On the Absence of Bound States for a Planar Massless Brown–Ravenhall-Type Operator

Abstract  We address the question of the existence of bound states for a suitably projected two-dimensional massless Dirac operator in the presence of a Bessel–Macdonald potential (also known as $K_0$-potential potential), raised in De Lima (Eur Phys J B 93:187, 2020). Based on Relativistic Hardy Inequality, we prove that this operator has no bound states if $\gamma \leq \gamma_{\text{crit}}$ (subcritical region), where $\gamma$ is a coupling constant.

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1 Introduction

In this short communication, we focus on the planar massless Dirac operator with Bessel–Macdonald potential restricted to its positive spectral subspace (a Brown–Ravenhall-type operator); namely, in order to describe a quasi-particle in the field of another quasi-particle and subject to relativistic effects, the suitably projected planar massless Dirac operator with a Bessel–Macdonald potential is

$$B(x) = \Lambda_+ (D_0(x) - \gamma K_0(\mu|x|)) \Lambda_+ ,$$

where $D_0$ is the free massless Dirac operator in $d = 2$, $\gamma > 0$ is the coupling parameter taken to be contained in the non-negative semi-axis $[0, \infty)$ and $K_0(\mu|x|)$ is the Bessel–Macdonald potential induced by

$$K_0(\mu|x|) = \frac{1}{2} \int_0^\infty e^{-\frac{\mu^2|x|^2}{\eta}} e^{-\frac{s}{4\pi\eta}} \frac{1}{\eta} d\eta .$$

Here, $\mu > 0$ is a real parameter. Specifically, we shall address the question of the existence of bound states for the operator (1.1) raised in Ref. [1].

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Remark 1 (The origin of the operator (1.1)) At this point, we recall that the physics of graphene has acted as a bridge between quantum field theory and condensed matter physics due to the special quality of graphene quasi-particles behaving as massless two-dimensional Dirac fermions [2]. In our case, the operator (1.1) emerges from a parity-preserving $U(1) \times U(1)$ massless QED$_3$ proposed in [1] as a pristine graphene-like planar quantum electrodynamics model, exhibiting two massive gauge bosons. In particular, the $K_0$-potential arises when analyzing the $s$- and $p$-wave Møller (electron-polaron–electron-polaron) scattering amplitudes. While the $p$-wave state fermion-fermion (or antifermion-antifermion) scattering potential shows to be repulsive whatever the values of the electric ($e$) and chiral ($g$) charges, for $s$-wave scattering of fermion-fermion (or antifermion-antifermion), the interaction potential might be attractive provided $g^2 > e^2$ (see in Ref. [1] the details). Thus, the question of whether or not the attractive $s$-wave state potential favors $s$-wave massless bipolarons (two-fermion bound states) shall be answered here by investigating into details the operator (1.1), together with necessary and sufficient conditions which guarantee relativistic two-particle massless bound states.

2 Positive spectral subspace and reduction of spinors

Any attempt to describe bound states of spin-$1/2$ relativistic quasi-particles via the potential $K_0$ must take into account the problem of continuum dissolution [3–5]. A solution to this problem is, according to Brown–Ravenhall [3], to consider the suitably projected planar massless Dirac operator in the presence of a $K_0$-potential given by

$$B(x) = \Lambda_+ \left( D_0(x) - \gamma K_0(\mu|x|) \right) \Lambda_+ .$$

Here, the operator $D_0$ is a first-order operator acting on spinor-valued functions $\Psi(x) = (\psi_1(x), \psi_2(x))$, with 2 components, of the space variable $x = (x_1, x_2)$. We denote by $\mathbb{C}^2$ the 2-dimensional complex vector space in which the values of $\Psi(x)$ lie. $D_0$ has the form

$$D_0 = -i \sigma \cdot \nabla ,$$

where $\sigma = (\sigma_1; \sigma_2)$ are the Pauli $2 \times 2$-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

$$\Lambda_+ \overset{\text{def.}}{=} \chi_{(0,\infty)}(D_0) ,$$

where $\chi_{(0,\infty)}$ is the characteristic function of the interval $(0, +\infty)$, denotes the projection of $L^2(\mathbb{R}^2; \mathbb{C}^2)$ onto the positive spectral subspace of $D_0$; namely,

