A fixed point theorem involving rational expressions without using Picard iteration

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1 Introduction and Preliminaries

Let $T$ be a self-mapping on a Banach space $(X, \| \cdot \|)$ and $x_0 \in X$. The sequence $\{x_p\}$ is called Picard sequence if $x_p = Tx_{p-1}$ for $n = 1, 2, \cdots$. In this case, the mapping $T$ is said to be a Picard operator.

In [1], the authors set out a different general principle for obtaining fixed points. We recall the main results of their paper.

Theorem 1. [1] Let $T$ be self-mapping on a nonempty closed convex subset $C$ of a Banach space $X$. If for some $x_0 \in C$ there exists a constant $c, 0 \leq c < 1$ such that

$\|x_{p+2} - x_{p+1}\| \leq c \|x_{p+1} - x_p\|$ for all $p = 0, 1, 2, \cdots$, \hfill (1.1)

where

$x_{p+1} := \frac{1}{2} (x_p + Tx_p)$. \hfill (1.2)

Then, $\{x_p\}$ converges to a point $u$ in $C$. If, in addition, there exist the nonnegative constants $a, \beta, \gamma, \delta, 0 \leq \gamma < 1$ such that

$\|Tx_p - Tu\| \leq a \|x_p - u\| + \beta \|x_p - Tx_p\| + \gamma \max\{\|u - Tu\|, \|x_p - Tu\|, \delta \|u - Tx_p\|\}$ \hfill (1.3)

for all $p$ sufficiently large, then $u$ is a fixed point of $T$.

A mapping $\phi : [0, \infty) \to [0, \infty)$ is called a comparison function if it is increasing and $\phi^n(t) \to 0, n \to \infty$, for any $t \in [0, \infty)$. We denote by $\Phi$, the class of the corporation function $\phi : [0, \infty) \to [0, \infty)$. For more details and examples, see e.g. [2]-[6]. Among them, we recall the following essential result.

Lemma 2. (Berinde [2], Rus [3]) If $\phi : [0, \infty) \to [0, \infty)$ is a comparison function, then:

(1) each iterate $\phi^k$ of $\phi, k \geq 1$, is also a comparison function;
(2) $\phi$ is continuous at 0;
(3) $\phi(t) < t$, for any $t > 0$.

Later, Berinde [2] introduced the concept of $(c)$-comparison function in the following way.

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Definition 3. (Berinde [2]) A function \( \varphi : [0, \infty) \rightarrow [0, \infty) \) is said to be a \((c)\)-comparison function if
\( (c_1) \) \( \varphi \) is monotone increasing,
\( (c_2) \) the series \( \sum_{j=1}^{\infty} \varphi^j(t) \) is convergent for any \( t \in [0, \infty) \) (here \( \varphi^j \) denotes the \( j \)-th iteration of \( \varphi \)).

By \( \Psi \) we will denote the set of all \((c)\)-comparison functions.

Very recently, Karapınar ([7]) announced the following interesting result.

Theorem 4. [7] On a nonempty closed convex subset \( C \) of a Banach space \( X \), let \( T \) be a self-mapping. If for some \( x_0 \in C \), there exists a \( \varphi \in \Psi \) such that
\[
\|x_{p+2} - x_{p+1}\| \leq \varphi\left(\|x_{p+1} - x_p\|\right) \quad \text{for all } p = 0, 1, 2, \ldots,
\]
where
\[
x_{p+1} := \frac{1}{2}(x_p + Tx_p).
\]
Then, \( \{x_n\} \) converges to a point \( u \) in \( C \). If, in addition, there exist the nonnegative constants \( a, \beta, \gamma, \delta, \ 0 \leq \gamma < 1 \), such that
\[
\|Tx_p - Tu\| \leq a\|x_p - u\| + \beta\|x_p - Tx_p\| + \gamma \max\{\|u - Tu\|, \|x - Tu\|, \|x - Tx_p\|\},
\]
for all \( p \) sufficiently large, then \( u \) is a fixed point of \( T \).

