Topological Conformal Gravity in Four Dimensions

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Abstract

In this paper, we present a new formulation of topological conformal gravity in four dimensions. Such a theory was first considered by Witten as a possible gravitational counterpart of topological Yang–Mills theory, but several problems left it incomplete. The key in our approach is to realise a theory which describes deformations of conformally self-dual gravitational instantons. We first identify the appropriate elliptic complex which does precisely this. By applying the Atiyah–Singer index theorem, we calculate the number of independent deformations of a given gravitational instanton which preserve its self-duality. We then quantise topological conformal gravity by BRST gauge-fixing, and discover how the quantum theory is naturally described by the above complex. Indeed, it is a process which closely parallels that of the Yang–Mills theory, and we show how the partition function generates an uncanny gravitational analogue of the first Donaldson invariant.

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1. Introduction

It is somewhat ironic to note that in recent years, physicists have applied the Feynman path integral formalism in quantum field theory to derive various non-trivial differential-topological results in mathematics. Such theories are generally known as topological quantum field theories (a review may be found in ref. [1]), and have largely arisen from the work of Witten.

In ref. [2], Witten introduced a type of supersymmetric Yang–Mills theory whose partition function generates the Donaldson polynomial invariants of smooth four-manifolds [3]. The essential point of this model is the introduction of ghost fields to cancel any local dynamical degrees of freedom. This produces a quantum field theory which does not depend on any local structure like the metric, but whose observables are manifold invariants. Such invariants characterise the global properties of the manifold in question, which may be its topology, or its underlying smooth or differential structure as in the case of the Donaldson polynomials. The analogue of this theory in two dimensions is a certain non-linear, supersymmetric sigma model [4], which Witten has shown to give rise to invariants of symplectic geometry due to Gromov. In a third paper [5], Witten also showed how the three-dimensional Chern–Simons action produces the Jones polynomial of knot theory.

While the techniques used in quantum field theory will probably not stand up to a completely mathematically rigorous treatment, mathematicians may still profitably utilise these physical constructions to gain insight into both presently known and new manifold invariants. A prime example of this is the use of Yang–Mills instantons by Donaldson [3] to construct his polynomial invariants. Conversely, topological quantum field theories provide fertile ground for physicists to deepen the understanding of the (in)formal manipulations of Feynman path integrals, especially when using them to “derive” rigorous results in mathematics like the Donaldson and Jones polynomials.

It is with this philosophy in mind that we write this paper. Back in four dimensions, Witten [6] has considered conformal gravity as a candidate for constructing a gravitational counterpart of the Donaldson polynomials. However, the model he constructed was unsatisfactory in a few ways, and he thus left open the question of whether his theory led to any invariants of four-manifolds. In this paper, we will attempt to rebuild this model from a new point of view. Throughout, we will draw upon the many remarkable similarities between topological Yang–Mills theory and its conformal gravity analogue. Indeed,
it is surprising that while self-dual Yang–Mills instantons over four-manifolds play a very important rôie in the recent study of these manifolds, self-dual gravitational instantons \textit{intrinsic} to four-manifolds have yet to make an impact on this area. We will thus propose suitable gravitational analogues of the Donaldson polynomials as possibly new invariants of four-manifolds.

Let us now give a lightning review of several important points of topological Yang–Mills theory \cite{2,4} that will be relevant to our discussion below. Consider the classical action

\[ \int_M d^4 x \ tr \{ F_{ab} \ast F^{ab} \}, \quad (1.1) \]

where \( F_{ab} \) is the usual Yang–Mills field strength of the gauge potential \( A_a \). The trace is over the gauge-group indices, and \( M \) is a compact four-manifold. \( \ast F_{ab} = \frac{1}{2} \epsilon_{abcd} F_{cd} \) is the dual of \( F_{ab} \). This action is invariant under arbitrary variations of the gauge field \( A_a \). In order to quantise it using the Feynman path integral formalism, the topological symmetry of the action has to be gauge-fixed. This is done using the BRST quantisation method, and it introduces three quantum ghost fields to completely break the symmetry. With an appropriate choice of gauge \cite{1}, the final gauge-fixed action turns out to be the ordinary Yang–Mills functional

\[ \int_M d^4 x \ tr \{ F_{ab} F^{ab} \}, \quad (1.2) \]

plus extra ghost interaction terms which cancel out all local degrees of freedom in (1.1).

Now, the classical action (1.2) is minimised by field strengths which satisfy either the self-duality condition \( F_{ab} = \ast F_{ab} \), or the anti-self-duality condition \( F_{ab} = - \ast F_{ab} \). Their corresponding gauge potentials are known as Yang–Mills instantons. Thus, in evaluating the partition function and quantum observables of topological Yang–Mills theory, we have to consider quantum fluctuations around these classical instanton solutions. In other words, we need a theory describing self-dual, infinitesimal \textit{deformations} around a given Yang–Mills instanton.

Such a theory was first studied by Atiyah, Hitchin and Singer \cite{8} in the more general context of the Yang–Mills moduli space, that is the space of all gauge-inequivalent instantons. This moduli space was a vital tool used by Donaldson in discovering his manifold invariants \cite{3}. For our present purposes, the most important property of it that we need to know is its dimension. Observe that the value of the dimension is precisely the number of independent non-trivial deformations that can be made around an instanton, which preserve its self-duality. (Physicists usually refer to this number as the number of
free parameters of the instanton solution.) It depends only on the topological properties of the manifold and fibre bundle in question.

The way to derive the value of this dimension is to introduce a sequence of mappings between vector bundles, known as an elliptic complex, which describes deformations of instantons [8]. (This has also been done by physicists in ref. [9].) It is closely related to the de Rham complex of forms. By invoking the Atiyah–Singer index theorem [10], one can then calculate the so-called index of this elliptic complex. The value of this index turns out to be the dimension of the moduli space.

The nature of the partition function of topological Yang–Mills theory depends crucially on the dimension of the Yang–Mills moduli space [2]. As it also turns out, the three gauge-fixing ghost fields of the quantum action belong to each of the three vector bundles of the elliptic complex. This is not too surprising by remarks we made earlier, that these ghost fields characterise quantum deformations about the classical instanton solutions of (1.2). Thus, the importance of the mathematical theory of instanton deformations to topological Yang–Mills theory simply cannot be overlooked.

In this paper, we will ask ourselves whether we can proceed analogously to construct a theory of topological gravity. In ref. [6], Witten proposed conformal gravity as the appropriate counterpart to the Yang–Mills theory. This theory has, as the classical action,

\[ \int_M d^4x \sqrt{g} C_{abcd} C^{abcd}, \]  

where \( C_{abcd} \) is the Weyl tensor of \( M \). This is the gravitational analogue of (1.2). It is minimised by manifolds possessing either a self-dual or anti-self-dual Weyl tensor, and such manifolds are known as conformally (anti-) self-dual gravitational instantons [11].

The corresponding topological action for conformal gravity is

\[ \int_M d^4x \sqrt{g} \, C_{abcd} \ast C^{abcd}, \]  

clearly the analogue of (1.1). We will construct its quantum theory by BRST gauge-fixing, and this introduces three ghost fields. Again with an appropriate choice of gauge, the total quantum action turns out to be the sum of (1.3) and a set of ghost interaction terms, which cancel out all local degrees of freedom in (1.3).

It is clear that we now need a deformation theory of conformally self-dual gravitational instantons, to characterise the quantum effects of the ghost fields. We write down an elliptic complex which realises this. Actually, this complex was considered by I.M. Singer in
unpublished work many years ago [12], but has since remained relatively unknown in both the mathematics and physics literature. We will refer to this elliptic complex as the gravitational instanton deformation complex. By applying the Atiyah–Singer index theorem to this elliptic complex, we derive an expression for the number of independent deformations of a given manifold that preserve its self-duality. This number $\varpi$ is essentially a topological quantity, which will be important in the quantum theory of topological gravity. Under certain conditions, one could then integrate these infinitesimal deformations to obtain a local moduli space of gravitational instanton metrics, whose dimension is $\varpi$.

Since Witten’s original work on topological gravity [6], there have been several papers written on such theories in four dimensions. In fact, it was very quickly demonstrated [14,15] how Witten’s quantum topological gravity theory arises from the BRST gauge-fixing of the classical action (1.4). However, as in the original paper itself, the significance of the gravitational instanton deformation complex in this theory was not appreciated. Consequently, topological gravity in four dimensions became a nightmare of long spinor expressions and complicated differential operators, and explicit invariants were not found. What we have succeeded in doing, in this paper, is to simplify the whole process of gauge-fixing topological gravity, by rewriting everything in terms of the two differential operators occurring in the deformation complex. This results in a procedure that is no more complicated than gauge-fixing topological Yang–Mills theory, and its interpretation becomes much clearer. In particular, we are able to demonstrate how to calculate a simple “manifold invariant”, which is just the value of the partition function itself, in the case when the gravitational moduli space is discrete. It turns out to be the number of points in this moduli space, counted with signs. This is strikingly similar to the first Donaldson invariant in the Yang–Mills case [3].

We perhaps should also mention that there has been some effort by Torre [16] to construct a topological field theory based on the deformation theory of gravitational instantons. This is in much the same spirit as this paper, and indeed he has constructed an elliptic complex to study these deformations. But while his use of Ashtekar variables is a very good idea, he did not apply the Atiyah–Singer index theorem to calculate the index of his complex, and so was not led to a complete theory.

\footnote{1 However, very recently, there emerged a rigorous mathematical study into such conformally self-dual deformations of gravitational instantons [13], in an effort to discover invariants of four-manifolds just as Donaldson used Yang–Mills instantons to construct his invariants.}
Several other groups [17,18] have also in fact studied four-dimensional topological gravity, not constructed from (1.4), but from topological combinations of the Riemann tensor. Hence their models are not conformally invariant like ours.

Before beginning in detail, let us clarify the spinor notation used in this paper. The analogue of the Lorentz group on a manifold with positive-definite signature is SO(4). Its two-fold covering group is Spin(4), which has the decomposition Spin(4) ≃ SU(2) × SU(2). So at least locally, we can introduce two vector bundles Ω_+ and Ω_- of spinor fields with positive and negative chirality. They are also known as the bundles of anti-self-dual and self-dual spinors respectively [8].