$$\Lambda_+ = \frac{1}{2} \left( \mathbb{1}_{2 \times 2} + \frac{-i \sigma \cdot \nabla}{\sqrt{-\Delta}} \right) ,$$

where $\Delta$ is the laplacian operator on $\mathbb{R}^2$. Note that $D_0 \Lambda_+ = \Lambda_+ D_0 = \sqrt{-\Delta} \Lambda_+$. The last equality is a consequence of the following fact: in Fourier variables, the projector $\Lambda_+$ is a multiplication operator given by the following expression:

$$\hat{\Lambda}_+ = \frac{1}{2} \left( \mathbb{1}_{2 \times 2} \mathbb{1} \sigma \cdot \mathbb{p} \right) .$$

Hence, the Brown–Ravenhall-type operator is given formally as

$$B(x) = \Lambda_+ \sqrt{-\Delta} \Lambda_+ - \gamma \Lambda_+ K_0(\mu|x|) \Lambda_+ .$$
acting in \( L_2(\mathbb{R}^2; \mathbb{C}^2) \), or, equivalently
\[
\mathcal{B}(x) = \Lambda_+ \sqrt{-\Delta} - \gamma \Lambda_+ K_0(\mu |x|) \, .
\]
acting in \( \mathcal{H}_+ \overset{\text{def}}{=} \Lambda_+(L_2(\mathbb{R}^2; \mathbb{C}^2)) \).

The first step to prove the existence or absence of bound states for the operator (1.1) is a reduction of spinors. We will follow the same strategy as Zelati-Nolasco [6]: we use the Foldy–Wouthuysen transformation (FW), given by a unitary transformation \( U_{\text{FW}} \) which transforms the free massless Dirac operator into the diagonal form (see details in [7] for the case in \( d = 1 + 2 \))
\[
\mathcal{D}_{\text{FW}} = U_{\text{FW}} \mathcal{D}_0 U_{\text{FW}}^{-1} = \begin{pmatrix} \sqrt{-\Delta} & 0 \\ 0 & -\sqrt{-\Delta} \end{pmatrix} = \sigma_3 \sqrt{-\Delta} \, ,
\]
where \( \sigma_3 \) is the Pauli \( 2 \times 2 \)-matrix
\[
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \, .
\]

**Remark 2** With the usual quantization rule \( p \mapsto -i \nabla \) (here the units are chosen so that \( \hbar = c = 1 \)), let us recall that to the operator \( \sqrt{-\Delta} \) can be defined for all \( \Psi \in H^1(\mathbb{R}^2; \mathbb{C}^2) \) as the inverse Fourier transform of the \( L_2 \)-function \( |p| \hat{\Psi}(p) \) (where \( \hat{\Psi} \) denotes the Fourier transform of \( \Psi \)). To \( \sqrt{-\Delta} \) we can associate the following quadratic form
\[
q_{\hbar}(\Phi, \Psi) \overset{\text{def}}{=} \langle \Phi, \sqrt{-\Delta} \Psi \rangle = \frac{1}{(2\pi \hbar)^2} \int_{\mathbb{R}^2} |p| \overline{\Phi}(p) \hat{\Psi}(p) \, dp \, .
\]
which can be extended to all functions \( \Phi, \Psi \) in the form domain \( \mathcal{Q}(\sqrt{-\Delta}) = H^{1/2}(\mathbb{R}^2; \mathbb{C}^2) \), where
\[
H^{1/2}(\mathbb{R}^2; \mathbb{C}^2) = \left\{ \Psi \in L_2(\mathbb{R}^2; \mathbb{C}^2) \mid \int_{\mathbb{R}^2} (1 + |p|^2)^{1/2} |\hat{\Psi}(p)|^2 \, dp < \infty \right\} \, .
\]

Under the FW-transformation the projector \( \Lambda_+ \) becomes simply
\[
\Lambda_{+\text{FW}} \overset{\text{def}}{=} U_{\text{FW}} \Lambda_+ U_{\text{FW}}^{-1} = \frac{1}{2} (I_{2 \times 2} + \sigma_3) \, .
\]
Therefore the positive energy subspace for \( \mathcal{D}_{\text{FW}} \) is simply given by
\[
\mathcal{H}_+ = \left\{ \Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix} \in L_2(\mathbb{R}^2; \mathbb{C}^2) \mid \psi \in L_2(\mathbb{R}^2; \mathbb{C}) \right\}
\]

