent to the \(L^1\) convergence of the sequence of these densities.

It is worth mentioning that the above results have been obtained with really simple, in comparison to the usual Young measure methods, apparatus: the change of variable theorem for multiple integrals plays central role. This can be important in applications.

In this article we prove the following result. Given a sequence of bounded oscillating functions being sums of functions having continuously differentiable inverses we show, that if the total slopes form monotonic sequence, than the sequence of the associated Young measures is weakly (in rca(\(K\))) convergent to a homogeneous Young measure. Moreover, this limit measure has a density which is a weak \(L^1\) limit of the sequence of densities of the Young measures associated with the sequence of oscillating functions.

2 Necessary facts about Young measures

In this section we gather information about Young measures that will be of use in the sequel. The literature concerning Young measures is large. An interested reader may consult, for example, besides the books that have already been cited, \([1, 2, 3, 4]\) and the references cited there.

We will denote by rca(\(K\)) the space of regular, countably additive scalar measures on a Borel \(\sigma\)-algebra of subsets of a compact set \(K\). We equip this space with the norm \(\|m\|_{\text{rca}(K)} := |m|(\Omega)\). Here \(|\cdot|\) stands for the total variation of the measure \(m\). Then the pair \((\text{rca}(K), \|\cdot\|_{\text{rca}(K)})\) is a Banach space. The Riesz representation theorem states that in this case \(\text{rca}(K)\) is conjugate space to the space \((C(K), \|\cdot\|_\infty)\).

Let \((X, \mathcal{A}, \rho)\) be a measure space, \(Y\) – a Banach space and denote by \(\langle \cdot , \cdot \rangle\) the duality pairing. Recall, that a function \(f: X \to Y^*\) is weakly*-measurable if for any \(y \in Y\) the function \(x \mapsto \langle f(x), y \rangle\) is \(\mathcal{A}\)-measurable.

The elements of a space \(L^\infty_{\text{w}^*}(\Omega; \text{rca}(K))\) are functions

\[ \nu: \Omega \ni x \to \nu(x) \in \text{rca}(K), \]

that are weakly*-measurable. We endow this space with a norm

\[ \|\nu\|_{L^\infty_{\text{w}^*}(\Omega; \text{rca}(K))} := \text{ess sup}\{\|\nu(x)\|_{\text{rca}(K)} : x \in \Omega\}, \]

where

\[ \text{ess sup}\{\|\nu(x)\|_{\text{rca}(K)} : x \in \Omega\} := \inf \{\alpha \in \mathbb{R} \cup \{\infty\} : \|\nu(x)\|_{\text{rca}(K)} \leq \alpha \text{ a.e. in } \Omega\}. \]

It is proved in \([13]\) that the space \((L^1(\Omega, C(K)))^*\) is isometrically isomorphic to the space \(L^\infty_{\text{w}^*}(\Omega; \text{rca}(K))\).
The set $\mathcal{Y}(\Omega, K)$ of the Young measures consists of those functions from the space $(L^\infty_w(\Omega; \text{rca}(K)), \|\cdot\|_{L^\infty_w(\Omega; \text{rca}(K))})$, whose values are probability measures on $K$.

It is seen that homogeneous Young measures are those functions from $(L^\infty_w(\Omega; \text{rca}(K)), \|\cdot\|_{L^\infty_w(\Omega, \text{rca}(K))})$, whose values are constant mappings (see also [11] for a very simple formal proof).

**Remark 2.1.** In [13] Young measures are defined as weakly measurable mappings, but it seems to be an inaccuracy, since $\text{rca}(K)$ is in fact a conjugate space. Compare footnote 80 on page 36 in [13] with, for example, Definition 2.1.1 (d) on page 109 in [6].

We are also interested in the subset of $\mathcal{Y}(\Omega, K)$ with elements being Young measures associated with Borel functions $u: \Omega \to K$. We can infer from the Theorem 3.1.6 in [13] that for any such function $u$ there exists a Young measure which is associated with it.

**Remark 2.2.** Due to the fact that homogeneous Young measures are constant functions, we will write $'\nu'$ instead of $'\nu = (\nu_x)_{x \in \Omega}'$.

