STABILITY AND ERRORS ANALYSIS OF TWO ITERATIVE SCHEMES OF FRACTIONAL STEPS TYPE ASSOCIATED TO A NONLINEAR REACTION-DIFFUSION EQUATION

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Abstract. We present the error analysis of two time-stepping schemes of fractional steps type, used in the discretization of a nonlinear reaction-diffusion equation with Neumann boundary conditions, relevant in phase transition and interface problems. We start by investigating the solvability of such boundary value problems in the class $W^{1,2}_p(Q)$. One proves the existence, the regularity and the uniqueness of solutions, in the presence of the cubic nonlinearity type. The convergence and error estimate results (using energy methods) for the iterative schemes of fractional steps type, associated to the nonlinear parabolic equation, are also established. The advantage of such method consists in simplifying the numerical computation. On the basis of this approach, a conceptual algorithm is formulated in the end. Numerical experiments are presented in order to validate the theoretical results (conditions of numerical stability) and to compare the accuracy of the methods.

1. Introduction. Consider a one dimensional nonlinear reaction-diffusion equation with respect to the unknown function $v(t, x)$:

$$
\begin{align*}
& p_1 \frac{\partial}{\partial t} v - p_2 \Delta v = p_3 F(v) + f(t, x) \quad \text{in } Q = [0, T] \times \Omega \\
& p_2 \frac{\partial}{\partial n} v = 0 \quad \text{on } \Sigma = [0, T] \times \partial \Omega \\
& v(0, x) = v_0(x) \quad \text{on } \Omega,
\end{align*}
$$

where:

- $\Omega$ is a bounded domain in $\mathbb{R}$ with smooth boundary $\partial \Omega = \Gamma$ and $T > 0$ stands for some final time;
- $v(t, x)$ is the unknown function; in particular, $v(t, x)$ is the phase function (used to distinguish between the states (phases) of a material which occupies the region $\Omega$ at every time $t \in [0, T]$);
- $p_1, p_2, p_3$ are positive values;
- $F : \mathbb{R} \rightarrow \mathbb{R}$ is a real function having the structure: $F(\varphi) = \varphi - \varphi^3 \forall \varphi \in \mathbb{R}$.
- $f(t, x) \in L^p(Q)$ is a given function and $p \geq 2$ (see (2)).

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\[ v_0 \in W^{2-\frac{2}{p}}_{\infty} (\Omega) \] verifying the compatibility condition \( p_2 \frac{\partial}{\partial n} v_0 = 0; \)
\[ n = n(x) \text{ is a vector of the outward (from } \Omega \text{) unit normal to the surface } \Sigma; \frac{\partial}{\partial n} \]
denotes differentiation along \( n. \)

Here we have used the standard notation for Sobolev spaces, namely, given a positive integer \( k \) and \( 1 \leq p \leq \infty, \) we denote by \( W^{k,2p}(Q) \) the usual Sobolev space on \( Q: \)
\[ W^{k,2p}(Q) = \left\{ y \in L^p(Q) : \frac{\partial^r}{\partial t^r} \frac{\partial^s}{\partial x^s} y \in L^p(Q), \text{ for } 2r + s \leq 2k \right\}, \]
i.e., the space of functions whose \( t \)-derivatives and \( x \)-derivatives up to the order \( k \) and \( 2k, \) respectively, belong to \( L^p(Q). \) Also, we have used the Sobolev spaces \( W^l_p(\Omega) \) with nonintegral \( l \) for the initial and boundary conditions, respectively (see [19] - Chapter 1 and references therein).

The nonlinear parabolic equation (1) occurs in the phase-field transition system [6], where the phase function \( v(t,x) \) describes the transition between the solid and liquid phases in the solidification process of a material occupying a region \( \Omega. \) The classical regular potential in Caginalp’s model (see [6]) is obtained for \( p_3 = \frac{1}{2} \), i.e. the nonlinearities in (1) becomes \( F(\varphi) = \frac{1}{2\xi} (\varphi - \varphi^3) \) (see also [22]). Further, the regular potential indicated above also includes the general nonlinearities \( F(z) = a_1 z + a_2 z^2 + \ldots + a_{2p-2} z^{2p-2} + a_{2p-1} z^{2p-1} \) with \( a_{2p-1} < 0 \) (see [28] and references therein). Endowed with dynamic boundary conditions and singular potentials, problem (1) was treated in [10].

(1) it is a particular instance of the Allen-Cahn equation [2], which was introduced to describe the motion of anti-phase boundaries in crystalline solids, and it has been widely applied to many [9] complex moving interface problems, e.g., the mixture of two incompressible fluids, the nucleation of solids, the vesicle membranes.

A great deal of work has been done on reaction-diffusion problems and Allen-Cahn equations [1], [11], [7], [12], [14], [17], [18], [21], [27], [32], [32]-[37]. For more general assumptions and with various types of boundary conditions, equation (1) has been numerically investigated in e.g., [3], [4], [12], [15], [16], [24]-[29], [32].

The error analysis for the implicit backward Euler and finite elements approximation is presented in [32], while a discontinuous-Galerkin in time method is analyzed in [11]. Computations with several different higher-order time-stepping schemes, such as BDF2-AB2 and Crank-Nicolson, are used for the sharp interface limit in [37].

The outline of the paper is as follows. In Section 2 we prove the existence, regularity, stability and uniqueness of the solution to the nonlinear problem (1) in the presence of homogeneous Neumann boundary conditions, while, in Section 3 we are concerned with the convergence of fractional steps type scheme associated to the nonlinear reaction-diffusion equation (1). In Section 4 we introduce two semi-discrete in time approximations, proving consistency and stability results for error equations by energy estimates arguments. The numerical experiments in Section 5 confirms the theoretical rates of convergence. The concluding remarks are formulated in Section 6.

2. Well-posedness of solutions to the nonlinear equation (1). In the present Section we will investigate the solvability of the first boundary value problems of the form (1) in the class \( W^{1,2}_p(Q). \) One proves the existence, the regularity and the uniqueness of solutions (Theorem 2.1 below) to the nonlinear parabolic problem (1)
considering the cubic nonlinearity \( p_s(v - v^3) \) which verifies for \( N = 1 \) and \( r = 3 \) the general assumptions \( H_0 \) and \( H_2 \) formulated in [30], [31], that is:

\[
H_0 : \quad (v - v^3)|v|^{3p-4}v \leq 1 + |v|^{3p-1} - |v|^{3p}.
\]

\[
H_2 : \quad \text{There exist a function } \bar{F} : \mathbb{R}^2 \rightarrow \mathbb{R} \text{ and a constant } b_0 > 0 \text{ verifying the relations:}
\]

\[
((v_1 - v_1^3) - (v_2 - v_2^3))^2 \leq \bar{F}(v_1, v_2)(v_1 - v_2)^2;
\]

\[
F(v_1, v_2) \leq b_0(1 + |v_1|^4 + |v_2|^4), \quad \forall v_1, v_2 \in \mathbb{R}.
\]

The proof of Theorem 2.1 was given in [5] and [27] noting that there formula-

The basic tools in the analysis of the problem (1) are the Leray-Schauder degree

The main result of this section establishes the dependence of the solution \( v(t, x) \) in the nonlinear parabolic equation (1) on the term \( f(t, x) \) in the right-hand side.

**Theorem 2.1** There exists a unique solution \( v \in W^{1,2}_p(Q) \) for (1) and \( v \) satisfies

\[
\|v\|_{W^{1,2}_p(Q)} \leq C \left\{ 1 + \|v_0\|_{W^{3,2-rac{2}{p}}_\infty(\Omega)}^{3-rac{2}{p}} + \|f\|_{L^p(Q)} \right\}, \tag{3}
\]

where the constant \( C \) depends on \( |\Omega|, T, M, p, p_1, p_2, \) and \( p_s \), but is independent of \( v \) and \( f \).