2 Main results

First, we demonstrate the following useful Lemma:

Lemma 5. Let \( C \) be a nonempty closed convex subset of a Banach space \( X \) and the sequence \( \{x_p\} \) in \( C \). If for some \( x_0 \in C \) there exists a function \( \varphi \in \Psi \) such that
\[
\|x_{p+2} - x_{p+1}\| \leq \varphi\left(\|x_{p+1} - x_p\|\right) \quad \text{for all } p = 0, 1, 2, \ldots,
\]
then the sequence \( \{x_p\} \) converges to a point \( u \in C \).

Proof. Let \( x_0 \in C \) be such that the inequality (2.1) holds. Thus, recursively, from the inequality (2.1) and taking into account the monotony of the function \( \varphi \), it follows that
\[
\|x_{p+1} - x_p\| \leq \varphi\left(\|x_1 - x_0\|\right) \quad \text{for all } p = 0, 1, 2, \ldots
\]
and by \((c_2)\), we have that \( \lim_{p \to \infty} \|x_{p+1} - x_p\| = 0 \), so the sequence \( \{x_p\} \) is asymptotically regular. Moreover, by the triangle inequality and taking \((c_2)\) into account, we get
\[
\|x_{p+n} - x_p\| \leq \sum_{j=p}^{p+n-1} \|x_{j+1} - x_j\| \leq \sum_{j=p}^{p+n-1} \varphi\left(\|x_1 - x_0\|\right)
\]
\[
= U_{n+p-1} - U_{p-1} \to 0 \quad \text{as } p \to \infty,
\]
where \( U_p = \sum_{j=n}^{p} \varphi\left(\|x_1 - x_0\|\right) \). Thereupon, \( \{x_p\} \) is a Cauchy sequence and since, the subset \( C \) is closed, we deduce that \( \{x_p\} \) converges to a point \( u \in C \). \( \square \)

Theorem 6. Let \( C \) be a nonempty closed convex subset of a Banach space \( X \), \( T \) be a self-mapping on \( C \), \( x_0 \) be arbitrary in \( C \) and the sequence \( \{x_n\} \) in \( C \) defined as
\[
x_{p+1} := \frac{1}{2}(x_p + Tx_p).
\]

(2.4)
Then the sequence \( \{x_p\} \) is convergent provided that there exists \( \varphi \in \Phi \) such that
\[
\|x_{p+2} - x_{p+1}\| \leq \varphi(\|x_{p+1} - x_p\|) \quad \text{for all } p = 0, 1, 2, \cdots.
\] (2.5)

Moreover if \( u = \lim_{p \to \infty} x_p \) and there exist the constants \( c_1, c_2 \geq 0, 0 \leq \lambda < 1 \), such that
\[
\|Tx_p - Tu\| \leq c_1 \|x_p - u\| + c_2 \max \left\{ \frac{\|x_{p+1} - u\|}{1 + \|x_{p+1} - u\|}, \|x_{p} - u\| \right\} + \lambda \max \left\{ \frac{\|x_{p} - u\|}{1 + \|x_{p} - u\|}, \|x_{p} - Tu\| \right\}
\] (2.6)

for \( p \) sufficiently large, then \( u \) is a fixed point of \( T \).

**Proof.** Let \( x_0 \in C \) and the sequence \( \{x_p\} \) be defined by (2.4). Thus, keeping in mind (2.10), by Lemma 5 it follows that \( \{x_p\} \) is a convergent sequence. Let \( u \in C \) be the limit of \( x_p \). We claim that \( u \) is a fixed point of the mapping \( T \). Indeed, we can easily see that (2.4) can be rewritten as \( Tx_p - x_p = 2(x_{p+1} - x_p) \). Therefore,
\[
\|x_{p} - x_p\| = 2\|x_{p+1} - x_p\| \leq 2\varphi^p(\|x_{1} - x_0\|) \to 0 \quad \text{as } p \to \infty
\]