### 3 Oscillating functions

Let $\Omega \subset \mathbb{R}^d$ be a nonempty, bounded open set of the Lebesgue measure $M > 0$. Consider $\{\Omega\}$ – an open partition of $\Omega$ into at most countable number of open subsets $\Omega_1, \Omega_2, \ldots, \Omega_n, \ldots$ such that

(i) the elements of $\{\Omega\}$ are pairwise disjoint;

(ii) $\bigcup_i \overline{\Omega_i} = \overline{\Omega}$, where $\overline{\Omega}$ denotes the closure of the set $\Omega$.

Let us consider functions $f_i: \Omega_i \to K \subset \mathbb{R}^d, i = 1, 2, \ldots$, with inverses $f_i^{-1}$ that are continuously differentiable on $f(\Omega_i)$ and let $K_i := \overline{f(\Omega_i)}$ be compact. Denote for each $i = 1, 2, \ldots$ the Jacobian of $f_i^{-1}$ by $J_{f_i^{-1}}$.

Let an oscillating function $f: \Omega \to K$, with $K := \overline{f(\Omega)}$ compact, be of the form

$$f(x) = \sum_i f_i(x) \chi_{\Omega_i}(x), \quad x \in \bigcup_i \Omega_i. \quad (3.1)$$

Then there holds the following theorem (see Proposition 5.1. in [7]).

**Theorem 3.1.** The Young measure associated with Borel function $f$ satisfying (3.1) is a homogeneous one and its density $g$ with respect to the Lebesgue measure on $K$ is of the following form

$$g(y) = \frac{1}{M} \sum_{i, y \in \Omega_i} |J_{f_i^{-1}}(y)| \quad (3.2)$$

**Definition 3.1.** Let an oscillating function $f$ be given by the equation (3.1). The total slope $J_t f$ of $f$ is defined by

$$J_t f(y) := \sum_{i, y \in \Omega_i} |J_{f_i^{-1}}(y)|.$$
Example 3.1. Let $f_n(t) = \sin(2n\pi t)$, $t \in (0, 1)$, $n \in \mathbb{N}$. Then
\[ \forall n \in \mathbb{N} \quad J_{f_n}(y) = \frac{1}{\pi \sqrt{1-y^2}}, \]
and thus the Young measure associated with each $f_n$ is homogeneous and absolutely continuous with respect to the Lebesgue measure. Its density is equal to $J_{f_n}$ (see Example (c) in section 5 of [7]).

4 Weak sequential convergence of functions and measures

We will now recall classical theorems concerning weak sequential convergence of functions and measures that will be needed in the sequel. The expression ‘weak convergence of the sequence of measures’ will be meant as the ‘weak convergence of the sequence of measures as elements of the Banach space $\text{rca}(K)^*$ (with the total variation norm).

Recall that if $\rho$ is a measure on $K$ and for some function $w: K \to \mathbb{R}$ integrable with respect to the measure $\xi$ there holds: for any Borel subset $A$ of $K$ we have $\rho(A) = \int_A w(y)d\xi(y)$, then the function $w$ is called a density of the measure $\rho$. In this case $\rho$ is absolutely continuous with respect to $\xi$ (shortly: $\xi$-continuous):
\[ \xi(A) = 0 \Rightarrow \rho(A) = 0. \]

Let $(X, \mathcal{A}, \rho)$ be a measure space and consider a sequence $(u_n)$ of scalar functions defined on $X$ and integrable with respect to the measure $\rho$ (that is, $\forall n \in \mathbb{N} f_n \in L^1(\rho)$) and a function $u \in L^1(\rho)$. Recall that $(u_n)$ converges weakly sequentially to $u$ if
\[ \forall g \in L^\infty(\rho) \quad \lim_{n \to \infty} \int_X u_n g d\rho = \int_X u g d\rho. \]

The following theorem characterizes weak sequential $L^1$ convergence of functions and weak convergence of measures. We refer the reader to [5, 3, 4].