If \( v_1, v_2 \) are two solutions of (1) corresponding to the data \( f^1, f^2 \in L^p(Q) \), respectively, for the same initial conditions, such that

\[
\|v_1\|_{W^{1,2}_p(Q)} \leq M, \quad \|v_2\|_{W^{1,2}_p(Q)} \leq M, \tag{4}
\]

then

\[
\|v_1 - v_2\|_{W^{1,2}_p(Q)} \leq C \|f^1 - f^2\|_{L^p(Q)}, \tag{5}
\]

where the constant \( C \) depends on \( |\Omega|, T, M, p, p_1, p_2, \) and \( b_0 \) but is independent of \( v_1, v_2, f^1 \) and \( f^2 \).

**Proof.** The proof of Theorem 2.1 was given in [5] and [27] noting that there formulation differs from this by certain physical parameters, which implies different values for the constant \( C \) in (3) and (4). Moreover, corresponding to different boundary conditions (including nonlinear and nonhomogeneous boundary conditions), similar results were proved in [8], [21], [23], [26] and [28].

**Corollary 2.2** Under hypotheses \( H_0 \) and \( H_2 \) the problem (1) possesses a unique solution \( v \in W^{1,2}_p(Q) \).

**Proof.** Let \( f^1 = f^2 = f \) in the Theorem 2.1. Then (5) shows that the conclusion of the corollary is true. 

\[\square\]
3. Approximating schemes. The aim of this Section is to use the fractional steps method in order to approximate the solution of the nonlinear problem (1), whose uniqueness is guaranteed by Corollary 2.2. To do that, let us associate to the time-interval \([0, T]\) the equidistant grid of length \(\varepsilon_M = \frac{T}{M}\), for any integer \(M \geq 1\). Corresponding to it, the following two approximating schemes can be written in order to approximate the solution of nonlinear boundary value problem (1):

\[
\begin{align*}
\frac{\partial}{\partial t} v_M(t, \cdot) - \frac{p_2}{p_1} \Delta v_M(t, \cdot) &= \frac{1}{p_1} f(t, \cdot) & t \in [i\varepsilon_M, (i + 1)\varepsilon_M], \\
\frac{p_2}{p_1} \frac{\partial}{\partial n} v_M(t, \cdot) &= 0 \\
v_M(i\varepsilon_M, \cdot) &= z_M(\varepsilon_M, \cdot), \ i = 0, ..., M - 1,
\end{align*}
\]

where \(z_M(\varepsilon_M, \cdot)\) is the solution of Cauchy problem

\[
\begin{align*}
z'_M(\tau, \cdot) + \frac{p_2}{p_1} (z^2_M(\tau, \cdot) - z_M(\tau, \cdot)) &= 0, & \tau \in [0, \varepsilon_M], \\
z_M(0, \cdot) &= v_M^-(i\varepsilon_M, \cdot), \ v_M^-(0, \cdot) = v_0(x) & i = 0, 1, ..., M - 1,
\end{align*}
\]

as well as

\[
\begin{align*}
\frac{\partial}{\partial t} V_M(t, \cdot) - \frac{p_2}{p_1} \Delta V_M(t, \cdot) - \frac{p_2}{p_1} V_M(t, \cdot) &= \frac{1}{p_1} f(t, \cdot) & t \in [i\varepsilon_M, (i + 1)\varepsilon_M], \\
\frac{p_2}{p_1} \frac{\partial}{\partial n} V_M(t, \cdot) &= 0 \\
V_M(i\varepsilon_M, \cdot) &= Z_M(\varepsilon_M, \cdot), \ i = 0, ..., M - 1,
\end{align*}
\]

with \(Z_M(\varepsilon_M, \cdot)\) - solution of the Cauchy problem

\[
\begin{align*}
Z'_M(\tau, \cdot) + \frac{p_2}{p_1} Z^2_M(\tau, \cdot) &= 0, & \tau \in [0, \varepsilon_M], \\
Z_M(0, \cdot) &= v^+_M(i\varepsilon_M, \cdot), \ v^+_M(0, \cdot) = v_0(x) & i = 0, 1, ..., M - 1,
\end{align*}
\]

while \(v^+_M(i\varepsilon_M, \cdot) = \lim_{t \uparrow i\varepsilon_M} v_M(t, \cdot) (V_M(t, \cdot), \) respectively).

We point out that the sequence of approximating problems (6)-(7) and (8)-(9), supplies a decoupling method for the original problem (1) into a linear parabolic boundary value problem (6) or (8) and, corresponding, a nonlinear evolution equation (7) or (9). The advantage of this approach consists in simplifying the numerical computation of the approximations to (1), due to that the fractional steps method avoids the iterative process in passing from a time level to the next one.

The main question is the convergence of the sequence \(\{v_M\} \) (\(\{V_M\}\), respectively) of solutions to the approximate problems (6)-(7) ((8)-(9), respectively) to the unique solution \(v \in W^{1,2}_p(Q)\) of problem (1) as \(M \to \infty\). We will treat the convergence of this numerical schemes on the basis of an abstract approximation result (see [24], [26] for a detailed discussion).

We have

**Theorem 3.1** Assume that the function \(y \mapsto p_2(y^3) + \omega y\) is strictly increasing on \(\mathbb{R}\) for some \(\omega > 0\). Then for all \(v_0 \in W^{2,\frac{2}{3}}_\infty(\Omega) \subset L^p(\Omega)\) and \(f \in L^p(Q)\), the
sequence \( \{v_M\} \) solving (6)-(7) converges to the unique solution \( v \) of problem (1) in the following sense
\[
\lim_{M \to \infty} v_M(t) = v(t) \quad \text{in } L^p(\Omega)
\]
uniformly with respect to \([0,T]\).

Proof. The proof of Theorem 3.1 can be found in [28]; we omit the details. \( \square \)

For convergence results of the approximating scheme (8)-(9), we recommend the works [23] and [24]-[27], where different boundary conditions (including dynamic boundary conditions) were used instead of (1)2, as well as different proof techniques.

We note that the difference between the above schemes is given by the position of the linear term from the nonlinearities \( F(v) \) in the original problem (1) which can appear in Cauchy problem (7) or in the linear problem (8), as was done in previous works: [3], [4], [15], [16], [23]-[29] and [32].

4. Two methods and error analysis. Let the time step \( \varepsilon_M = \frac{T}{2^M} \), \( M \geq 1 \), be fixed, arbitrary, \( t_n = n \varepsilon_M \), \( n = 0,1,\ldots,M \) (see the previous section), and assume that the initial data \( v_0 \) as well as the forcing term \( f^n := f(t_n) \) are given (superscripts denote the time level of approximation). We consider the following two first-order methods for semi-discretization in time:

- the first method we analyze is an implicit-explicit (IMEX) scheme, similar to the fractional time step method introduced by (8)-(9)
\[
\begin{align*}
\frac{p_1 v^{n+1} - \phi^n}{\varepsilon_M} - p_2 \Delta v^{n+1} - p_3 v^{n+1} &= f^{n+1} \\
\phi^n &= v^n \left( 1 + 2 \frac{p_3}{p_1} \varepsilon_M (v^n)^2 \right)^{-\frac{1}{2}}. 
\end{align*}
\] (10)

We note that \( \phi_n \) is the value at \( \varepsilon_M \) of the exact solution of the ordinary differential equation \( p_1 u' + p_3 u^3 = 0 \) on \([t_n, t_{n+1}]\), with the initial condition \( u(t_n) = v^n \).