General spin bundles may then be constructed:

\[ \Omega_{mn} = S^n \Omega_+ \otimes S^m \Omega_- , \]  

where \( S^n \Omega_\pm \) is the \( n \)-fold symmetric product of \( \Omega_\pm \). A field \( \Psi \) belonging to \( \Omega_{mn} \) has spin \((m,n)\), and transforms under the irreducible representation \((m,n)\) of SU(2) \( \times \) SU(2). In the notation of two-component spinors, it has \( m \) unprimed and \( n \) primed indices

\[ \Psi_{A_1...A_m A'_1...A'_n} , \]  

which are completely symmetric in \( A_1...A_m \) and \( A'_1...A'_n \). (By virtue of this symmetry, they are also completely trace-free in \( A_1...A_m \) and \( A'_1...A'_n \).) Such a field will be referred to as an \((m,n)\)-field. The dimension of the vector space \( \Omega_{mn} \) is

\[ \dim \Omega_{mn} = (m + 1)(n + 1) . \]  

Each spinor index takes either value 0 or 1. In general, a tensor \( \Phi_{\alpha_1...\alpha_n} \) with \( n \) indices can be written as a spinor with \( n \) unprimed and \( n \) primed indices \( \Phi_{A_1...A_n A'_1...A'_n} \). We can further decompose it into its irreducible components, which consist of a spinor \( \tilde{\Phi}_{A_1...A_n A'_1...A'_n} \in \Omega_{nn} \), completely symmetric in \( A_1...A_n \) and \( A'_1...A'_n \), plus other anti-symmetric terms. These anti-symmetric terms consist of lower-order irreducible spinors multiplied by an appropriate number of anti-symmetric epsilon symbols \( \epsilon_{AB} \) or \( \epsilon_{A'B'} \). Further details may be found in ref. [20].

Let us quickly run through a few examples of the types of spinors that will be used in this paper. A covector \( \xi_a \) is written as \( \xi_{AA'} \) in spinor notation. For a less trivial case, consider the symmetric two-index tensor \( h_{ab} \). It can be decomposed as

\[ h_{ab} \equiv h_{ABA'B'} = \tilde{h}_{ABA'B'} + \frac{1}{4} h_{AB} \epsilon_{A'B'} , \]  

where \( \tilde{h}_{ABA'B'} \) is the irreducible part of the \( h_{ab} \) tensor, and \( \frac{1}{4} h_{AB} \epsilon_{A'B'} \) are the anti-symmetric terms.
where the irreducible component \( \tilde{h}_{ABA'B'} \in \Omega_{22} \) describes the trace-free part of \( h_{ab} \), and \( h \in \Omega_{00} \) is the trace part of \( h_{ab} \). One can easily check to see that the number of degrees of freedom correspond. On the other hand, the anti-symmetric two-index tensor \( F_{ab} \) may be decomposed spinorally as
\[
F_{ab} \equiv F_{ABA'B'} = F_{AB} \epsilon_{A'B'} + F_{A'B'} \epsilon_{AB}.
\] (1.9)

It turns out that the symmetric spinor \( F_{AB} \in \Omega_{20} \) is anti-self-dual, while \( F_{A'B'} \in \Omega_{02} \) is self-dual. This exemplifies the well-known fact that a 2-form \( F_{ab} \) may be written as a direct sum of its self-dual and anti-self-dual parts.

From now on, all the spinors encountered in our study of topological gravity will be irreducible ones. The one exception is the metric tensor, which in spinor notation reads \( g_{ab} \equiv g_{ABA'B'} = \epsilon_{AB} \epsilon_{A'B'} \). It does not belong to \( \Omega_{22} \), although its trace-free variation does. The spinors we will use are also globally well-defined, even on a manifold which does not admit a spin structure.

Our spinor notation generally follows that of ref. [20], except our Ricci spinor \( \Phi_{ABA'B'} \) and Ricci scalar \( R \) both have opposite signs to their notation. This is to conform to what is more usual in the literature.

### 2. Deformations of a conformally self-dual manifold

On a four-dimensional Riemannian manifold, 2-forms are special in that the space of 2-forms, \( \Lambda^2 \), splits into the direct sum
\[
\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-,
\] (2.1)
of self-dual and anti-self-dual 2-forms respectively. The Riemann curvature tensor is a self-adjoint transformation \( \mathcal{R}: \Lambda^2 \rightarrow \Lambda^2 \) given by [21]
\[
\mathcal{R}(e^a \wedge e^b) = \frac{1}{2} \sum_{c,d} R^{ab}_{cd} e^c \wedge e^d,
\] (2.2)
where \( \{e^a\} \) is some basis of 1-forms. Hence we can block diagonalise the transformation \( \mathcal{R} \) with respect to the decomposition (2.1)
\[
\mathcal{R} = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix},
\] (2.3)
where \( X : \Lambda_+^2 \rightarrow \Lambda_+^2 \), \( Z : \Lambda_-^2 \rightarrow \Lambda_-^2 \) and \( Y : \Lambda_-^2 \rightarrow \Lambda_+^2 \). In terms of its irreducible components, \( \mathcal{R} \) may be decomposed into the trace-free part of \( X \), the trace-free part of \( Z \), the trace part, and \( Y \). In more familiar notation, this decomposition may be written as

\[
R^{ab}_{\ cd} = W_+^{\ ab}_{\ cd} + W_-^{\ ab}_{\ cd} + \frac{1}{6} \delta_{[c}^{[a} \delta_{d]}^{b]} R + 2 \delta_{[c}^{[a} \Phi^{b]}_{d]} .
\] (2.4)

\( R \) is the Ricci scalar, \( \Phi_{ab} \) is the trace-free part of the Ricci tensor, \( W_+^{ab}_{\ cd} \) is the self-dual part of the Weyl tensor, and \( W_-^{abcd} \) its anti-self-dual part. The complete Weyl tensor is

\[
C_{abcd} = W_+^{abcd} + W_-^{abcd} .
\] (2.5)

It is self-dual:

\[
C_{abcd} = *C_{abcd} \equiv \frac{1}{2} \epsilon_{ab}^{\ ef} C_{efcd} ,
\] (2.6)

if and only if \( W_-^{abcd} = 0 \), and is anti-self-dual:

\[
C_{abcd} = - *C_{abcd} \equiv -\frac{1}{2} \epsilon_{ab}^{\ ef} C_{efcd} ,
\] (2.7)

if and only if \( W_+^{abcd} = 0 \). Note that if the Riemann tensor is self-dual, then it follows from the irreducible decomposition (2.4) that the Weyl tensor is also self-dual, and that the full Ricci tensor vanishes. The converse is however untrue—the self-duality of the Weyl tensor does not imply the self-duality of the Riemann tensor. We will be interested in the more general case of the Weyl tensor being self-dual. A manifold with metric admitting such a property is known as a conformally self-dual manifold, and some examples will be discussed below.

I.M. Singer has, in unpublished work [12], derived an expression for the number \( \varpi \) of independent conformally self-dual deformations of a given compact manifold \( M \), that has a metric which is conformally self-dual. \( \varpi \) is the sum of a topological quantity and two terms dependent on the metric on \( M \), given by [21]

\[
\varpi = \frac{1}{2} (29|\tau| - 15 \chi) + \text{(dimension of the conformal group of } M\text{)}
\]

\[+ \text{(correction for absence of vanishing theorem if } R \leq 0\text{)} .\] (2.8)

\( \varpi \) was derived by applying the Atiyah–Singer index theorem to an appropriate elliptic complex. \( \chi \) and \( \tau \) are the Euler characteristic and Hirzebruch signature of the manifold respectively, both of which are topological invariants. They can be written in terms of the irreducible components of the Riemann tensor as

\[
\chi = \frac{1}{32 \pi^2} \int d^4 x \sqrt{g} \left( C_{abcd} C^{abcd} - 2 \Phi_{ab} \Phi^{ab} + \frac{1}{6} R^2 \right) ,
\] (2.9)
\[ \tau = \frac{1}{48\pi^2} \int d^4x \sqrt{g} C_{abcd} \ast C^{abcd}. \] (2.10)

The results hold if \( M \) has no boundary, and will be modified by appropriate boundary terms otherwise. The above expression for \( \varpi \) also excludes diffeomorphisms and conformal rescalings of the metric, which trivially leave the manifold conformally self-dual.

In the rest of this section, we will present a low-key derivation of (2.8) and apply it to a few examples.

### 2.1. The gravitational instanton deformation complex

Consider the following sequence of mappings, which we call the gravitational instanton deformation complex:

\[
0 \xrightarrow{D_{-1}} \left\{ \text{infinitesimal coordinate transformations} \right\} \xrightarrow{D_0} \left\{ \text{trace-free metric variations} \right\} \xrightarrow{D_1} \left\{ \text{anti-self-dual part of Weyl tensor} \right\} \xrightarrow{D_2} 0. \tag{2.11}
\]

This sequence is only defined on a manifold whose metric has a \textit{self-dual} Weyl tensor, although the theory of this section may be carried through analogously for the anti-self-dual case via a change in orientation.

\( D_{-1} \) is an operator which takes 0 to the zero vector field.