In the FW-representation the associated quadratic form acting on \( \mathcal{H}_+ \) is defined by
\[
\langle \varphi, \mathcal{B}_{\text{FW}} \psi \rangle_{L_2(\mathbb{R}^2; \mathbb{C})} = \langle \varphi, \sqrt{-\Delta} \psi \rangle_{L_2(\mathbb{R}^2; \mathbb{C}^2)} + \langle \varphi, V_{\text{FW}} \psi \rangle_{L_2(\mathbb{R}^2; \mathbb{C})} , \tag{2.1}
\]
for any \( \varphi, \psi \in H^{1/2}(\mathbb{R}^2; \mathbb{C}) \), where \( V_{\text{FW}} \psi = Q^* U_{\text{FW}} V U_{\text{FW}}^{-1} Q \psi \), with \( V(x) = -\gamma K_0(\beta |x|) \) and
\[
Q : \mathbb{C} \to \mathbb{C}^2 , \quad Q(z_1) = (z_1, 0) \, , \\
Q^* : \mathbb{C}^2 \to \mathbb{C} , \quad Q^*(z_1, z_2) = z_1 \, ,
\]
so that
\[
\langle \varphi, \sqrt{-\Delta} \psi \rangle_{L_2(\mathbb{R}^2; \mathbb{C}^2)} = \left\langle \Lambda_+ U_{\text{FW}}^{-1} Q \varphi, \mathcal{D}_0 \Lambda_+ U_{\text{FW}}^{-1} Q \psi \right\rangle_{L_2(\mathbb{R}^2; \mathbb{C}^2)} = \left\langle \Lambda_+ U_{\text{FW}}^{-1} \begin{pmatrix} \varphi \\ 0 \end{pmatrix}, \mathcal{D}_0 \Lambda_+ U_{\text{FW}}^{-1} \begin{pmatrix} \psi \\ 0 \end{pmatrix} \right\rangle_{L_2(\mathbb{R}^2; \mathbb{C}^2)} \, .
\]
Our investigation is based on the following result [8, Lemma 8.2]:

Theorem 3.1 (Relativistic Hardy Inequality)

Let \( Q \) be a unitary equivalence between the operators \( \Lambda_+ U_{FW}^{-1} Q \Phi \) and \( \Lambda_+ U_{FW}^{-1} \Phi \) with form domain \( \mathcal{D}(\sqrt{-\Delta}) = H^{1/2}(\mathbb{R}^d; \mathbb{C}) \). Hence, the transition from \( \Psi \in L_2(\mathbb{R}^d; \mathbb{C}) \) to the reduced spinor \( \bar{\Psi} \in L_2(\mathbb{R}^d; \mathbb{C}) \) through the introduction of the operator \( \mathcal{B}_{FW} \) is possible because we are working in \( \mathcal{H}_+ \). Naturally, the map \( \langle \Psi, \mathcal{B}_{FW} \Phi \rangle \to \langle \Psi, \mathcal{B}_{FW} \Phi \rangle \), where \( \Psi \in L_2(\mathbb{R}^d; \mathbb{C}) \) and \( \bar{\Psi} \in L_2(\mathbb{R}^d; \mathbb{C}) \), determines a unitary equivalence between the operators \( \mathcal{B} \) and \( \mathcal{B}_{FW} \). Hence, the absence of bound states for the operator \( \mathcal{B}_{FW} \) implies the absence of bound states for the operator \( \mathcal{B} \). So, from now on, we will work directly with the operator

\[
\mathcal{B}_{FW} = \sqrt{-\Delta} - \gamma K_0(\mu|x|). 
\]

### 3 Absence of bound states

A standard problem of spectral theory in quantum mechanics is to obtain conditions on a potential in order to guarantee that this potential has bound states. In this section, our goal is to investigate whether the potential of the Bessel–Macdonald type, \( V(x) = \gamma K_0(\mu|x|) \), can lead to bound states of massless Dirac quasi-particles. Our investigation is based on the following result [8, Lemma 8.2]:

**Theorem 3.1** (Relativistic Hardy Inequality) Let \( d \geq 2 \), and let \( \psi \) be a function in \( H^{1/2}(\mathbb{R}^d) \). Then

\[
\int_{\mathbb{R}^d} \frac{1}{|x|} |\psi(x)|^2 \, dx \leq C_d^2 \int_{\mathbb{R}^d} |p||\hat{\psi}(p)|^2 \, dp = C_d^2 \int_{\mathbb{R}^d} \bar{\psi}(x) \sqrt{-\Delta} \psi(x) \, dx,
\]

where the best possible value of the constant \( C_d \) is

\[
C_d = \frac{\Gamma\left(\frac{d}{2} - 1\right)}{\sqrt{2} \Gamma\left(\frac{d+1}{2}\right)}.
\]

The equality is only attained if \( \psi = 0 \), i.e., for any bigger constant the inequality fails for some function in \( H^{1/2}(\mathbb{R}^d) \).

**Remark 3** This inequality goes back to Kato [9, Eq.(V.5.33)] and Herbst [10, Theorem 2.5]. See also [11, Theorem 1.7.1].

By applying Theorem 3.1 to \( \langle \psi, \mathcal{B}_{FW} \psi \rangle \) we obtain

\[
\langle \psi, \mathcal{B}_{FW} \psi \rangle \geq \int_{\mathbb{R}^d} \bar{\psi}(x) \left( \frac{1}{C_d^2} \frac{1}{|x|} - \gamma K_0(\mu|x|) \right) \psi(x) \, dx.
\]

Next, we shall use the following rough estimate

\[
K_0(\mu|x|) = \frac{1}{2} \int_0^\infty e^{-\frac{\pi \mu^2 \eta^2}{2}} e^{-\frac{x}{\eta}} \frac{1}{\eta} \, d\eta
\]

\[
= \frac{1}{2|x|^\alpha} \int_0^\infty \frac{|x|^\alpha}{\eta^{\alpha/2}} e^{-\frac{\pi \mu^2 \eta^2}{2}} e^{-\frac{x}{\eta}} \frac{1}{\eta^{1-\alpha/2}} \, d\eta
\]

\[
\leq \frac{1}{2|x|^\alpha} \left( \sup_{t>0} t^{\alpha/2} e^{-\pi \mu^2 t} \right) \int_0^\infty e^{-\frac{x}{\eta}} \frac{1}{\eta^{1-\alpha/2}} \, d\eta.
\]

Now, since
On the Absence of Bound States

(i) $t^{\alpha/2}e^{-\pi\mu^2t} > 0$, $\forall \, t > 0$.
(ii) $\lim_{t \to 0^+} \left(t^{\alpha/2}e^{-\pi\mu^2t}\right) = 0 = \lim_{t \to +\infty} \left(t^{\alpha/2}e^{-\pi\mu^2t}\right)$.
(iii) $\frac{d}{dt} \left(t^{\alpha/2}e^{-\pi\mu^2t}\right) = 0 \iff t = \frac{\alpha}{2\pi\mu^2}$.

we conclude that

$$\sup_{t > 0} t^{\alpha/2}e^{-\pi\mu^2t} = t^{\alpha/2}e^{-\pi\mu^2t}\Bigg|_{t = \frac{\alpha}{2\pi\mu^2}} = \left(\frac{\alpha}{2\pi\mu^2}\right)^{\alpha/2} e^{-\alpha/2}.$$

Furthermore, according to Table of Integrals of Gradshtein-Ryzhik [12, 3.381, 4., p.346], it follows that

$$\int_0^\infty e^{-\frac{\mu}{4\pi}} \frac{1}{\eta^{1-\alpha/2}} d\eta = (4\pi)^{\alpha/2} \Gamma(\alpha/2).$$

Hence, we have

$$K_0(\mu|x|) \leq \frac{C_{\alpha,\mu}}{|x|^{\alpha}},$$

where

$$C_{\alpha,\mu} = \frac{1}{2} \left(\frac{\alpha}{2\pi\mu^2}\right)^{\alpha/2} (4\pi)^{\alpha/2} \Gamma(\alpha/2) e^{-\alpha/2}.$$

