A VISUAL TOUR VIA THE DEFINITE INTEGRATION $\int_a^b \frac{1}{x} \, dx$

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Abstract. Geometrically, $\int_a^b \frac{1}{x} \, dx$ means the area under the curve $\frac{1}{x}$ from $a$ to $b$, where $0 < a < b$, and this area gives a positive number. Using this area argument, in this expository note, we present some visual representations of some classical results. For example, we demonstrate an area argument on a generalization of Euler’s limit $\left( \lim_{n \to \infty} \left( \frac{(n+1)}{n} \right)^n = e \right)$. Also, in this note, we provide an area argument of the inequality $b^a < a^b$, where $e \leq a < b$, as well as we provide a visual representation of an infinite geometric progression. Moreover, we prove that the Euler’s constant $\gamma \in \left[ \frac{1}{2}, 1 \right)$ and the value of $e$ is near to 2.7.

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1. Introduction

It is well known that the function $\phi : (0, +\infty) \to \mathbb{R}$, defined by $\phi(x) = \frac{1}{x}$, is a monotone decreasing and continuous. Thus $\phi(x)$ is Riemann integrable on $[a, b]$ where $0 < a < b$. Geometrically, $\int_a^b \frac{1}{x} \, dx$ means the area under the curve $y = \frac{1}{x}$ from $a$ to $b$. Moreover, it is useful to observe that the function $f(t) = \int_1^t \frac{1}{x} \, dx$ is strictly increasing for $t \geq 1$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{\(\ln b - \ln a < \frac{1}{2} \cdot \left( \frac{1}{a} + \frac{1}{b} \right) \cdot (b - a)\).}
\end{figure}

Let $a$ and $b$ be two positive real numbers. Then the fact $\frac{1}{x} + \frac{x}{ab} \leq \frac{1}{a} + \frac{1}{b}$ for $a \leq x \leq b$ (as $(x-b)(x-a) \leq 0$) is equivalent to saying that the line $y = \frac{1}{a} + \frac{1}{b} - \frac{x}{ab}$ lies above the curve $y = \frac{1}{x}$ for $a \leq x \leq b$. Thus, Figure 1 shows that the area under the curve $y = \frac{1}{x}$ from $a$ to $b$ is less than the area of the trapeziums covering it, i.e.,

\[ \ln b - \ln a < \frac{1}{2} \cdot \left( \frac{1}{a} + \frac{1}{b} \right) \cdot (b - a). \]

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Again, the fact \( \frac{1}{x} + \frac{4x}{(a+b)^2} \geq \frac{4}{a+b} \) for \( x > 0 \) (follows from AM-GM inequality) is equivalent to saying that

\[
\ln b - \ln a > \frac{2}{a+b} \cdot (b-a).
\]

the curve \( y = \frac{1}{x} \) lies above its tangent line \( y = \frac{4}{a+b} - \frac{4x}{(a+b)^2} \) at the point \( \left( \frac{a+b}{2}, \frac{2}{a+b} \right) \).

Thus, figure 2 gives the visualization that the area under the curve \( y = \frac{1}{x} \) from \( a \) to \( b \) is greater than the area of the trapezium below it, i.e.,

\[
\ln b - \ln a > \frac{2}{a+b} \cdot (b-a).
\]

2. Tour-1

In a recent note of the American Mathematical Monthly, R. Farhadian \([6]\) made a beautiful generalization of Euler’s limit \( \left( \lim_{n \to \infty} \left( \frac{\binom{n+1}{n}}{n} \right)^n = e \right) \) as follows:

**Theorem 1.** \([6]\) Let \( A_n \) be a strictly increasing sequence of positive numbers satisfying the asymptotic formula \( A_{n+1} \sim A_n \), and let \( d_n = A_{n+1} - A_n \). Then

\[
\lim_{n \to \infty} \left( \frac{A_{n+1}}{A_n} \right)^{\frac{d_n}{A_n}} = e.
\]