But, since \( \lim_{p \to \infty} x_p = u \), taking into account the uniqueness of the limit we get that \( \lim_{p \to \infty} Tx_p = u \). Now, letting the limit of (2.11) as \( p \to \infty \), we have
\[
\|u - Tu\| = \lim_{p \to \infty} \|Tx_p - Tu\|
\]
\[
\leq \lim_{p \to \infty} \left[ c_1 \|x_p - u\| + c_2 \max \left\{ \frac{\|x_p - Tx_p\|}{1 + \|x_p - u\|}, \|x_{p} - u\| \right\} + \lambda \max \left\{ \frac{\|x_p - u\|}{1 + \|x_{p} - u\|}, \|x_{p} - Tu\| \right\} \right]
\] (2.7)

which is a contradiction. Thereupon, \( u = Tu \).

Adding a supplementary condition in Theorem 6, we can assure the uniqueness of the fixed point.

**Theorem 7.** If in Theorem 6 the constants \( c_1, c_2, \lambda \geq 0 \) are such that \( c_1 + c_2 + \lambda < 1 \) then the fixed point of the mapping \( T \) is unique.

**Proof.** Let \( \{x_p\} \) be the sequence defined by (2.4). We know by the previous proof, that \( \{x_p\} \) is convergent to a point \( u \in C \). More than that, \( \lim_{p \to \infty} x_p = \lim_{p \to \infty} Tx_p = u \) and \( Tu = u \).

Supposing than there exists a point \( v \in \mathbb{C} \) such that \( Tv = v \neq u \), by (2.11) we have
\[
\|Tx_p - v\| = \|Tx_p - Tu\| \leq c_1 \|x_p - v\| + c_2 \max \left\{ \frac{\|x_{p+1} - u\|}{1 + \|x_{p+1} - u\|}, \|v - Tx_p\| \right\} + \lambda \max \left\{ \frac{\|v - u\|}{1 + \|v - u\|}, \|v - Tu\| \right\}
\] (2.8)

\[
= c_1 \|x_p - v\| + c_2 \max \left\{ \frac{\|x_{p+1} - u\|}{1 + \|x_{p+1} - u\|}, \|v - Tx_p\| \right\} + \lambda \max \left\{ \frac{\|v - u\|}{1 + \|v - u\|}, \|x_{p+1} - v\| \right\}
\]

\[
= c_1 \|x_p - v\| + c_2 \max \left\{ \frac{\|x_{p+1} - u\|}{1 + \|x_{p+1} - u\|}, \|v - Tx_p\| \right\} + \lambda \max \left\{ 0, \|x_{p+1} - v\| \right\}.
\]
Taking the limit as $p \to \infty$ in (2.8) we get
\[
\|u - v\| \leq c_1 \|u - v\| + c_2 \|u - v\| + \lambda \|u - v\| = (c_1 + c_2 + \lambda) \|u - v\| < \|u - v\|.
\]
This is a contradiction, so that $u = v$. \hfill \Box

**Corollary 8.** Let $C$ be a nonempty closed convex subset of a Banach space $X$, $T$ be a self-mapping on $C$, $x_0$ be arbitrary in $C$ and the sequence $\{x_n\}$ in $C$ defined as
\[
x_{p+1} := \frac{1}{2}(x_p + Tx_p).
\]
Then the sequence $\{x_p\}$ is convergent provided that there exists $0 \leq \kappa < 1$ such that
\[
\|x_{p+2} - x_{p+1}\| \leq \kappa \cdot \|x_{p+1} - x_p\| \quad \text{for all} \quad p = 0, 1, 2, \ldots.
\]
Moreover if $u = \lim_{p \to \infty} x_p$ and there exist the constants $c_1, c_2 \geq 0$, $0 \leq \lambda < 1$, such that
\[
\|Tx_p - Tu\| \leq c_1 \|x_p - u\| + c_2 \max \left\{ \frac{\|x_p - Tx_p\| \|1 + \|u - Tu\|\|}{1 + \|x_p - u\|}, \|u - Tu\| \right\}
\]
\[
+ \lambda \max \left\{ \frac{\|u - Tu\| \|1 + \|x_p - u\|\|}{1 + \|x_p - u\|}, \|x_n - Tx\| \right\}
\]
for all $p$ sufficiently large, then $u$ is a fixed point of $T$. \hfill \Box

