THE ECLECTIC CONTENT AND SOURCES OF CHRISTOPHER
CLAVIUS’S GEOMETRIA PRACTICA

JOHN B. LITTLE

ABSTRACT. We consider the Geometria Practica of Christopher Clavius, S.J., a
surprisingly eclectic and comprehensive textbook of practical geometry, whose first
dition appeared in 1604. Our focus is on four particular sections from Books IV
and VI where Clavius has either used his sources in an interesting way or where
he has been uncharacteristically reticent about them. These include the treatments
of Heron’s Formula, Archimedes’ Measurement of the Circle, four methods for
constructing two mean proportionals between two lines, and finally an algorithm for
computing nth roots of numbers.

1. INTRODUCTION

1.1. Clavius. Christopher Clavius, S.J. (1538–1612) was certainly the preeminent Je-
suit mathematician of his era and an important mathematical astronomer He was
admitted into the Society of Jesus in Rome in 1555, studied at the University of Coim-
bra in Portugal for two years, then returned to Rome where he completed his studies.
In comments to some of his younger Jesuit colleagues, he claimed that he was largely
self-taught in mathematics. This is sometimes seen as doubtful since the well-known
mathematician Pedro Nunes (1502–1578) was active at Coimbra during Clavius’s time
there. However, while there might be traces of Nunes’s influence in some of Clav-
ius’s more algebraic works, no direct evidence of contact is known. Starting from 1563
through the end of his life (except for a short assignment in Naples in 1595–1596),
Clavius served as professor of mathematics at the Jesuit Collegio Romano. At the
start of this time, he taught the regular mathematics curriculum and led an “Academy”
in which exceptionally able and energetic students could pursue the study of math-
ematics beyond the basics. Around the time of his sojourn in Naples, he essentially
retired from regular teaching and devoted himself primarily to writing and mentoring
the mathematicians of the Academy. Many of the talented Jesuit mathematicians of
the generation around 1600 (Christoph Grienberger, Odo van Maelcote, Grégoire de
St. Vincent, Paulo Lembo, Paul Guldin, Orazio Grassi, and others) passed through
this Academy and some stayed on at the Collegio as professors of mathematics.

Clavius was fundamentally a commentator, expositor, and evaluator of the math-
ematical work of others, not primarily as an original mathematical researcher in the
modern sense. His mathematical outlook was essentially conservative and grounded
firmly in the geometry of the Elements of Euclid, with some “excursions” into parts
of algebra and what we would call discrete mathematics. Yet his view of the subject
was broad enough to acknowledge both the certainty of mathematical knowledge due
the subject’s reliance on strict standards of proof and the utility of mathematics for

Date: December 16, 2021.

1See [13, 1, 2]. The original documentary sources for many of the facts mentioned here are discussed
by Knobloch and Baldini.
Besides his teaching and mentoring of younger mathematicians, a large portion of Clavius’s energies were devoted to the production of numerous influential textbooks or source books for the teaching of a wide range of mathematical subjects. These included his extensively augmented edition of the Elements of Euclid (first edition 1574), the Epitome arithmeticae practicae (Summary of Practical Arithmetic, 1583), this Geometria Practica (Practical Geometry, first edition 1604), and the Algebra (1608). Clavius also wrote a well-known commentary on the Sphere of Sacrobosco, books on the astrolabe and the construction of sundials, and more elementary treatments of plane and spherical triangles. His collected mathematical works (the Opera Mathematica) were published in five volumes starting in 1611.

1.2. The Geometria Practica. The 1604 edition of the Geometria Practica was printed by the shop of Luigi Zanetti in Rome; just two years later, a second edition was produced by the printshop of Johann Albin in Mainz in 1606. The 1604 edition is slightly longer because of a different page format. However, there are no substantial differences between the texts. Moreover, very similar (but not identical) woodcut figures were used in both editions, so the overall appearance does not differ significantly. The version of the Geometria Practica included in the Opera Mathematica contains some corrections of typographical and mathematical errors in the previous editions, an expanded discussion of the quadrant constructed in Book I, and some other relatively minor additions. In this essay, page numbers refer to the page in the 1606 edition.

While a number of scholars have written on aspects of the book, as a whole, Clavius’s Geometria Practica has apparently received less scholarly attention than his edition of Euclid’s Elements. Yet this work has unexpected and surprising features. Clavius’s account certainly stands out in several ways within the whole genre of practical geometry texts, and these features form the major reasons this work remains interesting from the historical perspective. They have also furnished the main motivations for the author of this essay to undertake a translation of the entire Geometria Practica from the original Latin into English using the 1606 second edition.

First, the eclecticism—the sheer range of different types of topics that fall under the category of practical geometry or allied areas for Clavius and that make it into this book—is remarkable. In his Preface, Clavius discusses previous works in the practical geometry genre, and how he wants his work to stand out from the others:

... Many erudite men have pursued all of its [i.e. practical geometry’s] parts with accurate and diligent writing. Among them, Leonardo Pisano [“Fibonacci”], Brother Luca Paccioni, Nicolo Tartaglia, Oronce Finé, Girolamo Cardano and others have demonstrated preeminence

---

2This is expressed most explicitly in Clavius’s essay In disciplinas mathematicas prolegomena (Prolegomena on the mathematical disciplines) included in Volume I of the Opera Mathematica. Clavius sees mathematics as intermediate between metaphysics and natural philosophy, an idea that traces back at least to Proclus’s Commentary on Book I of Euclid’s Elements, a text Clavius mentions several times. See, for instance, Chapter 1.

3These have been digitized, see: https://clavius.library.nd.edu/mathematics/clavius.

4This is perhaps a reflection of certain derogatory attitudes toward “applied” or “practical” mathematics in general.

5See 21 and the Introduction to 12 by the translator, F. Homann, S.J.

6This translation is available at CrossWorks, the online faculty and student scholarship repository maintained by the Library of the College of the Holy Cross. All quotations of passages from the Geometria Practica in English are taken from this translation. The original Latin text from the 1606 edition of the Geometria Practica will be provided in footnotes, for purposes of comparison.
and flourished to exceptional praise. But I would judge Giovanni Antonio Magini one of the first in mathematical excellence. He has taught so much about the measurement of lines, and treated this subject so fully, so systematically and with such perspicacity, that he seems to have snatched away, not only the standing from those who wrote before him, but also the hope of equal, let alone greater, glory from those coming after him. But truly, Magini concerned himself only with this one part of this subject, and the others, although they undertook to present all of those parts, have left out much in writing their books. I decided, if possible, to complete the subject, so that \textit{whatever has been profitably handed down by others or found by myself in practical geometry is enclosed within the circle of one work}.\footnote{7} (\cite{7 Preface}; emphasis added)

Thus, in this work, Clavius aims for \textit{completeness} within the subject of practical geometry as understood by his contemporaries. The following rough outline of topics will demonstrate how wide-ranging and encyclopedic this book truly is:\footnote{8}

- Book I: Construction of a proportional compass and quadrant for measuring lengths and angles; summary of elementary plane trigonometry
- Book II: Measuring lengths, heights, and depths with the astronomical quadrant\footnote{9}
- Book III: A parallel discussion of measuring lengths, heights, and depths with the geometer’s square; other methods for the same sorts of problems
- Book IV: Measuring areas of plane regions, including an augmented translation of of Archimedes’ \textit{Measurement of the Circle}, and quoting from other works of Archimedes including the \textit{Quadrature of the Parabola}
- Book V: Measuring volumes of solid bodies, with extensive quotations of results from Archimedes’ works \textit{On the Sphere and Cylinder}, and \textit{On Conoids and Spheroids}.
- Book VI: Geodesy, that is, the division of rectilinear surfaces of whatever sort, either by lines drawn through some point, or by parallel lines\footnote{10} how plane or solid figures are increased or decreased in a given ratio; several methods for finding two mean proportionals between two given lines selected from the commentary on Archimedes’ \textit{On the Sphere and Cylinder} by Eutocius of Ascalon and Pappus’s \textit{Mathematical Collection}; finally, an algorithm for extracting all sorts of roots by hand calculations

\footnote{7... & multos, & eruditos viros ... , qui partes illius omnes accurata, & diligentii scriptione persecuti sunt: Inter quos, vt Leonhardus Pisanus, Frater Lucas Pacciolus, Nicolaus Tartalea, Orontius, Cardanus, aliique praeicipius obtinuerunt: ita eximia in caeteris laude floruerunt. Primas tamen adiudicarim Io. Antonio Magino praestanti Mathematico; qui tametsi tantum linearum dimensiones docuit, ea tamen copia, doctrina, perspicacitate cuncta tradidit, vt locum non modo iis, qui ante scripserrunt, sed spem posteris aequalis gloriarum, ne dum maioris, ademisse videatur. Verum quoniam & hic de vnica tantum parte fuit sollicitus: & aliis, quamuis aggressi omnia, multa tamen inter scribendum praeterierunt: decreui, si qua possem, perficere: vt, quicquid utiliter in Geometria practica ab aliis traditum, à me etiam inuentum est, vnius operis gyro clauderetur.}

\footnote{8At a higher degree of granularity, the complete list of chapter headings and propositions that serves as the table of contents is even more evidence here.}

\footnote{9Discussions of problems similar to those considered here can be seen in almost all practical geometry books. As Raynaud points out, \cite[p. 15]{23}, these are part of a long and surprisingly stable tradition with connections to propositions 19–22 from Euclid’s \textit{Optics}.}

\footnote{10Clavius’s sources and his treatment of this topic are discussed by E. Knobloch in \cite{15}.}
- Book VII: Isoperimetric figures and questions (drawing on material in Pappus and the commentary on Ptolemy’s *Almagest* by Theon of Alexandria), together with an appendix on the problem of squaring the circle via the *quadratrix* curve of Hippias, drawing from Pappus

- Book VIII: Various geometric theorems and constructions that Clavius says can be used to build mathematical power in problem-solving—several of these are drawn from Pappus’s *Mathematical Collection*, including one discussing the trisection of general angles using the *conchoid* curve of Nicomedes. A table of squares and cubes of all \( N \leq 1000 \) is included at the end, together with a discussion of how the table can be extended using facts about the first and second differences of the sequences of squares and cubes, with applications to extraction of square and cube roots.

As is true in all of his other works, Clavius also has clarity of exposition as a second main goal. A striking example of this commitment to completeness and clarity of exposition is Clavius’s treatment of Archimedes’ *Measurement of the Circle* in Chapter 6 of Book IV, which will be examined in detail in §3.

