The Building Blocks of Spatiotemporal Intermittency

Martin van Hecke

Center for Chaos and Turbulence Studies, The Niels Bohr Institute, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark.

(Received 1 January 1997; revised 1 July 1997; accepted 1 July 1997)

We obtain a family of uniformly propagating hole-solutions to the complex Ginzburg-Landau equation which differ from the well-known Nozaki-Bekki holes. They describe the spatial organization and velocity of the dominant local structures in the spatiotemporal intermittent regime. We discuss the phenomenology of these intermittent states in terms of the properties of the local structures.

47.52.+j, 05.45.+b, 47.54.+r, 03.40.Kf

A proper understanding of spatiotemporal chaos, i.e., deterministic chaos occurring in extended systems that are driven from equilibrium, is lacking. Since the number of effective degrees of freedom diverges with the system size, most of the tools developed for low-dimensional systems are inapplicable. Moreover, these tools do not provide a proper framework to describe the spatial organization of extended chaos. In many cases, the dynamical states appear to be built up from local, almost particle-like objects with well-defined dynamics and interactions (Fig. 1b-d). A description of spatiotemporal chaos in terms of these structures is therefore desirable [1].

In this Letter we will investigate the local structures that appear in the spatiotemporal intermittent regime of the complex Ginzburg-Landau equation (CGLE):

\[ A_t = A + (1 + i c_1) A_{xx} - (1 - i c_3)|A|^2 A. \]  

(1)

This amplitude equation describes pattern formation near a Hopf bifurcation and has been applied to describe patterns occurring in, e.g., fluid-convection, Faraday waves, optical systems, chemical oscillations, and turbulent flow past a wake [2]. As a function of the coefficients \( c_1 \) and \( c_3 \), which are determined by the underlying physical problem, behavior ranging from completely regular to strongly chaotic has been found [3,4] (Fig. 1a).

In the spatiotemporal intermittent regime [4,5], a plain wave attractor coexists with a chaotic attractor; most initial conditions evolve to the latter (Figs. 1b-d). The typical states consist of patches of plane waves, separated by various “holes”, i.e., local structures characterized by a depression of \( |A| \). Similar intermittent states have been reported for the damped Kuramoto-Sivashinsky equation, coupled CGLE’s, Rayleigh-Bénard convection and the printer instability [6,7]. It has been suggested that spatiotemporal intermittency should occur generally in the transition route from laminar to chaotic states, and the phenomenology suggests a relation to directed percolation [8,9].

The local structures in the intermittent regime can be divided into two groups depending on the wavenumbers \( q_l \) and \( q_r \) of the asymptotic waves they connect. The quasi-stationary structures in Fig. 1c and 1d have \( q_l \neq q_r \) and are related to the intensively studied Nozaki-Bekki holes [10]. However, the dominant local structures have velocities and asymptotic wavenumbers that are incompatible with the Nozaki-Bekki holes [11]. For example, the holes shown in Fig. 1b all have \( q_l \approx q_r \approx 0 \) since the plain waves (gray areas) have wavenumbers close to zero; similar holes are the dominant structures in the bi-chaotic regime. In the following we shall characterize these holes and their dynamical properties, and discuss their relevance for the chaotic states of the CGLE.

In Fig. 2, \(|A|\), the complex phase (\( \arg(A) \)) and the local wavenumber \( q := \partial_x \arg(A) \) for the left lower part of Fig. 1b are shown. The wavenumbers of the laminar patches are quite close to zero, while the cores of the local structures are characterized by a sharp phase-gradient (peak in \( q \)) and a dip of \(|A|\). The holes propagate with a speed of \( 0.95 \pm 0.1 \) and either their phase-gradient spreads out and the hole decays, or the phase-gradient steepens and the hole evolves to a phase-slip. As a first step in describing these local structures which have a slowly evolving velocity and spatial structure, we will study the coherent structures, i.e., structures with fixed spatial structure and velocity.