\( D_0 \) is defined by

\[
[D_0 \xi]_{ab} \equiv \nabla_a \xi_b + \nabla_b \xi_a - \frac{1}{2} g_{ab} \nabla_c \xi^c, \tag{2.12}
\]

where \( \xi^a \) is a vector field generating an infinitesimal coordinate transformation of \( M \) via \( x^a \to x^a + \xi^a \). For this transformation to be a conformal one, it is necessary and sufficient that \( \xi_a \) satisfies the conformal Killing vector equation

\[
\nabla_a \xi_b + \nabla_b \xi_a - \frac{1}{2} g_{ab} \nabla_c \xi^c = 0. \tag{2.13}
\]

Hence the kernel of \( D_0 \) is simply the conformal group of \( M \). The image of \( D_0 \) is the trace-free part of the metric variation under a change of coordinates:

\[
\delta g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a - \frac{1}{2} g_{ab} \nabla_c \xi^c. \tag{2.14}
\]

The trace component, which corresponds to conformal rescalings of the metric under which the Weyl tensor is invariant, has been projected out from the deformation complex.
We set \(D_1 \equiv p_- L\), where \(p_- = \frac{1}{2}(1 - \ast)\) is the anti-self-dual projection operator. \(L\) is the operator defined by
\[
[Lh]_{abcd} \equiv \delta C_{abcd} ,
\]
where \(\delta C_{abcd}\) is the first-order variation of the Weyl tensor under the trace-free metric variation \(\delta g_{ab} = h_{ab}\):
\[
\delta C_{abcd} = \left\{ \frac{1}{2} \nabla_d \nabla_a h_{bc} + \frac{1}{4} g_{da} (\Box h_{bc} + \nabla_b \nabla^e h_{ce} + \nabla_c \nabla^e h_{be}) - \frac{1}{12} g_{da} g_{bc} \nabla^e \nabla^f h_{ef} \\
+ \frac{1}{4} h_{da} \Phi_{bc} + \frac{1}{4} g_{da} (h^e \Phi_{ec} + 2 h^e \Phi_{eb}) - \frac{1}{6} g_{da} g_{bc} \epsilon^e \Phi_{ef} \\
+ \frac{1}{12} R g_{da} h_{bc} + \frac{1}{4} C_{cdae} \epsilon^e b - \frac{1}{2} g_{da} \epsilon^e f C_{cef} \right\} \\
- \{a \leftrightarrow b\} - \{c \leftrightarrow d\} + \{a \leftrightarrow b, c \leftrightarrow d\} .
\]

If the domain of \(L\) is enlarged to include trace perturbations, we have the relation \((L \Omega^2 g)_{abcd} = \Omega^2 C_{abcd}\) for any non-zero suitably differentiable function \(\Omega\). This follows from the conformal properties of the Weyl tensor. A special case of this is the following expression for the Weyl tensor:
\[
C_{abcd} = [Lg]_{abcd} .
\]

As can be seen, the operator \(L\), and consequently \(D_1\), is a rather complicated second-order differential operator. It is more compactly expressed in terms of spinors, as we will show below for the case of \(D_1\) acting on trace-free metric variations. The kernel of \(D_1\) is the set of metric perturbations which preserve the self-duality of the Weyl tensor.

Lastly, \(D_2\) is identically the zero operator.

In the spinor notation introduced earlier, \(\xi_a\) may be rewritten as \(\xi_{AA'}\). It is illuminating to check that the degrees of freedom correspond. \(\xi_a\) has four independent components and the dimension of \(\Omega_{11}\) is \((1 + 1)(1 + 1) = 4\), as from \((1.7)\). The trace-free symmetric metric variation \(h_{ab}\) may be written as \(h_{ABA'B'} \in \Omega_{22}\), and indeed \(\dim \Omega_{22} = 9\) as expected.

In terms of spinors, the orthogonal decomposition of the Weyl tensor \((2.5)\) into its self-dual and anti-self-dual parts reads \((20)\)
\[
C_{abcd} = W_{A'B'C'D'} \epsilon_{AB} \epsilon_{CD} + W_{ABCD} \epsilon_{A'B'} \epsilon_{C'D'} .
\]

\(W_{A'B'C'D'} \in \Omega_{04}\) is the self-dual part, while \(W_{ABCD} \in \Omega_{40}\) is the anti-self-dual part. Each of these terms is totally symmetric in its indices, and has 5 independent components. Together, they make up the 10 independent components of \(C_{abcd}\).
Thus in terms of spinor bundles, the instanton deformation complex (2.11) may be written as
\[ 0 \xrightarrow{D_{-1}} \Omega_{11} \xrightarrow{D_0} \Omega_{22} \xrightarrow{D_1} \Omega_{40} \xrightarrow{D_2} 0 \]. \hfill (2.19)
Note that the dimension of \( \Omega_{11} \) plus that of \( \Omega_{40} \) equals the dimension of \( \Omega_{22} \).

The two non-trivial operators in (2.19) may be recast as
\[ [D_0 \xi]_A^{A'} B^{B'} = \frac{1}{2} (\nabla_{A'} \xi_{B'} + \nabla_{B'} \xi_{A'} + \nabla_{AB'} \xi_{A'B'} + \nabla_{BA'} \xi_{AB'}) , \] \hfill (2.20)
and
\[ [D_1 h]_{ABCD} = \frac{1}{2} [\nabla^{(A'} (A' \nabla_{B')} + \Phi_{(AB'} A'B^) ] h_{CD)A'B'} , \] \hfill (2.21)
where the round brackets in the latter expression indicate symmetrisation of the indices between them. It is equal to \( \delta W_{ABCD} [1] \), the change in the anti-self-dual Weyl tensor under the trace-free metric variation \( \delta g_{ab} = h_{ab} \equiv h_{ABA'B'} \).

We can also introduce the adjoint operators \( D^*_0 : \Omega_{22} \to \Omega_{11} \) and \( D^*_1 : \Omega_{40} \to \Omega_{22} \) via the relations
\[ \int_M d^4x \sqrt{|g|} h_{ABA'B'} [D_0 \xi]_A^{A'} B^{B'} = \int_M d^4x \sqrt{|g|} [D^*_0 h]_{AA'} \xi^{AA'} , \] \hfill (2.22)
and
\[ \int_M d^4x \sqrt{|g|} \chi_{ABCD} [D_1 h]_{ABCD} = \int_M d^4x \sqrt{|g|} [D^*_1 \chi]_{ABCD} h^{ABA'B'} . \] \hfill (2.23)
Explicitly, they are given by
\[ [D^*_0 h]_{AA'} = -2 \nabla^{BB'} h_{ABA'B'} , \] \hfill (2.24)
and
\[ [D^*_1 \chi]_{ABCD} = \frac{1}{2} \left[ -\nabla^C (A' \nabla^D B') + \Phi^{CD} A'B' \right] \chi_{ABCD} . \] \hfill (2.25)

2.2. Ellipticity of the deformation complex

Before proceeding, we have to check that the instanton deformation complex (2.19) is an elliptic complex. It is a complex because at each step of the sequence, the image of \( D_{i-1} \) is contained in the kernel of \( D_i \), i.e., \( D_i D_{i-1} = 0 \). In particular, \( D_1 D_0 = 0 \) because the self-duality of Weyl tensor is preserved under a trace-free metric perturbation of the form (2.14), corresponding to an unphysical coordinate transformation. Indeed it may be
verified explicitly from the two expressions (2.20) and (2.21) via use of the Bianchi identity that

\[ [D_1 D_0 \xi]_{ABCD} = -\nabla_{EE'} W_{ABCD} \xi^{EE'} + W_{(ACD} E^{\nabla_B)B'} \xi_{E'E} . \]  

(2.26)

This vanishes on-shell, i.e., when the Weyl tensor is self-dual: \( W_{ABCD} = 0 \).

To prove ellipticity, we must first introduce the symbols of the operators \( D_0 \) and \( D_1 \). The symbol \( \tilde{D} \) of an operator \( D \) is obtained by replacing all partial derivatives in \( D \) with covectors \( k_a \), and retaining only the highest order terms in \( k_a \) [21]. Thus the symbols of \( D_0 \) and \( D_1 \) are, in terms of tensors,

\[ [\tilde{D}_0 (k) \xi]_{ab} = k_a \xi_b + k_b \xi_a - \frac{1}{2} g_{ab} k_c \xi^c , \]  

(2.27)

and

\[ [\tilde{D}_1 (k) h]_{abcd} = \left\{ \frac{1}{2} k_d k_a h_{bc} + \frac{1}{4} g_{da} (-k^2 h_{bc} + k_b k^e h_{ce} + k_c k^e h_{be}) \\
- \frac{1}{12} g_{da} g_{be} k^f h_{cf} \right\} \\
- \{ a \leftrightarrow b \} - \{ c \leftrightarrow d \} + \{ a \leftrightarrow b, c \leftrightarrow d \} . \]  

(2.28)

A way to think of the resulting symbol sequence:

\[ 0 \longrightarrow \Omega_{11} \xrightarrow{\tilde{D}_0} \Omega_{22} \xrightarrow{\tilde{D}_1} \Omega_{40} \longrightarrow 0 , \]  

(2.29)

is that it represents the special case of the full deformation complex (2.19) when all curvature vanishes, i.e., in flat space. In such a situation, it is easy to check that

\[ \tilde{D}_1 (k) \tilde{D}_0 (k) = 0 , \]  

(2.30)

for \( k \neq 0 \).

The instanton deformation complex (2.19) is elliptic if its symbol sequence is exact [21]:

\[ \text{Im} \tilde{D}_0 (k) = \text{Ker} \tilde{D}_1 (k) , \]  

(2.31)

for \( k \neq 0 \). Since (2.30) already implies \( \text{Im} \tilde{D}_0 (k) \subseteq \text{Ker} \tilde{D}_1 (k) \), ellipticity requires

\[ \text{Ker} \tilde{D}_1 (k) \subseteq \text{Im} \tilde{D}_0 (k) . \]  

(2.32)

This condition states that, if the vanishing Weyl tensor of flat space is left invariant under a trace-free metric perturbation \( h_{ab} \):

\[ [D_1 h]_{abcd} = 0 , \]  

(2.33)
then it must follow that $h_{ab}$ is a trivial coordinate transformation:

$$h_{ab} = [D_0 \xi]_{ab},$$

for some vector field $\xi^a$. (This is the analogue of the Poincaré lemma for the de Rham complex—where a closed form in flat space implies that it is exact.)

Now it can be shown that the map $\tilde{D}_0(k)$ is injective, while $\tilde{D}_1(k)$ is surjective, for all $k \neq 0$. Since $\Omega_{11}$ is four-dimensional, it follows that the image of $\tilde{D}_0(k)$ forms a four-dimensional subspace of the nine-dimensional vector space $\Omega_{22}$. On the other hand, since $\Omega_{40}$ is five-dimensional, the domain of $\tilde{D}_1(k)$ in $\Omega_{22}$ must be at least five-dimensional. Because of (2.30), the non-trivial domain of $\tilde{D}_1(k)$ is exactly five-dimensional. We have the situation as depicted in Fig. 1.