The second feature that has seemed surprising to this author is the resolutely *dual theoretical and practical focus* of much of this text on practical geometry. The practical side is signaled immediately in the author’s Preface where Clavius discusses his motivation for writing the book. After saying that his experience as a teacher has taught him that most students work and learn best when they understand that what they are learning will prove to be useful, Clavius addresses how the contents of this book may find uses in the real world:

> For of course as long as the methods by which we must make measurements to understand the lengths of fields, the heights of mountains, the depths of valleys, and the distances between all locations are presented, it is clear to anyone (in my opinion) how much that is of use in the construction of buildings, in agriculture, in the design of weapons, in the contemplation of the stars, and in all other arts and disciplines, can flow from the study of these things. (Preface)

Clavius consistently uses numerical examples in many sections and he presents a number of purely calculational methods (e.g. the methods for extraction of roots in Book VI and the material on differences of squares and cubes at the end of Book VIII). He discusses the use of different mechanical tools for measurements and is even willing to countenance “mechanical,” hence necessarily approximate, methods of measurement in geometric diagrams:

---

11As Knobloch writes, “... son approche démontre les limites d’une division trop tranchée entre géométrie pratique et géométrie savante,” [15, p. 60]. That is, “... [Clavius’s] approach shows the limitations of a too-definite distinction between practical geometry and theoretical geometry.” This applies to almost every section of the *Geometria Practica*, not just the discussion of geodesy.

12Et verò cum perpetua multorum annorum experientia compererim, admodum paucos esse, qui non in Mathematicis exercerantur eo consilio, vt quae didicerint, ad aliquem vsum trahant.

13Etenim dum certa ratio traditur, qua camporum longitudines, altitudines montium, vallium depressiones, locorum omnium inaequalitates inter se, & interualla deprehendere metiendo debeamus: cuilibet liquet, vt arbitror, quantum commodi, utilitatisque substructioni aedificiorum, cultui agrorum, armorum tractationi, contemplationi siderum, aliisque artibus, & disciplinis ex horum cogitatione manare possit.
No one should be troubled that we have said lines are sometimes to be measured mechanically with an iron chain, or by means of the instrumentum partium. For in this business, especially for fields and farms, this mechanical way of making measurements is wholly admissible, partly since this is the custom among all surveyors, partly because the geometric way is not always possible, but mostly because for the dimensions of farms or other areas it is sufficient to come close enough to the truth that no notable error is made. If someone does not approve of this way of measuring lines, it assuredly and necessarily takes away every possibility of measuring farms or other areas. For in what way is it certain that a given field or figure has known sides, if these have not been explored by some material measurement? If therefore mechanical measurement of lines (not straying far from the truth, as it were) is used by everyone, I do not see why we should think to reject it in measuring lines in figures.

There are indeed possible practical applications of many of the more theoretical topics treated in Books IV through VIII as well. Clavius includes a section on methods used by surveyors in Book IV and a section on measuring volumes of barrels or casks at the end of Book V. However, once he gets past the very basic material in Books I, II, and III, Clavius’s focus seems to shift to developing the mathematical theory along with a few practical applications, and he usually provides full proofs for the most important results.

Another hallmark of Clavius’s approach and theoretical orientation even within the practical discussions is his scrupulous attention to providing reasons for almost everything he writes and sources for the material he does not prove in detail. This applies even within Books I, II, and III. Throughout the text, an elaborate system of marginal notes identifies justifications for assertions and for the individual steps in proofs or computations. Over the course of the whole book, the justifications for the steps in those proofs span almost all of the 13 books of the canonical version of Euclid’s Elements, plus the 14th and 15th books added by later authors and included in Clavius’s edition of the Elements, as well as some of Clavius’s other texts, several works of Archimedes and Apollonius’s Conics (once).

A third feature that is clearly visible in the above outline, but that might be surprising, is the extent to which Clavius draws on Archimedes, Pappus, Claudius Ptolemy, Euclid’s Elements and Proclus’s commentary on Book I, and works of other ancient

---

14 That is, the proportional compass introduced in Book I.

15 Neminem autem moueat, aut perturbet, quod rectas dixerimus metiendas esse nonnunquam mechanicè per catenulam aliquam ferream, aut per instrumentum partium. Nam in hoc dimittiendi negotio, praesertim in campis, & agris admissenda omnino est huiusmodi mechanica linearum dimensio, tum quia apud omnes agrimensores hic mos est: tum quia non semper via Geometrica id praestare potest; tum vero maximè, quia in dimensionibus agrorum, siue figurarum satis est rem prope verum attingere, dum modo notabilis error non comittatur. Quod si haec dimensio quadrundem linearum alii non probetur, is profecto ê medio tollat, necesse est, omnem agrorum, figurarumque dimensionem. Vnde enim constat, agrum propositum, vel figuram habere latera cognita, nisi haec ipsa per mensuram aliquam materialem sint explorata? Si igitur laterum dimensionis mechanica, tanquam à vero parum aberrans, ab omnibus vsurpatur, cur eam in lineis intra figuram metiendas reiçiendam censeamus, non video.

16 Primarily Measurement of the Circle, On the Sphere and Cylinder, and On Conoids and Spheroids, but less commonly also the Quadrature of the Parabola.
and medieval mathematicians and his contemporaries. It is worthwhile to note here that some of the work of the ancient Greeks was just coming back into the European mathematical mainstream at precisely this time due to the work of humanist scholars such as Federico Commandino and others. Commandino’s Latin translation of the surviving portions of Pappus’s *Mathematical Collection*, for instance, only appeared in print in 1588.

1.3. This essay. Our plan in this essay is to flesh out this general description of the eclectic content and sources of Clavius’s *Geometria Practica* by focusing on four particular sections dealing with topics of particular interest. We have restricted ourselves to parts of the text not covered in detail by other authors. So for instance, we have not included a discussion of the Appendix to Book VII giving Clavius’s approach to the problem of squaring the circle via the quadratrix curve because that is analyzed deeply by Bos in [3, Chapter 9]. Similarly we have not considered the discussion of geodesy at the start of Book VI, since Clavius’s approach has been discussed by Knobloch in [15].

The sections we do discuss are ones where (in our judgment) Clavius has either used his sources in an interesting way, or he has been uncharacteristically reticent about those sources. We will look first at the beginning of his discussion of computing areas of triangles in Book IV, where Clavius presents what we now call Heron’s formula before the usual method based on finding an altitude of the triangle. He does not say anything about his sources there, but by comparing what Clavius says with the development in Leonardo Pisano’s *De Practica Geometrie* some insight may be gained. Second, we will look at Clavius’s treatment of Archimedes’ *Measurement of the Circle* in Book IV. Here we will see that Clavius has presented essentially a complete reworking of the extant Archimedean text incorporating many additional explanatory comments and details not found in other versions. Third, we will consider Clavius’s discussion of some of the Greek constructions for finding two mean proportionals between two given lines. This involved a very deliberate selection of only a few of the methods discussed in the commentary on Archimedes’ *On the Sphere and Cylinder* by Eutocius of Ascalon and by Pappus in the *Mathematical Collection*. Finally, we consider the discussion of an algorithm for extraction of nth roots discussed at the end of Book VI. Clavius does not explicitly identify his source here. But by considering what books would have been available to him, and comparing his treatment of extraction of roots with what appears in one of those books, we are able to propose what we believe is a very likely candidate. This may have been noted before, but if so, we are not aware of it.

---

17Interestingly enough, Clavius only refers to the *Conics* of Apollonius a handful of times.
18A more extensive version of this also appears in Clavius’s edition of Euclid.
19As a rule, Clavius is very careful to identify sources, and it stands out when he does not do so. Over the course of this book, the list of authors cited is quite extensive, including (but possibly not limited to) Apollonius, Archimedes, Archytas, Giovanni Battista Benedetti, Campanus de Novare, Girolamo Cardano, Federico Commandino, John Dee, Diostratus, Diocles, Albrecht Dürer, Eratosthenes, Euclid, Eutocius of Ascalon, Oronce Finé, François de Foix, Comte de Candale, Niccolo Fontana (“Tartaglia”), Gemma Frisius, Marino Ghetaldi, Christoph Grienberger, Hippocrates, Hypsicles, Ioannes Pediasimos, Leonardo Pisano (“Fibonacci”), Ludolph van Ceulen, Mohammad of Baghdad, Odo van Maelcote, Giovanni Antonio Magini, Francesco Maurolico, Menaechmus, Nicholas of Cusa, Nicomedes, Latino Orsini, Luca Pacioli, Pappus, Georg Peuerbach, Proclus, Ptolemy, Joseph Justus Scaliger, Sporus, Simon Stevin, Theon of Alexandra, Juan Bautista Villalpando, Johannes Werner. A fuller listing of all the authors cited by Clavius across his whole written output is given in [14].
2. Clavius’s treatment of “Heron’s formula” for triangles in Book IV.

In Chapter 2 of Book IV, Clavius discusses methods for finding the area of a plane triangle. He says there are two ways of doing this and he will first present the most accurate or precise one. He states this as a rule or procedure for doing the computation:

Let all the sides be added together in one sum; let each of the sides be subtracted from half of this sum, so that three differences between the semiperimeter and the sides are obtained; finally, let these three differences and the semiperimeter be multiplied together. The square root of the number produced will be the area of the triangle which is sought.\(^{20}\)

In modern algebraic terms, the procedure can be collapsed into the single formula

\[ A = \sqrt{s(s-a)(s-b)(s-c)}, \]

where \(a, b, c\) are the side lengths and \(s = \frac{(a + b + c)}{2}\) is the semiperimeter of the triangle. This usually goes by the name Heron’s Formula today, and indeed this is stated and proved in Proposition I.8 of the Metrica of Heron of Alexandria (ca. 10–ca. 70 CE\(^{21}\)). Clavius provides three numerical examples of triangles with integer side lengths, the last of which leads to a product \(s(s-a)(s-b)(s-c)\) that is not a square. He then gives a complete, detailed proof that this does in fact produce the area of the triangle.

Unusually for him, Clavius does not provide an attribution for this result, so several natural questions arise. First, what if any source(s) was he drawing on here and why did he not mention it (or them)? And even before that question: What source or sources for this result would have been available? This is an interesting question because in Clavius’s time Heron’s Metrica was not known; it was considered lost until 1896, when Richard Schöne recognized it as part of a manuscript kept in a library in Istanbul.\(^{22}\)

To help make some comparisons between various proofs, we begin with a version of the diagram in Heron’s proof for a specific triangle.\(^{23}\) Heron’s proof in outline consists of the following steps. First, let the circle \(Z\Delta E\) with center at \(H\) be inscribed in

\[ \text{Colligantur omnia latera in unam summam: Ex huius summa semisse subtrahantur singula latera, vt habeantur tres differentiae inter illam semissem, & latera singula: Postremo tres hae differentiae, & dicta semissis inter se mutuo multiplicentur. Producti enim numeri radix quadrata erit area trianguli quesita.} \]

\(^{20}\)Colligantur omnia latera in unam summam: Ex huius summa semisse subtrahantur singula latera, vt habeantur tres differentiae inter illam semissem, & latera singula: Postremo tres hae differentiae, & dicta semissis inter se mutuo multiplicentur. Producti enim numeri radix quadrata erit area trianguli quesita.