Moreover, the fractional step method (10) can be equivalently written as
\[
\begin{align*}
p_1 \frac{v^{n+1} - v^n}{\varepsilon_M} - p_2 \Delta v^{n+1} \\
-p_3 \left[ v^{n+1} - (v^n)^3 \left( 1 + 2 \frac{p_3}{p_1} \varepsilon_M (v^n)^2 \right)^{\frac{1}{2}} \left( 1 + 2 \frac{p_3}{p_1} \varepsilon_M (v^n)^2 \right)^{\frac{1}{2}+1} \right]
\end{align*}
\]
\( = f^{n+1}.
\]

- the second method we consider is also an IMEX scheme, similar to the fractional steps type method introduced by (6)-(7)
\[
\begin{align*}
p_1 \frac{v^{n+1} - \phi^n}{\varepsilon_M} - p_2 \Delta v^{n+1} &= f^{n+1} \\
\phi^n &= v^n \left( \frac{1}{z_3(0) + z_2(0) + z_1(0)} e^{\varepsilon_M (z^n)^2} \right)^{\frac{1}{2}}. 
\end{align*}
\] (11)

We note that \( \phi_n \) is the value at \( \varepsilon_M \) of the exact solution of the ordinary differential equation (see (7))
\[
z' = \frac{p_3}{p_1} \left( z - z^3 \right)
\]
on $[0, \varepsilon_M]$, with the initial condition $z_M(0, \cdot) = v^n = v^n_\eta(n\varepsilon_M, \cdot)$, $n = 0, 1, \ldots, M - 1$, $v^0 = v^0_M(0, \cdot) = v_0(x)$. Moreover, the fractional step method (11) can be equivalently written as

$$p_1 \frac{v^{n+1} - v^n}{\varepsilon_M} - p_2 \Delta v^{n+1} + \frac{p_1}{\varepsilon_M} (v^n)^3 = \frac{2}{\varepsilon_M} e^{-2\frac{p_2}{p_1} \varepsilon_M} \frac{2}{\varepsilon_M} e^{-2\frac{p_2}{p_1} \varepsilon_M} + \frac{2}{\varepsilon_M} e^{-2\frac{p_2}{p_1} \varepsilon_M} + \frac{2}{\varepsilon_M} e^{-2\frac{p_2}{p_1} \varepsilon_M} + \frac{2}{\varepsilon_M} e^{-2\frac{p_2}{p_1} \varepsilon_M} \frac{2}{\varepsilon_M} e^{-2\frac{p_2}{p_1} \varepsilon_M} = f^{n+1}.$$ 

**Error estimates** for the numerical schemes (10)-(11) will be presented in the sequel. In order to facilitate the reader’s easy tracking the next computations, let us introduce the following notation

$$S(\alpha, \beta) = a e^{-\frac{2p_2}{p_1} \varepsilon_M} + b e^{\frac{2p_2}{p_1} \varepsilon_M}.$$ 

We begin by defining the point-wise truncation errors with respect to the time discretization, that is:

$$e^n := v(t_n) - v^n, \quad \forall \ n = 0, 1, \ldots, M,$$

and the local truncation errors, respectively:

$$E_{old}^{n+1} := p_1 \frac{v(t_{n+1}) - v(t_n)}{\varepsilon_M} - p_2 \Delta v(t_{n+1})$$

$$= \frac{2}{\varepsilon_M} \left[ (1 + 2\frac{p_2}{p_1} \varepsilon_M) v^2(t_n) \right]^\frac{1}{2}$$

$$- f(t_{n+1}),$$

$$E_{new}^{n+1} := p_1 \frac{v(t_{n+1}) - v(t_n)}{\varepsilon_M} - p_2 \Delta v(t_{n+1})$$

$$= \frac{2}{\varepsilon_M} S(v(t_n), 1) S\left( z_M^3(0, \cdot), z_M^2(0, \cdot) \right) S\left( z_M^3(0, \cdot), z_M^2(0, \cdot) \right) + 1$$

$$- f(t_{n+1}),$$

\[\forall n = 0, 1, \ldots, M, \text{ with } z_M(0, \cdot) = v^{-}(n\varepsilon_M, \cdot) = \lim_{t \uparrow n\varepsilon} v(t, \cdot).\]

By subtracting (10)-(11) respectively from (12)-(13), we obtain the following equations in errors

$$\frac{p_1}{\varepsilon_M} (e^{n+1} - e^n) - p_2 \Delta e^{n+1} - p_3 e^{n+1} + p_3 D_{old}^{n+1} = E_{old}^{n+1},$$

$$\frac{p_1}{\varepsilon_M} (e^{n+1} - e^n) - p_2 \Delta e^{n+1} + p_3 D_{new}^{n+1} = E_{new}^{n+1},$$

where

$$D_{old}^{n+1} = \frac{2v^3(t_n)}{\left( 1 + 2\frac{p_2}{p_1} \varepsilon_M v^2(t_n) \right)^\frac{1}{2}} \left[ \left( 1 + 2\frac{p_2}{p_1} \varepsilon_M v^2(t_n) \right)^\frac{1}{2} + 1 \right]$$

$$- f(t_{n+1}).$$
\[ D_{n+1}^{new} = \frac{p_1}{\varepsilon_M} v^3(t_n) \frac{S(v(t_n), 1)}{S_{\frac{1}{2}} (z^3_M (0, \cdot), z^2_M (0, \cdot))} \left[ S_{\frac{1}{2}} (z^3_M (0, \cdot), z^2_M (0, \cdot)) + 1 \right] \]

4.1. Consistency result. In what follows we will assume that there exists a maximum principle result for solutions of the fractional steps methods (10)-(11) and \( v_0(x) \in [-1, 1] \) for a.e. \( x \in \Omega \). We have the following result regarding consistency.

**Lemma 4.1.** Assuming that the weak solution of (1) also satisfies \( v \in W^{1,2}_p(Q) \). Then the local truncation errors satisfy:

\[ \varepsilon_M \sum_{n=0}^{M-1} \| E_{old}^{n+1} \|_{L^p(\Omega)}^p \leq \varepsilon_M^{2(p-1)} \int_{0}^{t_M} \| p_1 |v''(\tau)| + 3p_1 |v'(\tau)| \|_{L^p(\Omega)}^p d\tau + T \left( \frac{3p_3}{2p} \right)^p |\Omega|, \]

\[ \varepsilon_M \sum_{n=0}^{M-1} \| E_{new}^{n+1} \|_{L^p(\Omega)}^p \leq \varepsilon_M^{2(p-1)} \int_{0}^{t_M} \left( p_1^p \|v''(\tau)\|_{L^p(\Omega)}^p + 4p_1^p \|v'(\tau)\|_{L^p(\Omega)}^p \right) d\tau + 2p-1p_1^p \varepsilon_M^{1-p}. \]