For the 1D CGLE, coherent structures have been described in a simple framework [12]. By substituting an ansatz for a uniformly propagating solution of the form \( A(x,t) = e^{-i\omega t} \hat{A}(\xi) \) into the CGLE (\( \xi := x - vt \)), we obtain a set of coupled first order ordinary differential equations (ODE’s)

\[ \begin{align*}
\partial_\xi a &= \kappa a, \\
\partial_\xi z &= -z^2 + \frac{1}{1 + i c_3} \left[ -1 - i\omega + (1 - ic_3)a^2 - vz \right],
\end{align*} \]  

(2a, 2b)

where \( a := |\hat{A}| \) and where the complex quantity \( z \) is defined as \( \partial_\xi \ln(|A|) := \kappa + i q \). Equation (2b) is equivalent to two real valued equations, so (2) can be seen as a 3D real-valued dynamical system [12]. Plain waves correspond to fixed points of (2), and the hole solutions we are interested in correspond to orbits connecting these fixed points. A rather complete study of the heteroclinic orbits which describe for instance the Nozaki-Bekki holes \( (q_l \neq q_r) \) has been made [13]. We are here interested in
local structures that have \( q_l = q_r \), i.e., \textit{homoclinic} orbits of the ODE’s \( (2) \).

In general the ODE’s \( (2) \) have \( \omega \) and \( v \) as free parameters, but since the wavenumber in the laminar patches is approximately zero, we demand \( q_l = q_r = 0 \), which fixes \( \omega = -c_3 \). The fixed point at \((a, z) = (1, 0)\) corresponds to the \( q = 0 \) plain waves and has a 1D outgoing manifold and a 2D spiraling ingoing manifold (Fig. 3a). To create a homoclinic orbit, we have to connect these manifolds. This amounts to satisfying only a single condition, and since we have one free parameter \((v)\), we can expect a discrete set of homoclines for \( q_l = q_r = 0 \). Performing a simple numerical integration of \( (2) \) and adjusting the free parameter \( v \), we obtain a homoclinic orbit for \( v \approx 0.916 \); the corresponding coherent structure (Fig. 3b) will be referred to as a \textit{homoclon}. For \( q_l = q_r \neq 0 \), i.e., \( \omega \neq -c_3 \) one can obtain similar homoclinic orbits, so in fact there exists a one-parameter family of homoclines.

In Fig. 3b the homoclines are compared to the local structures in the intermittent regime. The slight deviation between this particular local structure and the homoclin is mainly due to the fact that the local structure has a slowly evolving shape. The longer the lifetime of the holes, the better the fit is to the homoclines. For nearby values of \( c_1 \) and \( c_3 \) one finds similar correspondences between the \( q = 0 \) homoclines and the local structures.

It is instructive to compare the homoclines with the Nozaki-Bekki holes \( (1) \). The Nozaki-Bekki holes contradict naive counting arguments and have been shown to be structurally unstable, while the homoclines satisfy the counting arguments, and are structurally stable. One can verify that for \( c_1 = 0.6, c_3 = 1.4 \), a Nozaki-Bekki hole with \( q_l = 0 \) has \( q_r \approx 0.837 \) and a velocity of 1.673, which is completely different from the local structures and the homoclines. In some regimes, in particular for negative \( c_1 \), the Nozaki-Bekki holes are dynamically relevant, and as they are sources for waves with \( q \neq 0 \), we obtain grain boundaries between \( q \neq 0 \) and \( q \approx 0 \) waves (Fig. 1c,d). In the limit where \( q_l = q_r \), the width of the Nozaki-Bekki holes diverges, so we can only conclude that there are two distinct types of hole solutions: heteroclinic Nozaki-Bekki holes and homoclines.

Homoclinic orbits of \( (2) \) describe at least two other cases. Since the homoclinic orbit occurs as a saddle-focus loop, there are multi-circuit loops and a limit-cycle for nearby values of \( v \) or \( \omega \). The limit cycle collapses via a Hopf bifurcation on another fixed point of the ODE’s \( (2) \) for sufficiently small \( v \). The (quasi)periodic states of the CGLE that have been called “compression waves” \( (1) \) are described by these limit-cycles. Furthermore, the “riding chaos” recently reported in \( (1) \) has been described in the context of homoclinic orbits for the \textit{phase} equation of the CGLE, and these orbits also occur in the ODE’s \( (2) \). For example, the uniformly propagating structures shown in Fig. 11 of \( (1) \) occur in a background of wavenumber \( \approx 0.038 \) for \( c_1 = 1.75, c_3 = 0.8 \).

The ODE’s for the full CGLE \( (2) \) admit a homoclinic orbit for \( v \approx 0.557 \), that describes the riding chaotic states perfectly. For the same asymptotic wavenumbers and values of the coefficients, there exists also a homoclinic orbit for \( v \approx 1.124 \); this orbit is similar to the homoclines and cannot be obtained from the phase equations. More detailed studies of the homoclinic orbits are underway.