\(^{21}\)The Islamic mathematician al-Bīrūnī (973–1048) thought that the result was originally proved by Archimedes, and C. M. Taisbak has recently provided a conjectural reconstruction for how Archimedes might have stated the result. See \(^{28}\). The statement we have is quite unusual for Greek mathematics from the time of Archimedes or earlier because it is not clear what geometric significance should be attached to the product of four lengths or the product of two areas (which arises in the proof). In our formula \(A = \sqrt{s(s-a)(s-b)(s-c)},\) the quantities on the right would be interpreted as numbers; in fact Clavius says exactly this at one point in his version of the argument. But that is not what Greeks working in the strict Euclidean tradition would have done. Taisbak thinks the Archimedean form of the statement could have been that the triangle is the mean proportional between two rectangles. This is certainly possible. But it must be treated as a conjecture since no known Archimedean text deals with questions related to Heron’s formula.

\(^{22}\)This was first published in \(^{10}\). A modern study of this sole known surviving manuscript of the Metrica can be found in \(^{11}\).

\(^{23}\)See Figure 1. To generate these figures, we used the triangle with vertices at \(B = (0, 0), C = (5, 0)\) and \(A = (1, 2)\) in the Cartesian plane. This happens to have a right angle at \(A\) so some of the line segments in the figures are in rather special positions that facilitated the plotting. However, this does not affect the arguments. None of the authors we consider would have done things this way, of course.
the triangle $AB\Gamma$. (Proposition 4 in Book IV of Euclid gives a construction for this where $H$ is found as the intersection of two of the angle bisectors of the triangle, but Heron takes this as known and does not mention it explicitly.) $H\Delta = HZ = HE$ since these are all radii of the same circle. Then since $HE$, $H\Delta$ and $HZ$ are perpendicular to the sides of the triangle, the area of triangle $AB\Gamma$ will equal one half times $EH$ times the perimeter of the triangle. Heron then does further construction steps, first extending $B\Gamma$ to $\Theta\Gamma$, letting $\Theta B = A\Delta$. This makes $\Theta\Gamma$ equal to the semiperimeter of the triangle. Second, he takes $HA$ perpendicular to $HT$ and extends to $\Lambda$ which is the intersection with the line through $B$ perpendicular to $BT$. It follows that $HB\Lambda\Gamma$ is a cyclic quadrilateral and facts about the diagonals in such quadrilaterals imply triangle $B\Lambda\Gamma$ is similar to triangle $\Delta HA$. The proportionality of corresponding sides implies the square of the area of the triangle $AB\Gamma$ is equal to the square on the perpendicular $HE = H\Delta = HZ$ above times the square on the semiperimeter (using the fact that $B\Gamma$ and $\Theta B = A\Delta$ together equal the semiperimeter.)

Several of the previous practical geometry texts that Clavius mentions in his preface (see above) also include proofs of Heron’s process/formula for finding the area of a triangle. Significant for our question, and reflecting the fact that the Metrica was not known directly, none of them attributes this to Heron either. One of the earliest that does is the groundbreaking De Practica Geometrie of Leonardo of Pisa (“Fibonacci”) (ca. 1170–ca. 1250). The later Summa de arithmetica geometria proportioni et proportionalit`a by Luca Pacioli (1447–1517) does as well and Pacioli’s treatment is virtually a copy of what Fibonacci says (although written in the Tuscan dialect of Italian rather than Latin).

Marshall Clagett has written that Leonardo “borrows heavily and often in verbatim fashion” in the revised version of this work from the Verba filiorum Moysi filii Sekir.

\[24\]

I have consulted \[20\], a scanned version of the 1523 edition of Pacioli’s book at www.e-rara.ch.
This work, also known as the Liber trium fratrum de geometria, is a Latin translation of an Arabic work on mensuration by the 9th century Banū Mūsā brothers made by Gerard of Cremona (1114–1187). The original authors were key figures in the early translation movement by which Greek mathematics became known in the Islamic world and the Greek original of the Metrica may have been available in Baghdad at this time. But there are significant differences between the Metrica version and the Verba filiorum version. Figure 2 shows what the diagram in the Verba filiorum looks like for our triangle: After identifying the points $D$, $Z$, $U$ as the points of tangency of the inscribed circle, the side $AB$ is extended to $AH$ by making $BH = GU$, so $AH$ is equal to the semiperimeter. Similarly $AG$ is extended to $AK$ making $GK = BU = BZ$ and angle $AKT$ is a right angle. The point $T$ is chosen so that it lies on the angle bisector at $A$. It follows that triangle $EBU$ is similar to triangle $BTH$ and the proportionality of corresponding sides implies the square of the area of the triangle $ABΓ$ is equal to the square on the perpendicular $EU = EZ$ times to the square on the semiperimeter $AH$.

Clagett mentions that he believes Fibonacci’s debt to the Banū Mūsā applies specifically to the treatment of Heron’s formula in Fibonacci’s work. However, a close analysis of the argument and the diagrams provided shows that while the proof of Heron’s formula in Proposition VII of the Verba filiorum, and reproduced in Clagett’s book, has many features in common with Fibonacci’s proof, it also has other features in common with the proof from Heron’s Metrica that do not occur in Fibonacci. Probably the major example here is that the whole first phase of the argument in both the Metrica proof and in the Verba filiorum proof consists of considering the inscribed circle in the triangle (as in Proposition 4 from Book IV of Euclid). Fibonacci does not mention

\footnote{See [4, p. 224].}

\footnote{See [4, p. 224].}
the inscribed circle; in fact he ends up repeating a large portion of the Euclidean proof to show that if perpendiculars (or as Fibonacci says, “cathetes”) are dropped to the three sides from the intersection point of two angle bisectors in the triangle, then the three perpendicular segments are equal. There are also some less drastic differences in the way that similar triangles within the figure are used to deduce that the square of the area is equal to the square on the perpendicular above times the square on the semiperimeter (and the square on the semiperimeter equals the semiperimeter times the product of the three excesses of the semiperimeter over the sides). So it is surely not entirely accurate to characterize Fibonacci’s proof (at least as a whole) as “verbatim borrowing” even if the overall strategies of the proofs are similar and the final sections of the proofs do more or less converge.

On the other hand, Clavius’s version of the proof of Heron’s formula is different again, but significantly closer to the proof in Fibonacci than it is to the proof in the Verba filiorum. To discuss this in more detail it will be necessary to consider the diagrams from these two proofs. (See Figures 3 and 4.) These two figures show Clavius’s and Fibonacci’s constructions applied to the same particular triangle as in the previous figures. Both Clavius and Fibonacci start by considering angle bisectors (these are two of the dashed black lines) for the two vertices on the horizontal side in the diagram and their intersection point ($t$, and $D$, respectively). They both drop perpendiculars ($th$, $tz$, $te$, and $DG$, $DE$, $DF$ resp.) and use facts about congruent triangles in the figure to show that the three perpendiculars are equal, that $at$ (resp. $AD$) also bisects that

27There is an unfortunate mistranslation at the start of the proof of Heron’s formula in [22]. At the start of the first full paragraph on p. 81, the Hughes translation says, “To prove this: in triangle $abg$ bisect the two equal angles $abg$ and $agb$ ... ” This would make the proof apply only to isosceles triangles. But that is not correct. The Latin text in the 19th century version edited by B. Boncampagni, [21, p. 40] says at this point: “Ad cuius rei demonstrationem adiaceat trigonum $abg$: et dividantur in duo equa anguli, qui sub $abg$, et $agb$, a rectis $bt$, et $tg$ $...$ ” That is, “To prove this: in the triangle $abg$, let the angles $abg$ and $agb$ [each] be divided into two equal angles by the lines $bt$ and $tg$ $...$ ” Fibonacci is definitely not restricting his discussion to isosceles triangles.

28The difference in the diagrams is also mentioned in [22]. See the footnote on p. 83.
angle and, moreover, the two segments closest to each vertex are equal—that is $ae = az$, $be = bh$, $gz = gh$ (resp. $AE = EF$, $CF = CG$, and $BE = BG$). Neither mentions the inscribed circle, which would be tangent to the sides of the triangle in the points $c, z, h$ (resp. $E, F, G$). This implies that any one of the sides, together with one of the equal segments not meeting that side are together equal to the semiperimeter of the triangle – for example, side $ab$ (resp. $AC$) together with $gh$ or $gz$ (resp. $BG$ or $BE$). In addition, the three excesses of the semiperimeter over the sides of the triangle that feature in Heron’s formula coincide with the segments: $ae$ or $az$, $be$ or $bh$, $gh$ or $gz$ (resp. $AE$ or $AF$, $BG$ or $BE$, $CG$ or $CF$).

Then, in further parallel constructions, the sides $ag$, $ab$ (resp. $AB$ and $AC$) are extended to $am$, $bl$ (resp. $BH$, $AI$) by making $gm = hb$ and $bl = gh$ (resp. $BH = GC$ and $CI = BG$). As noted before, this makes both $am$ and $al$ (resp. $AH$ and $AI$) equal to the semiperimeter of the triangle, hence equal$^{29}$ At this point, Fibonacci says to produce the third angle bisector at until it meets the segment $lk$ making a right angle with $ab$ at $k$. Clavius, on the other hand (literally!), says to produce $AD$ to $K$ where it meets the line through $H$ perpendicular to $AH$. But either way, the next deduction is that by the SAS criterion, the triangles $amk$ and $alk$ (resp. $AHK$ and $AIK$) are congruent, so angle $amk$ (resp. $AIK$) is also a right angle, and moreover $mk = lk$ (resp. $HK = IK$).

In the final constructions, Fibonacci says to cut off the segment $bn$ from $gb$ so that $gn = gm = bh$, and hence $bn = bl = gh$. Clavius does the parallel operations making $BL = BH = CG$ and hence $CI = CI = BG$. But now Clavius does one further step that Fibonacci does not: He extends $AH$ to $AM$, making $HM = CL = CI$. With $k$, resp. $K$ joined to all of the newly constructed points, both proofs proceed to show that the lines $kn$ (resp. $KL$) meet the horizontal side in a right angle. The additional triangle $HMK$ introduced in Clavius’s argument is congruent to triangles $CIK$ and $CLK$ (resp. $bnk$, $blk$ in Fibonacci’s figure), and hence it is somewhat redundant. But what we have here would seem to be a typical kind of procedure for Clavius; at the cost of a few more steps, he furnishes a reader of his proof with another triangle $HKM$ that gives a perhaps easier way to understand why the angle at $n$ or $L$ is a right angle (this is not really clear visually in Clavius’s original diagram, where it seems no attempt has been made to show all the right angles accurately).

Finally, as in all of the proofs of Heron’s formula we have discussed, similar triangles can be identified in the figure such that the proportionality of corresponding sides implies that the square of the perpendicular (e.g. $DE$ in Clavius’s figure) times the semiperimeter (e.g. $AH$ in Clavius’s figure) is equal to the product of the three excesses of the semiperimeter over the sides, or $(DE)^2(AH) = (EB)(BH)(AE)$. Hence the square of the perpendicular times the square of the semiperimeter, that is, $(DE)^2(AH)^2$, equals $(AH)(EB)(BH)(AE)$. With some rearrangement of factors, this equals $s(s - a)(s - b)(s - c)$ in the modern algebraic form of Heron’s formula. Fibonacci uses the triangles $ebt$ and $kbl$; Clavius uses the triangles $AED$ and $AHK$. The details in this step are somewhat different, but the idea is analogous.