**Proof.** From the Taylor expansion we have

\[ \frac{v(t_{n+1}) - v(t_n)}{\varepsilon_M} = v'(t_{n+1}) + \int_{t_n}^{t_{n+1}} v''(\tau) \frac{t_n - \tau}{\varepsilon_M} d\tau, \]

which substituting into (12) and using the original equation (1) evaluated at \( t_{n+1} \), i.e.,

\[ p_1 v'(t_{n+1}) - p_2 \Delta v(t_{n+1}) + p_3 v^3(t_{n+1}) - p_3 v(t_{n+1}) = f(t_{n+1}), \]

we obtain

\[ E_{old}^{n+1} = p_1 \int_{t_n}^{t_{n+1}} v''(\tau) \frac{t_n - \tau}{\varepsilon_M} d\tau \]

\[ - p_3 \left[ v^3(t_{n+1}) - v^3(t_n) \right] \frac{2}{\left( 1 + 2p_3 \varepsilon_M v^2(t_n) \right)^\frac{1}{2}} \left[ \left( 1 + 2p_3 \varepsilon_M v^2(t_n) \right)^\frac{1}{2} + 1 \right]. \]
\[
= p_i \int_{t_n}^{t_{n+1}} v''(\tau) \frac{t_n - \tau}{\varepsilon_M} d\tau - p_3 \left( v^3(t_{n+1}) - v^3(t_n) \right) - p_3 v^3(t_n) \left[ 1 - \frac{2}{\left(1 + 2 \frac{p_3}{p_1} \varepsilon_M v^2(t_n)\right)^{\frac{1}{2}} + 1} \right] \\
\leq p_i \int_{t_n}^{t_{n+1}} v''(\tau) \frac{t_n - \tau}{\varepsilon_M} d\tau - p_3 v^3(t_n) \int_{t_n}^{t_{n+1}} \left( v^2(t_{n+1}) + v(t_{n+1})v(t_n) + v^2(t_n) \right) - p_3 v^3(t_n) \left[ \frac{1}{\left(1 + 2 \frac{p_3}{p_1} \varepsilon_M v^2(t_n)\right)^{\frac{1}{2}}} - 1 \right] \\
\leq p_i \int_{t_n}^{t_{n+1}} v''(\tau) \frac{t_n - \tau}{\varepsilon_M} d\tau - p_3 \int_{t_n}^{t_{n+1}} v'(\tau) d\tau \left( v^2(t_{n+1}) + v(t_{n+1})v(t_n) + v^2(t_n) \right) - p_3 v^3(t_n) \left[ \frac{1}{\left(1 + 2 \frac{p_3}{p_1} \varepsilon_M v^2(t_n)\right)^{\frac{1}{2}}} - 1 \right] \\
\leq p_i \int_{t_n}^{t_{n+1}} v''(\tau) \frac{t_n - \tau}{\varepsilon_M} d\tau - p_3 \int_{t_n}^{t_{n+1}} v'(\tau) d\tau \left( v^2(t_{n+1}) + v(t_{n+1})v(t_n) + v^2(t_n) \right) - 2 \varepsilon_M \frac{p_3^2}{p_1} v^5(t_n) \left[ \frac{1}{\left(1 + 2 \frac{p_3}{p_1} \varepsilon_M v^2(t_n)\right)^{\frac{1}{2}}} + 1 \right]^{\frac{1}{2}}.
\]

Using the maximum principle for the exact solution we have

\[
\| E_{n+1} \|_{L^p(\Omega)} \leq p_i \int_{t_n}^{t_{n+1}} |v''(\tau)| d\tau + 3 p_3 \int_{t_n}^{t_{n+1}} |v'(\tau)| d\tau + \varepsilon_M \frac{3 p_3^2}{2 p_1} |\Omega|^{\frac{1}{p}},
\]

and therefore using Hölder’s inequality we obtain

\[
\varepsilon_M \sum_{n=0}^{M-1} \| E_{n+1} \|_{L^p(\Omega)}^p \\
\leq \varepsilon_M \sum_{n=0}^{M-1} \left[ \int_{t_n}^{t_{n+1}} \left( p_i |v''(\tau)| + 3 p_3 |v'(\tau)| \right) d\tau \right]^{\frac{p}{1}} + \varepsilon_M \frac{3 p_3^2}{2 p_1} |\Omega|^{\frac{1}{p}} \\
\leq \varepsilon_M \sum_{n=0}^{M-1} \left[ 2^{p-1} \int_{t_n}^{t_{n+1}} \left( p_i |v''(\tau)| + 3 p_3 |v'(\tau)| \right) d\tau \right]^{\frac{p}{1}} + \varepsilon_M \frac{3 p_3^2}{2 p_1} |\Omega|^{\frac{1}{p}} \\
\leq \varepsilon_M 2^{p-1} \sum_{n=0}^{M-1} \varepsilon_M^{p-1} \int_{t_n}^{t_{n+1}} |p_i |v''(\tau)| + 3 p_3 |v'(\tau)| |\Omega|^{\frac{p}{1}} d\tau + \varepsilon_M \frac{3 p_3^2}{2 p_1} |\Omega|^{\frac{1}{p}}.
\]
Now, using \((t_{n+1} - t_n)\), we have

\[
\begin{align*}
\|E_{n+1}^{\text{new}}\|_{L^p(\Omega)} &\leq p_1 \left[ \int_{t_n}^{t_{n+1}} |v''(\tau)| \, d\tau \right]^{p^*} + 4p_3 \left[ \int_{t_n}^{t_{n+1}} |v'(\tau)| \, d\tau \right]^{p^*} + \frac{p_1}{\varepsilon_M}.
\end{align*}
\]

Now, using \((a + b)^m \leq 2^{m-1}(a^m + b^m)\) as well as the Hölder’s inequality, we obtain

\[
\begin{align*}
\|E_{n+1}^{\text{new}}\|_{L^p(\Omega)}^{p} &\leq 2^{p-1} \left[ \int_{t_n}^{t_{n+1}} (p_1 |v''(\tau)| + 4p_3 |v'(\tau)|) \, d\tau \right]^{p^*} + \left( \frac{p_1}{\varepsilon_M} \right)^p \right]^p
\leq 2^{p-1} \left[ 2^{p-1} \varepsilon_M^{-1} \int_{t_n}^{t_{n+1}} (p_1^p |v''(\tau)|^{p^*} + 4^p p_3^p |v'(\tau)|^{p^*} \, d\tau \right]^{p} + \left( \frac{p_1}{\varepsilon_M} \right)^p.
\end{align*}
\]
Finally, summing the previous inequality for \( n = 0, \ldots, M - 1 \) and then multiplying the result with \( \varepsilon_M \), we get (19) which conclude the proof of the Lemma 4.1.

4.2. Stability result. For future reference, we recall here the following form of the discrete Grönwall lemma. Assume \( w_n, \alpha_n, q_n \geq 0, \beta \in [0, 1) \) satisfy \( w_n + q_n \leq \alpha_n + \beta \sum_{\kappa=1}^{n} w_{\kappa} \ \forall \kappa \geq 0 \), where \( \{\alpha_n\} \) is non-decreasing. Then

\[
w_n + \frac{q_n}{1 - \beta} \leq \frac{\alpha_n - \beta w_0}{1 - \beta} \exp \left( \frac{n\beta}{1 - \beta} \right).
\]

(20)

To obtain convergence results for the numerical methods (10) and (11), we will prove a stability result using energy estimates. In this respect, we begin by testing (14) and (15) with \( e^{n+1} |e^{n+1}|^{p-2} \).

We get

\[
\begin{align*}
\mathbf{T1} & \quad p_1 \int_\Omega \frac{e^{n+1} - e^n}{\varepsilon_M} e^{n+1} |e^{n+1}|^{p-2} dx - p_2 \int_\Omega \Delta e^{n+1} e^{n+1} |e^{n+1}|^{p-2} dx \\
\mathbf{T2} & \quad - p_3 \int_\Omega |e^{n+1}|^p dx + p_3 \int_\Omega D^{n+1}_{old} e^{n+1} |e^{n+1}|^{p-2} dx = \int_\Omega E^{n+1}_{old} e^{n+1} |e^{n+1}|^{p-2} dx,
\end{align*}
\]

(21)

\[
\begin{align*}
\mathbf{T3} & \quad - p_3 \int_\Omega |e^{n+1}|^p dx + p_3 \int_\Omega D^{n+1}_{new} e^{n+1} |e^{n+1}|^{p-2} dx = \int_\Omega E^{n+1}_{new} e^{n+1} |e^{n+1}|^{p-2} dx, \\
\mathbf{T4} & \quad + p_3 \int_\Omega D^{n+1}_{new} e^{n+1} |e^{n+1}|^{p-2} dx = \int_\Omega E^{n+1}_{new} e^{n+1} |e^{n+1}|^{p-2} dx,
\end{align*}
\]

(22)

for any \( p \geq 2 \). In the following, we will analyze each term in (21)-(22) separately.