**Proof.** Put $\varphi(t) = \kappa t$, with $0 \leq \kappa < 1$, in Theorem 6. \hfill \Box

**Corollary 9.** Let $C$ be a nonempty closed convex subset of a Banach space $X$, and $T, G : C \to C$ be two mappings such that $Gx = \frac{x + Tx}{2}$, for any $x \in C$. Supposing that there exists $0 < \tau < 2$ such that
\[
\|Tx - Ty\| \leq \tau \max \left\{ \frac{\|x - y\|}{1 + \|x - y\|}, \frac{1}{2} \|x - y\|, \frac{1}{2} \|y - Tx\| \right\},
\]
for any $x, y \in C$. If there exist two positive real numbers $a, b$, with $a < \frac{1}{\tau}$ and $b < 1$ such that
\[
\|x - TGx\| \leq a\|T^2x - TGx\|, \quad \text{for any} \quad x \in C;
\]
\[
\|T^2x - \omega\| \leq b\|x - \omega\|, \quad \text{for} \quad \omega \in \{Tx, Gx\},
\]
then there exists at least one point $u \in C$ such that $Tu = u$. \hfill \Box

**Proof.** First of all, by considering the definition of the mapping $G$, we can easily obtain the following relations:
\[
\|x - Tx\| = \|x - 2Gx + x\| = 2\|Gx - x\|,
\]
\[
\|Gx - Tx\| = \|Gx - 2Gx + x\| = \|Gx - x\|,
\]
\[
\|Gx - TGx\| = \|Gx - 2G^2x + Gx\| = 2\|G^2x - Gx\|.
\]
Let $x_0$ be an arbitrary but fixed point in $C$ and the sequence $\{x_p\}$ defined as follows:
\[
x_1 = Gx_0, \ldots, x_p = Gx_{p-1} = G^p x_0.
\]
Using this notation, the relations (2.14) become
\[
\|x_p - Tx_p\| = 2\|x_p - Gx_{p-1}\|,
\]
\[
\|x_{p+1} - Tx_p\| = \|x_{p+1} - x_p\|,
\]
\[
\|x_{p+1} - Tx_{p+1}\| = 2\|x_{p+2} - x_{p+1}\|.
\]
Thus,
\[ \|x_{p+2} - x_{p+1}\| = \frac{1}{2}\|x_{p+1} - Tx_{p+1}\| = \frac{1}{2}\|2x_{p+1} - 2Tx_{p+1}\| = \frac{1}{2}\|(2x_{p+1} - Tx_{p+1}) - Tx_{p+1}\| \]
\[ \leq \frac{1}{4}\|(2x_{p+1} - Tx_{p+1}) - Tx_{p+1}\| + \|Tx_{p} - Tx_{p+1}\| \]
\[ = \frac{1}{4}\|(x_p + Tx_p - TX_p_{p+1}) + \|Tx_{p} - TX_{p+1}\| \]
\[ = \frac{1}{4}\|(x_p - TX_p_{p+1}) + \|Tx_{p} - TX_{p+1}\| \]
\[ \leq \frac{1}{4}\max \left\{ \|x_p - TX_p_{p+1}\|, \|Tx_{p} - TX_{p+1}\| \right\} . \]