The proof of (2.32) then follows immediately. Suppose $h_{ab}$ is a non-trivial metric perturbation which leaves the Weyl tensor invariant. Since it does not belong to the image of $\tilde{D}_0(k)$, it must reside in the non-trivial five-dimensional domain of $\tilde{D}_1(k)$. But this leads to a contradiction as $h_{ab}$ is supposed to leave the Weyl tensor invariant. Hence the instanton deformation complex is indeed elliptic.

We have seen that $\Omega_{22}$ of the symbol sequence (2.29) may be decomposed into two orthogonal subspaces, one of which whose elements may be written as $\tilde{D}_0(k)$ of some
(1, 1)-field, and the other whose elements may be expressed as \( \tilde{D}_i^* (k) \) acting on some \((4, 0)\)-field. Such a decomposition also holds for the full deformation complex, and this is the generalisation of the Hodge-decomposition theorem for the de Rham complex.

Let us introduce the associated Laplacians of the deformation complex, defined by

\[
\triangle (i) \equiv D_{i-1} D_{i-1}^* + D_i^* D_i,
\]

i.e.,

\[
\begin{align*}
\triangle (0) &= D_0^* D_0 , \\
\triangle (1) &= D_0 D_0^* + D_1^* D_1 , \\
\triangle (2) &= D_1 D_1^* .
\end{align*}
\]

(2.35)

\( \triangle (0) \) is a second-order differential operator defined on \((1, 1)\)-fields; \( \triangle (1) \) is a combination of a second-order differential operator and a fourth-order one, which acts on \((2, 2)\)-fields; while \( \triangle (2) \) is a second-order differential operator acting on \((4, 0)\)-fields. Of course, physically, there is here a mismatch of dimensions. This may be resolved by implicitly introducing some arbitrary scale factor with dimension inverse length squared, into the \( D_0 \) terms of (2.35). As a result, \( D_0 \) will have dimensions inverse length squared, and all our Laplacians will be “fourth-order” differential operators. They are elliptic operators because the deformation complex is elliptic. Note also that these Laplacians look nothing like the usual ones arising from the de Rham complex [19]. It may be possible to formulate a relationship between them, however.

The decomposition theorem for the deformation complex may now be stated. Denote \( \Omega (0) \equiv \Omega_{11} \), \( \Omega (1) \equiv \Omega_{22} \) and \( \Omega (2) \equiv \Omega_{40} \). If \( \omega_i \in \Omega (i) \), then it can be uniquely decomposed into the sum

\[
\omega_i = D_{i-1} \alpha_{i-1} + D_i^* \beta_{i+1} + \gamma_i ,
\]

(2.36)

where \( \alpha_{i-1} \) is some element of \( \Omega (i-1) \), \( \beta_{i+1} \) is some element of \( \Omega (i+1) \), and \( \gamma_i \) is an element of \( \Omega (i) \) satisfying \( \triangle (i) \gamma_i = 0 \). The first term is called the exact part of \( \omega_i \), the second term its coexact part, and the last term the harmonic part. Note that the harmonic part of \( \omega_i \) satisfies

\[
D_i \gamma_i = 0 , \quad D_i^* \gamma_i = 0 .
\]

(2.37)

Before leaving this subsection, let us make one more remark. As we have seen, the dimension of \( \Omega_{22} \) equals the sum of the dimensions of \( \Omega_{11} \) and \( \Omega_{40} \). This suggests that we can rewrite the deformation complex (2.19) as

\[
\Omega_{22} \xrightarrow{T} \Omega_{11} \oplus \Omega_{40} ,
\]

(2.38)

where the operator \( T \) is defined by \( T = D_0^* \oplus D_1 \), and its adjoint is \( T^* = D_0 \oplus D_1^* \). This form of the deformation complex maps trace-free metric variations at a point to those which are trivial, and to those which deform the Weyl tensor.
2.3. The index of the deformation complex

As we have showed, the operators $D_i$ of the deformation complex satisfy $D_iD_{i-1} = 0$. This implies that $\text{Im } D_{i-1} \subseteq \text{Ker } D_i$, but unlike its symbol sequence, equality between the sets does not hold in general. We may then define the \textit{cohomology groups} of the deformation complex to be \cite{21}

$$H^i \equiv \frac{\text{Ker } D_i}{\text{Im } D_{i-1}},$$

which consists of elements of the kernel of $D_i$, identified if they differ by an element in the image of $D_{i-1}$. They may be alternatively written as

$$H^i = \text{Ker } D_i \cap \text{Ker } D_{i-1}^*$$

$$= \text{Ker } \Delta_{(i)},$$

and so there is an isomorphism between $H^i$ and the harmonic subspace of $\Omega_{(i)}$. These cohomology groups are finite dimensional, and we set $h^i \equiv \dim H^i$. The \textit{index} of the deformation complex is defined to be

$$\text{index} \equiv h^0 - h^1 + h^2,$$

and it is a topological quantity which we will calculate in the next subsection.

Let us now examine these three cohomology groups.

The image of $D_{-1}$ consists of a single point, so it is zero-dimensional. Hence, $H^0 = \text{Ker } D_0$ is the conformal group of $M$. On a four-dimensional compact manifold $M$, the dimension of the conformal group can be at most 15, which corresponds to a conformally flat manifold such as the four-sphere $S^4$. This includes both the number of isometries, which are generated by vector fields $\xi^a$ satisfying

$$\nabla_a \xi_b + \nabla_b \xi_a = 0;$$

and the number of non-isometric or proper conformal transformations satisfying (2.13), for $\nabla_a \xi^a \neq 0$. The dimension of the isometry group is at most 10, leaving a maximum of 5 independent proper conformal transformations on $M$.

A manifold $M$ is called Einstein if it satisfies

$$R_{ab} = \Lambda g_{ab},$$
for some real value of the cosmological constant $\Lambda$. A theorem of Yano [22] states that if $M$ is a compact Einstein manifold and if it admits an infinitesimal proper conformal transformation, then $M$ is isometric to the four-sphere $S^4$. Hence if $M$ is Einstein but not isometric to $S^4$, it admits no proper conformal transformations, and $h^0 \leq 10$.

Another theorem of Yano [22] states that there are no conformal Killing vectors on a manifold with negative Ricci scalar, so that $h^0$ vanishes.

$H^1$ consists of self-dual variations of the Weyl tensor ($\text{Ker } D_1$) factored out by trace-free metric fluctuations of the form of $\text{Im } D_0$, as given in (2.14). The dimension of this space $h^1$ is the number of independent non-trivial self-dual deformations that can be made around the self-dual Weyl tensor of $M$. This is precisely the quantity $\varpi$ given by (2.8), which we are going to find an expression for.

By (2.40), we may alternatively write

$$H^1 = \text{Ker } D^*_0 \cap \text{Ker } D_1 .$$

(2.44)

The physical interpretation of this is that instead of modding out trivial metric variations $h_{ab} = [D_0 \xi]_{ab}$, we are fixing the gauge to be $[D^*_0 h]_a = 0$. Thus $h^1$ counts the number of independent solutions to

$$D^*_0 \psi = 0 , \quad D_1 \psi = 0 ,$$

(2.45)

for $\psi \in \Omega^{22}$.

$\text{Ker } D_2$ is the whole of the space $\Omega_{40}$. Hence $H^2$ is the subspace of $\Omega_{40}$ orthogonal to the mapping $D_1$, or equivalently just $L$. This cohomology group consists of fields belonging to the harmonic subspace of $\Omega_{40}$, i.e., the kernel of $D^*_1$. The dimension of $H^2$ can be at most 5, corresponding to that of $\Omega_{40}$ itself.

This number in fact vanishes if $R > 0$, as we will now show for the Einstein case. Any element of $H^2$ belongs to the kernel of $\triangle_{(2)}$. In the Einstein case, $\triangle_{(2)}$ may be explicitly evaluated from (2.21) and (2.25) to read

$$[D_1 D^*_1 \chi]_{ABCD} = \frac{1}{16} (-\Box + \frac{2}{3} \Lambda)(-\Box + 2 \Lambda) \chi_{ABCD} .$$

(2.46)

This operator is positive definite if $R = 4 \Lambda > 0$. In such a case, the kernel of $\triangle_{(2)}$ has to be trivial, and so $h^2$ vanishes. We will show below via an example that, in the case of a manifold with a self-dual Riemann tensor, $h^2$ counts the number of covariantly constant objects in $\Omega_{40}$.
2.4. Calculation of the index

The Atiyah–Singer index theorem may now be applied to the instanton deformation complex. It is a formula which simply relates the index of an elliptic complex to the topological properties (twisting) of the bundles \( \Omega_{mn} \) associated with it. On a four-dimensional Riemannian manifold \( M \), it reads \[10\]

\[
\text{index} = \int_M \frac{\text{ch}(\bigoplus(-1)^i E_i) \text{td}(TM \otimes \mathbb{C})}{e(TM)} .
\]

(2.47)

The integral over \( M \) indicates that we extract the 4-forms in the integrand, and integrate them over \( M \). \( TM \) is the tangent bundle of \( M \), and \( TM \otimes \mathbb{C} \) its complexification. The \( E_i \)'s are the vector bundles over \( M \) associated with the elliptic complex. In our case, the Whitney sum over the bundles as indicated in the index theorem is

\[
\bigoplus(-1)^i E_i = \Omega_{11} \ominus \Omega_{22} \oplus \Omega_{40} .
\]

(2.48)

\( \text{ch}, \text{td} \) and \( e \) are the Chern character, Todd class and Euler class of the various vector bundles involved.