The final step in both of these proofs is to note that by the usual “one half base times height” way of computing the areas of triangles, the sum of the areas of the three triangles $atb$, $btg$, $gta$ (resp. $ADC$, $CDB$, $BDA$), which equals the area of the whole triangle, is also equal to the product of any one of the three equal perpendiculars and

$^{29}$In the diagrams here, in fact, they are two sides of a square, but that is only true because our triangles have right angles at $a$, $A$.\
the semiperimeter. Since Clavius has not proved the basic method for computing areas of triangles yet in the *Geometria Practica*, he has to give a forward reference to the next section in Chapter 2 of his Book IV. Fibonacci puts the discussion of Heron’s formula after his treatment of the other method, so he has set up what he needs already.

To conclude this section, we can say that Clavius did not attribute this result to Heron because in his time it was simply not generally known that an ancient source for this result was the *Metrica* of Heron. On the other hand, let us offer the conjecture that here, as was often the case, Clavius reworked and amplified what he found in other sources so that his version has additional or alternate features intended to heighten clarity or to increase convenience for his readers. Here it seems very probable that he was looking at Fibonacci’s proof (or perhaps other proofs derived from that one, such as the proof in Paccioli’s text) but his version is not a verbatim copy, any more than Fibonacci’s was a verbatim copy of the proof in the *Verba filiorum*. Why Clavius chose not to say this explicitly at this point in his book is still somewhat mysterious, however. In analogous situations, Clavius did sometimes say explicitly how his account of a proof would differ from what was found in his source(s).

3. **Clavius’s treatment of Archimedes’ “Measurement of the Circle” in Book IV.**

After Greek versions of this work of Archimedes (including summaries from Book V of the *Mathematical Collection* of Pappus and the commentary on Ptolemy’s *Almagest* by Theon) were intensively studied in the Islamic world and the resulting Arabic translations were retranslated into Latin, the *Measurement of the Circle* was surely the best-known and most-copied Archimedean text throughout the medieval period in western Europe. A major part of the reason for this was certainly the utility of the results of this work for practical questions. On the other hand, the brevity of the work and its somewhat sketchy form have led Dijksterhuis to conjecture that “it is quite possible that the fragment we possess formed part of a larger work,” and Knorr to judge that the versions we have represent “at best an extract from the original composition.” In *Clagett* reproduces two translations of this work from Arabic into Latin, the first made (“perhaps”) by Plato of Tivoli (fl. 12th century), and the second made by the same Gerard of Cremona mentioned in the previous section. Clagett also reproduces six additional “emended” versions as well as the treatment of the results of this work in the *Verba Filiorum*, following the Banū Mūsā. Part III of Knorr, *Part III of Knorr*, contains a more complete study of the transmission including additional versions and reflecting more recent scholarship. Here we will simply say that the common elements of most of these are three propositions stated in this order and in something like these forms:

**Proposition 1** (Archimedes, *Measurement of the Circle*, 1). *Every circle is equal [in area] to a right triangle, one of whose sides containing the right angle is equal to the radius of the circle while the other side containing the right angle is equal to the circumference.*

**Proposition 2** (Archimedes, *Measurement of the Circle*, 2). *The ratio of the area of any circle to the square of its diameter is the ratio 11 to 14.*

---

30For example, he was very explicit about this in the introduction to Book VI on *geodesy*.
31See [p. 222].
32See [p. 375].
Proposition 3 (Archimedes, *Measurement of the Circle*, 3). The circumference of a circle exceeds three times its diameter by a quantity less than $\frac{1}{7}$ of the diameter and greater than $\frac{10}{71}$ of the diameter.\(^{33}\)

It is interesting to compare Clavius’s treatment of these results to what authors of other practical geometry texts say. One of the earliest, the *Practica Geometricae* attributed to Hugh of St. Victor (ca. 1096–1141), \(^{12}\) does not mention this topic at all. In his *De Practica Geometrie*, Fibonacci addresses the content of all three propositions in turn.\(^{34}\) However, Fibonacci does not really attempt to present the full proof for Proposition 1 that is found in other versions. Instead, what he says is as follows. First, considering a regular polygon circumscribed about the circle, Fibonacci argues that the product of the radius and the perimeter of the polygon is greater than the area of the circle by considering the triangles formed by joining the center and the vertices of the polygon. For example, the area of the pentagon in Figure 5 will be

\[
5 \cdot \frac{1}{2} \cdot (AB)(OC),
\]

which is greater than the area of the circle. Hence the product of the radius and a number greater than half the circumference of the circle gives an area greater than the circle. Fibonacci apparently takes it as obvious that the perimeter of the circumscribed polygon is greater than the circumference of the circle.\(^{35}\)

Next, by a clever observation, Fibonacci considers an inscribed \(n\)-gon and adds vertices bisecting the arcs between successive vertices of the \(n\)-gon to form an inscribed regular \(2n\)-gon. Fibonacci notes that the product of the radius and half the perimeter of the \(n\)-gon is equal to the area of the \(2n\)-gon, hence less than the area of the circle. This follows since, for instance, the product $\frac{1}{2}(OC)(AB)$ in Figure 6 is equal to the sum of the areas of the triangles $OAB$ and $ABC$. Hence four times this will equal the

\[5 \cdot \frac{1}{2} \cdot (AB)(OC)\]

That is, in modern terms, $3 \frac{22}{7} < \pi < 3 \frac{1}{7}$. Archimedes may well have used methods similar to the ones to be discussed to produce tighter estimates for the ratio of the circumference to the diameter. But if so, no text doing this has survived.

\[\]

\(^{33}\text{That is, in modern terms, }3 \frac{22}{7} < \pi < 3 \frac{1}{7}.\text{ Archimedes may well have used methods similar to the ones to be discussed to produce tighter estimates for the ratio of the circumference to the diameter. But if so, no text doing this has survived.}\]

\(^{34}\text{See }22\text{ pp. 152–158, paragraphs [191]–[200] of Chapter 3.}\]

\(^{35}\text{Clavius returns to this point in Book VIII of the *Practical Geometry*, discussing arguments by Archimedes from *On the Sphere and Cylinder*, and an alternate treatment by Girolamo Cardano.}\]
area of the octagon drawn with dotted lines. Similar arguments apply for all \( n \geq 4 \).
Therefore multiplying the radius by a number less than half the circumference of the circle gives an area less than the area of the circle. “Whence it is concluded that the product of the radius of the circle and half its circumference equals its area.”

Although it is certainly intuitively clear that the areas of the inscribed and circumscribed polygons converge to the area of the circle as \( n \to \infty \), it seems fair to characterize what Fibonacci says as more of a plausibility argument than a complete proof because he has not shown that the difference between a circumscribed polygon and an inscribed polygon can be made arbitrarily small.

Second, Fibonacci argues by way of Euclid XII.2 that the ratio of the square of the diameter to the area is the same for all circles. He essentially then does a “proof by numerical example” for Proposition 2, using the result of Proposition 3 and effectively taking \( \pi = \frac{22}{7} \). There is no indication that this ratio is only an approximation and that no actual circle has the square of the diameter exactly equal to 196 and area exactly 154.

Finally, Fibonacci turns to the Archimedean estimates \( 3 \frac{10}{71} < \pi < 3 \frac{1}{7} \) (and here mentions Archimedes explicitly for the first time). He says that he is not going to follow Archimedes’ proof exactly because smaller numbers will suffice to make the point. Interestingly, Fibonacci includes more about the individual details of the calculations paralleling the proof of Proposition 3 than most of the versions of the *Measurement of the Circle* do.
As was true for the discussion of Heron’s formula considered in the previous section, Luca Pacioli’s discussion of the results of the Measurement of the Circle follows Fibonacci very closely, even with virtually identical diagrams. Other practical geometry texts after Clavius’s time tended to include the Archimedean estimates and discuss how to compute areas of circles and parts of circles, but to omit proofs for these facts entirely.

By contrast with Fibonacci or Pacioli, Clavius explains what he aims to do in this introductory paragraph:

It will not be a digression, therefore, if I include [Archimedes’] truly most acute and precise book, partly because it is very brief (indeed, it consists of only three propositions), partly so that the student, in order to understand something so useful and so widely applied in the works of all authors, should not be forced to go to Archimedes himself, and finally mostly because the writings of Archimedes, as a result of their brevity, are somewhat obscure, and we hope to bring some light to them.

Clavius starts with an account of the rather subtle exhaustion proof of Proposition 1 found in most of the versions of the Measurement of the Circle mentioned above. It must be proved that the area of a circle is the same as the area of a right triangle with the two sides about the right angle equal to the radius of the circle and the circumference of the circle. This proof has much in common with the proof of Proposition 2 in Book XII of Euclid. The plan is to show that assuming the area of the circle is either greater than or less than the area of a right triangle as in the statement leads to a contradiction. A key role will be played by a statement introduced by Euclid in the proof of Proposition 1 in Book X of the Elements. Clavius uses this in the form: If an area at least half the area of the circle is taken away from the circle, and from the residual area again an area at least half of that remaining area is taken away, and so on, a there will eventually remain an area less than any positive magnitude $z$.

First suppose the circle is larger than the stated triangle by a certain positive magnitude (Clavius calls this $z$). Let a sequence of non-overlapping areas be removed from the interior of the circle. Specifically, the inscribed square is removed at the first step, then four triangles on the sides of the square inscribed in the circle, so at this point a regular octagon has been removed, then eight triangles on the sides of the octagon, so the total removed figure is a regular 16-gon, etc. Clavius actually stops with the octagon so he has a diagram that is virtually identical with the ones presented by...
Fibonacci and Pacioli.\footnote{See Figure 7. Many of Clavius’s point labels and additional lines constructed in Clavius’s very “busy” diagram have been omitted at this stage for clarity.} Note that the right triangle $EAB$ is half of the square with $E, A, B$ as three vertices. Four of these triangles make up the inscribed square in the circle and four of those small squares make up a square circumscribed about the circle. So the inscribed square is half of the circumscribed square and hence more than half of the circle. Similarly, the four triangles on sides of the inscribed square (four triangles congruent to $ABO$) are half the rectangles with bases equal to the sides of the inscribed square and heights equal to the perpendicular segments such as $OT$. Hence removing those four triangles removes more than half of the area between the inscribed square and the circle. It is not hard to show that this pattern continues indefinitely. So eventually the remaining region between a $2^m$-gon and the circle will have area less than the positive magnitude $z$. However, this leads to a contradiction if the construction is applied sufficiently many times. The area of the circle is supposed to equal (area of triangle) $+$ $z$, but the area of the circle also equals:

\[
\text{(area of inscribed polygon)} + \text{(remaining area)} < \text{(area of triangle)} + z,
\]

since the perimeter of the polygon is smaller than the circumference of the circle, and the apothem—the perpendicular from the center to the side of the inscribed polygon—is less than the radius, and hence the area of the inscribed polygon is less than the area of the triangle. Hence the area of the circle cannot be greater than the area of the triangle.