In case that \( \max \left\{ \|x_p - TX_p_{p+1}\|, \|Tx_{p} - TX_{p+1}\| \right\} = \|Tx_{p} - TX_{p+1}\| \), by (2.12) and keeping in mind (2.16), we obtain
\[ \|x_{p+2} - x_{p+1}\| \leq \frac{1}{2}\|Tx_{p} - TX_{p+1}\| \]
\[ \leq \frac{1}{2} \max \left\{ \|x_p - x_{p+1}\|, \frac{1}{2}\|x_p - x_{p+1}\| + \|x_{p+1} - x_p\|, \|x_{p+1} - Tx_{p+1}\|, \frac{1}{2}\|x_{p+1} - Tx_{p+1}\| \right\} \]
\[ \leq \frac{1}{2} \max \left\{ \|x_p - x_{p+1}\|, \frac{1}{2}\|x_p - x_{p+1}\| + \|x_{p+1} - x_p\|, \frac{1}{2}\|x_{p+1} - Tx_{p+1}\| \right\} \]
\[ \leq \frac{1}{2} \max \left\{ \|x_p - x_{p+1}\|, \|x_p - x_{p+1}\|, \|x_{p+1} - x_p\|, \frac{1}{2}\|x_{p+1} - x_p\| + \|x_{p+1} - x_{p+2}\| \right\} \]
\[ \leq \frac{1}{2} \max \left\{ \|x_p - x_{p+1}\|, \|x_p - x_{p+1}\|, \|x_{p+1} - x_p\|, \|x_{p+1} - x_{p+2}\| \right\} \]
\[ \leq \frac{1}{2} \max \left\{ \|x_p - x_{p+1}\|, \|x_{p+1} - x_{p+2}\| \right\} . \]

Of course, since in case that \( \max \left\{ \|x_p - x_{p+1}\|, \|x_{p+1} - x_{p+2}\| \right\} = \|x_{p+1} - x_{p+2}\| \), we get a contradiction, it follows that
\[ \|x_{p+2} - x_{p+1}\| \leq \frac{1}{2} \|x_{p+1} - x_p\| , \]
for \( p \in \mathbb{N} \).

In the second case, when \( \max \left\{ \|x_p - Tx_{p+1}\|, \|Tx_{p} - Tx_{p+1}\| \right\} = \|x_p - Tx_{p+1}\| \), we have
\[ \|x_{p+2} - x_{p+1}\| \leq \frac{1}{2}\|Tx_{p} - Tx_{p+1}\| . \]

But, from (2.13), we get
\[ \|x_p - Tx_{p+1}\| \leq a \|T^2x_p - Tx_{p+1}\| , \]
\[ \|T^2x_p - Tx_{p}\| \leq b \|x_p - Tx_p\| = 2b \|x_p - x_{p+1}\| , \]
\[ \|T^2x_p - x_{p+1}\| \leq b \|x_p - x_{p+1}\| , \tag{2.19} \]

and then
\[ |x_{p+2} - x_{p+1}| = \frac{1}{2} |x_p - T x_{p+1}| \leq \frac{a_1 T}{2} \left( |T x_{p+1} - x_{p+1}| + \frac{1}{2} |T^2 x_{p+1} - T x_{p+1}| \right) \]
\[ \leq \frac{a_1 T}{2} \max \left\{ |T x_{p+1} - x_{p+1}|, \frac{1}{2} |T^2 x_{p+1} - T x_{p+1}| \right\} \]
\[ = \frac{a_1 T}{2} \max \left\{ |x_p - x_{p+1}|, b |x_{p+1} - x_p|, |x_{p+1} - x_{p+2}| \right\} \]

Consequently, since \( \frac{a_1 T}{2} < 1 \), we get
\[ |x_{p+1} - x_{p+2}| \leq \frac{a_1 T}{2} |x_{p+1} - x_p|, \tag{2.20} \]

Thereupon, if we denote \( \kappa = \max \{ \frac{T}{2}, \frac{a_1 T}{2} \} \), from (2.18) and (2.18) we have
\[ |x_{p+1} - x_{p+2}| \leq \kappa |x_{p+1} - x_p|, \tag{2.21} \]

for any \( p \in \mathbb{N} \).

