Divergences in the Casimir energy

J.S. Dowker*

*Department of Theoretical Physics,
The University of Manchester, Manchester, England

Abstract

The divergence found by Nesterenko, Lambiase and Scarpetta is attributed to the existence of edges in the considered manifold.

*dowker@a13.ph.man.ac.uk
1. Introduction

In a recent work, [1], Nesterenko, Lambiase and Scarpetta encounter extra divergences in the Casimir energy of a semi-circular infinite cylinder. The present letter makes some brief comments on this calculation.

2. Divergences in the presence of boundaries

A general discussion, and some examples, can be found in Dowker and Kennedy, [2] sections 5 and 6 and some further remarks in Dowker and Banach [3]. Let us work in \( D \) spatial dimensions, and assume space-time is ultrastatic \( T \times M \). In the approach of Dowker and Critchley, [4], the one-loop effective Lagrangian for scalar fields is

\[
L^{(1)} = -\frac{i}{2} \lim_{s \to 1} \frac{\text{tr}_D[\zeta(s - 1)]}{s - 1} = -\frac{i}{2} \lim_{s \to 1} \left( \frac{\text{tr}_D[\zeta(0)]}{s - 1} + \text{tr}_D[\zeta'(0)] \right)
\]

(2.1)

in terms of the space-time \( \zeta - \) function, or, better, \( \zeta - \)operator, \( \zeta(s) \). The square brackets indicate that a time coincidence limit has been taken and the trace can be taken as an integration over \( M \) of the space diagonal elements. A scale dependent term could be added, but the pole is sufficient for now. It can be shown, [2], that the total vacuum energy (the Casimir energy), \( E_M \), equals \(-L^{(1)}\).

To connect \( E_M \) with the \( \zeta - \)function on the spatial section, \( M \), we relate this latter \( \zeta - \)function to the spacetime \( \zeta - \)function by [2]

\[
[\zeta(s)] = \frac{i}{(4\pi)^{1/2}} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \zeta_M(s - 1/2).
\]

(2.2)

It can be seen from (2.1) and (2.2) that if \( \zeta_M(s), \equiv \text{tr}_D \zeta_M(s) \), has no pole at \( s = -1/2 \) then \( E_M \) is given by the finite expression

\[
E_M = \frac{1}{2} \zeta_M(-1/2)
\]

used in [2,5,3] and many other places since.

We prefer this derivation of this result, rather than the more usual one of simply regularising the divergent eigenvalue sum form of the Casimir energy, since it arises via the effective action, a scalar, invariant quantity whose renormalisation (which we do not attempt) is more transparent when divergences appear, as they will, in general. In this case one finds the expression

\[
E_M = \frac{1}{2} \text{Pf} \zeta_M(-1/2) + \lim_{s \to 1} \frac{C_{(D+1)/2}(M)}{s - 1}
\]

(2.3)
the pole residue being given in terms of the \( \mathcal{M} \) heat-kernel expansion coefficient 
\( C_{(D+1)/2}(\mathcal{M}) \). (For a conformally invariant theory, this would be the conformal anomaly.)

For simplicity, let us now assume that \( \mathcal{M} \) is flat. The coefficients \( C_n \) are then composed of just boundary parts, which we write \( B_n(\partial \mathcal{M}) \), the point now being that if one combines the Casimir energies for the inside, \( \mathcal{M} \), and the outside, \( \mathcal{M}^* \), of the bounding surface, since \( \partial \mathcal{M} = -\partial \mathcal{M}^* \) the divergent pole terms in \( E \) and \( E^* \) from (2.3) will either cancel or double according to the property, used in [2],

\[
B_n(-\partial \mathcal{M}) = (-1)^{2n+1} B_n(\partial \mathcal{M})
\]

depending on whether \( D \) is even or odd. The explicit evaluations on the ball by Cognola, Elizalde and Kirsten, [6], illustrate this very nicely. It is to this particular calculation that the authors of Ref [1] refer and which they essentially repeat for the semi-circular cylinder. In this geometry extra divergences do occur, even for the outside–inside combination, because, as already pointed out in [2], the reflection symmetry, (2.4), no longer holds if the domain \( \mathcal{M} \) has edges or corners, or other singularities.

To the author’s knowledge, a general treatment of this situation does not yet exist but to proceed a little further we consider in more detail the particular arrangement in [1]. It is shown in [2] section 6 that, for a four-dimensional space-time, the vacuum energy per unit length of an infinite waveguide of uniform cross-section, \( \mathcal{D} \), is

\[
E_D = \frac{1}{8\pi} \left( \lim_{s \to 1} \frac{\zeta_D(-1)}{s-1} + \zeta_D(-1) + \zeta_D'(1) \right).
\]  

(2.5)

The pole residue is proportional to the heat-kernel coefficient \( C_2(\mathcal{D}) \) which is, as has been stated, all boundary part \( B_2(\partial \mathcal{D}) \), satisfying (2.4) if the boundary is smooth, which is the most commonly referred to case. As stated, this pole will therefore cancel on combining the outside and inside expressions. The finite term, \( \zeta_D(-1) \), whose coefficient is actually undetermined because of scale ambiguities, also cancels leaving just, [2],

\[
E = E_D + E_{D^*} - E_{\mathcal{D} \cup \mathcal{D}^*} = \frac{1}{8\pi} \left( \zeta_D(-1) + \zeta_{D^*}(-1) - \zeta_{\mathcal{D} \cup \mathcal{D}^*}(-1) \right)
\]

as the finite Casimir energy associated with the existence of the (infinitely thin) surface \( \partial \mathcal{D} \) in the region \( \mathcal{D} \cup \mathcal{D}^* \).

If the boundary is not smooth (say it is piecewise smooth) then we should enumerate the pieces, \( \partial \mathcal{D}_i \), which meet in the intersections \( I_{ij} \) (here simply points)
and one should properly say that $C_2$ takes contributions from the $\partial D_i$ and the $\mathcal{I}_{ij}$ on the same footing. It is these latter contributions that violate the reflection symmetry (2.4) and are responsible for any extra divergences found in [1]. They typically involve the squares of the extrinsic curvatures.

As the particular example under consideration, the heat-kernel expansion on a semi-disc (a hemi-one ball), and indeed on an arbitrary sector, is easily worked out either from the eigenmodes, à la Stewartson and Waechter, or Moss [7], or from images (when they apply) or from $\zeta$- functions or from a combination of techniques. In the case of the semicircular boundary, the extra term, over and above the ‘circular’ one, arises from the image contribution.

For the record the semi-circle expansion is

$$K(t) \sim \frac{|\mathcal{D}|}{8\pi t} - \frac{5}{16} \frac{|\partial \mathcal{D}|}{(\pi t)^{1/2}} + \frac{5}{24} + \frac{(\pi t)^{1/2}}{16} \left( \frac{1}{16} + \frac{1}{\pi} \right) + \frac{347}{10080} t + .$$

A tolerably comprehensive analysis of the general $C_2$ coefficient in the presence of boundary discontinuities in arbitrary dimensions has been attempted in Ref [8]. If it is required to eliminate the divergences by any form of renormalisation then it is necessary to motivate bare quantities of the general form given in [8]. About this we have nothing to say. Without physical justification for studying such singular manifolds (e.g. brane worlds), perhaps one should not strive too hard in this direction especially since in the generic case, where $C_{(D+1)/2}$ takes contributions from submanifolds of codimension up to the dimension of the manifold, dimensional arguments alone show that pole cancellation is impossible.

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4. References

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