A very useful tool used in evaluating the index theorem is the splitting principle, which allows us to treat any vector bundle as a direct sum of one-dimensional line bundles. For example, it implies that

\[
\text{ch}(\bigoplus(-1)^i E_i) = \text{ch}(\Omega_{11} \ominus \Omega_{22} \oplus \Omega_{40})
\]

\[
= \text{ch}(\Omega_{11}) - \text{ch}(\Omega_{22}) + \text{ch}(\Omega_{40}) .
\]

(2.49)

One can also use it to derive the expressions \[10\]

\[
\text{e}(TM) = \prod_{i=1}^{2} x_i = x_1 x_2 ,
\]

(2.50)

\[
\text{td}(TM \otimes \mathbb{C}) = \prod_{i=1}^{2} \frac{-x_i^2}{(1 - e^{-x_i})(1 - e^{x_i})} ,
\]

(2.51)

where \( x_i = \lambda_i / 2\pi \) are two-forms, for \( \lambda_1 \) and \( \lambda_2 \) the two independent eigenvalues of the curvature two-form (2.2). The latter expression can be simplified to give

\[
\text{td}(TM \otimes \mathbb{C}) = 1 - \frac{1}{12} p_1 ,
\]

(2.52)

where \( p_1 = x_1^2 + x_2^2 \) is the first Pontryagin class of \( TM \).
Römer [23] has calculated the Chern character of $\Omega_{mn}$ to fourth-order in $x_i$, and he finds that

$$\text{ch}(\Omega_{mn}) = A_m A_n + \frac{1}{2} A_m B_n (x_1 + x_2)^2 + \frac{1}{2} A_n B_m (x_1 - x_2)^2$$
$$+ \frac{1}{3} B_m B_n (x_1^2 - x_2^2)^2 + \frac{1}{24} A_m C_n (x_1 + x_2)^4 + \frac{1}{24} A_n C_m (x_1 - x_2)^4,$$

where

$A_n = n + 1$; \hspace{1em} $B_n = \sum_{k=0}^{n} (k - \frac{1}{2} n)^2$; \hspace{1em} $C_n = \sum_{k=0}^{n} (k - \frac{1}{2} n)^4$.

One can then readily show that

$$\text{ch}(\Omega_{11}) - \text{ch}(\Omega_{22}) + \text{ch}(\Omega_{40}) = -10 x_1 x_2 + \frac{15}{2} x_1^2 x_2^2 - \frac{17}{3} x_1 x_2 (x_1^2 + x_2^2).$$

(2.53)

Substituting all this into the index theorem gives

$$\text{index} = \int_M (-10 + \frac{15}{2} x_1 x_2 - \frac{17}{3} x_1 x_2) (1 - \frac{1}{12} p_1)$$
$$= \int_M \left( \frac{15}{2} e - \frac{29}{6} p_1 \right)$$
$$= \frac{1}{2} (15 \chi - 29 \tau).$$

(2.54)

Similarly, considering the case of opposite orientation ($\Omega_{04}$ instead of $\Omega_{40}$) gives the index to be

$$\text{index} = \frac{1}{2} (15 \chi + 29 \tau).$$

(2.55)

Bear in mind that a conformally self-dual manifold would have Weyl tensor belonging to $\Omega_{04}$, and have a positive signature $\tau$. On the other hand, a conformally anti-self-dual manifold would have Weyl tensor belonging to $\Omega_{40}$, and have negative $\tau$. Hence the index is actually given by

$$\text{index} = \frac{1}{2} (15 \chi - 29|\tau|).$$

(2.56)

Finally, recall that most of the known gravitational instantons are, in fact, non-compact. However, it is possible to treat them as compact manifolds with boundaries which recede to infinity. The presence of a boundary will modify the value of the index derived above. The new index may be calculated via an application of the Atiyah–Patodi–Singer index theorem [24], which is the extension of the Atiyah–Singer index theorem to manifolds with boundary, to the deformation complex. In fact, Römer and collaborators [23,25] have calculated such boundary corrections of certain index theorems for several gravitational instantons with boundary. In particular, they have treated the Eguchi–Hanson and other asymptotically locally Euclidean instantons, whose boundaries are $S^3$ identified under discrete subgroups of SU(2). One could try to generalise their calculations to the instanton deformation complex.
2.5. Moduli space of gravitational instanton metrics

To summarise, we have so far obtained the relation defined on a self-dual manifold $M$:

$$\varpi = \frac{1}{2}(29|\tau| - 15\chi) + \dim(\text{conformal group of } M) + h^2,$$  \hspace{1cm} (2.58)

where $h^2$ is some “small” correction which vanishes if $M$ is Einstein and has positive Ricci scalar.

Now consider the space of all metrics, modded out by conformal and coordinate transformations. Given a manifold $M$, we define the gravitational moduli space of $M$ to be that subset of the former space consisting of metrics on $M$ which are conformally self-dual.

Given a particular conformally self-dual metric $g_0$ on $M$, (2.58) counts the number of independent tangent vectors to $g_0$ in the moduli space. This is because in our derivation of (2.58), we have assumed that the metric variations are infinitesimal.

Because the last two terms of (2.58) depend on the metric, “integrating” $g_0$ to give a local manifold structure for the moduli space is in general not well defined. This means that the moduli space may have singularities around $g_0$. On the other hand, if a local manifold structure existed around $g_0$, its dimension would be given by $\varpi$.

Even if there are no obstructions to defining a local manifold structure around $g_0$, the moduli space still may not exist globally as a manifold. This could happen if there existed another conformally self-dual metric $g_1$ on $M$, which is “very different” from $g_0$. The number of independent tangent vectors at $g_1$ could be different from that at $g_0$. Thus the moduli space might consist of disconnected compact manifolds with different dimensions.

The gravitational moduli space is much more well behaved if the two non-topological terms of (2.58) are not present. Let us give a mathematical characterisation of this, which follows the analysis applied to the Yang–Mills moduli space. Consider the deformation complex in the form (2.38). For a given metric $g$ on $M$, tangent vectors to $g$ belong to the cohomology group $H^2$. This is equivalently the kernel of $T$. If $T$ is a surjective map, then the implicit function theorem can be invoked to deduce that the kernel of $T$ is smooth near $g$. $T$ is surjective if the kernel of $T^*$ vanishes:

$$\text{Ker } T^* = \text{Ker } D_0 \oplus \text{Ker } D_1^* = H^0 \oplus H^2 = 0.$$  \hspace{1cm} (2.59)

Thus the moduli space exists as a smooth manifold around $g$ if $h^0$ and $h^2$ both vanish.

An analogous analysis of the Yang–Mills case may then be applied here to show that these local manifolds can be patched together to give a global manifold structure to
the moduli space. This gravitational moduli space will have a topology induced by the space of conformally and physically inequivalent metrics, and whose dimension is equal to \( \frac{1}{2} (29|\tau| - 15\chi) \). It will in general be a smooth manifold, except perhaps for isolated instances of singularities.

\( \varpi \) is called the virtual dimension of the moduli space. It is virtual because it does not imply the existence of conformally self-dual metrics when \( \varpi > 0 \). What it means is that if such a metric exists, then it would have a moduli space of dimension \( \varpi \) around it, or more generally \( \varpi \) tangent vectors at that point in the moduli space.

### 2.6. Some examples

Presently, we do not know of any examples of compact conformally self-dual manifolds which are not conformally Einstein. Thus the following theorem of Hitchin [26] is of some importance to us:

Let \( M \) be a compact conformally self-dual Einstein manifold. Then

1. If \( R > 0 \), \( M \) is either isometric to the four-sphere \( S^4 \), or to the complex projective two-space \( \mathbb{C}P^2 \), with their standard metrics;
2. If \( R = 0 \), \( M \) is either flat (for example, the flat metric on the four-torus \( T^4 \)) or its universal covering is the K3 surface with the Calabi–Yau metric.

Bearing this uniqueness result in mind, let us now compute \( \varpi \) for the manifolds listed above.

The standard constant curvature metric on \( S^4 \) is conformally flat, so it has a vanishing Weyl tensor which is trivially self-dual. It has \( \chi = 2 \) and \( \tau = 0 \), giving the index to be 15. The conformal group of this manifold is 15-dimensional, and \( h^2 = 0 \) since \( R > 0 \). Substituting these values into (2.58) yields the value zero, which shows that the constant curvature metric on \( S^4 \) admits no conformally self-dual deformations except for a scale.

It can be shown [8] that the standard metric on \( S^4 \) is the unique conformally self-dual one. Hence the gravitational moduli space of \( S^4 \) consists of a single point.

The standard metric on \( \mathbb{C}P^2 \) is the Fubini–Study metric [21], which has a self-dual Weyl tensor. For this case \( \chi = 3 \) and \( \tau = 1 \), and so the index is 8. Its conformal group is 8-dimensional. Again \( R > 0 \), so \( h^2 \) vanishes. Thus applying the formula (2.58) shows that there are no conformally self-dual deformations of the standard metric on \( \mathbb{C}P^2 \), apart from a scale.

In this case, we cannot rule out the possibility of finding other examples of conformally self-dual metrics with topology \( \mathbb{C}P^2 \).
The K3 surface with the Calabi–Yau metric has a self-dual Riemann tensor, so it is conformally self-dual and $R$ vanishes. Now $\chi = 24$ and $\tau = 16$, giving an index of $-52$. Its conformal group is zero-dimensional. In this case, $h^2$ is non-vanishing and we will now calculate its value.

The number of anti-self-dual harmonic 2-forms on K3 is

$$b_2 = \frac{1}{2}(-\tau + \chi - 2) = 3$$

(2.60)

Such a harmonic 2-form which is anti-self-dual satisfies

$$\triangle F_{AB} = -\Box F_{AB} + \frac{4}{3}\Lambda F_{AB} + 2W_{ABCD}F^{CD}$$

(2.61)

But in the present case, this equation reduces down to simply $\Box F_{AB} = 0$. Now $F_{AB}$ may be decomposed into a symmetric product of two principle spinors $F_{AB} = \epsilon^{(1)}_{(A}\epsilon^{(2)}_{B)}$. The fact that there are exactly three of these objects on K3 implies that there are precisely two covariantly constant unprimed principle spinors $\iota_A$ and $\o_B$ that exist on K3 $[27]$. The three anti-self-dual harmonic 2-forms are then $\iota_{(A}^\iota_{B)}$, $\iota_{(A}^\o_{B)}$ and $\o_{(A}^\o_{B)}$.

From the two principle spinors, we can also construct five covariantly constant objects in $\Omega_{40}$: $\iota_{(A}^\iota_{B)}^\iota_{C}^{C}$, $\iota_{(A}^\iota_{B)}^\o_{C}^{C}$, $\iota_{(A}^\o_{B)}^\iota_{C}^{C}$, $\iota_{(A}^\o_{B)}^\o_{C}^{C}$ and $\o_{(A}^\o_{B)}^\o_{C}^{C}$. These five quantities clearly satisfy the equation (2.46), and so must belong to $H^2$. Hence, $h^2 = 5$ for K3. Applying the formula (2.58) shows that there are 57 independent conformally self-dual deformations of K3, aside from a scale. Altogether, the K3 instanton with the Calabi–Yau metric has 58 parameters.