**T1 & T2.** The following

\[
\frac{1}{p} \left( \|e^{n+1}\|^p_{L^p(\Omega)} - \|e^n\|^p_{L^p(\Omega)} \right) \leq \int_\Omega (e^{n+1} - e^n) e^{n+1} |e^{n+1}|^{p-2} dx,
\]

(23)

\[
(-\Delta e^{n+1}, e^{n+1} |e^{n+1}|^{p-2}) = (p - 1) \int_\Omega |\nabla e^{n+1}|^2 |e^{n+1}|^{p-2} dx.
\]

(24)

was established in [27] and [32].

**T3 & T5.** Now we will analyze the nonlinear terms \( D^{n+1}_{old} \) and \( D^{n+1}_{new} \) defined in (16)-(17).
Lemma 4.2. Assume that there exists a maximum principle result in the case of methods (10) and (11). Then

\[
\int_Ω D^{n+1}_\text{old} e^{n+1} |e^{n+1}|^{p-2} dx \geq -\left(3 + 4 \frac{p_ε}{p_1} \varepsilon_M\right) \int_Ω \left(\frac{p-1}{p} |e^{n+1}|^p + \frac{1}{p} |e^n|^p\right) dx,
\]

and

\[
\int_Ω D^{n+1}_\text{new} e^{n+1} |e^{n+1}|^{p-2} dx \geq \frac{p_1}{4} \varepsilon_M \left[\frac{1}{e^{8 \frac{p_ε}{p_1} \varepsilon_M}} + e^{4 \frac{p_ε}{p_1} \varepsilon_M}\right] \int_Ω \left(\frac{p-1}{p} |e^{n+1}|^p + \frac{1}{p} |e^n|^p\right) dx.
\]

Proof. See [27] and [32].

\(\Box\)

**T4 & T6.** Using the Young inequality

\[
ab \leq \frac{1}{m} a^m + \frac{1}{n} b^n, \quad \frac{1}{m} + \frac{1}{n} = 1, \quad a, b \in \mathbb{R}^+,
\]

with \(a = |E^{n+1}|, \ b = |e^{n+1}|^{p-1}, \ m = p\) and \(n = \frac{p}{p-1}\), the local truncation error term gives,

\[
(E^{n+1}, e^{n+1} |e^{n+1}|^{p-2}) \leq \frac{1}{p} ||E^{n+1}||_{L^p(Ω)}^p + \frac{p-1}{p} ||e^{n+1}||_{L^p(Ω)}^p,
\]

where \(E^{n+1} = E^{n+1}_\text{old}\) or \(E^{n+1} = E^{n+1}_\text{new}\).

As we have proposed previously, at this point we have the estimations (23)-(27), corresponding to the terms **T1 - T6.** Now we will continue with the evaluation of relations (21)-(22). We first substitute in (21) the relations (23)-(24) and (27) corresponding to \(E^{n+1} = E^{n+1}_\text{old}\) to obtain

\[
\frac{p_1}{p \varepsilon_M} \int_Ω \|e^{n+1}|^p - |e^n|^p\| dx
\]

\[
+ p_2(p-1) \int_Ω \|\nabla e^{n+1}\| e^{n+1} |e^{n+1}|^{p-2} dx + p_3 \int_Ω D^{n+1}_\text{old} e^{n+1} |e^{n+1}|^{p-2} dx
\]

\[
\leq p_3 ||e^{n+1}||_{L^p(Ω)}^p + \frac{1}{p} ||E^{n+1}_\text{old}||_{L^p(Ω)}^p + \frac{p-1}{p} ||e^{n+1}||_{L^p(Ω)}^p
\]

\[
= \frac{1}{p} ||E^{n+1}_\text{old}||_{L^p(Ω)}^p + \left(p_3 + \frac{p-1}{p}\right) ||e^{n+1}||_{L^p(Ω)}^p.
\]

Sum (28) for \(n = 0, \ldots, M - 1\)

\[
\frac{p_1}{p \varepsilon_M} \|e^n\|_{L^p(Ω)}^p + p_2(p-1) \sum_{n=1}^M \|\nabla e^n\| e^n |e^n|^{p-2} dx + p_3 \sum_{n=1}^M \int_Ω D^{n}_\text{old} e^n |e^n|^{p-2} dx
\]

\[
\leq \frac{p_1}{p \varepsilon_M} \|e^0\|_{L^p(Ω)}^p + \frac{1}{p} \sum_{n=1}^M \|E^{n}_\text{old}\|_{L^p(Ω)}^p + \left(p_3 + \frac{p-1}{p}\right) \sum_{n=1}^M ||e^n||_{L^p(Ω)}^p.
\]
and multiplying by \( \varepsilon_M \frac{p}{p_1} \) yields

\[
\| e^M \|_{L^p(\Omega)}^p + p(p - 1) \frac{p_2}{p_1} \varepsilon_M \sum_{n=1}^M \int_\Omega |\nabla e^n|^2 |e^n|^{p-2} \, dx + \frac{p_2}{p_1} \varepsilon_M \sum_{n=1}^M \int_\Omega D_{\text{old}}^n e^n |e^n|^{p-2} \, dx
\]

\[
\leq \| e^0 \|_{L^p(\Omega)}^p + \frac{1}{p_1} \varepsilon_M \sum_{n=1}^M \| E_{\text{odd}}^n \|_{L^p(\Omega)}^p + (p_3 + \frac{p-1}{p}) \frac{p_2}{p_1} \varepsilon_M \sum_{n=1}^M \| e^n \|_{L^p(\Omega)}^p.
\]

(29)

In the fractional steps method (10) we obtain from (25) and (29)

\[
\| e^M \|_{L^p(\Omega)}^p + p(p - 1) \frac{p_2}{p_1} \varepsilon_M \sum_{n=1}^M \int_\Omega |\nabla e^n|^2 |e^n|^{p-2} \, dx
\]

\[
\leq \left( 1 + \frac{p_2}{p_1} \left( 3 + 4 \frac{p_2}{p_1} \varepsilon_M \right) \varepsilon_M \right) \| e^0 \|_{L^p(\Omega)}^p + \frac{1}{p_1} \varepsilon_M \sum_{n=1}^M \| E_{\text{odd}}^n \|_{L^p(\Omega)}^p
\]

\[
+ \left[ 4p_3 \left( 1 + \frac{p_2}{p_1} \varepsilon_M \right) \left( 1 + \frac{p_2}{p_1} \right) \right] \frac{p_2}{p_1} \varepsilon_M \sum_{n=1}^M \| e^n \|_{L^p(\Omega)}^p.
\]

Therefore, assuming that the time-step is small enough

\[
\varepsilon_M \leq \varepsilon_{\text{old}} := \frac{p_1}{8 pp_3} \left( \sqrt{(4p_3 p + p - 1)^2 + 16 pp_2^2 - 4p_3 p - p + 1} \right), \tag{30}
\]

we obtain from Grönwall’s inequality (20) the following stability estimate

\[
\| e^M \|_{L^p(\Omega)}^p + p(p - 1) \frac{p_2}{p_1} \varepsilon_M \sum_{n=1}^M \int_\Omega |\nabla e^n|^2 |e^n|^{p-2} \, dx
\]

\[
\leq \frac{1 - \varepsilon_M}{\varepsilon_{\text{old}} - \varepsilon_M} \times \exp \left( \frac{M \varepsilon_M}{\varepsilon_{\text{old}} - \varepsilon_M} \right).
\]