Now, we will provide a second proof of it, which is purely pictorial. From Figure 3, it is clear that

\[
\frac{2}{A_n + A_{n+1}} \cdot (A_{n+1} - A_n) < \ln(A_{n+1}) - \ln(A_n) < \frac{1}{2} \left( \frac{1}{A_n} + \frac{1}{A_{n+1}} \right) \cdot (A_{n+1} - A_n),
\]

**Figure 2.** \( \ln b - \ln a > \frac{2}{a+b} \cdot (b-a) \).

**Figure 3.**
i.e.,
\[
\frac{2}{1 + \frac{A_{n+1}}{A_n}} < \ln \left( \frac{A_{n+1}}{A_n} \right) \frac{\Delta n}{\Delta n} < \frac{1}{2} \cdot \left( 1 + \frac{A_n}{A_{n+1}} \right).
\]
Since \( A_{n+1} \sim A_n \), thus \( \lim_{n \to \infty} \left( \frac{A_{n+1}}{A_n} \right) \frac{\Delta n}{\Delta n} = e \).

**Remark 1.** It is well-known that if \( a_n \) is a sequence of positive numbers satisfying \( \lim_{n \to +\infty} a_n = 0 \), then
\[
\lim_{n \to +\infty} (1 + a_n)^{\frac{1}{a_n}} = e.
\]
Here, we will provide a visual proof of it. From the figure, it is clear that

\[
\frac{2}{2 + a_n} \cdot a_n < \ln(1 + a_n) - \ln 1 < \frac{1}{2} \cdot \left( 1 + \frac{1}{1 + a_n} \right) \cdot a_n,
\]
i.e.,
\[
\lim_{n \to +\infty} (1 + a_n)^{\frac{1}{a_n}} = e.
\]

3. Tour-2

Next, we provide a pictorial description of a geometric series
\[
1 + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots = \frac{r}{r - 1}
\]
when \( r > 1 \). Since \( \int_1^r \frac{dx}{x} = \int_1^r \frac{dy}{y} \), thus the area covered by the rectangle \( \{(0,1); (1,1); (1, \frac{1}{r}); (0, \frac{1}{r})\} \) is same as the area covered by the rectangle \( \{(1,0); (r,0); (r, \frac{1}{r}); (1, \frac{1}{r})\} \). Thus
\[
\left( 1 - \frac{1}{r} \right) \cdot 1 = \left( \frac{1}{r} - \frac{1}{r^2} \right) \cdot (r - 1) + \left( \frac{1}{r^2} - \frac{1}{r^3} \right) \cdot (r - 1) + \cdots,
\]
\[
= \frac{(r - 1)^2}{r^2} \cdot \left( 1 + \frac{1}{r} + \frac{1}{r^2} + \cdots \right),
\]
i.e.,
\[
\left( 1 + \frac{1}{r} + \frac{1}{r^2} + \cdots \right) = \frac{r}{r - 1},
\]
which gives the required equality.
The two constants $e$ and $\pi$ have encouraged many visual proofs of the inequality $\pi^e < e^\pi$. In a recent Mathematical Intelligencer note ([2]), the author provided a visual proof of the inequality $\pi^e < e^\pi$. However, their visual proof can be used to show the more general inequality $b^a < a^b$, where $e \leq a < b$.

**Visual Proof-1**

Since $\ln a \geq 1$, thus $\frac{1}{x \ln a} \leq \frac{1}{x}$ for $x > 0$. Thus the Figure 6 shows that the area under the curve $y = \frac{1}{x \ln a}$ from $a$ to $b$ is less than the area of the rectangle PQRS, i.e.,

$$\ln b \ln a - 1 = \int_a^b \frac{dx}{x \ln a} \leq \frac{1}{a} (b-a) = \frac{b}{a} - 1,$$

i.e.,

$$b^a < a^b.$$
Visual Proof-2
Also, Figure 7 shows that the area under the curve \( y = \frac{1}{x} \) from \( a \ln a \) to \( b \ln a \) is less than the area of the rectangle covering it. Since \( e \leq a \), so \( 1 \leq \ln a \), i.e., \( a \leq a \ln a < b \ln a \).