3. Topological conformal gravity

Having developed a deformation theory of gravitational instantons in the previous section, we are now ready to tackle the theory of topological gravity. Starting with the classical action of topological conformal gravity, we analyse its symmetries and BRST-quantise it by introducing the appropriate ghost fields and gauge-fixing functions. These ghost fields are of the same nature as the fields that appear in the deformation complex (2.19), and their interaction terms involve the differential operators $D_0$ and $D_1$ of the deformation complex. We then evaluate the partition function of the quantum theory, and show how this produces an invariant which looks very much like the first Donaldson polynomial of topological Yang–Mills theory.
Throughout this section, we will be closely imitating the way one goes about in dealing with the topological Yang–Mills case, particularly the work of Baulieu and Singer [7]. Indeed, we will follow their notation as far as naming the fields is concerned, and not that of Witten [4].

3.1. Classical theory

Conformal gravity may be regarded as the theory which arises from the classical action

$$\int_M \frac{d^4x}{\sqrt{g}} C_{abcd}C^{abcd}. \quad (3.1)$$

It is a higher-derivative gravity theory which is not only invariant under coordinate and Lorentz transformations, but also conformal transformations. The theory is power-counting renormalisable, and it was once hoped that it could describe gravity at Planck lengths, while reducing to ordinary gravity at larger distances. However, because of the conformal anomaly and problems with unitarity, it has since become less fashionable. A review of conformal gravity and its quantisation may be found in ref. [28].

The important property of the action (3.1) that we will need below is that it is minimised by metrics whose Weyl tensors are (anti-) self-dual, as in (2.6) or (2.7). This follows immediately from the inequality [11]

$$(C_{abcd} \pm \ast C_{abcd})^2 \geq 0. \quad (3.2)$$

Such manifolds are known as conformally (anti-) self-dual gravitational instantons, and known examples [21] include the standard metrics on flat Euclidean four-space, four-torus, four-sphere, the Eguchi–Hanson metric, the Taub–NUT metric, the Fubini–Study metric on CP^2, the Gibbons–Hawking multi-centre metrics, and the Calabi–Yau metric on K3.

It is clear that the action (3.1) is the closest gravitational analogue to the Yang–Mills functional (1.2) available. Indeed, bearing in mind the action for topological Yang–Mills theory (1.1), the classical action for topological conformal gravity is very similarly,

$$\int_M \frac{d^4x}{\sqrt{g}} C_{abcd} \ast C^{abcd}, \quad (3.3)$$

which up to a factor is the Hirzebruch signature (2.10). The easiest way to see that (3.3) is invariant under arbitrary variations of the metric is to recognise that it is equivalent to

$$\int_M \frac{d^4x}{\sqrt{g}} R_{abcd} \ast R^{abcd}, \quad (3.4)$$
which in turn can be written in terms of forms as
\[
\int_M \text{tr}\{R \wedge R\} ,
\] (3.5)
where the curvature 2-form is given by \( R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b \), for connection 1-forms \( \omega^a_b \). Varying (3.3) with respect to \( \omega^a_b \) quickly shows that it is invariant under arbitrary changes in \( \omega^a_b \), and hence under arbitrary variations of the metric.

To be more precise about this metric variation, recall from ref. [29] that any metric perturbation \( \delta g_{ab} = h_{ab} \) may be decomposed into its trace-free part \( \tilde{\phi}_{ab} \), and its trace part \( \phi_{ab} \):
\[
h_{ab} = \tilde{\phi}_{ab} + \frac{1}{4} \phi g_{ab} , \quad \tilde{\phi}^a_a = 0 .
\] (3.6)
\( \phi \) corresponds to conformal rescalings of the metric, which we will as usual ignore. \( \tilde{\phi}_{ab} \) may be further decomposed orthogonally into its transverse trace-free part \( \phi^{TT}_{ab} \), and its longitudinal part \( \phi^L_{ab} \):
\[
\tilde{\phi}_{ab} = \phi^{TT}_{ab} + \phi^L_{ab} ,
\] (3.7)
where \( \phi^L_{ab} \) may be expressed in terms of a vector field \( \xi_a \) as
\[
\phi^L_{ab} = \nabla_a \xi_b + \nabla_b \xi_a - \frac{1}{2} g_{ab} \nabla_c \xi^c ,
\] (3.8)
and the transverse trace-free part satisfies
\[
\nabla^a \phi^{TT}_{ab} = 0 .
\] (3.9)

Hence to characterise an arbitrary metric variation of (3.3), consider the quantum metric fluctuation
\[
g_{ab} = g_{ab}^{cl} + h_{ab} ,
\] (3.10)
about a fixed classical background \( g_{ab}^{cl} \). We project out the trace part, so that \( h_{ab} \) takes the form (3.7). In spinor notation, \( h_{ab} = h_{ABA'B'} \) belongs to \( \Omega_{22} \). The longitudinal part of \( h_{ab} \) is of the form \( D_0 \) acting on some vector field. Thus, it characterises that part of \( \Omega_{22} \) which is exact, by our decomposition theorem of the instanton deformation complex. Furthermore, the transverse trace-free part of \( h_{ab} \) consists of the coexact sector of \( \Omega_{22} \); and (3.3) follows from the property that \( D_1 D_0 = 0 \):
\[
\nabla^b h^{TT}_{ab} \propto (D^*_b h^{TT})_a = (D^*_b D^*_1 \chi)_a = 0 .
\] (3.11)
Note that the transverse trace-free also consists of elements of \( \Omega_{22} \) which are harmonic.

Thus, we can write an arbitrary quantum metric fluctuation of (3.3) in the form
\[
h_{ABA'B'} = \rho_{ABA'B'} + [D_0 \omega]_{ABA'B'} ,
\] (3.12)
\( \rho_{ABA'B'} \) belongs to \( \Omega_{22} \), while \( \omega_{AA'} \) is an element of \( \Omega_{11} \).
3.2. BRST symmetry and gauge-fixing

Since the classical action is invariant under the metric variation (3.12), we must gauge-fix this topological symmetry in order for the Feynman path integral of (3.3) to make sense. To do so, we will use the BRST quantisation scheme. In the first step of this procedure, we introduce ghost fields \( \psi_{ABA'B'} \) and \( c_{AA'} \) corresponding to variations \( \rho_{ABA'B'} \) and \( \omega_{AA'} \) of (3.12) respectively. One may think of \( c_{AA'} \) as a ghost which gauges diffeomorphisms of the form (3.8), while \( \psi_{ABA'B'} \) as a ghost which gauges the topological symmetry. Both \( \psi_{ABA'B'} \) and \( c_{AA'} \) have ghost number +1 and are therefore anti-commuting.

At this stage, observe that \( \rho_{ABA'B'} \) in (3.12) is defined modulo a term of the form

\[
\rho_{ABA'B'} \sim \rho_{ABA'B'} + [D_0\Lambda]_{ABA'B'} .
\] (3.13)

One can say that the ghost field \( \psi_{ABA'B'} \) itself has its own gauge invariance, which we must also gauge-fix by introducing a second generation ghost field \( \phi_{AA'} \) corresponding to \( \Lambda_{AA'} \). It has ghost number +2, and is commuting.

The BRST version of the transformation (3.12) is

\[
s g_{ABA'B'} = s h_{ABA'B'} = \psi_{ABA'B'} + [D_0c]_{ABA'B'} ,
\] (3.14)

where \( s \) is the BRST operator. Note that \( s \) raises the ghost number of its operand by one unit, and is anti-commuting. The BRST version of the symmetry (3.13) is

\[
s \psi_{ABA'B'} = [D_0\phi]_{ABA'B'} .
\] (3.15)

From (3.14), one could proceed to calculate \( s \) acting on the inverse metric \( g^{ab} \), via the condition that \( s \{ g^{ab} g_{bc} \} = 0 \). Another quantity one could then compute is the action of \( s \) on the Christoffel symbol \( \Gamma^a_{bc} \). However, the contribution from these types of terms to the final gauge-fixed action are of at least third order in the quantum fields. We may neglect such higher order contributions since in this theory, like topological Yang–Mills theory, the semi-classical limit is exact [2]. Thus in what follows, we only write down the \( s \) transformations to lowest order. The BRST symmetry will still be preserved if one consistently incorporates higher order terms as necessary. In particular to lowest order, \( s \) commutes with the operators \( D_0 \) and \( D_1 \).

Now, because \( W_{ABCD} = (D_1g)_{ABCD} \) if we extend the domain of \( D_1 \) as in (2.17), we have

\[
s W_{ABCD} = [D_1s]_{ABCD} = [D_1(\psi + D_0c)]_{ABCD} .
\] (3.16)
Remember that $D_1 D_0 = 0$ only when on-shell, when the Weyl tensor is self-dual.

By construction, $s$ must be a nilpotent operator, i.e., it satisfies $s^2 = 0$. To enforce this, note that we immediately have from (3.15) that

$$s \phi_{AA'} = 0 .$$

(3.17)

It also follows from (3.15) and (3.14) that we must set

$$s c_{AA'} = -\phi_{AA'} .$$

(3.18)

Also observe that

$$s^2 W_{ABCD} = [D_1 s \psi + D_1 D_0 s c]_{ABCD}$$

$$= [D_1 D_0 \phi - D_1 D_0 \phi]_{ABCD}$$

$$= 0 .$$

(3.19)

Hence the operator $s$ defined by the above set of transformations is indeed nilpotent, even when off-shell.

Having introduced the three ghost fields and the BRST operator $s$, it is now time to choose the gauge-fixing terms. Corresponding to the first symmetry of (3.14), we may impose the choice of gauge that

$$W_{ABCD} = 0 ,$$

(3.20)

that is, the Weyl tensor be self-dual.