We will now return to the homoclines which appear in the spatiotemporal intermittent regime and study their stability. Following an isolated, slightly perturbed homoclon, one immediately finds that they are weakly unstable \( (2) \). In a similar fashion to the local structures found in the intermittent state, they either slowly decay or grow out to a phase-slip (Fig 4a-d). In the phase-space of the CGLE \( (1) \) we can think of the homoclines as \textit{unstable} equilibria separating plain waves and phase-slips.

In Fig. 4a we show the decay of a slightly perturbed homoclin (perturbation of order \( 10^{-6} \)). The total phase-difference \( \Delta \theta \) across the decaying homoclin is conserved since there are no phase-slips; for \( c_1 = 0.6 \) and \( c_3 = 1.4, \Delta \theta \approx 3.24 \). During the decay, the wavenumber-peak, amplitude-dip and apparent velocity decrease; for long times, the dynamics crosses over to a slow phase-diffusion by which \( \Delta \theta \) is smeared out. In Fig. 4b, the evolution of a slightly perturbed homoclin towards several phase-slips is shown. The wavenumber-peak and amplitude-dip slowly grow, and at \( t \approx 117 \) the first phase-slip occurs, from which a typical spatiotemporal intermittent state nucleates. Before this phase-slip, the wavenumber acquires a negative peak in order to conserve the total phase-difference across the structure (Fig. 4c). Both these peaks diverge at the phase-slip event, and just after the phase-slip, the winding-number \( \int dxq/2\pi \) has decreased by 1. Therefore, the negative phase-bump that corresponds to the new left moving hole, is quite steep (Fig. 4d), and this hole will quickly grow out to a new phase-slip, from which a strong right moving hole is generated etc. When we quench \( c_1 \) and \( c_3 \) in the direction of the transition to plain waves, these zigzag motions of the holes become very dominant (Fig. 5a).

The group-velocity for the plain waves is close to zero. Therefore, in the co-moving frame of a right moving hole, the wave to the right has its group-velocity pointing inward and the wave to the left has it pointing outward; hence the homoclines are neither sources nor sinks. Suppose we have a homoclin with a positive phase-difference \( \Delta \theta \), moving to the right into a plain wave with wavenumber \( q_r \). Because they are unstable, their dynamical fate depends strongly on \( q_r \). When \( q_r > 0 \), this leads to the “winding up” of the homoclin (increase of \( \Delta \theta \)), and a phase-slip is generated in which the total winding-number is decreased. The larger \( q_r \) is, the faster this process goes, and as a result, the total number of phase-windings in the system is driven to approximately zero. When \( q_r < 0 \), or equivalently, when a left moving ho-
moclon invades a state with positive $q$, this leads to the “winding down” of the homoclon, which then decays. Its associated $\Delta \theta$ is smeared out and this drives the wavenumbers in the laminar patches to zero. The winding up and down is shown in Fig. 5b, where we follow the evolution of an isolated homoclon in a background state with wavenumber 0.05. This strong sensitivity to the asymptotic waves has as a consequence that the fluctuations in the laminar patches of the intermittent regime have a strong effect on the homoclons. Small wave-packets, resulting from decaying homoclons, can trigger the decay or phase-slip of the propagating holes; this might result in correlations in the intermittent regime over long times and distances.

The homoclons have only one weakly unstable eigenmode. Consider the example illustrated in Fig. 6a. The initial condition here has $|A| = 1$ and a triangular wavenumber profile. In general, there is no reason to expect such an initial condition to evolve to a homoclon, but by adjusting only one parameter in the initial condition, this is precisely what happens; for the example shown in Fig. 6a, the height of the triangle was set to 0.437754 while its width was 20. Increasing the height or width of the initial wavenumber-blob leads to a steepening hole like in Fig. 4b, while a decrease leads to a decaying hole like in Fig. 4a. The total phase-difference across the triangular initial condition is 8.76, so a big positive wavenumber-packet is emitted from the homoclon. We have also tried this procedure for other initial conditions, and found that in general the adjustment of only one free parameter in the initial conditions leads to the generation of a long living homoclon; the dimension of the unstable manifold is therefore one. The contraction towards this unstable manifold is rather fast. This is reflected in the dynamics in the intermittent regime; the local structures evolve to a reasonable approximation along the 1D unstable manifold of the homoclons.

To substantiate this claim, we considered the relation between the values of the extrema of $q$ and the corresponding local minima of $|A|$. As shown in Fig. 6b, these quantities are strongly correlated. This indicates that a one-parameter family of profiles of $A$ is dominant. Note the increase of the density around values of $a$ and $q$ which correspond to the coherent homoclon: being an equilibrium state, the dynamics spend a relative long time there.