\[
\int_{a \ln a}^{b \ln a} \frac{dx}{x} < \frac{1}{a} \cdot (b - a) \ln a.
\]

\[
\ln b - \ln a < \ln a \cdot \left( \frac{b}{a} - 1 \right),
\]
i.e.,

\[
\frac{\ln b}{\ln a} - 1 < \frac{b}{a} - 1.
\]
Thus

\( a \ln b < b \ln a \), i.e., \( b^a < a^b \).

**Corollary 1.** If we take \( a = e \), then \((a, 0)\) and \((a \ln a, 0)\) will be coincided with \((e, 0)\). Also, \((a, \frac{1}{a})\) and \((a \ln a, \frac{1}{a})\) will be coincided with \((e, \frac{1}{e})\). Thus the figure 7 becomes:

\[
\int_{e}^{b} \frac{dx}{x} < \frac{1}{e}(b - e).
\]

Thus we get \( \ln b - 1 < \frac{b}{e} - 1 \), i.e., \( b^e < e^b \).

**Corollary 2.** By taking \( a = e \) and \( b = \pi \), we get \( \pi^e < e^\pi \) \((2)\).
5. Tour-4

Considering the definition of the number $e$ by the equation

$$1 = \int_{1}^{e} \frac{1}{x} dx,$$

we are explaining that why the value of $e$ is near to 2.7. Basically, we will show that $2.7 < e < 2.75$. Applying the lower bound of the integral $\int_{a}^{b} \frac{1}{x} dx$ (see, Figure 9), we have

\[
\int_{1}^{11} \frac{1}{x} dx = \int_{4}^{9} \frac{1}{x} dx + \int_{6}^{9} \frac{1}{x} dx + \int_{11}^{11} \frac{1}{x} dx > \frac{2}{5} + \frac{2}{5} + \frac{1}{5} = 1,
\]

i.e.,

$$\int_{1}^{11} \frac{1}{x} dx > \int_{1}^{e} \frac{1}{x} dx \Rightarrow e < 2.75.$$

Again, applying the upper bound of the integral $\int_{a}^{b} \frac{1}{x} dx$ (see, Figure 10), we have

\[
\int_{1}^{11} \frac{1}{x} dx < \frac{1}{2} \cdot (\frac{1}{a} + \frac{1}{b}) \cdot (b - a).
\]

Figure 9. $\int_{a}^{b} \frac{1}{x} dx > \frac{2(b-a)}{b+a}$.

Figure 10. $\int_{a}^{b} \frac{1}{x} dx < \frac{1}{2} \cdot (\frac{1}{a} + \frac{1}{b}) \cdot (b - a)$. 

\[
\int_{10}^{27} \frac{1}{x} \, dx = \int_{10}^{12} \frac{1}{x} \, dx + \int_{12}^{15} \frac{1}{x} \, dx + \int_{15}^{18} \frac{1}{x} \, dx + \int_{18}^{21} \frac{1}{x} \, dx + \int_{21}^{24} \frac{1}{x} \, dx + \int_{24}^{27} \frac{1}{x} \, dx < \frac{1}{2} \left( \frac{2}{10} + \frac{2}{12} \right) + \frac{1}{2} \left( \frac{3}{12} + \frac{3}{15} \right) + \frac{1}{2} \left( \frac{3}{15} + \frac{3}{18} \right) + \frac{1}{2} \left( \frac{3}{18} + \frac{3}{21} \right)
\]
\[
+ \frac{1}{2} \left( \frac{3}{21} + \frac{3}{24} \right) + \frac{1}{2} \left( \frac{3}{24} + \frac{3}{27} \right)
\]
\[
< 1,
\]
i.e.,
\[
\int_{1}^{2.7} \frac{1}{x} \, dx < \int_{1}^{e} \frac{1}{x} \, dx \Rightarrow e > 2.70.
\]