The second symmetry in (3.14) and the second generation one in (3.13) are similar in nature, and so would have the same type of gauge-fixing function. The normal choice of such a function would have the form $[D_0^* \lambda]_{AA'} = 0$ for some field $\lambda_{ABA'B'} \in \Omega_{22}$. By considering ghost numbers, we would have the two respective gauge-fixing functions to be

$$[D_0^* h]_{AA'} = 0 ,$$

(3.21a)

and

$$[D_0^* \psi]_{AA'} = 0 .$$

(3.21b)

Corresponding to the three ghost fields $\psi_{ABA'B'}$, $c_{AA'}$ and $\phi_{AA'}$, we have also to introduce their anti-ghost fields $\bar{\chi}_{ABCD}$, $\bar{c}_{AA'}$ and $\bar{\phi}_{AA'}$. The BRST transforms of these

\footnote{The former is the usual harmonic or de Donder gauge condition.}
Table 1. The fields of topological conformal gravity

| Field   | Meaning                              | Representation | Ghost Number | Dimension in [Length]$^n$ | Conformal Weight* |
|---------|--------------------------------------|----------------|--------------|----------------------------|-------------------|
| $h$     | metric variation                     | (2, 2)         | 0            | 0                          | 2                 |
| $C$     | Weyl tensor                          | (4, 0) $\oplus$ (0, 4) | 0            | -2                         | 2                 |
| $W_+$   | self-dual part of $C$                | (0, 4)         | 0            | -2                         | 0                 |
| $W_-$   | anti-self-dual part                  | (4, 0)         | 0            | -2                         | 0                 |
| $c$     | diffeomorphism ghost                 | (1, 1)         | +1           | +2                         | 2                 |
| $\bar{c}$ | anti-ghost of $c$                     | (1, 1)         | -1           | -2                         | $k$               |
| $b$     | Lagrange multiplier                  | (1, 1)         | 0            | -2                         | $k$               |
| $\psi$  | topological ghost                    | (2, 2)         | +1           | 0                          | 2                 |
| $\bar{\chi}$ | anti-ghost of $\psi$               | (4, 0)         | -1           | -2                         | 0                 |
| $B$     | Lagrange multiplier                  | (4, 0)         | 0            | -2                         | 0                 |
| $\phi$  | ghost for ghost $\psi$               | (1, 1)         | +2           | +2                         | 2                 |
| $\bar{\phi}$ | anti-ghost of $\phi$               | (1, 1)         | -2           | -2                         | $l$               |
| $\bar{\eta}$ | Lagrange multiplier                | (1, 1)         | -1           | -2                         | $l$               |

* (with all indices downstairs)

anti-ghosts are just three Lagrange multiplier fields, respectively denoted $B_{ABCD}$, $b_{AA'}$, and $\bar{\eta}_{AA'}$, for our gauge-fixing functions. In other words, we have

$$
\begin{align*}
    s\bar{\chi}_{ABCD} &= B_{ABCD} , & sB_{ABCD} &= 0 , \\
    s\bar{c}_{AA'} &= b_{AA'} , & sb_{AA'} &= 0 , \\
    s\bar{\phi}_{AA'} &= \bar{\eta}_{AA'} , & s\bar{\eta}_{AA'} &= 0 .
\end{align*}
$$

(3.22)

These therefore are the fields that characterise topological conformal gravity. It should be clear to the reader by now that spinor indices just form an unnecessary clutter in our equations. We will work towards an index-free notation, where the various fields we have introduced are understood to belong to the appropriate one of the three representations $\Omega_{11}$, $\Omega_{22}$ or $\Omega_{40}$. For future reference, we list these fields and their properties in Table 1.

3.3. Conformal weights

Note that in Table 1, we have listed down the conformal weights of our fields. This is
because as its name implies, conformal gravity is classically a conformally invariant theory and so we should be careful to check that no conformal anomaly develops in the quantum theory.

Our convention here follows that of Penrose and Rindler [20], who consider the metric rescaling given by

$$g_{ab} \mapsto \hat{g}_{ab} = \Omega^2 g_{ab},$$

where $\Omega$ is a non-zero suitably differentiable function. From this and the BRST transformations listed in the previous subsection, we can therefore derive the conformal weights of all the fields. Note that the choice of conformal weights for the anti-ghost fields (and hence the Lagrange multiplier fields) are essentially free, and should be chosen for later consistency.

At this stage, recall from sec. 5.9 of ref. [20] that the equation $\nabla^{BB'} \lambda_{ABA'B'}$ transforms as a conformal density (of weight $-2$) if and only if $\lambda_{ABA'B'}$ has conformal weight $-2$. Thus, both of the gauge-fixing conditions in (3.21) are not (locally) conformally invariant, since $h_{ABA'B'}$ and $\psi_{ABA'B'}$ do not have conformal weight $+2$. If we were to proceed blindly ahead, our resulting gauge-fixed action may not be conformally invariant.

Witten was faced a rather similar problem in ref. [6], which he tried to resolve by introducing a new field with the appropriate conformal weight to the theory. However, this naturally made the theory very arbitrary. Soon after, Labastida and Pernici [14] pointed out that introducing a new field was unnecessary, and one could obtain conformal invariance by inserting appropriate powers of $g = \text{det}(g_{ab})$, which has conformal weight $+8$, into the right places. For example, our gauge-fixing conditions (3.21) should be modified to read

$$[D_0^*(g^{-1/2}h)]_{AA'} = [D_0^*(g^{-1/2}\psi)]_{AA'} = 0.$$  

(3.24)

We will implicitly adopt this procedure, and insert appropriate powers of $g$ into the appropriate places. They will not be written down explicitly for notational simplicity. It will become clear below that the quantum theory indeed preserves conformal invariance.

### 3.4. Evaluation of the partition function

The time has arrived for the quantisation of (3.3), which may be rewritten in our index-free notation as

$$\int_M d^4x \sqrt{g} \left(W_+^2 - W_-^2\right).$$

(3.25)
The partition function for topological conformal gravity is

\[ Z = \int \mathcal{D}X \exp(-I_{GF}), \quad (3.26) \]

where \( \mathcal{D}X \) represents the path integral over the fields \( h, c, \bar{c}, b, \psi, \bar{\chi}, B, \phi, \bar{\phi} \) and \( \eta \). The gauge-fixed action consists of the classical action minus an \( s \)-exact part:

\[ I_{GF} = \int_M d^4x \sqrt{g} \left[ (W_+^2 - W_-^2) - s \{ \cdots \} \right]. \quad (3.27) \]

Since \( s \) is nilpotent, \( I_{GF} \) is BRST invariant for any choice of terms in the curly brackets (with vanishing ghost number). We judiciously choose

\[ s \{ \cdots \} = s \{ \bar{\chi}W_+ + \frac{1}{8} \alpha \bar{\chi}B + cD_0^* h + \frac{1}{4} \beta \bar{c}b + \bar{\phi}D_0^* \psi \}, \quad (3.28) \]

where \( \alpha \) and \( \beta \) are real gauge parameters. The first, third and last terms in the brackets have the form (anti-ghost)\( \times \) (gauge-fixing condition). \( s \) acting on this then gives terms of the form (Lagrange multiplier)\( \times \) (gauge-fixing condition) in the Lagrangian, plus other terms describing interactions amongst the ghost fields. The second and fourth terms in the brackets give rise to terms of the form (Lagrange multiplier)\(^2\). When expanded out using the properties of \( s \), (3.28) reads

\[ s \{ \cdots \} = BW_+ + \frac{1}{8} \alpha B^2 + \bar{\chi}D_1 \psi + \bar{\chi}D_1 D_0 c + bD_0^* h \]
\[ + \frac{1}{4} \beta b^2 + cD_0^* \psi + cD_0^* D_0 c + \bar{\eta}D_0^* \psi + \bar{\phi}D_0^* D_0 \phi. \quad (3.29) \]

Firstly, let us choose the gauge \( \alpha = \beta = 1 \). Since \( B \) is not a dynamical field, it may be eliminated via its equation of motion \( B = -4W_- \) to give

\[ BW_+ + \frac{1}{8} B^2 \sim -2W_-^2. \quad (3.30) \]

This term adds with the classical action of topological conformal gravity to give a term of the form \( W_+^2 + W_-^2 = C^2 \) in the Lagrangian. This indeed is just the classical action of ordinary conformal gravity.

Similarly, we can eliminate the \( b \) field via its equation of motion \( b = -2D_0^* h \) to yield a term of the form

\[ bD_0^* h + \frac{1}{4} b^2 \sim -(D_0^* h)^2 \sim hD_0 D_0^* h. \quad (3.31) \]
One can also absorb the term \( c D_0^* \psi \) into the term \( \eta D_0^* \psi \) via a redefinition of the field \( \eta \). Hence the total gauge-fixed action becomes

\[
I_{GF} = \int_M d^4x \sqrt{g} \left( C^2 - h D_0 D_0^* h - \bar{\chi} D_1 \psi - \bar{\chi} D_1 D_0 c - \bar{\eta} D_0^* \psi - c D_0^* D_0 c - \bar{\phi} D_0^* D_0 \phi \right). \tag{3.32}
\]

We are now ready to evaluate the partition function (3.26). Note that the gauge-fixed action (3.32) consists of classical conformal gravity together with quantum ghost fields that cancel out all the local degrees of freedom in the classical term. Hence classical minima of this theory are just conformally self-dual gravitational instantons as discussed earlier, for these form the dominant contribution to the partition function (3.26). To characterise the quantum fluctuations around these instanton solutions, we introduce the quadratic partition function

\[
Z^{(2)} = \int DX \exp(-I^{(2)}), \tag{3.33}
\]

corresponding to the part of the action quadratic in the quantum fields. The total partition function is then the sum of the quadratic partition functions \( Z^{(2)} \) over gravitational instantons, whose self-duality enforce the condition that \( D_1 D_0 = 0 \).

By performing the remaining Gaussian integrals over the commuting and anti-commuting quantum fields, we can represent \( Z^{(2)} \) as a ratio of determinants of the Laplacians involved. The Gaussian integral over the commuting \( \phi - \bar{\phi} \) set of fields in (3.32) yields the determinant \( \det^{-1} \Delta_{(0)} \), which cancels with the \( \det \Delta_{(0)} \) contribution coming from the anti-commuting set of fields \( c - \bar{c} \).