Theorem 4.1. (a) (J. Dieudonné, 1957) let $X$ be a locally compact Hausdorff space and $(X, \mathcal{A}, \rho)$ - a measure space with $\rho$ regular. A sequence $(u_n) \subset L^1(\rho)$ converges weakly to some $u \in L^1(\rho)$ if and only if $\forall A \in \mathcal{A}$ the limit
\[ \lim_{n \to \infty} \int_A u_n d\rho \]
exists and is finite;

(b) let $X$ be a locally compact Hausdorff space and denote by $\mathcal{B}(X)$ the $\sigma$-algebra of Borel subsets of $X$. A sequence $(\rho_n)$ of scalar measures on $\mathcal{B}(X)$ converges weakly to some scalar measure $\rho$ on $\mathcal{B}(X)$ if and only if $\forall A \in \mathcal{B}(X)$ the limit
\[ \lim_{n \to \infty} \rho_n(A) \]
exists and is finite.
Corollary 4.1. (a) let \((\rho_n)\) be a sequence of measures having respective densities \(u_n, n \in \mathbb{N}\). Then the sequence \((u_n)\) is weakly convergent in \(L^1(X)\) to some function \(h\) if and only if the sequence \((\rho_n)\) is weakly convergent to some measure \(\eta\);

(b) assume additionally, that \(X \subset \mathbb{R}^l\) is compact and let \((\rho_n)\) be a sequence of homogeneous Young measures having respective densities \(u_n\). Then the sequence \((u_n)\) is weakly convergent in \(L^1(X)\) to some function \(h\) if and only if the sequence \((\rho_n)\) is weakly convergent to some measure \(\eta\).

Proof. (\(\Rightarrow\)) Since \((u_n)\) is weakly convergent in \(L^1(X)\), then for any measurable \(A \subseteq X\) the limit 
\[
\lim_{n \to \infty} \int_A u_n \, dp = \lim_{n \to \infty} \rho_n(A)
\]
exists and is finite. This in turn is equivalent to the fact that the sequence \((\rho_n)\) of measures is weakly convergent to some measure \(\eta\).

(\(\Leftarrow\)) We proceed as above, but start the reasoning from the weak convergence of the sequence \((\rho_n)\).

Remark 4.1. (a) part (a) of the Corollary 4.1 is simple yet useful corollary of the Theorem 4.1. In particular, special form of its first part appears in the proof of the Corollary 1.58 in [5], Theorem 6.4.1 in [3], Theorem 11 in chapter VII in [4]. However, since this formulation is important in the sequel and we have not been able to trace it in that form in the literature, we formulate and prove it here;

(b) part (b) is generalization of the Theorem 4.3 in [11].

We now introduce the notion of a density of a Young measure. We remember, that Young measure is in fact a family \(\nu = (\nu_x)_{x \in \Omega}\). In a special case, when this family consists of one element only (i.e. it does not depend on the variable \(x\)) we say about homogeneous Young measure.

Let \(\Omega\) be a nonempty, bounded open subset of \(\mathbb{R}^n\), \(K\) – compact subset of \(\mathbb{R}^l\). The Borel \(\sigma\)-algebra of subsets of \(K\) will be denoted \(\mathcal{B}(K)\).

Definition 4.1. We say that a family \(h = (h_x)_{x \in \Omega}\) is a density of a Young measure \(\nu\) with respect to the measure \(\xi\) defined on \(\mathcal{B}(K)\) if for any \(x \in \Omega\) the function \(h_x\) is a density of the measure \(\nu_x\) i.e. for any \(A \in \mathcal{B}(K)\) there holds \(\nu_x(A) = \int_A h_x(y) d\xi(y)\).

Proposition 4.1. Let \(\nu\) be a Young measure and let \(h = (h_x)_{x \in \Omega}\) be a density of \(\nu\) with respect to the measure \(\xi\). Then \(\nu\) is a homogeneous Young measure if and only if the family \(h = (h_x)_{x \in \Omega}\) consists of one element only, up to the set of \(\xi\)-measure 0.

Proof. Necessity follows from the Definition 4.1 and the properties of the integral. For the sufficiency assume that \(\nu\) is not homogeneous. Then there
exist points $x_1, x_2$ in $\Omega$ and a set $A \in \mathcal{B}(K)$ of positive $\xi$-measure such that $\nu_{x_1} \neq \nu_{x_2}$ and

$$\nu_{x_1}(A) = \int_A h_{x_1}(y)d\xi(y) \neq \nu_{x_2}(A) = \int_A h_{x_2}(y)d\xi(y),$$

which contradicts the fact that $h_{x_1} = h_{x_2}$ $\xi$-a.e. \hfill \Box

Before formulating the next theorem of the article we will recall two classical theorems that will be needed in its proof. We again refer the reader to [5, 3, 4].