If we take $\alpha = 1$, then

$$K_0(\mu|x|) \leq \frac{C_{1,\mu}}{|x|}.$$

This implies, according to relativistic Hardy inequality, that we have for $d = 2$

$$\int_{\mathbb{R}^2} \overline{\psi}(x) \left(\frac{1}{C_1^2} \frac{1}{|x|} - \gamma K_0(\mu|x|)\right) \psi(x) \, dx \geq \frac{1}{C_1^2} \left(1 - \gamma C_1^2 C_{1,\mu}\right) \int_{\mathbb{R}^2} \frac{1}{|x|} |\psi(x)|^2 \, dx. \quad (3.3)$$

Note that

$$C_1^2 C_{1,\mu} = \frac{1}{\sqrt{2}} \Gamma(1/2)e^{-1/2} \left(\frac{\Gamma(1/4)}{\sqrt{2} \Gamma(3/4)}\right)^2 \frac{1}{\mu}.$$

We can simplify the expression of this constant, taking into account the relationship that exists between the gamma function and the beta function. Indeed, it follows that

$$C_1^2 C_{1,\mu} = \frac{[B(1/2, 1/4)]^2}{\sqrt{8\pi e}} \frac{1}{\mu} = \frac{[\Gamma(1/4)]^4}{2(2\pi)^{3/2} e^{1/2}} \frac{1}{\mu}.$$

In the last equality, we use the well-known expression

$$B(x, y) = 2 \int_0^{\pi/2} (\cos \varphi)^{2x-1} (\sin \varphi)^{2y-1} \, d\varphi,$$

and the Table of Integrals of Gradshtein-Ryzhik [12, 3.621, 7.*, p.395] in order to calculate the value of the function $B(1/2, 1/4)$.

At this point, we remark a number of interesting properties of planar massive Dirac operator with Bessel–Macdonald potential, restricted to its positive spectral subspace, which have been obtained in Ref. [13]. For instance, the following have been established.

1. $B_{FW}$ is shown to be bounded below if, and only if, $\gamma \leq \gamma_{crit} = (C_1^2 C_{1,\mu})^{-1}$ (a property also referred to as stability of matter). $B_{FW}$ is, in fact, shown to be positive in [13, Proposition 4.5], the estimate (in appropriate units) $B_{FW} \geq m(1 - \gamma\gamma_{crit}^{-1})$ being obtained.
2. $\mathcal{B}_{FW}$ is self-adjoint on the form domain $H^{1/2}(\mathbb{R}^2; \mathbb{C})$ if $\gamma < \gamma_{\text{crit}}$ [13, Proposition 3.4]. For the critical value $\gamma = \gamma_{\text{crit}}$, the Friedrichs Extension Theorem guarantees that the quadratic form $\langle \psi, \mathcal{B}_{FW} \psi \rangle$ is a closable quadratic form and its closure is the quadratic form of a unique self-adjoint operator associated with $\mathcal{B}_{FW}$. Thereby, the critical coupling constant $\gamma_{\text{crit}}$ can be mathematically thought of as that coupling constant where a natural definition of self-adjointness ceases to exist.

3. The essential spectrum $\sigma_{\text{ess}}(\mathcal{B}_{FW})$ is proved in [13, Theorem 4.1] to coincide with $[m, \infty)$ when $\gamma \leq \gamma_{\text{crit}}$. Possible embedded eigenvalues in the essential spectrum $\sigma_{\text{ess}}(\mathcal{B}_{FW})$ are absent [13, Lemma 4.4]. In particular, all eigenvalues are non-negative, i.e., in $[0, m)$ the discrete spectrum $\sigma_{\text{disc}}(\mathcal{B}_{FW})$ consists of an infinite number of isolated eigenvalues of finite multiplicity.