On the other hand, from the inequality (2.12), for \( x = x_p \) and \( y = u \), we get
\[ |T x_p - T u| \leq T \max \left\{ |x_p - u|, \frac{1}{2} \frac{|x_p - u|}{T + |x_p - u|}, \frac{|u - T x_p|}{T + |x_p - u|}, \frac{1}{2} |x_p - T u| \right\} \]
\[ \leq \frac{T}{2} |x_p - u| + \frac{T}{2} \max \left\{ \frac{|x_p - T x_p|}{1 + |x_p - u|}, \frac{|u - T x_p|}{1 + |x_p - u|} \right\} \]
\[ + \frac{T}{4} \max \left\{ \frac{|u - T x_p|}{1 + |x_p - u|}, \frac{|x_p - T u|}{1 + |x_p - u|} \right\} \]
\[ \leq \frac{T}{2} |x_p - u| + \frac{T}{2} \max \left\{ \frac{|x_p - T x_p|}{1 + |x_p - u|}, \frac{|u - T x_p|}{1 + |x_p - u|} \right\} \]
\[ + \frac{T}{4} \max \left\{ \frac{|u - T x_p|}{1 + |x_p - u|}, \frac{|x_p - T u|}{1 + |x_p - u|} \right\}. \]

Moreover, choosing \( c_1 = \frac{T}{2}, c_2 = \frac{T}{2} \) and \( \lambda = \frac{T}{2} \), for \( p \) sufficiently large, we get (2.11) and then, by Theorem 6 it follows that \( u \) is a fixed point of \( T \).

**Corollary 10.** Let \( C \) be a nonempty closed convex subset of a Banach space \( X \), and \( T : C \to C \) be a mapping such that \( T^2 = I \). If there exists \( 0 \leq \gamma < 2 \) such that
\[ |T x - T y| \leq \gamma \max \left\{ |x - y|, \frac{1}{2} \frac{|x - T x|}{1 + |x - y|}, \frac{1}{2} \frac{|y - T y|}{1 + |x - y|}, \frac{1}{2} \frac{|y - T x|}{1 + |y - T y|}, \frac{1}{2} \frac{|x - T y|}{1 + |x - T x|} \right\}, \tag{2.22} \]

for any \( x, y \in C \) then we can find at least one point \( u \in C \) such that \( T u = u \).
Thus, Case 1.

and we distinguish two cases:

Case 2.

Let $p \in \mathbb{N}$ (since $\frac{\tau}{2} < 1$).

Case 2. $\max \{ \|x_p - T x_{p+1}\|, \|T x_p - T x_{p+1}\| \} = \|x_p - T x_{p+1}\|$. We have

$$
\|x_{p+2} - x_{p+1}\| \leq \frac{1}{2} \|x_p - T x_{p+1}\| = \frac{1}{2} \|T x_p - T x_{p+1}\|
$$

$$
= \frac{\tau}{2} \max \left\{ \|x_p - T x_{p+1}\|, \|x_{p+1} - x_{p+2}\| \right\}
$$

Consequently, since $\frac{\tau}{2} < 1$, we get

$$
\|x_{p+1} - x_{p+2}\| \leq \frac{\tau}{2} \|x_{p+1} - x_p\|.
$$

and letting $\kappa = \max \{ \frac{\tau}{2}, \frac{\alpha}{2} \}$, from (2.23) and (2.24) we have

$$
\|x_{p+1} - x_{p+2}\| \leq \kappa \|x_{p+1} - x_p\|,
$$
for any $p \in \mathbb{N}$.

Again, letting $x = x_p$ and $y = u$ in (2.22), we get

$$
\|Tx - Tu\| \leq T \max \left\{ \frac{\|x_p - u\|}{1 + \|x_p - T\|}, \frac{\|u - Tu\|}{1 + \|x_p - Tu\|}, \frac{\|Tu - T\|}{1 + \|x_p - Tu\|} \right\}
$$

Moreover, choosing $c_1 = \frac{\|x_p - u\|}{1 + \|x_p - T\|}$ and $c_2 = \frac{\|u - Tu\|}{1 + \|x_p - Tu\|}$, for $p$ sufficiently large, we get (2.11) and then, by Theorem 6 it follows that $u$ is a fixed point of $T$.

\[\square\]

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