Now suppose that the circle is less than the stated triangle by a certain magnitude (again denoted $z$). Clavius and Archimedes now start from the square circumscribed about the circle (whose area is definitely greater than the area of the triangle since the perimeter of the square is greater than the circumference of the circle, and the apothem is the same as the radius). Clavius begins removing areas from the square: first the circle, then four exterior triangles (such as $KXV$ in Figure 8) with base tangent to the circle, then again eight exterior triangles with base tangent to the circle at the

\[\text{Figure 7. The essential portions of Clavius’s diagram for the proof of the first part of Proposition 1.}\]
midpoints of the arcs $BO$, $OV$, and so forth. The remaining regions (after the very first step when the circle is removed) now are collections of what Clavius calls "mixed triangles," with one side an arc of the circle. Again since the inscribed square in the circle is half of the circumscribed circle, after the circle has been removed, less than half of the circumscribed square remains. Similarly, each of the four exterior triangles such as $KXV$ is greater than half of the area between the circle and the lines $BK$, $KA$, and so forth. Since more than half the remaining area is removed at each step, eventually the remaining area becomes less than $z$. But this also leads to a contradiction. On the one hand

\[(\text{area of circumscribed polygon}) > (\text{area of triangle})\]

because the perimeter of the polygon is greater than the circumference and the apothems are now all equal to the radius of the circle. But on the other hand, by the process described above,

\[(\text{area of circumscribed polygon}) < (\text{area of circle}) + z = (\text{area of triangle}).\]

Hence, since the triangle in the statement of the proposition is neither greater nor less than the circle, it can only equal the circle. Clavius’s version of this proof incorporates many explanatory comments and justifications for the individual steps in the reasoning not found in other versions.

An amusing sidelight in the form of a long Scholium follows, in which Clavius refutes Joseph Justus Scaliger’s claim that Archimedes must have been mistaken in this proof.

43Scaliger (1540–1609) was an eminent French Protestant classical philologist and historian who also fancied himself a mathematician. He managed to convince himself both that the area of a circle is $\frac{4}{3}$ times the area of the inscribed regular hexagon, a statement that is equivalent to $\pi = \frac{24\sqrt{3}}{3}$ and also that the area of a square with side equal to the circumference of a unit circle is ten times the area of the square on the diameter, a statement equivalent to $\pi = \sqrt{10}$. These are not consistent with each other and neither is consistent with the Archimedean estimates $3\frac{1}{7} < \pi < 3\frac{10}{7}$. Moreover, either one would imply that the circle can be squared with straightedge and compass, which we know today is impossible. Clavius suggests these misunderstandings as one possible reason for Scaliger’s claim that the Archimedean argument cannot be correct. Scaliger’s faulty conclusions were included in
Indeed, Clavius takes Scaliger to task rather savagely over his misunderstandings. A small sample:

And I am really astonished that you, **Mathematicus** that you are, deny that some quantity is equal to another when it is neither greater nor less. For if it is not equal to the other, then it will be unequal to the other, therefore either greater or less, against that hypothesis. Or don’t you see that not only Archimedes but also Euclid used this way of arguing most frequently in Book XII of the *Elements* ([7, p. 185])

Scaliger had had another “run-in” with Clavius over the Gregorian calendar reform in which Clavius had taken a leading role, as well as ongoing controversies with other Jesuits on various subjects, so there was ample bad blood between them. Some of that is manifest in the scathing polemical tone of Clavius’s comments.

Following this, Clavius notes that the usual Proposition 3 is used in the proof of Proposition 2, and hence he has decided to reverse the order of Propositions 2 and 3 as found in other versions to maintain the chain of logical implications. In his account of this famous proof, Clavius essentially follows the plan used in most other versions. There are again two steps, the first (in effect) considering polygons circumscribed about the circle and triangles with one side along a tangent to the circle, the second (in effect) considering polygons inscribed in the circle and triangles inscribed a semicircle. In each phase, an angle is repeatedly bisected, until one side in the triangle comes from a regular 96-gon. The Pythagorean theorem is applied repeatedly to estimate ratios and the proof consists essentially of a complicated series of numerical calculations, quite different from many Greek geometric arguments.

Two aspects of Clavius’s version of the proof of his Proposition 2 (= usual Proposition 3), as compared with other versions, are notable. First, Clavius provides more details, more justification for individual steps, and a fuller treatment of the calculations leading to the estimates than Archimedes (or whoever wrote the versions of the Archimedean text that we have) did. Since the bisection steps in each half of the proof follow exactly the same plan, some versions work out the first step in detail, and then just present the numerical results for the subsequent steps. This is very much along the lines of how he treats information from his sources in other sections of the *Geometria Practica*. The second aspect is probably somewhat less mathematically significant, but still interesting. Namely, by showing the bisections from the first phase of the proof to the left of a vertical diameter in the circle, and the bisections from the second phase to the right, Clavius manages to condense all the steps of the constructions for both

---

his mathematical *magnum opus*, grandly titled *Cyclometrica Elementa*, published in 1594 in a lavish edition with statements of theorems in both Latin and ancient Greek. In 1609 Clavius published an 84-page pamphlet *Refutatio Cyclometriae Iosephi Scaligeri* (Refutation of the *Cyclometrica* of Joseph Scaliger), giving a blow-by-blow analysis of all of the (numerous) errors in Scaliger’s work. This is contained as an appendix in Volume V of the *Opera Mathematica*.

44Et sane miror, te, Mathematicus, cum sis, negare quantitatem aliquam illi esse aequalem, qua neque maior est, neque minor. Si enim aequalis non est, erit inaequalis. Igitur vel maior vel minor, contra hypothesim cum dicatur neque maior esse, neque minor. An non vides, non solum Archimedem, sed etiam Euclidem lib. 12. hunc argumentandi modum frequentissimé vsurpare?

45Clavius writes: Haec est Archimedis propositio 3. quam nos secundum facimus, vt doctrinae ordo servetur, quando quidem sequens propositio 3. quam ipse 2. facit, hanc nostram propositionem 2. in demonstrationem adhibet.

46The author used this expedient, in fact, in his translation of Clavius’s proof because the repetitive structure of the original makes it very tedious reading!
phases of the proof into a single diagram in a clever, but still entirely understandable way. Most versions provide separate diagrams for each phase.

Another quite significant deviation from sources such as Fibonacci is that in his Proposition 3 (= usual Proposition 2), Clavius explicitly adds the qualifier *approximately* (proximè in the Latin) to the usual statement that the ratio of a circle to the square on the diameter is the ratio 11 to 14. As we noted above, Fibonacci did not do this in his version of this part of the Archimedean text. Clavius’s proof is essentially the same, though, using the approximation $\pi \doteq 22/7$.

Finally, it is interesting to note that after his account of Archimedes’ results, Clavius also includes closer approximations to $\pi$ later in Book IV, quoting results of his contemporary Ludolph van Ceulen (1540–1610), and his student, Jesuit colleague, and successor as professor of Mathematics at the *Collegio Romano*, Christoph Grienberger (1561–1636). Clavius states the equivalent of the bounds

$$3 \frac{14159265358979323846}{100000000000000000000} < \pi < 3 \frac{14159265358979323847}{100000000000000000000}$$

and tries to give a practical “spin” for how these might be useful. If one of these estimates is used (for instance, the upper bound to parallel the $22/7$ value), then

... the area of the circle will differ less from the true value than the area found from the Archimedean ratio. But since it is more difficult to compute with large numbers than with small ones, in practice the Archimedean ratio is applied. However, when more accurate values are desired, the Ludolphine ratio above should be used, especially for large circles.

Even today, though, it is difficult to imagine where 20 decimal place accuracy might really be needed(!)

4. **Clavius’s Discussion of Methods for Finding Two Mean Proportionals between Two Given Lines in Book VI.**

In this section and the next, we will discuss two connected topics from Book VI of the *Geometrica Practica*. The first is the extended solution of the following Proposition 15 from pages 266–272:

**Proposition 4. To find two mean proportionals between two given lines approximately.**

This, the closely connected problem of duplicating the cube, plus the problems of squaring the circle and trisecting an arbitrary angle (which are discussed by Clavius

in Books VII and VIII), were tremendous stimuli to Greek geometry over hundreds of years.\footnote{For an extensive modern study of the surviving sources and the historical development, see \cite{17}.} As Clavius says,

We will first report what the ancient geometers have left to us in their writings concerning this problem. For this drove and tormented the talents of many, although up to this day, no one will truly and geometrically have constructed two mean proportionals between two given lines.\footnote{Quocirca prius in hac propos. in medium afferemus, quae antiqui Geometrae nobis haec de rescripta relinquerunt. Multorum enim ingenia res haec exercuit, atque torsit, quamuis nemo ad hanc vsque diem, verè, ac Geometricè duas medias proportionales inter duas rectas datas inuenerit.}\footnote{This is translated together with the Archimedean text in \cite{19}.}

Hippocrates of Chios (ca. 470–ca. 410 BCE) was traditionally credited with the reduction of the problem of duplicating the cube to the problem of constructing two mean proportionals between given lines. If $AB$ and $CD$ are the lines, two other lines $XY$ and $ZW$ are said to be two mean proportionals (in continued proportion) if

$$AB : XY :: XY : ZW \quad \text{and} \quad XY : ZW :: ZW : CD.$$ 

Representing the lengths by numbers and using algebra, this becomes the string of equations

$$\frac{AB}{XY} = \frac{XY}{ZW} = \frac{ZW}{CD},$$

from which it follows that

$$\left(\frac{ZW}{CD}\right)^3 = \frac{AB}{CD}.$$

So for instance if $CD = 1$ and $AB = 2$ in some units, a construction of the two mean proportionals gives the line $ZW$ which has length $\sqrt[3]{2}$, and that is the edge length of the cube with twice the volume of the cube with edge length $CD = 1$.

Clavius has included what might seem to be a surprising amount of the Greek work on this construction problem in the \textit{Geometria Practica}. Although the works of the authors involved did not survive from antiquity in their original forms, they were summarized and hence preserved in the commentary on Archimedes’ \textit{On the Sphere and Cylinder} by Eutocius of Ascalon (ca. 480–ca. 540 CE)\footnote{The same is true for the \textit{quadratrix} curve of Hippias that Clavius studies extensively in Book VII.} This is Clavius’s stated primary source for this material although he may also have consulted Book III of the \textit{Mathematical Collection} of Pappus of Alexandria (ca. 290–ca. 350 CE) where some of the same methods are surveyed.