Now consider the term \( \sqrt{g} C^2 \). One could in principle work out this term to second order in the metric variation \( \delta g_{ab} = h_{ab} \), but in practice this is an extremely tedious exercise. The trick is to rewrite \( \sqrt{g} C^2 = \sqrt{g} C * C + 2 \sqrt{g} W^2 \), and realise that since the first term is a topological invariant, it does not contribute to the quantum dynamics. The second term, of the form \( \sqrt{g} W^2 \), is positive definite and is minimised when the manifold is self-dual. To second order in the trace-free metric variation \( h_{ab} \), it is

\[
\sqrt{g}(1 - \frac{1}{4} h^2)(W_- + D_1 h)^2
= \sqrt{g} (W_+^2 + (D_1 h)^2 + 2W_- D_1 h - \frac{1}{4} h^2 W^2). \tag{3.34}
\]

But imposing the condition \( W_- = 0 \), we find that the second order quantum variation coming from \( C^2 \) term is

\[
(D_1 h)^2 \sim -h D_1^* D_1 h. \tag{3.35}
\]
Observe that the $h$ fields in (3.35) are transverse trace-free. By contrast, the $h$ fields in the second term of (3.32) are longitudinal. With both terms together, the Gaussian integral over the $h$ field just gives $\det^{-1/2} \Delta_{(1)}$ in the quadratic partition function.

Consider now the two remaining terms of (3.32):

$$\bar{\chi} D_1 \psi + \bar{\eta} D_0^* \psi.$$  \hfill (3.36)

These $\bar{\eta}-\psi-\bar{\chi}$ fields are precisely those encountered in the deformation complex (2.19). Observe that the $\bar{\chi}$ and $\bar{\eta}$ field equations from (3.36) respectively give

$$D_1 \psi = 0, \quad D_0^* \psi = 0.$$ \hfill (3.37)

These equations are nothing but the deformation equations. Hence if we want to eliminate fermionic zero-modes from this theory, we will have to choose our manifold $M$ such that its gravitational moduli space has vanishing dimension. This is what we will assume in the rest of this section.

A subtlety arises when trying to evaluate determinants for the $\bar{\eta}-\psi-\bar{\chi}$ system. Note that the terms in (3.36) define a differential operator $T$

$$T : \Omega_{22} \to \Omega_{11} \oplus \Omega_{40} ,$$ \hfill (3.38)

which is the same operator as that defined in (2.38). Since $T$ does not map $\Omega_{22}$ into itself, its determinant is not a priori well-defined. One could however consider the adjoint of $T$

$$T^* : \Omega_{11} \oplus \Omega_{40} \to \Omega_{22} ,$$ \hfill (3.39)

and define in the usual way

$$\det T \equiv \det^{1/2}(T^* T) ,$$ \hfill (3.40)

where $T^* T$ maps $\Omega_{22}$ into itself. Thus, in this case $\det T = \det^{1/2} \Delta_{(1)}$. Hence, our quadratic partition function reduces to simply

$$Z^{(2)} = \left( \frac{\det \Delta_{(1)}}{\det \Delta_{(1)}} \right)^{1/2} = \pm 1 .$$ \hfill (3.41)

3 In this paper, the word “fermionic” refers to the anti-commuting ghost fields, but they do not have half-integer spins.
The total partition function is then a sum of $\pm 1$’s over conformally self-dual gravitational instantons on $M$:

$$Z(M) = \sum_{\text{instantons}} \pm 1. \quad (3.42)$$

We could have started out by choosing instead the delta-function gauge: $\alpha = \beta = 0$. This gauge has the advantage over the previous choice of gauge in being computationally simpler, although it does not illustrate the physical interpretation of topological gravity as a type of conformal gravity theory without local degrees of freedom. But the final results are the same, as we now quickly show.

In the delta-function gauge, the quadratic partition function in (3.33) is

$$I^{(2)} = -\int_M d^4x \sqrt{g} s \{ \bar{\chi} W_- + \bar{c} D_0^* h + \bar{\phi} D_0^* \psi \}$$

$$= -\int_M d^4x \sqrt{g} (BW_- + \bar{\chi} D_1 \psi + \bar{\chi} D_1 D_0 c + b D_0^* h + \bar{c} D_0^* \psi$$

$$+ \bar{c} D_0^* D_0 c + \bar{\eta} D_0^* \psi + \bar{\phi} D_0^* D_0 \phi). \quad (3.43)$$

Again the determinants coming from the $\phi$–$\bar{\phi}$ term cancels with that of the $c$–$\bar{c}$ term, and we absorb the $\bar{\eta} D_0^* \psi$ term into the $\bar{\eta} D_0^* \psi$ term.

The functional integral over the $B$ field yields the delta function constraint $\delta (W_-) = 0$. This enforces the on-shell condition automatically, so that $D_1 D_0 = 0$. Recalling that the lowest order variation of $W_-$ is $D_1 h$, the quadratic partition becomes

$$I^{(2)} = -\int_M d^4x \sqrt{g} (BD_1 h + b D_0^* h + \bar{\chi} D_1 \psi + \bar{\eta} D_0^* \psi). \quad (3.44)$$

That this gives (3.33) value one up to a sign is unambiguous. The $\bar{\eta} \psi - \bar{\chi}$ system of anti-commuting fields, by previous arguments, yields the determinant term $\det^{1/2} \triangle_{(1)}$. Analogously, the $b$–$h$–$B$ system of commuting fields gives $\det^{-1/2} \triangle_{(1)}$, which cancels with the other determinant. Hence we arrive at (3.42) as before.

Note that this value of the partition function is conformally invariant, so our quantum theory of topological gravity is free of any conformal anomaly.

The expression (3.42) is our gravitational analogue of the first Donaldson invariant, which is the sum of $\pm 1$’s over discrete points of the Yang–Mills moduli space. To determine which sign to use for each instanton, it would not be unreasonable to proceed in the same way as in the Yang–Mills case [2]. We choose a particular instanton metric $g_0$ and declare it to have the positive sign. Given any other instanton $g_1$, we interpolate between the two
via a curve in the space of conformally and physically inequivalent metrics, and change the sign whenever the fermionic determinant in (3.42) has a zero-eigenvalue at some point along the curve. Spectral flow guarantees that this definition is unambiguous, i.e., it is independent of the curve chosen.

It is clear that (3.42) does not depend on the metric, since we are in effect “summing over metrics” with a particular topology. (By contrast, the metric independence of the corresponding Donaldson invariant is less obvious [3].) For now, we will conjecture that (3.42) is a differential invariant of conformally self-dual four-manifolds with a discrete gravitational moduli space. In other words, it is able to distinguish inequivalent smooth structures underlying the manifold, much like the Donaldson invariant. Because of the heuristic nature of our quantum field theoretic derivation, it would be very satisfying if a more rigorous derivation and study of (3.42) could be made.

In the case when the dimension of the moduli space is non-zero, there will be non-trivial solutions to (3.37), resulting in fermionic zero-modes in the theory. The general strategy we then need to adopt is to introduce non-vanishing path integrals of the form

$$Z(O) = \int D X \exp(-I) \cdot O,$$

where $O$ is some functional of the fields $X$. In the field theoretic sense, $O$ is called an observable and $Z(O)$ is the correlation function or vacuum expectation value of the observable. In order to preserve the topological nature of these correlation functions, we require $O$ to be $s$-invariant. We then absorb the zero-modes by demanding that $O$ has ghost number equal to the dimension of the moduli space [4].

4. Concluding remarks

In this paper, we have written down an elliptic complex (2.11) which describes deformations of conformally self-dual gravitational instantons. By applying the Atiyah–Singer index theorem to this elliptic complex, we derived an expression (2.58) for the number of independent non-trivial deformations that can be made about a given gravitational instanton, which preserve its self-duality.

For a given manifold $M$, we defined the gravitational moduli space of $M$ to be the set of conformally self-dual metrics on $M$, factored out by conformal and coordinate transformations. The virtual dimension of the moduli space is given by the number (2.58).
Armed with this mathematical theory of instanton deformations, we then proceeded to develop a theory of topological conformal gravity starting from the classical action (3.3). The BRST gauge-fixing procedure introduced three ghost fields which characterise quantum fluctuations about classical gravitational instantons. It turned out that these ghost fields are most naturally described by the above elliptic complex. In particular, when the gravitational moduli space consists of discrete points, we evaluated the partition function of topological gravity to obtain (3.42). This quantity may be regarded as the gravitational counterpart to the first Donaldson invariant of four-manifolds.

It may be possible to compute gravitational analogues of the higher-order Donaldson invariants when the dimension of the moduli space is greater than zero, by choosing appropriate observables $O$ in (3.45). However, it would probably be premature to do so here, at least until a proper mathematical theory utilising gravitational instantons to classify four-manifolds materialises. With such a theory, it would then provide an impetus for physicists to try to rederive the invariants from topological gravity. What we hoped was achieved in this paper, was to set up the basic quantum theory of topological conformal gravity, and thus put forward plausible physical evidence that gravitational instantons, like their Yang–Mills counterparts, may be used to study the differential topology of four-manifolds. To this effect, we calculated the value of the partition function in a simple case to give a flavour of how these invariants might look like.

In a separate light, one may conjecture that a theory of quantum gravity, if it exists, could be described by a topological quantum field theory. What we live in may then correspond to a phase of quantum gravity where the topological symmetry has been broken, possibly in a way not unlike the Higgs mechanism. To study such a scenario, we need to develop models of topological gravity in four dimensions. One such model has been proposed in this paper, and it is closely related to conformal gravity.

There is another way in which four-dimensional topological quantum field theories may be relevant to quantum gravity. In the Euclidean path integral approach [31], one is supposed to sum over all metrics and topologies in the path integral. However, the complete classification of four-manifolds still eludes us, so any sum over topologies necessarily cannot be complete. This does not mean that we should look for the key only under the lamp-post where there is light, to quote Hawking [32]. A way to get round this is to postulate that any manifold invariant admits a path integral representation via a topological field theory. Consequently, we could possibly define a sensible measure for the sum over topologies by considering topological Yang–Mills theory, topological gravity and
other four-dimensional topological quantum field theories, rather than working with the manifold invariants themselves.

Only time will tell.

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