**Theorem 4.2.** (a) *(Radon-Nikodym theorem for finite measures)*

Let $\mu$ and $\rho$ be finite measures on a measurable space $(X, \mathcal{A})$ and assume that $\rho$ is $\mu$-continuous. Then there exists a unique (up to the set of $\mu$-measure 0) $\mu$-integrable function $w \in L^1(X)$ such that

$$\forall A \in \mathcal{A} \quad \rho(A) = \int_A wd\mu;$$

(b) *(Vitali-Hahn-Saks theorem)*

Let $(X, \mathcal{A})$ be measurable space, $\mu$ - a nonnegative finite measure and let $(\rho_n)$ be a sequence of $\mu$-continuous scalar measures on $\mathcal{A}$. If for any $A \in \mathcal{A}$ the limit $\lim_{n \to \infty} \rho_n(A)$ exists, then the formula:

$$\forall A \in \mathcal{A} \quad \rho(A) := \lim_{n \to \infty} \rho_n(A)$$

defines a $\mu$-continuous scalar measure on $\mathcal{A}$.

The following theorem states that weak sequential convergence of homogeneous Young measures with densities is equivalent with the weak convergence of these densities and that in the limit we obtain the respective objects of the same nature.

**Theorem 4.3.** Let $K \subset \mathbb{R}^l$ be compact and denote by $\mu$ the Lebesgue measure on $\mathcal{B}(K)$. Let $(\rho_n)$ be a sequence of $\mu$-continuous homogeneous Young measures. Denote their respective densities by $u_n$.

Then the sequence $(u_n)$ is weakly convergent in $L^1(K)$ to some function $h$ if and only if the sequence $(\rho_n)$ is weakly convergent in the Banach space $(\text{rca}(K), \| \cdot \|_{\text{rca}(K)})$ to some measure $\eta$.

Moreover, $\eta$ is a homogeneous Young measure with density $h$.

**Proof.** The first part of the theorem is part (b) of the Corollary 4.1. For the second part, it follows from the Vitali-Hahn-Saks theorem that the measure $\eta$ is $\mu$-continuous, so by the Radon-Nikodym theorem it has a density $r$. Now
choose and fix set $A \in \mathcal{B}(K)$. We then have

$$
\left| \int_A r d\mu - \int_A h d\mu \right| \leq \left| \int_A r d\mu - \eta(A) \right| + \left| \eta(A) - \rho_n(A) \right| + \left| \rho_n(A) - \int_A u_n d\mu \right| + \\
\left| \int_A u_n d\mu - \int_A h d\mu \right|.
$$

The first and third term on the right hand-side vanish as $r$ is density of $\eta$ and $u_n$ is density of $\rho_n$, $n \in \mathbb{N}$. The second and fourth term tend to zero as $n \to \infty$ because of the respective weak convergence of $(\rho_n)$ and $(u_n)$. As the set $A$ has been chosen arbitrarily, we have $r = h$ up to the set of $\mu$-measure 0.

Further,

$$
\left| 1 - \int_K h d\mu \right| \leq \left| 1 - \int_K u_n d\mu \right| + \left| \int_K u_n d\mu - \int_K h d\mu \right|.
$$

The first term of the right-hand side is equal to 0 because the elements of the sequence $(\rho_n)$ are probability measures on $K$, while the second term tends to 0 as $n \to \infty$ because $(u_n)$ is weakly convergent in $L^1(K)$ to $h$. This means that $\eta$ is a probability measure on $K$. By the uniqueness of the weak limit the mapping

$$
\Omega \ni x \to \eta \in rca(K)
$$

is constant, so it is weakly*-measurable. Thus $\eta$ is a Young measure. The fact that it is homogeneous is now consequence of the Proposition 4.1.

Proof. We can assume that the sequence $(Jt f_n)$ is nondecreasing. By Theorem 3.3 the Young measures $\rho_n$ are homogeneous with densities given by the equation (3.2). Then for any $m, n \in \mathbb{N}$, $m \leq n$ and any $A \in \mathcal{B}(K)$ there holds

$$
\int_A Jt f_m d\mu \leq \int_A Jt f_n d\mu,
$$

which means that the limit of the sequence $(\int_A Jt f_n d\mu)$ exists and is finite. The result now follows from Theorems 4.1 and 4.3.

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