In the fractional steps method (11) we obtain from (23)-(24) and (27) corresponding to \( E^{n+1} = E_{\text{new}}^{n+1} \)

\[
\| e^M \|_{L^p(\Omega)}^p + p_2 (p - 1) \frac{p_2}{p_1} \varepsilon_M \sum_{n=1}^M \int_\Omega |\nabla e^n|^2 |e^n|^{p-2} \, dx
\]

\[
\leq \| e^0 \|_{L^p(\Omega)}^p + \frac{1}{p_1} \varepsilon_M \sum_{n=1}^M \| E_{\text{new}}^n \|_{L^p(\Omega)}^p + \frac{p_2}{p_1} \varepsilon_M \sum_{n=1}^M \| e^n \|_{L^p(\Omega)}^p.
\]

(31)

Therefore, assuming that the time-step is small enough

\[
\varepsilon_M \leq \varepsilon_{\text{new}} := \frac{p_1}{p - 1}, \tag{32}
\]
using Grönwall’s inequality (20), we obtain from (31) the following stability estimate

\[
\|e^M\|_{L^p(\Omega)}^p + p(p - 1) \frac{p_2}{p_1} \frac{\varepsilon_M}{1 - \varepsilon_M / \varepsilon_{new}} \sum_{n=1}^{M} \int_{\Omega} |\nabla e^n|^2 |e^n|^{p-2} \, dx \\
+ \frac{p_2}{4} \frac{\varepsilon_M}{\varepsilon_{new}} \left[ \frac{1}{e^{\frac{\varepsilon_M}{p_1}}} + e^{\frac{\varepsilon_M}{p_2}} \right] \sum_{n=1}^{M} \int_{\Omega} |e^n|^p \, dx \\
\leq \frac{\left(1 - \frac{\varepsilon_M}{\varepsilon_{new}}\right) \|e^0\|_{L^p(\Omega)}^p + \frac{1}{p_1} \frac{\varepsilon_M}{1 - \varepsilon_M / \varepsilon_{new}} \sum_{n=1}^{M} \|E^n_{old}\|_{L^p(\Omega)}^p}{1 - \frac{\varepsilon_M}{\varepsilon_{new}}} \times \exp\left(\frac{M\varepsilon_M}{\varepsilon_{new} - \varepsilon_M}\right).
\]

Now we collect the stability estimates for the error equations we proven so far in the following result.

Lemma 4.3. Assuming the time-step \(\varepsilon_M\) for methods (10) and (11) satisfy (30) and (32), respectively. Then the errors satisfy the following estimates

\[
\|e^M\|_{L^p(\Omega)}^p + p(p - 1) \frac{p_2}{p_1} \frac{\varepsilon_M}{1 - \varepsilon_M / \varepsilon_{new}} \sum_{n=1}^{M} \int_{\Omega} |\nabla e^n|^2 |e^n|^{p-2} \, dx \\
+ \frac{p_2}{4} \frac{\varepsilon_M}{\varepsilon_{new}} \left[ \frac{1}{e^{\frac{\varepsilon_M}{p_1}}} + e^{\frac{\varepsilon_M}{p_2}} \right] \sum_{n=1}^{M} \int_{\Omega} |e^n|^p \, dx \\
\leq \exp\left(\frac{M\varepsilon_M}{\varepsilon_{new} - \varepsilon_M}\right) \left[ \|e^0\|_{L^p(\Omega)}^p + \frac{1}{p_1} \frac{\varepsilon_M}{1 - \varepsilon_M / \varepsilon_{new}} \sum_{n=1}^{M} \|E^n_{old}\|_{L^p(\Omega)}^p \right].
\]

Finally we can prove that methods (10)-(11) are first order accurate in time.

Theorem 4.1 Assume that the time steps \(\varepsilon_M\) satisfies the assumption formulated in and that the exact solution \(v\) to problem (1) is also \(W^{1,2}_p(Q)\) regular. Then methods (10)-(11) satisfy the following error estimates

\[
\|e^M\|_{L^p(\Omega)}^p + p(p - 1) \frac{p_2}{p_1} \frac{\varepsilon_M}{1 - \varepsilon_M / \varepsilon_{new}} \sum_{n=1}^{M} \int_{\Omega} |\nabla e^n|^2 |e^n|^{p-2} \, dx \\
+ \frac{p_2}{4} \frac{\varepsilon_M}{\varepsilon_{new}} \left[ \frac{1}{e^{\frac{\varepsilon_M}{p_1}}} + e^{\frac{\varepsilon_M}{p_2}} \right] \sum_{n=1}^{M} \int_{\Omega} |e^n|^p \, dx \\
\leq \exp\left(\frac{M\varepsilon_M}{\varepsilon_{old} - \varepsilon_M}\right) \left[ (1 + \frac{p_2}{p_1} (3 + 4p_2 \varepsilon_M) \frac{\varepsilon_M}{1 - \varepsilon_M / \varepsilon_{old}}) \|e^0\|_{L^p(\Omega)}^p + \frac{1}{p_1} \frac{\varepsilon_M}{1 - \varepsilon_M / \varepsilon_{old}} \sum_{n=1}^{M} \|E^n_{old}\|_{L^p(\Omega)}^p \right].
\]
\[ \|e^M\|_{L^p(\Omega)}^p + p(p-1)p_2 \frac{\varepsilon_M}{p_1 - \varepsilon_M/\varepsilon_{new}} \sum_{n=1}^{M} \int_{\Omega} |\nabla e^n|^2 |e^n|^{p-2} \, dx \]  
\[ + \frac{p^2}{4} \varepsilon_M \left[ \frac{1}{e^{p_1 - M}} + e^{\frac{p_2}{p_1 - M}} \right] \sum_{n=1}^{M} \int_{\Omega} |e^n|^p \, dx \]  
\[ \leq \exp \left( \frac{M \varepsilon_M}{\varepsilon_{new} - \varepsilon_M} \right) \left[ \|e^0\|_{L^p(\Omega)}^p \right]  
\[ + \frac{1}{p_1 - \varepsilon_M/\varepsilon_{new}} \left( \int_{\Omega} \left| p_1 |u''(\tau)| + 3p_3 |u'(\tau)| \right|^p \, d\tau + T \left( \frac{3}{2} p_3 \right)^p |\Omega| \right) \].

Proof. Using Lemma 4.1 and Lemma 4.3, we can easy deduce that relations (35) and (36) are true.

5. Numerical approach.

5.1. Numerical methods and stability conditions. In this Subsection we are concerned with the numerical approximation of the solution \( v^{n+1}(t, x) \) in (10)-(11), denoted by \( V(t, x) \) in the sequel. As already stated, we will work in one dimension and then \( \Delta v^{n+1} = \Delta V = V_{xx} \). To fix the ideas, let \( \Omega = [0, b] \subset \mathbb{R}_+ \) and we introduce over it the grid with \( N \) equidistant nodes

\[ x_j = (j-1)dx \quad j = 1, 2, \ldots, N, \quad dx = \frac{b}{N-1}. \]

Given a positive value \( T \) and considering \( M \) as the number of equidistant nodes in which is divided the time interval \([0, T]\), we set

\[ t_i = (i-1)\varepsilon_M \quad i = 1, 2, \ldots, M, \quad \varepsilon_M = \frac{T}{M-1}. \]

Now we denote by \( V^i_j \) the approximate values in the point \((t_i, x_j)\) of the unknown functions \( V(t, x) \). More precisely

\[ V^i_j = V(t_i, x_j) \quad i = 1, 2, \ldots, M, \quad j = 1, 2, \ldots, N, \]

or, for later use

\[ V^i \triangleq (V^1_1, V^1_2, \ldots, V^1_N)^T \quad i = 1, 2, \ldots, M. \]  

(37)

We continue by explaining how we treat each term in (10)-(11). The Laplace operator will be approximated by a second order centred finite differences, which means:

\[ V_{xx}(t_i, x_j) = \Delta_{dx} V^i_j \approx \frac{V^i_{j+1} - 2V^i_j + V^i_{j-1}}{dx^2} \quad i = 1, 2, \ldots, M, \quad j = 1, 2, \ldots, N. \]  

(38)

(\( \Delta_{dx} \) is the discrete Laplacian depending on the step-size \( dx \)).