Note that Clavius explicitly says “approximately” in the statement of the problem (in the Latin: \textit{prope verum}, literally “near the truth”). This feature might seem curious for modern readers and it surely deserves some elaboration. Clavius makes this qualification because the solutions he will present all involve either limiting operations relying on the senses of the geometry (so-called \textit{neusis} ($\nu\varepsilon\sigma\varsigma$) constructions) or the use of auxiliary curves such as \textit{cissoid} of Diocles or the \textit{conchoid} of Nicomedes that cannot be drawn as a whole using only the straightedge and compass.\footnote{In his well-known methodological discussions of different solutions of construction problems from Books III, IV, and VII of the \textit{Mathematical Collection}, Pappus would say they are not \textit{planar} solutions.} Because these solutions use more than the traditional Euclidean tools, they don’t qualify as what Clavius means by “geometric” or exact solutions.\footnote{56}
It is understood today that no such purely “geometric” solutions are possible for the three problems mentioned above and it is primarily this methodological question—are the three problems solvable under the most severe restriction to the use of only the Euclidean tools?—that has survived in modern discussions. For instance, many undergraduate algebra courses discuss these problems via coordinate geometry and the characterization of points constructible with straightedge and compass as those whose coordinates lie in a field at the end of a tower of quadratic extensions starting with the rational numbers. The fact that the index $\left[ \mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q} \right]$ is equal to 3 shows that it is not possible to duplicate the cube with straightedge and compass, and that is often the end of the story.

But this is a very modern and “pure mathematical” way of looking at things. For Clavius, as for at least some of the Greeks before him, although the methodological question might be interesting, it was also important to find some reasonably accurate method for constructing the two mean proportionals even if it meant using approximate methods rather than an exact, “geometrical” solution. Perhaps surprisingly, this is actually a very practical problem that had important applications in architecture, military science and many of the other areas Clavius mentions in his Preface. It gives a method for determining the linear dimensions of a solid figure similar to a given figure whose volume has a given ratio to the volume of the given figure. Just as a procedure for finding one mean proportional lets one rescale a plane figure in a given ratio, a solution for this problem lets one rescale solid figures in any given ratio, and Clavius points this out explicitly a number of pages later, after Proposition 17 in the same Book VI:

> This establishes the method by which a cube is not only to be duplicated (which the ancients were seeking), but also increased or decreased in any given ratio. It also gives the method by which bores of cannons are to be made larger or smaller according to a given ratio.\footnote{Constat ex his, qua ratione Cubus non solum duplicandus sit (quod veteres inquirebant) sed etiam augendus minuendusque in quacunque proportione: Item quo facto pylae bombardarum maiores, aut minores fieri debeant secundum proportionem datam. In this connection we also point out the first part of the heading of the first method Clavius presents–Method of Heron in the introduction to the Mechanics and Making of Missile-throwing Machines(1)}

We note that Fibonacci also discusses methods for finding two mean proportionals in his De Practica Geometrie.\footnote{See [22, Chapter 5], paragraphs [12]-[15].} We will return to this point shortly and compare his approach with Clavius’s approach.

In introducing his discussion, Clavius says he is making a very deliberate choice from among the many solutions presented in Eutocius’s commentary:

> Although they are most elegant and acute, the solutions of Eratosthenes, Plato, Pappus of Alexandria, Sporus, Menacchmus by means of the hyperbola and parabola, then with the help of two parabolas, and Archytas of Tarentum will be omitted and we will explain only the four solutions from Heron and Apollonius of Perga, Philo of Byzantium and Philoponus, Diocles, and Nicomedes. \textit{We have judged these to be more useful, easier, and less prone to error.} Anyone who should want the other methods will be able to read them in the commentary of Eutocius of Ascalon in the second book of On the Sphere and Cylinder of...
Archimedes, and in the book of Johannes Werner of Nuremburg on the conic sections\textsuperscript{[59]} (\textit{I}, p. 266); emphasis added)

In other words, the methods discussed here are sufficient for the applications Clavius has in mind and they are the ones he thinks are easiest and best suited for practical implementation.

By way of contrast, Fibonacci makes a different selection and presents only the methods ascribed to Archytas, Philo, and Plato by Eutocius. Hence there is very little overlap between his account and Clavius’s account. Moreover, he presents the method of Archytas (which relies on some quite involved solid geometry) first, after saying that finding the two mean proportionals “... is not a thing that can be done easily, but this is how it must be done.”\textsuperscript{[60]}

Turning now to the details of Clavius’s account, the first method presented actually combines two very closely related approaches, ascribed to Heron and Apollonius and discussed separately by Eutocius. Clavius’s version is a very close copy of Eutocius’s text for Heron’s method, with the variation represented by Apollonius’s method inserted at one point. In Figure 9, suppose we wish to find two mean proportionals between the lines $AB$ and $BC$, which have been arranged as two sides of the rectangle $ABCD$. For Heron’s method, Clavius says

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{ClaviusDiagram.png}
\caption{The essential portions of Clavius’s diagram for the first two methods. For Heron’s method and Philo’s method, he dashed line $GBOF$ would need to be rotated about $B$ to reach the final desired position with $EG = EF$ or $OF = BG$.}
\end{figure}

\textsuperscript{\textsuperscript{59}}German mathematician, 1468–1522.

\textsuperscript{\textsuperscript{60}}\textit{Praetermissis autem modis Eratosthenis; Platonis; Pappi Alexandrini; Spori; menechmi tum beneficio Hyperbolae, ac parabolae, tum ope duarum parabolae; & Architae Tarentini, quamuis acutissimis, subtilissimisque; solum quatuor ab Herone, Apollonio Pergaeo, Philone Bysantio, Philoppono, Diocle, & Nicomede traditos explicabimus, quos commodiores, faciioresque, & errori minus obnoxios iudicauimus. Qui aliorum rationes desiderat, legere eas poterit in Commentariis Eutocij Ascalonitae in librum 2. Archimedis de Sphaera, & Cylindro: Item in libello Ioannis Vernerii Norimbergensis de sectionibus Conicis.}

\textsuperscript{\textsuperscript{61}}... hoc facili operari non possit, tamen, qualiter hoc fieri debat. $[21]$ p. 153.
With sides $DA$, $DC$ extended, it is understood that a straightedge [represented by the dashed line in the figure] placed at $B$ should be moved until it meets $DA$, $DC$, produced, in points $F$ and $G$ such that the lines $EF$ and $EG$ are equal.

When this is true, consideration of the various similar triangles in the figure shows that $AF$ and $CG$ are the two desired mean proportionals between $AB$ and $BC$. Apollonius’s variation of this method consists of finding a circle with center at $E$ which has a chord $GF$ passing through $B$, and hence $EF = EG$ again. Clavius includes a brief description of a trial-and-error method for finding the required circle not found in Eutocius.

The second method, ascribed to Philo and Philoponus, has been reworked and greatly simplified by Clavius based on the realization that it is again very closely related to the first one (in fact Clavius sets up the discussion so that the same diagram applies). Namely, with the circle $DABC$ described with center $E$ and radius $EA = EB = EC = ED$, the ruler at $B$ (that is, the dashed line in the figure) is moved until $BG = OF$, where $O$ is the second intersection with the circle above. Then it is easy to see we are back in exactly the same configuration as in the other methods, so the same reasoning applies to give the two mean proportionals. In our opinion, this family of methods would certainly be among the easiest to apply. They would probably be the most accurate as well. Given a sufficiently accurate diagram of the rectangle $ABCD$ and its diagonals, rotating a ruler passing through $B$, or using a compass to draw various test circles with center at $E$ would certainly give satisfactory results if they were performed with sufficient care: the chances of making large errors would be extremely small since it is so clear what is required.

Note that the geometer is required to rotate the line through $B$ or adjust the radius of a circle centered at $E$ until a certain condition is satisfied. As presented by Clavius, this involves approximation processes making use of the senses of the geometer, as we said earlier. The next two methods will be somewhat different in that they are set up to make use of auxiliary curves whose description (that is, the description of the whole curve and not just a finite set of points on the curve) requires tools besides the straightedge and compass.

The next method Clavius discusses is ascribed Eutocius to Diocles (ca. 240–ca. 180 BCE), and specifically to a book called On Burning Mirrors. The Greek original has not survived so this was known only from fragments preserved in other texts like Eutocius’s commentary. But an Arabic translation of the whole has survived and this has now been translated into English by G. J. Toomer. Clavius covers essentially the same ground as in the corresponding section from Eutocius’s commentary. However, as usual, he has reworked and augmented his source material significantly. Clavius begins by separating off what he calls the “Lemma of Diocles,” which identifies a geometric configuration containing two mean proportionals between given lines. (See Figure 10. An equivalent figure with Greek letter labels appears in Eutocius.) Let $AC$ and $BD$ be diameters of the circle meeting at right angles at $E$. Let arcs $DG$ and $DF$ be equal and join $CG$. Let $GL$ and $KF$ be drawn parallel to $BD$. Let $CG$ meet $KF$ at $H$. Then by considering relationships of the lines in the figure, Clavius essentially follows the proof given by Eutocius to show that $FK$ and $KC$ are two mean proportionals between $AK$ and $KH$. Similarly, if the arcs $DM$ and $DN$ are equal, then drawing $CM$ cutting the

\[62\text{Protractis autem lateribus, } DA, DC, \text{ intelligatur circa punctum } B, \text{ moueri regula hinc inde, donec ita secet } DA, DC \text{ productas in } F, & G, \text{ vt rectae emissae } EF, EG, \text{ aequales sint.}\]
vertical line $PN$ in $O$, it follows that $NP$ and $PC$ are two mean proportionals between $AP$ and $PO$.

Now given two lines $AB > BC$, we can apply the “Lemma of Diocles” in the following way: First construct a circle with radius $AB$ and lay off $BC$ along a perpendicular as in Figure 11. Provided that we can find a point $H$ and the vertical segment $KH$ (parallel to $EF$) so that the intersection $L$ of the extended line $AC$ and $KH$ makes the arcs $EH$ and $EM$ (formed by the line through $D$ and $L$) equal, then the “Lemma of Diocles” will imply that $KH$ and $DK$ are two mean proportionals between $AK$ and $KL$. But the triangles $ABC$ and $AKL$ are similar, and hence we can rescale all four lines by the ratio $AB : AK$ to get two mean proportionals between $AB$ and $BC$ as desired.

Finding the required point $H$ could be done by the same sort of approximate trial-and-error processes we saw in the previous methods. But Diocles and Clavius now
actually take this idea one step farther. Namely, start by considering the circle with radius $AB$ as before. If the locus of all points $L$ as in this figure for all possible arcs $EM$ is considered, the so-called cissoid of Diocles (a cuspidal cubic algebraic curve) is obtained. Namely, for each possible $M$ in the quadrant $AE$, consider the line $DM$ and then take $K$ so that the vertical line $KH$ makes the arcs $EH$ and $EM$ equal. Take the point $L$ corresponding to that choice of $M$ as the intersection of the lines $KH$ and $MD$.