Returning to the massless case, since the difference between the operator $\sqrt{-\Delta + m^2}$ and $\sqrt{-\Delta}$ is bounded, more precisely $-\Delta + m^2 \geq \sqrt{-\Delta} + m^2 \geq \sqrt{-\Delta}$, the stability described in Remark 1 is the same as the stability of operator (2.3). Note that for $m = 0$ the bound in Remark 1 shows positivity directly. Moreover, according to Remark 2, the operator (2.3) is self-adjoint on the form domain $H^{1/2}(\mathbb{R}^2; \mathbb{C})$ if $\gamma < \gamma_{\text{crit}}$. Finally, with respect to Remark 3, the operator (2.3) has the essential spectrum to coincide with $[0, \infty)$ when $\gamma \leq \gamma_{\text{crit}}$. In this case, all eigenvalues, $\lambda$, should be negative. But, according to (3.3), this would only be possible if $\gamma > \gamma_{\text{crit}}$ (supercritical region). We have nothing to say about this, since this would imply the non-self-adjointness of the operator $\mathcal{B}_{FW}$ and, therefore, the absence of dynamics by the celebrated Stone’s Theorem [14, Theorem VIII.8]. This leaves the possibility of $\lambda = 0$. Then, suppose that $0$ is an eigenvalue of $\mathcal{B}_{FW}$ with corresponding eigenfunction $\psi$. Thus, the right-hand side of (3.2) must be zero. But this would imply that there is equality in (3.1) with $\psi \neq 0$, which is not possible.

In short, bound states for the operator (2.3) should occur whenever a quasi-particle in the field of another quasi-particle cannot move to infinity. That is, the quasi-particle should be confined or bound at all energies to move within a finite and limited region of space. The operator (2.3) in this region admits only solutions with eigenvalues that are in the $\sigma_{\text{disc}}(\mathcal{B}_{FW})$, which in this case is empty for $\gamma \leq \gamma_{\text{crit}}$. Hence, for the planar massless Dirac operator with Bessel–Macdonald potential, restricted to its positive spectral subspace, there are no bound states in the subcritical region. Thus, with the analysis carried out above, we are in a position to answer the question raised in Ref. [1] by means of the following

**Theorem 3.2** For $\gamma \leq \gamma_{\text{crit}}$, the massless planar Dirac operator with Bessel–Macdonald potential, restricted to its positive spectral subspace, has no eigenvalues and therefore has no bound states.

**Author’s Contribution** All authors contributed equally to this work.

**Data Availability**

The data that support the findings of this study are available from the corresponding author upon reasonable request.

**Declarations**

**Conflict of interest**

On behalf of all authors, the corresponding author states that there is no conflict of interest.

**References**

1. W.B. De Lima, O.M. Del Cima, E.S. Miranda, On the electron-polaron-electron-polaron scattering and Landau levels in pristine graphene-like quantum electrodyamics. Eur. Phys. J. B 93, 187 (2020)
2. M.A.H. Vozmediano, M.I. Katsnelson, F. Guinea, Gauge fields in graphene. Phys. Rep. 496, 109 (2010)
3. G. Brown, D.G. Ravenhall, On the interaction of two electrons. Proc. R. Soc. London A208, 552 (1951)
4. J. Sucher, Foundations of the relativistic theory of many-electron atoms. Phys. Rev. A 22, 348 (1980). (Erratum Phys. Rev. A 23, 388 (1981))
5. J. Sucher, Continuum dissolution and the relativistic many-body problem: a solvable model. Phys. Rev. Lett. 55, 1033 (1985)
6. V.C. Zelati, M. Nolasco, A variational approach to the Brown–Ravenhall operator for the relativistic one-electron atoms. Nonlinear Anal. 136, 62 (2016)
7. B. Binegar, Relativistic field theories in three dimensions. J. Math. Phys. 23, 1511 (1982)
8. E.L. Lieb, R. Seiringer, The Stability of Matter in Quantum Mechanics (Cambridge University Press, 2010)
9. T. Kato, Perturbation Theory for Linear Operators (Springer, 1976)
10. I. Herbst, Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$. Commun. Math. Phys. 53, 285 (1977)
11. A.A. Balinsky, W.D. Evans, R.T. Lewis, The Analysis and Geometry of Hardy’s Inequality (Springer, 2015)
12. I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products*, 7th edn. (Academic Press, 2007)
13. M.B. Alves, O.M. Del Cima, D.H.T. Franco, On the stability and spectral properties of the two-dimensional Brown–Ravenhall operator with an attractive potential of the Bessel–Macdonald type. Submitted to Reviews in Mathematical Physics
14. M. Reed, B. Simon, *Modern Methods of Mathematical Physics. Functional Analysis*, vol. 1 (Academic Press, 1980)

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