In Fig. 6c,d we collapsed the profiles which correspond to certain isolated minima of $|A|$. The profiles shown in Fig. 6c correspond to the coherent homoclon, while the profiles in Fig. 6d correspond to states evolving towards a phase-slip.

This suggests a phenomenological model in terms of moving “homoclon” particles possessing an internal degree of freedom that parameterizes the location on the unstable manifold. More detailed studies of this are underway [19].

In conclusion, we have described a new class of coherent solutions which occur in the spatiotemporal intermittent regime and which are intimately connected to phase-slips; they are therefore one of the prime local structures of the 1D CGLE.

It is a pleasure to acknowledge discussions with Tomas Bohr, Lorentz Kramer, Wim van Saarloos, Emilio Hernández-García and Maxi San-Miguel. This work was supported by the Netherlands Organization for Scientific Research (NWO).
FIG. 1. (a) “Phase-diagram” of the CGLE. For small $c_1$ and $c_3$ all initial conditions evolve to plain waves. In the intermittent regime, a plain wave attractor and a chaotic attractor coexist. Beyond the full curve $c_1 c_3 = 1$, all plain waves are linearly unstable and all states are spatiotemporal chaotic. At a zero of $\mathcal{A}$ the complex phase is undefined and phase-slips occur (see Fig. 2); the chaotic state is then called defect-chaos. When $\mathcal{A}$ has no zeroes we speak of phase-chaos. In the bi-chaotic regime, a defect and phase-chaotic attractor coexist [4,11]. (b)-(d) Space-time plots (over a range of $200 \times 150$) of $|\mathcal{A}|$ (black corresponds to $|\mathcal{A}|=0$) showing chaotic states in the spatiotemporal intermittent regime, for coefficients $(c_1,c_3)=(0.6,1.4)$ (b), $(0,1.8)$ (c) and $(0,1.4)$ (d).

FIG. 2. Space-time (60x50) plots of the left-lower part of Fig. 1b, showing $|\mathcal{A}|$, arg($\mathcal{A}$) and $q:=\partial_x$ arg($\mathcal{A}$) in detail.

FIG. 3. (a): The homoclinic orbit of the ODE’s (2) in $a,q,\kappa$ space. (b) The amplitude and wavenumber profile of the corresponding coherent structures (curves). The circles correspond to a local structure obtained from simulations of the CGLE in the spatiotemporal intermittent regime. The bump in $a$ to the right of the core corresponds to the spiraling motion on the incoming manifold of the fixed point $(a,q,\kappa)=(1,0,0)$.

FIG. 4. (a,b) Evolution of the wavenumber profiles of slightly perturbed homoclon. Consecutive time-slices have a time difference of 5. (c,d) Wavenumber profiles (full curves) and winding-number $\int dxq/2\pi$ (dashed curves) just before (c) and after (d) the first phase-slip.

FIG. 5. (a) The dominance of zigzagging holes near the transition to plain waves $c_1 = 0.6, c_3 = 1.2$ for space $\times$ time $= 512 \times 1000$. (b) The evolution of a homoclon in a background state with wavenumber 0.05, for $c_1 =0.6, c_3 = 1.4$ and space $\times$ time $= 512 \times 250$.

FIG. 6. (a) The wavenumber profiles of a “triangular” initial condition evolving to a homoclon. (b) Collapse of the extrema of $a$ and $q$ in the intermittent regime. (c) Amplitude profiles of $|\mathcal{A}|$ with minima of $|\mathcal{A}|$ around the homoclon value. (d) Similar profiles for steeper minima.
This figure "fig1.jpg" is available in "jpg" format from:

http://arxiv.org/ps/chao-dyn/9707010v1
This figure "fig2.jpg" is available in "jpg" format from:

http://arxiv.org/ps/chao-dyn/9707010v1
This figure "fig3.jpg" is available in "jpg" format from:

http://arxiv.org/ps/chao-dyn/9707010v1
This figure "fig4.jpg" is available in "jpg" format from:

http://arxiv.org/ps/chao-dyn/9707010v1
This figure "fig5.jpg" is available in "jpg" format from:

http://arxiv.org/ps/chao-dyn/9707010v1
This figure "fig6.jpg" is available in "jpg" format from:

http://arxiv.org/ps/chao-dyn/9707010v1