Then for each point $C$ on the radius $BE$, the line $AC$, when extended, will intersect the cissoid at some uniquely determined $L$ and hence produce a line $KH$ making the arcs $EM$ and $EH$ equal. Then two mean proportionals between $AB$ and $AC$ will be found as above by rescaling $KH$ and $DK$, which are mean proportionals between $AK$ and $KL$. Thus the cissoid in effect solves the problem for all possible pairs of the fixed $AB$ and smaller segments $BC$ simultaneously. As usual, Clavius provides a much more specific description of how the cissoid curve, or more precisely, as many points on the curve as desired, can be constructed. Absent the whole curve, that is having only a finite collection of points on the curve, some approximation or judgment of the geometer would still be needed to connect the points into a continuous curve and find an appropriate point $L$ for an arbitrary given line $BC$ as above. Hence the qualifier “approximately” (the prope verum in the Latin) still applies.

The final method for constructing two mean proportionals between given line segments addressed by Clavius is the one attributed to Nicomedes, using the conchoid curve. (The name was apparently suggested by the similarity between its shape and the shells of some marine molluscs.) This discussion is probably the closest Clavius comes to simply reproducing what he finds in Eutocius or parts thereof. Clavius starts by saying the conchoid can be drawn with a certain instrument (which is described in the first section of Eutocius’s version of this method). But since Clavius does not have a copy of the instrument, he says it will be enough to give a construction by which as many points on the conchoid as desired can be produced. (Note the parallel with the discussion of the cissoid.) So let $AB$ be a line and let $CD$ be another perpendicular line meeting $AB$ at a right angle at $E$. Taking $D$ as a pole, consider all straight lines passing through $D$. All lines except the parallel to $AB$ through $D$ will intersect $AB$ (extended if necessary). Say the line $DS$ meets $AB$ at $S$. Then extending the line again in the direction of $S$, there will be another point $F$ on the line with $SF = EC$. The locus of all such points $F$ is the curve known as the conchoid. Next, following Eutocius, Clavius proves two “remarkable properties” considered by Nicomedes. First, the farther the point $S$ is from $E$, the smaller the vertical distance is from $F$ to the line $AB$ and second, the conchoid meets every line lying above $AB$, no matter how close. Following this Clavius shows how the conchoid gives a solution of the following problem that Eutocius credits to Nicomedes:

---

63 Clavius does not use this name, though. In the coordinate system suggested by placing the diameters along the coordinate axes and taking the circle to have radius 1, the equation of the cissoid is $(x^2 + (y + 1))^2(y + 1) = 2x^2$.
64 More precisely, if we introduce coordinates placing the $x$-axis along the line $AB$ and $E$ at the origin and take $CE = ED$, Nicomedes’ conchoid is one of the connected components of the real algebraic quartic curve defined by $(x^2 + (y + 1)^2)y^2 = (y + 1)^2$. There is also a second connected component below the line $AB$ with a cusp at the point $(0, -1)$, namely the point $D$.
65 duas proprietates huius lineae insignes, [7, p. 270].
66 In modern terms, the line $AB$ is a horizontal asymptote of the conchoid.
Given any rectilinear angle, and a point outside the lines making up the angle, to construct from this point, a line intersecting the lines containing the given angle, so that the portion of the line intercepted between the lines is equal to a given line.

Finally, Eutocius and Clavius show how the solution of this problem lets one construct the triangle $GDF$ in Figure 9 for which $AF$ and $CG$ are the two mean proportionals between the sides $AB$ and $BC$ of the rectangle as in that figure. There are several additional constructions of lines made starting from the rectangle and the problem above is used to produce a line intersecting two other lines such that the line intercepted is equal to one half of $AB$. Here Clavius adds a sort of mnemonic diagram intended to help the reader visualize some of the proportionalities between sides of similar triangles in the rather complicated figure.

These discussions give additional excellent examples of the dual practical and theoretical focus of Clavius's *Geometria Practica* that we identified in the Introduction. They show how Clavius engages with sources from ancient Greek geometry and how he seeks to adapt the results for practical purposes, while still providing a complete development of the theory involved. This complete development often includes added or modified features designed to smoothe the way for a student learning the material. In particular, the choices he makes of which methods to include certainly do address his criteria of *usefulness*, *ease of application*, and *lower susceptibility to error*. Moreover, the methods of Diocles and Nicomedes are certainly more involved than the previous ones, so there is a very clear progression from simpler methods to more complicated ones.

5. Clavius’s presentation of extraction of $n$th roots in Book VI.

While Clavius was a strong adherent and proponent of the geometrical methods found in Euclid’s *Elements*, he understood very well that many geometrical constructions corresponded to *algebraic or numerical* calculations. A key example is the construction of one or two mean proportionals between two lines $AB$ and $CD$. If $XY$ is one mean proportional, then $AB : XY :: XY : CD$, and in algebraic terms

$$\frac{AB}{XY} = \frac{XY}{CD}.$$

---

67Dato quouis angulo rectilineo, & puncto extra lineas angulum datum comprehendentes: Ab illo puncto educere rectam secantem rectas datum continentes angulum, ita vt eius portio inter illas rectas intercepta aequalis sit datae rectae.
Hence
\[
\left( \frac{XY}{CD} \right)^2 = \frac{AB}{CD},
\]
so, in numerical terms, finding \( XY \) is essentially the same thing as finding a square root. We have already seen that finding two mean proportionals is essentially the same thing as finding a cube root. The pattern would continue: if any number \( n \geq 1 \) of mean proportionals in continued proportion were found, that would be the same as finding an \((n + 1)\)st root. As a result many texts on practical geometry, including the texts of Fibonacci and Pacioli mentioned earlier, included extensive discussions of numerical algorithms for computing square and cube roots (at least). Clavius’s book is no exception. He points out this connection in Proposition 18 immediately following the material discussed in §4 above and he devotes the final section of his Book VI to this topic, starting with the statement of Proposition 19: “To extract a root of any sort.”

As we mentioned in the Introduction, this is another case where Clavius does not acknowledge a source explicitly. Indeed, he is almost coy about this, saying only that his treatment of a root extraction algorithm is “from a book of a certain remarkable German arithmetic.” The first treatment of this material in the German language was contained in the very well-known book Die Coss by Christoph Rudollf (1499–1545). A revised and much expanded edition of this book prepared by Michael Stifel (1487–1567) was published in 1553 and went through several later editions. It seems very probable that this (or possibly a later edition) is the book Clavius was drawing from, and the specific section he was looking at was the Anhang to Chapter 4 of Part I written by Stifel, found starting on folio 46 and going to folio 59.

As usual, Clavius’s account is not directly copied from Stifel’s version. Clavius’s explanations are rewritten and expanded. Different numerical examples are presented. Our conjecture that Clavius was consulting this source is based on the fact that the overall outline of the method Clavius presents is essentially exactly the same as what Stifel presents:

- Very similar terminology for the different species of roots is used, e.g. “zensizenic” roots are fourth roots, “surdesolidic” roots are fifth roots, and so forth. Variations of this terminology are found in many 16th century works dealing with algebra, through, so this is only a start.
- The digits from the number whose root is being found are grouped into “points” in the same way by marking certain digits with dots; each “point” will yield one digit of the root (Clavius writes the dots below the corresponding digits, while Stifel writes them above, though).

---

68 Radicem cuiuslibet generis extrahere.
69 ex libro eximij cuiusdam Arithmetici Germani
70 The word Coss in German was borrowed from the Italian cosa (i.e. “thing”). Both were used to represent the unknown in an algebraic equation before the development of symbolic forms of algebraic expressions. A generation of early German algebraists were known as cossists.
71 The title page says, in part: Die Coss Christophs Rudolffs Mit schönen Exempeln der Coss Durch Michael Stifel Gebessert und sehr gemehrt ... Zu Königsperr in Preussen, Gedrückt durch Alexandrum Lutomyslensem im jar 1553. (That is, “Christoph Rudolff’s Die Coss, with beautiful examples [of these techniques], improved and much augmented by Michael Stifel, ...” Digitized version from www.math.uni-bielefeld.de/~sieben/rudolff.pdf.
• Tables of \( n \)th powers of the digits 1, 2, 3, \ldots, 9 are provided for use with each “point” so that the largest \( n \)th power that can be subtracted from the “point” can be identified.

• The essential role of collection of “special numbers” for each species of root to be used in preparing the “divisor” at each step of the algorithm is the same in both.

• The calculations are laid out in a very similar (and also very easy) tabular format.

Clavius provides a table containing the binomial coefficients up to \( n = 17 \) on page 278 of [7]. This is not found explicitly in Stifel’s discussion, so Clavius might be taking this from another source he does not mention, or computing the entries himself.

Probably the best way to convince the reader of this identification of Clavius’s source is to quote from two extractions of cube roots, one from Clavius and one from Stifel’s Anhang. The process described finds the root decimal digit by decimal digit. The steps all follow the same pattern after the determination of the left-most digit. So the point will be made if we look at the determination of the first two digits of the root in the examples. We begin with the first two steps of this example from Clavius:

Let it be required to extract the cube root of

\[
2\;3\;9\;4\;8\;3\;1\;9\;0
\]

...
First, from the point 239, I subtract the 216 which is the largest cube contained in it. I write its cube root 6 in the margin in the quotient. And since 23 is left over, the next point will be 23483.

\[
\begin{array}{c}
36 \quad - - \quad 300 \\
6 \quad - - \quad 30
\end{array}
\]

Next I provide a divisor in this way. Over [the digit] 6 of the root found above, I put its square, 36. And on the right, I place the two particular numbers for cube roots, namely, 300 and 30. I multiply the numbers on the first row, yielding a product of 10800 and I add the product from multiplying the two numbers on the second row, 180. The sum 10980 will be the divisor. (It would be enough to take the product of the two numbers on the first row as the divisor, namely 10800, as must be understood in other root extractions.) I divide the point 23483 by 10980 and write the quotient 2 next to the digit 6 found first. I treat what comes after this digit as follows. At the right

\[
\begin{array}{c}
36 \quad - - \quad 300 \quad - - \quad 2. \\
6 \quad - - \quad 30 \quad - - \quad 4. \\
\end{array}
\]

of the numbers 36 and 300, I add this digit 2 [found in the quotient] and below it, its square, 4, and its cube, 8. Now, the three numbers on each of the first two rows are multiplied, and the products are 21600 and 720. Adding the cube 8 makes 22328. I subtract this from the point, leaving 1155, and the next point will be 1155190. ([7], pp. 280–281)

Clavius continues to find the (approximate) cube root 621 for 239483190. Note that \(621^3 = 239483061\), so this value is 129 “short.” Later in this section, Clavius also shows how to compute additional decimal digits in the fractional part, obtaining closer approximate cube roots.