6. Tour-5

The Euler’s constant is defined as
\[
\gamma = \lim_{n \to \infty} \gamma_n,
\]
where
\[
\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n.
\]

In this visual tour, we will show that the existence of the Euler’s constant \( \gamma \) and \( \gamma \in \left( \frac{1}{2}, 1 \right) \). Since
\[
1 - \gamma_n = \ln n - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n},
\]
thus \( 1 - \gamma_n \) can be described as the shaded area in the following figure. It is seen from the figure that

{\( 1 - \gamma_n \)} is strictly monotone increasing, and \( 1 - \gamma_n > 0 \). That is \( \{ \gamma_n \} \) is strictly monotone decreasing sequence and \( \gamma_n \) is bounded above by 1.
Next, we define a sequence \( \{A_n\} \), where \( A_n \) is described by the shaded area in the following figure. Then

\[
A_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln(n + 1)
\]

it is seen that \( \{A_n\} \) is strictly monotone increasing and \( A_n > 0 \). Since

\[
A_n = \gamma_n - \ln(n + 1) + \ln n,
\]

thus

\[
\gamma_n > \ln(n + 1) - \ln n,
\]

which means, by Figure 13, that

\[
\gamma_n > \ln(b + 1) - \ln a,
\]

i.e., \( \gamma_n \) is bounded below by 0. Thus the Euler’s constant

\[
\gamma = \lim_{n \to \infty} \gamma_n,
\]

exist, and \( \gamma \in [0, 1) \).
Next, we assume that
\[
\Gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln(n + \frac{1}{2}).
\]

Thus,
\[
\Gamma_{n+1} - \Gamma_n = \frac{1}{n+1} + \ln(n + \frac{1}{2}) - \ln(n + \frac{3}{2}).
\]

Thus by the figure 14,

\[\text{Figure 14. } \ln b - \ln a < \frac{b-a}{a}, \text{ where } b > a > 0.\]

\[
\Gamma_{n+1} - \Gamma_n < \frac{1}{n+1} - \frac{1}{n + \frac{3}{2}} < 0,
\]
i.e., \(\{\Gamma_n\}\) is strictly monotone decreasing sequence.

Again, Figure 14 shows that
\[
\int_{n}^{n+\frac{1}{2}} \frac{1}{x} \, dx < \frac{1}{2n},
\]
and, Figure 15 shows that

\[\text{Figure 15. } \ln b - \ln a < \frac{1}{2}(\frac{1}{a} + \frac{1}{b}) \cdot (b-a).\]
\[
\int_1^n \frac{1}{x} \, dx = \sum_{i=1}^{n-1} \int_i^{i+1} \frac{1}{x} \, dx < \sum_{i=1}^{n-1} \frac{1}{2} \left( \frac{1}{i} + \frac{1}{i+1} \right) = 1 + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n-1} + \frac{1}{2n}.
\]
Thus
\[
\ln \left( n + \frac{1}{2} \right) = \int_1^{n+1} \frac{1}{x} \, dx = \int_1^n \frac{1}{x} \, dx + \int_n^{n+\frac{1}{2}} \frac{1}{x} \, dx < 1 + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n-1} + \frac{1}{n},
\]
Thus \( \Gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln(n + \frac{1}{2}) > \frac{1}{2} \). Hence \( \lim \Gamma_n \) exist and \( \lim \Gamma_n \in \left[ \frac{1}{2}, 1 \right) \).

As \( \gamma_n - \Gamma_n = \ln(n + \frac{1}{2}) - \ln n \), thus, applying Figures 13 and 14, we get
\[
\frac{1}{2n+1} < \gamma_n - \Gamma_n < \frac{1}{2n},
\]
i.e.,
\[
\gamma = \lim_{n \to \infty} \gamma_n = \lim_{n \to \infty} \Gamma_n \text{ and } \gamma \in \left[ \frac{1}{2}, 1 \right).
\]

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