Involving the separation of variables method to solve the Cauchy problem (9) (see Morosanu [26]), we get

\[ \begin{cases} 
V^1_j = Z_M(\varepsilon_M, V_-(t_1, x_j)) = Z_M(\varepsilon_M, v_0(x_j)) = v_0(x_j) \sqrt{\frac{p_3}{p_3 + \varepsilon_M v_0(x_j)}}, \\
V^i_j = Z_M(\varepsilon_M, V_-(t_i, x_j)) = V_-(t_i, x_j) \sqrt{\frac{p_3}{p_3 + \varepsilon_M V_-(t_i, x_j)}} \quad i = 2, \ldots, M-1, \\
j = 1, 2, \ldots, N.
\]  

(39)
Corresponding to $\Omega$, already chosen in one dimension, the boundary $\partial \Omega$ is reduced to the set $\{0, b\}$. To approximate $V_x(0)$ ($V_x(b)$) we will use a backward (forward) finite differences; this leads to

$$V_0^i = V_1^i, \quad V_{N+1}^i = V_N^i \quad i = 1, 2, \ldots, M,$$

where $V_0^i$ and $V_{N+1}^i$ are dummy variables.

Finally we refer to the term $p_3 V(t_i, x_j)$ in (10). To approximate this quantity (the reaction term), we will involve an implicit formula, i.e.:

$$p_3 V(t_i, x_j) \approx p_3 V_j^i \quad i = 1, 2, \ldots, M, \quad j = 1, 2, \ldots, N.$$

We are now ready to build the approximation scheme, mentioned at the beginning.

**1-IMBDF - First-order Implicit Backward Difference Formula.** To develop such a scheme, we begin by replacing in (10) the approximations stated in (38)-(41). We deduce:

$$p_1 \frac{V_j^i - V_{j-1}^i}{\varepsilon M} = p_2 \Delta x V_j^i + p_3 V_j^i + f_j^i,$$

for $i = 2, 3, \ldots, M, \quad j = 1, 2, \ldots, N$ and $f_j^i = f(t_i, x_j)$.

Using in (42) the equalities from (38) and arranging convenient, we conclude that, via 1-IMBDF, the system (10) is discretized as follows

$$-p_3 \frac{\varepsilon M}{dx} V_{j-1}^i + \left[ p_1 + 2p_2 \frac{\varepsilon M}{dx^2} - p_3 \varepsilon M \right] V_j^i - p_2 \frac{\varepsilon M}{dx^2} V_{j+1}^i = p_1 V_{j-1}^i + \varepsilon M f_j^i,$$

for $i = 2, 3, \ldots, M, \quad j = 1, 2, \ldots, N$.

In order to compute the matrix $(V_j^i)$, the linear system (43) will be solved ascending with respect to time levels. For the first time level ($i = 1$), the value $V_1^i$ is computed by (39). Moreover, let us point out from (40) and (43) that we have $N$ unknowns for each time-level $i$, $i = 2, 3, \ldots, M$ (see also (37)).

If we set

$$c_1 = -p_2 \frac{\varepsilon M}{dx^2}, \quad c_2 = p_1 - 2c_1, \quad c_3 = \frac{p_3 \varepsilon M}{dx},$$

then the system (43), coupled with (40), can be rewritten in matrix form as

$$AV^i = B V^{i-1} + d^i \quad i = 2, 3, \ldots, M,$$

where

$$A = \begin{pmatrix} c_1 + c_2 & c_1 & 0 & \cdots & 0 & 0 \\ c_1 & c_2 & c_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_1 & c_2 \\ 0 & 0 & 0 & \cdots & c_1 & c_1 + c_2 \end{pmatrix}, \quad B = \begin{pmatrix} p_1 & 0 & \cdots & 0 & 0 \\ 0 & p_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p_1 \\ 0 & 0 & 0 & \cdots & 0 & p_1 \end{pmatrix},$$

$$d^i = \varepsilon M \left(f_1^i, f_2^i, \ldots, f_N^i\right)^T.$$

Therefore, the general design of the algorithm to calculate the approximate solution of nonlinear system (1), via fractional steps method and 1-IMBDF, is the following one.
Begin Alg. 1-IMBDF

Choose $T > 0$, $b > 0$;
Choose $M > 0$, $N > 0$ and compute $\varepsilon_M$, $dx$;
Choose $v_0(x)$, $f(t, x)$;
For $i = 2$ to $M$ do
    Compute $V^{i-1}$ using (39);
    Compute $V^i$ solving the linear system (44);
End-for;
End.

As it is well known, most initial value problems reduce to solving large sparse linear systems of the form (44). For later use (e.g., numerical implementation of conceptual algorithms), we will prove the following

**Lemma 5.1** If

$$ p_1 + p_2 \frac{\varepsilon_M}{dx^2} \neq p_3 \varepsilon_M, \quad (45) $$

then the matrix coefficients in linear system (44) can be factored into the product of a lower-triangular matrix and an upper-triangular matrix ($LU$-factorization).

*Proof.* Let denote by $a_{mn}$, $m, n = 1, 2, \ldots, N$, the elements of matrix coefficients in linear system (44). Analyzing the main diagonal elements of matrix $A$, we find that $c_1 + c_2 \neq 0$ reflect the assumptions expressed in (45), as well as that $c_2 \neq 0$. Thus, we find easily that $a_{mm} \neq 0 \forall m = 1, 2, \ldots, N$ and so the Gaussian elimination can be performed on the system (44) without interchanges; consequently $A$ has an $LU$ factorization. \hfill □

5.2. **Stability conditions.** To establish conditions of stability for the linear difference equation (44), introduced in the previous subsection, we will use in our analysis the Lax-Richtmyer definition of stability, expressed in terms of norm $\| \cdot \|_{\infty}$ (see [29] and references therein). The equation (44) may be rewritten in a more convenient form as

$$ V^i = A^{-1} BV^{i-1} + A^{-1} d^i \quad i = 2, 3, \ldots, M \quad (46) $$

(the existence of $A^{-1}$ will be proved in the Proposition 3.1 below). In addition, the matrix $A$ can be written in the form

$$ A = D(I + D^{-1}G) \quad (47) $$

where $D = \text{diag}(c_1, c_2, \ldots, c_2, c_1 + c_2)$ and $G = A - D$. Thus, noting $a_1 = c_1 + c_2$, we have

$$ D^{-1}G = \begin{pmatrix} 0 & c_1 & 0 & \cdots & 0 & 0 \\ c_1 & 0 & c_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_1 & 0 \\ 0 & 0 & 0 & \cdots & c_2 & c_2 \\ 0 & 0 & 0 & \cdots & c_2 & c_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{c_1}{a_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{c_1}{a_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{c_1}{a_1} \end{pmatrix} $$
and a simple analysis of all lines in matrix $D^{-1}G$ allows us to deduce that we only have two distinct lines. The sum of each such line is written in vector $w$ below

$$w = \begin{bmatrix} c_1, 2c_1, a_1, c_2, \end{bmatrix}.$$  \hfill (48)

Let’s denote by $v_{\text{min}} = \min\{|a_1|, |c_2|\}$.