We now translate a step of the computation from folios 47-49 in Stifel’s *Anhang*\(^{75}\)

\[8 \quad 0 \quad 6 \quad 2 \quad 1 \quad 5 \quad 6 \quad 8 \quad 0 \quad 0 \quad 0\]

Erstlich subtrahir ich von dem hindersten puncten (das ist von 80) die aller grösste cubic zal/ die ich subtrahiren kan. Die selbig ist 64. so bleybett nach vbrig davon 16 die gehören denn zum nehsten puncten hernach/ der selbig uverkompt denn dese figuren 16621. So setz nu die cubic wurzel von 6 in den quotient. facit 4. und is also der erst punct aufgericht.

So nehme ich nu fur mich den andern punct/ nemlich 16221. Den dividir ich mit 4800. (das kompt von 300 mal 16) Nu gibt das gedacht dividieren nur 3 in den quotient. Und ist also die newe figur gefunden.

Dem selbigen nach stehn die zwo zalen 300 und 30. mit jren zugezethnen zalen also.

\[
\begin{array}{c}
16 \quad - - \quad 300 \quad - - \quad 3 \\
4 \quad - - \quad 30 \quad - - \quad 9
\end{array}
\]

Denn erstlich ist gefunden in den quotient de figur 4. die steht neben 30 zur linken hand/ vnd drob neben 300 steht jr quadrat/ nemlich 16.

So is nu darnach gfunden in den quotient die figur 3. Die steht oben neben 300 zur rechten hand/ vnd darunter steht jr quadrat 9. neben 30. wie du alles vol sihest.

So multiplicir ich nu/ vnd sprich. 16 mal 300 mal 3. facit. 14400. vnd 4 mal 30 mal 9. facit 1080. Das addir ich/ so kompt 15480. Das subtrahir ich von 16621. Als vom andern puncten diser operation/ so bleyben denn 1141.
Example.

First I subtract the largest cube that I can from the leftmost point (that is, from 80). That is 64, leaving 16, which then belongs to the next point, which is composed of the digits 16621. So now I set the cube root of 64 in the quotient, and the first point is decided. So then I take the next point, namely 16621. I divide that by 4800 (which is 300 times 16) and that division gives 3 in the quotient. And so the new digit is found. I put this next to the two numbers 300 and 30 with the accompanying numbers in this way:

\[
\begin{array}{cccc}
16 & - & 300 & - \\
4 & - & 30 & - \\
\end{array}
\]

Since the digit 4 was found [first] in the quotient, that is placed next to the 30 on the left, and above, next to 300 goes its square, namely 16. On the right next to the 300 goes the next digit 3, and its square 9 goes below next to the 30, as you clearly see.

So now I multiply and say 16 times 300 times 3 makes 14400, and 4 times 30 times 9 makes 1080. I add those and obtain 15480. I subtract that from the 16621 as from the other points. The number 1141 remains. Last, I take the cube of the newly-found digit 3. Namely, 3 times 3 times 3 makes 27. I also subtract this and 1114 remains. This belongs to the following point.

Since there were four “points” in the original number, Stifel’s cube root will contain four decimal digits. After two more steps of the process, he finds the value 4320, an exact cube root of 80621568000.

If my conjecture that Clavius was following Stifel’s presentation of a root extraction algorithm here is correct (and I hope I have proved the point with the quotations above!), then there remains the question why Clavius did not make an explicit attribution to Rudolff and/or Stifel. It is certainly possible that Clavius thought he did not need to say any more to identify the source because Rudolff’s Die Coss was extremely well-known, at least in German-speaking areas because it was the first book on this material published in German. However, there is another circumstance that might just provide another component of an explanation. Namely, Stifel had started out as an Augustinian monk, but later became a Protestant minister and an outspoken supporter of Martin Luther. Unlike the citation of the Protestant Scaliger mentioned earlier in §4, where Clavius was being explicitly critical of the other’s work, Clavius was singling out Stifel’s algorithm for high praise and recommending its use. Under those circumstances, it may be that Clavius (or his Jesuit colleagues and superiors) thought it was not politic to mention Stifel’s name.

6. Conclusions

Clavius presented a tremendous amount of interesting and useful mathematics in his Geometria Practica and in his other writings. In assembling the material for this...
book, he drew on an extremely broad range of ancient, medieval, and contemporary sources. At the same time, his typical procedure was to rework, augment, and clarify the mathematical texts he dealt with. It seems arguable that he achieved his stated goal of presenting the whole range of practical geometry as understood in his time, and he did it in a form that would be useful for his readers.

The quality of this work was recognized very soon after it appeared, as evidenced (for instance) by the fact that mathematicians such as Kepler mentioned sections of this book in their writings. Recognition of Clavius's work was also evident in other ways. In the Jesuit mission in China, one of Clavius’s former students in the Collegio Romano, Mateo Ricci, S.J. (1552–1610), together with his Chinese collaborator Xu Guangqi (1562–1633), made translations of not only the first six books of Euclid’s Elements from Clavius’s version, but also material from the Geometria Practica. Later, Giacomo Rho, S.J. (1593–1638) made Chinese translations of additional sections of this work.

However, if I may be allowed to speculate in this last paragraph, in some ways, I would argue that Clavius’s Geometria Practica actually represents almost the end of the sub-genre of “theoretical practical geometry” in the style we have seen in our discussions. There were certainly many later practical geometry books, but they tended more toward the “practical,” and in many cases omitted proofs or theoretical developments. In addition, Clavius’s essential mathematical conservatism and his devotion to the synthetic Euclidean tradition in geometry would shortly come to seem very old-fashioned. The recovery of Pappus’s treatment of the Greek tradition of geometric analysis in Book VII of the Mathematical Collection, combined with the ever-growing influence of algebraic thinking was the impetus for an explosion of work starting in the late 16th century and continuing into the first half of the 17th century (this is discussed in a fascinating way in [3]). But this was largely orthogonal to the ways that Clavius approached geometry and he seemingly had little interest in or taste for that side of Pappus’s writings. Within 30 years of his death, the introduction and systematic use of analytic, or coordinate geometry by Descartes and others was well under way. That new way of harnessing the power of algebra to discover new geometrical results and prove them was fundamentally changing the practice of mathematics.

References

[1] Baldini, U. “Clavius and the Scientific Scene in Rome,” in Gregorian Reform of the Calendar; Proceedings of the Vatican Conference to Commemorate its 400th Anniversary, eds. G.V. Coyne, S.J., M.A.Hoskin, and O.Pederson, Rome: Pontificia Academia Scientiarum and Specola Vaticana, 1983.
[2] Baldini, U. “The Academy of Mathematics of the Collegio Romano from 1553 to 1612,” in Jesuit Science and the Republic of Letters, ed. M. Feingold, Boston: MIT Press, 2003.
[3] Bos, H. Redefining Geometrical Exactness: Descartes’ Transformation of the Early Modern Concept of Construction, New York: Springer, 2001.
[4] Clagett, M. Archimedes in the Middle Ages, Volume 1, The Arabo-Latin Tradition. Madison: University of Wisconsin Press, 1964.
[5] Dijksterhuis, E.J. Archimedes (trans. C. Dijkshoorn), Princeton, NJ: Princeton University Press, 1956.
[6] Clavius, C. Geometria Practica, first edition, Rome: Luigi Zannetti, 1604.

76 In Kepler’s Harmonice Mundi. The reference was to Clavius’s discussion of various approximate constructions of regular heptagons in Book VIII.

77 [18, pp. 318–319]

78 See in particular [27, Chapter 2, §3].
[7] Clavius, C. *Geometria Practica*, second edition, Mainz: Johann Albin, 1606.
[8] Clavius, C. *Opera Mathematica* (in five volumes), Mainz: Reinhard Eltz, 1611.
[9] *Diocles On Burning Mirrors* (trans. G.J. Toomer), New York: Springer, 1976.
[10] Heronis Alexandrini opera quae supersunt omnia. Volumen III. Rationes dimetiendi et Commentatio dioptica, ed. Hermann Schöne, Leipzig: B. G. Teubner, 1903.
[11] Heron d’Alexandrie, *Metrica. Introduction, édition critique, traduction française et commentaires par F. Acerbi et B. Vitrac*, Pisa: Fabrizio Serra editore, 2014.
[12] Hugh of St. Victor (attrib.) *Practical Geometry* (trans. F. Homann, S.J.), Milwaukee WI: Marquette University Press, 1991.
[13] Knobloch, E. “Sur la vie et l’oeuvre de Christophore Clavius (1538-1612),” *Revue d’histoire des sciences* 41, (1988), 331–356.
[14] Knobloch, E. “L’Oeuvre de Clavius et ses Sources Scientifiques,” in *Les jésuites à la Renaissance*, ed. Luce Giard, Paris: Presses Universitaires de France, 1995.
[15] Knobloch, E. “Clavius et la partition des polygones,” in *Géométrie pratique, Géomètres, ingénieurs et architectes, XVIe–XVIIIe siècle*, ed. D. Raynaud, Presses Universitaires de Franche-Comté, 2015.
[16] Knorr, W. *Textual Studies in Ancient and Medieval Geometry*, Boston, MA: Birkhauser, 1989.
[17] Martzloff, J.-C., “Clavius traduit en Chinois” in *Les jésuites à la Renaissance*, ed. Luce Giard, Paris: Presses Universitaires de France, 1995.
[18] Netz, R. *The Works of Archimedes: Translation and Commentary, Volume 1: The Two Books On the Sphere and the Cylinder*, Cambridge UK: Cambridge University Press, 2004.
[19] Pacioli, L. *Summa de arithmetica geometria proportioni et proportionalità*, 1523 edition, digitized version at [www.e-rara.ch](http://www.e-rara.ch).
[20] Pisano, L., *Scritti di Leonardo Pisano*, vol. 2, B. Boncampagni, ed., Rome: 1862.
[21] Pisano, L., *Fibonacci’s De Practica Geometrie*, B. Hughes ed. New York: Springer Science + Business, 2008.
[22] Proclus: *A Commentary on the First Book of Euclid’s Elements* (trans. G. Morrow), Princeton, NJ: Princeton University Press, 1970.
[23] Raynaud, D. “Introduction,” in *Géométrie pratique, Géomètres, ingénieurs, et architectes, XVIe–XVIIIe siècle*, Presses Universitaires de Franche-Comté, 2015.
[24] Rommevaux, S. *Clavius, une clé pour Euclide au XVIe siècle*, Paris: Librarie Philosophique J. Vrin, 2005.
[25] Rudolff, C. *Die Coss*, revised and augmented edition by Michael Stifel, Königsberg, 1553; digitized version from [www.math.uni-bielefeld.de/~zieben/rudolff.pdf](http://www.math.uni-bielefeld.de/~zieben/rudolff.pdf).
[26] Sasaki, C. *Descartes’ Mathematical Thought*, Dordrecht: Springer Science+Business Media, 2003.
[27] Taibak, C. “An Archimedean Proof of Heron’s Formula for the Area of a Triangle: Heuristics Reconstructed,” in *From Alexandria Through Baghdad*, ed. N. Sidoli and G. Van Brummelen, Heidelberg: Springer, 2014.

**DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, COLLEGE OF THE HOLY CROSS, WORCESTER, MA 01610**

*Email address: jlittle@holycross.edu*