Now we are able to prove the following result with respect to the stability in matrix equation (46).

**Proposition 5.1.** Suppose that $v_{\text{min}} - 2|c_1| > 0$. If

$$0 < \frac{p_1}{v_{\text{min}} - 2|c_1|} < 1,$$

then the equation (46) is stable. Otherwise, it is unstable.

**Proof.** The proof is reduced to checking the condition of stability which, based on the Lax-Richtmyer definition mentioned above and taking into account the relation (46), it reduces to check the inequality

$$\|A^{-1}B\|_\infty < 1.$$  

We begin our analysis by determining an estimate for $\|D^{-1}G\|_\infty$. As we have already noted (see relation (48)), this is equivalent with the following equality:

$$\|D^{-1}G\|_\infty = \max |w|,$$

wherefrom we easily derive the estimate

$$\|D^{-1}G\|_\infty < \frac{2|c_1|}{v_{\text{min}}}. \hfill (49)$$

The estimate (49) allows us now to prove the existence of $A^{-1}$. Indeed, since by hypothesis we have assumed that $2|c_1| < v_{\text{min}}$ than $\|D^{-1}G\|_\infty < 1$ which guarantees that there exist $(I+D^{-1}G)^{-1}$. Moreover, there exist $A^{-1}$ and $A^{-1} = (I+D^{-1}G)^{-1}D^{-1}$. Using the well known inequality: $\|(I + D^{-1}G)^{-1}\|_\infty \leq \frac{1}{1-\|D^{-1}G\|_\infty}$ and making use of (47), it follows that

$$\|A^{-1}\|_\infty \leq \|(I + D^{-1}G)^{-1}\|_\infty \|D^{-1}\|_\infty \leq \frac{1}{1-\|D^{-1}G\|_\infty} \|D^{-1}\|_\infty. \hfill (50)$$

How $\|D^{-1}G\|_\infty \leq 1$ imply that $1 - \|D^{-1}G\|_\infty \geq 1 - \frac{2|c_1|}{v_{\text{min}}} > 0$, we easily deduce from this that

$$0 < \frac{1}{1-\|D^{-1}G\|_\infty} \leq \frac{v_{\text{min}}}{v_{\text{min}} - 2|c_1|},$$

Since $\|D^{-1}\|_\infty \leq \frac{1}{v_{\text{min}}}$ and involving the above estimate, from (50) we finally obtain

$$\|A^{-1}\|_\infty \leq \frac{1}{v_{\text{min}} - 2|c_1|}. \hfill (51)$$

Now we turn our attention to matrix $B$. Analyzing the matrix $B$ lines, it follows that

$$\|B\|_\infty = p_1. \hfill (52)$$

Summing up and making use of (51)-(52) we derive the following estimate

$$\|A^{-1}B\|_\infty \leq \|A^{-1}\|_\infty \|B\|_\infty \leq \frac{1}{v_{\text{min}} - 2|c_1|} \|B\|_\infty,$$

which leads to the estimate $\|A^{-1}B\|_\infty < 1$ as we claimed at beginning. \hfill \square
5.3. Numerical experiments. We will compare the two numerical solutions obtained by (10) and (11) with the following exact solution to (1)

\[ v_e(t,x) := \exp(-2\omega^2 t) \cos \left( \frac{\pi x}{b} \right), \quad t \in [0,T], \quad x \in [0,b], \]

with the forcing term

\[ f_{v_e}(t,x) = e^{-2\omega^2 t} \cos \left( \frac{\pi x}{b} \right) \left[ -2\omega^2 p_1 - p_2 \left( \frac{\pi}{b} \right)^2 - p_3 \left( 1 - \exp(-4\omega^2 t) \cos^2 \left( \frac{\pi x}{b} \right) \right) \right] \]

In numerical tests we will consider a particular case of the nonlinear reaction-diffusion equation (1), namely, the Allen-Cahn equation ([1]), which means \( p_1 = \alpha * \xi, \ p_2 = \xi \) and \( p_3 = \frac{1}{2} \xi \).

The initial values \( v_0(x_j), \ j = 1,2,...,N \), were computed via Matlab function `csapi(v0)` - cubic spline interpolant to the given data:

\[
\begin{align*}
&v_0 = [-1.4 -1.4 -1.44 -1.42 -1.44 -1.43 -1.43 -1.42 -1.4 -1.4 -1.4 -1.4 -1.25 -1.2 -1.17 -1.15 \ldots \nonumber \\
&\quad -1.1 -1.08 -1.0 -0.95 -0.85 -0.8 -0.8 -0.75 -0.7 -0.7 -0.6 -0.59 -0.56 -0.52 -0.49 -0.47 -0.44 -0.42 -0.39 -0.36 -0.33 -0.3 \ldots \nonumber \\
&\quad .7 -0.79 -0.87 -0.88 -0.8 -0.81 \ldots -0.75 -0.68 -0.6 -0.57 -0.55 -0.53 -0.51 -0.49 -0.47 -0.45 -0.43 -0.41 -0.39 \ldots \nonumber \\
&\quad .59 1. 1.08 1.1 1.15 1.17 1.2 1.25 1.3 1.3 1.3 1.3 1.3 1.3 1.3 1.3 1.3 1.3 \ldots];
\end{align*}
\]

For beginning, taking \( T = 1, \omega = 0.5, b = 1, \alpha = 1.0 e + 2, \xi = .5, N = 31, \varepsilon_M = 0.1, M = T/\varepsilon_M \), the shape of the graphs plotted in Figures 1 shows the stability and accuracy of the numerical results obtained by implementing the conceptual algorithm Alg.1-IMBDF (see (45)), while, the errors \( \|v_e - V_N\|_\infty \) produced by three methods analyzed in [32] (the Newton method, the linearized method and the old fractional steps method - (10)), as well as the new fractional steps method (11), are shown in Figure 2.

The numerical experiment was performed further with \( T = 2, b = 1 \) (see Figure 3) and \( T = 2, b = 2 \) (see Figure 4). The approximate solution \( V_M^j, j = 1.2,\ldots,N \) was computed iteratively for \( \varepsilon_M = \varepsilon_M/k, \ k = 1,2,\ldots,5 \).

6. Conclusions.
- As a novelty of this work we refer firstly to the new scheme of fractional steps type, introduced by (11) in order to approximate the solution to the nonlinear reaction-diffusion problem (1). Corresponding, we have considered an IMEX scheme and we have proved stability result for the error equation.
which shows us that the new fractional steps method depend linearly on the small parameter, like as in the methods studied in [32].

- Secondly, in the numerical experiments we focus our attention on a particular case of (1) - the Allen-Cahn equation, which serves as a mathematical model for many complex moving interface problems and in which the challenge in terms of numerical analysis is due to the thickness of the interface separating different phases.

- Not least, let’s remark from the graphical representation of the errors (see Figures 2-4), produced by those four methods analyzed, that conditions of consistency and stability are sustained by both theory and numerical experiment and that are significantly influenced by the parameters of time and space (see [29] for detailed discussions regarding the dependence on the physical parameters). In addition, let’s remark that conditions of stability are sustained by both theory and numerical experiment (see Figure 1) and that are significantly dependent on all parameters (see Proposition 5.1).

Figure 2. Errors $\|v_e - V_j^M\|_\infty$ of the Newton, the linearized and the fractional steps methods: (10)-(11)

Figure 3. Errors $\|v_e - V_j^M\|_\infty$ of the Newton, the linearized and the fractional steps methods: (10)-(11)
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